A smooth bouncing cosmology with scale invariant spectrum

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Abstract. We present a bouncing cosmology which evolves from the contracting to the expanding phase in a smooth way, without developing instabilities or pathologies and remaining in the regime of validity of 4D effective field theory. A nearly scale invariant spectrum of perturbations is generated during the contracting phase by an isocurvature scalar with a negative exponential potential and then converted to adiabatic. The model predicts a slightly blue spectrum, $n_S \gtrsim 1$, no observable gravitational waves and a high (but model dependent) level of non-Gaussianities with local shape. The model represents an explicit and predictive alternative to inflation, although, at present, it is clearly less compelling.

Keywords: cosmological perturbation theory, physics of the early universe
1. Introduction

Bouncing cosmologies, in which the present era of expansion is preceded by a contracting phase, have been studied as potential alternatives to inflation in solving the problems of standard FRW cosmology. The cosmological history is continued to the infinite past, so that all the problems related to imposing extremely finely tuned initial conditions in the high curvature regime are avoided. The pre-big-bang scenario [1] and the ekpyrotic/cyclic models are different incarnations of these ideas [2,3].

So far, two main problems have prevented these models from becoming serious contenders for inflation.

1. No completely explicit model of a bouncing phase has been found. Both in the pre-big-bang scenario and in the ekpyrotic/cyclic one the bouncing phase lies outside the regime of validity of the effective field theory, so that the issue requires some input from a more fundamental theory as string theory. The study of time dependent solutions in string theory and of the way singularities are resolved is not sufficiently developed to describe a bouncing phase or even to assess whether a bounce is possible altogether [1], [4]–[10].

2. No compelling mechanism for producing an approximately scale invariant spectrum of adiabatic perturbations has been found. The pre-big-bang model predicted a dominant isocurvature mode, which is nowadays vastly incompatible with data. Although some alternatives have been studied, scale invariance does not come out naturally [1]. In the ekpyrotic/cyclic case it was argued that the same field leading to the fast contraction towards the bounce gives rise, with its quantum fluctuations, to an adiabatic, approximately scale invariant spectrum. The issue is not completely straightforward because, as discussed, the bounce is not under full control so that the way perturbations evolve through it cannot be made explicit. The presence of a dynamical attractor simplifies the problem and predictions can be made under very
general assumptions about the bounce [11]. Unfortunately the result is not scale invariant [12]–[14], [11]. One cannot rule out the possibility that the bouncing phase has very peculiar features which change the result [15], but in this case all predictions will depend on the details of the unknown phase.

Recently there has been progress on both these issues and the purpose of this paper is to put these new ingredients together to obtain an explicit and controllable bouncing model with an approximately scale invariant spectrum of density perturbations. Let us discuss how to address the two problems discussed above.

(1) In a recent paper [16] we studied explicit 4D models of a bouncing phase of the Universe. To induce the bounce, the stress energy tensor must violate the null energy condition. In a spatially flat FRW metric this condition corresponds to the inequality \( \dot{H} < 0 \), which is clearly violated at the point where we reverse from contraction to expansion: \( H = 0 \) and \( \dot{H} > 0 \). The violation of the null energy condition is usually associated with catastrophic instabilities which make the theory pathological. However this is not true in complete generality, as it can be shown with the quite general low energy effective field theory approach developed in [16]. An explicit realization can be obtained starting from the ghost condensate model [17] that we are now going to briefly describe. In an expanding Universe one expects that a scalar field \( \phi(t) \) evolving in time is progressively driven to rest by the Hubble friction. However, with a generic Lagrangian invariant under the shift symmetry \( \phi \rightarrow \phi + \text{const} \).

\[
\mathcal{L} = P((\partial \phi)^2),
\]

it is easy to realize that another possibility is that \( \dot{\phi} \) goes to a constant, \( \dot{\phi} \rightarrow M_{\text{gc}}^2 \), in correspondence of a minimum of the function \( P \). The peculiarity of this solution is that, although the scalar keeps on evolving at constant speed, the stress energy tensor is the one of a cosmological constant, so that the background metric can be de Sitter or Minkowski. Fluctuations around this background solution are healthy and this theory has been studied as a consistent modification of gravity in the IR. To understand the relevance of this model in our context one can study the stress energy tensor of the perturbations around the background solution: \( \phi(\vec{x}, t) = M_{\text{gc}}^2 (t + \pi(\vec{x}, t)) \).

\[
T_{\mu\nu} \text{ starts linearly in } \pi \text{ with a term [17]}
\]

\[
T_{\mu\nu} = M_{\text{gc}}^4 \pi \delta_{\mu0}\delta_{\nu0}.
\]

This shows that a fluctuation with \( \pi < 0 \) has negative energy! This property opens up the possibility of violating the null energy condition. In particular an explicit model of a bounce can be obtained adding to the Lagrangian above a suitable potential \( V(\phi) \). We will see for example that a bouncing phase with a constant positive \( \dot{H} \) can be obtained simply with a parabolic potential. In [16] we checked that, with a suitable choice of the model parameters, no dangerous instabilities are present. Besides giving an existence proof of a smooth bounce, our results allow to study the evolution of perturbations across this phase. Nothing exotic happens in these models, so that the general conclusions of [11] apply. Therefore the simple mechanism for producing a scale invariant spectrum of perturbations proposed in the ekpyrotic/cyclic scenarios does not work. The fluctuations in the field leading the contracting phase have a very blue spectrum, which is completely negligible on the scales of cosmological interest. This motivates us to look for an additional source of perturbations.
Recently Lehners et al [18] pointed out that a second scalar field whose energy density is negligible and that evolves along a negative exponential potential could be the source of a scale invariant spectrum of perturbations. These fluctuations can be easily converted to adiabatic and therefore match observations. Early studies of isocurvature perturbations in bouncing scenarios can be found in [19,20].

A possible emerging picture is therefore as follows. The Universe is contracting and its energy density is dominated by a field $\phi$, with a Lagrangian similar to the ghost condensate theory, with the addition of a suitable potential. As in the ekpyrotic/cyclic scenario the contraction satisfies $\dot{H} \ll -H^2$. At a certain point $H$ flips sign until $H$ gets to zero and the Universe starts expanding and smoothly connects, after reheating, to a standard FRW cosmology. During the contracting phase a second field $\psi$ evolves along a negative exponential potential. Perturbations in this ‘isocurvature’ field are then converted into adiabatic ones: as $\phi$ perturbations are very suppressed, this second source gives the leading contribution to cosmological inhomogeneities.

We are going to study the model starting in section 2 from the field $\psi$, which is the source of perturbations and therefore the sector of the model giving all the predictions. We will see that, neglecting gravity, an exactly scale invariant spectrum of perturbations is obtained for any negative exponential potential. Gravity slightly modifies the spectrum and the resulting tilt is blue. The deviation from scale invariance may be very small, but it is not possible to get a red spectrum. This prediction is disfavoured by recent data [21] and the model will therefore be ruled out in the very near future if the detection of a red tilt is confirmed. Additional predictions are the presence of a quite high (but model dependent) level of non-Gaussianity (with a local shape dependence) and an unobservably small level of primordial gravitational waves (as in the ekpyrotic/cyclic scenarios). In section 3 we study the unperturbed evolution led by the $\phi$ field. It is important to stress that all the details are very model dependent and that even the use of a ghost-condensate-like theory should be considered simply as a working example of an healthy bouncing cosmology. As all predictions do not depend on how the contracting and bouncing phases are realized, one can envisage other very different possibilities. Some remarks about the exponential form of the potential, which is the root of scale invariance in our model, is made in section 4 before the conclusions in section 5. In the appendix we study the full Lagrangian of perturbations of $\phi$ and $\psi$, including mixing terms.

2. Exponential is good

Let us for the moment completely neglect gravity. Consider a scalar $\psi$ with standard kinetic term and potential $V(\psi)$. The unperturbed space independent solution $\psi(t)$ will satisfy

$$\ddot{\psi}(t) + V'(\psi(t)) = 0.$$ (3)

Perturbations around it will be denoted by $\delta \psi(\vec{x}, t)$. We want to answer the following question: which form of the function $V(\psi)$ gives rise to a scale invariant spectrum of perturbations $\delta \psi$?

The equation of motion for $\delta \psi$ is

$$\delta \ddot{\psi} + [k^2 + V''(\psi(t))] \delta \psi = 0.$$ (4)
If \( V'' < 0 \), when the gradient term becomes negligible with respect to the mass term the mode ‘freezes’, i.e. the solutions of the equation above are no longer oscillatory. To have a power-law solution for \( \delta \psi(t) \) in this regime one requires \( V''(\psi(t)) \propto t^{-2} \). For \( t < 0 \) the mass term increases (in modulus) and more and more modes freeze.

As we are assuming \( V''(\psi(t)) \propto t^{-2} \), equation (4) has solutions of the form
\[
\psi_k \sim \frac{1}{\sqrt{k}} F(kt)
\]
in the conventional normalization.

Therefore to get a scale invariant spectrum the evolution after freezing must be of the form \( \delta \psi \propto t^{-1} \). This time evolution is solution only if
\[
V''(\psi(t)) = -\frac{2}{t^2}.
\]  
(6)

With this choice, besides the solution \( \delta \psi \sim t^{-1} \), we have a decaying solution \( \delta \psi \sim t^2 \). The explicit solution with the standard normalization is given by
\[
\delta \psi_k = \frac{1}{\sqrt{2k}} e^{-ikt} \left( 1 - i \frac{k}{kt} \right).
\]  
(7)

Clearly the result of equation (6) could have been anticipated, as equation (4) is now the same as the one describing a massless scalar living in de Sitter space (replacing \( t \) with the conformal time \( \eta \)).

Now we should look at equation (6) as an equation for the function \( V(\psi) \). Taking the time derivative of equation (3) we get
\[
\ddot{\psi} - \frac{2}{t^2} \dot{\psi} = 0.
\]  
(8)

Neglecting a decaying term this gives \( \dot{\psi}(t) = -2M/t \), with \( M \) an integration constant with dimension of a mass. Thus we have
\[
\psi(t) = -2M \log(-t) + c.
\]  
(9)

We can now get rid of time in the second derivative of the potential \( V''(\psi(t)) = -2/t^2 \) and then integrate to finally get
\[
V(\psi) = -V_0 e^{\psi/M}.
\]  
(10)

We thus reach the conclusion that, in the absence of gravity, a field moving along any negative exponential potential generates a scale invariant spectrum of perturbations [22]. Notice that while generating say 60 e-folds of scale invariant spectrum the potential varies by \( e^{60} \). This implies that we really need an exponential potential, and that our result is not an artifact of requiring exact scale invariance\(^3\).

We now want to see whether this way of generating a flat spectrum survives when we put gravity back into the game. If the field \( \psi \) is cosmologically relevant, i.e. if its contribution to the total stress energy tensor is significant, then the picture drastically changes due to its mixing with gravity [11]. Therefore we are going to assume on the

\(^3\) Actually we do not know if the perturbations are scale invariant over 60 e-folds, but only in a much narrower range. For sure we need an exponential potential in the observable window of \( \sim 10 \) e-folds.
contrary that the energy of $\psi$ is negligible so that we can forget about its mixing with gravity$^4$.

Let us assume that $\psi$ lives in a contracting Universe with $a \propto |t|^p$, $H = p/t$, with $t < 0$ and $p < 1/3$. When this last inequality is satisfied, the background solution blue-shifts so fast that gets rid of initial curvature, inhomogeneities and anisotropies. In this background the equation of motion for $\psi(t)$ is given by

$$\ddot{\psi} + \frac{3p}{t} \dot{\psi} - \frac{V_0}{M} e^{\psi/M} = 0.$$  \hspace{1cm} (11)

It is straightforward to check that we still have a power-law solution with

$$\dot{\psi} = -\frac{2M}{t} \quad V = -\frac{2M^2(1-3p)}{t^2}.$$ \hspace{1cm} (12)

The total energy of $\psi$ is obviously no longer conserved and it is given by $\rho_\psi = 6pM^2t^{-2}$. This contribution will be negligible with respect to the total energy density of the Universe $\rho = 3M^2_\mathrm{P}H^2 = 3p^2M^2t^{-2}$, for sufficiently small $M$. The solution described above is tuned, as the divergence of $\psi$ and $H$ both happen at $t = 0$. This case is clearly not generic, but if we shift the background solution $H = p(t + t_0)^{-1}$, we see that equation (12) is still an approximate solution of (11) going sufficiently back into the past: $|t| \gg |t_0|$. Let us study the behaviour of perturbations, that are now going to deviate from an exactly scale invariant spectrum.

**Tilt of the spectrum.** In the presence of gravity, the equation describing the perturbations is

$$\delta\ddot{\psi} + \frac{3p}{t} \delta\dot{\psi} + \left[ \frac{k^2}{t^2} - \frac{2(1-3p)}{t^2} \right] \delta\psi = 0.$$ \hspace{1cm} (13)

The equation can be simplified using the conformal time $\eta$ and the variable $u = a\delta\psi$. At first order in $p$ we obtain

$$\frac{d^2u}{d\eta^2} + \left( k^2 - \frac{2-3p}{\eta^2} \right) u = 0.$$ \hspace{1cm} (14)

The solution of this equation with the correct limit at short distance is given by

$$u_k(\eta) = \sqrt{\frac{\pi}{2}} e^{(\pi/4)(1+\sqrt{9-12p})} \sqrt{-\eta} H_{(1/2)}(\sqrt{9-12p}(-k\eta)).$$ \hspace{1cm} (15)

Taking the limit $\eta \to 0$ we get that the function goes, at first order in $p$, as $k^{-3/2+p}$. The tilt is thus blue and given by

$$n_s - 1 = 2p.$$ \hspace{1cm} (16)

As there is no natural, smooth transition between the rapidly contracting phase when perturbations are produced and the bounce, we cannot establish a ‘natural’ value for $n_s - 1$; the deviation from scale invariance can be arbitrarily tiny. On the contrary it is quite natural in inflationary models to have deviations from scale invariance (of both

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$^4$ The precise regime in which this approximation is justified depends not only on the relative contribution to the energy density, but also on the nature of the field that drives the expansion of the Universe. In the appendix we shall determine precisely the conditions for our approximation to be valid; for the moment we just assume that we are in such a regime.

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Journal of Cosmology and Astroparticle Physics 11 (2007) 010 (stacks.iop.org/JCAP/2007/i=11/a=010)
red and blue kind) of order $n_s - 1 \sim \pm N_e^{-1} \sim \pm \text{few \%}$, where $N_e$ is the number of e-folds to the end of inflation. Moreover here we do not have any analogue of the slow-roll expansion, so that there is no reason why the tilt should be approximately a constant or equivalently why the parameter $p$ should not change in time\footnote{In the text we consider the case in which the ratio $H^2/|V''(\psi)|$ is small and constant, which is analytically simpler. However it seems rather tuned to assume that the two quantities $H^2$ and $V''$ vary by many orders of magnitude keeping their ratio constant. To have approximate scale invariance we just need this ratio to remain small, though it can vary significantly while modes of cosmological interest freeze.}: in particular the tight constraints that we have on $p$ on large scales cannot be extrapolated to short scales. The only sharp prediction is that the tilt cannot be red.

Fluctuations in the $\psi$ field will be converted into adiabatic perturbations, which can be described by the usual variable $\zeta$. We postpone to the next section the discussion about possible conversion mechanisms. The conversion will happen when modes of cosmological interest are frozen and gradients are irrelevant, so that the tilt of the spectrum cannot change. At linear order

$$\zeta = \frac{\delta \psi}{M_e} \sim \frac{t_0^{-1}}{M_e},$$

where $M_e$ is a parameter with dimension of a mass, which depends on the conversion mechanism.

Let us now look at the other observables.

**Gravity waves.** Relic gravitational waves will have a very blue spectrum since they are just sensitive to the ratio $H/M_P$ during the contracting phase and will be therefore strongly suppressed on large scales. The situation is very close to ekpyrotic/cyclic models \[23\] and no detection is possible in the foreseeable future.

**Non-Gaussianities.** There are two sources of non-Gaussianity one has to consider. First of all, the exponential potential is non-linear and therefore it sources interactions among different $\delta \psi$ modes. This is similar to what happens in inflationary models with a second field: self-interactions of this isocurvature scalar induce a non-linear evolution, so that the statistics of the field becomes non-Gaussian. The level of non-Gaussianity, i.e. the correction to the linear theory, can be obtained comparing the mass term $\frac{1}{2} V''(\psi) \delta \psi^2$ with the cubic term $\frac{1}{3!} V'''(\psi) \delta \psi^3$. This gives

$$\text{NG} \sim \frac{\delta \psi}{M} \sim \frac{|t_0|^{-1}}{M}.$$ \hspace{1cm} (18)

As $\delta \psi$ increases with time, most of the non-linearity will develop at the very end. The resulting non-Gaussianity is thus of order $1/(|t_0|M)$, where $t_0$ is the latest time where the solution described in the previous section applies.

The leading non-Gaussianity is cubic (quartic and higher terms are suppressed with respect to the cubic by further powers of $\delta \psi/M$) and it will show up as a 3-point function correlator. As non-linearities are dominated by the late time evolution, when gradient terms can be neglected, the momentum dependence of the 3-point function will be of the ‘local’ form. The same happens in multi-field inflationary models where the out of the horizon evolution dominates the non-Gaussianity \[24, 25\].
The explicit calculation, neglecting $O(p)$ corrections, is quite simple. Following Maldacena [27] the 3-point correlator at the final time $t_0$ is given at tree level by\(^{6}\)

$$
\langle \delta \psi_{k_1}(t_0) \delta \psi_{k_2}(t_0) \delta \psi_{k_3}(t_0) \rangle = -i \int_{-\infty + i\epsilon}^{t_0} \langle \delta \psi_{k_1}(t_0) \delta \psi_{k_2}(t_0) \delta \psi_{k_3}(t_0) \rangle H_{\text{int}}(s) \, ds + \text{c.c.}
$$

(19)

where the interaction Hamiltonian is the cubic $\delta \psi$ self-interaction

$$
H_{\text{int}}(t) = \frac{V''(t)}{3!} \delta \psi^3 = -\frac{2}{Mt^2} \delta \psi^3.
$$

(20)

Using the normalized free field solution

$$
\delta \psi_k = \frac{1}{\sqrt{2k}} e^{-ikt} \left( 1 - \frac{i}{kt} \right)
$$

we get

$$
\langle \delta \psi_{k_1}(t_0) \delta \psi_{k_2}(t_0) \delta \psi_{k_3}(t_0) \rangle = (2\pi)^3 \delta \left( \sum_i \vec{k}_i \right) \prod_i 2k_i^3 \left( \frac{1}{t_0 + i k_1} \right) \left( \frac{1}{t_0 + i k_2} \right)
\times \left( \frac{1}{t_0 + i k_3} \right) e^{-i(k_1 + k_2 + k_3)t_0} \int_{-\infty + i\epsilon}^{t_0} \left( k_1 + \frac{i}{s} \right) \left( k_2 + \frac{i}{s} \right) \left( k_3 + \frac{i}{s} \right) \, ds + \text{c.c.}
$$

(22)

We are interested in the leading contribution for small $t_0$, which is given by

$$
\langle \delta \psi_{k_1}(t_0) \delta \psi_{k_2}(t_0) \delta \psi_{k_3}(t_0) \rangle = (2\pi)^3 \delta \left( \sum_i k_i^3 \right) \prod_i 2k_i^3 \left( \frac{1}{8|t_0|^3} \right) \frac{1}{M|t_0|}.
$$

(23)

We recognize in front the expected ‘local’ momentum dependence. With the usual definition of the power spectrum of $\delta \psi$ at time $t_0$: \(\langle \delta \psi_{k_1} \delta \psi_{k_2} \rangle \equiv (2\pi)^3 \delta(k_1 + k_2)k_1^{-3} \cdot \Delta_\psi\),\(\Delta_\psi = 1/(2t_0^2)\), we can rewrite the result as

$$
\langle \delta \psi_{k_1}(t_0) \delta \psi_{k_2}(t_0) \delta \psi_{k_3}(t_0) \rangle = (2\pi)^3 \delta \left( \sum_i k_i^3 \right) \prod_i 2k_i^3 \Delta_\psi^{3/2} \sqrt{\frac{2}{4}} \frac{1}{M|t_0|}.
$$

(24)

We see explicitly that the non-linear corrections are of order $1/(|M|t_0)$ as discussed above.

The local form of non-Gaussianity is usually defined through the relation

$$
\zeta(x) = \zeta_{NL}(x) - \frac{3}{5} \frac{\zeta_{NL}^2}{J_{\text{NL}}^2} \left( \zeta_{NL}(x)^2 - \langle \zeta_{NL}^2 \rangle \right),
$$

(25)

where $\zeta$ is the observed perturbation and $\zeta_{NL}$ is a Gaussian variable. Experimental limits are given in terms of the scalar variable $J_{\text{NL}}$\(^{\text{local}}\). The variable $\zeta$ will be related to $\delta \psi$ as $\zeta(\vec{x}) = \pm \delta \psi(\vec{x})/M_c$, with $M_c$ a mass scale depending on the conversion mechanism. Notice that we have a sign ambiguity: depending on the mechanism, positive $\delta \psi$ will correspond

\(^{6}\) The tree level calculation can be interpreted as the classical non-linearity among the modes generated by the background. There will be quantum corrections to this calculation coming from loop diagrams and thus suppressed by higher powers of $\hbar$ [28]. In our case an interaction of the form $V'' \delta \psi^n/n!$ will give a correction to the 3-point function of order $1/(t_0^2 |E/M|^{n+1})$, where $E$ is the typical energy of the process $E \sim |t|^{-1}$. The cubic term is the most important giving a correction $\sim 1/(|t_0| |M|)$, which is anyway small as the tree level calculation forces $1/(|t_0| |M|) \ll 1$. 

Journal of Cosmology and Astroparticle Physics 11 (2007) 010 (stacks.iop.org/JCAP/2007/i=11/a=010) 8
A smooth bouncing cosmology with scale invariant spectrum

to positive $\zeta$ or vice versa. Possible additional non-linearities in this relationship will be discussed below.

The 3-point function of $\zeta$ will thus be given by

$$\langle \zeta_{\vec{k}_1} \zeta_{\vec{k}_2} \zeta_{\vec{k}_3} \rangle = \pm (2\pi)^3 \delta \left( \sum_i \vec{k}_i \right) \frac{\prod_i k_i^3}{4} \frac{1}{M|t_0|} \Delta^3\zeta^2 \left( \sum_i k_i^3 \right),$$

(26)

which implies

$$f_{\text{local}} = \mp \frac{5\sqrt{2}}{24} \Delta^{-1/2} \frac{1}{M|t_0|}.$$  

(27)

The tightest experimental constraints on $f_{\text{local}}$ are presently coming from the analysis of WMAP 3yr data [21, 26]

$$-36 < f_{\text{local}} < 100 \quad \text{at 95\% C.L.}$$

(28)

This constraint gives

$$\frac{1}{M|t_0|} < 100 \times \frac{24}{5\sqrt{2}} \Delta^{1/2} \approx 6 \times 10^{-3} \quad \text{at 95\% C.L.}$$

(29)

The second source of non-Gaussianity is the conversion between isocurvature perturbations $\delta \psi$ and the adiabatic mode $\zeta$. As everything happens when gradients are irrelevant the relationship between the two variables will be local in real space, but in general not linear

$$\zeta(\vec{x}) = \delta \psi(\vec{x})/M_c + f_{\text{local}}^2 (\delta \psi(\vec{x})/M_c)^2 + \cdots.$$  

(30)

The quadratic correction gives an additional contribution to the local non-Gaussianity calculated in equation (27). What is experimentally constrained is the sum of the two parameters $f_{\text{local}}$. The contribution from the conversion mechanism cannot be further studied without specifying the model; only the local shape dependence of the 3-point function is model independent.

In this section we have derived all the predictions assuming that the energy density of the field $\psi$ is so small that its mixing with gravity can be neglected, at least when modes relevant for cosmology freeze. In the appendix we study the full action describing both $\psi$ and the field $\phi$, which dominates the stress energy tensor. In this way we can check under which conditions the mixing of $\psi$ with gravity can be neglected. If the mixing cannot be disregarded, predictions will change and become more model dependent, as they will depend on the Lagrangian of $\phi$. In this case our sharp prediction of a blue tilt might also change. We leave these scenarios for future study.

3. The whole story

Now that we have discussed the dynamics of perturbations sourced by the $\psi$ field, let us concentrate on the unperturbed history of our bouncing cosmology. We will assume that the entire history, until the beginning of the standard hot FRW era, is dominated by the scalar $\phi$, with the Lagrangian of a ghost condensate theory with the addition of a suitable potential. As it should be clear from the discussion above, this model should be regarded as an explicit example of a smooth evolution from contraction to the bounce
and then to the expanding phase, which is not affected by fast instabilities or other pathologies. As predictions do not depend on the explicit realization of the unperturbed history, alternative models may be found, either at the level of field theory or in which stringy effects are important.

### 3.1. Contracting phase

Although not required, we saw in the previous section that it is simpler to study an evolution with $a \sim |t|^p$ with $p$ constant. Furthermore we demand $p \ll 1$, so that the spectrum is close to scale invariance. We want to find out which form of the potential $V(\phi)$ must be added to the ghost condensate Lagrangian to obtain this evolution of the scale factor. Neglecting terms with more than one derivative acting on $\phi$ we have

$$
\mathcal{L} = \sqrt{-g} [P(X, \phi) - V(\phi)]
$$

For concreteness we can take a functional form $P(X, \phi) = \phi^{-2} \tilde{P}(X)$. In the introduction, in which the shift symmetry of $\phi$ is not only broken by the potential, but also in the generalized kinetic term $P$. The reason for this will become clear later on. For concreteness we can take a functional form $P(X, \phi) = \phi^{-2} \tilde{P}(X)$. In the absence of a potential, it is easy to check that, if $\tilde{P}$ has a minimum where it vanishes ($\tilde{P}'(\phi_0^2) = \tilde{P}(\phi_0^2) = 0$), we have the solution $\phi(t) = \phi_0 t$ with a Minkowski metric (or de Sitter in the presence of a positive cosmological constant). In this case the system behaves as the original ghost condensate, with the only difference that, as $P_{XX}(\phi_0^2, \phi_0(t)) = (\phi_0 t)^{-2} \tilde{P}'(\phi_0^2)$ depends on time, the dynamics of perturbations progressively changes.

The stress energy tensor is given by

$$
T_{\mu\nu} = [P(X, \phi) - V(\phi)]g_{\mu\nu} + 2P(X, \phi) \partial_\mu \phi \partial_\nu \phi.
$$

For sufficiently small departure from $\tilde{P}' = 0$, the part of the stress energy tensor that depends on the generalized kinetic term is linear in $\dot{\pi}$ and it contributes only to the energy density, without pressure. In the Friedmann equation for $\dot{H}$ the contribution of the potential cancels:

$$
M_P^2 \dot{H} = -\frac{1}{2} (\rho + p) = -\frac{1}{2} \times 4P_{XX} \phi_0^4 \dot{\pi} = -\frac{1}{2} \frac{\dot{M}_{gc}^2}{t^2} \dot{\pi},
\quad \dot{M}_{gc}^2 = 4\tilde{P}''(\phi_0^2).
$$

In the last equality we have assumed that the deviation from the solution in the absence of potential, $\phi(t) = \phi_0 t$, is small. Assuming this, we can solve for $\pi$ as a function of $t$:

$$
-M_P^2 \frac{p}{t^2} = -\frac{1}{2} \frac{\dot{M}_{gc}^2}{t^2} \dot{\pi} \Rightarrow \dot{\pi} = \frac{2p M_P^2}{\dot{M}_{gc}^2}.
$$

The velocity of $\phi$ is shifted by a constant amount. The deviation from the solution in the absence of a potential is therefore small if $\dot{\pi} = 2p M_P^2 / \dot{M}_{gc}^2 \ll 1$, as we are going to assume in the following. We can now solve the first Friedmann equation to find $V(\phi)$

$$
V(t) = 3M_P^2 H^2 + 2M_P^2 \dot{H} = \frac{M_P^2 (3p^2 - 2p)}{t^2}.
$$
In the approximation $\dot{\pi} \ll 1$ we have $\phi(t) = \dot{\phi}_0 t$ so that the potential turns out to be
\[
V = -\frac{(2p - 3p^2)M_p^2\dot{\phi}_0^2}{\dot{\pi}^2} + \mathcal{O}(\dot{\pi}).
\] (36)

We are interested in the regime $p \ll 1$ so that we obtain an inverse quadratic potential which is negative and going to $-\infty$ for $t \to 0$. This solution makes sense. In the absence of a potential, starting from $\dot{\pi} > 0$, we would have a matter-domination-like solution, as the contribution to $T_{\mu\nu}$ proportional to $\dot{\pi}$ does not have pressure. This would correspond to $p = \frac{2}{3}$ and in fact $V = 0$ in the equation above for $p = 2/3$. As $t^{-2}\dot{\pi} \propto \dot{H} \propto a^{-2/p}$, for smaller $p$ we need $t^{-2}\dot{\pi}$ to increase faster as a function of the scale factor and this corresponds to a negative tilted potential which induces an increase in the $\phi$ velocity. Vice versa for $p > \frac{2}{3}$.

It is important to stress that the solution above is a dynamical attractor in the same way as in the ekpyrotic/cyclic models. One can check that if one starts with a $\pi$ slightly different from the one of equation (34), this perturbation to the velocity dies as $t$ and the solution goes back to the unperturbed one. Also curvature and anisotropies become irrelevant, as the energy density increases with the scale factor faster than $a^{-6}$ [29]. For a local observer the evolution converges to a single unperturbed history, which turns out to be a crucial simplification to allow following perturbations across the bounce [11]. This last point is not so relevant in our case as we have an explicit bounce and we can follow the perturbations throughout.

To conclude this section we are now going to study perturbations of $\phi$; the purpose is just to check that it is consistent to neglect them and to have only the $\psi$ sector as source of the observed perturbations.

Following the general results of [16] in the appendix we derive the Lagrangian for the usual variable $\zeta$ which describes the scalar perturbations of the system\footnote{What is relevant for observations is the constant mode of $\zeta$ [11]. As we are interested in studying the behaviour of $\phi$ perturbations, we simply disregard the $\psi$ field in this section.}. We simply get
\[
S = \int d^3x \, dt \, a^3(t) \left[ \frac{1}{2} \frac{M_g^2 t^{-2}}{H^2} \dot{\zeta}^2 - \frac{M_p^2}{p} \left( \frac{\partial \zeta}{a} \right)^2 \right].
\] (37)

One can show by symmetry arguments [16] that the coefficient of the spatial kinetic term $(\partial \zeta/a)^2$ is always given by $M_p^2 H/H^2$, independently of the $\phi$ Lagrangian. The time kinetic term is not fixed by symmetries and in our case it is time independent; therefore $\zeta$ modes have a constant speed of sound $c_s = \sqrt{2p} M_p/M_g \ll 1$. The amplitude of a $\zeta$ mode at freezing will be
\[
\zeta \sim \frac{H}{M_g t^{-1}} \frac{1}{c_s^{3/2}},
\] (38)

where $t$ and $H$ must be evaluated at horizon crossing. The amplitude grows as $H$ so that the spectrum is strongly blue. Neglecting $p$-corrections $H_{\text{freezing}}(k) \propto k$, so that the final spectrum goes as $1/k$ instead of the usual $k^{-3}$ for the scale invariant case. This also tells us that the $\zeta$ perturbations are completely negligible on scales of cosmological interest.

As we discussed, in the limit $\dot{H} \to 0$ the spatial kinetic term in (37) vanishes. For this reason it is important to consider higher derivative terms which in the $\pi$ Lagrangian
A smooth bouncing cosmology with scale invariant spectrum
give [17, 16]
\[ -\frac{1}{2} \bar{M}_{gc}^2 (\partial^2 \pi / a^2)^2. \] (39)
This term is the leading spatial kinetic term in the original ghost condensate model for which \( \dot{H} = 0 \) and it is very important to control the stability of the system as we will discuss in the next section.

However, as a consequence of the mixing with gravity, the higher derivative term proportional to \( \bar{M}_{gc}^2 \) induces an additional 2-derivative term in equation (37) of the form
\[ \frac{\bar{M}_{gc}^2 \bar{M}_{gc}^2 t^{-2}}{2H^2 M_P^2} \left( \frac{\partial}{\partial a} \zeta \right)^2 \] (40)
which has the wrong, unstable sign and describes a Jeans-like instability [17]. The system is stable if the term \( (\partial \zeta)^2 \) in equation (37), which has the healthy sign, dominates. This happens for
\[ \bar{M}_{gc} \ll c_s M_P. \] (41)
Restricting to this regime, one can also see that around the time when a mode freezes the term \( (\partial^2 \zeta / a^2)^2 \) is completely negligible, and this justifies our use of the simplified Lagrangian (37) for the study of the fluctuations generated during the contracting phase.

3.2. The bounce

The very same procedure can be used to get the potential for the bouncing phase. For simplicity we come back to the usual ghost condensate with a non-standard kinetic term of the form \( P(X) \), with the shift symmetry broken only by a potential term. In this case we can assume without loss of generality that \( 4P' \dot{\phi}^4 = \dot{\phi}^2 \equiv M_{gc}^4 \).

As an example we take \( H \) to be a linear function of time across the bounce
\[ H(t) = \frac{t}{T^2} \quad \dot{H} = \frac{1}{T^2}. \] (42)
As \( \dot{H} \) is constant equation (33) implies a constant negative \( \dot{\pi} \) so that
\[ \pi = \frac{2M_P^2}{\bar{M}_{gc} T^2}, \] (43)
where we choose the integration constant such that \( \pi \) vanishes when the Universe reverses from contraction to expansion. In the limit \( |\dot{\pi}| \ll 1 \), we get the parabolic potential
\[ V(\phi) = \frac{2M_P^2}{T^2} + \frac{3M_P^2 \dot{\phi}^2}{(\bar{M}_{gc} T)^4}. \] (44)
We start in contraction with a positive potential energy and a negative contribution from the \( \dot{\pi} \). The potential decreases in time, while \( \dot{\pi} \) stays constant until the total energy density goes to zero and the Universe bounces to an expanding phase.

The dispersion relation for the fluctuations of the ghost condensate as calculated in [16] for a generic FRW background is
\[ \omega^2 = -\frac{\bar{M}_{gc}^2 M_{gc}^4}{2M_P^2 M_{gc}^4} k^2 + \frac{\bar{M}_{gc}^2 M_{gc}^4 k^4}{M_{gc}^4} \] (45)
As discussed in the last section, there is a $k^4$ term proportional to $\bar{M}_{gc}^2$, and two terms going as $k^2$. The difference with the last section is that now both the $k^2$ terms have the unstable sign. The one proportional to $\bar{M}_{gc}^2$ describes the Jeans instability which is a consequence of the mixing with gravity. The second term is proportional to $\dot{H}$, as we remarked in the last section, so that it has now the unstable sign.

It is straightforward to check [16] that one cannot make both these sources of instability arbitrarily small: the best compromise, i.e. the choice which minimizes the instability rate of the system, is to make the two instabilities comparable and this happens for $\bar{M}_{gc}^M = \bar{M}_{gc}^2 / \bar{M}_P^2 \simeq H^{1/2}$. In this case it is easy to check that the most unstable mode has a rate of instability $\omega_{\text{max}} \simeq H^{1/2} = 1/T$. As the duration of the bounce is of order $T$, this mode grows by order one and this is clearly not problematic.

One may worry about the relevance of this instability for predictions, as modes of cosmological interest evolve during the bouncing phase. Intuitively however one expects that if the bounce occurs sufficiently fast, the time during which the very long wavelength modes evolve is much shorter than their typical frequency, making the effect of this evolution irrelevant for cosmological observations. Let us check that this is indeed the case and the effect is completely negligible. As one can easily verify, the most unstable mode has a wavelength much smaller than $H^{-1}(T)$, so that for modes of cosmological interest, which are obviously with wavelengths much larger than $H^{-1}(T)$, one can neglect the $k^4$ term and take the dispersion relation to be

$$\omega^2 = -\left(\frac{\bar{M}_P}{\bar{M}_{gc}^2 T}\right)^2 k^2. \tag{46}$$

Each mode will grow by a factor $\omega(k) \cdot T$, where we have taken the duration of the bounce to be of order $T$. Assuming that just after the end of the bounce we begin a hot FRW cosmology, a given mode can be characterized as in inflationary cosmology with its number of e-folds. In this context this just tells us how much its wavelength is larger than the horizon at the beginning of the FRW era. We obtain

$$\omega T = \frac{\bar{M}_P}{\bar{M}_{gc}^2 T} \frac{1}{e^{-N}} \sim \frac{\bar{M}_{gc}}{\bar{M}_P} e^{-N}, \tag{47}$$

where in the second passage we have taken the relation $T^{-1} = \dot{H}^{1/2} = \bar{M}_{gc} M_{gc}^2 / \bar{M}_P^2$, the choice which minimizes the instability rate. The effect is in any case exponentially small and thus completely irrelevant for observations.

The two phases described above, the contraction with constant $p$ and the bounce, must be somehow glued together. One can imagine for concreteness a sudden transition, when the potential for $\phi$ jumps from the negative values required for the contracting phase to the positive ones required for the bounce. In the transition, energy is conserved so that the raise in the potential must be compensated by a negative contribution: $\dot{\pi}$ becomes negative, which is indeed necessary for the bounce to happen. A schematic representation of the potential is shown in figure 1. $\Delta t$ represents the interval of time during which the transition from the contracting to the bouncing solution occurs. Notice that one can make $\Delta t$ much shorter than $H^{-1}$, so that it can be treated as instantaneous for the cosmological history, and at the same time much longer than $M^{-1}$, so that the transition can be described within the regime of the effective theory.
A smooth bouncing cosmology with scale invariant spectrum

![Diagram of potential](image)

**Figure 1.** Schematic representation of the potential of $\phi$ during the contracting and the expanding phase.

The sharp jump in the potential is rather unpleasant. However it is important to stress that something so abrupt is unavoidable in a bouncing cosmology where $H$ must evolve in a short time (of order $H^{-1}$) from a large negative value before the bounce to a large positive value in the expanding phase.

Finally, as shown in figure 1, we imagine that at the end of the bounce the energy associated with the ghost condensate is converted to radiation, and the standard epoch of cosmology begins.

We have not yet discussed the conversion of $\delta \psi$ perturbations to adiabatic ones. One can envisage many possibilities. We assumed that when relevant modes freeze, the field $\psi$ gives a sufficiently small contribution to the energy density to neglect its mixing with gravity (see the appendix). If this ceases to be true later on, a mixing between $\delta \psi$ and $\zeta$ will occur. Alternatively a mixing between $\phi$ and $\psi$ in the potential can source the conversion. In both cases an approximately scale invariant spectrum for $\zeta$ is induced in a way similar to what shown in the appendix and this will be preserved until the expanding phase: the general approach advocated in [11] can in fact be applied and explicitly checked in our smooth bouncing solution. Another possibility is that the bouncing phase itself depends on the isocurvature scalar $\psi$, a sort of ‘variable bounce’ in analogy with the variable decay scenario in inflation [31]. In this particular setting, one could imagine a connection between the sudden transition in the potential and some coupling between $\phi$ and $\psi$. A further possibility is that $\delta \psi$ fluctuations are converted only in the expanding phase, as in the curvaton [30] and variable decay models [31]. Whatever is the mechanism, to find the final value of $\zeta$ one has just to solve the unperturbed history for different values of $\psi$ in the commonly used ‘parallel Universes’ or $\delta N$ approach. This simplification follows from the fact that $\delta \psi$ is relevant only when relevant modes are out of the horizon.

4. Naturalness of the exponential

In inflation the origin of an approximately scale invariant spectrum of perturbations can be traced back to the fact that the inflaton potential is very flat, which implies that the
A smooth bouncing cosmology with scale invariant spectrum

Hubble parameter is almost constant. In our scenario the origin of scale invariance is completely different, as it comes from a precise functional form of the potential \( V(\psi) \), i.e. a negative exponential. It is worth emphasizing that while relevant perturbations are produced, the potential of \( \psi \) changes by many orders of magnitude; an exponential is not a useful approximation to simplify the algebra but a physical request: we need a negative exponential potential over a large region.

One could object that this is not very satisfactory: in inflation we just need a sufficiently shallow potential, while here we have to require a precise functional form, which includes an infinite series of non-renormalizable operators! We are obviously interested in the perturbative region of the exponential, where the potential flattens out and the theory becomes weakly coupled. The flatness of the potential in this region is technically natural, in the sense that loop corrections will be suppressed by derivatives of the potential, which are small. In other words \( \psi \) can be thought as an approximate Goldstone boson in this region. However this does not help in selecting a specific function, among the infinitely many which flatten out at infinity.

The point is that an exponential potential is not a generic infinite series of non-renormalizable operators, but a particular one which emerges in many examples. There are at least two possible origins of exponential potentials. The first one is just a consequence of dimensional reduction as it can be seen in this very simple example. If we take a 5-d theory compactified on \( S^1 \), the dimensional reduction of the Einstein–Hilbert action gives at the level of zero modes

\[
S = \frac{1}{2} \int d^4x \sqrt{-g} M_5^2 TR, \tag{48}
\]

where \( g \) is the 4D metric and \( R \) its Ricci scalar. The radion field \( T \) multiplies the 5D Planck scale \( M_5 \). If we do a conformal rescaling to go to Einstein frame the radion gets a kinetic term of the form

\[
\int d^4x \sqrt{-\tilde{g}} M_4^2 \left( \partial \log T \right)^2, \tag{49}
\]

where \( M_4 \) is the 4D Planck mass. This implies that if the energy of the system depends polynomially on the radius \( T \), for example including a cosmological constant or just at loop level because of the Casimir effect, the canonically normalized radion will have an exponential potential, with a positive or negative overall sign depending on the model. For instance a negative cosmological constant induce a negative exponential potential. The generalization of this trivial example leads to the appearance of exponential potentials in supergravity compactifications and therefore in string theory.

Another possible origin of exponential potentials in through non-perturbative effects. In supersymmetric theories one can have directions in field space which are flat in perturbation theory. These can however be lifted by non-perturbative effects. For instance if the flat direction enters in the gauge coupling, instanton effects will give contribution going as powers of

\[
e^{-(1/\ell^2)}(\psi). \tag{50}
\]

We thus see another possible origin of an exponential potential.

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8 The situation resembles the one of inflationary models with a variation of the inflaton larger than the Planck scale. Without additional symmetries, we need a functional fine tuning to keep the potential sufficiently flat, i.e. we have to control an infinite series of non-renormalizable operators.
We stress that, although an exponential shape of the potential is rather ubiquitous in many scenarios, it is not clear whether the condition $M \ll M_P$ and the request of a negative potential can be fulfilled in string theory \cite{32,33} or in some other UV complete theory.

5. Conclusions

Inflation is a tremendously compelling scenario. However to make progress it is crucial to have alternative models and to work out their predictions, leaving the final word to experiments. It might be dangerous to think that experiments are just measuring the parameters of the only theory we have.

In this paper we described a bouncing cosmology, in which the bounce is smooth and without pathologies. This is indeed possible in a theory with a scalar with a modified kinetic term, like the ghost condensate model with the addition of a potential.

What is new in this scenario with respect to previous bouncing models, like the pre-big-bang scenario and the ekpyrotic/cyclic one, is that the physics of the bounce is explicit and under control. On the contrary, in the other models one must assume that the bounce occurs and its description must remain qualitative as it lies out of the regime of validity of effective field theory and of the present understanding of string theory. This is particularly important when one studies the evolution of perturbations across the bounce, which is crucial to assess the viability of a model. To this smooth model of a bounce we add the mechanism to produce density perturbations recently studied by Lehners et al \cite{18} in the context of the ekpyrotic/cyclic model, that is an isocurvature scalar moving along a negative exponential potential.

We therefore have an explicit and controllable model, in which predictions can be derived: no observable gravitational waves, a high (but model dependent) level of non-Gaussianity with a local shape and, most importantly, a slightly blue spectral index. This model will be ruled out if the present preliminary detection of a red tilt is confirmed by future data.

The model is clearly not very compelling as the Lagrangian of the $\phi$ field seems really ad hoc. This is the result of the fact that it is difficult to construct a non-pathological system which induces a bounce. The stress energy tensor must violate the null energy condition and this usually leads to catastrophic instabilities \cite{16}. Maybe there are much simpler systems leading to a bounce, either at the level of effective field theory or in which quantum gravity is relevant. Notice however that predictions depend only on the isocurvature scalar $\psi$ and therefore are not sensitive to the explicit realization of the contracting phase and of the bounce.

Alternatively, it might be that the complicated Lagrangian for $\phi$ signals the fact that theories which violate the null energy condition cannot be realized, in the sense that our effective field theory cannot be UV completed. A deeper understanding of the implications of the null energy condition is clearly needed.

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Appendix. Gravitational mixing between $\phi$ and $\psi$

In this appendix we provide the explicit Lagrangian for the system of the Ghost Condensate $\phi$ plus the scalar field $\psi$ during the contracting phase. With the resulting Lagrangian, we shall be able to study the conditions under which we can neglect the gravitational fluctuations in the $\delta \psi$ equation of motion, justifying more rigorously what assumed in section 2.

In order to construct the action, we follow closely [27]. We use ADM parametrization of the metric:

$$ds^2 = -N^2 dt^2 + \hat{g}_{ij}(dx^i + N^i dt)(dx^j + N^j dt),$$  \hspace{1cm} (A.1)

where our background solution is of the form:

$$ds^2 = -dt^2 + a^2(t) d\vec{x}^2,$$  \hspace{1cm} (A.2)

where $a(t) \propto |t|^p$, and we consider general metric fluctuations

$$N = 1 + \delta N, \quad N_j = \delta N_j, \quad \hat{g}_{ij} = a^2(t) \delta_{ij} + \delta \hat{g}_{ij}.$$  \hspace{1cm} (A.3)

In this language, the Einstein–Hilbert action takes the form:

$$S_{EH} = \frac{1}{2} M_p^2 \int d^4 x \sqrt{-g} R = \frac{1}{2} M_p^2 \int d^3 x dt \sqrt{\hat{g}} \left[ NR^{(3)} + \frac{1}{N} (E^{ij} E_{ij} - E_{ij}^2) \right],$$  \hspace{1cm} (A.4)

where $R^{(3)}$ is the Ricci scalar of the induced 3D metric. $E_{ij}$ is related to the extrinsic curvature $K_{ij}$ of hypersurfaces of constant $t$,

$$E_{ij} \equiv NK_{ij} = \frac{1}{2} (\partial_t \hat{g}_{ij} - \hat{\nabla}_i N_j - \hat{\nabla}_j N_i),$$  \hspace{1cm} (A.5)

where $\hat{\nabla}$ is the covariant derivative associated to the induced 3D metric $\hat{g}_{ij}$.

The ghost condensate $\phi$ and the scalar field $\psi$ has a background solution of the form:

$$\phi(\vec{x}, t) = \hat{\phi}_0 \cdot (t + \pi_0(t) + \delta \pi(\vec{x}, t)), \quad (A.6)$$
$$\psi(\vec{x}, t) = \psi_0(t) + \delta \psi(\vec{x}, t). \quad (A.7)$$

The action for the Ghost Condensate is given by\footnote{In this appendix we neglect effects due to higher derivative terms that in the main text are proportional to $\bar{M}_{gc}^2$. Following [16], one can show that in order for a comoving mode $\zeta_k$ to be stable before freezing ($\omega(k) \sim H$), $\bar{M}_{gc}$ must be so small that the $k^4$ term becomes irrelevant well before this time. Furthermore, as discussed in the main text, the Jeans instability that $\bar{M}_{gc}$ would induce is subdominant with respect to the $k^2$ term proportional to $\dot{H}$. Here we are interested in studying the gravitational mixing between $\phi$ and $\psi$, and this occurs only around or after the freezing time, so that we can safely neglect the higher derivative terms in our discussion.}:

$$S_{GC} = \int d^4 x \sqrt{-\hat{g}} P(X, \phi) = \int d^3 x dt \sqrt{\hat{g}} N P(X, \phi), \quad (A.8)$$
where $X = -g^{\mu\nu} \partial_{\mu} \phi \partial_{\nu} \phi$. The action for the field $\psi$ is given by:

$$S_\psi = \frac{1}{2} \int d^3 x \, dt \, \sqrt{g} \left[ N^{-1}(\dot{\psi} - N^i \partial_i \psi)^2 - N \ddot{g}^{ij} \partial_i \psi \partial_j \psi - 2NV(\psi) \right], \quad (A.9)$$

where $V(\psi) = V_0 e^{\psi/M}$. We see that $\psi$ and $\phi$ mix only through gravity. As we said in the main part of the paper, at a certain point during the contracting phase, or after the bounce, there must be some direct interaction between the $\phi$ and $\psi$ to allow for the conversion of the isocurvature fluctuations into adiabatic one. This however must occur at least after all the relevant cosmological modes have exit the horizon. This is a much later time with respect to the one we are considering here. For this reason, here we can neglect all direct interactions between $\psi$ and $\phi$.

The ADM formalism is designed so that one can think of $\hat{g}_{ij}$, $\phi$ and $\psi$ as dynamical variables, and $N$ and $N^i$ as Lagrange multipliers. We will choose a gauge for $\hat{g}_{ij}$, $\phi$ and $\psi$ that will fix time and spatial reparametrizations. We find convenient to define the gauge:

$$\delta \pi = 0, \quad \hat{g}_{ij} = a^2(t)(1 + 2\zeta)\delta_{ij}. \quad (A.10)$$

This gauge fixes completely time and space diffeomorphisms at non-zero momentum. It represent a gauge where the ghost condensate is uniform, and taken as the time variable. The dynamical degrees of freedom are $\zeta$ and $\delta \psi$. In the limit where the gravitational mixing of $\psi$ with $\phi$ is negligible, we can think of $\zeta$ as the scalar degree of freedom associated with the ghost condensate. In this gauge, the action for $\phi$ takes the form:

$$S_{GC} = \int d^3 x \, dt \, \sqrt{\hat{g}} \left[ \frac{(-2M^2 + M_p^2 p)}{t^2} \frac{1}{N} \right. \left. + \frac{(1 - 3p)}{t^2} \frac{(-2M^2 + M_p^2 p)}{N} + \frac{1}{2} \frac{\hat{M}_g^2}{t^2} (\delta N)^2 \right]. \quad (A.11)$$

The tadpole terms in (A.11) are chosen in order to ensure that the background solution has exactly the form $a(t) \propto |t|^p$, and $\psi_0(t) = -2M/t, V(\psi_0(t)) = -(2M^2(1 - 3p))/t^2$ as the one we used in the main part of the paper. This means that the form of the function $P(X, \phi)$ used here and the corresponding $\phi_0(t)$ solution will be slightly different (by order $p$ corrections) from the one used in the main part. Thought the physical implications do not change, considering an exact solution simplifies relevantly the study of the action of the system.

To find the action for $\zeta$ and $\delta \psi$, we solve for $N$ and $N^i$ through their equations of motion and plug the result back in the action. This procedure is allowed because $N$ and $N^i$ are Lagrange multipliers. The equations of motion for $N$ and $N^i$ read at first order:

$$\partial_t \left[ \frac{2M}{t} \delta \psi + 2M_p^2 H(t) \delta N - 2M_p^2 \dot{\zeta} \right] = 0,$$

$$2M_p^2 p \left( 1 - 3p \right) \frac{\hat{M}_g^2}{t^2} \delta N + \frac{2M(1 - 3p)}{t^2} \delta \psi + \frac{2M}{t} \dot{\delta \psi} - \frac{2M_p^2 p}{t} \chi$$

$$-\frac{6M_p^2 p}{t} \dot{\zeta} - 2M_p^2 \frac{\partial^2}{\alpha^2} \zeta = 0, \quad (A.12)$$
where for convenience we have defined:
\[ N^i = \partial_i \psi, \quad \partial^2 \psi = \chi. \] (A.13)

We can solve these equations to first order to obtain:
\[ N_1 = -\frac{M}{M_p^2} \delta \psi + \frac{t}{p} \tilde{\chi}, \]
\[ \chi = -\frac{M M_*}{2 M_p^2 p^2} t \delta \psi + \frac{M}{M_p^2 p} \delta \psi + \left( \frac{1}{p} + \frac{M_*}{2 M_p^2 p^2} \right) \tilde{\chi} - \frac{t}{p a^2} (\partial_i^2 \chi). \] (A.14)

In order to find the quadratic action for \( \zeta \) and \( \delta \psi \), we do not need the second order solutions for \( N \) and \( N^i \). The reason for this is that the second order term in \( N \) will multiply the constraint \( \partial L / \partial N \) evaluated at zeroth order, which vanishes since the zeroth order solution obeys the equations of motion. A similar argument holds for \( N^i \).

Substituting in the action \( S = S_{EH} + S_{GC} + S_{\delta \psi} \), after performing some integration by parts, we find an action of the form:
\[ S = \int d^3x \, dt \, a^3 [L_{\zeta} + L_{\psi} + L_{\text{Mixing}}], \] (A.15)

where:
\[ L_{\zeta} = \frac{(2 M_p^2 + M_*^2)}{2 p^2} \left( \frac{\dot{\zeta}}{a} \right)^2 - \frac{M_p^2}{p} \left( \frac{\partial_t \zeta}{a} \right)^2, \]
\[ L_{\delta \psi} = \frac{1}{2} \tilde{\chi} \dot{\psi}^2 - \frac{1}{2} \left( \frac{\partial_t \delta \psi}{a} \right)^2 + \left( 1 - 3 p - \frac{6 M^2}{M_p^2} + \frac{M^2 M_*^2}{2 M_p^2 p^2} \right) \frac{1}{t^2} \dot{\delta \psi}^2, \] (A.16)
\[ L_{\text{Mixing}} = -\frac{M \partial_t \delta \psi + 2 M_p (\partial_t^2 \delta \psi)}{M_p^2 p^2} + 2 \frac{M_p}{p^2} \left( \frac{\partial_t^2 \delta \psi}{a^2} \right) \delta \psi. \]

We see that, neglecting the mixing, the speed of sound of \( \zeta \) is given by:
\[ c_s^2 = \frac{2 p M_p^2}{(2 M_p^2 + M_*^2)} \approx \frac{2 p M_p^2}{M_*^2} \ll 1, \] (A.17)
where in the second and third passage we have used the fact that for the validity of the ghost condensate effective theory we need to have:
\[ \frac{p M_p^2}{M_*^2} \ll 1. \] (A.18)

The speed of sound of \( \zeta \) must be very small. It is useful to write the former Lagrangians in terms of \( c_s \). Keeping only leading order terms in \( c_s \), we obtain:
\[ L_{\zeta} = \frac{M_p^2}{2 p c_s^2} \left( \dot{\zeta}^2 - c_s^2 \left( \frac{\partial_t \zeta}{a} \right)^2 \right), \]
\[ L_{\delta \psi} = \frac{1}{2} (\tilde{\chi} \dot{\psi}^2 - \frac{1}{2} \left( \frac{\partial_t \delta \psi}{a} \right)^2 + \left( 1 - 3 p - \frac{6 M^2}{M_p^2} + \frac{M^2 M_*^2}{2 M_p^2 p^2} \right) \frac{1}{t^2} \dot{\delta \psi}^2, \] (A.19)
\[ L_{\text{Mixing}} = -\frac{2 M_p}{c_s^2 t} \dot{\zeta} \delta \psi + 2 \frac{M_p}{p^2} \left( \frac{\partial_t^2 \delta \psi}{a^2} \right) \delta \psi. \]
If we look at the Lagrangian for $\delta\psi$, we see that the inclusion of gravitational perturbations has produced two effects: the appearance of a direct coupling between $\zeta$ and $\psi$, and a correction to the mass term which is generated when we plug back into $L_{\delta\psi}$ the solutions for $N$ and $N^i$. Obviously, the form of the mixing terms does depend on the nature of the field which leads the contraction. However, there is a regime where $\psi$ is so subdominant that these mixing effects are negligible. In this case, the nature of the field which drives the contraction is irrelevant for most of the results. This is the case in which we concentrated in the paper. Now we are able to explicitly check what is the condition under which this is verified. The condition will turn out to be not exactly $\rho_\psi/\rho_{\text{tot}} \ll 1$, because we shall have to impose the mixing to be so small not to alter the calculation we did for the tilt, and also because the unmixed $\zeta$ has a small speed of sound. It is rather straightforward, comparing for example the terms of mixing with the diagonal terms for $\delta\psi$, that the effects of gravitational mixing are negligible for the $\delta\psi$ equation of motion if

$$\frac{\rho_\psi}{\rho_{\text{tot}}} = \frac{M^2}{pM_P^2} \ll \text{Min}\{p^2c_s^{-1}, p c_s^2\}. \quad (A.20)$$

It is interesting to note that even though this condition is satisfied, which means that $\zeta$ has no effect on $\delta\psi$, still at late times, $t \gg 1/k$, the effect of $\delta\psi$ on $\zeta$ is not negligible. This is just the effect that on large scales, after the particular mode has frozen, different values of $\delta\psi$ induce small differences in the expansion of the separate regions. The induced $\zeta$ is scale invariant and of size:

$$\zeta_{\text{induced}} \sim \frac{M}{M_P^2 t} \lesssim \frac{1}{100} \frac{M^2}{M_P^2} \approx p^2c_s^2 \lesssim 10^{-5}, \quad (A.21)$$

where the first inequality comes from the limit on non-Gaussianities, the second from (A.20), and the third from the fact that, because of the constraint on the tilt of the two point function, $p \lesssim \text{few} \times 10^{-2}$, and $c_s \ll 1$ for the validity of the effective field theory. The induced $\zeta$ is therefore too small, justifying our request for a separate mechanism for conversion of isocurvature perturbations into adiabatic ones.

An alternative interesting regime, different from the one on which we concentrated in this paper, is the one where the inequality (A.20) begins to be violated. In this case, two important things might occur. On the one hand, it is clear from the mass term in $L_{\delta\psi}$ that the factors of $p$ that determine the tilt might be overcome, inducing possibly a red tilt. On the other hand, the induced $\zeta$ might become large enough, without loosing its scale invariance, so that there might be no need for an explicit conversion mechanism between isocurvature and adiabatic perturbations. Though these results would be appealing, the predictions would depend also on the details of the field that drives the contracting phase, and therefore they would be much more model dependent. For this reason, in the main part of the paper we concentrated in the region where the condition (A.20) is satisfied and all the effects coming from mixing with gravity can be safely neglected.

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