A Note on a Picture-Hanging Puzzle

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Abstract

In the picture-hanging puzzle we are to hang a picture so that the string loops around \( n \) nails and the removal of any nail results in a fall of the picture. We show that the length of a sequence representing an element in the free group with \( n \) generators that corresponds to a solution of the picture-hanging puzzle must be at least \( n \sqrt[2]{\frac{\log n}{\log 2}} \).

In other words, this is a lower bound on the length of a sequence representing a non-trivial element in the free group with \( n \) generators such that if we replace any of the generators by the identity the sequence becomes trivial.

1 Introduction

If we hang a picture on a wall, to use more than one nail is useful in case a single nail is not strong enough to support the string so that the picture does not fall. We are interested in hanging a picture by using more than one nail, but assuming that each nail is strong enough to hold the picture’s weight. This is not as boring as it sounds, since we additionally assume that removing any nail results in a fall of the picture.

If such a picture-hanging can be done by using two nails was asked by A. Spivak [6] in 1997. The answer to this puzzle is “yes”, see [1, Figure 1(b)]. Werner Schräzler [5] observed that every such picture-hanging gives rise to Borromean rings [4, p. 10], that is, three loops in \( \mathbb{R}^3 \) that cannot be separated such that no two among them are linked. This leads to a generalization of the puzzle, which is called the 1-out-of-\( n \) puzzle. In the 1-out-of-\( n \) puzzle, we are to hang a picture using \( n \) nails so that removing any nail causes the picture to fall. We refer the reader to the paper by E. Demaine et al. [1] for a detailed account of the history of the problem, and its generalization to the setting of arbitrary monotone boolean functions.

According to [2], it was Neil Fitzgerald who first observed that the 1-out-of-\( n \) puzzle can be mathematically described as a problem about the free group on \( n \)-generators. Every element of such a group is a finite sequence \( s = x_1 \ldots x_n \), where each \( x_i \) is a symbol in the set \( A = \{ a_1, a_1^{-1}, \ldots, a_n, a_n^{-1} \} \). The sequence \( s \) corresponding to a picture supported by \( n \) nails is obtained as follows. Each symbol \( a_i \) in \( A \) represents a particular nail. We realize the construction in \( \mathbb{R}^2 \) so that every nail is represented by a unique point on the \( x \)-axis and the string supporting the picture is a closed curve \( C \) disjoint from the points representing nails intersecting the region \( R = \{(x, y) \in \mathbb{R}^2 | y < 0 \} \). We assume that \( C \) is piece-wise linear and in a general position with respect to the chosen coordinate axis. We start with an empty sequence and traverse the whole

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curve $C$ from a point in $R$ exactly once in an arbitrary sense. Each time we cross the vertical ray emanating from a nail $a_i$ upward from left to right we append the symbol $a_i$ to the so far constructed sequence, and each time we cross the vertical ray emanating from a nail $a_i$ upward from right to left we append the symbol $a_i^{-1}$ to the so far constructed sequence. Thus, in our model we do not allow the string to knot with itself.

A reduction of $s = x_1 \ldots x_n$ is an operation that outputs a sequence obtained from $s$ by deleting a pair of consecutive symbols $x_i$ and $x_{i+1}$, for some $i$, such that $\{x_i, x_{i+1}\} = \{a_j, a_j^{-1}\}$, for some $j$. The sequence $s$ is trivial if by successively applying the reduction to $s$ we can obtain an empty sequence. The picture supported by $n$ nails falls if the sequence $s$ is trivial. Hence, in the 1-out-of- $n$ puzzle we are to construct a non-trivial sequence $s$ containing at least one occurrence of $a_i$ or $a_i^{-1}$, for every $1 \leq i \leq n$ with the following property. For every $i$, $1 \leq i \leq n$, it holds that $s$ becomes trivial if we delete in it every occurrence of the symbol $a_i$ and $a_i^{-1}$. The question we are interested in is how short a sequence $s$ corresponding to a solution can be.

Solution to the 1-out-of-$n$ puzzle. In the trivial case $n = 1$, we deal with the traditional picture-hanging in which the string loops around a single nail. Thus, in this case the shortest solution corresponds to sequence $s_1 = a_1$. For $n = 2$, the shortest solution has length four and the corresponding sequence is $s_2 = a_1a_2a_1^{-1}a_2^{-1}$, in group theory known as commutator of $a_1$ and $a_2$. Already the case of $n = 2$, suggests how to construct a sequence $s_n$ of a solution for an arbitrary $n$ discovered by Fitzgerald and Taylor [3]. Namely, given $s_{n_1}$ and $s_{n_2}$ with disjoint symbol sets we put $s = s_{n_1} s_{n_2} s_{n_1}^{-1} s_{n_2}^{-1}$, where $s^{-1} = x_{m-1} \ldots x_1^{-1}$ for $s = x_1 \ldots x_m$. The shortest sequence $s_n$ obtained by this method is the one constructed recursively by taking $n_1 = \lfloor n/2 \rfloor$ and $n_2 = \lceil n/2 \rceil$. This idea leads to Chris Lusby Taylor’s solution of the 1-out-of-$n$ puzzle, whose corresponding sequence $s$ has length roughly quadratic in $n$, see [1] Section 3 for an analysis of the construction. A question posed therein asks if this is the family of shortest possible solutions. We remark that Michael Paterson showed, see [1] Figure 8, that if we allow the string to twist around itself, that is, not just around nails, solutions of linear complexity in $n$ are possible.

As a first step towards answering this question, we show that $s$ must be of length at least $n2\sqrt{\log_2 n}$. We are not aware of any better previously proved lower bound for this problem than the trivial lower bound of $2n$, for $n \geq 2$, which holds because every symbol must appear an even number of times in the sequence.

Theorem 1. A sequence $s$ corresponding to a solution of the 1-out-of- $n$ puzzle must have length at least $n2\sqrt{\log_2 n}$.

In Section 2 we introduce some additional terminology and observations that the proof of Theorem 1 is based on. We establish the theorem by proving its reformulation, Theorem 4 in Section 3.

2 Preliminaries

Let $s$ be a sequence $x_1x_2 \ldots x_m$ with symbols in $[\pm n] = \{\pm 1, \ldots, \pm n\}$ of cardinality $2n$. A cyclic shift of $s = x_1 \ldots x_m$ is a sequence $s_c = x_c x_{c+1} \ldots x_m x_1 \ldots x_{c-1}$ for some $c \in [m]$. The length of $s$ is $m$. For $I \subseteq [n]$ we denote by $\pm I$ the set $\{\pm i \mid i \in I\}$. The restriction of $s$ to $I \subset [n]$ is the sequence obtained from $s$ by deleting all the symbols that do not belong to $\pm I$, that is, it is the maximal sequence $x_{i_1} \ldots x_{i_k}$ such that $1 \leq i_1 < \ldots < i_k \leq m$ and $x_{i_j} \in \pm I$, for every $1 \leq j \leq k$. We denote by $I = \{i_1, \ldots, i_k\}$ the set of indices corresponding to the restriction of $s$ to $I$. We endow $I$ with the total order $\prec$ such that $i_1 \prec i_2 \prec \ldots \prec i_k$. A cyclic shift of $(I, \prec)$ is a pair $(I, \prec')$, where $\prec'$ is a total order of $I$ such that $i_{c+1} \prec' i_{c+2} \prec' \ldots \prec' i_k \prec' i_1 \prec' \ldots \prec' i_c$ for some $c \in [k]$. 

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Let us assume that the size $k$ of $I$ is even. A nested matching of $(I, <)$ is a set of pairs $M = \{j_1^j, \ldots, j_k^j\}$, where $j_l < j_l'$ for all $1 \leq l \leq k/2$, such that $I = \{j_1, \ldots, j_k/2, j_k/2\}$ and there does not exist $l$ and $p$ for which $j_l < j_p < j_l' < j_p'$, if a sequence $s = x_1x_2\ldots x_m$ is trivial, the successive reductions give rise to a nested matching $M = \{j_1j_1', \ldots, j_{m/2}j_{m/2}'\}$ of $(|m|, <)$ (with the total order $1 < \ldots < m$) such that $x_{j_l} = -x_{j_l}'$, for all $1 \leq l \leq m/2$.

The following definition captures the most crucial property of a solution to the simplified 1-out-of-$n$ puzzle. The symbol $i \in [n]$ is important in $s$ if $x_j = i$ or $x_j = -i$, for at least one value of $i$, $1 \leq i \leq m$, and the restriction of $s$ to $[n] \setminus \{i\}$ is a trivial sequence. Let $I \subseteq [n]$. The sequence $s$ is $I$-good if every $i \in I$ is important in $s$. A solution $s$ to the simplified 1-out-of-$n$ puzzle is a non-trivial sequence with symbols in $[\pm n]$ that is $[n]$-good.

By using the notion of the nested matching defined above it is not to see that $s$ is trivial if and only if its cyclic shift is trivial, and the same holds for the property of $s$ being $I$-good.

**Observation 2.** Let $I \subseteq [n]$ be such that $|I| \geq 2$. A cyclic shift of a nontrivial $I$-good sequence $s = x_1 \ldots x_m$ with symbols in $[\pm n]$ is also a nontrivial $I$-good sequence.

**Proof.** First, we observe that a nested matching $M$ witnessing that a cyclic shift $s$ is trivial, also witnesses that $s$ is trivial. This will prove that a cyclic shift preserves the (non)triviality. A nested matching $M$ of a cyclic shift $(|m|, <')$ of $(|m|, <)$ is also a nested matching of $(|m|, <)$, since a pair $j_lj_l'j_l'p \in M$, for which $j_l < j_p < j_l' < j_p'$, would yield a pair in $(|m|, <')$ contradicting the fact that $M$ is a nested matching in $(|m|, <')$. Thus, a cyclic shift $s_c$ of $s$ is again a nontrivial sequence.

Second, we observe that a cyclic shift preserves the $I$-goodness which will conclude the proof. Let $s_{i}$ be the restriction of $s$ to $[n] \setminus \{i\}$. Let $(J_i = \{j_1, \ldots, j_k\}, <)$ be such that $j_1 < \ldots < j_k$ and $s_i = x_{j_1} \ldots x_{j_k}$. Let $M_i$, for $i \in I$, denote a nested matching of $(J_i, <)$ witnessing the fact that $s_i$ is a trivial sequence. Since $M_i$ is also a nested matching of every cyclic shift of $(J_i, <)$, and the restriction of a cyclic shift $s_c$ of $s$ to $[n] \setminus \{i\}$ is a cyclic shift of $s_i$ the claim follows. 

We are interested only in shortest non-trivial $I$-good sequences. Thus, by the following observation we can assume that the first and last symbol in a sequence are not inverses of each other.

**Observation 3.** Let $I \subseteq [n]$ be non-empty. Let $s = x_1 \ldots x_m$ be a nontrivial $I$-good sequence of minimal length with symbols in $[n]$. It must be that $x_1 \neq -x_m$.

**Proof.** Since $|I| \geq 1$, we have that $m \geq 1$. For the sake of contradiction suppose that $x_1 = -x_m$. 

First, we assume that $|x_1| \in I$. We claim that there exists $x_i$ such that $i \neq 1, m$ and $|x_i| = |x_1| = |x_m|$. Indeed, otherwise $x_2 \ldots x_{m-1}$ is trivial, since $i \in I$, and hence, also $s$ is trivial (contradiction). It follows that $|I| \geq 2$, since otherwise $x_2 \ldots x_{m-1}$ contradicts the choice of $s$.

In the general case $|x_1| \in [n]$, and by the previous paragraph, $I \setminus \{|x_1|\} \neq \emptyset$. We show that $x_2 \ldots x_{m-1}$ is a non-trivial $I$-good sequence which contradicts the choice of $s$ and concludes the proof. Indeed, $x_2 \ldots x_{m-1}$ is not trivial by the argument from the previous paragraph. Furthermore, a nested matching $M_i$, for every $i \in I \setminus \{|x_1|\} \neq \emptyset$, witnessing the fact that the restriction of $s$ to $[n] \setminus \{i\}$ is trivial, must contain $1j$ and $j'm$. Possibly $j = m$ and $j' = 1$ in which case $1j = j'm$. Then $(M \cup \{jj'\}) \setminus \{1j, j'm\}$ is a nested matching witnessing that $i$ is also important in $x_2 \ldots x_{m-1}$. 

\[\square\]
3 Bounding the size of solutions

In this section, we prove the following theorem which immediately implies our main result, Theorem 1

**Theorem 4.** Let \( s = x_1 \ldots x_m \) be a nontrivial \([n]\)-good sequence with symbols in \([\pm n]\). The length of \( s \) is at least \( n 2^{\log_2 n} \).

**Proof.** We assume that the length of \( s \) is smallest possible. Let \( s' \) denote the sequence obtained from \( s \) by replacing in \( s \) every maximal subsequence of consecutive \( \pm i \) with a single \( i \), for every \( i \in [n] \). The replacement induces a natural correspondence between the subsequences of consecutive symbols in \( s \) and \( s' \).

Let \( s' = x'_1 \ldots x'_{m'} \). Let \( i \in [n] \) be a symbol minimizing the size \( k \) of \( J_i = \{ j \mid x'_j = i \} \). By Observation 2 we assume that \( x'_1 = i \). Let \( i_1 < \ldots < i_k \) denote the elements of \( J_i \). Let \( i_{k+1} = m + 1 \). Let \( s'_l = x'_{i_l+1} \ldots x'_{i_{l+1}-1} \), for \( 1 \leq l \leq k \). Let \( s_l \) denote the subsequence of consecutive symbols in \( s \) corresponding to \( s'_l \). By Observation 2 and 3 \( s \) does not contain two consecutive symbols that are inverses of each other. It follows that none of \( s_l \)'s is a trivial sequence, since an element \( jj' \) minimizing \( |j' - j| \) in a matching witnessing, that \( s_l \) is trivial, must have \( |j' - j| = 1 \).

Let \( I_l \) be a set consisting of important symbols in \( s_l \). Let \( m_l \) be the size of \( I_l \). Observe that for every \( j \in [n] \setminus \{ i \} \) there exists \( l \) such that \( j \in I_l \). Indeed, let \( M_j \) denote a nested matching witnessing the fact that the restriction of \( s \) to \([n] \setminus \{ j \} \) is trivial. Let \( j_1, j_2 \in M_j \), \( j_1 < j_2 \), be minimizing \( j_2 - j_1 \) subject to \( x_{j_1} = x_{j_2} = i \) or \( -x_{j_1} = x_{j_2} = i \). Due to the choice \( j_1, j_2 \) it must be that there exists \( l \) such that \( s_l = x_{j_1+1} \ldots x_{j_2-1} \). The subset of \( M_j \) consisting of pairs \( j_3, j_4 \) for which \( j_1 < j_3, j_4 < j_2 \) witnesses that \( j \) is important in \( s_l \). Let us apply a cyclic shift to \( s \) so that \( s_l \) becomes \( s_k \) in the resulting sequence. By Observation 2 the resulting sequence is still nontrivial and \([n]\)-good. Hence, there exist at least two distinct values \( l_1, l_2 \) such that \( j \in I_{l_1}, j \in I_{l_2} \).

By a double counting argument it will follow that

\[
\sum_{i=1}^{k} m_i \geq 2(n - 1) \tag{1}
\]

First, we count the pairs \((j, I_l)\) such that \( j \in I_l \) from the perspective of \( j \)'s. We obtain that \( \sum_j |\{(j, I_l) \mid I_l \ni j \}| \geq 2(n - 1) \). Second, by counting from the perspective of \( I_l \)'s we obtain that \( \sum_l |\{(j, I_l) \mid j \in I_l \}| = \sum_{l=1}^{k} m_l \).

Let \( f(|I|) \) be a function that returns the minimal length of a non-trivial \( I \)-good sequence over \([n]\). Since \( s_l \) is non-trivial we can lower bound the length of every \( s_l \) by \( f(m_l) \). We have \( f(1) = 1, f(2) = 4, f(3) = 10 \). The fact that \( f(2) = 4 \) follows since every symbol must have an even number of occurrences in an \( I \)-good sequence if \(|I| \geq 2 \). To see that \( f(3) = 10 \) we observe the following. Let \(|I| = 3 \). If an important symbol \( i \) has only two occurrences in an \( I \)-good word \( s \) over \([n]\) then a shortest desired word \( s = is_1(-i)s_2 \), where the length of both \( s_1 \) and \( s_2 \) is at least \( f(2) \). Indeed, \( m_1 = m_2 = 2 \) by (1).

Clearly, \( f(n) \geq \sum_{l=1}^{k} f(m_l) + k \), but also \( f(n) \geq nk \) by the choice of \( i \). We can assume that \((*) \quad k < 2^{\sqrt{\log_2 n}} \) since otherwise we are done. The claimed lower bound follows due to the following chain of inequalities.

\footnote{This cannot be improved for all values in \([n] \setminus \{ i \} \) since there exist minimal nontrivial \([n]\)-good sequences in which for some \( j \in [n] \setminus \{ i \} \) we have exactly two distinct such values for any \( n \).}
\[ f(n) \geq \sum_{l=1}^{k} f(m_l) + k \]

**Induction Hypothesis**

\[ \geq \sum_{l=1}^{k} m_l 2^{\sqrt{\log_2 m_l}} + k \]

Convexity of \( n 2^{\sqrt{\log_2 n}} \) and (1)

\[ \geq 2^{(n-1)2 \sqrt{\log_2 n} - 2(n-1) \sqrt{\log_2 n}} + k \]

\[ > 2(n-1)2^{\sqrt{\log_2 n} - 2(n-1) \sqrt{\log_2 n}} \] (2)

Let \( t(n) = 1 + \log_2(n-1) - \log_3 n \). It remains to show that the right hand side of (2) bounds the desired estimate in the last row from the above. To this end we proceed by the following sequence of inequalities, each inequality except for the first one being a consequence of the previous one. It is easy to check that the inequalities in the first row hold for every \( n \geq 4, n \in \mathbb{N} \). The second row is obtained by rearranging the terms in the first row.

\[
\begin{align*}
t(n) - t^2(n) &> 0 > \sqrt{\log_2 n} (2 \log_2(n) - 2 \log_2(n-1) - 1) \\
1 - \sqrt{\log_2 n} + \log_2(n-1) &> t(n) - 2 \sqrt{\log_2 n} n! + \log_3 n \\
\sqrt{1 - \sqrt{\log_2 n} + \log_2(n-1)} &> -t(n) + \sqrt{\log_2 n} \quad \text{(taking } \sqrt{\cdot} < \sqrt{\cdot}) \\
1 + \log_2(n-1) + \sqrt{1 - \sqrt{\log_2 n} + \log_2(n-1)} &> \log_2 n + \sqrt{\log_2 n} \\
\log_2(2(n-1)2^{1-\sqrt{\log_2 n} + \log_2(n-1)}) &> \log_2 \left( n 2^{\sqrt{\log_2 n}} \right) \quad \text{(taking } \log_2(\cdot) < \log_2(\cdot)) \\
2(n-1)2^{\sqrt{\log_2 n} - 2(n-1) \sqrt{\log_2 n}} &> n 2^{\sqrt{\log_2 n}}
\end{align*}
\]

\[
\square
\]

**Tightness of Theorem 4** We think that the construction of Taylor described in the introduction is optimal and therefore we suspect that the lower bound in Theorem 4 is far from the true minimal value, which should be quadratic in \( n \). Our believe is supported by a computer assisted verification of the optimality of the construction up to \( n = 5 \) carried out by Dumitrescu and Ghosh [2].

Note that the proof of Theorem 4 establishes a slightly stronger claim. Namely, the lower bound of \(|I|2^{\sqrt{\log_2 |I|}}\) on the length of a non-trivial I-good sequence \( s \) over \([\pm n]\), where \( I \subseteq [n] \). Therefore it is natural to ask if every (some) non-trivial I-good sequence over \([\pm n]\) of minimal length has symbols only in \( I \). We conjecture that this is the case.

**Conjecture 5.** Every non-trivial I-good sequence, where \( I \subseteq [n] \), over \([\pm n]\) of minimal length has symbols only in \( \pm I \).

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