Periodic solutions and numerical simulations for composite laminated circular cylindrical shell

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Abstract. Periodic solutions and numerical simulations for a composite laminated circular cylindrical shell under the parametric excitation of temperature are investigated in this paper. By introducing some transformations and defining a Poincaré displacement map, some results, including the existence condition for periodic solutions, least upper bound of the number of periodic solutions and the parameter control conditions, are obtained. To demonstrate the applicability and validity of our theoretical results, the phase portraits of the periodic solutions with different values of the detuning parameter are presented by numerical simulations.

1. Introduction

Composite materials are widely used in engineering applications because of the good mechanical properties [1-3]. Composite circular cylindrical shell structures, as light weight and high strength structural components, are often used in aeronautic, astronautic, mechanical engineering, etc. areas. The mechanical model of composite laminated circular cylindrical shell, described by high-dimensional nonlinear system, exhibits complicated vibrations [4-5] as the nonlinear modal interactions, which may lead to severe damages to the components of aircrafts, aerospace vehicles, etc. Hence, the study on the nonlinear dynamic behaviors of composite laminated circular cylindrical shell has important theoretical and application value for vibration controlling.

The periodic vibration of composite laminated circular cylindrical shell, as a very important type of vibration, has aroused great interest among researchers in recent years. For a high-dimensional system, the study on the theoretical analysis of periodic solutions bifurcation is much more sophisticated compared with the planar system, and numerical methods become important tools for the investigations. Amabili [6] investigated the nonlinear response, including periodic vibration, quasi-periodic vibration etc., in the case of 1:1:2 internal resonances of circular cylindrical shells by Amabili–Reddy shear deformation theory and numerical methods. Pellicano and Barbieri [7] studied the dynamic phenomenon of circular cylindrical shells under the inertial axial loads and found that there exists no periodic solution in the instability region by the method of path following numerical. Zhang et al established some models for the composite circular cylindrical shell clamped along the axial direction [8-9] and with double membranes [10-11] based on Hamilton’s principal, and discussed the periodic motions in the case of 1:1 and 1:2 internal resonance by numerical methods, respectively. However, the numerical methods for studying the periodic vibration of high-dimensional system, with
huge amount of computations, are difficult to provide a comprehensive understanding on the dynamic behaviors for all the parameters in different regions. Hence, it is urgent to develop a theoretical method for studying the periodic solutions of high-dimensional system to facilitate the real applications of composite laminated circular cylindrical shell.

In fact, some achievements have been made in the periodic solutions of high-dimensional system. Many effective methods, such as Poincaré map [12], Melnikov method [13], and averaging method [14] etc., were developed to detect the existence and number of periodic solutions. Barreira et al. [15-16] proved the existence of periodic solutions for a kind of three-dimensional system by generalizing Poincaré-Pontryagin theorem, and applied the result to Volterra system as an application. Sun et al. [18] studied the existence of periodic solutions for a degenerate four-dimensional perturbed Hamilton system by improving the Melnikov method, and applied it to a two-degree-of-freedom model in engineering application. Li et al. [19] studied the existence and number of periodic solutions for a four-dimensional slow-fast system and the application to a honeycomb sandwich plate by the methods of curvilinear coordinate and Poincaré map. However, the certain existing results can’t satisfy the need of the practical engineering, and it is important to provide a method to solve the problems appearing in real applications.

This overall structure shows as follows. In section 2, the model of composite laminated circular cylindrical shell is presented, and the four-dimensional averaged equation is obtained by multiple scales method. In section 3, the existence condition for the periodic solutions, least upper bound of the number periodic solutions and the parameter control conditions of composite laminated circular cylindrical shell are discussed. In section 4, the periodic solutions are shown by numerical simulations. In section 5, conclusions in this paper are presented.

2. The model of composite laminated circular cylindrical shell

In this section, we focus on the composite laminated circular cylindrical shell model under the parametric excitation of temperature which can be described by the following two-degree-of-freedom dynamic system [11],

\[
\dot{x}_i + (\omega_i^2 + \beta_{ij} T \cos(\Omega t)) x_i + (\beta_{j2} + \beta_{ij} T \cos(\Omega t)) x_{j-1} + \beta_{j4} x_i + \beta_{j5} \dot{x}_i + \beta_{j6} x_1 + \beta_{j7} \dot{x}_2 + \beta_{j8} x_2 + \beta_{j9} \dot{x}_2 + \beta_{j10} x_1 + \beta_{j11} \dot{x}_1 + \beta_{j12} x_3 + \beta_{j13} \dot{x}_3 + \beta_{j14} x_3 + \beta_{j15} \dot{x}_3 + \beta_{j16} x_2 + \beta_{j17} \dot{x}_2 + \beta_{j18} x_2 + \beta_{j19} \dot{x}_2 + \beta_{j20} x_3 + \beta_{j21} \dot{x}_3 + \beta_{j22} x_3 + \beta_{j23} \dot{x}_3 + \beta_{j24} x_2 + \beta_{j25} \dot{x}_2 = 0
\]

(1)

where \( i = 1,2, \) “..” denotes the derivative with respect to the variable \( t \). The notations \( \omega_i \) and \( \beta_{ij} \)

\( j = 1,2,\ldots,24 \) denote natural frequencies and non-dimensional coefficients, respectively.

Consider the following relationships

\[
\omega_i^2 = \frac{1}{4} \Omega^2 + \varepsilon \sigma_i, \quad \omega_j^2 = \frac{1}{4} \Omega^2 + \varepsilon \sigma_j
\]

(2)

where \( 0 < |\varepsilon| < 1 \), \( \sigma_i (i = 1,2) \) are two detuning parameters. Supposing \( \Omega = 2 \) for convenience and introducing the following scale transformations,

\[
\beta_{ij} \to \varepsilon \beta_{ij}, \quad \beta_{j24} \to \varepsilon^2 \beta_{j24}
\]

(3)

where \( i = 1,2, \) \( j = 1,2,\cdots,23 \), the four-dimensional averaged equation of system (1) can be obtained as follows based on multiple scales method

\[
y = By + F(y)
\]

(4)

where \( y = (y_1, y_2, y_3, y_4)^T \in \mathbb{R}^4 \), \( B \in \mathbb{R}^{2^4} \), \( B_i \in \mathbb{R}^{2^{2^2}} \), \( B = \begin{bmatrix} B_1 & 0 \\ 0 & B_2 \end{bmatrix}, \quad B_1 = \begin{bmatrix} 0 & -\alpha \\ \alpha & 0 \end{bmatrix}, \quad \alpha = \frac{\sigma_1}{2}, \)

\( F = (F_1, F_2, F_3, F_4)^T \), and the expressions of \( F_j(y) (j = 1,2,3,4) \), depending on the expressions of system (1), are exhibited in appendix.
Now system (1) is transformed into system (4), next we study the periodic solutions and numerical simulations of composite laminated circular cylindrical shell based on system (4).

3. Periodic solutions of composite laminated circular cylindrical shell

The model of composite laminated circular cylindrical shell exhibits extremely complicated periodic vibration behaviours. In this section, we will study the periodic solutions by introducing some transformations and defining a Poincaré displacement map.

3.1. Transformations and Poincaré displacement map

First we will transform system (4) into a new form by introducing a series of transformations, and the following lemma can be obtained.

**Lemma 1** By the following transformation,

\[
(\beta, \sigma, \tau) = (\varepsilon \beta, \varepsilon \sigma, \alpha^{-1} \tau)
\]

system (4) becomes

\[
\frac{dy}{d\tau} = J y + \alpha^{-1} \varepsilon F(y)
\]

where \(i = 1, 2, j = 1, 2, \ldots, 20, J = \begin{pmatrix} J_1 & 0 \\ 0 & J_1 \end{pmatrix}, J_1 = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} \).

System (6) can be reduced to a linear centre system when \(\varepsilon = 0\), and we are concerned with the periodic solutions bifurcating from centre singular point when \(\varepsilon \neq 0\). Writing system (6) in the cylindrical coordinates, the following lemma can be obtained.

**Lemma 2** By the following transformations,

\[
y_1 = r \cos \theta, \quad y_2 = r \sin \theta, \quad y_k = u_{k-1}, \quad k = 3, 4
\]

system (6) can be written as,

\[
\frac{du}{d\theta} = \varepsilon H(\theta, u) + O(\varepsilon^2)
\]

where

\[
u = (r, u_1, u_2)^T, \quad H = (H_1, H_2, H_3)^T
\]

\[
H_1(\theta, u) = \alpha^{-1} (F_1(\theta, u) \cos \theta + F_2(\theta, u) \sin \theta)
\]

\[
H_{k-1}(\theta, u) = \alpha^{-1} F_k(\theta, u)
\]

**Proof** Considering expressions (7), we obtain

\[
\frac{dr}{d\tau} = r \cos \theta \frac{d\theta}{d\tau} = -r \sin \theta + \varepsilon \alpha^{-1} F_1(\theta, u)
\]

\[
\frac{d\theta}{d\tau} = r \cos \theta \frac{d\theta}{d\tau} = r \cos \theta + \varepsilon \alpha^{-1} F_2(\theta, u)
\]

\[
\frac{du_{k-1}}{d\tau} = \varepsilon \alpha^{-1} F_k(\theta, u), \quad k = 3, 4
\]

Hence

\[
\frac{dr}{d\theta} = \varepsilon \alpha^{-1} (F_1(\theta, u) \cos \theta + F_2(\theta, u) \sin \theta)
\]

\[
\frac{d\theta}{d\tau} = 1 - \varepsilon^{-1} \alpha^{-1} (F_1(\theta, u) \sin \theta - F_2(\theta, u) \cos \theta)
\]

\[
\frac{du_{k-1}}{d\tau} = \varepsilon \alpha^{-1} F_k(\theta, u)
\]

Considering the above equations, we obtain

\[
\frac{dr}{d\theta} = \varepsilon \alpha^{-1} (F_1(\theta, u) \cos \theta + F_2(\theta, u) \sin \theta) \left(1 + \sum_{j=1}^{\infty} \left(\varepsilon^{-1} \alpha^{-1} (F_1(\theta, u) \sin \theta - F_2(\theta, u) \cos \theta) \right)^j \right)
\]
\[
\begin{align*}
\frac{du_{t+1}}{d\theta} & = e\alpha^{-1}F_1(\theta, u)\cos \theta + F_1(\theta, u)\sin \theta + O(\varepsilon^2) \\
& = e\alpha H_1(\theta, u) + O(\varepsilon^2)
\end{align*}
\]

This completes the proof.

Next we will define a Poincaré displacement map for system (8).

We denote \((\varepsilon, \theta, z)\) as the solution, with initial condition \(z(0, \varepsilon) = z\), of system (8), where \(\theta \in S^1, \ z = (r_0, u_{20}, u_{30}) \in \mathbb{R}^3\). Define a global cross section \(\Sigma\) as follows

\[
\Sigma = \{(u, \theta)\mid \theta = 0\} \in \mathbb{R}^3 \times S^1
\]

The Poincaré displacement map of system (8) is defined as follows,

\[
P(\varepsilon, \theta, z) : \Sigma \rightarrow \Sigma : u(0, \varepsilon, z) \rightarrow u(2\pi, \varepsilon, z)
\]

and the displacement function of system (8) is defined as follows

\[
\Phi(\varepsilon, \theta, z, \varepsilon) = e^{-1}(u(2\pi, \varepsilon, z) - u(0, \varepsilon, z))
\]

where \(\Phi = (\Phi_1, \Phi_2, \Phi_3)^T\). Based on the definitions, the study on the exact expression of displacement function is very important for our investigation, and the following result holds.

**Lemma 3** For the solution \(z = (r_0, u_{20}, u_{30}) \in \mathbb{R}^3\) of system (8), the expression of displacement function with respect to the solution \(z\) can be written as follows

\[
\Phi(\varepsilon, \theta, z) = h(z) + O(\varepsilon)
\]

where

\[
h = (h_1, h_2, h_3)^T
\]

\[
h_1(z) = \alpha^{-1}\left(-3\beta_{10}r_0^3 + 2\beta_{21}u_{20}^2 + \beta_{11}u_{30}^2\right)
\]

\[
h_2(z) = \alpha^{-1}\left(-6\beta_{21}u_{20}^2 + 2\beta_{31}u_{30}^2 - (4\beta_{21}u_{20}^2 + 2\beta_{25})u_{20}
\right.

\[
- (4\beta_{21}u_{20}^2 + 2\sigma_2 + 2\beta_{21}u_{30}^2 - \beta_2 T)u_{30})
\]

\[
h_3(z) = \alpha^{-1}\left(-6\beta_{21}u_{20}^2 + 2\beta_{21}u_{30}^2 + \beta_{25}u_{30}
\right)

\[
+ (4\beta_{21}u_{30}^2 + 2\beta_{25})u_{30})
\]

**Proof** Substituting \(u(\theta, z, \varepsilon)\) into system (8), we obtain

\[
\frac{\partial u(\theta, z, \varepsilon)}{\partial \theta} = eH(\theta, u(\theta, z, \varepsilon)) + O(\varepsilon^2)
\]

Taylor series expansion of the expression \(u(\theta, z, \varepsilon)\) can be written as

\[
u(\theta, z, \varepsilon) = z + au(\theta, z) + O(\varepsilon^2)
\]

where \(u = (r, u_{20}, u_{30})^T\). Since \(u(0, \varepsilon, z) = z\), we obtain \(u(0, \varepsilon) = 0\). Substituting equation (10) into system (9), the expansions of \(\frac{\partial u(\theta, z, \varepsilon)}{\partial \theta}\) and \(H(\theta, u(\theta, z, \varepsilon))\) for \(\varepsilon\), at \(\varepsilon = 0\), can be expressed as follows

\[
\frac{\partial u(\theta, z, \varepsilon)}{\partial \theta} = e^2u(\theta, z) + O(\varepsilon^2)
\]

and

\[
H(\theta, u(\theta, z, \varepsilon)) = H(\theta, z + au(\theta, z) + O(\varepsilon^2))
\]
\[ H(\theta, z) + \sum_{i=1}^{\infty} \partial^i H(\theta, z) = H(\theta, z) + O(\varepsilon) \] (12)

where

\[ \partial^i H(\theta, z) = \sum_{\gamma_1,\gamma_2,\gamma_3} \left( \partial^i H(\theta, z) \right) \gamma_1 \partial \varepsilon^\gamma \partial \varepsilon^\gamma \partial \varepsilon^\gamma \]

Based on (9), equating the coefficients of \( \varepsilon^1 \) at both sides of system (9), we obtain

\[ \frac{\partial u_i(\theta, z)}{\partial \theta} = H(\theta, z) \] (13)

Based on \( u_i(0, z) = 0 \), we obtain

\[ \Phi(z, \varepsilon) = \varepsilon^1 \left( a_i(2\pi, z) + O(\varepsilon^2) \right) = \int_0^{2\pi} H(\theta, z) d\theta + O(\varepsilon) \]

Write \( h(z) = \int_0^{2\pi} H(\theta, z) d\theta \) for convenience. With the aid of Maple and Matlab software, the exact expression of \( \Phi(z, \varepsilon) \) can be obtained.

This completes the proof.

3.2. The periodic solutions of composite laminated circular cylindrical shell

In this section, we will discuss the existence condition for periodic solutions, upper bound of the number of periodic solutions and the parameter control conditions of composite laminated circular cylindrical shell by discussing the zero solutions of displacement function.

First we give the following result on the existence condition for periodic solutions of system (8).

**Theorem 1**

When \( h(z) \) is not identically zero, if there exist \( z^* = (r_0^*, u_{20}^*, u_{30}^*) \) such that \( h(z^*) = 0 \) and \( \frac{\partial h(z)}{\partial z} \) at \( z^* \neq 0 \), there exist an isolated periodic solution of system (8) which is close to \( z^* \).

**Proof**

Note \( h(z^*) = 0 \), then \( \Phi(z^*, 0) = h(z^*) + O(0) = 0 \). Since

\[ \left. \frac{\partial \Phi(z, \varepsilon)}{\partial z} \right|_{z^*, 0} = \frac{\partial h(z)}{\partial z} \neq 0 \]

there exists a neighborhood \( V_0 \) of \( \varepsilon = 0 \) and \( U_0 \) of \( (z^*, 0) \) such that there exists a unique solution \((z^*(\varepsilon), \varepsilon)\) which satisfies \( \Phi(z^*(\varepsilon), \varepsilon) = 0 \) based on the implicit function theorem. By the definition of Poincaré displacement map, system (8) has a periodic solution which is close to \( z^* \).

This completes the proof.

Based on the theorem 1, the number of periodic solutions of system (8) can be estimated by the isolated solutions of \( h(z) = 0 \). Hence, to study the number of periodic solutions of system (6), we need to find the solutions of \( h(z) = 0 \) which satisfy \( r_0 > 0 \) based on the transformation in lemma 2. Next we will discuss the upper bound of the number of periodic solutions for system (6) and the parameter control conditions.

**Theorem 2**

Suppose \( -9\beta_{211}\beta_{110} + 4\beta_{210}\beta_{113} = 0 \). The following statements hold for system (6).

(a) The upper bound of the number of periodic solutions is 5.

(b) The 5 periodic solutions can be obtained when the following condition (C1) or (C2) holds for \( \beta_{10}\beta_{110} < 0 \), where

(C1) \( 2\beta_{211}\beta_{14} - 3\beta_{110}\beta_{25} \neq 0, (\beta_{211}T)^2 - 4a^2 > 0, -b_1c_1 > 0, g_i > 0, i = 1, 2 \).

(C2) \( 2\beta_{211}\beta_{14} - 3\beta_{110}\beta_{25} = 0, f_1 > 0, f_2 > 0, i = 1, 2 \).
where the expressions of $a_i, b_i, c_i, g_i, f_{i1},$ and $f_{i2},$ depending on the coefficients of system (6), are shown in appendix.

**Proof**
Based on theorem 1, we will discuss the solutions of $h(z) = 0$ by the following three cases:

1. $u_{20}u_{30} \neq 0$; $u_{20}u_{30} = 0, u_{20}^2 + u_{30}^2 \neq 0$; $u_{20} = u_{30} = 0$.

(1) $u_{20}u_{30} \neq 0$. Due to the physical significance of the parameter $\beta_{110}$ in real engineering applications, we obtain $\beta_{110} \neq 0$. The following expressions can be obtained based on $h(z) = 0$,

$$h_1(z) = 3\beta_{110}u_3^2 + (2\beta_{113}u_{30}^2 + 2\beta_{111}u_{20}^2 + \beta_{114})z = 0$$

$$h_2(z) = 3\beta_{211}(u_{20}^2 + u_{30}^2) + 2\beta_{213}u_{20}^2u_{30} + \beta_{214}(u_{20}^2 + u_{30}^2) - \beta_{21}Tu_{20}u_{30} = 0$$

$$h_3(z) = 6\beta_{212}u_{20}^2u_{30} + 6\beta_{211}u_{20}^2u_{30}u_{30} - (4\beta_{219}u_0^2 + 6\beta_{214}u_0^2 + \beta_{21}T + 2\sigma_z)n_0 = 0$$

$$+(6\beta_{211}u_{30}^2 + 4\beta_{212}u_0^2 + 2\beta_{215})n_{30} = 0$$

(14)

Hence, based on equation (14), we can obtain the following expressions when $2\beta_{211}\beta_{14} - 3\beta_{110}\beta_{25} \neq 0$,

$$r_0^2 = 2\beta_{113}u_{30}^2 + 2\beta_{111}u_{20}^2 + \beta_{114}$$

$$-3\beta_{110}$$

$$2\beta_{212}\beta_{14} - 3\beta_{215}\beta_{110}(u_{20}^2 + u_{30}^2) + \beta_{21}Tu_{20}u_{30} = 0$$

(15)

where $u_{20} = -c_i / b_i$, $i = 1,2$ and the expressions of $b_i$ and $c_i$ are shown in appendix. Hence, from equation (15), we obtain there are at most four sets of real solutions $(r_0, u_{20}, u_{30})$ of $h(z) = 0$, and all the four sets of solutions can be written as follows if condition (C1) is satisfied,

$$(r_{i1}, u_{20,i1}, u_{30,i}) = \left[\frac{j + 1}{2}, (-1)^{j-1} \frac{c_i}{b_i}, (-1)^{j-1} \frac{c_i}{b_i}\right]$$

where $j = 1,2,3,4, l = \left\lfloor \frac{j + 1}{2} \right\rfloor$. $[\cdot]$ denotes integral part.

When $2\beta_{212}\beta_{14} - 3\beta_{110}\beta_{25} \neq 0$, equation (14) can be reduced to $\beta_{21}Tu_{20}u_{30} = 0$, and there is no solution for $h(z) = 0$ due to $\beta_{21}T \neq 0$.

(2) $u_{20}u_{30} = 0, u_{20}^2 + u_{30}^2 \neq 0$. There is no solution for $h(z) = 0$ when $2\beta_{212}\beta_{14} - 3\beta_{110}\beta_{25} \neq 0$, and there are at most four sets of solutions for $h(z) = 0$ when $2\beta_{212}\beta_{14} - 3\beta_{110}\beta_{25} = 0$. If condition (C2) is satisfied, the four sets of solutions can be obtained as follows for $i = 1,2, j = 3,4$,

$$\left(r_{i1}, u_{20,i1}, u_{30,i1}\right) = \left(\frac{f_{i1}}{4}, 0, \frac{f_{i1}}{6}\right)$$

$$\left(r_{i2}, u_{20,i2}, u_{30,i2}\right) = \left(\frac{f_{i2}}{4}, -1, \frac{f_{i2}}{6}\right)$$

$$(3) u_{20} = u_{30} = 0. Now h(z) = 0 can be reduced to 3\beta_{110}u_0^2 + \beta_{114}u_0 = 0, and there is at most one solution. The solution can be obtained as follows when $\beta_{110} < 0$,

$$\left(r_{i3}, u_{20,i3}, u_{30,i3}\right) = \left(\frac{\beta_{114}}{-3\beta_{110}}, 0, 0\right)$$

Under the above three cases, the Jacobian of $h(z) = 0$ at each solution is not zero. Hence, when $\beta_{110} < 0$ and one of the conditions (C1)-(C2) is satisfied, there are five periodic solutions.

This completes the proof.

The number and relative positions of periodic solutions of system (6) are different under different parameter conditions. Detuning parameter is an important parameter for composite laminated circular cylindrical shell, next we discuss the influence of $\sigma_z$ on the number of periodic solutions and the following result holds.

**Theorem 3**
Considering system (6) and denoting
where the vector \( \mathbf{P} = (\beta_{14}, \beta_{10}, \beta_{13}, \beta_{25}, \beta_{211}, \beta_{212}, \beta_{218}, \beta_{219}, \beta_{21}T) \), the 5 periodic solutions can be obtained with the parameter values \( \mathbf{P} \in M \) for detuning parameter \( \sigma_2 \in (10.5, 33) \).

**Proof** For the parameters \( \mathbf{P} \in M \), we have

\[
2\beta_{21};\beta_{14} - 3\beta_{10};\beta_{25} = -120, \quad (\beta_{21}T)^2 - 4a^2 = 900, \quad b_1c_1 = 4200 \times (-792 + 24\sigma_2),
\]

\[
b_2c_2 = 262.5 \times (-378 + 6\sigma_2), \quad g_1 = \frac{39}{35} + \frac{2}{35}\sigma_2, \quad g_2 = \frac{3}{5} + \frac{2}{35}\sigma_2.
\]

Hence \( b_1c_2 < 0, \quad g_i > 0 \) for arbitrary \( i = 1, 2 \) for \( \sigma_2 \in (10.5, 33) \). There are 5 periodic solutions based on condition (C1) in theorem 2.

This completes the proof.

4. **Numerical simulations**

In this section, we discuss the periodic solutions of composite laminated circular cylindrical shell under the conditions of \( \mathbf{P} \in M \) and \( \sigma_2 \in (10.5, 33) \). By numerical simulations, the phase portraits of the periodic solutions can be obtained with different values of \( \sigma_2 \).

Figure 1 demonstrates the 5 periodic solutions of composite laminated circular cylindrical shell when \( \sigma_2 = 20 \in (10.5, 33) \). Figures 1 (a)-(b), respectively, demonstrate the projections and time-history diagram on the plane \((y_1, y_1)\) and \((t, y_1)\). Figures 1 (c)-(d) show the phase portraits in space \((y_1, y_2, y_3)\) and \((y_1, y_2, y_4)\).

![Figure 1](image)

**Figure 1.** The periodic solutions for \( \sigma_2 = 20 \)

Figure 2 illustrates the 5 periodic solutions of composite laminated circular cylindrical shell when \( \sigma_2 = 25 \in (10.5, 33) \), where the black, blue and red periodic solutions correspond to those in figure 1 with the detuning parameter condition changing from \( \sigma_2 = 20 \) to \( \sigma_2 = 25 \). With the value of \( \sigma_2 \) increasing, the amplitudes of the first-order mode and the relative positions of the periodic solutions are different.
Figure 2. The periodic solutions for $\sigma_2 = 25$

Figure 3 indicates the 5 periodic solutions when $\sigma_2 = 30 \in (10.5,33)$, where the periodic solutions with different colours correspond those in figures 1-2.

Figure 3. The periodic solutions for $\sigma_2 = 30$

In this section, we obtain the periodic solutions with different values of $\sigma_2$ under the conditions of $P \in M$ and $\sigma_3 \in (10.5,33)$. Based on the numerical simulations, we find the relative positions and amplitudes of the periodic solutions could produce variations with the changing of the value of detuning parameter $\sigma_2$. The amplitudes of the first-order mode which are closely related to $\sqrt{y_1^2 + y_2^2}$ show an increasing trend for the red and blue periodic solutions with the increasing of $\sigma_2$ compared with the phase portraits in figures 1-3.
5. Conclusions
In this paper, the periodic solutions and numerical simulations for a composite laminated circular cylindrical shell under the parametric excitation of changing temperature are investigated. Some theoretical results on the existence, least upper bound and parameter control conditions of periodic solutions are obtained and the numerical analysis by taking detuning parameter $\sigma_2$ as a control parameter verified the validity of our results. Numerical simulations are carried out to study the relative positions and amplitudes of the periodic solutions with different values of $\sigma_2 \in (10,5,33)$ under the conditions of $P \in M$.

Periodic solutions of system (6) we obtained correspond to periodic motions of composite laminated circular cylindrical shell. With the value of detuning parameter $\sigma_2$ changing, the relative positions and the amplitudes of the first-order mode of periodic solutions produce variations, while periodic solutions with high-amplitude and low-frequency may cause great damages to composite laminated circular cylindrical shell and the structures. Our theoretical results may provide a theoretical guidance on the vibration controlling.

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Appendices
The expressions of $F_j(y)$, $(j = 1,2,3,4)$ in system (4) are shown as follows

$$F_j(y) = F_j^1(y) + F_j^2(y)$$  \hspace{1cm} (A.1)

$$F_j^1(y) = -\frac{1}{2} \beta_{14} y_i + \frac{1}{4} \beta_{14} y_{s-4} - \frac{1}{2} \beta_{10} y_{2s-4} + \left( \frac{1}{4} \beta_{13} T + (-1)^i \frac{1}{2} \beta_{12} \right) y_{s-4}$$  \hspace{1cm} (A.2)

$$F_j^2(y) = -\frac{1}{2} \beta_{24} y_{s-2} + \left( \frac{1}{4} \beta_{23} T + (-1)^i \frac{1}{2} \beta_{22} \right) y_{s-4} - \frac{1}{2} \beta_{20} y_{4s} + \left( \frac{1}{4} \beta_{21} T + (-1)^i \frac{1}{2} \sigma_2 \right) y_{s-4}$$  \hspace{1cm} (A.3)

$$F_{2s-4}^3(y) = -\frac{3}{2} \beta_{11} y_{s-4} - \left( \frac{3}{2} \beta_{16} y_1 + \frac{3}{2} \beta_{13} y^2_1 + \frac{1}{2} \beta_{12} y^2_4 \right) y_1 + 3 \beta_{10} y_{4s} - \frac{3}{2} \beta_{17} y_3 - \frac{3}{2} \beta_{18} y_4 - \frac{3}{2} \beta_{19} y_4 - \frac{1}{2} \beta_{10} y_{4s} + \frac{3}{2} \beta_{20} y_{4s}$$  \hspace{1cm} (A.4)

$$F_{2s-4}^3(y) = -\frac{3}{2} \beta_{11} y_{s-4} + \left( \frac{3}{2} \beta_{16} y_1 + \frac{3}{2} \beta_{13} y^2_1 + \frac{1}{2} \beta_{12} y^2_4 \right) y_1 + 3 \beta_{10} y_{4s} - \frac{3}{2} \beta_{17} y_3 - \frac{3}{2} \beta_{18} y_4 - \frac{3}{2} \beta_{19} y_4 - \frac{1}{2} \beta_{10} y_{4s} + \frac{3}{2} \beta_{20} y_{4s}$$  \hspace{1cm} (A.5)

where $i = 1,2$, $k = 3,4$.

The expressions of $a$, $b_i$, $c_i$, $g_i$, $f_{3i}$, and $f_{2i}$ appeared in theorem 2 are shown as follows, where $i = 1,2$.

$$a = \frac{2 \beta_{14} \beta_{212} - 3 \beta_{12} \beta_{110}}{3 \beta_{110}}$$
\[ b_i = (18 \beta_{110} (\beta_{218} - k_i \beta_{211}) + 8 \beta_{113} (k_i \beta_{232} - 8 \beta_{219}) (1 + k_i^2)) \\
\]
\[ c_i = 3 \beta_{110} (\beta_{211} + 2 \kappa - 2 k_i \beta_{25}) + 4 \beta_{14} (k_i \beta_{212} - \beta_{219}) \\
\]
\[ g_i = \frac{2 \beta_{111} (1 + k_i^2) c_i - \beta_{25} k_i}{3 \beta_{110} b_i}, \\
\]
\[ k_i = -\beta_{25} T + (-1)^i \sqrt{(\beta_{25} T)^2 - 4a^2} \\
\]
\[ f_{i1} = \frac{(-1)^{i-1} \beta_{25} \beta_{211} T - 2 \beta_{211} \beta_{25} + 2 \beta_{25} \beta_{218}}{\beta_{211} \beta_{219} - \beta_{218} \beta_{212}}, \\
\]
\[ f_{i2} = \frac{(-1)^i \beta_{21} \beta_{211} T + 2 \beta_{211} \beta_{25} - 2 \beta_{25} \beta_{219}}{\beta_{218} \beta_{211} - \beta_{212} \beta_{219}} \\
\]

(A.6)

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