Painlevé III and a singular linear statistics in Hermitean random matrix ensembles I.

Yang Chen
Department of Mathematics
Imperial College London
180 Queen’s Gates
London SW7 2BZ UK

Alexander Its
Department of Mathematical Sciences
Indiana University-Purdue University Indianapolis
402 N. Blackford Street
Indianapolis, IN 46202-3216 USA

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Abstract

In this paper, we study a certain linear statistics of the unitary Laguerre ensembles, motivated in part by an integrable quantum field theory at finite temperature. It transpires that this is equivalent to the characterization of a sequence of polynomials orthogonal with respect to the weight

\[ w(x) = w(x, s) := x^\alpha e^{-x} e^{-s/x}, \quad 0 \leq x < \infty, \quad \alpha > 0, \quad s > 0, \]

namely, the determination of the associated Hankel determinant and recurrence coefficients. Here \( w(x, s) \) is the Laguerre weight \( x^\alpha e^{-x} \) 'perturbed' by a multiplicative factor \( e^{-s/x} \), which induces an infinitely strong zero at the origin.

For polynomials orthogonal on the unit circle, a particular example where there are explicit formulas, the weight of which has infinitely strong zeros, was investigated by Pollaczek and Szegő many years ago. Such weights are said to be 'singular' or irregular due to the violation of the Szegő condition.

In our problem, the linear statistics is a sum of the reciprocal of positive random variables \( \{x_j : j = 1, ..., n\} ; \sum_{j=1}^{n} 1/x_j \).

We show that the moment generating function, or the Laplace transform of the probability density function of this linear statistics is expressed as the ratio of Hankel determinants and as an integral of the combination of a particular third Painlevé function.
1 Introduction.

It is a well known fact that the joint probability density of the eigenvalues, \( \{ x_j : j = 1, \ldots, n \} \) of any Hermitean matrix ensemble is \([34]\)

\[
p(x_1, \ldots, x_n)dx_1 \ldots dx_n = \frac{1}{D_n[w]} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{l=1}^{n} w(x_l)dx_l,
\]

where \( D_n \) the normalization constant reads

\[
D_n[w] = \frac{1}{n!} \int_{\mathbb{R}^n_+} \prod_{1 \leq j < k \leq n} (x_j - x_k)^2 \prod_{l=1}^{n} w(x_l)dx_l.
\]

For the sake of concreteness the domain of integration is \( \mathbb{R}^n_+ := [0, \infty)^n \).

Here \( w(\geq 0) \) is a weight supported on \( \mathbb{R}_+ \). Furthermore we suppose \( w \) has moments of all order, that is,

\[
\mu_j := \int_{\mathbb{R}_+} x^j w(x)dx, \quad j \in \{0, 1, \ldots\},
\]

exist.

We shall see that \( D_n \) will play a fundamental role in this paper.

It is also well known that the normalization constant defined above has two more alternative representations; the first of which as the determinant of the Hankel or moment matrix,

\[
D_n[w] = \det (\mu_{j+k})_{j,k=0}^{n-1} = \prod_{j=0}^{n-1} h_j.
\]

The quantity \( h_j \) in the second equality of (1.4) is square of the weighted \( L^2 \) norm of monic polynomials of degree \( j \), orthogonal with respect to \( w \);

\[
\int_{\mathbb{R}_+} P_j(x)P_k(x)w(x)dx = h_j \delta_{j,k}.
\]

The random variable known as the linear statistics is a sum of a function of the random variables \( \{ x_j : j = 1, \ldots, n \} \);

\[
\sum_{j=1}^{n} f(x_j),
\]

whose probability density function is determined by the standard formula,

\[
P_f(Q) = \frac{1}{n!} \int_{\mathbb{R}^n_+} p(x_1, \ldots, x_n) \delta \left( Q - \sum_{j=1}^{n} f(x_j) \right) dx_1 \ldots dx_n.
\]
The moment generating function denoted as $M_f(s)$ (assuming $f(x) > 0$ for all $x \in \mathbb{R}_+$) is the Laplace transform of $P_f(Q)$ with respect to $s$ and has the following form:

\[
M_f(s) = \int_0^\infty P_f(Q)e^{-sQ}dQ = \frac{\det (\mu_{j+k}(s))_{j,k=0}^{n-1}}{\det (\mu_{j+k}(0))_{j,k=0}^{n-1}} = \frac{D_n[w(.,s)]}{D_n[w(.,0)]} = \frac{\prod_{j=0}^{n-1} h_j(s)}{\prod_{j=0}^{n-1} h_j(0)},
\]

where

\[
\mu_j(s) := \int_0^\infty x^j w(x)e^{-sf(x)}dx, \quad j \in \{0, 1, \ldots\}
\]

and

\[
h_j(s) = \int_0^\infty P_j^2(x)w(x)e^{-sf(x)}dx,
\]

are moments and the square of the $L^2$ norm of the polynomials $P_j$ orthogonal with respect to $w \exp(-sf)$, respectively. Therefore we see that the moment generating function is the ratio of the Hankel determinant generated by the ‘perturbed’ weight $w \exp(-sf)$ to the corresponding quantity generated by the ‘original’ weight $w$.

Because the moments depend on $s$, the coefficients of the polynomials $P_j(z)$ also depend on $s$, however, we shall not display this dependence most of the time.

Our monic polynomials are normalized so that:

\[
P_n(z,s) = z^n + p_1(n,s) z^{n-1} + \ldots + P_n(0,s),
\]

with $P_0(z,s) := 1$ and $p_1(0,s) := 0$.

We note here Heine’s multiple integral representation of $P_n(z)$,

\[
P_n(z) = \frac{1}{n!} \int_{\mathbb{R}_+^n} \prod_{j=1}^n (z - x_j)p(x_1, \ldots, x_n)dx_1 \ldots dx_n.
\]

In section 2, a description is given for a pair of ladder operators for smooth weights. These will lead to a linear second order ordinary differential equation satisfied by $P_n(z)$ and two fundamental compatibility conditions valid for all $z \in \mathbb{C} \cup \infty$.

We denote the compatibility conditions as $(S_1)$ and $(S_2)$.

The compatibility conditions are essentially a consequence of the recurrence relations;

\[
z P_n(z) = P_{n+1}(z) + \alpha_n(s)P_n(z) + \beta_n(s)P_{n-1}(z),
\]

together with the ‘initial’ conditions: $P_0(z) = 1$, and $\beta_0 P_{-1}(z) = 0$, and the Christoffel-Darboux formula (also a consequence of (1.13)).

In the (1.13),

\[
\alpha_n(s) \in \mathbb{R}, \quad n = 0, 1, \ldots
\]

and

\[
\beta_n(s) = \frac{h_n}{h_{n-1}} = \frac{D_{n+1}D_{n-1}}{D_n^2} > 0, \quad n = 1, 2, \ldots
\]
are the recurrence coefficients.

An easy consequence of (1.11) and (1.13) is that

\[ p_1(n, s) - p_1(n + 1, s) = \alpha_n(s). \]  

(1.14)

Taking a telescopic sum of (1.14) together with \( p_1(0, s) = 0 \), implies

\[ \sum_{j=0}^{n-1} \alpha_j(s) = -p_1(n, s). \]  

(1.15)

We refer the readers to [39] for basic facts about orthogonal polynomials.

In our problem,

\[ f(x) := \frac{1}{x}, \quad 0 \leq x < \infty, \]  

(1.16)

and the unperturbed weight is given by the equation,

\[ w_0(x) = x^\alpha e^{-x}, \quad \alpha > 0. \]

The compatibility conditions \((S_1)\) and \((S_2)\), and a combination of these, \((S'_2)\), produces a pair

non-linear difference equations, satisfied by the auxiliary quantities \(a_n\) and \(b_n\). See (2.16) and

(2.17).

The recurrence coefficients, \(\alpha_n\) and \(\beta_n\) are ultimately expressed in terms of \(a_n\) and \(b_n\). See

(2.9) and (2.14).

The linear statistics (1.16) leads to the weight

\[ w(x, s) = w_0(x)e^{-s/x} := x^\alpha e^{-x} e^{-s/x}, \quad \alpha > 0, \quad s \geq 0. \]

Such weights arise from a certain problem in mathematical physics: An integrable quantum field

theory at finite temperature [31].

In the theory of orthogonal polynomials, the effect of infinitely strong zeros on the Hankel
determinants, recurrence coefficients and polynomials themselves is of considerable interest.

For orthogonal polynomials with weight \(w\) supported on \([-1, 1]\), the classical Szegö theory gives a comprehensive account of the large \(n\) behavior of the recurrence coefficients and the polynomials (both outside \([-1, 1]\) and on \((-1, 1)\) if \(w\) is absolutely continuous and satisfies the Szegö condition,

\[ \int_{-1}^{1} \frac{|\ln w(x)|}{\sqrt{1-x^2}} dx < \infty. \]

See [pp. 296–312,[39]], [23] and [35] regarding Szegö’s theory.

However, there is a class of orthogonal polynomials discovered by Pollaczek and extended by

Szegö which is in some sense irregular. See [pp. 393–400,[39]] and [40] about this class of orthogonal
polynomials. The Pollaczek-Szegö weight behaves like

\[ \exp \left( -\frac{c}{\sqrt{1-x^2}} \right), \quad c > 0, \]

near \(\pm 1\), and consequently just violates the Szegö condition.
We reproduce here some of the results of [40] to illustrate the irregularity.

Associated with the weight,

\[ w(x; a, b) := \frac{e^{(2\phi - \pi)\phi(\theta)}}{\cosh(\pi\phi(\theta))} \]

where \( x := \cos \theta, \ 0 < \theta < \pi \) and

\[ \phi(\theta) := \frac{a \cos \theta + b}{2 \sin \theta}, \ \ a, \ b \in \mathbb{R}, \ a \geq |b|, \]

are the normalized Pollaczek polynomials \( \{p_n(x; a, b)\} \),

\[ \int_{-1}^{1} [p_n(x; a, b)]^2 w(x; a, b) dx = 1. \]

If \( x \to 1 \), then

\[ w(x; a, b) \approx 2 e^{(a+b)(1-\pi/\theta)}, \ \text{as} \ \theta \to 0, \]

which shows that the weight vanishes exponentially at \( x = 1 \). An easy computation demonstrates the same behavior at \( x = -1 \).

The large \( n \) behavior of \( p_n(x; a, b) \) are as follows:

(a) \[ p_n(1; a, b) \sim n^{1/4} e^{2\sqrt{a+b}\sqrt{n}}, \]

(b) \[ p_n(x; a, b) \sim n^K [x + (x^2 - 1)]^n, \ K = K(x) = \frac{ax + b}{2\sqrt{x^2 - 1}}, \ x \notin [-1, 1], \]

(c) \[ p_n(\cos \theta; a, b) = A(\theta) \cos [n\theta - \phi(\theta) \ln n + B(\theta)] + \epsilon_n(\theta), \ \lim_{n \to \infty} \epsilon_n(\theta) = 0, \ 0 < \theta < \pi. \]

where the \( n \)- independent functions \( A(\theta)(>0), \ B(\theta) \) are analytic in \( (0, \pi) \).

This is to be contrasted with the large \( n \) behavior of the normalized Jacobi polynomials \( \{p_n(x)\} \)

associated with the weight

\[(1-x)^\alpha (1+x)^\beta, \ \alpha > -1, \ \beta > -1, \ x \in [-1, 1],\]

(a') \[ p_n(1) \sim n^{\alpha + 1/2}, \]

(b') \[ p_n(x) \sim \left[ x + \sqrt{x^2 - 1} \right]^n, \ x \notin [-1, 1] \]

(c') \[ p_n(\cos \theta) = A_1(\theta) \cos [n\theta + B_1(\theta)] + \epsilon_n(\theta), \]

where \( A_1(\theta) \) and \( B_1(\theta) \) are functions of the same kind.

The symbol \( \sim \) indicates that the ratio of the given quantities approaches a non-zero limit, while \( \not\approx \) indicates that the limit is 1. This is the convention adopted by [40] and will not be used later.

Our paper is a first step in study of the Pollaczek-Szegö type orthogonal polynomials supported in infinite intervals.

With reference to the Heine formula, we see that,

\[ (-1)^n P_n(0, s) = \frac{D_n[w(.s, \alpha + 1)]}{D_n[w(.s, \alpha)]}. \]  \hspace{1cm} (1.17) 5
For the unperturbed or Laguerre weight, we have the explicit determination

\[ (-1)^n P_n(0,0) = \frac{\Gamma(n + \alpha + 1)}{\Gamma(\alpha + 1)}, \]  

(1.18)

since,

\[ D_n[w(., 0, \alpha)] = \frac{G(n + 1)G(n + \alpha + 1)}{G(\alpha + 1)}, \]  

(1.19)

where \( G(z) \) is the Barnes \( G- \) function that satisfies the functional relation \( G(z + 1) = \Gamma(z)G(z) \). In section 3, by taking derivative with respect to \( s \) on the orthogonality relations we obtain a pair of differential-difference equations or the Toda equations. Combining the Toda equations and the non-linear difference equations obtained in section 2 produce a particular Painlevé III satisfied by \( \alpha_n(s) \), up to linear shift in \( n \).

The \( \tau- \) function for this \( P_{III} \) turns out to be intimately related to the Hankel determinant

\[ D_n(s) = \det \left( \int_0^\infty x^{j+k} x^\alpha e^{-x-s/x} \right)_{j,k=0}^{n-1}. \]  

(1.20)

We also express the recurrence coefficients \( \alpha_n \) and \( \beta_n \) in terms of the logarithmic derivative of Hankel determinant

\[ H_n := s \frac{d}{ds} \ln D_n(s), \]

and obtain a functional equation involving \( H_n, H'_n \) and \( H''_n \). The resulting second order non-linear ordinary differential equation satisfied by \( H_n \), is recognized to be the Jimbo-Miwa-Okamoto \( \sigma- \) form of our \( P_{III} \).

In section 4, we show, with the aid of the non-linear difference equations derived in section 2, another functional equation involving \( H_n, H_{n+1} \) and \( H_{n-1} \). We call the resulting non-linear second order difference equation satisfied by \( H_n \) the discrete \( \sigma- \) form of \( P_{III} \).

In sections 5, the Riemann-Hilbert approach to orthogonal polynomials and the isomonodromy deformation theory of Jimbo and Miwa, are used to re-derive the \( P_{III} \) and thereby identify the auxiliary quantities, \( a_n \) and \( b_n \), introduced in section 2, with the objects of the Jimbo-Miwa isomonodromy theory of Painlevé equations.

In section 6, we show that the Hankel determinant is the isomonodromy \( \tau- \) function in the sense of Jimbo and Miwa, and put into context of the general theory of integrable systems the identities derived in sections 2 and 3.

As we are studying an example of orthogonal polynomials where the otherwise classical weight,

\[ w_0(x) = x^\alpha e^{-x}, \quad x \in \mathbb{R}_+, \quad \alpha > 0 \]

is perturbed by an infinitely strong zero,

\[ w(x, s) := w_0(x) e^{-s/x}, \quad s \geq 0, \]

the natural questions of interest are about the large \( n \) behavior of the Hankel determinant, recurrence coefficients and the orthogonal polynomials. Such investigations will therefore provide...
valuable insights into the asymptotic of the associated Painlevé transcendant. These results will be published in a forthcoming paper [14].

We want to emphasize that in this paper and in its follow-up we do not claim the introduction of new concepts. Our main aim is to investigate a concrete important example of the linear statistics which leads to a strong zero at \( x = 0 \) using the known techniques. In addition, taking this example as a “case study” we show how the apparatus which are used by the two communities - the orthogonal polynomial community and the integrable system community, match with each other.

2 Ladder operators and non-linear difference equations.

The pair of ladder operators has been known to various authors. See, for example, [3], [4], [7], [8], [9], [10], [33], [11], [18], [19], [32] and [37]. In fact, Magnus in [33] noted that such operators were known to Laguerre.

Because the associated fundamental compatibility conditions and their use in the derivation of the Painlevé transcendant [15],[16], [18], [20], [33] are perhaps less well known we summarize these findings in (2.1–2.5), (S_1), (S_2) and (S_2') in a form which we find particular easy to use. We note here that (S_1), (S_2) and (S_2') were also known to Magnus [33] and (S_2) also appeared in [27]. See also [20].

For polynomials orthogonal on the unite circle the analogues ladder operators can be found in [25]. See [21] for the circular case applicable to bi-orthogonal polynomials. The compatibility condition in the circular case can be found [1] where it was used to obtain in explicit form of the Toeplitz determinant with the pure Fisher-Hartwig symbol and the discriminant of the associated orthogonal polynomials.

The compatibility conditions can also be adapted to the situation where the weight has discontinuities. See [15] and [2].

The ladder operators are,

\[
\frac{d}{dz} + B_n(z) P_n(z) = \beta_n A_n(z) P_{n-1}(z) \tag{2.1}
\]

\[
\left( \frac{d}{dz} - B_n(z) - \nu'(z) \right) P_{n-1}(z) = -A_{n-1}(z) P_n(z) \tag{2.2}
\]

where

\[
A_n(z) = \frac{1}{h_n} \int_0^\infty \frac{\sqrt{z} - \sqrt{y}}{z-y} P_n^2(y) w(y) dy \tag{2.3}
\]

\[
B_n(z) = \frac{1}{h_{n-1}} \int_0^\infty \frac{\sqrt{z} - \sqrt{y}}{z-y} P_n(y) P_{n-1}(y) w(y) dy \tag{2.4}
\]

\[
\nu(z) := -\ln w(z), \tag{2.5}
\]

and the associated fundamental compatibility conditions are,

\[
B_{n+1}(z) + B_n(z) = (z - \alpha_n) A_n(z) - \nu'(z) \tag{S_1}
\]

\[
1 + (z - \alpha_n)(B_{n+1}(z) - B_n(z)) = \beta_{n+1} A_{n+1}(z) - \beta_n A_{n-1}(z), \tag{S_2}
\]
valid for all $z \in \mathbb{C} \cup \{\infty\}$. See [12] for a recent derivation of the compatibility conditions. To arrive at the equations (2.3) and (2.4) we have assumed that $w(0) = w(\infty) = 0$. This is certainly the situation for our problem since we have assume that $\alpha > 0$ and $s \geq 0$.

Combining suitably $(S_1')$ and $(S_2')$ gives an expression involving $\sum_{j=0}^{n-1} A_j(z)$, $B_n(z)$ and $\sqrt{z}$ from which further insight into recurrence coefficients may be gained.

The equation $(S_2')$ may be thought of as the first integral from $(S_1)$ and $(S_2)$.

Although $(S_2')$ first appeared in [33] in a slightly different form, we present here a derivation of a version which we find useful in practice.

Multiplying $(S_2')$ by $A_n(z)$ we see that the r.h.s. of the resulting equation is a first order difference, while the l.h.s., with $(z - \alpha_n)A_n(z)$ replaced by $B_{n+1}(z) + B_n(z) + \sqrt{z}$ is a first order difference plus $A_n(z)$. Taking a telescopic sum, together with the initial conditions $B_0(z) = A_{-1}(z) = 0$, produces the Lemma,

**Lemma 1**

$$B_n^2(z) + \sqrt{z} B_n(z) + \sum_{j=0}^{n-1} A_j(z) = \beta_n A_n(z) A_{n-1}(z).$$

(S2)

If $\sqrt{z}$ is rational then so are $A_n(z)$ and $B_n(z)$. See (2.3) and (2.4). Furthermore, eliminating $P_{n-1}(z)$ from (2.1) and (2.2) it is easy to show that $y(z) := P_n(z)$ satisfies the second order linear ordinary differential equation,

$$y''(z) - \left( \sqrt{z} + \frac{A_n(z)}{A_n(z)} \right) y'(z) + \left( B'_n(z) - B_n(z) \frac{A'_n(z)}{A_n(z)} + \sum_{j=0}^{n-1} A_j(z) \right) y(z) = 0.$$ (2.6)

Note that $(S_2')$ has been used to simplify the coefficient of $y(z)$ in (2.6).

The equation (2.6) can also be found in [37], albeit in a different form.

For the problem at hand, $f(x) = 1/x$, $x \geq 0$, the weight and associated quantities are,

$$w(x, s) = x^\alpha e^{-x-s/x}, \quad \nu(z) = z + s/z - \alpha \ln z, \quad \nu'(z) = 1 - s/z^2 - \alpha/z$$

$$\frac{\nu'(z)}{z} = \frac{1}{z} \left( \frac{\alpha}{y} + \frac{s}{y^2} \right) + \frac{s}{z^2}.$$ (2.7)

Using these we have the next Lemma.

**Lemma 2** The "coefficients" $A_n(z)$ and $B_n(z)$ appearing in the ladder operators are

$$A_n(z) = \frac{1}{z} + \frac{a_n}{z^2}, \quad \text{ (2.7)}$$

$$B_n(z) = -\frac{n}{z} + \frac{b_n}{z^2}, \quad \text{ (2.8)}$$

$$a_n := \frac{s}{h_n} \int_0^\infty \frac{p^2}{y} wdy, \quad a_n(0) = 0, \quad b_n := \frac{s}{h_{n-1}} \int_0^\infty \frac{P_n P_{n-1}}{y} wdy, \quad b_n(0) = 0.$$
Proof. From the definitions of $A_n(z)$ and $B_n(z)$ and with the identities,

$$1 = \frac{1}{h_n} \int_0^\infty \left( \frac{\alpha}{y} + \frac{s}{y^2} \right) P_n^2 w dy,$$

$$-n = \frac{1}{h_{n-1}} \int_0^\infty \left( \frac{\alpha}{y} + \frac{s}{y^2} \right) P_n P_{n-1} w dy,$$

obtained by integration by parts, we find (2.7) and (2.8).

We see that at this stage there are 4 unknowns, $\alpha_n$, $\beta_n$, $a_n$ and $b_n$. In what follows we will show how $(S_1)$ and $(S'_2)$ can be applied to obtain amongst other things the pair non-linear difference equations involving $a_n$ and $b_n$ mentioned earlier.

On equating the residues on both sides of $(S_1)$, we find,

$$\alpha_n = 2n + 1 + \alpha + a_n,$$  \hspace{0.5cm} (2.9)

$$b_{n+1} + b_n = s - \alpha_n a_n.$$  \hspace{0.5cm} (2.10)

Carrying out a similar calculation with $(S'_2)$, gives,

$$\beta_n = n(n+\alpha) + b_n + \sum_{j=0}^{n-1} a_j,$$  \hspace{0.5cm} (2.11)

$$\beta_n(a_n + a_{n-1}) = ns - (2n + \alpha)b_n,$$  \hspace{0.5cm} (2.12)

$$b_n^2 - sb_n = \beta_n a_n a_{n-1}.$$  \hspace{0.5cm} (2.13)

The upshot of these equations is that $\alpha_n$ and $\beta_n$ are entirely determined by $a_n$ and $b_n$, where $\alpha_n$ is simply $a_n$ plus $2n + 1 + \alpha$.

Eliminating $a_{n-1}$ from (2.12) and (2.13) we have the next Lemma.

**Lemma 3**

$$\beta_n a_n^2 = [ns - (2n + \alpha)b_n]a_n - (b_n^2 - sb_n).$$  \hspace{0.5cm} (2.14)

Therefore (2.14) expresses $\beta_n$ in terms of $a_n$ and $b_n$, and importantly bypasses the finite sum in (2.11). Eliminating $\beta_n$ from (2.11) and (2.14), an expression can be found for $\sum_{j=0}^{n-1} a_j$, in terms of $a_n$ and $b_n$.

We state this in the next Lemma

**Lemma 4**

$$\sum_{j=0}^{n-1} a_j = -n(n+\alpha) - b_n + \frac{ns - (2n + \alpha)b_n}{a_n} - \frac{b_n^2 - sb_n}{a_n^2}.$$  \hspace{0.5cm} (2.15)

Note that because $a_n(0) = b_n(0) = 0$, (2.9) and (2.11) reduce to $\alpha_n(0) = 2n + 1 + \alpha$, and $\beta_n(0) = n(n + \alpha)$, respectively, which we recognize to be the recurrence coefficients of the monic Laguerre polynomials.
In summary, with reference to (2.13) and (2.14), we obtain two non-linear difference equations, satisfied by \( a_n \) and \( b_n \),

\[
\begin{align*}
b_{n+1} + b_n & = s - (2n + 1 + \alpha + a_n)a_n, \quad (2.16) \\
(b_n^2 - sb_n)(a_n + a_{n-1}) & = [ns - (2n + \alpha)b_n]a_na_{n-1}, \quad (2.17)
\end{align*}
\]

to be iterated in \( n \) with the initial conditions,

\[
\begin{align*}
a_0(s) & = \sqrt{s} \frac{K_\alpha(2\sqrt{s})}{K_{\alpha+1}(2\sqrt{s})}, \quad (2.18) \\
b_0(s) & = 0, \quad (2.19)
\end{align*}
\]

where \( K_\alpha(z) \) is the MacDonald functions of the second kind.

We call (2.16) and (2.17) together with the initial conditions (2.18) and (2.19) the MacDonald’s hierarchy. See also [4] for the treatment of a class of semi-classical weights.

Solutions for \( a_n \) and \( b_n \) that are rational functions of \( 2\sqrt{s} \), are found for \( \alpha = p + 1/2, \ p \in \mathbb{Z} \), since

\[
K_{p+1/2}(z) = K_{-p-1/2}(z) = \sqrt{\frac{\pi}{2z}} e^{-z} \sum_{k=0}^{p} \frac{(p + k)!}{k!(p - k)!(2z)^{k}}.
\]

### 3 Toda evolution and Painl´eve III.

By taking derivatives with respect to \( s \) on the orthogonality relation gives rise to Toda type equations.

Because,

\[
h_n = \int_0^\infty P_n^2 w dy,
\]

we have,

\[
s \frac{d}{ds} h_n = -s \int_0^\infty \frac{P_n^2}{y} w dy,
\]

hence,

\[
\begin{align*}
s \frac{d}{ds} \ln h_n & = -a_n \quad (3.1) \\
s \frac{d}{ds} \ln \beta_n & = a_{n-1} - a_n. \quad (3.2)
\end{align*}
\]
We also have,

\[
0 = \frac{d}{ds} \int_0^\infty P_nP_{n-1}w dy \\
= h_{n-1} \frac{d}{ds}P_1(n) - \int_0^\infty \frac{P_nP_{n-1}}{y} w dy
\]

(3.3)

\[
\frac{dp_1(n)}{ds} = b_n
\]

(3.4)

\[
\frac{d\alpha_n}{ds} = \frac{s da_n}{ds} = b_n - b_{n+1} = \beta_n - \beta_{n+1} + \alpha_n.
\]

(3.5)

The equation (3.6) follows from (2.9), (1.14) and (2.11)

There is another identity involving \(\sum_{j=0}^{n-1} a_j\):

\[
s \frac{d}{ds} \sum_{j=0}^{n-1} a_j = -b_n,
\]

(3.7)

which is an immediate consequence of a telescopic sum of the second equality of (3.6).

We now show that the Hankel determinant \(D_n\), is up to scaling transformation the \(\tau\)– function of the Toda-equations. Let

\[
\tilde{D}_n(s) := s^{-n(n+\alpha)}D_n(s).
\]

We find by summing (3.1),

\[
s \frac{d}{ds} \ln D_n = -\sum_{j=0}^{n-1} a_j,
\]

(3.8)

since

\[
\sum_{j=0}^{n-1} \ln h_j = \ln D_n.
\]

Applying \(s \frac{d}{ds}\) to (3.8) and keeping in mind (3.7), (2.11) and (3.8) gives

\[
s \frac{d}{ds} \left( s \frac{d}{ds} \ln D_n \right) = b_n
\]

\[
= \beta_n - n(n+\alpha) - \sum_{j=0}^{n-1} a_j
\]

\[
= \beta_n - n(n+\alpha) + s \frac{d}{ds} \ln D_n.
\]

The last equation simplifies to

\[
s^2 \frac{d^2}{ds^2} \ln D_n(s) = \frac{D_{n+1}D_{n-1}}{D_n^2} - n(n+\alpha),
\]

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since
\[ \beta_n = \frac{D_{n+1}D_{n-1}}{D_n^2}. \]

In terms of \( \tilde{D}_n(s) \) we have,
\[ \frac{d^2}{ds^2} \ln \tilde{D}_n(s) = \frac{\tilde{D}_{n+1}\tilde{D}_{n-1}}{D_n^2}. \]  
(3.9)

The equation (3.9) is the Toda molecule equation [38] and shows that \( \tilde{D}_n(s) \) is the corresponding \( \tau \)– function of the Toda equations (3.2) and (3.6).

As the Hankel determinant is now identified with the \( \tau \)– function (see section 6 for more on this issue), we may expect the emergence of a Painlevé equation. In fact, \( a_n(s) \) satisfies a particular \( P_{III} \).

To see this, we first investigate the evolution of \( a_n(s) \) and \( b_n(s) \) as functions of \( s \).

Lemma 5 For a fixed \( n \), the auxiliary quantities \( a_n \) and \( b_n \) satisfy the following coupled Riccatti equations:

\[ s \frac{da_n}{ds} = 2b_n + (2n + 1 + \alpha + a_n)a_n - s \]  
(3.10)

\[ s \frac{db_n}{ds} = \frac{2}{a_n}(b_n^2 - sb_n) + (2n + \alpha + 1)b_n - ns. \]  
(3.11)

Proof. The equation (3.10) follows from applying \( s \frac{d}{ds} \) to (2.9) together with the first equality of (3.6) and with (2.10) to replace \( b_{n+1} \) by \( s - \alpha na_n - b_n \).

A little bit more work is required to prove (3.11). First apply \( s \frac{d}{ds} \) to (2.11),

\[ s \frac{d\beta_n}{ds} = s \frac{db_n}{ds} + s \frac{d}{ds} \sum_{j=0}^{n-1} a_j \]
\[ = \frac{db_n}{ds} - b_n \]
\[ = \beta_n a_{n-1} - \beta_n a_n \]
\[ = \frac{b_n^2 - sb_n}{a_n} - \left[ ns - (2n + \alpha)b_n - \frac{b_n^2 - sb_n}{a_n} \right], \]
where the last three equalities follow from (3.7), (3.2), (2.13) and (2.14). \( \square \)

The next theorem identifies \( a_n \) as a particular third Painlevé function.

Theorem 1 For a fixed \( n \in \{0, 1, 2, \ldots\} \) the auxiliary quantity \( a_n \) satisfies

\[ a_n'' = \frac{(a_n')^2}{a_n} - \frac{a_n'}{s} + (2n + 1 + \alpha)\frac{a_n^2}{s^2} + \frac{a_n^3}{s^2} + \frac{\alpha}{s} - \frac{1}{a_n}, \]  
(3.12)

with the initial conditions
\[ a_n(0) = 0, \quad a_n'(0) = \frac{1}{\alpha}, \quad \alpha > 0. \]  
(3.13)

If \( a_{n}(s) := -q(s) \), then \( q(s) \) is \( P_{III}'(-4(2n + 1 + \alpha), -4\alpha, 4, -4) \), following the convention of [36].
Proof. Eliminate $b_n$ from (3.10) and (3.11) gives (3.12). The initial conditions follows from a straightforward computation. □

**Remark I.** If $n = 0$, then $b_0 = 0$ and (3.10) is solved by

$$\sqrt{s} \frac{K_\alpha(2\sqrt{s})}{K_{\alpha+1}(2\sqrt{s})}$$

which is (2.18). We observe that the above also solve (3.12) for $n = 0$.

**Remark II.** An alternative form, obtained by,

$$a_n(s) =: \frac{s}{X_n(s)},$$

reads,

$$X''_n = \frac{(X'_n)^2}{X_n} - \frac{X'_n}{s} - \frac{\alpha X^2_n}{s^2} - \frac{2n + 1 + \alpha}{s} + \frac{X^3_n}{s^2} - \frac{1}{X_n}, \quad (3.14)$$

which is a $P_{III}'(−4\alpha,−4(2n + 1 + \alpha),4,−4)$. If the derivatives in (3.14) were neglected, then $X_n$ solves the quartic,

$$X^4 − \alpha X^3 − (2n + 1 + \alpha)sX − s^2 = 0. \quad (3.15)$$

We may interpret an appropriate solution (3.15) as the geometric mean of the end points of the support of a single interval equilibrium density. This appears in a potential theoretic minimization problem, the detail of which is in a forthcoming paper [14]. We note that another $P_{III}$ associated with the Toeplitz determinant

$$\det[I_{j-k+\nu}(\sqrt{t})]_{0 \leq j,k \leq n-1},$$

was found in [21] (see also [22]). Here $I_\nu(z)$ is the modified Bessel’s function of the first kind. We note that this Toeplitz determinant can be thought of as a Toeplitz-analog of the Hankel determinant we are studying in this paper. Also, the above Toeplitz determinant with $\nu = 0$ appeared [41] in connection with a certain ensemble of $n \times n$ unitary matrices and Ulam’s problem in combinatorics.

In the next theorem, we display two alternative integral representation of $D_n$, in terms of $a_n$ and $X_n$.

**Theorem 2**

$$\ln \frac{D_n(s)}{D_n(0)} = \int_0^s \left[ \frac{t}{2} - \frac{1}{4} \left( \frac{t}{a_n} - \alpha \right)^2 - a_n \left( n + \frac{\alpha}{2} \right) - \frac{a^2_n}{4} + \frac{1}{4} \left( 1 - ta'_n \right)^2 \right] \frac{dt}{t} \quad (3.16)$$

$$= \int_0^s \left[ \frac{t}{2} - \frac{1}{4} (X_n - \alpha)^2 - (n + \frac{\alpha}{2}) \frac{t}{X_n} - \frac{t^2}{4X^2_n} + \frac{t^2 X^2'_n}{4X^2_n} \right] \frac{dt}{t}. \quad (3.17)$$

**Proof:** From (3.8) and (2.15) we see that the logarithmic derivative of $D_n(s)$ is expressed in terms of $a_n$ and $b_n$. If we use (3.10) to eliminate $b_n$ in favor of $a_n$ and $a'_n$ from the resulting equation, then (3.16) follows after some simplification. The equation (3.17) follows from the substitution $a_n(s) =: s/X_n(s)$. Note that $D_n(0)$ is given by (1.19). □
Put
\[ H_n := s \frac{d}{ds} \ln D_n. \]  (3.18)

In section 6 we will show that the Hankel Determinant \( D_n(s) \) can be identified with the Jimbo-Miwa \( \tau \)-function corresponding to the solution \( a_n(s) \) of the Painlevé III’ equation (3.12). Namely, we will show that
\[ D_n(s) = \text{const} \ \tau(s) e^{\frac{s(n+\alpha)}{2}}. \]  (3.19)

In its turn, this relation yields the following formula for the quantity \( H_n \),
\[ H_n = \sigma(s) + s \frac{n(n+\alpha)}{2}, \]  (3.20)

where \( \sigma(s) \equiv s \frac{d}{ds} \ln \tau \) is the Jimbo-Miwa-Okamoto \( \sigma \)-function corresponding to the equation \( P_{III}' \). Therefore, \( H_n, H'_n \) and \( H''_n \) should satisfy a functional equation
\[ f(H_n, H'_n, H''_n, n, s, \cdot) = 0, \]
known as the Jimbo-Miwa-Okamoto \( \sigma \)-form of our \( P_{III}' \).

With \( H_n \) defined above, it is easy to see from (3.7), (3.8) and (2.11) that
\[ b_n = sH'_n \]  (3.21)
\[ \beta_n = n(n+\alpha) + sH'_n - H_n. \]  (3.22)

In the next theorem we state the non-linear second order ordinary differential equation satisfied by \( H_n \).

**Theorem 3**  If
\[ H_n := s \frac{d}{ds} \ln D_n(s), \]  (3.23)
then
\[ (sH''_n)^2 = [n - (2n + \alpha)H'_n]^2 - 4[n(n+\alpha) + sH'_n - H_n]H'_n(H'_n - 1). \]  (3.24)

Proof. First we re-write (2.14) and (3.11) as
\[ \beta_n a_n + \frac{b_n^2 - sb_n}{a_n} = ns - (2n+\alpha)b_n \]  (3.25)
\[ \frac{2}{a_n} (b_n^2 - sb_n) = sb'_n - b_n + ns - (2n+\alpha)b_n \]  (3.26)
respectively. Eliminate \( a_n \) from (3.25) and (3.26) produces
\[ (sb'_n - b_n)^2 = [ns - (2n+\alpha)b_n]^2 - 4\beta_n(b_n^2 - sb_n). \]  (3.27)

The equation (3.24) follows by substituting \( b_n \) and \( \beta_n \) from (3.21) and (3.22) into (3.27). \( \Box \)
Hence the recurrence coefficients $\alpha_n$ and $\beta_n$, of the orthogonal polynomials associated with our weight,

$$w(x, s) = x^\alpha e^{-x-s/x}, \quad x \in [0, \infty), \quad \alpha > 0, \quad s > 0,$$

are expressed in terms of $H_n$, $H'_n$ and $H''_n$, as follows:

$$\begin{align*}
\alpha_n &= 2n + 1 + \alpha + \frac{2s(H'_n)^2 - sH'_n}{sH''_n + n - (2n + \alpha)H'_n} \quad (3.28) \\
\beta_n &= n(n + \alpha) + sH'_n - H_n. \quad (3.29)
\end{align*}$$

and that $H_n$ itself satisfies (3.24).

**Remark III.** With the identification (3.20) of the function $H_n$ as the $\sigma$-function (up to a linear shift), equation (3.24) coincides, up to the changing the independent variable $s$ to the variable $t = \sqrt{s}$, with equation (C.29) of [30] $^1$.

## 4 Discrete $\sigma$– form

We may anticipate due to the recurrence relations and non-linear difference equations, (2.16) and (2.17) that, for a fixed $t$, $H_n$, and $H_{n\pm 1}$ would satisfy a discrete analog of (3.24). It turns out that in this instance $a_n$ has a simpler expression in term of $H_{n\pm 1}$. We shall obtain an expression for $b_n$ in terms of $H_n$ and $H_{n\pm 1}$. From

$$H_n = -\sum_{j=0}^{n-1} a_j,$$

we have

$$a_n = H_n - H_{n+1}, \quad (4.1)$$

and

$$a_n + a_{n-1} = H_{n-1} - H_{n+1} =: \delta^2 H_n.$$

Multiply the above by $\beta_n$ we find,

$$\beta_n(a_n + a_{n-1}) = \beta_n\delta^2 H_n \quad (4.2)$$

$$= ns - (2n + \alpha)b_n, \quad (4.3)$$

where the last equation follows from (2.12). Now (2.11) becomes

$$\beta_n = n(n + \alpha) + b_n - H_n.$$

$^1$When comparing equation (3.24) with equation (C.29) of [30] one has to also take into account that, if we denote the $\sigma$-function of [30], for which equation (C.29) is written, as $\sigma_{JM}$ then the exact relation with our $\sigma$ function is given by the equation, $\sigma_{JM}(t) = 2\sigma(t^2)$. 

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Substituting the above into (4.2) produces a linear equation in $b_n$ whose solution is

$$b_n = \frac{ns + \delta^2 H_n[H_n - n(n + \alpha)]}{2n + \alpha + \delta^2 H_n}.$$ (4.4)

Hence the auxiliary quantities $a_n$ and $b_n$, and $\beta_n$ are now expressed in terms of $H_n$, and $H_{n \pm 1}$. Substituting these into (2.13) give rise to the discrete $\sigma-$ form stated in the next Theorem.

**Theorem 4** If

$$H_n := s \frac{d}{ds} \ln D_n,$$

then

$$\{[H_n - n(n + \alpha)]\delta^2 H_n + ns\}\{[H_n - n(n + \alpha) - s]\delta^2 H_n - (n + \alpha)s\}$$

$$(2n + \alpha + \delta^2 H_n)\{ns + (2n + \alpha)[n(n + \alpha) - H_n]\}(H_n - H_{n+1})(H_{n-1} - H_n).$$ (4.5)

We also have the discrete analog of (3.28) and (3.29),

$$\alpha_n = 2n + 1 + \alpha + H_n - H_{n+1}$$ (4.6)

$$\beta_n = n(n + \alpha) + \frac{ns - n(n + \alpha)\delta^2 H_n - (2n + \alpha)H_n}{2n + \alpha + \delta^2 H_n}.$$ (4.7)

Now the obvious equalities, (4.6) = (3.28) and (4.7) = (3.29), imply two further differential-difference equations which $H_n$ must satisfy,

$$H_n - H_{n+1} = \frac{2s(H'_n)^2 - sH'_n}{sH'_n + n - (2n + \alpha)H'_n}$$ (4.8)

$$\frac{ns - n(n + \alpha)\delta^2 H_n - (2n + \alpha)H_n}{2n + \alpha + \delta^2 H_n} = sH'_n - H_n.$$ (4.9)

**Remark IV.** From the point of view of general Jimbo-Miwa isomonodromy theory of Painlevé equations, which we will be discussing in sections 5 and 6, equation (4.5) should be related to the Bäcklund - Schlesinger transformations of the $\tau$ - function. However, we failed to identified equation (4.5) with any of the difference equations for the $\tau$ - function discussed in [30] and describing the possible Schlesinger transformations. Since it is written for the logarithmic derivative of the $\tau$ - function, and not for the $\tau$ - function itself as in [30], equation (4.5) might be in fact of a different nature than the ones considered in [30]. We also want to mention that, equation (4.5) is an integrable discrete equation - its Lax pair is formed by the first and the third equations of the triple (5.17) of section 5, and the equation itself, as it has already been noticed, represents a Bäcklund - Schlesinger transformation of the third Painlevé equation. Therefore, we expect this equation to be equivalent to one of the known discrete Painlevé equations, which we have not yet identify. Apparently this identification is not quite straightforward. One of the referees has suggested that (4.5) could be a composition of the basic Schlesinger transformation $T_1$ and $T_2$ found in [22] (see Proposition (4.6) of [22]).
5 An alternative derivation of the Painlevé III equation.

In this section we present an alternative derivation of the third Painlevé equation (3.12) for the quantity $a_n(s)$. This derivation is based on the Riemann-Hilbert point of view [18], [19] on orthogonal polynomials and makes use of the general Jimbo-Miwa-Ueno theory of isomonodromy deformations. This in turn allows us to place some of the key identities of the preceding sections into a general framework of integrable systems.

The Riemann-Hilbert problem for the orthogonal polynomials at hand is the following

- $Y : \mathbb{C} \setminus \mathbb{R}_+ \to \mathbb{C}^{2 \times 2}$ is analytic.
- $Y(x) = Y_-(x) \begin{pmatrix} 1 & x^\alpha e^{-x-s/x} \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}_+ \setminus \{0\}$, with $\mathbb{R}_+$ oriented from left to right.
- $Y(z) = (I + O(1/z)) \begin{pmatrix} z^n & 0 \\ 0 & z^{-n} \end{pmatrix}$ as $z \to \infty$.
- $Y(z) = O(1)$ as $z \to 0$.

Here $Y_{\pm}(z)$ denote the non-tangential limiting values of $Y(z)$ on $\mathbb{R}_+$ taken (in the usual pointwise sense) from the $\pm$-side. The Riemann-Hilbert problem has the unique solution expressed in terms of the orthogonal polynomials $P_n(z)$,

$$Y(z) = \begin{pmatrix} P_n(z) & \frac{1}{2 \pi i} \int_{\mathbb{R}_+} \frac{P_n(x)x^\alpha e^{-x-s/x}}{x-z} \, dx \\ -\frac{2\pi i}{h_{n-1}} P_{n-1}(z) & -\frac{1}{h_{n-1}} \int_{\mathbb{R}_+} \frac{P_{n-1}(x)x^\alpha e^{-x-s/x}}{x-z} \, dx \end{pmatrix}.$$  

(5.1)

We also note that an immediate consequence of the uni-modularity of the jump matrix of the above Riemann-Hilbert problem is the identity\(^2\),

$$\det Y(z) \equiv 1.$$  

(5.2)

Equation (5.1) implies, in particular, that the asymptotic behavior of the function $Y(z)$ at $z = \infty$ and $z = 0$ can be specified as the following full asymptotic series,

$$Y(z) \sim \left( I + \sum_{k=1}^{\infty} Y_{-k} \frac{z^k}{z^k} \right) z^{n\sigma_3}, \quad z \to \infty,$$

(5.3)

and

$$Y(z) \sim Q \left( I + \sum_{k=1}^{\infty} Y_k z^k \right), \quad z \to 0,$$

(5.4)

\(^2\)A more conventional derivation of this identity is based on the use of the basic three-term recurrence equation (1.13) (see e.g. [13])
where $\sigma_3$ denotes, as usual, the third Pauli matrix

$$
\sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
$$

Moreover, the matrix coefficients $Y_{\pm k}$ and $Q$ of these series are the smooth functions of $n$ and $s$ (and of course of $\alpha$), and they all can be easily expressed in terms of the fundamental objects associated with the orthogonal polynomials $P_n(z)$, i.e. in terms of the functions $p_k(n,s), h_n(s)$, and the negative moments of the polynomials $P_n(z)$. Indeed, by a straightforward calculation we have from (5.1) the following expressions for the coefficient $Y_{-1}$ and for the matrix multiplier $Q \equiv Y(0)$.

$$
Y_{-1} = \begin{pmatrix} p_1(n) & -\frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_{n-1}} & -p_1(n) \end{pmatrix},
$$

(5.5)

$$
Q = \begin{pmatrix} 1 & p_n \\ -q_n & 1 - q_n p_n \end{pmatrix} P_n^{\sigma_3}(0),
$$

(5.6)

where

$$
p_n = \frac{P_n(0)}{2\pi i} \int_{\mathbb{R}_+} \frac{P_n(x) x^\alpha e^{-s} e^{-x/s} x}{x} dx,
$$

(5.7)

and we have taken into account the determinant identity (5.2).

We are now going to write down the triple of the differential and difference equations for the function $Y_n(z,s)$ following the standard procedure of the theory of integrable systems (see [17], [29], [30]; see also [18], [26], [13], and [28]).

Put

$$
\Psi(z) \equiv \Psi(z,n,s,\alpha) := Y(z) e^{-\frac{1}{2}(z^2 + s^2)} z^\alpha e^{-\frac{1}{2} s^2} \sigma_3 e^{-\frac{1}{2} z^2} \sigma_3,
$$

(5.8)

where the branch of the function $z^{\frac{\alpha}{2}}$ we define by the condition, $-\pi < \arg z < \pi$. The Riemann-Hilbert relations in terms of the function $\Psi(z)$ reads as follows,

- (i) $\Psi : \mathbb{C} \setminus \mathbb{R} \to \mathbb{C}^{2 \times 2}$ is analytic.

- (ii) $\Psi_+(x) = \Psi_-(x) \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}$ for $x \in \mathbb{R}$ and $x > 0$.

- (iii) $\Psi_+(x) = \Psi_-(x) e^{\pi i \alpha \sigma_3}$ for $x \in \mathbb{R}$ and $x < 0$.

- (iv) $\Psi(z) \sim (I + \sum_{k=1}^\infty \Psi_k z^k) z^{(n+\frac{s}{2})} \sigma_3 e^{-\frac{1}{2} s^2} \sigma_3$ as $z \to \infty$.

- (v) $\Psi(z) \sim Q (I + \sum_{k=1}^\infty \Psi_k z^k) z^{\frac{n}{2}} \sigma_3 e^{-\frac{1}{2} z^2} \sigma_3$ as $z \to 0$,

where the real line $\mathbb{R}$ is oriented as usual from left to right. The coefficients $\Psi_{\pm k}$ of the asymptotic series are easily evaluated via combinations of the coefficients $Y_{\pm k}$ and the coefficients of the expansions of the exponential function $e^{-\frac{1}{2} z^2}$ and $e^{-\frac{1}{2} s^2}$ near $z = \infty$ and $z = 0$, respectively. In particular, we have that,

$$
\Psi_{-1} = \begin{pmatrix} p_1(n) - \frac{s}{2} & -\frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_{n-1}} & -p_1(n) + \frac{s}{2} \end{pmatrix},
$$

(5.9)
The important feature of the $\Psi$-RH problem is that the jump matrices of the jump relations (ii) and (iii) do not depend on $z$, $s$, and $n$. Therefore, by standard arguments based on the Liouville theorem (cf. [17], [18]), we conclude that the logarithmic derivatives,

$$A(z) := \frac{\partial \Psi(z)}{\partial z} \Psi^{-1}(z), \quad B(z) := \frac{\partial \Psi(z)}{\partial s} \Psi^{-1}(z), \quad \text{and} \quad U(z) := \Psi(z, n + 1) \Psi^{-1}(z, n), \quad (5.10)$$

are rational functions of $z$. Using the asymptotic expansions (iv) and (v), we can evaluate the respective principal parts at the poles at the points $z = 0$ and $z = \infty$ and arrive, taking into account (5.9) and (5.6), at the following explicit formulae for the function $A(z)$, $B(z)$, and $U(z)$.

$$A(z) = -\frac{1}{2} (\sigma_3, \Psi) + \frac{A_1}{z} + \frac{A_2}{z^2}, \quad (5.11)$$

$$B(z) = -\frac{A_2}{sz}, \quad (5.12)$$

and

$$U(z) = z \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} + U_0, \quad (5.13)$$

where

$$A_1 = \frac{1}{2} [\sigma_3, \Psi] + \left( n + \frac{\alpha}{2} \right) \sigma_3$$

$$= \begin{pmatrix} n + \frac{\alpha}{2} - \frac{h_n}{2\pi i} \\ \frac{2\pi i}{h_n} - n - \frac{\alpha}{2} \end{pmatrix}, \quad (5.14)$$

$$A_2 = \frac{s}{2} Q \sigma_3 Q^{-1} = \frac{s}{2} \begin{pmatrix} 1 - 2p_n q_n & -2p_n \\ 2q_n (p_n q_n - 1) & 2p_n q_n - 1 \end{pmatrix}, \quad (5.15)$$

and

$$U_0 = \Psi^{-1}(n + 1) \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} - \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \Psi^{-1}(n)$$

$$= \begin{pmatrix} p_1(n + 1) - p_1(n) & \frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_n} & 0 \end{pmatrix} \equiv \begin{pmatrix} -\alpha_n & \frac{h_n}{2\pi i} \\ -\frac{2\pi i}{h_n} & 0 \end{pmatrix}. \quad (5.16)$$

According to the standard methodology (cf. [29], [30], [18]), relations (5.10) should be now re-interpreted as a system of linear differential-difference equations,

$$\begin{cases} \frac{\partial \Psi(z)}{\partial z} = A(z) \Psi(z) \\ \frac{\partial \Psi(z)}{\partial s} = B(z) \Psi(z) \\ \Psi(z, n + 1) = U(z) \Psi(z, n) \end{cases} \quad (5.17)$$

$^3$Notation $[M_1, M_2]$ means the usual commutator of two matrices, $[M_1, M_2] = M_1 M_2 - M_2 M_1$. 

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which we call the Lax triple.

The compatibility conditions of this system, i.e. the equations,

\[
\frac{\partial A(z)}{\partial s} - \frac{\partial B(z)}{\partial z} = [B(z), A(z)] \quad (\Psi_{zs} = \Psi_{sz})
\]

\[
\frac{\partial U(z)}{\partial s} = B(z, n + 1)U(z) - U(z)B(z, n), \quad ((\Psi(z, n + 1))_s = \Psi_s(z, n + 1))
\]

and

\[
\frac{\partial U(z)}{\partial z} = A(z, n + 1)U(z) - U(z)A(z, n), \quad ((\Psi(z, n + 1))_z = \Psi_z(z, n + 1))
\]

yield the Painlevé type (equ. (5.18)), the Toda type (equ. (5.19)) and the discrete Painlevé or
Freud type (equ. (5.20)) equations, respectively, for a proper combination of functions \(h_n, p_n\) and \(q_n\). Moreover, using the Jimbo-Miwa list of the Lax pairs for Painlevé equations [30] and noticing
that the “master equation”, i.e. the first equation of system (5.17) is a \(2 \times 2\) system with two
irregular singular points of Poincare rank 1, one concludes that the relevant Painlevé equation is,
in fact, the third Painlevé equation. In order to make a precise statement, i.e. to point out the
exact combination of functions \(h_n, p_n\) and \(q_n\) which makes the solution of Painlevé III equation,
we only need to perform a simple scaling transformation of system (5.17) which would bring it to
the normal form of [30]. To this end, we introduce the new indepndent variables,

\[
\lambda := s^{-1/2}z, \quad \text{and} \quad t := \sqrt{s},
\]

and pass from the function \(\Psi(z, s)\) to the function \(\Phi(\lambda, t)\) defined by the equation,

\[
\Phi(\lambda, t) := t^{-(n+\frac{\alpha}{2})\sigma_3}\Psi(t\lambda, t^2).
\]

We notice that in terms of the function \(\Phi(\lambda, t)\) the asymptotic relations (iv) and (v) transform
into the relations,

\[
\Phi(\lambda) \sim \left( I + \sum_{k=1}^{\infty} \frac{\Phi_{-k}}{\lambda^k} \right) \lambda^{(n+\frac{\alpha}{2})\sigma_3} e^{-\frac{t\lambda^2}{t} \sigma_3}, \quad \lambda \to \infty,
\]

and

\[
\Phi(\lambda) \sim R \left( I + \sum_{k=1}^{\infty} \Phi_k \lambda^k \right) \lambda^{\sigma_3} e^{-\frac{t}{2\lambda} \sigma_3}, \quad \lambda \to 0,
\]

with the new coefficients connected to the old ones by the equations,

\[
R = t^{-(n+\frac{\alpha}{2})\sigma_3} Q t^\sigma_3, \quad \Phi_{-k} = t^{-k} t^{-(n+\frac{\alpha}{2})\sigma_3} \Psi_{-k} t^{(n+\frac{\alpha}{2})\sigma_3}, \quad \Phi_k = t^k t^{-\frac{\alpha}{2}\sigma_3} \Psi_k t^{\frac{\alpha}{2}\sigma_3}.
\]

Simultaneously, the first two equations of system (5.17) transform into the Jimbo-Miwa Lax pair
for the third Painlevé equation,

\[
\left\{\begin{array}{l}
\frac{\partial \Phi(\lambda)}{\partial \lambda} = \left( -\frac{t}{2} \sigma_3 + \frac{A_{-1}}{\lambda} + \frac{A_{-2}}{\lambda^2} \right) \Phi(\lambda) \equiv A_{JM}(\lambda) \Phi(\lambda) \\
\frac{\partial \Phi(\lambda)}{\partial t} = \left( -\frac{\lambda}{2} \sigma_3 + B_0 + \frac{B_{-1}}{\lambda} \right) \Phi(\lambda) \equiv B_{JM}(\lambda) \Phi(\lambda).
\end{array}\right.
\]

(5.25)
Here, the matrix coefficients $A_{-1}, A_{-2}, B_0,$ and $B_{-1}$ are given by the equations,

$$A_{-1} = t^{-(n+\frac{\alpha}{2})}A_1 t^{(n+\frac{\alpha}{2})} = \begin{pmatrix} -\frac{\theta_\infty}{2} & u \\ v & \frac{\theta_\infty}{2} \end{pmatrix},$$

(5.26)

$$A_{-2} = \frac{1}{t} t^{-(n+\frac{\alpha}{2})}A_1 t^{(n+\frac{\alpha}{2})} = \begin{pmatrix} \zeta + \frac{t}{2} & -w\zeta \\ \frac{\zeta + t}{w} & -\zeta - \frac{t}{2} \end{pmatrix},$$

(5.27)

$$B_0 = \frac{1}{t} A_{-1} - \frac{1}{t} \left( n + \frac{\alpha}{2} \right) \sigma_3 = \frac{1}{t} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix},$$

(5.28)

$$B_{-1} = -\frac{1}{t} A_{-2} = -\frac{1}{t} \begin{pmatrix} \zeta + \frac{t}{2} & -w\zeta \\ \frac{\zeta + t}{w} & -\zeta - \frac{t}{2} \end{pmatrix},$$

(5.29)

where

$$\theta_\infty = -\alpha - 2n,$$

(5.30)

and the new scalar functional parameters $u, v, \zeta,$ and $w$ are defined in terms of the original functions $h_n, h_{n-1}, p_n,$ and $q_n$ via the formulae,

$$u = -\frac{h_n}{2\pi i} t^{-2n-\alpha}, \quad v = \frac{2\pi i}{h_{n-1}} t^{2n+\alpha},$$

$$\zeta = -t p_n q_n, \quad w = -\frac{1}{q_n} t^{-2n-\alpha}.$$

(5.31)

(5.32)

With the $u-w$ notations, the Lax pair (5.25) matches, up to the replacement $t \rightarrow -t$ and the use of the letter $z$ instead of the letter $\zeta$, the Lax pair presented on page 439 of [30], and hence we can use the general results of [30].

**Theorem 5 ([30])** Consider the over-determined linear system (5.25) with the matrix coefficients defined by the equations

$$A_{-1} = \begin{pmatrix} -\frac{\theta_\infty}{2} & u \\ v & \frac{\theta_\infty}{2} \end{pmatrix}, \quad A_{-2} = \begin{pmatrix} \zeta + \frac{t}{2} & -w\zeta \\ \frac{\zeta + t}{w} & -\zeta - \frac{t}{2} \end{pmatrix},$$

(5.33)

$$B_0 = \frac{1}{t} \begin{pmatrix} 0 & u \\ v & 0 \end{pmatrix}, \quad B_{-1} = -\frac{1}{t} \begin{pmatrix} \zeta + \frac{t}{2} & -w\zeta \\ \frac{\zeta + t}{w} & -\zeta - \frac{t}{2} \end{pmatrix},$$

(5.34)

i.e., by the right hand sides of the last equalities in the formulae (5.26) - (5.29) not necessarily assuming any specific, “orthogonal polynomial” choice of the parameters $\theta_\infty, u, v, \zeta$ and $w$. Then the following is true.
1. The compatibility condition of system (5.25), i.e. the matrix equation,
\[ \frac{\partial A_{JM}}{\partial t} - \frac{\partial B_{JM}}{\partial \lambda} = [B_{JM}, A_{JM}], \]  
(5.35)
is equivalent to the following set of scalar equations,
\[ t \frac{du}{dt} = \theta_{\infty} u + 2t \zeta w, \]  
(5.36)
\[ t \frac{dv}{dt} = -\theta_{\infty} v + \frac{2t}{w} (\zeta + t), \]  
(5.37)
\[ t \frac{d\zeta}{dt} = 2\zeta w \zeta + \frac{2u(\zeta + t)}{w}, \]  
(5.38)
\[ t \frac{d\ln w}{dt} = 2u w - 2wv - \theta_{\infty}, \]  
(5.39)
with the quantity,
\[ \theta_{0} := -\frac{\theta_{\infty}}{t} (2\zeta + t) + \frac{2u(\zeta + t)}{tw} - \frac{2\zeta}{t} w, \]  
(5.40)
being the first integral of system (5.36) - (5.39).

2. The quantity \( \theta_{0} \) is very similar to the parameter \( \theta_{\infty} \). Indeed, they describe the formal monodromy at the relevant irregular points; \( \theta_{\infty} \) – at \( \lambda = \infty \) and \( \theta_{0} \) – at \( \lambda = 0 \). This means that they appear as the branching exponents in the following formal matrix solutions of the Lax pair (5.25) at the points \( \lambda = \infty \) and \( \lambda = 0 \).

\[ \Phi_{\lambda}^{(\infty)} = \left( I + \sum_{k=1}^{\infty} \frac{\Phi_{k}^{(\infty)}}{\lambda^{k}} \right) \lambda^{-\frac{\theta_{\infty}}{2} \sigma_{3} e^{-\frac{t}{2} \sigma_{3}}}, \]  
(5.41)
and
\[ \Phi_{\lambda}^{(0)} = R^{(0)} \left( I + \sum_{k=1}^{\infty} \frac{\Phi_{k}^{(0)}}{\lambda^{k}} \right) \lambda^{\frac{\theta_{0}}{2} \sigma_{3} e^{-\frac{t}{2} \sigma_{3}}}. \]  
(5.42)

3. The function
\[ y := -\frac{u}{\zeta w}, \]  
(5.43)
satisfies the third Painlevé equation,
\[ \frac{d^{2}y}{dt^{2}} = \frac{1}{y} \left( \frac{dy}{dt} \right)^{2} - \frac{1}{t} \frac{dy}{dt} - \frac{1}{t} \left( 4\theta_{0} y^{2} + 4(1 - \theta_{\infty}) \right) + 4y^{3} - \frac{4}{y}, \]  
(5.44)

In the standard notations [24], this is the Painlevé III equation, \( P_{III}(-4\theta_{0}, -4(1-\theta_{\infty}), 4, -4) \).

\[ \text{All the coefficients } \Phi_{k}^{(\infty)} \text{ of series (5.41) are uniquely defined as rational functions of } u, v, \zeta, w, \text{ and } t \text{ via simple recurrence relations [29]. The matrix factor } R^{(0)} \text{ is defined, up to a right diagonal multiplier, } R^{(0)} \to R^{(0)} \Lambda, \text{ by the equations, } \frac{1}{4} R^{(0)} \sigma_{3} \left( R^{(0)} \right)^{-1} = A_{-2}, \det R^{(0)} = 1. \text{ The coefficients } \Phi_{k}^{(0)} \text{ of series (5.42) are defined as rational functions of } u, v, \zeta, w, \text{ and } t \text{ up to the conjugation } \Phi_{k}^{(0)} \to \Lambda^{-1} \Phi_{k}^{(0)} \Lambda. \text{ We will discuss this issue in more detail in the next section.} \]
Comparing the general formal expansions (5.41), (5.42) with the asymptotic series (5.23), (5.24), we conclude that in our case the parameters $\theta_{\infty}$ and $\theta_0$ assume the values (see also (5.30)),

$$\theta_{\infty} = -\alpha - 2n, \quad \text{and} \quad \theta_0 = \alpha. \quad (5.45)$$

Simultaneously, from (5.31)-(5.32) we see that in our case,

$$y = \frac{h_n}{2\pi i tp_n}. \quad (5.46)$$

In other words, relation (5.46) provides us with the combination of the functions $h_n, p_n, q_n$ which satisfies the third Painlevé equations and which we have been looking for, while relations (5.45) specifies the parameters of the Painlevé equation. We conclude then that the function $y$ defined in (5.46) satisfies the following Painlevé III equation,

$$\frac{d^2y}{dt^2} = \frac{1}{y} \left( \frac{dy}{dt} \right)^2 - \frac{1}{t} \frac{dy}{dt} - \frac{1}{t} \left( 4\alpha y^2 + 4(2n + 1 + \alpha) \right) + 4y^3 - \frac{4}{y}. \quad (5.47)$$

What is still left for us is to establish the connection between the function $y(t)$ defined in (5.46) and the function $a_n(s)$ defined in (2.7). This is easy; indeed, we have that

$$\int_0^\infty \frac{P_n^2}{x} wdx = \int_0^\infty P(x) \left( x^{n-1} + p_1(n)x^{n-2} + \ldots + \frac{P_n(0)}{x} \right) wdx = P_n(0) \int_0^\infty \frac{P_n}{x} wdx,$$

and hence (see (5.7)),

$$p_n = \frac{h_n a_n}{2\pi i s}; \quad (5.48)$$

which in turn implies that

$$y = \frac{t}{a_n} \equiv \frac{X_n}{t}. \quad (5.49)$$

It is an elementary exercise to check that the substitution,

$$y = \frac{t}{a_n}, \quad t = \sqrt{s},$$

transforms equation (5.47) into equation (3.12) for the quantity $a_n(s)$.

We conclude this section by revealing the connections of some of the key identities established in the previous sections with the general constructions of the isomonodromy theory of Painlevé equations discussed above.

We first notice that, similar to the derivation of equation (5.48), the following

$$q_n = \frac{2\pi i b_n}{h_n a_n}. \quad (5.50)$$

Therefore all the matrix coefficients of the Lax triple (5.17) can be expressed in terms of the functions $a_n, h_n, a_n, b_n$. Actually, we need only to re-write the coefficient $A_2$ from (5.15),

$$A_2 = \frac{s}{2} \begin{pmatrix} 1 - \frac{2h_n}{s} & -\frac{h_n a_n}{\pi i s} \\ \frac{4\pi i b_n}{h_n a_n} \left( \frac{b_n}{s} - 1 \right) & 2 \frac{h_n}{s} - 1 \end{pmatrix}. \quad (5.51)$$
Substituting this representation together with the similar formulae for $A_1$ and $U_0$ (see (5.14) and (5.16)) into the compatibility equation (5.20) yields the set of the scalar difference equations (2.9) - (2.13) of section 2. Similar operation with the compatibility equation (5.19) results in the differential-difference equations (3.1) - (3.6) of section 3. The scalar form of compatibility equation (5.18) we have already discussed in detail in this section. As we have seen, this equation is equivalent to the set of scalar equations (5.36) - (5.39) which, in terms of the functions $h_n(s), h_{n-1}(s), a_n(s), b_n(s)$, transforms to the system,

\begin{align*}
  s \frac{d h_n}{ds} &= -h_n a_n, \\
  s \frac{d h_{n-1}}{ds} &= \frac{h_{n-1}^2}{h_n} \frac{b_n}{a_n} (s - b_n), \\
  s \frac{d b_n}{ds} &= b_n - \frac{b_n}{a_n} (s - b_n) - \frac{h_n}{h_{n-1}} a_n, \\
  s \frac{d a_n}{ds} &= 2b_n + (2n + 1 + \alpha + a_n)a_n - s,
\end{align*}

which is equivalent to the system of equations derived in Lemma 5 of section 3.

We want to highlight the theoretical meaning of the important identity (2.14). It is, in fact, the formal monodromy identity (5.40) written in terms of the functions $h_n(s), h_{n-1}(s), a_n(s), b_n(s)$.

It is also worth mentioning, that the ladder equations (2.1) and (2.2) are just the first column of the master, $z$ - equation of the Lax triple (5.17). The first column of the second equation of the triple (5.17) yields the ladder operators in $s$,

\begin{align*}
  \left( zs \frac{d}{ds} - b_n \right) P_n(z) &= -\beta_n a_n P_{n-1}(z), \\
  \left( zs \frac{d}{ds} - b_{n-1} - \alpha_{n-1} a_{n-1} + za_{n-1} \right) P_{n-1}(z) &= a_{n-1} P_n(z).
\end{align*}

When deriving these equations we made use of the relations (2.10) and (2.13). The equations (5.56) and (5.57) of course can be derived using the orthogonality relations, bypassing the isomonodromy theory.

Finally, the third equation in (5.17) is equivalent to the basic recurrence relations (1.13) for the polynomials $P_n$. It should be pointed out, that for the general solutions of the Painlevé III equations, only the first two equations, which are equivalent under scaling to the Lax pair (5.25), take place. From the point of view of the general isomonodromy theory of Painlevé equations, the third equation of the triple (5.17) describes the Bäcklund-Schlesinger transformation of the third Painlevé equation (see [29]; see also Chapter 6 of [28]).

**6 The Hankel determinant as the isomonodromy $\tau$ - function.**

Let us remind the reader the Jimbo-Miwa definition of the $\tau$-function corresponding to the third Painlevé equation (5.44).
Consider the formal series (5.41) and (5.42) of theorem 5. In the footnote to this theorem we have already mentioned that all the coefficients of these series can be evaluated as rational functions of the parameters $u, v, \zeta, w,$ and $t$. For example, the first coefficients in each of the series are given by the relations (see p. 440 of [30]),

$$\Phi_1^{(\infty)} = \frac{1}{t} \begin{pmatrix} uv - t \zeta - \frac{t^2}{2} & u \\ -v & -uv + t \zeta + \frac{t^2}{2} \end{pmatrix},$$

and

$$\Phi_1^{(0)} = \frac{1}{t} \begin{pmatrix} \tilde{u}v - t \zeta - \frac{t^2}{2} & -\tilde{u} \\ \tilde{v} & -\tilde{u}v + t \zeta + \frac{t^2}{2} \end{pmatrix},$$

where the parameters $\tilde{u}$ and $\tilde{v}$ are related to the basic parameters $u$ and $v$ through the equation,

$$\begin{pmatrix} R^{(0)}(0) \\ R^{(0)}(\infty) \end{pmatrix} = \begin{pmatrix} -\frac{\theta_0}{2} & u \\ v & \frac{\theta_0}{2} \end{pmatrix} \begin{pmatrix} \tilde{u} \\ \tilde{v} \end{pmatrix},$$

which, in particular, means the identity,

$$\tilde{u}v = uv + \frac{\theta_0^2 - \theta_0^2}{4}. \quad (6.4)$$

Denote $Y_\infty(\lambda)$ and $Y_0(\lambda)$ the series in the brackets of formulae (5.41) and (5.42), i.e.

$$Y_\infty(\lambda) = \left( I + \sum_{k=1}^{\infty} \frac{\Phi_1^{(\infty)}}{\lambda^k} \right),$$

and

$$Y_0(\lambda) = \left( I + \sum_{k=1}^{\infty} \frac{\Phi_1^{(0)}}{\lambda^k} \right).$$

The Jimbo-Miwa-Ueno isomonodromy $\tau$-function [29], in the case of the Lax pair (5.25) is defined by the formula,

$$d \ln \tau = -\text{Trace Res}_{\lambda=0} Y_0^{-1}(\lambda) \frac{\partial Y_0}{\partial \lambda}(\lambda) dT_0(\lambda) - \text{Trace Res}_{\lambda=\infty} Y_\infty^{-1}(\lambda) \frac{\partial Y_\infty}{\partial \lambda}(\lambda) dT_\infty(\lambda), \quad (6.7)$$

where

$$dT_0(\lambda) = -\frac{1}{2\lambda} \sigma_3 dt, \quad \text{and} \quad dT_\infty(\lambda) = -\frac{\lambda}{2} \sigma_3 dt.$$

Substituting (6.5) and (6.6) into (6.7), we arrive at the equation,

$$d \ln \tau = \frac{1}{2} \text{Trace} \left( \Phi_1^{(0)} \sigma_3 + \Phi_1^{(\infty)} \sigma_3 \right) dt,$$

which, taking into account (6.1), (6.2), and (6.4), implies that,

$$d \ln \tau(t) = H_{III}(u(t), v(t), \zeta(t); t) dt,$$
where

\[ H_{III}(u, v, \zeta; t) = \frac{1}{t} \left( 2uv - 2t\zeta - t^2 - \frac{\theta_0^2 - \theta_\infty^2}{4} \right). \tag{6.9} \]

Let us now turn to the Hankel determinant \( D_n(s) \) and consider the quantity

\[ H_n := \frac{d}{ds} \ln D_n, \]

which has played a central role in sections 3 and 4. From (3.8) and (2.15) we have that

\[ H_n = n(n + \alpha) + b_n - \frac{ns - (2n + \alpha)b_n}{a_n} + \frac{b_n^2 - sb_n}{a_n^2}. \tag{6.10} \]

At the same time, in our case (see (5.45)),

\[ \theta_0 = \alpha, \quad \theta_\infty = -\alpha - 2n, \quad uv = -\beta_n, \quad \text{and} \quad \zeta = -\frac{b_n}{t}, \]

where for the last two formulae we have used (5.31), (5.32), (5.48) and (5.50). Substituting these formulae into (6.9) we obtain that for our special solution of system (5.36) - (5.39) the function \( H_{III} \) assumes the form,

\[ H_{III} = \frac{1}{t} \left( -2\beta_n + 2b_n - s + n(n + \alpha) \right). \tag{6.11} \]

Recalling now identity (2.14) (which we remind the reader is the formal monodromy identity (5.40) in disguise) we arrive, after some simple algebra, at the relation,

\[ H_{III} = \frac{2}{t} H_n - t - \frac{n(n + \alpha)}{t}. \tag{6.12} \]

This equation, in its turn, implies that

\[ s \frac{d}{ds} \ln \tau = \frac{t}{2} \frac{d}{dt} \ln \tau = \frac{t}{2} H_{III} = H_n - \frac{s}{2} - \frac{n(n + \alpha)}{2} \]

\[ = \frac{s}{ds} \ln D_n - \frac{s}{2} - \frac{n(n + \alpha)}{2}, \tag{6.13} \]

and hence we obtain the following relation between the Hankel determinant \( D_n \) and the \( \tau \)-function of the third Painlevé equation (compare with [26] where similar formula is derived for a class of Toeplitz determinants),

\[ D_n(s) = \text{const} \tau(s)e^{\frac{t}{4}s\frac{n(n + \alpha)}{2}}. \tag{6.14} \]

**Remark V.** The function \( H_{III}(u, v, \zeta; t) \) actually depends on \( \zeta \) and the product \( uv \). The latter, can be expressed, with the help of the formal monodromy relation (5.40) in terms of \( \zeta, t, \) and \( y \). Indeed, we have that

\[ 2uv = (\theta_0 + \theta_\infty)ty + 2y\theta_\infty + 2y^2(\zeta + t), \]

and expression (6.9) can be transformed to the following equation defined \( H_{III} \) as a function of \( y, \zeta \) and \( t \),

\[ H_{III}(y, \zeta; t) = \frac{1}{t} \left( 2y^2\zeta^2 + 2(ty^2 + \theta_\infty y - t)\zeta + (\theta_0 + \theta_\infty)ty - t^2 - \frac{\theta_0^2 - \theta_\infty^2}{4} \right). \tag{6.15} \]
This is the canonical (see again p. 440 of [30]) representation of the logarithmic derivative of the \( \tau \)-function for Painlevé III equation (5.44). The remarkable fact of the general Jimbo-Miwa-Ueno theory is that the function \( H_{III}(y, \zeta; t) \) is the Hamiltonian of the third Painlevé equation.

**Remark VI.** In sections 2 and 3, an important role has been played by equation (2.15) which transforms a *non-local* object - the sum of \( a_j \) from \( j = 0 \) to \( j = n - 1 \), to a *local* expression, which involves only \( a_n \) and \( b_n \). We can see now an intrinsic reason for that. Indeed, on one hand, the sum mentioned is, by its very nature, the logarithmic derivative of the Hankel determinant. On the other hand, the latter is a \( \tau \)-function and hence its logarithmic derivative *must* admit a local representation in view of the general formula (6.7).

**Remark VII.** The isomondromy context for orthogonal polynomials and, in particular, the interpretation of the Hankel determinants as isomonodromy \( \tau \)-functions have been well understood for some time, since the early nineties works [18] and [33]. For the most general semiclassical weights this fact was established in the recent paper [5] (see also [6])

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