PROLIFERATION OF OBSERVABLES AND MEASUREMENT IN QUANTUM-CLASSICAL HYBRIDS

HANS-THOMAS ELZE

Dipartimento di Fisica “Enrico Fermi”, Università di Pisa, Largo Pontecorvo 3, I-56127 Pisa, Italia
elze@df.unipi.it

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Following a review of quantum-classical hybrid dynamics, we discuss the ensuing proliferation of observables and relate it to measurements of (would-be) quantum mechanical degrees of freedom performed by (would-be) classical ones (if they were separable). – Hybrids consist in coupled classical (“CL”) and quantum mechanical (“QM”) objects. Numerous consistency requirements for their description have been discussed and are fulfilled here. We summarize a representation of quantum mechanics in terms of classical analytical mechanics which is naturally extended to QM-CL hybrids. This framework allows for superposition, separable, and entangled states originating in the QM sector, admits experimenter’s ‘Free Will’, and is local and non-signalling. – Presently, we study the set of hybrid observables, which is larger than the Cartesian product of QM and CL observables of its components; yet it is smaller than a corresponding product of all-classical observables. Thus, quantumness and classicality infect each other.

Keywords: quantum-classical hybrid dynamics; oscillator representation; classical control; measurement; hybrid observables

1. Introduction

The hypothetical direct coupling of quantum mechanical and classical degrees of freedom – “hybrid dynamics” – departs from quantum mechanics. We review the theory presented in Refs. [1,2], emphasizing here new aspects of relevant sets of observables related to typical hybrid interactions, which resemble measurements.

Hybrid dynamics has been researched for practical as well as theoretical reasons. For example, the Copenhagen interpretation, as in standard textbooks, entails the measurement problem which, together with the fact that quantum mechanics needs interpretation, in order to be operationally well defined, may indicate that it deserves amendments. It has been recognized earlier that a theory which dynamically crosses the quantum-classical border should have an impact on the measurement problem, as well as on attempts to describe consistently the interaction between quantum matter and classical spacetime.

Numerous works have appeared attempting to formulate a satisfactory hybrid dynamics. Generally, they show deficiencies in one or another respect. Which has
led to various no-go theorems, in view of a lengthy list of desirable properties or consistency requirements, see, for example, Refs. 5, 6:

- Conservation of energy.
- Conservation and positivity of probability.
- Separability of QM and CL subsystems in the absence of their interaction, recovering the correct QM and CL equations of motion, respectively.
- Consistent definitions of states and observables; existence of a Lie bracket structure on the algebra of observables that suitably generalizes Poisson and commutator brackets.
- Existence of canonical transformations generated by the observables; invariance of the classical sector under canonical transformations performed on the quantum sector only and vice versa.
- Existence of generalized Ehrenfest relations (i.e. the correspondence limit) which, for bilinearly coupled CL and QM oscillators, are to assume the form of the CL equations of motion (“Peres-Terno benchmark” test 7).

More recently, these have been followed by more sophisticated considerations trying to single out “t’he” hybrid theory. These require:

- ‘Free Will’ 8.
- Locality.
- No-signalling.
- QM / CL symmetries and ensuing separability carry over to hybrids.

These issues have been reviewed in recent works by Hall and Reginatto, who introduced a form of hybrid dynamics that conforms with the first group of points listed above 9, 10, 11. Their ensemble theory is based on configuration space, which requires a certain nonlinearity of the underlying action functional and entails effects that might allow to falsify this proposal experimentally 12.

We have proposed an alternative theory of hybrid dynamics based on notions of phase space 1. This is partly motivated by work on related topics of general linear dynamics and classical path integrals 13, 14 and extends work by Heslot, demonstrating that quantum mechanics can entirely be rephrased in the language and formalism of classical analytical mechanics 15. Introducing unified notions of states on phase space, observables, canonical transformations, and a generalized quantum-classical Poisson bracket, this has led to an intrinsically linear hybrid theory, which allows to fulfil all of the above consistency requirements.

It has been shown more recently by Burić and collaborators that dynamical aspects of our proposal can indeed be derived for an all-quantum mechanical composite system by imposing constraints on fluctuations in one subsystem, followed by suitable coarse-graining 16.

Objects that somehow reside between classical and quantum mechanics have been described also in a statistical theory, based on very different premises than the hybrid theories considered here 17. It remains to uncover their relation.
We point out that it is of great practical interest to better understand QM-CL hybrids appearing in QM approximation schemes. These typically address many-body systems or interacting fields, which are naturally separable into QM and CL subsystems, for example, representing fast and slow degrees of freedom, mean fields and fluctuations, etc. (keywords: Born-Oppenheimer approximation, mesoscopic systems, CL control of QM objects, “semiclassical quantum gravity”); for references see Ref. [1].

Furthermore, concerning the hypothetical emergence of quantum mechanics from some coarse-grained deterministic dynamics (see, for example, Refs. [18], [19], [20] with numerous references to earlier work), the quantum-classical backreaction problem might appear in new form, namely regarding the interplay of fluctuations among underlying deterministic and emergent QM degrees of freedom. Which can be rephrased succinctly as the question: “Can quantum mechanics be seeded?”

Besides constructing the QM-CL hybrid formalism and showing how it conforms with the above consistency requirements, we earlier discussed the possibility to have classical-environment induced decoherence, quantum-classical backreaction, a deviation from the Hall-Reginatto proposal in presence of translation symmetry, and closure of the algebra of hybrid observables [1]. Questions of locality, symmetry vs. separability, incorporation of superposition, separable, and entangled QM states, and ‘Free Will’ were considered in detail [2].

Presently, we briefly recollect, in Section 2., some of the earlier results, which will be useful in the following. In Section 3., we expand on our previous observation that genuine QM-CL hybrids have an algebra of observables, which is not simply formed by the Cartesian product of those pertaining to the QM and CL sectors, respectively, in the absence of the hybrid interaction [1]. In various ways, this has more recently been mentioned also in Refs. [21], [22]. We will show here that the characteristically enlarged hybrid algebra of observables can actually be understood as a measurement effect exerted by (would-be) CL degrees of freedom on (would-be) QM ones (if they were separable). Some concluding remarks are made in Section 4.

2. Linear quantum-classical hybrid dynamics – a review

The following is a synopsis of classical Hamiltonian mechanics, of its generalization incorporating quantum mechanics by Heslot [15], and of our extension which describes the hypothetical direct coupling between QM and CL degrees of freedom in hybrids [12]. Readers familiar with the earlier derivations may directly pass to Section 3.

2.1. Classical mechanics

Evolution of a classical object is described in relation to its 2n-dimensional phase space, which is its state space. A real-valued regular function on the state space defines an observable, i.e., a differentiable function on this smooth manifold.
There always exist (local) systems of so-called canonical coordinates, commonly denoted by \((x_k, p_k)\), \(k = 1, \ldots, n\), such that the Poisson bracket of any pair of observables \(f, g\) assumes the standard form (Darboux’s theorem) \(^{23}\):

\[
\{f, g\} = \sum_k \left( \frac{\partial f}{\partial x_k} \frac{\partial g}{\partial p_k} - \frac{\partial f}{\partial p_k} \frac{\partial g}{\partial x_k} \right).
\] (1)

This is consistent with \(\{x_k, p_l\} = \delta_{kl}\), \(\{x_k, x_l\} = \{p_k, p_l\} = 0\), \(k, l = 1, \ldots, n\), and reflects the bilinearity, antisymmetry, derivation-like product formula, and Jacobi identity which define a Lie bracket operation, \(f, g \rightarrow \{f, g\}\), mapping a pair of observables to an observable.

General transformations \(G\) of the state space are restricted by compatibility with the Poisson bracket structure to so-called canonical transformations, which do not change physical properties of an object; e.g., a translation, a rotation, a change of inertial frame, or evolution in time. Such \(G\) induces a change of an observable, \(f \rightarrow G(f)\), and is an automorphism of the state space compatible with the Poisson brackets. The set of canonical transformations has a Lie group structure. Therefore, it is sufficient to consider infinitesimal transformations generated by the elements of the corresponding Lie algebra. Then, an infinitesimal transformation \(G\) is canonical, if and only if for any observable \(f\) the map \(f \rightarrow G(f)\) is given by \(f \rightarrow f' = f + \{f, g\} \delta \alpha\), with some observable \(g\), the so-called generator of \(G\), and \(\delta \alpha\) an infinitesimal real parameter. – An infinitesimal canonical transformation of the canonical coordinates, for example, is given by:

\[
x_k \rightarrow x'_k = x_k + \frac{\partial g}{\partial p_k} \delta \alpha,
\] (2)

\[
p_k \rightarrow p'_k = p_k - \frac{\partial g}{\partial x_k} \delta \alpha,
\] (3)

where we employ the Poisson bracket of Eq. (1).

This illustrates the fundamental relation between observables and generators of infinitesimal canonical transformations in classical Hamiltonian mechanics.

2.2. Quantum mechanics

Following Heslot’s work, we learn that the previous analysis can be generalized and applied to quantum mechanics; this concerns the dynamical aspects as well as the notions of states, canonical transformations, and observables.

We recall that the Schrödinger equation and its adjoint can be derived from an action principle as related Hamiltonian equations of motion \(^{1}\). – We must add the normalization condition, for any state vector \(|\Psi\rangle\):

\[
\mathcal{C} := \langle \Psi(t)|\Psi(t)\rangle = \text{constant} \equiv 1,
\] (4)

an essential ingredient of the associated probability interpretation. Furthermore, state vectors that differ by an unphysical constant phase are to be identified. Which
reminds us that the quantum mechanical state space is formed by the rays of an underlying Hilbert space, i.e., forming a complex projective space.

2.2.1. Oscillator representation

A unitary transformation describes QM evolution, $|\Psi(t)\rangle = \hat{U}(t-t_0)|\Psi(t_0)\rangle$, with $U(t-t_0) = \exp[-i\hat{H}(t-t_0)/\hbar]$, which solves the Schrödinger equation. Thus, a stationary state, characterized by $\hat{H}|\phi_i\rangle = E_i|\phi_i\rangle$, with real energy eigenvalue $E_i$, performs a harmonic motion, i.e., $|\psi_i(t)\rangle = \exp[-iE_i(t-t_0)/\hbar]|\phi_i\rangle$. We assume a denumerable set of eigenstates of the Hamilton operator $\hat{H}$.

These findings and the Hamiltonian character of the underlying equations of motion suggest to introduce the oscillator representation. Consider expanding any state vector with respect to a complete orthonormal basis, $\{|\Phi_i\rangle\}$:

$$|\Psi\rangle = \sum_i |\Phi_i\rangle(X_i + iP_i)/\sqrt{2\hbar},$$

where the generally time dependent expansion coefficients are written in terms of real and imaginary parts, $X_i, P_i$. Employing this expansion allows to evaluate what we define as Hamiltonian function, i.e., $\mathcal{H} := \langle \Psi | \hat{H} | \Psi \rangle$:

$$\mathcal{H} = \frac{1}{2\hbar} \sum_{i,j} \langle \Phi_i | \hat{H} | \Phi_j \rangle (X_i - iP_i)(X_j + iP_j) =: \mathcal{H}(X_i, P_i).$$

Choosing the set of energy eigenstates, $\{|\phi_i\rangle\}$, as basis of the expansion, we obtain:

$$\mathcal{H}(X_i, P_i) = \sum_i \frac{E_i}{2\hbar}(P_i^2 + X_i^2),$$

hence the name oscillator representation.

Indeed, evaluating $|\dot{\Psi}\rangle = \sum_i |\dot{\Phi}_i\rangle(X_i + iP_i)/\sqrt{2\hbar}$ according to Hamilton’s equations with the Hamiltonian function of Eq. (6) or (7), gives back the Schrödinger equation. Furthermore, the constraint, Eq. (11), becomes:

$$C(X_i, P_i) = \frac{1}{2\hbar} \sum_i (X_i^2 + P_i^2) \overset{1}{=} 1.$$

Thus, the vector with components given by $(X_i, P_i)$, $i = 1, \ldots, N$, is confined to the surface of a $2N$-dimensional sphere with radius $\sqrt{2\hbar}$, which presents a major difference to CL Hamiltonian mechanics.

Our reasoning, so far, indicates that $(X_i, P_i)$ may serve as canonical coordinates for the state space of a QM object. Next, we introduce a Poisson bracket, similarly as in Subsection 2.1., for any two observables on the spherically compactified state space, i.e. real-valued regular functions $F, G$ of the coordinates $(X_i, P_i)$:

$$\{F, G\} = \sum_i \left( \frac{\partial F}{\partial X_i} \frac{\partial G}{\partial P_i} - \frac{\partial F}{\partial P_i} \frac{\partial G}{\partial X_i} \right).$$
Thus, time evolution of any observable $O$ is generated by the Hamiltonian:

$$\frac{dO}{dt} = \partial_t O + \{O, \mathcal{H}\}, \quad (10)$$

and, in particular, we find that the constraint of Eq. (8) is conserved:

$$\frac{dC}{dt} = \{C, \mathcal{H}\} = 0 \quad (11)$$

### 2.2.2. Canonical transformations and quantum observables

In the following, we explain the compatibility of the notion of observable introduced in passing above – as in classical mechanics – with the usual QM one. This can be demonstrated rigourously by the implementation of canonical transformations and analysis of the role of observables as their generators.

The Hamiltonian function has been defined as observable in Eq. (6), which relates it directly to the corresponding QM observable, namely the expectation of the self-adjoint Hamilton operator. This is indicative of the general structure. Refering to Refs. [11][15] for more details, three most important points are:

- **A)** *Compatibility of unitary transformations and Poisson structure.* – The canonical transformations discussed in Section 2.1. represent automorphisms of classical state space which are compatible with the Poisson brackets. Automorphisms of QM Hilbert space are implemented by unitary transformations. This implies a transformation of the canonical coordinates here, *i.e.*, of the expansion coefficients $X_i, P_i$ introduced in Eq. (5). From this, one derives the invariance of the Poisson brackets under unitary transformations. Consequently, *unitary transformations on Hilbert space are canonical transformations on the $(X,P)$ space.*

- **B)** *Self-adjoint operators as observables.* – Any infinitesimal unitary transformation $\hat{U}$ can be generated by a self-adjoint operator $\hat{G}$, such that:

$$\hat{U} = 1 - \frac{i}{\hbar}\hat{G}\delta\alpha, \quad (12)$$

which leads to the QM relation between an observable and a self-adjoint operator. By a simple calculation, one obtains:

$$X_i \rightarrow X_i' = X_i + \frac{\partial\langle\hat{\Psi}\hat{G}\hat{\Psi}\rangle}{\partial P_i}\delta\alpha, \quad (13)$$

$$P_i \rightarrow P_i' = P_i - \frac{\partial\langle\hat{\Psi}\hat{G}\hat{\Psi}\rangle}{\partial X_i}\delta\alpha. \quad (14)$$

Then, the relation between an observable $G$, defined in analogy to Section 2.1., and a self-adjoint operator $\hat{G}$ can be inferred from Eqs. (13)–(14):

$$G(X_i,P_i) = \langle\hat{\Psi}\hat{G}\hat{\Psi}\rangle, \quad (15)$$

*i.e.*, by comparison with the classical result. Hence, a *real-valued regular function $G$ of the state is an observable, if and only if there exists a self-adjoint operator $\hat{G}$
such that Eq. (15) holds. This implies that all QM observables are quadratic forms in the $X_i$'s and $P_i$'s, which are essentially fewer than in the corresponding CL case; for the generalization necessary when QM-CL hybrids interact, see Section 3.

• C) Commutators as Poisson brackets. – From the relation (15) between observables and self-adjoint operators and the Poisson bracket (9) one derives:

$$\{F,G\} = \langle \Psi | \frac{1}{i\hbar} [\hat{F},\hat{G}] | \Psi \rangle ,$$

with both sides of the equality considered as functions of the variables $X_i, P_i$ and with the commutator defined as usual. Therefore, the commutator is a Poisson bracket with respect to the $(X,P)$ state space and relates the CL algebra of observables, cf. Section 2.1., to the QM algebra of self-adjoint operators.

In conclusion, quantum mechanics shares with classical mechanics an even dimensional state space, a Poisson structure, and a related algebra of observables. It differs essentially by a restricted set of observables and the requirements of phase invariance and normalization, which compactify the underlying Hilbert space to the complex projective space formed by its rays.

2.3. Quantum-classical Poisson bracket, hybrid states and their evolution

The far-reaching parallel of classical and quantum mechanics, as we have seen, suggests a generalized Poisson bracket for any two observables $A, B$ defined on the Cartesian product state space of CL and QM sectors of a hybrid:

$$\{A, B\}_\times := \{A, B\}_{CL} + \{A, B\}_{QM}$$

$$:= \sum_k \left( \frac{\partial A}{\partial x_k} \frac{\partial B}{\partial p_k} - \frac{\partial A}{\partial p_k} \frac{\partial B}{\partial x_k} \right) + \sum_i \left( \frac{\partial A}{\partial X_i} \frac{\partial B}{\partial P_i} - \frac{\partial A}{\partial P_i} \frac{\partial B}{\partial X_i} \right).$$

It shares the usual properties of a Poisson bracket. – Note that due to the convention introduced by Heslot [15] to which we have adhered in Section 2.2, in particular, the QM variables $X_i, P_i$ have dimensions of $(\text{action})^{1/2}$ and, consequently, no $\hbar$ appears in Eqs. (17)–(18). At the expense of introducing appropriate rescalings, these variables could be made to have their usual dimensions and $\hbar$ to appear explicitly here. However, for the remainder of this article, we choose units conveniently such that $\hbar \equiv 1$.

Let an observable “belong” to the CL (QM) sector, if it is constant with respect to the canonical coordinates of the QM (CL) sector. Then, the $\{\ ,\ \}_\times$-bracket has the important properties:

• D) It reduces to the Poisson brackets introduced in Eqs. (1) and (9), respectively, for pairs of observables that belong either to the CL or the QM sector. • E) It reduces to the appropriate one of the former brackets, if one of the observables belongs only to either one of the two sectors. • F) It reflects the separability of CL and QM sectors, since $\{A, B\}_\times = 0$, if $A$ and $B$ belong to different sectors.
Hence, if a canonical transformation is performed on the QM (CL) sector only, then observables that belong to the CL (QM) sector remain invariant.

Next, we recall the hybrid density \( \rho \) defined in Ref.\(^1\) as expectation in a given state \( |\Psi\rangle \) of a self-adjoint density operator \( \hat{\rho} \):

\[
\rho(x_k, p_k; X_i, P_i) := \langle \Psi | \hat{\rho}(x_k, p_k) | \Psi \rangle = \frac{1}{2} \sum_{i,j} \rho_{ij}(x_k, p_k)(X_i - i P_i)(X_j + i P_j)
\]

(19)

using the oscillator expansion, Eq. (5), and \( \rho_{ij}(x_k, p_k) := \langle \Phi_i | \hat{\rho}(x_k, p_k) | \Phi_j \rangle = \rho_{ji}(x_k, p_k) \). It describes a quantum-classical hybrid ensemble by a real-valued, positive semi-definite, normalized, and possibly time dependent regular function, the probability distribution \( \rho \), on the Cartesian product state space canonically coordinated by \( 2(n + N) \)-tuples \( (x_k, p_k; X_i, P_i) \); the variables \( x_k, p_k \), \( k = 1, \ldots, n \) and \( X_i, P_i \), \( i = 1, \ldots, N \) are reserved for the CL and QM sector, respectively.

Expanding \( \hat{\rho} \) in terms of its eigenstates, \( \hat{\rho} = \sum_j w_j |j\rangle \langle j| \), one also obtains:

\[
\rho(x_k, p_k; X_i, P_i) = \sum_j w_j(x_k, p_k) \text{Tr}( |\Psi\rangle \langle j| |\Psi\rangle \langle j| )
\]

(20)

\[
= \sum_j w_j(x_k, p_k) |\langle j|\Psi\rangle|^2
\]

(21)

with \( 0 \leq w_j \leq 1 \) and \( \sum_j \int d^nx d^np w_j(x_k, p_k) = 1 \). This suggests that \( \rho(x_k, p_k; X_i, P_i) \), when properly normalized, is the probability density to find in the hybrid ensemble the QM state \( |\Psi\rangle \), parametrized by \( X_i, P_i \) through Eq. (5), together with the CL state given by a point in phase space, specified by coordinates \( x_k, p_k \).

The content of our definition of the hybrid density \( \rho \) has been further investigated in Ref.\(^2\) with respect to superposition, pure/mixed, or separable/entangled QM states, possibly present before or after QM and CL sectors of a hybrid interact.

Instead of pursuing this, we introduce the appropriate Liouville equation for the dynamical evolution of hybrid ensembles.\(^1\) Based on Liouville’s theorem and the generalized Poisson bracket defined in Eqs. (14–18), we are led to:

\[-\partial_t \rho = \left\{ \rho, \mathcal{H}_\Sigma \right\}_x \]

(22)

with \( \mathcal{H}_\Sigma \equiv \mathcal{H}_\Sigma(x_k, p_k; X_i, P_i) \) and:

\[
\mathcal{H}_\Sigma := \mathcal{H}_\text{CL}(x_k, p_k) + \mathcal{H}_\text{QM}(X_i, P_i) + \mathcal{I}(x_k, p_k; X_i, P_i)
\]

(23)

which defines the relevant Hamiltonian function, including a hybrid interaction; \( \mathcal{H}_\Sigma \) is required to be an observable, in order to have a meaningful notion of energy. Note that energy conservation follows trivially from \( \left\{ \mathcal{H}_\Sigma, \mathcal{H}_\Sigma \right\}_x = 0 \).

An important advantage of Hamiltonian flow and a general property of the Liouville equation in this context is: \(^2\)

- G) The normalization and positivity of the probability density \( \rho \) are conserved in presence of a hybrid interaction; hence, its interpretation remains valid.

However, the simple form of \( \rho \) as a bilinear function of QM “phase space” variables \( X_i, P_i \), stemming from the expectation of a density operator \( \hat{\rho} \), does not
hold generally for interacting QM-CL hybrids. As pointed out in Section 5.4 of Ref. 16, the oscillator expansion of observables, such as in the second of Eqs. (19), has to be generalized to allow for what we named almost-classical observables next.

3. Proliferation of observables by measurement-like interactions

This comes about, since the “classical part” of the bracket, \( \{ A, B \}_{\text{CL}} \), can generate terms which do not qualify as observable with respect to the QM sector; here we assume that \( A \) and \( B \) are both hybrid observables, as defined before. Such terms, in general, are of the form:

\[
\frac{1}{4} \sum_{i,i',j,j'} \{ A_{ij}, B_{i'j'} \}_{\text{CL}} (X_i - iP_i)(X_j + iP_j)(X_{i'} - iP_{i'})(X_{j'} + iP_{j'})
\]

where we used the oscillator expansion, Eq. (5), and:

\[
\{ A_{ij}, B_{i'j'} \}_{\text{CL}} = \sum_k \left( \frac{\partial A_{ij}}{\partial x_k} \frac{\partial B_{i'j'}}{\partial p_k} - \frac{\partial A_{ij}}{\partial p_k} \frac{\partial B_{i'j'}}{\partial x_k} \right),
\]

since, for example, \( A \equiv A(x_k, p_k; X_i, P_i) = \sum_{i,j} A_{ij}(x_k, p_k)(X_i - iP_i)(X_j + iP_j) \).

Thus, evolution of hybrid observables, of the density \( \rho \) in particular, can induce a structural change: while continuing to be CL observables, they do not remain QM observables (quadratic forms in \( X_i, P_i \)).

3.1. Enlarged “classical \times almost-classical algebra” of hybrid observables

In order to maintain formal consistency of the algebraic framework, we assume:

- H) The algebra of hybrid observables is closed under the QM-CL Poisson bracket operation, implemented by \( \{ , \} \times \).

This amounts to a physical hypothesis and its consequences will be discussed in the following.

The normalization constraint, cf. Eq. (8), is preserved under the evolution, since \( \{ C, H \Sigma \}_\times = 0 \), even in presence of QM-CL hybrid interaction. Consistently with closure of the enlarged algebra of hybrid observables, we also obtain:

\[
\{ C(X_i, P_i), G(x_k, p_k; X_i, P_i) \}_\times = \{ C(X_i, P_i), G(x_k, p_k; X_i, P_i) \}_\text{QM} = 0,
\]

where \( G(x_k, p_k; X_i, P_i) \) stands for any element of the enlarged algebra.[11]

We define an almost-classical observable as a real-valued bilinear function of the phase space coordinates \( (X_i, P_i) \) built from pairs of factors like \( (X_i - iP_i)(X_j + iP_j) \), such as in the left-hand side of Eq. (24). This implies:

- I) The QM observables (quadratic forms in phase space coordinates) form a subset of almost-classical observables which, in turn, form a subset of classical observables (real-valued regular functions of phase space coordinates).
Furthermore, elements of the algebra of hybrid observables, generally, are \textit{classical} with respect to coordinates \((x_k, p_k)\) and \textit{almost-classical} with respect to coordinates \((X_i, P_i)\).

The meaning of this enlarged “classical \(\times\) almost-classical algebra” for interacting QM-CL hybrids has been illustrated in Ref. \cite{2} by a Gedankenexperiment, which questions naive expectation that quantum and classical objects evolve separately in quantum and classical ways, when they no longer interact. – The enlargement of the algebra of observables might be a hint that features of QM-CL hybrids could be relevant for how QM emerges. One would like to understand how a large algebra of classical observables (regular functions on phase space) is reduced, via almost-classical observables at an intermediary stage, to a smaller QM algebra (self-adjoint operators on Hilbert space) for an object that becomes “quantized”.

3.2. \textit{Almost-classical observables in a toy model of an interacting QM-CL hybrid}

In order to further illuminate the necessity of an enlarged “classical \(\times\) almost-classical algebra” of observables, as compared to the separable case in the absence of a genuine hybrid interaction, we present a simple model.

Consider a CL object in one dimension together with a two-state QM object, described by phase space coordinates \((x, p)\) and \((X_{\pm}, P_{\pm})\), respectively, in our formalism. We recall the Eqs. \((22)\)–\((23)\): 

\[-\partial_t \rho = \{\rho, \mathcal{H}_\Sigma\} \times, \]

with the Hamiltonian function 

\[\mathcal{H}_\Sigma := \mathcal{H}_{CL}(x, p) + \mathcal{H}_{QM}(X_{\pm}, P_{\pm}) + \mathcal{I}(x, p, X_{\pm}, P_{\pm})\]

and with a particular hybrid interaction, 

\[\mathcal{I}(p, X_{\pm}, P_{\pm}) \equiv \langle \psi | \hat{I}(p) | \psi \rangle, \]

defined by:

\[\hat{I}(p) := g(t)p \hat{\sigma}_z, \tag{27}\]

where \(\hat{\sigma}_z\) is a Pauli matrix and \(g\) stands for a time-dependent coupling.

This model can be easily solved, if we make a few additional simplifications in due course. – Define \(f := \int_0^T dt g(t)\), where \(T\) represents the duration of the interaction. For sufficiently small \(T\) and a strong coupling, we may neglect the influence of \(\mathcal{H}_{CL} + \mathcal{H}_{QM}\) on the evolution of the hybrid density \(\rho\), in comparison with \(\mathcal{I}\). In this case, the Liouville equation can be integrated with the result:

\[\rho(x, p; X_{\pm}, P_{\pm}; T) = \rho(x - f \langle \psi | \hat{\sigma}_z | \psi \rangle, p; X_{\pm}, P_{\pm}; 0), \tag{28}\]

\[\text{i.e., in terms of the initial density}.\]

Let us assume that initially there were no correlations between CL and QM sectors. Therefore, the initial hybrid density is factorized, 

\[\rho(x, p; X_{\pm}, P_{\pm}; 0) = \rho_{CL}(x, p; 0)\rho_{QM}(X_{\pm}, P_{\pm}; 0). \]

Together with Eq. \((23)\), this implies:

\[\rho(x, p; X_{\pm}, P_{\pm}; T) = \rho_{CL}(x - f \langle \psi | \hat{\sigma}_z | \psi \rangle, p; 0)\rho_{QM}(X_{\pm}, P_{\pm}; 0) \]

\[\equiv \rho_{CL}(x - f \langle \psi | \hat{\sigma}_z | \psi \rangle, p; 0)\langle \psi | \hat{\rho}_{QM} | \psi \rangle. \tag{29}\]

The state vector can always be expanded with respect to a twodimensional orthonormal basis \(\{|\pm\rangle\}\), 

\[\sqrt{2}\langle \psi \rangle = (X_+ + iP_+)|+\rangle + (X_- + iP_-)|-\rangle.\]

Furthermore,
if the initial density matrix has a decohered form with respect to $\hat{\sigma}_z$, that is $\hat{\rho}_{QM} = w_+ |+\rangle \langle +| + w_- |-\rangle \langle -|$, with $w_+ + w_- = 1$, then $2 \langle \psi | \hat{\rho}_{QM} | \psi \rangle = w_+ (X_+^2 + P_+^2) + w_- (X_-^2 + P_-^2)$ and $2 \langle \psi | \hat{\sigma}_z | \psi \rangle = (X_+^2 + P_+^2) - (X_-^2 + P_-^2)$.

Recalling also the normalization condition, Eq. (8), here $X_+^2 + P_+^2 + X_-^2 + P_-^2 = 2$, we can interpret the result of Eq. (29) in a simple way. – Correlated with the probability to find a given $|\psi\rangle$ in the initial ensemble state of the QM sector, the position of the CL distribution is shifted by a certain amount to the left or right along the $x$-axis. In particular, if $|\psi\rangle$ has either $|+\rangle$- or $|-\rangle$-component only, then the shift amounts to $x \to x \pm f$, respectively. If, furthermore, the initial distribution is strongly peaked, such that $\rho_{CL}(x - f, p; 0) \rho_{CL}(x + f, p; 0) \approx 0$, then the CL degree of freedom acts like a “pointer” indicating the distribution of results of “spin-up/down”-measurements effected on the QM two-state object:

$$\rho(x, p; X_\pm, P_\pm; T) = \rho_{CL}(x \mp f, p; 0) \cdot w_\pm,$$

where either upper or lower signs apply. This describes an ideal measurement situation, where the CL sector of the hybrid “measures” the QM sector.

We do not expect qualitative features of this model to change, if one or the other of many possible generalizations is incorporated. In particular, looking back at the quite general result in Eq. (28), we see explicitly that $\rho$, as a result of the hybrid interaction chosen in Eq. (27), and for a generic initial $\rho_{CL}$, becomes unavoidably an element of the larger “classical $\times$ almost-classical algebra” of hybrid observables defined in the first part of this Section 3. The way this happens is in accordance with our general discussion above.

Similar effects must show up in the evolution of other hybrid observables $O$, determined by $\frac{dO}{dt} = \partial_t O + \{O, H_{\Sigma}\}_\times$, cf. Eq. (10), in the presence of a genuine hybrid interaction.

4. Concluding remarks

We have presented a review of the linear dynamics of QM-CL hybrids laid out earlier in Refs. 1-2.

We emphasize again that the CL sector of a hybrid does not necessarily present an approximation for some of the quantum mechanical degrees of freedom in a fully quantum mechanical multi-partite system. Rather, we have continued to study presently, whether such a hybrid theory can stand formally on its own and meet all the posed consistency requirements, cf. Section 1.

This has led us here to discuss in more detail the earlier observation that a consistent description of hybrids seems to entail an enlarged algebra of observables, in particular, as compared to the Cartesian product of sets of observables that belong to QM and CL sectors, in the separable case. This observation has been made more recently also in different context 21-22.

We recall that Man’ko and his collaborators repeatedly pointed out that classical states may differ from what could be obtained as the “$\hbar \to 0$” limit of quantum
mechanical ones. Furthermore, they show that all states can be classified by their “tomograms” as 

*either* CL or QM, CL and QM, and *neither* CL nor QM. Yet, in order to understand the origin of these “Man’ko classes”, a dynamical explanation has been missing.

We find the results of Section 3. interesting in this respect, namely that a consistent hybrid description enforces the enlarged algebra of observables, due to genuine hybrid interactions. More specifically, we have seen in detail how QM observables have to be generalized in the form intermediary *almost-classical observables*, which form a subset of corresponding classical observables. This corresponds to Man’ko’s findings in our approach, where all states are represented in phase space, and provides a dynamical explanation.

In a simple model, we have shown that this enlargement of the set of observables is not only necessary but very welcome. It is generated dynamically and accommodates the measurement-like effect of the hybrid interactions. Through them, the would-be (in the separable case) CL degrees of freedom of the hybrid perform measurements on the would-be QM ones.

It would be most interesting, if one could similarly find some underlying dynamical reason for such *structural change* that occurs when, conversely, a CL object is turned into a QM object, *i.e.*, when it becomes quantized.

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