Construction of Biorthogonal Wavelet Packets on Local Fields of Positive Characteristic

F. A. Shah* and M. Y. Bhat**

*Department of Mathematics, University of Kashmir
South Campus, Anantnag-192101
Jammu and Kashmir, India
fashah79@gmail.com

**Department of Mathematics, Central University of Jammu
Jammu-180011, Jammu and Kashmir, India
gyounusg@gmail.com

Abstract: Orthogonal wavelet packets lack symmetry which is a much desired property in image and signal processing. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality. In the present paper, we construct biorthogonal wavelet packets on local fields of positive characteristic and investigate their properties by means of the Fourier transforms. We also show how to obtain several new Riesz bases of the space $L^2(K)$ by constructing a series of subspaces of these wavelet packets. Finally, we provide the algorithms for the decomposition and reconstruction using these biorthogonal wavelet packets.

Keywords: Wavelet; multiresolution analysis; scaling function; wavelet packet; Riesz basis; local field; Fourier transform

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1. Introduction

A field $K$ equipped with a topology is called a local field if both the additive $K^+$ and multiplicative groups $K^*$ of $K$ are locally compact Abelian groups. The local fields are essentially of two types: zero and positive characteristic (excluding the connected local fields $\mathbb{R}$ and $\mathbb{C}$). Examples of local fields of characteristic zero include the $p$-adic field $\mathbb{Q}_p$ where as local fields of positive characteristic are the Cantor dyadic group and the Vilenkin $p$-groups. Even though the structures and metrics of local fields of zero and positive characteristics are similar, their wavelet and multiresolution analysis theory are quite different. In recent years, local fields have attracted the attention of several mathematicians, and have found innumerable applications not only to number theory but also to representation theory, division algebras, quadratic forms and algebraic geometry. As a result, local fields are now consolidated as part of the standard repertoire of contemporary mathematics. For more about local fields and their applications, we refer to the monographs [15, 21].
In recent years, there has been a considerable interest in the problem of constructing wavelet bases on various spaces other than \( \mathbb{R} \), such as abstract Hilbert spaces [20], locally compact Abelian groups [8], Cantor dyadic groups [11], \( p \)-adic fields [10] and zero-dimensional groups [14]. Recently, R.L. Benedetto and J.J. Benedetto [3] developed a wavelet theory for local fields and related groups. They did not develop the multiresolution analysis (MRA) approach, their method is based on the theory of wavelet sets. The concept of multiresolution analysis on a local field \( K \) of positive characteristic was introduced by Jiang et al. [9]. They pointed out a method for constructing orthogonal wavelets on local field \( K \) with a constant generating sequence and derived necessary and sufficient conditions for a solution of the refinement equation to generate a multiresolution analysis of \( L^2(K) \). Subsequently, the tight wavelet frames on local fields were constructed by Li and Jiang in [13]. They have established necessary condition and sufficient conditions for tight wavelet frame on local fields in frequency domain. Shah and Debnath [18] have constructed tight wavelet frames on local field \( K \) of positive characteristic via extension principles. They also provide a sufficient condition for finite number of functions to form a tight wavelet frame and established general principle for constructing tight wavelet frames on local fields. On the other hand, Behera and Jahan [1] have constructed biorthogonal wavelets on a local field \( K \) of positive characteristic and showed that if \( \phi \) and \( \tilde{\phi} \) are the dual scaling functions associated with dual MRA’s on local field \( K \) of positive characteristic such that their translates are biorthogonal, then the corresponding wavelet families are also biorthogonal.

It is well-known that the classical orthogonal wavelet bases have poor frequency localization. To overcome this disadvantage, Coifman et al. [7] constructed univariate orthogonal wavelet packets. Well known Daubechies orthogonal wavelets are a special case of wavelet packets. Later on, Chui and Li [5] generalized the concept of orthogonal wavelet packets to the case of non-orthogonal wavelet packets so that they can be applied to the spline wavelets and so on. The introduction of biorthogonal wavelet packets attributes to Cohen and Daubechies [6]. They have also shown that all the wavelet packets, constructed in this way, are not led to Riesz bases for \( L^2(\mathbb{R}) \). Shen [19] generalized the notion of univariate orthogonal wavelet packets to the case of multivariate wavelet packets. Other notable generalizations are the orthogonal version of vector-valued wavelet packets [4], multiwavelet packets [12], orthogonal and biorthogonal wavelet packets related to the Walsh polynomials on a positive half-line \( \mathbb{R}^+ \) [16,17].

Recently, Behera and Jahan [2] have constructed orthogonal wavelet packets and wavelet frame packets on local field \( K \) of positive characteristic and show how to construct an orthonormal basis from a Riesz basis. Orthogonal wavelet packets have many desired properties such as compact support, good frequency localization and vanishing moments. However, there is no continuous symmetry which is a much desired property in imaging the compression and signal processing. To achieve symmetry, several generalizations of scalar orthogonal wavelet packets have been investigated in literature. The biorthogonal wavelet packets achieve symmetry where the orthogonality is replaced by the biorthogonality. Therefore, the objective of this paper is to construct biorthogonal wavelet packets on local fields of positive characteristic and investigate their properties by means of the Fourier transforms and construct several new Riesz bases of space \( L^2(K) \). Finally, we
establish some algorithms for decomposition and reconstruction using these biorthogonal wavelet packets.

This paper is organized as follows. In Section 2, we discuss some preliminary facts about local fields of positive characteristic and also some results which are required in the subsequent sections including the definition of an MRA on local fields. In Section 3, we examined some of the properties of the biorthogonal wavelet packets via Fourier transforms. In Section 4, we generate Riesz bases of \(L^2(K)\) from these wavelet packets. Section 5, deals with the decomposition and reconstruction algorithms corresponding to these wavelet packets.

2. Preliminaries on Local Fields

Let \(K\) be a field and a topological space. Then \(K\) is called a local field if both \(K^+\) and \(K^*\) are locally compact Abelian groups, where \(K^+\) and \(K^*\) denote the additive and multiplicative groups of \(K\), respectively. If \(K\) is any field and is endowed with the discrete topology, then \(K\) is a local field. Further, if \(K\) is connected, then \(K\) is either \(\mathbb{R}\) or \(\mathbb{C}\). If \(K\) is not connected, then it is totally disconnected. Hence by a local field, we mean a field \(K\) which is locally compact, non-discrete and totally disconnected. The \(p\)-adic fields are examples of local fields. More details are referred to [15, 21]. In the rest of this paper, we use \(\mathbb{N}, \mathbb{N}_0\) and \(\mathbb{Z}\) to denote the sets of natural, non-negative integers and integers, respectively.

Let \(K\) be a fixed local field. Then there is an integer \(q = p^r\), where \(p\) is a fixed prime element of \(K\) and \(r\) is a positive integer, and a norm \(|.|\) on \(K\) such that for all \(x \in K\) we have \(|x| \geq 0\) and for each \(x \in K \setminus \{0\}\) we get \(|x| = q^k\) for some integer \(k\). This norm is non-Archimedean, that is \(|x + y| \leq \max\{|x|, |y|\}\) for all \(x, y \in K\) and \(|x + y| = \max\{|x|, |y|\}\) whenever \(|x| \neq |y|\). Let \(dx\) be the Haar measure on the locally compact, topological group \((K,+).\) This measure is normalized so that \(\int_K dx = 1\), where \(\mathcal{O} = \{x \in K : |x| \leq 1\}\) is the ring of integers in \(K\). Define \(\mathfrak{B} = \{x \in K : |x| < 1\}\). The set \(\mathfrak{B}\) is called the prime ideal in \(K\). The prime ideal in \(K\) is the unique maximal ideal in \(\mathcal{O}\) and hence as result \(\mathfrak{B}\) is both principal and prime. Therefore, for such an ideal \(\mathfrak{B}\) in \(\mathcal{O}\), we have \(\mathfrak{B} = \langle p \rangle = p\mathcal{O}\).

Let \(\mathcal{O}' = \mathcal{O} \setminus \mathfrak{B} = \{x \in K : |x| = 1\}\). Then, it is easy to verify that \(\mathcal{O}'\) is a group of units in \(K^+\) and if \(x \neq 0\), then we may write \(x = p^kx', x' \in \mathcal{O}^*\). Moreover, each \(\mathfrak{B}^k = p^k\mathcal{O} = \{x \in K : |x| < q^{-k}\}\) is a compact subgroup of \(K^+\) and usually known as the fractional ideals of \(K^+\) (see [15]). Let \(U = \{a_i\}_{i=0}^{q-1}\) be any fixed full set of coset representatives of \(\mathfrak{B}\) in \(\mathcal{O}\), then every element \(x \in K\) can be expressed uniquely as \(x = \sum_k c_k p^k\) with \(c_i \in U\). Let \(\chi\) be a fixed character on \(K^+\) that is trivial on \(\mathcal{O}\) but is non-trivial on \(\mathfrak{B}^{-1}\). Therefore, \(\chi\) is constant on cosets of \(\mathcal{O}\) so if \(y \in \mathfrak{B}^k\), then \(\chi_y(x) = \chi(yx), x \in K\). Suppose that \(\chi_u\) is any character on \(K^+\), then clearly the restriction \(\chi_u|\mathcal{O}\) is also a character on \(\mathcal{O}\). Therefore, if \(\{u(n) : n \in \mathbb{N}_0\}\) is a complete list of distinct coset representative of \(\mathcal{O}\) in \(K^+\), then it is proved in [21] that the set \(\{\chi_{u(n)} : n \in \mathbb{N}_0\}\) of distinct characters on \(\mathcal{O}\) is a complete orthonormal system on \(\mathcal{O}\).
The Fourier transform \( \hat{f} \) of a function \( f \in L^1(K) \cap L^2(K) \) is defined by
\[
\hat{f}(\xi) = \int_K f(x) \chi_{\xi}(x) dx.
\]
(2.1)

It is noted that
\[
\hat{f}(\xi) = \int_K f(x) \chi_{\xi}(x) dx = \int_K f(x) \chi(-\xi x) dx.
\]

Furthermore, the properties of Fourier transform on local field are much similar to those on the real line. In particular Fourier transform is unitary on \( L^2(K) \).

Let us now impose a natural order on the sequence \( \{u(n)\}_{n=0}^{\infty} \). Since \( \mathcal{D}/\mathcal{B} \cong GF(q) \) where \( GF(q) \) is a \( c \)-dimensional vector space over the field \( GF(q) \), we choose a set \( \{1 = \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1}\} \subset \mathcal{D}^* \) such that \( \text{span}\{\zeta_j\}_{j=0}^{c-1} \cong GF(q) \). Let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( n \in \mathbb{N}_0 \) such that \( 0 \leq n < q \), we have
\[
 n = a_0 + a_1 p + \cdots + a_{c-1} p^{c-1}, \ 0 \leq a_k < p, \ k = 0, 1, \ldots, c-1.
\]
Define
\[
u(n) = (a_0 + a_1 \zeta_1 + \cdots + a_{c-1} \zeta_{c-1}) p^{-1}.
\]
(2.2)

For \( n \in \mathbb{N}_0 \) and \( 0 \leq b_k < q, k = 0, 1, 2, \ldots, s \), we write
\[
 n = b_0 + b_1 q + b_2 q^2 + \cdots + b_s q^s,
\]
such that
\[
u(n) = \nu(b_0) + \nu(b_1) p^{-1} + \cdots + \nu(b_s) p^{-s}.
\]
(2.3)

If \( r, k \in \mathbb{N}_0 \) and \( 0 \leq s < q^k \), then it follows that
\[
u(rq^k + s) = \nu(r) p^{-k} + \nu(s).
\]

Further, it is easy to verify that \( \nu(n) = 0 \) if and only if \( n = 0 \) and \( \{\nu(\ell) + \nu(k) : k \in \mathbb{N}_0\} = \{\nu(k) : k \in \mathbb{N}_0\} \) for a fixed \( \ell \in \mathbb{N}_0 \). From now we will write \( \chi_n \) for \( \chi_{\nu(n)} \).

Let the local field \( K \) be of characteristic \( p > 0 \) and \( \zeta_0, \zeta_1, \zeta_2, \ldots, \zeta_{c-1} \) be as above. We define a character \( \chi \) on \( K \) as follows:
\[
\chi(\zeta_\mu) = \begin{cases} 
\exp(2\pi i/p), & \mu = 0 \text{ and } j = 1, \\
1, & \mu = 1, \ldots, c - 1 \text{ and } j \neq 1.
\end{cases}
\]
(2.4)

**Definition 2.1.** Let \( \{f_n\}_{n \in \mathbb{N}_0} \) be a sequence of a Hilbert space \( H \). Then, \( \{f_n\}_{n \in \mathbb{N}_0} \) is said to form a Riesz basis for \( H \) if

(a) \( \{f_n\}_{n \in \mathbb{N}_0} \) is linearly independent, and
(b) there exists constants $A$ and $B$ with $0 < A \leq B < \infty$ such that

$$A\|f\|_2^2 \leq \sum_{k \in \mathbb{N}_0} |\langle f, f_n \rangle|^2 \leq B\|f\|_2^2 \quad \text{for every } f \in H.$$  \hfill (2.5)

Let us recall the definition of an MRA on local fields of positive characteristics ([9]).

**Definition 2.2.** Let $K$ be a local field of positive characteristic $p > 0$ and $\mathfrak{p}$ be a prime element of $K$. A multiresolution analysis (MRA) of $L^2(K)$ is a sequence of closed subspaces $\{V_j : j \in \mathbb{Z}\}$ of $L^2(K)$ satisfying the following properties:

(a) $V_j \subset V_{j+1}$ for all $j \in \mathbb{Z}$;

(b) $\bigcup_{j \in \mathbb{Z}} V_j$ is dense in $L^2(K)$;

(c) $\bigcap_{j \in \mathbb{Z}} V_j = \{0\}$;

(d) $f(\cdot) \in V_j$ if and only if $f(\mathfrak{p}^{-j} \cdot) \in V_{j+1}$ for all $j \in \mathbb{Z}$;

(e) there is a function $\varphi \in V_0$, called the scaling function, such that $\{\varphi(\cdot - u(k)) : k \in \mathbb{N}_0\}$ forms a Riesz basis for $V_0$.

Since $\varphi \in V_0 \subset V_1$ and $\{\varphi_{1,k} : k \in \mathbb{N}_0\}$ is a Riesz basis of $V_1$, we have

$$\varphi(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k \varphi(\mathfrak{p}^{-1}x - u(k)), \quad (2.6)$$

where $a_k = \langle \varphi, \varphi_{1,k} \rangle$ and $\{a_k\}_{k \in \mathbb{N}_0} \in \ell^2(\mathbb{N}_0)$. Taking Fourier transform of equation (2.6), we get

$$\hat{\varphi}(x) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \chi_k(\mathfrak{p}x) \hat{\varphi}(\mathfrak{p}x)$$

$$= m_0(\mathfrak{p}x) \hat{\varphi}(\mathfrak{p}x), \quad (2.7)$$

where $m_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k \chi_k(\xi)$.

Let $W_j, j \in \mathbb{Z}$ be the direct complementary subspace of $V_j$ in $V_{j+1}$. Assume that there exists $q - 1$ functions $\{\psi_1, \psi_2, \ldots, \psi_{q-1}\}$ in $L^2(K)$ such that their translates and dilations form Riesz bases of $W_j$, i.e.,

$$W_j = \overline{\text{span}} \left\{ q^{j/2} \psi_\ell(\mathfrak{p}^{-j} \cdot - u(k)) \right\}, \quad j \in \mathbb{Z}, \quad k \in \mathbb{N}_0, \quad 1 \leq \ell \leq q - 1. \quad (2.8)$$

Since $\psi_\ell \in W_0 \subset V_1, 1 \leq \ell \leq q - 1$, there exist a sequence $\{a_k^\ell\} \in \ell^2(\mathbb{N}_0)$ such that

$$\psi_\ell(x) = \sum_{k \in \mathbb{N}_0} a_k^\ell q^{j/2} \varphi(\mathfrak{p}^{-1}x - u(k)), \quad 1 \leq \ell \leq q - 1. \quad (2.9)$$
Equation (2.9) can be written in the frequency domain as

\[ \hat{\psi}_\ell(x) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \chi_k(p\xi) \hat{\varphi}(p\xi) \]

\[ = m_\ell(p\xi) \hat{\varphi}(p\xi), \]

where \( m_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} a_k^\ell \chi_k(\xi), \quad 1 \leq \ell \leq q - 1. \)

**Definition 2.3.** Let \( f, \hat{f} \in L^2(K) \) be given. We say that they are biorthogonal if

\[ \left\langle f(\cdot), \hat{f}(\cdot - u(k)) \right\rangle = \delta_{0,k}, \quad (2.11) \]

where \( \delta_{0,k} \) is the Kronecker’s delta function.

If \( \varphi(\cdot), \hat{\varphi}(\cdot) \in L^2(K) \) are a pair of biorthogonal scaling functions, then we have

\[ \left\langle \varphi(\cdot), \hat{\varphi}(\cdot - u(k)) \right\rangle = \delta_{0,k}, \quad k \in \mathbb{N}_0. \quad (2.12) \]

Further, we say that \( \psi_\ell(\cdot), \hat{\psi}_\ell(\cdot) \in L^2(K), \quad 1 \leq \ell \leq q - 1 \) are a pair of biorthogonal wavelets associated with a pair of biorthogonal scaling functions \( \varphi(\cdot), \hat{\varphi}(\cdot) \in L^2(K) \) if, the set \( \{ \psi_\ell(\cdot - u(k)) : k \in \mathbb{N}_0, 1 \leq \ell \leq q - 1 \} \) forms a Riesz basis of \( W_0 \), and

\[ \left\langle \varphi(\cdot), \hat{\psi}_\ell(\cdot - u(k)) \right\rangle = 0, \quad k \in \mathbb{N}_0, \quad 1 \leq \ell \leq q - 1, \quad (2.13) \]

\[ \left\langle \hat{\varphi}(\cdot), \psi_\ell(\cdot - u(k)) \right\rangle = 0, \quad k \in \mathbb{N}_0, \quad 1 \leq \ell \leq q - 1, \quad (2.14) \]

\[ \left\langle \psi_\ell(\cdot), \psi_{\ell'}(\cdot - u(k)) \right\rangle = \delta_{\ell,\ell'} \delta_{0,k}, \quad k \in \mathbb{N}_0, \quad 1 \leq \ell, \ell' \leq q - 1. \quad (2.15) \]

For \( \ell = 1, 2, \ldots, q - 1 \), we have

\[ W^\ell_j = \text{span} \left\{ q^{j/2} \psi_\ell(p^{-j} \cdot - u(k)) \right\}, \quad j \in \mathbb{Z}, \quad k \in \mathbb{N}_0. \quad (2.16) \]

Using the definition of \( W_j \) and identities (2.13)-(2.15), we have the following result:

**Proposition 2.4** [1]. If \( \psi_\ell(\cdot), \hat{\psi}_\ell(\cdot) \in L^2(K), \quad 1 \leq \ell \leq q - 1 \) are a pair of biorthogonal wavelets associated with a pair of biorthogonal scaling functions \( \varphi(\cdot), \hat{\varphi}(\cdot) \in L^2(K) \), then

\[ L^2(K) = \bigoplus_{j \in \mathbb{Z}} W_j = \bigoplus_{j \in \mathbb{Z}} \bigoplus_{\ell=1}^{q-1} W_j. \quad (2.17) \]

In the biorthogonal setting, the refinement equation and wavelet equation are much similar to the equations (2.6) and (2.9).
\[ \hat{\varphi}(x) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \hat{a}_k \hat{\varphi}(p^{-1}x - u(k)), \] (2.18)

and

\[ \hat{\psi}_\ell(x) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \hat{a}_k \hat{\psi}(p^{-1}x - u(k)), \quad 1 \leq \ell \leq q - 1. \] (2.19)

Taking Fourier transform of equations (2.18) and (2.19), we obtain

\[ \hat{\varphi}(x) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \hat{a}_k \chi_k(\xi) \hat{\varphi}(p \xi) \]

\[ = \tilde{m}_0(p \xi) \hat{\varphi}(p \xi), \] (2.20)

where \( \tilde{m}_0(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \hat{a}_k \chi_k(\xi), \) and

\[ \hat{\psi}_\ell(x) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \hat{a}_k \chi_k(\xi) \hat{\varphi}(p \xi) \]

\[ = \tilde{m}_\ell(p \xi) \hat{\varphi}(p \xi), \] (2.21)

where \( \tilde{m}_\ell(\xi) = \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \hat{a}_k \chi_k(\xi), \quad 1 \leq \ell \leq q - 1. \)

Moreover, it is proved in [1] that if \( \varphi(\cdot), \hat{\varphi}(\cdot) \in L^2(K) \) are a pair of biorthogonal scaling functions associated with the MRA, then the system of functions \( \{ \varphi(\cdot - u(k)) : k \in \mathbb{N}_0 \} \) is biorthogonal to \( \{ \hat{\varphi}(\cdot - u(k)) : k \in \mathbb{N}_0 \} \) if and only if

\[ \sum_{k \in \mathbb{N}_0} \hat{\varphi}(\xi + u(k)) \hat{\varphi}(\xi + u(k)) = 1 \quad a.e. \] (2.22)

3. Biorthogonal Wavelet Packets on Local Fields

For \( n = 0, 1, \ldots, \) the basic wavelet packets associated with a scaling function \( \varphi(\cdot) \) on a local field \( K \) of positive characteristic are defined recursively by

\[ \omega_n(x) = \omega_{qr+s}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} a_k^s \omega_r(p^{-1}x - u(k)), \quad 0 \leq s \leq q - 1 \] (3.1)

where \( r \in \mathbb{N}_0 \) is the unique element such that \( n = qr + s, \) \( 0 \leq s \leq q - 1 \) holds (see [2]).

Similar to the orthogonal wavelet packets, the biorthogonal wavelet packets associated with \( \hat{\varphi}(\cdot) \) are given by
\[ \tilde{\omega}_n(x) = \tilde{\omega}_{qr+s}(x) = \sqrt{q} \sum_{k \in \mathbb{N}_0} \tilde{a}_k \tilde{\omega}_r(p^{-1}x - u(k)), \quad 0 \leq s \leq q - 1. \] (3.2)

Clearly, for \( r = 0 \) and \( 1 \leq s \leq q - 1 \), we have
\[ \omega_0(\cdot) = \varphi(\cdot), \quad \tilde{\omega}_0(\cdot) = \tilde{\varphi}(\cdot), \quad \omega_s(\cdot) = \psi_\ell(\cdot), \quad \tilde{\omega}_s(\cdot) = \tilde{\psi}_\ell(\cdot), \quad 1 \leq \ell \leq q - 1. \]

Also, the Fourier transform of (3.1) and (3.2) gives
\[ \hat{\omega}_{qr+s}(\xi) = m_s(p\xi) \hat{\omega}_r(p\xi), \quad \hat{\tilde{\omega}}_{qr+s}(\xi) = \tilde{m}_s(p\xi) \hat{\tilde{\omega}}_r(p\xi). \] (3.3)

We are now in a position to discuss the biorthogonality properties for these wavelet packets by means of the Fourier transform.

**Lemma 3.1.** Assume that \( \omega_s(x), \tilde{\omega}_s(x) \in L^2(K) \) are a pair of biorthogonal wavelets associated with a pair of biorthogonal scaling functions \( \omega_0(x), \tilde{\omega}_0(x) \). Then we have
\[ \sum_{\ell=0}^{q-1} m_r(p\xi + pu(\ell)) \overline{m_s(p\xi + pu(\ell))} = \delta_{r,s}, \quad 0 \leq r, s \leq q - 1. \] (3.5)

**Proof.** For given \( 0 \leq r, s \leq q - 1 \), we have
\[
\delta_{r,s} = \sum_{k \in \mathbb{N}_0} \omega_r(\xi + u(k)) \overline{\tilde{\omega}_r(\xi + u(k))} \\
= \sum_{k \in \mathbb{N}_0} m_r(p\xi + pu(k)) \overline{\tilde{m}_s(p\xi + pu(k))} \\
= \sum_{\ell=0}^{q-1} \sum_{k \in \mathbb{N}_0} m_r(p\xi + pu(qk + \ell)) \overline{\tilde{m}_s(p\xi + pu(qk + \ell))} \\
\times \left\{ \sum_{k \in \mathbb{N}_0} \tilde{\omega}_0(p\xi + pu(qk + \ell)) \overline{\tilde{\omega}_0(p\xi + pu(qk + \ell))} \right\} \\
= \sum_{\ell=0}^{q-1} \left( m_r(p\xi + pu(\ell)) \overline{m_s(p\xi + pu(\ell))} \right) \\
\times \left\{ \sum_{k \in \mathbb{N}_0} \tilde{\omega}_0(p\xi + pu(qk + \ell)) \overline{\tilde{\omega}_0(p\xi + pu(qk + \ell))} \right\} \\
= \sum_{\ell=0}^{q-1} m_r(p\xi + pu(\ell)) \overline{m_s(p\xi + pu(\ell))}. 
\]
Theorem 3.2. If \{ω_n(x) : n ∈ ℓ \} and \{\tilde{ω}_n(x) : n ∈ ℓ \} are wavelet packets associated with a pair of biorthogonal scaling functions ω₀(x) and \tilde{ω}_0(x), respectively. Then, we have

\[
\left\langle ω_n(·), \tilde{ω}_n(· - u(k)) \right\rangle = \delta_{0,k}, \quad k ∈ ℤ, \; n ∈ ℓ.
\] (3.6)

Proof. We will prove this result by using induction on \( n < t \), where \( t ∈ ℤ \). It follows from (2.12) and (2.15) that the claim is true for \( n = 0 \) and \( n = 1, 2, \ldots, q - 1 \). Assume (3.6) holds for \( n < t \), where \( t ∈ ℤ \). Then, we prove the result (3.6) for \( n = t \). Let \( n = qr + s \), where \( r ∈ ℤ, 0 ≤ s ≤ q - 1 \), and \( r < n \). Therefore, by the inductive assumption, we have

\[
\left\langle ω_t(·), \tilde{ω}_t(· - u(k)) \right\rangle = \delta_{0,k} \iff \sum_{k ∈ ℤ} ω_t(ξ + u(k)) \tilde{ω}_t(ξ + u(k)) = 1.
\]

Using Lemmas 2.5, 3.1 and equations (3.3) and (3.4), we obtain

\[
\left\langle ω_n(·), \tilde{ω}_n(· - u(k)) \right\rangle
= \left\langle \hat{ω}_n(·), \hat{ω}_n(· - u(k)) \right\rangle
= \int_K \tilde{ω}_{qr+s}(ξ) \tilde{ω}_{qr+s}(ξ) \chi_k(ξ) dξ
= \int_K m_s(θξ) \tilde{ω}_r(θξ) \overline{m_s(θξ)} \tilde{ω}_r(θξ) \chi_k(ξ) dξ
= \int_D \sum_{k ∈ ℤ} m_s(θξ + pu(k)) \tilde{ω}_r(θξ + pu(k)) \times \overline{m_s(θξ + pu(k))} \tilde{ω}_r(θξ + pu(k)) \chi_k(ξ) dξ
= \int_D \sum_{ℓ=0}^{q-1} \sum_{k ∈ ℤ} m_s(θξ + pu(qk + ℓ)) \tilde{ω}_r(θξ + pu(qk + ℓ)) \times \overline{m_s(θξ + pu(qk + ℓ))} \tilde{ω}_r(θξ + pu(qk + ℓ)) \chi_k(ξ) dξ
= \int_D \sum_{ℓ=0}^{q-1} m_s(θξ + pu(ℓ)) \overline{m_s(θξ + pu(ℓ))}
\times \left\{ \sum_{k ∈ ℤ} \tilde{ω}_r(θξ + pu(qk + ℓ)) \tilde{ω}_r(θξ + pu(qk + ℓ)) \chi_k(ξ) dξ \right\}
= \int_D \sum_{ℓ=0}^{q-1} m_s(θξ + pu(ℓ)) \overline{m_s(θξ + pu(ℓ))} \chi_k(ξ) dξ
= \delta_{0,k}.
\]
Theorem 3.3. Suppose \( \{\omega_n(x) : n \in \mathbb{N}_0\} \) and \( \{\tilde{\omega}_n(x) : n \in \mathbb{N}_0\} \) are the biorthogonal wavelet packets associated with a pair of biorthogonal scaling functions \( \omega_0(x) \) and \( \tilde{\omega}_0(x) \), respectively. Then, we have

\[
\left\langle \omega_{qr+s_1} \left( \cdot, -u(k) \right) \rightangle, \tilde{\omega}_{qr+s_2} \left( \cdot, -u(k) \right) \right \rangle = \delta_{0,k} \delta_{s_1,s_2}, \quad 0 \leq s_1, s_2 \leq q - 1, \quad r, k \in \mathbb{N}_0.
\] (3.7)

Proof. By Lemma 2.5, we have

\[
\left\langle \omega_{qr+s_1} \left( \cdot, -u(k) \right) \rightangle = \int_k^\omega \hat{\omega}_{qr+s_2} (\xi) \chi_k(\xi) d\xi
\]

\[
= \int_k^\omega m_{s_1} (p\xi) \hat{\omega}_r(p\xi) \overline{m_{s_2} (p\xi) \hat{\omega}_r(p\xi) \chi_k(\xi)} d\xi
\]

\[
= \int_k^\omega \sum_{k \in \mathbb{N}_0} m_{s_1} (p\xi + pu(k)) \hat{\omega}_r(p\xi + pu(k))
\]

\[
\times \overline{m_{s_2} (p\xi + pu(k)) \hat{\omega}_r(p\xi + pu(k)) \chi_k(\xi)} d\xi
\]

\[
= \int_k^\omega \sum_{\ell=0}^{q-1} \sum_{k \in \mathbb{N}_0} m_{s_1} (p\xi + pu(qk + \ell)) \hat{\omega}_r(p\xi + pu(qk + \ell))
\]

\[
\times \overline{m_{s_2} (p\xi + pu(qk + \ell)) \hat{\omega}_r(p\xi + pu(qk + \ell)) \chi_k(\xi)} d\xi
\]

\[
= \int_k^\omega \sum_{\ell=0}^{q-1} m_{s_1} (p\xi + pu(\ell)) \overline{m_{s_2} (p\xi + pu(\ell))}
\]

\[
\times \left\{ \sum_{k \in \mathbb{N}_0} \hat{\omega}_r(p\xi + pu(qk + \ell)) \overline{\hat{\omega}_r(p\xi + pu(qk + \ell))} \chi_k(\xi) d\xi \right\}
\]

\[
= \int_k^\omega \sum_{\ell=0}^{q-1} m_{s_1} (p\xi + pu(\ell)) \overline{m_{s_2} (p\xi + pu(\ell))} \chi_k(\xi) d\xi
\]

\[
= \delta_{0,k} \delta_{s_1,s_2}.
\]

Theorem 3.4. Suppose \( \{\omega_n(x) : n \in \mathbb{N}_0\} \) and \( \{\tilde{\omega}_n(x) : n \in \mathbb{N}_0\} \) are wavelet packets with respect to a pair of biorthogonal scaling functions \( \omega_0(x) \) and \( \tilde{\omega}_0(x) \), respectively. Then, we have

\[
\left\langle \omega_{\ell} \left( \cdot, -u(k) \right) \right\rangle, \tilde{\omega}_n \left( \cdot, -u(k) \right) \right\rangle = \delta_{\ell,n} \delta_{0,k}, \quad \ell, n, k \in \mathbb{N}_0.
\] (3.8)

Proof. For \( \ell = n \), the result (3.8) follows by Theorem 3.2. When \( \ell \neq n \), and \( 0 \leq \ell, n \leq q - 1 \), the result (3.8) can be established from Theorem 3.3. Assume \( \ell \) is not equal to
n and at least one of \( \ell, n \) does not belong to \( \{1, 2, \ldots, q - 1\} \), then we can write \( \ell, n \) as \( \ell = qr_1 + s_1, n = qu_1 + v_1, r_1, u_1 \in \mathbb{N}_0, s_1, v_1 \in \{0, 1, 2, \ldots, q - 1\} \).

**Case 1:** If \( r_1 = u_1 \), then \( s_1 \neq v_1 \). Therefore, (3.8) follows by virtue of the properties (3.3)-(3.5) and Lemma 2.5 i.e.,

\[
\left\langle \omega_k(\cdot), \tilde{\omega}_n(\cdot, -u(k)) \right\rangle \\
= \left\langle \omega_{qr_1+s_1}(\cdot), \tilde{\omega}_{qu_1+v_1}(\cdot, -u(k)) \right\rangle \\
= \left\langle \tilde{\omega}_{qr_1+s_1}(\cdot), \tilde{\omega}_{qu_1+v_1}(\cdot, -u(k)) \right\rangle \\
= \int_K \tilde{\omega}_{qr_1+s_1}(\xi) \tilde{\omega}_{qu_1+v_1}(\xi) \chi_k(\xi) d\xi \\
= \int_K m_{s_1}(p\xi) \tilde{\omega}_{r_1}(p\xi) \tilde{\omega}_{u_1}(p\xi) \chi_k(\xi) d\xi \\
= \int_D \sum_{k \in \mathbb{N}_0} m_{s_1}(p\xi + p\xi(k)) \tilde{\omega}_{r_1}(p\xi + p\xi(k)) \\
\quad \times \tilde{\omega}_{u_1}(p\xi + p\xi(k)) \chi_k(\xi) d\xi \\
= \int_D \sum_{\ell=0}^{q-1} \sum_{k \in \mathbb{N}_0} m_{s_1}(p\xi + pu(\ell)) \tilde{\omega}_{r_1}(p\xi + pu(\ell)) \\
\quad \times \tilde{\omega}_{u_1}(p\xi + pu(\ell)) \chi_k(\xi) d\xi \\
= \int_D \sum_{\ell=0}^{q-1} m_{s_1}(p\xi + pu(\ell)) \tilde{\omega}_{r_1}(p\xi + pu(\ell)) \\
\quad \times \tilde{\omega}_{u_1}(p\xi + pu(\ell)) \chi_k(\xi) d\xi \\
= \delta_{0,k}.
\]

**Case 2:** If \( r_1 \neq u_1 \), then \( r_1 = pr_2 + s_2, u_1 = pu_2 + v_2 \), where \( r_2, u_2 \in \mathbb{N}_0 \), and \( s_2, v_2 \in \{0, 1, \ldots, q - 1\} \). If \( r_2 = u_2 \), then \( s_2 \neq v_2 \). Similar to Case 1, (3.8) can be established. When \( r_2 \neq u_2 \), we order \( r_2 = pr_3 + s_3, u_2 = pu_3 + v_3 \), where \( r_3, u_3 \in \mathbb{N}_0 \), and \( s_3, v_3 \in \{0, 1, \ldots, q - 1\} \). Thus, after taking finite steps (denoted by \( h \)), we obtain \( r_h, u_h \in \mathbb{N}_0 \) and \( s_h, v_h \in \{0, 1, \ldots, q - 1\} \). If \( r_h = u_h \), then \( s_h \neq v_h \). Similar to Case 1, (3.8) can be established. When \( r_h \neq u_h \), it follows from equations (2.12)-(2.15) that
\[ \langle \omega_{rh}(\cdot), \tilde{\omega}_{uh}(\cdot - u(k)) \rangle = 0 \iff \sum_{k \in \mathbb{N}_0} \omega_{rh}(\xi + u(k)) \overline{\tilde{\omega}_{uh}(\xi + u(k))} = 0, \quad \xi \in K. \]

Also, we have
\[
\langle \omega_r(\cdot), \tilde{\omega}_u(\cdot - u(k)) \rangle = \langle \hat{\omega}_{rh}(\cdot), \hat{\tilde{\omega}}_{uh}(\cdot - u(k)) \rangle = \langle \hat{\omega}_{qr_1,s_1}(\cdot), \hat{\tilde{\omega}}_{qu_1,v_1}(\cdot - u(k)) \rangle = 0, \quad \xi \in K.
\]

For any \( n \in \mathbb{N}_0 \), define
\[
E_n = \left\{ f(x) : f(x) = \sum_{k \in \mathbb{N}_0} a_k \omega_n(x - u(k)), \quad \{a_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}, \quad (4.1)
\]
\[
\tilde{E}_n = \left\{ \tilde{f}(x) : \tilde{f}(x) = \sum_{k \in \mathbb{N}_0} \tilde{a}_k \tilde{\omega}_n(x - u(k)), \quad \{\tilde{a}_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}. \quad (4.2)
\]

Clearly \( E_0 = V_0 \) and \( E_s = W_0^s \), for any \( 1 \leq s \leq q - 1 \). Assume that \( \left\{ m_s(p\xi + pu(k)) \right\}_{s,k=0}^{q-1} \) is a unitary matrix.

4. Construction of Riesz Bases from Wavelet Packets

In this section, we will decompose the subspaces \( V_j, \tilde{V}_j, W_j \) and \( \tilde{W}_j \) by constructing subspaces of wavelet packets. We also present a direct decomposition for \( L^2(K) \).

For any \( n \in \mathbb{N}_0 \), define
\[
E_n = \left\{ f(x) : f(x) = \sum_{k \in \mathbb{N}_0} a_k \omega_n(x - u(k)), \quad \{a_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}, \quad (4.1)
\]
\[
\tilde{E}_n = \left\{ \tilde{f}(x) : \tilde{f}(x) = \sum_{k \in \mathbb{N}_0} \tilde{a}_k \tilde{\omega}_n(x - u(k)), \quad \{\tilde{a}_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}. \quad (4.2)
\]

Clearly \( E_0 = V_0 \) and \( E_s = W_0^s \), for any \( 1 \leq s \leq q - 1 \). Assume that \( \left\{ m_s(p\xi + pu(k)) \right\}_{s,k=0}^{q-1} \) is a unitary matrix.
Lemma 4.1. For \( n \in \mathbb{N}_0 \), the space \( \Delta E_n \) can be decomposed into the direct sum of \( U_{qn+s} \), \( 1 \leq s \leq q - 1 \), i.e.,

\[
\Delta E_n = \bigoplus_{s=0}^{q-1} U_{qn+s},
\]

where \( \Delta \) is the dilation operator such that \( \Delta f(x) = f(p^{-1}x) \), for any \( f \in L^2(K) \).

Proof. For \( n \in \mathbb{N}_0 \), we claim that

\[
\Delta E_n = \left\{ f(x) : f(x) = \sum_{s=0}^{q-1} \sum_{k \in \mathbb{N}_0} a_k^s \omega_{qn+s}(x - u(k)), \ {a_k^s}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \right\}. \tag{4.4}
\]

As for any \( 0 \leq s \leq q - 1 \), by (3.1) and (4.1), \( \omega_{qn+s}(x - u(k)) \in \Delta E_n \). Assume that \( f(x) \in \Delta E_n \), then there exists a sequence \( \{b_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \) such that

\[
f(x) = \sum_{k \in \mathbb{N}_0} b_k \omega_n(p^{-1}x - u(k)). \tag{4.5}
\]

Similarly, for each \( s = 0, 1, \ldots, q - 1 \), there exist a sequence \( \{a_k^s\}_{k \in \mathbb{N}_0} \) in \( l^2(\mathbb{N}_0) \) such that

\[
f(x) = \sum_{s=1}^{q-1} \sum_{k \in \mathbb{N}_0} a_k^s \omega_n(p^{-1}x - u(k)) \tag{4.6}
\]

provided \( f(x) \in \Delta E_n \).

Taking Fourier transform on the both sides of (4.5) and (4.6), respectively and by using (3.3), we obtain

\[
\hat{f}(\xi) = h(p\xi)\hat{\omega_n}(p\xi) = \sum_{s=1}^{q-1} g_s(\xi)m_s(p\xi)\hat{\omega_n}(p\xi), \tag{4.7}
\]

where \( h(\xi) = \sum_{k \in \mathbb{N}_0} b_k \chi_k(\xi), \ g_s(\xi) = \sum_{k \in \mathbb{N}_0} a_k^s \chi_k(\xi). \)

The above result (4.7) follows if the following equality holds:

\[
h(p\xi) = \sum_{s=1}^{q-1} g_s(\xi) m_s(p\xi). \tag{4.8}
\]

For any \( \{b_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \), we will prove that there exists a sequence \( \{a_k^s\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \) such that (4.8) is satisfied. Moreover, equation(4.8) is equal to the following identity:

\[
h(p\xi + pu(k)) = \sum_{s=1}^{q-1} g_s(\xi) m_s(p\xi + pu(k)). \tag{4.9}
\]
The solvibility of (4.9) for every sequence \( \{b_k\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \) follows from the fact that the matrix \( \{m_s(p\xi + pu(k))\}_{s,k=0}^{q-1} \) is unitary. Hence, equation (4.4) follows. Further, applying Theorem 3.3, it follows that

\[
\{\omega_{qn+s}(p^{-1}x - u(k)) \mid n \in \mathbb{N}_0, \ 0 \leq s \leq q-1, \ k \in \mathbb{N}_0\}
\]

is a Riesz bases of \( \Delta E_n \).

Similar to (4.3), we can establish the following results:

\[
\tilde{E}_0 = \tilde{V}_0, \ \tilde{E}_s = \tilde{W}_0^s, \ 1 \leq s \leq q-1,
\]

and

\[
\Delta \tilde{E}_n = \bigoplus_{s=0}^{q-1} \tilde{U}_{qn+s}, \ 1 \leq s \leq q-1. \tag{4.10}
\]

For \( \ell \in \mathbb{N} \), define \( \vartheta_{\ell} = \sum_{j=0}^{\ell} q^j \Lambda \), where \( \Lambda = \{0, 1, 2, \ldots, q-1\} \), \( \vartheta_{\ell} = \vartheta_{\ell} - \vartheta_{\ell-1} \). Now, we will establish the direct decomposition of the space \( L^2(K) \).

**Theorem 4.2.** The family of functions \( \{\omega_n(x - u(k)), n \in \vartheta_{\ell}, k \in \mathbb{N}_0\} \) constitutes Riesz basis of \( \Delta^\ell W_0 \). In particular \( \{\omega_n(x - u(k)), n \in \vartheta_{\ell}, k \in \mathbb{N}_0\} \) constitutes Riesz basis of \( L^2(K) \).

**Proof.** From equation (4.3), we have

\[
\Delta E_0 = \bigoplus_{s=0}^{q-1} E_s \text{ i.e., } \Delta E_0 = E_0 \bigoplus_{s=1}^{q-1} E_s.
\]

Since \( E_0 = V_0 \) and \( W_0 = \bigoplus_{s=1}^{q-1} W_0^s = \bigoplus_{s=1}^{q-1} E_s \), therefore, \( \Delta E_0 = V_0 \bigoplus W_0 \). It can be inductively inferred from (4.3) that

\[
\Delta^\ell E_0 = \bigoplus_{n \in \vartheta_{\ell}} E_n, \ \ell \in \mathbb{N}. \tag{4.11}
\]

Since \( V_{j+1} = V_j \bigoplus W_j, j \in \mathbb{Z} \), hence, \( \Delta^\ell E_0 = \delta^{\ell-1} E_0 \bigoplus_{n \in \vartheta} W_0, \ \ell \in \mathbb{N} \). Now, it follows from (4.3) and Proposition 2.4 that \( \Delta^\ell W_0 = \bigoplus_{n \in \vartheta} E_n \), and

\[
L^2(K) = V_0 \bigoplus \left( \bigoplus_{\ell \geq 0} \Delta^\ell W_0 \right) = E_0 \bigoplus \left( \bigoplus_{\ell \geq 0} \left( \bigoplus_{n \in \vartheta_{\ell}} E_n \right) \right) = \bigoplus_{n \in \mathbb{N}_0} E_n. \tag{4.12}
\]

In view of Theorem 3.3, the family of functions \( \{\omega_n(x - u(k)), n \in \vartheta_{\ell}, k \in \mathbb{N}_0\} \) is a Riesz basis of \( \Delta^\ell W_0 \). Thus, according to (4.12), the family \( \{\omega_n(x - u(k)), n \in \vartheta_{\ell}, k \in \mathbb{N}_0\} \) forms a Riesz basis of \( L^2(K) \).
Corollary 4.3. For every \( \ell \in \mathbb{N} \), the family of functions \( \{ \tilde{\omega}_n(x - u(k)), n \in \vartheta, k \in \mathbb{N}_0 \} \) forms a Riesz basis of \( \tilde{\Delta}^\ell W_0 \).

Corollary 4.4. For every \( \ell \in \mathbb{N} \), the family of functions \( \{ \omega_n(p^{-j}x - u(k)), n \in \vartheta, k \in \mathbb{N}_0 \} \) forms a Riesz basis of \( L^2(K) \).

5. Decomposition and Reconstruction Algorithms

We begin this section with the decomposition formulae for the biorthogonal wavelet packets on local fields of positive characteristic followed by an algorithm.

Theorem 5.1. Let \( \{ \omega_n \} \) and \( \{ \tilde{\omega}_n \} \) be the biorthogonal wavelet packets defined by (3.1) and (3.2), respectively. Then for all \( k \in \mathbb{N}_0 \), we have the following decomposition formulae:

\[
\begin{align*}
\omega_n(p^{-1}x - u(k)) &= 1 \frac{1}{\sqrt{q}} \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q \mu}^\nu \omega_{qn+\nu}(x - u(\mu)), \\
\tilde{\omega}_n(p^{-1}x - u(k)) &= 1 \frac{1}{\sqrt{q}} \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} a_{k-q \mu}^\nu \tilde{\omega}_{qn+\nu}(x - u(\mu)).
\end{align*}
\]

Proof. We will prove only (5.1). The second formula (5.2) being the dual of (5.1) will follow. Thus using equation (3.1), we have

\[
\begin{align*}
1 \frac{1}{\sqrt{q}} \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q \mu}^\nu \omega_{qn+\nu}(x - u(\mu)) &
= 1 \frac{1}{\sqrt{q}} \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q \mu}^\nu q^{1/2} \sum_{r \in \mathbb{N}_0} a_r^\nu \omega_n(p^{-1}(x - u(\mu)) - u(r)) \\
&= \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \sum_{r \in \mathbb{N}_0} a_{k-q \mu}^\nu \omega_n(p^{-1}x - u(q \mu - r)) \\
&= \sum_{\nu=0}^{q-1} \sum_{t \in \mathbb{N}_0} \omega_n(p^{-1}x - u(t)) \sum_{\mu \in \mathbb{N}_0} \tilde{a}_{k-q \mu}^\nu a_t^\nu \omega_{qn+\nu} \\
&= \omega_n(p^{-1}x - u(k)).
\end{align*}
\]

This completes the proof of the Theorem.

Given a level \( J \) and consider

\[
f \approx f_J = \sum_{k \in \mathbb{N}_0} c_k^J \omega_0(p^{-J}x - u(k)),
\]
where \( \{c^j_k\} \in l^2(\mathbb{N}_0) \). Using the fact
\[
V_J = W_{J-1} \oplus V_{J-1} = \cdots = W_{J-1} \oplus W_{J-2} \oplus \cdots W_{J-M} \oplus V_{J-M},
\]
one obtains
\[
f_J = g_{J-1} + g_{J-2} + \cdots + g_{J-M} + f_{J-M},
\]
where \( f_{J-M} \in V_{J-M} \) and \( g_j \in W_j, j = J - M, \ldots, J - 1 \).

Furthermore, by using Theorem 5.1, \( g_j \in W_j, j = J - M, \ldots, J - 1 \) can be further decomposed. To do this, let
\[
f_j(x) = \sum_{k \in \mathbb{N}_0} c^j_k \omega_0(p^{-j}x - u(k)), \quad (5.3)
\]
and
\[
g_j(x) = \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d^\nu_{j,k} \omega_\nu(p^{-j}x - u(k)), \quad (5.4)
\]
where \( \{c^j_k\}_{k \in \mathbb{N}_0}, \{d^\nu_{j,k}\}_{k \in \mathbb{N}_0} \in l^2(\mathbb{N}_0) \).

Implementation of equation (5.1) for \( n = 0 \) gives the decomposition of \( f_j(x) \) as
\[
\begin{align*}
f_j(x) &= \sum_{k \in \mathbb{N}_0} c^j_k \omega_0(p^{-j}x - u(k)) \\
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} c^j_k \sum_{\nu=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} \tilde{a}^\nu_{k-q\mu} \omega_{q\mu+\nu}(p^{-j}x - u(\mu)) \\
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\mu \in \mathbb{N}_0} c^j_\mu \sum_{\nu=0}^{q-1} \tilde{a}^\nu_{\mu-qk} \omega_\nu(p^{-j+1}x - u(k)) \\
&= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \left( \sum_{\mu \in \mathbb{N}_0} c^j_\mu \tilde{a}^0_{\mu-qk} \right) \omega_0(p^{-j+1}x - u(k)) \\
&\quad + \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\nu=1}^{q-1} \left( \sum_{\mu \in \mathbb{N}_0} c^j_\mu \tilde{a}^\nu_{\mu-qk} \right) \omega_\nu(p^{-j+1}x - u(k)) \\
&= \sum_{k \in \mathbb{N}_0} c^j_{k-1} \varphi(p^{-j+1}x - u(k)) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d^\nu_{j,k-1} \omega_\nu(p^{-j+1}x - u(k)) \\
&= f_{j-1}(x) + g_{j-1}(x),
\end{align*}
\]
where

\[
   c_k^{j-1} = \frac{1}{\sqrt{q}} \sum_{\mu \in \mathbb{N}_0} c_\mu^{j-1} \alpha^0_{\mu - qk}, \quad d_k^{j-1} = \frac{1}{\sqrt{q}} \sum_{\mu \in \mathbb{N}_0} c_\mu^j \alpha^\nu_{\mu - qk},
\]

\[k \in \mathbb{N}_0, j = J, J - 1, \ldots, J - M + 1.\]

For all \(r \in \mathbb{N}_0\), we have

\[
g_j \in W_j = \Delta^j W_1 = \Delta^j \mu \Delta^j W_1 = \Delta^j \bigoplus_{\nu = q^r}^q W_\nu.
\]

Using Theorem 5.1 for \(n = 1, 2, \ldots, q^{r+1} - 1\), yields

\[
g_j(x) = \sum_{\nu = 1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j} \omega_\nu (p^{-j} x - u(k))
\]

\[
= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\nu = 1}^{q-1} \sum_{\mu \in \mathbb{N}_0} d_k^{\nu,j} \sum_{s=0}^{q-1} \bar{a}_{\mu - qk}^s \omega_{q^r - q^s} (p^{-j} x - u(k))
\]

\[
= \frac{1}{\sqrt{q}} \sum_{k \in \mathbb{N}_0} \sum_{\nu = 1}^{q-1} \left( \sum_{\mu \in \mathbb{N}_0} d_k^{[\nu/q],j} \bar{a}_{\mu - qk}^{\nu/q} (p^{-j} x - u(k)) \right) \omega_\nu (p^{-j} x - u(k))
\]

\[
= \sum_{k \in \mathbb{N}_0} \sum_{\nu = 1}^{q^2 - 1} d_k^{\nu,j,1} \omega_\nu (p^{-j} x - u(k))
\]

\[
\vdots
\]

\[
= \sum_{k \in \mathbb{N}_0} \sum_{\nu = q^r}^{q^{r+1} - 1} d_k^{\nu,j,r} \omega_\nu (p^{-j} x - u(k)),
\]

where

\[
d_k^{\nu,j,i} = \frac{1}{\sqrt{q}} \sum_{\mu \in \mathbb{N}_0} d_k^{[\nu/q],j,i-1} \bar{a}_{\mu - qk}^{\nu/q} \omega_\nu (p^{-j} x - u(k)), \quad d_k^{\nu,j,0} = d_k^{\nu,j}.
\]

\[i = 1, 2, \ldots, r, \nu = q^i, q^i + 1, \ldots, q^{i+1} - 1.
\]

Therefore, for \(r \in \mathbb{N}_0\), \(f_J\) can be decomposed as:

\[
f_J = f_{J-M} + \sum_{j=J-M}^{J-1} g_j
\]

\[
= \sum_{k \in \mathbb{N}_0} c_k^{J-M} \omega_0 (p^{J-M} x - u(k)) + \sum_{j=J-M}^{J-1} \sum_{\nu = 1}^{q-1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j} \omega_\nu (p^{-j} x - u(k))
\]

\[
= \sum_{k \in \mathbb{N}_0} c_k^{J-M} \varphi (p^{J-M} x - u(k)) + \sum_{j=J-M}^{J-1} \sum_{\nu = q^r}^{q^{r+1} - 1} \sum_{k \in \mathbb{N}_0} d_k^{\nu,j,r} \omega_\nu (p^{-j} x - u(k)),
\]
where the coefficients are given by the equations (5.5) and (5.6).

On the other hand, by using equation (3.1), we can reconstruct \( g_j(\cdot) \) as follows:

\[
g_j(x) = \sum_{\nu=q}^{q+1} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j,r} \omega_{\nu}(p^{r-j}x - u(k))
\]

\[
= \sum_{\nu=q}^{q+1} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j,r} \sum_{\mu \in \mathbb{N}_0} a_{\mu}^{j} (\nu-q[\nu/q]) \omega_{[\nu/q]}(p^{r-j-1}x - u(qk - \mu))
\]

\[
= \sum_{\nu=q}^{q+1} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j,r-1} \omega_{\nu}(p^{r-j-1}x - u(k))
\]

\[
\vdots
\]

\[
= \sum_{\nu=1}^{q} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j} \omega_{\nu}(p^{j}x - u(k)),
\]

where

\[
d_{\nu}^{j,i-1} = \sum_{s=0}^{q-1} \sum_{\mu \in \mathbb{N}_0} d_{\nu}^{j+s,i} a_{\mu}^{j} d_{\nu}^{j}\omega_{\nu}, \quad d_{\nu}^{j} = d_{\nu}^{j,0}.
\]

(5.7)

\( i = 1, 2, \ldots, r, \nu = q^{i-1}, q^{i-1} + 1, \ldots, q^i - 1. \)

Thus, after obtaining the coefficients \( d_{\nu}^{j}, \nu = 1, 2, \ldots, q-1, j = J-M, \ldots, J-1, k \in \mathbb{N}_0, \)
we use Theorem 5.1 and (2.6) to construct \( f_j \) as follows:

\[
f_j = f_{j-1} + g_{j-1}
\]

\[
= \sum_{k \in \mathbb{N}_0} c_{k}^{j-1} \omega_{0}(p^{j-1}x - u(k)) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j-1} \omega_{\nu}(p^{j+1}x - u(k))
\]

\[
= \sum_{k \in \mathbb{N}_0} c_{k}^{j-1} \sum_{\mu \in \mathbb{N}_0} a_{\mu}^{0} \omega_{0}(p^{j}x - u(qk - \mu)) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j-1} \sum_{\mu \in \mathbb{N}_0} a_{\mu}^{\nu} \omega_{0}(p^{j}x - u(qk - \mu))
\]

\[
= \sum_{k \in \mathbb{N}_0} c_{k}^{j-1} \sum_{\mu \in \mathbb{N}_0} a_{\mu}^{0-qk} \omega_{0}(p^{j}x - u(\mu)) + \sum_{\nu=1}^{q-1} \sum_{k \in \mathbb{N}_0} d_{\nu}^{j-1} \sum_{\mu \in \mathbb{N}_0} a_{\mu}^{\nu-qk} \omega_{0}(p^{j}x - u(\mu))
\]

\[
= \sum_{k \in \mathbb{N}_0} \left( \sum_{\mu \in \mathbb{N}_0} c_{k}^{j-1} a_{\mu}^{0-qk} + \sum_{\nu=1}^{q-1} \sum_{\mu \in \mathbb{N}_0} d_{\nu}^{j-1} a_{\mu}^{\nu-qk} \right) \omega_{0}(p^{j}x - u(k))
\]
\[ = \sum_{k \in \mathbb{N}_0} c_{k}^j \varphi(p^{-j}x - u(k)), \]

where

\[ c_{k}^j = \sum_{\mu \in \mathbb{N}_0} c_{\mu-k}^{j-1} a_{\mu-1}^0 + \sum_{\nu=1}^{q-1} \sum_{\mu \in \mathbb{N}_0} d_{\mu-k}^{\nu,j-1} a_{\mu-1}^\nu, \quad j = J - M + 1, J - M + 2, \ldots, J, \quad k \in \mathbb{N}_0. \quad (5.8) \]

Therefore, with the given sequences \( \{c_{k}^{j-M}\}_{k \in \mathbb{N}_0} \) and \( \{d_{k}^{\nu,j-M}\}_{k \in \mathbb{N}_0}, \nu = 1, 2, \ldots, q - 1 \), and applying (5.8), one can reconstruct

\[ f \approx f_J = \sum_{k \in \mathbb{N}_0} c_{k}^j \omega_0(p^{-j}x - u(k)) \in V_J. \]

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