ON LANGLANDS PROGRAM, GLOBAL FIELDS AND SHTUKAS

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Abstract. The purpose of this paper is to survey some of the important results on Langlands program, global fields, $D$-shtukas and finite shtukas which have influenced the development of algebra and number theory. It is intended to be selective rather than exhaustive, as befits the occasion of the 80-th birthday of Yakovlev, 75-th birthday of Vostokov and 75-th birthday of Lurie.

Under assumptions on ground fields results on Langlands program have been proved and discussed by Langlands, Jacquet, Shafarevich, Parshin, Drinfeld, Lafforgue and others.

This communication is an introduction to the Langlands Program, global fields and to $D$-shtukas and finite shtukas (over algebraic curves) over function fields. At first recall that linear algebraic groups found important applications in the Langlands program. Namely, for a connected reductive group $G$ over a global field $K$, the Langlands correspondence relates automorphic forms on $G$ and global Langlands parameters, i.e. conjugacy classes of homomorphisms from the Galois group $\text{Gal}(\overline{K}/K)$ to the dual Langlands group $\hat{G}(\mathbb{Q}_p)$. In the case of fields of algebraic numbers, the application and development of elements of the Langlands program made it possible to strengthen the Wiles theorem on the Shimura-Taniyama-Weil hypothesis and to prove the Sato-Tate hypothesis.

V. Drinfeld and L. Lafforgue have investigated the case of functional global fields of characteristic $p > 0$ (V. Drinfeld for $G = GL_2$ and L. Lafforgue for $G = GL_r$, $r$ is an arbitrary positive integer). They have proved in these cases the Langlands correspondence.

Under the process of these investigations, V. Drinfeld introduced the concept of a $F$-bundle, or shtuka, which was used by both authors in the proof for functional global fields of characteristic $p > 0$ of the studied cases of the existence of the Langlands correspondence.

Along with the use of shtukas developed and used by V. Drinfeld and L. Lafforgue, other constructions related to approaches to the Langlands program in the functional case were introduced.

G. Anderson has introduced the concept of a $t$-motive. U. Hartl, his colleagues and students have introduced and have explored the concepts of finite, local and global $G$-shtukas.

In this review article, we first present results on Langlands program and related representation over algebraic number fields. Then we briefly present approaches by U. Hartl, his colleagues and students to the study of $D$-shtukas and finite shtukas. These approaches and our discussion relate to the Langlands program as well as to the internal development of the theory of $G$-shtukas.
This communication is an introduction to the Langlands Program and to (D-)shtukas and finite shtukas (over algebraic curves) over function fields. The Langlands correspondence over number fields in its full generality is facing with problems [1, 2, 3, 4, 5, 6, 7]. So results from Galois theory, algebraic number theory and function fields can help understand it.

0.1. Elements of algebraic number theory and field theory. The questions what is a Galois group of a given algebraic closure of the number field or the local field, embedding problems of fields and extensions of class field theory belong to fundamental questions of Galois theory and class field theory. A.V. Yakovlev, S.V. Vostokov, B.B. Lur’e works spans many areas of Galois theory, fields theory and class field theory. The results obtained indicate that these questions connect with module theory, homological algebra and with other topics of algebra and number theory [8, 9, 10, 11, 12, 14, 15]. The development and applications of these theories are described in papers by I.R. Shafarevich [4] and by F.N. Parshin [5] (and in references therein). For further details we refer the reader to papers themselves. By the lack of author’s competence we discuss here very shortly only connection of local fields with formal modules.

0.2. The Hensel-Shafarevich canonical basis in complete discrete valuation fields. Vostokov has constructed a canonical Hensel-Shafarevich basis in $\mathbb{Z}_p$—module of principle units for complete discrete valuation field with an arbitrary residue field [11]. Vostokov and Klimovitski in paper [13] give construction of primary elements in formal module. Ikonnikova, Shaverdova [16] and Ikonnikova [17] use these results under construction, respectively, the Shafarevich basis in higher-dimensional local fields and under proving two theorems on the canonical basis in Lubin-Tate formal modules in the case of local field with perfect residue field and in the case of imperfect residue field. These canonical bases are obtained by applying a variant of the Artin-Hasse function.

0.3. $L^G$ for reductive group $G$. Here we follow to [1, 2, 27, 28, 29]. At first recall that linear algebraic groups found important applications in the Langlands program. Namely, for a connected reductive group $G$ over a global field $K$, the Langlands correspondence relates automorphic forms on $G$ and global Langlands parameters, i.e. conjugacy classes of homomorphisms from the Galois group $\text{Gal}(\overline{K}/K)$ to the dual Langlands
group $\hat{G}(\overline{\mathbb{Q}}_p)$. Let $\overline{K}$ be an algebraic closure of $K$ and $K_s$ be the separable closure of $K$ in $\overline{K}$.

**Definition.** Let $G$ be the connected reductive algebraic group over $\overline{K}$. The root datum of $G$ is a quadruple $(X^*(T), \Delta, X_*(T), \Delta^v)$ where $X^*$ is the lattice of characters of the maximal torus $T$, $X_*$ is the dual lattice, given by the 1-parameter subgroups, $\Delta$ is the set of roots, $\Delta^v$ is the corresponding set of coroots.

The dual Langlands group $\hat{G}$ is a complex reductive group that has the dual root data: $(X_*(T), \Delta^v, X^*(T), \Delta)$. Here any maximal torus $\hat{T}$ of $\hat{G}$ is isomorphic to the complex dual torus $X^*(T) \otimes \mathbb{C}^* = \text{Hom}(X_*(T), \mathbb{C}^*)$ of any maximal torus $T$ in $G$. Let $\Gamma_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$.

Given $G$, the Langlands $L$-group of $G$ is defined as semidirect product $L^G = \hat{G} \rtimes \Gamma_{\mathbb{Q}}$.

In the case of fields of algebraic numbers, the application and development of elements of the Langlands program made it possible to strengthen the Wiles theorem on the Shimura-Taniyama-Weil hypothesis and to prove the Sato-Tate hypothesis. Langlands reciprocity for $GL_n$ over non-archimedean local fields of characteristic zero is given by Harris-Taylor [20].

**0.4. Langlands correspondence over functional global fields of characteristic $p > 0$.** V. Drinfeld [6] and L. Lafforgue [7] have investigated the case of functional global fields of characteristic $p > 0$ (V. Drinfeld for $G = GL_2$ and L. Lafforgue for $G = GL_r$, $r$ is an arbitrary positive integer). They have proved in these cases the Langlands correspondence.

In the process of these studies, V. Drinfeld introduced the concept of a $F$-bundle, or shtuka, which was used by both authors in the proof for functional global fields of characteristic $p > 0$ of the studied cases of the existence of the Langlands correspondence [19].

Along with the use of shtukas developed and used by V. Drinfeld and L. Lafforge, other constructions related to approaches to the Langlands program in the functional case were introduced.

G. Anderson has introduced the concept of a $t$-motive [23]. U. Hartl, his colleagues and postdoc students have introduced and have explored the concepts of finite, local and global $G$-shtukas [33, 35, 34, 36, 38, 39].

In this review, we first present results on Langlands program and related representation over algebraic number fields. Then we briefly present approaches by U. Hartl, his colleagues and students to the study of $G$.
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-shtukas. These approaches and our discussion relate to the Langlands program as well as to the internal development of the theory of G-shtukas. Some results on commutative formal groups and commutative formal schemes can be found in [46, 47, 48] and in references therein.

The content of the paper is as follows:

Introduction.
1. Some results of the implementation of the Langlands program for fields of algebraic numbers and their localizations.
2. Elliptic modules and Drinfeld shtukas.
3. Finite G-shtukas.

1. SOME RESULTS ON LANGLANDS PROGRAM OVER ALGEBRAIC NUMBER FIELDS AND THEIR LOCALIZATIONS

Langlands conjectured that some symmetric power $L$-functions extend to an entire function and coincide with certain automorphic $L$-functions.

1.1. Abelian extensions of number fields. In the case of algebraic number fields Langlands conjecture (Langlands correspondence) is the global class field theory:

Representations of the abelian Galois group $Gal(K^{ab}/K) = \text{characters}$ of the Galois group $Gal(K^{ab}/K)$

... correspond to automorphic forms on $GL_1$ that are characters of the class group of ideles. Galois group $Gal(K^{ab}/K)$ is the profinite completion of the group $\mathbb{A}^*(K)/K^*$ where $\mathbb{A}(K)$ denotes the adele ring of $K$.

If $K$ is the local field, then Galois group $Gal(K^{ab}/K)$ is canonically isomorphic to the profinite completion of $K^*$.

1.2. $l$-adic representations and Tate modules. Let $K$ be a field and $\overline{K}$ its separate closure, $E_n = \{ P \in E(\overline{K}) | nP = 0 \}$ the group of points of elliptic curve $E(\overline{K})$ order dividing $n$. When $char K$ does not divide $n$ then $E_n$ is a free $\mathbb{Z}/n\mathbb{Z}$-module of rank 2.

Let $l$ be prime, $l \neq char K$. The projective limit $T_l(E)$ of the projective system of modules $E_{lm}$ is free $\mathbb{Z}_{l}\text{-adic Tate module of rank 2}.

Let $V_l(E) = T_l(E) \otimes_{\mathbb{Z}_l} \mathbb{Q}_l$. Galois group $Gal(\overline{K}/K)$ acts on all $E_{lm}$, so there is the natural continuous representation ($l$-adic representation)

$$\rho_{E,l} : Gal(\overline{K}/K) \to Aut T_l(E) \subseteq Aut V_l(E).$$

$V_l(E)$ is the first homology group that is dual to the first cohomology group of $l$-adic cohomology of elliptic curve $E$ and Frobenius $F$ acts on
the homology and dually on cohomology. The characteristic polynomial \( P(T) \) of the Frobenius not depends on the prime number \( l \).

1.3. Zeta functions and parabolic forms. Let (in P. Deligne notations) \( X \) be a scheme of finite type over \( \mathbb{Z} \), \(|X|\) the set of its closed points, and for each \( x \in |X| \) let \( N(x) \) be the number of points of the residue field \( k(x) \) of \( X \) at \( x \). The Hasse-Weil zeta-function of \( X \) is, by definition

\[
\zeta_X(s) = \prod_{x \in |X|} (1 - N(x)^{-s})^{-1}.
\]

In the case when \( X \) is defined over finite field \( \mathbb{F}_q \), put \( q_x = N(x) \), \( \deg(x) = [k(x) : \mathbb{F}_q] \), so \( q_x = q^{\deg(x)} \). Put \( t = q^{-s} \). Then

\[
Z(X, t) = \prod_{x \in |X|} (1 - t^{\deg(x)})^{-1}.
\]

The Hasse-Weil zeta function of \( E \) over \( \mathbb{Q} \) (an extension of numerators of \( \zeta_E(s) \) by points of bad reduction of \( E \)) is defined over all primes \( p \):

\[
L(E(\mathbb{Q}), s) = \prod_p (1 - a_p p^{-s} + \epsilon(p) p^{1-2s})^{-1},
\]

here \( \epsilon(p) = 1 \) if \( E \) has good reduction at \( p \), and \( \epsilon(p) = 0 \) otherwise.

Put \( T = p^{-s} \). For points of good reduction we have

\[
P(T) = 1 - a_p T + p T^2 = (1 - \alpha T)(1 - \beta T)
\]

For symmetric power \( L \)-functions (functions \( L(s; E; \text{Sym}^n) \), \( n > 0 \); see below) we have to put

\[
P_p(T) = \prod_{i=0}^n (1 - \alpha^i \beta^{n-i} T)
\]

For \( GL_2(\mathbb{R}) \), let \( C \) be its center, \( O(2) \) the orthogonal group.

Upper half complex plane has the representation: \( \mathbb{H}^2 = GL_2(\mathbb{R})/O(2)C \). So it is the homogeneous space of the group \( GL_2(\mathbb{R}) \).

A cusp (parabolic) form of weight \( k \geq 1 \) and level \( N \geq 1 \) is a holomorphic function \( f \) on the upper half complex plane \( \mathbb{H}^2 \) such that

a) For all matrices

\[
g = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a,b,c,d \in \mathbb{Z}, a \equiv 1(N), d \equiv 1(N), c \equiv 0(N)
\]

and for all \( z \in \mathbb{H}^2 \) we have

\[
f(gz) = f((az+b)/(cz+d)) = (cz+d)^k f(z)
\]
(automorphic condition).

b) \[ |f(z)|^2 (Imz)^k \]
is bounded on \( \mathbb{H}^2 \).

Mellin transform \( L(f, s) \) of the parabolic form \( f \) coincides with Artin \( L \)-series of the representation \( \rho_f \).

The space \( \mathcal{M}_n(N) \) of cusp forms of weight \( k \) and level \( N \) is a finite dimensional complex vector space. If \( f \in \mathcal{M}_n(N) \), then it has expansion

\[ f(z) = \sum_{n=1}^{\infty} c_n(f) \exp(2\pi inz) \]

and \( L \)-function is defined by

\[ L(f, s) = \sum_{n=1}^{\infty} c_n(f)/n^s. \]

1.4. Modularity results. The compact Riemann surface \( \Gamma \backslash \mathbb{H}^2 \) is called the modular curve associated to the subgroup of finite index \( \Gamma \) of \( GL_2(\mathbb{Z}) \) and is denoted by \( X(\Gamma) \). If the modular curve is elliptic it is called the elliptic modular curve.

The modularity theorem states that any elliptic curve over \( \mathbb{Q} \) can be obtained via a rational map with integer coefficients from the elliptic modular curve.

By the Hasse-Weil conjecture (a cusp form of weight two and level \( N \) is an eigenform (an eigenfunction of all Hecke operators)). The conjecture follows from the modularity theorem.

Recall the main (and more stronger than in Wiles [21] and in Wiles-Taylor [22] papers) result by C. Breuil, B. Conrad, F. Diamond, R. Taylor [24].

Theorem. (Taniyama-Shimura-Weil conjecture - Wiles Theorem.) For every elliptic curve \( E \) over \( \mathbb{Q} \) there exists \( f \), a cusp form of weight 2 for a subgroup \( \Gamma_0(N) \), such that \( L(f, s) = L(E(\mathbb{Q}), s) \).

Here \( \Gamma_0(N) \) is the modular group

\[ \Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix}, a, b, c, d \in \mathbb{Z}, c \equiv 0 \pmod{N}, \det \begin{pmatrix} a & b \\ c & d \end{pmatrix} = 1 \right\}. \]

Recall that for projective closure \( \overline{E} \) of the elliptic curve \( E \) we have

\[ \overline{E}(\mathbb{F}_p) = 1 - a_p + p. \]

By H. Hasse

\[ a_p = 2\sqrt{p} \cos \varphi_p. \]
Conjecture. \textit{(Sato-Tate conjecture)} Let \( E \) be an elliptic curve without complex multiplication. Sato have computed and Tate gave theoretical evidence that angles \( \varphi_p \) in the case are equidistributed in \([0, \pi]\) with the Sato-Tate density measure

\[
\frac{2}{\pi} \sin^2 \varphi.
\]

We have two theorems from Serre \cite{serre} which give the theoretical explanation in terms of Galois representations. Here we recall the corollary of the theorems.

Corollary. \textit{(Serre \cite{serre})} The elements are equidistributed for the \( v \) normalized Haar measure of \( G \) if and only if \( c = 0 \) for every \( X \) irreducible character of \( G \), i. e. , if and only if the \( L \)-functions relative to the non trivial irreducible characters of \( G \) are holomorphic and non zero at \( s = 1 \).

The current state of Sato-Tate conjecture is now Clozel–Harris–Shepherd-Barron–Taylor Theorem \cite{clozel} \cite{harris}.

Theorem. \textit{(Clozel, Harris, Shepherd-Barron, Taylor).} Suppose \( E \) is an elliptic curve over \( \mathbb{Q} \) with non-integral \( j \) invariant. Then for all \( n > 0 \); \( L(s; E; Sym^n) \) extends to a meromorphic function which is holomorphic and non-vanishing for \( \Re(s) \geq 1 + n/2 \).

These conditions and statements are sufficient to prove the Sato-Tate conjecture.

Under the prove of the Sato-Tate conjecture the Taniyama-Shimura-Weil conjecture oriented methods of A. Wiles and R. Taylor are used.

Recall also that the proof of Langlands reciprocity for \( GL_n \) over non-archimedean local fields of characteristic zero is given by Harris-Taylor \cite{harris}.

2. Elliptic modules and Drinfeld shtukas.

Let

\[ F_q \] be the algebraic closure of \( F_q \),

\( C \) be a smooth projective geometrically irreducible curve over \( F_q \),

\( K \) be the function field \( F_q(C) \) of \( C \),

\( \nu \) be a close point of \( C \),

\( A \) be the ring of functions regular on \( C - \nu \),

\( K_\nu \) be the completion of \( K \) at \( \nu \) with valuation ring \( O_\nu \),

\( C_\nu \) be the completion of the algebraic closure of \( K_\nu \).

At first recall some known facts about algebraic curves over finite fields.

We will identify the set \( |C| \) of closed points of \( C \) with \( \mathcal{C}(\overline{F_q}) = \text{Hom}_{\mathbb{F}_q}(\text{Spec} \overline{F_q}, C) \).

Let \( k(\nu) \) be the residue field of \( \nu \). Then the degree of \( \nu \) is equal of the number of elements \([k(\nu) : F_q]\).

Below in this section we follow to \cite{serre} \cite{hartshorne} \cite{weil}.
2.1. Elliptic modules.

Lemma. Let $k$ be a field of characteristic $p > 0$ and let $R$ be a $k$-commutative ring with unit (there exists a morphism $k → R$). The additive scheme $\mathbb{G}_a$ over $R$ is represented by the polynomial ring $R[X]$ with structural morphism $α : R[X] → R[X] \otimes_R R[X]$, given by $α(X) = X \otimes 1 + 1 \otimes X$. A morphism $φ : \mathbb{G}_a → \mathbb{G}_a$ of additive schemes over $R$ is defined by an additive polynomial. If $ψ$ is another such morphism, then $φ \circ ψ = φ(ψ(T))$. So the set of (endo)morphisms of additive scheme has the structure of a ring.

Example 1. Let $a ∈ R[X]$, $pa = 0$. Then the morphism $φ(T) = aT^p^n$, $(n ≥ 0)$ is additive. Any additive morphism $φ(T)$ in characteristic $p$ has the form $φ(T) = a_0T + a_1T^p + ⋅⋅⋅ + a_nT^{p^n}$.

Proposition. Let $k$ be a field of characteristic $p > 0$. Put $τa = a^pτ$. There is an isomorphism between $\text{End}_k(\mathbb{G}_a)$ and the ring of noncommutative polynomials $k\{τ\}$.

For any $φ(T) = a_0T + a_1T^p + ⋅⋅⋅ + a_nT^{p^n} ∈ \text{End}_k(\mathbb{G}_a)$ and any $φ(τ) = a_0 + a_1τ + ⋅⋅⋅ + a_nτ^n ∈ k\{τ\}$ Lubin morphisms $[32]$ $c_0$ and $c$ are defined:

$$c_0(φ(T)) = a_0, c(φ(τ)) = a_0.$$

Respectively we define

$$\text{deg}(φ(T)) = p^n, d(φ(τ)) = n.$$

Proposition. Any ring morphism $A → \text{End}_k(\mathbb{G}_a)$ is either injective or has image contained in the constants $k ⊂ k\{τ\}$.

Sketch of the proof. $k\{τ\}$ is a domain. $\text{End}_k(\mathbb{G}_a)$ is isomorphic to $k\{τ\}$. $A$ is a ring with divisor theory $\mathcal{D}$ and for any prime divisor $p ∈ \mathcal{D}$ the residue ring $A/p$ is a field. From these statements the proposition follows.

Assume now that $k$ is an $A$-algebra, i.e. there is a morphism $i : A → k$.

Definition. An elliptic module over $k$ (of rank $r = 2$) is an injective ring homomorphism

$$φ : A → \text{End}_k(\mathbb{G}_a)$$

$$a ↦ φ_a,$$

such that for all $a ∈ A$ we have

$$d(φ(τ)) = 2 \cdot \text{deg}(a),$$

$$c(φ(τ)) = i(a).$$
Example 2. Let \( k = \mathbb{F}_q(T) \), \( A = \mathbb{F}_q[\mathbb{P}^1 - \nu] = \mathbb{F}_q[T] \). Let \( i(T) = T^2 + 1 \). In this case an elliptic module \( \varphi \) is given by
\[
\varphi = T^2 + 1 + c_1 \cdot \tau + c_2 \cdot \tau^2, c_1, c_2 \in k, c_2 \neq 0.
\]

*Remark.* By the same way it is possible to define a Drinfeld module (over a field) for any natural \( r \).

Now consider the case of Drinfeld modules over a base scheme. Let \( S \) be an \( A \)-scheme, \( \mathcal{L} \) a line bundle over \( S \), \( i^* : S \to \text{Spec} \ A \) be an \( A \) scheme morphism dual to the ring homomorphism \( i : A \to O_S \).

**Definition.** (Drinfeld module over a base scheme) A Drinfeld module over \( k \) of rank \( r \) is an \( A \)-module homomorphism
\[
\varphi : A \to \text{End}_S(\mathcal{L})
\]
\[
a \mapsto \varphi_a,
\]
such that for all \( a \in A \) we have
1) locally, as a polynomial in \( \tau \), \( \varphi_a \) has the degree
\[
d(\varphi(\tau)) = r \cdot \deg(a),
\]
2) a unit as its leading coefficient \( a_n \) and
\[
c(\varphi(\tau)) = i(a).
\]

2.2. **Drinfeld shtukas.** In notations of previous subsection let \( x \in k, a \in A, \varphi_a(\tau) \) be a Drinfeld module of rank \( r \). Put \( L = k\{\tau\}, f(\tau) \in L, k[A] = k \otimes_{\mathbb{F}_q} A, \deg_T f(\tau) \) the degree in \( \tau \) of \( f(\tau) \).

**Lemma.** Define the action of \( k[A] \) on \( L \) by the formula:
\[
x \otimes a \cdot f(\tau) = x \cdot f(\varphi_a(\tau)).
\]
Then \( L \) is a free \( k[A] \)-module of rank \( r \).

**Remark.** Let \( E_s = \{ f(\tau) \in L| \deg_T f(\tau) \leq s \}, E = \bigoplus_{s=0}^{\infty} E_s, E[1] = \bigoplus_{s=0}^{\infty} E_{s+1}. E, E[1] \) are graded modules over the graded ring and give rise to locally free sheaves \( \mathcal{F}, \mathcal{E} \) of rank \( r \) over \( C \).

Put \( \mathcal{C}_S = C \times_{\mathbb{F}_q} S, \sigma_q = id_C \otimes \text{Frob}_{q,S} : \mathcal{C}_S \to \mathcal{C}_S \)

**Definition.** A (right) \( \mathcal{D} \)-shtuka (\( F \)-sheaf \([19]\)) of rank \( r \) over an \( \mathbb{F}_q \)-scheme \( S \) is a diagram \( \mathcal{F} \xrightarrow{\alpha} \mathcal{E} \xrightarrow{\beta} (id_C \otimes \text{Frob}_{q,S})^* \mathcal{F} \), such that \( \text{coker} c_1 \) is supported on the graph \( \Gamma_\alpha \) of a morphism \( \alpha : S \to C \) and it is a line bundle on support, \( \text{coker} c_2 \) is supported on the graph \( \Gamma_\beta \) of a morphism \( \beta : S \to C \) and it is a line bundle on support.

If \( \Gamma_\alpha \cap \Gamma_\beta = \emptyset \) it is possible to give the next definition of \( \mathcal{D} \)-shtuka \([34, 37]\).
Definition. A global shtuka of rank $r$ with two legs over an $\mathbb{F}_q$-scheme $S$ is a tuple $N = (N, (c_1, c_2), \tau_N)$ consisting of 1) a locally free sheaf $N$ of rank $r$ on $\mathcal{C}_S$; 2) $\mathbb{F}_q$-morphisms $c_i : S \to \mathcal{C}$ ($i = 1, 2$), called the legs of $N$; 3) an isomorphism $\tau_N : \sigma_q^*N|_{\mathcal{C}_S - \Gamma_{c_1} \cup \Gamma_{c_2}} \simeq N|_{\mathcal{C}_S - \Gamma_{c_1} \cup \Gamma_{c_2}}$ outside the graphs $\Gamma_{c_i}$ of $c_i$, $\Gamma_{c_1} \cap \Gamma_{c_2} = \emptyset$.

Definition. A global shtuka over $S$ is a $D$-shtuka if $\tau_N$ satisfies $\tau_N(\sigma_q^*N) \subset N$ on $\mathcal{C}_S - \Gamma_{c_2}$ with cokernel locally free of rank 1 as $\mathcal{O}_S$-module, and $\tau_{N^{-1}}(N) \subset \sigma_q^*N$ on $\mathcal{C}_S - \Gamma_{c_1}$ with cokernel locally free of rank 1 as $\mathcal{O}_S$-module.

3. Finite $G$-shtukas.

We follow to [19, 33, 35, 36, 37]. We start with very short indication on the general framework of the section. In connection with Drinfeld’s constructions of elliptic modules Anderson [23] has introduced abelian $t$-modules and the dual notion of $t$-motives. Beside with mentioned papers these are the descent theory by A. Grothendieck [40], cotangent complexes by Illusie [44], by S. Lichtenbaum and M. Schlessinger [41], by Messing [42] and by Abrashkin [43]. In this framework to any morphism $f : A \to B$ of commutative ring objects in a topos is associated a cotangent complex $L_{(B/A)}$ and to any morphism of commutative ring objects in a topos of finite and locally free $\text{Spec}(A)$-group schemes $G$ is associated a cotangent complex $L_{(G/\text{Spec}(A))}$ as has presented in books by Illusie [44].

3.1. Finite shtukas and formal groups. Let $S$ be a scheme over $\text{Spec} \mathbb{F}_q$.

Definition. A finite $\mathbb{F}_q$-shtuka over $S$ is a pair $M = (M, F_M)$ consisting of a locally free $\mathcal{O}_S$-module $M$ on $S$ of finite rank and an $\mathcal{O}_S$-module homomorphism $F_M : \sigma_q^*M \to M$.

Author [36] investigates relation between finite shtukas and strict finite flat commutative group schemes and relation between divisible local Anderson modules and formal Lie groups. The cotangent complexes as in papers by S. Lichtenbaum and M. Schlessinger [41], by W. Messing [42], by V. Abrashkin [43] are defined and are proved that they are homotopically equivalent.

Then the deformations of affine group schemes follow to the mentioned paper of Abrashkin are investigated and strict finite $\mathcal{O}$-module schemes are defined.

Next step of the research is devoted to relation between finite shtukas by
3.2. Local shtukas and local Anderson modules. Recall some notions and notations. An ideal \( I \) in a commutative ring \( A \) is locally nilpotent at a prime ideal \( \mathfrak{p} \) if the localization \( I_{\mathfrak{p}} \) is a nilpotent ideal in \( A_{\mathfrak{p}} \). In the framework of smooth projective geometrically irreducible curves \( C \) over \( \mathbb{F}_q \) let \( \text{Nilp}_{A_{\nu}} \) denote the category of \( A_{\nu} \)-schemes on which the uniformizer \( \xi \) of \( A_{\nu} \) is locally nilpotent. Here \( A_{\nu} \cong \mathbb{F}_\nu[[\xi]] \) is the completion of the local ring \( \mathcal{O}_{C,\nu} \) at a closed point \( \nu \in C \).

Let \( \text{Nilp}_{\mathbb{F}_q[[\xi]]} \) be the category of \( \mathbb{F}_q[[\xi]] \)-schemes on which \( \xi \) is locally nilpotent. Let \( S \in \text{Nilp}_{\mathbb{F}_q[[\xi]]} \). Let \( M \) be a sheaf of \( \mathcal{O}_S[[z]] \)-modules on \( S \) and let \( \sigma^* M = M \otimes \mathcal{O}_S[[z]] \). \( M \)-local Anderson modules by Hartl \([33]\) with improvements in \([37]\) and local shtukas are investigated. The equivalence between the
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category of effective local shtukas over $S$ and the category of $z$-divisible local Anderson modules over $S$ is treated by the authors [36, 37]. The theorem about canonical $\mathbb{F}_p[[\xi]]$-isomorphism of $z$-adic Tate-module of $z$-divisible local Anderson module $G$ of rank $r$ over $S$ and Tate module of local shtuka over $S$ associated to $G$ is given. The main result of [36] is the following (section 2.5) interesting result: it is possible to associate a formal Lie group to any $z$-divisible local Anderson module over $S$ in the case when $\xi$ is locally nilpotent on $S$. We note that related with [36] and in some cases more general results have presented in the paper by U. Hartl, E. Viehmann [35].

References

[1] Langlands R. P. On the notion of an automorphic representation, Automorphic Forms, Representations, and L-Functions, Proc. Symp. Pure Math., vol. 33, Part I, American Mathematical Society, Providence, pp. 203–207, 1979.
[2] Langlands R. P., Base change for GL(2), Annals of Math. Studies, vol. 96, Princeton Univ. Press, Princeton, 1980.
[3] Jacquet, H., and R. P. Langlands, Automorphic Forms on GL(2), Lecture Notes in Mathematics, vol. 114, Springer-Verlag, Berlin, 1970.
[4] Shafarevich I.R. Abelian and nonabelian mathematics (in Russian), Sochineniya (Works), v.3, part 1, Moscow, Prima-B, 397–415, 1996.
[5] Parshin A.N. Questions and remarks to the Langlands program, Uspekhi Matem. Nauk, 67, n 3, 115–146, 2012.
[6] Drinfeld V. Langlands conjecture for GL(2) over functional elds. Proceedings of the International Congress of Mathematicians (Helsinki, 1978), pp. 565–574, Acad. Sci. Fennica, Helsinki, 1980.
[7] Lafforgue L. Chtoucas de Drinfeld, formule des traces d’Arthur-Selberg et correspondance de Langlands. Proceedings of the International Congress of Mathematicians, Vol. I (Beijing, 2002), pp. 383–400, Higher Ed. Press, Beijing, 2002.
[8] Shafarevich I.R. On embedding problem for fields. Math. USSR-Izv., vol. 18, P. 918–924, 1954.
[9] Yakovlev A.V. The embedding problem for number fields. Math. USSR-Izv., vol. 31, no. 2, P. 211–224, 1967.
[10] Yakovlev A.V. Galois group of the algebraic closure of local fields. Math. USSR-Izv., vol. 32, no. 6, P. 1283–1322, 1968.
[11] Vostokov S.V. Canonical Hensel-Shafarevich basis in complete discrete valuation fields, Zap. Nauchn. Semin. POMI, 394, 174–193, 2011.
[12] Vostokov S.V. Explicit construction of class field theory for a multidimensional local field Math. USSR-Izv., vol. 49, no. 2, 283–308, 1985.
[13] Vostokov S.V. Klimovitskii I.I. Primery elements in formal modules, Mathematics and Informatics, 2, Steklov Math. Inst., RAS, Moscow, 153–163, 2013.
[14] B. B. Lur’e. On embedding problem with kernel without center Math. USSR-Izv., vol. 28, 1135–1138,1964.
[15] B. B. Lur’e. Universally solvable embedding problems, Proc. Steklov Inst. Math., 183, 141–147, 1991.
[16] Ikonnikova E.V., Shaverdova E.V. The Shafarevich basis in higher-dimensional local field, *Zap. Nauchn. Semin. POMI*, 413, 115–133, 2013.

[17] Ikonnikova E.V. The Hensel-Shafarevich canonical basis in Lubin-Tate formal modules, *J. Math. Sci., New York* 219, No. 3, 462–72, 2016.

[18] Serre J.-P. Abelian l-Adic Representations and Elliptic Curves, Addison-Wesley Publishing Company, NY, 1989.

[19] Drinfeld V. Moduli varieties of $F$-sheaves. *Func. Anal. and Applications*, vol. 21, pp. 107-122, 1987.

[20] Harris M., Taylor R. The geometry and cohomology of some simple Shimura varieties, Ann. Math. Stud. 151, Princeton University Press, Princeton, 2001.

[21] Wiles, A., Modular elliptic curves and Fermat's last theorem. *Ann. of Math. (2)* 141, no. 3, 443–551, 1995.

[22] Taylor, R.; Wiles, A., Ring-theoretic properties of certain Hecke algebras. *Ann. of Math. (2)* 141, no. 3, 553–572, 1995.

[23] Anderson G. t-Motives. Duke math. J., vol. 53, pp. 457-551, 1986.

[24] Breuil, C., Conrad, B., Diamond F., Taylor R., On the modularity of elliptic curves over $\mathbb{Q}$: wild 3-adic exercises. *J. Amer. Math. Soc.* 14, no. 4, 843-939, 2001.

[25] Clozel L., Harris M., Taylor, R., Automorphy for some $l$-adic lifts of automorphic mod $l$ Galois representations, Pub. Math. I.H.E.S. 108 , 1-181, 2008.

[26] Harris M., Shepherd-Barron N., Taylor R. A family of Calabi-Yau varieties and potential automorphy, *Ann. of Math.* (2) vol. 171, no. 2, 779-813, 2010.

[27] Arthur J. The principle of functoriality, *Bulletin of the American Math. society*, Vol. 40, no. 1, pp. 39-53, 2003.

[28] Borel A. Automorphic $L$-functions, Proc. Symp. Pure Math., vol. 33, Part 2, pp. 27–61, 1979.

[29] Tate J. Number theoretic background, Proc. Symp. Pure Math., vol. 33, Part 2, pp. 3–26, 1979.

[30] Drinfeld V. Elliptic modules. *Mat. Sbornik*, vol. 94, no. 4, pp. 594-627, 1974.

[31] Deligne P., Husemoller D., Survey of Drinfeld’s modules, *Contemporary Math.*, vol. 67, pp. 25-91, 1987.

[32] Lubin J. One-parameter formal Lie groups over $p$-adic integer rings, *Ann. of Math.*, vol. 80, pp. 464–484, 1964.

[33] Urs Hartl. Number Fields and Function fields - Two Parallel Worlds, Papers from the 4th Conference held on Texel Island, April 2004, *Progress in Math.* 239, Birkhauser-Verlag, Basel, pp. 167-222, 2005.

[34] Hartl U., Araasteh Rad. Local $\mathbb{R}$-shtukas and their relation to global $G$-shtukas. *Münster J. Math.* 7, No. 2, 623-670, 2014.

[35] Hartl U., Viehmann E. J. reine angew. Math. (Crelle) 656, 87-129, 2011.

[36] Singh R., Local shtukas and divisible local Anderson-modules, Univ. Münster, Fachbereich Mathematik und Informatik (Diss.). 72 p. 2012.

[37] Hartl U., Singh R., Local Shtukas and Divisible Local Anderson Modules, *Canadian J. of Math.*, vol. 71, no. 5, 1163-1207, 2019.

[38] Araasteh Rad, Uniformizing the moduli stacks of global $G$-shtukas, Univ. Münster, Fachbereich Mathematik und Informatik, (Diss.). 85 p. 2012.

[39] Weiß A, Foliations in moduli spaces of bounded global $G$-shtukas. Univ. Münster, Fachbereich Mathematik und Informatik, (Diss.). 97 p. 2017.
[40] Grothendieck A. Catégories fibrées et descente, Exposè VI in Revètements étales et groupe fondamental (SGA 1), Troisième édition, corrigé, Institut des Hautes Études Scientifiques, Paris, 1963.

[41] Lichtenbaum S., Schlessinger M. The cotangent complex of morphisms, Transactions of the American Math. society, 128, pp. 41-70. 1967.

[42] Messing W. The Cristals Associated to Barsotti-Tate Groups, LNM 264, Springer-Verlag, Berlin etc.1973.

[43] Abrashkin V. Compositio Mathematika, 142:4, pp. 867-888, 2006.

[44] Illusie L. Complex cotangent et deformations. I, II, LNM, Vol.239, Vol. 283, Springer Verlag, Berlin-NY, 1971, 1972.

[45] Faltings G. Group schemes with strict O-action, Mosc. Math. J. 2, no. 2, 249-279, 2002.

[46] Glazunov N., Quadratic forms, algebraic groups and number theory, Chebyshevskii Sbornik, vol.16, no. 4, P.77–89, 2015.

[47] Glazunov N., Extremal forms and rigidity in arithmetic geometry and in dynamics, Chebyshevskii Sbornik, vol.16, no. 3, P.124–146, 2015.

[48] Glazunov N., Duality in abelian varieties and formal groups over local fields. I. Chebyshevskii Sbornik, vol. 19, no.1, P. 44-56, 2018.
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