Dispersive analysis of the $S$-, $P$-, $D$-, and $F$-wave $\pi\pi$ amplitudes

P. Bydžovský, R. Kamiński, V. Nazari

Nuclear Physics Institute, Czech Academy of Sciences, Řež, Czech Republic
Institute of Nuclear Physics, Polish Academy of Sciences, Kraków, Poland

(Dated: December 4, 2016)

A reanalysis of $\pi\pi$ amplitudes for all important partial-waves below about 2 GeV is presented. A set of once subtracted dispersion relations with imposed crossing symmetry condition is used to modify unitary multi-channel amplitudes in the $S$, $P$, $D$, and $F$ waves. So far, these specific amplitudes constructed in our works and many other analyzes have been fitted only to experimental data and therefore do not fulfill the crossing symmetry condition. In the present analysis, the self consistent, i.e. unitary and fulfilling the crossing symmetry, amplitudes for the $S$, $P$, $D$, and $F$ waves are formed. The proposed very effective and simple method of modification of the $\pi\pi$ amplitudes does not change their previous original mathematical structure and the method can be easily applied in various other analyzes.

PACS numbers: 11.55.Fv, 11.55.-m, 11.80.Et, 13.75.Lb
Keywords: scalar mesons, dispersion relations, multi-channel amplitudes

I. INTRODUCTION

The enthusiasm in the analysis of $\pi\pi$ interaction amplitudes has been increased significantly quite recently. Especially important were numerous works on dispersive analyzes of experimental data made by Bern [1] and Madrid-Kraków [2] group. The significant progress was made when the analyzes began to effectively use theoretical constrains i.e. crossing symmetry condition imposed on the amplitudes found in experimental analyzes. This was particularly important because of big differences between results obtained by various experimental groups and even between data sets found in the same experimental analysis [3, 4].

Those dispersive analyzes had immediately large impact on the spectroscopy of light scalar mesons (i.e. $f_0(500)$ and $f_0(980)$) which is evident comparing the tables of the Particle Data Group from the years 2010 and 2012 [5, 6]. The analyzes provided also a set of all important amplitudes ($S$, $P$, $D$ and $F$) well describing experimental data up to 1420 MeV and 2000 MeV in case of works done by the Madrid-Kraków and Bern group, respectively. In both those analyzes the $S$ and $P$ wave amplitudes were fitted also to dispersion relations with imposed crossing symmetry constrain below 1100 MeV. The Bern group was using Roy equations [7] which need two subtractions and found analytical solution below 800 MeV. The Madrid-Kraków group also used Roy equations and additionally Roy like ones, so called GKPY equations with only one subtraction what gave more precise output amplitudes. The amplitudes of higher partial-waves, $D$ and $F$, were also fitted to such dispersion relations but only indirectly by means of the Roy or GKPY equations, i.e. they were present in the kernel part of the equations written for the $S$ and $P$ waves.

An example of practical application of the GKPY equations is our last reanalysis of the $S$- and $P$-wave multi-channel amplitudes [8] constructed in a previous analysis by fitting the experimental data only [9]. One of the most spectacular effects given by the GKPY equations was a shift of the $f_0(500)$ pole by several hundred MeV towards the position indicated by the analyzes of the Bern and Madrid-Kraków group. It is important to note that this reanalysis was done keeping the original mathematical structure of the amplitudes proposed in the work [8].

The aim of this work is to perform similar but more extensive reanalysis of the multi-channel $\pi\pi$ amplitudes including the $S$, $P$, $D$, and $F$ partial waves important in the low energy region (below 2 GeV). The initial amplitudes, in the following denoted as “original”, were fitted only to the experimental data and are taken from our previous analysis ($S_0 [8]$ and $P_1 [9]$ hereafter we will use notation $\ell I$ if needed ($\ell$-meson-meson partial wave and $I$-isospin)) or are updated in this work utilizing a form from [10] ($D_0$ and $F_1$). To perform this analysis we use the GKPY equations for the $D$ and $F$ waves constructed and presented in [11], which have not been used so far in the analysis of amplitudes. The final amplitudes are constrained by both the experimental data and the GKPY dispersion relations (i.e. by crossing symmetry). As the reanalysis is not too much effective for higher partial-waves, particularly for $F_1$, parameters of this amplitude are expected to be only weakly changed. In the analysis, only some parameters of the amplitudes are changed which do not alter the mathematical structure of the original amplitudes, similarly as in our previous analysis of the $S$ and $P$ waves [8].

The reanalyzed partial-wave multi-channel amplitudes, constrained also by the crossing symmetry, can be utilized in accounting for the final-state interactions in the decays of heavy mesons and in photoproduction processes, for example, in the CLAS12 and GlueX experiments at JLab. The amplitudes can also be used in constructing a full $\pi\pi$ amplitude giving the cross section at low energies.

The paper is organized as follows: in Section II we
briebrly reminde the method and results of our previous analysis of the S0 and P1 partial-wave amplitudes done in [8]. In Section III we present new D0 and F1 amplitudes fitted to experimental data. In Section IV we detail the method and give results of the dispersive analysis for all considered partial-waves. Section V is devoted to discussion of obtained results. Here we also show results for the low-energy total and differential cross sections in the π⁺π⁻ scattering. Summary of results is in Section VI.

II. DISPERSIVE ANALYSIS OF THE S- AND P-WAVE AMPLITUDES

In our previous work [8] we reanalyzed the S- and P-wave amplitudes constructed in [8] refitting some of their parameters simultaneously to experimental data and to GKPY dispersion relations. These multi-channel amplitudes are unitary and analytic on the Riemann surface with free parameters that are just positions of poles on different Riemann sheets and few background parameters. Such simple and not biased mathematical structure makes interpretation of obtained results very easy and unambiguous. In analysis [8] these amplitudes were, however, fitted only to very dispersed experimental data from various experiments what resulted in pole positions sometimes very different from those obtained in other analyzes and from those in Particle Data Group Tables.

In the analysis [8] the original mathematical structure of the resonant and background parts of amplitudes from [8] was not changed. The only novelty was a new parameterization of the near threshold amplitudes, which was necessary due to the total lack of description of the phase shifts in this region [8]. Polynomials were, therefore, added to both S and P amplitudes and the phase shifts (values and the first derivatives) were smoothly matched below the K K̄ threshold, at about 400 and 600 MeV for the S and P waves, respectively. In Ref. [8] we also constructed a “new” S-wave isoscalar amplitude hereafter called New S-wave fitting its parameters only to the data, what improved behavior of the amplitude. In the subsequent analysis both “Old” [4] and New S-wave isoscalar amplitudes were used.

In analysis [8] the strategy of our work was following: the Old (New) S-wave isoscalar and the isovector P-wave amplitudes, both supplemented with the near threshold polynomials, were used as the input in the GKPY equations that have a general form

\[
\text{Re} t_\ell^{(\text{OUT})}(s) = \sum_{I'=0}^{2} C^{I'I'} t_0^{I'} (4m_\pi^2) + \sum_{I'=0}^{3} \sum_{\ell'=0}^{\infty} \int ds K_{I'I'}^{\ell}(s, s') \text{Im} t_\ell^{(IN)}(s'),
\]

where \( t_\ell^{(IN)}(s') \) and \( t_\ell^{(OUT)}(s) \) are the input and output amplitudes, respectively, in a given partial-wave \( \ell, \ell' \) with isospin \( I, I' \). The \( C^{I'I'} \) is the crossing matrix constant and \( K_{I'I'}^{\ell}(s, s') \) are kernels constructed for partial-wave projected amplitudes with the imposed \( s \leftrightarrow t \) crossing symmetry condition. The kernels for the S and P waves were presented in [2].

As it is seen in Eq. (1), one has to use the imaginary parts of all important partial-wave amplitudes as an input. We took, therefore, these important amplitudes, S2, D0, D2 and F1, directly from Ref. [2] and kept them fixed during the analysis. Finally we fitted some parameters of the S0 and P1 amplitudes simultaneously to the experimental data and to the dispersion relations [1]. As a result, the \( f_0(500) \) pole moved by several hundred MeV to a new position close to that found in dispersive analyzes [14, 16].

The minimized full \( \chi^2 \) function was composed of two data terms \( \chi^2_{\text{Data}}(k) \), related with data in the S and P waves, and of three terms \( \chi^2_{DR}(k) \) for the output amplitudes from the dispersion relations

\[
\chi^2 = \sum_{k=1}^{2} \chi^2_{\text{Data}}(k) + \sum_{k=1}^{3} \chi^2_{DR}(k),
\]

where \( k = 1, 2, 3 \) itemizes, respectively \( I \) partial-waves: the S0, P1 and S2. Corresponding \( \chi^2_{\text{Data}}(k) \) and \( \chi^2_{DR}(k) \) were expressed by

\[
\chi^2_{\text{Data}}(k) = \sum_{i=1}^{N_k} \left[ \delta_i^{\exp} - \delta_i^{\text{th}} \right]^2 + \sum_{i=1}^{N_k} \left[ \eta_i^{\exp} - \eta_i^{\text{th}} \right]^2
\]

and

\[
\chi^2_{DR}(k) = \sum_{i=1}^{N_{DR}} \left[ \text{Re} t_\ell^{(\text{OUT})}(s_i) - \text{Re} t_\ell^{(\text{IN})}(s_i) \right]^2,
\]

where \( \delta_i^{\exp} \) and \( \delta_i^{\text{th}} \) are experimental and theoretical phase shifts, respectively, and \( N_k \) are numbers of the data points for phase shifts (or inelasticities) of the S0 and P1 partial-waves in considered coupled channels. Symbol \( N_{DR} \) is a number of energy points between the \( \pi \pi \) threshold and 1100 MeV, at which we calculated \( \chi^2_{DR}(k) \) (for all three waves \( N_{DR} = 26 \) was chosen) and \( \Delta \text{Re} t_\ell^{(\text{OUT})}(s_i) \) are fixed to 0.01 in order to make the \( \chi^2_{DR}(k) \) comparable with the \( \chi^2_{\text{Data}}(k) \).

Let us here notice that terms \( \chi^2_{DR}(k) \) are in fact not \( \chi^2 \) functions but rather squared weighted differences between the input and output amplitudes. For simplicity we keep, however, the name \( \chi^2 \) for these terms.

III. NEW ANALYSIS OF THE D- AND F-WAVE EXPERIMENTAL DATA

The experimental data for the \( \pi \pi \) scattering in the D0 and F1 waves were analyzed in Refs. [3, 10] to study the \( f_2 \) and \( p_3 \) mesons. In presented here analysis the
S-matrix formalism for N coupled channels was utilized similarly as in our previous dispersive analysis of S and P waves [8]. Due to the large number of opened channels in the D and F waves the uniformizing variable (see Eq. (1) in [8]) could not be used and therefore, the Jost matrix determinant was constructed using the multi-channel Breit-Wigner forms.

In the present analysis we have used the same formalism for the D0 and F1 waves as in Refs. [8] [10] and updated the list of contributing resonance states for the D0 wave according to the latest issue of PDG [14]. The corresponding free parameters were fitted to experimental data. In the case of F1 wave we have found that enough is only one resonance state \( \rho_3(1690) \) and have constructed a reasonable description also in the threshold region. These updated D0 and F1 amplitudes were then used in the analysis with the GKPY equations taken from [11] (see [IV C]). In the following subsections we give more details on the formalism and construction of the New D0 and New F1 amplitudes.

### A. Formalism

The matrix elements \( S_{ij} \) of the N-channel S matrix \((i, j = 1, 2, ..., N)\) are expressed via the Jost matrix determinant, \( d(k_1, k_2, ..., k_N) \) \((k_i \text{ are the channel momenta})\), using the Le Couteur–Newton relations [9] [10]. These expressions together with the formulas of analytical continuation of the matrix elements to the unphysical sheets naturally generate the resonance poles and zeros on the Riemann surface. The Jost determinant is considered in a separable form \( d = d_{\text{bgr}} d_{\text{res}} \). The resonance part is described by the multi-channel Breit-Wigner form

\[
d_{\text{res}} = \prod_r \left[ M_r^2 - s - i \sum_{j=1}^N \rho_{rj}^{2j+1} R_{rj} f_{rj}^2 (s-s_j) \right],
\]

where \( s \) is the invariant total energy squared, \( M_r \) and \( J = \ell \) are the resonance mass and spin, respectively, \( \rho_{rj} = 2k_j / \sqrt{M_r^2 - s_j} \) with \( s_j \) the channel thresholds, \( R_{rj} \) are the Blatt-Weisskopf barrier factors, and the free parameter \( f_{rj} \) is related with a decay width of a resonance \( r \) into a channel \( j \).

The background part \( d_{\text{bgr}} \), which represents mainly an influence of neglected channels and resonances, adds in general an energy dependent phase in each channel.

### B. Fits for the D wave

In the analysis of the data in the tensor-isoscalar sector we have considered explicitly the channels: 1- \( \pi\pi \), 2-effective \( (2\pi)(2\pi) \), 3- \( K\bar{K} \), and 4- \( \eta\eta \). The resonant part of the Jost determinant, \( d(k_1, k_2, k_3, k_4) \), is then given by the four-channel Breit-Wigner form [8] with \( J = 2 \) and the barrier factor

\[
R_{rj} = \frac{9 + \frac{4}{3}(\sqrt{M_r^2 - s_j r_{rj}})^2 + \frac{1}{16}(\sqrt{M_r^2 - s_j r_{rj}})^4}{9 + \frac{4}{3}(\sqrt{s - s_j r_{rj}})^2 + \frac{1}{16}(\sqrt{s - s_j r_{rj}})^4},
\]

where the radii \( r_{rj} \) have a common value 0.943 fm [10] which was kept constant in our analysis.

In the set of resonance states contributing to the process we have considered eleven states presented in the PDG summary table [14]: \( f_2(1270), f_2(1430), f_2(1525), f_2(1640), f_2(1810), f_2(1910), f_2(1950), f_2(2010), f_2(2150), f_2(2300), f_2(2340) \). We have not included the broad state \( f_2(1565) \) which was not listed in the previous issue of PDG and which can be mimic by the nearby state \( f_2(1525) \). The masses of the resonances were taken from the PDG tables but in the curse of fitting \( f_2(1430), f_2(1525), f_2(1910), f_2(2300) \) and \( f_2(2340) \) resonance masses were allowed to change slightly within an interval of several standard deviations around the central value. The partial widths of the resonances, the parameters \( f_{rj} \) in [8], were fitted to the data.

The background part of the Jost determinant was taken from [8] and has the form

\[
d_{\text{bgr}} = \exp \left[ -i \sum_{j=1}^4 \frac{2k_j}{\sqrt{s}} \right] (a_j + ib_j),
\]

where \( a_2 = a_3 = a_4 = 0 \),

\[
a_1 = \alpha_{11} + \frac{s - s_3}{s} \alpha_{13} \theta(s - s_3) + \frac{s - s_v}{s} \alpha_{10} \theta(s - s_v),
\]

\[
b_j = \beta_j + \frac{s - s_v}{s} \gamma_j \theta(s - s_v), \quad \text{for } j = 1, 3, 4,
\]

and \( b_2 = 0 \). The threshold \( s_v = 2.274 \text{ GeV}^2 \) accounts for effects from the channels \( \eta\eta, \rho \rho, \) and \( \omega \omega \) not included explicitly in the analysis. The parameters \( \alpha_{11}, \alpha_{13}, \alpha_{10}, \beta_j, \) and \( \gamma_j \) were fitted to the data.

The experimental data for the \( \pi\pi \) scattering are from the energy-independent analysis by Hyams et al. [12] and the data for inelastic scattering \( \pi\pi \rightarrow K\bar{K}, \eta\eta \) from Ref. [13]. To warrant a right behavior of the elastic phase shifts in the threshold region, i.e. a consistency with the data for inelastic scattering are from [9], we have included in the data set also the sixteen points \( (\pi\pi) \) generated in the range 282-825 MeV by the phenomenological amplitudes [12] with errors about 10%.

In successive fitting of the parameters to the data we found a solution with \( \chi^2/n.d.f. = 242.28/(199 - 58) = 1.72 \) where the value without the pseudo data is \( \chi^2/n.d.f. = 239.28/(183 - 58) = 1.91 \). The parameters of resonances of the New D0 amplitude are given in Table [1] and a comparison with the old D0 amplitude [9] is shown in Figs. [1] and [2]. The background parameters are: \( \alpha_{11} = 0.00096, \alpha_{13} = -0.04105, \alpha_{10} = -0.186, \beta_1 = -0.0531, \beta_3 = -1.99, \beta_4 = -1.47, \gamma_1 = 0.00128, \gamma_3 = 1.99, \) and \( \gamma_4 = 1.43 \).
a reasonable data description. We have there-

This solution is a bit worse (the $\chi^2$) than that in Ref. [2] ($\chi^2$ was $156.62/(168 - 69) = 1.58$) but we have achieved the right behavior of $\delta_{11}$ for energies below 800 MeV (see the detail in Fig. 1(a)) which allows us to avoid a polynomial-like extension of the phase shift as in the case of the $S$ and $P$ amplitudes. In Figs. 1 and 2 for inelasticity $\eta_{11}$ and the squared modulus of the $S$ wave, there are only two resonance amplitudes. In Figs. 3 and 4 -4 for inelastic channel one can see even a slight improvement in description of the data. Please notice also that the set of resonances and their masses are in a good agreement with the PDG tables, see Table 1.

### TABLE 1: Parameters of the Breit-Wigner form (in MeV) for the New $D_0$ amplitude. The masses of the resonances from PDG [14] are also shown in the second column.

| state   | PDG    | $M_r$  | $f_{r1}$ | $f_{r2}$ | $f_{r3}$ | $f_{r4}$ |
|---------|--------|--------|----------|----------|----------|----------|
| $f_2(1270)$ | 1275.5 ± 0.8 | 1275.5 | 459.3 | 0.001 | 204.0 | 91.3 |
| $f_2(1430)$ | 1430 | 1463.2 | 42.3 | 0.12 | 346.8 | 0.02 |
| $f_2(1525)$ | 1525 ± 5 | 1570.7 | 0.01 | 207.5 | 128.4 | 96.3 |
| $f_2(1640)$ | 1639 ± 6 | 1639.0 | 145.3 | 524.4 | 430.5 | 233.5 |
| $f_2(1810)$ | 1815 ± 12 | 1815.0 | 163.5 | 279.2 | 497.2 | 590.3 |
| $f_2(1910)$ | 1903 ± 9 | 1903.0 | 0.077 | 65.3 | 0.067 | 371.3 |
| $f_2(1950)$ | 1944 ± 12 | 1944.0 | 5.01 | 59.5 | 625.7 | 97.9 |
| $f_2(2010)$ | 2011 ± 62 | 2027.0 | 0.001 | 146.4 | 457.1 | 0.5 |
| $f_2(2150)$ | 2157 ± 12 | 2157.0 | 0.015 | 445.8 | 148.1 | 354.6 |
| $f_2(2300)$ | 2297 ± 28 | 2181.6 | 78.14 | 74.9 | 818.3 | 169.5 |
| $f_2(2340)$ | 2345 ± 40 | 2383.3 | 46.20 | 7.1 | 633.2 | 163.8 |

C. Fits for the $F$ wave

In the isovector $F$ wave, there are only two resonance states listed in the PDG summary table which are relevant for the data description below 2 GeV: $\rho_2(1690)$ and $\rho_3(1990)$ [14]. For the former, the decay widths into the $\pi\pi$, $\pi^+\pi^-\pi^0$, $\omega\pi$, $KK$ and $K\bar{K}$ channels are well established whereas for the latter the partial widths are not known. To learn on importance of these resonances in description of data we performed fits in [17] (Tables V and VI). We showed that if both resonances are fitted simultaneously then the mass of $\rho_3(1990)$ turns into a huge number showing that one resonance state is enough to achieve a reasonable data description. We have therefore considered only $\rho_3(1990)$ in the analysis of the phase shift and inelasticity parameter in the $\pi\pi$ scattering [12]. This state is also apparently well seen in the pronounced data structure.

In the analysis we have included four channels: 1- $\pi\pi$, 2- effective $(2\pi)(2\pi)$, 3- $\omega\pi$, and 4- $KK$. The Blatt-Weisskopf barrier factor in the Breit-Wigner form [15] was in the case of $J = 3$

$$R_{1j} = \frac{225 + 45(\mu_j r_{1j})^2 + 6(\mu_j r_{1j})^4 + (\mu_j r_{1j})^6}{225 + 45(2\mu_j r_{1j})^2 + 6(2\mu_j r_{1j})^4 + (2\mu_j r_{1j})^6},$$

where $\mu_j$ is equal to $\sqrt{M_j^2 - s}$. The radii $r_{1j}$ possess a common value 0.927 fm [10] that were kept constant in our analysis. The resonance mass and the Breit-Wigner parameters $f_{1j}$ were fitted to the data.

In our analysis the dispersion relations directly affect only the energy region below 1100 MeV but the higher energy region, where the $\rho_3(1690)$ resonance is clearly seen, is influenced indirectly. The $F1$ amplitude is, therefore, almost entirely determined by the experimental data. Since the resonance state is described by the Breit-Wigner form, which works well only in a vicinity of the resonance, we had to take a particular care of behavior of the amplitude below about 1000 MeV. We have, therefore, chosen a simple modification of the phase shift by means of the background phase in the form of the quadratic polynomial of $s$

$$d_{gr} = \text{exp} \left[-i \left(\frac{2k_1}{\sqrt{s}}\right)^7 \left(\frac{a_\alpha + \frac{4k_1^2}{s_1}}{a_\beta + \frac{4k_1^2}{s_1}a_\gamma}\right)\right],$$

where $k_1$ is equal to $\sqrt{s - s_1}/2$ and the parameters $a_\alpha$, $a_\beta$, and $a_\gamma$ were fitted to the data. Similarly as in the case of the $D_0$ wave we have included pseudo data points but now 31 for energies 282 MeV < $\sqrt{s}$ < 895 MeV.
and the background parameters are $\chi^2$ channel fit with $1.58 \pm 0.80$ resonance: 23.6 $\pm$ 0.7% for $\rho(1690)$ mass, 16.4 $\pm$ 1.2% and 1.58 $\pm$ 0.26% into the $\pi\pi$, $\pi^+\pi^0$, $\omega\pi$, $K\bar{K}\pi$, and $KK$ channels, respectively. We did therefore a two-channel fit with $\chi^2/n.d.f. = 136.20/(108 - 6) = 1.34$ where the value without the pseudo data is $\chi^2/n.d.f. = 80.84/(77 - 6) = 1.38$ showing that the pseudo data are quite consistent with the experimental data used in the analysis. The resonance parameters are shown in Table II and the background parameters are $a_0 = 0.000008$, $a_3 = -0.000098$, and $a_7 = 0.000016$. These parameters are quite small but they play an important role. In the fit with only the first term in $\beta$, the $\chi^2/n.d.f. = 4.0$ where the main contribution comes from the pseudo data. This we consider as a strong evidence of the need for the nonzero additional terms in $\beta$. To verify that the experimental data can be described purely by one state $\rho_3(1690)$ we fitted only the data with one Breit-Wigner form without background and assum-
integrals, i.e. the low-energy resonances and the $\pi\pi$ background. Definitions of the $\chi^2$ functions were analogous to Eqs.\,(24).

The analysis was split into three steps. In the first step, \textit{(the SP analysis)}, only the S0 and P1 amplitudes were modified. In the fitting procedure, the matching energy and parameters of only $f_0(500)$ and $f_0(980)$ resonances and of the $\pi\pi$ background were free in the S0 wave. Likewise in the P1 wave the matching energy and parameters of background and only the $\rho(770)$ resonance were fitted. Note that contrary to Ref.\,[8] the parameters of $f_0(1500)$ were not changed. Therefore the number of free parameters in the SP analysis decreased to 31. To make a comparison with the results from \cite{8} we used in this step the phenomenological D0 and F1 amplitudes \cite{2} as in \cite{8}.

In the second step, \textit{(the DF analysis)}, we used the final S0 and P1 amplitudes from the SP analysis and the New D0 and F1 from Sect. III as initial amplitudes. In this step, only the latter two amplitudes were successively modified fitting the parameters of the Breit-Wigner forms of the $f_2(1270)$, $f_2(1525)$, $f_2(1640)$, $f_2(1810)$, $f_2(2125)$, and $f_2(2300)$ and $\rho_3(1690)$ resonances for the D0 and F1 partial-waves, respectively, together with the background parameters. The number of free parameters in the DF analysis was 31.

In the third step, \textit{(the SPDF analysis)}, we started with the S0 and P1 amplitudes from the SP analysis and with the D0 and F1 amplitudes from the DF analysis. In this step we fitted again all free parameters considered above and arrived at the final form of all four amplitudes that we denote “re-fitted". These amplitudes are optimized to the data and are consistent with the GKPY equations.

\begin{table}[h]
\centering
\begin{tabular}{l|cccccc}
\hline
& $\chi^2_{\text{Data}}$ & $\delta_{11}$ & $\eta_{11}$ & $\delta_{12}$ & $|S_{12}|$ & $|S_{13}|$ \\
\hline
initial & 321.8 & 132.9 & 23.0 & 126.4 & 35.6 & 3.89 \\
re-fitted-SP & 282.9 & 118.8 & 19.4 & 118.1 & 21.3 & 5.40 \\
\hline
\end{tabular}
\caption{Values of $\chi^2$ for data in the S0 and P1 waves before (initial) and after (re-fitted-SP) fitting in the SP analysis.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{l|cccccc}
\hline
& $\chi^2_{\text{Data}}$ & $\delta_{11}$ & $\eta_{11}$ \\
\hline
initial & 302.8 & 264.1 & 38.7 \\
re-fitted-SP & 301.7 & 262.4 & 39.3 \\
\hline
\end{tabular}
\caption{Values of $\chi^2$ for the D0 and F1 waves before (initial) and after (re-fitted-SP) fitting in the SP analysis.}
\end{table}

\section{Results of the DF analysis}

In order to improve agreement of the D0 and F1 wave amplitudes constructed in sections IIIB and IIIC with...
crossing symmetry condition, the New D0 and New F1 amplitudes have been fitted to the GKPY dispersion relations and to the data, where the S0- and P1-wave amplitudes were from the SP analysis (section IV.B) and remained fixed. Hence, the total $\chi^2$ in Eq. (2) was composed of eight parts: two parts for $\chi^2_{Data}(k)$ in Eq. (3) for D0 and F1 partial-waves and six parts for $\chi^2_{DR}(k)$ in Eq. (4) for all partial-waves, namely S0, S2, P1, D0, D2, and F1.

### D. Results of the SPDF analysis

In the third-last step of our analysis, the total $\chi^2$ in Eq. (2) was composed of ten parts. Four parts for $\chi^2_{Data}(k)$ in Eq. (3) for S0, P1, D0, and F1 partial-waves and six parts for $\chi^2_{DR}(k)$ in Eq. (4) for all considered partial-waves, namely S0, S2, P1, D0, D2, and F1.

| S0 wave | $\chi^2_{Data}$ | $\delta_{11}$ | $\eta_{11}$ | $|S_{12}|$ | $|S_{13}|$ |
|---------|-----------------|---------------|-------------|-----------|-----------|
| initial | 321.8           | 132.9         | 23.0        | 126.4     | 35.6      | 3.89      |
| re-fitted | 292.2           | 129.3         | 19.2        | 117.5     | 21.1      | 5.02      |

| P1 wave | $\chi^2_{Data}$ | $\delta_{11}$ | $\eta_{11}$ |
|---------|-----------------|---------------|-------------|
| initial | 302.8           | 264.1         | 38.7        |
| re-fitted | 299.3           | 260.7         | 38.6        |

### TABLE V: Values of $\chi^2$ for data in the S0 and P1 waves before (initial) and after (re-fitted) fitting in the SPDF analysis.

| D0 wave | $\chi^2_{Data}$ | $\delta_{11}$ | $\eta_{11}$ | $|S_{12}|$ | $|S_{13}|$ |
|---------|-----------------|---------------|-------------|-----------|-----------|
| initial | 242.3           | 137.0         | 76.7        | 21.1      | 7.44      |
| re-fitted | 218.7           | 128.3         | 64.1        | 18.1      | 8.1       |

| F1 wave | $\chi^2_{Data}$ | $\delta_{11}$ | $\eta_{11}$ |
|---------|-----------------|---------------|-------------|
| initial | 136.5           | 120.4         | 16.1        |
| re-fitted | 137.3           | 120.8         | 16.6        |

### TABLE VI: Values of $\chi^2$ for data in the D0 and F1 waves before (initial) and after (re-fitted) fitting in the SPDF analysis.

| $\chi^2_{DR}$ | S0 | S2 | P1 | D0 | D2 | F1 |
|---------------|----|----|----|----|----|----|
| initial       | 895.7 | 842.9 | 8.43 | 44.3 |
| re-fitted-SP  | 37.8 | 7.32 | 10.3 | 20.2 |

| $\chi^2_{DR}$ | S0 | S2 | P1 | D0 | D2 | F1 |
|---------------|----|----|----|----|----|----|
| initial       | 313.2 | 107.8 | 24.4 | 18.0 | 125.1 | 16.8 | 21.2 |
| re-fitted-SP  | 113.9 | 5.05 | 17.9 | 27.8 | 27.5 | 15.8 | 19.9 |

### TABLE VII: Values of $\chi^2$ for the dispersion relations for all waves before (initial) and after (re-fitted) fitting in the SPDF analysis.

Remind that in the SP analysis we used phenomenological parameterizations of the D0- and F1-wave $\pi\pi$ amplitudes from [2].

Tables VII show the $\chi^2$ of the data and the dispersion relations for all waves distinctly. The values of the $\chi^2_{Data}$ for the S0 and D0 waves have generally improved except in the $\pi\pi \rightarrow \eta\pi$ channel for the S0 wave and $\pi\pi \rightarrow \eta\eta$ channel for the D0 one. Description of data in this inelastic channel tends to be worse for both S0 and D0 waves (see $|S_{13}|$ in Table V and $|S_{14}|$ in Table VI) which can be attributed to the coupling between these two waves. Description of data in the P1 wave slightly improved for both phase shift and inelasticity. The $\chi^2_{Data}$ for the F1 wave almost did not change because the parameters of the New F1 amplitude are not too much affected by fitting the dispersion relations. All components of the $\chi^2_{DR}$, except that for P1, are smaller after fitting with a substantial improvement for the S0 and D0 waves.

Note that the initial values of the $\chi^2_{DR}$ for the S0 and P1 waves in Table VII differ from the final values in Table IV due to different D0 and F1 amplitudes used in the analysis. In the SP analysis, the D0 and F1 amplitudes are from [2] and not modified while in the SPDF analysis we used the New D0 and New F1 amplitudes. It is especially well seen for the initial value of the $\chi^2_{DR}$ for the P1 wave which is smaller in the SPDF analysis (18.0) than in the SP one (44.3). Finally it becomes larger but still comparable with the final value in the SP analysis. This suggests quite strong influence of the other amplitudes (especially New F1) on the P1 amplitude in the dispersive analysis.

Comparing results of the SPDF and SP analysis for the S0 wave we see that description of the phase shift $\delta_{11}$ is slightly worse in the SPDF analysis. This we attribute to the influence of the New D0 and New F1 amplitudes, especially of the former as there is stronger correlation between these two waves (S0 and D0). This can be particularly well seen in a comparison of the final (re-fitted-SP) values of $\chi^2_{DR}$ for S0 and S2 in Table IV with the initial values in Table VII.

In Table VIII we show influence of fitting parameters of the $f_0(1500)$ on the results in the SP and SPDF analyses to see significance of this resonance in the analysis. As it was expected from the previous analysis in Ref. [8] for the S and P waves, results are not too much sensitive to changes of the $f_0(1500)$-resonance parameters. Although the $\chi^2$ is smaller when parameters of $f_0(1500)$ resonance are free, the $\chi^2/n.d.f.$ do not change. This corroborates our previous results of the analysis in Ref. [8] where pa-
parameters of \( f_0(1500) \) changed very slightly.

\[
\begin{array}{ccc}
\chi^2/\text{n.d.f. (} f_0(1500) \, \text{fixed)} & \chi^2/\text{n.d.f. (} f_0(1500) \, \text{free)} \\
\chi^2(\text{SP}) & 622.4/541=1.15 & 609.7/529=1.15 \\
\chi^2(\text{SPDF}) & 1061.5/895=1.19 & 1053.6/883=1.19 \\
\end{array}
\]

TABLE VIII: Values of the \( \chi^2/\text{n.d.f.} \) for the SP and SPDF analyzes when parameters of the \( f_0(1500) \) resonance are fixed or free in fitting.

V. DISCUSSION

In Tables IX - XII we provide a comparison of parameters of the original ("New" amplitude from \[8\] in case of the \( S0 \) wave) and re-fitted \( S0- \) and \( P1 \)-wave amplitudes in the SPDF analysis. A substantial change of the parameters after the analysis is a shift of the position of the \( \sigma \) pole on sheet II. The new position change, \((477.6 \pm 14 - i 302.0 \pm 14 \text{ MeV})\) is within three standard deviations consistent with the result in Ref. \[8\], \((445.2 \pm 14 - i 296.4 \pm 14 \text{ MeV})\) demonstrating a stability of the results. This new value is also compatible with that presented by Particle Data Group \[14\] \((400 - 550) - i(200 - 350) \text{ MeV}\). The pole positions of the \( f_0(980) \) resonance did not change too much. The real and imaginary parts became only slightly smaller and are compatible with the values presented in \[8\]. In the \( P1 \) wave, the position of the \( \rho(770) \)-pole on the Riemann sheet VI was shifted significantly toward smaller energies and closer to the real axis. This, however, does not affect a data description as this pole is far from the physical region. Note, however, that the sheet VI is directly connected with the physical one above the \( \rho \sigma \) threshold and therefore a pole lying on the sheet VI near the real axis above the \( \rho \sigma \) threshold, in many cases, can influence appreciably description of the data.

The background parameters of the \( S0 \) and \( P1 \) waves changed moderately showing that the background part plays only a marginal role in the amplitude. The matching energy \( \sqrt{s_{00}} \) became bigger in comparison to that in \[8\]. One may attribute this to the influence of the New \( D0 \) wave. On the contrary the matching energy \( \sqrt{s_{01}} \) for the \( P1 \) wave is similar to that in \[8\]. For the \( P1 \) wave the background parameter \( b \) acquired a very small negative value which affects inelasticity of the \( P1 \) wave, slightly violating unitarity for energies near 1.8 GeV (see Fig. 6 b)).

Figures 3 - 6 illustrate the results of the re-fitted \( S0 \) and \( P1 \) amplitudes for the phase shift and inelasticities in the \( \pi \pi \to \pi \pi, K \bar{K} \) and \( \eta \eta' \) channels compared to the original amplitudes and available experimental data. Our final results describe the data very well in all considered channels. An improvement is especially apparent for the elastic phases \( \delta_{11} \) for both waves in the low-energy region. The turn observed at 1.28 GeV in Fig. 3 b) for inelasticity of the \( S0 \) wave in \( \pi \pi \to \pi \pi \) is due to opening of the \( \sigma \sigma \) channel included in the background part \[8\].

The new (re-fitted) parameters of the \( D0 \) and \( F1 \) waves in the SPDF analysis are given in Tables XIII and XIV. The re-fitted background parameters of the \( D0 \) amplitude are: \( \alpha_{11} = 0.0011853, \alpha_{13} = 0.037747, \alpha_{10} = -0.46722, \beta_1 = 0.15631, \beta_3 = -8.5280, \beta_5 = -11.4446, \gamma_1 = -0.31272, \gamma_3 = 9.9804, \) and \( \gamma_4 = 14.8899. \) The re-fitted background parameters of the \( F1 \) amplitude are: \( a_0 = 0.0000132, a_2 = -0.00102, \) and \( a_4 = 0.0000151. \) Note that contributions of the big magnitudes of the parameters \( \beta_3 \) and \( \gamma_3 \) in the \( K \bar{K} \) channel and \( \beta_4 \) and \( \gamma_4 \) in the \( \eta \eta' \) channel tend to cancel each other above the effective vector-vector channel (\( s_{0s} \)) in the background part, see the formulas and text below Eq. (7). Similar correlations between \( \beta_j \) and \( \gamma_j \) were observed also in the data analysis performed in Sect. IIIB.

A comparison of Tables III and XIV shows that the parameters of the \( F1 \) wave changed only slightly as it

\[
\begin{array}{ccc}
\text{Sheet} & \text{original} & \text{re-fitted} \\
\hline
\text{II} & E_r & 562.9 & 477.6 \\
\Gamma_r/2 & 417.1 & 302.0 \\
\text{III} & E_r & 594.7 & 717.6 \\
\Gamma_r/2 & 417.1 & 300.4 \\
\text{IV} & E_r & 615.1 & 422.2 \\
\Gamma_r/2 & 417.1 & 448.5 \\
\text{VII} & E_r & 583.3 & 602.7 \\
\Gamma_r/2 & 417.1 & 206.7 \\
\end{array}
\]

TABLE IX: Real (\( E_r \)) and imaginary (\( \Gamma_r/2 \)) parts of poles on the Riemann sheets of two lowest resonances in the \( S0 \) amplitude before (original) and after (re-fitted) the full analysis. The values are in MeV.

\[
\begin{array}{ccc}
\text{Parameter} & \text{original} & \text{re-fitted} \\
\hline
a_{11} & -0.0131 & -0.07870 \\
a_{1a} & 0.0 & 0.13210 \\
a_{1v} & 0.046 & -0.13380 \\
a_{1\eta} & -0.0302 & -0.01832 \\
b_{1a} & 0.0 & 0.09207 \\
b_{1v} & 0.0573 & 0.01855 \\
b_{1\eta} & 0.0 & -0.03753 \\
\sqrt{s_{00}} & 406.5 & 495.0 \\
\end{array}
\]

TABLE X: Values of the background parameters and the matching energy \( \sqrt{s_{00}} \) (in MeV) for the \( S0 \) wave before (original) and after (re-fitted) the full analysis.
can be also seen from a comparison of the initial and re-fitted values of the $\chi^2_{Data}$ in Table \textbf{VI}. Such comparison is more complicated for the $D_0$ amplitude since there are more resonance states with more free parameters. Analysis shows that some $D_0$ resonances are irrelevant and some of them have weighty decay widths ($f_{ri}$) only in two or three channels. For example, the decay widths of $f_2(1640)$ into the $\pi\pi$ and $\eta\eta$ channels are practically zero after the dispersive analysis (compare Tables \textbf{I} and \textbf{XIII}) which is compatible with PDG \cite{14} where only the $2\pi\pi$ and $KK$ decays are seen. Similarly the $f_2(2300)$ resonance reveals a very weak coupling/branching to the $2\pi\pi$ channel after the dispersive analysis. The $f_2(1525)$ resonance with a negligible decay width into the $\pi\pi$ channel \cite{14} has also a small value of the parameter $f_{r1}$ in our analysis. Note that in the latest issue of PDG \cite{14} only decays of the $f_2(1270)$ and $f_2(1525)$ resonances into the $\pi\pi$, $2\pi\pi$, $KK$, and $\eta\eta$ channels are precisely determined whereas for the other resonance states mostly a status “seen” is reported.

Results for the phase shifts and inelasticities for the $D_0$ and $F_1$ waves are presented in Figs. \textbf{1} - \textbf{3}. The dispersive analysis practically did not affect the description of the elastic phase shifts $\delta_{11}$ in both waves. The same holds true for the inelasticity parameter in the $F_1$ wave. On the contrary inelasticity in the $D_0$ wave for the $\pi\pi \rightarrow \pi\pi$ channel has improved significantly, especially around 1 and 1.3 GeV. In the 1 GeV region the New $D_0$ ampli-

| Sheet       | original | re-fitted |
|------------|----------|-----------|
| $\rho(770)$ |          |           |
| II $E_r$   | 766.0    | 765.4     |
| $\Gamma_r/2$ | 72.5     | 73.0      |
| III $E_r$  | 758.7    | 799.4     |
| $\Gamma_r/2$ | 72.5     | 53.7      |
| VI $E_r$   | 753.5    | 1.28      |
| $\Gamma_r/2$ | 72.5     | 0.49      |
| VII $E_r$  | 760.82   | 1051.5    |
| $\Gamma_r/2$ | 72.5     | 8.09      |

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
Parameter & original & re-fitted \\
\hline
$a$       & -0.2860  & -0.33148  \\
$b$       & 0.00012  & -0.00008  \\
$\sqrt{s_{01}}$ & 643.6    & 637.3     \\
\hline
\end{tabular}
\caption{The same as in Table \textbf{X} but for the $\rho(770)$ resonance in the $P1$ wave.}
\end{table}

\begin{table}[h]
\centering
\begin{tabular}{lcc}
\hline
Parameter & original & re-fitted \\
\hline
$a$       & -0.2860  & -0.33148  \\
$b$       & 0.00012  & -0.00008  \\
$\sqrt{s_{01}}$ & 643.6    & 637.3     \\
\hline
\end{tabular}
\caption{The same as in Table \textbf{X} but for the $P1$ wave.}
\end{table}

FIG. 4: Results of the original $S_0$ (dashed line) and re-fitted (after fitting to data and dispersion relations) $S_0$ (solid line) amplitudes for the phase shift (a) and inelasticity (b) of the $\pi\pi \rightarrow \pi\pi$ scattering are compared with experimental data.
Results show that not all first four lightest states, i.e. resonance states are dominant or ineffective in our approach. Noticeable changes are also apparent in behavior of $|S_{14}|^2$ ($\pi\pi \rightarrow \eta\eta$) in Fig. 2(b).

We have performed various fits to learn which $D0$ resonance states are dominant or ineffective in our approach. Results show that not all first four lightest states, i.e. $f_2(1270), f_2(1430), f_2(1525),$ and $f_2(1640),$ are the most relevant states in the analysis, as one would naively expect. This is demonstrated in Table XVII which shows the $\chi^2$ for the fits with one particular resonance omitted.

The $\chi^2$ was composed of the $\chi^2_{\text{data}}$ for the $D0$ wave and $\chi^2_{\text{DR}}$ for all partial-waves. Values of the $\chi^2$ in the fits No. 9, 11, and 12 are almost equal to the value in the fit No. 1 with all resonance states included. Accordingly, one can conclude that the $f_2(2010), f_2(2300),$ and $f_2(2340)$ resonances are insignificant in the description. Values of the $\chi^2$ in the fits No. 3, 5, 7, and 8 show that although the $f_2(1430), f_2(1640),$ and $f_2(1910)$ do not play a very important role in the data description one should keep them. The other states, $f_2(1525), f_2(1810),$ and $f_2(2150),$ influence behavior of the amplitude even more. Obviously the most significant resonance which dominates behavior of the amplitude is the $f_2(1270)$ resonance.

### Table XIII: Parameters of the Breit-Wigner form (in MeV) for the $D0$ wave after fitting in the SPDF analysis. The initial values of the parameters are in Table [III]. The mass of the resonance was not changed in the dispersive analysis.

| state    | $M$  | $f_{r1}$ | $f_{r2}$ | $f_{r3}$ | $f_{r4}$ |
|----------|------|----------|----------|----------|----------|
| $f_2(1270)$ | 1275.5 | 451.6    | 77.3     | 121.0    | 64.7     |
| $f_2(1525)$ | 1570.7 | 76.3     | 400.8    | 999.0    | 922.9    |
| $f_2(1640)$ | 1639.0 | 0.002    | 356.3    | 222.0    | 0.0002   |
| $f_2(1810)$ | 1815.0 | 120.4    | 813.0    | 999.0    | 873.2    |
| $f_2(2150)$ | 2157.0 | 11.7     | 842.8    | 803.4    | 999.0    |
| $f_2(2300)$ | 2181.6 | 27.0     | 0.0005   | 453.5    | 142.9    |

### Table XIV: Parameters of the Breit-Wigner form (in MeV) for the $F1$ wave after fitting in the SPDF analysis. The initial values of the parameters are in Table [III].

| state    | $M$  | $f_{r1}$ | $f_{r2}$ | $f_{r3}$ | $f_{r4}$ |
|----------|------|----------|----------|----------|----------|
| $\rho_0(1690)$ | 1713.7 | 291.6    | 497.8    | 0.0      | 0.0      |

$\chi^2/n.d.f.$

| Fit No. | All states included | $\chi^2/n.d.f.$ |
|---------|---------------------|-----------------|
| 1       | 326.21/330=0.9885   |                 |
| 2       | $f_2(1270)$         | 5838.8/330=17.69|
| 3       | $f_2(1430)$         | 327.39/330=0.9921|
| 4       | $f_2(1525)$         | 336.97/330=1.0211|
| 5       | $f_2(1640)$         | 327.76/330=0.9932|
| 6       | $f_2(1810)$         | 348.07/330=1.0548|
| 7       | $f_2(1910)$         | 327.36/330=0.9920|
| 8       | $f_2(1950)$         | 327.26/330=0.9917|
| 9       | $f_2(2010)$         | 326.68/330=0.9899|
| 10      | $f_2(2150)$         | 368.81/330=1.1176|
| 11      | $f_2(2300)$         | 326.54/330=0.9985|
| 12      | $f_2(2340)$         | 326.64/330=0.9898|

### Table XV: Values of $\chi^2$ after fitting with omitted some specific resonance state in the $D0$ wave.

#### A. Full $\pi\pi$ amplitude

The set of all important partial-wave amplitudes modified and re-fitted in the previous sections allowed us to construct the full $\pi\pi$ amplitude and to calculate the total and differential $\pi\pi \rightarrow \pi\pi$ cross section up to about 2 GeV. In the following we briefly summarize the basic formulas to clarify the normalization and the partial-
wave decomposition.

The partial-wave amplitudes are related to the phase shift and inelasticity as

$$t'_\ell(s) = \sqrt{\frac{s}{4ik_1}} [\eta'_\ell(s) e^{2\delta'_\ell(s)} - 1]$$

and summed with the Legendre polynomials $P_\ell(cos \theta)$ give the full invariant $\pi \pi$ amplitude $\mathcal{T}^I(s,t)$ in a given isospin channel $I$

$$\mathcal{T}^I(s,t) = 32\pi \sum_\ell (2\ell + 1) t'_\ell(s) P_\ell(cos \theta),$$

where $\theta$ is the scattering angle between two pions in the c.m. frame.

The full invariant amplitude for the $\pi^+\pi^-$ scattering is

$$\mathcal{T}_{\pi^+\pi^-}(s, t) = \frac{1}{3} \mathcal{T}^0(s, t) + \frac{1}{2} \mathcal{T}^1(s, t) + \frac{1}{6} \mathcal{T}^2(s, t)$$

which gives the differential cross section in the c.m. frame

$$\frac{d\sigma_{\pi^+\pi^-}}{d\Omega} = \frac{1}{64\pi^2 s} |\mathcal{T}_{\pi^+\pi^-}|^2,$$

and the total cross section

$$\sigma_{\pi^+\pi^-} = \frac{1}{2k_1\sqrt{s}} \Im \mathcal{T}_{\pi^+\pi^-}(s, \theta = 0).$$

The total cross section and its components from the individual partial-waves are presented on Fig. 7. Well seen is the dominance of the $f_0(500)$ and $\rho(770)$ below 1 GeV. Two maxima above this energy are formed mostly by $f_2(1270)$ and $\rho_3(1690)$. Other waves - $S2$ and $D2$ have minor influence on the cross section. Our theoretical predictions very well agree with data from [12, 18]. Worthy is to notice that such good agreement has been achieved without fitting directly to these data. Uncertainties of the total cross sections presented on the figure were calculated using the Monte Carlo method for 1000 randomly generated sets of all free, in minimizations, parameters taken within their 1 $\sigma$ deviation.

In many experimental analyses performed in the seventies (see some papers in [12, 19]) the $f_0(500)$ was replaced by a very broad and heavier (with mass around 1 GeV) resonance. Its influence on the phase shifts and cross sections was weaker and distributed in a wider energy range. Therefore, it is interesting to see how the cross section would look if the $f_0(500)$ has been removed from the analysis. We demonstrate it on Fig. 8. Of course the main difference, in comparison with Fig. 7, concerns region below 0.7 GeV which is almost completely determined by the $f_0(500)$. Small enhancement near the $\pi\pi$ threshold is caused by the fixed scalar-isoscalar scattering length and the slope parameter. It is worth to pay attention on completely different behavior of the cross section on Figs. 7 and 8 near the $KK$ threshold where one observes the $f_0(980)$. In the cross sections without the $f_0(500)$ clearly seen is a small peak instead of the deep minimum seen on Fig. 7. The peak appears due to absence of about 90 degree component in the $S$ wave phase shift generated by the $f_0(500)$ pole and zero. On Fig. 8 one can also see that the interference of the $f_0(500)$ with all other amplitudes is positive in the whole energy range except of the region between around 1.5 GeV and 1.7 GeV and vicinity of the small peak around 1 GeV.
FIG. 8: Prediction of the total and S-wave cross sections in the low-energy region with and without $f_0(500)$ are shown for the $\pi^+\pi^-\rightarrow\pi^+\pi^-$ scattering.

The elastic differential cross sections at two energies: 550 MeV and 770 MeV are presented on Fig. 9. Clearly seen is, as expected, the significant role of the S-wave at 550 MeV (dominated by the $f_0(500)$). An interference between the $S$ and $P$ waves producing the enhancement at forward angles is substantial and interference with other waves - only noticeable. In vicinity of the $\rho(770)$ the cross section is, of course, dominated by the $P$-wave. Again, the interference between the $S$ and $P$ waves is very important and interference with other waves barely noticeable.

VI. CONCLUSIONS

We have constructed new multi-channel $D_0$- and $F_1$-wave $\pi\pi$ amplitudes using the multi-channel Breit-Wigner formalism. When constructing the $D_0$ wave we utilized the tensor-isoscalar resonances presented in the latest issue of the PDG tables. We showed that the data in the $F_1$-wave can be satisfactorily described considering only one resonance $\rho_3(1690)$. The other state $\rho_3(1990)$ also listed in PDG appeared as unnecessary in the data description.

The previously constructed $S0$- and $P1$-wave multi-channel amplitudes and the new ones, $D0$ and $F1$, were modified in the dispersive analysis using the Roy-like equations (GKPY). The isotensor amplitudes $S2$ and $D2$, also used in the analysis, were taken in the phenomenological form and were not changed.

The modified (re-fitted) partial-wave multi-channel amplitudes $S0$, $P1$, $D0$, and $F1$ are optimized to the experimental data in the considered channels and they fulfill the crossing symmetry condition imposed by the GKPY dispersion equations. The overall description of the data is satisfactory. In the $S0$-wave amplitude a position of the $f_0(500)$ pole on the second Riemann sheet was changed to the value very well consistent with the PDG values and with our previous result. In the analysis of the $D0$ wave we concluded that apart of the dominant $f_2(1270)$ state the resonances $f_2(1525)$, $f_2(1810)$, and $f_2(2150)$ also do play an important role in the data description and are also required in the dispersive analysis.

The modified partial-wave amplitudes were utilized in constructing the full invariant $\pi\pi$ scattering amplitude and the cross sections were calculated.

ACKNOWLEDGEMENT

We want to thank Yurii S. Surovtsev for his help and many useful discussions with him. We are particularly grateful to George Rupp for many important comments and observations, and for very fruitful discussions with one of us. This work has been partially supported by the Polish Science Center (NCN) Grants No. Dec-
2013/09/B/ST2/04382 and DEC-2014/15/N/ST2/03504 and by the Grant Agency of the Czech Republic under the grant No. P203/15/04301.

[1] B. Ananthanarayan, G. Colangelo, J. Gasser and H. Leutwyler, Phys. Rept. 353, 207 (2001).
[2] R. Garcia-Martin, R. Kamiński, J.R. Pela’ez, J. Ruiz de Elvira and F. J. Yndurain, Phys. Rev. D 83, 074004 (2011).
[3] J. R. Pela’ez and F. J. Yndurain, Phys. Rev. D 71, 074016 (2005).
[4] R. Kamiński, Int. J. Mod. Phys. Conf. Ser. 39, 1560087 (2015).
[5] K. Nakamura et al. (Particle Data Group), J. Phys. G 37, 075021 (2010).
[6] "2013 Review of Particle Physics" J. Beringer et al. (Particle Data Group), Phys. Rev. D 86, 010001 (2012).
[7] S. M. Roy, Phys. Lett. B 36, 353 (1971).
[8] P. Bydżoński, R. Kamiński, V. Nazari, Phys. Rev. D 90, 116005 (2014).
[9] Yu. S. Surovtsev, P. Bydżoński, R. Kamiński and M. Nagy, Phys. Rev D 81, 016001 (2010).
[10] Yu. S. Surovtsev, P. Bydżoński, R. Kamiński and M. Nagy, arXiv:1104.0538 [hep-ph].
[11] R. Kamiński, Phys. Rev. D 83, 076008 (2011).
[12] B. Hyams et al., Nucl. Phys. B 64, 134 (1973).
[13] S.J. Lindenbaum and R.S. Longacre, Phys. Lett. B 274, 492 (1992); R.S. Longacre et al., Phys. Lett. B 177, 223 (1986).
[14] K. A. Olive et al. (Part. Data Group) Chin. Phys. C 38, 090001 (2014).
[15] I. Caprini, G. Colangelo, H. Leutwyler, Phys. Rev. Lett. 96, 132001 (2006).
[16] R. Garcia-Martin, R. Kamiński, J.R. Pela’ez and J. Ruiz de Elvira, Phys. Rev. Lett. 107, 072001 (2011).
[17] V. Nazari, R. Kamiński, Acta Phys. Polon. B 46, 1355 (2015).
[18] N.N. Biswas et al., Phys. Rev. Lett. 18, 273 (1967).
[19] P. Estabrooks and A. D. Martin, Nucl. Phys. B 79, 301 (1974); S. D. Protopopescu et al., Phys. Rev. D 7, 1279 (1973); R. Kamiński, L. Leśniak, and K. Rybicki, Z. Phys. C 74, 79 (1997); Eur. Phys. J. direct C 4, 1 (2002); B. Hyams et al., Nucl. Phys. B 100, 205 (1975); M. J. Losty et al., Nucl. Phys. B 69, 185 (1974); W. Hoogland et al., Nucl. Phys. B 126, 109 (1977); N. B. Durusoy et al., Phys. Lett. 45B, 517 (1973); G. Grayer et al., Nucl. Phys. B 75, 189 (1974).