On the Hydrodynamic Description of Holographic Viscoelastic Models

Martin Ammon, Matteo Baggioli, Scán Gray, Sebastian Grieninger, and Akash Jain

1 Theoretisch-Physikalisches Institut, Friedrich-Schiller-Universität Jena, Max-Wien-Platz 1, D-07743 Jena, Germany.
2 Instituto de Física Teórica UAM/CSIC, c/ Nicolás Cabrera 13-15, Universidad Autónoma de Madrid, Cantoblanco, 28049 Madrid, Spain.
3 Department of Physics, University of Washington, Seattle, WA 98195-1560, USA.
4 Department of Physics & Astronomy, University of Victoria, PO Box 1700 STN CSC, Victoria, BC, V8W 3Y2, Canada.

We investigate the near-equilibrium dynamics of homogeneous holographic models with spontaneously broken translations. We show that the dual hydrodynamic description must include the so-called “configuration pressure” – a novel transport coefficient proposed recently – to reproduce the quasinormal mode spectrum in such models faithfully. While being characteristic of strained equilibrium configurations that do not minimise free energy, we illustrate that configuration pressure is a crucial ingredient even in models with no background strain and affects the low-energy dispersion relations through its temperature derivatives.

**Introduction.**—Models with broken translational invariance have attracted a great deal of interest in the holographic community in recent years, especially in relation to their hydrodynamic description [1–9] and their possible relevance for strange metal phenomenology [10–13]. Particular emphasis has been given to the so-called homogeneous models, e.g. massive gravity [14–17]; Q-lattices [18, 19]; and helical lattices [20, 21]; due to their appealing simplicity. Despite the sustained activity in the field, there still remain a number of open questions. For instance, it has been unclear what hydrodynamic framework appropriately describes the near-equilibrium dynamics of field theories dual to these models. The authors of [3] wrote down a generic theory of linearised hydrodynamics with broken translations (see also [2, 22]), which is largely matched by results from holography [8–11, 19, 23–26]. However, the first indication that something might be amiss came from [9], in the form of a disagreement between the holographic results and the predictions of [3] regarding the longitudinal diffusion mode.

Recently, a new fully non-linear hydrodynamic theory for viscoelasticity was proposed in [5]. At the linear level, this formulation differs from [3] due to the presence of an additional transport coefficient, $\mathcal{P}$, called the lattice or configuration pressure. Physically, $\mathcal{P}$ can be understood as a contribution to the mechanical pressure as a result of working around a uniformly strained equilibrium state. In this sense the configuration pressure is analogous to the magnetisation pressure which appears in the presence of an external magnetic field [27, 28]. As it turns out, $\mathcal{P}$ is non-zero in the holographic models mentioned above (see [5]) and, as we illustrate in this letter, consistently explains the discrepancies reported in [9].

It is misleading, however, to dismiss this new coefficient purely as an artifact of background strain. While $\mathcal{P}$ certainly vanishes in an unstrained equilibrium state that minimises free energy, its temperature dependence still carries vital physical information and affects various modes through $\mathcal{P}' = \partial_T \mathcal{P}$. For scale invariant theories, this leads to a non-zero bulk modulus $B = -TP'/2$, which would be zero in the formulation of [3]. It is expected, therefore, that the discrepancy found in [9] would still be present for holographic models without background strain, e.g. in [19].

In this letter, we consider the most general isotropic Lorentz violating massive gravity theories in two spatial dimensions [17]. The dual field theories correspond to isotropic and conformal viscoelastic systems with spontaneously broken translations. By carefully studying the quasinormal modes of these systems, and in particular the dispersion relations of the hydrodynamic modes, we illustrate that the dual field theories are perfectly described by the hydrodynamic framework of [5].

**Viscoelastic Hydrodynamics.**—Let us start by briefly reviewing the formulation of viscoelastic hydrodynamics from [5]. We will work in $d = 2$ spatial dimensions. The fundamental ingredients in the theory are the fluid velocity $u^\mu$, temperature $T$, and translation Goldstone bosons $\Phi$. Moreover, we define $e^\mu_\nu = \partial_\nu \Phi^\mu$, which is used to further define $h^{IJ} = e^I_\mu e^{J\mu}$; $e^{I\mu} = h^{I\mu|}_\nu e^{\nu}_\nu$; $h_{\mu\nu} = h^{I\mu|}_\nu e_{\nu}$, and the strain tensor $u_{\mu\nu} = \frac{1}{2}(h_{IJ} - \delta_{IJ}/\alpha^2) e^I_\mu e^J_\nu$, for some constant $\alpha$. The constitutive relations of an isotropic neutral viscoelastic system, written in a small strain expansion, are given as

$$T^{\mu\nu} = (\epsilon + p + T \mathcal{P}' u^\lambda) u^\mu u^\nu + (p + \mathcal{P} u^\lambda) \eta^{\mu\nu} + \mathcal{P} h^{\mu\nu} - \eta \sigma^{\mu\nu} - \zeta \partial_\mu u^\nu - G(2u^{\mu\nu} - \delta_{IJ} h^{\mu\nu}) - B u^\lambda h^{\mu\nu}, \quad (1a)$$

with the thermodynamic identities $dp = \epsilon \, dT$, $\epsilon = Ts - p$. Prime denotes derivative with respect to $T$ for fixed $\alpha$. These are accompanied by the configuration (Josephson) equations for the Goldstones

$$u^\mu e^I_\mu = \frac{h^{IJ}}{\sigma} \partial_\nu (\mathcal{P} e^\nu_\nu - (B-G) u^\lambda e^\nu_\nu - 2G u^{\mu\nu} e_{IJ}). \quad (1b)$$

Here $p$ and $\mathcal{P}$ are the thermodynamic and configuration pressures respectively, $\epsilon$ and $s$ are energy and entropy.
The bulk modulus leads to a set of identities [5] in conformal viscoelastic fluids, wherein (up to contact-terms) retarded two point functions free-energy density for various coefficients appearing in (1) in terms of the $\omega$ all functions are evaluated at $\chi$. Here $\Gamma$ sectors, and a diffusion mode in the longitudinal sector sound modes, one each in longitudinal and transverse sectors, and a diffusion mode in the longitudinal sector

$$\omega = \pm v_{||} k - \frac{i}{2} \Gamma_{||} k^2 + \ldots, \quad \omega = -i D_{||} k^2 + \ldots. \quad (2)$$

Following [5], the sound velocities $v_{||}$, attenuation constants $\Gamma_{||}$, and diffusion constant $D_{||}$ are given as

$$v_{||}^2 = \frac{G}{\chi_\pi}, \quad v_{\perp}^2 = \frac{(s + P')^2}{s' \chi_\pi} + \frac{B + G - P}{\chi_\pi},$$

$$\Gamma_{\perp} = \frac{\eta + G s T^2}{\chi_\pi}, \quad D_{||} = \frac{s^2}{\sigma s'} B + G - P,$$

$$\Gamma_{||} = \frac{\eta + \chi_\pi}{\chi_\pi} + \frac{T^2 s^2 v_{||}^2}{\sigma \chi_\pi} \left( 1 - \frac{s + P'}{T s' v_{||}^2} \right)^2. \quad (3)$$

Here $\chi_\pi = \epsilon + p + P$ is the momentum susceptibility; all functions are evaluated at $T = T_0$. Note that the pair of transverse sound modes are not present when $G = 0$. Instead, they are replaced by a single shear diffusion mode $\omega = -i D_{||} k^2$ with $D_{||} = \eta/\chi_\pi$. We can obtain formulas for various coefficients appearing in (1) in terms of the free-energy density $\Omega$, stress-tensor one-point function, and (up to contact-terms) retarded two point functions

$$p = -\Omega, \quad \epsilon = \langle T^{tt} \rangle, \quad P = \langle T^{xx} \rangle + \Omega, \quad \chi_\pi v_{||}^2 = \lim_{\omega \to 0} \lim_{k \to 0} \Re G_{T^{xx} T^{xx}},$$

$$G = \chi_\pi v_{\perp}^2 = \lim_{\omega \to 0} \lim_{k \to 0} \Re G_{T^{xx} T^{xy}}, \quad \eta = -\lim_{\omega \to 0} \lim_{k \to 0} \frac{1}{\omega} \Im G_{T^{xx} T^{xy}}, \quad \frac{(\epsilon + p)^2}{\sigma^2 \chi_\pi} = \lim_{\omega \to 0} \lim_{k \to 0} \omega \Im G_{\Phi^* \Phi^*}. \quad (4)$$

The bulk modulus $B$ can be obtained indirectly using the $v_{||}^2$ Kubo formula.

For our application to holography, we shall be interested in conformal viscoelastic fluids, wherein $T_{\mu\nu}$ is 0. This leads to a set of identities [5]

$$\epsilon = 2(p + P), \quad TP' = 3P - 2B, \quad \zeta = 0. \quad (5)$$

Taking derivative of the first relation, we also find the specific heat $c_v = T s' = 2(s + P')$. Using these, we can derive a relation between sound velocities $v_{||}^2 = 1/2 + v_{\perp}^2 [29]$. For conformal theories $v_{\perp}$ and $\Gamma_{\perp}$ stay the same as in (3), however the expressions for the longitudinal sector simplify to

$$\Gamma_{||} = \frac{\eta}{\chi_\pi} + \frac{T^2 s^2 G^2}{\sigma s' \chi_\pi v_{||}^2}, \quad D_{||} = \frac{s^2}{\sigma s'} B + G - P. \quad (6)$$

Interestingly, apart from the implicit dependence in $\chi_\pi$, only $D_{||}$ depends explicitly on $P$, which explains the discrepancy reported in the diffusion mode in [9]. Note that using (5), the bulk modulus can be rewritten as $B = (3P - TP')/2$. Consequently, a conformal viscoelastic system only responds to bulk stress if $P, P' \neq 0$.2

**Unstrained Equilibrium Configurations.**—In equilibrium states with no background strain, the equilibrium configuration pressure is zero, $P(T_0) = 0$. In this case, momentum susceptibility reduces to a familiar expression $\chi_\pi = \epsilon + p$. Although not particularly useful in the generic case, in conformal limit this can be used to further simplify (6) to

$$\Gamma_{||} = \frac{\eta}{\chi_\pi} + \frac{2G^2/\sigma}{Ts + 2G}, \quad D_{||} = \frac{T s^2/\sigma}{Ts + 2G} B + G. \quad (7)$$

Note that $P'(T_0) \neq 0$, leading to a non-zero bulk modulus $B = -TP'/2$. The occurrence of $P'$ in the denominator of $D_{||}$ suggests that temperature dependence of configuration pressure still plays an important role in an unstrained equilibrium configuration. Another signature of configuration pressure in a conformal viscoelastic system is the specific heat $c_v = 2(s + P') \neq 2s$. In absence of conformal invariance, the effects of $P'$ will also contaminate the expression for the longitudinal sound mode.

**Holographic Model.**—Let us consider a simple holographic model with $(d + 2)$-dimensional Einstein-AdS gravity coupled to $d$ copies of St"uckelberg scalars $\phi^i$

$$S_{\text{bulk}} = \int d^{d+2} x \sqrt{-g} \left( \frac{R}{2} + \frac{d(d+1)}{2\ell^2} - m^2 V(T^{IJ}) \right). \quad (8)$$

Here $T^{IJ} = \phi^i \partial_\phi \phi^i \partial_\phi \phi^j$ is the kinetic matrix; $\ell$ is the AdS-radius, which we in the following set to one; and $m$ is a parameter related to the graviton mass. We have set $8\pi G_N/\ell^d = 1$. For the isotropic case in $d = 2$, we can generically take $V(T^{IJ}) = V(X, Z)$ where $X = 1/2 \tr Z$

---

1 The limit $G \to 0$ is subtle and must be performed at the level of the transverse sector dispersion relations, $\omega^2 s T + (1 + \frac{4T_{xy}}{\eta s}) (s^2 P - k^2 G) + i\omega k^2 \eta = 0$.

2 Nevertheless, the compressibility $\beta \equiv (-1/V) \partial T_{xx} / \partial V$ is finite even in the absence of the configuration pressure, and in the conformal case it is given by $\beta^{-1} = (3/4)x$ [30]. It is possible to show that in terms of the compressibility the longitudinal speed can be written as $v_{||}^2 = (\beta^{-1} + G)/\chi_\pi$ [9, 23].
and $Z = \det I$ \cite{16, 17, 23}. The scalars $\phi^I$ are dual to the boundary operators $\phi^I$ and break the translational invariance of the dual field theory (see \cite{31} and \cite{17} for the specifics of the symmetry breaking pattern). Depending on the boundary conditions imposed on $\phi^I$, this breaking can either be explicit, spontaneous, or pseudo-spontaneous \cite{5, 8, 15, 23, 32}. Presently, we shall be interested in models with spontaneously broken translations leading to phonon dynamics in the dual field theory \cite{9, 23, 24, 33, 34}.

We consider a black brane solution of (8) in Eddington-Finkelstein (EF) coordinates with the metric

$$ds^2 = \frac{1}{u^2} \left(-f(u)\, dt^2 - 2\, dt\, du + dx^2 + dy^2\right),$$

with a radially constant profile for the scalars, $\phi^I = \alpha \, x^I$, for some constant $\alpha$. The radial coordinate $u \in [0, u_h]$ spans from the boundary $u = 0$ to the horizon $u = u_h$. The emblackening factor $f(u)$ takes a simple form

$$f(u) = 1 - \frac{u^3}{u_h^3} - u^3 \int_u^{u_h} \frac{m^2}{\pi^2} V(\alpha^2 \, \delta^2, \alpha^4 \, \delta^4) \, d\delta.$$  

(10)

Linear perturbations around the black brane geometry capture near-equilibrium finite temperature fluctuations in the boundary field theory \cite{23, 30, 35–37}, which is expected to be described by the theory of viscoelastic hydrodynamics discussed previously \cite{5}.

Temperature and entropy density in the boundary field theory are identified with the Hawking temperature and area of the black brane respectively

$$T = -\frac{f'(u_h)}{4\pi} = \frac{3 - m^2 V_h}{4\pi u_h}, \quad s = \frac{2\pi}{u_h}. \quad (11)$$

Here $V_h = V(u_h^3, \alpha^2, u_h^4, \alpha^4)$. Free energy density is defined as the renormalized euclidean on-shell action \cite{38}. The expectation value $(T^{\mu\nu})$ can be read off using the leading fall off of the metric at the boundary. Using (4), this leads to the thermodynamic parameters

$$p = \frac{1}{2u_h^3} - m^2 u_h^3 \left(\frac{1}{2} V_h - U_h\right), \quad \epsilon = \frac{1}{u_h^3} - \frac{m^2}{u_h^3} U_h,$$

$$\mathcal{P} = \frac{m^2}{u_h^3} \left(\frac{1}{2} V_h - \frac{3}{2} U_h\right). \quad (12)$$

We have defined $U_h = -\frac{u_h^3}{2} \int \alpha^4 \delta^4 V(\alpha^2 \delta^2, \alpha^4 \delta^4) d\delta$, assuming $V(X, Z)$ to fall off faster than $\sim u^3$ at the boundary. Using these expressions with (5), we can find

$$B = \frac{m^2}{4u_h^3} \left(3V_h - 9U_h + \frac{u_h \partial_{u_h} V_h (m^2 V_h - 3)}{m^2 (V_h - u_h \partial_{u_h} V_h - 3)}\right). \quad (13)$$

Finally, using the results of \cite{4, 8, 11}, we can derive a horizon formula for $\sigma$, which reads

$$\sigma = \frac{m^2}{2\alpha^2 u_h^3} \partial V_h, \quad (14)$$

and agrees well with the numerical results obtained with the Kubo formula in (4). The remaining coefficients, $G$ and $\eta$, must be obtained numerically.

The non-trivial expression for $\mathcal{P}$ in (12) indicates the presence of background strain in these holographic models. This is associated with the equilibrium state $\phi = \alpha \, x^I$ not being a minimum of free energy \cite{19, 24, 39}. To wit, using (12) one can check that $d\Omega/d\alpha\mathcal{P} = -d\rho/d\alpha|_{T_\pi} = 0$ leads to $\mathcal{P} = 0$. However, as is evident from (3), the presence of $\mathcal{P}$ by itself does not lead to any linear instability or superluminality \cite{9, 23, 24}. Setting $\mathcal{P} = 0$ in (12), we can find a thermodynamically favored state $\alpha = \alpha_0$ as a non-zero solution of $V_h = 3U_h$. Notice that $\mathcal{P}'|_{\alpha=\alpha_0} \neq 0$, meaning that configuration pressure still plays a crucial role in the dual hydrodynamics through its temperature derivatives, as discussed around (7). In particular, these models can have non-zero bulk modulus despite being scale invariant.

Simple monomial models considered previously in the literature \cite{17, 23, 24, 30, 36, 37}, such as $V(X, Z) = X^N, Z^M$, do not admit $\mathcal{P} = 0$ states with non-zero $\alpha$.\footnote{However, the would-be preferred state $\alpha = 0$ is not a good vacuum of the theory, since the model is strongly coupled around that background \cite{24}. Therefore, in these theories, it is incorrect to compare free energies of states with $\alpha \neq 0$ against the state $\alpha = 0$.}

The simplest models admitting states with $\mathcal{P} = 0$ have polynomial potentials such as $V(X, Z) = X + \lambda X^2$. Unfortunately, this naive model has linear instabilities (imaginary speed of sound and negative diffusion constant). Nevertheless, it can be used as a toy model to illustrate the importance of $\mathcal{P}^I \neq 0$: we refer the reader to the supplementary material for more details.

In the remainder of this letter, we specialise to models with $V(X, Z) = X^N, Z^M$ and $N > 5/2$, $M > 5/4$ to numerically obtain $G$ and $\eta$, and test the agreement between quasinormal modes and the hydrodynamic predictions.

**Numerical Results.**—We can compute the full spectrum of quasinormal modes, in both the transverse and longitudinal sectors, using pseudo-spectral methods following \cite{9, 23, 24, 40, 41}. As we discussed around (6), the configuration pressure does not appear explicitly in the transverse sound modes, leading to the same predictions by \cite{3} and \cite{5} (modulo the definition of $\chi_{\pi\pi}$). Since the discrepancy in $\chi_{\pi\pi}$ has already been identified and tested against hydrodynamic predictions \cite{23, 24}, here we only focus on the longitudinal sector.

We start with $V(X, Z) = X^N$ models and compare the hydrodynamic prediction for the longitudinal attenuation constant $\Gamma_\parallel$ and diffusion constant $D_\parallel$ in (3) with the numerical results obtained from the holographic model. The results are shown in fig. 1. The agreement is extremely good and is valid independent of $N$. We no longer see a discrepancy in the diffusion mode.
Let us now consider models $V(X, Z) = Z^M$. Generically, $X$-independent potentials $V(X, Z) = V(Z)$ enjoy a larger symmetry group – the dual field theory is invariant under volume preserving diffeomorphisms, modeling a fluid. These models have $G = 0$, leading to the absence of transverse phonons [17], and $\eta$ saturating the Kovtun-Son-Starinets bound [35]. In fig. 2 we show a comparison between the hydrodynamic prediction and numerical results for quasinormal modes for $V(X, Z) = Z^2$. The excellent agreement confirms that the hydrodynamic framework of [5] is valid for a general class of viscoelastic models.

Conclusions. — In this letter we have illustrated that the theory of viscoelastic hydrodynamics formulated in [5] is the appropriate hydrodynamic description for the holographic models of [17] with spontaneously broken translations. The theory faithfully predicts all the transport coefficients and the behaviour of the low-energy quasinormal modes in the holographic setup. Moreover, it resolves the tensions between the previous hydrodynamic framework of [3] and the holographic results reported in [9]. This discrepancy had been previously attributed to the presence of background strain in these holographic models, i.e. that the equilibrium configurations do not lie at a minimum of free energy. However, here we argued that such discrepancies are present even in unstrained equilibrium configurations and are accounted for only after including the effects of configuration pressure proposed in [5]. More generally, we expect that the hydrodynamic formulation of [5] will continue to work for all homogeneous holographic models with spontaneously broken translations [6, 7, 19, 20], due to the same symmetry-breaking pattern.

The analysis in this letter opens up the stage for various interesting future explorations. An immediate goal would be to inspect various holographic models for viscoelasticity in the literature, with zero background strain, and identify the role of non-zero $P'$ on the quasinormal spectrum. It would also be interesting to better understand the physical role of configuration pressure, which is the crucial ingredient for the matching outlined in this letter.

The addition of a small explicit breaking of translations to the hydrodynamic framework of [5] could also provide an understanding of the universal phase relaxation relation $\Omega \sim M^2/\sigma$ (with $\Omega$ the Goldstone phase relaxation rate; $M$ the mass of the pseudo-Goldstone mode). This relation was proposed in [11] and was later verified for the models presented in this paper in [8]. It could also provide an explanation for the complex dynamics found in the pseudo-spontaneous limit in [23].

One may further consider the viscoelastic hydrodynamic theory of [5] beyond linear response in order to explore the full rheology of the holographic models considered in this letter, as initiated in [37].

In conclusion, this letter marks an important development in understanding the nature of the field theories dual to the widely used holographic models with spontaneously broken translational invariance, and provides another robust bridge between holography, hydrodynam-
ics (in its generalised viscoelastic form) and effective field theory.

Acknowledgments.—We thank Jay Armas, Blaise Goutéraux and Sean Hartnoll for several helpful discussions and comments. Séan G would like to thank IFT Madrid for hospitality during the initial stages of this work. Sebastian G thanks the University of Victoria for hospitality during the initial stages of this work.

MA is funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) – 406116891 within the Research Training Group RTG 2522/1. Sebastian G gratefully acknowledges financial support by the DAAD (German Academic Exchange Service) for a Jahrestipendium für Doktorandinnen und Doktoranden in 2019. AJ is supported by the NSERC Discovery Grant program of Canada.

* martin.ammon@uni-jena.de
† matteo.baggioli@uam.es
§ sean.grau@uni-jena.de
¶ sebastian.grieninger@gmail.com
\* ajain@uvic.ca

[1] B. I. Halperin and D. R. Nelson, Phys. Rev. Lett. 41, 121 (1978).
[2] P. C. Martin, O. Parodi, and P. S. Pershan, Phys. Rev. A 6, 2401 (1972).
[3] L. V. Delacrétaz, B. Goutéraux, S. A. Hartnoll, and A. Karlsson, Phys. Rev. B96, 195128 (2017), arXiv:1702.05104 [cond-mat.str-el].
[4] A. Donos, A. Martin, C. Pantelidou, and V. Ziogas, JHEP 10, 218 (2019), arXiv:1905.08039 [hep-th].
[5] J. Armas and A. Jain, (2019), arXiv:1908.01175 [hep-th].
[6] A. Donos, A. Martin, C. Pantelidou, and V. Ziogas, (2019), arXiv:1906.03132 [hep-th].
[7] S. Grozdanov and N. Poovuttikul, Phys. Rev. D97, 106005 (2018), arXiv:1801.03199 [hep-th].
[8] M. Ammon, M. Baggioli, and A. Jiménez-Alba, JHEP 09, 124 (2019), arXiv:1904.05785 [hep-th].
[9] M. Ammon, M. Baggioli, S. Gray, and S. Grieninger, JHEP 10, 064 (2019), arXiv:1905.09164 [hep-th].
[10] A. Amoretti, D. Areán, B. Goutéraux, and D. Musso, Phys. Rev. Lett. 120, 171603 (2018), arXiv:1712.07994 [hep-th].
[11] A. Amoretti, D. Areán, B. Goutéraux, and D. Musso, Phys. Rev. Lett. 123, 211602 (2019), arXiv:1812.08118 [hep-th].
[12] R. A. Davison, K. Schalm, and J. Zaanen, Phys. Rev. B89, 245116 (2014), arXiv:1311.2451 [hep-th].
[13] L. V. Delacrétaz, B. Goutéraux, S. A. Hartnoll, and A. Karlsson, SciPost Phys. 3, 025 (2017), arXiv:1612.04381 [cond-mat.str-el].
[14] D. Vegh, (2013), arXiv:1301.0537 [hep-th].
[15] T. Andrade and B. Withers, JHEP 05, 101 (2014), arXiv:1311.5157 [hep-th].
[16] M. Baggioli and O. Pujolas, Phys. Rev. Lett. 114, 251602 (2015), arXiv:1411.1003 [hep-th].
[17] L. Alberte, M. Baggioli, A. Khmelnitsky, and O. Pujolas, JHEP 02, 114 (2016), arXiv:1510.09080 [hep-th].
[18] A. Donos and J. P. Gauntlett, JHEP 04, 040 (2014), arXiv:1311.3292 [hep-th].
[19] A. Amoretti, D. Areán, B. Goutéraux, and D. Musso, Phys. Rev. D97, 086017 (2018), arXiv:1711.06610 [hep-th].
[20] T. Andrade, M. Baggioli, A. Krikun, and N. Poovuttikul, JHEP 02, 085 (2018), arXiv:1708.08306 [hep-th].
[21] T. Andrade and A. Krikun, JHEP 05, 119 (2019), arXiv:1812.08132 [hep-th].
[22] P. Chaikin and T. Lubensky, Principles of Condensed Matter Physics (Cambridge University Press, 2000).
[23] M. Baggioli and S. Grieninger, JHEP 10, 235 (2019), arXiv:1905.09488 [hep-th].
[24] L. Alberte, M. Ammon, M. Baggioli, A. Jiménez-Alba, and O. Pujolá, (2017), arXiv:1711.03100 [hep-th].
[25] A. Amoretti, D. Areán, B. Goutéraux, and D. Musso, JHEP 10, 068 (2019), arXiv:1904.11445 [hep-th].
[26] A. Amoretti, D. Areán, B. Goutéraux, and D. Musso, (2019), arXiv:1910.11330 [hep-th].
[27] S. A. Hartnoll, P. K. Kovtun, M. Muller, and S. Sachdev, Phys. Rev. B76, 144502 (2007), arXiv:0706.3215 [cond-mat.str-el].
[28] M. M. Caldarelli, A. Christodoulou, I. Papadimitriou, and K. Skenderis, JHEP 04, 001 (2017), arXiv:1612.07214 [hep-th].
[29] A. Esposito, S. Garcia-Saenz, A. Nicolas, and R. Penco, JHEP 12, 113 (2017), arXiv:1708.09391 [hep-th].
[30] M. Baggioli, V. C. Castillo, and O. Pujolá, (2019), arXiv:1910.05281 [hep-th].
[31] A. Nicolas, R. Penco, F. Piazza, and R. Rattazzi, JHEP 06, 155 (2015), arXiv:1501.03845 [hep-th].
[32] L. Alberte, M. Ammon, M. Baggioli, A. Jiménez, and O. Pujolà, JHEP 01, 129 (2018), arXiv:1708.08477 [hep-th].
[33] M. Baggioli, U. Gran, A. J. Alba, M. Tornsö, and L. Melgar, JHEP 10, 092 (2018), arXiv:1810.025 (2018), arXiv:1801.03199 [hep-th].
Supplementary Material

Holographic Models Without Background Strain.—In this appendix, we assemble details regarding holographic models with zero configuration pressure in equilibrium. Equivalently, there is no background strain in the equilibrium state. These are the so-called “thermodynamically favorable” models (see, for instance, the higher-derivative model of [19]), which admit translationally broken phases that minimise free energy. We illustrate that even for such models, the configuration pressure plays a crucial role in the dual hydrodynamics through its temperature derivatives and the hydrodynamic theory in [3] is not applicable.

Let us consider the simplest quadratic model $V(X, Z) = X + \lambda X^2$. Unfortunately, this model is unstable: (I) the shear modulus is negative, (II) the speed of transverse sound is imaginary, and (III) the longitudinal diffusion constant becomes negative at large $m/T$. It can be verified that all the models $V(X, Z)$ with spontaneous breaking of translations and $P = 0$ suffer from such linear instabilities or have ghostly excitations in the bulk. Clearly, the model $V(X, Z) = X + \lambda X^2$ cannot describe a stable physical system, but it can be used as a toy example to illustrate the importance of configuration pressure. It is easy to find that $V_h = \alpha^2 u_h^2 + \lambda \alpha^4 u_h^4$ and $U_h = \alpha^2 u_h^2 - \lambda \alpha^4 u_h^4$. Setting $P$ in (12) to zero, we find the preferred value of $\alpha \neq 0$ to be

$$\alpha^2 = \frac{1}{2\lambda u_h^2},$$

(A.1)

which matches the result of [19] in the zero charge density limit $\rho = 0$.4

We obtain the hydrodynamic parameters

$$T = \frac{3}{4\pi u_h} \left(1 - \frac{m^2}{4\lambda}\right), \quad s = \frac{2\pi}{u_h},$$

$$p = \frac{1}{2u_h^2} \left(1 - \frac{m^2}{4\lambda}\right), \quad \epsilon = \frac{1}{u_h^2} \left(1 - \frac{m^2}{4\lambda}\right),$$

$$P' = -\frac{4\pi}{3u_h^2} \left(\frac{m^2}{\lambda + 5m^2/12}\right), \quad B = \frac{m^2}{2\lambda u_h^2} \frac{m^2}{\lambda + 5m^2/12},$$

$$\sigma = \frac{2m^2}{u_h^2}, \quad c_v = \frac{4\pi}{u_h^2} \frac{\lambda - m^2/4}{\lambda + 5m^2/12}.$$  

(A.2)

Notice that the formulas (12)-(13) are not directly applicable to the potential considered in this appendix, since those formulae require that the potential falls of faster than $\sim u^3$ at the boundary. $G$ and $\eta$ have to be found numerically using (4). We see that $P' \neq 0$ leading to $B \neq 0$ and $c_v \neq 2s$ in these models, as discussed in the main text.

We can also compute the quasinormal models for this system numerically and compare them against the hydrodynamic predictions of [5] and that of [3] without $P'$. We see in fig. A.1 that the transverse speed of sound $v_{\perp}$ is imaginary due to negative shear modulus $G$; nevertheless predictions from hydrodynamics match perfectly. We again find a discrepancy in $D_{\parallel}$ similar to [9] compared to [3], which is resolved by including $P'$ contributions from [5]. See fig. A.2.

---

Figure A.1. $v_{\perp}^2$ for $V(X) = X + X^2/2$ model with $P = 0$ alongside the hydrodynamic predictions (solid lines). We have chosen $u_h = 1$ setting $\alpha = 1$.

Figure A.2. $D_{\parallel}$ for $V(X) = X + X^2/2$ model with $P = 0$ alongside the hydrodynamic predictions (solid lines). We have chosen $u_h = 1$ setting $\alpha = 1$.

Despite the simplicity and linear instability of this model, it shares various features of interest with similar holographic models without background strain, such as the one discussed in [19]. Similar models can also be constructed in the frameworks of [19, 20, 42-44]. The requirement of thermodynamic stability for isotropic models can be implemented as $\Omega = -(T_{xx})$ [45], which according to (4) is precisely $P = 0$. Irrespective of the particular model at play, while we might be able to set $P = 0$ by

---

4 The notational relationships are $\alpha \equiv k$ and $\lambda \equiv \lambda_2$, where the right-hand sides of the identifications are the notation of [19]. Notice also that eq. (45) in [19] contains typos; it should read $k^2 R_1(0) + 2\lambda_2 k^4 R_2(0) - \lambda_1 \rho^2 k^2 T_2(0) = 0$. 

---

[44] A. Donos and C. Pantelidou, JHEP 05, 079 (2019), arXiv:1903.05114 [hep-th].
[45] A. Donos and J. P. Gauntlett, JHEP 10, 038 (2013), arXiv:1306.4937 [hep-th].
judiciously choosing $\alpha$ in the equilibrium state, we will generically be left with a non-zero $P'$, which must be taken into account in the dual hydrodynamic theory. We leave a careful inspection of configuration pressure in these models to future work.

**Holographic Renormalisation.**—In this section we give some details regarding the holographic renormalisation underlying the models in this letter.

The bulk action (8) has to be supplemented with appropriate boundary counter terms to have a well-defined variational principle

$$S_{\text{counter}} = \int_{u=\epsilon} d^{d+1}x \sqrt{-\gamma} \left( K - \frac{d}{\ell} + m^2 \tilde{V}(\tilde{T}^I) \right). \tag{A.3}$$

Here $\gamma_{\mu\nu} = \lim_{u \to \epsilon} g_{\mu\nu}$ is the induced metric at the boundary, $K$ is the extrinsic curvature, and $\tilde{T}^I = \gamma^{\mu\nu} \partial_\mu \phi^I \partial_\nu \phi^J$. $\tilde{V}(\tilde{T}^I)$ is an appropriate boundary potential fixed by requiring that the on-shell action of the black brane solution (9) to be finite. For instance, in $d = 2$, for $V(X) = X^N$ models with $N > 3/2$ we have $\tilde{V}(X) = 0$, while for $N < 3/2$ we get $\tilde{V}(X) = X/(3 - 2N)$, where $X = \frac{1}{2} \text{tr} \tilde{Z}$. For $V(X) = X + \lambda X^2$ discussed in the appendix, we instead find $\tilde{V}(X) = X$. In this appendix, we focus on holographic renormalisation for $V(X) = X + \lambda X^2$.

To implement spontaneous symmetry breaking for models $V(X, Z) = X^N, Z^M$ with $N < 5/2, M < 5/4$, one needs to implement *alternative quantisation* (see [5]) for the scalars. More precisely, one needs to deform the boundary theory with a term

$$S_{\text{alt}} = \int_{u=\epsilon} d^{d+1}x \sqrt{-\gamma} \Pi_I \phi^I, \tag{A.4}$$

where

$$\Pi_I = \frac{1}{\sqrt{-\gamma}} \frac{\delta(S + S_{\text{counter}})}{\delta \phi^I}$$

$$= \delta_{IJ} \left( V'(X) n^a \partial_a \phi^J + \nabla_{(\gamma)} (V'(X) \partial_\mu \phi^J) \right). \tag{A.5}$$

$\nabla_{(\gamma)}$ is the covariant derivative associated with $\gamma_{\mu\nu}$ and $n_a$ is the outward pointing normal vector at the boundary.

The (A.4) term in the action turns $\Phi^I$ at the boundary into the dynamical operator, while the associated source is now given by the boundary value of $\Pi_I$. We are interested in dual hydrodynamic models in the absence of sources for the scalars. Hence, in alternative quantisation we impose the boundary conditions

$$\lim_{\epsilon \to 0} \frac{1}{\epsilon^{d+1}} \Pi_I = 0. \tag{A.6}$$

Finally, for the metric we always impose the standard boundary conditions

$$\lim_{\epsilon \to 0} \epsilon^2 \gamma_{\mu\nu} = \eta_{\mu\nu}. \tag{A.7}$$

Note that the background profile for the scalars, $\phi^I = x^I$, is no longer an external source providing the explicit breaking of translations in the alternative quantisation scheme. This is the fundamental reason why models like $V(X) = X$, using alternative quantization [5], realise the spontaneous (and not explicit [15]) breaking of translations.

---

5 For potentials $V(X, Z) = X^N, Z^M$ with $N > 5/2, M > 5/4$, one instead needs to follow *standard quantisation* in order to have spontaneous symmetry breaking as shown in [24].