SKYRMED MONOPOLES

D.Yu. Grigoriev\textsuperscript{1*}, P.M. Sutcliffe\textsuperscript{2} and D.H. Tchrakian\textsuperscript{1}

\textsuperscript{1} Mathematical Physics, National University of Ireland Maynooth (NUIM),
Maynooth, Co. Kildare, Ireland.
Email : dima@thphys.may.ie, tigran@maxwell.thphys.may.ie

\textsuperscript{2} Institute of Mathematics, University of Kent at Canterbury,
Canterbury, CT2 7NF, U.K.
Email : P.M.Sutcliffe@ukc.ac.uk

June 2002

Abstract

We investigate multi-monopole solutions of a modified version of the BPS Yang-Mills-Higgs model in which a term quartic in the covariant derivatives of the Higgs field (a Skyrme term) is included in the Lagrangian. Using numerical methods we find that this modification leads to multi-monopole bound states. We compute axially symmetric monopoles up to charge five and also monopoles with Platonic symmetry for charges three, four and five. The numerical evidence suggests that, in contrast to Skyrmions, the minimal energy Skyrmied monopoles are axially symmetric.

\textsuperscript{*}On leave of absence from Institute for Nuclear Research of Russian Academy of Sciences
1 Introduction

Two of the most interesting kinds of topological solitons in three space dimensions are BPS monopoles and Skyrmions. Although there are some similarities between monopoles and Skyrmions, which we shall discuss shortly, there are a number of important differences which we first recall. $SU(2)$ BPS monopoles are soliton solutions of a Yang-Mills-Higgs gauge theory (with a massless Higgs) in which the topological charge $N$ is an element of the second homotopy group of the two-sphere, identified as the Higgs field vacuum manifold. The topological charge is therefore associated with a winding of the Higgs field on the two-sphere at spatial infinity. In the BPS limit there is a $4N$-dimensional moduli space of static solutions which are degenerate in energy, so in this sense there are no stable bound states since any static charge $N$ solution has the same energy as $N$ well-separated charge one monopoles. Contrast these features with those of Skyrmions, which are soliton solutions of a nonlinear sigma model with target space $SU(2)$. The Skyrme field is constant on the two-sphere at spatial infinity and this yields a compactification of Euclidean three-space to a three-sphere. The integer-valued topological charge (baryon number) is an element of the third homotopy group of the target space and counts the number of times that the target space is covered by the Skyrme field throughout space. There are static forces between Skyrmions, which for a suitable relative internal orientation are attractive, and this leads to multi-Skyrmion bound states.

To summarize, three main differences between Skyrmions and monopoles are the basic fields of the model, the way the topological charge arises, and the existence (or not) of bound states. Given these facts it is rather surprising that there appears to be some similarity between various monopole and Skyrmion solutions. There are axially symmetric monopoles and Skyrmions for all charges greater than one (although above charge two these are not the minimal energy Skyrmions) and both have solutions with Platonic symmetries for the same certain charges. For example, there is a tetrahedral monopole for $N = 3$, a cubic monopole for $N = 4$ and a dodecahedral monopole for $N = 7$. All BPS monopoles of a given charge have the same energy but these particular monopole solutions are selected out by being mathematically more tractable than an arbitrary solution. For these three values of the charge $N = 3, 4, 7$ the minimal energy Skyrmion has precisely the same symmetry as the above monopoles and energy density isosurfaces are qualitatively similar. These and other similarities can be partially understood by relating both types of soliton to rational maps between Riemann spheres.

The obvious differences and yet remarkable similarities between monopoles and Skyrmions is the motivation for the present work, where we aim to modify the BPS monopole Lagrangian by the addition of a Skyrme-like term with the goal of breaking the energy degeneracy and producing monopole bound states. That such a modification might yield monopole bound states is suggested by the fact that more complicated models, involving Skyrme-like terms, have been shown to have this property.

We refer to the soliton solutions of our modified model as Skyrmed monopoles, though in the following for brevity we mainly use the term monopoles, and refer to monopole solutions of the unmodified model as BPS monopoles. Our numerical computations, for monopoles
up to charge five, show that the modified model does indeed have multi-monopole bound states, though perhaps surprisingly our numerical results suggest that the minimal energy multi-monopoles are all axially symmetric and do not share the Platonic symmetries of the corresponding minimal energy Skyrmions. Platonic monopole solutions are computed, and although they have low energies they are very slightly above those of the axially symmetric solutions.

Explicitly, the model we consider is defined by the following energy function (we deal only with static solutions in this letter but the extension to the relativistic Lagrangian is obvious)

\[
E = \frac{1}{8\pi} \int -\text{Tr} \left( \frac{1}{2} F_{ij}F_{ij} + D_i \Phi D_i \Phi + \frac{\mu^2}{2} [D_i \Phi, D_j \Phi][D_i \Phi, D_j \Phi] \right) d^3x.
\] (1)

Here Latin indices run over the spatial values 1,2,3, the Higgs field and gauge potential are \( \Phi, A_i \in su(2) \), the covariant derivative is \( D_i \Phi = \partial_i \Phi + [A_i, \Phi] \) and \( F_{ij} \) is the field strength.

The boundary condition is that \( |\Phi|^2 = -\frac{1}{2} \text{Tr} \Phi^2 \) equals one at spatial infinity. The Higgs field at infinity then defines a map between two-spheres and the winding number of this map is the monopole number \( N \).

If \( \mu = 0 \) then the energy (1) is the usual BPS Yang-Mills-Higgs energy and monopole solutions satisfy the first order Bogomolny equations. All members of the \( 4N \)-dimensional moduli space of solutions have energy \( E = N \), and include solutions describing \( N \) well-separated monopoles as well as axially symmetric \( N \)-monopoles. For \( \mu \neq 0 \) the additional term is the gauge analogue of the Skyrme term for the sigma model. In the sigma model context the presence of the Skyrme term is necessary to have stable soliton solutions but in the monopole context it is optional. Clearly the energy degeneracy of the BPS model will be broken for \( \mu \neq 0 \) and as we shall describe below this produces monopole bound states, rather than the familiar monopole-monopole repulsion induced by the addition of a Higgs potential, which is an alternative way to lift the energy degeneracy of the BPS model.

2 Numerical Methods and Results

In order to construct static solutions of the field equations which follow from the variation of the energy (1) we apply a simulated annealing algorithm \([11]\) to minimize the energy using a finite difference discretization on a grid containing \( 81^3 \) points with a lattice spacing \( dx = 0.25 \). Note that this grid is a little smaller than those currently in use to study similar problems for Skyrmions \([1]\) (although simulated annealing computations of Skyrmions on grids containing \( 80^3 \) points do provide accurate results \([3]\), but the large number of fields which need to be dealt with in studying a Yang-Mills-Higgs gauge theory make it difficult to handle grids much larger than this with our current resources. However, by testing our codes on the BPS limit (\( \mu = 0 \)) where exact results are known, we are able to estimate the numerical errors involved and have confidence in our results being accurate to the level that we discuss later.
In order to apply our annealing code we need to provide initial conditions which have the correct topological winding of the Higgs field at infinity. To provide these initial conditions, and be able to prescribe any particular symmetry that we may want to impose, we make use of a formula relating the asymptotic Higgs field to a rational map between Riemann spheres [8]. Explicitly, the initial Higgs field is given by

\[
\Phi = \frac{if(r)}{1 + |R|^2} \left( \frac{1 - |R|^2}{2R} \frac{2R}{|R|^2 - 1} \right)
\]

where \( f(r) \) is a real profile function, which depends on the radius \( r \), and satisfies the boundary conditions that \( f(0) = 0 \) and \( f = 1 \) on the boundary of the numerical grid. Here \( R(z) \) is a rational map of degree \( N \) in the complex variable \( z \), i.e. a ratio of two polynomials of degree no greater than \( N \), which have no common factors and at least one of the polynomials has degree precisely \( N \). The variable \( z \) is a Riemann sphere coordinate on the unit sphere around the origin in space i.e. it is given by

\[ z = e^{i\phi} \tan(\theta/2) \]

where \( \theta \) and \( \phi \) are the usual polar coordinates. In the BPS case there is a one-to-one correspondence between charge \( N \) monopole solutions and (an equivalence class of) degree \( N \) rational maps [9] and the existence of certain symmetric monopole solutions can be proved by the construction of the associated symmetric maps [5]. Although there is clearly no such correspondence in our modified model we shall make use of some of the relevant symmetric maps in our initial conditions. We take all gauge potentials to be zero initially and this preserves any symmetry that the Higgs field may initially have. On the boundary of the grid the Higgs field is fixed to the initial form (2), which in particular ensures that the winding number remains equal to \( N \), but the gauge potential is annealed to minimize the energy given the fixed boundary Higgs field.

As a test of the accuracy of our code we first compute several BPS monopoles. For the \( N = 1 \) monopole (with rational map \( R = z \)) we find an energy \( E = 1.007 \) whose deviation from unity is an indication of the error associated with the energy values we quote. Another important test is to compare the energies of different BPS multi-monopole solutions which have the same charge. Of course a perfect calculation would produce energies equal to the charge for any solution. As an example, using the rational map \( R = z^3 \) of the axially symmetric 3-monopole in the initial condition produces the energy \( E = 3.021 \), whereas the rational map \( R = (\sqrt{3}iz^2 - 1)/(z^3 - \sqrt{3}iz) \) anneals to produce a tetrahedrally symmetric 3-monopole with energy \( E = 3.018 \). This illustrates the fact that our energies are accurate to around 1% but that comparisons between different configurations are likely to be more accurate, in this case the error is around 0.1%. Similar results were obtained for other BPS examples.

We now turn to the modified model with \( \mu \neq 0 \), and the first issue to address is a suitable choice for the value of \( \mu \). To facilitate numerical comparisons it is useful to choose a value of \( \mu \) large enough so that the additional term raises the energy of the \( N = 1 \) monopole by something of the order of 50% from the BPS value, since it then has an effect significant enough to be calculated numerically but does not dominate over the usual terms. In fig. 1 we plot the energy of the \( N = 1 \) monopole as a function of \( \mu^2 \). This calculation is performed by using a hedgehog ansatz and computing the energy minimizing profile
functions. From fig. 1 we see that a reasonable choice is $\mu = 5$, which we use from now on, and this gives $E = 1.591$. Using the full three-dimensional annealing code we compute the 1-monopole energy to be $E_1 = 1.602$ which is in reasonable agreement with the more accurate one-dimensional calculation.

The crucial calculation is now to compute the energy of the axially symmetric 2-monopole. Using the rational map $R = z^2$ we compute the axially symmetric 2-monopole, whose energy density isosurface is displayed in fig. 2A, and find the energy $E_2 = 2.777$. The important point is that $E_2/2 = 1.388 < 1.602 = E_1$ so a 2-monopole bound state exists. It seems reasonable to conclude that the minimal energy 2-monopole is axially symmetric, though clearly we have not proved this. Note that $2E_1 - E_2 = 0.427$ and, as we mentioned above, this is expected to be significantly larger than the numerical errors present in our energy comparisons. If required a more accurate calculation of the 2-monopole energy could be performed by making use of the axial symmetry to reduce to an effective two-dimensional computation.

For higher charges we first look at axially symmetric monopoles by using the rational maps $R = z^N$. For $N = 2, 3, 4, 5$ the energies $E_N$ and energies per monopole $E_N/N$ are

![Figure 1: The 1-monopole energy as a function of $\mu^2$.](image)
| N  | G            | E       | E/N     |
|----|--------------|---------|---------|
| 1  | O(3)         | 1.602   | 1.602   |
| 2  | O(2) × Z₂    | 2.777   | 1.388   |
| 3  | O(2) × Z₂    | 3.807   | 1.269   |
| 3  | Td           | 3.869   | 1.290   |
| 4  | O(2) × Z₂    | 4.847   | 1.212   |
| 4  | Oh           | 4.974   | 1.244   |
| 5  | O(2) × Z₂    | 5.924   | 1.185   |
| 5  | D₂d          | 5.982   | 1.196   |
| 5  | Oh           | 5.987   | 1.197   |

Table 1: The monopole charge $N$, the symmetry group $G$ of the energy density, the energy $E$ and energy per monopole $E/N$ for several examples of Skyrmed monopoles.

presented in Table 1 and we display energy density isosurfaces in figs. 2A,2B,2C,2D. We also plot the energy per monopole for these axially symmetric solutions as a function of monopole number in fig. 3. This plot demonstrates that all these solutions are stable against the break-up into $N$ well separated monopoles, and also into any well-separated clusters containing single or axially symmetric monopoles. Note that for these axially symmetric solitons the energy per monopole decreases as the monopole number increases and this contrasts sharply with Skyrmions. For axially symmetric Skyrmions with $N \geq 2$ the energy per Skyrmion increases with the number of Skyrmions \[10\], and only the $N = 2$ minimal energy Skyrmion has an axial symmetry. Furthermore, for $N > 4$ the axially symmetric charge $N$ Skyrmion is not even bound against the break-up into $N$ well-separated single Skyrmions.

The fact that for the axially symmetric solutions the energy per monopole decreases as a function of increasing monopole number (we have also checked that this trend continues up to $N = 10$, using larger grids) makes it possible that the minimal energy monopole is axially symmetric for all $N \geq 2$. In order to test this we have computed some non-axially symmetric monopoles with $N > 2$ which have the symmetries of the known minimal energy Skyrmions, since these are the obvious non-axial contenders for minimal energy monopoles.

The minimal energy $N = 3$ Skyrmion has tetrahedral symmetry $T_d$ and the relevant rational map is the one mentioned earlier, $R = (\sqrt{3}iz^2 - 1)/(z^3 - \sqrt{3}iz)$. Annealing produces the tetrahedral 3-monopole displayed in fig. 2E which has an energy $E_3^T = 3.869$. This is very slightly higher than the energy of the axial 3-monopole $E_3 = 3.807$, and since $E_3^T - E_3 = 0.062$ we expect that even though this difference is almost as large as the likely overall error in the computation of each individual energy, it is an order of magnitude greater than the errors we estimate in the comparison between two energies. This calculation suggests that the axial 3-monopole has less energy than the tetrahedral 3-monopole, in contrast to Skyrmions, and hence that it is likely to be the minimal energy 3-monopole. Of course, since the energy differences are small it is desirable to have a more accurate calculation of both these energies using larger grids, but this is beyond our current resources. We
Figure 2: Energy density isosurfaces (to scale) of various Skyrmed monopoles. 
A) $N = 2$ axial, B) $N = 3$ axial, C) $N = 4$ axial, D) $N = 5$ axial, E) $N = 3$ tetrahedral, F) $N = 4$ octahedral, G) $N = 5$ dihedral, H) $N = 5$ octahedral.

have verified that the axial 3-monopole has less energy than the tetrahedral 3-monopole for a number of other values of the parameter $\mu$ and also performed another consistency check by computing the energy of the additional term given the two different BPS 3-monopoles. This will be a good approximation to the excess above the BPS bound in the limit where $\mu$ is small, so that the fields vary little from the BPS configurations. This result is in agreement with the full nonlinear computation since it yields an excess energy which is slightly less for the axial 3-monopole than for the tetrahedral 3-monopole, though we must point out that this calculation does appear to be very sensitive to obtaining the BPS solution to a very high accuracy. In principle, given the correspondence between BPS monopoles and rational maps, the additional energy contribution should provide an interesting energy function on the space of rational maps, though it does not seem possible to obtain any explicit information about this energy function without first computing the monopole fields, which can only be done numerically and is computationally expensive.

An interesting question, given that our results suggest that the tetrahedral 3-monopole is not the minimal energy solution, is whether this is a stable local minimum or a saddle point solution. We are unable to answer this question at this stage, since the algorithm requires the Higgs field to be fixed on the boundary of the grid with a prescribed form, and hence symmetry. In principle, since we have not explicitly fixed a gauge, any Higgs field which has a winding number equal to $N$ is equivalent to any other, so it should be possible to move between different configurations if the symmetry is initially broken by the gauge potentials, but in practice this does not happen since the energy differences between various configurations are too small and the gauge potentials quickly anneal to match the symmetry of the Higgs field. It is this technical difficulty which prevents us from simply
finding the minimal energy $N$-monopole by starting from an asymmetric initial condition, which is the method used for Skyrmions but in that case the Skyrme field is fixed on the boundary of the grid to be a constant and contains no information about the structure and symmetry of the Skyrmion.

The minimal energy 4-Skyrmion has octahedral symmetry $O_h$ and is described by the rational map $R = (z^4 + 2\sqrt{3}iz^2 + 1)/(z^4 - 2\sqrt{3}iz^2 + 1)$. Using this map we compute the cubic 4-monopole displayed in fig. 2F with energy $E_4^O = 4.974$. This is again slightly larger than the energy of the axial 4-monopole $E_4 = 4.847$ and further supports our findings that the minimal energy monopoles do not share the symmetries of the minimal energy Skyrmions.

The minimal energy $N = 5$ Skyrmion has only the dihedral symmetry $D_{2d}$ and corresponds to a rational map of the form $R = (z^5 + bz^3 + az)/(az^4 - bz^2 + 1)$ where $a$ and $b$ are particular real constants. Using the values associated with the minimal energy 5-Skyrmion produces the monopole displayed in fig. 2G with an energy $E_5^D = 5.982$ which is larger than the axial energy $E_5 = 5.924$. For charge 5 there is also another obvious minimal energy candidate, which is an octahedrally symmetric $O_h$ monopole associated with the above

Figure 3: The energy per monopole $E/N$ for the axially symmetric monopoles (crosses) and Platonic monopoles (stars).
rational map in which the parameter $b$ is zero and $a = -5$. The annealed monopole has energy $E^O_5 = 5.987$ and is presented in fig. 2H. Deforming the dihedral monopole to the octahedral monopole produces a tiny change in energy, and the difference is even within the numerical errors expected when comparing two energies, so we can only conclude that the numerical results suggest that both have higher energy than the axial $5$-monopole, but which of these two has the lower energy is not clear.

For all the charges and examples discussed above we have performed several other computations using both larger and smaller values for the parameter $\mu$ and found qualitatively similar results. In all cases the axially symmetric monopoles are always those with the lowest energy, suggesting that this is the case for all $\mu > 0$. We have also examined the replacement of the fourth-order Skyrme term by a sixth-order term and found similar results.

3 Conclusion

Motivated by the similarities and differences between BPS monopoles and Skyrmions we have investigated a modification of the usual BPS Yang-Mills-Higgs model by including a Skyrme-term formed from the covariant derivatives of the Higgs field. We found that this modification indeed produces monopoles which are more like Skyrmions, in the sense that bound states now exist, but that the numerical results suggest that the minimal energy monopoles for charges greater than two do not share the symmetries of the minimal energy Skyrmions, but instead appear to be axially symmetric. The energy differences we have found are not substantial, so further more accurate computations would be desirable, but we have demonstrated a significant difference (with values well beyond our expected numerical errors) between the behaviour of axially symmetric monopoles in our modified model and axially symmetric Skyrmions; in the axial monopole case the energy per soliton is a decreasing function of soliton number and in the Skyrmion case it is an increasing function. This property alone demonstrates that our modified monopoles have qualitative differences with Skyrmions.

There are a number of interesting properties of monopoles in the modified model which require further investigation. These include a study of the energy of a $2$-monopole configuration as a function of the monopoles separation and the related issue of how the interaction between two well-separated monopoles depends on their relative phase. The dynamics and scattering of monopoles in this model would also seem worth investigating, both using full field simulations and approximate techniques. In principle the moduli space approximation could be applied to Skyrmed monopoles by treating the modification as a perturbation to the BPS monopole metric together with an induced potential function on the BPS monopole moduli space.

Although the main motivation for this work is to explore connections between various types of three-dimensional topological solitons, the additional Skyrme term that we have included is a natural modification that might arise in an effective theory. In this context the value of $\mu$ is expected to be much smaller than the value we have studied for numerical
convenience, but the qualitative features of our results, such as monopole bound states, remain valid.

Acknowledgements

Many thanks to R. Flume, C.J. Houghton, B. Kleihaus, J. Kunz and N.S. Manton for useful discussions. We thank Enterprise–Ireland and the British Council for financial support under project BC/2001/021. PMS acknowledges the EPSRC for an Advanced Fellowship. The research of DG is supported by Enterprise–Ireland grant SC/2000/020.

References

[1] R.A. Battye and P.M. Sutcliffe, Phys. Rev. Lett. 79, 363 (1997); Phys. Rev. Lett. 86, 3989 (2001); Rev. Math. Phys. 14, 29 (2002).

[2] E. Braaten, S. Townsend and L. Carson, Phys. Lett. B 235, 147 (1990).

[3] M. Hale, O. Schwindt and T. Weidig, Phys. Rev. E 62, 4333 (2000).

[4] N.J. Hitchin, N.S. Manton and M.K. Murray, Nonlinearity 8, 661 (1995).

[5] C.J. Houghton, N.S. Manton and P.M. Sutcliffe, Nucl. Phys. B 510, 507 (1998).

[6] C.J. Houghton and P.M. Sutcliffe, Commun. Math. Phys. 180, 343 (1996); Nonlinearity 9, 385 (1996).

[7] B. Kleihaus, D. O’Keeffe and D.H. Tchrakian, Phys. Lett. B 427, 327 (1998); Nucl. Phys. B 536, 381 (1999).

[8] T. Ioannidou and P.M. Sutcliffe, J. Math. Phys. 40, 5440 (1999).

[9] S. Jarvis, J Reine Angew Math 524, 17 (2000).

[10] V.B. Kopeliovich and B.E. Stern, JETP Lett. 45, 203 (1987).

[11] P.J.M. van Laarhoven and E.H.L. Aarts, Simulated annealing: theory and applications, Kluwer Academic Publishers, (1987).