Three-dimensional matching is NP-Hard

Shrinu Kushagra*

Abstract
The standard proof of NP-Hardness of 3DM provides a power-4 reduction of 3SAT to 3DM. In this note, we provide a linear-time reduction. Under the exponential time hypothesis, this reduction improves the runtime lower bound from $2^{o(\sqrt[4]{m})}$ (under the standard reduction) to $2^{o(m)}$.

Keywords: 3DM, 3SAT, ETH, X3C, NP-Hard

1 Introduction
In this note, we first establish the hardness of the following decision problem.

Definition 1 (3DM).
Input: Sets $W,X$ and $Y$ and a set of matches $M \subseteq W \times X \times Y$ of size $m$.
Output: YES if there exists $M' \subseteq M$ such that each element of $W,X,Y$ appears exactly once in $M'$. NO otherwise.

To prove that 3DM is NP-Hard, we reduce an instance of 3SAT to the given problem. Next, we define the 3SAT decision problem.

Definition 2 (3-SAT).
Input: A boolean formulae $\phi$ in 3CNF form with $n$ literals and $m$ clauses.
Output: YES if $\phi$ is satisfiable, NO otherwise.

Given an instance of 3SAT with $n$ literals and $m$ clauses, [Garey and Johnson, 1979] construct a graph with $\Theta(nm)$ vertices and $\Theta(n^2m^2)$ edges. Thus, this is a power-4 reduction. In this note, we use a similar but a more efficient gadget and provide a linear-time reduction of the 3SAT instance to the given problem.

*Email: skushagr@uwaterloo.ca. The results in this note first appeared in [Kushagra et al., 2019] where the reduction was used to prove query lower bounds for a clustering problem.
2 Hardness of 3DM

Theorem 3. Three-dimensional matching is an NP-Hard problem.

Figure 1: Part of graph \( G \) constructed for the literal \( x_1 \). The figure is an illustration for when \( x_1 \) is part of four different clauses. The triangles (or hyper-edge) \((a_i, b_i, c_i)\) capture the case when \( x_1 \) is true and the other triangle \((b_i, c'_i, a_{i+1})\) captures the case when \( x_1 \) is false. Assuming that a clause \( C_j = \{x_1, x_2, x_3\} \), the hyper-edges containing \( tf_i, tf'_i \) and \( t_1, t'_1 \) capture different settings. The hyper-edges containing \( t_1, t'_1 \) ensure that at least one of the literals in the clause is true. The other two ensure that two variables can take either true or false values.

Our reduction is described in Fig. 1. For each literal \( x_i \), let \( m_i \) be the number of clauses in which the literal is present. We construct a “truth-setting” component containing \( 2m_i \) hyper-edges (or triangles). We add the following hyper-edges to \( M' \).

\[
\{(a_k[i], b_k[i], c_k[i]) : 1 \leq k \leq m_i\} \\
\cup \{(a_{k+1}[i], b_k[i], c'_k[i]) : 1 \leq k \leq m_i\}
\]

Note that one of \((a_k, b_k, c_k)\) or \((a_{k+1}, b_k, c'_k)\) have to be selected in a matching \( M' \). If the former is selected, that corresponds to the variable \( x_i \) being assigned true, the latter corresponds to false. This part is the same as the standard construction.
For every clause $C_j = \{x_1, x_2, x_3\}$ we add three types of hyper-edges. The first type ensures that at least one of the literals is true.

$$\{(c_k[i], t_1[j], t'_1[j]) : x_i' \in C_j\} \cup \{(c'_k[i], t_1[j], t'_1[j]) : x_i \in C_j\}$$

The other two types of hyper-edges (conected to the $tf_i$’s) say that two of the literals can be either true or false. Hence, we connect them to both $c_k$ and $c'_k$

$$\{(c_k[i], tf_1[j], tf'_1[j]) : x_i' \text{ or } x_i \in C_j\}$$

$$\cup \{(c_k[i], tf_2[j], tf'_2[j]) : x_i' \text{ or } x_i \in C_j\}$$

$$\cup \{(c'_k[i], tf_1[j], tf'_1[j]) : x_i' \text{ or } x_i \in C_j\}$$

$$\cup \{(c'_k[i], tf_2[j], tf'_2[j]) : x_i' \text{ or } x_i \in C_j\}$$

Note that in the construction $k$ refers to the index of the clause $C_j$ in the truth-setting component corresponding to the literal $x_i$. Using the above construction, we get that

$$W = \{c_k[i], c'_k[i]\}$$

$$X = \{a_k[i] \cup \{t_1[j], tf_1[j], tf_2[j]\}\}$$

$$Y = \{b_k[i] \cup \{t'_1[j], tf'_1[j], tf'_2[j]\}\}$$

Hence, we see that $|W| = 2 \sum_i m_i = 6m$. Now, $|X| = |Y| = \sum_i m_i + 3m = 6m$. And, we have that $|M| = 2 \sum_i m_i + 15m = 21m$. Thus, we see that this construction is linear in the number of clauses.

Now, if the 3-SAT formula $\phi$ is satisfiable then there exists a matching $M'$ for the 3DM problem. If a variable $x_i = T$ in the assignment then add $(c_k[i], a_k[i], b_k[i])$ to $M'$ else add $(c'_k[i], a_{k+1}[i], b_k[i])$. For every clause $C_j$, let $x_i$ (or $x_i'$) be the variable which is set to true in that clause. Add $(c_k[i], t_1[j], t'_1[j])$ (or $(c'_k[i], t_1[j], t'_1[j])$) to $M'$. For the remaining two clauses, add the hyper-edges containing $tf_1[j]$ and $tf_2[j]$ depending upon their assignments. Clearly, $M'$ is a matching.

Now, the proof for the other direction is similar. If there exists a matching, then one of $(a_k, b_k, c_k)$ or $(a_{k+1}, b_k, c'_k)$ have to be selected in a matching $M'$. This defines a truth assignment of the variables. Now, the construction of the clause hyper-edges ensures that every clause is satisfiable.

### 3 Exponential Time Hypothesis for 3DM

Before we start the discussion in the section, let’s review the definition of the exponential time hypothesis.

**Exponential Time Hypothesis (ETH)**

There does not exist an algorithm which decides 3-SAT and runs in $2^{o(m)}$ time.

If the exponential hypothesis is true, the standard reduction of 3-SAT to 3DM [Garey and Johnson, 1979] implies that any algorithm for 3DM runs in $2^{o(m^{1/4})}$. 


However, using the reduction in Section 2, we get a more tighter dependence on \( m \) stated as a theorem below.

**Theorem 4.** If the exponential time hypothesis holds then there does not exist an algorithm which decides the three-dimensional matching problem (3DM) and runs in time \( 2^{o(m)} \).

**Proof.** For the sake on contradiction, suppose that such an algorithm \( \mathcal{A} \) exists. Then, using the reduction from Section 2 and \( \mathcal{A} \), we get an algorithm for 3SAT that runs in \( 2^{o(m)} \) time which contradicts the ET hypothesis. \( \square \)

An immediate corollary of this result applies to another popular problem Exact Cover by 3-sets.

**Definition 5 (X3C).**

*Input:* \( U = \{x_1, \ldots, x_{3q}\} \). A collections of subsets \( S = \{S_1, \ldots, S_m\} \) such that each \( S_i \subseteq U \) and contains exactly three elements.

*Output:* YES if there exist \( S' \subseteq S \) such that each element of \( U \) occurs exactly once in \( S' \), NO otherwise.

**Corollary 6.** If the exponential time hypothesis holds then there does not exist an algorithm which decides exact cover by 3-sets problem (X3C) and runs in time \( 2^{o(m)} \).

**Proof.** The proof follows from the trivial reduction of 3DM to X3C where \( U = W \cup X \cup Y \) and \( S = M \). \( \square \)

**References**

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