Rotating dirty black hole and its shadow

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In this paper, we examine the effect of dark matter to a Kerr black hole of mass $m$. The metric is derived using the Newman-Janis algorithm, where the seed metric originates from the metric of a Schwarzschild black hole surrounded by a spherical shell of dark matter with mass $M$ and thickness $\Delta r_s$. We analyzed both the time-like and null geodesics and found out that if the dark matter density is considerably low, time-like geodesics shows more deviations from the Kerr case compared to null geodesics. Furthermore, energy extraction via the Penrose process remains unchanged. A high concentration of dark matter near the rotating black hole is needed to have considerable deviations on the horizons and photonsphere radius. With the dark matter configuration used in this study, we found that deriving an analytic estimate to determine the condition for dark matter to have a notable change in the shadow radius is inconvenient.

I. INTRODUCTION

Perhaps one of the most interesting objects in the universe is a black hole, at least for theoretical physicists, as it provides a theoretical playground in finding hints about the possible union of quantum theory and Einstein’s general relativity. In this quest, a remarkable breakthrough happened in 2019, where the Event Horizon Telescope collaborative efforts successfully imaged the silhouette of the supermassive black hole at the heart of galaxy M87 using a technique called Very Long Baseline Interferometry (VLBI). Future improvements in visualizing black holes might reveal the true geometry of black holes [1–52].

A study in Ref. [54] analytically estimated the specific condition for dark matter effects to occur notably in the shadow of a Schwarzschild black hole. In this paper, we follow the same model for the dark matter configuration, and motivations to investigate a more realistic scenario - dark matter effects on a rotating black hole. The dark matter configuration is described only through its mass, and span that can be adjusted to determine dark matter density, hence, making it less model-dependent.

For the rest of this paper, Sect. [II] introduces the Schwarzschild metric surrounded by dark matter as modeled in Ref. [54]. In Sect. [III] we derive the rotating solution by using the seed metric introduced in Sect. [II] In Sections [IV] to [VII] we investigate the effect of dark matter on the horizons, time-like and null circular orbits, black hole shadow, and its observables. Sect. [VIII] is devoted to summarizing the results and recommendations for future studies. Lastly, we consider the +2 metric signature, and $G = c = 1$.

II. SCHWARZSCHILD BLACK HOLE SURROUNDED BY DARK MATTER

The metric for a static, spherically-symmetric, uncharge, and non-rotating black hole is expressed as

$$ds^2 = -f(r)dt^2 + f(r)^{-1}dr^2 + r^2(d\theta^2 + \sin^2 \theta d\phi^2)$$  (1)

where the metric function $f(r)$ is given by

$$f(r) = 1 - \frac{2m}{r}.$$  (2)

In an attempt to make some estimates as to what extent does dark matter can affect the black hole geometry, a mass function incorporating dark matter as an effective mass is introduced in Ref. [54]. The metric function reads

$$f(r) = 1 - \frac{2}{r}(m + MG(r))$$  (3)

where $M$ is the dark matter mass and $G(r)$ describes its configuration. If the dark matter’s configuration is a spherical shell that surrounds the black hole, then $G(r)$ can be expressed as

$$G(r) = \left(3 - 2 \frac{r - r_s}{\Delta r_s}\right)\left(\frac{r - r_s}{\Delta r_s}\right)^2.$$  (4)

Here, $r_s$ is the inner radius of the dark matter shell and $\Delta r_s$ is its thickness. Indeed, (4) is chosen so that for a given value of $r_s$ and $\Delta r_s$, a piecewise mass function can be realized:

$$m(r) = \begin{cases} 
   m, & r < r_s; \\
   m + MG(r), & r_s \leq r \leq r_s + \Delta r_s; \\
   m + M, & r > r_s + \Delta r_s 
\end{cases}.$$  (5)

In this way, when observing certain black hole phenomena, theorists can have insights or alternative perspectives where to attribute the cause of deviations from a theory [68–69].
It is easy to see that (5) reduces to the Schwarzschild case if $M = 0$ and if $r < r_s$ due to (5). In the presence of dark matter, $\Delta r_s$ can never be zero, but it can have a value that is much larger than $r_s$. In this way, the dark matter density can be adjusted by tweaking the values of $M$ and $\Delta r_s$. Also, note that $M < 0$ and $r_s = 0$ are allowed. In this paper, we restrict the analysis to $M > 0$ and $r_s = 2m$. Results in Ref. \cite{m} estimated the thickness of dark matter in order have a noticeable effect on the shadow of Schwarzschild black hole:

$$\Delta r_s \sim \sqrt{3mM}. \quad (6)$$

III. KERR BLACK HOLE SURROUNDED BY DARK MATTER

Using the Newman-Janis algorithm \cite{1, 7}, we generalize (1) with metric function given by (3) to a Kerr black hole that is surrounded by spherical shell of dark matter. The standard formalism starts with the conversion of the coordinates in (1) to a horizon-penetrating coordinates (also known as Eddington-Finkelstein coordinates):

$$du = dt - dr^* = dt - \frac{dr}{f(r)} \quad (7)$$

and we obtain

$$ds^2 = -f(r)du^2 - 2du dr + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2. \quad (8)$$

The components of the contravariant metric tensor $g^{\mu\nu}$ in line element (3) can be expressed in terms of the null tetrad vector components which is

$$g^{\mu\nu} = -l^\mu n^\nu - l^\nu n^\mu + m^\mu \tilde{m}^\nu + m^\nu \tilde{m}^\mu \quad (9)$$

where

$$l = l^\mu \frac{\partial}{\partial x^\mu} = \delta^\mu_0 \frac{\partial}{\partial x^0},$$

$$n = n^\mu \frac{\partial}{\partial x^\mu} = \left(\delta^\mu_0 - \frac{f(r)}{2} \delta^\mu_1\right) \frac{\partial}{\partial x^0},$$

$$m = m^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{2r}} \left(\delta^\mu_2 + \frac{i}{\sin \theta} \delta^\mu_3\right) \frac{\partial}{\partial x^1},$$

$$\tilde{m} = \tilde{m}^\mu \frac{\partial}{\partial x^\mu} = \frac{1}{\sqrt{2}r} \left(\delta^\mu_2 - \frac{i}{\sin \theta} \delta^\mu_3\right) \frac{\partial}{\partial x^1}. \quad (10)$$

We then do a basic coordinate transformation by implementing

$$x'^\mu = x^\mu + ia(\delta^\mu_r - \delta^\mu_\phi) \cos \theta \rightarrow \begin{cases} u' = u - ia \cos \theta, \\ r' = r + ia \cos \theta, \\ \theta' = \theta, \\ \phi' = \phi \end{cases} \quad (11)$$

so that $f(r) \rightarrow f(r, \bar{r})$. Also, along with this transformation is the transformation of the null tetrad vector components via

$$e^\mu_a \rightarrow e'^\mu_a = \frac{\partial x'^\mu}{\partial x^\nu} e^\nu_a \equiv \left(l'^\mu, n'^\nu, m'^\mu, \tilde{m}'^\nu\right). \quad (12)$$

In particular, the transformation matrix in (12) is given by

$$\begin{pmatrix} \frac{\partial x'^\mu}{\partial x^\nu} \\ \frac{\partial x'^\nu}{\partial x^\mu} \end{pmatrix} = \begin{pmatrix} 1 & 0 & ia \sin \theta & 0 \\ 0 & 1 & -ia \sin \theta & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \quad (13)$$

and hence, the null tetrad vector components are now the following:

$$l'^\mu = \delta^\mu_1,$$

$$n'^\mu = \left(\delta^\mu_0 - \frac{f(r, \bar{r}) \delta^\mu_1}{2}\right),$$

$$m'^\mu = \frac{1}{\sqrt{2r}} \left(\delta^\mu_0 - \delta^\mu_1 \right) ia \sin \theta + \delta^\mu_1 + \frac{i}{\sin \theta} \delta^\mu_3,$$

$$\tilde{m}'^\mu = \frac{1}{\sqrt{2r}} \left(-\delta^\mu_0 - \delta^\mu_1 \right) ia \sin \theta + \delta^\mu_1 - \frac{i}{\sin \theta} \delta^\mu_3. \quad (14)$$

The components of the new contravariant metric tensor can now be constructed using

$$g'^{\mu\nu} = -l'^\mu n'^\nu - l'^\nu n'^\mu + m'^\mu \tilde{m}'^\nu + m'^\nu \tilde{m}'^\mu \quad (15)$$

which results to

$$g'^{\mu\nu} = \begin{pmatrix} \frac{a^2 \sin^2 \theta}{\Sigma} & 0 & 0 & 0 \\ 0 & \frac{a^2 \sin^2 \theta}{\Sigma} & 0 & 0 \\ 0 & 0 & \frac{1}{1 + a^2 \sin^2 \theta} & \frac{\Sigma}{1 + a^2 \sin^2 \theta} \\ 0 & 0 & \frac{\Sigma}{1 + a^2 \sin^2 \theta} & \frac{1}{1 + a^2 \sin^2 \theta} \end{pmatrix} \quad (16)$$

where $\Sigma = r^2 + a^2 \cos^2 \theta$ and $F$ is a function of $r$ and $\theta$:

$$F = \frac{\Delta(r)}{r^2 + a^2 (1 - \sin^2 \theta)} \quad (17)$$

We get the inverse metric as

$$g'^{\mu\nu} = \begin{pmatrix} -F & 0 & 0 & \frac{a \sin^2 \theta \left(F - 1\right)}{\Sigma} \\ 0 & 0 & 0 & \frac{\Sigma}{\Delta(r)} \\ 0 & 0 & \frac{\Sigma}{\Delta(r)} & a \sin^2 \theta \\ \frac{\Sigma}{\Delta(r)} & \frac{\Sigma}{\Delta(r)} & \frac{1}{\Sigma} & \frac{1}{\Delta(r)} \end{pmatrix} \quad (18)$$

where $A = \Sigma + a^2 \left(2 - F\right) \sin^2 \theta$. The final step in the Newman-Janis procedure is to revert back to Boyer-Lindquist coordinates by using the coordinate transformation

$$dt = du' - \frac{\Delta(r)}{\Delta(r)} dr', \quad d\phi = d\phi' - \frac{a}{\Delta(r)} dr' \quad (19)$$

where $\Delta(r)$ is defined in terms of the complexified metric function $f(r, \bar{r})$. In fact, following Ref. \cite{7} on how $r$ is complexified,

$$\Delta(r) = r^2 - 2m(r) r + a^2 \quad (20)$$

where $m(r) = m + MG(r)$. Thus, the line element of a rotating, uncharged, and axially-symmetric black hole surrounded
by a spherical shell of dark matter is given by

\[
\begin{align*}
    ds^2 &= -\left(1 - \frac{2m(r)r}{\Sigma}\right)dt^2 - \frac{4am(r)r \sin^2 \theta}{\Sigma} dtd\phi \\
    &\quad + \frac{\Sigma}{\Delta(r)} dr^2 + \Sigma d\theta^2 \\
    &\quad + \sin^2 \theta \left[r^2 + a^2 \left(1 + \frac{2m(r)r \sin^2 \theta}{\Sigma}\right)\right] d\phi^2.
\end{align*}
\] (21)

In what follows, we will always treat the inner radius of the spherical shell of dark matter, \(r_s\), to always coincide with the event horizon. Hence, it is expected for the value of \(r_s\) to change because the location of the event horizon depends on the spin parameter \(a\).

IV. HORIZONS

We now examine the horizons of the metric given in Eq. (21). Since the metric blows up when \(\Delta(r) = 0\), the locations of the horizons can be found by solving

\[
    r^2 - 2[m + MG(r)] r + a^2 = 0.
\] (22)

Figure 1 shows two plots about the behavior of \(\Delta(r)\) for specific spin parameter and as dark matter mass \(M\) varies. The location of the horizons for different values of spin parameter \(a\). For both cases, the event horizon is unaffected regardless of dark matter density. The radius of the Cauchy horizon decreases as dark matter mass increases. The overall effect is to increase the separation distance between the inner and outer horizons. Without dark matter, we know that the extreme condition occurs when \(a = m\) where the two horizons coincide. As Fig. 1(a) indicates, dark matter makes it possible for the two horizons to coincide even at \(a > m\).

\[
\begin{align*}
    (a) &\quad a = 0.99m, \Delta r_s = 100m, r_s = 1.14m. \\
    (b) &\quad a = 0.50m, \Delta r_s = 100m, r_s = 1.87m.
\end{align*}
\]

FIG. 1. Behavior of \(\Delta(r) = 0\).

For the radius of the ergosphere, one can find its location by
solving \( r \) as \( g_{tt} = 0 \). In particular,
\[
1 - 2\left(\frac{m + MG(r)r}{\Sigma}\right) = 0.
\]
(23)

If \( \theta = \pi/2 \), the Kerr black hole without dark matter surrounding it will give only one value of the ergoregion, which is at \( r = 2m \) because \( g_{tt} \) becomes independent of \( a \). When there is dark matter, the influence of the spin parameter remains because of \( r_s \). Hence, we observe the dark matter effect on the ergosphere in Fig. 2(a) as the black hole is near extremal. At \( a = 0.50m \), the dark matter effect is not very obvious, as Fig. 2(b) shows. Dark matter can also produce (though unphysical) a second ergoregion at large values of \( r \). The significance of this ergoregion becomes important if \( M >> \Delta r_s \).

V. TIME-LIKE CIRCULAR ORBITS

The geodesics of both particles and photons can be studied using the Hamilton-Jacobi approach to general relativity. The Hamilton-Jacobi equation reads
\[
\sum_{\mu} \xi_{\mu} \frac{\partial S}{\partial x^\mu} = 0,
\]
where \( S \) is the Jacobi action and defined in terms of an affine parameter \( \lambda \) and coordinates \( x^\mu \). In general relativity, the Hamiltonian is given by
\[
H = \frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu},
\]
and it follows that
\[
\frac{\partial S}{\partial \lambda} = -\frac{1}{2} g^{\mu\nu} \frac{\partial S}{\partial x^\mu} \frac{\partial S}{\partial x^\nu}.
\]
(26)
The metric in Eq. (21) is independent on \( t, \phi, \) and \( \lambda \), thus we can use the separability ansatzs given by
\[
S = \frac{1}{2} \mu^2 \lambda - E t + L \phi + S_r(r) + S_\theta(\theta),
\]
(27)
where \( \mu \) is proportional to the particle’s rest mass and \( S_r(r) + S_\theta(\theta) \) are both functions of \( r \) and \( \theta \). The equations of motions are then derived by combining (26) and (27). The results are
\[
\sum \frac{dt}{d\lambda} = \frac{r^2 + a^2}{\Delta(r)} P(r) - a(aE \sin^2 \theta - L),
\]
\[
\sum \frac{dr}{d\lambda} = \sqrt{R(r)},
\]
\[
\sum \frac{d\theta}{d\lambda} = \sqrt{\Theta(\theta)},
\]
\[
\sum \frac{d\phi}{d\lambda} = \frac{a}{\Delta(r)} P(r) - \left( aE - \frac{L}{\sin^2 \theta} \right),
\]
(28)
with \( P(r), R(r) \) and \( \Theta(\theta) \) given by
\[
P(r) = E(r^2 + a^2) - aL,
\]
\[
R(r) = P(r)^2 - \Delta(r)[Q + (aE - L) + \mu^2 r^2],
\]
\[
\Theta(\theta) = Q - \left[ \frac{a^2 (\mu^2 - E^2)}{2 + \sin^2 \theta} + \frac{L^2}{E^2} \right] \cos^2 \theta,
\]
(29)
with \( Q \) being the Carter constant: \( Q \equiv \mathcal{K} - (L - aE)^2 \) and \( \mathcal{K} \) is another constant of motion.

In order to see the effect of dark matter to the innermost stable circular orbit (ISCO) of particles, we introduce the following impact parameters \( [78, 80] \):
\[
\xi = \frac{L}{\sqrt{E^2 - \mu^2}}, \quad \eta = \frac{Q}{E^2 - \mu^2}.
\]
(30)
Using the substitution \( \epsilon = \frac{E}{\sqrt{E^2 - \mu^2}} \), the second line in Eq. (29) can be expressed as
\[
R(r) = \left[ \epsilon (r^2 + a^2) - a\xi \right]^2 - \Delta(r) \left[ \eta + (\epsilon a - \xi)^2 + \frac{\mu^2 \epsilon^2}{E^2} \right]
\]
(31)
For circular orbits, the conditions
\[
R(r) = \frac{dR(r)}{dr} |_{r=r_\epsilon} = 0
\]
(32)
must be satisfied. The simultaneous equations will allow us to derive the expressions for \( \xi \) and \( \eta \). By setting \( \eta = 0 \), it is possible to obtain solutions for the energy per unit mass \( E \) of the particle. The results are the following cumbersome equations (\( \Delta'(r) \) denotes a derivative with respect to \( r \)):
\[
\xi = \frac{\epsilon \left[ E\Delta'(r) \left( a^2 + r^2 \right) + \sqrt{2\mathcal{A} - 2E\Delta'(r)r} \right]}{Ea\Delta'(r)},
\]
(33)
\[
\eta = \frac{8r\epsilon^2}{\Delta'(r)^2Ea^2 \left\{ -\frac{1}{8} \frac{\Delta'(r)^2r \left( a^2 \mu^2 + E^2 r^2 \right) - \frac{1}{2} E \sqrt{2} \left( a^2 + \frac{\Delta'(r)r}{2} - \Delta(r) \right) \sqrt{\mathcal{A}}}{2} + \frac{1}{2} \frac{2 a \Delta'(r) \Delta(r)}{\Delta'(r) \Delta'(r) \Delta^2} \left[ \frac{E^2}{2} \left( \Delta(r) - a^2 \right) + E^2 r^2 \right] + E^2 \Delta(r) r \left( a^2 - \Delta(r) \right) \right\} }.
\]
(34)
\[
E = \left[ \frac{a^2 \Delta'(r)^2 r^2 + 2 \sqrt{2 \mathcal{B}} - 2 \Delta'(r) \Delta(r) r \left( a^2 - \Delta(r) \right) - 8 \Delta(r) \left( a^2 - \Delta(r) \right)^2}{16 \Delta(r) r^4 \left( a^2 - \Delta(r) \right) - \Delta'(r)^2 r^4 + 8 \Delta'(r) \Delta(r) r^4} \right]^{1/2}.
\]
(35)
where $\Delta'(r)$ denotes a derivative with respect to $r$,

$$A = \Delta(r)^2 r \left[ 2E^2 r - \Delta'(r)\mu^2 \right]$$

and

$$B = a^2 \Delta(r)^2 \left[ 2a^2 - \Delta(r) \right] + \Delta'(r)r^3.$$

The location of ISCO, which dictates the innermost part of an accretion disk, can be found by differentiating $E^2$ with respect to $r$ and solving the result for $r$. Unfortunately, analytic solution is inconvenient given that $\Delta(r)$ depends on $G(r)$ in $\Delta$ which gives additional complexity to the equation. Notice in Figure 3 that even if the dark matter density is very low, a time-like particle is very sensitive to dark matter effects. At very large $\Delta r$, one can see that the ISCO radius is increased relative to its value when $M = 0$ and specific spin parameter. Abnormalities arises as dark matter density increases because a second and large ISCO radius is produced. This larger ISCO radius is irrelevant because of how ISCO is defined. Nevertheless, this can constrain the extent as to what should be the value of dark matter density which provides physical sense and closely agrees with prevailing results.

Other types of circular orbits such as bound, stable, and unstable circular orbits can be studied qualitatively using the effective potential method. Following Ref. [81], the effective potential is given by

$$V_\pm = \frac{2m(r)aL}{r^3 + a^2 \left( 2m(r) + r \right)} \pm \left\{ \frac{\Delta(r) \left[ (r^2 + a^2)^2 - a^2 \Delta(r) + r^2 L^2 \right]}{\left[ r^3 + a^2 \left( 2m(r) + r \right) \right]^2} \right\}^{1/2}, \quad (36)$$

Figure 4 tells us plenty of information about other types of circular orbits. Here, the vertical line represents the location of the event horizon. The maxima of the effective potential represents the unstable circular orbit in which any perturbation in the particle’s orbit will dictate whether it will fall into the black hole or escape to infinity. The higher the spin of the black hole, the higher the energy requirement in this unstable orbit. The inset plot reveals that increasing dark matter mass decreases slightly the energy required in such an orbit. Furthermore, the radius where the peak is located decreases. The reverse happens in the stable circular orbit in which the radius where the minima occurs increases. Also, like the unstable orbit, the energy requirement decreases more obviously. For the low spin parameter, the available energy for elliptic bound orbits to occur is minimal, hence its easier to plunge into the black hole.

By convention, particles that revolve clockwise on a black hole have negative angular momentum $L$. Figure 5 shows that in a Schwarzschild case, the effective potential is always positive, and negative energy is not even allowed. If the black hole is rotating, negative effective potentials (or negative energies) of particles are allowed for $aL < 0$ and energy extraction from the black hole is allowed via the Penrose process. Regardless of dark matter density, the location where the Penrose process should occur remains unchanged. For a black hole that has a very high spin (Fig. 5(a)), the particle has more negative energy compared to a black hole that spins slowly (Fig. 5(b)). Hence, any deviations in the Penrose process due to dark matter is very negligible.

VI. NULL CIRCULAR ORBITS AND BLACK HOLE SHADOW

Null geodesics are of importance in studying the contour of a black hole silhouette. In particular, we need to determine and locate the unstable circular orbit for photons and do the backward ray tracing method to plot the contour of the resulting shadow in the celestial coordinates of a remote observer. For this case, we use (30) and take $\mu = 0$. Doing the same procedures on how we derived (31) and (32), we obtain the following:

$$\xi = \frac{\Delta(r)(r^2 + a^2) - 4\Delta(r)r}{a\Delta'(r)}, \quad (37)$$
FIG. 4. Effective potential for $\Delta r_s = 100m$ and $L = 2.75m$.

\[
\eta = -\frac{r^4 \Delta'(r)^2 + 8r^3 \Delta(r)\Delta'(r) + 16r^2 \Delta(r)(r^2 - \Delta(r))}{a^2 \Delta'(r)^2}
\]  

(38)

which appears to be general. If $M = 0$, one can obtain the very well known analytic formula for the prograde and retrograde orbit radii. However, it can be tedious or inconvenient to obtain an analytic formula for a Kerr black hole with dark matter configuration given in \[ 44 \] since the result of $\eta = 0$ involves a 5th power polynomial:

\[
16\Delta(r)r^2 (a^2 - \Delta(r)) - \Delta'(r)r^4 + 8\Delta'(r)\Delta(r)r^3 = 0
\]  

(39)

This inconvenience is also true even with the approximation as $\Delta r_s \to \infty$, which reduces the above equation to the 4th power. By numerical calculations and satisfying \[ 44 \] we can locate the unstable photon orbits and obtain some insights as to what happens when $\Delta r_s \to \infty$ (i.e., low dark matter density). Unlike the time-like particles, Figure 6 reveals that high dark matter density is needed in order to see deviations in the null orbits. In the near extremal case, Fig. 6(a) shows that the prograde is nearly unaffected, while the dark matter effect on the retrograde radius is to decrease its value relative to the Kerr case where $M = 0$. In Fig. 6(b) the change in the prograde orbit is evident. These changes, that the photon radius must decrease due to the presence of dark matter, agrees with the result in Ref. \[ 54 \]. As explained, the decrease in radius is due the dark matter’s full effect (both under and above the photon sphere) causing a new orbital equilibrium.

Any perturbations can lead the photons in the unstable orbit to escape the rotating black hole’s gravitational influence. These photons will travel in the intervening space between the black hole and the remote observer. In our case, the photons will pass through the dark matter configuration. Hence, we expect a difference in the resulting shadow when vacuum, and space with dark matter, are compared. The method on deriving the celestial coordinates with respect to the Zero Angular Momentum Observers (ZAMO) is very well established. The
celestial coordinates, in general, are given by [82]

\[
\alpha = -r_0 \frac{\xi}{\zeta \sqrt{g_{\phi\phi}} \left(1 + \frac{g_{\phi\phi}}{g_{\theta\theta}} \xi\right)},
\]

\[
\beta = r_0 \frac{\pm \sqrt{\Theta(t)}}{\zeta \sqrt{g_{\theta\theta}} \left(1 + \frac{g_{\theta\theta}}{g_{\phi\phi}} \xi\right)}
\]

and in the limit \( r \to \infty \), (40) reduces to

\[
\alpha = -\xi \csc \theta_0,
\]

\[
\beta = \pm \sqrt{\eta + a^2 \cos^2 \theta_0 - \xi^2 \cot^2 \theta_0}
\]

where \( \theta_0 \) is the polar orientation of the remote observer with respect to the equatorial plane, while \( \xi \) and \( \eta \) are given by (37) and (38). Figure 7 shows how different dark matter density affects the black hole shadow. If there is no dark matter, we find the almost D-shaped contour of the Kerr black hole when the spin parameter is near extremal. When dark matter is present, the contour that represents the retrograde photon orbit bulges more as dark matter density increases. These contours perfectly agree with Fig. 6(a). In effect, this increases the radius of the shadow. The contour that represents the prograde orbit deviates less as dark matter density increases. With the given values of dark matter mass in the contour plot, it seems that the change in the size of the shadow is kind of exaggerated. It is only to demonstrate, however, how dark matter changes the size of the shadow. The D-shaped contour is not changed at all, hence, the fundamental properties of the rotating black hole is retained in the presence of dark matter. Fig. 7(b) shows the shadow contour when the black hole spin is a bit lower.

When we consider different values of the polar angle \( \theta_0 \), Figure 8 shows how the remote observer sees the rotating black hole. As the observer gets near the poles, the shadow contour is becoming more of an ellipse-shaped.
Here, we show briefly a derivation of the shadow radius, and introduce the expression for the radius distortion parameter, which are known to be observables useful in extracting information about black hole shadows [83, 84]. In order to have insights about the shadow radius, a schematic diagram like in Figure 9 is very helpful (see also [85, 86]). Here, calling $x_c = O\alpha_o$, we can see that $R_s = \alpha_r - x_c$. A right triangle will be formed and by inspection, $R_s^2 = \beta^2 + (\alpha_t - x_c)^2$. After some basic algebra,

$$R_s = \frac{\beta^2 + (\alpha_t - \alpha_r)^2}{2|\alpha_t - \alpha_r|}. \quad (42)$$

As mentioned, $R_s$ corresponds to the line $\overline{\alpha_r\alpha_o}$, and this radius is used for the reference circle. The shadow distortion is then defined as $d_s = \hat{\alpha}_l - \alpha_l$. Thus, the radius distortion parameter can be expressed as

$$\delta_s = \frac{d_s}{R_s} = \frac{\hat{\alpha}_l - \alpha_l}{R} \quad (43)$$

Figure 10 shows how dark matter affects the shadow radius. The black dotted horizontal line represents the Schwarzschild case ($M = 0$). Indeed, dark matter increases the shadow radius and such increase is also amplified by the black hole’s spin parameter $a$. For both cases in the figure, the curve is asymptotic to the Schwarzschild case when $\Delta r_s \to \infty$. Due to (39) and the complexity looming in (42), we emphasize again that it is inconvenient to derive a formula to estimate the effective radius of the dark matter halo in order to have considerable effect on the shadow radius. This is unlike the Schwarzschild scenario where the estimate $\Delta r_s = \sqrt{3}mM$ was easily attained because $r_{ph}$ can be derived analytically as well as the expression for the shadow radius.

The radius distortion parameter is plotted in Figure 11. Here, we see the agreement in Figure 7 because as the spin parameter decreases, the more the shadow becomes close to a perfect circle. The radius distortion is indeed greater when the black hole spin is near the extremal case, which is also amplified by dark matter effect.

We can also use the shadow radius to determine the angular diameter of the rotating black hole. The angular radius is given by

$$\theta_s = 9.87098 \times 10^{-8} \frac{R sm}{D} \quad (44)$$

where $m$ must be measured in terms of solar mass and $D$ in parsec. Let’s consider supermassive black hole in M87 galaxy with mass $m = 6.9 \times 10^9 M_\odot$ and its distance from Earth is $D = 16.8 Mpc$. Figure 12 shows the plot with and without dark matter. In general, not only the dark matter influences the increase in angular diameter, but also the spin parameter.
Even for $M = 50m$ and $\Delta r_s = 100m$, the angular diameter increases drastically as $a$ increases.

The energy emission rate of a black hole is defined as

$$\frac{d^2E}{d\sigma dt} = \frac{2\pi^2}{\sigma^3} \Pi_{ilm} \left( e^{\pi \sigma T} - 1 \right),$$

where $T$ is the black hole temperature. Following [56], the temperature is given by

$$T = \frac{r_h}{4\pi (r_h^2 + a^2)^2} \left\{ 2a^2 (f(r_h) - 1) + r_h (r_h^2 + a^2) f'(r_h) \right\},$$

(46)
in which $r_h$ is the event horizon radius and $f(r_h) = -g_{tt}$ in the metric [21]. For a remote observer, the area of the shadow is approximately equal to the high energy absorption cross-section which oscillates around a constant, $\Pi_{ilm} = \pi R_s^2$. Figure [21] shows how the energy emission rate changes if the rotating black hole is surrounded by dark matter. The deviation from the Kerr case is not obvious when $M = 50m$.

However, in the third plot ($M = 100m$), the peak is slightly higher. Hence, the effect of dark matter is to increase the energy emission rate near the event horizon. Dark matter also has a negligible effect on the photon’s peak frequency because shifting to lower or higher frequency is not so evident, even in the case of high dark matter density.

**VIII. CONCLUSION**

In this paper, we extended the study in Ref. [54] to a rotating black hole by utilizing the Newman-Janis algorithm. Considering only the case where the initial configuration of the dark matter starts at the event horizon, we analyzed changes in some basic black hole properties such as the horizons, and the ergoregion. Along with the null geodesics, we have shown through numerical analysis that these require high dark matter density in order to have considerable changes. With the nature of how dark matter is configured, it is inconvenient to
FIG. 12. Angular diameter.

FIG. 13. Energy Emission rate.
provide an analytic estimate using the radius of the black hole shadow to determine notable changes due to the dark matter effect. The size of the shadow is seen to enlarge while the shape is maintained. We also showed that time-like geodesics are very sensitive to dark matter effects because the location of the ISCO radius drastically change even at low-density configuration. While the Penrose process remained unaffected, other types of orbits are seen to be affected by dark matter. Following this study, prospects might include the assumption of a hypothetical situation where dark matter penetrates the event horizon: coinciding with the Cauchy horizon, or even in the Schwarzschild case, remains an inconvenient task. It is interesting to find out whether finding such estimate is possible by using a different black hole phenomena.

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