Abstract

Hyper-cubic groups in any dimension are defined and their conjugate classifications and representation theories are derived. Double group and spinor representation are introduced. A detailed calculation is carried out on the structures of four-dimensional cubic group $O_4$ and its double group, as well as all inequivalent single-valued representations and spinor representations of $O_4$. All representations are derived adopting Clifford theory of decomposition of induced representations. Based on these results, single-valued and spinor representations of the orientation-preserved subgroup of $O_4$ are calculated.

Keywords: hyper-cubic group, double group, spinor representation, Clifford theory

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I Introduction

It is well-known that electrons stay in spinor representations of the symmetry group of a given lattice in condensed matter physics; it is reasonable to assume that quarks, leptons, as well as baryons, should reside in spinor representations of the symmetry group of a four-dimensional lattice in lattice field theory (the concept of “spinor representation” will be clarified in the next section). Accordingly, to explore the structure and representations (spinor representations especially) of such groups has important significance in high energy physics.

In this paper, we concentrate on the case of hyper-cubic lattices, though they are not the maximum symmetric lattices in four dimension [1]. In history, the first representation-theoretical consideration of symmetry group of such lattices was given by A. Young [2]. Then mathematicians worked in this field due to the interest of wreath product [3][4] to which A. Kerber gave a thorough review in his book [5]. Physicists took part in after K. G. Wilson introduced lattice gauge theory [6]. M. Baake et al first gave an explicit description of characters of four-dimensional cubic group [7]; J. E. Mandula et al derived the same results using a different method [8]. As for spinor representations, Mandula et al resolved this problem for what we call orientation-preserved four-dimensional cubic-group in [9].

In this paper, the power of Clifford theory on decomposition of induced representations (Sec.II.1) is fully applied. A systematic and schematic description of conjugate classification and representation theory of generalized cubic group $O_n$, as well as the concept of orientation-preserved subgroup of them $SO_n$, is given in Sec.II.2. Double group is introduced in Sec.II.3 to clarify the terms “single-valued representation” and “two-valued representation (spinor representation)”.

Then specifying these general results to four dimension, we give a detailed description of structure and conjugate classification of $O_4$ (Sec.III.1), its double $O_4$ (Sec.III.2), and those of $SO_4$, $SO_4$ (Sec.V). We derive all inequivalent single-valued representations as well as spinor representations of $O_4$, adopting Clifford theory (Sec.V). Based on these results, we reproduce representation theory of $SO_4$ in Sec.VI.
It should be pointed out that the “spinor” part for $O_4$ of our work is completely new and that although other results are well-known, our method to derive them is much more tidy and systematic than that used by other authors who gave the same results, thanks for the power of Clifford theory.

II Conceptual foundations

II.1 Clifford theory on decomposition of induced representations

Two results of Clifford theory, a powerful method for decomposing induced representations of a given group $E$ with a normal subgroup $N$, will be applied in this paper. We will use $\mathcal{C}[E]$ for the group algebra of $E$ in complex field and $G$ for $E/N$ below. The first result is

Theorem 1 (Clifford) Let $M$ be a simple $\mathcal{C}[E]$-module, and $L$ a simple $\mathcal{C}[N]$-submodule of $M_N$ s.t. $L$ is stable relative to $E$, i.e. $L$ is isomorphic to all of its conjugates. Then

$$M \cong L \otimes_{\mathcal{C}} I$$

for a left ideal $I$ in $\text{End}_{\mathcal{C}[E]}L^E$. The $E$-action on $L \otimes_{\mathcal{C}} I$ is given by

$$x \mapsto U(x) \otimes V(x), x \in E$$

where $U : E \to GL(L)$ is a projective representation of $E$ on $L$, and $V : E \to GL(I)$ is a projective representation of $G$, that is, $V(x)$ depends only on the coset $xN$ of $x$ in $G$, for each $x \in E$. The factor sets associated with $U$ and $V$ are inverse of each other.

The second result can be regarded as a special case of Theorem 1. Let $E = N \rtimes G$, $|E| < \infty$ and $N$ be abelian, then adjoint action of $G$ upon $N$ makes $N$ a $G$-module. This $G$-action can be extended naturally to a $G$-action upon $\mathcal{C}[N]$ by linearity. Define $\Pi(N) := \{\pi_\mu\} \subset \mathcal{C}[N],$

$$\pi_\mu := \sum_{a \in N} \chi_\mu(a^{-1})a$$  \hspace{1cm} (1)$$

where $\chi_\mu$ are all irreducible representations of $N$. The $G$-action on $\Pi(N)$ is closed and thus $\Pi(N)$ is separated into orbits $\Pi(N) = \coprod_{o \in \mathcal{I}} \Pi_o$ where $\mathcal{I}$ is a index set to label different orbits. For each $\Pi_o$,
choose one of its element and denote it as $\pi_{o,e}$. The stabilizer of each $\pi_{o,e}$ in $G$ (little group) is denoted as $S_o$. There is a bijection from $G/S_o = \{hS_o\}$ to $\Pi_o$ defined by

$$Ad_h(\pi_{o,e}) = h\pi_{o,e}h^{-1} =: \pi_{o,h}$$

(2)

where $\{h\}$ is a system of representatives of left cosets $G/S_o$. Define

$$\Pi_{o,h;\eta,i} \equiv \pi_{o,h} \otimes_{S_o} e^o_{\eta,i}$$

(3)

in which $\{e^o_{\eta,i}|i = 1, 2, ..., d^o_\eta\}$ with fixed $o, \eta$ and $\eta$ is the $\eta$th irreducible representation of $S_o$ whose dimension is $d^o_\eta$, then

**Proposition 1** (little group method)\[11\][12][13]

1. For each fixed $(o, \eta)$, $\{\Pi_{o,h;\eta,i}\}$ induces an irreducible representation of $E$, denoted as $D_{o,\eta}$;

2. If $(o, \eta) \neq (o', \eta')$, then $D_{o,\eta}$ and $D_{o',\eta'}$ are inequivalent;

3. $\{D_{o,\eta}\}$ gives all inequivalent irreducible representations of $E$.

**II.2 Cubic group in any dimension**

The symmetry group of a cube including inversions in three dimensional Euclidean space, which is denoted as $O_h$ in the theory of point groups \[13\], can be generalized into any $n$-dimensional Euclidean space $E^n$, along two different approaches whose results are equivalent. The first approach of generalization which is very natural and straightforward is geometrical. An $n$-cube (or hyper-cube in $E^n$) $C_n$ is defined to be a subset of $E^n$, $C_n = \{p|x^i(p) = \pm 1\}$, where $x^i : E^n \rightarrow \mathbb{R}$, $i = 1, 2, ..., n$ are coordinate functions of $E^n$, together with the distance inherited from $E^n$. $n$-Cubic group (hyper-cubic group of degree $n$) $O_n$ consists of all isometries of $E^n$ which stabilize $C_n$. While the second approach of generalization is algebraic. $O_h$ has a semi-direct product structure as $Z_2^3 >\triangleleft S_3$ \[11\]; we generalize this to $Z_2^n >\triangleleft S_n$ which is just a wreath product $Z_2 \wr S_n$ of $Z_2$ with $S_n$. We point out that these two generalizations are identical. Let $\{e_i\}$ be a standard orthogonal basis of $E^n$, namely $x^j(e_i) = \delta^i_j$. Define $n + 1$ points in $C_n$ to be $p_0 = (-1, -1, ..., -1), p_i = p_0 + 2e_i$. 
Lemma 1 \( \forall \epsilon \in O_n, \epsilon \) is entirely determined by images \( \epsilon(p_i), i = 0, 1, 2, ..., n. \)

Proof:
The fact that \( \epsilon \) is an isometry of \( E^n \) ensures the equality of Euclidean distances \( d(p, p_i) = d(\epsilon(p), \epsilon(p_i)), \)
\( i = 0, 1, ..., n \) for any other \( p \) in \( C_n \). If all \( \epsilon(p_i) \) are given, \( \epsilon(p) \) will be fixed for any other \( p \) accordingly due to the fundamental lemma of Euclidean geometry (lemma A-1 in Appendix A). In fact, the existence of solution in lemma A-1 is guaranteed by that \( \epsilon \) stabilizes \( C_n \) and lemma A-1 itself ensures the uniqueness.

\( \square \)
To fix \( \epsilon(p_0) \), there are \( 2^n \) ways; while for a fixed \( \epsilon(p_0) \), there are \( n! \) possibilities to fix \( \epsilon(p_i), i = 1, 2, ..., n. \) Therefore, \( |O_n| = 2^n \cdot n! \).

Proposition 2 \( (Structure \ of \ O_n) \)
\[ O_n \cong Z_2^n \rtimes S_n \] (4)

Proof:
Introduce a class of isometries in \( E^n \):
\[ \sigma(e_i) = e_{\sigma(i)}; I_i(e_j) = (1 - 2\delta_{ij})e_j, i = 1, 2, ..., n \] (5)
where \( \sigma \in S_n \) permutes the axes and \( I_i \) inverts the \( i \)th axis. Subjected to the relations
\[ I_i^2 = e, I_iI_j = I_jI_i, i, j = 1, 2, ..., n; \sigma I_i = I_{\sigma(i)}\sigma, \sigma \in S_n \] (6)
these isometries generate a sub-group of \( C_n \) isomorphic to \( Z_2^n \rtimes S_n \) whose order is \( 2^n n! = |O_n| \). So (4) follows.

\( \square \)
A. Kerber gave a detailed introduction on the conjugate classification and representation theory of a general wreath product \( N \wr G \) in [5]. We specify his general results to our case \( Z_2^n \rtimes S_n \cong Z_2 \wr S_n \).

Some fundamental facts about symmetrical groups \( S_n \) should be recalled [15]. Each element \( \sigma \in S_n \)
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has a cycle decomposition

$$\sigma = \begin{pmatrix} 1 & 2 & \ldots & n \\ \sigma(1) & \sigma(2) & \ldots & \sigma(n) \end{pmatrix} = \prod_{k=1}^{n} \prod_{\alpha=1}^{\nu_k} \tau_{k\alpha}$$

(7)

where $\tau_{k\alpha}$ are independent $k$-cycles, which can be expressed as $(a_1a_2\ldots a_k)$, and write $n(k, \alpha) = \{a_1, a_2, \ldots, a_n\}$. The cycle structure of $\sigma$ can be represented formally as

$$(\nu) = \prod_{k=1}^{n} (k^\nu_k)$$

(8)

where $\{\nu_k\}$ satisfies $\sum_{k=1}^{n} k \cdot \nu_k = n$. Two elements in $S_n$ are conjugate equivalent, iff they have the same cycle structure. The number of elements in class $(\nu)$ is equal to $N(\nu) = n! / \prod_{k=1}^{n} (k^{\nu_k} \nu_k !)$. Each cycle structure $(\nu)$ can be visualized by one unique Young diagram which is denoted also by $(\nu)$.

There is a one-one correspondence between all inequivalent irreducible representations of $S_n$ and all Young diagrams, which enable us to represent each irreducible representation by the corresponding Young diagram $(\nu)$. We write the basis of one of these representations $(\nu)$ in $d(\nu)$ dimension as $e_{(\nu)i}$, $i = 1, 2, \ldots, d(\nu)$.

We point out that the conjugate classification of $O_n$ has a deep relation with that of $S_n$. A generic element in $Z_2 \wr S_n$ can be written as

$$\sigma \cdot \prod_{i} I_{i}^{s_i} = \begin{pmatrix} 1 & 2 & \ldots & n \\ (-)^{s_1}\sigma(1) & (-)^{s_2}\sigma(2) & \ldots & (-)^{s_n}\sigma(n) \end{pmatrix}$$

(9)

in which $s_i \in \mathbb{Z}/2\mathbb{Z}$. We call the r.h.s. of (9) by permutation with signature. $\sigma \prod_{i} I_{i}^{s_i}$ can be decomposed according to (9), i.e.$\prod_{i} I_{i}^{s_i} = \prod_{k=1}^{n} \prod_{\alpha=1}^{\nu_k} \prod_{a \in n(k, \alpha)} I_{a}^{s_a}$ and

$$\sigma \prod_{i} I_{i}^{s_i} = \prod_{k=1}^{n} \prod_{\alpha=1}^{\nu_k} (\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_{a}^{s_a})$$

(10)

The cycle with signature is defined to be

$$\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I_{a}^{s_a} = \begin{pmatrix} a_1 & a_2 & \ldots & a_k \\ (-)^{s_{a_1}}a_2 & (-)^{s_{a_2}}a_3 & \ldots & (-)^{s_{a_k}}a_1 \end{pmatrix}$$
For two independent \((k, \alpha), (k', \alpha')\), it is easy to verify that
\[
\tau_k \tau_{k'} = \tau_{k'} \tau_k \quad \prod_{a \in n(k, \alpha)} I^s_a \tau_k \tau_{k'} = \tau_k \tau_{k'} \quad \prod_{a \in n(k', \alpha')} I^s_a \tau_{k'} = \tau_{k'} \tau_k \quad \prod_{a \in n(k', \alpha')} I^s_a
\]

**Proposition 3** \([3][4][5]\) We use \(\sim\) to denote conjugate equivalent.

1. **(descent rule)**
\[
\sigma \prod_i I_i^{s_i} \sim \sigma' \prod_i I_i^{s_i'} \Rightarrow \sigma \sim \sigma'
\]  \(\text{(11)}\)

2. **(permutation rule)** Let
\[
\tilde{\sigma} = \begin{pmatrix} 1 & 2 & \ldots & n \\ \tilde{\sigma}(1) & \tilde{\sigma}(2) & \ldots & \tilde{\sigma}(n) \end{pmatrix} = \begin{pmatrix} \sigma(1) & \sigma(2) & \ldots & \sigma(n) \\ \sigma'(1) & \sigma'(2) & \ldots & \sigma'(n) \end{pmatrix}
\]
then
\[
\tilde{\sigma}(\sigma \prod_i I_i^{s_i})\tilde{\sigma}^{-1} = \begin{pmatrix} \tilde{\sigma}(1) & \tilde{\sigma}(2) & \ldots & \tilde{\sigma}(n) \\ (-)^{s_1} \sigma'(1) & (-)^{s_2} \sigma'(2) & \ldots & (-)^{s_n} \sigma'(n) \end{pmatrix}
\]  \(\text{(12)}\)

3. **(signature rule within one cycle)** Let \(\tau_{k\alpha}\) be a \(k\)-cycle and \(a_0\) be a given number in \(n(k, \alpha)\), then
\[
\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I^s_a \sim \tau_{k\alpha} \prod_{a \in n(k, \alpha)} I^s_a + \delta_{a_0} + \delta_{\tau_{k\alpha}(a_0)}
\]  \(\text{(13)}\)

Note that \(\tau_{k\alpha}(a_0)\) is calculated modulo \(k\) (the subscripts of \(I_a\) are always understood in this way).

4. **(signature rule between two cycles)** Let \(\tau_{k\alpha}, \tau_{k\beta}\) be two independent \(k\)-cycles and we define a bijection \(\theta : n(k, \alpha) \rightarrow n(k, \beta), a_i \mapsto b_i\). Then
\[
\tau_{k\alpha} \prod_{a \in n(k, \alpha)} I^s_a \cdot \tau_{k\beta} \prod_{b \in n(k, \beta)} I^s_b \sim \tau_{k\alpha} \prod_{a \in n(k, \alpha)} I^s_{a\theta(a)} \cdot \tau_{k\beta} \prod_{b \in n(k, \beta)} I^s_{b-1(b)}
\]  \(\text{(14)}\)

This theorem ensures conjugate classification of \(Z_2 \wr S_n\) is totally determined by the structure of cycles with signature. We verify this statement by generalizing Young diagram technology. First, draw a Young diagram with numbers and signatures for each element \(\sigma \prod_i I_i^{s_i} \in Z_2 \wr S_n\) according to the decomposition Eq.(10) by the following rules:
1. Plot Young diagram of the class in $S_n$ to which $\sigma$ belongs and fill each column of this Young diagram with numbers in corresponding cycle by cyclic ordering from up-most box to down-most box.

2. Draw a slash in the Young box if the number in this box is mapped to a minus-signed number.

Secondly, partition elements in $Z_2 \wr S_n$ by their cycle structure in $S_n$ and Eq. (11) guarantees elements belong to different partitions can not be conjugate equivalent. Eq. (12) implies that all the numbers that we filled by rule I are unnecessary, so smear them out and leave boxes and slashes only. Within each column, Eq. (13) says that the positions of slashes make no difference. What’s more, in fact only that the total number of slashes is even or odd distinguishes different classes. Therefore we regulate each column to contain zero or one slash at the bottom box. Eq. (14) shows that we can not distinguish the case that one column without any slash (Mr. Zero) is put to the left to another column with one slash (Mr. One) from that Mr.Zero is to the right of Mr.One, if they have same cyclic length; thus we regulate that Mr.Zero shall always stand left to Mr.One. Therefore, conjugate classes of $Z_2 \wr S_n$ can be uniquely characterized by generalizing Young diagrams containing slashes. Following Eq. (8), we represent conjugate classes by

$$\begin{align*}
(\nu^+, \nu^-) = \prod_{k=1}^{n} (k^\nu_k^+ + \nu_k^-) 
\end{align*}$$

(15)

where $\nu_k^+$ is the number of Mr.Zero-type $k$-cycles and $\nu_k^-$ is that of Mr.One-type $k$-cycles, which satisfy $\nu_k^+ + \nu_k^- = \nu_k$. It is not difficult to check some numerical properties of conjugate classes of $Z_2 \wr S_n$.

**Corollary 1**

1. Given a class $(\nu)$ in $S_n$, there are

$$\begin{align*}
\prod_{k=1}^{n} (1 + \nu_k)
\end{align*}$$

(16)

classes in $Z_2 \wr S_n$ which descend to $(\nu)$.

2. The number of elements in a class $(\nu^+, \nu^-)$ is

$$\begin{align*}
N_{(\nu^+, \nu^-)} = N_{(\nu)} \prod_{k=1}^{n} (C \nu_k^+ \left( \sum_{i=0}^{\left[ \frac{k}{2} \right]} C^2_i \right) \nu_k^+ \left( \sum_{j=1}^{\left[ \frac{k-1}{2} \right]} C_{2j-1}^2 \nu_k^- \right))
\end{align*}$$

(17)
where \( C_n^m \) is combinatorial number defined to be \( m!/(n!(m-n)!) \).

3. The order of a class \((\nu^+, \nu^-)\) is

\[
lcm(\{ k \cdot 2^{\delta(\nu^-_k)} | \nu_k \neq 0 \})
\]

where \( \delta(\nu^-_k) = 0 \), if \( \nu^-_k = 0 \); \( \delta(\nu^-_k) = 1 \), if \( \nu^-_k > 0 \).

4. Determinant (signature, parity) of a class

\[
det((\nu^+, \nu^-)) = (-1)^{\sum_{k=1}^{n} \nu^-_k} \cdot det((\nu))
\]

where \( det((\nu)) \) is the determinant of \((\nu)\) in \( S_n \).

All inequivalent irreducible representations of \( Z_2^n \) can be expressed as

\[
\chi(s) := \bigotimes_{p=1}^{n} \chi_{(-)}^{s_p}
\]

in which \( s_p \in \mathbb{Z}/2\mathbb{Z}, p = 1, 2, ..., n \) and \( \chi_{(-)}, \chi_{(+)} \) are two irreducible representations of \( Z_2 \) with \( \chi_{(+)} \) being the unit representation. Thus \( \pi(s) \) can be defined by Eq.\( (\ref{eq:rep}) \) and \( \Pi(Z_2^n) = \{ \pi(s) \} \). Note that \( \pi(s) \) satisfy \( \pi(s)\pi(s') = \pi(s \cdot s') \) where \( (s \cdot s')(p) = s(p)s'(p) \). \( \Pi(Z_2^n) \) is divided into \( n + 1 \) orbits under the \( S_n \)-action, namely \( \Pi(Z_2^n) = \bigoplus_{p=0}^{n} \Pi_p \). For a given \( p \), \( \Pi_p \) consists of those \( \pi(s) \) who has \( p \) components in \( s \) equal to 1, other \( n - p \) components equal to 0; hence \( |\Pi_p| = C_n^p \). Each \( \pi_{p,e} \) is specified to a \( \pi(s) \) with \( s_p = 0, p = 1, 2, ..., n - p; s_p = 1, p = n - p + 1, ..., n \), whose stationary subgroup is just \( S_{n-p} \otimes S_p \), denoted as \( F_p \). Representatives of left-cosets in \( S_n/F_p \) are written as \( \sigma_r \), then according to Eqs.\( (\ref{eq:rep})-(\ref{eq:rep2}) \) and Theorem \( \ref{thm:rep} \).

**Proposition 4** (Representation theory of \( Z_2 \wr S_n \))

\[
\Pi_{p,\sigma_r,(\mu)i,(\nu)j} = \pi_{p,\sigma_r} \otimes F_p (e_{(\mu)i} \otimes e_{(\nu)j})
\]

give all inequivalent irreducible representations of \( Z_2 \wr S_n \) when \( (p, (\mu), (\nu)) \) runs over its domain.

where \( \pi_{p,\sigma_r} = Ad_{\sigma_r}(\pi_{p,e}) \) whose \( (s) \) will be denoted as \( (s^{p(\sigma_r)}) \).
Corollary 2  
1. (Burnside formula)  \( \sum_{(p, (\mu), (\nu))} (C^{p}_{n}d(\mu)d(\nu))^2 = 2^n n! \)

2. The number of conjugate classes is  \( \sum_{(p, (\mu), (\nu))} 1 \).

3. (Representation matrix element) Given  \( \sigma \prod_q I_q^n \in Z_2 \wr S_n \),

\[
D_{(p, (\mu), (\nu))}(\sigma \prod_q I_q^n)^{\sigma', i', j'}_{\sigma, i, j} = \delta_{\sigma, (\sigma_r)}D(\mu)(\sigma_{(n-p)}(\sigma\sigma_r))^{i''}_{i} D(\nu)(\sigma_p(\sigma\sigma_r))^{j''}_{j} \prod_q (-)^{s(\sigma_r)}t_q
\]

4. (Character)

\[
\chi_{(p, (\mu), (\nu))}(\sigma \prod_q I_q^n) = \delta_{\sigma, (\sigma_r)}\chi(\mu)(\sigma_{(n-p)}(\sigma_r))\chi(\nu)(\sigma_p(\sigma_r)) \prod_q (-)^{s(\sigma_r)}t_q
\]

where  \( \sigma_r, \sigma_{(n-p)}, \sigma_p \) map an element in  \( S_n \) to its decompositions according to  \( S_n/F_p, S_{n-p} \) and  \( S_p \) respectively.

At the end of this subsection, we introduce the orientation-preserved  \( n \)-cubic group  \( SO_n \) which is a normal subgroup of  \( O_n \)

\[
SO_n := (O_n \cap SO(n)) < O_n
\]  \( (21) \)

Define  \( Z_2^n|_e \) as a subgroup of  \( Z_2^n \) generated by  \( I_i I_j, i \neq j \) and  \( Z_2^n|_o := Z_2^n \setminus Z_2^n|_e \). Then

\[
SO_n = (Z_2^n|_e \cdot A_n) \bigsqcup (Z_2^n|_o \cdot (S_n \setminus A_n))
\]  \( (22) \)
in which  \( \cdot \) is product of two subsets in a group,  \( A_n \) stands for the alternative subgroup in  \( S_n \). Thus,  \( |SO_n| = (2^n \cdot (n!))/2 \).

II.3 Double group and spinor representation

Some fundamental facts of Clifford algebra are necessary for giving the definition and properties of double groups. Denote the Clifford algebra upon Euclidean space  \( V \) as  \( Cl(V) \); the isometry \( x \mapsto -x \) on  \( V \) extends to an automorphism of  \( Cl(V) \) denoted by  \( x \mapsto \bar{x} \) and referred to as the canonical
automorphism of $\text{Cl}(V)$. We use $\text{Cl}^*(V)$ to denote the multiplicative group of invertible elements in $\text{Cl}(V)$ and the Pin group is the subgroup of $\text{Cl}^*(V)$ generated by unit vectors in $V$, i.e.

$$\text{Pin}(V) := \{a \in \text{Cl}^*(V) : a = u_1 \cdots u_r, u_j \in V, \|u_j\| = 1\}$$

Proofs of the following four statements can be found in [19].

**Lemma 2** If $u \in V$ is nonnull, then $R_u$, reflection along $u$, is given in terms of Clifford multiplication by

$$R_u x = -uxu^{-1}, \forall x \in V$$

**Theorem 2** The sequence

$$0 \rightarrow \mathbb{Z}_2 \rightarrow \text{Pin}(V) \xrightarrow{\text{Ad}} O(V) \rightarrow 1$$

is exact, in which

$$\text{Ad}_a(x) := axa^{-1}, \forall x \in \text{Cl}(V), a \in \text{Pin}(V)$$

We will usually write $\text{Ad}$ just by $\pi$ as a surjective homomorphism.

**Proposition 5** $\text{Cl}(E^4)$, as an associative algebra with unit, is isomorphic to $M_2(\mathbb{H})$ where $\mathbb{H}$ denotes quaternions.

**Lemma 3** Under the above algebra isomorphism, the image of $\text{Pin}(E^4)$ is a subset of $\text{SU}(4)$.

Now we give the main definition of this paper.

**Definition 1** Let $\epsilon$ be an injective homomorphism from a group $G$ to $O(n)$, then the double group or the spin-extension of $G$ with respect to $\epsilon$ is defined to be $D_n(G, \epsilon) := \pi^{-1}(\epsilon(G))$.

An introduction to double groups in three dimension can be found in [17]. Following elementary facts in the theory of group extension [18], this diagram

$$\begin{align*}
0 & \longrightarrow \mathbb{Z}_2 \longrightarrow \text{Pin}(E^n) \xrightarrow{\pi} O(n) \longrightarrow 1 \\
\downarrow & \quad \quad \downarrow \pi \quad \quad \downarrow \epsilon \\
0 & \longrightarrow \mathbb{Z}_2 \longrightarrow \pi^{-1}(\epsilon(G)) \longrightarrow G \longrightarrow 1
\end{align*}$$
is commutative. If $\epsilon_1(G) \sim \epsilon_2(G)$, there is

$$
\begin{array}{c}
0 \longrightarrow Z_2 \longrightarrow \pi^{-1}(\epsilon_1(G)) \longrightarrow \epsilon_1(G) \longrightarrow 1 \\
\| \\
0 \longrightarrow Z_2 \longrightarrow \pi^{-1}(\epsilon_2(G)) \longrightarrow \epsilon_2(G) \longrightarrow 1
\end{array}
$$

Note that the double group is not a universal object for a given abstract group $G$ but a special type of $Z_2$-central extension of $G$ subjected to the embedding $\epsilon$. For example, the results of doubling two $Z_2$ subgroups in $O(2)$, $I := \{1, \sigma\}, R := \{1, R(\pi)\}$ where $\sigma$ denotes reflection along $y$-axis and $R(\pi)$ is the rotation over $\pi$, are $\pi^{-1}(I) \cong Z_2 \otimes Z_2$ while $\pi^{-1}(R) \cong Z_4$. Nevertheless, we will use symbol $\bar{G}$ to denote the double group at most cases where $n$ and $\epsilon$ are fixed, and will not distinguish $G$ from $\epsilon(G)$. Meanwhile, symbol $\bar{e}$ is adopted to refer $-1$ in Clifford algebra and is called central element.

Let $s : G \to \bar{G}$, s.t. $\pi s = \text{Id}_G$, namely $s$ is a cross-section of $\pi$. There is a property of the conjugate classes of $\bar{G}$ which is easy to verify.

**Lemma 4** Let $C$ be a conjugate class in $G$, then either will $\pi^{-1}(C)$ be one conjugate class in $\bar{G}$ satisfying $\forall g \in C, s(g) \sim -s(g)$; or it will split into two conjugate classes $C_1, C_2$ in $\bar{G}$ s.t. $\forall g \in C, s(g) \in C_1 \Leftrightarrow -s(g) \in C_2$.

We will give a more deep result on the splitting of conjugate classes when doubling $G$ to $\bar{G}$ in our another paper.

Let $r$ be an irreducible representation of $\bar{G}$ on $L$, then $r(-1) = \pm 1$.

**Definition 2** An irreducible representation of $\bar{G}$ with $r(-1) = 1$ is called a single-valued representation of $G$ while an irreducible representation with $r(-1) = -1$ is called a spinor representation or two-valued representation of $G$.

**Proposition 6** Let $\text{IRR}_C(G)$ be the class of all inequivalent irreducible representations of $G$ and $\text{IRR}_C(G)^* \equiv \text{IRR}_C(G)$ be the class of all inequivalent single-valued representations of $G$, define $\phi : \text{IRR}_C(G) \to \text{IRR}_C(G)^*, r \mapsto r \circ \pi$. Then $\phi$ is a bijection.
Proof:
One can check: $r \circ \pi$ is a representation of $\bar{G}$; if $r \cong r'$, then $r \circ \pi \cong r' \circ \pi$; that $r$ is irreducible implies that $r \circ \pi$ is irreducible and $r \circ \pi$ is single-valued. Therefore, $\phi$ is well-defined. If $r$ and $r'$ are inequivalent, then $r \circ \pi$ and $r' \circ \pi$ are two elements in $\text{IRR}_c(G)^s$, namely $\phi$ is injective. To prove that $\phi$ is a surjection, consider any $\tilde{r} \in \text{IRR}_c(G)^s : \bar{G} \to L$. Define $r : G \to L, g \mapsto \tilde{r}(s(g))$ where $s(g)$ is any element in $\pi^{-1}(g)$. One can check: $r$ is a well-defined map since $\tilde{r}$ is single-valued; $r$ is an irreducible representation of $G$ on $L$, accordingly $r \in \text{IRR}_c(G)$ and lastly, $\phi(r) = \tilde{r}$. So the result follows.

This proposition says that all single-valued representations of $G$ which are part of inequivalent irreducible representations of $\bar{G}$ are completely determined by the representation theory of $G$.

III Structure of $O_4$

III.1 Structure of $O_4$

It follows Proposition 2 that $O_4 \cong Z_2^4 \triangleleft S_4$; hence $|O_4| = 384$. In point group theory, rotation subgroup of $O_h$ is denoted as $O$; on the other hand, $S_4 \cong Z_2^2 \triangleleft S_3 \cong O$ \[1\]. We write the isomorphism explicitly. The structure of $Z_2^2 \triangleleft S_3$ is given by four generators $\alpha, \beta, \eta, t$ and the relations

\begin{align*}
\alpha^2 &= e, \beta^2 = e, \alpha \beta = \beta \alpha \quad (23) \\
t^3 &= e, \eta^2 = e, \eta t = t^2 \eta \quad (24) \\
t \alpha &= \alpha \beta t, t \beta = \alpha t, \eta \alpha = \beta \eta \quad (25)
\end{align*}

and the isomorphisms are defined to be

\[(12)(34) \leftrightarrow \alpha \leftrightarrow \text{diag}(-1, -1, 1), (13)(24) \leftrightarrow \beta \leftrightarrow \text{diag}(1, -1, -1)\]
Structure and Representations for $O_4$

(234) $\leftrightarrow t \leftrightarrow \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}$, (23) $\leftrightarrow \eta \leftrightarrow \begin{pmatrix} 0 & 0 & -1 \\ 0 & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$

The structure of $Z_4^1 \rtimes S_4$ is given by (23)(24)(25) together with (see Eq.(6))

$I^2 = e, I_i I_j = I_j I_i, i, j = 1..4, i \neq j$  \hspace{1cm} (26)

$\alpha I_1 = I_2 \alpha, \alpha I_3 = I_4 \alpha$

$\beta I_1 = I_3 \beta, \beta I_2 = I_4 \beta$

$t I_1 = I_1 t, t I_2 = I_4 t, t I_3 = I_2 t, t I_4 = I_3 t$  \hspace{1cm} (27)

$\eta I_1 = I_1 \eta, \eta I_2 = I_3 \eta, \eta I_3 = I_4 \eta$

The matrix representations of above generators are given by (see Eq.(5))

$\begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 \end{pmatrix}$ $\leftrightarrow \alpha$

$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\leftrightarrow \beta$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 0 \end{pmatrix}$ $\leftrightarrow t$

$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\leftrightarrow \eta$

$\begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ $\leftrightarrow \gamma$

In fact, if we introduce

then the generators of $O_4$ can be reduced to a smaller set $\{I_i, \gamma, t|i = 1, 2, 3, 4\}$ whose generating relations are (26)(27) together with

$\gamma^2 = e, t^3 = e, (t \gamma)^4 = e$  \hspace{1cm} (30)

$\gamma I_1 = I_3 \gamma, \gamma I_2 = I_2 \gamma, \gamma I_4 = I_4 \gamma$  \hspace{1cm} (31)
while $\alpha = (t^2\gamma)^2, \beta = t\gamma t^2\gamma t, \eta = \gamma t\gamma t^2\gamma$.

Applying the general results on conjugate classification of $O_n$ Eqs. (15)(16)...(19), we give the table of conjugate classes of $O_4$ (see Tab. [1]).

### III.2 Construction of $\overline{O_4}$

We will denote $s(g)$ still as $g$ for all $g \in G$. $\overline{O_4}$ is generated by the equations below.

**Proposition 7**

$$I_i^2 = -1, I_i I_j = -I_j I_i, i, j = 1..4, i \neq j$$

$$\gamma^2 = -1, t^3 = -1, (t\gamma)^4 = -1$$

$$\gamma I_1 = -I_3\gamma, \gamma I_2 = -I_2\gamma, \gamma I_4 = -I_4\gamma$$

$$tI_1 = I_1 t, tI_2 = I_4 t, tI_3 = I_3 t, tI_4 = I_3 t$$

**Proof:**

First Eqs. (32)...(35) are valid. In fact, the standard orthogonal bases in $E^4$ satisfy Clifford relations $e_i e_j + e_j e_i = -2\delta_{ij}$ which is equivalent to (32); therefore, one can take $I_i = e_i$. Following lemma 2, we set $\gamma = \frac{1}{\sqrt{2}}(e_3 - e_1)$ and check that (34) is satisfied. Let $t = \frac{1}{2}(1 - e_2 e_3 + e_2 e_4 - e_3 e_4)$ which is the product of $\frac{1}{\sqrt{2}}(e_2 - e_3)$ and $\frac{1}{\sqrt{2}}(e_4 - e_2)$, and (33) can be verified. Finally, one can check that (33) is also satisfied.

Second, notice that above equations are just Eqs. (26)(27)(30)(31), which generate $O_4$, twisted by a $Z_2$ factor set. So due to the validity of the above equations, $\forall g \in \overline{O_4}$, either $g$ or $-g$ will be generated. But $-1$ can be generated. Therefore, the above equation set generations $\overline{O_4}$.

$\blacksquare$

We can give another proof of this result by proposition 3. In fact, we introduce $\gamma$-matrices in $E^4$ as

$$\gamma_i = \begin{pmatrix} 0_{2 \times 2} & i\sigma_i \\ i\sigma_i & 0_{2 \times 2} \end{pmatrix}, i = 1, 2, 3; \gamma_4 = \begin{pmatrix} 0_{2 \times 2} & -1_{2 \times 2} \\ 1_{2 \times 2} & 0_{2 \times 2} \end{pmatrix}$$
in which $\sigma_i$ stand for three Pauli matrices

\[
\sigma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad \sigma_2 = \begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \quad \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\]

Note that $\sigma_2$ in our convention is different from the usual definition in physics. $\gamma_i (i = 1..4)$ satisfy Clifford relations

$$\gamma_i \gamma_j + \gamma_j \gamma_i = -2\delta_{ij} \mathbf{1}_{4 \times 4}$$

and

$$\gamma_i^\dagger = -\gamma_i, \quad \gamma_i \gamma_i^\dagger = \mathbf{1}_{4 \times 4}, \quad det(\gamma_i) = 1.$$

We use $S(g)$ as the image of $s(g)$ in $M_2(H)$. Let

$$S(I_i) = \gamma_i, S(\gamma) = \frac{i}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, S(t) = \frac{e^{i\pi/4}}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & i & 1 \\ 0 & 0 & -i & 1 \end{pmatrix}$$

then one can check that these matrices give correct images under $\tilde{A}d$ and satisfy the corresponding relations in (32)...(35). It should be noticed that the $\tilde{A}d$-map condition can fix these matrices up to a non-vanishing scalar and that by using lemma 3 the scalar can be fixed up to a $Z_4$ uncertainty, namely if one searches out a $S(g)$ then $iS(g), -S(g), -iS(g)$ will also work. One can figure out two of them by calculating the projections on the basis of $Cl(E^4)$ and ruling out those whose projections are pure imaginary.

We point out that the generating relations in proposition 7 are not unique, due to the canonical automorphism of $Cl(E^4)$. In fact from the second proof of this proposition, we have notified that at last there is still a $Z_2$ uncertainty. Consequently, we can change the cross-section $s$ to another one $s'$ by a “local” $Z_2$ transformation and the underlined equations in Eqs.(32)...(35) may gain or lose some $(-1)$-factors accordingly. Anyway, they are equivalent to the former ones.

To classify the elements in $O_4$, Lemma 4 will enable us to use the same symbols for the conjugate classes of $O_4$ and to use a “$r$” for those splitting classes. Except for classes 1, 8, 14, 15, 20 which
split into two classes for each, any other class in $O_4$ is lifted to one class. Therefore, there are totally 25 classes in $\overline{O_4}$ (see table [I]).

IV Representations of $\overline{O_4}$

IV.1 Single-valued representations of $O_4$

Due to Theorem 6, there are totally 20 inequivalent single-valued representations of $O_4$ corresponding to the 20 inequivalent irreducible representations of $O_4$; the representation theory of $O_4$ can be systematically solved by applying little group method (Proposition [I]).

All inequivalent irreducible characters of $Z_2^4$ are listed in Table [II]. Following Theorem [I], $\Pi(Z_2^4)$ are partitioned into orbits with index set defined in a physical convention $I := \{S, P, V, A, T\}$.

\[
\begin{align*}
\Pi_S &= \{\pi_{0000}\}, F_S \cong S_4; \\
\Pi_P &= \{\pi_{1111}\}, F_P \cong S_4; \\
\Pi_V &= \{\pi_{0001}, \pi_{0010}, \pi_{0100}, \pi_{1000}\}, F_V \cong S_3; \\
\Pi_A &= \{\pi_{1110}, \pi_{1101}, \pi_{1011}, \pi_{0111}\}, F_A \cong S_3; \\
\Pi_T &= \{\pi_{0011}, \pi_{0101}, \pi_{1001}, \pi_{0110}, \pi_{1010}, \pi_{1100}\}, F_T \cong Z_2^2
\end{align*}
\]

We will use $[\lambda]$ instead of $(\nu)$ to denote Young diagrams where $[\lambda] = [\lambda_1 \lambda_2 ... \lambda_n], \lambda_k = \sum_{i=k}^{n} \nu_i$.

Orbit $S$

All inequivalent irreducible representations of $S_4$ is labeled by $[4], [31], [2^2], [21^2], [1^4]$; accordingly,

\[
\Pi_S \cdot ([4], [31], [2^2], [21^2], [1^4])
\]

provide two one-dimensional, one two-dimensional and two three-dimensional representations. As for representation matrices, all $I_i, i = 1..4$ are mapped to identity, while $\alpha, \beta, t, \eta$ take the same matrix
form as they have in $S_4$, i.e. $\Pi_S \cdot [4] : I_i, \alpha, \beta, t, \eta \rightarrow 1$; $\Pi_S \cdot [1^4] : I_i, \alpha, \beta, t \rightarrow 1, \eta \rightarrow -1$;

$$\Pi_S \cdot [2^2] : I_i, \alpha, \beta \rightarrow 1_{2\times2}, t \rightarrow \begin{pmatrix} e^{i\frac{2\pi}{3}} & 0 \\ 0 & e^{i\frac{4\pi}{3}} \end{pmatrix}, \eta \rightarrow \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix};$$

$$\Pi_S \cdot [31] : I_i \rightarrow 1_{3\times3}, \alpha \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \beta \rightarrow \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}, t \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \eta \rightarrow \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix};$$

$\Pi_S \cdot [21^2]$: $I_i, \alpha, \beta, t$ take the same form of $\Pi_S \cdot [31]$ and $\eta$ gains a minus sign compared to $\Pi_S \cdot [31]$.

**Orbit** $P$

$$\Pi_P \cdot ([4], [31], [2^2], [21^2], [1^4])$$

The only difference from **Orbit** $S$ is that $I_i$ are mapped to $-1$.

**Orbit** $V$

All inequivalent irreducible representations of $S_3$ can be written as $[3], [21], [1^3]$ and it has no difficulty, using our generating relations, to check

$$\pi_{1000} \alpha^{-1} = \pi_{0100}, \eta \pi_{0100} \eta^{-1} = \pi_{0010}, \alpha \pi_{0010} \alpha^{-1} = \pi_{0001}$$

Hence, this orbit gives two four-dimensional representations and one eight-dimensional representation.

$$\Pi_V \cdot ([3], [21], [1^3]) = (e, \alpha, \beta \eta, \alpha \beta \eta) \cdot \pi_{1000} \cdot ([3], [21], [1^3])$$

The representation matrices of $\Pi_V \cdot [3]$ are coincident with those in Eqs.(28)(29). Representation matrices of $\Pi_V \cdot [1^3]$ are the same as those in $\Pi_V \cdot [3]$, except that $\eta$ picking on a minus sign.

$$\Pi_V \cdot [21] :$$

$$e_i \rightarrow \begin{pmatrix} \Pi_V \cdot [3](e_i) & 0_{4\times4} \\ 0_{4\times4} & \Pi_V \cdot [3](e_i) \end{pmatrix},$$
\[
\begin{align*}
\alpha & \rightarrow \begin{pmatrix}
\Pi_V \cdot [3](\alpha) & 0_{4 \times 4} \\
0_{4 \times 4} & \Pi_V \cdot [3](\alpha)
\end{pmatrix}, \\
\beta & \rightarrow \begin{pmatrix}
0_{4 \times 4} & \Pi_V \cdot [3](\beta) \\
\Pi_V \cdot [3](\beta) & 0_{4 \times 4}
\end{pmatrix}, \\
t & \rightarrow \begin{pmatrix}
e^{\frac{2\pi i}{3}} \cdot \Pi_V \cdot [3](t) & 0_{4 \times 4} \\
0_{4 \times 4} & e^{\frac{2\pi i}{3}} \cdot \Pi_V \cdot [3](t)
\end{pmatrix}, \\
\eta & \rightarrow \begin{pmatrix}
0_{4 \times 4} & \Pi_V \cdot [3](\eta) \\
\Pi_V \cdot [3](\eta) & 0_{4 \times 4}
\end{pmatrix},
\end{align*}
\]

**Orbit A**

Similar to **Orbit V**, there are two four-dimensional representations and one eight-dimensional representation.

\[
\Pi_A \cdot ([3], [21], [1^3]) = (e, \alpha, \beta \eta, \alpha \beta \eta) \cdot \pi_{0111} \cdot ([3], [21], [1^3])
\]

while the representation matrices for \( I_i \) pick on a minus sign, without changing the others.

**Orbit T**

The stationary subgroup \( F_T \) leaving \( \pi_{0110} \) invariant is \( \{ e, \eta, \alpha \beta, \alpha \beta \eta \} \) with four one-dimensional irreducible representations, denoted by \( \pi_{(a,b)}, a, b = 0, 1 \). Therefore, there are four six-dimensional representations given by this orbit. Notice that

\[
\alpha \pi_{0110} \alpha^{-1} = \pi_{1001}, t\pi_{1001}t^{-1} = \pi_{1010}, t\pi_{1010}t^{-1} = \pi_{1100},
\]

\[
\alpha \pi_{1010} \alpha^{-1} = \pi_{0101}, \beta \pi_{1100} \beta^{-1} = \pi_{0011},
\]

the four representations can be labeled as

\[
\Pi_T \cdot (\pi_{00}, \pi_{01}, \pi_{10}, \pi_{11}) = (e, \alpha, \alpha \beta t, \beta t^2) \cdot \pi_{0110} \cdot (\pi_{00} + \pi_{01} + \pi_{10} + \pi_{11})
\]

Then we enumerate the matrices for the four representations.

\[
\Pi_T \cdot \pi_{00} : I_1 \rightarrow diag(1, -1, -1, -1, 1, 1), I_2 \rightarrow diag(-1, 1, 1, -1, -1, 1),
\]

\[
I_3 \rightarrow diag(-1, 1, -1, 1, 1, -1), I_4 \rightarrow diag(1, -1, 1, -1, -1, -1),
\]
\[ \begin{align*}
\alpha &\rightarrow \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix} & & \beta &\rightarrow \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 
\end{pmatrix} \\
\tau &\rightarrow \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix} & & \eta &\rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 
\end{pmatrix} \\
\Pi_T \cdot \pi_{01} : I_1 &\rightarrow \Pi_T \cdot \pi_{00}(I_1), I_2 &\rightarrow \Pi_T \cdot \pi_{00}(I_2), I_3 &\rightarrow \Pi_T \cdot \pi_{00}(I_3), I_4 &\rightarrow \Pi_T \cdot \pi_{00}(I_4)
\end{align*} \]
\[ \begin{align*}
\alpha &\rightarrow \begin{pmatrix}
0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 
\end{pmatrix} & & \beta &\rightarrow \begin{pmatrix}
0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 
\end{pmatrix} \\
\tau &\rightarrow \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 
\end{pmatrix} & & \eta &\rightarrow \begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 
\end{pmatrix} \\
\Pi_T \cdot \pi_{10} : I_1 &\rightarrow \Pi_T \cdot \pi_{00}(I_1), I_2 &\rightarrow \Pi_T \cdot \pi_{00}(I_2), I_3 &\rightarrow \Pi_T \cdot \pi_{00}(I_3), I_4 &\rightarrow \Pi_T \cdot \pi_{00}(I_4)
\end{align*} \]
Here we find all 20 inequivalent irreducible representations corresponding to the 20 conjugate classes of $O_4$, which satisfy Burside formula

$$2 \times (1^2 + 1^2 + 2^2 + 3^2 + 3^2) + 2 \times (4^2 + 4^2 + 8^2) + 4 \times 6^2 = 384$$

Following proposition $\S$, we have found all of the single-valued representations of $O_4$.

### IV.2 Spinor representations of $O_4$

Notice the following facts that $\mathbb{Z}_2^4 \triangleleft O_4$, $O_4/\mathbb{Z}_2^4 \cong S_4$ and Eqs.($\S\S\S\S$) generate a spinor representation of $O_4$ which is denoted still as $S$; what’s more, its restriction to $\mathbb{Z}_2^4$ is also a two-valued representation of $\mathbb{Z}_2^4$. These facts ensure two conditions in Theorem $\S$. To apply Theorem $\S$ to deduce spinor representations of $O_4$, we develop a calculation method. The matrices of a spinor representation of $O_4$ for $I, \gamma, t$, denoted as $\tilde{S}(I), \tilde{S}(\gamma), \tilde{S}(t)$, can be decomposed as

$$\tilde{S}(I_i) = S(I_i) \otimes 1, i = 1, 3, 4; \tilde{S}(I_2) = -S(I_2) \otimes 1;$$

$$\tilde{S}(\gamma) = \Gamma \otimes \tilde{\gamma}; \tilde{S}(t) = T \otimes \tilde{t}$$

where $\Gamma, T$ and $S(I_i)$ act on the same module, $\tilde{\gamma}, \tilde{t}$ have the same texture (zero matrix elements) of the representation matrices of five inequivalent irreducible representations of $S_4$ (the minus added before $S(I_2)$ is for a physical convention). There are five spinor representations of dimension 4, 4, 8, 12 and 12 respectively and the second half of Burside formula is satisfied.

$$4^2 + 4^2 + 8^2 + 12^2 + 12^2 = 384$$

Corresponding the generating equations ($\S\S\S\S$)...($\S\S\S\S$), there are a system of matrix equations.

$$\tilde{S}(\gamma)^2 = \tilde{S}(t)^3 = -1, (\tilde{S}(\gamma) \tilde{S}(t))^4 = -1$$

(37)
\[ S(\gamma) \tilde{S}(I_2) = -\tilde{S}(I_2) S(\gamma), \quad S(\gamma) \tilde{S}(I_4) = -\tilde{S}(I_4) S(\gamma), \quad S(\gamma) \tilde{S}(I_1) = -\tilde{S}(I_3) S(\gamma), \quad (38) \]

\[ \tilde{S}(t) \tilde{S}(I_1) = \tilde{S}(I_1) \tilde{S}(t), \quad \tilde{S}(t) \tilde{S}(I_2) = -\tilde{S}(I_4) \tilde{S}(t), \quad \tilde{S}(t) \tilde{S}(I_3) = -\tilde{S}(I_2) \tilde{S}(t), \quad \tilde{S}(t) \tilde{S}(I_4) = \tilde{S}(I_3) \tilde{S}(t), \quad (39) \]

plus a unitary condition

\[ \tilde{S}(\gamma) \dag \tilde{S}(\gamma) = 1, \quad \tilde{S}(t) \dag \tilde{S}(t) = 1 \quad (40) \]

Note that we add a minus sign to the second and the third equations in Eq. (39) compared with Eq. (35) according to the same physics convention, though they are completely equivalent.

Solving Eqs. (37)...(40) for four-dimensional case gives two solutions

\[ 4_+ : \tilde{S}(\gamma) = \Gamma \cdot \tilde{\gamma}_+, \Gamma = \frac{1}{\sqrt{2}} \begin{pmatrix} 0 & 0 & 1 & -1 \\ 0 & 0 & -1 & -1 \\ 1 & -1 & 0 & 0 \\ -1 & -1 & 0 & 0 \end{pmatrix}, \quad \tilde{\gamma}_+ = i \]

\[ \tilde{S}(t) = T \cdot \tilde{t}, T = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i & 0 & 0 \\ 1 & i & 0 & 0 \\ 0 & 0 & \tilde{d} & 1 \\ 0 & 0 & -i & 1 \end{pmatrix}, \quad \tilde{t} = e^{i\frac{\pi}{4}} \]

\[ 4_- : \tilde{S}(\gamma) = \Gamma \cdot \tilde{\gamma}_-, \tilde{\gamma}_- = -i, \tilde{S}(t) = T \cdot \tilde{t} \]

Note that \( 4_+ \) is just the representation \( S \) with \( \tilde{S}(I_2) = -S(I_2) \).

As for eight-dimensional case we can suppose

\[ \tilde{\gamma} = \begin{pmatrix} 0 & \tilde{c} \\ \tilde{d} & 0 \end{pmatrix}, \tilde{t} = \begin{pmatrix} \tilde{a} & 0 \\ 0 & \tilde{d} \end{pmatrix} \]

The solution \( 8 \) is given by

\[ \tilde{c} = e^{i\frac{\pi}{4}}, \tilde{d} = e^{i\frac{\pi}{4}}, \tilde{a} = e^{i\frac{\pi}{12}}, \tilde{d} = e^{i\frac{11\pi}{12}} \]
Finally, we set for the twelve-dimensional case

\[
\tilde{\gamma} = \begin{pmatrix}
0 & 0 & \tilde{z} \\
0 & \tilde{y} & 0 \\
\tilde{x} & 0 & 0
\end{pmatrix}, \tilde{t} = \begin{pmatrix}
0 & 0 & \tilde{n} \\
\tilde{l} & 0 & 0 \\
0 & \tilde{m} & 0
\end{pmatrix}
\]

Such that

\[
12_+ : \tilde{x} = 1, \tilde{y} = i, \tilde{z} = -1, \tilde{l} = 1, \tilde{m} = 1, \tilde{n} = e^{i \frac{5\pi}{4}}
\]

\[
12_- : \tilde{x} = 1, \tilde{y} = -i, \tilde{z} = -1, \tilde{l} = 1, \tilde{m} = -1, \tilde{n} = e^{i \frac{3\pi}{4}}
\]

So far, we obtain all inequivalent irreducible representations of $\overline{O}_4$ and we summarize our results in Table IV.

**V Structure of $\overline{SO}_4$**

Specify $n = 4$ in Eq.(21) then we know immediately that $|SO_4| = 192$. Introduce

\[
\eta = \gamma t \gamma t^2 \gamma; \quad (41)
\]

\[
\alpha = (t^2 \gamma)^2, \beta = t \gamma t^2 \gamma t; \quad (42)
\]

\[
x = e_1 \eta, y = e_4 \eta, q = e_2 \eta \quad (43)
\]

then the structure of $SO_4$ can be summarized as

\[
x^2 = y^2 = q^4 = e, yx = xy, qx = xq^3, qy = yq^3 \quad (44)
\]

\[
\alpha^2 = \beta^2 = t^3 = e, \beta \alpha = \alpha \beta, t \alpha = \alpha \beta t, t \beta = \alpha t \quad (45)
\]

\[
\alpha x = q \beta, \alpha y = q^3 \beta, \alpha q = x \beta \quad (46)
\]

\[
\beta x = q^3 \alpha, \beta y = q \alpha, \beta q = y \alpha \quad (47)
\]

\[
tx = xt^2, ty = q^3 t^2, tq = yt^2 \quad (48)
\]

Together with

\[
x \alpha = \beta q^3, xt = t^2 x; y \beta = \alpha q^3, yt = t^2 q^3; qt = t^2 y \quad (49)
\]
Accordingly, each group element can be expressed as a “normal ordering” product of $x, y \to q \to \alpha, \beta \to t$ and their powers from left to right. Throwing away all classes which belong to $O_4$ but not to $SO_4$, there are 11 left which are 1, 3, 5, 7, 8, 11, 12, 14, 15, 18, 20 in Table I. The 14th and the 20th will part into two classes with equal numbers of elements each under adjoint action of $SO_4$ which are denoted as 14, 14’, 20, 20’. Therefore, there are 13 conjugate classes in $SO_4$.

Due to the fact that $SO_4 \triangleright O_4$, the diagram

$$
\begin{array}{c}
0 \longrightarrow Z_2 \longrightarrow O_4 \longrightarrow O_4 \\
\| \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad 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brief observation gives some important information. Firstly, \(1_1 \cong 1_4, 1_2 \cong 1_3, 2_1 \cong 2_2, 3_1 \cong 3_4, 3_2 \cong 3_3, 4_1 \cong 4_2 \cong 4_3, 8_1 \cong 8_2, 6_1 \cong 6_3\). Secondly, omitting equivalence, \(1_1, 1_2, 2_1, 3_1, 3_2, 4_1, 4_2, 8_1, 6_1\) remain irreducible within \(\overline{SO_4}\), while other 7 become reducible. Thirdly, as for each of these reducible ones, the inner product of the character with itself equals to 2, implying that it can be reduced to two inequivalent irreducible representations, thus there are 13 single-valued and 10 spinor representations as we expected. Finally, it is one possible solution to Burside theorem that each of these seven reducible representations splits into 2 inequivalent irreducible representations with equal dimensions. We conjecture that it is the solution to our representation theory of \(\overline{SO_4}\) and try to verify it below.

Summarily speaking, there are nine single-valued inequivalent irreducible representationss inherited from \(\overline{O_4}\)

\[
1_1, 1_2, 2 \equiv 2_1, 3_1, 3_2, 4_1, 4_2, 8 \equiv 8_1, 6 \equiv 6_1
\]

and we conjecture the splitting relations

\[
\begin{align*}
6_2, 6_4 & \rightarrow 3_\alpha, 3_\beta, 3_\gamma, 3_\delta \\
4_1, 4_2 & \rightarrow 2_\alpha, 2_\beta, 2_\gamma, 2_\delta \\
8 & \rightarrow 4_\alpha, 4_\beta \\
12_1, 12_2 & \rightarrow 6_\alpha, 6_\beta, 6_\gamma, 6_\delta
\end{align*}
\]
VI.1 Hidden single-valued representations

The representation matrices of $x$, $y$, $q$ in $6_2$ are written as

$$
x \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix},
$$

$$
y \mapsto \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix},
$$

$$
q \mapsto \begin{pmatrix} -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & -1 & 0 \end{pmatrix}.
$$

The textures of these matrices inspire us to such a hypotheses that in $3_{\alpha,\beta,\gamma,\delta}$, $x, y, q$ take on a form like

$$
x, y \mapsto \begin{pmatrix} \pm 1 \\ \pm H \end{pmatrix}, q \mapsto \begin{pmatrix} \pm 1 \\ \pm Q \end{pmatrix},
$$

where $H \equiv \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, Q \equiv \begin{pmatrix} 0 & -1 \\ -1 & 0 \end{pmatrix}$. After taking account of the conjugate equivalence, only four possibilities survive from the totally 64, i.e.

$$
I : x \to \begin{pmatrix} 1 \\ H \end{pmatrix}, y \to \begin{pmatrix} 1 \\ -H \end{pmatrix}, q \to \begin{pmatrix} -1 \\ Q \end{pmatrix} \tag{50}
$$

$$
II : x \to \begin{pmatrix} -1 \\ H \end{pmatrix}, y \to \begin{pmatrix} -1 \\ -H \end{pmatrix}, q \to \begin{pmatrix} 1 \\ Q \end{pmatrix} \tag{51}
$$

$$
III : x \to \begin{pmatrix} 1 \\ -H \end{pmatrix}, y \to \begin{pmatrix} 1 \\ H \end{pmatrix}, q \to \begin{pmatrix} -1 \\ Q \end{pmatrix} \tag{52}
$$

$$
IV : x \to \begin{pmatrix} -1 \\ -H \end{pmatrix}, y \to \begin{pmatrix} -1 \\ H \end{pmatrix}, q \to \begin{pmatrix} 1 \\ Q \end{pmatrix} \tag{53}
$$

which also satisfy Eq.(44). Then we regard $\alpha, \beta, t$ as unknowns, (45)-(49) as constraint, and solve these matrix equations. Modulo similarity, each of Eqs.(50)-(53) gives two solutions, labeled
as $I, I', II, II', III, III', IV, IV'$; however, there is no difficulty to find out that $I \cong III$, $I' \cong III'$, $II \cong IV$, $II' \cong IV'$. Thus $I, I', II, II'$ are what we need.

\[
3_\alpha \equiv I : \alpha \to \text{diag}(-1, 1, -1), \beta \to \text{diag}(1, -1, -1), t \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
3_\beta \equiv I' : \alpha \to \text{diag}(1, -1, -1), \beta \to \text{diag}(-1, 1, -1), t \to \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}
\]

\[
3_\gamma \equiv II : \alpha \to \text{diag}(-1, 1, -1), \beta \to \text{diag}(1, -1, -1), t \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}
\]

\[
3_\delta \equiv II' : \alpha \to \text{diag}(1, -1, -1), \beta \to \text{diag}(-1, 1, -1), t \to \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \end{pmatrix}
\]

\section*{VI.2 Spinor representations}

It is more straightforward to reduce out the spinor representations. We recall that the spinor representation matrices of $O_4$ are of the form of tensor product

\[
S_i(g) = S(g) \otimes s_i(g), \forall g \in O_4, i = 4_1, 4_2, 8, 12_1, 12_2
\]

where $S$ is given by the algebraic isomorphism from $Cl(E^4)$ to $M_2(H)$ and $s_i$ has the same texture (zero matrix element positions) of irreducible representation $i$ of $S_4$. Additionally, for $g$ in $SO_4$, $S(g)$ takes on a 2-by-2 block diagonal form

\[
S(g) = \begin{pmatrix} S_{up}(g) & 0 \\ 0 & S_{down}(g) \end{pmatrix}
\]
So it is just what we want

\[ S_{up}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\frac{3}{4})\pi} & e^{i(-\frac{5}{4})\pi} \\ e^{i(\frac{1}{4})\pi} & e^{i(-\frac{1}{4})\pi} \end{pmatrix}, S_{up}(y) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\frac{3}{4})\pi} & e^{i(\frac{1}{4})\pi} \\ e^{i(-\frac{5}{4})\pi} & e^{i(-\frac{1}{4})\pi} \end{pmatrix} \] (55)

\[ S_{up}(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\frac{1}{4})\pi} & e^{i(\frac{3}{4})\pi} \\ e^{i(-\frac{1}{4})\pi} & e^{i(-\frac{3}{4})\pi} \end{pmatrix}, S_{up}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & -i \\ -i & 1 \end{pmatrix} \] (56)

\[ S_{up}(\alpha) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix}, S_{up}(\beta) = \begin{pmatrix} -i & 0 \\ 0 & i \end{pmatrix} \] (57)

\[ S_{down}(x) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\frac{1}{4})\pi} & e^{i(-\frac{5}{4})\pi} \\ e^{i(\frac{3}{4})\pi} & e^{i(-\frac{1}{4})\pi} \end{pmatrix}, S_{down}(y) \rightarrow \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(-\frac{1}{4})\pi} & e^{i(\frac{1}{4})\pi} \\ e^{i(-\frac{3}{4})\pi} & e^{i(\frac{5}{4})\pi} \end{pmatrix} \] (58)

\[ S_{down}(q) = \frac{1}{\sqrt{2}} \begin{pmatrix} e^{i(\frac{1}{4})\pi} & e^{i(\frac{3}{4})\pi} \\ e^{i(-\frac{1}{4})\pi} & e^{i(-\frac{3}{4})\pi} \end{pmatrix}, S_{down}(t) = \frac{1}{\sqrt{2}} \begin{pmatrix} i & 1 \\ -i & 1 \end{pmatrix} \] (59)

\[ S_{down}(\alpha) = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}, S_{down}(\beta) = \begin{pmatrix} 0 & -i \\ -i & 0 \end{pmatrix} \] (60)

Keeping the second factor unchanged, each spinor representation in \( \overline{O}_4 \) splits into two spinor representations in \( \overline{SO}_4 \), denoted as \( 2_\alpha, 2_\beta, 2_\gamma, 2_\delta, 4_\alpha, 4_\beta, 6_\alpha, 6_\beta, 6_\gamma, 6_\delta \).

In fact, \( S(g) \) falls in the so-called “chiral”-representation of \( Cl(E^4) \) in physical language. Due to \( Cl(V) = Cl(V)_e \oplus Cl(V)_o \), and the choice of chiral-representation, there are

\[ Cl(E^4)_e \cong \begin{pmatrix} H & O \\ 0 & H \end{pmatrix}, Cl(E^4)_o \cong \begin{pmatrix} 0 & H \\ H & 0 \end{pmatrix} \]

Notice that \( \overline{SO}_4 < Spin(4) \subset Cl(E^4)_e \), so our reducing process for spinor representations roots in the structure of Clifford algebra.

Conclusively, our conjecture gives all inequivalent irreducible representations of \( \overline{SO}_4 \) whose characters are summarized in Table [VI].
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A Fundamental lemma of n-dimensional Euclidean geometry

Lemma A-1 (Weak form) Let \( p_i, i = 0, 1, 2, ..., n \) be \( n + 1 \) points in \( n \)-dimensional Euclidean space \( E^n \) which are non-collinear and give \( n + 1 \) non-negative real numbers \( d_i, i = 0, 1, 2, ..., n \), then there exists at most one point \( p \in E^n \) s.t. \( d(p, p_i) = d_i \).

Proof:
Without losing generality, set \( p_0 = (0, 0, ..., 0) \) and understand \( p, p_i, i = 1, 2, ..., n \) as vectors in \( E^n \). Consider equation set

\[
(p - p_i, p - p_i) = d_i^2, \quad i = 1, 2, ..., n \tag{A-1}
\]
\[
(p, p) = d_0^2 \tag{A-2}
\]

where \((,)\) is standard inner product in \( E^n \). Substitute (A-2) into (A-1)

\[
(p_i, p) = \frac{1}{2}(d_0^2 - d_i^2 + (p_i, p_i)), \quad i = 1, 2, ..., n \tag{A-3}
\]

The non-collinearity implies that (A-3) has a solution \( p \). The weak form of fundamental lemma of Euclidean geometry follows.
\( \square \)

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|   |   |
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"SplitNo" reflects the relation between the classes of $Z_2 \rtimes S_4$ and those of $S_4$. "ord" means order of each class. "num" is the number of elements in each class. "det" is the signature of each class. See Eqs.(16)…(19).

Table I: Conjugate Classes of $Z_2^4 >\triangleleft S_4$

| No | SplitNo | YoungDiagram | ord | num | det | No | SplitNo | YoungDiagram | ord | num | det |
|----|---------|--------------|-----|-----|-----|----|---------|--------------|-----|-----|-----|
| 1  | 1-1     |               | 1   | 1   | 1   | 2  | 1-2     |               | 2   | 4   | -1  |
| 3  | 1-3     |               | 2   | 6   | 1   | 4  | 1-4     |               | 2   | 4   | -1  |
| 5  | 1-5     |               | 2   | 1   | 1   |    |         |               |     |     |     |
| 6  | 2-1     |               | 2   | 12  | -1  | 7  | 2-2     |               | 2   | 24  | 1   |
| 8  | 2-3     |               | 4   | 12  | 1   | 9  | 2-4     |               | 2   | 12  | -1  |
| 10 | 2-5     |               | 4   | 24  | -1  | 11 | 2-6     |               | 4   | 12  | 1   |
| 12 | 3-1     |               | 2   | 12  | 1   | 13 | 3-2     |               | 4   | 24  | -1  |
| 14 | 3-3     |               | 4   | 12  | 1   |    |         |               |     |     |     |
| 15 | 4-1     |               | 3   | 32  | 1   | 16 | 4-2     |               | 6   | 32  | -1  |
| 17 | 4-3     |               | 6   | 32  | -1  | 18 | 4-4     |               | 6   | 32  | 1   |
| 19 | 5-1     |               | 4   | 48  | -1  | 20 | 5-2     |               | 8   | 48  | 1   |
Table II: Conjugate Classes of $O_4$

| No. | 1   | 1'  | 2   | 3   | 4   | 5   | 6   | 7   | 8   | 8'  | 9   | 10  | 11  | 12  | 13  | 14  | 14' | 15  | 15' | 16  | 17  | 18  | 19  | 20  | 20' |
|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| num.| 1   | 1   | 8   | 12  | 8   | 2   | 24  | 48  | 12  | 12  | 24  | 48  | 24  | 24  | 48  | 12  | 12  | 32  | 32  | 64  | 64  | 64  | 96  | 48  | 48  |
| ord.| 1   | 2   | 4   | 4   | 2   | 2   | 4   | 4   | 8   | 8   | 2   | 8   | 8   | 4   | 8   | 4   | 4   | 6   | 3   | 12  | 6   | 6   | 4   | 8   | 8   |

The labels of classes are descended from those of $O_4$ with "r" for those classes split when lifted into $\overline{O}_4$. 
Table III: Character Table of $Z_2^4$

| $\overline{Z}_2$ | $[e]$ | $[I_1]$ | $[I_2]$ | $[I_3]$ | $[I_4]$ | $[I_{12}]$ | $[I_{13}]$ | $[I_{14}]$ | $[I_{23}]$ | $[I_{24}]$ | $[I_{234}]$ | $[I_{134}]$ | $[I_{124}]$ | $[I_{123}]$ | $[I_{1234}]$ |
|------------------|------|--------|--------|--------|--------|----------|----------|----------|----------|----------|----------|----------|----------|----------|----------|
| $\chi_{0000}$   | 1    | 1      | 1      | 1      | 1      | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        | 1        |
| $\chi_{0001}$   | 1    | 1      | 1      | -1     | 1      | 1        | -1       | -1       | -1       | -1       | 1        | -1       | -1       | 1        | -1       |
| $\chi_{0010}$   | 1    | 1      | 1      | -1     | 1      | 1        | -1       | -1       | -1       | -1       | 1        | -1       | -1       | 1        | -1       |
| $\chi_{0100}$   | 1    | 1      | -1     | 1      | 1      | -1       | 1        | -1       | 1        | -1       | 1        | -1       | -1       | 1        | -1       |
| $\chi_{1000}$   | 1    | -1     | 1      | 1      | 1      | -1       | -1       | 1        | 1        | 1        | -1       | -1       | -1       | 1        | -1       |
| $\chi_{0011}$   | 1    | 1      | 1      | -1     | 1      | -1       | -1       | -1       | -1       | 1        | 1        | 1        | -1       | -1       | 1        |
| $\chi_{0101}$   | 1    | 1      | -1     | 1      | -1     | 1        | -1       | 1        | -1       | 1        | -1       | 1        | -1       | 1        | 1        |
| $\chi_{1001}$   | 1    | -1     | 1      | 1      | -1     | -1       | 1        | 1        | -1       | -1       | 1        | 1        | -1       | 1        | 1        |
| $\chi_{0110}$   | 1    | 1      | -1     | -1     | 1      | -1       | 1        | 1        | -1       | -1       | 1        | -1       | -1       | 1        | 1        |
| $\chi_{1010}$   | 1    | -1     | 1      | -1     | 1      | -1       | 1        | -1       | 1        | -1       | 1        | -1       | 1        | 1        | 1        |
| $\chi_{1100}$   | 1    | -1     | -1     | 1      | 1      | -1       | -1       | 1        | -1       | -1       | 1        | 1        | 1        | -1       | -1       |
| $\chi_{1110}$   | 1    | -1     | -1     | -1     | 1      | 1        | 1        | -1       | -1       | 1        | 1        | 1        | -1       | -1       | -1       |
| $\chi_{1101}$   | 1    | -1     | -1     | 1      | -1     | 1        | 1        | -1       | 1        | -1       | 1        | 1        | -1       | -1       | 1        |
| $\chi_{1011}$   | 1    | -1     | 1      | -1     | -1     | 1        | 1        | -1       | 1        | 1        | -1       | 1        | 1        | -1       | -1       |
| $\chi_{1111}$   | 1    | -1     | -1     | -1     | -1     | 1        | 1        | 1        | 1        | 1        | -1       | -1       | -1       | 1        | -1       |

$I_{i_1i_2...i_n} := I_{i_1} \cdot I_{i_2} \cdot ... \cdot I_{i_n}$. Irreducible characters are labeled as $\chi_{s_1,s_2,s_3,s_4}$, $s_i \in \mathbb{Z}/2\mathbb{Z}$ (see Eq.(20)).
Table IV: Character Table of $\overline{O}_4$

|   | $\Pi_S$ | $\Pi_P$ | $\Pi_V$ | $\Pi_A$ | $\Pi_T$ | spinor rep. |
|---|---------|---------|---------|---------|---------|-------------|
| (1)  | $\chi_1$ | $\chi_1$ | $\chi_1$ | $\chi_1$ | $\chi_1$ | $\chi_1$ |
| (2)  | $\chi_2$ | $\chi_2$ | $\chi_2$ | $\chi_2$ | $\chi_2$ | $\chi_2$ |
| (3)  | $\chi_3$ | $\chi_3$ | $\chi_3$ | $\chi_3$ | $\chi_3$ | $\chi_3$ |
| (4)  | $\chi_4$ | $\chi_4$ | $\chi_4$ | $\chi_4$ | $\chi_4$ | $\chi_4$ |
| (5)  | $\chi_5$ | $\chi_5$ | $\chi_5$ | $\chi_5$ | $\chi_5$ | $\chi_5$ |
| (6)  | $\chi_6$ | $\chi_6$ | $\chi_6$ | $\chi_6$ | $\chi_6$ | $\chi_6$ |
| (7)  | $\chi_7$ | $\chi_7$ | $\chi_7$ | $\chi_7$ | $\chi_7$ | $\chi_7$ |
| (8)  | $\chi_8$ | $\chi_8$ | $\chi_8$ | $\chi_8$ | $\chi_8$ | $\chi_8$ |
| $\pi_00$ | $\pi_00$ | $\pi_00$ | $\pi_00$ | $\pi_00$ | $\pi_00$ | $\pi_00$ |
| $\pi_{10}$ | $\pi_{10}$ | $\pi_{10}$ | $\pi_{10}$ | $\pi_{10}$ | $\pi_{10}$ | $\pi_{10}$ |
| $\pi_{11}$ | $\pi_{11}$ | $\pi_{11}$ | $\pi_{11}$ | $\pi_{11}$ | $\pi_{11}$ | $\pi_{11}$ |
| $4_+$ | $4_+$ | $4_+$ | $4_+$ | $4_+$ | $4_+$ | $4_+$ |
| $4_-$ | $4_-$ | $4_-$ | $4_-$ | $4_-$ | $4_-$ | $4_-$ |
| $8_+$ | $8_+$ | $8_+$ | $8_+$ | $8_+$ | $8_+$ | $8_+$ |
| $12_+$ | $12_+$ | $12_+$ | $12_+$ | $12_+$ | $12_+$ | $12_+$ |
| $12_-$ | $12_-$ | $12_-$ | $12_-$ | $12_-$ | $12_-$ | $12_-$ |

Character is labeled by a superscript showing its dimension where an underline shows a spinor representation and a subscript distinguishing different representations with same dimension.
| No. | 1  | 11 | 8  | 12 | 11 | 12 | 11 | 12 | 12 | 12 | 12 | 12 |
|-----|----|----|----|----|----|----|----|----|----|----|----|----|
| num. | 1  | 12 | 1  | 1  | 48 | 12 | 12 | 12 | 24 | 6  | 6  | 6  | 6  |
| ord. | 1  | 2  | 4  | 2  | 4  | 8  | 8  | 8  | 8  | 4  | 4  | 4  | 4  |

| No. | 14 | 14 | 14 | 14' | 15 | 15 | 18 | 18 | 20 | 20 | 20 | 20' |
|-----|----|----|----|-----|----|----|----|----|----|----|----|-----|
| num. | 12 | 12 | 12 | 12  | 12 | 12 | 12 | 12 | 24 | 24 | 24 | 24 |
| ord. | 24 | 24 | 24 | 24  | 24 | 24 | 24 | 24 | 24 | 24 | 24 | 24 |

Table V: Conjugate Classes of $SO_4$
Table VI: Character Table of $\overline{O_1}$ (with respect to the classes of $SO_4$)

|     | 1 | 1 | 3 | 5 | 7 | 8 | 8 | 11 | 11 | 12 | 14 | 14' | 14' | 15 | 18 | 18 | 20 | 20' | 20' | $(\chi, \chi)$ |
|-----|---|---|---|---|---|---|---|----|----|----|----|-----|-----|----|----|----|----|----|----|----------|
| $\chi_1^{(1)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2^{(1)}$ | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_1^{(2)}$ | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | -1 | -1 | -1 | 0 | 0 | 0 |
| $\chi_2^{(2)}$ | 3 | 3 | 3 | 3 | 3 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 |
| $\chi_1^{(3)}$ | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_2^{(3)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_1^{(4)}$ | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | -1 | -1 | -1 | 0 | 0 | 0 |
| $\chi_2^{(4)}$ | 3 | 3 | 3 | 3 | 3 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 | 1 |
| $\chi_3^{(5)}$ | 4 | 4 | 0 | -4 | -4 | 0 | 2 | 2 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 |
| $\chi_4^{(6)}$ | 4 | 4 | 0 | -4 | -4 | 0 | -2 | -2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 |
| $\chi_1^{(8)}$ | 8 | 8 | 0 | -8 | -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | -1 | 1 | 1 | 0 | 0 |
| $\chi_2^{(8)}$ | 4 | 4 | 0 | -4 | -4 | 0 | -2 | -2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 |
| $\chi_3^{(10)}$ | 6 | 6 | -2 | 6 | 6 | 0 | 0 | 0 | 0 | 2 | -2 | -2 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_4^{(10)}$ | 6 | 6 | -2 | 6 | 6 | 2 | -2 | -2 | -2 | -2 | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 |
| $\chi_1^{(12)}$ | 6 | 6 | -2 | 6 | 6 | 2 | -2 | -2 | 2 | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_2^{(12)}$ | 4 | -4 | 0 | 0 | 0 | 0 | -2 $\sqrt{2}$ | 2 $\sqrt{2}$ | 0 | 0 | 0 | 2 | -2 | -2 | -2 | 2 | -2 | -2 | 2 | -2 | 0 | $\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2}$ | $\sqrt{2}$ |

$(\chi, \chi)$ evaluates the inner product of a character with itself.
Table VII: Character Table of $SO_4$

| num. | 1   | 1   | 12  | 1   | 48  | 12  | 12  | 12  | 12  | 24  | 6   | 6   | 6   | 6   | 32  | 32  | 32  | 32  | 24  | 24  | 24  | 24  |
|------|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\chi_0^{(1)}$ | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 |
| $\chi_1^{(1)}$ | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | -1 | -1 |
| $\chi_1^{(2)}$ | 2 | 2 | 2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | 2 | 2 | 2 | 2 | -1 | -1 | -1 | 0 | 0 | 0 | 0 |
| $\chi_1^{(3)}$ | 3 | 3 | 3 | 3 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | 0 | 0 | 0 | -1 | -1 | -1 | -1 | 0 | 0 | 0 |
| $\chi_1^{(4)}$ | 4 | 4 | 0 | -4 | -4 | 0 | 2 | 2 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 | 0 |
| $\chi_1^{(5)}$ | 4 | 4 | 0 | -4 | -4 | 0 | -2 | -2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 1 | 1 | -1 | -1 | 0 | 0 | 0 |
| $\chi_1^{(6)}$ | 8 | 8 | 0 | -8 | -8 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -1 | -1 | 1 | 1 | 0 | 0 | 0 |
| $\chi_1^{(7)}$ | 6 | 6 | -2 | 6 | 6 | 0 | 0 | 0 | 0 | 0 | 0 | 2 | -2 | -2 | -2 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_2^{(1)}$ | 3 | 3 | 3 | 3 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | 0 | 0 | 0 | -1 |
| $\chi_2^{(2)}$ | 3 | 3 | -1 | 3 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | -1 | -1 | 1 | 1 |
| $\chi_2^{(3)}$ | 3 | 3 | -1 | 3 | 1 | -1 | -1 | -1 | -1 | 1 | 1 | 1 | 1 | 1 | 1 | 3 | 3 | 3 | -1 | -1 | 1 | 1 |
| $\chi_3^{(1)}$ | 3 | 3 | -1 | 3 | -1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 3 | 3 | 0 | 0 | 0 | 0 | 1 |
| $\chi_3^{(2)}$ | 3 | 3 | -1 | 3 | -1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 3 | 3 | -1 | -1 | 0 | 0 | 0 |
| $\chi_3^{(3)}$ | 3 | 3 | -1 | 3 | -1 | 1 | 1 | 1 | 1 | 1 | -1 | -1 | -1 | -1 | -1 | 3 | 3 | -1 | -1 | 0 | 0 | 0 |
| $\chi_4^{(1)}$ | 2 | 2 | 0 | -2 | 0 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_4^{(2)}$ | 2 | 2 | 0 | -2 | 0 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_4^{(3)}$ | 2 | 2 | 0 | -2 | 0 | -2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_4^{(4)}$ | 4 | 4 | 0 | 4 | 4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | -4 | -4 | -1 | 1 | 1 | -1 |
| $\chi_4^{(5)}$ | 4 | 4 | 0 | -4 | -4 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 4 | 4 | -1 | 1 | 1 | -1 |
| $\chi_4^{(6)}$ | 6 | 6 | -6 | -6 | 0 | -2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_4^{(7)}$ | 6 | 6 | -6 | -6 | 0 | -2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |
| $\chi_4^{(8)}$ | 6 | 6 | -6 | -6 | 0 | -2 | 2 | 2 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 | 0 |