Improving quantum parameter estimation by monitoring quantum trajectories

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Quantum-enhanced parameter estimation has widespread applications in many fields. An important issue is to protect the estimation precision against the noise-induced decoherence. Here we develop a general theoretical framework for improving the precision for estimating an arbitrary parameter by monitoring the noise-induced quantum trajectory (MQT) and establish its connections to the purification-based approach to quantum parameter estimation. MQT can be achieved in two ways: (i) Any quantum trajectories can be monitored by directly monitoring the environment, which is experimentally challenging for realistic noises; (ii) Certain quantum trajectories can also be monitored by frequently measuring the quantum probe alone via ancilla-assisted encoding and error detection. This establishes an interesting connection between MQT and the full quantum error correction protocol. Application of MQT to estimate the level splitting and decoherence rate of a spin-1/2 under typical decoherence channels demonstrate that it can avoid the long-time exponential loss of the estimation precision and, in special cases, recover the Heisenberg scaling.

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I. INTRODUCTION

The precise estimation of parameters characterizing physical processes [1,2] has applications in many fields, such as gravitational-wave detection [3,4], frequency spectroscopy [5,6], magnetometry [7,8], optical phase estimation [9], and atomic clocks [10]. With classical probes, repeated measurements can be used to improve the estimation precision according to the classical $1/\sqrt{N}$ scaling with respect to the number $N$ of repetitions. With quantum probes, quantum resources (such as entanglement) can be utilized to improve the estimation beyond the classical scaling and even attain the fundamental Heisenberg $1/N$ scaling allowed by quantum mechanics, where $N$ is the number of probes used in the estimation. However, the inevitable presence of environmental noises decoheres the quantum probes [11], limits the available quantum resources, and severely degrades the estimation precision. This poses a critical challenge to the practical realization of quantum-enhanced parameter estimation.

To address this problem, several methods have been developed, such as dynamical decoupling [12,16] (see Refs. 2 and 17 for a review), time optimization [13,20], and quantum error correction (QEC) [21–28] or feedback control [29–31]. The idea of dynamical decoupling is to apply pulsed [13,16,32–39] or continuous [15,40–42] control on the quantum probe to reduce its coupling to the noise and hence prolong its coherence time. It has achieved remarkable success in detecting alternating signals [43,44], noises [40,41,45–49], and other quantum objects [50,51], but it is only applicable to non-Markovian noises [57,60]. The idea of time optimization is to mitigate decoherence by shortening the evolution time of the quantum probe. It can improve the scaling of the estimation precision beyond the classical scaling, but requires vanishingly short evolution time and large-scale entanglement. Note that many Markovian environments only allow the classical $1/\sqrt{N}$ scaling even if the most general scheme is employed [61,62]. In this case, using short-range correlated states, which can be modeled by matrix product states [65], already gives almost optimal performance. The idea of QEC is to detect and then correct the noise-induced erroneous evolution. This is a powerful method applicable to both Markovian and non-Markovian noises [21–28]. For Hamiltonian parameter estimation, recent works [68,69] show that when the unitary Hamiltonian evolution can be distinguished from the noise-induced evolution, QEC can even recover the ultimate Heisenberg scaling; otherwise only a constant-factor improvement over the classical scaling is possible.

Very recently, an interesting method was proposed [70–72] to improve the estimation precision. The idea is to monitor the environment [73,75] continuously to (fully or partially) extract the information that leaks into the environment. For certain Markovian environment, this method can recover the Heisenberg scaling [70–72], but previous studies focus on specific Markovian environments and measurements and usually relies on Gaussian approximation or numerically solving the stochastic master equations. Moreover, this method requires direct measurement on the environment, which is very challenging for realistic noise processes.

In this work, we try to address the above problems. First, we develop a general theoretical framework for improving the precision of parameter estimation via continuous monitoring of a general (either Markovian or non-Markovian) environment and further establish its connection to the purification-based approach to quantum parameter estimation [61], which has motivated many works that derive fundamental bounds on the estimation precision [20,62,69,76,77]. Second, for Markovian environment, we provide a superoperator approach to determining the fundamental bounds on the estimation precision. This approach corresponds to an exact integration of the stochastic master equation [71,72], and may provide exact analytical expressions for some simple models. Third, we relax the conceptually simple but experimentally challenging requirement of monitoring the environment to the
concept of monitoring the quantum trajectories (referred to as MQT for brevity): any quantum trajectories can be monitored by monitoring the environment, but certain quantum trajectories can also be monitored by frequently measuring the quantum probe (without monitoring the environment) via ancilla-assisted encoding and error detection [21,23], i.e., the first two steps of QEC. This QEC-based MQT not only makes certain MQT experimentally feasible, but also establishes an interesting connection between MQT and the full QEC-based metrology [21,23]. The QEC-based MQT can be regarded as a QEC protocol without corrective operations, so it is less powerful than QEC when perfect error correction is available. Nevertheless, MQT itself provides an insight into how the information leaks into the distinct quantum trajectories and how they are recovered. Moreover, for certain models where corrective operations are not necessary, the MQT becomes advantageous because it avoids faulty corrective operations that may degrade the estimation precision significantly [22].

We apply this method to the estimation of the level splitting $\omega$ and the decoherence rate $\gamma$ of a spin-1/2 under three decoherence channels: spin relaxation, spin flip, and spin dephasing. We find that it can significantly improve the precision for estimating $\omega$ under the spin relaxation channel, avoid the exponential loss of the precision for estimating $\gamma$ (estimating $\omega$) under all the decoherence channels (under the spin flip channel) and even recover the Heisenberg scaling for estimating $\omega$ under the spin dephasing channel.

This paper is organized as follows. In Sec. II, we give the general theory of MQT. In Sec. III, we apply MQT to estimate the level splitting and decoherence rate of a spin-1/2. In Sec. IV, we draw the conclusions.

II. GENERAL IDEA AND THEORY

To estimate an unknown parameter $\theta$, the quantum probe starts from certain initial state $\rho_0$ and then undergoes certain $\theta$-dependent evolution for an interval $T$ into the final state $\rho(\theta)$, followed by an optimal measurement on $\rho(\theta)$ to transfer all the information about $\theta$ from the quantum probe into the measurement outcome. After repeating the above procedures for $\nu \gg 1$ times, we can use the $\nu$ measurement outcomes to construct an optimal unbiased estimator to $\theta$, such as the maximum likelihood estimator or the Bayesian estimator [78]. The estimation precision for $\theta$ is determined by the quantum Cramér-Rao bound [79, 80] as

$$\delta \theta \equiv \frac{1}{\sqrt{\mathcal{F} [\rho(\theta)]}}.$$  \hspace{1cm} (1)

where $\mathcal{F} [\rho(\theta)]$ is the quantum Fisher information (QFI) [80] about $\theta$ provided by a single copy of $\rho(\theta)$, while $\nu \mathcal{F} [\rho(\theta)]$ is the total QFI provided by $\nu$ copies of $\rho(\theta)$. In Appendix A, we provide a detailed introduction to all the relevant concepts, such as QFI, classical Fisher information (CFI), optimal measurements, and optimal unbiased estimators.

A. Non-unitary evolution and purification

When the environment (or the quantum trajectories of the quantum probe) is not monitored, the noise-induced decoherence during the $\theta$-dependent evolution is a “black box” [see Fig. 1(a)], so our state of the knowledge about the quantum probe is described by the non-selective density matrix $\bar{\rho}(t)$. The most general non-unitary evolution of $\bar{\rho}(t)$ is described by a time-local master equation [81, 82]

$$\frac{d}{dt} \bar{\rho}(t) = \mathcal{L}(t) \bar{\rho}(t),$$

where $\mathcal{L}(t)$ is a $\theta$-dependent Liouvillian, e.g., $\mathcal{L}(t) \bar{\rho} = -i[H(t), \bar{\rho}]$ in the absence of decoherence or $\mathcal{L}(t) \bar{\rho} = -i[H(t), \bar{\rho}] + \sum \gamma(c,t) D[c,\bar{\rho}](c,\bar{\rho})$ under a general decoherence channel, where $D[c] \equiv c\rho c^\dagger - |c^\dagger c,\rho|/2$ describes the decoherence in the Lindblad form, $\{c,\bar{\rho}\}$ are time-dependent quantum jump operators, and $\gamma(c,t)$ are time-dependent decoherence rates. The final state of the quantum probe is

$$\bar{\rho}(\theta) \equiv \mathcal{T} e^{i \int_0^T \mathcal{L}(t) dt} \rho_0 \equiv \mathcal{N}_0(\rho_0),$$  \hspace{1cm} (2)

where $\mathcal{T}$ is the time-ordering superoperator and $\mathcal{N}_0$ stands for the $\theta$-dependent non-unitary evolution – the quantum channel, which maps a $\theta$-independent initial state $\rho_0$ to a $\theta$-dependent final state $\bar{\rho}(\theta)$. The non-unitary nature of the quantum channel is manifested in the fact that the final state is mixed even if the initial state is pure.

Recently, there were remarkable progress in establishing practical bounds on the achievable estimation precision in the presence decoherence [20,61,69,76,77]. The key idea is to purify the non-unitary quantum channel $\mathcal{N}_0$ of the quantum probe into a unitary evolution of an extended system consist-
ing of the quantum probe and an environment and then minimize the QFI of the extended system [61, 62]. In the following, we use this purification formalism to establish a general theory of MQT for an arbitrary environment.

When the environment causing the decoherence is included, the joint evolution of the extended system (consisting of the quantum probe and the environment) during $[0, T]$ is described by a unitary evolution operator $U_{\text{ext}}(\theta)$ and the final state of the extended system is

$$\rho_{\text{ext}}(\theta) \equiv U_{\text{ext}}(\theta) (|E_0\rangle\langle E_0| \otimes \rho_0) U_{\text{ext}}^\dagger(\theta),$$  \hspace{1cm} (3)

where $|E_0\rangle$ is the $\theta$-independent initial state of the environment. Therefore, including the environment purifies the non-unitary quantum channel $\mathcal{N}_0$ of the quantum probe into a unitary evolution of the extended system, which maps a pure initial state $|E_0\rangle \otimes |\psi_0\rangle$ into a pure final state $U_{\text{ext}}(\theta) |E_0\rangle \otimes |\psi_0\rangle$. Tracing out the environmental degree of freedom in an arbitrary orthonormal complete, $\theta$-independent basis $\{|E_l\rangle\}$ gives the reduced density matrix of the quantum probe:

$$\tilde{\rho}(\theta) = \text{Tr}_E \rho_{\text{ext}}(\theta) = \sum_l \Pi_l(\theta) \rho_0 \Pi_l^\dagger(\theta) \equiv \sum_l \tilde{\rho}_l(\theta),$$  \hspace{1cm} (4)

where

$$\Pi_l(\theta) \equiv \langle E_l | U_{\text{ext}}(\theta) | E_0 \rangle$$  \hspace{1cm} (5)

are Kraus operators acting on the quantum probe. Equation (4) gives a representation of the non-unitary quantum channel $\mathcal{N}_0$ in terms of a set of Kraus operators $\{|\Pi_l(\theta)\rangle\}$ [61], or equivalently a representation of $\tilde{\rho}(\theta) = \mathcal{N}_0(\rho_0)$ [see Eq. (2)] in terms of a set of quantum trajectories $\tilde{\rho}_l(\theta) \equiv \Pi_l(\theta) \rho_0 \Pi_l^\dagger(\theta)$, which occurs with a probability $P_l(\theta) \equiv \text{Tr} \tilde{\rho}_l(\theta)$. The completeness of the environmental basis $\sum_l |E_l\rangle \langle E_l| = 1$ leads to the completeness of the Kraus operators: $\sum_l \Pi_l^\dagger(\theta) \Pi_l(\theta) = 1$ and hence the normalization $\text{Tr} \tilde{\rho}(\theta) = \sum_l P_l(\theta) = 1$. For a pure initial state $\rho_0 = |\psi_0\rangle \langle \psi_0|$ of the quantum probe, the final state of the extended system is

$$|\Psi_{\text{ext}}(\theta)\rangle = U_{\text{ext}}(\theta) |E_0\rangle \otimes |\psi_0\rangle = \sum_l |E_l\rangle \otimes \Pi_l(\theta) |\psi_0\rangle,$$  \hspace{1cm} (6)

where $\Pi_l(\theta) |\psi_0\rangle$ is a pure-state quantum trajectory of the quantum probe.

Replacing $U_{\text{ext}}(\theta)$ by $u_{\text{ext}}(\theta) U_{\text{ext}}(\theta)$ with $u_{\text{ext}}(\theta)$ being an arbitrary unitary operator acting on the environment leaves the quantum channel $\mathcal{N}_0$ and hence the final state $\tilde{\rho}(\theta) = \mathcal{N}_0(\rho_0)$ of the quantum probe invariant, but changes $\Pi_l(\theta)$ to

$$\pi_l(\theta) \equiv \langle E_l | u_{\text{ext}}(\theta) U_{\text{ext}}(\theta) | E_0 \rangle = \sum_{l'} u_{l'\theta}(\theta) \Pi_{l'}(\theta),$$

where $u_{l'\theta}(\theta) \equiv \langle E_l | u_{\text{ext}}(\theta) U_{\text{ext}}(\theta) | E_{l'} \rangle$ is a unitary matrix, so it gives a different representation of the non-unitary quantum channel $\mathcal{N}_0$ in terms of a different set of Kraus operators $\{|\pi_l(\theta)\rangle\}$, or equivalently, a representation of $\tilde{\rho}(\theta) = \mathcal{N}_0(\rho_0)$ in terms of a different set of quantum trajectories: $\tilde{\rho}(\theta) = \sum_l \pi_l(\theta) \rho_0 \pi_l^\dagger(\theta)$. Therefore, the purification (and hence representation) of the non-unitary quantum channel $\mathcal{N}_0$ is not unique: given a purification $U_{\text{ext}}(\theta)$ and hence a representation $\{\Pi_l(\theta) \equiv \langle E_l | U_{\text{ext}}(\theta) | E_0 \rangle\}$ for $\mathcal{N}_0$, exhausting all possible unitaries $u_{\text{ext}}(\theta)$ exhausts all possible unitary purifications $u_{\text{ext}}(\theta) U_{\text{ext}}(\theta)$ and hence all possible Kraus operator representations $\{\pi_l(\theta) \equiv \sum_{l'} u_{l'\theta}(\theta) \Pi_{l'}(\theta)\}$ of $\mathcal{N}_0$. Physically, this means that there are an infinite number of different environments that lead to the same reduced evolution $\mathcal{N}_0$ of the quantum probe. Next we consider a hierarchy of constraints on our ability to measure the joint system and derive an hierarchy of inequalities for the precision of estimating $\theta$. In the following, we omit the dependences of various quantities on $\theta$ for brevity.

### B. An hierarchy of estimation precision

Here we consider a fixed environment that purifies the non-unitary evolution $\mathcal{N}_0$ of the extended system consisting of the quantum probe and the environment. Correspondingly, the final state $\tilde{\rho} = \mathcal{N}(\rho_0)$ of the quantum probe is purified into $\rho_{\text{ext}}$ in Eq. (3) for the extended system.

First, when arbitrary joint measurements on the extended system are available, we can make an optimal joint measurement (see Appendix A) on the extended system to extract all the QFI $\mathcal{F}[\rho_{\text{ext}}]$ in the final state $\rho_{\text{ext}}$, so the fundamental precision follows from Eq. (1) as

$$\delta \theta_{\text{ext}} \equiv \frac{1}{\sqrt{\mathcal{F}[\rho_{\text{ext}}]}}.$$  \hspace{1cm} (7)

This fundamental estimation precision was considered in Refs. [71, 72, 74] for the special case of time-homogeneous Markovian quantum channel, as described by a time-homogeneous master equation.

Second, when arbitrary joint measurements are not available, but arbitrary separate measurements on the quantum probe and the environment are available, we can utilize different measurements on the environment to unravel $\tilde{\rho}$ into different sets of quantum trajectories [see Fig. 1(b)]. Specifically, a projective measurement on the environment in an arbitrary orthonormal complete, $\theta$-independent basis $\{|E_l\rangle\}$ has a probability $P_l \equiv \text{Tr} \tilde{\rho}_l$ to yield an outcome $|E_l\rangle$ and the occurrence of this outcome collapses the quantum probe into the corresponding quantum trajectory $\tilde{\rho}_l \equiv \Pi_{l\rho_0} \Pi_l^\dagger$ [with $\Pi_l$ given by Eq. (5)], which can be normalized as $\tilde{\rho}_l \equiv \tilde{\rho}_l / P_l$. The average amount of information in the measurement outcome is quantified by the CFI

$$F[\{P_l\}] = \frac{\sum_l (\delta \theta_l P_l)^2}{P_l},$$  \hspace{1cm} (8)

while the average amount of information in the quantum trajectory $\tilde{\rho}_l$ is quantified by the QFI $\mathcal{F}[\rho_l]$. The latter can be fully extracted by an optimal measurement on the quantum probe (see Appendix A). Therefore, the average amount of information extracted from a measurement on the environment in the basis $\{|E_l\rangle\}$ and an optimal measurement on the quantum
probe is

$$\mathbb{F} \equiv F[P_i] + \sum_i P_i F[\rho_i], \quad (9)$$

which coincides with the QFI $\mathcal{F}[\rho_{\text{ext}}|E_i]$ in the joint state

$$\rho_{\text{ext}}|E_i⟩ \equiv \sum_i |E_i⟩⟨E_i| \otimes \Pi_0 \rho_0 \Pi_1^i = \sum_i P_i |E_i⟩⟨E_i| \otimes \rho_i \quad (10)$$

after measuring the environment in the basis $|E_i⟩$. The joint state before the measurement Eq. (5) can be written as

$$\rho_{\text{ext}} = \sum_{i'} |E_{i'}⟩⟨E_{i'}| \otimes \Pi_0 \rho_0 \Pi_1^{i'},$$

so the measurement on the environment removes all off-diagonal coherences in the measurement basis $|E_i⟩$. After repeating this procedure for $n \gg 1$ times, we can use the $n$ outcomes from the measurements on the environment and the $n$ outcomes from the optimal measurements on the quantum probe to construct an optimal unbiased estimator to $\theta$ (see Appendix A). The fundamental estimation precision of this MQT method follows from Eq. (11) as

$$\delta \theta_{\text{MQT}} \equiv \frac{1}{\sqrt{\mathbb{F}}}. \quad (11)$$

In a previous work, Albarelli et al. [71] considered homodyne measurement on a Markovian bosonic environment (leading to time-homogeneous Markovian dynamics) and arrived at Eq. (11) for this specific model through straightforward (but somewhat tedious) derivation with the assistance of both the classical Cramér-Rao bound and the quantum Cramér-Rao bound. Here our analysis shows that: (i) Eq. (11) is valid for general non-unitary dynamics and general (projective) measurements on the environment; (ii) Eq. (11) follows directly from the quantum Cramér-Rao bound [Eq. (11)]. A similar analysis has been used to discuss quantum parameter estimation with post-selection [83].

Third, if only the quantum probe can be measured, then we can use an optimal measurement (see Appendix A) on the quantum probe to extract all the QFI $\mathcal{F}[\tilde{\rho}]$ in $\tilde{\rho}$, so the estimation precision follows from Eq. (11) as

$$\delta \tilde{\theta} \equiv \frac{1}{\sqrt{\mathcal{F}[\tilde{\rho}]}}, \quad (12)$$

Since the evolution $\rho_{\text{ext}} \rightarrow \rho_{\text{ext}}|E_i⟩$ and $\rho_{\text{ext}}|E_i⟩ \rightarrow \tilde{\rho} = \text{Tr}_E \rho_{\text{ext}}|E_i⟩$ are both non-unitary, while any $\theta$-independent quantum operation cannot increase the QFI [84], we have

$$\mathcal{F}[\rho_{\text{ext}}] \geq \mathbb{F} = \mathcal{F}[\rho_{\text{ext}}|E_i⟩] \geq \mathcal{F}[\tilde{\rho}] \quad (13)$$

and hence

$$\delta \theta_{\text{ext}} \leq \delta \theta_{\text{MQT}} \leq \delta \tilde{\theta}.$$
where $|A|$ is the maximal eigenvalue of $\sqrt{A^*A}$. When in addition to the quantum probe, access to some ancillas is available, this upper bound is attainable; otherwise this upper bound is not necessarily attainable \([61, 87]\).

To make connection to the MQT method, we notice that when $\rho_0 = |\psi_0\rangle\langle\psi_0|$, the quantum trajectories are pure states: $\tilde{\rho}_t = |\tilde{\psi}_t\rangle\langle\tilde{\psi}_t| = P_t|\psi_t\rangle\langle\psi_t|$, where $|\tilde{\psi}_t\rangle \equiv \Pi_t|\psi_0\rangle$ is the un-normalized trajectory. $P_t = \langle\tilde{\psi}_t|\tilde{\psi}_t\rangle$ is the occurrence probability, and $|\tilde{\psi}_t\rangle \equiv |\tilde{\psi}_t\rangle/\sqrt{P_t}$ is the normalized trajectory. The QFI $\mathcal{F}[\Psi_{\text{ext}}]$ in the final state $\rho_{\text{ext}} = |\Psi_{\text{ext}}\rangle\langle\Psi_{\text{ext}}|$ [see Eq. \((6)\)] of the extended system is

$$\mathcal{F}[\Psi_{\text{ext}}] = 4\langle\rho_0|a|\psi_0\rangle^2 - \langle\rho_0|b|\psi_0\rangle^2,$$

where $a$ and $b$ are two real vectors: $(a)_t = \sqrt{P_t}$ and $(b)_t = \sqrt{P_t}\langle\psi_t|\partial_t\rho|\psi_t\rangle$. Therefore, the inequality $\mathcal{F}[\Psi_{\text{ext}}] \geq \mathcal{F}_0$ simply follows from the Cauchy-Schwarz inequality. This inequality is saturated if and only if $\mathbf{u} \propto \mathbf{v}$, i.e., when

$$\langle\psi_t|\partial_t\rho_t\rangle = A(\theta),$$

where $A(\theta)$ is an arbitrary $l$-independent constant.

Now we discuss the connection and distinction between the MOP technique and the MQT method. In the context of MOP, the ultimate goal is to derive the tightest upper bound for the maximal QFI $\max_{\rho_0} \mathcal{F}[\tilde{\rho}]$, which characterizes the fundamental precision for estimating the parameter $\theta$ of a given quantum channel $\mathcal{N}$ without any access to the environment. As a result, the environment, the purification $|\Psi_{\text{ext}}\rangle$, and the QFI $\mathcal{F}[\Psi_{\text{ext}}]$ are not physical objects but instead mathematical tools for converting the direct evaluation of the mixed-state QFI $\mathcal{F}[\tilde{\rho}]$ to a minimization problem: $\mathcal{F}[\tilde{\rho}] = \min_{\rho_0} \mathcal{F}[\Psi_{\text{ext}}]$. The key physics is that the quantum channel $\mathcal{N}$ and hence the final state $\tilde{\rho} = \mathcal{N}(\rho_0)$ can be generated by an infinite number of fictitious environments. Each distinct environment corresponds to a distinct joint unitary evolution $U_{\text{ext}}$ and hence a distinct Kraus operator representation $\{\Pi_l\}$ [see Eq. \((5)\)] of $\mathcal{N}$ and a distinct purification $|\Psi_{\text{ext}}\rangle$ [see Eq. \((6)\)] of $\tilde{\rho}$. The purification-based definitions of the QFI [Eq. \((13)\) or \((15)\)] dictates: (i) $\mathcal{F}[\Psi_{\text{ext}}] \geq \mathcal{F}[\tilde{\rho}]$, i.e., the environment never decreases the QFI; (ii) there exists QFI-preserving environments for which the joint state $|\Psi_{\text{ext}}\rangle$ contains the same QFI as the reduced state: $\mathcal{F}[\Psi_{\text{ext}}] = \mathcal{F}[\tilde{\rho}]$.

By contrast, in the context of the MQT method, we assume that we have access to the physical environment that is coupled to the quantum probe. In this case, the quantum probe and the physical environment undergoes physical unitary evolution $U_{\text{ext}}$ as determined by their physical Hamiltonians and mutual couplings, so we no longer have any degree of freedom to choose the environment. Here the joint state $|\Psi_{\text{ext}}\rangle$ and its QFI $\mathcal{F}[\Psi_{\text{ext}}]$ are completely determined by the initial state $|\psi_0\rangle$ of the quantum probe, while $\mathcal{F} = \mathcal{F}[\rho_{\text{ext}}|E_l]\}$ also depends on the basis $||E_l}\rangle$ of the measurement on the environment. The ultimate goal is to optimize the initial state $|\psi_0\rangle$ and the measurement basis $||E_l}\rangle$ for maximal $\mathcal{F}$, e.g., if we can find a suitable basis that satisfies Eq. \((16)\), then $\mathcal{F}$ attains its maximum $\mathcal{F}[\Psi_{\text{ext}}]$. Interestingly, the purification-based definition of the QFI suggests that when the physical environment happens to be QFI-preserving, i.e., $\mathcal{F}[\Psi_{\text{ext}}] = \mathcal{F}[\tilde{\rho}]$, then Eq. \((13)\) dictates $\mathcal{F}[\Psi_{\text{ext}}] = \mathcal{F}[\tilde{\rho}]$ and hence $\delta\theta_{\text{ext}} = \delta\theta_{\text{MQT}} = \delta\theta$, i.e., including the environment provides no advantage in improving the estimation precision.

### D. Superoperator approach for Markovian dynamics

Here we consider homogeneous Markovian quantum channel $\mathcal{N} = e^{\mathcal{L}t}$ described by a time-independent Liouvillian $\mathcal{L}$, e.g., $\mathcal{L}\tilde{\rho} = -i[H, \tilde{\rho}]$ in the absence of decoherence or $\mathcal{L}\tilde{\rho} = -i[H, \tilde{\rho}] + \sum a \mathcal{D}[c_a]\tilde{\rho}$ in the presence of decoherence. With all detectable quantum jumps \([88–95]\) denoted by the superoperator $\mathcal{J}$ (see the next subsection for an example), $\mathcal{L}$ becomes the sum of $\mathcal{J}$ and $\mathcal{L}_0 \equiv \mathcal{L} - \mathcal{J}$ and the non-selective final state of the quantum probe unravels into all possible quantum trajectories [cf. Eq. \((3)\)]:

$$\tilde{\rho} = \tilde{\rho}_0 + \int_0^T dt_1 \tilde{\rho}_{t_1} + \int_0^T dt_2 \tilde{\rho}_{t_2} + \cdots,$$

where

$$\tilde{\rho}_0 \equiv e^{\mathcal{L}_0 t} \rho_0$$

is the trajectory with no quantum jump,

$$\tilde{\rho}_{t_1} \equiv e^{\mathcal{L}(T-t_1)} \mathcal{J} e^{\mathcal{L}_0 t_1} \rho_0$$

is the trajectory with one exclusive quantum jump at $t_1$,

$$\tilde{\rho}_{t_{t_1 t_2}} \equiv e^{\mathcal{L}(T-t_2)} \mathcal{J} e^{\mathcal{L}_0 (t_2-t_1)} \mathcal{J} e^{\mathcal{L}_0 t_1} \rho_0$$

is the trajectory with two exclusive quantum jumps at $t_1$ and $t_2$, etc.
a probability density $p_{\rho_{\theta}} \equiv \text{Tr} \rho_{\theta} \rho_{\theta}$, etc. The normalization $\text{Tr} \rho = 1$ leads to the normalization of all the probabilities:

$$P_0 + \int_0^T p_t dt_1 + \int_0^T dt_2 \int_0^{t_2} dt_1 p_{\rho_{\theta}} + \cdots = 1.$$ 

These analytical expressions for the quantum trajectories in terms of the quantum jump superoperator $J$ and jumpless evolution superoperator $L_0$ correspond to an exact integration of the stochastic master equation $[71, 72]$.

In MQT, we not only track the quantum trajectory, but also record the timing of every observed quantum jump during the evolution [see Fig. 11(b)]. At the end of the evolution, we make an optimal measurement on the quantum probe to transfer all the QFI in the quantum probe into the measurement outcome. After repeating this measurement cycle for $v \gg 1$ times, we can use all the observed timings and the $v$ measurement outcomes to construct an optimal unbiased estimator to $\theta$ (see Appendix A). The fundamental precision is given by Eq. (11), where $F = F + \tilde{F}$ [cf. Eq. (9)] is the sum of the CFI [cf. Eq. (8)]

$$F = \frac{(\partial_\theta p_0)^2}{P_0} + \int_0^T \frac{(\partial_\theta p_{\rho_1})^2}{p_1} dt_1 + \cdots \quad (18)$$

contained in all the timings of the quantum jumps and the trajectory-averaged QFI:

$$\tilde{F} \equiv P_0 F [p_0] + \int_0^T p_{\rho_1} F [p_{\rho_1}] dt_1 + \cdots, \quad (19)$$

where $p_0 \equiv \tilde{p}_0 / P_0, p_{\rho_1} \equiv \tilde{p}_{\rho_1} / p_{\rho_1},$ etc. are normalized quantum trajectories.

For the estimation of Hamiltonian parameters, the jumpless trajectory $\tilde{\rho}_0$ is less (usually not) influenced by the decoherence, so its QFI is much higher than other trajectories and the non-selective state $\tilde{\rho}$. When $[J, L_0] = 0$, we have $\tilde{\rho}_{\rho_{\theta}} = J^T \tilde{\rho}_0$ and $\tilde{\rho} = e^{t \tilde{\rho}} \rho_0$, i.e., all the quantum jumps can be deferred after the jumpless evolution. If $J$ further preserves the QFI, then all the quantum trajectories contain the same QFI as the jumpless trajectory, so $F \geq \tilde{F} [p_0]$ and $\delta \theta_{\text{MQT}} \leq 1 / \sqrt{\tilde{F} [p_0]}$.

The capability of MQT to resolve different quantum trajectories motivates a probabilistic protocol by post-selection of quantum trajectories. In this protocol, after resolving the quantum trajectories, we only perform optimal measurements on high-QFI trajectories and then construct an optimal unbiased estimator based on the outcomes of these measurements, while discarding all zero-QFI and even low-QFI trajectories. On one hand, this treatment reduces the workload of performing a large number of optimal measurements and constructing optimal unbiased estimators from a large number of measurement outcomes (see Appendix A). On the other hand, discarding any trajectory with nonzero QFI will degrade the fundamental estimation precision. Therefore, one can balance between the estimation precision and the cost of measurement and data post-processing to optimize the whole parameter estimation process. A similar situation has been encountered in other probabilistic metrology protocols such as weak-value amplification [96, 97], where post-selection gains technical advantages in data processing [98] at the cost of degrading the fundamental estimation precision [83, 99].

### E. Monitoring quantum trajectories via noiseless ancillas: connection to QEC

In most cases, MQT requires direct measurement of the noisy environment—an experimentally challenging task for realistic noise processes. Fortunately, when the Hamiltonian evolution and the decoherence channel satisfy certain conditions, it is possible to achieve MQT by quantum error encoding and error detection assisted by noiseless ancillas [21–23, 69]. Nevertheless, MQT itself provides an interesting insight into how the information leaks into the distinct quantum trajectories and how they are recovered. Moreover, for very special cases where corrective operations are not necessary (see Appendix B for an example), MQT becomes advantageous as it avoids faulty corrective operations [22].

We begin with a simple example: monitoring the spin-flip channel of a spin-1/2 in the Lindblad form: $d \rho(t) / dt = -\gamma D_S \rho(t)$. Here the quantum jump operator $S$, induce random jumps between $| \uparrow \rangle$ and $| \downarrow \rangle$. These random jumps can be monitored via the QEC protocol $[22]$, which adds an ancilla that is not affected by the noise. Physically, this can be realized in a nitrogen-vacancy center $[8, 100]$, where the electron spin serves as the quantum probe spin-1/2 and the 15N nuclear spin serves as the ancilla. We use $| \uparrow \rangle, | \downarrow \rangle$ for the spin-1/2, $| 0 \rangle, | 1 \rangle$ for the ancilla, $\sigma_z$ for the Pauli matrix on the spin-1/2, and $\sigma_z^{\omega a}$ for the Pauli matrix on the ancilla: $\sigma_z^{\omega a}(0) = | 0 \rangle \langle 0 |$ and $\sigma_z^{\omega a}(1) = -| 1 \rangle \langle 1 |$. The syndrome operator is $\Sigma \equiv \sigma_z^{\omega a}$. The code subspace spanned by $| \uparrow\downarrow \rangle | 0 \rangle$ and $| \uparrow\downarrow \rangle | 1 \rangle$ is the eigensubspace of the syndrome operator with eigenvalue $+1$, so we denote the code subspace by $\Sigma_+$. The occurrence of a spin flip maps the code subspace $\Sigma_+$ onto an orthogonal error subspace as spanned by $| \downarrow\uparrow \rangle | 0 \rangle$ and $| \downarrow\uparrow \rangle | 1 \rangle$. This error subspace is an eigensubspace of the syndrome operator $\Sigma$ with eigenvalue $-1$, so we denote it by $\Sigma_-$. The occurrence of another spin flip maps $\Sigma_+ \rightarrow \Sigma_+$, back to $\Sigma_+$. Therefore, frequently measuring the syndrome operator $\Sigma$ allows us to monitor the spin flip in real time. For Hamiltonian parameter estimation, if the Hamiltonian commutes with $\Sigma$ and hence leaves $\Sigma_+$ invariant, then monitoring the spin flip does not affect the coherent Hamiltonian evolution. As an example, we consider $H = \omega S_z$ with $\omega$ the unknown parameter to be estimated. The total non-selective evolution is $d \rho(t) / dt = -i[H, \rho(t)] + \gamma D_S \rho(t)$. Starting from an initial state $| \phi_0 \rangle = a | \uparrow \rangle | 0 \rangle + b | \downarrow \rangle | 1 \rangle \in \Sigma_+$, in the absence of quantum jumps, the Hamiltonian evolution keeps the state inside $\Sigma_+: | \phi(t) \rangle = a | \uparrow \rangle | 0 \rangle + e^{i\omega t} b | \downarrow \rangle | 1 \rangle$. The occurrence of a quantum jump at $t_{QJ}$ maps $| \phi(t_{QJ}) \rangle$ to $| \phi(t_{QJ}) \rangle = a | \downarrow \rangle | 0 \rangle + e^{i\omega t_{QJ}} b | \uparrow \rangle | 1 \rangle \in \Sigma_-$, which can be detected as a sign switch of $\Sigma$. The subsequent Hamiltonian evolution keeps the
the component inside the Lindblad span of the decoherence channel \[68, 69\]) and a “parallel” component (i.e., the component outside the Lindblad span of the decoherence channel). However, frequent measurement on the syndrome operator to another eigenstate with an opposite eigenvalue, i.e., (ii) The quantum jump operator anti-commutes with the syndrome operator \(\Sigma\) (see Fig. 2 for an example for \(N\) with distinct eigenvalues. However, frequent measurement on the syndrome operator invariant, while the latter (as well as the quantum jump) can induce transitions between different eigensubspaces. \[\Sigma\] and three error subspaces \(\Sigma_+\), \(\Sigma_-\), \(\Sigma_{zz}\). Different subspaces are connected by the flip of individual spins.

The key benefits of this QEC-based MQT are: (i) The code subspace \(\Sigma_+\) and the error subspace \(\Sigma_\omega\) are eigensubspaces of the syndrome operator \(\Sigma\) with distinct eigenvalues. (ii) The quantum jump operator anti-commutes with the syndrome operator, so it maps an eigenstate of the syndrome operator to another eigenstate with an opposite eigenvalue, i.e., it induces transition between \(\Sigma_+\) and \(\Sigma_\omega\) and hence can be detected by measuring the syndrome operator \[101\]. If, in addition, the Hamiltonian commutes with the syndrome operator, so that the Hamiltonian evolution leaves \(\Sigma_+\) and \(\Sigma_\omega\) invariant, then the detection of the quantum jump does not affect the Hamiltonian evolution, similar to the full QEC-based metrology \[22 \cdot 26\]. For example, a single noiseless ancilla allows us to monitor the spin flip of an arbitrary number \(N\) of spin-1/2's by using the \(N\)-component syndrome operator \(\Sigma = (\sigma_+^{(1)} \sigma_+^{(2)} \cdots \sigma_+^{(N)} \sigma_-^{(N)})\), where \(\sigma_+^{(i)}\) is the Pauli operator for the \(i\)th spin-1/2 and the flip of the \(i\)th spin-1/2 is detected as the sign switch of the \(i\)th component \(\sigma_+^{(i)} \sigma_-^{(i)}\) (see Fig. 2 for an example for \(N = 2\). When the Hamiltonian leaves each eigensubspace invariant, e.g., \(H = \omega S_+ + \cdots + S_N^{(N)}\), MQT does not affect the Hamiltonian evolution.

The requirement that MQT leave the Hamiltonian evolution intact can be satisfied only when the Hamiltonian is completely “transversal” to the quantum jump operator \[2\] (e.g., \(\omega S_z\) is transversal to \(S_z\) in our example). Generally, the Hamiltonian is the sum of a “transversal” component \(H_\perp\) (i.e., the component outside the Lindblad span of the decoherence channel \[68, 69\]) and a “parallel” component \(H_\parallel\) (i.e., the component inside the Lindblad span of the decoherence channel \[68, 69\]): the former keeps each eigensubspace of the syndrome operator invariant, while the latter (as well as the quantum jump) can induce transitions between different eigensubspaces. However, frequent measurement on the syndrome operator leads to quantum Zeno effect that effectively suppresses \(H_\parallel\), so that only \(H_\perp\) survives \[69\]. In this case, MQT changes the intrinsic Hamiltonian evolution and hence cannot reveal the intrinsic evolution of the quantum trajectories.

The discussions above suggest that for Hamiltonian parameter estimation, there is an interesting connection between QEC-based MQT and the full QEC-based metrology \[68, 69\]: (i) When \(H_\perp = 0\), the QEC-based MQT will completely freeze the Hamiltonian evolution, so the full QEC protocol is not applicable to improve the estimation precision. (ii) When \(H_\perp \neq 0\), the QEC-based MQT will suppress \(H_\parallel\) but leave \(H_\perp\) intact, so the full QEC protocol can recover the Heisenberg scaling of the estimation precision in the noiseless case \[68, 69\]. In particular, when \(H_\perp = 0\), the QEC-based MQT leaves the Hamiltonian evolution intact, so the full QEC protocol can fully recover the estimation precision in the noiseless case \[68, 69\].

III. APPLICATION TO SPIN-1/2

We consider a spin-1/2 S undergoing the non-selective evolution \(\rho(t) = L \rho(t)\) starting from a general initial state

\[
\rho_0 = \begin{pmatrix} \rho_{\uparrow\uparrow} & \rho_{\uparrow\downarrow} \\ \rho_{\downarrow\uparrow}^* & \rho_{\downarrow\downarrow}^* \end{pmatrix}
\]

with \(\rho_{\uparrow\uparrow} = 1 - \rho_{\downarrow\downarrow}, \) where \(L \rho = -i(\omega S_\omega \rho) + \gamma D(\rho)\), \(S_\omega\) is the \(\omega\) component of the spin, \(\omega\) is the level splitting, \(c\) represents an arbitrary quantum jump operator, and \(\gamma\) is the decoherence rate. Monitoring the quantum jump \(J_\rho = c \rho c^\dagger\) amounts to decomposing \(L\) into the quantum jump \(J\) and the jumpless evolution \(e^{L_\rho \tau} = e^{-iH \rho \tau},\) where

\[
\tilde{H} = \omega S_z - i \frac{\gamma}{2} c^\dagger c
\]

is an effective non-Hermitian Hamiltonian that governs the jumpless trajectory \(\tilde{\rho}_0 = e^{\tilde{H} T} \rho_0\). We consider the estimation of the level splitting \(\omega\) or the decoherence rate \(\gamma\) (at \(\omega = 0\)) under three decoherence channels: spin relaxation \(c = S_\perp = | \downarrow \rangle \langle \uparrow |\), spin flip \(c = S_x\), and spin dephasing \(c = S_z\). We use a subscript \(\omega\) (or \(\gamma\)) to denote the information about \(\omega\) (or \(\gamma\)), e.g., the CFI about \(\omega\) (or \(\gamma\)) is \(F_\omega\) (or \(F_\gamma\)), and the total information about \(\omega\) (or \(\gamma\)) from MQT is \(\mathbb{F}_\omega\) (or \(\mathbb{F}_\gamma\)).

For the spin-relaxation channel, QEC-based MQT is not possible, so MQT requires direct measurement of the environment, which is experimentally challenging for many realistic environments. For the special environment – single-mode cavity, the spin relaxation due to the cavity field is always accompanied by the emission of a cavity photon, so it can be monitored by using a photon detector to measure the cavity output \[29\]. For the spin flip channel, the Hamiltonian \(H = \omega S_\perp\) is completely “transversal” to the quantum jump \(S_z\),
so QEC-based MQT is applicable and the syndrome operator is \( \Sigma = \sigma_y \sigma_z^\text{act} \) (see Sec. II D). For the spin dephasing channel, the Hamiltonian \( H = \omega S_z \) completely lies in the Lindblad span \( \{ \mathcal{L} \} \) of the decoherence channel, i.e., \( H_{\alpha} = 0 \), so QEC-based MQT will completely freeze the Hamiltonian evolution. Therefore, only when \( \omega = 0 \) can the QEC-based MQT be used to estimate \( \gamma \) and, in this case, the results for \( \mathbb{F}_\gamma \) are identical to the spin flip channel since \( S_z \) and \( S_x \) are connected by a unitary \( \pi/2 \)-rotation around the \( y \) axis. In other cases, MQT requires directly monitoring the environment, which may be experimentally challenging. In Appendix B, we give an example for using QEC-based MQT to recover the Heisenberg scaling of frequency estimation for multiple qubits.

In a recent work, Albarelli et al. [71] considered the fundamental estimation precision of frequency by directly monitoring the radiation field from an ensemble of atoms undergoing Markovian collective dephasing. During the first-round revision of this manuscript after submission, we became aware of another work by Albarelli et al. [72], which further include the Markovian spin-flip channel and demonstrated the interesting possibility of recovering the Heisenberg scaling of the estimation precision with respect to the number \( N \) of atoms. These works focus on frequency estimation and its scaling with respect to the number \( N \) of atoms and rely on either the Gaussian approximation in the limit of a large number of atoms [71] or numerical integration of the stochastic master equation [72]. Here we focus on the time scaling of the estimation precision for both the frequency \( \omega \) and the dissipation rate \( \gamma \) of a single spin-1/2. Moreover, the superoperator formalism in Sec. II D allows us to obtain explicit analytical expressions.

In the following, we first give the quantum trajectories and their information content for each decoherence channel and then discuss the estimation precision for \( \omega \) and \( \gamma \).

### A. Quantum trajectories and Fisher information

For the spin relaxation channel, \( \tilde{H} = \omega S_z - i(\gamma/2) \uparrow\downarrow \) and \( \mathcal{J}_\gamma^2 = 0 \), so the spin can undergo at most one quantum jump. The jumpless trajectory is

\[
\tilde{\rho}_\text{0} = \begin{bmatrix}
    e^{-\gamma T/4} \rho_{\uparrow\uparrow} & e^{-\gamma T/2} e^{-i\omega T} \rho_{\uparrow\downarrow} \\
    e^{-\gamma T/2} e^{i\omega T} \rho_{\downarrow\uparrow} & \rho_{\downarrow\downarrow}
\end{bmatrix}.
\]

(20)

The trajectory with an exclusive quantum jump at \( t_1 \) is \( \tilde{\rho}_{t_1} = \gamma e^{-\gamma t_1} \downarrow\downarrow \). Other trajectories \( \tilde{\rho}_{t_1-\cdots-t_n} \) with \( n \geq 2 \) quantum jumps are absent. The sum of all the quantum trajectories gives the non-selective density matrix

\[
\tilde{\rho} = \begin{bmatrix}
    e^{-\gamma T/4} \rho_{\uparrow\uparrow} & e^{-\gamma T/2} e^{-i\omega T} \rho_{\uparrow\downarrow} \\
    e^{-\gamma T/2} e^{i\omega T} \rho_{\downarrow\uparrow} & 1 - \rho_{\downarrow\downarrow} e^{-\gamma T}
\end{bmatrix}.
\]

(21)

For the spin flip channel, \( \tilde{H} = \omega S_z - i(\gamma/8) \) and \( \mathcal{J}_\gamma^2 = (\gamma/4)^2 \), so an arbitrary number of quantum jumps is possible. The jumpless trajectory is

\[
\tilde{\rho}_\text{0} = e^{-\gamma T/4} \begin{bmatrix}
    \rho_{\uparrow\uparrow}^\text{act} & e^{-i\omega T} \rho_{\uparrow\downarrow} \\
    e^{i\omega T} \rho_{\downarrow\uparrow}^\text{act} & \rho_{\downarrow\downarrow}
\end{bmatrix}.
\]

(22)

The trajectory with \( n \) quantum jumps at \( t_1, \ldots, t_n \) is

\[
\tilde{\rho}_{t_{n-1}} = (\frac{\gamma}{4})^n e^{-\gamma T/4} \begin{bmatrix}
    \rho_{\uparrow\uparrow}^\text{act} & e^{-i\omega T} \rho_{\uparrow\downarrow} \\
    e^{i\omega T} \rho_{\downarrow\uparrow}^\text{act} & \rho_{\downarrow\downarrow}
\end{bmatrix}.
\]

(23)

for even \( n \) and \( \sigma_x \tilde{\rho}_{t_{n-1}} \sigma_x \) for odd \( n \), where \( \sigma_x \) is the Pauli matrix and \( s(t) \) starts from \( +1 \) at \( t = 0 \) and reverses its sign at \( t_1, \ldots, t_n \). The sum of all the trajectories gives the non-selective final state

\[
\tilde{\rho} = \frac{1}{2} + \frac{e^{i\gamma T/2}}{2} (\rho_{\uparrow\uparrow} - \rho_{\downarrow\downarrow}) (\rho_{\uparrow\downarrow} + \gamma^* \rho_{\downarrow\uparrow} e^{-i\omega T} / 4 - \frac{e^{i\gamma T/2}}{2} (\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow})).
\]

(24)

where \( f = \cos(\omega T) - i(\gamma \omega) \sin(\omega T) \) and \( g = \gamma \sin(\omega T) / (4 \omega) \) with \( \omega = \sqrt{\omega^2 - (\gamma/4)^2} \).

For the spin dephasing channel, \( \tilde{H} = \omega S_z - i(\gamma/8) \) and \( \mathcal{J}_\gamma^2 = (\gamma/4)^2 \), so an arbitrary number of quantum jumps is possible. The jumpless trajectory is the same as the spin flip channel. The trajectory with \( n \) quantum jumps at \( t_1, \ldots, t_n \) is

\[
\tilde{\rho}_{t_{n-1}} = (\frac{\gamma}{4})^n e^{-\gamma T/4} \begin{bmatrix}
    \rho_{\uparrow\uparrow}^\text{act} & (\gamma^* \rho_{\downarrow\uparrow} e^{-i\omega T} / 4 - \frac{e^{i\gamma T/2}}{2} (\rho_{\uparrow\downarrow} - \rho_{\downarrow\uparrow})).
\end{bmatrix}
\]

(25)

which is independent of the timings of the \( n \) quantum jumps. Summing all the trajectories gives the non-selective final state

\[
\tilde{\rho} = e^{-\gamma T/2} e^{i\omega T} \rho_{\uparrow\downarrow} e^{-\gamma T/2} e^{-i\omega T} \rho_{\downarrow\downarrow}.
\]

(26)

In Table II, we list the information for estimating \( \omega \) and for estimating \( \gamma \) at \( \omega = 0 \) for each decoherence channel. In the MQT approach, after \( \nu \) repeated measurement cycles, the estimation precision

\[
\delta \theta = \frac{1}{\sqrt{2 \nu}}
\]

(27)

is determined by the total information \( \mathbb{F}_\theta (\theta = \omega \text{ or } \gamma) \) from each measurement cycle. In the conventional approach, after \( \nu \) repeated measurement cycles, the estimation precision

\[
\delta \tilde{\theta} = \frac{1}{\sqrt{\nu \mathbb{F}_\theta [\tilde{\rho}]}}
\]

(28)

is determined by the QFI \( \mathcal{F}_\theta[\tilde{\rho}] \) (\( \theta = \omega \text{ or } \gamma \)) of the non-selective final state \( \tilde{\rho} \) in each measurement cycle. According to Eq. (13), we always have \( \mathbb{F}_\theta \geq \mathcal{F}_\theta [\tilde{\rho}] \), so MQT always improve the estimation precision, but the degree of improvement depend on the parameter to be estimated and the decoherence channel.
| Information about $\omega$ | Spin relaxation channel ($c = S_{\perp}$) | Spin flip channel ($c = S_{\perp}$) | Spin dephasing channel ($c = S_{\perp}$) |
|---------------------------|---------------------------------|---------------------------------|---------------------------------|
| $\mathcal{F}_s[\rho_0]$  | $\frac{4T^2|\rho_{11}|^2 e^{-\gamma T}}{(\rho_{11} e^{-\gamma T} + \rho_{11})^2}$ | $4T^2|\rho_{11}|^2$ | $4T^2|\rho_{11}|^2$ |
| $\mathcal{F}_s[\rho_{11}]$ | 0 | $4\left[\int_0^T s(t) dt\right]^2 |\rho_{11}|^2$ | $4T^2|\rho_{11}|^2$ |
| $\mathcal{F}_0$ | 0 | 0 | 0 |
| $\mathcal{F}_s[\bar{\rho}]$ | $4T^2|\rho_{11}|^2 e^{-\gamma T}$ | $4T^2|\rho_{11}|^2 e^{-\gamma T/2}$ | $4T^2|\rho_{11}|^2 e^{-\gamma T}$ |

From Table I we see that estimating the Hamiltonian parameter $\omega$ is very different from estimating the decoherence parameter $\gamma$ (at $\omega = 0$): 

1. For all the decoherence channels, the CFI about $\omega$ ($\gamma$) is zero (nonzero), i.e., the timings of the quantum jumps contain information about $\gamma$, but no information about $\omega$. Physically, $\gamma$ characterizes the rate of the quantum jump, so the occurrence probabilities of all the quantum trajectories depend on $\gamma$, but are independent of $\omega$, leading to nonzero $F_\gamma$, but vanishing $F_\omega$ according to Eq. (13). Therefore, for estimating $\omega$, we need not record the timings of the quantum jumps.

2. For the spin flip and spin dephasing channels, the QFI about $\omega$ ($\gamma$) is nonzero (zero) for all the trajectories. Physically, the dependence of the normalized quantum trajectories on $\omega$ and $\gamma$ originate from the jumpless evolution $L_0$ or equivalently $\tilde{H} = \omega S_z - i\gamma/8$, which imprints the $\omega$ dependence but no $\gamma$ dependence onto the normalized quantum trajectories. Therefore, for estimating $\gamma$, we need only record the timings of the quantum jumps, while measurements over the final state are not necessary.

3. For the spin relaxation channel, the trajectory with quantum jumps has vanishing QFI about $\omega$ and $\gamma$, because a single quantum jump projects an arbitrary state into $|\downarrow\rangle$ and hence eliminate all the information. Therefore, once a quantum jump is detected, we can immediately stop the current measurement cycle and start the next measurement cycle to reduce the total time cost.

4. The information about $\omega$ are all proportional to $|\rho_{11}|^2$, i.e., the projection of the initial spin in the $xy$ plane, because $\omega$ is imprinted onto the quantum probe through the Larmor precession around the $z$ axis. The positive-definiteness of the density matrix dictates $|\rho_{11}|^2 \leq \rho_{11} |\rho_{11}|$, so in the following we set $|\rho_{11}|^2 = \rho_{11} |\rho_{11}|$ to optimize the estimation precision for $\omega$.

### B. Estimation of $\omega$

For the spin relaxation channel, the jumpless trajectory [Eq. (20)] contains the QFI

$$\mathcal{F}_s[\rho_0] = 4T^2\frac{|\rho_{11} e^{-\gamma T} \rho_{11}|}{(\rho_{11} e^{-\gamma T} + \rho_{11})^2}$$

and occurs with a probability $P_\omega(T) = \rho_{11} |\rho_{11}| + \rho_{11}$, while the trajectories with a quantum jump contains no QFI. Given an arbitrary evolution time $T$, preparing the initial state $\rho_{11} =
1 - \rho_{\uparrow\uparrow} = 1/(e^{\gamma T} + 1) makes the QFI of the jumpless trajectory attains its maximum \( T^2 \). This motivates a parameter estimation protocol based on the probabilistic preparation of the jumpless trajectory (which contains all the information about \( \omega \)): \( \nu \) successful preparation of the jumpless trajectory heralds the estimation precision \( \delta \omega = 1/(\sqrt{\nu} T) \), which attains the Heisenberg scaling. The drawback is that the success probability \( P_{\text{s}}(T) = 2/(e^{\gamma T} + 1) \) for each preparation decreases with increasing \( T \). As discussed in Sec. [111] such probabilistic protocol reduces the workload of data processing at the cost of reducing the fundamental estimation precision.

For deterministic parameter estimation, we should take into account the occurrence probabilities of the quantum trajectories. The estimation precision of MQT is determined by

\[
\mathcal{F}_\omega = \frac{\mathcal{F}_\omega[\hat{\rho}]}{\rho_{\text{null}} e^{-\gamma T} + \rho_{\uparrow\downarrow}},
\]

where \( \mathcal{F}_\omega[\hat{\rho}] = 4T^2 \rho_{\uparrow\uparrow} e^{-\gamma T} \rho_{\text{null}} \) is the QFI of the conventional approach. For small \( \rho_{\text{null}} \), the enhancement of \( \mathcal{F}_\omega \) relative to \( \mathcal{F}_\omega[\hat{\rho}] \) could be very large when \( \gamma T \gg 1 \). In addition, MQT also shortens the total time cost: once a quantum jump is detected, we should immediately stop the current measurement cycle (because the trajectory with a quantum jump has no QFI) and start the next measurement cycle. In this case, the average time cost for each measurement cycle is

\[
T_{\text{ave}} = P_{\text{s}} T + \int_0^T t_1 \text{Tr} \hat{\rho}_I dt_1 = \frac{\rho_{\uparrow\downarrow}}{\gamma} (1 - e^{-\gamma T}) + \rho_{\uparrow\downarrow} T,
\]

which is shorter than \( T \), especially when \( \rho_{\uparrow\downarrow} \) is small. Since the total time cost of \( \nu \) repeated measurement cycles is \( \nu T_{\text{ave}} \), the estimation precision per unit time, i.e., the sensitivity \( \delta \omega = 1/\sqrt{T_{\text{ave}}} \) [cf. Eq. (22)], is determined by the information extraction rate

\[
\frac{\mathcal{F}_\omega}{T_{\text{ave}}} = \frac{4 \gamma T^2}{(\sqrt{\nu} e^{\gamma T} - \gamma T)^2} \rightarrow 4 \gamma T^2 e^{-\gamma T},
\]

where the inequality is saturated at \( \rho_{\uparrow\downarrow}/\rho_{\uparrow\uparrow} = \sqrt{(1 - e^{-\gamma T})} / (\gamma T e^{\gamma T}) \). For comparison, the information extracting rate of the conventional method is

\[
\frac{\mathcal{F}_\omega[\hat{\rho}]}{T} \leq T e^{-\gamma T},
\]

where the inequality is saturated at \( \rho_{\uparrow\downarrow} = \rho_{\uparrow\uparrow} = 1/2 \). Therefore, MQT enhances the long-time information extraction rate by a factor \( 4\gamma T \) although both \( \mathcal{F}_\omega \) and \( \mathcal{F}_\omega[\hat{\rho}] \) exhibit the same exponential decay, as shown in Fig. 3(a).

For the spin flip channel, the jumpless trajectory [Eq. (22)] attains the Heisenberg scaling with respect to \( T \), while the trajectory with \( n \) quantum jumps at \( t_1, \ldots, t_n \) [Eq. (23)] contains the QFI

\[
\mathcal{F}_\omega[\hat{\rho}_{t_1, \ldots, t_n}] = 4 \left( \int_0^T s(t)^2 dt \right) \rho_{\uparrow\downarrow} \rho_{\text{null}},
\]

where \( s(t) \) starts from \( +1 \) at \( t = 0 \) and reverses its sign at \( t_1, \ldots, t_n \). Physically, every quantum jump reverses the direction of the phase accumulation [manifested as the sign reversal of \( s(t) \)] and \( \int_0^T s(t)^2 dt \) is the “net” phase accumulation time. For example, we consider the trajectory with a single quantum jump at \( t_1 \). Starting from an initial state \( |\uparrow\rangle + |\downarrow\rangle \), the quantum probe first evolves as \( |\uparrow\rangle + e^{i\omega_0 t_1} |\downarrow\rangle \) and then undergoes a spin flip into the state \( |\downarrow\rangle + e^{-i\omega_0 t_1} |\uparrow\rangle \), and then evolve into the final state \( |\downarrow\rangle + e^{-i\omega_1 (T-t_1)} e^{i\omega_0 t_1} |\uparrow\rangle \), so its “net” phase accumulation time is \( t_1 - (T - t_1) \), which coincides with \( \int_0^T s(t) dt \).

Here the Heisenberg scaling of \( \mathcal{F}_\omega[\hat{\rho}_{t_1, \ldots, t_n}] \) again motivates a probabilistic parameter estimation protocol based on preparing the jumpless trajectory: \( \nu \) successful preparations with evolution time \( T \) and \( \rho_{\uparrow\downarrow} = \rho_{\uparrow\downarrow} = 1/2 \) herald the estimation precision \( \delta \omega = 1/(\sqrt{\nu} T) \), which attains the Heisenberg scaling with respect to \( T \). The drawback is that the success probability \( P_{\text{s}}(T) = e^{-\gamma T/4} \) for each preparation decreases with increasing \( T \), so this probabilistic protocol reduces the workload of data processing at the cost of reducing the fundamental estimation precision. For deterministic parameter estimation, we average the QFIs of every trajectory over their occurrence probabilities to obtain

\[
\frac{\mathcal{F}_\omega}{T} \rightarrow \frac{16}{\gamma} \rho_{\uparrow\downarrow} \rho_{\text{null}}.
\]

Thus MQT avoids the long-time exponential decay of the sensitivity (i.e., precision per unit time), as shown in Fig. 3(b). Physically, the jumpless trajectory has a negligible occurrence probability, so the long-time linear scaling \( \mathcal{F}_\omega = \frac{16}{\gamma} \rho_{\uparrow\downarrow} \rho_{\text{null}} \). The drawback is that the success probability \( P_{\text{s}}(T) = e^{-\gamma T/4} \) for each preparation decreases with increasing \( T \), so this probabilistic protocol reduces the workload of data processing at the cost of reducing the fundamental estimation precision. For deterministic parameter estimation, we average the QFIs of every trajectory over their occurrence probabilities to obtain

\[
\frac{\mathcal{F}_\omega[\hat{\rho}]}{T} = 4T \rho_{\uparrow\downarrow} \rho_{\text{null}} e^{-\gamma T/2}.
\]
\( \rho \) \( \text{s(t)} \text{nd(t)}^2 \) and hence the QFI of each trajectory increase linearly with \( T \) on average, similar to a random walk. MQT gives access to this trajectory-resolved QFI, as opposed to the ensemble QFI \( \mathcal{F}_\omega[\bar{\rho}] \) from the conventional method.

When we take into account the finite interval \( \Delta \) between successive syndrome measurements in QEC-based MQT for the spin-flip channel, we obtain
\[
\mathbb{P}_\omega = \frac{32\rho_{\uparrow\uparrow}\rho_{\downarrow\downarrow}}{\gamma^2} e^{-\gamma T/4} e^{2\zeta} \left[ \left(1 - \xi e^{-\gamma} \right)^N + \left(1 + \xi e^{-\gamma} \right)^N \left(\frac{\gamma T}{2} e^{-\gamma} - 1\right) \right]
\]  
(30)
under the condition \( 1/\Delta \gg \omega \gg \gamma \) and \( \gamma T \ll 1/\gamma \Delta \), where \( \zeta \equiv \gamma \Delta/4 \) and \( N \equiv T/\Delta \) is the total number of syndrome measurements during the interval \( T \). When the syndrome measurements are much faster than the decoherence \( (\zeta \ll 1) \), \( \mathbb{P}_\omega \) approaches the ideal results in Table I. When each syndrome measurement is imperfect, we can combine several adjacent imperfect syndrome measurements into a composite measurement to suppress the measurement error exponentially.

Finally, we turn to the spin dephasing channel. Interestingly, each quantum jump merely flips the phase between \( \uparrow \) and \( \downarrow \) without affecting the phase accumulation [Eq. (25)], so every quantum trajectory contain the same QFI
\[
\mathbb{P}_\omega = \mathcal{F}_\omega[\rho_{0}] = \mathcal{F}_\omega[\rho_{-\gamma,-\delta}] = 4T^2 \rho_{\uparrow\uparrow} \rho_{\downarrow\downarrow},
\]  
(31)
in contrast to the conventional method:
\[
\mathcal{F}_\omega[\bar{\rho}] = 4T^2 \rho_{\uparrow\uparrow} \rho_{\downarrow\downarrow} e^{-\gamma T}.
\]
So MQT can restore the Heisenberg scaling of the estimation precision, as shown in Fig. 3(c). Unfortunately, QEC-based MQT for the dephasing channel of a single spin-1/2 is not possible, so direct measurement over the environment is necessary. For multiple qubits, the key to the recovery of the Heisenberg scaling for estimating \( \omega \) is \([\mathcal{J}, L_0] = 0\) and the preservation of the QFI under the quantum jump \( \mathcal{J} \) [see the discussions after Eq. (18)]. When this condition is satisfied, QEC-based MQT can be used to recover the Heisenberg scaling (see Appendix B for an example).

### C. Estimation of \( \gamma \)

For the spin relaxation channel, the estimation precision of MQT is determined by \( \bar{\rho}_{\gamma} \approx \rho_{\uparrow\uparrow}/\gamma^2 \) at \( \gamma T \gg 1 \). By contrast, the estimation precision of the conventional method is determined by \( \mathcal{F}_\gamma[\bar{\rho}] \approx T^2 \rho_{\uparrow\uparrow} e^{-2\gamma T} \) at \( \gamma T \gg 1 \). Thus MQT avoids the exponential loss of the QFI in the conventional method. In addition, it also shortens the total time cost from \( T \) to \( T_{\text{ave}} \) [Eq. (29)]. The information extraction rate, which determines the sensitivity \( \delta \gamma \sqrt{T_{\text{ave}}} = 1/\sqrt{\mathcal{F}_\gamma[T_{\text{ave}}]} \), is
\[
\frac{\bar{\rho}_{\gamma}}{T_{\text{ave}}} \leq \frac{\gamma}{\gamma},
\]  
(32)
where the inequality is saturated at \( \rho_{\uparrow\downarrow} = 0 \). The information extracting rate of the conventional method is
\[
\frac{\mathcal{F}_\gamma[\bar{\rho}]}{T} \leq \frac{T}{e^{\gamma T} - 1},
\]  
(33)
where the inequality is saturated at \( \rho_{\uparrow\downarrow} = \rho_{\downarrow\uparrow} = 0 \). Therefore, MQT avoids the long-time exponential loss of the sensitivity, as shown in Fig. 3(d).

For the spin flip channel, the information about \( \gamma \) is entirely the CFI in the timings of the quantum jumps: \( \mathbb{P}_\gamma = F_\gamma = T/(4\gamma) \). The information extraction rate from the MQT is
\[
\frac{\mathbb{P}_\gamma}{T} = \frac{1}{4\gamma},
\]  
(34)
The information extraction rate of the conventional method is
\[
\frac{\mathcal{F}_\gamma[\bar{\rho}]}{T} \leq \frac{T}{4(e^{\gamma T} - 1)},
\]  
(35)
where \( \rho_{++}, \rho_{--} \) are the populations of \(|\pm\rangle \equiv (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2} \) in the initial state \( \rho_{0} \) and the inequality is saturated at \( |\rho_{--}|^2 = \rho_{++}, \rho_{--} \) and \( \rho_{++} = \rho_{--} = 1/2 \). Equation (35) is also the ultimate precision bound [102] for adaptive estimation of the dephasing rate without MQT. Comparing Eq. (34) to Eq. (35), we see that MQT avoids the long-time exponential loss of the sensitivity. At \( \omega = 0 \), including the finite interval \( \Delta \) between successive syndrome measurements in QEC-based MQT for the spin-flip channel amounts to a multiplicative factor \( 4\zeta/(e^{\gamma T} - 1) \) (with \( \zeta \equiv \gamma \Delta/4 \) and \( N \equiv T/\Delta \)) to \( \mathbb{P}_\gamma \). When \( \zeta \ll 1, \mathbb{P}_\gamma \) approaches the ideal results in Table I. For the spin dephasing channel, we obtain exactly the same results, with \(|\uparrow\rangle (|\downarrow\rangle) \) playing the role of \(|\uparrow\rangle (|\downarrow\rangle)\).

### IV. CONCLUSION

Quantum enhanced parameter estimation has widespread applications in many fields. An important issue is to protect the estimation precision against the noise-induced decoherence. For this purpose, we have developed a general theoretical framework for improving the precision of parameter estimation by monitoring the noise-induced quantum trajectories (MQT) of the quantum probe and further establish its connection to the purification-based approach to quantum parameter estimation [61]. For Markovian environment, we provide a superoperator approach to determining the fundamental bounds on the estimation precision. This approach may provide exact analytical expressions for some simple models. MQT can be achieved in two ways: (i) Any quantum trajectories can be monitored by directly monitoring the environment, which is experimentally challenging for realistic noises. (ii) Certain quantum trajectories can also be monitored by occasionally measuring the quantum probe (without monitoring the environment) via ancilla-assisted encoding and error detection, as used in quantum error correction (QEC). This QEC-based MQT makes certain MQT feasible and further establishes an interesting connection between MQT and the metrology protocols based on full QEC [21–28]. We apply MQT to the estimation of the level splitting \( \omega \) and the decoherence rate \( \gamma \) of a spin-1/2 under three decoherence channels: spin relaxation, spin flip, and spin dephasing. We find that it can significantly improve the precision for estimating \( \omega \) under the spin relaxation channel, avoid the exponential loss of the...
2. The quantum probe undergoes a measurement, which produces an outcome according to certain probability distribution. In this step, the quantum Fisher information contained in \( \rho \) is transferred into the classical information in the measurement outcome. The information contained in each outcome is quantified by the classical Fisher information (CFI) \( F \), which obeys

\[
F \leq \mathcal{F}.
\]  

(A1)

3. Steps 1-2 are repeated \( \nu \) times and the \( \nu \) outcomes are processed to yield an estimator \( \hat{\theta}_{\text{est}} \) to the unknown parameter \( \theta \). In this step, the total CFI \( \nu F \) contained in the \( \nu \) outcomes is converted to the estimation precision, as quantified by the statistical error of the estimator:

\[
\delta \theta \equiv \sqrt{\langle (\hat{\theta}_{\text{est}} - \theta)^2 \rangle},
\]  

(A2)

where \( \langle \cdots \rangle \) denotes the average over a lot of estimators obtained by repeating steps 1-3 many times. For unbiased estimators obeying \( \langle \hat{\theta}_{\text{est}} \rangle = \theta \), the precision \( \delta \theta \) is fundamentally limited by the inequality

\[
\delta \theta \geq \frac{1}{\sqrt{\nu F}} \Leftrightarrow (\delta \theta)^2 \leq \nu F,
\]  

(A3)

known as the Cramér-Rao bound \[78, 80\].

### 1. Quantum Fisher Information

The amount of information about \( \theta \) contained in a general \( \theta \)-dependent quantum state \( \rho \) is quantified by its QFI [80]

\[
\mathcal{F} \equiv \text{Tr} \rho L^2,
\]

where \( L \) is the so-called symmetric logarithmic derivative operator: it is an Hermitian operator defined through [79]

\[
\delta \rho = \frac{1}{2}(L \rho + \rho L).
\]

The QFI is invariant under any \( \theta \)-independent unitary transformations. For a pure state \( \rho = |\Phi\rangle \langle \Phi| \), we have \( L = 2 \delta \theta \rho \) and hence

\[
\mathcal{F} = 4\langle (\delta \rho \Phi | \delta \theta \Phi) - (|\Phi \rangle \langle \Phi | \delta \theta \Phi) \rangle^2.
\]  

(A4)

For a general mixed state with the spectral decomposition \( \rho = \sum_n p_n |\Phi_n\rangle \langle \Phi_n| \), its QFI is [103–106]

\[
\mathcal{F}[\rho] = \sum_n \frac{(\delta \rho p_n)^2}{p_n} + \sum_n p_n \mathcal{F}[|\Phi_n\rangle] - \sum_{m,n} \frac{8p_mp_n}{p_m + p_n} \langle |\Phi_m\rangle \langle \Phi_n | \delta \theta \Phi \rangle^2,
\]

where \( \{p_n\} \) are nonzero eigenvalues of \( \rho \), \( \{\Phi_n\} \) are the corresponding orthonormalized eigenstates, and \( \mathcal{F}[\Phi_n] \) is the QFI of the pure state \( |\Phi_n\rangle \) [see Eq. (A3)]. For a two-level system, its density matrix can always be expressed in terms of the Pauli matrices \( \sigma \) as \( \rho = (1/2)(1 + \sigma \cdot \mathbf{n}) \), where \( \mathbf{n} \equiv \text{Tr} \sigma \rho \) is the Bloch vector. The QFI for such a state is \[15, 107, 108\]

\[
\mathcal{F} = |\mathbf{n} \cdot \delta \mathbf{n}|^2 + \frac{(\mathbf{n} \cdot \delta \mathbf{n})^2}{1 - |\mathbf{n}|^2}.
\]

where the second term is absent when \( |\mathbf{n}| = 1 \), i.e., when \( \rho \) is a pure state. When \( \rho = \rho^{(1)} \otimes \cdots \otimes \rho^{(N)} \) is the direct product state of \( N \) quantum probes, its QFI is additive: \( \mathcal{F} = \sum_{n=1}^{N} \mathcal{F}[\rho^{(n)}] \).

The importance of the QFI for parameter estimation is manifested in the inequalities Eqs. (A1) and (A3). Namely, given \( \rho \) and hence \( \mathcal{F} \), the precision of any unbiased estimator from \( \nu \) repetitions of any measurement is limited by the inequality

\[
\delta \theta \geq \frac{1}{\sqrt{\nu F}},
\]  

(A5)

known as the quantum Cramér-Rao bound \[79, 80\]. Saturating this bound requires saturating Eqs. (A1) and (A3) simultaneously, i.e., using optimal measurements to convert all the QFI into the CFI and using optimal unbiased estimators to convert all the CFI into the precision of the estimator.

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Appendix A: Framework of quantum parameter estimation

Here we provide a self-contained introduction to the typical framework and important concepts in quantum parameter estimation. A typical (non-adaptive) parameter estimation protocol using a quantum probe to estimate an unknown, real parameter \( \theta \) consists of three steps (Fig. 3):

1. The quantum probe starts from an initial state \( \rho_{\text{in}} \) and undergoes certain \( \theta \)-dependent evolution into a final state \( \rho \) that depends on \( \theta \). This step imprints the information about \( \theta \) into the final state \( \rho \). The information contained in \( \rho \) is quantified by the quantum Fisher information (QFI) \( \mathcal{F} \).

2. The quantum probe undergoes a measurement, which produces an outcome according to certain probability distribution. In this step, the quantum Fisher information contained in \( \rho \) is transferred into the classical information in the measurement outcome. The information contained in each outcome is quantified by the classical Fisher information (CFI) \( F \), which obeys

\[
F \leq \mathcal{F}.
\]  

(A1)

3. Steps 1-2 are repeated \( \nu \) times and the \( \nu \) outcomes are processed to yield an estimator \( \hat{\theta}_{\text{est}} \) to the unknown parameter \( \theta \). In this step, the total CFI \( \nu F \) contained in the \( \nu \) outcomes is converted to the estimation precision, as quantified by the statistical error of the estimator:

\[
\delta \theta \equiv \sqrt{\langle (\hat{\theta}_{\text{est}} - \theta)^2 \rangle},
\]  

(A2)

where \( \langle \cdots \rangle \) denotes the average over a lot of estimators obtained by repeating steps 1-3 many times. For unbiased estimators obeying \( \langle \hat{\theta}_{\text{est}} \rangle = \theta \), the precision \( \delta \theta \) is fundamentally limited by the inequality

\[
\delta \theta \geq \frac{1}{\sqrt{\nu F}} \Leftrightarrow (\delta \theta)^2 \leq \nu F,
\]  

(A3)

known as the Cramér-Rao bound \[79, 80\]. Saturating this bound requires saturating Eqs. (A1) and (A3) simultaneously, i.e., using optimal measurements to convert all the QFI into the CFI and using optimal unbiased estimators to convert all the CFI into the precision of the estimator.
2. Classical Fisher information and optimal measurements

A general measurement with discrete outcomes \(|u\rangle\) is described by the positive-operator valued measure (POVM) elements \(|M_u\rangle\) satisfying the completeness relation \(\sum_u M_u^\dagger M_u = I\). Given a quantum state \(\rho\), it yields an outcome \(u\) according to the probability distribution \(P(u|\theta) \equiv \text{Tr} M_u^\dagger \rho M_u\) that depends on \(\theta\). The amount of information about \(\theta\) contained in each outcome is quantified by the CFI [78]:

\[
F \equiv \sum_u P(u|\theta) \left( \frac{\partial \ln P(u|\theta)}{\partial \theta} \right)^2 .
\]

For continuous outcomes, we need only replace \(\sum_u\) by \(\int d\theta\) everywhere. The inequality Eq. (A1) expresses the simple fact that no new information about \(\theta\) can be generated in the measurement process: those measurements that convert all (part) of the QFI into the CFI are called optimal (non-optimal). Given \(\rho\), the optimal measurement is not unique. The projective measurement on the symmetric logarithmic derivative operator \(L\) has been identified [80] as an optimal measurement.

3. Optimal unbiased estimators

Given the measurement distribution \(P(u|\theta)\) and hence the CFI \(F\) of each outcome, the precision \(\delta \theta\) of any unbiased estimator \(\theta_{\text{un}}(u)\) constructed from the outcomes \(u \equiv (u_1, \cdots, u_v)\) of \(v\) repeated measurements is limited by the Cramér-Rao bound Eq. (A3), which expresses the simple fact that no new information about \(\theta\) can be generated in the data processing: optimal (non-optimal) unbiased estimators convert all (part) of the CFI into the useful information \((\delta \theta)^{-2}\) quantified by the precision \(\delta \theta\). In the limit of large \(v\), two kinds of estimators are known to be unbiased and optimal: the maximum likelihood estimator and the Bayesian estimator [78], as we introduce now.

Before any measurements, our prior knowledge about the unknown parameter \(\theta\) is quantified by certain probability distribution \(P_0(\theta)\), e.g., a \(\delta\)-like distribution corresponds to knowing \(\theta\) exactly, a flat distribution corresponds to completely no knowledge about \(\theta\), while a Gaussian distribution \(P_0(\theta) \propto e^{-(\theta-\theta_0)^2/(2\sigma_0^2)}\) corresponds to knowing \(\theta\) to be \(\theta_0\) with a typical uncertainty \(\sigma_0\).

Upon getting the first outcome \(u_1\), our knowledge about \(\theta\) is immediately refined from \(P_0(\theta)\) to

\[
P_{u_1}(\theta) = \frac{P_0(\theta)P(u_1|\theta)}{N(u_1)}
\]

according to the Bayesian rule [109], where \(N(u_1) \equiv \int d\theta P_0(\theta)P(u_1|\theta)\) is a normalization factor ensuring \(P_{u_1}(\theta)\) is normalized to unity: \(\int P_{u_1}(\theta)d\theta = 1\). Here \(P_{u_1}(\theta)\) is the posterior probability distribution of \(\theta\) conditioned on the outcome of the measurement being \(u_1\); its parametric dependence on \(u_1\) means that different measurement outcomes leads to different refinement of knowledge about \(\theta\).

Upon getting the second outcome \(u_2\), our knowledge is immediately refined from \(P_{u_1}(\theta)\) to

\[
P_{u_1u_2}(\theta) = \frac{P_0(\theta)P(u_1|\theta)P(u_2|\theta)}{N(u_1,u_2)},
\]

where \(N(u_1,u_2) = \int P_0(\theta)P(u_1|\theta)P(u_2|\theta)d\theta\) is a normalization factor for the posterior distribution \(P_{u_1u_2}(\theta)\). If we omit the trivial normalization factors, then the measurement-induced knowledge refinement becomes

\[
P_0(\theta) \xrightarrow{u_1} P_0(\theta)P(u_1|\theta) \xrightarrow{u_2} P_0(\theta)P(u_1|\theta)P(u_2|\theta) \xrightarrow{u_1} \cdots .
\]

Upon getting \(v\) outcomes \(u \equiv (u_1, \cdots, u_v)\), our knowledge about \(\theta\) is quantified by the posterior distribution

\[
P_u(\theta) \sim P_0(\theta)P(u|\theta)
\]

up to a trivial normalization factor, where \(P(u|\theta) = P(u_1|\theta) \cdots P(u_v|\theta)\) is the probability for getting the outcome \(u\). The posterior distribution \(P_u(\theta)\) completely describe our state of knowledge about \(\theta\). Nevertheless, sometimes a single number, i.e., an unbiased estimator, is required as the best guess to \(\theta\). There are two well-known estimators: the maximum likelihood estimator [78]

\[
\theta_{\text{ML}}(u) \equiv \arg \max \theta P_u(\theta)
\]

is the peak position of \(P_u(\theta)\) as a function of \(\theta\), while the Bayesian estimator [78]

\[
\theta_{\text{Bay}}(u) \equiv \int \theta P_u(\theta)d\theta
\]

is the average of \(\theta\). For large \(v\), both estimators are unbiased and optimal: \(\langle \theta_{\text{ML}} \rangle = \theta\) and \(\delta \theta_{\text{Bay}} = 1/\sqrt{vF(\theta)}\), where \(\sigma = M\) or \(B\), and \(\langle \cdots \rangle\) denotes the average over a large number of estimators obtained by repeating the \(v\)-outcome estimation scheme many times and \(\delta \theta_{\text{Bay}}\) is defined as Eq. (A8) or

\[
\delta \theta_{\text{Bay}} = \sqrt{\int (\theta - \theta_{\text{Bay}}(u))^2 P_u(\theta)d\theta}.
\]

Appendix B: Monitoring dephasing of logical qubits

We consider an odd number \(m\) of spin-1/2’s and each spin-1/2 is subjected to an independent spin dephasing channel. The quantum jump operator \(S_z\) of the dephasing channel leads to random transitions between the two eigenstates \(|\pm\rangle \equiv (|\uparrow\rangle \pm |\downarrow\rangle)/\sqrt{2}\) of \(\sigma_z\) during the evolution. We define the \((m-1)\)-component syndrome operator \(\Sigma = (\sigma_z^{(1)} \sigma_z^{(2)} \cdots \sigma_z^{(m-1)} \sigma_z^{(m)})\), where \(\sigma_z^{(i)}\) is the Pauli matrix for the \(i\)th spin-1/2. The code subspace as spanned by \(|\pm\rangle\) [21][24]

\[
|\pm\rangle \equiv \frac{|+\rangle^{\otimes m} \pm |-\rangle^{\otimes m}}{\sqrt{2}}.
\]
is an eigensubspace of $\Sigma$ with eigenvalue $+1$ for every component of $\Sigma$. This allows us to monitor the simultaneous quantum jump of at most $(m-1)/2$ qubits, e.g., the quantum jump of the $i$th spin-1/2 switches the sign of the $(i-1)$th and the $i$th component of $\Sigma$, while leaves other components of $\Sigma$ intact. When the Hamiltonian commutes with the syndrome operator, monitoring the quantum jump does not affect the Hamiltonian evolution. When the Hamiltonian further commutes with all the quantum jump operators $S^{(1)}_z, \cdots, S^{(m)}_z$, the quantum jump does not affect the Hamiltonian evolution. In this case, MQT can fully recover the estimation precision in the noiseless case.

As an example, consider $H = (\omega/2)\vec{\sigma}_z$, where $\vec{\sigma}_z \equiv \sigma^{(m)}_z$ maps $|\uparrow\rangle$ to $|\downarrow\rangle$ and $|\downarrow\rangle$ to $|\uparrow\rangle$. This Hamiltonian commutes with the syndrome operator and all the quantum jump operators. The initial state $|\psi_0\rangle = (|\uparrow\rangle + |\downarrow\rangle)/\sqrt{2}$ lies inside the code subspace. In the absence of quantum jumps, the evolution under $H$, which always maps $|\uparrow\rangle$ to $|\downarrow\rangle$ to another eigenstate of $H$ with the same eigenvalue, thus all the detectable quantum jumps do not cause the loss of the QFI. For example, a quantum jump of the $k$th qubit at time $t$ maps the $m$-qubit state $|\psi_0\rangle$ to $|\psi_0\rangle$ by $\sqrt{2}$ to $(\sigma^{(k)}_z |\uparrow\rangle + e^{i\omega t} \sigma^{(k)}_z |\downarrow\rangle) / \sqrt{2}$, where $|\psi_0\rangle$ is still an eigenstate of $\vec{\sigma}_z$ with the same eigenvalue $+1$ ($-1$). Afterwards, the evolution under $H = (\omega/2)\vec{\sigma}_z$ leads to the final state $(\sigma^{(k)}_z |\uparrow\rangle + e^{i\omega t} \sigma^{(k)}_z |\downarrow\rangle) / \sqrt{2}$, which contains the same QFI as the jumpless trajectory. Therefore, MQT allows us to continuously track the quantum trajectory and hence recover the Heisenberg scaling of the estimation precision without error correction, as long as in between two round of syndrome measurements, the number of phase-flipped qubits do not exceed $(m - 1)/2$. By contrast, when the quantum trajectories are not monitored, the QFI of the non-selective final state would decay to zero on time scales $\gg$ single-qubit decoherence time.

The results above are consistent with the conclusion of Refs. [68, 69] since the Hamiltonian $H \propto \vec{\sigma}_z = \sigma^{(m)}_z$ lies outside the Lindblad span of all the decoherence channels. The results above can also be generalized to $N$ logical qubits, where each logical qubit is composed of $m$ qubits through the $m$-qubit phase-flip code and is driven by the Hamiltonian $(\omega/2)\vec{\sigma}_z$. Starting from the initial state $(|\uparrow\rangle^{\otimes N} + |\downarrow\rangle^{\otimes N})/\sqrt{2}$, all the detectable quantum trajectories contain the same QFI $N^2T^2$, which gives the Heisenberg scaling for the estimation precision about $\omega$.

Appendix C: Estimation precision for spin-flip channel

1. Ideal continuous monitoring

For ideal, continuous monitoring of the quantum jump (i.e., spin flip), the normalized quantum trajectory with $n$ quantum jumps at $t_1, \cdots, t_n$ has a QFI

$$\mathcal{F}_n = \left\{ \begin{array}{ll}
4|\rho_{t_1}|^2(T - 2t_1 + 2t_2 - \cdots - 2t_{2k+1})^2 & (n = 2k + 1), \\
4|\rho_{t_1}|^2(T + 2t_1 - 2t_2 - \cdots - 2t_{2k})^2 & (n = 2k), 
\end{array} \right.$$

with occurrence probability $p_{t_1 \cdots t_n} = (\gamma/4^n) e^{-\gamma T/4}$. For $n = 0$, they reduce to the QFI and the occurrence probability of the jumpless trajectory. The trajectory-averaged QFI follows by straightforward calculation:

$$\mathcal{F}_\omega = 4T^2|\rho_{t_1}|^2 e^{-\omega T/4} \sum_{k=0}^{\infty} \frac{(\gamma T/4)^k}{(2k + 1)!} \left( 1 + \frac{\gamma T/4}{2k + 3} \right).$$

The sum $\sum_{k=0}^{\infty} = (e^{-\omega T/4} - 1)/\omega$ is equal to $(e^{-x} - \sinh x)/x^2$ with $x \equiv \gamma T/4$, so we obtain the $\mathcal{F}_\omega$ listed in Table I. Since $p_{t_1 \cdots t_n}$ is independent of $\omega$ and the timings $(t_1, \cdots, t_n)$ of the quantum jumps, Eq. (18) gives $F_\omega = 0$ and

$$P_n = \int_0^T dt_1 \cdots \int_0^{t_1} dt_n p_{t_1 \cdots t_n} = (\gamma T/4)^n n! e^{-\gamma T/4}$$

is the probability for $n$ quantum jumps during $[0, T]$.

2. Realistic QEC-based monitoring

When we use QEC-based MQT with the syndrome operator $\Sigma = \sigma^{(m)}_z$ with eigenvalues $\pm 1$ (see Sec. II E), the interval $\Delta$ between successive syndrome measurements is finite. The total evolution interval $[0, T]$ is divided into $N = T/\Delta$ intervals of length $\Delta$ sandwiched by $N$ syndrome measurements at $t = \Delta, 2\Delta, \cdots, N\Delta$. At $t = 0$, the initial state of the spin-1/2 and the ancilla is an eigenstate of the syndrome operator with eigenvalue $\lambda_0 = +1$. For clarity we use $\lambda_k (= +1$ or $-1$) to denote the outcome of the $k$th syndrome measurement, where $k = 1, 2, \cdots, N$. If the $k$th syndrome measurement reports a sign switch, i.e., $\lambda_k = -\lambda_{k-1}$, then the spin-1/2 undergoes an odd number of quantum jumps during $[(k - 1)\Delta, k\Delta]$, otherwise (i.e., $\lambda_k = \lambda_{k-1}$) the spin-1/2 undergoes an even number of quantum jumps during $[(k - 1)\Delta, k\Delta]$. If $\lambda\Delta \ll 1$, then the possibility of multiple quantum jumps between neighboring syndrome measurement is exponentially small, so the syndrome measurements can well resolve individual quantum jumps. Otherwise, the syndrome measurements only give little information about the quantum jumps.

First, we derive the evolution of the quantum trajectories between the $(k - 1)$th and the $k$th syndrome measurements. Suppose the previous syndrome measurements give the outcomes $\lambda_1, \cdots, \lambda_{k-1}$ and the un-normalized quantum trajectory $\tilde{\rho}$ immediately after the $(k - 1)$th syndrome measurement is

$$\tilde{\rho} = \begin{bmatrix}
\tilde{\rho}_{11} & \tilde{\rho}_{12} \\
\tilde{\rho}_{21} & \tilde{\rho}_{22}
\end{bmatrix},$$
which is an eigenstate of $\Sigma$ with eigenvalue $\lambda_{k-1}$. The non-selective evolution during $[(k-1)\Delta,k\Delta]$ gives the non-selective final state:

$$e^{\Delta \hat{p}} = M_{k} \hat{p} + M_{k} \hat{p},$$

where $L\rho = -i[H,\rho] + \gamma D[S_{\omega}]\rho$ and $M_{k}$ are superoperators:

$$M_{k} \hat{p} \equiv e^{-\gamma \rho \uparrow \downarrow \cos \theta \rho \uparrow \downarrow f(\Delta)} \left[ \rho \uparrow \downarrow + \left( \rho \uparrow \downarrow f(\Delta) \rho \uparrow \downarrow \sin \theta \right) \right],$$

$$M_{k} \hat{p} \equiv e^{-\gamma \rho \uparrow \downarrow \sin \theta \rho \uparrow \downarrow g(\Delta)} \left[ \rho \uparrow \downarrow \right] \rho \uparrow \downarrow \sin \theta \hat{p} \right],$$

$\zeta \equiv \gamma \Delta / 4$, $f(\Delta) \equiv \cos(w \Delta) - i(\omega / w) \sin(w \Delta)$, $g(\Delta) \equiv \gamma \sin(w \Delta) / (4w)$, and $w \equiv \sqrt{\omega^{2} - (\gamma / 4)^{2}}$. Since each quantum jump leads to the exchanges $| \uparrow \rangle \leftrightarrow | \downarrow \rangle$ and hence $\rho \uparrow \downarrow \leftrightarrow \rho \downarrow \uparrow$, we identify $\hat{p}_{+} \equiv M_{k} \hat{p} (\hat{p}_{-} \equiv M_{k} \hat{p})$ as the un-normalized quantum trajectory containing an even (odd) number of quantum jumps during $[(k-1)\Delta,k\Delta]$. In other words, the un-normalized quantum trajectory at $t = k\Delta$ is $\hat{p}_{-}$ if the $k$th syndrome measurement reports a sign switch, or $\hat{p}_{+}$ otherwise. Their occurrence probabilities are $\text{Tr} \hat{p}_{\pm} = [(1 \pm e^{-2\gamma \Delta}) / 2] \text{Tr} \hat{p}$, so the non-selective evolution is trace-preserving: $\text{Tr} M_{k} \hat{p} + \text{Tr} M_{k} \hat{p} = \text{Tr} \hat{p}$. When $\hat{p} = \hat{p}$, the occurrence probabilities of $\hat{p}_{\pm}$ are $p \equiv (1 + e^{-2\gamma \Delta}) / 2$ and $1 - p$, respectively.

This single-step evolution can be iterated $N$ times to yield the quantum trajectories at $t = T$. Specifically, if only the syndrome measurements at $t = k_{1}\Delta, \ldots, k_{m}\Delta$ report sign switches, then the un-normalized quantum trajectory at $t = T$ is

$$\hat{p}_{k_{1} \cdots k_{m}} = M_{k_{m}} \cdots M_{k_{1}} \hat{p} \rho_{0},$$

and its occurrence probability

$$P_{k_{1} \cdots k_{m}} \equiv \text{Tr} \hat{p}_{k_{1} \cdots k_{m}} = p^{N-m}(1 - p)^{m}$$

is independent of $k_{1}, \ldots, k_{m}$. Since $p$ is independent of $\omega$, the CFI about $\omega$ vanishes: $F_{\omega} = 0$, and the CFI about $\gamma$ follows from Eq. (18) as

$$F_{\gamma} = \frac{\sum_{m=0}^{N} \sum_{k_{1} \cdots k_{m}} \left( \partial_{\gamma} \rho \right)^{2}}{p_{k_{1} \cdots k_{m}}} \left( \frac{N}{4} \gamma \right) \approx \frac{1}{4} \frac{4 \gamma}{e^{2\gamma} - 1},$$

where $p_{m} \equiv \sum_{k_{1}, \ldots, k_{m}} P_{k_{1} \cdots k_{m}} = C_{N}^{m} p^{N-m}(1 - p)^{m}$ is the probability for $m$ quantum jumps during $[0,T]$ and $C_{N}^{m} = N!/(m!(N - m)!)$ is the binomial coefficient. Next, we calculate the QFIs in the normalized quantum trajectories: $\rho_{k_{1} \cdots k_{m}} \equiv \hat{p}_{k_{1} \cdots k_{m}} / P_{k_{1} \cdots k_{m}}$.

When $\omega = 0$, we have $f(\Delta) = \cos \theta$, $g(\Delta) = \sin \theta$, and hence $M_{k} \rho_{0} = \rho_{0}$ and $M_{k} \rho_{0} = (1 - p)\rho_{0}$. So $\rho_{k_{1} \cdots k_{m}} = \rho_{0}$ (for even $m$) or $\rho_{k_{1} \cdots k_{m}} = \sigma_{i} \rho_{0} \sigma_{i}$ (for odd $m$) is independent of $\gamma$ and hence give zero trajectory QFI about $\gamma$. In this case, the total information from MQT is $F_{\gamma} = F_{\gamma}$.

When $\omega \neq 0$, to obtain explicit analytical expression, we assume $1/\Delta \gg \omega \gg \gamma$ and $\gamma T \ll 1/(\gamma \Delta)$, so that $f(\Delta) \approx e^{-i \omega \Delta}$, $g(\Delta) \approx \zeta$, and hence

$$\rho_{k_{1} \cdots k_{m}} = \begin{cases} \rho_{\uparrow \downarrow} a & \text{(for } m = 2n + 1), \\ \rho_{\downarrow \uparrow} b & \text{(for } m = 2n), \\ \end{cases}$$

where

$$a \equiv \frac{\rho_{\uparrow \downarrow} e^{-i \omega \Delta/4}}{p^{2n-2p+1}((1 - p)\rho_{0})^{2n+1}} e^{-i \omega \Delta + (2n)^{2} \sum_{i=1}^{N/m} (1/f (k_{i}))},$$

$$b \equiv \frac{\rho_{\downarrow \uparrow} e^{-i \omega \Delta/4}}{p^{2n-2p+1}((1 - p)\rho_{0})^{2n+1}} e^{-i \omega \Delta + (2n)^{2} \sum_{i=1}^{N/m} (1/f (k_{i}))}.$$
