Historically, understanding thermal conductivity has played an important role in the development of materials theory. There has recently been a growth of interest in studying this basic property in strongly correlated materials.\textsuperscript{1-3} One of the recurrent consequences of strong interactions is the development of anisotropic gaps in the excitation spectrum: possible examples include heavy fermion superconductors, Kondo insulators and the chevrel superconductor \textit{V}_3\textit{Si}.\textsuperscript{3,4} In such systems, the thermal conductivity has an important role to play in the elucidation of the gap symmetry.

Pioneering work on the theory of thermal conductivity was carried out in the sixties\textsuperscript{5-7}; by and large, the theoretical approaches taken today are a direct application of this early work\textsuperscript{8-10}. In this paper, we show how these classic approaches require modification to take account of new thermal conduction channels that are introduced by interactions. As part of this discussion, we shall show how thermal conductivity, like particle or charge conductivity, can be regarded as a boundary condition response which is most directly computed as a response to a fictitious gauge field. We illustrate this approach in the context of anisotropic superconductivity, showing how interaction contributions to the thermal current significantly modify the flow of heat produced by excitations near a gap node.

Thermal conductivity is directly related, via the Kubo formula, to thermal current fluctuations
\begin{equation}
\kappa^{ab} = -\beta \lim_{\omega \to 0} \frac{d}{d\omega} \Re \left\{ \int_0^\infty dt e^{i\omega t} \langle [a^b(t), b^a(0)] \rangle \right\}.
\end{equation}

There is an important distinction between heat and charge conductivity. Unlike charge density, the energy density does not commute with the interactions: this means that heat is directly transmitted by interactions, introducing new interaction contributions to the microscopic thermal current operator. The effect of these new thermal conduction channels is important when interactions severely modify the dispersion of the electrons, as in the case of a Mott insulator, or a superconductor with gap nodes.\textsuperscript{11-13}

To derive the thermal current we appeal to Noether’s theorem, which relates continuity of energy flow to the covariance of the action under co-ordinate transformations in time \( t \to t' = t + \phi(x, t') \). Consider an electronic system described by the Lagrangian
\begin{equation}
\mathcal{L} = \int d^3x (i \dot{\psi}^\dagger \gamma^\mu \partial_\mu \psi - H),
\end{equation}
where \( H \) is the Hamiltonian and \( \partial_\mu = \frac{1}{2} (\partial_x - \partial_y) \) is the antisymmetrized time-derivative. By Noether’s theorem, the change in the action \( S = \int \mathcal{L}(x,t) dt \) is the energy continuity equation
\begin{equation}
\delta S/\delta \phi(x, t) = - \left[ \partial_t \epsilon(x, t) + \nabla \cdot \bar{J}(x, t) \right],
\end{equation}
where \( \epsilon \) and \( \bar{J} \) are the energy and thermal current density respectively. Continuity of energy flow follows from the invariance of the action under co-ordinate transformations \( \delta S/\delta \phi(x, t) = 0 \). Writing \( \delta S = \int [\delta S/\delta \phi(\phi(x, t), t) \phi(x, t) \cdot \bar{J}(x, t)] \) and integrating by parts, we find
\begin{equation}
\delta S = \int dt d\bar{x} \left[ \phi(\bar{x}, t) \epsilon(\bar{x}, t) + \nabla \phi(\bar{x}, t) \cdot \bar{J}(\bar{x}, t) \right]
\end{equation}
so that \( \epsilon(x) = \delta S/\delta \phi \), while the heat current is \( \bar{J} = \delta S/\delta \nabla \phi \). The calculation of the thermal current requires special care because interactions are non-local and all higher derivatives of the fields must be taken into account when taking the functional derivative. This important point was overlooked in the classic treatment by Langer.\textsuperscript{8} We can take these effects into account by noting that the derivative operators inside the action are evaluated at constant time \( t \), so that under the transformation \( t \to t' \), they acquire a covariant form
\begin{equation}
-\nabla \to -\nabla - \hat{\bar{A}} \tilde{\omega},
\end{equation}
where we denote \( \tilde{\omega} = i \partial_{\mu} \). The field \( \hat{\bar{A}} = \nabla \phi \) adjusts for the fact that spatial derivative \( \nabla \equiv \nabla |_{t} \) is taken at constant \( t' \). It follows that the energy current is
\begin{equation}
\bar{J} = \delta S/\delta \nabla \phi = - \partial H[\hat{\bar{A}}]/\partial \hat{\bar{A}},
\end{equation}
where \( H[\hat{\bar{A}}] \) is the “gauged” Hamiltonian with the replacement \( \hat{k} \to \hat{k} - \hat{\bar{A}} \tilde{\omega} \) in both the kinetic and interaction terms. We see that \( \hat{\bar{A}} \) plays the role of a “fictitious” gauge field conjugate to the thermal current.
To illustrate this procedure, consider a fluid of electrons with kinetic energy $\varepsilon_{\mathbf{k}}$ and an exchange interaction $J_k$, where

$$H[\hat{A}] = \sum_{\mathbf{k}} \left[ \psi^\dagger_{\mathbf{k}} \varepsilon_{\mathbf{k}} \psi_{\mathbf{k}} + \frac{1}{2} \hat{J}_{\mathbf{k}} \cdot \hat{A}_{\mathbf{k}} \right],$$

where $\hat{A}_{\mathbf{k}}$ is the Fourier transform of the spin density at wavevector $\mathbf{k}$. Differentiating with respect to $\hat{A}$ gives

$$\hat{j} = i \sum_{\mathbf{k}} \left[ (\nabla_{\mathbf{k}} \varepsilon_{\mathbf{k}}) \psi^\dagger_{\mathbf{k}} \hat{A}_{\mathbf{k}} + \frac{1}{2} (\nabla_{\mathbf{k}} J_{\mathbf{k}}) \hat{\sigma}_{\mathbf{k}} \cdot \hat{\sigma}_{\mathbf{k}} \right].$$

The second term reflects the additional heat flow created by the exchange interaction. In an antiferromagnetic Mott insulator, it is this term which is responsible for the conduction of heat by spin-waves. The approach illustrated here can be used on any Hamiltonian, without the need to develop the equations of motion for that particular model. Past efforts to compute the thermal current contribution from interactions have adopted an equation of motion approach on a case-by-case basis. This yields long expressions for the thermal current, where the time derivatives are expanded in their full glory.$^{11,12}$

The usefulness of the gauge-theoretic derivation of the thermal current lies partly in its invariance under the renormalization group. This means that it can be directly applied to the effective Lagrangian that describes the low energy physics of an interacting system. In any case where the low energy physics is described by weakly interacting quasiparticles with energy $E_{\mathbf{k}}$, application of the same procedure yields

$$\hat{j} = i \sum_{\mathbf{k}} (\nabla_{\mathbf{k}} E_{\mathbf{k}}) \hat{a}^\dagger_{\mathbf{k}} \hat{\sigma}_{\mathbf{k}} \cdot \hat{\sigma}_{\mathbf{k}} = \sum_{\mathbf{k}} (E_{\mathbf{k}} \nabla_{\mathbf{k}} E_{\mathbf{k}}) n_{\mathbf{k}},$$

where $n_{\mathbf{k}} = \hat{a}^\dagger_{\mathbf{k}} \hat{a}_{\mathbf{k}}$ is the quasiparticle number operator and we have used the equation of motion to make the last substitution. When the mass renormalization of the electrons is highly anisotropic, the quasiparticle thermal current contains a large interaction component. Microscopic calculations that ignore these contributions fail to recover the correct quasiparticle description of the heat current at low temperatures.

A particularly important illustration of this effect occurs in an anisotropic superconductor. In this case, the BCS Hamiltonian takes the form

$$H[\hat{A}] = \frac{1}{2} \sum_{\mathbf{k}} \Psi^\dagger_{\mathbf{k}} \Delta_{\mathbf{k}} \Psi_{\mathbf{k}},$$

where $\Delta_{\mathbf{k}}$ is the anisotropic gap function, $\Psi^\dagger_{\mathbf{k}}$ is the four-component Nambu spinor for the electrons. The thermal current operator is then

$$\hat{j} = \frac{1}{2} \sum_{\mathbf{k}} \Psi^\dagger_{\mathbf{k}} \nabla_{\mathbf{k}} \hat{E}_{\mathbf{k}} \hat{\omega} \Psi_{\mathbf{k}}.$$

The off-diagonal terms in this expression derive from the interaction contribution to the thermal current. In the presence of weak scattering, we can use the equation of motion to replace

$$\nabla_{\mathbf{k}} \Delta_{\mathbf{k}} \hat{\omega} \rightarrow \frac{1}{2} \{ \hat{E}_{\mathbf{k}}, \nabla_{\mathbf{k}} \hat{E}_{\mathbf{k}} \} = E_{\mathbf{k}} \nabla_{\mathbf{k}} E_{\mathbf{k}} \hat{1},$$

where $E_{\mathbf{k}} = (\varepsilon^2_{\mathbf{k}} + \Delta^2_{\mathbf{k}})^{1/2}$. This recovers the quasiparticle form (9). Note however, that the Fermi velocity is replaced by the quasiparticle group velocity $\nabla_{\mathbf{k}} = \nabla_{\mathbf{k}} E_{\mathbf{k}}$, which has important components parallel to the Fermi surface. These can significantly affect the anisotropy of the thermal conductivity. We now use (11) and apply the Kubo formula, to obtain

$$\kappa^{ij} = \frac{\pi}{2T} \sum_{\mathbf{k}} \int_{-\infty}^\infty d\omega \omega^2 \left( -\frac{\partial f}{\partial \omega} \right) \Lambda^{ij}(\mathbf{k}, \omega),$$

where

$$\Lambda^{ij}(\mathbf{k}, \omega) = \text{Re} \left\{ Tr \left[ (\nabla_{\mathbf{k}} \hat{E}_{\mathbf{k}})^i A_{\mathbf{k}}(\omega) (\nabla_{\mathbf{k}} \hat{E}_{\mathbf{k}})^j A_{\mathbf{k}}(\omega) \right] \right\},$$

and

$$A_{\mathbf{k}}(\omega)$$

is the matrix spectral function and $f$ is the Fermi function. Carrying out the trace we find

$$\kappa^{ij} = \frac{1}{2T} \int_{-\infty}^\infty d\omega \omega^2 \left( -\frac{df}{d\omega} \right) \frac{N(\omega)}{T(\omega)} \langle \psi^i \psi^j \rangle \omega,$$
where $\Gamma(\omega)$ is the quasiparticle scattering rate and
\[
N(\omega)\langle \tilde{v}^i \tilde{v}^j \rangle = \sum_k \tilde{v}_{ik}^j \tilde{v}_{kj}^j \delta(\omega - \tilde{E}_k)
\]  
(16)
describes the quasiparticle velocity distribution.

To show how the interaction contribution to the thermal current modifies the thermal conductivity, consider the case of a “polar” superconductor, with a spherical Fermi surface and gap function $|\Delta_k|^2 = \Delta_0^2 k_z^2$, which leads to a line of gap nodes in the basal plane $k_z = 0$. The kinetic contribution to the thermal current operator always lies normal to the electron Fermi surface, and if we omit the effects of interaction, the low temperature thermal conductivity lies predominantly in the basal plane.

In the correct calculation, the current carried by each excitation lies in the direction of the quasiparticle velocity $v^Q_k$, which lies normal to the surfaces of constant energy (Fig. 1). The line-node in the basal plane is surrounded by elliptical surfaces of constant energy and consequently, in the vicinity of the gap node there are quasiparticles which propagate perpendicular to the basal plane, giving rise to a increased thermal conductivity in this direction.

At low temperatures the thermal conductivity in both directions is proportional to $T^2$, with a finite ratio at $T = 0$
\[
\frac{\kappa^{zz}}{\kappa^{xx}} = \frac{1}{2} \left( \frac{\Delta_0}{\epsilon_F} \right)^2.
\]  
(17)
Without interaction effects, this ratio vanishes at $T = 0$.

In Figure 2 we show how the anisotropy in the thermal conductivity varies as the ratio between the maximum gap and the Fermi energy is increased from zero. Two effects of interest are observed: (i) the anisotropy is reduced by these interaction effects; (ii) the thermal conductivity is enhanced in the region close to the transition temperature. These effects become important when the transition temperature is a substantial fraction of the Fermi energy. Such considerations may be particularly relevant to heavy fermion superconductors, of which two cases are particularly worthy of mention: $UBe_{13}$ and $UPt_3$. In both cases, NMR measurements suggest the presence of line nodes. In the former case, superconductivity develops before a Fermi liquid regime has been established, so it is very likely that $\Delta/\epsilon_F$ is large. In the latter case, the conventional wisdom is that $\Delta/\epsilon_F \sim 0.05 - 0.1$. However, measurements of the thermal conductivity on very pure samples of this material show that $\kappa^{zz}/\kappa^{xx} \sim$ constant at low temperatures; they also display a marked peak in the ratio $\kappa^{zz}/\kappa^{nn}$ near $T_c$. Existing approaches attribute these effects to a combination of both elastic and inelastic scattering. While such an explanation is surely feasible, an alternative explanation might be obtained by assuming a larger ratio $\Delta/\epsilon_F$, and attributing these features to the interaction contribution to the thermal current that has been hitherto ignored.

The main point of this paper has been to emphasize that interactions actually modify the thermal current, which can have important effects on the thermal conductivity. As an illustration of these ideas, we have considered the effect of gap anisotropy in polar superconductors. Our basic approach can be applied to other gap anisotropic systems. One very interesting case in this respect may be the small gap Kondo insulators, where the insulating gap appears to exhibit a point node\textsuperscript{14}. Similar considerations can also be applied to the momentum current, a point which is important in the interpretation of measurements of the ultrasound attenuation in strongly correlated systems. This could explain why early ultrasound measurements in $UBe_{13}$\textsuperscript{15} found no anisotropy, despite the clear suggestion of line nodes from NMR measurements\textsuperscript{16}.

A secondary aspect of our work concerns the observation that thermal currents can be treated using a gauge theoretic approach. Just as momentum currents, or stress ($\sigma$) in a material result from the gauge field we call strain ($u$), $\sigma = R u$, energy currents result from a “temporal strain” which we represent by the gauge field $\vec{A}$. If we calculate the heat current response to this fictitious field
\[
\vec{j}_T(\omega) = -Q(\omega) \vec{A}(\omega),
\]  
(18)
we find that the “London Kernel” is directly related to the thermal conductivity by the relation $Q(\omega) = i \omega T \kappa(\omega)$, from which we deduce that
\[ \frac{\partial \vec{A}}{\partial t} = -\vec{\nabla} T \cdot T. \]  

(19)

In other words, \( \vec{A} \) is the gauge field conjugate to thermal gradients. By analogy with the case of momentum currents, the quantity \( \vec{A} \) may be regarded as a sheer in time. Just as broken spatial translation symmetry enables a material to support a persistent stress or "momentum superflow", we are led to speculate that if it were possible to produce a state of condensed matter with spontaneously broken time translation symmetry, such a system would exhibit "heat superflow", manifested by an infinite thermal conductivity.

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