On the nonexistence of Einstein metric on 4-manifolds

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Abstract
By using the gluing formula of the Seiberg-Witten invariant, we show the nonexistence of Einstein metric on manifolds obtained from a 4-manifold with nontrivial Seiberg-Witten invariant by performing sufficiently many connected sums or appropriate surgeries along circles or homologically trivial 2-spheres with closed oriented 4-manifolds with negative definite intersection form.

1 Introduction
A smooth Riemannian manifold \((M, g)\) is called Einstein if it satisfies

\[ Ric_g = cg, \]

where \(Ric_g\) denotes the Ricci curvature of \(g\), and \(c\) is a constant. When the dimension of \(M\) is less than 4, any Einstein manifold is a space form whose classification is well-known. In higher dimensions, it is in general difficult to decide whether a manifold admits an Einstein metric. Unlike the dimension greater than 4 where no topological obstruction is known, any closed orientable 4 manifold \(M\) admitting an Einstein metric must satisfy the Hitchin-Thorpe inequality \([3, 6, 13]\)

\[ 2\chi(M) + 3|\tau(M)| \geq 0 \]

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with equality held only by a quotient of $K3$ surface or 4-torus, where $\chi(M)$ and $\tau(M)$ respectively denote the Euler characteristic and the signature of $M$. This well-known inequality is the consequence of the 4-dimensional Chern-Gauss-Bonnet formula.

Since the 4-dimensional geometry is complicated by the possible existence of many smooth structures, the condition for the existence of Einstein metric on 4-manifolds inevitably involve the underlying smooth structure. It was the Seiberg-Witten theory that has brought a remarkable improvement of the Hitchin-Thorpe condition. LeBrun exploited the curvature estimate coming from the Seiberg-Witten theory to derive that any closed Einstein 4-manifold $M$ with a monopole class satisfies

$$\chi(M) \geq 3\tau(M)$$

with equality held only by a compact complex hyperbolic 2-space or a flat 4-manifold ([7]), and

**Theorem 1.1 (LeBrun [9])** Let $M$ be a smooth closed oriented 4-manifold with a nontrivial Seiberg-Witten invariant. Then $M \# k\mathbb{C}P^2 \# l(S^1 \times S^3)$ does not admit Einstein metric if $k + 4l > 0$ and $k + 4l \geq \frac{1}{3}(2\chi(M) + 3\tau(M))$.

In this article, we generalize this theorem to :

**Theorem 1.2** Let $M$ be a smooth closed oriented 4-manifold with a nontrivial Seiberg-Witten invariant and $N$ be a smooth closed oriented 4-manifold with $b_2^+(N) = 0$. Then $M \# N$ does not admit Einstein metric if

$$b_2(N) + 4b_1(N) > 0$$

and

$$b_2(N) + 4b_1(N) \geq \frac{1}{3}(2\chi(M) + 3\tau(M)).$$

**Definition 1** Let $M_1$ and $M_2$ be smooth n-manifolds and suppose that $k$-spheres $c_1$ and $c_2$ are embedded into $M_1$ and $M_2$ respectively with trivial normal bundle. A surgery of $M_1$ and $M_2$ along $c_i$‘s are defined as the result of deleting tubular neighborhood of each $c_i$ and gluing the remainders by identifying two boundaries $S^k \times S^{n-k-1}$ using a diffeomorphism of $S^k$ and the reflection map of $S^{n-k-1}$.
Note that the surgery on $M$ with $(S^1 \times S^3) \# N$ along a null-homotopic circle in $M$ and a circle representing $[S^1] \times \{\text{pt}\} \in H_1(S^1 \times S^3, \mathbb{Z})$ gives $M \# N$. More generally, we will prove:

**Theorem 1.3** Let $M$ be a smooth closed oriented 4-manifold with a nontrivial Seiberg-Witten invariant and $N_i$ be a smooth closed oriented 4-manifold with $b_2^+(N_i) = 0$ and $b_1(N_i) \geq 1$ for $i = 1, \cdots, m$. Suppose that $c_i \subset N_i$ is an embedded circle nontrivial in $H_1(N_i, \mathbb{R})$ for $i = 1, \cdots, m$, and $\tilde{M}$ is a manifold obtained from $M$ by performing a surgery with $\bigcup_{i=1}^m N_i$ along $\bigcup_{i=1}^m c_i$.

Then $\tilde{M}$ does not admit Einstein metric if

$$\sum_{i=1}^m (b_2(N_i) + 4(b_1(N_i) - 1)) > 0$$

and

$$\sum_{i=1}^m (b_2(N_i) + 4(b_1(N_i) - 1)) \geq \frac{1}{3} (2\chi(M) + 3\tau(M)).$$

Most generally, we can also allow surgeries along homologically trivial 2-spheres to give:

**Theorem 1.4** Let $M$ be a smooth closed oriented 4-manifold with a nontrivial Seiberg-Witten invariant, and $N_i, \tilde{N}_j$ for $i = 1, \cdots, m$ and $j = 1, \cdots, n$ be smooth closed oriented 4-manifolds such that $b_2^+(N_i) = b_2^+(\tilde{N}_j) = 0$ and $b_1(N_i) \geq 1$. Suppose that $c_i \subset N_i$ for $i = 1, \cdots, m$ is an embedded circle nontrivial in $H_1(N_i, \mathbb{R})$, and $F_j \subset M$ and $\tilde{F}_j \subset \tilde{N}_j$ for $j = 1, \cdots, n$ are embedded 2-spheres trivial in $H_2(M, \mathbb{R})$ and $H_2(\tilde{N}_j, \mathbb{R})$ respectively.

If $\tilde{M}$ is a manifold obtained from $M$ by performing a surgery with $\bigcup_{i=1}^m N_i$ along $\bigcup_{i=1}^m c_i$, and with $\bigcup_{j=1}^n \tilde{N}_j$ along $\bigcup_{j=1}^n F_j$ and $\bigcup_{j=1}^n \tilde{F}_j$, then $\tilde{M}$ does not admit Einstein metric if

$$\sum_{i=1}^m (b_2(N_i) + 4(b_1(N_i) - 1)) + \sum_{j=1}^n (b_2(\tilde{N}_j) + 4(b_1(\tilde{N}_j) + 1)) \geq \frac{1}{3} (2\chi(M) + 3\tau(M)).$$

## 2 Computation of Seiberg-Witten invariant

We will give a brief definition of the Seiberg-Witten invariant. Let $M$ be a smooth oriented Riemannian 4-manifold and $\mathfrak{s}$ be a Spin$^c$ structure on it.
We assume that $M$ is closed or noncompact with a cylindrical-end metric. Let $A(M)$ be the graded algebra over $\mathbb{Z}$ defined by

$$\mathbb{Z}[H_0(M; \mathbb{Z})] \otimes \wedge \ast H_1(M; \mathbb{Z})$$

with $H_0(M; \mathbb{Z})$ grading two and $H_1(M; \mathbb{Z})$ grading one. An element in $A(M)$ canonically gives a cocycle of the Seiberg-Witten moduli space, i.e. the solution space modulo gauge transformations of the Seiberg-Witten equations of $(M, s)$. Thus the evaluation on the fundamental cycle of the moduli space is the Seiberg-Witten invariant as a function

$$SW_{M,s} : A(M) \to \mathbb{Z}.$$  

When $b^+_2(M) > 1$, this is independent of a Riemannian metric and a perturbation term, thus giving a topological invariant. (If $b^+_2(M) = 1$, it may depend on the chamber.) The first Chern class of a Spin$^c$ structure on $M$ whose Seiberg-Witten invariant is nontrivial is called a basic class of $M$. For more details on the Seiberg-Witten invariant, the readers are referred to [10, 11, 12].

We will need the following gluing formulae of the Seiberg-Witten invariant.

**Lemma 2.1** Let $N$ be a closed oriented smooth 4-manifold with negative-definite intersection form $Q$. Then there exists a Spin$^c$ structure $s'$ on $N$ satisfying $c_1^2(s') = -b_2(N)$.

**Proof.** By the Donaldson’s theorem, $Q$ is diagonalizable. (The original Donaldson’s theorem [4] is stated for the simply-connected case, but a simple application of the Mayer-Vietoris argument gives this generalization.) Let $\{\alpha_1, \cdots, \alpha_{b_2(N)}\}$ be a basis of $H^2(N, \mathbb{Z}) \otimes \mathbb{Q}$ diagonalizing $Q$.

We have to show that there exists an element $x \in H^2(N, \mathbb{Z})$ such that $Q(x, x) = -b_2(N)$, and $x$ is characteristic, i.e. $Q(x, \alpha) \equiv Q(\alpha, \alpha) \mod 2$ for any $\alpha \in H^2(N, \mathbb{Z})$. This is done by taking $x = \sum_{i=1}^{b_2(N)} \pm \alpha_i$. 

**Theorem 2.2** Let $M$ and $N$ be smooth closed oriented 4-manifolds such that $b^+_2(M) > 0$, $b^+_2(N) = 0$, and $b_1(N) \geq 1$. Let $c \subset N$ be an embedded circle nontrivial in $H_1(N, \mathbb{R})$ and $\tilde{M}$ be the manifold obtained by performing a surgery on $M$ with $N$ along $c$. 


If \( \tilde{s} \) is the Spin\(^c\) structure on \( \tilde{M} \) obtained by gluing a Spin\(^c\) structure \( s \) on \( M \) and a Spin\(^c\) structure \( s' \) on \( N \) satisfying \( c^2_1(s') = -b_2(N) \), then

\[
SW_{\tilde{M}, \tilde{s}}(a \cdot [d_1] \cdots [d_{b_1(N)-1}]) = \pm SW_{M, s}(a)
\]

for \( a \in A(M) \), where \([d_1], \cdots, [d_{b_1(N)-1}]\) along with \( r[c] \) for some \( r \in \mathbb{Q} \) form a basis for the non-torsion part of \( H_1(N, \mathbb{Z}) \).

**Proof.** See [12].

**Theorem 2.3** Let \( M \) and \( N \) be smooth closed oriented 4-manifolds such that \( b_2^+(M) > 0 \), and \( b_2^+(N) = 0 \). Suppose that \( F \subset M \) and \( \tilde{F} \subset N \) are embedded 2-spheres trivial in \( H_2(M, \mathbb{R}) \) and \( H_2(N, \mathbb{R}) \) respectively, and \( \tilde{M} \) is the manifold obtained by performing a surgery on \( M \) with \( N \) along \( F \) and \( \tilde{F} \).

If \( \tilde{s} \) is the Spin\(^c\) structure on \( \tilde{M} \) obtained by gluing a Spin\(^c\) structure \( s \) on \( M \) and a Spin\(^c\) structure \( s' \) on \( N \) satisfying \( c^2_1(s') = -b_2(N) \), then

\[
SW_{\tilde{M}, \tilde{s}}(a \cdot [\gamma] \cdot [d_1] \cdots [d_{b_1(N)}]) = \pm SW_{M, s}(a)
\]

for \( a \in A(M) \), where \( \gamma \) is a circle \( \{ \text{pt} \} \times D^2 \) in a small tubular neighborhood \( F \times D^2 \) of \( F \), and \([d_1], \cdots, [d_{b_1(N)}]\) form a basis for the non-torsion part of \( H_1(N, \mathbb{Z}) \).

**Proof.** Perform a surgery on \( M \) with \( S^4 \) along \( F \) to obtain \( M' \). In the same way, we get \( N' \). The surgery on \( M' \) with \( N' \) along the circle \( \gamma \) gives \( \tilde{M} \).

**Lemma 2.4** Let \( \tilde{M} \) be the manifold obtained from \( M \) by deleting a small tubular neighborhood of \( F \). Then

\[
H_1(M', \mathbb{R}) \cong H_1(\tilde{M}, \mathbb{R}) \cong H_1(M, \mathbb{R}) \oplus \mathbb{R},
\]

and

\[
H_2(M', \mathbb{R}) \cong H_2(\tilde{M}, \mathbb{R}) \cong H_2(M, \mathbb{R}),
\]

where the additional \( \mathbb{R} \)-factor is generated by \([\gamma]\), and the isomorphisms are induced by the obvious inclusions. Likewise for \( N' \).
Proof. Obviously \( H_1(M', \mathbb{R}) \simeq H_1(\hat{M}, \mathbb{R}) \), because \( \pi_1(M') \simeq \pi_1(\hat{M}) \) by the Seifert-Van Kampen theorem. To see \( H_1(\hat{M}, \mathbb{R}) \simeq H_1(M, \mathbb{R}) \oplus \mathbb{R} \), it is enough to show that \( i_* \) in the following commutative diagram of exact sequences is injective.

\[
\begin{array}{ccccccccc}
H_2(\hat{M}, \partial\hat{M}) & \xrightarrow{\partial_*} & H_1(\partial\hat{M}) & \xrightarrow{i_*} & H_1(\hat{M}) \\
P D & & & & & & & & P D \\
H^2(\hat{M}) & \xrightarrow{i^*} & H^2(\partial\hat{M}) & \xrightarrow{\partial^*} & H^3(\hat{M}, \partial\hat{M}).
\end{array}
\]

Suppose not. Then \( i^* \) in the above diagram is surjective. This means that there exists a nonzero element in \( H^2(M) \), which is dual to \([F]\), yielding a contradiction. This also means that \([F]\) is zero in \( H_2(M, \mathbb{R}) \), which will be used just below.

The fact \( H_2(\hat{M}, \mathbb{R}) \simeq H_2(M, \mathbb{R}) \) follows from the exact sequence

\[
H_2(\partial\hat{M}) \xrightarrow{i_*} H_2(\hat{M}) \oplus H_2(S^2 \times D^2) \xrightarrow{\phi} H_2(M) \to 0,
\]

and similarly the fact \( H_2(\hat{M}, \mathbb{R}) \simeq H_2(M', \mathbb{R}) \) follows from the exact sequence

\[
H_2(\partial\hat{M}) \xrightarrow{i_*} H_2(\hat{M}) \oplus H_2(D^3 \times S^1) \xrightarrow{\phi} H_2(M') \to 0,
\]

where the sequences end with 0, because \( i_* : H_1(\partial\hat{M}) \to H_1(\hat{M}) \) is injective.

Note that \( s \) and \( s' \) restrict to be trivial on \( F \) and \( \bar{F} \) respectively. Thus we abuse the notation to let \( s \) and \( s' \) be the induced Spin\(^c\) structures on \( M' \) and \( N' \) respectively. By Ozsváth and Szabó [11],

\[
SW_{M', s}(a \cdot [\gamma]) = \pm SW_{M, s}(a)
\]

for \( a \in \mathbb{A}(M) \). Applying the previous theorem [2.2],

\[
SW_{M', s}(a \cdot [\gamma]) = \pm SW_{\tilde{M}', \tilde{s}}(a \cdot [\gamma] \cdot [d_1] \cdots [d_{b_1(N)}])
\]

for \( a \in \mathbb{A}(M) \).
3 Proof of Theorem 1.3

We need to have a basic class on \( \tilde{M} \). Let \( s \) be the \( \text{Spin}^c \) structure on \( M \) with a nontrivial Seiberg-Witten invariant. Applying theorem 2.2 successively, \( \tilde{M} \) has nontrivial Seiberg-Witten invariant for \( \tilde{s} \). Write \( c_1(\tilde{s}) \) as \( c_1(s) + E \) where \( E = c_1(s') \) coming from \( \cup_{i=1}^m N_i \).

Then the proof proceeds in a similar way to [9]. First,

\[
\chi(\tilde{M}) = \chi(M) + \sum_{i=1}^m \chi(N_i)
= \chi(M) + \sum_{i=1}^m (2 - 2b_1(N_i) + b_2(N_i)),
\]

and

\[
H_2(\tilde{M}, \mathbb{Z}) \cong H_2(M, \mathbb{Z}) \oplus (\oplus_{i=1}^m H_2(N_i, \mathbb{Z}))
\]

by a simple Mayer-Vietoris argument. (Here, we use the fact that \( c_i \)'s are all non-torsion.) Thus

\[
2\chi(\tilde{M}) + 3\tau(\tilde{M}) = 2\chi(M) + 3\tau(M) - \sum_{i=1}^m (b_2(N_i) + 4(b_1(N_i) - 1)).
\]

Lemma 3.1 Any Riemannian metric \( g \) on \( \tilde{M} \) satisfies

\[
\frac{1}{4\pi^2} \int_{\tilde{M}} \left( \frac{s_g^2}{24} + 2|W_g|^2 \right) \, d\mu_g \geq \frac{2}{3} \left( 2\chi(M) + 3\tau(M) \right).
\]

Proof. Since \( c_1(\tilde{s}) + E \) and \( c_1(\tilde{s}) - E \) are basic classes of \( \tilde{M} \), LeBrun’s estimate [9] gives

\[
\frac{1}{4\pi^2} \int_{\tilde{M}} \left( \frac{s_g^2}{24} + 2|W_g|^2 \right) \, d\mu_g \geq \frac{2}{3} \left( (c_1(\tilde{s}) \pm E)^+ \right)^2,
\]

where \((\cdot)^+\) denotes the self-dual harmonic part. On the other hand,

\[
((c_1(\tilde{s}) \pm E)^+)^2 = (c_1(\tilde{s})^+ \pm E^+)^2 \\
= (c_1(\tilde{s})^+)^2 \pm 2c_1(\tilde{s})^+ \cdot E^+ + (E^+)^2 \\
\geq (c_1(\tilde{s})^+)^2 \pm 2c_1(\tilde{s})^+ \cdot E^+.
\]

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Thus at least one of \(((c_1(s) + E)^+)\) and \(((c_1(s) - E)^+)\) should be greater than or equal to \((c_1(s)^+)\). Say \(((c_1(s) + E)^+) \geq (c_1(s)^+)\). Then

\[
((c_1(s) + E)^+) \geq c_1^2(s) \geq 2\chi(M) + 3\tau(M),
\]

where we used the fact that \(d(s) := \frac{1}{4}(c_1^2(s) - (2\chi(M) + 3\tau(M)))\), the dimension of the Seiberg-Witten moduli space of \((M, s)\) is nonnegative. □

Now suppose that \(g\) is an Einstein metric on \(\tilde{M}\). Then the Chern-Gauss-Bonnet formula gives:

\[
2\chi(\tilde{M}) + 3\tau(\tilde{M}) = \frac{1}{4\pi^2} \int_{\tilde{M}} \left( \frac{s^2}{24} + 2|W^+|^2 + \frac{|\tilde{r}|^2}{2} \right) d\mu_g
\]

\[
= \frac{1}{4\pi^2} \int_{\tilde{M}} \left( \frac{s^2}{24} + 2|W^+|^2 \right) d\mu_g
\]

\[
\geq \frac{2}{3} (2\chi(M) + 3\tau(M)).
\]

Combined with (1), it follows that

\[
\frac{1}{3} (2\chi(M) + 3\tau(M)) \geq \sum_{i=1}^{m} (b_2(N_i) + 4(b_1(N_i) - 1)) . \quad (3)
\]

It remains to deal with the equality case in the above inequality. Suppose the equality holds. Then from the above we have

\[
((c_1(s) + E)^+) = c_1^2(s) = 2\chi(M) + 3\tau(M). \quad (4)
\]

Suppose \(\sum_{i=1}^{m} (b_2(N_i) + 4(b_1(N_i) - 1)) > 0\), which implies

\[
((c_1(s) + E)^+) \geq 0
\]

by (3) and (4).

From the the equality in (2), LeBrun’s result [9] says that \((\tilde{M}, g)\) must be almost-Kähler with almost-Kähler form a multiple of \((c_1(s) + E)^+\) such that the basic class \(c_1(s) + E\) being the anti-canonical class of the associated almost-complex structure, and the almost-Kähler form is an eigenvector of \(W^+\) everywhere.
Applying Armstrong’s result \cite{Armstrong} that any closed almost-Kähler Einstein 4-manifold whose almost-Kähler form is an eigenvector of $W_+$ everywhere is Kähler, or Apostolov-Armstrong-Drăghici’s result \cite{Apostolov-Armstrong-Drăghici} that any closed almost-Kähler 4-manifold which saturates (2) and whose Ricci tensor is invariant under the almost-complex structure is Kähler, we conclude that $(\tilde{M}, g)$ is Kähler.

Since $(\tilde{M}, g)$ is Kähler-Einstein, we can apply the Enriques-Kodaira classification of compact complex surfaces. Since $\tilde{M}$ has a nontrivial Seiberg-Witten invariant, its Kodaira dimension is nonnegative. Then it is minimal, because it admits a Kähler-Einstein metric.

Now the anti-canonical class is non-torsion, because $c_2^1(s) > 0$ from (4). Then the basic classes of such a minimal Kähler surface are numerically equivalent to $rc_1(K)$, where $|r| \leq 1$ is a rational number, and $K$ is the canonical line bundle. (See \cite{Wu}.) But $\pm(c_1(s) \pm E)$ are basic classes of $\tilde{M}$. This means that $E = 0$, implying that

$$b_2(N_i) = 0 \quad \forall i.$$

Finally using Wu’s formula \cite{Wu} for a closed almost-complex 4-manifold, and (4),

$$0 = (c_1(s) + E)^2 - (2\chi(\tilde{M}) + 3\tau(\tilde{M}))$$

$$= c_1(s)^2 - \sum_{i=1}^{m} b_2(N_i) - (2\chi(M) + 3\tau(M) - \sum_{i=1}^{m} (b_2(N_i) + 4(b_1(N_i) - 1)))$$

$$= -\sum_{i=1}^{m} 4(b_1(N_i) - 1),$$

implying that

$$b_1(N_i) = 1 \quad \forall i.$$  

Thus $\sum_{i=1}^{m} (b_2(N_i) + 4(b_1(N_i) - 1)) = 0$, yielding a contradiction.

### 4 Proof of Theorem 1.4

By successively applying theorem 2.2 and 2.3 the Seiberg-Witten invariant of $(\tilde{M}, \tilde{s})$ is nontrivial, where $\tilde{s}$ is the Spin$^c$ structure gotten by gluing $s$ on $M$ which has nontrivial Seiberg-Witten invariant and $s'$ on $(\bigcup_{i=1}^{m} N_i) \cup (\bigcup_{j=1}^{n} \bar{N}_j)$ such that $c_2^1(s'|_{N_i}) = -b_2(N_i)$ and $c_2^1(s'|_{\bar{N}_j}) = -b_2(\bar{N}_j)$ for all $i, j$. 

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As before, we have

\[
\chi(\tilde{M}) = \chi(M) + \sum_{i=1}^{m} \chi(N_i) + \sum_{j=1}^{n} (\chi(\tilde{N}_j) - 4)
\]

\[
= \chi(M) + \sum_{i=1}^{m} (2 - 2b_1(N_i) + b_2(N_i)) + \sum_{j=1}^{n} (-2 - 2b_1(\tilde{N}_j) + b_2(\tilde{N}_j)),
\]

and

\[
H_2(\tilde{M}, \mathbb{R}) \simeq H_2(M, \mathbb{R}) \oplus (\oplus_{i=1}^{m} H_2(N_i, \mathbb{R})) \oplus (\oplus_{j=1}^{n} H_2(\tilde{N}_j, \mathbb{R}))
\]

by a simple Mayer-Vietoris argument. (Here, we use the fact that $c_i$’s are non-torsion, and $F_j$’s and $\bar{F}_j$’s are all torsion.) Thus

\[
2\chi(\tilde{M}) + 3\tau(\tilde{M}) = 2\chi(M) + 3\tau(M) - \sum_{i=1}^{m} (b_2(N_i) + 4(b_1(N_i) - 1))
\]

\[
- \sum_{j=1}^{n} (b_2(\tilde{N}_j) + 4(b_1(\tilde{N}_j) + 1)).
\]

Now proceeding in the same way as theorem 1.3, the existence of an Einstein metric on $\tilde{M}$ dictates that

\[
\frac{1}{3}(2\chi(M) + 3\tau(M)) \geq \sum_{i=1}^{m} (b_2(N_i) + 4(b_1(N_i) - 1)) + \sum_{j=1}^{n} (b_2(\tilde{N}_j) + 4(b_1(\tilde{N}_j) + 1)),
\]

and if the equality holds, then the left hand side of the above inequality is positive, and the same argument as theorem 1.3 gives that

\[b_2(N_i) = b_2(\tilde{N}_j) = 1 \quad \forall i, j,\]

and

\[
\sum_{i=1}^{m} 4(b_1(N_i) - 1) + \sum_{j=1}^{n} 4(b_1(\tilde{N}_j) + 1) = 0
\]

which is a contradiction.
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