Infinite Product Representation for the Szegö Kernel for an Annulus

Nuraddeen S. Gafai\(^1\),\(^2\) Ali H. M. Murid\(^2\), and Nur H. A. A. Wahid\(^3\)

\(^1\)Department of Mathematics and Statistics, Umaru Musa Yar’adua University Katsina, Nigeria
\(^2\)Department of Mathematical Sciences, Faculty of Science, Universiti Teknologi Malaysia (UTM), 81310 Johor Bahru, Johor, Malaysia
\(^3\)Faculty of Computer and Mathematical Sciences, Universiti Teknologi MARA, 40450 Shah Alam, Selangor, Malaysia

Correspondence should be addressed to Ali H. M. Murid; alihassan@utm.my

Received 23 November 2021; Revised 15 January 2022; Accepted 22 March 2022; Published 12 April 2022

1. Introduction

The Ahlfors map is a branching \(n\)-to-one map from an \(n\)-connected region onto the unit disk. It is intimately tied to the Szegö kernel of an \(n\)-connected region [1]. The boundary values of the Szegö kernel satisfy the Kerzman-Stein integral equation, which is a Fredholm integral equation of the second kind for a region with a smooth boundary [2]. The boundary values of the Ahlfors map are completely determined from the boundary values of the Szegö kernel [1–3]. For an annulus region \(\Omega\), the Szegö kernel can be expressed as a bilateral series from which the zero can be determined analytically [4]. The Kerzman-Stein integral equation has been solved using the Adomian decomposition method in [5] to give another bilateral series form for the Szegö kernel for \(\Omega\) that converges faster. There are various special functions in the form of bilateral and basic bilateral series [6–8]. For example, the bilateral basic hypergeometric series contain, as special cases, many interesting identities related to infinite products, theta functions, and Ramanujan’s identities. It is therefore natural to ask if the bilateral series for the Szegö kernel for \(\Omega\) can be summed as special functions or an infinite product that exhibits clearly its zero.

In this paper, we show how to express the bilateral series for the Szegö kernel for \(\Omega\) as a basic bilateral series (also known as \(q\)-bilateral series). Ramanujan’s sum is then applied to obtain the infinite product representation for the Szegö kernel for \(\Omega\). The product clearly exhibits the zero of the Szegö kernel for \(\Omega\) and its connection with the \(q\)-gamma function and the modified Jacobi theta function is shown. Using the symmetry of Ramanujan’s sum, we show how to easily transform the bilateral series for the Szegö kernel in [4] to the bilateral series in [5].

The plan of the paper is as follows: After the presentation of some preliminaries in Section 2, we derive the basic bilateral series and infinite product representations for the Szegö kernel for \(\Omega\) in Section 3. We then derive a closed form of the Szegö for \(\Omega\) in terms of \(q\)-gamma function and the modified Jacobi theta function. In Section 4, we show how to extend the representations in Section 3 to the general
annulus using the transformation formula for the Szegö kernel under conformal mappings. Similar $q$-analysis for the weighted Szegö kernel for $\Omega$ is presented in Section 5. In Section 6, we give numerical comparisons for computing the Szegö kernel for $\Omega$ using bilateral series, infinite product, and integral equation formulations.

2. Preliminaries

Let $\Omega = \{z : \rho < |z| < 1\}$ be an annulus with $0 < \rho < 1$ and a point $a \in \Omega$. The boundary $\Gamma$ of $\Omega$ consists of two smooth Jordan curves with the outer curve $\Gamma_0$, oriented counterclockwise, and the inner curve $\Gamma_1$ oriented clockwise. The positive direction of the contour $\Gamma = \Gamma_0 \cup \Gamma_1$ is usually that for which the region is on the left as one traces the boundary.

Let $\{\varphi_n(z)\}_{n=1}^{\infty}$ be an orthonormal basis for the Hardy spaces $H^2(\Gamma)$. Since the Szegö kernel $S(z, a)$ is the reproducing kernel for $H^2(\Gamma)$, it can be written as [4]

$$S(z, a) = \sum_{n=0}^{\infty} \varphi_n(z)\varphi_n(\bar{a}), a \in \Omega,$$

(1)

with absolute and uniform convergence on compact subsets of $\Omega$. An orthogonal basis for $H^2(\Gamma)$ is $\{z^n\}_{n=-\infty}^{\infty}$. Thus

$$||z^n||^2 = \int_{\Gamma}|z^n|dz = 2\pi(1 + \rho|z|),$$

(2)

where $|dz|$ is the arc length measure. Therefore, an orthonormal basis for $H^2(\Gamma)$ is [3, 4]

$$\left\{\frac{z^n}{\sqrt{2\pi(1 + \rho|z|)}}\right\}_{n=-\infty}^{\infty} \tag{3}$$

Using (1) and (3), the series representation for the Szegö kernel for $\Omega$ is given by [4]

$$S(z, a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \left(\frac{z\bar{a}^n}{1 + \rho^n|z|}\right), a \in \Omega, z \in \Omega \cup \Gamma. \tag{4}$$

Series (4) is a bilateral series. It has a zero at $z = -\rho/a$ [4]. Another bilateral series representation for the Szegö kernel for $\Omega$ is given by [5] (in an equivalent form)

$$S(z, a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} (-1)^n\rho^n, z \in \Omega \cup \Gamma, a \in \Omega, \tag{5}$$

which is initially obtained by solving the Kerzman-Stein integral equation using the Adomian decomposition method. It is also shown in [5] how to derive (5) directly from (4) using geometric series. It is illustrated in [5] that series (5) converges faster than (4).

More generally, if $\Omega_1$ is any doubly connected region with the smooth boundary $\Gamma_1$, and $f(z)$ is a biholomorphic map of $\Omega_1$ onto $\Omega$, then the Szegö kernel for $\Omega_1$ can be obtained via the transformation formula as [1]

$$S_1(z, a) = \sqrt{f'(z)f(f(z), f(a))}, \tag{6}$$

where $\rho$ is unknown but can be computed.

The Szegö kernel $S_1(z, a)$ can also be computed without using conformal mapping. The boundary values of the Szegö kernel $S_1(z, a)$ on $\Gamma_1$ satisfy the Kerzman-Stein integral equation [2, 4],

$$S_1(z, a) + \int_{\Gamma} A(z, w)S_1(w, a)dw = g(z), z \in \Gamma_1, \tag{7}$$

where

$$A(z, w) = \begin{cases} \frac{1}{2\pi} \left(\frac{T(w)}{z-w} - \frac{T(z)}{z-w}\right), & z \neq w \in \Gamma_1, \\ 0, & z = w \in \Gamma_1, \end{cases} \tag{8}$$

and $z(t)$ is a parametrization of $\Gamma_1$. The function $A(z, w)$ is known as the Kerzman-Stein kernel, and it is continuous on the boundary of $\Omega_1$ [9, 10]. In fact, the integral equation (7) is also valid for an $n$-connected region.

Since bilateral series and basic bilateral series will be used throughout this paper, we recall some facts about $q$-series notations and results.

Let $0 < q < 1$ and $a \in \mathbb{C}$. The $q$-shifted factorial is defined as [7]

$$(q^n ; q)_n = \begin{cases} 1, & n = 0, \\ (1 - q^n)(1 - q^{n+1}) \cdots (1 - q^{n-n-1}), & n = 1, 2, \ldots, \\ \frac{1}{(1 - q^{n-1})(1 - q^{n-2}) \cdots (1 - q^{n-n})}, & n = -1, -2, \ldots. \end{cases} \tag{9}$$

This notation yields the shifted factorial as a special case through

$$\lim_{q \to 1} \frac{(q^n ; q)_n}{(q ; q)_n} = a(a+1) \cdots (a+n-1), n = 1, 2, \ldots. \tag{10}$$
If \( \alpha \) is written in place of \( q^n \), then (9) becomes
\[
(\alpha ; q)_n = \begin{cases} 
1, & n = 0, \\
(1 - \alpha)(1 - aq) \cdots (1 - aq^{n-1}), & n = 1, 2, \ldots, \\
\frac{1}{(1 - aq^{-1})(1 - aq^{-2}) \cdots (1 - aq^{-n})}, & n = -1, -2, \ldots.
\end{cases}
\]
(11)

It can be shown that [7]
\[
\frac{1 - \alpha}{1 - aq^n} = \frac{(\alpha ; q)_n}{(aq^n ; q)_n}, \quad n = 0, \pm 1, \pm 2, \ldots
\]
(12)

If \( n \rightarrow \infty \), it is standard to write
\[
(\alpha ; q)_\infty = \prod_{n=0}^{\infty} (1 - aq^n),
\]
(13)

which is absolutely convergent for all finite values of \( \alpha \), real or complex, when \( |q| < 1 \) [6]. This yields
\[
(\alpha ; q)_n = \frac{(\alpha ; q)_\infty}{(aq^n ; q)_\infty}.
\]
(14)

Observe that \((\alpha ; q)_\infty\) would have zero as a factor if \( \alpha = 1 \). It would be zero also if \( \alpha = q^{-1}, q^{-2}, q^{-3}, \ldots \), but these are all outside the circle \(|z| = 1\) since \(|q| < 1 \) [8].

The bilateral basic hypergeometric series in base \( q \) with one numerator and one denominator parameters is defined by [6–8]
\[
\psi_1(\alpha; \beta; q; z) = \sum_{n=0}^{\infty} \frac{(\alpha ; q)_n}{(\beta ; q)_n} z^n.
\]
(15)

The series is convergent for \(|q| < 1\) and \(|\beta/\alpha| < |z| < 1\).

The classical Ramanujan’s \( \psi_1 \) summation is given by [7, 8]
\[
\psi_1(\alpha; \beta; q; z) = \frac{(az ; q)_\infty(q/az ; q)_\infty(\beta/\alpha ; q)_\infty(\alpha ; q)_\infty}{(z ; q)_\infty(z/az ; q)_\infty(\beta ; q)_\infty(q/\alpha ; q)_\infty} \beta/\alpha |z| < 1.
\]
(16)

The special case \( \beta = aq \) of Ramanujan’s \( \psi_1 \) summation yields [8]
\[
\sum_{n=-\infty}^{\infty} \frac{z^n}{1 - aq^n} = \frac{(az ; q)_\infty((q/az ; q)_\infty(q/\alpha ; q)_\infty(\alpha ; q)_\infty)}{(z ; q)_\infty((z/az ; q)_\infty(\beta ; q)_\infty(q/\alpha ; q)_\infty(\alpha ; q)_\infty)},
\]
(17)

also known as Cauchy’s formula. Due to symmetry in \( \alpha \) and \( z \) on the right-hand side of (17), it implies [8]
\[
\sum_{n=-\infty}^{\infty} \frac{z^n}{1 - aq^n} = \sum_{n=-\infty}^{\infty} \frac{\alpha^n}{1 - oz^n}.
\]
(18)

The \( q \)-gamma function is defined as [7]
\[
\Gamma_q(x) = \frac{(q ; q)_\infty}{(q^x ; q)_\infty} (1 - q)^{-x}, \quad 0 < q < 1, x = C - \{0, -1, -2, \ldots\}.
\]
(19)

Another important special function that is used in this paper is the modified Jacobi theta function defined by [7]
\[
\theta(x ; q) = (x ; q)_\infty(q/x ; q)_\infty,
\]
(20)

where \( x \neq 0 \) and \(|q| < 1\). For a more detailed discussion on \( q \)-series and historical perspectives, see, for example, [6–8] and the references therein.

3. Szegő Kernel for an Annulus and Basic Bilateral Series

In this section, we express the bilateral series (4) as a basic bilateral series and derive the infinite product representation of the Szegő kernel for \( \Omega \). It is given in the following theorem.

**Theorem 1.** Let \( \Omega \) be the annulus \( \{ z : \rho < |z| < 1 \} \) bounded by \( \Gamma \). For \( a \in \Omega \), \( z \in \Omega \cup \Gamma \), the Szegő kernel for \( \Omega \) can be represented by
\[
S(z, a) = \frac{1}{2\pi(1 + \rho)} \psi_1(-\rho; \rho^3; \rho^2; az),
\]
(21)

\[
= \frac{1}{2\pi} \prod_{n=0}^{\infty} \frac{(1 + \rho^2 (\rho + 1)) (az + \rho^{2n+1}) (1 - \rho^{2n+2})^2}{(1 - az (a + 1)) (a - \rho^{2n+2}) (1 + \rho^{2n+2})^2}. \tag{22}
\]

The zero of \( S(z, a) \) in \( \Omega \) is the zero of the factor \( az + \rho \), that is, \( z = -\rho/\alpha \).

**Proof.** From (4), we have
\[
S(z, a) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(az)^n}{1 + \rho^{2n+1}} = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(az)^n}{1 - (-\rho)^{2n}}, \tag{23}
\]

Letting \( \alpha = -\rho \) and \( q = \rho^2 \) yields
\[
S(z, a) = \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(az)^n}{1 - az^n}, \tag{24}
\]

\[
= \frac{1}{2\pi(1 - \alpha)} \sum_{n=0}^{\infty} \frac{1 - \alpha}{1 - az^n} (az)_n. \tag{25}
\]

Applying (12) and (15) gives
\[
S(z, a) = \frac{1}{2\pi(1 - \alpha)} \psi_1(\alpha; az; q; \bar{a}z). \tag{26}
\]
Note that the $\psi_1$ series above is convergent because $|q| = \rho^2 < 1$ and $|b/a| = |aq/a| = |q| = \rho^2 < |a\bar{z}| < 1$. Substituting $\alpha = -\rho$ and $q = \rho^2$ into (26) gives (21).

Applying Ramanujan’s sum (16) to (26), gives

$$S(z, a) = \frac{1}{2\pi(1-a)} \frac{(a\bar{z}; q)\co (q/a; q)\co (q; q)_{\co}^2}{(a\bar{z}; q)\co (q/a; q)\co (q; q)_{\co}^2}.$$  

(27)

But from (14), with $n = 1$, we have

$$(1-a)(aq; q)\co = (a; q)\co.$$  

(28)

Thus, (27) becomes

$$S(z, a) = \frac{1}{2\pi(1-a)} \frac{(a\bar{z}; q)\co (q/a; q)\co (q; q)_{\co}^2}{(a\bar{z}; q)\co (q/a; q)\co (q; q)_{\co}^2},$$  

(29)

$$= \frac{1}{2\pi} \sum_{n=0}^{\infty} \frac{(1-a\bar{z}q^n)(1-q^{n+1}/a\bar{z})(1-q^{n+1})^2}{(1-a\bar{z}q^n)(1-q^{n+1}/a\bar{z})(1-q^{n+1}/a)(1-qa^n)}.$$  

(30)

Substituting $\alpha = -\rho$ and $q = \rho^2$ into (30) gives (22). The infinite product (22) would have poles if

$$1-\bar{a}\rho^{2n} = 0 \text{ or } \bar{a}z - \rho^{2n+2} = 0,$$  

(31)

which implies

$$z = \frac{1}{a}\frac{\rho^{2n}}{\rho} \text{ or } z = \frac{\rho^{2n+2}}{a}.$$  

(32)

But

$$\frac{1}{|a\rho^{2n}|} > 1, \quad \left|\frac{\rho^{2n+1}}{\rho}\right| < \rho^{2n+1} < \rho.$$

(33)

Therefore, the poles are all outside $\Omega$.
The infinite product (22) would have zeros if

$$1+\bar{a}z\rho^{2n+1} = 0 \text{ or } \bar{a}z + \rho^{2n+1} = 0,$$  

(34)

which implies

$$z = -\frac{1}{a}\frac{\rho^{2n+1}}{\rho} \text{ or } z = -\frac{\rho^{2n+1}}{a}.$$  

(35)

For the first case

$$\frac{1}{|a\rho^{2n+1}|} > 1, \quad \left|\frac{\rho^{2n+1}}{\rho}\right| > 1,$$

(36)

which is outside $\Omega$. For the second case, observe that

$$\rho^{2n+1} < \left|\frac{\rho^{2n+1}}{a}\right| < \rho^{2n},$$  

which clearly has a zero inside $\Omega$ when $n = 0$. Thus, the infinite product (22) for $S(z, a)$ has only one zero inside $\Omega$ at $z = -\rho/a$. This completes the proof.

We note that the series representation (21) for $S(z, a)$ is valid only for $|z| < 1$, while the infinite product representation (22) for $S(z, a)$ is meaningful for all $z \in \mathbb{C}$ except for the infinitely many poles at $z = 0, \rho^{-2n}/a, \rho^{2n+2}/a$.

We next show that the Szegő kernel for $\Omega$ can also be expressed in terms of the basic gamma function and modified Jacobi theta function. By applying (20) to (29) and substituting $\alpha = -\rho$ and $q = \rho^2$, we have

$$S(z, a) = \frac{1}{2\pi} \frac{\theta(a\bar{z}; q)\co (q/a; q)\co (q; q)_{\co}^2}{\theta(a\bar{z}; q)\co (q/a; q)\co (q; q)_{\co}^2}.$$  

(38)

Applying (19) with $q = \rho^2$, observe that

$$\frac{\rho^2; \rho^2_{\co}}{(1-\rho^2; \rho^2_{\co})} = \frac{\rho^2; \rho^2_{\co}}{(1-\rho^2; \rho^2_{\co})} = \frac{\Gamma(\rho)(\rho)}{(1-\rho^2)^{1/2}},$$

(39)

where $x$ satisfies $\rho^{2n} = -\rho$. This equation may be written as

$$e^{(2n-1)\ln \rho} = e^n,$$

(40)

which yields a solution

$$x = \frac{1}{2} + \frac{in}{\ln \rho}.$$  

(41)

Thus, (38) becomes

$$S(z, a) = \frac{\Gamma(\rho)(\lambda)}{2\pi(1-\rho^2)^{1/2}} \cdot \frac{\theta(-\rho a; \rho^2)}{\theta(a\bar{z}; \rho^2)}.$$  

(42)

This can be regarded as a closed-form expression for the Szegő kernel for $\Omega$.

In the following, we show how to easily transform series (4) to series (5) using (18). Letting $\alpha = -\rho$ and $q = \rho^2$, (4) becomes

$$S(z, a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(a\bar{z})^n}{1-\alpha q^n} = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{a^n}{1-(a\bar{z})q^n},$$

(43)

where in the last step we have used (18). By replacing $\alpha = -\rho$ and $q = \rho^2$, we get

$$S(z, a) = \frac{1}{2\pi} \sum_{n=-\infty}^{\infty} \frac{(-1)^n\rho^n}{1-(a\bar{z})\rho^{2n}}.$$  

(44)
Letting \( n = -m \) yields
\[
S(z, a) = \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \phi^{-m}}{1 - (a \bar{z}) \theta^{2m}} = \frac{1}{2\pi} \sum_{m=1}^{\infty} \frac{(-1)^m \phi^{-m}}{\theta^{2m} - a \bar{z}}
\] (45)
which is the same as (5).

4. Szegö Kernel for General Annulus

Consider the general annulus \( \Omega_2 = \{ z : r_2 < |z - z_0| < r_1 \} \) with boundary denoted by \( \Gamma_2 \). The region \( \Omega_2 \) reduces to \( \Omega \) if \( z_0 = 0, r_2 = \rho, \) and \( r_1 = 1 \).

**Theorem 2.** Let \( z_0 \in \mathbb{C}, z \in \Omega_2 \cup \Gamma_2, \) and \( a \in \Omega_2 \). The Szegö kernel for \( \Omega_2 \) can be represented by the bilateral series as
\[
S_2(z, a) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \left( \frac{\alpha - \bar{z}_0}{\rho} \right)^n (z - z_0)^n,
\] (46)
\[
= \frac{1}{2\pi} \sum_{n=1}^{\infty} \left( \frac{\rho^{n+1} - r_2^n (z - z_0)}{\rho^{n+1} - r_2^n (a - \bar{z}_0)} \right).
\] (47)
The zero of \( S_2(z, a) \) in \( \Omega_2 \) is \( z = z_0 - r_1 r_2 \alpha - \bar{z}_0 \).
**Proof.** Observe that the function \( f(z) = (z - z_0)/r_1 \) maps \( \Omega_2 \) onto \( \Omega \) with \( \rho = r_1 r_2 \).

Applying the transformation formula (6) yields
\[
S_2(z, a) = \sqrt{\frac{f'(z)S(f(z), f(a))}{f'(a)}}
= \frac{1}{\sqrt{r_1}} S\left( \frac{z - z_0}{r_1}, \frac{a - \bar{z}_0}{r_1} \right) \frac{1}{\sqrt{r_1}}
= \frac{1}{r_1} S\left( \frac{z - z_0}{r_1}, \frac{a - \bar{z}_0}{r_1} \right).
\] (48)
Applying (4) to (48) with \( z \) and \( a \) replaced by \( (z - z_0)/r_1 \) and \( (a - \bar{z}_0)/r_1 \) respectively, gives
\[
S_2(z, a) = \frac{1}{2\pi r_1} \sum_{n=1}^{\infty} \left( \frac{(z - z_0)(\alpha \bar{z}_0)}{r_1} \right)^n \frac{1}{1 + (r_2/r_1)^{2n+1}},
\] (49)
which simplifies to (46).

Applying (5) to (48) instead of \( z \) and \( a \) replaced by \( (z - z_0)/r_1 \) and \( (a - \bar{z}_0)/r_1 \) respectively, gives
\[
S_2(z, a) = \frac{1}{2\pi r_1} \sum_{n=1}^{\infty} \left( \frac{r_1^2/r_1}{(r_2/r_1)^{2n} - (z - z_0)(\alpha \bar{z}_0)/r_1^2} \right)^n,
\] (50)
which simplifies to (47).

Using the fact that \( S(z, a) \) has a zero at \( z = \rho \alpha \) for \( \Omega \), the zeros of \( S_2(z, a) \) for \( \Omega_2 \) is \( z = z_0 - r_1 r_2 \alpha \) which implies \( z = z_0 - (\rho \alpha)(\alpha - \bar{z}_0) \).
This completes the proof.

Similarly, the infinite product representation of \( S_2(z, a) \) for \( \Gamma_2 \) can be obtained by applying (22) to (48) with \( z \) and \( a \) replaced by \( (z - z_0)/r_1 \) and \( (a - \bar{z}_0)/r_1 \), respectively.

5. The Weighted Szegö Kernel for an Annulus and Basic Bilateral Series

The weighted Szegö kernel is defined in [11] as
\[
K^f_q(z, w) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(\bar{w}z)^n}{1 + t_q \theta^{2n}}, \quad t > 0, q < |z|, |w| < 1.
\] (51)
To adopt the notations used in this paper, we change \( t \) to \( \rho \), \( w \) to \( a \), and \( K^f_q(z, w) \) to \( S^f_\rho(z, a) \) in (51), which gives
\[
S^f_\rho(z, a) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(\bar{a}z)^n}{1 + t \theta^{2n}}, \quad t > 0, \rho < |z|, |a| < 1.
\] (52)
Note that \( S^f_\rho(z, a) \) is exactly the kernel \( S(z, a) \) for \( \Omega \) discussed in Section 1. The zeros of the kernel \( S^f_\rho(z, a) \) are not discussed in [11] but have expressed interest on the effect of the weight on the location of its zeros. In the following theorem, we express the weighted Szegö kernel \( S^f_\rho(z, a) \) as a basic bilateral series and derive its associated infinite product representation as well as its zeros.

**Theorem 3.** Let \( \Omega \) be the annulus \( \{ z : \rho < |z| < 1 \} \) bounded by \( \Gamma_r \). For \( a \in \Omega, z \in \Omega \cup \Gamma_r, \) and \( t > 0 \), the weighted Szegö kernel \( S^f_\rho(z, a) \) for \( \Omega \) can be represented by
\[
S^f_\rho(z, a) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(\bar{a}z)^n}{1 - (t \theta^{2n})},
\] (53)
\[
= \frac{1}{2\pi} \prod_{n=1}^{\infty} \frac{1}{1 - (q \theta^{2n})^2} (1 + t_q \theta^{2n+1})^2 \prod_{n=1}^{\infty} \frac{\alpha \bar{z}_0 (1 + \theta^{2n} \alpha \bar{z}_0)}{1 - (t \theta^{2n+1})^2}.
\] (54)
The kernel \( S^f_\rho(z, a) \) has a zero in \( \Omega \) only if \( t \) takes the form \( t = \rho^{2(n+1)} \), \( m = 0, 1, 2, \ldots \). In both cases, the zero is \( z = -\rho \alpha \).
**Proof.** Observe that
\[
S^f_\rho(z, a) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(\bar{a}z)^n}{1 - (t \rho^{2n})},
\] (55)
Letting \( \alpha = -t \) and \( q = \rho^2 \), the above equation becomes
\[
S^f_\rho(z, a) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(\bar{a}z)^n}{1 - \theta^{2n}},
\] (56)
which is exactly the same form as (24). Applying the result (26) with \( \alpha = -t \), the above equation becomes
\[
S^f_\rho(z, a) = \frac{1}{2\pi} \sum_{n=1}^{\infty} \frac{(\bar{a}z)^n}{1 - \theta^{2n}}.
\] (57)
Series (57) is convergent because \( |q| = \rho^2 < 1 \) and \( |\beta| = |\rho| < 1 \). Substituting \( q = \rho^2 \) gives (41).
Applying the result (29) with $\alpha = -t$ to (57) yields
\[
S'_p(z, a) = \frac{1}{2\pi} \left( -t\bar{a}z ; q \right)_{\infty} \left( q(-t)\bar{a}z ; q \right)_{\infty} \left( q ; q \right)_{\infty}^2.
\]
(58)

Replacing $q = \rho^2$ and applying (13) give (54).
In the proof of Theorem 1, we have shown that the factors $(1 - \bar{a}z\rho^{2n})(\bar{a}z - \rho^{2n+2})$ have no zeros in $\Omega$. The factors $(1 + \rho^{2n+2}t)(1 + t\rho^{2n})$ would have zeros if
\[
\rho^{2n+2}/t = -1 \text{ or } t\rho^{2n} = -1.
\]
(59)

Since $t > 0$, we conclude that the kernel $S'_p(z, a)$ has no poles in $\Omega$ for any $t > 0$. The factors $(1 + \bar{a}z\rho^{2n})(\bar{a}z + \rho^{2n+2}/t)$ would have zeros if
\[
1 + \bar{a}z\rho^{2n} = 0 \text{ or } \bar{a}z + \rho^{2n+2}/t = 0,
\]
(60)

which implies
\[
z = -\frac{1}{\bar{t}a\rho^{2n}} \text{ or } z = -\frac{\rho^{2n+2}}{\bar{t}a}.
\]
(61)

For the first case, observe that
\[
\frac{1}{t\rho^{2n}} < \frac{1}{|\bar{t}a\rho^{2n}|} < \frac{1}{t\rho^{2n+1}}.
\]
(62)

To have a zero in $\Omega$, we must have the condition
\[
\rho \leq \frac{1}{t\rho^{2n}} < \frac{1}{|\bar{t}a\rho^{2n}|} < \frac{1}{t\rho^{2n+1}} \leq 1,
\]
(63)

which means
\[
t \leq \frac{1}{\rho^{2n+1}} \text{ and } t \geq \frac{1}{\rho^{2n+1}}.
\]
(64)

Hence, we must have $t = \rho^{-(2n+1)}$. In this case, the zero of $S'_p(z, a)$ in $\Omega$ is $z = -\rho/\bar{a}$.

For the second case, observe that
\[
\frac{\rho^{2n+2}}{t} < \frac{\rho^{2n+2}}{|\bar{t}a|} < \frac{\rho^{2n+1}}{t}.
\]
(65)

To have a zero in $\Omega$, we must have the condition
\[
\rho \leq \frac{\rho^{2n+2}}{t} < \frac{\rho^{2n+2}}{|\bar{t}a|} < \frac{\rho^{2n+1}}{t} \leq 1,
\]
(66)

which means
\[
t \leq \rho^{2n+1} \text{ and } t \geq \rho^{2n+1}.
\]
(67)

Hence, we must have $t = \rho^{2n+1}$. In this case, the zero of $S'_p(z, a)$ in $\Omega$ is $z = -\rho/\bar{a}$. This completes the proof.

The weighted Szegö kernel can also be expressed in terms of the basic gamma function and the modified Jacobi theta function. By applying (20) to (58) with $q = \rho^2$, we have
\[
S'_p(z, a) = \frac{1}{2\pi} \frac{\theta(-t\bar{a}z ; \rho^2)}{\theta(-t ; \rho^2)}\frac{\rho^2}{\rho^2(-t ; \rho^2)} (\rho^2 ; \rho^2)^2_{\infty}.
\]
(68)

Observe that
\[
\frac{(\rho^2 ; \rho^2)_{\infty}}{(-t ; \rho^2)_{\infty}} = \frac{(\rho^2 ; \rho^2)_{\infty}}{(\rho^2 \omega ; \rho^2)_{\infty}} = \frac{\Gamma_{\rho^2}(x)}{(1 - \rho^2)^{1-x}},
\]
(69)

where $x$ satisfies $\rho^{2x} = -t$. This equation may be written as
\[
2x \ln \rho = \ln (-t) = \ln (-t) + i \arg (-t) = \ln t + i\pi,
\]
(70)

which yields a solution
\[
x = \frac{\ln t + i\pi}{2 \ln \rho}.
\]
(71)

Observe also that
\[
\frac{(\rho^2 ; \rho^2)_{\infty}}{(-t ; \rho^2)_{\infty}} = \frac{(\rho^2 ; \rho^2)_{\infty}}{(\rho^2 \omega ; \rho^2)_{\infty}} = \frac{\Gamma_{\rho^2}(y)}{(1 - \rho^2)^{1-y}},
\]
(72)

where $y$ satisfies $\rho^{2y} = -t$. This equation may be written as
\[
(2y - 2) \ln \rho = \ln \left( -\frac{1}{\rho^2} \right) = \ln \left( -\frac{1}{\rho^2} \right) + i \arg \left( -\frac{1}{\rho^2} \right) = -\ln t + i\pi,
\]
(73)

which yields a solution
\[
y = 1 + \frac{-\ln t + i\pi}{2 \ln \rho}.
\]
(74)

Thus, (68) becomes
\[
S'_p(z, a) = \frac{\Gamma_{\rho^2}(y)}{2\pi(1 - \rho^2)^{1-y}} \theta(-t\bar{a}z ; \rho^2)_{\infty},
\]
(75)

This can be regarded as a closed-form expression for the weighted Szegö kernel for an annulus $\Omega$. Observe that (75) reduces to (42) when $t = \rho$. 
6. Numerical Computation of the Szegö Kernel for an Annulus

In this section, we compare the speed of convergence of the three formulas for computing the Szegö kernel for Ω based on the two bilateral series (4) and (5) and the infinite product (22).

To approximate (4) numerically, we calculate

\[ S(z, a) \approx S_{10}(z, a) = \frac{1}{2\pi} \sum_{k=-10}^{10} \frac{(za)^k}{1 + \rho^{2k+1}}, \quad (76) \]

and \( S_{20} \) and \( S_{100} \).

To approximate (5) numerically, we calculate

\[ S(z, a) \approx S^*_1(z, a) = \frac{1}{2\pi} \sum_{k=-10}^{10} \frac{(-1)^k \rho^k}{z - \bar{a}}, \quad (77) \]

and \( S^*_{20} \).

To approximate (22) numerically, we compute

\[ S(z, a) \approx S^*_{15}(z, a) = \frac{1}{2\pi} \prod_{k=0}^{15} \frac{(1 + \bar{a} \rho^{2k+1}) (z - \bar{a}) \rho^{2k+1} (1 - \rho^{2k+2})^2}{(1 - \bar{a} \rho^{2k}) (z - \rho^{2k}) (1 + \rho^{2k+2})^2}, \quad (78) \]

and \( S^*_{20} \) and \( S^*_{15} \).

The approximations are then compared with the numerical solution of the Kerzman-Stein Equation (7). To solve (7), we used the Nyström method [5] with the trapezoidal rule with \( n \) selected nodes on each boundary component \( \Gamma_0 \) and \( \Gamma_1 \). The approximate solution is represented by \( \tilde{S}_n \) where \( n \) is the number of nodes. All the computations were done using MATHEMATICA 12.3. Four numerical examples are given for different values of \( a \) and \( \rho \). The results for the error norms are presented for each example.

We consider an annulus \( \Omega \) bounded by

\[ \Gamma_0 : z_0(t) = e^{i\theta}, \quad (79) \]

\[ \Gamma_1 : z_1(t) = \rho e^{-i\theta}, \]

with \( 0 \leq t \leq 2\pi \).
We consider an annulus \( \Omega \) with \( a = 0.8 \) and \( \rho = 0.4 \). The results for the error norms are presented in Tables 7–9.

Example 4. We consider an annulus \( \Omega \) with \( a = -0.4 - 0.5i \) and \( \rho = 0.1 \). The results for the error norms are presented in Tables 10–12.

The numerical results presented in Tables 1–12 show that computations using the infinite product formula (22) converge faster than the bilateral series formulas (4) and (5).

### 7. Conclusion

This paper has shown that the bilateral series for the Szegö kernel for \( \Omega \) is a disguised bilateral basic hypergeometric series \( \sum_{k=0}^{\infty} \frac{1}{(1 + a)^k} \). Ramanujan’s sum for \( \sum_{k=0}^{\infty} \frac{1}{(1 + a)^k} \) is then applied to obtain the infinite product representation for the Szegö kernel for \( \Omega \). The product clearly exhibits the zero of the Szegö kernel for an \( \Omega \). The Szegö kernel can also be expressed as a closed form in terms of the \( q \)-gamma function and the modified Jacobi theta function. Similar \( q \)-analysis has also been conducted for the Szegö kernel for general \( \Omega \) and for the weighted Szegö kernel for \( \Omega \). The numerical comparisons have shown that the infinite product method converges faster than the bilateral series methods for computing the Szegö kernel for \( \Omega \).

For future work, it is natural to devote further investigation on the infinite product representation for the Szegö kernel for doubly connected regions via the transformation formula (6) and Theorem 1. This however requires knowledge of conformal mapping of doubly connected regions to annulus [12–15]. For some ideas on numerical methods for computing the zero of the Szegö kernel for doubly connected regions, see [16]. Alternatively, perhaps some computational intelligence algorithms can also be considered to compute the zero, like the monarch butterfly optimization (MBO) [17], earthworm optimization algorithm (EWA) [18], elephant herding optimization (EHO) [19], moth search (MS) algorithm [20], slime mould algorithm (SMA) [21], and Harris hawks optimization (HHO) [22].

### Data Availability

No data were used to support this study.

### Conflicts of Interest

The authors declare that there are no conflicts of interest regarding the publication of this paper.

### Acknowledgments

The authors wish to thank the Universiti Teknologi Malaysia for supporting this work. This work was supported by the Ministry of Higher Education Malaysia under Fundamental Research Grant Scheme (FRGS/1/2019/STG06/UTM/02/20). This support is gratefully acknowledged. The first author would also like to acknowledge the Tertiary Education Trust Fund (TETFund) Nigeria for overseas scholarship award. The authors thank the referees for comments and suggestions which improved the paper.

### Table 8: Error norms between \( S_{10}^* \) and \( \hat{S}_n \) and \( S_{50}^* \) and \( \hat{S}_n \).

| \( n \) | \( \| S_{10}^* - \hat{S}_n \|_{\infty} \) | \( \| S_{50}^* - \hat{S}_n \|_{\infty} \) |
|------|-----------------|-----------------|
| 16   | 8.28737 (-03)   | 8.28061 (-03)   |
| 32   | 2.33562 (-04)   | 2.2673 (-04)    |
| 64   | 1.79806 (-05)   | 1.79491 (-07)   |
| 128  | 1.78806 (-05)   | 1.1287 (-15)    |

### Table 9: Error norms between \( S_{128}^* \) and \( \hat{S}_n \), \( S_{10}^* \) and \( \hat{S}_n \), and \( S_{15}^* \) and \( \hat{S}_n \).

| \( n \) | \( \| S_{128}^* - \hat{S}_n \|_{\infty} \) | \( \| S_{10}^* - \hat{S}_n \|_{\infty} \) | \( \| S_{15}^* - \hat{S}_n \|_{\infty} \) |
|------|-----------------|-----------------|-----------------|
| 16   | 8.28737 (-03)   | 8.28061 (-03)   | 8.28061 (-03)   |
| 32   | 2.33562 (-04)   | 2.2673 (-04)    | 2.2673 (-04)    |
| 64   | 1.79806 (-05)   | 1.79491 (-07)   | 1.79491 (-07)   |
| 128  | 1.78806 (-05)   | 1.1287 (-15)    | 1.1287 (-15)    |

### Table 10: Error norms between \( S_{10} \) and \( \hat{S}_n \), \( S_{50} \) and \( \hat{S}_n \), and \( S_{100} \) and \( \hat{S}_n \).

| \( n \) | \( \| S_{10} - \hat{S}_n \|_{\infty} \) | \( \| S_{50} - \hat{S}_n \|_{\infty} \) | \( \| S_{100} - \hat{S}_n \|_{\infty} \) |
|------|-----------------|-----------------|-----------------|
| 16   | 3.15879 (-03)   | 2.61429 (-04)   | 2.61429 (-04)   |
| 32   | 3.22447 (-03)   | 2.08805 (-07)   | 2.08805 (-07)   |
| 64   | 3.28124 (-03)   | 5.91022 (-11)   | 5.91022 (-11)   |
| 128  | 3.28124 (-03)   | 5.91168 (-11)   | 5.91168 (-11)   |

### Table 11: Error norms between \( S_{10}^* \) and \( \hat{S}_n \), \( S_{50}^* \) and \( \hat{S}_n \), and \( S_{15}^* \) and \( \hat{S}_n \).

| \( n \) | \( \| S_{10}^* - \hat{S}_n \|_{\infty} \) | \( \| S_{50}^* - \hat{S}_n \|_{\infty} \) | \( \| S_{15}^* - \hat{S}_n \|_{\infty} \) |
|------|-----------------|-----------------|-----------------|
| 16   | 2.61429 (-04)   | 2.61429 (-04)   | 2.61429 (-04)   |
| 32   | 2.08805 (-07)   | 2.08805 (-07)   | 2.08805 (-07)   |
| 64   | 6.46416 (-13)   | 1.3313 (-13)    | 1.33121 (-13)   |
| 128  | 6.77069 (-13)   | 1.49882 (-15)   | 1.55654 (-15)   |

### Example 1.
We consider an annulus \( \Omega \) with \( a = 0.7i \) and \( \rho = 0.5 \). The results for the error norms are presented in Tables 1–3.

### Example 2.
We consider an annulus \( \Omega \) with \( a = -0.4 - 0.6i \) and \( \rho = 0.3 \). The results for the error norms are presented in Tables 4–6.
References

[1] S. R. Bell, The Cauchy Transform, Potential Theory, and Conformal Mapping, CRC Press, Boca Raton, 1992.

[2] S. R. Bell, "Numerical computation of the Ahlfors map of a multiply connected planar domain," Journal of Mathematical Analysis and Applications, vol. 120, no. 1, pp. 211–217, 1986.

[3] T. J. Tegtmeyer, The Ahlfors Map and Szegő Kernel in Multiply Connected Domains, [Ph.D. thesis], Purdue University, 1998.

[4] T. J. Tegtmeyer and A. D. Thomas, “The Ahlfors map and Szegő kernel for an annulus,” Rocky Mountain Journal of Mathematics, vol. 29, no. 2, pp. 709–723, 1999.

[5] N. H. A. A. Wahid, A. H. M. Murid, and M. I. Muminov, "Convergence of the series for the Szegő kernel for an annulus region," AIP Conference Proceedings, vol. 1974, article 030025, p. 8, 2018.

[6] J. S. Lucy, Generalized Hypergeometric Functions, Cambridge University Press, 1996.

[7] G. Gasper and M. Rahman, Basic Hypergeometric Series, Encyclopedia of Mathematics and Applications, vol. 96, Cambridge University Press, 2nd edition, 2004.

[8] P. J. Warren, An Introduction to q-Analysis, American Mathematical Society, 2020.

[9] N. Kerzman and E. M. Stein, "The Cauchy kernel, the Szegő kernel, and the Riemann mapping function," Mathematische Annalen, vol. 236, no. 1, pp. 85–93, 1978.

[10] N. Kerzman and M. R. Trummer, “Numerical conformal mapping via the Szegő kernel,” Journal of Computational and Applied Mathematics, vol. 14, no. 1-2, pp. 111–123, 1986.

[11] S. McCullough and L. Shen, "On the Szegő kernel of an annulus," Proceedings of the American Mathematical Society, vol. 121, no. 4, pp. 1111–1121, 1994.

[12] G. T. Symm, "Conformal mapping of doubly-connected domains," Numerische Mathematik, vol. 13, no. 5, pp. 448–457, 1969.

[13] K. Amano, "A charge simulation method for the numerical conformal mapping of interior, exterior and doubly-connected domains," Journal of Computational and Applied Mathematics, vol. 53, no. 3, pp. 353–370, 1994.

[14] M. M. S. Nasser, "A boundary integral equation for conformal mapping of bounded multiply connected regions," Computational Methods and Function Theory, vol. 9, no. 1, pp. 127–143, 2009.

[15] A. W. K. Sangawi, A. H. M. Murid, and M. M. S. Nasser, "Annulus with circular slit map of bounded multiply connected regions via integral equation method," Bulletin of the Malaysian Mathematical Sciences Society, vol. 35, no. 4, pp. 945–959, 2012.

[16] A. H. M. Murid, N. H. A. A. Wahid, and M. I. Muminov, "Methods and comparisons for computing the zeros of the Ahlfors map for doubly connected regions," AIP Conference Proceedings, vol. 2423, article 020026, 2021.

[17] G.-G. Wang, S. Deb, and Z. Cui, "Monarch butterfly optimization," Neural Computing and Applications, vol. 31, no. 7, pp. 1995–2014, 2019.

[18] G.-G. Wang, S. Deb, and L. D. S. Coelho, "Earthworm optimisation algorithm: a bio-inspired metaheuristic algorithm for global optimisation problems," International Journal of Bio-Inspired Computation, vol. 12, no. 1, pp. 1–22, 2020.