Representation theorems for generators of Reflected BSDEs with continuous and linear-growth generators

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Abstract: In this paper, we establish some representation theorems for generators of reflected backward stochastic differential equations (RBSDEs in short) in the space of random variables and the space of stochastic processes, respectively, when generators are continuous with linear-growth, which are extensions of some representation theorems for generators of BSDEs. Using the representation theorems, some general converse comparison theorems for RBSDEs are obtained.

Keywords: Backward stochastic differential equation; representation theorem for generator; converse comparison theorem; obstacle.

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1 Introduction

The theory of backward stochastic differential equations (BSDEs in short) has gone through rapid development in many different research areas in recent 20 years, one can see an excellent survey given by Peng (2010). One of important results of BSDE theory is representation theorem of generator, which establishes a relation between generator and solutions of BSDEs in limit form, can be used to resolve many problems in BSDEs theory and nonlinear expectation theory. Representation theorem of generator is firstly obtained by Briand et al. (2000), then generalized to the case generator $g$ only satisfy Lipschitz condition and applied to study general converse comparison theorems of BSDE and uniqueness theorem, translation invariance, convexity, etc, for $g$-expectation by Jiang (2005a, b, c, 2008). Since then, representation theorem of BSDE is studied further in the space of random variables (see Liu et al. (2007), Jia (2008), Song et al. (2012), etc) and in the space of stochastic processes (see Fan and Hu (2008), Fan and Jiang (2010), Fan et al. (2011), etc), respectively.

El Karoui et al. (1997) introduced the notion of reflected backward stochastic differential equations (RBSDEs in short), which is a BSDE with an additional continuous, increasing process $K$ in this equation to keep the solution above a given continuous process $L$, called obstacle. In the following years, RBSDEs are studies by many papers, due to the wide applications of RBSDE to the pricing of America options, mixed control, partial differential equations, etc.

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A nature problem is how to establish a representation theorem for generators of RBSDEs? A basic difficulty to solve this problem is that the solution of RBSDE is restricted by obstacle, and the form of RBSDE is also not the same as BSDE. In this paper, we solve this problem using a localization method dependent on stopping times. The main results of this paper is that we establish some representation theorems for generators of RBSDEs in the space of random variables and the space of stochastic processes, respectively, when the generators satisfy continuous and linear growth condition, which generalize the representation theorems for BSDEs in Jia (2008) and Fan and Jiang (2010) to RBSDEs case.

Converse comparison theorems for BSDEs is firstly studied by Chen(1997), then by Briand et al. (2001), Jiang (2005a, 2005c) and Jia (2008), etc, which shows that we can compare the generators of BSDEs through comparing the solution of BSDEs. Converse comparison theorems for BSDEs is firstly studied by Li and Tang (2007), then by Li and Gu (2007), when generators $g$ is continuous in $t$ and satisfy Lipschitz condition. Using representation theorems obtained in this paper, we obtain some general converse comparison theorems RBSDEs, whose generators are only continuous with linear-growth in $(y, z)$.

This paper is organized as follows. In the next section, we will recall the definition of RBSDEs and show some important lemmas. In section 3, we will establish some representation theorems for generators of RBSDEs under continuous and linear growth condition. In section 4, we will give some applications of representation theorems obtained in this paper. In the Appendix, we will give the proof of Lemma 2.1.

2 Preliminaries

Let $(\Omega, \mathcal{F}, P)$ be a complete probability space carrying a $d$-dimensional standard Brownian motion $(B_t)_{t \geq 0}$, starting from $B_0 = 0$, let $(\mathcal{F}_t)_{t \geq 0}$ denote the natural filtration generated by $(B_t)_{t \geq 0}$, augmented by the $P$-null sets of $\mathcal{F}$, let $|z|$ denote its Euclidean norm, for $z \in \mathbb{R}^n$, let $T > 0$ be a given real number. We define the following usual spaces:

$$L^p(\mathcal{F}_t) = \{ \xi : \mathcal{F}_t\text{-measurable random variable; } \mathbb{E}(|\xi|^p) < \infty, \ t \in [0, T], \ p \geq 1; \}
$$

$$S^2(0, T; \mathbb{R}) = \{ \psi : \text{continuous predictable process; } ||\psi||_{S^2}^2 = \mathbb{E}\left[\sup_{0 \leq t \leq T}|\psi_t|^2\right] < \infty; \}
$$

$$\mathcal{H}^p(0, T; \mathbb{R}^d) = \{ \psi : \text{predictable process; } ||\psi||_{\mathcal{H}^p}^p = \mathbb{E}\left[\int_0^T|\psi_t|^p dt\right] < \infty, \ p \geq 1. \}
$$

Let us consider a function $g$

$$g(\omega, t, y, z) : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$$

such that $(g(t, y, z))_{t \in [0, T]}$ is progressively measurable for each $(y, z) \in \mathbb{R} \times \mathbb{R}^d$. In this paper, we will make the following assumptions on $g$:

(A1). (Linear-growth) There exists a constant $\lambda \geq 0$, and non-negative stochastic process $\gamma_t \in \mathcal{H}^2(0, T; \mathbb{R})$ such that $P$-a.s., $\forall t \in [0, T], \ \forall (y, z) \in \mathbb{R} \times \mathbb{R}^d, \ (i = 1, 2) :$

$$|g(t, y, z)| \leq \lambda (\gamma_t + |y| + |z|).$$

(A2). (Continuity) $P$-a.s., $\forall t \in [0, T], \ (y, z) \mapsto g(t, y, z)$ is continuous.

(A3). $P$-a.s., $\forall (t, y) \in [0, T] \times \mathbb{R}^d,$

$$g(t, y, 0) = 0.$$
(A4). This assumption is Assumption (A1) in which the non-negative stochastic process $\gamma_t$ satisfying

$$E \left[ \sup_{0 \leq s \leq T} |\gamma_t|^2 \right] < +\infty.$$ 

Note that the assumptions (A1) and (A2) have been made in many papers related to BSDE theory such as Matoussi (1997), Hamadène et al. (1997) and Bahlali et al. (2005), etc. The assumption (A3) is very interesting condition under which the BSDE has many interesting properties.

**Definition 2.1** A RBSDE is associated with a terminal condition $\xi \in L^2(\mathcal{F}_T)$, a generator $g$, a lower obstacle $\{L_t\}_{0 \leq t \leq T}$, which is continuous progressively measurable real-valued process such that $\{L_t\}_{0 \leq t \leq T} \in \mathcal{S}^2(0, T; \mathbb{R})$ and $L_T \leq \xi$. A solution of this equation is a triple $(Y, Z, K)$ of progressively measurable processes taking values in $\mathbb{R} \times \mathbb{R}^d \times \mathbb{R}_+$ and satisfying

$$ \begin{cases} 
(i) & Z \in \mathcal{H}^2(0, T; \mathbb{R}^d), \quad Y, K \in \mathcal{S}^2(0, T; \mathbb{R}) \\
(ii) & Y_t = \xi + \int_t^T g(s, Y_s, Z_s) \, ds + K_T - K_t - \int_t^T Z_s \, dB_s, \quad \forall t \in [0, T] \\
(iii) & P - \text{a.s.}, \quad L_t \leq Y_t, \quad \forall t \in [0, T] \text{ and } \int_0^T (Y_t - L_t) \, dK_t = 0 \\
(iv) & K \text{ is continuous and increasing, } K_0 = 0.
\end{cases} $$

We call the RBSDE in Definition 2.1 is a RBSDE with parameter $(g, T, \xi, L)$, which is introduced in El Karoui et al. (1997). By Matoussi (1997), Hamadène et al. (1997) or Bahlali (2005), we can get, under assumptions (A1) and (A2), the RBSDE has at least a solution. In particular, it has a minimal solution $(\underline{Y}, \underline{Z}, \underline{K})$ and a maximal solution $(\overline{Y}, \overline{Z}, \overline{K})$ in the sense that, for any solution $(Y, Z, K)$ of this equation, we have $P - \text{a.s.}, \underline{Y} \leq Y \leq \overline{Y}$ and $\underline{K} \geq K \geq \overline{K}$.

**Remark 2.1** In fact, by the Definition 2.1, we get that, the solution of RBSDE with parameter $(g, T, \xi, L)$ is only dependent on generator $g(t, y, z)|_{[t, y, z] \in [0, T] \times (\mathcal{L} \cup +\infty) \times \mathbb{R}^d}$, not dependent on $g(t, y, z)|_{[t, y, z] \in [0, T] \times (-\infty, \mathcal{L} \cup \mathbb{R}^d}$, respectively, and $P - \text{a.s.}$.

$$ g_1(t, y, z) = g_2(t, y, z), \quad \forall (t, y, z) \in [0, T] \times [L_t, +\infty) \times \mathbb{R}^d, $$

then the solutions of this two RBSDEs are the same.

The following Lemma 2.1 gives a priori estimation for RBSDEs, under assumptions (A1) and (A2), which can be proved using a standard argument given in Briand et al (2000, Proposition 2.2) for BSDEs. Its proof is given in the Appendix for convention.

**Lemma 2.1** Let $g$ satisfy the assumptions (A1) and (A2), $(Y, Z, K)$ be an arbitrary solution of RBSDE with parameter $(g, \xi, T, L)$, then there exists a constant $C$ only dependent on $\lambda$ in (A1) and $T$, such that for $0 \leq r < t \leq T$,

$$ E \left[ \sup_{s \in [r, t]} |Y_s|^2 + \int_r^t |Z_s|^2 \, ds + |K_t - K_r|^2 |\mathcal{F}_r \right] \leq CE \left[ |Y_t|^2 + \left( \int_r^t \gamma_s \, ds \right)^2 + \sup_{s \in [r, t]} (L_s^+)^2 |\mathcal{F}_r \right]. $$

Now, we introduce a stochastic differential equation (SDE). Suppose $b(\cdot, \cdot, \cdot) : \Omega \times [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^n$ and $\sigma(\cdot, \cdot, \cdot) : \Omega \times [0, T] \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times d}$ and always satisfy the following two conditions (H1) and (H2) in this paper.
In this section, we will firstly establish a representation theorem for generators of RBSDEs in the space of random variables.

(H1) (Lipschitz condition) there exists a constant $\mu \geq 0$ such that $P$-a.s., $\forall t \in [0, T]$, $\forall x, y \in \mathbb{R}^n$,

$$|b(t, x) - b(t, y)| + |\sigma(t, x) - \sigma(t, y)| \leq \mu |x - y|.$$ 

(H2) (Linear-growth) there exists a constant $\nu \geq 0$ such that $P$-a.s., $\forall t \in [0, T]$, $\forall x \in \mathbb{R}^n$,

$$|b(t, x)| + |\sigma(t, x)| \leq \nu (1 + |x|).$$

Given $(t, x) \in [0, T] \times \mathbb{R}^p$, by SDE theory, the following SDE:

$$\begin{cases}
X^t_{s,x} = x + \int_t^s b(u, X^t_{u,x})du + \int_t^s \sigma(u, X^t_{u,x})dB_u, & s \in [t, T], \\
X^t_{s,x} = x & s \in [0, t],
\end{cases}$$

have a unique continuous adapted solution $(X^t_{s,x})_{s \geq 0}$.

**Remark 2.2** In fact, from SDE theory, we know $E \left[ |X^t_{s,x} - x|^2 \right]$ is continuous in $s$ and

$$E \left[ \sup_{0 \leq s \leq T} |X^t_{s,x}|^2 \right] < +\infty.$$ 

By the proof of Jiang (2008, Proposition 2.2), we can get the following Lemma 2.2.

**Lemma 2.2** Let $q > p \geq 1$. Let $(\psi_t)_{t \in [0, T]}$ be a $\mathbb{R}$-valued, progressively measurable process and $E \left[ \int_0^T |\psi_t|^q dt < \infty \right]$, then for almost every $t \in [0, T]$ and any stopping time $\tau \in [0, T - t]$, we have

$$\psi_t = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t + \varepsilon \wedge \tau} \psi_u ds.$$ 

**Lemma 2.3** (Hewitt and Stromberg (1978, Lemma 18.4)) Let $f$ be a Lebesgue integrable function on the interval $[0, T]$. Then for almost every $t \in [0, T]$, we have

$$\lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t + \varepsilon} |f(u) - f(t)| ds = 0.$$ 

By the proof of Fan and Jiang (2010, Corollary 1), we can get the following Lemma 2.4.

**Lemma 2.4** Let $g$ satisfy the assumptions (A1) and (A2), $(\eta, x, q) \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathbb{R}^n \times \mathbb{R}^n$, then there exists a non-negative process sequence $\{\psi_t^n\}_{t \in [0, T]}$ such that $\lim_{n \to \infty} \|\psi_t^n\|_{\mathcal{H}^2} = 0$, and $dP \times dt$ - a.s., for any $n \in \mathbb{N}$ and $(\bar{y}, \bar{z}, \bar{x}) \in \mathbb{R}^{1+d+n}$,

$$|g(t, \bar{y}, \bar{z} + \sigma^*(t, \bar{x})q) - g(t, \eta_t, \sigma^*(t, x)q)| \leq 2(n + \kappa) \left(|\bar{y} - \eta_t| + |\bar{z}| + |\bar{x} - x|\right) + \psi_t^n,$$

where $\kappa = \lambda (1 + |q| \nu)$, $\lambda$ is the constant in (A1) and $\mu$ is the constant in (H1).

3 **Representation theorems for generators**

In this section, we will firstly establish a representation theorem for generators of RBSDEs in the space of random variables.
Theorem 3.1 Let $1 \leq p < 2$ and $g$ satisfy the assumptions (A1) and (A2), then for each $\eta \in \mathcal{S}^2(0, T; \mathbb{R})$ satisfying $\eta > L$, $(x, q) \in \mathbb{R}^n \times \mathbb{R}^n$ and for almost every $t \in [0, T]$, there exists a stopping time $\tau \in [0, T - t]$ depending on $(t, \eta, x, q)$, such that

$$g(t, \eta, \sigma(t, x)q) + q \cdot b(t, x) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_{t+\varepsilon}^{t+\varepsilon} - \eta - E \left[ K_{t+\varepsilon}^{t+\varepsilon} - K_t^{t+\varepsilon} | F_t \right] \right), \quad (1)$$

where $(Y_s^{t+\varepsilon, \eta}, Z_s^{t+\varepsilon, \eta}, K_s^{t+\varepsilon, \eta})$ is an arbitrary solution of RBSDE with parameter $(g(t + \varepsilon \wedge \tau, \eta_t + q \cdot (X_{t+\varepsilon}^{t, x} - x), L)$. Moreover, if $g$ also satisfies (A4), then (1) holds for $p = 2$.

Proof. For each $\eta \in \mathcal{S}^2(0, T; \mathbb{R})$ satisfying $\eta > L$, and $(t, x, q) \in [0, T] \times \mathbb{R}^n \times \mathbb{R}^n$, we define the following stopping time:

$$\tau := \inf \left\{ s \geq 0 : \eta_t + q \cdot (X_{t+s}^{t,x} - x) \leq L_t \right\} \wedge (T - t). \quad (2)$$

By $\eta > L$ and the continuity of $X_{t+s}^{t,x}$, we have $0 < \tau \leq T - t$ and

$$\eta_t + q \cdot (X_{t+s}^{t,x} - x) \geq L_{t+s} \wedge \tau, \forall s \in [0, T]. \quad (3)$$

For $\varepsilon \in [0, T - t]$, let $(Y_s^{t+\varepsilon, \eta}, Z_s^{t+\varepsilon, \eta}, K_s^{t+\varepsilon, \eta})$ be a solution of RBSDE with parameter $(g(t + \varepsilon \wedge \tau, \eta_t + q \cdot (X_{t+\varepsilon}^{t, x} - x), L)$ and set

$$\tilde{Y}_s^{t+\varepsilon, \eta} := Y_s^{t+\varepsilon, \eta} - (\eta_t + q \cdot (X_s^{t,x} - x)), \quad s \in [t, t + \varepsilon \wedge \tau],$$

$$\tilde{Z}_s^{t+\varepsilon, \eta} := Z_s^{t+\varepsilon, \eta} - \sigma^*(s, X_s^{t,x})q, \quad s \in [t, t + \varepsilon \wedge \tau],$$

$$\tilde{L}_s := L_{t+s} \wedge \tau - (\eta_t + q \cdot (X_{t+s}^{t, x} - x)), \quad s \in [t + \varepsilon \wedge \tau, T],$$

$$\tilde{L}_s := L_{t+s} - (\eta_t + q \cdot (X_{t+s}^{t, x} - x)), \quad s \in [t, t + \varepsilon \wedge \tau],$$

$$\tilde{L}_s := L_{s} - (\eta_s + q \cdot (X_s^{t,x} - x)) = L_s - \eta_s, \quad s \in [0, t].$$

Then by $\eta > L$ and (3), we have

$$\tilde{L}_s \leq 0, \quad s \in [0, T]. \quad (4)$$

Applying Itô formula to $\tilde{Y}_s^{t, \varepsilon}$, for $s \in [t, t + \varepsilon \wedge \tau]$, we have

$$\tilde{Y}_s^{t+\varepsilon, \eta} = \int_s^{t+\varepsilon} \left( g(r, \tilde{Y}_r^{t+\varepsilon, \eta} + \eta_t + q \cdot (X_r^{t,x} - x), \tilde{Z}_r^{t+\varepsilon, \eta} + \sigma^*(r, X_r^{t,x})q) + q \cdot b(r, X_r^{t,x}) \right) dr$$

$$+ K_{t+s}^{t+\varepsilon, \eta} - K_{t}^{t+\varepsilon, \eta} - \int_s^{t+\varepsilon} \tilde{Z}_r^{t+\varepsilon, \eta} dB_r. \quad (5)$$

Set

$$\tilde{g}(r, \tilde{y}, \tilde{z}) := \begin{cases} 0, & r \in [t + \varepsilon \wedge \tau, T], \\
\frac{g(r, \tilde{y} + \eta_t + q \cdot (X_r^{t,x} - x), \tilde{z} + \sigma^*(r, X_r^{t,x})q) + q \cdot b(r, X_r^{t,x})}{\eta_t + q \cdot (X_r^{t,x} - x) + \sigma^*(r, X_r^{t,x})q) + q \cdot b(r, X_r^{t,x})}, & r \in [t, t + \varepsilon \wedge \tau], \\
\frac{g(r, \tilde{y} + \eta_t + q \cdot (X_r^{t,x} - x) + \sigma^*(r, X_r^{t,x})q) + q \cdot b(r, X_r^{t,x})}{\eta_t + q \cdot (X_r^{t,x} - x) + \sigma^*(r, X_r^{t,x})q) + q \cdot b(r, X_r^{t,x})}, & r \in [0, t]. \end{cases}$$

By (A1) and (H2), we have

$$\tilde{g}(r, \tilde{y}, \tilde{z}) \leq \lambda(\gamma_r + |\tilde{y}| + \sup_{u \in [0, T]} |\eta_u| + |q \cdot (X_r^{t,x} - x)| + |\tilde{z} + \sigma^*(r, X_r^{t,x})q) + q \cdot b(r, X_r^{t,x})|)$$

$$\leq (1 + \lambda)(\gamma_r + |\tilde{y}| + |\tilde{z}|) \quad (6)$$
where $\tilde{\gamma}_r = (\gamma_r + \sup_{u \in [0,T]} |\eta_u| + |q((|X_r^{t,x}| + |x|)) + 2r|q|(1 + |X_r^{t,x}|)).$ By Remark 2.2, $\eta \in S^2(0,T;\mathbb{R})$ and $\gamma_t \in H^2(0,T;\mathbb{R}),$ we get $\tilde{\gamma}_t \in H^2(0,T;\mathbb{R}).$ Then by (6) and (A2), we can check that $\tilde{g}$ also satisfy the assumptions (A1) and (A2).

Let $(\tilde{g}_t, \tilde{z}_t, \tilde{k}_t)$ be a solution of RBSDE with parameter $(\tilde{g}, \tilde{Z}_{t}^{t+\varepsilon \wedge \tau}, \tilde{L}).$ By (5), it is not difficult to check that there exists a solution $(y_s^{t+\varepsilon \wedge \tau}, z_s^{t+\varepsilon \wedge \tau}, k_s^{t+\varepsilon \wedge \tau})$ of RBSDE with parameter $(\tilde{g}, T, 0, \tilde{L})$ such that, for $s \in [t, t + \varepsilon \wedge \tau],$

$$y_s^{t+\varepsilon \wedge \tau} = \tilde{y}_s^{t+\varepsilon \wedge \tau}, \quad z_s^{t+\varepsilon \wedge \tau} = \tilde{Z}_s^{t+\varepsilon \wedge \tau}, \quad k_s^{t+\varepsilon \wedge \tau} = K_s^{t+\varepsilon \wedge \tau} - K_{t+\varepsilon \wedge \tau} + \tilde{k}_t, \quad (7)$$

for $s \in [t + \varepsilon \wedge \tau, T],$

$$y_s^{t+\varepsilon \wedge \tau} = 0, \quad z_s^{t+\varepsilon \wedge \tau} = 0, \quad k_s^{t+\varepsilon \wedge \tau} = K_s^{t+\varepsilon \wedge \tau} - K_{t+\varepsilon \wedge \tau} + \tilde{k}_t,$$

and for $s \in [0, t],$

$$y_s^{t+\varepsilon \wedge \tau} = y_s, \quad z_s^{t+\varepsilon \wedge \tau} = z_s, \quad k_s^{t+\varepsilon \wedge \tau} = k_s.$$

Therefore by (4), (6), (7) and Lemma 2.1, we get there exists a constant $\tilde{C}$ only dependent on $T$ and $\lambda,$ such that

$$E\left[ \sup_{s \in [t, t+\varepsilon \wedge \tau]} |Y_s^{t+\varepsilon \wedge \tau}|^2 + \int_t^{t+\varepsilon \wedge \tau} |Z_s^{t+\varepsilon \wedge \tau}|^2 ds \right] \leq \tilde{C} \left[ \int_t^{t+\varepsilon \wedge \tau} \tilde{\gamma}_s ds \right]^2. \quad (8)$$

Set

$$M_t^{\varepsilon, \tau} := \frac{1}{\varepsilon} E\left[ \int_t^{t+\varepsilon \wedge \tau} g(r, \tilde{Y}_r^{t+\varepsilon \wedge \tau} + \eta_t + q \cdot (X_r^{t,x} - x), \tilde{Z}_r^{t+\varepsilon \wedge \tau} + \sigma^*(r, X_r^{t,x})q) dr \right],$$

$$P_t^{\varepsilon, \tau} := \frac{1}{\varepsilon} E\left[ \int_t^{t+\varepsilon \wedge \tau} g(r, \eta_t, \sigma^*(r, x)q) dr \right],$$

$$U_t^{\varepsilon, \tau} := \frac{1}{\varepsilon} E\left[ \int_t^{t+\varepsilon \wedge \tau} q \cdot b(r, X_r^{t,x}) dr \right].$$

Since $Y_t^{t+\varepsilon \wedge \tau} - \eta_t = \tilde{Y}_t^{t+\varepsilon \wedge \tau},$ then by (5), we have

$$\begin{aligned}
\frac{1}{\varepsilon} \left( Y_t^{t+\varepsilon \wedge \tau} - \eta_t - E\left[ K_t^{t+\varepsilon \wedge \tau} - K_{t+\varepsilon \wedge \tau} |F_t \right] \right) - g(t, \eta_t, \sigma^*(t, x)q) - q \cdot b(t, x) \\
= \frac{1}{\varepsilon} \left( \tilde{Y}_t^{t+\varepsilon \wedge \tau} - E\left[ \tilde{K}_t^{t+\varepsilon \wedge \tau} - \tilde{K}_{t+\varepsilon \wedge \tau} |F_t \right] \right) - g(t, \eta_t, \sigma^*(t, x)q) - q \cdot b(t, x) \\
= M_t^{\varepsilon, \tau} + U_t^{\varepsilon, \tau} - g(t, \eta_t, \sigma^*(t, x)q) - q \cdot b(t, x) \\
= (M_t^{\varepsilon, \tau} - P_t^{\varepsilon, \tau}) + (P_t^{\varepsilon, \tau} - g(t, \eta_t, \sigma^*(t, x)q)) + (U_t^{\varepsilon, \tau} - q \cdot b(t, x)). \quad (9)
\end{aligned}$$

Thus, in the following, we only need prove that (9) converge to 0 in $L^p,$ for $1 \leq p < 2.$

By Jensen inequality, Hölder inequality and Lemma 2.4, we get that there exists a non-negative process sequence $\{(\psi_t^n)_{t \in [0,T]}\}_{n=1}^\infty \in H^2(0,T;\mathbb{R})$ depending on $(\eta_t, x, q)$ and $\lim_{n \to \infty} \|\psi_t^n\|_{\mathcal{H}^2} = 0,$ such that, for any $n \geq 1,$

$$E[M_t^{\varepsilon, \tau} - P_t^{\varepsilon, \tau}]^2 \leq \frac{1}{\varepsilon^2} E\left[ \int_t^{t+\varepsilon \wedge \tau} \left| g(r, \tilde{Y}_r^{t+\varepsilon \wedge \tau} + \eta_t + q \cdot (X_r^{t,x} - x), \tilde{Z}_r^{t+\varepsilon \wedge \tau} + \sigma^*(r, X_r^{t,x})q) - g(r, \eta_t, \sigma^*(r, x)q) \right|^2 dr \right].$$
Then by (10)-(13), Fubini Theorem, Lemma 2.3 and (14), we get that for almost every $t$

Thus, for almost every $t$

By Fatou Lemma, Fubini Theorem and $\lim_{\varepsilon \to 0^+} \varepsilon = 0$

By Fubini Theorem and Remark 2.2, we have

By Fatou Lemma, Fubini Theorem and $\lim_{n \to \infty} \|\psi^n_t\|_{H^2} = 0$

Thus, for almost every $t \in [0, T]$, we have

Then by (10)-(13), Fubini Theorem, Lemma 2.3 and (14), we get that for almost every $t \in [0, T]$,

$$
\lim_{\varepsilon \to 0^+} E|M_t^{\varepsilon, \tau} - P_t^{\varepsilon, \tau}|^2 \leq \lim_{n \to \infty} \lim_{\varepsilon \to 0^+} \varepsilon E[|\psi^n_t|^2] = 0.
$$
For $1 \leq p < 2$, by Jensen inequality and Lemma 2.2, we have, for almost every $t \in [0, T]$, 

$$
\lim_{\varepsilon \to 0^+} E\left| P^{t, \tau}_\varepsilon - g(t, \eta_t, \sigma^*(t, x)q) \right|^p
= \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} (g(r, \eta_r, \sigma^*(r, x)q) dr - g(t, \eta_t, \sigma^*(t, x)q)) |F_t| \right]^p
\leq \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} g(r, \eta_r, \sigma^*(r, x)q) dr - g(t, \eta_t, \sigma^*(t, x)q) \right]^p
= 0.
$$

By Jensen inequality, we have,

$$
\lim_{\varepsilon \to 0^+} E\left| U^{t, \tau}_\varepsilon - q \cdot b(t, x) \right|^2
= \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} (q \cdot b(r, X^t_{r,x}) - q \cdot b(r, x) + q \cdot b(r, x)) dr - q \cdot b(t, x) |F_t| \right]^2
\leq \lim_{\varepsilon \to 0^+} 2E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, X^t_{r,x}) - q \cdot b(r, x) dr - q \cdot b(t, x) |F_t| \right]^2
+ \lim_{\varepsilon \to 0^+} 2E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) |F_t| \right]^2
\leq \lim_{\varepsilon \to 0^+} \frac{2}{\varepsilon^2} E \left( \int_t^{t+\varepsilon \land T} |q \cdot b(r, X^t_{r,x}) - q \cdot b(r, x)| dr \right)^2
+ \lim_{\varepsilon \to 0^+} 2E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) \right]^2.
$$

By (H2), Lebesgue dominated convergence theorem, Fubini theorem and Lemma 2.3, we have

$$
\int_0^T \lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) \right]^2 dt
= \int_0^T E \left[ \lim_{\varepsilon \to 0^+} \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) \right]^2 \right] dt
= \int_0^T E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) \right]^2 dt
= E \left[ \int_0^T \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) \right]^2 dt
= 0.
$$

Then we have, for almost every $t \in [0, T]$, 

$$
\lim_{\varepsilon \to 0^+} E \left[ \frac{1}{\varepsilon} \int_t^{t+\varepsilon \land T} q \cdot b(r, x) dr - q \cdot b(t, x) \right]^2 = 0.
$$

Then by (17), (18), Hölder inequality, Fubini Theorem, (H1) and Remark 2.2, we have, for almost every $t \in [0, T]$, 

$$
\lim_{\varepsilon \to 0^+} E\left| U^{t, \tau}_\varepsilon - q \cdot b(t, x) \right|^2
\leq \lim_{\varepsilon \to 0^+} \frac{2}{\varepsilon^2} E \left( \int_t^{t+\varepsilon \land T} |q \cdot b(r, X^t_{r,x}) - q \cdot b(r, x)| dr \right)^2
$$
By Fubini theorem and Lemma 2.3, we have

\begin{align}
\leq \lim_{\varepsilon \to 0^+} & \frac{2}{\varepsilon} |g|^2 \int_t^{t+\varepsilon} E[b(r, X^{t,x}_r) - b(r, x)]^2 dr \\
\leq \lim_{\varepsilon \to 0^+} & \frac{2}{\varepsilon} |q|^2 \int_t^{t+\varepsilon} E\mu^2 |X^{t,x}_r - x|^2 dr \\
\leq \lim_{\varepsilon \to 0^+} & \frac{2}{\varepsilon} |q|^2 \mu^2 \int_t^{t+\varepsilon} (E|X^{t,x}_r - x|^2) dr \\
= & 0. \quad (19)
\end{align}

By (9), (15), (16) and (19), we get (1) holds.

Moreover, if \( g \) also satisfies (A4), then we have
\[ E \left[ \sup_{0 \leq \tau \leq T} |g(r, \eta_r, \sigma^*(r, x) q)|^2 \right] < +\infty, \]
from (A1), (H2) and \( \eta \in S^2(0, T; \mathbb{R}) \). Then by (16), Lebesgue dominated convergence theorem, Fubini theorem and Lemma 2.3, we have

\begin{align}
\int_0^T \lim_{\varepsilon \to 0^+} E|P^{t,\tau} - g(t, \eta_t, \sigma^*(t, x) q)|^2 dt & \\
\leq & \int_0^T \lim_{\varepsilon \to 0^+} E \left| \frac{1}{\varepsilon} \int_t^{t+\varepsilon} g(r, \eta_r, \sigma^*(r, x) q) dr - g(t, \eta_t, \sigma^*(t, x) q) \right|^2 dt \\
= & \int_0^T E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} g(r, \eta_r, \sigma^*(r, x) q) dr - g(t, \eta_t, \sigma^*(t, x) q) \right]^2 dt \\
= & \int_0^T E \left[ \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \int_t^{t+\varepsilon} g(r, \eta_r, \sigma^*(r, x) q) dr - g(t, \eta_t, \sigma^*(t, x) q) \right]^2 dt \\
= & 0.
\end{align}

Then we have, for almost every \( t \in [0, T] \),
\[ \lim_{\varepsilon \to 0^+} E|P^{t,\tau} - g(t, \eta_t, \sigma^*(t, x) q)|^2 = 0. \quad (20) \]

By (9), (15), (19) and (20), we get that (1) holds for \( p = 2 \). The proof is complete. \( \square \)

**Theorem 3.2** Let \( 1 \leq p < 2 \), \( g \) satisfy assumptions (A1)-(A3) and there exists a constant \( C \) such that \( \sup_{t \in [0, T]} L_t \leq C \), then for each \( (y, x, q) \in \mathbb{C}, +\infty) \times \mathbb{R}^n \times \mathbb{R}^n \) and for almost every \( t \in [0, T] \), there exists a stopping time \( \tau > 0 \) depending on \( (t, y, x, q) \), such that
\[ g(t, y, \sigma^*(t, x) q) + q \cdot b(t, x) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y^{t+\varepsilon \wedge \tau}_{t} - y \right), \quad (21) \]
where \( (Y^{t+\varepsilon \wedge \tau}_{s}, Z^{t+\varepsilon \wedge \tau}_{s}, K^{t+\varepsilon \wedge \tau}_{s}) \) is the maximal solution of RBSDE with parameter \( (g, t + \varepsilon \wedge \tau, y + q \cdot (X^{t,x}_{t+\varepsilon \wedge \tau} - x), L) \). Moreover, if \( g \) also satisfies (A4), then (21) holds for \( p = 2 \).

**Proof.** For each \( (t, x, q) \in [0, T] \times \mathbb{R}^d \times \mathbb{R}^n \), and \( y > C \), we define the following stopping time:
\[ \tau = \inf \left\{ s \geq 0 : y + q \cdot (X^{t,x}_{t+s} - x) \leq C \right\} \wedge (T - t), \]
By \( y > C \) and the continuity of \( X^{t,x}_{t+s} \), we have \( 0 < \tau \leq T - t \) and
\[ y + q \cdot (X^{t,x}_{t+s} - x) \geq C \geq L_{t+s \wedge \tau}, \quad \forall s \in [0, T]. \quad (22) \]
For \( \varepsilon \in ]0, T-t[ \), we denote the maximal solution of RBSDE with parameter \( (g, t+\varepsilon \wedge \tau, y+q \cdot (X_{t+\varepsilon \wedge \tau} - x), L) \) and \( (g, t+\varepsilon \wedge \tau, C, L) \) by \( (Y_s^{t+\varepsilon \wedge \tau}, Z_s^{t+\varepsilon \wedge \tau}, K_s^{t+\varepsilon \wedge \tau}) \) and \( (Y_s^{C,t+\varepsilon \wedge \tau}, Z_s^{C,t+\varepsilon \wedge \tau}, K_s^{C,t+\varepsilon \wedge \tau}) \), respectively. By (A3) and sup_{t \in [0,T]} L_t \leq C, we can check that

\[
(Y_s^{C,t+\varepsilon \wedge \tau} = C, Z_s^{C,t+\varepsilon \wedge \tau} = 0, K_s^{C,t+\varepsilon \wedge \tau} = 0), \quad \forall s \in [0, t+\varepsilon \land \tau],
\]

is a solution of RBSDE with parameter \( (g, t+\varepsilon \wedge \tau, C, L) \). By (22) and comparison theorem of RBSDEs with continuous and linear-growth generators (see Bahlali et al. (2005, Proposition 2.6)), we have

\[
\begin{align*}
Y_s^{t+\varepsilon \wedge \tau} &\geq Y_s^{C,t+\varepsilon \wedge \tau} \geq Y_s^{t+\varepsilon \wedge \tau} = C, \quad s \in [0, t+\varepsilon \wedge \tau], \\
K_s^{t+\varepsilon \wedge \tau} &\leq K_s^{C,t+\varepsilon \wedge \tau} \leq K_s^{t+\varepsilon \wedge \tau} = 0, \quad s \in [0, t+\varepsilon \wedge \tau].
\end{align*}
\]

Since \( K_s^{t+\varepsilon \wedge \tau} \geq 0 \), we have,

\[
K_s^{t+\varepsilon \wedge \tau} = 0, \quad s \in [0, t+\varepsilon \wedge \tau].
\] (23)

By (23) and Theorem 3.1, we can complete the proof. \( \square \)

Let \( g = z \), \( b(t, x) = 0 \), \( \sigma(t, x) = 1 \), \( x = 0 \) in Theorem 3.1 and Theorem 3.2, then we have the following Corollary 3.1 and Corollary 3.2 immediately.

**Corollary 3.1** Let \( 1 \leq p < 2 \), \( g \) satisfy assumptions (A1) and (A2), for each \( \eta \in \mathcal{S}^2(0,T; \mathbb{R}) \) satisfying \( \eta > L \), \( z \in \mathbb{R}^d \) and for almost every \( t \in ]0, T[ \), there exists a stopping time \( \tau > 0 \) depending on \( (t, \eta, z) \), such that

\[
g(t, \eta, z) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_t^{t+\varepsilon \land \tau} - \eta - E \left[ K_t^{t+\varepsilon \land \tau} - K_t^{t+\varepsilon \land \tau} \mid \mathcal{F}_t \right] \right),
\] (24)

where \( (Y_s^{t+\varepsilon \land \tau}, Z_s^{t+\varepsilon \land \tau}, K_s^{t+\varepsilon \land \tau}) \) is an arbitrary solution of RBSDE with parameter \( (g, t+\varepsilon \land \tau, \eta, t \leq \tau, z \cdot (B_{t+\varepsilon \land \tau} - B_t), L) \). Moreover, if \( g \) also satisfies (A4), then (24) holds for \( p = 2 \).

**Corollary 3.2** Let \( 1 \leq p < 2 \), \( g \) satisfy assumptions (A1)-(A3) and there exists a constant \( C \) such that \( \sup_{t \in [0,T]} L_t \leq C \), then for each \( (y, z) \in ]C, +\infty[ \times \mathbb{R}^d \) and for almost every \( t \in ]0, T[ \), there exists a stopping time \( \tau > 0 \) depending on \( (t, y, z) \), such that

\[
g(t, y, z) = L^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_t^{t+\varepsilon \land \tau} - y \right),
\] (25)

where \( (Y_s^{t+\varepsilon \land \tau}, Z_s^{t+\varepsilon \land \tau}, K_s^{t+\varepsilon \land \tau}) \) is the maximal solution of RBSDE with parameter \( (g, t+\varepsilon \land \tau, y + z \cdot (B_{t+\varepsilon \land \tau} - B_t), L) \). Moreover, if \( g \) also satisfies (A4), then (25) holds for \( p = 2 \).

Obviously, Theorem 3.1, Theorem 3.2, Corollary 3.1 and Corollary 3.2 are representation theorems in the space of random variables. Now, we will give two representation theorems for generators of RBSDEs with one obstacle in the space of processes.

**Theorem 3.3** Let \( 1 \leq p < 2 \) and \( g \) satisfy the assumptions (A1) and (A2), then for each \( \eta \in \mathcal{S}^2(0,T; \mathbb{R}) \) satisfying \( \eta > L \), \( (x, q) \in \mathbb{R}^n \times \mathbb{R}^n \) and there exists a set of stopping times \( \{\tau_i\}_{i \in [0, T]} \) where \( \tau_i > 0 \) is dependent on \( (t, \eta_i, x, q) \), such that

\[
g(t, \eta_i, \sigma^*(t, x) + q \cdot b(t, x) + q \cdot b(t, x) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_t^{t+\varepsilon \land \tau_i} - \eta_i - E \left[ K_t^{t+\varepsilon \land \tau_i} - K_t^{t+\varepsilon \land \tau_i} \mid \mathcal{F}_t \right] \right),
\] (26)
where \((Y^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t,Z^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t,K^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t)\) is an arbitrary solution of RBSDE with parameter \((g,t+\varepsilon\wedge\tau_t,\eta_t+q\cdot(X^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t}-x),L)\). Moreover, if \(g\) also satisfies (A4), then (26) holds for \(p=2\).

**Proof.** The proof is dependent on the argument of Theorem 3.1 and Fan and Jiang (2010, Theorem 1). We sketch this proof. For each \(\eta\in\mathcal{S}^2(0,T;\mathbb{R})\) satisfying \(\eta>L\), and \((t,x,q)\in[0,T]\times\mathbb{R}^n\times\mathbb{R}^n\), let \(\tau\) be the stopping time defined in (2). Since \(\tau\) is dependent on \((t,\eta_t,x,q)\), we can rewrite \(\tau\) by \(\tau_t\), and the corresponding \(M^\varepsilon,\pi\), \(P^\varepsilon,\pi\), \(U^\varepsilon,\pi\) defined in the proof of Theorem 3.1 by \(M^\varepsilon,\pi\), \(P^\varepsilon,\pi\), \(U^\varepsilon,\pi\), respectively, for convention. Then (7) can be rewritten as follows

\[
\frac{1}{\varepsilon}(Y^\varepsilon_{t\varepsilon}-\eta_t-E[K^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t-\varepsilon\wedge\tau_t|\mathcal{F}_t]) = g(t,\eta_t,\sigma^\varepsilon(t,x)q-q\cdot b(t,x)
\]

\[
= (M^\varepsilon,\tau_t-P^\varepsilon,\tau_t) + (P^\varepsilon,\tau_t-g(t,\eta_t,\sigma^\varepsilon(t,x)q)) + (U^\varepsilon,\tau_t-q\cdot b(t,x)).
\]

Then, the similar argument as the proof of (20) in Fan and Jiang (2010) (or (3.11) in Fan and Hu (2008)) gives,

\[
\lim_{\varepsilon\to 0+} E\left[\int_0^T |U^\varepsilon,\tau_t-q\cdot b(t,x)|^2 dt\right] = 0.
\]

By Fubini Theorem, (10) and the similar argument of (25) in Fan and Jiang (2010), we can deduce that

\[
\lim_{\varepsilon\to 0+} E\left[\int_0^T |M^\varepsilon,\tau_t-P^\varepsilon,\tau_t|^2 dt\right] = \lim_{\varepsilon\to 0+} \int_0^T E|M^\varepsilon,\tau_t-P^\varepsilon,\tau_t|^2 dt = 0.
\]

By the similar argument of (21) in Fan and Jiang (2010), we can deduce that, for \(1\leq p<2\),

\[
\lim_{\varepsilon\to 0+} E\left[\int_0^T |P^\varepsilon,\tau_t-g(t,\eta_t,\sigma^\varepsilon(t,x)q)|^p dt\right] = 0.
\]

Then from (27)-(30), we get (26). Moreover, if \(g\) also satisfies (A4), then by the similar argument of (31) in Fan and Jiang (2010), we get

\[
\lim_{\varepsilon\to 0+} E\left[\int_0^T |P^\varepsilon,\tau_t-g(t,\eta_t,\sigma^\varepsilon(t,x)q)|^2 dt\right] = 0.
\]

Then from (27)-(29) and (31), we get that (26) holds for \(p=2\). The proof is complete. \(\square\)

By Theorem 3.3 and the proof of Theorem 3.2, we can get the following representation theorem in the space of processes. We omit its proof.

**Corollary 3.3** Let \(1\leq p<2\), \(g\) satisfy assumptions (A1)-(A3) and there exists a constant \(C\) such that \(\sup_{t\in[0,T]} L_t \leq C\), then for each \((y,x,q)\in\mathcal{S}^2(0,T;\mathbb{R})\times\mathbb{R}^n\times\mathbb{R}^n\) and there exists a set of stopping times \(\{\tau_t\}_{t\in[0,T]}\), where \(\tau_t>0\) is dependent on \((t,\eta_t,x,q)\), such that

\[
g(t,y,\sigma^\varepsilon(t,x)q+q\cdot b(t,x) = \mathcal{H}^p - \lim_{\varepsilon\to 0+} \frac{1}{\varepsilon} (\mathcal{Y}^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t-y),
\]

where \((\mathcal{Y}^\varepsilon_{t\varepsilon},\mathcal{Z}^\varepsilon_{t\varepsilon},\mathcal{F}^\varepsilon_{t\varepsilon})\) is the maximal solution of RBSDE with parameter \((g,t+\varepsilon\wedge\tau_t,y+q\cdot(X^\varepsilon_{t\varepsilon}+\varepsilon\wedge\tau_t}-x),L)\). Moreover, if \(g\) also satisfies (A4), then (32) holds for \(p=2\).
Remark 3.1 All representation theorems for generators in this section are true in local space. In fact, from Remark 2.1, it follows that they may be not true in whole space. This is not the same as BSDEs.

Remark 3.2 Let \((Y, Z, K)\) be a solution of RBSDE with parameter \((g, T, \xi, L)\). In fact, if \(L \equiv -\infty\), then we can check that \(K \equiv 0\). Therefore, when \(L \equiv -\infty\), RBSDE with parameter \((g, T, \xi, L)\) will become the following standard BSDE with parameter \((g, T, \xi)\):

\[
Y_t = \xi + \int_t^T g(s, Y_s, Z_s) \, ds - \int_t^T Z_s \cdot dB_s, \quad t \in [0, T],
\]

and the corresponding Theorem 3.1 and Theorem 3.3 will become to the following representation theorem for standard BSDEs in the space of random variables obtained by Jia (2008, Theorem 2.3.5):

\[
g(t, y, \sigma^*(t, x))q + q \cdot b(t, x) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_t^{t+\varepsilon} - y \right),
\]

and the following representation theorem for standard BSDEs in the space of stochastic processes obtained by Fan and Jiang (2010, Theorem 1):

\[
g(t, y, \sigma^*(t, x))q + q \cdot b(t, x) = \mathcal{H}^p - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} \left( Y_t^{t+\varepsilon} - y \right),
\]

respectively, where \((Y_s^{t+\varepsilon}, Z_s^{t+\varepsilon})\) is an arbitrary solution of BSDE with parameter \((g, t + \varepsilon, y + q \cdot (X_s^{t+\varepsilon} - x))\).

Remark 3.3 Representation theorems for generators of BSDEs in the space of random variables are established in Jiang (2005c) under the additional condition that \(b(t, x)\) and \(\sigma(t, x)\) in SDEs are both right continuous in \(t\). In fact, one can see that this condition is eliminated in all representation theorems in this section.

4 Some applications

In this section, we will give some applications of representation theorem for RBSDEs obtained in Section 3. Firstly, we will establish two converse comparison theorems for RBSDEs.

Theorem 4.1 (Converse comparison theorem I) Let generators \(g_1\) and \(g_2\) satisfy assumptions (A1) and (A2), and for any stopping time \(\tau \in [0, T]\), \(\xi \in L^2(F_{\tau})\) and \(\xi \geq L_\tau\), RBSDEs with parameters \((g_1, \tau, \xi, L)\) and \((g_2, \tau, \xi, L)\) exist solution \((Y^1_\tau, Z^1_\tau, K^1_\tau)\) and \((Y^2_\tau, Z^2_\tau, K^2_\tau)\), respectively, such that \(\forall t \in [0, T]\),

\[
P - a.s., \quad Y^1_{t\land \tau} \geq Y^2_{t\land \tau}, \quad E[(K^1_{t\land \tau} - K^2_{t\land \tau})|\mathcal{F}_t] \leq E[(K^2_{\tau} - K^2_{t\land \tau})|\mathcal{F}_t],
\]

then \(\forall \eta \in \mathcal{S}^2(0, T; \mathbb{R})\) satisfying \(\eta \geq L, \forall z \in \mathbb{R}^d\), and almost every \(t \in [0, T]\), we have

\[
P - a.s., \quad g_1(t, \eta, z) \geq g_2(t, \eta, z).
\]

Proof. By Corollary 3.1, for each \(\eta \in \mathcal{S}^2(0, T; \mathbb{R})\) satisfying \(\eta \geq L, \forall z \in \mathbb{R}^d\), and for almost every \(t \in [0, T]\), there exists a stopping time \(\delta > 0\) depending on \((t, \eta, z)\), such that

\[
g_i(t, \eta, z) = L^1 - \lim_{\varepsilon \to 0^+} \frac{1}{\varepsilon} (Y^i_{t+\varepsilon \land \delta} - \eta - E[(K^i_{t+\varepsilon \land \delta} - K^i_{t+\varepsilon \land \delta})|\mathcal{F}_t]), \quad i = 1, 2.
\]
where \((Y_{t+\varepsilon\wedge\delta,i}, Z_{t+\varepsilon\wedge\delta,i}, K_{t+\varepsilon\wedge\delta,i})\) is an arbitrary solution of RBSDE with parameter \((g_i, t+\varepsilon\wedge\delta, \eta_t + \varepsilon \cdot (B_{t+\varepsilon\wedge\delta} - B_t), L), \ i = 1, 2, \) respectively. Then by (34), there exists a sequence \(\{n_k\}_{k \geq 1}\) such that, for almost every \(t \in [0, T]\),

\[
P - a.s., \quad g_i(t, \eta_t, z) = \lim_{n_k \to +\infty} n_k(Y_{t+\frac{1}{n_k} \wedge \delta,i} - \eta_t - E[(K_{t+\frac{1}{n_k} \wedge \delta,i} - K_t)|\mathcal{F}_t]), \ i = 1, 2. \tag{35}
\]

By (33) and (35), we can deduce, for almost every \(t \in [0, T]\),

\[
P - a.s., \quad g_1(t, \eta_t, z) \geq g_2(t, \eta_t, z).
\]

From above inequality and (A2), The proof is completed. □

By Corollary 3.2 and the same argument of Theorem 4.1, we have the following converse comparison theorem, we omit its proof.

**Theorem 4.2** (Converse comparison theorem II) Let generators \(g_1\) and \(g_2\) satisfy assumptions (A1)-(A3), there exists a constant \(C\) such that \(\sup_{t \in [0, T]} L_t \leq C\), and for any stopping time \(\tau \in [0, T]\), \(\xi \in L^2(\mathcal{F}_\tau)\), the maximal solutions \((\nabla_{t,\tau}^{\tau,i}, Z_{t,\tau}^{\tau,i}, K_{t,\tau}^{\tau,i})\) of RBSDE with parameter \((g_i, \tau, \xi, L), \ i = 1, 2,\) satisfy \(\forall t \in [0, T]\),

\[
P - a.s., \quad \nabla_{t,\tau}^{\tau,1} \geq \nabla_{t,\tau}^{\tau,2},
\]

then \(\forall (y, z) \in [C, +\infty) \times \mathbb{R}^d\) and almost every \(t \in [0, T]\), we have

\[
P - a.s., \quad g_1(t, y, z) \geq g_2(t, y, z).
\]

**Remark 4.1** Theorem 4.1 and Theorem 4.2 are both established for generators which are continuous with linear-growth in \((y, z)\), while converse comparison theorems of RBSDEs in Li, Tang (2007) and Li, Gu (2007) are obtained under the conditions that generators satisfy Lipschitz condition and are continuous in \(t\).

**Remark 4.2** Theorem 4.1 and Theorem 4.2 both show that generators can be compared in local space. In fact, by Remark 2.1, we get that the generators can not be compared in whole space. This is not the same as BSDEs.

Self-financing condition and Zero-interest condition are considered in Peng (2006), Jia (2008), Fan and Jiang (2010), and Fan et al. (2011), respectively, in BSDEs case. we will discuss them in RBSDE case.

**Theorem 4.3** (Self-financing condition) Let \(g\) satisfy assumptions (A1),(A2) and \(\sup_{t \in [0, T]} L_t < 0\), then the following two statements are equivalent:

(i) For almost every \(t \in [0, T]\),

\[
P - a.s., \quad g(t, 0, 0) = 0;
\]

(ii) There exist a solution \((Y_t, Z_t, K_t)\) of RBSDE with parameter \((g, T, 0, L)\) such that

\[
P - a.s., \quad Y_t = 0, \forall t \in [0, T].
\]
Proof. We can easily check that (i) implies (ii) by setting \((Y_t, Z_t, K_t) = (0, 0, 0)\). If (ii) holds, then we can check that for any stopping time \(\tau \in [0, T]\), there exist a solution \((y_t^\tau, z_t^\tau, k_t^\tau)\) of RBSDE with parameter \((g, \tau, 0, L)\) such that

\[
P - a.s., \quad y_t^\tau = 0, \quad z_t^\tau = Z_t, \quad k_t^\tau = 0, \quad \forall t \in [0, \tau].
\]

Then by Corollary 3.1, we complete the proof. \(\square\)

**Theorem 4.4 (Zero-interest condition)** Let \(g\) satisfy assumptions \((A1),(A2)\) and there exist a constant \(C\) such that \(\sup_{t \in [0, T]} L_t < C\), then the following two statements are equivalent:

(i) For almost every \(t \in [0, T]\),

\[
P - a.s., \quad g(t, y, 0) = 0, \quad \forall y \in [C, +\infty);
\]

(ii) For any \(y \geq C\), there exist a solution \((Y_t, Z_t, K_t)\) of RBSDE with parameter \((g, T, y, L)\) such that

\[
P - a.s., \quad Y_t = y, \quad \forall t \in [0, T].
\]

Proof. We can easily check that (i) implies (ii) by setting \((Y_t, Z_t, K_t) = (y, 0, 0)\). If (ii) holds, then we can check that for \(y \geq C\), any stopping time \(\tau \in [0, T]\), there exist a solution \((y_t^\tau, z_t^\tau, k_t^\tau)\) of RBSDE with parameter \((g, \tau, y, L)\) such that

\[
P - a.s., \quad y_t^\tau = y, \quad z_t^\tau = Z_t, \quad k_t^\tau = 0, \quad \forall t \in [0, \tau].
\]

Then by Corollary 3.1, we complete the proof. \(\square\)

**Appendix**

In this Appendix, we will give the proof of Lemma 2.1.

**Proof of Lemma 2.1**

For \(0 \leq r < t \leq T\), applying Itô formula to the process \(|Y_s|^2 e^{\beta s}\) in \([r, t]\), we have

\[
|Y_r|^2 e^{\beta r} + \int_r^t (\beta |Y_s|^2 + |Z_s|^2) e^{\beta s} ds
= |Y_t|^2 e^{\beta t} + 2 \int_r^t Y_s g(s, Y_s, Z_s) e^{\beta s} ds + 2 \int_r^t Y_s e^{\beta s} dK_s - 2 \int_r^t Y_s Z_s e^{\beta s} dB_s
\leq |Y_t|^2 e^{\beta t} + 2 \int_r^t |Y_s||g(s, Y_s, Z_s)| e^{\beta s} ds + 2 \int_r^t Y_s e^{\beta s} dK_s - 2 \int_r^t Y_s Z_s e^{\beta s} dB_s
\leq |Y_t|^2 e^{\beta t} + 2 \lambda \int_r^t |Y_s| |\gamma_s| e^{\beta s} ds + 2 \int_r^t (\lambda |Y_s|^2 + \lambda |Y_s||Z_s|) e^{\beta s} ds + 2 \int_r^t Y_s e^{\beta s} dK_s - 2 \int_r^t Y_s Z_s e^{\beta s} dB_s
\leq |Y_t|^2 e^{\beta t} + 2 \lambda \int_r^t |Y_s| |\gamma_s| e^{\beta s} ds + \int_r^t ((2\lambda + 2\lambda^2)|Y_s|^2 + \frac{1}{2} |Z_s|^2) e^{\beta s} ds + 2 \int_r^t Y_s e^{\beta s} dK_s - 2 \int_r^t Y_s Z_s e^{\beta s} dB_s
\]

\[14\]
where we have used (A1). Setting $\beta = 2\lambda^2 + 2\lambda$, then we have

$$|Y_r|^2 e^{\beta r} + \frac{1}{2} \int_r^t |Z_s|^2 e^{\beta s} ds \leq |Y_t|^2 e^{\beta t} + 2\lambda \int_r^t |Y_s||\gamma_s|e^{\beta s} ds + 2 \int_r^t Y_s e^{\beta s} dK_s - 2 \int_r^t Y_s e^{\beta s} dB_s.$$ 

Then using the identity $\int_r^t (Y_s - L_s) dK_s = 0$ and $L \leq L^+$, we have

$$E \left[ \frac{1}{2} \int_r^t |Z_s|^2 e^{\beta s} ds \right] |F_t \geq E \left[ |Y_t|^2 e^{\beta t} + 2\lambda \int_r^t |Y_s||\gamma_s|e^{\beta s} ds + 2 \int_r^t L_s^+ e^{\beta s} dK_s |F_t \right],$$

and

$$\sup_{s \in [r,t]} |Y_s|^2 e^{\beta s} \leq |Y_t|^2 e^{\beta t} + 2\lambda \int_r^t |Y_s||\gamma_s|e^{\beta s} ds + 2 \int_r^t L_s^+ e^{\beta s} dK_s + 4 \sup_{u \in [r,t]} \int_r^u Y_s e^{\beta s} dB_s.$$ 

By BDG inequality, we have

$$E \left[ \sup_{s \in [r,t]} |Y_s|^2 e^{\beta s} |F_r \right] 
\leq E \left[ |Y_t|^2 e^{\beta t} + 2\lambda \int_r^t |Y_s||\gamma_s|e^{\beta s} ds + 2 \int_r^t L_s^+ e^{\beta s} dK_s |F_r \right] + CE \left[ \left( \int_r^t |Y_s|^2 |Z_s|^2 e^{2\beta s} ds \right)^{\frac{1}{2}} |F_r \right] 
\leq E \left[ |Y_t|^2 e^{\beta t} + 2\lambda \int_r^t |Y_s||\gamma_s|e^{\beta s} ds + 2 \int_r^t L_s^+ e^{\beta s} dK_s |F_r \right] + \frac{C^2}{2} E \left[ \int_r^t |Z_s|^2 e^{\beta s} ds |F_r \right] 
+ \frac{1}{2} E \left[ \sup_{s \in [r,t]} |Y_s|^2 e^{\beta s} |F_r \right],$$

where $C$ is a constant, which may change from line to line. Combining above one and (36), we can deduce

$$E \left[ \sup_{s \in [r,t]} |Y_s|^2 e^{\beta s} + \int_r^t |Z_s|^2 e^{\beta s} ds |F_r \right] 
\leq CE \left[ |Y_t|^2 e^{\beta t} + 2\lambda \int_r^t |Y_s||\gamma_s|e^{\beta s} ds + 2 \int_r^t L_s^+ e^{\beta s} dK_s |F_r \right] 
\leq CE \left[ |Y_t|^2 e^{\beta t} + \frac{1}{2C} \sup_{s \in [r,t]} |Y_s|^2 e^{\beta s} + 2C \left( \int_r^t |\gamma_s| e^{(\beta s)/2} ds \right)^2 + 2 \int_r^t L_s^+ e^{\beta s} dK_s |F_r \right]$$

Then we have

$$E \left[ \sup_{s \in [r,t]} |Y_s|^2 + \int_r^t |Z_s|^2 ds |F_r \right] \leq CE \left[ |Y_t|^2 + \left( \int_r^t |\gamma_s| ds \right)^2 + \int_r^t L_s^+ dK_s |F_r \right],$$

where $C$ is only dependent on $T$ and $\lambda$. Since

$$K_t - K_r = Y_r - Y_t - \int_r^t g(s, Y_s, Z_s) ds + \int_r^t Z_s dB_s,$$

then by (A1), BDG inequality and (37), we can deduce

$$E \left[ (K_t - K_r)^2 |F_r \right] \leq CE \left[ |Y_t|^2 + \left( \int_r^t |\gamma_s| ds \right)^2 + \int_r^t L_s^+ dK_s |F_r \right].$$

(38)
Since
\[ E \left[ \int_r^t L^+_s dK_s | \mathcal{F}_r \right] \leq E \left[ \frac{C}{2} \sup_{s \in [r,t]} (L^+_s)^2 + \frac{1}{2C} |K_t - K_r|^2 | \mathcal{F}_r \right]. \]

(39)

Then by (38) and (39), we have
\[ E \left[ \int_r^t L^+_s dK_s | \mathcal{F}_r \right] \leq CE \left[ |Y_t|^2 + \left( \int_r^t \gamma_s ds \right)^2 + \sup_{s \in [r,t]} (L^+_s)^2 | \mathcal{F}_r \right]. \]

(40)

By (37), (38) and (40), we can complete the proof. \(\square\)

References

[1] Bahlali, K., Hamadne, S., Mezerdi, B., 2005. Backward stochastic differential equations with two reflecting barriers and continuous with quadratic growth coefficient. Stochastic Processes and their Applications, 115(7), 1107-1129.

[2] Briand, P., Coquet, F., Hu, Y., Mémin, J., Peng, S., 2000. A converse comparison theorem for BSDEs and related properties of \(g\)-expectation. Electron. Comm. Probab. 5, 101-117.

[3] Chen, Z., 1998. A property of backward stochastic differential equations. C. R. Acad. Sci. Paris, Ser. I, 326 (4), 483-488.

[4] El Karoui, N., Kapoudjian, C., Pardoux, E., Peng, S., Quenez, M.C., 1997. Reflected backward SDE’s, and related obstacle problems for PDE’s. Ann. Probab. 25(2), 702-737.

[5] Fan, S. J., Hu, J. H., 2008. A limit theorem for solutions to BSDEs in the space of processes. Statistics and Probability Letters, 78(8), 1024-1033.

[6] Fan, S. J., Jiang, L., 2010. A representation theorem for generators of BSDEs with continuous linear-growth generators in the space of processes. Journal of Computational and Applied mathematics, 235(3), 686-695.

[7] Fan, S., Jiang, L., Xu, Y., 2011. Representation theorem for generators of BSDEs with monotonic and polynomial-growth generators in the space of processes. Electronic Journal of Probability, 16(27), 830-834.

[8] Hamadène, S., Lepeltier, J.P., Matoussi, A., 1997. Double barrier backward SDEs with continuous coefficient, In: El Karoui N., Mazliak L.(eds.), Backward Stochastic Differential Equations, Pitman Research Notes in Mathematics Series, No. 364. Longman, Harlow, pp. 161-175.

[9] Hewitt, E., Stromberg, K.R., 1978. Real and Abstract Analysis. Springer-Verlag, New York.

[10] Jia, G., 2008. Backward stochastic differential equations, \(g\)-expectations and related semilinear PDEs. PH.D Thesis, ShanDong University, China, 2008.

[11] Jiang, L., 2005a. Representation theorems for generators of backward stochastic differential equations. Comptes Rendus Mathematique, 340(2), 161-166.

[12] Jiang, L., 2005b. Representation theorems for generators of backward stochastic differential equations and their applications. Stochastic Process. Appl. 115 (12), 1883-1903.

[13] Jiang, L., 2005c. Nonlinear expectation \(g\)-expectation theory and its applications in finance. Ph.D Thesis. ShanDong University, China.

[14] Jiang, L., 2008. Convexity, translation invariance and subadditivity for \(g\)-expectations and related risk measures. Annals of Applied Probability 18(1), 245-258.

[15] Li, J, Gu, Y., 2007. Converse comparison problems for reflected backward stochastic differential equations. Chinese Journal of Contemporary Mathematics, 28(2), 201-210.
[16] Li, J, Tang, S., 2007. A local strict comparison theorem and converse comparison theorems for reflected BSDEs. Stochas. Proce. Appli. 117, 1234-1250.

[17] Liu, Y., Jiang, L., Xu, Y., 2008. A local limit theorem for solutions of BSDEs with Mao’ non-Lipschitz generator. Acta Math. Appli. Sinica, English Series 24, 329-336.

[18] Matoussi, A., 1997. Reflected solutions of backward stochastic differential equations with continuous coefficient. Statistics and probability letters, 34(4), 347-354.

[19] Peng, S., 2010. Backward stochastic differential equation, nonlinear expectation and their applications. Proceedings of The International Congress of Mathematicians. Hyderabad, India, 393-432.

[20] Peng, S., 2006. Modelling derivatives pricing mechanisms with their generating functions. arXiv preprint arXiv:0605599.

[21] Song, L., Hu, F., Chen, Z., 2012. Representation theorems for generators of BSDEs in $L^p$ spaces. Acta Mathematicae Applicatae Sinica, English Series, 28(2), 255-264.