Random Dirac Fermions:  
The \text{su}(N) \text{ Gauge Potential and } \mathbb{Z}_N \text{ Twists}  

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We obtain the disorder averaged (critical) four-point correlations for \(N\) species of (two-dimensional Euclidean) Dirac fermions subject to a (Gaussian) random \text{su}(N) gauge field. The replica approach and the strong disorder approach yield identical results, as one might expect in the light of recent developments in the \(N = 2\) case [1]. We establish a connection with ‘dressed’ \(\mathbb{Z}_N\) twist fields in the \(c = -2\) logarithmic conformal field theory (LCFT), thereby extending the recent connection to ‘dressed’ \(\mathbb{Z}_2\) twists [1,4]. We draw attention to the fact that ‘dressed’ \(\mathbb{Z}_2\) twist operators have previously appeared in the description of half flux-quantum quasiholes in the bulk excitations of the Haldane–Rezayi Quantum Hall state [3].

I. INTRODUCTION

A remarkable connection between the problem of Dirac fermions in a random \text{su}(2) gauge field and the \(c = -2\) LCFT has recently emerged [1,4]. The disorder averaged correlation functions of the so-called Q-matrix in the \text{su}(2) random gauge problem — from which one obtains correlation functions of the local density of states (LDOS) [1,5,6] — are those of \(\mathbb{Z}_2\) twist fields in the \(c = -2\) LCFT ‘dressed’ (or augmented) by the exponential of a free bosonic field. In this paper we demonstrate that the connection holds beyond the \(N = 2\) case in that disorder averaged correlation functions in the \text{su}(N) problem are obtained by ‘dressing’ those of \(\mathbb{Z}_N\) twists.

The problem of random Dirac fermions in an Abelian vector potential was originally introduced in the context of the plateau transitions in the Integer Quantum Hall Effect [1]. The non-Abelian version of this problem appeared in a treatment of disordered two-dimensional d-wave superconductivity [1,2,3] and has been the subject of deeper investigation [4,5,6]. The \(c = -2\) LCFT has also been studied extensively in connection with dense polymers [1,2,3], the Haldane–Rezayi Quantum Hall state [8] and, somewhat more generally, as an instructive example of a LCFT [16,19]. That the random gauge field problem and the \(c = -2\) LCFT bear a close relationship is intriguing, and warrants further investigation.

The structure of this paper is as follows: in section II we use the replica approach to study the four-point function of the Q-matrix for the \text{su}(N) problem. The reliability of the

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replica approach is confirmed by independent calculation in the limit of infinite disorder strength (Appendix A) as one might expect given the recent convergence of results for three distinct treatments of the $N = 2$ case [1]. The Q-matrix is governed by a non-linear sigma model (familiar in the conventional theory of localization) with the addition of a WZNW term (which ensures criticality.) The Q-matrix correlation functions are obtained from the solution of the $\text{su}(r)_N$ Knizhnik–Zamolodchikov equations (in the replica $r \to 0$ limit) ‘dressed’ by the exponential of a free bosonic field, as was first elucidated in [3]. We demonstrate that this correlation function assumes a simple form in terms of a ($N$-species) generalization of the (complete) elliptic integrals. The ordinary (complete) elliptic integrals have played an instrumental rôle in establishing the connection between the random gauge field problem and ‘dressed’ $Z_2$ twist operators of the $c = -2$ LCFT [1,2,21]. In an analogous manner, the generalized elliptic integrals enable one to establish a connection to ‘dressed’ $Z_N$ twist operators of the form appearing in [4,22]. Finally, we present concluding remarks and a brief appendix on the most significant identities which appear in the strong disorder treatment of the $\text{su}(N)$ problem — the fermionic ‘dressing’ of the $\text{su}(N)_{-2N}$ WZNW model [8,11,13].

II. REPLICAS AND $Z_N$ TWISTS

As is discussed in section III of reference [7], the four-point correlation function of the Q-matrix (with all replica indices set to unity) reads

$$ \langle Q_{11}(1)Q_{11}^\dagger(2)Q_{11}(3)Q_{11}^\dagger(4) \rangle \sim |z_{13}z_{24}|^{-2/N^2}|z(1-z)|^{-2/N^2}[F_{11} + F_{12} + F_{21} + F_{22}] $$

(2.1)

where $z = z_{12}z_{34}/z_{13}z_{24}$ and

$$ F_{ij} = F_i^{(1)}(z)F_j^{(2)}(\bar{z}) + F_i^{(2)}(z)F_j^{(1)}(\bar{z}) $$

(2.2)

are single-valued combinations of the solutions to the $\text{su}(r)_N$ Knizhnik–Zamolodchikov equations in the $r \to 0$ limit [7,23]. In the replica limit these equations are of the form

$$ Nz \frac{dF_1}{dz} = -F_2, \quad N(1-z) \frac{dF_2}{dz} = F_1 $$

(2.3)

and admit solutions in terms of the ordinary hypergeometric functions.

1Namely, supersymmetry, replicas, and in the limit of infinite disorder strength where a special treatment is possible.

2The prescribed normalization of these solutions ensures the single-valuedness (monodromy invariance) of the combination (2.2).
\[ F_1^{(1)} = 2F_1[-\frac{1}{N}, \frac{1}{N}; 1; z] \]  
\[ F_1^{(2)} = (1 - z) \ 2F_1[1 - \frac{1}{N}, 1 + \frac{1}{N}; 2; 1 - z] \]  
\[ F_2^{(1)} = \frac{z}{N} \ 2F_1[1 - \frac{1}{N}, 1 + \frac{1}{N}; 2; z] \]  
\[ F_2^{(2)} = N \ 2F_1[-\frac{1}{N}, \frac{1}{N}; 1; 1 - z] \]  
\[ (2.4a) \]
\[ (2.4b) \]
\[ (2.4c) \]
\[ (2.4d) \]

It is a straightforward exercise in the application of the Gauss recursion relations for the ordinary hypergeometric function to demonstrate that
\[ F_{11} + F_{12} + F_{21} + F_{22} = \frac{4N}{\pi^2} [K_N(z)K_N(1 - \bar{z}) + K_N(\bar{z})K_N(1 - z)] \]  
\[ (2.5) \]
where we have defined the generalized (complete) elliptic integral of the first kind
\[ K_N \equiv \frac{\pi}{2} 2F_1[1 - \frac{1}{N}, \frac{1}{N}; 1; z]. \]  
\[ (2.6) \]

That is to say, upto overall normalization,
\[ \langle Q_{11}(1) Q_{11}^\dagger(2) Q_{11}(3) Q_{11}^\dagger(4) \rangle \sim |z_{13}z_{24}|^{-2/N^2} |z(1 - z)|^{-2/N^2} [K_N(z)K_N(1 - \bar{z}) + K_N(1 - z)K_N(\bar{z})]. \]  
\[ (2.7) \]

We invite the reader to compare this four-point function with that of $Z_N$ twist operators of the $c = -2$ LCFT as discussed in the context of dense polymers — see equations (83) and (84) of reference [14] and note the small typing error in the prefactor:
\[ \langle \sigma_+(1) \sigma_-(2) \sigma_+(3) \sigma_-(4) \rangle \sim |z_{13}z_{24}|^{2k/N(1-k/N)} |z(1 - z)|^{2k/N(1-k/N)} [F_{k/N}(z)F_{k/N}(1 - \bar{z}) + F_{k/N}(1 - z)F_{k/N}(\bar{z})]. \]  
\[ (2.8) \]

Here $\sigma_+$ stands for the $Z_N$ twist operator $\sigma_{k/N}$, $\sigma_-$ stands for the $Z_N$ anti-twist operator, both of conformal dimension
\[ h = -\frac{1}{2} \frac{k}{N} \left(1 - \frac{k}{N}\right) \]  
\[ (2.9) \]
and where
\[ F_{k/N} = 2F_1[1 - \frac{k}{N}, \frac{k}{N}; 1; z]. \]  
\[ (2.10) \]

For more details we refer the reader to §3.2 of [14]. In particular setting $k = 1$ in equations (2.8) and (2.10) one obtains
\[ \langle \sigma_+(1) \sigma_-(2) \sigma_+(3) \sigma_-(4) \rangle \sim |z_{13}z_{24}|^{2/N(1-1/N)} |z(1 - z)|^{2/N(1-1/N)} [K_N(z)K_N(1 - \bar{z}) + K_N(\bar{z})K_N(1 - z)] \]  
\[ (2.11) \]

\[ ^3 \text{The factor of } \pi/2 \text{ ensures that } K_2 \text{ agrees with the usual definition of } K. \text{ We note that the generalized (complete) elliptic integral of the second kind assumes the form } E_N = \frac{\pi}{2} 2F_1[-\frac{1}{N}, \frac{1}{N}; 1; z]. \]
It is readily seen that equations (2.7) and (2.11) coincide up to a simple prefactor to which
we now turn our attention. Adopting the conventions appearing in chapter 9 of [24] we
consider the four-point correlation functions of the exponentials of free bosonic fields
\[
\nu_\alpha = e^{i\sqrt{2}\alpha \varphi}
\]  
(2.12)
governed by the action
\[
S = \frac{1}{8\pi} \int d^2x \partial_\mu \varphi \partial^\mu \varphi.
\]  
(2.13)
and of conformal dimension \( h = \alpha^2 \). The four-point function of such fields is
\[
\langle \nu_\alpha(1)\nu_\alpha(2)\nu_\alpha(3)\nu_\alpha(4) \rangle = \left| (z_{13}z_{24}) \right|^{-4\alpha^2} \left| z(1-z) \right|^{-4\alpha^2}
\]  
(2.14)
The necessary compensating prefactor between equations (2.7) and (2.11) is obtained by
setting \( \alpha^2 = 1/2N \). That is to say, one obtains the following identification between the
Q-matrix and the ‘dressed’ \( Z_N \) twist operator of \( c = -2 \):
\[
Q \sim e^{i\varphi/\sqrt{N}} \sigma_+, \quad Q^\dagger \sim e^{-i\varphi/\sqrt{N}} \sigma_-
\]  
(2.15)
Indeed, the conformal dimensions of the bosonic exponent and the \( Z_N \) twist operator (with
\( k=1 \)) add up to
\[
\frac{1}{2N} - \frac{1}{2N} \left( 1 - \frac{1}{N} \right) = \frac{1}{2N^2}
\]  
(2.16)
which is the known conformal dimension of the Q-matrix/LDOS [3].

III. CONCLUSIONS

In this paper we have discussed the explicit connection between Dirac fermions in a
random su(\( N \)) gauge field and \( Z_N \) twists in the \( c = -2 \) LCFT. This generalizes the ‘dressing’
of the \( h = -1/8 \) twist field of relevance in the \( N = 2 \) case [1, 2]. We note that we have not
discussed the \( N > 2 \) analogue of ‘dressing’ the \( h = 3/8 \) twist field, but one anticipates a
connection to the excited twist fields \( \tau_{k/N} \) disussed in [4].

Given that dressed \( Z_2 \) twist operators are deemed responsible for the creation of half
flux-quantum quasiholes in the bulk excitations of the Haldane–Rezayi Quantum Hall state
— see section V of reference [3] — one might expect a \( Z_N \) generalization to shed some light
on the su(\( N \)) random gauge field problem.

These issues are under current investigation.

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APPENDIX A: STRONG DISORDER

In the limit of strong disorder one may obtain the correlation functions of the Q-matrix by a fermionic ‘dressing’ of the solutions to the su(N)−2N Knizhnik–Zamolodchikov equations \[8,11,13\]. These solutions may be written in the form $F^{(a)}_i = z^{1-1/N^2} (1-z)^{1-1/N^2} \mathcal{F}^{(a)}_i$ where

\begin{align*}
\mathcal{F}^{(1)}_1 &= (1-z) \, _2F_1[2 - \frac{1}{N}, 2 + \frac{1}{N}; 2; z] \\
\mathcal{F}^{(2)}_1 &= (1-z) \, _2F_1[2 - \frac{1}{N}, 2 + \frac{1}{N}; 3; 1-z] \\
\mathcal{F}^{(1)}_2 &= -\frac{1}{2N} z \, _2F_1[2 - \frac{1}{N}, 2 + \frac{1}{N}; 3; z] \\
\mathcal{F}^{(2)}_2 &= -2N z \, _2F_1[2 - \frac{1}{N}, 2 + \frac{1}{N}; 2; 1-z].
\end{align*}

Dressing with fermions leads one to consider single-valued combinations of the functions

\[ g^{(a)} = N \mathcal{F}^{(a)}_1 + \mathcal{F}^{(a)}_2 + \frac{z}{1-z} [N \mathcal{F}^{(a)}_2 + \mathcal{F}^{(a)}_1]. \quad (A2) \]

A straightforward but tedious application of the Gauss recursion relations for hypergeometric functions allows one to recover the generalized (complete) elliptic integrals:

\[ g^{(1)} = \frac{2N}{\pi(1-z)} K_N(z), \quad g^{(2)} = -\frac{4N^2}{\pi(1-z)} K_N(1-z) \quad (A3a) \]

In this manner one is able to recover the results of the replica approach.

\[ \text{[1]} \ M. J. Bhaseen, J.-S. Caux, I. I. Kogan, and A. M. Tsvelik, \texttt{cond-mat/0012240}. \]
\[ \text{[2]} \ A. W. W. Ludwig, \texttt{cond-mat/0012189}. \]
\[ \text{[3]} \ V. Gurarie, M. Flohr, and C. Nayak, Nucl. Phys. \textbf{B498}, 513 (1997). \]
\[ \text{[4]} \ A. W. W. Ludwig, M. P. A. Fisher, R. Shankar, and G. Grinstein, Phys. Rev. B \textbf{50}, 7526 (1994). \]
\[ \text{[5]} \ A. A. Nersesyan, A. M. Tsvelik, and F. Wenger, Phys. Rev. Lett \textbf{72}, 2628 (1994). \]
\[ \text{[6]} \ A. A. Nersesyan, A. M. Tsvelik, and F. Wenger, Nucl. Phys. \textbf{B438}, 561 (1995). \]
\[ \text{[7]} \ J.-S. Caux, I. I. Kogan, and A. M. Tsvelik, Nucl. Phys. \textbf{B466}, 444 (1996). \]
\[ \text{[8]} \ C. Chamon, C. Mudry, and X.-G. Wen, Nucl. Phys. \textbf{B466}, 383 (1996). \]
\[ \text{[9]} \ I. I. Kogan, C. Mudry, and A. M. Tsvelik, Phys. Rev. Lett. \textbf{77}, 707 (1996). \]
\[ \text{[10]} \ J.-S. Caux, Phys. Rev. Lett. \textbf{81}, 4196 (1998). \]
\[ \text{[11]} \ J.-S. Caux, N. Taniguchi, and A. M. Tsvelik, Phys. Rev. Lett (1998). \]
\[ \text{[12]} \ J.-S. Caux, N. Taniguchi, and A. M. Tsvelik, Nucl. Phys. \textbf{B525}, 621 (1998). \]
\[ \text{[13]} \ D. Bernard, \texttt{hep-th/9509137}. \]
\[ \text{[14]} \ H. Saleur, Nucl. Phys. \textbf{B382}, 486 (1992). \]
\[ \text{[15]} \ E. V. Ivashkevich, J. Phys. \textbf{A32}, 1691 (1999). \]
\[ \text{[16]} \ V. Gurarie, Nucl. Phys. \textbf{B410}, 535 (1993). \]
[17] H. G. Kausch, hep-th/9510143.
[18] H. G. Kausch, Nucl. Phys. B570, 699 (2000).
[19] M. R. Gaberdiel and H. G. Kausch, Nucl. Phys. B538, 631 (1999).
[20] I. I. Kogan and J. F. Wheater, Phys. Lett. B486, 353 (2000).
[21] M. J. Bhaseen, cond-mat/0011223.
[22] L. Dixon, D. Friedan, E. Martinec, and S. Shenker, Nucl. Phys. B282, 13 (1987).
[23] V. G. Knizhnik and A. B. Zamolodchikov, Nucl. Phys. B247, 83 (1984).
[24] P. D. Francesco, P. Mathieu, and D. Sénéchal, *Conformal Field Theory* (Springer, 1997).