EXTENSION AND TRACE THEOREMS FOR NONCOMPACT DOUBLING SPACES

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ABSTRACT. We generalize the extension and trace results of Björn-Björn-Shanmugalingam [3] to the setting of complete noncompact doubling metric measure spaces and their uniformized hyperbolic fillings. This is done through a uniformization procedure introduced by the author that uniformizes a Gromov hyperbolic space using a Busemann function instead of the distance functions considered in the work of Bonk-Heinonen-Koskela [7]. We deduce several new corollaries for the Besov spaces that arise as trace spaces in this fashion, including density of Lipschitz functions with compact support, embeddings into Hölder spaces for appropriate exponents, and the existence of Lebesgue points quasieverywhere with respect to the Besov capacity under an additional reverse doubling hypothesis on the measure. Under this same reverse doubling hypothesis we also obtain a hyperbolic Poincaré inequality for Besov functions that controls the mean oscillation of a Besov function over a ball through an integral over an extended ball of an upper gradient for an extension of this function to a uniformized hyperbolic filling of the space.

1. INTRODUCTION

In recent breakthrough work of Björn-Björn-Shanmugalingam [3] an extensive array of properties of Besov spaces on a compact doubling metric measure space \( Z \) were established by exhibiting these spaces as the trace space of a Newton-Sobolev space on an associated incomplete metric graph \( X_\rho \) having \( Z \) as its boundary. These properties include density of Lipschitz functions, the existence of quasicontinuous representatives of Besov functions, embeddings into Hölder spaces under appropriate hypotheses on the regularity exponents, and the existence of Lebesgue points quasieverywhere under an additional reverse-doubling hypothesis on the measure [3, Section 13]. In this work we extend these results to the setting of complete doubling metric measure spaces by removing the compactness hypothesis on \( Z \).

This is done through the use of a new uniformization construction for Gromov hyperbolic spaces introduced by the author in previous work [14] to similarly exhibit \( Z \) as the boundary of an incomplete metric graph \( X_\rho \), even in the case that \( Z \) is not compact. For the compact case this uniformization is done in [3] using the procedure of Bonk-Heinonen-Koskela [7]; our construction generalizes this procedure.

To state our main theorems we need to introduce some important additional concepts, which are similar in nature to those considered in [3]. For precise definitions of the notions that follow we refer to Sections 2, 3, and 6. A metric measure space \( (Z, d, \nu) \) is a metric space \( (Z, d) \) equipped with a Borel regular measure \( \nu \) such that for every ball \( B \subset Z \) we have \( 0 < \nu(B) < \infty \). This space is doubling if there is a constant \( C_\nu \geq 1 \) such that for each ball \( B \) we have \( \nu(2B) \leq C_\nu \nu(B) \), where \( 2B \) denotes the ball with the same center as \( B \) and twice the radius. Given a complete doubling metric measure space \( (Z, d, \nu) \), in Section 6 we construct a metric graph \( X \) that is a proper geodesic Gromov hyperbolic space such that \( Z \) can be canonically identified with the complement of a distinguished point \( \omega \) in the Gromov boundary \( \partial X \) of \( X \). Such a graph is known as a hyperbolic filling of \( Z \); these
constructions were first considered by Bonk-Kleiner \[9\] and Bourdon-Pajot \[12\] for compact \( Z \) (see also Bonk-Schramm \[8\] for an alternative construction known as the hyperbolic cone). Our construction for noncompact \( Z \) is inspired by a construction due to Buyalo-Schroeder \[15\] Chapter 6).

We then \textit{uniformize} \( X \) to produce a uniform metric space \( X_\rho \) such that the metric boundary \( \partial X_\rho \) of \( X_\rho \) has a canonical biLipschitz identification with \( Z \). After a biLipschitz change of coordinates on \( Z \) we can then assume that \( Z \) is isometrically identified with \( \partial X_\rho \). In Section 4 we lift \( \nu \) to a \textit{uniformly locally doubling} measure \( \mu \) on the hyperbolic graph \( X \). We uniformize this measure for each parameter \( \beta > 0 \) to obtain a new measure \( \mu_\beta \) on \( X_\rho \) such that the resulting metric measure spaces \((X_\rho,d_\rho,\mu_\beta)\) and \((\bar{X}_\rho,d_\rho,\mu_\beta)\) are each doubling and satisfy a 1-Poincaré inequality, where \( \mu_\beta \) is extended to \( \bar{X}_\rho \) by setting \( \mu_\beta(\partial X_\rho) = 0 \). Newtonian functions on these metric measure spaces will serve as counterparts to Besov functions on \((Z,d,\nu)\). In the process of establishing these properties of the measure \( \mu_\beta \), we generalize several results of Björn-Björn-Shanmugalingam \[2\] concerning the transformation of doubling measures and Poincaré inequalities under the Bonk-Heinonen-Koskela uniformization to the setting of our uniformization.

For a given \( p \geq 1, \theta > 0 \), and \( u \in L^p_\text{loc}(Z) \), the Besov norm on \( Z \) is defined by

\begin{equation}
\|u\|_{B^\theta_p(Z)} = \int_Z \int_Z \frac{|u(x) - u(y)|^p}{d(x,y)^{p\theta}} \frac{d\nu(x)d\nu(y)}{\nu(B(x,\theta d(x,y)))}.
\end{equation}

We write \( B^\theta_p(Z) \subset L^p_\text{loc}(Z) \) for the subspace of all functions \( u \) such that \( \|u\|_{B^\theta_p(Z)} < \infty \). We note that, properly speaking, (1.1) is actually a seminorm on \( L^p_\text{loc}(Z) \) since any constant function has Besov norm 0. We define \( B^\theta_p(Z) = L^p(Z) \cap B^\theta_p(Z) \) to be the subspace of \( B^\theta_p(Z) \) consisting of those functions that are \( p \)-integrable over \( Z \). We equip this subspace with the norm

\begin{equation}
\|u\|_{B^\theta_p(Z)} = \|u\|_{L^p(Z)} + \|u\|_{B^\theta_p(Z)},
\end{equation}

With this norm \( B^\theta_p(Z) \) is a Banach space by the arguments in \[3\] Remark 9.8. We always have \( B^\theta_p(Z) = B^\theta_p(Z) \) whenever \( Z \) is bounded. For bounded \( Z \) these Besov spaces were introduced by Bourdon-Pajot in a more restrictive setting in \[12\]. In \[17\] it was shown by Gogatishvili-Koskela-Shanmugalingam that this definition coincides with a more classical formulation for Besov spaces on metric spaces. We will use the notation of Bourdon-Pajot in this paper.

On \( X_\rho \) and its completion \( \bar{X}_\rho = X_\rho \cup Z \), we will consider the \textit{Newtonian} spaces defined through upper gradients. These spaces are based on a notion of gradient in the metric space setting introduced by Heinonen-Koskela \[20\], and their formal study was initiated by Shanmugalingam \[28\]. We give an abbreviated description of these spaces here; a more detailed description can be found in Section 8. Beginning with a metric measure space \((Y,d,\mu)\), a Borel function \( g : Y \rightarrow [0,\infty] \) is an \textit{upper gradient} of a function \( u : Y \rightarrow [-\infty,\infty] \) if for each nonconstant compact rectifiable curve \( \gamma \) in \( Y \) joining two points \( x, y \in Y \) we have that

\begin{equation}
|u(x) - u(y)| \leq \int_\gamma g \, ds.
\end{equation}

By convention we require that \( \int_\gamma g \, ds = \infty \) if \( u(x) = \pm \infty \) or \( u(y) = \pm \infty \). The \textit{Newtonian norm} of \( u \) for a given \( p \geq 1 \) is then defined to be

\begin{equation}
\|u\|_{N^{1,p}(Y)} = \|u\|_{L^p(Y)} + \inf_g \|g\|_{L^p(Y)},
\end{equation}
where the infimum is taken over all upper gradients \( g \) of \( u \). We write \( \tilde{N}^{1,p}(Y) \) for the collection of functions such that \( \|u\|_{\tilde{N}^{1,p}(Y)} < \infty \). Similarly the Dirichlet norm of \( u \) for a given \( p \geq 1 \) is defined to be the Dirichlet \( p \)-energy,
\[
\|u\|_{D^{1,p}(Y)} = \inf_g \|g\|_{L^p(Y)},
\]
with the infimum taken over all upper gradients \( g \) of \( u \). We then write \( \tilde{D}^{1,p}(Y) \) for the collection of functions such that \( \|u\|_{\tilde{D}^{1,p}(Y)} < \infty \), which can equivalently be thought of as the collection of functions \( u \) with a \( p \)-integrable upper gradient. We again note that \((1.4)\) and \((1.5)\) actually define seminorms on \( \tilde{N}^{1,p}(Y) \) and \( \tilde{D}^{1,p}(Y) \) and one must pass to a quotient to obtain a true norm. This issue will not be relevant to any of the statements we make in this first section.

We can now state our main theorem. Given \( p \geq 1 \) and \( 0 < \theta < 1 \), the uniformized hyperbolic filling \( \tilde{X}_p \) is considered to be equipped with the uniformized measure \( \mu_\beta \) for the specific parameter value \( \beta = p(1 - \theta) \). We refer to Section 11 for a precise description of the measure \( \mu_\beta \) for each \( \beta > 0 \). We warn the reader that our definition of \( \beta \) is slightly different than the one used in [3]; our \( \beta \) corresponds to \( \beta/\epsilon \) in their work.

The linear operators below are maps between seminormed spaces, and should be understood as being bounded with respect to these seminorms; to simplify terminology we will generally refer to the seminorms \((1.1), (1.4), \) and \((1.5)\) as norms on \( B^\theta_p(Z) \), \( \tilde{N}^{1,p}(Y) \), and \( \tilde{D}^{1,p}(Y) \) respectively, even though one must pass to a quotient of these spaces to actually obtain a norm. The expression \((1.2)\) does define a genuine norm on \( B^\theta_p(Z) \), however. These matters are treated in greater detail in Section 8.

**Theorem 1.1.** There are bounded linear operators \( T : \tilde{D}^{1,p}(X_p) \to B^\theta_p(Z) \) and \( P : B^\theta_p(Z) \to \tilde{D}^{1,p}(\tilde{X}_p) \) such that for each \( f \in B^\theta_p(Z) \) we have \( T(Pf) = f \) \( \nu \)-a.e., i.e., \( T \circ P \) is the identity on \( B^\theta_p(Z) \).

Furthermore \( T \) restricts to a bounded linear operator \( T : \tilde{N}^{1,p}(X_p) \to B^\theta_p(Z) \) and there is a truncation \( P_0 \) of \( P \) that defines a bounded linear operator \( P_0 : B^\theta_p(Z) \to \tilde{N}^{1,p}(\tilde{X}_p) \) with \( T \circ P_0 \) being the identity on \( B^\theta_p(Z) \).

The operators \( T \) and \( P \) are known as extension and trace operators respectively. We refer to Sections 10 and 11 for a description of these operators, as well as a description of the parameters on which the norms of these linear operators depend. The truncation \( P_0 \) of \( P \) is defined in Section 11; we remark that \( P \) itself does not define a bounded linear operator from \( B^\theta_p(Z) \) to \( \tilde{N}^{1,p}(X_p) \), as an adaptation of the arguments of Proposition 7.7 shows. We emphasize that the domain of \( T \) consists of functions defined on \( X_p \), not on \( \tilde{X}_p \); even though \( \partial X_p \) has measure zero with respect to \( \mu_\beta \), it is not at all obvious that functions in \( \tilde{D}^{1,p}(X_p) \) and \( \tilde{N}^{1,p}(X_p) \) can naturally be extended to functions in \( \tilde{D}^{1,p}(\tilde{X}_p) \) and \( \tilde{N}^{1,p}(\tilde{X}_p) \). For this we rely on work of J. Björn and Shanmugalingam [6]. Theorem 1.1 can be interpreted as saying that the trace space of \( \tilde{D}^{1,p}(X_p) \) on \( Z \) is \( B^\theta_p(Z) \), and the trace space of \( \tilde{N}^{1,p}(X_p) \) on \( Z \) is \( B^\theta_p(Z) \).

Theorem 1.1 yields numerous corollaries for the Besov space \( B^\theta_p(Z) \), all of which follow from this theorem in essentially the same way that the corollaries of [3] Corollary 1.2 follow from [3] Theorem 1.1. We split these consequences into three corollaries. We refer to Section 11 for more precise formulations of these corollaries; here we have endeavored to avoid overburdening the corollaries with additional definitions. For an extensive discussion of predecessors to these corollaries we refer to the discussion after [3] Corollary 1.2.
remark that all of these corollaries were established in the case of compact $Z$ in [3], however the generalization from compact $Z$ to noncompact $Z$ is highly nontrivial due to the nonlocal nature of the Besov norm.

Our first corollary shows that Lipschitz functions with compact support are dense in $\tilde{B}^\theta_p(Z)$ and that every function in $\tilde{B}^\theta_p(Z)$ is quasicontinuous with respect to the Besov capacity. Quasicontinuity is a strong refinement of Lusin’s theorem, as the Besov capacity is always absolutely continuous with respect to the measure $\nu$ on $Z$ (see Propositions [9] and 10.7). For each of the corollaries below we always assume that $p \geq 1$ and $0 < \theta < 1$.

**Corollary 1.2.** Lipschitz functions with compact support are dense in $\tilde{B}^\theta_p(Z)$. Each function in $\tilde{B}^\theta_p(Z)$ has a representative which is quasicontinuous with respect to the Besov capacity.

Our second corollary concerns embeddings of Besov spaces into Hölder spaces. For this we will use the notion of relative lower volume decay of order $Q$ for a doubling measure $\nu$, defined by the inequality (7.17) for a given exponent $Q > 0$. The exponent $Q$ is also known as a doubling dimension for $\nu$; we remark that the relative volume decay estimate always holds with $Q = \log_2 C_\nu$, where $C_\nu$ is the constant in the doubling inequality for $\nu$, but it can also hold for smaller $Q$. The notation $Q_\beta$ below is motivated by the equality $\beta = p(1 - \theta)$ that we assumed in Theorem 1.1. See also Lemma 7.6 and Proposition 11.4.

**Corollary 1.3.** Suppose that $\nu$ has relative lower volume decay of order $Q > 0$. Set $Q_\beta = \max\{1, Q + p(1 - \theta)\}$ and assume that $p > Q_\beta$. Then every function in $\tilde{B}^\theta_p(Z)$ has a representative that is locally $(1 - Q_\beta/p)$-Hölder continuous.

Finally we consider the case in which $\nu$ additionally satisfies a reverse-doubling property (11.1), which holds in particular when $Z$ is uniformly perfect [20, Lemma 4.1].

**Corollary 1.4.** Suppose that $\nu$ has relative lower volume decay of order $Q > 0$ and that $\nu$ satisfies the reverse-doubling property (11.1). We assume further that $p\theta < Q$ and set $Q_* = Qp/(Q - p\theta)$. Then functions in $\tilde{B}^\theta_p(Z)$ belong to $L^{Q_*}(Z)$ and have representatives which have $L^{Q_*}(Z)$-Lebesgue points outside a set of zero Besov capacity.

It is an interesting question what further hypotheses on $Z$ and $\nu$ are needed to obtain $f \in L^{Q_*}(Z)$ when $f \in \tilde{B}^\theta_p(Z)$. Such an estimate could be regarded as a type of Sobolev embedding theorem for $\tilde{B}^\theta_p(Z)$. We remark that the proof of Corollary 1.4 leads to a notable inequality that we call a hyperbolic Poincaré inequality for functions in $\tilde{B}^\theta_p(Z)$, which we state in Proposition 11.4 at the end of the paper. Roughly speaking, this inequality shows that upper gradients of extensions of a function $f \in \tilde{B}^\theta_p(Z)$ to a function $u \in D^{1,p}(\bar{X}_p)$ can be used to control the deviations of $f$ from its mean on a given ball in $Z$ in a formally similar manner to the classical Poincaré inequalities that we discuss in Section 8.

As we mentioned previously, this work is a direct descendant of a number of recent works connecting function spaces on doubling metric spaces with function spaces on hyperbolic fillings associated to the space. In addition to all of the works we mentioned previously, other notable works include Bonk-Saksman [10] (from whom we’ve taken a significant amount of direct inspiration), Bonk-Saksman-Soto [11] (who consider a wider class of function spaces on $Z$ including the Triebel-Lizorkin spaces), and Saksman-Soto [20] (who consider traces onto Ahlfors-regular subsets of $Z$). We also highlight the work of Malý [24], who obtains many similar results via different methods in the setting of John domains with compact closure (as well as several more general settings) in metric measure spaces.

The reader will find that the structure of our paper is broadly similar to that of [2] and [3]. This is because the bulk of the theory in [3] relies only on the fact that the uniformized
hyperbolic filling \( X_p \) equipped with the lifted measure \( \mu_\beta \) for a given \( \beta > 0 \) is a doubling metric measure space satisfying a 1-Poincaré inequality. We show that these claims carry over to the setting of noncompact \( Z \). Nevertheless the boundedness of \( Z \) (and consequently of \( X_p \) in their setting) enters into the proofs at several critical junctures, requiring us to modify their arguments at many steps. In particular there is no distinction between the spaces \( \tilde{D}^{1,p}(X_p) \) and \( \tilde{N}^{1,p}(X_p) \) when \( X_p \) is bounded, as is noted in Section 8. We are also able to give independent (and in some cases, simpler) proofs of several of their claims due to the more restrictive nature of our hyperbolic filling construction; our hyperbolic fillings in general contain more edges than the hyperbolic fillings considered in [4]. A drawback of our approach is that while we are able to recover their results on Besov spaces in [4, Section 13] via specialization of our corollaries to the case of compact \( Z \), we are not able to recover their results on trace and extension theorems for trees (see [4, Section 7]) due to the fact that our hyperbolic filling construction never yields a tree unless \( Z \) is a single point (see [4, Remark 7.2]; the same reasoning applies in our setting due to our parameter restriction (6.1)).

We now give an overview of the structure of the paper. In Sections 2 and 3 we review results from our previous work [15] as well as basic definitions concerning Gromov hyperbolic spaces and uniform metric spaces. Sections 4 and 5 are devoted to generalizing the results of [2] concerning global doubling and global Poincaré inequalities to the uniformization procedure we introduced in [14]. In Section 2 we construct a hyperbolic filling \( X \) of a complete doubling metric space \( Z \) and then construct a uniformization \( X_p \) of \( X \) whose boundary can be biLipschitz identified with \( Z \). This primarily relies on our previous results in [14]. In Section 7 we construct the lifts \( \mu_\beta \) of the measure \( \nu \) on \( Z \) for each \( \beta > 0 \) and establish important properties of the resulting metric measure space \((X_p, d_p, \mu_\beta)\). Section 8 consists of a detailed overview of the theory of Newtonian spaces for doubling metric spaces satisfying a Poincaré inequality with an emphasis on the Newtonian capacity. We then prove Theorem 1.1 in Sections 9 and 10. Finally in Section 11 we prove Corollaries 1.2, 1.3, and 1.4.

We offer extensive thanks to Nageswari Shanmugalingam, who provided us with several early drafts of the work [4] that inspired both our work here and our previous papers [14], [13].

2. Hyperbolic metric spaces

In this section we review some standard results regarding Gromov hyperbolic spaces, as well as some facts regarding Busemann functions established in our previous papers [14], [13]. Standard references for this material are [15], [16].

2.1. Definitions. Let \( X \) be a set and let \( f, g \) be real-valued functions defined on \( X \). For \( c \geq 0 \) we will write \( f \asymp_c g \) if

\[ |f(x) - g(x)| \leq c, \]

for all \( x \in X \). If the exact value of the constant \( c \) is not important or implied by context we will often just write \( f \asymp g \). The relation \( f \asymp g \) will sometimes be referred to as a rough equality between \( f \) and \( g \). Similarly for \( C \geq 1 \) and functions \( f, g : X \to (0, \infty) \), we will write \( f \asymp_C g \) if for all \( x \in X \),

\[ C^{-1}g(x) \leq f(x) \leq Cg(x). \]

We will write \( f \asymp g \) if the value of \( C \) is implied by context. We will write \( f \asymp_C g \) if \( f(x) \leq Cg(x) \) for all \( x \in X \) and \( f \asymp_C g \) if \( f(x) \geq C^{-1}g(x) \) for \( x \in X \). Thus \( f \asymp_C g \) if
and only if \( f \leq_K g \) and \( f \geq_K g \). As with the other notation, we will drop the constant \( C \)
and just write \( f \leq g \) or \( f \geq g \) if the value of \( C \) is implied by context. We will generally
stick to the convention of using \( c \geq 0 \) for additive constants and \( C \geq 1 \) for multiplicative
constants. To indicate on what parameters – such as \( \delta \) – the constants depend on we will
write \( c = c(\delta) \), etc. At the beginning of each section we will indicate on what parameters the
implied constants of the inequalities \( \leq \) and \( \geq \), the comparisons \( \asymp \), and the rough equalities
\( \approx \) are allowed to depend. We will often reiterate these conditions for emphasis.

For a metric space \((X, d)\) we will write \( B_X(x, r) = \{ y \in X : d(x, y) < r \} \)
for the open ball of radius \( r > 0 \) centered at a point \( x \in X \). We write \( \overline{B}_X(x, r) = \{ y \in X : d(x, y) \leq r \} \)
for the closed ball of radius \( r > 0 \) centered at \( x \). We note that the inclusion \( \overline{B}_X(x, r) \subset B_X(x, r) \)
of the closure of the open ball into the closed ball can be strict in general. By convention
all balls \( B \subset X \) are considered to have a fixed center and radius, even though it may be
the case that we have \( B_X(x, r) = B_X(x', r') \) as sets for some \( x \neq x' \), \( r \neq r' \). All balls
\( B \subset X \) are also considered to be open balls unless otherwise specified. We will write \( r(B) \)
for the radius of a ball \( B \). For a ball \( B = B_X(x, r) \) in \( X \) and a constant \( c > 0 \) we write
\( cB = B(x, cr) \) for the corresponding ball with radius scaled by \( c \). For a subset \( E \subset X \) we
write \( \text{diam}(E) = \sup\{ d(x, y) : x, y \in E \} \) for the diameter of \( E \).

Let \( f : (X, d) \to (X', d') \) be a map between metric spaces. We say that \( f \) is isometric
if \( d'(f(x), f(y)) = d(x, y) \) for \( x, y \in X \). We recall that a curve \( \gamma : I \to X \) is a geodesic
if it is an isometric mapping of the interval \( I \subset \mathbb{R} \) into \( X \). We say that \( X \) is geodesic
if any two points in \( X \) can be joined by a geodesic. A geodesic triangle \( \Delta \) in \( X \) consists of
three points \( x, y, z \in X \) together with geodesics joining these points to one another. Writing
\( \Delta = \gamma_1 \cup \gamma_2 \cup \gamma_3 \) as a union of its edges, we say that \( \Delta \) is \( \delta \)-thin
for a given \( \delta \geq 0 \) if for each point \( p \in \gamma_i, i = 1, 2, 3 \), there is a point \( q \in \gamma_j \)
with \( d(p, q) \leq \delta \) and \( i \neq j \). A geodesic metric space \( X \) is Gromov hyperbolic
if there is a \( \delta \geq 0 \) such that all geodesic triangles in \( X \) are \( \delta \)-thin; in this case we will also say that \( X \) is \( \delta \)-hyperbolic. When considering Gromov
hyperbolic spaces \( X \) we will usually use the generic distance notation \( |xy| := d(x, y) \)
for the distance between \( x \) and \( y \) in \( X \) and the generic notation \( xy \) for a geodesic connecting two
points \( x, y \in X \), even when this geodesic is not unique.

A metric space \((X, d)\) is proper if its closed balls are compact. The Gromov boundary \( \partial X \)
of a proper geodesic \( \delta \)-hyperbolic space \( X \) is defined to be the collection of all geodesic rays
\( \gamma : [0, \infty) \to X \) up to the equivalence relation of two rays being equivalent if they are at a
bounded distance from one another. We will often refer to the point \( \omega \in \partial X \) corresponding
to a geodesic ray \( \gamma \) as the endpoint of \( \gamma \). Using the Arzela-Ascoli theorem it is easy to see in
a proper geodesic \( \delta \)-hyperbolic space that for any points \( x, y \in X \cup \partial X \) there is a geodesic
\( \gamma \) joining \( x \) to \( y \). We will continue to write \( xy \) for any such choice of geodesic joining \( x \) to \( y \).
We will allow our geodesic triangles \( \Delta \) to have vertices on \( \partial X \), in which case we will still
write \( \Delta = xyz \) if \( \Delta \) has vertices \( x, y, z \). We remark that geodesic triangles with vertices in
\( X \cup \partial X \) are \( 10\delta \)-thin by [13, Lemma 2.2].

As in our previous work [14], we will use the notation \( \partial X \) for the Gromov boundary
of \( X \) even though it conflicts with the notation \( \partial \Omega = \Omega \setminus \Omega \) for the metric boundary of
a metric space \((\Omega, d)\) inside its completion \( \Omega \). Since we always assume that \( X \) is proper we will always
have \( \partial X = X \), so the metric boundary of \( X \) will always be trivial. Thus there will be no
ambiguity in using \( \partial X \) for the Gromov boundary as well.

For a geodesic ray \( \gamma : [0, \infty) \to X \), the Busemann function \( b_\gamma : X \to \mathbb{R} \)
associated to \( \gamma \) is defined by the limit

\[
(2.1) \quad b_\gamma(x) = \lim_{t \to \infty} |\gamma(t)x| - t.
\]
We then define
\begin{equation}
\mathcal{B}(X) = \{ b, + s : \gamma \text{ a geodesic ray in } X, s \in \mathbb{R} \},
\end{equation}
and refer to any function \( b \in \mathcal{B}(X) \) as a Busemann function on \( X \). See \cite{14} (1.4-1.5) for further details on these definitions. The Busemann functions \( b \in \mathcal{B}(X) \) are all 1-Lipschitz functions on \( X \). For a Busemann function \( b \) of the form \( b = b_\gamma + s \) for some \( s \in \mathbb{R} \), we define the endpoint \( \omega \in \partial X \) of \( \gamma \) to be the basepoint of \( b \) and say that \( b \) is based at \( \omega \).

We next state a useful fact regarding Busemann functions. Let \( X \) be a proper geodesic \( \delta \)-hyperbolic space and let \( b : X \to \mathbb{R} \) be a Busemann function based at some point \( \omega \in \partial X \).

By \cite{14} Lemma 2.5, if \( b' \) is any other Busemann function based at \( \omega \) then there is a constant \( s \in \mathbb{R} \) such that
\begin{equation}
b \backsimeq_{\gamma} b' + s,
\end{equation}
with \( s = 0 \) if the geodesic rays associated to \( b \) and \( b' \) have the same starting point. Thus all Busemann functions based at \( \omega \) differ from each other by an additive constant, up to an additive error of \( 72\delta \).

2.2. Gromov products. For \( x, y, z \in X \) the Gromov product of \( x \) and \( y \) based at \( z \) is defined by
\begin{equation}
(x|y)_z = \frac{1}{2}(|xz| + |yz| - |xy|).
\end{equation}

We can also take the basepoint of the Gromov product to be any Busemann function \( b \in \mathcal{B}(X) \). For \( b \in \mathcal{B}(X) \) the Gromov product to be any Busemann function \( b \in \mathcal{B}(X) \) based at \( b \) is defined by
\begin{equation}
(x|y)_b = \frac{1}{2}(b(x) + b(y) - |xy|).
\end{equation}

The following statements briefly summarize a more extensive discussion of Gromov products in \cite{14} Section 2. In particular we give details there for how the precise forms of these statements follow from the corresponding statements in the literature. For these statements it is useful to conceive of the Gromov boundary in an alternative way using Gromov products. Fix \( z \in X \). A sequence \( \{x_n\} \subset X \) converges to infinity if \( (x_n|z)_n \to \infty \) as \( m, n \to \infty \).

Two sequences \( \{x_n\} \) and \( \{y_n\} \) are equivalent if \( (x_n|y_n)_z \to \infty \). These notions do not depend on the choice of basepoint \( z \), as can easily be checked by the triangle inequality. For a proper geodesic \( \delta \)-hyperbolic space \( X \) the set of equivalence classes of sequences converging to infinity gives an equivalent definition of the Gromov boundary \( \partial X \), with the equivalence being given by sending a geodesic ray \( \gamma : [0, \infty) \to X \) to the sequence \( \{\gamma(n)\}_n \). For \( \xi \in \partial X \) and a sequence \( \{x_n\} \) that converges to infinity we will \( \{x_n\} \in \xi \) if \( \{x_n\} \in \xi \) belongs to the equivalence class of \( \xi \). For a geodesic ray \( \gamma : [0, \infty) \to X \) in \( X \) we similarly write \( \gamma \in \xi \) if \( \{\gamma(n)\}_n \in \xi \). We will also consider geodesic rays \( \gamma : (-\infty, 0] \to X \) with the opposite orientation, for which we write \( \gamma \in \xi \) if \( \{\gamma(-n)\}_n \in \xi \).

These notions may be extended to Busemann functions \( b \in \mathcal{B}(X) \) based at a given point \( \omega \in \partial X \). As above a sequence \( \{x_n\} \) converges to infinity with respect to \( \omega \) if \( (x_n|\omega)_b \to \infty \) as \( m, n \to \infty \), and two sequences \( \{x_n\} \) and \( \{y_n\} \) are equivalent with respect to \( \omega \) if \( (x_n|y_n)_b \to \infty \) as \( n \to \infty \). These definitions do not depend on the choice of Busemann function based at \( \omega \) by \cite{14} (2.4). The Gromov boundary relative to \( \omega \) is defined to be the set \( \partial_\omega X \) of all equivalence classes of sequences converging to infinity with respect to \( \omega \).

By \cite{15} Proposition 3.4.1 we have a canonical identification of \( \partial_\omega X \) with the complement \( \partial X \setminus \{\omega\} \) of \( \omega \) in the Gromov boundary \( \partial X \). We will thus use the notation \( \partial_\omega X = \partial X \setminus \{\omega\} \) throughout the rest of the paper.
Gromov products based at Busemann functions \( b \in \mathcal{B}(X) \) can be extended to points of \( \partial X \) by defining the Gromov product of equivalence classes \( \xi, \zeta \in \partial X \) based at \( b \) to be
\[
(\xi|\zeta)_b = \inf \liminf_{n \to \infty} (x_n|y_n)_b,
\]
with the infimum taken over all sequences \( \{x_n\} \in \xi \), \( \{y_n\} \in \zeta \); if \( b \in \mathcal{B}(X) \) has basepoint \( \omega \) then we leave this expression undefined when \( \xi = \zeta = \omega \). As a consequence of [15] Lemma 2.2.2, [15] Lemma 3.2.4, and the discussion in [14] Section 2.2, for any choices of sequences \( \{x_n\} \in \xi \) and \( \{y_n\} \in \zeta \) we have
\[
(\xi|\zeta)_b = \liminf_{n \to \infty} (x_n|y_n)_b \leq \limsup_{n \to \infty} (x_n|y_n)_b \leq (\xi|\zeta)_b + c(\delta),
\]
with the constant \( c(\delta) \) depending only on \( \delta \). One may take \( c(\delta) = 600\delta \). For \( x \in X \) and \( \xi \in \partial X \) the Gromov product based at \( b \) is defined analogously as
\[
(x|\xi)_b = \inf \liminf_{n \to \infty} (x_n|y_n)_b,
\]
and the analogous inequality (2.7) holds with the same constants. We remark that, with the extended definition (2.8), a sequence \( \{x_n\} \) belongs to the equivalence class of \( \xi \in \partial X \) if and only if \( (x_n|\xi)_b \to \infty \) for some (hence any) \( z \in X \). By [14] (2.10)], for all \( x, y \in X \cup \partial X \) with \( (x, y) \neq (\omega, \omega) \) we have
\[
(x|y)_b \leq \min\{b(x), b(y)\} + c(\delta),
\]
where we set \( b(\omega) = -\infty \) and \( b(\xi) = \infty \) for \( \xi \in \partial X \setminus \{\omega\} \). We may also take \( c(\delta) = 600\delta \) here.

We will require the following observation regarding Busemann functions in the proof of Lemma 3.7 in the next section.

**Lemma 2.1.** Let \( X \) be a proper geodesic Gromov hyperbolic space. Suppose that \( \partial X \) contains at least two points. Let \( b : X \to \mathbb{R} \) be a Busemann function based at \( \omega \in \partial X \). Then \( b(X) = \mathbb{R} \), i.e., \( b \) is surjective.

*Proof.* Since \( \partial X \) contains at least two points we can find a geodesic line \( \gamma : \mathbb{R} \to X \) starting from \( \omega \) and ending at some \( \zeta \in \partial X \) with \( \zeta \neq \omega \). By [14] Lemma 2.6] we can parametrize \( \gamma \) such that \( b(\gamma(t)) \equiv 1_{445} t \) for \( t \in \mathbb{R} \). For \( t \in \mathbb{R} \) we then set \( \tilde{b}(t) = b(\gamma(t)) \). The function \( \tilde{b} : \mathbb{R} \to \mathbb{R} \) is continuous and satisfies \( \tilde{b}(t) \to \pm \infty \) as \( t \to \pm \infty \). By the intermediate value theorem we conclude that \( \tilde{b}(\mathbb{R}) = \mathbb{R} \). By the construction of \( \tilde{b} \) this immediately implies that \( b(X) = \mathbb{R} \). \( \square \)

The requirement that \( \partial X \) contain at least two points is necessary for Lemma 2.1 to hold, as the example \( X = [0, \infty) \) shows.

2.3. Visual metrics. Let \( X \) be a proper geodesic \( \delta \)-hyperbolic space. Gromov products based at Busemann functions \( b \in \mathcal{B}(X) \) can be used to define visual metrics on the Gromov boundary \( \partial X \). We refer to [15] Chapters 2-3] as well as [14] Section 2.3] for precise details on this topic. We will summarize the results we need here. For \( b \in \mathcal{B}(X) \) we let \( \omega \) denote the basepoint of \( b \). We recall that we write \( \partial_\omega X = \partial X \setminus \{\omega\} \).

For \( b \in \mathcal{B}(X) \) and \( q > 0 \) we define for \( \xi, \zeta \in \partial_\omega X \),
\[
\alpha_{b,q}(\xi, \zeta) = e^{-q(\xi|\zeta)_b}.
\]
This may not define a metric on \( \partial_\omega X \), since the triangle inequality may not hold. However there is always \( q_0 = q_0(\delta) > 0 \) depending only on \( \delta \) such that for \( 0 < q \leq q_0 \) the function \( \alpha_{b,q} \) is 4-biLipschitz to a metric \( \alpha \) on \( \partial_\omega X \). We refer to any metric \( \alpha \) on \( \partial_\omega X \) that is biLipschitz
to \( a_{\omega,q} \) as a *visual metric* on \( \partial_\omega X \) based at \( b \) and refer to \( q \) as the *parameter* of \( \alpha \). We give \( \partial_\omega X \) the topology associated to a visual metric based at \( b \) for any Busemann function \( b \) based at \( \omega \). When equipped with a visual metric \( \partial_\omega X \) is a locally compact metric space.

### 3. Uniformization

In this section we will review the results of [14] concerning uniformizing Gromov hyperbolic spaces with Busemann functions. We begin with two essential definitions. For the first one we consider an incomplete metric space \((\Omega, d)\) and write \( \partial \Omega = \overline{\Omega} \setminus \Omega \) for the metric boundary of \( \Omega \) in its completion \( \overline{\Omega} \). We write \( d_\Omega(x) := \text{dist}(x, \partial \Omega) \) for the distance of a point \( x \in \Omega \) to the boundary \( \partial \Omega \). An important observation that we will use without comment is that \( d_\Omega \) defines a 1-Lipschitz function on \( \Omega \), i.e., for \( x, y \in \Omega \) we have

\[
|d_\Omega(x) - d_\Omega(y)| \leq d(x, y).
\]

For a curve \( \gamma : I \to \Omega \) we write \( \ell(\gamma) \) for the length of \( \gamma \); if \( \ell(\gamma) < \infty \) then we say that \( \gamma \) is *rectifiable*. For an interval \( I \subset \mathbb{R} \) and \( t \in I \) we write \( I_{s,t} = \{ s \in I : s \leq t \} \) and \( I_{s,t} = \{ s \in I : s \geq t \} \).

**Definition 3.1.** Let \((\Omega, d)\) be an incomplete, locally compact metric space. For a constant \( A \geq 1 \) and a compact interval \( I \subset \mathbb{R} \), a curve \( \gamma : I \to \Omega \) with endpoints \( x, y \in \Omega \) is *\( A \)-uniform* if

\[
(3.1) \quad \ell(\gamma) \leq Ad(x, y),
\]

and if for every \( t \in I \) we have

\[
(3.2) \quad \min\{\ell(\gamma|_{I_{s,t}}), \ell(\gamma|_{I_{s,t}})\} \leq Ad_\Omega(\gamma(t)).
\]

The metric space \( \Omega \) is *\( A \)-uniform* if any two points in \( \Omega \) can be joined by an \( A \)-uniform curve.

For the second key definition we consider a metric space \((X, d)\) and a continuous function \( \rho : X \to (0, \infty) \). Such a positive continuous function \( \rho \) will be referred to below as a *density* on \( X \). For a rectifiable curve \( \gamma \) in \( X \) we write

\[
\ell_\rho(\gamma) = \int_{\gamma} \rho \, ds,
\]

for the line integral of \( \rho \) along \( \gamma \). For reference below we say that \( X \) is *rectifiably connected* if any two points in \( X \) can be joined by a rectifiable curve.

**Definition 3.2.** Let \((X, d)\) be a rectifiably connected metric space and let \( \rho : X \to (0, \infty) \) be a density on \( X \). The *conformal deformation of \( X \) with conformal factor \( \rho \) is the metric space \( X_\rho = (X, d_\rho) \) with metric

\[
(3.3) \quad d_\rho(x, y) = \inf \ell_\rho(\gamma),
\]

with the infimum taken over all rectifiable curves \( \gamma \) joining \( x \) to \( y \).

If \( X \) is geodesic then we say further that the density \( \rho \) is *admissible* for \( X \) with constant \( M \geq 1 \) if for any \( x, y \in X \) and any geodesic \( \gamma \) joining \( x \) to \( y \) we have

\[
(3.4) \quad \ell_\rho(\gamma) \leq Md_\rho(x, y).
\]

Now consider a proper geodesic \( \delta \)-hyperbolic space \( X \). We define \( X \) to be *\( K \)-roughly starlike* from a point \( \omega \in \partial X \) if for each \( x \in X \) there is a geodesic line \( \gamma : \mathbb{R} \to X \) with \( \gamma|_{(-\infty,0]} \in \omega \) and \( \text{dist}(x, \gamma) \leq K \). We remark for use later that the rough starlikeness condition from \( \omega \) immediately implies that \( \partial X \) contains at least two points.
We fix a Busemann function $b$ based at $\omega$ and let $\varepsilon > 0$ be such that the density $\rho_\varepsilon(x) = e^{-e b(x)}$ is admissible on $X$ with constant $M$. Since $b$ is 1-Lipschitz we have the Harnack type inequality for $x, y \in X$,

\begin{equation}
(3.5) \quad e^{-\varepsilon|xy|} \leq \frac{\rho_\varepsilon(x)}{\rho_\varepsilon(y)} \leq e^{\varepsilon|xy|}.
\end{equation}

We remark that by [14, Theorem 5.1] the Harnack inequality (3.5) implies that there is always an $\varepsilon_0 = \varepsilon_0(\delta)$ such that $\rho_\varepsilon$ is admissible for $X$ for any $0 < \varepsilon \leq \varepsilon_0$ with constant $M = 20$. Thus the admissibility hypothesis on $\rho_\varepsilon$ is not particularly restrictive.

We write $X_\varepsilon = X_{\rho_\varepsilon}$ for the conformal deformation of $X$ with conformal factor $\rho$ and write $d_\varepsilon$ for the resulting distance on $X_\varepsilon$. We write $\ell_\varepsilon(\gamma) := \ell_{\rho_\varepsilon}(\gamma)$ for the lengths of curves measured in the metric $d_\varepsilon$. The properness of $X$ implies that $X_\varepsilon$ is locally compact. By [14, Theorem 1.4] the metric space $X_\varepsilon$ is incomplete and unbounded, and bounded geodesics in $X$ are $\Lambda$-uniform curves in $X_\varepsilon$. In particular the metric space $(X_\varepsilon, d_\varepsilon)$ is $\Lambda$-uniform. We write $B_\varepsilon(x,r)$ for the open ball of radius $r > 0$ centered at $x$ in the metric $d_\varepsilon$, and $B_X(x,r)$ for the corresponding ball of radius $r$ centered at $x$ in $X$.

For $x \in X_\varepsilon$ write $d_\varepsilon(x) = d_{X_\varepsilon}(x)$ for the distance to the metric boundary $\partial X_\varepsilon$ of $X_\varepsilon$. By [13, Theorem 1.6] there is a canonical identification $\iota : \partial \omega X \to \partial X_\varepsilon$ of the boundary of $X$ with respect to $\omega$ and the metric boundary $\partial X_\varepsilon$ of $X_\varepsilon$. The correspondence is given by showing that any sequence $\{x_n\}$ in $X$ converging to a point $\xi \in \partial \omega X$ is a Cauchy sequence in $X_\varepsilon$ converging to a point of $\partial X_\varepsilon$. In particular for $\xi, \zeta \in \partial \omega X$ we can define their distance with respect to the metric $d_\varepsilon$ to be $d_\varepsilon(\iota(\xi),\iota(\zeta))$. We will drop $\iota$ from the notation and simply write $d_\varepsilon(\xi, \zeta)$ for this quantity.

The local compactness of $X_\varepsilon$ implies by the Arzela-Ascoli theorem that, for a given $x, y \in X$, a minimizing curve $\gamma$ for the right side of (3.3) always exists. It is easy to see that such a curve must be a geodesic in $X_\varepsilon$, from which we conclude that $X_\varepsilon$ is always geodesic. By [7, Proposition 2.20] the completion $\bar{X}_\varepsilon$ of $X_\varepsilon$ is proper, and in particular is also locally compact. A second application of Arzela-Ascoli then shows that $\bar{X}_\varepsilon$ is also geodesic.

We collect here three important quantitative results regarding the uniformization $X_\varepsilon$ from our previous works [14, 13]. The standing assumptions for the rest of this section are that $X$ is a proper geodesic $\delta$-hyperbolic space that is $K$-roughly starlike from a point $\omega \in \partial X$, that $b : X \to \mathbb{R}$ is a Busemann function based at $\omega$, and that for a given $\varepsilon > 0$ the density $\rho_\varepsilon(x) = e^{-\varepsilon b(x)}$ on $X$ is admissible with constant $M$. All implied constants will depend only on $\delta$, $K$, $\varepsilon$, and $M$.

For this first lemma we set $|xy| = \infty$ if $x \neq y$ and either $x \in \partial \omega X$ or $y \in \partial \omega X$, and we set $|xy| = 0$ if $x = y \in \partial \omega X$.

**Lemma 3.3.** [13, Lemma 2.10] For $x, y \in X \cup \partial \omega X$ we have

\begin{equation}
(3.6) \quad d_\varepsilon(x,y) \asymp e^{-\varepsilon|x|y|} \min\{1,|xy|\}.
\end{equation}

In this second lemma we have absorbed the constant $\varepsilon^{-1}$ on the right in the reference into the implied constant.

**Lemma 3.4.** [14, Proposition 4.7] For $x \in X$ we have

\begin{equation}
(3.7) \quad d_\varepsilon(x) \asymp \rho_\varepsilon(x).
\end{equation}

The first comparison in the following result is given by the reference. The second comparison then follows directly from Lemma 3.4.
Lemma 3.5. [13, Lemma 6.2] There exists $0 < \lambda < 1$ with $\lambda = \lambda(\delta, K, \varepsilon, M)$ such that for any $x \in X$ and any $y, z \in B_\varepsilon(x, \lambda d_\varepsilon(x))$ we have that $|yz| \leq 1$ and that

$$d_\varepsilon(y, z) \preceq d_\varepsilon(x)|yz| \preceq \rho_\varepsilon(x)|yz|. \tag{3.8}$$

We conclude this section by adapting two key claims from [2] to our setting. The first claim adapts [2, Theorem 2.10]. The proof is essentially the same.

Lemma 3.6. There is a constant $C_* = C_*(\delta, K, \varepsilon, M) \geq 1$ such that for any $x \in X$ and any $0 < r \leq \frac{1}{2}d_\varepsilon(x)$ we have the inclusions,

$$B_X \left( x, \frac{C_*^{-1}r}{\rho_\varepsilon(x)} \right) \subset B_\varepsilon(x, r) \subset B_X \left( x, \frac{C_*r}{\rho_\varepsilon(x)} \right). \tag{3.9}$$

Proof. Let $y \in B_X(x, C_*^{-1}r/\rho_\varepsilon(x))$, for a constant $C_* \geq 1$ to be determined. Let $\gamma$ be a geodesic in $X$ joining $x$ to $y$ and let $z \in \gamma$. Then, since $r \leq \frac{1}{2}d_\varepsilon(x)$, we have by Lemma 3.4,

$$|xz| \leq \frac{C_*^{-1}d_\varepsilon(x)}{2\rho_\varepsilon(x)} \leq C_*^{-1}C,$$

with $C = C(\delta, K, \varepsilon, M) \geq 1$. This then implies by the Harnack inequality,

$$\rho_\varepsilon(z) \preceq e^{C_*^{-1}C} \rho_\varepsilon(x).$$

Choosing $C_*$ large enough that $e^{C_*^{-1}C} < 2$, we then obtain that

$$\rho_\varepsilon(z) \succeq 2 \rho_\varepsilon(x),$$

for $z \in \gamma$. We conclude that

$$d_\varepsilon(x, y) \leq \int_\gamma \rho_\varepsilon \, ds \leq 2\rho_\varepsilon(x)|xy| \leq 2C_*^{-1}r < r,$$

provided we take $C_* > 2$. This gives the inclusion on the left side of (3.9).

For the inclusion on the right side of (3.9), let $y \in B_\varepsilon(x, r)$ and let $\gamma_\varepsilon$ be a geodesic in $X_\varepsilon$ connecting $x$ to $y$. For $z \in \gamma_\varepsilon$ we then have $z \in B_\varepsilon(x, r)$ and therefore $d_\varepsilon(z) \geq \frac{1}{2}d_\varepsilon(x)$ by the triangle inequality since $r \leq \frac{1}{2}d_\varepsilon(x)$. Applying Lemma 3.4 we then have

$$\rho_\varepsilon(z) \geq C_*^{-1}d_\varepsilon(z) \geq \frac{1}{2}C_*^{-1}d_\varepsilon(x) \geq C^{-1}\rho_\varepsilon(x),$$

for a constant $C = C(\delta, K, \varepsilon, M) \geq 1$. Using this we conclude that

$$r > d_\varepsilon(x, y) \geq C_*^{-1}\rho_\varepsilon(x)|xy|.$$

Choosing $C_*$ to be greater than the constant $C$ on the right side of this inequality, we then conclude that

$$|xy| < \frac{C_*r}{\rho_\varepsilon(x)},$$

which gives the right side inclusion in (3.9). □
Following [2], the balls $B_z(x, r)$ for $x \in X_\varepsilon$, $0 < r \leq \frac{1}{2}d_\varepsilon(x)$ will often be referred to as subWhitney balls.

The second claim adapts [2, Lemma 4.8] to our setting. The proof given in [2] strongly relies on the uniformization $X_\varepsilon$ having finite diameter in their setting, so we have to take an approach that is somewhat different.

**Lemma 3.7.** There is a constant $\kappa_0 = \kappa_0(\delta, K, \varepsilon, M)$ such that if $0 < \kappa \leq \kappa_0$ then for every $x \in X_\varepsilon$ and every $r > 0$ we can find a ball $B_\varepsilon(z, \kappa r) \subset B_\varepsilon(x, r)$ with $d_\varepsilon(z) \geq 2\kappa r$.

**Proof.** Let $x \in X_\varepsilon$ and $r > 0$ be given. We first assume that $x \in X_\varepsilon$. By Lemma 2.1 we have $b(X) = \mathbb{R}$, which immediately implies that $\rho_\varepsilon(x) = (0, \infty)$. Thus we can find a point $z_0 \in X$ such that $\rho_\varepsilon(z_0) = r$. Let $\sigma$ be a geodesic in $X$ that is oriented from $x$ to $z_0$, which we will assume is parametrized by $d_\varepsilon$-arclength. Then $\sigma$ is an $A$-uniform curve in $X_\varepsilon$ with $A = A(\delta, K, \varepsilon, M) \geq 1$.

We first assume that $\ell_\varepsilon(\sigma) \geq \frac{2}{3}r$. In this case we set $z = \sigma(\frac{1}{4}r)$. Then since $\sigma$ is $A$-uniform we have $d_\varepsilon(z) \geq \frac{r}{2A}$ and

$$B_\varepsilon\left(z, \frac{r}{6A}\right) \subset B_\varepsilon\left(x, \frac{r}{3} + \frac{r}{6A}\right) \subset B_\varepsilon(x, r).$$

So in this case we can use any $\kappa \leq \frac{1}{6A}$.

Now consider the case in which $\ell_\varepsilon(\sigma) < \frac{2}{3}r$. We then set $z = z_0$ and observe that

$$B_\varepsilon\left(z, \frac{r}{3}\right) \subset B_\varepsilon\left(x, \ell_\varepsilon(\sigma) + \frac{r}{3}\right) \subset B_\varepsilon(x, r).$$

By construction we have

$$d_\varepsilon(z) \geq C^{-1}\rho_\varepsilon(z) = C^{-1}r,$$

with $C = C(\delta, K, \varepsilon, M)$ being the implied constant from Lemma 3.4. Thus any $\kappa > 0$ such that $2\kappa \leq C^{-1}$ will work. We then set $\kappa_0' = \max\left(\frac{1}{6A}, \frac{1}{2C}\right)$ with $C$ being the implied constant of Lemma 3.4. Then the conclusions of the lemma hold for $x \in X_\varepsilon$, $r > 0$, and any $0 < \kappa \leq \kappa_0'$. We set $\kappa_0 = \frac{\kappa_0'}{2}$.

Now let $x \in \partial X_\varepsilon$ and $0 < \kappa \leq \kappa_0$ be given. Set $r' = \kappa r / \kappa_0'$. Then $r' < r$ since $\kappa_0 < \kappa_0'$, so we can find $x' \in X_\varepsilon$ sufficiently close to $x$ such that $B_\varepsilon(x', r') \subset B_\varepsilon(x, r)$. We then apply the previous claims to $x'$, $r'$, and $\kappa_0'$ to find $z \in X_\varepsilon$ such that $B_\varepsilon(z, \kappa_0 r') \subset B_\varepsilon(x', r')$ and $d_\varepsilon(z) \geq 2\kappa_0' r'$. It follows that

$$B_\varepsilon(z, \kappa r) = B_\varepsilon(z, \kappa_0' r') \subset B_\varepsilon(x', r') \subset B_\varepsilon(x, r),$$

and

$$d_\varepsilon(z) \geq 2\kappa_0' r' = 2\kappa r.$$

This completes the proof of the lemma. \(\Box\)

The conclusion of Lemma 3.7 is closely related to the corkscrew condition for domains in metric spaces. See [3, Definition 2.4].

4. Global doubling

We will use some standard notation from analysis on metric spaces starting in this section. Some of these notions were previously defined in the introduction; we repeat the definitions for reference here. Throughout this paper a metric measure space $(X, d, \mu)$ will be a triple consisting of a metric space $(X, d)$ together with a Borel regular measure $\mu$ on $X$ such that $0 < \mu(B) < \infty$ for all balls $B \subset X$. We make the standard caveat that a ball may be describable using more than one center and radius, and for this reason we stipulate that all
of our balls come with a fixed choice of center and radius. The measure \( \mu \) is **doubling** on \( X \) if there is a constant \( C_\mu \geq 1 \) such that for any \( x \in X \) and any \( r > 0 \),
\[
\mu(B_X(x, 2r)) \leq C_\mu \mu(B_X(x, r)).
\]

We say that \( \mu \) is **uniformly locally doubling** if there is an \( R_0 > 0 \) such that (4.1) holds for any \( x \in X \) and any \( 0 < r \leq R_0 \). In this case we will also say that \( \mu \) is **doubling on balls of radius at most \( R_0 \)**. We will frequently make use of the following consequence of the doubling estimate (4.1): if \( \mu \) is doubling on balls of radius at most \( R_0 \) with constant \( C_\mu \) and \( 0 < r \leq R \leq R_0 \) then
\[
\mu(B_X(x, R)) \approx C \mu(B_X(x, r)),
\]
with constant \( C \) depending only on \( C_\mu \) and the ratio \( R/r \). This estimate follows by iterating the estimate (4.1) and then also noting that \( \mu(B_X(x, R)) \geq \mu(B_X(x, r)) \) since \( B_X(x, r) \subset B_X(x, R) \).

We will require the following proposition from [2], which is stated there in a more general form.

**Proposition 4.1.** [2, Proposition 3.2] Let \((X, d)\) be a geodesic metric space and let \( \mu \) be a measure on \( X \) that is doubling on balls of radius at most \( R_0 \), with doubling constant \( C_\mu \). Then for any \( R_1 > 0 \) the measure \( \mu \) is doubling on balls of radius at most \( R_1 \), with doubling constant depending only on \( R_1/R_0 \) and \( C_\mu \).

Thus if \( \mu \) is doubling on balls of radius at most \( R_0 \) then given any \( R_1 > 0 \) we can assume that \( \mu \) is also doubling on balls of radius at most \( R_1 \) at the cost of increasing the uniform local doubling constant of \( \mu \) by an amount depending only on \( R_1/R_0 \) and \( C_\mu \).

We now describe the setting of this section. We begin with a proper geodesic \( \delta \)-hyperbolic \( X \) that is \( K \)-roughly starlike from a point \( \omega \in \partial X \) and let \( b \) be a Busemann function on \( X \) based at \( \omega \). We let \( \varepsilon > 0 \) be such that the associated density \( \rho_\varepsilon \) is admissible for \( X \) with constant \( M \). As in the previous section we write \( X_\varepsilon \) for the uniformization of \( X \), \( d_\varepsilon \) for the distance on \( X_\varepsilon \), etc. We let \( \mu \) be a Borel regular measure on \( X \) such that \( 0 < \mu(B) < \infty \) for all balls \( B \subset X \). We will assume that there is an \( R_0 > 0 \) such that \( \mu \) is doubling on balls of radius at most \( R_0 \) with doubling constant \( C_\mu \).

For each \( \beta > 0 \) we define a measure \( \mu_\beta \) on \( X \) by
\[
d\mu_\beta(x) = \rho_\beta(x) d\mu(x) = e^{-\beta b(x)} d\mu(x),
\]
for \( x \in X \). We will consider \( \mu_\beta \) as a measure on \( X_\varepsilon \) and extend it to the completion \( \bar{X}_\varepsilon \) by setting \( \mu_\beta(\partial X_\varepsilon) = 0 \). In this section we will be establishing a criterion (Definition 4.9) for \( \mu_\beta \) to be doubling on \( \bar{X}_\varepsilon \).

**Remark 4.2.** Our notation for \( \mu_\beta \) is different from the notation used in the analogous setting in [2]; our \( \mu_\beta \) corresponds to \( \mu_{\beta \varepsilon} \) in [2]. We suppress the dependence on \( \varepsilon \) in the notation for \( \mu_\beta \) because \( \varepsilon \) will be considered to be fixed in all settings that we consider.

In the claims in the rest of this section all implicit constants will depend only on \( \delta, K, \varepsilon, M, \beta, R_0 \), and \( C_\mu \). We will refer to this collection of seven parameters as the **data**. We will refer to the specific parameters \( \delta, K, \varepsilon, M, \beta \) as the **uniformization data** and say that a constant depends only on the uniformization data if it depends only on these five parameters. At several points we will need to increase the radius \( R_0 \) by an amount depending only on the uniformization data in order to ensure that \( \mu \) is doubling at a larger scale using Proposition 4.1. When we do this we will also need to increase \( C_\mu \) by a corresponding amount depending only on the uniformization data.
Remark 4.3. We will also often refer to just the four parameters $\delta, K, \varepsilon, M$ as the uniformization data. It will be clear from context when $\beta$ can and cannot be excluded from the list. This distinction will only be important in the proof of Proposition 4.12, which isn’t needed for any of our primary claims.

Our first claim corresponds to [2, Lemma 4.5]. It provides an important estimate on the measure of subWhitney balls in $X$.

**Lemma 4.4.** Let $x \in X$ and $0 < r \leq \frac{1}{2} d_\varepsilon(x)$. Then

$$
\mu_\beta(B_\varepsilon(x, r)) \asymp \rho_\beta(x) \mu\left(B_X\left(x, \frac{r}{\rho_\varepsilon(x)}\right)\right),
$$

with comparison constant depending only on the data.

**Proof.** By Lemma 3.4 we have for all $y \in B_\varepsilon(x, r)$,

$$
\rho_\beta(y) = \rho_\varepsilon(y)^\beta \asymp d_\varepsilon(y)^\beta \asymp d_\varepsilon(x)^\beta \asymp \rho_\beta(x),
$$

with the comparison $d_\varepsilon(y) \asymp_2 d_\varepsilon(x)$ following from the condition on $r$. Applying Lemma 3.6 and the chain of comparisons (4.4), we conclude that

$$
\mu_\beta(B_\varepsilon(x, r)) \asymp \rho_\beta(x) \mu(B_\varepsilon(x, r)) \lesssim \rho_\beta(x) \mu\left(B_X\left(x, \frac{C_r r}{\rho_\varepsilon(x)}\right)\right),
$$

with $C_r = C_r(\delta, K, \varepsilon, M)$ being the constant from Lemma 3.6. A similar argument shows that

$$
\mu_\beta(B_\varepsilon(x, r)) \gtrsim \rho_\beta(x) \mu\left(B_X\left(x, \frac{C_r^{-1} r}{\rho_\varepsilon(x)}\right)\right).
$$

We thus conclude that

$$
(4.5) \quad \rho_\beta(x) \mu\left(B_X\left(x, \frac{C_r^{-1} r}{\rho_\varepsilon(x)}\right)\right) \lesssim \mu_\beta(B_\varepsilon(x, r)) \lesssim \rho_\beta(x) \mu\left(B_X\left(x, \frac{C_r r}{\rho_\varepsilon(x)}\right)\right).
$$

The condition on $r$ implies that

$$
(4.6) \quad \frac{r}{\rho_\varepsilon(x)} \leq \frac{1}{2} \frac{d_\varepsilon(x)}{\rho_\varepsilon(x)} \leq C,
$$

with $C$ depending only on the uniformization data by Lemma 3.4. By Proposition 4.1 we can, at the cost of increasing the local doubling constant $C_\mu$ of $\mu$ by an amount depending only on the data, assume that $R_0 > C C_r$ for the constant $C$ in inequality (4.5) and the constant $C_r$ in Lemma 3.6. Then the comparison (4.2) allows us to conclude that

$$
\mu\left(B_X\left(x, \frac{C_r^{-1} r}{\rho_\varepsilon(x)}\right)\right) \asymp \mu\left(B_X\left(x, \frac{r}{\rho_\varepsilon(x)}\right)\right) \asymp \mu\left(B_X\left(x, \frac{C_r r}{\rho_\varepsilon(x)}\right)\right).
$$

Combining this comparison with inequality (4.5) proves the lemma. \(\square\)

By combining Lemma 4.4 with Lemma 3.7, we obtain the following estimate for $\mu_\beta(B_\varepsilon(x, r))$ when $0 < r \leq \frac{1}{2} d_\varepsilon(x)$. Below we let $\kappa_0 = \kappa_0(\delta, K, \varepsilon, M)$ be defined as in Lemma 3.7.

**Lemma 4.5.** Let $x \in X$ and $0 < r \leq \frac{1}{2} d_\varepsilon(x)$. Let $0 < \kappa < \kappa_0$ and $z \in X$ be given such that $B_\varepsilon(z, \kappa r) \subset B_\varepsilon(x, r)$ and $d_\varepsilon(z) \geq 2 \kappa r$. Then

$$
\mu_\beta(B_\varepsilon(x, r)) \asymp \mu_\beta(B_\varepsilon(z, \kappa r)),
$$

with comparison constants depending only on the data and $\kappa$.\(\square\)
Proof. By Lemmas 3.4 and 3.6 we have
\[
|z| \leq \frac{C_r r}{\rho_\epsilon(z)} \leq \frac{C_s d_\epsilon(x)}{2\rho_\epsilon(x)} \lesssim 1,
\]
with implied constant depending only on the uniformization data, where \(C_s\) is the constant from Lemma 3.6. Since \(z \in B_\epsilon(x, r)\) and \(r \leq \frac{1}{2} d_\epsilon(x)\), we conclude that we have \(d_\epsilon(z) \approx_2 d_\epsilon(x)\). We thus obtain from Lemma 3.4 that \(\rho_\epsilon(z) = \rho_\epsilon(x)\) with comparison constant depending only on the uniformization data. Since \(d_\epsilon(z) \geq 2\kappa r\), we have by Lemma 3.4 that
\[
1 \gtrsim \frac{\kappa r}{\rho_\epsilon(z)} \approx \frac{r}{\rho_\epsilon(z)} \approx \frac{r}{\rho_\epsilon(x)} \approx \frac{C_s r}{\rho_\epsilon(x)},
\]
with all implied constants depending only on the uniformization data and \(\kappa\). We can thus apply Proposition 4.1 to conclude that we can assume that \(\mu\) is doubling on balls of radius at most any of the terms appearing in (4.8), at the cost of increasing the doubling constant of \(\mu\) by an amount depending only on the uniformization data and \(\kappa\). It follows that
\[
\mu(B_X(z, \frac{\kappa r}{\rho_\epsilon(z)})) \approx \mu(B_X(z, \frac{r}{\rho_\epsilon(z)})) \approx \mu(B_X(z, \frac{r}{\rho_\epsilon(x)})) \approx \mu(B_X(x, \frac{C_s r}{\rho_\epsilon(x)})) \approx \mu(B_X(x, \frac{r}{\rho_\epsilon(x)}))
\]
with implied constants depending only on the data and \(\kappa\). The third comparison above follows from the fact that \(z \in B_X\left(x, \frac{C_s r}{\rho_\epsilon(x)}\right)\) by (4.7). Since the comparison \(\rho_\beta(z) \approx \rho_\beta(x)\) follows from the comparison \(\rho_\epsilon(z) \approx \rho_\epsilon(x)\) (with comparison constants depending only on the uniformization data), applying Lemma 4.4 to \(B_\epsilon(z, \kappa r)\) and \(B_\epsilon(x, r)\) (note that \(\kappa r \leq \frac{1}{2} d_\epsilon(z)\) by assumption) then gives
\[
\mu_\beta(B_\epsilon(z, \kappa r)) \approx \mu_\beta(B_\epsilon(x, r)),
\]
with comparison constants depending only on the data and \(\kappa\). \(\square\)

In order to obtain doubling of \(\mu_\beta\) on \(\tilde{X}_\epsilon\), we need to be able to extend Lemma 4.5 to cover \(x \in \tilde{X}_\epsilon\) and all \(r > 0\). The following definition axiomatizes this condition. We will use this technical definition to verify doubling of the lifted measures in Section 7. We let \(\kappa_0 > 0\) be as defined in Lemma 3.7. The existence of \(z\) as specified below is given by Lemma 3.7. We recall that the measure \(\mu_\beta\) on \(X_\epsilon\) is extended to the completion \(\tilde{X}_\epsilon\) by setting \(\mu_\beta(\partial X_\epsilon) = 0\).

**Definition 4.6.** The measure \(\mu_\beta\) is \(\partial\)-controlled if for each \(0 < \kappa \leq \kappa_0\) there exists a constant \(C_\beta(\kappa) \geq 1\) such that for any \(\xi \in \partial X_\epsilon\), \(r > 0\), and \(z \in X\) such that \(B_\epsilon(z, \kappa r) \subset B_\epsilon(\xi, r)\) and \(d_\epsilon(z) \geq 2\kappa r\), we have
\[
\mu_\beta(B_\epsilon(\xi, r)) \approx C_\beta(\kappa) \mu_\beta(B_\epsilon(z, \kappa r)).
\]

Observe that the lower bound in (4.9) follows trivially from the inclusion \(B_\epsilon(z, \kappa r) \subset B_\epsilon(\xi, r)\). Thus, it is only the upper bound that is of interest in (4.9). To motivate Definition 4.6 we note that being \(\partial\)-controlled is necessary for \(\mu_\beta\) to be doubling on \(\tilde{X}_\epsilon\).

**Proposition 4.7.** Suppose that \(\mu_\beta\) is doubling on \(\tilde{X}_\epsilon\) with constant \(C_\mu_\beta\). Then \(\mu_\beta\) is \(\partial\)-controlled with \(C_\beta(\kappa)\) depending only on \(\kappa\) and \(C_\mu_\beta\) for each \(0 < \kappa \leq \kappa_0\).
Proof. Let $0 < \kappa \leq \kappa_0$, $\xi \in \partial X_\varepsilon$, $r > 0$, and $z \in X$ be given as in Definition 4.6. Then $B_\varepsilon(\xi, r) \subset B_\varepsilon(z, 2r)$ since $z \in B_\varepsilon(\xi, r)$, which implies that
\[
\mu_\beta(B_\varepsilon(z, \kappa r)) \leq \mu_\beta(B_\varepsilon(\xi, r)) \leq \mu_\beta(B_\varepsilon(z, 2r)).
\]

By (4.2) applied to $\mu_\beta$ we conclude that $\mu_\beta(B_\varepsilon(z, \kappa r)) \asymp \mu_\beta(B_\varepsilon(\xi, r))$ with comparison constant depending only on $\kappa$ and $C_{\mu_\beta}$. \□

When $\mu_\beta$ is $\partial$-controlled we can improve Lemma 4.3 to hold for all $x \in \bar{X}_\varepsilon$ and $r > 0$.

**Lemma 4.8.** Suppose that $\mu_\beta$ is $\partial$-controlled. Let $x \in \bar{X}_\varepsilon$, $r > 0$, $0 < \kappa \leq \kappa_0$ and $z \in X$ be given such that $B_\varepsilon(z, \kappa r) \subset B_\varepsilon(x, r)$ and $d_\varepsilon(z) \geq 2\kappa r$. Then
\[
\mu_\beta(B_\varepsilon(x, r)) \asymp \mu_\beta(B_\varepsilon(z, \kappa r)),
\]
with comparison constant depending only on the data, $\kappa$, and $C_\beta(\kappa/3)$.

**Proof.** The lower bound $\mu_\beta(B_\varepsilon(x, r)) \geq \mu_\beta(B_\varepsilon(z, \kappa r))$ follows from the inclusion $B_\varepsilon(z, \kappa r) \subset B_\varepsilon(x, r)$, so we only need to establish the upper bound in (4.10). By Lemma 4.5 it suffices to consider the case $x \in \bar{X}_\varepsilon$, $r > \frac{1}{2}d_\varepsilon(x)$. We can then find $\xi \in \partial X_\varepsilon$ such that $B_\varepsilon(x, r) \subset B_\varepsilon(\xi, 3r)$. Applying (4.9) with $\kappa$ replaced by $\frac{1}{4}\kappa$, we conclude that
\[
\mu_\beta(B_\varepsilon(x, r)) \leq \mu_\beta(B_\varepsilon(\xi, 3r)) \asymp_{C_\beta(\kappa/3)} \mu_\beta(B_\varepsilon(z, \kappa r)).
\]

This gives the desired upper bound in the case $r > \frac{1}{2}d_\varepsilon(x)$ with comparison constant $C_\beta(\kappa/3)$. \□

We will now show that $\mu_\beta$ is doubling on $\bar{X}_\varepsilon$ when it is $\partial$-controlled, with doubling constant depending only on the data and the particular constants $C_\beta(\kappa_0/6)$ and $C_\beta(\kappa_0/12)$. In fact our proof will show that it suffices to assume that (4.9) holds only for $\kappa = \kappa_0/6$ and $\kappa = \kappa_0/12$.

**Proposition 4.9.** Suppose that $\mu_\beta$ is $\partial$-controlled. Then $\mu_\beta$ is doubling on $\bar{X}_\varepsilon$ with doubling constant depending only on the data and max\{$C_\beta(\kappa_0/6), C_\beta(\kappa_0/12)$\}.

**Proof.** For $x \in \bar{X}_\varepsilon$ we use Lemma 3.1 to fix $\kappa = \frac{\kappa_0}{6} > 0$ and $z \in X$ such that the ball $B_\varepsilon(z, \kappa r)$ is contained in $B_\varepsilon(x, r)$ (and therefore also in $B_\varepsilon(x, 2r)$) and $d_\varepsilon(z) \geq 2\kappa r$. We set $\kappa' = \frac{\kappa}{2}$. Since $d_\varepsilon(z) \geq 2\kappa r = 2\kappa' \cdot 2r$, we can apply Lemma 4.8 twice to obtain
\[
\mu_\beta(B_\varepsilon(x, 2r)) \asymp \mu_\beta(B_\varepsilon(z, 2\kappa' r)) = \mu_\beta(B_\varepsilon(z, \kappa r)) \asymp \mu_\beta(B_\varepsilon(x, r)),
\]
with implied constants depending only on the data, $\kappa_0$, and max\{$C_\beta(\kappa_0/6), C_\beta(\kappa_0/12)$\}, which means they only depend on the data and max\{$C_\beta(\kappa_0/6), C_\beta(\kappa_0/12)$\} since $\kappa_0$ is determined by the uniformization data. \□

In practice it is often easier to verify the following condition when attempting to show that $\mu_\beta$ is $\partial$-controlled.

**Lemma 4.10.** Suppose that for some $0 < \kappa \leq \kappa_0$ there is a constant $C'_\beta(\kappa) \geq 1$ such that, for any $\xi \in \partial X_\varepsilon$, $r > 0$, and $z \in X$ with $B_\varepsilon(z, \kappa r) \subset B_\varepsilon(\xi, r)$ and $d_\varepsilon(z) \geq 2\kappa r$, we have
\[
\mu_\beta(B_\varepsilon(\xi, r)) \lesssim_{C'_\beta(\kappa)} \rho_\beta(\varepsilon) \mu(B_X(z, 1)).
\]

Then the comparison (4.9) holds for this value of $\kappa$ with comparison constant $C_\beta(\kappa)$ depending only on the data, $\kappa$, and $C'_\beta(\kappa)$.

Consequently if (4.11) holds for all $0 < \kappa \leq \kappa_0$ then $\mu_\beta$ is $\partial$-controlled with constants $C_\beta(\kappa)$ depending only on the data, $\kappa$, and $C'_\beta(\kappa)$.\]
Proof. As remarked after Definition 4.6, it suffices to verify the upper bound in (4.9). Since \( \kappa r \leq \frac{1}{4}d_{e}(z) \), we can apply Lemma 3.4 to obtain
\[
\mu_{\beta}(B_{e}(z, \kappa r)) \approx \rho_{\beta e}(z) \mu \left( B_{X} \left( z, \frac{\kappa r}{\rho_{e}(z)} \right) \right),
\]
with comparison constants depending only on the data. We then observe by Lemma 3.4 and the fact that \( d_{e}(z) < r \) since \( z \in B_{e}(\xi, r) \),
\[
(4.12) \quad \kappa C^{-1} \leq \frac{\kappa d_{e}(z)}{\rho_{e}(z)} \leq \frac{\kappa r}{\rho_{e}(z)} \leq \frac{d_{e}(z)}{2 \rho_{e}(z)} \leq C,
\]
for a constant \( C = C(\delta, K, \varepsilon, M) \geq 1 \) depending only on the uniformization data. By Proposition 4.1 we can, by increasing the local doubling constant \( C_{\mu} \) of \( \mu \) by an amount depending only on the data, assume that \( \mu \) is doubling on balls of radius at most \( C \). We then conclude that
\[
\mu \left( B_{X} \left( z, C^{-1} \kappa \right) \right) \approx \mu(B_{X}(z, C)) \approx \mu(B_{X}(z, 1)),
\]
with comparison constants depending only on the data and \( \kappa \), which implies by (4.12) that
\[
\mu \left( B_{X} \left( z, \frac{\kappa r}{\rho_{e}(z)} \right) \right) \approx \mu(B_{X}(z, 1)),
\]
again with comparison constants depending only on the data and \( \kappa \). We thus conclude that
\[
\mu_{\beta}(B_{e}(z, \kappa r)) \approx \rho_{\beta e}(z) \mu(B_{X}(z, 1)),
\]
with comparison constant depending only on the data and \( \kappa \). Our assumption then implies that
\[
\mu_{\beta}(B_{e}(\xi, r)) \lesssim C_{\beta}(\kappa) \rho_{\beta e}(z) \mu(B_{X}(z, 1)) \approx \mu_{\beta}(B_{e}(z, \kappa r)).
\]
We conclude that the comparison (4.10) holds with constant \( C_{\beta}(\kappa) \) depending only on the data, \( \kappa \), and \( C_{\beta}(\kappa) \).
\[ \square \]

The next result will not be used in the main theorems of this paper, since we will be verifying the doubling property for measures on hyperbolic fillings in Section 7 in a different fashion. We will show, in analogy to [2] Proposition 4.7], that \( \mu_{\beta} \) is always doubling on \( \hat{X}_{e} \) for \( \beta \) sufficiently large. We will need the following refinement of Proposition 4.11.

Lemma 4.11. [2] Lemma 3.5] Let \((X, d, \mu)\) be a geodesic metric measure space such that \( \mu \) is doubling on \( X \) of radius at most \( R_{0} \) with constant \( C_{\mu} \). Let \( n \in \mathbb{N} \) be a given integer.

1. For \( x, y \in X \) and \( 0 < r \leq R_{0} \) satisfying \( d(x, y) < nr \), we have
\[
\mu(B_{X}(x, r)) \leq C_{\mu}^{n} \mu(B_{X}(y, r)).
\]

2. For \( 0 < r \leq \frac{1}{4}R_{0} \), every ball \( B \subset X \) of radius \( nr \) can be covered by at most \( C_{\mu}^{n(n+4)/4} \)
balls of radius \( r \).

Proposition 4.12. There is \( \beta_{0} = \beta_{0}(\delta, K, \varepsilon, M, R_{0}, C_{\mu}) > 0 \) such that if \( \beta \geq \beta_{0} \) then \( \mu_{\beta} \) is doubling on \( \hat{X}_{e} \) with constant \( C_{\mu_{\beta}} \) depending only on the data.

Proof. By the proof of Proposition 4.10 it suffices to verify that the comparison (4.11) holds for \( \kappa = \kappa_{0}/6 \) and \( \kappa = \kappa_{0}/12 \). By Lemma 3.4 it suffices to show that the inequality (4.12) holds for \( \kappa = \kappa_{0}/6 \) and \( \kappa = \kappa_{0}/12 \). Let \( \xi \in \partial X_{e} \) and \( r > 0 \) be given, and suppose that \( \kappa > 0 \) and \( z \in X \) are given such that \( B_{e}(z, \kappa r) \subset B_{e}(\xi, r) \) and \( d_{e}(z) \geq 2\kappa r \), where \( \kappa = \kappa_{0}/c_{0} \) and \( c_{0} = 6 \) or \( c_{0} = 12 \). We define for \( n \geq 1 \),
\[
A_{n} = \{ x \in B_{e}(\xi, r) \cap X : e^{-n}r \leq d_{e}(x) < e^{1-n}r \}.
\]
Since $x \in B_\varepsilon(\xi, r)$ implies that $d_\varepsilon(x) < r$, we have $B_\varepsilon(\xi, r) \cap X = \bigcup_{n=1}^\infty A_n$. Since $\mu_\beta$ is extended to $\partial X_\varepsilon$ by setting $\mu_\beta(\partial X_\varepsilon) = 0$, we conclude that

$$\mu_\beta(B_\varepsilon(\xi, r)) = \sum_{n=1}^\infty \mu_\beta(A_n).$$

For any given $x \in A_n$ we either have $|xz| < 1$ or $|xz| \geq 1$. In the second case we use Lemma 3.3 to obtain

$$
\varepsilon |xz| = \frac{e^{-2\varepsilon|z|_b}}{\rho_\varepsilon(x)\rho_\varepsilon(z)} \geq \frac{d_\varepsilon(x, z)^2}{d_\varepsilon(x)d_\varepsilon(z)} \leq \frac{(d_\varepsilon(x, \xi) + d_\varepsilon(\xi, z))^2}{2\kappa_0 e^{-n\tau^2}} \leq \frac{\kappa_0}{\kappa_0} \lesssim e^n,
$$

with implied constant depending only on $\delta$, $K$, $\varepsilon$, and $M$, since $\kappa_0 = \kappa_0(\delta, K, \varepsilon, M)$ and $c_0 \leq 12$. We then conclude that $\varepsilon |xz| \leq n + c'$, with $c' = c'(\delta, K, \varepsilon, M) \geq 0$. Since this inequality trivially holds with $c' = \varepsilon$ when $|xz| < 1$, we in fact obtain the inequality $\varepsilon |xz| \leq n + c'$ in both cases. Since $n \geq 1$, we thus obtain that

$$|xz| \leq \varepsilon^{-1}(1 + c')n.$$ 

We set $c_* = \varepsilon^{-1}(1 + c')$, observing that $c_* = c_*(\delta, K, \varepsilon, M)$. Then $|xz| \leq c_* n$ for $x \in A_n$.

We now apply Proposition 4.1 to ensure that $\mu$ is doubling on balls of radius at most $R_1 = \max\{4R_0 + c_*, 1\}$. The doubling constant $C'_\mu$ for $\mu$ on balls of radius at most $R_1$ then depends only on $\delta$, $K$, $\varepsilon$, $M$, $R_0$, and $C_\mu$. In particular $C'_\mu$ does not depend on $\beta$. Applying (2) of Lemma 4.11 we cover $A_n \subset B_X(z, c_* n)$ with $N_n \lesssim e^{c_n}$ many balls $B_{n,j}$ of radius $R_0$, where

$$\alpha = \alpha(\delta, K, \varepsilon, M, R_0, C_\mu) = \frac{7}{4}\log C'_\mu.$$ 

Hence $\alpha$ also does not depend on $\beta$. We set $\beta_0 := 2\alpha$ and assume that $\beta \geq \beta_0$.

We can clearly assume that each ball $B_{n,j}$ intersects $A_n$, from which we conclude that the centers $x_{n,j}$ of the balls $B_{n,j}$ satisfy

$$|x_{n,j}| \leq R_0 + c_* n < R_1 n,$$

since $n \geq 1$. Applying (1) of Lemma 4.11 then gives that

$$\mu(B_{n,j}) \leq (C'_\mu)^n \mu(B_X(z, R_1)) \leq e^{c_n} \mu(B_X(z, R_1)).$$

Since $d_\varepsilon(z) \asymp_2 r$, we obtain that

$$\rho_{\beta\varepsilon}(z) \asymp d_\varepsilon(z)^\beta \asymp r^\beta,$$

with comparison constants depending only on the uniformization data. Likewise for $x \in A_n$, (4.13)

$$\rho_{\beta\varepsilon}(x) \asymp d_\varepsilon(x)^\beta \asymp (e^{-n}r)^\beta,$$

again with comparison constants depending only on the uniformization data. The Harnack inequality (3.5) implies that $\rho_{\beta\varepsilon}(y) \asymp C \rho_{\beta\varepsilon}(x_{n,j})$ for each $y \in B_{n,j}$, with $C$ depending only on the data (since each ball $B_{n,j}$ has radius $R_0$). Furthermore, since there is some point $y \in A_n$ such that $|x_{n,j} y| \leq R_0$, it follows from the comparison (4.13) that $\rho_{\beta\varepsilon}(x_{n,j}) \asymp (e^{-n}r)^\beta$, which completes the proof.
with comparison constant depending only on the uniformization data. Thus we conclude that
\[
\mu_\beta(A_n \cap B_n,j) \leq \mu_\beta(B_n,j) \\
x \sim \rho_\beta(x_n,j) \mu(B_n,j) \\
\lesssim (e^{-n}\rho_\beta(B_n,j)) \\
\lesssim e^{-\beta n} \rho_\beta(z) \mu(B_X(z,R_1)),
\]
with implied constants depending only on the data. By our restriction $\beta \geq \beta_0 = 2\alpha$, we conclude that
\[
\mu_\beta(A_n \cap B_n,j) \lesssim e^{-2\alpha n} \rho_\beta(z) \mu(B_X(z,R_1)).
\]
It then follows from this inequality and the bound $N_n \lesssim e^{\alpha n}$ that
\[
\mu_\beta(B_{\varepsilon}(\xi,r)) \leq \int_{E} \left| u - u_{E} \right|^{p} d\mu \\
\lesssim \rho_\beta(z) \mu(B_X(z,R_1)) \sum_{n=1}^{\infty} e^{-\alpha n} \\
\lesssim \rho_\beta(z) \mu(B_X(z,R_1)) \sum_{n=1}^{\infty} N_n e^{-2\alpha n} \\
\lesssim \rho_\beta(z) \mu(B_X(z,R_1)) \\
\lesssim \rho_\beta(z) \mu(B_X(z,1)),
\]
with comparison constants depending only on the data, with the final inequality following by summing the geometric series. Finally, since $R_1$ depends only on the data and $R_1 \geq 1$, we have $\mu(B_X(z,R_1)) \sim \mu(B_X(z,1))$, from which it follows that
\[
\mu_\beta(B_{\varepsilon}(\xi,r)) \lesssim \rho_\beta(z) \mu(B_X(z,1)),
\]
with implied constant depending only on the data. As noted at the beginning of this proof, this implies the desired comparisons (4.9) for $\kappa = \kappa_0/6$ and $\kappa = \kappa_0/12$ by Lemma 4.10.

5. Global Poincaré Inequality

Let $(X,d,\mu)$ be a metric measure space. For a measurable subset $E \subset X$ satisfying $0 < \mu(E) < \infty$ and a function $u$ that is $\mu$-integrable over $E$ we write
\[
u_{E} = \frac{1}{\mu(E)} \int_{E} u \ d\mu
\]
for the mean value of $u$ over $E$. We record the following simple lemma for use later.

**Lemma 5.1.** [Lemma 4.17] Let $u : X \to \mathbb{R}$ be integrable, let $1 \leq p < \infty$, let $\alpha \in \mathbb{R}$, and let $E \subset X$ be a measurable set with $0 < \mu(E) < \infty$. Then
\[
\left( \int_{E} \left| u - u_{E} \right|^{p} d\mu \right)^{1/p} \leq 2 \left( \int_{E} \left| u - \alpha \right|^{p} d\mu \right)^{1/p}
\]

Let $u : X \to \mathbb{R}$ be given. We recall from the introduction that a Borel function $g : X \to [0,\infty]$ is an upper gradient for $u$ if for each rectifiable curve $\gamma$ joining two points $x,y \in X$ we have
\[
|u(x) - u(y)| \leq \int_{\gamma} g \ ds.
\]
A measurable function $u : X \to \mathbb{R}$ is integrable on balls if for each ball $B \subset X$ we have that $u$ is integrable over $B$. For a given $p \geq 1$ and $\lambda \geq 1$ we say that $X$ supports a $p$-Poincaré inequality if there are constants $\lambda \geq 1$ and $C_{PI} > 0$ such that for each measurable function $u : X \to \mathbb{R}$ that is integrable on balls, for each ball $B \subset X$, and each upper gradient $g$ of $u$ we have

$$
\int_B |u - u_B| \, d\mu \leq C_{PI} \text{diam}(B) \left( \int_{\lambda B} g^p \, d\mu \right)^{1/p},
$$

for a constant $C_{PI} > 0$. The constant $\lambda$ is called the dilation constant. If there is a constant $R_0 > 0$ such that (5.1) only holds on balls of radius at most $R_0$, then we will say that $X$ supports a $p$-Poincaré inequality on balls of radius at most $R_0$. We will also say that $X$ supports a uniformly local $p$-Poincaré inequality. For more details on Poincaré inequalities we refer to Section 8.

For this section we carry over the same standing hypotheses and notation as discussed at the start of Section 4. We will assume in addition that we are given $p \geq 1$ such that the Gromov hyperbolic space $X$ equipped with the uniformly locally doubling measure $\mu$ supports a $p$-Poincaré inequality on balls of radius at most $R_0$, where $R_0$ is the same radius up to which $\mu$ is doubling on $X$. We will also assume that $\mu_\beta$ is doubling on $\hat{X}_\varepsilon$ for some constant $C_{\mu_\beta}$. We will show under these hypotheses that the metric measure space $(\hat{X}_\varepsilon, d_\varepsilon, \mu_\beta)$ supports a $p$-Poincaré inequality with dilation constant $\lambda = 1$ and constant $C_{PI}^*$ depending only on the uniformization data and the constants $R_0$, $C_\mu$, $C_\beta$, $p$, $\lambda$, and $C_{PI}$. We will also assume that $\mu_\beta$ is doubling associated to the uniformly local doubling of $\mu$, the global doubling of $\mu_\beta$, and the uniformly local $p$-Poincaré inequality on $X$.

The proof splits into two steps. In the first step we show that (5.1) holds on sufficiently small subWhitney balls in $X_\varepsilon$. The proof is essentially identical to [2, Lemma 6.1]. In the statement and proof of Lemma 5.2 “the data” refers to the uniformization data and the constants $R_0$, $C_\mu$, $C_\beta$, $p$, $\lambda$, and $C_{PI}$. For Lemma 5.1 we do not need to assume that $\mu_\beta$ is doubling.

**Lemma 5.2.** There exists $c_0 > 0$ depending only on the uniformization data and $R_0$ such that for all $x \in X_\varepsilon$ and all $0 < r \leq c_0 d_\varepsilon(x)$ the $p$-Poincaré inequality (5.1) holds on the ball $B_\varepsilon(x,r)$ with dilation $\hat{\lambda} \geq 1$ and constant $\hat{C}_{PI}$ depending only on the data.

**Proof.** Put $B_\varepsilon = B_\varepsilon(x,r)$ with $0 < r \leq c_0 d_\varepsilon(x)$, where $c_0 > 0$ is a constant to be determined. Let $C_\varepsilon$ be the constant of Lemma 5.6. We choose $c_0 > 0$ small enough that $c_0 C_\varepsilon^2 \leq \frac{1}{2}$. We conclude by applying Lemma 3.6 twice that

$$
B_\varepsilon \subset B := B_X \left( x, \frac{C_\varepsilon r}{\rho_\varepsilon(x)} \right) \subset B_\varepsilon \left( x, C_\varepsilon^2 r \right) = \hat{\lambda} B_\varepsilon,
$$

with $\hat{\lambda} = C_\varepsilon^2$, since

$$
C_\varepsilon^2 r \leq c_0 C_\varepsilon^2 d_\varepsilon(x) \leq \frac{1}{2} d_\varepsilon(x).
$$

Moreover by (4.4) we see that for all $y \in \hat{\lambda} B_\varepsilon$ we have $\rho_\beta(y) \simeq \rho_\beta(x)$ with comparison constant depending only on the uniformization data.

Now let $u$ be a function on $X_\varepsilon$ that is integrable on balls and let $g_\varepsilon$ be an upper gradient of $u$ on $X_\varepsilon$. By the same basic calculation as in [2, (6.3)] we have that $g := g_\varepsilon \rho_\varepsilon$ is an upper gradient of $u$ on $X$. For $c_0$ sufficiently small (depending only on the uniformization data
and $R_0$) we will have by Lemma 8.3 that
\[ \frac{C_r}{\rho_\varepsilon(x)} \leq \frac{C_\epsilon c_0 d_\varepsilon(x)}{\rho_\varepsilon(x)} \leq R_0. \]

Thus the $p$-Poincaré inequality \((5.1)\) (for $\mu$) holds on $B$. Since $\rho_{\beta \varepsilon}(y) \asymp \rho_{\beta \varepsilon}(x)$ on $\lambda B_\varepsilon$ with comparison constant depending only on the uniformization data (by \((4.4)\)) we have that
\[ \mu_\beta(B) \asymp \rho_{\beta \varepsilon}(x) \mu(B), \]
with comparison constant depending only on the uniformization data, and the same comparison holds with either $B_x$ or $\lambda B_\varepsilon$ replacing $B$. Writing $u_{B_x} = \int_B u \, d\mu$, we conclude by using the inclusions of \((5.2)\), the measure comparison \((5.3)\), and the $p$-Poincaré inequality for $\mu$ on $B$,
\[
\int_{B_\varepsilon} |u - u_{B_x}| \, d\mu_\beta \lesssim \int_B |u - u_{B_x}| \, d\mu \\
\leq 2C_{p1}C_{\epsilon} r \left( \int_B g_\varepsilon \, d\mu \right)^{1/p} \\
\asymp \frac{r}{\rho_\varepsilon(x)} \left( \int_B (g_\varepsilon \rho_\varepsilon)^p \, d\mu_\beta \right)^{1/p} \\
\lesssim r \left( \int_{\lambda B_\varepsilon} g_\varepsilon^{p} \, d\mu_\beta \right)^{1/p},
\]
where all implied constants depend only on the data. By Lemma 5.1 we can replace $u_{B_x}$ with $u_{B_x, \mu_\beta} = \int_{B_x} u \, d\mu_\beta$ on the left to conclude the proof of the lemma. \qed

The second part of the proof is the following key proposition.

**Proposition 5.3.** [2, Proposition 6.3] Let $\Omega$ be an $A$-uniform metric space equipped with a globally doubling measure $\nu$ such that there is a constant $0 < c_0 < 1$ for which the $p$-Poincaré inequality \((5.1)\) holds for fixed constants $C_{p1}$ and $\lambda$ on all subWhitney balls $B$ of the form $B = B_\Omega(x, r)$ with $x \in \Omega$ and $0 < r \leq c_0 d_\Omega(x)$. Then the metric measure space $(\Omega, d, \nu)$ supports a $p$-Poincaré inequality with dilation constant $A$ and constant $C'_{p1}$ depending only on $A$, $c_0$, $C_{p1}$, $\lambda$, and the doubling constant $C_\nu$ for $\nu$.

This proposition is stated for bounded $A$-uniform metric spaces in [2] but the proof works without modification for unbounded $A$-uniform metric spaces provided that the doubling property of $\nu$ holds at all scales and the $p$-Poincaré inequality on sub-Whitney balls hold at all appropriate scales.

We can now verify the global $p$-Poincaré inequality on $\bar{X}_\varepsilon$. Below “the data” includes all the constants from Lemma 5.2 as well as the doubling constant $C_{\mu_\beta}$ for $\mu_\beta$.

**Proposition 5.4.** The metric measure spaces $(X_\varepsilon, d_\varepsilon, \mu_\beta)$ and $(\bar{X}_\varepsilon, d_\varepsilon, \mu_\beta)$ support a global $p$-Poincaré inequality with dilation constant $1$ and constant $C'_{p1}$ depending only on the data.

**Proof.** By Lemma 5.2 there is a $c_0 > 0$ determined only by the data such that the $p$-Poincaré inequality holds on subWhitney balls of the form $B_\varepsilon(x, r)$ with $0 < r \leq c_0 d_\varepsilon(x)$ for $x \in X$, with uniform constants $C_{\bar{p}_1}$ and $\bar{\lambda}$. Since $(X_\varepsilon, d_\varepsilon)$ is an $A$-uniform metric space with $A = A(\delta, K, \varepsilon, M)$ and we assumed $\mu_\beta$ is globally doubling on $X_\varepsilon$ with constant $\mu_\beta$, it follows from Proposition 5.3 that the metric measure space $(X_\varepsilon, d_\varepsilon, \mu_\beta)$ supports a $p$-Poincaré inequality with constant $C'_{p1}$ depending only on the data and dilation constant $A$. \[\square\]
Since $X_z$ is geodesic it follows that the $p$-Poincaré inequality \([5.1]\) in fact holds with dilation constant 1, with constant $C^p_1$ depending only on the data \([19]\) Theorem 4.18.

By \([21]\) Lemma 8.2.3 it follows that the completion $(\bar{X}_z, \bar{d}_\beta, \mu_\beta)$ (with $\mu_\beta(\partial X_z) = 0$) also supports a $p$-Poincaré inequality with constants depending only on the constants for the $p$-Poincaré inequality on $X_z$ and the doubling constant of $\mu_\beta$. Since $\bar{X}_z$ is also geodesic it follows by the same reasoning \([19]\) Theorem 4.18\] that we can take the dilation constant to be 1 in this case as well. \(\square\)

6. Hyperbolic fillings

Let $(Z, d)$ be a metric space and let $0 < a < 1$ and $\tau > 1$ be given parameters. We will assume these parameters satisfy

\[
\tau > \left\{ 3, \frac{1}{1-a} \right\}.
\]

We write $B_Z(z, r)$ for the ball of radius $r$ centered at $z$ in $Z$. We construct a hyperbolic filling $X$ of $Z$ as in \([1]\). A subset $S \subset Z$ is $r$-separated for a given $r > 0$ if for each $x, y \in S$ we have $d(x, y) \geq r$. For each $n \in Z$ we select a maximal $a^n$-separated subset $S_n$ of $Z$. Then for each $n \in Z$ the balls $B_Z(z, a^n)$, $z \in S_n$, cover $Z$.

The vertex set of $X$ has the form

\[
V = \bigcup_{n \in \mathbb{Z}} V_n, \quad V_n = \{(z, n) : z \in S_n\}.
\]

To each vertex $v = (z, n)$ we associate the dilated ball $B(v) := B_Z(z, \tau a^n)$. We define a projection $\pi : V \to Z$ by setting $\pi(z, n) = z$ and define the height function $h : V \to \mathbb{Z}$ by $h(z, n) = n$. For a vertex $v \in V$ we will sometimes write $v$ in place of $\pi(v)$ and consider $v$ both as a point of $Z$ and a vertex of $X$, except in places where this could cause confusion.

We place an edge between two vertices $v$ and $w$ if $|h(v) - h(w)| \leq 1$ and $B(v) \cap B(w) \neq \emptyset$. For vertices $v, w \in V$ we will write $v \sim w$ if there is an edge joining $v$ to $w$. We write $E$ for the set of edges in $X$. The resulting graph $X$ is connected by \([14]\) Proposition 5.5. We give $X$ the geodesic metric in which all edges have unit length. We will write $B_X(x, r)$ for the ball of radius $r > 0$ centered at $x \in X$, as in the previous sections. We extend the height function piecewise linearly to the edges of $X$ to define a 1-Lipschitz function $h : X \to \mathbb{R}$.

We say that an edge $e$ is vertical if it connects two vertices of different heights and horizontal if it connects two vertices of the same height. A vertical edge path in $X$ is a sequence of edges joining a sequence of vertices $\{v_k\}$ with $h(v_{k+1}) = h(v_k) + 1$ for each $k$ or $h(v_{k+1}) = h(v_k) - 1$ for each $k$. In the first case we say that the edge path is ascending and in the second case we say that the edge path is descending. We observe vertical edge paths are always geodesics in $X$ since edges in $X$ can only join two vertices of the same or adjacent heights. Thus we will also refer to vertical edge paths as vertical geodesics.

The following is an easy observation from the construction of $X$.

Lemma 6.1. \([14]\) Lemma 5.3\] Let $v, w \in V$ with $h(v) \neq h(w)$ and $B(v) \cap B(w) \neq \emptyset$. Then there is a vertical geodesic connecting $v$ to $w$.

The metric space $Z$ is doubling if there is an integer $D \geq 1$ such that for each $z \in Z$ and $r > 0$ any $r$-separated subset of $B_Z(z, 2r)$ has at most $D$ points. We say that the graph $X$ has bounded degree if there is an integer $N \geq 1$ such that each vertex is connected by an edge to at most $N$ other vertices. Note that if $X$ has bounded degree then it is proper, i.e., closed balls in $X$ are compact. These two concepts are closely linked to one another, as the following proposition shows.
Proposition 6.2. The metric space $Z$ is doubling if and only if the hyperbolic filling $X$ has bounded degree, and this equivalence is quantitative in the doubling constant, vertex degree, $a$, and $\tau$.

Proof. We give a proof that closely follows the proof of an analogous proposition in the work of Buyalo-Schroeder [15 Proposition 8.3.3]. We first show that $X$ has bounded degree if $Z$ is doubling. It’s an easy standard fact that if $Z$ is doubling with constant $D$ then there is a control function $T : [1, \infty) \to \mathbb{N}$, quantitative in $D$, such that for each $r > 0$ every ball of radius $sr$ contains at most $T(s)$ points that are $r$-separated (see [15 Exercise 8.3.1]). This follows by simply applying the doubling condition repeatedly across multiple scales.

Let $v \in V_n$ be any vertex and consider the associated ball $B(v)$. For each $m \in \mathbb{Z}$ we let $S_m(v) \subset S_m$ denote the set of points $z \in S_m$ such that the associated vertex $w \in V_m$ satisfies $v \sim w$. Then we can only have $S_m(v) \not= \emptyset$ when $|m-n| \leq 1$. If $w \in S_m(v)$ then $B(v) \cap B(w) \not= \emptyset$ and $m \leq n-1$. This implies that $d(\pi(v), \pi(w)) < 2\tau a^{n-1}$. Thus $S_{n-1}(v)$, $S_n(v)$, and $S_{n+1}(v)$ each form an $a^{n+1}$-separated set inside of the ball $B(\pi(v), 2\tau a^{n-1})$ and thus each have cardinality bounded by $T(2\tau a^{-2})$. We conclude that $X$ has bounded degree with degree bound $N = 3T(2\tau a^{-2})$.

Let’s now assume that $X$ has bounded degree, so that each vertex of $X$ is connected to at most $N$ other vertices. Consider a ball $B_Z(z, 2r)$ in $Z$. Let $k \in \mathbb{Z}$ be such that $a^{k+1} < 2r \leq a^k$. Let $l \in \mathbb{N}$ be the minimal integer such that $a^{l-1} \leq \frac{r}{2}$. Then any $r$-separated subset of $B_Z(z, 2r)$ is also $a^{k+l}$-separated. Let $v \in V_k$ be such that $d(z, \pi(v)) < a^k$. Then for $y \in B_Z(z, 2r)$ we have

$$d(y, \pi(v)) \leq d(y, z) + d(z, \pi(v)) < 2a^k < \tau a^k,$$

since $\tau > 3$, which implies that $B_Z(z, 2r) \subseteq B(v)$. It thus suffices to show that there is a uniform bound on the size of an $a^{k+l}$-separated subset of $B(v)$, quantitative in $N$, $a$, $l$, and $\tau$ (note $l$ is quantitative in $a$).

If $\{z_n\}$ is an $a^{k+l}$-separated subset of $B(v)$ then the balls $B_Z(z_n, a^{k+l})$ are all disjoint since $a^l < \frac{r}{2}$. We select for each $z_n$ a corresponding vertex $v_n \in V_{k+2l}$ such that $\pi(v_n) \in B_Z(z_n, a^{k+2l})$; these vertices are all distinct since these balls are disjoint. We then have

$$B_Z(z_n, a^{k+2l}) \subset B_Z(\pi(v_n), 2a^{k+2l}) \subset B(v_n),$$

since $\tau > 3$, which implies that the vertex $v_n$ then satisfies $B(v_n) \cap B(v) \not= \emptyset$. It thus suffices to produce a uniform bound on the number of vertices $w \in V_{k+2l}$ such that $B(w) \cap B(v) \not= \emptyset$. Given such a vertex $w$, since $B(w) \cap B(v) \not= \emptyset$ and $l \geq 1$, by Lemma 6.1 we can find a vertical geodesic from $w$ to $v$ of length $2l$. We conclude that any vertex $w \in V_{k+2l}$ with $B(w) \cap B(v)$ is joined to $v$ by a vertical geodesic of length $2l$. Since the number of vertices joined to $v$ by a vertical geodesic of length $2l$ is at most $N^{2l}$, this produces our desired bound. \qed

We will assume for the rest of this section that $Z$ is a complete doubling metric space, from which it follows that $X$ is a proper geodesic metric space. Note that the doubling condition on $Z$ implies that bounded subsets of $Z$ are totally bounded, from which it follows that any closed and bounded subset of $Z$ is compact. We conclude in particular that $Z$ is proper. The following consequence of the doubling condition will be used frequently in subsequent sections.

Lemma 6.3. Let $Z$ be doubling with doubling constant $D$. Let $S$ be an $r$-separated subset of $Z$ for a given $r > 0$. Then for any $\tau \geq 1$ and $z \in Z$ we have that $z \in B_Z(x, \tau r)$ for at most $D^l+1$ points $x \in S$, where $l$ is the minimal integer such that $2^l \geq \tau$. 

Proof. Suppose that \( z \in B_Z(x, \tau r) \) for some \( x \in S \). Then \( y \in B_Z(x, 2\tau r) \) for any other \( y \in S \) such that \( z \in B_Z(y, \tau r) \). Thus \( S \cap B_Z(x, 2\tau r) \) defines an \( r \)-separated subset of the ball \( B_Z(x, 2\tau r) \), which is also an \( r \)-separated subset of the ball \( B_Z(x, 2^{l+1}r) \). The doubling property then implies that the cardinality of \( S \cap B_Z(x, 2\tau r) \) is bounded above by \( D^{l+1} \). \( \square \)

Throughout the rest of this section all implied constants are considered to only depend on \( a \) and \( \tau \). By [14] Proposition 5.9 we have that the hyperbolic filling \( X \) is \( \delta \)-hyperbolic with \( \delta = \delta(a, \tau) \).

A vertical geodesic \( \gamma \) in \( X \) is anchored at a point \( z \in Z \) if for each vertex \( v \) on \( \gamma \) we have that \( z \in B(\pi(v), a^{h(v)}) \). Note this implies that \( z \in B(v) \). If we do not need to mention the point \( z \) then we will just say that \( \gamma \) is anchored. The following simple lemma shows that for any \( z \in Z \) we can find a geodesic line anchored at \( z \).

**Lemma 6.4.** For any \( z \in Z \) there is a vertical geodesic line \( \gamma: \mathbb{R} \to X \) anchored at \( z \). If \( v \in V \) satisfies \( z \in B(\pi(v), a^{h(v)}) \) then we can choose \( \gamma \) such that \( v \in \gamma \).

**Proof.** For each \( n \in Z \) the balls \( B(\pi(v), a^n) \) for \( v \in V_n \) cover \( Z \). Thus for each \( n \) we can choose a vertex \( v_n \in Z \) such that \( z \in B(\pi(v_n), a^n) \). Then \( z \in B(v_n) \) as well. It follows that \( B(v_{n+1}) \cap B(v_n) \neq \emptyset \) for each \( n \in Z \), i.e., \( v_n \sim v_{n+1} \). We can then find a vertical geodesic \( \gamma: \mathbb{R} \to X \) such that \( v_n \in \gamma \) for each \( n \). If \( v \in V \) is a given vertex such that \( z \in B(\pi(v), a^{h(v)}) \) then we can choose \( v_{h(v)} = v \) in this construction to guarantee that \( v \in \gamma \). \( \square \)

**Remark 6.5.** The definition of anchored geodesics in [14] is less restrictive, requiring only that \( z \in B(\pi(v), a^{h(v)}) \) for each \( v \in \gamma \) instead. This was because we were not assuming that \( Z \) was complete in that paper. We will use the more restrictive inclusion \( z \in B(\pi(v), a^{h(v)}) \) here instead, with the understanding that all of the claims proved regarding anchored geodesics in [14] hold for these geodesics in particular.

By [14] Lemma 5.11] all descending anchored geodesic rays in \( X \) are at a bounded distance from one another and therefore define a common point \( \omega \in \partial X \). By [14] Proposition 5.13] we have a canonical identification \( \partial_w X \cong Z \) that can be realized by identifying a point \( z \in Z \) with the collection of ascending geodesic rays anchored at \( z \). Under this identification the metric \( d \) on \( Z \) defines a visual metric on \( \partial_w X \) with parameter \( q = -\log a \). Furthermore by [14] Lemma 6.1] we have that \( X \) is \( \frac{1}{2} \)-roughly starlike from \( \omega \).

By [14] Lemma 5.12] there is a Busemann function \( b \) based at \( \omega \) such that the height function \( h \) satisfies \( h \leq 3 b \). Thus the height function \( h \) can be thought of as a Busemann function on \( X \) based at \( \omega \) up to an additive error of 3. We define the Gromov product based at \( h \) by

\[ (x|y)_h = \frac{1}{2}(h(x) + h(y) - 2|xy|), \]

for \( x, y \in X \). We then observe that \( (x|y)_h \) is \( 3 \) \( (x|y)_b \) as well. Since \( h \) is 1-Lipschitz we also have

\[ (x|y)_h \leq \min\{h(x), h(y)\}, \]

for \( x, y \in X \). The Gromov product based at \( h \) is then extended to \( \partial_w X \) by the same formulas (2.0) and (2.3) as were used for \( b \), and the same estimates (2.7) and (2.9) hold for this extension with \( h \) replacing \( b \). We have the following key estimate relating Gromov products based at \( h \) to the distance on \( Z \). We recall that all implied constants depend only on \( a \) and \( \tau \). We also recall that \( X \) is \( \delta \)-hyperbolic with \( \delta = \delta(a, \tau) \).
Lemma 6.6. [13, Lemma 5.6] For \( v, w \in V \) we have
\[
a^{(v|w)_\rho} = d(v, w) + \max\{a^{h(v)}, a^{h(w)}\}.
\]

The following observation will also be useful.

Lemma 6.7. For any vertex \( v \in V \) we have
\[
(v|\pi(v))_\rho \approx h(v).
\]

Proof. By Lemma 6.4 we can find an ascending geodesic ray \( \gamma : [0, \infty) \to X \) starting from \( v \) that is anchored at \( \pi(v) \). Then a straightforward calculation shows that we have \( (\gamma(t)|v)_\rho = h(v) \) for all \( t \geq 0 \). Since the sequence \( \{\gamma(n)\}_{n=0}^\infty \) converges to \( \pi(v) \in \partial_\omega X \cong Z \), the conclusion then follows from inequality (2.7).

We define a density \( \rho : X \to (0, \infty) \) by \( \rho(x) = a^{h(x)} \) and write \( X_\rho \) for the conformal deformation of \( X \) with conformal factor \( \rho \). We write \( d_\rho \) for the metric on \( X_\rho \). By [13, Theorem 1.9] the density \( \rho \) is admissible for \( X \) with constant \( M = M(a, \tau) \) and bounded geodesics in \( X \) are \( A \)-uniform curves in \( X_\rho \) with \( A = A(a, \tau) \). In particular \( X_\rho \) is an \( A \)-uniform metric space. Furthermore we have a canonical \( L \)-biLipschitz identification of \( \partial X_\rho \) with \( Z \), with \( L = L(a, \tau) \). Thus after a biLipschitz change of metric on \( Z \) we can assume that \( Z \) is isometrically identified with \( \partial X_\rho \). We will make this biLipschitz change of metric and thus consider \( Z \) as an isometrically embedded subset of \( X_\rho \) with \( Z = \partial X_\rho \). We will then use the notation \( Z \) and \( \partial X_\rho \) interchangeably for \( Z \), with the choice of notation depending on the context. For \( x \in X_\rho \) and \( r > 0 \) we will write \( B_\rho(x, r) \) for the ball of radius \( r \) centered at \( x \) in the metric \( d_\rho \).

Set \( \varepsilon = -\log a \). Let \( b \) be a Busemann function based at \( \omega \) such that \( b \eqsim h \) and define a density \( \rho_\varepsilon : X \to (0, \infty) \) on \( X \) by
\[
(6.2) \quad \rho_\varepsilon(x) = e^{-\varepsilon b(x)} = a^{b(x)}.
\]

Then \( X_\varepsilon \) is biLipschitz by the identity map to \( X_\rho \) with biLipschitz constant \( a^{-3} \). In particular \( \rho_\varepsilon \) is also an admissible density on \( X \) with constant \( M = M(a, \tau) \). We thus deduce that the results of Section 3 also hold for \( X_\rho \). In particular we have the following two key lemmas, in which the implied constants depend only on \( a \) and \( \tau \). We recall below that we define \( |x\bar{y}| = \infty \) if \( x \neq y \) and either \( x \in \partial_\omega X \) or \( y \in \partial_\omega X \), and set \( |x\bar{y}| = 0 \) if \( x = y \in \partial_\omega X \). Recall also that we have canonically identified \( \partial_\omega X \) with \( Z \).

Lemma 6.8. Let \( x, y \in X \cup \partial_\omega X \). Then we have
\[
d_\rho(x, y) \asymp a^{(x|y)_\rho} \min\{1, |x\bar{y}|\}.
\]

Lemma 6.9. For \( x \in X \) we have
\[
d_\rho(x) \asymp a^{h(x)}.
\]

In connection with Lemma 6.9 the following observation will be useful.

Lemma 6.10. For \( v \in V \) we have
\[
d_\rho(v, \pi(v)) \asymp d_\rho(v) \asymp a^{h(v)}.
\]

Proof. We trivially have \( d_\rho(v, \pi(v)) \geq d_\rho(v) \). On the other hand, by Lemma 6.4 we can find an ascending vertical geodesic ray \( \gamma : [0, \infty) \to X \) starting at \( v \) and anchored at \( \pi(v) \). A straightforward computation shows that
\[
\ell_\rho(\gamma) = \int_0^\infty a^{\gamma(t)h(v)} dt \lesssim a^{h(v)},
\]
from which it follows that \(d_p(v, \pi(v)) \lesssim a^{h(v)}\). The conclusion of the lemma then follows from Lemma 6.9 \(\square\)

We now introduce a concept inspired by work of Lindquist \[23\] Definition 4.10. Our definition is somewhat different from the one given there. For this definition we recall that balls are always considered to have a fixed center and radius, even if a particular subset can be described as a ball in multiple different ways. The quantity \(2r\) below comes from the upper bound \(\text{diam}(B) \leq 2r\).

**Definition 6.11.** Let \(B = B_Z(z, r)\) be any ball in \(Z\). The hull \(H^B \subset \mathcal{X}_p\) of \(B\) in \(\mathcal{X}_p\) is the union \(H^B = \bigcup \mathcal{B}_X(v, \frac{1}{2})\) over all vertices \(v \in V\) such that \(a^{h(v)} \leq 2r\) and \(B(v) \cap B \neq \emptyset\).

We consider \(H^B\) as being equipped with the uniformized metric \(d_p\). For \(n \in \mathbb{Z}\) we write \(H^B_n = H^B \cap V_n\) for the set of vertices in \(H^B\) at height \(n\).

Since balls are considered to come assigned with a center and radius, it may be the case for two balls \(B\) and \(B’\) in \(\mathcal{X}_p\) with different centers and radii that \(B = B’\) as sets but \(H^B \neq H^{B’}\). By construction each vertex \(v \in H^B\) has the property that \(a^{h(v)} \leq 2r\) and \(B(v) \cap B \neq \emptyset\), and for each \(x \in H^B\) there is a vertex \(v \in H^B\) such that \(|xv| \leq \frac{1}{2}\). An edge \(e\) in \(\mathcal{X}\) satisfies \(e \subset H^B\) if and only if the endpoints of \(e\) both belong to \(H^B\). For each \(n \in \mathbb{Z}\) such that \(a^n \leq 2r\) we have by construction that the balls \(B(v)\) for \(v \in H^B_n\) cover \(B\). In this paper we will primarily be using hulls of balls in \(\mathcal{X}\) as a convenient approximation to balls in \(\mathcal{X}_p\) centered at points of \(\mathbb{Z}\). We first show that the metric boundary of \(H^B\) coincides with the closure \(\bar{B}\) of \(B\) in \(\mathbb{Z}\).

**Lemma 6.12.** Let \(B \subset \mathcal{X}_p\) be any ball and let \(H^B \subset \mathcal{X}_p\) be its hull. Then

\[\partial H^B = \overline{\mathcal{B}_X^{(1)}} \setminus H^B = \bar{B} \subset \mathcal{X}_p.\]

**Proof.** Let \(r = r(B)\). Let \(\{x_n\} \subset H^B\) be any sequence for which there is some \(y \in \mathcal{X}_p\) such that \(d_p(x_n, y) \to 0\). If \(y \in \mathcal{X}_p\) then, since the metric \(d_p\) is locally bilipschitz to the hyperbolic metric on \(\mathcal{X}\), we must have \(|x_n y| \to 0\) as well. Since the closed balls \(\mathcal{B}_X(v, \frac{1}{2})\) for \(v \in V\) cover \(\mathcal{X}\), we can find a vertex \(v\) such that \(y \in \mathcal{B}_X(v, \frac{1}{2})\). If \(y \in \mathcal{B}_X(v, \frac{1}{2})\) then we must have \(x_n \in \mathcal{B}_X(v, \frac{1}{2})\) for \(n\) sufficiently large. Since \(x_n \in H^B\) this implies that \(v \in H^B\), from which it follows that \(y \in H^B\), contradicting our assumptions. If \(|yv| = \frac{1}{2}\) then we let \(w\) be the other vertex on the edge containing \(y\). Then for \(n\) sufficiently large we must have either \(x_n \in \mathcal{B}_X(v, \frac{1}{2})\) or \(x_n \in \mathcal{B}_X(w, \frac{1}{2})\). Thus we must have either \(v \in H^B\) or \(w \in H^B\), both of which imply that \(y \in H^B\). This again contradicts our assumptions.

Thus we must have \(y \in \mathcal{X}_p = \partial \mathcal{X}_p\). Then \(\{x_n\}\) converges to a point of \(\partial \mathcal{X}_p\) and therefore \(h(x_n) \to \infty\) as \(n \to \infty\). By the definition of \(H^B\) we can find a sequence of vertices \(\{v_n\} \subset H^B\) such that \(|x_n v_n| \leq \frac{1}{2}\). By Lemma 6.3 it follows that \(d_p(x_n, v_n) \to 0\), so we also have \(d_p(v_n, y) \to 0\). Set \(z_n = \pi(v_n)\). Then \(d_p(z_n, v_n) \lesssim a^{h(v_n)}\) by Lemma 6.10 and therefore \(d_p(z_n, y) \to 0\). But since \(B \cap \mathcal{B}_X(v_n) \neq \emptyset\), we can find points \(y_n\) in \(B \cap \mathcal{B}_X(v_n)\) for each \(n\) that satisfy \(d(y_n, z_n) < \tau a^{h(v_n)}\). This implies that \(d(y_n, y) \to 0\) as well, which implies that \(y \in \bar{B}\) since \(y_n \in B\) for each \(n\).

We conclude that \(\partial H^B \subset \bar{B}\). To obtain equality, let \(y \in \bar{B}\) be any point and let \(\gamma\) be an ascending vertical geodesic ray anchored at \(y\) as constructed in Lemma 6.4 starting from a vertex \(v_0 \in V_0\). Let \(\{v_n\}_{n \geq 0}\) be the sequence of vertices on \(\gamma\) with \(h(v_n) = n\). For each \(n\) we then have \(y \in B(v_n)\) by construction and therefore \(B(v_n) \cap B \neq \emptyset\) by the definition of the closure \(\bar{B}\). For \(n\) sufficiently large we will have that \(a^n \leq 2r\) and therefore \(v_n \in H^B\). Then \(d_p(v_n, y) \to 0\) since \(\gamma\) has \(y\) as its endpoint in \(\partial \mathcal{X}_p\). It follows that \(y \in \overline{H^B}\). Since \(H^B \subset \mathcal{X}_p\), we in fact have \(y \in \partial H^B\). Thus \(\partial H^B = \bar{B}\). \(\square\)
We remind the reader that in general $\bar{B} = \overline{B_Z(z,r)}$ may be a proper subset of the closed ball $B_Z(z,r) = \{x \in Z : d(x, z) \leq r\}$.

We next show that the closure of the hull of a ball $B \subset Z$ can be approximated by balls in $\bar{X}_\rho$ centered at the center of $B$. We then use this to show that balls in $\bar{X}_\rho$ centered at points of $Z$ can be approximated by the closures of hulls of balls in $Z$. For a ball $B = B_Z(z,r)$ in $Z$ we write $\bar{B} = B_\rho(z,r)$ for the corresponding ball centered at $z$ in $\bar{X}_\rho$.

**Lemma 6.13.** There is a constant $C = C(a, \tau) \geq 1$ such that if $B$ is any ball in $Z$ then

$$C^{-1}B \subset \overline{H^B} \subset CB.$$  

**Proof.** Let $B = B_Z(z,r)$ be a given ball. Let $0 < \lambda_0 < 1$ be a given parameter to be tuned in the proof and suppose that $v \in V$ is a vertex satisfying $d_\rho(v,z) < \lambda_0 r$. Then by Lemma 6.8 we have that

$$a^{(v|z)h} \lesssim d_\rho(v,z) < \lambda_0 r.$$ 

By inequality (2.9) it follows that

$$a^{h(v)} \lesssim \lambda_0 r,$$

as well. We have by inequality (6.4) and Lemma 6.10

$$d(\pi(v),z) \leq d_\rho(\pi(v),v) + d_\rho(v,z) \lesssim a^{h(v)} + \lambda_0 r \lesssim \lambda_0 r.$$

Thus $d(\pi(v),z) \leq C_0 \lambda_0 r$ and $a^{h(v)} \leq C_0 \lambda_0 r$ for a constant $C_0 = C_0(a,\tau) \geq 1$ depending only on $a$ and $\tau$. We set $\lambda_0 = C_0^{-1}/2$. It then follows that we have $d(\pi(v),z) < r$ and $a^{h(v)} \leq 2r$. This implies that $\pi(v) \in B$, which means that $B(v) \cap B \neq \emptyset$. Since $a^{h(v)} \leq 2r$ it then follows that $v \in H^B$.

We conclude that all vertices $v \in \lambda_0 \bar{B}$ satisfy $v \in H^B$. Let $0 < \lambda < \lambda_0$ be another given parameter. Let $x \in \lambda \bar{B} \cap X_\rho$ be any given point and let $v$ be a vertex satisfying $|xv| \leq \frac{1}{2}$. Then by Lemma 6.8 and inequality (6.4) (for $\lambda$ instead of $\lambda_0$) we have

$$d_\rho(v,x) \lesssim a^{(v|z)h} \lesssim a^{h(v)} \lesssim \lambda r,$$

from which it follows that

$$d_\rho(v,z) \leq d_\rho(v,x) + d_\rho(x,z) \lesssim \lambda r.$$ 

Thus there is a constant $C = C(a,\tau)$ such that $d_\rho(v,z) < C\lambda r$ for any vertex $v$ such that there is a point $x \in \lambda \bar{B}$ with $|xv| \leq \frac{1}{2}$. We set $\lambda = C^{-1}\lambda_0$. Then it follows that $v \in \lambda_0 \bar{B}$ and therefore $v \in H^B$. By the definition of the hull we then conclude that $x \in H^B$. Thus $\lambda \bar{B} \cap X_\rho \subset H^B$. Finally, since $\lambda < 1$ and $\partial X_\rho = Z$, if $x \in \lambda \bar{B} \cap \partial X_\rho = \lambda B$ then $x \in B$ and therefore $x \in H^B$ by Lemma 6.12. This proves the inclusion $\lambda \bar{B} \subset H^B$.

Now let $v \in H^B$ be any vertex. Let $y \in B(v) \cap B$ be a point in this intersection. Then

$$d(\pi(v),z) \leq d(\pi(v),y) + d(y,z) < \tau a^{h(v)} + r \lesssim r,$$

since $a^{h(v)} \leq 2r$. Then by Lemma 6.10

$$d_\rho(v,z) \leq d_\rho(v,\pi(v)) + d(\pi(v),z) \lesssim a^{h(v)} + r \lesssim r.$$ 

Thus there is a constant $C = C(a,\tau) \geq 1$ such that $d_\rho(v,z) \leq Cr$. It follows that $v \in C\bar{B}$.

If $x \in H^B$ is an arbitrary point then we can find a vertex $v \in H^B$ such that $|xv| \leq \frac{1}{2}$. Then $v \in C\bar{B}$ by our previous calculations. By Lemma 6.8 we have

$$d_\rho(x,v) \lesssim a^{(x|v)h} \lesssim a^{h(v)} \leq 2r.$$
Thus
\[ d_\rho(x, z) \leq d_\rho(x, v) + d_\rho(v, z) \lesssim r. \]
It follows that \( x \in CB \) as well, for a possibly larger constant \( C = C(a, \tau) \). Finally since \( \partial HB = \bar{B} \subset 2B \) by Lemma 6.12, we conclude that there is a constant \( C = C(a, \tau) \) such that \( HB \subset CB \). This completes the proof of the first assertion.

It remains to obtain the chain of inclusions (6.3). Let \( B \) be a given ball in \( Z \) and let \( C = C(a, \tau) \geq 1 \) be the constant obtained above. Applying the conclusion of the first part to the ball \( CB \) gives that \( \bar{B} \subset HB \subset HB^{-1} \subset \bar{B} \). This establishes the inclusions (6.3). \( \square \)

7. Lifting doubling measures

We start with a complete doubling metric measure space \((Z, d, \nu)\), meaning that \((Z, d)\) is a metric space equipped with a doubling Borel measure \( \nu \) satisfying \( 0 < \nu(B) < \infty \) for all balls \( B \) in \( Z \). We will write \( C_\nu \) for the doubling constant of \( \nu \). It’s easy to see for a doubling metric measure space \((Z, d, \nu)\) that the underlying metric space \((Z, d)\) is doubling with constant \( D = D(C_\nu) \), see for instance [21, Chapter 4.1]. Thus by Proposition 6.2 any hyperbolic filling \( X \) of \( Z \) with parameters \( 0 < a < 1 \) and \( \tau > \max\{3, (1 - a)^{-1}\} \) will have vertex degree bounded above by \( N = N(a, \tau, C_\nu) \).

We fix parameters \( 0 < a < 1 \) and \( \tau > \max\{3, (1 - a)^{-1}\} \) and let \( X \) be a hyperbolic filling of \( Z \) with these parameters as constructed in the previous section. We carry over all concepts and notation from the previous section. In particular we let \( X_\rho \) denote the conformal deformation of \( X \) with conformal factor \( \rho(x) = a^{h(x)} \) and isometrically identify \( Z \) with \( \partial X_\rho \) via a biLipschitz change of metric on \( Z \). Throughout the first part of this section (until Lemma 7.3) all implied constants will depend only on \( a, \tau \), and the doubling constant \( C_\nu \) for \( \nu \).

Our first task in this section will be to lift the doubling measure \( \nu \) to a uniformly locally doubling measure \( \mu \) on \( X \) that supports a uniformly local \( 1 \)-Poincaré inequality. To this end we adapt the construction in [3, Section 10]. As before we write \( V = \bigcup_{n \in \mathbb{Z}} V_n \) for the vertices of \( X \) and write \( E \) for the set of edges of \( X \). We let \( \mathcal{L} \) denote the Borel measure on \( X \) given by Lebesgue measure on each edge of \( X \), recalling that each edge of \( X \) has unit length. The measure \( \mathcal{L} \) can equivalently be thought of as the 1-dimensional Hausdorff measure on \( X \).

We define a measure \( \hat{\mu} \) on \( V \) by setting for each \( v \in V \),
\[
\hat{\mu}(\{v\}) = \nu(B(v)),
\]
where we recall that if \( v = (z, n) \) then \( B(v) = B_Z(z, \tau a^n) \). To simplify notation we will write \( \hat{\mu}(v) := \hat{\mu}(\{v\}) \). Our first lemma shows that adjacent vertices have comparable \( \hat{\mu} \)-measure.

Lemma 7.1. Let \( v, w \in V \) satisfy \( v \sim w \). Then
\[
\hat{\mu}(v) \asymp \hat{\mu}(w).
\]
Proof. By symmetry it suffices to verify the upper bound \( \hat{\mu}(v) \lesssim \hat{\mu}(w) \). Since \( v \sim w \) we must have \( |h(v) - h(w)| \leq 1 \), which implies that \( h(v) \geq h(w) - 1 \). Since \( B(v) \cap B(w) \neq \emptyset \), we must then have \( d(v, w) < 2\tau a^{h(w)-1} \). Thus if \( z \in B(v) \) then
\[
d(z, w) \leq d(v, w) + d(z, v) < 3\tau a^{h(w)-1}.
\]
Thus $B(v) \subset B_Z(w, 3\tau a^{b(w)}-1)$. Writing $3\tau a^{b(w)}-1 = 3a^{-1}(\tau a^{b(w)})$, the doubling condition on $\nu$ implies that

$$\nu(B(w)) \geq B_Z(w, 3\tau a^{b(w)}-1) \geq \nu(B(v)).$$

This implies that $\hat{\mu}(v) \lesssim \hat{\mu}(w)$. \qed

We next smear out $\hat{\mu}$ to a measure $\mu$ on $X$ by setting, for a Borel set $A \subset X$,

$$\mu(A) = \sum_{v \in V} \sum_{w \sim v} (\hat{\mu}(v) + \hat{\mu}(w)) \mathcal{L}(A \cap vw).$$

(7.2)

Here $vw$ denotes the edge connecting $v$ to $w$. By Lemma (6.3) and the fact that $X$ has vertex degree bounded by $N = N(a, \tau, C_\nu)$, we obtain the useful comparison for any vertex $v \in V$ and any $w \sim v$,

$$\nu(B(v)) = \hat{\mu}(v) \asymp \mu(vw).$$

We can now apply [3, Theorem 10.2] to directly obtain uniformly local doubling of $\mu$ and a uniformly local 1-Poincaré inequality on $X$.

**Proposition 7.2.** For each $R_0 > 0$ there is a constant $C_0 = C_0(a, \tau, C_\nu, R_0) \geq 1$ such that for all balls $B = B_X(x, r)$ in $X$ with $0 < r \leq R_0$ and every integrable function $u$ on $B$ with upper gradient $g$ on $B$ we have

$$\mu(2B) \leq C_0 \mu(B),$$

and

$$\int_B |u - u_B| \, d\mu \leq C_0 r \int_B g \, d\mu.$$

(7.4)

(7.5)

We let $\beta > 0$ be given and define a measure $\mu_\beta$ on $X_\rho$ by, for $x \in X$,

$$d\mu_\beta(x) = a^{\beta h(x)} d\mu(x).$$

(7.6)

We extend $\mu_\beta$ to a measure on $\bar{X}_\rho$ by setting $\mu_\beta(\partial X_\rho) = 0$. We observe that we can apply the results of Sections 4 and 5 in this setting, as if we set $\varepsilon = -\log a$ and let $b$ be a Busemann function on $X$ based at $\omega$ such that $b = \varepsilon h$ as in (6.2) then $X_{\varepsilon}$ will be biLipschitz to $X_{\rho}$ by the identity map with constant $a^{-3}$, as noted before. If we define a measure $\tilde{\mu}_\beta$ on $X$ by

$$d\tilde{\mu}_\beta(x) = e^{-\beta b(x)} d\mu_\beta(x) = a^{\beta b(x)} d\mu(x),$$

then the results of Sections 4 and 5 can be applied directly to $\tilde{\mu}_\beta$. But since

$$\frac{d\tilde{\mu}_\beta}{d\mu_\beta} \asymp a^{-3\beta},$$

it immediately follows that these results can be applied to $\mu_\beta$ as well. We note for applying these results that $X$ is $\delta$-hyperbolic with $\delta = \delta(a, \tau)$, that $K = \frac{1}{3}$, that $\varepsilon = -\log a$, and that $M = M(a, \tau)$ by [14, Theorem 1.9].

Our next goal will be to show that $\mu_\beta$ is $\partial$-controlled in the sense of Definition 4.6 for any $\beta > 0$. We start by estimating the measure of the hull of a ball in $Z$ and use this to estimate the measure of balls centered at the boundary. We recall for a ball $B = B_Z(z, r)$ in $Z$ that we write $\bar{B} = B_\rho(z, r)$ for the corresponding ball in $\bar{X}_\rho$. Throughout the rest of this section all implied constants will depend only on $a, \tau, C_\nu,$ and $\beta$. By the estimate (7.3) and the fact that $h$ is 1-Lipschitz, we obtain for any vertex $v \in V$ and any edge $e$ with $v \in e$,

$$\mu_\beta(e) \asymp a^{\beta h(e)} \nu(B(v)).$$

(7.7)
A half-edge $e_*$ in $X$ is a geodesic segment in $X$ of length $\frac{1}{2}$, starting from a vertex $v \in V$. By applying the definition (7.2) and using Lemma 7.1, we similarly obtain for any half-edge $e_* \subset X$:

$$\mu_B(e_*) \asymp a^{\beta h(v)} \nu(B(v)).$$  \hspace{1cm} (7.8)

Since $X$ has vertex degree bounded by $N = N(a, \tau, C_{\nu})$, we obtain the useful estimate for a vertex $v \in V$:

$$\mu_B \left( \bar{B}_X \left( v, \frac{1}{2} \right) \right) = \sum_{v \in e_*} \mu_B(e_*) \asymp a^{\beta h(v)} \nu(B(v)),$$  \hspace{1cm} (7.9)

where the sum is taken over all half-edges $e_*$ starting from $v$.

**Lemma 7.3.** Let $B$ be any ball in $Z$. Then we have

$$\mu_B(H^B) \asymp r^\beta \nu(B),$$  \hspace{1cm} (7.10)

and

$$\mu_B(\bar{B}) \asymp r^\beta \nu(B).$$  \hspace{1cm} (7.11)

**Proof.** We first obtain the comparison (7.10). Let $B = B_z(z, r)$ be the given ball. Let $m$ be the minimal integer such that $a^m \leq 2r$, so that we have $a^m \asymp r$. We then sum the estimate (7.8) over all vertices $v \in H^B$, noting that each point $x \in H^B$ satisfies $|xv| \leq \frac{1}{\tau}$ for at least one vertex $v \in H^B$ and at most two vertices. We then obtain that

$$\mu_B(H^B) \asymp \sum_{n=m}^{\infty} a^{\beta n} \left( \sum_{v \in H_n(B)} \nu(B(v)) \right).$$  \hspace{1cm} (7.12)

To estimate the inner sum on the right, note by Lemma 5.3 that there is a constant $M = M(a, \tau, C_{\nu})$ such that for any $x \in Z$ we have that $x$ belongs to at most $M$ balls $B(v)$ for $v \in V_n$. Furthermore each ball $B(v)$ for $v \in H^B$ has radius at most $2\tau r$ by the definition of $H^B$, so for each $n \in Z$ we must have that

$$B \subset \bigcup_{v \in H_n(B)} B(v) \subset 5\tau B.$$  \hspace{1cm} (7.13)

By the doubling property of $\nu$ and the bounded overlap of the balls $B(v)$ associated to vertices $v \in H_n(B)$ we conclude that

$$\sum_{v \in H_n(B)} \nu(B(v)) \asymp \nu(B).$$  \hspace{1cm} (7.14)

Applying this comparison, summing the resulting geometric series in (7.12), and then using $a^m \asymp r$ gives

$$\mu_B(H^B) \asymp r^\beta \nu(B),$$  \hspace{1cm} (7.15)

as desired. To obtain the corresponding result for $\bar{B}$, we use Lemma 6.13 together with the fact that $\mu_\beta(\partial X_\rho) = 0$ to obtain that

$$\mu_\beta(H^B) \leq \mu_\beta(\bar{B}) \leq \mu_\beta(H^B),$$  \hspace{1cm} (7.16)

with $C = C(a, \tau)$. By (7.10) and the doubling property for $\nu$ we have $\mu_\beta(H(C^{-1}B)) \asymp r^\beta \nu(B)$ and $\mu_\beta(H(CB)) \asymp r^\beta \nu(B)$, so the comparison for $\mu_\beta(\bar{B})$ follows.  \hspace{1cm} (7.17)

We let $\kappa_0 = \kappa_0(a, \tau)$ be the constant determined by Lemma 5.7. We recall the key Definition 4.6 for the measure $\mu_\beta$ to be $\partial$-controlled.
We thus obtain that \( \kappa \) on \( \mathbb{R}^5.4 \) together with Proposition 7.2 with \( C \) and therefore \( \nu \) property of \( B \).

Since \( \lambda(x) \geq C \), we observe that, since the edges of \( \nu \) the ball \( B \)

Let’s now analyze the right side of (7.13). Since \( d_\rho(x) \simeq r \), Lemma 6.9 gives us that \( a^{h(x)} \simeq r^\beta \). It thus suffices to find a constant \( C'_\nu(\kappa) \) satisfying the same conditions such that

\[
(7.13) \quad r^\beta \nu(B) \lesssim C'_\nu(\kappa) a^{h(x)} \mu(B_X(x,1))
\]

For this final inequality we observe that, since the edges of \( X \) have unit length, for any \( x \in X \) the ball \( B_X(x,1) \) must contain at least one half-edge \( e_\tau \) in \( X \). Let \( v_0 \) be the vertex on a half-edge \( e_\tau \) that is contained in \( B_X(x,1) \). Then by Lemma 7.1 and the formula (7.2) for the initial lift \( \mu \) of \( \nu \) to \( X \), we conclude that

\[
\nu(B_0) \approx \mu(e_\tau) \leq \mu(B_X(x,1)).
\]

Since \( a^{h(x)} \simeq r \) and \( h(x) = h(v_0) \), it then follows that \( a^{h(v_0)} \simeq r \). Let \( y = \pi(v_0) \in Z \) be the point underlying \( v \) in \( Z \). Then by the doubling property of \( \nu \) we have that \( \nu(B(v_0)) \approx \nu(B_Z(y,r)) \). We have from Lemma 6.8 and Lemma 6.10 that

\[
d_\rho(x,y) \lesssim d_\rho(x,v_0) + d_\rho(v_0,y) \lesssim a^{h(v_0)} \lesssim r,
\]

and therefore \( d_\rho(y,z) \lesssim r \) since \( x \in B_\rho(z,r) \). There is thus a constant \( C = C(a,\tau,C_\nu,\beta,\kappa) \geq 1 \) such that \( B_Z(y,r) \subset B_Z(z,Cr) \) and \( B_Z(z,r) \subset B_Z(y,Cr) \). This implies by the doubling property of \( \nu \) that \( \nu(B_Z(y,r)) \approx \nu(B_Z(z,r)) \), with comparison constant depending only on \( \kappa \). We conclude that inequality (7.14) holds with \( C'_\nu(\kappa) \) of the desired form.

We thus obtain that \( \mu_\beta \) is \( \partial \)-controlled for all \( \beta > 0 \) with constant \( C'_\nu(\kappa) \) depending only on \( \kappa \).

We now combine Proposition 7.2 with \( R_0 = 1 \) and Proposition 4.1.9 to conclude that the metric measure space \((X_\rho,d_\rho,\mu_\beta)\) is doubling with doubling constant \( C_\mu(\beta) \) depending only on \( \kappa \). a, \( C_\nu, \beta, \) and the values of the function \( \kappa \rightarrow C_\rho(\kappa) \) at \( \kappa = \kappa_0/6 \) and \( \kappa = \kappa_0/12 \). Since \( \kappa_0 = \kappa_0(a,\tau) \) and the function \( \kappa \rightarrow C_\rho(\kappa) \) depends only on \( a, \tau, C_\nu, \) and \( \beta, \) we conclude that the doubling constant \( C_\mu(\beta) \) depends only on \( a, \tau, C_\nu, \beta, \) and \( \beta \). Lastly we apply Proposition 5.4 together with Proposition 7.2 with \( R_0 = 1 \) to conclude that the metric measure spaces \((X_\rho,d_\rho,\mu_\beta)\) and \((X_\rho,d_\rho,\mu_\beta)\) each support a 1-Poincaré inequality with dilation constant 1 and constant \( C_\mu(\beta) \) depending only on \( a, \tau, C_\nu, \) and \( \beta. \)
We note the following corollary of Lemma 7.3 and Proposition 7.4 which provides an estimate for the $\mu_\beta$-measure of balls centered at any point of $X_\rho$. Closeness below is taken with respect to the metric $d_\rho$.

**Corollary 7.5.** Let $x \in X_\rho$ and $r > 0$ be given. If $r \geq d_\rho(x)$ then we let $z \in Z$ be a point closest to $x$, while if $r \leq d_\rho(x)$ then we let $v \in V$ be a vertex of $X_\rho$ nearest to $x$. Then in the case $r \geq d_\rho(x)$ we have

$$
\mu_\beta(B_\rho(x, r)) \approx r^\beta \nu(B_\rho(z, r)),
$$

while in the case $r \leq d_\rho(x)$ we have

$$
\mu_\beta(B_\rho(x, r)) \approx r d_\rho(x)^{\beta-1} \nu(B(v)).
$$

**Proof.** We first consider the case $r \geq d_\rho(x)$. In this case we have that $d_\rho(x, z) \leq r$ and therefore

$$
B_\rho(z, r) \subset B_\rho(x, r) \subset B_\rho(z, 3r).
$$

The desired estimate then follows from Lemma 7.3 and the fact that $\nu$ is doubling.

We now consider the case $r \leq d_\rho(x)$. We can then use Lemmas 7.3 and 6.8 to obtain for some constants $C_\tau = C_\tau(a, \tau)$ and $C_0 = C_0(a, \tau)$ that

$$
B_\rho \left( x, \frac{1}{2} r \right) \subset B_X \left( x, \frac{C_\tau r}{d_\rho(x)} \right) \subset B_X(x, C_0).
$$

Let $G$ be a minimal subgraph of $X$ such that $B_X(x, C_0) \subset G$. Since $X$ has vertex degree bounded by $N = N(a, \tau, C_\nu)$, $G$ has a number of edges $M = M(a, \tau, C_\nu)$ uniformly bounded in terms $a$, $\tau$, and $C_\nu$. Furthermore by Lemma 7.1 the measure $\mu_\beta$ restricted to $G$ is uniformly comparable to the measure $a^{\beta h(v)} \hat{\mu}(v)\mathcal{L}|_G$ with comparison constants depending only on $a$, $\tau$, $C_\nu$, and $\beta$, recalling that $\mathcal{L}$ denotes the measure on $X$ that restricts to Lebesgue measure on each edge of $X$. The ball $B_\rho \left( x, \frac{1}{2} r \right)$ can be written as a union of at most $M$ $d_\rho$-geodesics starting from $x$. These geodesic segments have length comparable to $ra^{-h(v)}$ in $X$ since $h(y) \approx h(v)$ for $y \in G$. Applying $\mu_\beta$ to these segments gives the upper bound

$$
B_\rho \left( x, \frac{1}{2} r \right) \lesssim r a^{(\beta-1)h(v)} \hat{\mu}(v).
$$

On the other hand the ball $B_\rho \left( x, \frac{1}{2} r \right)$ must contain at least one $d_\rho$-geodesic segment $\sigma$ of length $\frac{1}{2} r$ (with respect to $d_\rho$). Evaluating $\mu_\beta$ on $\sigma$ gives the lower bound,

$$
B_\rho \left( x, \frac{1}{2} r \right) \gtrsim r a^{(\beta-1)h(v)} \hat{\mu}(v).
$$

Applying Lemma 6.8 again, we obtain $a^{(\beta-1)h(v)} \approx a^{(\beta-1)h(x)} \approx d_\rho(x)^{\beta-1}$. Combining this with the equality $\hat{\mu}(v) = \nu(B(v))$ gives the second estimate (7.16). \qed

For any doubling measure $\nu$ on $Z$ there is always some $Q > 0$ such that we have for any $z \in Z$ and $0 < r' \leq r$,

$$
\nu(B(z, r')) \gtrsim \nu(B(z, r)) \geq C_{low}^{-1} \left( \frac{r'}{r} \right)^Q,
$$

for some constant $C_{low} \geq 1$. See \cite{21} Lemma 8.1.13. One may always take $Q = \log_2 C_\nu$, but it is possible that (7.17) holds for smaller values of $Q$. We say that $\nu$ has relative lower volume decay of order $Q$ if inequality (7.17) holds for all $z \in Z$, all $0 < r' \leq r$, and some implied constant. The exponent $Q$ functions as a kind of dimension for $\nu$, especially for the purpose of embedding theorems \cite{18} Section 5].
In the next lemma we obtain an estimate on the relative lower volume decay exponent for \( \mu_{\beta} \) in terms of the corresponding exponent for \( \nu \). Compare [3, Lemma 10.6].

**Lemma 7.6.** Suppose that \( \nu \) has relative lower volume decay of order \( Q > 0 \). Then \( \mu_{\beta} \) has relative lower volume decay of order \( Q_{\beta} = \max\{1, Q + \beta\} \) on \( \bar{X}_\rho \) with constant \( C_{\text{low}}' \) depending only on \( a, \tau, \beta, Q \), and the constant \( C_{\text{low}} \) in (7.17).

**Proof.** This is a direct consequence of the estimates of Corollary 7.5. Throughout this proof implied constants are allowed to also depend on \( Q \) and the constant \( C_{\text{low}}' \) in (7.17). Let \( x \in \bar{X}_\rho \) be given. If \( r \leq d_{\rho}(x) \) then \( r' \leq d_{\rho}(x) \) as well and applying the estimate (7.13) for both of them gives

\[
\frac{\mu_{\beta}(B_{\rho}(x, r'))}{\mu_{\beta}(B_{\rho}(x, r))} \leq \frac{r'}{r}.
\]

Thus in this case (7.17) holds with exponent 1.

Similarly if \( r' \geq d_{\rho}(x) \) then \( r \geq d_{\rho}(x) \) and we can apply the estimate (7.13) to both of them. This gives, for a nearest point \( z \in Z \) to \( x \) with respect to the metric \( d_{\rho} \),

\[
\frac{\mu_{\beta}(B_{\rho}(x, r'))}{\mu_{\beta}(B_{\rho}(x, r))} \leq \left(\frac{r'}{r}\right)^{Q + \beta} \frac{\nu(B(z, r'))}{\nu(B(z, r))} \geq \left(\frac{r'}{r}\right)^{Q + \beta}.
\]

Thus (7.17) holds with exponent \( Q + \beta \).

Finally we consider the case that \( r' \leq d_{\rho}(x) \) and \( r \geq d_{\rho}(x) \). We set \( r'' = d_{\rho}(x) \) and write

\[
\frac{\mu_{\beta}(B_{\rho}(x, r'))}{\mu_{\beta}(B_{\rho}(x, r))} = \frac{\mu_{\beta}(B_{\rho}(x, r''))}{\mu_{\beta}(B_{\rho}(x, r''))} \frac{\mu_{\beta}(B_{\rho}(x, r'))}{\mu_{\beta}(B_{\rho}(x, r'))}.
\]

The first case can be applied to the first ratio and the second case can be applied to the second ratio. This gives

\[
\frac{\mu_{\beta}(B_{\rho}(x, r'))}{\mu_{\beta}(B_{\rho}(x, r))} \geq \frac{r'}{r''} \left(\frac{r''}{r}ight)^{Q + \beta}.
\]

If \( Q + \beta \geq 1 \) then since \( r' \leq r'' \) this implies that

\[
\frac{r'}{r''} \left(\frac{r''}{r}ight)^{Q + \beta} \geq \left(\frac{r'}{r}ight)^{Q + \beta}.
\]

Thus (7.17) holds with exponent \( Q + \beta \) in this subcase. If \( Q + \beta \leq 1 \) then we instead use the fact that \( r'' \leq r \) to obtain

\[
\frac{r'}{r''} \left(\frac{r''}{r}ight)^{Q + \beta} \geq \frac{r'}{r}.
\]

Thus (7.17) holds with exponent 1 in this subcase. \( \square \)

Finally we remark that we always have \( \mu_{\beta}(X_{\rho}) = \infty \), even when \( \nu(Z) < \infty \). For each \( n \in \mathbb{Z} \) we let \( E_{n} \) denote the set of all edges in \( V \) which have at least one vertex in \( V_{n} \). We note that the definition of a metric measure space forces \( 0 < \nu(Z) \leq \infty \), since any ball \( B \subset Z \) must satisfy \( 0 < \nu(B) < \infty \).

**Proposition 7.7.** We have \( \mu_{\beta}(E_{n}) \asymp a^{\beta n} \nu(Z) \) for each \( n \in \mathbb{Z} \). Consequently \( \mu_{\beta}(X_{\rho}) = \infty \).

**Proof.** By the estimate (7.14) we have for any edge \( e \in E_{n} \) that \( \mu_{\beta}(e) \asymp a^{\beta n} \nu(B(v)) \) for a vertex \( v \in e \cap V_{n} \). Since \( X \) has vertex degree bounded by \( N = N(a, \tau, C_{\rho}) \), since each vertex
v ∈ V_n is attached to at least one edge, and since the balls B(v) for v ∈ V_n cover Z and have bounded overlap by Lemma 6.3 we conclude that
\[ \sum_{e ∈ E_n} μ_β(e) ≍ \sum_{v ∈ V_n} a^{2n} μ(B(v)) \approx a^{2n} μ(Z). \]
This proves the main estimate. The conclusion μ_β(X_β) = ∞ follows by letting n → −∞. □

8. Function spaces and capacities

In this section we review some material from [3, Section 9] and make some adjustments to deal with the fact that the spaces we will be considering are unbounded.

8.1. Newtonian spaces. For this first part we will be expanding on the discussion from the introduction; we refer to [21] for more details. We start with a metric measure space \((Y, d, μ)\). For \(p ≥ 1\) and a measurable function \(u : Y → [−∞, ∞]\) we will use the notation \(∥u∥_{L^p(Y)} = (\int_Y |u|^p dμ)^{1/p}\) for the \(L^p\) norm of \(u\), and write \(L^p(Y)\) for the associated \(L^p\) space on \(Y\). We will also say that functions \(u ∈ L^p(Y)\) are \(p\)-integrable. We write \(L^p_{loc}(Y)\) for the local \(L^p\) space on \(Y\), defined to be the set of all measurable functions \(u : Y → [−∞, ∞]\) such that \(u|_B ∈ L^p(B)\) for all balls \(B ⊂ Y\), where we consider \(B\) as being equipped with the restricted measure \(μ|_B\). For a subset \(G ⊂ Y\) we will always write \(χ_G\) for the characteristic function of \(G\).

Remark 8.1. In order to condense notation throughout the paper we will be omitting the measure from the notation for the function spaces that we consider. Thus we write \(L^p(Y) = L^p_{loc}(Y, μ)\), etc. We will always assume that \(p ≥ 1\) wherever it appears as an exponent, and we will always exclude the case \(p = ∞\). We will always be using a fixed choice of measure on each metric space that we consider. Balls \(B ⊂ Y\) in a metric measure space \((Y, d, μ)\) will always be considered as metric measure spaces \((B, d, μ|_B)\) with the restriction of the distance and measure on \(Y\) to \(B\). Similarly we will sometimes write “a.e.” for “\(μ\)-a.e.” when the measure \(μ\) is understood.

We next define the \(p\)-modulus for \(p ≥ 1\). Let \(Γ\) be a family of curves in \(Y\). A Borel function \(ρ : Y → [0, ∞]\) is admissible for \(Γ\) if for each curve \(γ ∈ Γ\) we have \(\int_γ ρ \, ds ≥ 1\). The \(p\)-modulus of \(Γ\) is then defined as
\[ \text{Mod}_p(Γ) = \inf_ρ \int_Y ρ^p \, dμ, \]
with the infimum taken over all admissible Borel functions \(ρ\) for \(Γ\). We say that \(Γ\) is \(p\)-exceptional if \(\text{Mod}_p(Γ) = 0\). A property \(P\) is said to hold for \(p\)-a.e. curve if the collection of curves for which \(P\) fails is \(p\)-exceptional. A subset \(G ⊂ Y\) is \(p\)-exceptional if the family of all nonconstant curves meeting \(G\) is \(p\)-exceptional.

We recall the definition (1.3) of upper gradients in the introduction. It is natural to relax this definition by allowing an exceptional set of curves on which the inequality (1.3) can potentially fail. For a function \(u : Y → [−∞, ∞]\) and an exponent \(p ≥ 1\) we say that a Borel function \(g : Y → [0, ∞]\) is a \(p\)-weak upper gradient for \(u\) if the upper gradient inequality (1.3) holds for \(p\)-a.e. curve in \(Y\). The following standard lemma shows that \(p\)-weak upper gradients of \(u\) are not far from being true upper gradients of \(u\) from the perspective of the \(L^p\) norm.

Lemma 8.2. [21 Lemma 6.2.2] Let \(u : Y → [−∞, ∞]\) be a function and suppose that \(g\) is a \(p\)-weak upper gradient of \(u\). Then there is a \(a\) a monotone decreasing sequence of upper gradients \(\{g_n\}\) of \(u\), with \(g_n ≥ g\) for each \(n\), such that \(∥g_n − g∥_{L^p(Y)} → 0\).
In particular if $u$ has a $p$-integrable $p$-weak upper gradient then it has a $p$-integrable upper gradient. We will be using Lemma \[8.2\] implicitly in many of the statements that follow.

If $u$ has a $p$-integrable upper gradient $g$ then it has a minimal $p$-weak upper gradient $g_u$ that satisfies $g_u \leq g$ a.e. for all $p$-integrable $p$-weak upper gradients $g$ of $u$. This minimal $p$-weak upper gradient is unique up to a set of measure zero. The Newtonian space $N^{1,p}(Y)$ is the space of all measurable functions $u : Y \to [-\infty, \infty]$ such that $\|u\|_{L^p(Y)} < \infty$ and such that $u$ has a $p$-weak upper gradient $g \in L^p(Y)$. This space comes equipped with the seminorm

$$\|u\|_{N^{1,p}(Y)} = \|u\|_{L^p(Y)} + \|g_u\|_{L^p(Y)},$$

where $g_u$ is a minimal $p$-weak upper gradient for $u$. Lemma \[8.2\] shows that this definition is equivalent to the previous definition \[11.4\] that we gave for the norm $\|u\|_{N^{1,p}(Y)}$. We emphasize that functions in $\tilde{N}^{1,p}(Y)$ are required to be defined pointwise everywhere, in contrast to what is typically required in the standard Sobolev space theory. We write $N^{1,p}(Y) = \tilde{N}^{1,p}(Y)/\sim$ be the quotient by the equivalence relation $u \sim v$ if $\|u-v\|_{N^{1,p}} = 0$. The space $N^{1,p}(Y)$ will also be referred to as the Newtonian space, and we will engage in the standard practice from the theory of $L^p$ spaces of not distinguishing the notation between a function $u \in \tilde{N}^{1,p}(Y)$ and its corresponding equivalence class $[u] \in N^{1,p}(Y)$. Equipped with the norm \[8.1\] the space $N^{1,p}(Y)$ is a Banach space \[25\]. As with $L^p$ and Sobolev spaces, we also define a local version $N_{loc}^{1,p}(Y)$ consisting of those functions $u : Y \to [-\infty, \infty]$ such that for each ball $B \subset Y$ we have $u|_B \in \tilde{N}^{1,p}(B)$.

The $C^p_Y$-capacity of a set $G \subset Y$ is defined as

$$C_p^Y(G) = \inf_u \|u\|_{N^{1,p}(Y)},$$

with the infimum taken over all functions $u \in \tilde{N}^{1,p}(Y)$ satisfying $u \geq 1$ on $G$. A property $P$ is said to hold quasi-everywhere (q.e.) if the set $G$ of points at which it fails has zero $p$-capacity, i.e., it satisfies $C_p^Y(G) = 0$. By \[21\] Proposition 7.2.8 a set $G$ has zero $p$-capacity if and only if $\mu(G) = 0$ and $G$ is $p$-exceptional. Two functions $u, v \in \tilde{N}^{1,p}(Y)$ satisfy $u \sim v$ if and only if $u = v$ quasi-everywhere \[21\] Proposition 7.1.31]. Furthermore if $u = v$ a.e. then $u = v$ q.e. Thus the $C^p_Y$-capacity captures the degree of ambiguity one is allowed in the definition of Newtonian functions in $N^{1,p}(Y)$.

The Dirichlet space $\tilde{D}^{1,p}(Y)$ consists of all measurable functions $u : Y \to [-\infty, \infty]$ such that $u$ has an upper gradient $g \in L^p(Y)$, or equivalently, such that $u$ has a $p$-weak upper gradient $g \in L^p(Y)$. We equip $\tilde{D}^{1,p}(Y)$ with the seminorm

$$\|u\|_{\tilde{D}^{1,p}(Y)} = \|g_u\|_p = \inf_g \|g\|_p,$$

with $g_u$ denoting a minimal $p$-weak upper gradient for $u$ and the infimum being taken over all $p$-integrable upper gradients of $u$, or equivalently, over all $p$-integrable $p$-weak upper gradients of $u$. This matches our first definition \[8.3\] by Lemma \[8.2\]. Similarly to $N^{1,p}(Y)$, we have for $u, v \in \tilde{D}^{1,p}(Y)$ that $u = v$ a.e. if and only if $u = v$ q.e. \[21\] Lemma 7.1.6]. As with $N^{1,p}(Y)$, in order to obtain a norm we set $\tilde{D}^{1,p}(Y) = \tilde{D}^{1,p}(Y)/\sim$, with $u \sim v$ if $\|u-v\|_{\tilde{D}^{1,p}(Y)} = 0$. We define a local version $\tilde{D}_{loc}^{1,p}(Y)$ of the Dirichlet space exactly as we did for the Newtonian space, writing $u \in \tilde{D}_{loc}^{1,p}(Y)$ for a measurable function $u : Y \to [-\infty, \infty]$ if $u|_B \in \tilde{D}^{1,p}(B)$ for each ball $B \subset Y$.

**Remark 8.3.** When $Y$ is proper the spaces $\tilde{N}_{loc}^{1,p}(Y)$ and $\tilde{D}_{loc}^{1,p}(Y)$ can equivalently be described as the set of measurable functions $u : Y \to [-\infty, \infty]$ such that each $x \in Y$ has an
open neighborhood \(U_x\) on which \(u|_{U_x} \in \hat{N}^{1,p}(U_x)\) or \(u|_{U_x} \in \hat{D}^{1,p}(U_x)\) respectively. This is the definition of the local spaces given in [21, Chapter 7]. The definition of the local spaces that we use here is more restrictive when \(Y\) is not proper. In the next section we will be applying this definition in the particular case that \(Y\) is incomplete.

In order to obtain further properties of these function spaces we need to make some additional assumptions on \(Y\). We will be assuming that \(Y\) is a geodesic metric space that contains at least two points, that \(\mu\) is a doubling measure on \(Y\), and that the metric measure space \((Y,d,\mu)\) supports a \(p\)-Poincaré inequality for all measurable functions \(u : Y \to [-\infty, \infty]\) that are integrable on balls, all \(p\)-integrable upper gradients \(g : Y \to [0, \infty]\) for \(u\), and all balls \(B \subset Y\),

\[
\left(\int_B |u - u_B| \, d\mu\right)^{1/p} \leq C_{PI} \text{diam}(B) \left(\int_B g^p \, d\mu\right)^{1/p},
\]

for a constant \(C_{PI} > 0\). This corresponds to the previously considered \(p\)-Poincaré inequality (5.1) with dilation constant \(\lambda = 1\). If \(Y\) is geodesic then the stronger inequality eqrefweak Poincare follows from the weaker inequality (5.1) [19, Theorem 4.18]. We note that if \((Y,d,\mu)\) supports a \(p\)-Poincaré inequality then \((Y,d,\mu)\) supports a \(q\)-Poincaré inequality for all \(q \geq p\) by Hölder’s inequality, with new constants depending only on \(p, q\), and \(C_{PI}\).

With these assumptions \(Y\) supports the following stronger form of the Poincaré inequality for any ball \(B \subset Y\), any integrable function \(u : B \to \mathbb{R}\), and any \(p\)-integrable upper gradient \(g\) of \(u\) in \(B\),

\[
\left(\int_B |u - u_B|^p \, d\mu\right)^{1/p} \leq C_0 \text{diam}(B) \left(\int_B g^p \, d\mu\right)^{1/p},
\]

with the constant \(C_0 > 0\) depending only on the constants of the \(p\)-Poincaré inequality for \(Y\) and the doubling constant for \(\mu\) [21, Remark 9.1.19]. This follows by applying Hölder’s inequality to a class of Sobolev-Poincaré inequalities for \(Y\) [21, Theorem 9.1.15] (see also [18]). This same application of Hölder’s inequality also shows that (8.3) similarly holds when \(u : B \to [-\infty, \infty]\) is defined only on \(B\) and \(g : B \to [0, \infty]\) is a \(p\)-integrable upper gradient of \(u\) on \(B\). We will usually use the inequality (8.4) in the reformulated form,

\[
\int_B |u - u_B|^p \, d\mu \leq C_0^p \text{diam}(B)^p \int_B g^p \, d\mu.
\]

In keeping with standard conventions, we will refer to both (8.3) and (8.4) as \((p,p)\)-Poincaré inequalities. We will use a generic constant \(C > 0\) in place of the specific constants \(C_{PI}\) and \(C_0\) in (8.3) and (8.4) when applying these inequalities.

The \((p,p)\)-Poincaré inequality (8.5) implies by a straightforward truncation argument that all functions \(u \in \hat{D}^{1,p}_{loc}(Y)\) are \(p\)-integrable over balls. The proof is very similar to the proof of [21, Lemma 8.1.5].

**Proposition 8.4.** Let \(B \subset Y\) be any ball and let \(u : B \to \mathbb{R}\) be a measurable function such that \(u\) has an upper gradient \(g : B \to [0, \infty]\) with \(g \in L^p(B)\). Then \(u \in L^p(B)\). Consequently \(\hat{D}^{1,p}(B) = \hat{N}^{1,p}(B)\) (as sets). Therefore \(\hat{D}^{1,p}_{loc}(Y) = \hat{N}^{1,p}_{loc}(Y)\) and \(\hat{D}^{1,p}(Y) \subset \hat{N}^{1,p}_{loc}(Y)\).

**Proof.** For each \(n \in \mathbb{N}\) we let \(u_n = \max\{-n, \min\{u, n\}\}\). Then \(|u_n| \leq n\) and consequently \(u_n\) is integrable over \(B\). Furthermore \(g\) is also an upper gradient for \(u_n\) on \(B\).
6.3.23]. Thus by the $p$-Poincaré inequality \([8.5]\) we have
\[
\int_B |u_n - (u_n)_B| \, d\mu \leq C \text{diam}(B) \left(\int_B g^p \, d\mu\right)^{1/p}.
\]
Since
\[
\int_B \int_B |u_n(x) - u_n(y)| \, d\mu(x)d\mu(y) \leq 2\int_B |u_n - (u_n)_B| \, d\mu,
\]
we can apply the monotone convergence theorem to the sequence of functions $\varphi_n : B \times B \to \mathbb{R}$ given by $\varphi_n(x, y) = |u_n(x) - u_n(y)|$ to obtain that
\[
\int_B \int_B |u(x) - u(y)| \, d\mu(x)d\mu(y) \leq 2C\text{diam}(B) \left(\int_B g^p \, d\mu\right)^{1/p} < \infty.
\]
It follows that the function $x \mapsto |u(x) - u(y)|$ is integrable over $B$ for a.e. $y \in B$, which immediately implies that $u$ is integrable over $B$. Applying the $(p, p)$-Poincaré inequality \([8.5]\) then gives $u \in L^p(B)$. We conclude that $\tilde{D}^{1, p}(B) = \tilde{N}^{1, p}(B)$. The equality $\tilde{D}_{\text{loc}}^{1, p}(Y) = N_{\text{loc}}^{1, p}(Y)$ then follows from the definitions. The inclusion $\tilde{D}^{1, p}(Y) \subset \tilde{N}_{\text{loc}}^{1, p}(Y)$ then follows from the inclusion $\tilde{D}^{1, p}(Y) \subset \tilde{D}^{1, p}(B)$. \(\square\)

The $(p, p)$-Poincaré inequality \([8.5]\) allows us to show that $D^{1, p}(Y)$ is a Banach space. We will actually be able to show that $D^{1, p}(B)$ is a Banach space for any ball $B \subset Y$ as well.

**Proposition 8.5.** The normed space $D^{1, p}(Y)$ is a Banach space. The same is true of $D^{1, p}(B)$ for any ball $B \subset Y$.

**Proof.** We prove the second claim first. By Proposition \([8.4]\) we have $\tilde{D}^{1, p}(B) = \tilde{N}^{1, p}(B)$ as sets. Since $\|u\|_{D^{1, p}(B)} \leq \|u\|_{N^{1, p}(B)}$ for any $u \in \tilde{D}^{1, p}(B)$, it follows that the quotient projection $N^{1, p}(Y) \to \tilde{D}^{1, p}(B)$ is continuous. Let $\{u_n\}$ be a Cauchy sequence in $D^{1, p}(B)$, and choose a sequence of representatives $\{\tilde{u}_n\} \subset \tilde{D}^{1, p}(B) = \tilde{N}^{1, p}(B)$. Since constant functions have norm 0 in $\tilde{D}^{1, p}(B)$, by adding an appropriate constant to $\tilde{u}_n$ for each $n$ we can assume that $\langle \tilde{u}_n \rangle_B = 0$ for all $n$.

For each $m, n \in \mathbb{N}$ we let $g_{m, n}$ be a minimal $p$-weak upper gradient of $\tilde{u}_m - \tilde{u}_n$ in $B$. Then inequality \([8.5]\) implies for $m, n \in \mathbb{N}$ that
\[
\int_B |\tilde{u}_m - \tilde{u}_n|^p \, d\mu \leq C \text{diam}(B)^p \int_B g_{m, n}^p \, d\mu.
\]
By hypothesis the right side converges to 0 as $m, n \to \infty$. Thus $\{\tilde{u}_n\}$ defines a Cauchy sequence in $L^p(B)$, which implies that $\{\tilde{u}_n\}$ defines a Cauchy sequence in $\tilde{N}^{1, p}(B)$. Since $N^{1, p}(B)$ is a Banach space, it follows that there is some function $\tilde{u} \in \tilde{N}^{1, p}(B)$ such that $\tilde{u}_n \to \tilde{u}$ in $\tilde{N}^{1, p}(B)$. Letting $u$ denote the projection of $\tilde{u}$ to $D^{1, p}(B)$, this implies that $u_n \to u$ in $D^{1, p}(B)$. It follows that $D^{1, p}(B)$ is a Banach space.

Now let $\{u_n\}$ be a Cauchy sequence in $D^{1, p}(Y)$ and choose a sequence of representatives $\{\tilde{u}_n\} \subset \tilde{D}^{1, p}(Y)$. Fix a point $x \in Y$ and for each $k \in \mathbb{N}$ let $B_k = B(x, k)$ be the ball of radius $k$ centered at $x$. By adding appropriate constants to $\tilde{u}_n$ we can arrange that $\langle \tilde{u}_n \rangle_{B_k} = 0$ for each $n$. For each $n, k \in \mathbb{N}$ we then set $\tilde{u}_{n, k} = \tilde{u}_n - (\tilde{u}_n)_{B_k}$. Since $\{\tilde{u}_{n, k}\}$ also defines a Cauchy sequence in $\tilde{D}^{1, p}(B_k)$, the argument in the first part of the proof then shows that for each $k \in \mathbb{N}$ there is a function $v_k \in \tilde{N}^{1, p}(B_k)$ such that $\|(\tilde{u}_{n, k} - v_k)\|_{B_k} \to 0$ as $n \to \infty$.

Restricting $\tilde{u}_{n, k}$ to $B_j$ for some $j < k$ shows that
\[
\|(\tilde{u}_{n, k} - v_k)\|_{N^{1, p}(B_j)} = \|(\tilde{u}_n - (\tilde{u}_n)_{B_k} - v_k)\|_{B_j} \to 0.
\]
Since \( \| (\tilde{u}_{n,j} - v_j) \|_{\mathcal{N}^1,p(B_j)} \to 0 \) as well, it follows by the triangle inequality that
\[
(8.6) \quad \lim_{n \to \infty} \| (\tilde{u}_n)_{B_k} - (\tilde{u}_n)_{B_k} - v_j) B_k \|_{\mathcal{N}^1,p(B_j)} = 0.
\]
Applying this to the special case \( j = 1 \), we obtain that
\[
\lim_{n \to \infty} \| (\tilde{u}_n)_{B_k} - v_k + v_1) B_k \|_{\mathcal{N}^1,p(B_k)} = 0.
\]
It follows that for each \( k \in \mathbb{N} \) there is a constant \( c_k \in \mathbb{R} \) such that \( v_k - v_1 = c_k \) a.e. on \( B_1 \) and \( c_k = \lim_{n \to \infty} (\tilde{u}_n)_{B_k} \). Note that \( c_1 = 0 \). Applying this to \( \tilde{B} \), we conclude that \( v_k - v_j = c_k - c_j \) a.e. on \( B_j \). We define \( \bar{u} : Y \to \mathbb{R} \) by setting \( \bar{u} = v_k - c_k \) on \( B_k \); since for \( j \leq k \) we have \( v_k - c_k = v_j - c_j \) a.e. on \( B_j \subset B_k \) it follows that \( \bar{u} \) is a well-defined measurable function on \( Y \).

It remains to show that \( \bar{u} \in \mathcal{D}^1(p)(Y) \) and that, denoting its projection to \( \mathcal{D}^1(p)(Y) \) by \( u \), we have \( u_n \to u \) in \( \mathcal{D}^1(p)(Y) \). By construction we have that \( u_n \to u \) in \( \mathcal{D}^1(p)(B_k) \) for each \( k \in \mathbb{N} \) since \( u_n |_{B_k} \sim \tilde{u}_{n,k} \) and \( \tilde{u}_{n,k} \sim v_k \) in \( \mathcal{D}^1(p)(B_k) \) for each \( k \). Letting \( g_k \) denote a minimal \( p \)-weak upper gradient of \( \tilde{u} \) in \( \mathcal{D}^1(p)(B_k) \), it follows that we will have
\[
\| g_k \|_{L^p(B_k)} \leq \limsup_{n \to \infty} \| u_n \|_{\mathcal{D}^1(p)(B_k)} \leq \limsup_{n \to \infty} \| u_n \|_{\mathcal{D}^1(p,y)} < \infty,
\]
with the final inequality following from the assumption that \( \{u_n\} \) is a Cauchy sequence in \( \mathcal{D}^1(p)(Y) \). By uniqueness of minimal \( p \)-weak upper gradients we have that \( g_j = g_k \) a.e. on \( B_j \) for each \( j < k \). We then extend \( g_k \) to \( Y \) by setting \( g_k(x) = 0 \) for \( x \notin B_k \) and define a Borel function \( g : Y \to [0, \infty] \) by \( g(x) = \sup_{n \in \mathbb{N}} g_k(x) \). Then \( g \) defines a \( p \)-weak upper gradient for \( u \) on \( Y \). Furthermore we have \( g = g_k \) a.e. on \( B_k \) for each \( k \). By the monotone convergence theorem applied to the sequence of functions \( g(k)(x) = \sup_{1 \leq j \leq k} g_j(x) \), we then conclude that
\[
\| g \|_{L^p(Y)} \leq \limsup_{n \to \infty} \| u_n \|_{\mathcal{D}^1(p,y)} < \infty.
\]
Thus \( \bar{u} \in \mathcal{D}^1(p)(Y) \).

We thus conclude that \( \bar{u} - \tilde{u}_n \in \mathcal{D}^1(p)(Y) \) for each \( n \in \mathbb{N} \). We let \( f_n \) be a minimal \( p \)-integrable \( p \)-weak upper gradient of \( \bar{u} - u_n \) on \( Y \) for each \( n \). To show that \( u_n \to u \) in \( \mathcal{D}^1(p)(Y) \) it suffices to show that \( \| f_n \|_{L^p(Y)} \to 0 \). For each \( k \in \mathbb{N} \) we have that \( f_n |_{B_k} \) is a minimal \( p \)-weak upper gradient of \( (\tilde{u}_{n,k} - v_k) |_{B_k} \). It follows that
\[
\| f_n \|_{L^p(B_k)} = \| (\tilde{u}_{n,k} - v_k) |_{B_k} \|_{\mathcal{D}^1(p)(B_k)} \leq \limsup_{m \to \infty} \| (\tilde{u}_{n,k} - \tilde{u}_{m,k}) |_{B_k} \|_{\mathcal{D}^1(p)(B_k)} = \limsup_{m \to \infty} \| (\tilde{u}_n - \tilde{u}_m) |_{B_k} \|_{\mathcal{D}^1(p)(B_k)} \leq \limsup_{m \to \infty} \| \tilde{u}_n - \tilde{u}_m \|_{\mathcal{D}^1(p)(Y)}.
\]
By the monotone convergence theorem it follows that
\[
\| f_n \|_{L^p(Y)} \leq \limsup_{m \to \infty} \| \bar{u}_n - \bar{u}_m \|_{\mathcal{D}^1(p,Y)}.
\]
The right side converges to 0 as \( n \to \infty \) since \( \{u_n\} \) is a Cauchy sequence in \( \mathcal{D}^1(p)(Y) \). We conclude that \( \| f_n \|_{L^p(Y)} \to 0 \) as \( n \to \infty \), which implies that \( u_n \to u \) in \( \mathcal{D}^1(p)(Y) \).

We next discuss quasicontinuity. A function \( u \) on \( Y \) is \( C^\gamma_p \)-quasicontinuous if for each \( \eta > 0 \) there is an open set \( U \subset Y \) such that \( C^\gamma_p(U) < \eta \) and \( u |_{Y \setminus U} \) is continuous. We will make use of the following theorem of Björn-Björn-Shanmugalingam.
Theorem 8.6. [1] Let \((Y, d, \mu)\) be a metric measure space that is complete, doubling, and supports a \(p\)-\(\text{Poincaré inequality}\). Then every \(u \in \tilde{N}^{1,p}(Y)\) is \(C_p^Y\)-quasicontinuous. Moreover \(C_p^Y\) is an outer capacity, i.e., for any subset \(G \subset Y\),

\[
C_p^Y(G) = \inf_U C_p^Y(U),
\]

with the infimum being taken over all open subsets \(U\) with \(G \subset U\).

The quasicontinuity statement of Theorem 8.6 can be extended to \(u \in \tilde{D}^{1,p}(Y)\). For this we introduce an important construction that we will also use in the next section. For a function \(\psi : Y \to \mathbb{R}\) defined on a metric space \((Y, d)\) we write

\[
\text{supp}(\psi) := \{y \in Y : \psi(y) \neq 0\},
\]

for the support of \(\psi\), defined as the closure of the points on which \(\psi\) is nonzero. We say that \(\psi\) has bounded support if \(\text{supp}(\psi)\) is a bounded subset of \(Y\), and we say that \(\psi\) is compactly supported if \(\text{supp}(\psi)\) is compact. These two notions coincide when \(Y\) is proper.

Proposition 8.7. [21, Lemma B.7.4] Let \((Y, d)\) be a doubling metric space with doubling constant \(D\). Let \(r > 0\) be given. Let \(\{y_n\}_{n \in J}\) be a maximal \(r\)-separated subset of \(Y\) indexed by \(J \subset \mathbb{N}\). Then there is a corresponding collection \(\{\psi_n\}_{n \in J}\) of functions \(\psi_n : Y \to [0, 1]\) such that for each \(n \in J\) we have

\[
\text{supp}(\psi_n) \subset B(y_n, 2r),
\]

and for each \(y \in Y\),

\[
\sum_{n \in J} \psi_n(y) = 1,
\]

and \(\psi_n\) is \(C r^{-1}\)-Lipschitz for each \(n \in S\) with \(C = C(D)\) depending only on the doubling constant \(N\).

The collection of functions \(\{\psi_n\}_{n \in J}\) will be referred to as a Lipschitz partition of unity. As noted after [10, (10)] the condition on \(\text{supp}(\psi_n)\) can be obtained by a slight modification of the proof in the reference. We remark that by Lemma 6.3 the sum (8.8) always has only finitely many nonzero terms for each fixed choice of \(y \in Y\). We also note that if \(Y\) is unbounded then we can always choose the index set \(J\) to satisfy \(J = \mathbb{N}\) by renumbering the indices.

Proposition 8.8. Let \((Y, d, \mu)\) be a geodesic metric measure space that is complete, doubling, and supports a \(p\)-\(\text{Poincaré inequality}\). Then every \(u \in \tilde{D}^{1,p}(Y)\) is \(C_p^Y\)-quasicontinuous.

Proof. If \(Y\) is bounded then \(\tilde{D}^{1,p}(Y) = \tilde{N}^{1,p}(Y)\) by Proposition 8.4 and the claim then follows from Theorem 8.6. We can thus assume that \(Y\) is unbounded. Let \(u \in \tilde{D}^{1,p}(Y)\) be given and let \(g\) be a \(p\)-integrable upper gradient for \(u\) on \(Y\). Let \(\{\psi_n\}_{n=1}^\infty\) be a Lipschitz partition of unity corresponding to a maximal \(1\)-separated subset \(\{y_n\}_{n=1}^\infty\) of \(Y\). Let \(B_n = B(y_n, 2)\). Then \(u \in L_p^f(B_n)\) for each \(n \in \mathbb{N}\) by Proposition 8.4.

Let \(L\) be the upper bound for the Lipschitz constants of \(\psi_n\) for \(n \in \mathbb{N}\) given by Proposition 8.7. We conclude that \(\psi_n u \in \tilde{D}^{1,p}(B_n)\) for each \(n \in \mathbb{N}\) since it is easy to see from the product rule for upper gradients [21, Proposition 6.3.28] that \(\psi_n g + Lu\) defines a \(p\)-integrable upper gradient of \(\psi_n u\) on \(B_n\). Applying Proposition 8.4 again, we conclude that \(\psi_n u \in N^{1,p}(B_n)\) and therefore \(\psi_n u \in \tilde{N}^{1,p}(Y)\) since \(\text{supp}(\psi_n) \subset B_n\). Thus each function \(\psi_n u\) is \(C_p^Y\)-quasicontinuous by Theorem 8.6.
Now let $\eta > 0$ be given. For each $n \in \mathbb{N}$ we can find an open set $U_n \subset Y$ with

$$C^Y_p(Y\setminus U_n) < 2^{-n-1}\eta,$$

such that $\psi_n u$ is continuous on $Y\setminus U_n$. Setting $U = \bigcup_{n=1}^{\infty} U_n$, we conclude that $U$ is an open subset of $Y$ for which we have $C^Y_p(Y\setminus U) < \eta$ since $C^Y_p$ is an outer measure on $Y$. Thus each function $\psi_n u$ is continuous on $Y\setminus U$. By Lemma [21, Lemma 7.2.4] we have $\psi_j = 0$ on $B_n$ for all but finitely many indices $j \in \mathbb{N}$. Since $u = \sum_{n \in \mathbb{N}} \psi_n u$, it follows that $u$ is continuous on $B_n\setminus U$ for each $n$ and therefore $u$ is continuous on $Y\setminus U$ since each point $x \in Y\setminus U$ has an open neighborhood $U_x$ with $U_x \subset B_n\setminus U$ for some index $n \in \mathbb{N}$. \hfill \Box

We will use the product rule for upper gradients frequently in the rest of the paper without further citation. We refer the reader to [21 Proposition 6.3.28] for the precise hypotheses under which the product rule holds; for our purposes it suffices to note that the product rule holds for products of functions $u$ and $v$ that each have locally $p$-integrable $p$-weak upper gradients $f$ and $g$ respectively, in which case any Borel representative of $\|g u| + f|v|\$ defines a $p$-weak upper gradient for the product $uv$ (note that the function $g u| + f|v|$ need not itself be $p$-integrable). The reference requires that $u$ and $v$ are absolutely continuous along $p$-a.e. compact curve in $Y$, which follows immediately from the local $p$-integrability of $f$ and $g$ and [21 Proposition 6.3.2] since each compact curve in $Y$ is contained in some ball $B \subset Y$ on which $f$ and $g$ are $p$-integrable.

8.2. Besov spaces. In this section we will be dropping the requirement that our space supports a Poincaré inequality, so we will be switching to different symbols $Z$ for the metric space and $\nu$ for the measure to reflect this. We will assume that $(Z, d, \nu)$ is a complete doubling metric measure space. As before we write $C_p$ for the doubling constant of the measure $\nu$. For a given $p \geq 1$ and $\theta > 0$ we recall for a function $u \in L^{p}_{\text{loc}}(Z)$ the definitions (1.1) and (1.2) of the Besov norms $\|u\|_{B^p_p(Z)}$ and $\|u\|_{B^{\theta, p}_p(Z)}$ of $u$. The space $B^0_p(Z)$ is the subspace of $L^{p}_{\text{loc}}(Z)$ for which we have $\|u\|_{B^0_p(Z)} < \infty$ and the space $\tilde{B}^0_p(Z) = L^p(Z) \cap B^0_p(Z)$ is characterized by the finiteness of the norm $\|u\|_{B^0_p(Z)}$.

We let $B^0_p(Z) = \tilde{B}^0_p(Z)/\sim$ be the quotient of $\tilde{B}^0_p(Z)$ by the equivalence relation $u \sim v$ if $\|u - v\|_{B^0_p(Z)} = 0$. It is easy to see from the definition (1.1) that $\|u\|_{B^0_p(Z)} = 0$ if and only if there is a constant $c \in \mathbb{R}$ such that $u \equiv c$ a.e.; hence we can equivalently characterize $B^0_p(Z)$ as being the quotient of $\tilde{B}^0_p(Z)/\sim$ by the equivalence relation $u \sim v$ if $u - v$ is constant a.e. on $Z$. It turns out that $B^0_p(Z)$ is a Banach space; we will deduce this from Proposition 5.5 in Proposition 11.1 at the end of the paper.

We define the Besov capacity $C^{Z}_{B^p_p} \equiv \text{capacity}$ for a subset $G \subset Z$

$$C^{Z}_{B^p_p}(G) = \inf_{u} \|u\|_{B^0_p(Z)}^p,$$

with the infimum being taken over all $u \in \tilde{B}^0_p(Z)$ such that $u \geq 1$ $\nu$-a.e. on a neighborhood of $G$. We have to take a neighborhood of $G$ due to the lack of pointwise control over functions $u \in \tilde{B}^0_p(Z)$. A function $u \in \tilde{B}^0_p(Z)$ is $C^{Z}_{B^p_p}$-quasicontinuous if for each $\eta > 0$ there is an open set $U \subset Z$ such that $C^{Z}_{B^p_p}(U) < \eta$ and $u|_{Z \setminus U}$ is continuous.

In connection with the Besov capacity, the following truncation lemma is useful:

**Lemma 8.9.** Let $f \in \tilde{B}^0_p(Z)$ be given and define

$$\hat{f} = \max\{0, \min\{1, f\}\}.$$
Then \( \hat{f} \in \dot{B}_p^\alpha(Z) \) with \( \|\hat{f}\|_{\dot{B}_p^\alpha(Z)} \leq \|f\|_{\dot{B}_p^\alpha(Z)} \) and \( \|\hat{f}\|_{\dot{B}_p^\alpha(Z)} \leq \|f\|_{\dot{B}_p^\alpha(Z)} \).

Proof. For \( \nu \)-a.e. \( x, y \in Z \) we have

\[
|\hat{f}(x) - \hat{f}(y)| \leq |f(x) - f(y)|.
\]

This immediately implies that \( \|\hat{f}\|_{\dot{B}_p^\alpha(Z)} \leq \|f\|_{\dot{B}_p^\alpha(Z)} \). Similarly we have \( |\hat{f}(x)| \leq |f(x)| \) for \( \nu \)-a.e. \( x \in Z \), which implies that \( \|\hat{f}\|_{L^p(Z)} \leq \|f\|_{L^p(Z)} \) and therefore \( \|\hat{f}\|_{\dot{B}_p^\alpha(Z)} \leq \|f\|_{\dot{B}_p^\alpha(Z)} \). \( \square \)

We have the following key estimate, which follows from the estimates of [17, Theorem 5.2] with \( \alpha^{-1} > 1 \) in place of 2; see also [3] Lemma 9.9 which gives the analogous estimate on a bounded space with \( \alpha = a^{-1} \).

**Lemma 8.10.** For any \( 0 < a < 1 \) and \( u \in B_p^\alpha(Z) \) we have

\[
\|u\|_{B_p^\alpha(Z)} \lesssim C \sum_{n \in \mathbb{Z}} \int_Z \int_{B_{2}(x,a^n)} \frac{|u(x) - u(y)|^p}{a^{np}} \, d\nu(y) \, d\nu(x),
\]

with \( C = C(a, p, \theta, C_\nu) \) depending only on \( a, p, \theta, \) and the doubling constant \( C_\nu \) of \( \nu \).

We have the following useful consequence of Lemma 8.10 which shows for \( 0 < \theta < 1 \) that Lipschitz functions on \( Z \) with bounded support belong to \( \dot{B}_p^\alpha(Z) \).

**Lemma 8.11.** Suppose that \( 0 < \theta < 1 \). Let \( u : Z \to \mathbb{R} \) be an L-Lipschitz function for which there is some ball \( B = B_{2}(z,R) \) such that \( \text{supp}(u) \subset B \). Then

\[
(8.9) \quad \|u\|_{B_p^\alpha(Z)} \lesssim C L(R^{1-\theta} + R^\theta) \nu(B)^{1/p},
\]

with \( C = C(p, \theta, C_\nu) \). In particular \( u \in \dot{B}_p^\alpha(Z) \).

Proof. Let \( k \in \mathbb{Z} \) be the greatest integer such that \( 2^{-k} \geq R \). Observe that we then have \( R \approx 2^{-k} \). For \( m \geq 1 \) we put

\[
A_m = \{ x \in Z : 2^{m-k} \leq d(x,z) < 2^{m-k+1} \}.
\]

We apply Lemma 8.10 with \( a = \frac{1}{2} \) to obtain

\[
\|u\|_{B_p^\alpha(Z)} \lesssim C \sum_{n \in \mathbb{Z}} \int_Z \int_{B_{2}(x,2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x)
\]

\[
= \sum_{n \in \mathbb{Z}} \int_{B_{2}(z,2^{-k+1})} \int_{B_{2}(x,2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x)
\]

\[
+ \sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_{A_m} \int_{B_{2}(x,2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x),
\]

with \( C = C(p, \theta, C_\nu) \). For \( x \in B_{2}(z,2^{-k+1}) \) and \( n \geq k - 1 \) we then have from the fact that \( u \) is L-Lipschitz,

\[
\int_{B(x,2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \leq L p 2^{n(\theta - 1)p}.
\]
For $n \leq k - 2$ we observe that $B_Z(z, R) \subset B_Z(x, 2^{-k+2})$ since $x \in B_Z(z, 2^{-k+1})$ and $2^{-k} \geq R$. Thus $u(y) = 0$ for $y \notin B_Z(x, 2^{-k+2})$. It follows that

$$\int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) = \frac{1}{\nu(B_Z(x, 2^{-n}))} \int_{B_Z(x, 2^{-k+2})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) \leq \nu(B_Z(x, 2^{-k+2})) \int_{B_Z(x, 2^{-k+2})} \frac{|u(x) - u(y)|^p}{L^{p\theta}} d\nu(y) \leq L^{p\theta} \frac{\nu(B_Z(x, 2^{-n}))}{\nu(B_Z(x, 2^{-k+2}))} \int_{B_Z(x, 2^{-k+2})} \frac{|u(x) - u(y)|^p}{L^{p\theta}} d\nu(y),$$

since $\nu(B_Z(x, 2^{-k+2})) \leq \nu(B_Z(x, 2^{-n}))$ because $n \leq k - 2$. Thus, since $0 < \theta < 1$ and $R \approx 2^{-k}$, we have for $x \in B_Z(z, 2^{-k+1}),$

$$\sum_{n \in \mathbb{Z}} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) \leq \sum_{n=k-1}^{\infty} L^{p\theta (\theta - 1)} + \sum_{n=-\infty}^{k-2} L^{p\theta (n+k-1)/2} \leq C L^{p(2(\theta - 1)/2 + 2^{-\theta p})} \leq C L^{p(R^{-\theta - 1})p + \theta p},$$

with $C = C(p, \theta)$ depending only on $p$ and $\theta$. It follows that

$$(8.10) \sum_{n \in \mathbb{Z}} \int_{B_Z(x, 2^{-k+1})} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) d\nu(x) \leq C L^{p(R^{-\theta - 1})p + \theta p}) \nu(B),$$

with $C = C(p, \theta, C_p)$, using the doubling property of $\nu$ to obtain that

$$\nu(B_Z(z, R)) \approx \nu(B_Z(z, 2^{-k+1})),$$

since $R \approx 2^{-k}$.

We now split into two cases. The first case is that in which $Z = B_Z(z, R)$. In this case we have $A_m = \emptyset$ for each $m \geq 1$ since $d(x, z) < R < 2^{-k+1}$ for all $x \in Z$. We thus derive from (8.10) that

$$\|u\|_{L^p(B_Z(x, 2^{-n}))} \leq C L^{p(R^{-\theta - 1})p + \theta p}) \nu(B),$$

with $C = C(p, \theta, C_p)$. Taking the $p$th root of each side and using the inequality $(s + t)^{1/p} \leq s^{1/p} + t^{1/p}$ for real numbers $s, t \geq 0$ then gives (S.4).

We now consider the case $Z \notin B_Z(z, R)$. Let $x \in A_m$ for some $m \geq 1$. Then $u(x) = 0$. If $y \in B_Z(x, 2^{-k+m-1}) \cap B_Z(z, 2^{-k})$ then

$$d(x, z) \leq d(x, y) + d(y, z) < 2^{-k+m-1} + 2^{-k} \leq 2^{-k+m-1} + 2^{-k+m-1} = 2^{-k+m},$$

contradicting the definition of $A_m$. Thus $B_Z(x, 2^{-n}) \cap B_Z(z, 2^{-k}) = \emptyset$ whenever $n \geq k - m + 1$. Since $R \leq 2^{-k}$, this implies that

$$\int_{A_m} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) d\nu(x) = 0,$$

for $n \geq k - m + 1$. Since $u(x) = 0$, for $n < k - m + 1$ we can use the upper bound

$$\int_{A_m} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} d\nu(y) d\nu(x) \leq 2^{n\theta p} \|u\|_{L^p(B_Z(x, 2^{-n})))}.$$
Since for \( x \in A_m \) we have \( A_m \subset B_Z(x, 2^{m-k+1}) \subset B_Z(x, 2^{m-k+2}) \), it follows from the doubling property of \( nu \) and the fact that \( n \leq k - m \) that
\[
\frac{\nu(A_m)}{\nu(B_Z(x, 2^{-n}))} \leq \frac{\nu(B_Z(x, 2^{m-k+2}))}{\nu(B_Z(x, 2^{-k+m}))} \leq C_{\nu}^2,
\]
and therefore
\[
\int_{A_m} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x) \leq C_{\nu}^2 2^{n\theta p} \|u\|_{L^p(Z)}^p.
\]
Summing the geometric series over \( n \leq k - m \) then gives
\[
\sum_{n \in \mathbb{Z}} \int_{A_m} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x) \lesssim C \, 2^{(k-m)\theta p} \|u\|_{L^p(Z)}^p,
\]
with \( C = C(p, \theta, C_{\nu}) \). Summing this geometric series over \( m \geq 1 \) and recalling that \( R \approx 2^{-k} \) then gives
\[
\sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_{A_m} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x) \lesssim_C \, R^{-\theta p} \|u\|_{L^p(Z)}^p,
\]
with \( C = C(p, \theta, C_{\nu}) \). Since \( u \) is \( L \)-Lipschitz on \( Z \) and since \( u(z) = 0 \) for \( z \in Z \setminus B_Z(z, R) \) (note we’ve assumed in this case that \( Z \setminus B_Z(z, R) \neq \emptyset \)), we conclude that \( |u(z)| \leq 2LR \) for \( z \in Z \). It follows that
\[
\|u\|_{L^p(Z)}^p \leq 2^p L^p R^p \nu(B).
\]
We thus conclude that
\[
\sum_{m=1}^{\infty} \sum_{n \in \mathbb{Z}} \int_{A_m} \int_{B_Z(x, 2^{-n})} \frac{|u(x) - u(y)|^p}{2^{-n\theta p}} \, d\nu(y) \, d\nu(x) \lesssim_C \, L^p R^{(1-\theta)p} \nu(B),
\]
with \( C = C(p, \theta, C_{\nu}) \). Combining the estimates (8.10) and (8.12) together and then taking the \( p \)th root of each side gives the desired estimate for \( \|u\|_{B^p(Z)}^p \).

9. Trace theorems

In this section we carry over the concepts and notation from Section 7; we refer the reader back to the start of Section 7 for an overview of the setting and notation. Since we showed in Proposition 7.4 that the geodesic doubling metric measure spaces \((X_p, d_p, \mu_\beta)\) and \((\bar{X}_p, d_p, \mu_\beta)\) each support a 1-Poincaré inequality, they also support a \( p \)-Poincaré inequality for each \( p \geq 1 \). Hence the results of the previous section apply to the metric measure spaces \((X_p, d_p, \mu_\beta)\) and \((\bar{X}_p, d_p, \mu_\beta)\).

We will generalize the trace results of [3, Section 11] to the case of a potentially unbounded complete doubling metric space \( Z \). Our treatment of the trace results in this context will be slightly different than that of [3], falling closer in spirit to the work of Bonk-Saksman [10] in that we will express the trace as a \( \nu \)-a.e. limit of Lipschitz functions on \( Z \). Throughout this section our parameter \( \beta \) corresponds to the ratio \( \beta/\varepsilon \) in [3, Section 11].

Remark 9.1. The statements from [3, Remark 9.5] carry through without modification to our setting, as they only rely on the fact that the metric measure space \((X_p, d_p, \mu_\beta)\) is a metric graph with \( \mu_\beta \) being comparable to a multiple of Lebesgue measure on each edge. Given \( p \geq 1 \), the only family of nonconstant compact rectifiable curves in \( X_p \) with zero \( p \)-modulus (with respect to \( \mu_\beta \)) is the empty family. Thus any \( p \)-weak upper gradient for \( u \) on \( X_p \) is an upper gradient for \( u \). Since functions in \( N^1_{\text{loc}}(X_p) \) are absolutely continuous,
along $p$-a.e. curve \cite[Proposition 6.3.2]{21} this also implies that any function $u \in \tilde{N}_{loc}^{1,p}(X_\rho)$ is continuous on $X_\rho$ and absolutely continuous along each edge of $X_\rho$. Restricted to each edge the minimal upper gradient $g_u$ of $u$ is given by $g_u = \left| \frac{du}{ds_\rho} \right|$, with $ds_\rho$ denoting arclength with respect to the distance $d_\rho$ and $\frac{du}{ds_\rho}$ denoting the metric differential of $u$ on this edge given by the absolute continuity of $u$ on this edge (see \cite[Theorem 4.4.8]{21}). All points in $X_\rho$ have positive capacity, and each equivalence class in $N^{1,p}(X_\rho) = N^{1,p}(X_\rho)$ consists of a single function that is continuous. Lastly all of these statements remain true of any metric subgraph of $X_\rho$, equipped with the restriction of the measure $\mu_\beta$ to this subgraph.

The following invaluable inequality is an immediate consequence of the convexity of the function $t \to t^p$ for $p \geq 1$ on $[0, \infty)$. We will use it throughout this section largely without comment.

Lemma 9.2. Let $\{x_i\}_{i=1}^k$ be nonnegative real numbers. Then for any $p \geq 1$ we have

$$\left( \sum_{i=1}^k x_i \right)^p \leq k^{p-1} \sum_{i=1}^k x_i^p.$$  

Using Proposition \cite{87} we fix, for each $n \in \mathbb{Z}$, a Lipschitz partition of unity $\{\psi_i\}_{i \in V_n}$ associated to the $a^n$-separated subset $V_n$ of $\mathbb{Z}$, considering these vertices of $X$ as points of $\mathbb{Z}$. Since we require $\tau > 3$ the condition \cite{87} implies that $\text{supp}(\psi_v) \subset B(v)$ for all $v \in V$. Since $0 \leq \psi_v \leq 1$ we have the bound

$$\|\psi_v\|_{L^1(\mathbb{Z})} \leq \nu(B(v)).$$  

We make some additional definitions here for future reference. For $n \in \mathbb{Z}$ we write $X_{\geq n} = X \cap h^{-1}([n, \infty))$ for the set of all points in $X$ of height at least $n$ and write $X_{\leq n} = X \cap h^{-1}((-\infty, n])$ for the set of all points in $X$ of height at most $n$. We consider each of these subsets as being equipped with the metric $d_\rho$. For a ball $B \subset \mathbb{Z}$, its hull $H^B \subset X_\rho$, and any integer $n \in \mathbb{Z}$ we then set $H^B_{\geq n} = H^B \cap X_{\geq n}$ and $H^B_{\leq n} = H^B \cap X_{\leq n}$.

All implied constants throughout this section will depend only on $a, \tau, C_\nu, \beta$, and the exponent $p \geq 1$. We will always assume that $p > \beta$. By Remark 9.1 we have $\tilde{N}^{1,p}_{loc}(X_\rho) = N^{1,p}_{loc}(X_\rho)$ and consequently $\tilde{N}^{1,p}(X_\rho) = N^{1,p}(X_\rho)$.

Our first proposition constructs the trace of a Newtonian function defined on the hull $H^B \subset X_\rho$ of a ball $B \subset \mathbb{Z}$. We recall from Lemma 6.12 that for a ball $B \subset \mathbb{Z}$ we have $\partial H^B = \partial B$. We will need the following lemma.

Lemma 9.3. Let $e \in E$ be any edge of $X$ and let $v \in e$ denote either vertex on $e$. Then restricted to $e$ we have the comparison

$$ds_\rho|_e \asymp \frac{a^{(1-\beta)\rho(v)} d\mu_\beta|_e}{\nu(B(v))},$$

where $ds_\rho$ denotes arc length with respect to the metric $d_\rho$.

Proof. Let $e \in E$ be a given edge and let $v \in e$ denote a vertex on $e$. By the definition of $X_\rho$ we have on $e$,

$$ds_\rho|_e \asymp a^{\rho(v)} d\ell|_e,$$

where we recall that $\ell$ is the measure on $X$ given by Lebesgue measure on each unit length edge of $X$. On $e$ we also have by the definition \cite{72} of the measure $\mu$ and by Lemma 7.7

$$d\ell|_e \asymp \frac{d\mu|_e}{\mu(v)} = \frac{d\mu|_e}{\nu(B(v))} \asymp \frac{d\mu_\beta|_e}{a^{\beta\rho(v)} \nu(B(v))}.$$
The comparison (9.2) follows by combining the comparisons (9.3) and (9.4).

For defining $T_n u$ in (9.5) below we recall that functions in Newtonian spaces are required to be defined pointwise everywhere; in this particular case the functions are actually continuous by Remark 9.1. We consider $H^B$ as being equipped with the restriction $\mu_\beta|_{H^B}$ of the measure $\mu_\beta$ to this subset.

**Proposition 9.4.** Let $p > \beta$ be given. Let $B \subset Z$ be any ball in $Z$ of radius $r > 0$ and let $u \in N^{1,p}(H^B)$ be given. Then $u$ has a trace $Tu \in L^p(B)$ given as follows: for each $n \in \mathbb{Z}$ such that $a^n \leq 2r$ and each $z \in B$ we set

$$
T_n u(z) = \sum_{v \in V_n} u(v) \psi_v(z) = \sum_{v \in H^B_n} u(v) \psi_v(z),
$$

then we have $T_n u \to Tu$ in $L^p(B)$. Furthermore, letting $k$ be the minimal integer such that $a^k \leq 2r$, we have the following estimate for any $p$-integrable upper gradient $g$ of $u$ on $H^B$ and any integer $n \geq k$,

$$
\|Tu - T_n u\|_{L^p(B)} \lesssim a^{(p-\beta)/p} \|g\|_{L^p(H^B_n)}.
$$

We remark that the second equality in (9.5) follows from the fact that $\psi_v(z) \neq 0$ implies that $z \in B(v)$ and therefore $B(v) \cap B \neq \emptyset$ if $z \in B$, which implies that $u \in H^B$ provided that $a^h(v) \leq 2r$.

**Proof.** We will first consider the case $p = 1$. Then $0 < \beta < 1$. We will show in this case that the claim actually holds for $\beta = 1$ as well. Let $u \in \tilde{N}^{1,1}(H^B)$ be a given function, which is continuous by Remark 9.1. We let $g \in L^1(H^B)$ be an integrable upper gradient for $u$. For $z \in B$ we write $H^B_n(z)$ for the set of $v \in H^B_n$ such that $z \in B(v)$. We note that by the discussion in the previous paragraph we have for all $z \in B$ and $n \in \mathbb{Z}$ such that $a^n \leq 2r$,

$$
\sum_{v \in H^B_n(z)} \psi_v(z) = L.
$$

We can then estimate

$$
|T_{n+1} u(z) - T_n u(z)| \leq \sum_{v' \in H^B_{n+1}(z)} \sum_{v \in H^B_n(z)} |u(v') - u(v)| \psi_v(z) \psi_{v'}(z).
$$

Observe that we can only have $\psi_{v'}(z) \psi_v(z) \neq 0$ if $B(v) \cap B(v') \neq \emptyset$, which implies that there is a vertical edge $e_{v,v'}$ from $v$ to $v'$. Since $g$ is an upper gradient for $u$ on $H^B$, we thus conclude in this case that

$$
|u(v') - u(v)| \leq \int_{e_{v,v'}} g \, ds, \quad e \in \mathcal{U}(v)
$$

with the sum being taken over the set $\mathcal{U}(v)$ of all upward directed vertical edges $e$ starting from $v$, and with $ds$ denoting arclength with respect to the metric $d_{\rho}$. By Lemma 9.3 this implies that

$$
|u(v') - u(v)| \lesssim \frac{a_{(1-\beta)n}}{\nu(B(v))} \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta.
$$

Applying the estimate (9.9) to the inequality (9.7) gives

$$
|T_{n+1} u(z) - T_n u(z)| \lesssim a_{(1-\beta)n} \sum_{w \in H^B_n(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta \right) \frac{\psi_w(z)}{\nu(B(v))}.
$$
Let $m > n$ be a given integer. Summing inequality (9.10) above from $n$ to $m - 1$, we obtain that

\[
|T_m u(z) - T_n u(z)| \leq \sum_{j=n}^{m-1} a^{(1-\beta)j} \sum_{v \in H^B_j(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_e \right) \psi_v(z)
\]

(9.11)

\[
\leq \sum_{j=n}^{m-1} a^{(1-\beta)j} \sum_{v \in H^B_j(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_e \right) \psi_v(z) \frac{\nu(B(v))}{\nu(B(v))}.
\]

(9.12)

Integrating (9.11) over $B$ and using the bound (9.1) then gives

\[
\|T_m u - T_n u\|_{L^1(B)} \lesssim \sum_{j=n}^{\infty} a^{(1-\beta)j} \sum_{v \in H^B_j(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_e \right).
\]

(9.13)

If $\beta = 1$ then it immediately follows that

\[
\|T_m u - T_n u\|_{L^1(B)} \lesssim \|g\|_{L^1(H^B_{\geq n})}.
\]

If $0 < \beta < 1$ then we bound each of the inner sums in (9.13) by $\|g\|_{L^1(H^B_{\geq n})}$ and sum the resulting geometric series to obtain

\[
\|T_m u - T_n u\|_{L^1(B)} \lesssim \left( \sum_{j=n}^{\infty} a^{(1-\beta)j} \right) \|g\|_{L^1(H^B_{\geq n})} \lesssim a^{(1-\beta)n} \|g\|_{L^1(H^B_{\geq n})}.
\]

(9.14)

Since the sequence of sets $\{H_{\geq n}\}_{n=k}^{\infty}$ (for $k$ the minimal integer such that $a^k \leq 2r$) satisfies $\mu_\beta(H_{\geq n}) \to 0$ as $n \to \infty$ (since this is a nested sequence with empty intersection for which each set has finite measure) it follows that $\|g\|_{L^1(H^B_{\geq n})} \to 0$ as $n \to \infty$. Thus $\{T_n u\}$ defines a Cauchy sequence in $L^1(B)$ which converges in $L^1(B)$ to a function $Tu \in L^1(B)$. It follows that for $\nu$-a.e. $z \in Z$ we have $\lim_{n \to \infty} T_n u(z) = Tu(z)$. Letting $m \to \infty$ in the above inequalities, we deduce for any $n \geq k$ that

\[
\|Tu - T_n u\|_{L^1(B)} \lesssim a^{(1-\beta)n} \|g\|_{L^1(H^B_{\geq n})}.
\]

This establishes (9.1) in the case $p = 1$. We conclude in particular that the proposition holds in the case $p = 1$.

We now consider the case $p > 1$. We let $u \in \tilde{N}^{1,1}(H^B)$ be a given function and let $g \in L^p(H^B)$ be a $p$-integrable upper gradient for $u$. We start from the estimate (9.11) for $m > n$ and $z \in B$. We let $\lambda > 0$ be a given parameter. By applying Hölder’s inequality for sequences on the right side of (9.11) with the conjugate exponents $p$ and $q = \frac{p}{p-1}$, we
obtain for \( m > n \) and \( z \in B \),

\[
|T_m u(z) - T_n u(z)| \lesssim \sum_{j=n}^{\infty} \sum_{v \in H_j B(z)} \left( a^{(1-\beta-\lambda)j} \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta \right)^{\frac{\psi_v(z)}{\nu(B(v))}} a^{j\lambda} \\
\leq \left( \sum_{j=n}^{\infty} a^{p(1-\beta-\lambda)j} \left( \sum_{v \in H_j B(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta \right) \frac{\psi_v(z)}{\nu(B(v))} \right) \right)^{\frac{p}{1/p}} \left( \sum_{j=n}^{\infty} a^{\frac{qj\lambda}{q}} \right)^{1/q} \\
\lesssim a^{n\lambda} \left( \sum_{j=n}^{\infty} a^{p(1-\beta-\lambda)j} \left( \sum_{v \in H_j B(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta \right) \frac{\psi_v(z)}{\nu(B(v))} \right) \right)^{\frac{p}{p}} ,
\]

where now the implied constants also depend on \( \lambda \). By Lemma 6.3 the sets \( H_j B(z) \) have a number of elements uniformly bounded in \( a, \tau, \) and \( C_v \) for each \( j \geq n \) and \( z \in B \). Using Lemma 9.2 it follows from this and the fact that \( X \) has vertex degree uniformly bounded in terms of this same data that we have

\[
\left( \sum_{v \in H_j B(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta \right) \frac{\psi_v(z)}{\nu(B(v))} \right)^{\frac{p}{p}} \lesssim \sum_{v \in H_j B(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g \, d\mu_\beta \right)^{p} \frac{\psi_v(z)^p}{\nu(B(v))^p} \\
\lesssim \sum_{v \in H_j B(z)} \sum_{e \in \mathcal{U}(v)} \left( \int_e g \, d\mu_\beta \right)^{p} \frac{\psi_v(z)}{\nu(B(v))^p} .
\]

where we have used \( 0 \leq \psi_v(z) \leq 1 \) so that \( \psi_v(z)^p \leq \psi_v(z) \). By Jensen’s inequality for integrals of convex functions on probability spaces we have for any edge \( e \in X \),

(9.15)  
\[
\left( \int_e g \, d\mu_\beta \right)^{p} = \mu_\beta(e)^p \left( \int_e g \, d\mu_\beta \right)^{p} \\
\leq \mu_\beta(e)^p \int_e g^p \, d\mu_\beta \\
= \mu_\beta(e)^{p-1} \int_e g^p \, d\mu_\beta
\]

Using this estimate together with the comparison (7.7), we conclude that we have the inequality

(9.18)  
\[
|T_m u(z) - T_n u(z)|^p \lesssim a^{\lambda p \eta} \sum_{j=n}^{\infty} a^{(p-\beta-\lambda)j} \sum_{v \in H_j B(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g^p \, d\mu_\beta \right) \frac{\psi_v(z)}{\nu(B(v))} ,
\]

again with the implied constant additionally depending on \( \lambda \). For future reference we note that inequality (9.18) also holds for \( p = 1 \) as it is implied by the inequality (9.11).

We now set \( \lambda = (p-\beta)/p \). This simplifies (9.18) to

(9.19)  
\[
|T_m u(z) - T_n u(z)|^p \lesssim a^{(p-\beta)n} \sum_{j=n}^{\infty} \sum_{v \in H_j B(z)} \left( \sum_{e \in \mathcal{U}(v)} \int_e g^p \, d\mu_\beta \right) \frac{\psi_v(z)}{\nu(B(v))} .
\]
Integrating each side over $B$ and using (9.21), we conclude that for all $m > n \geq k$,
\[
\|T_m u - T_n u\|_{L^p(B)}^p \lesssim a^{(p-1)\beta n} \sum_{j=n}^{\infty} \sum_{v \in H_j} \left( \sum_{e \in U(v)} \int g^p \, d\mu_{\beta} \right) 
\lesssim a^{(p-\beta)n} \|g\|_{L^p(H^B_{\geq n})}^p.
\]
By taking the $p$th root of each side, we obtain that
\[
\|T_m u - T_n u\|_{L^p(B)} \lesssim a^{(p-\beta)/p} \|g\|_{L^p(H^B_{\geq n})}.  \tag{9.20}
\]
In particular we have
\[
\|T_m u - T_n u\|_{L^p(B)} \lesssim a^{((p-\beta)/p)n} \|g\|_{L^p(H^B)}.  
\]
The right side converges to 0 as $n \to \infty$ since $p > \beta$. We conclude that $\{T_n u\}$ defines a Cauchy sequence in $L^p(B)$ which therefore converges in $L^p(B)$ to a function $Tu \in L^p(B)$. Letting $m \to \infty$ in (9.20) then gives (9.6). \hfill \Box

The proposition below is an immediate consequence of Proposition 9.4. We recall that $\tilde{N}^1_{\text{loc}}(X_\rho) = \tilde{D}^1_{\text{loc}}(X_\rho)$, so we will be formulating our theorems in terms of $\tilde{N}^1_{\text{loc}}(X_\rho)$.

**Proposition 9.5.** Let $u \in \tilde{N}^1_{\text{loc}}(X_\rho)$ be given with $p > \beta$. Then $u$ has a trace $Tu \in L^p_{\text{loc}}(Z)$ given as follows: for each $n \in \mathbb{Z}$ and each $z \in Z$ we set
\[
T_n u(z) = \sum_{v \in V_n} u(z) \beta_v(z).  \tag{9.21}
\]
Then for each ball $B \subset Z$ we have $T_n u \to Tu$ in $L^p(B)$. Furthermore, for a given ball $B$ of radius $r > 0$ and $k$ the minimal integer such that $a^k \leq 2r$, we have the following estimate for any $p$-integrable upper gradient $g$ of $u$ on $H^B$ and any integer $n \geq k$,
\[
\|Tu - T_n u\|_{L^p(B)} \lesssim a^{((p-\beta)/p)n} \|g\|_{L^p(H^B_{\geq n})}.  \tag{9.22}
\]
If furthermore $u$ has a $p$-integrable upper gradient $g$ on $X_\rho$ then we have for each $n \in \mathbb{Z}$,
\[
\|Tu - T_n u\|_{L^p(Z)} \lesssim a^{((p-\beta)/p)n} \|g\|_{L^p(H^B_{\geq n})}.  \tag{9.23}
\]

**Proof.** Let $u \in \tilde{N}^1_{\text{loc}}(X_\rho)$ be given. If $B \subset Z$ is any ball of radius $r > 0$ then by Lemma 6.19 we have that $H^B \subset CB$ for some $C = C(a, \tau) \geq 1$, where $\hat{B} = B_\rho(z, r) \subset \hat{X}_\rho$, which implies for any $x \in H^B$ that $H^B \subset B_\rho(x, 2Cr) \cap X_\rho$. It follows that if $u \in N^1_{\text{loc}}(X_\rho)$ then $u|_{H^B} \in N^1_{\text{loc}}(H^B)$ for any ball $B \subset Z$. All of the claims of the proposition except for inequality (9.23) then follow immediately from the corresponding claims of Proposition 9.4 upon observing that the formulas (9.5) and (9.21) defining $T_n u$ in each proposition are the same.

We now assume that $u$ has a $p$-integrable upper gradient $g$ on $X_\rho$ and let $n \in \mathbb{Z}$ be given. For each vertex $v \in V_n$ we let $B(v)$ be the associated ball of radius $r(B(v)) = 7a^n$, so that in particular we have $a^n \leq 2r(B(v))$. Then $g$ is a $p$-integrable upper gradient of $u$ on $H^B(v)$, so inequality (9.22) implies that for each $v \in V_n$ we have
\[
\|(Tu - T_n u)\chi_{B(v)}\|_{L^p(Z)} \lesssim a^{((p-\beta)/p)n} \|g\|_{L^p(H^B_{\geq n})}.  
\]
Since the balls $B(v)$ for $v \in V_n$ cover $Z$ with bounded overlap by Lemma 6.3, we deduce from the triangle inequality in $L^p(Z)$ that
\[ \|Tu - T_n u\|_{L^p(Z)} \lesssim a^{(p-\beta)/p} \sum_{v \in V_n} \|g\|_{H^p(B(v))}. \]

To prove (9.23) it thus suffices to show that the subsets $H_{\geq n}^{B(v)}$ of $X_{\geq n}$ for $v \in V_n$ also have uniformly bounded overlap.

Suppose that $x \in H_{\geq n}^{B(v)} \cap H_{\geq n}^{B(w)}$ for two vertices $v, w \in V_n$. Since the midpoints of all edges in $X$ have $\mu_\beta$-measure zero, we can assume for the purpose of these estimates that $x$ is not the midpoint of an edge in $X$. We then let $v'$ be the closest vertex to $x$ in $X$. We must then have $v' \in H_{\geq n}^{B(v)} \cap H_{\geq n}^{B(w)}$ as well, by the definition 6.11 of the hull. Thus $B(v') \cap B(v) \neq \emptyset$ and $B(v') \cap B(w) \neq \emptyset$. Since $r(B(v')) \leq r(B(v))$, this implies that $B(w) \subset 5B(v)$. Consequently $\pi(w) \in 5B(v)$ for each $w \in V_n$ such that $H_{\geq n}^{B(v)} \cap H_{\geq n}^{B(w)}$ contains a point of $X$ that is not the midpoint of an edge. The points $\pi(w)$ then define an $a^\beta$-separated subset of a ball $5B(v)$ that has radius $5\tau a^n$. The desired uniform bound on the number of such points $\pi(w)$ then follows from the doubling property of $\nu$ (and therefore of $Z$).

By Proposition 9.5 we have a linear trace operator
\[ T : N_{1,p}^1(X_p) \to L^p_\text{loc}(Z), \]
defined by $u \to Tu$. The domain of $T$ depends on both $p$ and $\beta$, however we will suppress this dependence in the notation.

We next show that $T$ restricts to a bounded linear operator $T : N_{1,p}^1(X_p) \to L^p(Z)$; we recall from Remark 9.1 that each equivalence class in $N_{1,p}^1(X_p)$ consists of a single continuous function, so we can consider $N_{1,p}^1(X_p)$ to be canonically identified with $N_{1,p}^1(X_p)$.

**Proposition 9.6.** Let $u \in N_{1,p}^1(X_p)$ with $p > \beta$. Then $Tu \in L^p(Z)$ with the estimate
\[ \|Tu\|_{L^p(Z)} \lesssim \|u\|_{N_{1,p}^1(X_p)}. \]

**Proof.** We define a function $\xi : X_p \to [0,1]$ by setting $\xi(x) = 1$ for $x \in X_{\geq 1}$, $\xi(x) = 0$ for $x \in X_{\leq 0}$, and linearly interpolating (with respect to the metric $d_\rho$) the values of $\xi$ on each vertical edge connecting $X_{\leq 0}$ to $X_{\geq 1}$. On a given vertical edge $e$ connecting $v$ with $h(v) = 0$ and $h(w) = 1$ we have from Lemma 6.3 that $d_\rho(v, w) \simeq 1$. Thus there is a constant $L = L(a, \tau)$ such that $\xi$ is $L$-Lipschitz on $X_p$. Since $\xi|_{X_{\leq 0}} = 0$, it follows that the scaled characteristic function $LX_{X_{\leq 0}}$ defines an upper gradient for $\xi$ on $X$.

Now set $u_* = \xi u$. Then $|u_*| \leq |u|$ and therefore $\|u_*\|_{L^p(X_p)} \leq \|u\|_{L^p(X_p)}$. Let $g_u$ be a minimal $p$-weak upper gradient for $u$ on $X_p$, which is an upper gradient for $u$ on $X_p$ by Remark 9.1. The product rule for upper gradients implies that
\[ g_* := L|u| + g_u \geq LX_{X_{\leq 0}}|u| + \xi \cdot g_u, \]
is an upper gradient of $u_*$. It follows that
\[ \|g_*\|_{L^p(X_p)} \lesssim \|u\|_{L^p(X_p)} + \|g_u\|_{L^p(X_p)} = \|u\|_{N_{1,p}^1(X_p)}. \]
We conclude from the above that we have $\|u_*\|_{N_{1,p}^1(X_p)} \lesssim \|u\|_{N_{1,p}^1(X_p)}$.

Since $u_*|_{X_{\geq 1}} = u|_{X_{\geq 1}}$, it follows immediately from the defining formula (9.21) for the trace that we have $Tu_* = Tu$. On the other hand we have by construction that $T_0 u_* \equiv 0$ since $u_*(v) = 0$ for all $v \in V_0$. It then follows from (9.23) applied in the case $n = 0$ that
\[ \|Tu\|_{L^p(Z)} \lesssim \|u\|_{N_{1,p}^1(X_p)} \lesssim \|u\|_{N_{1,p}^1(X_p)}. \]
Lipschitz functions on $X_\rho$ belong to $\tilde{N}^{1,p}_{\text{loc}}(X_\rho)$ and have a canonical extension by continuity to $\partial X_\rho = Z$. We show below that this extension agrees with the trace $T$. The equality $\hat{u}|_Z = Tu$ below should be understood as holding for the distinguished $L$-Lipschitz representative of $Tu$ in $L^p_{\text{loc}}(Z)$.

**Proposition 9.7.** Let $u : X_\rho \to \mathbb{R}$ be $L$-Lipschitz and let $\hat{u} : \tilde{X}_\rho \to \mathbb{R}$ denote the canonical $L$-Lipschitz extension of $u$ to the completion $\tilde{X}_\rho$ of $X_\rho$. Then $\hat{u}|_Z = Tu$. In particular $Tu$ is $L$-Lipschitz.

**Proof.** We define $T_n u$ for each $n \in \mathbb{Z}$ as in (9.3). Let $z \in Z$ be a given point and let $\gamma_z : \mathbb{R} \to X$ be an ascending geodesic line anchored at $z$ as given by Lemma 6.4. Let $\{v_n\}_{n \in \mathbb{Z}}$ be the sequence of vertices on $\gamma_z$ with $h(v_n) = n$. Then $d(\pi(v_n), z) < a^n$ for each $n \in \mathbb{Z}$. For each $n \in \mathbb{Z}$ we then have by Lemma 6.10,

$$d_\rho(v_n, z) \leq d_\rho(v_n, \pi(v_n)) + d(\pi(v_n), z) \lesssim a^n.$$ 

Since $\hat{u}$ is $L$-Lipschitz on $\tilde{X}_\rho$, it then follows that

$$|u(v_n) - \hat{u}(z)| \lesssim L a^n.$$

On the other hand we have

$$|u(v_n) - T_n u(z)| \leq \sum_{v \in V_n} |u(v_n) - u(v)| \psi_n(z)$$

The terms in the sum on the right are nonzero only when $z \in B(v)$. Since $z \in B(v_n)$, this implies that $z \in B(v_n) \cap B(v)$. Thus there is a horizontal edge $e$ from $v_n$ to $v$. It then follows from Lemma 3.3 that $d_\rho(v_n, v) \lesssim a^n$. We thus conclude in this case that

$$|u(v_n) - u(v)| \lesssim L d_\rho(v_n, v) \lesssim L a^n,$$

which implies that

$$|u(v_n) - T_n u(z)| \lesssim \sum_{v \in V_n} L a^n \psi_n(z) = L a^n.$$ 

Thus for all $n \in \mathbb{Z}$ and $z \in Z$ we have

$$|T_n u(z) - \hat{u}(z)| \lesssim L a^n.$$

Since $T_n u \to Tu$ in $L^p(\tilde{B})$, it follows in particular that $T_n u \to Tu$ pointwise a.e. on $Z$. We thus obtain that for any $z \in Z$ such that $\lim_{n \to \infty} T_n u(z) = Tu(z)$ we in fact have $Tu(z) = \hat{u}(z)$. Thus $Tu$ agrees $\nu$-a.e. on $Z$ with the $L$-Lipschitz function $\hat{u}|_Z$, as desired. \qed

We now estimate the Besov norm of the trace $Tu$ for $u \in \tilde{D}^{1,p}(X_\rho) \subset \tilde{N}^{1,p}_{\text{loc}}(X_\rho)$. We note below that if $p > \beta$ and we set $\theta = (p - \beta)/p$ then $0 < \theta < 1$.

**Proposition 9.8.** Let $u \in \tilde{D}^{1,p}(X_\rho)$ be given with $p > \beta$. Let $\theta = (p - \beta)/p$. Then

$$\|Tu\|_{B^\theta_p(Z)} \lesssim \|u\|_{\tilde{D}^{1,p}(X_\rho)}.$$ 

Consequently if $u \in \tilde{N}^{1,p}(X_\rho)$ then $Tu \in B^\theta_p(Z)$ with the estimate

$$\|Tu\|_{B^\theta_p(Z)} \lesssim \|u\|_{\tilde{N}^{1,p}(X_\rho)}.$$
Proof. We start with \( u \in \overline{D}^{1,p}(X_p, \mu_\beta) \) as specified. Since \( u \in \overline{D}^{1,p}(X_p) = N^{1,p}_{\text{loc}}(X_p) \) it follows from Proposition 9.5 that the trace \( Tu \) given by formula (9.21) exists and satisfies \( Tu \in L^p_{\text{loc}}(Z) \). We have the estimate for any \( n \in \mathbb{Z} \) and \( \nu \text{-a.e. } x, y \in Z \),

\[
|Tu(x) - Tu(y)|^p \lesssim |Tu(x) - T_n u(x)|^p + |T_n u(x) - T_n u(y)|^p + |Tu(y) - T_n u(y)|^p.
\]

For \( n \in \mathbb{Z} \) and \( x \in Z \) we define

\[
A_n(x) = \{ y \in Z : a^{n+1} \leq d(x, y) < a^n \}.
\]

We then have the following estimate for the Besov norm of \( Tu \), using the doubling property of \( \nu \),

\[
||Tu||_{B^p(Z)}^p \lesssim \int_Z \sum_{n \in \mathbb{Z}} \int_{A_n(x)} \frac{|Tu(x) - T_n u(x)|^p}{a^{n \theta p}} \frac{d\nu(y) d\nu(x)}{\nu(B(x, a^n))} + \int_Z \sum_{n \in \mathbb{Z}} \int_{A_n(x)} \frac{|T_n u(x) - T_n u(y)|^p}{a^{n \theta p}} \frac{d\nu(y) d\nu(x)}{\nu(B(x, a^n))} + \int_Z \sum_{n \in \mathbb{Z}} \int_{A_n(y)} \frac{|Tu(y) - T_n u(y)|^p}{a^{n \theta p}} \frac{d\nu(x) d\nu(y)}{\nu(B(y, a^n))}.
\]

Similarly to the proof of [3, Theorem 11.1], we label the three summands on the right sequentially as (I), (II), and (III), and estimate each one separately. Since (I) and (III) are related by switching the roles of \( x \) and \( y \), it suffices to estimate (I) and (II).

We begin with (I). Since none of the terms depend on \( y \), we can integrate with respect to this variable and use the fact that \( \nu(A_n(x)) \leq \nu(B(x, a^n)) \) (since \( A_n(x) \subset B(x, a^n) \)) to obtain

\[
(I) \lesssim \int_Z \sum_{n \in \mathbb{Z}} \frac{|Tu(x) - T_n u(x)|^p}{a^{n \theta p}} d\nu(x).
\]

We let \( g \) be a minimal \( p \)-weak upper gradient for \( u \) on \( X_p \), which is an upper gradient for \( u \) by Remark 9.1. We then apply inequality (9.18) to obtain for any choice of \( \lambda > 0 \)

\[
(I) \lesssim \int_Z \sum_{n \in \mathbb{Z}} a^{(\lambda - \theta)pn} \sum_{j=n}^{\infty} a^{(p - \beta - p \lambda)j} \sum_{v \in V_j} \left( \sum_{e \in \mathcal{U}(v)} \int e g^p d\mu_\beta \right) \frac{\psi_v(x)}{\nu(B(v))} d\nu(x),
\]

with the implied constant additionally depending on \( \lambda \). Using the bound (9.21) we conclude from inequality (9.24) that

\[
(I) \lesssim \sum_{n \in \mathbb{Z}} a^{(\lambda - \theta)pn} \sum_{j=n}^{\infty} a^{(p - \beta - p \lambda)j} \sum_{v \in V_j} \left( \sum_{e \in \mathcal{U}(v)} \int e g^p d\mu_\beta \right) a^{(\lambda - \theta)pn}.
\]

Using Tonelli’s theorem we can switch the order of summation to obtain

\[
(I) \lesssim \sum_{j \in \mathbb{Z}} a^{(p - \beta - p \theta)j} \sum_{v \in V_j} \left( \sum_{e \in \mathcal{U}(v)} \int e g^p d\mu_\beta \right) \sum_{n=-\infty}^{j} a^{(\lambda - \theta)pn}.
\]

We set \( \lambda = \theta/2 = (p - \beta)/2p > 0 \). Summing the geometric series on the far right above then gives

\[
(I) \lesssim \sum_{j \in \mathbb{Z}} a^{(p - \beta - p \theta)j} \sum_{v \in V_j} \left( \sum_{e \in \mathcal{U}(v)} \int e g^p d\mu_\beta \right).
\]

Since we assumed that \( \theta = (p - \beta)/p \), this simplifies to the desired estimate \( (I) \lesssim \|g\|_{L^p(X_p)} \).
We now estimate (II). For this we observe that
\[
|T_n u(x) - T_n u(y)|^p \leq \sum_{v \in V_n} \sum_{v' \in V_n} |u(v) - u(v')|^p \psi(x) \psi(y)
\]
\[
\leq \sum_{v \in V_n} \sum_{v' \in V_n} |u(v) - u(v')|^p \psi(x) \psi(y)
\]
using \( \psi, \leq 1 \), since this sum contains a number of terms uniformly bounded in terms of the number of \( v \in V_n \) such that \( \psi(x) \neq 0 \) and \( \psi(y) \neq 0 \), which is uniformly bounded in terms of \( \alpha, \tau \), and \( C_\tau \) by Lemma 5.3. Now suppose in addition that \( y \in B(x, a^\alpha) \), which follows from \( y \in A_n(x) \). If \( \psi(x) \psi(y) \neq 0 \) for some \( v, v' \in V_n \) then we must have \( y \in B(v', 2a^\alpha) \) by the construction of the Lipschitz partition of unity in Proposition 8.7. Since \( d(x, y) \leq a^\alpha \) it then follows that \( x \in B(v', 3a^\alpha) \subset B(v') \) (since \( \tau > 3 \)). Since we also have \( x \in B(v) \), it then follows that \( B(v) \cap B(v') \neq \emptyset \). This implies that there is a horizontal edge in \( X \) from \( v \) to \( v' \). Then, writing \( \mathcal{H}(v) \) for the set of all horizontal edges having \( v \) as a vertex, using the comparison (9.2) and using the Jensen inequality estimate (9.1), and then using the comparison (7.7), we conclude that
\[
|T_n u(x) - T_n u(y)|^p \lesssim \sum_{v \in V_n} \sum_{v' \in V_n} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x) \psi(y)
\]
\[
= \sum_{v \in V_n} \sum_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x)
\]
\[
\lesssim a^{p(1-\beta)} \sum_{v \in V_n} \sum_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x) \psi(y)
\]
\[
\lesssim a^{p(1-\beta)} \sum_{v \in V_n} \sum_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x)
\]
\[
\lesssim a^{-\beta} \sum_{v \in V_n} \sum_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x)
\]
Using this estimate in (II) and using the fact that \( p - \beta = p \theta \), we conclude that
\[
(II) \lesssim \int_{\mathbb{Z}} \int_{A_n(x)} \int_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x) \psi(y) \, d\nu(\mathcal{H}(v)) \, d\nu(x)
\]
\[
\lesssim \int_{\mathbb{Z}} \int_{A_n(x)} \int_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \psi(x) \psi(y) \, d\nu(x)
\]
which, upon integrating with respect to \( y \) followed by \( x \) and using \( \nu(A_n(x)) \leq \nu(B(x, a^\alpha)) \) and the bound (9.1), gives
\[
(II) \lesssim \int_{\mathbb{Z}} \int_{A_n(x)} \int_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \, d\nu(x)
\]
\[
\lesssim \int_{\mathbb{Z}} \int_{A_n(x)} \int_{v' \in \mathcal{H}(v)} \left( \int_{e} g \, ds_\alpha \right)^p \, d\nu(x)
\]
Since \( \|g\|_{L^p(Z)} = \|u\|_{L^p(X_n)} \), we conclude the desired estimate for \( \|Tu\|_{B_p^\theta(Z)} \). The corresponding estimate for \( \|Tu\|_{B_p^\theta(Z)} \) then follows from Proposition 9.6.

We conclude from Proposition 9.8 that the trace \( T : N^{1,p}_{\text{loc}}(X_n) \to L^p_{\text{loc}}(Z) \) restricts to bounded linear operators \( T : \Lambda^{1,p}(X_n) \to B^\theta_p(Z) \) and \( T : D^{1,p}(X_n) \to B^\theta_p(Z) \) when we set \( \theta = (p - \beta)/p \). In the next section we will show that these linear operators are surjective.
We can now generalize [3] Proposition 11.2] and [3] Theorem 11.3] to our setting. The compactness of $Z$ and the uniformization $\bar{X}_p$ are only used to show that the metric measure spaces $(X_p, d_p, \mu_\beta)$ and $(\bar{X}_p, d_p, \mu_\beta)$ are doubling and support a $p$-Poincaré inequality, and to then derive the corresponding estimates of Proposition 9.3 in [3]. Once all of these claims have also been verified in for the case of noncompact $Z$ and $\bar{X}_p$, the proofs for [3] Proposition 11.2] and [3] Theorem 11.3] carry over verbatim to the noncompact setting. Since the proofs are short, we reproduce abbreviated versions of them here for convenience for the reader.

The first proposition shows that the boundary measure $\nu$ is absolutely continuous with respect to the $C_p^{\bar{X}_p}$-capacity when $p > \beta$. We recall that our parameter $\beta$ corresponds to $\beta/\epsilon$ in [3]. By Remark 9.1] any subset $G \subset \bar{X}_p$ with $C_p^{\bar{X}_p}(G) = 0$ must satisfy $G \subset Z$.

**Proposition 9.9.** [3] Proposition 11.2] Let $G \subset Z$. If $p > \beta$ and $C_p^{\bar{X}_p}(G) = 0$ then $\nu(G) = 0$.

**Proof.** By Theorem 8.6] we can find open sets $U_n \subset \bar{X}_p$ for each $n \in \mathbb{N}$ with $G \subset U_n$ and $C_p^{\bar{X}_p}(U_n) < 1/n$. The intersection $G' = \bigcap_{n=1}^{\infty} U_n$ then defines a Borel subset of $\bar{X}_p$ with $C_p^{\bar{X}_p}(G') = 0$, hence $G' \subset Z$. Let $K \subset G'$ be compact. Since $(\bar{X}_p, d_p, \mu_\beta)$ supports a $p$-Poincaré inequality, by [1] Theorem 6.7(xi)] we can find Lipschitz functions $u_k$ on $\bar{X}_p$ such that $u_k = 1$ on $K$ and $\|u_k\|_{N^1_p(\bar{X}_p)} < 1/k$. By Proposition 9.7] we have $Tu_k = u_k|_{Z}$, hence in particular $Tu_k = 1$ on $K$ as well. Thus by Proposition 9.6] we have for each $k$,

$$\nu(K)^{1/p} \leq \|Tu_k\|_{L^p(Z)} \lesssim \|u_k\|_{N^1_p(\bar{X}_p)} < \frac{1}{k}.$$ 

By letting $k \to \infty$ we conclude that $\nu(K) = 0$. Since $G'$ is a Borel set and $\nu$ is a Borel regular measure on $Z$, we conclude that

$$\nu(G) \leq \nu(G') = \sup_{K \subset G'} \nu(K) = 0,$$

where the supremum is taken over all compact subsets $K$ of $G'$.

Since $(X_p, d_p, \mu_\beta)$ is a uniform geodesic metric measure space that is doubling and satisfies a $p$-Poincaré inequality, by work of J. Björn and Shanmugalingam [6] Proposition 5.9] there is a bounded linear operator assigning to any $u \in N^1_p(X_p)$ an extension $\tilde{u} \in N^1_p(\bar{X}_p)$. We can then consider the restriction $\tilde{u}|_{Z}$ of this extension to $Z$. The next theorem gives a characterization of the trace $Tu \in B^p_0(Z)$ for $u \in N^1_p(X_p)$ and shows that $\tilde{u}|_{Z}$ agrees $\nu$-a.e. with $Tu$.

**Proposition 9.10.** [3] Theorem 11.3] Let $u \in N^1_p(X_p)$ with $p > \beta$ and set $\theta = (p - \beta)/p$. Then $u$ has an extension $\tilde{u} \in N^1_p(\bar{X}_p)$. This extension satisfies $\tilde{u}|_{Z} = Tu$ $\nu$-a.e. . Consequently we have the estimates

$$\|\tilde{u}\|_{B^p_0(Z)} \lesssim \|u\|_{D^1_p(X_p)},$$

and

$$\|\tilde{u}\|_{L^p(Z)} \lesssim \|u\|_{N^1_p(X_p)}.$$ 

Moreover, for $C_p^{\bar{X}_p}$-a.e. (and thus $\nu$-a.e.) $z \in Z$ we have

$$(9.25) \quad \lim_{r \to 0^+} \int_{X_p \cap B^\rho(z,r)} |u - \tilde{u}(z)|^p \, d\mu_\beta = 0.$$
The limit \( r \to 0^+ \) indicates taking the limit through positive values of \( r \). The extension \( \hat{u} \) is unique up to sets of zero \( p \)-capacity in \( \bar{X}_p \): if \( \hat{u}_1 \) and \( \hat{u}_2 \) are two such extensions then \( \hat{u}_1 = \hat{u}_2 \) \( \mu_\beta \)-a.e. on \( \bar{X}_p \) since \( \mu_\beta(\partial X_p) = 0 \), which implies that they are equal q.e. on \( \bar{X}_p \) (see the discussion after (5.2)). This implies by Proposition 9.9 that the restriction \( \hat{u}|_Z \) is unique up to sets of \( \nu \)-measure zero. In particular the restriction \( \hat{u}|_Z \) is well-defined as an element of \( L^p(Z) \).

For a metric measure space \((Y,d,\mu)\) and an exponent \( p \geq 1 \) we say that a point \( x \in Y \) is an \( L^p(Y)\)-Lebesgue point of a measurable function \( u : Y \to [\pm \infty] \) if

\[
\lim_{r \to 0^+} \int_{B_Y(x,r)} |u - u(x)|^p \, d\mu = 0.
\]

If \( x \) is an \( L^p(Y)\)-Lebesgue point of \( u \) then by Hölder’s inequality it is also an \( L^q(Y)\)-Lebesgue point of \( u \) for each \( 1 \leq q \leq p \). Since \( \mu_\beta(\partial X_p) = 0 \), (9.25) can be rephrased as saying that \( C_p^{\nu}\)-q.e. point of \( Z \) is an \( L^p(X_p)\)-Lebesgue point for \( \hat{u} \).

**Proof.** By [23] Theorem 4.1 and Corollary 3.9 we can find a sequence of Lipschitz functions \( u_k \in N^{1,p}(X_p) \) such that \( \|u_k - \hat{u}\|_{N^{1,p}(X_p)} \to 0 \) and \( u_k(x) \to \hat{u}(x) \) for q.e. \( x \in X_p \) as \( k \to \infty \).

For Proposition 9.7 we have \( (uj - u_k) = \hat{u}_j - \hat{u}_k \) for each \( j,k \in \mathbb{N} \). It follows from Proposition 9.6 that

\[
\|\hat{u}_j - \hat{u}_k\|_{B_p^0(Z)} \lesssim \|u_j - u_k\|_{N^{1,p}(X_p)} \leq \|u_j - u_k\|_{N^{1,p}(\bar{X}_p)}.
\]

Thus \( \{\hat{u}_k\} \) defines a Cauchy sequence in \( \tilde{B}_p^\theta(Z) \). Since \( \tilde{B}_p^\theta(Z) \) is a Banach space by \[3\] Remark 9.8, we conclude that this sequence converges to \( \hat{u} \) in \( \tilde{B}_p^\theta(Z) \) to a function \( u' \). In particular we have \( \hat{u}_k \to u' \) in \( L^p(Z) \), which implies that \( \hat{u}_k \to u' \) \( \nu\)-a.e. on \( Z \). Since we also have \( \hat{u}_k \to \hat{u} \) \( \nu\)-a.e. on \( Z \), we conclude that \( u' = \hat{u} = \hat{u}|_Z \) \( \nu\)-a.e. Since the trace \( T \) defines a bounded linear operator \( T : N^{1,p}(X_p) \to \tilde{B}_p^\theta(Z) \) it follows that \( Tu = \hat{u}|_Z \) in \( \tilde{B}_p^\theta(Z) \), i.e., \( Tu = \hat{u}|_Z \) \( \nu\)-a.e. The estimates for \( \|\hat{u}|_Z\|_{B_p^0(Z)} \) and \( \|\hat{u}|_Z\|_{L^p(Z)} \) then follow from the corresponding estimates for \( Tu \) since \( Tu = \hat{u}|_Z \) \( \nu\)-a.e.

For \( p > 1 \) the equality (9.25) follows from the fact that \( C_p^\nu\)-q.e. point of \( \bar{X}_p \) is an \( L^p(X_p)\)-Lebesgue point of \( \hat{u} \) [21] Theorem 9.2.8]. The same claim also holds for \( p = 1 \) [22] Theorem 4.1 and Remark 4.7]; note in our case that \( \mu_\beta(X_p) = \infty \) by Proposition 7.7. This proves (9.25).

Proposition 9.10 has a natural generalization to functions \( u \in \tilde{D}^{1,p}(X_p) \).

**Proposition 9.11.** Let \( u \in \tilde{D}^{1,p}(X_p) \) with \( p > \beta \) and set \( \theta = (p - \beta)/p \). Then \( u \) has an extension \( \hat{u} \in \tilde{D}^{1,p}(X_p) \). This extension satisfies \( \hat{u}|_Z = Tu \) \( \nu\)-a.e. . Consequently we have the estimate

\[
\|\hat{u}|_Z\|_{B_p^\theta(Z)} \lesssim \|u\|_{D^{1,p}(X_p)},
\]

Moreover, for \( C_p^\nu\)-q.e. \( (\text{and thus } \nu\text{-a.e.}) \) \( z \in Z \) we have

\[
\lim_{r \to 0^+} \int_{X_p \cap B_{\rho}(z,r)} |u - \hat{u}(z)|^p \, d\mu_\beta = 0.
\]

**Proof.** Let \( u \in \tilde{D}^{1,p}(X_p) \) be given. We extend \( u \) to \( \hat{u} : \bar{X}_p \to [\pm \infty] \) by setting for \( z \in Z \),

\[
\hat{u}(z) = \limsup_{r \to 0^+} \int_{X_p \cap B_{\rho}(z,r)} u \, d\mu_\beta.
\]
The integrability of $u$ over $X_p \cap B_p(z, r)$ follows from the fact that this is a bounded subset of $X_p$ and that functions $u \in \tilde{D}^1(X_p)$ are integrable on balls by Proposition 8.4.

We fix $z_0 \in Z$ and for each $n \in \mathbb{N}$ let $\zeta_n : [0, \infty) \to [0, 1]$ be a piecewise linear function such that $\zeta_n(t) = 1$ for $0 \leq t \leq n$, $\zeta_n(t) = 2 - n^{-t}$ for $n \leq t \leq 2n$, and $\zeta_n(t) = 0$ for $t \geq 2n$. We then define $\kappa_n : \hat{X}_p \to [0, 1]$ by $\kappa_n(x) = \zeta_n(d_p(x, z_0))$. We note that $\kappa_n$ is $n^{-1}$-Lipschitz for each $n$. The rescaled characteristic function $n^{-1} \chi_{B_p(z_0, 2n)}$ defines an upper gradient for $\kappa_n$ on $\hat{X}_p$.

For each $n \in \mathbb{N}$ we set $f_n = \kappa_n u$. The product rule for upper gradients implies that if $g$ is any upper gradient for $u$ on $X_p$ then for each $n \in \mathbb{N}$,

$$g_n := n^{-1} \chi_{B_p(z_0, 2n)} u + g,$$

defines an upper gradient of $f_n$ on $X_p$. We claim that if $g$ is $p$-integrable then $g_n$ is $p$-integrable as well. For this it suffices to show that the function $n^{-1} \chi_{B_p(z_0, 2n)} u$ is $p$-integrable on $X_p$ for each $n \in \mathbb{Z}$. But since for any point $x \in B_p(z_0, 2n) \cap X_p$ we have $B_p(z_0, 2n) \subset B_p(x, 4n)$, this claim follows from the fact that functions in $\tilde{D}^1(X_p)$ are $p$-integrable on balls by Proposition 8.4. Thus $f_n \in \tilde{N}^1(X_p)$ for each $n \in \mathbb{N}$.

We can therefore apply Proposition 9.10 to $f_n$ for each $n \in \mathbb{N}$. We thus obtain an extension $\hat{f}_n \in \tilde{N}^1(X_p)$ of $f_n$ for each $n$ that satisfies for $C_p^{X_p}$-q.e. (in particular $\nu$-a.e.) $z \in Z$,

$$\lim_{r \to 0^+} \int_{X_p \cap B_p(z, r)} |f_n - \hat{f}_n(z)|^p \, d\mu_\beta = 0.$$

We let $G \subset Z$ be the set of points such that (9.29) holds for each $n \in \mathbb{N}$, which satisfies $C_p^{X_p}(Z \setminus G)$ by the countable subadditivity of the capacity. If $z \in G$ then for sufficiently large $n$ we will have from (9.29) and the fact that $f_n = u$ on $B_p(z_0, n)$,

$$\lim_{r \to 0^+} \int_{X_p \cap B_p(z, r)} |u - \hat{f}_n(z)|^p \, d\mu_\beta = 0.$$

Applying Hölder’s inequality then gives

$$\lim_{r \to 0^+} \int_{X_p \cap B_p(z, r)} |u - \hat{f}_n(z)| \, d\mu_\beta = 0,$$

from which we conclude that $\hat{f}_n(z) = \hat{u}(z)$. We conclude in particular that the equality (9.27) holds. We also conclude that $\lim_{n \to \infty} \hat{f}_n(z) = \hat{u}(z)$ for $C_p^{X_p}$-q.e. $z \in Z$, hence also for $\nu$-a.e. $z \in Z$.

It is clear from the defining formula (9.22) for the trace $Tu$ that we have $T f_n = T u$ $\nu$-a.e. on $B_p(z_0, n/2)$ since $f_n = u$ on $B_p(z_0, n)$. Thus we also have $\lim_{n \to \infty} T f_n(z) = T u(z)$ for $\nu$-a.e. $z \in Z$. Since $\hat{f}_n|_Z = T f_n$ $\nu$-a.e. by Proposition 9.10 we conclude that $\hat{u}|_Z = T u$ $\nu$-a.e. The estimate (9.20) immediately follows from this equality.

Lastly we need to show that $\hat{u} \in D^1(X_p)$. Let $g_u$ be a minimal $p$-weak upper gradient of $u$ on $X_p$. For each $n$ we let $\tilde{g}_n$ denote a minimal $p$-weak upper gradient of $f_n$ on $\hat{X}_p$. We then define a Borel function $\tilde{g} : \hat{X}_p \to [0, \infty]$ by setting $\tilde{g}(x) = \sup_{n \geq 0} \tilde{g}_n(x)$ for $x \in B_p(z_0, n/2) \setminus B_p(z_0, (n - 1)/2)$ and $n \in \mathbb{N}$; here by definition $B(z_0, 0) = \emptyset$.

We claim that $\tilde{g}$ is a $p$-integrable $p$-weak upper gradient for $\hat{u}$ on $\hat{X}_p$. Let $G \subset Z = \partial X_p$ be the set we constructed earlier on which (9.29) holds for each $n \in \mathbb{N}$. We then let $\hat{G} = G \cup X_p$. Since $C_p^{X_p}(\hat{X}_p \setminus \hat{G}) = C_p^{X_p}(Z \setminus G) = 0$ we have that $\hat{X}_p \setminus \hat{G}$ is $p$-exceptional, i.e., we have that $p$-a.e. curve in $\hat{X}_p$ belongs entirely to $\hat{G}$. Now let $x, y \in \hat{G}$ be given and let
γ be a curve joining them in \( \hat{G} \). We can then choose \( n \) large enough that \( \gamma \subset B_\rho(z_0, n/2) \). Restricted to \( B_\rho(z_0, n/2) \cap \hat{G} \) we have that \( \hat{f}_n = \hat{u} \) by the construction of \( \hat{f}_n \) and \( \hat{u} \). Thus since \( \gamma \subset B_\rho(z_0, n/2) \cap \hat{G} \) and \( \bar{g}_n \leq \bar{g} \) on \( B_\rho(z_0, n/2) \),

\[
|\hat{u}(x) - \hat{u}(y)| = |\hat{f}_n(x) - \hat{f}_n(y)| \leq \int_{\gamma} \bar{g}_n \, ds \leq \int_{\gamma} \bar{g} \, ds.
\]

It follows that \( \bar{g} \) is a \( p \)-weak upper gradient for \( \hat{u} \) on \( \hat{X}_\rho \).

It remains to show that \( \bar{g} \) is \( p \)-integrable on \( \hat{X}_\rho \). Since \( \mu_\beta(\partial X_\rho) = 0 \) it suffices to show that \( \bar{g} \) is \( p \)-integrable on \( X_\rho \). Since \( X_\rho \) is open in \( \hat{X}_\rho \) it follows from [21] Proposition 6.3.22 that for each \( n \in \mathbb{N} \) the minimal \( p \)-weak upper gradient \( \bar{g}_n \) for \( f_n \) on \( X_\rho \) coincides \( \mu_\beta \)-a.e. on \( X_\rho \) with the minimal \( p \)-weak upper gradient \( \bar{g}_n \) of \( f_n \) on \( X_\rho \). On the open set \( B_\rho(z_0, n/2) \cap X_\rho \) we have that \( f_n = u \) by construction. By using [21] Proposition 6.3.22 again it follows that the minimal \( p \)-weak upper gradient \( \bar{g}_n \) for \( f_n \) on \( X_\rho \) coincides \( \mu_\beta \)-a.e. with the minimal \( p \)-weak upper gradient \( g_u \) for \( u \) on \( X_\rho \) when each of these upper gradients are restricted to \( B_\rho(z_0, n/2) \cap X_\rho \). It follows that on \( B(z_0, n/2) \cap X_\rho \) we have \( \bar{g}_n = g_u \) \( \mu_\beta \)-a.e. By the defining formula for \( \bar{g} \) we conclude that we have \( \bar{g} = g_u \) \( \mu_\beta \)-a.e. on \( X_\rho \). It follows that \( \bar{g} \) is \( p \)-integrable on \( X_\rho \) and therefore on \( \hat{X}_\rho \).

\[ \blacksquare \]

10. Extension theorems

In this section we establish analogues of the extension theorems in [3] Section 11] in the case of noncompact \( Z \). We carry over all conventions, notation, and hypotheses from the previous section. Following Bonk-Saksman [10], for any function \( f \in L^1_{\text{loc}}(Z) \) we define the Poisson extension \( Pf : X_\rho \to \mathbb{R} \) by setting for any vertex \( v \in V \),

\[
Pf(v) = \int_{B(v)} f \, d\nu,
\]

and then extending \( Pf \) to the edges of \( X_\rho \) by linearly interpolating (with respect to the metric \( d_\rho \)) the values of \( Pf \) on the vertices of each edge. Then \( Pf \) defines a continuous function on \( X_\rho \). We extend \( Pf \) to \( \hat{X}_\rho \) by defining for \( z \in Z \),

\[
Pf(z) = \limsup_{r \to 0^+} \int_{B_\rho(z,r)} Pf \, d\mu_\beta.
\]

Then \( Pf : \hat{X}_\rho \to [-\infty, \infty] \). The resulting function defines a linear operator \( P : L^1_{\text{loc}}(Z) \to L^1_{\text{loc}}(X_\rho) \) since \( \mu_\beta(\partial X_\rho) = 0 \). This operator is similar to the one used in [3] Section 11], however we use a different notation to avoid conflict with the notation \( E \) for the set of edges in \( X \).

A computation similar to the one done in Proposition 7.7 shows that we can only have \( Pf \in L^p(\hat{X}_\rho) \) for some \( p \geq 1 \) if \( f \equiv 0 \) \( \nu \)-a.e. on \( Z \). We rectify this issue by defining truncations \( P_n f \) of \( Pf \) for each \( n \in \mathbb{Z} \) as follows: we let \( \xi_n : \hat{X}_\rho \to [0, 1] \) be the function defined by setting \( \xi(x) = 1 \) for \( x \in X_{\geq n+1} \) and \( x \in Z \), \( \xi(x) = 0 \) for \( x \in X_{\leq n} \), and linearly interpolating (with respect to the metric \( d_\rho \)) the values of \( \xi_n \) on each vertical edge connecting \( X_{\leq n} \) to \( X_{\geq n+1} \). We then define \( P_n f = \xi_n Pf \) for \( x \in X_\rho \). The operators \( f \to P_n f \) on \( L^1_{\text{loc}}(Z) \) are also linear for each \( n \in \mathbb{Z} \). We have \( P_n f|_{X_{\leq n}} = 0 \) and \( P_n f|_{X_{\geq n+1}} = Pf|_{X_{\geq n+1}} \) as well as \( P_n f|_Z = Pf|_Z \).

In our first proposition of this section we analyze the \( L^p \) norm of \( Pf \) on the subsets \( X_{\geq n} \) for \( n \in \mathbb{Z} \). This estimate does not require \( p > \beta \).
Proposition 10.1. Let \( f \in L^p(Z) \) be given. Then for each \( n \in \mathbb{Z} \),
\[
\|P \|_{L^p(X_{\geq n})} \lesssim a^{\beta n} \|f\|_{L^p(Z)}.
\]
Consequently,
\[
\|P_n f\|_{L^p(X_{\geq n})} \lesssim a^{\beta n} \|f\|_{L^p(Z)}
\]

Proof. If \( v \) and \( w \) are two distinct vertices of the same edge \( e \) then, using Jensen’s inequality twice,
\[
\int_e |Pf|^p d\mu_{\beta} \leq \mu_{\beta}(e)(|Pf(v)|^p + |Pf(w)|^p)
\]
\[
\lesssim \mu_{\beta}(e) \left( \int_{B(v)} |f|^p d\nu + \int_{B(w)} |f|^p d\nu \right).
\]
Since \( v \sim w \), we have \( B(v) \cap B(w) \neq \emptyset \) and \( r(B(v)) \approx r(B(w)) \). Thus \( B(w) \subset 4a^{-1}B(v) \).
It then follows from the doubling property of \( \nu \) and the comparison (7.7) that
\[
\int_e |Pf|^p d\mu_{\beta} \lesssim \mu_{\beta}(e) \int_{4a^{-1}B(v)} |f|^p d\nu \lesssim a^{\beta h(v)} \int_{4a^{-1}B(v)} |f|^p d\nu.
\]
As in Proposition 10.1, we let \( E_j \) be the set of all edges in \( X \) with at least one vertex in \( V_j \).
The balls \( 4a^{-1}B(v) \) for \( v \) a vertex of some \( e \in E_j \) then cover \( Z \) with bounded overlap by Lemma 6.3. Using the doubling of \( \nu \) again, it follows from summing over all such vertices \( v \) that for any \( j \in \mathbb{Z} \),
\[
\int_{E_j} |Pf|^p d\mu_{\beta} \lesssim a^{\beta j} \int_Z |f|^p d\nu.
\]
Summing the geometric series over all integers \( j \geq n \) and observing that each edge \( e \in X \) belongs to at most two of the sets \( E_n \) for \( n \in \mathbb{Z} \) then gives the first desired estimate.
The estimate for \( P_n f \) follows by observing that \( |P_n f| \leq |Pf| \) and \( P_n f |_{X_{\leq n}} = 0 \). \( \square \)

We next show that \( Pf \) belongs to \( \tilde{D}^{1,p}(X,\rho) \) when \( f \in \tilde{B}^\theta_p(Z) \), where \( \theta = (p - \beta)/p \).
We deduce from this that for \( f \in \tilde{B}^\theta_p(Z) \) the truncations \( P_n f \) belong to \( \tilde{N}^{1,p}(\tilde{X},\rho) \) for each \( n \in \mathbb{Z} \).

Proposition 10.2. If \( f \in \tilde{B}^\theta_p(Z) \) with \( p > \beta \) and \( \theta = (p - \beta)/p \) then
\[
\|Pf\|_{D^{1,p}(X,\rho)} \lesssim \|f\|_{B^\theta_p(Z)}.
\]
If \( f \in \tilde{B}_{p,\beta}(Z) \) then we further have for each \( n \in \mathbb{Z} \),
\[
\|P_n f\|_{D^{1,p}(X,\rho)} \lesssim a^{(\beta-1)n} \|f\|_{L^p(Z)} + \|f\|_{B^\theta_p(Z)},
\]
and
\[
\|P_n f\|_{N^{1,p}(X,\rho)} \lesssim (a^{\beta n} + a^{(\beta-1)n}) \|f\|_{L^p(Z)} + \|f\|_{B^\theta_p(Z)}.
\]

Proof. For each edge \( e \) of \( X \) with vertices \( v \) and \( w \) we set
\[
ge_e = \frac{|Pf(v) - Pf(w)|}{d_\rho(v, w)}.
\]
The function \( g(x) = g_e \) for \( x \in e \) belonging to the interior of \( e \) defines an upper gradient for \( Ef \) on the edge \( e \) by the construction of \( Ef \). We define \( g : X,\rho \to \mathbb{R} \) by setting \( g(v) = 0 \) for each \( v \in V \) and setting \( g(x) = g_e \) for each point \( x \in X \) belonging to the interior of an edge \( e \).
Since the vertices \( V \) of \( X \) have measure zero with respect to the 1-dimensional Hausdorff
measure on $\mathcal{X}_p$, it follows that $g$ defines an upper gradient for $Pf$ on $\mathcal{X}_p$. We note that $g$ is constant on the interior of each edge of $X$.

As noted in the proof of Proposition 10.1, if $e$ is an edge of $X$ with vertices $v$ and $w$ then $B(w) \subset 4a^{-1}B(v)$. Since $|vw| = 1$, we also have from Lemma 6.3,

$$d_\rho(v, w) \asymp a^{(v,w)_h} \asymp a^{h(v)}.$$  

Consequently by the doubling of $\nu$,

$$g_e \lesssim a^{-h(v)} \int_{4a^{-1}B(v) \setminus 4a^{-1}B(v)} |f(x) - f(y)| \, d\nu(x) \, d\nu(y).$$

By Jensen’s inequality we then have

$$g_e^p \lesssim a^{-ph(v)} \int_{4a^{-1}B(v) \setminus 4a^{-1}B(v)} |f(x) - f(y)|^p \, d\nu(x) \, d\nu(y).$$

Multiplying and dividing by $a^{p\theta h(v)}$ and noting that $p\theta = p - \beta$, we conclude that

$$g_e^p \lesssim a^{-\beta h(v)} \int_{4a^{-1}B(v) \setminus 4a^{-1}B(v)} \frac{|f(x) - f(y)|^p}{a^{p\theta h(v)}} \, d\nu(x) \, d\nu(y).$$

Recall that $\mathcal{L}$ denotes the measure on $X$ that restricts to 1-dimensional Lebesgue measure on each unit length edge of $X$. We have from the definition (17.2) of $\mu$ and Lemma (7.1) that for any edge $e$ in $X$ and either vertex $v$ on $e$,

$$d\mu_{\beta|e} \asymp a^{\beta h(v)} \nu(B(v)) d\mathcal{L}|e.$$  

We thus have for any edge $e$ in $X$ and either vertex $v$ on $e$,

$$\int g_e^p \, d\mu_{\beta} \lesssim \int_{4a^{-1}B(v) \setminus 4a^{-1}B(v)} \frac{|f(x) - f(y)|^p}{a^{p\theta h(v)}} \, d\nu(x) \, d\nu(y).$$

Now suppose that $v \in V_n$. If $x, y \in 4a^{-1}B(v)$ then $y \in B_\mathcal{L}(x, 8\tau a^{-1})$. Thus

$$\int g_e^p \, d\mu_{\beta} \lesssim \int_{4a^{-1}B(v) \setminus 4a^{-1}B(v)} \frac{|f(x) - f(y)|^p}{a^{p\theta n}} \, d\nu(y) \, d\nu(x).$$

Summing this inequality over the set $E_n$ of edges in $X$ having at least one vertex in $V_n$ and using the fact that the balls $4a^{-1}B(v)$ for $v$ a vertex of some $e \in E_n$ cover $\mathcal{Z}$ with bounded overlap by Lemma 6.3 we obtain in a similar manner to what was done in Proposition 10.1

$$\int_{E_n} g_e^p \, d\mu_{\beta} \lesssim \int_{B_\mathcal{L}(x, 8\tau a^{-1})} \frac{|f(x) - f(y)|^p}{a^{p\theta n}} \, d\nu(y) \, d\nu(x).$$

By summing this inequality over $n \in \mathbb{Z}$ and using the fact that each edge $e$ belongs to at most two sets $E_n$, we conclude that

$$\|g\|_{L_p^\mu(X_n)}^p \lesssim \sum_{n \in \mathbb{Z}} \int_{B_\mathcal{L}(x, 8\tau a^{-1})} \frac{|f(x) - f(y)|^p}{a^{p\theta n}} \, d\nu(y) \, d\nu(x).$$

Let $m = m(a, \tau) \in \mathbb{N}$ be the minimal integer such that $a^{-m} \geq 8\tau a^{-1}$. Then by the doubling property for $\nu$ we have for each $n \in \mathbb{Z}$,

$$\int_{B_\mathcal{L}(x, a^{-m})} \frac{|f(x) - f(y)|^p}{a^{p\theta n}} \, d\nu(x) \lesssim \int_{B_\mathcal{L}(x, a^{-m})} \frac{|f(x) - f(y)|^p}{a^{p\theta (n-m)}} \, d\nu(x).$$

It then follows by Lemma 8.10 that

$$\|g\|_{L_p^\mu(X_n)}^p \lesssim \sum_{n \in \mathbb{Z}} \int_{B_\mathcal{L}(x, a^{-m})} \frac{|f(x) - f(y)|^p}{a^{p\theta n}} \, d\nu(y) \, d\nu(x) \asymp \|f\|_{B_p^\mathcal{L}(\mathcal{Z})}^p.$$  

(10.6)
We conclude in particular that \( Pf \in \tilde{D}^{1,p}(X_p) \).

Let \( u \in \tilde{D}^{1,p}(\tilde{X}_p) \) be an extension of \( Pf \) to \( \tilde{X}_p \) given by Proposition 9.11. Then, by (9.27) and the definition of \( Pf \) on \( Z \), we have that \( u = Pf \ C_p \)-q.e. on \( Z \), from which it follows that \( u = Pf C_{p}^{\tilde{X}_{p}} \)-q.e. on \( \tilde{X}_p \). Let \( g_u \) be a minimal \( p \)-weak upper gradient for \( u \) on \( \tilde{X}_p \). Then \( g_u \) must also be a \( p \)-weak upper gradient for \( Pf \) on \( \tilde{X}_p \) since \( p \)-a.e. curve in \( \tilde{X}_p \) does not meet the \( p \)-exceptional set \( \{ u(x) \neq Pf(x) \} \subset \tilde{X}_p \). This shows that \( Pf \in \tilde{D}^{1,p}(\tilde{X}_p) \).

Now let \( g_{Pf} \) be a minimal \( p \)-weak upper gradient for \( Pf \) on \( \tilde{X}_p \). Then \( g_{Pf} \) is also a \( p \)-integrable \( p \)-weak upper gradient of \( Pf \) on \( X_p \). Since every compact curve in \( X_p \) has positive \( p \)-modulus (when considered as a family of curves with one element, see Remark 9.1) it follows that \( g_{Pf} \) is actually a \( p \)-integrable upper gradient of \( Pf \) on \( X_p \). Thus by (10.6) and the fact that \( \mu (\partial X_p) = 0 \),

\[
\| g_{Pf} \|_{L^p(\tilde{X}_p)} = \| g_{Pf} \|_{L^p(X_p)} \lesssim \| f \|_{B^q_p(X)}.
\]

This proves the first estimate (10.2).

Let \( n \in \mathbb{Z} \) be given and consider the truncated extension \( P_n f = \xi_n Pf \). By Lemma 6.8 if we have vertices \( v \in X_{\leq n} \) and \( w \in X_{\geq n+1} \) connected by a vertical edge then \( d_p(v, w) \simeq a^n \).

Thus \( \xi_n \) is \( La^{-n} \)-Lipschitz on \( \tilde{X}_p \) for some constant \( L = L(a, \tau) \geq 1 \). Since \( \xi_n |_{X_{\leq n}} = 0 \) and \( \xi |_{X_{\geq n+1} \cup Z} = 1 \), we conclude that \( La^{-n} \chi_{X \geq n \cup Z} \) defines an upper gradient for \( \xi_n \) on \( \tilde{X}_p \). Let \( g_{Pf} \) be a minimal \( p \)-weak upper gradient for \( Pf \) on \( \tilde{X}_p \). By the product rule for upper gradients we then conclude from \( |\xi_n| \leq 1 \) that we have that

\[
g_n := La^{-n} \chi_{X \geq n \cup Z} |Pf| + g_{Pf},
\]

is an upper gradient for \( P_n f \) on \( \tilde{X}_p \). Since \( \mu (\partial X_p) = 0 \), we conclude from Proposition 10.1 and the bound (10.2) that

\[
\| g_n \|_{L^p(\tilde{X}_p)} \lesssim a^{-n} \| Pf \|_{L^p(X_{\geq n})} + \| g_{Pf} \|_{L^p(\tilde{X}_p)} \lesssim a^{-n} \| Pf \|_{L^p(X_{\geq n})} + \| Pf \|_{D^{1,p}(\tilde{X}_p)} \lesssim a^{(\beta-1)n} \| f \|_{L^p(Z)} + \| f \|_{B^q_p(Z)}.
\]

The bound (10.3) follows. The bound (10.4) is then a consequence of Proposition 10.1. 

We have thus defined a bounded linear operator \( P : \tilde{B}^{q}_p(Z) \to D^{1,p}(\tilde{X}_p) \) as well as bounded linear operators \( P_n : \tilde{B}^{q}_p(Z) \to \tilde{N}^{1,p}(\tilde{X}_p) \) for each \( n \in \mathbb{Z} \). We will relate these operators to the trace operator \( T \) of the previous section using Proposition 10.3 below. This proposition shows, for each \( q \geq 1 \), that \( L^q(Z) \)-Lebesgue points for \( f \in L^q_{loc}(Z) \) are \( L^q(\tilde{X}_p)\)-Lebesgue points for \( Pf \).

**Proposition 10.3.** Let \( f \in L^1_{loc}(Z) \). Let \( z \in Z \) be an \( L^q(Z) \)-Lebesgue point for \( f \) for a given \( q \geq 1 \). Then \( z \) is an \( L^q(\tilde{X}_p) \)-Lebesgue point for \( P \) and \( Pf(z) = f(z) \). Consequently the same is true with \( P_n f \) replacing \( Pf \) for each \( n \in \mathbb{Z} \).

**Proof.** We closely follow the proof from the final part of [3 Theorem 12.1]. We let \( z \in Z \) be an \( L^q(Z) \)-Lebesgue point for \( f \). We fix an arbitrary integer \( N \geq 0 \) and consider all \( x \in X \) such that \( d_p(x, z) < a^N \). If \( x \) belongs to an edge \( e \) with vertices \( v \) and \( w \) then either \( v \) or \( w \) must belong to \( B_p(z, a^N) \) since the metric on \( \tilde{X}_p \) is geodesic. Let \( v \) be the vertex belonging to \( B_p(z, a^N) \). Then

\[
a^{h(v)} \simeq d_p(v) \leq d_p(v, z) \leq a^N, \tag{10.7}
\]
so that we have \( a^h(v) \lesssim a^N \). Then by Lemma 6.10 we have
\[
d_{p}(\pi(v), z) \leq d_{p}(\pi(v), v) + d_{p}(v, z) \lesssim a^{h(v)} + a^N \lesssim a^N.
\]
A similar estimate shows that the other vertex \( w \) also satisfies \( d(\pi(w), z) \lesssim a^N \). Since \( Pf(x) \) is a convex combination of \( Pf(v) \) and \( Pf(w) \), we have by Jensen’s inequality,
\[
\int_{e}|Pf - f(z)|^q d \mu_{\beta} \leq (|Pf(v) - f(z)|^q + |Pf(w) - f(z)|^q) \mu_{\beta}(e).
\]
Using Jensen again gives
\[
|Pf(v) - f(z)|^q \leq \int_{B(v)} |f - f(z)|^q d \nu,
\]
and the same with \( w \) replacing \( v \). Combining this with the comparison (7.7) and using the fact that \( B(w) \subset 4a^{-1}B(v) \) gives
\[
\int_{e}|Pf - f(z)|^q d \mu_{\beta} \lesssim a^{\beta n} \int_{4a^{-1}B(v)} |f - f(z)|^q d \nu.
\]
Summing this over all edges \( e \) such that at least one vertex of \( e \) belongs to \( B_p(z, a^N) \) and then using the fact that \( X \) has bounded degree and \( b(v) \geq N - c \) for some constant \( c = c(a, \tau) \) (by (10.7)), we conclude that
\[
\int_{B_p(z, a^N)} |Ef - f(z)|^q d \mu_{\beta} \lesssim \sum_{n \geq N-c} \sum_{v \in V \cap B_p(z, a^N)} a^{\beta n} \int_{4a^{-1}B(v)} |f - f(z)|^q d \nu.
\]
Let \( B = B_Z(z, a^N) \) and let \( C = C(a, \tau) \geq 1 \) be chosen large enough that Lemma 6.13 implies that \( B_p(z, a^N) \subset H^{CB} \). Then for a given \( n \geq N - c \), the balls \( 4a^{-1}B(v) \) for \( v \in H^{CB}_n \) have bounded overlap by Lemma 7.3 and will be contained in \( C'B \) for a constant \( C' = C'(a, \tau) \).
Thus
\[
\int_{B_p(z, a^N)} |Pf - f(z)|^q d \mu_{\beta} \lesssim \sum_{n \geq N-c} a^{\beta n} \int_{C'B} |f - f(z)|^q d \nu.
\]
By combining Lemma 7.3 and the doubling property of \( \nu \) we conclude that
\[
\mu_{\beta}(B_p(z, a^N)) \asymp a^{\beta N} \nu(B(Z, a^N)) \asymp a^{\beta N} \nu(B(Z, C'a^N)).
\]
Thus
\[
\int_{B_p(z, a^N)} |Pf - f(z)|^q d \mu_{\beta} \lesssim \int_{B_Z(z, C'a^N)} |f - f(z)|^q d \nu.
\]
Letting \( N \to \infty \), we conclude that \( z \) is also an \( L^q(X_Z) \)-Lebesgue point for \( Pf \) and that \( Pf(z) = f(z) \) (by the definition (10.1) of \( Pf(z) \)). The conclusions for \( P_n f \) for each \( n \in Z \) then follow from the fact that \( P_n f = Pf \) on \( X_{\geq n+1} \cup Z \), which implies that the \( L^q(X_Z) \)-Lebesgue points for \( P_n f \) and \( P f \) on \( Z \) are the same.

When specialized to Lipschitz functions, Propositions 10.2 and 10.3 show that the extension \( Pf \) is also Lipschitz and restricts to \( f \) on \( Z \).

**Proposition 10.4.** Let \( f : Z \to \mathbb{R} \) be \( L \)-Lipschitz. Then there is a constant \( C = C(a, \tau) \geq 1 \) such that \( Pf : \bar{X}_p \to \mathbb{R} \) is \( CL \)-Lipschitz. Furthermore we have \( Pf|_Z = f \).
Proof. For any vertex $v \in V$ we have

$$|Pf(v) - f(\pi(v))| \leq \int_{B(v)} |f - f(\pi(v))| \, d\nu \leq L\tau a^{h(v)}.$$ 

Let $e$ be an edge of $X$ with vertices $v$ and $w$. Then $B(v) \cap B(w) \neq \emptyset$, so we must have $d(\pi(v), \pi(w)) \leq 2\tau a^{h(v) - 1}$ since $|h(v) - h(w)| \leq 1$. We then conclude that

$$|Pf(v) - Pf(w)| \leq |Pf(v) - f(\pi(v))| + |f(\pi(v)) - f(\pi(w))| + |f(\pi(w)) - Pf(w)|$$

$$\leq 4\tau a^{-1}L\alpha^{h(v)}.$$ 

We have $d_p(v, w) = a^{h(v)}$ by Lemma 6.3 with comparison constant depending only on $\alpha$ and $\tau$. Thus if we define $g_e$ as in (10.6) then we conclude that $g_e \leq CL$ with $C = C(a, \tau) \geq 1$. We conclude that $Pf$ is CL-Lipschitz in the metric $d_p$ on each edge of $X_p$. Since $X_p$ is geodesic it follows that $Pf$ is CL-Lipschitz on $X_p$.

Since $f$ is Lipschitz we have that every point of $Z$ is an $L^1(Z)$-Lebesgue point for $f$. Proposition 10.3 then implies that every point of $Z$ is an $L^1(X_p)$-Lebesgue point for $Pf$. Fix a point $z \in Z$ and let $\gamma : [0, \infty) \to X$ be an ascending vertical geodesic ray anchored at $z$ as constructed in Lemma 6.3 with vertices $v_n = \gamma(n)$ on $\gamma$ for each $n \geq 0$ satisfying $h(v_n) = n$. Let $r > 0$ be given and choose $n$ large enough that $v_n \in B_p(z, r)$. Since $Pf$ is CL-Lipschitz on $X_p$ and since $z \in B(v_n)$, we have

$$\int_{B_p(z, r)} |Pf - f(z)| \, d\mu_\beta \leq 2CLr + |Pf(v_n) - f(z)|$$

$$\leq 2CLr + \int_{B(v_n)} |f - f(z)| \, d\nu$$

$$\leq 2CLr + CL\alpha^n.$$ 

As $r \to 0$ we have $n \to \infty$, so we conclude that

$$\lim_{r \to 0} \int_{B_p(z, r)} |Pf - f(z)| \, d\mu_\beta = 0.$$ 

Since $z$ is an $L^1(\bar{X}_p)$-Lebesgue point for $Pf$, it follows that $Pf(z) = f(z)$. Since this holds for any $z \in Z$ we conclude that $Pf|_Z = f$ and that every point of $\bar{X}_p$ is an $L^1(\bar{X}_p)$-Lebesgue point for $Pf$.

Let $\hat{u}$ denote the unique $CL$-Lipschitz extension of $Pf|_X$ to $\bar{X}_p$. Then every point of $\bar{X}_p$ is also an $L^1(\bar{X}_p)$-Lebesgue point for $\hat{u}$. Since $\mu_\beta(Z) = 0$, we have $\hat{u} = Pf$ $\mu_\beta$-a.e. on $\bar{X}_p$. Since every point of $\bar{X}_p$ is an $L^1(\bar{X}_p)$-Lebesgue point for both $\hat{u}$ and $Pf$, we conclude that $\hat{u} = Pf$. This implies in particular that $Pf$ is $CL$-Lipschitz on $\bar{X}_p$, as desired. \hfill \Box

Remark 10.5. For Lipschitz functions $f : Z \to \mathbb{R}$ one can define a simpler Lipschitz extension $\tilde{f} : \bar{X}_p \to \mathbb{R}$ that does not make use of the measure $\nu$ by setting $\tilde{f}(v) = f(\pi(v))$ for each vertex $v \in V$ and linearly interpolating the values of $\tilde{f}$ on the edges between vertices (with respect to the metric $d_p$). This extension will have the same properties as the extension $Pf$ of $f$ used in Proposition 10.4.

We can now relate the trace and extension operators.

**Proposition 10.6.** Let $f \in \tilde{B}^\theta_p(Z)$ for $p > \beta$ and $\theta = (p - \beta)/p$. Then $T(Pf) = f$ $\nu$-a.e. Consequently the induced trace operators $T : D^1,\nu(X_p) \to B^\theta_p(Z)$ and $T : N^1,\nu(X_p) \to \tilde{B}^\theta_p(Z)$ are surjective.
Proof. Let \( f \in \tilde{B}_p^\theta(Z) \) be given. Since \( f \in L^1_{\text{loc}}(Z) \) we have that \( \nu \)-a.e. point of \( Z \) is an \( L^1(Z) \)-Lebesgue point of \( f \) by the Lebesgue differentiation theorem \[10\] Theorem 1.8. By Proposition \[10.2\] we have \( Pf \in \tilde{D}^{1,p}(\bar{X}_p) \), and by Proposition \[10.3\] we have that each \( L^1(Z) \)-Lebesgue point \( z \in Z \) for is an \( L^1(\bar{X}_p) \)-Lebesgue point for \( Pf \) that satisfies \( Pf(z) = f(z) \). We write \( u = Pf|_{X_u} \) and let \( \tilde{u} \) denote the extension of \( u \) to \( \bar{X}_p \) given by Proposition \[9.11\]. Then \( \tilde{u}|_Z = Tu \) \( \nu \)-a.e. On the other hand we have by \[9.27\] and Hölder’s inequality that each \( L^1(\bar{X}_p) \)-Lebesgue point of \( Pf \) on \( Z \) (and therefore each \( L^1(Z) \)-Lebesgue point of \( Z \)) satisfies \( Pf(z) = \tilde{u}(z) \). We conclude that \( Pf|_Z = \tilde{u}|_Z = T(Pf) \) \( \nu \)-a.e. Since \( Pf|_Z = f \) \( \nu \)-a.e. we conclude that \( T(Pf) = f \) \( \nu \)-a.e., as desired.

Let \( f \in \tilde{B}_p^\theta(Z) \) be arbitrary. Then \( u = Pf|_{X_u} \) defines an element of \( \tilde{D}^{1,p}(X_p) \) such that \( Tu = f \) \( \nu \)-a.e. on \( Z \) and therefore \( Tu = f \) in \( B^\theta_p(Z) \). We conclude that the trace operator \( T : D^{1,p}(X_p) \to B^\theta_p(Z) \) is surjective.

Now let \( f \in \tilde{B}^\theta_p(Z) \) be arbitrary. We set \( u = P_0f|_{X_u} \). Then since \( u|_{X_{u \geq 1}} = Pf|_{X_{u \geq 1}} \), we conclude that \( u \) defines an element of \( N^{1,p}(X_p) \) such that \( Tu = f \) \( \nu \)-a.e. on \( Z \), which implies that \( Tu = f \) in \( \tilde{B}^\theta_p(Z) \). Thus the trace operator \( T : N^{1,p}(X_p) \to B^\theta_p(Z) \) is also surjective. \( \square \)

Proposition \[10.6\] completes the proof of Theorem \[10.6\] The representative \( Pf|_Z \) of \( f \) in \( \tilde{B}^\theta_p(Z) \) constructed in the proof of Proposition \[10.6\] is better behaved than the original function \( f \) in many respects. We elaborate on this in the next section.

As a first application of Proposition \[10.6\] we show that the Besov capacity can be computed in terms of the \( C^\rho_p \)-capacity and vice versa. We recall that the implied constants below depend only on \( a, \tau, C_\nu, p, \) and \( \beta \).

**Proposition 10.7.** Let \( p > \beta \) and set \( \theta = (p - \beta)/p \). Then for any set \( G \subset Z \) we have \( C^\rho_p(G) \asymp C^\theta_p(G) \).

**Proof.** Let \( G \subset Z \) be a given subset. Let \( f \in \tilde{B}^\theta_p(Z) \) be given such that \( f \geq 1 \) \( \nu \)-a.e. on a neighborhood \( U \) of \( G \). By truncating \( f \) using Lemma \[8.9\] and then redefining \( f \) on a \( \nu \)-null set, we can assume that we in fact \( f \equiv 1 \) on \( U \). Let \( P_0f \in N^{1,p}(\bar{X}_p) \) be the extension of \( f \) given by Proposition \[10.2\]. Since every point of \( U \) is an \( L^1(Z) \)-Lebesgue point for \( f \), we conclude by Proposition \[10.3\] that \( P_0f|_U = f \). Thus \( Pf = 1 \) on \( U \). It follows that \( Pf \) is admissible for the \( C^\theta_p \)-capacity of \( G \) and therefore

\[
C^\theta_p(G) \leq \|P_0f\|_{N^{1,p}(\bar{X}_p)}^p \lesssim \|f\|_{B^\theta_p(Z)}^p.
\]

Minimizing over all admissible \( f \) for the \( C^\theta_p \)-capacity of \( G \) then gives \( C^\rho_p(G) \lesssim C^\theta_p(G) \).

For the other direction, let \( \varepsilon > 0 \) be given and let \( U \subset \bar{X}_p \) be an open set containing \( G \) such that \( C^\rho_p(U) < \varepsilon \), which we can find by Theorem \[8.6\]. We let \( u \in \tilde{N}^{1,p}(\bar{X}_p) \) be a function such that \( u \geq 1 \) on \( U \) and \( \|u\|_{N^{1,p}(\bar{X}_p)} < C^\rho_p(U) + \varepsilon \). Then \( u|_Z \in B^\theta_p(Z) \) by Proposition \[10.10\] and \( u \geq 1 \) on an open subset \( U \cap Z \subset Z \) of \( Z \) that contains \( G \). Thus

\[
C^\theta_p(G) \leq \|u|_Z\|_{B^\theta_p(Z)} \lesssim \|u\|_{N^{1,p}(\bar{X}_p)}< C^\rho_p(U) + \varepsilon < C^\rho_p(G) + 2\varepsilon.
\]
Letting $\varepsilon \to 0$ gives the desired result. \hfill \square

11. Properties of Besov spaces

In this final section we apply the results of Sections 9 and 10 to establish a number of properties of the Besov spaces $\dot{B}_p^\theta(Z)$ on a complete doubling metric measure space $(Z, d, \nu)$ for $p \geq 1$ and $0 < \theta < 1$. Most of these properties were previously established by Björn-Björn-Shanmugalingam in the case that $Z$ is compact [3, Section 13]. In particular we prove Corollaries 12 and 13 and 14.

For this section we consider a complete doubling metric measure space $(Z, d, \nu)$. We let $X$ be a hyperbolic filling of $Z$ with parameters $a = \frac{1}{2}$ and $\tau = 4$; these parameters satisfy the requirement (4.1), as can be easily checked. We let $\mu$ be the lift of the measure $\nu$ to $X$ defined in (4.2). We then let $X_p$ be the uniformized hyperbolic filling corresponding to these parameters as defined in Section 4. For a given $p \geq 1$ and $0 < \theta < 1$ we set $\beta = p(1 - \theta)$, noting that we then have $p > \beta$ and $\theta = (p - \beta)/p$. Note that in contrast to Sections 9 and 10 we are considering $\beta$ here as depending on $p$ and $\theta$ instead of considering $\theta$ as depending on $p$ and $\beta$. We then let $\mu_\beta$ be the measure on $\tilde{X}_p$ defined by (7.6).

Throughout the rest of this section all implied constants will depend only on the doubling constant $C_\nu$ for $\nu$ as well as the exponents $p \geq 1$ and $0 < \theta < 1$. The uniformized filling $X_p$ will always be considered to be equipped with the measure $\mu_\beta$ for $\beta = p(1 - \theta)$, where $p$ and $\theta$ are given in the hypotheses of each proposition.

Our first application of this section establishes the fact that the homogeneous Besov space $B_p^\theta(Z) = \dot{B}_p^\theta(Z)/\sim$ is a Banach space.

**Proposition 11.1.** $B_p^\theta(Z)$ is a Banach space for $p \geq 1$ and $0 < \theta < 1$.

*Proof.* Let $\{f_n\}$ be a Cauchy sequence in $B_p^\theta(Z)$, and let $\{\tilde{f}_n\}$ be a sequence of representatives of these functions in $\dot{B}_p^\theta(Z)$. The sequence of functions $\{P\tilde{f}_n\}$ in $\dot{D}^{1,\theta}(\tilde{X}_p)$ then defines a Cauchy sequence in $D^{1,\theta}(X_p)$ by Proposition 10.2 and the linearity of the extension operator $P$. Since $D^{1,\theta}(X_p)$ is a Banach space by Proposition 8.3 we conclude that there is a function $u \in \dot{D}^{1,\theta}(\tilde{X}_p)$ such that $\|P\tilde{f}_n - u\|_{D^{1,\theta}(X_p)} \to 0$. Then $u$ has a trace $Tu \in \dot{B}_p^\theta(Z)$ by Proposition 9.8. By Proposition 9.8 we also conclude that

$$
\|T(P\tilde{f}_n) - Tu\|_{B_p^\theta(Z)} = \|T(P\tilde{f}_n - u)\|_{B_p^\theta(Z)} \to 0.
$$

Since $T(P\tilde{f}_n) = \tilde{f}_n$ $\nu$-a.e. on $Z$ by Proposition 10.6 this implies that $\|\tilde{f}_n - Tu\|_{B_p^\theta(Z)} \to 0$. Letting $f$ denote the projection of $Tu$ to $B_p^\theta(Z)$, we conclude that $f_n \to f$ in $B_p^\theta(Z)$. It follows that $B_p^\theta(Z)$ is a Banach space. \hfill \square

Our next application concerns the density of Lipschitz functions in $B_p^\theta(Z)$. We note that $Z$ is proper since it is complete and doubling, which implies that closed and bounded subsets of $Z$ are compact.

**Proposition 11.2.** Lipschitz functions with compact support are dense in $B_p^\theta(Z)$ for $p \geq 1$ and $0 < \theta < 1$.

*Proof.* We first remark that Lipschitz functions with compact support belong to $\dot{B}_p^\theta(Z)$ by Proposition 8.1. We next note that functions with compact support are dense in $N^{1,\theta}(\tilde{X}_p)$ by Proposition 7.1.35 since the metric measure space $(\tilde{X}_p, d_p, \mu_\beta)$ is complete, doubling, and supports a $p$-Poincaré inequality. If $u \in N^{1,\theta}(\tilde{X}_p)$ has compact support then the proof of Theorem 8.2.1 shows that one can find a sequence of Lipschitz functions $\{u_n\}$ on $\tilde{X}_p$
with compact support such that \( u_n \to u \) in \( N^{1,p}(\overline{\mathcal{X}}_\rho) \). It follows that Lipschitz functions with compact support are dense in \( N^{1,p}(\overline{\mathcal{X}}_\rho) \).

Let \( f \in \tilde{B}^\theta_p(Z) \) be given. Let \( P_0 f \in N^{1,p}(\overline{\mathcal{X}}_\rho) \) be the extension of \( f \) constructed in Proposition 10.2. We can then find a sequence of Lipschitz functions \( \{ u_n \} \) on \( \overline{\mathcal{X}}_\rho \) with compact support such that \( \| u_n - P_0 f \|_{N^{1,p}(\overline{\mathcal{X}}_\rho)} \to 0 \). By Proposition 9.8 we then have \( \| Tu_n - T(P_0 f) \|_{\tilde{B}^\theta_p(Z)} \to 0 \). By Proposition 9.7 we have \( Tu_n = u_n|_Z \) for each \( n \) and by Proposition 10.6 we have \( T(P_0 f) = f \) \( \nu \)-a.e. It follows that \( \| u_n|_Z - f \|_{\tilde{B}^\theta_p(Z)} \to 0 \). Since each of the restrictions \( u_n|_Z \) are Lipschitz functions on \( Z \) with compact support, we conclude that Lipschitz functions with compact support are dense in \( \tilde{B}^\theta_p(Z) \).

As we remarked after the proof of Proposition 10.6 given \( f \in \tilde{B}^\theta_p(Z) \) it is possible to find a representative of \( f \) with better regularity properties. Proposition 11.3 below makes this precise.

**Proposition 11.3.** For each \( f \in \tilde{B}^\theta_p(Z) \) there is a \( C^Z_{\tilde{B}^\theta_p} \)-quasicontinuous function \( \tilde{f} \) such that \( f = \tilde{f} \) \( \nu \)-a.e. in \( Z \).

**Proof.** We let \( f = Pf|_Z \) be the restriction of \( Pf \) to \( Z \) that was considered in the proof of Proposition 10.6 as shown in the proof we have \( \tilde{f} = f \) \( \nu \)-a.e. on \( Z \). Since \( Pf \in \tilde{D}^{1,p}(\overline{\mathcal{X}}_\rho) \), we have by Proposition 8.8 that \( Pf \) is \( C^X_{\overline{\mathcal{X}}_\rho} \)-quasicontinuous. Thus for each \( \eta > 0 \) we can find an open subset \( U \subset \overline{\mathcal{X}}_\rho \) such that \( C^X_{\overline{\mathcal{X}}_\rho}(U) < \eta \) and \( Pf|_{\overline{\mathcal{X}}_\rho \setminus U} \) is continuous. Setting \( W = U \cap Z \), we conclude that \( W \) is open in \( Z \) and that \( \tilde{f}|_Z \setminus W \) is continuous. By Proposition 10.7 we have

\[
C^Z_{\tilde{B}^\theta_p}(W) \preceq C^\overline{\mathcal{X}}_{\overline{\mathcal{X}}_\rho}(W) \preceq C^X_{\overline{\mathcal{X}}_\rho}(U) < \eta.
\]

Since \( \eta > 0 \) was arbitrary, we conclude that \( \tilde{f} \) is \( C^Z_{\tilde{B}^\theta_p} \)-quasicontinuous. \( \square \)

We next consider embeddings of Besov spaces into Hölder spaces. For this next proposition we recall the notion of relative lower volume decay defined in (7.17), and recall that every doubling measure \( \nu \) satisfies this condition for \( Q = \log_2 C_\nu \).

**Proposition 11.4.** Suppose that \( \nu \) has relative lower volume decay of order \( Q > 0 \). Let \( p \geq 1 \) and \( 0 < \theta < 1 \) be given. We set \( \beta = p(1-\theta) \) and let \( Q_\beta = \max\{1, Q + \beta\} \). If \( p > Q_\beta \) then every \( f \in \tilde{B}^\theta_p(Z) \) has a \( \nu \)-a.e. representative that is \( (1-Q_\beta/p) \)-Hölder continuous on each ball \( B \subset Z \).

**Proof.** Let \( f \in \tilde{B}^\theta_p(Z) \) be given with \( p \) satisfying \( p > Q_\beta \). As in the proof of Proposition 11.3 we let \( f = Pf|_Z \) denote the restriction of the extension \( Pf \) to \( Z \), which satisfies \( Pf|_Z = f \) \( \nu \)-a.e. by Proposition 10.6. By Lemma 7.4 the metric measure space \( (\overline{\mathcal{X}}_\rho, d_\rho, \mu_\beta) \) has relative lower volume decay of order \( Q_\beta \). Since \( p > Q_\beta \), we conclude from [21] Theorem 9.2.4] that functions in \( \tilde{D}^{1,p}(\overline{\mathcal{X}}_\rho) \) are \( (1-Q_\beta/p) \)-Hölder continuous on each ball \( \tilde{B} \subset \overline{\mathcal{X}}_\rho \). Since the metric \( d_\rho \) on \( \overline{\mathcal{X}}_\rho \) restricts to the metric \( d \) on \( Z \), we conclude that \( Pf|_Z \) is \( (1-Q_\beta/p) \)-Hölder continuous on any ball \( B = B_Z(z,r) \subset Z \), viewing this ball as a subset of the ball \( \tilde{B} = B_\rho(z,r) \subset \overline{\mathcal{X}}_\rho \). \( \square \)

The calculations after [3] Proposition 13.7] give more precise details about the settings in which Proposition 11.4 applies.

We can also show that functions \( f \in \tilde{B}^\theta_p(Z) \) have Lebesgue points quasi-everywhere with respect to the Besov capacity. For this we will need to assume that the measure \( \nu \) also
satisfies a reverse-doubling condition for an exponent $\eta > 0$ and any $0 < r' \leq r$,

$$\frac{\nu(B_z(r'))}{\nu(B_z(r))} \leq C_{rev} \left(\frac{r'}{r}\right)^{\eta},$$

for some constant $C_{rev} \geq 1$. Under the hypothesis that $\nu$ is doubling, the reverse-doubling condition (11.1) on $\nu$ is equivalent to $Z$ being uniformly perfect [25, Lemma 7], which in turn implies that for each ball $B \subset Z$ we have

$$r(B) \asymp \text{diam}(B),$$

with implied constants depending only on the doubling constant for $\nu$ and the constant $C_{rev}$ and exponent $\eta$ in (11.1).

We then have the following two-weighted Poincaré inequality [3, Proposition 13.5], which is a special case of an inequality of Björn-Kalajdija [5, Theorem 3.1]; we note that this inequality still applies in our setting since we assume that the reverse-doubling condition (11.1) holds at all scales and since we have that the metric measure spaces $(X, d, \mu)$ and $(\tilde{X}, \rho, \mu_{\beta})$ support a $p$-Poincaré inequality. Below we set $\beta = p(1 - \theta) > 0$ and $B = B_Z(z, r)$ for $z \in Z$ and $r > 0$. We recall that $\tilde{B} = B_{\rho}(z, r)$ then denotes the ball centered at $z$ of the same radius in $\tilde{X}_\rho$. We will allow $X$ to be a hyperbolic filling of $Z$ for any parameter $0 < a < 1$ and $\tau > \max\{3, (1 - a)^{-1}\}$, with the understanding that the implied constant in (11.3) further depends on $a$ and $\tau$.

**Proposition 11.5.** Let $1 \leq p < q < \infty$ and $u \in \dot{N}^{1,p}(\tilde{X}_\rho)$ be such that $\nu$-a.e. point of $Z$ is an $L^1(\tilde{X}_\rho)$-Lebesgue point for $u$. Let $g$ be a $p$-integrable $p$-weak upper gradient for $u$ on $\tilde{X}_\rho$. Suppose further that $\nu$ satisfies the reverse-doubling condition (11.1). Then for all balls $B = B_Z(z, r) \subset Z$ with $z \in Z$ and $r > 0$,

$$\left(\int_B |u - u_B|^q \, d\nu\right)^{1/q} \lesssim \Theta_q(r) \left(\int_{2B} g^p \, d\mu_{\beta}\right)^{1/p},$$

with implied constant depending additionally on the constant $C_{rev}$ and exponent $\eta > 0$ in (11.1), where

$$\Theta_q(r) = \sup_{0 < s \leq r} \sup_{z \in B} \frac{\nu(B_z(s))^{1/q}}{\mu_{\beta}(B_{\rho}(z, s))^{1/p}}.$$

We remark that our expression for $\Theta_q(r)$ is equivalent to the one given in [3, Proposition 13.5], since they consider $\nu$ as a measure on $\tilde{X}_\rho$ by setting $\nu(X, \rho) = 0$. In (11.3) $u_B$ denotes the mean value of $u$ over the ball $B$ when $B$ is equipped with the measure $\mu_{\beta}$. By Lemma 7.3 we have

$$\Theta_q(r) \asymp \sup_{0 < s \leq r} \sup_{z \in B} s^{1 - \beta/p} \nu(B_z(s))^{1/q - 1/p}.$$

**Proposition 11.6.** Suppose that $\nu$ has relative lower volume decay of order $Q > 0$ and that $\nu$ satisfies the reverse-doubling condition (11.1) for some $\eta > 0$. Let $p \geq 1$ and $0 < \theta < 1$ be given such that $p\theta < Q$ and set $Q_* = Qp/(Q - p\theta)$. Let $f \in \dot{B}^p_{\nu}(Z)$ be given. Then there is a function $\tilde{f} \in \dot{B}^p_{\nu}(Z)$ such that $\tilde{f} = f$ $\nu$-a.e. and $C_{\tilde{B}^p_{\nu}}^Z$-q.e. point $z \in Z$ is an $L^{Q_*}(Z)$-Lebesgue point of $\tilde{f}$. In particular we have $f \in L^{Q_*}_{\text{loc}}(Z)$.

The conclusion means that there is a set $G \subset Z$ with $C_{\tilde{B}^p_{\nu}}^Z(Z \setminus G) = 0$ for which every point $z \in G$ is an $L^{Q_*}(Z)$-Lebesgue point of $\tilde{f}$. Note by Hölder’s inequality that the conclusion
implies that $C_{B_p}^Z$-q.e. point $z \in Z$ is an $L^q(Z)$-Lebesgue point of $\hat{f}$ for $1 \leq q \leq Q_*$. We also note that we always have $Q_* > p$ when $p \theta < Q$.

Proof. Throughout this proof all implied constants will be allowed to additionally depend on the constant $C_{rev}$ and exponent $\eta > 0$ in (11.4). For the sake of the proof of Proposition 11.7 afterwards, we will consider here a hyperbolic filling $X$ of $Z$ for arbitrary $0 < a < 1$ and $\tau > \max\{3, (1-a)^{-1}\}$, so we will be allowing implied constants to also depend on these parameters $a$ and $\tau$.

Let $f \in \hat{D}^{1,p}(\hat{X}_\rho)$ be given as in the hypotheses. We let $u \in \hat{D}^{1,p}(\hat{X}_\rho)$ be any function such that $u|_Z = f$ $\nu$-a.e. on $Z$; by Proposition 10.4 we can take $u = Pf$, but we will allow for a more general choice of $u$ in the proof. We then set $\hat{f} = u|_Z$. We note that this function $u$ agrees $\mu_\beta$-a.e. with the extension $\hat{u}$ of $u|_{X_\rho}$ to $\hat{X}_\rho$ given by Proposition 9.11 since $\mu_\beta(\partial X_\rho) = 0$, which implies that $\hat{u} = u \in C_p^{X_\rho}$-q.e. since both of these functions belong to $\hat{D}^{1,p}(\hat{X}_\rho)$. By Proposition 9.11 and Hölder’s inequality we then have for $C_p^{X_\rho}$-q.e. $z \in Z$,

$$\lim_{r \to 0+} \int_{B_r(z,r)} |u - \hat{f}(z)| d\mu_\beta = 0,$$

where we have used that $\mu_\beta$ is extended to $Z = \partial X_\rho$ by $\mu_\beta(\partial X_\rho) = 0$. By Proposition 9.9 we then have that $\nu$-a.e. point of $Z$ is an $L^1(X_\rho)$-Lebesgue point for $u$. Let $g$ be any $p$-integrable $p$-weak upper gradient for $u$ on $X_\rho$. Then by [21] Lemma 9.2.4 we have for $C_p^{X_\rho}$-q.e. $z \in Z$,

$$\lim_{r \to 0+} r^p \int_{B_r(z,r)} g^p d\mu_\beta = 0.$$

By Proposition 11.7 it follows that each of these assertions also hold for $C_{B_p}^Z$-q.e. point of $Z$.

We will show that any point $z \in Z$ for which (11.5) and (11.6) hold is an $L^{Q_*}(Z)$-Lebesgue point of $\hat{f}$.

Let $z \in Z$ be a point for which (11.5) and (11.6) hold. By Proposition 11.5 applied with $q = Q_* > p$ we have for each ball $B = B_z(z,r) \subset Z$,

$$\left( \int_B |\hat{f} - u_B|^{Q_*} dv \right)^{1/Q_*} \lesssim \Theta_Q.(r) \left( \int_{2B} g^p d\mu_\beta \right)^{1/p}.$$ 

By combining this with the triangle inequality in $L^{Q_*}(Z)$ we conclude that

$$\left( \int_B |\hat{f} - \hat{f}(z)|^{Q_*} dv \right)^{1/Q_*} \lesssim \Theta_Q.(r) \left( \int_{2B} g^p d\mu_\beta \right)^{1/p} + \nu(B)^{1/Q_*} |u_B - \hat{f}(z)|.$$ 

By dividing through by $\nu(B)^{1/Q_*}$, we conclude that

$$\left( \int_B |\hat{f} - \hat{f}(z)|^{Q_*} dv \right)^{1/Q_*} \lesssim \Theta_Q.(r) \frac{\mu_\beta((2B)^1/p)}{\nu(B)^{1/Q_*}} \left( \int_{2B} g^p d\mu_\beta \right)^{1/p} + \int_B |u - \hat{f}(z)| d\mu_\beta.$$ 

The second term on the right converges to 0 as $r \to 0$ by (11.3). To show that the first term on the right converges to 0 it suffices by (11.6) to show that we have

$$\frac{\Theta_Q.(r) \mu_\beta((2B)^1/p)}{\nu(B)^{1/Q_*}} \lesssim r.$$
By (11.3), Lemma (7.3) and the lower volume decay bound (7.17) (here we are using an equivalent uncentered version of this bound applied to the containment of balls $B_{Z}(z,s) \subset 2B$ for $z \in B$, $0 < s \leq r$, see (9.1.14)) we have
\[
\frac{\Theta_{Q_{*}}(r)\mu_{\beta}(2B)^{1/p}}{\nu(B)^{1/p}} \lesssim r^{Q/Q_{*}-Q/p+\beta/p} \sup_{0<s\leq r} \sup_{z \in B} s^{1-\beta/p-Q/Q_{*}+Q/p} \\
= r^{Q/Q_{*}-Q/p+\beta/p+2\theta} \\
= r
\]
since, recalling that $\beta = p(1-\theta)$, the exponent of $s$ simplifies to
\[
1 - \frac{\beta}{p} - \frac{Q}{Q_{*}} + \frac{Q}{p} = 1 - \frac{p(1-\theta)}{p} - \frac{Q-p\theta}{p} + \frac{Q}{p} = 2\theta > 0,
\]
and the exponent of $r$ in the first line simplifies to
\[
\frac{Q}{Q_{*}} - \frac{Q}{p} + \frac{\beta}{p} = \frac{Q-p\theta}{p} - \frac{Q}{p} + \frac{Q(1-\theta)}{p} = 1 - 2\theta.
\]
We conclude that $C_{B}^{p,q,E}$, q.e. point of $Z$ is an $L^{Q_{*}}(Z)$-Lebesgue point of $\tilde{f}$, as desired. Lastly, for use in Proposition (11.7) below we note that combining (11.7), (11.9), and the radius estimate (11.2),
\[
(11.10) \left( \int_{B} |f-u_{B}|^{Q_{*}} \, d\nu \right)^{1/Q_{*}} \lesssim \text{diam}(B) \left( \int_{2B} g^{p} \, d\mu_{\beta} \right)^{1/p},
\]
where we have used that $f = \tilde{f}$ $\nu$-a.e. on $Z$. Here $g$ denotes any $p$-integrable $p$-weak upper gradient for the extension $u \in \tilde{D}^{q}(\tilde{X}_{\rho})$, and the implied constant depends on the constants and exponents in (7.17) and (11.1) as well as the constants $a$ and $\tau$ associated to the hyperbolic filling $X$ and the exponents $p$ and $\theta$. \qed

By Hölder’s inequality the inequality (11.10) implies the following important inequality for functions in $\tilde{B}_{p}^{q}(Z)$, which we will call a hyperbolic $(q,p)$-Poincaré inequality since it relates the $q$-means of functions on balls in $Z$ to the $p$-norms of upper gradients of extensions of those functions to the uniformized hyperbolic filling $\tilde{X}_{\rho}$. The second inequality follows from the first by applying Lemma (5.1) with $\alpha = u_{B}$.

**Proposition 11.7.** Suppose that $\nu$ has relative lower volume decay of order $Q > 0$ and that $\nu$ satisfies the reverse-doubling condition (11.11) for some $\eta > 0$. Let $p \geq 1$ and $0 < \theta < 1$ be given such that $p\theta < Q$ and set $Q_{*} = Qp/(Q-p\theta)$. Let $f \in B_{p}^{q}(Z)$ be given and let $g : \tilde{X}_{\rho} \to [0,\infty]$ be a Borel function that is a $p$-integrable $p$-weak upper gradient of some function $u \in \tilde{D}^{q}(\tilde{X}_{\rho})$ such that $u|_{Z} = f$ $\nu$-a.e. Then for any $1 \leq q \leq Q_{*}$ and any ball $B \subset Z$ we have
\[
\left( \int_{B} |f-u_{\tilde{B}}|^{q} \, d\nu \right)^{1/q} \lesssim \text{diam}(B) \left( \int_{2B} g^{p} \, d\mu_{\beta} \right)^{1/p},
\]
with implied constant depending only on the constants and exponents in (7.17) and (11.1), the constants $a$ and $\tau$ associated to the hyperbolic filling $X$, and the exponents $p$ and $\theta$. Consequently we have
\[
\left( \int_{B} |f-f_{B}|^{q} \, d\nu \right)^{1/q} \lesssim \text{diam}(B) \left( \int_{2B} g^{p} \, d\mu_{\beta} \right)^{1/p}.
\]
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