A proof that square ice entropy is $\frac{3}{2} \log_2(4/3)$

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Abstract
In this text, we provide a fully rigorous and complete proof of E.H.Lieb’s statement that (topological) entropy of square ice (or six vertex model, XXZ spin chain for anisotropy parameter $\Delta = 1/2$) is equal to $\frac{3}{2} \log_2(4/3)$. For this purpose, we gather and expose in full detail various arguments dispersed in the literature on the subject, and complete several of them that were left partial.

1 Introduction

Separate the earth from the fire, the subtle from the raw, sweetly with great industry. - The Emeralt tablet

1.1 Computing entropy of multidimensional subshifts of finite type

This work is the consequence of a renewal of interest from the field of symbolic dynamics to entropy computation methods developed in quantum and statistical physics for lattice models. This interest comes from constructive methods for multidimensional subshifts of finite type (some equivalent formulation in symbolic dynamics of lattice models), that are involved in the characterization by M. Hochman and T. Meyerovitch of the possible values of topological entropy for these dynamical systems (where the dynamics are provided by the action of the $\mathbb{Z}^2$ shift action) with a recursion-theoretic criterion. The consequences of this theorem are not only that entropy may be algorithmically uncomputable for a multidimensional subshift of finite type, which was previously known, but also a strong evidence that the study of these systems as a class is intertwined with computability theory. Moreover, it is an important tool in order to localize sub-classes for which the entropy is computable in a uniform way, as ones defined by strong dynamical constraints. Some current research attempts to understand the frontier between the uncomputability and the computability of entropy for multidimensional SFT. For instance, approaching the frontier from the uncomputable, the author, together with M. Sablik, proved that the characterization of M. Hochman and T. Meyerovitch stands under a relaxed form of the constraint studied in, which includes notably all square lattice models considered exactly solvable in quantum and statistical physics. Solvable means here that some exact (not necessary proved) values are provided for some characteristics of the model, such as entropy. In order to approach the frontier from the computable, it is natural to attempt understanding (in particular proving) and extending the computation methods developed for these models.
1.2 Content of this text

Our study in the present text focuses on square ice (or equivalently the six vertex model, or the
XXZ spin chain for anisotropy parameter $\Delta = 1/2$). Since it is central amongst quantum solvable
models, this work will serve as a ground for further connections between entropy computation
methods and constructive methods coming from symbolic dynamics. The entropy of square ice was
argued by E.H. Lieb [Lieb 1967] to be exactly $\frac{3}{2} \log_2 \left( \frac{4}{3} \right)$. However, his proof was not complete,
as it relied on a non verified hypothesis (the condensation of Bethe roots, defined in the text,
according to a density function, proved in Section 6). Moreover, some arguments of an article of
C.N Yang and C.P. Yang [Yang Yang I] on which it relied were left partial (the analyticity of the
roots according to the anisotropy parameter, proved in Section 5.3). We propose a rigorous proof
of E.H.Lieb’s statement:

**Theorem 1.** The entropy of square ice is equal to $\frac{3}{2} \log_2 \left( \frac{4}{3} \right)$.

This proof relies on the argumentation of E.H.Lieb and on some ideas, developed in order to
prove the hypothesis of E.H.Lieb, that one can find in [Kozlowski]. Besides partial arguments, var-
ious hurdles prevent the readers (in particular with mathematical background) to have an overview
of the subject in reasonable time. That is why we include some exposition of what can be considered
as background material. The proof is thus self-contained, except that it relies on the coordinate
Bethe ansatz, exposed in a clear way by H. Duminil-Copin et al. [Duminil-Copin et al.]. In proving Theorem 1
an important difficulty was at first to connect and assemble the arguments that
were found in the literature; we encountered many obstacles, that are related to the form of the
literature itself. This could be explained by the fact that mathematics and mathematical physics,
although they share the same language, are different discursive formations, notion introduced in
the Archeology of knowledge by M.Foucault. The mathematical text is (in our view) in partic-
ular serve a neat separation between statements that have different natures (theorem and proof,
or comments); the hermetic axiom quoted above reflects this separation, where the earth could
represents the ground in thought and statements (as proven theorems), and the fire the continuing
transformation. In Section 8 one can find a short analysis on this matter and also some comments
on the limits of the computing method presented in this text. We hope this short analysis could
enlight the nature of the difficulties of this work, and will be followed by further developments.

One can find a summary of the proof of Theorem 1 in Section 3 after some recall of definitions
related to symbolic dynamics and representations of square ice in Section 2.

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2 Background: square ice and its entropy

2.1 Subshifts of finite type

2.1.1 Definitions

Let $\mathcal{A}$ be some finite set, called alphabet. For all $d \geq 1$, the set $\mathcal{A}^{\mathbb{Z}^d}$, whose elements are called
configurations, is a topological space with the infinite power of the discrete topology on $\mathcal{A}$. Let us
denote $\sigma$ the shift action of $\mathbb{Z}^d$ on this space defined by the following equality for all $u \in \mathbb{Z}^d$ and $x$
element of the space: $(\sigma^u.x)_v = x_{v+u}$. A compact subset $X$ of this space is called a $d$-dimensional
subshift when this subset is stable under the action of the shift, which means that for all $u \in \mathbb{Z}^d$:
$\sigma^u.X \subset X$. For any finite subset $U$ of $\mathbb{Z}^d$, an element $p$ of $\mathcal{A}^U$ is called a pattern on the alphabet
$\mathcal{A}$ and on support $U$. We say that this pattern appears in a configuration $x$ when there exists
a translate $V$ of $U$ such that $x_V = p$. We say that it appears in another pattern $q$ on support
containing $U$ such that the restriction of $q$ on $U$ is $p$. We say that it appears in a subshift $X$ when
it appears in a configuration of $X$. Such a pattern is also called globally admissible for $X$. For
all \( d \geq 1 \), the number of patterns on support \( U^d_N \equiv [1,N]^d \) that appear in a \( d \)-dimensional subshift \( X \) is denoted \( N_N(X) \). When \( d = 2 \), the number of patterns on support \( U^2_{M,N} \equiv [1,M] \times [1,N] \) that appear in \( X \) is denoted \( N_{M,N}(X) \). A \( d \)-dimensional subshift \( X \) defined by forbidding patterns in some finite set \( F \) to appear in the configurations, formally:

\[
X = \left\{ x \in A^Z : \forall U \subset Z, x_U \notin F \right\}
\]

is called a subshift of finite type (SFT). In a context where the set of forbidden patterns defining the SFT is fixed, a pattern is called locally admissible for this SFT when no forbidden pattern appears in it. A morphism between two \( Z^d \)-subshifts \( X,Z \) is a continuous map \( \varphi : X \rightarrow Z \) such that \( \varphi \circ \sigma^v = \sigma^v \circ \varphi \) for all \( v \in Z^d \) (the map commutes with the shift action). An isomorphism is an invertible morphism.

### 2.1.2 Topological entropy

**Definition 1.** Let \( X \) be a \( d \)-dimensional subshift. The topological entropy of \( X \) is defined as:

\[
h(X) \equiv \inf_{N \geq 1} \frac{\log_2(N_N(X))}{N^d}.
\]

It is a well known fact in topological dynamics that this infimum is a limit:

\[
h(X) = \lim_{N \geq 1} \frac{\log_2(N_N(X))}{N^d}
\]

It is a topological invariant, meaning that when there is an isomorphism between two subshifts, these two subshifts have the same entropy [Lind Marcus].

**Definition 2.** Let \( X \) be a bidimensional subshift \((d = 2)\). For all \( n \geq 1 \), we denote \( X_N \) the subshift obtained from \( X \) by restricting to the width \( N \) infinite strip \( \{1,...,N\} \times \mathbb{Z} \). Formally, this subshift is defined on alphabet \( A^N \) and by that \( z \in X_N \) if and only if there exists \( x \in X \) such that for all \( k \in \mathbb{Z} \), \( z_k = (x_{1,k},...,x_{N,k}) \). See Figure 1

![Figure 1: Illustration of Definition 2 for \( N = 3 \).](image)

**Proposition 1.** The entropy of \( X \) can be computed through the sequence \( (h(X_N))_N \):

\[
h(X) = \lim_{N} \frac{h(X_N)}{N}
\]

We include a proof of this statement, for completeness:
Proof. From the definition of $X_N$:

$$h(X) = \lim_{N} \lim_{M} \frac{\log_2(\mathcal{N}_{M,N}(X))}{NM}.$$ 

We prove this by an upper bound on the $\lim sup_N$ and a lower bound on the $\lim inf_N$ of the sequence in this formula.

- **Upper bound by cutting squares in rectangles:** Since for any $M, N, k$, the set $\cup_{kM,kN}^{(2)}$ is the union of $MN$ translates of $\cup_{k}^{(2)}$, a pattern on support $\cup_{kM,kN}^{(2)}$ can be seen as an array of patterns on $\cup_{k}^{(2)}$. As a consequence,

$$\mathcal{N}_{kM,kN}(X) \leq (\mathcal{N}_{k,k}(X))^{MN},$$

and using this inequality, we get:

$$\lim_{M} \frac{\log_2(\mathcal{N}_{M,N}(X))}{NM} = \lim_{M} \frac{\log_2(\mathcal{N}_{kM,kN}(X))}{k^2NM} \leq \lim_{M} \frac{\log_2(\mathcal{N}_{k,k}(X))}{k^2} = \frac{\log_2(\mathcal{N}_{k,k}(X))}{k^2}.$$ 

As a consequence, for all $k$,

$$\lim_{N} \lim_{M} \frac{\log_2(\mathcal{N}_{M,N}(X))}{NM} \leq \frac{\log_2(\mathcal{N}_{k,k}(X))}{k^2},$$

and this implies

$$\lim_{N} \lim_{M} \frac{\log_2(\mathcal{N}_{M,N}(X))}{NM} \leq h(X),$$

by taking $k \to +\infty$ in the last inequality.

- **Lower bound by cutting rectangles in squares:** For all $M, N$, by considering a pattern on $\cup_{MN,NM}^{(2)}$ as a union of translates of $\cup_{M,N}^{(2)}$, we get that:

$$\mathcal{N}_{MN,NM}(X) \leq (\mathcal{N}_{M,N}(X))^{MN}.$$ 

Thus,

$$h(X) = \lim_{M} \frac{\log_2(\mathcal{N}_{MN,NM}(X))}{M^2N^2} \leq \lim_{M} \frac{\log_2(\mathcal{N}_{M,N}(X))}{MN}.$$ 

As a consequence,

$$h(X) \leq \lim_{N} \lim_{M} \frac{\log_2(\mathcal{N}_{M,N}(X))}{NM}.$$ 

These two inequalities prove that the sequence $\left(\frac{h(X_N)}{N}\right)_N$ converges, and that the limit is $h(X)$. □

In the following, for all $N$ and $M$, we assimilate patterns on $\cup_{M}^{(1)}$ of $X_N$ with patterns of $X$ on $\cup_{M,N}^{(2)}$.

### 2.2 Representations of square ice

The square ice can be defined as an isomorphic class of subshifts of finite type, whose elements can be thought as its representations. The most widely used is the six vertex model (whose name derives from that the elements of the alphabet represent vertices of a regular grid) and is presented in Section 2.2.1. In this text, we will use another representation, presented in Section 2.2.2 whose configurations consist of drifting discrete curves, representing possible particle trajectories. In Section 2.2.3 we provide a proof that one can restrict to a particular subset of patterns in order to compute entropy of square ice.
2.2.1 The six vertex model

The six vertex model is the subshift of finite type described as follows:

**Symbols:** 
\[
\begin{array}{ccccccc}
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
  + & + & + & + & + & + & + \\
\end{array}
\]

**Local rules:** Considering two adjacent positions in \( Z^2 \), the arrows corresponding to the common edge of the symbols on the two positions have to be directed the same way. For instance, the pattern \( + + + + \) is allowed, while \( + + + + \) is not.

**Global behavior:** The symbols draw a lattice whose edges are oriented in such a way that all the vertices have two incoming arrows and two outgoing ones. This is called an Eulerian orientation of the square lattice. See an example of admissible pattern on Figure 2.

![Figure 2: An example of locally and thus globally admissible pattern of the six vertex model.](image)

**Remark 1.** The name of square ice of the class of SFT appears clearly when considering the following application on the alphabet of the six vertex model to local configurations of dihydrogen monoxide:

![Application example](image)

2.2.2 Drifting discrete curves

From the six vertex model, we derive another representation of square ice through an isomorphism, which consist in transforming the letters via an application \( \pi_s \) on the alphabet of the six vertex model, described as follows:

![Application example](image)

The pattern on Figure 2 can be represented as on Figure 3. In this SFT, the local rules are that any outgoing segment of curve in a non-blank symbol is extended in its direction on the next position.

In the following, we denote \( X^* \) this SFT.

**Remark 2.** One can see straightforwardly that locally admissible patterns of this SFT are always globally admissible, since any locally admissible pattern can be extended into a configuration by extending the curves in a straight way.
2.2.3 Entropy of $X^s$ and cylindric stripes subshifts of square ice

Consider some alphabet $\mathcal{A}$, and $X$ a bidimensional subshift of finite type on this alphabet. For all $N \geq 1$, we denote $\Pi_N = \mathbb{Z}/(NZ) \times \mathbb{Z}$. Let us also denote $\pi_N : [1, N] \times \mathbb{Z} \to \Pi_N$ the canonical projection, and $\phi_N : \mathcal{A}^{[1,N]} \times \mathbb{Z} \to \mathcal{A}^{\Pi_N}$ the application that wraps configurations of $X_N$ on the infinite cylinder $\Pi_N$. Formally, for all $u \in [1, N] \times \mathbb{Z}$ and $x \in \mathcal{A}^{[1,N]} \times \mathbb{Z}$,

$$\left(\phi_N(x)\right)_{\pi_N(u)} = xu.$$

We say that a pattern $p$ on support $U \subset [1, N] \times \mathbb{Z}$ appears in a configuration $x$ on $\Pi_N$ when there exists a configuration in $X_N$ whose image by $\pi_N$ is $x$ and there exists an element $u \in \Pi_N$ such that for all $v \in U$, $x_{u+v} = x_v$.

**Notation 1.** Let us denote $X_N$ the set of configurations in $X_N$ whose image by $\phi_N$ does not contain any forbidden pattern for $X$ (in other words this pattern can be wrapped on an infinite cylinder without breaking the rules defining $X$).

We also call $(M, N)$-cylindric pattern of $X$ a pattern on $U_{M,N}$ that can be wrapped on a finite cylinder $\mathbb{Z}/N\mathbb{Z} \times \{1, \ldots, M\}$. Let us prove a preliminary result on entropy of square ice, which relates the entropy of $X^s$ to the sequence $(h(X_N^s))_N$:

**Lemma 1.** The subshift $X^s$ has entropy equal to

$$h(X^s) = \lim_{N \to \infty} \frac{h(X_N^s)}{N}.$$

**Remark 3.** In order to prove this lemma, we use a technique that first appeared in a work of S.Friedland [Friedland], that relies on a symmetry of the alphabet and rules of the SFT.

**Proof.**

1. **Lower bound:** Since for all $N$, $X_N^s \subset X_N^*$, then $h(X_N^s) \leq h(X_N^*)$. We deduce by Proposition 1 that

$$\limsup_N \frac{h(X_N^s)}{N} \leq h(X^*).$$

2. **Upper bound:**

Consider the transformation $\tau$ on the six-vertex model alphabet that consists in a horizontal symmetry of the symbols and then the inversion of all the arrows. The symmetry can be represented as follows:

```
  + + + + + + + + + + + +
  |   |   |   |   |   |   |
  + + + + + + + + + + + +
```

The inversion is represented:
As a consequence $\tau$ is:

We define then a horizontal symmetry operation $T_N$ (see Figure 4 for an illustration) on patterns whose support is some $U^{(2)}_{M,N}$, with $M \geq 1$, such that for all $M \geq 1$, $p$ having support $U^{(2)}_{M,N}$, $T_M(p)$ has also support $U^{(2)}_{M,N}$ and for all $(i,j) \in U^{(2)}_{M,N}$,

$$T_N(p_{i,j}) = \tau(p_{N-i,j}).$$

We define also the applications $\partial r_N \equiv \partial r_N(p)$ (resp. $\partial l_N \equiv \partial l_N(p)$) that acts on patterns of the six vertex model whose support is some $U^{(2)}_{M,N}$, $M \geq 1$ and such that for all $M \geq 1$ and $p$ on support $U^{(2)}_{M,N}$, $\partial N(p)$ is a length $M$ word and for all $j$ between 1 and $M$ (resp. $M$), $\partial N(p_j)$ is the east (resp. west) arrow in the symbol $p_{N,j}$ (resp. $p_{1,j}$). For instance, if $p$ is the pattern on the left on Figure 4 then $\partial r_N(p)$ (resp. $\partial l_N(p)$) is the word:

$$←←→→ \quad \text{(resp. } →→→→).$$

Figure 4: Illustration of the definition of $T_3$: the pattern on left (on support $U^{(2)}_{3,4}$) is transformed into the pattern on the right via this transformation.

For the purpose of notation, we denote also $\pi_s$ the application that transforms patterns of the six vertex model into patterns of $X_s$ via the application of $\pi_s$ letter by letter. Let us consider the transformation $T_N \equiv \pi_s \circ T_N \circ \pi_s^{-1}$ on patterns of $X_s$ on some $U^{(2)}_{M,N}$. We also denote $\partial N_s \equiv \partial N \circ \pi_s^{-1}$, $\partial N^s_s \equiv \partial N \circ \pi_s^{-1}$. Let us prove some properties of these transformations. For any word $w$ on the alphabet \{←→\} (or \{↑,↓\}), we denote $\bar{w}$ the word obtained by exchanging the two letters in the word $w$.

(a) **Preservation of global admissibility:**

For any $p$ globally admissible, $T_N(p)$ is also locally admissible, and as a consequence globally admissable: indeed, it is sufficient to check that for all $u,v$ in the alphabet, if $uv$ is not a forbidden pattern in the six vertex model, then $\tau(v)\tau(u)$ is also not a forbidden pattern and that if $u \atop v$ is not forbidden, then $\tau(u) \atop \tau(v)$ is also not forbidden.
The first assertion is verified because $uv$ is not forbidden if and only if the arrows of these symbols attached to their adjacent edge are pointing in the same direction, and this property is conserved when changing $uv$ into $\tau(v)\tau(u)$. The second one is verified for a similar reason.

(b) **Gluing patterns:**

Let us consider any $N, M \geq 1$ and $p, p'$ two patterns of $X^s$ on support $U_{M,N}^{(2)}$, such that $\partial_{N}^{l,s}(p) = \partial_{N}^{l,s}(p')$ and $\partial_{N}^{r,s}(p) = \partial_{N}^{r,s}(p')$. Let us denote pattern $p''$ on support $U_{M,2N}^{(2)}$ such that the restriction of $p''$ on $U_{M,N}^{(2)}$ is $p$ and the restriction on $(0,N) + U_{M,N}^{(2)}$ is $T_N(p')$.

- **This pattern is admissible** (locally and thus globally). Indeed, this is sufficient to check that gluing the two patterns $p$ and $p'$ does not make appear forbidden patterns, and this comes from that for all letter $u$, $u\tau(u)$ is not forbidden. This can be checked directly, letter by letter.

- **Moreover, $p''$ is in $N_{M}(\overline{X_{2N}})$**. Indeed, this pattern can be wrapped on a cylinder, and this comes from the fact that if $u$ is a symbol of the six vertex model, $\tau(u)$ is not forbidden.

3. **From the gluing property to an upper bound:** Given $w = (w^l, w^r)$ some pair of words on $\{\rightarrow, \leftarrow\}$, we denote $N_{M,N}^w$ the number of patterns of $X^s$ on support $U_{M,N}^{(2)}$ such that $\partial_{N}^{l,s}(p) = w^l$ and $\partial_{N}^{r,s}(p) = w^r$. Since $T_N$ is a bijection, denoting $\overline{w} = (\overline{w^l}, \overline{w^r})$, we have

$$N_{M,N}^w = N_{M,N}^{\overline{w}}.$$ 

From last point, for all $w$,

$$N_{M}(\overline{X_{2N}}) \geq N_{M,N}^{w} \cdot N_{M,N}^{\overline{w}} = (N_{M,N}^w)^2 \geq N_{M,2N}^w.$$ 

By summing over all possible $w$:

$$2^{2M} \cdot (N_{M}(\overline{X_{2N}}))^\frac{1}{2} = \sum_w (N_{M}(\overline{X_{2N}}))^\frac{1}{2} \geq \sum_w N_{M,N}^w = N_{M,N}(X^s).$$

As a consequence for all $N$,

$$2 + \frac{1}{2} h(\overline{X_{2N}}) \geq h(X^s_{2N}).$$

This implies that

$$\liminf_N h(\overline{X_{2N}})/2N \geq \liminf_N h(X^s_{N})/N = h(X^s).$$

For similar reasons

$$\liminf_N h(X^s_{2N+1})/2N+1 \geq \liminf_N h(X^s_{N})/N = h(X^s),$$

and thus:

$$\liminf_N h(X^s_{N})/N \geq \liminf_N h(X^s_{N})/N = h(X^s).$$
3 Overview of the text

In the following, we provide a complete proof of the following theorem:

**Theorem 1.** The entropy of square ice is equal to

\[ h(X^s) = \frac{3}{2} \log_2 \left( \frac{4}{3} \right) \]

The proof of Theorem 1 can be overviewed as follows:

- The strategy is primarily:
  1. to compute the entropies \( h(X^s_N) \),
  2. then to use Lemma 1 in order to compute \( h(X^s) \).

- The first point is derived from the **transfer matrix** method, which allows to express \( h(X^s_N) \) with a formula involving a sequence of numbers defined implicitly through a system of non-linear equations called **Bethe equations**. This method is itself decomposed in several steps:

  1. **Formulation with transfer matrices** [Section 4]: it is usual, when dealing with unidimensional subshifts of finite type, to express their entropy as the greatest eigenvalue of a matrix which relates which couples of rows of symbols can be adjacent. In this text, we use the adjacent matrix \( V_N^{*} \) of a factor subshift, thought as acting on \( \Omega_N = C^2 \otimes ... \otimes C^2 \). Lemma 1 tells that one can compute \( h(X^s_N) \) by computing the maximal eigenvalue of the adjacency matrix of the factor of \( X^s_N \).

  2. **Lieb path - transport of information through analyticity** [Section 3]:

     In quantum physics, **transfer matrices**, which are complexifications of the adjacency matrix in a local way (in the sense that the coefficient relative to a couple of rows is a product of coefficients in \( C \) relative to the symbols in the two rows) are used to derive properties of the system. In our study, the adjacency matrix is seen as a particular value of an analytic path of such transfer matrices, \( t \mapsto V_N(t) \) such that for all \( t \), \( V_N(t) \) is an irreducible non-negative and symmetric matrix, and such that \( V_N(1) = V_N^{*} \). Such a path is called (for the clarity of the exposition) a **Lieb path** in this text. In Section 3 we define the Lieb path that will be used in the following. This part is a detailed exposition of notions defined in the article of E.H.Lieb [Lieb 1967].

  3. **Coordinate Bethe ansatz** [Section 5]:

     We use the coordinate Bethe ansatz (exposed in [Duminil-Copin et al.] and related in the present text), that provides candidate eigenvectors for the matrix \( V_N(t) \) for all \( t \) in each of spaces \( \Omega^{(n)}_N \), \( n \leq N \) that form a decomposition of \( \Omega_N \):

     \[ \Omega_N = \bigoplus_{n=0}^{N} \Omega^{(n)}_N \]

     The candidate eigenvectors and eigenvalues depend each on a sequence \( (p_j)_{j=1...n} \) that verifies a non-linear system of equations called Bethe equations.

     It is shown that the system of Bethe equations on the parameters \( p_j \) admits a unique solution for each \( n, N \) and \( t \), denoted \( (p_j(t))_j \) for all \( t \in (0, \sqrt{2}) \), in a context where \( n, N \) are fixed, using convexity arguments on an auxiliary function. The analyticity of the two types of paths and the convexity of the auxiliary function ensures that \( t \mapsto (p_j(t))_j \) is analytic. This part completes the proof of an argument left uncomplete in [Yang Yang 1]. In order to identify the greatest eigenvalue, we use the fact that \( V_N(\sqrt{2}) \) commutes with some Hamiltonian \( H_N \) that is completely diagonalised (following...
The vector is non zero and associated to the maximal eigenvalue of $H_N$ on $\Omega_N^{(n)}$. By Perron-Frobenius theorem, the vector has positive coordinates, and by the same theorem, the associated Bethe value is effectively an eigenvalue of the transfer matrix and it is equal to the greatest eigenvalue of the restriction to $\Omega_N^{(n)}$. By continuity, this is true also for $t$ in a neighborhood of $\sqrt{2}$. By analyticity, this identity is true for all $t \in (0, \sqrt{2})$.

- The second point is derived in two steps:

1. **Asymptotic condensation of Bethe roots [Section 6]:**
   The sequences $(p_j(t))_j$ are transformed into sequences $(\alpha_j(t))_j$ through an analytic bijection. The values of these sequences are called **Bethe roots**.
   We first prove that the sequences of Bethe roots are condensed according to a density function $\rho_t$ over $\mathbb{R}$, relatively to any continuous decreasing and integrable function $f : (0, +\infty) \to (0, +\infty)$, which means that the Cesaro mean of the finite sequence $(f(\alpha_j(t)))_j$ converges towards $\int_0^t f(x)f(x)dx$. This part involves rigorous proofs, some simplifications and adaptations of arguments that appeared in [Kozlowski]. The density $\rho_t$ is defined through Fredholm integral equation, corresponding the asymptotic version of the Bethe equations. This equation is solved through Fourier analysis, following a computation done in [Yang Yang II].

2. **Computation of integrals [Section 7]:** The condensation property proved in the last point implies that the formula obtained for $\frac{1}{N}h(X_N)$ converges to an integral involving $\rho_1$. The formula obtained for $\rho_1$ allows the computation of this integral, through lace integrals techniques. This part is a detailed version of computations exposed in [Lieb 1967].

4 **A Lieb path for square ice**

In this section, we define the matrices $V_N^*$ [Section 4.1] and define an example of Lieb path $t \mapsto V_N(t)$ for the discrete curves shift $X^*$ [Section 4.2], and relate $h(X^*)$ to $V_N(1) = V_N^*$ [Section 4.3].

4.1 **The interlacing relation and the matrices $V_N^*$**

In the following, for a square matrix $M$, we will denote $M[u, v]$ its entry on $(u, v)$. Moreover, we denote $\{0, 1\}_N^*$ the set of length $N$ words on $\{0, 1\}$.

**Notation 2.** Consider $u, v$ two words in $\{0, 1\}_N^*$, and $w$ some $(N, 1)$-cylindric pattern of the subshift $X$. We say that the pattern $w$ connects $u$ to $v$ (we denote this $uR^w[v]$), when for all $k \in [1, N]$, $w_k = 1$ (resp. $v_k = 1$) if and only if $w$ has an incoming (resp. outgoing) curve on the bottom (resp. top) of its $k$th symbol. This notation is illustrated on Figure 4.

**Definition 3.** Let us denote $R \subset \{0, 1\}_N^* \times \{0, 1\}_N^*$ the relation defined by $uR^v$ if and only if there exists a $(N, 1)$-cylindric pattern $w$ of the discrete curves shift $X^*$ such that $uR^w[v]$.

**Notation 3.** For all $u \in \{0, 1\}_N^*$, we denote $|u|_1$ the number of $k \in [1, N]$ such that $u_k = 1$. If $|u|_1 = n$, we denote $q_1[u] < ... < q_n[u]$ the integers such that $u_k = 1$ if and only if $k = q_i[u]$ for some $i \in [1, n]$.

Let us also notice that $uR^v$ implies that the number of 1 symbols in $u$ is equal to the number of 1 symbols in $v$.

**Definition 4.** We say that two words $u, v$ in $\{0, 1\}_N^*$ such that $|u|_1 = |v|_1 \equiv n$ are **interlaced** when one of the two following conditions is satisfied:

\[
q_1[u] \leq q_1[v] \leq q_2[u] \leq ... \leq q_n[u] \leq q_n[v] \\
q_1[v] \leq q_1[u] \leq q_2[v] \leq ... \leq q_n[v] \leq q_n[u].
\]
Proposition 2. For two length $N$ words $u, v$, we have $u \mathcal{R} v$ if and only if $|u_1| = |v_1| \equiv n$ and $u, v$ are interlaced.

Proof. (⇒): assume that $u \mathcal{R}[w]v$ for some $w$.

First, since $w$ is a $(N, 1)$-cylindric pattern, each of the curves that crosses its bottom side also crosses its top side, which implies that $|u_1| = |v_1|$.

Let us assume that $u$ and $v$ are not interlaced. Without loss of generality, one can assume that $q_1[u] \leq q_1[v]$.

1. The position $q_1[u]$ is connected to $q_1[v]$:
   Indeed, if it did not, another curve would connect another position $q_k[u]$, $k \neq 1$ of $u$ to $q_1[v]$. Since $q_k[u] > q_1[u]$ (by definition), this curve would cross the left border of $w$. It would imply that in the $q_1[u]$th symbol of $w$, two pieces of curves would appear: one horizontal, corresponding to the curve connecting the position $q_k[u]$ to $q_1[v]$, and the one that connects $q_1[u]$ to another position in $u$, which is not possible, by the definition of the alphabet of $X^*$: this is illustrated on Figure 5.

2. The position $q_2[u]$ is connected to $q_2[v]$:
   The curve crossing the position $q_1[u]$ at the bottom of $w$ can not cross the position $q_2[u] + 1$ of $w$ (since it would imply symbols that are not in the alphabet). Thus $q_1[v] \leq q_2[u]$. Moreover, $q_2[v] \geq q_2[u]$, since if it was not the case, there would be a curve connecting some $q_k[u] > q_2[v]$ to $q_2[v]$, thus crossing the left border of $w$, which would imply non-existent symbols in position $q_1[u]$ of $w$. Thus, for the same reason as in the first point argument, $w$ connects position $q_2[u]$ to $q_2[v]$.

3. Repetition:
   We repeat the argument of the last point in order to obtain:
   $$q_1[u] \leq q_1[v] \leq q_2[u] \leq \ldots \leq q_n[u] \leq q_n[v],$$
   meaning that $u$ and $v$ are interlaced.

(⇐): if $|u_1| = v_1$ and $u, v$ are interlaced, then we define $w$ by connecting $q_i[u]$ to $q_i[v]$ for all $i \in [1, n]$. We thus have directly $u \mathcal{R}[w]v$.  

Figure 5: Illustration for the definition of the notation $u \mathcal{R}[w]v$.

Figure 6: Illustration of (impossible) crossing situation, which would imply non-authorized symbols.
Proposition 3. When $u \mathcal{R} v$ and $u \neq v$, there exists a unique $w$ such that $u \mathcal{R} \{w\} v$. When $u = v$ there are exactly two possibilities, either the word $w$ that connects $q_i[u]$ to itself for all $i$, or the one connecting $q_i[u]$ to $q_{i+1}[u]$ for all $i$.

Proof. Consider two words $u \neq v$ such that $u \mathcal{R} v$. There exists at least one $i$ such that $q_i[u]$ is different from any $q_j[v]$. This forces any $w$ such that $u \mathcal{R} \{w\} v$ to connect the position $q_i[u]$ to:

- $q_{j_0}[v]$ minimal amongst the positions $q_j[v] \geq q_i[u]$, if the set of $j$ such that this is verified is not empty;
- else the position $q_1[v]$.

If $j < n$, then since $u$ and $v$ are interlaced (Proposition 2), $q_{j+1}[u] \geq q_{j_0}[v]$. Thus it has to be connected to $q_{j_0}[v]$ if $j_0 < n$, else to $q_1[v]$. If $j = n$, then for the same reason $q_1[u]$ has to be connected to $q_{j_0+1}[v]$ if $j_0 < n$, else to $q_1[v]$. Repeating this argument, we get the unicity of $w$. \[ \square \]

4.2 The Lieb path $t \mapsto V_N(t)$

Notation 4. Let $N \geq 1$ be an integer, and $t > 0$. Let us denote $\Omega_N$ the space $\mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$, tensor product of $N$ copies of $\mathbb{C}^2$, whose canonical basis elements are denoted indifferently by $\epsilon = (\epsilon_1 \ldots \epsilon_N)$ or the words $\epsilon_1 \ldots \epsilon_N$, for $(\epsilon_1, \ldots, \epsilon_N) \in \{0, 1\}^N$, according to quantum mechanics notations, in order to distinguish them from the coordinate definition of vectors of $\Omega_N$.

Notation 5. For all $N$ and $(N, 1)$-cylindric pattern $w$, let $|w|$ denote the number of symbols

in this pattern. For instance, for the word $w$ on Figure 5, $|w| = 6$.

Definition 5. For all $t \geq 0$, let us define $V_N(t) \in \mathcal{M}_{2N}(\mathbb{C})$ the matrix such that for all $\epsilon, \eta \in \{0, 1\}^N$,

$$V_N(t)[\epsilon, \eta] = \sum_{w \in \mathcal{R} \{w\} \eta} \epsilon^{|w|}$$

For all $N$ and $n \leq N$, let us denote $\Omega_N^{(n)} \subset \Omega_N$ the vector space generated by the $\epsilon = (\epsilon_1 \ldots \epsilon_N)$ such that $|\epsilon|_1 = n$.

Proposition 4. For all $N$ and $n \leq N$, the matrix $V_N(t)$ stabilizes the vector subspaces $\Omega_N^{(n)}$:

$$V_N(t).\Omega_N^{(n)} \subset \Omega_N^{(n)}.$$

Proof. This is a direct consequence of Proposition 2 since if $V_N(t)[\epsilon, \eta] \neq 0$ for $\epsilon, \eta$ two elements of the canonical basis of $\Omega_N$, then $|\epsilon|_1 = |\eta|_1$.

Let us recall that a non-negative matrix $A$ is called irreducible when there exists some $k \geq 1$ such that all the coefficients of $A^k$ are positive. Let us also recall the Perron-Frobenius theorem for symmetric, non-negative and irreducible matrices.

Theorem 2 (Perron-Frobenius). Let $A$ be a symmetric, non-negative and irreducible matrix. Then $A$ has a positive eigenvalue $\lambda$ such that any other eigenvalue $\mu$ of $A$ satisfies $|\mu| \leq \lambda$. Moreover, there exists some eigenvector $u$ for the eigenvalue $\lambda$ with positive coordinates such that if $v$ is another eigenvector (not necessarily for $\lambda$) with positive coordinates, then $v = \alpha u$ for some $\alpha > 0$.

Let us prove the unicity of the positive eigenvector up to a multiplicative constant:
Proof. Let us denote \( u \in \Omega_N \) the Perron-Frobenius eigenvector and \( v \in \Omega_N \) another vector whose coordinates are all positive, associated to the eigenvalue \( \mu \). Then
\[
\mu u^t.v = (Au)^t.v = u^tAv = \lambda u^t.v
\]
Thus, since \( u^t.v > 0 \), then \( \mu = \lambda \), and by (usual version of) Perron-Frobenius, there exists some \( \alpha \in \mathbb{R} \) such that \( v = \alpha u \). Since \( v \) has positive coordinates, \( \alpha > 0 \). \( \square \)

**Lemma 2.** The matrix \( V_N(t) \) is symmetric, non-negative and for all \( n \leq N \), its restriction to \( \Omega_{N}^{(n)} \) is irreducible whenever \( t > 0 \).

**Proof.**

- **Symmetry:** since the interlacing relation is symmetric, for all \( \epsilon, \eta \in \{0,1\}^N \), we have that \( V_N(t)(\epsilon, \eta) > 0 \) if and only if \( V_N(t)(\eta, \epsilon) > 0 \). When this is the case, and \( \epsilon \neq \eta \) (the case \( \epsilon = \eta \) is trivial), there exists a unique (Proposition 3) \( w \) connecting \( \epsilon \) to \( \eta \). The coefficient of this word is exactly \( t^{2(n - |\{k : \epsilon_k = m-1\}|)} \), where \( n = |\{k : \epsilon_k = 1\}| = |\{k : \eta_k = 1\}| \), and this coefficient is indifferent to the exchange of \( \epsilon \) and \( \eta \).

- **Irreducibility:** Let \( \epsilon, \eta \) be two elements of the canonical basis of \( \Omega_N \) such that \( |\epsilon|_1 = |\eta|_1 = n \). We shall prove that \( V_N^N(t)[\epsilon, \eta] > 0 \).

1. **Interlacing case:**
   - If they are interlacing, \( V_N(t)[\epsilon, \eta] > 0 \). As a consequence, since \( V_N(t)[\epsilon', \epsilon'] > 0 \) for all \( \epsilon' \), one keeps the positivity by repeating the action of \( V_N(t) \). Thus \( V_N^N(t)[\epsilon, \eta] > 0 \).

2. **Non-interlacing case:**
   - Decreasing the interlacing degree:
     - If they are not interlaced, let us denote \( [q_j]\eta \) the maximal number of \( q_j[\eta] \) that lie in some \( [q_j][\epsilon], [q_{j+1}][\epsilon] \]. This number is greater or equal to 2. Let us see that there exists some \( \epsilon' \) such that \( \epsilon R \epsilon' \) and \( \omega(\epsilon, \eta) < \omega(\epsilon', \eta) \). Let us consider some \( i \) such that \( [q_i][\epsilon], [q_{i+1}][\epsilon] \cap [q_j][\eta] : j \in [1, n] \) is empty and \( [q_{i+1}][\epsilon], [q_{i+2}][\epsilon] \cap [q_j][\eta] : j \in [1, n] \) has more than one element (this case happens because \( \epsilon, \eta \) are not interlaced and \( |\epsilon|_1 = |\eta|_1 \)), and consider the word \( w \) that connects the curve crossing position \( q_{i+1}[\epsilon] \) to the maximal \( q_j[\eta] \in [q_{i+1}][\epsilon], [q_{i+2}][\epsilon] \) and fixes the other positions. Let us call \( \epsilon' \) the vector such that \( w \) connects \( \epsilon \) to \( \epsilon' \). We have indeed that \( \omega(\epsilon', \eta) < \omega(\epsilon, \eta) \).
   - A sequence with decreasing interlacing degree: As a consequence, since \( \omega(\epsilon, \eta) \leq N \), one can construct a finite sequence of words \( \epsilon(k), k = 1...m \) such that \( m \leq N \), \( \epsilon(1) = \epsilon \), \( \epsilon(m) \) and \( \eta \) are interlaced, and for all \( k < m \), \( \epsilon(k) R \epsilon(k+1) \). This means that for all \( k < m \), \( V_N[\epsilon(k), \epsilon(k+1)] > 0 \) and \( V_N[\epsilon(m), \eta] > 0 \). As a consequence, \( V_N(t)^N[\epsilon, \eta] > 0 \).

Since for all \( \epsilon, \eta \) with same number of curves, \( V_N(t)^N[\epsilon, \eta] > 0 \), this means that \( V_N(t) \) is irreducible on \( \Omega_N^{(n)} \) for all \( n \leq N \). \( \square \)

### 4.3 Relation between \( h(X^*) \) and the matrices \( V_N(1) \)

**Notation 6.** For all \( N \) and \( n \leq N \), let us denote \( \overline{X}_{n,N} \) the subset (which is also a subshift) of \( \overline{X}_N \) which consists in the set of configurations of \( \overline{X}_N \) such that the number of curves that cross each of its rows is \( n \), and \( \overline{X}_{n,N} \) the subset of \( \overline{X}_N \) such that the number of arrows pointing south in the south part of the symbols in any row is \( n \).

**Notation 7.** Let us denote, for all \( N \) and \( n \leq N \), \( \lambda_{n,N}(t) \) the greatest eigenvalue of \( V_N(t) \) on \( \Omega_N^{(n)} \).

**Proposition 5.** For all \( N \) and \( n \leq N \): \( h(\overline{X}_{n,N}) = \log_2(\lambda_{n,N}(1)) \).
Proof. • Correspondance between $X_{n,N}$ patterns and trajectories under action of $V_N(1)$:

Since for all $N$, $n \leq N$ and $\epsilon, \eta$ in the canonical basis of $\Omega_N^{(n)}$, $V_N(1)[\epsilon, \eta]$ is the number of ways to connect $\epsilon$ to $\eta$ by a $(N,1)$-cylindric pattern, and that there is a natural invertible map from the set of $(M,N)$-cylindric patterns to the sequences $(w_i)_{i=1}^M$ of $(N,1)$-cylindric patterns such that there exists some $(\epsilon_i)_{i=1}^{M+1}$ such that for all $i$, $|\epsilon_i| = n$ and for all $i \leq M$, $\epsilon_i R[w_i]\epsilon_{i+1}$,$$
\|(V_N(1)^M_{\Omega_N^{(n)}})||_1 = N_M(\mathbb{X}_{n,N}).$$

• Gelfand’s formula:

It is known (Gelfand’s formula) that: $\|(V_N(1)^M_{\Omega_N^{(n)}})||_1^{1/M} \to \lambda_{n,N}(1)$.

As a consequence of the first point: $h(\mathbb{X}_{n,N}) = \log_2(\lambda_{n,N}(1))$. □

Proposition 6. For all $N$: $h(X^s) = \lim_N \frac{1}{N} \max_{n \leq N} h(\mathbb{X}_{n,N})$.

Proof. We have the decomposition

$$\mathbb{X}_N = \bigcup_{n=0}^N \mathbb{X}_{n,N}.$$  

Moreover, these subshifts are disjoint. As a consequence:

$$h(\mathbb{X}_N) = \max_{n \leq N} h(\mathbb{X}_{n,N}).$$

From this we deduce the statement. □

As a consequence of Lemma 1

Lemma 3. For all $N \geq 1$ and $n \leq N$, $h(\mathbb{X}_{n,N}) = h(\mathbb{X}_{N-n,N})$.

Proof. For the purpose of notation, we also denote $\pi_s$ the application from $\mathbb{X}_{n,N}$ to $\mathbb{X}_{n,N}$ that consists in an application of $\pi_s$ letter by letter. This map is invertible. Let us consider the application $\pi_s$ from $\mathbb{X}_{n,N}$ to $\mathbb{X}_{N-n,N}$ that inverts all the arrows. This map is an isomorphism, and thus the map $\pi_s \circ \pi_s^{-1}$ is also an isomorphism from $\mathbb{X}_{n,N}$ to $\mathbb{X}_{N-n,N}$. As a consequence, the two subshifts have the same entropy:

$$h(\mathbb{X}_{n,N}) = h(\mathbb{X}_{N-n,N}).$$ □

The following corollary is a straightforward consequence of Lemma 3.

Corollary 1. The entropy of $X^s$ is given by the following formula:

$$h(X^s) = \lim_N \frac{1}{N} \max_{n \leq N/2+1} \log_2(\lambda_{n,N}(1)).$$

Lemma 4. We deduce that:

$$h(X^s) = \lim_N \frac{1}{N} \max_{n \leq (N-1)/4} \log_2(\lambda_{2n+1,N}(1)).$$
Figure 7: Illustration of the curve suppressing operation; the leftmost position of the bottom raw crossed by a curve is colored gray on the left pattern.

Proof. Let us fix some integer $N$ and for all $n$ between $1$ and $N/2 + 1$, and consider the application that from the set of patterns of $X^s_{n,N}$ on $U_M(1)$ associates a pattern of $X^s_{n-1,N}$ on $U'_M(1)$ by suppressing the curve that crosses the leftmost symbol in the bottom row of the pattern crossed by a curve [See an schema on Figure 7]

For each pattern of $X^s_{n-1,N}$, the number of patterns in its pre-image by this transformation is bounded from above by $N^{N/2}$ As a consequence, for all $M$:

$$N_M(X^s_{n-1,N}) \leq N_M(X^s_{n,N}),$$

and thus

$$h(X^s_{n-1,N}) + \log_2(N) \geq h(X^s_{n,N}).$$

As a consequence:

$$h(X^s) = \lim_{N \to \infty} \frac{1}{N} \max_n \left( \max_{2n+1 \leq N/2+1} h(X^s_{2n+1,N}), \max_{2n \leq N/2+1} h(X^s_{2n,N}) \right)$$

$$\leq \lim_{N \to \infty} \frac{1}{N} \max_n \left( \max_{2n+1 \leq N/2+1} h(X^s_{2n+1,N}), \max_{2n-1 \leq N/2+1} h(X^s_{2n-1,N}) + \log_2(N) \right)$$

$$\leq \lim_{N \to \infty} \frac{1}{N} \max_{n \leq N/4} h(X^s_{2n+1,N})$$

Moreover,

$$h(X^s) \geq \lim_{N \to \infty} \frac{1}{N} \max_{n \leq N/4} h(X^s_{2n+1,N}),$$

thus we have the following equality:

$$h(X^s) = \lim_{N \to \infty} \frac{1}{N} \max_{n \leq (N-1)/4} \log_2(\lambda_{2n+1,N}(1)).$$

With a similar argument, we get:

$$h(X^s) = \lim_{N \to \infty} \frac{1}{N} \max_{n \leq (N-1)/4} \log_2(\lambda_{2n+1,N}(1)).$$

5 Coordinate Bethe ansatz

In this section, we recall the statement of the coordinate Bethe ansatz [Section 5.2], after defining some auxiliary functions [Section 5.1]. We then prove the existence and analyticity of solutions of the system of equations $(E_j)(t,n,N)$, $j \leq n$. [Section 5.3]. Then, following [Lieb Shultz Mattis],
we diagonalise a Hamiltonian related to the transfer matrix $V_N(\sqrt{2})$ [Section 5.4]. In the end, we use this analysis in order to identify the largest eigenvalue of the restriction of $V_N(t)$ to $\Omega_N^{(n)}$ for $t \in (0, \sqrt{2})$ and $n \leq N/2 + 1$ [Section 5.5].

5.1 Auxiliary functions

5.1.1 Notations

Let us denote $\mu : (-1,1) \to (0,\pi)$ the inverse of the function $\cos : (0,\pi) \to (-1,1)$. For all $t \in (0,\sqrt{2})$, we will denote $\Delta_t = \frac{2\pi^2}{2t^2}$, $\mu_t = \mu(-\Delta_t)$, and $I_t = ((\pi - \mu_t), (\pi - \mu_t))$.

**Notation 8.** Let us denote $\Theta$ the unique analytic function $(t,x,y) \mapsto \Theta_t(x,y)$ from the set \{$(t,x,y) : x,y \in I_t$\} to $\mathbb{R}$ such that $\Theta_{\sqrt{2}}(0,0) = 0$ and for all $t,x,y$,

\[
\exp(-i\Theta_t(x,y)) = \exp(i(x-y)) \frac{e^{-ix} + e^{iy} - 2\Delta_t}{e^{-iy} + e^{ix} - 2\Delta_t}.
\]

By a unicity argument, one can see that for all $t,x,y$, $\Theta_t(x,y) = -\Theta_t(y,x)$. As a consequence, for all $x$, $\Theta_t(x,x) = 0$. For the same reason, $\Theta_t(x,-y) = -\Theta_t(-x,y)$ and $\Theta_t(-x,-y) = -\Theta_t(x,y)$. Moreover, $\Theta_t$ and all its derivatives can be extended by continuity on $I_t^2 \setminus \{(x,x) : x \in \partial I_t\}$. For the purpose of notation, we will denote also $\Theta_t$ the extended function. We will use the following:

**Computation 1.** For all $y \neq (\pi - \mu_t)$, $\Theta_t((\pi - \mu_t), y) = 2\mu_t - \pi$.

**Proof.** From the definition of $\Theta_t$:

\[
\exp(-i\Theta_t((\pi - \mu_t), y)) = e^{i(\pi - \mu_t - y)} \frac{e^{iy} - e^{i\mu_t} - 2\Delta_t}{e^{-iy} - e^{-i\mu_t} - 2\Delta_t} = e^{i(\pi - \mu_t - y)} \frac{e^{iy} + e^{-i\mu_t}}{e^{-iy} + e^{i\mu_t}}
\]

As a consequence,

\[
\exp(-i\Theta_t((\pi - \mu_t), y)) = e^{i(\pi - \mu_t - y)} \frac{e^{iy} 1 + e^{-i(\mu_t + y)} - e^{-iy} \mu_t}{e^{iy} e^{-i(\mu_t + y)} + 1} = e^{i(\pi - 2\mu_t)}.
\]

This yields the statement as a consequence. \[\square\]

**Notation 9.** Let us denote $\kappa$ the unique analytic map $(t,\alpha) \mapsto \kappa_t(\alpha)$ from $(0,\sqrt{2}) \times \mathbb{R}$ to $\mathbb{R}$ such that $\kappa_{\sqrt{2}}(0) = 0$ and for all $t,\alpha$,

\[
\frac{e^{i\kappa_t(\alpha)}}{e^{i\mu_t + \alpha} - 1} = e^{i\mu_t - e^{i\alpha}}.
\]

With the argument of unicity, we have that for all $t,\alpha$, $\kappa_t(-\alpha) = -\kappa_t(\alpha)$, and as a consequence, $\kappa_t(0) = 0$. We also denote, for all $t,\alpha, \beta$,

$\theta_t(\alpha, \beta) = \Theta_t(\kappa_t(\alpha), \kappa_t(\beta))$.

5.1.2 Properties of the auxiliary functions

5.1.2.1 Computation of the derivative $\kappa'_t$:

**Computation 2.** Let us fix some $t \in (0, \sqrt{2})$. For all $\alpha \in \mathbb{R}$,

\[
\kappa'_t(\alpha) = \frac{\sin(\mu_t)}{\cosh(\alpha) - \cos(\mu_t)}.
\]
Proof.  

- **Computation of** $\cos(\kappa_t(\alpha))$ **and** $\sin(\kappa_t(\alpha))$: 

\[
e^{i\kappa_t(\alpha)} = \frac{(e^{-i\mu t + \alpha} - 1) (e^{i\mu t} - e^\alpha)}{|e^{i\mu t + \alpha} - 1|^2} = \frac{e^\alpha + e^{2\alpha} e^{-i\mu t} - e^{i\mu t} + e^\alpha}{(\cos(\mu_t) e^\alpha - 1)^2 + (\sin(\mu_t) e^\alpha)^2}.
\]

Thus by taking the real part,

\[
\cos(\kappa_t(\alpha)) = \frac{2e^\alpha + (e^{2\alpha} - 1) \cos(\mu_t)}{\cos^2(\mu_t) e^{2\alpha} - 2 \cos(\mu_t) e^\alpha + 1 + (1 - \cos^2(\mu_t)) e^{2\alpha}}
\]

\[
\cos(\kappa_t(\alpha)) = \frac{2e^\alpha + (e^{2\alpha} - 1) \cos(\mu_t)}{e^{2\alpha} - 2 \cos(\mu_t) e^\alpha + 1} = \frac{1 - \cos(\mu_t) \cosh(\alpha)}{\cosh(\alpha) - \cos(\mu_t)} = \frac{\sin^2(\mu_t) + \cos^2(\mu_t) - \cos(\mu_t) \cosh(\alpha)}{\cosh(\alpha) - \cos(\mu_t)} - \cos(\mu_t).
\]

A similar computation gives

\[
\sin(\kappa_t(\alpha)) = \frac{\sin(\mu_t) \sinh(\alpha)}{\cosh(\alpha) - \cos(\mu_t)}
\]

- **Deriving the expression** $\cos(\kappa_t(\alpha))$: 

As a consequence, for all $\alpha$:

\[
-k'_t(\alpha) \sin(\kappa_t(\alpha)) = -\frac{\sin^2(\mu_t) \sinh(\alpha)}{(\cosh(\alpha) - \cos(\mu_t))^2} = -\frac{\sin(\kappa_t(\alpha))^2}{\sinh(\alpha)}.
\]

Thus, for all $\alpha$ but in a discrete subset of $\mathbb{R}$,

\[
k'_t(\alpha) = \frac{\sin(\mu_t)}{\cosh(\alpha) - \cos(\mu_t)}.
\]

This identity is thus verified on all $\mathbb{R}$, by continuity.

\[
\square
\]

5.1.2.2 **Domain and invertibility**

**Proposition 7.** For all $t$, $\kappa_t(\mathbb{R}) \subset I_t$. Moreover, $\kappa_t$ considered as a function from $\mathbb{R}$ to $I_t$ is bijective.

**Proof.**  

- **Injectivity:**

Since $\mu_t \in (0, \pi)$, then $\sin(\mu_t) > 0$ and we have the inequality $\cosh(\alpha) \geq 1 > \cos(\mu_t)$. As a consequence, $\kappa_t$ is strictly increasing, and thus injective.

- **The equality** $\kappa_t(\alpha) = n\pi$ **implies** $\alpha = 0$:

Assume that for some $\alpha$, $\kappa_t(\alpha) = n\pi$ for some integer $n$. If $n$ is odd, then:

\[
e^\alpha - e^{i\mu t} = e^{i\mu t + \alpha} - 1.
\]

\[
e^\alpha + 1 = e^{i\mu t} (e^\alpha + 1),
\]

and thus $e^{i\mu t} = 0$, which is impossible, since $\mu_t \in (0, \pi)$. If $n$ is even, then

\[
-e^\alpha + e^{i\mu t} = e^{i\mu t + \alpha} - 1.
\]

As a consequence, since $e^{i\mu t} \neq -1$, we have $e^\alpha = 1$, and thus $\alpha = 0$. 

\[17\]
• Extension of the images:

Since when $\alpha$ tends towards $+\infty$ (resp. $-\infty$), the function tends towards $-e^{i\mu}$ (resp. $e^{i\mu}$), $\kappa_t(\alpha)$ tends towards some $n\pi - \mu_t$ (resp. $m\pi + \mu_t$), and from the above property, $n = 1$ (reps. $m = -1$). Thus the image of $\kappa_t$ is the set $I_t$.

Thus $\kappa_t$ is an invertible map from $\mathbb{R}$ to $I_t$.

\[ \square \]

5.1.2.3 A relation between $\theta_t$ and $\kappa_t$: The following equality originates in [Yang Yang 1]. We provide some details of a relatively simple way to compute it.

**Computation 3.** For any numbers $t, \alpha, \beta$:

\[
\frac{\partial \theta_t}{\partial \alpha}(\alpha, \beta) = \frac{\partial \theta_t}{\partial \beta}(\alpha, \beta) = -\frac{\sin(2\mu_t)}{\cosh(\alpha - \beta) - \cos(2\mu_t)}
\]

**Proof.**

• Deriving the equation that defines $\Theta_t$:

Let us denote, for all $x, y$:

\[ G_t(x, y) = \frac{x(1 - 2\Delta_t y) + y}{x + y - 2\Delta_t} \]

Then we have that for all $x, y$

\[
\frac{\partial G_t}{\partial x}(x, y) = \frac{(1 - 2\Delta_t y)(x + y - 2\Delta_t) - (x(1 - 2\Delta_t y) + y)}{(x + y - 2\Delta_t)^2}
\]

\[
\frac{\partial G_t}{\partial x}(x, y) = -2\Delta_t \frac{1 + y^2 - 2\Delta_t y}{(x + y - 2\Delta_t)^2}
\]

For all $t, \alpha$, let us denote $\alpha_t \equiv \kappa_t(\alpha)$. By definition of $\Theta_t$, for all $\alpha, \beta$,

\[ \exp(-i\Theta_t(\alpha_t, \beta_t)) = G(e^{i\alpha_t}, e^{-i\beta_t}). \]

Thus we have, by deriving this equality:

\[ -i \frac{d}{d\alpha}(\Theta_t(\alpha_t, \beta_t)) \exp(-i\Theta_t(\alpha_t, \beta_t)) = i\kappa_t'(\alpha) e^{i\alpha_t} \frac{\partial}{\partial x} G_t(e^{i\alpha_t}, e^{-i\beta_t}). \]

\[ \frac{d}{d\alpha}(\Theta_t(\alpha_t, \beta_t)) = -\kappa_t'(\alpha) e^{i\alpha_t} \frac{\partial}{\partial x} G_t(e^{i\alpha_t}, e^{-i\beta_t}) \]

\[ \frac{d}{d\alpha} \Theta_t(\alpha_t, \beta_t) = \frac{(\kappa_t'(\alpha)2\Delta_t e^{i\alpha_t})(1 + e^{-2i\beta_t} - 2\Delta_t)}{(e^{i\alpha_t} + e^{-i\beta_t} - 2\Delta_t)(e^{i\alpha_t} + e^{-i\beta_t} - 2\Delta_t, e^{i(\alpha_t - \beta_t)})} \]

Factoring by $e^{i(\alpha - \beta)}$:

\[ \frac{d}{d\alpha} \Theta_t(\alpha_t, \beta_t) = 2\Delta_t \kappa_t'(\alpha) \frac{e^{i\beta_t} + e^{-i\beta_t} - 2\Delta_t}{(e^{i\alpha_t} + e^{-i\beta_t} - 2\Delta_t)(e^{i\beta_t} + e^{-i\beta_t} - 2\Delta_t, e^{i(\alpha_t - \beta_t)})} \]

• Simplification of a term $e^{i\alpha_t} + e^{-i\beta_t} - 2\Delta_t$:

Let us denote the function $F$ defined by

\[ F_t(\alpha, \beta) = e^{i\alpha_t} + e^{-i\beta_t} - 2\Delta_t. \]

By definition of $\kappa_t$ and $-2\Delta_t = e^{-i\mu_t} + e^{i\mu_t}$ we have:
We have left to see that

\[ F_1(\alpha, \beta) = \frac{e^{i\mu t} - e^{\alpha}}{e^{i\mu t} - e^{\beta}} + \frac{e^{i\mu t - \beta} - 1}{e^{i\mu t} - e^{\beta}} + e^{i\mu t} + e^{-i\mu t}. \]

\[ F_1(\alpha, \beta) = \frac{(e^{i\mu t} - e^{\alpha})(e^{i\mu t} - e^{\beta}) + (e^{i\mu t} + \alpha - 1)(e^{i\mu t + \beta} - 1) + (e^{i\mu t} + e^{-i\mu t} + e^{i\mu t} - 1)(e^{i\mu t} - e^{\beta})}{(e^{i\mu t} + \alpha - 1)(e^{i\mu t} - e^{\beta})}. \]

\[ F_1(\alpha, \beta) = \frac{e^{3i\mu t + \alpha} + e^{\beta - i\mu t} - e^{i\mu t}(e^{\alpha} + e^{\beta})}{(e^{i\mu t} - e^{\beta})}. \]

- **Simplification of \( \Theta_t \)'s derivative:**

For all \( \alpha, \beta \), we have

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = 2\Delta \frac{F_1(\beta, \beta_t)}{F_1(\alpha, \beta_t)}. \]

As a consequence of last point,

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = -(e^{i\mu t} + \alpha - 1)(e^{i\mu t} - e^{\alpha})(e^{i\mu t + \beta} + e^{\beta} - 2e^{i\mu t}e^{\beta}). \]

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = -(e^{i\mu t} + \alpha - 1)(e^{i\mu t} - e^{\alpha})(e^{2i\mu t} - 1)(e^{2i\mu t} + e^{\beta} - 2e^{i\mu t}e^{\beta}). \]

Since in the denominator of the fraction in square of the modulus of some number, we rewrite it.

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = -(e^{i\mu t} + \alpha - 1)(e^{i\mu t} - e^{\alpha})(e^{2i\mu t} + e^{\beta} - 2e^{i\mu t}e^{\beta}). \]

We rewrite also the other terms, by splitting the \( e^{2i\mu t} \) in the denominator in two parts, one makes appear \( \sin(\mu t) \), and the other one, the square modulus:

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = -4|e^{i\mu t + \alpha} - 1|^2 \frac{e^{\beta} \sin(\mu t) \cdot \sin(2\mu t)}{(e^{\alpha} + e^{\beta})^2 \cos(2\mu t) - 1} + (e^{\alpha} - e^{\beta} + e^{\alpha} \sin^2(2\mu t)). \]

By writing \( \sin^2(2\mu t) = 1 - \cos^2(2\mu t) \) and then factoring by \( 1 - \cos(2\mu t) \):

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = -4|e^{i\mu t + \alpha} - 1|^2 \frac{e^{\beta} \sin(\mu t) \cdot \sin(2\mu t)}{1 - \cos(2\mu t)} (e^{\alpha} + e^{\beta})(1 - \cos(2\mu t)) + (e^{\alpha} - e^{\beta})^2 (1 + \cos(2\mu t)). \]

Developing the denominator and factoring it by \( 4e^{\alpha + \beta} \), we obtain:

\[ \frac{1}{\kappa'_t(\alpha)} \frac{d}{d\alpha} \Theta_t(\alpha, \beta_t) = -\frac{|e^{i\mu t + \alpha} - 1|^2 \sin(\mu t) \cdot \sin(2\mu t)}{e^{\alpha} (1 - \cos(2\mu t))} \frac{\sin(\mu t) \cdot \sin(2\mu t)}{\cosh(\alpha - \beta) - \cos(2\mu t)}. \]

We have left to see that

\[ \frac{\sin(\mu t) \kappa'_t(\alpha) |e^{i\mu t + \alpha} - 1|^2}{e^{\alpha} (1 - \cos(2\mu t))} = 1. \]

This derives directly from \( 1 - \cos(2\mu t) = 2 \sin^2(\mu t) \) and the value of \( \kappa'_t(\alpha) \) given by Computation 2.
The other equality:

We obtain the value of \( \frac{d}{dx}(\Theta_t(\alpha, \beta)) \) through the equality \( \Theta(x, y) = -\Theta(y, x) \) for all \( x, y \).

**Lemma 5.** For all \( t, \alpha, \beta \),

\[
\theta_t(\alpha + \beta, \alpha) = \theta_t(\beta, 0).
\]

**Proof.** Let us fix some \( \alpha \in \mathbb{R} \). By Computation 3, the derivative of the function \( \beta \mapsto \theta_t(\alpha + \beta, \alpha) \) is equal to the derivative of the function \( \beta \mapsto \theta_t(\beta, 0) \). As a consequence, these two functions differ by a constant. Since they have the same value in \( \beta = 0 \), they are equal.

### 5.2 Statement of the ansatz

**Notation 10.** For all \( (p_1, ..., p_n) \in I^n \), let us denote \( \psi_{\mu, n, N}(p_1, ..., p_n) \) the vector in \( \Omega_N \) such that for all \( \epsilon \in \{0, 1\}^N \),

\[
\psi_{\mu, n, N}(p_1, ..., p_n)[\epsilon] = \sum_{\sigma \in \Sigma_n} C_\sigma(t)[p_1, ..., p_n] \prod_{k=1}^n e^{ip_k(\sigma_k)\epsilon_k}[\epsilon],
\]

where (for \( \epsilon(\sigma) \) denote the signature of \( \sigma \)):

\[
C_\sigma(t)[p_1, ..., p_n] = \epsilon(\sigma) \prod_{1 \leq k < l \leq n} \left( 1 + e^{i(p_k(\sigma_k) + p_l(\sigma_l))} - 2\Delta_t e^{ip_k(\sigma_k)} \right).
\]

**Theorem 3.** For all \( N \) and \( n \leq N/2 \), and \( (p_1, ..., p_n) \in I_t \) distinct such that for all \( j \) the following equation is verified:

\[
(E_j)[t, n, N] : \quad np_j(t) = 2\pi j - (n + 1)\pi - \sum_{k=1}^n \Theta_t(p_j(t), p_k(t)).
\]

Then we have:

\[
V_N(t).\psi_{\mu, n, N}(p_1, ..., p_n) = \Lambda_{n, N}(t)[p_1, ..., p_n] \psi_{\mu, n, N}(p_1, ..., p_n),
\]

where \( \Lambda_{n, N}(t)[p_1, ..., p_n] \) is equal to

\[
\prod_{k=1}^n L_t(e^{ip_k}) + \prod_{k=1}^n M_t(e^{ip_k})
\]

when all the \( p_k \) are distinct from 0. Else, it is equal to:

\[
\left( 2 + t^2(N - 1) + \sum_{k \neq 1} \frac{\partial \Theta_t}{\partial x}(0, p_k) \right) \prod_{k=1}^n M_t(e^{ip_k})
\]

for \( l \) such that \( p_l = 0 \).

In [Duminil-Copin et al.] (Theorem 2.2), the equations (BE) are implied by the equations \((E_j)[t, n, N]\) in Theorem 3 by taking the exponential of the members of (BE). In order to make the connection easier with [Duminil-Copin et al.], here is a list of correspondences between the notations: in [Duminil-Copin et al.], the notation \( t \) corresponds to \( c \), and it is fixed in the formulation of the theorem. Thus, \( \Delta_t \) corresponds to \( \Delta, \mathcal{L}_t \) to \( \mathcal{D}_\Delta \), \( V_N(t) \) to \( V \), \( \psi_{1, n, N}(p_1, ..., p_n) \) to \( \psi \), \( L_t \) and \( M_t \) to \( L \) and \( M \), \( \Theta_t \) to \( \Theta \), \( \Lambda_{n, N}(t)[p_1, ..., p_n] \) to \( \Lambda \), \( C_\sigma(t)[p_1, ..., p_n] \) to \( A_\sigma \) and the sequence \((x_k)_k \) to the sequence \((q_k[\epsilon])_k \) for some \( \epsilon \).
5.3 Existence of solutions of Bethe equations and analyticity

In this section, we will prove the following, which is a rigorous and complete version of an argument in [Yang Yang I]:

**Theorem 4.** There exists a unique sequence of analytic functions $p_j : (0, \sqrt{2}) \mapsto (-\pi, \pi)$ such that for all $t \in (0, \sqrt{2})$, $p_j(t) \in I_t$ and we have the system of Bethe equations:

$$(E_j)[t, n, N] : \quad Np_j(t) = 2\pi j - (n + 1)\pi - \sum_{k=1}^{n} \Theta_t(p_j(t), p_k(t)).$$

Moreover, for all $t$ and $j$, $p_{n-j+1}(t) = -p_j(t)$; for all $t$, the $p_j(t)$ are all distinct.

**Idea of the proof:** Following C.N. Yang and C.P. Yang [Yang Yang I], we use an auxiliary multivariate function $\zeta_t$ whose derivative is zero exactly when the equations $(E_j)[t, n, N]$ are verified. We prove that this function is convex, which means that it admits a minimum (this relies on the properties of $\theta_t$ and $\kappa_t$). Since we rule out the possibility that the minimum is on the border of the domain, this function admits a point where its derivative is zero, and thus the system of equations $(E_j)[t, n, N]$ admits a unique solution. In order to prove the analyticity, we then define a function of $t$ that verifies an analytic differential equation (and thus is analytic), whose value in some point coincides with the minimum of $\zeta_t$. Since the differential equation ensures that $\zeta_t'$ is null on the values of this function, this means that for all $t$, its value in $t$ is the minimum of $\zeta_t$.

**Proof.**

- The solutions are critical points of an auxiliary function $\zeta_t$:

  Let us denote, for all $t, p_1, \ldots, p_n$:

  $$\zeta_t(p_1, \ldots, p_n) = N \sum_{j=1}^{n} \int_{0}^{\kappa_t^{-1}(p_j)} \kappa_t(x)dx + \pi(n + 1 - 2j) \sum_{j=1}^{n} \kappa_t^{-1}(p_j) + \sum_{k<j} \int_{0}^{\kappa_t^{-1}(p_j) - \kappa_t^{-1}(p_k)} \theta_t(x, 0)dx.$$  

  The interest of this function lies in the fact that for all $j$ (here the argument in each of the sums is $k$):

  $$\frac{\partial \zeta_t}{\partial p_j}(p_1, \ldots, p_n) = \left(\kappa_t^{-1}\right)'(p_j) \left( Np_j - 2\pi j + (n + 1)\pi - \sum_{k<j} \Theta_t(\kappa_t^{-1}(p_j) - \kappa_t^{-1}(p_k), 0) \right) + \sum_{j<k} \theta_t(\kappa_t^{-1}(p_k) - \kappa_t^{-1}(p_j), 0).$$

  $$= \left(\kappa_t^{-1}\right)'(p_j) \left( Np_j - 2\pi j + (n + 1)\pi - \sum_{k<j} \Theta_t(\kappa_t^{-1}(p_k), \kappa_t^{-1}(p_j)) \right) + \sum_{j<k} \theta_t(\kappa_t^{-1}(p_j), \kappa_t^{-1}(p_k)).$$

  $$= \left(\kappa_t^{-1}\right)'(p_j) \left( Np_j - 2\pi j + (n + 1)\pi + \sum_{k} \Theta_t(p_j, p_k) \right),$$

  since for all $x, y$, $\Theta_t(x, y) = -\Theta_t(y, x)$.

  Hence, the system of Bethe equations is verified for the sequence $(p_j)_j$ if and only for all $j$, $\frac{\partial \zeta_t}{\partial p_j}(p_1, \ldots, p_n) = 0$.  

• Convexity of $\zeta$:  
Let us denote $\tilde{\zeta} : \mathbb{R}^n \to \mathbb{R}$ such that for all $\alpha_1, \ldots, \alpha_n$:

$$\tilde{\zeta}(\alpha_1, \ldots, \alpha_n) = \zeta(\kappa_t(\alpha_1), \ldots, \kappa_t(\alpha_n)) .$$

From the last point, we have that for all sequence $(\alpha_k)_k$ and all $j$:

$$\frac{\partial \tilde{\zeta}}{\partial p_j}(\alpha_1, \ldots, \alpha_n) = N \kappa_t(\alpha_j) - 2\pi j + (n + 1)\pi + \sum_k \theta_t(\alpha_j, \alpha_k).$$

As a consequence, for all $k \neq j$:

$$\frac{\partial^2 \tilde{\zeta}}{\partial p_k \partial p_j}(\alpha_1, \ldots, \alpha_n) = \frac{\partial \theta_t}{\partial \alpha}(\alpha_j, \alpha_k) = \frac{\sin(2\mu_t)}{\cosh(\alpha_j - \alpha_k) - \cos(2\mu_t)}.$$

Moreover, for all $j$:

$$\frac{\partial^2 \tilde{\zeta}}{\partial^2 p_j}(\alpha_1, \ldots, \alpha_n, t) = N \kappa_t'(\alpha_j) + \sum_{k \neq j} \frac{\partial \theta_t}{\partial \alpha}(\alpha_j, \alpha_k) = N \kappa_t'(\alpha_j) - \sum_{k \neq j} \frac{\partial \theta_t}{\partial \beta}(\alpha_j, \alpha_k).$$

Let us denote $\tilde{H}_t(\alpha_1, \ldots, \alpha_n)$ the Hessian matrix of $\tilde{\zeta}_t$. For any $(x_1, \ldots, x_n) \in \mathbb{R}^n$, we have

$$\tilde{H}_t(\alpha_1, \ldots, \alpha_n) = N \sum_j \kappa_t'(\alpha_j) x_j^2 + \sum_{j \neq k} \left( \frac{\partial \theta_t}{\partial \beta}(\alpha_j, \alpha_k) x_j (x_j - x_k) \right) .$$

As a consequence $\tilde{\zeta}_t$ is a convex function. As a consequence, if it has a (local) minimum, it is unique. Since $\kappa_t$ is increasing, this property is also true for $\zeta_t$.

• The function $\zeta$ has a minimum in $I^\ast$:

Let us consider $(C_l)$, an increasing sequence of compact intervals such that $\bigcup C_l = I_t$.

Let us assume that $\zeta$ has no minimum in $I^\ast$. As a consequence, for all $j$, the minimum $p^{(l)}_j$ of $\zeta_t$ on $(C_l)^{n}$ is on its border. Without loss of generality, we can assume that there exists some $p^{(\infty)} \in T_l$ such that $p^{(l)} \to p^{(\infty)}$.

We can assume without loss of generality that there exists some $j_0 \in [1, n]$ such that $j \leq j_0$ if and only if $p^{(\infty)}_j = \pi - \mu_t$. In this case, there exists $l_0$ such that for all $l$ and $j \leq j_0$, $p^{(l)}_j \geq 0$. Since $\tilde{\zeta}_t$ is convex and that $p^{(l)}$ is a minimum for this function on the compact $(C_l)^{n}$, then for all $j \leq j_0$,

$$\frac{\partial \tilde{\zeta}(p^{(l)}_1, \ldots, p^{(l)}_n)}{\partial p_j} \leq 0.$$

This is a particular case of the fact that for a convex and continuously differentiable function $f : I \to \mathbb{R}$, where $I$ is a compact interval of $\mathbb{R}$, if its minimum on $I$ is the maximal element of this compact, then $f'$ is negative on this point, as illustrated on Figure S.

Since $\Theta_t$ cannot be defined on $\{ (x, x) : x \in \partial I \}$, in order to have an inequality that can be transformed by continuity into an inequality on $p$, we sum these inequalities:
Figure 8: Illustration of the fact that the minimum of a convex continuously differentiable function on a real compact interval has non-positive derivative.

$$\sum_{j=1}^{j_0} \frac{\partial \zeta}{\partial p_j}(p_1^{(l)}, \ldots, p_n^{(l)}) \leq 0.$$  

According to the first point of the proof, this inequality can be re-written:

$$N \sum_{j=1}^{j_0} p_j - 2\pi \sum_{j=1}^{j_0} j + j_0(n+1)\pi + \sum_{j=1}^{j_0} \sum_{k} \Theta_t(p_j^{(l)}, p_k^{(l)}) \leq 0.$$  

For all $j, j' \leq j_0$, the terms $\Theta_t(p_j^{(l)}, p_{j'}^{(l)})$ and $\Theta_t(p_{j'}^{(l)}, p_j^{(l)})$ cancel out in this sum. As a consequence:

$$N \sum_{j=1}^{j_0} p_j - 2\pi \sum_{j=1}^{j_0} j + j_0(n+1)\pi + \sum_{j=1}^{j_0} \sum_{k>j_0} \Theta_t(p_j^{(l)}, p_k^{(l)}) \leq 0.$$  

This time, the inequality can be extended by continuity and we obtain:

$$N \sum_{j=1}^{j_0} p_j - 2\pi \sum_{j=1}^{j_0} j + j_0(n+1)\pi + \sum_{j=1}^{j_0} \sum_{k>j_0} \Theta_t(p_j^{(\infty)}, p_k^{(\infty)}) \leq 0.$$  

From Computation I we have:

$$Nj_0(\pi - \mu_t) - 2\pi \sum_{j=1}^{j_0} j + j_0(n+1)\pi + j_0(n-j_0)(2\mu - \pi) \leq 0.$$  

Since $\mu_t \leq \pi$ and that $2j_0(n-j_0) - Nj_0 = -2j_0^2 < 0$, this last inequality implies:

$$j_0(n+1)\pi + j_0(n-j_0)\pi \leq 2\pi \sum_{j=1}^{j_0} j.$$  

On the other hand, we have:

$$\sum_{j=1}^{j_0} j \leq n j_0 - \sum_{j=1}^{j_0} j = n j_0 - \frac{j_0(j_0 + 1)}{2}.$$  

As a consequence:
Since this last inequality is impossible, this means that \( \zeta_t \) has a minimum in \( I^n \).

- **Characterization of the solutions with an analytic differential equation:**

Let us denote \( \mathbf{p}(t) = (p_1(t), \ldots, p_n(t)) \), for all \( t \in (0, \sqrt{2}) \), the unique minimum of the function \( \zeta_t \) in \( I^n \). Let us denote \( t \mapsto s(t) \) the unique solution of the differential equation:

\[
\frac{\partial^2 \zeta_t}{\partial t \partial p_j}(s_1(t), \ldots, s_n(t)) = 0
\]

such that \( s(t) \) is the minimum of the function \( \zeta_t \) when \( t = \sqrt{2}/2 \), where \( H_t \) is the Hessian matrix of \( \zeta_t \). Since this is an analytic differential equation, it solution \( s \) is analytic.

Let us rewrite the equation:

\[
H_t(s_1(t), \ldots, s_n(t)) \cdot s'(t) = -\left( \frac{\partial^2 \zeta_t}{\partial t \partial p_j}(s_1(t), \ldots, s_n(t)) \right)_j
\]

This means that for all \( j \),

\[
\frac{\partial \zeta_t}{\partial p_j}(s_1(t), \ldots, s_n(t))
\]

is a constant. Since \( s(t) \) is the minimum of \( \zeta_t \) when \( t = \sqrt{2}/2 \), this constant is zero. As a consequence, by unicity of the minimum of \( \zeta_t \) for all \( t \), \( s(t) = \mathbf{p}(t) \). This means that \( t \mapsto \mathbf{p}(t) \) is analytic.

- **Antisymmetry of the solutions:**

For all \( t, j \), since \( \mathbf{p}(t) \) is the minimum of \( \zeta_t \):

\[
N \mathbf{p}_{n-j+1} - 2\pi (n - j + 1) + (n + 1) \pi + \sum_k \Theta_k(\mathbf{p}_{n-j+1}, \mathbf{p}_{n-k+1}) = 0.
\]

\[
N \mathbf{p}_{n-j+1} + 2\pi j - (n + 1) \pi + \sum_k \Theta_k(\mathbf{p}_{n-j+1}, \mathbf{p}_{n-k+1}) = 0.
\]

\[
-N \mathbf{p}_{n-j+1} - 2\pi j + (n + 1) \pi - \sum_k \Theta_k(\mathbf{p}_{n-j+1}, \mathbf{p}_{n-k+1}) = 0.
\]

\[
-N \mathbf{p}_{n-j+1} + 2\pi j + (n + 1) \pi + \sum_k \Theta_k(-\mathbf{p}_{n-j+1}, -\mathbf{p}_{n-k+1}) = 0.
\]

This means that the sequence \( (-\mathbf{p}_{n-j+1}(t))_j \) is a minimum for \( \zeta_t \), and as a consequence, for all \( j \), \( \mathbf{p}_{n-j+1} = -\mathbf{p}_j \).
• The numbers $p_j(t)$ are all distinct:

Let us consider the function

$$
\chi_t : \alpha \mapsto N\kappa_t(\alpha) + \sum_{k=1}^{n} \theta_t(\alpha, \alpha_k(t)),
$$

where for all $k$, $\alpha_k(t)$ is equal to $\kappa_t^{-1}(p_k(t))$. For all $j$, this function has value $\pi(2j - (n + 1))$ in $\alpha_j(t)$ (by the Bethe equations). The finite sequence $(\pi(2j - (n + 1)))_j$ is increasing, thus it is sufficient to prove that the function $\chi_t$ is increasing. Its derivative is:

$$
\chi'_t : \alpha \mapsto N\kappa'_t(\alpha) + \sum_{k=1}^{n} \frac{\partial \theta_t}{\partial \alpha}(\alpha, \alpha_k(t)).
$$

Since $t \in (0, \sqrt{2})$, $\sin(\mu_t) < 0$, and thus this function is positive. As a consequence $\chi_t$ is increasing.

5.4 Diagonalisation of some Heisenberg Hamiltonian

In this section, following the technique introduced by Lieb, Schultz and Mattis, we diagonalise some Hamiltonian (which is a matrix acting on $\Omega_N$).

5.4.1 Bosonic creation and annihilation operators

Let us recall that $\Omega_N = \mathbb{C}^2 \otimes \ldots \otimes \mathbb{C}^2$. In this section, for the purpose of notation, we identify $\{1, \ldots, N\}$ with $\mathbb{Z}/NZ$.

Notation 11. Let us denote $a$ and $a^\ast$ the matrices in $\mathcal{M}_2(\mathbb{C})$ equal to

$$
a \equiv \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad a^\ast \equiv \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}.
$$

For all $j \in \mathbb{Z}/NZ$, we denote $a_j$ (creation operator at position $j$) and $a_j^\ast$ (annihilation operator at position $j$) the matrices in $\mathcal{M}_{2^N}(\mathbb{C})$ equal to

$$
a_j \equiv id \otimes \ldots \otimes a \otimes \ldots \otimes id, \quad a_j^\ast \equiv id \otimes \ldots \otimes a^\ast \otimes \ldots \otimes id.
$$

where $id$ denotes the identity, and $a$ acts on the $j$th copy of $\mathbb{C}^2$.

In other words, the image of a vector $|\epsilon_1 \ldots \epsilon_N\rangle$ in the basis of $\Omega_N$ by $a_j$ (resp. $a_j^\ast$) is as follows:

• if $\epsilon_j = 0$ (resp. $\epsilon_j = 1$), then the image vector is $0$;
• if $\epsilon_j = 1$ (resp. $\epsilon_j = 0$), then the image vector is $|\eta_1 \ldots \eta_N\rangle$ such that $\eta_j = 0$ (resp. $\eta_j = 1$) and for all $k \neq j$, $\eta_k = \epsilon_k$.

Remark 4. The term creation (resp. annihilation) refer to the fact that for two elements $\epsilon$, $\eta$ of the basis of $\Omega_N$, $a_j[\epsilon, \eta] \neq 0$ (resp. $a_j^\ast[\epsilon, \eta] \neq 0$) implies that $|\eta| = |\epsilon| + 1$ (resp. $|\eta| = |\epsilon| - 1$). If we think of 1 symbols as particles, this operator acts by creating (resp. annihilating) a particle.

Lemma 6. The matrices $a_j$ and $a_j^\ast$ verify the following properties, for all $j$ and $k \neq j$:

• $a_j a_j^\ast + a_j^\ast a_j = id$.
• $a_j^2 = a_j^\ast 2 = 0$.
• $a_j$, $a_j^\ast$ commute both with $a_k$ and $a_k^\ast$. 

25
Proof. • By straightforward computation, we get
\[
aa^* = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix}
\]
and
\[
a^*a = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}
\]
Thus \(aa^* + a^*a\) is the identity of \(\mathbb{C}^2\). As a consequence, for all \(j\),
\[
a_ja_j^* + a_j^*a_j = \text{id} \otimes ... \otimes \text{id},
\]
which is the identity of \(\Omega_N\).

• The second set of equalities comes directly from
\[
a^2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}
\]
\[
(a^*)^2 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}.
\]

• The last set derives from the fact that any operator on \(\mathbb{C}^2\) commutes with the identity.

5.4.2 Definition and properties of the heisenberg hamiltonian

Notation 12. Let us denote \(H_N\) the matrix in \(\mathcal{M}_{2^N}(\mathbb{C})\) defined as:
\[
H_N = \sum_{j \in \mathbb{Z}/N\mathbb{Z}} (a_j^*a_{j+1} + a_ja_{j+1}^*)
\]

Lemma 7. This matrix \(H_N\) is non-negative, symmetric and for all \(n\), its restriction to \(\Omega_N^{(n)}\) is irreducible.

The proof of Lemma 7 is similar to the one of Lemma 2, following the interpretation of the action of \(H_N\) described in Remark 5.

Remark 5. For all \(j\), \(a_j^*a_{j+1} + a_ja_{j+1}^*\) acts on a vector \(\epsilon\) in the basis of \(\Omega_N\) by exchanging the symbols in positions \(j\) and \(j+1\) if they are different. If they are not, the image of \(\epsilon\) by this matrix is \(0\). As a consequence, for two vectors \(\epsilon\) and \(\eta\) in the basis of \(\Omega_N\), \(H_N[\epsilon, \eta] \neq 0\) if and only if \(\eta\) is obtained from \(\epsilon\) by exchanging a 1 symbol of \(\epsilon\) with a 0 in its neighborhood. The Hamiltonian \(H_N\) thus corresponds to \(H\) in [Duminil-Copin et al.] for \(\Delta = 0\).

5.4.3 Fermionic creation and anihilation operators

Notation 13. Let us denote \(\sigma\) the matrix of \(\mathcal{M}_2(\mathbb{C})\) defined as:
\[
\sigma = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}.
\]

Let us denote, for all \(j \in \mathbb{Z}/N\mathbb{Z}\), \(c_j\) and \(c_j^*\) the matrices
\[
c_j = \sigma \otimes ... \sigma \otimes a \otimes \text{id} \otimes ... \otimes \text{id}, \quad c_j^* = \sigma \otimes ... \sigma \otimes a^* \otimes \text{id} \otimes ... \otimes \text{id}.
\]
Let us recall that two matrices \(P, Q\) anticommute when \(PQ = -QP\).

Lemma 8. These operators verify the following properties for all \(j\) and \(k \neq j\):
• $c_j^* c_j + c_j^* c_j = id$.

• $c_j^*$ and $c_j$ anticommute with both $c_k^*$ and $c_k$.

• $a_{j+1}^* a_j = -c_j^* c_{j+1}$ and $a_j^* a_{j+1} = -c_j^* c_{j+1}$.

Proof. • Since $\sigma^2 = id$, for all $j$,

$$c_j^* c_j + c_j^* c_j = a_j a_j^* + a_j^* a_j.$$

From Lemma 3, we now that this operator is equal to identity.

• We can assume without loss of generality that $j < k$. Let us prove that $c_j$ anticommutes with $c_k$ (the other cases are similar):

$$c_j c_k = id \otimes \ldots \otimes a \sigma \otimes \sigma \otimes \ldots \otimes \sigma \otimes \sigma \otimes id \otimes \ldots \otimes id.$$

$$c_j c_k = id \otimes \ldots \otimes \sigma a \otimes \sigma \otimes \ldots \otimes \sigma \otimes \sigma \otimes id \otimes \ldots \otimes id.$$

Hence it is sufficient to see:

$$\sigma a = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ -1 & 0 \end{pmatrix}$$

$$a \sigma = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} = -\sigma a$$

• Let us prove the first equality (the other one is similar):

$$c_{j+1}^* c_j = id \otimes \ldots \otimes id \otimes \sigma a \otimes a^* \otimes id \otimes \ldots \otimes id.$$

We have just seen in the last point that $\sigma a = -a$. As a consequence $c_{j+1}^* c_j = -a_{j+1}^* a_j$. \qed

5.4.4 Action of a symmetric orthogonal matrix

Let us denote $c^*$ is the vector $(c_1^*, \ldots, c_N^*)$ and $c^t$ is the transpose of the vector $(c_1, \ldots, c_N)$. Let us consider a symmetric and orthogonal matrix $U = (u_{i,j})_{i,j}$ in $\mathcal{M}_N(\mathbb{R})$ and denote $b$ and $b^*$ the matrices:

$$b = U . c^t = (b_1, \ldots, b_N), \quad b^* = c^* . U^t = (b_1^*, \ldots, b_N^*).$$

Notation 14. For all $\alpha \in \{0, 1\}^N$, we denote:

$$\psi_\alpha = (b_1^* \alpha^1) \ldots (b_N^* \alpha^N) \nu_N,$$

where $\nu_N = |0, \ldots, 0\rangle$.

Lemma 9. For all $j$ and $k \neq j$:

• $b_j$ and $b_j^*$ anticommute with both $b_k$ and $b_k^*$ and $b_j^* b_j + b_j b_j^* = id$.

• For all $\alpha \in \{0, 1\}^N$, $\psi_\alpha \neq 0$.

• For all $j$ and $\alpha$, we have:

1. $b_j^* b_j \psi_\alpha = 0$ if $\alpha_j = 0$,
2. $b_j^* b_j \psi_\alpha = \psi_\alpha$ if $\alpha_j = 1$. 

27
Proof.  

- **Anticommutation relations:**

Let us prove that $b_j$ and $b_k^*$ anticommute (the other statements of the first point have a similar proof). We rewrite the definition of $b_j$ and $b_k^*:

$$b_j = \sum_i u_{i,j} c_i \quad \text{and} \quad b_k^* = \sum_i u_{i,k} c_i^* = \sum_i u_{i,k} c_i^*.$$ 

Thus

$$b_j b_k^* = \sum_i \sum_{j \neq i} u_{i,j} u_{i,k} c_i c_i^* + \sum_i u_{i,j} u_{i,k} c_i c_i^*.$$ 

From Lemma \[\text{S}\]

$$b_j b_k^* = - \sum_i \sum_{j \neq i} u_{i,j} u_{i,k} c_i c_i^* - \sum_i u_{i,j} u_{i,k} (id - c_i^* c_i).$$

Since the matrix $U$ is orthogonal,

$$b_j b_k^* = - \sum_i \sum_{j \neq i} u_{i,j} u_{i,k} c_i c_i^* - \sum_i u_{i,j} u_{i,k} c_i c_i^* = - b_k b_j.$$ 

Let us notice that this step is the reason why we use the operators $c_i$ instead of the operators $a_i$.

- For all $k$, $b_k^* = \sum s u_{k,s} a_s^*$. As a consequence, for a sequence $l_1, ..., l_s$, 

$$b_{k_1}^* ... b_{k_s}^* \nu_N = \sum_{l_1} ... \sum_{l_s} \left( \prod_{j=1}^s u_{k_j,l_j} \right) \left( \prod_{j=1}^s a_{l_j} \right) \nu_N.$$ 

Since $(a^*)^2 = 0$, the sum can be considered on the integers $l_1, ..., l_s$ such that they are two by two distinct. The operator $a_{l_1} a_{l_2} ... a_{l_s}$ acts on $\nu_N$ by changing the 0 on positions $l_1, ..., l_s$ into symbols 1. The coefficient of the image of $\nu_N$ by this operator in the vector $b_{k_1}^* ... b_{k_s}^* \nu_N$ is thus:

$$\sum_{\sigma \in S_n} \prod_{j=1}^s u_{k_j,l_{\sigma(j)}}.$$ 

If this coefficient was equal to zero for all $\sigma$, it would mean that any size $s$ sub-matrix of $U$ have determinant equal to zero, which is impossible since $U$ is orthogonal, and thus invertible. As a consequence, none of the vectors $\psi_\alpha$ is to zero.

- When $\alpha_j = 0$, from the fact that when $j \neq k$, $b_j$ and $b_k^*$ anticommute, we get that 

$$b_j \psi_\alpha = (-1)^{l} (b_1^*)^{a_1} ... (b_{k_s}^*)^{a_{k_s}} \nu_N b_j \nu_N,$$

and $b_j \nu_N = 0$, since for all $j$, $a_j \nu_N = 0$. As a consequence $b_j^* b_j \nu_N = 0$. When $\alpha_j = 1$, by the anticommutation relations:

$$b_j^* b_j \psi_\alpha = (b_1^*)^{a_1} ... (b_{j-1}^*)^{a_{j-1}} b_j^* b_j (b_j^*)^{a_j} ... (b_{k_s}^*)^{a_{k_s}} \nu_N,$$

since the coefficients $-1$ introduced by anticommutation are canceled out by the fact that we use it for $b_j$ and $b_j^*$. From the first point:

$$b_j^* b_j \psi_\alpha = (b_1^*)^{a_1} ... (b_{j-1}^*)^{a_{j-1}} b_j^* (id - b_j^* b_j) (b_j^*)^{a_j} ... (b_{k_s}^*)^{a_{k_s}}.$$

$$b_j^* b_j \psi_\alpha = \psi_\alpha - (b_1^*)^{a_1} ... (b_{k_s}^*)^{a_{k_s}} b_j^* \nu_N = \psi_\alpha.$$ 

$$\square$$

28
5.4.5 Diagonalisation of the Hamiltonian

**Theorem 5.** The eigenvalues of $H_N$ are exactly the numbers:

$$2 \sum_{\alpha_j=1}^{\alpha} \cos \left(\frac{2\pi j}{N}\right),$$

for $\alpha \in \{0, 1\}^N$.

**Proof.**

1. **Rewriting $H_N$:**

   From Lemma 8, we can write $H_N$ as:

   $$H_N = \sum_j c_j^* c_{j+1} + c_{j+1}^* c_j.$$

   The Hamiltonian $H_N$ can be then rewritten as $H_N = e^* M e^t$, where $M$ is the matrix defined by blocks

   $$M = \frac{1}{2} \begin{pmatrix}
   0 & id & id \\
   id & \ddots & \ddots \\
   \ddots & \ddots & id \\
   id & id & 0
   \end{pmatrix},$$

   where $id$ denotes the identity matrix on $\mathbb{C}^2$, and $0$ denotes the null matrix. Let us denote $M'$ the matrix of $M_{2N}(\mathbb{R})$ obtained from $M$ by replacing $0, id$ by $0, 1$.

2. **Diagonalisation of $M$:** The matrix $M'$ is symmetric and thus can be diagonalised in $M_{2N}(\mathbb{R})$ in an orthogonal basis. It is rather straightforward to see that the vectors $\psi_k$, for any $k \in \{0, ..., N-1\}$ are an orthonormal family of eigenvectors of $M'$ for the eigenvalue $\lambda_k = \cos \left(\frac{2\pi k}{N}\right)$, where for all $j \in \{1, ..., N\}$,

   $$\psi^k_j = \sqrt{\frac{2}{N}} \left(\sin \left(\frac{2\pi kj}{N}\right), \cos \left(\frac{2\pi kj}{N}\right)\right).$$

   This comes from the equalities

   $$\cos(x - y) + \cos(x + y) = 2 \cos(x) \cos(y),$$

   $$\sin(x - y) + \sin(x + y) = 2 \cos(x) \sin(y),$$

   applied to $x = k(j-1)$ and $y = k(j+1)$. This family of vectors is free, since the Vandermonde matrix with coefficients $e^{2\pi kj/N}$ is invertible. As a consequence, one can write

   $$U'M'U^t = D',$$

   where $D'$ is the diagonal matrix whose diagonal coefficients are the numbers $\lambda_k$, and $U'$ is the orthogonal matrix given by the vectors $\psi^k$. Replacing any coefficient of these matrices by the product of this coefficient with the identity, one gets an orthogonal matrix $U$ and a diagonal one $D$ such that:

   $$UMU^t = D.$$

3. **Some eigenvectors of $H_N$:** Let us consider the vectors $\psi_\alpha$ constructed in Section 5.4.4 for the matrix $U$ of the last point, which is symmetric and orthogonal. From the expression of $H_N$, we get that

   $$H_N \psi_\alpha = \left(2 \sum_{j, \alpha_j=1}^{\alpha} \cos \left(\frac{2\pi j}{N}\right)\right) \cdot \psi_\alpha.$$

   Since $\psi_\alpha$ is non zero, this is an eigenvector of $H_N$. 

29
4. The family \( (\psi_\alpha) \) is a basis of \( \Omega_N \):

From cardinality of this family (the number of possible \( \alpha \), equal to \( 2^N \)), this is sufficient to prove that this family is free. For this purpose, let us assume that there are exists a sequence \( (x_\alpha)_{\alpha \in \{0,1\}^N} \) such that

\[
\sum_{\alpha \in \{0,1\}^N} x_\alpha \cdot \psi_\alpha = 0.
\]

We apply first \( b_1^* b_1 \ldots b_N^* b_N \) and get that \( x(1,\ldots,1)\psi(1,\ldots,1) = 0 \), and thus \( x(1,\ldots,1) = 0 \). By repeating this argument, we obtain that all the coefficient \( x_\alpha \) are null. As a consequence \( (\psi_\alpha) \) is a base of eigenvectors for \( H_H \), and the eigenvalues obtained in the last point cover all the eigenvalues of \( H_N \).

5.5 Identification

The proofs for the following two lemmas can be found in [Duminil-Copin et al.] (respectively Lemma 5.1 and Theorem 2.3). In Lemma 10, our notation \( H_N \) corresponds to their notations \( H \) for \( \Delta = 0 \), and \( V_N(\sqrt{2}) \) corresponds to \( V \) for \( \Delta = 0 \). In Lemma 11, the equations \( (E_j)[\sqrt{2}, n, N] \) correspond to their \( (BE) \), \( \psi \) to \( \psi \) for \( \Delta = 0 \).

Lemma 10. For all \( N \geq 1 \), the Hamiltonian \( H_N \) and \( V_N(\sqrt{2}) \) commute:

\[
H_N V_N(\sqrt{2}) = V_N(\sqrt{2}) H_N.
\]

Lemma 11. For all \( N \) and \( n \leq N \), let us denote \( (p_j)_j \) the solution of the system of equations \( (E_j)[\sqrt{2}, n, N] \), then denoting \( \psi \equiv \psi_{\sqrt{2},n,N}(p_1, \ldots, p_n) \):

\[
H_N \psi = \left( 2 \sum_{k=1}^n \cos(p_k) \right) \psi.
\]

Let us prove that for all \( t \in (0, \sqrt{2}) \), the greatest eigenvalue of \( V_N(t) \) is given by the algebraic Bethe ansatz:

Theorem 6. For all \( N \) and \( n \leq N / 4 \), and \( t \in (0, \sqrt{2}) \),

\[
\lambda_{2n+1,N}(t) = \Lambda_{2n+1,N}(t)[p_1(t), \ldots, p_{2n+1}(t)].
\]

Proof. 1. The Bethe vector is \( \neq 0 \) for \( t \) in a neighborhood of \( \sqrt{2} \):

- Limit of the Bethe vector in \( \sqrt{2} \):

Let us denote \( (p_j(t))_j \) the solution of the system of equations \( (E_j)[t, 2n+1, N] \). Let us recall [Theorem 3] that for all \( t \) and \( \epsilon \) in the canonical basis of \( \Omega_N \),

\[
\psi_{t,2n+1,N}(p_1(t), \ldots, p_{2n+1}(t))[\epsilon] = \sum_{\sigma \in \Sigma_{2n+1}} C_{\sigma}(t)[p(t)] \prod_{k=1}^{2n+1} e^{ip_k(t)\epsilon_k} q_k[\epsilon].
\]

This expression admits a limit when \( t \to \sqrt{2} \), given by:

\[
\sum_{\sigma \in \Sigma_{2n+1}} C_{\sigma}(\sqrt{2})[p(\sqrt{2})] \prod_{k=1}^{2n+1} e^{ip_k(\sqrt{2})\epsilon_k} q_k[\epsilon],
\]

where \( (p_k(\sqrt{2}))_k \) is solution of the system of equations \( (E_k)[\sqrt{2}, 2n+1, N] \).
• The term $\epsilon(\sigma)C_\sigma(\sqrt{2})[p(\sqrt{2})]$ is independent from $\sigma$:

Indeed, we have:

$$\prod_{1 \leq k < l \leq 2n+1} (1 + e^{i[p_1(\sqrt{2})+p_\sigma(l)(\sqrt{2})]}) = \prod_{1 \leq \sigma^{-1}(k) < l \leq 2n+1} (1 + e^{i[p_k(\sqrt{2})+p_\sigma(l)(\sqrt{2})]})$$

Indeed, for all $l \neq k$, one of the conditions $\sigma^{-1}(k) < l$ or $\sigma^{-1}(l) < \sigma^{-1}(k)$ is verified, exclusively. This means that $(1 + e^{i[p_k(\sqrt{2})+p_\sigma(l)(\sqrt{2})]})$ appears exactly once in the product for each $l, k$ such that $l \neq k$.

• This term is not equal to zero:

Indeed, none of the $p_k(\sqrt{2}) + p_\sigma(l)(\sqrt{2})$ can be equal to $\pm \pi$. This comes from the fact that the system of Bethe equations $(E_k)[\sqrt{2}, 2n + 1, N]$ has a unique simple solution given by:

$$p_k(\sqrt{2}) = \frac{\pi}{N} \left(2k - \frac{(2n + 1 + 1)}{2}\right) = \frac{2\pi}{N} (k - (n + 1)).$$

These numbers are enframed by:

$$p_1(\sqrt{2}) = -2\pi N - 1, \quad p_n(\sqrt{2}) = 2\pi N - 1.$$

Since $n \leq N/4$ these numbers are in $[-\pi/2, \pi/2]$, and the possible sums of two different of these numbers is in $]-\pi, \pi[.$

• The limit of Bethe vectors is non-zero: As a consequence of last points, we have that the limit of Bethe vectors when $t \to \sqrt{2}$ is, up to a non-zero constant (last point):

$$\sum_{\sigma \in \Sigma_{2n+1}} \epsilon(\sigma) \prod_{k=1}^{2n+1} e^{i[p_\sigma(k)(\sqrt{2})q_k(k)]},$$

which is the determinant of the matrix $\left(e^{i[p_\sigma(l)(\sqrt{2})q_l(k)]}\right)_{k,l}$, which is a submatrix of the matrix $\left(e^{i[s_k(l)]}\right)_{k,l}$. where $(s_k)$ is a sequence of distinct numbers in $]-\pi/2, \pi/2[$ such that for all $k \leq 2n + 1$,

$$s_k = p_\sigma(k)(\sqrt{2}),$$

and $(s'_l)$ is a sequence of distinct integers such that for all $l \leq n$,

$$s'_l = q_l[k].$$

If the determinant is non-zero, then the sum above is non zero. This is the case since this last matrix is obtained from the Vandermonde matrix $\left(e^{i[s_k(l)]}\right)_{k,l}$ whose determinant is

$$\prod_{k<l} (e^{is_k} - e^{is_l}) \neq 0,$$

by a permutation of the columns.

2. From the Hamiltonian to the transfer matrix:

• Eigenvector of $V_N(\sqrt{2})$ and $H_N$: Since that limit of Bethe vector is not equal to zero, it is an eigenvector of the matrix $V_N(\sqrt{2})$. It is also an eigenvector of the Hamiltonian $H_N$, for the eigenvalue

$$2 \left(\sum_{k=1}^{n-1} \cos \left(\frac{2\pi k}{N}\right) + \sum_{k=N-n+1}^{N} \cos \left(\frac{2\pi k}{N}\right)\right).$$
This is a consequence of Lemma 11 since for all \( j \), \( N \mathbf{p}_j(\sqrt{2}) = 2\pi(j - (n + 1)) \): the eigenvalue is
\[
2 \sum_{k=1}^{2n+1} \cos \left( \mathbf{p}_k(\sqrt{2}) \right) = 2 \sum_{k=1}^n \cos \left( \mathbf{p}_k(\sqrt{2}) \right) + 2 \sum_{k=n+1}^{2n+1} \cos \left( N - \mathbf{p}_k(\sqrt{2}) \right) 
= 2 \left( \sum_{k=1}^{n-1} \cos \left( \frac{2\pi k}{N} \right) + \sum_{k=N-n+1}^{N} \cos \left( \frac{2\pi k}{N} \right) \right)
\]

- **Comparison with the other eigenvalues of \( H \):** From Theorem 5 we know that this is the largest eigenvalue of \( H_N \) on \( \Omega_N^{(2n+1)} \). Indeed, it is straightforward that \( \psi_{\alpha} \) is in \( \Omega_N^{(2n+1)} \) if and only if the number of \( k \) such that \( \alpha_k = 1 \) is \( 2n + 1 \). The sum in the statement of Theorem 5 is maximal amongst these sequences when:
\[
\alpha_1 = ... = \alpha_{n-1} = 1 = \alpha_{N-n+1} = ... = \alpha_N
\]
and the other \( \alpha_k \) are equal to 0.

- **Identification:**

As a consequence, from Perron-Frobenius theorem, the limit of Bethe vectors in \( \sqrt{2} \) is positive, thus this is also true for \( t \) sufficiently close to \( \sqrt{2} \). From the same theorem, it is associated to the maximal eigenvalue of \( \psi_N(t) \). As a consequence, the Bethe value \( \Lambda_{2n+1,N}(t) | \mathbf{p}_1(t), ..., \mathbf{p}_{2n+1}(t) \rangle \) is equal to the largest eigenvalue \( \lambda_{2n+1,N}(t) \) of \( \psi_N(t) \) on \( \Omega_N^{(2n+1)} \) for these values of \( t \). Since these two functions are analytic in \( t \) (by the Implicit functions theorem on the characteristic polynomial, using the fact that the largest eigenvalue is simple), one can identify these two functions on the interval \( (0, \sqrt{2}) \). 

\[\square\]

6 **Asymptotic properties of Bethe roots**

Let us fix some \( d \in [0, 1/2] \), and \( (N_k)_k \) and \( (n_k)_k \) some sequences of integers such that for all \( k \), \( n_k \leq N_k/2 + 1 \) and \( n_k/N_k \to d \). In this section, we study the asymptotic behavior of the sequences \( (\alpha_j^{(k)}(t))_j \), where
\[
(\mathbf{p}_j^{(k)}(t))_j \equiv (\kappa_t(\alpha_j^{(k)}(t)))_j
\]
is solution of the system of Bethe equations \( (E_j)[t, n_k, N_k], j \leq n_k \), when \( k \) tends towards \( +\infty \). For this purpose, we introduce in Section 6.1 the counting functions \( \xi_i^{(k)} \) associated to the corresponding Bethe roots. Roughly, these functions ‘represent’ the density of Bethe roots in the real line. In Section 6.2 we prove that the sequence of functions \( \xi_i^{(k)} \) converges uniformly on any compact to a function \( \xi_i \). In Section 6.3 we then prove the following, which will be used in the last Section 7 in order to compute entropy of square ice: for all function \( f : \mathbb{R} \to \mathbb{R} \) which is continuous and bounded,
\[
\frac{1}{N_k} \sum_{j=\lfloor n_k/2 \rfloor + 1}^{n_k} f(\alpha_j^{(k)}(t)) \to \int_0^{\xi_i^{(d)}(d)} f(\alpha) \xi_i(\alpha) d\alpha.
\]

6.1 **The counting functions associated to Bethe roots**

In this section, we define the counting functions and prove some additional preliminary facts on the auxiliary functions \( \theta_t \) and \( \kappa_t \) that we will use in the following [Section 6.1.2]. We prove also that the number of Bethe roots vanishes as one get close to \( +\infty \), with a speed that does not depend on \( k \) [Section 6.1.2].
6.1.1 Definition

Notation 15. For all $t \in (0, \sqrt{2})$, and all integer $k$, let us denote $\xi_t^{(k)} : \mathbb{R} \to \mathbb{R}$ the counting function defined as follows:

$$\xi_t^{(k)} : \alpha \mapsto \left( \frac{1}{2\pi} \kappa_t(\alpha) + \frac{n_k + 1}{2N_k} + \frac{1}{2\pi N_k} \sum_j \theta_t(\alpha, \alpha_j^{(k)}(t)) \right),$$

Fact 1. Let us notice some properties of these functions, that we will use in the following:

1. By Bethe equations, for all $j$ and $k$,

$$\xi_t^{(k)}(\alpha_j^{(k)}(t)) = \frac{j}{N_k} \equiv \rho_j^{(k)}.$$

2. For all $k, t$, the derivative of $\xi_t^{(k)}$ is the function

$$\alpha \mapsto \frac{1}{2\pi} \kappa'_t(\alpha) + \frac{1}{2\pi N_k} \sum_j \frac{\partial \theta_t}{\partial \alpha}(\alpha, \alpha_j(t)) > 0.$$

Indeed, this comes directly from the fact that $\mu_t \in \left(\frac{\pi}{2}, \pi\right)$. As a consequence, the counting functions are increasing.

We will use also the following:

Proposition 8. We have the following limits for the functions $\kappa_t$ and $\theta_t$ on the border of their domains:

$$\lim_{+\infty} \kappa_t = -\lim_{-\infty} \kappa_t = \pi - \mu_t$$

and that for all $\beta \in \mathbb{R}$,

$$\lim_{+\infty} \theta_t(\alpha, \beta) = -\lim_{-\infty} \theta_t(\alpha, \beta) = 2\mu_t - \pi.$$

Proof. Let us prove this property for $\kappa_t$, the limits for $\theta_t$ are obtained applying the same reasoning. Let us recall that for all $\alpha \in \mathbb{R}$,

$$\kappa'_t(\alpha) = \frac{\sin(\mu_t)}{\cosh(\alpha) - \cos(\mu_t)}.$$

Since this function is positive, $\kappa_t$ is increasing, and thus admits a limit in $\pm \infty$. Since $\kappa'_t$ is integrable, these limits are finite. Since for all $\alpha$,

$$e^{i\kappa_t(\alpha)} = \frac{e^{i\mu_t} - e^{i\alpha}}{e^{i\mu_t + \alpha} - 1},$$

and the limit of this expression when $\alpha$ tends to $+\infty$ is $-e^{-i\mu_t}$, then there exists some $k \in \mathbb{Z}$ such that:

$$\lim_{+\infty} \kappa_t = 2k\pi + \pi - \mu_t$$

Since $\kappa_t$ is a bijective map from $\mathbb{R}$ to $I_t$ [Proposition 7], then $k = 0$. Thus we have

$$\lim_{+\infty} \kappa_t = \pi - \mu_t.$$

The limit in $-\infty$ is obtained by symmetry. \qed

Notation 16. For any compact interval $I \subset \mathbb{R}$, we denote

$$\mathcal{V}_I(\epsilon, \eta) = \{ z \in \mathbb{C} : |\text{Im}(z)| < \eta, d(\text{Re}(z), I) < \epsilon \}.$$
6.1.2 Rarefaction of Bethe roots near infinities

For all $k$, $t$ and $M > 0$, we denote

$$P_t^{(k)}(M) = \left\{ j \in [1, n_k] : \alpha_j^{(k)}(t) \notin [-M, M] \right\}.$$  

**Theorem 7.** For all $t \in (0, \sqrt{2})$, $\epsilon > 0$, there exists some $M > 0$ and $k_0$ such that for all $k \geq k_0$,

$$\frac{1}{N_k} \left| P_t^{(k)}(M) \right| \leq \epsilon.$$

**Idea of the proof:** In order to prove this statement, we formulate it as the equality of a number $q_t$ defined as a lim sup (of an expression depending on an integer and an interval) to zero. We extract a sequence of integers $(\nu(k))_t$ and $(l_t) = ([M_1, M_2])_t$ that realises this lim sup. For these sequences, we enframe the smallest (resp. greatest) integer such that the corresponding Bethe root is greater than $M_1$ (resp. smaller than $-M_2$), by a lower bound and an upper bound. Using Bethe equations and properties of $\kappa_t$ and $\theta_t$ (boundedness and monotonicity), we prove an inequality relating these two bounds. Taking the limit $l \to +\infty$, we obtain an inequality that forces $q_t = 0$.

**Proof.** In this proof, we assume, in order to simplify the computations, that for all $k$, $n_k$ is even, and we denote $n_k = 2m_k$. However, similar arguments are valid for any sequence $(n_k)_k$. Moreover, if $d = 0$, the statement is trivial, and as a consequence, we assume in the remaining of the proof that $d > 0$. It is sufficient to prove then that for all $\epsilon > 0$, there exists some $M$ and $k_0$ such that for all $k \geq k_0$

$$\frac{1}{n_k} \left| P_t^{(k)}(M) \right| \leq \epsilon.$$

- **Formulation with superior limits:**
  If $\limsup_n \alpha_{n_k}^{(k)}$ is finite, then the Bethe roots are bounded independently from $k$ (from below this comes from the asymmetry of $\alpha^{(k)}$), and thus the statement is verified.

  Let us thus assume that $\limsup_k \alpha_{n_k}^{(k)} = +\infty$, meaning that there exists some $\nu : \mathbb{N} \to \mathbb{N}$ such that

  $$\alpha_{\nu(k)}^{(k)} \to +\infty.$$  

  Let us denote for all $k$, $t$ and $M > 0$ the proportion $q_t^{(k)}(M)$ of positive Bethe roots $\alpha_j^{(k)}$ that are greater than $M$. Since for all $k$, $\alpha^{(k)}$ is an antisymmetric and increasing sequence, $\alpha_j^{(k)} > 0$ implies that $j \geq m_k + 1$, and we define this proportion as:

  $$q_t^{(k)}(M) = \frac{1}{m_k} \left| \left\{ j \in [m_k + 1, 2m_k] : \alpha_j^{(k)} \geq M \right\} \right|.$$  

  We also denote $q_t(M) = \limsup_k q_t^{(\nu(k))}(M)$ and

  $$q_t = \limsup_M q_t(M) \leq 1.$$  

  By construction, there exists an increasing sequence $(M_t)_t$ of real numbers and a sequence $(k_t)_t$ of integers such that for all $\epsilon > 0$, there exists some $l_0$ and for all $l \geq l_0$:

  $$q_t - \epsilon < q_t(M_l) - \frac{\epsilon}{2} < q_t^{(\nu(k))}(M_l) < q_t(M_l) + \frac{\epsilon}{2} < q_t + \epsilon.$$  

  The proof of the statement reduces to prove that $q_t = 0$.

- **Bounds for the cutting integers sequence:**
By summing values of the counting function, we have

$$\sum_{k=\pi}^{2m_{\nu}(k)} \xi_k^{(\nu(k))}(\alpha_k^{(\nu(k))}) = \frac{1}{2\pi} \sum_{k=\pi}^{2m_{\nu}(k)} \kappa_k^{(\nu(k))} + \frac{2m_{\nu}(k) + 1}{2N_{\nu}(k)} (2m_{\nu}(k) + 1 - \overline{a}_l)$$

and

$$+ \frac{1}{2\pi N_{\nu}(k)} \sum_{k=\pi}^{2m_{\nu}(k)} \sum_{k' \neq k} \theta_{l}(\alpha_k^{(\nu(k))}, \alpha_{k'}^{(\nu(k))}).$$

Figure 9: Illustration of the definition and lower bound of the cutting integer.

1. **Lower bound:**
   As a consequence of the first point,

   $$ (q_l + \epsilon) m_{\nu(k)} \geq \left| \left\{ j \in [2m_{\nu(k)} + 1, 2m_{\nu(k)}] : \alpha_j^{(\nu(k))} \geq M_l \right\} \right|.$$

   $$\left| \left\{ j \in [m_{\nu(k)} + 1, 2m_{\nu(k)}] : \alpha_j^{(\nu(k))} < M_l \right\} \right| = m_{\nu(k)} - \left| \left\{ j \in [m_{\nu(k)} + 1, 2m_{\nu(k)}] : \alpha_j^{(\nu(k))} \geq M_l \right\} \right| \geq m_{\nu(k)} \cdot (1 - q_l - \epsilon).$$

   Thus the cutting integer (which separates the Bethe roots according to their position relative to $M_l$, or equivalently the greatest $j$ such that the associated Bethe root satisfies the inequality $\alpha_j^{(\nu(k))} < M_l$) is bounded from below by:

   $$m_{\nu(k)} + m_{\nu(k)} \cdot \max(0, 1 - \epsilon - q_l) \geq \max(0, 2m_{\nu(k)}(1 - \epsilon - q_l)).$$

   Since it is an integer, it is also greater than

   $$\underline{a}_l \equiv \max(0, [2m_{\nu(k)}(1 - \epsilon - q_l)]).$$

2. **Upper bound:**
   Let us also denote $\overline{a}_l = [2m_{\nu(k)}(1 + \epsilon - q_l)] + 1$. For a similar reason, the cutting integer is smaller than $\overline{a}_l$. See a schema on Figure 9.

3. **Another similar bound:**
   Moreover, since $l \geq l_0$,

   $$q_l^{(\nu(k))}(M_{l_0}) \geq q_l^{(\nu(k))}(M_l) > q_l(M_l) - \frac{\epsilon}{2}.$$

   As a consequence of a reasoning similar to the first point,

   $$\left| \left\{ j \in [m_{\nu(k)} + 1, 2m_{\nu(k)}] : \alpha_j^{(\nu(k))} < M_{l_0} \right\} \right| \geq m_{\nu(k)} \cdot (1 - q_l - \epsilon),$$

   and thus for all $j \leq \underline{a}_l$, $\alpha_j^{(\nu(k))} < M_{l_0}$.

- **Inequality involving $\underline{a}_l$ and $\overline{a}_l$ through Bethe equations:**

By summing values of the counting function,
By Bethe equations,
\[
\sum_{k=\pi_t}^{2m_\nu(k_t)} \zeta_t^{(\nu(k))}(\alpha_k^{(\nu(k))}) = \frac{1}{N_\nu(k_t)} \sum_{k=\pi_t}^{2m_\nu(k_t)} k = \frac{(2m_\nu(k_t) + \pi_t)(2m_\nu(k_t) - \pi_t + 1)}{2N_\nu(k_t)}.
\]
As a direct consequence, and since \(\theta_l\) is increasing in its first variable and \(\theta_l(a,\alpha) = 0\) for all \(a\),
\[
\frac{(2m_\nu(k_t) - \pi_t + 1)(\pi_t - 1)}{2N_\nu(k_t)} \geq \frac{1}{2\pi} \sum_{k=\pi_t}^{2m_\nu(k_t)} \kappa_t(\alpha_k^{(\nu(k))}) + \frac{1}{2\pi N_\nu(k_t)} \sum_{k=\pi_t}^{2m_\nu(k_t)} \theta_l(\alpha_k^{(\nu(k))}, \alpha_{k'}^{(\nu(k'))})
\]
As well, using again the fact that \(\theta_l\) is increasing in its first variable, we use the bound
\[
\theta_l(\alpha_k^{(\nu(k))}, \alpha_{k'}^{(\nu(k'))}) \geq \theta_l(M_1, M_{l_0})
\]
when \(k \geq \pi_t\) and \(k' \leq \pi_t\) (this is a consequence of the third bound proved in the last point).
The terms corresponding to other pairs \((k, k')\) are bounded by 0. We also use the fact that \(\kappa_t\) is increasing. This is written:
\[
\frac{(2m_\nu(k_t) - \pi_t + 1)(\pi_t - 1)}{2N_\nu(k_t)} \geq (2m_\nu(k_t) - \pi_t + 1)\frac{1}{2\pi} \kappa_t(M_1) + (2m_\nu(k_t) - \pi_t + 1)\frac{\mu}{2\pi N_\nu(k_t)} \theta_l(M_1, M_{l_0}).
\]
We take the limit when \(l \to +\infty\), and obtain:
\[
\frac{d}{2} (1 - \epsilon - q_l) \geq \frac{\pi - \mu_t}{2\pi} + \frac{d}{2} \frac{2\mu_t - \pi}{\pi} (1 - q_l).
\]
Taking the limit when \(\epsilon \to 0\),
\[
\frac{d}{2} (1 - q_l) \geq \frac{\pi - \mu_t}{2\pi} + \frac{d}{2} \frac{2\mu_t - \pi}{\pi} (1 - q_l).
\]
This inequality can be rewritten:
\[
(1 - q_l) \left( \frac{d}{2} - \frac{2\mu_t - \pi}{\pi} \right) \geq \frac{\pi - \mu_t}{2\pi}
\]
Finally: \(1 - q_l \geq \frac{1}{\pi t} \geq 1\), and thus \(q_l = 0\).

\[\square\]

### 6.2 Convergence of the sequence of counting functions \((\xi^{(k)})_k\)

In this section, we prove that the sequence of functions \((\xi^{(k)})_k\) converges uniformly on any compact to a function \(\xi_{t,a}\). After some recalls on complex analysis [Section 6.2.1], we prove that if a subsequence of this sequence of functions converge on any compact of their domain towards a function, then this function verifies a Fredholm integral equation [Section 6.2.2], which is solved
through Fourier analysis, and the solution is proved to be unique, in Section 6.2.3 by solving a similar equation verified by the derivative of this function. We deduce in Section 6.2.4 that this fact implies that the sequence of counting functions converge to $\xi_{\nu}$.

For all $t$, there exists $\tau_t > 0$ such that for all $k$, the functions $\kappa_k$, $\Theta_k$ and $\xi_{\nu}^{(k)}$ can be extended analytically on the set $\mathcal{I}_{\gamma_t} \equiv \{ z \in \mathbb{C} : \text{Im}(z) < \tau_t \} \subset \mathbb{C}$. For the purpose of notation, the extended functions are denoted like their restriction on $\mathbb{R}$.

6.2.1 Some complex analysis background

Let us recall some results of complex analysis that we will use in the following of this section. Let $U$ be an open subset of $\mathbb{C}$.

**Definition 6.** We say that a sequence $(f_m)_m$ of functions $U \to \mathbb{C}$ is **locally bounded** when for all $z \in U$, the sequence $(|f_m(z)|)_m$ is bounded.

**Theorem 8** (Montel). Let $(f_m)_m$ be a locally bounded sequence of holomorphic functions $U \to \mathbb{C}$. There exists a subsequence of $(f_m)_m$ which converges uniformly on any compact subset of $U$.

**Lemma 12.** Let $(f_m)_m$ be a locally bounded sequence of continuous functions $U \to \mathbb{C}$ and $f : U \to \mathbb{C}$ such that any subsequence of $(f_m)_m$ which converges uniformly on any compact subset of $U$ towards some function, then this limit is $f$. Then $(f_m)_m$ converges uniformly on any compact towards $f$.

**Proof.** Let us assume that $(f_m)_m$ does not converge towards $f$. Then there exists some $\epsilon > 0$, compact $K \subset U$ and a non-decreasing function $\nu : \mathbb{N} \to \mathbb{N}$ such that for all $m$,

$$|| (f_{\nu(m)} - f)_K ||_{\nu} \geq \epsilon.$$

From Montel theorem, one can extract a subsequence of $(f_{\nu(m)})_m$ which converges towards $f$ uniformly on any compact of $U$, and in particular on the compact $K$. This is in contradiction to the above inequality, and we deduce that $(f_m)_m$ converges towards $f$. \hfill $\square$

**Theorem 9** (Cauchy formula). Let us assume that $U$ is simply connected and let $f : U \to \mathbb{C}$ be a holomorphic function and $\gamma$ a lace included in $U$ that is homeomorphic to a circle positively oriented. Then for all $z$ which in the interior domain of the lace,

$$f(z) = \frac{1}{2\pi i} \int_{\gamma} \frac{f(s)}{s - z} ds.$$

Let us also recall a sufficient condition for a holomorphic function to be biholomorphic:

**Theorem 10.** Let $f : U \to \mathbb{C}$ be a holomorphic function onto an open and simply connected set $U$. Let $V \subset U$ and $\gamma$ a lace included in $U$ that is homeomorphic to a circle positively oriented, and such that $V$ is included in the interior domain of $\gamma$. We assume that:

1. for all $z \in V$ and $s \in \gamma$, $f(z) \neq f(s)$,
2. and for all $z \in V$, $f'(z) \neq 0$.

Then $f$ is a **biholomorphism** from $V$ onto its image, meaning that there exists some holomorphic function $g : f(V) \to U$ such that for all $z \in f(V)$, $g(f(z)) = z$ and for all $z \in U$, $g(f(z)) = z$. Moreover, for all $z \in f(V)$,

$$g(z) = \frac{1}{2\pi i} \int_{\gamma} s \frac{f'(s)}{f(s) - z} ds.$$
6.2.2 The limits of subsequences of \((\xi_t^{(k)})_k\) satisfy a Fredholm integral equation

In this section, we prove the following:

**Theorem 11.** Let \(\nu : \mathbb{N} \to \mathbb{N}\) be a non-decreasing function, and assume that \((\xi^{(\nu(k))}_t)_m\) converges uniformly on any compact of \(I_t\) towards a function \(\xi_t\). Then this function satisfies the following equation for all \(\alpha \in I_t\):

\[
\xi_t'(\alpha) = \frac{1}{2\pi} \kappa_t'(\alpha) + \int_{\mathbb{R}} \frac{\partial}{\partial \alpha} \alpha' \xi_t' d\beta.
\]

Moreover, \(\xi_t(0) = d/2\).

**Proof.**

- **Convergence of the derivative of the counting functions:** Since any compact of \(I_t\), can be included in the interior domain of a rectangle triangle, through derivation of Cauchy formula, the derivative of \((\xi^{(\nu(k))}_t)'\) converges also uniformly on any compact, towards \(\xi_t'\). Since \(\left|\xi^{(k)}_t\right|'\) is bounded by a constant that does not depend on \(m\), and that \(s \mapsto \theta_t(\alpha, s)\) is integrable on \(\mathbb{R}\) for all \(\alpha\), then \(s \mapsto \theta_t(\alpha, s)\xi_t'(s) ds\) is integrable on \(\mathbb{R}\).

- **Some notations:**

Let us fix some \(\varepsilon > 0\), and \(\alpha_0 \in \mathbb{R}\). In the following, we consider some *irrational* number (and as a consequence not the image of a Bethe root) \(M > 1\) such that:

1. \(M \in \xi_t(\mathbb{R})\)
2. such that: \(\left|P_t^{(k)}(M)\right| \leq \frac{\varepsilon}{\pi(4M - 2)}\) for all \(k\) greater than some \(k_0\) (in virtue of Theorem 7).
3. and \(\alpha_0 \in \xi_t^{-1}([-M, M])\).

Since \(\mathcal{X}_t(\mathbb{R})\) is an interval (this function is increasing on \(\mathbb{R}\)), one can take \(M\) arbitrarily close to the supremum of this interval. When \(M\) tends towards this supremum, \(\xi_t^{-1}(M)\) tends to \(+\infty\): if it did not, then this would contradict the fact that this is the supremum (again by monotonicity). One can assume that \(M\) is such that

\[
\frac{1}{2\pi} \left|\int_{\xi_t^{-1}([-M, M])} \theta_t(\alpha, \beta)\xi_t'(\beta) d\beta\right| \leq \frac{\varepsilon}{4}.
\]

Let us also denote \(J_t = \xi_t^{-1}([-M, M])\).

- **The derivative of \(\xi_t\) relative to the axis \(i\mathbb{R}\) is non-zero when close enough to \(\mathbb{R}\):**

Indeed, for all \(\alpha, \lambda \in \mathbb{R}\),

\[
\xi_t^{(k)}(\alpha + i\lambda) = \frac{1}{2\pi} \kappa_t(\alpha + i\lambda) + \frac{n_k + 1}{2N_k} \sum_j \theta_t(\alpha + i\lambda, \alpha_j^{(k)}(t)).
\]

As a direct consequence the derivative of the function \(\lambda \mapsto -i\xi_t^{(k)}(\alpha + i\lambda)\) in 0 is:

\[
\frac{1}{2\pi} \kappa_t'(\alpha) + \frac{1}{2\pi N_k} \sum_j \theta_t(\alpha, \alpha_j^{(k)}(t)) = (\xi_t^{(k)})'(\alpha) \geq \frac{1}{2\pi} s\kappa_t'(\alpha) > 0.
\]

Thus for all \(\alpha\), the derivative of the function \(\lambda \mapsto -i\xi_t(\alpha + i\lambda)\) in 0 is greater than

\[
\frac{1}{2\pi} \kappa_t'(\alpha).
\]
Moreover, since the second derivative of $\lambda \mapsto -i\xi_t^{(k)}(\alpha + i\lambda)$ is a bounded function of $\alpha$, with a bound that is independent from $k$, through Taylor integral formula, there exists a constant $p_t > 0$ such that for all $\lambda \in \mathbb{R}$ and $\alpha \in \mathbb{R}$:

$$|\xi_t(\alpha + i\lambda) - i\xi_t'(\alpha).\lambda - \xi_t(\alpha)| \leq p_t\lambda^2,$$

which implies:

$$|\text{Im}(\xi_t(\alpha + i\lambda)) - \xi_t'(\alpha).\lambda| \leq p_t\lambda^2.$$ 

$$\text{Im}(\xi_t(\alpha + i\lambda)) \geq \xi_t'(\alpha).\lambda - p_t\lambda^2.$$

- **The restriction of $\xi_t$ on some $\mathcal{V}_{J_t}(\eta_t, \epsilon_t)$ is a biholomorphism onto its image:**

Since $M$ is defined so that $M \in \xi_t(\mathbb{R})$, then $J_t$ is compact. This means, as a consequence of last point, that there exists some positive number $\sigma_t < \tau_t$ such that for all $z \in \mathcal{V}_{J_t}(\sigma_t, 1) \setminus \mathbb{R}$, then $\xi_t(z) \notin \mathbb{R}$.

Let us consider the lace $\gamma_t = \partial \mathcal{V}_{J_t}(\sigma_t, 1)$ (see an illustration on Figure 10).

Let us prove that there exist some $\epsilon_t > 0$ and $\eta_t > 0$ such that the values taken by the function $\xi_t$ on $\mathcal{V}_{J_t}(\eta_t, \epsilon_t)$ are distinct from any value taken by the same function on the lace $\gamma_t$. This is done in two steps, as follows:

1. First, we consider open neighbourhoods (illustrated by dashed squares on Figure 10) for the two points of $\gamma_t \cap \mathbb{R}$ such that the values taken by $\xi_t$ on these sets are distant by more than a positive constant from the values taken on $J_t$. This is possible since $\xi_t$ is strictly increasing on $\mathbb{R}$.

2. On the part of $\gamma_t$ that is not included in these two open sets, the function $\xi_t$ takes non-real values, and the set of values taken is compact, by continuity. As a consequence, the set of values taken on the lace $\gamma_t$ is included into a compact that does not intersect the set of values taken on $J_t$. Thus one can separate these two sets of values with open sets, meaning that there exist some $\epsilon_t > 0$ and $\eta_t > 0$ such that the set of values taken by $\xi_t$ on $\mathcal{V}_{J_t}(\eta_t, \epsilon_t)$ does not intersect the set of values taken by this function on $\gamma_t$.

In virtue of Theorem 10 this means that $\xi_t$ is a biholomorphism from $\mathcal{V}_{J_t}(\eta_t, \epsilon_t)$ onto its image on this set. As a consequence, it is also an open function, and its image on $\mathcal{V}_{J_t}(\eta_t, \epsilon_t)$ contains the image of $J_t$, which $[-M, M]$ by definition.
• Asymptotic biholomorphism property for $\xi_t^{(\nu(k))}$:
  It derives from the last point that there exists some $k_1 \geq k_0$ such that for all $k \geq k_1$, the values of $\xi_t^{(\nu(k))}$ on $\gamma_t$ are distinct from the values of $\xi_t^{(k)}$ on $\mathcal{V}_J_t(\eta_t, \epsilon_t)$, and as a consequence, for the same reason as the last point, $\xi_t^{(\nu(k))}$ is a biholomorphism from $\mathcal{V}_J_t(\eta_t, \epsilon_t)$ onto its image on this set. Moreover, since $\xi_t^{(\nu(k))}$ converges uniformly to $\xi_t$ on $\mathcal{V}_J_t(\eta_t, \epsilon_t)$, it converges also uniformly on $\mathcal{V}_J_t(\eta_t, \epsilon_t)$, and $\xi_t^{(\nu(k))}(\mathcal{V}_J_t(\eta_t, \epsilon_t))$ contains $\mathcal{V}_{[-M,M]}(\eta'_t, \epsilon'_t)$, then there exists some $\eta'_t, \epsilon'_t > 0$ and some $k_2 \geq k_1$ such that for all $k \geq k_2$, $\xi_t^{(\nu(k))}(\mathcal{V}_J_t(\eta_t, \epsilon_t))$ contains $\mathcal{V}_{[-M,M]}(\eta'_t, \epsilon'_t)$.

• Lace integral expression of the counting functions and approximation of $\xi_t^{(\nu(k))}$:
  We deduce that for all $k \geq k_2$, and $\sigma < \eta'_t$, positive such that the lace:
  $$\Gamma_t^\sigma \equiv \{-M, M\} \times [-\sigma, \sigma] \bigcup [-M, M] \times \{-\sigma, \sigma\}$$
  is included into $\xi_t^{(\nu(k))}(\mathcal{V}_J_t(\eta_t, \epsilon_t))$. See Figure 11 for an illustration.

Figure 11: Illustration for the definition of the lace $\Gamma_t^\sigma$.

We then have, since $\alpha_0 \in J_t$ the following equation for all $k$, $t$, $\sigma$:

$$\frac{1}{2\pi N_{\nu(k)}} \sum_{j \in \mathcal{P}_t^{(\nu(k))}(M)} \theta_t(\alpha_0, \alpha_j^{(\nu(k))}(t)) = \frac{1}{2\pi} \oint_{\Gamma_t^\sigma} \theta_t\left(\alpha_0, \left(\xi_t^{(\nu(k))}\right)^{-1}(s)\right) \frac{e^{2i\pi N_{\nu(k)}s}}{(e^{2i\pi N_{\nu(k)}s} - 1)} ds$$

Indeed, there are no poles for $\xi_t^{(\nu(k))}$ on $\Gamma_t^\sigma$ since $M$ is irrational. The poles of the function inside the domain delimited by $\Gamma_t^\sigma$ are exactly the numbers $\rho_j^{(\nu(k))}$. By the residues theorem, and since for all $j$, $\xi_t^{(\nu(k))}(\alpha_j(t)) = \rho_j^{(\nu(k))}$:

$$\oint_{\Gamma_t^\sigma} \theta_t\left(\alpha_0, \left(\xi_t^{(\nu(k))}\right)^{-1}(s)\right) \frac{e^{2i\pi N_{\nu(k)}s}}{(e^{2i\pi N_{\nu(k)}s} - 1)} ds = 2\pi i \sum_{j \in \mathcal{P}_t^{(\nu(k))}(M)} \frac{1}{2\pi N_{\nu(k)}} \theta_t(\alpha_0, \alpha_j^{(\nu(k))}(t))$$

• Approximations:
  We deduce that for all $k \geq k_2$ and all $\sigma < \eta'_t$,

$$\left|\xi_t^{(\nu(k))}(\alpha_0) - \frac{1}{2\pi} \kappa_t(\alpha_0) - \frac{n_{\nu(k)} + 1}{2N_{\nu(k)}} - \frac{1}{2\pi} \oint_{\Gamma_t^\sigma} \theta_t\left(\alpha_0, \left(\xi_t^{(\nu(k))}\right)^{-1}(s)\right) \frac{e^{2i\pi N_{\nu(k)}s}}{(e^{2i\pi N_{\nu(k)}s} - 1)} ds\right|$$

40
is smaller than
\[
\sum_{j \notin \mathcal{P}_1^{(v(k))}(M)} \left| \theta_t(\alpha_0, \alpha_j^{(v(k))}(t)) \right| \leq (2\mu_t - \pi) \left\{ j \in [1, n_{v(k)}] : \alpha_j^{(v(k))}(t) \notin [-M, M] \right\} \leq \frac{\epsilon}{2}.
\]
by notations of the second point of this proof. Let us also note \(k_3 \geq k_2\) some integer such that for all \(m \geq k_3\),
\[
\left| \frac{n_{v(k)} + 1}{2N_{v(k)}} - \frac{d}{2} \right| \leq \frac{\epsilon}{8}.
\]
We then evaluate convergence of various terms:

1. Convergence of the bottom part of the lace integral to an integral on a real segment when \(\sigma \to 0\):
   By continuity of \(\xi_t^{-1}\), there exists some \(\sigma_0 > 0\) such that for all \(k \geq k_3\), \(\sigma \leq \sigma_0\),
   \[
   \left| \int_{[-M, M]} \theta_t(\alpha_0, \xi_t^{-1}(\beta - i\sigma))d\beta - \int_{[-M, M]} \theta_t(\alpha_0, \xi_t^{-1}(\beta))d\beta \right| \leq \frac{\epsilon}{16}.
   \]
   By change of variable in the second integral:
   \[
   \left| \int_{[-M, M]} \theta_t(\alpha_0, \xi_t^{-1}(\beta - i\sigma))d\beta - \int_{\xi_t^{-1}([-M, M])} \theta_t(\alpha_0, \xi_t(\beta))d\beta \right| \leq \frac{\epsilon}{16}.
   \]

2. Bounding the lateral parts of the lace integral for \(\sigma \to 0\):
   There exists some \(\sigma_1 > 0\) such that \(\sigma_1 \leq \sigma_0\) such that for all \(\sigma \leq \sigma_1\), \(k \geq k_3\),
   \[
   \left| \frac{1}{2\pi} \int_{-\sigma}^{\sigma} \theta_t(\alpha_0, (\xi_t^{(v(k))})^{-1}(\pm M + i\lambda)) \frac{e^{2i\pi N_{v(k)}(\pm M + i\lambda)}}{e^{2i\pi N_{v(k)}(\pm M + i\lambda)} - 1} d\lambda \right| \leq \frac{\epsilon}{64}.
   \]

3. Convergence of the top and bottom parts of the lace integral when \(k \to +\infty\):
   Then there exists some \(k_4 \geq k_3\) such that for all \(k \geq k_4\),
   \[
   \left| \frac{1}{2\pi} \int_{-M}^{M} \theta_t(\alpha_0, (\xi_t^{(v(k))})^{-1}(\beta + i\sigma)) \frac{e^{2i\pi N_{v(k)}(\beta + i\sigma)}}{e^{2i\pi N_{v(k)}(\beta + i\sigma)} - 1} d\beta \right| \leq \frac{\epsilon}{64}
   \]
   \[
   \left| \frac{1}{2\pi} \int_{-M}^{M} \theta_t(\alpha_0, (\xi_t^{(v(k))})^{-1}(\beta - i\sigma)) \left( \frac{e^{2i\pi N_{v(k)}(\beta - i\sigma)}}{e^{2i\pi N_{v(k)}(\beta - i\sigma)} - 1} - 1 \right) d\beta \right| \leq \frac{\epsilon}{64}.
   \]
   All these inequalities together with
   \[
   \left| \frac{1}{2\pi} \int_{(\xi_t^{-1}([-M, M]))^c} \theta_t(\alpha_0, \xi_t(\beta))d\beta \right| \leq \frac{\epsilon}{4}
   \]
   imply, by multiple applications of the triangular inequality, that for all \(k \geq k_4\),
   \[
   \left| \xi_t^{(v(k))}(\alpha_0) - \frac{1}{2\pi} \int_{k_4(\alpha_0)} - \frac{d}{2} - \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta_t(\alpha_0, \beta) \xi_t(\beta)d\beta \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{8} + 2 \frac{\epsilon}{16} + 3 \frac{\epsilon}{64} = \epsilon.
   \]
• Integral equations:
As a consequence, since this is true for all $\epsilon > 0$ we have the following equality for all $\alpha \in \mathbb{R}$:

$$
\xi_t(\alpha_0) = \frac{1}{2\pi} \kappa_t(\alpha) + \frac{d}{2} + \frac{1}{2\pi} \int_{-\infty}^{\infty} \theta_t(\alpha_0, \beta) \xi'_t(\beta) d\beta.
$$

Moreover, this equality is verified for any $\alpha$, and differentiating it relatively to $\alpha$:

$$
\xi'_t(\alpha) = \frac{1}{2\pi} \kappa'_t(\alpha) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \theta_t}{\partial \alpha}(\alpha, \beta) \xi'_t(\beta) d\beta.
$$

• Value of $\xi_t(0)$:
Since $\xi_t^{(k)}$ is increasing for all $k$, we have directly:

$$
\left\lfloor \frac{n_k}{2} \right\rfloor N_k = \xi_t^{(k)}(\alpha_{\lfloor n_k/2 \rfloor}(t)) \leq \xi_t^{(k)}(\alpha_{\lceil n_k/2 \rceil + 1}(t)) = \left\lceil \frac{n_k}{2} \right\rceil + 2 N_k.
$$

As a consequence $\xi_t(0) = d/2$.

6.2.3 Solution of the Fredholm equation

In this section, we prove that the integral equation on $\xi_t$ in the statement of Theorem 11 is unique and compute its solution:

Proposition 9. Let $t \in (0, \sqrt{2})$ and $\rho$ a continuous function in $L^1(\mathbb{R}, \mathbb{R})$ such that for all $\alpha \in \mathbb{R}$,

$$
\rho(\alpha) = \frac{1}{2\pi} \kappa_t(\alpha) + \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\partial \theta_t}{\partial \alpha}(\alpha, \beta) \rho(\beta) d\beta.
$$

Then for all $\alpha$,

$$
\rho(\alpha) = \frac{1}{4\mu_t \cosh (\pi \alpha/2 \mu_t)}.
$$

Proof. The proof consists essentially in the application of Fourier transform techniques. We will denote, for convenience, for all $\alpha$ and $\mu$,

$$
\Xi_{\mu}(\alpha) = \frac{\sin(\mu)}{\cosh(\alpha) - \cos(\mu)}.
$$

• Application of Fourier transform:
Let us denote $\hat{\rho}$ the Fourier transform of $\rho$: for all $\omega$,

$$
\hat{\rho}(\omega) = \int_{-\infty}^{\infty} \rho(\alpha) e^{i\omega \alpha} d\alpha,
$$

which exists since $\rho$ is $L^1(\mathbb{R})$. As well, denote $\hat{\Xi}_{\mu}$ the Fourier transform of $\Xi_{\mu}$. Thus, since

$$
\int_{-\infty}^{\infty} \frac{\partial \theta_t}{\partial \alpha}(\alpha, \beta) \rho(\beta) d\beta = - \int_{-\infty}^{\infty} \Xi_{\mu}(\alpha - \beta) \rho(\beta) d\beta,
$$

this defines a convolution product, which is transformed in a simple product through the Fourier transform, so that for all $\omega$:

$$
\hat{\rho}(\omega) = \frac{1}{2\pi} \hat{\Xi}_{\mu}(\omega) - \frac{1}{2\pi} \hat{\Xi}_{2\mu}(\omega) \hat{\rho}(\omega).
$$

$$
2\pi \hat{\rho}(\omega) = \frac{\hat{\Xi}_{\mu}(\omega)}{1 + \frac{1}{2\pi} \hat{\Xi}_{2\mu}(\omega)}
$$
• Computation of $\hat{\Xi}_\mu$:

- **Singularities of this function:** The singularities of the function $\Xi_\mu$ are exactly the numbers $i(\mu + 2k\pi)$ for $k \geq 0$ and $i(-\mu + 2k\pi)$ for $k \geq 1$, since for $\alpha \in \mathbb{C}$, $\cosh(\alpha) = \cos(\mu)$ if and only if $\cos(i\alpha) = \cos(\mu)$, and this implies that $\alpha = i(\pm \mu + 2k\pi)$ for some $k$.

- **Computation of the residues:**
  For all $k$, the residue of $\Xi_\mu$ in $i(\mu + 2k\pi)$ is
  \[
  \text{Res}(\Xi_\mu, i\mu + 2k\pi) = \left. \frac{e^{i\gamma \cdot (i\mu + 2k\pi)}}{i} \right|_{\alpha = i\mu + 2k\pi} = \frac{1}{i} e^{-\gamma(\mu + 2k\pi)}.
  \]
  As well,
  \[
  \text{Res}(\Xi_\mu, -i\mu + 2k\pi) = \left. \frac{e^{i\gamma \cdot (-i\mu + 2k\pi)}}{i} \right|_{\alpha = -i\mu + 2k\pi} = -\frac{1}{i} e^{-\gamma(-\mu + 2k\pi)}.
  \]
  We have, for all $\gamma$:
  \[
  \int_{-\infty}^{+\infty} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi \frac{\sinh[(\pi - \mu)\gamma]}{\sinh(\pi\gamma)}
  \]

- **Residue theorem:**
  Let us denote, for all integer $n$, the lace $\Gamma_n = [-n, n] + i[0, n]$. The residues $\Xi_\mu$ inside the domain delimited by this lace are the $i(\mu + 2k\pi)$ with $k \geq 0$, and the $i(-\mu + 2k\pi)$ with $k \geq 1$. For all $n$,
  \[
  \int_{\Gamma_n} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = \int_{\Gamma_n} \frac{\sinh(i\mu)}{i(\cosh(\alpha) - \cosh(i\mu))}e^{i\alpha\gamma}d\alpha
  \]
  By the residue theorem,
  \[
  \int_{\Gamma_n} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi i \left( \sum_{k \geq 0} \text{Res}(\Xi_\mu, i(\mu + 2k\pi)) - \sum_{k \geq 1} \text{Res}(\Xi_\mu, i(-\mu + 2k\pi)) \right).
  \]

- **Asymptotic behavior:**
  Since only the contribution on $[-n, n]$ of the integral is non zero asymptotically, and by convergence of the integral and the sums,
  \[
  \int_{-\infty}^{+\infty} \Xi_\mu(\alpha)e^{i\alpha\gamma}d\alpha = 2\pi e^{-\gamma\mu} + 2\pi \sum_{k=1}^{+\infty} (-e^{-\gamma\mu} + e^{-\gamma\mu})e^{-2\gamma k\pi}
  \]
  \[
  = 2\pi e^{-\gamma\mu} + 2\pi(-e^{-\gamma\mu} + e^{-\gamma\mu}) \left( \frac{1}{1 - e^{-2\gamma\pi}} - 1 \right)
  \]
  \[
  = 2\pi e^{-\gamma\mu} + 2\pi(-e^{-\gamma\mu} + e^{-\gamma\mu}) \frac{e^{-\gamma\pi} - e^{-\gamma\pi}}{e^{\gamma\pi} - e^{-\gamma\pi}}
  \]
  \[
  = 2\pi e^{-\gamma(\pi + \mu)} - e^{-\gamma(\pi - \mu)} - e^{\gamma(\mu - \pi)} + e^{\gamma(-\pi - \mu)}
  \]
  \[
  = 2\pi \frac{\sinh(\gamma(\pi - \mu))}{\sinh(\gamma\pi)}.
  \]

• Computation of $\hat{\rho}$:
Using this expression of the Fourier transform of $\Xi_\mu$, for all $\omega$, 
\[
2\pi \rho(\omega) = \frac{2\pi \sinh(\omega(\pi - \mu))}{\sinh(\pi\omega) + \sinh(\omega(\pi - 2\mu))}
= \frac{\pi}{4\pi \sinh(\mu\omega)} \frac{e^{\pi\mu}(1 + e^{-2\mu\omega}) - e^{-\pi\mu}(1 + e^{2\mu\omega})}{e^{\pi(\pi - \mu)}(e^{\mu\omega} + e^{-\mu\omega}) - e^{-\pi(\pi - \mu)}(e^{-\mu\omega} + e^{\mu\omega})}
= \frac{\pi}{\cosh(\mu\omega)}.
\]

- Inverse transform:

We thus have for all $\alpha$:
\[
2\pi \rho(\alpha) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{\pi}{\cosh(\mu\omega)} e^{-i\omega\alpha} d\omega = \frac{1}{\mu t} \int_{-\infty}^{\infty} \frac{1}{2 \cosh(u)} e^{-i\pi\mu} du,
\]
where we used the variable change $u = \mu t \omega$. Using the computation of the Fourier transform of $\Xi_\mu$ for $\mu = \pi/2$,
\[
\int_{-\infty}^{\infty} \frac{1}{\cosh(\alpha)} e^{i\alpha t} d\alpha = 2\pi \frac{\sinh(\pi/2)}{\sinh(\pi\gamma)} = \frac{\pi}{\cosh(\pi\gamma/2)}.
\]
Thus we have
\[
2\pi \rho(\alpha) = \frac{1}{2\mu t} \frac{\pi}{\cosh(\pi\alpha/2\mu t)} = \frac{\pi}{2\mu t} \frac{1}{\cosh(\pi\alpha/2\mu t)}.
\]

6.2.4 Convergence of $\xi_t^{(k)}$:

**Theorem 12.** There exists a function $\xi_{t,d} : \mathbb{R} \rightarrow \mathbb{R}$ such that $\xi_t^{(k)}$ converges uniformly on any compact towards $\xi_{t,d}$. Moreover, this function satisfies the following equation for all $\alpha$:
\[
\xi_{t,d}(\alpha) = \frac{1}{2\pi} \xi_t(\alpha) + \frac{1}{2\pi} \int_{\mathbb{R}} \frac{\partial}{\partial \alpha} (\alpha, \beta) \xi_{t,d}(\beta) d\beta,
\]
and $\xi_{t,d}(0) = d/2$.

**Proof.** Consider any subsequence of $(\xi_t^{(k)})_k$ which converges uniformly on any compact of $I_{\tau_t}$ to a function $\xi_t$. Via Cauchy formula, the derivative of $\xi_t^{(k)}$ converges uniformly on any compact to $\xi_t'$. Since the functions $\xi_t^{(k)}$ are uniformly bounded by a constant which is independant from $k$, and that for all $k$, $(\xi_t^{(k)})$, $\xi_t'$ is positive and $\xi_t$ is bounded, and thus $\xi_t$ is in $L^1(\mathbb{R}, \mathbb{R})$. From Theorem 11, we get that $\xi_t'$ verifies a Fredholm equation, which has a unique solution in $L^1(\mathbb{R}, \mathbb{R})$ [Proposition 0]. From Theorem 11, $\xi_t$, as a function on $\mathbb{R}$, is the unique primitive function of this one which has value $d/4$ on 0. Since this function is analytic, it determines its values on the whole stripe $I_{\tau_t}$. In virtue of Lemma 12, $(\xi_t^{(k)})_k$ converge towards this function. \[\square\]

**Proposition 10.** The limit of the function $\xi_{t,d}$ in $+\infty$ is $d/2 + \frac{1}{4}$, and the limit in $-\infty$ is $d/2 - 1/4$.

**Proof.** For all $\alpha$,
\[
\xi_{t,d}(\alpha) = \frac{d}{2} + \frac{1}{4\mu_t} \int_{0}^{\infty} \frac{1}{\cosh(\pi x/2\mu t)} dx = \frac{d}{2} + \frac{1}{2\pi} \int_{0}^{2\mu_t \alpha/\pi} \frac{1}{\cosh(x)} dx.
\]
This converges in $+\infty$ to:
\[
\frac{d^2}{2} + 1 + \int_0^{+\infty} \frac{e^x}{e^{2x} + 1} \, dx = \frac{d^2}{2} + \frac{1}{\pi} \int_0^{+\infty} (\arctan(\exp))'(x) \, dx = \frac{d^2}{2} + \frac{1}{\pi} - \frac{d}{2} + \frac{1}{4}.
\]
For the same reason, the limit in $-\infty$ is $d/2 - 1/4$.

**Remark 6.** As a consequence, this limit is $>d$ when $d < 1/2$ and equal to $d$ when $d = 1/2$.

### 6.3 Condensation of Bethe roots relative to some functions

In this section, we prove that if $f$ is a continuous function $(0, +\infty) \to (0, +\infty)$, decreasing and integrable, then the scaled sum of the values of $f$ on the Bethe roots converges to an integral involving $f$ and $\xi_{t,d}$ [Theorem 13]. Let us denote, for all $t, m$ and $M > 0$:
\[
Q_t^{(k)}(M) \equiv \left\{ j \in [1, n_k] : \xi_{t,d}^{-1} \left( \frac{j}{N_k} \right) \notin [-M, M] \right\},
\]
and for two fine sets $S, T$, we denote $S \Delta T = S \setminus T \cup T \setminus S$. For a compact set $K \subset \mathbb{R}$, we denote its diameter $\delta(K) \equiv \max_{x, y \in K} |x - y|$. For $I$ a bounded interval of $\mathbb{R}$, we denote $l(I)$ its length. When
\[ J = \bigcup_j I_j \]
with $I_k$ bounded and disjoint intervals, the length of $J$ is
\[ l(J) = \sum_j l(I_j). \]

**Theorem 13.** Let $f : (0, +\infty) \to (0, +\infty)$ a continuous, decreasing and integrable function. Then:
\[
\frac{1}{N_k} \sum_{j=\lceil n_k/2 \rceil + 1}^{n_k} f(\alpha_{(k)}^{(t)}(j)) \to \int_0^{\xi_{t,d}^{-1}(d)} f(\alpha)\xi'_{t,d}(\alpha) \, d\alpha,
\]
where we denote $\xi_{t,1/2}^{-1}(1/2) = +\infty$.

**Remark 7.** This is another version of a statement proved in [Kozlowski] for bounded continuous and Lipshitz functions, which is not sufficient for the proof of Theorem 1.

**Proof.** In all the proof, the indexes $j$ in the sums are in $[\lceil n_k/2 \rceil + 1, n_k]$.

- **Setting:** Let $\epsilon > 0$ and $t \in (0, \sqrt{2})$. Let us fix some $M$ such that for all $k$ greater than some $k_0$:
  \[
  \frac{1}{N_k} |P_t^{(k)}(M)| \leq \frac{\epsilon}{2\|f_{[M,+\infty]}\|_{\infty} + 1},
  \]
  \[
  \left| \int_{[M,+\infty)} f(\alpha)\xi'_{t,d}(\alpha) d\alpha \right| \leq \frac{\epsilon}{2},
  \]
  and if $d < 1/2$,
  \[
  M > \xi_{t,d}^{-1}(d),
  \]
  which is possible in virtue of Proposition 10.
• Using the rarefication of Bethe roots:

\[
\frac{1}{N_k} \sum_{j=\lfloor n_k/2 \rfloor + 1}^{n_k} f(\alpha_j^{(m)}(t)) = \frac{1}{N_k} \sum_{j=\lfloor n_k/2 \rfloor + 1}^{n_k} f\left(\left(\zeta_t^{(k)}\right)^{-1} \left(\frac{j}{N_k}\right)\right) \\
= \frac{1}{N_k} \sum_{j \notin P_t^{(k)}(M)} \left(\zeta_t^{(k)}\right)^{-1} \left(\frac{j}{N_k}\right) \\
+ \frac{1}{N_k} \sum_{j \in P_t^{(k)}(M)} \left(\zeta_t^{(k)}\right)^{-1} \left(\frac{j}{N_k}\right)
\]

As a consequence of the first point

\[
\left| \frac{1}{N_k} \sum_{j=\lfloor n_k/2 \rfloor + 1}^{n_k} f(\alpha_j^{(k)}) - \frac{1}{N_k} \sum_{j \notin P_t^{(k)}(M)} \left(\zeta_t^{(k)}\right)^{-1} \left(\frac{j}{N_k}\right) \right| \leq \frac{1}{N_k} \|P_t^{(k)}(M)\|_{\infty} ||f|_{M,\infty}||_{\infty} \\
\leq \frac{\epsilon}{2} ||f|_{M,\infty}||_{\infty} + 1 \\
\leq \frac{\epsilon}{2},
\]

since by definition, if \( j \in P_t^{(k)}(M) \) and \( j \geq \lfloor n_k/2 \rfloor + 1 \), then

\[
\left(\zeta_t^{(k)}\right)^{-1} \left(\frac{j}{N_k}\right) \geq M.
\]

• On the asymptotic cardinality of \((P_t^{(k)}(M))^c \Delta (\tilde{Q}_t^{(k)}(M))^c\):

\[
\frac{1}{N_k} \| (P_t^{(k)}(M))^c \Delta (\tilde{Q}_t^{(k)}(M))^c \| \to 0.
\]

Indeed, \((P_t^{(k)}(M))^c \Delta (\tilde{Q}_t^{(k)}(M))^c\) is equal to the set

\[
\left\{ j \in [1,n_k] : \frac{j}{N_k} \in \left(\zeta_t^{(k)}([-M,M]) \Delta (\xi_{t,d}([-M,M]))\right) \right\},
\]

thus its cardinality is smaller than

\[
\delta \left(N_k \left( \left(\zeta_t^{(k)}([-M,M]) \Delta (\xi_{t,d}([-M,M]))\right) \right) + 1, \right.
\]

which is equal to

\[
N_k \delta \left( \left(\zeta_t^{(k)}([-M,M]) \Delta (\xi_{t,d}([-M,M]))\right) \right) + 1.
\]

As a consequence:

\[
\frac{1}{N_k} \| (P_t^{(k)}(M))^c \Delta (\tilde{Q}_t^{(k)}(M))^c \| \leq \delta \left( \left(\zeta_t^{(k)}([-M,M]) \Delta (\xi_{t,d}([-M,M]))\right) \right) + \frac{1}{N_k}.
\]

Since \( \zeta_t^{(k)} \) converges to \( \xi_{t,d} \) on any compact, and in particular \([-M,M]\), the diameter on the right of this inequality converges to 0 when \( k \) tends towards \(+\infty\).
• **Replacing** $P_t^{(k)}(M)$ **by** $Q_t^{(k)}(M)$ **in the sum:**

Since $f$ is decreasing and positive, for all $j \in [[n_k/2] + 1, n_k]$,

$$\frac{1}{N_k} \left| f \left( \left( \xi_t^{(k)} \right)^{-1} \left( \frac{j}{N_k} \right) \right) \right| \leq \int_{\left[ \frac{j}{N_k}, \frac{j}{N_k} \right]} f \left( \left( \xi_t^{(k)} \right)^{-1} (x) \right) dx.$$  

As a consequence, the difference

$$\frac{1}{N_k} \left| \sum_{j \not\in P_t^{(k)}(M)} f \left( \left( \xi_t^{(k)} \right)^{-1} \left( \frac{j}{N_k} \right) \right) - \sum_{j \not\in Q_t^{(k)}(M)} f \left( \left( \xi_t^{(k)} \right)^{-1} \left( \frac{j}{N_k} \right) \right) \right|$$

is smaller than

$$\int_{J_k} \left| f \left( \left( \xi_t^{(k)} \right)^{-1} (x) \right) \right| dx = \int_{\xi_t^{(k)}(J_k)} f(x) \left( \xi_t^{(k)} \right)'(x) dx,$$

where $J_k$ is the union of the intervals

$$\left[ \frac{j - 1}{N_k}, \frac{j}{N_k} \right],$$

for $j \in (P_t^{(k)}(M))^c \Delta (Q_t^{(k)}(M))^c$. Since the functions $\xi_t^{(k)}$ are uniformly bounded independently of $k$, there exists a constant $C_t > 0$ such that for all $k$:

$$\int_{\xi_t^{(k)}(J_k)} f(x) \left( \xi_t^{(k)} \right)'(x) dx \leq C_t \int_{\xi_t^{(k)}(J_k)} f(x) dx,$$

Since $f$ is decreasing,

$$\int_{\xi_t^{(k)}(J_k)} f(x) \leq \int_{[0,l(\xi_t^{(k)}(J_k))]} f(x),$$

From the fact that $\xi_t^{(k)}$ is increasing:

$$l(\xi_t^{(k)}(J_k)) = \int_{J_k} (\xi_t^{(k)})'(\alpha) d\alpha$$

Since the derivative of $\xi_t^{(k)}$ is bounded uniformly and independently of $k$, and that the length of $J_k$ is smaller than $\frac{1}{N_k} \left| (P_t^{(k)}(M))^c \Delta (Q_t^{(k)}(M))^c \right|$, 

$$l(\xi_t^{(k)}(J_k)) \to 0.$$  

From the integrability of $f$ on $(0, +\infty)$:

$$\int_{[0,l(\xi_t^{(k)}(J_k))]} f(x) \to 0.$$  

As a consequence, there exists some $k_1 \geq k_0$ such that for all $k \geq k_1$,

$$\frac{1}{N_k} \left| \sum_{j \not\in P_t^{(k)}(M)} f \left( \left( \xi_t^{(k)} \right)^{-1} \left( \frac{j}{N_k} \right) \right) - \sum_{j \not\in Q_t^{(k)}(M)} f \left( \left( \xi_t^{(k)} \right)^{-1} \left( \frac{j}{N_k} \right) \right) \right| \leq \frac{\epsilon}{4}.$$

• **Approximating** $\xi_t^{(k)}$ **by** $\xi_t,d$ **in the sum:**
1. **Bounding the contribution in a neighborhood of 0:**

With an argument similar to the one used in the last point (bounding with integrals), there exists $\sigma > 0$ smaller than $M$ such that for all $k$,

$$\frac{1}{N_k} \sum_{j \in (Q^{(k)}(\sigma)) \cap (Q^{(k)}(M))} \left| f \left( \left( \xi^{(k)}_t \right)^{-1} \left( \frac{j}{N_k} \right) \right) \right| \leq \frac{\varepsilon}{8}.$$  

2. **Using the convergence of $\xi^{(k)}_t$ on a compact away from 0:**

There exists some $k_2 \geq k_1$ such that for all $k \geq k_2$:

$$\frac{1}{N_k} \sum_{j \in Q^{(k)}(\sigma) \cap (Q^{(k)}(M))} \left| f \left( \left( \xi^{(k)}_t \right)^{-1} \left( \frac{j}{N_k} \right) \right) - f \left( \xi^{-1}_{t,d} \left( \frac{j}{N_k} \right) \right) \right| \leq \frac{\varepsilon}{16}.$$

Indeed, for all the integers $j$ in the sum, $\xi^{-1}_{t,d} \left( \frac{j}{N_k} \right) \in [\sigma,M]$, and by uniform convergence of $\left( \xi^{(k)}_t \right)^{-1}$ on the compact $\xi_{t,d}([\sigma,M])$, for $k$ great enough, the real numbers $\left( \xi^{(k)}_t \right)^{-1} \left( \frac{j}{N_k} \right)$ and $\xi^{-1}_{t,d} \left( \frac{j}{N_k} \right)$ for these integers $k$ and these indexes $j$ all lie in the same compact interval. Since $f$ is continuous, there exists some $\eta > 0$ such that whenever for $x, y$ lie in this compact interval and $|x - y| \leq \eta$, then $|f(x) - f(y)| \leq \frac{\varepsilon}{8}$. Since $\left( \xi^{(k)}_t \right)^{-1}$ converges uniformly towards $\xi^{-1}_{t,d}$ on the compact $\xi_{t,d}([\sigma,M])$, there exists some $k_3 \geq k_2$ such that for all $k \geq k_3$, and for all $j$ such that $j \in Q^{(k)}\sigma$ and $j \notin Q^{(k)}M$,

$$\left| \left( \xi^{(k)}_t \right)^{-1} \left( \frac{j}{N_k} \right) - \xi^{-1}_{t,d} \left( \frac{j}{N_k} \right) \right| \leq \eta.$$

As a consequence, we obtain the announced inequality.

- **Convergence of the remaining sum:**

The following sum is a Riemann sum:

$$\sum_{j \notin Q^{(k)}(M)} f \left( \xi^{-1}_{t,d} \left( \frac{j}{4m} \right) \right),$$

and if $d = 1/2$ it converges towards

$$\int_{\xi_{t,d}(0)}^{\xi_{t,d}(M)} f \left( \xi^{-1}_{t,d}(\alpha) \right) d\alpha = \int_0^M f(\alpha) \xi^{-1}_{t,d}(\alpha) d\alpha,$$

by a change of variable. If $d < 1/2$, it converges towards

$$\int_{\xi_{t,d}(0)}^d f \left( \xi^{-1}_{t,d}(\alpha) \right) d\alpha = \int_0^d f(\alpha) \xi^{-1}_{t,d}(\alpha) d\alpha,$$

since in this case, $M$ was chosen such that $M > \xi^{-1}_{t,d}(d)$.

As a consequence, there exists some $k_4 \geq k_3$ such that for all $k \geq k_4$, if $d = 1/2$:

$$\left| \sum_{j \notin Q^{(k)}(M)} f \left( \xi^{-1}_{t,d} \left( \frac{j}{N_k} \right) \right) - \int_0^M f(\alpha) \xi^{-1}_{t,d}(\alpha) d\alpha \right| \leq \frac{\varepsilon}{16}.$$

If $d < 1/2$:  

48
\[ \left| \sum_{j \notin \mathcal{Q}_n^k(M)} f \left( \xi_{l,d}^{-1} \left( \frac{j}{N_k} \right) \right) - \int_0^d f(\alpha) \xi_{l,d}(\alpha) d\alpha \right| \leq \frac{\epsilon}{16} \]

- Assembling the inequalities:

All put together, we have for all \( k \geq k_4, \)

\[ \left| \frac{1}{N_k} \sum_{j=[n_k/2]+1}^{n_k} f(\alpha_j^{(k)}(t)) - \int_0^{\xi_j^{-1}(d)} f(\alpha) \xi_j(\alpha) d\alpha \right| \leq \frac{\epsilon}{2} + \frac{\epsilon}{4} + \frac{\epsilon}{8} + \frac{\epsilon}{16} = \epsilon. \]

Since for all \( \epsilon > 0 \) there exists such an integer \( k_4, \) this proves the statement.

\[ \square \]

7 Computation of square ice entropy

In this last section, we compute the entropy of the square ice model.

**Notation 17.** For all \( d \in [0, 1/2], \) we denote:

\[ F(d) = -2 \int_0^{\xi_j^{-1}(d)} \log_2 (2 |\sin(\kappa_t(\alpha)/2)|) \rho_t(\alpha) d\alpha. \]

**Lemma 13.** Let us consider \( (N_k) \) some sequence of integers, and \( (n_k) \) another sequence such that for all \( k, n_k \leq (N_k - 1)/4, \) and \( (2n_k + 1)/N_k \to d \in [0, 1/2]. \) Then

\[ \log_2(\lambda_{2n_k+1,N_k}(1)) \to F(d). \]

**Proof.** In this proof, for all \( k \) we denote:

\[ (p_j^{(k)})_j = (\alpha_j^{(k)})_j, \]

the solution of the system of Bethe equations \( (E_k)[1, 2n_k + 1, N_k]. \) The for all \( k: \)

\[ \lambda_{2n_k+1,N_k}(1) = \Lambda_{2n_k+1,N_k}[p^{(k)}] = \left( 2 + (N_k - 1) + \sum_{j \neq (n_k+1)} \frac{\partial \Theta_1}{\partial x} (0, p_j^{(k)}) \right) \prod_{j=1}^{n_k} M_1(e^{p_j^{(k)}}), \]

since by antisymmetry of \( p^{(k)} \) and that \( 2n_k + 1 \) is odd, \( p_{n_k+1}^{(k)} = 0. \)

For all \( z \) such that \( |z| = 1, \)

\[ M_1(z) = \frac{z}{z-1}. \]

By antisymmetry of the sequences \( p^{(k)}, \) for all \( k: \)

\[ \prod_{j=1}^{n_k} e^{p_j^{(k)}} = \prod_{j=1}^{n_k} e^{p_j^{(k)}/2} = 1. \]

As a consequence:

\[ \Lambda_{2n_k+1,N_k}[p^{(k)}] = \left( 2 + (N_k - 1) + \sum_{j \neq (n_k+1)} \frac{\partial \Theta_1}{\partial x} (0, p_j^{(k)}) \right) \prod_{j=1}^{n_k} \frac{1}{e^{p_j^{(k)}} - 1} = \left( 2 + (N_k - 1) + \sum_{j \neq (n_k+1)} \frac{\partial \Theta_1}{\partial x} (0, p_j^{(k)}) \right) \prod_{j=1}^{n_k} \frac{e^{-p_j^{(k)}/2}}{e^{p_j^{(k)}/2} - e^{-p_j^{(k)}/2}}. \]
Since this eigenvalue is positive,
\[
\Lambda_{2n_k+1,N_k}[p^{(k)}] = |\Lambda_{2n_k+1,N_k}[p^{(k)}]| = 2 + (N_k - 1) + \sum_{j \neq (n_k+1)} \frac{\partial \Theta_1}{\partial x} (0, p^{(k)}_j) \prod_{j=1}^{n_k} \frac{1}{\sin \left( \frac{p^{(k)}_j}{2} \right)}.
\]
As a consequence, since \( \partial \Theta_1/\partial x \) is a bounded function,
\[
\lim_{k \to \infty} \log_2(\lambda_{2n_k+1,N_k}(1)) = -\lim_k \left( \frac{1}{N_k} \sum_{j=1}^{n_k} \log_2 \left( 2 \left| \sin \left( \frac{p^{(k)}_j}{2} \right) \right| \right) + O \left( \frac{\log_2(N_k)}{N_k} \right) \right)
\]
\[
= -2 \lim_k \frac{1}{N_k} \sum_{j=\lceil n_k/2 \rceil}^{n_k} \log_2 \left( 2 \left| \sin \left( \kappa_t(\alpha)^{(k)} / 2 \right) \right| \right)
\]
\[
= -2 \int_0^{\xi_{t,j}^{(d)}} \log_2(2|\sin(\kappa_t(\alpha)/2)|) \rho_t(\alpha) d\alpha
\]
\[
= F(d).
\]
where \( \rho_t = \xi_{t,j}^{(d)} \), and we used the antisymmetry of the Bethe roots vectors in the second equality.

For the other equalities, they are a consequence of Theorem 1.3, since the function defined as

\[
\lim_{k \to \infty} \log_2(\lambda_{2n_k+1,N_k}(1)) = -\log_2(2|\sin(\kappa_t(\alpha)/2)|)\biggr|_{(0,\infty)}
\]

is continuous, integrable, decreasing and positive:

1. **Positive**: For all \( \alpha > 0 \), \( \kappa_t(\alpha) \) is in
\[
(0, \pi - \mu_t) = \left(0, \frac{\pi}{3}\right).
\]

As a consequence, \( 2 \sin(\kappa_t(\alpha)/2) \) is in \((0, 1)\), and this implies that for all \( \alpha > 0 \),
\[
-\log_2(2|\sin(\kappa_t(\alpha)/2)|) > 0.
\]

2. **Decreasing**: This comes from the fact that \( -\log_2 \) is decreasing, and \( \kappa_t \) is increasing, and the sinus is increasing on \((0, \pi/6)\).

3. **Integrable**: Since \( \kappa_t(0) = 0 \) and \( \kappa_t'(0) > 0 \), for \( \alpha \) positive sufficiently close to 0 \( 2 \sin(\kappa_t(\alpha)/2) \leq 2\kappa_t'(0)\alpha \).

As a consequence,
\[
-\log_2(2|\sin(\kappa_t(\alpha)/2)|) \leq -\log_2(2\kappa_t'(0)\alpha).
\]

Since the logarithm is integrable on any bounded neighborhood of 0, the function \( \alpha \mapsto -\log_2(2|\sin(\kappa_t(\alpha)/2)|) \) is integrable.

The other limit is obtained by antisymmetry of \( \kappa_t \).

\textbf{Theorem 1.} The entropy of square ice is
\[
h(X^t) = \frac{3}{2} \log_2 \left( \frac{4}{3} \right).
\]

\textbf{Remark 8.} This value corresponds to \( \log_2(W) \) in \cite{Lieb1967}.

\textbf{Proof.} Here we fix \( t = 1 \in (0, \sqrt{2}) \). As a consequence, \( \mu_t = 2\pi/3 \).
Entropy of $X^*$ and asymptotics of the maximal eigenvalue:

Let us recall that the entropy of $X^*$ is given by:

$$h(X^*) = \lim_{N \to \infty} \frac{1}{N} \max_{n \leq (N-1)/4} \log_2(\lambda_{2n+1,N}(1)).$$

For all $N$, we denote $\nu(N)$ the smallest $n \leq (N-1)/4$ such that for all $n \leq (N-1)/4$,

$$\lambda_{2\nu(N)+1,N}(1) \geq \lambda_{2n+1,N}(1).$$

By compacity, there exists an increasing sequence $(N_k)$ such that $(2\nu(N_k)+1)/N_k$ converges towards some non-negative real number $d$. Since for all $k$, $\nu(N_k) \leq (N_k-1)/4$, then $d \leq 1/2$.

In virtue of Lemma 13, $h(X^*) = F(d)$.

Comparison with the asymptotics of other eigenvalues:

Moreover, if $d$ is another number $d \in [0, 1/2]$, there exists $\nu' : \mathbb{N} \to \mathbb{N}$ such that

$$(2\nu'(N) + 1)/N \to d.$$ 

For all $k$,

$$\lambda_{2\nu'(N_k)+1,N_k}(1) \leq \lambda_{2\nu(N_k)+1,N_k}(1).$$

Also in virtue of Lemma 13, $h(X^*) \geq F(d)$, and thus

$$F(d) = \max_{d \in [0, 1/2]} F(d).$$

This maximum is realized only for $d = 1/2$. As a consequence $d = 1/2$.

Rewritings:

As a consequence,

$$h(X^*) = -2 \int_{0}^{+\infty} \log_2(2|\sin(\kappa t(\alpha)/2)|)\rho_t(\alpha)d\alpha$$

Let us rewrite this expression of $h(X^*)$ using

$$|\sin(x/2)| = \sqrt{1 - \cos(x)/2}.$$ 

This leads to:

$$h(X^*) = -\frac{\log_2(2)}{2} \int_{-\infty}^{+\infty} \rho_t(\alpha)d\alpha - \frac{1}{2} \int_{-\infty}^{+\infty} \log_2(1 - \cos(\kappa t(\alpha)))\rho_t(\alpha)d\alpha.$$ 

Thus,

$$h(X^*) = -\frac{1}{2} \int_{-\infty}^{+\infty} \log_2(2 - 2\cos(\kappa t(\alpha)))\rho_t(\alpha)d\alpha.$$ 

Let us recall that for all $\alpha$,

$$\rho(\alpha) = \frac{1}{4\mu_t \cosh(\pi \alpha/2\mu_t)} = \frac{3}{8\pi \cosh(3\alpha/4)},$$

$$\cos(\kappa_t(\alpha)) = \frac{\sin^2(\mu_t)}{\cosh(\alpha) - \cos(\mu_t)} - \cos(\mu_t) = \frac{3}{4(\cosh(\alpha) + 1/2)} + \frac{1}{2}.$$
We have that:

\[ h(X) = -\frac{3}{16\pi} \int_{-\infty}^{+\infty} \log_2 \left( 1 - \frac{3}{2\cosh(\alpha) + 1} \right) \frac{1}{\cosh(3\alpha/4)} d\alpha. \]

Using the variable change \( e^\alpha = x^4 \), \( dx = 4dx \),

\[ h(X) = -\frac{3}{16\pi} \int_{0}^{+\infty} \log_2 \left( 1 - \frac{3}{x^4 + 4} \right) \frac{2}{(x^4 + 1/x^4)} \frac{4}{x} dx. \]

By symmetry of the integrand:

\[ h(X) = -\frac{3}{4\pi} \int_{-\infty}^{+\infty} \frac{x^2 dx}{x^6 + 1} \log_2 \left( \frac{2x^4 - 1 - x^8}{1 + x^4 + x^8} \right) dx \]

\[ h(X) = -\frac{3}{4\pi} \int_{-\infty}^{+\infty} \frac{x^2 dx}{x^6 + 1} \log_2 \left( \frac{(x^2 - 1)^2(x^2 + 1)^2}{1 + x^4 + x^8} \right) dx. \]

**Application of the residues theorem:**

In the following, we use the standard determination of the logarithm on \( \mathbb{C} \setminus \mathbb{R}_- \).

We apply the residue theorem to obtain (the poles of the integrand are \( e^{i\pi/6}, e^{i\pi/2}, e^{i5\pi/6} \)):

\[
\int_{-\infty}^{+\infty} \frac{x^2 \log_2(x + i)}{x^6 + 1} dx = 2\pi i \left( \sum_{k=1,3,5} \frac{e^{ik\pi/3} \log_2(e^{ik\pi/6} + i)}{6e^{ik\pi/6}} \right),
\]

\[
\int_{-\infty}^{+\infty} \frac{x^2 \log_2(x - i)}{x^6 + 1} dx = -2\pi i \left( \sum_{k=7,9,11} \frac{e^{ik\pi/3} \log_2(e^{ik\pi/6} - i)}{6e^{ik\pi/6}} \right).
\]

By summing these two equations, we obtain that \( \int_{-\infty}^{+\infty} \frac{x^2 \log_2(x^2 + 1)}{x^6 + 1} dx \) is equal to:

\[
\frac{\pi}{3} \left[ \log_2(e^{i\pi/6} + i) - \log_2(e^{i\pi/2} + i) + \log_2(e^{i5\pi/6} + i) + \log_2(e^{i7\pi/6} - i) - \log_2(e^{i3\pi/2} - i) + \log_2(e^{i3\pi/6} - i) \right]
\]

This is equal to

\[
\frac{\pi}{3} \log_2(\left| e^{i\pi/6} + i \right|^2) - \log_2(\left| e^{i\pi/2} + i \right|) + \log_2(\left| e^{i5\pi/6} + i \right|^2) = \frac{2\pi}{3} \log_2 \left( \frac{3}{2} \right).
\]

**Other computations:**

We do not include the following computation, since it is very similar to the previous one:

\[
\int_{-\infty}^{+\infty} \frac{x^2}{x^6 + 1} \log_2(1 + x^4x^8) dx = \frac{2\pi}{3} \log_2 \left( \frac{3}{2} \right).
\]

For the last integral, we write \( \log_2((x^2 - 1)^2) = 2\text{Re}(\log_2(x - 1) + \log_2(x + 1)) \) and obtain:

\[
\int_{-\infty}^{+\infty} \frac{x^2}{x^6 + 1} \log_2((x^2 - 1)^2) = 2\text{Re} \left( \int_{-\infty}^{+\infty} \frac{x^2}{x^6 + 1} \log_2(x - 1) + \int_{-\infty}^{+\infty} \frac{x^2}{x^6 + 1} \log_2(x + 1) \right) = \frac{2\pi}{3} \log_2 \left( \frac{1}{2} \right).
\]
Summing these integrals:

As a consequence

\[ h(X^*) = -\frac{3}{4\pi} \left( \frac{2\pi}{3} \left( \log_2 \left( \frac{1}{2} \right) + 2 \log_2 \left( \frac{3}{2} \right) - \log_2 \left( \frac{8}{3} \right) \right) \right) = \frac{1}{2} \log_2 \left( \frac{4^3}{3^3} \right) = \frac{3}{2} \log_2 \left( \frac{4}{3} \right). \]

8 Comments

8.1 On the limits of the computing method

This text is meant as a ground for further research, that would aim at extending the computation method that we exposed to a broader set of multidimensional SFT, including for instance Kari-Culik tilings [Culik], the monomer-dimer model [see for instance [Friedland Peled]], subshifts of square ice [Gangloff Sablik], the hard square shift [Pavlov], the eight-vertex model [Baxter] or a three-dimensional version of the six vertex model. Adaptations for these models may be possible, but would not be immediate at all. We explain here at which points the method has limitations, each of them coinciding with a specific property of square ice.

8.1.1 Symmetry and irreducibility

Let us recall that we called Lieb path an analytic function of transfer matrices \( t \mapsto V_N(t) \) such that for all \( t \), \( V_N(t) \) is an irreducible non-negative and symmetric matrix on \( \Omega_N \).

8.1.1.1 Implications of the symmetry and mixing properties of square ice. Although the definition of transfer matrices admits straightforward generalization to multidimensional SFT and their non-negativity does not seem difficult to achieve, the property of symmetry of the matrices \( V_N(t) \) relies on symmetries of the alphabet and local rules of the SFT. S.Friedland [Friedland] proved that under these symmetry constraints (which are verified for instance by the monomer-dimer and hard square models, but a priori not by Kari-Culik tilings), entropy is algorithmically computable, through a generalisation of the gluing argument exposed in Lemma 1. Outside of the class of SFT defined by these symmetry restrictions, as far as we know, only strong mixing or measure theoretic conditions ensure algorithmic computability of entropy, leading for instance to relatively efficient algorithms approximating the hard square shift entropy [Pavlov]. On the other hand, the irreducibility of the matrices \( V_N(t) \) derives from the irreducibility property of the stripes subshifts \( X_{\delta}^N \) [Definition 2], that can be derived from the linear block gluing property of \( X^* \) [Gangloff Sablik]. This property consists in the possibility for any pair of pattern on \( \Gamma_N^{(2)} \) to be glued in any relative positions, provided that the distance between the two patterns is greater than a minimal distance, which is \( O(N) \).

8.1.1.2 Possible relaxations of some arguments. Lemma 1 which relies on a horizontal symmetry of the model, is a simplification in the proof of Theorem 1 whose implication is that entropy of \( X^* \) can be computed through entropies of subshifts \( \overline{X}_N \), and thus simplifies Algebraic Bethe ansatz, that we will expose in another text. One can see in [Viéira Lima-Santos] that it is possible to use the ansatz without Lemma 1. However, this application of the ansatz would lead to different Bethe equations, and it is not clear if these equations admits solutions, and if we can evaluate their asymptotic behavior. The symmetry is also involved in the equality of entropy of \( \overline{X}_{n,N} \) and entropy of \( X_{N-n,N} \). Without this equality, we don’t know how to identify the greatest eigenvalue of \( V_N(t) \) with the candidate eigenvalue obtained via the ansatz.
8.2 On the gap between mathematics and mathematical physics

The difficulties that were encountered in proving Theorem 1 besides partial arguments, were related primarily to the form of the literature on the subject, as a field of research in mathematical physics, and the gap that there exists with mathematical literature. In Section 8.2.1 we provide a short analysis of this gap, that relies on the concept of discursive formation developed by M. Foucault in the Archaeology of knowledge, which generalises the particularly organised and socially structured forms of discourse that are sciences, or philosophy, and important aspects of discursive formations that are the mode of creation, existence and coexistence of concepts within it and the conception of units of meaning. We describe there mathematics and mathematical physics as distinct discursive formation: we analyse, from our point of view, their contemporary form and the consequence it has on how the units of meaning and objects of discourse (theorems, proofs) are conceived. A significant part of the work done in the core of the present text is an analog of translation, from discursive formation of to another, and we provide some examples, in Section 8.2.2 of the difficulties implied in the translation process by the distinction of mathematics and mathematical physics as discursive formations.

8.2.1 Distinct discursive formations

From our point of view, the most salient difference that separate mathematics from mathematical physics - which use the same elements of meanings, the same formalism - lies on the use and the structuration of language.

8.2.1.1 On mathematics: As far as we understand it, contemporary mathematics could be conceived as a space of constant exchange, accumulation and communication of proof techniques (communicated as proofs of theorems) that derive from the impossibility to foresee the use that tools which emerged in another area can have to solve a problem. This implies a logic of accumulation of the texts, and their (cognitive) content. This constant exchange and the inflation of accessible information that follows seems to imply a shift in the function of the mathematician, who has to understand primarily which technique is suitable to which problem. The universality of the language of mathematics is fundamental for these exchanges, but there is also a necessity for the mathematical text to match the functions of memory, in particular the optimisation of the time and attention allocated to reading, and thus the accessibility of the (cognitive) content of the text. The text thus is assumed to allow the extraction of pertinent information, from the point of view of the reader, whose background is a priori unknown. The text relates how the various techniques involved are articulated and the context in which they are applicable, etc. It exhibits a hierarchisation of its content, from an overall point of view that includes motivations for the reader to get into the text, to the many details, and including markers of the function of each articulation in this hierarchy (including the functions of lemma, definition, comments, section abstract, etc), but also markers for the possibility of further development. The necessity for a motivation for the reader implies in particular that the ability of a technique or a set of techniques to actually prove a theorem defines the unity of discourse (the article). The dynamics of contemporary mathematics seems also to have an impact on the way concepts are formed (or equivalently named), since the name is a marker (that helps for bibliographical orientation) of depthness and level of connectivity to other notions, and on the inclusion in the pair theorem-proof of the a priori implementability in another context of the articulation of techniques involved in the proof, given as prerequisite only the understanding of elementary mathematical objects involved.

8.2.1.2 On mathematical physics: On the other hand, as far as we understand it, physics are structured around the contradiction of general theories explaining domains of phenomena appearing in the world, and the value of a theory is subjected to experimentation, which selects (as in natural selection) the technical tools that are adapted to explanation. Any theory is thus temporary, as well as any technical development within it, which is supposed to provide technical tools to compute some characteristics of the physical system studied. These technical developments,
when appearing in a recurrent way, are then turned into an intuition on the behavior of these systems, by essentialisation. The same principle of optimisation of information treatment that leads in mathematics to memory-structured texts seems to lead in physics to meaning units (texts) that are centered on isolated (computing) non-rigorous techniques that presuppose a knowledge of their context. These techniques are selected by their recurrence in various directions of research within this context, before an attempt of a rigorous version, the expectation of which relies on the close relation to objects that have a meaning in the reality.

8.2.2 Translation difficulties

From the point of view of information treatment, an aspect of the relation between mathematics and mathematical physics is analogical to the relation in the brain between myelin and neurons. While neurons seem to be formed by an a priori production followed by selection, the myelin is constructed selectively around some neurons in order to accelerate their information processing. Following this analogy, the general process of myelinisation, of which the present text is an element, exhibits a lot of difficulties, that are not related to grammar of the respective languages of mathematics and mathematical physics (since they are the same) but to their structures as discursive formations.

Indeed, a primary effect of the conception of meaning units as defined by isolated techniques is the non-neglectable distance that there exists between an existing group of techniques and an actual proof of a theorem using these techniques, for a subject within the space of mathematics. This effect is due in part to the fluctuation of notations (there is an interesting analogy between the use of common notations and prototyping in programming) and terms used from a text to another to designate the same objects that come along with non-explicit one-to-one transformations of the objects considered (as it is for instance from [Yang Yang I] to [Kozlowski] for the definition of the function \( \theta_t \) [Section 5.1.1]), in part also to non-explicit reference to definitions or other techniques, the knowledge of which is presupposed in the text. The difficulties that come from the fluctuation of notations have a particular effect on cases distinctions, since we tend, in order to accelerate reading, to identify cases (for an example, the phases of the six-vertex model, that corresponds to domains for the parameter \( t \) in the present text) to properties of the objects considered in these cases: this acceleration becomes an obstacle in the presence of a change of notations.

Some other difficulties come from the multiplicity of methods whose mode of coexistence is not explicit in the literature (for instance the coordinate Bethe ansatz, exposed for instance in [Duminil-Copin et al.], and the algebraic Bethe ansatz (does it worth to invest time in understanding the other technique when one is more directly accessible?). Also, the absence of markers for the generality of the techniques used can be misleading, as well as a formulation of the generality of the method, where the degree of generality is ambiguous, since it is dependant upon implicit prerequisite of the knowledge of the field (for the algebraic Bethe ansatz in the literature for instance). Some particular difficulties come more directly from the absence of separation of statements having different functions (definition or lemma), underlocalised autoreference to parts of the text, in particular in the case of multiple references, or the absence of parastructural comments or object typing for the various mathematical objects considered, which swipe off the possible ambiguities in writing.

At a higher level, the ambiguity of the distinction between mathematics and mathematical physics is itself a difficulty, that demand specific tools, such as the ones developed by M. Foucault in the Archaeology of knowledge, to make visible the border between the discursive formations and explain it, in order to develop general translation tools for this border, from discursive formation to discursive formation, such as, simply, a change of perception of the unity of the text or the multiplication of the sources in order to understand the concepts involved.

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56