Linkage of Finite Gorenstein Dimension Modules

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Introduction

The theory of linkage of algebraic varieties introduced by Peskine and Szpiro (1974).

Martsinkovsky and Strooker (2004) give its analogous definition for modules over non–commutative semiperfect Noetherian rings by using the composition of the two functors:

transpose and syzygy.
These functors and their compositions were studied by Auslander and Bridger in “Stable module theory” (1969).

The Gorenstein (or $G$-) dimension was introduced by Auslander (1966–7) and studied by Auslander and Bridger (1969).

In this work, we study the theory of linkage for class of modules which have finite Gorenstein dimensions. In particular, for a horizontally linked module $M$ of finite and positive $G$-dimension, we study the role of its reduced grade, $\text{r.grade} (M)$, on the depth of its linked module $\lambda M$. 
Organization of Talk

- Notations and elementary definitions.
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- Linkage and the reduced grade.
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Let $R$ be a ring. Consider an $R$-modules $M$ with a submodule $N$. The module $M$ is said to be a \textit{superfluous extension} of $N$ if for every submodule $H$ of $M$, $H + N = M \implies H = M$. 
Let $R$ be a ring. Consider an $R$-modules $M$ with a submodule $N$. The module $M$ is said to be a superfluous extension of $N$ if for every submodule $H$ of $M$, $H + N = M \iff H = M$.

Let $X$ be an $R$-module. A projective cover of $X$ is a pair $(P, f)$ such that $P$ is a projective $R$-module and $f : P \rightarrow X$ is an epimorphism with $P$ is a superfluous extension of $\ker f$. 
Projective covers and their superfluous epimorphisms, when they exist, are unique up to isomorphism. The main effect of \( f \) having a superfluous kernel is the following: if \( K \) is any proper submodule of \( P \), then \( f(K) \neq X \). If \((P, f)\) is a projective cover of \( M \), and \( P' \) is another projective module with an epimorphism \( f' : P' \rightarrow X \), then there is an epimorphism \( \alpha \) from \( P' \) to \( P \) such that \( f\alpha = f' \).

Unlike injective envelopes, which exist for every left (right) \( R \)-module regardless of the ring \( R \), left (right) \( R \)-modules do not in general have projective covers.

A ring \( R \) is called left (right) \textit{perfect} if every left (right) \( R \)-module has a projective cover.
A ring is called **semiperfect** if every finitely generated left (right) $R$-module has a projective cover.

Any commutative noetherian local ring is semiperfect.
Notations and Definitions

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**Commutative semiperfect noetherian rings**

Throughout, $R$ is a commutative semiperfect noetherian ring and all modules are finite (i.e. finitely generated) $R$–modules so that any such module has a projective cover.
Transposes

Let

\[ P_1 \xrightarrow{f} P_0 \rightarrow M \rightarrow 0 \]

be a finite projective presentation of \( M \). The transpose of \( M \), \( \text{Tr} M \), is \( \text{Coker } f^* \), where \((-)^* := \text{Hom}_R(–, R)\), which satisfies in the exact sequence

\[ 0 \rightarrow M^* \rightarrow P_0^* \xrightarrow{f^*} P_1^* \rightarrow \text{Tr } M \rightarrow 0. \]

and is unique up to projective equivalence; that is if \( P \oplus M = Q \oplus M' \) (denoted by \( M = M' \)) with \( P \) and \( Q \) are projective then \( \text{Tr } M \cong \text{Tr } M' \).

A stable module \( M \) is a module which has no non–trivial projective summands.

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Linkage of Finite Gorenstein Dimension Modules
This is a result from Auslander (1965) that there is an exact sequence

\[ 0 \rightarrow \text{Ext}^1_R(\text{Tr} M, R) \rightarrow M \xrightarrow{e_M} M^{**} \rightarrow \text{Ext}^2_R(\text{Tr} M, R) \rightarrow 0. \]

where \( e_M : M \rightarrow M^{**} \) is the natural map.
Let \( P \xrightarrow{\alpha} M \) be an epimorphism such that \( P \) is a projective. The syzygy module of \( M \), denoted by \( \Omega M \), is the kernel of \( \alpha \) which is unique up to projective equivalence. Thus \( \Omega M \) is uniquely determined, up to isomorphism, by a projective cover of \( M \).
Linkage of ideals

Let $R$ be Gorenstein local and let $c$ be an ideal of $R$ such that $R/c$ is Gorenstein. Two ideals $a$ and $b$ of $R$ are said to be linked by $c$ if $c \subseteq a \cap b$, $a = c : b$ and $b = c : a$. Martsinkovsky and Strooker (MS) have introduced the operator $\lambda := \Omega \mathrm{Tr}$. They showed that over a ring $R$, the ideals $a$ and $b$ are linked by zero ideal if and only if $R/a \cong \lambda(R/b)$ and $R/b \cong \lambda(R/a)$. Arash Sadeghi and M.T.D.
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Linkage of modules

(MS): Two $R$–modules $M$ and $N$ are said to be *horizontally linked* if $M \cong \lambda N$ and $N \cong \lambda M$. 

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In this situation the module $M$ is called a *horizontally linked module* and one has $M \cong \lambda^2 M$. 
Gorenstein class

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Notations and Definitions

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Definition
An $R$–module $M$ is said to belong to the $G$-class, $G(R)$, whenever

(i) the biduality map $e_M : M \to M^{**}$ is an isomorphism;

(ii) $\text{Ext}^i_R(M, R) = 0$ for all $i > 0$;

(iii) $\text{Ext}^i_R(M^*, R) = 0$ for all $i > 0$. 
Notations and Definitions

**G–dimension**

Any projective module is in $G$-class. Trivially any $R$–module $M$ has a $G$-resolution which is a right acyclic complex of modules in $G(R)$ whose 0th homology module is $M$. The module $M$ is said to have finite $G$-dimension, denoted by $\text{Gdim}_R(M)$, if it has a $G$-resolution of finite length. Note that $\text{Gdim}_R(M) \leq \text{pd}_R(M)$.

**Theorem** (Masiek) If $\text{Gdim}_R(M) < \infty$, then

(i) $\text{Gdim}_R(M) = \sup\{i \geq 0 | \text{Ext}^i_R(M, R) \neq 0\}$

and (ii) if $R$ is local, then

$\text{Gdim}_R(M) = \text{depth } R – \text{depth } R(M)$ (Auslander-Bridger).
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**Theorem**

*(Masiekh) If $G\dim_R(M) < \infty$, then*

(i) $G\dim_R(M) = \sup \{ i \geq 0 \mid \ Ext^i_R(M, R) \neq 0 \}$; and (ii) if $R$ is local, then $G\dim_R(M) = \text{depth } R - \text{depth } R(M)$ (Auslander-Bridger).
Linkage and Reduced Grade

Definition

The reduced grade of an $R$–module $M$ is defined to be

$$r.\text{grade} (M) = \inf \{ i > 0 \mid \text{Ext}^i_R(M, R) \neq 0 \}.$$ 

introduced by Hoshino (1990).

Note that grade $R(M) = r.\text{grade} R(M)$ if grade $R(M) > 0$. Moreover, if Gdim $R(M) = 0$ then r.grade $(M) = \infty$. For modules of finite and positive G-dimension, one has r.grade $(M) \leq \text{Gdim} R(M)$ and so it is finite.
Lemma

Let $M$ be a horizontally linked $R$–module of finite and positive $G$-dimension. Set $n = r.\text{grade}(M)$. Then

$$\text{Ass}_R(\text{Ext}_R^n(M, R)) = \{p \in \text{Spec } R \mid Gdim_{R_p}(M_p) \neq 0, \text{depth}_{R_p}((\lambda M)_p) = n = r.\text{grade}_{R_p}(M_p)\}.$$
Lemma

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Proposition

Let $M$ be a horizontally linked $R$–module of finite $G$-dimension. Then

$$\text{Gdim}_R(M) = 0 \text{ if and only if } \text{depth}_{R_p}(M_p) + \text{depth}_{R_p}((\lambda M)_p) > \text{depth}_{R_p} \text{ for all } p \in \text{Spec } R \setminus X^0(R).$$

Here $X^i(R) = \{ p \in \text{Spec } R \mid \text{depth } R_p \leq i \}$. 

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Corollary

Let $M$ be a horizontally self-linked $R$–module of finite $G$-dimension. Then $Gdim_R(M) = 0$ if and only if $\operatorname{depth}_{R_p}(M_p) > \frac{1}{2}(\operatorname{depth}_{R_p})$ for all $p \in \operatorname{Spec} R \setminus X^0(R)$. 
Corollary

Let $M$ be a horizontally self-linked $R$–module of finite $G$-dimension. Then $G\dim_R(M) = 0$ if and only if $\text{depth}_{R_p}(M_p) > \frac{1}{2}(\text{depth}_{R_p})$ for all $p \in \text{Spec } R \setminus X^0(R)$.

Proposition

Let $M$ be a horizontally linked $R$–module of finite and positive $G$-dimension. Set $t_M = \text{r.grade}(M) + \text{r.grade}(\lambda M)$, then $M$ is of $G$-dimension zero on $X^{t_M - 1}(R)$.
Auslander shows that, for a positive integer $k$, the following statements are equivalent.

(i) $r.\text{grade} (Tr M) > k$.

(ii) $M$ is a $k$th syzygy.

Here we show that:

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(i) $r.grade(M) \geq k$.

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Let $M$ be an $R$–modules. If $M$ is horizontally linked module, then

$$\text{Ext}^i_R(M, M) \cong \text{Ext}^i_R(\lambda M, \lambda M)$$

for all $i$, $1 \leq i < \inf\{r.\text{grade}(M), r.\text{grade}(\lambda M)\}$.

In particular, if $G\text{dim}_R(M) = 0$ then $\text{Ext}^i_R(M, M) \cong \text{Ext}^i_R(\lambda M, \lambda M)$ for all $i > 0$. 
Let $M$ be an $R$–module of finite positive $G$–dimension. The following inequalities are just mentioned:

$$\text{grade}_R(M) \leq \text{r.grade}(M) \leq \text{Gdim}_R(M) \leq \text{pd}_R(M).$$
Reduced $G$–Perfect Modules

Let $M$ be an $R$–module of finite positive $G$–dimension. The following inequalities are just mentioned:

$$\text{grade}_R(M) \leq r.\text{grade}(M) \leq \text{Gdim}_R(M) \leq \text{pd}_R(M).$$

- $M$ is called perfect if $\text{grade}_R(M) = \text{pd}_R(M)$. 

- $M$ is called $G$–perfect if $\text{grade}_R(M) = \text{Gdim}_R(M)$. 

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Let $M$ be an $R$–module of finite positive $G$–dimension. The following inequalities are just mentioned:

$$\text{grade } R(M) \leq r.\text{grade } (M) \leq \text{Gdim } R(M) \leq \text{pd } R(M).$$

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- We define $M$ is reduced $G$–perfect if $r.\text{grade } (M) = \text{Gdim } R(M)$. 
Theorem

Let \((R, \mathfrak{m})\) be a local Cohen-Macaulay ring of dimension \(d\). If \(M\) is reduced \(G\)-perfect of \(G\)-dimension \(n\), then

\[
\text{depth}_R(M) + \text{depth}_R(\lambda M) = d + \text{depth}_R(\operatorname{Ext}^n_R(M, R)).
\]

Proposition

Let \(M\) be a reduced \(G\)-perfect \(R\)-module of \(G\)-dimension \(n\), then the following statements hold true.

(i) \(\operatorname{Ext}^i_R(\lambda M, R) \cong \operatorname{Ext}^{n+i}_R(\operatorname{Ext}^n_R(M, R), R)\) for all \(i > 0\).

(ii) Assume that \(M\) is stable \(R\)-module. Then \(M\) is horizontally linked if and only if \(r.\text{grade}(M) + r.\text{grade}(\lambda M) = \text{grade}_R(\operatorname{Ext}^n_R(M, R))\).
Let $R$ be local. Recall from Evan-Griffith “Syzygies” (LMS Lecture Notes 1985) that

$$\text{syz}(M) = \text{Sup}\{n \mid M \text{ is } n\text{th syzygy in a minimal free resolution of an } R\text{-module } N\}.$$ 

Note that $\text{syz}(M) = \infty$, whenever $G\text{dim}_R(M) = 0$. If $M$ is a horizontally linked of finite and positive $G$-dimension then we have $\text{syz}(M) = r.\text{grade}(\lambda M)$. 
Theorem

Let $R$ be a Cohen-Macaulay local ring of dimension $d$, and let $M$ be a horizontally linked module of finite and positive $G$-dimension. If $\lambda M$ is reflexive then

$$\text{depth } R (M) = \text{syz } (M) = r.\text{grade } (\lambda M);$$

$$\text{Ext } r.\text{grade } (\lambda M) R (\lambda M, R) \sim = \text{Ext } d R (\text{Ext } G\dim R (M) R (M, R), R).$$
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Reduced $G$–Perfect Modules

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- $\text{Ext}_{R}^{r.\text{grade}(\lambda M)}(\lambda M, R) \cong \text{Ext}_{R}^{d}(\text{Ext}_{R}^{G\text{dim}_R(M)}(M, R), R)$. 

Arash Sadeghi and M.T.D. Linkage of Finite Gorenstein Dimension Modules
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Arash Sadeghi and M.T.D. Linkage of Finite Gorenstein Dimension Modules
Let $\alpha$ and $\beta$ be ideals in a Gorenstein local ring $R$ which are linked by a Gorenstein ideal $\gamma$. Schenzel (1983) proved that the Serre condition $(S_r)$ for $R/\alpha$ is equivalent to the vanishing of the local cohomology groups $H^i_{m}(R/\beta) = 0$ for all $i$, $\dim (R/\beta) - r < i < \dim (R/\beta)$. Here we extend this result for any horizontally linked module of finite $G$-dimension over a Cohen-Macaulay local ring.
Let \( \alpha \) and \( \beta \) be ideals in a Gorenstein local ring \( R \) which are linked by a Gorenstein ideal \( \mathfrak{c} \). Schenzel (1983) proved that the Serre condition \((S_r)\) for \( R/\alpha \) is equivalent to the vanishing of the local cohomology groups \( H^i_{m}(R/\beta) = 0 \) for all \( i, \dim (R/\beta) - r < i < \dim (R/\beta) \). Here we extend this result for any horizontally linked module of finite \( G \)-dimension over a Cohen-Macaulay local ring.

First we bring the following lemma which is clear if the ground ring is Gorenstein by using the Local Duality Theorem.
Lemma

Let $R$ be a Cohen-Macaulay local ring of dimension $d$ and let $M$ be an $R$–module of dimension $d$ which is not maximal Cohen-Macaulay. If $G\dim_R(\lambda M) < \infty$ then 
$$\sup\left\{ i \mid H^i_{\mathfrak{m}}(M) \neq 0, i \neq d \right\} = d - \text{grade}(M).$$

Now we are able to generalize a result of Schenzel (1983) for modules of finite Gorenstein dimension.

Theorem

Let $R$ be a Cohen-Macaulay local ring of dimension $d$, and let $M$ be a horizontally linked $R$–module of finite G-dimension. Let $k$ be a non-negative integer. Then $M$ satisfies the Serre condition $(S_k)$ if and only if $H^i_{\mathfrak{m}}(\lambda M) = 0$ for all $i$, $d - k + 1 \leq i \leq d - 1$. 

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Let $R$ be a Cohen-Macaulay local ring of dimension $d$ and let $M$ be an $R$–module of dimension $d$ which is not maximal Cohen-Macaulay. If $Gdim_R(\lambda M) < \infty$ then $\sup\{i \mid H^i_m(M) \neq 0, i \neq d\} = d - r.grade(M)$. 

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D-Gheibi-Hassanzadeh-Sadeghi (2011) have shown that
\[ \text{Ext}^i_R(M, R) \cong H^i_m(\lambda M) \]
for all \( i, 1 \leq i < \dim R \) whenever \( R \) is Cohen-Macaulay with canonical module \( \omega_R \), \( \text{Tor}^R_i(M, \omega_R) = 0 \) for all \( i > 0 \) and \( \text{Ext}^i_R(M, R) \) is of finite length for all \( i, 1 \leq i < \dim R \).
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D-Gheibi-Hassanzadeh-Sadeghi (2011) have shown that $\operatorname{Ext}^i_R(M, R) \cong H^i_m(\lambda M)$ for all $i$, $1 \leq i < \dim R$ whenever $R$ is Cohen-Macaulay with canonical module $\omega_R$, $\operatorname{Tor}^R_i(M, \omega_R) = 0$ for all $i > 0$ and $\operatorname{Ext}^i_R(M, R)$ is of finite length for all $i$, $1 \leq i < \dim R$. In the following, we prove the same result without assuming that $R$ is Cohen-Macaulay and without condition on torsion modules of the canonical module.

**Theorem**

Let $R$ be a local ring with depth $R \geq 2$ and let $M$ be an $R$–module. Assume that $n$ is an integer such that $1 < n \leq \text{depth } R$ and that $\operatorname{Ext}^i_R(M, R)$ is of finite length for all $i$, $1 \leq i < n$. Then $\operatorname{Ext}^i_R(M, R) \cong H^i_m(\lambda M)$ for all $i$, $1 \leq i < n$. 
An $R$–module $M$ is said to be linked to an $R$–module $N$, by an ideal $\mathfrak{c}$ of $R$, if $\mathfrak{c} \subseteq \text{Ann}_R(M) \cap \text{Ann}_R(N)$ and $M$ and $N$ are horizontally linked as $R/\mathfrak{c}$–modules. In this situation we denote $M \sim_{\mathfrak{c}} N$.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring, $\mathfrak{c}_1$ and $\mathfrak{c}_2$ Gorenstein ideals. Let $M_1$, $M$ and $M_2$ be $R$–modules such that $M_1$ is linked to $M$ by $\mathfrak{c}_1$ and $M$ is linked to $M_2$ by $\mathfrak{c}_2$. Martsinkovsky and Strooker prove that $\text{Gdim}_R(M_1) = \text{Gdim}_R(M_2)$ and also

$$\text{Ext}^i_{R/\mathfrak{c}_1}(M_1, R/\mathfrak{c}_1) \cong \text{Ext}^i_{R/\mathfrak{c}_2}(M_2, R/\mathfrak{c}_2)$$

for all $i > 0$. 
Definition

An $R$–module $M$ is said to be linked to an $R$–module $N$, by an ideal $c$ of $R$, if $c \subseteq \text{Ann}_R(M) \cap \text{Ann}_R(N)$ and $M$ and $N$ are horizontally linked as $R/c$–modules. In this situation we denote $M \sim_c N$.

Let $(R, \mathfrak{m})$ be a Gorenstein local ring, $c_1$ and $c_2$ Gorenstein ideals. Let $M_1$, $M$ and $M_2$ be $R$–modules such that $M_1$ is linked to $M$ by $c_1$ and $M$ is linked to $M_2$ by $c_2$. Martsinkovsky and Strooker prove that $\text{Gdim}_R(M_1) = \text{Gdim}_R(M_2)$ and also

\[
\Ext^i_{R/c_1}(M_1, R/c_1) \cong \Ext^i_{R/c_2}(M_2, R/c_2) \text{ for all } i > 0.
\]
In this part we establish this isomorphism, without assuming $R$ is Gorenstein, but we assume some conditions on the modules $M_1, M, M_2$ and on ideals $c_1, c_2$.

Throughout this section $R$ is a local ring, $K$ and $M$ are $R$–modules. Denote $M^\dagger = \text{Hom}_R(M, K)$. The module $M$ is called $K$-reflexive if the canonical map $M \to M^{\dagger\dagger}$ is bijective.
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**Definition**

*The module $M$ is said to have $G_K$-dimension zero if*

1. $M$ is $K$-reflexive;

2. $\text{Ext}^i_R(M, K) = 0$, for all $i > 0$;

3. $\text{Ext}^i_R(M^\dagger, K) = 0$, for all $i > 0$. 

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The module $M$ is said to have $G_K$-dimension zero if

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(ii) $\text{Ext}^i_R(M, K) = 0$, for all $i > 0$;

(iii) $\text{Ext}^i_R(M^\dagger, K) = 0$, for all $i > 0$. 
A $G_K$-resolution of a finite $R$–module $M$ is a right acyclic complex of modules of $G_K$-dimensions zero whose 0th homology module is $M$. The module $M$ is said to have finite $G_K$-dimension, denoted by $G_K$-dim$_R(M)$, if it has a $G_K$-resolution of finite length.
An \( R \)-module \( K \) is called a semidualizing module (suitable), if

(i) the homothety morphism \( R \to \text{Hom}_R(K, K) \) is an isomorphism;

(ii) \( \text{Ext}^i_R(K, K) = 0 \) for all \( i > 0 \).
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(i) the homothety morphism $R \to \text{Hom}_R(K, K)$ is an isomorphism;

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Semidualizing modules are studied by Foxby, Golod, and many others. It is obvious that $R$ itself is a semidualizing $R$–module. Also it is well known that if $R$ is Cohen-Macaulay then its canonical module (if exists) is a semidualizing module.
Theorem (Golod)

Let $K$ be a semidualizing $R$–module. For an $R$–module $M$ of finite $G_K$-dimension the following statements hold true.

(i) $G_K\text{-dim}(R(M)) = \sup\{i \mid \text{Ext}_i^R(M, K) \neq 0, i \geq 0\}$.

(ii) If $G_K\text{-dim}(R(M)) < \infty$ then $G_K\text{-dim}(R(M)) = \text{depth } R - \text{depth } R(M)$. 

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Semidualizing Modules and Evenly Linked Modules

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We recall the following definitions (Golod).
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**Definition**

An $R$–module $M$ is called $G_K$-perfect if $\text{grade}_R(M) = G_K\text{-dim}_R(M)$. An ideal $I$ is called $G_K$-perfect if $R/I$ is $G_K$-perfect as $R$–module. An $R$–module $M$ is called $G_K$-Gorenstein if $M$ is $G_K$-perfect and $\text{Ext}_R^n(M, K)$ is cyclic, where $n = G_K\text{-dim}_R(M)$. An ideal $I$ is called $G_K$-Gorenstein if $R/I$ is $G_K$-Gorenstein as $R$–module.
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An $R$–module $M$ is called $G_K$-perfect if $\text{grade}_R(M) = G_K\text{-dim}_R(M)$.

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Note that if $K$ is a semidualizing $R$–module and $I$ is a $G_K$-Gorenstein ideal of $G_K$-dimension $n$, then $\text{Ext}^n_R(R/I, K) \cong R/I$. (Golod)
Proposition

Let $K$ be a semidualizing $R$–module, $c_1$ and $c_2$ two $G_K$-Gorenstein ideals. Assume that $M_1, M,$ and $M_2$ are $R$–modules such that $M_1 \sim_{c_1} M$ and $M \sim_{c_2} M_2$. Denote the common value of grade$(c_1)$ and grade$(c_2)$ by $n$. Then $\text{Ext}_R^i(M_1, K) \cong \text{Ext}_R^i(M_2, K)$ for all $i, i > n$. 
Corollary

Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring with canonical module \(\omega_R\). Assume that \(\mathfrak{c}_1\) and \(\mathfrak{c}_2\) are Gorenstein ideals and that \(M_1, M, \text{ and } M_2\) are \(R\)-modules such that \(M_1 \sim c_1 M\) and \(M \sim c_2 M_2\). Set \(n = \dim_R(M_1) = \dim_R(M_2)\). Then \(H^i_{\mathfrak{m}}(M_1) \cong H^i_{\mathfrak{m}}(M_2)\), for all \(i, i < n\).
Corollary

Let \((R, \mathfrak{m})\) be a Cohen-Macaulay local ring with canonical module \(\omega_R\). Assume that \(\mathfrak{c}_1\) and \(\mathfrak{c}_2\) are Gorenstein ideals and that \(M_1,M,\) and \(M_2\) are \(R\)-modules such that \(M_1 \sim_{\mathfrak{c}_1} M\) and \(M \sim_{\mathfrak{c}_2} M_2\). Set \(n = \dim_R(M_1) = \dim_R(M_2)\). Then \(H^i_{\mathfrak{m}}(M_1) \cong H^i_{\mathfrak{m}}(M_2)\), for all \(i, i < n\).

http://arxiv.org/abs/1109.6528v1
Corollary

Let $(R, \mathfrak{m})$ be a Cohen-Macaulay local ring with canonical module $\omega_R$. Assume that $\mathfrak{c}_1$ and $\mathfrak{c}_2$ are Gorenstein ideals and that $M_1, M,$ and $M_2$ are $R$–modules such that $M_1 \sim_{\mathfrak{c}_1} M$ and $M \sim_{\mathfrak{c}_2} M_2$. Set $n = \dim_R(M_1) = \dim_R(M_2)$. Then $H^i_{\mathfrak{m}}(M_1) \cong H^i_{\mathfrak{m}}(M_2)$, for all $i, i < n$.

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