Covariant approach to the conformal dynamical equivalence in astrophysics

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Abstract

We use covariant techniques to examine the implications of the dynamical equivalence between geodesic motions and adiabatic hydrodynamic flows. Assuming that the metrics of a geodesically and a non-geodesically moving fluid are conformally related, we calculate and compare their mass densities. The density difference is then expressed in terms of the fundamental physical quantities of the fluid, such as its energy and isotropic pressure. Both the relativistic and the non-relativistic case are examined and their differences identified. Our analysis suggests that observational determinations of astrophysical masses based on purely Keplerian motions could underestimate the available amount of matter.

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1 Introduction

Geodesic motions are central when it comes to mass measurements. In particular, the observational determination of the masses of various astrophysical objects is based on the assumption of Keplerian, that is of purely geodesic, motions. For example, the central mass concentration in various galaxies is determined by Doppler-shift measurements of radiative sources which are assumed to move along geodesic trajectories (e.g. see [1]). Nevertheless, there are known cases where non-gravitational forces are strong enough to affect the purely geodesic nature of these trajectories and where a hydrodynamic description of the motion is more appropriate [2]. In these cases one would like to know whether the standard measurements overestimate or underestimate the available amount of matter. This is the question the present article will attempt to address.

We approach the problem by treating the geodesic and the hydrodynamic descriptions of motion as dynamically equivalent [3]. Then, by introducing a physical fluid, with an adiabatic hydrodynamic flow, and a virtual one, that follows a geodesic trajectory, we can calculate and compare their corresponding mass densities. Any difference found between the virtual and the physical densities can be interpreted as a correction to mass determinations based on the geodesic motion approximation.\textsuperscript{1} In [3] this was achieved by assuming that the metrics of the two fluids are re-

\textsuperscript{1}With the exception of Sec. 5, by mass we will always imply the total energy of the matter. We will therefore use the terms mass and energy interchangeably. In Sec. 5, however, this is no longer the case and by mass we will specifically indicate the rest mass of a Newtonian medium.
lated through a simple conformal transformation (see also [4] for further discussion). Moreover, when this approach was applied to the central mass of certain Active Galactic Nuclei (AGN), it showed that mass determinations based on the geodesic motion approximation underestimated the available amount of matter [3].

Here we provide a covariant generalisation of the analysis introduced in [3] that allows for a unified treatment of both the Newtonian and the relativistic cases. Among the advantages of the covariant formalism are its compactness and the use of variables with a straightforward geometrical and physical interpretation [5]. This facilitates a mathematically clear and physically transparent treatment of the problem at hand. Our analysis does not impose any a priori symmetries on the spacetime metric and therefore applies to a range of physically realistic situations. The only restrictions are on the nature of the medium, which is assumed to be a perfect fluid, and the adiabaticity of the flow. Following [3], we translate the dynamical equivalence between the geodesic motion of the virtual fluid and the hydrodynamic flow of its physical counterpart into a simple conformal relation between their respective metrics. This allows us to calculate the corresponding mass densities and express their difference as a function of fundamental fluid variables, like its energy density and pressure. As in [3], we interpret this difference as a correction to mass determinations based on the assumption of purely geodesic motions. Our analysis points towards a mass deficit every time the geodesic motion approximation is used for mass measurements.

2 Energy difference between geodesic and hydrodynamic flows

Consider a spacetime \((\mathcal{M}, g_{\mu\nu})\) and the conformal geometry of \(\mathcal{M}\), namely the set of all metrics \(\tilde{g}_{\mu\nu}\) conformal to the physical metric \(g_{\mu\nu}\) so that \(\tilde{g}_{\mu\nu} = \Omega^2 g_{\mu\nu}\), where \(\Omega\) is a nonzero suitably differentiable function.\(^2\) Under such transformations geodesic curves do not generally remain geodesics unless they are null [6]. On these grounds, one could consider the adiabatic hydrodynamic flow of a fluid in a given spacetime, and assume that it is represented by the geodesic motion of a virtual fluid in another spacetime, which is conformally related to the first. Then, adiabaticity and the dynamical equivalence of the two motions specifies the conformal factor to be [3]

\[
\Omega = \frac{\mathcal{E} + p}{\rho c^2}, \tag{1}
\]

where \(\mathcal{E} = \rho c^2 + \rho \Pi\) is the energy density of the fluid, with \(\rho\) being its rest-mass density and \(\Pi\) the internal energy per unit mass [3, 7]. According to (1), \(\Omega \geq 1\) for all conventional types of matter with \(p \geq 0\). Note that the limit \(\Omega = 1\) corresponds to “cold matter” (i.e. “dust”) with \(p = 0 = \Pi\). Obviously, in that case \(\tilde{g}_{\mu\nu} = g_{\mu\nu}\) and the two spacetimes coincide.

The aforementioned dynamical equivalence translates into a relation between the variables of the two fluids, namely the physical and the virtual one. In particular, if \(\tilde{\mathcal{E}}\) is the energy density of the virtual fluid and \(\mathcal{E}\) is that of the physical fluid, then [3]

\[
\tilde{\mathcal{E}} = \Omega^{-2}(\mathcal{E} + \mathcal{E}_1). \tag{2}
\]

Therefore, \(\mathcal{E}_1\) provides a measure of the energy density difference that results from the dynamical equivalence of the two motion representations. This difference, which will be interpreted as a

\(^2\)Greek indices run from 0 to 3, while Latin indices take the values 1,2,3. Also, throughout this paper we employ a Lorenzian metric with signature \((+ - - -)\).
correction to any mass determination based on purely geodesic trajectories, is given by
\[ \Delta \mathcal{E} \equiv \mathcal{E} - \check{\mathcal{E}} = (1 - \Omega^{-2}) \mathcal{E} - \Omega^{-2} \mathcal{E}_1, \quad (3) \]
with the right-hand side expressed relative to the quantities of the physical fluid. In particular, following [3] we have
\[ \mathcal{E}_1 = \frac{2}{\kappa} \Box \Omega + 2 \frac{1}{\kappa \Omega} u^\mu u^\nu \nabla_\nu \nabla_\mu \Omega - \frac{4}{\kappa \Omega^2} u^\mu u^\nu \nabla_\mu \Omega \nabla_\nu \Omega + \frac{1}{\kappa \Omega^2} g^{\mu \nu} \nabla_\mu \Omega \nabla_\nu \Omega, \quad (4) \]
where \( u_\mu \) is the dimensionless timelike 4-velocity vector of the matter normalised so that \( u_\mu u^\mu = 1 \), \( \nabla_\mu \) is the standard covariant derivative operator and \( \kappa = \frac{8 \pi G}{c^4} \). As expected, both \( \mathcal{E}_1 \) and \( \Delta \mathcal{E} \) vanish at the \( \Omega = 1 \) limit. Also, from Eq. (3) it becomes clear that, since \( \Omega > 1 \), \( \Delta \mathcal{E} \) is positive if \( \mathcal{E}_1 \) is negative. Moreover, even when \( \mathcal{E}_1 > 0 \) the energy difference will remain positive provided \( \mathcal{E}_1 < (\Omega^2 - 1) \mathcal{E} \), as Eq. (3) guarantees. In any such case the energy density of the virtual fluid will be less than that of the actual one. Whenever this happens mass determination based on purely geodesic motions will underestimate the matter content of the medium. In what follows we will investigate this possibility in more detail while trying to maintain the generality of our discussion.

3 Covariant description of the energy difference

Using \( u_\mu \), the timelike 4-velocity vector of the matter, we define \( h_{\mu \nu} = g_{\mu \nu} - u_\mu u_\nu \) as our projection tensor orthogonal to \( u_\mu \). Then, the four terms in the right-hand side of Eq. (4) are respectively recast as

\[ \Box \Omega \equiv g^{\mu \nu} \nabla_\mu \nabla_\nu \Omega = D^2 \Omega + \check{\Omega} - \dot{u}^\mu D_\mu \Omega, \quad (5) \]

\[ u^\mu u^\nu \nabla_\nu \nabla_\mu \Omega = \dot{\Omega} - \dot{u}^\mu D_\mu \Omega, \quad (6) \]

\[ u^\mu u^\nu \nabla_\mu \Omega \nabla_\nu \Omega = \ddot{\Omega}^2, \quad (7) \]

\[ g^{\mu \nu} \nabla_\mu \Omega \nabla_\nu \Omega = D_\mu \Omega \nabla^\mu \Omega + \dot{\Omega}^2, \quad (8) \]

where \( \dot{\Omega} = u^\nu \nabla_\nu \Omega \), \( D_\mu \Omega = h_\mu^\nu \nabla_\nu \Omega \) and \( D^2 \Omega = h^{\mu \nu} \nabla_\mu \nabla_\nu \Omega \) by definition, while \( \dot{u}_\mu = u^\nu \nabla_\nu u_\mu \) is the 4-acceleration. Note that an overdot indicates time derivatives, that is covariant derivatives projected along the timelike 4-velocity vector \( u_\mu \). On the other hand, \( D_\mu = h_\mu^\nu \nabla_\nu \) is the covariant derivative operator orthogonal to \( u_\mu \), which is used to denote the local spatial gradients. Finally, \( D^2 = h^{\mu \nu} \nabla_\mu \nabla_\nu \) is the projected Laplacian operator.

Substituting the results (5)-(8) into Eq. (4) we arrive at the following alternative, covariant expression for the key quantity \( \mathcal{E}_1 \)

\[ \mathcal{E}_1 = -\frac{1}{\kappa} \left( \frac{2}{\Omega} D^2 \Omega + \frac{3}{\Omega^2} \dot{\Omega}^2 - \frac{1}{\Omega^2} D_\mu \Omega D^\mu \Omega \right). \quad (9) \]

Of the three terms on the right-hand side, the second is negative definite while the third is positive definite. The former of these two terms suggests that mass determinations based on the geodesic motion approximation will underestimate the total energy density of the matter, whereas the latter indicates the opposite. The sign of the first term, on the other hand, is not a priori fixed but depends on the sign of the spatial Laplacian \( D^2 \Omega \). Clearly, the overall effect of these three terms depends on their relative strength. Note that the third term in Eq. (9) is quadratic in \( D_\mu \Omega \), which suggests that it should become strong only in highly inhomogeneous situations.
4 Energy difference and the fluid parameters

For a better insight into the physical implications of the equivalence between geodesic and hydrodynamic motions, it helps to recast expression (9) with respect to the fundamental physical quantities of the fluid, namely $\rho$, $p$ and $\mathcal{E}$. As before, we will do so by restricting our analysis to isentropic (i.e. adiabatic) flows, which means that

$$\nabla_\mu \Pi = \frac{p}{\rho^2} \nabla_\mu \rho. \tag{10}$$

without any extra constraints on the metric [3, 4]. Then, taking the covariant derivative of Eq. (1) and using result (10) we obtain

$$\nabla_\mu \Omega = \frac{1}{\rho c^2} \nabla_\mu p. \tag{11}$$

Contracted along the timelike direction $u_\mu$ and projected orthogonal to $u_\mu$ the above gives

$$\dot{\Omega} = \frac{\dot{p}}{\rho c^2} \quad \text{and} \quad D_\mu \Omega = \frac{1}{\rho c^2} D_\mu p, \tag{12}$$

respectively. Moreover, the latter of these two results leads to

$$D^2 \Omega = \frac{1}{\rho c^2} D^2 p - \frac{1}{\rho^2 c^2} D_\mu D^\mu p. \tag{13}$$

On using (12) and (13) we may recast expression (9) as

$$\mathcal{E}_1 = -\frac{1}{\kappa \rho c^2} \left( 2D^2 p + \frac{3}{\Omega \rho c^2} \frac{\rho}{\rho c^2} - \frac{2}{\rho} D_\mu p D^\mu p - \frac{1}{\Omega \rho c^2} D_\mu p D^\mu p \right), \tag{14}$$

where $\Omega \rho c^2 = \mathcal{E} + p$ (see Eq. (1)). The above expresses the difference, between the observationally determined (i.e. the virtual) and the actual energy density, as a function of key physical quantities like the fluid rest-mass density and isotropic pressure. Moreover, it shows that both $\mathcal{E}_1$ and $\Delta \mathcal{E}$ vanish when the pressure is zero. As expected, this means that the use of geodesic motions will lead to a noticeable underestimation (or overestimation) of the available mass mainly in domains where the fluid pressure and its gradients are appreciable. Such domains could be the highly dense central regions of certain active galaxies.

Expression (14) also provides a physical interpretation to some of the mathematical comments made immediately after Eq. (9). For example, the sign of the Laplacian term in (9) depends on whether $D^2 p$ is positive or negative. This, in turn, depends on whether we are looking at an overdense or an underdense region respectively. In other words, the Laplacian term, alone, will underestimate the fluid energy density whenever the geodesic motion approximation is applied to an overdense region. Note that for our purposes we will always consider overdense regions, which means that the Laplacian term in the right-hand side of (14) will always be treated as positive. Also, according to (14), time variations in the fluid pressure (and therefore in its energy density) will also underestimate the available energy. Spatial gradients in the fluid pressure, on the other hand, could lead to an energy overestimation. This latter effect, which depends on the specific nature of the fluid, will be addressed in more detail in the following sections.
5 The case of a Newtonian medium

Equation (14) applies to a general relativistic fluid in strong gravity environments. Most astrophysical situations, however, are adequately described by the Newtonian theory. In this respect, it is worth recovering the classical limit of Eq. (14). We will do that next for the case of a polytropic gas with $p = k \rho^\gamma$, where the parameters $k$ and $\gamma$ will be treated as constants. For such a medium we have

$$E_1 = - \frac{v_s^2}{\kappa (E + p)} \left[ 2D^2 \rho + \frac{1}{\rho} \left( 2(\gamma - 2) - \frac{v_s^2 \rho}{E + p} \right) D_\mu \rho D^\mu \rho + \frac{3v_s^2}{E + p} \rho^2 \right],$$

(15)

where $v_s^2 = dp/d\rho = \gamma p/\rho$ is the square of the (adiabatic) polytropic sound speed. For a non-relativistic gas, we have $\Pi/c^2 \ll 1$ and therefore $E \approx \rho c^2$. Also, $p \ll \rho c^2$ implying that $E + p \approx \rho c^2$ and that $v_s^2/c^2 \ll 1$. At this limit the above equation reduces to

$$\rho_1 \approx - \frac{2v_s^2}{\kappa \rho c^4} \left[ \partial^2 \rho + \frac{\gamma - 2}{\rho} \partial_\alpha \rho \partial^\alpha \rho \right],$$

(16)

having dropped all terms quadratic in $p/\rho c^2$ and $v_s^2/c^2$. Note that for a Newtonian fluid expression (3) reduces to $\Delta \rho \approx - \rho_1$ to lowest order in $p/\rho c^2$ and $v_s^2/c^2$. In addition, at the weak gravity limit we can replace the projected covariant derivative operators with ordinary partial derivatives (i.e. $D_\mu \to \partial_\mu$). Then,

$$\rho_1 \approx - \frac{k \gamma \rho^{\gamma - 2}}{4\pi G} \left[ \partial^2 \rho + \frac{\gamma - 2}{\rho} \partial_\alpha \rho \partial^\alpha \rho \right],$$

(17)

where $v_s^2 = k \gamma \rho^{\gamma - 1}$ is the sound speed of our polytropic gas and $\partial^2 \equiv \partial_\alpha \partial^\alpha$ is the standard 3-dimensional Laplacian. The above equation, which has been derived by reducing the fully relativistic covariant formula (14) to the Newtonian limit, agrees completely with the result obtained in [3] via a metric-based approach. Following result (17) we conclude that $\rho_1 < 0$, and therefore $\Delta \rho > 0$, whenever $\gamma \geq 2$. Note that, due to the Laplacian term in the right-hand side of Eq. (17), which is assumed positive, the aforementioned condition on the polytropic index for $\Delta \rho$ to be positive is sufficient but not necessary. In practise, this means that $\Delta \rho > 0$ for almost all matter distributions where the pressure increases with the density. Moreover, the $\gamma$-dependence effectively disappears when the inhomogeneity is weak and the second term in the brackets becomes negligible. Based on these results one may argue that determining the mass content in a ‘Newtonian region’ on the assumption of geodesic motions will generally underestimate the available amount of matter.

6 The case of a relativistic medium

Let us now turn our attention to a relativistic fluid and use expression (14) to identify the differences from the above treated Newtonian case. To begin with, for a perfect fluid the standard energy density conservation means that

$$\dot{\mathcal{E}} = -(1 + w) \Theta \mathcal{E},$$

(18)

where $w = p/\mathcal{E}$ and $\Theta = \nabla_\mu u^\mu = D_\mu u^\mu$. Note that the scalar $\Theta$ describes the volume expansion (or contraction) between the worldlines of two neighbouring fluid particles (e.g. see [5]). Moreover, the barotropic equation of state of the fluid, namely the fact that the pressure is a function of the energy density alone (i.e. $p = p(\mathcal{E})$), implies that

$$D_\mu p = \epsilon_s^2 D_\mu \mathcal{E} \quad \text{and} \quad \dot{\mathcal{E}} = \epsilon_s^2 \dot{\mathcal{E}},$$

(19)
where \( c_s^2 = \frac{dp}{dE} \) is the dimensionless adiabatic sound speed. Then, assuming that the equation of state of the fluid does not change (i.e. setting \( w \), \( c_s^2 = \) constant) in the region under consideration, we obtain

\[
D^2 p = c_s^2 D^2 E.
\]  

(20)

Finally, in addition to constraint (10), adiabaticity also guarantees that

\[
\frac{1}{E(1+w)} \nabla_\mu E = \frac{1}{\rho} \nabla_\mu \rho.
\]  

(21)

Note that, when the parameters \( w \), \( c_s^2 \) are not constant, one needs to specify the equation of state of the fluid.

On using results (18)-(21), expression (14) reads

\[
\mathcal{E}_1 = -\frac{c_s^2}{\kappa c_s^2(1+w)^2} \left[ 2\mathcal{E}(1+w)D^2 \mathcal{E} + 3c_s^2(1+w)^2 \Theta^2 \mathcal{E}^2 - (2 + c_s^2)D_\mu \mathcal{E} D^\mu \mathcal{E} \right].
\]  

(22)

Accordingly \( \mathcal{E}_1 = 0 \) when \( c_s^2 = 0 \). The latter implies that the fluid pressure is a covariantly constant quantity (i.e. isobaric flow with \( \nabla_\mu p = 0 \) - see Eqs. (19)). Then, for a barotropic fluid, isobaric flow means that \( \mathcal{E} \) is also covariantly constant. We will return to the implications of Eq. (22) for a relativistic medium in the next section. For the moment, we simply point out that the sign of the right-hand side of (22) depends on the relative contribution of the three terms in the brackets.

Note that when \( D_\mu \mathcal{E} = 0 \) but \( \dot{\mathcal{E}} \neq 0 \), namely for a fluid with a spatially homogeneous energy density distribution, expression (22) reduces to \( \mathcal{E}_1 = -3c_s^4 \Theta^2 / \kappa < 0 \). In this case we have a positive energy difference (i.e. \( \Delta \mathcal{E} > 0 \) - see Eq. (2)), which implies that mass calculations based on geodesic motions will lead to an energy deficit. Interestingly, this last result is independent of whether there is expansion or contraction, but vanishes when we consider a stationary region.

7 The energy deficit

The Newtonian case was addressed in [3] by considering a spherically symmetric region and assuming an outwardly decreasing mass density distribution of the Plummer-type

\[
\rho = \rho_0 \left[ 1 + \left( \frac{r}{r_0} \right)^2 \right]^{-n/2},
\]  

(23)

where \( r \) is the radius of the region in question, \( \rho_0 \) and \( r_0 \) are constants and \( n \) is a positive integer with typical values either 3 or 5 (e.g. see [8]). Note that the Plummer parameters are related by \( \rho_0 = 2^{n/2} \rho(r_0) \), an expression that will prove useful later in this section. For \( \gamma \leq 1 + 1/n \) it can be shown that \( \rho_1 \), as given by (17), is always negative which implies that the geodesic motion approximation will underestimate the available matter density in the region [3].

In the Newtonian limit one could reasonably well ignore terms quadratic in \( p/\rho c^2 \) and \( v_s^2/c^2 \), given that the pressure is a negligible fraction of the matter density. This means that we can disregard the contribution of the last two terms in Eq. (15). When dealing with a relativistic medium, however, this is no longer an option, since the pressure forms a considerable fraction of the matter-energy density. In this case one should include the last two terms in Eq. (22). Their relative contribution depends on whether the region is stationary or not, and also on the degree of the inhomogeneity. In principle, one could apply expression (22) to any situation and to any
given density distribution, and then calculate the energy deficit or surplus due to the geodesic motion approximation. However, the complexity of the relativistic equations means that analytic quantitative results are very difficult to extract, even for the relatively simple Plummer distribution. Nevertheless, one can still obtain some general qualitative results and this is what we will try to do next.

As mentioned before, the last term in the right-hand side of Eq. (22) will become important only in highly inhomogeneous situations. Indeed, assuming Euclidean geometry for simplicity, a dimensional analysis shows that

$$\frac{D^2 \mathcal{E}}{\delta \lambda^2} \quad \text{and} \quad D_\mu \mathcal{E} D^\mu \mathcal{E} \sim \left( \frac{\delta \mathcal{E}}{\delta \lambda} \right)^2,$$

(24)

where $\mathcal{E}$ is the average energy density of the medium and $\delta \lambda$ is the characteristic length scale associated with the spatial variation of $\mathcal{E}$. So, as long as the density contrast $\delta = \delta \mathcal{E} / \mathcal{E}$ is small compared to unity, we have

$$\frac{1}{\mathcal{E}} D^2 \mathcal{E} \sim \frac{\delta \mathcal{E} / \mathcal{E}}{\delta \lambda^2} \gg \left( \frac{\delta \mathcal{E} / \mathcal{E}}{\delta \lambda} \right)^2 \sim \frac{1}{\mathcal{E}^2} D_\mu \mathcal{E} D^\mu \mathcal{E},$$

(25)

and the Laplacian term in the right-hand side of (22) dominates over the quadratic term $D_\mu \mathcal{E} D^\mu \mathcal{E}$. Then,

$$\mathcal{E}_1 \simeq -\frac{c_s^2}{\kappa \mathcal{E}(1 + w)} \left[ 2D^2 \mathcal{E} + 3c_s^2(1 + w)\Theta^2 \mathcal{E} \right] < 0.$$

(26)

where the last term contributes only in non-stationary regions. Therefore, in a weakly inhomogeneous environment, the use of geodesic motions for the observational determination of the matter content will underestimate the available energy density in the region.

In highly inhomogeneous situations, however, $\delta > 1$ and the same dimensional argument shows that this time it is the quadratic term that dominates over the Laplacian in the right-hand side of (22). In this case,

$$\mathcal{E}_1 \simeq -\frac{c_s^2(2 + c_s^2)}{\kappa \mathcal{E}^2(1 + w)^2} \left[ 3c_s^2(1 + w)^2\Theta^2 \mathcal{E}^2 - (2 + c_s^2)D_\alpha \mathcal{E} D^\alpha \mathcal{E} \right].$$

(27)

Since both $w$ and $c_s^2$ are of order unity, the sign of the above depends primarily on two parameters. These are the contraction scalar $\Theta$, which provides a measure of the average collapse timescale, and the ratio $D_\alpha \mathcal{E} / \mathcal{E} \sim \delta / \delta \lambda$, which determines the degree ($\delta$) and the scale ($\delta \lambda$) of the inhomogeneity. Therefore, provided the collapse timescale is short enough, that is as long as $1 < \delta < \Theta \delta \lambda$, we will have $\mathcal{E}_1 < 0$ and a deficit in the estimated energy density (i.e. $\Delta \mathcal{E} > 0$). It should be emphasised that $\delta > \Theta \delta \lambda$ does not automatically imply a negative energy difference and an overestimated energy density. Indeed, suppose that the region in question is stationary. In that case the right-hand side of (27) is dominated by the inhomogeneous term and

$$\mathcal{E}_1 \simeq \frac{c_s^2(2 + c_s^2)}{\kappa \mathcal{E}^2(1 + w)^2} D_\alpha \mathcal{E} D^\alpha \mathcal{E} > 0.$$  

(28)

Following (3), the above leads to $\Delta \mathcal{E} < 0$ only if

$$\frac{1}{\mathcal{E}^2} D_\mu \mathcal{E} D^\mu \mathcal{E} > \frac{\kappa(\Omega^2 - 1)(1 + w)^2}{c_s^2(2 + c_s^2)} \mathcal{E},$$

(29)
which gives a measure of the inhomogeneity required for a negative energy density difference. Given that \( c_s^2(2 + c_s^2)/(\Omega^2 - 1)(1 + w)^2 \) is of order unity, the above translates into the following order of magnitude condition on \( \delta, \delta \lambda \)

\[
\delta > \sqrt{\kappa \delta \lambda}.
\]  

(30)

for \( \Delta E < 0 \). In other words, mass determinations based on the assumption of geodesic motions in a region filled with a highly inhomogeneous relativistic medium, will underestimate the available amount of energy unless condition (30) is satisfied. It is therefore interesting to examine whether the above condition is typically satisfied or not. Assuming a Plummer-type density distribution for the relativistic fluid, then in absolute values the associated density contrast is \( \delta = n x^2 (\delta r/r)/(1 + x^2) \), with \( x = r/r_0 \) by definition. In this environment, condition (30) becomes

\[
x(1 + x^2)^{n-1} > \frac{r_0}{nc} \sqrt{8\pi G \rho_0},
\]  

(31)

since \( \mathcal{E} = \rho c^2 \) and \( \delta \lambda = \delta r \) given the symmetries of the Plummer distribution. Then, for \( x \gg 1 \) the above translates into the following condition involving the Plummer parameters

\[
\frac{r_0}{nc} \sqrt{8\pi G \rho_0} \gg 1,
\]  

(32)

where \( n > 2 \) (with typical values \( n = 3, 5 \)) [9]. Consider now a supergiant elliptical galaxy, with mass \( M \) of the order of \( 10^{13} - 10^{14} M_\odot \), and assume that \( r_0 = 3 R_S \), where \( R_S = 2GM/c^2 \) is the Schwarzschild radius of the central black hole. In that case \( \rho(r_0) \sim 10^{-17} \text{gr/cm}^{-3} \) [10], which provides a value for \( \rho_0 \) (recall that \( \rho_0 = 2^{n/2} \rho(r_0) \) for a Plummer-type density profile). Substituting the above into condition (32) we obtain

\[
M \gg n 2^{-n/2} \times 8.261 \times 10^{15} M_\odot,
\]  

(33)

imposing a lower bound of approximately \( 10^{16} M_\odot \) on the mass of the central compact object. Such a value is clearly way above any accepted mass estimate for the core region of a galaxy, which means that our original assumption that condition (30) holds is false. In other words, as in the Newtonian case, one can argue that mass determinations based on purely geodesic motions will generally underestimate the energy content in 'relativistic regions' as well.

8 Discussion

Estimating the masses of the various astrophysical bodies is central to almost all astronomical problems. Such mass determinations use observational techniques which are based on the assumption of geodesic motions, that is motions under the effect of gravity alone. In the presence of non-gravitational forces, however, due to, say, pressure gradients, viscosity or an inhomogeneous magnetic field, a purely geodesic motion is no longer sustainable. Then, a hydrodynamical description of the motion is more appropriate, as it appears to be the case near the core regions of typical AGNs. Therefore, the emerging question is how accurate the standard mass measurements are and in particular whether they underestimate or overestimate the available amount of matter. Answering the question to the full is not an easy task and is further complicated by a number of unknown parameters. The latter are related to the nature of the astrophysical medium in the region under consideration, its distribution, etc. Nevertheless, one could still try to address the problem in a series of qualitative steps. The first is to investigate whether, and under what conditions, the
standard methods underestimate or overestimate the available amount of matter. Here, we have attempted to address this issue by generalising the work of [3]. At the core of our approach is the conformal dynamical equivalence between the motions of a virtual fluid, which follows a geodesic trajectory, and of its physical counterpart, which has an adiabatic hydrodynamical flow. The assumption of adiabaticity is typical in analytical studies, which by nature cannot address highly complicated physical systems. In our case adiabaticity means that we consider systems which are isolated enough to maintain their thermodynamical content unchanged or slowly varying in time.

The dynamical equivalence between geodesic and hydrodynamic motions translates into a simple conformal relation between the metrics of the respective fluids. This in turn provides a mathematical framework where one can calculate and compare the matter content of the two fluids. Any difference found could then be interpreted as a correction to mass determinations based on the geodesic motion approximation. Based on that one can then proceed to examine whether the assumption of geodesic motions underestimates, or overestimates, the amount of matter available in certain astrophysical formations. Here, we have considered the conformal dynamical equivalence between geodesic motions and hydrodynamical flows due to non-zero pressure gradients. However, the same principle could be used to analyse more general situations. One should be able to consider, for example, the conformal dynamical equivalence between geodesic motions and magneto-hydrodynamical flows.

Our approach offers a unified and fully covariant treatment of both the Newtonian and the relativistic cases. The first task was to provide the general mathematical framework where one can select and study specific individual cases. For this reason we did not impose any a priori constraints on the spacetime geometry, which made our approach applicable to a range of physically realistic situations. The second objective was to investigate whether or not mass measurements using purely geodesic motions underestimate the available amount of matter. According to our analysis, mass determinations based on geodesic motions in a Newtonian environment will underestimate the available matter of a polytropic fluid, provided the associated polytropic index satisfies the constraint \( \gamma \geq 2 \). Crucially, this \( \gamma \)-dependence can only become important in highly inhomogeneous situations. The relativistic analysis also led to similar conclusions. We found, in particular, that measurements based on purely Keplerian motions will underestimate the available amount of energy unless the region in question is both highly inhomogeneous and stationary. Moreover, the latter possibility appears unattainable in practise, at least for Plummer-type matter distributions. Note that the Plummer-type profile, and others like e.g. the Navarro-Frenk-White and the Hernquist profiles, are particular cases of a more general "universal" density distribution, which are supported by observations, and, so, they have been widely applied in the literature [11]-[14]. Thus, based on the conformal dynamical equivalence scenario, we feel confident enough to argue that in the majority of astrophysical situations the use of geodesic motions for the observational determination of masses will underestimate the available amount of matter. Given that, one may also want to consider the potential implications of the dynamical equivalence principle for the large-scale properties of the universe. In the present article, however, we have focused on astrophysical situations and therefore we refer the reader to [15, 16] for a discussion on the possible global applications of this approach. In astrophysics the questions we would like to address further are whether the mass underestimation identified here for Plummer-type density profiles is typical, and whether there are cases where a gross underestimation of the available matter can occur. If it so proves, it could force a radical revision of the current views on issues as important as the amount of baryonic matter in our universe and its present distribution.
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