On the Convergence of Nested Decentralized Gradient Methods with Multiple Consensus and Gradient Steps

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Abstract—In this paper, we consider minimizing a sum of local convex objective functions in a distributed setting, where the cost of communication and/or computation can be expensive. We extend and generalize the analysis for a class of nested gradient-based distributed algorithms (NEAR-DGD, [1]) to account for multiple gradient steps at every iteration. We show the effect of performing multiple gradient steps on the rate of convergence and on the size of the neighborhood of convergence, and prove R-Linear convergence to the exact solution with a fixed number of gradient steps and increasing number of consensus steps. We test the performance of the generalized method on quadratic functions and show the effect of multiple consensus and gradient steps in terms of iterations, number of gradient evaluations, number of communications and cost.

Index Terms—Distributed Optimization, Communication, Optimization Algorithms, Network Optimization.

I. INTRODUCTION

The focus of this paper is on designing and analyzing distributed optimization algorithms that employ multiple agents in a connected network with the collective goal of minimizing

\[ \min_{x \in \mathbb{R}^p} \ h(x) = \sum_{i=1}^{n} f_i(x), \tag{I.1} \]

where convex function \( h : \mathbb{R}^p \rightarrow \mathbb{R} \) is the global objective function, convex function \( f_i : \mathbb{R}^p \rightarrow \mathbb{R} \) for each \( i \in \{1, 2, \ldots, n\} \) is the local objective function available only to agent \( i \), and vector \( x \in \mathbb{R}^p \) is the decision variable that the agents are optimizing cooperatively. Such problems arise in a plethora of applications such as wireless sensor networks [2–5], smart grids [6, 7], multi-vehicle and multi-robot networks [8–10] and machine learning [11–14], to mention a few.

In order to optimize (I.1) it is natural to employ a distributed optimization algorithm, where the agents iteratively perform local computations based on a local objective function and local communications, i.e., information exchange with their one-step neighbors in the underlying network. To decouple the computation of individual agents, problem (I.1) is often reformulated as the following consensus optimization problem [15, 16],

\[ \min_{x_i \in \mathbb{R}^p} \ \sum_{i=1}^{n} f_i(x_i) \tag{I.2} \]

s.t. \( x_i = x_j, \ \forall i, j \in \mathcal{N}_i \),

where \( x_i \in \mathbb{R}^p \) for each agent \( i \in \{1, 2, \ldots, n\} \) is a local copy of the decision variable, and \( \mathcal{N}_i \) denotes the set of (one-step) neighbors of the \( i \)th agent. The consensus constraint imposed in problem (I.2) enforces that local copies of neighboring nodes are equal; assuming that the underlying network is connected, the constraint ensures that all local copies are equal and as a result problems (I.1) and (I.2) are equivalent.

For compactness, we express problem (I.2) as

\[ \min_{x \in \mathbb{R}^p} \ f(x) = \sum_{i=1}^{n} f_i(x_i) \tag{I.3} \]

s.t. \( (W \otimes I_p)x = x \),

where \( x \in \mathbb{R}^{np} \) is a concatenation of all local \( x_i \)’s, \( W \in \mathbb{R}^{n \times n} \) is a matrix that captures information about the underlying graph, \( I_p \) is the identity matrix of dimension \( p \), and the operator \( \otimes \) denotes the Kronecker product operation, with \( W \otimes I_p \in \mathbb{R}^{np \times np} \). Matrix \( W \), known as the consensus matrix, is a symmetric, doubly-stochastic matrix with \( w_{ii} > 0 \) and \( w_{ij} > 0 \) \((i \neq j)\) if and only if \( i \) and \( j \) are neighbors in the underlying communication network. This matrix has the property that \((W \otimes I_p)x = x\) if and only if \( x_i = x_j \) for all \( i \) and \( j \) in the connected network, i.e., problems (I.2) and (I.3) are equivalent. Moreover, the matrix \( W \) has exactly one eigenvalue equal to 1 and the rest of eigenvalues have absolute values strictly less than 1. We use \( \beta \), with \( 0 < \beta < 1 \), to denote the second largest, in magnitude, eigenvalue of \( W \).

In this paper, we investigate a class of first-order primal methods that perform nested communication and computation steps and that are adaptive. Our work is closely related to a few lines of research that we delineate below:

1) distributed first-order primal algorithms [15–27]: methods that use only gradient information and operate in primal space (i.e., directly on problem (I.3));

2) separated communication/computation [1, 26–31]: methods that decompose the communication and computation steps and perform them sequentially;

3) communication efficient [14, 32–41]: methods that incorporate communication considerations in the design;
4) **exact** [1, 42–46]: methods that converge to the optimal solution using a fixed step length on strongly convex functions;

5) **time-varying** [1, 28, 30]: methods that do not perform a fixed number of communication and/or gradient steps per iteration.

For a more extensive literature review of the above methods see [1, 15, 18, 42, 47] and the references therein.

There has been a recent surge of interest by the machine learning community in Federated Learning (FL) [48–54], which can be viewed as a distributed optimization framework over a star graph. FL operates in a \( n \)-client-server setup where clients do not communicate with each other directly, rather they communicate with the server who aggregates information and send averages to the clients (i.e., every round of communication all clients (nodes) have the same information). Thus, the effective communication pattern (in the notation of this paper) is a complete graph with weights \( 1/n \) (where \( n \) is the number of clients). FL is a special case of the distributed optimization problems considered in this paper.

It is common practice in FL to design communication efficient methods where the clients take multiple gradient steps towards minimizing local cost function before communicating to the server [37, 49, 50, 53–56]. This is due to the fact that there are numerous problems that arise in machine learning where local computations are cheap relative to the cost of communication. Most of these algorithms can be viewed as a special case of the class of the nested algorithms considered in this paper. Moreover, much of the current analysis considers algorithms that employ diminishing sequences of step sizes, whereas we consider a fixed step size algorithm allowing us to prove linear convergence rates to either an error neighborhood with a constant number of gradient steps or the exact solution with decreasing gradient steps (bounded below by one). We should note that we are unable to prove exact convergence if more than one (but finitely many) gradient steps are employed at every iteration which is consistent with the recent results in [49, 50, 57]. Finally, another advantage of our framework is the flexibility of adjusting the number of computation and communication steps depending on the applications. In many applications, e.g., [58], even the computation of inexact gradient direction can be expensive and thus may favor a method with more communication steps.

The main innovation of this paper is to extend and generalize the existing analysis for a class of nested gradient-based distributed algorithms to account for multiple gradient steps at every iteration (per round of communication). More specifically, we focus on variants of the NEAR-DGD method proposed in [1] and analyze a general algorithm that (potentially) takes both multiple consensus and gradient steps at every iteration. The main challenge here is that with multiple gradient steps each agent makes good progress towards minimizers with respect to their local objective functions, which may be far away from the global optimal solution. We note that even if we initialize the algorithm at the global optimal solution, the iterates will first move away before they converge back. We show the effect (theoretically and empirically) of performing multiple gradient steps on the rate of convergence and the size of the neighborhood. Moreover, we prove \( R \)-Linear convergence to the exact solution for the NEAR-DGD method that employs a decreasing number of gradient steps and an increasing number of consensus steps using a constant step length on strongly convex functions.

The paper is organized as follows. In Section II we introduce the NEAR-DGD method with multiple consensus and gradient steps per iteration. We then provide a convergence analysis for the method in Section III. In Section IV we illustrate the empirical performance of the method, and in Section V we provide some concluding remarks.

**II. THE NEAR-DGD METHOD WITH MULTIPLE CONSENSUS AND GRADIENT STEPS**

We consider an algorithm that performs multiple consensus and gradient steps at each iteration. More specifically, we analyze the generalized form of the NEAR-DGD method proposed in [1]. The most general form of the algorithm – which we call NEAR-DGD\( ^{t_{c},t_{g}} \) – can be expressed in terms of two operators:

- **Consensus Operator:** \( \mathcal{W}[x] = \mathcal{Z}x \).
- **Gradient Operator:** \( \mathcal{T}[x] = x - \alpha \nabla f(x) \), where \( \mathcal{Z} = \mathbb{W} \otimes I_p \in \mathbb{R}^{np \times np} \) and \( \nabla f(x_k) \in \mathbb{R}^{np} \) is a concatenation of the local gradients. The \( k \)-th iterate of the NEAR-DGD\( ^{t_{c},t_{g}} \) can be expressed as

\[
x_k = \mathcal{W}^{t_{c}(k)}[\mathcal{T}^{t_{g}(k)}[x_{k-1}]]
\]

where \( \mathcal{W}^{t_{c}(k)}[x] \) denotes \( t_{c}(k) \) nested consensus operations (steps)

\[
\mathcal{W}^{t_{c}(k)}[x] = \underbrace{\mathcal{W} \cdots \mathcal{W}[\mathcal{W}[x]]}_{t_{c}(k) \text{ operations}}.
\]

and \( \mathcal{T}^{t_{g}(k)}[x] \) denotes \( t_{g}(k) \) nested gradient operations (steps). One can describe the iterations of the NEAR-DGD\( ^{t_{c},t_{g}} \) method in terms of an intermediate variable \( y_k \in \mathbb{R}^{np} \) as

\[
y_k = \mathcal{T}^{t_{g}(k)}[x_k] = x_k - \alpha \sum_{j=1}^{t_{g}(k)} \nabla f(x_j^{k-1}), \quad (II.1)
\]

\[
x_{k+1} = \mathcal{W}^{t_{c}(k)}[y_k] = (\mathbb{W} \otimes I_p)^{t_{c}(k)}y_k = \mathcal{Z}^{t_{c}(k)}y_k, \quad (II.2)
\]

where \( x_j^{k} = x_{j-1}^{k} - \alpha \nabla f(x_{j-1}^{k}) \) for \( j = 1, \ldots, t_{g} \) with \( x_0^{k} = x_k \in \mathbb{R}^{np} \), and \( \nabla f(x_j^{k}) \in \mathbb{R}^{np} \) is a concatenation of the local gradients \( \nabla f(x_j^{k}) \) for \( i = 1, \ldots, n \). The three indices \((i, k, j)\) of \( x_{j,k}^{i} \) indicate the agent index \( i \), the iteration count \( k \) and the gradient step index \( j \). In the case where the superscript \( j \) is dropped (e.g., \( x_k \)) this denotes the iterate after \( t_{c}(k) \) consensus steps have been performed. Moreover, note that \( y_k = x_k^{t_{g}} \).

By setting the parameters \( t_{c} \) and \( t_{g} \) appropriately, one can recover several methods from the literature; Table I summarizes these methods. We should note that some of the methods summarized in the table (e.g., [28, 30]) do not exactly fit in the NEAR-DGD\( ^{t_{c},t_{g}} \) algorithmic framework, nevertheless, these methods decouple the consensus and gradient steps and perform multiple consensus and/or gradient steps.
III. CONVERGENCE ANALYSIS

In this section, we analyze the NEAR-DGD\(^{t_e,t_g}\) method with both multiple communication and computation steps. We begin by assuming that the method takes a fixed number of consensus (\(t_c\)) and gradient (\(t_g\)) steps per iteration. We then generalize the results to the case where the number of steps vary at every iteration. We make the following assumptions that are standard in the distributed optimization literature [1, 16].

Assumption III.1. Each local objective function \(f_i\) has \(L_i\)-Lipschitz continuous gradients. We define \(L = \max_i L_i\).

Assumption III.2. Each local objective function \(f_i\) is \(\mu_i\)-strongly convex.

Moreover, for both the theoretical and numerical results presented in this paper, we initialize the iterate \(x_{i,0} = s_0\) for each \(i \in [1, 2, \ldots, n]\), where \(s_0 \in \mathbb{R}^p\) is any vector; however, we should note that our theoretical results would hold with different initialization. Our analysis depends on the constant \(0 < \beta < 1\); the second largest, in magnitude, eigenvalue of the consensus matrix \(W\).

For notational convenience, we introduce the following quantities that are used in the analysis

\[
\bar{x}_k = \frac{1}{n} \sum_{i=1}^{n} x_{i,k}, \quad \bar{y}_k = \frac{1}{n} \sum_{i=1}^{n} y_{i,k}, \quad g_k = \sum_{j=1}^{t_c(k)} \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(x_{i,j-1}^{(k)}), \quad \tilde{g}_k = \sum_{j=1}^{t_c(k)} \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\bar{x}_j^{(k)}),
\]

where

\[
\tilde{x}_k^{j-1} = \tilde{x}_k^{j-1} - \alpha \frac{1}{n} \sum_{i=1}^{n} \nabla f_i(\tilde{x}_k^{j-1}) \quad \text{for } j = 1, \ldots, t_g \quad \text{(III.1)}
\]

and \(\bar{x}_k\). The vectors \(\bar{x}_k \in \mathbb{R}^p\) and \(\bar{y}_k \in \mathbb{R}^p\) correspond to the average of local estimates, \(g_k \in \mathbb{R}^p\) represents the average of local gradients at the current local estimates, and \(\tilde{g}_k \in \mathbb{R}^p\) is the average gradient at \(\bar{x}_k\). The vectors \(\bar{x}_k \in \mathbb{R}^p\) represent the iterates produced by taking gradient steps on the average objective function \(\tilde{f}(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)\) starting from \(\bar{x}_k\). We should note that these iterates are never explicitly computed and are solely defined for analysis purposes.

We note that the gradient steps II.1 in the NEAR-DGD\(^{t_e,t_g}\) method can be viewed as \(t_g(k)\) gradient iterations on the following unconstrained problem

\[
\min_{x_i \in \mathbb{R}^p} \sum_{i=1}^{n} f_i(x_i). \quad \text{(III.2)}
\]

We use this observation to bound the iterates \(x_k\) and \(y_k\).

Lemma III.3. (Bounded iterates) Suppose Assumptions III.1 and III.2 hold, and let the steplength satisfy \(\alpha < \frac{1}{\mu}\). Then, the iterates generated by the NEAR-DGD\(^{t_e,t_g}\) method (II.1)-(II.2) are bounded, namely,

\[
\|x_k\| \leq D, \quad \|y_k\| \leq D,
\]

where

\[
D = \|y_0 - u^*\| + \frac{\mu + 1}{\nu} \|u^*\|, \quad u^* = [u^*_1; u^*_2; \ldots; u^*_n] \in \mathbb{R}^{np}, \quad u^*_i = \arg \min_{u_i} f_i(u_i), \quad u^* \text{ is the optimal solution of (III.2), } \nu = 2\alpha\gamma, \quad \gamma = \min_i \gamma_i, \text{ and } \gamma_i = \frac{\mu_i\gamma}{\mu_i + \gamma_i} \text{ for } 1 \leq i \leq n.
\]

| Method                  | \(t_c(k)\) | \(t_g(k)\) | Gradient/Functions | Communication | Convergence | Convergence Rate | Reference |
|-------------------------|------------|------------|--------------------|---------------|-------------|-----------------|----------|
| D-NC                    | \(O(\log k)\) | 1          | deterministic/convex | full          | exact       | Sub-linear      | [30]     |
| APG-MSC                 | \(k\)      | 1          | deterministic/convex | full          | exact       | Sub-linear      | [28]     |
| NEAR-DGD                | 1          | 1          | deterministic/convex | full          | neighborhood | R-Linear        | [1]      |
| NEAR-DGD\(^{t_e}\)      | \(t_c\)    | 1          | deterministic/strongly convex | full          | neighborhood | R-Linear        | [1]      |
| NEAR-DGD\(^{t_g}\)      | \(k\)      | 1          | deterministic/strongly convex | full          | exact       | R-Linear        | [1]      |
| NEAR-DGD\(^{t_e,t_g}\)  | \(t_c\)    | \(t_g\)   | deterministic/strongly convex | full          | neighborhood | R-Linear        | this paper |
| Choco-SGD               | 1          | 1          | stochastic/strongly convex | quantized     | (in expectation) | Sub-linear      | [37]     |
| Local SGD               | 1          | \(t_g\)   | stochastic/strongly convex | full          | (in expectation) | Sub-linear      | [53]     |
| SG-NEAR-DGD\(^{t_e}\)   | \(t_c\)    | 1          | stochastic/strongly convex | full          | neighborhood | R-Linear        | [31]     |
| SG-NEAR-DGD\(^{t_g}\)   | \(k\)      | 1          | stochastic/strongly convex | full          | neighborhood | R-Linear        | [31]     |
where the second inequality is due to converting a finite sum to an infinite sum, the third inequality is due to the fact that 
$0 < 1 - \nu < 1$, and the last inequality is due to using an upper bound on the fraction in the second term. Thus, we bound the iterate as

$$
\| y_k \| \leq \| y_0 - u^\ast \| + \| u^\ast \| \\
\leq \| y_0 - u^\ast \| + \frac{\nu}{\nu - 1} \| u^\ast \|.
$$

We now show that the same result is true for the iterates. Using the definition of $x_k$ (II.2)

$$
\| x_{k+1} \| = \| Z^t \cdot y_k \| \\
\leq \| Z^t \| \| y_k \| \\
\leq \| y_k \| \leq D.
$$

Notice that the average iterates defined in (III.1) are a sequence of gradient descent steps on the function $f(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x)$. Under Assumptions III.1 and III.2, it can be shown that the function $f(x)$ is $\bar{\mu}$-strongly convex and has $L_f$-Lipschitz continuous gradients. Therefore, following the same procedure as above, we have

$$
\| x_k - x^\ast \| \leq \sqrt{\| y_0 - x^\ast \| + \| x^\ast \|} \\
\leq \sqrt{\| z \| + \| x^\ast \|} \\
\leq \sqrt{\| x^\ast \| + \| x_k \|} \\
\leq \sqrt{\| x^\ast \| + \frac{D}{\sqrt{n}},}
$$

where $\bar{\nu} = 2\alpha \cdot \frac{\mu_f}{\mu_f + L_f}$.

Lemma III.3 shows that the iterates generated by the NEAR-DGD$c,t_\nu$ method, where the number of consensus and gradient steps are fixed (and possibly greater than 1), are bounded. These results can be extended to show that the iterates generated by the NEAR-DGD$c,t_\nu$ method with varying number of consensus and gradient steps at every iteration (i.e., $t_c(k)$, $t_g(k)$) are also bounded.

For notational convenience, we define the quantity

$$
\eta = 1 + \alpha L,
$$

which is bounded from above and below by 2 and 1, respectively, as $0 < \alpha \leq 1/L$. Before we proceed, we provide a technical lemma that bounds the deviation between the individual gradients and the average gradient at any iterate within a compact set.

**Lemma III.4.** Suppose Assumptions III.1 and III.2 hold. Then, for any given $x \in D(x^\ast)$, there exists a constant $M \geq 0$ such that

$$
\| \nabla f_i(x) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x) \| \leq M, \quad (III.4)
$$

for $1 \leq i \leq n$, where $x^\ast$ is the optimal solution of (I.3), $D(x^\ast) = \{ z : \| z - x^\ast \| \leq D \}$, $D$ is defined in Lemma III.3, and $M = 2L \tilde{D} + \sum_{i=1}^{n} \| \nabla f_i(x^\ast) \|$.

1Note, $\mu_f = \frac{1}{n} \sum_{i=1}^{n} \mu_i$, and $\bar{L}_f = \frac{1}{\eta} \sum_{i=1}^{n} L_i$. 

Proof. We have,
\[
\left\| \nabla f_i(x) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x) \right\|
\leq \left\| \nabla f_i(x) - \nabla f_i(x^*) + \nabla f_i(x^*) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x) \right\|
+ \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x^*) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x^*)
\leq \left\| \nabla f_i(x) - \nabla f_i(x^*) \right\| + \| \nabla f_i(x^*) \|
+ \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(x^*)
\leq L_i \hat{D} + \| \nabla f_i(x^*) \| + L_f \hat{D}
\leq 2L \hat{D} + \sum_{i=1}^{n} \| \nabla f_i(x^*) \| = M,
\]
where the third inequality is due to Assumption III.1 and Lemma III.3 and \( \sum_{j=1}^{n} \nabla f_j(x^*) = 0 \). \( \square \)

The result of Lemma III.4 is independent of our algorithm and is valid for any finite set of functions.

Lemma III.5. (Bounded deviation from mean) Suppose Assumptions III.1 and III.2 hold. Then, the total deviation of each agent’s estimates \((x_{i,k} \text{ and } y_{i,k})\) from the mean are bounded, namely,
\[
\left\| x_{i,k} - \bar{x}_k \right\| \leq \beta^i \ast D, \tag{III.5}
\]
\[
\left\| y_{i,k} - \bar{y}_k \right\| \leq \beta^i \ast D + 2D. \tag{III.6}
\]

for all \( k = 1, 2, \ldots \) and \( 1 \leq i \leq n \). Moreover,
\[
\left\| x_{i,k} - \hat{x}_k \right\| \leq \eta^n \beta^i \ast D + \alpha M \left( \frac{\eta^n - 1}{\eta - 1} \right), \tag{III.7}
\]
\[
\left\| y_{i,k} - \hat{y}_k \right\| \leq \beta^i \ast D L \left( \frac{\eta^n - 1}{\eta - 1} \right) + M \left( \frac{\eta^n - 1}{\eta - 1} - t_g \right), \tag{III.8}
\]
for all \( k = 1, 2, \ldots, \) \( j = 0, \ldots, \) \( t_g \) and \( 1 \leq i \leq n \).

Proof. Consider,
\[
\left\| x_{i,k} - \bar{x}_k \right\| = \left\| x_{i,k} - \bar{y}_{k-1} \right\|
\leq \left\| x_{i,k} - \bar{y}_{k-1} \right\|
\leq \left\| \left( W^i \ast I \right) y_{k-1} - \frac{1}{n} \sum_{j=1}^{n} \left( \left( W^i \ast I \right) y_{k-1} \right) \right\|
\leq \left\| \left( W^i \ast I \right) y_{k-1} - \frac{1}{n} \sum_{j=1}^{n} \left( \left( W^i \ast I \right) y_{k-1} \right) \right\| \| y_{k-1} \|
\leq \beta^i \ast \| y_{k-1} \| \leq \beta^i \ast D,
\]
where the first equality is due to the fact that \( \bar{x}_k = Z^i \bar{y}_{k-1} = \bar{y}_{k-1} \) and the last inequality is due to Lemma III.3.

For the local \( y_{i,k} \) iterates in (III.6), consider
\[
\left\| y_{i,k} - \bar{y}_k \right\| \leq \left\| y_{i,k} - \bar{x}_{k+1} \right\| + \left\| \bar{x}_{k+1} - \bar{y}_k \right\|
\leq \beta^i \ast D + \| \bar{x}_{k+1} - \bar{y}_k \|
\leq \beta^i \ast D + \| \bar{y}_k - (W^i \ast I) y_{k+1} \|
\leq \beta^i \ast D + \| (I - W^i \ast I) y_{k+1} \|
\leq \beta^i \ast D + 2D,
\]
where the second inequality is due to (III.5) and the last inequality is due to Lemma III.3.

We prove result (III.7) by induction. The statement is true for \( j = 0 \). Now, assume that it is true for some \( j = l \), and consider,
\[
\left\| x_{i,k} - \hat{x}_k^l \right\|
= \left\| x_{i,k} - \bar{x}_k^l - \alpha \left( \nabla f_i(x_{i,k}^l) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\hat{x}_k) \right) \right\|
\leq \left\| x_{i,k} - \bar{x}_k^l \right\| + \alpha \left\| \nabla f_i(x_{i,k}^l) - \nabla f_i(\hat{x}_k) \right\|
+ \alpha \left\| \nabla f_i(\hat{x}_k) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\hat{x}_k) \right\|
\leq \left\| x_{i,k} - \bar{x}_k^l \right\| + \alpha \left| \nabla f_i(x_{i,k}^l) - \nabla f_i(\hat{x}_k) \right|
+ \alpha \left\| \nabla f_i(\hat{x}_k) - \frac{1}{n} \sum_{j=1}^{n} \nabla f_j(\hat{x}_k) \right\|
\leq \left( 1 + \alpha L \right) \left\| x_{i,k} - \bar{x}_k^l \right\| + \alpha M
\leq \eta^{l+1} \beta^i \ast D + \alpha M \left( \frac{\eta^{l+1} - 1}{\eta - 1} \right) + \alpha M
= \eta^{l+1} \beta^i \ast D + \alpha M \frac{\eta^{l+1} - 1}{\eta - 1},
\]
where the first equality is due to the definitions given in (II.1) and (III.1), the second inequality is due to Assumptions III.1 and Lemma III.4, the third inequality is due to the definition of \( L \) and the last inequality is due to the definition of \( \eta \) and (III.5).

To establish (III.8), we have
\[
\left\| g_k - \hat{y}_k \right\| = \sum_{j=1}^{t_g} \sum_{i=1}^{n} \left( \nabla f_i(x_{i,k}^{j-1}) - \nabla f_i(\hat{x}_k^{j-1}) \right)
\leq \sum_{j=1}^{t_g} \sum_{i=1}^{n} \left( \nabla f_i(x_{i,k}^{j-1}) - \nabla f_i(\hat{x}_k^{j-1}) \right)
\leq LD \beta^i \ast \sum_{j=1}^{t_g} \eta^{j-1} + M \sum_{j=1}^{t_g} \left( \eta^{j-1} - 1 \right)
\leq LD \beta^i \ast \sum_{j=1}^{t_g} \eta^{j-1} + M \left( \eta^{j-1} - 1 \right),
\]
where the first inequality is due to Assumption III.1, the definition of \( \eta \), and (III.7) and the second inequality is due to the definition of \( L \) and \( \eta \), and the fact that \( \alpha L \leq 1 \). \( \square \)

Lemma III.5 shows that the distance between the local iterates \( x_{i,k} \text{ and } y_{i,k} \) are bounded from their means. Similar to the results in Lemma III.3, these results can be extended to account for a varying number of consensus and gradient steps at every iteration since these results are for each iteration \( k \).
We now investigate the optimization error of the NEAR-DGD$^{c,t_g}$ method. To this end, we make use of an observation made in [1, Section V]. Namely,
\begin{equation}
\bar{y}_{k+1} = \bar{y}_k - \alpha g_k,
\end{equation}

(III.9)
can be viewed as a sequence of $t_g(k)$ inexact gradient descent steps on the following unconstrained problem
\begin{equation}
\min_{x \in \mathbb{R}^p} \tilde{f}(x) = \frac{1}{n} \sum_{i=1}^{n} f_i(x),
\end{equation}

(III.10)
where $\tilde{y}_k$ is the sequence of $t_g$ exact gradient descent steps.

We should mention that contrary to the analysis in [1], in this work we consider the error instead of the square of the error, and as such we are able to obtain tighter bounds.

**Theorem III.6. (Bounded distance to minimum)** Suppose Assumptions III.1 and III.2 hold, and let the steplength satisfy $\alpha \leq \min \left\{ \frac{1}{L}, c_4 \right\}$, where $c_4 = \frac{2}{\mu_j + L_j}$. Then, the iterates generated by the NEAR-DGD$^{c,t_g}$ method (II.1)-(II.2) satisfy
\begin{align*}
\|\bar{x}_k - x^*\| &\leq c_1 t_g(\eta^s - 1) + c_3 \beta \eta^s \left( \frac{1}{\eta - 1} - t_g(\eta - 1) \right)
\end{align*}
\begin{align*}
&+ c_5 \eta^s \left( 1 - \frac{1}{1-c_i^s} \right),
\end{align*}

where
\begin{align*}
c_1 &= \sqrt{1 - \alpha c_2}, \quad c_2 = \frac{2\mu_j L_j}{\mu_j + L_j},
\end{align*}
\begin{align*}
c_3 &= \frac{\alpha D L}{\eta - 1}, \quad c_5 = \frac{\alpha M}{\eta - 1},
\end{align*}

$x^*$ is the optimal solution of (I.3), $D$ is defined in Lemma III.3 and $\eta = 1 + \alpha L$.

**Proof.** Using the definitions of the $\bar{x}_k$, $g_k$, (III.9) and the fact that $W$ is doubly-stochastic, we have
\begin{equation}
\|\bar{x}_{k+1} - x^*\| \leq \|\bar{x}_k - x^* - \alpha \bar{g}_k\| + \alpha \|\bar{g}_k - g_k\|. \tag{III.11}
\end{equation}
The result of Lemma III.5 bounds the quantity $\|\bar{g}_k - g_k\|$.

Consider the first term on the right hand side of (III.11), and observe that this is precisely the distance to optimality after performing $t_g$ gradient steps on the function $\tilde{f}$. Therefore, by [59, Theorem 2.1.15, Chapter 2], we have
\begin{equation}
\|\bar{x}_k - x^* - \alpha \bar{g}_k\| \leq \sqrt{(1 - \alpha c_2)t_g} \|\bar{x}_k - x^*\|. \tag{III.12}
\end{equation}
Combining (III.11), (III.12) and using (III.8),
\begin{align*}
\|\bar{x}_{k+1} - x^*\| &\leq \sqrt{(1 - \alpha c_2)t_g} \|\bar{x}_k - x^*\| + \alpha LD \beta \eta^s \left( \frac{1}{\eta - 1} - t_g(\eta - 1) \right) \\
&+ c_3 \beta \eta^s \left( \frac{1}{\eta - 1} - t_g \right).
\end{align*}
\begin{equation}
\tag{III.13}
\end{equation}
Recursive application of (III.13), and using the definitions of $c_1$ and $c_3$ yields
\begin{align*}
\|\bar{x}_k - x^*\| &\leq c_1 t_g \|\bar{x}_0 - x^*\| + c_3 \beta \eta^s \left( \frac{1}{\eta - 1} - t_g(\eta - 1) \right) \\
&+ c_5 \eta^s \left( 1 - \frac{1}{1-c_i^s} \right),
\end{align*}
which concludes the proof. \hfill \Box

Theorem III.6 shows that the average of the iterates generated by the NEAR-DGD$^{c,t_g}$ converge to a neighborhood of the optimal solution whose size is defined by the steplength, the second largest eigenvalue of $W$, the number of consensus steps and the number of gradient steps. We observe that as the number of gradient steps $t_g$ increase, the rate constant $c_1 t_g$ in the first term of the right hand side decreases, thereby increasing the speed of convergence to the neighborhood. The second and third terms on the right hand side represent the size of this neighborhood. As the number of gradient steps increases, the numerators of these terms increases at geometric rate without any bound and the denominator also increases (but is bounded above by 1), and so the size of the neighborhood increases. Thus, there is a clear trade-off between the speed of convergence and the size of the neighborhood, with respect to the number of gradient steps taken. On the other hand, as the number of consensus steps $t_c$ increase, the neighborhood of convergence decreases and the rate is not affected. Table II summarizes the results for different $t_g$ and $t_c$. We observe that the rate depends on the constant $c_1$ which can be bounded using the bound on $\alpha$ by $\sqrt{\frac{1 - \alpha c_2}{\eta - 1}}$, where $\kappa = \frac{\eta^s}{1-c_i^s}$. Therefore, the dependence on the condition number is similar to that of gradient methods in the centralized setting. We should also note that in the case of federated learning, where the equivalence is a complete graph, we have $\beta = 0$. Therefore, the neighborhood term does not depend on the consensus steps and so it suffices to choose $t_c = 1$.

We now provide a convergence result for the local agent estimates of the NEAR-DGD$^{c,t_g}$ method.

**Corollary III.7. (Local agent convergence)** Suppose Assumptions III.1-III.2 hold, and let the steplength satisfy $\alpha \leq \min \left\{ \frac{1}{L}, c_4 \right\}$. Then, for $k = 0, 1, \ldots
\begin{align*}
\|x_{i,k} - x^*\| &\leq c_1 t_g \|x_0 - x^*\| + \beta \delta + c_5 \eta^s \left( 1 - \frac{1}{1-c_i^s} \right) \\
&+ c_3 \beta \eta^s \left( \frac{1}{\eta - 1} - t_g(\eta - 1) \right) + 2D,
\end{align*}
where $c_1, c_3, c_4$ and $c_5$ are given in Theorem III.6, $\eta = 1 + \alpha L$, $D$ is defined in Lemma III.3 and $\delta = \left( c_5 \eta^s - 1 \right) + D > 0$.
Proof. Using the results from Lemma III.5 and Theorem III.6, we have
\[ \|x_{i,k} - x^*\| \leq \|\bar{x}_k - x^*\| + \|x_{i,k} - \bar{x}_k\| \]
\[ \leq c_1^{k}\|\bar{x}_0 - x^*\| + c_3\beta c_2 (\eta^s(k) - 1) \]
\[ + c_5 \left(\frac{\eta_{i,k}^s - 1}{1 - c_1^{k}}\right) + \beta^{t_{k}}D, \]
and
\[ \|y_{i,k} - x^*\| \leq \|y_k - x^*\| + \|y_{i,k} - y_k\| \]
\[ = \|\bar{x}_{k+1} - x^*\| + \|y_{i,k} - y_k\| \]
\[ \leq c_1^{k+1}\|\bar{x}_0 - x^*\| + c_3\beta c_2 (\eta^s(k) - 1) \]
\[ + c_5 \left(\frac{\eta_{i,k}^s - 1 - t_g(\eta - 1)}{1 - c_1^{k}}\right) + \beta^{t_{k}}D + 2D. \]
Using the definition of \( \delta \) completes the proof. \( \square \)

Similar to the analysis of the NEAR-DGD+ method [1], and under the same conditions as in Theorem III.6, one can show that for any increasing sequence (of integers) of consensus steps \( \{t_c(k)\}_k \) and decreasing sequence (of integers) bounded by 1 of gradient steps \( \{t_g(k)\}_k \), the iterates produced by the NEAR-DGD\(^{t_c,t_g}\) method converge to \( x^* \) (the optimal solution of (1.3)). Specifically, if
\[ \lim_{k \to \infty} t_c(k) \to \infty \quad \text{and} \quad \lim_{k \to \infty} t_g(k) = 1, \]
then the iterates produced by the NEAR-DGD\(^{t_c,t_g}\) method converge to \( x^* \).

We now show that the iterates produced by the NEAR-DGD\(^{t_c(t_g(k))}\) method converge to the optimal solution at an R-Linear rate, with appropriately chosen sequences \( \{t_c(k)\}_k \) and \( \{t_g(k)\}_k \).

**Theorem III.8. (R-Linear convergence NEAR-DGD\(^{t_c(t_g(k))}\) method)** Suppose Assumptions III.1 and III.2 hold, let the steplength satisfy \( \alpha \leq \min \left\{ \frac{1}{2}c_1, c_4 \right\} \), and let \( t_c(k) = k \) and \( t_g(k) = \max\{t_g(0) - 1, k\} \), where \( t_g(k) \in \mathbb{Z}_+ \). Then, the iterates generated by the NEAR-DGD\(^{t_c(t_g(k))}\) method (II.1)-(II.2) converge at an R-Linear rate to the solution. Namely,
\[ \|\bar{x}_k - x^*\| \leq C \rho^k \quad (\text{III.14}) \]
for all \( k = 0, 1, 2, \ldots \), where
\[ C = \max \left\{ \|x_0 - x^*\|, \frac{8(\hat{c}_3 + \hat{c}_5)}{(\alpha c_2)^2} \right\}, \]
\[ \rho = \max \left\{ \beta, \tau, 1 - \frac{\alpha c_2}{2} \right\}, \]
\[ \tau = \max_{i=0, \ldots, t_g(0)-1} \left\{ \frac{T_{i+1}}{T_i} \right\}, \]
\[ T_k = \eta^s(k) - 1 - t_g(k)(\eta - 1), \]
\[ \hat{c}_3 = c_3(\eta^s(0) - 1), \quad \hat{c}_5 = c_5T_0, \]
c\(_2\), c\(_3\), c\(_4\), c\(_5\) and \( \eta \) are given in Theorem III.6.

Proof. We first consider the term
\[ T_k = \eta^s(k) - 1 - t_g(k)(\eta - 1), \]
and note that \( \{T_k\} \) is a decreasing sequence for all \( k \leq t_g(0) - 1 \) and for any \( k \geq t_g(0) \), \( T_k = 0 \) because of the definition of \( t_g(k) \). Hence, by the definition of \( \tau \), we have
\[ T_k \leq T_0 \tau^k. \quad (\text{III.15}) \]
We prove the result by induction. By the definition of \( C \) the base case \( k = 0 \) holds. Assume that the result is true for the \( k^{th} \) iteration, and consider the \( (k+1)^{th} \) iteration. Starting from (III.13) and using the definitions of \( c_3 \) and \( c_5 \), we have
\[ \|\bar{x}_{k+1} - x^*\| \leq c_1^{k+1}\|\bar{x}_k - x^*\| + c_3\beta c_2 (\eta^s(k) - 1) \]
\[ + c_5 T_k \]
\[ \leq c_1^{k+1}\|\bar{x}_k - x^*\| + c_3\beta c_2 (\eta^s(k) - 1) \]
\[ + c_5 T_0 \tau^k \]
\[ \leq c_1 C \rho^k + c_3 \beta c_2 (\eta^s(0) - 1) + c_5 T_0 \tau^k \]
\[ = C \rho^k \left[ c_1 + \hat{c}_3 \beta c_2 + \hat{c}_5 \tau^k \right] \]
\[ \leq C \rho^k \left[ c_1 + \hat{c}_3 + \hat{c}_5 \right] \]
\[ \leq C \rho^k \left[ \sqrt{1 - \alpha c_2} + \frac{(\alpha c_2)^2}{8} \right] \]
\[ \leq C \rho^k \left[ 1 - \alpha c_2 \right] \]
\[ \leq C \rho^{k+1}, \]
where the second inequality is due to (III.15), the third inequality is due to the inductive hypothesis and \( \eta^s(k) \leq \eta^s(0) \) (since \( \eta > 1 \) and \( t_g(k) \geq t_g(0) \)), the first equality is by the definitions of \( \hat{c}_3 \) and \( \hat{c}_5 \), fourth inequality is due to the fact that \( \rho \geq \beta \) and \( \rho \geq \tau \), the fifth inequality is due to the definitions of \( C \) and \( c_1 \), the sixth inequality is due to the Taylor expansion around \( \sqrt{1 - \alpha c_2} \), and the last inequality is due to the definition of \( \rho \). \( \square \)

Theorem III.8 illustrates that when the number of consensus steps is increased at the appropriate rate \( (t_c(k) = k) \) and the number of gradient steps is decreased at the appropriate rate \( (t_g(k) = \max\{t_g(0) - k, 1\}) \), where \( t_g(k) \in \mathbb{Z}_+ \), then the NEAR-DGD\(^{t_c(k),t_g(k)}\) method converges to the solution at an R-Linear rate.

We now provide a convergence result for the local agent estimates of the NEAR-DGD\(^{t_c(k),t_g(k)}\) method.

**Corollary III.9. (Local agent convergence)** Suppose the conditions of Theorem III.8 are satisfied. Then, for \( k = 0, 1, \ldots \)
\[ \|x_{i,k} - x^*\| \leq C \rho^k + \beta^k D \]
\[ \|y_{i,k} - x^*\| \leq C \rho^{k+1} + \beta^{k+1} D + 2D, \]
where \( C \) and \( \rho \) are defined in Theorem III.8, and \( D \) is defined in Lemma III.3.

Proof. Using the results from Lemma III.5 and Theorem III.8,
\[ \|x_{i,k} - x^*\| \leq \|\bar{x}_k - x^*\| + \|x_{i,k} - \bar{x}_k\| \]
\[ \leq C \rho^k + \beta^k D, \]
and
\[ \|y_{i,k} - x^*\| \leq \|\bar{y}_k - x^*\| + \|y_{i,k} - \bar{y}_k\| \]
\[
= \|\bar{x}_{k+1} - x^*\| + \|y_{i,k} - \bar{y}_k\| \\
\leq C\rho^{k+1} + \beta^k + 1 D + 2D.
\]
\[\blacksquare\]

This result implies that the local iterates $x_{i,k}$ generated by NEAR-DGD$^{c_i,t_g}$ method converge to the optimal solution, whereas the local iterates $y_{i,k}$ do not.

We now investigate the work complexity of the method. By work complexity we mean the total amount of work (gradient evaluations and communication steps) required to get an $\epsilon$-accurate solution (i.e., $\|\bar{x}_k - x^*\| \leq \epsilon$).

**Corollary III.10. (Work Complexity)** If the conditions in Theorem III.8 are satisfied, then the work complexity (total number of gradient evaluations $\tau_g$ and rounds of communications $\tau_c$) to get an $\epsilon$-accurate solution (for $\epsilon$ sufficiently small), that is $\|\bar{x}_k - x^*\| \leq \epsilon$, for the algorithm are given as follows,
\[
\tau_g = \bar{O}
\left( \max \left\{ \frac{1}{1 - \beta^t} \frac{L_f}{\mu_f} \log \left( \frac{1}{\epsilon} \right) \right\} \right),
\]
\[
\tau_c = \bar{O}
\left( \left( \max \left\{ \frac{1}{1 - \beta^t} \frac{L_2}{\mu_f} \log \left( \frac{1}{\epsilon} \right) \right\} \right)^2 \right).
\]

**Proof.** For simplicity, we consider the asymptotic complexity where $\epsilon$ is sufficiently small such that the total number of iterations to get an $\epsilon$-accurate solution, $k$, is larger than $t_g(0) - 1$ and the effect of $\tau$ vanishes. By Theorem III.8, we require
\[
k \geq \frac{\log(C \tau)}{\log(\frac{1}{\rho})}.
\]
iterations to get an $\epsilon$-accurate solution ($C\rho^k \leq \epsilon$). Using the fact that $\alpha < \frac{1}{L}$ and the definition of $c_2$, we have that $1 - \frac{\alpha^k}{2} \approx 1 - \frac{\mu^k}{L}$. Now, using this in the definition of $\rho$ (and ignoring the logarithmic dependence on the parameters $C$ and $\tau$) and approximating $-\log(\rho) \approx 1 - \rho$, we have
\[
k = \bar{O}
\left( \max \left\{ \frac{1}{1 - \beta^t} \frac{L_f}{\mu_f} \log \left( \frac{1}{\epsilon} \right) \right\} \right).
\]

Since we require $k$ communications at the $k$th iterate, the total number of communications ($\tau_c$) required is
\[
\tau_c = \sum_{i=1}^{k} = \frac{(k)(k+1)}{2} = \mathcal{O}(k^2) = \mathcal{O}\left(\left(\log\left(\frac{1}{\epsilon}\right)\right)^2\right).
\]

In the asymptotic region we are considering with $k$ larger than $t_g(0) - 1$, using definition of $t_g(k)$, we have,
\[
\tau_g = \sum_{i=0}^{t_g(0) - 1} (t_g(0) - i) + \sum_{i=t_g(0)}^{k} 1 = k + \frac{t_g^2(0) - t_g(0) + 2}{2}.
\]
Therefore, for sufficiently small $\epsilon$, we have $\tau_g = \mathcal{O}(k)$ which completes the proof. \[\blacksquare\]

We make the following observations about this result. The bound on $\tau_g$ matches with the bound for gradient descent in the centralized setting (which can be viewed as a complete graph with $\beta = 0$). As the graph topology changes, the bounds on $\tau_c$ and $\tau_g$ depend on the tradeoff between the condition number $\frac{\mu^k}{L^t}$ and the graph dependent parameter $\beta$.

Similar analysis can be done to show the work complexity required to get an $\epsilon$-accurate solution for the local iterates. Note, that this can only be done for the local iterates $x_{i,k}$, but not the local iterates $y_{i,k}$ as these iterates do not converge.

### IV. Numerical Results

In this section, we present numerical results demonstrating the performance of the NEAR-DGD$^{c_i,t_g}$ method, and the effect of performing both multiple consensus and gradient steps. The performance of the methods was evaluated via relative error ($\|\bar{x}_k - x^*\|^2/\|x^*\|^2$) in terms of: (i) iterations, (ii) cost\(^2\), (iii) number of gradient evaluations, and (iv) number of communications. The aim of this section is to show the practical performance of the class of methods and to highlight that the theoretical results are realized in practice.

We investigated the performance of different variants of the NEAR-DGD$^{c_i,t_g}$ on quadratic functions of the form
\[
f(x) = \frac{1}{2} \sum_{i=1}^{n} x^T A_i x + b_i^T x,
\]
where each node $i = \{1, \ldots, n\}$ has local information $A_i \in \mathbb{R}^{p \times p}$ and $b_i \in \mathbb{R}^p$. The problem was constructed as described in [60]; we considered a 4-cyclic graph topology (i.e., each node is connected to its 4 immediate neighbors), we chose the dimension size $p = 10$, the condition number ($\kappa = \frac{L}{\mu}$) was set to $10^4$ and the number of agents in the network ($n$) was 10.

We define variants of the NEAR-DGD$^{c_i,t_g}$ method as NEAR-DGD$^+$ ($(g_1, g_2), (c_1, c_2)$), where $g_1$ denotes the initial number of gradient steps and $g_2$ is the interval used for decreasing the number of gradient steps (the minimum number of gradient steps was 1), and $c_1$ denotes the initial number of consensus steps and $c_2$ describes if/how the number of communication steps was increased. Note, $g_2 = “-”$ and/or $c_2 = “-”$ indicates that the number of gradient and consensus steps, respectively, was kept constant. Moreover, NEAR-DGD$^+ ($(g_1, g_2), (c_1, k)$) indicates that the number of consensus steps was chosen as $t_c(k) = k$, NEAR-DGD$^+ ($(g_1, g_2), (c_1, 500+))$ indicates that the initial number of consensus steps was $c_1$ and that the number of consensus steps was increased by 1 every 500 iterations, and NEAR-DGD$^+ ($(g_1, 10—), (c_1, c_2))$ indicates that the initial number of gradient steps was equal to $g_1$ and that the number of gradient steps is reduced by 1 every 10 iterations. The markers in the Figures 1, 2, 3 and 4 are placed every 500 iterations. In this

\(^2\)We measure cost as proposed in [1]; namely,
\[
\text{Cost} = \#\text{Communications} \times c_c + \#\text{Computations} \times c_g,
\]
where $c_c$ and $c_g$ are exogenous application-dependent parameters reflecting the costs of communication and computation, respectively.
regard, one can clearly see the effect of the cost per iteration for the different methods.  

Figure 1 illustrates the performance of DGD as well as several variant of the NEAR-DGD method. For this plot, we used $c_c = c_g = 1$. The results show the rates of convergence and the neighborhoods of convergence of the methods. As predicted by the theory, the methods that do not increase the number of consensus steps converge only to a neighborhood of the solution, whereas methods that increase the number of consensus steps converge to the solution. Moreover, as predicted by the theory, methods that perform multiple gradient steps have a faster initial convergence rate. In terms of iterations, the NEAR-DGD$^{+}((1, -), (1, k))$ method is the fastest. However, this is not the case when comparing the methods in terms of number of communications or cost. This motivated us to investigate practical variant of the methods (see Figures 3 and 4).

In Figure 2, we illustrate the performance of the methods on three different quadratic problems for different cost structures: (i) $c_c = 1, c_g = 100$; (ii) $c_c = 1, c_g = 1$; (iii) $c_c = 100, c_g = 1$. Each row represents a different problem and each column a different cost structure. As is clear, the performance of the methods is highly dependent on the specific cost structure of the application. When the cost of gradient computations is large as compared to the cost of communications, the NEAR-DGD$^{+}((1, -), (1, k))$ method performs the best. This is not the case when the converse is true ($c_c > c_g$), where the best performing methods appear to be the standard DGD and NEAR-DGD$((1, -), (1, -))$ methods.

In Figures 3 and 4 we investigate the performance of practical variants of the NEAR-DGD method. Specifically, in these experiments we illustrate the behavior of methods that do not increase the number of communication steps as aggressively, and concurrently do not decrease the number of gradient steps as aggressively. Figure 3 show the performance of the methods in terms of iterations, gradient evaluations, communication and cost (with $c_g = c_c = 1$), and Figure 4 we show the performance in terms of cost for three different settings (left: $c_g = 100, c_c = 1$; center: $c_g = 1, c_c = 1$; right: $c_g = 1, c_c = 100$). One can clearly observe from both figures that there are benefits to employing the practical variants of the methods. This is especially apparent in terms of cost for all three different cost structures.

V. FINAL REMARKS & FUTURE WORK

Distributed optimization methods that decouple the communication and computation steps have sound theoretical properties and are efficient over a wide variety of distributed optimization problems. The NEAR-DGD method is one such method that performs nested communication and gradient steps at every iteration. In this paper, we generalized the analysis of the NEAR-DGD method to account for both multiple gradient and multiple consensus steps at every iteration. More specifically, we showed both theoretically and empirically the
effect of performing multiple gradient steps on the rate of convergence and the size of the neighborhood of convergence, and proved R-Linear convergence to the exact solution for a method that performs a decreasing number of gradient steps per iteration and an increasing number of consensus steps. We believe that this analysis completes the picture for the class of NEAR-DGD algorithms, and provides a theoretical justification for the common practice of using multiple local gradients by the federated learning community. The studies here could also guide the algorithm design choice of the number of communication and gradient steps performed per iteration. Future work includes extensions to directed networks, the setting with stochastic gradient, accelerated variants, and the development of schemes that adaptively select the number of communication and computation steps at every iteration depending on the application.

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