Combined Effect of Load Waviness and Auxeticity on the Shear Deformation in a Class of Rectangular Plates

Teik-Cheng Lim
School of Science and Technology, SIM University, Singapore.
tclim@unisim.edu.sg

Abstract. Auxetic materials are solids that exhibit negative Poisson’s ratio. Due to its uniqueness, it is expected that auxetic solids and structures manifest certain mechanical responses that are distinct from conventional materials. So far classical plate theory (CPT) has been used to investigate the behavior of thin auxetic plates and, for thick auxetic plates, the first-order shear deformation theory (FSDT) has been recently employed. The third-order shear deformation theory (TSDT) is adopted herein as it employs a realistic transverse shear stress and strain distribution across the plate cross section, and therefore removes the need for a correction factor. The results obtained herein reveal that as a plate material’s Poisson’s ratio becomes more negative, the plate deflection characteristic mimics that of (a) reduced plate thickness, (b) reduced plate aspect ratio, and (c) reduced load waviness. The results of this investigation support (i) the use of auxetic materials in applications where there is a need to reduce the extent of transverse shear deformation, and (ii) the use of the simpler CPT instead of any shear deformation plate theory if the Poisson’s ratio of the plate material is sufficiently negative.

1. Introduction
The use of the classical plate theory (CPT) [1, 2] for predicting the deflection of plates is valid when the plate is thin due to the insignificant amount of transverse shear deformation. Its deflection modeling neglects transverse shear strain and stress. However the transverse shear deformation increases with the plate thickness, thereby increasing the extent of deflection. To take into consideration the transverse shear deformation, the first-order shear deformation theory (FSDT) [3, 4] was developed. Since FSDT assumes constant transverse shear strain (and hence constant transverse shear stress) along the plate thickness, a shear correction factor is a required compensation because the actual transverse shear stress is parabolically distributed along the plate thickness. On the other hand, the third-order shear deformation theory (TSDT) does away with the shear correction factor, as it caters to a quadratic distribution of the transverse shear strains (and hence transverse shear stress) with diminishing transverse shear stresses at the top and bottom surfaces of the plate [5]. Although TSDT of plates have been increasingly investigated (e.g. [6,7]), it has not been used for analyzing auxetic (negative Poisson’s ratio) plates. The reader is referred to analyses of auxetic plates using CPT [8-11] and FSDT [12-15], as well as a recent monograph for an introduction to auxetic materials [16]. In a departure of previous analysis that adopt FSDT for understanding the negativity effect of a rectangular plate’s Poisson’s ratio.

2. Analysis
Using the superscript notations \((K)\) and \((R)\) for indicating Kirchhoff and Reddy plates respectively, while \(M^0\) and \(M\) indicate the marcus moment and the bending moment respectively, Wang et al. [17] showed that the Kirchhoff and Reddy plate deformations for a simply supported rectangular plate are related as

\[
\nabla^2 w^{(R)} - \lambda_0^2 w^{(R)} = -\lambda_0^2 \left( w^{(R)} + \frac{C}{Gh} M^{(K)} \right)
\]

(1a)

where

\[
\lambda_0^2 = \frac{70Gh}{D}
\]

(1b)

and \(C = 17/14\), in which \(G\) and \(D\) denote the shear modulus and flexural rigidity of the plate, respectively. To re-cast the above relationship in a form that is more readily executed, we substitute

\[
\frac{G}{D} = \frac{6(1-\nu)}{h^3}
\]

(2)

so that

\[
\lambda_0^2 = \frac{420(1-\nu)}{h^2}
\]

(3)

and

\[
M^{(K)} = -D\nabla^2 w^{(K)}
\]

(4)

to give the general deflection relationship between Reddy and Kirchhoff simply supported rectangular plates as

\[
w^{(R)} = \frac{h^2}{420(1-\nu)} \nabla^2 w^{(R)} = w^{(K)} - \frac{17h^2}{84(1-\nu)} \nabla^2 w^{(K)}
\]

(5)

with the error from using the CPT, with reference to the TSDT, as

\[
error = w^{(K)} - w^{(R)} = \frac{h^2}{420} \left( \frac{85\nabla^2 w^{(K)} - \nabla^2 w^{(R)}}{1-\nu} \right).
\]

(6)

For a general sinusoidal transverse load of

\[
q(x, y) = q_0 \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\]

acting on a simply supported rectangular plate of sides \(a\) and \(b\) measured along the \(x\) and \(y\) axes, respectively, where \(m\) and \(n\) are positive odd integers, the CPT and TSDT plate deflections of the form

\[
\begin{align*}
\{ w^{(K)} \} &= \{ A^{(K)}_{mn} \} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b} \\
\{ w^{(R)} \} &= \{ A^{(R)}_{mn} \} \sin \frac{m\pi x}{a} \sin \frac{n\pi y}{b}
\end{align*}
\]

(8)

satisfies all boundary conditions:

\[
w_{x=0,a} = w_{y=0,b} = \frac{\partial^2 w}{\partial x^2} \bigg|_{x=0,a} = \frac{\partial^2 w}{\partial y^2} \bigg|_{y=0,b} = 0.
\]

(9)

Substituting Eq.(8) into Eq.(5) leads to

\[
A^{(R)}_{mn} \left[ 1 + \frac{\pi^2 h^2}{420(1-\nu)} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right] = A^{(K)}_{mn} \left[ 1 + \frac{17\pi^2 h^2}{84(1-\nu)} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right) \right]
\]

(10)

where the solution for \(A^{(K)}_{mn}\) has been given as [18]

\[
A^{(K)}_{mn} = \frac{d_0}{\pi^4 D} \left( \frac{m^2}{a^2} + \frac{n^2}{b^2} \right)^{-2}
\]

(11)
The ratio of deflection by TSDT to that by CPT is independent from the coordinate position \((x, y)\) of the plate
\[
\frac{w^{(R)}}{w^{(K)}} = \frac{A^{(R)}_{mn}}{A^{(K)}_{mn}} = \frac{420a^2b^2(1-v)+85\pi^2h^2(a^2n^2+b^2m^2)}{420a^2b^2(1-v) + \pi^2h^2(a^2n^2+b^2m^2)},
\]
indicating that an amount of \(84\pi^2h^2(a^2n^2+b^2m^2)\) is attributed to the transverse shear deformation according to TSDT. As is well known, the transverse shear deformation reduces as the plate becomes thinner. The second terms on at the numerator and denominator on the RHS of Eq.(12) also suggests that transverse shear deformation reduces as the load is less wavy (i.e. smaller values of \(m\) and \(n\)). Alternatively, the transverse shear deformation can be reduced by using auxetic materials, as implied in Eq.(12).

3. Results and Discussion

Figure 1(a) shows the plots of Reddy-to-Kirchhoff deflection ratio for square plates \((a = b = l)\) under half sine wave load \((m = n = 1)\) for various dimensionless plate thickness \(h/l\). Although it is known that thicker plates encounter greater transverse shear deformation, it is of interest to note that for a thick auxetic plate with \(h/l = 0.18\) and \(v = -0.65\), the transverse shear deformation is lower than that of an incompressible plate \((v = 0.5)\) of lower dimensionless thickness of \(h/l = 0.1\). In addition, a thick plate of \(h/l = 0.18\) that is auxetic at \(v = -0.95\) is mechanically equivalent to a thin plate of \(h/l = 0.1\) that is conventional with \(v = 0.4\), as the Reddy-to-Kirchhoff deflection ratio for both cases approximate \(w^{(R)}/w^{(K)} \approx 1.0656\). Similarly, a highly auxetic plate \((v = -0.95)\) of dimensionless thickness \(h/l = 0.1\) is mechanically equivalent to a very thin conventional plate \((v = 0.3, h/l = 0.06)\) as both plates exhibit \(w^{(R)}/w^{(K)} = 1.02\), i.e. a transverse shear deformation that exceeds the bending deformation by 2%.

![Figure 1](image-url)

**Figure 1.** Plots of TSDT-to-CPT deflection ratio for square plates under half sine wave load for various dimensionless thicknesses (left), and at dimensionless thickness of 0.1 for various load waviness (right).

A family of Reddy-to-Kirchhoff deflection ratio for square plates \((a = b = l)\) and dimensionless thickness of \(h/l = 0.1\) is plotted in figure 1(b) for different patterns of load distribution, in which the extent of load waviness is denoted by the number of \(m\) and \(n\), i.e. \(m\) and \(n\) refer to the waviness along the \(x\) and \(y\) axes of the plate, respectively. It is readily seen from Eq.(15), and verified from figure 1(b), that the waviness controls the extent of transverse shear deformation via \((m^2 + n^2)\) for a...
square plate or \((b^2m^2 + a^2n^2)\) for a rectangular plate. It can be observed that an auxetic plate under a load of higher waviness deflects as if for a conventional plate under a load of lower waviness. For example, a highly auxetic plate of \(v = -0.7\) under a very high waviness load pattern of \(m = 3, n = 5\) (i.e. \(m^2 + n^2 = 34\)), a moderately auxetic plate of \(v = -0.3\) under a moderately high waviness load pattern of \(m = 1, n = 5\) (i.e. \(m^2 + n^2 = 26\)), a conventional plate of \(v = 0.1\) under a moderately low waviness load pattern of \(m = 3, n = 3\) (i.e. \(m^2 + n^2 = 18\)), and an incompressible plate (i.e. \(v = 0.5\)) under a very low waviness load pattern of \(m = 1, n = 3\) (i.e. \(m^2 + n^2 = 10\)) give equal Reddy-to-Kirchhoff deflection ratio of \(w(R)/w(K)\) at \(z = 0\). It is also noticeable that an equal Reddy-to-Kirchhoff deflection ratio for rectangular plates with dimensionless thickness of 0.1 under various half sine load pattern.

Curves of Reddy-to-Kirchhoff deflection ratio for rectangular plates are displayed in figure 2 for dimensionless plate thickness of \(h/\sqrt{ab} = 0.1\) under half sine load pattern (i.e. \(m = n = 1\)). As before, the Reddy-to-Kirchhoff deflection ratio decreases as the Poisson’s ratio of the plate becomes more negative. It is interesting to note that at the Reddy-to-Kirchhoff deflection ratio of \(w(R)/w(K) =1.049319\) is valid for a negative Poisson’s ratio (\(v = -0.7\)) plate with aspect ratio of \(a/b = 4\), a zero Poisson’s ratio plate with aspect ratio \(a/b = 2\), and a positive Poisson’s ratio plate (\(v = 0.2\)) with aspect ratio \(a/b = 1\). It is also noticeable that an equal Reddy-to-Kirchhoff deflection ratio can be obtained at \(v = \pm 1/4\), i.e. \(w(R)/w(K) =1.052605\) when \(a/b = 1\) at \(v = 1/4\) and when \(a/b = 3\) at \(v = -1/4\); for completeness’ sake, the same Reddy-to-Kirchhoff deflection ratio can be obtained when \(a/b = 5\) at \(v = -0.95\).

As a design guideline, a 10% deviation of the more accurate TSDT deflection from the less accurate CPT is shown in figures 1 and 2, such that the CPT remains valid under the dashed lines even for thick and high aspect ratio plates under loads of high waviness pattern.

4. Conclusions
The deflection characteristic of an auxetic plate, vis-à-vis a conventional plate, has been herein investigated using a third-order shear deformation theory (TSDT), and investigated for a simply supported rectangular plates for the entire range of Poisson’s ratio of isotropic solids, with variations to the plate thickness, plate aspect ratio and load pattern waviness. Results from the present study imply that, should there be a requirement to use thick plates and/or long narrow plates with very wavy load pattern such that significant the transverse shear deformation takes places, then the choice of auxetic materials will not only reduces the extent of transverse shear deformation with respect to
bending deformation, but also justifies the use of classical plate theory (CPT). Hence a geometrically thick plate becomes a mechanically thin plate should the Poisson’s ratio of the plate material be sufficiently negative to effectively diminish the extent of shear deformation vis-à-vis bending deformation. Auxetic plates are therefore useful for state of the art pressure sensor plates due to the enhanced in-plane deformation at the surface arising from decreased shearing-to-bending deformation ratio. It is herein suggested that experimental work be performed in future to validate Eq.(12).

References

[1] Kirchhoff G 1850 Über das gleichgewicht und die bewegung einer elastischen scheibe Journal für die Reine und Angewandte Mathematik 40 pp 51-88
[2] Love A E H 1888 The small free vibrations and deformation of a thin elastic shell Philosophical Transactions of the Royal Society of London A 179 pp 491-546
[3] Reissner E 1945 The effect of transverse shear deformation on the bending of elastic plates Journal of Applied Mechanics 12 pp 68-77
[4] Mindlin R D 1951 Influence of rotatory inertia and shear on flexural motions of isotropic, elastic plates Journal of Applied Mechanics 18 pp 31-38
[5] Reddy J N and Wang C M 1998 Deflection relationships between classical and third-order plate theories Acta Mechanica 130 pp 199-208
[6] Gao X L, Huang J X and Reddy J N 2013 A non-classical third-order shear deformation plate model based on a modified couple stress theory Acta Mechanica 224 pp 2699-2718
[7] Gao X L and Zhang G Y 2015 A microstructure- and surface energy-dependent third-order shear deformation beam model Zeitschrift für Angewandte Mathematik und Physik 66 pp 1871-1894
[8] Lim T C 2013 Circular auxetic plates Journal of Mechanics 29 pp 121-133
[9] Lim T C 2013 Optimal Poisson's ratios for laterally loaded rectangular plates Journal of Materials: Design and Applications 227 pp 111-123
[10] Lim T C 2016 Bending stresses in triangular auxetic plates Journal of Engineering Materials and Technology 138 article 014501
[11] Lim T C 2016 Simply-supported elliptical auxetic plates Journal of Mechanics 32 pp 413-419
[12] Lim T C 2013 Shear deformation in thick auxetic plates Smart Materials and Structures 22 article 084001
[13] Lim T C 2014 Shear deformation in rectangular auxetic plates Journal of Engineering Materials and Technology 136 article 031007
[14] Lim T C 2014 Elastic stability of thick auxetic plates Smart Materials and Structures 24 article 045004
[15] Lim T C 2014 Vibration of thick auxetic plates Mechanics Research Communications 61 pp 60-66
[16] Lim T C 2015 Auxetic Materials and Structures (Singapore: Springer)
[17] Wang C M, Reddy J N and Lee K H 2000 Shear Deformable Beams and Plates (New York: Elsevier)
[18] Timoshenko S P and Woinowsky-Krieger S 1970 Theory of Plates and Shells (Singapore: McGraw-Hill)
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Teik-Cheng Lim
School of Science and Technology, SIM University, Singapore.

There is a typographical error in Equation (1a).

Page 2, RHS of Eq.(1a):

\( w^{(R)} \rightarrow w^{(K)} \)