A Theoretical Framework for Bayesian Nonparametric Regression: Orthonormal Random Series and Rates of Contraction

FANGZHENG XIE\textsuperscript{*}, WEI JIN\textsuperscript{**}, and YANXUN XU\textsuperscript{†}

Department of Applied Mathematics and Statistics
Johns Hopkins University
Baltimore, Maryland 21218, USA
E-mail: \textsuperscript{*}fxie5@jhu.edu; \textsuperscript{**}wjin@jhu.edu; \textsuperscript{†}yanxun.xu@jhu.edu

We develop a unifying framework for Bayesian nonparametric regression to study the rates of contraction with respect to the integrated $L_2$-distance without assuming the regression function space to be uniformly bounded. The framework is built upon orthonormal random series in a flexible manner. A general theorem for deriving rates of contraction for Bayesian nonparametric regression is provided under the proposed framework. Three non-trivial applications of the proposed framework are provided: The finite random series regression of an $\alpha$-Hölder function, with adaptive rates of contraction up to a logarithmic factor, given independent and uniform design points; The un-modified block prior regression of an $\alpha$-Sobolev function, with adaptive-and-exact rates of contraction, given independent and uniform design points; The squared-exponential Gaussian process regression of a supersmooth function with a near-parametric rate of contraction, under the condition that the design points are fixed and reasonably spread. These applications serve as generalization or complement of the their respective results in the literature. Extension to sparse additive models in high dimensions is discussed as well.

Keywords: Bayesian nonparametric regression, integrated $L_2$-distance, orthonormal random series, rate of contraction.

1. Introduction

Consider the standard nonparametric regression problem $y_i = f(x_i) + \epsilon_i$, $i = 1, \cdots, n$, where the predictors $(x_i)_{i=1}^n$ are referred to as design (points) and take values in $[0, 1]^p \subset \mathbb{R}^p$, $\epsilon_i$’s are independent and identically distributed (i.i.d.) mean-zero Gaussian noises with $\text{var}(\epsilon_i) = \sigma^2$, and $y_i$’s are the responses. We follow the popular Bayesian approach by assigning $f$ a carefully-selected prior, and perform inference tasks by finding the posterior distribution of $f$ given the observations $(x_i, y_i)_{i=1}^n$.

We develop a theoretical framework for Bayesian nonparametric regression to study the rates of contraction with respect to the integrated $L_2$-distance

$$\|f - g\|_2 = \left\{ \int_{\mathcal{X}} [f(x) - g(x)]^2 \, dx \right\}^{1/2}.$$
The regression function $f$ is assumed to be represented using a set of appropriate orthonormal basis functions $(\psi_k)_{k=1}^\infty$: $f(x) = \sum_k \beta_k \psi_k(x)$. The coefficients $(\beta_k)_{k=1}^\infty$ are allowed to range over the entire real line $\mathbb{R}$ and the space of regression functions is allowed to be unbounded, including the renowned Gaussian process priors as special examples.

Rates of contraction of posterior distributions for Bayesian nonparametric priors have been studied extensively. Following the earliest framework on generic rates of contraction theorems with i.i.d. data proposed by [12], specific examples for density estimation via Dirichlet process mixture models [4, 13, 16, 37] and location-scale mixture models [22, 46] are discussed. For nonparametric regression, the rates of contraction had not been discussed until [14], who develop a generic framework for fixed-design to study rates of contraction with respect to the empirical $L_2$-distance. There are extensive studies for various priors that fall into this framework, including location-scale mixture priors [9], conditional Gaussian tensor-product splines [10], and Gaussian processes [42, 44], among which adaptive rates are obtained in [9, 10, 44].

Although it is interesting to achieve adaptive rates of contraction with respect to the empirical $L_2$-distance for nonparametric regression, this might be restrictive since the empirical $L_2$-distance quantifies the convergence of functions only at the given design points. In nonparametric regression, one also expects that the error between the estimated function and the true function can be globally small over the whole design space [47], i.e., small mean-squared error for out-of-sample prediction. Therefore the integrated $L_2$-distance is a natural choice. For Gaussian processes, [40], [48] provide contraction rates for nonparametric regression with respect to the integrated $L_2$, and $L_\infty$-distance, respectively. A novel spike-and-slab wavelet series prior is constructed in [52] to achieve adaptive contraction with respect to the stronger $L_\infty$-distance. These examples however, take advantages of their respective prior structures and may not be easily generalized. A closely related reference is [26], which discusses the rates of contraction of the rescaled-Gaussian process prior for nonparametric random-design regression with respect to the integrated $L_1$-distance, which is weaker than the integrated $L_2$-distance. Although a generic framework for the integrated $L_2$-distance is presented in [20], the prior there is imposed on a uniformly bounded function space and hence rules out some popular priors, e.g., the popular Gaussian process prior [29].

It is therefore natural to ask the following fundamental question: for Bayesian nonparametric regression, can one build a unifying framework to study rates of contraction for various priors with respect to the integrated $L_2$-distance without assuming the uniform boundedness of the regression function space? In this paper we provide a positive answer to this question. Inspired by [11], we build the general framework upon series expansion of functions with respect to a set of orthonormal basis functions. The prior is not necessarily supported on a uniformly bounded function space. A general rate of contraction theorem with respect to the integrated $L_2$-distance for Bayesian nonparametric regression is provided under the proposed framework. Examples of applications falling into this framework include the finite random series prior [30, 35], the (un-modified) block prior [11], and the classical squared-exponential Gaussian process prior [29]. In particular, for the block prior regression, rather than modifying the block prior by conditioning
on a truncated function space as in [11] with a known upper bound for the unknown true regression function, we prove that the un-modified block prior automatically yields rate-exact Bayesian adaptation for nonparametric regression without such a truncation. We further extend the proposed framework to sparse additive models in high dimensions. The analyses of the above applications and extension under the proposed framework also generalize their respective results in the literature. This is made clear in Sections 3 and 4.

The literatures regarding rates of contraction for Bayesian nonparametric models are largely based on the cutting-edge prior-concentration-and-testing procedure proposed by [2] and [12]. The major challenge of establishing a general framework for Bayesian nonparametric regression is that unlike the Hellinger distance for the density estimation and the empirical $L_2$-distance for fixed-regression problem, the integrated $L_2$-distance for random-design regression problem is not locally testable, complicating the construction of certain test functions. The exact definition of local testability of a distance is provided later, but a locally testable distance is the key ingredient for the prior-concentration-and-testing procedure to work. It turns out that by modifying the testing procedure, we only require the integrated $L_2$-distance to be locally testable over an appropriate class of functions. The above arguments are made precise in Section 2.

The layout of this paper is as follows. In Section 2 we introduce the random series framework for Bayesian nonparametric regression and present the main result concerning rates of contraction. As applications of the main result, we derive the rates of contraction of various renowned priors for nonparametric regression in the literature with substantial improvements in Section 3. Section 4 elaborates on extension of the proposed framework to sparse additive models in high dimensions. The technical proofs of the main results are deferred to Section 5.

Notations

For $1 \leq r \leq \infty$, we use $\| \cdot \|_r$ to denote both the $\ell_r$-norm on any finite dimensional Euclidean space and the integrated $L_r$-norm of a measurable function (with respect to the Lebesgue measure). In particular, for any function $f \in L_2([0,1]^p)$, we use $\| f \|_2$ to denote the integrated $L_2$-norm defined to be $\| f \|_2^2 = \int_{[0,1]^p} f^2(x) dx$. We follow the convention that when $r = 2$, the subscript is omitted, i.e., $\| \cdot \|_2 = \| \cdot \|$. The Hilbert space $l^2$ denotes the space of sequences that are squared-summable. We use $[x]$ to denote the maximal integer no greater than $x$, and $\lfloor x \rfloor$ to denote the minimum integer no less than $x$. The notations $a \lesssim b$ and $a \gtrsim b$ denote the inequalities up to a positive multiplicative constant, and we write $a \asymp b$ if $a \lesssim b$ and $a \gtrsim b$. Throughout capital letters $C,C_1,C,C',D,D_1,D,D',\cdots$ are used to denote generic positive constants and their values might change from line to line unless particularly specified, but are universal and unimportant for the analysis.

We refer to $P$ as a statistical model if it consists of a class of densities on a sample space $\mathcal{X}$ with respect to some underlying $\sigma$-finite measure. Given a (frequentist) statistical model $P$ and the i.i.d. data $(w_i)_{i=1}^n$ from some $P \in P$, the prior and the posterior
distribution on \( \mathcal{P} \) are always denoted by \( \Pi(\cdot) \) and \( \Pi(\cdot \mid \mathbf{w}_1, \cdots, \mathbf{w}_n) \), respectively. Given a function \( f : \mathcal{X} \to \mathbb{R} \), we use \( \mathbb{P}_n f = n^{-1} \sum_{i=1}^n f(x_i) \) to denote the empirical measure of \( f \), and \( \mathcal{G}_n f = n^{-1/2} \sum_{i=1}^n \left[ f(x_i) - \mathbb{E} f(x_i) \right] \) to denote the empirical process of \( f \), given the i.i.d. data \( (x_i)_{i=1}^n \). With a slight abuse of notations, when applying to a set of design points \( (x_i)_{i=1}^n \), we also denote \( \mathbb{P}_n f = n^{-1} \sum_{i=1}^n f(x_i) \) and \( \mathcal{G}_n f = n^{-1/2} \sum_{i=1}^n \left[ f(x_i) - \mathbb{E} f(x_i) \right] \) to be the empirical measure and empirical process, even when the design points \( (x_i)_{i=1}^n \) are deterministic. In particular, \( \phi \) denotes the probability density function of the (univariate) standard normal distribution, and we use the shorthand notation \( \phi_\sigma(y) = \phi(y/\sigma)/\sigma \) to denote the density of \( N(0, \sigma^2) \). For a metric space \((\mathcal{F}, d)\), for any \( \epsilon > 0 \), the \( \epsilon \)-covering number of \((\mathcal{F}, d)\), denoted by \( \mathcal{N}(\epsilon, \mathcal{F}, d) \), is defined to be the minimum number of \( \epsilon \)-balls of the form \( \{g \in \mathcal{F} : d(f, g) < \epsilon \} \) that are needed to cover \( \mathcal{F} \).

## 2. The framework and main results

Consider the nonparametric regression model: \( y_i = f(x_i) + e_i \), where \( (e_i)_{i=1}^n \) are i.i.d. mean-zero Gaussian noises with \( \text{var}(e_i) = \sigma^2 \), and \( (x_i)_{i=1}^n \) are design points taking values in \( [0, 1]^p \). Unless otherwise stated, the design points \( (x_i)_{i=1}^n \) are assumed to be independently and uniformly sampled for simplicity throughout the paper. The framework presented in this paper naturally adapts to the case where the design points are independently sampled from a density function that is bounded away from 0 and \( \infty \). We assume that the responses \( y_i \)'s are generated from \( y_i = f_0(x_i) + e_i \) for some unknown \( f_0 \in L_2([0, 1]^p) \), thus the data \( \mathcal{D}_n = (x_i, y_i)_{i=1}^n \) can be regarded as i.i.d. samples from a distribution \( \mathbb{P}_0 \) with joint density \( p_0(x, y) = \phi_\sigma(y - f_0(x)) \). Throughout we assume that the variance \( \sigma^2 \) is known, but the framework can be easily extended to the case where \( \sigma \) is unknown by placing a prior on \( \sigma \) that is supported on a compact interval contained in \((0, \infty)\) with a density bounded away from 0 and \( \infty \).

The regression model is parametrized by the function \( f \), and we follow the standard nonparametric Bayes procedure by assigning a prior distribution \( \Pi \) on \( f \) as follows. Let \( (\psi_k)_{k \in \mathcal{K}} \) be a set of orthonormal basis functions in \( L_2([0, 1]^p) \), where \( \mathcal{K} \) is a countably infinite index set. Since \( f_0 \in L_2([0, 1]^p) \), we also impose the prior \( \Pi \) on a sub-class of \( L_2([0, 1]^p) \) and write \( f \) in terms of the orthonormal series expansion \( f(x) = \sum_{k \in \mathcal{K}} \beta_k \psi_k(x) \), where the coefficients \( (\beta_k)_{k \in \mathcal{K}} \) are imposed a prior distribution such that \( \sum_{k \in \mathcal{K}} \beta_k^2 < \infty \) a.s.. The prior on \( (\beta_k)_{k \in \mathcal{K}} \) can be very general as long as it yields realizations that are squared-integrable with probability one. We shall also assume that the true function \( f_0 \) yields a series expansion \( f_0(x) = \sum_{k \in \mathcal{K}} \beta_0 k \psi_k(x) \), where \( \sum_{k \in \mathcal{K}} \beta_0 k^2 < \infty \), i.e., \( f_0 \in L_2([0, 1]^p) \). We require that \( \sup_{k \in \mathcal{K}} \| \psi_k \|_\infty < \infty \). The widely used tensor-product Fourier basis satisfies this requirement:

\[
\psi_{k_1, \cdots, k_p}(x_1, \cdots, x_p) = \prod_{j=1}^p \psi_{k_j}^j(x_j),
\]

where \( \psi_1^1(x) = 1 \), \( \psi_{2k}^1(x) = \sin k \pi x \), \( \psi_{2k+1}^1(x) = \cos k \pi x \), and the index set is the \( p \)-times cartesian product of the set of all positive integers \( \mathcal{K} = \mathbb{N}_+^p \).

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Remark 2.1. The prior $\Pi$ is supported on a function space that may not be uniformly bounded and the setup here is more general than that in [20]. The proposed framework includes the well-known Gaussian process prior as a special case. Let $K : [0, 1]^p \times [0, 1]^p \to [0, \infty)$ be a positive definite covariance function that yields an eigenfunction expansion $K(x, x') = \sum_{k=1}^{\infty} \lambda_k \psi_k(x) \psi_k(x')$, where $(\psi_k)_{k=1}^{\infty}$ is a set of orthonormal eigenfunctions of $K$, and $(\lambda_k)_{k=1}^{\infty}$ are the corresponding eigenvalues. For a Gaussian process prior $\text{GP}(0, K)$ on $f$, the Karhunen-Loève theorem [43] yields the following representation of $f$: $f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x)$, where $\beta_k \sim N(0, \lambda_k)$ independently.

Before presenting the main result for studying convergence of Bayesian nonparametric regression under the proposed framework, let us first recall the extensively used prior-concentration-and-testing procedure for deriving rates of contraction proposed by [12]. The general setup is as follows. Suppose $(w_i)_{i=1}^{n} \subset \mathcal{W}$ are i.i.d. data sampled from a distribution $P_0$ that yields a density $p_0$ with respect to some underlying $\sigma$-finite measure. Let $\Theta$ be the parameter space (typically infinite dimensional for nonparametric problems) such that $P_0 = p_{\theta_0}$ for some $\theta_0 \in \Theta$. By imposing a prior $\Pi$ on the parameter $\theta \in \Theta$ (equipped with a suitable measurable structure), the posterior distribution can be written as

$$\Pi(A \mid D_n) = \frac{\int_A \exp \left[ \Lambda_n(\theta \mid D_n) \right] \Pi(d\theta)}{\int_{\Theta} \exp \left[ \Lambda_n(\theta \mid D_n) \right] \Pi(d\theta)},$$

for any measurable $A$ in $\Theta$, where $\Lambda_n$ is the log-likelihood ratio

$$\Lambda_n(\theta \mid D_n) = \sum_{i=1}^{n} \log \frac{p_0(w_i)}{p_{\theta}(w_i)}.$$

An appropriate distance $d$ on $\Theta$ is crucial to study rates of contraction, since typically different distances would yield different rates, and when $\Theta$ is infinite dimensional, different distances are possibly not equivalent. In order that the posterior distribution $\Pi(\cdot \mid D_n)$ contracts to $\theta_0$ at rate $\epsilon_n$ with respect to a distance $d$ on $\Theta$, i.e., $\Pi(d(\theta, \theta_0) > M\epsilon_n \mid D_n) \to 0$ in $P_0$-probability for some large constant $M > 0$, the prior-concentration-and-testing procedure requires that the following hold with some constants $D, D' > 0$ for sufficiently large $n$:

1. The prior concentration condition holds:

$$\Pi(E_0 \left( \log \frac{p_0}{p_{\theta}} \right) \leq \epsilon_n^2, \ E_0 \left[ \left( \log \frac{p_0}{p_{\theta}} \right)^2 \right] \leq \epsilon_n^2) \geq e^{-Dn\epsilon_n^2}. \quad (2.1)$$

2. There exists a sequence $(\Theta_n)_{n=1}^{\infty}$ of subsets of $\Theta$ (often referred to as the sieves) and test functions $(\phi_n)_{n=1}^{\infty}$ such that

$$\Pi(\Theta_n) \leq e^{-(D+4)n\epsilon_n^2}, \ E_0 \phi_n \to 0, \text{ and } \sup_{\theta \in \Theta_n \cap \{d(\theta, \theta_0) > M\epsilon_n\}} E_\theta(1 - \phi_n) \leq e^{-D'Mn\epsilon_n^2}.$$
The major technical barrier for establishing a general framework to study rates of contraction for nonparametric regression with respect to $\| \cdot \|_2$ lies in the construction of suitable aforementioned test functions $(\phi_n)_{n=1}^\infty$. Typically, the existence of suitable test functions can be guaranteed by elaborating on the covering numbers of certain function classes (see, for example, [12, 13, 16, 42] among others), provided that the metrics of interest satisfies the local testability condition (see Section 7 of [12]). Formally, we say that a distance $d : \Theta \times \Theta \to [0, \infty]$ is locally testable, if there exists a sequence of test functions $\phi_n : W^n \to [0, 1]$ such that the following holds for some constant $C > 0$ when $n$ is sufficiently large whenever $\theta_1 \neq \theta_0$:

$$E_0 \phi_n \leq \exp(-Cnd^2(\theta_0, \theta_1)) \quad \text{and} \quad \sup_{d(\theta, \theta_1) \leq \xi} E_\theta (1 - \phi_n) \leq \exp(-Cnd^2(\theta_0, \theta_1)),$$

where $\xi \in (0, 1)$ is a fixed constant. When the parameter space $\Theta$ itself is a class of densities, i.e., $\{p_\theta : \theta \in \Theta\}$ is parametrized by $p$ itself, it is proved that the Hellinger distance is locally testable [23]. The widely-used empirical $L_2$-distance for fixed-design nonparametric regression is also locally testable (see, for example, [14]). The integrated $L_2$-distance is, nonetheless, not locally testable when the space of functions is not uniformly bounded. To resolve this issue, we modify the general procedure above by imposing more assumptions on the sieves.

Since $\mathcal{K}$ is a countable set, we may assume without loss of generality that $\mathcal{K} = \mathbb{N}_+$. For a fixed integer $m$ and a positive constant $\delta$, we denote $\mathcal{F}_m(\delta)$ by a class of functions in $L_2([0, 1]^p)$ that satisfies the following property:

$$\mathcal{F}_m(\delta) \subset \left\{ f(x) = \sum_{k=1}^\infty \beta_k \psi_k(x) : \sum_{k=m+1}^\infty |\beta_k - \beta_{0k}| \leq \delta \right\}. \quad (2.2)$$

The sieves $(\mathcal{F}_n)_{n=1}^\infty$ are constructed by letting $\mathcal{F}_n = \mathcal{F}_{m_n}(\delta)$ for certain sequences $(m_n)_{n=1}^\infty \subset \mathbb{N}_+$ and $(\delta)_{n=1}^\infty \subset (0, \infty)$.

Rather than considering the supremum over $\{f : \|f - f_0\| \leq \xi\|f_1 - f_0\|\}$ as in [12], in the proposed framework we consider the supremum over the smaller function class $\{f : \|f - f_0\| \leq \xi\|f_1 - f_0\|\} \cap \mathcal{F}_m(\delta)$ and obtain the following weaker result analogous to the local testability condition.

**Lemma 2.1.** Let $\mathcal{F}_m(\delta)$ satisfies (2.2). Then for any $f_1 \in \mathcal{F}_m(\delta)$ with $\sqrt{n}\|f_1 - f_0\|_2 > 1$, there exists a test function $\phi_n : (\mathcal{X} \times \mathcal{Y})^n \to [0, 1]$ such that

$$E_0 \phi_n \leq \exp\left(-\frac{Cn\|f_1 - f_0\|_2^2}{m}\right),$$

$$\sup_{\{f \in \mathcal{F}_m(\delta) : \|f - f_1\|_2 \leq \xi\|f_0 - f_1\|_2\}} E_f (1 - \phi_n) \leq \exp\left(-\frac{Cn\|f_1 - f_0\|_2^2}{m}\right) + 2 \exp\left(-\frac{Cn\|f_1 - f_0\|_2^2}{m\|f_1 - f_0\|_2^2 + \delta^2}\right)$$

for some constant $C > 0$ and $\xi \in (0, 1)$. 


Based on Lemma 2.1, we are able to establish the following lemma that tests against the large composite alternative $\mathcal{F}_{m_n}(\delta) \cap \{\|f - f_0\|_2 > M_0\}$. In particular, it is instructive to the construction of aforementioned suitable test functions with respect to the integrated $L_2$-distance $\| \cdot \|_2$.

**Lemma 2.2.** Suppose that $\mathcal{F}_{m_n}(\delta)$ satisfies (2.2) for an $m \in \mathbb{N}_+$ and a $\delta > 0$. Let $(\epsilon_n)_{n=1}^\infty$ be a sequence with $n\epsilon_n^2 \to \infty$. Then there exists a sequence of test functions $(\phi_n)_{n=1}^\infty$ such that

$$E_0 \phi_n \leq \sum_{j=M}^\infty N_{nj} \exp \left(-Cnj^2\epsilon_n^2\right),$$

$$\sup_{(f \in \mathcal{F}_{m_n}(\delta), \|f - f_0\|_2 > M_0)} E_f (1 - \phi_n) \leq \exp(-CM^2n\epsilon_n^2) + 2 \exp\left(-\frac{CM^2n\epsilon_n^2 + \delta^2}{mM^2\epsilon_n^2 + \delta^2}\right),$$

where $N_{nj} = N(\xi_j\epsilon_n, S_{nj}(\epsilon_n), \| \cdot \|_2)$ is the covering number of

$$S_{nj}(\epsilon_n) = \{f \in \mathcal{F}_{m_n}(\delta) : j\epsilon_n < \|f - f_0\|_2 \leq (j + 1)\epsilon_n\},$$

and $C$ is some positive constant.

The prior concentration condition (2.1) is a very important ingredient in the study of Bayes theory. It guarantees that the denominator appearing in the posterior distribution $\int_\Theta \exp[\Lambda_n(\theta | D_n)] \Pi(d\theta)$ can be lower bounded by $e^{-D'nm\epsilon_n^2}$ for some constant $D' > 0$ with large probability (see, for example, Lemma 8.1 in [12]). In the context of normal regression, the Kullback-Leibler divergence is proportional to the integrated $L_2$-distance between two regression functions. Motivated by this observation, we establish the following lemma that yields an exponential lower bound for the denominator $\int \exp(\Lambda_n) \Pi(df)$ under the proposed framework.

**Lemma 2.3.** Denote

$$B(m, \epsilon, \omega) = \left\{ f(x) = \sum_{k=1}^\infty \beta_k \psi_k(x) : \|f - f_0\|_2 < \epsilon, \sum_{k=m+1}^\infty |\beta_k - \beta_0| \leq \omega \right\}$$

for any $\epsilon, \delta > 0$ and $m \in \mathbb{N}_+$. Suppose sequences $(\epsilon_n)_{n=1}^\infty$ and $(k_n)_{n=1}^\infty$ satisfy $\epsilon_n \to 0$, $n\epsilon_n^2 \to \infty$, $k_n\epsilon_n^2 = O(1)$, and $\omega$ is some constant. Then for any constant $C > 0$,

$$P_0 \left(\int \exp(\Lambda_n) \Pi(df) \leq \Pi(B(k_n, \epsilon_n, \omega)) \exp \left[-\left(C + \frac{1}{\sigma^2}\right)n\epsilon_n^2\right]\right) \to 0.$$

In some cases it is also straightforward to consider the prior concentration with respect to the stronger $\| \cdot \|_\infty$-norm. For example, for a wide class of Gaussian process priors, the prior concentration $\Pi(\|f - f_0\|_\infty < \epsilon)$ has been extensively studied (see, for example, [15, 42, 44] for more details).
Lemma 2.4. Suppose the sequence \((\epsilon_n)_{n=1}^\infty\) satisfies \(\epsilon_n \to 0\) and \(n\epsilon_n^2 \to \infty\). Then for any constant \(C > 0\),
\[
\mathbb{P}_0 \left( \int \exp(\Lambda_n) \Pi(df) \leq \Pi(\|f - f_0\|_\infty < \epsilon_n) \exp \left[ - \left( C + \frac{1}{\sigma^2} \right) n\epsilon_n^2 \right] \right) \to 0.
\]

Now we present the main result regarding the rates of contraction for Bayesian non-parametric regression under the orthonormal random series framework. The proof is based on the modification of the prior-concentration-and-testing procedure.

Theorem 2.1 (Generic Contraction). Let \((\epsilon_n)_{n=1}^\infty\) and \((\xi_n)_{n=1}^\infty\) be sequences such that \(\min(n\epsilon_n^2, n\epsilon_n^4) \to \infty\) as \(n \to \infty\), with \(0 \leq \xi_n \leq \epsilon_n \to 0\). Assume that the sieve \((F_{m_n}(\delta))_{n=1}^\infty\) satisfies (2.2) with \(m_n\epsilon_n^2 \to 0\) for some constant \(\delta\). In addition, assume that there exists another sequence \((k_n)_{n=1}^\infty \subset \mathbb{N}_+\) such that \(k_n \xi_n^2 = O(1)\). Suppose the following conditions hold for some constant \(\omega, D > 0\) and sufficiently large \(n\) and \(M\):
\[
\sum_{j=M}^{\infty} N_{nj} \exp \left( -Dn\epsilon_n^2 \right) \to 0,
\]
where \(N_{nj} = N(j\epsilon_n, S_{nj}(\epsilon_n), \|\cdot\|_2)\) is the covering number of \(S_{nj} = \{f \in F_{m_n}(\delta) : j\epsilon_n < \|f - f_0\|_2 \leq (j+1)\epsilon_n\}\), and \(B(k_n, \xi_n, \omega)\) is defined in Lemma 2.3. Then \(E_0 \left[ \Pi(\|f - f_0\|_2 > M\epsilon_n | D_n) \right] \to 0\).

Remark 2.2. In light of Lemma 2.4, by exploiting the proof of theorem 2.1 we remark that when the assumptions and conditions in theorem 2.1 hold with (2.5) replaced by \(\Pi(\|f - f_0\|_\infty < \xi_n) \geq \exp(-Dn\xi_n^2)\), the same rates of contraction also hold: \(E_0 \left[ \Pi(\|f - f_0\|_2 > M\epsilon_n | D_n) \right] \to 0\) for sufficiently large \(M > 0\).

3. Applications

In this section we consider three concrete priors on \(f\) for the nonparametric regression problem \(y_i = f(x_i) + \epsilon_i, i = 1, \cdots, n\). For simplicity the design points are assumed to independently follow the one-dimensional uniform distribution \(\text{Unif}(0, 1)\). For some of the examples, the results presented in this section can be easily generalized to the case where the design points are multi-dimensional by considering the tensor-product Fourier basis. The results for these applications under the proposed framework also generalize their respective counterparts in the literature.
3.1. Finite random series regression with adaptive rate

In this subsection the true regression function \( f_0 \) is assumed to be in the \( \alpha \)-Hölder ball

\[
\mathcal{C}_\alpha(Q) = \left\{ f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x) : \sum_{k=1}^{\infty} k^n |\beta_k| \leq Q \right\},
\]

where \( \alpha > 1/2 \) is the smoothness level, and \( Q > 0 \) is some positive constant. In particular, we do not assume that the smoothness level \( \alpha \) of \( f_0 \) is known \textit{a priori}. In the literature of Bayes theory, such a procedure is referred to as \textit{adaptive}. We shall also assume that a lower bound \( \alpha^* \) for \( \alpha \) is known: \( \alpha \geq \alpha^* > 1/2 \).

The finite random series prior \([1, 30, 35, 45]\) is a popular prior in the literature of Bayesian nonparametric theory. It is a class of hierarchical priors that first draw an integer-valued random variable serving as the number of “terms” to be used in a finite sum, and then sample the “term-wise” parameters given the number of “terms.” The finite random series prior typically does not depend on the smoothness level of the true function, often yields minimax-optimal rates of contraction (up to a logarithmic factor) in many other nonparametric problems (e.g., density estimation \([30, 35]\) and fixed-design regression \([1]\)), and hence is fully adaptive. However, the adaptive rates of contraction for the finite random series prior in the random-design regression with respect to the integrated \( L_2 \)-distance has not been established. In this subsection we address this issue within the orthonormal random series framework proposed in Section 2.

We now formally construct the finite random series prior for nonparametric regression. Let \( f(x) \) be represented by a set of orthonormal basis functions \( (\psi_k)_{k=1}^{\infty} \) in \( L_2([0, 1]) \):

\[
f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x).
\]

The coefficients \((\beta_k)_{k=1}^{\infty}\) are imposed a prior distribution as follows: first sample an integer-valued random variable \( N \) from a density function \( \pi_N(m) \) (with respect to the counting measure on \( \mathbb{N}_+ \)), and then given \( N = m \) the coefficients \( \beta_k \)'s are independently sampled according to

\[
\Pi(d\beta_k \mid N = m) = \left\{ \begin{array}{ll} k^\gamma g(k^\gamma \beta_k) d\beta_k, & \text{if } 1 \leq k \leq m, \\
\delta_0(d\beta_k), & \text{if } k \geq m,
\end{array} \right.
\]

where \( g \) is an exponential power density \( g(x) \propto \exp(-\tau_0 |x|^\gamma) \) for some \( \tau, \tau_0 > 0 \) \([36]\) and \( \gamma \in (1/2, \alpha^*) \). We further require that

\[
\pi_N(m) \geq \exp(-b_0 m \log m) \quad \text{and} \quad \sum_{N=m+1}^{\infty} \pi_N(m) \leq \exp(-b_1 m \log m) \quad (3.1)
\]

for some constants \( b_0, b_1 \). For instance, the zero-truncated Poisson distribution \( \pi_N(m) = (e^\lambda - 1)^{-1}\lambda^m / m! \mathbb{1}(m \geq 1) \) satisfies condition (3.1) \([46]\).

The following theorem shows that the finite random series prior constructed above is adaptive and the rate of contraction \( n^{-\alpha/(2\alpha+1)}(\log n)^{\delta} \) with respect to the integrated \( L_2 \)-distance is minimax-optimal up to a logarithmic factor \([38]\).
Theorem 3.1. Suppose the true regression function $f_0 \in \mathcal{C}_\alpha(Q)$ for some $\alpha > 1/2$ and $Q > 0$, and $f$ is imposed the prior $\Pi$ given above. Then there exists some sufficiently large constant $M > 0$ such that $\mathbb{E}_0 \left[ \Pi \left( \|f - f_0\|_2 > Mn^{-\alpha/(2\alpha + 1)}(\log n)^t \mid D_n \right) \right] \to 0$ for any $t > \alpha/(2\alpha + 1)$.

3.2. Block prior regression with adaptive and exact rate

In the literature of adaptive Bayesian procedure, the minimax-optimal rates of contraction are often obtained with an extra logarithmic factor. It typically requires extra work to obtain the exact minimax-optimal rate. Gao and Zhou [11] elegantly construct a modified block prior that yields rate-adaptive (i.e., the prior does not depend on the smoothness level) and rate-exact contraction for a wide class of nonparametric problems, without the logarithmic factor. Nevertheless, for nonparametric regression, [11] modifies the block prior by conditioning on the space of uniformly bounded functions. Requiring a known upper bound for the unknown $f_0$, when constructing the prior is restrictive since it eliminates the popular Gaussian process priors. Besides the theoretical concern, the block prior itself is also a conditional Gaussian process and such a modification is inconvenient for implementation. In this section, we address this issue by showing that for nonparametric regression such a modification is not necessary. Namely, the un-modified block prior itself also yields the exact minimax-optimal rate of contraction for $f_0$ in the $\alpha$-Sobolev ball

$$\mathcal{H}_\alpha(Q) = \left\{ f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x) : \sum_{k=1}^{\infty} k^{2\alpha} \beta_k^2 \leq Q \right\},$$

and it does not depend on the smoothness level $\alpha$ of the true regression function.

We now describe the block prior. Given a sequence $\beta = (\beta_1, \beta_2, \cdots)$ in the square-summable sequence space $l^2$, define the $\ell$th block $B_\ell$ to be the integer index set $B_\ell = \{k_\ell, \cdots, k_{\ell+1} - 1\}$ and $n_\ell = |B_\ell| = k_{\ell+1} - k_\ell$, where $k_\ell = [e^{\ell}]$. We use $\beta_\ell = (\beta_j : j \in B_\ell) \in \mathbb{R}^{n_\ell}$ to denote the coefficients with index lying in the $\ell$th block $B_\ell$. Let $(\psi_k)_{k=1}^{\infty}$ be the Fourier basis, i.e., $\psi_1(x) = 1$, $\psi_{2k}(x) = \sqrt{2} \sin \pi kx$, and $\psi_{2k+1}(x) = \sqrt{2} \cos \pi kx$, $k \in \mathbb{N}_+$. The block prior is a prior distribution on $f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x)$ induced by the following distribution on the Fourier coefficients $(\hat{\beta}_k)_{k=1}^{\infty}$:

$$\beta_\ell \mid A_\ell \sim \mathcal{N}(0, A_\ell I_{n_\ell}), \quad A_\ell \sim g_\ell, \quad \text{independently for each } \ell,$$

where $(g_\ell)_{\ell=0}^{\infty}$ is a sequence of densities satisfying the following properties:

1. There exists $c_1 > 0$ such that for any $\ell$ and $t \in [e^{-\ell^2}, e^{-\ell}]$,

$$g_\ell(t) \geq \exp(-c_1 e^t). \tag{3.2}$$

2. There exists $c_2 > 0$ such that for any $\ell$,

$$\int_0^{\infty} t g_\ell(t) dt \leq 4 \exp(-c_2 \ell^2). \tag{3.3}$$
A theoretical framework for Bayesian regression

3. There exists $c_3 > 0$ such that for any $\ell$,
\[
\int_{-\infty}^{\infty} g_\ell(t) dt \leq \exp(-c_3 \ell^c).
\] (3.4)

The existence of a sequence of densities $(g_\ell)_{\ell=0}^{\infty}$ satisfying (3.2), (3.3), and (3.4) is verified in [11] (see proposition 2.1 in [11]).

Our major improvement for the block prior regression is the following theorem, which shows that the (un-modified) block prior yields rate-exact Bayesian adaptation for non-parametric regression.

**Theorem 3.2.** Suppose the true regression function $f_0 \in H_\alpha(Q)$ for some $\alpha > 1/2$ and $Q > 0$, and $f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x)$ is imposed the block prior $\Pi$ as described above. Then $E_0 \left[ \|f - f_0\|_2 > M n^{-\alpha/(2\alpha + 1)} \mid D_n \right] \to 0$ for some sufficiently large constant $M > 0$.

Rather than using the sieve $F_n$ proposed in theorem 2.1 in [11], which does not necessarily satisfy (2.2), we construct $F_m(\delta)$ in a slightly different fashion:

\[
F_m(\delta) = \left\{ f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x) : \sum_{k=1}^{\infty} (\beta_k - \beta_{0k})^2 k^{2\alpha} \leq Q^2 \right\}
\]

with $m \approx n^{1/(2\alpha + 1)}$ and $\delta = Q$. The covering number $N_n$ can be bounded using the metric entropy for Sobolev balls (for example, see lemma 6.4 in [3]), and the rest conditions in 2.1 can be verified using similar techniques as in [11].

As discussed in Section 4.2 in [11], the block prior can be easily extended to the wavelet basis functions and wavelet series. The wavelet basis functions are another widely-used class of orthonormal basis functions for $L_2([0, 1])$. Let $(\psi_{jk})_{j \in \mathbb{N}, k \in I_j}$ be an orthonormal basis of compactly supported wavelets for $L_2([0, 1])$, with $j$ referring to the so-called “resolution level,” and $k$ to the “dilation” (see Section E.3 in [15]). We adopt the convention that the index set $I_j$ for the $j$th resolution level runs through $\{0, 1, \cdots, 2^j - 1\}$. The exact definition and specific formulas for the wavelet basis are not of great interest to us, and for a complete and thorough review of wavelets from a statistical perspective, we refer to [17]. We shall assume that the wavelet basis $\psi_{jk}$’s are appropriately selected such that for any $f(x) = \sum_{j=0}^{\infty} \sum_{k \in I_j} \beta_{jk} \psi_{jk}(x)$, the following inequalities hold [6, 7, 19]:

\[
\|f\|_2 \leq \sum_{j=0}^{\infty} \left( \sum_{k \in I_j} \beta_{jk}^2 \right)^{1/2} \quad \text{and} \quad \|f\|_\infty \leq \sum_{j=0}^{\infty} 2^{j/2} \max_{k \in I_j} |\beta_{jk}|.
\]

It is worth noticing that unlike the Fourier basis, the wavelet basis functions are typically not uniformly bounded in $\| \cdot \|_\infty$: $\sup_{k,j} \|\psi_{j,k}\|_\infty = \infty$. However, the supremum norm of the wavelet series can be upper bounded in terms of the wavelet coefficients, which allows us to modify the framework in Section 2 for wavelet basis functions.
Given a function $f$ with wavelet series expansion $f(x) = \sum_{j=0}^{\infty} \sum_{k \in I_j} \beta_{jk} \psi_{jk}(x)$, the block prior for the wavelet series is introduced through the wavelet coefficients $\beta_{jk}$’s as follows:

$$\beta_j | A_j \sim N(0, A_k I_{n_k}), \quad A_j \sim g_j,$$

where $\beta_j = (\beta_{jk} : k \in I_j)$, $n_k = |I_k| = 2^j$, and $g_j$ is given by

$$g_j(t) = \begin{cases} e^{j^2 \log^2 2 (e^{-2^j \log^2 2} - T_j)} t + T_j, & 0 \leq t \leq e^{-j^2 \log^2 2}, \\ e^{-2^j \log^2 2}, & e^{-j^2 \log^2 2} < t \leq e^{-j^2 \log^2 2}, \\ 0, & t > e^{-j^2 \log^2 2}. \end{cases}$$

$$T_j = \exp \left[ (1 + j^2) \log 2 \right] - \exp \left[ (-2^j + j^2 - j) \log 2 \right] + e^{-2^j \log^2 2}.$$

We further assume that $f_0$ lies in the $(2, 2, \alpha)$-Besov ball $\mathbb{B}_2^\alpha(Q)$ defined as follows [11] for some $\alpha > 1/2$ and $Q > 0$:

$$\mathbb{B}_2^\alpha(Q) = \left\{ f(x) = \sum_{j=0}^{\infty} \sum_{k \in I_j} \beta_{j,k} \psi_{j,k}(x) : \sum_{j=0}^{\infty} 2^{2\alpha j} \sum_{k \in I_j} \beta_{j,k}^2 \leq Q^2 \right\},$$

which turns out to be equivalent to the aforementioned $\alpha$-Sobolev ball. For the block prior regression via wavelet series, the rate-exact Bayesian adaptation also holds.

**Theorem 3.3.** Suppose the true regression function $f_0 \in \mathbb{B}_2^\alpha(Q)$ for some $\alpha > 1/2$ and $Q > 0$, and $f(x) = \sum_{j=0}^{\infty} \sum_{k \in I_j} \beta_{j,k} \psi_{j,k}(x)$ is imposed the block prior for wavelet series $\Pi$ as described above. Then there exists some sufficiently large constant $M > 0$ such that $E_0 \left[ \Pi \left( \|f - f_0\|_2 > M n^{-\alpha/(2\alpha + 1)} | D_n \right) \right] \to 0$.

### 3.3. Squared-exponential Gaussian process regression with fixed design

So far, the design points $(x_i)_{i=1}^n$ in this paper for the nonparametric regression problem $y_i = f(x_i) + \epsilon_i$ are assumed to be randomly sampled over $[0, 1]^p$. This is referred to as the random-design regression problem. There are, however, many cases where the design points $(x_i)_{i=1}^n$ are fixed and can be controlled. One of the examples is the design and analysis of computer experiments [8, 34]. To emulate a computer model, the design points are typically manipulated so that they are reasonably spread. In some physical experiments the design points can also be required to be fixed [39].

In this subsection we consider one of the most widely used and perhaps the most popular Gaussian processes ([29]) GP(0, $K$) with the covariance function $K(x, x') = \exp \left[ - (x - x')^2 \right]$ of the squared-exponential form, for the fixed-design nonparametric regression problem. We show that optimal rates of contraction with respect to the integrated $L_2$-distance is also attainable in such a scenario when the design points are
reasonably selected, in contrast to the most Bayesian literatures that obtain rates of contraction with respect to the empirical $L_2$-distance. This can be done by slightly extending the framework in Section 2.

We first present an assumption for the design points. Suppose that the design points $(x_i)_{i=1}^n \subset [0, 1]$ are one-dimensional and are fixed instead randomly sampled. Intuitively, the design points need to be relatively “spread” so that the global behavior of the true signal $f_0$ can be recovered as much as possible. Formally, we require that the design points satisfy [51, 52]

$$\sup_{x \in [0, 1]} \left| \frac{1}{n} \sum_{i=1}^n 1(x_i \leq x) - x \right| = O \left( \frac{1}{n} \right). \quad (3.5)$$

A simple example of such design $(x_i)_{i=1}^n$ is the univariate equidistance design, i.e., $x_i = (i - 1/2)/n$. It clearly satisfies (3.5) (see, for example, [52]).

Now we extend the orthonormal random series framework in Section 2 to the (one-dimensional) fixed-design regression problem with Fourier basis: Assume that $f$ yields Fourier series expansion $f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x)$, where $\psi_1(x) = 1, \psi_2(x) = \sqrt{2} \sin k\pi x$, and $\psi_{2k+1}(x) = \sqrt{2} \cos k\pi x, k \in \mathbb{N}_+$. We also assume that $f_0(x) = \sum_{k=1}^{\infty} \beta_{0k} \psi_k(x)$ lies in the $\alpha$-Hölder space $\mathcal{C}_\alpha(Q)$ with $\alpha > 1$. Consider the sieve

$$\mathcal{F}_{m_n}(\delta) \subset \left\{ f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x) : \sum_{k=m_n+1}^{\infty} |\beta_k - \beta_{0k}|^\alpha \leq \delta \right\} \quad (3.6)$$

with $\alpha > 1$. Notice that the requirement (3.6) is stronger than the condition (2.2), as $\mathcal{F}_{m_n}(\delta)$ contains functions that permit term-by-term differentiation.

With the above ingredients, we are in a position to present the following modification of Theorem 2.1 for the fixed-design regression, which might be of independent interest as well.

**Theorem 3.4 (Generic Contraction, Fixed-design).** Suppose the design points $(x_i)_{i=1}^n$ are fixed and satisfy (3.5). Let $(\epsilon_n)_{n=1}^\infty$ and $(\xi_n)_{n=1}^\infty$ be sequences such that $\min(n\epsilon_n^2, n\xi_n^2) \to \infty$ as $n \to \infty$ with $0 \leq \xi_n \leq \epsilon_n \to 0$. Let the sieves $(\mathcal{F}_{m_n}(\delta))_{n=1}^\infty$ satisfy (3.6) for some constant $\delta > 0$, where $m_n \to \infty$ and $m_n/n \to 0$. Suppose the conditions (2.3), (2.4), and $\Pi(\|f - f_0\|_\infty < \xi_n) \geq \exp(-Dm_n^2)$ hold for some constant $D > 0$ and sufficiently large $n$ and $M$. Then $\mathbb{E}_0 \left[ \Pi(\|f - f_0\|_2 > M\epsilon_n \mid \mathcal{D}_n) \right] \to 0$.

Several fundamental concepts and properties of the squared-exponential Gaussian process are needed in order to apply Theorem 3.4. The eigen-system of the squared-exponential covariance function has been studied in the literature (see, for example, [49]). Under the aforementioned Fourier basis, the covariance function $K$ yields the following eigen-expansion $K(x, x') = \sum_{k=1}^{\infty} \lambda_k \psi_k(x)\psi_k(x')$, where the eigenvalues $(\lambda_k)_{k=1}^\infty$ decay at $\lambda_k \asymp \exp(-k^2/4)$. It follows by the Karhunen-Loève theorem that $f$ yields a series expansion $f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x)$, where $\beta_k \sim N(0, \lambda_k)$. 


Given constant $c, Q > 0$, define the function class 

$$A_c(Q) = \left\{ f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x) : \sum_{k=1}^{\infty} \beta_k^2 \exp \left( \frac{k^2}{c} \right) \leq Q^2 \right\}.$$ 

The function class $A_c(Q)$ is closely related to the reproducing kernel Hilbert space (RKHS) associated with $GP(0, K)$, a concept playing a fundamental role in the literature of Bayes theory. For a complete and thorough review of RKHS from a Bayesian perspective, we refer to [43]. A key feature of the functions in $A_c(Q)$ is that they are “supersmooth,” i.e., they are infinitely differentiable. For the squared-exponential Gaussian process regression, the following property regarding the corresponding RKHS is available by applying theorem 4.1 in [43].

**Lemma 3.1.** Let $H$ be the RKHS associated with the squared-exponential Gaussian process $GP(0, K)$, where $K(x, x') = \exp[-(x - x')^2]$. Then $A_4(Q) \subset H$ for any $Q > 0$.

Under the squared-exponential Gaussian process prior $\Pi$, the rate of contraction of a supersmooth $f_0 \in A_4(Q)$ is $1/\sqrt{n}$ up to a logarithmic factor.

**Theorem 3.5.** Assume that the design points $(x_i)_{i=1}^n$ are fixed and satisfy (3.5). Suppose the true regression function $f_0 \in A_4(Q)$ for some $Q > 0$, and $f$ follows the squared-exponential Gaussian process prior $\Pi$. Then there exists some sufficiently large constant $M > 0$ such that $E_0 \left[ \Pi(\|f - f_0\|_2 > M n^{-1/2} (\log n) \mid D_n) \right] \to 0$.

**Remark 3.1.** For the squared-exponential Gaussian process regression with random design, the rate of contraction with respect to the integrated $L_2$-distance for $f_0 \in A_4(Q)$ has been studied in the literature. In contrast, we remark that for the fixed-design regression problem, Theorem 3.5 is new and original, and provides a stronger result compared to the existing literature (see, for example, Theorem 10 in [40]).

The major technique for proving Theorem 3.4 through Theorem 3.5 is the construction of an appropriate sieve $F_{m_n}(\delta)$. Specifically, the sieve $F_{m_n}(\delta)$ is taken as 

$$F_{m_n}(\delta) = \left\{ f(x) = \sum_{k=1}^{\infty} \beta_k \psi_k(x) : \sum_{k=m_n+1}^{\infty} (\beta_k - \beta_{0k})^2 e^{k^2/8} \leq Q^2 \right\},$$

with $m_n \asymp (\log n)^2$ and $\delta = Q$. The probability $\Pi(F_{m_n}(\delta))$ can be bounded by the Markov’s inequality, and the covering number $N_{n,j}$ is bounded by the following lemma:

**Lemma 3.2.** For all $\epsilon > 0$, it holds that 

$$\log N \left( \epsilon, \left\{ (\beta_1, \beta_2, \cdots) \in l^2 : \sum_{k=1}^{\infty} \beta_k^2 e^{k^2/c} \leq Q^2 \right\}, \| \cdot \|_2 \right\} \lesssim \left( \log \frac{1}{\epsilon} \right)^{3/2}.$$
Remark 3.2. The contraction rate $n^{-1/2}(\log n)$ is the same as the parametric rate $1/\sqrt{n}$ up to a logarithmic factor (see, for example, chapter 10 in [41]). This is in accordance with the following interesting fact: the space of functions that yield analytic extension to an open subset of the complex plane is only slightly larger than finite-dimensional spaces in terms of their covering numbers (see, for example, proposition C.9 in [15]).

4. Extension to sparse additive models in high dimensions

We have so far considered the design space that is low dimensional with fixed $p$. Nonetheless, the rapid development of technology has been enabling scientists to collect high-dimensional data, where the number of covariates $p$ can be much larger than the sample size $n$, to explore the potentially nonlinear relationship between these covariates and certain outcome of interest. It naturally motivates the study of nonparametric regression in high dimensions [50]. In this section, we focus on one class of high-dimensional nonparametric regression problem, known as sparse additive models, and illustrate that with suitable prior specification, the framework for low-dimensional Bayesian nonparametric regression naturally extends to such a high-dimensional scenario.

We first briefly review the sparse additive models. Still consider the nonparametric regression model $y_i = f(x_i) + e_i$, where the design space $\mathcal{X} = [0, 1]^p$ is multidimensional ($p > 1$). For any design point $x \in \mathcal{X}$, we use the unbolded notation $x_j$ to denote the $j$th coordinate of $x$, and similarly, $x_{ij}$ to denote the $j$th coordinate of $x_i$, given $n$ design points $(x_i)_{i=1}^n$ in $\mathcal{X}$. Now assume that the regression function $f(x)$ is of an additive structure: $f(x) = \mu + \sum_{j=1}^p f_j(x_j)$. Without loss of generality, one can assume that each component $f_j(x_j)$ is centered: $\int_0^1 f_j(x_j) dx_j = 0$, $j = 1, \cdots, p$. For sparse additive models in high dimensions, $p$ is typically much larger than $n$, and the underlying true regression function $f_0$ depends only on a small number of covariates, say, $x_{j_1}, \cdots, x_{j_q}$, i.e., $f_0(x) = \mu_0 + \sum_{j=1}^q f_{0,j}(x_{j_j})$, where each $f_{0,j} : [0, 1] \to \mathbb{R}$ is a univariate function, and $q$ is the number of active covariates that does not change with sample size. Furthermore, the indices of these active covariates $\{j_1, \ldots, j_q\}$ and $q$ are unknown. This is referred to as the sparse additive models in high dimensions [18, 21, 25, 27, 28]. There has also been several works regarding Bayesian modeling of sparse additive models in high dimensions, see, for example, [24, 33, 50]. Nonetheless, the posterior contraction with respect to the integrated $L_2$-distance remains unsolved. We address this issue in this section.

To model the sparsity occurring in the high-dimensional additive regression model and meanwhile incorporating the orthonormal series framework in Section 2, we consider the following parameterization of $f$ by expanding each component $f_j$ using Fourier basis functions:

$$f(x) = \mu + \sum_{j=1}^p z_j f_j(x_j), \quad f_j(x_j) = \sum_{k=1}^\infty \beta_{jk} \psi_k(x_j), \quad z_j \in \{0, 1\}, \quad j = 1, \cdots, p, \quad (4.1)$$
where \( z_j = 1 \) indicates that the \( j \)th covariate is active, \( z_j = 0 \) otherwise, and \((\psi_k)_{k=1}^\infty\) are the Fourier basis functions: \( \psi_1(x) = 1 \), \( \psi_2(x) = \sqrt{2}\sin k\pi x \), and \( \psi_{2k+1}(x) = \sqrt{2}\cos k\pi x \), \( k = 1, 2, \ldots \). To ensure that \( \int_0^1 f_j(x)dx = 0 \), we require that

\[
\beta_{j1} = -\sum_{k=2}^\infty \beta_{jk} \int_0^1 \psi_k(x_j)dx_j, \quad j = 1, \ldots, p.
\]

Following the strategy in Section 2, the coefficients \((\beta_{jk})_{k=1}^\infty\) are imposed certain prior distributions such that \( \sum_{k=2}^\infty \beta_{jk}^2 < \infty \) with prior probability one for each \( j = 1, \ldots, p \), and \( \mu \) is imposed a prior distribution \( \pi(\mu) \). We complete the prior distribution \( \Pi \) by imposing the selection variables \( z_j \) with a Bernoulli distribution \( z_j \sim \text{Bernoulli}(1/p) \). The Bernoulli prior for sparsity has been widely adopted in other high-dimensional Bayesian models with variable selection structures (see, for example, \([5, 31, 32]\)).

We now extend Theorem 2.1 to sparse additive models. Assume that \( f_0(x) = \mu_0 + \sum_{j=1}^p f_{0j}(x_j) \), where each \( f_{0j} \) yields a series expansion \( f_{0j}(x_j) = \sum_{k=1}^\infty \beta_{0jk}\psi_k(x_j) \). When \( j \notin \{j_1, \ldots, j_q\} \), this should be interpreted as \( \beta_{0jk} = 0 \) for all \( k = 1, 2, \ldots \). Denote \( z = [z_1, \ldots, z_p]^T \in \{0,1\}^p \). Let \( A \) be an positive integer. In light of the spirit of (2.2), consider the sieve of the form \( G_m(\delta) = \bigcup_{||\mathbf{z}||_1 \leq Aq} G_m(\delta, \mathbf{z}) \), where \( G_m(\delta, \mathbf{z}) \) satisfies

\[
G_m(\delta, \mathbf{z}) \subset \left\{ f(\mathbf{x}) = \mu + \sum_{j=1}^p \sum_{k=1}^\infty z_j \beta_{jk}\psi_k(x_j) : \beta_{j1} = -\sum_{k=2}^\infty \beta_{jk} \int_0^1 \psi_k(x_j)dx_j, \right. \\
\left. \sum_{j=1}^p \sum_{k=\max(m+1)}^\infty |z_j \beta_{jk} - \beta_{0jk}| \leq \delta \right\}
\] (4.2)

**Theorem 4.1** (Generic Contraction, Sparse Additive Models). Consider the aforementioned sparse additive model in high dimensions. Let \((\epsilon_n)_{n=1}^\infty\) and \((\xi_n)_{n=1}^\infty\) be sequences such that \( \min(n\epsilon_n^2, n\xi_n^2) \to \infty \) as \( n \to \infty \), with \( 0 \leq \xi_n \leq \epsilon_n \to 0 \). Assume that there exist sieves of the form \( G_{m_n}(\delta) = \bigcup_{||\mathbf{z}||_1 \leq A_n} G_m(\delta, \mathbf{z}) \), where \( G_m(\delta, \mathbf{z}) \) satisfies (4.2), \((m_n)_{n=1}^\infty\), \((A_n)_{n=1}^\infty\) are sequences such that \( A_n m_n \epsilon_n^2 \to 0 \), and \( \delta \) is some constant. Let \((k_n)_{n=1}^\infty\) be another sequence such that \( k_n \epsilon_n^2 = O(1) \). Suppose the following conditions hold for some constant \( \omega, D > 0 \) and sufficiently large \( n \) and \( M \):

\[
\sum_{j=M}^\infty N_{n_j}^{A_n} \exp \left[ -Dn_j^2 \epsilon_n^2 \right] \to 0,
\] (4.3)

\[
\Pi \left( G_{m_n}(\delta) \right) \lesssim \exp \left[ - \left( 2D + \frac{1}{\sigma^2} \right) n\epsilon_n^2 \right],
\] (4.4)

\[
\Pi \left( B_n(k_n, \xi_n, \omega) \cap \{|\mathbf{z}|_1 \leq 2q\} \right) \geq \exp \left[ -Dn\xi_n^2 \right],
\] (4.5)
Remark 4.1. Yang and Tokdar [50] showed that the minimax rate of convergence with respect to \( \| \cdot \|_2 \) for sparse additive models is \( n^{-\alpha/(2\alpha+1)} + \sqrt{(\log p)/n} \), provided that \( \log p \lesssim n^c \) for some \( c < 1 \). The first term \( n^{-\alpha/(2\alpha+1)} \) is the usual rate for estimating a one-dimensional \( \alpha \)-smooth function, and the second term \( \sqrt{(\log p)/n} \) comes from the
complexity of selecting the $q$ active covariates $x_{j_1}, \cdots, x_{j_q}$ among all $p$ covariates. Under the assumption that $\log p \lesssim \log n$, the minimax rate of convergence is dominated by the first term $n^{-\alpha/(2\alpha+1)}$. Thus Theorem 4.2 states that with the aforementioned finite random series prior for sparse additive model in high dimensions, the rate of contraction is also adaptive and minimax-optimal modulus an logarithmic factor, generalizing the result in Section 3.1.

5. Proofs of the main results

Proof of Lemma 2.1. We first observe the following fact: for any $f(x) = \sum_k \beta_k \psi_k(x) \in \mathcal{F}_m(\delta)$, the following holds:

$$\|f - f_0\|_\infty^2 \lesssim m\|f - f_0\|_2^2 + \delta^2. \quad (5.1)$$

In fact, by the Cauchy-Schwartz inequality,

$$\|f - f_0\|_\infty \lesssim \sum_{k=1}^\infty |\beta_k - \beta_{0k}| \leq \sum_{k=1}^m |\beta_k - \beta_{0k}| + \sum_{k=m+1}^\infty |\beta_k - \beta_{0k}| \leq \sqrt{m}\|f - f_0\|_2 + \delta,$$

and hence, $\|f - f_0\|_\infty^2 \lesssim m\|f - f_0\|_2^2 + \delta^2$. Let us take $\xi = 1/(4\sqrt{2})$. Define the test function to be $\phi_n = 1\{T_n > 0\}$, where

$$T_n = \sum_{i=1}^n y_i(f_1(x_i) - f_0(x_i)) - \frac{1}{2} n\mathbb{P}_n(f_1^2 - f_0^2) - \frac{\sqrt{n}}{8\sqrt{2}} \|f_1 - f_0\|_2 \sqrt{\mathbb{P}_n(f_1 - f_0)^2}.$$

We first consider the type I error probability. Under $\mathbb{P}_0$, we have $y_i = f_0(x_i) + e_i$, where $e_i$'s are i.i.d. $N(0, \sigma^2)$ noises. Therefore, there exists a constant $C_1 > 0$ such that $\mathbb{P}_0(e_i > t) \leq \exp(-4C_1 t^2)$ for all $t > 0$. Then for a sequence $(a_i)_{i=1}^n \in \mathbb{R}^n$, the Chernoff bound yields $\mathbb{P}_0\left(\sum_{i=1}^n a_i e_i \geq t\right) \leq \exp\left(-4C_1 t^2 / \sum_{i=1}^n a_i^2\right)$. Now we set $a_i = f_1(x_i) - f_0(x_i)$ and

$$t = \frac{1}{2} n\mathbb{P}_n(f_1 - f_0)^2 + \frac{\sqrt{n}}{8\sqrt{2}} \|f_1 - f_0\|_2 \sqrt{\mathbb{P}_n(f_1 - f_0)^2}.$$

Clearly,

$$t^2 \geq n\mathbb{P}_n(f_1 - f_0)^2 \left[\frac{1}{4} n\mathbb{P}_n(f_1 - f_0)^2 + \frac{1}{128} n\|f_1 - f_0\|_2^2\right] \geq n\mathbb{P}_n(f_1 - f_0)^2 \left[\frac{1}{128} n\|f_1 - f_0\|_2^2\right].$$

Then under $\mathbb{P}_0(\cdot \mid x_1, \cdots, x_n)$, we have

$$\mathbb{E}_0(\phi_n \mid x_1, \cdots, x_n) \leq \exp\left(-\frac{C_1}{32} n\|f_1 - f_0\|_2^2\right).$$

It follows that the unconditioned error can be bounded:

$$\mathbb{E}_0\phi_n \leq \exp\left(-\frac{C_1}{32} n\|f_1 - f_0\|_2^2\right).$$
We next consider the type II error probability. Under $P_f$, we have $y_i = f(x_i) + e_i$ with $e_i$'s being i.i.d. mean-zero Gaussian. Then for any $f$ with $\|f - f_1\|_2 \leq \|f_0 - f_1\|_2 / (4\sqrt{2}) \leq \|f_0 - f_1\|_2 / 4$, we have

$$E_f(1 - \phi_n) \leq E \left( \mathbb{1}_{P_n(f - f_1)^2 \leq \frac{1}{16} P_n(f_1 - f_0)^2} E_f(1 - \phi_n | x_1, \ldots, x_n) \right) + P \left( P_n(f - f_1)^2 > \frac{1}{16} P_n(f_1 - f_0)^2 \right).$$

When $P_n(f - f_1)^2 \leq \frac{1}{16} P_n(f_1 - f_0)^2$, we have

$$T_n + \frac{\sqrt{n}}{8\sqrt{2}} \|f_1 - f_0\|_2 \sqrt{nP_n(f_1 - f_0)^2} = \sum_{i=1}^n e_i [f_1(x_i) - f_0(x_i)] + nP_n(f - f_1)(f_1 - f_0) + \frac{1}{2} nP_n(f_1 - f_0)^2 \geq \sum_{i=1}^n e_i [f_1(x_i) - f_0(x_i)] + \frac{1}{4} nP_n(f_1 - f_0)^2.$$

Now set

$$R = R(x_1, \ldots, x_n) = \frac{1}{4} nP_n(f_1 - f_0)^2 - \frac{\sqrt{n}}{8\sqrt{2}} \|f_1 - f_0\|_2 \sqrt{nP_n(f_1 - f_0)^2}.$$

Then given $R \geq \sqrt{n}\|f_1 - f_0\|_2 \sqrt{nP_n(f_1 - f_0)^2} / (8\sqrt{2})$, we use the Chernoff bound to obtain

$$P_f(T_n < 0 | x_1, \ldots, x_n) \leq P \left( \sum_{i=1}^n e_i [f_1(x_i) - f_0(x_i)] \leq -R | x_1, \ldots, x_n \right) \leq \exp \left( -\frac{4C_1 R^2}{nP_n(f_1 - f_0)^2} \right) \leq \exp \left( -\frac{C_1 \|f_1 - f_0\|_2^2}{32} \right).$$

On the other hand,

$$P \left( R < \frac{\sqrt{n}}{8\sqrt{2}} \|f_1 - f_0\|_2 \sqrt{nP_n(f_1 - f_0)^2} \right) = P \left( G_n(f_1 - f_0)^2 < -\frac{\sqrt{n}}{2} \|f_1 - f_0\|_2^2 \right).$$
It follows that
\[
E \left[ I \left\{ \mathbb{P}_n(f - f_1)^2 \leq \frac{1}{16} \mathbb{P}_n(f_1 - f_0)^2 \right\} \mathbb{E}_f(1 - \phi_n | x_1, \ldots, x_n) \right] \\
\leq E \left[ I \left\{ R \geq \frac{\sqrt{n}}{8\sqrt{2}} \| f_1 - f_0 \|_2 \sqrt{n \mathbb{P}_n(f_1 - f_0)^2} \right\} \mathbb{P}_n(f - f_1)^2 \leq \frac{1}{16} \mathbb{P}_n(f_1 - f_0)^2 \right\} \mathbb{P}_f(T_n < 0 | x_1, \ldots, x_n) \right] \\
+ \mathbb{P} \left( R < \frac{\sqrt{n}}{8\sqrt{2}} \| f_1 - f_0 \|_2 \sqrt{n \mathbb{P}_n(f_1 - f_0)^2} \right) \\
\leq \exp \left( -\frac{C_1}{32} n \| f_1 - f_0 \|^2_2 \right) + \mathbb{P} \left( \mathcal{G}_n(f_1 - f_0)^2 < -\frac{\sqrt{n}}{2} \| f_1 - f_0 \|^2_2 \right).
\]
Using Bernstein’s inequality (Lemma 19.32 in [41]), we obtain the tail probability of the empirical process \( \mathcal{G}_n(f_1 - f_0)^2 \)
\[
\mathbb{P} \left( \mathbb{P}_n(f_1 - f_0)^2 < -\frac{\sqrt{n}}{2} \| f_1 - f_0 \|^2_2 \right) \leq \exp \left( -\frac{1}{4} E(f_1 - f_0)^4 + \| f_1 - f_0 \|^2_2 \| f_1 - f_0 \|^2_\infty / 2 \right) \\
\leq \exp \left( -\frac{C'n \| f_1 - f_0 \|^2_2}{m \| f_1 - f_0 \|^2_2 + \delta^2} \right),
\]
for some constant \( C' > 0 \), where we use the relation (5.1). On the other hand, when
\[
\mathbb{P}_n(f - f_1)^2 > \mathbb{P}_n(f_1 - f_0)^2 / 16,
\]
we again use Bernstein’s inequality and the fact that \( f \in \{ f \in F_m(\delta) : \| f - f_1 \|^2_2 \leq 2^{-5} \| f_0 - f_1 \|^2_2 \} \) to compute
\[
\mathbb{P} \left( \mathbb{P}_n(f - f_1)^2 > \frac{1}{16} \mathbb{P}_n(f_1 - f_0)^2 \right) \leq \exp \left( -\frac{1}{4} \| g \|^2_2 + \| f_1 - f_0 \|^2_2 \| g \|_\infty / 32 \right),
\]
where \( g = (f - f_1)^2 - (f_1 - f_0)^2 / 16 \). We further compute
\[
\| g \|^2_2 \leq \left( \| (f - f_1)^2 \|^2_2 + \frac{1}{16} \| (f_1 - f_0)^2 \|^2_2 \right)^2 \\
\leq \left( \| f - f_1 \|^\infty \| f - f_1 \|_2 + \frac{1}{16} \| f_1 - f_0 \|^\infty \| f_1 - f_0 \|_2 \right)^2 \\
\leq \| f - f_1 \|^\infty \| f - f_1 \|_2^2 + \| f_1 - f_0 \|^\infty \| f_1 - f_0 \|_2^2 \\
\leq (m \| f_1 - f_0 \|^2_2 + \delta^2) \| f_0 - f_1 \|^2_2,
\]
where we use (5.1), the fact that \( \| f - f_1 \|_2 \leq \| f_0 - f_1 \|_2 \), and that
\[
\| f - f_1 \|^2_\infty \leq 2 \| f - f_0 \|^2_\infty + 2 \| f_0 - f_1 \|^2_\infty \leq m \| f_1 - f_0 \|^2_2 + \delta^2.
\]
Similarly, we obtain on the other hand,
\[
\| g \|_\infty = \| f - f_1 \|^2_\infty + \frac{1}{16} \| f_1 - f_0 \|^2_\infty \leq m \| f_0 - f_1 \|^2_2 + \delta^2.
\]
Therefore, we end up with
\[
P_n(f - f_1)^2 > \frac{1}{16} P_n(f_1 - f_0)^2 \leq \exp \left( -\frac{\tilde{C}_2 f_1 - f_0}{m f_1 - f_0^2 + \delta^2} \right),
\]
where $\tilde{C}_2 > 0$ is some constant. Hence we obtain the following exponential bound for type I and type II error probabilities:
\[
E_0 \phi_n \leq \exp(-C n \| f_1 - f_0 \|^2),
\]
\[
E(1 - \phi_n) \leq \exp(-C n \| f_1 - f_0 \|^2) + 2 \exp \left( -\frac{C n \| f_1 - f_0 \|^2}{m \| f_1 - f_0 \|^2 + \delta^2} \right)
\]
for some constant $C > 0$ whenever $\| f_1 - f_0 \|^2 / 32 \leq \| f_1 - f_0 \|^2 / 32$. Taking the supremum of the type II error over $f \in \{ f \in F_m(\delta) : \| f_1 - f_0 \|^2 \leq \| f_1 - f_0 \|^2 / 32 \}$ completes the proof.

**Proof of Lemma 2.2.** We partition the alternative set into disjoint union
\[
\{ f \in F_m(\delta) : \| f - f_0 \|^2 > M \epsilon_n \}
\subset \bigcup_{j=M}^{\infty} \{ f \in F_m(\delta) : j \epsilon_n < \| f - f_0 \|^2 \leq (j + 1) \epsilon_n \} := \bigcup_{j=M}^{\infty} S_{nj}(\epsilon_n).
\]
For each $S_{nj}(\epsilon_n)$, we can find $N_{nj} = N(\| \cdot \|_2^{\epsilon_n}, S_{nj}(\epsilon_n), \| \cdot \|_2)$-many functions $f_{njl} \in S_{nj}(\epsilon_n)$ such that
\[
S_{nj}(\epsilon_n) \subset \bigcup_{i=1}^{N_{nj}} \{ f \in F_m(\delta) : \| f - f_{njl} \|^2 \leq \xi j \epsilon_n \}.
\]
Since for each $f_{njl}$, we have $f_{njl} \in S_{nj}(\epsilon_n)$, implying that $\| f_{njl} - f_0 \|^2 > j \epsilon_n$, we obtain the final decomposition of the alternative
\[
S_{nj}(\epsilon_n) \subset \bigcup_{i=1}^{N_{nj}} \{ f \in F_m(\delta) : \| f - f_{njl} \|^2 \leq \xi \| f_0 - f_{njl} \|^2 \}.
\]
Now we apply Lemma 2.1 to construct individual test function $\phi_{njl}$ for each $f_{njl}$ satisfying the following property
\[
E_0 \phi_{njl} \leq \exp(-C n j^2 \epsilon_n^2),
\]
\[
\sup_{\{ f \in F_m(\delta) : \| f - f_{njl} \|^2 \leq \xi^2 \| f_0 - f_{njl} \|^2 \}} E_f (1 - \phi_{njl}) \leq \exp(-C n j^2 \epsilon_n^2) + 2 \exp \left( -\frac{C n j^2 \epsilon_n^2}{m j^2 \epsilon_n^2 + \delta^2} \right).
\]
where we have used the fact that $\|f_{njl} - f_0\|_2 > \epsilon_n$. Now define the global test function to be $\phi_n = \sup_{j \geq M} \max_{1 \leq l \leq N_{nj}} \phi_{njl}$. Then the type I error probability can be upper bounded using the union bound

$$E_0 \phi_n \leq \sum_{j=M}^{\infty} \sum_{l=1}^{N_{nj}} E_0 \phi_{njl} \leq \sum_{j=M}^{\infty} \sum_{l=1}^{N_{nj}} \exp(-Cn_j^2 \epsilon_n^2) = \sum_{j=M}^{\infty} N_{nj} \exp(-Cn_j^2 \epsilon_n^2).$$

The type II error probability can also be upper bounded:

$$\sup_{\{f \in \mathcal{F}_m(\delta): \|f - f_0\|_2 > M \epsilon_n\}} E_f(1 - \phi_n) \leq \sup_{j \geq M} \sup_{l=1, \ldots, N_{nj}} \sup_{\{f \in \mathcal{F}_m(\delta): \|f - f_0\|_2 \leq \epsilon_0 - f_{njl}\|_2\}} E_f(1 - \phi_{njl}) \leq \exp(-CM^2 n \epsilon_n^2) + 2 \exp\left(\frac{Cn_j^2 \epsilon_n^2}{M^2 \epsilon_n^2 + \delta^2}\right).$$

The proof is thus completed. \hfill \Box

**Proof of Lemma 2.3.** Denote the re-normalized restriction of $\Pi$ on $B_n = B(k_n, \epsilon_n, \omega)$ to be $\Pi(\cdot \mid B_n)$, and the random variables $(V_{ni})_{i=1}^n, (W_{ni})_{i=1}^n$ to be

$$V_{ni} = f_0(x_i) - \int f(x_i) (\Pi(df \mid B_n), \quad W_{ni} = \frac{1}{2} \int (f(x_i) - f_0(x_i))^2 \Pi(df \mid B_n).$$

Let

$$\mathcal{H}_n := \left\{ \int \exp(\Lambda_n(f \mid \mathcal{D}_n)) \Pi(df) > \Pi(B_n) \exp \left[-\left(C + \frac{1}{\sigma^2}\right) n \epsilon_n^2\right] \right\}.$$

Then by Jensen’s inequality

$$\mathcal{H}_n^c \subset \left\{ \int \exp(\Lambda_n(f \mid \mathcal{D}_n)) \Pi(df \mid B_n) \leq \exp \left[-\left(C + \frac{1}{\sigma^2}\right) n \epsilon_n^2\right] \right\}$$

$$\subset \left\{ \frac{1}{\sigma^2} \sum_{i=1}^n (e_i V_{ni} + W_{ni}) \geq \left(C + \frac{1}{\sigma^2}\right) n \epsilon_n^2 \right\}$$

$$\subset \left\{ \frac{1}{\sigma^2} \sum_{i=1}^n e_i V_{ni} \geq Cn \epsilon_n^2 \right\} \cup \left\{ \sum_{i=1}^n W_{ni} \geq n \epsilon_n^2 \right\}.$$

Now we use the Chernoff bound for Gaussian random variables to obtain the conditional probability bound for the first event given the design points $(x_i)_{i=1}^n$:

$$\mathbb{P}_0 \left( \sum_{i=1}^n e_i V_{ni} \geq Cn \sigma^2 n \epsilon_n^2 \mid x_1, \ldots, x_n \right) \leq \exp \left(-\frac{C^2 \sigma^4 n \epsilon_n^2}{\mathbb{P}_n V_{ni}^2}\right).$$
Since over the function class $B_n$, we have $\|f - f_0\|_2 \leq \epsilon_n$, $\kappa_n \epsilon_n^2 = O(1)$, and
\[
\|f - f_0\|_\infty \lesssim \sum_{k=1}^{\infty} |\beta_k - \beta_{0k}| \leq \sqrt{k_n} \sum_{k=1}^{k_n} (\beta_k - \beta_{0k})^2 + \sum_{k=k_n+1}^{\infty} |\beta_k - \beta_{0k}| = O(1),
\]
it follows from Fubini’s theorem that
\[
\mathbb{E}(V_n^2) \leq \int \|f_0 - f\|_2^2 \Pi(df \mid B_n) \leq \epsilon_n^2,
\]
\[
\mathbb{E}(V_{n}^{2}) \leq \mathbb{E} \left[ \int (f_0(x) - f(x))^4 \Pi(df \mid B_n) \right] \leq \int \|f - f_0\|^4_\infty \|f - f_0\|^2_2 \Pi(df \mid B_n) \lesssim \epsilon_n^2.
\]
Hence by the Chebyshev’s inequality,
\[
\mathbb{P} \left( |P_n V_{n}^2 - \mathbb{E}(V_{n}^2)| > \epsilon_n^2 \right) \leq \frac{1}{n\epsilon_n^4} \text{var}(V_{n}^2) \leq \frac{1}{n\epsilon_n^4} \mathbb{E}(V_{n}^2) \lesssim \frac{1}{n\epsilon_n^2} \rightarrow 0
\]
for any $\epsilon > 0$, i.e., $P_n V_{ni}^2 \leq \mathbb{E}V_{ni}^2 + o_P(\epsilon_n^2) \leq \epsilon_n^2 (1 + o_P(1))$, and hence,
\[
\exp \left( -\frac{C^2 \sigma^4 n \epsilon_n^4}{P_n V_{ni}^2} \right) = \exp \left( -\frac{C^2 \sigma^4 n \epsilon_n^2}{1 + o_P(1)} \right) \rightarrow 0
\]
in probability. Therefore by the dominated convergence theorem the unconditioned probability goes to 0:
\[
P_0 \left( \sum_{i=1}^{n} e_i V_{ni} \geq C \sigma^2 n \epsilon_n^2 \right) = \mathbb{E} \left[ P_0 \left( \sum_{i=1}^{n} e_i V_{ni} \geq C \sigma^2 n \epsilon_n^2 \mid x_1, \ldots, x_n \right) \right]
\leq \mathbb{E} \left[ \exp \left( -\frac{C^2 \sigma^4 n \epsilon_n^4}{P_n V_{ni}^2} \right) \right] \rightarrow 0.
\]
For the second event we use the Bernstein’s inequality. Since
\[
\mathbb{E}W_{ni} = \frac{1}{2} \int \|f - f_0\|^2_2 \Pi(df \mid B_n) \leq \frac{1}{2} \epsilon_n^2,
\]
\[
\mathbb{E}W_{ni}^2 \leq \frac{1}{4} \mathbb{E} \left[ (f(x) - f_0(x))^4 \Pi(df \mid B_n) \right] \leq \frac{1}{4} \int \|f - f_0\|^4_\infty \|f - f_0\|^2_2 \Pi(df \mid B_n) \lesssim \epsilon_n^2,
\]
then
\[
P \left( \sum_{i=1}^{n} W_{ni} > n \epsilon_n^2 \right) \leq P \left( \sum_{i=1}^{n} (W_{ni} - \mathbb{E}W_{ni}) > \frac{1}{2} n \epsilon_n^2 \right) \leq \exp \left( -\frac{1}{4 \mathbb{E}W_{ni}^2 + \epsilon_n^2 \|W_{ni}\|_\infty / 2} \right)
\leq \exp \left( -\frac{\tilde{C} n \epsilon_n^2}{1 + \|W_{ni}\|_\infty} \right),
\]
where \( \|W_{ni}\|_\infty = \sup_{x \in [0,1]^p} (1/2) \int (f(x) - f_0(x))^2 \Pi(df \mid B_n). \) Since for any \( f \in B_n, \|f - f_0\|_\infty = O(1), \) it follows that \( \|W_{ni}\|_\infty = O(1), \) and hence, \( \mathbb{P}(\sum_i W_{ni} > n\epsilon_n^2) \to 0. \)

To sum up, we conclude that

\[
\mathbb{P}(\mathcal{H}_n^c) \leq \mathbb{P}_0 \left( \sum_{i=1}^n e_i V_{ni} \geq C\sigma^2 n\epsilon_n^2 \right) + \mathbb{P} \left( \sum_{i=1}^n W_{ni} > n\epsilon_n^2 \right) \to 0.
\]

\[\square\]

**Proof of lemma 2.4.** Denote \( \Pi(\cdot \mid B_n) = \Pi(\cdot \cap B_n) / \Pi(B_n) \) to be the re-normalized restriction of \( \Pi \) on \( B_n = \{ \|f - f_0\|_\infty < \epsilon \} \), and

\[
V_{ni} = f_0(x_i) - \int f(x_i) \Pi(df \mid B_n), \quad W_{ni} = \frac{1}{2} \int (f(x_i) - f_0(x_i))^2 \Pi(df \mid B_n).
\]

Similar to the proof of lemma 2.3, we obtain

\[
\mathcal{H}_n^c = \left\{ \int \exp(\Lambda_n(f \mid D_n))\Pi(df) \leq \Pi(B_n) \exp \left[ - \left( C + \frac{1}{\sigma^2} \right) n\epsilon_n^2 \right] \right\}
\subset \left\{ \frac{1}{\sigma^2} \sum_{i=1}^n (e_i V_{ni} + W_{ni}) \geq \left( C + \frac{1}{\sigma^2} \right) n\epsilon_n^2 \right\} \subset \left\{ \frac{1}{\sigma^2} \sum_{i=1}^n e_i V_{ni} \geq \left( C + \frac{1}{\sigma^2} \right) n\epsilon_n^2 \right\},
\]

where we use the fact that \( W_{ni} \leq (1/2) \|f - f_0\|_\infty^2 \Pi(df \mid B_n) \leq \epsilon_n^2/2 \) for all \( f \in B_n \) in the last step. Conditioning on the design points \( (x_i)_{i=1}^n \), we have

\[
\mathbb{P}_0(\mathcal{H}_n^c \mid x_1, \cdots, x_n) \leq \exp \left[ - \left( C + \frac{1}{\sigma^2} \right)^2 \frac{\sigma^4 n\epsilon_n^4}{\mathbb{P}_0 V_{ni}^2} \right] \leq \exp \left[ - \left( C + \frac{1}{\sigma^2} \right)^2 \frac{\sigma^4 n\epsilon_n^4}{\mathbb{P}_0 V_{ni}^2} \right] \leq \exp \left[ - \left( C + \frac{1}{\sigma^2} \right)^2 \frac{\sigma^4 n\epsilon_n^2}{\mathbb{P}_0 V_{ni}^2} \right] \to 0.
\]

The proof is completed by applying the dominated convergence theorem. \[\square\]

**Proof of Theorem 2.1.** Let \( \phi_n \) be the test function given by lemma 2.2 and

\[
\mathcal{H}_n = \left\{ \int \exp(\Lambda_n(f \mid D_n))\Pi(df) \geq \exp \left[ - \left( \frac{3D}{2} + \frac{1}{\sigma^2} \right) n\omega^2 \right] \right\}.
\]

It follows from condition (2.5) that

\[
\mathcal{H}_n^c \subset \left\{ \int \exp(\Lambda_n)\Pi(df) < \Pi(B_n(k_n, \omega, \omega)) \exp \left[ - \left( \frac{D}{2} + \frac{1}{\sigma^2} \right) n\omega^2 \right] \right\}.
\]
and hence, \( \mathbb{P}_0(\mathcal{H}_n^c) = o(1) \) by lemma 2.3. Now we decompose the expected value of the posterior probability

\[
\mathbb{E}_0 \left[ \mathbb{P}(f \in \mathcal{D}_n \mid f \notin \mathcal{D}_n) \right] \\
\leq \mathbb{E}_0 \left[ (1 - \phi_n) \mathbb{P}(\mathcal{H}_n \mid f \notin \mathcal{D}_n) \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
\leq \mathbb{E}_0 \left[ (1 - \phi_n) \mathbb{P}(\mathcal{H}_n \mid f \notin \mathcal{D}_n) \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
\leq \mathbb{E}_0 \left[ (1 - \phi_n) \mathbb{P}(\mathcal{H}_n \mid f \notin \mathcal{D}_n) \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right].
\]

By (2.3) and lemma 2.2 the type I error probability \( \mathbb{E}_0 \phi_n \to 0 \). It suffices to bound the first term. Observe that on the event \( \mathcal{H}_n \), the denominator in the square bracket can be lower bounded:

\[
\mathbb{E}_0 \left[ (1 - \phi_n) \mathbb{P}(\mathcal{H}_n \mid f \notin \mathcal{D}_n) \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
\leq \exp \left( \frac{3D}{2} + \frac{1}{\sigma^2} \right) \mathbb{E}_0 \left[ (1 - \phi_n) \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
+ \exp \left( \frac{3D}{2} + \frac{1}{\sigma^2} \right) \mathbb{E}_0 \left[ \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right].
\]

By Fubini’s theorem, lemma 2.2 we have

\[
\mathbb{E}_0 \left[ (1 - \phi_n) \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
\leq \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0 \left[ \sup_{f \in \mathcal{F}_m(\delta)} \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
\leq \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0 \left[ \frac{CM^2 n\epsilon_n^2}{m_n \epsilon_n^2 + \delta} \right] \\
\leq \exp(-\bar{C} M^2 n\epsilon_n^2).
\]

for some constant \( \bar{C} > 0 \) for sufficiently large \( n \), since \( m_n \epsilon_n^2 \to 0 \) and \( \delta = O(1) \) by assumption. For the integral on \( \mathcal{F}_m(\delta) \), we apply Fubini’s theorem to obtain

\[
\mathbb{E}_0 \left[ \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] = \int_{\mathcal{F}_m(\delta)} \mathbb{E}_0 \left[ \prod_{i=1}^{n} \frac{p_f(X_i, Y_i)}{p_0(X_i, Y_i)} \right] \Pi(df) \leq \Pi(\mathcal{F}_m(\delta)).
\]

Hence we proceed to compute

\[
\mathbb{E}_0 \left[ \left( \frac{3D}{2} + \frac{1}{\sigma^2} \right) \mathbb{E}_0(\phi_n + \mathbb{E}_0((1 - \phi_n) \mathbb{P}(\mathcal{H}_n^c))) \right] \\
\leq \exp \left( \frac{3D}{2} + \frac{1}{\sigma^2} \right) n\epsilon_n^2 - \bar{C} M^2 n\epsilon_n^2 \\
+ \exp \left( \frac{3D}{2} + \frac{1}{\sigma^2} \right) n\epsilon_n^2 - \left( 2D + \frac{1}{\sigma^2} \right) n\epsilon_n^2 \to 0
\]

\[
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\]
as long as $M$ is sufficiently large, where (2.4) is applied.

**Supplementary Material**

Supplement to “A theoretical framework for Bayesian nonparametric regression: orthonormal random series and rates of contraction” (). The supplementary material contains the remaining proofs and additional technical results.

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