REGULAR AND EFFECTIVE REGULAR CATEGORIES OF LOCALES

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Abstract. We examine the analogues for the respective categories of locales of two well-known results about regularity and effectiveness of some categories of spaces. We show that the category of compact regular locales is effective regular (=Barr-exact). We also show that the category of compactly generated Hausdorff locales is regular, provided that it is coreflective within Hausdorff locales. We do not appeal to the existence of points (which would render the first of the two results trivial) but rely on the treatment of the subject by methods that are valid in the internal logic of a topos. In doing that we examine some questions about the tensor product of sup-lattices, in particular about the interplay of the tensor product with epimorphic inverse directed limits of sup-lattices. This in turn relies on the 2-categorical nature of the tensor product of sup-lattices investigated by Kenney and Wood.

1. Introduction

While regular and, even more so, effective regular categories occur more frequently in the realm of algebra there are two well-known cases of categories of spaces that have these features. The category of compact Hausdorff spaces is effective regular and the category of compactly generated (weakly) Hausdorff spaces is regular [Cagliari, Mantovani, Vitale 1995]. It is a rather natural question to ask whether the corresponding categories of locales maintain these features.

For the case of compact Hausdorff locales we know from [Townsend 1998] that they form a regular category. We show here that it is also effective. The extra step, effectiveness of equivalence relations, almost exists implicitly in the work of [Vermeulen 1994] on proper maps of locales, in particular his result that proper closed equivalence relations on compact locales are effective.

The situation concerning compactly generated Hausdorff locales is much more complicated. First of all we adopt the definition of compactly generated locales introduced in [Escardo 2006], which constitutes the main, if not the only, study of such locales: a Hausdorff locale is compactly generated if it is isomorphic to the colimit of the (directed, extremal monomorphic) diagram of its compact sublocales (via the canonical comparison as a co-cone for that diagram). The major question that is left open in that work is whether such locales form a coreflective subcategory of that of Hausdorff locales. This would be the case if, for every Hausdorff locale, the canonical comparison described above...
were monomorphic, in which case the colimit in question would be Hausdorff and the comparison map the co-unit of the adjunction. The question of coreflectivity is important for the way products (and hence also pullbacks) are calculated in that category, namely whether they are calculated by applying coreflection to the localic product. This in turn affects our strategy for approaching the question of regularity of compactly generated Hausdorff locales. For that we adapt the argument due to [Day and Street 1989] for deriving regularity of the inductive completion of a category from the regularity of the given category. The argument is familiar in the theory of locally presentable categories but its essential ingredients do not require local presentability. One key step is the “uniformity lemma”, namely that if objects (like the compactly generated locales) are built up from building blocks (like their compact sublocales), then the vertices of a finite diagram of such can be expressed as colimits of the building blocks over the same indexing category. This uses only the density of the building blocks and their closure under finite colimits in the broader category. The other step has to do with the existence of regular epi-mono factorizations and the stability of regular epis under pullback. For that only the commutation of pullbacks with a particular type of colimits is needed (directed extremal monomorphic ones, in the case at hand). This is where the nature of the products plays a role. If we assume coreflectivity we arrive at that commutation result and subsequently at the regularity of compactly generated Hausdorff locales.

The desired commutation described above hinges on that of products with directed extremal monomorphic colimits. When stated as a question about frames the result seems to fail in general but the canonical morphism from the coproduct of a frame with an inverse directed epimorphic limit to the limit of the coproducts with the factors is surjective. This holds more generally as a result about tensor products and inverse directed epimorphic limits of sup-lattices. We arrive at that exploiting the description and, more importantly, the 2-categorical nature of the tensor product of sup-lattices given in [Kenney and Wood 2010].

Our terminology is, we believe, standard. A map of locales \( f: X \to Y \) is determined by a map \( f^*: OY \to OX \) between the respective frames that preserves finite infima and all suprema. Hence it has a right adjoint \( f_* \vdash f^* \). The map is a surjection if \( f^* \) reflects order. It is proper if \( f_* \) preserves directed suprema and, for all \( U \in OX, V \in OY, f_*(U \lor f^*V) = f_*U \lor V \). Under the equivalence of the category of locales over \( X \) with that of locales internal in sheaves on \( X \), proper maps in the former correspond to compact locales in the latter [Johnstone 2002]. A locale \( X \) is Hausdorff if its diagonal \( X \to X \times X \) is closed.

2. Tensor product and inverse filtered limits of sup-lattices

**Funtorial character of tensor product:** We review here the main points of [Kenney and Wood 2010] that are used for the development of the necessary properties of the tensor product of sup-lattices. We rely on the description of the tensor product of sup-lattices given there. It has the advantage that, rather than being presented in terms of generators and relations,
the tensor product of two sup-lattices is given as a certain subset of the set of downwards closed subsets (downsets, for short) of their product. In particular, denoting by $DM$ the set of downsets of $M$ ordered by inclusion,

$$M \otimes N = \{W \in D(M \times N) \mid \forall S \in DM \forall T \in DN (S \times T \subseteq W \Rightarrow (\bigvee S, \bigvee T) \in W)\}.$$  

This renders calculating with elements of the tensor product easier. In [Kenney and Wood 2010] it is not discussed how the functorial character of the tensor product can be recast in such terms. However the detailed analysis provided by the authors of the 2-dimensional nature of the tensor product in the category of partially ordered sets leads directly to the answer. Recall that the above description of the tensor product is derived by the fact that it occurs as an inverter in the 2-category of posets

$$M \otimes N \rightarrow D(M \times N) \downarrow D(DM \times DN)$$

of a 2-arrow (inequality) between the upper and lower parallel arrows. In order to be more precise, recall first that the downset formation defines a 2-functor on posets, sending an order-preserving map $f: X \rightarrow Y$ to $Df: DX \rightarrow DY$ with

$$Df(S) = \downarrow f[S] = \{y \in Y \mid \exists x \in S \ y \leq f(x)\}.$$  

The upper arrow here is $D(\downarrow_M \times \downarrow_N)$ for $\downarrow_M \times \downarrow_N: M \times N \rightarrow DM \times DN$ the map induced by the inclusion of a poset into the respective set of downsets via down-segment. The lower arrow is also induced by taking $f = \downarrow_M \times \downarrow_N$ and applying the construction that sends $S \in DX$ to $\{y \in Y \mid \forall x \in X (f x \leq y \Rightarrow x \in S)\}$ The inclusion

$$\{y \in Y \mid \exists x \in S \ y \leq f(x)\} \subseteq \{y \in Y \mid \forall x \in X (f x \leq y \Rightarrow x \in S)\}$$

is obvious when $f$ is order-reflecting, as it is the case with $\downarrow_M \times \downarrow_N$.

The key result for accounting in such terms for the functorial character of the tensor product, i.e the action of $f \otimes g: M \otimes N \rightarrow M' \otimes N'$ induced by some $f: M \rightarrow M'$ and $g: N \rightarrow N'$, is the following ([Kenney and Wood 2010], Lemma 3.3): Given a 2-arrow $g \leq f: A \rightarrow X$ in the category of sup-lattices with $f \dashv \varphi$, $g \dashv \gamma$ in the 2-category of posets and an inverter diagram in posets

$$I \xrightarrow{\kappa} X \xleftarrow{\varphi} A$$

then the fully faithful $\kappa: I \rightarrow A$ has a left adjoint which provides the co-inverter of the diagram of adjoints, i.e
with \( k \vdash \kappa \) is a co-inverter diagram and for each \( h: X \to J \) that co-inverts \( g \leq f \), the induced by the universal property of the co-inverter unique \( l: I \to J \) with \( l \cdot k = h \) is given as \( h \cdot \kappa \).  

2.1. Proposition. The map \( f \otimes g: M \otimes N \to M' \otimes N' \) induced by some \( f: M \to M' \) and \( g: N \to N' \) in the category of sup-lattices, acts on any \( S \in M \otimes N \) as  

\[
(f \otimes g)(S) = (D(f \times g)(S))^\#, 
\]

where \((-)^\#\) denotes the action of the reflection to the inclusion \( \kappa': M' \otimes N' \to D(M' \times N') \).  

Proof. Following the above discussion, in the diagram (where the two rows are inverter diagrams)  

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{\kappa} & D(M \times N) \xrightarrow{D(f \times g)} D(DM \times DN) \\
\downarrow D(f \times g) & \downarrow D(f \times g) & \downarrow D(f \times g) \\
M' \otimes N' & \xrightarrow{\kappa'} & D(M' \times N') \xrightarrow{D(Df \times Dg)} D(DM' \times DN')
\end{array}
\]

an arrow from \( M \otimes N \) to \( M' \otimes N' \) will be induced by the universal property of the upper row below  

\[
\begin{array}{ccc}
M \otimes N & \xrightarrow{k} & D(M \times N) \xleftarrow{D(f \times g)} D(DM \times DN) \\
\downarrow D(f \times g) & \downarrow D(f \times g) & \downarrow D(f \times g) \\
M' \otimes N' & \xrightarrow{k'} & D(M' \times N') \xleftarrow{D(Df \times Dg)} D(DM' \times DN')
\end{array}
\]

as a co-inverter diagram, provided that the two squares to the right are commutative. Moreover, in that case, the induced arrow will be given as  

\[
(f \otimes g)(S) = (D(f \times g)(S))^\#, 
\]

where \((-)^\#\) denotes the action of the reflection to the inclusion \( \kappa': M' \otimes N' \to D(M' \times N') \).  

Now coming to the commutativity of the two squares, in order to describe the action of the parallel maps, note first that we have natural isomorphisms \( DM \otimes DN \cong D(M \times N) \) and \( D^2M \otimes D^2N \cong D(DM \times DN) \) induced by the bi-sup-preserving map \( DM \times DN \to D(M \times N) \) sending \( (S, T) \) to \( S \times T \) ([Kenney and Wood 2010], Lemma 3.2). As explained in loc. cit. \( M \otimes N \) occurs as a co-inverter of  

\[
\bigcup_{M} \otimes \bigcup_{N} \leq D \bigvee_{M} \otimes D \bigvee_{N}: D^2M \otimes D^2N \to DM \otimes DN
\]
and via the isomorphism, eventually, \( M \otimes N \) occurs as the co-inverter of \( \bigcup(- \times -) \leq D(M \times N) \):

\[
\bigcup(- \times -) \leq D(M \times N)
\]

The commutativity of the upper square then follows by functoriality of \( D \) in combination with the fact that \( f \) and \( g \) are sup-preserving. Finally, inspecting the lower square we find that a downset of pairs of downsets \( \{(S_\alpha, T_\alpha) \mid \alpha \in A\} \in D(M \times N) \) is sent by the lower composite to

\[
\bigcup\{U_\alpha \times V_{\alpha'} \mid U_\alpha \subseteq Df(S_\alpha), V_{\alpha'} \subseteq Dg(T_{\alpha'})\}
\]

while by the upper composite to

\[
\{W \in D(M \times N) \mid W \subseteq \bigcup_{(\alpha, \alpha')} Df(S_\alpha) \times Dg(T_{\alpha'})\}
\]

which are both equal to

\[
\{(m', n') \in M' \times N' \mid \exists \alpha \in A \exists \alpha' \in A \exists s \in S_\alpha \exists t \in T_{\alpha'} (m', n') \leq (fs, gt)\}.
\]

We close the discussion of the functorial behaviour of the tensor product of sup-lattices by noting that, not only is the inverse image of a downset under an order-preserving map a downset, but moreover

2.2. Proposition. Given \( f: M \rightarrow M' \) and \( g: N \rightarrow N' \) in the category of sup-lattices and \( W \in M' \otimes N' \), then \( (f \times g)^{-1}[W] \in M \otimes N \), hence when \( f \) and \( g \) are surjective \( W = (f \otimes g)((f \times g)^{-1}[W]) \).

Proof. We show that \( (f \times g)^{-1}[W] \) has the defining property of the elements of \( M \otimes N \) as a subset of \( D(M \times N) \). Let \( (S, T) \in D(M \times DN) \) be such that \( S \times T \subseteq (f \times g)^{-1}[W] \). Then \( f[S] \times g[T] = (f \times g)[S \times T] \subseteq W \) and since \( f, g \) are sup-preserving

\[
(f \times g)(\bigvee S, \bigvee T) = (f(\bigvee S), g(\bigvee T)) = (\bigvee f[S], \bigvee g[T]) \in W
\]

by the defining property of \( W \).

For the second claim, \( (f \otimes g)((f \times g)^{-1}[W]) = ((f \times g)((f \times g)^{-1}[W]))^\# = W^\# = W \), where the second equation holds by surjectivity of \( f \times g \) and the third by the fact that \( W \) is already in the reflective sub-poset of \( D(M' \times N') \).
Interplay of tensor product with inverse directed limits: The following appears in the proof of [Joyal and Tierney 1984] Proposition I.2.

2.3. Lemma. Let \((t_{ij}: A_i \to A_j)\) be an inverse directed diagram in the category of sup-lattices, such that all the transition maps \(t_{ij}\) are surjective. Then the projections \(p_i: \lim_i A_i \to A_i\) are also surjective.

Proof. Given an element \(a_i \in A_i\) and denoting by \(t^*_{ij}\) the right adjoint (for notational convenience, contrary to the customary use of upper star to denote a left adjoint) of the suprema-preserving \(t_{ij}\), construct a compatible family \((a_i)_{i \in I} \in \lim_i A_i\) by setting

\[
a_j = t_{kj}t_{ki}^*(a_i)
\]

where \(A_k\) is mapping to \(A_i\) and \(A_j\) because of directedness. The definition is independent of any particular \(A_k\) since if \(A_k, A_m\) were mapping to \(A_i\) and \(A_j\) then there would be a further \(A_l\) mapping to \(A_k\) and \(A_m\) and then, taking into account that for surjective \(t_{ij}\) with \(t_{ij} \dashv t^*_{ij}\) we have \(x = t_{ij}t^*_{ij}(x)\),

\[
t_{mj}t_{mi}^*(a_i) = t_{mj}t_{lim}t_{im}^*t_{mi}^*(a_i) = t_{kj}t_{lk}t_{kl}^*(a_i) = t_{kj}t_{kl}^*(a_i).
\]

Notice that the definition gives \(a_j = t_{ij}(a_i)\) when \(j \geq i\) and \(a_j = t^*_{ji}(a_i)\) when \(j \leq i\). This way we have indeed defined a compatible family because for \(j \leq i\), \(a_i = t_{ji}t_{ji}^*(a_i)\).

\[\square\]

2.4. Proposition. Let \((t_{ij}: A_i \to A_j)\) be an inverse directed diagram in the category of sup-lattices, such that all the transition maps \(t_{ij}\) are surjective. Let \((U_i)_{i \in I}\) be a compatible family of down-sets, in the sense that for all transitions \(t_{ij}, \downarrow t_{ij}[U_i] = U_j\) (which means that \((U_i)_{i \in I}\) is an element of \(\lim_i D(A_i)\)). Then for any two projections \(p_i: \lim_i A_i \to A_i, p_j: \lim_i A_i \to A_j\) we have that \(p_i^{-1}[U_i] = p_j^{-1}[U_j]\).

Proof. Clearly it suffices to show that the equality holds for \(i \leq j\).

We show first the inclusion \(\subseteq\): Let \((a_i)_{i \in I} \in p_i^{-1}[U_i]\), meaning in particular that \(a_i \in U_i\). We set \(u_j = t_{ij}(a_i) \in t_{ij}[U_i] \subseteq t_{ij}[U_i] = U_j\). Let \(\hat{u}\) denote the compatible family constructed in the previous Lemma so that \(\hat{u}_j = u_j\). We want to show that \((a_i)_{i \in I} \leq \hat{u}\).

In the special case of a \(k\) such that \(j \leq k\) it is obvious from the definition of \(\hat{u}\) that \(a_k = t_{jk}(a_j) = t_{jk}(t_{ij}(a_i)) = t_{jk}(u_j) = \hat{u}_k\).

In the special case where \(k \leq j\) we want to have \(a_k \leq t_{kj}^*(u_j) = \hat{u}_k\). This is equivalent to \(a_j = t_{kj}(a_k) \leq u_j\), which holds as equality, by the definition of \(u_j\).

Finally, for a general \(k\), considering an \(l\) such that \(A_l\) maps to both \(A_j\) and \(A_k\), we have

\[
\hat{u}_k = t_{lk}t_{ij}^*(u_j) = t_{lk}(\hat{u}_l) \geq t_{lk}(a_l) = a_k,
\]

where we have used that \(\hat{u}_l \geq a_l\) for \(l \leq j\).
For the converse inclusion, consider \((a_i)_{i \in I} \in p^{-1}_j[U_j]\), hence such that \(a_j \in U_j\). Set \(u_i = t_{i}^*(a_j)\) and consider again the compatible family \(\hat{u}\) with \(\hat{u}_i = u_i\), provided by the previous Lemma, so that \(\hat{u} \in p^{-1}_i[U_i]\). We want to show that \((a_i)_{i \in I} \leq \hat{u}\).

In the special case where \(i \leq k\), \(\hat{u}_k = t_{ik}(u_i) = t_{ik}t_{ij}^*(a_j)\), so we want to have \(a_k \leq t_{ik}t_{ij}^*(a_j)\). Indeed, \(t_{ij}^*(a_j) = t_{ij}^*t_{ij}(a_i) \geq a_i\) by adjunction, hence \(t_{ik}t_{ij}^*(a_j) \geq t_{ik}(a_i) = a_k\).

In the special case where \(k \leq i\) we want to show that \(\hat{u}_k = t_{ki}^*(u_i) \geq a_k\). But

\[ \hat{u}_k = t_{ki}^*(u_i) = t_{ki}^*t_{ij}^*(\hat{u}_j) = t_{kj}^*(\hat{u}_j), \]

hence the inequality is equivalent to \(a_j = t_{kj}(a_k) \leq \hat{u}_j\), which is known to hold by the previous step since \(i \leq j\).

Finally for a general \(k\), considering a step \(l\) that precedes both in the poset \(I\) we have that \(\hat{u}_k = t_{ik}(\hat{u}_i)\) and since \(l \leq i\) the previous case gives that \(a_l \leq \hat{u}_i\), hence \(a_k = t_{ik}(a_l) \leq t_{ik}(\hat{u}_i) = \hat{u}_k\).

\[\Box\]

2.5. **Corollary.** If \((t_{ij}: A_i \to A_j)\) is an inverse directed diagram in the category of sup-lattices, such that all the transition maps \(t_{ij}\) are surjective, then the induced map \((Dp_i)_{i \in I}: D(\lim_i A_i) \to \lim_i D(A_i)\) is surjective.

**Proof.** An element of \(\lim_i D(A_i)\) amounts to a compatible family \((U_i)_{i \in I}\) as in the Proposition, hence setting \(W = p^{-1}_i[U_i]\) we get a downset of \(\lim_i A_i\) whose image under \(Dp_i = \downarrow p_i[-]\) is \(U_i\), for all \(i \in I\), by surjectivity of each \(p_i\).

\[\Box\]

2.6. **Theorem.** Let \((t_{ij}: A_i \to A_j)\) be an inverse directed diagram in the category of sup-lattices, such that all the transition maps \(t_{ij}\) are surjective and \(B\) any sup-lattice. Then the canonical map

\[(p_i \otimes id_B)_{i \in I}: (\lim_i A_i) \otimes B \to \lim_i (A_i \otimes B)\]

is surjective.

**Proof.** By the previous Corollary the induced map

\[ D(\lim_i A_i \times B) \cong D(\lim_i (A_i \times B)) \to \lim_i D(A_i \times B) \]

is surjective so an element in \(\lim_i (A_i \otimes B)\), seen as an element of \(\lim_i D(A_i \times B)\) is in the image of \((D(p_i \times id_B))_{i \in I}\). The element that maps to it is already in \((\lim_i A_i) \otimes B\) by Proposition 2.2.

\[\Box\]

3. **Some exactness properties of categories of locales**

It is obvious that a limit of a diagram \(\mathcal{I} \to \text{Frm}\) in the category of frames is calculated by considering the limit of the diagram \(\mathcal{I} \to \text{Frm} \to \text{SupLat}\) in the category of sup-lattices and endowing it with a frame structure componentwise, while the coproduct of frames is given by their tensor product as sup-lattices [Joyal and Tierney 1984]. Hence translating the above theorem for the dual of category of frames and taking into account that surjective maps of frames correspond to extremal monomorphisms of locales, we obtain the
3.1. **Theorem.** Let \((t_{ij}: X_i \rightarrow X_j)\) be a directed diagram of inclusions in the category of locales and \(Y\) any locale. Then the canonical map

\[
\text{colim}_i(X_i \times Y) \rightarrow (\text{colim}_i X_i) \times Y
\]

is an inclusion.

Since the arguments presented so far are valid in the internal logic of a topos, exploiting the well-known equivalence

\[
\text{Loc}/X \simeq \text{Loc}(\text{Shv}X)
\]

we get

3.2. **Corollary.** Let \((t_{ij}: X_i \rightarrow X_j)\) be an directed diagram of inclusions in the category of locales over a locale \(Z\) and \(Y \rightarrow Z\) any map of locales. Then the canonical map

\[
\text{colim}_i(X_i \times_Z Y) \rightarrow (\text{colim}_i X_i) \times_Z Y
\]

is an inclusion.

Let us recall from [Escardo 2006] that a Hausdorff locale (i.e one whose diagonal is closed) is called compactly generated if the canonical comparison map

\[
\varepsilon_X: \text{colim} C_i \rightarrow X,
\]

where the colimit is taken over all the compact sublocales of \(X\) (hence is a directed diagram of inclusions), is an isomorphism. For an arbitrary Hausdorff locale \(X\) the above map is not known to be a monomorphism in the category of locales. In case it is, Escardo shows that it constitutes the counit of an adjunction, rendering the category CGHLoc of compactly generated Hausdorff locales a coreflective subcategory of Hausdorff locales. Let us refer to the assumption that CGHLoc is coreflective in Hausdorff locales as the **coreflectivity hypothesis**. We have

3.3. **Proposition.** Under the coreflectivity hypothesis, directed colimits of inclusions are stable under product in CGHLoc, i.e if \((t_{ij}: X_i \rightarrow X_j)\) is a directed diagram of inclusions in CGHLoc and \(Y\) any locale in that category, then the canonical map

\[
\text{colim}_i(X_i \times Y) \rightarrow (\text{colim}_i X_i) \times Y
\]

\((\ast)\) is an isomorphism.

**Proof.** We begin with the special case, where the directed system of the locales \(X_i\) is that of the inclusions of compact sublocales into a compactly generated one. We argue first that the canonical map, considered in CGHLoc, is an inclusion. (One has to be aware of the fact that even under the coreflectivity hypothesis a directed colimit of inclusions of Hausdorff locales that is calculated in the category of locales need not be Hausdorff, while
a product of compactly generated locales, calculated in the category of locales, need not be compactly generated.) But we have by 3.1 an inclusion

$$\text{colim}_i(X_i \times Y) \rightarrow (\text{colim}_i X_i) \times Y \quad (**)$$

(colimits and product calculated in the category of locales). The colimit \(\text{colim}_i X_i\) calculated in the category of locales is by assumption compactly generated Hausdorff hence it is a colimit in the sense of CGHLoc. Its product with the Hausdorff locale \(Y\) remains Hausdorff hence so is the sublocale \(\text{colim}_i(X_i \times Y)\). The products \(X_i \times Y\), again calculated in the category of locales, have a factor which is compact Hausdorff hence they are compactly generated Hausdorff themselves (so they are products in CGHLoc). Since, as argued, their colimit is Hausdorff, this remains a colimit in CGHLoc. Now the product in the right hand side of the desired isomorphism (*) has to be the one in CGHLoc which means that we have (under the coreflectivity hypothesis) to apply the coreflection functor to the right hand side of (**). So we consider the colimit \(\text{colim}_i C_k\) of the system of all compact sublocales of \((\text{colim}_i X_i) \times Y\). Every compact sublocale of \(\text{colim}_i(X_i \times Y)\) participates in that system hence their colimit will be a sublocale of the colimit of the latter. This justifies the existence of the exhibited inclusion.

On the other hand, the inclusion of each \(C_k\) into the localic product followed by the projection to \(\text{colim}_i X_i\) factors through a compact sublocale of that colimit, in particular \(C_k \rightarrow X_{i(k)} \rightarrow \text{colim}_i X_i\) and similarly for the projection to the other factor, \(C_k \rightarrow D_k \rightarrow Y\). Hence the compact sublocales \(X_{i(k)} \times D_k\) are final among the \(C_k\) so their colimit is \((\text{colim}_i X_i) \times Y\). Since \(\text{colim}_i(X_i \times Y)\) is a cone for the diagram of the \(X_{i(k)} \times D_k\) we get a factorization \((\text{colim}_i X_i) \times Y \rightarrow \text{colim}_i(X_i \times Y)\) and hence the desired isomorphism in the special case.

We can now extend the result to all directed colimits of inclusions in CGHLoc by virtue of the following calculation:

$$\text{colim}_i X_i \times Y \cong \text{colim}_i X_i \times \text{colim}_j K_j$$
$$\cong \text{colim}_j(\text{colim}_i X_i \times K_j)$$
$$\cong \text{colim}_j \text{colim}_i(X_i \times K_j)$$
$$\cong \text{colim}_i \text{colim}_j(X_i \times K_j)$$
$$\cong \text{colim}_i(X_i \times \text{colim}_j K_j)$$
$$\cong \text{colim}_i(X_i \times Y)$$

In the second as well as in the second last isomorphism above we used the special case of the result. Some more care is required to justify the third isomorphism. Here we use commutation of product by compact regular locales with all colimits. In particular, since the colimit \(\text{colim}_i X_i\) is meant in the sense of CGHLoc, we use the commutation of product by the \(K_j\) with the monomorphic directed colimit as formed in the category of locales, as well as with the quotient by the closure of the diagonal that gives the Hausdorff reflection of that directed colimit. 

\[\square\]
Remark: The above argument would have been simpler if we had not have to deal with the possible discrepancy between forming the colimit of a directed system of inclusions of Hausdorff locales in the category of all locales and forming it in the category of Hausdorff locales. So we would like to pose the question: is the colimit of a directed system of inclusions of Hausdorff locales a Hausdorff locale? A positive answer to that question would have of course rendered the coreflectivity hypothesis superfluous.

3.4. Proposition. Under the coreflectivity hypothesis, directed colimits of inclusions are stable under pullback in CGHLoc, i.e if $Z$ is a compactly generated Hausdorff locale $(t_{ij}: f_i \to f_j)$ is a directed diagram of inclusions over it $(f_i: X_i \to Z)$ and $Y \to Z$ another map in the same category, then the canonical map $$\text{colim}_i(X_i \times_Z Y) \to (\text{colim}_iX_i) \times_Z Y$$ is an isomorphism.

Proof. We want to apply the previous result relativized over a base $Z$, that is to exploit the previous result as a statement about products in $\text{Loc}(\text{Shv}Z)$. In order to do that, we need to make sure that the data of this Proposition, in particular the map $f: \text{colim}_iX_i \to Z$, give data of the previous Proposition when relativized over $Z$, more specifically that it corresponds to a compactly generated locale in $\text{Shv}(Z)$. This means that $f: \text{colim}_iX_i \to Z$ is a colimit in $\text{Loc}/Z$ of proper maps. Indeed, for each $X \to Z$ with $X, Z$ compactly generated, the fact that $X \cong \text{colim}_{j \in J}K_j$ gives, for each $j$, a factorization $K_j \to L_j \hookrightarrow Z$ which is proper: The map $K_j \to L_j$ is proper being one between compact Hausdorff locales ([Townsend 1998] 3.6.1), while $L_j \hookrightarrow Z$ is proper being a closed inclusion (the image $L_j$ is closed as a compact sublocale of a Hausdorff one). Obviously the colimit of these composites, for all $$K_j \to L_j \hookrightarrow Z = K_j \hookrightarrow \text{colim}_jK_j \cong X \to Z$$ is $\text{colim}_jK_j \cong X \to Z$ hence that latter is compactly generated in $\text{Loc}(\text{Shv}Z)$ and we can apply the previous Proposition.

4. Regularity of the category of compactly generated Hausdorff locales

We begin by generalizing a lemma due to B. Day and R. Street that is well-known for the case of locally presentable categories [Day and Street 1989]. Its statement has only to do with density assumptions (of the presentable objects in the original case) and the closure of the dense subcategory under certain colimits. We include the proof for the sake of completeness of exposition.

4.1. Lemma. Let $\mathcal{K}$ be a cocomplete category containing a dense subcategory $\mathcal{C}$ which is closed in $\mathcal{K}$ under finite colimits. Then for any small category with finite hom-sets $\mathcal{D}$ and diagram $D \in [\mathcal{D}, \mathcal{K}]$ we have that $$D \cong \text{colim} \left( [\mathcal{D}, \mathcal{C}] \downarrow D \Rightarrow [\mathcal{D}, \mathcal{K}] \Rightarrow [\mathcal{D}, \mathcal{K}] \right)$$
Proof. We show that, for all \( d \in D \), the evaluation at \( d \) of the canonical morphism from the colimit to \( D \) is an isomorphism in \( K \). Colimits in \([D, K]\) are given object-wise so, writing \( i: C \to K \) the inclusion and \( \partial_0: [D, C] \downarrow D \to [D, C] \) the domain functor

\[
\text{colim} \left( [D, C] \downarrow D \to [D, K] \right)(d) \cong \text{ev}_d \left( \text{colim} \left( [D, i] \cdot \partial_0: [D, C] \downarrow D \to [D, C] \to [D, K] \right) \right) \cong \text{colim} \left( ev_d \cdot [D, i] \cdot \partial_0: [D, C] \downarrow D \to [D, C] \to [D, K] \to K \right) \tag{1}
\]

On the other hand the density of \( C \) in \( K \) means that for all \( d \in D \)

\[
Dd \cong \text{colim}(C \downarrow Dd \to C \to K),
\]

while inspection gives that the composite

\[
ev_d \cdot [D, i] \cdot \partial_0: [D, C] \downarrow D \to [D, C] \to [D, K] \to K \tag{2}
\]

is naturally isomorphic to the composite

\[
i \cdot \partial_0 \cdot (ev_d \downarrow D): [D, C] \downarrow D \to C \downarrow Dd \to C \to K \tag{3}
\]

Moreover the functor \( ev_d \downarrow D: [D, C] \downarrow D \to C \downarrow Dd \) is final, essentially because \( ev_d: [D, C] \to C \) has a left adjoint given by \( C \mapsto \bigcup_{D(d, -)} C \) (whose existence is granted by the fact that \( D \) has finite hom-sets and \( C \) is closed under finite colimits in \( C \).) Hence combining the isomorphisms (1), (2), (3) with the latter finality result we get the desired isomorphism.

4.2. Proposition. Let \( K \) be a cocomplete and finitely complete category, such that it contains a dense subcategory \( C \) which is closed in \( K \) under finite limits and finite colimits. Assume that the objects of \( K \) are expressed (by density of \( C \)) as colimits of objects from \( C \) of such kind that commute with pullbacks in \( K \). Then, if \( C \) is regular, \( K \) is also regular.

Proof. First we apply the above lemma for \( D \) the category \( \bullet \to \bullet \) so that we express every morphism \( K \to K' \) in \( K \) as a colimit of morphisms \( C_i \to C_i' \) between objects in the full subcategory \( C \). Using the regularity of \( C \) we take the regular epi - mono factorization \( C_i \to C_i'' \to C_i' \) of every such morphism. Taking colimit of the appropriate kind we get a factorization

\[
K \cong \text{colim}_i C_i \to \text{colim}_i C_i'' \to \text{colim}_i C_i' \cong K',
\]

where the first morphism is obviously regular epi while the second one is mono because its kernel-pair consists of equal legs: it is obtained by applying the appropriate colimit to the equal legs of the kernel-pairs of the monos \( C_i'' \to C_i' \), using the commutation of the appropriate colimits with pullbacks.

Then we show stability of regular epis under pullbacks by applying the lemma to the category \( D \) given as

\[
\begin{array}{c}
\bullet \\
\downarrow \\
\bullet
\end{array}
\]
Again, using the commutation of the appropriate kind of colimit with pullbacks, the proof given in the Corollary of section 1 in [Day and Street 1989] applies to this situation.

Our intention is to apply the above to the category of compactly generated Hausdorff locales. We have seen in the previous section that, under the coreflectivity hypothesis, monomorphic directed colimits are stable under pullback in that category. We need commutation of monomorphic directed colimits with pullbacks in order to use the above.

4.3. Lemma. Assume that in a category monomorphic directed colimits are stable under pullback and that the injections into such colimits are monomorphisms. Then monomorphic directed colimits commute with pullbacks.

Proof. Let $\mathcal{I} \to \mathcal{K}$ be a monomorphic directed diagram in $\mathcal{K}$ and consider the pullback of the diagram

\[
\begin{array}{ccc}
\text{colim}_i Y_i & \to & \text{colim}_i Z_i \\
\downarrow & & \downarrow \\
\text{colim}_i X_i & \to & \text{colim}_i X_i \times \text{colim}_i Z_i \\
\end{array}
\]

Then

\[
\text{colim}_i X_i \times \text{colim}_i Z_i \cong \text{colim}_i \text{colim}_i \left( X_i \times \text{colim}_i Z_i \right)
\cong \text{colim}_i \left( X_i \times \text{colim}_i Z_i \right).
\]

where the first isomorphism is due to pullback-stability of monomorphic directed colimits and the second is due to directedness of $\mathcal{I}$. Finally, since each $Z_i \to \text{colim}_i Z_i$ is monomorphism

\[
X_i \times \text{colim}_i Z_i \cong X_i \times Z_i.
\]

as the following diagram of pullbacks indicates

\[
\begin{array}{ccc}
X_i \times \text{colim}_i Z_i & \to & \text{colim}_i Z_i \\
\downarrow & & \downarrow \\
X_i & \to & \text{colim}_i Z_i
\end{array}
\]

In view of Proposition 3.4, Proposition 4.2 and the previous lemma we get

4.4. Theorem. Under the coreflectivity hypothesis, namely that it is coreflective in the category of Hausdorff locales, the category of compactly generated Hausdorff locales is regular.

Proof. Apply Proposition 4.2 for $\mathcal{K}$ the category of compactly generated Hausdorff locales, $\mathcal{C}$ the category of compact Hausdorff locales, which is regular by [Townsend 1998] 3.6.3.
5. Effectivity of the category of compact Hausdorff locales

Recall that a locale $X$ is regular if every element of its frame of opens is the supremum of all the elements of the frame that are well inside it. An element of a frame $U$ is well inside $V$, written $U \preceq V$, if there exists a $W$ such that $U \land W = 0$ and $W \lor V = X$. Recall also that a locale is compact Hausdorff if it is compact regular (a result due to [Vermeulen 1991] but for see also [Townsend 1998] 3.4.2). A surjective map of locales is one where the inverse image of the corresponding map between the respective frames reflects order.

5.1. Proposition. The image of a compact locale by a proper surjection is compact

Proof. Let $q: X \to Q$ be a proper surjection of locales, $q^*: OQ \toOX$ its inverse image and assume that $Q = \bigvee U_i$, where the union is directed. Then

$$X = q^* Q = q^* (\bigvee U_i) = \bigvee \{ q^* U_i \}
$$

hence there is an $i$ such that $X = q^* U_i$. It follows that

$$Q = q_* q^* U_i = U_i,
$$

where the last equation follows by the fact that $q^*$ reflects order.

5.2. Proposition. The image of a regular locale by a proper surjection is regular.

Proof. For a proper surjection $q: X \to Q$ with $X$ is regular we have that for every $V \in OQ$

$$q^* V = \bigvee \{ U \in OX \mid U \preceq q^* V \}
$$

from which we get

$$V = q_* q^* V = q_* (\bigvee \{ U \in OX \mid U \preceq q^* V \}) = \bigvee \{ q_* U \in OX \mid U \preceq q^* V \}
$$

since the involved supremum is directed hence preserved by $q_*$ ([Johnstone 1982], III 1.1). Now $U \preceq q^* V$ implies $q_* U \preceq V$ because if $W \in OX$ is a witness for the first relation, i.e we have

$$U \land W = 0 \quad \text{and} \quad q^* V \lor W = X
$$

then

$$q_* U \land q_* W = q_* 0 = 0
$$

(the latter because $Z \leq q_* 0$ iff $q^* Z \leq 0 = q^* 0$ and $q^*$ reflects $\leq$) and also

$$Q = q_* X = q_* (q^* V \lor W) = V \lor q_* W
$$

by properness of $q$. We conclude that $q_* W$ is a witness for $q_* U \preceq V$, hence

$$V = \bigvee \{ q_* U \in OQ \mid U \preceq q^* V \} \leq \bigvee \{ q_* U \in OQ \mid q_* U \preceq V \}.
$$
5.3. Theorem. The category of compact Hausdorff locales is effective regular (\(=\) Barr-exact)

Proof. First of all the category CHausLoc of compact Hausdorff locales is regular by [Townsend 1998] 3.6.3. Equivalence relations in this category are proper and closed, as every map between compact Hausdorff locales is proper. We know from [Vermeulen 1994] 5.17 that closed, proper equivalence relations on compact locales are effective, so they are the kernel pairs of their coequalizers in the category of locales. But the coequalizer of a proper equivalence relation is proper by [Vermeulen 1994] 5.5. Hence the coequalizer in the category of locales of a (proper as it will be) equivalence relation between compact regular locales is compact regular, by the above two propositions. Since limits in the category in question are constructed as in the category of locales, we conclude that every equivalence relation in CHausLoc, being proper, is the kernel pair of its coequalizer in the category of locales, which lives in CHausLoc.

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