REPRESENTATIONS OF DUAL SPACES

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Abstract. We give a nonlinear representation of the duals for a class of Banach spaces. This leads to classroom-friendly proofs of the classical representation theorems \( H' = H \) and \( (L^p)' = L^q \). Our proofs extend to a family of Orlicz spaces, and yield as an unexpected byproduct a version of the Helly–Hahn–Banach theorem.

1. A bijection between normed spaces and their duals

Let \( X \) be a real normed space and \( X' \) its dual. We recall the following two notions:

Definition.

- \( X \) is uniformly convex if for every \( \varepsilon > 0 \) there exists a \( \delta > 0 \) such that
  \[
  \| x + y \| \geq 1 - \delta, \quad \text{then} \quad \| x - y \| \leq \varepsilon.
  \]
- A function \( F : X \to \mathbb{R} \) is Gateaux differentiable in \( x \) if there exists a continuous linear functional \( F' \in X' \) such that for all \( u \in X \),
  \[
  F(x + tu) = F(x) + tF'(x)u + o(t) \quad \text{as} \quad t \to 0.
  \]

Let us denote by \( S \) and \( S' \) the unit spheres of \( X \) and \( X' \), respectively. We recall (see Lemma 3 below) that if \( X \) is a uniformly convex Banach space, then the restriction to \( S \) of every linear form \( y \in S' \) achieves its maximum in a unique point \( M(y) \); furthermore, the map \( M : S' \to S \) is continuous.

**Theorem 1.** Let \( X \) be a uniformly convex Banach space. If its norm \( N : X \to \mathbb{R} \) is Gateaux differentiable on the unit sphere \( S \), then \( M : S' \to S \) is a continuous bijection, and it inverse is the derivative \( N' : S \to S' \) of the norm.

Proof. If \( x, z \in S \) and \( t > 0 \), then by the triangular inequality we have
  \[
  \frac{N(x + tz) - N(x)}{t} \leq \frac{N(tz)}{t} = N(z) = 1,
  \]
so that \( N'(z) \leq 1 \), with equality if \( z = x \). It follows that \( \| N'(z) \| = 1 \).

By uniform convexity, the linear form \( N'(x) \) achieves its maximum at a unique point (see Lemma 3 below), so that \( N'(z) \leq 1 \) for all \( z \in S \), different from \( x \). Hence every \( x \in S \) is uniquely determined by \( N'(x) \), and thus \( N' \) is injective.

To prove that \( N' \) is onto, pick \( y \in S' \) arbitrarily. Then \( x := M(y) \) is a point of minimal norm of the affine hyperplane \( \{ z \in X : y(z) = 1 \} \), and therefore
  \[
  \frac{d}{dt} \| x + t(z - y(z)x) \| |_{t=0} = 0, \quad \text{i.e.,} \quad N'(x)(z - y(z)x) = 0
  \]
for every \( z \in X \). Since \( N'(x)x = 1 \), this is equivalent to \( y = N'(x) \). \( \square \)

*Date: Version 2019-02-18.
2010 Mathematics Subject Classification. Primary 46B10; Secondary 46E30.
Key words and phrases. Banach space, dual space, Hilbert space, Orlicz space, uniform convexity, Lagrange multiplier, Riesz representation theorem, Lebesgue integral.*
Remarks 2.

(i) The above proof is essentially an application of the Lagrange multiplier theorem to maximize a linear functional on the unit sphere. Since the norm is not assumed to be a $C^1$ function, we have considered an equivalent extremal problem to minimize the norm on a closed affine hyperplane.

(ii) The homogeneous extension of $N' : S \to S'$ yields a bijection between $X$ and $X'$. Theorem 4. Helly–Hahn–Banach theorem for special spaces: the norm.

We recall the short proof of the lemma used above:

Lemma 3. Let $X$ be a uniformly convex Banach space, and $y \in S'$. There exists a (unique) point $x = M(y) \in S$ such that

$$y(x) = 1, \quad \text{and} \quad y(z) < 1 \quad \text{for all other} \quad z \in S.$$ 

Furthermore, the map $M : S' \to S$ is a continuous.

Proof. Since $X$ is complete and $S$ is closed, it suffices to show that every sequence $(x_n) \subset S$ satisfying $y(x_n) \to 1$ is a Cauchy sequence. Given $\varepsilon > 0$ arbitrarily, choose $\delta > 0$ according to the uniform convexity, and then choose a large integer $n_0$ such that $y(x_n) > 1 - \delta$ for all $n \geq n_0$. If $m, n \geq n_0$, then

$$\left\| \frac{x_n + x_m}{2} \right\| \geq y \left( \frac{x_n + x_m}{2} \right) > 1 - \delta$$

and therefore $\|x_n - x_m\| < \varepsilon$.

For the continuity of $M$ we have to show that if $y_n \to y$ in $S'$, then $M(y_n) \to M(y)$. Writing $x_n := M(y_n)$ and $x := M(y)$ for brevity, since

$$|y(x_n) - 1| = |y(x_n) - y_n(x_n)| \leq \|y_n - y\| \to 0,$$

$y(x_n) \to 1$, and therefore $(x_n)$ converges to a point $\tilde{x} \in S$ satisfying $y(\tilde{x}) = 1$ as in the first part of the proof. Since $y(x) = 1$ we have $\tilde{x} = x$ by uniqueness. □

2. Applications of Theorem 1

Theorem 1 has important consequences. We start with a strengthened version of the Helly–Hahn–Banach theorem for special spaces:

Theorem 4. Let $X$ be a Banach space satisfying the hypotheses of Theorem 1, and $y_1$ a continuous linear functional defined on some subspace $X_1$ of $X$.

Then $y_1$ extends to a unique continuous linear functional $y \in X'$ with preservation of the norm.
Proof. We may assume that \( \|y_1\| = 1 \), and we may assume by a continuous extension to the closure of \( X_1 \) that \( X_1 \) is closed. Then \( X_1 \) also satisfies the hypotheses of Theorem 1, so that \( y_1 = N'(x_1)|_{X_1} \) for a unique \( x_1 \in S \cap X_1 \). It follows that \( y := N'(x_1) \) is an extension of \( y_1 \), and \( \|N'(x_1)\| = 1 = \|y_1\| \).

If \( \tilde{y} \in S' \) is an arbitrary extension of \( y_1 \), then \( \tilde{y} = N'(\tilde{x}_1) \) for a unique \( \tilde{x}_1 \in S \), characterized by the equality \( \tilde{y}(\tilde{x}_1) = 1 \). Since \( x_1 \in S \) and \( \tilde{y}(x_1) = y_1(x_1) = 1 \), \( \tilde{x}_1 = x_1 \) by uniqueness.

Remark 5. Geometrically the proof is based on the observation that the unique hyperplane separating \( x_1 \) from the unit ball is the tangent hyperplane at \( x_1 \).

We know that every uniformly convex Banach space is reflexive. This can be seen easily under the further assumption that \( X' \) satisfies the hypotheses of Theorem 1.

Proposition 6. If \( X, X' \) are uniformly convex Banach spaces and the norm of \( X' \) is Gâteaux differentiable on \( S' \), then \( X \) is reflexive.

Proof. Given \( \Phi \in X'' \) arbitrarily, we have to find \( x \in X \) satisfying \( \Phi(y) = y(x) \) for all \( y \in X' \).

We may assume that \( \|\Phi\| = 1 \). By Lemma 3 there exists a unique \( y \in S' \) satisfying \( \Phi(y) = 1 \), and then a unique \( x \in S \) satisfying \( y(x) = 1 \). Defining \( \Phi_x \in X'' \) by \( \Phi_x(y) := y(x) \) for all \( y \in X' \), we have \( \Phi, \Phi_x \in S'' \) and \( \Phi(y) = 1 = \Phi_x(y) \). Applying Theorem 1 to \( X' \) we conclude that \( \Phi = \Phi_x \).

Now we turn to the description of the duals of Hilbert and \( L^p \) spaces.

Theorem 7 (Riesz–Fréchet [1,10]). If \( X \) is a Hilbert space with scalar product \( \langle \cdot, \cdot \rangle \), then the formula

\[ \Phi(x)u := \langle x, u \rangle, \quad x, u \in X \]

defines an isometric isomorphism \( \Phi \) of \( X \) onto \( X' \).

Proof. Since \( \Phi \) is an isometric isomorphism of \( X \) into \( X' \) by the Cauchy–Schwarz inequality, it suffices to show that \( \Phi \) maps \( S \) onto \( S' \).

Every Hilbert space is uniformly convex by the parallelogram law, and its norm is Gâteaux differentiable in every \( x \in S \) with \( N'(x) = \Phi(x) \) because the following relation holds for each \( u \in X \):

\[ \|x + tu\| = \sqrt{\|x\|^2 + 2t \langle x, u \rangle + t^2 \|u\|^2} = 1 + t \langle x, u \rangle + o(t) \quad \text{as} \quad t \to 0. \]

Applying Theorem 1 we conclude that \( \Phi \) is a bijection between \( S \) and \( S' \).

As we shall see in the more general context of Orlicz spaces (Lemma 12), the \( L^p \) norms satisfy the hypotheses of Theorem 1.

Theorem 8 (Riesz [11,13]). If \( X = L^p \) with \( 1 < p < \infty \) on some measure space and \( q = p/(p - 1) \) is the conjugate exponent, then the formula

\[ \Phi(g)f := \int g f \, dx, \quad g \in L^q, \quad f \in L^p \]

defines a linear isomorphism \( \Phi \) of \( L^q \) onto \( (L^p)' \).

Proof. Since \( \Phi \) is an isometric isomorphism of \( L^q \) into \( (L^p)' \) by the Hölder inequality, it suffices to show that each \( \varphi \in (L^p)' \) of norm one has the form \( \varphi = \Phi(g) \) with some \( g \in L^q \).
Due to Lemma 12, the hypotheses of Theorem 1 are fulfilled, and the Gâteaux derivative of the norm of \( L^p \) in any \( h_0 \in S \) is given by the formula
\[
N'(h_0)u = \int |h_0|^\frac{p}{q} \text{sign}(h_0) u \, dx.
\]
This implies \( \varphi = \Phi(g_0) \) with \( g_0 = |h_0|^\frac{p}{q} \text{sign}(h_0). \)

**Remarks 9.**

(i) The formula \( F(h) := |h|^\frac{p}{q} \text{sign} h \) defines a bijection between \( L^p \) and \( L^q \) whose inverse is \( F^{-1}(g) := |g|^\frac{q}{p} \text{sign} g \). Since \( \Phi = N' \circ F^{-1} \) and \( N' \), \( \Phi \) are homeomorphisms by Theorems 1, 8 and Remark 2 (iii), we conclude that \( F \) is a homeomorphism between \( L^p \) and \( L^q \). This is a special case of Mazur’s theorem \([4]\) stating that the spaces \( L^p \) are homeomorphic for all finite \( p \).

(ii) We recall the extensions of Theorems 7 and 8 to complex Hilbert and \( L^p \) spaces: the formulas
\[
\Phi(g) := \langle g, f \rangle \quad \text{and} \quad \Phi(f) := \int \overline{f} g \, dx
\]
define conjugate linear isometries of \( X \) onto \( X' \) and of \( L^q \) onto \( (L^p)' \), respectively. Only the surjectivity needs an additional argument, and this follows by the classical method of Murray \([7]\). For example, if \( \varphi \in (L^p)' \), then \( \Phi(g) = \varphi \) with \( g := g_1 + ig_2 \) by a direct computation, where the real-valued functions \( g_1, g_2 \in L^q \) are defined by the equalities
\[
\text{Re} \varphi(h) = \int g_1 h \, dx \quad \text{and} \quad \text{Re} \varphi(ih) = \int g_2 h \, dx
\]
for all real-valued functions \( h \in L^p \).

(iii) Another proof of Theorem 8 was given recently in \([14]\).

3. **Duals of Orlicz spaces**

In this section we generalize the proof of Theorem 8 to a class of Orlicz spaces \([2, 3, 8, 9, 15]\). First we briefly recall some basic facts. Let \( P : \mathbb{R} \to \mathbb{R} \) be an even convex function satisfying the conditions
\[
\lim_{u \to 0} \frac{P(u)}{u} = 0 \quad \text{and} \quad \lim_{u \to \infty} \frac{P(u)}{u} = \infty,
\]
and \( Q : \mathbb{R} \to \mathbb{R} \) its convex conjugate, defined by the formula
\[
Q(v) := \sup_{u \in \mathbb{R}} \{ u |v| - P(u) \}.
\]
For example, if \( P(u) = p^{-1} |u|^p \) for some \( 1 < p < \infty \), then \( Q(v) = q^{-1} |v|^q \) with the conjugate exponent \( q = p/(p-1) \).

Given a measure space \((\Omega, \mathcal{M}, \mu)\), henceforth we consider only measurable functions \( f, g, h : \Omega \to \mathbb{R} \). The **Orlicz spaces** \( L^P \) and \( L^Q \) are defined by
\[
f \in L^P \iff fg \text{ is integrable whenever } \int Q(g) \, dx < \infty
\]
and
\[
g \in L^Q \iff fg \text{ is integrable whenever } \int P(f) \, dx < \infty.
\]
If we endow \( L^p \) with the Luxemburg norm
\[
\|f\|_{P,Lux} := \inf \left\{ k > 0 : \int P(k^{-1} f) \, dx \leq 1 \right\}
\]
and \( L^Q \) with the Orlicz norm
\[
\|g\|_{Q,Orl} := \sup \left\{ \int fg \, dx : \int P(f) \, dx \leq 1 \right\},
\]
then they become Banach spaces,
\[
\int |fg| \, dx \leq \|f\|_{P,Lux} \|g\|_{Q,Orl}
\]
for all \( f \in L^p \) and \( g \in L^Q \) because \( \|f\|_{P,Lux} \leq 1 \iff \int P(f) \, dx \leq 1 \), and the formula
\[
\Phi(g)(f) := \int fg \, dx
\]
defines a linear isometry \( \Phi : L^Q \rightarrow (L^p)' \).

**Theorem 10.** Assume that \( P \) is differentiable, and satisfies the following conditions:

(i) there exists a constant \( \alpha \) such that \( P(2u) \leq \alpha P(u) \) for all \( u \in \mathbb{R} \);

(ii) for each \( \varepsilon > 0 \) there exists a \( \rho(\varepsilon) > 0 \) such that
\[
|u - v| \geq \varepsilon (|u| + |v|) \implies \frac{P(u) + P(v)}{2} \leq P\left( \frac{u + v}{2} \right) \geq \rho(\varepsilon) \frac{P(u) + P(v)}{2}.
\]

Then \( \Phi : L^Q \rightarrow (L^p)' \) is an isometrical isomorphism of \( L^Q \) onto \( (L^p)' \).

**Example 11.** If \( 1 < p < \infty \) and \( P(u) := p^{-1} |u|^p \), then Theorem 10 reduces to Riesz’s theorem \((L^p)' = L^q \) with \( q = p/(p-1) \). Indeed, \( P \) is differentiable with \( P'(u) = |u|^{p-1} \text{sign } u \), and the condition (i) is satisfied with \( \alpha = 2p \). To check (ii) we may assume by homogeneity that \( (u, v) \) belongs to the compact set
\[
K := \{(u, v) \in \mathbb{R}^2 : |u| + |v| = 1\}.
\]
The inequality follows by observing that the left hand side has a positive minimum on \( K \) by continuity and strict convexity, while \( P(u) + P(v) \) has a finite maximum here. Finally, on the classical spaces \( L^p \) the Orlicz and Luxemburg norms coincide.

For the proof of Theorem 10 we admit temporarily the following

**Lemma 12.** Assume that the conditions of Theorem 10 are satisfied. Then

(i) \( L^p \) is uniformly convex;

(ii) \( P'(h) \in L^Q \) for every \( h \in L^p \);

(iii) for any fixed \( f, h \in L^p \) the function \( t \mapsto \int P(h + tf) \, dx \) is differentiable in zero, and its derivative is equal to \( \int P'(h)f \, dx \).

**Proof of Theorem 10.** As before, it suffices to show that every \( \varphi \in (L^p)' \) of unit norm is of the form \( \Phi(h) \) for a suitable \( h \in L^p \).

As in the proof of Theorem 10 by the uniform convexity of \( L^p \) there exists a function \( h \in L^p \) such that \( \varphi(h) = 1 \), and \( \|f\| > 1 \) for all other functions \( f \) in the closed affine hyperplane \( H := \{ f \in L^p : \varphi(f) = 1 \} \).

By the definition of the Luxemburg norm we have
\[
\int P(h) \, dx = 1, \quad \text{and} \quad \int P(f) \, dx > 1 \quad \text{for all other } f \in H,
\]
so that $h$ also minimizes the functional $f \mapsto \int P(f) \, dx$ on $H$. Hence for any fixed $f \in L^p$ the function

$$t \mapsto \int P(h + t[f - \varphi(f)h]) \, dx$$

has a minimum in $0$, and therefore its derivative vanishes here:

$$\int P'(h) (f - \varphi(f)h) \, dx = 0.$$

This is equivalent to $\varphi = \varphi_g$ with

$$g = \frac{P'(h)}{\int P'(h)h \, dx} \in L^q. \quad \square$$

**Proof of Lemma 12.** (i) Following McShane [5] first we prove for all $\varepsilon \in (0, 1)$ the inequality

$$P\left(\frac{u - v}{2}\right) \leq \varepsilon \cdot P\left(\frac{u + v}{2}\right) + \frac{1}{\rho(\varepsilon)} \left( P\left(\frac{u + v}{2}\right) - P\left(\frac{u - v}{2}\right) \right).$$

In case $|u - v| \leq \varepsilon(|u| + |v|)$ this readily follows from the convexity and evenness of $P$ and from the equality $P(0) = 0$ because

$$P\left(\frac{u - v}{2}\right) \leq \varepsilon \cdot \left( \frac{\varepsilon}{2} |u| + \frac{\varepsilon}{2} |v| + (1 - \varepsilon) \cdot 0 \right) \leq \varepsilon \cdot \frac{P(u) + P(v)}{2}.$$

If $|u - v| \geq \varepsilon(|u| + |v|)$, then we infer from the convexity of $P$ and from the condition (ii) of Theorem 10 that

$$P\left(\frac{u - v}{2}\right) \leq \frac{P(u) + P(v)}{2} \leq \frac{1}{\rho(\varepsilon)} \left( P(u) + P(v) - P\left(\frac{u + v}{2}\right) \right).$$

We complete the proof of the lemma by showing that if

$$\int P(f) \, dx \leq 1, \quad \int P(g) \, dx \leq 1 \quad \text{and} \quad \int P\left(\frac{f + g}{2}\right) \, dx \geq 1 - \varepsilon \rho(\varepsilon),$$

then

$$\int P(f - g) \, dx \leq 2\alpha \varepsilon$$

with $\alpha$ given by the condition (i) of Theorem 10. This follows by using the inequality (2):

$$\frac{1}{\alpha} \int P(f - g) \, dx = \int P\left(\frac{f - g}{2}\right) \leq \varepsilon \cdot \int \frac{P(f) + P(g)}{2} \, dx + \frac{1}{\rho(\varepsilon)} \int P(f) + P(g) \, dx - P\left(\frac{f + g}{2}\right) \, dx \leq \varepsilon + \frac{1}{\rho(\varepsilon)} \left( 1 - \int P\left(\frac{f + g}{2}\right) \, dx \right) \leq \varepsilon + \frac{1 - [1 - \varepsilon \rho(\varepsilon)]}{\rho(\varepsilon)} = 2\varepsilon.$$
(ii) By definition we have to show that $P'(h)f$ is integrable for every $f \in L^P$. Setting $g := |h| + |h| \in L^P$, this follows from the inequalities
\[
\int |P'(h)f| \, dx \leq \int |P'(g)g| \, dx \leq \alpha \int P(g) \, dx < \infty.
\]
The last inequality is a consequence of the condition (i) of Theorem 10: we have
\[
u P'(u) \leq \int P'(t) \, dt = P(2u) - P(u) \leq \alpha P(u)
\]
for all $u \geq 0$, and $u P'(u)$ is an even function. (The equality holds because, as a convex function, $P$ is absolutely continuous.)

(iii) We have
\[
\lim_{t \to 0} \frac{P(f + th) - P(f)}{t} = P'(f)h
\]
almost everywhere (where both $f$ and $h$ are defined and are finite). Furthermore, if $0 < |t| \leq 1$, then
\[
\left| \frac{P(f + th) - P(f)}{t} \right| = \left| P'(f + th)h \right| \leq \left| P'(g)g \right|
\]
with $g := |f| + |h| \in L^P$ by the Lagrange mean value theorem with some $t'$ between 0 and $t$ (depending on $x$). The last function is integrable because
\[
\int |P'(g)g| \, dx \leq \|g\|_{P,\text{Lux}} \|P'(g)\|_{Q,\text{Orl}} < \infty.
\]
Applying Lebesgue’s dominated convergence theorem we conclude that
\[
\lim_{t \to 0} \frac{\int P(f + th) \, dx - \int P(f) \, dx}{t} = \int P'(f)h \, dx.
\]

Remark 13. A complete, but necessarily rather technical characterization of uniformly convex Orlicz space was given by different tools in [3].

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