Trouble for MAC

B. Holdom and F. S. Roux

Department of Physics, University of Toronto
Toronto, Ontario, M5S1A7, CANADA

Abstract

We show that the next-to-leading corrections to the kernel of the gap equation can be large and of opposite sign to the lowest order kernel, in the presence of a gauge boson mass. This calls into question the reliability of the Most Attractive Channel hypothesis.

There are few rigorous results for the case of dynamical symmetry breaking in theories which are neither supersymmetric or QCD-like. Efforts to build realistic models based on this large class of theories have then relied on various dynamical assumptions. One dynamical assumption in particular has had a major impact on model building. This is the most attractive channel (MAC) hypothesis [1], which assumes that symmetry breaking will be dictated by a fermion bilinear condensing in the channel which is most attractive under the exchange of one gauge boson. This is a simple prescription which receives support from QCD and which can be easily applied to other theories. It ties in closely with the ladder approximation to the gap equation, which is the basis of many studies of chiral symmetry breaking.

The gap equation analysis does in fact provide some support for the MAC hypothesis, in the sense that it provides an estimate of the critical coupling required for symmetry breaking to occur. This critical coupling is often significantly less [2] than what might be expected for a truly strong gauge coupling, with the latter being $\alpha \approx 4\pi$. For a critical coupling smaller than this, it is then somewhat plausible that corrections beyond the ladder graphs could be rather small. This possibility was reinforced by the study in [3], where corrections to the kernel of the gap equation were considered to second (next-to-leading) order. The corrections were found to be at the 20% level or less.
In the case of an unbroken strong gauge interaction with small $\beta$-function (walking theory) was considered, and the second order kernel was explored in a certain momentum region (one momentum much larger than another) in order to obtain analytical results. We will complement that study by instead considering the second order kernel for the momentum region expected to dominate in the loop integrations in the effective action. Our main object is to consider the implications of a possible gauge boson mass, associated with the breakdown of the strong gauge interaction. The effect of a gauge boson mass in the fermion gap equation should be considered for consistency, since the MAC hypothesis claims to differentiate between different symmetry breaking patterns, some of which cause the gauge symmetry itself to break. We find that a gauge boson mass can cause the order $\alpha^2$ term in the kernel to be as large or larger than the order $\alpha$ term, and of opposite sign. This makes the use of the MAC prescription to decide which channel actually does condense very uncertain.

We will consider $n_f$ flavors of fermions all in the fundamental representation of the gauge group $SU(N)$. The effective action in euclidean space is

$$\Gamma(S) = -\text{Tr}(\ln(S^{-1})) + \text{Tr}([S^{-1} - \partial\partial]S) - (2\Pi \text{ diagrams})$$ \hspace{1cm} (1)

where the fermion propagator is

$$S(p) = (Z(p)[\partial + \Sigma(p)])^{-1}.$$ \hspace{1cm} (2)

Flavor and color indices are implicit.

It is typical that the dominant contributions to this effective potential come from momentum scales larger than $\Sigma(p)$, in which case we can perform an expansion in powers of $\Sigma(p)/p$. This can be rigorously justified in the case of a walking theory. A different situation in which this expansion may be justified will emerge below. If we expand the effective potential to second order in $\Sigma(p)$, we find that the leading piece (zeroth order in $\Sigma(p)/p$) of the equation

$$\frac{\delta \Gamma(S)}{\delta Z(p)} = 0$$ \hspace{1cm} (3)

reads

$$(Z(p) - 1)\dot{\phi} = \frac{\delta(2\Pi \text{ diagrams})_{S_0}}{\delta S_0(p)}.$$ \hspace{1cm} (4)
where $S^0(p)$ is the massless fermion propagator.

By making use of (1) [3], the piece quadratic in $\Sigma(p)$ in the effective action becomes

$$\Gamma(\Sigma^2) = \frac{n_f N}{4\pi^2} \left( \int_0^\infty dp p \Sigma(p)^2 - \frac{1}{2} \int_0^\infty \int_0^\infty dk \Sigma(p) \Sigma(k) F(p,k) \right)$$  \hspace{1cm} (5)

$$F(p,k) = \frac{p^k}{2\pi^2 n_f N Z(p) Z(k)} \int_0^\pi d\theta \sin^2(\theta) d_{ij} d_{kl} K_{ijkl}(p,k)$$  \hspace{1cm} (6)

$$K_{ijkl}(p,k) = \frac{\delta(2PI \text{ diagrams}) S^0_{ij}(p) S^0_{kl}(k)}{\delta S^0_{ij}(p) \delta S^0_{kl}(k)}$$  \hspace{1cm} (7)

$K$ is a truncated 4-point function which incorporates the appropriate symmetry factors of the original 2PI diagrams. The $d$’s are constant matrices in color/flavor/spinor space and are nontrivial only in color space, where they reflect the representation $R$ of $SU(N)$ in which the fermion mass lies.

When the momentum integrals in (3) are defined with an infrared cutoff $\kappa$, for example $\kappa \approx \Sigma(\kappa)$, then $\Sigma(p)$ is determined nonperturbatively from the gap equation

$$\frac{\delta \Gamma(S)}{\delta \Sigma(p)} = 0.$$  \hspace{1cm} (8)

$Z$ and $K$ are determined perturbatively from (4) and (7), where massless fermion propagators are used. We renormalize in the $\overline{\text{MS}}$ scheme, so that $Z$ and $K$ become functions of the renormalization scale $\mu$, the gauge coupling $\alpha(\mu)$ and the gauge parameter $\xi(\mu)$. We will choose Feynman gauge $\xi(\mu) = 1$. The usual analysis we are exploring displays a gauge dependence, as explicitly obtained in [3]. This due to the presence of a nonlocal fermion mass and only a local gauge boson-fermion vertex in the effective action. Gauge independence would require additional contributions to gauge boson-fermion vertices proportional to the nonlocal mass. In the case of a massive gauge boson there is a Goldstone boson-fermion vertex which is also proportional to the nonlocal fermion mass. We shall restrict ourselves to the usual analysis and not consider any diagram which introduces a dependence on the fermion mass through a vertex.

We shall investigate the perturbative expansion of the kernel

$$F(p,k) = \frac{2}{\pi} (C_2(r_1) + C_2(r_2) - C_2(R))(F_1(p,k)\alpha + \mathcal{R} F_2(p,k)\alpha^2) + ...$$  \hspace{1cm} (9)
Notice that the well known group theory factor appearing in the first order term also multiplies the second order term. This factor includes all the dependence on the representation $R$ of the condensing channel ($r_1$ and $r_2$ are the representations of the two fermions). The appearance of this at second order is a result of keeping only the diagrams which are leading in either $N$ or $n_f$, as displayed in Fig. (1).

At first order

$$F_1(p, k) = (p^2 + k^2 + M^2 - \sqrt{(p^2 + k^2 + M^2)^2 - 4p^2k^2})/(2pk) \quad (10)$$

$$= \min\left(\frac{p}{k}, \frac{k}{p}\right) \quad \text{for} \quad M = 0 \quad (11)$$

where $M$ is a possible gauge boson mass. The $R$ in (9) will be defined by normalizing $F_2(p, k)$ relative to $F_1(p, k)$ at some momenta which dominates the integrations in the effective action (5), as follows.

The integrals over $p$ and $k$ in (5) could be exchanged for integrals over $\sqrt{(p^2 + k^2)/2}$ and $p/k$. There is then some momentum $q_{\text{dom}}$ which dominates the integral over $\sqrt{(p^2 + k^2)/2}$. This scale is determined dynamically by the form of $\Sigma(p)$, and it makes sense to set the renormalization scale $\mu = q_{\text{dom}}$. Note that for an unbroken gauge symmetry in the walking limit, $q_{\text{dom}}$ is much larger than $\Sigma(\kappa)$ [4]. There is another scale in the problem when the gauge boson has mass $M$. We do not expect that $q_{\text{dom}}$ will be much smaller than $M$, since $M$ acts as a natural infrared cutoff. On the other hand if $q_{\text{dom}}$ is much larger than $M$ then we revert to the previous case in which the gauge boson mass plays no role in the dynamics. Thus in the massive case we assume that the dynamics is such that $q_{\text{dom}}$ and $M$ are of the same order, and so we set $q_{\text{dom}} = M = \mu$.

We are therefore led to the following normalization of $F_2(p, k)$ at the point $p = k = q_{\text{dom}}$, where we note that the maximum value of $F_1$ occurs for $p = k$ for a fixed $p^2 + k^2$.

$$F_2 = F_1 \quad \text{for} \quad p = k = \mu = M \quad (12)$$

$$F_2 = F_1 = 1 \quad \text{for} \quad p = k = \mu \quad \text{when} \quad M = 0 \quad (13)$$

\[1\] When $R$ is the singlet channel, the nonleading diagrams are suppressed by $1/N^2$; otherwise the suppression is $1/N$. And we note in particular that the nonleading crossed-ladder graph is small even without the group theory suppression.
With this normalization, the size of $\mathcal{R}$ provides a measure of the importance of higher order effects. It is a conservative estimate, since the $\alpha$ in (9) could be larger than unity and closer to $4\pi$ if the interaction is truly strong. Our measure of the higher order effects differs somewhat from that in [3]; there the simplifying assumption $F_2(p, k) \propto \min(\frac{p}{k}, \frac{k}{p})$ was made, in which case it was sufficient to evaluate the kernel at the point $p \gg k$.

To obtain $\mathcal{R}$ we perform the angular integration in (6) numerically. When the gauge boson is massless we find

$$R_{\text{massless}} = .33C_2(G) - .21n_f T(r) + C_2(r)/2\pi.$$  \hspace{1cm} (14)

Here $T(r) \equiv \frac{(T(r_1) + T(r_2))}{2}$ and $C_2(r) \equiv \frac{(C_2(r_1) + C_2(r_2))}{2}$. The last term arises from the $Z$ factors in (8), the second term from the fermion loop diagram, and the first term from the other diagrams in Fig. (1). Thus when the number of fermions is small, the second order kernel reinforces the attraction found at lowest order in the singlet channel. For $N = 3$ and $n_f = 3$, with fermions in the fundamental representation, we have $R_{\text{massless}} = .9$. For increasing numbers of fermions, $R_{\text{massless}}$ decreases, and it becomes negative when the number of fermions are large enough to result in a small $\beta$-function, corresponding to the condition $11N = 4n_f T(R)$. For example for $N = 3$ and $n_f = 16$ we have $R_{\text{massless}} = -.4$. We find that the dependence of $F_2(p, k)$ on $p/k$ for fixed $p^2 + k^2$ is quite different from that displayed by $F_1(p, k)$. This is the main source of difference between our result and that in [3].

We now consider the case that the gauge boson is massive. We will continue to use Feynman gauge and will ignore the additional diagrams involving Goldstone bosons which are all subleading in $1/N$. To be specific we consider the breakdown of $SU(N)$ to $SU(N - 1)$, corresponding to a fermion mass in the symmetric tensor representation, with the single massive fermion being a $SU(N - 1)$ singlet. We find

$$R_{\text{massive}} = -.26C_2(G) - .045n_f T(r) - .12C_2(r).$$  \hspace{1cm} (15)

For example for $N = 3$ and $n_f = (3, 16)$ we have $R_{\text{massive}} = (-1.0, -1.3)$. Compared to the massless case we also find that $F_1(p, k)$ and $F_2(p, k)$ have a much more similar dependence on $p/k$ for fixed $p^2 + k^2$. Thus we conclude that the order $\alpha^2$ term in the kernel of the gap equation is large and of opposite sign to the order $\alpha$ term.

5
The main uncertainty in \( \mathcal{R}_{\text{massive}} \) is that the true values of \( q_{\text{dom}} \), \( M \), and \( \mu \) may in fact deviate from our assumption of equality. We stress though that \( q_{\text{dom}} \) is dynamically determined from the form of \( \Sigma(p) \), and its value should be such as to maximize the attraction in the preferred channel. A more minor source of uncertainty is in the relation of the two gauge boson masses corresponding to diagonal and nondiagonal generators respectively (we have denoted the latter mass by \( M \)). The numbers appearing in (15) assume that the former is \( \sqrt{\frac{4}{3}} \) times as large as the latter, as may be expected when \( N = 3 \).

It appears that the possibility of a symmetry breaking solution is self-consistent, in the sense that attraction in this channel is a consequence of the gauge boson mass which in turn is consistent with the symmetry breaking fermion mass. On the other hand other conditions must be satisfied before gauge symmetry breaking can occur; in particular the theory cannot be purely vectorlike \([5]\). Besides theories which are explicitly chiral, additional gauge interactions or nonrenormalizable interactions generated by physics at a higher scale may make a theory effectively chiral. Our analysis also applies to situations where the source of the gauge boson mass is something other than the strong dynamics in question.

We now return to the validity of the expansion in \( \Sigma(p)/p \), which has been justified so far only for the case of an unbroken, walking theory. The expansion would be justified in the broken theory if a fermion mass smaller than \( M \) emerged, since \( M \) sets the scale for the dominant momenta in the loops. We now see that there could be a dynamical reason for this to occur. The point is that from (15) we see that the fermion loop (the term with \( n_f \)) tends to enhance the strength of the second order kernel. We therefore see that a fermion mass small compared to \( M \) enhances the chance that the mass forms, since a large fermion mass damps the fermion loop contribution. This is in contrast to QCD where a large quark mass and the resulting damping of the fermion loop enhances the lowest order attraction in the color singlet channel (see (14)). Of course we are only pointing out a possible mechanism, since higher order effects are important and unknown.

We have explored the case of a broken gauge symmetry and have found that there is little reason to believe the lowest order most attractive channel hypothesis. Although this result runs counter to conventional wisdom concerning MAC, it is not
terribly surprising—when the coupling is large, higher order effects can be important. The problem of gauge dependence also plagues the usual analysis, but there is no reason to expect that additional contributions present in a gauge invariant treatment would cancel those that we have found. Our results are sufficient to call into question the use of the MAC hypothesis as a test of whether or not gauge symmetries break. More powerful techniques are needed to study the symmetry breaking patterns of interest for the construction of realistic theories of mass and flavor.

**Acknowledgment**

This research was supported in part by the Natural Sciences and Engineering Research Council of Canada. BH thanks the hospitality and support of the KEK theory group, where this work was completed.

**References**

[1] S. Raby, S. Dimopoulos, and L. Susskind, Nucl. Phys. **B169**, 373 (1980).

[2] B. Holdom, Phys. Rev. **D56** (1997) 7461.

[3] T. Appelquist, K. Lane and U. Mahanta, Phys. Rev. Lett. **61** (1988) 1553.

[4] B. Holdom, Phys. Lett. **B213** (1988) 365; Phys. Rev. Lett. **62** (1989) 997.

[5] C. Vafa and E. Witten, Nucl. Phys. **B234** (1984) 173.
Figure (1): The diagrams leading in $N$ or $n_f$ which contribute to the second order kernel.