RATIONAL TRACER:
a Tool for Faster Rational Function Reconstruction

Vitaly Magerya

Institute for Theoretical Physics (ITP), Karlsruhe Institute of Technology (KIT), 76128 Karlsruhe, Germany
E-mail: vitalii.maheria@kit.edu

Abstract

Rational Tracer (RATRACER) is a tool to simplify complicated arithmetic expressions using modular arithmetics and rational function reconstruction, with the main idea of separating the construction of expressions (via tracing, i.e. recording the list of operations) and their subsequent evaluation during rational reconstruction.

RATRACER can simplify arithmetic expressions (provided as text files), solutions of linear equation systems (specifically targeting Integration-by-Parts (IBP) relations between Feynman integrals), and even more generally: arbitrary sequences of rational operations, defined in C++ using the provided library ratracer.h. Any of these can also be automatically expanded into series prior to reconstruction.

This paper describes the usage of RATRACER specifically focusing on IBP reduction, and demonstrates its performance benefits by comparing with KIRA+FIREFLY and FIRE6.

Contents

1 Introduction ........................................... 2
2 The RATRACER approach ............................... 3
3 Setting up ............................................. 4
4 Using the command-line tool ............................ 5
  4.1 List of provided commands .............................. 5
  4.2 Reconstructing a single expression ..................... 8
    4.2.1 Expanding an expression into a series .......... 8
  4.3 Solving IBP relations (together with KIRA) ............. 9
    4.3.1 Specifying preferred master integrals .......... 10
    4.3.2 Master-wise and sector-wise reconstruction .......... 12
    4.3.3 Reconstruction with kinematics set to constants ..... 13
    4.3.4 Reconstructing a truncated series in $\epsilon$ .......... 14
1 Introduction

Reconstruction of rational functions based on modular arithmetic methods is an established field of mathematics. Its main idea is that a rational function in multiple variables can be reconstructed from the knowledge of its values modulo a prime number with the variables set to fixed numbers. Because modular evaluation is a comparatively fast operation, these methods enable one to sidestep the complexity and the intermediate expression swell associated with simplifying the expressions symbolically (i.e. using the approaches of the traditional computer algebra systems).

As an illustration of a simpler case (polynomial reconstruction), with the knowledge of
\[ f(11) = 606 \pmod{997} \quad \text{and} \quad f(38) = 827 \pmod{997} \] (1)
one can use Lagrange interpolation and rational number reconstruction [1, 2] to conclude that
\[ f(x) = 606 \frac{x - 38}{11 - 38} + 827 \frac{x - 11}{38 - 11} = 996 + 599x = -1 + \frac{4}{5} x \pmod{997}. \] (2)
This form can then be double-checked (or possibly improved) with more evaluations of \( f(x) \pmod{n} \).

Similar methods have also been developed for rational functions [3–10], with the goal of minimizing the number of “probes” (black-box evaluations of the target function modulo a prime).

Rational function reconstruction is actively used in high-energy physics following the work of [11, 12] as a method of solving large systems of linear equations generated as a part of solving Integration-by-Parts (IBP) relations between the Feynman integrals using the Laporta algorithm [13]. Multiple state-of-the-art IBP solvers are using modular arithmetics: FIRE6 [14], KIRA [15, 16] when used with FIREFLY [17, 18], FINITFLOW [19], CARAVEL [20], and FINRED (a private implementation by von Manteuffel et al). Solving IBP relations is one of the bottlenecks for computing higher order perturbative corrections in high-energy physics, and a major motivation for RATRACER.
2 The RATRACER approach

A typical IBP-solving tool based on modular arithmetics generates the needed linear equations, and then repeatedly solves them via Gaussian elimination modulo a 63-bit prime with the variables set to large random values. The (integer) values of the coefficients in the final row-reduced form are used to reconstruct them as rational expression.

It is our observation that a typical IBP solver spends the majority of its time—as much as 90% of it—managing its data structures used to represent the equations, rather than performing modular arithmetics. Fortunately, it is possible to forgo these data structures altogether: if instead of solving the linear system each time anew, one would trace it and record each arithmetic operation performed during the Gaussian elimination (as e.g. “add location 1 to location 2 and store into location 3”), then for subsequent evaluations there is no need to re-run the original algorithm—only this list of operations. We shall call such a list “a trace”. Evaluating a trace instead of the original algorithm immediately eliminates the overhead of managing data structures, and allows one to store all the needed intermediate values in a tightly packed linear region of memory, improving the performance further.

RATRACER is an implementation of this idea. It consists of a small C++ library, ratracer.h, used to record traces of arbitrary user-defined calculations, and the command-line tool, ratracer, used to simplify and reconstruct traces, as well as parse arithmetic expressions and solve linear systems of equations.

The trace files RATRACER works on are stored in a custom binary format optimized for performance and convenience; essence the format simply contains a list of arithmetic operations with the output values specially marked. For example, an expression like “$2 \times x + 3$” could be encoded as

```
location #1 = integer 2
location #2 = variable 1
location #3 = product of location #1 and #2
location #4 = integer 3
location #5 = sum of location #3 and #4
save location #5 as output 1
```

Having recordings of operations as a standalone object comes with multiple advantages. With a trace file RATRACER can:

- Optimize a trace by eliminating operations that don't contribute to the output (“dead code”, e.g. redundant equations), folding constants (i.e. replacing $2 \times 3$ by 6), simplifying operations (i.e. replacing $1 \times x$ by just $x$), merging common subexpressions, and reshuffling the temporary locations to minimize the memory usage.

- Reconstruct a trace with a subset of its variables set to integers, rationals, or arbitrary expressions.

- Select a subset of a trace's outputs and drop the rest, so that instead of reconstructing the whole set of resulting coefficients it would be possible to reconstruct any subset. The idea here is that evaluating a subset is faster than evaluating the whole set (after dead code elimination); this also allows to easily parallelize the whole reconstruction across multiple computers: each can work on reconstructing its own subset of outputs. This way one can achieve an equivalent of master-wise reduction, sector-wise reduction, or any mixture of these by just choosing which outputs to keep in which trace files.
• Expand the outputs of a trace in a series (without performing reconstruction). Because the outputs in a trace file are just trees of arithmetic expressions, any operation that makes sense on arithmetic expressions makes sense on a trace file too. Expanding a trace in a series has two immediate uses:

– automatically calculating derivatives of IBP coefficients without reconstructing them;
– increasing the reconstruction performance by expanding IBP coefficients into series in a small parameter (e.g. the dimensional regulator \( \epsilon \), or a small mass ratio), and throwing away higher orders—if those are deemed to not be needed in practice.

The main disadvantage is the size of the trace files: because it grows with the number of the required operations, for problems of interest it can go into the gigabyte range, making it impractical to keep it in the memory. For this reason R\_ATRACER makes sure that:

1. During trace recording (i.e. expression tracing or equations solving) the operations are not kept in the memory, but are progressively written to disk instead.

2. During optimization transformations the whole trace is never fully loaded into the memory—it is read in chunks from the disk instead—and the memory requirement of temporary data structures used during the transformation is never proportional to the number of operations, but is at most proportional to the memory size required to evaluate the trace (which is always assumed to fit into the memory).

This makes the algorithms used for these optimizations more complicated, and limits transformations like common expression elimination to work on a subset of the code at a time, rather than being global. Still, R\_ATRACER manages to work within this constraint well.

3. During the evaluation the trace is never fully loaded into memory (unless specifically allowed by the user), and is instead read in chunks from the disk.

We have found that the time to read a very large trace from a solid-state drive is less than the time needed to evaluate the corresponding trace, so the overhead is not great, but the parallel evaluation is limited by this for larger traces. We do not however believe this to be a principal limitation, as additional ways to improve the parallelizability of very large traces can be developed.

In our experience complicated IBP reductions like massive 5-point 2-loop problems (e.g. Section A.3) fit well under several gigabytes, and thus are not limited by the trace sizes.

3 Setting up

To use R\_ATRACER first get its source code from its repository over at

https://github.com/magv/ratracer

R\_ATRACER depends on FIREFLY [17] for rational function reconstruction, and on FLINT [21] for modular arithmetics. These libraries will be automatically downloaded when building R\_ATRACER; their dependencies GMP [22], MPFR [23], and JEMALLOC [24] as well.

To download the dependent libraries, compile them, and build the ratracer executable, run:

\$ make
4 Using the command-line tool

The command-line tool ratracer contains commands to solve linear equation systems, operate on traces, and ultimately reconstruct them. Its general usage pattern is:

$ ratracer command args ... command args ... ...

The logic behind the commands is that the tool maintains its internal state, and each command modifies it in a specific way. The internal state includes:

- the current trace: the set of inputs, outputs, and the set of instructions; the instructions come in two kinds:
  - the low-level instructions (suitable for evaluation, but not for optimization),
  - the high-level instructions (suitable for optimization passes, but not for evaluation);
- the current set of linear equations and the list of integral families;
- the current set of variable substitutions.

4.1 List of provided commands

- **load-trace file.trace**
  Load the given trace. Automatically decompress the file if the filename ends with “.gz”, “.bz2”, “.xz”, or “.zst”.

- **save-trace file.trace**
  Save the current trace to a file. Automatically compress the file if the filename ends with “.gz”, “.bz2”, “.xz”, or “.zst”.

- **show**
  Show a short summary of the current trace.

- **list-outputs [--to=filename]**
  Print the full list of outputs of the current trace.

- **stat**
  Collect and show the current code statistics.

- **disasm [--to=filename]**
  Print a disassembly of the current trace.

- **measure**
  Measure the evaluation speed of the current trace.

- **set name expression**
  Set the given variable to the given expression in the further traces created by `trace-expression`, `load-equations`, or loaded via `load-trace`.

- **unset name**
  Remove the mapping specified by `set`.
• **load-trace file.trace**  
Load the given trace.

• **trace-expression filename**  
Load a rational expression from a file and trace its evaluation.

• **keep-outputs filename**  
Read a list of output name patterns from a file, one pattern per line; keep all the outputs that match any of these patterns, and erase all the others.  
The pattern syntax is simple: "*" stands for any sequence of characters, all other characters stand for themselves.

• **drop-outputs filename**  
Read a list of output names from a file, one name per line; erase all outputs contained in the list.

• **optimize**  
Optimize the current trace by propagating constants, merging duplicate expressions, and erasing dead code.

• **finalize**  
Convert the (not yet finalized) code into a final low-level representation that is smaller, and has drastically lower memory usage. Automatically eliminate the dead code while finalizing.

• **unfinalize**  
The reverse of finalize (i.e. convert low-level code into high-level code), except that the eliminated code is not brought back.

• **reconstruct**  
Reconstruct the rational form of the current trace using the FIRELY library.  
If the **--inmem** flag is set, load the whole code into memory during reconstruction; this increases the performance especially with many threads, but comes at the price of higher memory usage.  
This command uses the FIRELY library for the reconstruction; **--factor-scan** and **--shift-scan** flags enable FIRELY's factor scan and/or shift scan (which are normally recommended); and **--bunches** sets its maximal bunch size.

• **reconstruct0**  
Same as reconstruct, but assumes that there are 0 input variables needed, and is therefore faster.  
This command does not use the FIRELY library. The code is always loaded into memory (as with the **--inmem** option of reconstruct).

• **evaluate**  
Evaluate the trace in terms of rational numbers.  
Note that all the variables must have been previously substituted, e.g. using the **set** command.
• **define-family** *name* [*--indices=n*]
  Predefine an indexed family with the given number of indices used in the equation parsing. This is only needed to guarantee the ordering of the families, otherwise they are auto-detected from the equation files.

• **load-equations** *file.eqns*
  Load linear equations from the given file in KIRA format, tracing the expressions. Automatically decompress the file if the filename ends with “.gz”, “.bz2”, “.xz”, or “.zst”.

• **drop-equations**
  Forget all current equations and families.

• **solve-equations**
  Solve all the currently loaded equations by gaussian elimination, tracing the process.
  Do not forget to **choose-equation-outputs** after this.

• **choose-equation-outputs** [*--family=name*] [*--maxr=n*] [*--maxs=n*] [*--maxd=n*]
  Mark the equations defining the specified integrals as the outputs, so they could be later reconstructed.
  That is, for each selected equation of the form $-I_0 + \sum_i I_i C_i = 0$, add each of the coefficients $C_i$ as an output with the name $CO[I_0, I_i]$.
  This command will fail if the equations are not in the fully reduced form (i.e. after **solve-equations**).
  The equations are filtered by the family name, maximal sum of integral’s positive powers (*--maxr*), maximal sum of negative powers (*--maxs*), and/or maximal sum of powers above one (*--maxd*).

• **show-equation-masters** [*--family=name*] [*--maxr=n*] [*--maxs=n*] [*--maxd=n*]
  List the unreduced items of the equations filtered the same as in **choose-equation-outputs**.

• **dump-equations** [*--to=filename*]
  Dump the current list of equations with numeric coefficients. This should only be needed for debugging.

• **to-series** *varname maxorder*
  Re-run the current trace treating each value as a series in the given variable, and splitting each output into separate outputs per term in the series.
  The given variable is eliminated from the trace as a result. The variable mapping is also reset.

• **sh** *command*
  Run the given shell command.

• **help**
  Show a help message and quit.
4.2 Reconstructing a single expression

To reconstruct a single arithmetic expression, prepare a file with this expression:

```
$ echo "1/(x+y) + 1/(x-y)" > expression.txt
```

Then trace the expression from the file, optimize the trace, and reconstruct it:

```
$ ratracer trace -expression expression.txt \ 
    optimize finalize \ 
    reconstruct --inmem --to=reconstruction.txt

[...]
```

```
$ cat reconstruction.txt
expression.txt =
  (x)/((-1/2)*y^2+1/2*x^2);
```

This tells us that

\[
\frac{1}{x+y} + \frac{1}{x-y} = \frac{x}{2x^2-\frac{1}{2}y^2}.
\]  
(3)

4.2.1 Expanding an expression into a series

Assuming that \( x \) is small in the same expression as before, it is possible to expand it into a series, keeping the first several orders in \( x \):

```
$ ratracer trace -expression expression.txt \ 
    optimize finalize \ 
    to-series x 5 \ 
    optimize finalize \ 
    reconstruct --inmem --to=series.txt

$ cat series.txt
ORDER[expression.txt , x^1] =
  ((-2))/(y^2);
ORDER[expression.txt , x^2] =
  ((0))/(1);
ORDER[expression.txt , x^3] =
  ((-2))/(y^4);
ORDER[expression.txt , x^4] =
  ((0))/(1);
ORDER[expression.txt , x^5] =
  ((-2))/(y^6);
```

This gives us the series expansion of

\[
\frac{1}{x+y} + \frac{1}{x-y} = -\frac{2}{y^2}x - \frac{2}{y^4}x^3 - \frac{2}{y^6}x^5 + O(x^6).
\]  
(4)

Note that it might be convenient to separately save the trace of `expression.txt`,

```
$ ratracer trace -expression expression.txt optimize finalize \ 
    save -trace expression.trace.gz
```

Here we have asked to automatically compress the trace with `gzip`; the recommended (and much faster) compressor however is `zstd`, which we can request using the file extension `.zst` (if the `zstd` program is installed).

In any case, the obtained trace can now be either restored fully as
Or expanded into series and reconstructed

```bash
$ ratracer load -trace expression.trace.gz \
  to-series x 5 optimize finalize reconstruct
```

Of course, it is also possible to similarly save the trace after the series expansion.

### 4.3 Solving IBP relations (together with KIRA)

![Figure 1: Massive 2-loop diamond with 2 masses, 2 massive legs, 5 propagators, and the total of 2 scaleless ratios.](image)

**RATRACER** can import linear equations in KIRA format and solve them, so any IBP reduction KIRA allows for can also be performed in combination with RATRACER for better performance and flexibility.

Let us illustrate this by reducing integrals in the two-loop diamond topology of Figure 1. First start by preparing the KIRA configuration:

```bash
$ mkdir config
$ cat >config/integralfamilies.yaml <<EOF
integralfamilies:
  - name: "diamond"
    loop_momenta: ["l1", "l2"]
    top_level_sectors: [b11111]
    propagators:
      - ["l1 - l2", "0"]
      - ["l1", "m1sq"]
      - ["l2", "m1sq"]
      - ["l1 + q", "m2sq"]
      - ["l2 + q", "m2sq"]
EOF
```

```bash
$ cat >config/kinematics.yaml <<EOF
kinematics:
  incoming_momenta: ["q"]
  outgoing_momenta: []
  kinematic_invariants:
    - ["s", 2]
    - ["m1sq", 2]
    - ["m2sq", 2]
  scalarproduct_rules:
    - [["q", "q"], "s"]
  symbol_to_replace_by_one: "s"
EOF
```
Next, prepare the KIRA job specification, e.g. requesting the export for all equations containing integrals with up to 3 numerators and up to 2 dots, and run it:

```
$ cat > export-equations.yaml <<EOF
jobs:
  - reduce_sectors:
      reduce:
      - { topologies: [diamond], sectors: [b11111], r:8, s:3 }
    select_integrals:
      select_mandatory_recursively:
      - { topologies: [diamond], sectors: [b11111], r:8, s:3, d:2 }
    run_symmetries: true
    run_initiate: input
    run_triangular: false
    run_back_substitution: false
run_firefly: false
EOF
```

```
$ kira export-equations.yaml
```

The equations will be exported to a set of `input_kira/diamond/SYSTEM_*`.kira.gz files (compressed with gzip). We can now load them into RATRACER and solve them (note that RATRACER automatically handles compressed equation files):

```
$ ratracer \\
  $(for x in input_kira/*/*kira.gz; do echo load-equations $x; done) \\
  solve-equations \\
  choose-equation-outputs --family=diamond --maxr=8 --maxs=3 --maxd=2 \\
  drop-equations \\
  optimize finalize \\
  reconstruct --inmem --threads=8 --factor-scan --shift-scan \\
  --to=ibp-tables.txt
```

The solution file will look like this:

```
$ cat ibp-tables.txt
CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]] =
((-16/3)*m1sq*d^2+( -7/3)*d^2*m2sq ^2+1/3* m1sq*d^3+( -7/3)*d^2+...  
CO[diamond[1,1,1,1,3],diamond[1,-1,1,1,0]] =
((-5147/40)*m1sq*3^d^2*m2sq^3+1/6*d^4*m2sq^6+37/32*m1sq^d^4*...  
...
```

Here $CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]]$ stands for “coefficient of the integral diamond$_{1,1,1,1,3}$ w.r.t. the master integral diamond$_{0,1,1,1,1}$”, so

\[
\text{diamond}_{1,1,1,3} = \left( \frac{16}{3} m_1^2 d^2 - \frac{7}{3} d^2 m_2^4 + \frac{1}{3} m_1^2 d^3 - \frac{7}{3} d^2 + \ldots \right) \text{diamond}_{0,1,1,1,1} + \\
\left( \frac{5147}{40} m_1^6 d^2 m_2^6 + \frac{1}{6} d^4 m_2^{12} + \frac{37}{32} m_1^2 d^4 \times \ldots \right) \text{diamond}_{1,-1,1,0} + \\
+ \ldots \ldots \tag{5}
\]

### 4.3.1 Specifying preferred master integrals

By default RATRACER will choose some integrals as masters, but selecting a better master basis can turn a hard problem into an easy one. Of particular note are the $d$-factorizing bases [25,
26], where the dependence of the reduction coefficients on the dimensional parameter $d$ in the denominator is factorized, which greatly simplifies the reduction process.

To force RATRACER to choose a particular basis one can exploit the fact that RATRACER maintains a strict order of integrals when solving an equation system: the integrals of the families defined first are eliminated last. Thus, defining an auxiliary family, e.g. named “master”, and adding equations setting master, to the desired integral (or a linear combination of integrals), will make sure that these integrals will be eliminated last, becoming the preferred master integrals.

To achieve this, first create a file with the additional equations:

```bash
$ cat >master-equations.txt <EOF
master@1*1
diamond[0,1,1,0,0]*-1

master@2*1
diamond[0,0,1,1,0]*-1

master@3*1
diamond[1,0,1,1,0]*-1

master@4*1
diamond[1,-1,1,1,0]*-1

master@5*1
diamond[1,0,1,1,-1]*-1

master@6*1
diamond[0,1,1,1,0]*-1

master@7*1
diamond[0,0,0,1,1]*-1

master@8*1
diamond[0,1,0,1,1]*-1

master@9*1
diamond[0,1,1,1,1]*-1
EOF
```

Then, solve the equations as before, but with additionally defining the master integral family (to make sure it is defined first), and loading this equation file in addition to what KIRA has exported:

```bash
$ ratracer \
define-family master \nload-equations master-equations.txt \n$(for x in input_kira/*/*kira.gz; do echo load-equations $x; done) \nsolve-equations \nchoose-equation-outputs --family=diamond --maxr=8 --maxs=3 --maxd=2 \ndrop-equations \noptimize finalize \nreconstruct --inmem --threads=8 --factor-scan --shift-scan \n--to=ibp-tables.txt
```

This time the result will look like the following:
\begin{verbatim}
$ cat ibp-tables.txt
CO[diamond[1,1,1,1,3],master9] =
  (-16/3)*m1sq*d^2+( -7/3)*d^2*m2sq^2+1/3*m1sq*d^3+( -7/3)*d^2+...
...
\end{verbatim}

Naturally, specifying more master integrals than there are, or less than there should be is harmless, because all we are doing is modifying the priority of the elimination.

### 4.3.2 Master-wise and sector-wise reconstruction

Instead of reconstructing the equation system solutions right away one can first operate on the trace. Let us start by saving it to a file.

\begin{verbatim}
$ ratracer \\
  $(for x in input_kira/*/*kira.gz; do echo load-equations $x; done) \\
  solve-equations \\
  choose-equation-outputs --family=diamond --maxr=8 --maxs=3 --maxd=2 \\
  optimize finalize \\
  save-trace trace.zst
\end{verbatim}

Now let us select only the outputs with the first master integral (diamond[0,1,1,0,0]).

\begin{verbatim}
$ cat >output-list.txt <<EOF
CO[* ,diamond[0,1,1,0,0]]
EOF
$ ratracer \\
  load-trace trace.zst \\
  keep-outputs output-list.txt \\
  unfinalize finalize \\
  save-trace trace-master1.zst
\end{verbatim}

Now trace-master1.zst only contains the coefficients with the first master, and all the remaining computations are removed from it, and it is therefore faster to evaluate (by a factor of 3.6 in this case):

\begin{verbatim}
$ ratracer load-trace trace.zst measure \\
  ...
Average time: 0.0001809s after 4095 evals \\
  ...
$ ratracer load-trace trace-master1.zst measure \\
  ...
Average time: 4.959e-05s after 16383 evals \\
  ...
\end{verbatim}

By saving traces of coefficients with each of the masters into separate files and reconstructing them separately we can achieve “master-wise” reduction. In other systems the same could be achieved by setting other master integrals to zero and re-running the IBP solution; with RATRACER we only need to select the appropriate outputs, and drop the computations that don’t contribute to them.

Similarly, we can split the computation by the integral being reduced:

\begin{verbatim}
$ cat >output-list.txt <<EOF
CO[diamond[1,1,1,1,3],*]
EOF
$ ratracer \\
\end{verbatim}
The resulting trace is, as expected, faster than the original one (by a factor of 9.3x this time):

```bash
$ ratracer load-trace trace-integral1.zst measure

Average time: 1.955e-05s after 32767 evals
```

By selectively choosing which integrals should go into which files, we can achieve sector-wise reduction.

Finally, we don’t need to stop at splitting by master or by integral, we can do both and have a separate trace for each coefficient, or any subset of them if we wish.

### 4.3.3 Reconstruction with kinematics set to constants

With `RATRACER` it is possible to set variables to some expressions (e.g. constants) before loading a trace, so that the trace would use that as its input. One use of this functionality is to evaluate the IBP reduction coefficients at particular fixed values of the kinematic invariants. This can be done the following way:

```bash
$ ratracer
    set m1sq 3
    set m2sq 5/2
    load-trace trace.zst
    reconstruct

... CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]] =
    ((-8/19773)*d^3+464/59319*d^2+(-2536/59319)*d+1360/19773)/((-1/6)*d+1);
...
```

Note that the reconstruction is performed in only one variable: $d$, because all the other variables are now fixed.

Another use of this feature is to perform reconstruction on a particular line (or, more generally, hypersurface) in the kinematic parameter space. For example, we can set $m_1^2 = 3/5\lambda$ and $m_2^2 = 7/11\lambda$, and then reconstruct only in $\lambda$ (and $d$):

```bash
$ ratracer
    set m1sq '3/5*lambda'
    set m2sq '7/11*lambda'
    load-trace trace.zst
    reconstruct

... CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]] =
    ((-15)+(-38/165)*lambda^2+1/6*lambda^3+21/2*lambda^2+(-12/605)*lambda^32+...
```
4.3.4 Reconstructing a truncated series in $\epsilon$

A major increase in reconstruction performance can be obtained by noting that fairly often the dimensional parameter $d$ enters the reconstructed expressions at a fairly high power (as $d^6$ in the case of Figure 1), but in practice most of this information is not needed: what is needed is the expansion of the coefficients in $\epsilon$ (with $d = 4 - 2\epsilon$), and only the first few terms of it, up to $O(\epsilon^1)$ or $O(\epsilon^0)$ usually, so restoring the dependence on $d^6$ (i.e. $\epsilon^6$) is a waste of time. Fortunately, we can avoid this waste by expanding the coefficients into a series in $\epsilon$ before the reconstruction, and then only reconstruct the required orders.

Here let us expand up to $O(\epsilon^0)$ terms:

```bash
$ ratracer
  set d '4-2*eps'
  load -trace trace.zst 
  to-series eps 1 
  optimize finalize 
  reconstruct

... ORDER[CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]],eps^0] =
(1+4*m1sq+m2sq^2+(-2)*m2sq^*m1sq^2+(-2)*m1sq^*m2sq)/(15*m1sq^4*... ORDER[CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]],eps^1] =
((-2)*m1sq)/(15*m1sq^4*m2sq^2+15*m2sq^2+15*m1sq^2+15*m2sq^4+... ORDER[CO[diamond[1,1,1,1,3],diamond[1,-1,1,1,0]],eps^-1] =
(6*m1sq^3*m2sq^3/2*m1sq^*m2sq^4+9*m1sq^3*m2sq^*2+(-6)*m1sq^4*m2sq^... ...
```

The performance effect of this expansion is two-fold:

- First, the variable in which the expansion is made is removed from the reconstruction, making it faster (fewer probes are required).
- Second, more outputs are needed and their per-probe evaluation time might be larger than that of the original problem.

In practice this works out so that expanding to $O(\epsilon^0)$ improves the overall reconstruction time by a factor of 2–5; less so for $O(\epsilon^1)$, and if a high enough expansion order is requested, then there is no performance benefit. For this reason it is important to specify the expansion order as low as is practically needed. For example, if $CO[diamond_{1,1,1,3},diamond_{0,1,1,1,1}]$ is only needed to $O(\epsilon^0)$, but the other coefficients are needed to $O(\epsilon^1)$, we can expand everything up to $O(\epsilon^1)$ and then remove the coefficients we don't need:

```bash
$ cat >droplist.txt <<EOF
ORDER[CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]],eps^1]
EOF
$ ratracer
  set d '4-2*eps'
  load -trace trace.zst 
  to-series eps 1 
  drop-outputs droplist.txt 
  optimize finalize 
  reconstruct

... ORDER[CO[diamond[1,1,1,1,3],diamond[0,1,1,1,1]],eps^0] =
```

14
(1+4*m1sq+m2sq^2+(-2)*m2sq+m1sq^4)/(15*m1sq^4*...
ORDER[CO[diamond[1,1,1,1,3],diamond[1,-1,1,1,0]],eps^-1] =
(6*m1sq^3*m2sq+3/2*m1sq*m2sq^4+9*m1sq^3*m2sq^2+(-6)*m1sq^4*m2sq+...
... 

Speaking more generally, the series does not need to be in $\epsilon$, and any variable can be used for the expansion. For example, if one is interested in some particular high-energy region, expansion in a kinematic invariant may be appropriate. Multiple expansions are possible too.

5 Using the C++ library

In addition to the command-line tool RATRACER provides a library ratracer.h, which allows users to construct traces of arbitrary computations defined as C++ code. The command-line tool itself is built on top of this library. A program using it first needs to make sure the FLINT [21] and GMP [22] libraries are installed, then include the header file as

```cpp
#include "ratracer.h"
```

and link the resulting program together with the FLINT and GMP libraries, for example:

```cpp
c++ -o custom_program custom_program.cpp -lflint -lgmp
```

5.1 List of provided functions

The interface provided by ratracer.h consist of an opaque structure representing a value (a rational modulo a prime):

```cpp
struct Value { }; 
```

and a class that contains all the arithmetic operations one can perform on these values:

```cpp
struct Tracer {
    void clear();
    size_t checkpoint();
    void rollback(size_t checkpoint);
    Value var(size_t idx);
    Value of_int(int64_t x);
    Value of_fmpz(const fmpz_t x);
    bool is_zero(const Value &a);
    bool is_minus1(const Value &a);
    Value mul(const Value &a, const Value &b);
    Value mulint(const Value &a, int64_t b);
    Value add(const Value &a, const Value &b);
    Value addint(const Value &a, int64_t b);
    Value sub(const Value &a, const Value &b);
    Value addmul(const Value &a,
                 const Value &b1,
                 const Value &b2);
    Value inv(const Value &a);
    Value neginv(const Value &a);
    Value neg(const Value &a);
    Value pow(const Value &base, long exp);
};
```
Value shoup_precomp(const Value &a);
Value shoup_mul(const Value &a,
    const Value &aprecomp,
    const Value &b);
Value div(const Value &a, const Value &b);
void assert_int(const Value &a, int64_t n);
void add_output(const Value &src , const char *name);
size_t input(const char *name , size_t len);
size_t input(const char *name);
int save(const char *path);
};

The Tracer structure should be initialized by:

    Tracer tracer_init();

5.2 An example

A very simple usage example would be this fragment that records a trace with

\[
\text{expr} = x^2 + 3y
\]

```
#include "ratracer.h"

int main() {
    Tracer tr = tracer_init();
    Value x = tr.var(tr.input("x"));
    Value y = tr.var(tr.input("y"));
    Value expr = tr.add(tr.pow(x, 2), tr.mulint(y, 3));
    tr.add_output(expr , "expr");
    tr.save("example.trace.gz");
    return 0;
}
```

To compile and run the program on a normal Unix machine, execute:

```
$ c++ -o example example.cpp -lflint -lgmp
$ ./example
```

The resulting trace `example.trace.gz` can be examined with `ratracer`:

```
$ ratracer load -trace example.trace.gz show
[...]
0.0002 +0.0001 * load-trace
0.0002 +0.0000 Importing 'example.trace.gz'
0.0055 +0.0053 * show
0.0055 +0.0000 Current trace:
0.0055 +0.0000  - inputs: 2
0.0055 +0.0000  0) x
0.0055 +0.0000  1) y
0.0055 +0.0000  - outputs: 1
0.0055 +0.0000  0) expr
0.0055 +0.0000  - big integers: 0
0.0055 +0.0000  - instructions: 0B final, 16kB temp
0.0055 +0.0000  - locations: 0 final (0B), 1024 temp (8.00kB)
```

16
It can also be disassembled into human-readable format:

```bash
$ ratracer load-trace example.trace.gz disasm

0.0055 +0.0000 * disasm
# ninputs = 2
# noutputs = 1
# nconstants = 0
# nfinlocations = 0
# nlocations = 1024
# low-level code (0B)
# high-level code (16384B)
0 = var #0 'x'
1 = var #1 'y'
2 = int #3
3 = mul 1 2
4 = mul 0 0
5 = add 4 3
6 = output 5 #0 'expr'
```

The resulting expression can be reconstructed too. For best results, we first optimize the trace, and perform the reconstruction in memory:

```bash
$ ratracer load-trace example.trace.gz \
   optimize \
   finalize \
   reconstruct --inmem

expr =
    (x^2+3*y)/(1);
```

## 6 Benchmarks

To put the performance of RATRACER into context, we have compared it with two state-of-the-art IBP solvers: KIRA and FireFly. Specifically, we have obtained IBP reduction tables for several example topologies (see Figure 2, Figure 3, Figure 4, Figure 5, and Figure 6 from Appendix A) using the following methods:

- “RATRACER”: using KIRA\(^1\) to generate the equations, then using RATRACER to solve them, and finally using RATRACER and FireFly\(^2\) to reconstruct the answer.

---

\(^1\)KIRA version 2.2 as of 2022-05-04, built with the default options plus the JEMALLOC [24] memory allocator.
\(^2\)FireFly version 2.0.3 as of 2022-05-30, built with the default options.
• “RATRACER+$O(\epsilon^n)$”: same as the previous method, but additionally asking RATRACER to expand the results in $\epsilon$ up to (and including) terms of the order $\epsilon^n$ before proceeding with the reconstruction.

• “RATRACER+Scan”: same as RATRACER, but with the “shift scan” and “factor scan” options of FIREFly enabled.

• “KIRA+FIREFly”: using KIRA together with its FIREFly backend. Following [16] we set the “maximal bunch size” parameter to 4 (RATRACER uses the same value).

• “KIRA”: using KIRA in its default configuration (i.e. using FERMAT [30] for symbolic expression simplification). This method does not directly use modular arithmetics (aside from a preparation step, and possibly inside FERMAT), so we only provide these measurements as a reference.

• “FIRE6”: using FIRE6 and LITERED [31] as described in [14]. This method also does not use modular arithmetics (aside from a preparation step), so we only provide these measurements as a reference.

• “FIRE6+Hints”: same as FIRE6, but with with a separate step preparing the “hints file” as recommended in [14].

Among the benchmarks we consider two to be realistic “hard” problems: the two-loop massive hexatriangle described in Section A.1 and the two-loop massive non-planar box from Section A.2. The remaining ones are provided for comparison.

6.1 What is measured?

In all cases the performance is measured on the same machine running openSUSE 15.3 with an AMD EPYC 7282 processor, an SSD disk, and sufficient RAM. The programs are asked to use 8 threads, and hyperthreading is effectively disabled via a Linux thread CPU affinity setting. The code for running the benchmarks has is publicly available on GitHub.

The detailed benchmark result tables can be found in Appendix A. The reported performance figures are:

• “Total time”: time from start to finish (from the plain configuration files to the final IBP substitution tables), including any preparation time that the method requires.

• “FIREFly time”: pure reduction time, excluding any preparation.

• “Probes”: number of black-box probes FIREFly has evaluated to get the final reconstruction.

• “Probe time”: average time of a single black-box probe, as reported by FIREFly.

• “FIREFly eff.”: the efficiency of FIREFly’s rational reconstruction implementation, measured as the ratio of the total probe time to the total FIREFly time scaled by the thread count.

• “Memory”: peak memory usage during the reduction.

• “Disk”: peak disk usage during the reduction.

3FIRE6 version 6.4.2, as of 2022-02-29, build with the default options.
4Using HYPOTHREAD, https://github.com/magv/hypothread.
5See https://github.com/magv/ibp-benchmark.
### 6.2 Benchmark summary

Here is a short summary of RADRACER performance in all the benchmarks from Appendix A:

| Section | Probe time speedup vs. KIRA | Total time speedup vs. KIRA+FIRELY | Total time speedup vs. KIRA | $O(\mathcal{e}^0)$ speedup | $O(\mathcal{e}^1)$ speedup |
|---------|----------------------------|-----------------------------------|-----------------------------|---------------------------|---------------------------|
| A.1     | 20                         | 5.2                               | 1.2                         | 3.2                       | 2.4                       |
| A.2     | 7.8                        | 6.0                               | 37                          | 2.7                       | 1.4                       |
| A.3     | 8.9                        | 2.1                               | 1/5.1                       | 5.5                       | 4.0                       |
| A.4     | 26                         | 1.7                               | 1/3.3                       | 2.3                       | 1.7                       |
| A.5     | 9.6                        | 5.2                               | 2.6                         | 4.3                       | 2.3                       |

To summarize the benchmark results:

- **RADRACER** consistently improves the probe time by a factor of 3–20 compared to **KIRA**.

- At the same time the overall runtime is improved by a factor of 5 for the “hard” examples compared to **KIRA+FIRELY**. This is not always as big as the probe time improvement would suggest; the reason is that with probes this fast the overhead of the rational reconstruction in **FIRELY** begins to dominate. In fact, on some examples like Section A.4 **RADRACER** probes are so fast that **FIRELY** efficiency drops below 3%.

- Interestingly, plain **KIRA** (i.e. **KIRA+FERMAT**) is normally faster than **KIRA+FIRELY**, with the exception of the non-planar box example (Section A.2), where it spends a big part of its reduction time waiting for a single **FERMAT** process to finish.

- A speedup by a factor of 2.5–5.5 is possible if one expands the reduction coefficients to $O(\mathcal{e}^0)$, or only 1.4–4 if $O(\mathcal{e}^1)$ is needed.

### 7 Conclusions

**RADRACER** shows consistent performance improvement over **KIRA+FIRELY** (sometimes by an order of magnitude), and sporadic improvement over **KIRA+FERMAT**. We believe the 5x improvement demonstrated for challenging reductions such as massive five-point two-loop topologies is already enough for **RADRACER** to be a valuable option for problems at the boundary of what is practical.

In the future we envision several ways to improve the performance of **RADRACER** further. First, seeing that with the fast probe times that **RADRACER** often provides the overhead in the **FIRELY** library becomes a limiting factor, we see improvements in **FIRELY** to reduce that overhead as an important potential speedup source. Second, when solving linear equation systems it is possible to achieve smaller (and therefore faster) traces by spending the time to devise better equation elimination orders. Third, optimizing the traces further to make the memory access pattern more linear could improve the performance of traces with large memory usage.
Acknowledgments

The author would like to thank Gudrun Heinrich, Sven Yannick Klein, Matthias Kerner, and Lisa Biermann for reading through a draft of this paper and providing comments and suggestions; Sven Yannick Klein and Fabian Lange for discussions related to FIRELY usage and inner workings. This work was partially funded by the Deutsche Forschungsgemeinschaft (DFG, German Research Foundation) under grant 396021762—TRR 257.

References

[1] P. S. Wang, M. J. T. Guy, and J. H. Davenport. “p-Adic Reconstruction of Rational Numbers”. In: SIGSAM Bull. 16.2 (May 1982), pp. 2–3. ISSN: 0163-5824. DOI: 10.1145/1089292.1089293.

[2] M. Monagan. “Maximal Quotient Rational Reconstruction: An Almost Optimal Algorithm for Rational Reconstruction”. In: Proceedings of the 2004 International Symposium on Symbolic and Algebraic Computation. ISSAC ’04. 2004, pp. 243–249. ISBN: 158113827X. DOI: 10.1145/1005285.1005321.

[3] E. Kaltofen and B. M. Trager. “Computing with polynomials given by black boxes for their evaluations: Greatest common divisors, factorization, separation of numerators and denominators”. In: Journal of Symbolic Computation 9.3 (1990). Computational algebraic complexity editorial, pp. 301–320. ISSN: 0747-7171. DOI: 10.1016/S0747-7171(08)80015-6.

[4] D. Y. Grigoriev, M. Karpinski, and M. F. Singer. “Interpolation of sparse rational functions without knowing bounds on exponents”. In: Proceedings [1990] 31st Annual Symposium on Foundations of Computer Science. 1990, 8409–8406 vol.2. DOI: 10.1109/FSCS.1990.89616.

[5] D. Y. Grigoriev and M. Karpinski. “Algorithms for Sparse Rational Interpolation”. In: Proceedings of the 1991 International Symposium on Symbolic and Algebraic Computation. ISSAC ’91. 1991, pp. 7–13. ISBN: 0897914376. DOI: 10.1145/120694.120696.

[6] J. de Kleine, M. Monagan, and A. Wittkopf. “Algorithms for the Non-Monic Case of the Sparse Modular GCD Algorithm”. In: Proceedings of the 2005 International Symposium on Symbolic and Algebraic Computation. ISSAC ’05. 2005, pp. 124–131. ISBN: 1595930957. DOI: 10.1145/1073884.1073903.

[7] S. Khodadad and M. Monagan. “Fast Rational Function Reconstruction”. In: Proceedings of the 2006 International Symposium on Symbolic and Algebraic Computation. ISSAC ’06. 2006, pp. 184–190. ISBN: 1595932763. DOI: 10.1145/1145768.1145801.

[8] E. Kaltofen and Z. Yang. “On Exact and Approximate Interpolation of Sparse Rational Functions”. In: Proceedings of the 2007 International Symposium on Symbolic and Algebraic Computation. ISSAC ’07. 2007, pp. 203–210. ISBN: 9781595937438. DOI: 10.1145/1277548.1277577.

[9] A. Cuyt and W.-S. Lee. “Sparse interpolation of multivariate rational functions”. In: Theoretical Computer Science 412.16 (2011). Symbolic and Numerical Algorithms, pp. 1445–1456. ISSN: 0304-3975. DOI: 10.1016/j.tcs.2010.11.050.

[10] Q.-L. Huang and X.-S. Gao. “Sparse Rational Function Interpolation with Finitely Many Values for the Coefficients”. In: Mathematical Aspects of Computer and Information Sciences. 2017, pp. 227–242. ISBN: 978-3-319-72453-9. DOI: 10.1007/978-3-319-72453-9_16.
[11] A. von Manteuffel and R. M. Schabinger. “A novel approach to integration by parts reduction”. In: Phys. Lett. B 744 (2015), pp. 101–104. DOI: 10.1016/j.physletb.2015.03.029. arXiv: 1406.4513 [hep-ph].

[12] T. Peraro. “Scattering amplitudes over finite fields and multivariate functional reconstruction”. In: JHEP 12 (2016), p. 030. DOI: 10.1007/JHEP12(2016)030. arXiv: 1608.01902 [hep-ph].

[13] S. Laporta. “High precision calculation of multiloop Feynman integrals by difference equations”. In: Int. J. Mod. Phys. A 15 (2000), pp. 5087–5159. DOI: 10.1142/S0217751X00002159. arXiv: hep-ph/0102033.

[14] A. V. Smirnov and F. S. Chuharev. “FIRE6: Feynman Integral REduction with Modular Arithmetic”. In: Comput. Phys. Commun. 247 (2020), p. 106877. DOI: 10.1016/j.cpc.2019.106877. arXiv: 1901.07808 [hep-ph].

[15] P. Maierhöfer, J. Usovitsch, and P. Uwer. “KIRA—A Feynman integral reduction program”. In: Comput. Phys. Commun. 230 (2018), pp. 99–112. DOI: 10.1016/j.cpc.2018.04.012. arXiv: 1705.05610 [hep-ph]. URL: https://gitlab.com/kira-pyred/kira.

[16] J. Klappert et al. “Integral reduction with KIRA 2.0 and finite field methods”. In: Comput. Phys. Commun. 266 (2021), p. 108024. DOI: 10.1016/j.cpc.2021.108024. arXiv: 2008.06494 [hep-ph].

[17] J. Klappert and F. Lange. “Reconstructing rational functions with FIREFLY”. In: Comput. Phys. Commun. 247 (2020), p. 106951. DOI: 10.1016/j.cpc.2019.106951. arXiv: 1904.00009 [cs.SC]. URL: https://gitlab.com/firefly-library/firefly.

[18] J. Klappert, S. Y. Klein, and F. Lange. “Interpolation of dense and sparse rational functions and other improvements in FIREFly”. In: Comput. Phys. Commun. 264 (2021), p. 107968. DOI: 10.1016/j.cpc.2021.107968. arXiv: 2004.01463 [cs.MS].

[19] T. Peraro. “FINITEFLOW: multivariate functional reconstruction using finite fields and dataflow graphs”. In: JHEP 07 (2019), p. 031. DOI: 10.1007/JHEP07(2019)031. arXiv: 1905.08019 [hep-ph].

[20] S. Abreu et al. “CARAVEL: A C++ framework for the computation of multi-loop amplitudes with numerical unitarity”. In: Comput. Phys. Commun. 267 (2021), p. 108069. DOI: 10.1016/j.cpc.2021.108069. arXiv: 2009.11957 [hep-ph].

[21] W. Hart et al. FLINT: Fast Library for Number Theory. Version 2.9.0. 2022. URL: https://flintlib.org/.

[22] T. Granlund et al. GMP: The GNU Multiple Precision Arithmetic Library. Version 6.2.1. 2020. URL: https://gmplib.org/.

[23] L. Fousse et al. “MPFR: A Multiple-Precision Binary Floating-Point Library with Correct Rounding”. In: ACM Trans. Math. Softw. 33.2 (June 2007), 13–es. ISSN: 0098-3500. DOI: 10.1145/1236463.1236468. URL: https://www.mpfr.org/.

[24] J. Evans et al. JEMALLOC: A general purpose malloc(3) implementation that emphasizes fragmentation avoidance and scalable concurrency support. Version 5.3.0. 2022. URL: http://jemalloc.net/.

[25] A. V. Smirnov and V. A. Smirnov. “How to choose master integrals”. In: Nucl. Phys. B 960 (2020), p. 115213. DOI: 10.1016/j.nuclphysb.2020.115213. arXiv: 2002.08042 [hep-ph].

[26] J. Usovitsch. “Factorization of denominators in integration-by-parts reductions”. In: (Feb. 2020). arXiv: 2002.08173 [hep-ph].
A Benchmark tables

A.1 Two-loop massive hexatriangle

\begin{center}
\includegraphics[width=0.2\textwidth]{hexatriangle.png}
\end{center}

Figure 2: Two-loop hexatriangle with one mass, 2 massless and 3 massive legs, 8 propagators, and the total of 6 scaleless ratios.

In this task integrals from a massive two-loop hexatriangle family of Figure 2 are reduced to its 168 master integrals. Reduction is requested for all 9719 integrals with at most one numerator and one dot. The solvers are asked to use a fixed master integral basis which is specifically chosen to be \(d\)-factorizing. Because the reduction is fairly complicated, the values of all kinematic and mass parameters are set to numerical values on a particular line in the parameter space, leaving only the dimensional regulator \(d\) and the line parameter \(\lambda\) for the reconstruction (i.e. 2 parameters instead of 6). This task simulates the reduction needed to obtain differential equations for the master integrals along a particular line. The family itself appears in \(q\bar{q} \rightarrow t\bar{t}H\) production at two loops.

Note that we have imposed a 5-hour time limit on each calculation; the measurement for FIRE6 was aborted because of this limit.
|                | Total time | FIREFLY time | Probe time | Probe count | FIREFLY eff. | Memory | Disk |
|----------------|------------|--------------|------------|-------------|--------------|--------|------|
| **KIRA+FireFly** | 13407 s    | 13367 s      | 1.2 s      | $7.7\times 10^4$ | 82.8%        | 4.2 GB | 1.6 GB |
| **RATRACER**    | 2579 s     | 2451 s       | 59.0 ms    | $7.6\times 10^4$ | 23.0%        | 3.3 GB | 2.2 GB |
| **RATRACER+Scan** | 2759 s    | 2659 s       | 57.1 ms    | $5.3\times 10^4$ | 14.3%        | 2.4 GB | 2.2 GB |
| **RATRACER+$\mathcal{O}(\epsilon^0)$** | 799 s     | 697 s        | 75.0 ms    | $6.1\times 10^3$ | 8.2%         | 2.0 GB | 2.4 GB |
| **RATRACER+$\mathcal{O}(\epsilon^0)+Scan$** | 3368 s    | 3268 s       | 74.5 ms    | $1.7\times 10^4$ | 4.7%         | 2.4 GB | 2.2 GB |
| **RATRACER+$\mathcal{O}(\epsilon^1)$** | 1088 s    | 970 s        | 90.6 ms    | $6.1\times 10^3$ | 7.1%         | 3.1 GB | 2.2 GB |
| **RATRACER+$\mathcal{O}(\epsilon^1)+Scan$** | 4906 s    | 4789 s       | 88.9 ms    | $1.7\times 10^4$ | 3.9%         | 3.7 GB | 2.2 GB |
| **RATRACER+$\mathcal{O}(\epsilon^2)$** | 1324 s    | 1193 s       | 106 ms     | $6.1\times 10^3$ | 6.7%         | 4.0 GB | 2.4 GB |
| **RATRACER+$\mathcal{O}(\epsilon^2)+Scan$** | 6032 s    | 5905 s       | 103 ms     | $1.7\times 10^4$ | 3.6%         | 4.7 GB | 2.4 GB |
| **KIRA**        | 3156 s     | —            | —          | —           | —            | 4.4 GB | 1.4 GB |
| **Fire6+Hints** | crash      | —            | —          | —           | —            | —      | —     |
| **Fire6**       | >5 h       | —            | —          | —           | —            | >2 GB  | >90 GB |

### A.2 Non-planar two-loop box with two masses

![Non-planar 2-loop box with 2 masses](image)

Figure 3: Non-planar 2-loop box with 2 masses, 4 massless legs, and 7 propagators, and the total of 3 scaleless ratios.

In this task integrals from a two-loop non-planar massive box family from Figure 3 are reduced to its 76 master integrals. Reduction is requested for all 3177 integrals with up to two numerators and no dots. All the solvers are asked to use the same master integrals as proposed by KIRA (these have only numerators and no dots). In [16] this family is called “topo5”.
|                     | Total time | FIREFly time | Probe time | Probe count | FIREFly eff. | Memory | Disk       |
|---------------------|------------|--------------|------------|-------------|--------------|--------|------------|
| KIRA+FireFly        | 2482 s     | 2470 s       | 182 ms     | 1.0 \times 10^5 | 95.9% | 2.1 GB | 74.8 MB   |
| Ratracer            | 713 s      | 692 s        | 23.4 ms    | 1.7 \times 10^5 | 72.3% | 1.8 GB | 489 MB    |
| Ratracer+Scan       | 412 s      | 393 s        | 23.3 ms    | 1.0 \times 10^5 | 77.0% | 1.3 GB | 413 MB    |
| Ratracer+\(\mathcal{O}(\epsilon^0)\) | 190 s      | 163 s        | 47.8 ms    | 2.4 \times 10^4 | 89.3% | 865 MB | 2.7 GB    |
| Ratracer+\(\mathcal{O}(\epsilon^0)\)+Scan | 153 s      | 126 s        | 47.6 ms    | 1.7 \times 10^4 | 81.9% | 720 MB | 2.7 GB    |
| Ratracer+\(\mathcal{O}(\epsilon^1)\) | 382 s      | 351 s        | 58.5 ms    | 4.2 \times 10^4 | 88.2% | 1.3 GB | 2.8 GB    |
| Ratracer+\(\mathcal{O}(\epsilon^1)\)+Scan | 299 s      | 269 s        | 58.0 ms    | 3.0 \times 10^4 | 80.2% | 1.2 GB | 2.7 GB    |
| Ratracer+\(\mathcal{O}(\epsilon^2)\) | 710 s      | 673 s        | 70.1 ms    | 6.6 \times 10^4 | 86.3% | 2.2 GB | 2.9 GB    |
| Ratracer+\(\mathcal{O}(\epsilon^2)\)+Scan | 532 s      | 497 s        | 69.7 ms    | 4.6 \times 10^4 | 80.6% | 1.8 GB | 2.9 GB    |
| KIRA                | 15199 s    | —            | —          | —           | —            | 9.0 GB | 549 MB    |
| Fire6+Hints         | >5 h       | —            | —          | —           | —            | >4 GB  | >28 GB    |
| Fire6               | >5 h       | —            | —          | —           | —            | >4 GB  | >28 GB    |

### A.3 Two-loop massive pentabubble

![Figure 4](image.png)

Figure 4: Two-loop pentabubble with one mass, 2 massless and 3 massive legs, 6 propagators, and the total of 6 scaleless ratios.

In this task integrals from a massive two-loop hexatriangle family of Figure 4 are reduced to its 38 master integrals. Reduction is requested for all 2196 integrals with at most one numerator and one dot. The solvers are asked to use a fixed master integral basis which is specifically chosen to be \(d\)-factorizing; this basis change is essential for performance in this reduction, as a naive Laporta basis increases the reduction time by orders of magnitude.
|                  | Total time | FIRE FLY time | Probe time | Probe count | FIRE FLY eff. | Memory | Disk |
|------------------|------------|---------------|------------|-------------|---------------|--------|------|
| **KIRA+FireFly** | 231 s      | 224 s         | 4.7 ms     | $1.5 \times 10^5$ | 40.3%         | 741 MB | 20.1 MB |
| **RATRACER**     | 210 s      | 204 s         | 555 us     | $1.9 \times 10^5$ | 6.5%          | 1.0 GB | 42.3 MB |
| **RATRACER+Scan**| 112 s      | 106 s         | 531 us     | $1.2 \times 10^5$ | 7.5%          | 537 MB | 30.2 MB |
| **RATRACER+$\mathcal{O}(\epsilon^0)$** | 20.3 s | 15.0 s | 578 us | $3.5 \times 10^4$ | 16.6% | 501 MB | 20.1 MB |
| **RATRACER+$\mathcal{O}(\epsilon^0)$+Scan** | 27.1 s | 22.2 s | 571 us | $2.8 \times 10^4$ | 9.1% | 374 MB | 18.7 MB |
| **RATRACER+$\mathcal{O}(\epsilon^1)$** | 28.1 s | 22.5 s | 686 us | $3.5 \times 10^4$ | 13.2% | 818 MB | 29.0 MB |
| **RATRACER+$\mathcal{O}(\epsilon^1)$+Scan** | 41.6 s | 36.1 s | 691 us | $2.9 \times 10^4$ | 6.9% | 608 MB | 26.8 MB |
| **RATRACER+$\mathcal{O}(\epsilon^2)$** | 32.8 s | 26.8 s | 767 us | $3.5 \times 10^4$ | 12.4% | 971 MB | 33.5 MB |
| **RATRACER+$\mathcal{O}(\epsilon^2)$+Scan** | 50.6 s | 44.8 s | 773 us | $2.9 \times 10^4$ | 6.2% | 741 MB | 31.0 MB |
| **KIRA**         | 21.8 s     | —             | —          | —           | —             | 1.1 GB | 14.7 MB |
| **Fire6+Hints**  | crash      | —             | —          | —           | —             | —      | —    |
| **Fire6**        | >5 h       | —             | —          | —           | —             | >4 GB  | >36 GB |

### A.4 Three-loop diamond with two masses

![Three-loop diamond with two masses](image)

Figure 5: Massive 3-loop diamond with 2 masses, 2 massive legs, 8 propagators, and the total of 2 scaleless ratios.

In this task integrals from a three-loop massive diamond family from Figure 5 are reduced to its 40 master integrals. Reduction is requested for all 24777 integrals with up to two numerators and one dot.
|                  | Total time | FIREFly time | Probe time | Probe count | FIREFly eff. | Memory | Disk   |
|------------------|------------|--------------|------------|-------------|--------------|--------|--------|
| KIRA+FireFly     | 116 s      | 99.7 s       | 57.8 ms    | 3.310^3     | 23.8%        | 2.8 GB | 116 MB |
| Ratracer         | 69.1 s     | 48.2 s       | 2.3 ms     | 6.710^3     | 3.9%         | 4.1 GB | 184 MB |
| Ratracer+Scan    | 72.6 s     | 55.7 s       | 2.2 ms     | 6.710^3     | 2.2%         | 3.7 GB | 103 MB |
| Ratracer+O(\epsilon^0) | 29.7 s  | 12.6 s       | 3.2 ms     | 4.110^2     | 1.3%         | 3.7 GB | 99.5 MB |
| Ratracer+O(\epsilon^0)+Scan | 42.6 s | 25.9 s       | 3.3 ms     | 6.710^2     | 1.1%         | 3.7 GB | 99.5 MB |
| Ratracer+O(\epsilon^1) | 41.7 s  | 21.7 s       | 5.8 ms     | 4.110^2     | 1.4%         | 4.9 GB | 159 MB |
| Ratracer+O(\epsilon^1)+Scan | 66.3 s | 47.0 s       | 5.1 ms     | 6.610^2     | 0.9%         | 3.8 GB | 154 MB |
| Ratracer+O(\epsilon^2) | 51.1 s  | 28.5 s       | 6.7 ms     | 4.110^2     | 1.2%         | 6.7 GB | 208 MB |
| Ratracer+O(\epsilon^2)+Scan | 84.9 s | 63.2 s       | 6.8 ms     | 6.810^2     | 0.9%         | 5.0 GB | 202 MB |
| KIRA             | 21.3 s     | —            | —          | —           | —            | 858 MB | 118 MB |
| Fire6+Hints      | 134 s      | —            | —          | —           | —            | 749 MB | 11.4 GB |
| Fire6            | 112 s      | —            | —          | —           | —            | 805 MB | 11.4 GB |

### A.5 Massive two-loop box

![Diagram of a two-loop box with one mass, 4 massless legs, 7 propagators, and the total of 2 scaleless ratios.](image)

In this task, integrals from a family of the massive two-loop box from Figure 6 are reduced to master integrals. Reduction is requested for all 35097 integrals with up to two numerators and two dots.
| Algorithm | Total Time | FireFly Time | Probe Time | Probe Count | FireFly Eff. | Memory | Disk |
|-----------|------------|--------------|------------|-------------|--------------|--------|------|
| KIRA+FireFly | 3933 s | 3899 s | 225 ms | $6.3 \times 10^4$ | 45.8% | 28.8 GB | 3.1 GB |
| RATRACER | 579 s | 518 s | 23.9 ms | $2.3 \times 10^4$ | 11.6% | 14.2 GB | 1.0 GB |
| RATRACER+Scan | 435 s | 383 s | 23.4 ms | $1.3 \times 10^4$ | 10.2% | 12.2 GB | 833 MB |
| RATRACER+$O(e^0)$ | 101 s | 61.5 s | 23.3 ms | $2.5 \times 10^3$ | 11.8% | 12.3 GB | 1.3 GB |
| RATRACER+$O(e^0)$+Scan | 134 s | 95.6 s | 22.6 ms | $2.3 \times 10^3$ | 6.9% | 12.4 GB | 1.3 GB |
| RATRACER+$O(e^1)$ | 189 s | 136 s | 38.7 ms | $2.8 \times 10^3$ | 9.9% | 12.6 GB | 1.4 GB |
| RATRACER+$O(e^1)$+Scan | 256 s | 205 s | 37.7 ms | $2.6 \times 10^3$ | 5.9% | 12.1 GB | 1.4 GB |
| RATRACER+$O(e^2)$ | 272 s | 205 s | 50.9 ms | $2.8 \times 10^3$ | 8.6% | 18.1 GB | 1.5 GB |
| RATRACER+$O(e^2)$+Scan | 366 s | 304 s | 49.4 ms | $2.6 \times 10^3$ | 5.2% | 15.5 GB | 1.5 GB |
| KIRA | 1126 s | — | — | — | — | 5.8 GB | 1.9 GB |
| FIRE6+Hints | >5 h | — | — | — | — | >3 GB | >80 GB |
| FIRE6 | 3848 s | — | — | — | — | 3.9 GB | 21.1 GB |