Universal infrared scaling of gravitational wave background spectra

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We study the general infrared behavior of the power spectrum of a stochastic gravitational wave background produced by stress tensor in the form bilinear in certain dynamical degrees of freedom. We find $\Omega_{GW} \propto k^3$ for a very wide class of the sources which satisfy a set of reasonable conditions. Namely, the $k^3$ scaling is universally valid when the source term is bounded in both frequency and time, is effective in a radiation-dominated stage, and for $k$ smaller than all the physical scales associated with the source, like the peak frequency, peak width, and time duration, etc. We also discuss possible violations of these conditions and their physical implications.

Introduction. The discovery of gravitational waves (GWs) from mergers of binary black holes (BBHs) and binary neutron stars (BNSs) by LIGO/VIRGO \cite{LIGO1, LIGO2} has brought the dawn to GW cosmology. Once produced, GWs propagate almost freely in the universe, they therefore carry the information about their origins as well as the evolution of the universe. The stochastic gravitational wave backgrounds (SGWB) may originate from many different physical sources such as BH/NS binaries \cite{LIGO1, LIGO2, BNS1, BNS2, BNS3, BNS4}, first order phase transitions during the evolution of the universe \cite{FO1, FO2, FO3, FO4}, spectator field(s) \cite{Spec1, Spec2, Spec3}, reheating/preheating after inflation \cite{RH1, RH2, RH3, RH4, RH5}, topological defects \cite{Tod1, Tod2}, primordial magnetic field \cite{PMF1, PMF2}, and primordial scalar and tensor perturbations from inflation.

Different GW experiments are sensitive to different frequencies. GWs with cosmological wavelengths, i.e., frequencies of order $10^{-16}$ Hz, can be indirectly detected by the B-mode polarization of the cosmic microwave background (CMB) \cite{CMB1, CMB2}. For GWs with frequency of order $10^{-9}$ Hz, pulsar timing array (PTA) is the most effective detector \cite{PTA1, PTA2, PTA3}. LIGO and LIGO-like interferometers can detect GWs of $10 \sim 10^9$ Hz \cite{LIGO1, LIGO2, LIGO3}, while space-based interferometers like LISA \cite{LISA1, LISA2, LISA3}, Taiji \cite{Taiji1, Taiji2}, Tianqin \cite{Tianqin1, Tianqin2}, Decigo \cite{Decigo1, Decigo2}, and beyond \cite{Beyond1, Beyond2} are sensitive to smaller frequencies $10^{-4} \sim 1$ Hz.

Cosmologists describe the SGWBs by their energy density per logarithmic frequency normalized by the current critical energy density of the universe, $\Omega_{GW}$. Most of the GW signals generated by given physical processes exhibit a power-law structure on the red side of a characteristic frequency $f_*$, e.g., $\Omega_{GW} \propto f^\beta$ for $f < f_*$. Searching and identification of a SGWB spectrum also rely on the ansatz of such a power-law scaling \cite{PPV1, PPV2}. For instance, an incoherent superposition of GWs emitted by BBHs has a characteristic scaling $\beta = 2/3$ \cite{BBH1, BBH2}. Secondary GWs induced by scalar curvature perturbations with a broad peak scales as $f^3$, while a $\delta$-function peak gives $f^2$. For SGWB from first order phase transitions, Ref. \cite{FO5} has noticed the universal $f^3$-scaling for small $f$, and justified it by a causality argument. But some steeper powers are observed later, especially for long-lived highly oscillating sources \cite{LIGO5, LIGO6}. Therefore it is interesting to study the infrared scaling of SGWB spectra in a general way that can be applied to a wide class of sources, and find out under which conditions the $k^3$-law may be obtained. This is the main task of this work.

Gravitational Waves from Bilinear Source. The metric we assume is

$$ds^2 = a^2(\eta)(-dt^2 + (\delta_{ij} + h_{ij})dx^i dx^j),$$

where $h_{ij}(\eta, \mathbf{x})$ is a transverse traceless (i.e., tensor) perturbation. We expand it as

$$h_{ij}(\eta, \mathbf{x}) = \int \frac{d^3k}{(2\pi)^{3/2}} \sum_{\lambda = +, -} e^\lambda_{ij}(\mathbf{k})h_{k,\lambda}(\eta) e^{ik\cdot x},$$

where $e^\pm_{ij}(\mathbf{k})$ are two orthonormal polarization tensors of GWs perpendicular to $\mathbf{k}$-direction with $e^{\lambda}_{ij} e^{\mu^*}_{ij} = \delta^\lambda_{\mu}$ and $\sum_{\lambda} e^{\lambda}_{ij} e^{\lambda^*}_{lm} = \Lambda_{ij}^{lm}$. $\Lambda_{ij}^{lm}$ is the transverse traceless projector defined as $\Lambda_{ij}^{lm} \equiv \frac{1}{2}(\pi_i^l \pi_j^m + \pi_j^l \pi_i^m - \pi_i^m \pi_j^l)$, where $\pi_{ij} \equiv \delta_{ij} - \hat{k}_i \hat{k}_j$ is the transverse projector perpendicular to $\mathbf{k}$ direction. It is customary to describe the SGWB by the energy density of GWs per logarithmic frequency normalized by the critical density, $\Omega_{GW} = (\rho_{\text{crit}})^{-1} d\rho_{GW}/d\ln k$, where $\rho_{\text{crit}} = 3H^2/(8\pi G)$ and

$$\rho_{GW} = \frac{1}{64\pi G a^2} \int \frac{dk}{k} \frac{k^3}{2\pi^2} \sum_{\lambda = +, -} \langle h^\dagger_{k,\lambda}(\eta) h_{p,\lambda}^*(\eta) \rangle + k^2 h_{k,\lambda}(\eta) h^*_{p,\lambda}(\eta) \rangle_{p=k}.$$
where a prime on $h_k$ denotes the derivative with respect to the conformal time $\eta$, and a prime on angular bracket means an overall $\delta^{(3)}(k - p)$ factor is removed. The angular brackets mean an ensemble average. We start from the equation of motion for the Fourier mode $h_k$,

$$h_{k,\lambda}'' + 2Hh_{k,\lambda}' + k^2h_{k,\lambda} = 16\pi Ga^2c_s^2\ell^4\Lambda_{ij}(k)T_{ab,k}, \quad (3)$$

where $H \equiv a'/a = aH$ is the conformal Hubble parameter, and $T_{ab}$ is the spatial components of the energy-momentum tensor whose transverse traceless part sources the tensor perturbation. We will discuss the origins of $T_{ab}$ later. The solution to (3) can be written as

$$h_{k,\lambda}(\eta) = \frac{2}{a(\eta)}M^2_{Pl} \int_0^\eta d\tilde{\eta} a^3(\tilde{\eta})G_k(\eta; \tilde{\eta})e^{i\ell a^3(\tilde{\eta})T_{ab,k}(\tilde{\eta})},$$

where $G_k(\eta; \tilde{\eta})$ is the retarded Green function of the equation, $v_k'' + (k^2 - a''/a)v_k = \delta(\eta - \tilde{\eta})$ with $v_k = ah_k$, which in the radiation dominated universe takes the form,

$$G_k(\eta; \tilde{\eta}) = \frac{\sin k(\eta - \tilde{\eta})}{k} \Theta(\eta - \tilde{\eta}),$$

where $\Theta$ is the Heaviside step function. The GW energy density parameter is

$$\Omega_{GW}(k) = \frac{k^3}{12\pi^2a^4H^2M^2_{Pl}} \int_0^\eta d\eta_1 \int_0^{\eta_1} d\eta_2 a^3(\eta_1)a^3(\eta_2) \times \cos k(\eta_1 - \eta_2)\Lambda_{cd}(k)T_{ab}(\eta_1)T_{ab}(\eta_2)P_{p=k},$$

where a prime on the correlator means omitting an overall $\delta$-function $\delta^{(3)}(k - p)$ as before.

The energy-momentum tensor that sources GWs is definitely model dependent. However, for a broad class of models, it is in the bilinear form,

$$T_{ab}(\eta, x) = \partial_\eta \phi(\eta, x)\partial_\eta \phi(\eta, x) + v_a(\eta, x)v_b(\eta, x),$$

up to a total derivative, where $\phi$ is a scalar field, and $v_a$ is a vector field which can be decomposed into the divergence and transverse parts as $v_a = \partial_\eta v + v_x$. This is the most general form of the stress tensor in the form bilinear in scalar or vector degrees of freedom. After transforming it to momentum space, (7) becomes a sum of convolutions, with its two-point function given by

$$\langle T_{ab}^{\eta}(\eta_1)T_{cd}^{\eta}(\eta_2) \rangle = \int \frac{d^3dq}{(2\pi)^3} \left[ \left( v^{\eta \eta}_q(\eta_1) v^{\eta \eta}_q(\eta_2) \right) \delta^{ds}(q) \right]$$

$$+ \ell^4(\ell - k)^4g^q(\mathbf{p} - \mathbf{q})^d(\phi_\eta(\eta_1)\phi_\eta(\eta_2)) \delta^\eta_\phi(\eta_2)$$

The 4-point function of the vector $v^a$ may be expressed as

$$\langle v^c_\eta(\eta_1)v^b_\eta(\eta_2)v^c_\phi(\eta_2)v^d_\phi(\eta_1) \rangle$$

$$= \langle v^c_\eta(\eta_1)v^c_\phi(\eta_2) \rangle \langle v^b_\eta(\eta_1)v^d_\phi(\eta_1) \rangle$$

$$+ \langle v^c_\eta(\eta_1)v^c_\phi(\eta_2) \rangle \langle v^b_\eta(\eta_1)v^d_\phi(\eta_1) \rangle$$

$$+ \langle v^c_\eta(\eta_1)v^b_\eta(\eta_1)v^c_\phi(\eta_2) \rangle$$

$$+ \langle v^c_\eta(\eta_1)v^b_\eta(\eta_1)v^c_\phi(\eta_2) \rangle,$$

where the contribution of the connected 4-point function to the GW energy density will vanish by symmetry. A similar expression holds for the scalar $\phi$. The two-point function $\langle \phi^\eta_\phi(\eta_1)v^c_\eta(\eta_2) \rangle$ can be decomposed into the parts longitudinal and perpendicular to the $\ell$-direction, while the two-point function of the scalar $\langle \phi_\eta(\eta_1)v^c_\eta(\eta_2) \rangle$ as usual,

$$\langle \phi^\eta_\phi(\eta_1)v^c_\phi(\eta_2) \rangle = \delta^{(3)}(\ell - q) \frac{2\pi^2}{3} \hat{\ell} \cdot \hat{q} \hat{w} \pi^{ac}(\ell),$$

$$\langle \phi_\eta(\eta_1)v^c_\phi(\eta_2) \rangle = \delta^{(3)}(\ell - q) \frac{2\pi^2}{3} \hat{\ell} \cdot \hat{q} \hat{w} \delta_\phi(\eta_1, \eta_2) \hat{w} \pi^{ac}(\ell) \delta_\phi(\eta_1, \eta_2),$$

where $\pi^{ac}(\ell) = \delta^{ab} - \hat{\ell} \hat{a} \hat{c}$. Note that since (10) and (11) are unequal-time correlators, $P_\phi$, $P_w$, and $P_w$ may not be positive definite unless $\eta_1 = \eta_2$. As $v_a$ and $\phi$ are independent variables, we have

$$\langle \phi^\eta_\phi(\eta_1)v^b_\phi(\eta_2) \rangle \langle \phi^c_\phi(\eta_2) \rangle$$

$$\rightarrow \delta^{(3)}(\ell - q) \frac{4\pi^4}{3\ell^2} P_\phi(\ell) P_\phi(\ell),$$

$$\rightarrow \delta^{(3)}(\ell - q) \delta^{(3)}(k - p) \frac{4\pi^4}{3\ell^2} [P_{w}(\ell)P_w(|k - \ell|)\pi_\ell^2(\ell)\pi_{\eta_1}^2(\ell) + P_{\ell}(\ell)P_{\ell}(|k - \ell|)\pi_\ell^2(\ell)\pi_{\eta_1}^2(\ell) + P_{\ell}(\ell)P_{\eta_1}(|k - \ell|)\pi_\ell^2(\ell)\pi_{\eta_1}^2(\ell)],$$

$$\rightarrow \delta^{(3)}(\ell - q) \delta^{(3)}(k - p) \frac{4\pi^4}{3\ell^2} \delta_\phi(\eta_1, \eta_2) \delta_\phi(\eta_1, \eta_2) \delta_\phi(\eta_1, \eta_2),$$

where $\ell \equiv \ell/\ell$ and $\eta \equiv |k - \ell|$. In the infrared limit, $\ell$ is smaller than any scale in the source, thus $\eta \rightarrow -\ell$. This is equivalent to picking up the leading order in the multipole expansion around $k$ in (12) and (13). As there is no $k$ dependence in the leading order, we can expand the transverse projectors to combinations of Kronecker $\delta$ and unit vector $\hat{\ell}$,

$$\langle \phi^\eta_\phi(\eta_1)v^b_\phi(\eta_2) \rangle \langle \phi^c_\phi(\eta_2) \rangle$$

$$\rightarrow \delta^{(3)}(\ell - q) \delta^{(3)}(k - p) \frac{4\pi^4}{3\ell^2} [P_{w}(\ell)P_w(|k - \ell|)^2 \hat{\ell} \cdot \hat{q} \hat{w} \delta_\phi(\ell) \delta_d]$$

$$+ \delta^{(3)}(\ell - q) \delta^{(3)}(k - p) \frac{4\pi^4}{3\ell^2} [P_{w}(\ell)P_w(|k - \ell|)^2 \hat{\ell} \cdot \hat{q} \hat{w} \delta_\phi(\ell) \delta_d]$$

$$+ \delta^{(3)}(\ell - q) \delta^{(3)}(k - p) \frac{4\pi^4}{3\ell^2} [P_{w}(\ell)P_w(|k - \ell|)^2 \hat{\ell} \cdot \hat{q} \hat{w} \delta_\phi(\ell) \delta_d]$$

$$+ \delta^{(3)}(\ell - q) \delta^{(3)}(k - p) \frac{4\pi^4}{3\ell^2} [P_{w}(\ell)P_w(|k - \ell|)^2 \hat{\ell} \cdot \hat{q} \hat{w} \delta_\phi(\ell) \delta_d].$$

Here the arguments of the functions $P_{w,\ell}(\eta_1, \eta_2, l)$ are not explicitly written. Then we substitute (11) and (14) into (8) and (9). The overall $\delta$-function, $\delta^{(3)}(k - \ell)$, will be kept till the end, while $\delta^{(3)}(\ell - q)$ can be integrated by $\int d^3q$ in (8), making $q \rightarrow \ell$. The integral over $d^3\ell$ can be written as $\ell^2d\ell d\Omega_\ell$, which can greatly simplify
the multipole moments as
\[
\int d\Omega_k \hat{\ell}_i \hat{\ell}_j = \frac{4\pi}{3} \delta_{ij}, \tag{16}
\]
\[
\int d\Omega_k \hat{\ell}_i \hat{\ell}_j \hat{\ell}_m = \frac{4\pi}{15} (\delta_{ij} \delta_{tm} + \delta_{it} \delta_{jm} + \delta_{im} \delta_{jt}). \tag{17}
\]
All these results will be contracted with $\Lambda_{ab}^{cd}$ in (4). By using the traceless property of $\Lambda$ as well as $\Lambda_{ab} = 2$, we have
\[
\Omega_{GW}(\eta, k) = \frac{k^3}{45a^3 H^2 M^4_{\text{pl}}} \int_0^{\eta_0} d\eta_1 \int_0^{\eta_0} d\eta_2 a^3(\eta_1) a^3(\eta_2) \times \cos(k(\eta_1 - \eta_2)) \int d\ell \left[(2P_v + 3P_w)^2 + 5P_w^2 + 4P_\phi^2\right]. \tag{18}
\]
The solution (18) is valid only in the radiation-dominated universe, so (18) holds only until the matter-radiation equality, $\eta < \eta_{eq}$. For a wide class of models, the source exists only for a finite duration of time $\Delta \eta_s$. When going to the small $k$ limit, we have $k \ll \Delta \eta_s^{-1}$, and the cosine function can be approximately taken to be 1. Then taking account of the redshift factor from the equal time to present, we obtain
\[
\Omega_{GW}(\eta_0, k) = \frac{k^3}{45a^3 a^2 \eta_0^2 H^2 M^4_{\text{pl}}} \int_0^{\eta_0} d\eta_1 \int_0^{\eta_0} d\eta_2 a^3(\eta_1) a^3(\eta_2) \times \int d\ell \left[(2P_v + 3P_w)^2 + 5P_w^2 + 4P_\phi^2\right]. \tag{19}
\]
Note that the quadratic forms in (19) are always positive. Thus assuming the integral is finite, we have
\[
0 < \int d\ell \left[(2P_v + 3P_w)^2 + 5P_w^2 + 4P_\phi^2\right] < \infty. \tag{20}
\]
Since this integral is $k$-independent, we find the universal scaling $\Omega_{GW}(\eta_0, k) \propto k^3$ in the limit $k \to 0$.

To summarize, the above can be stated as the following theorem: The GW spectrum we observe today scales as $\Omega_{GW}(\eta_0, k) \propto k^3$ in the infrared regime if the source satisfies the conditions:

(I) The integral (20) is finite.

(II) $k$ is smaller than all the scales associated with the source term, for instance $k_s$ (characteristic frequency), $\Delta k$ (peak width), and $\Delta \tau_s^{-1} \equiv (a_s \Delta \eta_s)^{-1}$ (duration of the source).

(III) Modes of interest reenter the Hubble horizon during the radiation dominated era.

We note that $k$ is connected to the GW frequency today by $f = ck/(2\pi a_0)$.

Let us list below a few specific examples of the sources in the bilinear form (4), and see if they satisfy the above conditions and give the $k^3$ scaling. The violation of these conditions will be discussed later.

(a) **Secondary GWs induced by scalar perturbations** [92–126]. Tensor perturbations are decoupled from the scalar perturbation at linear order, but they are coupled at second order. Such induced secondary GWs are negligible on CMB scales as the curvature perturbation $\mathcal{R}$ is highly constrained to be of order $10^{-5}$ [127]. However, the power spectrum of the curvature perturbation may have peak(s) on small scales and it is suggested that a substantial amount of primordial black holes (PBHs) may form when the rare, very high peaks reenter the horizon $\mathcal{R} \equiv \bar{\mathcal{R}}$ [128, 133], which, according to their constraints of different masses [134–148], can be a candidate of dark matter [149–165], the merging black holes detected by LIGO/VIRGO [7, 12, 160, 173], or seeds for structure formation [174, 178]. A simple estimate gives that the density parameter of GWs generated from the second-order scalar perturbations are roughly of order $R^2$, where $\mathcal{R}$ is the amplitude of the conserved comoving curvature perturbation on superhorizon scales. The GW spectrum peaks at the characteristic frequency proportional to $M_{\text{PBH}}^{-1/2}$, and may have unique features around the peak. Neglecting the isotropic components, the source stress tensor takes the form,
\[
\frac{T_{ij}}{M^4_{\text{pl}}} = -2\Phi 2\frac{\partial_i \partial_j \Phi}{a^2} \Phi + \frac{\partial_i}{a} \left(\Phi + \frac{\Phi'}{\mathcal{H}}\right) \frac{\partial_j}{a} \left(\Phi + \frac{\Phi'}{\mathcal{H}}\right) \tag{21}
\]
where $\Phi$ is the curvature perturbation on the Newton slices. During the radiation-domination, $\Phi$ is given by
\[
\Phi_0 = 2\mathcal{R}(\ell) \frac{j_1(c_s \ell \eta)}{c_s \ell \eta}, \tag{22}
\]
where $\mathcal{R}(\ell)$ is the conserved comoving curvature perturbation on superhorizon scales and $c_s = 1/\sqrt{3}$. When the scale enters the horizon, $\ell \eta \gg 1$, the source term $\Phi_0(\eta)$ decays rapidly as $(\ell \eta)^{-2}$. Thus the $(\partial_i \Phi \partial_j \Phi')$-term dominates on subhorizon scales. Then when the power spectrum of $\Phi$ has a peak at a certain scale $k_s$ with a width $\sigma$, which is usually the case for PBH formation scenarios, the finiteness of the integral (21) is assured. We thus reach the conclusion that $\Omega_{GW} \propto k^3$ for $k \ll \min(k_s, \sigma)$. This picture will break down when the width is infinitesimally small, i.e., for the $\delta$-function peak. We will discuss this case later.

(b) **GWs from first order phase transitions** [13, 14]. A first order phase transition may take place in various particle physics models [15–20]. When it happens, bubbles nucleate at a rate $\Gamma(t)$. The bubbles expand and collide after some characteristic time duration $\Delta t \approx 1/\beta \equiv \Gamma/t$. So the characteristic radius at collision is $R_c = v_w/\beta$, where $v_w$ is the velocity of the bubble wall in the rest frame of the bubble center. Besides the bubble collisions [179, 187], there are also compressible modes
Among these three processes, bubbles collide in a short period of time \( \Delta \eta_{\text{coll}} = a_s/\beta \), while the sound waves and/or turbulence may exist for a longer period of time \( \Delta \eta_{\text{bw}}, \Delta \eta_{\text{turb}} \). The GW spectrum scales as \( \Omega_{\text{GW}} \propto k^3 \) for \( k \ll \min(a_s/R_s, \Delta \eta_{\text{bw}}^{-1}, \Delta \eta_{\text{turb}}^{-1}) \). There will be some subtlety when \( \Delta \eta_{\text{bw}} \) or \( \Delta \eta_{\text{turb}} \) is larger than \( a_s/R_s \), which we will discuss later. Magnetic field itself can induce GWs, which is highly constrained by the CMB B-mode polarization \[43, 45\].

(c) GWs induced by an inflaton or a spectator scalar field in preheating after inflation \[26, 41\]. The source term is \( T_{ij} = \partial_i \phi \partial_j \phi \), where \( \phi \) is the inflaton or some other intermediate scalar field. The parametric resonance may largely enhance the amplitude of \( \phi \), which in turn may generate substantial GWs. Detailed study should be done by lattice simulations, but the infrared scaling is \( k^3 \) as expected, as was first pointed out in \[29\]. An related case is GWs from domain walls \[45, 47\].

Violation of the conditions. In this section we discuss the cases when any of the three conditions may be violated, and clarify the physical reasons behind such violations. Condition \[11\] breaks down if \[20\] diverges or vanishes. Since it vanishes only for the trivial case of vanishing sources, we may focus on the divergent case. For simplicity, let us consider a power-law divergence in the limit \( k \to 0 \). In this case if we evaluate \[6\] while keeping \( k \) small but finite, the resulting integral would behave as \( k^{-\gamma} (\gamma > 0) \). This would mean \( \Omega_{\text{GW}}(\eta_0) \propto k^{3-\gamma} \). In passing, it is worth mentioning that in this case of the power-law divergence, the power spectrum index of \( \Omega_{\text{GW}}(\eta_0) \) will be always smaller than \( 3 \).

A simplest example is the SGWB induced by scale-invariant scalar perturbations at second order. In this case, the curvature perturbation on Newton slices \( \Phi \) in the radiation-dominated stage is given by Eq. \[22\], and the dominant term of \( T_{ij} \) in \[21\] will be proportional to \( \langle (\eta \Phi')^2 \rangle \sim \langle R(t) \cos(\Theta(t))/\eta \rangle^2 \propto R^2(t)(\eta)^{-2} \), where we have focused on the \( \ell \)-dependence of the term. Thus if \( R \) is scale-invariant, the resultant power spectrum \( P_{\Phi \Phi} \) will have a power-law index of \( -4 \). This would give rise to a cubic divergence of the integral. In other words, the integral would behave as \( k^{-2} \) and cancel the \( k^3 \) coefficient in \[6\], resulting in a scale-invariant \( \Omega_{\text{GW}}(k, \eta_0) \).

Another example is the SGWB induced by scalar perturbations with a \( \delta \)-function spectrum \[92, 94, 100-103, 107, 112, 114, 116\]. For \( P_{\Phi \Phi} = A \delta (\ln(k/k_0)) \), we have the square of it in \[20\], integrating over \( dk \) will leave a \( \delta(0) \) which diverges, so Condition \[11\] is broken and we know the scaling is slower than \( k^3 \). In fact since \( \delta(0) \) is a linear divergence in momentum space, the resultant spectrum is expected to behave as \( k^{3-1} = k^2 \), and it indeed turns out to be the case. We note that, in addition to the divergence, as the \( \delta \)-function has a zero width, Condition \[11\] is also violated.

Another, rather important case of divergence is the case when \( k \to 0 \) limit is not quite meaningful, to begin with. Such a case appears if \( P_{\phi \phi}(\eta, \eta_0, \ell) \) is highly oscillating and the oscillation resonates with \( \cos k(\eta_0 - \eta) \). In this case, we cannot directly take the \( k \to 0 \) limit in \[18\], as the main contributions to the integral come from the resonances, and render the resulting \[18\] \( k \)-dependent, and the dependence is completely model-dependent.

One of the most important examples is the stochastic GWs from the incoherent superpositions of the compact binaries. In this case, we may estimate the \( k \)-dependence as follows. Consider a compact star binary in circular orbit with radius \( R \). The angular frequency of the orbit is \( \omega_0 = \sqrt{GM/R^3} \), which varies slowly with time due to GW emission. This gives a highly oscillating source power spectrum with frequency \( 2\omega_0 \), with the main contribution to the Fourier modes around \( k = 2\omega_0 \). Thus the simple infrared limit \( k \to 0 \) in \[18\] becomes meaningless. The time integral in \[18\] may be easily evaluated by the stationary phase approximation, and one obtains the correct scaling \( \Omega_{\text{GW}} \propto k^{2/3} \[80, 85\]. The \( k^3 \) scaling is, however, still valid for scales larger than the maximum orbital radius of the binaries, i.e., on scales greater than the Hubble horizon at their formation epoch. This corresponds to a current frequency \( f \ll a_{eq}(\rho_{eq}/M_{\text{BH}})^{1/3} \approx (M_\odot/M_{\text{BH}})^{1/3} \times 10^{-10} \text{Hz} \[196\], which is far smaller than the detectable band of any experiment.

A similar situation happens for GWs generated from the sound waves and the MHD turbulence at first-order phase transitions, provided that they last for a period longer than the characteristic time of bubble collision. In such a case, the source term can be decomposed into harmonics with frequency \( c_s \ell \) inside the horizon, with the main contribution coming from \( k \sim c_s \ell \). If the duration is large, \( \Delta \eta_{\text{bw}} \gg a_s/R_s \), the GW spectrum may have a steeper power spectrum, which is at most \( \propto k^0 \) for \( \Delta \eta_{\text{bw}} \ll k \ll a_s/R_s \[83, 91\]. Nevertheless, for \( k \ll \Delta \eta_{\text{bw}}^{-1} \), the \( k^3 \) law still holds. Note that when \( \Delta \eta_{\text{bw}} \) is larger than the Hubble horizon scale at the
collision time, $(a, H_*)^{-1}$, the duration time $\Delta \eta_{\text{gw}}$ is approximately equal to the Hubble horizon scale when the source ceases to exist, i.e., $\Delta \eta_{\text{gw}} \simeq (a, H_*)^{-1}$.

**Conclusion.** In this paper we studied the spectrum of stochastic GWs generated by the energy-momentum tensor bilinear in the source, and showed that the infrared of stochastic GWs generated by the energy-momentum of the source ceases to exist, i.e., $\Delta \eta_{\text{gw}} \simeq (a, H_*)^{-1}$. This can be explained by causality [197, 199]. Suppose the source of the GWs peaks at a characteristic scale $1/k$. For an infrared scale $1/k > 1/k_*$, there are $N = (k_*/k)^3$ causally disconnected patches where GW amplitudes is $h_*$. As a random variable, $h_\i$ obeys the Poisson distribution, which means on scales of $1/k$, $h_k = \sum_i h_{\i}(i)/N$. The two-point function is then $\langle h_k h_k \rangle = N^{-2} \sum_i \langle h_{\i}(i) h_{\i}(i) \rangle = N^{-1} |h_k|^2 = (k/k_*)^3 |h_\i|^2$. This is the two-point correlation of $h_k$ at their formation on superhorizon scales, which evolves to $h_k(\eta_0)$ at present by a redshift factor $a_k/a_0 \sim 1/k$. This redshift factor is canceled by the $k$ factor in the definition of the energy density of GWs: $\rho_{\text{GW}} \sim \langle h_k^2 \rangle \sim (k/a)^3 \langle h_k^2 \rangle$. Thus we obtain the scaling $\Omega_{\text{GW}} \propto \rho_{\text{GW}} \propto k^3 |h_k|^2$.

Our discussion actually shows that during the radiation dominated era, $\Omega_{\text{GW}} \propto k^3$ can still hold well inside the horizon, as long as $k$ is smaller than all the scales of the source, especially the inverse time duration for which the source exists. The condition that integral [20] is positive and finite can be guaranteed easily if the source is spiky and transient. But there will be more subtleties if the source is highly oscillating for a long period of time. We also discuss such physical cases when the integral [20] contains a highly oscillating part, and we find that the resonance between the Green function and the oscillating source term becomes crucial. However, $k^3$ law will be valid for those modes which are superhorizon at the moment when the source term disappears or stops oscillating. We believe our result can clarify some confusing understanding on the infrared power of SGWB, which is helpful in the future search for the SGWB signals.

Finally, we comment on the role of Condition [111]. It is easy to extend our discussion to the universe with a constant equation of state parameter $w = P/\rho$. We see that in the causality argument the $k^3$ factor from causality and $k^2$ factor from the definition of $\rho_{\text{GW}}$ are universal, yet the redshift factor $a_k/a_0$ depends on the background evolution. Assuming both Conditions [1] and [111] hold, since $\eta_k \sim 1/k$ and $a(\eta) \sim \eta^{2/(1+3w)}$, we immediately obtain

$$\Omega_{\text{GW}}(\eta_0, k) \propto k^{3+2\sw/3w}. \tag{25}$$

For instance in the induced GW case, if the scalar perturbation reenters the horizon in the matter dominated epoch when $w = 0$, we see from (25) that $\Omega_{\text{GW}} \propto k$. This may be seen in [80, 90, 95, 107, 126]. An interesting case is when $−1/3 < w < −1/15$, where the power-law index becomes negative. Note, however, that (25) is invalid for an accelerated universe, i.e. $w < −1/3$. Nevertheless, some models predict an enhanced production of GWs during inflation by spectator fields, and it seems the $k^3$ scaling is still valid under some conditions [21, 22]. It is interesting to see if there also exists a universal infrared scaling for an accelerated universe, which we leave for future studies.

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