Chiral zero modes of the $SU(n)$ Wess-Zumino-Novikov-Witten model

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Abstract

We define the chiral zero modes’ phase space of the $G = SU(n)$ Wess-Zumino-Novikov-Witten (WZNW) model as an $(n-1)(n+2)$-dimensional manifold $\mathcal{M}_q$ equipped with a symplectic form $\Omega_q$ involving a Wess-Zumino (WZ) term $\rho$ which depends on the monodromy $M$ and is implicitly defined (on an open dense neighbourhood of the group unit) by

$$d\rho(M) = \frac{1}{3} \text{tr} (M^{-1}dM)^3.$$ (\*)

This classical system exhibits a Poisson-Lie symmetry that evolves upon quantization into a $U_q(\mathfrak{sl}_n)$ symmetry for $q$ a primitive even root of 1. For each (non-degenerate, constant) solution of the classical Yang-Baxter equation we write down explicitly a $\rho(M)$ satisfying Eq.(\*) and invert the form $\Omega_q$, thus computing the Poisson bivector of the system. The resulting Poisson brackets (PB) appear as the classical counterpart of the exchange relations of the quantum matrix algebra studied previously in [32]. We argue that it is advantageous to equate the determinant $D$ of the zero modes’ matrix $(a^\alpha_j)$ to a pseudo-invariant under permutations $q$-polynomial in the $SU(n)$ weights, rather than to adopt the familiar convention $D = 1$. A finite dimensional "Fock space" operator realization of the factor algebra $\mathcal{M}_q/I_h$, where $I_h$ is an appropriate ideal in $\mathcal{M}_q$ for $q^h = -1$, is briefly discussed.
1 Introduction

Two-dimensional conformal current algebra models are known to lead to an unconventional problem of classical symplectic dynamics defined in terms of a multivalued action [49, 44, 50] or, equivalently, by a closed – but not exact – 3-form [35], depending on a group valued field. It has been noted at an early stage of this development [1] that the most interesting new features of the theory already appear in a finite dimensional ”toy model”. The present paper is devoted to a study of a version of such a finite dimensional ”chiral zero modes’” model. We display its precise relation to the (infinite dimensional) WZNW theory, reformulate it as a constrained dynamical system in the case when the underlying group is $SU(n)$, compute Poisson brackets among the basic dynamical variables for a given non-degenerate solution of the classical (dynamical) Yang-Baxter equations and demonstrate that they appear as a (quasi)classical limit of quantum exchange relations considered earlier [32].

1.1 The zero modes’ manifold

Let $G$ be a semisimple compact Lie group of $n \times n$ matrices with Lie algebra $\mathfrak{g}$. The zero modes’ manifold of a chiral WZNW model is not uniquely determined by the corresponding 2-dimensional (2D) conformal theory. It depends on the splitting of the $G$-valued field $g(x^0, x^1)$ into chiral factors,

$$g(x^0, x^1) = g_L(x^1 + x^0)g_R^{-1}(x^1 - x^0),$$

which obey a twisted periodicity condition (involving monodromy degrees of freedom),

$$g_C(x + 2\pi) = g_C(x)M, \quad C = L, R, \quad M \in G,$$

implying that the 2D field is periodic: $g(x^0, x^1 + 2\pi) = g(x^0, x^1)$. A further arbitrariness is involved in the factorization of the chiral fields $g_C(x)$ into (classical counterparts of) chiral vertex operators $u(x)$ and zero modes $a$; we shall write, in particular, the left movers’ field in the form

$$g_L(x)^A = u(x)^A j^A_{\alpha} \quad (A, j, \alpha = 1, \ldots, n).$$

The chiral vertex operators have, by definition, diagonal monodromies so that the ($x$-independent) matrix $a = (a^j_{\alpha})$ is chosen to diagonalize $M$ :

$$aM = M_p a, \quad M_p = q^{2p}, \quad q = e^{-i\hat{p}}, \quad \bar{q} = e^{i\hat{p}}.$$
Here \( k \) is the *Kac-Moody level* appearing as a coupling constant in the WZNW model \([50]\) and \( \hat{p} \) is a diagonal matrix whose entries define a weight vector belonging to the Weyl alcove \( A_n \) of the dual to the Cartan subalgebra of \( G \). For \( G = su(n) \)

\[
\hat{p} = \begin{pmatrix} p_1 & 0 & \cdots & 0 \\
0 & p_2 & \cdots & 0 \\
\cdots & \cdots & \cdots & \cdots \\
0 & 0 & \cdots & p_n \end{pmatrix}
\]

(1.5)

and the Weyl alcove can be conveniently identified with

\[
A_n = \left\{ p = \{p_i\}_{i=1}^n, \quad p_{ij} := p_i - p_j > 0 \text{ for } i < j, \quad P := \frac{1}{n} \sum_{i=1}^n p_i = 0 \right\},
\]

(1.6)

\( p_j \) playing thus the role of barycentric coordinates.

While the weights \( p_{\alpha_j} \) corresponding to the simple roots \( \alpha_j \) of \( G \) (\( p_{\alpha_j} = p_j \) for \( G = su(n) \)) provide an intrinsic characteristic of the state space of (both the chiral and the 2D) WZNW model, the zero mode matrix \( a_{\alpha}^j \) is gauge dependent. We shall use this freedom to work in a "covariant but not unitary gauge" (discussed in Section 3 below) and to equate, for \( G = SU(n) \), the determinant \( \Delta \) of \( (a_{\alpha}^j) \) to a pseudo-invariant under permutations of \( p_j \) function of \( p \) (cf. \([32]\)),

\[
\Delta := \det(a_{\alpha}^j) = \Delta_q(p) := \prod_{i<j} [p_{ij}] \quad \text{for } G = SU(n),
\]

(1.7)

\[
[p] := \frac{q^p - \bar{q}^p}{q - \bar{q}} \quad (q\bar{q} = 1),
\]

rather than to 1 as done in most related studies \([1, 4, 8, 15, 9, 10, 19]\).

**Remark 1.1** We use on purpose different notation for the indices like \( A, j, \alpha \) of \( u \) and \( a \) that run in the same range (1.3) since they have rather different nature. While the chiral model is invariant under left shifts of \( G \) (acting on \( A \)), it only admits a Poisson-Lie (or quantum group) symmetry with respect to \( \alpha \), while \( j \) labels the diagonal elements of \( M_p \).

### 1.2 The case \( n = 2 \) and its \( k \to \infty \) limit. The form \( \Omega_q \) for \( SU(2) \)

The advantage of the Ansatz (1.7) (as compared to the conventional \( \Delta = 1 \)) is exhibited on the simple example of the \( SU(2) \) model space and its \( q \)-deformation which we proceed to sketch. It can be also viewed as an introduction to the general case.
The realization of all irreducible representations (IR) of $SU(2)$ with multiplicity 1 in the Fock space of a pair of creation and annihilation operators is half a century old (see [46] and [12]). Its classical counterpart is the space $\mathbb{C}^2$ regarded as a Kähler manifold with a symplectic form

$$\Omega_1 = (idz_\alpha \wedge d\bar{z}^\alpha \equiv idz_\alpha d\bar{z}^\alpha \equiv i(dz_1 d\bar{z}^1 + dz_2 d\bar{z}^2)).$$

(1.8)

(We omit throughout this paper the wedge sign $\wedge$ for exterior product of differentials but keep it for the skew product of vector fields.) The corresponding Poisson bivector,

$$\mathcal{P}_1 = i \frac{\partial}{\partial z_\alpha} \wedge \frac{\partial}{\partial \bar{z}^\alpha},$$

(1.9)

yields the PB counterpart of the canonical commutation relations for (bosonic) creation and annihilation operators:

$$\{z_1, z_2\} = 0 = \{\bar{z}^1, \bar{z}^2\}, \quad \{z_\alpha, \bar{z}^\beta\} = i\delta_\alpha^\beta.$$  

(1.10)

In order to express $\Omega_1$ (1.8) in terms of the above "group like" variable $a = (a_\alpha^\alpha)$ and "weight" $p \equiv p_{12}$, we set

$$a = \begin{pmatrix} z_1 \\ -\bar{z}^2 \\ z_2 \\ \bar{z}^1 \end{pmatrix}, \quad p := \det a = z_1 \bar{z}^1 + z_2 \bar{z}^2 (> 0 \Leftrightarrow p \in \mathcal{A}_2),$$

(1.11)

$$\hat{p} = \frac{1}{2} \sigma_3 p.$$  

(1.12)

A simple calculation allows then to rewrite $\Omega_1$ as an exact 2-form:

$$\Omega_1 = -id\text{tr} (\hat{p} d\!a a^{-1}) .$$

(1.13)

The symplectic form $\Omega_q$ for the $SU(2)_k$ WZNW zero modes (derived for the general $SU(n)_k$ case in Section 2 below) appears as a 1-parameter deformation of (1.13):

$$\Omega_q(a, M_p) = \frac{k}{4\pi} \{\text{tr} (d\!a a^{-1}(2dM_p M_p^{-1} + M_p d\!a a^{-1} M_p^{-1})) - \rho(a^{-1} M_p a) \} .$$

(1.14)

Here $M_p$ is the diagonal matrix defined in (1.4), and $\rho$ is the WZ term:

$$\frac{k}{2\pi} dM_p M_p^{-1} = i d\hat{p}, \quad d\rho(M) = \frac{1}{3} \text{tr} (dM M^{-1})^3.$$  

(1.15)

(The 3-form in the right hand side is closed but not exact on $G$; the complex 2-form $\rho$ can only be defined on an open dense neighbourhood $G_0$ of the identity of $G$.)
The phase space $\mathcal{M}_q$ is a 4-dimensional surface in the 5-dimensional space of variables $a_\alpha^j$ and $p$, singled out by the equation (1.7):

\begin{equation}
\det a \equiv D = [p] \quad (\rightarrow p \text{ for } k \to \infty, \text{ resp. } q \to 1). \tag{1.16}
\end{equation}

To summarize: for (undeformed) $SU(2)$ creation and annihilation operators the determinant (1.11) plays the role of a number operator. More precisely, in the quantum theory $p \in \mathbb{N}$ is the dimension of the IR of $SU(2)$ spanned by all homogeneous polynomials of the creation operators $a_\alpha^1$ of degree $p - 1$ (acting on the Fock space vacuum). For $p > 0$ we can introduce new matrix variables with determinant 1,

\begin{equation}
g_\alpha^j := \frac{1}{\sqrt{p}} a_\alpha^j, \quad \det(g_\alpha^j) = 1 \tag{1.17}
\end{equation}

preserving the form of $\Omega_1$ ($= -i d \text{tr}(\hat{p}dgg^{-1})$). The new variables $(g_\alpha^j)$ obeying (1.17), however, would not satisfy the canonical PB relations for creation and annihilation operators. For $q \neq 1$ ($k$ finite) a change of variables $a_\alpha^j \to g_\alpha^j = [p]^{-1/2}a_\alpha^j$ (that would again give $\det a = 1$) may become singular, as $[k] = 0$ for $q$ given by (1.4). From this point of view, the convention $\det a = 1$ is neither convenient nor always possible.

### 1.3 Outlook and references

Although the WZNW model was introduced [50] in terms of a multivalued action, its solution was first given in the axiomatic approach to conformal current algebra models [41, 47]. The canonical (Lagrangean) approach had to wait the discovery of the link between the quantum exchange relations and the Yang-Baxter equation [5]. It was initiated for the WZNW model in [14] and was given a strong impetus by [27]. Among early subsequent work ([8, 6, 35, 17, 18, 29, 7, 33, 34]) we would like to single out the development by Gawędzki and coworkers [35, 29, 36] of a truly canonical first order formalism adapted to the problem. The present paper is devoted to a self-contained study of the finite dimensional zero modes’ problem (without recurrent appeal to its infinite dimensional origin). This problem was first singled out in [1] followed by [4, 15, 30] – among others. It has an interest of its own, exhibiting in a nutshell a number of properties that attract the attention of both physicists and mathematicians: Poisson-Lie symmetry [45, 4, 9, 10, 2, 11], $r$- ($R$-) matrices (classical and quantum) [13, 45, 28], dynamical $r$- ($R$-) matrices [37, 31, 26, 43, 42, 25, 24, 3]. The study of the $SU(2)$ case in [33] was extended to $SU(n)$ in [38] and [32], $\mathfrak{sl}(n)$ being singled out among other simple Lie algebras by the fact that the corresponding quantum $R$-matrices satisfy quadratic (Hecke algebra) relations. The gauge freedom in the very definition of the zero mode phase space was discussed in
and its BRS (co)homology was studied in [22, 23] (for a concise review see [21]). The presence of such a freedom allows, in particular, to avoid the complications of the unitary gauge advocated in [9, 10].

1.4 Outline of the paper

After sketching (in Section 2.1) the derivation of the expression (2.10) for $\Omega_q$ that generalizes (1.14) to any compact semisimple Lie group $G$, we study in Section 2.2 an extension $M_q^{ex}$ of the phase space $M_q$ for $G = SU(n)$ for which one derives a more manageable symplectic form $\Omega_q^{ex}$. In Section 2.3 we display the undeformed limit $k \to \infty$ ($q \to 1$) in which the WZ term disappears. The resulting form $\Omega_1$ can be easily inverted. We also display the Hamiltonian vector fields corresponding to the constraints $\chi := \log \frac{D_q(p)}{p}$ and $P := \frac{1}{n} \sum_{s=1}^{n} p_s$. In particular,

$$i \frac{\hat{\partial}}{\partial P} \Omega_q^{ex} = i \sum_{s=1}^{n} \frac{\hat{\partial}}{\partial p_s} \Omega_q^{ex} = d\chi = \frac{dD}{D} - \frac{dD_q(p)}{D_q(p)}.$$

Here $\hat{X} \Omega$ means the contraction of the vector field $X$ with the form $\Omega$; we have, e.g.,

$$\frac{\hat{\partial}}{\partial p_s} dp_j = \delta^s_j - dp_j \frac{\hat{\partial}}{\partial p_s}.$$

It is important that these "momentum maps" remain valid after $q$-deformation (i.e., for finite $k$). Section 3 is devoted to inverting the form $\Omega_q^{ex}$ (and $\Omega_q$), thus computing PB among zero modes. In Section 4.1 we demonstrate that the quasiclassical limit ($k \gg n$, $p_{j\ell} \gg 1$, $\frac{p_{j\ell}}{k}$ finite) of the quantum exchange relations of [38, 32, 39] reproduces the PB relations of Section 3. In the rest of Section 4 we review the $U_q(s\ell_n)$ symmetry of the quantum matrix algebra and its operator realization.

2 Zero modes’ phase space from chiral WZNW 2-form

2.1 From 2D canonical 3-form to zero modes’ symplectic form

The canonical approach to a field theory in $D$-dimensional space-time formulated in [35] (where its sources are cited and reviewed) starts with a closed $(D + 1)$-form $\omega$ ($= dL(x)$ if a Lagrangean $D$-form $L(x)$ exists). It allows to read off the equations of motion while the integral over a $(D - 1)$-dimensional space-like surface provides the symplectic form of the theory. A form of this
type, called symplectic density, was recently (partly rediscovered and) applied to Yang-Mills, general relativity, Chern-Simons and supergravity theories [40]. In the case of the WZNW model the 3-form $\omega$ can be written as the sum of an exact form and the canonical invariant closed 3-form on the group $G$,

$$\omega = d \left\{ \frac{1}{2} \text{tr} \left( i g^{-1} dg + \frac{\pi}{k} J \right)^* J \right\} + \frac{k}{12 \pi} \text{tr} (g^{-1} dg)^3, \quad (2.1)$$

where $J$ is the current 1-form and $^* J$ is its Hodge dual:

$$J(x) = j_\mu(x) dx^\mu, \quad ^* J(x) = \varepsilon_{\mu\nu} j^\nu(x) dx^\mu \quad (\varepsilon_{\mu\nu} = -\varepsilon_{\nu\mu}, \varepsilon_{01} = 1 = \varepsilon^{10}). \quad (2.2)$$

We shall sum up without derivation the implications of Eq.(2.1). The equations of motion, obtained as the pull-back of the contractions of $\omega$ with the vertical vector fields $\frac{\delta}{\delta j_\mu(x)}$ and $g(x) X \frac{\delta}{\delta g(x)}$, read

$$J = \frac{k}{2\pi i} g^{-1} dg, \quad dJ + \frac{2\pi i}{k} J^2 = 0 \quad \Rightarrow \quad d(J + ^* J) = 0. \quad (2.3)$$

They imply the existence of left and right (Nöther) currents depending on a single light cone variable,

$$j_R = \frac{1}{2}(j^0 + j^1), \quad j_L = \frac{1}{2} g^{(j^1 - j^0)g^{-1}}, \quad \partial_+ j_R = 0 = \partial_- j_L \quad \text{for} \quad \partial_\pm = \frac{1}{2}(\partial_1 \pm \partial_0), \quad (2.4)$$

and the factorization (1.1) of $g(x^0, x^1)$.

The symplectic form $\Omega^{(2)}$ can be expressed in terms of either of the two chiral currents:

$$\Omega^{(2)} = \int_\pi^{-\pi} \omega dx^1 = \quad (2.5)$$

$$= - \int_\pi^{-\pi} dx \text{tr} \left( i d(j_L dg^{-1}) + \frac{k}{4\pi} d g^{-1} (dg g^{-1})' \right) =$$

$$= \int_\pi^{-\pi} dx \text{tr} \left( i d(j_R g^{-1} dg) + \frac{k}{4\pi} g^{-1} d g (g^{-1} dg)' \right).$$

Inserting the factorized expression (1.1) for $g$ in (2.5), one can split $\Omega^{(2)}$ into chiral symplectic forms

$$\Omega^{(2)} = \Omega(g_L, M) - \Omega(g_R, M), \quad (2.6)$$

where

$$\Omega(g_C, M) = \frac{k}{4\pi} \left\{ \text{tr} \left( \int_\pi^{-\pi} d x (g_C^{-1} d g_C (g_C^{-1} d g_C)') + b_{C}^{-1} d b_C d M M^{-1} \right) - \rho(M) \right\} \quad (2.7)$$
with

\[ b_C := g_C(-\pi), \quad M = b_C^{-1} g_C(\pi) \left( = g_C^{-1}(x) g_C(x + 2\pi) \right), \quad C = L, R. \] (2.8)

The cumbersome (ill defined) WZ term \( \rho(M) \) (satisfying (1.15)) has been added and subtracted from the two chiral terms to ensure \( d\Omega = 0 \). An alternative approach, introducing quasi-Poisson manifolds [2] (for which the Jacobi identity satisfied by proper PB is replaced by a weaker condition) is developed in [11].

Finally, substituting \( g_L(x) \) by its expression (1.3), we find

\[ \Omega(g_L, M) = \Omega(u, M_p) + \omega_q(M_p) + \Omega_q(a, M_p) \] (2.9)

where

\[ \Omega_q(a, M_p) = \] (2.10)

\[ = \frac{k}{4\pi} \{ \text{tr} (daa^{-1}(2dM_pM_p^{-1} + M_pdaa^{-1}M_p^{-1})) - \rho(a^{-1}M_pa) \} - \omega_q(M_p) \]

and \( \omega_q \) is an arbitrary closed 2-form (which will be restricted further by some symmetry conditions). For \( G = SU(2) \) there is a single variable \( p \), hence \( \omega_q(M_p) \equiv 0 \) and (2.10) coincides with (1.14).

A detailed derivation of the results formulated in this subsection will be presented elsewhere.

2.2 Basis of right invariant 1-forms. An extended phase space and a privileged choice of \( \omega_q \) for \( G = SU(n) \)

We shall now write down the first two terms in the expression (2.10) as sums of products of right invariant forms. To this end we shall use the Cartan-Weyl basis \( \{ h_i, e_\alpha \} \), \( \alpha \) running through the positive roots of \( \mathcal{G}_\mathbf{a} \) (in its \( n \)-dimensional fundamental representation) satisfying

\[ [h_i, h_j] = 0, \quad [h_i, e_{\pm\alpha}] = \pm 2\frac{(\alpha|\alpha_j)}{|\alpha_j|^2} e_{\pm\alpha}, \quad [e_{\alpha}, e_{-\alpha}] = \delta_{\alpha\beta} h_j \] (2.11)

\( (i, j = 1, \ldots, r := \text{rank } \mathcal{G}, \) and shall write

\[ \hat{p} = \sum_{j=1}^{r} p_{\alpha_j} h^j \quad \text{with} \quad \text{tr} (h^i h_j) = \delta^i_j \quad (\text{and} \quad \text{tr} (e_{\alpha} e_{-\beta}) = \delta_{\alpha\beta}) \] (2.12)

(thus \( \{ h_i \} \) and \( \{ h^j \} \) define dual bases of diagonal matrices). Let further \( \Theta^j, \Theta^{\pm\alpha} \) and \( \frac{dD}{\theta} \) be the corresponding right invariant 1-forms in \( T^* G^e_{\mathbf{a}} \),

\[ G^e_{\mathbf{a}} := (G \times \mathbb{R}_+)^g, \] (2.13)
defined by
\[ \Theta^j = -i \text{tr} (a^{-1} h^j da), \quad \Theta^\pm = -i \text{tr} (a^{-1} e_\pm da), \quad \frac{dD}{D} = \text{tr} (daa^{-1}). \]

(2.14)

It then follows that
\[ -idaa^{-1} = \sum_{j=1}^r \Theta^j h_j + \sum_{\alpha > 0} (\Theta^\alpha e_\alpha + \Theta^{-\alpha} e_{-\alpha}) - \frac{i}{n} \frac{dD}{D} \mathbb{I} \]
where \( D = \det a > 0 \) and \( \mathbb{I} \) is the \( n \times n \) unit matrix.

If \( G \) is compact, then the forms \( \Theta^j \) are real while \( \Theta^{-\alpha} \) are complex conjugate to \( \Theta^\alpha \). We also note that the (Lie algebra valued) 1-form (2.15) is not closed but defines a flat connection, the \( \Theta \)'s satisfying the Cartan-Maurer relations. We shall use, in particular,
\[ d\Theta^j = i \sum_{\alpha > 0} (\Lambda^j | \alpha) \Theta^\alpha \Theta^{-\alpha} \quad (\Lambda^j | \alpha_\ell) = \delta^j_\ell, \]
(2.16)

\( \Lambda^j \) being the fundamental weights of \( G \).

Inserting (2.12) into the first term in the right hand side of (2.10) and using (1.15) and (2.14), we deduce
\[ \frac{k}{2\pi} \text{tr} (daa^{-1} dM_p M^{-1}_p) = i \text{tr} (daa^{-1} \hat{p}) = \sum_{j=1}^r dp_\alpha, \Theta^j. \]
(2.17)

The second term is expressed as a sum of products of conjugate off-diagonal forms:
\[ \frac{k}{4\pi} \text{tr}(daa^{-1} M_p daa^{-1} M^{-1}_p) = \frac{k}{4\pi} (\mathcal{F} - q) \sum_{\alpha > 0} [2p_\alpha] \Theta^\alpha \Theta^{-\alpha}, \]
(2.18)

where \( p_\alpha \) is a linear functional on the roots:
\[ p_\alpha = \sum_{j=1}^r (\Lambda^j | \alpha) p_{\alpha_j} \quad \text{for} \quad \alpha = \sum_{j=1}^r (\Lambda^j | \alpha) \alpha_j; \]
(2.19)

here \( (\Lambda^j | \alpha) \in \mathbb{Z}_+ \) and we have the relation
\[ \text{Ad}_{M_p} e_\alpha := M_p e_\alpha M_p^{-1} = \overline{q^{2p_\alpha}} e_\alpha. \]
(2.20)

At this point we shall specialize to the case \( G = SU(n) \) and will view the \( (n-1)(n+2) \)-dimensional symplectic manifold \( \mathcal{M}_q = \mathcal{M}_q(n) \) as a submanifold of codimension 2 in the extended \( (n(n+1)) \)-dimensional phase space \( \mathcal{M}_q^\text{ex} \) spanned by \( p_i \) and \( a^j_\alpha \) \( (i, j, \alpha = 1, \ldots, n) \) regarded as independent variables:
\[ \mathcal{M}_q = \left\{ (p_i, a^j_\alpha) \in \mathcal{M}_q^\text{ex} ; \quad P := \frac{1}{n} \sum_{s=1}^n p_s = 0, \quad \chi := \log \frac{D}{D_q(p)} = 0 \right\}. \]
(2.21)
We introduce the Weyl basis \( \{ e^j_i \} \) of \( n \times n \) matrices satisfying
\[
e^j_i e^\ell_k = \delta^j_k \delta^\ell_i, \quad (e^j_i)^\ell_k = \delta_i^\ell \delta^j_k, \quad i, j, k, \ell = 1, \ldots, n.
\] (2.22)

The positive roots \( \alpha_{ij} \) \((i < j)\) of \( su(n) \) correspond to raising operators, \( e^j_i \), while \(-\alpha_{ij} \) are associated with lowering ones, \( e^i_j \). Eq. (2.15) now assumes a simple explicit form:
\[
-ida^{-1} = \Theta^j_k e^k_j \left( \equiv \sum_{j,k=1}^n \Theta^j_k e^k_j \right), \quad \Theta^j_k = -itr \left( e^j_k daa^{-1} \right) = -ida^j_k (a^{-1})^k_j.
\] (2.23)

The general Cartan-Maurer relations (which incorporate (2.16)) are written simply as
\[
d\Theta^j_k = i \Theta^s_j \Theta^k_s.
\] (2.24)

Recalling that the relation (1.7) is invariant under simultaneous permutation of the rows of the matrix \( (a^j_i) \) and of \( p_j \) (i.e., under the action on both sides of the \( su(n) \) Weyl group), we shall also require permutation invariance of the extended form \( \omega_{q}^{ex}(p) \). We shall determine \( \omega_{q}(M_p) = \omega_{q}^{ex}(p) \mid_{P=0} \) by further demanding that the symplectic form \( \Omega_q^{ex} \) on \( \mathcal{M}_{q}^{ex} \),
\[
\Omega_q^{ex} = \sum_{s=1}^n dp_s \Theta^s_k \left\{ (q - \bar{q}) \sum_{j<\ell} \left[ 2p_{j\ell} \Theta^j_k \Theta^k_j + \rho(a^{-1}M_p a) \right] \right\} - \omega_{q}^{ex}(p)
\] (2.25)
will reduce to \( \Omega_q \) on the surface \( \mathcal{M}_q \subset \mathcal{M}_q^{ex} \). In order to implement this last condition, we shall require that the terms involving \( dP \) cancel in the difference
\[
-\omega_q(M_p) = \left( -idP \frac{dD_q(p)}{D_q(p)} \right) - \omega_q^{ex}(p).
\] (2.26)

Inserting the expression (cf. (1.7)) for \( D_q(p) \) which implies
\[
\frac{dD_q(p)}{D_q(p)} = \frac{\pi}{k} \sum_{j<\ell} \cotg \left( \frac{\pi}{k} p_{j\ell} \right) dp_{j\ell},
\] (2.27)
we find a form \( \omega_{q}^{ex}(p) \) satisfying all above conditions:
\[
\omega_{q}^{ex}(p) = i\frac{\pi}{k} \sum_{j<\ell} \cotg \left( \frac{\pi}{k} p_{j\ell} \right) dp_{j\ell} dp_{\ell}.
\] (2.28)

Indeed, using the relation
\[
p_j = P + \frac{1}{n} \sum_{s=1}^n p_{js},
\] (2.29)
we deduce

$$\omega_q(M_p) = \frac{i\pi}{nk} \sum_{1 \leq j < \ell < m \leq n} \left( \cotg(\frac{\pi}{k} p_{j\ell}) + \cotg(\frac{\pi}{k} p_{\ell m}) - \cotg(\frac{\pi}{k} p_{jm}) \right) dp_{j\ell} dp_{\ell m}$$

(2.30)

(note that for \(n = 2\) there is no triple \(j, \ell, m\) satisfying the above inequalities so that the form \(\omega_q(M_p)\) vanishes, as it should, while \(\omega^e_q(p)\) (2.28) reduces to a single term: \(\omega^e_q(p) = i \frac{\pi}{k} \cotg(\frac{\pi k p}{p}) dp_1 dp_2\)).

We observe the relative simplicity of the extended symplectic form (2.25), (2.28) as compared with \(\Omega_q\) (obtained from (2.10) by inserting (2.17) with \(p_\alpha_j = p_{jj} + 1, \Theta_j = n - j \sum_{s=1}^{j} \Theta_s - \frac{j}{n} \sum_{s=j+1}^{n} \Theta_s, (2.31)\) (2.18) and (2.30)). It is, therefore, rewarding to know that the PB we are interested in can be computed using the simpler expression \(\Omega^e_q\), as we shall see in Section 3. In the next subsection we shall display this property for the \(k \to \infty\) limit theory.

### 2.3 Right invariant vector fields. The limit \(k \to \infty\).

**Dirac brackets**

It is easy to display the basis of right invariant vector fields \(\{ \frac{\partial}{\partial p_\ell}, V^k_j \}\) dual to the basis \(\{ dp_\ell, \Theta^k_j \}\) of 1-forms:

$$V^k_j = i \text{tr} \left( e^k_\ell a \frac{\partial}{\partial a} \right) = ia^k_\sigma \frac{\partial}{\partial a^\sigma}. \quad (2.32)$$

Indeed, contracting the form \(\Theta^k_m\) (2.23) with \(V^k_j\), we find

$$\hat{V}^k_j \Theta^k_m = \text{tr} (e^k_\ell a a^{-1} e^\ell_m) = \delta^k_j \delta^k_m, \quad \hat{V}^k_j dp_\ell = 0; \quad (2.33)$$

obviously,

$$\frac{\partial}{\partial p_\ell} \Theta^k_m = 0, \quad \frac{\partial}{\partial p_\ell} dp_\ell = \delta^k_\ell.$$ 

This would allow to invert the form \(\Omega^e_q\) but for the WZ term.

We shall profit from the above remark taking the limit \(k \to \infty\) in which the WZ term disappears. Indeed, using the expression for \(q\) in (1.4), we find

$$\lim_{k \to \infty} \frac{k}{2\pi} (\bar{q} - q) = i, \quad \frac{1}{2} \lim_{k \to \infty} \lfloor 2p \rfloor = p \quad (2.34)$$

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and hence,

\[
\Omega^{ex}_1(a,p) = \sum_{s=1}^{n} dp_s \Theta^s - i \sum_{1 \leq j < \ell \leq n} p_{j \ell} \Theta^j \Theta^\ell - \omega_1(p) =
\]

\[
d \sum_{s=1}^{n} p_s \Theta^s - i \sum_{1 \leq j < \ell \leq n} \frac{dp_j dp_\ell}{p_{j \ell}}.
\]  

(2.35)

Here we have set

\[
\lim_{k \to \infty} \frac{k^4}{4 \pi} \rho(a^{-1} M_p a) = 0 .
\]  

(2.36)

In fact, since the right hand side of (2.35) is a closed 2-form, it follows that

\[
\lim_{k \to \infty} \frac{k^4}{4 \pi} \text{tr} (dM M^{-1})^3 = 0 \quad \text{for} \quad M = a^{-1} M_p a ;
\]  

(2.37)

we conclude that \( \frac{k^4}{4 \pi} \rho \) can be also chosen to vanish in this limit – a property that can be derived from the expression for \( \rho(a^{-1} M_p a) \) given in Section 3.

As anticipated, it is straightforward to invert the 2-form (2.35). The result can be encoded in the Poisson bivector

\[
P = \sum_{s=1}^{n} V^s \wedge \frac{\partial}{\partial p_s} + i \sum_{1 \leq j < \ell \leq n} \frac{1}{p_{j \ell}} (V^j \wedge V^\ell - V^\ell \wedge V^j ),
\]  

(2.38)

which gives rise to the following PB:

\[
\{ p_j, p_\ell \} = 0 , \quad \{ a^j_\alpha, p_\ell \} = i \delta^j_\ell \delta^j_\ell \]

and

\[
\{ a^j_\alpha , a^\ell_\beta \} = r^{(1)}(p)^{j \ell}_{j' \ell'} a^j_\alpha a^\ell_\beta \quad \text{(i.e.,} \quad \{ a_1 , a_2 \} = r^{(1)}(p)_{12} a_1 a_2 \}
\]  

(2.39)

(2.40)

where the undeformed classical dynamical \( r \)-matrix is given by

\[
r^{(1)}(p)^{j \ell}_{j' \ell'} = \begin{cases} i \frac{1}{p_{j \ell}} (\delta^j_\ell \delta^\ell_{j'} - \delta^j_{j'} \delta^\ell_{\ell'}) & \text{for } j \neq \ell \\ 0 & \text{for } j = \ell \end{cases}.
\]  

(2.41)

For a general Poisson manifold \( \mathcal{M} \) with a pair of second class constraints \( P \) and \( \chi \) the Dirac brackets \( \{ f, g \}_D \) [20] of two arbitrary functions on \( \mathcal{M} \) are expressed in terms of their PB as

\[
\{ f, g \}_D = \{ f, g \} + \frac{1}{\{ P, \chi \}} (\{ f, P \} \{ \chi, g \} - \{ f, \chi \} \{ P, g \} ) .
\]  

(2.42)

We shall verify that in the case at hand

\[
\{ p_{j \ell} , P \} = 0 = \{ p_{j \ell} , \chi \} , \quad \{ a^j_\alpha , \chi \} = 0 .
\]  

(2.43)
The first pair of equations implies that $p_{j\ell}$ are ”observables” on $M_q \subset M_q^{ex}$, so that $\{p_{j\ell}, f\}_D = \{p_{j\ell}, f\}$ for any function $f$ on $M_q^{ex}$; in particular,
\[
\{p_{j\ell}, a^m_\alpha\} = i(\delta^m_j - \delta^m_\ell)a^m_\alpha = \{p_{j\ell}, a^m_\alpha\}_D.
\]
(2.44)

The last equation (2.43) is sufficient to assert that the PB (2.40) coincide with the corresponding Dirac brackets.

Although it is easy to verify (2.43) directly, using (2.38)-(2.40), we shall give a more general derivation that will apply to the case of finite $k (q \neq 1)$ as well. To this end we shall use the momentum maps
\[
i\frac{\partial}{\partial P}\Omega_1 = i\sum_{s=1}^{n} \frac{\partial}{\partial p_s}\Omega_1 = i\sum_{s=1}^{n} \Theta^s - \sum_{1 \leq j < \ell \leq n} \frac{dp_{j\ell}}{p_{j\ell}} = d\chi; \quad \frac{1}{n}\sum_{s=1}^{n} \hat{V}^s\Omega_1 = dP.
\]
(2.45)

Displaying the Hamiltonian vector fields corresponding to $\chi$ and $P$, Eq. (2.45) allows to compute any PB of the constraints; in particular,
\[
\{\chi, a^j_\alpha\} = i\frac{\partial}{\partial P} da^j_\alpha = 0, \quad \{\chi, p_{j\ell}\} = i\frac{\partial}{\partial P} dp_{j\ell} = 0, \\
\{p_{j\ell}, P\} = \frac{1}{n}\sum_{s=1}^{n} \hat{V}^s dp_{j\ell} = 0, \quad \{P, \chi\} = -i.
\]
(2.46)

We find, on the other hand,
\[
\{a^j_\alpha, p_{\ell}\}_D = \{a^j_\alpha, p_{\ell}\} + i\{a^j_\alpha, P\}\{\chi, p_{\ell}\} = ia^j_\alpha(\delta^j_\ell - \frac{1}{n}).
\]
(2.47)

3 Inverting $\Omega_q^{ex}$. PB in $M_q(n)$

3.1 The WZ form

It was Gawędzki [35] (see also [29]) who introduced in the early 1990’ies the WZ 2-form $\rho(M)$ and described its relation to the non-degenerate (constant) solutions of the classical Yang-Baxter equation (CYBE). Gradually, a more general and complete understanding of such a relation has been worked out [10, 30]. We shall only deal here with a special case of the outcome of [30] corresponding essentially to the early discussion in [29].

We shall again start with an arbitrary semisimple matrix Lie group $G$ with Lie algebra $\mathfrak{g}$. For an arbitrary pair $\{t^a\}, \{T_b\}$ of dual bases in $\mathfrak{g}$, we can write the Killing metric tensor $\eta_{ab}$ and its inverse, $\eta^{ab}$, as
\[
\eta_{ab} = \text{tr} (T_a T_b), \quad \eta^{ab} = \text{tr} (t^a t^b) \quad \text{for} \quad \text{tr} (t^a T_b) = \delta^a_b.
\]
(3.1)

In the Cartan-Weyl basis $\{T_a\} = \{h_i, e_{\pm\alpha}\}$ we have $\{t^a = h^i, e_{\pm\alpha}\}$ and the nonzero elements of $\eta$ are
\[
\eta_{ij} = \text{tr} h_i h_j = (\alpha_i|\alpha_j), \quad \eta_{\alpha\beta} = \text{tr} e_\alpha e_\beta = \delta_{\alpha,-\beta}
\]
(3.2)
(where the norm square of the highest root is fixed to 2).

The polarized Casimir invariant $C_{12} \in \text{Sym}\left(G \otimes G\right)$, given (in Faddeev’s notation [28]) by

$$C_{12} = \eta_{ab} t_1^a t_2^b = T_a \otimes t^a \equiv t^a \otimes T_a = h_1^a h_2^b + \sum_\alpha e_{\alpha 1} e_{-\alpha 2} \quad (3.3)$$

where the sum is taken over all, positive and negative, roots $\alpha$, plays the role of the unit operator on $G$:

$$CX := \text{tr}_2 (C_{12} X_2) = X (\equiv X_1) \quad \text{for} \quad X \in G. \quad (3.4)$$

Let $r_{12} = -r_{21} (\in G \wedge G)$ be a solution of the modified CYBE

$$[r_{12}, r_{13} + r_{23}] + [r_{13}, r_{23}] = [C_{12}, C_{23}] = -f_{abc} t^a t^b t^c, \quad (3.5)$$

and let $r$ be the corresponding operator ($r : G \to G$) defined by taking the trace in the second argument as in (3.4):

$$rX := \text{tr}_2 (r_{12} X_2) \quad \text{for} \quad X \in G \quad \Rightarrow \quad r_{12} = r C_{12}. \quad (3.6)$$

**Proposition 3.1** Let the 2-form $\rho(M)$ be written in terms of a skew-symmetric kernel $K(M)_{12} \in G \wedge G$ for $M \in G_0$ where $G_0$ is an open dense neighbourhood of the group unit in which the operator $(1 - \text{Ad}_M) r + 1 + \text{Ad}_M$, $\text{Ad}_M X := MXM^{-1}$, is invertible, and let $K(M)$ be the corresponding operator $K(M) : G \to G$ defined as in (3.6),

$$\rho(M) = \frac{1}{2} \text{tr} (dMM^{-1} K(M) dMM^{-1}) , \quad (\text{tr} (XK(M)X) = 0 \ \forall X \in G). \quad (3.7)$$

Assume further that

$$K(M) = ((1 + \text{Ad}_M) r + 1 - \text{Ad}_M)((1 - \text{Ad}_M) r + 1 + \text{Ad}_M)^{-1} \quad (3.8)$$

so that $K(1) = r$. Then $\rho(M)$ satisfies

$$d\rho(M) = \frac{1}{3} \text{tr} (dMM^{-1})^3 \quad (3.9)$$

iff $r_{12}$ (related to $r$ by (3.6)) satisfies the modified CYBE (3.5).

The statement is a corollary of Propositions 1 and 2 of [30]; see also the earlier discussion in [29].

**Remark 3.1** The modified CYBE (3.5) for $r_{12}$ is equivalent to the standard CYBE

$$[r_{12}^+, r_{13}^+ + r_{23}^+] + [r_{13}^+, r_{23}^+] = 0 \quad \text{for} \quad r_{12}^+ = r_{12} \pm C_{12}. \quad (3.10)$$
In fact, Ref. [29] deals with Eq. (3.10).

**Remark 3.2** Using (3.8), it is easy to check that the skewsymmetry of \( K(M) \) is equivalent to that of \( r \), \( \text{tr} K(M) = -K(M) \) ⇔ \( \text{tr} r = -r \) where the transposition \( \text{tr} \) is w.r. to the invariant bilinear form \( \text{tr} \).

**Remark 3.3** One can consider a more general Ansatz of type (3.8) allowing the operator \( r \) to depend on \( M \). Then one has to deal with a "dynamical" version of the (modified) CYBE including differentiation in the group parameters – see Eqs. (1.3) and (3.8) of [30]. One argues in [10] that a constant classical \( r \)-matrix cannot correspond to a compact group \( G \). It is well known, indeed, that the modified CYBE (3.5) has no real solution in \( G \wedge G \) for \( G \) compact. We shall however stick to the above simple choice which uses a complex 2-form \( \rho \).

Using the operators
\[
A_\pm := 1 \pm \text{Ad}_{M_p}, \quad A_- dM_p M_p^{-1} = 0 = (A_+ - 2)dM_p M_p^{-1},
\]
while \( K^a \) is given by
\[
K^a := \text{Ad}_a K(a^{-1}M_p a) \text{Ad}_{a^{-1}} = (A_+ r^a + A_-)(A_- r^a + A_+)^{-1},
\]
\[
r^a := \text{Ad}_a r \text{Ad}_{a^{-1}}
\]
\( (K^a \) and \( r^a \) are skewsymmetric together with \( K(M) \) and \( r \)).

Using (3.7) and (3.8), we can present the WZ 2-form in (2.10) as
\[
\rho(a^{-1}M_p a) = \frac{1}{2} \text{tr} \{ (dM_p M_p^{-1} - A_-(daa^{-1})) K^a (dM_p M_p^{-1} - A_-(daa^{-1})) \}.
\]
(3.11)

Here and below we are using the operators
\[
A_\pm := 1 \pm \text{Ad}_{M_p}, \quad A_- dM_p M_p^{-1} = 0 = (A_+ - 2)dM_p M_p^{-1},
\]
while \( K^a \) is given by
\[
K^a := \text{Ad}_a K(a^{-1}M_p a) \text{Ad}_{a^{-1}} = (A_+ r^a + A_-)(A_- r^a + A_+)^{-1},
\]
\[
r^a := \text{Ad}_a r \text{Ad}_{a^{-1}}
\]
\( (K^a \) and \( r^a \) are skewsymmetric together with \( K(M) \) and \( r \)).

We note that \( \rho(a^{-1}M_p a) \) coincides with its extension on \( \mathcal{M}_{q}^\text{ex}(n) \). This is obviously true for the 3-form \( d\rho(M) \) (3.9) for \( M = a^{-1}M_p a \). Indeed, the contribution of the term proportional to \( dP \) is given by
\[
d\rho^\text{ex}(a^{-1}M_p a) - d\rho(a^{-1}M_p a) = \frac{2\pi i}{k} dP \text{tr} (dM M^{-1})^2 \equiv 0.
\]
(3.14)

(Remember that the diagonal monodromy \( M_p \) enters in \( \Omega_q^\text{ex} \) through
\[
\]
\[
dM_p M_p^{-1} = \frac{2\pi i}{k} \sum_{s=1}^{n} dp_s \epsilon_s^s \equiv \frac{2\pi i}{k} (dP \mathbf{1} + d\hat{p}),
\]
(3.15)
cf. (2.12), and \( \det a \) is not set equal to \( D_q(p) \). Since \( \rho \) is defined by Eq. (3.9), it can be left unchanged in the extended phase space. This is certainly true for the expression (3.11)-(3.13) provided we take – as we will – the standard solution of the modified CYBE (3.5)

\[
r_{12} = \sum_{\alpha > 0} (e_\alpha \otimes e_{-\alpha} - e_{-\alpha} \otimes e_\alpha) \equiv \sum_{\alpha > 0} (e_{1\alpha}e_2^\alpha - e_1^\alpha e_{2\alpha}) \quad (e^\alpha := e_{-\alpha})
\]

for which

\[
r_{\pm \alpha} = \pm e_{\pm \alpha} \quad \text{for} \quad \alpha > 0, \quad r_h = 0 = r I.
\]

### 3.2 Poisson bivector for \( \Omega^e_q \)

We shall first establish the relation (1.18) which, according to (2.46), is sufficient to prove that the PB of \( a_j^q \) can be computed using the form \( \Omega^e_q \) (2.25), (2.28) on \( M^e_q(n) \). Eq. (1.18) follows from (2.26)-(2.28) and (3.14) (together with the subsequent argument), which imply

\[
\hat{\partial} \frac{\partial}{\partial P} \rho(a^{-1}M_p \alpha) = 0.
\]

Similarly, one can deduce

\[
\sum_{s=1}^n \hat{V}^s \rho(a^{-1}M_p \alpha) = 0 \implies -\frac{1}{n} \sum_{s=1}^n \hat{V}^s \Omega^e_q = dP,
\]

which extends the second equation (2.45) to \( q \neq 1 \).

Using (3.11)-(3.13), we can write the extended symplectic form (2.25) as

\[
\Omega^e_q = \sum_{s=1}^n dp_s \Theta^s + \frac{k}{2\pi} \sum_{j \neq \ell, r \neq s} \Theta^j \Theta^r (\omega - X)^{-1} t_{jr} + \sum_{j, r \neq s, t \neq q} dp_j \Theta^r X_{jq}^s (\omega - X)^{-1} t_{sr} - \frac{\pi}{2k} \sum_{j \neq \ell} dp_j dp_\ell \left( \omega^{j\ell} + X_{j\ell}^s + \sum_{s \neq t, s' \neq t'} X_{jt}^{s's'} (\omega - X)^{-1} t_{s't'} X_{t'\ell}^s \right)
\]

where

\[
\omega^{j\ell} := i \cotg \frac{\pi}{k} p_{j\ell}, \quad \omega^{nj}_{m\ell} = -\omega^{j\ell} \delta^j_m \delta^i_{\ell} = \omega^{nj}_{m\ell} \delta^j_m
\]

so that

\[
\omega^e_q(p) = \frac{\pi}{k} \sum_{j \neq \ell} \omega^{j\ell} dp_j dp_\ell, \quad \frac{A^*}{A} e^j = -\omega^{j\ell} e^j = \omega^{nj}_{m\ell} e^m
\]
(because $Ad_{M_p} e^j_\ell = q^{2p j} e^j_\ell$ ) and $X^n_{m\ell}$ is defined as
\[
r^a e^j_\ell = -X^n_{m\ell} e^m_n \quad \Rightarrow \quad X_{12} = -Ad_{a_1\alpha_2} r_{12} \quad (X_{12} = -X_{21}).
\] (3.23)

To derive (3.21), one uses (2.10), (3.11), (2.23) and (3.15) as well as (3.21), (3.22). This allows to present $\Omega^{ex}_{n}$ as
\[
\Omega^{ex}_{n} = -\omega^{ex}_{n}(p) - \frac{k}{8\pi} \text{tr} \{ (A_- daa^{-1})(A_+ - K^a A_-)(daa^{-1}) + \\
2 dM_p M_p^{-1} (2 - K^a A_-)(daa^{-1}) + dM_p M_p^{-1} K^a (dM_p M_p^{-1}) \} \quad (3.24)
\]

Note that the only nonzero contribution of the diagonal elements of $daa^{-1}$ comes through the term
\[
-\frac{k}{2\pi} \text{tr} (dM_p M_p^{-1} daa^{-1}) = \sum_{s=1}^{n} dp_s \Theta^s_s. \quad (3.25)
\]

One also uses relation (3.13) and its corollary
\[
(A_+ - K^a A_-) r^a = K^a A_+ - A_- \quad \Rightarrow \quad A_+ - K^a A_- = 4 Ad_{M_p} (r^a A_- + A_+)^{-1} \quad (3.26)
\]
as well as
\[
\frac{k}{4\pi} \text{tr} dM_p M_p^{-1} K^a A_- (daa^{-1}) = \frac{k}{2\pi} \text{tr} dM_p M_p^{-1} r^a (r^a + \frac{A_+}{A_-})^{-1} daa^{-1} = \\
\sum_{j, r \neq s, t \neq q} dp_j \Theta^r_s X^{j\ell}_{tq} [ (\omega - X)^{-1}]_{tr}^{qs} \quad (3.27)
\]
and
\[
\text{tr} dM_p M_p^{-1} K^a (dM_p M_p^{-1}) = \text{tr} dM_p M_p^{-1} r^a (dM_p M_p^{-1}) + \\
+ \text{tr} dM_p M_p^{-1} r^a [(1 + \frac{A_-}{A_+} r^a)^{-1} - 1] (dM_p M_p^{-1}), \quad (3.28)
\]

where the second term in the right hand side gives
\[
\text{tr} (r^a dM_p M_p^{-1}) [(1 + \frac{A_-}{A_+} r^a)^{-1} \frac{A_-}{A_+} r^a] (dM_p M_p^{-1}) = \\
= \frac{4\pi^2}{k^2} \sum_{j \neq \ell} dp_j dp_\ell \sum_{s \neq t, s' \neq t'} X^{j\ell}_{st} [ (\omega - X)^{-1}]_{ss'} X_{t' t}. \quad (3.29)
\]

The PB derived from $\Omega^{ex}_{n}$ can be compactly written in terms of the Poisson bivector
\[
\mathcal{P} = \sum_{m=1}^{n} V^m_m \wedge \frac{\partial}{\partial p_m} + \\
+ \frac{\pi}{2k} \left( \sum_{n \neq m} \omega^{mn} (V^m_m \wedge V^n_n - V^m_m \wedge V^n_n) - \sum_{n, m, s, t} X^{ns}_{mt} V^m_m \wedge V^n_n \right) \quad (3.30)
\]
obeying the operator equation
\[ P_{12} (\Omega_q^{ex})_{23} = I_{13}. \] (3.31)

Here \( I \) is the mixed \((1,1)\)-tensor
\[ I = \sum_{j=1}^{n} (\frac{\partial}{\partial p_j} \otimes dp_j + V_j^j \otimes \Theta_j^j) + \sum_{j \neq \ell} V_j^\ell \otimes \Theta_j^\ell, \] (3.32)

which plays the role of the identity operator in the space of 1-forms \( \Theta \), resp. vector fields \( X \) in the sense
\[ \Theta I := \sum_{j=1}^{n} \Theta \left( \frac{\partial}{\partial p_j} \right) dp_j + \sum_{j,\ell=1}^{n} \Theta (V_j^j) \Theta_\ell^\ell = \Theta \quad (\Theta (X) \equiv \hat{X} \Theta), \]
\[ I X := \sum_{j=1}^{n} \frac{\partial}{\partial p_j} (X) + \sum_{j,\ell=1}^{n} V_j^\ell \Theta_\ell^j (X) = X. \] (3.33)

We find, in particular,
\[ \{a_1, a_2\} \equiv P_{12}(a_1, a_2) = r_{12}(p)a_1a_2 - \frac{\pi}{k} a_1a_2r_{12} \] (3.34)

where
\[ r(p)_{j\ell}^{j\ell} = \begin{cases} i\frac{\pi}{k} \cotg(\frac{\pi}{k} p_{j\ell}) (\delta_{j\ell}^j \delta_{j\ell}^\ell - \delta_{j\ell}^\ell \delta_{j\ell}^j) & \text{for } j \neq \ell \\ 0 & \text{for } j = \ell \end{cases} \] (3.35)

and
\[ r_{\alpha\beta}^{\alpha\beta} = -\epsilon_{\alpha\beta} \delta_{\beta}^\alpha \delta_{\alpha}^\beta \] (3.36)
(cf. (3.23) for the standard solution (3.16), (3.17)).

The other two basic PB coincide with those in (2.39) (and the Dirac bracket \( \{a^+_a, p_\ell\} \) – with (2.47)).

The operators in the triple tensor product \( \mathbb{C}^n \times \mathbb{C}^n \times \mathbb{C}^n \)
\[ r_{ab}^\pm (p) = r_{ab}(p) \pm \frac{\pi}{k} C_{ab}, \quad a, b = 1, 2, 3 \quad a < b \] (3.37)
satisfy the dynamical CYBE [26]
\[ [r_{12}^\pm (p), r_{13}^\pm (p) + r_{23}^\pm (p)] + [r_{13}^\pm (p), r_{23}^\pm (p)] + \text{Alt} (dr^\pm) = 0, \] (3.38)

where
\[ \text{Alt} (dr^\pm) := -i \sum_{j=1}^{n} \frac{\partial}{\partial p_j} \left( c_{j1}^j r_{23}^\pm (p) - c_{j2}^j r_{13}^\pm (p) + c_{j3}^j r_{12}^\pm (p) \right) \equiv \text{Alt} (dr). \] (3.39)

As the verification of (3.38) requires some work, we sketch the main steps in the Appendix.
4 Quantization

4.1 Quantum exchange relations and their quasiclassical limit

The exchange relations for the quantum matrix algebra – which we shall again denote by $\mathcal{M}_q$ – have been derived earlier on the basis of an analysis of the braiding properties of $SU(n)_k$ WZNW 4-point blocks [32, 39] satisfying the Knizhnik-Zamolodchikov equations [41, 48]. They have the form [32]

$$[q^{p_{ij}}, q^{p_{k\ell}}] = 0, \quad q^{p_{ij}a_{\alpha}^\ell} = a_{\alpha}^\ell q^{p_{ij}+\delta_\ell^\alpha-\delta_j^\alpha},$$

(4.1)

$$\hat{R}(p)^{\pm 1} a_{1} a_{2} = a_{1} a_{2} \hat{R}^{\pm 1},$$

(4.2)

where

$$\left(q^{\frac{1}{2}} \hat{R}\right)^{\pm 1}_{i_{i}+1} = q^{\pm 1} I_{i_{i}+1} - A_{i_{i}+1}, \quad A_{\beta_1,\beta_2}^{\alpha_1,\alpha_2} = q^{\delta_2^{\alpha_1} \delta_1^{\alpha_2} - \delta_2^{\alpha_2} \delta_1^{\alpha_1}},$$

(4.3)

$$q^{\alpha_2 \alpha_1} = \begin{cases} q^{-1} & \text{for } \alpha_1 < \alpha_2 \\ 1 & \text{for } \alpha_1 = \alpha_2 \\ q & \text{for } \alpha_1 > \alpha_2 \end{cases}, \quad q = e^{-i\pi}, \quad h = k + n$$

(4.4)

$$\left(q^{\frac{1}{2}} \hat{R}(p)\right)^{\pm 1}_{i_{i}+1} = q^{\pm 1} I_{i_{i}+1} - A_{i_{i}+1}(p),$$

(4.5)

$$A_{j_{j},j_{2}}^{i_{i},i_{2}}(p) = \frac{[p_{i_{i}i_{2}} - 1]}{[p_{i_{i}i_{2}}]} (\delta_{j_{1}j_{2}}^{i_{1}i_{2}} - \delta_{j_{2}j_{1}}^{i_{1}i_{2}}).$$

Both $A_{i_{i}+1} =: A_i$ and $A_{i_{i}+1}(p) =: A_i(p)$ satisfy the Hecke algebra relations

$$A_i A_{i+1} A_i - A_i = A_{i+1} A_i A_{i+1} - A_{i+1}, \quad A_i^2 = [2] A_i, \quad [A_i, A_j] = 0 \quad \text{for } |i-j| > 1,$$

(4.6)

It remains to verify that the quasiclassical limit of these relations indeed reproduces the PB relations of Section 3.

One can introduce two deformation parameters: $\frac{1}{k}$ and the (implicit in common notation) Planck constant $\hbar$ – see [1]. If one ascribes to the physical quantities $\hat{k}$ and $\hat{p}$ the dimension of action, then our dimensionless numbers $k$ and $p$ shall be written as $k = \frac{k}{\hbar}$ and $p = \frac{p}{\hbar}$. We shall distinguish the quasiclassical limit ($\hbar \to 0$) from the undeformed limit ($k \to \infty$) without using the parameter $\hbar$, by characterizing the second one by

$$k \to \infty, \quad p_{j\ell} \text{ finite}, \quad \frac{p_{j\ell}}{k} \to 0,$$

(4.7)

while setting for the first one of interest

$$\frac{k}{n} \to \infty, \quad p_{j\ell} \to \infty, \quad \frac{p_{j\ell}}{k} \text{ finite} \quad (j < \ell).$$

(4.8)

The substitution of the level $k$ by the height $h = k + n$ in the quantum expression for $q$ (4.4) is consistent with (4.8) but we are only aware of an
explanation of its necessity that uses the full (with infinite number of degrees of freedom) WZNW model which involves the Sugawara formula expressing the stress energy tensor as a norm square of the $SU(n)$ current (see [41, 47]).

Let $P$ be the permutation operator for either set of indices, $j, \ell, \ldots$ or $\alpha, \beta, \ldots:

\begin{align*}
P_{12} = (p_{i_1 i_2}) = (\delta_{j_1 i_2} \delta_{j_2 i_1}) \quad \text{or} \quad P_{12} = (p_{\alpha_1 \beta_2}) = (\delta_{\beta_2 \alpha_1} \delta_{\alpha_2 \beta_1}) \quad (4.9)
\end{align*}

and let $I_{12}$ be the corresponding unit operator (e.g., $I_{i_1 i_2} = \delta_{i_1 i_2} \delta_{i_2 i_1}$). Then we can write

\begin{align*}
R_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} &= (\hat{R} P)_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} = q^j \left((q - q^\alpha \alpha_1) P_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} + I_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}\right), \quad (4.10) \\
R(p)^{j_1 j_2}_{i_1 i_2} &= (\hat{R} P)^{i_1 i_2} = q^j \left(\frac{q^{p_{j_1 j_2}}}{[p_{j_1 j_2}]} P_{i_1 i_2}^{j_1 j_2} + \frac{[p_{j_1 j_2}]}{[p_{j_1 j_2}]} I_{i_1 i_2}^{j_1 j_2}\right). \quad (4.11)
\end{align*}

Setting now

\begin{align*}
q = 1 - \frac{i \pi}{k} + O\left(\frac{\pi^2}{k^2}\right) \quad (4.12)
\end{align*}

we find

\begin{align*}
R_{12} = I_{12} + i \frac{\pi}{k} r_{12} + O\left(\frac{\pi^2}{k^2}\right), \quad R(p)_{12} = I_{12} + i r_{12}(p) + O\left(\frac{\pi^2}{k^2}\right), \quad (4.13)
\end{align*}

where

\begin{align*}
r_{12} = r_{12} - C_{12}, \quad r_{\beta_1 \beta_2}^{\alpha_1 \alpha_2} = -\epsilon_{\alpha_1 \alpha_2} P_{\beta_1 \beta_2}^{\alpha_1 \alpha_2}, \quad C_{12} = P_{12} - \frac{1}{n} I_{12}, \quad (4.14) \\
r_{12}(p) = r_{12} - C_{12}, \quad r(p)^{i_1 i_2} = i \frac{\pi}{k} \cotg \left(\frac{\pi}{k} p\right) (\delta_{i_1 i_2} \delta_{i_2 i_1} - \delta_{i_1 i_2} \delta_{i_2 i_1}). \quad (4.15)
\end{align*}

The reason why we are keeping the factor $\frac{\pi}{k}$ in the definition of $r_{12}(p)$ is that it has a nonzero undeformed limit since

\begin{align*}
\lim_{k \to \infty} \frac{\pi}{k} \cotg \left(\frac{\pi}{k} p\right) = \frac{1}{p}. \quad (4.16)
\end{align*}

Taking into account that $[C_{12}, a_1 a_2] = 0$, we thus recover the PB relations of Section 3.2.

### 4.2 $U_q(s\ell_n)$ symmetry of the exchange relations

Let, for $G_0 \ni M \equiv (M_{i,j})_{i,j=1}^n$, $M_n = 0 \neq \det \begin{pmatrix} M_{n-1} & M_{n-1}^{n-1} \\ M_n & M_n \end{pmatrix}$ etc. and

\begin{align*}
M = q_{\alpha}^{n-1} M_+ M_-^{n-1}, \quad M_+ = N_+ D, \quad M_- = N_- D, \quad D = (d_{\alpha}^\alpha \delta_{\alpha}^\alpha), \quad (4.17)
\end{align*}
We note that the centralizer of \( q \) under the coaction of \( U \) (see [32]). It follows that these exchange relations are invariant under the coaction of \( U \).

\[
N_+ = \begin{pmatrix}
1 & f_1 & f_{12} & \cdots \\
0 & 1 & f_2 & \cdots \\
0 & 0 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad N_- = \begin{pmatrix}
1 & 0 & 0 & \cdots \\
e_1 & 1 & 0 & \cdots \\
e_{21} & e_2 & 1 & \cdots \\
\vdots & \vdots & \vdots & \ddots
\end{pmatrix}, \quad (4.18)
\]

where the common diagonal matrix \( D \) has unit determinant: \( d_1 d_2 \cdots d_n = 1 \).

It can be deduced from (4.1), (4.2) and \( M = a^{-1} M_p a \) that

\[
[\hat{R}^\pm, M_{2\pm} M_{1\pm}] = 0, \quad \hat{R} M_{2-} M_{1+} = M_{2+} M_{1-} \hat{R}. \quad (4.19)
\]

It is known that Eqs. (4.19) for the matrices \( M_{\pm} \) are equivalent to the defining relations of the quantum universal enveloping algebra \( U_q := U_q(\mathfrak{sl}_n) \) [16] that is paired by duality to \( \text{Fun} (SL_q(n)) \) [28]. The Chevalley generators of \( U_q \) are related to the elements of the matrices (4.17), (4.18) by ([28], see also [34])

\[
d_i = q^{\Lambda_{i-1} - \Lambda_i} \quad (i = 1, \ldots , n, \ \Lambda_0 = 0 = \Lambda_n), \quad e_i = (\overline{q} - q)E_i, \quad f_i = (\overline{q} - q)F_i,
\]

\[
(\overline{q} - q)f_{ij} = f_jf_i - qf_i f_j = (\overline{q} - q)^2(F_2F_1 - qF_1 F_2) \quad \text{etc.},
\]

\[
(\overline{q} - q)e_{21} = e_1 e_2 - q e_2 e_1 = (\overline{q} - q)^2(E_1 E_2 - qE_2 E_1) \quad \text{etc.} \quad (4.20)
\]

Here \( \Lambda_i \) are the fundamental co-weights of \( \mathfrak{sl}(n) \) related to the co-roots \( H_i \) by \( H_i = 2\Lambda_i - \Lambda_{i-1} - \Lambda_{i+1} \); \( E_i \) and \( F_i \) are the raising and lowering operators satisfying

\[
[E_i, F_j] = [H_i]\delta_{ij}, \quad q^{\Lambda_i} E_j = E_j q^{\Lambda_{i+1}}, \quad q^{\Lambda_i} F_j = F_j q^{\Lambda_{i-1}},
\]

\[
[E_i, E_j] = 0 = [F_i, F_j] \quad \text{for} \quad |j - i| \geq 2,
\]

\[
[2]X_i X_{i \pm 1} X_i = X_{i \pm 1} X_i^2 + X_i^2 X_{i \pm 1} \quad \text{for} \quad X = E, F. \quad (4.21)
\]

The exchange relations (4.1), (4.2) imply

\[
M_{1\pm} Pa_1 = a_2 \hat{R}^+ M_{2\pm} \quad (4.22)
\]

(see [32]). It follows that these exchange relations are invariant under the coaction of \( U_q \),

\[
[E_a, q^j a^i] = \delta_{a a -1} a^{i -1} q^H a, \quad [q^H a F_a, a^i] = \delta_{a a} q^H a^{i +1},
\]

\[
q^H a^i = \alpha^j q^{H_a + \delta_{a a} - \delta_{a a -1}}, \quad a = 1, \ldots , n - 1. \quad (4.23)
\]

We note that the centralizer of \( q^{p_1} \) (\( \prod_{i=1}^{n} q^{p_i} = 1 \)) in the algebra (4.1), (4.2) (i.e., the maximal subalgebra commuting with all \( q^{p_i} \)) is spanned by \( U_q \) over the field \( \mathbb{Q}(q, q^{p_i}) \) of rational functions of \( q^{p_i} \).
4.3 Operator realization

We shall sketch and briefly discuss the finite-dimensional Fock-like space realization of the quantum matrix algebra of \[32\].

The ”Fock space” $\mathcal{F}$ and its dual $\mathcal{F}'$ are defined as $\mathcal{M}_q$-modules with 1-dimensional $U_q$-invariant subspaces of multiples of (non-zero) bra and ket vacuum vectors $|0\rangle$ and $\langle 0|$ (such that $\langle 0|\mathcal{M}_q = \mathcal{F}'$, $\mathcal{M}_q|0\rangle = \mathcal{F}$) satisfying

\[
a_{i\alpha}^\dagger|0\rangle = 0 \quad \text{for} \quad i > 1, \quad \langle 0|a_i^\dagger = 0 \quad \text{for} \quad j < n, \quad q^{p_{ij}}|0\rangle = q^{j-i}|0\rangle, \quad \langle 0|q^{p_{ij}} = q^{j-i}\langle 0|, \quad (X - \varepsilon(X))|0\rangle = 0 = \langle 0|(X - \varepsilon(X)), \quad \forall X \in U_q,
\]

with $\varepsilon(X)$ the counit. The duality between $\mathcal{F}$ and $\mathcal{F}'$ is established by a bilinear pairing $\langle .|\, . \rangle$ such that

\[
\langle 0|0\rangle = 1, \quad \langle \Phi|A|\Psi \rangle = \langle \Psi|A^\dagger|\Phi \rangle
\]

where $A \rightarrow A^\dagger$ is a linear antiinvolution (transposition) of $\mathcal{M}_q$ defined for generic $q$ by

\[
D_i(p)(a_{i\alpha}^\dagger) = \tilde{a}_i^\alpha := \frac{1}{[n-1]!}E^{\alpha_1...\alpha_{n-1}}_{\epsilon i_1...i_{n-1}}a_{i_1}a_{i_2}...a_{i_{n-1}}^\dagger, \quad (q^{p_{i\alpha}})^\dagger = q^{-p_{i\alpha}}.
\]

Here $D_i(p)$ stands for the product

\[
\mathcal{D}_i(p) = \prod_{j<\ell,j\neq i} [p_{ij}] \quad (\Rightarrow [\mathcal{D}_i(p), a_{i\alpha}^\dagger] = 0 = [\mathcal{D}_i(p), \tilde{a}_i^\alpha])
\]

for the definition of the $U_q$- and, respectively, the ”dynamical” Levi-Civita tensors $E^{\alpha_1...\alpha_n}_{\epsilon i_1...i_n}$, see [38, 32]. The antiinvolution (4.26) extends the known transposition of $U_q$ determined by its action on the Chevalley generators (see Section 3 of [34]),

\[
E_i' = F_i q^{H_{i-1}}, \quad F_i' = q^{1-H_{i}}E_i, \quad (q^{H_{i}})^\dagger = q^{-H_{i}}
\]

to the quantum matrix algebra (cf. Section 3.1 and Appendix B of [32]).

The space $\mathcal{F}$ admits a canonical basis of weight vectors whose inner product can be computed (see Section 3.2 of [32]). For $n = 2$ the basis has the simple form

\[
|p, m\rangle = (a_1^\dagger)^m(a_2^\dagger)^{p-1-m}|0\rangle, \quad 0 \leq m \leq p - 1 \quad (p \equiv p_{12}),
\]

and the inner product is given by

\[
\langle p', m'|p, m \rangle = \delta_{pp'}\delta_{mm'}q^{m(p-1-m)}[m]![p - 1 - m]!
\]
For the deformation parameter $q$ appearing in (4.4),
\[ q = e^{-i\hat{q}}, \quad h = k + n \quad \Rightarrow \quad q^h = -1 \] (4.31)
i.e., $q$ a (here, even) root of unity, the Fock space has an infinite dimensional $U_q$ invariant subspace of null vectors orthogonal to any vector in $\mathcal{F}$. In the $n = 2$ case all null vectors belong to the set $\mathcal{I}_h|0\rangle$ where $\mathcal{I}_h$ is the ideal generated by $[hp], [hH], q^{hp} + q^{Hh}, (a^i_\alpha)^h, i, \alpha = 1, 2$. The definition of the ideal $\mathcal{I}_h$ can be generalized to any $n \geq 2$ assuming that it includes the $h$-th powers of all minors of the quantum matrix $(a^i_\alpha)$. For $n = 2$ the factor space $\mathcal{F}_h$ is spanned by vectors of the form (4.29) with $0 < p < 2h$ and $m$ in the range $0 \leq m \leq p - 1$, for $1 \leq p \leq h$, and $p - h \leq m \leq h - 1$ for $h + 1 \leq p \leq 2h - 1$. It splits into a direct sum of $2h - 1$ irreducible representations of $U_q(s\ell_2)$ of total dimension $h^2$.

For general $n$ and generic $q$ ($q$ not a root of unity) the space $\mathcal{F}$ has been proven to be a model space for $U_q$ (see Section 3.1 of [32]). The question of what should be viewed as a model space for the reduced $U_q$ ($U_q$ factored by its maximal ideal) for $q$ satisfying (4.31) appears to be unsettled. If we define it as the direct sum of integrable representations (those with $0 < p < h$, for $n = 2$) of multiplicity 1, then the question arises whether there is a natural (say, a BRS type) procedure that would reduce $\mathcal{F}_h$ to such a sum. A BRS procedure was introduced in [22, 23] for the tensor product of two copies of $\mathcal{F}_h$ – corresponding to the left and right movers’ zero modes of a $SU(2)$ WZNW model – but this changes the problem.

5 Concluding remarks

We have tried to make the present study of the chiral zero modes’ phase space reasonably self-contained and have, hence, included some known material. It may be, therefore, useful to list at this point what appears to us as the main new features in our treatment.

We find explicitly the correspondence between the WZ term $\rho(a^{-1}M_pa)$ (rendering the zero modes’ symplectic form (2.10) closed) and the solutions of the CYBE.

It is essential for the present treatment of the $SU(n)$ case that the determinant $\det(a^i_\alpha)$ of the zero modes’ $n \times n$ matrix is set equal to an (essentially unique) pseudoinvariant $q$-polynomial in the $su(n)$ weights - see (1.7). Accordingly, the symplectic form (2.10) in the $(n-1)(n+2)$-dimensional zero modes’ phase manifold $\mathcal{M}_q$ necessarily contains, for $n > 2$, a term $\omega_q(M_p)$ depending only on the diagonal monodromy.

The counterpart of $\omega_q(M_p)$ in the symplectic form of the chiral WZNW model with diagonal monodromy, being closed by itself, is often omitted. This additional term is necessary in order to reproduce upon quantization the basic exchange relations involving the dynamical $R$-matrix of [37, 1, 15, 32].
The expression for $\omega_q$ is simpler – and easier to derive – in the extended $(n(n + 1)\text{-dimensional})$ phase space $\mathcal{M}_q^{ex}$ spanned by $p_i$ and $a^\alpha_i$, $i,j,\alpha = 1,\ldots,n$, the form $\omega_q^{ex}$ (2.28) being nontrivial even for $n = 2$ (yielding, in the undeformed limit, the standard symplectic structure on $\mathbb{C}^2$ viewed as a Kähler manifold in that case).

The Dirac brackets of the physically interesting quantities $a^\alpha_i$ and $p_{j\ell}$ coincide with their Poisson brackets since they Poisson commute with one of the constraints and can be, hence, derived working in the (more symmetric) extended phase space.

The expression (3.30) for the Poisson bivector in $\mathcal{M}_q^{ex}$ allows to directly compute the Poisson brackets of interest.

The quantum theory of chiral zero modes has been only briefly reviewed in Section 4 concluding with the formulation of an open problem related to the concept of a model space for the quantum universal enveloping algebra $U_q(s\ell_n)$ for $q$ a root of unity.

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**Appendix A**

We begin by reproducing the properties of the polarized Casimir operators $C_{mn}$ relevant for the proof of the CYBE (for both constant and ”dynamical” – i.e., $p$-dependent – $r^\pm$):

\[
[C_{12}, C_{13} + C_{23}] = 0 = [C_{12} + C_{13}, C_{23}] .
\]  
(A.1)

For $G = SU(n)$ $C_{12}$ is given, essentially, by the permutation operator (4.9):

\[
C_{12} = P_{12} - \frac{1}{n} I_{12} , \quad \text{or} \quad C_{t_1t_2}^{j_1j_2} = \delta_{t_2t_1}^{j_1j_2} - \frac{1}{n} \delta_{t_1t_2}^{j_1j_2} \quad \left( \delta_{t_m}^{j_k} := \delta_{t_k}^{j_m} \right),
\]  
(A.2)

and Eq. (A.1) gives

\[
[C_{12}, C_{13} + C_{23}] + [C_{13}, C_{23}] = -[C_{12}, C_{23}] = \left( \delta_{t_2t_3}^{j_1j_2} - \delta_{t_3t_2}^{j_1j_2} \right) .
\]  
(A.3)
Next we verify that the mixed \((r-C)\) terms in the CYBE (3.38) (or (3.10)) vanish,

\[
[r_{12}(p), C_{13} + C_{23}] + [r_{12}(p), C_{23} - C_{12}] + [C_{12} + C_{13}, r_{23}(p)] = 0 , \tag{A.4}
\]

using, e.g., the general identities \(P_{ab}r_{bc} = r_{ac}P_{ab}\) for \(a, b, c\) all different, as well as skewsymmetry of \(r_{ab} = -r_{ba}\). Computing the sum of commutators in (3.38), we find

\[
( [r_{12}(p), r_{13}(p) + r_{23}(p)] + [r_{13}(p), r_{23}(p)] )^{j_i j_2 j_3}_{\ell_1 \ell_2 \ell_3} = \frac{\pi^2}{k^2}((c_{j_1 j_2} + c_{j_2 j_3})c_{j_1 j_3} - c_{j_1 j_2}c_{j_2 j_3})(\delta_{\ell_2 \ell_3 \ell_1}^{j_1 j_2 j_3} - \delta_{\ell_3 \ell_1 \ell_2}^{j_1 j_2 j_3}), \tag{A.5}
\]

where

\[
c_{j\ell} := \cotg \frac{\pi}{k} p_{j\ell} = -c_{\ell j} , \quad j \neq \ell , \quad c_{\ell \ell} := 0 . \tag{A.6}
\]

On the other side, (3.39) gives

\[
\text{Alt}(dr) = \frac{\pi}{k} \left(\delta_{j_1 j_2} c'_{j_2 j_3} + \delta_{j_2 j_3} c'_{j_1 j_3} + \delta_{j_1 j_3} c'_{j_1 j_2}\right)(\delta_{\ell_2 \ell_3 \ell_1}^{j_1 j_2 j_3} - \delta_{\ell_3 \ell_1 \ell_2}^{j_1 j_2 j_3} ), \tag{A.7}
\]

where

\[
c'_{j\ell} := -\frac{\pi}{k} \frac{1}{\sin^2 \frac{\pi}{k} p_{j\ell}} = c'_{\ell j} , \quad j \neq \ell , \quad c'_{\ell \ell} := 0 . \tag{A.8}
\]

To prove (3.38), it suffices to combine (A.3)-(A.8) with one of the following relations (depending on whether all the three indices \(j_1, j_2, j_3\) are different or not):

\[
\left( \cotg \alpha + \cotg \beta \right) \cotg (\alpha + \beta) - \cotg \alpha \cotg \beta = -1 , \tag{A.9}
\]

\[
\cotg^2 \alpha - \frac{1}{\sin^2 \alpha} = -1 . \tag{A.10}
\]

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