Duality for positive opetopes and tree complexes

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Abstract

We show that the (positive) zoom complexes, here called tree complexes, with fairly natural morphisms, form a dual category to the category of positive opetopes with contraction epimorphisms. We also show how this duality can be slightly generalized to thicket complexes and opetopic cardinals.

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1 Introduction

Opetopes are higher dimensional shapes that were originally invented in [BD] as shapes that can be used to define a notion of higher dimensional category. By now there are more than a dozen of definitions of opetopes. Some definitions use very abstract categorical machinery [Bu, Z3], some are more concrete using one way or another some kinds of operads and/or polynomial or analytic monads [BD, HMP, Let, Ch, Z3, SZ, FS],
some definitions describe the ways opetopes can be generated \cite{HMP}, \cite{CTM}, \cite{T}, and finally there are also some purely combinatorial definitions \cite{P}, \cite{KJBM}, \cite{Z1}, \cite{Z2}, \cite{St}.

So it is not surprising that it is easier to show a picture of an opetope then to give a simple definition that will leave the reader with no doubts as to what an opetope is. In this paper we will deal with positive opetopes (p-opetopes, for short) only. This means that each face in opetopes we consider has at least one face of codimension 1 in its domain. As a consequence such opetopes have no loops.

The definitions of opetopes mentioned above seem to agree in a ‘reasonable sense’. However, the morphisms between opetopes are not always treated the same way. In some approaches even face maps between opetopes do not seem to be natural. In \cite{Z1}, \cite{Z3} it is shown how opetopes can be treated as some special kinds of $\omega$-categories and therefore all the $\omega$-functors (i.e. all face maps and all degeneracies) between opetopes can be considered. Of all the definitions of opetopes, the one given in \cite{KJBM}, through so called zoom complexes, seem to be very different from any other. In fact, to describe this definition even pictures are of a very different kind, as the reader can notice below. In this paper we shall show that there is an explanation of this phenomenon. Namely, the (positive) zoom complexes, here called tree complexes, with fairly natural morphisms, form a dual category to the category of positive opetopes with contraction epimorphisms (often called $\iota$-epis).

The contraction epimorphisms are some kind of degeneracies of opetopes that can send a face only to a face but possibly of a lower dimension (still preserving usual constraints concerning both domains and codomains).

Below we draw some pictures of positive opetopes of few low dimensions and corresponding dual\footnote{The fact that these structures are dual one to another is actually the main claim of the paper.} tree complexes, a simplified version of zoom complexes in \cite{KJBM}.

An opetope $O_1$ of dimension 1

\[
\begin{array}{c}
t_2 \xrightarrow{y} t_1
\end{array}
\]

and its dual tree complex $T_1$

\[
\begin{array}{c}
dim 0 \quad \text{dim} 1
\end{array}
\]

\[
\begin{array}{c}
\bullet t_2) \quad t_1 \quad \bullet y
\end{array}
\]

An opetope $O_2$ of dimension 2

\[
\begin{array}{c}
t_2 \xrightarrow{t_3} \xrightarrow{b} y_2 \xrightarrow{y_1} t_1
\end{array}
\]

and its dual tree complex $T_2$

\[
\begin{array}{c}
dim 0 \quad \text{dim} 1 \quad \text{dim} 2
\end{array}
\]

\[
\begin{array}{c}
\bullet t_3/t_2/t_1 \quad \bullet \quad \bullet y_1 \quad \bullet b
\end{array}
\]

An opetope $O_3$ of dimension 3
and its dual tree complex $T_3$

The drawings of the above tree complexes are, in fact, drawings of consecutive (non-empty) constellations of tree complexes. In particular, it is not an accident that the partial order of nesting of circles in one constellation is isomorphic to the partial order vertices of the next constellation.

Opetopic cardinals still consist of cells that can be meaningfully composed in a unique way, but have a bit more general shape. Here is an example

To illustrate how the duality works on morphisms is a bit more involved. We shall present a $\iota$-epimorphism from $O_3$ to $O_2$ by naming faces of $O_3$

\(^{2}\)The name ‘thicket’ was suggested to me by Karol Szumiło.
by the cells on $O_2$

they are sent to. For example, there are three faces in $O_3$ sent to the 1-face $y_2$. Two of them are 1-faces and one of them is a $y_2$ face. The dual of this morphism is a collection of three embeddings of trees sending leaves (vertices) to leaves, and inner nodes (circles) to inner nodes and respecting the relation between circles at one dimension and the vertices on the next dimension. We present the dual morphism of tree complexes from $T_2$ to $T_3$ drawing $T_3$ and naming its nodes

by the names of the nodes of the tree complex $T_2$

that are sent to those faces.

This duality could be compared to a restricted version of the duality between simple categories and discs [Be], [On], [MZ]. In that duality we have on one side some pasting diagrams described in terms of (simple) $\omega$-categories and all $\omega$-functors, and on the other some combinatorial structures called (finite) discs and some natural morphisms of disks. Roughly speaking, a finite disc is a finite planar tree extended by dummy/sink nodes at the ends of any linearly ordered set of sons of each node. These sink nodes do not bring any new information about the object but they are essential to get the right notion of a morphism between ‘such structures’, i.e. all those that correspond to $\omega$-functors on the other side. If we were to throw away the sink nodes, i.e. we would consider trees instead of discs, we could still have a duality but we would need to revise the notion of a morphism on both sides. In fact, we would have duality for degeneracy maps only. On the planar tree side we would not be able to dump a ‘true’ node onto a dummy node. This corresponds on the side of simple categories to the fact that we consider only some $\omega$-functors. If we think about $\omega$-functors between simple $\omega$-categories as a kind of ‘partial
composition of a part of the pasting diagram’, we would need to restrict to those \(\omega\)-functors that represent partial composition of the whole pasting diagram. In other words, if we have trees and do not have sink nodes around, our operations cannot drop any part of the pasting diagram before they start to compose them. The duality presented in this paper can be understood through this analogy. Namely, at the level of objects positive opetopes correspond to tree complexes but when we look at the morphism, the natural morphisms of tree complexes correspond only to degeneracies (contraction epimorphisms) on the side of positive opetopes. This leaves of course an open question of whether we can extend tree complexes one way or the other, introducing some kind of sink nodes, so that we could have duality for more maps, e.g. all \(\iota\)-maps, and not only \(\iota\)-epimorphisms, or even all \(\omega\)-functors?

The paper is organized as follows. In Section 2 we define a simplification of both constellations and zoom complex that were originally introduced in [KJBM], here called tree complexes, and the maps of both constellations and tree complexes. In Section 3, we describe duality for the category \(\text{pOpe}_\iota\) of opetopes with contraction epimorphisms on one side and the category \(\text{cTree}\) of tree complexes and tree complex maps. In Section 4, we present the extension of this duality to larger categories of \(\text{pOpeCard}_\iota\) of opetopic cardinals with contraction epimorphisms and \(\text{cThicket}\) of thicket complexes and thicket complex maps. The paper ends with an appendix where the relevant notions and facts concerning positive opetopes are recalled from [Z1] and [Z4].

2 The category of tree complexes

2.1 Trees

A **tree** is a finite poset with binary sups and no infs of non-linearly ordered non-empty subsets. In particular, tree can be empty but if it is not, it has the largest element, called root. A tree embedding is a one-to-one function that preserves and reflects order. **Tree** is the category of trees and tree embeddings. We will also consider other kinds of (monotone) morphisms of trees: sup-morphisms (= preserving suprema), monotone maps (automatically preserving infs), onto maps.

**Construction.** Let \(S, T\) be trees. Let \(S_\bot\) denotes the poset obtained by adding bottom element to \(S\). Then \(S_\bot\) has both sups and infs, i.e. it is a lattice. If \(D \subseteq S\) is a proper downward closed subset of \(S\), then \(S - D\) is again a tree. Any monotone map \(f : S - D \to T\) can be extended to an infs preserving map \(f_* : S_\bot \to T_\bot\) (sending \(D\) and \(\bot\) to \(\bot\)). \(f_*\) has a left adjoint \(f^* : T_\bot \to S_\bot\). If \(f\) is onto, then \(f^*\) reflects \(\bot\), and hence it restricts to a sup-preserving morphism again named \(f^* : T \to S\).

**Some notions and notation concerning trees.** Let \(T\) be a tree, \(t, t' \in T\).

1. \(t < t'\) means that \(t\) is a son of \(t'\) or \(t'\) is the successor of \(t\).

2. The suprema (infima) of a subset \(X\) of a poset \(T\) will be denoted by \(\sup^T(X)\) or \(\inf^T(X)\) (or \(\inf(X)\)) and if \(X = \{t, t'\}\), we can also write \(t \lor^T t'\) of \(t \land^T t'\) of \(t \land t'\).

3. A subposet \(X\) of a tree \(T\) is a **convex subtree** of \(T\) iff \(X\) has the largest element, and whenever \(x, x' \in X, s \in T\) and \(x < s < x'\), then \(s \in X\). Clearly a convex subset of a tree is in particular a non-empty tree. Let \(\text{St}(T)\) denote the poset of the convex sub-trees of the tree \(T\).

4. \(t \perp t'\) means that either \(t \leq t'\) or \(t' \leq t\).
5. \( t \) is a leaf in \( T \) iff the set \( \{ s \in T : s \prec t \} \) is empty. \( \text{lvs}(T) \) denotes the set of leaves of the tree \( T \).

6. \( \text{lvs}^T(t) \) is the set of leaves of the tree \( T \) over the element \( t \), i.e.

\[
\text{lvs}^T(t) = \{ s \in \text{lvs}(T) : s \leq t \}.
\]

7. \( \text{cvr}^T(t) \) is the cover of the element \( t \) in the tree \( T \)

\[
\text{cvr}^T(t) = \{ s \in T : s \prec t \},
\]

i.e. the set of elements of the tree \( T \) whose successor is \( t \).

8. If \( X \) is a convex subtree of \( T \), then the cover of \( X \) in the tree \( T \) is the set

\[
\text{cvr}^T(X) = \bigcup_{x \in X} \text{cvr}^T(x) - X.
\]

Note that \( \text{cvr}^T(t) = \text{cvr}^T(\{t\}) \), so the notation for value of \( \text{cvr}^T \) on elements and convex subsets is compatible.

We have an easy Lemma establishing some relation between the above notions. It will be needed for the proof of the duality.

**Lemma 2.1.** Let \( S \) be a tree.

1. Let \( T \) be a convex subtree of \( S \) not containing leaves. Then the family set \( \{ \text{lvs}^S(s) \}_{s \in \text{cvr}^S(T)} \) is a partition of the set \( \text{lvs}^S(\text{sup}(T)) \).

2. Let \( T \) be a convex subtree of \( S \) not containing leaves and let \( \{ T_i \}_{i \in I} \) be a partition of \( T \). Then, for \( t \in S \)

   (a) \( t \in \text{cvr}^S(T_i) \), for some \( i \in I \), iff either \( t \in \text{cvr}^S(T) \) or there is \( j \in I \) such that \( t = \text{sup}(T_j) \);

   (b) \( t = \text{sup}^S(T_i) \) iff either \( t = \text{sup}(T) \) or there is \( j \in I \) such that \( t \in \text{cvr}^S(T_j) \). \( \square \)

### 2.2 Constellations

A *constellation* is a triple \((S_1, \sigma, S_0)\), where \( S_0 \) and \( S_1 \) are trees and \( \sigma \) is a monotone function

\[
\sigma : S_1 \to \text{St}(S_0)
\]

such that

1. it preserves top element;

2. if \( s, s' \in S_1 \) and \( \sigma(s) \cap \sigma(s') \neq \emptyset \), then \( s \perp s' \).

Let \( \sigma : S_1 \to \text{St}(S_0) \) be a constellation. Then the constellation tree \((S_1 \triangleleft_\sigma S_0, \leq^{\text{co}})\) is the tree arising by extension of the tree \( S_1 \) by nodes of the tree \( S_0 \) added as leaves, so that if \( s_0 \in S_0 \) and \( s_1 \in S_1 \), then \( s_0 \) is a leaf over \( s_1 \) iff \( s_0 \in \sigma(s_1) \).

Formally, the set \( S_1 \triangleleft_\sigma S_0 \) is a disjoint sum of \( S_1 \) and \( S_0 \). If \( s_0 \in S_0 \), and \( s_1 \in S_1 \), then the corresponding elements in \( S_1 \triangleleft_\sigma S_0 \) are denoted by \( s^*_0 \) and \( s^*_1 \), respectively. The constellation order \( \leq^{\text{co}} \) in \( S_1 \triangleleft_\sigma S_0 \) is defined as follows. If \( s_0, t_0 \in S_0 \) and \( s_1, t_1 \in S_1 \), then

1. \( s^*_1 \leq^{\text{co}} t^*_1 \) iff \( s_1 \leq^T t_1 \);
2. \( s_0^\bullet \leq^{co} s_1^\circ \) iff \( s_0 \in \sigma(s_1) \);

3. \( s_0^\bullet \leq^{co} t_0^\circ \) iff \( s_0 = t_0 \);

4. \( s_1^\circ \leq^{co} s_0^\bullet \) never holds.

Clearly \((S_1 \lhd_{\sigma} S_0, \leq^{co})\) is again a tree. We often drop the index \( \sigma \) in \( S_1 \lhd_{\sigma} S_0 \) when it does not lead to a confusion.

Let \((S_1, \sigma, S_0)\) and \((T_1, \tau, T_0)\) be two constellations. Any pair of tree embeddings \( f_0 : S_0 \rightarrow T_0, f_1 : S_1 \rightarrow T_1, \) such that \( f_0(\sigma(s)) \subseteq \tau(f_1(s)) \) for \( s \in S_1 \), induces a tree embedding of constellation orders

\[ f_1 \lhd f_0 : S_1 \lhd_{\sigma} S_0 \rightarrow T_1 \lhd_{\tau} T_0. \]

Such a pair is a morphism of constellations

\[ (f_1, f_0) : (S_1, \sigma, S_0) \rightarrow (T_1, \tau, T_0) \]

iff the induced map \( f_1 \lhd f_0 \) preserves binary sups. The category of constellations will be denoted by \textbf{Constell}.

**Remarks and notation.**

1. The fibers of any constellation \( \sigma \) are linearly ordered.

2. Let \( \sigma_\perp : (S_1)_\perp \rightarrow \text{St}(S_0)_\perp \) be the extension of the function \( \sigma \) sending \( \perp \) to \( \perp \). Then \( \sigma \) is a constellation iff \( \sigma_\perp \) preserve all infs iff \( \sigma \) preserves top element and preserves and reflects binary infs, i.e.

   \[ \begin{align*}
   &\text{(a)} \quad \sigma(\text{sup}(S_1)) = S_0; \\
   &\text{(b)} \quad \text{and for } s, s' \in S_1, \quad \sigma(s) \cap \sigma(s') \neq \emptyset \text{ iff } s \land s' \text{ is defined and then } \sigma(s) \cap \sigma(s') = \sigma(s \land s').
   \end{align*} \]

3. One can look at a single constellation \((S_1, \sigma, S_0)\) as a way of gluing elements of a poset \( S_0 \) as new leaves in the posted \( S_1 \) along the function \( \sigma \). Thus the order of \( S_0 \) is not essential for building one constellation order. The constellation order can be graphically drawn with leaves from \( S_0 \), also called vertices, marked as dots, and inner nodes from \( S_1 \), also called circles, marked as circles, enclosing all the leaves under them and all the smaller circles. General elements of constellation orders are often called nodes.

4. Let \((T_1, \tau_0, T_0)\) and \((T_2, \tau_1, T_1)\) be two (consecutive) constellations. We can form a diagram

\[ T_2 \stackrel{\infty}{\longrightarrow} T_2 \lhd T_1 \stackrel{\bullet}{\longrightarrow} T_1 \lhd T_0 \stackrel{\cdot}{\longrightarrow} T_0 \]

consisting of two coproduct diagrams.

   \[ \begin{align*}
   &\text{(a)} \quad \text{If } t \in T_1, \text{ then } t^\bullet \in T_1 \lhd T_0 \text{ and } t^\circ \in T_2 \lhd T_1. \text{ So the node } t, \text{ depending on the order in which we consider it, can be either a vertex (leaf) or a circle (inner node). As we will deal with this situation very often, we will usually be careful to distinguish these two roles, when it may cause confusions, by putting either circle of dot over the node considered.}
   \end{align*} \]

   \[ \begin{align*}
   &\text{(b)} \quad \text{Moreover, for } X \subseteq T_1 \text{ we use very often the notation } X^\bullet \subseteq T_1 \lhd T_1 \text{ and } X^\circ = X^\bullet \subseteq T_1 \lhd T_0.
   \end{align*} \]
2.3 Tree complexes

A tree complex \((T, \tau)\) is a sequence of constellations:

\[
\begin{align*}
\tau_0 : T_1 &\to \text{St}(T_0), \\
\tau_1 : T_2 &\to \text{St}(T_1), \\
&\vdots \\
\tau_i : T_{i+1} &\to \text{St}(T_i), \\
&\vdots
\end{align*}
\]

for \(i \in \omega\), with almost all sets \(T_i\) empty. The dimension \((T, \tau)\) is \(n\) iff \(T_n\) be the last non-empty set. We write \(\text{dim}(T)\) for dimension of the tree complex \((T, \tau)\). \(T_0\) as well as \(T_n\) are required to be singletons. We sometimes tacitly assume that \(T_0 = \{\bullet_T\}\) and \(T_n = \{m_T\}\).

A morphism of tree complexes \(f : (S, \sigma) \to (T, \tau)\) is a family of tree embeddings \(f_i : S_i \to T_i\), for \(i \in \omega\), such that, for \(i \in \omega\),

\[
(f_{i+1}, f_i) : (S_{i+1}, \sigma_i, S_i) \longrightarrow (T_{i+1}, \tau_i, T_i)
\]

is a morphism of constellations, i.e. the tree embeddings

\[
\vec{f}_i = f_{i+1} \triangleleft f_i : S_{i+1} \triangleleft_{\sigma_i} S_i \longrightarrow T_{i+1} \triangleleft_{\tau_i} T_i
\]

that preserve binary sups.

The category of tree complexes and their morphisms will be denoted by \(\text{cTree}\).

3 Duality

3.1 From positive opetopes to tree complexes

In this section we define a functor

\[
\text{pOpe}_{i, \text{epi}} \xrightarrow{(-)^*} \text{cTree}^{\text{op}}
\]

Let \(P\) be a p-opetope. We shall define a tree complex \((P^*, \pi)\). For \(i \in \omega\), the set

\[
P_i^* = (\delta \gamma(i+1)(m_P), \leq^-) = (P_i^* - \gamma(P_{i+1}), \leq^-),
\]

is the set of faces of the \(i\)-th tree of the tree complex \((P^*, \pi)\). The \(i\)-th constellation map

\[
\pi_i : P_{i+1}^* \longrightarrow \text{St}(P_i^*)
\]

is given, for \(p \in P_{i+1}^*\), by

\[
\pi_i(p) = \{ s \in P_i^* : s \prec^+ \gamma(p) \}.
\]

NB. If \(p, p' \in P_{i+1}^*\), then \(p \Prec^{-} p'\) iff \(\gamma(p) \prec^+ \gamma(p')\).

With the notation as above, we have

**Proposition 3.1.** For \(i \in \omega\), the triple \((P_{i+1}^*, \pi_{i+1}, P_i^*)\) defined above is a constellation. Thus \((P^*, \pi)\) is a tree complex.
Proof. First we show that, for \( p \in P_{i+1}^* \), \( \pi_i(p) \) is a convex subtree of \((P_i^*, \prec)\). The \( \prec^+ \)-least element in the \( \gamma \)-pencil of \( \gamma(p) \) is the largest element of \( \pi_i(p) \). Let \( p_1, p_3 \in \pi_i(p) \) and \( p_2 \in P_{i+1}^* \) such that \( p_1 \prec p_2 \prec p_3 \). Thus there is a maximal lower \( P_i^* = P_i - \gamma(P_{i+1}^*) \)-path containing \( p_1, p_2, p_3 \). Thus, by Path Lemma (cf. [Z1] or Appendix), \( p_2 \prec^+ \gamma(p) \).

Next we show that \( \pi_i : P_{i+1}^* \to P_i^* \) is monotone. Let \( p, p' \in P_{i+1}^* \) so that \( p \perp p' \). Then \( \gamma(p) \prec^+ \gamma(p') \) and hence
\[
\pi(p) = \{ s \in P_i^* : s \prec^+ \gamma(p) \} \subseteq \{ s \in P_i^* : s \prec^+ \gamma(p') \} = \pi(p'),
\]
as required.

Finally, we will show that if \( p, p' \in P_{i+1}^* \) and \( s \in \pi_i(p) \cap \pi_i(p') \), then \( p \perp p' \).

Thus \( s \prec^+ \gamma(p) \) and \( s \prec^+ \gamma(p') \). Let \( r_1, \ldots, r_k \) be a maximal lower \( P_{i+1}^* \)-path such that, for some \( j, s \in \delta(r_j) \). Then both \( p \) and \( p' \) must occur in this path. So \( p \perp p' \), as required. \( \square \)

We define a poset morphism
\[
\varepsilon_{P,i} : (P_{i+1}^*, \prec) \to (P_i, \prec)
\]
as follows. For \( p \in P_{i+1}^* \), we put
\[
\varepsilon_{P,i}(p) =\begin{cases} s & \text{if } p = \ast s \text{ for some } s \in P_i^*, \\ \gamma(s) & \text{if } p = \ast s \text{ for some } s \in P_{i+1}^*. \end{cases}
\]

Lemma 3.2. The morphism \( \varepsilon_{P,i} \) defined above is an order isomorphism.

Proof. If \( p_1, p_2 \in P_{i+1}^* \) and \( p_1 \not\prec p_2 \), then \( \gamma(p_1) \neq \gamma(p_2) \). Moreover, if \( p \in P_i^* = P_i - \gamma(P_{i+1}) \), then \( \varepsilon_{P,i}(p) = p \neq \gamma(p_1) = \varepsilon_{P,i}(p_1) \). Thus \( \varepsilon_{P,i} \) is one-to-one. It is onto as well, since \( P_i = (P_i - \gamma(P_{i+1}) \cup \gamma(P_{i+1} - \gamma(P_{i+2}))) \).

It remains to show that \( \varepsilon_{P,i} \) preserves and reflects order. We have that
\[
p \prec^+ p_1 \Longleftrightarrow \gamma(p_1) \Longleftrightarrow \gamma(p) \Longleftrightarrow p \prec^+ p \Longleftrightarrow \varepsilon_{P,i}(p) \Longleftrightarrow \varepsilon_{P,i}(p_1).
\]
Moreover
\[
p_1 \prec p_2 \Longleftrightarrow \gamma(p_1) \Longleftrightarrow \gamma(p_2) \Longleftrightarrow p_2 \prec p \Longleftrightarrow \varepsilon_{P,i}(p_2) \Longleftrightarrow \varepsilon_{P,i}(p_2).
\]
The other cases are obvious. \( \square \)

Let \( f : P \to Q \) be a \( \ast \)-epimorphism of \( \rho \)-opetopes. We define a tree complex map
\[
f = \{ f_i : (Q_i^*, \pi) \to (P_i^*, \pi) \}.
\]
For \( i \in \omega \), the map \( f_i : Q_i^* \to P_i^* \) is defined as follows. Let \( q \in Q_i^* = Q_i - \gamma(Q_{i+1}) \), \( p \in P_i^* = P_i - \gamma(P_{i+1}) \), \( 0 \leq i \). Then
\[
f_i(q) = p
\]
iff \( p \) is the unique element of \( P_i - \gamma(P_{i+1}) \) so that \( f(p) = q \). Such an element exists since \( f_i \) is epi.

We can also describe the above map using the construction from Section 2.1 as follows. We have a monotone onto map \( f_i : P_i - \ker(f), \leq^+ \to (Q_i, \leq^+) \). As \( P_i \cap \ker(f) \) is a proper downward closed subset of \( P_i \), \( f_i \) extends to an all \( \inf \) preserving map \( f_{i,*} : P_{i,\perp} \to Q_{i,\perp} \) sending \( P_i \cap \ker(f) \) to \( \perp \). Thus it has a left adjoint \( f_i : Q_{i,\perp} \to P_{i,\perp} \). Clearly, \( f_i \), defined this way, preserves \( \sup \). For \( q \in Q_i \), \( f_i(q) \) picks the least element in the fiber of the function \( f_i \) over element \( q \). We have
Lemma 3.3. With the notation as above, the following diagram

\[
\begin{array}{c}
(P_{i+1}^* \triangleleft P_i^*)_{\perp} \\
(Q_{i+1}^* \triangleleft Q_i^*)_{\perp} \\
(P_i)_{\perp} \\
(Q_i)_{\perp}
\end{array}
\]

commutes. In particular, \( f_i^* \triangleleft f_i^* \) preserves binary sups.

Proof. Let \( q^* \in Q_{i+1}^* \triangleleft Q_i^* \) and let \( f_i^* \triangleleft f_i^*(q^*) = p^* \). Thus \( f_i^*(p) = q \in Q_{i+1}^* \) and, as \( p \) is a leaf, \( \overline{f}_i(q) = p \). Hence the square commutes in this case.

Now, let \( q^0 \in Q_{i+1}^* \triangleleft Q_i^* \) and let \( f_i^* \triangleleft f_i^*(q^0) = p^0 \). Thus \( f_{i+1}^*(p) = q \). So \( p \) is indeed in the fiber of \( f_{i+1} \) over \( q \). We need to show that \( p \) is minimal in this fiber. Suppose to the contrary that there is \( p' \in P_{i+1} \) such that \( p' <^+ \gamma(p) \) and yet \( f_{i+1}^*(p') = q \). Then, as \( p \in P_{i+1} \triangleleft \gamma(P_{i+2}) \) and \( p' <^+ \gamma(p) \), there is a \( p'' \in \delta(p) \) such that \( p' \leq^+ p'' \). We have \( q = f_{i+1}^*(p') \leq^+ f_{i+1}^*(p'') \leq^+ f_{i+1}^*(\gamma(p)) = q \). But then \( f_{i+1}^*(p'') = q = f_{i+1}^*(\gamma(p)) \) and hence \( p \in \ker(f) \), contrary to the supposition. Thus the diagram commutes in this case, as well. \( \square \)

As a corollary of Lemma 3.3, we get

Proposition 3.4. Let \( f : (P, \gamma, \delta) \to (Q, \gamma, \delta) \) be an epi \( \iota \)-map of positive opetopes. Then the family of maps \( f^* = \{ f_i^* \}_{i \in \omega} : (Q^*, \pi) \to (P^*, \pi) \) defined above is a morphism of tree complexes. \( \square \)

Examples. We explain below in more detail the correspondence from the introduction between the \( \iota \)-epimorphism \( f : O_3 \to O_2 \) and its dual tree complex embedding.

1. The dual of the opetope \( Q = O_2 \) of dimension 2

\[
\begin{array}{c}
y_3 \\
\downarrow b \\
y_2 \\
y_1 \\
t_1
\end{array}
\]

is the tree complex \( Q^* = T_2 \) with nodes

\[
\begin{align*}
Q_0^* &= \{ t_3 \}, \\
Q_1^* &= \{ y_2 > y_3 \}, \\
Q_2^* &= \{ b \},
\end{align*}
\]

and the constellation maps

\[
\begin{align*}
\pi_0(y_2) &= \{ t_3 \}, \\
\pi_0(y_3) &= \{ t_3 \}, \\
\pi_1(b) &= \{ y_2, y_3 \}.
\end{align*}
\]

Such a tree complex \( Q^* \) can be drawn as follows:
2. The dual of the opetope $P = O_3$ of dimension 3

is a tree complex $P^*$ with nodes

- $P_0^* = \{t_4\}$,
- $P_1^* = \{y_1 > y_4 > y_5 > y_6\}$,
- $P_2^* = \{b_1 > b_2 > b_3\}$,
- $P_3^* = \{\beta\}$,

and the constellation maps

- $\pi_0(y_1) = \{t_4\}$,
- $\pi_0(y_4) = \{t_4\}$,
- $\pi_0(y_5) = \{t_4\}$,
- $\pi_0(y_6) = \{t_4\}$,
- $\pi_1(b_1) = \{y_1, y_4, y_5, y_6, \}$,
- $\pi_1(b_2) = \{y_4, y_5, y_6, \}$,
- $\pi_1(b_3) = \{y_4, y_5, \}$,
- $\pi_2(\beta) = \{b_1, b_2, b_3, \}$.

Such a tree complex $P^*$ can be drawn as follows:
3. The dual of the \( \iota \)-epimorphism \( f : P \to Q \) from opetope \( P = O_3 \) to the opetope \( Q = O_2 \) given by

\[
\begin{align*}
    f_0(t_0) &= f_1(y_1) = f_0(t_1) = t_1, \\
    f_0(t_2) &= f_1(y_5) = f_0(t_3) = t_2, \\
    f_0(t_4) &= t_3, \\
    f_1(y_2) &= f_2(b_1) = f_1(y_0) = y_1, \\
    f_1(y_4) &= f_2(b_3) = f_1(y_3) = y_2, \\
    f_1(y_6) &= y_3, \\
    f_2(b_2) &= f_3(\beta) = f_2(b_0) = b,
\end{align*}
\]

is a morphism of tree complexes \( f^* : Q^* \to P^* \) such that

\[
\begin{align*}
    f_0^*(t_3) &= t_4, \\
    f_1^*(y_2) &= y_4, \\
    f_1^*(y_3) &= y_6, \\
    f_2^*(b) &= b_2.
\end{align*}
\]

3.2 From tree complexes to positive opetopes

In this section we define a functor

\[
p\text{Ope}_{\iota, \text{epi}} \xrightarrow{(-)^*} \text{cTree}^{\text{op}}
\]

Let \( (S, \sigma) = \{S_i, \sigma_i\}_{i \in \omega} \) be a tree complex. We define the p-opetope \( \{S^*_i, \gamma^i, \delta^i\}_{i \in \omega} \), as follows. We put

\[
(S^*_i, <^{co}) = (S_{i+1} \triangleleft \sigma, S_i, <^{co}),
\]

i.e. the set \( S^*_i \) of \( i \)-dimensional faces of the p-opetope \( S^* \) is the universe of the \( i \)-th constellation poset of \( (S, \sigma) \). Later we shall prove that the constellation order \( <^{co} \) agree with the upper order \( <^+ \), defined using the operations \( \gamma \) and \( \delta \) below.

Let \( i \in \omega \). The \((i-th)\) codomain operation

\[
\gamma : S^*_{i+1} \to S^*_i
\]

is defined, for \( p \in S^*_{i+1} \), as follows

\[
\gamma(p) = \sup_{S^*_i}(\text{lvs}^{S^*_{i+1}}(p)^o).
\]

In words

1. if the face \( p \) is a node, i.e. \( p = t^* \), then its codomain \( \gamma(t^*) = t^o \), i.e. it is ‘the same’ \( t \) but considered as a circle one dimension below;
2. if the face $p$ is a circle, i.e. $p = t^o$, then its codomain $\gamma(p)$ is the circle $s^o$ whose corresponding node $s^*$ is the supremum $\sup_{i+1}^{S_i^*}(lvs_{i+1}^{S_i^*}(t^o)^o)$ in $S_i^*$ of the leaves in $S_{i+1}^*$ over $t^o$ considered as circles one dimension below, in $S_i^*$.

The $(i-th)$ domain operation

$$\delta : S_{i+1}^* \longrightarrow P_{\neq \emptyset}(S_i^*)$$

is defined, for $p \in S_{i+1}^*$, as follows

$$\delta(p) = cvr_i^{S_i^*}(lvs_{i+1}^{S_i^*}(p)^o).$$

In words

1. if the face $p$ is a node, i.e. $p = t^\bullet$, then its domain $\delta(t^\bullet) = cvr_i^{S_i^*}(t^o)$, i.e. it is the cover of ‘the same’ $t$ but considered as a circle one dimension below;

2. if the face $p$ is a circle, i.e. $p = t^o$, then its domain $\delta(t^o)$ is the sum of $\delta$’s applied to the leaves over $t^o$ in $S_{i+1}^*$ considered as circles one dimension below in $S_i^*$ minus these leaves considered as circles.

Lemma 3.5. Let $(S, \sigma)$ be a tree complex. Then the face structure $(S, \sigma)^* = (S^*, \gamma, \delta)$ defined above is a positive opetope.

Proof. Let $(S^*, \gamma, \delta)$ a face structure as defined above. We shall check that it satisfies the axioms of positive opetopes.

Globularity. We shall use Lemma 2.1

Fix $s \in S_{i+2}^*$, for some $i \geq 0$. Then $lvs_{i+2}^i(s)^o$ is a convex subtree of $S_{i+1}^*$ not containing leaves. Thus, by Lemma 2.1.1, the family of sets

$$\{lvs_{i+1}^i(r)\}_{r \in cvr_{i+1}^i(lvs_{i+2}^i(s)^o)}$$

is a partition of the set

$$lvs_{i+1}^i(\sup_{i+1}^i(lvs_{i+2}^i(s)^o)).$$

If $r = r'^\bullet \in cvr_{i+1}^i(lvs_{i+2}^i(s)^o)$, for some $r' \in S_{i+2}$, then $lvs_{i+1}^i(r)^o = r'^o$. If $r = r'^o \in cvr_{i+1}^i(lvs_{i+2}^i(s)^o)$, for some $r' \in S_{i+2}$, then $lvs_{i+1}^i(r)^o = \sigma_{i+1}(r)$. Thus in any case $lvs_{i+1}^i(r)^o$ is a convex subtree, and hence the partition, give rise to the partition of a convex subtree

$$lvs_{i+1}^i(\sup_{i+1}^i(lvs_{i+2}^i(s)^o))^o,$$

into a family of convex subtrees

$$\{lvs_{i+1}^i(r)^o\}_{r \in cvr_{i+1}^i(lvs_{i+2}^i(s)^o)}.$$ 

Then, using Lemma 2.1.2, we get

$$\gamma \gamma(s) =$$

$$= \sup_{i}(lvs_{i+1}^i(\sup_{i+1}^i(lvs_{i+2}^i(s)^o))^o) =$$

$$= \bigcup_{r \in cvr_{i+1}^i(lvs_{i+2}^i(s)^o)} \sup_{i}(lvs_{i+1}^i(r)^o) - \bigcup_{r \in cvr_{i+1}^i(lvs_{i+2}^i(s)^o)} cvr_i^i(lvs_{i+1}^i(r)^o) =$$
\[
= \gamma \delta(s) - \delta \delta(s),
\]
and
\[
\delta \gamma(s) = 
= \text{cvr}^i(\text{lvs}^{i+1}(\sup^{i+1}(\text{lvs}^{i+2}(s)^0))^0) = 
= \bigcup_{r \in \text{cvr}^{i+1}(\text{lvs}^{i+2}(s)^0)} \text{cvr}^i(\text{lvs}^{i+1}(r)^0) - \bigcup_{r \in \text{cvr}^{i+1}(\text{lvs}^{i+2}(s)^0)} \sup^i(\text{lvs}^{i+1}(r)^0) = 
= \delta \delta(s) - \gamma \delta(s),
\]
as required.

**Strictness.**

We shall show that the transitive relation \(<^+\), defined using \(\gamma\)'s and \(\delta\)'s, coincides with the constellation order \(<^0\).

Let \(s, s^0 \in S_{i+1} \triangleleft S_i\), for some \(0 \leq i \leq \text{dim}(S)\). Then \(\gamma(s^*) = s^0\). Moreover, \(s \not<^0 s^0\) iff \(s \in \text{cvr}^i(s^0) = \delta(s^*)\). The latter condition means that \(s <^+ s^0\). Thus \(<^0 \subseteq <^+\).

It remains to show that \(<^+ \subseteq <^0\). Assume \(s, s' \in S_{i+1} \triangleleft S_i\) and that there is \(r \in S_{i+2} \triangleleft S_{i+1}\) such that \(s \in \delta(r)\) and \(\gamma(r) = s'\). We shall show that \(s <^0 s'\).

If \(r = r^*\), for some \(r' \in S_{i+1}\), then \(s' = r^0\) and \(s \in \text{cvr}^i(r^0)\) so \(s <^0 s'\) indeed.

Now assume that \(r = r^0\), for some \(r' \in S_{i+2}\). Let \(s = s_0 \in \delta(r^0) = \text{cvr}^i(\text{lvs}^{i+1}(r^0)^0)\). Let \(s_1\) be the \(<^0\)-successor of \(s_0\), i.e. \(s_0 <^0 s_1\). Since \(\text{lvs}^{i+1}(r^0)\) is a convex tree, there is a path \(s_1, \ldots, s_k\) in \(\text{lvs}^{i+1}(r^0)^0\) such that \(s_i <^0 s_{i+1}\) and
\[
s_k = \sup^i(\text{lvs}^{i+1}(r^0)^0) = \gamma(r^0) = s'.
\]
Thus \(s = s_0 <^0 s_k = s'\), as required.

**Disjointness.**

Let \(s, t \in S_{i+1}^*\) such that \(s <^+ t\). Then \(\text{lvs}^{i+1}(s) \subseteq \text{lvs}^{i+1}(t)\). Hence
\[
\gamma(s) = \sup^i(\text{lvs}^{i+1}(s)^0) \not\in \text{cvr}^i(\text{lvs}^{i+1}(t)^0) = \delta(t),
\]
i.e. \(s \not<^- t\).

On the other hand, if \(s' \in \delta(s)\), then
\[
s' <^+ \gamma(s) = \sup^i(\text{lvs}^{i+1}(s)^0) \leq^+ \sup^i(\text{lvs}^{i+1}(t)^0) = \gamma(t),
\]
i.e. \(\gamma(t) \not\in \delta(s)\) and \(t \not<^- s\), as well.
Thus if \(s <^+ t\), then \(t \not<^- s\), as required.

**Pencil linearity.**

Let \(s, t \in S_i^*\), for some \(0 \leq i \leq \text{dim}(S)\).

Assume that \(s \not= t\) and \(\gamma(s) = \gamma(t)\). Then \(s\) and \(t\) cannot be leaves at the same time. If \(s\) is a leaf, then \(s <^0 t\) and hence \(s <^+ t\), by the above. If both \(s\) and \(t\) are inner nodes, then
\[
\sigma_i(s)^0 \ni \sup^{i-1}(\text{lvs}^i(s)^0) = \sup^{i-1}(\text{lvs}^i(t)^0) \in \sigma_i(t)^0.
\]
Thus \(\sigma_i(s) \cap \sigma_i(t) \neq \emptyset\), and, as \(\sigma_i\) is a constellation, we have \(s \bot^+ t\).
Now assume that there is \( r \in \delta(s) \cap \delta(t) \). Let \( s_1^0 \) be the successor of \( r \), i.e. \( r <^\omega s_1^0 \). Hence \( s_1^i \in \lvs^i(s)^0 \cap \lvs^i(t)^0 \) and hence

\[
s_1^i \in \lvs^i(s) \cap \lvs^i(t).
\]

If \( s \) and \( t \) were leaves, then we would have \( s = t \).

If \( s \) is a leaf and \( t \) is an inner node then \( s \in \lvs^i(t) \) and hence \( s <^+ t \).

If both \( s \) and \( t \) are inner nodes then

\[
s_1^i \in \lvs^i(s) \cap \lvs^i(t) = \sigma_{i+1}(s)^* \cap \sigma_{i+1}(t)^*,
\]

and as \( \sigma_{i+1} \) is a constellation, \( s \perp^+ t \). \( \square \)

Let \( f: (S, \sigma) \to (T, \tau) \) be a map of tree complexes. It gives rise to maps of faces, for \( k \in \omega \),

\[
\tilde{f}_k = f_{k+1} < f_k: S_k^* = S_{k+1} < S_k \to T_k^* = T_{k+1} < T_k,
\]

that, by definition, preserve sups. Note that the maps \( \tilde{f}_k \)'s do not preserve the domains or codomains just defined above, in general. These maps induce the morphism of p-opetopes \( f^*: T^* \to S^* \) as follows. Let \( t \in T_i^* \) and \( s \in S_j^* \), with \( 0 \leq j \leq i \leq \dim(T) \). Then

\[
f^*_i(t) = s
\]

iff

1. \( j \) is the maximal such number such that there is \( s' \in S_j^* \) such that

\[
\tilde{f}_j(s') \leq^\omega \delta_{\gamma(j+1)}(t);
\]

2. and \( s \) is the \( \leq^\omega \)-maximal \( s' \in S_j^* \) satisfying the above inequality.

**Lemma 3.6.** Let \( f: (S, \sigma) \to (T, \tau) \) be a morphism of tree complexes. Then the set of maps \( f^* = \{ f_i^* \}_{i \in \omega} : (T, \tau)^* \to (S, \sigma)^* \) is a \( \iota \)-epimorphism of positive opetopes.

**Proof.** Let us fix \( i \in \omega \) and \( s \in S_i^* = S_{i+1} < S_i \). Let \( t = \tilde{f}_i(s) \). Then \( \tilde{f}_i(s) \leq^\omega t \) and, as \( \tilde{f}_i \) is one-to-one, it is the largest such \( s \). Thus \( f_i^*(t) = s \). Since \( s \) was arbitrary, \( f_i^* \) is onto, for any \( i \in \omega \) and hence \( f^* \) epi.

For preservation of both codomains and domains by \( f^* \), we fix \( i > 0 \) and \( t \in T_i^* \) and we consider three cases:

1. \( \tilde{f}_i(t) \in S_i^* \);
2. \( \tilde{f}_i(t) \in S_{i-1}^* \);
3. \( \tilde{f}_i(t) \in S_j^* \), for some \( j < i - 1 \).

**Preservation of codomains** \( \gamma \).

**Case** \( \gamma.1: \) \( \tilde{f}_i(t) = s \in S_i^* \).

First we shall show that \( \tilde{f}_{i-1}(\gamma(s)) \leq^\omega \gamma(t) \). Since \( \tilde{f}_i \)'s are monotone and preserve leaves, we have

\[
\tilde{f}_i(\lvs^i(s)) \subseteq \lvs^i(\tilde{f}_i(s)) \subseteq \lvs^i(t).
\]

Using the above and the fact that \( \tilde{f}_i \)'s preserve sups, we have

\[
\tilde{f}_{i-1}(\gamma(s)) =
\]
Now, contrary to the claim we want to prove, we assume that there is $s_1^* \in S_{i-1}^*$ such that $\gamma(s) \prec co s_1$ and

$$\bar{f}_{i-1}(\gamma(s)) \prec co \bar{f}_{i-1}(s_1^*) \leq co \gamma(t) \in lvs^i(t)^o \subseteq T_{i-1}^*.$$  

Thus $\bar{f}_{i-1}(s_1^*) \in T_i^* - \bar{f}_{i-1}(lvs^i(s)^o)$. As $\bar{f}_{i}(s) \leq co t$, we have $\bar{f}_{i}(lvs^i(s)) \subseteq lvs^i(t)$. Hence $\bar{f}_{i-1}(\gamma(s)) \in lvs^i(t)^o$. Since $lvs^i(t)^o$ is a convex subtree, we have

$$\bar{f}_{i-1}(s_1^*) \in lvs^i(t)^o.$$  

As $\bar{f}_{i}(s_1^*) = \bar{f}_{i-1}(s_1^*)$, we have $\bar{f}_{i}(s_1^*) \leq co t$. Since we also have $s_1^* \not\in lvs^i(s)$, we get that

$$s < \sup \{\{s_1^*\} \cup lvs^i(s)\} = s_2^*$$

and

$$\bar{f}_{i}(s) \prec co \bar{f}_{i}(s_2^*) =$$

$$\bar{f}_{i}(\sup \{\{s_1^*\} \cup lvs^i(s)\}) =$$

$$\sup \{\{\bar{f}_{i}(s_1^*)\} \cup \bar{f}_{i}(lvs^i(s))\} \leq co t.$$

This is a contradiction with the fact that $f_1^*(t) = s$.

**Case $\gamma.2$:** $f_1^*(t) = s_1 \in S_{i-1}^*$.  

Thus we have a $t_1 \in \delta(t)$ such that

$$\bar{f}_{i-1}(s_1) \leq co t_1 \prec co \gamma(t).$$

We need to show that $s_1$ is the largest such an element of $S_{i-1}^*$ that $\bar{f}_{i-1}(s_1) \leq co \gamma(t)$. Suppose to the contrary that there is $s_2^* \in S_{i-1}^*$ such that $s_1 \prec co s_2^*$ and

$$\bar{f}_{i-1}(s_1) \prec co \bar{f}_{i-1}(s_2^*) \leq co \gamma(t).$$

We have $\bar{f}_{i-1}(s_1) \leq co t_1$ and $\bar{f}_{i-1}(s_1) \prec co \bar{f}_{i-1}(s_2^*)$, and, as we cannot have $\bar{f}_{i-1}(s_2^*) \leq co t_1$, we have

$$\bar{f}_{i-1}(s_1) \leq co t_1 < co \bar{f}_{i-1}(s_2^*) \leq co \gamma(t).$$

Since $lvs^i(t)^o$ is a convex subtree of $T_{i-1}^*$, it follows that $f_1^*(s_2^*) \in lvs^i(t)^o$. Thus $f_1^*(s_2^*) \in lvs^i(t)$, i.e. $f_1^*(s_2^*) \leq co t$. Hence $f_1^*(t) \in S_i^*$, contrary to the supposition. End of Case 2.

**Case $\gamma.3$:** $f_1^*(t) = s_1 \in S_j^*$, for some $j < i - 1$.

Suppose there is $s_2 \in S_{i-1}^*$ such that $\bar{f}_{i-1}(s_2) \leq co \gamma(t)$. Then, as $j < i - 1$, $\bar{f}_{i-1}(s_2) \not\leq co t'$, for all $t' \in \delta(t)$. Thus there is $t_1 \in \delta(t)$ and $s_3 \in S_i^*$ so that $s_2 = s_3$ and

$$t_1 < co \bar{f}_{i-1}(s_3) \leq co \gamma(t).$$

As $lvs^i(t)$ is a convex subtree, we have $f_1^*(s_3) \in lvs^i(t)^o$ and then $f_1^*(s_3) \in lvs^i(t)$. This contradicts the fact that $j < i - 1$. Thus $f_1^*(t) = s_4 \in S_{i-j}^*$ such that $j' < i - 1$. If $j' > j$, then

$$\bar{f}_{j'}(s_4) \leq co \delta \gamma(j'+1)(\gamma(t)) = \delta \gamma(j'+1)(t)$$
and this contradicts the choice of \(s_1 \in S^*_t\). If \(j > j'\), then
\[
\delta \gamma^{(j+1)}(t) = \delta \gamma^{(j+1)}(\gamma(t))
\]
contradicting the choice of \(s_4 \in S^*_t\). Thus \(j = j'\) and \(s_1 = s_3\), as required.

**Preservation of domains** \(\delta\).

*Case \(\delta.1\): \(f^*_t(t) = s \in S^*_t\).

We shall show that \(f^*_t\) restricts to a bijection
\[
f^*_t : \delta(t) - \ker(f^*) \rightarrow \delta(s).
\]
Let \(t_1 \in \delta(t) - \ker(f^*)\). Thus there is \(s_1 \in S^*_t\) such that \(f^*_t-1(t_1) = s_1\) and hence \(f^*_t(s_1) \leq \delta t_1\). Since \(f^*\) preserves codomains \(\gamma(s) = f^*_t(\gamma(t))\).

Since \(f^*_t(s_1) \not\in \text{lvs}(t)\subseteq f^*_t-1(\text{lvs}(s)\circ\gamma)\), it follows that \(s_1 \not\in \text{lvs}(s)\circ\gamma\).

We shall show that \(s_1 <^\co \gamma(s)\). Suppose not. Then \(\gamma(s) <^\co s_1 \vee \gamma(s)\) and
\[
f^*_t(s_1 \vee \gamma(s)) = f^*_t(s_1) \vee f^*_t(\gamma(s)) \leq \delta \gamma(t).
\]
This means that
\[
\gamma(f^*_t(t)) = \gamma(s) <^\co s_1 \vee \gamma(s) \leq \delta f^*_t-1(\gamma(t)).
\]
and that the codomains are not preserved. Thus \(s_1 <^\co \gamma(s)\) indeed.

Next we show that \(s_1 \in \delta(s)\). Again, we suppose that this is not the case. Then there is \(s_2 \in \delta(s)\) such that \(s_1 <^\co s_2\). We have
\[
f^*_t(s_1) <^\co f^*_t(s_2) <^\co f^*_t(\gamma(s)) \leq \delta \gamma(t),
\]
and
\[
f^*_t(s_1) \leq \delta t_1 <^\co \gamma(t).
\]
As \(f^*_t(t_1) = s_1\), we have
\[
f^*_t(s_1) <^\co t_1 <^\co f^*_t(s_2) <^\co f^*_t(\gamma(s)) \leq \delta \gamma(t).
\]
As the set \(\text{lvs}(t)\circ\gamma\) is a convex subtree of \(T^*_t\), we have \(f^*_t(s_2) \in \text{lvs}(t)\circ\gamma\). Hence \(s_2 \not\in \text{lvs}(s)\) and \(f^*_t(s_2) \in \text{lvs}(t)\). Thus we have
\[
f^*_t(s_2 \vee s) = f^*_t(s_2) \vee f^*_t(s) \leq \delta t,
\]
and \(s <^\co s_2 \vee s\). This contradicts the fact \(f^*_t(t) = s\). Thus \(s_1 \in \delta(s)\), as claimed.

So far we have shown that \(f^*_t\) restricts to a well defined function
\[
f^*_t : \delta(t) - \ker(f^*) \rightarrow \delta(s).
\]
We shall show that it is a bijection.

Let \(t_1, t_2 \in \delta(t)\) and \(s \in S^*_t\) and \(f^*_t(t_1) = f^*_t(t_2)\). Hence \(s_1 <^\co f^*_t-1(t_1)\) and \(s_1 <^\co f^*_t-1(t_2)\) and then \(t_1 \perp t_2\) or \(t_2 = t_1\). As \(\delta(t)\) is an antichain in \(S^*_t\), \(t_1 = t_2\). Thus \(f^*_t\) is one-to-one.

To see that \(f^*_t\) is onto, let us fix an arbitrary \(s_1 \in \delta(s)\). Then \(s_1 \not\in \text{lvs}(s)\circ\gamma\).

We shall show that \(f^*_t(s_1) \not\in \text{lvs}(t)\circ\gamma\). Suppose to the contrary that \(s_1 = s_3\), for some \(s_3 \in S_t\) and that
\[
\text{lvs}(t)\circ\gamma \ni f^*_t(s_3) \not\in f^*_t-1(\text{lvs}(s)\circ\gamma).
\]
Hence $s_3^* \notin \text{lvs}^i(s)$ and $f(s_3^*) \in \text{lvs}^i(t)$. Thus

$$\bar{f}_i(s_3^* \lor s) = \bar{f}_i(s_3^*) \lor \bar{f}_i(s) \leq^\text{co} t$$

and $s < s_3^* \lor s$. This contradicts the fact that $f_1^*(t) = s$. Thus $f_1^*(s_1) \notin \text{lvs}^i(t)^o$ indeed.

There is $t_1 \in \delta(t)$ such that $\bar{f}_{i-1}(s_1) \leq^\text{co} t_1$. Let $s_2 \in \text{lvs}^i(t)^o$ such that $s_1 <^\text{co} s_2$. Then

$$\bar{f}_{i-1}(s_2) \in \bar{f}_{i-1}(\text{lvs}^i(s)^o) \subseteq \text{lvs}^i(t)^o.$$ 
Hence $s_1$ is the largest element of $S_i^*$ such that $\bar{f}_i(s_1) \leq^\text{co} t$, and hence $f_1^*(t_1) = s_1$, as required. End of proof of Case $\delta.1$.

**Case $\delta.2$:** $f_1^*(t) = s_1 \in S_i^*$. 
In this case we have a $t_1 \in \delta(t)$ such that $\bar{f}_{i-1}(s_1) \leq^\text{co} t_1$. Clearly $f_1^*(t_1) = s_1 = f_1^*(t)$. It remains to show that $\delta(t) - \{t_1\} \subseteq \ker(f^*)$. Suppose to the contrary that there is $t_2 \in \delta(t)$, $t_2 \neq t_1$ such that $f_1^*(t_2) \in S_i^*$. Thus there is $s_2 \in S_i^*$ such that $\bar{f}_{i-1}(s_2) \leq^\text{co} t_2 \leq \gamma(t)$. Hence $s_1 < s_1 \lor s_2$ and

$$\bar{f}_{i-1}(s_1 \lor s_2) = \bar{f}_{i-1}(s_1) \lor \bar{f}_{i-1}(s_2) \leq^\text{co} \gamma(t).$$
But then

$$\gamma^{(i-1)}(f_1^*(t)) = \gamma^{(i-1)}(s_1) = s_1 <^\text{co} s_1 \lor s_2 \leq^\text{co} f_1^*(\gamma^{(i-1)}(t))$$
and this contradicts the fact that $f^*$ preserves codomains.

**Case $\delta.3$:** $f_1^*(t) = s_1 \in S_j^*$, for some $j < i - 1$.
We need to show that $\delta(t) \subseteq \ker(f^*)$. Suppose not. Then there is $t_1 \in \delta(t)$ and $s_2 \in S_i^*$ such that $\bar{f}_i(s_2) \leq^\text{co} t_1$. But this means that $f_1^*(t) \in S_j^*$, for some $j \geq i - 1$, contrary to the supposition.

$\square$

For the proof of duality we need the following observations. We use the notation introduced above.

**Lemma 3.7.** Let $P$ be a positive opetope, $i \in \omega$, $p \in P_{i+1} - \gamma(P_{i+2})$, $p_{\text{root}} = \sup_{<}(\pi(\gamma(p)))$. Then

1. $\gamma \gamma(p) = \gamma(p_{\text{root}})$;
2. the map

$$\xi_p : (\pi(\gamma(p)), <^\gamma) \longrightarrow (\text{lvs}^i(p)^o, <^\text{co}),$$

such that, for $q \in \pi(\gamma(p)) \subseteq P_i - \gamma(P_{i+1})$, $\xi_p(q) = q^o$ is an order isomorphism.
3. In particular, $p_{\text{root}}^* = \xi_p(p_{\text{root}}) = \sup^i(\text{lvs}^{i+1}(p^o)).$

**Proof.** Exercise (use Path Lemma). $\square$

### 3.3 The main theorem

In this section we shall prove that the functors defined in previous sections are essential inverse one to the other.

**Theorem 3.8.** The functors

$$\text{pOpe}_{i,\text{epi}} \xrightarrow{(-)^*} \text{cTree}^\text{op}$$

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defined above, establish a dual equivalence of categories between categories of positive opetopes with \( \iota \)-epimorphisms and tree complexes with embeddings.

**Proof of Theorem 3.3**

We shall define two natural isomorphisms \( \eta \) and \( \varepsilon \).

Let \((S, \sigma)\) be a tree complex. Recall that

\[
S_i^* = S_{i+1} \triangleleft S_i, \quad \text{and} \quad S_i^{**} = (S_{i+1} \triangleleft S_i) - \gamma(S_{i+2} \triangleleft S_{i-1}).
\]

For \( i \in \omega \), the \( i \)-th component

\[
\eta_{S,i} : S_i \rightarrow S_i^{**}
\]

of \( \eta : (S, \sigma) \rightarrow (S^{**}, \sigma^{**}) \) is defined as

\[
\eta_{S,i}(s) = s^*,
\]

i.e. it is a vertex in \( S_{i+1} \triangleleft S_i \). Clearly \( \eta_{S,i} \) is one-to-one and, as all circles in \( S_{i+1} \triangleleft S_i \) are of form \( \gamma(S_{i+2} \triangleleft S_{i-1}) \), \( \eta_{S,i}(s) \) is onto, as well. To see that \( \eta_{S,i} \) is an order isomorphism, consider \( s_1, s_2 \in S_i \). Then

\[
s_1^* \prec s_2^* \quad \text{iff} \quad \gamma(s_1^*) \in \delta(s_2^*)
\]

\[
\text{iff} \quad s_1^0 \in \text{cvr}^{-1}(s_2^0)
\]

\[
\text{iff} \quad s_1^0 <_{\text{co}} s_2^0
\]

\[
\text{iff} \quad s_1 <_{S_i} s_2.
\]

To see that \( \eta \) is an isomorphism of tree complexes, it is enough to show that, for \( i \in \omega \),

\[
(\eta_{S,i+1}, \eta_{S,i} : (S_{i+1}, \sigma_i, S_i) \rightarrow (S_{i+1}^{**}, \sigma_i^{**}, S_i^{**}))
\]

is an isomorphism of constellations. To this aim, it is enough to show that the square

\[
\begin{array}{ccc}
S_{i+1} & \xrightarrow{\sigma_i} & St(S_i) \\
\eta_{S,i+1} & & \downarrow \eta_{S,i} \\
S_{i+1}^{**} & \xrightarrow{\sigma_i^{**}} & St(S_i^{**})
\end{array}
\]

commutes, where \( \eta_{S,i} \) is the image function induced by the function \( \eta_{S,i} \). Let \( s \in S_{i+1}^* \).

We have

\[
\eta_{S,i}(s) = \{ t^* : t \in S_i, \ t \in \sigma_i(s) \} = \{ t^* \in S_i^{**} : t^* <_{\text{co}} s^0 \} = \{ t^* \in S_i^{**} : t^* <^+ \gamma(s^*) \} = \sigma_i^{**}(s^*) = \sigma_i^{**}(\eta_{S,i}(s)).
\]

The naturality of \( \eta \) is clear.

Now we turn to the natural isomorphism \( \varepsilon \). Let \( P \) be a p-opetope, \( i \in \omega \). The maps

\[
\varepsilon_{P,i} : (P_{i+1}^* \triangleleft P_i^*, <_{\text{co}}) \rightarrow (P_i, <^+)
\]
Then we shall consider four cases, one by one.

Using Lemma 3.7, we have

\[ \varepsilon_{P,i}(p_1^*) = p_1 \] and, for \( p_2 \in P_{i+1} - \gamma(P_{i+2}) \), \( p_2^* \in P_{i+1} - \gamma(P_i) \), we have \( \varepsilon_{P,i}(p_2^*) = \gamma(p_2) \).

We need to show that \( \varepsilon_P \) preserves both codomains \( \gamma \) and domains \( \delta \). Naturlality of \( \varepsilon \) is again clear.

Preservation of codomains. Let \( p_1 \in P_{i+1} - \gamma(P_{i+2}) \). We have

\[ \varepsilon_{P,i}(\gamma(p_1^*)) = \varepsilon_{P,i}(p_1^*) = \] \[ = \varepsilon_{P,i+1}(p_1^*) = \gamma(\varepsilon_{P,i+1}(p_1^*)) \]

Let \( p_2 \in P_{i+2} - \gamma(P_{i+3}) \) and \( p_{\text{root}} \in P_{i+1} - \gamma(P_{i+2}) \) such that \( p_{\text{root}} = \sup^j(lvs^{i+1}(p_2^*)) \). Using Lemma 3.7 we have

\[ \varepsilon_{P,i}(\gamma(p_2^*)) = \varepsilon_{P,i}(\sup^j(lvs^{i+1}(p_2^*))) = \] \[ = \varepsilon_{P,i}(p_{\text{root}}) = \gamma(p_{\text{root}}) = \] \[ = \gamma(p_2^*) = \gamma(\varepsilon_{P,i+1}(p_2^*)) \]

Preservation of domains. Let \( p \in P_{i+1} \) and \( q \in P_i \). We need to verify that

\[ q \in \delta(p) \quad \text{iff} \quad \varepsilon_{P,i}(q) \in \delta(\varepsilon_{P,i+1}(p)) \]. \hspace{1cm} (1)

We shall prove the above equivalence by cases depending on the form of \( p \) and \( q \).

Let \( p_1 \in P_{i+1} - \gamma(P_{i+2}) \), \( p_2 \in P_{i+2} - \gamma(P_{i+3}) \), \( q_1 \in P_i - \gamma(P_{i+1}) \), \( q_2 \in P_{i+1} - \gamma(P_{i+2}) \). Then we shall consider four cases, one by one.

**Case 1:** \( p = p_1^*, q = q_1^* \). We have

\[ q_1^* \in \delta(p_1^*) \]

\[ \text{iff} \]

\[ q_1^* \in \text{cvt}^{-1}(P_1^*) \]

\[ \text{iff} \]

\[ q_1^* \prec^\text{co} P_1^* \]

\[ \text{iff} \]

\[ q_1 \in \delta(p_1) \]

\[ \text{iff} \]

\[ \varepsilon_{P,i}(q_1^*) \in \delta(\varepsilon_{P,i+1}(p_1^*)) \]

**Case 2:** \( p = p_1^*, q = q_2^* \).

\[ q_2^* \in \delta(p_1^*) \]

\[ \text{iff} \]

\[ q_2^* \prec^\text{co} P_1^* \]

\[ \text{iff} \]

\[ q_2 \prec P_1 \]

\[ \text{iff} \]

\[ \gamma(q_2) \in \delta(p_1) \]

\[ \text{iff} \]

\[ \varepsilon_{P,i}(q_2^*) \in \delta(\varepsilon_{P,i+1}(p_1^*)) \].
Case 3: $p = p^0_2, q = q^\bullet_1$.

$q^\bullet_1 \in \delta(p^0_2)$

iff

$q^\bullet_1 \in \text{cvr}^i \circ (\text{lvs}^i(p^0_2)^\circ)$

iff

$q^\bullet_1 \not\in \text{lvs}^i(p^0_2)^\circ$ and there is $q_3 \in P_i - \gamma(P_{i+1})$ such that $q^\bullet_1 <^\circ q_3^0$ and $q_3^0 \in \text{lvs}^i(p^0_2)^\circ$

iff

there is $q_3 \in P_i - \gamma(P_{i+1})$ such that $q_1 \in \delta(q_3)$ and $q_3 \preceq^+ \gamma(p_2)$

iff (Path Lemma)

$q_1 \in \delta \gamma(p_2)$

iff

$\varepsilon_{P,i}(q^\bullet_1) \in \delta(\varepsilon_{P,i+1}(p^0_2))$.

Case 4: $p = p^0_2, q = q^\circ_2$.

$q^\circ_2 \in \delta(p^0_2)$

iff

$q^\circ_2 \in \text{cvr}^i \circ (\text{lvs}^i(p^0_2)^\circ)$

iff

$q^\circ_2 \not\in \text{lvs}^i(p^0_2)^\circ$ and there is $q_3 \in P_i - \gamma(P_{i+1})$ such that $q^\circ_2 <^\circ q_3^0$ and $q_3^0 \in \text{lvs}^i(p^0_2)^\circ$

iff

$q_2 \not\preceq^+ \gamma(p_2)$ and there is $q_3 \in P_i - \gamma(P_{i+1})$ such that $q_2 <^\circ q_3$ and $q_3 \preceq^+ \gamma(p_2)$

iff (Path Lemma)

$\gamma(q_2) \in \delta \gamma(p_2)$

iff

$\varepsilon_{P,i}(q^\circ_2) \in \delta(\varepsilon_{P,i+1}(p^0_2))$.

4 Thicket complexes

4.1 Thicket constellations

A thicket is a finite poset $(P, \leq)$ such that, for any $p \in P$, the subposet $\{p' \in P : p' \leq p\}$ is a tree. A morphism of thickets is a one-to-one function that preserves and reflects order. Thicket is the category of thickets. $\text{St}(P)$ is the poset of convex sub-trees of thicket $P$.

A thicket-constellation (or t-constellation or even constellation for short) is a triple $(T', \tau, T)$ where $\tau$ is a monotone function

$\tau : T' \to \text{St}(T)$

such that if $t, t' \in T'$ and $\sigma(t) \cap \sigma(t') \neq \emptyset$, then $t \perp t'$.
Let $\tau : T' \rightarrow \text{St}(T)$ be a t-constellation. Then the thicket $T' \triangleleft_\tau T$, the extension of $T'$ by $T$ along $\tau$, is the thicket $T'$ with nodes of $T$ added as leaves so that if $x \in T$ and $y \in T'$, then $x <^{co} y$ in $T' \triangleleft_\tau T$ iff $x \in \tau(y)$. The order $<^{co}$ is called the constellation order of the t-constellation $\tau : T' \rightarrow \text{St}(T)$, or just constellation order if the t-constellation is understood. Any pair of tree maps $f : S \rightarrow T$, $f' : S' \rightarrow T'$ induces a monotone map $f' \triangleleft f : S' \triangleleft_\sigma S \rightarrow T' \triangleleft_\tau T$. Such a pair

$$(f, f') : (S', \sigma, S) \rightarrow (T', \tau, T)$$

is a morphism of t-constellations if the induced map $f' \triangleleft f : S' \triangleleft_\sigma S \rightarrow T' \triangleleft_\tau T$ preserves binary sups.

### 4.2 Thicket complexes and duality

A thicket complex $(T, \tau)$ is a sequence of t-constellations:

$$\tau_0 : T_1 \rightarrow \text{St}(T_0),$$

$$\tau_1 : T_2 \rightarrow \text{St}(T_1),$$

$$\cdots$$

$$\tau_i : T_{i+1} \rightarrow \text{St}(T_i),$$

$$\cdots$$

$i \in \omega$, with almost all sets $T_i$ empty. The dimension $(T, \tau)$ is $n$ iff $T_n$ be the last non-empty set. We write $\text{dim}(T)$ for dimension of the tree complex $(T, \tau)$. $T_0$ is required to be a singleton.

A morphism of thicket complexes $f : (S, \sigma) \rightarrow (T, \tau)$ is a family of thicket embeddings $f_i : S_i \rightarrow T_i$, for $i \in \omega$, such that, for $i \in \omega$,

$$(f_{i+1}, f_i) : (S_{i+1}, \sigma_i, S_i) \rightarrow (T_{i+1}, \tau_i, T_i)$$

is a morphism of t-constellations, i.e. the thicket embeddings

$$\tilde{f}_i = f_i \triangleleft f_i : S_{i+1} \triangleleft_\sigma S_i \rightarrow T_{i+1} \triangleleft_\tau T_i$$

that preserve binary sups.

Size of a w-complex $(T, \tau)$ is a sequence of natural numbers $\text{size}(T, \tau) = \{s_i\}_{i \in \omega}$ so that $s_i = \text{size}(T, \tau)_i$ is the number of maximal elements in the thicket $T_i$. A w-complex $(T, \tau)$ is a tree complex iff $\text{size}(T, \tau)_i \leq 1$, for all $i \in \omega$.

The category of thicket complexes and their morphisms will be denoted by $\mathbf{cThicket}$. Clearly $\mathbf{cTree}$ is a full subcategory of $\mathbf{cThicket}$.

**Theorem 4.1.** The functors

$$\text{pOpeCard}_{\iota, \text{epi}} \xrightarrow{(-)^*} \mathbf{cThicket}^{\text{op}}$$

defined as those for trees, establish a dual equivalence of categories between categories of opetopic cardinals with $\iota$-epimorphisms and thicket complexes with embeddings.

**Proof** This is an easy extension of the corresponding fact concerning tree complexes and p-opetopes. □
Appendix: Positive opetopes and positive opetopic cardinals

In this appendix we recall the notion of positive opetopes, positive opetopic cardinals, their morphisms: face maps and $\iota$-maps. We also quote without proofs some facts from [Z1] and [Z4].

5.1 Positive hypergraphs

A positive hypergraph $S$ is a family $\{S_k\}_{k \in \omega}$ of finite sets of faces, a family of functions $\{\gamma_k : S_{k+1} \to S_k\}_{k \in \omega}$, and a family of total relations $\{\delta_k : S_{k+1} \to S_k\}_{k \in \omega}$. Moreover, $\delta_0 : S_1 \to S_0$ is a function and only finitely many among sets $\{S_k\}_{k \in \omega}$ are non-empty. As it is always clear from the context, we shall never use the indices of the functions $\gamma$ and $\delta$. We shall ignore the difference between $\gamma(x)$ and $\{\gamma(x)\}$ and in consequence we shall consider iterated applications of $\gamma$’s and $\delta$’s as sets of faces, e.g. $\delta(x) = \bigcup_{y \in \delta(x)} \delta(y)$ and $\gamma\delta(x) = \{\gamma(y) \mid y \in \delta(x)\}$.

A morphism of positive hypergraphs $f : S \to T$ is a family of functions $f_k : S_k \to T_k$, for $k \in \omega$, such that, for $k > 0$ and $a \in S_k$, we have $\gamma(f(a)) = f(\gamma(a))$ and $f_{k-1}$ restricts to a bijection $f_a : \delta(a) \to f(\delta(a))$.

The category of positive hypergraphs is denoted by $\mathbf{pHg}$.

We define a binary relation of lower order on $\langle S_k, - \rangle$ for $k > 0$ as the transitive closure of the relation $\langle S_k, - \rangle$ on $S_k$ such that, for $a, b \in S_k$, $a \langle S_k, - \rangle b$ iff $\gamma(a) \in \delta(b)$. We write $a \perp b$ iff either $a \perp - b$ or $b \perp - a$, and we write $a \leq - b$ iff either $a = b$ or $a \perp - b$.

We also define a binary relation of upper order on $\langle S_k, + \rangle$ for $k \geq 0$ as the transitive closure of the relation $\langle S_k, + \rangle$ on $S_k$ such that, for $a, b \in S_k$, $a \langle S_k, + \rangle b$ iff there is $a \in S_{k+1}$ so that $a \in \delta(a)$ and $\gamma(a) = b$. We write $a \perp + b$ iff either $a \perp + b$ or $b \perp + a$, and we write $a \leq + b$ iff either $a = b$ or $a \perp + b$.

5.2 Positive opetopic cardinals

A positive hypergraph $S$ is a positive opetopic cardinal if it is non-empty, i.e. $S_0 \neq \emptyset$ and it satisfies the following four conditions

1. Globularity: for $a \in S \geq 2$:

\[\gamma\gamma(a) = \gamma\delta(a) - \delta\delta(a), \quad \delta\gamma(a) = \delta\delta(a) - \gamma\delta(a);\]

2. Strictness: for $k \in \omega$, the relation $\langle S_k, + \rangle$ is a strict order; $\langle S_0, + \rangle$ is linear;

3. Disjointness: for $k > 0$,

\[\perp S_k, - \cap \perp S_k, + = \emptyset\]

4. Pencil linearity: for any $k > 0$ and $x \in S_{k-1}$, the sets

\[\{a \in S_k \mid x = \gamma(a)\} \quad \text{and} \quad \{a \in S_k \mid x \in \delta(a)\}\]

are linearly ordered by $\langle S_k, + \rangle$.

The category of positive opetopic cardinals is the full subcategory of $\mathbf{pHg}$ whose objects are positive opetopic cardinals. It is denoted by $\mathbf{pOpeCard}$. 

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5.3 Positive opetopes

The size of positive opetopic cardinal $S$ is a sequence of natural numbers $\text{size}(S) = \{|S_n - \delta(S_{n+1})|\}_{n \in \omega}$, with all elements above $\dim(S)$ being equal to 0. We have an order $<$ on such sequences of natural numbers so that $\{x_n\}_{n \in \omega} < \{y_n\}_{n \in \omega}$ iff there is $k \in \omega$ such that $x_k < y_k$ and, for all $l > k$, $x_l = y_l$. This order is well founded and hence facts about positive opetopic cardinals can be proven by induction on their size.

Let $P$ be a positive opetopic cardinal. We say that $P$ is a positive opetope if $\text{size}(P)l \leq 1$, for $l \in \omega$. By $\text{pOpe}$ we denote full subcategory of $\text{pHg}$ whose objects are positive opetopes.

Some notions and notation.

1. Let $S$ be a positive hypergraph $S$, $x$ a face of $S$. Then $S[x]$ denotes the least subhypergraph of $S$ containing face $x$. The dimension of $S$ is maximal $k$ such that $S_k \neq \emptyset$. We denote by $\dim(S)$ the dimension of $S$.

2. Let $P$ be an opetope. If $\dim(P) = n$, then the unique face in $P_n$ is denoted by $\text{mp}$.

3. The function $\gamma^{(k)} : P \rightarrow P_{\leq k}$ is an iterated version of the codomain function $\gamma$ defined as follows. For any $k, l \in \omega$ and $p \in P_l$,

$$\gamma^{(k)}(p) = \begin{cases} \gamma\gamma^{(k+1)}(p) & \text{if } l > k \\ p & \text{if } l \leq k. \end{cases}$$

4. Let $a, b \in P_k$. A lower path $a_0, \ldots, a_m$ from $a$ to $b$ in $P$ is a sequence of faces $a_0, \ldots, a_m \in S_k$ such that $a = a_0, b = a_m$ and, for $\gamma(a_{i-1}) \in \delta(a_i), i = 1, \ldots, m$.

5. Let $x, y \in P_k$. An upper path $x, a_0, \ldots, a_m, y$ from $x$ to $y$ in $S$ is a sequence of faces $a_0, \ldots, a_m \in P_{k+1}$ such that $x \in \delta(a_0), y = \gamma(a_m)$ and $\gamma(a_{i-1}) \in \delta(a_i)$, for $i = 1, \ldots, m$.

Lemma 5.1. Let $P$ be a positive opetopic cardinal, $n \in \omega$, $a, b \in P_n$, $a <^+ b$. Then, there is an upper $P_{n+1} - \gamma(P_{n+2})$-path from $a$ to $b$. □

Lemma 5.2 (Path Lemma). Let $P$ be an opetope. Let $k \geq 0, B = (a_0, \ldots, a_k)$ be a maximal $S_n$-lower path in a positive opetopic cardinal $P$, $b \in S_n$, $0 \leq s \leq k, a_s <^+ b$. Then there are $0 \leq l \leq s \leq p \leq k$ such that

1. $a_i <^+ b$ for $i = l, \ldots, p$;
2. $\gamma(a_p) = \gamma(b)$;
3. either $l = 0$ and $\delta(a_0) \subseteq \delta(b)$ or $l > 0$ and $\gamma(a_{l-1}) \in \delta(b)$;
4. $\gamma(a_l) \in \iota(S)$, for $l \leq i < p$. □

5.4 The embedding of $\text{pOpeCard}$ into $\omega\text{Cat}$

There is an embedding

$(-)^* : \text{pOpeCard} \rightarrow \omega\text{Cat}$

defined as follows, c.f. [Z1]. Let $T$ be an opetopic cardinal. The $\omega$-category $T^*$ has as $n$-cells pairs $(S, n)$, where $S$ is a subopetopic cardinal of $T$, $\dim(S) \leq n$, and $n \geq 0$. A cell $(S, n)$ is called proper iff $n = \dim(S)$. $\diamondsuit$
For $k < n$, the domain and codomain operations
\[ d^{(k)}, c^{(k)} : T^*_n \rightarrow T^*_k \]
are given, for $(S, n) \in T^*_n$, by
\[ (d^{(k)}(S, n)) = (d^{(k)}(S), k), \quad (c^{(k)}(S, n)) = (c^{(k)}(S), k) \]
where
\[ (d^{(k)}(S))_l = \begin{cases} \emptyset & \text{if } l > k \\ S_k - \gamma(S_{k+1}) & \text{if } l = k \\ S_l & \text{if } l < k \end{cases} \]
and
\[ (c^{(k)}(S))_l = \begin{cases} \emptyset & \text{if } l > k \\ S_k - \delta(S_{k+1}) & \text{if } l = k \\ S_{k-1} - \nu(S_{k+1}) & \text{if } l = k - 1 \geq 0 \\ S_l & \text{if } l < k - 1 \end{cases} \]
The identity operation
\[ id : T^*_n \rightarrow T^*_{n+1} \]
is given by
\[ (S, n) \mapsto (S, n + 1). \]
The composition operation is defined, for pairs of cells $(S, n), (S', n') \in T^*$ with $k \leq n, n'$ such that $d^{(k)}(S, n) = c^{(k)}(S', n')$, as the sum
\[ (S, n) \circ (S', n') = (S \cup S', \max(n, n')). \]

Now $T^*$ together with operations of domain, codomain, identity and composition is an $\omega$-category. If $f : S \rightarrow T$ is a map of opetopic cardinals and $S'$ is a sub-opetopic cardinal of $S$, then the image $f(S')$ is a sub-opetopic cardinal of $T$. This defines the functor $(-)^*$ on morphisms. We recall from [Z1]

**Theorem 5.3.** The embedding
\[ (-)^* : pOpeCard \rightarrow \omega\text{Cat} \]
is well defined and full on isomorphisms and it factorises through $\text{Poly} \rightarrow \omega\text{Cat}$ via full and faithful functor, $(-)^* : pOpeCard \rightarrow \text{Poly}$, into the category of polygraphs. $\square$

### 5.5 $\iota$-Maps of positive opetopes

The embedding $(-)^* : pOpe \rightarrow \omega\text{Cat}$ is not full. The morphisms $P^* \rightarrow Q^*$ in $\omega\text{Cat}$ between images of opetopes are $\omega$-functors that send generators to generators. The category $pOpe_{\omega}$ with the same objects as $pOpe$ will be so defined that the embedding $(-)^* : pOpe_{\omega} \rightarrow \omega\text{Cat}$ (denoted the same way) will be full on $\omega$-functors that send generators to either generators or identities on generators of a smaller dimension.

Let $P$ and $Q$ be positive opetopes. A **contraction morphism of opetopes** (or $\iota$-map, for short), $h : Q \rightarrow P$, is a function $h : |Q| \rightarrow |P|$ between faces of opetopes such that
1. $\dim(q) \geq \dim(h(q))$, for $q \in Q$;
2. (preservation of codomains) $h(\gamma^{(k)}(q)) = \gamma^{(k)}(h(q))$, for $k \geq 0$ and $q \in Q_{k+1}$;
3. (preservation of domains)
(a) if \( \dim(h(q)) = \dim(q) \), then \( h \) restricts to a bijection
\[
(\delta(q) - \ker(h)) \xrightarrow{h} \delta(h(q))
\]
for \( k \geq 0 \) and \( q \in Q_{k+1} \), where the kernel of \( h \) is defined as
\[
\ker(h) = \{ q \in Q | \dim(q) > \dim(h(q)) \}.
\]
(b) if \( \dim(h(q)) = \dim(q) - 1 \), then \( h \) restricts to a bijection
\[
(\delta(q) - \ker(h)) \xrightarrow{h} \{h(q)\}
\]
for \( k \geq 0 \) and \( q \in Q_{k+1} \);
(c) if \( \dim(h(q)) < \dim(q) - 1 \), then \( \delta^{(k)}(q) \subseteq \ker(h) \).

We call a face \( q \in Q_m \) \( k \)-collapsing iff \( h(q) \in P_{m-k} \). A 0-collapsing face is called non-collapsing.

We have an embedding \( \kappa : pOpe \rightarrow \hat{pOpe}_l \) that induces the usual adjunction \( \kappa_! \dashv \kappa^* \)
\[
\begin{array}{c}
pOpe
\end{array}
\xrightarrow{\kappa!}
\begin{array}{c}
\hat{pOpe}_{\leq l}
\end{array}
\xleftarrow{\kappa^*}
\begin{array}{c}
pOpe
\end{array}
\]

**Lemma 5.4.** Let \( h : Q \rightarrow P \) be a \( \iota \)-map, \( q_1, q_2 \in Q - \ker(h) \) and \( l < k \in \omega \) such that
\[
\begin{array}{c}
\gamma^{(k+1)}(q_1) \quad \gamma^{(k+1)}(q_2)
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(k)}(q_1) \quad \gamma^{(k)}(q_2)
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(l+1)}(q_1) \quad \gamma^{(l+1)}(q_2)
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(l)}(q_1) \quad \gamma^{(l)}(q_2)
\end{array}
\]
Then there is \( l \leq l' < k \) such that
\[
\begin{array}{c}
\gamma^{(k+1)}(q_1) \quad h(\gamma^{(k+1)}(q_1))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(k)}(q_1) \quad h(\gamma^{(k)}(q_1))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(l+1)}(q_1) \quad h(\gamma^{(l+1)}(q_1))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(l)}(q_1) \quad h(\gamma^{(l)}(q_1))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(k+1)}(q_2) \quad h(\gamma^{(k+1)}(q_2))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(k)}(q_2) \quad h(\gamma^{(k)}(q_2))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(l+1)}(q_2) \quad h(\gamma^{(l+1)}(q_2))
\end{array}
\]
\[
\begin{array}{c}
\gamma^{(l)}(q_2) \quad h(\gamma^{(l)}(q_2))
\end{array}
\]

**Proof.** Simple check. \( \square \)

From the above we get immediately

**Corollary 5.5.** Let \( h : Q \rightarrow P \) be a \( \iota \)-map, \( q_1, q_2 \in Q - \ker(h) \). Then
\[
\begin{array}{l}
1. \quad q_1 \prec q_2 \text{ iff } h(q_1) \prec h(q_2); \\
2. \quad \text{if } q_1 \prec q_2, \text{ then } h(q_1) \succeq h(q_2); \\
3. \quad \text{if } h(q_1) \prec h(q_2), \text{ then } q_1 \prec q_2; \\
4. \quad \text{if } h(q_1) = h(q_2), \text{ then } q_1 \perp q_2. \quad \square
\end{array}
\]

A set \( X \) of \( k \)-faces in a positive opetope \( P \) is a \( \prec^+ \)-interval (or interval, for short) if it is either empty or there are two \( k \)-faces \( x_0, x_1 \in P_k \) such that \( x_0 \leq^+ x_1 \) and \( X = \{ x \in P_k | x_0 \leq^+ x \leq^+ x_1 \} \). Any interval in any positive opetope is linearly ordered by \( \leq^+ \).
Corollary 5.6. Let \( h : Q \to P \) be a contraction of positive opetopes, \( p \in P_k \). Then the fiber of \( k \)-faces (of non-degenerating faces) \( h^{-1}(p) - \ker(h) \) of \( h \) over \( p \) is an interval.

Proof. From Corollary 5.5.4 we get that \( h^{-1}(p) - \ker(h) \) is linearly ordered. And from Corollary 5.5.2 that this linear order is an interval. \( \square \)

5.6 The embedding of \( \mathbf{pOpe} \) into \( \mathbf{\omega Cat} \)

We extend the embedding functor \((-)^*\) to contractions

\[
(-)^*: \mathbf{pOpe} \to \mathbf{\omega Cat}.
\]

Let \( h : Q \to P \) be a contraction morphism in \( \mathbf{pOpe} \). Then

\[
h^*: Q^* \to P^*
\]

is an \( \omega \)-functor such that

\[
h^*(k, A) = (k, \bar{h}(A))
\]

where \((k, A) \in Q^*_k\), and \( \bar{h}(A) \) is the set-theoretic image of the opetopic cardinal \( A \) under \( h \).

Theorem 5.7. The functor

\[
(-)^*: \mathbf{pOpe} \to \mathbf{\omega Cat}
\]

is well defined. The objects of \( \mathbf{pOpe} \) are sent under \((-)^*\) to positive-to-one polygraphs. \((-)^*\) is faithful, conservative and full on those \( \omega \)-functors that send generators to either generators or to (possibly iterated) identities on generators of smaller dimensions. In particular, it is full on isomorphisms. \( \square \)

References

[BD] J. Baez, J. Dolan, Higher-dimensional algebra III: n-Categories and the algebra of opetopes. Advances in Math. 135 (1998), pp. 145-206.

[Be] C. Berger, A Cellular Nerve for Higher Categories. Adv. in Mathematics 169, (2002), pp. 118-175.

[Bu] A. Burroni, Higher-dimensional word problems with applications to equational logic, Theoretical Computer Science 115 (1993), no. 1, pp. 43–62.

[Ch] E. Cheng, The Category of Opetopes and the Category of Opetopic Sets, Theory and Applications of Categories, Vol. 11, No. 16, 2003, pp. 353–374.

[CTM] P-L. Curien, C. Ho Thanh, S. Mimram, Syntactic approaches to opetopes, arXiv:1903.05848 [math.CT], 2019.

[FS] M. Fiore, P. Saville, List Objects with Algebraic Structure, In Proceedings of the 2nd International Conference on Formal Structures for Computation and Deduction (FSCD 2017), No. 16, pages 1-18, 2017.

[HMP] C. Hermida, M. Makkai, J. Power, On weak higher dimensional categories, I Parts 1,2,3, J. Pure and Applied Alg. 153 (2000), pp. 221-246, 157 (2001), pp. 247-277, 166 (2002), pp. 83-104.
[T] C. Ho Thanh, Opetopes. Syntactic and Algebraic Aspects. Doctoral thesis at the University of Paris (Denis Diderot) (2020), pp. 1-324.

[KJBM] J. Kock, A. Joyal, M. Batanin, J-F. Mascari, Polynomial Functors and opetopes, Adv. Math. 224 (2010) pp. 2690-2737.

[Lei] Tom Leinster, Higher Operads, Higher Categories, London Mathematical Society Lecture Note Series 298, Cambridge University Press, Cambridge 2004. Preprint available as arXiv:math/0305049v1 [math.CT].

[MZ] M. Makkai, M. Zawadowski, Disks and duality, Theory and Applications of Categories, Vol. 8, 2001, No. 7, pp 114-243.

[Ou] D. Oury, On the duality between trees and disks, Theory and Applications of Categories, Vol. 24, 2010, No. 16, pp 418-450.

[P] T. Palm, Dendrotopic sets, Hopf algebras, and semiabelian categories, Fields Inst. Commun. vol. 43 (2004), 411-461 AMS, Providence, RI.

[St] R. Steiner, Opetopes and chain complexes, Theory and Applications of Categories, Vol. 26, 2012, No. 19, pp 501-519.

[SZ] S. Szawiel, M. Zawadowski, The web monoid and opetopic sets, J. of Pure and Applied Algebra 217 (2013), pp. 1105-1140.

[Z1] M. Zawadowski, On positive face structures and positive-to-one computads. ArXiv:0708.2658 [math.GT], 2006, pp. 1-77.

[Z2] M. Zawadowski, On ordered face structures and many-to-one computads. ArXiv:0708.2659 [math.CT], (2007), pp. 1-95.

[Z3] M. Zawadowski, Lax Monoidal Fibrations, in Models, Logics, and Higher-Dimensional Categories: A Tribute to the Work of Mihály Makkai (B. Hart, et al., editors) (CRM Proceedings 53, 2011), pp. 341-424.

[Z4] M. Zawadowski, Positive Opetopes with Contractions form a Test Category, ArXiv:1712.06033 [math.CT], (2017).