Global well-posedness, stability and instability for the non-viscous Oldroyd-B model

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Abstract. In this paper we consider the 3-dimensional incompressible Oldroyd-B model. First, we establish two results of the global existence for different kinds of the coupling coefficient \( k \). Then, we prove that the solutions \((u, \tau)\) are globally steady when \( k^m \rightarrow k > 0 \), though \((u, \tau)\) corresponds to different decays for different kinds of \( k > 0 \). Finally, we show that the energy of \( u(t, x) \) will have a jump when \( k \rightarrow 0 \) in large time, which implies a non-steady phenomenon. In a word, we find an interesting physical phenomenon of (1.2) such that smaller coupling coefficient \( k \) will have a better impact for the energy dissipation of \((u, \tau)\), but \( k \) can't be too small to zero, or the dissipation will vanish instantly. While the damping term \( \tau \) and \( D(u) \) always bring the well impact for the energy dissipation.

Keywords: Oldroyd-B model; global existence; stability; instability; decay; energy dissipation.

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1 Introduction and main results

In this paper, we study the incompressible Oldroyd-B model of the non-Newtonian fluid in \( \mathbb{R}^+ \times \mathbb{R}^d \)

\[
\begin{aligned}
\partial_t u + (u \cdot \nabla) u - \nu \Delta u + \nabla p &= k \text{div}(\tau), \\
\partial_t \tau + (u \cdot \nabla) \tau - \eta \Delta \tau + \mu \tau + Q(\nabla u, \tau) &= \alpha D(u), \\
\text{div} u &= 0, \\
u(x, 0) &= u_0(x), \quad \tau(0, x) = \tau_0(x),
\end{aligned}
\]

(1.1)

where \( u \) denotes the velocity, \( \tau = \tau_{i,j} \) is the non-Newtonian part of the stress tensor (\( \tau \) is a \( d \times d \) symmetric matrix here) and \( p \) is a scalar pressure of fluid. \( D(u) \) is the symmetric part of the velocity gradient,

\[
D(u) = \frac{1}{2}(\nabla u + (\nabla u)^T).
\]

The \( Q \) above is a given bilinear form:

\[
Q(\tau, \nabla u) = \tau \Omega(u) - \Omega(u) \tau + b(D(u) \tau + \tau D(u)),
\]

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where $b$ is a parameter in $[-1, 1]$, $\Omega(u)$ is the skew-symmetric part of $\nabla u$, i.e.

$$\Omega(u) = \frac{1}{2}(\nabla u - (\nabla u)^T).$$

The parameters $\nu, \eta, \mu, \alpha$ are non-negative and they are specific to the characteristic of the considered material, $\nu$ is the viscous coefficient, while $\eta$ is the stress coefficient. In [19], $\mu$ and $\alpha$ correspond respectively to $1/We$ and $2(1 - \theta)/(WeRe)$, where $Re$ is the Reynolds number, $\theta$ is the ratio between the relaxation and retardation times and $We$ is the Weissenberg number. $k$ is the coupling coefficient connecting the velocity $u$ (kinetic energy) and the stress tensor $\tau$ (elastic potential energy).

The Oldroyd-B model describes the motion of some viscoelastic flows. Formulations about viscoelastic flows of Oldroyd-B type are first established by Oldroyd in [21]. For more detailed physical background and derivations about this model, we refer the readers to [2, 8, 18, 21].

When $\nu > 0$ and $\eta = 0$, Chemin and Masmoudi [5] first obtained the local solutions and global small solutions in the critical Besov spaces when $\nu > 0, \mu_1 > 0, \alpha > 0$, and $\eta = 0$. They get the global small solutions when the initial and coupling parameters is small, i.e.($\mu_1 \alpha \leq c \mu_2 \nu$). The condition $\mu_1 \alpha \leq c \mu_2 \nu$ means that coupling effect between the two equation is less important than the viscosity. Inspired by the work [13, 7], Zi, Fang and Zhang improved their results in the critical $L^p$ framework for the case of non-small coupling parameters in [31]. Zhu [30] got small global smooth solutions of the 3D Oldroyd-B model with $\eta = 0, \mu = 0$ by observing the linearization of the system satisfies the damped wave equation. Inspired by the work of Zhu [30] and Danchin in [10], Chen and Hao [6] extended this small data global solution in Sobolev spaces to the critical Besov spaces. Moreover, Zhai [27] constructs global solutions for a class of highly oscillating initial velocities by observing the special structure of the system. In the corotational case, i.e. $b = 0$, Lions and Masmoudi established the existence of global weak solution in [19].

When $\nu = 0$ and $\eta > 0$, Elgindi and Rousset [14] established a global large solution in a certain sense by building a new quantity to avoid singular operators. Later, Liu and Elgindi [13] extend these results in 3d for totally small initial data $\|u_0, \tau_0\|_{H^s(\mathbb{R}^3)}$, $s > \frac{3}{2}$. Recently, Constantin, Wu, Zhao and Zhu [9, 26] established these small data global solutions in the case of no damping mechanism and general tensor dissipation.

In this paper, we consider the global well-posedness, stability and instability for the Oldroyd-B model (1.1) with $\nu = 0$ and $\eta > 0$. Without lose of generality, we let $\nu = 0, \alpha = 1, \mu = 1$ and $\eta = 1$. Since the coupling coefficient $k$ is finite, we set $0 \leq k \leq 10$ in this paper, then (1.1) becomes:

$$\begin{cases}
\partial_t u + (u \cdot \nabla) u + \nabla p = k \text{div}(\tau), \\
\partial_t \tau + (u \cdot \nabla) \tau - \Delta \tau + \tau + Q(\nabla u, \tau) = \mathbb{D} u, \\
\text{div} u = 0, \\
u(x, 0) = u_0(x), \quad \tau(0, x) = \tau_0(x),
\end{cases}$$

When $k > 0$, since $\tau$ and $\mathbb{D} u$ are the damping terma, some dissipations will appear on $\|\tau\|_{H^s}$ and $\|\mathbb{D} u\|_{H^{s-1}}$. However, when $k = 0$, since the system (1.2) decouples, all the dissipations will vanish. This implies that the coupling coefficient $k$ plays a key role in energy dissipation, which is what we study on this paper.

Firstly, we introduce the global existence of (1.2). Recall that, for $d = 2$, by building a new quantity $\Gamma = w - \frac{\text{curl} div}{\Delta} \tau$ Elgindi and Rousset [14] established a class of global solutions for
\[\textbf{[12]},\text{ which need the following initial conditions:}\]

\[\|u_0, \tau_0\|_{H^1(\mathbb{R}^2)} + \|\text{curl}u_0, \tau_0\|_{B^0_{\infty,1}(\mathbb{R}^2)} \leq \epsilon_0, \quad (u_0, \tau_0) \in H^s(\mathbb{R}^2) \times H^s(\mathbb{R}^2), \quad s > 2.\]

Since \( H^s \to B^{s+1}_{2,1} \to B^0_{\infty,1} \) with \( s > 1 \), their result means some large initial data for the global existence. However, when \( d = 3 \), it seems to be a challenge for the same conditions of initial data. Because a new term \( w \nabla u \) appears in the equation of \( w(t, x) \) in dimensional three, so as the equation of \( \Gamma(t, x) \). This cause the main difficulty to obtain the global existence for \([12]\).

To overcome this difficulty, we observe that the damping term \( D \) will help us prove the global existence for a more general class of initial data such that:

\[\|u_0\|_{B^1_{\infty,1}(\mathbb{R}^3)} + \|\tau_0\|_{B^0_{\infty,1}(\mathbb{R}^3)} \leq k^4\epsilon_0, \quad \forall k \in (0, \frac{1}{C^2 + 1}], \tag{1.3}\]

By \([12]\) we obtain the global existence of \([1.2]\) without \((u_0, \tau_0) \in H^s\). Indeed, \([14]\) used the following estimation \((\hat{R} := -(\Delta)^{-1}\text{curl}(\text{div} \cdot), \quad \forall \epsilon > 0):\]

\[\|\hat{R}, u \cdot \nabla \tau\|_{L^2(B^0_{\infty,1})} \leq C\|u\|_{L^1(\infty)}\|\tau\|_{L^2(B^0_{\infty,1})} \leq C\|u\|_{L^1(\infty)}\|\tau\|_{L^2(B^0_{\infty,1})}, \tag{1.4}\]

where \( H^2(\mathbb{R}^d) \to B^\infty_{\infty,1}(\mathbb{R}^d) \). With the help the convective term \( u \nabla \Gamma \), we find that the \( H^s \) norms for \( w, \tau \) are not required in \([12]\). So our condition \([13]\) implies a more general class of large initial data for global existence (see Remark \([1.1]\)). Moreover, for sufficient small \( k \), we obtain the exponential decay in the critical Besov spaces. Here are two results of global existence.

**Theorem 1.1.** Let \((u_0, \tau_0) \in B^{1+\frac{d}{p}}_{p,1}(\mathbb{R}^3) \times B^{\frac{d}{p}}_{p,1}(\mathbb{R}^3) \) with \( p \in [1, \infty] \). If there exists a \( \epsilon_0 \) small enough such that

\[\|u_0\|_{B^{1+\frac{d}{p}}_{p,1}} + \|\tau_0\|_{B^{\frac{d}{p}}_{p,1}} \leq k^4\epsilon_0 := \frac{k^4}{4(C^2 + 1]}, \quad \forall k \in (0, \frac{1}{C^2 + 1}], \]

then the solution \((u, \tau)\) of \([1.2]\) exists globally in \( C([0, \infty); B^{1+\frac{d}{p}}_{p,1}(\mathbb{R}^3)) \times C([0, \infty); B^{\frac{d}{p}}_{p,1}(\mathbb{R}^3)) \cap L^1([0, \infty); B^{2+\frac{d}{p}}_{p,1}(\mathbb{R}^3)) \). Moreover, one have

\[\|\nabla u(t)\|_{B^{1+\frac{d}{p}}_{p,1}} + k\|\tau(t)\|_{B^{\frac{d}{p}}_{p,1}} \leq C\|\nabla u_0\|_{B^{1+\frac{d}{p}}_{p,1}} e^{-\frac{4}{\epsilon}t}. \tag{1.5}\]

**Remark 1.1.** Since \( B^{\frac{d}{p}}_{p,1} \to B^0_{\infty,1}, \quad p < \infty. \) By Theorem \([12]\) we claim that our result includes some large initial data. For example, choose \( \varphi \) be a smooth, radial and non-negative function in \( \mathbb{R}^2 \) such that

\[\phi = \begin{cases} 1, & \text{for } |\xi| \leq 1, \\ 0, & \text{for } |\xi| \geq 2. \end{cases} \tag{1.6}\]

Let \((u_0, \tau_0) := \frac{1}{N}(\psi, \varphi), \text{ where } \psi, \varphi \in S^3, \div \psi = 0 \text{ and } F(\varphi) = (\phi(\xi - 2^N e), \phi(\xi - 2^N e), \phi(\xi - 2^N e)) \) with \( e = (1, 1), N \in \mathbb{N}^+ \). Then, one can easily deduce that

\[\Delta_j \varphi = \varphi \quad \text{when } j = N; \quad \Delta_j \varphi = 0 \quad \text{when } j \neq N.\]

So for sufficient large \( N \) and \( p < \infty \), we have

\[\|\tau_0\|_{B^{\frac{d}{p}}_{p,1}} \approx \frac{2^p N}{N}, \quad \text{but } \|\tau_0\|_{B^0_{\infty,1}} \leq \frac{C}{N}. \]

This implies the global existence for some large initial data, which is different from the result in \([14]\) \([12]\).
Theorem 1.2. Let \((u_0, \tau_0) \in (H^s(\mathbb{R}^3), H^s(\mathbb{R}^3))\) with \(s > \frac{5}{2}\). If there exists a \(\epsilon_0\) small enough such that
\[
\|\nabla u_0\|_{H^{s-1}} + \|\tau_0\|_{H^s} \leq k^6 \epsilon_0 = \frac{k^6}{4(C^0 + 1)}, \quad \forall k \in (0, 10)
\] (1.7)
then the solution \((u, \tau)\) of (1.2) exists globally in \(C([0, \infty); H^s(\mathbb{R})) \times C([0, \infty); H^s(\mathbb{R}^3)) \cap L^2([0, \infty); H^{s+1}(\mathbb{R}^3))\).
Moreover, one have
\[
\|\tau(t)\|_{H^s} + \|\nabla u(t)\|_{H^{s-1}} \leq C \epsilon_0 (1 + t)^{-\frac{1}{2}}.
\] (1.8)

Remark 1.2. \([13]\) proved the global existence when \(\|u_0, \tau_0\|_{H^s} \leq \epsilon_0\) \((s > \frac{5}{2})\), and the polynomial decay of \(\|\tau(t)\|_{H^s} + \|\nabla u(t)\|_{H^{s-1}}\). Theorem 1.2 just attenuates the condition such that \(\|u_0\|_{L^2}\) could be large, a small improvement.

Combining Theorem 1.2 and Theorem 1.1, we find an interesting phenomenon. When the coupling coefficient \(k\) is large \((k \in (0, 10])\), by Theorem 1.2, we obtain the polynomial decay of \(\|\nabla u, \tau\|_{L^2}\). However, when \(k\) is small \((k \in (0, \frac{5}{2} + 1])\), by Theorem 1.1, we obtain the exponential decay of \(\|\nabla u, \tau\|_{L^2}\). This implies that the size of the coupling coefficient \(k\) determines the extent of the decay of the velocity field \(u\) and the stress tensor \(\tau\). There will be a better decay for sufficient small \(k\), since small \(k\) means small distraction for the equation of \(u(t, x)\), while the damping term \(\mathcal{D} u\) can develop a larger impact.

Next, by Remark 1.2, \(\bar{k} = \frac{1}{C^0 + 1}\) seems to be a boundary between these two kinds of attenuation. Furthermore, one will ask whether the solutions are close to each other when \(k \to \bar{k}\)? The answer is true. Now we give a more general theorem to verify that all the solutions in above theorem will be close to each other when \(k^m \to k\) for any fixed \(k > 0\) and \(t > 0\).

Theorem 1.3. Let \((u_0, \tau_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)\) with \(s > \frac{5}{2}\). Assume
\[
\lim_{m \to \infty} |k^m - k| = 0 \quad \text{for any fixed } k, k^m \in (0, 10).
\] (1.9)
If the initial data satisfies
\[
\|\nabla u_0\|_{H^{s-1}} + \|\tau_0\|_{H^s} \leq k^6 \epsilon_0,
\]
then we have
\[
\lim_{m \to \infty} \|u^m - u\|_{L^\infty([0, \infty); H^s)} + \|\tau^m - \tau\|_{L^\infty([0, \infty); H^s) \cap L^2([0, \infty); H^{s+1})} = 0,
\] (1.10)
where \((u^m, \tau^m)\) are the global solutions of (1.2) with the coefficient \(k^m\) \((m \in \mathbb{N} \cap \infty)\) and \((u^\infty, \tau^\infty) := (u, \tau)\).

However, the damping effect cannot be better when the coupling coefficient \(k\) is too small that \(k \to 0\). Because (1.2) will decouple as \(k = 0\), which means the damping effect will vanish! As a result, (1.10) is no longer valid. Indeed, we will prove that the energy of \(u^k\) will have a jump when \(k \to 0\) for large time. This implies the system (1.2) is not globally steady for \(k \to 0\), while for local time (1.2) is steady in [20].

Set \(A := \{(u_0, \tau_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3), \ s > \frac{5}{2}\}\) (1.2) has a unique solution for any fixed \(k\). Here is the unsteady result.
Theorem 1.4. Let \((u^k, \tau^k)\) be the corresponding solutions for (1.2) with every \(k \in [0, \varepsilon_0]\). Then there exists a large time \(T(k)\) and a sequence \((u_0, \tau_0)(k) \in A\) as initial data such that when \(t \geq T(k)\), we have
\[
\|u - u^k\|_{L^2} \geq \frac{\varepsilon_0}{2},
\]
where \(\varepsilon_0 = \frac{1}{4(C^6+1)}\), a fixed constant.

Remark 1.3. When \(k \in (0, \frac{1}{C+1}]\), by Theorem 1.1, we obtain the exponential decay of \(\|u\|_{L^2}\). However, when \(k = 0\), by the classical Euler equation we deduce that \(\|u(t)\|_{L^2}\) is conservative, while \(\|\tau\|_{H^s}\) doesn’t decay anymore. This implies that the sign of the coupling coefficient \(k\) determines whether the norm of the velocity field \(u(t,x)\) has decay. In fact, when \(k > 0\), since \(\tau\) is a heat type equation with damping mechanisms \(\tau\) and \(Du\), the coupling term \(k\text{div}\tau\) passes the decay of \(\tau\) to \(u\), but this process of transformation is transient for \(k \to 0\) in large time (see Theorem 1.4).

All in all, combining Theorem 1.1-Theorem 1.4 we conclude that larger coupling coefficient \(k\) will have a worse impact to the extent of the decay of \((u,\tau)\), but it is necessary for the appearance of decay (\(k\) must be positive, or the decay will vanish instantly), while the damping term \(\tau\) and \(Du\) always bring the well impact for the decay.

The paper is organized as follows. In section 2, we will give the tools (Littlewood-Paley decomposition and paradifferential calculus) and Besov spaces. In section 3, we prove the global existence of (1.2) for different kinds of \(k\). In section 4, we prove the stability of (1.2) when \(k^m \to k > 0\). In section 5, we show that the energy of \(u^k(t,x)\) will have a jump when \(k \to 0\) for large time, which implies the (1.2) is not globally steady for \(k \to 0\).

Notation Throughout the paper, we denote the norms of usual Lebesgue space \(L^p(\mathbb{R}^3)\) by \(\|u\|_{L^p}^p = \int_\Omega |u|^p dx\), for \(1 \leq p < \infty\). \(C_i\) and \(C\) denote different positive constants in different places.

2 Preliminaries

In this section, we will recall some properties about the Littlewood-Paley decomposition and Besov spaces.

Proposition 2.1. Let \(C\) be the annulus \(\{\xi \in \mathbb{R}^d : \frac{2}{3} \leq |\xi| \leq \frac{8}{3}\}\). There exist radial functions \(\chi\) and \(\varphi\), valued in the interval \([0,1]\), belonging respectively to \(\mathcal{D}(B(0,\frac{2}{3}))\) and \(\mathcal{D}(C)\), and such that
\[
\forall \xi \in \mathbb{R}^d, \quad \chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1,
\]
\[
\forall \xi \in \mathbb{R}^d \setminus \{0\}, \quad \sum_{j \in \mathbb{Z}} \varphi(2^{-j}\xi) = 1,
\]
\[
|j - j'| \geq 2 \Rightarrow \text{Supp } \varphi(2^{-j}\cdot) \cap \text{Supp } \varphi(2^{-j'}\cdot) = \emptyset,
\]
\[
j \geq 1 \Rightarrow \chi(\cdot) \cap \text{Supp } \varphi(2^{-j}\cdot) = \emptyset.
\]
The set \(\tilde{C} = B(0,\frac{2}{3}) + C\) is an annulus, and we have
\[
|j - j'| \geq 5 \Rightarrow 2^j\tilde{C} \cap 2^{j'}\tilde{C} = \emptyset.
\]
Further, we have
\[ \forall \xi \in \mathbb{R}^d, \frac{1}{2} \leq \chi^2(\xi) + \sum_{j \geq 0} \varphi^2(2^{-j} \xi) \leq 1, \]
and
\[ \forall \xi \in \mathbb{R}^d \setminus \{0\}, \frac{1}{2} \leq \sum_{j \in \mathbb{Z}} \varphi^2(2^{-j} \xi) \leq 1. \]

**Definition 2.1.** \[1\] Let \( u \) be a tempered distribution in \( \mathcal{S}'(\mathbb{R}^d) \) and \( \mathcal{F} \) be the Fourier transform and \( \mathcal{F}^{-1} \) be its inverse. For all \( j \in \mathbb{Z} \), define
\[ \Delta_j u = 0 \text{ if } j \leq -2, \quad \Delta_{-1} u = \mathcal{F}^{-1}(\chi \mathcal{F} u), \quad \Delta_j u = \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \mathcal{F} u) \text{ if } j \geq 0, \quad S_j u = \sum_{j' < j} \Delta_{j'} u. \]

Then the Littlewood-Paley decomposition is given as follows:
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u \text{ in } \mathcal{S}'(\mathbb{R}^d). \]

Let \( s \in \mathbb{R}, 1 \leq p, r \leq \infty \). The nonhomogeneous Besov space \( B^s_{p,r}(\mathbb{R}^d) \) is defined by
\[ B^s_{p,r} = B^s_{p,r}(\mathbb{R}^d) = \{ u \in \mathcal{S}'(\mathbb{R}^d) : \| u \|_{B^s_{p,r}(\mathbb{R}^d)} = \left\| \left\{ 2^{js} \| \Delta_j u \|_{L^p(\mathbb{R}^d)} \right\}_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})} < \infty \}. \]

**Definition 2.2.** \[1\] The homogeneous dyadic blocks \( \dot{\Delta}_j \) are defined on the tempered distributions by
\[ \dot{\Delta}_j u = \varphi(2^{-j} D) u := \mathcal{F}^{-1}(\varphi(2^{-j} \cdot) \hat{u}). \]

\[ \dot{S}_j u = \sum_{j' \leq j-1} \dot{\Delta}_{j'} u. \]

**Definition 2.3.** We denote by \( S'_h \) the space of tempered distributions \( u \) such that
\[ \lim_{j \to -\infty} \dot{S}_j u = 0 \text{ in } \mathcal{S}' . \]

The homogeneous Littlewood-Paley decomposition is defined as
\[ u = \sum_{j \in \mathbb{Z}} \Delta_j u, \quad \text{for } u \in S'_h. \]

**Definition 2.4.** For \( s \in \mathbb{R}, 1 \leq p \leq \infty \), the homogeneous Besov space \( \dot{B}^s_{p,r} \) is defined as
\[ \dot{B}^s_{p,r} := \{ u \in S'_h, \| u \|_{B^s_{p,r}} < \infty \}, \]
where the homogeneous Besov norm is given by
\[ \| u \|_{\dot{B}^s_{p,r}} := \left\| \left\{ 2^{js} \| \dot{\Delta}_j u \|_{L^p} \right\}_{j \in \mathbb{Z}} \right\|_{l^r(\mathbb{Z})}. \]

In this paper, we use the "time-space" Besov spaces or Chemin-Lerner space first introduced by Chemin and Lerner in \[4\].

**Definition 2.3.** Let \( s \in \mathbb{R} \) and \( 0 < T \leq +\infty \). We define
\[ \| u \|_{\dot{L}^2_s(\mathbb{R}^d)} := \sum_{j \in \mathbb{Z}} 2^{js} \left( \int_0^T \| \Delta_j u(t) \|_{L^p}^2 \, dt \right)^{\frac{1}{2}}, \]
for \( p, q \in [1, \infty) \) and with the standard modification for \( p, q = \infty \).

By the Minkowski’s inequality, it is easy to verify that
\[
\|u\|_{L^2(B^s_{p,r})} \leq \|u\|_{L^2(B^2_{p,r})} \quad \text{if } \lambda \leq r,
\]
and
\[
\|u\|_{L^2(B^s_{p,r})} \geq \|u\|_{L^2(B^0_{p,r})} \quad \text{if } \lambda \geq r.
\]

The following Bernstein’s lemma will be repeatedly used in this paper.

**Lemma 2.1.** \( \{1\} \) Let \( B \) is a ball and \( C \) is a ring of \( \mathbb{R}^d \). There exists constant \( C \) such that for any positive \( \lambda \), any non-negative integer \( k \), any smooth homogeneous function \( \sigma \) of degree \( m \), any couple \((p, q)\) \( \in [1, \infty]^2 \) with \( q \geq p \geq 1 \), and any function \( u \in L^p \), there holds
\[
supp \hat{u} \subset \lambda B \Rightarrow \sup_{|\alpha| = k} \|\partial^{\alpha} u\|_{L^p} \leq C^{k+1} \lambda^{k+\frac{d}{p}-\frac{1}{q}} \|u\|_{L^p},
\]
\[
supp \hat{u} \subset \lambda C \Rightarrow \|\sigma(D)u\|_{L^p} \leq \sup_{|\alpha| = k} \|\partial^{\alpha} u\|_{L^p} \leq C^{k+1} \lambda^{d} \|u\|_{L^p},
\]
\[
supp \hat{u} \subset \lambda C \Rightarrow \sup_{|\alpha| = k} \|\sigma(D)u\|_{L^p} \leq C_{\sigma, m} \lambda^{m+d(\frac{1}{p} - \frac{1}{q})} \|u\|_{L^p}.
\]

Next, we will give the paraproducts and product estimates in Besov spaces. Recalling the paraproduct decomposition
\[
u v = T_u v + T_v u + R(u, v),
\]
where
\[
T_u v := \sum_q S_{q-1} u \Delta_v, \quad R(u, v) := \sum_q \Delta_q u \Delta_q v, \quad \text{and } \Delta_q = \Delta_{q-1} + \Delta_q + \Delta_{q+1}.
\]

The paraproduct \( T \) and the remainder \( R \) operators satisfy the following continuous properties.

**Proposition 2.2.** \( \{1\} \) For all \( s \in \mathbb{R}, \sigma > 0, \) and \( 1 \leq p, p_1, p_2, r, r_1, r_2 \leq \infty \), the paraproduct \( T \) is a bilinear, continuous operator from \( L^{\infty} \times B^s_{p,r} \) to \( B^0_{p,r} \), and from \( B^{-\sigma}_{p_1, r_1} \times B^s_{p_2, r_2} \) to \( B^{-\sigma}_{p, r} \) with \( \frac{1}{p} = \min\{\frac{1}{p_1} + \frac{1}{p_2}\}, \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \). The remainder \( R \) is bilinear continuous from \( B_{p_1, r_1}^s \times B_{p_2, r_2}^s \) to \( B_{p, r}^{s_1 + s_2} \) with \( s_1 + s_2 > 0 \), \( \frac{1}{p} = \frac{1}{p_1} + \frac{1}{p_2} \leq 1 \), and \( \frac{1}{r} = \frac{1}{r_1} + \frac{1}{r_2} \leq 1 \). In particular, if \( r = \infty \), the continuous property for the remainder \( R \) also holds for the case \( s_1 + s_2 = 0, r = \infty, \frac{1}{r_1} + \frac{1}{r_2} = 1 \).

Combining the above proposition with Lemma \( 2.1 \) yields the following product estimates:

**Corollary 2.1.** \( \{1\} \) Let \( a \) and \( b \) be in \( L^\infty \cap B^s_{p,r} \) for some \( s > 0 \) and \((p, r) \in [1, \infty]^2\). Then there exists a constant \( C \) depending only on \( d, p \) and such that
\[
\|ab\|_{B^s_{p,r}} \leq C(\|a\|_{L^\infty} \|b\|_{B^s_{p,r}} + \|b\|_{L^\infty} \|a\|_{B^s_{p,r}}).
\]

Finally, we introduce some useful results about the following heat conductive equation and the transport equation
\[
\begin{align*}
\begin{cases}
u_t - \Delta u + \beta u = G, \quad x \in \mathbb{R}^d, \quad \beta \geq 0, \quad t > 0, \\
u(0, x) = \nu_0(x), \quad x \in \mathbb{R}^d,
\end{cases}
\end{align*}
\]
\[
\begin{align*}
\begin{cases}
u_t + v \cdot \nabla f + \beta f = g, \quad x \in \mathbb{R}^d, \quad \beta \geq 0, \quad t > 0, \\
f(0, x) = f_0(x), \quad x \in \mathbb{R}^d,
\end{cases}
\end{align*}
\]
which are crucial to the proof of our main theorem later.
Lemma 2.2. [1] Let $1 \leq p \leq q \leq \infty$ and $k \geq 0$, it holds that
$$\|\nabla^k e^{t\Delta} f\|_{L^q} \leq C t^{-\frac{k}{2}-\frac{1}{p}+\frac{k}{q}}\|f\|_{L^p}.$$  

Lemma 2.3. Let $s \in \mathbb{R}$, $\beta \geq 0$, $1 \leq q, q_1, p, r \leq \infty$ with $q_1 \leq q$. Assume $u_0 \in B^s_{p,r}$, and $G$ in $\hat{L}^\beta_{T}(p,r)$. Then (2.1) has a unique solution $u$ in $\hat{L}^\beta_{T}(B^s_{p,r})$ and satisfies
$$\|u\|_{\hat{L}^\beta(T)(B^s_{p,r})} \leq C_1 \left(\|u_0\|_{B^s_{p,r}} + \|G\|_{\hat{L}^\beta(T)(B^{s+\frac{2}{p}}_{p,r})}\right).$$  
Moreover, if $\beta > 0$, without loss of generality we set $\beta = 1$, one have
$$\|u\|_{\hat{L}^1(T)(B^s_{p,r})} \leq C_1 \left(\|u_0\|_{B^s_{p,r}} + \|G\|_{\hat{L}^1(T)(B^{s+\frac{2}{p}}_{p,r})}\right),$$
and
$$\|ue^{\theta t}\|_{\hat{L}^1(T)(B^s_{p,r})} \leq \frac{C_1}{1-\theta} \left(\|u_0\|_{B^s_{p,r}} + \|e^{\theta t}G\|_{\hat{L}^1(T)(B^{s+\frac{2}{p}}_{p,r})}\right),$$
where $0 \leq \theta < 1$.

Proof. (2.3) can be founded in [1], we should only prove (2.3). Indeed, since
$$\Delta_j u = e^{-t\Delta} \Delta_j u_0 + \int_0^t e^{(t-s)\Delta} \Delta_j G ds,$$
when $j \geq 0$, by $\|e^{t\Delta} \Delta_j u\|_{L^p} \leq C e^{-2t^2} \|\Delta_j u\|_{L^p}$ one can easily get
$$\|2^{s+\frac{2}{p}} \|u\|_{L^\beta(T)L^p}\|_{L^\beta_{T}(L^p)} \leq C_1 \left(\|u_0\|_{B^s_{p,r}} + \|G\|_{\hat{L}^\beta(T)(B^{s+\frac{2}{p}}_{p,r})}\right).$$
When $j = -1$, by $\|e^{t\Delta} \Delta_{-1} u\|_{L^p} \leq C \|\Delta_{-1} u\|_{L^p}$ we have
$$\|\Delta_{-1} u\|_{L^\beta_{T}(L^p)} \leq C_1 \left(\|\Delta_{-1} u_0\|_{L^p} + \|\Delta_{-1} G\|_{\hat{L}^\beta(T)(L^p)}\right).$$
Combining the above two inequality, we obtain (2.3). To prove (2.4), since
$$(e^{\theta t} \Delta_j u) = e^{-(1-\theta)t} e^{t\Delta} \Delta_j u_0 + \int_0^t e^{-(1-\theta)(t-s)} e^{(t-s)\Delta} (e^{\theta s} \Delta_j G) ds,$$
one can take the similar operators to obtain (2.5).  

Lemma 2.4. [1] Let $s \in [\max\{-\frac{d}{p}, -\frac{d}{p'}\}, \frac{d}{p} + 1](s = 1 + \frac{d}{p}, r = 1; s = \max\{-\frac{d}{p}, -\frac{d}{p'}\}, r = \infty).$ Then there exists a constant $C$ such that for all solutions $f \in L^\infty([0, T]; B^s_{p,r})$ of (2.2) with initial data $f_0$ in $B^s_{p,r}$, and $g$ in $L^1([0, T]; B^s_{p,r})$ we have, for a.e. $t \in [0, T]$,
$$\|f(t)\|_{B^s_{p,r}} \leq C \left(\|f_0\|_{B^s_{p,r}} + \int_0^t V(t') \|f(t')\|_{B^s_{p,r}} + \|g(t')\|_{B^s_{p,r}} dt'\right) \leq e^{C_2 V(t)} \left(\|f_0\|_{B^s_{p,r}} + \int_0^t e^{-C_2 V(t')} \|g(t')\|_{B^s_{p,r}} dt'\right),$$  
where $V(t) = \int_0^t \|\nabla v\|_{B^s_{p,r} \cap L^\infty}^d ds$ if $s = 1 + \frac{1}{p}, r = 1$, $V(t) = \int_0^t \|\nabla v\|_{B^s_{p,1}}^\frac{d}{p} ds$.

Remark 2.1. [1] If $\text{div} v = 0$, we can get the same result with a better indicator: $\max\{-\frac{d}{p}, -\frac{d}{p'}\} - 1 < s < \frac{d}{p} + 1$ (or $s = \max\{-\frac{d}{p}, -\frac{d}{p'}\} - 1, r = \infty$).
Lemma 2.5. Let $\beta > 0$. There exists a constant $C$ such that for all smooth solutions of (2.2) with initial data $f_0$ in $L^p$, and $\nabla v, g$ in $L^1([0,T];L^p)$, we have, for all $1 \leq p \leq \infty$ and $t \in [0,T],$

$$\|f(t)\|_{L^1_t L^p_x} \leq C e^{V(t)} (\|f_0\|_{L^p} + \int_0^t \|g(t')\|_{L^p} dt'),$$

(2.7)

where $V(t) = \int_0^t \|\nabla v(t)\|_{L^\infty} ds$.

Proof. With loss of generality, we set $\beta = 1$. (2.2) can be rewrite as

$$dt(e^t f) + u \nabla (e^t f) = (e^t g).$$

Then one can easily deduce that

$$\|f(t)\|_{L^p} \leq C e^{V(t)} (\|f_0\|_{L^p} + \int_0^t \|e^{-(t-t')} g(t')\|_{L^p} dt'),$$

(2.8)

which implies (2.7) by Young inequality. \hfill \Box

3 Global existence

The proof of Theorem 1.1

Proof. Generally speaking, the bootstrap argument starts with an assumption. Let $T^*$ be the maximal existence time of the solution, for any $0 < t < T^*$,

$$\|\nabla u\|_{L^1_t L^p_x(B^{0}_{\infty,1})} \leq k^2 \epsilon_0, \quad \|\tau\|_{L^p_t (L^p_x(B^{0}_{\infty,1})) \cap L^4_t (L^2_x(B^{2}_{\infty,1}))} \leq k \epsilon_0, \quad \epsilon_0 := \frac{1}{4^6 (C^6 + 1)^{1/2}}.$$

(3.1)

where $C$ is a fixed positive constant, and $0 \leq k \leq \frac{1}{4(C^2 + 1)}$. Let the initial data $(u_0, \tau_0)$ be small enough such that

$$\|u_0\|_{B^{1}_{\infty,1}} + \|b_0\|_{B^{0}_{\infty,1}} \leq k^4 \epsilon_0.$$

(3.2)

We will divide the proof into 4 sections.

(1). First, we give the estimation of $\|\nabla u\|_{L^\infty_t (L^p_x(B^{0}_{\infty,1}))}$.

Applying Lemma 2.4 and (3.1) to the first equation of (1.2), we have

$$\|u\|_{L^\infty_t L^p_x(B^{0}_{\infty,1})} \leq C e^{Ck^2 \delta} (\|u_0\|_{B^{1}_{\infty,1}} + k \|\tau\|_{L^1_t (B^{2}_{\infty,1})}) \leq C(k^4 \epsilon_0 + k^2 \epsilon_0) \leq C k^2 \epsilon_0,$$

(3.3)

(2). Then, we estimate $\|\tau\|_{L^p_t (L^p_x(B^{0}_{\infty,1})) \cap L^4_t (L^2_x(B^{2}_{\infty,1}))}$.

Applying the Lemma 2.3 to the second equation of (1.2), it implies that

$$\|\tau\|_{L^p_t (L^p_x(B^{0}_{\infty,1})) \cap L^4_t (L^2_x(B^{2}_{\infty,1}))} \leq \|\tau_0\|_{L^p_x(B^{0}_{\infty,1})} + \int_0^t \|Q(\nabla u, \tau)\|_{L^p_x(B^{0}_{\infty,1})} + \|u \cdot \nabla \tau\|_{L^p_x(B^{0}_{\infty,1})} + \|\nabla u\|_{L^p_x(B^{0}_{\infty,1})} ds$$

$$\leq C (\|\tau_0\|_{B^{0}_{\infty,1}} + \|u\|_{L^p_t (B^{0}_{\infty,1})} + \|\tau\|_{L^4_t (B^{2}_{\infty,1})} + \|\nabla u\|_{L^p_x(B^{0}_{\infty,1}))})$$

$$\leq C (k^4 \epsilon_0 + k^2 \epsilon_0 + k^2 \epsilon_0 + \|\tau\|_{L^1_t (B^{2}_{\infty,1}))})$$

$$\leq C (k^4 \epsilon_0 + k^2 \epsilon_0) \leq C k^2 \epsilon_0,$$

(3.4)

where the last inequality holds by (3.1) and (3.2).
We establish a new quantity $f$: 
\[
\Gamma = w - k\tilde{R}_\tau, \quad \tilde{R} = -\frac{1}{(-\Delta)^{-1}}\text{curl}(\text{div}()) ,
\]
and get the following equation of $\Gamma$:
\[
\begin{align*}
\begin{cases}
\partial_t \Gamma + k\Gamma + u \cdot \nabla \Gamma - k[\tilde{R}, u \cdot \nabla] = w\nabla u - k\tilde{R}(Q(\nabla u, \tau)), \\
\Gamma(0, x) = w_0 - k\tilde{R}_\tau_0.
\end{cases}
\end{align*}
\]
Applying the $\Delta_j$ to the (3.6), note that
\[
\Delta_j(u \cdot \nabla \Gamma - k[\tilde{R}, u \cdot \nabla] = \Delta_j(u \cdot \nabla w - k\tilde{R}(u \cdot \nabla))
\]
\[
= \Delta_j T_u \nabla w - k\Delta_j\tilde{R} T_u \nabla + f_j
\]
\[
= S_{j-1} u\nabla \Delta_j w + (\Delta_j T_u \nabla w - S_{j-1} u\nabla \Delta_j w) - kT_u \nabla \Delta_j \tilde{R} + k[\Delta_j\tilde{R}, T_u \nabla] + f_j
\]
(3.7)
\[
= S_{j-1} u\nabla \Delta_j \Gamma + (\Delta_j T_u \nabla w - S_{j-1} u\nabla \Delta_j w) - k[\Delta_j\tilde{R}, T_u \nabla] + f_j
\]
where
\[
f_j = \Delta_j T_u \nabla w + \Delta_j R(\nabla w, u) - k\Delta_j\tilde{R} T_u \nabla + k\tilde{R} \Delta_j R(u, \nabla).
\]
So we have
\[
\partial_t \Delta_j \Gamma + K\Delta_j \Gamma + S_{j-1} u\nabla \Delta_j \Gamma + (\Delta_j T_u \nabla w - S_{j-1} u\nabla \Delta_j w) - K[\Delta_j\tilde{R}, T_u \nabla] + f_j = G_j,
\]
where $G_j := \Delta_j(w \nabla u) - k\tilde{R} \Delta_j(Q(\nabla u, \tau))$.

Firstly, we estimate the nonlinear terms of (3.8).

By Lemma 10.25 in [4], we get the commutator estimations:
\[
\sum_j \|\Delta_j T_u \nabla w - S_{j-1} u\nabla \Delta_j w\|_{L^\infty} + k\|\Delta_j\tilde{R}, T_u \nabla]\|_{L^\infty} \leq C(k + 1)\|\nabla u\|_{B^0_{\infty,1}} (\|w\|_{B^0_{\infty,1}}^1 + \|\tau\|_{B^2_{\infty,1}}^1),
\]
(3.9)

By Bony decomposition $f_j$ and $G_j$ can be estimated as
\[
\sum_j \|f_j\|_{L^\infty} \leq \sum_j \|\Delta_j T_u \nabla w\|_{L^\infty} + k\sum_j \|\Delta_j R(\nabla u, \nabla)\|_{L^\infty} + \|\Delta_j R(u, \tau)\|_{L^\infty} \leq C(1 + k)\|\nabla u\|_{B^0_{\infty,1}} (\|\tau\|_{B^2_{\infty,1}} + \|u\|_{B^1_{\infty,1}})
\]
(3.10)

and
\[
\sum_j \|G_j\|_{L^\infty} \leq C(1 + k)\|\nabla u\|_{B^0_{\infty,1}} (\|\tau\|_{B^2_{\infty,1}} + \|u\|_{B^1_{\infty,1}}),
\]
(3.11)

where we use the fact that $w \nabla u = \text{div}(w \otimes u)$ with $\text{div} w = \text{div} \text{curl} u = 0$ in three dimensions.

Then, applying Lemma 2.5 with $p = \infty$ to (3.8) and taking $\sum_{j \geq -1}$, by (3.9)-(3.11) we deduce that
\[
\|\Gamma\|_{L^p_\infty(B^0_{\infty,1})} + k\|\Gamma\|_{L^p_\infty(B^0_{\infty,1})} \leq C(\|\Gamma_0\|_{B^0_{\infty,1}} + \int_0^1 C(1 + k)(\|\nabla u\|_{B^0_{\infty,1}} + \|\tau\|_{B^2_{\infty,1}})\|\nabla u\|_{B^0_{\infty,1}}^2 ds
\]
\[
\leq C(k^4 \epsilon_0 + (1 + k)(k^2 \epsilon_0)^2) \leq C k^4 \epsilon_0
\]
(3.12)
So we have
\[ \| \Gamma \|_{L^1_t(B^{p,q}_{\infty,1})} \leq C k^3 \epsilon_0. \]
Combining (3.5), we deduce that
\[ \| u \|_{L^1_t(B^{p,q}_{\infty,1})} = C(k \| \tau \|_{L^1_t(B^{p,q}_{\infty,1})} + \| \Gamma \|_{L^1_t(B^{p,q}_{\infty,1})}) \]
\[ \leq C(k^3 \epsilon_0 + k^3 \epsilon_0) \]
\[ \leq \frac{1}{2} k^2 \epsilon_0, \]
where \( \epsilon_0 \) and \( k \) satisfies (3.1) and (3.2). Using the bootstrap argument for (3.5) and (3.13), we obtain that
\[ \| \nabla u \|_{L^1_t(B^{p,q}_{\infty,1})} \leq k^2 \delta \quad and \quad \| \tau \|_{L^{\infty}_t(B^{p,q}_{\infty,1}) \cap L^1(B^{p,q}_{\infty,1})} \leq k \delta, \quad \forall t \in [0, T^*). \]

Now, one can obtain the global existence of \((u, \tau)\) in \(C([0, \infty); B^{1+\frac{2}{p}}_{p,1}) \times (C([0, \infty); B^{\frac{2}{p}}_{p,1}) \cap L^1([0, \infty); B^{2+\frac{2}{p}}_{p,1})) \) easily, since (3.14) can be the blow-up criteria for (1.2). Indeed, applying Lemma 2.3–2.4 to (1.2), we have
\[ \| u \|_{L^\infty_t(B^{1+\frac{2}{p}}_{p,1})} \leq \| u_0 \|_{B^{1+\frac{2}{p}}_{p,1}} + C \int_0^t \| u \|_{B^{1+\frac{2}{p}}_{p,1}} + k \| \tau \|_{B^{1+\frac{2}{p}}_{p,1}} ds, \]
and
\[ \| \tau \|_{L^\infty_t(B^{2+\frac{2}{p}}_{p,1}) \cap L^1_t(B^{\frac{2}{p}}_{p,1})} \leq C \left( \| u_0 \|_{B^{1+\frac{2}{p}}_{p,1}} + \| \tau_0 \|_{B^{1+\frac{2}{p}}_{p,1}} \right) e^{Ct} \leq C e^t, \quad \forall t \in [0, T^*). \]
This implies \( T^* = \infty \).

(4) Finally, to complete the proof of Theorem ??, we now prove the exponential decay. Rewrite (3.6):
\[ \partial_t (e^{kt} \Gamma) + u \cdot \nabla (e^{kt} \Gamma) = k [\tilde{R}, u \cdot \nabla] (e^{kt} \tau) = w \nabla (e^{kt} u) - k \tilde{R} (Q(\nabla u, (e^{kt} \tau))). \]
Since \( w = \Gamma + k \tilde{R} \tau \), applying (2.4) in Lemma 2.3 one can obtain
\[ \| e^{kt} \Gamma(t) \|_{B^{\frac{2}{p}+\frac{2}{q}}_{p,1}} \leq C(\| \Gamma_0 \|_{B^{\frac{2}{p}+\frac{2}{q}}_{p,1}} + \| u \|_{B^{\infty,1}_{p,1}} (\| e^{ks} \nabla u \|_{B^{\frac{2}{p}+\frac{2}{q}}_{p,1}} + \| e^{ks} \tau \|_{B^{\frac{2}{p}+\frac{2}{q}}_{p,1}})) ds \]
\[ \leq C(k^4 \epsilon_0 + k^2 \epsilon_0) e^{kt} \| \nabla u \|_{B^{\infty,1}_{p,1}} + \| e^{kt} \tau \|_{L^1(B^{\frac{2}{p}+\frac{2}{q}}_{p,1})} + \int_0^t \| \nabla u \|_{B^{\infty,1}_{p,1}} \| e^{ks} \Gamma \|_{B^{\frac{2}{p}+\frac{2}{q}}_{p,1}} ds \]
where \( 0 < k \leq \frac{1}{C(2+4)} \leq \frac{1}{10} \) and \( \| \nabla u \|_{L^1_t(B^{p,q}_{\infty,1})} \leq k^4 \epsilon_0. \)
Recall the equation of \( \tau \) in (1.2), one have
\[ \partial_t (e^t \tau) - \Delta (e^t \tau) + u \cdot \nabla (e^t \tau) + Q(\nabla u, (e^t \tau)) = e^t \nabla u. \]
That is
\[ e^{kt} \tau = e^{t \Delta} e^{-(1-k)(t-s)} \tau_0 + \int_0^t e^{(t-s) \Delta} e^{-(1-k)(t-s)} (e^{ks} F), \]
where \( F := -u \cdot \nabla (\tau) - Q(\nabla u, (\tau)) + D u \). Applying (2.5) with \( k = \theta \) in Lemma 2.3, we have
\[
\| e^{kt} \tau(t) \|_{B_{p,1}^{\frac{n}{p}}} + \| e^{ks} \tau \|_{L_t^1(B_{p,1}^{\frac{n}{p}})} \leq C(\| \tau_0 \|_{B_{p,1}^{\frac{n}{p}}} + \int_0^t \| u \|_{B_{\infty,1}^1} \| e^{kt} \tau \|_{B_{p,1}^{\frac{n}{p}}} + \| e^{kt} \nabla u \|_{B_{p,1}^{\frac{n}{p}}} ds)
\]
\[
\leq C(\| \tau_0 \|_{B_{p,1}^{\frac{n}{p}}} + Ck \| e^{ks} \tau \|_{L_t^1(B_{p,1}^{\frac{n}{p}})}) + \int_0^t \| e^{kt} \tau \|_{B_{p,1}^{\frac{n}{p}}} ds)
\]
\[
\leq C(k^4 \epsilon_0 + \int_0^t \| e^{kt} \tau \|_{B_{p,1}^{\frac{n}{p}}} ds),
\]
(3.21)

Combining (3.20) with \( k = \frac{k}{4(C^2 + 1)} \times (3.21) \), and applying Gronwall inequality, we obtain
\[
e^{kt} \| \tau(t) \|_{B_{p,1}^{\frac{n}{p}}} + \| \Gamma \|_{B_{p,1}^{\frac{n}{p}}} \leq C(\| \tau_0 + \nabla u_0 \|_{B_{p,1}^{\frac{n}{p}}})e^{\frac{k}{4(C^2 + 1)}t} \leq Ck^6 e^{\frac{k}{4}t},
\]
(3.22)

where we use \( \epsilon_0 \leq \frac{4}{4(C^2 + 1)} \) and \( k \leq \frac{1}{C^2 + 1} \) by (3.1). Since \( w = \Gamma + k \tilde{R} \tau \), we obtain
\[
k\| \tau(t) \|_{B_{p,1}^{\frac{n}{p}}} + \| u(t) \|_{B_{p,1}^{\frac{n}{p}}} \leq C(\| \tau_0 + \nabla u_0 \|_{B_{p,1}^{\frac{n}{p}}})e^{-\frac{k}{4}t}.
\]

This complete the proof of Theorem 1.1

The proof of Theorem 1.2

The proof is similar to [13], we give the proof briefly.

\textbf{Proof.} For \( 0 < k \leq 10 \), assume that for any \( 0 \leq t < T < T^* \) we have
\[
\| \nabla u, \nabla \tau \|_{L_t^\infty(H^{s-1})}^2 + \| \nabla \tau \|_{L_t^2(H^1)}^2 \leq k^4 \delta^2 \quad \text{and} \quad \| \nabla u \|_{L_t^2(H^{s-1})}^2 \leq k^4 \delta^2,
\]
where \( \delta := 16(C^2 + 1) \epsilon_0 \) and \( \epsilon_0 := \frac{4}{4(C^2 + 1)} \) for a fixed large constant \( C \). Set the initial data such that
\[
\| \nabla u_0 \|_{H^{s-1}}^2 + \| \tau_0 \|_{H^s}^2 \leq k^6 \epsilon_0^2.
\]

Firstly, taking the \( L^2 \) and \( H^1 \) inner product of (1.2), we have
\[
\frac{1}{2} \| \tau \|_{L_t^\infty(L^2)}^2 + \| \tau \|_{L_t^2(H^1)}^2 \leq \| \tau_0 \|_{L^2(L^2)}^2 + \int_0^t \| \nabla u \|_{L^\infty} \| \tau \|_{L^2}^2 + \frac{1}{2} \| \nabla u \|_{L^2}^2 + \frac{1}{2} \| \tau \|_{L^2}^2 ds
\]
\[
\leq C k^6(\epsilon_0 + \delta^2),
\]
(3.23)

\[
\| u \|_{L_t^\infty(L^2)}^2 + k \| \tau \|_{L_t^2(L^2)}^2 + k \| \tau \|_{L_t^2(H^1)}^2 \leq \| u_0 \|_{L^2(L^2)}^2 + k \| \tau_0 \|_{L^2(L^2)}^2 + k \int_0^t \| \nabla u \|_{L^\infty} \| \tau \|_{L^2}^2 ds
\]
\[
\leq (\| u_0 \|_{L^2(L^2)}^2 + k \| \tau_0 \|_{L^2(L^2)} + k \| \tau \|_{L_t^2(L^2)}^2)
\]
\[
\leq C k^4(\| u_0 \|_{L^2(L^2)} + k \| \tau_0 \|_{L^2(L^2)}^2)
\]
(3.24)

\[
\int_0^t \| \nabla u \|_{L^2(L^2)}^2 ds \leq C \int_0^t (\| \tau \|_{L^2(L^2)} + \| \nabla u \|_{W^{1,\infty}} \| \tau \|_{L^2(L^2)} + \| \nabla u \|_{L^2(L^2)} \| \nabla u \|_{L^2(L^2-1)} + k \| \nabla \tau \|_{L^2(L^2)}^2 + \| \tau \|_{L^2(L^2)} \| \nabla u \|_{L^2(L^2)}^2 + \| \nabla u \|_{L^2(L^2)}^2 ds
\]
\[
\leq C k^4 \delta^2
\]
(3.25)

and
\[
\| \nabla u \|_{L_t^\infty(L^2)}^2 + k \| \nabla \tau \|_{L_t^\infty(L^2)}^2 + k \| \nabla \tau \|_{L_t^2(H^1)}^2 \leq \| u_0 \|_{L^2(L^2)}^2 + k \| \tau_0 \|_{L^2(L^2)}^2 + \int_0^t a \| \nabla u \|_{L^\infty} \| \nabla u \|_{L^2(L^2)}^2 + k \| \nabla u \|_{L^\infty} \| \tau \|_{L^2(L^2)}^2 ds
\]
\[
\leq C k^6(\epsilon_0 + \delta^2).
\]
(3.26)
Then, taking the $\dot{H}^s$ inner product of (1.2), we have
\[\begin{align*}
\|u\|_{L^2_t(\dot{H}^s)}^2 + k\|\tau\|_{L^2_t(\dot{H}^s)}^2 + k\|\tau\|_{L^2_t(\dot{H}^s \cap \dot{H}^{s+1})}^2
\leq & \|u_0\|_{\dot{H}^s}^2 + k\|\tau_0\|_{\dot{H}^s}^2 + \int_0^t a\|\nabla u\|_{L^\infty} \|\nabla u\|_{\dot{H}^s}^2 + k\|\nabla u\|_{H^{s-1}} \|\tau\|_{\dot{H}^{s+1}}^2 \, ds \\
\leq & CK^6 (\epsilon_0^2 + \delta^2). \tag{3.27}
\end{align*}\]

\[\begin{align*}
\int_0^t \|\nabla u\|_{\dot{H}^{s-1}}^2 \, ds & \leq C \int_0^t (\|\tau\|_{\dot{H}^s}^2 + \|u\|_{W^{1,\infty}}^2 \|\nabla \tau\|_{\dot{H}^s}^2) + \|\nabla u\|_{H^{s-1}} \|\nabla \tau\|_{\dot{H}^s}^2 \\
+ & k\|\nabla \tau\|_{L^2}^2 \, ds + \|\tau\|_{\dot{H}^s} \|\nabla u\|_{H^{s-1}} \\
\leq & CK^4 \delta^2 \tag{3.28}
\end{align*}\]

Combining (3.25)-(3.28) with the bootstrap argument, we finally obtain for any $0 \leq t < T^*$
\[\|\nabla u, \nabla \tau\|_{L^2_t(\dot{H}^s)}^2 + \|\nabla \tau\|_{L^2_t(\dot{H}^s)}^2 \leq \frac{1}{2} k^4 \delta^2 \text{ and } \|\nabla u\|_{L^2_t(\dot{H}^{s-1})}^2 \leq \frac{1}{2} k^4 \delta^2, \]

Finally, since the proof of (1.8) can refer to [13], this complete the proof.

\[\square\]

4 Global stability for $0 < k \leq 10$

In this section, we will give the prove of Theorem 1.3. Firstly, we give the global stability for (1.2) in a weaker space $H^{s-1}(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)$.

**Lemma 4.1.** Let $(u_1, \tau_1), (u_2, \tau_2)$ be two global strong solutions of (1.2) in Theorem 1.2 with fixed $0 < k \leq 10$, then for any $t \geq 0$ we have
\[\begin{align*}
\|u^1 - u^2, \tau^1 - \tau^2\|_{L^2_t(H^{s-1})}^2 + \|\tau^1 - \tau^2\|_{L^2_t(\dot{H}^s)}^2 + \|\nabla(u^1 - u^2)\|_{L^2_t(\dot{H}^{s-2})}^2 \\
\leq & \frac{C}{k} \|u_1^1 - u_0^1, \tau_1^1 - \tau_0^1\|_{\dot{H}^{s-1}}^2. \tag{4.1}
\end{align*}\]

**Proof.** Give the equation of $(u^1 - u^2, \tau^1 - \tau^2)$:
\[\begin{align*}
(u^1 - u^2)_{t} + u^1 \nabla (u^1 - u^2) + (u^1 - u^2) \nabla u^2 + \nabla (P^1 - P^2) = k \text{div}(\tau^1 - \tau^2), \\
(\tau^1 - \tau^2)_{t} + (\tau^1 - \tau^2)_{\cdot} - \Delta (\tau^1 - \tau^2) + u^1 \nabla (\tau^1 - \tau^1) + (u^1 - u^2) \nabla \tau^1 \\
+ Q(\nabla (u^1 - u^2), \tau^1) + Q(\nabla u^1, (\tau^1 - \tau^2)) = D(u^1 - u^2) \tag{4.2}
\end{align*}\]

By Theorem 1.2 we have
\[\|\nabla u_i, \nabla \tau_i\|_{L^2_t(\dot{H}^{s-1})}^2 + \|\nabla \tau_i\|_{L^2_t(\dot{H}^s)}^2 \leq C \delta^2 \text{ and } \|\nabla u_i\|_{L^2_t(\dot{H}^{s-1})}^2 \leq C \delta^2, \quad i = 1, 2, \quad \forall t > 0, \]
where $\delta := \epsilon_0^{\frac{1}{2}}$ and $\epsilon_0 := \frac{1}{4(\epsilon+1)}$.

Similarly to (3.24), (3.25), (3.27) and (3.28) in the proof of Theorem 1.2 we also use the energy
method and have:
\[
\|u^1 - u^2\|_{L^\infty(L^2)}^2 + k\|\tau^1 - \tau^2\|_{L^\infty(L^2)}^2 + k\|\tau^1 - \tau^2\|_{L^2(H^1)}^2 \\
\leq \|u_0^1 - u_0^2\|_{H^s} + Ck\|\tau_0^1 - \tau_0^2\|_{H^s} \\
+ \int_0^t \|u^2\|_{L^6}\|\nabla(u_1 - u_2)\|_{L^6}\|u_1 - u_2\|_{L^3} + k(\|\tau_2\|_{H^s}\|u_1 - u_2\|_{L^6}\|\tau_1 - \tau_2\|_{H^1} + \|u_2\|_{H^s}\|\tau_1 - \tau_2\|_{H^1})ds \\
\leq \|u_0^1 - u_0^2\|_{L^2}^2 + k\|\tau_0^1 - \tau_0^2\|_{L^2}^2 \\
+ \int_0^t \|u^2\|_{L^6}^2\|u_1 - u_2\|_{L^2}^2 + \|u^2\|_{L^6}^2\|u_1 - u_2\|_{L^2}^2 + k\|\tau_1 - \tau_2\|_{L^2}^2 + \|u^1 - u^2\|_{L^\infty(L^2)}^2)ds \\
\leq C\|u_0^1 - u_0^2\|_{L^2}^2 + \|\tau_0^1 - \tau_0^2\|_{L^2}^2 + k\|\tau_1 - \tau_2\|_{L^2}^2 + \|u^1 - u^2\|_{L^\infty(L^2)}^2),
\]

(4.3)

\[
\int_0^t \|\nabla(u^1 - u^2)\|_{L^2}^2 ds \leq C\int_0^t \|\tau^1 - \tau^2\|_{H^s}^2 + \|u_1 - u_2\|_{L^2(H^s-1)}^2(\|\tau_2\|_{H^s} + \|\nabla u_2, \nabla u_1\|_{H^s-1}) \\
+ \|\tau_1 - \tau_2\|_{H^s}^2 + \|u_1 - u_2\|_{L^2(H^s-1)}^2 + \|\nabla u_1 - \nabla u_2\|_{H^s-1} \\
\leq C(\|\tau^1 - \tau^2\|_{L^2(H^s) + \|u_1 - u_2\|_{L^2(H^s-1)}^2 + \|\tau_1 - \tau_2\|_{L^2(H^s-1)}^2 + \|u_1 - u_2\|_{L^2(H^s-1)}^2). \\
(4.4)
\]

and
\[
\int_0^t \|\nabla(u^1 - u^2)\|_{H^s-2}^2 ds \leq C(\|\tau_1 - \tau_2\|_{L^2(H^s-1)}^2 + \|u^1 - u^2\|_{L^2(H^s-1)}^2),
\]

(4.5)

Then, let \([5.8] + [4.3] + \frac{\delta}{10(k+1)(k+1)}[4.4] + [4.6]\), since \(0 < k \leq 10\), we deduce that
\[
\|u^1 - u^2\|_{L^\infty(H^s-1)}^2 + k\|\tau^1 - \tau^2\|_{L^\infty(H^s-1)}^2 + k\|\tau^1 - \tau^2\|_{L^2(H^s)}^2 + \|\nabla(u^1 - u^2)\|_{L^2(H^s-2)}^2 \\
\leq C(\|\tau_0^1 - \tau_0^2\|_{H^s-1}^2 + \|u_0^1 - u_0^2\|_{H^s-1}^2).
\]

(4.7)

This implies [4.1].

Secondly, we give the global stability in the original space \(H^{s-1}(\mathbb{R}^3) \times H^{s-1}(\mathbb{R}^3)\).

**Theorem 4.1.** Let \((u_0, \tau_0) \in H^s(\mathbb{R}^3) \times H^s(\mathbb{R}^3)\) with \(s > \frac{5}{2}\). Assume that \((u_0, \tau_0)\) satisfies the conditions in Theorem 2.2 such that
\[
\|\nabla u_0\|_{H^{s-1}} + \|\tau_0\|_{H^s} \leq k^6 \epsilon_0, \text{ for fixed } k \in (0, 10).
\]

If there exists a sequence \((u_0^n, \tau_0^n) \in (H^s(\mathbb{R}^3), H^s(\mathbb{R}^3))\) such that
\[
\lim_{n \to \infty} \|u_0^n - u_0, \tau_0^n - \tau_0\|_{H^s} = 0,
\]

then for any \(t > 0\) we have
\[
\lim_{n \to \infty} \|u^n - u\|_{L^\infty((0,\infty);H^s)} + \|u^n - u\|_{L^\infty((0,\infty);H^s)} = 0.
\]

(4.8)
Proof. Since the smallness of \((u_0, \tau_0)\) and \(\lim_{n \to \infty} \|u^n_0 - u_0, \tau^n_0 - \tau_0\|_{H^s} = 0\), let \((u^n_j, \tau^n_j)\) be the solutions of (1.2) with the initial data \((S_j u^n_0, S_j \tau^n_0)\) \((n \in \mathbb{N} \cup \infty)\), then by Theorem 1.2 (also use the bootstrap argument), \((u^n_j, \tau^n_j)\) are global solutions. Moreover, one can deduce that

\[
\|\nabla u^n_j, k\nabla \tau^n_j\|_{L^\infty_t(H^{s+1})}^2 + k\|\nabla \tau^n_j\|_{L^2_t(H^{s+1})}^2 \leq C\delta^2 \quad \text{and} \quad \|\nabla u^n_j\|_{L^2_t(H^{s+1})}^2 \leq C\delta^2,
\]

(4.9)

Since (4.9) is the blow-up criterion of (1.2), we easily obtain

\[
\|\nabla u^n_j, k\nabla \tau^n_j\|_{L^\infty_t(H^{s})}^2 + k\|\nabla \tau^n_j\|_{L^2_t(H^{s+1})}^2 + \|\nabla u^n_j\|_{L^2_t(H^{s})}^2 \leq C\|S_j u_0, S_j \tau_0\|_{L^\infty_t(H^{s+1})}^2
\]

\[
\leq C(2\|u_0, \tau_0\|_{L^\infty_t(H^{s})})^2.
\]

(4.10)

Now, by Lemma 4.11 for fixed \(0 < k \leq 10\) we already have

\[
\|u^n - u, \tau^n - \tau\|_{L^\infty_t([0,\infty);H^{s+1})} \leq \frac{C}{k}\|u^n_0 - u_0, \tau^n_0 - \tau_0\|_{L^\infty_t([0,\infty);H^{s+1})} \to 0.
\]

(4.11)

In order to verify (4.13), we should only prove the high frequency estimation \(\|u^n - u, \tau^n - \tau\|_{L^\infty_t(H^{s})} \to 0\). Our main idea is to estimate:

\[
\|u^n - u, \tau^n - \tau\|_{L^\infty_t(H^{s})}^2 \leq \|u^n - u^n, \tau^n - \tau^n\|_{L^\infty_t(H^{s})}^2 + \|u^n - u^n, \tau^n - \tau^n\|_{L^\infty_t(H^{s})}^2
\]

\[
+ \|u^n - u^n, \tau^n - \tau^n\|_{L^\infty_t(H^{s})}^2
\]

(4.12)

where \(u^n := u, \tau^n := \tau\). The proof will be divided into three parts.

(1) **estimate** \(\|u^n_j - u^n_j, \tau^n_j - \tau^n_j\|_{L^\infty_t(H^{s})}^2\) for fixed \(j\)

Firstly, we give the equation of \((u^n_j - u^n_j, \tau^n_j - \tau^n_j)\):

\[
\begin{aligned}
(u^n_j - u^n_j)t + u^n_j \nabla (u^n_j - u^n_j) + (\nabla (u^n_j - u^n_j)) \nabla (u^n_j - u^n_j) + \|u^n_j - u^n_j\|_{L^\infty_t(H^{s})}^2
\end{aligned}
\]

\[
\|\nabla (u^n_j - u^n_j)\|_{L^\infty_t(H^{s})}^2 + k\|\nabla (u^n_j - u^n_j)\|_{L^2_t(H^{s+1})}^2 + k\|\nabla (u^n_j - u^n_j)\|_{L^2_t(H^{s})}^2
\]

\[
+ Q(\nabla (u^n_j - u^n_j), \tau^n_j) + Q(\nabla u^n_j, \tau^n_j) = \mathbb{D}(u^n_j - u^n_j)
\]

(4.13)

By Lemma 4.11 we easily get

\[
\|u^n - u^n, \tau^n - \tau^n\|_{L^\infty_t(H^{s+1})}^2 + \|\nabla (u^n - u^n)\|_{L^2_t(H^{s+1})}^2
\]

\[
\leq C\|S_j u^n_0 - S_j u^n_0, S_j \tau^n_0 - S_j \tau^n_0\|_{H^{s-1}}^2
\]

\[
\leq C\|u^n_0 - u^n_0, \tau^n_0 - \tau^n_0\|_{H^{s-1}}^2 \to 0, \quad n \to \infty.
\]

(4.14)

Then, taking the \(H^s\) inner product of (1.2) (similar to (4.5) and (4.6)), by (4.10) we have

\[
\|u^n_j - u^n_j\|_{L^\infty_t(H^{s})}^2 + k\|\tau^n_j - \tau^n_j\|_{L^\infty_t(H^{s})}^2 + k\|\tau^n_j - \tau^n_j\|_{L^2_t(H^{s+1})}^2
\]

\[
\leq C\|u^n_0 - u^n_0\|_{H^s} + \|\tau^n_0 - \tau^n_0\|_{H^s} + \delta^2(\|\nabla (u^n_j - u^n_j)\|_{L^2_t(H^{s+1})}^2 + 2\|u^n_j - u^n_j\|_{L^2_t(H^{s})}^2)
\]

(4.15)

and

\[
\int_0^t \|\nabla (u^n_j - u^n_j)\|_{H^{s-1}}^2 ds \leq C(\|\tau^n_j - \tau^n_j\|_{L^\infty_t(H^{s+1})}^2 + \|u^n_j - u^n_j\|_{L^\infty_t(H^{s})}^2)
\]

(4.16)
Combing (4.15) with \(\frac{k}{16(C+1)(k+1)}\) (4.16), we finally obtain
\[
\|u^n_j - u^n_j\|_{L^\infty_t(H^s)} + k\|\tau^n_j - \tau^n_j\|_{L^\infty_t(H^s)} \leq C(2^j + 1)(\|u_0^n - u_0^n\|_{H^s} + \|\tau_0^n - \tau_0^n\|_{H^s} + \|u^n_j - u^n_j\|_{L^\infty_t(H^s)})
\]
\[+ k\|\tau^n_j - \tau^n_j\|^2_{L^2_t(H^s)} + k\|\tau^n_j - \tau^n_j\|^2_{L^2_t(H^s \cap H^{s+1})}) \to 0, \quad \text{for fixed } j. \tag{4.17}
\]

(2) estimate \(\|u^n - u^n_j, \tau^n - \tau^n_j\|_{L^\infty_t(H^s)}^2\) for any \(n \in \mathbb{N} \cup \infty\)
We give the equation of \((u^n_j - u^n, \tau^n_j - \tau^n)\)
\[
\begin{cases}
(u^n_j - u^n)t + u^n_j \nabla(u^n_j - u^n) + (u^n_j - u^n) \nabla u^n + \nabla (P^n_j - P^n) = k \text{div}(\tau^n_j - \tau^n), \\
(\tau^n_j - \tau^n)_t + (\tau^n_j - \tau^n) - \Delta(\tau^n_j - \tau^n) + u^n_j \nabla(\tau^n_j - \tau^n) + (u^n_j - u^n) \nabla \tau^n \\
+ Q(\nabla(u^n_j - u^n), \tau^n) + Q(\nabla u^n_j, (\tau^n_j - \tau^n)) = \mathbb{D}(u^n_j - u^n)
\end{cases}
\tag{4.18}
\]
The operators are similar to Lemma 4.1. The only difference is the high order term:
\[
\begin{align*}
\int_0^t < \nabla^s[(u^n - u^n_j)\nabla u^n_j], \nabla^s(u^n - u^n_j) > \, ds \\
\leq C\int_0^t \|\nabla u^n_j\|_{L^\infty} \|u^n - u^n_j\|_{H^s}^2 + C\|\nabla u^n_j\|_{H^s} \|u^n - u^n_j\|_{L^\infty} \|u^n - u^n_j\|_{H^s} \, ds \\
\leq C\int_0^t \delta \|u^n - u^n_j\|_{H^s}^2 + \frac{1}{\delta} \|\nabla u^n_j\|_{H^s}^2 \|u^n - u^n_j\|_{L^\infty}^2 \, ds \\
\leq C\int_0^t \delta \|u^n - u^n_j\|_{H^s}^2 + \frac{1}{\delta} \|\nabla u^n_j\|_{H^s}^2 \|\nabla(u^n - u^n_j)\|_{H^{s-2}}^2 \, ds \\
\leq C\delta \|u^n - u^n_j\|_{L^\infty_t(H^s)}^2 + \frac{1}{\delta} (2^j \delta)^2 \|u^n_j - S_ju^n_j\|_{H^{s-1}}^2 \\
\leq C\delta \|u^n - u^n_j\|_{H^s}^2 + \delta \|u^n_j - S_ju^n_j\|_{H^s}^2.
\end{align*}
\tag{4.19}
\]
where the fourth inequality holds by Lemmas 4.1 and 4.10, and we use the fact that \(\|u^n_0 - S_j u^n_0\|_{H^{s-1}} \leq C 2^{-j} \|u^n_0 - S_j u^n_0\|_{H^s}^2\).

Then, similar to (4.15) and (4.16) in Lemma 4.1, we have
\[
\|u^n - u^n_j\|_{L^\infty_t(H^s)}^2 \leq C(\|u^n_0 - S_j u^n_0\|_{H^s} + \|\tau^n_0 - S_j \tau^n_0\|_{H^s} + \delta^2 (\|\nabla(u^n - u^n_j)\|_{L^2_t(H^{s-1})}^2 + \|u^n - S_j u^n_0\|_{H^s}^2)).
\tag{4.20}
\]
and
\[
\int_0^t \|\nabla(u^n - u^n_j)\|_{H^{s-1}}^2 \, ds \leq C(\|\tau^n - \tau^n_j\|_{L^2_t(H^s \cap H^{s+1})} + \|u^n - u^n_j\|_{L^\infty_t(H^s)}^2) \tag{4.21}
\]
Combining (4.20) with \(\frac{k}{16(C+1)(k+1)} \times (4.21)\), we obtain
\[
\|u^n - u^n_j\|_{L^\infty_t(H^s)}^2 + k\|\tau^n - \tau^n_j\|_{L^\infty_t(H^s)}^2 + \frac{1}{\delta} \|\nabla(u^n - u^n_j)\|_{H^{s-1}}^2 \, ds
\leq C(\|u^n_0 - S_j u^n_0\|_{H^s} + \|\tau^n_0 - S_j \tau^n_0\|_{H^s} + \delta \|\nabla(u^n - u^n_j)\|_{L^2_t(H^{s-1})}) \\
\to 0, \quad j \to \infty, \quad \forall n \in \mathbb{N} \cup \infty. \tag{4.22}
\]

(3) Complete the proof
Combining (4.22), (4.17) with (4.12), one obtain that
\[
\|u^n - u^n_\infty, \tau^n - \tau^n_\infty\|_{L^\infty_t(H^s)} \to 0, \quad n \to \infty.
\]
In fact, for any \( \epsilon > 0 \), by (4.22), there exists a \( M(\epsilon) \) such that, when \( j \geq M \), we have
\[
\|u^n - u^n_j, \tau^n - \tau^n_j\|_{L^\infty_t(H^s)} \leq \frac{\epsilon}{3}, \quad \forall n \in \mathbb{N}^+ \cup \infty.
\]
Then, for this \( j \), by (4.11), there exists a \( \bar{M}(j, \epsilon) \) such that, when \( n \geq \bar{M} \), we have
\[
\|u^n_j - u^\infty_j, \tau^n_j - \tau^\infty_j\|_{L^\infty_t(H^s)} \leq \frac{\epsilon}{3},
\]
where \( \bar{M} \) is dependent on \( j, \epsilon \), since \( j \) is dependent on \( M(\epsilon) \), this implies that \( \bar{M} \) is dependent on \( \epsilon \). Finally, we have
\[
\|u^n - u, \tau^n - \tau\|_{L^\infty_t(H^s)}^2 \leq \|u^n - u^n_j, \tau^n - \tau^n_j\|_{L^\infty_t(H^s)}^2 + \|u^n_j - u^\infty_j, \tau^n_j - \tau^\infty_j\|_{L^\infty_t(H^s)}^2 + \|u^\infty_j - u, \tau^\infty_j - \tau\|_{L^\infty_t(H^s)}^2
\]
\[
\leq \frac{\epsilon}{3} + \frac{\epsilon}{3} + \frac{\epsilon}{3} = \epsilon.
\]
Combining with (4.11), that is
\[
\|u^n - u^\infty, \tau^n - \tau^\infty\|_{L^\infty_t(H^s)} \to 0, \quad n \to 0,
\]
which completes the proof.

Thanks to the globally steady result in Theorem 4.1 now we can prove Theorem 1.3 easily.

**Proof of Theorem 1.3**

*Proof.* To prove
\[
\lim_{m \to \infty} \|u^m - u, \tau^m - \tau\|_{L^\infty_t([0, \infty); H^s)} = 0
\]
for \( k^m \to k \), \( m \to \infty \). Our main idea is to estimate
\[
\|u^m - u, \tau^m - \tau\|_{L^\infty_t(H^s)}^2 \leq \|u^m - u^m_j, \tau^m - \tau^m_j\|_{L^\infty_t(H^s)}^2 + \|u^m_j - u^\infty_j, \tau^m_j - \tau^\infty_j\|_{L^\infty_t(H^s)}^2 + \|u^\infty_j - u, \tau^\infty_j - \tau\|_{L^\infty_t(H^s)}^2,
\]
where \( (u^m, \tau^m) \) are the solutions of (1.2) with the coefficient \( k^m \) and the same initial data \((u_0, \tau_0)\); \( (u^m_j, \tau^m_j) \) are the solutions of (1.2) with the coefficient \( k^m \) and the same initial data \((S_j u_0, S_j \tau)\) \((m \in \mathbb{N} \cup \infty, k^\infty = k, u^\infty := u, \tau^\infty := \tau_j)\).

Firstly, we estimate the term \( \|u^m_j - u^\infty_j, \tau^m_j - \tau^\infty_j\|_{L^\infty_t(H^s)}^2 \) with fixed \( j \). We have
\[
\begin{align*}
(u^m_j - u_j)_t + u^m_j \nabla (u^m_j - u_j) + (u^m_j - u_j) \nabla u_j + \nabla (P^m_j - P_j) &= k \text{div}(\tau^m_j - \tau_j) + (k^m - k) \text{div}\tau^m_j, \\
(\tau^m_j - \tau_j)_t + (\tau^m_j - \tau_j) - \Delta (\tau^m_j - \tau_j) + u^m_j \nabla (\tau^m_j - \tau_j) + (u^m_j - u_j) \nabla \tau_j \\
+ Q(\nabla (u^m_j - u_j), \tau_j) + Q(\nabla u^m_j, (\tau^m_j - \tau_j)) &= \mathbb{D}(u^m_j - u_j)
\end{align*}
\]
\[
(4.25)
\]
Similar to the proof of Theorem 4.1 by the energy estimations we have
\[
\|u^m_j - u^\infty_j, \tau^m_j - \tau^\infty_j\|_{L^\infty_t(H^s)}^2 \leq C(2^l + 1)(\|u^m_0 - u^\infty_0\|_{H^s} + \|\tau^m_0 - \tau^\infty_0\|_{H^s} + (k^m - k))\|\text{div}\tau^m_j\|_{L^2_t(H^s)}^2
\]
\[
\leq C(2^l + 1)(\|u^m_0 - u^\infty_0\|_{H^s} + \|\tau^m_0 - \tau^\infty_0\|_{H^s} + (k^m - k))
\]
\[
\to 0, \quad m \to \infty, \quad \text{for fixed } j.
\]
\[
(4.26)
\]
Then, by Theorem 4.1, we see that system (1.2) is globally steady for small initial data. Since

\[ \| u_0 - S_j u_0, \tau_0 - S_j \tau_0 \|_{L^\infty} \to 0, \quad j \to \infty, \]

so we have

\[ \| u^m - u_j^m, \tau^m - \tau_j^m \|_{L^\infty}^2 \to 0, \quad j \to \infty, \quad \text{for any } m \in \mathbb{N} \cap \infty. \quad (4.27) \]

Finally, combining (4.23), (4.27) with (1.4), we deduce that

\[ \lim_{n \to \infty} \| u^m - u \|_{L^\infty([0,\infty);L^\infty)} + \| u^m - u \|_{L^\infty([0,\infty);H^2)} = 0. \]

\[ \square \]

5 Instability for \( k \to 0 \).

Indeed, by (4.1) in Lemma 4.1, one can see that \( \| u^1 - u^2, \tau^1 - \tau^2 \|_{L^\infty} \) cannot be controlled by their initial data as \( k \to 0 \). In this section, we will prove that the system (1.2) is really unsteady as \( k \to 0 \) by showing that the \( L^2 \) norm of \( u^k(t,x) \) will have a jump for large time. Proof of Theorem 1.4: Let \( \epsilon_0 = \frac{1}{6k(4^k+1)} \) be the fixed small constant in Theorem 1.1 and Theorem 4.1. Recall the system:

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= k \text{div}(\tau), \\
\partial_t \tau + (u \cdot \nabla) \tau - \Delta \tau + \tau + Q(\nabla u, \tau) &= \mathbb{D} u, \quad k \in (0, \epsilon_0], \\
div u &= 0, \\
u(x,0) &= u_0(x), \quad \tau(0,x) = \tau_0(x),
\end{align*}
\]

(5.1)

and

\[
\begin{align*}
\partial_t u + (u \cdot \nabla) u + \nabla p &= 0, \\
\partial_t \tau + (u \cdot \nabla) \tau - \Delta \tau + \tau + Q(\nabla u, \tau) &= \mathbb{D} u, \\
div u &= 0, \\
u(x,0) &= u_0(x), \quad \tau(0,x) = \tau_0(x),
\end{align*}
\]

(5.2)

To prove the instability when \( k \to 0 \), we first give the definition of the global stability:

\[ \lim_{k \to 0} \| u^0 - u^k \|_{L^\infty(B_{2,1}^\frac{5}{2})} + \| \tau^0 - \tau^k \|_{L^\infty(B_{2,1}^\frac{5}{2})} = 0, \quad \forall (u_0, \tau_0) \in \mathbb{A} \text{ and } \forall t \in [0, \infty), \]

where \( \mathbb{A} := \{ (u_0, \tau_0) \in (B_{2,1}^\frac{5}{2} (\mathbb{R}^3), B_{2,1}^\frac{5}{2} (\mathbb{R}^3)) \} \) has a unique solution for any fixed \( k \). In order to prove the instability in large time, we should prove that for any \( k > 0 \) small enough, there exists a common initial sequence \( (u_0, \tau_0)(k) \) and a \( T(k) \) such that, when \( t \geq T \), we have

\[ \|(u^0 - u^a)(t)\|_{B_{2,1}^\frac{5}{2}} \geq \frac{\epsilon_0}{2}. \quad (5.3) \]

Now, let an axisymmetric vector field \( \phi \in S^3 \) with \( \text{div}\phi = 0 \). Set the initial data

\[ (u_0, \tau_0)(a) = k^6 \epsilon_0 \phi(k^4 x) \|\phi\|^{-2}_{L^2}, 0). \]

For any \( 0 < k \leq \epsilon_0 \), we have

\[ \|u_0\|_{L^2} = \epsilon_0, \quad \text{and} \quad \|u_0\|_{B_{\infty,1}^1} \leq \|u_0\|_{L^\infty} + \|u_0\|_{H^3} \leq C k^6 \epsilon_0, \quad w_0 = \text{curl} u_0. \]
These satisfy the conditions in Theorem 1.2 and Theorem 1.1, which means $\mathbf{(u_0, \tau_0)} \in A$. On one hand, by Theorem 1.1, (5.1) has a unique global strong solution $\mathbf{(u_k, \tau_k)}$ with the initial data $\mathbf{(u_0, \tau_0)(k)}$ ($\tau_0 = 0 \Rightarrow \tau = 0$). We also obtain the $L^2$ decay such that ($p = 2$):

$$\|\nabla u^k(t)\|_{L^2} \leq C\|u^k(t)\|_{L^2} \leq Ce^{-\frac{k}{4}t}.$$ 

(5.4)

Moreover, since $u_0 \in S^3 \in \dot{H}^{-1}$, combining Lemma 2.4 with (3.14), one can easily get that

$$\|u\|_{L^\infty_t(\dot{H}^{-1})} \leq C(\|u_0\|_{\dot{H}^{-1}} + \epsilon_0) \leq C.$$ 

(5.5)

By interpolation inequality, we obtain that

$$\|u^k(t)\|_{L^2} \leq C\|u^k(t)\|_{\dot{H}^{-1}}\|w^k(t)\|_{L^2} \leq Ce^{-\frac{k}{4}t}.$$ 

(5.6)

On the other hand, in (5.2), since $u_0$ is axisymmetric, by [22] one can easily obtain a unique global solution $\mathbf{(u_0, \tau_0)}$ with the same initial data $\mathbf{(u_0, \tau_0)(k)}$. Although the coefficients of (5.2) are independent of $k$, one can still look for the initial data which is dependent on $k$. Then, using the first equation (the classical Euler equation) of (5.1), we have

$$\|u^0(t)\|_{L^2} = \|u_0\|_{L^2} = \epsilon_0.$$ 

(5.7)

Therefore, there exists a $T = \frac{1}{k^2}$ such that when $t \geq T$, we have

$$\|u^0(t) - u^k(t)\|_{L^2} \geq \|u^0(t)\|_{L^2} - \|u^k(t)\|_{L^2} \geq \epsilon_0 - Ce^{-\frac{k}{4}t} \geq \epsilon_0 - Ck^2 \geq \epsilon_0 - C\epsilon_0^2 = \frac{\epsilon_0}{2},$$

(5.8)

where the second inequality is based on (5.6). This implies (5.3) ($B^\frac{5}{2,1} \hookrightarrow L^2$) and completes the proof Theorem 1.3.

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