Branes at Orbifolded Conifold Singularities

and Supersymmetric Gauge Field Theories

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Abstract

We consider D3 branes at orbifolded conifold singularities which are not quotient singularities. We use toric geometry and gauged linear sigma model to study the moduli space of the gauge theories on the D3 branes. We find that topologically distinct phases are related by a flop transition. It is also shown that an orbifold singularity can occur in some phases if we give expectation values to some of the chiral fields.
1 Introduction

Last years have witnessed great insights into understanding of supersymmetric gauge theory and supergravity theory. We now found that these are complementary descriptions of a single theory on solitonic brane solutions of M theory and string theory. Configurations containing NS fivebranes and D branes in string theory are tools for studying supersymmetric gauge field theory in various dimensions with different supersymmetries (see [5] for a complete set of references up to February 1998).

On the other hand, Maldacena’s conjecture proposes that M or string theory on the \( AdS_p \times S^k \), with \( N \) units of supergravity \( k \)-form field through \( S^k \) is dual to a \( p-1 \) specific conformal field on the boundary of the \( AdS_p \) space [6] (see [7] for an extensive review and a complete set of references). The initial proposal gave conformal field theories with maximal supersymmetry, \( \mathcal{N} = 4 \) in four dimensions. This was obtained by studying D3-branes in flat space. An immediate generalization to D3-branes at orbifold singularities breaks more supersymmetry [8, 9].

Another important class is obtained by studying D3-branes at non-orbifold singularities like conifold singularity. The conifold singularity has been analyzed in [22] where an infrared theory on the worldvolume of D3 branes was proposed. Other results for the case of non-orbifold singularities and their connection to field theories in three and four dimensions have been obtained in [1, 2, 23, 24, 25, 26, 27, 28, 29, 30, 31, 32, 33, 34, 35, 36].

In [1, 2], the authors have exploited the fact that the conifold singularity is dual to a system of perpendicular NS5 fivebranes intersecting over a 3+1 dimensional worldvolume. Their result was generalized in [29, 30] for more general conifolds. A duality between D3 branes on these general conifolds and configurations of NS and D4 branes was proposed together with relation between different resolutions of the singularity and displacements of NS branes. In [30], a mirror symmetry was proposed between orbifolded conifolds and generalized conifolds.

In the present paper we consider branes at orbifolded conifolds \( C_{kl} \) which is an orbifold of the three dimensional conifold \( xy - uv = 0 \) by a discrete group \( \mathbb{Z}_k \times \mathbb{Z}_l \). We show that the Higgs branch of the moduli space of the gauge theory is the resolved or (partially) resolved conifold singularity, depending on the values of the FI parameters as holomorphic quotients. The moduli spaces for \( \mathcal{N} = 2 \) theories has been interpreted in terms of symplectic quotients in a linear sigma model approach in [10], and in terms of holomorphic quotients in the mathematical approach in [11]. In [17], the latter approach has been extended to \( \mathcal{N} = 1 \) theories utilizing some ideas from [11, 18]. We also use toric geometry to study in detail the correspondence between D3 branes at orbifolded conifolds and brane configurations obtained after T-dualities (For details on toric geom-
etry see [11]). In [12, 13, 14], D-branes on various other singularities have been studied in the lines of [17]. The paper of [29] dealt with generalized conifolds $G_{kl}$: $xy = u^k v^l$.

The content of the paper is as follows: in section 2 we give a toric description of the quotient singularity of the three dimensional conifold. In section 3 we review relevant field theory and brane configurations. We argue why brane box model is more suitable for orbifolded conifolds. In section 4 we derive toric data for the simplest orbifolded conifold $C_{2,2}$. In section 5 we derive toric data for the orbifolded conifold $C_{2,3}$. In section 6 we describe different phases of the vacuum moduli space.

## 2 Toric Geometry of Orbifolded Conifolds

In this section, we will briefly review toric singularity and its physical realization by the moduli space of the D-brane world-volume gauge theory on it via gauged linear sigma models to fix notations and terminologies. For detailed review, we refer to [17]. A toric variety is a space which contains algebraic torus $(\mathbb{C}^*)^d$ as an open dense subset. For example, a projective space $\mathbb{P}^d = (\mathbb{C}^d+1 - \{0\})/\mathbb{C}^*$ is a toric variety because it contains $(\mathbb{C}^*)^d \cong (\mathbb{C}^*)^{d+1}/\mathbb{C}^* \subset (\mathbb{C}^{d+1} - \{0\})/\mathbb{C}^*$. As in the case of the projective space, we will express our toric varieties as a quotient space (this can be thought of as a holomorphic quotient in the sense of the Geometric Invariant Theory [19] or as a symplectic reduction as in gauged linear sigma model. In our cases, these two will be the same [20]):

$$V_\Delta = (C^q - F_\Delta)/(C^*)^{q-d}$$

(2.1)

where $q, F_\Delta$ and the action of $(C^*)^{q-d}$ on $C^q$ are determined by a combinatorial data $\Delta$. Now we give a description of the combinatorial data $\Delta$ for Gorenstein canonical singularity (i.e. a singularity with a trivial canonical class, $K$). Consider vectors $v_1, \ldots, v_q$ in a lattice $N = \mathbb{Z}^d \subset N_{\mathbb{R}} = N \otimes \mathbb{R} = \mathbb{R}^d$ in general position. We introduce the corresponding homogeneous coordinates $x_i$ for of $C^q - F_\Delta$ in the holomorphic quotients. In gauged linear sigma model, these correspond to matter multiplets. There will be $(q-d)$ independent relations

$$\sum_{i=1}^q Q_i^{(a)} v_i = 0, \quad a = 1, \ldots, q-d.$$  

(2.2)

Here $Q_i^{(a)}$'s correspond to the charges of the matter fields under $U(1)^{q-d}$ which is the maximal compact subgroup of $(C^*)^{q-d}$. The D-term equations will be

$$\sum_{i=1}^q Q_i^{(a)} |x_i|^2 = r_a, \quad a = 1, \ldots, q-d.$$  

(2.3)
In the holomorphic quotient, the charge matrix whose column vectors consist of $Q^{(a)}$ determines the action of $(\mathbb{C}^*)^{q-d}$ on $\mathbb{C}^q$ i.e. the action of $(\lambda_1, \lambda_2, \ldots, \lambda_{q-d}) \in (\mathbb{C}^*)^{q-d}$ on $(x_1, \ldots, x_q) \in \mathbb{C}^q$ is given by

$$(\lambda_1^{Q_1^{(1)}}, \lambda_2^{Q_2^{(1)}}, \ldots, \lambda_{q-d}^{Q_{q-d}^{(1)}}, x_1, \lambda_1^{Q_1^{(2)}}, \lambda_2^{Q_2^{(2)}}, \ldots, \lambda_{q-d}^{Q_{q-d}^{(2)}}, x_2, \ldots, \lambda_1^{Q_1^{(q-d)}}, \lambda_2^{Q_2^{(q-d)}}, \ldots, \lambda_{q-d}^{Q_{q-d}^{(q-d)}}, x_q) \quad (2.4)$$

Here the action can be carried out as written or in two steps, an $(\mathbb{R}^+)^{q-d}$ action and a $U(1)^{q-d}$ action if Kähler. The quotient will depend on the gauge fixing determined by the $(\mathbb{R}^+)^{q-d}$ action i.e. the moment map. In the holomorphic approach, this corresponds to different spaces $F_\Delta$ which give rise to (partial) resolutions of the original space $V_\Delta$. In toric diagram, this corresponds to different triangulations of a convex cone in $\mathbb{R}^d$ determined by $\{v_1, \ldots, v_q\}$. The collection of these combinatorial data is denoted by $\Delta$ called a fan. The quotient space $V_\Delta$ will have Gorenstein canonical singularity if there exists $u \in \mathbb{Z}^d$ such that $u \cdot v_i = 1$ for all $i \in \mathbb{Z}$. Thus $v_i$'s will lie on the hyperplane with normal $u$ at a distance $1/\|u\|$ from the origin in $\mathbb{R}^d$. This imposes the following condition on the charge vectors $Q^{(a)}$:

$$\sum_{i=1}^q Q_i^{(a)} = 0, \quad a = 1, \ldots, q-d \quad (2.5)$$

To put our discussions in the language of the gauged linear sigma model, recall that $\mathbb{C}^q$ is a symplectic manifold with the standard symplectic form $\omega = i \sum_{i=1}^q dz_i \wedge d\bar{z}_i$. The maximal compact subgroup $G := U(1)^{q-d}$ of $(\mathbb{C}^*)^{q-d}$ acts covariantly on a symplectic manifold $(\mathbb{C}^q, \omega)$ by symplectomorphisms. The infinitesimal action will give rise to a moment map $\mu : \mathbb{C}^q \rightarrow g^*$ by Poisson brackets. In coordinates, the components of $\mu : \mathbb{C}^q \rightarrow \mathbb{R}^{q-d}$ are given by

$$\mu_a = \sum_{i=1}^q Q_i^{(a)} |x_i|^2 - r_a \quad (2.6)$$

where $r_a$ are undetermined additive constants. The symplectic reduction is then defined as

$$V(r) \equiv \mu^{-1}(0)/G. \quad (2.7)$$

The structure of $V(r)$ will depend on $r$. It follows from $\mu^{(b)}$ that every $(\mathbb{C}^*)^{q-d}$-orbit in $\mathbb{C}^q$ will contribute at most one point to $V(r)$. The value of $r$ will determine the contributing orbits. For a fixed $r$, the set of $(\mathbb{C}^*)^{q-d}$-orbits which do not contribute is precisely $F_\Delta$. The quotient space $V(r)$ carries a symplectic form $\omega_r$ by reducing $\omega$. The symplectic reduction carries a natural complex structure, in which the reduced symplectic form becomes a Kähler form.

Now we will consider quotient singularities of the conifold (i.e. orbifolded conifold). The conifold is a three dimensional hypersurface singularity in $\mathbb{C}^4$ defined by:

$$C : \ xy - uv = 0. \quad (2.8)$$
The conifold can be realized as a holomorphic quotient of \( \mathbb{C}^4 \) by the \( \mathbb{C}^* \) action given by

\[
(A_1, A_2, B_1, B_2) \mapsto (\lambda A_1, \lambda A_2, \lambda^{-1} B_1, \lambda^{-1} B_2) \quad \text{for} \quad \lambda \in \mathbb{C}^*.
\]

Thus the charge matrix is the transpose of \( Q' = (1, 1, -1, -1) \) and \( \Delta = \sigma \) will be a convex polyhedral cone in \( \mathbb{N}^R = \mathbb{R}^3 \) generated by \( v_1, v_2, v_3, v_4 \in \mathbb{N}' = \mathbb{Z}^3 \) where

\[
v_1 = (1, 0, 0), \quad v_2 = (0, 1, 0), \quad v_3 = (0, 0, 1), \quad v_4 = (1, 1, -1).
\]

The isomorphism between the conifold \( C \) and the holomorphic quotient is given by

\[
x = A_1 B_1, \quad y = A_2 B_2, \quad u = A_1 B_2, \quad v = A_2 B_1.
\]

We take a further quotient of the conifold \( C \) by a discrete group \( \mathbb{Z}_k \times \mathbb{Z}_l \). Here \( \mathbb{Z}_k \) acts on \( A_1, B_j \) by

\[
(A_1, A_2, B_1, B_2) \mapsto (e^{-2\pi i/k} A_1, A_2, e^{2\pi i/k} B_1, B_2),
\]

and \( \mathbb{Z}_l \) acts by

\[
(A_1, A_2, B_1, B_2) \mapsto (e^{-2\pi i/l} A_1, A_2, B_1, e^{2\pi i/l} B_2).
\]

Thus they will act on the conifold \( C \) by

\[
(x, y, u, v) \mapsto (x, y, e^{-2\pi i/k} u, e^{2\pi i/k} v)
\]

and

\[
(x, y, u, v) \mapsto (e^{-2\pi i/l} x, e^{2\pi i/l} y, u, v).
\]

Its quotient is called the hyper-quotient of the conifold or the orbifolded conifold and denoted by \( C_{kl} \). To put the actions (2.11), (2.12) and (2.13) on an equal footing, consider the over-lattice \( N \):

\[
N = N' + \frac{1}{k} (v_3 - v_1) + \frac{1}{l} (v_4 - v_1).
\]

Now the lattice points \( \sigma \cap N \) of \( \sigma \) in \( N \) is generated by \( (k + 1)(l + 1) \) lattice points as a semigroup (These lattice points will be referred as a toric diagram.). The charge matrix \( Q \) will be \( (k + 1)(l + 1) \) by \( (k + 1)(l + 1) - 3 \). The discrete group \( \mathbb{Z}_k \times \mathbb{Z}_l \cong N/N' \) will act on the conifold \( C^4/\!/U(1) \) and its quotient will be the symplectic reduction \( C^{(k+1)(l+1)}/\!/U(1)^{(k+1)(l+1)-3} \) with the moment map associated with the charge matrix \( Q \). The new toric diagram for \( C_{kl} \) will also lie on the plane at a distance \( 1/\sqrt{3} \) from the origin with a normal vector \( (1, 1, 1) \) and we draw a toric diagram on the plane for \( C_{57} \):

The action (2.12), (2.13) of \( \mathbb{Z}_k \times \mathbb{Z}_l \) on the conifold \( C \) can be lifted to an action on \( \mathbb{C}^4 \) whose coordinates are \( x, y, u, v \). The ring of invariants will be \( \mathbb{C}[x^l, y^l, xy, u^k, v^k, uv] \) and the orbifolded conifold \( C_{kl} \) will be defined by the ideal \( (xy - uv) \mathbb{C}[x^l, y^l, xy, u^k, v^k, uv] \). Thus after renaming variables, the defining equation for the orbifolded conifold will be

\[
C_{kl} : xy = z^l, \quad uv = z^k.
\]
3 Branes at the singularities and Gauge Theory

We now put branes to probe the geometric background space. Consider a system of $M$ D3 branes sitting at the orbifolded conifold $C_{kl}$ in the transversal direction. In the spirit of [9], the corresponding gauge field theory on the world-volume of $D3$ has been obtained in [1] by investigating the action of the discrete group on the field theory of the conifold developed in [22].

The discrete group $\mathbb{Z}_k \times \mathbb{Z}_l$ acts on the fields $A_i, B_i$ of the conifold theory as in (2.12) and (2.13). By starting with a conifold theory with a group $SU(klM) \times SU(klM)$, we obtain via the projection induced by the actions the following $\mathcal{N} = 1$ supersymmetric generically chiral gauge theory for a specific choice for the Chan-Paton matrices:

$$\prod_{i=1}^k \prod_{j=1}^l SU(M)_{i,j} \times \prod_{i=1}^k \prod_{j=1}^l SU(M)'_{i,j}$$

(3.18)

with matter content
Field | Repr.
---|---
\((A_1)_{i+1,j+1;i,j}\) | \(\mathcal{D}_{i+1,j+1,i,j}\)
\((A_2)_{i,j;i,j}\) | \(\mathcal{D}_{i,j,i,j}\)
\((B_1)_{i,j;i,j+1}\) | \(\mathcal{D}_{i,j,i,j+1}\)
\((B_2)_{i,j;i,j+1}\) | \(\mathcal{D}_{i,j,i,j+1}\)

as explained in [1]. The superpotential is obtained by substituting the surviving fields into the conifold superpotential:

\[
W = \sum_{i,j} (A_1)_{i+1,j+1;i,j} (B_1)_{i,j;i,j+1} + (A_2)_{i,j;i,j+1} (B_2)_{i,j;i,j+1} + (B_1)_{i,j;i,j+1} (B_2)_{i,j;i,j+1} + (B_2)_{i,j;i,j+1}
\]  \(3.19\)

Moreover, by giving a vev to all the fields \((A_2)_{i,j;i,j+1}\), we obtain an \(\prod_{i,j} SU(M)_{i,j}\) gauge theory with surviving chiral multiplets \((A_1)_{i,j;i,j+1}\), \((B_1)_{i,j;i,j+1}\), \((B_2)_{i,j;i,j+1}\). The superpotential for these fields will be

\[
W = \sum_{i,j} (A_1)_{i,j;i,j+1} (B_1)_{i,j;i,j+1} + (A_2)_{i,j;i,j+1} (B_2)_{i,j;i,j+1} + (B_1)_{i,j;i,j+1} (B_2)_{i,j;i,j+1}
\]  \(3.20\)

This field theory is that appearing on D3 branes on an orbifold \(\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l\).

We now discuss how to arrive from the configurations with D3 branes at conifold singularities to configurations with D4 or D5 branes together with both types of NS branes. As from (2.17), we see that the orbifolded conifold can be viewed as a \(\mathbb{C}^* \times \mathbb{C}^*\) fibration over the \(z\) plane. In other words, for generic values of \(z\), the pairs of variables \((x, y)\) and \((u, v)\) describe \(\mathbb{C}^* \times \mathbb{C}^*\). Because we have \(\mathbb{C}^* \times \mathbb{C}^*\) fibration over the \(z\) plane, we have two different kind \(U(1)\) orbits, one in each \(\mathbb{C}^*\) fiber. So we can perform one T-duality or two T-dualities along each of these orbits. If we make one T-duality we obtain a configuration with \(k\) NS branes on a circle and all the configuration is at a \(Z_l\) singularity. As first explained in [4] and developed in [1], this is a chiral theory. Because we still have a singularity which cannot be controlled by removing NS branes, it is more advantageous to do both T-dualities in order to use all the geometrical information. By making these, we arrive to brane box configurations with two compact direction, containing D5 branes together with both types of NS branes. So by using the geometry, we study the Kähler deformation of the orbifolded conifold with brane boxes. As explained in [30], in order to account the number of Kähler structure parameters necessary to completely solve the singularity, we need to modify the intersections of the NS branes by so-called diamonds. By closing the diamonds we turn off the B field and by rotating the diamonds on a plane perpendicular on the D5 brane we resolve the singularity to \(\mathbb{C}^3/\mathbb{Z}_k \times \mathbb{Z}_l\).
4 The Orbifolded Conifold $C_{22}$

Consider a system of $M$ D3 branes sitting at the orbifolded conifold

$$C_{22} : xy = uv = z^2$$

(4.21)

As explained before, this is a chiral theory with the gauge group:

$$\prod_{i,j=1}^{2} SU(M)_{i,j} \times \prod_{i,j=1}^{2} SU(M)_{i,j}'.$$

(4.22)

Because the T-dual theory contains NS branes which are perpendicular, the adjoint fields become massive and they are integrated out, leaving only quadratic terms in the superpotential. For simplicity we denote the 16 fields by:

$$A_{11} = (A_1)_{22;11}, \ A_{12} = (A_1)_{21;12}, \ A_{13} = (A_1)_{12;21}, \ A_{14} = (A_1)_{11;22},$$

$$A_{21} = (A_2)_{11;11}, \ A_{22} = (A_2)_{12;12}, \ A_{23} = (A_2)_{21;21}, \ A_{24} = (A_2)_{22;22},$$

$$B_{11} = (B_1)_{11;12}, \ B_{12} = (B_1)_{12;11}, \ B_{13} = (B_1)_{21;22}, \ B_{14} = (B_1)_{22;21},$$

$$B_{21} = (B_2)_{11;21}, \ B_{22} = (B_2)_{12;22}, \ B_{23} = (B_2)_{21;11}, \ B_{24} = (B_2)_{22;12}.$$  

(4.23)

The D term equations are:

$$|A_{14}|^2 + |A_{21}|^2 - |A_{23}|^2 - |B_{23}|^2 = \xi_1$$

$$|A_{13}|^2 + |A_{22}|^2 - |B_{11}|^2 - |B_{24}|^2 = \xi_2$$

$$|A_{12}|^2 + |A_{23}|^2 - |B_{14}|^2 - |B_{21}|^2 = \xi_3$$

$$|A_{11}|^2 + |A_{24}|^2 - |B_{13}|^2 - |B_{22}|^2 = \xi_4$$

$$|B_{21}|^2 + |B_{11}|^2 - |A_{11}|^2 - |A_{21}|^2 = \xi_5$$

$$|B_{22}|^2 + |B_{12}|^2 - |A_{12}|^2 - |A_{22}|^2 = \xi_6$$

$$|B_{23}|^2 + |B_{13}|^2 - |A_{13}|^2 - |A_{23}|^2 = \xi_7$$

$$|B_{24}|^2 + |B_{14}|^2 - |A_{14}|^2 - |A_{24}|^2 = \xi_8$$

(4.24)

where the FI parameters satisfy the constraint

$$\sum_{i=1}^{8} \xi_i = 0$$

(4.25)

The superpotential is

$$W = A_{11}B_{11}A_{22}B_{22} + A_{12}B_{12}A_{21}B_{21} + A_{13}B_{13}A_{24}B_{24} + A_{14}B_{14}A_{23}B_{23} -$$

$$- A_{11}B_{21}A_{23}B_{13} - A_{12}B_{22}A_{24}B_{14} - A_{13}B_{23}A_{21}B_{11} - A_{14}B_{24}A_{22}B_{12}$$

(4.26)
There are 16 F-term constraints derived from this superpotential, not all of them independent. As opposed to other field theories considered previously in the literature, our case involves chiral fields so the F term equations will give equality between two products of three fields as for example the one obtained after taking the derivative with $A_{11} : B_{11}A_{22}B_{22} = B_{21}A_{23}B_{13}$ and the rest of 15 equations are similar. After solving the independent F-term equations, we arrive at 10 independent fields, the rest of 6 fields being expressed in terms of these. We chose $A_{24}, A_{13}, A_{14}, B_{12}, B_{13}, B_{14}, B_{21}, B_{22}, B_{23}, B_{24}$ as the independent variables. The solution for the F-term equations is:

$$
\begin{array}{cccccccc}
A_{24} & A_{13} & A_{14} & B_{12} & B_{13} & B_{14} & B_{21} & B_{22} & B_{23} & B_{24} \\
A_{11} & 0 & 0 & 1 & 0 & -1 & 1 & -1 & 0 & 1 & 0 \\
A_{12} & 0 & 1 & 0 & 0 & 1 & -1 & 0 & -1 & 0 & 1 \\
A_{21} & 1 & 0 & 0 & -1 & 0 & 1 & -1 & 1 & 0 & 0 \\
A_{22} & 1 & 1 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 \\
A_{23} & 1 & 1 & -1 & 0 & 1 & -1 & 0 & 0 & -1 & 1 \\
B_{11} & 0 & 0 & 0 & 1 & 1 & -1 & 1 & -1 & -1 & 1 \\
\end{array}
$$

We now proceed to obtain the vacuum moduli space in the usual way, i.e. by imposing the F-term constraints and the D-term constraints in the form of symplectic quotients as the gauged linear sigma model. If we impose only F-term constraints, we can identify the moduli, denoted by $\mathcal{M}_F$, of the 16 fields as a cone $M_+ = Z^{10}$ by expressing them in terms of 10 independent fields. To construct this as a symplectic quotient, we consider the dual cone $N_+$ in $N = \text{Hom}(\mathcal{M}, Z)$. It turns out that the dual cone $N_+$ is generated by 24 lattice points. Thus we have a map

$$T : Z^{24} \rightarrow N,$$

which is shown in the Figure 2.

$$T : \mathbb{Z}^{24} \rightarrow \mathbb{N}, \quad (4.28)$$

The transpose of the kernel of $T$ is then a $14 \times 24$ charge matrix $Q$ which is shown Figure 3. Thus we have an exact sequence:

$$0 \rightarrow \mathbb{Z}^{14} \xrightarrow{tQ} \mathbb{Z}^{24} \xrightarrow{T} \mathbb{N} \rightarrow 0.$$ 

(4.29)

Thus we have an exact sequence:

$$0 \rightarrow \mathbb{Z}^{14} \xrightarrow{tQ} \mathbb{Z}^{24} \xrightarrow{T} \mathbb{N} \rightarrow 0.$$ 

(4.29)

From this sequence, one can see that the moduli space $\mathcal{M}_F$ can be expressed as a holomorphic quotient of $\mathbb{C}^{24}$ by $(\mathbb{C}^*)^{14}$ whose action is specified by $Q$ (or a symplectic quotient by $U(1)^{14}$) via the map induced by $T$. To incooperate the D-term constraints,
Figure 2: The $10 \times 24$ matrix $T$

\[
\begin{pmatrix}
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 & 1 \\
0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 1 \\
0 & 0 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
\end{pmatrix}
\]

Figure 3: The $14 \times 24$ charge matrix $Q$

\[
\begin{pmatrix}
2 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\
1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]
we need to see how the action of \((\mathbf{C}^*)^{10}\) on the toric variety \(\mathcal{M}_F\) is represented in these terms. Since the action of \((\mathbf{C}^*)^{10}\) on the open subset \((\mathbf{C}^*)^{10} \subset \mathcal{M}_F\) must be the obvious multiplication, the action of \((\mathbf{C}^*)^{10}\) on \(\mathbf{C}^{24}\) is specified the transpose of a 10 \times 24 matrix \(U\) such that

\[
    T^\dagger U = \text{Id}_k. \tag{4.30}
\]

\(U\) is shown in the Figure 4.

The D-term equations are represented by a matrix \(V\) in the Figure 5. We ignored the charges on the dependent fields because they are already encoded in \(Q\). Thus on \(\mathbf{C}^{24}\), the D-term constraints are represented by the charge matrix \(VU\). Finally the full set of charges is given by a 21 \times 24 charge matrix \(\tilde{Q}\) (Figure 6.) by concatenating \(Q\) and \(VU\). The cokernel of its transpose gives toric data for our vacuum moduli space, denoted by \(\mathcal{M}\). After eliminating redundant variables, it is give in the form of a map \(T_M : \mathbf{Z}^3 \to \mathbf{Z}^3\):

\[
    T_M = \begin{pmatrix}
        2 & 1 & 0 & 1 & 0 & -1 & 0 & -1 & -2 \\
        0 & 1 & 2 & 0 & 1 & 2 & 0 & 1 & 2 \\
        -1 & -1 & -1 & 0 & 0 & 0 & 1 & 1 & 1
    \end{pmatrix}. \tag{4.31}
\]
The lattice points given by $T_M$ lie on the plane with normal $(1, 1, 1)$ at a distance $1/\sqrt{3}$ from the origin. We depict these lattice points

$$v_1 = (2, 0, -1), \quad v_2 = (1, 1, -1), \quad v_3 = (0, 2, -1),$$

$$v_4 = (1, 0, 0), \quad v_5 = (0, 1, 0), \quad v_6 = (-1, 2, 0),$$

$$v_7 = (0, 0, 1), \quad v_8 = (-1, 1, 1), \quad v_9 = (-2, 2, 1)$$

in the planar diagram (Figure 7). This is exactly a toric diagram for the orbifolded conifold $C_{22}$: $xy = uv = z^2$. The corresponding charge matrix $Q_M$ for the toric data $T_M$ with the Fayet-Iliopoulos D-term parameters from (4.24) is as follows:

$$Q_M = \begin{pmatrix}
0 & 0 & 0 & 2 & -2 & 0 & -1 & 0 & 1 & 2\xi_1 + \xi_2 + \xi_4 + \xi_5 - \xi_7 \\
0 & 0 & 0 & 1 & -1 & 0 & -1 & 1 & 0 & \xi_1 - \xi_7 \\
0 & 0 & 0 & 1 & -2 & 1 & 0 & 0 & 0 & \xi_1 - \xi_3 - \xi_6 - \xi_7 \\
1 & 0 & 0 & -2 & 0 & 0 & 1 & 0 & 0 & -\xi_1 - \xi_2 - \xi_5 - \xi_6 \\
0 & 0 & 1 & 0 & -2 & 0 & 1 & 0 & 0 & -\xi_2 - \xi_3 - \xi_6 - \xi_7 \\
0 & 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & -\xi_2 - \xi_6 \\
\end{pmatrix}$$

(4.33)

For this choice of redundant variables, Fayet-Iliopoulos D-term parameters must satisfy

$$\xi_1 > 0, \quad \xi_4 > 0, \quad -\xi_6 > 0, \quad -\xi_7 > 0, \quad -\xi_3 - \xi_6 - \xi_7 > 0,$$

$$-\xi_2 - \xi_6 - \xi_7 > 0, \quad -\xi_2 - \xi_3 - \xi_6 - \xi_7 > 0, \quad -\xi_1 - \xi_2 - \xi_5 - \xi_6 > 0,$$

$$-\xi_1 - \xi_2 - \xi_3 - \xi_5 - \xi_6 - \xi_7 > 0, \quad \xi_1 + \xi_4 + \xi_5 > 0, \quad \xi_1 + \xi_2 + \xi_4 + \xi_5 > 0.$$
\[
\begin{pmatrix}
 2 & -1 & 0 & -1 & -1 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\
 1 & 0 & 1 & 0 & -1 & 0 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & 0 & -1 & -1 & 0 & 1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & 0 & -1 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & -1 & 1 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & -1 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\
 1 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & 0 & -1 & -1 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & -1 & -1 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & -1 & 0 & -1 & 0 & 0 & 1 & 0 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 1 & 0 & -1 & -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 0 & 0 & 0 & -1 & 0 & 1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 1 & 1 & -1 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 -1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
 0 & -1 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
\end{pmatrix}
\]

Figure 6: The $21 \times 24$ matrix $\tilde{Q}$
5 The Orbifolded Conifold $xy = z^2, uv = z^3$

In this case we start with a system of D branes sitting at the orbifold conifold singularity

$$C_{23} : xy = z^2, uv = z^3.$$ \hspace{1cm} (5.35)

By putting on the $M$ D3 branes on $C_{23}$, we obtain the field theory with the gauge group:

$$\prod_{i=1}^{2} \prod_{j=1}^{3} SU(M)_{i,j} \times \prod_{i=1}^{2} \prod_{j=1}^{3} SU(M)'_{i,j}$$ \hspace{1cm} (5.36)

The matter content for the theory with the gauge group (5.36) is similar to the one encountered for the previous orbifolded conifold but we have 24 fields now instead of 16 as before. For simplicity we denote the 24 fields by:

$$A_{11} = (A_1)_{22;11}, \; A_{12} = (A_1)_{21;12}, \; A_{13} = (A_1)_{32;21}, \; A_{14} = (A_1)_{31;22}$$

$$A_{15} = (A_1)_{12;31}, \; A_{16} = (A_1)_{11;32}, \; A_{21} = (A_2)_{11;11}, \; A_{22} = (A_2)_{12;12}$$

$$A_{23} = (A_2)_{21;21}, \; A_{24} = (A_2)_{22;22}, \; A_{25} = (A_2)_{31;31}, \; A_{26} = (A_2)_{32;32}$$

$$B_{11} = (B_1)_{11;12}, \; B_{12} = (B_1)_{12;11}, \; B_{13} = (B_1)_{21;22}, \; B_{14} = (B_1)_{22;21}$$

$$B_{15} = (B_1)_{31;32}, \; B_{16} = (B_1)_{32;31}, \; B_{21} = (B_2)_{11;21}, \; B_{22} = (B_2)_{12;22}$$

$$B_{23} = (B_2)_{21;31}, \; B_{24} = (B_2)_{22;32}, \; B_{25} = (B_2)_{31;11}, \; B_{26} = (B_2)_{32;12}$$
The superpotential is then:

\[
W = A_{11} B_{11} A_{22} B_{22} + A_{12} B_{12} A_{21} B_{21} + A_{13} B_{13} A_{24} B_{24} + A_{14} B_{14} A_{23} B_{23} + \quad (5.38)
\]

\[
+ A_{15} B_{15} A_{26} B_{26} + A_{16} B_{16} A_{25} B_{25} - A_{11} B_{21} A_{23} B_{13} + A_{12} B_{22} A_{24} B_{14}
\]

\[
- A_{13} B_{23} A_{25} B_{15} - A_{14} B_{24} A_{26} B_{16} - A_{15} B_{25} A_{21} B_{11} - A_{16} B_{26} A_{22} B_{12}
\]

There are 24 F-term constraints derived from this superpotential, not all of them independent and by solving them we arrive at 14 independent fields, the rest of 10 fields being expressed in terms of these. We choose

\[
A_{16}, A_{26}, B_{11}, B_{12}, B_{13}, B_{14}, B_{15}, B_{16}, B_{21}, B_{22}, B_{23}, B_{24}, B_{25}, B_{26} \quad (5.39)
\]

as the independent fields.

The D term equations are:

\[
|A_{16}|^2 + |A_{21}|^2 - |B_{11}|^2 - |B_{25}|^2 = \xi_1 \quad (5.40)
\]

\[
|A_{15}|^2 + |A_{22}|^2 - |B_{11}|^2 - |B_{26}|^2 = \xi_2
\]

\[
|A_{12}|^2 + |A_{23}|^2 - |B_{14}|^2 - |B_{21}|^2 = \xi_3
\]

\[
|A_{11}|^2 + |A_{24}|^2 - |B_{13}|^2 - |B_{22}|^2 = \xi_4
\]

\[
|A_{14}|^2 + |A_{26}|^2 - |B_{16}|^2 - |B_{23}|^2 = \xi_5
\]

\[
|A_{13}|^2 + |A_{25}|^2 - |B_{15}|^2 - |B_{24}|^2 = \xi_6
\]

\[
|B_{21}|^2 + |B_{11}|^2 - |A_{11}|^2 - |A_{21}|^2 = \xi_7
\]

\[
|B_{22}|^2 + |B_{12}|^2 - |A_{12}|^2 - |A_{22}|^2 = \xi_8
\]

\[
|B_{23}|^2 + |B_{13}|^2 - |A_{13}|^2 - |A_{23}|^2 = \xi_9
\]

\[
|B_{24}|^2 + |B_{14}|^2 - |A_{14}|^2 - |A_{24}|^2 = \xi_{10}
\]

\[
|B_{25}|^2 + |B_{15}|^2 - |A_{15}|^2 - |A_{25}|^2 = \xi_{11}
\]

\[
|B_{26}|^2 + |B_{16}|^2 - |A_{16}|^2 - |A_{26}|^2 = \xi_{12}
\]

where the FI parameters satisfy the constraint

\[
\sum_{i=1}^{12} \xi_i = 0. \quad (5.41)
\]

We want to implement the same procedure as in the previous \( \mathbb{Z}_2 \times \mathbb{Z}_3 \) orbifolded conifold \( \mathcal{C}_{22} \). As before, we can identify the moduli space \( \mathcal{M}_F \) of 24 fields under the F-term constraints as a cone \( \mathbb{M}_+ \) in \( \mathbb{M} = \mathbb{Z}^{14} \). The dual cone \( \mathbb{N}_+ \) is generated by 80 lattice points represented by \( T \). Thus \( \mathcal{M}_F \) can be expressed as a symplectic quotient \( \mathbb{C}^{80} // U(1)^{66} \) whose action is specified by \( Q \). This can be expressed as the following exact sequence:

\[
0 \rightarrow \mathbb{Z}^{66} \xrightarrow{iQ} \mathbb{Z}^{80} \xrightarrow{T} \mathbb{N} \rightarrow 0. \quad (5.42)
\]
By further imposing 11 D-term equations from (5.40), we obtain toric data for the vacuum moduli space $\mathcal{M}$ as a three dimensional toric variety $\mathbb{C}^{80}/U(1)^{77}$. Because of huge sizes of the matrices involved, we only write the final toric data after eliminating redundant variables. It is given in the form of a map $T_{\mathcal{M}}: \mathbb{Z}^{12} \rightarrow \mathbb{Z}^3$:

$$T_{\mathcal{M}} = \begin{pmatrix}
0 & 1 & 2 & 3 & -1 & 0 & 1 & 2 & -1 & 0 & 1 & -2 \\
0 & -1 & -2 & -3 & 2 & 1 & 0 & -1 & 3 & 2 & 1 & 4 \\
1 & 1 & 1 & 1 & 0 & 0 & 0 & 0 & -1 & -1 & -1 & -1
\end{pmatrix} \quad (5.43)$$

The lattice points given by $T_{\mathcal{M}}$ are as follows:

$$v_1 = (0, 0, 1), \quad v_2 = (1, -1, 1), \quad v_3 = (2, -2, 1), \quad v_4 = (3, -3, 1),$$

$$v_5 = (-1, 2, 0), \quad v_6 = (0, 1, 0), \quad v_7 = (1, 0, 0), \quad v_8 = (2, -1, 0), \quad (5.44)$$

$$v_9 = (-1, 3, -1), \quad v_{10} = (0, 2, -1), \quad v_{11} = (1, 1, -1), \quad v_{12} = (-2, 4, -1),$$

which are drawn in Figure 8. This is exactly the toric data for the $\mathbb{Z}_3 \times \mathbb{Z}_2$ orbifolded conifold $\mathcal{C}_{32}$. The corresponding charge matrix is given by

$$Q_{\mathcal{M}} = \begin{pmatrix}
2 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -4 & 1 & 0 & 0 \\
-3 & 0 & 0 & 0 & 0 & 0 & -1 & 1 & 3 & 0 & 0 & 0 \\
1 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & -3 & 0 & 0 & 0 \\
-2 & 1 & 0 & 0 & 0 & 0 & -1 & 0 & 2 & 0 & 0 & 0 \\
-2 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 1 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & -2 & 0 & 0 & 1 \\
-1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 1 & 0 & 0 & 0 \\
-1 & 0 & 1 & 0 & 0 & 0 & 1 & 0 & -1 & 0 & 0 & 0
\end{pmatrix}. \quad (5.45)$$
The Fayet-Iliopoulos D-term parameters corresponding to each row is given by

\[
\begin{pmatrix}
-3\xi_1 + \xi_3 - 2\xi_4 - \xi_5 + 2\xi_6 - 3\xi_7 + \xi_9 - 2\xi_{10} - \xi_{11} \\
3\xi_1 - \xi_3 + 2\xi_4 + \xi_5 - 2\xi_6 + 2\xi_7 - \xi_8 - 2\xi_9 + \xi_{10} \\
-3\xi_1 - \xi_2 - 2\xi_4 - \xi_5 + \xi_6 - 3\xi_7 - \xi_8 - 2\xi_{10} - \xi_{11} \\
2\xi_1 - \xi_3 + \xi_4 - 2\xi_6 + \xi_7 - \xi_8 - 2\xi_9 \\
\xi_1 - \xi_2 - 3\xi_6 - \xi_8 - 2\xi_9 - \xi_{11} \\
-2\xi_1 - \xi_4 - \xi_5 + \xi_6 - 2\xi_7 - \xi_{10} - \xi_{11} \\
-\xi_1 - \xi_4 - \xi_5 - \xi_7 - \xi_{10} - \xi_{11} \\
\xi_1 + \xi_4 + \xi_5 + \xi_7 + \xi_{10} \\
-\xi_1 - \xi_2 - \xi_3 - \xi_4 - \xi_5 - \xi_6 - \xi_7 - \xi_8 - \xi_9 - \xi_{10} - \xi_{11}
\end{pmatrix}
\]

(5.46)

6 Partial Resolutions

In order to see (partial) resolutions of the singularities in the formalism used above, we need to turn on the Fayet-Iliopoulos terms. This will correspond to triangulations of the convex cone in toric geometry and moving the center of the moment map in symplectic reduction.

Before starting the actual discussion, we make some observations about the general cases. In [29] it was considered the case of generalized conifolds of type \(xy = u^kv^k\) and their resolutions. Their partial resolutions are conifold singularities, pinch point singularities and orbifold singularities and are obtained for different values of the FI parameters. In the T dual picture, D3 branes at \(xy = u^kv^k\) singularities transform into \(k\) NS branes, \(k\) NS’ branes on circle together with D4 branes having the circle as one of
the worldvolume coordinates. Partial resolutions of the singularity are obtained in the T-dual picture by moving one NS brane in the $x^7$ direction (in field theory this means to give expectation values to one field thus breaking the product of two gauge groups to a diagonal one). This smoothen the singularity to $xy = u^{k-1}v^k$. By removing a NS' brane, the singularity is smoothen to $xy = u^k v^{k-1}$. In [29], the starting point was D3 at $xy = u^2 v^2$ singularity whose T dual contains 2 NS and 2 NS' branes. By removing the two NS branes one arrives at the conifold singularity, by removing one NS and one NS' one arrives at the conifold and by removing either one NS or one NS' the pinch point singularity is obtained. This of course means that we resolve the initial “worse” singularity to a “smoother” one. By removing NS branes we have complete control on the spacetime singularity.

In the case of orbifolded conifolds we need to use brane box models obtained by making two T-dualities. In this case, the resolutions are obtained either by moving NS and NS' branes with respect to each other or by opening diamonds at the intersections of the NS and NS' branes.

The discussion is similar for both types of $\mathbb{Z}_k \times \mathbb{Z}_l$ orbifolded conifolds discussed in this paper. Let us consider the $\mathbb{Z}_2 \times \mathbb{Z}_2$ orbifolded conifold case. From ($4.33$), we have a moment map $\mu_M : \mathbb{C}^9 \rightarrow \mathbb{R}^6$:

$$
\mu_M = \begin{pmatrix}
2|p_3|^2 - 2|p_4|^2 - |p_6|^2 + |p_8|^2 - 2\xi_1 - \xi_2 - \xi_4 - \xi_5 + \xi_7 \\
|p_3|^2 - |p_4|^2 - |p_6|^2 + |p_7|^2 - \xi_1 + \xi_7 \\
|p_3|^2 - 2|p_4|^2 + |p_5|^2 - \xi_1 + \xi_2 + \xi_6 + \xi_7 \\
|p_4|^2 - 2|p_3|^2 + |p_6|^2 + \xi_1 + \xi_2 + \xi_5 + \xi_6 \\
|p_4|^2 - 2|p_4|^2 + |p_6|^2 + \xi_2 + \xi_3 + \xi_6 + \xi_7 \\
|p_4|^2 - |p_3|^2 - |p_4|^2 + |p_6|^2 + \xi_2 + \xi_6 
\end{pmatrix}
$$

(6.47)

where $p_i$ are homogeneous coordinates of $\mathbb{C}^9$. Then the $\mathcal{M}$ is the symplectic reduction $\mu_M^{-1}(0)/U(1)^6$. From the conditions of ($4.34$), Fayet-Iliopoulos parameters of the resulting $U(1)^6$ gauged linear sigma model satisfy inequalities

$$
-2\xi_1 - \xi_2 - \xi_4 - \xi_5 + \xi_7 < 0, \quad -\xi_1 + \xi_7 < 0, \\
-\xi_1 + \xi_3 + \xi_6 + \xi_7 < 0, \\
\xi_1 + \xi_2 + \xi_5 + \xi_6 < 0, \\
\xi_2 + \xi_3 + \xi_6 + \xi_7 < 0.
$$

(6.48)

But the condition ($4.34$) does not determine the sign of the last coordinate $\xi_2 + \xi_6$ of the center of the moment map $\mu_M$. Notice that the last coordinate of the moment map $\mu_M$
which is flopped as the sign of $\xi_2 + \xi_6$ changes. When $\xi_2 + \xi_6 > 0$, it is parameterized by the homogeneous coordinates $p_3$ and $p_4$. When $\xi_2 + \xi_6 < 0$, it is parameterized by the homogeneous coordinates $p_1$ and $p_6$. These two phases are topologically different. Thus the D-brane vacuum moduli space $\mathcal{M}$ does have topologically distinct phases which are related by a flop transition. This phenomenon has been observed for orbifold singularities [12, 13]. We can see this flop in the toric diagram which is shown in Figure 9.

For special values of $\xi_i$, there are several singularity types. Of course, we get the orbifolded conifold $C_{22}$ when all $\xi$ are zero. But the singularity becomes partially resolved, when fields get expectation values in terms of the FI parameters. One of the most interesting case is when we give expectation values to the fields $A_{2i}$, $i = 1, \cdots, 4$. This region corresponds to $\xi_5 + \xi_1 = \xi_6 + \xi_2 = \xi_7 + \xi_3 = \xi_8 + \xi_4$. Hence the last three coordinates of the center of the moment map $\mu_\mathcal{M}$ are zeros. Thus one can see that the lower left half triangle of the toric diagram will not be triangulated. So we will have an orbifold singularity $C^3/Z_2 \times Z_3$ for generic values of $\xi_i$ under these circumstances (Figure 10).

The configuration of D3 branes at this singularity is T-dual to a $2 \times 2$ brane box with trivial identification of the unit cell. In the language of [30], giving expectation values to the fields $A_{2i}$, i.e. going to a baryonic branch, means to rotate the diamonds which lie at the intersections of the NS and NS' branes. One can have similar discussions for the $Z_2 \times Z_3$ orbifolded conifold.

7 Conclusions

In this paper, we have used the toric geometry and Witten’s gauged linear sigma model to identify the Higgs moduli space of the field theory on the world volume of branes at
the orbifolded conifold singularity of type $C_{kl}$ which is a $\mathbb{Z}_k \times \mathbb{Z}_l$ quotient of the conifold $xy - uv = 0$. We have shown that the Higgs moduli space does have phases related by a flop transition and topology change can occur. It is also observed that the orbifold singularity can be obtained as one of the phases of the Higgs moduli space. In field theory, this corresponds to giving expectation value to some hypermultiplets.

Moreover, we have studied a correspondence between brane configurations and brane at singularities for the case of orbifolded conifolds of type $C_{kl}$.

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