A tight bound on the collection of edges in MSTs of induced subgraphs

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Let $G = (V, E)$ be a complete $n$-vertex graph with distinct positive edge weights. We prove that for $k \in \{1, 2, \ldots, n-1\}$, the set consisting of the edges of all minimum spanning trees (MSTs) over induced subgraphs of $G$ with $n - k + 1$ vertices has at most $nk - \binom{k+1}{2}$ elements. This proves a conjecture of Goemans and Vondrak [1]. We also show that the result is a generalization of Mader’s Theorem, which bounds the number of edges in any edge-minimal $k$-connected graph.

1 Introduction

Let $G = (V, E)$ be a complete $n$-vertex graph with distinct positive edge weights. For any set $X \subseteq V$, denote by $G[V \setminus X]$ the subgraph of $G$ induced by $V \setminus X$. We will also sometimes write this graph as $(V \setminus X, E)$, ignoring edges in $E$ incident on vertices in $X$. $\text{MST}(G[V \setminus X])$ denotes the set of edges in the graph’s minimum spanning tree. (The MST is unique due to the assumption that the edge weights are distinct.)

For $k \in \{1, 2, \ldots, n-1\}$, define

$$M_k(G) = \bigcup_{X \subseteq V, |X| = k-1} \text{MST}(G[V \setminus X]).$$

Note that for $k = 1$ we have $M_1(G) = \text{MST}(G)$. In [1], Goemans and Vondrak considered the problem of finding a sparse set of edges which, with high probability, contain the MST of a random subgraph of $G$. In this context they proved an upper bound on $M_k(G)$, namely that $|M_k(G)| < (1 + \frac{e}{2})kn$, and they conjectured that one should be able to improve the bound to $|M_k(G)| \leq nk - \binom{k+1}{2}$. In this paper we prove this conjecture.
Theorem 1

For any complete graph $G$ on $n$ vertices with distinct positive edge weights,

$$|M_k(G)| \leq nk - \binom{k+1}{2}. \quad (1)$$

As Goemans and Vondrak recognized, the bound is tight: for any $n$ and $k$ it is easy to produce edge weights giving equality in (1). One way is to fix an arbitrary set $V' \subseteq V$ with cardinality $k$, and partition the edges $E$ into three sets $E_0$, $E_1$ and $E_2$ where, for $i \in \{0, 1, 2\}$, $E_i$ contains all edges of $E$ having exactly $i$ endpoints in $V'$. Assign arbitrary distinct positive weights to the edges in $E_0$ such that all weights on $E_2$ are smaller than those on $E_1$, which in turn are smaller than those on $E_0$. It can easily be verified that $M_k(G) = E_2 \cup E_1$ and thus $|M_k(G)| = nk - \binom{k+1}{2}$.

Theorem 1’s assumption that $G$ is complete is not meaningfully restrictive. If $G$ is such that deletion of some $k - 1$ vertices leaves it disconnected, then the notion of $M_k(G)$ does not make sense; otherwise, it does not matter if other edges of $G$ are simply very costly or are absent.

The bound of Theorem 1 applies equally if we consider the edgeset of MSTs of induced subgraphs of size at most $n - k + 1$ (rather than exactly that number). This is an immediate consequence of the following remark.

Remark 2

For any complete graph $G$ on $n$ vertices with distinct positive edge weights, and $k \in \{1, 2, \ldots, n - 2\}$, $M_{k+1}(G) \supseteq M_k(G)$.

Proof. We will show that any edge $e$ in $M_k(G)$ is also in $M_{k+1}(G)$. By definition, $e \in M_k(G)$ means that there is some vertex set $X$ of cardinality $|X| = k - 1$ for which $e \in \text{MST}(G_k)$, where $G_k = G[V \setminus X]$.

Consider any leaf vertex $v$ of MST($G_k$), with neighbor $u$. We claim that deleting $v$ from $G_k$ (call the resulting graph $G_{k+1}$) results in the same MST less the edge $\{u, v\}$, i.e., that $\text{MST}(G_{k+1}) = \text{MST}(G_k) \setminus \{u, v\}$. This follows from considering the progress of Kruskal’s algorithm on the two graphs. Before edge $\{u, v\}$ is added to MST($G_k$), the two processes progress identically: every edge added to MST($G_k$) is also a cheapest edge for the smaller graph $G_{k+1}$. The edge $e$, added to MST($G_k$), of course has no parallel in $G_{k+1}$. As further edges are considered in order of increasing cost, again, every edge added to MST($G_k$) will also be added to MST($G_{k+1}$), using the fact that none of these edges is incident on $v$.

Thus, if $v$ is not a vertex of $e$, then $e \in \text{MST}(G_{k+1})$. Since MST($G_k$) has at least two leaves, it has at least one leaf $v$ not in $e$, unless MST($G_k$) = $e$, which is impossible since $G_k$ has at least 3 vertices. \qed
Outline of the paper

In Section 2 we define a “$k$-constructible” graph, and show that every graph $(V, M_k(G))$ is $k$-constructible, and every $k$-constructible graph is a subgraph of some graph $(V, M_k(G))$. This allows a simpler reformulation of Theorem 1 as Theorem 6, which also generalizes a theorem of Mader [3]. We prove Theorem 6 in Section 3.

2 $k$-constructible graphs

We begin by recalling Menger’s theorem for undirected graphs, which motivates our definition of $k$-constructible graphs. Two vertices in an undirected graph are called $k$-connected if there are $k$ (internally) vertex-disjoint paths connecting them.

**Theorem 3** (Menger’s theorem)
Let $s, t$ be two vertices in an undirected graph $G = (V, E)$ such that $\{s, t\} \notin E$. Then $s$ and $t$ are $k$-connected in $G$ if and only if after deleting any $k - 1$ vertices (distinct from $s$ and $t$), $s$ and $t$ are still connected.

**Definition 4** ($k$-constructible graph)
A graph $G = (V, E)$ is called $k$-constructible if there exists an ordering $O = \langle e_1, e_2, \ldots, e_m \rangle$ of the edges in $E$ such that for all $i \in \{1, 2, \ldots, m\}$ the graph $(V, \{e_1, e_2, \ldots, e_{i-1}\})$ contains at most $k - 1$ vertex-disjoint paths between the two endpoints of $e_i$. We say that $O$ is a $k$-construction order for the graph $G$.

Note that 1-constructible graphs are forests, and edge-maximal 1-constructible graphs are spanning trees. We therefore have in particular that graphs of the form $M_1(G)$ (i.e., MSTs, recalling the $G$ is complete) are edge-maximal 1-constructible graphs. A slightly weaker statement is true for all $k$: every graph $M_k(G)$ is $k$-constructible (Theorem 5.i), and every $k$-constructible graph is a subgraph of some graph $M_k(G)$ (Theorem 5.ii).

Note that a stronger statement, that the graphs of the form $M_k(G)$ are exactly the edge-maximal $k$-constructible graphs, is not true. To see this consider a cycle $C_4$ of length four. Assign weights $1, \ldots, 4$ to these four edges (in arbitrary order) and weights 5, 6 to the remaining edges of the complete graph on four vertices. It is easily checked that $M_2(G) = C_4$. But $M_2(G)$ is not edge-maximal, as a diagonal to the cycle $C_4$ can be added without destroying 2-constructibility.

**Theorem 5**

i) For every complete graph $G = (V, E)$ with distinct positive edge weights, $(V, M_k(G))$ is $k$-constructible.

ii) Let $G = (V, E)$ be $k$-constructible. Then there exist distinct positive edge weights for the complete graph $\bar{G} = (V, \bar{E})$ such that $E \subseteq M_k(\bar{G})$.
Proof. Part (i): Let \( G = (V, E) \) be a complete graph on \( n \) vertices with distinct positive edge weights. Let \( \langle e_1, e_2, \ldots, e_\binom{n}{2} \rangle \) be the ordering of the edges in \( E \) by increasing edge weights and \( O = \langle e_{r_1}, e_{r_2}, \ldots, e_{r_{\binom{n}{2}}(G)} \rangle \) be the ordering of the edges in \( M_k(G) \) by increasing edge weights. We will now show that \( O \) is a \( k \)-construction order for \( (V, M_k(G)) \). Let \( i \in \{1, 2, \ldots, |M_k(G)|\} \). As \( e_{r_i} \in M_k(G) \) there exists a set \( X \subseteq V \) with \( |X| = k - 1 \) and \( e_{r_i} \in \text{MST}(G \setminus X) \), implying that the two endpoints of \( e_{r_i} \) are not connected in the graph \( (V \setminus X, \{e_1, e_2, \ldots, e_{r_i-1}\}) \). By Menger’s theorem, this implies that there are at most \( k - 1 \) vertex-disjoint paths between the two endpoints of \( e_{r_i} \) in \( (V, \{e_1, e_2, \ldots, e_{r_i-1}\}) \). This statement remains thus true for the subgraph \( (V, \{e_{r_1}, e_{r_2}, \ldots, e_{r_{i-1}}\}) \). The ordering \( O \) is thus a \( k \)-construction order for \( (V, M_k(G)) \).

Part (ii): Conversely let \( G = (V, E) \) be a \( k \)-constructible graph with \( k \)-construction order \( O = \langle e_1, e_2, \ldots, e_{|E|} \rangle \). Let \( (V, \tilde{E}) \) be the complete graph on \( V \). We assign the following edge weights \( \tilde{w} \) to the edges in \( \tilde{E} \). We assign the weight 1 to \( e_1 \), 2 to \( e_2 \) and so on. The remaining edges \( \tilde{E} \setminus E \) get arbitrary distinct weights greater than \( |E| \). In order to show that the graph \( \tilde{G} = (V, \tilde{E}, \tilde{w}) \) satisfies \( E \subseteq M_k(G) \) consider an arbitrary edge \( e_i \in E \) and let \( C \subseteq V \) with \( |C| = k - 1 \) be a vertex set separating the two endpoints of \( e_i \) in the graph \( G_{i-1} = (V, \{e_1, e_2, \ldots, e_{i-1}\}) \). Applying Kruskal’s algorithm to \( \tilde{G}[V \setminus C] \), the set of all edges considered before \( e_i \) is contained in \( E(G_{i-1}) \), leaving the endpoints of \( e_i \) separated, so \( e_i \) will be accepted: \( e_i \in \text{MST}(\tilde{G}[V \setminus C]) \subseteq M_k(G) \).

We remark that the first part of the foregoing proof shows an efficient construction of \( M_k(G) \): follow a generalization of Kruskal’s algorithm, considering edges in order of increasing weight, adding an edge if (prior to addition) its endpoints are at most \( (k-1) \)-connected. Connectivity can be tested as a flow condition, so that the algorithm runs in polynomial time — far more efficient than the naive \( \Omega \left( \binom{n}{2} \right) \) protocol suggested by the definition of \( M_k(G) \). This again was already observed in [1].

By Theorem 5 the following theorem is equivalent to Theorem 1.

**Theorem 6**

For \( k \geq 1 \), every \( k \)-constructible graph \( G = (V, E) \) with \( n \geq k + 1 \) vertices satisfies

\[
|E| \leq nk - \binom{k+1}{2}.
\]

(2)

Theorem 6 generalizes a result of Mader [3], based on results in [2], concerning “\( k \)-minimal” graphs (edge-minimal \( k \)-connected graphs). Every \( k \)-minimal graph is \( k \)-constructible, since every order of its edges is a \( k \)-construction order. The following theorem is thus a corollary of Theorem 6.

**Theorem 7** (Mader’s theorem)

Every \( k \)-minimal graph with \( n \) vertices has at most \( nk - \binom{k+1}{2} \) edges.

Note that Mader’s theorem (Theorem 7) is weaker than Theorem 6 because while every \( k \)-minimal graph is \( k \)-constructible, the converse is false: not every \( k \)-constructible graph
is \( k \)-minimal. An example with \( k = 2 \) is a cycle \( C_4 \) with length four with an additional diagonal \( e \). The vertex set remains 2-connected even upon deletion of the edge \( e \), so the graph is not 2-minimal, but it is 2-constructible (by any order where \( e \) is not last).

3 Proof of the main theorem

In this section we prove Theorem 3. We fix \( k \) and prove the theorem by induction on \( n \). The theorem is trivially true for \( n = k+1 \), so assume that \( n \geq k+2 \) and that the theorem is true for all smaller values of \( n \). We prove (2) for a \( k \)-constructible graph \( G = (V, E) \) on \( n \) vertices and \( m \) edges which, without loss of generality, we may assume is edge-maximal (no edges may be added to \( G \) leaving it \( k \)-constructible). Fix a \( k \)-construction order

\[
O = \langle e_1, e_2, \ldots, e_m \rangle
\]

of \( G \) and (for any \( i \leq m \)) let \( G_i = (V, \{e_1, e_2, \ldots, e_i\}) \). Also fix a set \( C \subseteq V \) of size \( |C| = k - 1 \) such that the two endpoints of \( e_m \) lie in two different components \( Q^1, Q^2 \subseteq V \) of \( G_{m-1}[V \setminus C] \) (the set \( C \) exists by \( k \)-constructibility of \( G \) and Menger’s theorem). The edge maximality of \( G \) implies that \( Q^1, Q^2, C \) form a partition of \( V \). Let \( V^1 = Q^1 \cup C \) and \( V^2 = Q^2 \cup C \). (If there was a third component \( Q^3 \) then, even after adding \( e_m \), any \( v_1 \in Q^1 \) and \( v_3 \in Q^3 \) are at most \((k-1)\)-connected and so the edge \( \{v_1, v_3\} \) could be added, contradicting maximality.)

Our goal is to define two graphs \( G^1 = (V^1, E^1) \) and \( G^2 = (V^2, E^2) \) that satisfy the following property.

**Property 8**

- \( G^1 \) and \( G^2 \) are both \( k \)-constructible.
- \( E^1 \) contains all edges of \( G[V^1] \).
- \( E^2 \) contains all edges of \( G[V^2] \).
- For every pair of vertices \( c_1, c_2 \in C \) not connected by an edge in \( G \), there is an edge \( \{c_1, c_2\} \) in either \( E^1 \) or in \( E^2 \) (but not both).

If we can find graphs \( G^1 \) and \( G^2 \) satisfying Property 8 then the proof can be finished as follows. Note that we have the following equality:

\[
|E^1| + |E^2| = (m - 1) + |G[C]| + \left(\binom{k-1}{2} - |G[C]|\right).
\]

The term \( m - 1 \) comes from the fact that \( E^1 \cup E^2 \) covers all edges of \( G \) except \( e_m \), the term \( |G[C]| \) represents the double counting of edges contained in \( C \), and the last term counts the edges which are covered by \( E^1 \) and \( E^2 \) but not in \( G \).

We therefore have

\[
m = 1 + |E^1| + |E^2| - \binom{k-1}{2}.
\]
Applying the inductive hypothesis on \(G^1\) and \(G^2\) (which by Property \(\mathbb{S}\) are \(k\)-constructible) we get the desired result:

\[
m \leq 1 + \left(|V^1|k - \binom{k+1}{2}\right) + \left(|V^2|k - \binom{k+1}{2}\right) - \binom{k-1}{2}
\leq 1 + (n + k - 1)k - 2\binom{k+1}{2} - \binom{k-1}{2}
= nk - \binom{k+1}{2},
\]

where in the second inequality we have used \(|V_1| + |V_2| = n + |C| = n + k - 1\).

We will finally concentrate on finding \(G^1 = (V^1, E^1)\) and \(G^2 = (V^2, E^2)\) satisfying Property \(\mathbb{S}\).

Let \(B = \left(\binom{C}{2}\right) \setminus E\) be the set of all anti-edges in \(G[C]\). (\(\binom{C}{2}\) denotes the set of unordered pairs of elements of \(C\).) For \(\{c_1, c_2\} \in B\), let \(\ell(c_1, c_2)\) be the smallest value of \(i\) such that \(c_1\) and \(c_2\) are \(k\)-connected in \(G_i\). (Considering \(k\) vertex-disjoint paths between \(c_1\) and \(c_2\) in \(G_i\), and noting that deletion of the single edge \(e_i\) leaves them at least \(k - 1\) connected, it follows that \(c_1\) and \(c_2\) are precisely \((k - 1)\)-connected in \(G_{i-1}\).) Define \(B_i = \{\{c_1, c_2\} : \ell(c_1, c_2) = i\}\). Since by edge maximality of \(G\) every pair \(\{c_1, c_2\}\) is \(k\)-connected in \(G_m\), it follows that \(B_1, B_2, \ldots, B_m\) form a partition of \(B\).

Our basic strategy to define the graphs \(G^1\) and \(G^2\) (and appropriate orderings of their edges which prove that they are \(k\)-constructible) is as follows. In a particular way, we will partition each \(B_i\) as \(B^1_i \cup B^2_i\), and determine orders \(O^1_i\) and \(O^2_i\) on their respective edges. Let \(G^1\) be the graph constructed by the order

\[
O^1 = \langle e_1, O^1_1, e_2, O^1_2, \ldots, e_m, O^1_m \rangle, \tag{3}
\]

where (recalling that \(G^1\) has vertex set \(V^1\)) we ignore any edge \(e_i \notin \binom{V^1}{2}\). (There is no issue with edges from \(O^1_i\), as these belong to \(\binom{C}{2} \subseteq \binom{V^1}{2}\).) Define \(G^2\) symmetrically.

We need to show that the graphs \(G^1\) and \(G^2\) satisfy Property \(\mathbb{S}\) the central point will be to ensure that \(O^1\) is a \(k\)-construction order for \(G^1\), and \(O^2\) for \(G^2\). (By definition of the edges \(B_i\), note that every edge \(e \in O^1_i\) when added after \(e_i\) in the order \(O\) violates \(k\)-constructibility, but in the following we show how \(O^1_i, O^2_i\) can be chosen such that it will not violate \(k\)-constructibility in \(G^1\); likewise for edges \(e \in O^2_i\) and \(G^2\).)

To show that \(O^1\) and \(O^2\) are \(k\)-construction orders we need to check that, just before an edge is added, its endpoints are at most \((k - 1)\)-connected. To prove this, we distinguish between edges \(e_i \in E\) and edges \(e \in B\). We first dispense with the easier case of an edge \(e_i \in E\). Proposition \(\mathbb{S}\) shows that (for any orders \(O_i\) of \(B_i\)) in the edge sequence \(\langle e_1, O_1, \ldots, e_m, O_m \rangle\), every edge \(e_i\) has endpoints which are at most \((k - 1)\)-connected upon its addition to the graph \((V, \{e_1, O_1, \ldots, e_{i-1}, O_{i-1}\})\). It follows that the endpoints are also at most \((k - 1)\)-connected upon the edge’s addition to \(G^1\) (respectively, \(G^2\)), i.e., in the graph \((V^1, \{e_1, O^1_1, \ldots, e_{i-1}, O^1_{i-1}\})\), where as usual we disregard edges not in \(\binom{V}{2}\).
Proposition 9

Let \( i \in \{1, 2, \ldots, m\} \) and \( v_1, v_2 \in V \) such that \( \{v_1, v_2\} \) is not an edge in \( G_{i-1} \). If the maximum number of vertex-disjoint paths between \( v_1 \) and \( v_2 \) in \( G_{i-1} \) is \( r \leq k - 1 \), then the maximum number of vertex-disjoint paths between \( v_1 \) and \( v_2 \) in the graph \((V, \{e_1, e_2, \ldots, e_{i-1}\} \cup \bigcup_{l=1}^{i-1} B_l)\) is \( r \), too.

Proof. For any \( i, v_1, v_2 \) as above, let \( S \subseteq V, |S| = r \), be a set separating \( v_1 \) and \( v_2 \) in \( G_{i-1} \). As \( |S| = r < k \), \( S \) cannot separate two \( k \)-connected vertices in \( G_{i-1} \). This implies that any two vertices in \( V \setminus S \) that are \( k \)-connected in \( G_{i-1} \) lie in the same connected component of \( G_{i-1}[V \setminus S] \). As every edge in \( \bigcup_{l=1}^{i-1} B_l \) connects two vertices that are \( k \)-connected in \( G_{i-1} \), adding the edges \( \bigcup_{l=1}^{i-1} B_l \) to \( G_{i-1}[V \setminus S] \) does not change the component structure of \( G_{i-1}[V \setminus S] \). The set \( S \) thus remains a separating set for \( v_1 \) and \( v_2 \) in the graph \((V, \{e_1, e_2, \ldots, e_{i-1}\} \cup \bigcup_{l=1}^{i-1} B_l)\), proving that \( v_1 \) and \( v_2 \) are at most \( r \)-connected in this graph.

\( \square \) Proposition 9

With Proposition 9 addressing edges \( e_i \in E \), to ensure \( k \)-constructibility of \( O_1 \) and \( O_2 \), it suffices to choose for \( j \in \{1, 2\} \) and \( i \in \{1, 2, \ldots, m\} \) the orders \( O_{ij} \) in such a way that successively adding any edge \( e \in O_{ij} \) to the graph \( G_i[V] \) connects two vertices which were at most \((k-1)\)-connected.

Let \( C_i \subseteq V \) with \( |C_i| = k - 1 \) a set separating the endpoints of \( e_i \) in the graph \( G_{i-1} \). Let \( U, W \subseteq V \) be the two components of \( G_{i-1}[V \setminus C_i] \) containing the two endpoints of the edge \( e_i \). We define \( C^U = C \cap U, C^W = C \cap W \). Figure 1 illustrates these sets.

![Figure 1](image)

Figure 1: Sets defined to prove Propositions 10-12

The following proposition shows that the edges \( B_i \) form a bipartite graph.

Proposition 10

\[ B_i \subseteq C^U \times C^W \]
Proof. Suppose by way of contradiction that \( \exists e \in B_i \setminus (C^U \times C^W) \). Let
\[
O' = \langle e_1, \ldots, e_{i-1}, e, e_i, \ldots, e_m \rangle,
\]
the edge order obtained by inserting \( e \) immediately before \( e_i \) in the original order \( O = \langle e_1, e_2, \ldots, e_m \rangle \). We will show that \( O' \) is a \( k \)-construction order, thus contradicting the edge maximality of \( G \). For edges up to \( e_{i-1} \) this is immediate from the fact that \( O \) is a \( k \)-construction order. Proposition \([\text{II}] \) shows that edges \( e_{i+1} \) and later do not violate \( k \)-constructibility. (Literally, Proposition \([\text{II}] \) applies to the order \( \langle e_1, \ldots, e_i, e, e_{i+1}, \ldots, e_m \rangle \) rather than to \( O' \), but for edges \( e_{i+1} \) and later the swap of \( e_i \) and \( e \) is irrelevant.) The edge \( e \) itself does not violate \( k \)-constructibility, since by the definition of \( B_i \) its two endpoints are at most \( k-1 \) connected in \( G_{i-1} \). This leaves only edge \( e_i \) to check, but since \( e \notin U \times W \), \( C_i \) remains a separating set with cardinality \( k-1 \) for the two endpoints of \( e_i \) in the graph \( (V, \{e_1, e_2, \ldots, e_{i-1}, e\}) \). Thus \( O' \) is a \( k \)-construction order, giving the desired contradiction. \( \square \)

We will now describe a method for constructing the orders \( O^1_i, O^2_i \). Our approach is to define an order \( L = \langle v_1, v_2, \ldots, v_p \rangle \) on (a subset of) the vertices of \( C^U \cup C^W \) and to assign to every vertex \( v \in C^U \cup C^W \) a label \( \alpha(v) \in \{1, 2\} \). The two orders \( O^1_i, O^2_i \) are then defined as follows. We begin with \( O^1_i, O^2_i = \emptyset \) and add all edges in \( B_i \) which are incident to \( v_1 \) at the end of \( O^\alpha(v_1) \) in any order. In the next step all edges of \( B_i \) which are incident to \( v_2 \) and not already assigned to one of the orders \( O^1_i, O^2_i \) are added at the end of \( O^\alpha(v_2) \) in any order. This is repeated until all edges are assigned.

In what follows we show how to choose a vertex order \( L \) and labels \( \alpha \) so that \( O^1 \) and \( O^2 \) are \( k \)-construction orders. Just as \( O^1 \) and \( O^2 \) are built iteratively, so is \( L \), starting with \( L = \emptyset \).

For any \( X \subseteq C^U \cup C^W \), we define \( B_i(X) \) to be the set of edges in \( B_i \) incident on vertices in \( X \), i.e., \( B_i(X) = \{e \in B_i \mid e \cap X \neq \emptyset \} \).

**Proposition 11**

Let \( j \in \{1, 2\} \) and \( X \subseteq C^U \cup C^W \). We then have that \( \forall e \in B_i \setminus B_i(X) \) there are at most \( |C_i \cap V^j| + |X| \) vertex-disjoint paths between the two endpoints of \( e \) in the graph \( (V, \{e_1, e_2, \ldots, e_i \} \cup B_i(X))[V^j] \).

**Proof.** Observe that the set \( (C_i \cap V^j) \cup X \) separates the two endpoints of the edge \( e \) in the graph \( (V, \{e_1, e_2, \ldots, e_i \} \cup B_i(X))[V^j] \). As this set has cardinality \( |C_i \cap V^j| + |X| \) the result follows by Menger’s theorem. \( \square \)

Let \( X^1 \) be the set of vertices labeled 1 contained in the partially constructed \( L \), and \( X^2 \) those labeled 2. If we can find a vertex \( v \in (C^U \cup C^W) \setminus (X^1 \cup X^2) \) where the number of “new” edges incident on \( v \) satisfies
\[
|B_i(v) \setminus (B_i(X^1 \cup X^2))| \leq k - 1 - \min\{|C_i \cap V^1| + |X^1|, |C_i \cap V^2| + |X^2|\} \tag{4}
\]
then by Proposition 11, adding $v$ at the end of the current order $L$ and labeling it $\arg \min_{j \in \{1, 2\}} \{|C_i \cap V^j| + |X^j|\}$ does not violate $k$-constructibility of the orders $O^1$ and $O^2$.

The following proposition shows that, until the process is complete (until $B_i(X^1 \cup X^2) = B_i$), such a vertex $v$ can always be found.

**Proposition 12**

Let $X^1, X^2 \subseteq C^U \cup C^W$ be two disjoint sets. If $B_i(X^1 \cup X^2) \subsetneq B_i$, then there exists a vertex $v \in (C^U \cup C^W) \setminus (X^1 \cup X^2)$ that satisfies (7).

**Proof.** Note that $C^U$, $C^W$, and $C \cap C_i$ are disjoint and contained in $C$, so

$$|C^U| + |C^W| + |C \cap C_i| \leq |C| = k - 1,$$

where $|C| = k - 1$ by definition. Also,

$$|V^1 \cap C_i| + |V^2 \cap C_i| - |C \cap C_i| = |C_i| = k - 1.$$

From the fact that the right side of (5) is equal to $2(k - 1)$ minus that of (6), we get

$$|C^U| + |C^W| \leq (k - 1 - |V^1 \cap C_i|) + (k - 1 - |V^2 \cap C_i|).$$

By disjointness of $C^U$ and $C^W$,

$$|C^U \setminus (X^1 \cup X^2)| + |C^W \setminus (X^1 \cup X^2)|$$

$$= |C^U| + |C^W| - |X^1| - |X^2|$$

$$\leq (k - 1 - |V^1 \cap C_i| - |X^1|) + (k - 1 - |V^2 \cap C_i| - |X^2|),$$

using (7) in the last inequality. Thus, the smaller summand in (8) is at most the larger summand in (9), and without loss of generality we suppose that

$$|C^U \setminus (X^1 \cup X^2)| \leq k - 1 - |V^1 \cap C_i| - |X^1|.$$  

By the hypothesis $B_i(X^1 \cup X^2) \subsetneq B_i$, there is an edge $e \in B_i \setminus B_i(X^1 \cup X^2)$; by Proposition 10, $e = \{u, w\}$ with $u \in C^U$ and $w \in C^W$; and by definition of $B_i(X^1 \cup X^2)$, $u, w \notin X^1 \cup X^2$, i.e., $u \in C^U \setminus (X^1 \cup X^2)$ and $w \in C^W \setminus (X^1 \cup X^2)$. Then $v = w$ satisfies (4) because the new edges on $w$ must go to so-far-unused vertices in $C^U$:

$$|B_i(w) \setminus B_i(X^1 \cup X^2)| \leq |C^U \setminus (X^1 \cup X^2)|,$$

whence (10) closes the argument.

Therefore there always exist two $k$-construction orders $O^1, O^2$ as desired, which completes the proof of Theorem 6.

9
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