NOTES ON THE SCHWARZIAN TENSOR AND MEASURED
FOLIATIONS AT INFINITY OF QUASIFUCHSIAN MANIFOLDS

JEAN-MARC SCHLENKER

Abstract. The boundary at infinity of a quasifuchsian hyperbolic manifold is equipped with a holomorphic quadratic differential. Its horizontal measured foliation $f$ can be interpreted as the natural analog of the measured bending lamination on the boundary of the convex core. This analogy leads to a number of questions. We provide a variation formula for the renormalized volume in terms of the extremal length $\text{ext}(f)$ of $f$, and an upper bound on $\text{ext}(f)$.

We then describe two extensions of the holomorphic quadratic differential at infinity, both valid in higher dimensions. One is in terms of Poincaré-Einstein metrics, the other (specifically for conformally flat structures) of the second fundamental form of a hypersurface in a “constant curvature” space with a degenerate metric, interpreted as the space of horospheres in hyperbolic space. This clarifies a relation between linear Weingarten surfaces in hyperbolic manifolds and Monge-Ampère equations.

Contents

1. Introduction 2
1.1. The measured foliation at infinity 2
1.2. A variational formula for the renormalized volume 2
1.3. From the boundary of the convex core to the boundary at infinity 3
1.4. Surfaces associated to metrics at infinity 3
1.5. Geometric structures on a hyperbolic end 3
1.6. Main relations 4
1.7. Linear Weingarten surfaces 4
1.8. The Thurston metric at infinity 4
1.9. Content 5

Acknowledgements 8

2. Background material 8
2.1. The Schwarzian derivative 8
2.2. The Schwarzian tensor 9
2.3. Complex projective structures 9
2.4. The holomorphic quadratic differential at infinity of quasifuchsian manifolds 10
2.5. Hyperbolic ends 10
2.6. Poincaré-Einstein manifolds 11
2.7. Hypersurfaces 11
2.8. The space of horospheres 12

3. The measured foliation at infinity 12
3.1. The Fischer-Tromba metric 12
3.2. The energy of harmonic maps and the Gardiner formula 13
3.3. Extremal lengths of measured foliations 13
3.4. The renormalized volume of quasifuchsian manifolds 13
3.5. The measured foliation at infinity 14

Date: v1, March 30, 2018.
1. Introduction

1.1. The measured foliation at infinity. Consider a quasifuchsian manifold $M$ homeomorphic to $S \times \mathbb{R}$, where $S$ is a closed oriented surface of genus at least 2. We call $\mathcal{T}_S$ the Teichmüller space of $S$, $\mathcal{ML}_S$ the space of measured laminations on $S$, and $Q_S$ the space of holomorphic quadratic differential on $S$, which can be considered as a bundle over $\mathcal{T}_S$ with fibre $Q_c$ over $c \in \mathcal{T}_S$. We denote by $\mathbb{CP}^1_S$ the space of complex projective structures on $S$, which can through the Schwarzian derivative be considered as an affine bundle over $\mathcal{T}_S$ with fiber $Q_c$ over $c \in \mathcal{T}_S$ (see §2.1).

We also denote by $\mathcal{T}_{\partial M}, \mathcal{ML}_{\partial M}$, etc, the corresponding notions but on $\partial M$ rather than on $S$. If $M$ is a quasifuchsian manifold homeomorphic to $S \times \mathbb{R}$ then $\partial M$ is the disjoint union of two copies of $S$, which we denote by $\partial^- M$ and $\partial^+ M$, one with the opposite orientation.

Recall that the boundary at infinity of $M$, $\partial_\infty M$, can be identified with the quotient by the action of $\pi_1(M) = \pi_1(S)$ of the domain of discontinuity of $M$:

$$\partial_\infty M = \Omega_\rho / \rho(\pi_1(S)) = (\partial_\infty H^3 \setminus \Lambda_\rho) / \rho(\pi_1(S)).$$

Here $\rho : \pi_1(S) \to \text{Isom}(H^3)$ is the holonomy representation of $M$, and $\Lambda_\rho \subset \partial_\infty H^3$ is its limit set.

Since $\rho$ acts on $\partial_\infty H^3$ by complex projective transformations, $\partial_\infty M$ is endowed with a $\mathbb{CP}^1$-structure $\sigma \in \mathcal{CP}_{\partial M}$. Denote by $c \in \mathcal{T}_{\partial M}$ the underlying complex structure, and by $\sigma(c)$ the complex projective structure obtained by applying to $(\partial M, c)$ the Uniformization Theorem. The Schwarzian derivative of the holomorphic map isotopic to the identity between $(\partial M, \sigma(c))$ and $(\partial M, \sigma)$ is a holomorphic quadratic differential $-q \in Q_c$ (see §2.1).

We will consider a naturally defined measured foliation $f$ at infinity on $\partial_\infty M$. In the point of view developed here, $f$ is an analog at infinity of the measured bending lamination on the boundary of the convex core $C(M)$ of $M$.

**Definition 1.1.** The foliation at infinity of $M$, denoted by $f \in \mathcal{MF}$, is the horizontal foliation of the holomorphic quadratic differential $q$ of $M$.

1.2. A variational formula for the renormalized volume. We consider here the renormalized volume of quasifuchsian hyperbolic manifolds, see §3.4. There is a simple variational formula for the renormalized volume, in terms of $q$ and of the variation of the conformal structure at infinity, Equation 16 below. Here we write this variational formula in another way, involving the measured foliation at infinity.
Theorem 1.2. In a first-order variation of $M$, we have

\begin{equation}
\dot{V}_R = -\frac{1}{2}(d\text{ext}(f))(\dot{c}).
\end{equation}

Here $\text{ext}(f)$ is the extremal length of $f$, considered as a function over the Teichmüller space of the boundary $\mathcal{T}_{\partial M}$. The right-hand side is the differential of this function, evaluated on the first-order variation of the complex structure on the boundary.

Equation (1) is remarkably similar to the dual Bonahon-Schläfli formula. The dual volume of the convex core of $M$ is defined as

\[ V^*_C(M) = V_C(M) - \frac{1}{2}L_m(l), \]

where $m$ and $l$ are the induced metric and measured bending lamination on the boundary of the convex core of $M$. The dual Bonahon-Schläfli formula is then:

\[ \dot{V}^*_C = -\frac{1}{2}(dL(l))(\dot{m}). \]

This statement, taken from [29], is a consequence of the Bonahon-Schläfli formula, which is a variational formula for the (non-dual) volume of the convex core of $M$, see [4, 3].

1.3. From the boundary of the convex core to the boundary at infinity. Theorem 1.2, and its analogy to the dual Bonahon-Schläfli formula, suggests an analogy between the properties of quasifuchsian manifolds considered from the boundary of the convex core and from the boundary at infinity. For instance, on the boundary of the convex core, we have the following upper bound on the length of the bending lamination, see [6, Theorem 2.16].

Theorem 1.3 (Bridgeman, Brock, Bromberg). $L_{m}(l) \leq 6\pi|\chi(S)|$.

Similarly, on the boundary at infinity, we have the following result, proved in §3.7.

Theorem 1.4. $\text{ext}_{c}(f) \leq 3\pi|\chi(S)|$.

| On the convex core | At infinity |
|-------------------|-------------|
| Induced metric $m$ | Conformal structure at infinity $c$ |
| Thurston’s conjecture on prescribing $m$ | Bers’ Simultaneous Uniformization Theorem |
| Measured bending lamination $l$ | measured foliation $f$ |
| Hyperbolic length of $l$ for $m$ | Extremal length of $f$ for $c$ |
| Volume of the convex core $V_C$ | Renormalized volume $V_R$ |
| Dual Bonahon-Schläfli formula | Theorem 1.2 |
| $\dot{V}^*_C = -\frac{1}{2}(dL(l))(\dot{m})$ | $\dot{V}_R = -\frac{1}{2}(d\text{ext}(f))(\dot{c})$ |
| Bound on $L_m(l)$ [8] [9] | Theorem 1.4 |
| $L_{m}(l) \leq 6\pi|\chi(S)|$ | $\text{ext}_{c}(f) \leq 3\pi|\chi(S)|$ |
| Brock’s upper bound on $V_C$ [9] | Upper bound on $V_R$ [34] |

Table 1. Infinity vs the boundary of the convex core.

This analogy, briefly described in Table 1 suggests a number of questions (see 3.8) since it could be expected that, at least up to some point, phenomena known to hold on the boundary of the convex core might hold also on the boundary at infinity, and conversely.

Another series of questions arises from comparing the data on the boundary of the convex core to the corresponding data on the boundary at infinity. For instance,
it is well known that $m$ is uniformly quasi-conformal to $c$ (see [16, 17]), and one can ask whether similar statements hold for other quantities. We do not expand on those questions here.

1.4. Surfaces associated to metrics at infinity. We now consider another point of view on the Schwarzian derivative at infinity.

Let $\Omega \subset \partial_\infty H^3$ be an open domain, and let $h$ be a Riemannian metric on $\Omega$ compatible with the conformal structure of $\partial_\infty H^3$. We can associate to $h$ two distinct but related surfaces, each immersed in a 3-dimensional manifold.

1. (C. Epstein [14, 15] defined from $h$ (a non-smooth) surface $S_h \subset H^3$, which can be defined as the envelope of a family of horospheres associated to $h$ at each point of $\Omega$.)

2. One can associate to $h$ a smooth surface $S_h^*$ in a the space of horospheres of $H^3$, see [33]. This surface $S^*_h$ is dual (see [28]) to the Epstein surface $S_h$. The space of horospheres, denoted by $C^3_+$ below, has a degenerate metric but a rich geometric structure, and $S^*_h$ is equipped with an induced metric, $I^*_c$, and a "second fundamental form" $\Pi^*_c$. They satisfy the Codazzi equation, $d^2 \Pi^*_c = 0$, and a modified form of the Gauss equation, $\text{tr} I^*_c \Pi^*_c = -1 - K_{I^*_c}$. There is a natural embedding $\phi_h$ of $\Omega$ in $C^3_+$ with image $S^*_h$. The pull-back $\phi_h^* I^*_c$ is equal to $h$, while $\phi_h^* \Pi^*_c$ is a bilinear symmetric tensor field on $\Omega$ naturally associated to $h = I^*_c$.

The second geometric data, given by $I^*_c$ and $\Pi^*_c$, is perhaps less obvious than the first. However it is also quite natural and, as we will see below, it is an efficient tool in relating (1) to (3), (4) and (4') below.

1.5. Geometric structures on a hyperbolic end. Consider now a hyperbolic end $E$, for instance an end of a quasifuchsian or convex co-compact hyperbolic 3-manifold (the notion of hyperbolic end is recalled in [28]). We are interested here in three geometric structures that occur quite naturally on the boundary at infinity $\partial_\infty E$ of $E$. They are related to (1) and (2) above when $\Omega$ is the universal cover of the boundary at infinity $\partial_\infty E$ and $h$ is invariant under the action of $\pi_1 E$ on $\Omega$.

3. Extending to hyperbolic ends the construction made in [14], $\partial_\infty E$ is equipped with a complex structure $c$, with a complex projective structure $\sigma$, and with a holomorphic quadratic differential $q$, defined as the Schwarzian derivative of the holomorphic map isotopic to the identity between $(\partial_\infty E, \sigma)$ and $(\partial_\infty E, \sigma_F)$, where $\sigma_F$ is the Fuchsian complex projective structure associated to $c$.

4. Given any metric $I^*$ in the conformal class at infinity of $\partial_\infty E$, there is a section $\Pi^*$ if the bundle of bilinear symmetric forms on $T \partial_\infty E$. In [28], $I^*$ and $\Pi^*$ are defined in terms of equidistant foliations of a neighborhood of infinity in $E$: given $I^*$, there is a unique foliation such that the hyperbolic metric can be written, in a neighborhood of infinity, as

\[ dr^2 + \frac{1}{2}(e^{2r} I^* + 2 \Pi^* + e^{-2r} \mathcal{I}^*) \]

$I^*$ and $\Pi^*$ are called the induced metric and second fundamental form at infinity of $E$, since they satisfy the Codazzi equation, $d^2 \Pi^* = 0$, and a modified version of the Gauss equation for surfaces in 3-dimensional space-forms: $\text{tr} I^* \Pi^* = -K_{I^*}$. $I^*$ and $\Pi^*$ completely characterize $E$.

4' The hyperbolic metric on $E$ can be written as

\[ dx^2 + h(x) \]
where \((h_x)_{x \in (0, \epsilon)}\) is a one-parameter family of metrics on \(\partial_\infty E\). Moreover \(h_x\) can be written as

\[
h_x = h_0 + h_2 x^2 + h_4 x^4.
\]

The metric \(h_0\) is always in the conformal class on \(\partial_\infty E\) determined by the complex structure \(c\). Conversely, any such metric \(h_0\) is obtained in a unique way. The bilinear form \(h_4\) depends on \(h_0\) and \(h_2\) in a simple way (see [2.6]), so the geometry of \(E\) is encoded solely in \(h_0\) and \(h_2\).

There are some well-known relations between the geometric structures above. First, (4) and (4') are related in a particularly simple way. Given \(h_0\), it defines a unique equidistant foliation near infinity such that \([3]\) and \([4]\) hold. If both \((h_0, h_2)\) and \((I^*, \mathcal{II}^*)\) are determined by the same equidistant foliation, they are related by:

**Proposition 1.5.** \(I^* = 2h_0, \mathcal{II}^* = h_2\).

The proof is a direct consequence of the definition of \(h_0\) and \(h_2\) in [5] and [4], and of \(I^*\) and \(\mathcal{II}^*\) in [2]. The geometric quantities (1)–(4') defined above extend, to various extents, in higher dimension. In particular:

- (1) and (2) extend to higher dimensions, with \(\Omega \subset \partial_\infty H^{d+1}\), for \(d \geq 2\).
- (3) extends (in a way) to the situation where \(\partial_\infty E\) is replaced by any conformally flat metric, for instance a Riemannian metric in the conformal class at infinity of a hyperbolic end in dimension \(d + 1\). The Schwarzian derivative is then replaced by the Schwarzian tensor, defined in [2.22].
- (4) extends to hyperbolic ends in higher dimension.
- (4') extends to the setting where \(E\) is replaced by an end of a Poincaré-Einstein manifold, as recalled in [2.6].

### 1.6. Main relations.

We will show that the geometric structures (1)-(4') above are strongly related, in particular when \(h_0\) is hyperbolic. We also intend to clarify the notions of “induced metric” and “second fundamental forms” at infinity, denoted by \(I^*\) and \(\mathcal{II}^*\) here, and the corresponding notions for surfaces in \(C_+^3\), denoted by \(I^*_c\) and \(\mathcal{II}^*_c\) here. (The index \(c\) is not present in [33] but is introduced here to limit ambiguities.) Those relations lead to a simple conformal transformation rule for \(\mathcal{II}^*\), Theorem 1.6, and Corollary 1.7 below, which in turn provides a potentially useful relation between special surfaces in \(H^3\) and Monge-Ampère equations on surfaces.

We now consider a hyperbolic end \(E\), along with a metric \(I^*\) in the conformal class at infinity. This metric \(I^*\) determines an equidistant foliation of \(E\) near infinity by surfaces \((S_t)_{t \geq \epsilon}\), which in turns determines a metric \(h_0\) and a field of bilinear symmetric forms \(h_2\) on \(\partial_\infty E\).

The following can be found e.g. in [28, Lemma 8.3], but we will provide here a much simpler proof.

**Theorem 1.6.** Suppose that \(I^*\) is the hyperbolic metric in the conformal class on \(\partial_\infty E\). Then the traceless part \(\mathcal{II}^*_0\) of \(\mathcal{II}^*\) is equal to \(\mathcal{II}^*_0 = \text{Re}(q)\).

The proof can be found in [14.2].

Together with Proposition 1.5, we obtain the following direct consequence.

**Corollary 1.7.** Suppose again that \(I^*\) is the hyperbolic metric at infinity of \(E\). Then \(2h_0 = I^*\), while \(h_{2,0} = \text{Re}(q)\).

The relation between the data \(I^*, \mathcal{II}^*\) at infinity and the description by the induced metric and second fundamental form of the dual surface in \(C_+^3\) is quite simple.

**Theorem 1.8.** \(I^*_c = 2I^*, \text{ while } \mathcal{II}^*_c = \mathcal{II}^* + I^*\).
The proof is in \[\text{(4.1)}\]

A key tool in the paper is a simple variational formula for \(h_2\) under a conformal deformation of \(h_0\). We state here directly in the setting of Poincaré-Einstein manifolds. Here \(B\) is the Schwarzian tensor of Osgood and Stowe \[\text{(32)},\] a generalization of the Schwarzian derivative recalled in \[\text{(2.2)}\].

**Theorem 1.9.** Let \((M,g)\) be a \(d+1\)-dimensional Poincaré-Einstein manifold, \(d \geq 2\), and let \(h_0\) and \(h'_0 = e^{2u} h_0\) be two metrics in the conformal class at infinity on \(\partial M\). Let \((h_x)_{x > 0}\) and \((h'_x)_{x > 0}\) be the one-parameter families of metrics on \(\partial M\) determined by \(h_0\) (see above) and let \(h_2\) and \(h'_2\) be the second terms in the asymptotic developments of \((h_x)_{x > 0}\) and \((h'_x)_{x > 0}\). Then:

\[
h'_2 = h_2 + \text{Hess}(u) - du \otimes du + \frac{1}{2} \|du\|^2_{h_0} h_0 .
\]

As a consequence, the traceless part of \(h_2\) and \(h'_2\) are related by:

\[
h^2_{2,0} = h_{2,0} + B(h_0, h'_0) .
\]

The proof can be found in \[\text{(5)}\]. For \(d \geq 3\) it is a direct consequence of an explicit relation between \(h_2\) and \(h_0\), while for \(d = 2\) it uses the relation with surfaces in the space of horospheres.

As a consequence, we can describe \(\mathcal{B}\) or \(\mathcal{B}^e\) when \(I^*\) is any metric in the conformal class at infinity of \(E\). We will see some interesting examples below. To simplify notations, we use the following notation. If \(h\) is a Riemannian metric on a surface \(S\) and \(u : S \to \mathbb{R}\) is a smooth function, then

\[
\mathcal{B}(h, e^{2u} h) = \text{Hess}_h (u) - du \otimes du + \frac{1}{2} \|du\|_{h}^2 .
\]

Note that \(\mathcal{B}\) is a kind of non trace-free version of the Schwarzian tensor, and \(B(h, e^{2u} h)\) is the traceless part of \(\mathcal{B}(h, e^{2u} h)\).

**Corollary 1.10.** Suppose that \(\hat{I}^* = e^{2u} I^*\), where \(I^*\) is a metric in the conformal class at infinity \(c\) of \(E\). Then \(\hat{I}^*_c = e^{2u} I^*_c\), while

\[
\mathcal{B}^e = \mathcal{B}^e + \mathcal{B}(I^*, \hat{I}^*) ,
\]

\[
\mathcal{B}^e = \mathcal{B}^e + \mathcal{B}(I^*_c, \hat{I}^*_c) + \frac{1}{2} (I^*_c - I^*_c) .
\]

Those relations extend without change to conformally flat metrics in higher dimension, we do not elaborate on this point here.

1.7. **Linear Weingarten surfaces.** We consider linear Weingarten surfaces in \(H^3\), or in hyperbolic 3-manifolds, defined as a smooth surface \(S\) satisfying an equation of the form

\[
a K_c + b H + c = 0 ,
\]

where \(a, b, c \in \mathbb{R}\) are constants. Here \(H = \text{tr} (B)/2\) is the mean curvature of \(S\), where \(B\) is its shape operator, while \(K_c = \det(B)\) is its extrinsic curvature, related to the Gauss curvature \(K\) by the Gauss equation, \(K = -1 + K_c\).

We are particularly interested in some well-behaved surfaces that play a particular role in some situations, in particular when studying quasifuchsian 3-manifolds. We will say that a smooth hypersurface \(S \subset H^{d+1}\) is *horospherically tame*, or h-tame for short, if its principal curvatures are everywhere in \((-1,1)\). Note that the hyperbolic Gauss map of a complete h-tame surface in \(H^{d+1}\) is injective, so that it defines a data at infinity \((I^*, \mathcal{B}^e)\) on an open domain \(\Omega \subset \partial_{\infty} H^3\), see \[\text{(2.7)}\].
Proposition 1.11. If an oriented hypersurface \( S \subset H^3 \) is \( h \)-tame then the corresponding second fundamental form at infinity \( \mathcal{I}^* \) is positive definite. Any admissible pair \((\mathcal{I}^*, \mathcal{K}^*)\) in an open subset \( \Omega \subset \partial \infty H^3 \) with \( \mathcal{I}^* \) positive definite determines a smooth \( h \)-tame surface.

The definition of an “admissible pair” is given in \([2,5]\).

Proposition 1.12. Let \( S \) be a \( h \)-tame surface in \( H^3 \), and let \( \mathcal{I}^*, \mathcal{K}^* \) be the corresponding data at infinity. Suppose that \( a - b + c \neq 0 \). Then \( S \) satisfies \((7)\) if and only if the data at infinity satisfies the relation

\[
(8) \quad \det((a - b + c)\mathcal{I}^* + (c - a)\mathcal{K}^*) = b^2 - 4ac .
\]

Here \( \mathcal{I}^* \) is the “shape operator at infinity”, the unique bundle morphism self-adjoint for \( \mathcal{I}^* \) such that \( \mathcal{I}^* = \mathcal{I}^*(\mathcal{I}^*, \cdot) \), and \( \mathcal{K}^* \) is the identity.

Note that the case when \( a - b + c = 0 \) (or \( a + b + c = 0 \), after changing the orientation) corresponds to the case treated e.g. in \([20]\).

Together with \((5)\), this leads to the following characterization of linear Weingarten surfaces in terms of solutions of Monge-Ampère equations.

Proposition 1.13. Let \((\mathcal{I}^*, \mathcal{K}^*)\) be an admissible pair defined on an open domain \( \Omega \subset \partial \infty H^3 \), and let \( u : \Omega \to \mathbb{R} \). The surface defined by the metric at infinity \( e^{2u} \mathcal{I}^* \) is \( h \)-tame and satisfies \((7)\) if and only if:

1. \( \mathcal{I}^* + \mathcal{K}(\mathcal{I}^*, \mathcal{I}^*) \) is positive definite,
2. \( u \) satisfies the Monge-Ampère equation

\[
(9) \quad \det((a - b + c)(\mathcal{I}^* + \text{Hess}(u) - du \otimes du + \frac{1}{2} \|du\|^2_\mathcal{I}^*) + (c - a)e^{2u} \mathcal{I}^*) = (b^2 - 4ac)e^{4u} .
\]

Equation \((9)\) can be written as

\[
(10) \quad \det((a - b + c)(\mathcal{I}^* + \text{Hess}^2(u) - du \otimes D u + \frac{1}{2} \|du\|^2_\mathcal{I}^*) + (c - a)e^{2u} \mathcal{I}^*) = (b^2 - 4ac)e^{4u} ,
\]

where \( Du \) is the gradient of \( u \) for \( \mathcal{I}^* \) and \( \text{Hess}^2(u) = DDu \) is the Hessian of \( u \) considered as a 1-form with values in the tangent space of \( \Omega \).

The behavior of those linear Weingarten surfaces will likely be simpler if the following two conditions are satisfied:

- \( b^2 - 4ac > 0 \), since \((10)\) is then of elliptic type,
- \( (c - a)(a - b + c) \leq 0 \), since the elliptic solutions of \((10)\) will then always satisfy the first condition in Proposition 1.13.

We now outline three interesting special cases.

1.7.1. Minimal surfaces. We can take \( a = 0, b = 1, c = 0 \). In this case \( b^2 - 4ac > 0 \) and \( (c - a)(a - b + c) = 0 \), and \((9)\) becomes simply \( \det(\mathcal{I}^*) = 1 \). Therefore \((10)\) is simply:

\[
\det(\mathcal{I}^* + \text{Hess}^2(u) - du \otimes D u + \frac{1}{2} \|du\|^2_\mathcal{I}^*) = e^{4u} .
\]

1.7.2. CMC-1 surfaces. Here we can take \( a = 0, b = c = 1 \) (this corresponds to changing the orientation of the surface). Then \( a - b + c = 0 \), so Proposition 1.13 cannot directly be used. However following the same computations as in \([6,2]\) shows that \((9)\) becomes simply

\[
\text{tr}(\mathcal{I}^*) + 2 = 0 .
\]

As a consequence, \((10)\) is quasilinear, a fact that is not surprising since those surfaces are related to minimal surfaces in Euclidean space \([10]\) and have a Weierstrass representation.
1.7.3. Convex constant Gauss curvature surfaces. This is the case of surfaces of constant curvature $K_e = k \in (0, 1)$. We can then take $a = 1, b = 0, c = -k \in (-1, 0)$. Then $a + b + c \neq 0, b^2 - 4ac > 0,$ and $(c - a)(a + b + c) \leq 0.$

1.8. The Thurston metric at infinity. We outline here another special case of the relations described above, that was also a motivation for writing these notes. It is based on the work of Dumas [12].

Let $E$ be a hyperbolic end and let $S$ be the convex pleated surface which is the non-ideal boundary of $E$. The data at infinity $(I^*, II^*)$ corresponding to $S$ is quite interesting: $I^*$ is the “Thurston metric” on $\partial_\infty E$ associated to the pleated surface $S$, while $II^*$ has rank at most 1 at each point, and determines a measured lamination on $\partial_\infty E$. It is zero on the subset of points projecting to the totally geodesic part of the boundary of the convex core, and, on regions projecting to the support of the measured bending lamination, it is zero in directions of the lamination.

We can also consider on $\partial_\infty E$ the hyperbolic metric $\bar{I}^*$, and the corresponding second fundamental form at infinity $\bar{II}^*$. Then

$$\bar{I}^* = e^{2u}I^*,\tag{11}$$

where $u: \partial_\infty E \to \mathbb{R}$ is the solution of the equation

$$\Delta u = -K - e^{2u},$$

where $K$ is the curvature of the Thurston metric $I^*$, which takes values in $(-1,0)$. Moreover,

$$\bar{II}^* = II^* + B(I^*, \bar{I}^*).$$

Taking the trace-free part of this relation leads precisely to [12, Theorem 7.1].

A bound on solutions of (11) can then lead to a bound on the difference between $II^*$ and $\bar{II}^*$, as done (for the trace-free components) in [12, Theorem 11.4]. We do not elaborate more in this direction here.

1.9. Content. Section 2 contains background material used in the rest of the paper. Section 3 then contains details on the measured foliation at infinity and the proof of Theorems 1.2 and 1.4. Section 4 focuses on hypersurfaces in the space of horospheres and contains the proofs of Theorems 1.6 and 1.8. Finally, Section 5 presents the proof of Theorem 1.9 and Section 6 gives some details on the application to linear Weingarten surfaces.

Acknowledgements. I am grateful to Sergiu Moroianu for helpful remarks.

2. Background material

2.1. The Schwarzian derivative. Let $\Omega \subset \mathbb{C}$, and let $f: \Omega \to \mathbb{C}$ be holomorphic. The Schwarzian derivative of $f$ is a meromorphic quadratic differential defined as

$$S(f) = \left( \left( \frac{f''}{f'} \right)' - \frac{1}{2} \left( \frac{f''}{f'} \right)^2 \right) dz^2.$$

It has two remarkable properties.

- $S(f) = 0$ if and only if $f$ is a Möbius transformation,
- $S(g \circ f) = f^* S(g) + S(f)$.

As a consequence of those two properties, the Schwarzian derivative is defined for any holomorphic map from a surface equipped with a complex projective structure to another (see next section). It is a meromorphic quadratic differential on the domain, holomorphic if $f' \neq 0$ everywhere.

There are several nice geometric interpretations of the Schwarzian derivative, in particular in [10], [15] and in [12].
2.2. The Schwarzian tensor. Osgood and Stowe [32] generalized the Schwarzian derivative to the notion of Schwarzian tensor, associated to a conformal map between two Riemannian manifolds of the same dimension.

**Definition 2.1.** Let \((M, g)\) be a Riemannian \(d\)-dimensional manifold, and let \(u : M \to \mathbb{R}\). The Schwarzian tensor associated to the metrics \(g\) and \(e^{2u}g\) on \(M\) is defined as

\[
B(g, e^{2u}g) = (\text{Hess}_g(u) - du \otimes du)_0,
\]

where the index 0 denotes the traceless part with respect to \(g\).

The Schwarzian tensor is a natural generalization of the Schwarzian derivative in the sense that if \(\Omega \subset \mathbb{C}\) and \(f : \Omega \to \mathbb{C}\) is a holomorphic map, then

\[
B(|dz|^2, f^*(|dz|^2)) = \text{Re}(S(f)).
\]

The Schwarzian tensor also shares some key properties of the Schwarzian derivative:

- It behaves well under compositions of conformal maps: if \((M, g)\) is a Riemannian manifold and \(u, v : M \to \mathbb{R}\) are smooth functions, then

\[
B(g, e^{2u+2v}g) = B(g, e^{2u}g) + B(e^{2u}g, e^{2v+2u}g).
\]

- It behaves well under diffeomorphism: if \(\phi : N \to M\) is a diffeomorphism and \(h, h'\) are two conformal metrics on \(M\), then

\[
B(\phi^*h, \phi^*h') = \phi^*B(h, h').
\]

- If \(h_H\) is the hyperbolic metric on the ball \(B^d \subset \mathbb{R}^d\) given by the Poincaré model, and \(h_E\) is the Euclidean metric on \(\mathbb{R}^d\), then \(B(h_E, h_H) = 0\).

- Similarly, if \(h_S\) is the spherical metric on \(\mathbb{R}^d\) given as the push-forward of the spherical metric by the stereographic projection, then \(B(h_E, h_S) = 0\).

Here we give two simple interpretations of the Schwarzian tensor:

- as the difference between the second terms in the asymptotic development of metrics on Poincaré-Einstein manifolds, when one conformally varies the first term,
- as the variation in second fundamental forms of certain hypersurface in a \((d + 1)\)-dimensional space associated to conformal (and conformally flat) metrics.

The second interpretation is related to the interpretation in [28], but more in the setting of isometric embeddings in the space of horospheres as developed in [33]. The hypersurfaces that appear are dual, in a sense that will be made precise below, to the Epstein hypersurfaces of the metrics.

2.3. Complex projective structures. We will need to consider complex projective structures on closed surfaces. Recall that a complex projective structure (also called \(\mathbb{C}P^1\)-structure) is a \((G, X)\)-structure (see [39, 21]), where \(X = \mathbb{C}P^1\) and \(G = \text{PSL}(2, \mathbb{C})\). In other terms, they are defined by atlases with values in \(\mathbb{C}P^1\), with change of coordinates in \(\text{PSL}(2, \mathbb{C})\). We denote by \(\mathcal{CP}_S\) the space of \(\mathbb{C}P^1\)-structures on \(S\).

Given a complex structure \(c \in \mathcal{T}_S\) on \(S\), there is by the Riemann Uniformization Theorem a unique hyperbolic metric on \(S\) compatible with \(c\). This hyperbolic metric determines a complex projective structure on \(S\), because the hyperbolic plane can be identified with a disk in \(\mathbb{C}P^1\), on which hyperbolic isometries act by elements of \(\text{PSL}(2, \mathbb{C})\) fixing the boundary circle. We denote this complex projective structure by \(\sigma_F(c)\), and call it the Fuchsian complex projective structure of \(c\).

Let \(\sigma \in \mathcal{CP}_S\), and let \(c \in \mathcal{T}_S\) be the underlying complex structure. There is a unique map \(\phi : (S, \sigma) \to (S, \sigma_F(c))\) holomorphic for the underlying complex
structure. Let \( q(\sigma) = S(\phi) \) be its Schwarzian derivative. This construction defines a map \( Q : CP_S \to T^*T_S \), sending \( \sigma \) to \((c,q)\), with the holomorphic quadratic differential \( q \) considered as a cotangent vector to \( T_S \) at \( c \). The map \( Q \) is known to be a homeomorphism [13].

### 2.4. The holomorphic quadratic differential at infinity of quasifuchsian manifolds.

We now consider a quasifuchsian manifold \( M \) homeomorphic to \( S \times \mathbb{R} \), where \( S \) is a closed surface of genus at least 2. We note its boundary at infinity by \( \partial_{\infty} M \), so that \( \partial_{\infty} M \) is the disjoint union of two surfaces \( \partial_{\pm} M \), each homeomorphic to \( S \).

The boundary at infinity \( \partial_{\infty} M \) is the quotient of the complement of the limit set of \( M \) by the action of \( \pi_1 M \) on \( H^3 \). Since hyperbolic isometries act on \( \partial_{\infty} H^3 \) by complex projective transformations, \( \partial_{\infty} M \) is equipped with a complex projective structure, that we denote by \( \sigma \). We denote by \( c \) the complex structure underlying \( \sigma \).

**Definition 2.2.** We denote by \( q = q(\sigma) \) the “holomorphic quadratic differential at infinity” of \( M \).

Therefore, \( q \) can be considered as minus the “difference” between the quasifuchsian complex projective structure on \( \partial_{\infty} M \) and the Fuchsian complex projective structure obtained by applying the Riemann uniformization theorem to the complex structure \( c \).

### 2.5. Hyperbolic ends.

A hyperbolic end as considered here is non-complete hyperbolic manifold, diffeomorphic to \( S \times \mathbb{R}_{>0} \), where \( S \) is a closed surface of genus at least 2, which is complete on the side corresponding to \( \infty \) but has a metric completion obtained by adding a concave pleated surface on the boundary corresponding to \( 0 \). A typical example is a connected component of the complement of the convex core in a convex co-compact hyperbolic manifold.

Given a hyperbolic end \( E \), we denote by \( \partial_{\infty} E \) its ideal boundary, corresponding to \( \infty \) in the identification of \( E \) with \( S \times \mathbb{R}_{>0} \), and by \( \partial_0 E \) the concave pleated surface which is the boundary of its metric completion corresponding to \( 0 \). We will denote by \( \mathcal{E}_S \) the space of hyperbolic ends homeomorphic to \( S \times \mathbb{R}_{>0} \), considered up to isotopy.

Let \( E \) be a hyperbolic end. Its ideal boundary \( \partial_{\infty} E \) is equipped with a \( CP^1 \)-structure \( \sigma \). This is clear in the simpler case when the developing map of \( E \) is injective, since in this case \( \partial_{\infty} E \) is the quotient of a domain in \( CP^1 \) (identified with \( \partial_{\infty} H^3 \)) by an action of \( \pi_1 E \) by elements of \( PSL(2,\mathbb{C}) \). In the general case this picture has to be slightly generalized, and \( \partial_{\infty} E \) is a quotient of a simply connected surface which has a (not necessarily injective) projection to \( CP^1 \), see [38, 39].

In fact, this map from \( \mathcal{E}_S \) to \( CP_S \) is one-to-one, and a hyperbolic end can be constructed for any \( CP^1 \)-structure on \( S \), see [38].

Given a metric \( I^* \) in the conformal class at infinity of \( E \), it defines an equidistant foliation of a neighborhood of infinity in \( E \) in the following way. Given a real number \( r \), consider the Epstein surface \( S_r \) of \( e^{2r}I^* \). Then for \( r \) large enough, \( S_r \) is smooth and embedded, and even locally convex. Moreover, if \( r, r' \) are large enough, then \( S_r \) and \( S_r' \) are at fixed distance \( |r' - r| \). The hyperbolic metric then has the asymptotic expansion [4], and the term \( I^* \) does not change if one replaces \( I^* \) with \( e^{2r}I^* \).

In addition, \( I^* \) and \( I^{**} \) satisfy two relations (see [25, §5]): \( I^* \) is Codazzi for \( I^* \), that is, \( d^D I^{**} = 0 \), where \( D^* \) is the Levi-Civita connection of \( I^* \), and \( \text{tr} r.I^{**} = -K^* \), where \( K^* \) is the curvature of \( I^* \). We will say that \((I^*, I^{**})\) is an admissible pair if it satisfies those two equations.
2.6. Poincaré-Einstein manifolds. A Poincaré-Einstein manifold \((M^{d+1}, g)\) is a complete Riemannian manifold such that the Riemannian metric is Einstein and can be written as
\[
g = \frac{\bar{g}}{x^2},
\]
where \(\bar{g}\) is a smooth metric on a compact manifold with boundary \(\bar{M}\), \(M\) is the interior of \(\bar{M}\), and \(\|dx\|_{\bar{g}} = 1\) on \(\partial M\), see [18].

Poincaré-Einstein manifolds have a well-defined boundary at infinity \(\partial_{\infty} M\), identified with \(\partial M\), endowed with a conformal class of metrics, defined as the conformal class of the restriction of \(\bar{g}\) to \(\partial_{\infty} M\). In the neighborhood of each connected component of the boundary at infinity, one can write \(\bar{g} = dx^2 + h_x\), where \(h_x\) is a one-parameter family of metrics on \(\partial_{\infty} M\).

When \(d\) is even, \((h_x)_{x > 0}\) has the asymptotic expansion
\[
h_x \sim 0 \sum_{\ell=0}^{\infty} h_{x,\ell}(x^d \log x)^\ell.
\]
where \(h_{x,\ell}\) are one-parameter families of tensors on \(M\) depending smoothly on \(x\).

The tensor \(h_{x,0}\) has a Taylor expansion at \(x = 0\) given by
\[
h_{x,0} \sim 0 \sum_{j=0}^{\infty} x^{2j} h_{2j},
\]
where \(h_{2j}\) are formally determined by \(h_0\) if \(j < d/2\) and formally determined by the pair \((h_0, h_d)\) for \(j > d/2\); for \(\ell \geq 1\), the tensors \(h_{x,\ell}\) have a Taylor expansion at \(x = 0\) formally determined by \(h_0\) and \(h_d\).

When \(d\) is odd, \((h_x)_{x > 0}\) has the simpler asymptotic expansion
\[
h_x \sim 0 h_0 + x^2 h_2 + \cdots x^{d-1} h_{d-1} + x^d h_d + O(x^{d+1}),
\]
where \(\text{tr} h_j = 0\) for \(k < d\) are formally determined by \(h_0\), while the traceless part of \(h_d\) is “free”. All other terms in the asymptotic development are determined by \(h_0\) and by the traceless part of \(h_d\).

2.7. Hypersurfaces. Let \(S \subset H^{d+1}\) be an oriented hypersurface. We will denote by \(I\) its induced metric (classically called its “first fundamental form”).

Let \(N\) be the oriented unit normal vector field to \(S\). The shape operator of \(S\) is the bundle morphism \(B: TS \to TS\) defined as follows:
\[
\forall x \in S, \forall u \in T_x S, Bu = D_u N,
\]
where \(D\) is the Levi-Civita connection on \(H^{d+1}\). Then \(B\) is self-adjoint with respect to \(I\).

The second fundamental form of \(S\) is defined as
\[
\forall x \in S, \forall u, v \in T_x S, II(u, v) = I(Bu, v) = I(u, Bv),
\]
and its third fundamental form as
\[
\forall x \in S, \forall u, v \in T_x S, III(u, v) = I(Bu, Bv).
\]

The hyperbolic Gauss map of \(S\) is the map \(G: S \to \partial_{\infty} H^{d+1}\) sending a point \(x \in S\) to the endpoint of the geodesic ray starting from \(x\) in the direction of the oriented normal \(x\).

Definition 2.3. We say that \(S\) is horospherically tame, or \(h\)-tame, if its principal curvatures are everywhere in \((-1, 1)\).

Remark 2.4. If \(S\) is complete and \(h\)-tame, than its hyperbolic Gauss map is a diffeomorphism between \(S\) and a connected component of \(\partial_{\infty} H^{d+1} \setminus \partial_{\infty} S\).
Definition 2.5. Suppose that the hyperbolic Gauss map $G$ of $S$ is injective. The data at infinity associated to $S$ is the pair $(I^*, II^*)$ defined on $G(S)$ by

$$I^* = \frac{1}{2} G_*(I + 2II + III),$$
$$II^* = \frac{1}{2} G_*(I - III).$$

The shape operator at infinity is the bundle morphism $B^* : T(G(S)) \rightarrow T(G(S))$ which is self-adjoint for $I^*$ and such that

$$\forall y \in G(S), \forall u, v \in T_y G(S), II^*(u, v) = I^*(u, B^*v) = I^*(B^*u, v).$$

One can then prove (see [28]) that $(I^*, II^*)$ is an admissible pair, as defined above.

2.8. The space of horospheres. We denote by $C^3_+$ the space $S^2 \times \mathbb{R}$, with the degenerate metric $g = e^{2t}g_0 + 0 \times dt^2$, where $t \in \mathbb{R}$. The notation comes from the fact that it can be identified with the future light cone of a point in the 4-dimensional de Sitter space. Equivalently, it can be identified with the space of horospheres in the 3-dimensional hyperbolic space with the natural metric defined in terms of intersection angles, see [33, §2].

This space has a number of features that are strongly reminiscent of a 3-dimensional space of constant curvature.

- $PSL(2, \mathbb{C})$ acts by isometries on $C^3_+$. The simplest way to see this is by considering $C^3_+$ as the space of horospheres in $H^3$.
- There is a notion of “totally geodesic planes”, which are the 2-dimensional spheres with induced metric isometric to the round metric on $S^2$. Those planes can be identified with the set of horospheres going through a given point in $H^3$, and the action of $PSL(2, \mathbb{C})$ on those totally geodesic planes is transitive.
- There is a 2-dimensional space of totally geodesic planes going through each point in $C^3_+$. In addition, there is at each point a distinguished “vertical” direction, corresponding to the kernel of the metric. Moreover the integral lines of those vertical directions have a canonical affine structure.

Although the metric is degenerate, it is possible to define an analog of the Levi-Civita connection at a point $x \in C^3_+$. However it depends on the choice of a non-degenerate plane $H \subset T_x C^3_+$, and is defined for vector fields tangent to $H$ (see the last paragraph of [33, §2]).

Using this connection, one can define a notion of second fundamental form $II$ of a surface $S \subset C^3_+$ which is nowhere vertical (see [33, §5]). Using the induced metric $I$, one can then define the shape operator $B : TS \rightarrow TS$ as the self-adjoint operator such that $II = I(B\cdot \cdot)$. It satisfies two equations (see [33, §6]):

- the Codazzi equation $\nabla^B B = 0$, where $\nabla$ is the Levi-Civita connection of $I$ on $S$,
- a modified form of the Gauss equation: the curvature of $I$ is equal to $K = 1 - \text{tr}(B)$.

3. The measured foliation at infinity

3.1. The Fischer-Tromba metric. Let $g$ be a hyperbolic metric on $S$. The tangent space $T_g T$ can be identified with the space of symmetric 2-tensors on $S$ that are traceless and satisfy the Codazzi equation for $g$. (In other terms, the real parts of holomorphic quadratic differentials in $Q_g$.) We call $TT_g$ the space of those traceless Codazzi symmetric 2-tensors for $g$. 

Let $h, k$ be two such tensors and let $[h], [k]$ be the corresponding vectors in $T_g \mathcal{T}$. Then the Weil-Petersson metric between $[h]$ and $[k]$ can be expressed as

$$\langle [h], [k] \rangle_{WP} = \frac{1}{8} \int_S \langle h, k \rangle_g da_g.$$ 

The right-hand side of this equation is sometimes called the Fischer-Tromba metric on $\mathcal{T}$.

We can also relate the scalar product on symmetric 2-tensors to the natural bracket between holomorphic quadratic differentials and Beltrami differentials as follows.

**Lemma 3.1.** Let $X$ be a closed Riemann surface, and let $h$ be the hyperbolic metric compatible with its complex structure. Let $\dot{h}$ be a first-order deformation of $h$, and let $\mu$ be the corresponding Beltrami differential. Then for any holomorphic quadratic differential $q$ on $X$,

$$\int_X \langle \Re(q), \dot{h} \rangle_h da_h = 4 \Re \left( \int_X q \mu \right).$$

The proof is in Appendix A.

### 3.2. The energy of harmonic maps and the Gardiner formula

Let $f \in \mathcal{MF}_S$, and let $T_f$ be its dual real tree. For each $c \in T_S$, there is a unique equivariant harmonic map $u$ from $\tilde{S}$ to $T_f$. Let $E_f(c) = E(u, c)$ be its energy, and let $\Phi_f$ be its Hopf differential. Then

$$dE_f(c) = -4 \Re \langle \Phi_f, \dot{c} \rangle.$$

Here $\dot{c}$ is considered as a Beltrami differential, and $\langle \cdot, \cdot \rangle$ is the duality product between Beltrami differentials and holomorphic quadratic differentials. (See e.g. [13, Theorem 1.2].)

We use below the same notations, but with $S$ replaced by $\partial M$.

### 3.3. Extremal lengths of measured foliations

Let $f$ be a measured foliation on $S$ and, for given $c \in \mathcal{T}$, let $Q$ be the holomorphic quadratic differential on $S$ with horizontal foliation $f$.

**Definition 3.2.** The **extremal length** of $f$ at $c$ is the integral over $S$ of $Q$,

$$\text{ext}_c(f) = \int_S |Q|.$$

A more classical definition can be given in terms of modulus of immersed annuli, see [1].

**Theorem 3.3** ([14]). $Q = -\Phi_f$. Moreover,

$$E_f(c) = 2 \int_S |\Phi_f| = 2 \int_S |Q| = 2 \text{ext}_c(f).$$

### 3.4. The renormalized volume of quasifuchsian manifolds

The renormalized volume of quasifuchsian manifolds is closely related to the Liouville functional in complex analysis, see [36, 35, 37, 27]. However it can also be considered as a special case, in dimension 3, of the renormalized volume of conformally compact Einstein manifolds as seen in [35], see [26, 23, 22].

A definition of the renormalized volume of quasifuchsian manifolds can be found in [28, Def 8.1]. It satisfies a simple variational formula, which can be written as

$$\tilde{V}_R = -\frac{1}{4} \int_{\partial_C M} \langle \Pi_0^*, \dot{i}^* \rangle da_I.$$
where $R_0^*$ is the traceless part of the “second fundamental form at infinity” which, together with the metric at infinity $I^*$, completely characterizes a hyperbolic end.

However we know from Theorem 1.6 (see [28, Lemma 8.3]) that

$$R_0^* = \text{Re}(q).$$

So, applying Lemma 3.1 we find that in a first-order variation,

$$\dot{V}_R = -\text{Re}((q, \dot{c})),$$

where $q$ is considered as a vector in the complex cotangent to $T_S$ at $c$, and $\langle \cdot , \cdot \rangle$ is the duality bracket.

### 3.5. The measured foliation at infinity

We now introduce a measured foliation at infinity, which can be thought of as an analog at infinity of the measured bending lamination on the boundary of the convex core.

**Definition 3.4.** The measured foliation at infinity is the horizontal measured foliation of $q$. We denote it by $f$.

It follows from Theorem 3.3 that $\Phi_f = -q$.

**Lemma 3.5.** Let $c \in T\partial M$, and let $F \in MF_{\partial M}$. Then $F$ is the horizontal measured foliation of the quasifuchsian hyperbolic metric determined by $c$ if and only if the function $\Psi_F$ defined as

$$\Psi_F = V_R - \frac{1}{4} E_F : T\partial M \to \mathbb{R}$$

is critical at $c$.

**Proof.** Suppose first that $F$ is the horizontal measured foliation of $q$, the holomorphic quadratic differential at infinity of the quasifuchsian manifold $M(c)$.

It follows from (15) and (16) that, in a first-order deformation $\dot{c}$,

$$d\Psi_F(\dot{c}) = dV_R(\dot{c}) - \frac{1}{4} dE_F(\dot{c}) = \text{Re}(\langle q + \Phi_F, \dot{c} \rangle).$$

But we have seen that $q = -\Phi_F$, and it follows that $d\Psi_F = 0$.

Conversely, if $d\Psi_F = 0$, the same argument as above shows that $q = -\Phi_F$, so that $F$ is the horizontal measured foliation of $q$. \qed

### 3.6. Proof of Theorem 1.2

Equation (16) states that, in a first-order deformation of $M$,

$$\dot{V}_R = -\text{Re}((q, \dot{c})),$$

and using Theorem 3.3 we obtain that

$$\dot{V}_R = \text{Re}((\Phi_f, \dot{c})).$$

Using (15), this can be written as

$$\dot{V}_R = -\frac{1}{4} dE_f(\dot{c}).$$

Using Theorem 3.3 again, we finally find that

$$\dot{V}_R = -\frac{1}{2}(d\text{ext}(f))(\dot{c}).$$
3.7. Proof of Theorem 1.4. Nehari [31] proved that if \( f : D \to \mathbb{C} \) is a univalent holomorphic function defined on the unit disk, then its Schwarzian derivative can be written as

\[
S(f) = g \frac{dz}{\rho}^2
\]

where \( \rho |dz|^2 \) is the complete hyperbolic metric on the disk \( D \).

As a consequence,

\[
\int_{\partial \pm M} |q| \leq \int_{\partial \pm M} \frac{3}{2} dh_{\pm} ,
\]

where \( da_{h_{\pm}} \) is the area form of the hyperbolic metric \( h_{\pm} \) in the conformal class at infinity. Since the area of \( (\partial \pm M, h_{\pm}) \) is \( 2\pi |\chi(S)| \), the result follows.

3.8. Questions. We list here a number of questions motivated by the correspondence between the convex core and the boundary at infinity.

**Question 3.6.** Can Theorem 1.4 and Theorem 1.2 be extended to convex co-compact hyperbolic 3-manifolds? To geometrically finite hyperbolic 3-manifolds?

The definition of the renormalized volume can be extended to convex co-compact hyperbolic manifolds, and the main estimates also apply for convex co-compact manifolds with incompressible boundary, see [7]. We can expect Theorem 1.2 to apply to convex co-compact hyperbolic manifolds, and Theorem 1.4 to extend to convex co-compact hyperbolic manifolds with incompressible boundary, while the estimate for manifolds with compressible boundary might involve the injectivity radius of the boundary.

**Question 3.7.** Can Theorem 1.4 and Theorem 1.2 be extended to geometrically finite hyperbolic 3-manifolds?

Again, the definition and some key properties of the renormalized volume extend to geometrically finite hyperbolic 3-manifolds, see [24]. It could be expected that Theorems 1.2 and 1.4 extends to this setting.

**Question 3.8.** Suppose that \( M \) is not Fuchsian (that is, it does not contain any closed totally geodesic surface). Do \( f_- \) and \( f_+ \) fill?

This would be the analog of the well-known (and relatively easy) corresponding statement for \( l_- \) and \( l_+ \), the measured bending lamination on the boundary of the convex core.

**Question 3.9.** Let \( (f_-, f_+) \in \mathcal{ML}_S \times \mathcal{ML}_S \) with \( (f_-, f_+) \neq 0 \). Is there at most one quasifuchsian manifold with measured foliation at infinity \( (f_-, f_+) \)?

This is the analog at infinity of the uniqueness part of a conjecture of Thurston on the existence and uniqueness of a quasifuchsian manifold having given measured bending lamination \( (l_-, l_+) \) on the boundary of the convex core. In this case \( (l_-, l_+) \) are requested to fill and to have no closed leaf of weight larger than \( \pi \). The existence part of this conjecture was proved in [5], as well as the uniqueness for rational measured laminations.

A related question would be whether infinitesimal rigidity holds, that is, whether any non-zero first-order deformation of \( M \) induces a non-zero deformation of either the \( f_- \) or \( f_+ \). The analog question for \( l_- \) and \( l_+ \) is open.

**Question 3.10.** Given \( (f_-, f_+) \in \mathcal{ML}_S \times \mathcal{ML}_S \), what conditions should it satisfy so that there exists a quasifuchsian manifold \( M \) with measured foliation at infinity \( (f_-, f_+) \)?
If the answer to Question 3.11 is positive, then one should ask that (if \((f_-, f_+) \neq 0\)) \(f_-\) and \(f_+\) should fill. However other conditions might be necessary.

**Question 3.11.** Are there any extensions of the measured foliation at infinity in higher dimension, for quasifuchsian (or convex co-compact) hyperbolic \(d\)-dimensional manifolds?

For those manifolds, there is a well-defined notion of convex core, and the boundary of the convex core also has a “pleating”. However the pleating lamination might have a more complex structure than for \(d = 3\), with codimension 1 “pleating hypersurfaces” of the boundary meeting along singular strata of higher codimension. Other aspects of the renormalized volume of quasifuchsian manifold have a partial extension in higher dimensions, see e.g. [25].

**Question 3.12.** Is the renormalized volume convex in any reasonable sense?

It seems unlikely that the renormalized volume is convex for the Weil-Petersson metric, since this does not seem to be compatible with the behavior of its gradient close to the Weil-Petersson boundary of \(T_{\partial M}\), see [6]. However the renormalized volume is convex in the neighborhood of the Fuchsian locus, see [30, 11, 41].

Note that it has been proved recently that the renormalized volume is minimal at the Fuchsian locus (for quasifuchsian manifolds) and for metrics containing a convex core with totally geodesic boundary (for acylindrical boundary), see [12, 6].

4. The Second Fundamental Form at Infinity and the Space of Horospheres

4.1. Proof of Theorem 1.8. After replacing \(I^*\) by \(e^{2r}I^*\), for \(r\) large enough, the Epstein surface of \(I^*\) is smooth and embedded. We will suppose that this is the case, since the general case then follows by scaling.

Let \(S\) be the Epstein surface of \(I^*\), that is, \(S\) is a surface in \(E\) such that the hyperbolic Gauss map \(G\) of \(S\) is a diffeomorphism between \(S\) and \(\partial_{\infty} E\), and the pull-back \(G^*I^*\) is equal to \(\frac{1}{2}(I + 2\Pi + \mathbb{III})\) (see [28] Definition 5.3)). We can then consider the dual surface \(S^*\) in \(C^2\). Its induced metric is equal to \(I^*_{\epsilon} = I + 2\Pi + \mathbb{III}\) under the identification between \(S\) and \(S^*\) through the duality map (see [33] Lemma 3.5)). Note that the duality map followed by the projection in \(C^2\) along the vertical (degenerate) direction is equal to the hyperbolic Gauss map, and we therefore obtain that \(I^*_{\epsilon} = I + 2\Pi + \mathbb{III} = 2I^*\).

We denote by \(B^*\) and \(B^*_{\epsilon}\) the “shape operators” associated to \(I^*\) and \(I^*_{\epsilon}\), respectively. That is, \(B^*\) and \(B^*_{\epsilon}\) are self-adjoint with respect to \(I^*\) and \(I^*_{\epsilon}\), and

\[
B^* = I^* (B^* \cdot, \cdot) , \quad B^*_{\epsilon} = I^*_{\epsilon} (B^* \cdot, \cdot) .
\]

We also know that \(B^*_{\epsilon} = (E + B)^{-1} (E - B)\) (see [33] Eq. (31)), while \(B^* = (E + B)^{-1} (E - B)\) (see [28] Eq. (31)). So \(E + B^* = 2(E + B)^{-1} = 2B^*_{\epsilon}\), and the result follows.

4.2. Proof of Theorem 1.6. We now turn to the proof of Theorem 1.6, but will use Corollary 1.10 which is proved in the next section.

We consider the Riemann uniformization map \(\phi : \partial_{\infty} E \to D\), where \(D \subset \mathbb{C}\) is the disk. By definition \(\phi\) is a conformal diffeomorphism. The following metrics can be considered:

- on the domain \(\partial_{\infty} E\), the restriction of either a spherical metric \(h_S\) on \(\mathbb{C}P^1\), or a flat metric \(|dz|^2\) on \(\mathbb{C}P^1\) minus a point,
- on the target \(D\), either the hyperbolic metric \(h_D\), or the restriction of a flat metric \(|dz|^2\) defined on \(\mathbb{C}\).
Given a metric $h$ on $\partial_{\infty}E$, we denote by $\mathcal{H}_h^*$ the second fundamental form at infinity obtained when taking $I^* = h$, and similarly for $\mathcal{H}_{\ast h}$ and for their traceless components.

The spherical metric $h_S$ corresponds to a totally geodesic surface in $C^3$, so $\mathcal{H}_{\ast h_S} = 0$. It follows from Theorem 1.8 that the traceless part $\mathcal{H}_{h_S,0} = 0$. Applying (5) then shows that

$$\mathcal{H}_{\ast h_D,0} = B(h_S, \phi^*h_D).$$

It follows that

$$\mathcal{H}_{\ast h_D,0} = B(|dz|^2, h_S) + B(h_S, \phi^*h_D) + \phi^*B(h_D, |dz|^2)$$

because the first and third term on the right-hand side are zero. Thus

$$\mathcal{H}_{\ast h_D,0} = B(|dz|^2, \phi^*|dz|^2).$$

It now follows from (12) that

$$\mathcal{H}_{\ast h_D,0} = \text{Re}(\text{Sch}(\phi)),$$

as claimed.

5. Poincaré-Einstein ends

We now turn to the proof of Theorem 1.9. Suppose first that $d \geq 3$. Then the second term in the asymptotic development of $h_x$ near infinity has a simple expression in terms of $h_0$, see [19] (3.18)].

**Proposition 5.1** (Fefferman, Graham). $h_2$ is minus the Schouten tensor of $h_0$, see $h_2 = -\text{Sch}_{h_0}$.

Recall that the Schouten tensor is defined as

$$\text{Sch}_h = \frac{1}{d-2} \left( \text{Ric}_{h_0} - \frac{1}{2(n-1)} \text{Scal}_{h_0} h_0 \right).$$

Moreover, the Schouten tensor obeys the following transformation under a conformal transformation of the metric, see [2] (1.159):

$$\text{Sch}_{\ast h} = \text{Sch}_h - \text{Hess}(u) + du \otimes du - \frac{1}{2} \|du\|^2 h.$$

Theorem 1.9 follows for $d \geq 3$.

We now focus on $d = 2$. We have seen in Proposition 1.5 and Theorem 1.8 that $h_2 = \mathcal{H}^* = \mathcal{H}^*_c - I^*$, so it is sufficient to understand the variation of $\mathcal{H}^*_c$ in a conformal variation of $I^*$. We will prove first an infinitesimal version of Equation 6, and obtain the general result as a consequence.

**Lemma 5.2.** Let $(S, g)$ be a surface with a Riemannian metric, and let $\phi : S \to S^2$ be a conformal diffeomorphism. For each $u : S \to \mathbb{R}$, let $\Phi_u : S \to C^3$ be the isometric embedding such that $\pi \circ \Phi_u = \phi$, and let $\mathcal{H}^*_c(u)$ denote the pull-back by $\Phi_u$ of the second fundamental form of the image by $\Phi_u$. Then the differential of $\mathcal{H}^*_c(u)$ corresponding to a first-order variation $\dot{u}$ is

$$\mathcal{H}^*_c = \text{Hess}_{I^*_c}(\dot{u}) + \dot{u} I^*_c.$$

**Proof.** Let $x_0 \in S$, and let $P_0$ be the tangent plane to $\Phi_u(S)$ at $y_0 = \Phi_u(x_0)$. Then the second fundamental form $I^*_c$ of $\Phi_u(S)$ at $y_0$ can be defined (see [33, Lemma 5.2]) as the Hessian at $y_0$ of the function $v$ defined on $P_0$ such that $\Phi_u(S)$ is the graph of $v$ over $P_0$. By definition, $v(y_0) = 0$ and $de(v(y_0)) = 0$.

Now consider a first-order variation $\dot{u}$ of $u$, and let $u_0$ be the first-order vertical deformation of $P_0$, among totally geodesic planes in $C^3$, chosen such that the deformed totally geodesic plane remains tangent to the first-order deformation of
\[ \Phi_u(S) \] at the point corresponding to \( y_0 \). Since totally geodesic planes in \( C^3_+ \) correspond to constant curvature conformal metrics on \( S^2 \), \( \hat{u}_0 \) is uniquely determined by the condition that

\[ \hat{u}_0(y_0) = \hat{u}(y_0), \quad d\hat{u}_0(y_0) = d\hat{u}(y_0), \]

\[ \text{Hess}_{P_0}(\hat{u}_0) + \hat{u}_0 h_0 = 0, \]

where \( h_{P_0} \) is the induced metric on \( P_0 \).

Therefore, we have at \( y_0 \)

\[ \hat{\mathcal{II}}_c^* = \text{Hess}_{P_0}(\hat{u} - \hat{u}_0) \]
\[ = \text{Hess}_{P_0}(\hat{u}) - \text{Hess}(\hat{u}_0) \]
\[ = \text{Hess}_{P_0}(\hat{u}) + \hat{u}_0 h_{P_0} \]
\[ = \text{Hess}_{\hat{\mathcal{II}}_c^*}(\hat{u}) + u \mathcal{I}_c^*, \]

where the last equality follows from the fact that \( h_0 \) is equal to \( I_c^* \) on \( T_{y_0} \Phi_u(S) = T_{y_0} P_0 \) and because \( u(y_0) = du(y_0) = 0 \) so that the Hessian at \( y_0 \) for \( I_c^* \) is the same as the Hessian at \( y_0 \) for \( h_{P_0} \).

We can then integrate this first-order variation formula, to obtain Equation \( \text{(6)} \).

**Lemma 5.3.** Let \( \Omega \subset \mathbb{C}P^1 \), let \( h \) be a Riemannian metric on \( \Omega \) compatible with the complex structure, and let \( u : \Omega \to \mathbb{R} \) be a smooth function. Let \( \mathcal{II}_{c,h}^* \) and \( \mathcal{II}_{c,e^{2u}h}^* \) be the second fundamental forms of the isometric embeddings in \( C^3_+ \) of \( h \) and \( e^{2u}h \), respectively. Then

\[ \mathcal{II}_{c,e^{2u}h}^* = \mathcal{II}_{c,h}^* + \text{Hess}_h(u) - du \otimes du + \frac{1}{2} \| du \|^2 h + \frac{1}{2} (e^{2u} - 1) h. \]

**Proof.** Recall the conformal transformation rule for the Hessian: for any functions \( u, v : \Omega \to \mathbb{R} \),

\[ \text{Hess}_{e^{2u}h}(v) = \text{Hess}_h(v) - 2 du \otimes dv + \langle du, dv \rangle h. \]

This follows from the conformal transformation of the Levi-Civita connection, see [2] (1.159 a)).

This leads to a first-order variation formula for \( \mathcal{II}_{c,e^{2u}h}^* \) under a first-order variation of \( h \), based on the previous lemma.

\[ d\mathcal{II}_{c,e^{2u}h}^*(\hat{u}) = \text{Hess}_{e^{2u}h}(\hat{u} + \hat{u} e^{2u}) \]
\[ = \text{Hess}_h(\hat{u} - 2 du \otimes d\hat{u} + \langle du, d\hat{u} \rangle e^{2u} h + \hat{u} e^{2u} h) \]
\[ = \text{Hess}_h(\hat{u} - 2 du \otimes d\hat{u} + \langle du, d\hat{u} \rangle h + \hat{u} e^{2u} h), \]

and the result follows by integration:

\[ \mathcal{II}_{c,e^{2u}h}^* - \mathcal{II}_{c,h}^* = \int_{t=0}^1 d\mathcal{II}_{c,e^{2u}h}^*(u) dt \]
\[ = \text{Hess}_h(u) - du \otimes du + \frac{1}{2} \langle du, du \rangle h + \frac{1}{2} (e^{2u} - 1) h. \]

**Lemma 5.3** is equivalent to Equation \( \text{(6)} \), Equation \( \text{(5)} \) then follows by Theorem \( \text{(1.8)} \). We can then use Proposition \( \text{(1.5)} \) to prove Theorem \( \text{(1.9)} \) in dimension \( d = 2 \).

Sergiu Moroianu has suggested another proof of Theorem \( \text{(1.9)} \) that works both for \( d = 2 \) and for higher dimensions. It is perhaps conceptually simpler but computationally a bit more involved.
6. Linear Weingarten surfaces and Monge-Ampère equations

6.1. Tame hypersurfaces. We consider a hypersurface $S \subset H^{d+1}$, and use the same notations $I, II, III, B$ as above. The corresponding data at infinity are determined by $I^* = \frac{1}{2}(I + 2II + III), B^* = (E + B)^{-1}(E - B)$, and therefore

\[ B = (E + B^*)^{-1}(E - B^*) . \]

The proof of Proposition 1.11 is a direct consequence of this equation, since the eigenvalues of $B$ are in $(-1,1)$ if and only if the eigenvalues of $B^*$ are positive.

6.2. Relations on the data at infinity. A simple computation using (17) shows that

\[
\det(B) = \frac{\det(B^*) - \det(B^*) + 1}{\det(B^*) + \det(B^*) + 1}, \quad \tr(B) = 2 - \frac{1 - \det(B^*)}{\det(B^*) + \det(B^*) + 1} .
\]

Therefore, $S$ satisfies the Weingarten equation (4), $aK_v + bH + c = 0$, if and only if

\[
a(\det(B^*) - \det(B^*) + 1) + b(1 - \det(B^*)) + c(\det(B^*) + \det(B^*) + 1) = 0 ,
\]

so if and only if

\[
(a - b + c)\det(B^*) + (c - a)\tr(B^*) + (a + b + c) = 0 .
\]

This is the case if and only if

\[
\det((a - b + c)B^* + (c - a)E) = (a - b + c)(a + b + c) = 0,
\]

that is, if and only if

\[
\det((a - b + c)B^* + (c - a)E) = b^2 - 4ac .
\]

This proves Proposition 1.12.

6.3. Monge-Ampère equations. We now turn to the proof of Proposition 1.13. We set $I^* = e^{2u}I^*$, and denote by $H^*$ the second fundamental form at infinity associated to $I^*$. We have seen that

\[
H^* = \frac{e^{2u}I^*}{K} + B(I^*, I^*) .
\]

The surface corresponding to $e^{2u}I^*$ is h-tame if and only if $H^*$ is positive definite, that is, if and only if $H^* + B(I^*, I^*)$ is positive definite.

Moreover,

\[
\det((a - b + c)H^* + (c - a)I^*) = e^{-4u} \det((a - b + c)(H^* + B(I^*, I^*)) + (c - a)e^{2u}I^*)
\]

\[
= e^{-4u} \det((a - b + c)(H^* + B(I^*, I^*)) + (c - a)e^{2u}I^*) .
\]

The second point of Proposition 1.13 therefore follows from Proposition 1.12.

Appendix A. Proof of Lemma 3.1

We denote by $u : TS \rightarrow TS$ the h-self-adjoint morphism such that $\hat{h} = h(u, \cdot)$, and suppose that $u$ is traceless. We also denote by $J$ the complex structure of $h$, by $\hat{J}$ the first-order variation of $J$.

Statement A.1. $u = J\hat{J} = -\hat{J}J$. Note also that, since $u$ is traceless, $\hat{J}$ is self-adjoint.

\footnote{Hypothesis necessary on $B$, no eigenvalue 1?}
Proof. Note that $J^2 = -I$ so $J \dot{J} + \dot{J}J = 0$.

To prove the statement we have to check that, with this choice of $u$, the following two defining properties of $J$ remain valid at first order:

\begin{align*}
    h(Jx, y) &= -h(x, Jy), \\
    h(Jx, Jy) &= h(x, y).
\end{align*}

This translates as

\begin{align*}
    \dot{h}(Jx, y) + h(\dot{J}x, y) + h(x, \dot{J}y) + h(x, Jy) &= 0, \\
    \dot{h}(Jx, Jy) + h(\dot{J}x, Jy) + h(Jx, \dot{J}y) + h(Jx, Jy) &= \dot{h}(x, y),
\end{align*}

or in equivalent terms:

\begin{align*}
    h(\dot{J}Jx, y) + h(\dot{J}x, y) + h(x, \dot{J}Jy) + h(x, \dot{J}y) &= 0, \\
    h(J\dot{J}x, Jy) + h(J\dot{J}x, Jy) + h(\dot{J}x, \dot{J}y) + h(J\dot{J}x, Jy) &= \dot{h}(x, y),
\end{align*}

and both equations are clearly satisfied, the second because $\dot{J}$ is self-adjoint. \hfill \Box

Statement A.2. The Beltrami differential associated to $\dot{J}$ can be written as an antilinear morphism $\mu : TS \to TS$ as

$$
\mu = \frac{1}{2} J J - \frac{1}{2} \dot{J} J.
$$

Proof. Let $(J_t)_{t \in [0,1]}$ be a one-parameter family of complex structures, with $J_0 = J$ and $(d/dt)J_t = \dot{J}$ at $t = 0$. Then the Beltrami differential associated to the identity map from $(X, J)$ to $(X, J_t)$ is

$$
\mu_t = (\partial id)^{-1} \circ (\partial id) = \left(\frac{id - J_t J}{2}\right)^{-1} \circ \left(\frac{id + J_t J}{2}\right) = \left(id - J_t J\right)^{-1} \circ \left(id + J_t J\right).
$$

Differentiating this at $t = 0$ shows the result. \hfill \Box

Statement A.3. Let $z = x + iy$ be a complex coordinate. If the matrix of $u$ in the basis $(\partial_x, \partial_y)$ is

$$
\begin{pmatrix}
    a & b \\
    b & -a
\end{pmatrix},
$$

then

$$
\mu = \frac{a + ib}{2} \frac{dz}{dz}.
$$

Proof. Follows from checking explicitly that this expression leads to the correct matrix for $\mu$ seen as an antilinear morphism $TS \to TS$. \hfill \Box

Proof of the lemma. Write $\mu = (\mu_x + i\mu_y)dz/dz, q = (q_x + iq_y)dz^2, h = \rho^2|dz|^2$. Then the matrix of $u$ is

$$
\begin{pmatrix}
    2\mu_x & 2\mu_y \\
    2\mu_y & -2\mu_x
\end{pmatrix},
$$

so

$$
\dot{h} = (2\mu_x(dx^2 - dy^2) + 4\mu_y dxdy)\rho^2.
$$

On the other hand, $Re(q) = q_x(dx^2 - dy^2) - 2q_y dxdy$. So

$$
(\dot{h}, Re(q))_h = \frac{4\mu_x q_x - 4\mu_y q_y}{\rho^2},
$$

and

$$
\int_X (\dot{h}, Re(q))_h da_h = \int_X (4\mu_x q_x - 4\mu_y q_y) dxdy = 4Re \left( \int_X q_\mu \right).
$$

\hfill \Box
References

[1] Lars V. Ahlfors. *Conformal invariants*. AMS Chelsea Publishing, Providence, RI, 2010. Topics in geometric function theory, Reprint of the 1973 original, With a foreword by Peter Duren, F. W. Gehring and Brad Osgood.

[2] Arthur Besse. *Einstein Manifolds*. Springer, 1987.

[3] Francis Bonahon. A Schlafli-type formula for convex cores of hyperbolic 3-manifolds. *J. Differential Geom.*, 50(1):25–58, 1998.

[4] Francis Bonahon. Variations of the boundary geometry of 3-dimensional hyperbolic convex cores. *J. Differential Geom.*, 50(1):1–24, 1998.

[5] Francis Bonahon and Jean-Pierre Otal. Laminations mesurées de plissage des variétés hyperboliques de dimension 3. *Ann. Math.*, 160:1013–1055, 2004.

[6] M. Bridgeman, J. Brock, and K. Bromberg. Schwarzian derivatives, projective structures, and the Weil-Petersson gradient flow for renormalized volume. *ArXiv e-prints*, April 2017.

[7] M. Bridgeman and R. Canary. Renormalized volume and the volume of the convex core. *ArXiv e-prints*, February 2015. To appear, *Ann. Institut Fourier*.

[8] Martin Bridgeman. Average bending of convex pleated planes in hyperbolic three-space. *Invent. Math.*, 132(2):381–391, 1998.

[9] Jeffrey F. Brock. The Weil-Petersson metric and volumes of 3-dimensional hyperbolic convex cores. *J. Amer. Math. Soc.*, 16(3):495–535 (electronic), 2003.

[10] Robert L. Bryant. Surfaces of mean curvature one in hyperbolic space. *Astérisque*, (154-155):12, 321–347, 353 (1988), 1987. Théorie des variétés minimales et applications (Palaiseau, 1985–1984).

[11] Corina Ciobotaru and Sergiu Moroianu. Positivity of the renormalized volume of almost-Fuchsian hyperbolic 3-manifolds. *Proc. Amer. Math. Soc.*, 144(1):151–159, 2016.

[12] David Dumas. The Schwarzian derivative and measured laminations on Riemann surfaces. *Duke Math. J.*, 140(2):203–243, 2007.

[13] David Dumas. Complex projective structures. In *Handbook of Teichmüller theory. Vol. II*, volume 13 of *IRMA Lect. Math. Theor. Phys.*, pages 455–508. Eur. Math. Soc., Zürich, 2008.

[14] Charles L. Epstein. The hyperbolic Gauss map and quasiconformal reflections. *J. Reine Angew. Math.*, 372:96–135, 1986.

[15] Charles L. Epstein. Envelopes of horospheres and Weingarten surfaces in hyperbolic 3-space. Preprint, 1984.

[16] Charles L. Epstein. The hyperbolic Gauss map and quasiconformal reflections. *J. Reine Angew. Math.*, 372:96–135, 1986.

[17] D. B. A. Epstein and A. Marden. Convex hulls in hyperbolic spaces, a theorem of Sullivan, and measured pleated surfaces. In D. B. A. Epstein, editor, *Analytical and geometric aspects of hyperbolic space*, volume 111 of *L.M.S. Lecture Note Series*. Cambridge University Press, 1986.

[18] D. B. A. Epstein and V. Markovic. The logarithmic spiral: a counterexample to the $K = 2$ conjecture. *Ann. of Math.* (2), 161(2):925–957, 2005.

[19] Charles Fefferman and C. Robin Graham. Conformal invariants. *Astérisque*, (Numero Hors Série):95–116, 1985. The mathematical heritage of Elie Cartan (Lyon, 1984).

[20] Charles Fefferman and C. Robin Graham. The ambient metric, volume 178 of *Annals of Mathematics Studies*. Princeton University Press, Princeton, NJ, 2012.

[21] José Antonio Gálvez, Antonio Martínez, and Francisco Milán. Complete linear Weingarten surfaces of Bryant type. A Plateau problem at infinity. *Trans. Amer. Math. Soc.*, 356(9):3405–3428 (electronic), 2004.

[22] William M. Goldman. Geometric structures on manifolds and varieties of representations. In *Geometry of group representations (Boulder, CO, 1987)*, volume 74 of *Contemp. Math.*, pages 169–198. Amer. Math. Soc., Providence, RI, 1988.

[23] C. Robin Graham. Volume and area renormalizations for conformally compact Einstein metrics. In *The Proceedings of the 19th Winter School “Geometry and Physics” (Srní, 1999)*, number 63, pages 31–42, 2000.

[24] C. Robin Graham and Edward Witten. Conformal anomaly of submanifold observables in $AdS/CFT$ correspondence. *Nuclear Phys. B*, 546(1-2):52–64, 1999.

[25] Colin Guillarmou, Sergiu Moroianu, and Jean-Marc Schlenker. The renormalized volume and uniformization of conformal structures. *Journal of the Institute of Mathematics of Jussieu*, pages 1–60, 2016.

[26] Mark Henningson and Kostas Skenderis. The holographic weyl anomaly. *JHEP*, 9807:023, 1998.

[27] Kirill Krasnov. Holography and Riemann surfaces. *Adv. Theor. Math. Phys.*, 4(4):929–979, 2000.
Kirill Krasnov and Jean-Marc Schlenker. On the renormalized volume of hyperbolic 3-manifolds. *Comm. Math. Phys.*, 279(3):637–668, 2008.

Kirill Krasnov and Jean-Marc Schlenker. A symplectic map between hyperbolic and complex Teichmüller theory. *Duke Math. J.*, 150(2):331–356, 2009.

S. Moroianu. Convexity of the renormalized volume of hyperbolic 3-manifolds. *ArXiv e-prints*, March 2015.

Kirill Krasnov and Jean-Marc Schlenker. A symplectic map between hyperbolic and complex Teichmüller theory. *Duke Math. J.*, 150(2):331–356, 2009.

S. Moroianu. Convexity of the renormalized volume of hyperbolic 3-manifolds. *ArXiv e-prints*, March 2015.

Zeev Nehari. The Schwarzian derivative and schlicht functions. *Bull. Amer. Math. Soc.*, 55:545–551, 1949.

Brad Osgood and Dennis Stowe. The Schwarzian derivative, conformal connections, and Möbius structures. *J. Anal. Math.*, 76:163–190, 1998.

Jean-Marc Schlenker. Hypersurfaces in $H^n$ and the space of its horospheres. *Geom. Funct. Anal.*, 12(2):395–435, 2002.

Jean-Marc Schlenker. The renormalized volume and the volume of the convex core of quasi-fuchsian manifolds. *Math. Res. Lett.*, 20(4):773–786, 2013. v4 available as arXiv:1109.6663v4.

L. Takhtajan and P. Zograf. On uniformization of Riemann surfaces and the Weil-Petersson metric on the Teichmüller and Schottky spaces. *Mat. Sb.*, 132:303–320, 1987. English translation in *Math. USSR Sb.* 60:297-313, 1988.

Leon Takhtajan and Peter Zograf. Hyperbolic 2-spheres with conical singularities, accessory parameters and Kähler metrics on $M_{0,n}$. *Trans. Amer. Math. Soc.*, 355(5):1857–1867 (electronic), 2003.

Leon A. Takhtajan and Lee-Peng Teo. Liouville action and Weil-Petersson metric on deformation spaces, global Kleinian reciprocity and holography. *Comm. Math. Phys.*, 239(1-2):183–240, 2003.

Harumi Tanigawa. Grafting, harmonic maps and projective structures on surfaces. *J. Differential Geom.*, 47(3):399–419, 1997.

William P. Thurston. Three-dimensional geometry and topology. Originally notes of lectures at Princeton University, 1979. Recent version available on http://www.msri.org/publications/books/gt3m/, 1980.

William P. Thurston. Zippers and univalent functions. In *The Bieberbach conjecture (West Lafayette, Ind., 1985)*, volume 21 of *Math. Surveys Monogr.*, pages 185–197. Amer. Math. Soc., Providence, RI, 1986.

F. Vargas Palleţe. Local convexity of renormalized volume for rank-1 cusped manifolds. *ArXiv e-prints*, May 2015.

F. Vargas Palleţe. Continuity of the renormalized volume under geometric limits. *ArXiv e-prints*, May 2016.

Richard A. Wentworth. Energy of harmonic maps and Gardiner’s formula. In *In the tradition of Ahlfors-Bers. IV*, volume 432 of *Contemp. Math.*, pages 221–229. Amer. Math. Soc., Providence, RI, 2007.

Michael Wolf. On realizing measured foliations via quadratic differentials of harmonic maps to R-trees. *J. Anal. Math.*, 68:107–120, 1996.

University of Luxembourg, Department of mathematics, University of Luxembourg, Maison du nombre, 6 avenue de la Fonte, L-4364 Esch-sur-Alzette, Luxembourg

E-mail address: jean-marc.schlenker@uni.lu