BPS States, Torus Links and Wild Character Varieties

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Abstract: A string theoretic framework is constructed relating the cohomology of wild character varieties to refined stable pair theory and torus link invariants. Explicit conjectural formulas are derived for wild character varieties with a unique irregular point on the projective line. For this case, this leads to a conjectural colored generalization of existing results of Hausel, Mereb and Wong as well as Shende, Treumann and Zaslow.

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1. Introduction

The main goal of this paper is to develop a string theoretic framework for the cohomology of wild character varieties. Previous such constructions [14–16] have been carried out for regular and tamely ramified character varieties, leading to a physical derivation of the main conjectures of Hausel and Rodriguez-Villegas [40], respectively Hausel, Letellier and Rodriguez-Villegas [38]. Very briefly, using the \( P = W \) conjecture of de Cataldo, Hausel and Migliorini [17], the string theoretic approach places these conjectures in the framework of motivic Donaldson–Thomas theory developed by Kontsevich and Soibelman [52]. The conjectural formulas of [38,40] are then identified in [14,16] with refined Gopakumar–Vafa expansions for certain Calabi–Yau threefolds. An important part of this program, namely the refined stable pair formula for local curves without marked points, has been recently proven by Maulik [58].

The main outcome of the present work is a conjectural generalization of recent results of Hausel, Mereb and Wong [39] as well as Shende, Treumann and Zaslow [73] in the context of wild character varieties with one singular point on the projective line. The string theoretic construction employed in the process provides compelling evidence for the wild variant of the \( P = W \) conjecture of de Cataldo, Hausel, and Migliorini [17].

For completeness, note that topological and motivic invariants of moduli spaces of Higgs bundles and flat connections have been intensively studied in the recent mathematical literature employing different approaches. Arithmetic methods have been used [10,11,26,54,62,70], leading to complete results for Poincaré polynomials of Higgs bundle moduli spaces. Moreover, the motives of the moduli stacks of irregular Higgs bundles, as well as irregular connections over arbitrary fields have been recently computed in [29]. An alternative approach based on wallcrossing for moduli spaces of linear chains on curves was developed in [32], and used in [31] to compute the Hirzebruch genus of moduli spaces of \( PGL(r, \mathbb{C}) \) Higgs bundles. Finally, a different class of character varieties defined using Zariski closures of conjugacy classes at the marked points was studied in [55]. It is not clear at the moment if there is any conceptual relation between these results and the physical approach developed here. This remains an important open question for future research.