INVERSION FORMULA FOR THE HYPERGEOMETRIC FOURIER TRANSFORM ASSOCIATED WITH A ROOT SYSTEM OF TYPE $BC$

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Abstract. We give the inversion formula and the Plancherel formula for the hypergeometric Fourier transform associated with a root system of type $BC$, when the multiplicity parameters are not necessarily non-negative.

Introduction

Heckman and Opdam [13, 10, 11, 12, 21, 23, 14] have developed a theory of hypergeometric functions associated with root systems. When the multiplicity function $k$ takes some particular values, the Heckman-Opdam hypergeometric function coincides with the restriction to a Cartan subspace $a$ of the zonal spherical function on a Riemannian symmetric space of the non-compact type.

In this group case, the associated harmonic analysis have been developed by Harish-Chandra, Gindikin-Karpelevich, Helgason, Gangolli, Rosenberg, and other researchers (cf. [9, 15]). In particular, an explicit inversion formula, the Paley-Wiener theorem, and the Plancherel theorem for the spherical Fourier transform have been established.

Opdam [21] generalized these results to the hypergeometric Fourier transform, where the multiplicity function may take arbitrary non-negative values. In this case there are only continuous spectra. More precisely, the Plancherel measure is $c(\lambda, k)^{-1} c(-\lambda, k)^{-1} d\mu(\lambda)$ on $\sqrt{-1}a^*$, where $c(\lambda, k)$ is Harish-Chandra’s $c$-function and $\mu$ is a normalized Lebesgue measure. In [22] Opdam also studied the case of negative multiplicity functions when the root system is reduced. In such a situation, spectra with supports of lower dimensions appear in the Plancherel measure, in addition to the most continuous spectra described above.

If the root system is of type $BC_1$, then the associated Heckman-Opdam hypergeometric function is nothing but the Jacobi function, which was introduced and studied by Flensted-Jensen and Koornwinder [7, 18, 19]. The

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Jacobi function is an even eigenfunction of a second order ordinary differential operator and can be written by the Gauss hypergeometric function. When the multiplicity function corresponds to a rank one Riemannian symmetric space of the noncompact type, the second order differential operator is the radial part of the Laplace-Beltrami operator and the Jacobi function is the restriction of the zonal spherical function to a Cartan subspace \( a \simeq \mathbb{R} \). Let \( k_s \) and \( k_\ell \) denote the multiplicity parameters for the short roots \( \pm \beta \) and the long roots \( \pm 2\beta \), respectively. Flensted-Jensen \cite{7} Appendix 1 proved the inversion formula for the Jacobi transform (the hypergeometric Fourier transform in the \( BC_1 \) case) under the conditions \( k_s, k_\ell \in \mathbb{R} \) and \( k_s + k_\ell > -\frac{1}{2} \). If \( k \) further satisfies \( k_s + k_\ell + \frac{1}{2} - |k_\ell - \frac{1}{2}| \geq 0 \), then only continuous spectra appear. In the other cases, finite series of discrete spectra appear in addition to the continuous spectra. These discrete spectra and the corresponding Plancherel measure are obtained from residues of \( c(\lambda, k)^{-1}c(-\lambda, k)^{-1} \). Flensted-Jensen \cite{7} applied his results on the Jacobi transform to harmonic analysis of spherical functions on the universal covering group of \( SU(n,1) \) associated with a one-dimensional \( K \)-type. The Jacobi transform also can be applied to the harmonic analysis on some homogeneous vector bundles over hyperbolic spaces (cf. \cite{5,29}). In these group cases, discrete spectra correspond to relative discrete series representations.

The Heckman-Opdam hypergeometric function is a real analytic joint eigenfunction of a commuting family of differential operators. In a group case, the differential operators are radial parts of the invariant differential operators. The inversion formula for the hypergeometric Fourier transform gives an expansion of an arbitrary Weyl group invariant function in terms of the Heckman-Opdam hypergeometric functions. Heckman-Opdam theory gives a simultaneous generalization of the Euclidean Fourier analysis (the case of \( k = 0 \), the theory of spherical functions, and the Jacobi analysis, and it provides a rich framework of harmonic analysis associated with root systems.

In this paper we will study the case of the root system of type \( BC \) and arbitrary rank when the multiplicity function is not necessarily non-negative. Except for the case of type \( BC_1 \) mentioned above or a group case studied by the third author \cite{28}, this study has remained open for many years. We have decided to study this case for its application to harmonic analysis of spherical functions associated with certain \( K \)-types on connected semisimple Lie groups of finite center (cf. \cite{20} final comment).

For the root system of type \( BC_r \) with \( r \geq 2 \), there are three multiplicity parameters \( k_s, k_m \) and \( k_\ell \) corresponding to the short, medium, and long
roots respectively. We prove the inversion formula, the Paley-Wiener theorem, and the Plancherel theorem for the hypergeometric Fourier transform under the conditions $k_s, k_m, k_t \in \mathbb{R}$, $k_s + k_t > -\frac{1}{2}$ and $k_m \geq 0$. These conditions on $k$ make the spectral problem well-posed and cover the known group cases. An explicit expression of the Plancherel measure is obtained by calculus of residues of $c(\lambda, k)^{-1}c(-\lambda, k)^{-1}$. In particular, the square integrable hypergeometric functions are classified and their $L^2$-norms are calculated explicitly. Our method using residue calculus closely follows that of [28], where the third author studied spherical functions associated with a one-dimensional $K$-type on an irreducible Hermitian symmetric space and obtained the inversion formula for the spherical transform.

This paper is organized as follows. In Section 1 we review the Heckman-Opdam hypergeometric function. In Section 2 we define the hypergeometric Fourier transform associated with a root system of type $BC$ for various $k$ and derive a first form of inversion formula (Theorem 2.5 and Theorem 2.9) from Opdam’s result for the non-negative multiplicity functions. In Section 3 we define some sets of tempered spectra (3.12) that are crucial to the main results of this paper and study tempered and square integrable hypergeometric functions. In Section 4 we introduce a partial sum of Harish-Chandra series in order to simplify the residue calculus in the subsequent section. In Section 5 we prove the final form of inversion formula (Theorem 5.3), the Paley-Wiener theorem (Theorem 5.5), the Plancherel theorem (Theorem 5.7), and give the classification of square integrable hypergeometric functions and their $L^2$-norms explicitly (Corollary 5.9). Explicit formulas of the residues of $c(\lambda, k)^{-1}c(-\lambda, k)^{-1}$ that contribute to the Plancherel measure are given by (5.1), (5.2) and Proposition 5.4.

1. HECKMAN-OPDAM HYPERGEOMETRIC FUNCTIONS

In this section, we give basic notations for a root system of type $BC$ and review on the Heckman-Opdam hypergeometric function associated with a root system of type $BC$. We refer the reader to original papers by Heckman and Opdam [10, 13, 21] and survey articles [11, 12, 23, 1, 14] for details.

Let $\mathbb{N}$ denote the set of non-negative integers. Let $r$ denote a positive integer and $\mathfrak{a}$ an $r$-dimensional Euclidean space with an inner product $\langle \cdot, \cdot \rangle$. We often identify $\mathfrak{a}$ and $\mathfrak{a}^*$ by using the inner product. We use the same notation $\langle \cdot, \cdot \rangle$ for the inner product on $\mathfrak{a}^*$ and the complex symmetric bilinear form on $\mathfrak{a}_c^\times \mathfrak{a}_c^\times$. For $\lambda \in \mathfrak{a}_c^\times$ define $||\lambda|| = \sqrt{\langle \lambda, \bar{\lambda} \rangle}$, where $\bar{\cdot}$ denotes complex conjugation. Let $\mathcal{R} \subset \mathfrak{a}^*$ be a root system of type $BC_r$. It is of the form

$$\mathcal{R} = \{ \pm \beta_i, \pm 2\beta_i, \pm (\beta_p \pm \beta_q) : 1 \leq i \leq r, 1 \leq q < p \leq r \}.$$
where \( \{ \beta_1, \beta_2, \ldots, \beta_r \} \) is an orthogonal basis of \( \mathfrak{a}^* \) with \( ||\beta_p|| = ||\beta_q|| \) (1 \( \leq q < p \leq r \)). Moreover, we assume \( ||\beta_1|| = 2 \), so \( \{ \frac{1}{2} \beta_1, \ldots, \frac{1}{2} \beta_r \} \) forms an orthonormal basis of \( \mathfrak{a}^* \). Let \( \mathcal{R}^+ \) denote the positive system of \( \mathcal{R} \) defined by
\[
\mathcal{R}^+ = \{ \beta_i, 2\beta_i, \beta_p \pm \beta_q \; : \; 1 \leq i \leq r, 1 \leq q < p \leq r \}.
\]
Let \( \alpha_1, \ldots, \alpha_r \) denote the positive roots defined by
\[
\alpha_i = \beta_{r+1-i} - \beta_{r-i} \quad (1 \leq i \leq r - 1), \quad \alpha_r = \beta_1.
\]

Then
\[
\mathcal{B} = \{ \alpha_1, \alpha_2, \ldots, \alpha_r \}
\]
is the set of the simple roots in \( \mathcal{R}^+ \). We call \( \pm \beta_i, \pm (\beta_p \pm \beta_q) \) (\( p < q \)), and \( \pm 2\beta_i \), short, medium, and long roots, respectively. Let \( \mathcal{R}_s^+, \mathcal{R}_m^+, \) and \( \mathcal{R}_l^+ \) denote the subset of \( \mathcal{R}^+ \) consisting of the short, medium, and long roots, respectively. Let \( W \) denote the Weyl group for \( \mathcal{R} \). It is the semidirect product of \( \mathbb{Z}_2 \) and \( \mathfrak{S}_r \). The group \( \mathbb{Z}_2 \) and \( \mathfrak{S}_r \) act on \( \mathfrak{a}^* \) as sign changes and permutations of \( \{ \beta_1, \ldots, \beta_r \} \), respectively.

Let \( \mathbf{k} \) be a complex-valued \( W \)-invariant function on \( \mathcal{R} \), which is called a multiplicity function. Let \( \mathcal{K}_C \) be the space of the multiplicity functions. Let \( k_s, k_m, \) and \( k_l \) be the values of \( \mathbf{k} \) for the short, medium, and long roots respectively. We identify \( \mathbf{k} \in \mathcal{K}_C \) with the 3-tuple \( (k_s, k_m, k_l) \in \mathbb{C}^3 \), so \( \mathcal{K}_C \cong \mathbb{C}^3 \). Let
\[
\rho(\mathbf{k}) = \frac{1}{2} \sum_{\alpha \in \mathcal{R}^+} k_\alpha \alpha.
\]

For \( \alpha \in \mathcal{R} \) let \( \alpha^\vee \) denote the corresponding coroot \( \alpha^\vee = 2\alpha/\langle \alpha, \alpha \rangle \). Any \( \lambda \in \mathfrak{a}_C^* \) can be written in the form
\[
\lambda = \frac{1}{2} \sum_{i=1}^r \lambda_i \beta_i \quad \text{with} \quad \lambda_i = \langle \lambda, \beta_i^\vee \rangle \quad (1 \leq i \leq r).
\]
We identify \( \lambda \in \mathfrak{a}_C^* \) with the \( r \)-tuple \( (\lambda_1, \ldots, \lambda_r) \in \mathbb{C}^r \), so \( \mathfrak{a}_C^* \cong \mathbb{C}^r \). In particular,
\[
\rho(\mathbf{k}) = (k_s + 2k_l, k_s + 2k_l, 2k_m, \ldots, k_s + 2k_l + 2(r - 1)k_m).
\]

For \( \mathbf{k} \in \mathcal{K}_C \) let \( L(\mathbf{k}) \) denote the differential operator on \( \mathfrak{a} \) defined by
\[
L(\mathbf{k}) = \frac{1}{4} \sum_{i=1}^r \partial_{\beta_i}^2 + \sum_{\alpha \in \mathcal{R}^+} k_\alpha \coth \alpha \frac{\alpha}{2} \partial_{\alpha}.
\]
Here \( \partial_{\alpha} \) denotes the directional derivative corresponding to \( \alpha \). There exists an algebra \( \mathbb{D}(\mathbf{k}) \) of \( W \)-invariant differential operators on \( \mathfrak{a} \) with the properties: \( L(\mathbf{k}) \in \mathbb{D}(\mathbf{k}) \) and there exists an algebra isomorphism \( \gamma_k : \mathbb{D}(\mathbf{k}) \cong S(\mathfrak{a}_C)^W \), where \( S(\mathfrak{a}_C)^W \) denotes the set of the \( W \)-invariant elements in the symmetric algebra \( S(\mathfrak{a}_C) \). For \( \lambda \in \mathfrak{a}_C^* \) we consider the system of differential equations
\[
Df = \gamma_k(D)(\lambda)f \quad \text{for any} \quad D \in \mathbb{D}(\mathbf{k}).
\]
In particular, it contains the equation
\begin{equation}
L(k)f = \langle (\lambda, \lambda) - \langle \rho(k), \rho(k) \rangle \rangle f.
\end{equation}

Let
\[
\alpha_+ = \{x \in a^* ; \alpha(x) > 0 \text{ for all } \alpha \in \mathcal{R}^+\}.
\]
Let \(Q_+\) denote the subset of \(a^*\) spanned by \(\mathcal{B}\) over \(\mathbb{N}\). There exists a series solution \(\Phi(\lambda, k)\) of (1.8) of the form
\begin{equation}
\Phi(\lambda, k; x) = \sum_{\kappa \in Q_+} a_\kappa(\lambda, k) e^{(\lambda - \rho(k) - \kappa)(x)} \quad (x \in \alpha_+),
\end{equation}
with \(a_0(\lambda, k) = 1\). For a generic \(\lambda\) all the coefficients \(a_\kappa(\lambda, k)\) \((\kappa \in Q_+)\) are determined uniquely by (1.9) and \(\Phi(\lambda, k)\) converges on \(\alpha_+\). We call \(\Phi(\lambda, k)\) the Harish-Chandra series. As a function of the spectral parameter \(\lambda \in a^*_C\), \(\Phi(\lambda, k)\) is meromorphic with simple poles along hyperplanes of the form
\begin{equation}
\langle \lambda, \alpha^\vee \rangle = j \quad \text{for some } \alpha \in \mathcal{R}^+, \ j = 1, 2, 3, \ldots.
\end{equation}
See [11, Proposition 4.2.5]. Moreover, if
\begin{equation}
\langle \lambda, \alpha^\vee \rangle \notin \mathbb{Z} \quad \text{for all } \alpha \in \mathcal{R},
\end{equation}
then \(\{\Phi(w\lambda, k) ; w \in W\}\) forms a basis of the solution space of (1.8) on \(\alpha_+\).

The condition (1.12) can be written explicitly as
\begin{equation}
\lambda_i \notin \mathbb{Z} \quad (1 \leq i \leq r), \quad \frac{\lambda_p + \lambda_q}{2} \notin \mathbb{Z} \quad (1 \leq p \neq q \leq r).
\end{equation}

Define
\[
a_+^* = \{\lambda \in a^* ; \langle \lambda, \alpha \rangle > 0 \quad \text{for all } \alpha \in \mathcal{R}^+\}
\]
\[
\simeq \{(\lambda_1, \lambda_2, \ldots, \lambda_r) \in \mathbb{R}^r ; 0 < \lambda_1 < \lambda_2 < \cdots < \lambda_r\}.
\]
Let \(\text{Cl}(a^*_+)\) denote the closure of \(a^*_+\).

**Lemma 1.1.** Suppose \(T \subset a^*\) and \(U \subset K_C \simeq \mathbb{C}^3\) are compact sets. Define a polynomial \(p(\lambda)\) to be the product of \(\langle 2\lambda - \kappa, \kappa \rangle\) taken over \(\kappa \in Q_+ \setminus \{0\}\) such that \(\langle 2\lambda - \kappa, \kappa \rangle = 0\) for some \(\lambda \in T\). (Note that this is a finite product.) Then for any \(\varepsilon > 0\) there exist constants \(C > 0\) and \(n \in \mathbb{N}\) such that
\[
|p(\lambda)a_\kappa(\lambda, k)| \leq C(1 + ||\lambda||)^n e^{\varepsilon(x)}
\]
whenever \(\kappa \in Q_+\), \(\Re \lambda \in T\), \(k \in U\) and \(x \in a\) satisfies \(\alpha(x) > \varepsilon\) for any \(\alpha \in \mathcal{B}\). Hence for any compact subset \(V \subset a_+\) and any \(q \in S(a_C)\) there exist constants \(C' > 0\) and \(n' \in \mathbb{N}\) such that
\[
|q(\partial_{\lambda})p(\lambda)\Phi(\lambda, k; x)| \leq C'(1 + ||\lambda||)^{n'}
\]
for \((\lambda, k, x) \in (T + \sqrt{-1}a^*) \times U \times V\).

**Proof.** The lemma follows from a natural extension of the estimates for the coefficients \(a_\kappa(\lambda, k)\) due to Gangolli. See [15, Ch. IV Lemma 5.6] and [9, Theorem 4.5.4]. See also [2] Ch. I Lemma 5.1] and [28, Corollary 3.11]. □
Let $\Gamma(\cdot)$ denote the Gamma function. Define the meromorphic functions $\tilde{c}_\alpha(\lambda, k)$ ($\alpha \in \mathcal{R}$), $\tilde{c}(\lambda, k)$, and $c(\lambda, k)$ on $\mathfrak{a}^\times \times K_C$ by
\begin{equation}
\tilde{c}_\alpha(\lambda, k) = \frac{\Gamma(\langle \lambda, \alpha \rangle + \frac{1}{2} k_{\frac{1}{2}\alpha})}{\Gamma(\langle \lambda, \alpha \rangle + \frac{1}{2} k_{\frac{1}{2}\alpha} + k_\alpha)}.
\end{equation}
\begin{equation}
\tilde{c}(\lambda, k) = \prod_{\alpha \in \mathcal{R}^+} \tilde{c}_\alpha(\lambda, k),
\end{equation}
and
\begin{equation}
c(\lambda, k) = \frac{\tilde{c}(\lambda, k)}{\tilde{c}(\rho(k), k)}
\end{equation}
with the convention $k_{\frac{1}{2}\alpha} = 0$ if $\frac{1}{2}\alpha \notin \mathcal{R}$. We call $c(\lambda, k)$ Harish-Chandra’s $c$-function. By [11] (3.4.6) and [12] Proposition 5.1,
\begin{equation}
\tilde{c}(\lambda, k) = \prod_{1 \leq q < p \leq r} \frac{\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q)\right) \Gamma\left(\frac{1}{2}(\lambda_p + \lambda_q)\right)}{\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q + 2k_m)\right) \Gamma\left(\frac{1}{2}(\lambda_p + \lambda_q + 2k_m)\right)}
\end{equation}
\begin{equation}
\times \prod_{i=1}^{r} \frac{2^{-k_i} \Gamma\left(\frac{1}{2}\lambda_i\right) \Gamma\left(\frac{1}{2}(\lambda_i + 1)\right)}{\Gamma\left(\frac{1}{2}(\lambda_i + k_s + 1)\right) \Gamma\left(\frac{1}{2}(\lambda_i + k_s + 2k_{\ell})\right)}
\end{equation}
and
\begin{equation}
\tilde{c}(\rho(k), k) = \prod_{\ell=1}^{r} \frac{\Gamma(k_s + (i-1)k_m + k_{\ell}) \Gamma(k_m)}{\Gamma(2k_s + (i-1)k_m + k_{\ell}) \Gamma(ik_m)}.
\end{equation}
The latter is regular on the entire $K_C$. Note
\begin{equation}
c(\lambda, 0) = \frac{1}{2^{|\mathcal{R}|} |W|} \quad \text{for} \quad k_s = k_m = k_{\ell} = 0.
\end{equation}

Let $\mathcal{K}_{\text{reg}}$, $\mathcal{K}$, and $\mathcal{K}_+$ denote the subsets of $K_C \simeq \mathbb{C}^3$ given by
\begin{align*}
\mathcal{K}_{\text{reg}} &= \{k \in K_C ; \tilde{c}(\rho(k), k) \neq 0\}, \\
\mathcal{K} &= \{k \in K_C ; k_s, k_m, k_{\ell} \in \mathbb{R}\}, \\
\mathcal{K}_+ &= \{k \in \mathcal{K} ; k_\alpha \geq 0 \quad \text{for all} \quad \alpha \in \mathcal{R}\}.
\end{align*}

Thus $\mathcal{K}_+ \subset \mathcal{K}_{\text{reg}}$.

Throughout the paper we repeatedly use the following type of estimate which is applicable to $c(-\lambda, k)^{-1}$ and other similar functions.

**Lemma 1.2.** Let $\{(v_j, a_j(k), b_j(k))\}_{j=1}^{k}$ be a sequence of triples consisting of $v_j \in \mathfrak{a} \setminus \{0\}$ and polynomials $a_j(k), b_j(k)$ in $k$. Suppose $T \subset \mathfrak{a}^*$ and $U \subset K_C$ are compact sets such that
\begin{equation}
\psi(\lambda, k) = \prod_{j=1}^{k} \frac{\Gamma(\lambda(v_j) + a_j(k))}{\Gamma(\lambda(v_j) + b_j(k))}
\end{equation}
is regular on $(T + \sqrt{-1} \mathfrak{a}^*) \times U$. Then there exists constants $C > 0$ and $n \in \mathbb{N}$ such that
\begin{equation}
|\psi(\lambda, k)| \leq C(1 + ||\lambda||)^n
\end{equation}
on $(T + \sqrt{-1} \mathfrak{a}^*) \times U$.

**Proof.** Choose an arbitrary $v \in \{v_1, \ldots, v_k\}$ and let $\psi_v(\lambda, k)$ be the product of $\Gamma(\lambda(v_j) + a_j(k))/\Gamma(\lambda(v_j) + b_j(k))$ for all $j$ such that $v_j$ is proportional to
v. Then by the local theory of meromorphic functions, \( \psi_v(\lambda, k) \) is regular on \((T + \sqrt{-1}a^*) \times U\). By Stirling’s formula

\[
\Gamma(z) = \left( \frac{2\pi}{z} \right)^z \left( 1 + O\left( \frac{1}{z} \right) \right) \quad (|\arg z| < \pi - \delta, |z| \to \infty; \forall \delta > 0),
\]

there exist constants \( C_v > 0 \) and \( n_v \in \mathbb{N} \) such that

\[
|\psi_v(\lambda, k)| \leq C_v(1 + |\lambda(v)|)^{n_v}
\]
on \((T + \sqrt{-1}a^*) \times U\). By Stirling’s formula

\[
\Gamma(z) = \frac{2\pi}{z} \left( 1 + O\left( \frac{1}{z} \right) \right) \quad (|\arg z| < \pi - \delta, |z| \to \infty; \forall \delta > 0),
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\]

there exist constants \( C_v > 0 \) and \( n_v \in \mathbb{N} \) such that

\[
|\psi_v(\lambda, k)| \leq C_v(1 + |\lambda(v)|)^{n_v}
\]
on \((T + \sqrt{-1}a^*) \times U\). Since \( \psi \) is a product of \( \psi_v \)'s, we are done. (See [15, Ch. IV, Proposition 7.2], [9, Proposition 4.7.15], [28, Lemma 3.9], and [18, Lemma 2.2].)

\[\Box\]

For \( \lambda \in a_C^\circ \) satisfying (1.12) and \( k \in K_{reg} \), define

\[
(1.20) \quad F(\lambda, k; x) = \sum_{w \in W} c(w\lambda, k) \Phi(w\lambda, k; x).
\]
The coefficients \( c(w\lambda, k) \) and terms \( \Phi(w\lambda, k) \) on the right hand side are meromorphic function on \( a_C^\circ \times K_{reg} \) and indeed regular for \( \lambda \) and \( k \) satisfying (1.12) and \( k \in K_{reg} \). A deep theorem due to Heckman and Opdam says that \( F(\lambda, k; x) \) extends to an analytic function on \( a_C^\circ \) and \( F(\lambda, k; 0) = 1 \). It satisfies

\[
(1.21) \quad F(w\lambda, k; x) = F(\lambda, k; x) \quad \text{for all } w \in W,
\]

\[
(1.22) \quad F(\lambda, kw; x) = F(\lambda, k; x) \quad \text{for all } w \in W,
\]

\[
(1.23) \quad F(\lambda, k; x) = F(\bar{\lambda}, \bar{k}; x).
\]

\( F(\lambda, k; x) \) is the unique \( W \)-invariant real analytic solution of the hypergeometric system \([18]\) on \( a \) satisfying \( F(\lambda, k; 0) = 1 \). We call \( F(\lambda, k; x) \) the Heckman-Opdam hypergeometric function associated with the root system \( \mathcal{R} \).

Remark 1.3. (1) The Heckman-Opdam hypergeometric function is a generalization of the zonal spherical function on a Riemannian symmetric space of the non-compact type. Let \( G \) be a connected non-compact semisimple Lie group of finite center with the Cartan decomposition \( G = K \exp aK \).

Let \( \Sigma \subset a^\circ \) be the restricted root system for \( (g, a) \) and \( m_\alpha \) the dimension of the root space corresponding to \( \alpha \in \Sigma \). Set

\[
\mathcal{R} = 2\Sigma, \quad k_\alpha = m_{\alpha/2} \quad (\alpha \in \mathcal{R}).
\]

Then \( L(k) \) is the radial part of the Laplace-Beltrami operator on \( G/K \), \( c(\lambda, k) \) is Harish-Chandra’s \( c \)-function, \( F(\lambda, k) \) is the restriction to \( a \) of the zonal spherical function on \( G/K \). (Here \( \mathcal{R} \) and \( \Sigma \) are not necessarily of type \( BC \).)

The zonal spherical function is a bi-\( K \)-invariant function on \( G \). More generally, elementary spherical functions associated with some \( K \)-types are expressed by the Heckman-Opdam hypergeometric function. The case of one-dimensional \( K \)-types when \( G \) is of Hermitian type is studied by [11].
Section 5] and [23]. More generally, the cases of small $K$-types are studied by [20]. We call these cases “the group case” collectively.

(2) When the root system $\mathcal{R}$ is of type $BC_1$, the Hecke-Opdam hypergeometric function is the Jacobi function studied by Flensted-Jensen and Koornwinder:

$$F(\lambda, k; x) = \phi^{(\alpha, \beta)}_{\sqrt{-1}}(z)$$

$$: = 2F_1\left(\frac{1}{2}(\lambda + \rho(k)), \frac{1}{2}(-\lambda + \rho(k)); \alpha + 1; -(\sinh z)^2\right),$$

with $z = \frac{1}{2}\beta_1(x)$, $\alpha = k_s + k_\ell - \frac{1}{2}$, $\beta = k_\ell - \frac{1}{2}$, and $\rho(k) = \alpha + \beta + 1$, where $2F_1$ denote the Gauss hypergeometric function (cf. [7, 18, 19]).

(3) The hypergeometric system (1.8) has regular singular points at infinity in $\mathfrak{a}_+$ and is holonomic of rank $|W|$. The leading exponents at infinity of (1.8) are of the form $w\lambda - \rho(k)$ ($w \in W$). If $\lambda \in \mathfrak{a}^\bullet_+$ satisfies (1.12), then $\Phi(w\lambda, k)$ ($w \in W$) are solutions of (1.8) with the leading exponents $w\lambda - \rho(k)$ ($w \in W$). Even if (1.12) does not hold, there are still $|W|$ linearly independent real analytic solutions on $\mathfrak{a}_+$ with leading exponents $w\lambda - \rho(k)$ (counting the multiplicity on the wall in $\mathfrak{a}^\bullet$). They may have polynomial terms (or logarithmic terms when taking $\exp x$ as a coordinate) as in the case of the Gauss hypergeometric differential equation. For general $\lambda \in \mathfrak{a}^\bullet_+$, the asymptotic expansion (1.20) becomes a convergent expansion on $\mathfrak{a}_+$ of the form

$$F(\lambda, k; x) = \sum_{\mu \in W} \sum_{\kappa \in Q_+} p_{\mu, \kappa}(\lambda, k; x) e^{(\mu - \rho(k) - \kappa)(x)},$$

where $p_{\mu, \kappa}$ are polynomials in $x$. See [13, 10], [11, §4.2], and [24, 25, 26].

For $k \in \mathcal{K}_C$ let $\delta_k$ denote the weight function on $\mathfrak{a}$ given by

$$\delta_k = \prod_{\alpha \in \mathcal{R}^+} |e^{\frac{1}{2} \alpha} - e^{-\frac{1}{2} \alpha}|^{2k_\alpha}$$

$$= \prod_{\alpha \in \mathcal{R}_+^+} |e^{\frac{1}{2} \alpha} - e^{-\frac{1}{2} \alpha}|^{2k_\alpha + 2k_\ell} (e^{\frac{1}{2} \alpha} + e^{-\frac{1}{2} \alpha})^{2k_\ell} \prod_{\beta \in \mathcal{R}_+^0} |e^{\frac{1}{2} \beta} - e^{-\frac{1}{2} \beta}|^{2k_\beta}.$$

In order to describe a symmetry of $F(\lambda, k)$ with respect to $k$, we introduce the multiplicity function $\hat{k}$ associated with $k$, which is defined by

$$\hat{k}_s = k_s + 2k_\ell - 1, \quad \hat{k}_m = k_m, \quad \hat{k}_\ell = 1 - k_\ell.$$

By [11] Theorem 2.1.1,

$$\delta_{\hat{k}}^{1/2} \circ (L(k) + \langle \rho(k), \rho(k) \rangle) \circ \delta_k^{-1/2}$$

$$= \delta_{\hat{k}}^{1/2} \circ (L(\hat{k}) + \langle \rho(\hat{k}), \rho(\hat{k}) \rangle) \circ \delta_k^{-1/2}.$$

It follows that

$$\Phi(\lambda, k) = \delta_{\hat{k}}^{1/2} \delta_k^{-1/2} \Phi(\lambda, \hat{k}).$$
On the other hand, in view of the characterization of the hypergeometric functions we have
\begin{equation}
(1.26) 
F(\lambda, k) = 2^{2k} e^{-\frac{1}{2} \beta_1} e^{\frac{1}{2} \beta_2} F(\lambda, \hat{k}).
\end{equation}
Here
\begin{equation}
2^{2k} e^{-\frac{1}{2} \beta_1} e^{\frac{1}{2} \beta_2} = \prod_{i=1}^{r} \left( \cosh \frac{\beta_i}{2} \right)^{1-2k_i}
\end{equation}
is a nowhere vanishing analytic function on \( \mathfrak{a} \). For the elementary spherical functions associated with a one-dimensional \( K \)-type, the above formula was given by [11, Theorem 5.2.2] and [28, Proposition 2.6, Remark 3.8]. From (1.17) and (1.18) we have \( \tilde{c}(\lambda, k) = 2^{2k} e^{-\frac{1}{2} \beta_1} e^{\frac{1}{2} \beta_2} c(\lambda, \hat{k}) \) and \( \tilde{c}(\rho(k), k) = \tilde{c}(\rho(\hat{k}), \hat{k}) \). It follows that
\begin{equation}
(1.27) 
c(\lambda, k) = 2^{2k} e^{-\frac{1}{2} \beta_1} e^{\frac{1}{2} \beta_2} c(\lambda, \hat{k}).
\end{equation}

2. HYPERGEOMETRIC FOURIER TRANSFORM

Let \( dx \) and \( d\lambda \) denote the Lebesgue measures on the Euclidean spaces \( \mathfrak{a} \) and \( \mathfrak{a}^* \), respectively. Then
\begin{equation}
(2.1) 
d\lambda = d\lambda_1 d\lambda_2 \cdots d\lambda_r,
\end{equation}
in terms of the coordinates (1.3). (Recall that we assume \( ||\beta_1|| = 2 \). We should put the factor \( (||\beta_1||/2)^r \) on the right hand side of (2.1) if we consider a general inner product on \( \mathfrak{a}^* \).) For any \( \eta \in \mathfrak{a}^* \) let \( d\mu(\lambda) \) denote the measure on \( \eta + \sqrt{-1} \mathfrak{a}^* \) given by
\begin{equation}
(2.2) 
d\mu(\lambda) = (2\pi)^{-r} d(\mathrm{Im} \lambda) = (2\pi \sqrt{-1})^{-r} d\lambda.
\end{equation}
If \( \eta = 0 \) then the measures \( dx \) on \( \mathfrak{a} \) and \( d\mu(\lambda) \) on \( \sqrt{-1} \mathfrak{a}^* \) are normalized so that the inversion formula for the Euclidean Fourier transform
\begin{equation}
\hat{f}(\lambda) = \int_{\mathfrak{a}} f(x) e^{-\lambda(x)} dx \quad (f \in C_0^\infty(\mathfrak{a}))
\end{equation}
is given by
\begin{equation}
f(x) = \int_{\sqrt{-1} \mathfrak{a}^*} \hat{f}(\lambda) e^{\lambda(x)} d\mu(\lambda).
\end{equation}

A necessary and sufficient condition for the local integrability of the weight function \( \delta_k \) is given by [31, Section 2] and [22, Proposition 5.1]. Clearly \( \delta_k \) is locally integrable if and only if \( \delta_{\text{Re} k} \) is. For a real \( k \in \mathbb{K} \), \( \delta_k \) is locally integrable if and only if
\begin{equation}
k_+ + k_\ell > -\frac{1}{2} + \max\{0, -(r-1)k_m\} \quad \text{and} \quad k_m > -\frac{1}{r}.
\end{equation}
(Since \( e^t - e^{-t} = 2t + o(t) \), \( \delta_k \) is locally integrable if and only if the Selberg integral \( S_r(k_+ + k_\ell + \frac{1}{2}, 1, k_m) \) converges (cf. [8]).) Moreover, by (1.18), (2.3) holds if and only if \( k \) is in the connected component of \( \mathbb{K}_\text{reg} \cap \mathbb{K} \) containing \( \mathbb{K}_+ \). Let \( \mathbb{K}_1 \) denote the set of all \( k \in \mathbb{K} \) that satisfy (2.3). If \( k \in \mathbb{K}_1 \) then the problem of giving the spectral decomposition of \( L^2(\mathfrak{a}; \frac{1}{2i} \delta_k(x) dx) \) with respect to the hypergeometric function \( F(\lambda, k) \) makes sense.
Now for \( k \in K_1 + \sqrt{-1}K \) we define the hypergeometric Fourier transform \( \mathcal{F}_k \) of \( f \in C^{\infty}(a)^W \) by
\[
\mathcal{F}_k f(\lambda) = \int_{a_+} f(x)F(\lambda, k; x)\delta_k(x)dx = \frac{1}{|W|} \int_a f(x)F(\lambda, k; x)\delta_k(x)dx,
\]
which is a holomorphic function in \( \lambda \in a^*_C \). (Note \( F(\lambda, k; -x) = F(-\lambda, k; x) = F(\lambda, k; x) \) since \( \id_a \in W \).) Observe that \( \mathcal{F}_k f(\lambda) \) is also holomorphic in \( k \).

Given a \( W \)-invariant non-empty convex compact subset \( C \subset a \), consider the function \( H_C(\lambda) = \max_{x \in C} \lambda(x) \) of \( \lambda \in a^* \).

**Proposition 2.1.** Suppose the support of \( f \in C^{\infty}_0(a)^W \) is contained in \( C \). Then for any compact subset \( U \subset K_1 + \sqrt{-1}K \) and any \( n \in \mathbb{N} \)
\[
\sup_{\lambda \in a^*_C, k \in U} (1 + ||\lambda||)^n e^{-H_C(\text{Re}\lambda)}|\mathcal{F}_k f(\lambda)| < +\infty.
\]

For the proof of the proposition we need some preparation. First, by the same method as in the proof of [21, Proposition 6.1] we can prove

**Lemma 2.2.** Suppose \( k \in K_1 + \sqrt{-1}K \) and put \( \kappa = \frac{1}{2} \sum_{\alpha \in R^+} |\text{Im}(k_{\alpha})| \alpha \). Then for any \( x \in a \) and \( \lambda \in a^*_C \) it holds that
\[
|F(\lambda, k; x)| \leq |W|^\frac{1}{2} e^{\max_{\alpha \in W} \text{Re}_\lambda(x) + \max_{\alpha \in W} \text{Im}_\lambda(x)}.
\]

Next, for \( k \in K_C \) and \( \xi \in a \) we define the Cherednik operator
\[
T(k, \xi) = \partial_\xi + \sum_{\alpha \in R^+} k_{\alpha}(\xi) \frac{1}{1 - e^{-\alpha}}(1 - r_{\alpha}) - \rho(k)(\xi)
\]
where \( r_{\alpha} \in W \) is the reflection corresponding to \( \alpha \). This acts on \( C^{\infty}(a) \) as a differential-difference operator. If \( f \in C^{\infty}_0(a) \) is supported in \( C \) then \( T(k, \xi)f \) is also. The operators \( T(k, \xi) (\xi \in a) \) mutually commute and define a unique action \( T(k, p) \) on \( C^{\infty}(a) \) for any \( p \in S(a_C) \). If \( p \in S(a_C)^W \) then \( T(k, p) \) commutes with the \( W \)-action and there exists a differential operator \( D(k, p) \in \mathbb{D}(k) \) with \( \gamma_k(D(k, p)) = p \) such that \( T(k, p)f = D(k, p)f \) for any \( f \in C^{\infty}(a)^W \). In particular
\[
T(k, p)F(\lambda, k; x) = p(\lambda)F(\lambda, k; x) \quad \text{for any} \quad p \in S(a_C)^W.
\]

Let \( \mathcal{R}^0 = \mathcal{R}_0 \cup \mathcal{R}_m \) and put
\[
\Delta = \prod_{\alpha \in \mathcal{R}\cap \mathcal{R}^+} (e^{\frac{\alpha}{2}} - e^{-\frac{\alpha}{2}}) \in C^{\infty}(a),
\]
\[
\pi^\pm(k) = \prod_{\alpha \in \mathcal{R}^+} \left( \alpha^\pm \pm \left( k_\ell \pm \frac{k_s}{2} \right) \right) \prod_{\alpha \in \mathcal{R}^+} (\alpha^\pm \pm k_m) \in S(a_C),
\]
\[
G^+(k) = \Delta(x)^{-1} \circ T(k, \pi^+),
\]
\[
G^-(k) = T(k - 1, \pi^-) \circ \Delta(x).
\]
Here \( 1 \) denotes the multiplicity function which takes \( 1 \) on \( \mathcal{R}^0 \) and \( 0 \) on \( \mathcal{R}_s \). \( G^\pm(k) \) act on \( C^{\infty}(a)^W \) and there exist \( W \)-invariant differential operators
\( \hat{G}^\pm(k) \) on \( a \) such that \( G^\pm(k)f = \hat{G}^\pm(k)f \) for \( f \in C^\infty(a)^W \). \( \hat{G}^\pm(k) \) are called the hypergeometric shift operators with shift \( \pm 1 \) (cf. [14, §8.4.3]). All coefficients of \( \hat{G}^\pm(k) \) are analytic at least on \( a_+ \). It is known that \( \mathcal{K}_{\text{reg}} + 1 \subset \mathcal{K}_{\text{reg}} \) and that

\[
\tag{2.7} G^-(k + 1)F(\lambda, k + 1; x) = \frac{\tilde{c}(\rho(k), k)}{\tilde{c}(\rho(k + 1), k + 1)} F(\lambda, k; x)
\]

(cf. [11, Corollaries 3.6.5 and 3.6.7]). Also

\[
\tag{2.8} \hat{G}^+(k)\phi(\lambda, k; x) = (-1)^r \frac{\tilde{c}(-\lambda, k)}{\tilde{c}(-\lambda, k + 1)} \phi(\lambda, k + 1; x)
\]

(cf. [11, Ch. 3]). Here

\[
(-1)^r \frac{\tilde{c}(-\lambda, k)}{\tilde{c}(-\lambda, k + 1)} = (-1)^r \pi^+(k)(-\lambda) = \pi^-(k)(\lambda)
\]

is a polynomial.

**Lemma 2.3.** Suppose \( k \in \mathcal{K}_1 + \sqrt{-1}\mathcal{K}, \ f \in C^\infty_0(a)^W \) and \( g \in C^\infty(a)^W \). Then

\[
\int_a f(x)(G^-(k + 1)g(x))\delta_k(x)dx = (-1)^r \int_a (G^+(k)f(x))g(x)\delta_{k+1}(x)dx.
\]

**Proof.** We use the graded Hecke algebra \( H_k \), which is the unique unital \( \mathbb{C} \)-algebra with the following properties:

1. \( H_k \) contains \( S(a_\mathbb{C}) \) and the group algebra \( CW \) as subalgebras;
2. the multiplication map \( S(a_\mathbb{C}) \otimes CW \to H_k \) is a linear isomorphism;
3. \( r_\alpha \cdot \xi = r_\alpha(\xi) \cdot r_\alpha - (k_\alpha + 2k_{2\alpha})\alpha(\xi) \) for \( \xi \in a \) and \( \alpha \in B \).

Importantly the \( S(a_\mathbb{C}) \)-action \( T(k, \cdot) \) and the usual \( W \)-action on \( C^\infty(a_\mathbb{C}) \) are integrated into an \( H_k \)-action on \( C^\infty(a_\mathbb{C}) \). Put

\[
\epsilon^+ = |W|^{-1} \sum_{w \in W} w, \quad \epsilon^- = |W|^{-1} \sum_{w \in W} (\text{sgn} \ w)w \in \mathbb{C}W.
\]

We assert that

\[
\tag{2.9} \epsilon^- \pi^- \epsilon^+ = \pi^+ \epsilon^+ \text{ in } H_k.
\]

Indeed, it follows from Property (2) of \( H_k \) that the map \( S(a_\mathbb{C}) \to H_k \epsilon^+ \) defined by \( p \mapsto px^+ \) is a linear isomorphism. Let \( V \) be the subspace of \( S(a_\mathbb{C}) \) consisting of the elements of degree \( \leq |R^0 \cap R^+| \). Then \( V \epsilon^+ \) is a left \( W \)-module and it easily follows from the theory of \( W \)-harmonic polynomials that a skew element in \( V \epsilon^- \) is unique up to a scalar multiple. On the other hand, by Property (3), both the sides of (2.9) are skew and have the same top degree term \( \prod_{\alpha \in R^0 \cap R^+, \alpha^+} \epsilon^+ \). Thus the assertion is proved.

Let \( w^* = -\text{id}_a \) be the longest element of \( W \). Then \( \text{sgn}(w^*) = (-1)^r \). By [21, Lemma 7.8] it holds that

\[
\tag{2.10} \int_a \varphi(x)(T(k, p)\psi(x))\delta_k(x)dx = \int_a (w^* T(k, p)w^* \varphi(x))\psi(x)\delta_k(x)dx
\]

for \( p \in S(a_\mathbb{C}), \ \varphi \in C^\infty_0(a) \) and \( \psi \in C^\infty(a) \). Now we calculate

\[
\int_a f(x)(T(k, \pi^-)\Delta(x)g(x))\delta_k(x)dx
\]
Proof of Proposition 2.1. Let $k \in K_1 + \sqrt{-1}K$. In view of (2.3) we have $k, k + 1 \in K_1 + \sqrt{-1}K$. By (1.26)

$$(2.11) \quad \mathcal{F}_k f = \mathcal{F}_k 4^{2k-1} \prod_{i=1}^r \left( \cosh \frac{\beta_i}{2} \right)^{2k-1} f.$$ 

Also, by (2.7) and Lemma 2.3

$$(2.12) \quad \mathcal{F}_k f = \mathcal{F}_{k+1} \left( (-1)^r \frac{\hat{c}(\rho(k+1), k + 1)}{c(\rho(k), k)} G^+(k) f \right).$$

Let $p \in S(a_C)^W$. Then by (2.6) and (2.10)

$$(2.13) \quad p(\lambda) \mathcal{F}_k f(\lambda) = \mathcal{F}_k(T(k, p)f)(\lambda).$$

Now let us choose $a, b \in \mathbb{N}$ so that $k' := (k + a1) + b1 \in K_1 + \sqrt{-1}K$ for all $k \in U$. By the three formulas above there exists a $C^\infty$-function $g(k, x)$ on $(K_1 + \sqrt{-1}K) \times a$ which is $W$-invariant in $x$, supported in $(K_1 + \sqrt{-1}K) \times C$, and satisfying $p(\lambda) \mathcal{F}_k f(\lambda) = \mathcal{F}_{k'}(g(k, \cdot))(\lambda)$. Hence the proposition follows from Lemma 2.2 (cf. the proof of [22, Theorem 4.1]).

Remark 2.4. An argument similar to the above is used in [16] in the setting of the Jacobi polynomial associated with the root system of type $BC$.

Let $\mathcal{P}W(a_C^\circ)$ denote the space of all holomorphic functions $\phi$ on $a_C^\circ$ such that

$$(2.14) \quad \sup_{\lambda \in a_C^\circ} (1 + ||\lambda||)^n e^{-Hc(\Re \lambda)} |\phi(\lambda)| < +\infty \quad (\forall n \in \mathbb{N})$$

for some $W$-invariant non-empty convex compact subset $C \subset a$. It coincides with the Paley-Wiener space in the Euclidean Fourier analysis (the case of $k = 0$). By Proposition 2.1, $\mathcal{F}_k f \in \mathcal{P}W(a_C^\circ)^W$ for $f \in C^\infty_0(a)^W$.

Now suppose $k \in K_1 + \sqrt{-1}K$. By (1.17) we can choose $\eta \in -\text{Cl}(a_C^\circ)$ so that $c(-\lambda, k)^{-1}$ is regular on $\{ \lambda \in a_C^\circ; \Re \lambda \in \eta - \text{Cl}(a_C^\circ) \}$. For $\phi \in \mathcal{P}W(a_C^\circ)^W$ define

$$(2.15) \quad \mathcal{J}_k \phi(x) = \int_{\eta + \sqrt{-1}a_C^\circ} \phi(\lambda) \phi(\lambda, k; x) c(-\lambda, k)^{-1} d\mu(\lambda) \quad (x \in a_C^\circ).$$

If $T \subset a_C^\circ$ is a small compact neighborhood of $\eta$ then the polynomial $p(\lambda)$ in Lemma 1.1 is 1. Hence by Lemma 1.1 Lemma 1.2 and (2.14), the integral
on the right hand side of (2.15) does not depend on the choice of \( \eta \) and converges to a \( C^\infty \)-function on \( a_+ \). Moreover \( J_k \phi(x) \) is holomorphic in \( k \) for each fixed \( x \). By (2.25) and (1.27)

\[
4^{2k+1} \prod_{i=1}^r \left( \cosh \frac{\beta_i}{2} \right) J_k \phi(x) = J_k \phi(x).
\]

Also, by (2.8)

\[
(2.17) \quad (-1)^p \frac{\hat{c}(\rho(k+1), k+1)}{\hat{c}(\rho(k), k)} \hat{G}^+(k) J_k \phi(x) = J_{k+1} \phi(x).
\]

**Theorem 2.5** (Inversion formula, first form). Suppose \( k \in \mathcal{K}_1 + \sqrt{-1} \mathcal{K} \) and choose \( \eta \) as above. Suppose \( f \in C_0^\infty(a)^W \) and \( x \in a_+ \). Then \( f(x) = J_k F_k f(x) \), namely

\[
(2.18) \quad f(x) = \int_{q+\sqrt{-1}a^*} F_k f(\lambda) \Phi(\lambda, k; x) c(-\lambda, k)^{-1} d\mu(\lambda).
\]

**Proof.** Lemma 1.1, Lemma 1.2 and Proposition 2.1 imply that the integral on the right hand side of (2.18) is holomorphic in \( k \). By [21, Theorem 9.13] (2.18) holds for \( k \in \mathcal{K}_+ \). The general case follows by analytic continuation. \( \square \)

**Proposition 2.6.** For any \( \phi \in \mathcal{P}_W(a_k^\times)^W \) the support of \( J_k \phi \) is bounded. \( \square \)

**Proof.** Take \( y \in a_+ \) so that \( \phi \) satisfies (2.14) with \( C \) being the convex hull of \( W y \). If \( k \in \mathcal{K}_+ \) then \( J_k \phi(x) = 0 \) for any \( x \in a_+ \setminus C \) by [21, Theorem 8.6]. This holds for all \( k \in \mathcal{K}_1 + \sqrt{-1} \mathcal{K} \) by analytic continuation. \( \square \)

Now suppose \( k \in \mathcal{K}_1 \). Since \( 2k_s + 2k_x + 1 > 0 \) by (2.9), \( (k_s + 1, k_s + 2k_x) \notin (-N)^2 \). Hence from (1.17) we see that \( c(-\lambda, k)^{-1} \) is regular on \( \sqrt{-1}a^* \) as a function in \( \lambda \). For \( \phi \in \mathcal{P}_W(a_k^\times)^W \) and \( x \in a_+ \) we define

\[
(2.19) \quad J_{k, \phi} \phi(x) = \int_{\sqrt{-1}a^*} \phi(\lambda) \Phi(\lambda, k; x) c(-\lambda, k)^{-1} d\mu(\lambda)
\]

The right hand side of the first line converges to a \( C^\infty \)-function on \( a_+ \) by Lemma 1.1 and Lemma 1.2 (with \( a_j(k) \) and \( b_j(k) \) being constant functions) and becomes the second line by changes of variables and (1.20). The meaning of the symbol \( J_{k, \phi} \) will be clear in Section 3. \( J_{k, \phi} \) extends to a \( W \)-invariant \( C^\infty \)-function on \( a \) by the following lemma.

**Lemma 2.7.** Let \( k \in \mathcal{K}_{reg} \cap \mathcal{K} \). Let \( V \subset a \) be a compact set and \( p \in S(a_\mathcal{C}) \). Then there exist constants \( C > 0 \) and \( n \in \mathbb{N} \) such that

\[
|p(\hat{c}_x) F(\lambda, k; x) | \leq C(1 + |\lambda|)^n e^{\max_{w \in W} \text{Re} \lambda w} \text{Re} \lambda(x)
\]

for all \( \lambda \in a_k^\times \) and \( x \in V \).

**Proof.** This is a version of [22, Theorem 2.5] for the type \( BC_r \) root system and the same proof can apply. If \( k \in \mathcal{K}_+ \) then the estimate is true by [21].
Corollary 6.2]. From this we can deduce the general case using (1.26) and (2.7).

Remark 2.8. Ho and Ólafsson [17, Appendix] proved an estimate as in the above lemma for $k \in K$ with $k_s + k_\ell \geq 0$, $k_s \geq 0$, and $k_m \geq 0$ by generalizing the proof of [21, Corollary 6.2].

Theorem 2.9. Suppose $k \in K_1$ satisfies
\begin{equation}
(2.20) \quad k_s \geq -1, \quad k_m \geq 0, \quad k_s + 2k_\ell \geq 0.
\end{equation}

Then for $f \in C_0^\infty(\mathfrak{a})^W$ we have $f = \mathcal{F}_k f$, namely
\begin{equation}
(2.21) \quad f(x) = \frac{1}{|W|} \int_{\mathfrak{a}^*} \mathcal{F}_k f(\lambda) F(\lambda, k; x) \left|e(\lambda, k)\right|^{-2} d\mu(\lambda) \quad (x \in \mathfrak{a}).
\end{equation}

Proof. If $k \in K$ satisfies (2.20), then we can choose $\eta = 0$ in Theorem 2.5 (cf. (1.17)).

3. Tempered hypergeometric functions

In this section, we define the notion of tempered hypergeometric functions by the growth condition (3.15) and give a sufficient condition for temperedness (Corollary 3.3). We will see in Section 5 that those tempered hypergeometric functions contribute to the inversion formula and the Plancherel theorem for the hypergeometric Fourier transform. Though combinatorial features are different, our argument and result are similar to those of [22], where tempered hypergeometric functions are studied for some negative multiplicity functions on reduced root systems.

If $k \in K_1$ satisfies (2.20), then there are only continuous spectra in the inversion formula (2.21) for the hypergeometric Fourier transform. In this case, only the tempered hypergeometric functions $F(\lambda, k)$ with $\lambda \in \sqrt{-1}\mathfrak{a}^*$ contribute. If $k \in K_1$ does not satisfy (2.20), then $c(-\lambda, k)^{-1}$ has non-negligible singularities and we must take account of residues to shift the domain of integration in the right hand side of (2.18) as $\eta \to 0$. The most continuous part of the spectral decomposition is given by the right hand side of (2.21). Besides it, there appear spectra whose supports have dimensions lower than $r$.

Recall $K_1$ consists of all $k \in K$ satisfying (2.3). In this paper, we exclude the case of $k_m < 0$ in (2.3) and study the case of
\begin{equation}
(3.1) \quad k_s + k_\ell > -\frac{1}{2}, \quad k_m \geq 0.
\end{equation}

Under (3.1) the residue calculus can be handled explicitly by the same method as in the proof of [28, Theorem 6.7], where the inversion formula for the spherical transform associated with a one-dimensional $K$-type on a simple Lie group of Hermitian type is given. The case of
\begin{equation}
(3.2) \quad k_s + k_\ell > -\frac{1}{2}, \quad k_m = 0
\end{equation}


Inversion Formula for the Hypergeometric Fourier Transform

reduces to the case of $BC_1$ and is easy to analyze. If $k_m < 0$, then we might need a different kind of combinatorial argument as in [22] and we do not go further into this case in this paper. Let us denote the set of all $k \in \mathcal{K}$ that satisfy (3.1) by $\mathcal{K}'$. It is convenient to adopt the parameters $\alpha$ and $\beta$ for the Jacobi function (cf. [7, 18, 19]) as in the case of $r = 1$. We substitute

\[
\alpha = k_s + k_\ell - \frac{1}{2}, \quad \beta = k_\ell - \frac{1}{2}
\]

for the multiplicity parameters $k_s$ and $k_\ell$. Then $k \in \mathcal{K}$ belongs to $\mathcal{K}'$ if and only if

\[
\alpha > -1, \quad k_m \geq 0
\]

and $\rho(k)$ is written as

\[
\rho(k) = (\alpha + \beta + 1, \alpha + \beta + 2k_m + 1, \ldots, \alpha + \beta + 2(r - 1)k_m + 1).
\]

For $1 \leq i \leq r$ let $c_i(\lambda, k)$ denote the product of the factors of (1.17) for the roots $\beta_i$ and $2\beta_i$. That is,

\[
c_i(\lambda, k) := c_{\beta_i}(\lambda, k)c_{2\beta_i}(\lambda, k)
\]

\[
= \frac{2^{-\alpha + \beta} \Gamma(\frac{1}{2} \lambda_i) \Gamma(\frac{1}{2} (\lambda_i + 1))}{\Gamma(\frac{1}{2} (\lambda_i + \alpha - \beta + 1)) \Gamma\left(\frac{1}{2} (\lambda_i + \alpha + \beta + 1)\right)}
\]

\[
= \frac{2^{-\alpha + \beta - \lambda_i + 1} \sqrt{\pi} \Gamma(\lambda_i)}{\Gamma(\frac{1}{2} (\lambda_i + \alpha - |\beta| + 1)) \Gamma\left(\frac{1}{2} (\lambda_i + \alpha + |\beta| + 1)\right)}.
\]

Let $\Theta$ be a subset of $\mathcal{B}$, $\langle \Theta \rangle$ the subset of $\mathcal{R}$ consisting of the linear combinations of elements in $\Theta$, and $W_\Theta$ the subgroups of $W$ generated by \( \{r_\alpha; \alpha \in \Theta\} \). Define

\[
a_\Theta = \{ x \in a; \alpha(x) = 0 \text{ for all } \alpha \in \Theta \},
\]

\[
a(\Theta) = \{ x \in a; \langle x, y \rangle = 0 \text{ for all } y \in a_\Theta \}.
\]

Then $\langle \Theta \rangle$ is a root system in $a(\Theta)^*$ with a positive system $\langle \Theta \rangle^+ = \langle \Theta \rangle \cap R^+$, the set $\Theta$ of simple roots, and the Weyl group $W_\Theta$. For $\lambda \in a_\Theta^*$ we write $\lambda = \lambda_{\phi(\theta)} + \lambda_{\theta}$ with $\lambda_{\phi(\theta)} \in a(\Theta)^*$ and $\lambda_{\theta} \in a_\Theta^*$. Thus $\rho(k)_{a(\Theta)}$ is “$\rho(k)$” for the root system $\langle \Theta \rangle$.

\[
\tilde{c}_\Theta(\lambda, k) = \prod_{\alpha \in \langle \Theta \rangle^+} c_\alpha(\lambda, k), \quad \tilde{c}_\Theta(\lambda, k) = \prod_{\alpha \in R^+ \setminus \langle \Theta \rangle^+} c_\alpha(\lambda, k).
\]

In particular $\tilde{c}_\Theta(\lambda, k) = 1$. Moreover, define

\[
c_\Theta(\lambda, k) = \frac{\tilde{c}_\Theta(\lambda, k)}{\tilde{c}_\Theta(\rho(k), k)}, \quad c_\Theta(\lambda, k) = \frac{\tilde{c}_\Theta(\lambda, k)}{\tilde{c}_\Theta(\rho(k), k)}.
\]

Thus

\[
c(\lambda, k) = c_\Theta(\lambda, k)c_\Theta(\lambda, k)
\]

and $c_\Theta(\lambda, k)$ is the $c$-function for the root system $\langle \Theta \rangle$.

For $0 \leq i \leq r$ let $\Theta_i$ denote the subset of $\mathcal{B}$ given by

\[
\Theta_i = \{ \alpha_j; r - i + 1 \leq j \leq r \}.
\]
We have
\[
a(\Theta_i)^{-} = \text{Span}_\mathbb{R}\{\beta_j ; 1 \leq j \leq i\},
\]
\[
a_k^{-} = \text{Span}_\mathbb{R}\{\beta_j ; i+1 \leq j \leq r\}.
\]
Define the subset \(W^{\Theta_i}\) of \(W\) by
\[
W^{\Theta_i} = \{w \in W ; w(\Theta_i)^{+} \subset \mathbb{R}^{+}\}.
\]
This is a complete set of representatives for \(W/W_{\Theta_i}\). For \(1 \leq i \leq r\), \(\langle \Theta_i \rangle \subset a(\Theta_i)^{-}\) is a root system of type \(BC_i\), its Weyl group is \(W_{\Theta_i} \cong \mathbb{Z}_2 \times \tilde{S}_i\), and
\[
W^{\Theta_i} = \{(\varepsilon, \sigma) \in \mathbb{Z}_2 \times \tilde{S}_i ; \sigma(1) < \cdots < \sigma(i),
\]
\[
\varepsilon(\beta_j) = \beta_j (\forall j \in \{\sigma(1), \ldots, \sigma(i)\})\}.
\]

Hereafter in this section we assume \(k \in \mathcal{K}_1\). For \(1 \leq i \leq r\) let \(D_k(\Theta_i)\) denote the finite subset of \(a(\Theta_i)^{-} \cong \mathbb{R}^i\) given by
\[
D_k(\Theta_i) = \{(\lambda_1, \ldots, \lambda_i) \in \mathbb{R}^i ; \lambda_1 + |\beta| - \alpha - 1 \in 2\mathbb{N}, \lambda_i < 0, 
\lambda_{j+1} - \lambda_j - 2k_m \in 2\mathbb{N} (1 \leq j \leq i-1)\}.
\]
Note \(D_k(\Theta_i) \neq \emptyset\) if and only if \(\alpha - |\beta| + 2(i-1)k_m + 1 < 0\). In particular \(\beta \neq 0\) if \(D_k(\Theta_i) \neq \emptyset\). We set \(D_k(\emptyset) = \mathbb{R}^0 = \{0\}\) for convenience.

**Theorem 3.1.** Suppose \(0 \leq i \leq r\), \(\lambda_{a(\Theta_i)} \in D_k(\Theta_i)\) and \(x \in a_+\). If we write \(\lambda = \lambda_{a(\Theta_i)} + \lambda_{a_{\Theta_i}}\) for \(\lambda_{a_{\Theta_i}} \in a_{\Theta_i, C}^+\) then
\[
F(\lambda, k; x) = \sum_{w \in W^{\Theta_i}} c(w\lambda, k) \Phi(w\lambda, k; x)
\]
as a meromorphic function in \(\lambda_{a_{\Theta_i}}\). In particular, if \(\lambda \in D_k(B)\), then
\[
F(\lambda, k; x) = c(\lambda, k) \Phi(\lambda, k; x).
\]

**Remark 3.2.** When \(\langle \lambda_{a(\Theta_i)}, \alpha^{-} \rangle \in \mathbb{Z}\) for some \(\alpha \in \langle \Theta_i \rangle^{+}\) the right hand side of (3.13) should be interpreted as follows. We assume \(\beta > 0\). (The argument in the case of \(\beta < 0\) is similar.) Then \(\alpha - |\beta| + 1 = k_s + 1\). For \(\Delta k = (\Delta k_s, \Delta k_m, \Delta k_t)\) in a sufficiently small neighborhood of \(0 \in \mathcal{K}_C\) put
\[
\Delta \lambda_{a(\Theta_i)} = (\Delta k_s, \Delta k_s + 2\Delta k_m, \ldots, \Delta k_s + 2(i - 1)\Delta k_m) \in a(\Theta_i)^{-}_{\mathbb{Z}},
\]
so that \(\lambda_{a(\Theta_i)} + \Delta \lambda_{a(\Theta_i)} \in D_{k+\Delta k}(\Theta_i)\) when \(\Delta k\) is real. We see from (3.12) that \(\langle \lambda + \Delta \lambda_{a(\Theta_i)}, \alpha^{-} \rangle \notin \mathbb{Z} (\forall \alpha \in \mathbb{R})\) for generic \(\Delta k\) and \(\lambda_{a_{\Theta_i}}\). Thus
\[
\sum_{w \in W^{\Theta_i}} c(w(\lambda + \Delta \lambda_{a(\Theta_i)}), k + \Delta k) \Phi(w(\lambda + \Delta \lambda_{a(\Theta_i)}), k + \Delta k; x)
\]
is a meromorphic function in \(\Delta k\) and \(\lambda_{a_{\Theta_i}}\) and is equal to the holomorphic function \(F(\lambda + \Delta \lambda_{a(\Theta_i)}, k + \Delta k; x)\) by the proof below. The right hand side of (3.13) is the restriction of this function to \(\Delta k = 0\).

**Proof of Theorem 3.1.** First we assume that \(k\) and \(\lambda_{a_{\Theta_i}} \in a_{\Theta_i, C}^{+}\) are generic so that \(\lambda = \lambda_{a(\Theta_i)} + \lambda_{a_{\Theta_i}}\) satisfies \((1.12)\). (We consider \(\lambda_{a(\Theta_i)}\) varies with \(k\) as in the remark above.) We will show that all terms for \(w \in W \setminus W^{\Theta_i}\) in the right hand side of \((1.20)\) vanish. So suppose \(w \in W \setminus W^{\Theta_i}\). By (3.10) there
exists $\alpha \in \Theta_i$ such that $w\alpha \in -\mathcal{R}^+$. If $\alpha = \alpha_r = \beta_1$ and $w\beta_1 = -\beta_j$ with $1 \leq j \leq r$, then $c_j(w, \lambda, k) = 0$ by (3.6) and (3.12). If $\alpha = \alpha_{r-j} = \beta_{j+1} - \beta_j$ with $1 \leq j \leq i - 1$, then by (3.11) $c(w, \lambda, k)$ contains a factor
\[
\frac{\Gamma\left(\frac{1}{2}(\lambda_j - \lambda_{j+1})\right)}{\Gamma\left(\frac{1}{2}(\lambda_j - \lambda_{j+1} + 2k_m)\right)},
\]
which vanishes because $\frac{1}{2}(\lambda_j - \lambda_{j+1} + 2k_m)$ in the denominator becomes a non-positive integer by (3.12). Hence (3.13) holds. We may drop our assumption on $k$ and $\lambda_{\alpha_{\Theta}}$ by analytic continuation. 

We say that $F(\lambda, k)$ is tempered if there exist $C \geq 0$ and $d \in \mathbb{N}$ such that
\[
|F(\lambda, k; x)| \leq C(1 + ||x||)^d e^{-\rho(k)(x)} \quad \text{for all} \quad x \in \text{Cl}(a_+).
\]

**Corollary 3.3.** Let $k \in \mathcal{K}'$. For $\lambda \in D_k(\Theta_i) + \sqrt{-1}a_{\Theta_i}^*$, $F(\lambda, k)$ is tempered. Moreover, $F(\lambda, k)$ is a real-valued square integrable function (with respect to $\delta_k(x)dx$) for any $\lambda \in D_k(\mathcal{B})$.

**Proof.** By Theorem 3.1 we have a convergent expansion on $a_+$ of the form
\[
F(\lambda, k; x) = \sum_{\mu \in W^\Theta} \sum_{\kappa \in Q_+} p_{\mu, \kappa}(\lambda, k; x) e^{(\mu - \rho(k) - \kappa)},
\]
where $p_{\mu, \kappa}$ are polynomials in $x$. The leading exponents of $F(\lambda, k; x)$ for $\lambda \in D_k(\Theta_i) + \sqrt{-1}a_{\Theta_i}^*$ on $a_+$ are $w\lambda - \rho(k)$ ($w \in W^\Theta$). For $w = (\epsilon, \sigma) \in W^\Theta$, we have
\[
\langle \text{Re} w\lambda, \beta_j \rangle = \langle w\lambda_{\alpha(\Theta_i)}, \beta_j \rangle < 0 \quad \text{for any} \quad j \in \{\sigma(1), \ldots, \sigma(i)\},
\]
\[
\langle \text{Re} w\lambda, \beta_j \rangle = 0 \quad \text{for any} \quad j \notin \{\sigma(1), \ldots, \sigma(i)\}.
\]
Our results follows from the criterion of Casselman and Milićić (4, Corollary 7.2, Theorem 7.5). For $\lambda \in D_k(\mathcal{B})$, $F(\lambda, k)$ is real-valued by (3.14). 

In Section 5 we shall show that $F(\lambda, k)$ ($\lambda \in D_k(\mathcal{B})$) exhaust the square integrable hypergeometric functions after establishing the Plancherel theorem (see Corollary 5.9).

Now let $\Pi_k$ denote the set of real affine subspaces of $a_+^e$ of the form
\[
L = w(\lambda_{\alpha(\Theta_i)}) + \sqrt{-1}a_{\Theta_i}^*
\]
for some $w \in W$, $i \in \{0, \ldots, r\}$ and $\lambda_{\alpha(\Theta_i)} \in D_k(\Theta_i)$. After [22] we call each $L \in \Pi_k$ a tempered residual subspace. It is easy to see that two distinct tempered residual subspaces are disjoint. For $i = 0, \ldots, r$ let $W(\Theta_i)$ denote the stabilizer of $a(\Theta_i)$ in $W$. Then $W(\Theta_i)$ acts on $a_{\Theta_i}^*$ and if $i < r$ then it is identified with the Weyl group for the type $BC_{r-i}$ root system
\[
\{\pm \beta_j, \pm 2\beta_j, \pm (\beta_p \pm \beta_q); i < j \leq r, \ i < q < p \leq r\}.
\]
Put
\[
\text{Cl}(a_{\Theta_i}^+, \alpha) = \{\lambda \in a_{\Theta_i}^*; \langle \lambda, \alpha \rangle \geq 0 \quad \text{for any positive root} \ \alpha \ \text{in} \ (3.17)\}
\]
\[
\simeq \{(\lambda_{i+1}, \lambda_{i+2}, \ldots, \lambda_r) \in \mathbb{R}^{r-i}; \ 0 \leq \lambda_{i+1} \leq \lambda_{i+2} \leq \cdots \leq \lambda_r\}.
\]
In particular $\text{Cl}(a_{\emptyset,+}^*) = \text{Cl}(a_{+}^*)$ and $\text{Cl}(a_{B,+}^*) = \{0\}$. The following is immediate.

**Lemma 3.4.** The set $\bigcup_{i} P_{\lambda} \subset a_{C}^*$ is stable under $W$. As a complete set of representatives for the $W$-orbits in $\bigcup_{i} P_{\lambda}$ we can take $$\bigcup_{i=0}^{r} (D_{k}(\Theta_{i}) + \sqrt{-1} \text{Cl}(a_{B,+}^*)).$$

**Lemma 3.5.** For any $L \in P_{\lambda}$ there exists a polynomial $p_{L} \in S(a_{C})$ that does not vanish on $L$ and identically vanishes on each $L' \in P_{\lambda} \setminus \{L\}$ with $\dim L' \leq \dim L$.

**Proof.** Without loss of generality we may assume $L = (\eta_{1}, \ldots, \eta_{p}) + \sqrt{-1} a_{B,+}^*$ with $(\eta_{1}, \ldots, \eta_{p}) \in D_{k}(\Theta_{i})$. Let $X = \{\xi \in \alpha - |\beta| + 1 + 2Nk_{m} + 2N; \xi < 0\}$. Then

$$p_{L}(\lambda) := \left( \prod_{j=1}^{i} \frac{\prod_{\xi \in X} (\lambda_{j}^{2} - \xi^{2})}{\lambda_{j} - \eta_{j}} \right) \prod_{j=i+1}^{r} (\lambda_{j}^{2} - \xi^{2})$$

has the desired properties. \hfill \square

For each $\lambda_{a(\theta_{i})} \in D_{k}(\Theta_{i})$ ($0 \leq i \leq r$) let $S_{k,\lambda_{a(\theta_{i})}}$ denote the Schwartz space on $\lambda_{a(\theta_{i})} + \sqrt{-1} a_{B,+}^*$, $\mathbb{R}^{r-i}$ (the space of rapidly decreasing smooth functions on $\lambda_{a(\theta_{i})} + \sqrt{-1} a_{B,+}^*$). By the Euclidean Fourier analysis one sees that $\{\phi | \lambda_{a(\theta_{i})} + \sqrt{-1} a_{B,+}^*; \phi \in \mathcal{P}W(a_{C}^*)\}$ is a dense subspace of $S_{k,\lambda_{a(\theta_{i})}}$ with respect to the Schwartz space topology. Identifying $S_{k,\lambda_{a(\theta_{i})}}$ with a space of functions on $\bigcup_{i=0}^{r} (D_{k}(\Theta_{i}) + \sqrt{-1} a_{B,+}^*)$ put

$$S_{k} = \bigoplus_{i=0}^{r} S_{k,\lambda_{a(\theta_{i})}}^{W(\Theta_{i})}.$$  

The next lemma will be used in Section 5.

**Lemma 3.6.** $\{\phi | \bigcup_{i=0}^{r} (D_{k}(\Theta_{i}) + \sqrt{-1} a_{B,+}^*); \phi \in \mathcal{P}W(a_{C}^*)\}$ is a dense subspace of $S_{k}$.

**Proof.** The method below is almost the same as Opdam’s proof of [22 Theorem 5.5], though the claim of the lemma has a subtle difference with what was shown in his proof. First, clearly $\mathcal{P}W(a_{C}^*)$ is identified with a subspace of $S_{k}$ by restriction. We claim that for $-1 \leq i \leq r$, $\mathcal{P}W(a_{C}^*)$ contains a dense subspace of

$$\bigoplus_{j=0}^{i} S_{k,\lambda_{a(\theta_{j})}}^{W(\Theta_{j})}.$$  

This is trivial if $i = -1$. We proceed by induction. Let $i \geq 0$ and assume the claim is true for $i - 1$. Take an arbitrary $\lambda_{a(\theta_{i})} \in D_{k}(\Theta_{i})$ and put $L = \lambda_{a(\theta_{i})} + \sqrt{-1} a_{B,+}^*$. Let $p_{L}$ be the polynomial in Lemma 3.5. Take an arbitrary $\phi \in C_{0}^{\infty}(L)^{W(\Theta_{i})}$. Then there exists a sequence $\{\psi_{n}\} \subset \mathcal{P}W(a_{C}^*)$...
such that $\psi_n|_L \to (p_L|_L)^{-1}\phi$ in $S_{k, \lambda_0(\Theta)}$. Thus

$$\left( |W(\Theta_i)|^{-1} \sum_{w \in W(\Theta_i)} p_L(w\lambda)\psi_n(w\lambda) \right)|_L \to \phi$$

in $S_{k, \lambda_0(\Theta)}$. Note that

$$\left( |W(\Theta_i)|^{-1} \sum_{w \in W(\Theta_i)} p_L(w\lambda)\psi_n(w\lambda) \right)|_L = \left( |W^{\lambda(\Theta_0)}|^{-1} \sum_{w \in W} p_L(w\lambda)\psi_n(w\lambda) \right)|_L$$

and that $|W^{\lambda(\Theta_0)}|^{-1} \sum_{w \in W} p_L(w\lambda)\psi_n(w\lambda)$ belongs to

$$\mathcal{P}W(a^*_c)^W \cap \left( \bigoplus_{j=0}^{i-1} \frac{S_{k, \lambda(\Theta_0)}}{\bigoplus_{j=0}^{i} D_{k, \lambda_0(\Theta_0)}} \right).$$

Here $W^{\lambda(\Theta_0)}$ is the stabilizer of $\lambda_0(\Theta_0)$ in $W$. Hence by the assumption there exists a sequence $\{\tau_n\}$ in (3.19) such that $\tau_n \to \phi$ in $S_k$. This readily proves the claim for $i$. \[\square\]

## 4. A PARTIAL SUM OF HARIŞ-CHANDRA SERIES.

For a while we assume $k \in K_{\text{reg}}$. Let $\Xi$ denote the subset of $B$ given by

$$\Xi = \{\alpha_j \mid 1 \leq j \leq r - 1\}.$$ 

Then $\langle \Xi \rangle$ is a root type of system $A_{r-1}$ and $W_\Xi = \mathfrak{S}_r$. We define

$$F_\Xi(\lambda, k) = \sum_{s \in W_\Xi} c_\Xi(s\lambda, k) \Phi(s\lambda, k).$$

If $r = 1$, then $\Xi = \emptyset$. In this case we set $F_\Xi(\lambda, k) = \Phi(\lambda, k)$, $c_\Xi(\lambda, k) = 1$, and $W_\Xi = \{1_W\}$. In this section we rewrite the inverse hypergeometric Fourier transform $\mathcal{J}_k$ in terms of $F_\Xi(\lambda, k)$ and study some properties of $F_\Xi(\lambda, k)$. By the definition, $F_\Xi(s\lambda, k) = F_\Xi(\lambda, k)$ for any $s \in W_\Xi$. Note

$$\hat{c}_\Xi(\lambda, k) = \prod_{1 \leq q < p \leq r}^r \frac{\Gamma\left(\frac{1}{2} \lambda_p - \lambda_q\right)}{\Gamma\left(\frac{1}{2} \lambda_p + \lambda_q + 2k_m\right)},$$

$$\hat{\mathcal{C}}_\Xi(\rho(k), k) = \prod_{j=2}^r \frac{\Gamma(k_m)}{\Gamma(jk_m)}$$

$$\hat{c}_\Xi(\lambda, k) = \prod_{1 \leq q < p \leq r}^r \frac{\Gamma\left(\frac{1}{2} \lambda_p + \lambda_q\right)}{\Gamma\left(\frac{1}{2} \lambda_p + \lambda_q + 2k_m\right)} \prod_{j=1}^r \hat{c}_j(\lambda, k).$$

An argument similar to the proof of [11] Theorem 4.3.14] shows that the singularities of $F_\Xi(\lambda, k)$ are at most simple poles along hyperplanes of the form

$$\langle \lambda, \alpha^\vee \rangle = j \quad \text{for some} \quad \alpha \in R^+ \setminus \langle \Xi \rangle \quad \text{and} \quad j = 1, 2, \ldots.$$ 

Such a result for $F_{\Theta}$ with a general $\Theta \subset B$ is given by [27] Theorem 3.5].
Lemma 4.2. Let $k \in K_{\text{reg}}$ and $x \in a_+$. For each $\eta$ in
\[ Z_\eta = \{ \lambda \in a^* ; \langle \lambda, \alpha^\vee \rangle < 1 \text{ for any } \alpha \in R^+ \setminus \langle \Xi \rangle \} \]
there exist a neighborhood $T \subset Z_\eta$ of $\eta$ and constants $C > 0, n \in \mathbb{N}$ such that
\[ |F_\Xi(\lambda, k; x)| \leq C(1 + ||\lambda||)^n \text{ for } \lambda \in T + \sqrt{-1}a^*. \]

Proof. As a function in $\lambda$, $F_\Xi(\lambda, k; x)$ is holomorphic on $Z_\Xi + \sqrt{-1}a^*$. Take any compact neighborhood $T \subset Z_\Xi$ of $\eta$. We assert that there exists a sequence of pairs $\{(v_j, a_j)\}_{j=1}^k \subset (a \setminus \{0\}) \times \mathbb{R}$ such that
\[ \psi(\lambda) := F_\Xi(\lambda, k; x) \prod_{j=1}^k (\lambda(v_j) - a_j) \]
satisfies
\[ \exists C > 0 \exists n \in \mathbb{N} \forall \lambda \in T + \sqrt{-1}a^* |\psi(\lambda)| \leq C(1 + ||\lambda||)^n. \]
Indeed, we can take a sufficiently large $N \in \mathbb{N}$ so that
\[ \prod_{1 \leq q < p \leq r} \frac{\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q) + N\right)}{\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q + 2k_m)\right)} \]
is regular on $W_\Xi T + \sqrt{-1}a^*$. Put
\[ P(\lambda) = p(\lambda) c_\Xi(\lambda, k)^{-1} \prod_{1 \leq q < p \leq r} \frac{\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q) + N\right)}{\Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q + 2k_m)\right)} \]
where $p(\lambda)$ is the polynomial in Lemma 1.1 for the compact set $W_\Xi T \subset a^*$. Choose $\{(v_j, a_j)\}_{j=1}^k$ so that
\[ \prod_{j=1}^k (\lambda(v_j) - a_j) = \prod_{s \in W_\Xi} P(s\lambda). \]
Then the assertion follows from Lemma 1.1, Lemma 1.2 and (4.2).

If $k = 0$ then we are done. So assume $k > 0$. Let us show that even if $\psi(\lambda)$ is replaced with
\[ \phi(\lambda) := \frac{\psi(\lambda)}{\lambda(v_1) - a_1} = F_\Xi(\lambda, k; x) \prod_{j=2}^k (\lambda(v_j) - a_j), \]
(3.7) still holds for a smaller $T$. Since this is obvious when $\eta(v_1) \neq a_1$, we assume $\eta(v_1) = a_1$. We identify $v_1$ with a vector in $a^*$ and take a small $\varepsilon > 0$ so that
\[ T' := \eta + \{ \zeta v_1 ; -\varepsilon \leq \zeta \leq \varepsilon \} + \{ \lambda \in a^* ; \langle \lambda, v_1 \rangle = 0, ||\lambda|| \leq \varepsilon \} \subset T. \]
Now let $\lambda \in T^r + \sqrt{-1}a^r$. Suppose first $|\lambda(v_1) - a_1| \geq \varepsilon ||v_1||^2$. Then we have $|\phi(\lambda)| \leq \varepsilon^{-1}||v_1||^{-2}||\psi(\lambda)|| \leq \varepsilon^{-1}||v_1||^{-2}C(1 + ||\lambda||)^n$. Next, suppose $|\lambda(v_1) - a_1| \leq \varepsilon ||v_1||^2$ and put

$$\lambda_\zeta = \lambda + \left( \zeta - \frac{\lambda(v_1) - a_1}{||v_1||^2} \right)v_1$$

for $\zeta \in \mathbb{C}$ with $|\zeta| \leq \varepsilon$. Then $||\lambda_\zeta|| \leq ||\lambda|| + 2\varepsilon||v_1||$, $\lambda_\zeta(v_1) - a_1 = \zeta||v_1||^2$ and $\lambda_\zeta \in T^r + \sqrt{-1}a^r$. Hence by the maximum modulus principle

$$|\phi(\lambda)| \leq \max_{|\zeta| = \varepsilon} |\phi(\lambda_\zeta)| = \max_{|\zeta| = \varepsilon} |\phi(\lambda_\zeta)| = \varepsilon^{-1}||v_1||^{-2} \max_{|\zeta| = \varepsilon} |\psi(\lambda_\zeta)|$$

$$\leq \varepsilon^{-1}||v_1||^{-2}C(1 + 2\varepsilon||v_1|| + ||\lambda||)^n.$$ 

Thus our claim is shown. Repeating the same argument we can remove all the factors $(\lambda(v_j) - a_j)$ from $\psi(\lambda)$.

□

**Proposition 4.3** (Inversion formula, second form). Let $k \in K_1'$. If $r > 1$ then we further assume $k_m > 0$. Let $\phi \in \mathcal{PW}(a^k_\alpha)^W$ and $x \in a_\alpha$. Then

$$\mathcal{J}_k \phi(x) - \mathcal{J}_{k, \zeta} \phi(x) = \frac{-1}{(2\pi\sqrt{-1})^{r-1}(r-1)!} \sum_{\lambda \in D_k(\Theta_1)} \int_{\xi + \sqrt{-1}a^r_{\alpha_1}} (\phi(\lambda)F_\Xi(\lambda, k; x))|_{\lambda_1 = \xi} \times \text{Res}_{\lambda_1 = \xi} (c_{\Xi}(\lambda, k)^{-1}c(-\lambda, k)^{-1}) d\lambda_{\alpha_1}.$$ 

Here $d\lambda_{\alpha_1} = d\lambda_2 d\lambda_3 \cdots d\lambda_r$ and $\zeta := (\xi, \xi, \ldots, \xi) \in \mathbb{R}^{r-1} \simeq a^r_{\alpha_1}$ for $\xi \in D_k(\Theta_1) = \{\xi \in \mathbb{R}; \xi < 0, \xi - \alpha + |\beta| - 1 \in 2\mathbb{N}\}$. (If $r = 1$ then $\zeta = 0$ for each $\xi \in D_k(\Theta_1)$ and $d\lambda_{\alpha_1}$ is the counting measure on $\zeta + \sqrt{-1}a^r_{\alpha_1} = \{0\}$.)

**Proof.** The singularities of the function $\lambda \mapsto c(-\lambda, k)^{-1}$ in the region $\{\lambda \in a^k_\alpha; \Re \lambda = c(-\lambda, k)^{-1}\}$ are precisely the simple poles along $\lambda_j = \xi$ for $1 \leq j \leq r$ and $\xi \in D_k(\Theta_1)$ (cf. (4.11)). Let $\{\xi^{(1)} < \xi^{(2)} < \cdots < \xi^{(k)} = 0\} = D_k(\Theta_1) \cup \{0\}$. With a sufficiently small $\varepsilon > 0$ put

$$\eta^{(1,0)} = (\xi^{(1)} - \varepsilon, \xi^{(1)} - \varepsilon, \xi^{(1)} - \varepsilon, \ldots, \xi^{(1)} - \varepsilon),$$

$$\eta^{(1,1)} = (\xi^{(2)} - \varepsilon, \xi^{(1)} - \varepsilon, \xi^{(1)} - \varepsilon, \ldots, \xi^{(1)} - \varepsilon),$$

$$\eta^{(1,2)} = (\xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon, \xi^{(1)} - \varepsilon, \ldots, \xi^{(1)} - \varepsilon),$$

$$\vdots$$

$$\eta^{(1,r)} = \eta^{(2,0)} = (\xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon, \ldots, \xi^{(2)} - \varepsilon),$$

$$\eta^{(2,1)} = (\xi^{(3)} - \varepsilon, \xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon, \ldots, \xi^{(2)} - \varepsilon),$$

$$\vdots$$

$$\eta^{(k-1,r)} = \eta^{(k,0)} = (\xi^{(k)} - \varepsilon, \xi^{(k)} - \varepsilon, \xi^{(k)} - \varepsilon, \ldots, \xi^{(k)} - \varepsilon)$$

$$= (-\varepsilon, -\varepsilon, -\varepsilon, \ldots, -\varepsilon).$$
All these points belong to \(-\text{Cl}(a_+^\times)\). Define
\[
I^{(\ell,j)} = \int_{\Gamma^{(\ell,j)} + \sqrt{-1}\tau a^\times} \phi(\lambda) \, \Phi(\lambda, k; x) \, c(-\lambda, k)^{-1} d\mu(\lambda).
\]
Then \(I^{(1,0)} = J_k \phi(x)\) and \(I^{(k,0)} = J_{k,\emptyset} \phi(x)\).

Let \(1 \leq \ell < k\) and \(1 \leq j \leq r\). By Cauchy’s residue theorem, \(I^{(\ell,j)} - I^{(\ell,j-1)}\) is
\[
\frac{1}{(2\pi \sqrt{-1})^{r-1}} \int_{\Gamma^{(\ell,j)} + \sqrt{-1}\mathbb{R}^{r-1}} \left( \phi(\lambda) \, \Phi(\lambda, k; x) \right) \big|_{\lambda_j = \xi^{(\ell)}} \times \Res_{\lambda_j = \xi^{(\ell)}} c(-\lambda, k)^{-1} d\lambda_{\{1,2,\ldots,r\}\setminus\{j\}},
\]
where
\[
\xi^{(\ell)} = \left( (\xi^{(\ell+1)} - \varepsilon, \ldots, \xi^{(\ell+1)} - \varepsilon), (\xi^{(\ell)} - \varepsilon, \ldots, \xi^{(\ell)} - \varepsilon) \right) \in \mathbb{R}^{j-1} \times \mathbb{R}^{r-j}
\]
and \(d\lambda_{\{1,2,\ldots,r\}\setminus\{j\}} = \prod_{i \neq j} d\lambda_i\). The integrand as a function of \((\lambda_i)_{i \in \{1,2,\ldots,r\}\setminus\{j\}}\) is regular near \(\xi^{(\ell)} + \sqrt{-1}\mathbb{R}^{r-1}\) because the pole of \(\tilde{c}_i(-\lambda, k)^{-1}\big|_{\lambda_j = \xi^{(\ell)}}\) along \(\lambda_i = \xi^{(\ell)} (i \neq j)\) is canceled by a zero locus of \(\tilde{c}_i(-\lambda, k)^{-1}\big|_{\lambda_j = \xi^{(\ell)}}\) for either \(\alpha = \beta_i - \beta_j\) or \(\beta_j - \beta_i\) in \(\mathbb{R}^+\) (recall the assumption that \(k_m > 0\) if \(r > 1\)). Hence we can shift the domain of integration in (4.8) from \(\xi^{(\ell)} + \sqrt{-1}\mathbb{R}^{r-1}\) to \(\xi^{(\ell)} + \sqrt{-1}\mathbb{R}^{r-1}\).

Now the function \(\lambda \mapsto c_{\Xi}(\lambda, k)^{-1}\) is regular on
\[
\{ \lambda \in \mathbb{C}^r; \Re \lambda_1 \leq \Re \lambda_2 \leq \cdots \leq \Re \lambda_r \leq 0 \}.
\]
Also, (4.7) and (4.3) yield
\[
c_{\Xi}(s\lambda, k)^{-1}c(-s\lambda, k)^{-1} = c_{\Xi}(\lambda, k)^{-1}c(-\lambda, k)^{-1} \quad \text{for } s \in W_{\Xi}.
\]
Therefore, by (4.2) and changes of variables, \(I^{(\ell+1,0)} - I^{(\ell,0)} = \sum_{j=1}^r (I^{(\ell,j)} - I^{(\ell,j-1)})\) reduces to
\[
\frac{1}{(2\pi \sqrt{-1})^{r-1}(r-1)!} \int_{\Gamma^{(\ell)} + \sqrt{-1}\tau a^\times_1} \left( \phi(\lambda) \, F_{\Xi}(\lambda, k; x) \right) \big|_{\lambda_1 = \xi^{(\ell)}} \times \Res_{\lambda_1 = \xi^{(\ell)}} c_{\Xi}(\lambda, k)^{-1}c(-\lambda, k)^{-1} d\lambda_{\{1,2,\ldots,r\}}.
\]
This proves the proposition. \(\square\)

Suppose \(k \in K_1\), \(0 \leq i \leq r\) and \(\xi \in D_k(\Theta_i)\). For a meromorphic function \(\phi(\lambda)\) on \(a_{\xi}^\times\), we define its successive restriction
\[
\sigma_{\xi}(\phi(\lambda)) = \phi(\lambda) \big|_{\lambda_1 = \xi_1, \lambda_2 = \xi_2, \ldots, \lambda_i = \xi_i}
\]
when the right hand side is justified. (If \(i = 0\) and \(\xi = 0\) then \(\sigma_{\xi}(\phi(\lambda)) = \phi(\lambda)\).)

\textbf{Lemma 4.4.} Both \(\sigma_{\xi}(c_{\Xi}(\lambda, k)^{-1})\) and \(\sigma_{\xi}(c_{\Xi}^\times(\lambda, k))\) are well-defined meromorphic functions on \(a_{\Theta_i, \mathbb{C}}^\times\). More precisely, the latter can be written as
\[
\sigma_{\xi}(c_{\Xi}^\times(\lambda, k)) = a(\xi, k; \lambda_{\Theta_i}) \prod_{1 \leq q \leq i < p \leq r} \frac{\Gamma\left(\frac{1}{2}(\lambda_p + \xi_q)\right)}{\Gamma\left(\frac{1}{2}(\lambda_p + \xi_q + 2k_m)\right)}
\]
Thus, in either case, the lemma is also true for $\lambda_{\alpha_1} = (\lambda_{i+1}, \ldots, \lambda_r) \in \mathbb{C}^r - i; \alpha - |\beta| + 1 \leq \Re \lambda_j \leq 0$ ($i < j \leq r$).

Proof. If $i = 0$ then the lemma is obvious by (3.5) and (4.5). Assume the lemma is true for $i$ and let $(\xi, \xi_{i+1}) \in D_k(\Theta_{i+1})$. If $i = 0$ then the restriction of

$$
\frac{\Gamma(\lambda_1)}{\Gamma\left(\frac{1}{2}(\lambda_1 + \alpha - |\beta| + 1)\right)}
$$

to $\lambda_1 = \xi_1$ is a non-zero number $C$ because for $\xi_1 \in D_k(\Theta_1)$

$$
\xi_1 \in -\mathbb{N} \Leftrightarrow \xi_1 + \alpha - |\beta| + 1 \in -2\mathbb{N}.
$$

If $i > 1$ then

$$
\frac{\Gamma(\lambda_{i+1})}{\Gamma\left(\frac{1}{2}(\lambda_{i+1} + \alpha - |\beta| + 1)\right)} \prod_{1 \leq q < i} \frac{\Gamma\left(\frac{1}{2}(\lambda_{i+1} + \xi_q)\right)}{\Gamma\left(\frac{1}{2}(\lambda_{i+1} + \xi_q + 2k_m)\right)}
$$

and the restriction of this function to $\lambda_{i+1} = \xi_{i+1}$ is a non-zero number $C$ because for $\xi_{i+1} \in \mathbb{C}$ with $(\xi, \xi_{i+1}) \in D_k(\Theta_{i+1})$ we have

$$
\xi_{i+1} + \xi_1 \in -2\mathbb{N} \Leftrightarrow \xi_{i+1} + \alpha - |\beta| + 1 \in -2\mathbb{N},
$$

$$
\xi_{i+1} - \xi_1 \in -\mathbb{N} \Leftrightarrow \xi_{i+1} + \xi_1 + 2k_m \in -2\mathbb{N},
$$

$$
\xi_{i+1} + \xi_q \in -2\mathbb{N} \Leftrightarrow \xi_{i+1} + \xi_q + 2k_m \in -2\mathbb{N} \quad (1 \leq q < i).
$$

Thus, in either case, the lemma is also true for $i + 1$ with

$$
a((\xi, \xi_{i+1}), k; \lambda_{\alpha_{i+1}}) = C a((\xi, k; \lambda_{\alpha_i})|_{\lambda_i = \xi_i}). \quad \square
$$

We define the subgroup $W^\Xi := \mathbb{Z}_2^r$ of $W$. This is a complete set of representatives for both $W/W^\Xi$ and $W^\Xi \setminus W$. By (1.20) and (3.8),

$$
F(\lambda, k) = \sum_{w \in W^\Xi} \mathfrak{c}^\Xi(w\lambda, k)F_\Xi(w\lambda, k).
$$

For $0 \leq i \leq r$ define $W_i^\Xi = W^\Xi \cap W(\Theta_i)$. Then $W_i^\Xi \cong \mathbb{Z}_2^{r-i}$, whose elements act as changes of signs for $\beta_{i+1}, \ldots, \beta_r$.

Proposition 4.5. Suppose $k \in K'_1$, $0 \leq i \leq r$, $x \in a_+$ and $\xi \in D_k(\Theta_i)$.

Then as a meromorphic function on $a_{0,1}^2$,

$$
F(\lambda, k; x)|_{\lambda(\alpha_1) = \xi} = \sum_{w \in W_i^\Xi} \sigma_\xi(\mathfrak{c}^\Xi(w\lambda, k))F_\Xi(w\lambda, k; x)|_{\lambda(\alpha_1) = \xi}.
$$


Moreover, the coefficient of each term on the right hand side of (4.14) is regular and non-zero at \( \lambda_{a_{\ell i}} \in \sqrt{-1}a_{\ell i} \), such that

\[
(4.15) \quad \lambda_j \neq 0 (i < \forall j \leq r) \text{ and } \lambda_p \pm \lambda_q \neq 0 (i < \forall q < \forall p \leq r).
\]

**Proof.** Since the restriction operator \( \sigma_\xi \) and the \( W_i^\Xi \)-action commute, the second assertion of the proposition follows from Lemma 4.4.

We prove (4.14) in a similar way as Theorem 5.1. If \( \Delta \xi \in a(\Theta_i) \Xi \) varies with \( \Delta k \in K\Xi \) as \( \Delta \lambda_{a_{\ell i}} \) does in Remark 4.4, then it is easy to see from the proof of Lemma 4.3 that \( a(\xi + \Delta \xi, k + \Delta k; \lambda_{a_{\ell i}}) \) is holomorphic in \( \Delta k \) around 0. Thus, for any fixed \( \lambda_{a_{\ell i}} \in \sqrt{-1}a_{\ell i} \), satisfying (4.15), \( \sigma_\xi (c^\Xi(\omega, k)) \) \(( w \in W_i^\Xi \) analytically depends on \( k \). Hence we have only to prove (4.14) for generic \( k \) and \( \lambda_{a_{\ell i}} \) such that (4.12) holds for \( \lambda = \xi + \lambda_{a_{\ell i}} \). For such \( k \) and \( \lambda_{a_{\ell i}} \),

\[
\sigma_\xi (c^\Xi(\omega, k)) = c^\Xi(w(\xi + \lambda_{a_{\ell i}}), k) \quad ( w \in W_i^\Xi )
\]

and (4.14) holds for \( \lambda = \xi + \lambda_{a_{\ell i}} \). Thus (4.14) follows if we can show

\[
c^\Xi(w(\xi + \lambda_{a_{\ell i}}), k) = 0 \quad \text{for any } w \in W_i^\Xi \setminus W_i^\Xi.
\]

Let \( w \in W_i^\Xi \setminus W_i^\Xi \). Then there exists \( j \in \{1, \ldots, i\} \) such that \( w\beta_j = -\beta_j \). Let \( j \) be the smallest one. If \( j = 1 \) then \( c^\Xi(w(\xi + \lambda_{a_{\ell i}}), k) = 0 \) by (3.1) and (3.12). If \( j > 1 \) then \( c^\Xi(w(\xi + \lambda_{a_{\ell i}}), k) \) contains the factor

\[
\frac{\Gamma\left(\frac{1}{2}(-\xi_j + \xi_{j-1})\right)}{\Gamma\left(\frac{1}{2}(-\xi_j + \xi_{j-1} + 2k_m)\right)}
\]

which vanishes by (3.12).

\( \square \)

**Remark 4.6.** Corollary 3.3 follows also from Proposition 4.5.

---

5. **Inversion and Plancherel Formula**

In this section, we assume \( k \in K_1 \), namely, \( k_s, k_m \) and \( k_\ell \) are real numbers satisfying (3.1). Recall \( \alpha \) and \( \beta \) are given by (3.3). The sets \( D_k(\Theta_i) \subset a(\Theta_i)^* \) \((0 \leq i \leq r) \) are defined by (3.12). For \( i = 0, \ldots, r \) and \( \lambda_{a(\Theta_i)} = (\lambda_1, \ldots, \lambda_i) \in D_k(\Theta_i) \) we define a positive number \( d_{\Theta_i}(\lambda_{a(\Theta_i)}, k) \) as follows.

If \( k_m > 0 \) then we put

\[
(5.1) \quad d_{\Theta_i}(\lambda_{a(\Theta_i)}, k) = \tilde{c}_{\Theta_i}(\rho(k), k)^2 = \prod_{j=1}^{i} \frac{2^{2\alpha-2\beta-1}|\lambda_j|^{\frac{1}{2}(\lambda_j + \alpha + |\beta| + 1)} \Gamma\left(\frac{1}{2}(\lambda_j + \alpha + |\beta| + 1)\right) \Gamma\left(\frac{1}{2}(-\lambda_j + \alpha + |\beta| + 1)\right)}{\pi \Gamma\left(\frac{1}{2}(\lambda_j - \alpha + |\beta| + 1)\right) \Gamma\left(\frac{1}{2}(-\lambda_j - \alpha + |\beta| + 1)\right)} \times \prod_{1 \leq q < p \leq i} \frac{(\lambda_q - \lambda_p)^2 \Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q + 2k_m)\right) \Gamma\left(\frac{1}{2}(-\lambda_q - \lambda_p + 2k_m)\right)}{4 \Gamma\left(\frac{1}{2}(\lambda_p - \lambda_q + 2k_m)\right) \Gamma\left(\frac{1}{2}(-\lambda_q - \lambda_p + 2k_m)\right)}.
\]
If \( k_m = 0 \) or \( r = 1 \) we put
\[
(5.2) \quad d_{\Theta_i}(\lambda_{a(\Theta_i)}, \mathbf{k}) = c_{\Theta_i}(\rho(\mathbf{k}), \mathbf{k})^2 |W_{\Theta_i}^{\lambda_{a(\Theta_i)}}|^{-1}
\]
\[
\times \prod_{j=1}^i \frac{-2^a2^{-b-1} \lambda_j \Gamma(\frac{1}{2}(\lambda_j + a + |\beta| + 1)) \Gamma(\frac{1}{2}(-\lambda_j + a + |\beta| + 1))}{\pi \Gamma(\frac{1}{2}(\lambda_j - a + |\beta| + 1)) \Gamma(\frac{1}{2}(-\lambda_j - a + |\beta| + 1))}
\]
where \( W_{\Theta_i}^{\lambda_{a(\Theta_i)}} \) is the stabilizer of \( \lambda_{a(\Theta_i)} \) in \( W_{\Theta_i} \). Observe that \( d_{\Theta_i}(\lambda, \mathbf{k}) \) is actually positive. In particular \( d_{\Theta_i}(0, \mathbf{k}) = 1 \).

**Remark 5.1.** If \( \lambda_{a(\Theta_i)} \) varies with \( \mathbf{k} \) as in Remark 3.2 then (5.2) is the limit of (5.1) as \( k_m \to 0^+ \).

**Lemma 5.2.** Suppose \( 1 \leq i < r \) and \( \lambda_{a(\Theta_i)} = (\lambda_1, \ldots, \lambda_i) \in D_k(\Theta_i) \). If we write \( \lambda = \lambda_{a(\Theta_i)} + \lambda_{a(\Theta_i)} \) for \( \lambda_{a(\Theta_i)} = (\lambda_{i+1}, \ldots, \lambda_r) \in \mathbb{C}^{r-i} \approx a_{\Theta_i}^{\mathbb{C}} \) then \( e^{\Theta_i}(\lambda, \mathbf{k})^{-1} \) is a well-defined meromorphic function in \( \lambda_{a(\Theta_i)} \). The singularities of this function in the region
\[
(5.3) \quad \left\{ \begin{array}{l}
\lambda_i \leq \Re \lambda_{i+1} \leq \cdots \leq \Re \lambda_r \leq 0, \text{ and } \\
\Re \lambda_r - \Re \lambda_{i+1} < 2k_m \text{ when } k_m > 0
\end{array} \right\}
\]
only (5.3) are precisely the simple poles along \( \lambda_j = \xi \) for \( j = i+1, \ldots, r \) and \( \xi \in \mathbb{R} \) such that \( (\lambda_1, \ldots, \lambda_i, \xi) \in D_k(\Theta_{i+1}) \). In particular \( e^{\Theta_i}(\lambda, \mathbf{k})^{-1} \) is regular on \( \sqrt{-1}a_{\Theta_i}^{\mathbb{C}} \).

**Proof.** The first assertion is clear from the definition of \( e^{\Theta_i}(\lambda, \mathbf{k}) \). By (3.6) the function
\[
(5.4) \quad \prod_{j=i+1}^r \frac{\Gamma(\frac{1}{2}(-\lambda_j + a + |\beta| + 1)) \Gamma(\frac{1}{2}(-\lambda_j + a + |\beta| + 1))}{\Gamma(-\lambda_j)}
\]
\[
\times \prod_{1 \leq q < p \leq r} \frac{\Gamma(\frac{1}{2}(-\lambda_q - \lambda_p + 2k_m)) \Gamma(\frac{1}{2}(\lambda_q - \lambda_p + 2k_m))}{\Gamma(\frac{1}{2}(-\lambda_q - \lambda_p)) \Gamma(\frac{1}{2}(\lambda_q - \lambda_p))}
\]
has the same singularities as \( e^{\Theta_i}(\lambda, \mathbf{k})^{-1} \). We assert for \( j = i+1, \ldots, r \) that the singularities of the partial product
\[
(5.5) \quad \frac{\Gamma(\frac{1}{2}(-\lambda_j + a + |\beta| + 1))}{\Gamma(-\lambda_j)} \prod_{q=1}^i \frac{\Gamma(\frac{1}{2}(\lambda_q - \lambda_j + 2k_m))}{\Gamma(\frac{1}{2}(\lambda_q - \lambda_j))}
\]
on the region (5.3) are precisely the simple poles along \( \lambda_j = \xi \) for \( \xi \in \mathbb{R} \) such that \( (\lambda_1, \ldots, \lambda_i, \xi) \in D_k(\Theta_{i+1}) \). In fact, we can rewrite (5.5) as the product of
\[
(5.6) \quad \frac{\Gamma(\frac{1}{2}(-\lambda_j + a + |\beta| + 1))}{\Gamma(\frac{1}{2}(-\lambda_j + \lambda_1))} \prod_{q=1}^{i-1} \frac{\Gamma(\frac{1}{2}(-\lambda_j + \lambda_q + 2k_m))}{\Gamma(\frac{1}{2}(-\lambda_j + \lambda_q + 1))}
\]
and
\[
(5.7) \quad \frac{\Gamma(\frac{1}{2}(-\lambda_j + \lambda_i + 2k_m))}{\Gamma(-\lambda_j)}
\]
On the one hand, (5.10) is regular and non-vanishing on (5.3) since for \( \lambda_j \in \mathbb{C} \) with \( \lambda_1 \leq \text{Re} \lambda_j \) we have
\[
\begin{align*}
-\lambda_j + \alpha - |\beta| + 1 & \in -2\mathbb{N} \iff -\lambda_j + \lambda_1 \in -2\mathbb{N}, \\
-\lambda_j + \lambda_q + 2k_m & \in -2\mathbb{N} \iff -\lambda_j + \lambda_{q+1} \in -2\mathbb{N} \quad (1 \leq q < i).
\end{align*}
\]
On the other hand, the poles of (5.7) are just as stated above.

It is easy to check that the other factors in (5.4) produce neither any pole nor any zero locus that cancels a pole of (5.5).

Let
\[
d\lambda_{\alpha, \theta} = d\lambda_{i+1} \cdots d\lambda_r
\]
denote the Euclidean measure on \( a_{\theta_1}^\ast \) and \( \mu_{\theta_1} \) the measure on \( \sqrt{-1}a_{\theta_1}^\ast \) given by
\[
d\mu_{\theta_1}(\lambda_{\alpha, \theta}) = (2\pi)^{-r+i} d(\text{Im} \lambda_{\alpha, \theta}) = (2\pi \sqrt{-1})^{-r+i} d\lambda_{\alpha, \theta},
\]
which coincides with (2.2) if \( i = 0 \).

For \( 0 \leq i \leq r \), let \( \nu_{k, \theta_i} \) denote the measure on \( D_k(\Theta_i) + \sqrt{-1}a_{\theta_i}^\ast \) defined by
\[
\int_{D_k(\Theta_i) + \sqrt{-1}a_{\theta_i}^\ast} \psi(\lambda) d\nu_{k, \theta_i}(\lambda) = \sum_{\lambda_{\alpha, \theta} \in D_k(\Theta_i)} d\theta_i(\lambda_{\alpha, \theta}, k) \int_{\sqrt{-1}a_{\theta_i}^\ast} \psi(\lambda) |c^{\Theta_1}(\lambda, k)|^{-2} d\mu_{\theta_1}(\lambda_{\alpha, \theta}).
\]
In particular,
\[
\int_{\sqrt{-1}a_{\theta_i}^\ast} \psi(\lambda) d\nu_{k, \Theta_i}(\lambda) = \int_{\sqrt{-1}a_{\theta_i}^\ast} \psi(\lambda) \frac{d\mu(\lambda)}{|c(\lambda, k)|^2}
\]
and
\[
\int_{D_k(\mathbb{B})} \psi(\lambda) d\nu_{k, \mathbb{B}}(\lambda) = \sum_{\lambda \in D_k(\mathbb{B})} d\mu(\lambda, k) \psi(\lambda).
\]

Recall \( \mathcal{S}_k \) is defined by (3.13). For \( \phi \in \mathcal{S}_k \) let \( \mathcal{J}_{k, \theta_i}(\phi)(x) \) \( (0 \leq i \leq r) \) denote the functions defined by
\[
\mathcal{J}_{k, \theta_i}(\phi)(x) = \frac{1}{|W(\Theta_i)|} \int_{D_k(\Theta_i) + \sqrt{-1}a_{\theta_i}^\ast} \phi(\lambda)F(\lambda, k; x) d\nu_{k, \theta_i}(\lambda).
\]
By (2.14), Theorem 2.7, Lemma 5.2 and Lemma 1.2 the integral on the right hand side of (5.10) converges and defines an element of \( C^\infty(a)^W \). Note that \( \mathcal{J}_{k, \Theta_i}(\phi)(x) \) for \( \phi \in \mathcal{P}W(a_{\Theta_i}^\ast)^W \) coincides with (2.14).

Now we state the main result of this paper:

**Theorem 5.3** (Inversion formula, final form). Let \( k \in K_1^\ast \). Then for \( \phi \in \mathcal{P}W(a_{\Theta_i}^\ast)^W \)
\[
\mathcal{J}_k(\phi)(x) = \sum_{i=0}^r \mathcal{J}_{k, \Theta_i}(\phi)(x) \quad (x \in a_+).
\]
Thus, it holds for \( f \in C^\infty_0(a)^W \) that
\[
f(x) = \sum_{i=0}^{r} J_{k,\Theta_i}F_k f(x) \quad (x \in a).
\]

**Proof.** By Theorem 2.5 the latter formula follows from the former one.

First we assume \( r = 1 \) or \( k_m > 0 \). For \( 1 \leq i \leq r \) and \( \xi = (\xi_1, \ldots, \xi_i) \in D_k(\Theta_i) \) put
\[
I_{\Theta_i}(\xi, \lambda_{\alpha_\xi}; x) = \frac{(-1)^i}{(2\pi \sqrt{-1})^{r-i}(r-i)!} \times \left( \phi(\lambda) F_\xi(\lambda, k; x) \right)_{\lambda_{\alpha_\xi} = \xi} \times \text{Res} \cdots \text{Res} \text{Res} \left( -\lambda, k \right)^{-1}
\]
and
\[
\phi(\xi, x) = \sum_{\xi \in D_k(\Theta_i)} \int_{\sqrt{-1}a_{\Theta_i}^*} I_{\Theta_i}(\xi, \lambda_{\alpha_\xi}; x) \, d\lambda_{\alpha_\xi} \quad (x \in a_+).
\]
The function \( \lambda \mapsto \phi(\lambda) F_\xi(\lambda, k; x) \left( -\lambda, k \right)^{-1} \) is regular in the region \( \{ 1 \} \). By \( \text{(5.11)} \) and \( \text{(5.8)} \) we have
\[
\text{Res} \cdots \text{Res} \text{Res} \left( -\lambda, k \right)^{-1} = c^{\Theta_i}( -\lambda, k )^{-1} \big|_{\lambda_{\alpha_\xi} = \xi} \text{Res} \cdots \text{Res} \text{Res} \left( -\lambda, k \right)^{-1}
\]
and
\[
\text{Res}_{\lambda_1 = \xi_1} \cdots \text{Res}_{\lambda_2 = \xi_2} \text{Res}_{\lambda_1 = \xi_1} c^{\Theta_i}( -\lambda, k )^{-1}
\]
is a constant that depends only on \( k \) and \( \xi \in D_k(\Theta_i) \). Hence each integral on the right hand side of
\[
\text{(5.11)}\]
converges by Lemma 1.2, Lemma 4.2 and Lemma 5.2. Let us prove by induction that
\[
J_k \phi(x) - J_{k,\Theta_i} \phi(x) - \sum_{\jmath=1}^{i-1} \phi(\xi, x) = 0
\]
for \( i = 1, \ldots, r \). Here \( \xi = (\xi_1, \ldots, \xi_i) \in a_{\Theta_i}^* \). We already proved the case of \( i = 1 \) by Proposition 4.3. If \( r = 1 \) then we are done. So let \( 1 \leq i < r \) and \( \xi = (\xi_1, \ldots, \xi_i) \in D_k(\Theta_i) \). Let \( \{ \xi^{(1)} < \xi^{(2)} < \cdots < \xi^{(k)} = 0 \} = \{ \xi_{i+1} \in \mathbb{R} : (\xi_1, \ldots, \xi_i, \xi_{i+1}) \in D_k(\Theta_{i+1}) \} \cup \{ 0 \} \). With a sufficiently small \( \varepsilon > 0 \) put
\[
\eta^{(1,0)} = (\xi^{(1)} - \varepsilon, \ldots, \xi^{(1)} - \varepsilon, \xi^{(1)} - \varepsilon, \xi^{(1)} - \varepsilon),
\]
\[
\eta^{(1,1)} = (\xi^{(1)} - \varepsilon, \ldots, \xi^{(1)} - \varepsilon, \xi^{(1)} - \varepsilon, \xi^{(1)} + \varepsilon),
\]
\[
\eta^{(1,2)} = (\xi^{(1)} - \varepsilon, \ldots, \xi^{(1)} - \varepsilon, \xi^{(1)} + \varepsilon, \xi^{(1)} - \varepsilon),
\]
\[
\vdots
\]
\[
\eta^{(1,r-1)} = (\xi^{(1)} + \varepsilon, \ldots, \xi^{(1)} + \varepsilon, \xi^{(1)} + \varepsilon, \xi^{(1)} + \varepsilon),
\]
\[
\eta^{(2,0)} = (\xi^{(2)} - \varepsilon, \ldots, \xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon),
\]
\[
\eta^{(2,1)} = (\xi^{(2)} - \varepsilon, \ldots, \xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon, \xi^{(2)} - \varepsilon),
\]
All these points belong to the region (5.3) with \( \lambda_i = \xi_i \). We shift the domain of the integration
\[
\int_{\xi_i + \sqrt{-1}a_{\xi_i}} I_{\Theta_i}(\xi, \lambda_{\alpha_{\xi_i}} ; x) \, d\lambda_{\alpha_{\xi_i}}
\]
successively as
\[
\xi_i + \sqrt{-1}a_{\xi_i} \rightarrow \eta^{(1,0)} + \sqrt{-1}a_{\eta^{(1,0)}} \rightarrow \eta^{(1,1)} + \sqrt{-1}a_{\eta^{(1,1)}} \rightarrow \ldots \rightarrow \eta^{(k,0)} + \sqrt{-1}a_{\eta^{(k,0)}} \rightarrow \sqrt{-1}a_{\eta^{(k,0)}}
\]
while picking up residues as in the proof of Proposition 4.3. Let \( 1 \leq \ell < k \) and \( 1 \leq j \leq r - i \). Using Cauchy’s residue theorem and changes of variables in view of (4.10), Lemma 5.2 and (5.12), we have
\[
\int_{\eta^{(i,j)} + \sqrt{-1}a_{\eta^{(i,j)}}} I_{\Theta_i}(\xi, \lambda_{\alpha_{\xi_i}} ; x) \, d\lambda_{\alpha_{\xi_i}} - \int_{\eta^{(i,j-1)} + \sqrt{-1}a_{\eta^{(i,j-1)}}} I_{\Theta_i}(\xi, \lambda_{\alpha_{\xi_i}} ; x) \, d\lambda_{\alpha_{\xi_i}} = 2\pi \sqrt{-1} \sum_{\lambda_{\alpha_{\xi_{i}}} = \xi^{(i,j)}} \text{Res}_{\lambda_{\alpha_{\xi_{i}}} = \xi^{(i,j)}} I_{\Theta_i}(\xi, \lambda_{\alpha_{\xi_i}} ; x) \, d\lambda_{\alpha_{\xi_i}}.
\]
Note that this is independent of \( j \). Hence
\[
\int_{\xi_i + \sqrt{-1}a_{\xi_i}} I_{\Theta_i}(\xi, \lambda_{\alpha_{\xi_i}} ; x) \, d\lambda_{\alpha_{\xi_i}} - \int_{\sqrt{-1}a_{\xi_i}} I_{\Theta_i}(\xi, \lambda_{\alpha_{\xi_i}} ; x) \, d\lambda_{\alpha_{\xi_i}} = \sum_{\ell=1}^{k-1} \int_{\eta^{(i)} + \sqrt{-1}a_{\eta^{(i)}}} I_{\Theta_{i+\ell}}((\xi, \xi^{(i)}), \lambda_{\alpha_{\xi_{i+\ell}}} ; x) \, d\lambda_{\alpha_{\xi_{i+\ell}}}.
\]
This shows that the difference of (5.13) and (5.11) is (5.13) with \( i \) replaced by \( i + 1 \), completing the induction. If \( i = r \) then (5.13) equals (5.11). Therefore
\[
J_k \phi(x) - J_{k+1} \phi(x) = \sum_{i=1}^{r} \phi_{\Theta_i}(x).
\]
Now let \( 1 \leq i \leq r \) and \( \xi \in D_k(\Theta_i) \). From Lemma 4.4 and (3.8) it holds that as a meromorphic function on \( a_{\xi_i} \subset \mathbb{C} \)
\[
c_{\xi}(\lambda, k)^{-1}|_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} \prod_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} \text{Res}_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} c(-\lambda, k)^{-1} = \sigma_{\xi} c_{\xi}(\lambda, k)|_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} \prod_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} \text{Res}_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} c(-\lambda, k)^{-1}.
\]
\[
\sigma_{\xi} c_{\xi}(\lambda, k)|_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} \prod_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} \text{Res}_{\lambda_{\alpha(\xi_i)} = \xi_{\lambda_i = \xi_i}} c_{\Theta_i}(\lambda, k)^{-1} c_{\Theta_i}(\lambda, k)^{-1}.
\]
Here $\sigma_\xi$ is the restriction operator defined by (4.12). As we shall see in Proposition 5.4, the factor in the last line reduces to $(-1)^i d\sigma_{\Theta_i}(\xi, k)$. Furthermore $c^{\Theta_i}(\lambda, k) c^{\Theta_i}(-\lambda, k)$ is $W(\Theta_i)$-invariant and restricts to $|c^{\Theta_i}(\lambda, k)|^2$ on $\xi + \sqrt{-1}a^*_\Theta_i$ (cf. [28] Lemma 6.6). Hence by changes of variables and Proposition 4.5

$$\int_{\sqrt{-1}a^*_\Theta_i} I_{\Theta_i}(\xi, \lambda a_{\Theta_i}; x) d\lambda a_{\Theta_i}$$

$$= \frac{d\sigma_{\Theta_i}(\xi, k)}{|W(\Theta_i)|} \int_{\sqrt{-1}a^*_\Theta_i} \frac{\phi(\lambda) F(\lambda, k; x)}{|c^{\Theta_i}(\lambda, k)|^2} \bigg|_{\lambda a_{\Theta_i}=\xi} d\mu_{\Theta_i}(\lambda a_{\Theta_i}).$$

Thus $\phi_{\Theta_i}(x) = J_{k, \Theta_i} \phi(x)$ for $x \in a_+$ and $i = 1, \ldots, r$, proving the theorem under the assumption that $r = 1$ or $k_m > 0$.

Next, let us assume $r > 1$ and $k_m = 0$. In this case Proposition 4.3 is not applicable. Instead, we use the result in the case of $r = 1$ (the case of the Jacobi transform) that is shown above. Let $F_0(\lambda_1; x_1)$, $\Phi_0(\lambda_1; x_1)$, $c_0(\lambda_1)$ and $d_0(\lambda_1)$ be respectively $F(\lambda, k; x)$, $\Phi(\lambda, k; x)$, $c(\lambda, k)$ and $d_B(\lambda, k)$ in the case of $r = 1$. Then we have

$$F(\lambda, k; x) = \frac{1}{r!} \sum_{s \in \mathfrak{S}_r} \prod_{j=1}^r F_0(\lambda_j; x_{s(j)}), \quad \Phi(\lambda, k; x) = \prod_{j=1}^r \Phi_0(\lambda_j; x_j),$$

$$d\sigma_{\Theta_i}(\lambda a_{\Theta_i}; k) = \frac{(i!)^2}{|W_{\Theta_i}(\lambda a_{\Theta_i})|} \int_{\sqrt{-1}a^*_\Theta_i} d_0(\lambda_j), \quad c^{\Theta_i}(\lambda, k) = \frac{i!}{r!} \prod_{j=i+1}^r c_0(\lambda_j).$$

The first and second formulas are respectively (2.7) and (2.4) of [30]. To deduce the third and fourth formulas use (1.18). Hence if $x = (x_1, \ldots, x_r) \in a_+$, $\eta \ll 0$ and $s \in \mathfrak{S}_r$ then $J_{s} \phi(x)$ is equal to

$$(5.14) \int_{(\eta, \ldots, \eta) + \sqrt{-1}a^*} \phi(\lambda) \Phi((\lambda_s, \ldots, \lambda_s), \lambda, k; x) c(-\lambda, k)^{-1} d\mu(\lambda)$$

$$= \int_{(\eta, \ldots, \eta) + \sqrt{-1}a^*} \phi(\lambda) \Phi(\lambda, k, (x_{s(1)}, \ldots, x_{s(s)})) c(-\lambda, k)^{-1} d\mu(\lambda)$$

$$= \frac{r!}{(2\pi\sqrt{-1})^r} \int_{\eta-\sqrt{-1}x}^{\eta+\sqrt{-1}x} \Phi_0(\lambda_1; x_{s(1)}) d\lambda_1 \int_{\eta-\sqrt{-1}x}^{\eta+\sqrt{-1}x} \Phi_0(\lambda_2; x_{s(2)}) d\lambda_2$$

$$\cdots \int_{\eta-\sqrt{-1}x}^{\eta+\sqrt{-1}x} \Phi_0(\lambda_r; x_{s(r)}) d\lambda_r \quad c_0(-\lambda_r).$$

Each step in the iterated integral is the inverse Jacobi transform of a Paley-Wiener function. If $\psi(\lambda_j)$ is a Paley-Wiener function of one variable, then by the result in the case of $r = 1$ we have

$$\frac{1}{2\pi\sqrt{-1}} \int_{\eta-\sqrt{-1}x}^{\eta+\sqrt{-1}x} \psi(\lambda_j) \Phi_0(\lambda_j; x_{s(j)}) d\lambda_j$$

$$= \frac{1}{4\pi\sqrt{-1}} \int_{-\sqrt{-1}x}^{\sqrt{-1}x} \psi(\lambda_j) F_0(\lambda_j; x_{s(j)}) d\lambda_j \quad c_0(-\lambda_j).$$
Therefore (5.14) becomes

\[
(5.15) \quad \sum_{i=0}^{r} \sum_{J=(j_1, \ldots, j_i)} \sum_{\lambda_j \in \Delta_i} \prod_{j=1}^{r} F_0(\lambda_j; x_{s(j)}) \frac{r! \prod_{j \in J} d_0(\lambda_j)}{(4\pi \sqrt{-1})^{r-1}} \left( \int \phi(\lambda) \prod_{j=1}^{r} \frac{d\lambda_j}{|c_0(\lambda_j)|^2} \right) \]

Since \( J_k \phi(x) \) (which is equal to (5.15)) is independent of \( s \in \mathcal{S}_r \), we can replace \( \prod_{j=1}^{r} F_0(\lambda_j; x_{s(j)}) \) in (5.15) with \( F(\lambda, k; x) \). By changes of variables, (5.15) reduces to

\[
\sum_{i=0}^{r} \sum_{\lambda_i(\xi_1) \in D_k(\Theta_i)} \frac{(i!)^2 \prod_{j=1}^{i} d_0(\lambda_j)}{(2\pi \sqrt{-1})^{r-i} |W(\Theta_i)| |W_{\Theta_i}|} \left( \int \phi(\lambda) F(\lambda, k; x) \left( \frac{r!}{r!} \right)^2 \prod_{j=i+1}^{r} \frac{d\lambda_j}{|c_0(\lambda_j)|^2} \right) = \sum_{i=0}^{r} J_{k, \Theta_i} \phi(x). \]

\[ \square \]

**Proposition 5.4.** Assume \( k \in K_1' \) and let \( 1 \leq i \leq r \). If \( r > 1 \) then we further assume \( k_m > 0 \). For \( \xi \in D_k(\Theta_i) \),

\[
(5.16) \quad (-1)^i \sum_{\lambda_i=\xi_1} \sum_{\lambda_i=\xi_1} (c_{\Theta_i}(\lambda, k)^{-1} c_{\Theta_i}(-\lambda, k)^{-1}) = d_{\Theta_i}(\xi, k).
\]

**Proof.** By (3.6) and the following formulas

\[
(5.17) \quad \Gamma(z+1) = z \Gamma(z),
\]

\[
(5.18) \quad \Gamma(z)\Gamma(1-z) = \frac{\pi}{\sin \pi z}
\]

for the Gamma function, we have

\[
(5.19) \quad \tilde{c}_i(\lambda, k)^{-1} \tilde{c}_i(-\lambda, k)^{-1}
\]

\[
= \frac{\sin \frac{\pi}{2} (\lambda_i + \alpha - |\beta| + 1) \sin \frac{\pi}{2} (-\lambda_i + \alpha - |\beta| + 1)}{\sin \frac{\pi}{2} (\lambda_i + \xi_1) \sin \frac{\pi}{2} (\lambda_i - \xi_1)} p(\lambda_i)
\]

where

\[
p(z) := \frac{\Gamma(\frac{1}{2}(z + \alpha + |\beta| + 1)) \Gamma(\frac{1}{2}(-z + \alpha + |\beta| + 1))}{\Gamma(\frac{1}{2}(z - \alpha + |\beta| + 1)) \Gamma(\frac{1}{2}(-z - \alpha + |\beta| + 1))}
\]

Here the second equality in (5.19) holds since \( \xi_1 - \alpha + |\beta| - 1 \in 2\mathbb{N} \). Hence if \( i = 1 \) then

\[
- \sum_{\lambda_1=\xi_1} \sum_{\lambda_1=\xi_1} (c_{\Theta_i}(\lambda, k)^{-1} c_{\Theta_i}(-\lambda, k)^{-1})
\]

\[
= -2^{\alpha-\beta-2} p(\xi_1) \cdot \frac{\sin \frac{\pi}{2} \lambda_1}{\sin \frac{\pi}{2} (\lambda_1 + \xi_1)} \sum_{\lambda_1=\xi_1} \frac{1}{\sin \frac{\pi}{2} (\lambda_1 - \xi_1)}
\]
Let $F$ that binning (5.21) with (5.19) we have

\[ \frac{-2^{\alpha-\beta-1}i}{\pi} p(\xi_i). \]

This proves (5.16) for $i = 1$.

Now let $i > 1$. We will prove (5.16) assuming the equality holds for $i - 1$. Let $\xi = (\xi', \xi_i) \in D_k(\Theta_i)$ with $\xi' \in D_k(\Theta_{i-1})$. It is sufficient to show

\[ \frac{\tilde{c}_{\alpha'}(\rho(k), k)^{-2} de_\alpha(\xi', k)}{\tilde{c}_{\theta_{i-1}}(\rho(k), k)^{-2} de_{\alpha, i}(\xi', k)} = \prod_{\lambda_i = \xi_i, \alpha \in \langle \Theta_i \rangle \cup \langle \Theta_{i-1} \rangle} \tilde{c}_\alpha((\xi', \lambda_i), k)^{-1} \tilde{c}_{\alpha, i}((-\xi', -\lambda_i), k). \]

The set $\langle \Theta_i \rangle \cup \langle \Theta_{i-1} \rangle$ consists of roots $2\beta_i, \beta_i, \beta_i \pm \beta_j (1 \leq j < i)$. By (1.14), (5.17) and (5.18) we have for $j = 1, \ldots, i - 1$

\[ \tilde{c}_{\beta_i - \beta_j}((\xi', \lambda_i), k)^{-1} \tilde{c}_{\beta_i + \beta_j}((\xi', \lambda_i), k)^{-1} \times \tilde{c}_{\beta_i - \beta_j}((-\xi', -\lambda_i), k)^{-1} \tilde{c}_{\beta_i + \beta_j}((-\xi', -\lambda_i), k)^{-1} \]

\[ = \frac{-\sin \frac{\pi}{2}(\lambda_i + \xi_j) \sin \frac{\pi}{2}(\lambda_i - \xi_j)}{\sin \frac{\pi}{2}(\lambda_i + \xi_j + 2k_m) \sin \frac{\pi}{2}(\lambda_i - \xi_j + 2k_m)} q(\lambda_i, \xi_j) \]

\[ = \frac{-\sin \frac{\pi}{2}(\lambda_i + \xi_j) \sin \frac{\pi}{2}(\lambda_i - \xi_j)}{\sin \frac{\pi}{2}(\lambda_i + \xi_{j+1}) \sin \frac{\pi}{2}(\lambda_i - \xi_{j+1})} q(\lambda_i, \xi_{j+1}) \]

with

\[ q(z, w) := \frac{(w^2 - z^2)\Gamma\left(\frac{1}{2}(z - w + 2k_m)\right)\Gamma\left(\frac{1}{2}(z + w - 2k_m)\right)}{4\Gamma\left(\frac{1}{2}(z - w + 2k_m + 2)\right)\Gamma\left(\frac{1}{2}(z + w - 2k_m + 2)\right)}. \]

Here the second equality in (5.21) holds since $\xi_{j+1} - \xi_j - 2k_m \in 2\mathbb{N}$. Combining (5.21) with (5.19) we have

\[ \prod_{\alpha \in \langle \Theta_i \rangle \cup \langle \Theta_{i-1} \rangle} \tilde{c}_\alpha((\xi', \lambda_i), k)^{-1} \tilde{c}_{\alpha, i}((-\xi', -\lambda_i), k)^{-1} \]

\[ = \frac{2^{\alpha-\beta-2}\lambda_i \sin \pi \lambda_i}{\sin \frac{\pi}{2}(\lambda_i + \xi_i) \sin \frac{\pi}{2}(\lambda_i - \xi_i)} p(\lambda_i) \prod_{j=1}^{i-1} q(\lambda_i, \xi_j), \]

whose residue at $\lambda_i = \xi_i$ equals

\[ 2^{\alpha-\beta-2}\xi_i p(\xi_i) \prod_{j=1}^{i-1} q(\xi_i, \xi_j) \cdot \frac{\sin \pi \lambda_i}{\sin \frac{\pi}{2}(\lambda_i + \xi_i)} \bigg|_{\lambda_i = \xi_i} \cdot \frac{1}{\sin \frac{\pi}{2}(\lambda_i - \xi_i)} \]

\[ = \frac{2^{\alpha-\beta-1}2^i}{\pi} \frac{1}{\pi} p(\xi_i) \prod_{j=1}^{i-1} q(\xi_i, \xi_j). \]

Thus (5.20) is proved. \qed

**Theorem 5.5** (Paley-Wiener theorem). Let $k \in \mathbb{K}_1'$. Then the hypergeometric Fourier transform $F_k$ is a bijection of $C_0^\infty(\mathbb{A})^W$ onto $\mathcal{P}W(\mathbb{A}_C^*)^W$.

**Proof.** Let $\phi \in \mathcal{P}W(\mathbb{A}_C^*)^W$. By Proposition 2.6 and Theorem 5.3, $J_k \phi(x) \in C^\infty(\mathbb{A}_+)$ extends to a function in $C_0^\infty(\mathbb{A})^W$. Thus it only remains to prove that $F_k J_k \phi(\lambda) = \phi(\lambda)$. By [21 Theorem 9.13] the equality holds if $k \in \mathbb{K}_4$. 


The general result follows from this because by (2.11), (2.12), (2.16) and (2.17) it holds that \( \mathcal{F}_k \mathcal{J}_k \phi = \mathcal{F}_k \mathcal{J}_k \phi = \mathcal{F}_{k+1} \mathcal{J}_{k+1} \phi \) for any \( k \in \mathcal{K}_1' \). □

Now we define for \( \phi \in \mathcal{S}_k \)

\[
\mathcal{J}_k \phi(x) = \int_{\|a\|} J_k, \Theta_i, \phi(x) \in C^\infty (a)^W.
\]

Let \( \nu_k \) denote the measure on \( \bigcup_{i=0}^r (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+)) \) that coincides with \( \nu_k, \Theta_i \) on each \( D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+) \). By Lemma 1.2 and Lemma 3.6 both \( \mathcal{P}W(a_k^+)^W \) and \( \mathcal{S}_k \) are naturally identified with dense subspaces of \( L^2(\bigcup_{i=0}^r (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+); d\nu_k) \) by restriction.

**Lemma 5.6.** Suppose \( \phi \in \mathcal{S}_k \) and \( f \in C_0^\infty (a)^W \). Then

\[
\frac{1}{|W|} \int_a \mathcal{J}_k \phi(x)|f(x)\delta_k(x)dx = \int_{\|a\|} (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+)) \phi(\lambda) \mathcal{F}_k f(\lambda) d\nu_k(\lambda).
\]

**Proof.** The result follows from Lemma 1.2, Lemma 2.7, and Fubini’s theorem. □

**Theorem 5.7 (Plancherel theorem).** Let \( k \in \mathcal{K}_1' \). Then for \( f \in C_0^\infty (a)^W \),

\[
\frac{1}{|W|} \int_a |f(x)|^2 \delta_k(x)dx = \int_{\|a\|} (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+)) \mathcal{F}_k f(\lambda)^2 d\nu_k(\lambda).
\]

Moreover, the hypergeometric Fourier transform \( \mathcal{F}_k \) extends to an isometry of \( L^2(a; \frac{1}{|W|} \delta_k(x)dx)^W \) into \( L^2(\bigcup_{i=0}^r (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+); d\nu_k) \). For any \( \phi \in \mathcal{S}_k \) it holds that \( \mathcal{F}_k^{-1} \phi = \mathcal{J}_k \phi \).

**Proof.** Applying Lemma 5.6 to \( \phi = \mathcal{F}_k f \) we obtain (5.22) by Theorem 5.3. Thus \( \mathcal{F}_k \) extends to an isometry of \( L^2(a; \frac{1}{|W|} \delta_k(x)dx)^W \) into \( L^2(\bigcup_{i=0}^r (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+); d\nu_k) \). This is onto since the image is dense.

Now let \( \phi \in \mathcal{S}_k \) and put \( g = \mathcal{F}_k^{-1} \phi \in L^2(a; \frac{1}{|W|} \delta_k(x)dx)^W \). Then for any \( f \in C_0^\infty (a)^W \)

\[
(g, f) = (\phi, \mathcal{F}_k f) = \int_{\|a\|} (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+)) \phi(\lambda) \mathcal{F}_k f(\lambda) d\nu_k(\lambda).
\]

Here \((\cdot, \cdot)\) stands for the inner products of two Hilbert spaces. Comparing this with Lemma 5.6 one sees \( (g - \mathcal{J}_k \phi) \delta_k = 0 \) as a distribution on \( a \). Thus \( g(x) = \mathcal{J}_k \phi(x) \) almost everywhere. □

**Remark 5.8.** Let \( \nu_k \) denote the \( W \)-invariant measure on \( \bigcup \mathcal{H}_k \) whose restriction to \( \lambda_\Theta(\Theta_i) + \sqrt{r-1} a_{\Theta_i}^+ (0 \leq i \leq r; \lambda_\Theta(\Theta_i) \in D_k(\Theta_i)) \) is \n
\[
\frac{|W_\Theta(\Theta_i)|}{|W||W(\Theta)|} \nu_k = \frac{|W_\Theta(\Theta_i)|}{|W|} \nu_k.
\]

By Lemma 3.3 \( L^2(\bigcup_{i=0}^r (D_k(\Theta_i) + \sqrt{r-1} \text{Cl}(a_{\Theta_i}^+); d\nu_k) \) is naturally identified with \( L^2(\bigcup \mathcal{H}_k; d\nu_k)^W \).
We expect \( \mathcal{F}_k S_k \) would coincide with the Schwartz space \( C(a, k)^W \) introduced in [16]. In fact, Delorme [16] proved, for each reduced root system, that \( \mathcal{F}_k C(a, k)^W \) coincides with a version of the Schwartz space which is defined in a similar way to \( S_k \).

**Corollary 5.9.** Let \( k \in K' \). Then the hypergeometric function \( F(\lambda, k) \) is square integrable if and only if \( \lambda \in W D_k(B) \). Each square integrable hypergeometric function is of the form

\[
F(\lambda, k) = c(\lambda, k) \Phi(\lambda, k) \quad \text{for some } \lambda \in D_k(B).
\]

For any \( \lambda \in D_k(B) \),

\[
\frac{1}{|W|} \int_a F(\lambda, k; x)^2 \delta_k(x) \, dx = \frac{1}{d_B(\lambda, k)}.
\]

For any \( \lambda, \mu \in D_k(B) \) with \( \lambda \neq \mu \),

\[
\frac{1}{|W|} \int_a F(\lambda, k; x)F(\mu, k; x) \delta_k(x) \, dx = 0.
\]

**Proof.** Let \( \xi \in D_k(B) \) and let \( \phi_\xi(\lambda) \in S_k \) be the function which takes 1 at \( \lambda = \xi \) and 0 elsewhere. Then \( \mathcal{J}_k \phi_\xi(x) = d_B(\xi, k) F(\xi, k; x) \). Hence by Theorem 5.7, \( F(\xi, k, x) \) is square integrable and \( \mathcal{F}_k F(\xi, k) = d_B(\xi, k)^{-1} \phi_\xi \).

(We already know the square integrability by Corollary 3.3.) Now (5.22) and (5.23) are immediate.

Let us prove \( F(\xi, k) \)’s exhaust the square integrable hypergeometric functions. Assume that \( F(\mu, k) \) is square integrable for some \( \mu \in a_\omega^C \). Let \( p \in S(a_\omega) \) and let \( f \in C_0^\infty(a)^W \). By (2.20), (2.10), (2.13) and Theorem 5.7 we have

\[
\int_{\bigcup_{i=0}^q(D_k(\Theta_i) + \sqrt{-1} Cl(a^*_{\Theta_i}))} p(\mu) \mathcal{F}_k F(\mu, k)(\lambda) \mathcal{F}_k f(\lambda) \, d\nu_k(\lambda)
\]

\[
= \frac{p(\mu)}{|W|} \int_a F(\mu, k; x) f(x) \delta_k(x) \, dx
\]

\[
= \frac{1}{|W|} \int_a (T(k, p) F(\mu, k; x)) f(x) \delta_k(x) \, dx
\]

\[
= \frac{1}{|W|} \int_a F(\mu, k; x) (T(k, p) f(x)) \delta_k(x) \, dx
\]

\[
= \int_{\bigcup_{i=0}^q(D_k(\Theta_i) + \sqrt{-1} Cl(a^*_{\Theta_i}))} \mathcal{F}_k F(\mu, k)(\lambda) \mathcal{F}_k T(k, p) f(\lambda) \, d\nu_k(\lambda)
\]

\[
= \int_{\bigcup_{i=0}^q(D_k(\Theta_i) + \sqrt{-1} Cl(a^*_{\Theta_i}))} p(\lambda) \mathcal{F}_k F(\mu, k)(\lambda) \mathcal{F}_k f(\lambda) \, d\nu_k(\lambda).
\]

Here \( p^\vee(\lambda) = p(\lambda) \). Note that \( p(\lambda) = p(\lambda) \) on \( \bigcup_{i=0}^q(D_k(\Theta_i) + \sqrt{-1} Cl(a^*_{\Theta_i})) \) by the \( W \)-invariance. Thus by Lemma 3.4 (\( p(\mu) - p(\lambda) \)) \( \mathcal{F}_k F(\mu, k)(\lambda) = 0 \) except on a null set of \( \nu_k \). Since \( p \) can be chosen arbitrarily, this is possible only when \( \mu = w \xi \) for some \( w \in W \) and \( \xi \in D_k(B) \).

The second statement of the corollary follows from Theorem 3.1. \( \square \)
Remark 5.10. The square integrable hypergeometric functions are analytic continuation of the Jacobi polynomials. This fact was observed by [28, Remark 5.12] for the group case and mentioned without proof in [3, §6].

If $\beta < 0$, then $\lambda \in D_k(B)$ if and only if $\lambda_r < 0$ and $\mu = \lambda - \rho(k)$ satisfies $\mu_i \in 2\mathbb{N}, \mu_{i+1} - \mu_i \in 2\mathbb{N} \,(1 \leq i \leq r - 1)$.

Thus, $D_k(B)$ can be written as
\begin{equation}
D_k(B) = \{ \mu + \rho(k) \in a^* : \mu + \rho(k) \in -w_\Xi^* a^*_\alpha \}
\end{equation}
and $\langle \mu, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha \in R^+$.

Here $w_\Xi^*$ denotes a longest element of $W_\Xi = \mathcal{E}_r$.

If $k \in K_+$ and $\mu \in a^*$ satisfies $\langle \mu, \alpha^\vee \rangle \in \mathbb{N}$ for all $\alpha \in \mathcal{R}^+$, then $\Phi(\mu + \rho(k), k) = P(\mu, k)$, which is the Jacobi polynomial of the highest weight $\mu$ (cf. [13, 10, 11]). Note $k \in K_+$ if and only if $\alpha - \beta \geq 0, \beta \geq -\frac{1}{2}$, and $k_m \geq 0$. The function $F(\mu + \rho(k), k) = c(\mu + \rho(k), k)P(\mu, k)$ forms a one parameter family of polynomials with analytic parameter $\beta$. If
\begin{equation}
\beta < -\alpha - 2(r - 1)k_m - 1,
\end{equation}
then $D_k(B) \neq \emptyset$ and $F(\mu + \rho(k), k) (\mu + \rho(k) \in D_k(B))$ is square integrable. On the one hand, for $k \in K_+$ the $L^2$-norm of the constant multiple $c(\mu + \rho(k), k)P(\mu, k)$ of the Jacobi polynomial is an integral of $c(\mu + \rho(k), k)^2P(\mu, k)^2 \delta_k$ over a compact torus and its explicit formula is given by [11 Corollary 3.5.3]. On the other hand, for $k$ satisfying (3.4) and $\mu + \rho(k) \in D_k(B)$ the $L^2$-norm of $F(\mu + \rho(k), k)$ is an integral of $\frac{1}{|\mu|}F(\mu + \rho(k), k)^2 \delta_k$ over $a$ and its explicit formula is given by (5.24) and (5.1). Comparing these formulas we can observe that the two norms coincide up to a constant multiple that depends only on $k$.

We can reduce the case of $\beta > 0$ to the case of $\beta < 0$ by (1.26).

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