Abstract. For any non-degenerate, quasi-homogeneous hypersurface singularity $W$ and an admissible group of diagonal symmetries $G$, Fan, Jarvis, and Ruan have constructed a cohomological field theory which is a candidate for the mathematical structure behind the Landau-Ginzburg A-model. When using the orbifold Milnor ring of a singularity $W$ as a B-model, and the Frobenius algebra $\mathcal{H}_{W,G}$ constructed by Fan, Jarvis, and Ruan, as an A-model, the following conjecture is obtained: For a quasi-homogeneous singularity $W$ and a group $G$ of symmetries of $W$, there is a dual singularity $W^T$ such that the orbifold A-model of $W/G$ is isomorphic to the B-model of $W^T$. I will show that this conjecture holds for a two-dimensional invertible loop potential $W$ with its maximal group of diagonal symmetries $G_W$.

1. Introduction

In a recent paper [FJR], Fan, Jarvis, and Ruan constructed the mathematical theory (FJRW-theory) behind the Landau-Ginzburg A-model. Their construction gives, among other things, a Frobenius algebra $\mathcal{H}_{W,G}$ which is determined by a non-degenerate quasi-homogeneous polynomial $W$ and an admissible group $G$ of diagonal symmetries of $W$.

Landau-Ginzburg models have been studied extensively in the physics literature (see for example [GP]). In particular, a mirror construction for these models was suggested by Berglund and Hübsch [BH]. This mirror construction used the so-called ‘invertible’ potentials: quasi-homogeneous polynomials with the same number of monomials and variables. In the FJRW theory, these potentials must satisfy non-degeneracy conditions given by,

1. The potential $W$ must have an isolated singularity at the origin.
2. The weights (charges) of $W$ must be uniquely determined.

These non-degeneracy conditions imply that the invertible potentials will be of two kinds [KS],

\begin{align*}
W_{\text{loop}} &= X_1^{a_1}X_2 + \cdots + X_{N-1}^{a_{N-1}}X_N + X_N^{a_N}X_1 \\
W_{\text{chain}} &= X_1^{a_1}X_2 + \cdots + X_{N-1}^{a_{N-1}}X_N + X_N^{a_N}
\end{align*}

In [K], Kreuzer proved that the Berglund-Hübsch construction can be used to show that the types of potentials previously described satisfy a certain type of mirror symmetry. Based on Kreuzer’s work, Krawitz [Kr] conjectured that the FJRW construction satisfies the following,

Conjecture. (Landau-Ginzburg Mirror Symmetry Conjecture): For a non-degenerate, quasi-homogeneous, invertible singularity $W$ and its maximal group of diagonal symmetries $G_W$, there is a dual singularity $W^T$ such that the FJRW-ring of $\mathcal{H}_{W,G_W}$ is isomorphic to the (unorbifolded) Milnor ring of $W^T$.

This conjecture has been verified for simple singularities by Fan, Jarvis, and Ruan in [FJR], and for unimodal and bimodal singularities by Priddis et al. in [Pr]. Recently, Fan and Shen [FS] proved that this conjecture is true for all chain-type potentials in two dimensions ($N = 2$). Following some of the ideas of Kreuzer, I will show that the Landau-Ginzburg Mirror Symmetry conjecture is true for loop-type potentials in two dimensions.

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Combined with the work of Fan and Shen, this completes the proof of the conjecture for all superpotentials in two dimensions. In a recent preprint [Kr], Krawitz has shown us a proof of the conjecture for $N = 3$ and higher.

1.1. Outline of the Paper. The organization of the paper is as follows. First, a review of the FJRW construction will be given in Section 1.3. Section 1.4 describes some additional notation that will be used throughout the paper. In Section 2, a proof of the conjecture will be given for loop potentials in two dimensions.

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1.3. Review of the Construction. In this section, I will present a simplified description of the FJRW-theory, similar to the one outlined in [Pr]. For a full description of the theory, the reader is referred to the original paper of Fan, Jarvis, and Ruan [FJR].

A quasi-homogeneous polynomial $W \in \mathbb{C}[X_1, \ldots, X_N]$ is defined as a polynomial for which there exist rational positive degrees $q_1, \ldots, q_N \in \mathbb{Q}^{>0}$, such that for any $\lambda \in \mathbb{C}^*$

$$W(\lambda^{q_1} X_1, \ldots, \lambda^{q_N} X_N) = W(X_1, \ldots, X_N).$$

For $i \in \{1, ..., N\}$, we will call $q_i$ the weight of the variable $X_i$.

Let $W : \mathbb{C}^N \rightarrow \mathbb{C}$ be a quasi-homogeneous polynomial satisfying the non-degeneracy conditions (1) and (2). We define the local algebra of $W$, also known as the Milnor ring, by

$$\mathcal{O}_W := \mathbb{C}[X_1, \ldots, X_N]/\text{Jac}(W),$$

where Jac($W$) is the Jacobian ideal of $W$, generated by partial derivatives

$$\text{Jac}(W) := \left( \frac{\partial W}{\partial X_1}, \ldots, \frac{\partial W}{\partial X_N} \right).$$

It is easy to see that the local algebra is generated by monomials of the form $\prod X_i^{b_i}$. The local algebra is graded by the weighted-degree of each monomial, where $X_i$ has weight $q_i$.

The local algebra contains a unique highest-degree element given by $\det \left( \frac{\partial^2 W}{\partial X_i \partial X_j} \right)$, whose degree is given by

$$\hat{c}_W = \sum_{i=1}^N \left( 1 - 2q_i \right).$$

Also, the dimension of the local algebra as a vector space over $\mathbb{C}$ is given by

$$\mu = \prod_{i=1}^N \left( \frac{1}{q_i} - 1 \right).$$

For $f, g \in \mathcal{O}_W$, we define the residue pairing $\langle f, g \rangle : \mathcal{O}_W \times \mathcal{O}_W \rightarrow \mathbb{C}$ by

$$fg = \frac{\langle f, g \rangle}{\mu} \det \left( \frac{\partial^2 W}{\partial X_i \partial X_j} \right) + \text{lower order terms.}$$

This pairing endows the local algebra with the structure of a Frobenius algebra. This structure is known as the unorbifolded Landau-Ginzburg B-model.

In order to define the FJRW-ring $\mathcal{H}_{W,G}$, or A-model, we need both a potential $W$ and an admissible group of diagonal symmetries $G$ of $W$. For the definition of admissible, see [FJR]. We define the maximal group $G_W$ of diagonal symmetries of $W$ by

$$G_W := \{ (\alpha_1, \ldots, \alpha_N) \in (\mathbb{C}^*)^N \mid W(\alpha_1 X_1, \ldots, \alpha_N X_N) = W(X_1, \ldots, X_N) \}$$

(7)
This group is known to be admissible and, as shown in [FJR], when $W$ satisfies the non-degeneracy conditions, the group $G_W$ is finite. Also note that the element $J$, defined by

$$J := (e^{2\pi i q_1}, \ldots, e^{2\pi i q_n})$$

always belongs to $G_W$, and the group $\langle J \rangle$ generated by $J$ is always admissible.

The original definition of the state space of $\mathcal{H}_{W,G}$ is in terms of Lefschetz thimbles. However, it can be described more simply in terms of sums of local algebras.

For $g \in G$, let $\text{Fix } g \subseteq \mathbb{C}^N$ be the set of fixed points of $g$, and let $N_g$ be its dimension. Also, let $W|\text{Fix } g$ be the potential restricted to the fixed point locus of $g$. Define the vector space $\mathcal{H}_g$ by

$$\mathcal{H}_g := \mathcal{D}_{W|\text{Fix } g} \cdot \omega,$$

where $\omega = dX_{i_1} \wedge dX_{i_2} \wedge \cdots \wedge dX_{i_{N_g}}$.

The state space of $\mathcal{H}_{W,G}$ is defined as the $G$-invariant subspace of the sum of the $\mathcal{H}_g$:

$$\mathcal{H}_{W,G} := \left( \bigoplus_{g \in G} \mathcal{H}_g \right)^G.$$

This space can be graded by the so-called $W$-degree. In order to define the $W$-degree of each element in $\mathcal{H}_{W,G}$, we note that any element $g \in G$ can be written in the form

$$g = (e^{2\pi i \theta_1^g}, \ldots, e^{2\pi i \theta_N^g}).$$

If the phases $\theta_j^g$ satisfy the condition $0 \leq \theta_j^g < 1$, then we denote them by $\Theta_j^g$. Note that any element $g \in G$ can be uniquely written in the form

$$g = (e^{2\pi i \Theta_1^g}, \ldots, e^{2\pi i \Theta_N^g}), \quad \text{with} \quad 0 \leq \Theta_j^g < 1.$$

Let $\alpha_g \in (\mathcal{H}_g)^G$, then we define the $W$-degree of $\alpha_g$ by

$$\deg_W(\alpha_g) := N_g + 2 \sum_{j=1}^N (\Theta_j^g - q_j).$$

From (11), it is easy to check that $\text{Fix } g = \text{Fix } g^{-1}$ and thus there is a canonical isomorphism $I : \mathcal{H}_g \to \mathcal{H}_{g^{-1}}$. From this we can see that the pairing on $\mathcal{D}_{W|\text{Fix } g}$ induces a pairing $\eta_g$

$$\eta_g : (\mathcal{H}_g)^G \otimes (\mathcal{H}_{g^{-1}})^G \to \mathbb{C}, \quad \text{given by} \quad \eta_g(a,b) = \langle a, I^{-1}(b) \rangle.$$

The pairing on $\mathcal{H}_{W,G}$ is the direct sum of the pairings $\eta_g$. Fixing a basis for $\mathcal{H}_{W,G}$, we denote the pairing by a matrix $\eta_{\alpha,\beta} = \langle \alpha, \beta \rangle$, with inverse $\eta^{\alpha,\beta}$.

For each pair of non-negative integers $g$ and $n$, with $2g - 2 + n > 0$, the Fan-Jarvis-Ruan-Witten (FJRW) cohomological field theory produces classes $A^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \in H^*(\overline{\mathcal{M}}_{g,n})$ of complex codimension $D$ for each $n$-tuple $(\alpha_1, \alpha_2, \ldots, \alpha_n) \in (\mathcal{H}_{W,G})^n$. Here, $\overline{\mathcal{M}}_{g,n}$ is the stack of stable curves of genus $g$ with $n$ marked points, and the codimension $D$ is given by

$$D := \hat{c}_W(g-1) + \frac{1}{2} \sum_{n=1}^n \deg_W(\alpha_i),$$

We define three-point correlators by

$$\langle \alpha_1, \alpha_2, \alpha_3 \rangle := \int_{\overline{\mathcal{M}}_{0,3}} A^W_{0,3}(\alpha_1, \alpha_2, \alpha_3).$$
The three-point correlator \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) vanishes unless \( D \) is zero. These three-point correlators can be used to define structure constants for a multiplication on \( \mathcal{H}_{W,G} \). If \( r, s \in \mathcal{H}_{W,G} \), this multiplication is defined by

\[
r \star s := \sum_{\alpha, \beta} \langle r, s, \alpha \rangle \eta^{\alpha, \beta},
\]

where the sum is taken over all choices of \( \alpha \) and \( \beta \) in a fixed basis of \( \mathcal{H}_{W,G} \).

As described in [EP], in genus zero with three marked points, the class \( \Lambda^W_{g,3}(\alpha_1, \alpha_2, \alpha_3) \) satisfies the following axioms that allow us to compute most of the three-point correlators \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle \) explicitly.

**Axiom 1. Dimension:** If \( 2D \notin \mathbb{Z} \), then \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \). Otherwise, \( 2D \) is the real codimension of the class \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \). In particular, if \( g = 0 \) and \( n = 3 \), then \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle = 0 \) unless \( D = 0 \).

Notice that in the case where \( g = 0 \) and \( n = 3 \), if \( D \) is the codimension of the class \( \Lambda^W_{0,3}(\alpha_1, \alpha_2, \alpha_3) \), then \( D = 0 \) if and only if \( \sum_{i=1}^3 \deg W \alpha_i = 2\mathcal{E} \).

**Axiom 2. Symmetry:** Let \( \sigma \in S_n \). Then

\[
\Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = \Lambda^W_{g,n}(\alpha_{\sigma(1)}, \alpha_{\sigma(2)}, \ldots, \alpha_{\sigma(n)})
\]

The next few axioms rely on the degrees of line bundles \( \mathcal{L}_1, \ldots, \mathcal{L}_N \) endowing an orbicurve with a so-called \( W \)-structure; however, this can be reduced to a simple numerical criterion. Consider the class \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_k) \), with \( \alpha_j \in (\mathcal{H}_g)^{G} \) for each \( j \). For each variable \( X_j \), define \( l_j \) to be the degree of the line bundle \( \mathcal{L}_j \). By [FJR], this degree is given by the equation

\[
l_j = q_j(2g - 2 + k) - \sum_{i=1}^k \Theta^i_j.
\]

**Axiom 3. Integer degrees:** If \( l_j \notin \mathbb{Z} \) for some \( j \in \{1, \ldots, N\} \), then \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) = 0 \).

**Axiom 4. Concavity:** If \( l_j < 0 \) for all \( j \in \{1, 2, 3\} \), then \( \langle \alpha_1, \alpha_2, \alpha_3 \rangle = 1 \).

The next axiom is related to the Witten map:

\[
W : \bigoplus_{j=1}^N H^0(\mathcal{E}, \mathcal{L}_j) \to \bigoplus_{j=1}^N H^1(\mathcal{E}, \mathcal{L}_j)
\]

\[
W = \left( \frac{\partial W}{\partial x_1}, \frac{\partial W}{\partial x_2}, \ldots, \frac{\partial W}{\partial x_N} \right)
\]

The dimensions of the cohomologies \( H^0(\mathcal{E}, \mathcal{L}_j) \) and \( H^1(\mathcal{E}, \mathcal{L}_j) \) are \( h^0_j \) and \( h^1_j \) respectively. These dimensions are known to be given by

\[
h^0_j := \begin{cases} 0 & \text{if } l_j < 0 \\ l_j + 1 & \text{if } l_j \geq 0 \end{cases}
\]

\[
h^1_j := \begin{cases} -l_j - 1 & \text{if } l_j < 0 \\ 0 & \text{if } l_j \geq 0 \end{cases}
\]

so that both are non-negative integers satisfying \( h^0_j - h^1_j = l_j + 1 \). Moreover, we have that \( D = \sum_{j=1}^N (h^0_j - h^1_j) \).

In \( \mathcal{H}_{g,n} \), if \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) is a class of codimension zero, then these classes are constant and so, abusing notation, we will simply consider \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \) to be a complex number. We will use this convention through the rest of the paper.

**Axiom 5. Index Zero:** Consider the class \( \Lambda^W_{g,n}(\alpha_1, \alpha_2, \ldots, \alpha_n) \), with \( \alpha_i \in \mathcal{H}_{g_i,G} \). If \( \text{Fix } \gamma_i = \{0\} \) for each \( i \in \{1, 2, \ldots, n\} \) and

\[
D = \sum_{j=1}^N (h^0_j - h^1_j) = 0,
\]
Theorem 2.1. For an arbitrary two-dimensional loop potential \( \Lambda \), the group of diagonal symmetries \( G \) is of codimension zero and \( \Lambda_{g,n}(\alpha_1,\alpha_2,\ldots,\alpha_n) \) is equal to the degree of the Witten map.

Axiom 6. Composition: If the four-point class, \( \Lambda^W_{g,n}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) \) is of codimension zero, then it decomposes as sums of three-point correlators in the following way:

\[
\Lambda^W_{0,4}(\alpha_1,\alpha_2,\alpha_3,\alpha_4) = \sum_{\beta,\delta} \langle \alpha_1,\alpha_2,\beta \rangle \eta_{\beta,\delta} \langle \delta,\alpha_3,\alpha_4 \rangle = \sum_{\beta,\delta} \langle \alpha_1,\alpha_2,\beta \rangle \eta_{\beta,\delta} \langle \delta,\alpha_2,\alpha_4 \rangle.
\]

Note that \( \text{Fix} J = \{0\} \) so \( \mathcal{H}_J \cong \mathbb{C} \). Let \( \mathbf{1} \) be the element in \( \mathcal{H}_J \) corresponding to \( 1 \in \mathbb{C} \). This element has \( \text{deg} \mathbf{1} = 0 \) and it turns out to be the identity element in the FJRW-ring. The next axiom deals with this element.

Axiom 7. Pairing: For any \( \alpha_1,\alpha_2 \in \mathcal{H}_{W,G} \), we have \( \langle \alpha_1,\alpha_2,\mathbf{1} \rangle = \eta_{\alpha_1,\alpha_2} \).

Axiom 8. Sums of singularities: If \( W_1 \in \mathbb{C}[X_1,\ldots,X_r] \) and \( W_2 \in \mathbb{C}[Y_1,\ldots,Y_s] \) are two non-degenerate, quasi-homogeneous polynomials with maximal symmetry groups \( G_1 \) and \( G_2 \), then the maximal symmetry group of \( W = W_1 + W_2 \) is \( G = G_1 \times G_2 \), and there is an isomorphism of Frobenius algebras

\[
\mathcal{H}_{W,G} \cong \mathcal{H}_{W_1,G_{W_1}} \otimes \mathcal{H}_{W_2,G_{W_2}}
\]

1.4. Additional Notation. Throughout this paper we adopt the following notation. Let \( g \in G_W \). If \( \text{Fix} g = \{0\} \), define

\[
e_g := 1 \in \mathcal{H}_g \cong \mathbb{C},
\]

otherwise, if \( g \) fixes the variables \( X_{i_1},\ldots,X_{i_{N_g}} \), define

\[
e_g := dX_{i_1} \wedge dX_{i_2} \wedge \cdots \wedge dX_{i_{N_g}} \in \mathcal{H}_g.
\]

The identity element of \( G_W \) will be denoted by \( i_W \).

2. Two-Dimensional Loop Potentials

In this section we show that the Landau-Ginzburg Mirror Symmetry Conjecture holds for the so-called loop potentials in two dimensions. These potentials are of the form

\[
W = X_1^{a_1}X_2 + X_2^{a_2}X_1,
\]

where \( a_1, a_2 \in \mathbb{N}^{>1} \). Note that quasi-homogeneity implies that \( a_1q_1 + q_2 = 1 \) and \( a_2q_2 + q_1 = 1 \) for the weights \( q_1, q_2 \) of the loop potential, so \( q_1 = \frac{|G_W| - 1}{a_1 a_2 - 1} \) and \( q_2 = \frac{|G_W| - 1}{a_1 a_2 - 1} \).

Theorem 2.1. For an arbitrary two-dimensional loop potential

\[
W = X_1^{a_1}X_2 + X_2^{a_2}X_1
\]

the group of diagonal symmetries \( G_W \) is cyclic of order

\[
|G_W| = a_1 a_2 - 1.
\]

Moreover, \( G_W \) can be generated by either one of the following elements: \( g_1 := (e^{2 \pi i \theta_1^{(1)}}, e^{2 \pi i \theta_2^{(1)}}) \) or \( g_2 := (e^{2 \pi i \theta_1^{(2)}}, e^{2 \pi i \theta_2^{(2)}}) \), where

\[
\theta_1^{(1)} := \frac{-1}{|G_W|}, \quad \theta_2^{(1)} := \frac{a_1}{|G_W|}, \quad \theta_1^{(2)} := \frac{a_2}{|G_W|}, \quad \theta_2^{(2)} := \frac{-1}{|G_W|}.
\]

Proof. From the definition of the maximal group of diagonal symmetries, we see that any \( g \in G_W \) can be expressed in the form \( g = (\alpha_1,\alpha_2) \), satisfying the conditions

\[
\alpha_1^{a_1} = 1 \quad \text{and} \quad \alpha_2^{a_2} = 1.
\]
This implies that $a_1^{a_1 a_2 - 1} = 1$ and $a_2^{a_1 a_2 - 1} = 1$. Thus, $a_1$ and $a_2$ are primitive roots of unity of order $a_1 a_2 - 1$. From (16) it is clear that determining the value of $a_1$ fixes the value of $a_2$, and therefore $G_W$ must be isomorphic to the additive group of integers modulo $a_1 a_2 - 1$. Hence, $G_W$ is cyclic of order $|G_W| = a_1 a_2 - 1$.

From (15) it is clear that $g_2, g_2 \in G_W$, and that they are generators of the group.

\[\square\]

Remark 2.2. Note that $g_1^{-a_2} = g_2$ and $g_2^{-a_1} = g_1$.

We now want to construct the state space of $\mathcal{H}_{G,W}$. As mentioned in Section 1.3 it is possible to do this in terms of Milnor rings.

**Theorem 2.3.** The state space of the FJRW-ring of a two-dimensional loop potential $W$ with maximal group diagonal of symmetries $G_W$ is given by

$$\mathcal{H}_{W,G_W} = \left( X_1^{a_1^{-1}} \right) e_{i_W} \oplus \left( X_2^{a_2^{-1}} \right) e_{i_W} \bigoplus_{g \in G_W} \mathbb{C} e_g$$

Proof. Since $G_W = \langle g_i \rangle$ for $i \in \{1, 2\}$, we have that we can construct the state space of the FJRW-ring by taking powers of any of the generators $g_i$, i.e.

$$\mathcal{H}_{W,G_W} = \left( \bigoplus_{k=0}^{|G_W|-1} \mathcal{H}_{g_i}^k \right)^{G_W}.$$

By looking at (15), it is not hard to see that $|G_W|$ and $|G_W| \theta_j^{(i)}$ are relatively prime for any of the phases $\theta_j^{(i)}$ of $g_i$. From this result we see that $\text{Fix} g_k^i = \{0\}$ for $k \neq 0$, and therefore $N_{g_k}^i = 0$. Thus, we only need to find the elements in $\mathcal{H}_{g_i}$ that are invariant under the action of $G_W$. We first note that $\text{Fix} g_0^i = \text{Fix} i_W = \mathbb{C}^2$, where $i_W$ is the identity of $G_W$. In order to find $\mathcal{H}_{i_W}$ then, we need to compute the Milnor ring of $W$. We recall that this ring is given by,

$$\mathcal{D}_W = \mathbb{C}[X_1, X_2]/\text{Jac}(W)$$

where $\text{Jac}(W)$ is the Jacobian ideal of $W$. From these relations, we find that a basis for $\mathcal{D}_W$ as a vector space over $\mathbb{C}$ is given by monomials of the form

$$X_1^{b_1} X_2^{b_2}$$

(17)

where $0 \leq b_1 < a_1$ and $0 \leq b_2 < a_2$.

As described in Section 1.3 a basis for $\mathcal{H}_{i_W}$ is therefore given by elements of the form

$$\left( X_1^{b_1} X_2^{b_2} \right) e_{i_W}, \quad 0 \leq b_1 < a_1, \quad 0 \leq b_2 < a_2.$$

The elements in $\mathcal{H}_{i_W}$ invariant under the action of $G_W$ must satisfy

$$\sum_{j=1}^2 \theta_j^{(i)} b_j + \sum_{j=1}^2 \theta_j^{(i)} = m, \quad \text{where} \quad m \in \mathbb{Z}$$

(18)

This relation must hold true for any generator $g_i$ of $G_W$, so in particular we can pick $i = 1$ in the above condition. Since $\theta_1^{(1)}$ is negative and $\theta_2^{(1)}$ is positive, we have that the maximum and minimum values of (20) will be attained by $(X_2^{a_2^{-1}}) e_{i_W}$ and $(X_1^{a_1^{-1}}) e_{i_W}$, respectively. The value of $m$ in these cases will be 1 and 0 respectively. If we take any other element of $\mathcal{H}_{i_W}$ the value of (18) will be strictly between 0 and 1. Therefore, the only elements of $\mathcal{H}_{i_W}$ fixed under the action of $G_W$ are $(X_1^{a_1^{-1}}) e_{i_W}$ and $(X_2^{a_2^{-1}}) e_{i_W}$.

\[\square\]

Remark 2.4. The dimension of $\mathcal{H}_{W,G_W}$ as a vector space over $\mathbb{C}$ is given by $a_1 a_2$. 

Now that we have constructed the state space for $\mathcal{H}_{W,G_W}$, we would like to find the potential $W^T$ that will be the mirror dual of $W$. From the Berglund-Hübsch mirror construction [BH], this dual potential is given by

$$W^T = \overline{X}_1 X_2 + \overline{X}_2 X_1.$$  \hfill (19)

Let $\overline{q}_i$ be the weight of the variable $\overline{X}_i$ in $W^T$. It is not hard to check that

$$a_1 \overline{q}_2 + \overline{q}_1 = 1 \quad \text{and} \quad a_2 \overline{q}_1 + \overline{q}_2 = 1,$$  \hfill (20)

and that

$$\overline{q}_i := \theta_1^{(i)} + \theta_2^{(i)}, \quad i \in \{1, 2\}.$$  \hfill (21)

We claim that this new potential $W^T$ is the mirror dual of $W$, which means that $(\mathcal{H}_{W,G_W}, *) \cong \mathcal{D}_{W^T}$ and $(\mathcal{H}_{W^T,G_{W^T}}, *) \cong \mathcal{D}_W$, where $*$ is the multiplication defined in (13).

**Theorem 2.5.** For a loop potential $W = X_1^{a_1} X_2 + X_1 X_2^{a_2}$ with maximal group of diagonal symmetries $G_W$, there is a dual potential $W^T$ given by

$$W^T = \overline{X}_1 X_2 + \overline{X}_2 X_1$$

such that $\mathcal{H}_{W,G_W} \cong \mathcal{D}_{W^T}$ as graded Frobenius algebras, where $\mathcal{H}_{W,G_W}$ is graded by $W$-degree and $\mathcal{D}_{W^T}$ is graded by the weighted degree of monomials.

In order to prove Theorem 2.5, we must first prove a series of results:

**Lemma 2.6.** Every element $g \in G_W$ can be written in the form $g = Jg_1^{\alpha} g_2^\beta$ in a unique way, where $0 \leq \alpha \leq a_2 - 1$ and $0 \leq \beta \leq a_1 - 1$, with the exception of $i_W = Jg_1^{a_2 - 1} = Jg_2^{a_1 - 1}$.

**Proof.** First, note that

$$Jg_1^{a_2 - 1} = (e^{2\pi i (q_1+(a_2-1)\theta_1^{(1)})}, e^{2\pi i (q_2+(a_2-1)\theta_2^{(1)})}) = i_W,$$

and that

$$Jg_2^{a_1 - 1} = (e^{2\pi i (q_1+(a_1-1)\theta_1^{(2)})}, e^{2\pi i (q_2+(a_1-1)\theta_2^{(2)})}) = i_W.$$

Now suppose that for some $g \in G_W$ with $g \neq i_W$, we have that $g = Jg_1^{\alpha_1} g_2^{\beta_1} = Jg_1^{\alpha_2} g_2^{\beta_2}$, where $0 \leq \alpha_1, \alpha_2 \leq a_2 - 1$ and $0 \leq \beta_1, \beta_2 \leq a_1 - 1$. Assume without loss of generality that $\alpha_2 \geq \alpha_1$. If we divide one representation by the other, we find that

$$i_W = g_1^{\alpha_2 - \alpha_1} g_2^{\beta_2 - \beta_1} = g_1^{\alpha_2 - \alpha_1} g_1^{\alpha_2 - \alpha_1} g_1^{\alpha_2 - \alpha_1} g_1^{\alpha_2 - \alpha_1},$$

where the last equality comes after invoking Remark 2.2. This implies that $\alpha_2 - \alpha_1 - a_2(\beta_2 - \beta_1) = m|G_W|$, where $m$ is an integer. It is not hard to show that

$$-a_2(a_1 - 1) \leq \alpha_2 - \alpha_1 - a_2(\beta_2 - \beta_1) \leq a_2 - 1 + a_2(a_1 - 1) = |G_W|,$$

and therefore, the only possible values that $m$ can take are 0 and 1. The only way in which $m = 1$ is by letting $\alpha_2 = a_2 - 1, \alpha_1 = 0, \beta_2 = 0$, and $\beta_1 = a_1 - 1$. However, this would mean that $g = i_W$, which is impossible.

It is straightforward to show that the only way in which $m = 0$ is that $\alpha_1 = \alpha_2$ and that $\beta_1 = \beta_2$, but this means that the representation of $g$ in the form $Jg_1^{\alpha} g_2^{\beta}$ is unique.

We have thus far shown that there are exactly $a_1 a_2 - 1$ different elements of $G_W$ that can be written in the form $Jg_1^{\alpha} g_2^{\beta}$, with $0 \leq \alpha \leq a_2 - 1$ and $0 \leq \beta \leq a_1 - 1$. Since the order of $G_W$ is also $a_1 a_2 - 1$, then every element of $G_W$ can be written uniquely in the form $Jg_1^{\alpha} g_2^{\beta}$, with the exception of $i_W$. \hfill $\Box$

**Corollary 2.7.** Let $\gamma \in \mathcal{H}_{W,G_W}$, and suppose that $\gamma \in \mathcal{H}_g$ for some $g \in G_W$, where $g = Jg_1^{\alpha} g_2^\beta$, $0 \leq \alpha \leq a_2 - 1$, $0 \leq \beta \leq a_1 - 1$. Then the $W$-degree of $\gamma$ is given by

$$\deg_W(\gamma) = 2(\alpha \overline{q}_1 + \beta \overline{q}_2).$$
Proof. We divide the proof in two cases: $g = i_W$ and $g \neq i_W$.

Case 1: Suppose that $\gamma \in \mathcal{H}_i$. Then, the W-degree of $\gamma$ will be given by

$$
\deg_W(\gamma) = N_i + 2(0 - q_1) + 2(0 - q_2) = 2 - 2q_1 - 2q_2.
$$

A simple computation shows that $\deg_W(\gamma) = 2(a_2 - 1)\bar{q}_1 = 2(a_1 - 1)\bar{q}_2$.

Case 2: Suppose that $g \neq i_W$ and that $\gamma \in \mathcal{H}_g$. Let $g = Jg_1^a g_2^b$, where $0 \leq \alpha \leq a_2 - 1$, $0 \leq \beta \leq a_1 - 1$. Then

$$
\theta_1^a = q_1 + \alpha \theta_1^{(1)} + \beta \theta_1^{(2)} \quad \text{and} \quad \theta_2^b = q_2 + \alpha \theta_2^{(1)} + \beta \theta_2^{(2)}.
$$

It is not hard to show that $0 \leq q_1 + \alpha \theta_1^{(1)} + \beta \theta_1^{(2)} \leq 1$ and that $0 \leq q_2 + \alpha \theta_2^{(1)} + \beta \theta_2^{(2)} \leq 1$. Note that the only time that $\theta_1^a = 1$ or $\theta_2^b = 1$ is when $g = i_W$, but this was considered in Case 1, so we will assume that $0 \leq \theta_1^a, \theta_2^b < 1$. Therefore, we have that $\Theta_1^a = \theta_1^a$ and $\Theta_2^b = \theta_2^b$. We can now use (12) to compute the W-degree of $\gamma$

$$
\deg_W(\gamma) = N_\gamma + 2(q_1 + \alpha \theta_1^{(1)} + \beta \theta_1^{(2)} - q_1) + 2(q_2 + \alpha \theta_2^{(1)} + \beta \theta_2^{(2)} - q_2) = 0 + 2\alpha(\theta_1^{(1)} + \theta_2^{(2)}) + 2\beta(\theta_1^{(2)} + \theta_2^{(1)}) = 2(\alpha \bar{q}_1 + \beta \bar{q}_2).
$$

Lemma 2.8. For any integer $c$ with $0 \leq c < a_1 - 1$, we have that $(e_{Jg_i})^c = e_{Jg_i^c}$, where $i \in \{1, 2\}$.

Proof. For $c = 0$ the result is trivial since

$$
(e_{Jg_i^c})^0 = 1 = e_J = e_{Jg_i^0}.
$$

Now suppose that for $1 \leq c < a_1 - 1$, we have that $(e_{Jg_i})^{c-1} = e_{Jg_i^{c-1}}$, and consider the product $(e_{Jg_i})^{c-1} \ast e_{Jg_i}$. By definition, this product will be given by $\sum_{\alpha, \beta} \langle (e_{Jg_i})^{c-1}, e_{Jg_i}, \alpha \rangle \eta^{\alpha, \beta}$. Using our assumption, we find that

$$
\langle (e_{Jg_i})^{c-1} \ast e_{Jg_i}, \alpha \rangle = \sum_{\alpha, \beta} \langle e_{Jg_i^{c-1}}, e_{Jg_i}, \alpha \rangle \eta^{\alpha, \beta}.
$$

(22)

For these correlators to be non-zero we need $\deg_W(e_{Jg_i^{c-1}}) + \deg_W(e_{Jg_i}) + \deg_W(\alpha) = 2c$. Using Corollary 2.7, we find that this last relation is equivalent to

$$
2c\bar{q}_i + \deg_W(\alpha) = 2(a_i - 1)\bar{q}_i + 2(a_i - 1)\bar{q}_i + 2(a_i - 1)\bar{q}_i + 2(a_i - 1)\bar{q}_i,
$$

$$
\Rightarrow \deg_W(\alpha) = 2(a_i - 1 - c)\bar{q}_i + 2(a_i - 1)\bar{q}_i.
$$

From Corollary 2.7, it is easy to see that if $\gamma = Jg_i^{a_i - 1 - c} g_{i+1}^{-1}$, then $\deg_W(e_{\gamma}) = \deg_W(\alpha)$, and since $\gamma \neq i_W$, there is only one basis element coming from $\mathcal{H}_g$, and thus, the sum in equation (22) reduces to a single term. Also note that $\gamma g_i^c = i_W$, and therefore, $e_\gamma$ pairs up with $e_{Jg_i^c}$, which gives

$$
\langle (e_{Jg_i})^{c-1} \ast e_{Jg_i}, \alpha \rangle = \langle e_{Jg_i^{c-1}}, e_{Jg_i}, e_\gamma \rangle \eta^{e_\gamma, e_{Jg_i^c}}.
$$

To find the value of $\langle e_{Jg_i^{c-1}}, e_{Jg_i}, e_\gamma \rangle$ we must compute the degrees of its line bundles. From Section 1.3, we see that

$$
l_1 = q_1 - (q_1 + (c - 1)\theta_1^{(2)} + q_1 + (a_2 - 1)\theta_1^{(1)} + (a_1 - c - 1)\theta_1^{(2)} = -1.
$$

$$
l_2 = q_2 - (q_2 + (c - 1)\theta_2^{(2)} + q_2 + (a_2 - 1)\theta_2^{(1)} + (a_1 - c - 1)\theta_2^{(2)} = -1.
$$

Therefore, by Axioms 3 and 4 we have that $\langle e_{Jg_i^{c-1}}, e_{Jg_i}, e_\gamma \rangle = 1$. In a similar way it can be shown that $\eta^{e_\gamma, e_{Jg_i^c}} = 1$, and thus we find that $(e_{Jg_i})^c = e_{Jg_i^c}$, which concludes the proof of Lemma 2.8. \qed
Lemma 2.9. In $(\mathcal{H}_{W,G_{W}}, \ast)$
\[
ed_{h_{2}} + a_{2} e_{h_{2}} \ast \ned_{h_{1}} = 0 \quad \text{and} \quad \ned_{h_{2}} + a_{1} e_{h_{1}} \ast \ned_{h_{2}} = 0,
\]
where $h_{i} = Jg_{i}$, $i \in \{1, 2\}$.

Proof. Using the Lemma 2.8, we see that $\ned_{h_{2}} = e_{h_{2}}^{{a_{2}}^{-2}}$, and thus
\[
\ned_{h_{2}}^{-1} = \sum_{\alpha, \beta} \langle e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{2}}, \alpha} \rangle \eta^{\alpha, \beta},
\]
where we need $\deg_{W}(e_{Jg_{1}^{{a_{1}}^{-2}}}) + \deg_{W}(e_{Jg_{2}}) + \deg_{W}(\alpha) = 2\hat{c}$. Making use of Corollary 2.7, we note that this last relation is equivalent to $\deg_{W} \alpha = \hat{c}$. However, this is only possible if $\alpha \in \mathcal{H}_{W}$. Therefore, we have that
\[
\ned_{h_{2}}^{-1} = \sum_{\alpha, \beta} \langle e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{2}}, \alpha} \rangle \eta^{\alpha, \beta}, \quad \text{where} \ \alpha, \beta \in \{(X_{1}^{{a_{1}}^{-1}}) e_{i_{W}}, (X_{2}^{{a_{2}}^{-1}}) e_{i_{W}}\}
\]
\[
\Rightarrow \ned_{h_{2}} = \sum_{\alpha, \beta} \langle e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{2}}, \alpha} \rangle \eta^{\alpha, \beta} \langle \beta, e_{Jg_{2}}, e_{Jg_{1}^{{a_{1}}^{-2}}} \rangle \eta^{e_{Jg_{1}^{{a_{1}}^{-2}}, \delta}}
\]
\[
= \Lambda_{W}(e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{2}}, e_{Jg_{1}^{{a_{1}}^{-2}}}) \eta^{e_{Jg_{1}^{{a_{1}}^{-2}}, \delta}}.
\]
The inverse of $Jg_{1}^{{a_{1}}^{-2}}$ is given by $Jg_{1}^{{a_{2}}^{-1}}$, and so $\delta = e_{Jg_{1}^{{a_{2}}^{-1}} g_{2}}$. It is not hard to show that $\eta^{e_{Jg_{1}^{{a_{2}}^{-1}}}} = 1$, which gives us that
\[
\ned_{h_{2}} = \Lambda_{W}(e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{2}}, e_{Jg_{1}^{{a_{1}}^{-2}}}) e_{Jg_{1}^{{a_{2}}^{-1}} g_{2}}.
\]
To find the value of this four-point class, we compute the degrees of its line bundles,
\[
\begin{align*}
l_{1} &= 2q_{1} - (q_{1} + (a_{1} - 2)\theta_{1}^{(1)}) + q_{1} + \theta_{1}^{(1)} + q_{1} + (a_{1} - 2)\theta_{1}^{(1)} = -2, \\
l_{2} &= 2q_{2} - (q_{2} + (a_{1} - 2)\theta_{2}^{(1)}) + q_{2} + \theta_{2}^{(1)} + q_{2} + (a_{1} - 2)\theta_{2}^{(1)} = 0.
\end{align*}
\]
Using Axiom 5, we find that $\Lambda_{W}(e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{2}}, e_{Jg_{1}^{{a_{1}}^{-2}}}) = a_{2}$, and thus
\[
\ned_{h_{2}} = -a_{2} e_{Jg_{1}^{{a_{2}}^{-1}} g_{2}}.
\]
In the same way (25) was obtained, one can show that
\[
\ned_{h_{1}}^{-1} = \sum_{\alpha, \beta} \langle e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}}, \alpha} \rangle \eta^{\alpha, \beta}, \quad \text{where} \ \alpha, \beta \in \{(X_{1}^{{a_{1}}^{-1}}) e_{i_{W}}, (X_{2}^{{a_{2}}^{-1}}) e_{i_{W}}\}
\]
\[
\Rightarrow \ned_{h_{1}} \ast \ned_{h_{1}}^{-1} = \sum_{\alpha, \beta} \langle e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}}, \alpha} \rangle \eta^{\alpha, \beta} \langle \beta, e_{Jg_{1}}, e_{Jg_{1}^{{a_{1}}^{-2}}} \rangle \eta^{e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}^{{a_{1}}^{-2}}}}}
\]
\[
= \Lambda_{W}(e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}}, e_{Jg_{1}^{{a_{1}}^{-2}}}) \eta^{e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}^{{a_{1}}^{-2}}}}}
\]
It is not hard to show that $\eta^{e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}^{{a_{1}}^{-2}}}} = 1$. Now, to find the value of the four-point class we compute its line bundle degrees,
\[
\begin{align*}
l_{1} &= 2q_{1} - (q_{1} + (a_{2} - 2)\theta_{1}^{(1)}) + q_{1} + \theta_{1}^{(1)} + q_{1} + (a_{1} - 2)\theta_{1}^{(1)} = -1, \\
l_{2} &= 2q_{2} - (q_{2} + (a_{2} - 2)\theta_{2}^{(1)}) + q_{2} + \theta_{2}^{(1)} + q_{2} + (a_{1} - 2)\theta_{2}^{(1)} = -1,
\end{align*}
\]
and so by Axioms 3 and 4 we have that $\Lambda_{W}(e_{Jg_{1}^{{a_{1}}^{-2}}, e_{Jg_{1}}, e_{Jg_{1}^{{a_{1}}^{-2}}}) = 1$. Therefore, we have that
\[
\ned_{h_{1}} \ast \ned_{h_{1}}^{-1} = e_{Jg_{1}^{{a_{2}}^{-1}} g_{2}}.
\]
Putting this together with (26) allows us to show that $\ned_{h_{2}} + a_{2} e_{h_{2}} \ast \ned_{h_{1}} = 0$. Following the steps that led us to this relation, one can show that $\ned_{h_{1}} + a_{1} e_{h_{1}} \ast \ned_{h_{2}} = 0$. \qed
We are now in a position to prove Theorem 2.5,

**Proof of Theorem 2.5:**

Consider the map \( \varphi : Q_{W,\tau} \rightarrow H_{W,\mathcal{G}W} \) given by

\[
X^\alpha_1 X^\beta_2 \mapsto e_j g_{\alpha_1} g_{\beta_2}^T, \quad X^\alpha_1 \mapsto (X^\alpha_1) e_{i_1}, \quad X^\alpha_2 \mapsto (X^\alpha_2 - 1) e_{i_2},
\]

where \( 0 \leq \alpha \leq a_2 - 1 \) and \( 0 \leq \beta \leq a_1 - 1 \).

From Lemma 2.6, it is easy to see that this map is surjective, and because the dimensions of \( Q_{W,\tau} \) and \( H_{W,\mathcal{G}W} \) are equal, \( \varphi \) must be bijective.

Note that \( e_j, i \in \{1, 2\} \), and that the relations in \( Q_{W,\tau} \) are given by its Jacobian ideal, i.e.

\[
X^\alpha_2 + a_2 X^\alpha_1 X^\alpha_2 - 1 = 0, \quad X^\alpha_1 + a_1 X^\alpha_2 X^\alpha_2 - 1 = 0.
\]

Therefore, by Lemma 2.9, \( \text{Jac}(W^T) \subseteq \ker(\varphi) \), and we have that \( \varphi \) is the desired degree preserving isomorphism.

**References**

BH. P. Berglund and T. Hubsch, *A generalized construction of mirror manifolds*. Nuclear Physics B. 393:377, (1993).

FS. H. Fan and Y. Shen, *Quantum ring of singularity x^p + xy^q*. [arXiv:0902.2327v1 [math.AG]], February 2009.

FJR. Huijun Fan, Tyler J. Jarvis, and Yongbin Ruan, *The witten equation, mirror symmetry and quantum singularity theory*. [arXiv:0712.4021v3 [math.AG]], January 2009.

GP. B.R. Greene and M.R. Plesser. Nuclear Physics B. 338:15, (1990).

Kr. M. Kreuzer, *FJRW-ring and Landau-Ginzburg mirror symmetry*. In preparation.

K. M. Kreuzer, *The mirror map for invertible LG models*. Physics Letters B. 328:312-318, (1994).

KS. M. Kreuzer and H. Skarke. Commun. Math. Phys. B 411:559, (1992).

Pr. N. Prididd, M. Kreuzer, P. Acosta, N. Wilde N. and H. Rathnakamura, *FJRW-rings and mirror symmetry*. [arXiv:0903.3220v1 [math.AG]], March 2009.

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