Shrinking rates of horizontal gaps for generic translation surfaces

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Received: date / Accepted: date

Abstract A translation surface is given by polygons in the plane, with sides identified by translations to create a closed Riemann surface with a flat structure away from finitely many singular points. Understanding geodesic flow on a surface involves understanding saddle connections. Saddle connections are the geodesics starting and ending at these singular points and are associated to a discrete subset of the plane. To measure the behavior of saddle connections of length at most $R$, we obtain precise decay rates as $R \to \infty$ for the difference in angle between two almost horizontal saddle connections.

Keywords Translation Surfaces · Dynamical Systems · Saddle Connections · Mixing

Mathematics Subject Classification (2010) 37E35 · 32G15 · 37A25

Statements and Declarations:

1. Funding: The first author is supported by NSF grants DMS-2055354 and DMS-452762, the Sloan foundation, Poincaré chair, and Warnock chair.
   The second author was partially supported by the Deutsche Forschungsgemeinschaft (DFG) – Projektnummer 445466444 and 507303619.
2. Author Contributions: All authors wrote and reviewed the manuscript

The first author is supported by NSF grants DMS-2055354 and DMS-452762, the Sloan foundation, Poincaré chair, and Warnock chair. The second author is supported by the Deutsche Forschungsgemeinschaft (DFG) – Projektnummer 445466444.

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3. Data Availability Statement: Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

1 Introduction

Consider a finite collection of polygons in the plane, where all sides come in pairs of equal length with opposite orientations on the boundary of the polygons. Identifying these sides gives a compact finite type Riemann surface. The form $dz$ on the plane endows this Riemann surface with a holomorphic 1-form $\omega$. This structure is called a translation surface. Translation surfaces are stratified by their zeros and each connected component of each stratum supports a natural probability measure supported on unit area surfaces called Masur–Smillie–Veech (MSV) measure. More background on translation surfaces can be found, for example in [21, 25]. A saddle connection $\gamma$ is a geodesic starting and ending at the zeroes of $\omega$ with no zeroes in between. Associated to $\gamma$, we define the holonomy vector $v_\gamma = \int_\gamma d\omega \in \mathbb{C}$. To understand the geometry of a typical surface, much effort has gone into understanding the asymptotic behavior of holonomy vectors of saddle connections [13, 14, 15, 16, 23, 24, 26, 30, 31], and more recently the asymptotic behavior of pairs of saddle connections [4, 6]. We will consider

$$\Lambda_\omega(R) = \{v_\gamma \in \mathbb{C} \cap B(0, R) : \gamma \text{ is a saddle connection}\}$$

and

$$\Theta_\omega(R) = \{\text{arg}(v_\gamma) : v_\gamma \in \Lambda_\omega(R)\},$$

where $\text{arg}(v) \in [-\pi, \pi)$ is the angle $v$ makes with the horizontal. The sets $\Lambda_\omega(R)$ and $\Theta_\omega(R)$ are discrete subsets of $\mathbb{C}$ and $[-\pi, \pi)$, respectively, so we can define

$$\zeta_\omega(R) = \min\{\phi \in \Theta_\omega(R) : \phi \geq 0\} - \max\{\phi \in \Theta_\omega(R) : \phi < 0\}.$$ 

The main result of this paper is:

**Theorem 1.1** Let $\psi : [1, \infty) \to [1, \infty)$ be a nondecreasing function. In any connected component of a stratum of translation surfaces of genus at least 2,

1. If $\int_1^\infty \frac{1}{t^2 \psi(t)^2} \, dt < \infty$, then for MSV almost every $\omega$,

$$\liminf_{R \to \infty} \psi(R)R^2\zeta_\omega(R) = \infty.$$

2. If $\int_1^\infty \frac{1}{t^2 \psi(t)^2} \, dt = \infty$, then for MSV almost every $\omega$,

$$\liminf_{R \to \infty} \psi(R)R^2\zeta_\omega(R) = 0.$$
Remark 1.2 Note that the choice of a horizontal gap is a convenience. Apply a rotation to the full measure set in Theorem 1.1, and we obtain the same result in a different direction. Consider a countable subset $\mathcal{D}_n = \{\theta_n\}_{n \in \mathbb{N}} \subseteq [0, 2\pi)$. Since a countable union of measure 0 subsets is still measure zero, we obtain a natural corollary that the smallest gap of any of the directions in $\mathcal{D}_n$ has the same decay rate as given in Theorem 1.1.

We first provide an explanation for the scaling factor of $R^2$. Masur ([23, 24]) showed $|\Lambda_\omega(R)|$ has quadratic growth in the sense that for each $\omega$ there exist constants $c_1, c_2$ so that

$$c_1 R^2 \leq |\Lambda_\omega(R)| \leq c_2 R^2$$

for all large enough $R$. Because the total angle about the singular points for the flat metric is at most $4\pi(2g-2)$ and every saddle connection begins at ends at a singular point, there are at most $4g-4$ saddle connections in the same direction, so $|\Theta_\omega(R)|$ also has quadratic growth. This result explains the scaling factor of $R^2$ in Theorem 1.1. The quadratic growth of saddle connections was subsequently built on in [14, 16, 26, 30, 31].

For every translation surface, [22] shows that $\Theta_\omega = \bigcup_R \Theta_\omega(R)$ is dense in $[-\pi, \pi)$. If we order the points $\theta_1 \leq \cdots \leq \theta_{|\Theta_\omega(R)|}$ in $\Theta_\omega(R)$, then by density the adjacent differences $\theta_{j+1} - \theta_j \to 0$ as $R \to \infty$ for every $\omega$. Thus $\zeta_\omega(R) \to 0$ as $R \to \infty$ for every $\omega$. If we reduce to almost every translation surface, Theorem 1.1 give a rate of convergence for a single direction, yielding partial information on how $\Theta_\omega(R)$ is distributed in $[-\pi, \pi)$. For example, by considering $\psi(t) = \log(t)$ and $\psi(t) = \sqrt{t}$, we know for a typical surface that the rate of convergence is shrinking faster than $(\log(R)R^2)^{-1}$ along a subsequence and slower than $R^{-\frac{5}{2}}$. However, we cannot expect Theorem 1.1 to hold for every translation surface. Indeed when $\omega$ is a lattice surface (see [21, Sections 5 and 7] for a definition), [2] showed for every unbounded $\psi$ we have $\lim \inf_{R \to \infty} \psi(R)R^2\zeta_\omega(R) = \infty$.

It is natural question to ask if Theorem 1.1 can be extended to measures supported on SL(2, $\mathbb{R}$)-orbit closures which are not closed (lattice surfaces) or dense (connected component of a stratum). Though many pieces of our argument have analogous results for orbit closures, the local coordinate computations require great care centered on a very specific surface. In light of this, we save this question for future work.

The results of Theorem 1.1 consider the behavior of a single gap. There is also substantial work done on studying the behavior of the family of gaps. Namely one can study the entire set $\Theta_\omega(R)$. The distribution of normalized gaps exists for almost every $\omega$ by [2]. In many cases, the distribution has also been computed [3, 9, 20, 28, 29]. Considering the behavior of a single gap, which is the focus of the current paper, is orthogonal because the behavior of a single gap does not affect the distribution of gaps.
1.1 Outline of proof

The proof follows the now standard strategy of relating a problem about the geometry of a translation surface \( \omega \) to the orbit of \( \omega \) under Teichmüller geodesic flow, \( g_t = \begin{pmatrix} e^t & 0 \\ 0 & e^{-t} \end{pmatrix} \). In Section 2 we reduce our problem about gaps to a shrinking target problem for \( g_t \). The shrinking targets are sets \( A_t \) obtained by relating Theorem 1.1 to whether or not \( g_t \omega \in A_t \) for arbitrarily large \( t \). To prove Theorem 1.1 (2) we use independence results for the sets \( g_{-t} A_t \). A key tool to do this is the fact that the \( g_t \) action is exponentially mixing (Section 3). However, because our targets are not SO(2)-invariant, the estimates from exponential mixing are not sufficient to treat all non-increasing sequences. See Remark 2.12 and Assumptions (3) and (4) in Proposition 2.11. (c.f. [18, Proposition 1.1]). Section 4 establishes that Assumption (4) of Proposition 2.11 is satisfied and provides a different argument to overcome the limitations from exponential mixing. In fact, in Section 4 even the targets we consider are different.

2 Reductions

In this section we present a series of reductions:

1. In Section 2.1 and Section 2.2 we use renormalization to relate gaps to a shrinking target problem.
2. In Section 2.3, we consider the convergence case Theorem 1.1(1) and prove it suffices to show that for almost every \( \omega \) we have

\[
\liminf_{R \to \infty} \psi(R) R^2 \zeta_\omega(R) > 0.
\]

3. In Section 2.4, we consider the divergence case Theorem 1.1(2) and prove it suffices to show that for a positive measure set of \( \omega \) we have

\[
\liminf_{R \to \infty} \psi(R) R^2 \zeta_\omega(R) < \infty.
\]

4. In Section 2.5 we state and prove a partial converse to the Borel–Cantelli lemma (Proposition 2.11) that we use as the framework for proving Theorem 1.1.
5. We conclude in Section 2.6 by proving Theorem 1.1 under one additional assumption. This additional assumption is in Proposition 2.15, which states the existence of the sets needed for Proposition 2.11.

Section 3 and Section 4 are devoted to proving Proposition 2.15.
2.1 Definition and measure of the shrinking target sets

Fix $H$ a connected component of a stratum of translation surfaces, unmarked in the sense of [11, Section 2.3]. Note $H$ has complex dimension $2g+s-1$ with $s$ the number of distinct singularities. Fix $0 < \delta < 1$ as given by exponential mixing of the geodesic flow (Theorem 3.1), which depends only on $H$. We will sample $\psi$ along a discrete set $\{\psi(b_k)\}$ for $k \in \mathbb{N}$ where $b = e^{\ell_0}$, and $\ell_0 \geq 1$ is as in Corollary 4.11, for the given choice of $\delta$ and $I = (-\frac{\pi}{12}, \frac{\pi}{12})$. The definition of $\ell_0$ in turn depends on other constants in Section 4.2, which is a self-contained section with no dependency on the previous statements.

We now define the primary sets that we use to develop our shrinking target problem.

Definition 2.1 (Definition of the A’s) Fix $0 < \sigma < 1$, and let $0 \leq c < 1$. Define $T_{c,\sigma,j}^\pm = T_{c,\sigma,j,\psi,b}^\pm \subseteq \mathbb{C}$ to be the trapezoids with corners given by

$$c, 1, 1 \pm i \frac{\sigma}{\psi(b')}, c \pm ic \frac{\sigma}{\psi(b')}.$$

Set

$$H_{c,\sigma,j} = \{\omega \in H : \omega \text{ has a holonomy vector in } T_{c,\sigma,j}^+ \text{ and a holonomy vector in } T_{c,\sigma,j}^-\}.$$

Finally for $k \in \mathbb{N}$, define $A_k = A_k(c,\sigma) = A_k(c,\sigma,\psi,b) = g_{\log(\psi^k)} H_{c,\sigma,k}$.

Remark 2.2 In the following we drop the dependence on $\psi$ and $b$ as they are fixed whenever we consider these sets. When the choice of $c$ or $\sigma$ is clear, or arbitrary, then for clarity we will suppress the dependence and simply write $A_k$ instead of $A_k(c,\sigma)$.

The remainder of this subsection is devoted to obtaining measure bounds for $A_k$. Let $\mu$ denote the MSV probability measure on $H$, whose support is the locus of unit area surfaces in $H$. Since $\mu$ is $\text{SL}(2,\mathbb{R})$-invariant, it suffices to understand $\mu(H_{c,\sigma,j})$.

Lemma 2.3 Given a stratum $H$, there exists positive finite constants $m = m(H)$, $M = M(H)$, $c_H < 1$, and $\sigma_H < 1$ chosen so that for all $0 < \sigma < \sigma_H$

$$\frac{m\sigma^2}{\psi(b')^2} \leq \mu(H_{c_H,\sigma,j}) \leq M\sigma^2 \frac{1}{\psi(b')^2}.$$

Before proceeding with the proof, we quote the following result of Masur–Smillie (as quoted from [2]), and then develop the idea behind their result before proceeding to the proof of Lemma 2.3.

Lemma 2.4 There is a constant $M$ so that for all $\epsilon, \kappa > 0$, the subset of $H$ consisting of flat surfaces which have a saddle connection of length at most $\epsilon$ has measure at most $M\epsilon^2$. The subset of flat surfaces which have a saddle connection of length at most $\epsilon$ and another nonhomologous saddle connection of length at most $\kappa$ has measure at most $M\epsilon^2\kappa^2$. 

We remark that all uses of $M$ here are not necessarily the same, but vary by at most a multiplicative constant, which we can see for example in the proof of Lemma 2.3.

The upper bound of Lemma 2.3 will follow from Lemma 2.4. In order to obtain the lower bound, we first give some background on the construction of the measure $\mu$ following [25, p.464-5].

First, we allow $\mu$ to be a finite measure for the rest of the subsection, and note that the normalization to a probability measure will only affect the multiplicative constants up to dividing by the measure of the stratum. Now we can define $\mu$ on the flat structures of area 1 via a cone measure $\tilde{\mu}$ over flat structures with area at most 1. The cone measure $\tilde{\mu}$ is inherited from a measure on relative cohomology, defined via charts coming from the developing map. For $\omega \in H$ not in a proper orbifold locus, there exists $r_\omega > 0$ so that the ball of radius $r_\omega$ about the pre-image of $\omega$ in relative cohomology is sent injectively to the coordinate chart about $\omega$. For a point in a proper orbifold locus of $H$, we may choose a chart as in [11, Section 2.3]. Such an orbifold chart has the identification of $\tilde{\mu}$ and Lebesgue measure on relative cohomology on the image of the chart via an almost everywhere $k$-to-1 mapping, where $k$ is the cardinality of the local group of the orbifold substrata. In this situation, $r_\omega$ is chosen so that the map is $k$-to-1, off of any orbifold substrata intersected with the ball.

In order to obtain lower bounds on the measure of a set, it suffices to obtain measure bounds on the cone measure $\tilde{\mu}$ in a fixed coordinate chart. To do this, we work in local coordinates by writing $\omega = (x_1, x_2, x_3) \in \mathbb{C} \times \mathbb{C} \times \mathbb{C}^{2g+s-3}$ and $\tilde{\mu}$ as Lebesgue measure on $\mathbb{C}^{2g+s-1}$. Consider the Euclidean metric using the notation $| \cdot |$ on each component by identifying $\mathbb{C}^\ell$ with $\mathbb{R}^{2\ell}$. Define $B_{\mathbb{C}^\ell}(z, r) = \{ w \in \mathbb{C}^\ell : |w - z| < r \}$. We are now ready to prove Lemma 2.3.

Proof of Lemma 2.3 For the upper bound, flowing by geodesic flow which preserves measure, $g_{\log} (\sqrt{\sigma/\psi(b_k)}) H_{0,\sigma,k}$ has two non-homologous vectors of length at most $\sqrt{\frac{2\sigma}{\psi(b_k)}}$, so by Lemma 2.4 we obtain the desired upper bound.

For the lower bound, we work in local coordinates around a surface $u_0$ with two horizontal saddle connections of length 1. That is, in local coordinates, let $\omega_0 = (x_0^1, x_0^2, x_0^3)$ be an area 1 surface, and let $r > 0$ be the injectivity radius, or $k$-to-1 radius as described above. One can find such a surface in every connected component of every stratum. Indeed, [19] explicitly constructs surfaces, which are a single horizontal cylinder, in each connected component of every stratum of genus at least two, and then after flowing by $g_t$ if necessary so the cylinder has circumference at least 1, we can obtain the desired representative. From such a surface one can vary the length of a pair of boundary horizontal saddle connections to obtain $\omega$. As remarked at the end of [32, Section 1], the two horizontal vectors are in fact non-homologous, and we can pick a basis of homology with the period coordinates represented by saddle connections (see [17, Appendix A] and [10, Proof of Theorem 4.1]). To work with local coor-
dinates, we will first prove the lower bound for $\tilde{\mu}$ and then derive the lower bound for $\mu$.

If necessary, shrink $r$ so that $r < 1$ and the image of the chart is contained in a compact subset of $H$. Notice that for any $c$, since $\psi(t) \geq 1$, $T^\pm_{c,\sigma,j}$ is always contained in the trapezoid $T_{c,\sigma,j}$ with vertices $c \pm i\sigma, 1 \pm i\sigma$. So we can guarantee $T^\pm_{c,\sigma,j}$ is always contained in the chart whenever $T_{c,\sigma,j} \subset B_c(1,r)$. This geometric condition can be satisfied for some $0 < \sigma_H < r, c_H$ close to 1, and $\sigma < \sigma_H$. Define the set in $C \times C \times C^{2g+s-3}$ by

$$\tilde{H}_{c,\sigma,j} = T^+_{c_H,\sigma,j} \times T^-_{c_H,\sigma,j} \times B \quad \text{for} \quad B = B_{C^{2g+s-3}}(x^0, r).$$

By our choice of $r$ the measure $\tilde{\mu}$ on $H$ is locally Lebesgue. Hence if $m_j$ is Lebesgue measure on $C^1$, by symmetry of $T^\pm_{c_H,\sigma,j}$, there is a constant $\tilde{m}$ so that

$$\tilde{\mu}(\tilde{H}_{c,\sigma,j}) \geq \tilde{m} \cdot m_1(T^+_{c,\sigma,j})^2 m_{2g+s-3}(B).$$

We compute the Lebesgue measure of the trapezoids by

$$m_1(T^\pm_{c,\sigma,j}) = \frac{1}{2} \left( \frac{\sigma}{\psi(b')} + c_H \frac{\sigma}{\psi(b')} \right) (1 - c_H) = \frac{\sigma}{\psi(b')} \frac{(1 - c_H^2)}{2}.$$

The other $2g - 3$ coordinates and the possibility that we are in a coordinate patch determined by an orbifold substrata only change these bounds by a multiplicative constant. Thus there are constants $m, m'$ depending on the dimension of $H$ and $c_H$ so that

$$\tilde{\mu}(\tilde{H}_{c,\sigma,j}) \geq m' \cdot \tilde{m} \cdot m_1(T^\pm_{c,\sigma,j})^2 = m \sigma^2 \psi(b')^2.$$

To complete the proof, we need to pass from $\tilde{\mu}$ to the cone measure $\mu$. Specifically, recall on a measurable set $X$ of area 1 translations surfaces,

$$\mu(X) = \tilde{\mu} \left( \{tx : x \in X, 0 < t < 1\} \right).$$

Notice that scaling each of the coordinates by a number $0 < t < 1$ decreases the area of the surface. Notice also that the area of the translation surface changes continuously as a function of the fixed local coordinates, and $\omega_0$ has area 1 with coordinates $((1, 0), (1, 0), x^0_0)$. Thus, given $r, c_H < 1, \sigma_H$ as above, we can choose $\epsilon > 0, r', c'_H$ and $\sigma'_H$ satisfying $0 < r' < r - |x_0|\epsilon, c_H < c'_H < 1$ and $0 < \sigma'_H < \sigma_H$ so that every $\omega'$ in our coordinate patch with local coordinates in

$$T^+_{c_H,\sigma,j} \times T^-_{c_H,\sigma,j} \times B((1 - \epsilon)x^0, r')$$

with $\sigma < \sigma'_H$ has area less than 1 and arises as $\omega' = t\omega$ for $\omega$ in our coordinate patch, with area 1 and with local coordinates in

$$T^+_{c_H,\sigma,j} \times T^-_{c_H,\sigma,j} \times B.$$

We conclude by observing that the above estimate of $\tilde{\mu}$ in (2.2) works to estimate

$$\tilde{\mu} \left( T^+_{c_H,\sigma,j} \times T^-_{c_H,\sigma,j} \times B((1 - \epsilon)x, r') \right)$$

for all $\sigma < \sigma'_H$ so long as $\epsilon > 0$ is small enough and $r'$, $c'_H$ and $\sigma'_H$ satisfy $0 < r' < r - |x_0|\epsilon, c_H < c'_H < 1$ and $0 < \sigma'_H < \sigma_H$. \qed
2.2 A lemma on targets

For the following lemma and corollary, for generality we allow any \( b > 1 \). Note that \( b = e^{\ell_0} \) as defined before Definition 2.1 satisfies \( b > 1 \) since we choose \( \ell_0 \geq 1 \).

**Lemma 2.5** Let \( \phi : [1, \infty) \to [1, \infty) \) be nondecreasing. We have

\[
\int_1^\infty (t\phi(t))^{-1} \, dt = \infty
\]

if and only if

\[
\sum_{j=1}^\infty \phi(b^j)^{-1} = \infty
\]

for any \( b > 1 \).

**Proof** For ease of exposition we assume \( b = 2 \), the general case is similar, but requires using floor and ceiling functions. For each \( k \in \mathbb{N} \) we have

\[
2^k \frac{1}{2^k \phi(2^k)} \geq \int_{2^k}^{2^{k+1}} (t\phi(t))^{-1} \, dt \geq 2^k \frac{1}{2^{k+1} \phi(2^{k+1})}.
\]

It follows that

\[
\sum_{j=1}^\infty \phi(2^j)^{-1} \geq \int_2^\infty (t\phi(t))^{-1} \, dt \geq \frac{1}{2} \sum_{j=2}^\infty \phi(2^j)^{-1}.
\]

\[\Box\]

**Corollary 2.6** Let \( \psi : [1, \infty) \to [1, \infty) \) be nondecreasing. We have

\[
\int_1^\infty (t\psi(t))^2 \, dt = \infty
\]

if and only if

\[
\sum_{j=1}^\infty \psi(b^j)^{-2} = \infty
\]

for any \( b > 1 \).

2.3 Convergence reduction

**Lemma 2.7** Suppose \( \psi : [1, \infty) \to [1, \infty) \) is nondecreasing and also that

\[
\int_1^\infty \frac{1}{\psi(t)} \, dt < \infty.
\]

If

\[
\mu \left( \{ \omega : \lim \inf_{R \to \infty} \psi(R)R^2 \zeta_\omega(R) > 0 \} \right) = 1,
\]

then

\[
\mu \left( \{ \omega : \lim \inf_{R \to \infty} \psi(R)R^2 \zeta_\omega(R) = \infty \} \right) = 1.
\]
Proof We first construct a slightly smaller nondecreasing function \( \psi_0(t) \) so that
\[
\int_1^\infty \frac{1}{t \psi_0(t)^2} dt < \infty \quad \text{and} \quad \lim_{t \to \infty} \frac{\psi(t)}{\psi_0(t)} = \infty.
\]
To do this let \( n_1 = 1 \) and set
\[
n_j = \sup \left\{ T > n_{j-1} - 1 : \int_{n_{j-1}}^T \frac{1}{t \psi(t)} dt \leq 2^{-j} \int_1^\infty \frac{1}{t \psi(t)^2} dt \right\}.
\]
For \( j \in \mathbb{N} \) we piecewise define
\[
\psi_0(t) = j^{-1} \psi(t) \quad \text{whenever} \quad n_j \leq t < n_{j+1}.
\]
Observe that \( \psi_0 \) is also nondecreasing. Then since \( j \to \infty \) as \( t \to \infty \), we have
\[
\lim_{t \to \infty} \psi_0(t) = \infty.
\]
By the assumption of Lemma 2.7 there is a full measure set of \( \omega \) so that
\[
\liminf_{t \to \infty} \frac{\psi(t)}{\psi_0(t)} > 0.
\]
From this we have the desired result that for a full measure set of \( \omega \),
\[
\liminf_{t \to \infty} \frac{\psi(t)}{\psi_0(t)} = \infty.
\]
\[\square\]

2.4 Divergence Case

Recall for a sequence of sets \( (A_k)_{k=1}^\infty \), the limit superior is given by
\[
\limsup_{k \to \infty} A_k = \cap_{N=1}^\infty \cup_{i=N}^\infty A_i.
\]

Proposition 2.8 If for all \( \sigma > 0 \), \( \mu(\limsup A_k(0, \sigma, \psi, b)) > 0 \) then
\[
\liminf_{R \to \infty} \frac{\psi(R)}{R^2} > 0
\]
for \( \mu \)-a.e. \( \omega \).

Remark 2.9 Note that \( A_k(0, \sigma) \subseteq A_k(0, \sigma') \) whenever \( \sigma < \sigma' \). Thus the assumption of Proposition 2.8 is satisfied as long as \( \mu(\limsup A_k(0, \sigma)) > 0 \) for all \( \sigma \) small enough.

Proof Fix \( \sigma \) and write \( A_k \) for \( A_k(0, \sigma) \). We first claim \( \mu(\limsup A_k) = 1 \). We will show \( \limsup A_k \) is invariant under forward geodesic flow \( g_{\log(b^t)} \) for \( t > 0 \). Let
\[
\omega \in g_{\log(b^{-t})} \limsup_{k \to \infty} g_{\log(b^t)} H_{0, \sigma, k}.
\]
For \( n \in \mathbb{N} \), there exists \( k \geq n + t \) so that the monotonicity of \( \psi \) implies
\[
\omega \in g_{\log(b^{-t-k})} H_{0, \sigma, k} \subseteq g_{\log(b^{-t-k})} H_{0, \sigma, -t+k}.
\]
Hence $\omega \in \limsup A_k$. By ergodicity of the geodesic flow and the assumption that $\mu(\limsup A_k) > 0$, we conclude that $\mu(\limsup A_k) = 1$.

We now translate $\limsup A_k$ and claim that for $s_0 = \left\lceil \frac{\log(2)}{2 \log(b)} \right\rceil > 0$, any $\tilde{\omega} \in g_{-s_0 \log(b)} \limsup A_k$ satisfies

$$\lim_{R \to \infty} \inf R^2 \zeta_\omega(R) = 0.$$ 

First note that we are still working with a full measure set since $\mu$ is invariant under geodesic flow

$$\mu(g_{-s_0 \log(b)} \limsup A_k) = \mu(\limsup A_k) = 1.$$ 

Let $\tilde{\omega} = g_{-s_0 \log(b)} \omega$ for some $\omega \in \limsup A_k$. Since $\omega \in \limsup A_k$, for any $m \in \mathbb{N}$ we can find $\rho_m \geq m$ so that $\omega \in g_{\rho_m \log(b)} H_{0, \sigma, \rho_m}$. The choice of $s_0$ guarantees that the longest possible holonomy vectors in $g_{-s_0 \log(b)} T_{0, \sigma, \rho_m}^\pm$ are at most $b^{\rho_m}$ since the choice of $s_0$ is sufficient so that

$$b^{2(\rho_m - s_0)} + \frac{\sigma^2}{b^{2(\rho_m - s_0)} \psi(b^{\rho_m})^2} \leq b^{\rho_m}.$$ 

Thus the holonomy vectors detected by $g_{-s_0 \log(b)} T_{0, \sigma, \rho_m}^\pm$ have length at most $b^{\rho_m}$ and an upper bound on the angle around zero, giving the following upper bound

$$\zeta_\omega(b^{\rho_m}) \leq \arctan \frac{\sigma}{\psi(b^{\rho_m}) b^{2\rho_m - s_0}} < \frac{\sigma b^{s_0}}{\psi(b^{\rho_m}) b^{2\rho_m}}.$$ 

Since $s_0$ is fixed, we can take $\sigma \to 0$ to obtain $\liminf_{R \to \infty} \psi(R) R^2 \zeta_\omega(R) = 0$. \qed

2.5 Axiomatic framework

We will first recall the Borel–Cantelli lemma, and then spend the remainder of the section stating and proving a partial converse.

**Lemma 2.10 (Borel–Cantelli lemma)** Suppose $(A_k)_{k=1}^\infty$ are measurable sets with $\sum_{k=1}^\infty \mu(A_k) < \infty$. Then $\mu(\limsup A_i) = 0$.

**Proposition 2.11 (Exponential decay converse to Borel–Cantelli)** Let $C \geq 1, 0 < \delta < 1$, and $(A_k)_{k=1}^\infty$, $(B_k)_{k=1}^\infty$, $(C_k)_{k=1}^\infty$ be measurable sets. Suppose the following hold.

1. $\sum_{k=1}^\infty \mu(A_k) = \infty$.
2. For all $i \leq j$, $\mu(A_i) \geq \mu(A_j)$.
3. For all $i$, for all $j$ so that $j > i + C \log \left( \frac{1}{\mu(A_i)} \right)$ we have

$$\mu(A_i \cap A_j) \leq C \mu(A_i) \left[ \mu(A_j) + e^{-\frac{1}{4} |i-j|} \right].$$

4. For all $i$, for all $j$ so that $i < j \leq i + C \log \left( \frac{1}{\mu(A_i)} \right)$
(a) $B_i \subset A_i$ and $A_j \subset C_j$,
(b) $\mu(B_i) > \frac{1}{\tilde{c}} \mu(A_i)$,
(c) $\mu(C_j) < C \mu(A_j)^{\delta}$,
(d) $\mu(B_i \cap C_j) < C \mu(B_i) \left(2^{-(i-j)(1-\delta)} + \mu(C_j)^{1+\delta}\right)$.

Then $\limsup A_i$ has positive measure.

**Remark 2.12** One can observe the decay of correlations in Assumption (3) is insufficient to handle the generality of Theorem 1.1. Indeed consider $\psi(t) = \sqrt{\log(t+3) \log(\log(t+3))}$. Since $\int_1^\infty \frac{1}{t \log(t)} \, dt = \infty$, $\psi$ satisfies the second assumption of Theorem 1.1. Motivated by Lemma 2.3 we assume $\mu(A_k)$ is proportional to $\psi(b^k)^{-2} \approx \frac{1}{\log(b^k+4) \log(\log(b^k+4))} < \frac{1}{k \log(k) \log(b)}$. If $C' > 0$ is small enough, assumption (3) gives no bounds on $\mu(A_i \cap A_j)$ when $i < j < C' \log(i) + 1$. Define $n_k$ recursively by $n_1 = 2$ and $n_{k+1} = \lceil n_k + C' \log(n_k) \rceil$. Observe $\sum_{i=1}^\infty \mu(A_n_i) < \infty$, so $\mu(\limsup A_n_i) = 0$ by the Borel–Cantelli lemma. Thus we cannot draw any conclusions on $\mu(\limsup A_k)$ without Assumption (4).

We want to prove Proposition 2.11. This proof is inspired by [27, Theorem 2.1], which invokes the Chung–Erdős inequality.

**Lemma 2.13 (Chung–Erdős inequality [12])** Suppose $(A_k)_{k=1}^\infty$ is a sequence of measurable sets with $\mu\left(\bigcup_{k=1}^N A_k\right) > 0$, then

$$\mu\left(\bigcup_{k=1}^N A_k\right) \geq \frac{\left(\sum_{k=1}^N \mu(A_k)\right)^2}{\sum_{j,k=1}^N \mu(A_j \cap A_k)} \tag{2.3}$$

The next lemma will also be used to prove Proposition 2.11, which states the conditions on the sets $B_k \subset A_k$ that we use to show $\limsup \mu(B_k) > 0$, and thus $\limsup A_k > 0$.

**Lemma 2.14** We retain the notation of Proposition 2.11. There exists some $C \geq 1$ depending only on the constant $C$ as in Proposition 2.11 large enough so that if $\tilde{m}_i = i + \tilde{C} + \tilde{C} \log \left(\frac{1}{\mu(B_i)}\right)$, the following hold.

(a) $\sum_{k=1}^\infty \mu(B_k) = \infty$.
(b) For all $i \leq j$, $\mu(B_i) \geq \tilde{C} \mu(B_j)$.
(c) For all $i$ and for all $j$ so that $j > \tilde{m}_i$,

$$\mu(B_i \cap B_j) \leq \tilde{C} \mu(B_i) \left[\mu(B_j) + e^{-\frac{1}{\tilde{C}}(i-j)}\right].$$

(d) For all $i$ and for all $j$ with $i < j < \tilde{m}_i$,

$$\mu(B_i \cap B_j) < \tilde{C} \mu(B_i) \left[2^{-\frac{i-j}{\tilde{C}}} + \mu(B_j)^{1+\delta}\right].$$
Moreover there exists constants $D, D' > 0$ so that for any $n > 0$ and $N \geq n$,

$$\sum_{i,j=n}^{N} \mu(B_i \cap B_j) \leq \tilde{C} \left[ D \sum_{k=n}^{N} \mu(B_k) + D' + \left( \sum_{k=n}^{N} \mu(B_k) \right)^2 \right]. \quad (2.4)$$

Proof of Lemma 2.14. The first 4 parts use assumptions on $A_k$ in Proposition 2.11.

(a) Follows from Assumptions (1) and (4b).
(b) Follows from Assumptions (2) and (4b).
(c) Combine Assumption (3) with Assumptions (4a) and (4b).
(d) Combine Assumptions (4b) and (4c) with that fact that $\frac{\delta}{1 - e^{-2}} \in \left( \frac{1}{4}, \frac{1}{2} \right)$ and $C \geq 1$ to obtain

$$\mu(C_j) \frac{\delta}{1 - e^{-2}} \leq C \mu(A_j) \frac{\delta}{1 - e^{-2}} \leq C(C \mu(B_j)) \frac{\delta}{1 - e^{-2}} \leq \tilde{C}^2 \mu(B_j) \frac{\delta}{1 - e^{-2}}.$$

Now we move to Equation (2.4) where we want to find an upper bound for the denominator on the right hand side of Equation (2.3) applied to the sets $B_k$. First since $\tilde{C} \geq 1$,

$$\sum_{i,j=n}^{N} \mu(B_i \cap B_j) \leq 2 \sum_{i=n}^{N} \sum_{j>i}^{N} \mu(B_i \cap B_j) + \tilde{C} \sum_{k=n}^{N} \mu(B_k). \quad (2.5)$$

We split the double sum on the right hand side of Equation (2.5) into cases.

In the first case when $j > \tilde{m}_1$, part (c) implies

$$2 \sum_{i=n}^{N} \sum_{j>\tilde{m}_1}^{N} \mu(B_i \cap B_j) \leq 2\tilde{C} \sum_{i=n}^{N} \sum_{j>\tilde{m}_1}^{N} \mu(B_i) \mu(B_j) + \mu(B_i) e^{-\frac{\delta}{4(1-j)}}. \quad (2.6)$$

In the first sum of Equation (2.6), we re-expand the square so that

$$2 \sum_{i=n}^{N} \sum_{j>\tilde{m}_1}^{N} \mu(B_i) \mu(B_j) \left( \sum_{i=n}^{N} \mu(B_i) \right)^2 - \sum_{i=n}^{N} \mu(B_i)^2 \left( \sum_{i=n}^{N} \mu(B_i) \right)^2 \leq \left( \sum_{i=n}^{N} \mu(B_i) \right)^2. \quad (2.7)$$

In the second sum of Equation (2.6) we making the change of variables $k = j-i$, the geometric series formula gives

$$2 \sum_{i=n}^{N} \sum_{j>\tilde{m}_1}^{N} \mu(B_i) e^{-\frac{\delta}{4|j-i|}} \leq 2 \sum_{i=n}^{N} \mu(B_i) \frac{1}{1 - e^{-\frac{\delta}{4}}}. \quad (2.8)$$

Combining Equation (2.6), Equation (2.7), and Equation (2.8), we obtain

$$2 \sum_{i=n}^{N} \sum_{j>\tilde{m}_1}^{N} \mu(B_i \cap B_j) \leq \tilde{C} \left[ \left( \sum_{k=n}^{N} \mu(B_k) \right)^2 + \frac{2}{1 - e^{-\frac{\delta}{4}}} \sum_{k=1}^{N} \mu(B_k) \right]. \quad (2.9)$$
We now consider the case for \( i < j \leq \tilde{m}_i \). By (d)

\[
2 \sum_{i=n}^{N} \sum_{i<j \leq \tilde{m}_i} \mu(B_i \cap B_j) \\
\leq 2\tilde{C} \sum_{i=n}^{N} \sum_{i<j \leq \tilde{m}_i} \left( \mu(B_i)2^{-|i-j|(1-\delta)} + \mu(B_i)\mu(B_j) \frac{1}{1+\delta} \right). \tag{2.10}
\]

We again use a geometric series formula so that the first sum is bounded by

\[
2 \sum_{i=n}^{N} \sum_{i<j \leq \tilde{m}_i} \mu(B_i)2^{-|i-j|(1-\delta)} \leq \frac{2}{1-2^{-1-\delta}} \sum_{i=n}^{N} \mu(B_i). \tag{2.11}
\]

Note that \( \frac{1}{1+\delta} \in \left( \frac{5}{4}, \frac{4}{5} \right) \) so \( \frac{1}{1+\delta} \) is a power bigger than 1 with \( \mu(B_i) \leq 1 \). Therefore \( \mu(B_i)\frac{1}{1+\delta} \leq \mu(B_i) \). Combining this fact with (b),

\[
2 \sum_{i=n}^{N} \sum_{1<j \leq \tilde{m}_i} \mu(B_i)\mu(B_j) \frac{1}{1+\delta} \\
\leq 2\tilde{C} \sum_{i=n}^{N} \mu(B_i) \frac{1}{1+\delta} (\tilde{m}_i - i) \\
\leq 2\tilde{C} \sum_{i=n}^{N} \mu(B_i) + 2\tilde{C} \sum_{i=n}^{N} \mu(B_i) \frac{1}{1+\delta} \log \left( \frac{1}{\mu(B_i)} \right). \tag{2.12}
\]

Now choose \( n_0 \) large enough so that for all \( i \geq n_0 \),

\[
\log \left( \frac{1}{\mu(B_i)} \right) \leq \mu(B_i) \frac{1}{1+\delta}.
\]

Then if \( n \geq n_0 \),

\[
\sum_{i=n}^{N} \mu(B_i) \frac{1}{1+\delta} \log \left( \frac{1}{\mu(B_i)} \right) \leq \sum_{i=n}^{N} \mu(B_i).
\]

Otherwise if \( n \leq n_0 \)

\[
\sum_{i=n}^{N} \mu(B_i) \frac{1}{1+\delta} \log \left( \frac{1}{\mu(B_i)} \right) \leq \sum_{i=n}^{n_0-1} \mu(B_i) \log \left( \frac{1}{\mu(B_i)} \right) + \sum_{i=n_0}^{N} \mu(B_i)
\]

\[
\leq C' + \sum_{i=n_0}^{N} \mu(B_i).
\]

where \( C' > 0 \) is the bound for the finite sum. Therefore there is a constant \( C' \) so that

\[
\sum_{i=n}^{N} \mu(B_i) \frac{1}{1+\delta} \log \left( \frac{1}{\mu(B_i)} \right) \leq C' + \sum_{i=n}^{N} \mu(B_i). \tag{2.13}
\]
Thus we conclude by combining Equation (2.10), Equation (2.11), Equation (2.12), and Equation (2.13) so that

\[
2 \sum_{i=n}^{N} \sum_{i<j \leq \tilde{m}_i} \mu(B_i \cap B_j) \leq \tilde{C} \left[ 2CC'\tilde{C} + \left( \frac{2}{1 - 2^{-(1-\delta)}} + 4C\tilde{C} \right) \sum_{i=n}^{N} \mu(B_i) \right].
\]  

(2.14)

The proof of Equation (2.4) is completed by combining Equation (2.5), Equation (2.9), and Equation (2.14) where \(D' = 2CC'\tilde{C} \) and \(D = 1 + \frac{2}{1 - e^{-\delta}} + \frac{2}{1 - 2^{-(1-\delta)}} + 4C\tilde{C}. \)

\[ \Box \]

**Proof of Proposition 2.11.** It suffices show that the measure of \( \limsup B_k \) has positive measure. By Chung–Erdős inequality (Lemma 2.13), Lemma 2.14 (a), and Equation (2.4),

\[
\liminf_{N \to \infty} \mu \left( \bigcup_{k=n}^{N} B_k \right) \geq \liminf_{N \to \infty} \frac{\left( \sum_{k=n}^{N} \mu(B_k) \right)^2}{\tilde{C} \left[ D \sum_{k=n}^{N} \mu(B_k) + D' + \left( \sum_{k=n}^{N} \mu(B_k) \right)^2 \right]} = \liminf_{N \to \infty} \frac{1}{\tilde{C} \left( \sum_{k=n}^{N} \mu(B_k) \right) + \frac{D'}{\left( \sum_{k=n}^{N} \mu(B_k) \right) + 1}} = \frac{1}{C}.
\]

Hence \( \mu \left( \bigcup_{k=n}^{\infty} B_k \right) \geq \frac{1}{C} \). Notice \( \bigcup_{k=n}^{\infty} B_k \) is a nested decreasing sequence of sets, so we conclude

\[
\mu(\limsup B_n) = \mu \left( \bigcap_{n=1}^{\infty} \bigcup_{k=n}^{\infty} B_k \right) = \lim_{n \to \infty} \mu \left( \bigcup_{k=n}^{\infty} B_k \right) \geq \frac{1}{C} > 0.
\]

\[ \Box \]

2.6 Proof of Theorem 1.1

We now prove Theorem 1.1 conditional on the next proposition, whose proof is completed in Section 3 and Section 4.

**Proposition 2.15** For some \( c > 0 \) and for arbitrarily small \( \sigma > 0 \), there exist sets \( B_k, C_k \) so that along with the sets \( A_k = A_k(c, \sigma) \) as defined in Definition 2.1, the assumptions of Proposition 2.11 are satisfied.

Before outlining how to prove Proposition 2.15, we first show that it is sufficient to obtain Theorem 1.1.
Proof of Theorem 1.1. We first prove the convergence case. By Corollary 2.6, 
\[ \sum_{j=1}^{\infty} \psi(e^j)^{-2} < \infty, \]
and thus \( \lim_{j \to \infty} \psi(e^j) = \infty. \) Set
\[ L_k = \{ \omega : \exists R \in [e^k, e^{k+1}] \text{ so that } R^2 \zeta_\omega(R) < \psi(e^k)^{-1} \}. \]
For \( k \) large enough, if \( \zeta_\omega(R) < \frac{\psi(e^k)^{-1}}{R^2} \), then there are two saddle connections on \( \omega \) with horizontal holonomy at most \( R \) and vertical holonomy of magnitude at most \( \frac{\psi(e^k)^{-1}}{R} \). It follows that for \( \omega \in L_k \), \( g_{-\log(e^k \sqrt{\psi(e^k)})} \omega \) has two saddle connections of length at most \( \psi(e^k)^{-1/2} \varepsilon \). By Lemma 2.4, there is some constant \( C' \) so that \( \mu(L_k) \leq C' \psi(e^k)^{-2} \) so that \( \sum_{k=1}^{\infty} \mu(L_k) < \infty. \) By the Borel–Cantelli lemma (Lemma 2.10), \( \mu(\lim \sup L_k) = 0. \) Taking the complement, we have a full measure set of \( \omega \) so that for all \( k \geq k_0 \) (where \( k_0 \) depends on \( \omega \)), and for all \( R \in [e^k, e^{k+1}] \),
\[ \psi(R) R^2 \zeta_\omega(R) \geq \psi(e^k) R^2 \zeta_\omega(R) \geq 1. \]
Therefore
\[ \mu \left( \{ \omega : \lim \inf_{R \to \infty} \psi(R) R^2 \zeta_\omega(R) > 0 \} \right) \geq \mu \left( \{ \omega : \lim \inf_{R \to \infty} \psi(R) R^2 \zeta_\omega(R) \geq 1 \} \right) = 1. \]
By Lemma 2.7, the convergence case of Theorem 1.1 is verified.

We now prove the divergence case. By Corollary 2.6, \( \sum_{k=1}^{\infty} \psi(b^k)^{-2} = \infty. \) By Lemma 2.3 for any \( \sigma < \sigma_H \) our set \( A_k \) has measure proportional to \( \psi(b^k)^{-2}. \) By Proposition 2.11 and Proposition 2.15, for a positive measure set of \( \omega \) we have \( g_\omega \omega \in A_k \) for infinitely many \( k. \) Following Remark 2.9 we satisfy the assumptions of Proposition 2.8, which implies the divergence case of Theorem 1.1. \( \Box \)

Proof outline of Proposition 2.15.

We verify or state where each of the assumptions of Proposition 2.11 are verified.

1. Assumption (1) follows by the measure bounds of Lemma 2.3, Corollary 2.6
   and the fact that \( g_t \) preserves measure:
   \[ \sum_{k=1}^{\infty} \mu(A_k) \geq \sum_{k=1}^{\infty} \frac{m \sigma^2}{\psi(b^k)^2} = \infty. \]
2. Assumption (2) follows by Lemma 2.3: \( i \leq j \) implies \( \psi(b^i) \leq \psi(b^j) \), so
   \[ \mu(A_i) = \frac{m \sigma^2}{\psi(b^i)^2} \geq \frac{m \sigma^2}{\psi(b^j)^2} = \mu(A_j). \]
3. Assumption (3) is proved in Section 3.1 using Corollary 3.4, Lemma 3.5,
   and Lemma 3.7.
4. Assumption (4) is proved as follows. The construction of the \( B_k \) and \( C_k \)
   sets along with proofs of Assumptions (4a), (4b), and (4c) are given in
   Section 4.1. The proof is completed by verifying Assumption (4d) in
   Section 4.3. \( \Box \)
3 Verifying Proposition 2.11 Assumption (3): exponential decay of correlations for far away pairs

We begin by stating our key exponential mixing result:

**Theorem 3.1 (Stated from [5] Theorem C.4, see [8])** Fix $\mathcal{H}$ a connected component of the stratum and let $\mu$ be MSV measure as above. There exist $C > 0$ and $\delta > 0$ so that for all $h_1, h_2$ Lipschitz and compactly supported in a compact set $K$, there exists a constant $C_K$ depending only on the compact set $K$ so that for all $t \geq 0$

$$\left| \int h_1(h_2 \circ g_t) \, d\mu - \int h_1 \, d\mu \int h_2 \, d\mu \right| \leq C(C_K + \|h_1\|_\infty + \|h_2\|_{\text{Lip}})(C_K + \|h_2\|_\infty + \|h_2\|_{\text{Lip}})e^{-\delta t}.$$  

In order to apply Theorem 3.1, we need to use bump functions to approximate the sets $A_i$ and $A_j$. By $g_t$-invariance of $\mu$,

$$\mu(A_i \cap A_j) = \mu(H_{c_{\mathcal{H}}, \sigma, i} \cap g_{\log(b_{j-i})}H_{c_{\mathcal{H}}, \sigma, j}),$$

and in particular, using Theorem 3.1 to bound $\mu(H_{c_{\mathcal{H}}, \sigma, i} \cap g_{\log(b_{j-i})}H_{c_{\mathcal{H}}, \sigma, j})$ in turn gives bounds for $\mu(A_i \cap A_j)$. We will thus define our bump function to approximate $H_{c_{\mathcal{H}}, \sigma, i}$ and $H_{c_{\mathcal{H}}, \sigma, j}$.

**Definition 3.2** Recall we fixed above (Section 2.1) $0 < \delta < 1$ as in Theorem 3.1. For each $i$ and each $j > i + 4 \log \left(\frac{1}{\mu(A_j)}\right)$, define

$$\epsilon_{i,j} = e^{-\frac{\delta}{4}|i-j|}.$$  

Then for $\ell \in \{i, j\}$, define $\rho_{\ell}^{i,j} : \mathcal{H} \to \mathbb{R}$ to be zero outside of the coordinate patch for $H_{c_{\mathcal{H}}, \sigma, i}$ and $H_{c_{\mathcal{H}}, \sigma, j}$. Notice that by our choice, all the $H_{c_{\mathcal{H}}, \sigma, k}$ are contained in a single coordinate patch, $x_1, x_2 \in \mathbb{C}$, and $x_3 \in \mathbb{C}^{2g+3}$ we equip each copy of $\mathbb{C}^\ell$ with the standard $L^2$ Euclidean metric on $\mathbb{R}^{2\ell}$. On this coordinate patch we set

$$\rho_{i,j}^\ell(x_1, x_2, x_3) = f_1^\ell(x_1) f_2^\ell(x_2) f_3^\ell(x_3)$$

where

$$f_1^\ell(x_1) = \min \left\{ 1, \frac{1}{\epsilon_{i,j}} \text{dist} \left(x_1, \partial T^+_{c_{\mathcal{H}}, \sigma, \psi(b')} \right) \right\} \cdot \mathbf{1}_{T^+_{c_{\mathcal{H}}, \sigma, \psi(b')}},$$

$$f_2^\ell(x_2) = \min \left\{ 1, \frac{1}{\epsilon_{i,j}} \text{dist} \left(x_2, \partial T^-_{c_{\mathcal{H}}, \sigma, \psi(b')} \right) \right\} \cdot \mathbf{1}_{T^-_{c_{\mathcal{H}}, \sigma, \psi(b')}},$$

and

$$f_3^\ell(x_3) = \min \left\{ 1, \frac{1}{\epsilon_{i,j}} \text{dist} \left(x_3, \partial B(0, 1) \right) \right\} \cdot \mathbf{1}_{B(0, 1)}.$$
We want to show that our approximation functions are Lipschitz, so we first discuss some options of norms. The sup-norm metric, which we denote \( d_{\text{AGY}} \), is used to define the Lipschitz norm in Theorem 3.1 which is given in [8, Section 2.2.2]. For our purposes it is more convenient to use a metric coming from period coordinates, \( d_{\text{per}} \). We refer the reader to [17, Section 3.2] (where \( d_{\text{per}} \) is denoted \( d_{\text{Euclidean}} \)) for a detailed description of the construction on the space of all quadratic differentials with a fixed genus. We note that the arguments in [17, Section 3.2] can be repeated verbatim for a fixed stratum.

On each compact set this metric is uniformly comparable to the sup-norm metric, so for the estimate above the only effect on the Lipschitz functions we consider, which are all supported in a fixed compact set, is a multiplicative constant in the Lipschitz norms.

We now briefly outline that these two metrics are uniformly comparable on compact sets. Both \( d_{\text{AGY}} \) and \( d_{\text{per}} \) are defined as path metrics coming from norms on relative cohomology, where the norms depend on the point. Let us call these \textit{AGY norms} and \textit{Period norms}. The AGY norms vary continuously [8, Proposition 2.11] and so for each compact \( K \) all the AGY norms corresponding to points in \( K \) are uniformly comparable. On each compact set there are only finitely many Period norms (see the Remark at the end of [17, Section 3.2]), so all the norms we consider are uniformly comparable. The diameter of each compact set, \( K \), is bounded in both \( d_{\text{per}} \) and \( d_{\text{AGY}} \), so it suffices to prove there is a \( \tilde{\delta} > 0 \) and \( C \) (both depending on \( K \)) so that if \( x, y \in K \) then

\[
\frac{1}{C} \min\{\tilde{\delta}, d_{\text{per}}(x, y)\} < \min\{\tilde{\delta}, d_{\text{AGY}}(x, y)\} < C \min\{\tilde{\delta}, d_{\text{per}}(x, y)\}. \tag{3.1}
\]

To see (3.1), we choose a slightly larger compact set \( K' \) so that \( K \subset \text{Int}(K') \) and in particular \( \min\{d_{\text{per}}(K, \partial K'), d_{\text{AGY}}(K, \partial K')\} = \delta' > 0 \). Let \( C' \) be so that all the AGY and Period norms for all \( \omega \in K' \) are comparable by \( C' \). That is, if \( \omega, \omega' \in K' \) and \( \| \cdot \|_{\omega}, \| \cdot \|_{\omega'} \) are Period/AGY norms at \( \omega, \omega' \) then

\[
\frac{1}{C'} \| x - y \|_{\omega} \leq \| x - y \|_{\omega'} \leq C' \| x - y \|_{\omega}.
\]

We choose \( \tilde{\delta} < \frac{1}{C'} \delta' \) so that for any \( \omega \in K \), the \( \tilde{\delta} \)-neighborhood of \( \omega \) in both metrics is contained in a coordinate patch. Let \( \omega \in K \) be given and \( \text{dev} : \mathcal{U} \to H^1(\hat{\Sigma}, \Sigma, \mathbb{C}) \) be a coordinate chart so that the \( \tilde{\delta} \)-neighborhood of \( \omega \) in both metrics are contained in \( \mathcal{U} \). Now if \( \min\{d_{\text{per}}(\omega, \omega'), d_{\text{AGY}}(\omega, \omega')\} < \tilde{\delta} \) we have that

\[
\frac{1}{C'} \| \text{dev}(\omega) - \text{dev}(\omega') \|_{\omega''} < d_{\text{per}}(\omega, \omega') < C' \| \text{dev}(\omega) - \text{dev}(\omega') \|_{\omega''},
\]

and

\[
\frac{1}{C'} \| \text{dev}(\omega) - \text{dev}(\omega') \|_{\omega''} < d_{\text{AGY}}(\omega, \omega') < C' \| \text{dev}(\omega) - \text{dev}(\omega') \|_{\omega''},
\]

where \( \omega'' \in K' \) is arbitrary and \( \| \cdot \|_{\omega''} \) is either of the AGY or Period norms at \( \omega'' \). Because (3.1) is true outside of a \( \tilde{\delta} \) neighborhood in both metrics, we have (3.1) with \( C = C' \).
Lemma 3.3 There exists a constant $C'$ so that the functions $\rho_{i,j}^k$ are $\frac{C'}{\epsilon_{i,j}}$-Lipschitz with respect to $d_{\text{per}}$.

Proof Note that $f_k$ for $k = 1, 2, 3$ are all $\frac{1}{\epsilon_{i,j}}$-Lipschitz as the Euclidean distance function being 1-Lipschitz implies

$$|f_k^i(x_k) - f_k^j(y_k)| \leq \frac{1}{\epsilon_{i,j}} |\text{dist} (x_k, \partial T_{c_{n,\sigma}}(u_1)) - \text{dist} (y_k, \partial T_{c_{n,\sigma}}(u_1))|$$

$$\leq \frac{1}{\epsilon_{i,j}} |x_k - y_k|.$$

Now we claim that the function $\rho_{i,j}^k$ is $\frac{1}{\epsilon_{i,j}}$-Lipschitz with respect to the metric on $H$ given by

$$d_H((x_1, x_2, x_3), (y_1, y_2, y_3)) = |x_1 - y_1| + |x_2 - y_2| + |x_3 - y_3|.$$

To see this, let $(x_1, x_2, x_3)$ and $(y_1, y_2, y_3)$ be fixed. We compute

$$|\rho(x_1, x_2, x_3) - \rho(y_1, y_2, y_3)|$$

$$\leq |f_1(x_1)f_2(x_2)f_3(x_3) - f_1(y_1)f_2(y_2)f_3(y_3)|$$

$$+ |f_1(y_1)f_2(x_2)f_3(x_3) - f_1(y_1)f_2(y_2)f_3(y_3)|$$

$$+ |f_1(y_1)f_2(y_2)f_3(x_3) - f_1(y_1)f_2(y_2)f_3(y_3)|$$

$$\leq \frac{1}{\epsilon_{i,j}} (f_2(x_2)f_3(x_3)|x_1 - y_1| + f_1(y_1)f_3(x_3)|x_2 - y_2| + f_1(y_1)f_2(y_2)|x_3 - y_3|)$$

$$\leq \frac{1}{\epsilon_{i,j}} d_H((x_1, x_2, x_3) - (y_1, y_2, y_3)).$$

(Since $f_i \leq 1$)

As in the proof of the uniform comparability of $d_{\text{per}}$ and $d_{\text{AGY}}$, since $d_H$ comes from a single norm, $d_H$ is uniformly comparable to $d_{\text{per}}$. \(\square\)

We can now use the definition of the $\rho_{i,j}^k$ to state a corollary of Theorem 3.1.

Corollary 3.4 Fix $0 < \delta < 1$. Then there exists a constant $C$ so that for all $j > m_i$,

$$\int \rho_{i,j}^k(\rho_{i,j}^k \circ g_{\phi(j-i)}) \, d\mu \leq C\mu(A_i) \left[ \mu(A_j) + e^{-\frac{4}{\delta}|i-j|} \right].$$

Proof By definition $\|\rho_{i,j}^k\|_{\infty} = 1$, and by Lemma 3.3 $\|\rho_{i,j}^k\| = \frac{C'}{\epsilon_{i,j}} = e^{\frac{4}{\delta}|i-j|}$.

By Theorem 3.1, for perhaps a different constant $C$ coming from the uniformly comparable metrics and $C'$, the fact that $b \geq e$, and writing $C_K = 1 + C_K^+$,

$$\int \rho_{i,j}^k(\rho_{i,j}^k \circ g_{\phi(j-i)}) - \int \rho_{i,j}^k \int \rho_{i,j}^k$$

$$\leq C(C_K^+ e^{\frac{4}{\delta}|i-j|})^2 e^{-\delta|i-j|}$$

$$= C(C_K^+)^2 e^{-\delta|i-j|} + 2CC_K^+ e^{\frac{4}{\delta}|i-j|} + Ce^{-\frac{4}{\delta}|i-j|}$$

(Since $e^{-\frac{4}{\delta}|i-j|} \geq e^{-\frac{4}{\delta}|i-j|} \geq e^{-\delta|i-j|}$ as $e^{-\delta|i-j|} < 1$)

$$\leq \tilde{C} e^{-\frac{4}{\delta}|i-j|}.$$
By construction of $\rho_{i,j}^l$, $\int \rho_{i,j}^l = \mu(H_{c_H, \sigma, k}) = \mu(A_i)$, so
\[
\int \rho_{i,j}^l(\rho_{i,j}^l \circ g_{a_0(j-i)}) \leq \int \rho_{i,j}^l \int \rho_{i,j}^l + \tilde{C} e^{-\frac{1}{4}(|i-j|)} \leq \mu(A_i) \mu(A_j) + \tilde{C} e^{-\frac{1}{4}(|i-j|)}.
\]

Since $j > m_i$, we have $e^{-\frac{1}{4}(|i-j|)} < \mu(A_i)$. Thus, using $\tilde{C} > 1$,
\[
\int \rho_{i,j}^l(\rho_{i,j}^l \circ g_{a_0(j-i)}) \leq \tilde{C} \mu(A_i) \left[\mu(A_j) + e^{-\frac{1}{4}(|i-j|)}\right].
\]

☐

Next our goal is to get a relationship between $\mu(A_i \cap A_j)$ and $\int \rho_{i,j}^l(\rho_{i,j}^l \circ g_{a_0(j-i)})$.

**Lemma 3.5** For all $j > m_i$,
\[
\mu(A_i \cap A_j) \leq \int \rho_{i,j}^l(\rho_{i,j}^l \circ g_{a_0(j-i)}) \mu(E_j) + \mu(E_i)
\]
where $E \in \{\omega: \rho_{i,j}^l \in (0, 1)\}$.

**Proof** We first make a general claim.

**Claim** If $0 \leq g \leq 1$ then
\[
\int f \leq \int g + \mu\{f \neq g\}.
\]

To see this is true, we write
\[
\int f = \int g + f - g = \int g + \int_{\{f \neq g\}} f - g \leq \int g + \mu\{f \neq g\}
\]
where the last inequality follows since $f - g \leq 1$.

The proof now follows from the claim where $f = 1_{H_{c_H, \sigma, i} \cap g_{a_0(j-i)} H_{c_H, \sigma, j}}$ and $g = \rho_{i,j}^l(\rho_{i,j}^l \circ g_{a_0(j-i)})$, combined with the fact that $f \neq g$ occurs when $\rho_{i,j}^l \in (0, 1)$ or $\rho_{i,j}^l \in (0, 1)$. So $\mu\{f \neq g\} \leq \mu(E_i) + \mu(E_j)$. ☐

3.1 Proof that Proposition 2.11 Assumption (3) holds

To finish the proof of Proposition 2.11 Assumption (3) we will relate $\int \rho_{i,j}^l \int \rho_{i,j}^l$ to $\mu(E_i) + \mu(E_j)$ from Lemma 3.5. In order to do so, we will show it suffices to add a technical assumption about the behavior of $\psi$.

**Lemma 3.6** We may assume that for all $i, j$ with $j > i + \frac{1}{2} \log \left(\frac{1}{\mu(A_i)}\right)$
\[
eq \frac{1}{4}(|i-j|) \leq \min \left\{ \frac{\sigma^4}{7^4} \frac{1}{\psi(b)^4}, \frac{r}{2} 2^{-(n_H + 2)} \right\}
\]
where $r$ is as in the proof of Lemma 2.3 and $n_H = -\log_2(1 - c_H)$ for $c_H$ as in Lemma 2.3.
Lemma 3.7 Under the assumptions of Lemma 3.6 there exists a constant $C > 1$ so that

$$C \int \rho_{i,j}^1 d\mu \int \rho_{i,j}^1 d\mu > \mu(E_i) + \mu(E_j).$$

Indeed Lemma 3.7 is sufficient to conclude the proof of (3) as follows:

Proof Proposition 2.15, part (3) Fix $i$ and let $j > m_i$. Then

$$\mu(A_i \cap A_j) \leq \int \rho_{i,j}^1 (\rho_{i,j}^1 \circ g_{a(i-j)}) + \mu(A_i) + \mu(A_j) \quad \text{(By Lemma 3.5)}$$

$$\leq \int \rho_{i,j}^1 (\rho_{i,j}^1 \circ g_{a(i-j)}) + C \int \rho_{i,j}^1 \quad \text{(By Lemma 3.7)}$$

$$\leq C \mu(A_i) \left[ \mu(A_j) + e^{-\frac{3}{4}|j-i|} \right] + C \mu(A_i) \mu(A_j) \quad \text{(by Corollary 3.4, and since } \int \rho_{i,j}^1 \mu \leq \mu(A_i))$$

$$\leq \tilde{C} \mu(A_i) \left[ \mu(A_j) + e^{-\frac{3}{4}|j-i|} \right] \quad \text{(Setting } \tilde{C} = 2C.)$$

□

Since Lemma 3.7 depends on the additional assumptions in Lemma 3.6, we will first prove Lemma 3.6 by replacing the function $\psi$ with a function $\tilde{\psi}$ which satisfies Equation (3.2) and is still sufficient to prove the desired conclusion for $\psi$. The proof of Lemma 3.6 uses Lemma 3.8. After stating Lemma 3.8, we will then give a proof of Lemma 3.6, then Lemma 3.7, and the section concludes with the proof of Lemma 3.8.

Lemma 3.8 Let $\{a_j\}_{j \in \mathbb{N}}$ be a non-increasing sequence of positive numbers with $\sum_{j=1}^{\infty} a_j = \infty$. For any $\rho, k, \tau > 0$ with $\tau < 1$, there exists a constant $C > 0$ and a non-increasing sequence $\{c_j\}_{j \in \mathbb{N}}$ with $c_j = \min \left\{ \frac{1}{j}, \max \{a_j, \frac{1}{\tau} \} \right\}$, so that $\sum_{j=1}^{\infty} c_j = \infty$, $\sum_{j:c_j > a_j} a_j < \infty$, and

whenever $i \geq 3$, for all $j > \max \{i - C \log(c_i), 9 \}$, we have $e^{-\rho(j-i)} < \tau c_j^k$.

(3.3)

Proof Lemma 3.6 We first note that, from the proof of Lemma 2.3, we can choose $c_H$ close enough to 1 so that we always have $2^{-nH} < 2\tau$, and thus

$$\min \left\{ \frac{\sigma^4}{\tau^4} \frac{1}{\psi(b)^2} \frac{1}{2} 2^{-(nH+2)} \right\} = \min \left\{ \frac{\sigma^4}{\tau^4} \frac{1}{\psi(b)^2} \cdot 2^{-(nH+2)} \right\}$$

We next construct a modification replacing $\psi$ by $\tilde{\psi}$ so that $e^{-\frac{3}{4}|i-j|} \leq \frac{\sigma^4}{\tau^4} \frac{1}{\psi(b)^4}$ implies $e^{-\frac{3}{4}|i-j|} \leq \min \left\{ \frac{\sigma^4}{\tau^4} \frac{1}{\psi(b)^4} \cdot 2^{-(nH+2)} \right\}$. Namely using the constants from Lemma 2.3, define

$$\tilde{\psi}(b^i) = \begin{cases} \psi(b^i) & \frac{M \sigma^4}{\psi(b^i)} < 2^{-(nH+2)} \\ M \sigma^4 2^{(nH+2)} & \text{otherwise.} \end{cases}$$
We now have the upper bounds of $r/2$ and $2^{-n}n+2$ are trivially satisfied for $	ilde{\psi}$ for all $j > i + \frac{3}{4}\log\left(\frac{1}{\mu(A_j)}\right)$. Moreover for sets $\tilde{A}_j$ corresponding to $\tilde{\psi}$, $\hat{A}_j \subseteq A_j$, and so $\mu(\limsup A_j) \geq \mu(\limsup \hat{A}_j)$. Since our goal is to prove positive measure, we may now always assume the upper bounds of $r/2$ and $2^{-n}n+2$ hold.

We now construct $\tilde{\psi}$ from $\psi$ which satisfies Equation (3.2) by defining $\tilde{\psi}(b^j) = c_j^{-1/2}$ where $c_j = \min\{1/j, \max\{a_j, 1/j^2\}\}$ is the sequence from Lemma 3.8 for $a_j = \frac{1}{\psi(b^j)^2}$, $\rho = \frac{4}{7}$, $k = 2$, and $\tau = \frac{\sigma^4}{7^4} < 1$. Let $\tilde{\mathcal{A}}_i$ be the set corresponding to $\tilde{\psi}$.

Notice that $c_i = \frac{1}{\psi(b^i)}$, so by Lemma 3.8 whenever $i \geq 3$ and $j > \max\{i - C\log(\tilde{\psi}(2^i)^2), 9\}$, we indeed have for $i$ large enough that making $C$ larger if necessary
\[ e^{-\frac{4}{7^4}\psi(2^j)} < \tilde{\psi}(2^j)^2 < \mu(\tilde{\mathcal{A}}_i) \]

and moreover we have the desired inequality that
\[ e^{-\frac{4}{7^4}\psi(2^j)} < \frac{\sigma^4}{7^4} \tilde{\psi}(2^j)^4. \]

The restrictions on $i \geq 3$ and $j > 9$ do not play a role in the conclusion for the limsup sets. For $j \geq n > 9$, notice $c_j \leq a_j$ implies $\hat{A}_j \subseteq A_j$, so
\[ \bigcup_{j=n}^\infty A_j \supseteq \bigcup_{j=n}^\infty \tilde{A}_j \cup \bigcup_{j=n}^\infty A_j \supseteq \bigcup_{j=n}^\infty \hat{A}_j. \]

On the other hand since $c_j > a_j$ exactly when $c_j = 1/j^2$,
\[ \mu\left(\bigcup_{j=n}^\infty \hat{A}_j\right) \leq \mu\left(\bigcup_{j=n}^\infty \tilde{A}_j\right) + \mu\left(\bigcup_{j=n}^\infty \hat{A}_j\right) \leq \mu\left(\bigcup_{j=n}^\infty \hat{A}_j\right) + \sum_{j=n}^\infty \frac{m\sigma^2}{j^2}. \]

Since the tail sum of convergent series goes to zero, for any $\epsilon$ we can choose $n$ large enough so that $\sum_{j=n,c_j > a_j} \frac{m\sigma^2}{j^2} < \epsilon$. Hence,
\[ \mu\left(\bigcup_{j=n}^\infty A_j\right) \geq \mu\left(\bigcup_{j=n}^\infty \tilde{A}_j\right) \geq \mu\left(\bigcup_{j=n}^\infty \hat{A}_j\right) - \epsilon. \]

Thus in the limit we conclude $\mu(\limsup A_j) \geq \mu(\limsup \hat{A}_j)$. \hfill $\square$

With our preliminary work, Lemma 3.7 follows from geometric estimates on measures of balls and trapezoids.
Proof Lemma 3.7 Since \( \rho_{i,j}^j \leq \rho_{i,j}^l \), we have

\[
\int \rho_{i,j}^l \int \rho_{i,j}^j \geq \left( \int \rho_{i,j}^l \right)^2 \geq \mu(\rho_{i,j}^l = 1)^2.
\] (3.4)

Now we want to get a lower bound for the measure of the set where \( \rho_{i,j}^j = 1 \). To do this, we need to find the area of the subset of the trapezoid \( T_{c_H, \sigma,j}^+ \) where the \( \rho_{i,j}^l = 1 \). To do this, note the horizontal line \( y = \epsilon_{i,j} \) from \( c_H + \epsilon \) to \( 1 - \epsilon \) give the height of the inner trapezoid. The line which is length \( \epsilon_{i,j} \) away from the line \( y = \psi(b^j) x \) is given by

\[
y = \frac{\sigma}{\psi(b^j)} x - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}}.
\]

Thus the four corners of the trapezoid where \( \rho_{i,j}^j = 1 \) are given by \((c_H + \epsilon) + i \epsilon, (1 - \epsilon) + i \epsilon, (1 - \epsilon) + i(\sigma \psi(b^j)(1 - \epsilon) - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}}), (c_H + \epsilon) + i(\sigma \psi(b^j)(c_H + \epsilon) - \epsilon \sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2}})\).

Hence the area of subset of \( T_{c_H, \sigma,j}^+ \) where \( \rho_{i,j}^j = 1 \) is given by

\[
\frac{(1 - c_H - 2\epsilon_{i,j})}{2} \left( \frac{\sigma}{\psi(b^j)}(1 + c_H) - 2\epsilon_{i,j} \sqrt{1 + \frac{\sigma^2}{\psi(b^j)^2} - 2\epsilon_{i,j}} \right).
\]

By symmetry, the area of \( T_{c_H, \sigma,j}^- \) where \( \rho_{i,j}^l = 1 \) is the same as the area for \( T_{c_H, \sigma,j}^+ \). Thus the total area is the product of the two trapezoids with the product of the ball where \( n + 4 = 2g + s - 1 \) and \( \sigma(n) \) is gives the volume of
the $n$-ball. That is

$$\mu(\{\rho_{i,j}^i = 1\}) = \frac{(1 - c_H - 2\epsilon_{i,j})^2 \sigma(n)(r - \epsilon_{i,j})^n}{4}$$

$$\cdot \left( \frac{\sigma}{\psi(b')} (1 + c_H) - 2\epsilon_{i,j} \sqrt{1 + \frac{\sigma^2}{\psi(b')^2}} - 2\epsilon_{i,j} \right)^2$$

$$\geq d_n(1 - c_H - 2\epsilon_{i,j})^2 \left( \frac{\sigma}{\psi(b')} - 2\epsilon_{i,j} \sqrt{1 + \frac{\sigma^2}{\psi(b')^2}} - 2\epsilon_{i,j} \right)^2$$

(by (3.2), $\epsilon_{i,j} \leq \frac{c}{n}$ so $d_n$ depends only on $n$, and since $1 + c_H \geq 1$)

$$= d_n (2^{-nH} - 2\epsilon_{i,j})^2 \left( \frac{\sigma}{\psi(b')} - 6\epsilon_{i,j} \right)^2$$

(substituting $1 - c_H = 2^{-nH}$ from Lemma 2.3 and $\sigma^2/\psi(b')^2 < 1 < 3$)

$$\geq \frac{d_n}{2^{2(nH+1)}} \left( \frac{\sigma}{\psi(b')} - 6\epsilon_{i,j} \right)^2$$

(assuming by (3.2) $\epsilon_{i,j} < 2^{-(nH+2)}$, which implies $2^{-nH} - 2\epsilon_{i,j} > 2^{-(nH+1)}$)

$$\geq \frac{d_n}{2^{2(nH+1)}} \epsilon_{i,j}^2$$

(assuming by (3.2) that $\epsilon_{i,j} \leq \frac{\sigma}{\psi(b')}$, thus $6\epsilon_{i,j} + \epsilon_{i,j} \leq 7\epsilon_{i,j} \leq \frac{\sigma}{\psi(b')}$)

Combining this fact with (3.4), we obtain

$$\int \rho_{i,j}^i \int \rho_{i,j}^j \geq \frac{d_n^2}{2^{4(nH+1)}} \epsilon_{i,j}. \quad (3.5)$$

Now from the other end we want an upper bound for $\mu(\rho^i \in (0, 1]) + \mu(\rho^j \in (0, 1]) \leq 2C\epsilon_{i,j}$.

Given $\ell = i$ or $\ell = j$, we have the area of the $\epsilon_{i,j}$-boundary of one of the trapezoids $T_{c_H, \sigma, 2\epsilon}$ is given by

$$\mu(\partial_{i,j} T_{c_H, \sigma, 2\epsilon})$$

$$= \frac{\sigma}{2\psi(b')} (1 - c_H^2) - \left( \frac{1 - c_H}{2} - \epsilon_{i,j} \right) \left( \frac{\sigma}{\psi(b')} (1 + c_H) - 2\epsilon_{i,j} \sqrt{1 + \frac{\sigma^2}{\psi(b')^2}} \right)$$

(taking area of $T_{c_H, \sigma, \ell}$ less the area where $\rho_{i,j}^i = 1$)

$$= \epsilon_{i,j} \left[ \frac{\sigma}{\psi(b')} (1 + c_H) + (1 - c_H - 2\epsilon_{i,j}) \left[ 1 + \sqrt{1 + \frac{\sigma^2}{\psi(b')^2}} \right] \right]$$

(assuming $\sigma < \psi(b')$ which is easy since $\sigma < 1$ is fixed and $\psi \geq 1$ is nondecreasing)

$$\leq \epsilon_{i,j} \left[ 1 + c_H + (1 - c_H)(1 + \sqrt{2}) \right]$$

(where $C$ depends on $c_H$)

Thus

$$\mu(\{\rho_{i,j}^i \in (0, 1]\}) + \mu(\{\rho_{i,j}^j \in (0, 1]\}) \leq 2C\epsilon_{i,j}. \quad (3.6)$$
Combining Equation 3.6 with Equation 3.5,
\[
\int \rho_i \int \rho_i^j \geq \frac{d_i^2}{2^{4(n_H+1)}} e_{i,j}^j
\geq \frac{d_i^2}{2^{4(n_H+1)}} \left[ \mu\{\rho_i^j \in (0, 1)\} + \mu\{\rho_i^j \in (0, 1)\} \right].
\tag{3.7}
\]

Setting \( \tilde{C} = \frac{2^{4(n_H+1)}(2C)}{d_n^2} \), we can assume \( \tilde{C} > 1 \) since \( d_n \) is bounded above by a fixed constant, and we can make \( C \) larger if necessary. \( \Box \)

**Proof Lemma 3.8.** We first claim the following:

**Claim:** The sequence \( c_j \) is non-increasing, has divergent sum and
\[
\sum_{j : c_j > a_j} a_j < \infty, \quad \sum_{j : c_j > a_j} c_j < \infty.
\]

**Proof of Claim.** The maximum of two non-increasing sequences is non-increasing and so \( \max\{a_j, \frac{1}{j}\} \) is a non-increasing sequence (in \( j \)). Similarly the minimum of two non-increasing sequences is non-increasing and so \( c_j \) is non-increasing.

If \( \max\{a_j, \frac{1}{j}\} = \frac{1}{j} \), then \( c_j = j^{-2} > a_j \). Otherwise \( c_j = \min\{1/j, a_j\} \leq a_j \).

If \( c_j > a_j \) then \( a_j < \frac{1}{j} \) and so clearly \( \sum_{j > 1/j} a_j < \sum_j \frac{1}{j^2} < \infty \). On the other hand, since \( c_j > a_j \) is only possible when \( c_j = j^{-2} \),
\[
\sum_{j > a_j} c_j = \sum_{j > a_j} j^{-2} < \infty.
\]

Now observe that
\[
\sum_{j=2^k}^{2^{k+1}-1} c_j \geq \min \left\{ \frac{1}{2}, 2^k a_{2^k+1} \right\}.
\tag{3.8}
\]

Indeed we are estimating the sum from below by \( 2^k c_{2^k+1} \) and considering the different possibilities of \( c_{2^k+1} \). Notice that \( \sum a_j \) diverges and so \( \sum a_j \) diverges and thus \( \sum k \min\{\frac{1}{2}, 2^k a_{2^k+1}\} \) diverges. So \( \sum c_j = \sum_k \sum_{j=2^k}^{2^{k+1}-1} c_j \) diverges. \( \Box \)

We have proved the Claim and now proceed with the remainder of the proof of Lemma 3.8. We now show if \( C > \frac{4k}{\rho}(\frac{1}{k} + 1) \) then Equation (3.3) holds where \( \tilde{\tau} = \min\{\tau, 1\} \) and \( \tilde{k} = \max\{\frac{1}{2}, k\} \). Indeed it suffices to show that for all \( j > i + C \log(i) \) we have that
\[
e^{-\rho(j-i)} < \frac{\tau}{j^{2k}}.
\tag{3.9}
\]
Clearly smaller \( \tau \) and larger \( 2k \) make (3.9) harder to satisfy. So from here we assume \( \tau \leq 1 \) and \( 2k \geq 1 \), which motivated our choice of \( \tilde{\tau} \) and \( \tilde{k} \). Under these assumptions, Equation (3.9) is implied by \( j - i > \frac{2k}{\rho} \log(\frac{1}{\tau}) \).
If \( j \leq 2i \) this follows from our condition on \( C \). Indeed

\[
j - i > 4k \rho \left( \frac{1}{\tau} + 1 \right) \log(i) > 4k \rho \left( \frac{1}{\tau} + 1 \right) \left( \log(j) - \log(2) \right).
\]

And so

\[
j - i > 4k \rho \left( \log\left( j^{\frac{1}{\tau} + 1} \right) - \log(2^{\frac{j}{\tau + 1}}) \right) > 2k \rho \left( \frac{j}{\tau} \right).
\]

Note that the second inequality uses that \( \log\left( j^{\frac{1}{\tau} + 1} \right) > 2 \log\left( \frac{j}{\tau} \right) \) because \( j \geq 9 > 2^2 \) and the third inequality uses that \( j^{\frac{1}{\tau} + 1} > \frac{j}{\tau} \) for all \( j \geq 9 \) and \( \tau > 0 \).

For the case when \( j > 2i \), set \( f(x) = x - i \) and \( g(x) = 2k \rho \left( \frac{x}{\tau} \right) \). Note that \( f(2i) > g(2i) \) from the case where \( j \leq 2i \). Moreover, \( f'(x) = 1 > g'(x) = \frac{2k \rho}{x^2} \) for all \( x > \frac{2k \rho}{\tau} \).

Since \( i \geq 3 \) and \( \log(3) > 1 \),

\[
j > i + C \log(i) > 3 + \frac{4k \rho}{\tau} \left( \frac{1}{\tau} + 1 \right) \log(3) > \frac{2k \rho}{\tau}.
\]

So for all \( j > 2i \) we have \( f'(j) > g'(j) \) and \( f(2i) > g(2i) \). Hence \( f(j) > g(j) \) for all \( j \geq 2i \) as desired. \( \Box \)

4 Verifying Proposition 2.11 Assumption (4)

We begin this section by defining the sets \( B_i \) and \( C_i \) required for Proposition 2.15, and then verify these sets satisfy Assumption (4) of Proposition 2.11.

Definition 4.1 (Definition of the \( B \)'s) Set \( I \defeq \left( -\frac{\pi}{12}, \frac{\pi}{12} \right) \). For \( k \in \mathbb{N} \) define

\[
B_k = g_{\log(b^k)} g_{\log\left( \sqrt{\psi(b^k)} \right)} \bigcup_{\theta \in I} r_\theta W_k
\]

where we have the following definitions for the sets shown in Figure 1. We first pull back the set \( A_k \) so that trapezoids in \( H_{c,\sigma,k} \) makes a 45 degree angle and define

\[
\tilde{W}_k = g_{\log\left( \sqrt{\psi(b^k)} \right)} g^{-\log(b^k)} A_k.
\]

Then we restrict to a smaller subset of \( \tilde{W}_k \) denoted \( W_k \) so that \( r_\theta W_k \subset \tilde{W}_k \) for \( \theta \in I \). That is \( W_k \) is the set of \( \omega \) with two holonomy vectors \( v_1 \) and \( v_2 \) satisfying

\[
c_H \sqrt{\frac{2\sigma}{\psi(b^k)}} \leq |v_1|, |v_2| \leq \sqrt{\frac{\sigma}{\psi(b^k)}},
\]

and

\[
\arg(v_1) \in \left( \frac{\pi}{12}, \frac{\pi}{6} \right), \quad \arg(v_2) \in \left( -\frac{\pi}{6}, -\frac{\pi}{12} \right).
\]
**Definition 4.2 (Definition of the C’s)** Define

\[ C_k = g_{\log(b^k)} g_{\log(\sqrt{\psi(t)})} S(c_H, \sigma, b^k) \]

where

\[ S(c_H, \sigma, t) = \{ \omega \in H : \omega \text{ has a holonomy vector } v \text{ with } |v| \in (c_H \sqrt{\sigma(\psi(t))}, \sqrt{2\sigma(\psi(t))}) \} \].

4.1 Proof that (4) (a)-(c) hold

We now verify assumptions (4) (a), (b) and (c), where we set \( c = c_H \). Fix \( i \) and take \( j \) so that \( i < j \leq i + C \log \left( \frac{1}{\mu(A_i)} \right) \).

(a) As constructed \( B_i \subseteq A_i \) and \( A_j \subseteq C_j \).
(b) Following the proof of Lemma 2.3 where in Equation 2.1, we replace the trapezoids by sectors of annuli, we obtain a constants \( m', m > 0 \) so that

\[ \mu(W_i) \geq m' \frac{\sigma^2}{\psi'(b_j)^2} \left( \frac{\pi}{24} (1 - 2c_H) \right)^2 m_{2g+s-3}(B) > m \frac{\sigma^2}{\psi'(b_j)^2}. \]

Thus by Lemma 2.3, \( \mu(B_i) \geq \mu(W_i) \geq m\mu(A_i)/M \).
(c) Since the measure is invariant under geodesic flow and by Lemma 2.3,

\[ \mu(A_j) = \mu(H_{c_H, \sigma, j}) \geq m \frac{\sigma^2}{\psi'(b_j)^2}. \]
By Masur–Smillie Lemma 2.4,
\[ \mu(C_j) = \mu(S(c_H, \sigma, 2^j)) \leq M \frac{\sigma}{\psi(b^j)}. \]

Thus
\[ \mu(C_j) \leq M \frac{\sqrt{m \sigma^2}}{\psi(b^j)^2} = M \sqrt{m \mu(A_j)^{\frac{1}{2}}}. \]

4.2 Construction and circle averages of logsmooth functions

The main goal of this section is to prove Corollary 4.11, which extends the statements of [13] (giving averages over intervals) to include so-called logsmooth functions from [1] (which gives averages over the full circle).

Definition 4.3 A complex \( K \) in \( \omega \) is a closed subset of \( X \) whose boundary \( \partial K \) consists of a union of disjoint (in the interior) saddle connections such that if \( \partial K \) contains three saddle connections bounding a triangle, then the interior of that triangle is in \( K \). Given a complex \( K \) the complexity of \( K \) is the number of saddle connections needed to triangulate \( K \) and \( |\partial K| \) is the length of the longest saddle connection in \( \partial K \). For any \( \delta > 0 \) and \( k \in \mathbb{N} \), if \( M \) is the complexity of \( \omega \),
\[ \alpha_k(\omega) = \max_{K} \left( \frac{1}{\text{area}(K)^{\frac{1}{2}} |\partial K|^{1+\delta}} \right). \]

If the set over which we take the maximum is empty, then we set \( \alpha_k(\omega) = 0 \).

Definition 4.4 Given a function \( f \) on \( \mathcal{H} \) and a point \( \omega \in \mathcal{H} \), we let
\[ \text{Ave}_t(f)(\omega) = \frac{1}{2\pi} \int_{0}^{2\pi} f(g_T r_\theta \omega) \, d\theta. \]

Note that \( \alpha_1(\omega) = \frac{1}{\ell(\omega)^{\frac{1}{2}}} \), where \( \ell(\omega) \) is the length of the shortest saddle connection. Since \( M \) is finite for all \( k \) large enough, \( \alpha_k(\omega) = 0 \). For more information and intuition for the \( \alpha_k \), see Section 5.3 of [13]. From Proposition 5.5 of [13], we have

Proposition 4.5 Fix a stratum \( \mathcal{H} \), and \( 0 < \delta < \frac{1}{2} \). We can find a constant \( b \) such that for any interval \( I \subseteq S^1 \), there exists a constant \( c_I \) such that for all \( \omega \in \mathcal{H} \) and \( T \geq 0 \),
\[ \int_I \alpha_k(g_T r_\theta \omega) \, d\theta < c_I e^{-(1-2\delta)T} \sum_{j \geq k} \alpha_j(\omega) + b|I|. \]

The strategy we will take is to extend this theorem to a function \( V_\delta \) which is a weighted average of the \( \alpha_k \) functions. The choice of weights for the \( \alpha_k \) ensures the functions \( V_\delta \) have the following properties stated in Lemma 4.6.
Lemma 4.6 (Lemma 6.2 \[7\], Proof in \[1\]) Let \( \mathcal{V} \) be a neighborhood of the identity in \( \text{SL}(\mathbb{R}) \). Fix \( \mathcal{H} \) a connected component of a stratum. For every \( 0 < \delta < 1 \) there exists \( c_1 > 0 \) so that for all \( t > 0 \) there exists a function \( V^{(t)}_\delta : \mathcal{H} \to [1, \infty) \) and a scalar \( b_t \) satisfying the following properties. For all \( \omega \in \mathcal{H} \),

\[
\text{Ave}_t(V^{(t)}_\delta)(\omega) = \int_0^{2\pi} V^{(t)}_\delta(g_{\theta \omega}) d\theta \leq c_1 e^{-(1-\delta)t} V^{(t)}_\delta(\omega) + b_t.
\]

Moreover, \( V^{(t)}_\delta \) is logsmooth. That is
\[V^{(t)}_\delta(g_\omega) \leq c_3 V^{(t)}_\delta(\omega)\] for all \( \omega \in \mathcal{H} \) and \( g \in \mathcal{V} \).

Finally, there exists a constant \( C_{\delta,t} \) so that
\[
\frac{V^{(t)}_\delta(\omega)}{V^{(t)}(\omega)} \in [C_{\delta,t}^{-1}, C_{\delta,t}]
\]
where \( V_\delta = \max\{1, \alpha_1(\omega)\} = \max\{1, \frac{1}{t(\omega)^{1+\delta}}\} \).

We will build up to Corollary 4.11, which is similar to the conclusion of Lemma 4.6 where we average over an interval \( I \) instead of \([0, 2\pi)\). We now explicitly construct \( V_\delta \) using the following result.

Proposition 4.7 (Proposition 5.6 \[13\]) Fix \( \mathcal{H} \) and \( 0 < \delta < 1 \). There exists \( C > 0 \) so that for any \( t > 0 \), there exists constants \( b_t \) and \( w_t \) so that for any \( k \) and any \( \omega \in \mathcal{H} \),

\[
\text{Ave}_t(\alpha_k)(\omega) \leq C e^{-(1-\delta)t} \alpha_k(\omega) + w_t \sum_{j > k} \alpha_j(\omega) + b_t.
\]

Definition 4.8 Fix \( \delta \) and \( t > 0 \). Define
\[
\lambda_k^{(t)} = \left( \frac{w_t}{C} + 1 \right)^k
\]
where \( w_t \) and \( C \) are the constants of (4.3). Define
\[
V^{(t)}_\delta(\omega) = \sum_{k=0}^{M} \lambda_k^{(t)} \alpha_k(\omega)
\]
where \( M \) is the maximum complexity of \( \omega \).

Proof of Lemma 4.6. We first claim
\[
\lambda_k^{(t)} C e^{-(1-\delta)t} + w_t \sum_{j=0}^{k-1} \lambda_j^{(t)} \leq 2C \lambda_k^{(t)} e^{-(1-\delta)t}.
\]
To see this holds, note that \( e^{-\delta t} \geq 1 \). Thus since \( \lambda_k(t) \geq 1 \), we have

\[
1 - \frac{1}{\lambda_k(t)} \leq 1 \leq \left( \frac{w_t}{C} + 1 - 1 \right) e^{-(1-\delta)t}.
\]

Simplifying and using the finite geometric series formula, this implies

\[
\sum_{j=0}^{k-1} \lambda_j(t) = \frac{\lambda_k(t) - 1}{\lambda_k(t)} \leq \lambda_k(t) \frac{C}{w_t} e^{-(1-\delta)t}.
\]

Multiplying by \( w_t \) and adding \( \lambda_k(t) C e^{-(1-\delta)t} \) to each side yields (4.4).

We now want to prove that on average \( V_\delta(t) \) shrinks over circles of radius \( t \). To see this, we compute

\[
\text{Ave}_t(V_\delta(t))(\omega) = \sum_{k=0}^{M} \lambda_k(t) \text{Ave}_t(\alpha_k)(\omega)
\]

\[
\leq \sum_{k=0}^{M} \lambda_k(t) \left( C e^{-t(1-\delta)} \alpha_k(\omega) + w_t \sum_{j>k} \alpha_j(\omega) + b_t \right)
\]

(by (4.3))

\[
= \left( \sum_{k=0}^{M} \lambda_k(t) C e^{-t(1-\delta)} \alpha_k(\omega) \right) + w_t \sum_{k=1}^{M} \alpha_k(\omega) \left( \sum_{j=0}^{k-1} \lambda_j(t) \right) + b_t
\]

(replacing \( b_t = b_t \left( \sum_{k=0}^{M} \lambda_k(t) \right) \))

\[
\leq 2 C e^{-t(1-\delta)} \lambda_0(t) \alpha_0(\omega) + \sum_{k=1}^{M} \alpha_k(\omega) \left( \lambda_k(t) C e^{-t(1-\delta)} + w_t \sum_{j=0}^{k-1} \lambda_j(t) \right) + b_t
\]

(by (4.4) and replacing \( C \) with \( 2C \))

\[
\leq C e^{-t(1-\delta)} \sum_{k=0}^{M} \lambda_k(t) \alpha_k(\omega) + b_t
\]

\[
= C e^{-t(1-\delta)} V_\delta(t)(\omega) + b_t.
\]

The logsmoothness of the \( V_\delta(t) \) follows from [14, Proof of Proposition 7.2]. \( \square \)

Now that we have defined \( V_\delta(t) \) with the logsmooth property, we proceed to extending the results of [13] to include the \( V_\delta(t) \) function.

**Lemma 4.9** There exists a constant \( c_2 > 0 \) so that for any \( \tau > 0 \) and \( I \subseteq S^1 \) an interval, there exists \( t_0(\tau, |I|) \geq 0 \) so that for any \( \omega \in \mathcal{H} \) and \( t > t_0 \), we have

\[
\int_I V_\delta^{(\tau)}(g_{t+r\tau \hat{\omega}}) d\theta \leq c_2 \int_I \text{Ave}_{t}(V_\delta^{(\tau)})(g_{t+r\hat{\omega}}) d\theta
\]
where $J \subseteq S^1$ is an interval (that could depend on all other parameters) with $|J| = |I|$.

**Proof** Note that this result would follow directly from linearity combined with Lemma 5.2 of [13], except as stated in Lemma 5.2 the interval $J$ could depend on $\alpha_i$. However following the proof exactly using linearity to replace each $\alpha_i$ with $V_\delta^{(\tau)}$, we take the interval $2I$ with the same center as $I$ and twice the length. Then in the last 5 lines of the proof, we write $2I = J_1 \cup J_2$ as a union of two intervals with $|J_1| = |J_2| = |I|$. Then

$$
\max_{j=1,2} \int_{J_j} \text{Ave}_\tau (V_\delta^{(\tau)} (gr_\theta \omega)) \geq \frac{1}{2} \int_{2I} \text{Ave}_\tau (V_\delta^{(\tau)} (gr_\theta \omega)).
$$

Now define $J$ (which now depends on $V_\delta^{(\tau)}$ instead of individual $\alpha_k$) to be the interval on which the maximum is achieved, and the proof follows by linearity as desired. \qed

Now we state Proposition 5.5 of [13] for the $V_\delta^{(t)}$ functions.

**Proposition 4.10** Fix a stratum $H$ and $0 < \delta < 1$. Let $c_1$ and $c_2$ be the constants of Lemma 4.6 and Lemma 4.9, respectively. Choose $\tau > 0$ large enough so that

$$
c_1 c_2 e^{-(1-\delta)\tau} < \frac{1}{2}.
$$

Let $I \subseteq S^1$ be an interval and by Lemma 4.9 let $m$ be the smallest possible integer so that $(m-1)\tau > t_0(\tau, |I|)$. That is $m = 1 + \left\lceil \frac{t_0(\tau, |I|)}{\tau} \right\rceil$. There are constants $c = c(\tau, \delta, |I|) > 0$ and $b_\tau = b(\tau, \delta)$ so that for all $n \geq m$ and for any $\omega \in H$,

$$
\int_I V_\delta^{(\tau)}(g_n \tau r_\theta \omega) \ d\theta < ce^{-(1-\delta)n\tau} V_\delta^{(\tau)}(\omega) + b_\tau |I|.
$$

(4.5)

**Proof** Let $n \geq m$ and $\omega \in H$. Our goal is to construct the constants $c$ and $b_\tau$ so that Equation 4.5 holds. Indeed applying Lemma 4.9 followed by Lemma 4.6, we have

$$
\int_I V_\delta^{(\tau)}(g_n \tau r_\theta \omega) \ d\theta \leq c_2 \int_{J_{n-1}} \text{Ave}_\tau (V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega)) \ d\theta
$$

$$
\leq c_2 \left( \int_{J_{n-1}} c_1 e^{-(1-\delta)\tau} V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega) \ d\theta + b_\tau \right)
$$

$$
= c_2 c_1 e^{-(1-\delta)\tau} \int_{J_{n-1}} V_\delta^{(\tau)}(g_{(n-1)\tau} r_\theta \omega) \ d\theta + c_2 b_\tau |I|
$$

where the last equality follows from the fact that $|J_{n-1}| = |I|$. 

where we note $J$, replaced by $c$, in the last inequality we replaced $\tau > 0$, set $V_0$ in Proposition 4.10, assumption $\lambda > 0$, set $\mathcal{V}$ in the KAK decomposition to be a left $K$-invariant sets which contains $g_{r_0}$, and note $n_0 \tau - \ell < 0$. Then $r_0 \leq \lambda$, which implies $g_{r_0} \in \mathcal{V}$ and we can apply (4.1). We also want a lower bound on $r_0$. Assume $r_0 < \frac{1}{3\lambda}$, in which case $k > 3r/\lambda \geq 3(k-1)$ and we conclude $k \leq 3/2$ and thus $k = 1$. Thus if $k \geq 2$ we have $r_0 \geq \frac{1}{k}$, or $k = 1$. Then from Proposition 4.10,

$$\int_I V_\delta^{(r)}(g_{r_0} \tau r\omega) d\theta = \int_I V_\delta^{(r)}(g_{-k\tau} g_{n_0 \tau} \tau r\omega) d\theta$$

$$\leq c_3^k \int_I V_\delta^{(r)}(g_{n_0 \tau} \tau r\omega) d\theta$$

(by (4.1))

$$\leq c_3^k \left[ ce^{-(1-\delta)\lambda n_0 \tau} V_\delta^{(r)}(\omega) + b_r|I| \right]$$

(by (4.5))

$$\leq c_3^k \left[ ce^{-(1-\delta)\lambda n_0 \tau} V_\delta^{(r)}(\omega) + b_r|I| \right]$$

(since $n_0 \tau \geq \ell$.)

where in the last inequality we replaced $b_r$ with $2c_2 b_r$. By the logsmooth property of $V_\delta$ from Lemma 4.6, splitting into small steps, there exists some $k(m, \tau)$ so that

$$V_\delta^{(r)}(g_{(m-1)\tau} \tau r\omega) \leq c_3^{m} V_\delta^{(r)}(\omega).$$

Thus we can write our constant $c$ as

$$c = (c_1 c_2)^{n-m+1} (e^{-(1-\delta)\tau})^{-m+1} c_3^{m} |I|$$

where we note $m$ depends on $\tau$ and $|I|$, so $c$ depends only on $\delta$, $\tau$ and $|I|$. □

Corollary 4.11 Fix a connected component of a stratum $\mathcal{H}$ and $0 < \delta < 1$. There exists $\tau > 0$ so that for any interval $I \subseteq S^1$, there exists constants $c = c(\tau, \delta, |I|) > 0$ and $b_r = b(\tau, \delta)$ so that there exists an $\ell_0 > 0$ so that for all $\ell \geq \ell_0$ and for any $\omega \in \mathcal{H}$,

$$\int_I V_\delta^{(r)}(g_{r\theta} \omega) d\theta \leq ce^{-(1-\delta)\ell} V_\delta^{(r)}(\omega) + b_r|I|.$$
Thus if \( k = 1 \), we obtain the final constant which is independent of \( \ell \). Otherwise we assume \( k \geq 2 \), and

\[
\int_{I} V_{\delta}^{(\tau)}(g_{\tau}(\theta)\omega) \, d\theta \leq c_{\delta,\tau}^{-1} \left[ ce^{-(1-\delta)\ell} V_{\delta}^{(\tau)}(\omega) + b_{\tau} |I| \right] \quad \text{(since } r \leq \tau) \\
\leq c_{\delta,\tau}^{-1} \left[ ce^{-(1-\delta)\ell} V_{\delta}^{(\tau)}(\omega) + b_{\tau} |I| \right].
\]

Thus given the lower bound on \( r_0 \geq \frac{\lambda}{3} \), the final constants only depend on \( \tau \), \( \lambda \) and not \( \ell \) and we obtain the desired result. \( \square \)

4.3 Completion of the verification of (4) (d)

To obtain an upper bound for \( \mu(B_{i} \cap C_{j}) \), by \( g_{r} \)-invariance of \( \mu \), it suffices to find an upper bound for \( \mu(\tilde{B}_{i} \cap \tilde{C}_{j}) \) where \( \tilde{B}_{i} = \bigcup_{t \in I} r_{0} W_{i} = I \cdot W_{i} \) and \( \tilde{C}_{j} = g_{f(i,j)}S(c_{H}, \sigma, b^{j}) \) for \( f(i, j) = \log \left( b^{j-i} \sqrt{\frac{\psi(b^{j})}{\psi(b)}} \right) \).

Note, the shortest saddle connection on \( \omega \in S(c_{H}, \sigma, b^{j}) \) is at most \( \sqrt{\frac{2\sigma^2}{\psi(b)}} \) and so by (4.2) for such an \( \omega \) we have

\[
C_{\delta,\tau}^{-1} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{\frac{1}{1+4}} \leq V_{\delta}^{(\tau)}(\omega).
\]

So, if \( \omega' \in \tilde{C}_{j} \), we have \( V_{\delta}^{(\tau)}(g_{f(i,j)}\omega') \geq C_{\delta,\tau}^{-1} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{\frac{1}{1+4}} \). Thus, because

\[
\mu(\{ x \in S : h(x) > C \}) \leq C^{-1} \int_{S} |h(x)| \, d\mu(x)
\]

we have

\[
\mu(\tilde{B}_{i} \cap \tilde{C}_{j}) \leq \mu \left( \left\{ \omega \in \tilde{B}_{i} : V_{\delta}^{(\tau)}(g_{f(i,j)}\omega) \geq C_{\delta,\tau}^{-1} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{\frac{1}{1+4}} \right\} \right)
\]

(4.6)

\[
\leq C_{\delta,\tau} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{-\frac{1}{1+4}} \int_{I \cdot W_{i}} V_{\delta}^{(\tau)}(g_{f(i,j)}\omega) \, d\mu(\omega).
\]

Disintegrating the measure \( \mu = d\theta \, d\tilde{\mu} \) on \( SO(2) \times (H/SO(2)) \) and increasing to a full \( SO(2) \) orbit

\[
(4.6) \leq C_{\delta,\tau} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{-\frac{1}{1+4}} \int_{SO(2) \cdot W_{i}/SO(2)} \int_{I \cdot W_{i}} V_{\delta}^{(\tau)}(g_{f(i,j)} \omega) \, d\theta \, d\tilde{\mu}(\omega)
\]

\[
\leq C_{\delta,\tau} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{-\frac{1}{1+4}} \int_{SO(2) \cdot W_{i}/SO(2)} ce^{-(1-\delta)(f(i,j))} V_{\delta}^{(\tau)}(\omega) + b_{\tau} |I| \, d\tilde{\mu}(\omega)
\]

(by Corollary 4.11 and monotonicity of \( \psi \), \( f(i, j) \geq \log(b^{j-i}) = \ell_{0}(j - i) > \ell_{0} \))

\[
= C_{\delta,\tau} \left( \frac{\psi(b^{j})}{2\sigma} \right)^{-\frac{1}{1+4}} \left( ce^{-(1-\delta)(f(i,j))} \int_{W_{i}} V_{\delta}^{(\tau)}(\omega) \, d\mu(\omega) + b_{\tau} |I| \mu(W_{i}) \right).
\]

(since \( SO(2) W_{i} \) is \( SO(2) \)-invariant)
Shrinking rates of horizontal gaps for generic translation surfaces

Since all holonomy vectors in $W_i$ are contained in the circle we apply the upper bound for $V_\delta^{(\tau)}$

\[(4.6) \leq C_{\delta,\tau} \left( \frac{\psi(b)}{2\sigma} \right)^{\frac{1+\delta}{1-\delta}} \mu(W_i) \left( e^{-((1-\delta)/(f(i,j)))} \frac{C_{\delta,\tau}}{c_H} \left( \frac{\psi(b)}{\sigma} \right)^{\frac{1+\delta}{1-\delta}} + b_\tau |I| \right) \]

\[
\leq \mu(\tilde{B}_i) \left( \frac{c}{c_H} e^{-(1-\delta)f(i,j)} C_{\delta,\tau}^2 \frac{1+\delta}{1-\delta} \left( \frac{2\sigma}{\psi(b)} \right)^{\frac{1+\delta}{1-\delta}} + b_\tau |I| C_{\delta,\tau} \left( \frac{\mu(C_j)}{m} \right)^{\frac{1+\delta}{1-\delta}} \right),
\]

(since $W_i \subseteq \tilde{B}_i$)

Our goal is to compare the equation on the right hand side to the volume of $C_j$. Note by the construction of the MSV measure, there is a constant $m$ so that

\[\mu(C_j) = \mu(S(c_H, \sigma, 2')) \geq m \left( \frac{\sqrt{2\sigma}}{\psi(b')} \right)^2 = \left( \frac{2\sigma}{\psi(b')} \right)^2.
\]

Thus we have

\[\mu(B_j \cap C_j) = \mu(\tilde{B}_j \cap \tilde{C}_j)
\]

\[\leq \mu(\tilde{B}_j) \left( \frac{c}{c_H} C_{\delta,\tau}^2 \frac{1+\delta}{1-\delta} e^{-(1-\delta)f(i,j)} \left( \frac{\psi(b)}{\psi(b')} \right)^{\frac{1+\delta}{1-\delta}} + b_\tau |I| C_{\delta,\tau} \left( \frac{\mu(C_j)}{m} \right)^{\frac{1+\delta}{1-\delta}} \right),
\]

(by the definition of $f(i,j)$)

\[\leq \mu(\tilde{B}_j) \left( \frac{c}{c_H} C_{\delta,\tau}^2 \frac{1+\delta}{1-\delta} 2^{-((j-i)/(1-\delta))} + b_\tau |I| C_{\delta,\tau} \left( \frac{\mu(C_j)}{m} \right)^{\frac{1+\delta}{1-\delta}} \right),
\]

(since $\psi(R)$ is a nondecreasing sequence and $i < j, \psi(b') \leq \psi(b')$)

\[\leq \mu(\tilde{B}_j) \left( c_1 2^{-(j-i)/(1-\delta)} + c_2 \mu(C_j) \right)^{\frac{1+\delta}{1-\delta}} .
\]

(defined $c_1 = \frac{c}{c_H} C_{\delta,\tau}^2 \frac{1+\delta}{1-\delta}$ and $c_2 = \frac{b_\tau |I| C_{\delta,\tau}}{m^{\frac{1+\delta}{1-\delta}}}$)

Picking $C > \max\{c_1, c_2\}$ we obtain the desired inequality.

**Acknowledgements** We would like to thank Jayadev Athreya, Osama Khalil, and Howard Masur for useful discussions. We also thank the anonymous referee for improving the readability of the paper.

**Conflict of interest**

The authors declare that they have no conflict of interest.
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