Triaxial shape and rotation phases and transitions predicted by cranking models for light nuclei

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Abstract
Using Hartree-Fock variational method, we derive a quantal, microscopic cranking model Hamiltonian for triaxial rotation (MSCRM3) from the application of a rotation operator (with appropriately-chosen rotation angles) to a deformed nuclear ground-state wavefunction. The derived MSCRM3 Hamiltonian is identical in form to that of the conventional cranking model (CCRM3) except for residual correction terms, some of them are negligibly small. In the MSCRM3, the angular velocity is not a constant parameter, but is microscopically determined resulting in a self-consistent, and time-reversal and $D_2$ invariant Hamiltonian. The governing equations of the MSCRM3/CCRM3 are determined in closed algebraic forms, and solved iteratively using the MSCRM3/CCRM3 Newton’s equations of motion and Feynman’s theorem. The MSCRM3/CCRM3 are used to investigate the stability of the rotational states and nuclear shape and rotation mode transitions in the light nuclei $^{20}\text{Ne}$, $^{24}\text{Mg}$, and $^{28}\text{Si}$. The impact of spin-orbit and residuals of the square of the angular momentum and quadrupole-quadrupole interaction is examined. It is shown that the MSCRM3 predicts rotational relaxation of the intrinsic system, uniform rotation, triaxial rotation of a triaxial nucleus, observed reduced rotational-energy-level spacing due to wobbling quenching, and nutating, figure-skater’s slow-fast spinning, and tumbling rotations, whereas CCRM3 does not, uniform and figure-skating cases excepted. The MSCRM3 rotational states are found to be more unstable than those of the CCRM3. The MSCRM3 also rules out a free self-sustaining rotation.

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Keywords: triaxial rotation; closed-form 3D cranked deformed-oscillator solution; impact of spin-orbit and residual two-body interaction and square of angular momentum; strong coupling of intrinsic and collective rotational motions; positive and negative intrinsic-rotation feedback mechanisms; unstable cranked rotational states; planar, wobbly, principal-axis, uniform, non-uniform tilted-axis, nutating, tumbling, figure-skater’s rotations; rotational-band termination; time-reversal and $D_2$ invariance; rigid-flow angular momentum constraint; optimally intrinsic rotor Hamiltonian; corrections to the conventional cranking model; rotation relaxation; shape and rotation-mode transitions; wobbly rotation quenching, reduced yrast energy-level spacing in $^{20}\text{Ne}$; self-sustaining rotation

1. Introduction
It can be argued [1,2,3,4] that a reasonable separation of intrinsic and collective rotational motions as envisioned in the Bohr rotational model [5-8] cannot be achieved from first principles except under extreme conditions. Therefore, the nuclear shell model and the conventional cranking model combined with the projection methods seem to be the most fruitful approaches for the description of the nuclear rotational motion. The conventional cranking model for collective rotation about a single principal axis is frequently used to investigate the observed nuclear rotational properties (such as rotational bands, Coriolis anti-pairing, angular momentum...
alignment, back-bending, band termination, etc.) [9-34]. Other nuclear rotational features (such as precession, wobbling, tilted rotation, etc.), which involve a rotation about an arbitrary axis, have been studied [15,34-49] using a straightforward triaxial generalization of the uniaxial cranking model and solving numerically the model Schrödinger equation for a self-consistent potential energy. It is generally recognized [7,8,50] that the cranking model is phenomenological, semi-classical, and time-reversal non-invariant since the three components of the angular-velocity vector in the model are constant parameters. There have been many investigations [8,15,50-79] that derive the model from first principles to reveal the implicit model assumptions and approximations, using various methods (such as canonical transformations, angular-momentum constrained Hartree-Fock, time-dependent Hartree-Fock, time-dependent Lagrangian variation, angular momentum projection, generator co-ordinate, RPA, and density matrix, etc.) and approximations (such as expansion in powers of angular momentum operator, large deformation, using lowest order density matrix, etc). A number of studies have used the uniaxial [7,14,15,16,25,80,81,82] and triaxial [35-38,40,41,42,44] cranking models with a self-consistent deformed harmonic-oscillator potential to predict general rotational properties of nuclei to obtain physical insight in a physically transparent way using respectively analytical and numerical solution methods.

In Section 2 of this article, we use the solution of the Newton’s equation of motion of the CCRM3 with a self-consistent deformed harmonic-oscillator potential energy\(^1\) and Feynman’s theorem [83] to determine the expectation of the operators in CCRM3. The orientation angles of the angular-velocity vector and the self-consistent oscillator-potential frequencies are determined from the usual minimization of intrinsic energy with respect to the angels and the oscillator frequencies subject to a constant-volume condition. The resulting coupled CCRM3 algebraic equations are solved iteratively for each desired value of the angular-momentum quantum number \(J\) using the magnitude of the angular-velocity vector as an-adjustable parameter to achieve the desired \(J\).

Section 3 presents the predictions of CCRM3 for the light nuclei \(^{20}\)Ne, \(^{24}\)Mg, and \(^{28}\)Si.

In Section 4, we derive from first principles a microscopic, self-consistent, time-reversal and \(D_2\) invariant cranking model for triaxial rotation (MSCRM3) in two steps, as described in detail in [78,79]. In the first step, we apply a rotationally-invariant rotation operator to a deformed nuclear ground state to derive, from the nuclear Hamiltonian, a microscopic optimally-intrinsic rotor Hamiltonian\(^2\). The three rotation angles in the rotation operator are chosen to satisfy rigid-
flow velocity field conditions, which eliminate any explicit coupling of the angular-momentum
operators to any intrinsic operators yielding a tractable rotor Hamiltonian that is quadratic in the
angular-momentum operators. In the second step, we apply HF variational and second
quantization methods to the rotor Hamiltonian to derive MSCRM3 as the mean-field part of the
rotor Hamiltonian, and residual correction terms associated with the square of the angular-
momentum operators and two-body nucleon interaction, and negligibly small terms in product of
quadrupole-moment and angular-momentum operators, and angular velocities. MSCRM3 is a
microscopic generalization of CCRM3 and of the Thouless-Valatin semi-classical model for
uniaxial rotation [50]. The MSCRM3 and CCRM3 Schrodinger equations look identical except
that the angular velocity in MSCRM3 is not a free parameter but is determined by a microscopic
collective rigid-flow velocity field.

Section 5 presents the solution of MSCRM3 Schrödinger equation and its predictions for
\( ^{20}\text{Ne} \), \( ^{24}\text{Mg} \), and \( ^{28}\text{Si} \). In Section 5, we also investigate the impact of spin-orbit and residuals of
the square of the angular momentum operator and quadrupole-quadrupole interaction.

Section 6 summarizes the model development, predictions, and conclusions, and presents
future plans.

2. Derivation of coupled algebraic CCRM3 equations and their solution

The analysis in [35] solved numerically the CCRM3 Schrödinger equation, for a deformed
harmonic oscillator potential, to study triaxial rotation of an ellipsoid \( ^{24}\text{Mg} \). The analyses in
[41,44] also solved numerically the equation of the harmonic-oscillator CCRM3 and studied
stability of tilted rotation of nuclei over a range of mass numbers. The analysis in [44] concluded
that in even-even nuclei tilted rotations occur if and only if the nucleus has a triaxial shape in its
ground state. This conclusion is compared with the predictions of CCRM3 in Section 3.

Reference [36] used a tilted deformed harmonic-oscillator potential in CCRM3 to show that
tilted rotation is unstable, but it assumed that the off-diagonal frequencies in the tilted potential
can be varied independently of the diagonal frequencies. This assumption is not supported by our
results in [79, Eqs (42) and (43) page 11]. Nevertheless, the CCRM3 predictions in Section 3
show that tilted rotation is generally unstable, with the exception of uniform planar rotations.
Assuming a uniform rotation and using an isotropic harmonic-oscillator potential combined with
HF mean-field part of the separable quadrupole-quadrupole two-body interaction, Reference [38]
claims to show the existence of a time-dependent HF tilted rotation. This result seems to support
the MSCRM3 result in Section 5 that only a planar rotation can explain the observed yrast-line
curvature in \( ^{20}\text{Ne} \).

Instead of solving numerically the CCRM3 Schrödinger equation combined with the self-
consistency and constant volume equations, we reduce, in this section, these equations to coupled
algebraic equations and solved them iteratively as follows. We first solve the deformed-
harmonic-oscillator CCRM3 Newton’s equation of motion and determine the three normal-mode
frequencies \( \alpha_k \) using the CCRM3 Hamiltonian:

\[
\text{intrinsic, (iii) from the rotor Hamiltonian we derive a self-consistent time-reversal and } D_2
\text{ invariant cranking model Hamiltonian supplemented by residuals of the square of the angular-momentum operator and two-body interaction, and negligibly small terms in product of quadrupole-moment, angular-momentum operators, and angular velocities, and (iv) we model triaxial rotation.}
\]

\[3\] There have been also analyses of the conventional cranking model for triaxial rotation and tilted and wobbly
rotation using more realistic potentials including two-body interactions [27,38,42,47].

\[4\] It is much easier and more transparent to use this approach than numerically diagonalize the Hamiltonian matrix
obtained from Eq (1) in the phase space [84] coupled to the self-consistency equations as done in [44].
\[ \hat{H}_{cr} = \hat{H}_o - \vec{\Omega} \cdot \vec{J} \equiv \sum_{n=1}^{A} \hat{h}_n(n) \equiv \sum_{n=1}^{A} \left[ \hat{h}_n(n) - \vec{\Omega} \cdot \vec{j}(n) \right] \]  

where \( \hat{h}_n(n) \) and \( \vec{j}(n) \) are respectively the \( n \)th particle deformed harmonic-oscillator Hamiltonian and angular-momentum vector operator given by:

\[ \hat{h}_o \equiv \sum_{k=1}^{3} \left( \frac{p_k^2}{2M} + \frac{M}{2} \omega_k^2 x_k^2 \right), \quad \vec{j} \equiv \vec{x} \times \vec{p} \] 

and \( \vec{\Omega} \) is the angular velocity vector with the components, magnitude, and orientation given by:

\[ \vec{\Omega} = (\Omega_x, \Omega_y, \Omega_z) = \Omega \left( \sin \theta \cdot \cos \phi, \sin \theta \cdot \sin \phi, \cos \theta \right), \quad \Omega^2 = \Omega_x^2 + \Omega_y^2 + \Omega_z^2 \]  

The time evolution of a particle co-ordinate and momentum for the cranked Hamiltonian \( \hat{h}_{cr} \) is given in the Heisenberg representation by:

\[ \vec{x}(t) = e^{i\hat{h}_{cr}t/\hbar} \vec{x} e^{-i\hat{h}_{cr}t/\hbar} \]  

Substituting Eqs (1) and (2) into Eq (4), we obtain the particle acceleration or Newton’s equation of motion:

\[ M \frac{d^2}{dt^2} x_k(t) = -M \omega_k^2 x_k(t) - 2M \left( \vec{\Omega} \times \frac{d}{dt} \vec{x}(t) \right) + M \left( \vec{\Omega} \times (\vec{\Omega} \times \vec{x}(t)) \right) \]  

A periodic solution in the form \( \vec{x}(t) = \vec{x}_o e^{i\alpha t} \) of the three coupled Eq (5) gives three coupled algebraic equations for \( \vec{x}_o \). These equations have non-trivial solutions if the following cubic characteristic equation is satisfied:

\[ \alpha^6 - A_o \alpha^4 + B_o \alpha^2 - C_o = 0 \]  

where:

\[ A_o = \sum_{k=1}^{3} (\omega_k^2 + 2 \Omega_k^2), \quad B_o = \sum_{k=1}^{3} \left( \omega_k^2 \omega_j^2 - \omega_j^2 \Omega_k^2 \right) + 2 \sum_{k=1}^{3} \omega_k^2 \Omega_k^2 + \left( \sum_{k=1}^{3} \Omega_k^2 \right)^2 \] 

\[ C_o = \omega_1^2 \omega_2^2 \omega_3^2 - \omega_1^2 \omega_2^2 \Omega_3^2 - \omega_1^2 \omega_3^2 \Omega_2^2 - \omega_2^2 \omega_1^2 \Omega_3^2 - \omega_2^2 \omega_3^2 \Omega_1^2 - \omega_3^2 \omega_1^2 \Omega_2^2 - \omega_1^2 \omega_2^2 \Omega_3^2 - \omega_1^2 \omega_3^2 \Omega_2^2 - \omega_2^2 \omega_3^2 \Omega_1^2 + \omega_2^2 \Omega_1^2 \Omega_3^2 + \omega_3^2 \Omega_1^2 \Omega_2^2 + \omega_1^2 \Omega_2^2 \Omega_3^2 + \omega_1^2 \Omega_3^2 \Omega_2^2 + \sum_{k=1}^{3} \omega_k^2 \Omega_k^4 \]  

The three roots of Eq (6) are the normal-mode frequencies \( \alpha_k \), which are readily obtained in closed forms as functions of \( A_o, B_o, \) and \( C_o \) from the literature.

The intrinsic energy is then given by:

\[ E_{\text{int}} \equiv \langle H_{cr} \rangle = \frac{\hbar}{2} \sum_{k=1}^{3} \alpha_k \Sigma_k, \quad \Sigma_k \equiv \sum_{n=1}^{n_{ik}} (n_k + 1) \]  

where \( \langle H_{cr} \rangle \equiv \langle |H_{cr}| \rangle \) is a CCRM3 ground-state rotational-band eigenfunction, \( \Sigma_k \) is the

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\(^5\text{We have not included the } \tilde{t} S \text{ and } t^2 \text{ interaction terms from the Nilsson’s model in } \hat{h}_o \text{ because, as we show in Section 5, the } t^2 \text{ term is negligibly small for sd-shell nuclei [8 pages 133 and 137, 21, 25 page 217]. The } \tilde{t} S \text{ term increases the excitation energy (by about 10%) improving the agreement with the measurement, but otherwise it does not affect significantly the other predicted results.}\)
ground-state nucleon oscillator-shell total phonon occupation number along the $k$ axis, and $n_{kF}$ is the number of oscillator quanta along the $k$ axis at the Fermi surface.

We then use Eqs (1), (2), (6) to (9) and the Feynman’s theorem [8,16,83,84] to derive closed-form algebraic expressions for the expectation of the following CCRM3 operators with respect to $\hat{J}$:

$$\langle \hat{J}_k \rangle = -\frac{\partial E_{\text{int}}}{\partial \Omega_k} = -\hbar \sum_{l=1}^{3} \frac{\partial \alpha_l}{\partial \Omega_k} \Sigma_l,$$

$$\frac{M}{2} \langle x_k^2 \rangle = \frac{\partial E_{\text{int}}}{\partial \omega_k^2} = \hbar \sum_{l=1}^{3} \frac{\partial \alpha_l}{\partial \omega_k} \Sigma_l \tag{10}$$

Eqs (10) must also satisfy the constant nuclear volume and the density-potential shapes self-consistency conditions. As is normally done, these conditions imply, in the HF mean-field sense, a minimization of $E_{\text{int}}$ with respect to the oscillator frequencies $\omega_k^2$, yielding the results:

$$\omega_k^2 \langle x^2 \rangle = \omega_z^2 \langle y^2 \rangle = \omega_y^2 \langle z^2 \rangle,$$

$$\omega_x \omega_y \omega_z = \omega_o^3 \tag{11}$$

where $\omega_o$ is the isotropic nuclear harmonic-oscillator frequency: $\hbar \omega_o = PA^{-1/3}$ with $P = 35.43$ for light nuclei and $P = 41$ for heavy nuclei.

$\tilde{Q}$ in Eq (3) is determined in the usual manner as follows. The angles $(\theta, \phi)$ are chosen to minimize the intrinsic energy $E_{\text{int}}$, whence we obtain:

$$\tan \phi = \frac{\langle \hat{J}_2 \rangle}{\langle \hat{J}_1 \rangle}, \quad \tan \theta = \frac{\langle \hat{J}_1 \rangle \cdot \sqrt{1 + \tan^2 \phi \langle \hat{J}_3 \rangle}}{\langle \hat{J}_3 \rangle} \tag{12}$$

The magnitude $\Omega$ of $\tilde{Q}$ is adjusted to satisfy the condition that the wavefunction $\langle \hat{J}^2 \rangle$ yields the desired value of $\langle \hat{J} \rangle$ for a given value of the angular-momentum quantum number $J$ as follows:

$$\langle \hat{J}^2 \rangle = \langle \hat{J}_1^2 \rangle + \langle \hat{J}_2^2 \rangle + \langle \hat{J}_3^2 \rangle = \hbar^2 J(J + 1), \quad \text{for a triaxial rotation} \tag{13}$$

$$\langle \hat{J}^2 \rangle = \langle \hat{J}_1^2 \rangle + \langle \hat{J}_2^2 \rangle + \langle \hat{J}_3^2 \rangle = \hbar^2 J^2, \quad \text{for a principal-axis rotation} \tag{14}$$

We observe that, for the same $J$ value, the $\langle \hat{J}^2 \rangle$ value obtained from Eq (13) is higher than that from Eq (14), and hence so are the corresponding excitation energies.

The rotational-band excited-state ($E_J$) and excitation ($\Delta E_J$) energies are then determined from:

$$E_J \equiv \langle \hat{H}_o \rangle \equiv \langle \hat{H}_{\text{cr}} + \tilde{Q} \cdot \hat{J} \rangle = E_{\text{int}} + \tilde{Q} \cdot \langle \hat{J} \rangle, \quad \Delta E_J \equiv E_J - E_{J=0} \tag{15}$$

3. Predictions of CCRM3 for $^{20}\text{Ne}$, $^{24}\text{Mg}$, $^{28}\text{Si}$

The coupled algebraic Eqs (6), (9), (10), (11), (12), (13), and (14) are solved iteratively to calculate $\Omega_k$, $\langle \hat{J}_k \rangle$, $\alpha_k$, $\omega_k^2$, $\langle x_k^2 \rangle$, $E_J$, and $\Delta E_J$ over the entire allowed range of $J$ in $^{20}\text{Ne}$, $^{24}\text{Mg}$, and $^{28}\text{Si}$.

Two general conclusions that apply to all the rotational states predicted by CCRM3 are as follows.
The first conclusion is that the orientation \( (\theta, \phi) \) of \( \vec{\Omega} \) in Eq (12) is the same as the orientation \( (\theta_j, \phi_j) \) of \( \langle \vec{J} \rangle \) defined by:

\[
\tan \phi_j = \frac{\langle J_2 \rangle}{\langle J_1 \rangle}, \quad \tan \theta_j = \frac{\sqrt{\langle J_1 \rangle^2 + \langle J_2 \rangle^2}}{\langle J_3 \rangle} = \frac{\langle J_1 \rangle}{\langle J_3 \rangle} \sqrt{1 + \frac{\langle J_2 \rangle^2}{\langle J_1 \rangle^2}}
\]

Therefore, in CCRM3, the rotation is always uniform as concluded in \([27,72]\), i.e., the vector \( \langle \vec{J} \rangle = (\langle J_1 \rangle, \langle J_2 \rangle, \langle J_3 \rangle) \) is always parallel to the vector \( \vec{\Omega} \), and it can be along any one of the principal axes or along a tilted \( \vec{\Omega} \) axis. Eq (12) shows that as \( \langle J_k \rangle \) varies with \( J \), so do \( \phi \) and \( \theta \). We indicate by \( \phi_o \) and \( \theta_o \) the values of \( \phi \) and \( \theta \) at the start of the iterative solution procedure of the coupled algebraic CCRM3 equations.

The second conclusion is that the CCRM3 rotational states are inherently unstable because of a positive feedback mechanism between the variations in \( \Omega_k \) and \( \langle J_k \rangle \). This feedback is inferred from the CCRM3 predicted variation of the excited-state energy \( E_J \) in Eq (15) and the intrinsic energy \( E_{int} \) in Eq (9) with \( J \). Fig 1 shows that typically \( E_J \) increases and \( E_{int} \) decreases with increasing \( J \). This result implies that the increase in the rotational energy is supplied by a corresponding decrease in the intrinsic energy through changes in the intrinsic kinetic and deformation energies. Since, in Fig 1, \( \frac{\partial E_{int}}{\partial \Omega_k} < 0 \), and because \( \Omega_k \) increases with \( J \), we conclude, from Eq (10) that \( \langle J_k \rangle = -\frac{\partial E_{int}}{\partial \Omega_k} > 0 \), and hence \( \langle J_k \rangle \) decreases when \( \Omega_k \) decreases. Therefore, any decrease in \( \langle J_k \rangle \) causes \( \Omega_k \) to decrease, which in turn causes \( \langle J_k \rangle \) to decrease further thereby reducing \( \langle J_k \rangle \) to zero unless this free fall is prevented by some constraint. Therefore, the CCRM3 rotational states are inherently unstable because of the strong coupling between the rotational and intrinsic motions, and the resulting positive feedback mechanism between \( \langle J_k \rangle \) and \( \Omega_k \). This result implies that a free self-sustaining (stable) cranked rotational state is not possible, which is demonstrated using MSCRM3 in Section 4, footnote 11.

The strong coupling between the rotational and intrinsic motions and hence the rotational-state instability is caused by the self-consistency condition, which couples the oscillator-potential frequencies and the quadrupole moments (as in Eq (11)). Therefore, we expect this instability for any self-consistent mean-field potential and not just for the oscillator potential.

A CCRM3 rotational state does not rotationally collapse totally (i.e., all three \( \langle J_k \rangle \) components decrease to zero for a given non-zero \( J \)) because of the externally imposed constraint on \( \langle J^2 \rangle \) in Eqs (13) and (14). For this reason, \( \langle J_3 \rangle \) normally reduces to zero and hence \( \theta \) increases to 90° after a few iteration steps at \( J = 2 \) or at a higher \( J \), resulting in a planar uniform rotation. Then, for any small decrease in, for example, \( \langle J_2 \rangle \) in the iteration process, the positive
feedback causes \( \langle \hat{J}_z \rangle \) and hence \( \phi \) to monotonically decrease to zero and \( \langle \hat{J}_1 \rangle \) to increase to \( J \) in subsequent iteration steps to satisfy the wavefunction angular-momentum constraint in Eq (13) or (14), resulting in a uniform rotation about the principal \( x \) axis. Similarly, any small decrease in \( \langle \hat{J}_1 \rangle \) results in a monotonic decrease in \( \langle \hat{J}_1 \rangle \) and an increase in \( \langle \hat{J}_2 \rangle \) and hence \( \phi \), resulting in a uniform rotation along the principal \( y \) axis. An exception to these two cases is when the planar uniform rotation is along the \( \phi = 45^\circ \) line where \( \langle \hat{J}_1 \rangle = \langle \hat{J}_2 \rangle \), in which case the planar uniform rotation is stable in CCRM3 (but not in MSCR3 as shown in Section 5). Thus, Eq (13) or (14) does not allow all three components \( \langle \hat{J}_k \rangle \) to simultaneously decrease to zero. Therefore, we may consider Eq (13) or (14) to provide a negative feedback mechanism and counter the positive feedback mechanism, and stabilize the CCRM3 rotational state. However, unlike the positive feedback mechanism, which is inherently characteristic of the intrinsic system, the negative feedback mechanism is externally imposed by manually adjusting \( \Omega \) until either Eq (13) or Eq (14) is satisfied.

From the above results, we conclude that CCRM3 predicts only planar (in the \( x \)-\( y \) plane, i.e. \( \theta = 90^\circ \)) uniform rotation including uniform rotation along \( x \) or \( y \) axis, and it does not predict any triaxial rotations where none of \( \langle \hat{J}_k \rangle \) is zero.

For the \( ^{20}\text{Ne} \) ground state, the deformed harmonic-oscillator single-particle model predicts an axially-symmetric prolate shape with a nucleon configuration having the total oscillator-phonon quantum numbers: \( \Sigma_k = (14,14,22) \) [16,25,85,86,87]. For \( ^{24}\text{Mg} \) ground state, it predicts an axially-symmetric prolate shape with \( \Sigma_k = (20,20,24) \), and a triaxial shape with \( \Sigma_k = (16,20,28) \) [25,85,86,87]. For \( ^{28}\text{Si} \) ground state, it predicts a prolate axially-symmetric shape with \( \Sigma_k = (22,22,34) \), an oblate axially-symmetric shape with \( \Sigma_k = (30,30,18) \), and two ellipsoidal shapes with \( \Sigma_k = (26,22,30) \) and \( \Sigma_k = (30,26,22) \) [85,86,88,89]. For these nucleon configurations and using an iterative solution of the CCRM3 coupled algebraic Eqs (6), (9), (10), (11), (12), (13), and (14), the CCRM3 predictions for \( ^{20}\text{Ne} \), \( ^{24}\text{Mg} \), and \( ^{28}\text{Si} \) are as follows.

For prolate \( ^{20}\text{Ne} \), \( \phi_0 = 45^\circ \) and \( \phi_0 > 0^\circ \) (\( \theta \) increases to \( 90^\circ \) in the first few iteration steps), CCRM3 predicts a planar uniform rotation in the \( x \)-\( y \) plane that is symmetric about the \( z \) axis at all \( J \), and no band termination. For \( \phi_0 \neq 45^\circ \) and \( \phi_0 > 0^\circ \) (\( \theta \) increases to \( 90^\circ \) in the first few iteration steps), CCRM3 predicts planar uniform rotation in the \( x \)-\( y \) plane at \( J = 2 \), and uniform rotation along either the \( x \) axis (for \( \phi_0 < 45^\circ \)) or the \( y \) axis (for \( \phi_0 > 45^\circ \)) at \( J \geq 4 \). At \( J = 8 \), \( ^{20}\text{Ne} \) becomes axially symmetric about these axes and the rotational band terminates (as in the uniaxial CCRM model [7,14,16,25,78,80,81,82,90,91]). CCRM3 predicts that the value \( J = 8 \) is given by the difference \( \Sigma_3 - \Sigma_1 \) between the oscillator phonons each with a unit angular momentum:

\[
J = \langle \hat{J}_1 \rangle / \hbar = -\frac{\partial E_{\text{tot}}}{\partial \Omega} = -\sum_{i=1}^{3} \frac{\partial \alpha_i}{\partial \Omega} \Sigma_i = -[1 \Sigma_1 + 0 \Sigma_2 + (-1) \Sigma_3] = (\Sigma_3 - \Sigma_1) = (22 - 14) = 8
\]

In this case, the difference \( \Sigma_3 - \Sigma_1 \) is also equal to the number of valence nucleons (which is 4) multiplied by 2, which is the maximum angular momentum of each nucleon in the valence \( d \) sub-shell, giving the number 8. This last equality is not valid for some of the other cases considered below. The predicted/measured excitation energies and quadrupole moments are: \( \Delta E_2 = \)
0.948/1.6338, \( \Delta E_x = 3.254/4.247 \), \( \Delta E_y = 7.273/8.775 \), \( \Delta E_z = 14.956/11.948 \) (MeV), and \( Q_2 = 54.702 \), \( Q_4 = 50.822 \), \( Q_6 = 43.409 \), \( Q_8 = 41.721 \) (e·fm\(^2\))\(^6\). The excitation energy is underpredicted by about 1 MeV at \( J < 8 \) and overpredicted by about 3 MeV at \( J = 8 \). The underprediction occurs mainly because the residuals of the two-body interactions, particularly the residual of the square of the angular momentum operator, are neglected (as indicated by the estimation of their impact in Section 5). The overprediction occurs mainly because fluctuations in the angular velocity are neglected (as indicated by their impact estimated in footnote 6). The predicted excitation energies are significantly lower than the measured energies (1.39, 4.123, 8.113, and 13.213 MeV) at \( J > 4 \). At \( \phi_o \neq 45^\circ \), the excitation energies are quite low: \( \Delta E_z = 0.537 \) and \( \Delta E_y = 2.17 \) MeV, and the band terminates at \( J = 4 \) (predicted to be given by \( (\Sigma_j - \Sigma_i) = (24 - 20) = 4 \), which is smaller than that obtained from the maximum alignment of the angular momenta of either 4 or 8 valence nucleons in the \( d \) sub-shell) and the experimentally observed \( J = 8 \). We conclude that, for prolate \( ^{24}\text{Mg} \), the oscillator CCRM3 model predicts excitation spectra that are unrealistic and none of them resemble those predicted by some of the models that use cranking HF method with angular momentum projection and various nuclear interactions [92,93,95]. The oscillator CCRM3 prediction of uniform principal-axis rotation of triaxial \( ^{24}\text{Mg} \) is an example of a nucleus that is triaxial in its ground state but does not exhibit any triaxial or tilted-axis rotation. This result may not be in conflict with the result in [44] because [44] used only one kind of nucleons (since [44] was more concerned with determining local and global minima of \( E_{\text{int}} \) in the space of \( (\Omega, \theta, \phi) \) and mass

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\(^6\) The spin-orbit interaction increases the predicted excitation energies (by about 10%) so that they are in a better agreement with the measurement, but otherwise it does not change significantly the other predicted results, as shown in Section 5.
number), and we have found the results to be sensitive to the number of nucleons since it affects the values of $\Sigma_k$.

For prolate $^{28}$Si, $\phi_o=45^\circ$, and $\theta_o>0^\circ$ ($\theta$ increases to $90^\circ$ in the first few iterations), CCRM3 predicts an axially-symmetric planar uniform rotation along the $45^\circ$ line in the $x$-$y$ plane of an axially symmetric (about $z$ axis) nucleus and no band termination, unlike the result in [94] where the band is predicted to terminate at $J=12$. The predicted/measured excitation energies are: $\Delta E_2= 0.55/1.78$, $\Delta E_4= 1.89/4.62$, $\Delta E_6= 3.945/8.54$, $\Delta E_8= 6.99/13.21$, $\Delta E_{10}= 11.33/18.5$, $\Delta E_{12}= 18.19/26.8$, … (MeV). For $\phi_o \neq 45^\circ$ and $\theta_o>0^\circ$ ($\theta$ increases to $90^\circ$ in the first few iterations), CCRM3 predicts a nearly axially-symmetric nucleus (about $z$ axis) and a planar uniform rotation along the $45^\circ$ line in the $x$-$y$ plane up to $J=4$. During the iteration at $J=6$, the nucleus becomes triaxial, and $\phi$ continues to decrease (for $\phi_o < 45^\circ$) and to increase (for $\phi_o > 45^\circ$) causing the rotation to eventually become uniform along the $x$ axis and $y$ axis respectively at $J=10$. At $J=12$, the nucleus becomes axially symmetric about the $x$ axis for $\phi_o < 45^\circ$ and the band terminates with the excitation energies: $\Delta E_2= 0.55/1.78$, $\Delta E_4= 1.84/4.62$, $\Delta E_6= 3.87/8.54$, $\Delta E_8= 6.72/13.21$, $\Delta E_{10}= 9.48/18.5$, $\Delta E_{12}= 14.06/26.8$ (MeV). But for $\phi_o > 45^\circ$ the rotational band along the $y$ axis does not terminate and nucleus remains triaxial with the excitation energies: $\Delta E_2= 0.55/1.78$, $\Delta E_4= 1.84/4.62$, $\Delta E_6= 4.12/8.54$, $\Delta E_8= 9.71/13.21$, $\Delta E_{10}= 9.47/18.5$, $\Delta E_{12}= 14.19/26.8$, … (MeV). Note that for $\phi_o < 45^\circ$ the band terminates at $J=12$, which is predicted to be given by $(\Sigma_3^\prime-\Sigma_3)= (34-22)=12$. This $J$ value is smaller than that obtained from the maximum alignment of the angular momenta of 8 valence nucleons in the $d$ sub-shell each having 2 units of angular momentum. The excitation energies for $\phi_o < 45^\circ$ and $\phi_o > 45^\circ$ are similar except that for $\phi_o > 45^\circ$, excitation energy at $J=8$ is higher than that at $J=10$. This anomaly seems to arise from a peculiarity in the cranked harmonic oscillator frequencies. Note that for an initially prolate $^{28}$Si, CCRM3 does not predict any triaxial or tilted-axis rotation even when the nucleus becomes triaxial. The predicted excitation energies for prolate $^{28}$Si are significantly lower than the measured energies mainly because the residuals of the two-body interactions, particularly the residual of the square of the angular momentum operator, are neglected (as indicated by their impact estimated in Section 5, footnote 11).

For oblate $^{28}$Si, CCRM3 does not predict any collective rotational states for any value of $\phi_o$ and $\theta_o$, which may be expected since a collective rotation of an oblate shape is known to occur in conjunction with a collective rotation of a prolate shape [96]. Specifically, for $\theta_o = 0^\circ$ and for any value of $\Omega^2$, i.e., for a rotation about the symmetry $z$ (shortest) axis, CCRM3 predicts a non-collective individual-particle high (30 MeV or more) energy states with $\langle \hat{J}_1 \rangle= \langle \hat{J}_2 \rangle=0$, and $\langle \hat{J}_3 \rangle=-\hbar [0\Sigma_1+1\Sigma_2+(-1)\Sigma_3]=-\hbar (30-18)=-12\hbar$, which may be considered as the sum of 6 $d$-shell valence nucleons each with two units of angular momentum and aligned in the direction opposite to (positive value of) $\Omega^2$ (i.e., in the negative $z$-axis direction), as noted in [7 (pages 43,287), 8 (pages 102,142), 25 (pages 222), 37,97]. As $\Omega^2$ is increased, the nucleus becomes increasingly prolate and $\langle \hat{J}_3 \rangle$ remains mostly constant at $-12\hbar$ and intermittently increases or
decreases from -12 $\hbar$ and the nucleus intermittently becomes prolate and oblate. For $\theta_o = 90^\circ$ and $\phi_o = 0^\circ$ or $90^\circ$, i.e., for rotation along the $x$ or $y$ axis, CCRM3 predicts discontinuities in the derivative of the normal-mode frequency $\alpha_k^2$ with respect to $\Omega_1$ or $\Omega_2$, and negative values of $\langle \hat{J}_1 \rangle$ or $\langle \hat{J}_2 \rangle$ and positive values of $\Omega_1$ or $\Omega_2$ and large fluctuations in $\langle \hat{J}_1 \rangle$ or $\langle \hat{J}_2 \rangle$ between 0 and -12 $\hbar$. Therefore, we conclude that CCRM3 does not predict any collective rotation for oblate $^{28}\text{Si}$ nucleus.

For triaxial $^{28}\text{Si}$ at $J = 2$, and $\phi_o < 13.41^\circ$ and $13.41^\circ < \phi_o \leq 90^\circ$, and $\theta_o = 90^\circ$, $\langle \hat{J}_3 \rangle$ and $\Omega_3$ vanish in the first few iterations (the number 13.41$^\circ$ arises as a consequence of the particular asymmetry of $^{28}\text{Si}$ and the unstable nature of the CCRM3 rotational states) resulting in a planar rotation in the $x$-$y$ plane. Figs 2, 3 and 4 show the variations in the remaining variables in subsequent iterations. For $13.41^\circ < \phi_o \leq 90^\circ$ and after initial transitory oscillations in all the variables, $\langle \hat{J}_1 \rangle$ and $\Omega_1$ reduce to zero (because of the instability of the CCRM3 rotation states discussed above), and $\langle \hat{J}_2 \rangle$ increases to $J = 2$ and remains steady at this value thereafter. Therefore, the planar rotation transitions to a uniform rotation along the $y$ axis. $\Omega_2$, the expectation of the quadrupole moments $\langle x_k^2 \rangle$, and the excitation energy remain oscillatory. Since the rotation angular velocity $\Omega_2$ speeds up and slows down at fixed $\langle \hat{J}_2 \rangle = 2$, the rotation along the $y$ axis resembles a figure-skater’s fast and slow spinning. In this rotational mode, the skater speeds up and slows down the rotation at fixed (conserved – no torque) angular momentum by stretching and collapsing the arms. In the $^{28}\text{Si}$ case, the speeding and slowing rotation is achieved by oscillations in the moment of inertia about the $y$ axis, which is caused by the coupling between rotational motion and intrinsic harmonic-oscillator potential generated by the self-consistency condition. For $\phi_o < 13.41^\circ$, CCRM3 predicts a rotation behaviour similar to that for $13.41^\circ < \phi_o \leq 90^\circ$ with the exception that the rotation occurs along $x$ (and not the $y$) axis. At $J = 4$, the rotation along the $x$ and $y$ axes becomes steady, the nucleus becomes axially symmetric along these axes, and the band terminates. The excitation energy is oscillatory with the unrealistic average values $\Delta E_2 = 4.23/1.78$ and $\Delta E_2 = 1.54/4.62$ (MeV)$^7$. For $\theta_o \neq 90^\circ$, CCRM3 predicts rotation states where $\Omega_1$, $\Omega_2$, $\langle \hat{J}_1 \rangle$ and $\langle \hat{J}_2 \rangle$ vanish, and $\langle \hat{J}_3 \rangle$ and $\Omega_1$ point in opposite directions along the $z$ axis. From any initial value less than $90^\circ$, $\theta$ eventually decreases to zero resulting in a non-collective rotation along the symmetry $z$ axis, and with a high excitation energy that decreases with $J$. Therefore, CCRM3 does not predict any realistic rotation for triaxial $^{28}\text{Si}$ at any $J$ value.

We summarize the above results as follows.

For nuclei $^{20}\text{Ne}$, $^{24}\text{Mg}$, and $^{28}\text{Si}$ that are prolate axially-symmetric in their ground state, and for $\phi_o = 45^\circ$ and $\theta_o > 0^\circ$, CCRM3 predicts uniform planar rotation along the $45^\circ$ line in the $x$-$y$

---

7 This oscillatory rotational state may also be interpreted as describing an oscillation between two minimum-energy rotational states about 1 MeV apart similar to the two-neighbouring minimum-energy rotational states predicted in a 3-D cranked HFB with pairing plus $Q\bar{Q}$ interaction calculation [97].
plane with no band termination. For $\phi_{o} \neq 45^\circ$, CCRM3 predicts a planar uniform rotation at lower $J$ values (generally at $J = 2, 4$), and uniform rotation along either the $x$ (for $\phi_{o} < 45^\circ$) or $y$ (for $\phi_{o} > 45^\circ$) axis of a triaxial nucleus at higher $J$ values, which becomes a symmetry axis and the band terminates at $J = 8, 4$, and 12 for $^{20}\text{Ne}$, $^{24}\text{Mg}$, and $^{28}\text{Si}$ respectively. Considering planar rotation as a special case of triaxial rotation, we have here examples of nuclei that are axially symmetric in their ground state but exhibiting triaxial rotations. The predicted excitation energies are generally significantly lower than the measured energies mainly because the residuals of the two-body interactions, particularly the residual of the square of the angular momentum operator, are neglected (as indicated by their impact estimated in Section 5, footnote). For $^{20}\text{Ne}$, the excitation energy at $J = 8$ is overpredicted mainly because the fluctuations in the angular velocity are not modeled. These fluctuations are accounted for in MSCRM3 in Section 5.

CCRM3 predicts no collective rotation for oblate $^{28}\text{Si}$.

For triaxial $^{24}\text{Mg}$, CCRM3 predicts only uniform principal-axis rotation along the $x$ axis (for $\phi_{o} < 90^\circ$ and $\theta_{o} > 0^\circ$) with band termination at $J = 8$, and along the $y$ axis (for $\phi_{o} = 90^\circ$ and $\theta_{o} > 0^\circ$) with no band termination.

For triaxial $^{28}\text{Si}$ and $\theta_{o} = 90^\circ$, CCRM3 predicts figure-skater’s fast-slow rotation along the $x$ axis for $\phi_{o} < 13.41^\circ$ and along the $y$ axis for $13.41^\circ < \phi_{o} \leq 90^\circ$ at fixed $J = 2$ and steady rotation along these axes at $J = 4$, at which the band terminates when the nucleus becomes axially symmetric along the axes. For $\theta_{o} \neq 90^\circ$, CCRM3 predicts a $\theta$ that decreases to zero, and a non-collective rotation along the symmetry $z$ axis at $J = 2$ and 4 at which the band terminates, and an excitation energy that decreases with $J$. Therefore, CCRM3 predicts no realistic collective rotation for triaxial $^{28}\text{Si}$.

CCRM3 predicts no triaxial rotations for either triaxial $^{24}\text{Mg}$ or $^{28}\text{Si}$ nucleus, which may not be in conflict with the analysis result in [44] because [44] used only one kind of nucleons and the results are sensitive to the number of nucleons.
Fig 2: CCRM3 predicted $\varphi$ and $< J_k >$ versus iteration-step # for initial $\varphi_0 = 20^\circ$ at $J = 2$ in triaxial Si-28

transition to uniform rotation from planar

Fig 3: CCRM3 predicted angular velocities $\Omega_1$ and $\Omega_2$ for $J = 2$ and $\varphi_0 = 20^\circ$ for triaxial Si-28
4. Derivation of MSCRM3

In this section, we present briefly the derivation of the microscopic, quantum, self-consistent, time-reversal and $D_2$ invariant cranking model for triaxial rotation (MSCRM3) and residual corrections to it. More details are given in [78,79]. MSCRM3 is derived in two steps. In the first step, we apply the rotationally-invariant exponential rotation operator $e^{i\theta \cdot J/\hbar}$ to a deformed nuclear ground state $|\Phi_{gs}\rangle$ to obtain the rotated state:

$$ |\Phi (\theta)^{'}\rangle = e^{i\theta \cdot J/\hbar} \cdot |\Phi_{gs}\rangle $$

(16)

where $|\Phi_{gs}\rangle$ satisfies the nuclear Schrodinger equation:

$$ \hat{H}^o |\Phi_{gs}\rangle \equiv \left(\sum_{n=1}^{A} \frac{\hat{p}_n^2}{2M} + \frac{1}{2} \sum_{n,m} \hat{V} (|\vec{r}_n - \vec{r}_m|) \right) |\Phi_{gs}\rangle = E_{Jgs} |\Phi_{gs}\rangle $$

(17)

(for a rotationally invariant two-body interaction $\hat{V}$ or some approximation to it, with the spin, isospin, and exchange dependence left out for now). Substituting Eq (16) into Eq (17), we obtain the following microscopic, quantum, rotationally invariant, optimally-intrinsic rotor Schrodinger equation for triaxial rotation:

$$ \hat{H} |\Phi (\emptyset')\rangle \equiv e^{i\emptyset' \cdot J/\hbar} \cdot \hat{H}^o \cdot e^{-i\emptyset \cdot J/\hbar} \cdot |\Phi (\emptyset')\rangle $$

$$ \equiv \left(\hat{H}^o - \sum_{k=1}^{3} \frac{\hat{J}_k^2}{2M \cdot \hat{J}_k} - \frac{3}{2} \sum_{l \neq k=1}^{3} \frac{\hat{\Omega}_h \cdot \hat{J}_l \cdot \hat{J}_k}{2M \cdot \hat{J}_k} \right) |\Phi (\emptyset')\rangle = E_{Jgs} |\Phi (\emptyset')\rangle $$

(18)
where \( \hat{Q}_k = \sum_{n=1}^{A} x_{nl} \cdot x_{nk} \) is an element of the quadrupole-moment tensor, and \( \hat{J}_k \) is the rigid-flow moment of inertia tensor operator component defined in Eq (20). All the operator components appearing in Eq (18) and in the equations to follow, are along space-fixed (and not body-fixed) co-ordinate-system axes. We obtain the Hamiltonian \( \hat{H} \) in Eq (18) only when we use the following rigid-flow prescription for the three rotation angles \( \theta'_k \):

\[
\frac{\partial \theta'_k}{\partial x_{nj}} = -\sum_{k=1}^{3} i \chi_{jk} x_{nk}, \quad i \chi_{jk} = -i \chi_{kj} = 0 \quad \text{for} \quad j, k = l \quad (l, j, k = 1, 2, 3)
\]

where each of the three \( 3 \times 3 \) matrices \( \chi_k \) is real and anti-symmetric. Each of the three Eqs (19) adds a collective rigid-flow component to each nucleon velocity field. The prescription in Eq (19) renders Eq (18) quadratic in the total angular-momentum operators \( \hat{J}_k \). The non-zero elements of the matrices \( \chi_k \) are determined by choosing \( \theta'_k \) and \( \hat{J}_k \) to be canonically conjugate, and we obtain:

\[
\chi_{jk} = \frac{1}{\hat{J}_k}, \quad \hat{J}_1 = \sum_{n=1}^{A} (x_{nj}^2 + x_{nk}^2), \quad i, j, k = 1, 2, 3 \quad \text{and in cyclic order}
\]

where \( A \) is the mass number and \( M \hat{J}_k \) is the \( l \)th principal-axis component of the rigid-flow moment of inertia tensor in the space-fixed frame.

Eq (18) shows that, for each \( \theta'_k \) in Eq (16), the rotated deformed wavefunction \( |\Phi(\theta'_k)\rangle \) corresponds to the same ground-state energy \( E_{gs} \). That is, for all orientations \( \theta'_k \), the Hamiltonian \( \hat{H} \) and \( |\Phi(\theta'_k)\rangle \) describe degenerate (or collapsed) rotational states with the same energy \( E_{gs} \), a feature well-known for an anisotropic distribution of a system of particles [27 page 475]. Hence, we may conclude that \( \hat{H} \) is an intrinsic Hamiltonian and \( |\Phi(\theta'_k)\rangle \) is an intrinsic deformed wavefunction that is a superposition of angular-momentum eigenstates for a system with the kinematic moments of inertia \( M \hat{J}_k \). Indeed, we observe in Eq (18) that the rotational kinetic energy is subtracted from the nuclear Hamiltonian \( \hat{H}'_n \), leaving us with the intrinsic Hamiltonian \( \hat{H}' \). One can easily show that:

---

8 Otherwise, higher order terms in \( \hat{J}_k \) coupled to other (such as shear) operators would appear in Eq (18) [78,79]. Therefore, the rigid-flow choice in Eq (19) accounts in an optimum way for the interaction between the rotation and intrinsic motions, which is manifested in the rigid-flow kinematic moment of inertia \( \hat{J}_k \) in Eq (18). The absence of such coupling terms between the rotation and intrinsic motions in Eq (18) obtained using the unitary transformation in Eq (16) and the rigid-flow choice in Eq (19) is also obtained in [98] in a canonical transformation of the kinetic energy using a projection operator for the rotation Lie algebra so(3) that decomposes a single-particle momentum into a rigid-flow collective momentum component and its complementary orthogonal intrinsic momentum.

9 In many studies [99-103], the operator \( \sum_{k=1}^{3} \hat{J}_k^2 / s_k' \) with some inertia parameters \( s_k' \) has been used to remove spurious rotational excitation energy. The above results then show that for this removal to be optimal, \( s_k' \) must be replaced by \( M \hat{J}_k \).
\[
\left[ \mathbf{\hat{g}}_k, \hat{H} \right] = \frac{-i\hbar}{\hat{\mathbf{g}}_k} \left( \frac{\hat{Q}_k}{\hat{\mathbf{g}}_k} \cdot \hat{J}_i + \frac{\hat{Q}_k}{\hat{\mathbf{g}}_k} \cdot \hat{J}_k' \right) \approx 0, \quad (k \neq l \neq k' = 1, 2, 3)
\]  

(21)

Since \( \hat{Q}_k \) for \( k \neq l \) is very small compared to \( \hat{J}_i \) (as shown below). The result in Eq (21) is a further indication that \( \hat{H} \) in Eq (18) is optimally or ideally intrinsic. The Hamiltonian \( \hat{H} \) is also time-reversal and rotationally invariant since the rotation operator \( e^{i\theta \hat{J}_i} \) in Eq (16) is rotationally invariant.

The inverse of the rigid-flow moment of inertia \( \hat{\mathbf{I}} \) is a many-body operator. To render Eq (18) solvable, we replace \( \hat{\mathbf{I}} \) in Eq. (18) by its expectation value:

\[
\hat{\mathbf{I}} \equiv \langle \Phi | \hat{\mathbf{I}} | \Phi \rangle 
\]

(22)

This is a reasonably good approximation since \( \hat{\mathbf{I}} \) is a relatively large number and varies little and gradually with \( J \). Eq (18) then becomes:

\[
\hat{H} \left| \Phi \left( \vec{\theta} \right) \right\rangle = \left( \hat{H}_o^' - \sum_{k=1}^{3} \frac{j_k^2}{2M}\hat{\mathbf{Q}} \cdot \hat{\mathbf{Q}}_k \right) \left| \Phi \left( \vec{\theta} \right) \right\rangle = E_{jgs} \left| \Phi \left( \vec{\theta} \right) \right\rangle 
\]

(23)

We now apply to Eq (23) Hartree-Fock (HF) variational and second quantization methods to obtain:

\[
\hat{H} \left| \Phi \right\rangle = \left[ \hat{H}_{oHF}' + \hat{V}_{res} - \sum_{k=1}^{3} \frac{(\hat{j}_k^2)_{res}}{2M}\hat{\mathbf{Q}} \cdot \hat{\mathbf{Q}}_k \right] \left| \Phi \right\rangle = E_{jgsHF} \left| \Phi \right\rangle 
\]

(24)

where \( \hat{H}_{oHF}' \) is the Hartree-Fock mean-field independent-particle part of \( \hat{H} \) and satisfies the Schrödinger equation:

\[
\hat{H}_{oHF} \left| \Phi_{oHF} \right\rangle = \left[ \hat{H}_{oHF} - \vec{\Omega} \cdot \hat{\mathbf{J}} - \sum_{k=1}^{3} \frac{Q_{k}}{\mathbf{g}_k} \cdot \Omega_{k} \cdot \hat{J}_{k} - \frac{1}{2} \sum_{k=1}^{3} \Omega_{k} \cdot \hat{\mathbf{M}} \cdot \hat{\mathbf{Q}}_{k} \right] \left| \Phi_{oHF} \right\rangle = E_{jgsHF} \left| \Phi_{oHF} \right\rangle 
\]

(25)

where \( Q_{k} \equiv \langle \Phi_{oHF} | \hat{Q}_k | \Phi_{oHF} \rangle \equiv \langle \hat{Q}_k \rangle \), and \( \langle \rangle \) denotes the expectation value over the HF state \( | \Phi_{oHF} \rangle \). \( \hat{H}_{oHF} \) in Eq (25) is the HF mean-field independent-particle part of the nuclear Hamiltonian \( \hat{H}_o' \). \( \vec{\Omega} \cdot \hat{\mathbf{J}} \) in Eq (25) is the single-particle direct HF mean-field part of \( \sum_{k=1}^{3} \frac{\hat{j}_k^2}{2M}\hat{\mathbf{Q}} \cdot \hat{\mathbf{Q}}_k \), and \( \vec{\Omega} \) is the angular-velocity vector with the magnitude \( \Omega \) and orientation \((\theta, \phi)\), and is defined by its three Cartesian components:

\[
\vec{\Omega} \equiv (\Omega_x, \Omega_y, \Omega_z) \equiv \Omega (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \phi) = \left( \frac{\langle j_1 \rangle}{M \mathbf{g}_1}, \frac{\langle j_2 \rangle}{M \mathbf{g}_2}, \frac{\langle j_3 \rangle}{M \mathbf{g}_3} \right)
\]

(26)

\( (\hat{j}_i^2)_{res} \) in Eq (24) is the residual of the square of the angular-momentum operator given by [78,79]:

15
\[
\begin{align*}
(\mathbf{j}_i^2)_{\text{res}} & = \mathbf{j}_i^2 - 2\langle \Phi_{\text{crHF}} | \mathbf{j}_i | \Phi_{\text{crHF}} \rangle \cdot \mathbf{j}_i = \sum_{v \neq v'} (\mathbf{j}_i^2)_{v v'} - 2 \sum_{k=1}^{A} (\mathbf{j}_i)_{v k} \cdot (\mathbf{j}_i)_{v' k} + \sum_{l, k=1}^{A} (\mathbf{j}_i)_{l k} \cdot (\mathbf{j}_i)_{l k'} + \sum_{\mu, \mu', \nu', \nu} (\mathbf{j}_i)_{\mu \nu} \cdot (\mathbf{j}_i)_{\nu' \nu'} \cdot a_\mu^* a_\lambda a_\mu a_\lambda : \\
& \quad \text{(27)}
\end{align*}
\]

For the residual of nuclear interaction \( \nabla_{\text{res}} \) in Eq (24), we may use the separable effective quadrupole-quadrupole (long range) interaction (which is often used in nuclear structure calculations):

\[
\nabla_{\text{res}} = -\frac{\hbar}{2} Q' \cdot Q, \quad \hat{Q}_\mu \equiv \sum_{n=1}^{A} \left( r_n^2 \hat{q}_{n \mu} - \left< r_n^2 \hat{q}_{n \mu} \right> \right), \quad \hat{q}_{n \mu} \equiv Y_{2 \mu}(\theta_n, \phi_n), \quad \mu = 0, \pm 1, \pm 2 \tag{28}
\]

where the parameter \( \chi \) is the interaction strength. In Eq (24) we do not show the residual of the three-body operator \(-\sum_{l<k=1}^{A} \frac{3}{2M} \hat{q}_{l k}^* \cdot \mathbf{j}_l \cdot \mathbf{j}_k \) because it is negligibly small, complicated, and is not used in this article.

The terms in \( Q \) in the Hamiltonian \( \hat{H}_{\text{crHF}} \) in Eq (25) are very small, contributing less than 1\%\(^{10} \), and therefore they are neglected, and Eq (25) becomes:

\[
\hat{H}_{\text{crHF}} | \Phi_{\text{crHF}} \rangle = \left( \hat{H}_{\text{odHF}} - \hat{\Omega} \cdot \mathbf{j}_i \right) | \Phi_{\text{crHF}} \rangle = E_{J_{\text{gs crHF}}} | \Phi_{\text{crHF}} \rangle \tag{29}
\]

MSCRM3 Eq (29) is identical in form to CCRM3 Eq (1) except for the microscopically determined angular velocity vector \( \hat{\Omega} \). Taking the expectation of Eq (29), we obtain the intrinsic energy:

\[
E_{\text{int}} \equiv E_{J_{\text{gs crHF}}} = \left< \hat{H}_{\text{odHF}} - \hat{\Omega} \cdot \mathbf{j}_i \right> = \hbar \sum_{k=1}^{3} \alpha_k \Sigma_k \tag{30}
\]

where the normal-mode frequencies \( \alpha_k \) are given in Eq (6) since Eqs (29) and (1) are identical in form.

In Eq (29), \( | \cdot \rangle \equiv | \Phi_{\text{crHF}} \rangle \) is required to satisfy Eqs (31) instead of Eq (13) or (14). The value \( J_{\text{cal}} \) of the true (rather than the imposed \( J \)) angular momentum number, calculated from Eqs (10) and (30), is given by:

\[
J_{\text{cal}} = \begin{cases} 
0.5 \left[ -1 + \sqrt{1 + 4 \left( \left< \mathbf{j}_1^2 \right> + \left< \mathbf{j}_2^2 \right> + \left< \mathbf{j}_3^2 \right> \right)} \right], & \text{for a triaxial rotation} \\
\sqrt{\left< \mathbf{j}_1^2 \right> + \left< \mathbf{j}_2^2 \right> + \left< \mathbf{j}_3^2 \right>}, & \text{for a principal-axis rotation}
\end{cases} \tag{31}
\]

where the principal axes referred to here are simply the space-fixed axes. It is clear from Eqs (13), (14), and (31) that \( J_{\text{cal}} \) and \( J \) do not generally have the same value.

\(^{10}\) We now determine \( Q_{\text{ik}} = \langle \hat{Q}_{\text{ik}} \rangle \) in Eq (25) by substituting \( \hat{H}_{\text{crHF}} \) in Eq (25) into \( \left< \hat{H}_{\text{crHF}} \cdot \mathbf{j}_i \right> = 0 \), which is valid because \( | \cdot \rangle \) is an eigenstate of \( \hat{H}_{\text{crHF}} \). We then obtain \( Q_{\text{ik}} = -\left( \mathbf{j}_i \cdot \mathbf{j}_k \right) \Omega_i \Omega_k (\alpha_i^* - \alpha_k^* \delta^{(1)}) \) (\( k \neq l \)) when we ignore the terms in \( Q \) in \( \hat{H}_{\text{odHF}} \) in Eq (25), and determine that \( Q_i / \alpha_i < 0.1\% \). With the terms \( Q \) in \( \hat{H}_{\text{odHF}} \) included, the expressions for \( Q_{\text{ik}} \) are complicated but we expect them to yield a slightly larger value but still less than 0.2\%.
The orientation \((\theta, \phi)\) and the magnitude \(\Omega\) of the angular-velocity vector \(\mathbf{\Omega}\) are readily determined from the rigid-flow angular-velocity prescription in Eq (26), and are given by:

\[
\begin{align*}
\tan \phi &= \frac{\langle \hat{J}_z \rangle_{j_i}}{\langle \hat{J}_1 \rangle_{j_i}}, \\
\tan \theta &= \frac{\langle \hat{J}_1 \rangle_{j_i}}{\langle \hat{J}_1 \rangle_{j_i} \sqrt{1 + \tan^2 \phi}}
\end{align*}
\]

(32)

\[
\begin{align*}
\Omega^2 &= \Omega_1^2 + \Omega_2^2 + \Omega_3^2 = \frac{\langle \hat{J}_1 \rangle^2}{M^2 \sigma_1^2} + \frac{\langle \hat{J}_2 \rangle^2}{M^2 \sigma_2^2} + \frac{\langle \hat{J}_3 \rangle^2}{M^2 \sigma_3^2} \\
&= \frac{\langle \hat{J}_1 \rangle^2}{M^2 \sigma_1^2} + \left(\frac{1}{M^2 \sigma_1^2} - \frac{1}{M^2 \sigma_2^2} - \frac{1}{M^2 \sigma_3^2}\right) \langle \hat{J}_2 \rangle^2 + \left(\frac{1}{M^2 \sigma_2^2} - \frac{1}{M^2 \sigma_3^2}\right) \langle \hat{J}_3 \rangle^2
\end{align*}
\]

(33)

where, in the second line in Eq (33), \(\langle \hat{J}^2 \rangle = \langle \hat{J}_1 \rangle^2 + \langle \hat{J}_2 \rangle^2 + \langle \hat{J}_3 \rangle^2\). In Eq (33), we set \(\langle \hat{J}^2 \rangle\) to either \(\hbar^2 J^2\) for principal-axis rotation or to \(\hbar^2 J(J + 1)\) for triaxial rotation, with \(J\) taking on the desired value to be imposed on the wavefunction, as in Eqs (13) and (14). Eq (33) is examined further in Section 5 where we discuss rotational instability and its stabilization.

The independent-particle cranking model Schrödinger Eq (29), the angular velocity Eqs (32) and (33), the wavefunction angular-momentum constraint (i.e., the true angular momentum quantum number \(J_{cal}\)) in Eq (31) constitute a quantum, microscopic, self-consistent, time-reversal and \(D_2\) invariant, parameter-free cranking model for triaxial rotation (MSCRM3). MSCRM3 is considered a quantum, microscopic parameter-free analogue of CCRM3.

The above derivation of MSCRM3 reveals the approximations and assumptions implied by CCRM3, and its limitations. Specifically in MSCRM3: (i) the rigid-flow angular-velocity prescription in Eqs (19) and (20) ensures that the Hamiltonian \(\hat{H}\) in Eq (18) is optimally intrinsic as defined in Eq. (21); otherwise, \(\hat{H}\) in Eq (18) and hence \(\hat{H}_{crHF}\) in Eq (29) will contain higher order terms in \(\hat{J}_k\) coupled perhaps strongly to other operators, complicating the solution method of its Schrödinger equation and possibly requiring using additional approximations and assumptions, thereby complicating a comparison of MSCRM with CCRM3; clearly, the CCRM3 Hamiltonian \(\hat{H}_{cr}\) in Eq (1) is not optimally intrinsic since \(\mathbf{\dot{\Omega}}\) in Eq (1) is an arbitrary parameter, but perhaps it is close to being so since CCRM1 predicts a moment of inertia close to that of a rigid body, (ii) as in CCRM3, the MSCRM3 rotational states are inherently unstable because of the intrinsic-system destabilizing positive feedback mechanism between \(\Omega_k\) and \(\langle \hat{J}_k \rangle\) generated by the first of Eqs (10) and the self-consistency-condition-induced strong coupling between the rotation and intrinsic motions in Eq (11), (iii) the prescription for \(\Omega_k\) in Eq (26) allows \(\Omega_k\) and \(\langle \hat{J}_k \rangle\) to be determined self-consistently and completely\(^{11}\) via Eqs (32) and (33); Eq (33) provides

\(^{11}\) Note that, if we use Eq (26) to determine \(\Omega_k\) instead of the constraint in Eq (33), then the rotational states would decay to the ground state because of the strong coupling between the rotational and intrinsic motions and the resulting flow of energy between them and the resulting positive feedback between \(\Omega_k\) and \(\langle \hat{J}_k \rangle\). Indeed, when we keep the oscillator frequencies constant at their values at some \(J_{\text{stop}}\) and in the subsequent iteration steps, the decay
the negative feedback mechanism needed to prevent the intrinsic-system destabilizing positive feedback from reducing the three $\langle \hat{J}_k \rangle$ to zero simultaneously; the self-consistency between $\Omega_k$ and $\langle \hat{J}_k \rangle$ also allows the intrinsic system to rotationally relax after the $J$ imposed in the first iteration step is removed and replaced by Eq (33); this rotational relaxation yields a value of $J_{cal}$ in Eq (31) that is smaller or larger than $J$ as Fig 5 shows; in contrast, the CCRM3 forces the intrinsic system to accept a value of $J$ of the angular momentum that may be inconsistent with the nucleonic motion; (iv) the self-consistency between $\Omega_k$ and $\langle \hat{J}_k \rangle$ also causes fluctuations in these variables, which can result in the decay of one of the components of the angular momentum in an axially symmetric rotation in MSCRM3 but not in CCRM3; this in fact happens in MSCRM3 application to $^{20}\text{Ne}$, and this decay reduces the excitation energy; (v) a consequence of the interaction between positive and negative feedback mechanisms is oscillations in all the MSCRM3 variables $\alpha_k^2$, $\omega_k^2$, $\langle \hat{J}_k \rangle$, $J_{cal}$, $\Omega_k$, $\langle \hat{\xi}_k \rangle$, $\langle \delta_k \rangle$ etc., which occur because the positive feedback tends to increase and negative feedback tends to decrease their values; the oscillations are more prominent at low than high $J$ values because at high $J$, the stabilizing impact of negative feedback is stronger; the oscillations are avoided by appropriately averaging the values of $\alpha_k^2$, $\omega_k^2$, and $\langle \hat{J}_k \rangle$ over adjacent iteration steps; we use this averaging in all our calculations below; (vi) MSCRM3 captures a broader range of rotational phenomena than CCRM3, such as instability of axially symmetric rotation, nutation and figure-skater’s slow-fast spinning, tumbling of the principal axes of the quadrupole moment, more cases of rotational band termination, etc.; (vii) CCRM3 Eq (12) is inconsistent with MSCRM3 Eq (32): Eq (32) shows that, in MSCRM3, $\vec{\Omega}$ and $\langle \vec{J} \rangle$ are not parallel vectors, meaning that the rotation is generally not uniform (i.e., the orientation $(\theta,\phi)$ of $\vec{\Omega}$ defined in Eq (32) is not the same as the orientation $(\theta_j,\phi_j)$ of $\langle \vec{J} \rangle$ defined by $\tan \phi_j = \langle J_2 \rangle / \langle J_1 \rangle$ and $\tan \theta_j = \langle J_3 \rangle / (1 + \tan^2 \phi_j)$) unless $\langle J_2 \rangle = \langle J_3 \rangle = 0$, $\langle \hat{\xi} \rangle = \langle \hat{\delta} \rangle$ and $\langle \hat{\xi}_J \rangle = 0$, or $\langle J_1 \rangle = \langle J_3 \rangle = 0$, $\langle \hat{\xi} \rangle = \langle \hat{\delta} \rangle$. On the hand, Eq (12) shows that in CCRM3 $\vec{\Omega}$ and $\langle \vec{J} \rangle$ are always parallel vectors; (viii) MSCRM3 Schrodinger equation is self-consistent, parameter-free, time-reversal and $D_2$ and hence signature invariant, (because $\Omega_k$ in Eq (26) changes sign under time reversal and rotation through $\pi$ about each principal axis, and hence MSCRM3 wavefunction is a superposition of either even or odd $J_{cal}$ angular momentum eigenstates, whereas CCRM3 wavefunction contains even and odd $J_{cal}$ values. This MSCRM3 feature resolves the difficulty of relating the predicted and experimentally observed rotational bands as pointed out in [37]; and (ix) the above MSCRM3 derivation indicates that stops when $J_{cal}$ reaches $J_{stop}$. We have shown in previous studies [78,79] that a complete separation between collective rotation and other nuclear degrees of freedom is not possible except under some extreme conditions such as very low angular velocities and/or large deformations. Therefore, MSCRM3 does not describe a free or self-sustaining rotation. Indeed, one may wonder if free rotation at a constant angular momentum in a self-consistent system of particles is possible.
CCRM3 neglects the correction terms
\[ \sum_{l \neq k=1}^{3} \frac{\hat{Q}_k}{2M \hat{J}_k \hat{J}_l} \cdot \hat{J}_l \cdot \hat{J}_k \] (which is in fact negligibly small) and
\[ \sum_{k=1}^{3} \left( \hat{J}_k^2 \right)_{\text{res}} \] (which includes a part of the HF one-body direct and exchange terms and other residual parts of the square of the angular momentum operator), which is estimated in the footnote 11 in Section 5, to have significant impact on the excitation energies.

5. Solution and predictions of MSCRM3 algebraic equations for deformed self-consistent harmonic-oscillator potential

MSCRM3 Eq (29) is identical in form to CCRM3 Eq (1). Therefore, for the deformed self-consistent harmonic-oscillator potential, we solve MSCRM3 Eqs (29), (32), and (33), supplemented by Eqs (6), (9), (10), (11), (13), (14), (20), and (30), using the CCRM3 iteration method described in Section 3\(^\text{12}\). For the values of \( \hat{J}_k \), \( \hat{J}_k \), and \( \hat{J}_k \) in Eqs (32) and (33) at the start of the first iteration step, we use the corresponding values in the first iteration step in CCRM3. In the first iteration step, we adjust the value of \( \hat{J}_k \) until the value of \( \hat{J}_k \) computed from either Eq (13) or (14) is equal to the desired \( \hat{J}_k \) value imposed on the wavefunction. In the subsequent iteration steps, \( \hat{J}_k \) is computed from Eq (33). The MSCRM3 iteration program then determines the final converged values of \( \hat{J}_k \), \( \hat{J}_k \), \( \hat{J}_k \), \( \hat{J}_k \), and \( \hat{J}_k \) in the subsequent iteration steps.

Since MSCRM3 and CCRM3 equations have identical forms, MSCRM3 predictions are similar to those of CCRM3 given in Section 3 but there are some significant differences. We now present the predictions of MSCRM3 for nuclei \(^{20}\text{Ne}\), \(^{24}\text{Mg}\), and \(^{28}\text{Si}\).

For prolate \(^{20}\text{Ne}\), \( \phi_0 = 45^\circ \), and \( \theta_0 > 0^\circ \) (\( \theta \) increases to \( 90^\circ \) in the first few iteration steps), MSCRM3 predicts an axially-symmetric planar rotation in the \( x-y \) plane of the axially-symmetric (about \( z \) axis) \(^{20}\text{Ne}\) at \( J_{\text{cal}} \leq 6 \) (relaxed from \( J = 6.6 \)) as does CCRM3. At \( J_{\text{cal}} = 7.76 \) (relaxed from \( J = 8 \)), \(^{20}\text{Ne}\) becomes triaxial and the rotation becomes uniform along the \( x \) axis (this transition is not predicted by CCRM3 in Section 3). At \( J_{\text{cal}} = 8 = (\Sigma_3 - \Sigma_1) \) (relaxed from \( J = 10 \)), the rotational band terminates when \(^{20}\text{Ne}\) becomes axially symmetric about the \( x \) axis (this band termination is not predicted by CCRM3 in Section 3). The transition to a uniform rotation in MSCRM3 reduces the excitation energy at \( J = 8 \) (as a consequence of the switch between the two angular-momentum constraint forms in Eq (31)). As Fig 6 shows, this reduction in energy level spacing improves the agreement with the measurement, which is not predicted by CCRM3 in Section 3. The reduction in energy-level spacing has been known \([7,25]\), and it is explained here to be caused by the transition from wobbly planar to uniform rotation (in other words, by quenching of wobbling). This transition occurs in MSCRM3 but not in CCRM3 because in MSCRM3 \( \hat{J}_k \) and \( \hat{J}_k \) are determined self-consistently and hence the resulting fluctuations in one of the components of \( \hat{J}_k \) in the axially symmetric rotation and the positive feedback

\(^{12}\) We have also added the spin-orbit \( \hat{J} \hat{S} \) term in this formalism. An estimation of the impact of the spin-orbit from this analysis is presented below.
between $\vec{Q}$ and $\vec{J}$ cause one of the components of $\vec{J}$ in the planar rotation to decay to zero. Fig 7 shows that the addition of the spin-orbit interaction to the harmonic-oscillator potential increases the excitation energy by about 10% closer to the measured energy without significantly changing the other predicted features. Figs 6 and 7 indicate that MSCRM3 and CCRM3 predict excitation energies that are systematically lower than the measured energies by 0.7 to 2 MeV at $J \leq 6$. At $J \leq 6$, MSCRM3 excitation energy is lower by 1.3 MeV and that of CCRM3 higher by 3 MeV. The discrepancies in MSCRM3 predictions are expected to be corrected by accounting for the residuals of the square of the angular momentum and the quadrupole-quadrupole interaction in the model. We have not yet found a way to include these residues in the iterative procedure in the model. But we have obtained an estimate of their impact by including them instead into MSCR for uniaxial rotation [78] using Tamm-Dancoff approximation and cranked 1-particle1-hole basis states. The result in Fig 8 shows that these residues increase the excitation energies between 1 and 2 MeV, and would therefore improve the agreement between the measured and MSCRM3 predicted excitation energies in Fig 7. Fig 9 shows that, within the measurement uncertainties, MSCRM3 predicts reasonably well the measured intrinsic quadrupole moment [7,86,88] barring the significant discrepancy at $J_{\text{cal}} = 4$, which is not well understood. We note in Fig 9 that the rotation-mode transition discussed above improves the agreement between the MSCRM3 predicted and measured quadrupole moments, in contrast to the CCRM3 prediction. The predicted and measured quadrupole moments decrease with increasing $J$. This anti-stretching behaviour has been observed elsewhere [89,104] and is a consequence of the combined effects of centrifugal stretching and density-potential self-consistency and constant volume conditions. For $\phi_o \neq 45^\circ$ and $\theta_o > 0^\circ$ ($\theta$ increases to 90$^\circ$ in the first few iterations), MSCRM3 predicts a wobbly planar rotation in the $x$-$y$ plane at $J_{\text{cal}} = 2$ and 4 (relaxed from $J = 2.013,4.046$). At $J_{\text{cal}} = 6$ (relaxed from $J = 6.14$), $^{20}$Ne becomes triaxial and the rotation becomes uniform along the $x$ axis for $\phi_o < 45^\circ$ and $y$ axis for $\phi_o > 45^\circ$. At $J_{\text{cal}} = 8 = \Sigma_3 - \Sigma_1$ (relaxed from $J = 10$), $^{20}$Ne becomes axially symmetric about the $x$ and $y$ axis respectively, and the band terminates, similar to that predicted by CCRM3.

For prolate $^{24}$Mg, MSCRM3 and CCRM3 predict rotations similar to those for $^{20}$Ne with the exception that $^{24}$Mg band terminates at $J_{\text{cal}} = 4 = \Sigma_3 - \Sigma_1$ (relaxed from $J = 5$) at each value of $\phi_o$.

For triaxial $^{24}$Mg, and for all values of $\phi_o$ and $\theta_o$, MSCRM3 predicts a steady triaxial rotation (i.e., with non-zero components of the angular momentum) of a triaxial $^{24}$Mg at $J_{\text{cal}} = 2, 4$, and 6 (relaxed from $J = 1.998,3.976,6.092$). At $J_{\text{cal}} = 8$ (relaxed from $J = 8.2$), $^{24}$Mg and its rotation become axially symmetric about the $y$ axis, and the rotation develops a nutation, and the deformation and other variables become oscillatory as a function of the iteration-step number as observed in Figs 10 and 11. At $J_{\text{cal}} = 10$ (relaxed from $J = 10.8$), the axial rotation becomes steady, as in Fig 12. At $J_{\text{cal}} = 11.51$ (relaxed from $J = 14.5$), $^{24}$Mg and its rotation become

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13 The nutation is similar to that in the rotation of a fixed-point axially-symmetric rigid top in the earth's gravity [105], and is caused by the gravitational force or potential. In the case of the nucleus, the nutation arises because of the harmonic-oscillator potential and the imposition of self-consistency-constant volume conditions. The nutating rotational axially-symmetric state at $J = 8$ may be interpreted as describing two minimum energy rotational states about 0.5 MeV apart similar to that predicted in a 3-D cranked HFB with pairing plus $Q.Q$ interaction calculation [98].
spherically symmetric as in Figs 13 and 14, and the rotational band terminates. In contrast, CCRM3 in Section 3 predicts steady uniform rotation of a triaxial $^{24}\text{Mg}$ about either $x$ or $y$ axis. MSCRM3 and CCRM3 predict similar values for the intrinsic quadrupole moment $Q$ up to $J_{\text{cal}}=6$ as shown in Fig 15. Beyond $J_{\text{cal}}=6$, MSCRM3 predicts that the nucleus tumbles as $\phi_o$ is varied, i.e., the quadrupole-moment ellipsoidal shape rotates about one of the principal axes causing the quadrupole moment to possibly change sign. Therefore, beyond $J_{\text{cal}}=6$, MSCRM3 predicts much lower and negative values of $Q$ for $\phi_o = 89^\circ$, and positive $Q$ values for $\phi_o = 40^\circ$ as shown in Figs 15 and 16 because of the particular tumbling motion of the principal axes of the quadrupole moment. The MSCRM3 and CCRM3 predicted excitation energies are systematically about 1 to 3 MeV lower than the measured energies because the residues of the square of the angular momentum and two-body interaction are neglected as discussed above for $^{20}\text{Ne}$, but MSCRM3 energies are closer to the measured values.

For prolate $^{28}\text{Si}$, $\phi_o = 45^\circ$, and $\theta_o > 0^\circ$ ($\theta$ increases to $90^\circ$ in the first few iterations), MSCRM3 predicts a wobbly planar rotation in the $x$-$y$ plane of a $^{28}\text{Si}$ that is axially symmetric about the $z$ axis up to $J_{\text{cal}}=8$ (relaxed from $J=8.14$). At $J_{\text{cal}}=10$ (relaxed from $J=10.33$), $^{28}\text{Si}$ becomes triaxial and the rotation becomes uniform along the $y$ axis. At $J_{\text{cal}}=12 = \Sigma_2^2\Sigma_1^2$ (relaxed from $J=18.7$), $^{28}\text{Si}$ becomes axially symmetric about the $y$ axis and the rotational band terminates. In contrast, in CCRM3 in Section 3 predicts a planar uniform rotation of a $^{28}\text{Si}$ that remains axially symmetric about the $z$ axis at all $J$, and the band does not terminate. The MSCRM3 predicted/measured excitation energies and quadrupole moments are: $\Delta E_x = 0.55/1.78$, $\Delta E_y = 1.81/4.62$, $\Delta E_z = 3.88/8.54$, $\Delta E_6 = 5.96/12$, $\Delta E_{10} = 9.49/18.5$, $\Delta E_{12} = 14.06/26.5$ (MeV), and $Q_2 = 72.14$, $Q_4 = 68.48$, $Q_6 = 59.28$, $Q_8 = 35.77$ (e$\cdot$fm$^2$). For $\phi_o \neq 45^\circ$, and $\theta_o > 0^\circ$ ($\theta$ increases to $90^\circ$ in the first few iterations), MSCRM3 predicts a wobbly planar rotation in the $x$-$y$ plane of a triaxial $^{28}\text{Si}$ up to $J_{\text{cal}}=6$ (relaxed from $J=6.55$) for $\phi_o < 45^\circ$, and up to $J_{\text{cal}}=10$ (relaxed from $J=10.33$) for $\phi_o > 45^\circ$, at which the rotation becomes uniform along the $x$ and $y$ axes respectively. $^{28}\text{Si}$ remains triaxial up to $J_{\text{cal}}=12$ (relaxed from $J=18.7$) at which it becomes axially symmetric about $x$ and $y$ axes respectively, and the rotational band terminates. Therefore, for $\phi_o \neq 45^\circ$, $^{28}\text{Si}$ is triaxial over the entire range of $J_{\text{cal}}$ until the rotational band terminates unlike that for $\phi_o = 45^\circ$. The excitation energies and quadrupole moments are the same as those for $\phi_o = 45^\circ$. These results are similar to those predicted by CCRM3 in Section 3 except that in CCRM3 the $^{28}\text{Si}$ rotation remains planar and uniform up to $J=6$ (for $\phi_o < 45^\circ$ and $\phi_o > 45^\circ$) at which the nucleus becomes triaxial, and at $J=10$ the uniform planar rotation becomes uniform along the $x$ axis for $\phi_o < 45^\circ$ and uniform along the $y$ axis for $\phi_o > 45^\circ$. At $J=12$ the nucleus becomes axially symmetric along the $x$ axis and the band terminates, but the rotation along the $y$ axis does not terminate. The above differences between the predictions of MSCRM3 and CCRM3 arise from the self-consistency between $\vec{Q}$ and $\{\vec{J}\}$, which we also observe in $^{20}\text{Ne}$ and $^{24}\text{Mg}$ cases above.

For oblate $^{28}\text{Si}$, MSCRM3 predicts no collective rotation as does CCRM3.

For triaxial $^{28}\text{Si}$, $\phi_o < 13.41^\circ$ and $13.41^\circ < \phi_o \leq 90^\circ$, and $\theta_o = 90^\circ$ at $J_{\text{cal}}=2$, MSCRM3 predicts figure-skater’s slow and fast spinning rotation similarly to that predicted by CCRM3 in
Section 3 except that the spinning also nutates because not only $\Omega_1$ and $\Omega_2$ are oscillatory but also $\langle \dot{J}_1 \rangle$ and $\langle \dot{J}_2 \rangle$ are oscillatory for $\phi_o < 13.41^\circ$, and $13.41^\circ < \phi_o \leq 90^\circ$ respectively. These oscillations have higher amplitudes at higher $J$ so that the figure-skater’s spinning nutates more. As a result, at $J_{\text{cal}} = 6$, the rotation terminates on unphysical model parameter values. The predicted/measured average excitation energies are: $\Delta E_z = 4/1.78$, $\Delta E_z = 5/4.62$. For $\phi_o \neq 90^\circ$, MSCRM3 predicts a $\theta_o$ that decreases to zero, and a steady non-collective rotation along the symmetry $z$ axis at $J = 2$ and 4 at which the band terminates, and an excitation energy that decreases with $J$ as does CCRM3. Therefore, MSCRM3 and CCRM3 predict no realistic collective rotation for triaxial $^{28}\text{Si}$.

The above results are summarized as follow.

For prolate $^{20}\text{Ne}$ and $\phi_o = 45^\circ$, CCRM3 predicts planar (in the $x$-$y$ plane) uniform rotation (which is a special case of a triaxial rotation) of an axially symmetric nucleus at all $J$ values, and no band termination. In contrast, MSCRM3 predicts that the initial planar rotation of axially symmetric nucleus transitions to a uniform rotation of a triaxial nucleus along the $x$ axis at $J_{\text{cal}} = 7.76$ (relaxed from $J = 8$). At $J_{\text{cal}} = 8$ (relaxed from $J = 10$), the nucleus becomes axially symmetric about the $x$ axis, and the band terminates. The transition from $x$-$y$ planar to uniform rotation along $x$ axis reduces the excitation energy at $J_{\text{cal}} = 8$ improving the agreement between predicted and measured excitation energies. It is demonstrated that the systematic underprediction of 1 to 3 MeV in the excitation energy occurs because the residues in the square of the angular momentum and two-body interaction are neglected in the models. For $\phi_o \neq 45^\circ$, CCRM3 and MSCRM3 predict that the planar rotation transitions to a uniform rotation along the $x$ axis (for $\phi_o < 45^\circ$) and the $y$ axis (for $\phi_o > 45^\circ$) at higher $J$ and $J_{\text{cal}}$ values (but in MSCRM3, the transition to a rotation along the $x$ and $y$ axes can occur at different $J_{\text{cal}}$ values), and the band terminates at $J = 8$ and $J_{\text{cal}} = 8$ (relaxed from $J = 10$). For prolate $^{24}\text{Mg}$ and $^{28}\text{Si}$, CCRM3 and MSCRM3 predict shape and rotation type similar to those for $^{20}\text{Ne}$ except that the bands terminate at $J$ and $J_{\text{cal}} = 4$ and 12 respectively ($= \Sigma_3 - \Sigma_4$).

For triaxial $^{24}\text{Mg}$, CCRM3 predicts steady uniform rotation of a triaxial $^{24}\text{Mg}$ along (i) the $x$ axis for $\phi_o < 90^\circ$ followed by band termination at $J = 8$ when $^{24}\text{Mg}$ becomes axially symmetric about the $x$ axis, and (ii) along the $y$ axis for $\phi_o = 90^\circ$ with no band termination. In contrast, MSCRM3 predicts a steady triaxial rotation (with non-zero components of the angular momentum) at $J_{\text{cal}} < 8$, and a nutating and tumbling axially-symmetric rotation at $J_{\text{cal}} = 8$ (with oscillating shape and excitation energy), and a steady axially-symmetric rotation at $J_{\text{cal}} = 10$, and spherically symmetric shape and rotation and band termination at $J_{\text{cal}} = 12$. MSCRM3 predicts that the nucleus tumbles as $\phi_o$ is varied, i.e., the quadrupole-moment ellipsoidal shape rotates about one of the principal axes causing the quadrupole moment to possibly change sign. The MSCRM3 predicts systematically lower energies by about 1 to 2 MeV for all $J$, whereas CCRM3 predicts lower energies by about 1 to 2 MeV for $J < 8$ and predicts higher energy by 5 MeV at $J = 8$. These discrepancies occur because the models ignore the residues of the square of the angular momentum and two-body interaction as discussed above for $^{20}\text{Ne}$, but MSCRM3 predictions are closer to the measured values.

For oblate $^{28}\text{Si}$, CCRM3 and MSCRM3 predict no collective rotation.
For triaxial $^{28}\text{Si}$ and for $\phi_o < 13.41^\circ$ and $13.41^\circ < \phi_o \leq 90^\circ$ and $\theta_o = 90^\circ$, MSCRM3 and CCRM3 predict at $J = 2$ figure-skater’s fast-slow spinning, which also nutates in MSCRM3 but not in CCRM3. For $\theta_o \neq 90^\circ$, MSCRM3 and CCRM3 predict steady non-collective rotation about the symmetry $z$ axis, (i.e., $\theta_o$ decreases to zero) and the rotation terminates at $J = 4$. Therefore for $\theta_o \neq 90^\circ$, MSCRM3 and CCRM3 do not predict any realistic rotations for $^{28}\text{Si}$ nucleus, which may not be in conflict with the analysis result in [44] because [44] used only one kind of nucleons and the results are sensitive to the number of nucleons.

We conclude that except for planar rotation in the $x$-$y$ plane at $\phi_o = 45^\circ$ at all $J$ (which is a special case of triaxial rotation), CCRM predicts only principal-axis rotation and no triaxial rotations for nuclei $^{20}\text{Ne}$, $^{24}\text{Mg}$, and prolate $^{28}\text{Si}$. For triaxial $^{28}\text{Si}$, CCRM3 predicts no realistic rotations (although it does predict figure-skater’s rotation). These results may not be in conflict with the analysis result in [44] because [44] used only one kind of nucleons and the results are sensitive to the number of nucleons. For prolate $^{20}\text{Ne}$, $^{24}\text{Mg}$, and $^{28}\text{Si}$ and $\phi_o = 45^\circ$, MSCRM3 predicts wobbly planar rotation in the $x$-$y$ plane at $J_{cal} < 8, 4, \text{and 12}$ respectively and uniform rotation along either $x$ or $y$ axis. For triaxial $^{24}\text{Mg}$, MSCRM3 predicts steady triaxial rotation at $J_{cal} < 8$ and axially symmetric nutating rotation at $J_{cal} = 8$, followed by steady axially symmetric shape and rotation at $J_{cal} = 10$, and spherically symmetric shape and band termination at $J_{cal} = 12$. For triaxial $^{28}\text{Si}$, MSCRM3 does not predict any realistic rotations (although it does predict nutating figure-skater’s slow-fast spinning uniform rotation along either the $x$ or $y$ axis.

![Fig 5: MSCRM3 calculated $J_{cal}$ versus iteration # for $\phi_o=40^\circ$ showing rotational relaxation from higher initial $J_{cal}$ in $^{20}\text{Ne}$](image-url)
Fig 6: MSCRM3, CCRM3, \& measured excitation energies versus $J$
(versus relaxed $J_{cal}$) for $\varphi_o = 45^\circ$ in $^{20}$Ne

Fig 7: Spin-obrit MSCRM3, CCRM3, \& measured excitation energies
versus $J$ (relaxed $J_{cal}$) for $\varphi = 45^\circ$ in $^{20}$Ne
Fig. 8: MSCRIM1 predicted excitation energy versus $J$ for $^{20}$Ne-20 for uniform rotation without and with $Q$, $Q$ and $J^2$ residual interactions using cranked 1-particle and 1 hole states.

Fig. 9: MSCRIM3 & CCRM3 predicted & measured quadrupole moments versus $J$ (relaxed $J_{\text{cal}}$) for $\phi_0 = 45^\circ$ in $^{20}$Ne.
Fig 10: MSCR M3 predicted angular momentum $<J_k>$ versus iteration-step number for $\varphi_o=89^0$ at $J_{cal}=8$ in ellipsoidal $^{24}Mg$

Fig 11: MSCR M3 predicted quadrupole moment $<x_k^2>$ versus iteration step number for $\varphi_o=89^0$ at $J_{cal}=8$ in ellipsoidal $^{24}Mg$
Fig 12: MSCRIM3 predicted relaxed angular momentum $\langle J_k \rangle$ and $J_{cal}$ versus iteration number for $\varphi_o=89^\circ$ and $J_{cal}=10$ in ellipsoidal $^{24}Mg$.

Fig 13: MSCRIM3 predicted relaxed $J_{cal}$ and angular momentum $\langle J_k \rangle$ versus iteration step number for $\varphi_o=89^\circ$ and max $J_{cal}=11.51$ in ellipsoidal $^{24}Mg$. 
**Fig 14:** MSCRM3 predicted quadrupole moments $<x^2>$ versus iteration step # for $\varphi_o = 89^\circ$ and $J_{cal} = 11.51$ in ellipsoidal $^{24}Mg$

$\wedge$, $\vee$, $\wedge^2$, $\vee^2$, $\wedge^2$, and $\vee^2$

$<x^2>$ and $<z^2>$

**Fig 15:** MSCRM3 & CCRM3 predicted and measured quadrupole moments versus $J$ and $J_{cal}$ for $\varphi_o = 89^\circ$ in triaxial $^{24}Mg$

MSCRM3

CCRM3

measured

angular momentum $J$ and $J_{cal}$
6. Summary and conclusions

A quantum, microscopic, cranking-model Hamiltonian for triaxial rotation (MSCRM3) is derived by applying a rotationally-invariant exponential rotation operator to a deformed nuclear ground-state wavefunction and using Hartree-Fock variational method. The angles in the rotation operator are chosen to be canonically conjugate to the angular-momentum operator and impart a collective rigid-flow velocity field to each nucleon. This prescription renders the transformed nuclear Hamiltonian quadratic in the angular momentum with no explicit coupling to the intrinsic motion. As a result, the MSCRM3 Hamiltonian is identical in form to that of the conventional cranking model (CCRM3) except for terms associated with the residuals of the square of the angular momentum and two-body interaction and other negligibly small angular-momentum-quadrupole-moment dependent terms. In MSCRM3, the angular-velocity component \( \Omega_k \) \( (k = 1, 2, 3) \) is given by ratio of the expectation \( \langle J_k \rangle \) of the angular momentum to the expectation of the corresponding component of the rigid-flow moment of inertia, unlike that in CCRM3 where \( \Omega_k \) is a constant parameter. Consequently, MSCRM3 Hamiltonian is self-consistent, and time-reversal and \( D_2 \) invariant, rendering the wavefunction a superposition of either even or odd angular momentum eigenstates, unlike the case in CCRM3.

Feynman’s theorem is applied to the CCRM3 Schrodinger equation to relate the expectations of angular momentum and quadrupole moments to the three CCRM3 normal-mode frequencies. These frequencies are obtained from the solution of the CCRM3 Newton’s equations of motion for a deformed harmonic-oscillator potential, noting that these frequencies are the same in a classical and quantum formulation. The resulting expressions are used in the equations of the potential-quadrupole-moment self-consistency and constant-volume conditions. The resulting algebraic equations together with the equations for the usual cranking-model angular-momentum constraint, and those relating the angular-velocity vector to the expectation
of the angular momentum (obtained from minimization of the intrinsic energy) are solved iteratively. This solution procedure is also used for MSCRM3 except that in MSCRM3 each component of the angular-velocity vector is given by the ratio of the corresponding component of the expectation of the angular momentum to the expectation of the rigid-flow moment of inertia.

From the CCRM3 equations we deduce that the vectors \( \Omega \) and \( \left\langle \hat{J}_k \right\rangle \) are parallel and hence the rotation in CCRM3 is uniform. From the solution of CCRM3 equations, we show that the intrinsic energy decreases and the excited-state energy increases with increasing \( \left\langle J_k \right\rangle \) because the self-consistency and constant volume conditions couple strongly the intrinsic and rotational motions. This implies a positive feedback mechanism between the variations in \( \Omega_k \) and \( \left\langle \hat{J}_k \right\rangle \), so that any decrease in \( \left\langle \hat{J}_k \right\rangle \) causes \( \Omega_k \) to decrease, which in turn causes \( \left\langle \hat{J}_k \right\rangle \) to decrease further thereby reducing \( \left\langle \hat{J}_k \right\rangle \) to zero unless this free fall is prevented by the CCRM3 angular momentum constraint. Therefore, CCRM3 rotational states are inherently unstable causing two of the three components \( \left\langle \hat{J}_k \right\rangle \) to vanish (axially-symmetric rotation excepted) and hence CCRM3 generally favours uniform planar and principal-axis rotations\(^{14}\). These results also apply equally to MSCRM3 except that the rotation in MSCRM3 is generally not uniform, the rotational states are more unstable, and the intrinsic state exhibits rotational relaxation so that the value of the imposed angular momentum quantum number \( J \) differs from the calculated value \( J_{\text{cal}} \) all because of the fluctuations in \( \Omega_k \) and \( \left\langle \hat{J}_k \right\rangle \) since they proportional to each other in MSCRM3. MSCRM3 determines the three components of \( \Omega_k \) completely. Therefore, MSCRM3 can describe a free rotation, i.e., one where a \( J \) value is not imposed. However, because the rotational states are inherently unstable, this free-rotation state decays to the ground (\( J_{\text{cal}} = 0 \)) state (as happens in the decay of the excited rotational states in the experiments). Therefore, for a sustained rotation, we must and we do impose a \( J \) value, whose value differs from \( J_{\text{cal}} \) value.

Since MSCRM3 and CCRM3 equations have identical forms, their predictions for the light nuclei \( ^{20}\text{Ne}, \, ^{24}\text{Mg}, \, \text{and} \, ^{28}\text{Si} \) are similar with some exceptions. Therefore, we briefly describe the MSCRM3 predictions for these nuclei, and highlight where they differ significantly from the CCRM3 predictions. The predicted variables are presented as functions of \( J_{\text{cal}} \) or \( J \) and the polar \( \theta \) and azimuthal \( \phi \) angles of \( \vec{Q} \). These angles vary with \( J_{\text{cal}} \) or \( J \) and iteration-step number. We indicate by \( \phi_o \) and \( \theta_o \) the values of \( \phi \) and \( \theta \) in the first iteration step.

For prolate \( ^{20}\text{Ne}, \, ^{24}\text{Mg}, \, \text{and} \, ^{28}\text{Si} \) and \( \phi_o = 45^\circ \), MSCRM3 predicts a wobbly planar (in the \( x-y \) plane) non-uniform rotation at \( J_{\text{cal}} \leq 6, 4, 8 \) (relaxed from \( J = 6.6, 4, 8 \)) respectively. At \( J_{\text{cal}} = 7.76, 4, \) and 10 (relaxed from \( J = 8, 5, 10 \)) respectively the nuclei become triaxial and the rotation becomes uniform along the \( x \) or \( y \) axis. This shape and rotation transition occurs because of the positive feedback and self-consistency between \( \Omega_k \) and \( \left\langle \hat{J}_k \right\rangle \) and the resulting fluctuations in \( \left\langle \hat{J}_k \right\rangle \). CCRM3 does not predict these shape and rotation transitions because fluctuations in \( \left\langle \hat{J}_k \right\rangle \) in not modeled in CCRM3. At \( J_{\text{cal}} = 8, 4, 12 \) (relaxed from \( J = 10, 5, 18.7 \)) respectively the

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\(^{14}\) This instability is expected for other self-consistent mean-field potentials and not just for harmonic oscillator.
nuclei become axially symmetric about these axes and the rotational bands terminate. For reasons given above, this band termination is not predicted by CCRM3. For \( \phi_o \neq 45^\circ \), MSCRM3 predicts that at \( J_{\text{cal}} = 6, 4, 8 \) (relaxed from \( J = 6.6, 4, 8 \)) respectively the planar rotation transitions to a uniform rotation along the \( x \) axis (for \( \phi_o < 45^\circ \)) and the \( y \) axis (for \( \phi_o > 45^\circ \)). At \( J_{\text{cal}} = 8, 4, 12 \) (relaxed from \( J = 10, 4, 12 \)) respectively, the nuclei become axially symmetric about these axes and the bands terminate. Considering planar rotation as a special case of triaxial rotation, we have here examples of nuclei that are axially symmetric in their ground state but exhibiting triaxial rotations at higher \( J \) values. CCRM3 and MSCRM3 predict similar rotational-band excitation energies, which are about 1 to 2 \( \text{MeV} \) lower than those measured (CCRM3 predicted energy at \( J = 8 \) is about 3 \( \text{MeV} \) higher than that measured). However, the transition from planar to uniform rotation in MSCRM3 lowers by 4 \( \text{MeV} \) the excitation energy at \( J = 8 \) thereby explaining this phenomenon observed experimentally in \(^{20}\text{Ne}\). The addition of the spin-orbit interaction to the harmonic-oscillator potential increases the excitation energy by about 10% closer to the measured energy without changing the other predicted features. Accounting for the residuals of the square of the angular momentum and quadrupole-quadrupole interaction using Tamm-Dancoff approximation and cranked 1-particel 1-hole as basis states is estimated to increase the excitation energy by about 0.7 to 1.7 \( \text{MeV} \). MSCRM3 and CCRM3 predict similar values of the quadrupole moment and in reasonable agreement with the measurement within its uncertainties, except for \( \phi_o = 45^\circ \) where MSCRM3 prediction is closer to the measurement at band-termination \( J_{\text{cal}} = 8 \).

For oblate \(^{28}\text{Si}\), MSCRM3 and CCRM3 predict no collective rotation.

For triaxial \(^{24}\text{Mg}\), and for all values of \( \phi_o \) and \( \theta_o \), MSCRM3 predicts a steady triaxial rotation (with none of the three components of the angular momentum being zero) of a triaxial \(^{24}\text{Mg}\) at \( J_{\text{cal}} = 2, 4, 6 \) (relaxed from \( J = 1.998, 3.976, 6.092 \)). At \( J_{\text{cal}} = 8 \) (relaxed from \( J = 8.2 \)), \(^{24}\text{Mg}\) and its rotation become axially symmetric about the \( y \) axis, and the rotation develops a nutation, and the deformation and other variables become oscillatory as a function of the iteration-step number as observed in Figs 10 and 11\(^{15}\). At \( J_{\text{cal}} = 10 \) (relaxed from \( J = 10.8 \)), the axial rotation becomes steady, as in Fig 12. At \( J_{\text{cal}} = 11.51 \) (relaxed from \( J = 14.5 \)), \(^{24}\text{Mg}\) and its rotation become spherically symmetric as in Figs 13 and 14, and the rotational band terminates. In contrast, CCRM3 predicts steady uniform rotation of a triaxial \(^{24}\text{Mg}\) along either \( x \) or \( y \) axis. MSCRM3 predicts that the nucleus tumbles as \( \phi_o \) is varied, i.e., the quadrupole-moment ellipsoidal shape rotates about one of the principal axes causing the quadrupole moment to change in magnitude and possibly in sign. MSCRM3 and CCRM3 predict similar values for the intrinsic quadrupole moment \( Q \) up to \( J_{\text{cal}} \) and \( J = 6 \) as in Fig 15. Beyond \( J_{\text{cal}} \) and \( J = 6 \) MSCRM3 predicts much lower and negative values of \( Q \) for \( \phi_o = 89^\circ \), and positive \( Q \) values for \( \phi_o = 40^\circ \) as in Figs 15 and 16 because of the particular nature of the tumbling motion of the principal axes of the quadrupole moment. For triaxial \(^{24}\text{Mg}\), MSCRM3 and CCRM3 predict excitation energies similar to those the prolate nuclei discussed above.

\(^{15}\) The nutation is similar to that in the rotation of a fixed-point axially-symmetric rigid top in the earth’s gravity [105], and is caused by the gravitational force or potential. In the case of the nucleus, the nutation arises because of the harmonic-oscillator potential and the imposition of self-consistency-constant volume conditions. The nutating rotational axially-symmetric state at \( J = 8 \) may be interpreted as describing two minimum energy rotational states about 0.5 \( \text{MeV} \) apart similar to that predicted in a 3-D cranked HFB with pairing plus \( Q \) \( Q \) interaction calculation [98].
For triaxial $^{28}\text{Si}$ and for $\phi_0 < 13.41^\circ$ and $13.41^\circ < \phi_0 \leq 90^\circ$ and $\theta_0 = 90^\circ$, and at $J = 2$, MSCRM3 and CCRM3 predict figure-skater’s fast-slow spinning, which also nutates in MSCRM3 but not in CCRM3. For $\theta_0 \neq 90^\circ$, MSCRM3 and CCRM3 predict steady non-collective rotation about the symmetry $z$ axis, (i.e., $\theta_0$ decreases to zero) and the rotation terminates at $J = 4$. Therefore for $\theta_0 \neq 90^\circ$, MSCRM3 and CCRM3 do not predict any realistic rotations for $^{28}\text{Si}$ nucleus, which may not be in conflict with the analysis result in [44] because [44] used only one kind of nucleons and the results are sensitive to the number of nucleons.

In future articles, we will apply CCRM3 and MSCRM3 to other nuclei to learn more about the nature of their predictions, and we will examine further the impact of spin-orbit and residual two-body interactions. We will also study the impact of nuclear shape oscillations on the CCRM3 and MSCRM3 predictions.

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