NEW LOWER BOUNDS FOR TRACE RECONSTRUCTION

ZACHARY CHASE

Abstract. We improve the lower bound on worst case trace reconstruction from $\Omega(n^{5/4}/\sqrt{\log n})$ to $\Omega(n^{3/2}/\sqrt{\log n})$. As a consequence, we improve the lower bound on average case trace reconstruction from $\Omega(\log n/\sqrt{\log \log n})$ to $\Omega(\log n/(\log \log n)\sqrt{\log \log \log n})$.

1. Introduction

Given a string $x \in \{0, 1\}^n$, a trace of $x$ is obtained by deleting each bit of $x$ with probability $q$, independently, and concatenating the remaining string. For example, a trace of 11001 could be 101, obtained by deleting bits 2 and 3. The goal of the trace reconstruction problem is to determine an unknown string $x$, with high probability, by looking at as few independently generated traces of $x$ as possible.

More precisely, fix $\delta, q \in (0, 1)$. Take $n$ large. For each $x \in \{0, 1\}^n$, let $\mu_x$ be the probability distribution on $\{0, 1\}^\leq n$ given by $\mu_x(w) = (1 - q)^{|w|}q^{n - |w|}f(w; x)$, where $f(w; x)$ is the number of times $w$ appears as a subsequence in $x$, that is, the number of strictly increasing tuples $(i_1, \ldots, i_{|w|})$ such that $x_{i_j} = w_j$ for $1 \leq j \leq |w|$. The problem is to determine the minimum value of $T = T(n)$ for which there exists a function $f : (\{0, 1\}^\leq n)^T \to \{0, 1\}^n$ satisfying $\Pr_{\mu_x^T}[f(\tilde{U}_1, \ldots, \tilde{U}_T) = x] \geq 1 - \delta$ for each $x \in \{0, 1\}^n$ (where the $\tilde{U}_j$ denote the $T$ independently generated traces).

The problem of trace reconstruction was introduced by Batu, Kannan, Khanna, and McGregor [1] as “an abstraction and simplification of a fundamental problem in bioinformatics, where one desires to reconstruct a common ancestor of several organisms given genetic sequences from those organisms.” [2]

Holenstein, Mitzenmacher, Panigrahy, and Wieder [3] established an upper bound, that $\exp(\tilde{O}(n^{1/2}))$ traces suffice. Nazarov and Peres [4] and De, O’Donnell, and Servedio [5] simultaneously obtained the best known upper bound, that $\exp(O(n^{1/3}))$ traces suffice. The lower bound of $\Omega(n)$ was established in [1]. Holden and Lyons [2] obtained the (previous) best known lower bound, by presenting two strings $x'_n \neq y'_n \in \{0, 1\}^n$ for which $\Omega(n^{5/4}/\sqrt{\log n})$ traces are needed to distinguish between. In this paper, we improve the lower bound, exhibiting two strings $x_n \neq y_n \in \{0, 1\}^n$ for which $\Omega(n^{3/2}/\log^{16} n)$ traces are needed to distinguish between. In fact, our
methods show that $\Omega(\sqrt[3]{n}/\log^{16} n)$ traces are needed to distinguish between $x'_n$ and $y'_n$ as well.

Let $k \geq 1$, $n = 4k + 3$, and $x_n = (01)^k 1 (01)^{k+1}$, $y_n = (01)^{k+1} 1 (01)^k$, i.e.

$$x_n = 0101...0101 \ 1 \ 01 \ 0101...0101$$

$$y_n = 0101...0101 \ 01 \ 1 \ 0101...0101$$

**Theorem 1.** Fix $q, \delta \in (0, 1)$. Then there exists some constant $c = c(q, \delta) > 0$ so that at least $cn^{3/2}/\log^{16} n$ traces are required to distinguish between $x_n$ and $y_n$ with probability at least $1 - \delta$, under trace reconstruction with deletion probability $q$.

The main reason we are able to obtain an improvement over $n^{5/4}$ is that we explicitly compute the quantities relevant to determining the number of samples needed, rather than relying on a coupling argument to determine only the total variation distance.

A variant of the trace reconstruction problem is, instead of being required to reconstruct any string $x$, one must reconstruct, with high probability, a string $x$ chosen uniformly at random. The best known upper bound, due to Holden, Pemantle, and Peres, is that $\exp(O(\log^{1/3} n))$ traces suffice [6]. The (previous) best known lower bound was $\Omega(\log^{9/4} n)/\sqrt{\log \log n}$ [2]. Proposition 4.1 of [2] together with Theorem 1 implies

**Theorem 2.** For all $q \in (0, 1)$, there is $c = c(q) > 0$ so that for all large $n$, the probability of reconstructing a random $n$-bit string from $c \log^{5/2} n/(\log \log n)^{16}$ traces is at most $\exp(-n^{0.15})$, under trace reconstruction with deletion probability $q$.

Very recently, other variants of the trace reconstruction problem have been considered. The interested reader should refer to [7], [8], [9], and [10].

Here is an outline of the paper. In section 2, we determine exactly which quantity we must estimate in order to determine the number of samples needed, and we deduce Theorem 1 assuming an appropriate estimate. In section 3, we prove the estimate by obtaining closed form expressions for the probability distributions induced by the traces of $x_n$ and $y_n$ and related expressions. In section 4, we give the proofs of some lemmas used throughout section 3.

2. A Warmup to the Proof of Theorem 1

Throughout the proof, $A \lesssim B$ means $A \leq CB$ for some absolute constant $C$, and $A \gg B$ means $A \gtrsim B$ and $B \lesssim A$. We take $q = 1/2$ for ease. The (analogous) proof works for any $q \in (0, 1)$.

Fix $n \equiv 3 \pmod{4}$ large. Let $\mu$ be the probability measure for the traces of $x_n$ and $\nu$ be the probability measure for the traces of $y_n$. Let $A$ be a subset of $\{0, 1\}^{\leq n}$
with $\mu(A), \nu(A) \geq 1 - O(e^{-\log^2 n})$. Let $\mu|_A = \frac{\mu(A)}{\mu(A)}$. We will specify $A$ in section 3.2.

Define $Z : \{0, 1\}^n \to \mathbb{R}$ by $Z(w) := \frac{\mu(w)}{\nu(w)}$. We later establish the following 3 inequalities.

\begin{align*}
(1) \quad & \sup_{w \in A} |\log Z(w)| \lesssim \frac{\log^2(n)}{\sqrt{n}} \\
(2) \quad & \mathbb{E}_{\mu|_A}[\log Z] \lesssim \frac{\log^7(n)}{n^{3/2}} \\
(3) \quad & \frac{\log^{-2}(n)}{n^{3/2}} \lesssim \text{Var}_{\mu|_A}[\log Z] \lesssim \frac{\log^7(n)}{n^{3/2}}
\end{align*}

2.1. Deduction of Theorem 1 from the Three Inequalities. Let’s assume the three inequalities for now. Let $Z_1, \ldots, Z_T$ be $T$ independent copies of $Z$, $X_j(w) = \log Z_j(w) - \mathbb{E}_{\mu|_A}[\log Z_j]$ for $1 \leq j \leq T$, and $Y_T = \frac{X_1 + \cdots + X_T}{T}$. The Berry-Esseen theorem implies that, if $F_T$ is the cumulative distribution function of $\frac{Y_T \sqrt{T}}{\sqrt{\text{Var}_{\mu|_A}[X]}}$ with respect to $A$, i.e. $F_T(x) = \mathbb{P}_{\mu^T|_A}(\frac{Y_T \sqrt{T}}{\sqrt{\text{Var}_{\mu|_A}[X]}} < x)$, and $\Phi$ is the cumulative distribution of the standard normal distribution, then, for each $x \in \mathbb{R}$,

$$|F_T(x) - \Phi(x)| \leq \frac{C \mathbb{E}_{\mu|_A}[X^3]}{(\text{Var}_{\mu|_A}[X])^{3/2} \sqrt{T}}$$

for some absolute constant $C$, where $X := \log Z - \mathbb{E}_{\mu|_A}[\log Z]$.

As is well known, the optimal algorithm for distinguishing between $x_n$ and $y_n$ is provided by examining the log-likelihood ratios, that is, to guess $x_n$ if and only if it is more likely to have generated the observed traces. Therefore, if our success probability threshold is $\delta$, then $T$ samples suffice only if $\mathbb{P}_{\mu^T}(\prod_{t=1}^T \mu(w_t) < \prod_{t=1}^T \nu(w_t)) < \delta$.

Observe that

$$\mathbb{P}_{\mu^T} \left[ \frac{1}{T} \sum_{t=1}^T \log Z_t < 0 \right] \geq \mu(A^T) \mathbb{P}_{\mu^T|_A} \left[ \frac{X_1 + \cdots + X_T}{T} + \mathbb{E}_{\mu|_A}[\log Z] < 0 \right]$$

$$= \mu(A^T) \mathbb{P}_{\mu^T|_A} \left[ \frac{Y_T \sqrt{T}}{\sqrt{\mathbb{E}_{\mu|_A}[X^2]}} + \frac{\sqrt{T} \mathbb{E}_{\mu|_A}[\log Z]}{\sqrt{\text{Var}_{\mu|_A}[X]}} < 0 \right]$$

$$= \mu(A^T) F_T \left( -\frac{\sqrt{T} \mathbb{E}_{\mu|_A}[\log Z]}{\sqrt{\text{Var}_{\mu|_A}[\log Z]}} \right)$$

$$\geq \mu(A^T) \left[ \Phi \left( -\frac{\sqrt{T} \mathbb{E}_{\mu|_A}[\log Z]}{\sqrt{\text{Var}_{\mu|_A}[\log Z]}} \right) - \frac{C \mathbb{E}_{\mu|_A}[X^3]}{\mathbb{E}_{\mu|_A}[X^2]^{3/2} \sqrt{T}} \right].$$
Now, (2) and (3) imply that
\[
\frac{\sqrt{T \mathbb{E}_{\mu|A} [\log Z]}}{\sqrt{\text{Var}_{\mu|A} [\log Z]}} \leq \frac{\sqrt{T \log^2(n)}}{\sqrt{\log^2(n)}} = \sqrt{\frac{T}{n^{3/2}}} \log^3(n),
\]
and (1) and (3) imply that
\[
\mathbb{E}_{\mu|A} [X^2] \leq \left( \sup_{w \in A} |X(w)| \right) \cdot \mathbb{E}_{\mu|A} [X^2] \leq \frac{\log^2(n)}{\sqrt{n}} \leq n^{1/4} \log^3(n).
\]
Therefore,
\[
\mathbb{P}_{\mu^T} \left[ \sum_{t=1}^{T} \log Z_t < 0 \right] \geq \mu(A^T) \left[ \Phi \left( -C' \sqrt{\frac{T}{n^{3/2}}} \log^3(n) \right) - \frac{C'' n^{1/4} \log^3(n)}{\sqrt{T}} \right].
\]
If \( T = c_0 \frac{n^{3/2}}{\log^6(n)} \) for some small constant \( c_0 \) independent of \( n \) (but dependent on \( \delta \)), then
\[
\mathbb{P}_{\mu^T} \left[ \sum_{t=1}^{T} \log Z_t < 0 \right] \geq (1 - O(e^{-\log^2(n)})) c_0 n^{3/2} / \log^6(n) \left[ \Phi(-C' \sqrt{c_0}) - \frac{C'' \log^{11} n}{c_0 n^{1/2}} \right]
\]
is greater than \( \delta \) if \( c_0 \) is small enough and \( n \) large enough. Since the optimal probability of success is monotone in \( T \), Theorem 1 is deduced.

**Remark.** Since the error term in our application of Berry-Esseen is small enough, the argument just given can be easily adapted to show that \( \tilde{O}(n^{3/2}) \) samples are sufficient to distinguish between \( x_n \) and \( y_n \).

### 2.2. A Warmup to Proving the Three Inequalities.

For (1), it suffices to show
\[
(4) \quad \left| \frac{\mu(w) - \nu(w)}{\nu(w)} \right| \lesssim \frac{\log^2(n)}{\sqrt{n}}
\]
for each \( w \in A \). We do this in section 3.4.

Using \( \log(1 + t) \leq t + t^2 \) and \( \log^2(1 + t) \leq t^2 \) for small \( t \), we get that
\[
\mathbb{E}_{\mu|A} [\log Z] = \frac{1}{\mu(A)} \sum_{w \in A} \mu(w) \log \left( \frac{\mu(w)}{\nu(w)} \right) \leq \sum_{w \in A} \frac{\mu(w)}{\nu(w)} |\mu(w) - \nu(w)| + \sum_{w \in A} \frac{\mu(w) (\mu(w) - \nu(w))^2}{\nu(w)}
\]
and
\[
\text{Var}_{\mu|A} [\log Z] = \frac{1}{\mu(A)} \sum_{w \in A} \mu(w) \log^2 \left( \frac{\mu(w)}{\nu(w)} \right) \approx \sum_{w \in A} \frac{\mu(w) (\mu(w) - \nu(w))^2}{\nu(w)}.
\]
Since $\sum_{w \in A} \mu(w) - \nu(w) = - \sum_{w \notin A} \mu(w) - \nu(w) = O(e^{-\log^2 n})$ and $\frac{\mu(w)}{\nu(w)} \approx 1$ for $w \in A$, the equality

$$\sum_{w \in A} \frac{\mu(w)}{\nu(w)} [\mu(w) - \nu(w)] = \sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} + \sum_{w \in A} \mu(w) - \nu(w)$$

implies

$$\mathbb{E}_{\mu|A} [\log Z] \lesssim \sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} + O(e^{-\log^2 n}).$$

Also, $\frac{\mu(w)}{\nu(w)} \approx 1$ implies

$$\text{Var}_{\mu|A} [\log Z] \approx \sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)}.$$

Therefore, to prove (2) and (3), it suffices to establish

$$\frac{\log^{-2}(n)}{n^{3/2}} \lesssim \sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} \lesssim \frac{\log^7(n)}{n^{3/2}}.$$

By Proposition 1.3 in [2] and the fact that $\mu(A), \nu(A) \geq 1 - O(e^{-\log^2 n})$,

$$\frac{\log^{-2}(n)}{n^{3/2}} \lesssim \left( \sum_{w \in A} |\mu(w) - \nu(w)| \right)^2 \leq \left( \sum_{w \in A} \frac{|\mu(w) - \nu(w)|}{\nu(w)} \right) \left( \sum_{w \in A} \nu(w) \right).$$

Therefore, to establish (2) and (3), it suffices to prove the upper bound

$$\sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} \approx \frac{\log^7(n)}{n^{3/2}}.$$

3. Proving Inequalities (4) and (5)

3.1. Obtaining Closed Form Expressions for $\mu$ and $\nu$.

In this subsection, we obtain closed form expressions for the probability distributions of the traces of $x_n$ and $y_n$. Let $s_k = (01)^k = 0101\ldots01$ be of length $2k$. Let $f_c(w)$ denote the number of contiguous 01 appearances in $w$.

We will use the following simple and fortuitous combinatorial lemma. It is the main reason we are able to obtain a simple(r) closed form expression.

**Lemma 1.** Let $f(w; s)$ denote the number of times $w$ appears as a subsequence in $s$, that is, the number of strictly increasing tuples $(i_1, \ldots, i_{|w|})$ such that $s_{i_j} = w_j$ for $1 \leq j \leq |w|$. Then, for any $k \geq 0$, $f(w; s_k) = \binom{k + f_c(w)}{m}$ if $|w| = m$. 

Proof. We prove by induction on $k$ that the equality holds for all $m \geq 1$. For $k = 0$ or 1, the result is easy. Now assume the result holds for $k - 1$ for some $k \geq 2$. Take $m \geq 1$ and some $w$ of length $m$. If $w$ starts with ‘01’, then $f(w; s_k) = f(w_{3, m}; s_{k-1}) + f(w_{2, m}; s_{k-1}) + f(w; s_{k-1})$ depending on whether we choose the first ‘01’ as part of the subsequence, only the ‘0’, or neither the ‘0’ nor the ‘1’. By induction and two applications of Pascal’s identity, this is

\[
\binom{k-1 + f_c(w) - 1}{m-2} + \binom{k-1 + f_c(w) - 1}{m-1} + \binom{k-1 + f_c(w)}{m} = \binom{k-1 + f_c(w)}{m-1} + \binom{k-1 + f_c(w)}{m} = \binom{k + f_c(w)}{m}.
\]

And, if $w$ starts with ‘1’ or ‘00’, then $f(w; s_k) = f(w_{2, m}; s_{k-1}) + f(w; s_{k-1}) = \binom{k-1 + f_c(w)}{m-1} + \binom{k-1 + f_c(w)}{m} = \binom{k + f_c(w)}{m}$. □

Doing casework on whether $w$ includes the “special 1”, and if so, where it appears, Lemma 1 implies that

\[
2^n \mu(w) = \binom{2k + f_c(w)}{|w|} + \sum_{1 \leq j \leq |w| \atop w_j = 1} \binom{k + f_c(w_{1, j-1})}{j-1} \binom{k + f_c(w_{j+1, m})}{m-j}.
\]

\[
2^n \nu(w) = \binom{2k + f_c(w)}{|w|} + \sum_{1 \leq j \leq |w| \atop w_j = 1} \binom{k + 1 + f_c(w_{1, j-1})}{j-1} \binom{k + f_c(w_{j+1, m})}{m-j}.
\]

3.2. Defining the Set $A$.

We now define the “high probability” set used in section 2. Let

\[
A = \{w \in \{0, 1\}^n : |w| - 2k \leq \sqrt{k} \log(k) \text{ and } |f_c(w) - \frac{2k}{3}| \leq \sqrt{k} \log(k)\}.
\]

We show $\mu(A), \nu(A) \geq 1 - O(e^{-\log^2 n})$. To this end, and for the purposes of proving inequalities (4) and (5), we make frequent use of the following technical Lemma, used to estimate binomial coefficients. It is proven in the appendix.

**Lemma 2.** For $\eta$ bounded away from 0 and 1 and any $A, B, \Delta, \text{ and } \sigma$, it holds that

\[
\left[ \frac{(A + \Delta)(B - \Delta)}{(A \Delta)} \right]^{-\frac{1}{2}} (1 + O(\frac{\sigma^3}{A^2}))(1 + O(\frac{\Delta^3}{A^2}))(1 + O(\frac{1}{A}))(1 + O(\frac{(\Delta - \sigma)^2}{A^2})) \exp \left( \frac{1}{2} (\Delta - \sigma)^2 + \frac{1}{2} \frac{\sigma^2}{A^2} - \frac{1}{2} \Delta^2 \right) \\
\times (1 + O(\frac{\sigma^3}{B^2}))(1 + O(\frac{\Delta^3}{B^2}))(1 + O(\frac{1}{B}))(1 + O(\frac{(\Delta - \sigma)^2}{B^2})) \exp \left( \frac{1}{2} (\Delta - \sigma)^2 + \frac{1}{2} \frac{\sigma^2}{B^2} - \frac{1}{2} \Delta^2 \right).
\]
A corollary of Lemma 2 we will use frequently is that, for fixed \( A, B \) with, say, \( A \leq B \), and for fixed \( \eta \), as \( \Delta \) and \( \sigma \) range in \( [-\sqrt{A} \log A, \sqrt{A} \log A] \), the product \((A+\Delta)(B-\Delta)\) is, up to a \((1+O(\log^3 A)))\) multiplicative error, maximized at \( \sigma = \Delta = 0 \).

For instance, the corollary implies that for any \( w \in \{0, 1\}^m \),

\[
2^n \mu(w) \leq \binom{2k + f}{m} + m \max_{j,f_c(w_{1,j-1})} \left( \binom{k + f_c(w_{1,j-1})}{j-1} \right) \left( \frac{k + 1 + f - f_c(w_{j+1,m})}{m-j} \right)
\]

\[
\lesssim m \left( \frac{k + \frac{f}{2}}{m} \right) \lesssim \sqrt{k} \left( \frac{2k + f}{m} \right).
\]

The following is another simple combinatorial Lemma. It is proven in the appendix.

**Lemma 3.** For positive integers \( a \) and \( l \), the number of \( 0-1 \) strings \( w \) of length \( l \) such that \( f_c(w) = a \) is \( \left( \frac{l+1}{2a+1} \right) \).

Lemma 3 thus implies

\[
\mu(\{ w \in \{0, 1\}^m : f_c(w) = f \}) \lesssim 2^{-n} \sqrt{k} \left( \frac{2k + f}{m} \right) \left( \frac{m + 1}{2f + 1} \right).
\]

By apriori probabilistic reasoning,

\[
\mu \left( \bigcup_{m \notin [2k-\sqrt{k} \log(k), 2k+\sqrt{k} \log(k)]} \{0, 1\}^m \right) = O(e^{-\log^2 n}).
\]

Writing \( m = 2k + \delta \) and \( f = \frac{2k}{3} + \epsilon \), we see that

\[
\left( \frac{2k + f}{m} \right) \left( \frac{m}{2f} \right) = \left( \frac{\frac{8k}{3} + \epsilon}{2k + \delta} \right) \left( \frac{\frac{2k}{3} + 2\epsilon}{\frac{4k}{3} - 2\epsilon + \delta} \right).
\]

Using Lemma 2 with \( A = \frac{8k}{3}, B = \frac{4k}{3}, \eta = \frac{2k+\delta}{4k} = \frac{1}{2} + O\left(\frac{\log k}{\sqrt{k}}\right), \Delta = \epsilon \), and \( \sigma = 2\epsilon - \frac{2k}{3} \), we see that \( |f - \frac{2k}{3}| \geq \sqrt{k} \log(k) \) implies

\[
\mu(\{ w \in \{0, 1\}^m : f_c(w) = f \}) \lesssim 2^{-4k} \sqrt{k} e^{-\log^2 n} \left( \frac{8k/3}{4k/3} \right) \left( \frac{4k/3}{2k/3} \right)
\]

\[
= O(e^{-\log^2 n}).
\]

Hence, since there are at most \( n^2 \) values of \( (m, f) \), it holds that

\[
\mu(A) \geq 1 - O(e^{-\log^2 n}).
\]

The same argument shows that

\[
\nu(A) \geq 1 - O(e^{-\log^2 n}).
\]
3.3. Proving Inequality (4).

In this short subsection, we establish inequality (4). The explicit formula for \( \nu \) gives the lower bound

\[
2^n \nu(w) \geq \left( \frac{2k + f_c(w)}{|w|} \right),
\]

and so for any \( w \in A \) with \( |w| = m, f_c(w) = f \), we have

\[
\left| \frac{\mu(w) - \nu(w)}{\nu(w)} \right| \leq \frac{1}{(2k+f)^m} \sum_{1 \leq j \leq m} \left( k + f_c(w_{1,j-1}) \right) \left( k + f - f_c(w_{1,j-1}) \right) \frac{1}{k} |\delta_j - \epsilon_j|.
\]

The following (technical) lemma allows us to focus on the probablistically relevant ranges of the parameters involved.

**Lemma 4.** Let \( f \) and \( m \) be such that \( |f - \frac{2k}{3}|, |m - 2k| \leq \sqrt{k} \log(k) \). Then

\[
(k+a)^{(k+1+f-a)} \leq e^{-\log k} \left( \frac{4k}{m^2} \right)^2 \text{ unless } |a - \frac{f}{2}| \leq \sqrt{k} \log(k) \text{ and } |j - \frac{m}{2}| \leq \sqrt{k} \log(k).
\]

**Proof.** Lemma 2 implies, for any \( \lambda, \beta = O(A^{1/6}) \) and \( \eta \) bounded away from 0 and 1,

\[
\begin{pmatrix}
A + \lambda \sqrt{A} \\
\eta A + \beta \sqrt{A}
\end{pmatrix}
\begin{pmatrix}
A - \lambda \sqrt{A} \\
\eta A - \beta \sqrt{A}
\end{pmatrix}
\leq e^{\lambda^2 - \frac{\eta}{2} (\lambda - \beta)^2 (1 - \eta)}
\begin{pmatrix}
A \\
\eta A
\end{pmatrix}
\begin{pmatrix}
A \\
\eta A
\end{pmatrix}.
\]

We use \( A = k + \frac{f}{2}, \eta = \frac{m/2}{k+f} = \frac{3}{4} + O(\frac{\log k}{\sqrt{k}}), \lambda = \frac{a-f}{\sqrt{k+f}}, \) and \( \beta = \frac{j-m}{\sqrt{k+f}} \). \( \square \)

Using this Lemma together with Lemma 2, we obtain

\[
\left| \frac{\mu(w) - \nu(w)}{\nu(w)} \right| \leq \left( \frac{k+f}{2k+f} \right)^2 \log(k) \sqrt{k} \frac{1}{k} \sqrt{k} \log(k) \leq \frac{\log^2(k)}{\sqrt{k}} \leq \frac{\log^2(n)}{\sqrt{n}}.
\]

This establishes (4).

3.4. A Closed Form Expression for \( \sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} \).

In this subsection, we obtain a closed form expression for \( \sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} \), up to an acceptable (for the purposes of proving (5)) error. The lower bound obtained above on \( \nu \) together with the definition of \( A \) gives the upper bound

\[
\sum_{w \in A} \frac{(\mu(w) - \nu(w))^2}{\nu(w)} \leq \sum_{m \in [2k - \sqrt{k} \log(k), 2k + \sqrt{k} \log(k)]} \frac{1}{2^n (2k+f)^m} \sum_{|w|=m, f_c(w)=f} (2^n \mu(w) - 2^n \nu(w))^2.
\]

We fix \( m \) and \( f \) and focus on estimating

\[
\sum_{|w|=m, f_c(w)=f} (2^n \mu(w) - 2^n \nu(w))^2 =
\]

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we may assume \( j, t \) the case of be made shows the diagonal \( t > j \) is upper bounded by

\[
\sum_{\substack{|w| = m \\
 w \in A}} \sum_{1 \leq |w| = n \leq |w_j|} \left( k + f_c(w_{1,j-1}) \right) \left( k + f_c(w_{j+1,m}) \right) - \left( k + f_c(w_{1,j-1}) \right) \left( k + f_c(w_{j+1,m}) \right)
\]

\[
\times \left[ \left( k + f_c(w_{1,t-1}) \right) \left( k + f_c(w_{t+1,m}) \right) - \left( k + f_c(w_{1,t-1}) \right) \left( k + f_c(w_{t+1,m}) \right) \right].
\]

Up to a multiplicative factor of 2, we may assume \( t > j \) (the argument about to be made shows the diagonal \( t = j \) term is sufficiently small). Lemma 4 implies that we may assume \( j, t \in [\frac{m}{2} - \sqrt{k}\log(k), \frac{m}{2} + \sqrt{k}\log(k)] \). So we may in fact assume \( t > j + 5 \); indeed, by Lemmas 2 and 3, the sum over pairs \((j, t)\) with \( j \leq t < j + 5 \) is upper bounded by

\[
5 \sum_{\substack{|w| = m \\
 f_c(w) = f}} \sum_{j \in [k - \sqrt{k}\log(k), k + \sqrt{k}\log(k)]} \left( \frac{m}{2} \right)^4 \left( \frac{2f}{m} \right)^4 \frac{k}{\sqrt{k}} \left( \frac{m}{2} \right)^4 \left( \frac{m}{2} \right),
\]

and so summing this over \(|m - 2k| \leq \sqrt{k}\log(k)\) and \(|f - \frac{2k}{3}| \leq \sqrt{k}\log(k)\) with weights \( \frac{1}{2^n(2^{k+1})} \), we obtain an upper bound of

\[
(\sqrt{k}\log(k))^2 \frac{\log(k)^2}{\sqrt{k}} \frac{(k + \frac{4f}{m})^2}{24k} \frac{(m + \frac{4f}{m})^2}{24k} \sim \frac{50^3(k)}{k^3/2} \sim \frac{\log^3(n)}{n^{3/2}}.
\]

The reader should note that the estimates above indicate that we merely need a savings of \( \sqrt{k}\) over the trivial (magnitude) bound for \( \sum_{w \in A} (2^n\mu(w) - 2^n\nu(w))^2 \).

Fix some \( t \) and \( j \) with \( t > j + 5 \). We will now separate the sum over \( w \) based on \( f_c(w_{1,j-1}) \) and \( f_c(w_{1,t-1}) \). To relate \( f_c(w_{1,j-1}) \) to \( f_c(w_{j+1,m}) \) and \( f_c(w_{1,t-1}) \) to \( f_c(w_{t+1,m}) \), given \( f_c(w) \), we need to do casework on \( w_{j-1} \) and \( w_{t-1} \). We first do the case of \( w_{j-1} = w_{t-1} = 0 \). In this case, \( f_c(w_{j+1,m}) = f - f_c(w_{1,j-1}) - 1 \) and \( f_c(w_{t+1,m}) = f - f_c(w_{1,t-1}) - 1 \). This gives

\[
\sum_{\substack{|w| = m \\
 f_c(w) = f}} \sum_{w_{j-1} = 0, w_{t-1} = 0} \left[ \left( k + f_c(w_{1,j-1}) \right) \left( k + f_c(w_{j+1,m}) \right) - \left( k + f_c(w_{1,j-1}) \right) \left( k + f_c(w_{j+1,m}) \right) \right]
\]

\[
\times \left[ \left( k + f_c(w_{1,t-1}) \right) \left( k + f_c(w_{t+1,m}) \right) - \left( k + f_c(w_{1,t-1}) \right) \left( k + f_c(w_{t+1,m}) \right) \right] \]
Removing the product from the inner sum, we wish to count the set of \( \delta \) that may restrict attention to the case \((w, w)\). So, the case of \( w = 0, w = 1 \) affects our proceeding arguments. That is, our argument for a \( -1 \) savings for the other 3 cases. Therefore, we may restrict attention to the case \((w, 1, w) = (0, 0)\).

3.5. **Finishing the Proof of (5).**

In this final subsection, we use the closed form expression from subsection 3.4 to prove inequality (5). We start by noting that

\[
\left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a}{m - j} \right) - \left( \frac{k + 1 + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) = \\
\left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left[ \frac{m - j}{k + f - a - (m - j)} - \frac{j}{k + a - j} + O \left( \frac{1}{k} \right) \right].
\]

Let \( \delta_j \) and \( \epsilon_j \) be defined so that

\[
j = \frac{m}{2} + \delta_j\]
and
\[ a = \frac{j}{3} + \frac{f}{2} - \frac{m}{6} + \epsilon_j. \]

Observe that
\[ \frac{m - j}{k + f - a - (m - j)} - \frac{j}{k + a - j} = \frac{-2k\delta_j + \frac{m\delta_k}{3} + m\epsilon_j - f \delta_j}{(k + f - a - \frac{m}{2} + \delta_j)(k + a - \frac{m}{2} - \delta_j)}. \]

By Lemma 4, we may assume \( a \in [\frac{f}{2} - \sqrt{k \log(k)}, \frac{f}{2} + \sqrt{k \log(k)}] \), so that \( \epsilon_j = O(\sqrt{k \log(k)}) \). Since also \( m = 2k + O(\sqrt{k \log(k)}) \) and \( f = \frac{2k}{3} + O(\sqrt{k \log(k)}) \), we see that
\[ \frac{m - j}{k + f - a - (m - j)} - \frac{j}{k + a - j} = 18\left[\frac{1}{k}\epsilon_j - \delta_j\right] + O\left(\frac{\log^2(k)}{k}\right). \]

Therefore, defining \( \delta_t \) and \( \epsilon_t \) so that
\[ t = \frac{m}{2} + \delta_t \]
and
\[ b = \frac{t}{3} + \frac{f}{2} - \frac{m}{6} + \epsilon_t, \]
we see that (6) takes the form
\[
\frac{324}{k^2} \sum_{a,b,t,j} \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t]
\]
up to an acceptable error.

We now claim that \( b = a + \frac{t - j}{3} + O(\sqrt{t - j \log(k)}) \) or otherwise the magnitude of the summand corresponding to \( a, b, j, t \) is sufficiently small. Note that \( \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t] \)
\]
up to an acceptable error.

We now claim that \( b = a + \frac{t - j}{3} + O(\sqrt{t - j \log(k)}) \) or otherwise the magnitude of the summand corresponding to \( a, b, j, t \) is sufficiently small. Note that \( \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t] \)
\]
up to an acceptable error.

We now claim that \( b = a + \frac{t - j}{3} + O(\sqrt{t - j \log(k)}) \) or otherwise the magnitude of the summand corresponding to \( a, b, j, t \) is sufficiently small. Note that \( \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t] \)
\]
up to an acceptable error.

We now claim that \( b = a + \frac{t - j}{3} + O(\sqrt{t - j \log(k)}) \) or otherwise the magnitude of the summand corresponding to \( a, b, j, t \) is sufficiently small. Note that \( \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t] \)
\]
up to an acceptable error.

We now claim that \( b = a + \frac{t - j}{3} + O(\sqrt{t - j \log(k)}) \) or otherwise the magnitude of the summand corresponding to \( a, b, j, t \) is sufficiently small. Note that \( \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t] \)
\]
up to an acceptable error.

We now claim that \( b = a + \frac{t - j}{3} + O(\sqrt{t - j \log(k)}) \) or otherwise the magnitude of the summand corresponding to \( a, b, j, t \) is sufficiently small. Note that \( \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \times \left( \frac{k + b}{t - 1} \right) \left( \frac{k + f - b - 1}{m - t} \right) \left[ \delta_j - \epsilon_j \right] \cdot [\delta_t - \epsilon_t] \)
\]
where the sum is restricted to $b = a + \frac{t - j}{3} + O(\sqrt{f - j \log(k)}).

We split up

$$[\delta_t - \epsilon_t] = [\delta_t - \epsilon_j] + \left[ \epsilon_j + \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{m - 2 \delta_j} - \frac{1}{3} \right) (\delta_t - \delta_j) - \epsilon_t \right]$$

$$- \left[ \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{m - 2 \delta_j} - \frac{1}{3} \right) (\delta_t - \delta_j) \right].$$

For any fixed $a, j, t$, by Lemma 2 with $A = \delta_t - \delta_j, B = \frac{m}{2} - \delta_t, \eta = \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{\frac{m}{2} - \delta_j} = \frac{2}{3} + O(\log \frac{k}{\sqrt{k}}), \Delta = 0, \sigma = 2 \epsilon_t - 2 \epsilon_j - \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{\frac{m}{2} - \delta_j} - \frac{2}{3} \right) (\delta_t - \delta_j)$, we have that

$$\left( \frac{2}{3} (\delta_t - \delta_j) + 2 \epsilon_t - 2 \epsilon_j \right) \left( \frac{\frac{m}{2} - \delta_t}{f - \frac{2}{3} \delta_j - 2 \epsilon_j} \right) = \left( 1 + O \left( \frac{\log^3(k)}{\sqrt{\delta_t - \delta_j}} \right) \right) \left( \frac{1}{3} (\delta_t - \delta_j) + 2 \epsilon_t - 2 \epsilon_j \right) \left( \frac{\frac{m}{2} - \delta_t}{f - \frac{2}{3} \delta_j - 2 \epsilon_t} \right),$$

where $\epsilon^*_t$ is the reflection\(^1\) of $\epsilon_t$ about $\epsilon_j + \frac{1}{2} \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{\frac{m}{2} - \delta_j} - \frac{2}{3} \right) (\delta_t - \delta_j).$ And therefore, since

$$\epsilon_j + \frac{1}{2} \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{\frac{m}{2} - \delta_j} - \frac{2}{3} \right) (\delta_t - \delta_j) - \epsilon_t = O \left( \sqrt{\delta_t - \delta_j \log(k) + \log^2(k)} \right),$$

we deduce that

$$\left| \sum_b \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left[ \epsilon_j + \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{m - 2 \delta_j} - \frac{1}{3} \right) (\delta_t - \delta_j) - \epsilon_t \right] \right|$$

$$\lesssim \sum_b \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \log^5(k)$$

is small enough. And clearly the term

$$\sum_b \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) \left[ \left( \frac{f - \frac{2}{3} \delta_j - 2 \epsilon_j}{m - 2 \delta_j} - \frac{1}{3} \right) (\delta_t - \delta_j) \right]$$

is small enough. We therefore may focus on the term

$$\sum_b \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) [\delta_t - \epsilon_j].$$

\(^1\)We might have to round $\epsilon^*_t$ a bit (so that $\frac{t}{3} + \frac{j}{3} - \frac{m}{6} + \epsilon^*_t$ is an integer), but the induced error in this rounding is negligible, by Lemma 2.
If \( t > j + 5 \), Lemma 5, proven in the appendix, states that
\[
\sum_b \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) = \left( \frac{1}{2} + O \left( \frac{\log^2(k)}{t - j} \right) \right) \sum_b \left( \frac{t - j - 1}{b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - b - 1} \right) .
\]

And using the general combinatorial identity
\[
\sum C \left( \begin{array}{c} D \\ C \end{array} \right) \left( \begin{array}{c} E \\ F - C \end{array} \right) = \left( \begin{array}{c} D + E \\ F \end{array} \right) ,
\]
we see that
\[
\sum_b \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) = \left( \frac{1}{2} + O \left( \frac{\log^2(k)}{t - j} \right) \right) \sum_b \left( \frac{t - j - 1}{b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - b - 1} \right) .
\]

Therefore, (6) is, up to a negligible error, equal to
\[
\frac{162}{k^2} \sum_{a,t,j} \left( 1 + O \left( \frac{\log^2(k)}{t - j} \right) \right) \left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \left( \frac{j - 1}{2a + 1} \right) \left( \frac{m - j}{2f - 2a - 2} \right) \times \left( \frac{k + a + \frac{t - j}{3}}{t - 1} \right) \left( \frac{k + f - a - \frac{t - j}{3} - 1}{m - t} \right) [\delta_j - \epsilon_j] \cdot [\delta_t - \epsilon_j] ,
\]
where the sum is restricted to \( t > j + 5 \).

We can rid of the \( O(\log^2(k) / t - j) \) term trivially. Indeed, using Lemma 2, we can upper bound
\[
\left( \frac{k + a}{j - 1} \right) \left( \frac{k + f - a - 1}{m - j} \right) \lesssim \left( \frac{k + f}{m} \right)^2 ,
\]
and
\[
\left( \frac{j - 1}{2a + 1} \right) \left( \frac{m - j}{2f - 2a - 2} \right) \lesssim \left( \frac{m}{f} \right)^2 ;
\]
noting that for each \( \Delta \geq 5 \), the number of pairs \((t, j) \in [m/2 - \sqrt{k \log(k)}, m/2 + \sqrt{k \log(k)}] \) with \( t - j = \Delta \) is at most \( \sqrt{k \log(k)} \), we thus obtain an upper bound of
\[
\frac{162}{k^2} \sqrt{k \log(k)} \sqrt{k \log(k)} \left( \frac{k + f}{m} \right)^4 \left( \frac{m}{f} \right)^2 \sum_{\Delta=5}^{\infty} \frac{\log^2(k)}{\Delta} ,
\]
which is small enough.

Let
\[
f(\delta_j, \epsilon_j) = \frac{162}{k^2} \left( \begin{array}{c} k + a \\ j \end{array} \right) \left( \begin{array}{c} k + f - a \\ m - j \end{array} \right) \left( \begin{array}{c} j \\ 2a \end{array} \right) \left( \begin{array}{c} m - j \\ 2f - 2a \end{array} \right)
\]
Lemma 2. The first approximation is the content of Lemma 6. For the second, use $|\delta - \epsilon_j| \leq \sqrt{k \log k}$ and for any $\delta_t, \epsilon_j$ with $|\delta_t|, |\epsilon_j| \leq \sqrt{k \log k}$, it holds that

$$g(\delta_t, \epsilon_j) = \left( k + a + \frac{t - 2}{3} \right) \left( k + f - a - \frac{t - 2}{3} - 1 \right).$$

We break up the remaining expression as follows:

$$\sum_{\epsilon_j \delta_j} g(\delta_t, \epsilon_j) [\delta_j - \epsilon_j] = \sum_{\epsilon_j \delta_j} \left[ f(\epsilon_j, \delta_j) [\delta_j - \epsilon_j] + g(\delta_t, \epsilon_j)[\delta_j - \epsilon_j] \right].$$

We claim that $g$ has symmetry in $\delta_t$ about $\epsilon_j$ and $f$ has symmetry in $\delta_j$ about $\epsilon_j$: $g(\delta_t, \epsilon_j) \approx g(2\epsilon_j - \delta_t, \epsilon_j)$ and $f(\epsilon_j, \delta_j) \approx f(\epsilon_j, 2\epsilon_j - \delta_j)$. This is the content of the quite fortuitous Lemmas 6 and 7.

**Lemma 6.** For any $f$ and $m$ with $|f - \frac{2k}{3}|, |m - 2k| \leq \sqrt{k \log k}$ and for any $\delta_t, \epsilon_j$ with $|\delta_t|, |\epsilon_j| \leq \sqrt{k \log k}$, it holds that

$$\left( k + \frac{f}{2} + \frac{\delta_j}{3} \right) \left( k + \frac{f}{2} - \frac{\delta_j}{3} \right) = \left( 1 + O \left( \frac{\log^3(k)}{\sqrt{k}} \right) \right) \left( k + \frac{f}{2} - \frac{\delta_j}{3} \right).$$

**Proof.** Lemma 2, with $A = k + \frac{f}{2}, B = k + \frac{f}{2}, g = \frac{m/2}{k + \frac{f}{2}} = \frac{3}{4} + O\left( \frac{\log k}{\sqrt{k}} \right), \delta = \frac{\delta_j}{3} + \epsilon_j, \sigma = \delta_t$ and $\Delta = \frac{\delta_j}{3} - \frac{5\epsilon_j}{3}, \sigma = \delta_t - 2\epsilon_j$, shows that both products of binomial coefficients are $(1 + O\left( \frac{\log^3(k)}{\sqrt{k}} \right)) \exp\left( -\frac{3(\delta_t - \epsilon_j)^2}{k + f/2} \right) \left( k + f/2 \right)^2$. \hfill \(\square\)

**Lemma 7.** For any $f$ and $m$ with $|f - \frac{2k}{3}|, |m - 2k| \leq \sqrt{k \log k}$ and for any $\delta_j, \epsilon_j$ with $|\delta_j|, |\epsilon_j| \leq \sqrt{k \log k}$, it holds that

$$\left( k + \frac{f}{2} + \frac{\delta_j}{3} \right) \left( k + \frac{f}{2} - \frac{\delta_j}{3} \right) = \left( 1 + O \left( \frac{\log^3(k)}{\sqrt{k}} \right) \right) \left( k + \frac{f}{2} - \frac{\delta_j}{3} - 2\epsilon_j \right) \left( k + \frac{f}{2} - \delta_j + 2\epsilon_j \right).$$

and

$$\left( \frac{m + \delta_j}{2} + \frac{f}{2} \right) \left( \frac{m + \delta_j}{2} - \frac{f}{2} \right) = \left( 1 + O \left( \frac{\log^3(k)}{\sqrt{k}} \right) \right) \left( \frac{m + 2\epsilon_j - \delta_j}{f} + \frac{10\epsilon_j}{3} - \frac{2\delta_j}{3} \right) \left( \frac{m - 2\epsilon_j + \delta_j}{f} - \frac{10\epsilon_j}{3} + \frac{2\delta_j}{3} \right).$$

**Proof.** The first approximation is the content of Lemma 6. For the second, use Lemma 2 with $A = \frac{m}{2}, B = \frac{m}{2}, \eta = \frac{2f}{m} = \frac{2}{3} + O\left( \frac{\log k}{\sqrt{k}} \right), \Delta = \delta_j, \sigma = \frac{2\delta_j}{3} + 2\epsilon_j$ and $\Delta = 2\epsilon_j - \delta_j, \sigma = \frac{10\epsilon_j}{3} - \frac{2\delta_j}{3}$ to see that both products of binomial coefficients are $(1 + O\left( \frac{\log^3(k)}{\sqrt{k}} \right)) \exp\left( -\frac{18\epsilon_j^2}{m/2} \right).$ \hfill \(\square\)

---

2See footnote 1 on page 12.
Indeed, without loss of generality, we may assume \( \delta_j < \epsilon_j \), for which

\[
\sum_{\delta_t > \delta_j + 5} g(\delta_t, \epsilon_j)(\delta_t - \epsilon_j) = \sum_{\delta_t > \delta_j} g(\delta_t, \epsilon_j)(\delta_t - \epsilon_j) + O \left( \frac{\log^3(k)}{\sqrt{k}} \right) \sum_{\delta_t} g(\delta_t, \epsilon_j)|\delta_t - \epsilon_j|.
\]

Lemma 7 then allows us to write our expression as

\[
\sum_{\epsilon_j > \delta_j} \sum_{\delta_t > \delta_j + 5} f(\epsilon_j, \delta_j) \left[ (\epsilon_j - \delta_j) \sum_{\delta_t > \delta_j + 5} g(\delta_t, \epsilon_j)(\delta_t - \epsilon_j) + (\epsilon_j - \delta_j) \sum_{\delta_t > \delta_j + 5} g(\delta_t, \epsilon_j)(\delta_t - \epsilon_j) \right]
\]

up to a negligible error. But this is just 0, and so we’ve established (5).

4. Remaining Proofs of Lemmas

**Lemma 3.** For positive integers \( a \) and \( l \), the number of \( 0-1 \) strings \( w \) of length \( l \) such that \( f_c(w) = a \) is \( \binom{l+1}{2a+1} \).

**Proof.** Let \( g_l(a) \) be the desired quantity and \( g_l'(a) \) be the number of \( 0-1 \) strings of length \( l+1 \) that begin with a 0 and have \( f_c(w) = a \). Then if \( l \geq 2 \) and \( a \geq 1 \),

\[
g_l(a) = g_{l-2}(a-1) + g_{l-2}(a) + 2g_{l-2}'(a),
\]

where \( g_0(0) := 1 \), and

\[
g_l'(a) = g_{l-1}'(a) + g_{l-1}(a-1),
\]

the first equality following from doing casework on strings starting with 01; 11; and, 10 or 00; the second equality following from doing casework on strings start with 00; and 01. The result follows from checking that \( g_l(a) = \binom{l+1}{2a+1} \) and \( g_l'(a) = \binom{l+1}{2a} \) satisfy the recurrence relations with the right initial conditions. \( \square \)

**Lemma 5.** For any fixed \( a, j, t, m, f \) with \( |m - 2k|, |j - \frac{m}{2}|, |t - \frac{m}{2}|, |f - \frac{2k}{3}|, |a - \frac{t}{2}| \leq \sqrt{k} \log(k) \) and \( t > j \), the following holds:

\[
\sum_{b} \left( \frac{t - j - 1}{2b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - 2b - 1} \right) = \left( \frac{1}{2} + O \left( \frac{\log^2(k)}{t - j} \right) \right) \sum_{b} \left( \frac{t - j - 1}{b - 2a - 1} \right) \left( \frac{m - t + 1}{2f - b - 1} \right).
\]

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Lemma 2. For \( \eta \) bounded away from 0 and 1 and any \( A, B, \Delta, \) and \( \sigma \), it holds that
\[
\left[ \begin{pmatrix} (\frac{A+\Delta}{\eta A}) & (\frac{B-\Delta}{\eta B}) \\ (\frac{A}{\eta A}) & (\frac{B}{\eta B}) \end{pmatrix} \right]^{-1} = (1 + O(\sigma^3 A^2))(1 + O(\frac{\Delta}{A^2}))(1 + O(\frac{1}{A}))(1 + O(\frac{(\Delta - \sigma)^3}{A^2}))(1 + O(\frac{1}{A}))(1 + O(\frac{(\Delta - \sigma)^3}{A^2})) \exp \left( \frac{1}{2} \frac{(\Delta - \sigma)^2}{(1 - \eta)A} + \frac{1}{2} \frac{\sigma^2}{\eta A} - \frac{1}{2} \frac{\Delta^2}{A} \right) \\
\times (1 + O(\frac{\sigma^3}{B^2}))(1 + O(\frac{\Delta^3}{B^2}))(1 + O(\frac{1}{B}))(1 + O(\frac{(\Delta - \sigma)^3}{B^2}))(1 + O(\frac{1}{B}))(1 + O(\frac{(\Delta - \sigma)^3}{B^2})) \exp \left( \frac{1}{2} \frac{(\Delta - \sigma)^2}{(1 - \eta)B} + \frac{1}{2} \frac{\sigma^2}{\eta B} - \frac{1}{2} \frac{\Delta^2}{B} \right).
\]

Proof. Using Stirling’s approximation
\[ n! = \left(1 + O\left(\frac{1}{n}\right)\right) n^n e^{-n} \sqrt{2\pi n}, \]
we obtain
\[
\left[ \begin{pmatrix} (\frac{A+\Delta}{\eta A}) & (\frac{B-\Delta}{\eta B}) \\ (\frac{A}{\eta A}) & (\frac{B}{\eta B}) \end{pmatrix} \right]^{-1} = (1 + O(\frac{1}{A}))(1 + O(\frac{1}{B})){n!}.
\]
\[
\times \left[ \frac{(1 - \eta)A + (\Delta - \sigma)}{A + \Delta} \right] B - \frac{\Delta}{(1 - \eta)B - (\Delta - \sigma)} \right]^\Delta (1 + \frac{\Delta - \sigma}{\eta A}(1 - \frac{\Delta - \sigma}{(1 - \eta)A})^B
\]

\[
\times (1 + \frac{\Delta - \sigma}{\eta B})^B(1 - \frac{\Delta - \sigma}{A + \Delta})^A(1 - \frac{\Delta - \sigma}{(1 - \eta)B})^B(1 + \frac{\Delta - \sigma}{B - \Delta})^B.
\]

Now, using that \(\log(1 + x) = x - \frac{x^2}{2} + O(x^3)\) for small \(x\),

\[
(1 + \frac{\Delta - \sigma}{\eta A})^B(1 - \frac{\Delta - \sigma}{A + \Delta})^A(1 - \frac{\Delta - \sigma}{(1 - \eta)B})^B(1 + \frac{\Delta - \sigma}{B - \Delta})^B
\]

\[
= \exp(\eta A) \left( \frac{\sigma}{\eta A} - \frac{1}{2} \frac{\sigma^2}{\eta^2 A^2} + O(\frac{\sigma^3}{A^3}) \right) \exp((1 - \eta)A(\frac{\Delta - \sigma}{(1 - \eta)A} - \frac{1}{2} (\frac{\Delta - \sigma}{(1 - \eta)A})^2 + O(\frac{\Delta - \sigma}{A^3})) \times \exp(-A(\frac{\Delta - \sigma}{A + \Delta} + \frac{1}{2} \frac{\Delta^2}{(A + \Delta)^2} + O(\frac{\Delta^3}{(A + \Delta)^3})) \times \exp(-1(\frac{\Delta - \sigma}{1 - \eta}B + \frac{1}{2} (\frac{\Delta - \sigma}{(1 - \eta)B})^2 + O(\frac{\Delta - \sigma}{B^3})) \times (1 + O(\frac{\sigma^3}{A^3}(1 + \frac{\Delta^3}{A^3})) \exp(-1(\frac{\sigma^2}{2 \eta A} - \frac{1}{2} (\frac{\Delta - \sigma}{(1 - \eta)A})^2 + \frac{\Delta^2}{A})) \times (1 + O(\frac{\sigma^3}{B^3})(1 + O(\frac{\Delta^3}{B^3})) \exp(-1(\frac{\sigma^2}{2 \eta B} - \frac{1}{2} (\frac{\Delta - \sigma}{(1 - \eta)B})^2 + \frac{\Delta^2}{B}))
\]

And,

\[
\left[ \frac{\eta A + \sigma}{(1 - \eta)A + (\Delta - \sigma)} \right] \frac{1}{\eta B - (\Delta - \sigma)} = \left[ 1 + \frac{(1 - \eta)A + (\Delta - \sigma)}{(1 - \eta)B + (\Delta - \sigma)} \right] \frac{\sigma}{\eta A} - \frac{\Delta - \sigma}{(1 - \eta)B} + \frac{\sigma}{(1 - \eta)A} + O(\frac{\sigma^2}{A^2}) + O(\frac{\Delta - \sigma}{B^2}) + O(\frac{\sigma^3}{B^2}))
\]

Additionally,

\[
\left[ \frac{(1 - \eta)A + (\Delta - \sigma)}{A + \Delta} \right] = \left[ 1 + (\Delta - \sigma)B - (1 - \eta)B + (\Delta - \sigma)A - (1 - \eta)A \right] \frac{\Delta}{(1 - \eta)A} + O(\frac{\Delta - \sigma}{A^2}) + O(\frac{\Delta^2}{A^2}) + O(\frac{\Delta - \sigma}{B^2}) + O(\frac{\Delta^2}{B^2})
\]

Combining everything yields the Lemma.

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Department of Mathematics, California Institute of Technology, Pasadena, CA, 91125

E-mail address: zchase@caltech.edu