A NOTE ON CORRELATION AND LOCAL DIMENSIONS

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Abstract. Under very mild assumptions, we give formulas for the correlation and local dimensions of measures on the limit set of a Moran construction by means of the data used to construct the set.

1. Introduction

The correlation dimension was introduced in [18]. It is widely used in numerical investigations of dynamical systems. The properties of the correlation dimension and the $L^q$-spectrum has been studied for various types of attractors of iterated function systems; for example, see [3, 15, 17, 19, 22]. We continue this line of research and in Proposition 2.3 we complement the study initiated in [7, 10].

The other important object in this note is the local dimension of a measure. It has a close connection with the theory of Hausdorff and packing dimensions of a set. Therefore it is a classical problem to try to express the local dimension by means of the data used to construct the set; for example, see [1, 2, 7, 11, 13, 21]. Our main result in this note is Theorem 3.5. Under a natural separation condition, the finite clustering property, it solves this problem completely.

2. Correlation dimension via general filtrations

Let $(X, d)$ be a compact metric space and $\mu$ a locally finite Borel regular measure supported in $X$. Since the metric will always be clear from the content, we simply denote $(X, d)$ by $X$. Recall that the support of a measure $\mu$, denoted by $\text{spt}(\mu)$, is the smallest closed subset of $X$ with full $\mu$-measure. For $s \geq 0$ and $x \in X$, define the $s$-potential of $\mu$ at the point $x$ to be

$$\phi_s(x) = \int d(x, y)^{-s} \, d\mu(y),$$

where $d(x, y)$ is the distance between two points $x$ and $y$ in $X$. Furthermore, define the $s$-energy of $\mu$ to be

$$I_s(\mu) = \int \phi_s(x) \, d\mu(x) = \iint d(x, y)^{-s} \, d\mu(x) \, d\mu(y).$$

For the basic properties of the $s$-energy, the reader is referred to the Mattila’s book [14, §8]. The quantity

$$\dim_{\text{cor}}(\mu) = \inf \{s : I_s(\mu) = \infty\} = \sup \{s : I_s(\mu) < \infty\}$$

is called the correlation dimension of the measure $\mu$. Measure-theoretical properties of this dimension map are studied in [15].

We now recall the definition of the local dimension of measures. Let $\mu$ be a locally finite Borel regular measure on metric space $X$. The lower and upper local dimensions of the measure $\mu$ at a
point \( x \in X \) are defined respectively by
\[
\dim_{\text{loc}}(\mu, x) = \liminf_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}, \\
\dim_{\text{loc}}(\mu, x) = \limsup_{r \downarrow 0} \frac{\log \mu(B(x, r))}{\log r}.
\]
Here \( B(x, r) \) is the closed ball of radius \( r > 0 \) centered at \( x \in X \). We also define the lower Hausdorff dimension of the measure \( \mu \) by setting
\[
\dim_{\text{H}}(\mu) = \text{ess inf}_{x, \mu} \dim_{\text{loc}}(\mu, x).
\]
The correlation dimension of a measure \( \mu \) is at most the lower Hausdorff dimension of the measure \( \mu \). We recall the proof of this simple fact in the following lemma.

**Lemma 2.1.** If \( X \) is a compact metric space and \( \mu \) is a finite Borel regular measure on \( X \), then
\[
\dim_{\text{loc}}(\mu, x) = \inf \{ s : \phi_s(x) = \infty \} = \sup \{ s : \phi_s(x) < \infty \}
\]
for all \( x \in X \). Furthermore,
\[
\dim_{\text{cor}}(\mu) = \liminf_{r \downarrow 0} \frac{\log \int \mu(B(x, r)) \, d\mu(x)}{\log r} \leq \dim_{\text{H}}(\mu).
\]

**Proof.** Fix \( x \in X \). If \( s \) is so that \( \phi_s(x) < \infty \), then
\[
r^{-s} \mu(B(x, r)) \leq \int_{B(x, r)} d(x, y)^{-s} \, d\mu(y) \leq \phi_s(x) \quad \text{for all } r > 0.
\]
(2.1)
It follows that \( \dim_{\text{loc}}(\mu, x) \geq s \) and thus,
\[
\dim_{\text{loc}}(\mu, x) \geq \sup \{ s : \phi_s(x) < \infty \}.
\]

To show that \( \dim_{\text{cor}}(\mu) \leq \dim_{\text{H}}(\mu) \), fix \( s > \dim_{\text{H}}(\mu) \). Notice that there exists a set \( A \) with \( \mu(A) > 0 \) such that \( \dim_{\text{loc}}(\mu, x) < s \) for all \( x \in A \). The above reasoning implies that \( \phi_s(x) = \infty \) for all \( x \in A \). Therefore \( I_s(\mu) = \infty \) and the claim follows. Similarly, if \( s < \dim_{\text{cor}}(\mu) \), then, by integrating (2.1), we see that
\[
r^{-s} \int \mu(B(x, r)) \, d\mu(x) \leq I_s(\mu) < \infty \quad \text{for all } r > 0.
\]
Therefore
\[
\liminf_{r \downarrow 0} \frac{\log \int \mu(B(x, r)) \, d\mu(x)}{\log r} \geq \dim_{\text{cor}}(\mu). \tag{2.2}
\]
To show the remaining inequalities, fix \( t < s < \dim_{\text{loc}}(\mu, x) \). Observe that now there exists \( r_0 > 0 \) such that \( \mu(B(x, r)) < r^t \) for all \( 0 < r < r_0 \). Thus
\[
\phi_t(x) = \int \, d(x, y)^{-t} \, d\mu(y) = t \int_0^\infty r^{-t-1} \mu(B(x, r)) \, dr \\
\leq t \int_0^{r_0} r^{s-t-1} \, dr + t \int_{r_0}^\infty r^{-t-1} \mu(B(x, r)) \, dr < \infty
\]
and \( \inf \{ s : \phi_s(x) = \infty \} \geq t \). The proof of the converse inequality of (2.2) is similar and thus omitted. \( \square \)

**Remark 2.2.** (1) If there exist \( A \subset X \) and \( s, r_0, c > 0 \) such that \( \mu(B(x, r)) \leq cr^s \) for all \( 0 < r < r_0 \) and \( x \in A \), then Lemma 2.1 implies that \( \dim_{\text{cor}}(\mu|_A) \geq s \). In particular, if \( \mu \) is a finite measure, then for every \( \varepsilon > 0 \) there exists a compact set \( A \) with \( \mu(X \setminus A) < \varepsilon \) such that \( \dim_{\text{cor}}(\mu|_A) \geq \dim_{\text{H}}(\mu) \). To see this, fix \( \varepsilon > 0 \) and let \( \{s_i\}_{i \in \mathbb{N}} \) be a strictly increasing sequence converging to \( \dim_{\text{H}}(\mu) \). Egorov’s theorem implies that for every \( i \) there are \( r_i > 0 \) and a compact set \( A_i \subset X \) with \( \mu(X \setminus A_i) < 2^{-i} \varepsilon \) such that \( \mu(B(x, r)) < r^{s_i} \) for all \( 0 < r < r_i \) and \( x \in A_i \). Defining \( A = \bigcap_{i=1}^{\infty} A_i \) we have \( \mu(X \setminus A) \leq \sum_{i=1}^{\infty} \mu(X \setminus A_i) < \varepsilon \). Fix \( N \in \mathbb{N} \) and let \( B_N = \bigcap_{i=1}^{N} A_i \). Then \( \mu(B(x, r)) < r^{s_N} \)
for all $0 < r < \min_{i \in \{1, \ldots, N\}} r_i$ and $x \in B_N \supset A$. This gives $\dim_{\text{cor}}(\mu|_A) \geq s_N$ and, as $N$ was arbitrary, finishes the proof.

(2) Let us consider the standard $\frac{1}{3}$-Cantor set and define $\mu_p$ to be the Bernoulli measure associated to the probability vector $(p, 1 - p)$. It is well known that $\dim_{\text{cor}}(\mu_p) = -\log_3(p^2 + (1 - p)^2)$; for example, see Proposition 2.3. Recalling e.g. [4, Proposition 10.4], we see that $\dim_{\text{cor}}(\mu_p) < \dim_{\text{H}}(\mu_p)$ for all $p \in (0, 1) \setminus \{1/2\}$.

If the metric space $X$ is doubling, then we can define the correlation dimension via a discrete process. More precisely, we will see that the definition can be given in terms of general filtrations. These filtrations can be considered to be generalized dyadic cubes. This gives a way to calculate the correlation dimension in many Moran constructions; see Corollary 3.2.

Before stating the theorem, we recall the definitions of the doubling metric space and the general filtration. A metric space $X$ is said to be doubling, if there is a doubling constant $N = N(X) \in \mathbb{N}$ such that any closed ball $B(x, r)$ with center $x \in X$ and radius $r > 0$ can be covered by $N$ balls of radius $r/2$. A doubling metric space is always separable and the doubling property can be stated in several equivalent ways. For instance, a metric space $X$ is doubling if and only if there are $0 < s, C < \infty$ such that if $B$ is an $r$-packing of a closed ball $B(x, R)$ with $0 < r < R$, then the cardinality of $B$ is at most $C(R/r)^s$. Here the $r$-packing $B$ of a set $A$ is a collection of disjoint closed balls having radius $r$. We write $A B(x, r) = B(x, \lambda r)$ for $\lambda \in (0, \infty)$.

The definition of the general filtration is introduced in [7]. We assume that $(\delta_n)_{n \in \mathbb{N}}$ and $(\gamma_n)_{n \in \mathbb{N}}$ are two decreasing sequences of positive real numbers satisfying

(F1) $\delta_n \leq \gamma_n$ for all $n \in \mathbb{N},$

(F2) $\lim_{n \to \infty} \gamma_n = 0,$

(F3) $\lim_{n \to \infty} \log \delta_n / \log \delta_{n+1} = 1,$

(F4) $\lim_{n \to \infty} \log \gamma_n / \log \delta_n = 1.$

For each $n \in \mathbb{N}$, let $Q_n$ be a collection of disjoint Borel subsets of the doubling metric space $X$ such that each $Q \in Q_n$ contains a ball $B_0$ of radius $\delta_n$ and is contained in a ball $B^Q$ of radius $\gamma_n$. Define

$$ E = \bigcap_{n \in \mathbb{N}} \bigcup_{Q \in Q_n} Q. $$

The collection $\{Q_n\}_{n \in \mathbb{N}}$ is called the general filtration of $E$.

The classical dyadic cubes of the Euclidean space is an example of a general filtration. Such kind of nested constructions can also be defined on doubling metric spaces (see e.g. [8]) and these constructions also serve as examples. It should be noted that the nested structure is not always a necessity; consult [9] for such examples. In Lemma 3.1, we show that certain Moran constructions are general filtrations. These constructions include, for example, all the self-conformal sets satisfying the strong separation condition.

Besides giving the desired discrete version of the definition, the following result states also that the correlation dimension is in fact the $L^2$-spectrum of the measure; see [7, Proposition 3.2]. In $\mathbb{R}^n$, this is of course well known – the point here is that the equivalence now covers the general filtrations case too.

**Proposition 2.3.** Let $X$ be a compact doubling metric space. If $\{Q_n\}_{n \in \mathbb{N}}$ is a general filtration of $E$ and $\mu$ is a finite Borel regular measure on $E$, then

$$ \dim_{\text{cor}}(\mu) = \lim_{n \to \infty} \inf \frac{\log \sum_{Q \in Q_n} \mu(Q)^2}{\log \delta_n}. $$

**Proof.** Observe that for each $Q \in Q_n$ we have $Q \subset B(x, 2\gamma_n)$ for all $x \in Q$. Therefore

$$ \int \mu(B(x, 2\gamma_n)) \, d\mu(x) = \sum_{Q \in Q_n} \int_Q \mu(B(x, 2\gamma_n)) \, d\mu(x) \geq \sum_{Q \in Q_n} \mu(Q)^2 $$


and it follows from Lemma 2.1 and (F4) that
\[ \dim_{\text{cor}}(\mu) \leq \lim_{n \to \infty} \frac{\log \int \mu(B(x, 2\gamma_n)) \, d\mu(x)}{\log 2\gamma_n} \leq \lim_{n \to \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n} \mu(Q)}{\log \delta_n}. \]

To show the other inequality, fix \( r > 0 \) and let \( n \in \mathbb{N} \) be such that \( \gamma_{n+1} \leq r < \gamma_n \). Choose for each \( Q \in \mathcal{Q}_n \) balls \( B_Q \) of radius \( \delta_n \) and \( B^Q \) of radius \( \gamma_n \) so that \( B_Q \subset Q \subset B^Q \). Now for each \( Q \in \mathcal{Q}_n \) we have
\[ Q \subset B(x, 2\gamma_n) \subset B_Q[4\gamma_n] \subset \bigcup_{Q' \in \mathcal{C}_Q} \bigcup_{r < \gamma_n} B_{Q'}, \]
for all \( x \in Q \), where \( \mathcal{C}_Q = \{ Q' \in \mathcal{Q}_n : Q' \cap B_Q[4\gamma_n] \neq \emptyset \} \) and \( B[r] \) denotes the ball of radius \( r \) having the same center as the ball \( B \). Thus
\[ \int \mu(B(x, 2\gamma_n)) \, d\mu(x) = \sum_{Q \in \mathcal{Q}_n} \int_Q \mu(B(x, 2\gamma_n)) \, d\mu(x) \leq \sum_{Q \in \mathcal{Q}_n} \mu(Q) \sum_{Q' \in \mathcal{C}_Q} \mu(Q'). \tag{2.3} \]

Observe that if \( Q \in \mathcal{Q}_n \), then
\[ \mu(Q) \sum_{Q' \in \mathcal{C}_Q} \mu(Q') \leq \left( \sum_{Q' \in \mathcal{C}_Q} \mu(Q') \right)^2 \leq \# \mathcal{C}_Q \sum_{Q' \in \mathcal{C}_Q} \mu(Q')^2. \tag{2.4} \]

Here \( \# \mathcal{C}_Q \) is the cardinality of the set \( \mathcal{C}_Q \). Let us next estimate this cardinality.

Fix \( Q \in \mathcal{Q}_n \) and \( Q' \in \mathcal{C}_Q \). Since \( Q' \cap B_Q[4\gamma_n] \neq \emptyset \) we have \( B^{Q'} \cap B_Q[4\gamma_n] \neq \emptyset \) and
\[ B_{Q'} \subset Q' \subset B^{Q'} \subset B_Q[6\gamma_n]. \]
Thus the collection \( \{ B_{Q'} : Q' \in \mathcal{C}_Q \} \) is a \( \delta_n \)-packing of the ball \( B_Q[6\gamma_n] \). It follows from the doubling property of \( X \) that
\[ \# \mathcal{C}_Q = \# \{ B_{Q'} : Q' \in \mathcal{C}_Q \} \leq C \left( \frac{6\gamma_n}{\delta_n} \right)^s. \]

Recalling (2.3) and (2.4), this estimate gives
\[ \int \mu(B(x, 2\gamma_n)) \, d\mu(x) \leq C \left( \frac{6\gamma_n}{\delta_n} \right)^s \sum_{Q \in \mathcal{Q}_n} \sum_{Q' \in \mathcal{C}_Q} \mu(Q')^2. \tag{2.5} \]

To get rid of the double-sum, we need to estimate the cardinality of the set \( \mathcal{D}_{Q'} = \{ Q \in \mathcal{Q}_n : Q' \in \mathcal{C}_Q \} \). If \( Q \in \mathcal{Q}_n \) is such that \( Q' \in \mathcal{C}_Q \), then \( B^{Q'} \cap B_Q[4\gamma_n] \neq \emptyset \) and \( B_Q \subset B^{Q'} \subset B_Q[9\gamma_n] \). Thus the collection \( \{ B_Q : Q \in \mathcal{D}_{Q'} \} \) is a \( \delta_n \)-packing of the ball \( B^{Q'}[9\gamma_n] \) and, as above,
\[ \# \mathcal{D}_{Q'} = \# \{ B_Q : Q \in \mathcal{D}_{Q'} \} \leq C \left( \frac{9\gamma_n}{\delta_n} \right)^s. \]
Therefore, it follows from (2.5) that
\[ \int \mu(B(x, 2\gamma_n)) \, d\mu(x) \leq C^2 \left( \frac{6\gamma_n}{\delta_n} \right)^s \left( \frac{9\gamma_n}{\delta_n} \right)^s \sum_{Q \in \mathcal{Q}_n} \mu(Q)^2. \]

This, together with (F3) and (F4) yields
\[ \dim_{\text{cor}}(\mu) \geq \lim_{n \to \infty} \frac{\log \int \mu(B(x, 2\gamma_n)) \, d\mu(x)}{\log \gamma_{n+1}} \geq \lim_{n \to \infty} \frac{\log \sum_{Q \in \mathcal{Q}_n} \mu(Q)^2}{\log \delta_n} \]
finishing the proof. \( \square \)
3. Dimensions in Moran constructions

We adopt the following notation for the rest of the paper. For each \( j \in \mathbb{N} \) we fix a finite set \( I_j \) and define \( \Sigma = \prod_{j=1}^{\infty} I_j \). Thus, if \( \sigma \in \Sigma \), then \( \sigma = i_1 i_2 \cdots \) where \( i_j \in I_j \) for all \( j \in \mathbb{N} \). With the product discrete topology, \( \Sigma \) is compact. We let \( \sigma|_n = i_1 \cdots i_n \) and \( \Sigma_n = \{ \sigma|_n : \sigma \in \Sigma \} \) for all \( n \in \mathbb{N} \). We also define \( \Sigma_* = \bigcup_{n=1}^{\infty} \Sigma_n \). The concatenation of \( \sigma \in \Sigma_\ast \) and \( \delta \in \Sigma_\ast \cup \Sigma \) is denoted by \( \sigma \delta \). The length of \( \sigma \in \Sigma_* \) is denoted by \( |\sigma| \). For \( \sigma \in \Sigma_* \) we set \( \sigma^- = \sigma|_{|\sigma|-1} \) and \( [\sigma] = \{ \omega \in \Sigma : \omega|_{|\sigma|} = \sigma \} \). Finally, we define \( \sigma_0 \) to be \( \varnothing \).

Let \( X \) be a complete separable metric space. A Moran construction is a collection \( \{ E_\sigma : \sigma \in \Sigma_* \} \) of compact subsets of \( E_\varnothing \subset X \) satisfying the following two properties:

1. \( E_\sigma \subset E_{\sigma^-} \) for all \( \sigma \in \Sigma_* \),
2. diam\( (E_{\sigma|_n}) \to 0 \) as \( n \to \infty \) for all \( \sigma \in \Sigma \).

Here diam\( (A) \) denotes the diameter of the set \( A \). The limit set of the Moran construction is the compact set

\[
E = \bigcap_{n \in \mathbb{N}} \bigcup_{\sigma \in \Sigma_n} E_\sigma.
\]

Each \( E_\sigma \) is called a construction set of level \( |\sigma| \). We emphasize that the placement of the construction sets at each level of the Moran construction can be arbitrary, and they need not be disjoint. Clearly, self-similar sets introduced in [6], (finitely generated) self-conformal sets studied in [16], and Moran sets defined in [2, 7, 11, 12, 13, 20, 23] are all special cases of the limit sets of Moran constructions. If \( \sigma \in \Sigma \), then the single point in the set \( \bigcap_{n \in \mathbb{N}} E_{\sigma|_n} \subset E \) is denoted by \( \pi(\sigma) \). This defines a surjective mapping \( \pi : \Sigma \to E \) which is continuous in the product discrete topology.

The following lemma connects Moran constructions to general filtrations considered in [2].

Lemma 3.1. Let \( \{ E_\sigma : \sigma \in \Sigma_* \} \) be a Moran construction on a complete separable metric space satisfying [M1], [M2],

1. \( E_{\sigma_i} \cap E_{\sigma_j} = \emptyset \) for all \( \sigma_i, \sigma_j \in \Sigma_* \) with \( i \neq j \),
2. there exists \( C_0 > 0 \) such that for each \( \sigma \in \Sigma_* \) there is \( x \in E_\sigma \) for which
   \[
   B(x, C_0 \text{diam}(E_\sigma)) \subset E_\sigma,
   \]
3. it holds that
   \[
   \lim_{n \to \infty} \frac{\log \min\{\text{diam}(E_\omega) : \omega \in \Sigma_{n+1} \text{ such that } \omega^- = \sigma|_n \}}{\log \text{diam}(E_{\sigma|_n})} = 1,
   \]
where the convergence is uniform for all \( \sigma \in \Sigma \).

If \( Q_n = \{ E_\sigma : \text{diam}(E_\sigma) \leq \gamma_n < \text{diam}(E_{\sigma^-}) \} \), where \( \gamma_n = \min\{\text{diam}(E_\sigma) : \sigma \in \Sigma_n \} \), then there exists a sequence \( (\delta_n)_{n \in \mathbb{N}} \) such that \( \{ Q_n \}_{n \in \mathbb{N}} \) is a general filtration for \( E \).

Proof. The claim is proved in the course of the proof of [7, Lemma 4.2].

A self-conformal set satisfying the strong separation condition is a simple example of a limit set of a Moran construction satisfying the assumptions of Lemma 3.1. This can be easily seen since the condition

1. \( \text{diam}(E_\sigma) \geq c \text{diam}(E_{\sigma^-}) \) for all \( \sigma \in \Sigma_* \setminus \{ \varnothing \} \),

is satisfied by any self-conformal set, together with [M1] implies [M5].

Recall that in complete separable metric spaces locally finite Borel regular measures are Radon. If \( X \) and \( Y \) are complete separable spaces and \( \mu \) is a Radon measure with compact support on \( Y \), then the pushforward measure of \( \mu \) under a continuous mapping \( f : Y \to X \) is denoted by \( f\mu \). In this case, \( f\mu \) is a Radon measure and \( \text{spt}(f\mu) = f(\text{spt}(\mu)) \).

Corollary 3.2. Let \( \{ E_\sigma : \sigma \in \Sigma_* \} \) be a Moran construction on a compact doubling metric space satisfying [M1] and [M3] and
(M7) there exist $C \geq 1$ and a sequence $(\beta_n)_{n \in \mathbb{N}}$ with $\log \beta_n/\log \beta_{n+1} \to 1$ such that $C^{-1}\beta_n \leq \text{diam}(E_\sigma) \leq C\beta_n$ for all $\sigma \in \Sigma_n$ and $n \in \mathbb{N}$.

If $\mu$ is a finite Borel regular measure on $\Sigma$, then

$$\dim_{co}(\pi \mu) = \lim\inf_{n \to \infty} \frac{\log \sum_{\omega \in \Sigma_n} \mu([\omega])^2}{\log \beta_n}. $$

Proof. Observe that (M7) clearly implies (M5). Furthermore, it also guarantees that $C^{-1}\beta_n \leq \gamma_n \leq C\beta_n$ for all $n \in \mathbb{N}$, where $\gamma_n$ is as in Lemma 3.1. Therefore, the claim follows immediately from Lemma 3.1 and Proposition 2.3.

We will next start studying local dimensions of a measure. For that we need the following separation condition. Given a Moran construction $\{E_\sigma : \sigma \in \Sigma_n\}$, we set

$$N(x, r) = \{\sigma \in \Sigma_n : \text{diam}(E_\sigma) \leq r < \text{diam}(E_\sigma) - E_\sigma \cap B(x, r) \neq \emptyset\}$$

for all $x \in E$ and $r > 0$. We say that the Moran construction satisfies the finite clustering property if

$$\sup_{x \in E} \limsup_{r \downarrow 0} \#N(x, r) < \infty.$$

Observe that a Moran construction on a compact doubling metric space satisfying (M3) (M4) and (M6) satisfies the finite clustering property. For a self-conformal set, the finite clustering property is equivalent to the open set condition; see e.g. [12] Corollary 5.8. For more examples, the reader is referred to [11]. Also, Moran sets considered in [23] (with $c_0 > 0$) satisfy the finite clustering property.

The following proposition plays an important role in the proof of Theorem 3.5.

Proposition 3.3. Suppose that $X$ and $Y$ are complete separable metric spaces and $f : Y \to X$ is continuous. If $\mu$ is a locally finite Borel measure with compact support on $Y$ and $A \subset Y$ is such that $\mu(A) > 0$, then

$$\dim_{loc}(f \mu, f(y)) = \dim_{loc}(f|_A, f(y)),$$

$$\dim_{loc}(f \mu, f(y)) = \dim_{loc}(f|_A, f(y)),$$

for $\mu$-almost all $y \in A$.

Proof. Clearly always $\dim_{loc}(f \mu, f(y)) \leq \dim_{loc}(f|_A, f(y))$. Let us assume that there are a bounded set $A' \subset A$ and $\gamma > 0$ such that

$$\dim_{loc}(f \mu, f(y)) + \gamma < \dim_{loc}(f|_A, f(y))$$

for all $y \in A'$. Then for every $y \in A'$ there is a sequence $r_j \downarrow 0$ such that

$$\mu(A \cap f^{-1}(B(f(y), 5r_j))) < (5r_j)^\gamma f \mu(B(f(y), r_j))$$

for all $j \in \mathbb{N}$. Since $f \mu$ is a Radon measure we find an open set $U \subset X$ with $U \supset f(A')$ and $f \mu(U) < \infty$. Let $\varrho > 0$ and, by relying on the $5r$-covering theorem (see e.g. [5] Theorem 1.2), choose a countable disjoint subcollection $B_\varrho$ of

$$\{B(f(y), r) : y \in A' \text{ and } 0 < r < \varrho \text{ such that } B(f(y), r) \subset U \text{ and } \mu(A \cap f^{-1}(B(f(y), 5r))) < (5r)^\gamma f \mu(B(f(y), r)) \}$$

such that $5B_\varrho$ covers $f(A')$. Since now

$$\mu(A') \leq \sum_{B \in B_\varrho} \mu(A \cap f^{-1}(5B)) < \sum_{B \in B_\varrho} (5\varrho)^\gamma f \mu(B) \leq (5\varrho)^\gamma f \mu(U)$$

the claim follows by letting $\varrho \downarrow 0$. The proof for the upper local dimension is similar and thus omitted.
Remark 3.4. We say that a measure \( \mu \) on \( X \) has the \textit{density point property} if

\[
\lim_{r \downarrow 0} \frac{\mu(A \cap B(x, r))}{\mu(B(x, r))} = 1
\]

for \( \mu \)-almost all \( x \in A \) whenever \( A \subset X \) is \( \mu \)-measurable. It was demonstrated in [3] Example 5.6] that the density point property is not necessarily valid for all measures even if the space \( X \) is doubling. Despite of this, Proposition 3.3 implies that

\[
\dim_{\text{loc}}(\mu, x) = \dim_{\text{loc}}(\mu|_A, x),
\]

\[
\dim_{\text{loc}}(\mu, x) = \dim_{\text{loc}}(\mu|_A, x),
\]

for \( \mu \)-almost all \( x \in A \).

We will next show that under the finite clustering property, the lower local dimension of the pushforward measure can be obtained symbolically. This result generalizes [7] Lemma 4.2 and [11], Proposition 3.10]. The proof of [7, Lemma 4.2] used (M5) and assumed that the construction sets of the same level were disjoint and had a certain shape. In [11, Proposition 3.10], it was assumed that the Moran construction is so rigid that

(M8) there exists \( D \geq 1 \) such that \( \text{diam}(E_{ij}) \leq D \text{diam}(E_i) \) \( \text{diam}(E_j) \) for all \( i, j \in \Sigma \).

Although constructions given by iterated function systems satisfy (M8), it rules out many interesting Moran constructions. The proof also used uniform version of the finite clustering property. In the main part of the following theorem, we do not require any of these assumptions.

**Theorem 3.5.** If \( \{ E_{\sigma} : \sigma \in \Sigma_+ \} \) is a Moran construction on a complete separable metric space satisfying (M1) [M2] and the finite clustering property, and \( \mu \) is a finite Borel regular measure on \( \Sigma \), then

\[
\dim_{\text{loc}}(\pi \mu, \pi(\sigma)) = \lim_{n \to \infty} \frac{\log \mu([\sigma|_n])}{\log \text{diam}(E_{\sigma|_n})},
\]

(3.1)

\[
\dim_{\text{loc}}^{\ast}(\pi \mu, \pi(\sigma)) \geq \limsup_{n \to \infty} \frac{\log \mu([\sigma|_n])}{\log \text{diam}(E_{\sigma|_n})},
\]

(3.2)

for \( \mu \)-almost all \( \sigma \in \Sigma \). Furthermore, if the Moran construction also satisfies (M5) then (3.2) holds with an equality.

**Proof.** We may clearly assume that \( \mu \) has no atoms. Since \( E_{\sigma|_n} \subset B(\pi(\sigma), \text{diam}(E_{\sigma|_n})) \) and \( \mu([\sigma|_n]) \leq \pi \mu(E_{\sigma|_n}) \) for all \( \sigma \in \Sigma \) and \( n \in \mathbb{N} \), the right-hand side of (3.1) is an upper bound for the lower local dimension. Furthermore, if \( \varepsilon > 0 \), then (M5) implies that there exists \( n_0 \in \mathbb{N} \) such that \( \text{diam}(E_{\sigma|_n}) < \text{diam}(E_{\sigma|_{n+1}})^{1-\varepsilon} \). Thus

\[
\log \frac{\pi \mu(B(\pi(\sigma), r))}{\log r} \leq \log \frac{\pi \mu(B(\pi(\sigma), \text{diam}(E_{\sigma|_{n+1}})))}{\log \text{diam}(E_{\sigma|_{n+1}})} \leq \frac{\log \mu([\sigma|_{n+1}])}{(1 - \varepsilon) \log \text{diam}(E_{\sigma|_{n+1}})}
\]

for all \( r > 0 \) and \( n \geq n_0 \) with \( \text{diam}(E_{\sigma|_{n+1}}) < r \leq \text{diam}(E_{\sigma|_n}) \). This shows that, assuming (M5), the right-hand side of (3.2) is an upper bound for the upper local dimension.

To show that the right-hand side of (3.2) is also an lower bound, suppose to the contrary that there is a set \( A' \subset E \) with \( \mu(A') > 0 \) and \( s > 0 \) such that

\[
\dim_{\text{loc}}(\pi \mu, \pi(\sigma)) < s < \liminf_{n \to \infty} \frac{\log \mu([\sigma|_n])}{\log \text{diam}(E_{\sigma|_n})}
\]

for all \( \sigma \in A' \). By Egorov’s theorem, the limit above is uniform in a set \( A \subset A' \) with \( \mu(A) > 0 \): there is \( n_0 \in \mathbb{N} \) such that

\[
\mu([\omega|_n]) < \text{diam}(E_{\omega|_n})^s \quad \text{(3.3)}
\]

for all \( \omega \in A \) and \( n \geq n_0 \).
Let $\sigma \in A$ and, relying on the finite clustering property, choose $M \in \mathbb{N}$ and $r_0 > 0$ such that
\[#N(\pi(\sigma), r) \leq M\] for all $0 < r < r_0$. We trivially have
\[\pi(\mu|A)(B(\pi(\sigma), r)) \leq \sum_{\omega \in N(\pi(\sigma), r)} \mu|A([\omega])\]
for all $0 < r < r_0$. Observe that if $\omega \in \Sigma_\ast$ satisfies $\text{diam}(E_\omega) < \min\{\text{diam}(E_\tau) : \tau \in \Sigma_m\}$, then $|\omega| > n_0$. Therefore, assuming $r_0 < \min\{\text{diam}(E_\tau) : \tau \in \Sigma_m\}$, we may apply the estimate (3.3) for each $\omega \in N(\pi(\sigma), r)$ whenever $[\omega] \cap A \neq \emptyset$ and $0 < r < r_0$. Thus
\[\pi(\mu|A)(B(\pi(\sigma), r)) \leq \sum_{\omega \in N(\pi(\sigma), r)} \text{diam}(E_\omega)^s \leq Mr^s\]
and, consequently, $\dim_{\text{loc}}(\pi(\mu|A), \pi(\sigma)) \geq s > \dim_{\text{loc}}(\pi(\mu, \pi(\sigma))$ for all $\sigma \in A$. This contradicts with Proposition 3.3. The proof of (3.2) is similar and thus omitted. \hfill $\Box$

**Remark 3.6.** It would be interesting to know if the inequality in (3.2) can be strict under the assumptions of Theorem 3.5. We remark that a non-uniform version of (M5) suffices for the equality.

**Example 3.7.** For each $j \in \mathbb{N}$ let $I_j$ be a finite set, $N_j = |I_j|$, and $\Sigma = \prod_{j=1}^{\infty} I_j$. Let $p = (p_{j_1}, \ldots, p_{j_N})$ be a positive probability vector for all $j \in \mathbb{N}$ and $\mu$ the probability measure for which $\mu([\sigma]) = \prod_{j=1}^{n_j} p_{j_i}$, for all $\sigma = i_1 \cdots i_n \in \Sigma_n$ and $n \in \mathbb{N}$. We assume that $\{E_\sigma : \sigma \in \Sigma_\ast\}$ is a Moran construction on a compact doubling metric space satisfying (M3) (M4) (M7) and the finite clustering property. It follows from Theorem 3.5 that
\[\dim_{\text{loc}}(\pi(\mu, \pi(\sigma)) = \liminf_{n \to \infty} \sum_{\sigma \in \Sigma_n} \log p_{j_i} \log \beta_n^{j_i}\]
for $\mu$-almost all $\sigma = i_1 i_2 \cdots \in \Sigma$. On the other hand, since $\sum_{\sigma \in \Sigma_n} \mu([\sigma])^2 = \prod_{j=1}^{n} \sum_{i=1}^{N_j} p_{j_i}^2$, for all $n \in \mathbb{N}$, Corollary 3.2 implies that
\[\dim_{\text{coa}}(\pi(\mu)) = \liminf_{n \to \infty} \sum_{\sigma \in \Sigma_n} \log \left(\sum_{i=1}^{N_j} p_{j_i}^2\right) \log \beta_n^{j_i}.\]
In particular, if the measure $\mu$ is the uniform distribution, that is, $p_{j_i} = N_j^{-1}$ for all $i \in I_j$ and $j \in \mathbb{N}$, then $\dim_{\text{coa}}(\pi(\mu)) = \dim_{\text{H}}(\pi(\mu))$.

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