New motion algorithm in the particle-in-cell method

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Abstract. The article proposes a new method for solving the equations of motion of charged particles in electromagnetic fields. In the algorithm, the velocity of a charged particle at a time step is computed in accordance with the exact solution of the differential equation. The method has been compared with various known modifications of the Boris method. The comparison was carried out both in terms of the accuracy of the methods and the time of their operation. A new modification of the method uses the fact that the electric and magnetic fields are constant at a time step. It allows one to compute more accurately the trajectory and velocity of a charged particle without significant increasing the complexity of the computations. It has been shown that when choosing a modification of the Boris method to solve a problem, one should pay attention first of all to the accuracy of the solution, since a simpler and faster scheme may not give a gain in time.

1. Introduction

To solve numerically many physical problems, including the problems of plasma physics, the particle-in-cell (PIC) method developed by Harlow F. [1] is widely used. In the PIC method, plasma is represented as a large set of model charged particles moving in a self-consistent electromagnetic field. The electromagnetic field is determined from Maxwell’s equations, taking into account charges and currents as sources. Their densities are computed as functions of the coordinates and velocities of the particles using any accurate algorithm. Modeling the plasma behavior requires a large series of computations and processing a huge amount of data, therefore, an important criterion for choosing an algorithm to calculate the particle motion is the simplicity of the algorithm [2].

Currently, the most commonly used algorithm to find the trajectory and velocity of a particle is the Boris method [3]. The properties of this method had been described in detail in the literature (see, for example, [2], [4]): it belongs to the schemes of the “leapfrog” type, has the second order of accuracy, works equally well at short and long process development times, and is quite economical. A theoretical proof of the stability of the Boris method and the property of conservation of energy from the point of view of methods for solving ordinary differential equations in the absence of an electric field are presented in [5].

Recently, the interest to the Boris method and its modifications has been increased. They allow to simplify, refine, or accelerate the computation of the trajectory and velocity of a charged particle. The most famous and widely used modifications had been considered in [2] – [4]. In the paper [6] there had been proposed a three-step algorithm with the same accuracy as in the
Boris method. The \( n \)-step algorithm with the same accuracy as in the Boris method had been considered in [7]. In [8], a method that uses the Boris two-step algorithm \( n \) times, having a lower error than the Boris method had been proposed. The modification of the method based on the exact solution of the motion equations and determining precisely the rotation of a particle under the influence of a magnetic field had been considered in [9]. In [10], the implicit methods for solving energy-conserving relativistic problems had been considered, including the method based on the approximation of the guiding center. The algorithm proposed in [11] has a high order of accuracy and is based on a combination of the Boris method, other well-known algorithms, and high-order schemes. In [12], the methods of high (3rd and 4th) orders had been investigated, which allow using a larger time step and reducing the program runtime when solving relativistic problems.

In this paper, we present a new method for solving the equations of motion of charged particles in electromagnetic fields and compare it with the most frequently used modifications of the Boris method. The developed two-dimensional and three-dimensional algorithms are based on the use of an exact solution of the differential equation for the velocity of a charged particle at a time step. The considered methods are compared in terms of the accuracy and efficiency of computations using the solution of classical test problems in two-dimensional and three-dimensional cases in the nonrelativistic limit. In each case, the variants of analytically and discretely specified electric and magnetic fields have been used. For discretely defined fields, the 2D or 3D linear interpolation on shifted grids has been used. Our new modification of the Boris method makes it possible to more accurately compute the trajectory and velocity of a charged particle avoiding a significant increase in the number of arithmetic operations.

2. Statement of the problem

There is the motion of one charged particle of a mass \( m \) with a charge \( q \) in an electromagnetic field [13], [14]. The state of the particle in the Cartesian coordinates is determined by the coordinates in space \( \vec{x} = (x, y, z) \) and the velocity vector \( \vec{v} = (u, v, w) \) at a given point in space at a particular time. The particle motion is under the action of an external electromagnetic field, described by the Lorentz force:

\[
\vec{F} = q\vec{E} + \frac{q}{c}\vec{v} \times \vec{B},
\]

where \( \vec{E} \) is the vector of the electric field intensity, \( \vec{B} \) is the magnetic field intensity, and \( c \) is the speed of light. Within the framework of the PIC method, the motion of a charged particle satisfies the laws of mechanics, so it can be described by the following Cauchy problem for \( t \in [t_0, T] \):

\[
\begin{cases}
  \frac{d}{dt}\vec{x}(t) = \vec{v}(\vec{x}, t), \\
  m\gamma(\vec{v})\frac{d}{dt}\vec{v}(\vec{x}, t) = q \left( \vec{E}(\vec{x}, t) + \frac{\vec{v} \times \vec{B}(\vec{x}, t)}{c} \right), \\
  \vec{x}(t_0) = \vec{x}^0, \\
  \vec{v}(t_0) = \vec{v}^0,
\end{cases}
\]

where \( \gamma(\vec{v}) = 1/\sqrt{1 - (v/c)^2} \) is a relativistic factor. In this paper, we consider the nonrelativistic motion of a particle \( (\gamma = 1) \). In the general case, problem (2) cannot be solved analytically, which makes it necessary to use numerical methods for solving the problem.
3. Numerical methods to solve the problem

3.1. The Boris method

To compute the trajectory and velocity of a particle, the Boris method is usually used [3]. The main idea of the method consists in replacing the system of ODEs (2) with a system of finite-difference equations and solving it using the splitting by physical processes. Let us introduce the computational grid \( \{ t_k = t_0 + k \Delta t, \ k = 1, \ldots , n_t \} \), where \( \Delta t \) is a time step. The approximate values of the particle coordinate \( \vec{x}(t_k) \approx \vec{x}_k \), velocity vector \( \vec{v}(t_k) \approx \vec{v}_k \), magnetic intensity \( \vec{B}(\vec{x}_k, t_k) \approx \vec{B}_k \) and electric field intensity \( \vec{E}(\vec{x}_k, t_k) \approx \vec{E}_k \) are determined at the grid nodes sequentially. To improve the accuracy, a grid shifted by half a step is used. Here an approximate velocity vector \( \vec{v}(t_{k+1/2}) \approx \vec{v}_{k+1/2} \) is introduced, which is computed at the grid nodes \( \{ t_{k+1/2} = t_0 + (k + 1/2) \Delta t, \ k = 1, \ldots , n_t \} \).

Then, within the framework of the Boris method, the problem is reduced to solving the following system of finite-difference equations:

\[
\begin{align*}
\frac{\vec{x}_{k+1} - \vec{x}_k}{\Delta t} &= \vec{v}_{k+1/2},
\frac{\vec{v}_{k+1/2} - \vec{v}_{k-1/2}}{\Delta t} &= \frac{q}{m} \left( \frac{\vec{E}_k}{2c} \vec{v}_k + \frac{(\vec{v}_{k+1/2} + \vec{v}_{k-1/2}) \times \vec{B}_k}{2c} \right),
\end{align*}
\]

(3)

Thus, in the Boris scheme, the position of the particle at the next moment in time is explicitly computed from the velocity, the value of which is determined at an intermediate moment in time, which guarantees the second order of accuracy in determining the trajectory.

To solve the system (3), it is necessary to specify an additional initial condition for the velocity. The value \( \vec{v}_{-1/2} \) is introduced at a fictitious time \( t_{-1/2} \) and, then, in the second equation of the system (3) at time \( t_0 \) the velocity derivative is replaced by a one-sided finite difference:

\[
\frac{\vec{v}_0 - \vec{v}_{-1/2}}{\Delta t/2} \approx \left. \frac{d \vec{v}}{dt} \right|_{t=t_0} = \frac{q}{m} \vec{E}(t_0) + \frac{q}{mc} \left[ \vec{v}(t_0) \times \vec{B}(t_0) \right].
\]

Therefore, the required additional condition has the form:

\[
\vec{v}_{-1/2} = \vec{v}_0 - \frac{\Delta tq}{2m} \left( \vec{E}(t_0) + \frac{1}{c} \left[ \vec{v}_0 \times \vec{B}(t_0) \right] \right).
\]

At the next stage, the effect of the Lorentz force (1) is split into the contribution of the electric field at half-step (4), (6) and the contribution of the magnetic field (5):

\[
\vec{v}^- = \vec{v}_{k-1/2} + \frac{q}{m} \vec{E}_k \frac{\Delta t}{2},
\]

(4)

\[
\frac{\vec{v}^+ - \vec{v}^-}{\Delta t} = \frac{q}{2mc} (\vec{v}^+ + \vec{v}^-) \times \vec{B}_k,
\]

(5)

\[
\vec{v}_{k+1/2} = \vec{v}^+ + \frac{q}{m} \vec{E}_k \frac{\Delta t}{2}.
\]

(6)

The vector equation (5) reflects the rotation of the vector \( \vec{v}^- \) to \( \vec{v}^+ \) in a time step under the influence of the vector \( \vec{B} \). The geometric process of transforming the velocity vector at a time step is shown in Figure 1.

The solution of the equation (5) is reduced to a simple procedure for solving a \( 3 \times 3 \) SLAE by the Gauss method. However, during the code running the SLAE solving can be called thousands of times and, in total, increase the computation time significantly.
3.2. Modifications of the Boris method

The modifications of the Boris method are aimed to simplify the computations or decrease an error in the numerical solution of the problem (2). The replacement of the SLAE (5) with simpler computations or transfer from a three-step algorithm to less cumbersome methods is used.

3.2.1. Modification A

The modification proposed by Boris in 1970 [3] is based on the idea of avoiding to solve the SLAE. Additional vectors $\vec{p}$ and $\vec{v}'$ are introduced, due to which one can find $\vec{v}^+$ explicitly:

$$\vec{p} = \tan \left( \frac{\theta}{2} \right) b,$$

$$\vec{b} = \frac{\vec{B}_k}{|\vec{B}_k|},$$

$$\vec{v}' = \vec{v}^- + \vec{v}^- \times \vec{p},$$

$$\vec{v}^+ = \vec{v}^- + \frac{2}{1 + p^2} (\vec{v}' \times \vec{p}),$$

where $\theta = \frac{q \Delta t}{m} |\vec{B}_k|$ is the angle of rotation of the velocity vector per time step.

3.2.2. Modification B

The algorithm proposed by Birdsall and Langdon [4] differs from the above algorithm A in the way of representing the vector $\vec{p}$:

$$\vec{p} = \frac{\theta}{2} \vec{b}.$$

Such representation of the vector $\vec{p}$ allows to reduce the code running time, but may lead to the increase in the error. The schematic representation of the second stage of the modifications of the Boris method is shown in Fig. 2. The black lines correspond to the rotation through the angle $\theta$ of the vector $\vec{v}^-$ up to $\vec{v}^+_A$ in modification A without intermediate steps; the gray lines correspond to the transformation in modification B of the vector $\vec{v}^-$ to $\vec{v}^+_B$. The difference between the vectors $\vec{v}^+_A$ and $\vec{v}^+_B$ can be characterized by the value $\delta \theta$, which is the difference between the angles of the velocity vector rotation in the above modifications.

Figure 1. Speed conversion based on the Boris method.
3.2.3. Modification $C$  In [4], an alternative modification of the Boris method is proposed. It makes possible to determine the rotation of the velocity vector accurately. For that, a component of the vector $\vec{v}^-$ parallel to the magnetic intensity vector is introduced:

$$\vec{v}_{||}^-=\left(\vec{v}^-\cdot\vec{b}\right)\vec{b}.$$ 

Then

$$\vec{v}^+=\vec{v}_{||}^-+\left(\vec{v}^--\vec{v}_{||}^-\right)\cos\theta+\left(\vec{v}^-\times\vec{b}\right)\sin\theta.$$ 

As in the modification $A$, it allows one to rotate the velocity vector in a time step accurately, however, increases the code running time due to the use of trigonometric functions.

3.3. Test problem
To check the properties of the considered algorithms, a test problem of the charged particle motion has been used, where the magnetic field $\vec{B}$ and the potential electric field $\vec{E}$ depend on the position of the particle in space and are computed by the formulas [4]:

$$\vec{B} = \left(x^2 + y^2\right)^{1/2} \hat{e}_z,$$

$$\phi = 0.01 \left(x^2 + y^2\right)^{-1/2},$$

$$\vec{E} = -\nabla\phi = 0.01 \left(\frac{x}{(x^2 + y^2)^{3/2}}, \frac{y}{(x^2 + y^2)^{3/2}}, 0\right),$$

where $\phi$ is the potential of the electric field. The initial data for solving the equations of motion are: $\vec{x}_0 = (0.9, 0, 0)$, $\vec{v}_0 = (0.1, 0, 0)$. In the test problem, all quantities are given in dimensionless form.

In this statement of the problem, the particle will move in the plane $z = 0$. The rotation of the particle under the influence of a magnetic field occurs with a drift along the isolines of the electric field. It corresponds to the trajectory in Figure 3, obtained using the Boris method.

In Figure 3 the dashed line shows the isolines of the potential of the electric field $\phi = \text{const}$, along which the particle rotates. Here the color scale reflects the values of the potential. Over time, the trajectory will be locked and the particle will continue to move without changing the rotation radii. Similar results have been obtained using all the considered algorithms. We emphasize that the Boris method and its known modifications conserve the kinetic energy of a particle.
Figure 3. Trajectory of the charged particle in the electromagnetic field at the time interval [0, 600], Δt = 0.05.

4. New scheme for calculating the motion of a charged particle (algorithm D)

In the three-dimensional case \( \vec{B} = (B_x, B_y, B_z) \), \( \vec{E} = (E_x, E_y, E_z) \), the initial ODE system to find the components of the velocity vector \( \vec{v} = (u, v, w) \) of a charged particle is as follows:

\[
\begin{align*}
\frac{du}{dt} &= \frac{q}{m} E_x + \frac{q}{m c} B_z v - \frac{q}{m c} B_y w, \\
\frac{dv}{dt} &= \frac{q}{m} E_y + \frac{q}{m c} B_x w - \frac{q}{m c} B_z u, \\
\frac{dw}{dt} &= \frac{q}{m} E_z + \frac{q}{m c} B_y u - \frac{q}{m c} B_x v.
\end{align*}
\] (7)

If the electric and magnetic fields are constant, then the ODE system has an exact solution. First, we find the exact solution for the case when \( \vec{B} = (0, 0, B_z) \), \( \vec{E} = (E_x, E_y, E_z) \). The system (7) has the form:

\[
\begin{align*}
\frac{du}{dt} &= \frac{q}{m} E_x + \omega v, \\
\frac{dv}{dt} &= \frac{q}{m} E_y - \omega u, \\
\frac{dw}{dt} &= \frac{q}{m} E_z,
\end{align*}
\] (8)

where \( \omega = qB_z/(mc) \).

If the velocity value at the previous fractional step \( \vec{v}_{k-1/2} \) has been used as the initial condition, then the solution of the ODE system (8) at the time step \( t \in [t_k, t_k + \Delta t] \) has the
form:

\[
\begin{align*}
  u(t) &= \frac{E_y}{B_z} + \left( u_{k-1/2} - \frac{E_y}{B_z} \right) \cos \omega t + \left( v_{k-1/2} + \frac{E_x}{B_z} \right) \sin \omega t, \\
  v(t) &= -\frac{E_x}{B_z} + \left( v_{k-1/2} + \frac{E_x}{B_z} \right) \cos \omega t - \left( u_{k-1/2} - \frac{E_y}{B_z} \right) \sin \omega t, \\
  w(t) &= w_{k-1/2} + \frac{q}{m} E_z t.
\end{align*}
\]  

(9)

Note that if $B_z$ is small or equal to zero, then the formulas (9) have to be transformed. To do it, the trigonometric functions are expanded in the Taylor series remaining the first non-zero term. The first two formulas for the solution (9) take the form

\[
\begin{align*}
  u(t) &= u_{k-1/2} \cos \omega t + \frac{q}{m} E_x t + \frac{q}{2m} E_y \omega t^2, \\
  v(t) &= v_{k-1/2} \cos \omega t - \frac{q}{m} E_y t - u_{k-1/2} \sin \omega t - \frac{q}{2m} E_x \omega t^2.
\end{align*}
\]  

(10)

To find the exact solution of the system (8), we have used the formulas (9) and the expansion of the velocity vector into components parallel and perpendicular to the magnetic intensity vector.

In the general case, the longitudinal components of the vectors of the velocity and electric field intensity are computed by the formulas:

\[
\begin{align*}
  \vec{v}_{||} &= \left( \vec{v} \cdot \vec{b} \right) \vec{b}, \\
  \vec{E}_{||} &= \left( \vec{E} \cdot \vec{b} \right) \vec{b}, \\
  \vec{b} &= \frac{\vec{B}_k}{|\vec{B}_k|}.
\end{align*}
\]

The longitudinal velocity changes according to the formula:

\[
\vec{v}_{||}(t) = \vec{v}_{||}(0) + \vec{E}_{||} t.
\]

Transverse components appear as:

\[
\begin{align*}
  \vec{v}_{\perp} &= \vec{v} - \vec{v}_{||}, \\
  \vec{E}_{\perp} &= \vec{E} - \vec{E}_{||}.
\end{align*}
\]

By passing to arbitrary coordinates in the system (9), one can find a solution for the perpendicular component of the velocity:

\[
\begin{align*}
  u_{\perp}(t) &= E_y b_z - E_z b_y + \left[ u_{\perp}(0) - E_y b_z + E_z b_y \right] \cos \omega t + \\
  &+ \left[ v_{\perp}(0) b_z - w_{\perp}(0) b_y + E'_{x} \right] \sin \omega t, \\
  v_{\perp}(t) &= E_z b_x - E_x b_z + \left[ v_{\perp}(0) + E_z b_z + E'_{y} b_z \right] \cos \omega t + \\
  &+ \left[ w_{\perp}(0) b_x - u_{\perp}(0) b_z + E'_{y} \right] \sin \omega t, \\
  w_{\perp}(t) &= E_x b_y - E_y b_x + \left[ w_{\perp}(0) - E_x b_y + E'_{y} b_x \right] \cos \omega t + \\
  &+ \left[ u_{\perp}(0) b_y - v_{\perp}(0) b_x + E'_{z} \right] \sin \omega t,
\end{align*}
\]  

(11)
where $\vec{E}' = c\vec{E}_1 / B$. Adding the longitudinal velocity component and, also, taking into account the fact that the solution is considered at a time step and the value of the velocity at the step $t_{k-1/2}$ is considered as an initial condition, the discrete analogue of the equations (10) in the general three-dimensional case:

\[
\begin{align*}
    u_{k+1/2} &= E_y b_z - E_z b_y + \left[ u_{k-1/2}^+ - E'_y b_z + E'_z b_y \right] \cos \omega \Delta t + u_{||}, \\
    v_{k+1/2} &= E'_x b_x - E'_z b_y + \left[ v_{k-1/2}^+ - E'_x b_x + E'_z b_y \right] \cos \omega \Delta t + v_{||}, \\
    w_{k+1/2} &= E'_z b_y - E'_y b_z + \left[ w_{k-1/2}^+ - E'_z b_y + E'_y b_z \right] \cos \omega \Delta t + w_{||}.
\end{align*}
\]

There is the possible case when the magnetic field in the process of solving the problem becomes zero or is very small. In this case, one should use a difference analogue of the equations (10) in the general three-dimensional case:

\[
\begin{align*}
    u_{k+1/2} &= u_{k-1/2}^+ \cos \omega \Delta t + \frac{E_z^+ q \Delta t}{m} + \left( E_y^+ b_z - E_z^+ b_y \right) \frac{q}{2m} \omega \Delta t^2 + u_{||}, \\
    v_{k+1/2} &= v_{k-1/2}^+ \cos \omega \Delta t + \frac{E_y^+ q \Delta t}{m} + \left( E_z^+ b_y - E_x^+ b_y \right) \frac{q}{2m} \omega \Delta t^2 + v_{||}, \\
    w_{k+1/2} &= w_{k-1/2}^+ \cos \omega \Delta t + \frac{E_z^+ q \Delta t}{m} + \left( E_x^+ b_y - E_y^+ b_z \right) \frac{q}{2m} \omega \Delta t^2 + w_{||}.
\end{align*}
\]

To replace the basic formulas with the formulas (12), one can use the condition \( |\vec{E}| / |\vec{B}| > 10^{-5} \).

4.1. 3D test problem

To test the scheme \( D \), we have used the modernized test problem from the Section 3.3. Here all quantities are written in the dimensionless form.

Let in some coordinate system \( \{x', y', z'\} \) with basis vectors \( \vec{e}_x', \vec{e}_y', \vec{e}_z' \) and with the origin at point \( O \), the magnetic intensity vector and the electric field intensities coincide with those specified in the test problem:

\[
\begin{align*}
    \vec{B}'(\vec{x}') &= \vec{B}'(x', y', z') = (x'^2 + y'^2)^{1/2} \vec{e}_z', \\
    \vec{E}'(\vec{x}') &= \vec{E}'(x', y', z') = -\nabla \phi = 0.01 \left( \frac{x'}{(x'^2 + y'^2)^{3/2}}, \frac{y'}{(x'^2 + y'^2)^{3/2}}, 0 \right),
\end{align*}
\]
Initial data are \( \vec{x}'_0 = (0.9, 0, 0) \), \( \vec{v}'_0 = (0.1, 0, 0) \).

The current coordinate system \( \{x', y', z'\} \) is obtained from \( \{x, y, z\} \) by rotation around the \( Ox \) axis by the angle \( \alpha = 30^\circ \), which is characterized by the transition matrix:

\[
M = \begin{pmatrix}
1 & 0 & 0 \\
0 & \sqrt{3}/2 & 1/2 \\
0 & -1/2 & \sqrt{3}/2
\end{pmatrix}.
\]

The ratio between the vectors of velocity and position of a charged particle in the considered coordinate systems is

\[
\vec{x} = M \vec{x}', \\
\vec{v} = M \vec{v}'.
\]

The dependence of the electromagnetic field on the coordinates in the current coordinate system is determined as follows:

\[
\vec{B}(\vec{x}) = M \vec{B}'(\vec{x}') = M \vec{B}'(M^{-1} \vec{x}), \tag{16}
\]

\[
\vec{E}(\vec{x}) = M \vec{E}'(\vec{x}') = M \vec{E}'(M^{-1} \vec{x}). \tag{17}
\]

Thus, we have defined a statement of the three-dimensional problem (2) with the electromagnetic field given by formulas (13) - (17). Its solution can be compared with the solution of the plane problem by rotating the already known results around the \( Ox \) axis by the angle \( \alpha = 30^\circ \). The trajectory of a particle in the electromagnetic field is shown in Figure 4.

\[\text{Figure 4. Trajectory of a particle in the electromagnetic field, the modification } D, \Delta t = 0.0125 \]

\[n_t = 8 \times 10^5.\]

It shows that the resulting trajectory coincides with the trajectory obtained when solving the 2D problem (Fig. 3) in the coordinate system rotated by the angle \( \alpha = 30^\circ \).

Numerical analysis has shown that the three-dimensional version of the problem holds the property of the kinetic energy conservation. It has been demonstrated by all four considered algorithms.

4.2. Speed comparison of the Boris method modifications

Earlier in the Section 3, it has been shown that the considered methods to determine the trajectory of a charged particle differ in the underlying hypotheses and the accuracy of the computations. They also differ in the complexity and number of arithmetic operations, what
in turn affects the speed and running time of the code. Therefore, in order to compare the modifications of the Boris method in terms of the operation speed, a series of computations of the test problem in both formulations using each method have been carried out. Here the time of the method operation had been fixed separately from the rest of the code in \( n_t = 8 \cdot 10^7 \) time steps with the value of \( \Delta t = 0.0125 \). Since the computations of the same variant for different test runs show different times, the average time given in the Table 1 had been used as a comparative characteristic.

| Boris method | Scheme A | Scheme B | Scheme D |
|--------------|----------|----------|----------|
| 2.708        | 2.432    | 2.324    | 2.577    |

Thus, the times spent on the Boris method for modifications A, B, D are related as 1.165 : 1.046 : 1 : 1.108, respectively. The table shows that the Boris method runs 16% slower than the modification B, and the presence of a trigonometric function in the modification A slows down the method by 4% only. In both cases, the new scheme D does not give a large gain in time in comparison with other algorithms, but it does not turn out to be the slowest one, also.

5. Conclusion
In this paper, we have proposed a new modification of the Boris method to solve the equations of the charged particle motion, based on the idea of using the exact solution of the differential equation for the charged particle velocity at a time step. The comparative analysis of the Boris method and its modifications with the new scheme in a three-dimensional case has been carried out. It has been shown that the new modification has the smallest error. Also, a separate analysis for the cases of discretely specified the electric and magnetic fields has been carried out. The linear interpolation has been used to determine the fields at a given point of the particle trajectory. The results of the computations have shown that in the general case it was not always advisable to justify the choice of a specific modification of the Boris method by its speed. When choosing a modification of the Boris method for solving a specific problem, first of all, one should pay attention to the accuracy, since a simpler and faster scheme may not give a gain in time.

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