HIGHEST COEFFICIENTS OF WEIGHTED EHRHART QUASI-POLYNOMIALS FOR A RATIONAL POLYTOPE

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Abstract. We describe a method for computing the highest degree coefficients of a weighted Ehrhart quasi-polynomial for a rational simple polytope.

1. Introduction

Let \( p \) be a rational polytope in \( V = \mathbb{R}^d \) and \( h(x) \) a polynomial function on \( V \). A classical problem is to compute the sum of values of \( h(x) \) over the set of integral points of \( p \),

\[
S(p, h) = \sum_{x \in p \cap \mathbb{Z}^d} h(x).
\]

The function \( h(x) \) is called the weight. When \( p \) is dilated by an integer \( n \in \mathbb{N} \), we obtain a function of \( n \) which is quasi-polynomial, the so-called weighted Ehrhart quasi-polynomial of the pair \((p, h)\)

\[
S(np, h) = \sum_{m=0}^{d+M} E_m n^m.
\]

It has degree \( d+M \), where \( N = \text{deg } h \). The coefficients \( E_m \) are periodic functions of \( n \in \mathbb{N} \), with period the smallest integer \( q \) such that \( q p \) is a lattice polytope.

In [4], Barvinok obtained a formula relating the kth coefficient of the (unweighted) Ehrhart quasi-polynomial of a rational polytope to volumes of sections of the polytope by affine lattice subspaces parallel to k-dimensional faces of the polytope. As a consequence, he proved that the \( k_0 \) highest degree coefficients of the unweighted Ehrhart quasi-polynomial of a rational simplex can be computed by a polynomial algorithm, when the dimension \( d \) is part of the input, but \( k_0 \) is fixed.

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The sum $S(p, h)$ has natural generalizations, the intermediate sums $S^L(p, h)$, where $L \subseteq V$ is a rational vector subspace. For a polytope $p \subset V$ and a polynomial $h(x)$

$$S^L(p, h) = \sum_x \int_{p \cap (x+L)} h(y)\,dy,$$

where the summation index $x$ runs over the projected lattice in $V/L$. In other words, the polytope $p$ is sliced along lattice affine subspaces parallel to $L$ and the integrals of $h$ over the slices are added up. For $L = V$, there is only one term and $S^V(p, h)$ is just the integral of $h(x)$ over $p$, while, for $L = \{0\}$, we recover $S(p, h)$. Barvinok’s method was to introduce particular linear combinations of the intermediate sums,

$$\sum_{L \in \mathcal{L}} \rho(L)S^L(p, h).$$

It is natural to replace the polynomial weight $h(x)$ with an exponential function $x \mapsto e^{\langle \xi, x \rangle}$, and consider the corresponding holomorphic functions of $\xi$ in the dual $V^*$. Moreover, one can allow $p$ to be unbounded, then the sums

$$S^L(p)(\xi) = \sum_x \int_{p \cap (x+L)} e^{\langle \xi, y \rangle}\,dy$$

still make sense as meromorphic functions on $V^*$. The map $p \mapsto S^L(p)(\xi)$ is a valuation. In [2], we proved that Barvinok’s valuation $\sum_{L \in \mathcal{L}} \rho(L)S^L(p)(\xi)$ approximates $S(p)(\xi)$ in a sense which is made precise below. As a consequence, we recovered Barvinok’s main theorem of [4] and we sketched a method for computing the highest degree coefficients of the Ehrhart quasipolynomial of a rational simplex which is hopefully easier to implement. The proof in [2] relied on our Euler-Maclaurin expansion of these functions.

The main interest of the present article is to give a simpler formulation and an elementary proof of the approximation result of [2], in the case of a simple polytope.

In a forthcoming article, in collaboration with J. De Loera and M. Köppe, we plan to apply the results of this article to derive a polynomial algorithm for computing the $k_0$ highest degree coefficients of a weighted Ehrhart quasi-polynomial relative to a simplex, when $k_0$ and the degree of the weight $h(x)$ are fixed, but the dimension of the simplex is part of the input. The article [1] dealt with the integral $\int_p h(x)\,dx$, which is of course the highest coefficient, if $h(x)$ is homogeneous.

To explain the main idea, let us assume in this introduction that the vertices $s$ of $p$ are integral. Using a theorem of Brion, one writes
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$S(p)(\xi)$ as a sum of the generating functions of the supporting cones at the vertices $s$ of $p$,

$$S(p)(\xi) = \sum_{s \in V(p)} S(s + c_s)(\xi).$$

Then the dilated polytope $np$ has vertices $ns$ with the same cone of feasible directions $c_s$, thus

$$S(np)(\xi) = \sum_{s \in V(p)} e^{n\langle \xi, s \rangle} S(c_s)(\xi).$$

The generating function of a cone $c$ is a meromorphic function of a particular type, namely, near $\xi = 0$, it is a quotient of a holomorphic function by a product of linear forms. Hence, it admits a decomposition into homogeneous components. One shows that the lowest degree is $\geq -d$.

Let us fix an integer $k_0$, $0 \leq k_0 \leq d$. Our goal is to compute the $k_0+1$ highest degree coefficients of the Ehrhart quasi-polynomial of $p$ for the weight $h(x) = \langle \xi, x \rangle^M$. As we explain in Section 3, this computation amounts to computing the lowest homogeneous components

$$S(c_s)[-d+k](\xi), \ k = 0, \ldots, k_0,$$

of the generating functions of the cones $c_s$.

Our main result, Theorem 16, is an expression, depending on $k_0$, for the components $S(s + c)[-d+k](\xi), \ k = 0, \ldots, k_0$ in the particular case where the cone $s + c$ is simplicial. For a unimodular cone, the generating function is given by a ”short” formula, thus its lowest degree components are readily computed. In general, let $v_i \in \mathbb{Z}^d, i = 1, \ldots, d$, be integral generators of the edges of $c$. The finite sum

$$f(\xi) = \sum_{x \in (\sum_{i=1}^d [0,1] v_i) \cap \mathbb{Z}^d} e^{\langle \xi, x \rangle}$$

is an analytic function of $\xi$. (If $c$ is unimodular and the generators $v_i$ are primitive, then $f(\xi) = 1$). The term of degree $k$ of $f(\xi)$ is given by

$$\sum_{x \in (\sum_{i=1}^d [0,1] v_i) \cap \mathbb{Z}^d} \frac{\langle \xi, x \rangle^k}{k!}.$$

A monomial of degree $k$ in $(\xi_1, \ldots, \xi_d)$ can involve at most $k$ variables among the $\xi_i$. From this elementary remark, we deduce that the terms of degree $\leq k_0$ of $f(\xi)$ can be computed using a Moebius type combination of sums similar to $f(\xi)$, in dimension $\leq k_0$, and determinants. As a consequence, we obtain an expression for the terms
$S(s + c)[−d+k](ξ), \ k = 0, \ldots, k_0$, which involves the generating functions of cones in dimension $\leq k_0$ only. This feature is useful because, when the dimension is fixed, there is an efficient algorithm for decomposing a simplicial cone into a signed combination of unimodular cones, due to Barvinok.

2. Notations and basic facts

2.1. We consider a rational vector space $V$ of dimension $d$, that is to say a finite dimensional real vector space with a lattice denoted by $Λ$. We will need to consider subspaces and quotient spaces of $V$, this is why we cannot just let $V = \mathbb{R}^d$ and $Λ = \mathbb{Z}^d$. The set $Λ \otimes \mathbb{Q}$ of rational points in $V$ is denoted by $V_\mathbb{Q}$. A subspace $L$ of $V$ is called rational if $L \cap Λ$ is a lattice in $L$. If $L$ is a rational subspace, the image of $Λ$ in $V/L$ is a lattice in $V/L$, so that $V/L$ is a rational vector space. The image of $Λ$ in $V/L$ is called the projected lattice.

A rational space $V$, with lattice $Λ$, has a canonical Lebesgue measure $dx$, for which $V/Λ$ has measure 1.

A convex rational polyhedron $p$ in $V$ (we will simply say polyhedron) is, by definition, the intersection of a finite number of half spaces bounded by rational affine hyperplanes. We say that $p$ is solid (in $V$) if the affine span of $p$ is $V$.

In this article, a cone is a polyhedral cone (with vertex 0) and an affine cone is a translated set $s + c$ of a cone $c$.

A polytope $p$ is a compact polyhedron. The set of vertices of $p$ is denoted by $V(p)$. For each vertex $s$, the cone of feasible directions at $s$ is denoted by $c_s$.

A cone $c$ is called simplicial if it is generated by independent elements of $V$. A simplicial cone $c$ is called unimodular if it is generated by independent integral vectors $v_1, \ldots, v_k$ such that $\{v_1, \ldots, v_k\}$ can be completed to an integral basis of $V$. An affine cone $a$ is called simplicial (resp. simplicial unimodular) if it is the translate of a simplicial (resp. simplicial unimodular) cone.

2.2. Generating functions.

Definition 1. We denote by $\mathcal{H}(V^*)$ the ring of holomorphic functions defined around $0 \in V^*$. We denote by $\mathcal{M}(V^*)$ the ring of meromorphic functions defined around $0 \in V^*$ and by $\mathcal{M}_\ell(V^*) \subset \mathcal{M}(V^*)$ the subring consisting of those meromorphic functions $\phi(ξ)$ such that there exists a product of linear forms $D(ξ)$ such that $D(ξ)\phi(ξ)$ is holomorphic.
A function \( \phi(\xi) \in \mathcal{M}_\ell(V^*) \) has a unique expansion into homogeneous rational functions

\[
\phi(\xi) = \sum_{m \geq m_0} \phi_{[m]}(\xi).
\]

If \( P \) is a homogeneous polynomial on \( V^* \) of degree \( p \), and \( D \) a product of \( r \) linear forms, then \( \frac{P}{D} \) is an element in \( \mathcal{M}_\ell(V^*) \) homogeneous of degree \( m = p - r \). For instance, \( \xi_1 \xi_2 \) is homogeneous of degree 0. On this example, we observe that a function in \( \mathcal{M}_\ell(V^*) \) which has no negative degree terms need not be analytic.

Let us recall the definition of the functions \( I(p) \) and \( S(p) \in \mathcal{M}_\ell(V^*) \) associated to a polyhedron \( p \), (see for instance the survey \([5]\))

**Proposition 2.** There exists a unique map \( I \) which to every polyhedron \( p \subset V \) associates a meromorphic function \( I(p) \in \mathcal{M}_\ell(V^*) \), so that the following properties hold:

(a) If \( p \) is not solid or if \( p \) contains a straight line, then \( I(p) = 0 \).

(b) If \( \xi \in V^* \) is such that \( e^{(\xi, x)} \) is integrable over \( p \), then

\[
I(p)(\xi) = \int_p e^{(\xi, x)} dx.
\]

(c) For every point \( s \in V_Q \), one has

\[
I(s + p)(\xi) = e^{(\xi, s)} I(p)(\xi).
\]

**Proposition 3.** There exists a unique map \( S \) which to every polyhedron \( p \subset V \) associates a meromorphic function \( S(p) \in \mathcal{M}_\ell(V^*) \), so that the following properties hold:

(a) If \( p \) contains a straight line, then \( S(p) = 0 \).

(b) If \( \xi \in V^* \) is such that \( e^{(\xi, x)} \) is summable over the set of lattice points of \( p \), then

\[
S(p)(\xi) = \sum_{x \in p \cap \Lambda} e^{(\xi, x)}.
\]

(c) For every point \( s \in \Lambda \), one has

\[
S(s + p)(\xi) = e^{(\xi, s)} S(p)(\xi).
\]

Moreover the maps \( p \mapsto I(p) \) and \( p \mapsto S(p) \) have additivity properties, with consequence the fundamental Brion’s theorem.

**Theorem 4** (Brion, \([7]\)). Let \( p \) be a polyhedron with set of vertices \( V(p) \). For each vertex \( s \), let \( c_s \) be the cone of feasible directions at \( s \). Then

\[
S(p) = \sum_{s \in V(p)} S(s + c_s) \text{ and } I(p) = \sum_{s \in V(p)} I(s + c_s).
\]
2.3. Notations and basic facts in the case of a simplicial cone. Let \( v_i, i = 1, \ldots, d \) be linearly independent integral vectors and let 
\[ c = \sum_{i=1}^{d} \mathbb{R}^+ v_i \]
be the cone they span.

**Definition 5.** The semi-closed unit cell \( B \) of the cone (with respect to the generators \( v_i, i = 1, \ldots, d \)) is the set

\[ B = \sum_{i=1}^{d} [0, 1] v_i. \]

We recall the following elementary but crucial lemma.

**Lemma 6.**
(i) The affine cone \( (s + c) \cap \Lambda \) is the disjoint union of the translated cells \( s + B + v \), for \( v \in \sum_{j=1}^{d} \mathbb{N} v_j \).
(ii) The set of lattice points in the affine cone \( s + c \) is the disjoint union of the sets \( x + \sum_{i=1}^{d} \mathbb{N} v_i \) when \( x \) runs over the set \( (s + B) \cap \Lambda \).
(iii) The number of lattice points in the cell \( s + B \) is equal to the volume of the cell with respect to the Lebesgue measure defined by the lattice, that is

\[ \text{Card}((s + B) \cap \Lambda) = |\det(v_i)|. \]

Let \( s \in V_\mathbb{Q} \). We have immediately

\[ I(s + c)(\xi) = e^{\langle \xi, s \rangle} \frac{(-1)^d |\det(v_i)|}{\prod_{i=1}^{d} \langle \xi, v_i \rangle}. \]

The study of the function \( S(s + c)(\xi) \) will be the main point of this article. It reduces to the study of the holomorphic function \( S(s + B)(\xi) \) defined by the following finite sum, over the lattice points of the unit cell.

**Definition 7.**

\[ S(s + B)(\xi) = \sum_{x \in (s + B) \cap \Lambda} e^{\langle \xi, x \rangle}. \]

**Lemma 8.**

\[ S(s + c)(\xi) = S(s + B)(\xi) \frac{1}{\prod_{j=1}^{d} (1 - e^{\langle \xi, v_j \rangle})}. \]

In particular, \( S(s + c) \in \mathcal{M}_\ell(V^*) \), thus it admits a decomposition into homogeneous components,

\[ S(s + c)(\xi) = S_{[-d]}(s + c)(\xi) + S_{[-d+1]}(s + c)(\xi) + \ldots, \]

and the lowest degree term \( S_{[-d]}(s + c)(\xi) \) is equal to \( I(c)(\xi) \).
Proof. (2) follows from Lemma 6 (ii). Next, we write
\[
\prod_{j=1}^{d} \frac{1}{1 - e^{\langle \xi, v_j \rangle}} = \prod_{j=1}^{d} \frac{\langle \xi, v_j \rangle}{1 - e^{\langle \xi, v_j \rangle}} \prod_{j=1}^{d} \langle \xi, v_j \rangle.
\]
The function \( \frac{x}{1 - e^x} \) is holomorphic with value \(-1\) for \( x = 0 \). Thus \( S(s + c) \in \mathcal{M}_\ell(V^*) \). The value at \( \xi = 0 \) of the sum over the cell is the number of lattice points of the cell, that is the volume \( |\det(v_i)| \). This proves the last assertion. \( \square \)

3. Weighted Ehrhart quasipolynomials

Let \( p \subset V \) be a rational polytope and let \( h(x) \) be a polynomial function of degree \( M \) on \( V \). We consider the following weighted sum over the set of lattice points of \( p \),
\[
\sum_{x \in p \cap \Lambda} h(x).
\]
When \( p \) is dilated by a non negative integer \( n \), we obtain the quasipolynomial of the pair \((p, h)\),
\[
\sum_{x \in np \cap \Lambda} h(x) = \sum_{m=0}^{d+M} E_m n^m.
\]
The coefficients \( E_m \) actually depend on \( n \), but they depend only on \( n \) mod \( q \), where \( q \) is the smallest integer such that \( qp \) is a lattice polytope. If \( h(x) \) is homogeneous of degree \( M \), the highest degree coefficient \( E_{d+M} \) is equal to the integral \( \int_p h(x)dx \).

Let us fix a number \( k_0 \). Our goal is to compute the \( k_0 + 1 \) highest degree coefficients \( E_m \), for \( m = M + d, \ldots, M + d - k_0 \).

We concentrate on the special case where the polynomial \( h(x) \) is a power of a linear form
\[
h(x) = \frac{\langle \xi, x \rangle^M}{M!}.
\]
Of course, any polynomial can be written as a linear combination of powers of linear forms.

We will explain our results with the simplifying assumption that the vertices of the polytope are lattice points.

**Definition 9.** We define the coefficients \( E_m(p, \xi, M), m = 0, \ldots, M + d \) by
\[
\sum_{x \in np \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(p, \xi, M) n^m.
\]
Proposition 10. Let $\mathfrak{p}$ be a lattice polytope. Then, for $k \geq 0$, we have

$$E_{M+d-k}(\mathfrak{p}, \xi, M) = \sum_{s \in \mathcal{V}(\mathfrak{p})} \langle \xi, s \rangle^{M+d-k} (M+d-k)! S_{[-d+k]}(c_s)(\xi).$$

The highest degree coefficient is just the integral

$$E_{M+d}(\mathfrak{p}, \xi, M) = \int_{\mathfrak{p}} \frac{\langle \xi, x \rangle^M}{M!} dx.$$

Remark 11. As functions of $\xi$, the coefficients $E_m(\mathfrak{p}, \xi, M)$ are polynomial, homogeneous of degree $M$. However, in (5), they are expressed as linear combinations of rational functions of $\xi$, whose poles cancel out.

Proof. The starting point is Brion’s formula. As the vertices are lattice points, we have

$$\sum_{x \in \mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} S(s + c_s)(\xi) = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{\langle \xi, s \rangle} S(c_s)(\xi).$$

When $\mathfrak{p}$ is replaced with $n\mathfrak{p}$, the vertex $s$ is replaced with $ns$ but the cone $c_s$ does not change. We obtain

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} e^{\langle \xi, x \rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{n\langle \xi, s \rangle} S(c_s)(\xi).$$

We replace $\xi$ with $t\xi$,

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} e^{t\langle \xi, x \rangle} = \sum_{s \in \mathcal{V}(\mathfrak{p})} e^{nt\langle \xi, s \rangle} S(c_s)(t\xi).$$

The decomposition into homogeneous components gives

$$S(c_s)(t\xi) = t^{-d} I(c_s)(\xi) + t^{-d+1} S_{[-d+1]}(c_s)(\xi) + \cdots + t^k S_{[k]}(c_s)(\xi) + \cdots.$$

Hence, the $t^M$-term in the right-hand side is equal to

$$\sum_{k=0}^{M+d} (nt)^{M+d-k} t^{-d+k} \frac{\langle \xi, s \rangle^{M+d-k}}{(M+d-k)!} S_{[-d+k]}(c_s)(\xi).$$

Thus we have

$$\sum_{x \in n\mathfrak{p} \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{s \in \mathcal{V}(\mathfrak{p})} n^{M+d} \frac{\langle \xi, s \rangle^{M+d}}{(M+d)!} I(c_s)(\xi)$$

$$+ n^{M+d-1} \frac{\langle \xi, s \rangle^{M+d-1}}{(M+d-1)!} S_{[-d+1]}(c_s)(\xi) + \cdots + S_{[M]}(c_s)(\xi).$$
On this relation, we read immediately that $\sum_{x \in \mathcal{X} \cap \Lambda} \frac{(\langle \xi, x \rangle)^M}{M!}$ is a polynomial function of $n$ of degree $M + d$, and that the coefficient of $n^{M + d - k}$ is given by (5). The highest degree coefficient is given by

$$E_{M+d}(p, \xi, M) = \sum_{s \in V(p)} \frac{(\langle \xi, s \rangle)^{M+d}}{(M+d)!} I(s)(\xi).$$

Applying Brion’s formula for the integral, this is equal to the term of $\xi$-degree $M$ in $I(p)(\xi)$, which is indeed the integral $\int_p \frac{(\langle \xi, x \rangle)^M}{M!} dx$. □

From Proposition 10, we draw an important consequence: in order to compute the $k_0 + 1$ highest degree terms of the weighted Ehrhart polynomial for the weight $h(x) = \frac{(\langle \xi, x \rangle)^M}{M!}$, we need only the $k_0 + 1$ lowest degree homogeneous terms of the meromorphic function $S(c_s)(\xi)$, for every vertex $s$ of $p$. We compute such an approximation in the next section.

4. Approximation of the generating function of a simplicial affine cone

Let $c \subset V$ be a simplicial cone with integral generators $(v_j, j = 1, \ldots, d)$, and let $s \in V_\mathbb{Q}$. Let $k_0 \leq d$. In this section we will obtain an expression for the $k_0 + 1$ lowest degree homogeneous terms of the meromorphic function $S(s + c)(\xi)$. Recall that if $c$ is unimodular, the function $S(s + c)(\xi)$ has a ”short” expression, given by (2),

$$S(s + c)(\xi) = e^{\langle \xi, \bar{s} \rangle} \prod_{j=1}^{d} \frac{1}{1 - e^{\langle \xi, v_j \rangle}},$$

where $(v_i, i = 1, \ldots, d)$ are the primitive integral generators of the edges and $\bar{s}$ is the unique lattice point in the corresponding cell $s + B$. Thus in the unimodular case, computing the lowest degree components is immediate.

When $c$ is not unimodular, it is not possible to compute efficiently the Taylor expansion of the function $S(s + B)(\xi)$ at order $k_0$, if the order is part of the input as well as the dimension $d$. In contrast, if the order $k_0$ is fixed, we are going to obtain an expression for the terms of degree $\leq k_0$ which involves discrete summation over cones in dimension $\leq k_0$ only, and determinants. For example, for $k_0 = 0$, the constant term $S(s + B)(0)$ is the number of lattice points in the cell, which is equal to a determinant, by Lemma 6 (iii).

We need some notations. For $I \subseteq \{1, \ldots, d\}$, we denote by $L_I$ the linear span of the vectors $(v_i, i \in I)$. We denote by $B_I = \sum_{i \in I} [0, 1][v_i]$ the unit cell in $L_I$. 

We denote by $I^c$ the complement of $I$ in $\{1, \ldots, d\}$. We have $V = L_I \oplus L_{I^c}$. For $x \in V$ we denote the components by

$$x = x_I + x_{I^c}.$$ 

Thus we identify the quotient $V/L_{I^c}$ with $L_I$. Note that $L_I \cap \Lambda \subseteq \Lambda_I$, but the inclusion is strict in general.

**Example 12.** $v_1 = (1, 0), v_2 = (1, 2)$. Take $I = \{1\}$. Then $\Lambda_I = \mathbb{Z} v_1 + \mathbb{Z} v_2$.

We denote by $c_I$ the projection of the cone $c$ on the space $L_I$. Its edges are generated by $v_j, j \in I$, and the corresponding unit cell $B_I$ is the projection of $B$. Remark that $v_j$ may be non primitive for the projected lattice $\Lambda_I$, even if it is primitive for $\Lambda$, as we see in the previous example. This is the reason why in Lemma 6, we did not make the (unnecessary) assumption that the generators $v_j$ are primitive.

For $u = (u_1, \ldots, u_d)$, we denote $u_I = \sum_{i \in I} u_i$.

We denote the binomial coefficient $\frac{p!}{k!(p-k)!}$ by $\binom{p}{k}$.

**Definition 13.** Given a function $I \mapsto \lambda(I)$ on the set of subsets $I \subseteq \{1, \ldots, d\}$ with cardinal $|I| \leq k_0$, we denote

$$T(s, c, k_0, \lambda)(\xi) = \sum_{|I| \leq k_0} \lambda(I) \operatorname{vol}(B_{I^c}) S(s_I + c_I)(\xi)(-1)^{|I|} \prod_{j \in I^c} \frac{1}{\langle \xi, v_j \rangle}.$$ 

**Remark 14.** The function $S(s_I + c_I)(\xi)$ is a meromorphic function on the space $L_I^*$. We extend it to $V^*$ by the decomposition $V = L_I \oplus L_{I^c}$.

It is easy to see that the function $T(s, c, k_0, \lambda)(\xi)$ lies in $\mathcal{M}_\ell(V^*)$, and its expansion into homogeneous components has lowest degree $-d$. Thus

$$T(s, c, k_0, \lambda)(\xi) = T_{[-d]}(s, c, k_0, \lambda)(\xi) + T_{[-d+1]}(s, c, k_0, \lambda)(\xi) + \cdots.$$ 

We will use functions $I \mapsto \lambda(I)$ which have the following property.

**Definition 15.** A $(d, k_0)$-patchfunction is a function $I \mapsto \lambda(I)$ on the set of subsets $I \subseteq \{1, \ldots, d\}$ of cardinal $|I| \leq k_0$ which satisfies the following condition.

$$e^{u_1 + \cdots + u_d} \equiv \sum_{|I| \leq k_0} \lambda(I) e^{u_I} \mod \text{terms of u-degree} \geq k_0 + 1.$$
Theorem 16. Let $I \mapsto \lambda(I)$ be a $k_0$-patchfunction. Then we have
\begin{align*}
(9) \quad S(s + B)(\xi) & \equiv \sum_{|I| \leq k_0} \lambda(I) \text{vol}(B_{I^c})S(s_I + B_I)(\xi) \\
& \quad \text{mod terms of } \xi\text{-degree } \geq k_0 + 1.
\end{align*}
\begin{align*}
(10) \quad S(s + c)(\xi) & \equiv T(s, c, k_0, \lambda)(\xi) \text{ mod terms of } \xi\text{-degree } \geq -d + k_0 + 1.
\end{align*}

Proof. Using (2), we write
\[
S(s + c)(\xi) = \left( S(s + B)(\xi) \prod_{j=1}^d \frac{\langle \xi, v_j \rangle}{1 - e^{\langle \xi, v_j \rangle}} \right) \frac{1}{\prod_{j=1}^d \langle \xi, v_j \rangle}.
\]
Thus we need only the terms of $\xi$-degree at most $k_0$ in the Taylor expansion of the holomorphic function $S(s + B)(\xi) \prod_{j=1}^d \frac{\langle \xi, v_j \rangle}{1 - e^{\langle \xi, v_j \rangle}}$, and finally we need only the terms of $\xi$-degree at most $k_0$ in the Taylor expansion of $S(s + B)(\xi)$. Applying (3) to each term $e^{\langle \xi, x \rangle} = e^{\xi_1 x_1 + \cdots + \xi_d x_d}$ of the finite sum $S(s + B)(\xi)$, we have
\[
S(s + B)(\xi) = \sum_{|I| \leq k_0} \lambda(I) \sum_{x \in (s + B) \cap \Lambda} e^{\langle \xi, x \rangle} \text{ mod terms of } \xi\text{-degree } \geq k_0 + 1.
\]
For each $I$, the term $\sum_{x \in (s + B) \cap \Lambda} e^{\langle \xi, x \rangle}$ is the sum, over $x \in (s + B) \cap \Lambda$, of a function of $x$ which depends only on the projection $x_I$ of $x$ in the decomposition $x = x_I + x_{I^c} \in L^I \oplus L^{I^c}$. When $x$ runs over $(s + B) \cap \Lambda$, its projection $x_I$ runs over $(s_I + B_I) \cap \Lambda_I$. Let us show that the fibers have the same number of points, equal to $\text{vol}(B_{I^c})$. For a given $x_I \in (s_I + B_I) \cap \Lambda_I$, let us compute the fiber
\[
\{ y \in (s + B) \cap \Lambda; y_I = x_I \}.
\]
Fix a point $x_I + x_{I^c}$ in this fiber. Then $y = x_I + y_{I^c}$ lies in the fiber if and only if $y_{I^c} - x_{I^c} \in (s_{I^c} - x_{I^c} + B_{I^c}) \cap \Lambda$. By Lemma (6) ii, the cardinal of the fiber is equal to $\text{vol}(B_{I^c})$. Thus, we have obtained (9).

Next we write the sum $S(s_I + c_I)(\xi)$ over the projected cone $s_I + c_I$ in terms of the sum over the projected cell $s_I + B_I$. We obtain
\begin{align*}
S(s + c)(\xi) & \equiv \sum_{|I| \leq k_0} \lambda(I) \text{vol}(B_{I^c})S(s_I + c_I)(\xi) \prod_{j \in I^c} \frac{1}{(1 - e^{\langle \xi, v_j \rangle})} \\
& \equiv \sum_{|I| \leq k_0} \lambda(I) \text{vol}(B_{I^c})S(s_I + c_I)(\xi)(-1)^{d-|I|} \prod_{j \in I^c} \frac{1}{\langle \xi, v_j \rangle} \\
& \quad \text{mod terms of } \xi\text{-degree } \geq -d + k_0 + 1.
\end{align*}
Next we compute an explicit \((d, k_0)\)-patchfunction.

**Lemma 17.** If \(I \subseteq \{1, \ldots, d\}\) has cardinal \(|I| \leq k_0\), let

\[
\lambda_{d,k_0}(I) = (-1)^{|I|/2}(d - |I| - 1) \left( d - k_0 - 1 \right).
\]

Then \(\lambda_{d,k_0}\) satisfies Condition (\(\Box\)).

**Proof.** The trick is to write \(e^u = 1 + t(e^u - 1)|_{t=1}\). Thus

\[
e^{u_1 + \cdots + u_d} = \prod_{1}^{d} e^{u_i} = \prod_{1}^{d} (1 + t(e^{u_i} - 1))|_{t=1}
\]

Let us consider \(P(t) := \prod_{1}^{d}(1 + t(e^{u_i} - 1)) = \sum_{q=0}^{d} C_q(u)t^q\) as a polynomial in the indeterminate \(t\). As \(e^{u_i} - 1\) is a sum of terms of \(u_i\)-degree \(> 0\), we have

\[
e^{u_1 + \cdots + u_d} \equiv \sum_{q=0}^{k_0} C_q(u) \mod \text{terms of } u\text{-degree } \geq k_0 + 1.
\]

In order to compute the coefficient \(C_q(u)\), we write

\[
P(t) = \prod_{1}^{d}(1 + t(e^{u_i} - 1)) = \prod_{1}^{d}(1 + t + t^2 + \cdots + t^{e^{u_i} - 1}).
\]

By expanding the product, we obtain

\[
C_q(u) = \sum_{|I| \leq q} (-1)^{|I|/2}(d - |I|) \left( d - k_0 - 1 \right) e^{u_I}.
\]

Summing up these coefficients for \(0 \leq q \leq k_0\), we obtain

\[
\sum_{q=0}^{k_0} C_q(u) = \sum_{|I| \leq k_0} \left( \sum_{q=|I|}^{k_0} (-1)^{q-|I|}(d - |I|) \left( d - k_0 - 1 \right) \right) e^{u_I}.
\]

For \(m_0 \leq d_0\), let us denote

\[
F(m_0, d_0) = \sum_{j=0}^{m_0} (-1)^{j} \binom{d_0}{j}.
\]

Thus

\[
\sum_{q=0}^{k_0} C_q(u) = \sum_{|I| \leq k_0} F(k_0 - |I|, d - |I|) e^{u_I}.
\]
The sum $F(m_0, d_0)$ is easy to compute by induction on $m_0$, using the recursion relation
\[
\binom{d_0}{j} = \binom{d_0 - 1}{j} + \binom{d_0 - 1}{j - 1}.
\]
We obtain
\[
F(m_0, d_0) = (-1)^{m_0} \binom{d_0 - 1}{m_0}.
\]
Hence,
\[
F(k_0 - |I|, d - |I|) = (-1)^{k_0 - |I|} \binom{d - |I| - 1}{k_0 - |I|} = (-1)^{k_0 - |I|} \binom{d - |I| - 1}{d - k_0 - 1}.
\]
The claim follows now from Equation (11).

\[\square\]

**Remark 18.** As promised, the main feature of Formula (11) is that the right-hand-side $T(s, c, k_0, \lambda)$ involves discrete summations in dimension $|I| \leq k_0$ only.

Theorem 16 can be reformulated in terms of the intermediate valuations introduced by Barvinok in [4]. The reformulation relies on the next lemma, which shows that the $(d, k_0)$-patchfunction condition is equivalent to a Moebius-type condition for the function $I \mapsto \lambda(I)$.

**Lemma 19.** Let $0 \leq k_0 \leq d$ be two integers. Let $\lambda$ be a function on the set of subsets $I \subseteq \{1, \ldots, d\}$ of cardinal $|I| \leq k_0$. The following conditions are equivalent.

(i) For every $I_0$ of cardinal $|I_0| \leq k_0$,
\[
\sum\{I : |I| \leq k_0, I_0 \subseteq I\} \lambda(I) = 1.
\]

(ii) For every integer $k$ such that $0 \leq k \leq k_0$, we have the equality of polynomials
\[
(u_1 + \cdots + u_d)^k = \sum_{|I| \leq k_0} \lambda(I) u_I^k.
\]

(iii) The function $\lambda$ is a $(d, k_0)$-patchfunction.

**Proof.** Conditions (ii) and (iii) are clearly equivalent. Let us show that (i) and (ii) are equivalent. We expand $(u_1 + \cdots + u_d)^k$ into a sum of monomials. A monomial of degree $k$ can involve at most $k$ variables $u_i$, with $k \leq k_0$. Therefore we obtain
\[
\frac{1}{k!} (u_1 + \cdots + u_d)^k = \sum_{|I| \leq k_0} \sum_{(k_i) \in I} \prod_{i \in I} \frac{u_i^{k_i}}{k_i!}.
\]
We expand similarly each term in the right-hand side of (12). A given monomial $\prod_{i \in I_0} u^{k_i}_{ki}$, with $k_i \neq 0$ for all $i \in I_0$, occurs in the right-hand side of (12) exactly for the subsets $I$ such that $I_0 \subseteq I$. Thus (i) implies (ii). Conversely, Equation (12) for $k = k_0$ implies (i).

5. Computation of Ehrhart quasi-polynomials

We now apply the approximation of the generating functions of the cones at vertices to the computation of the highest coefficients for a weighted Ehrhart polynomial, when the weight is a power of a linear form, as we explained in section 3.

Corollary 20. Let $p$ be a simple lattice polytope. Fix $\xi \in \mathbb{R}^d$ and $M \in \mathbb{N}$. Let $E_m(p, \xi, M), m = 0, \ldots, d + M$, be the coefficients of the weighted Ehrhart polynomial

$$\sum_{x \in np \cap \Lambda} \langle \xi, x \rangle^M = \sum_{m=0}^{M+d} E_m(p, \xi, M)n^m.$$ 

Fix $0 \leq k_0 \leq d$. Let $\lambda$ be a $(d, k_0)$-patchfunction. Then, for $k = 0, \ldots, k_0$, the Ehrhart coefficient $E_{M+d-k}(p, \xi, M)$ is given by the following formula.

$$E_{M+d-k}(p, \xi, M) = \sum_{s \in V(p)} (n - n_s)^{M+d-k} \langle \xi, s \rangle^{M+d-k} \frac{(M + d - k)!}{(M + d - k)!} T(0, c_s, k_0, \lambda)[-d+k](\xi).$$

In the general case, when the vertices are not assumed to be lattice points, we state the result without going through the details of the computation.

Theorem 21. Let $p$ be a simple polytope. For each vertex $s$ of $p$, let $q_s \in \mathbb{N}$ be the smallest integer such that $q_s s \in \Lambda$. For $n \in \mathbb{N}$, let $n_s$ be the residue of $n$ mod $q_s$, so that $0 \leq n_s \leq q_s - 1$. Fix $\xi \in V^*$ and $M$ a nonnegative integer. Fix $0 \leq k_0 \leq d$. Let $\lambda$ be a $(d, k_0)$-patchfunction.

Then the Ehrhart quasi-polynomial

$$\sum_{x \in np \cap \Lambda} \langle \xi, x \rangle^M = \sum_{n=0}^{M+d} E_m(p, \xi, M)n^m$$

coincides in degree $\geq M + d - k_0$ with the following quasi-polynomial

$$\sum_{k=0}^{k_0} \sum_{s \in V(p)} (n - n_s)^{M+d-k} \langle \xi, s \rangle^{M+d-k} \frac{(M + d - k)!}{(M + d - k)!} T[-d+k](n_s s, c_s, k_0, \lambda)(\xi).$$
Observe that (15) is clearly a quasi-polynomial in $n$ with coefficients which depend only on the residues $(n \mod q_s) = n_s$, $s \in V(p)$.

Remark 22. In practice, we first reduce the vertices $s \mod \Lambda$ by using
\[ S(s + c_s)(\xi) = e^{(\xi,v)}S(s - v + c_s)(\xi), \quad \text{for } v \in \Lambda. \]
Then we write an approximation similar to (15).

6. Conclusion

Let $p \subset \mathbb{R}^d$ be a rational simplex. Let $\langle \xi, x \rangle$ be a rational linear form on $\mathbb{R}^d$, and consider a power $\langle \xi, x \rangle^M$. Let $E_m(p, \xi, M), m = 0, \ldots, d + M$, be the coefficients of the weighted Ehrhart quasi-polynomial
\[ \sum_{x \in p \cap \Lambda} \frac{\langle \xi, x \rangle^M}{M!} = \sum_{m=0}^{M+d} E_m(p, \xi, M) n^m. \]

Fix an integer $k_0$, $0 \leq k_0 \leq d$. The main consequence of this study is a method for efficiently computing the $k_0 + 1$ highest degree coefficients $E_m(p, \xi, M)$, for $m = M + d, \ldots, M + d - k_0$. The method relies on expanding (15) in Theorem 21 as a power series in $\xi$.

Furthermore, one can write any homogeneous polynomial weight $h(x)$ as a linear combination of powers of linear forms,
\[ h(x) = \sum_k c_k \langle \xi_k, x \rangle^M. \]

In a forthcoming article by the authors of [1], we will show how to derive
- first, an algorithm for computing $E_m(p, \xi, M)$, for $m = M + d, \ldots, M + d - k_0$. Hopefully this algorithm is polynomial, when the input consists of the dimension $d$ and the degree $M$, the rational simplex $p \subset \mathbb{R}^d$, the rational linear form $\xi$ on $\mathbb{R}^d$, provided $k_0$ is fixed;
- second, an algorithm for computing the $k_0 + 1$ highest degree coefficients of a weighted Ehrhart quasi-polynomial relative to a simplex. Hopefully this algorithm is polynomial when $k_0$ and the degree of the weight $h(x)$ are fixed, but the dimension of the simplex is part of the input.

[1] dealt with the case of the highest Ehrhart coefficient which is just the integral of the weight over the simplex.

References

[1] Baldoni V., Berline N., De Loera J., Koppe M., Vergne M. How to Integrate a Polynomial over a Simplex. [arXiv:0809.2083]
[2] Baldoni V., Berline N. and Vergne M. Euler-Maclaurin expansion of Barvinok valuations and Ehrhart coefficients of rational polytopes

[3] Barvinok A. I., Computing the Ehrhart polynomial of a convex lattice polytope, Discrete Comput. geom. 12 (1994), 35-48.

[4] Barvinok A. I., Computing the Ehrhart quasi-polynomial of a rational simplex, Mathematics of Computation, 75 (2006), 1449–1466.

[5] Barvinok A. I. and Pommersheim J., An algorithmic theory of lattice points in polyhedra, New Perspectives in Algebraic Combinatorics (Berkeley, CA, 1996-97), Math. Sci. Res. Inst. Public 38, Cambridge University Press, Cambridge, (1999), pp 91-147.

[6] Berline N. and Vergne M. Local Euler-Maclaurin formula for polytopes (2005), arXiv:math CO/0507256. To appear in Moscow Math. J.

[7] Brion M., Points entiers dans les polyèdres convexe, Ann. Sci. Ecole Norm. Sup. 21 (1988), 653-663.

[8] De Loera J.A., Haws D., Hemmecke R., Huggins H., Tauzer J. and Yoshida R., A Users Guide for LattE v1.1, 2003, software package LattE, available at http://www.math.ucdavis.edu/~latte

[9] Stanley R. Enumerative combinatorics. Vol 1 (1997), Cambridge Studies in Advanced Math. 49.

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