Global stability of systems related to the Navier-Stokes equations

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Abstract

A generalized Lyapunov method is outlined which predicts global stability of a broad class of dissipative dynamical systems. The method is applied to the complex Lorenz model and to the Navier-Stokes equations. In both cases one finds compact domains in phase space which contain the $\omega$ sets of all trajectories, in particular the fixed points, limit cycles, and strange attractors.

1 Introduction

In the theory of ordinary differential equations, the method of Lyapunov function in general serves to examine the stability of a fixed point and its domain of attraction, for an overview see e.g. [1]. The method can be naturally extended to the case where, instead of a fixed point, one is interested in the stability of a compact domain which has finite measure in phase space and is invariant under the flow.

In this contribution we essentially illustrate the power of generalized Lyapunov functions, which are not discussed in the standard textbooks on dynamical systems theory. In particular we are interested in finding globally attractive domains for a certain class of nonlinear models. The corresponding systems turn out to be globally stable in the sense that no trajectory which starts within a certain domain can leave it and the trajectories which start outside of the domain will end in it after sufficiently large times. The method presented cannot give details on the nature of the attractors contained within an attractive domain. Furthermore, we will only partially succeed to determine minimal attractive domains. This, on the other hand, opens the chance of finding attractive domains in an analytical way. As a matter of fact, in his famous paper, Lorenz showed [2] that (nonminimal) attractive domains can be found in an elementary way by linear methods provided the nonlinearities of the dissipative dynamical system are quadratic only and do not contribute to the overall energy balance.

In the next section the method of generalized Lyapunov functions will be introduced together with a class of dynamical systems as proposed by Lorenz [2], which allow for quadratic Lyapunov functions. As a first example, the method is applied to the real Lorenz model according to [3]. We discuss then in the third section a more detailed application to the five-dimensional or complex Lorenz model [4]. As compared with recent work [5] where the attractive domain has been successfully minimized to some extent for parameter values relevant in infra-red laser physics, further new results [6] are presented here. In the last section the generalized Lyapunov method serves to prove the boundedness of the velocity field of the incompressible Navier-Stokes equations. This is a known result and was shown in different ways elsewhere, for the case of periodic boundary conditions see e.g. section 5.3 in [8].

2 Generalized Lyapunov functions

Let us consider an autonomous dynamical system

$$\frac{dx}{dt} = f(x); \quad f : \mathbb{R}^n \rightarrow \mathbb{R}^n$$  (1)
where the vector field $f$ is supposed to be sufficiently smooth. Be $L(x)$ a positively definite, sufficiently smooth, scalar function with $L(0) = 0$ and $L(x) > 0$ for $x \neq 0$. Furthermore, be $G$ a domain which contains the point $x = 0$, and $\bar{G}$ its complement. Then we call $L$ a generalized Lyapunov function, if the following properties hold for a compact domain $G$

\[
L(x) > 0 \quad \text{for} \quad x \neq 0 \quad \text{and} \quad L(0) = 0 \quad (2)
\]

\[
\frac{dL}{dt} := \frac{d}{dt} L(x(t,x_0)) = \sum_{i=1}^{n} \frac{\partial L}{\partial x_i} f_i(x) < 0
\]

for $x \in \bar{G}$, \quad (3)

\[
\text{grad}_x L(x) \neq 0 \quad \text{for} \quad x \neq 0,
\]

and Lipschitz continuous. \quad (4)

The manifolds $L(x) = C = \text{constant}$ are closed hypersurfaces, which surround the point $x \neq 0$ and foliate the phase space. To see this, one starts with a sufficiently small constant $C$ which because of $L(x) > 0$ is connected with an ellipsoidal surface. Then one constructs the one-dimensional curves normal to the surfaces $L(x) = C$. If $\lambda$ is a suitable curve parameter, the curves $x(\lambda)$ with $x : \mathbb{R} \to \mathbb{R}^n$ can be defined through the property that the curve tangents are parallel to the surface normal in every point. We can thus consider the curves as trajectories of the following dynamical system

\[
\frac{dx_i}{d\lambda} = \frac{\partial L(x(\lambda))}{\partial x_i}, \quad i = 1, 2, \ldots, n. \quad (5)
\]

Since this system is autonomous and because of (4), we obtain unique curves $x(\lambda,x_0)$ which do not intersect or touch each other; $x_0 \in \mathbb{R}^n$ is an arbitrary initial point which can be chosen, for instance, on an ellipsoid close to the origin. In this way we have constructed a 1-1 map between the points of an arbitrary surface $L = C$ and the points of an ellipsoid close to the origin $x = 0$. In other words, the surfaces $L = C$ are homeomorphic to an ellipsoid surrounding $x = 0$. Moreover, any surface with a given constant $C_0$ separates the phase space into an inner part which contains $x = 0$ and an outer part foliated by the surfaces $C > C_0$.

At the boundary of $G$ we have points where $dL/dt = 0$. We remark that this set contains the critical points of the system where $f(x) = 0$, because $dL/dt = \sum_i f_i \partial L/\partial x_i = 0$. The surface $L = C^*$ which both bounds a domain containing the points with $dL/dt = 0$ and is minimal with respect to $C$, is called the critical one. Now we are ready to draw conclusions for the trajectories of the system (1). If $x_0$ is the initial point of a trajectory with the property $L(x_0) = C_0 > C^*$, then because of (3) we have $dL(x_0)/dt < 0$. Therefore the trajectory wanders towards inner points with smaller $C$ until the critical surface with $L(x) = C^*$ is reached. This tells that the critical surface is attractive from the outside. Simultaneously there can be no escape of a trajectory which starts inside the domain bounded by the critical surface. For illustration see Fig.1. The existence of a generalized Lyapunov function guarantees therefore that the trajectories of the dynamical system asymptotically are confined to the domain $G^*$ bounded by the critical surface. Clearly, if we find several Lyapunov functions with attractive domains $G_1^*, G_2^*, \ldots$, then the intersection $G^* = \cap_i G_i^*$ contains the minimal attractive domain.

\[\text{Figure 1: Illustration of a generalized Lyapunov function. The dashed curve shows the surface } \frac{dL}{dt} = 0, \text{which is tangent to the critical surface } L = C^* \text{ (dot-dashed). The latter confines a domain of attraction. The solid curves refer to surfaces } L = C \text{ with } C > C^* \text{ and } dL/dt < 0.\]

As a rather general example Lorenz considered the following dynamical system, for $i = 1, \ldots, n$,

\[
\frac{dx_i}{dt} \equiv \dot{x}_i = \sum_{j,k} a_{ijk} x_j x_k - \sum_j b_{ij} x_j + c_i \quad (6)
\]
with
\[ \sum_{i,j,k} a_{ijk} x_i x_j x_k \equiv 0 \quad \text{and} \quad \sum_{i,j} x_i b_{ij} x_j > 0. \quad (7) \]

He proposed the Lyapunov function
\[ L(x) = \frac{1}{2}(x_1^2 + x_2^2 + \ldots + x_n^2) \quad (8) \]
which gives rise to
\[ \dot{L} = -\sum_{i,j} x_i b_{ij} x_j + \sum_i c_i x_i. \quad (9) \]

Now, because the symmetric part of the matrix \( b_{ij} \) is positively definite (all eigenvalues are positive), \( dL/dt < 0 \) for sufficiently large \( |x| \). Therefore \( L \) fulfils all conditions (2)-(4) of a generalized Lyapunov function.

As an elementary example we consider the Lorenz model \[ 2 \]
\[ \dot{x} = \sigma(y - x); \quad \dot{y} = -xz + rx - y; \quad \dot{z} = xy - bz \quad (10) \]
with \( r, \sigma, b > 0 \). Sparrow, see Appendix C in \[ 3 \], proved the boundedness of this model with the aid of the following function
\[ \tilde{L} = rx^2 + \sigma y^2 + \sigma(z - 2r)^2. \quad (11) \]
After the coordinate shift \( x_1 := x, x_2 := y, x_3 := z - 2r \), we obtain
\[ L(x) := rx_1^2 + \sigma x_2^2 + \sigma x_3^2 \quad (12) \]
and
\[ \dot{L} = -2\sigma(rx_1^2 + x_2^2 + bx_3^2 + 2brx_3) \quad (13) \]
which is negatively definite for sufficiently large distances \( \sqrt{x_1^2 + x_2^2 + x_3^2} \). Thus \( L(x) \) as defined in (12) fulfils the conditions of a generalized Lyapunov function with the implication that a bounded domain exists which attracts all trajectories.

3 Application to the complex Lorenz model

The complex Lorenz model reads in standard form \[ 4 \]
\[ \dot{X} = -aY + rX - XZ \quad (14) \]
\[ \dot{Z} = -bZ + \frac{1}{2}(X^*Y + XY^*) \]
where \( X, Y \) and \( Z \) are complex variables and real, respectively. Furthermore, \( a = 1 - ie, r = r_1 + ir_2 \) with real parameters \( e, r_1, r_2, \sigma, \) and \( b \). In the case of modeling a detuned laser, the constants \( r_1, r_2 \) are related to the pumping rate and to the detuning, respectively. Furthermore \( \sigma = \kappa/\gamma_\perp \) and \( b = \gamma ||/\gamma_\perp \) where \( \kappa, \gamma_\perp, \gamma || \) denote the relaxation constants of the cavity, of the polarization, and of the inversion. The variable \( X \) is proportional to the complex electric field amplitude, \( Y \) is a linear combination of electric field and polarization, which are both complex, while \( Z \) is related to the so-called population inversion, for details see e.g. \[ 1 \]. As is well known \[ 4 \], this model has nontrivial stationary solutions only in the so-called laser case with the parameter constraint \( e = -r_2 \).

It is convenient to introduce real variables \( x_i \), with \( i = 1, \ldots, 5 \), by \( X = x_1 + i x_2, Y = x_3 + i x_4 \) and \( Z = x_5 \). The real version of (14) then reads
\[ \dot{x}_1 = -\sigma x_1 + x_3 \]
\[ \dot{x}_2 = -\sigma x_2 + x_4 \]
\[ \dot{x}_3 = r_1 x_1 - x_3 - r_2 x_2 - e x_4 - x_1 x_5 \]
\[ \dot{x}_4 = x_1 x_2 - x_4 + r_2 x_1 + e x_3 - x_2 x_5 \]
\[ \dot{x}_5 = -b x_5 + x_1 x_3 + x_2 x_4. \]

In \[ 4 \] the following Lyapunov function was proposed
\[ \tilde{L} = D^2(x_1^2 + x_2^2) + x_3^2 + x_4^2 + (x_5 - r_1 - D^2 \sigma)^2 \quad (16) \]
which has the Lie derivative
\[ \frac{1}{2} \frac{d\tilde{L}}{dt} = -\sigma D^2(x_1^2 + x_2^2) - x_3^2 - x_4^2 - r_2(x_2 x_3 - x_1 x_4) - b x_5(x_5 - r_1 - D^2 \sigma). \quad (17) \]
Here \( D \) is an arbitrary parameter at our disposition. The latter expression turns out to be negatively definite for sufficiently large distances \( \sqrt{x_1^2 + \ldots + x_5^2} \) provided \( D \) obeys the condition
\[ r_2^2/(4D^2 \sigma) < 1. \quad (18) \]
After the coordinate shift $x'_5 := x_5 - r_1 - D^2 \sigma$, the function $L(x_1, x_2, x_3, x_4, x'_5) := \dot{L}(x_1, \ldots, x_5)$ fulfills all requirements (2)-(4) of a generalized Lyapunov function. It is thus proved that also the complex Lorenz model is bounded for all parameters, with and without the laser condition $e = -r_2$. For quantitative results one determines the ellipsoid $L = C^*$ which touches the (geometrically different) ellipsoid $dL/dt = 0$ from the outside. This amounts to a five-dimensional secular problem which in the given case happens to be feasible analytically. The attractive domain is then minimized with respect to the parameter $D$ with due attention paid to the constraint (18). Details can be found in [5]. Numerical evaluations for physically relevant parameters give upper bounds for the laser electric field which exceed the maximum values reached by asymptotic solutions of (15) by factors of between 2 and 6. In extreme cases of transient evolution, the solutions approach within 20% of the upper bounds predicted by the Lyapunov method, see Fig.4 and 5 in [3].

As a remark, we have examined the more general Lyapunov function $\tilde{L}$

\[ L = D^2(x_1^2 + x_2^2) + x_3^2 + x_4^2 + 2\xi (x_2x_3 - x_1x_4) + (x_5 - \nu)^2; \]

\[ \nu = r_1 + D^2 \sigma + \xi r_2; \quad \xi^2 < D^2, \]

with the further disposable parameters $\xi$ in addition to $D$. $L$ fulfills the properties (2) and (4) as is immediately seen after the coordinate transformation $x_1 \rightarrow x_1 - \xi/D^2 x_4$, $x_2 \rightarrow x_2 + \xi/D^2 x_3$, $x_3 \rightarrow x_3$, $x_4 \rightarrow x_4$, $x_5 \rightarrow x_1 - \nu$. The Lie-derivative is given as

\[ \frac{1}{2} \frac{dL}{dt} = -(D^2 \sigma + \xi r_2)(x_1^2 + x_2^2) - x_3^2 - x_4^2 - [r_2 + (\sigma + 1)\xi](x_2x_3 - x_1x_4) - bx_3^2 + bx_5. \]

This derivative turns out to be negatively definite for sufficiently large distances $\sqrt{x_1^2 + \ldots + x_5^2}$, and thus obeying (3), provided

\[ \frac{[r_2 + (\sigma + 1)\xi]^2}{D^2 \sigma + \xi r_2} < 4. \]

The determination of upper bounds of the electric field amplitude $|X|^2$ is carried out in a similar way as in [5]. The main challenge consists in the task of simplifying rather involved analytical expressions for different parameter regions. In a physically relevant parameter domain

\[ r_2^2 \leq \frac{4\sigma r_1 (2 - b)(2\sigma - b)}{(\sigma + 1)^2 + (2 - b)(2\sigma - b)}. \]

with $b < 2$ and $2\sigma > b$, the following upper bound is found [5] which is minimized with respect to the two parameters $D$ and $\xi$

\[ |X|^2 \leq 4\sigma r_1 \frac{(\sigma + 1)^2}{(\sigma + 1)^2 + (2 - b)(2\sigma - b)}. \]

It is smaller than the upper bound $|X|^2 \leq 4\sigma r_1$ as found previously in [5] with one disposable parameter only, namely $D$.

4 Application to the Navier-Stokes equations

We consider the incompressible Navier-Stokes equations (NSE)

\[ \rho_0 \left[ \frac{\partial \mathbf{v}}{\partial t} + (\mathbf{v} \cdot \nabla) \mathbf{v} \right] = -\nabla p + \mu \Delta \mathbf{v} + \rho_0 \mathbf{f} \]

in the space domain $\Omega$ which, at least in one dimension, has a finite diameter so that it can be located between two parallel planes with finite distance $l$. Because of $\text{div} (\mathbf{v}) = 0$, the density $\rho_0$ is constant. We are not concerned here with weak solutions, which are discussed in [5], and assume that the solutions are sufficiently smooth. As boundary conditions we adopt the no-slip case with $\mathbf{v}|_{\partial \Omega} = 0$.

In order to obtain a dynamical system of ordinary differential equations, we represent the velocity field $\mathbf{v}$ in terms of an orthonormalized system $\Phi_n \in D(\Omega)$, with $\text{div} (\Phi_n) = 0, n=1,2,\ldots$, where $D(\Omega)$ denotes the space of $C^\infty$ functions with compact support in $\Omega$. We write

\[ \mathbf{v}(\mathbf{x}, t) = \sum_{n=1,2,\ldots} c_n(t) \Phi_n(\mathbf{x}) \quad \text{with} \quad c_n \in \mathbb{R} \]
and define the Lyapunov function as follows

\[
L(c_1, c_2, \ldots) := \rho_0 \sum_{n=1,2,\ldots} c_n(t) c_n(t) = \rho_0 \int_{\Omega} dV \cdot \mathbf{v}.
\]  

(26)

This function, obviously fulfills the conditions (2) and (4). To verify the property (3) we scalarly multiply the NSE (24) with \( \mathbf{v} \) and integrate over the space \( \Omega \). On the left hand side we get

\[
\frac{1}{2} \frac{dL}{dt} = \rho_0 \int_{\Omega} dV \mathbf{v} \cdot \frac{\partial \mathbf{v}}{\partial t}.
\]  

(27)

Because of \( \text{div}(\mathbf{v}) = 0 \), the cubic term can be transformed into the surface integral

\[
\int_{\partial \Omega} dS \mathbf{v}^2 \hat{n} \cdot \mathbf{v} = 0.
\]

The viscosity term, which is negatively definite, is estimated with the aid of the Poincaré inequality [10] as follows

\[
\int_{\Omega} dV \mathbf{v} \Delta \mathbf{v} = - \int_{\Omega} dV \frac{\partial v_k}{\partial x_i} \frac{\partial v_k}{\partial x_i} \leq - \frac{2}{l^2} \| \mathbf{v} \|^2
\]

\[
= - \frac{2}{l^2} \sum_{n=1,2,\ldots} c_n c_n
\]

(28)

where \( \| \mathbf{v} \|^2 = \int_{\Omega} dV \mathbf{v} \cdot \mathbf{v} \). The pressure term drops out after partial integration. When the last term with the force density \( f \) is estimated by the Schwarz inequality, we obtain

\[
\frac{1}{2} \frac{dL}{dt} \leq - \mu \frac{2}{l^2} \| \mathbf{v} \|^2 + \rho_0 \| \mathbf{v} \| \| f \|.
\]  

(29)

This proves that \( dL/dt < 0 \) for sufficiently large \( \| \mathbf{v} \|^2 = \sum c_n c_n \). Thus \( L \) possesses also the property (3), and as a consequence \( \| \mathbf{v} \| \) is asymptotically bounded provided the norm of \( f \) is finite for all times \( t \). From \( \dot{L} = 0 \) we obtain as asymptotic bound

\[
\| \mathbf{v} \| \leq \frac{l^2}{2\mu} \rho_0 \max_{t>0} \| f(t) \|.
\]  

(30)

As a remark, the problem of possible singularities in the solutions of the NSE are connected with the space gradient of \( \mathbf{v} \) rather than to the velocity itself, see [3]. The generalized Lyapunov method is related to so-called energy methods, see e.g. [10] and [1].

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References

[1] LaSalle J.P. and Lefschetz S., Stability by Lyapunov’s Direct Method with Applications, Academic Press (1961), New York

[2] Lorenz E.N., J. Atmos. Sci. 20 (1963), 130

[3] Sparrow C., The Lorenz equations: bifurcations, chaos, and strange attractors, Springer-Verlag (1982), Berlin

[4] Fowler A.C., Gibbon J.D., and McGuinness M.J., Physica D 4 (1982), 139

[5] Rauh A., Hannibal L., and Abraham N.B., Physica D 99 (1996), 45

[6] Rauh A., Remarks on unsolved problems of the incompressible Navier-Stokes equations, (this conference proceedings)

[7] F.Buss, Diploma thesis, University of Oldenburg (1997)

[8] Doering Ch.R. and Gibbon J.D., Applied Analysis of the Navier-Stokes Equations, Cambridge University Press (1995), Cambridge USA

[9] Bakasov A.A. and Abraham N.B., Phys. Rev. A 48 (1993), 1633

[10] Joseph D.D. Stability of Fluid Motions I, Springer-Verlag (1976), Berlin, p.13

[11] Straughan B., The energy method, stability, and nonlinear convection, Springer-Verlag (1992), New York