Communication cost of quantum channels

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Abstract

In distributed computing, it is common for a server to execute a program of a remote client, consuming the minimum amount of communication. We extend such a setting to the quantum regime and consider the task of communicating quantum channels for the purpose of executing them a given number of times. We derive a general lower bound for the amount of required communication. The bound shows that the amount of required communication is positively correlated to the performance of the channels in quantum metrology. We also propose a protocol achieving the bound for a type of channels.

Index Terms

quantum communication, quantum channel, quantum metrology, Heisenberg limit

I. INTRODUCTION

Quantifying the communication cost for remotely executing a desired computation is a fundamental issue in classical distributed computing \cite{1}, with applications in wireless sensor networks \cite{2} and in the design of distributed algorithms \cite{3}. Such an issue arises also in quantum computing, in particular in the tasks of quantum gate teleportation \cite{4}, \cite{5}, and delegated quantum computation \cite{6}, \cite{7}, \cite{8}, \cite{9}, when one party is asked to apply a quantum channel on the input state of another remote party. More generally, the quantification of the communication cost of quantum operations, both reversible and irreversible, is relevant to a number of information-theoretic tasks, including the design of programmable quantum devices \cite{10}, \cite{11}, \cite{12}, \cite{13}, \cite{14}, the manipulation and conversion of quantum gates \cite{15}, \cite{16}, \cite{17}, \cite{18}, \cite{19}, \cite{20}, quantum process tomography \cite{21}, \cite{22}, \cite{23}, and quantum reading \cite{24}, \cite{25}.

The task of communicating quantum gates arises naturally in the following distributed computing scenario. A client is connected to a remote server via a quantum communication link, and would like the server to perform a quantum gate. To fulfill such a task, the client sends a program, which is a quantum or classical message that describes the gate, to the server. The server then plugs in the program into a quantum computer which executes the desired gate. The target is to minimize the communication cost, while still implementing the desired computation with high precision.

In this work, we address the issue of minimizing the cost of communicating a quantum channel selected from a parametric family. The purpose of the communication is to enable the receiver to execute the channel \( n \) times with high accuracy. We show, by both a deriving fundamental bound and designing a concrete protocol, that the communication cost in such a task is related to the precision limits of quantum metrology \cite{26}, \cite{27}, \cite{28}, \cite{29}. For quantum channels that can be estimated to a root-mean-square error \( n^{-\beta} \), the bound implies that the communication cost is at least \((v\beta/2)\log n\) in the leading order of \( n \), where \( v \) is the dimension of the parametric family. As a consequence, channels that achieve the Heisenberg limit \( n^{-1} \) \cite{28}, \cite{30}, \cite{31}, \cite{32}, \cite{33}, \cite{34} have a higher communication cost than channels with the standard scaling \( n^{-1/2} \). Our bound also applies to a more general setting of remote channel simulation, where the server executes a compressed version of the original channel, from which the original channel can be retrieved by applying suitable pre- and post-processing operations.

II. THE CHANNEL COMMUNICATION TASK

Here we study the task of communicating \( n \) uses of a channel \( C \) in parallel. As depicted in Figure\textsuperscript{1} the task is for a server to perform \( n \) uses of the channel \( C \), based on the instructions provided by a client through the communication of a program (a quantum state) \( \eta_{\text{Tr},n} \) of a quantum system of the smallest possible size. In other words, a client requires the application of \( n \) uses of the channel \( C \) with the transmission of a program \( \eta_{\text{Tr},n} \) of the smallest possible size. The channel is randomly drawn from
a set \( \{ \mathcal{G}_t \}_{t \in \mathcal{T}} \) of quantum channels, with the parameter set \( \mathcal{T} \) being a bounded subset of \( \mathbb{R}^v \) for some \( v \in \mathbb{N} \). For simplicity, we assume that \( \mathcal{T} = \prod_{i=1}^{v} [a_i, b_i] \) with each \( [a_i, b_i] \) being an interval and denote by \( |\mathcal{T}| \) the volume of the parameter space. To evaluate the error of the task, we adopt the diamond norm of a CP map \( \mathcal{A} : L(\mathcal{H}_{in}) \rightarrow L(\mathcal{H}_{out}) \), defined as

\[
\| \mathcal{A} \|_D := \sup_{\Psi \in \text{St}(\mathcal{H}_{in} \otimes \mathcal{R})} \| \mathcal{A} \otimes I_{\mathcal{R}}(\Psi) \|_1,
\]

where \( \| \cdot \|_1 \) denotes the trace norm and \( \mathcal{R} \) is a reference system that can be assumed to be isomorphic to \( \mathcal{H}_{in} \). For any error threshold \( \epsilon > 0 \), we consider protocols such that \( (1/2) \| \mathcal{G}_{n,t} - \mathcal{G}_{\epsilon,n} \|_D \) is upper bounded by \( \epsilon \) for every \( t \in \mathcal{T} \), where \( \mathcal{G}_{n,t} \) is the approximate version of \( \mathcal{G}_{\epsilon,n} \) realized by the communication protocol.

Given an error threshold, the goal is to minimize the communication cost, namely the total number of qubits transmitted in the protocol. Explicitly, the communication cost is \( N := \log d_{\text{prog}} \), where \( d_{\text{prog}} \) is the dimension of the program in Figure 1 and \( \log \) denotes the base-2 logarithm. More rigorously, \( d_{\text{prog}} \) is the dimension of the space spanned by \( \{ \eta_{n,t} \}_t \).

The task is then to identify the smallest possible program for a multiple uses of a quantum channel drawn from a given parametric family. Multiple-use quantum channels are typical in quantum metrology and they can be classified by their parametric family. The set of quantum states (positive matrices with unit trace) on a Hilbert space \( \mathcal{H} \) is independent of \( t \). Consider a parametric family of quantum channels \( \{ \mathcal{G}_{\epsilon,n} \}_t \in \mathcal{T} \subseteq \text{Chan}(\mathcal{H}_{in}, \mathcal{H}_{out}) \) where \( \mathcal{T} \) is the parameter manifold. We assume the following conditions:

(C1) The range of \( \text{Choi}(\mathcal{G}_t) \) is independent of \( t \).

(C2) The minimum of Fisher information over the channel family is non-zero, i.e. \( J^R_{\text{opt}} := \inf_{t \in \mathcal{T}} J^R_{\mathcal{G}_t} > 0 \).

Our protocol of communicating \( \mathcal{G}_{\epsilon,n} \) works as the following:
Protocol 1 Communicating multiple uses of a channel.

(Preparation) The client and the server share the knowledge of the following discretization $\mathcal{D}_t$ of $\mathcal{T}$:

$$\mathcal{D}_t := \left\{ \frac{n^{-\alpha - 1/2}}{\sqrt{v J_R}} z \in \mathbb{Z}^v \right\} \cap \mathcal{T}. \quad (5)$$

1: (Description.) For any channel $\mathcal{C}^{\otimes n}_t$ to be simulated, instead of $t$ the client transmits the server a rounded value $t_r \in \mathcal{D}_t$ to satisfy

$$\|t_r - t\| < \frac{n^{-\alpha - 1/2}}{\sqrt{J_R}}. \quad (6)$$

2: (Implementation.) The server implements $\mathcal{C}^{\otimes n}_{t_r}$ on the input state.

As one can see from the above protocol, Condition (C2) is needed to ensure that the discretization $\mathcal{D}_t$ is well-defined. On the other hand, Condition (C1) implies that the channel is standard quantum limited [35], and thus it can be efficiently communicated by the protocol. For Heisenberg limited channels, the RLD Fisher information [4] becomes infinite [36], and thus the protocol fails.

The cost and the error rate of the above protocol is summarized by the following theorem. Proof of this theorem is postponed to the end of the paper.

**Theorem 1.** Under the conditions (C1) and (C2), for any set $\{\mathcal{C}^{\otimes n}_t\}_{t \in \mathcal{T}}$ of $n$-use quantum channels and for arbitrary $\alpha > 0$, Protocol 1 has an error $O(n^{-\alpha})$ and costs $(1/2 + \alpha)v \log n$ bits of communication at the leading order in $n$, where $v$ is the dimension of the parameter set $\mathcal{T}$.

V. A LOWER BOUND ON THE COST OF REMOTE CHANNEL SIMULATION AND THE OPTIMALITY OF PROTOCOL 1

In this section, we prove that the cost of Protocol 1 as given in Theorem 1 is minimal for the category of channels under its consideration. To this purpose, we prove a lower bound on the communication cost for a more general task of remote channel simulation, from which the optimality of Protocol 1 is immediate.

The task of simulating $n$ parallel uses of a channel $\mathcal{C}_t$, depicted in Figure 2, can be described as follows. A client, equipped with a small quantum computer, wants a server to execute $n$ parallel uses of a quantum channel $\mathcal{C}_t$, randomly drawn from a set $\{\mathcal{C}_t\}_{t \in \mathcal{T}}$. To this purpose, the server encodes the input of the desired channel $\mathcal{C}^{\otimes n}_t$ using an encoder $E$. The encoder outputs two systems: a small system, which is sent to the client through a quantum communication link, and a larger system, which is stored in a quantum memory $M$ at the server’s end. Then, the client uses its quantum computer to execute a channel $B_{n,t}$ on the small system, and sends the output back to the server. Finally, the server applies a decoder $D$ on its local quantum memory and on the system received by the client. The protocol is designed in such a way that the total transformation $D(B_{n,t} \otimes I_M)E$ is close to the desired channel $\mathcal{C}^{\otimes n}_t$.

The insertion of a quantum channel between an encoder and a decoder, using a quantum memory as in Figure 2, represents the most general transformation from quantum channels to quantum channels [37], [38]. This transformation represents the “transmission of a quantum channel” from the client to the server. The channel communication task described in Section III is a special case of the channel simulation task in Figure 2 corresponding to the situation in which the input of the channel $B_{n,t}$ is trivial, and therefore the channel $B_{n,t}$ is simply a state $\eta_{l,t}$. Hence, every lower bound on the total amount of communication required in the channel simulation scenario is also a lower bound on the amount of communication required in the channel communication scenario.

In the following, we assume that the parameter set $\mathcal{T}$ is a bounded subset of $\mathbb{R}^v$ for some $v \in \mathbb{N}$. For any error threshold $\varepsilon > 0$, we consider protocols such that $(1/2)\|\widehat{\mathcal{C}}^{\otimes n}_t - \mathcal{C}^{\otimes n}_t\|$ is upper bounded by $\varepsilon$ for every $t \in \mathcal{T}$, where $\widehat{\mathcal{C}}^{\otimes n}_t$ is the approximate version of $\mathcal{C}^{\otimes n}_t$ realized by the simulation protocol. The communication cost, on the other hand, is $N_{\text{tot}} := \log d_{\text{in}} + \log d_{\text{out}}$, where $d_{\text{in}}$ and $d_{\text{out}}$ are the dimensions of the input and output systems in Figure 2.

Our main result is an information-theoretic bound on the cost of $n$-use channel simulation.

**Theorem 2** (Fundamental limit on remote channel simulation). If a set $\{\mathcal{C}^{\otimes n}_t\}_{t \in \mathcal{T}}$ of $n$-use quantum channels can be estimated to inaccuracy $O(n^{-\beta})$, the leading order of its simulation cost is lower bounded by $(1 - \varepsilon')(\sqrt{\beta}/2) \log n$, for any $\varepsilon'$ greater than the simulation error. Here $v$ is the dimension of the parameter set $\mathcal{T}$.

The cost of remote channel simulation was determined by Fang et al. [39] in terms of the channel’s maximum output mutual information, whereas Theorem 2 establishes a connection between simulating multiple uses of a channel and quantum metrology. It can be immediately seen that for standard quantum limited (Heisenberg limited) channels, their simulation requires at least $(1/2) \log n (\log n)$ (qu)bits of communication. On the other hand, Theorem 2 reveals a general relation between quantum
channel simulation and quantum metrology, with standard quantum limited channels ($\beta = 1/2$) and Heisenberg limited channels ($\beta = 1$) as two special cases.

Theorem 2 also yields a lower bound on the cost of communicating channels as we desire. Indeed, a protocol that communicates $C_t^{\ominus \pi}$ from the server to the client can always be used for simulation, as sending programs (i.e. setting $d_{\infty} = 1$) is a method of fulfilling the simulation task. Comparing the costs listed in Theorem 1 and Theorem 2 yields the following corollary:

**Corollary 1.** If a set $\{C_t^{\ominus \pi}\}_{t \in T}$ of $n$-use quantum channels can be estimated to inaccuracy $O(n^{-\beta})$, the leading order of the length of its program is lower bounded by $(1 - \epsilon')(\sqrt{\beta}/2) \log n$, for any $\epsilon'$ greater than the programming error. Here $v$ is the dimension of the parameter set $T$.

As a consequence, Protocol 1 is optimal. That is, we can optimally communicate channels satisfying: 1) the range of $\text{Choi}(\mathcal{G}_t)$ is independent of $t$ and 2) $J_{C_t}^R := \inf_{t \in T} J_{C_t}^R > 0$.

A set of quantum channels $\{C_t^{\ominus \pi}\}_{t \in T}$ is called **programmable** if there exist a set of quantum states $\{\eta_t\}_{t \in T}$ and a quantum channel $\mathcal{W}$ so that $\mathcal{G}(\rho) = \mathcal{W} (\rho \otimes \eta_t)$ holds for every $t \in T$. When this is the case, the state $\eta_t$ is called a program for the quantum channel $\mathcal{G}_t$. Another immediate consequence of Theorem 2 is that programmable channels require $(1/2) \log n$ (qu)bits of communication, since they are standard quantum limited [40]. Programmability of quantum channels has been adapted to studying their communication capacities [41], [42] and to evaluating their performance in metrology [43], [40], [44], [45]. Theorem 2 goes further than previous works, in that it holds not only for perfectly programmable channels but also for approximate programming.

Before showing the proof for Theorem 2, we prepare a generic relation between the mutual information and the inaccuracy. Let us consider estimation of a random variable $X$ using an estimator $\hat{X}$. The inaccuracy of $\hat{X}$, defined as

$$\delta_p := \arg \min_\delta \max_x \Pr_{X \rightarrow \hat{X}}[|X - \hat{X}| \leq \delta] = p,$$

quantifies the size of the region in which the estimator has confidence $p \in (0, 1)$. Here $|y| := \sqrt{\langle y \mid y \rangle}$ for any vector $y$. In Appendix, we prove the following lemma on the relation between the inaccuracy and the mutual information between the random variable $X$ and its estimate $\hat{X}$.

**Lemma 1.** For a $v$-dimensional bounded continuous random variable $X$ with inaccuracy $\delta_p$, the following inequality holds

$$I(\hat{X} : X) \geq H(X) - p \log \left( \frac{B_{v, \delta_p}}{p} \right) - (1 - p) \log \left( \frac{2^v |X| - B_{v, \delta_p}}{1 - p} \right).$$

Here $H(X)$ denotes the differential entropy of the random variable $X$ and $B_{v, \delta_p} = \frac{(\sqrt{\delta_p})^v}{\Gamma(v/2 + 1)}$ (the Gamma function) is the volume of a $v$-ball with radius $\delta_p$.

Here we are interested in the case of small inaccuracy. Expanding the right hand side of the inequality in Lemma 1 for $\delta_p \ll 1$ and constant $p \in (0, 1)$ yields

$$I(\hat{X} : X) \geq -pv \log(\sqrt{\delta_p}) + H(X) + p \log \Gamma \left( \frac{v}{2} + 1 \right) + h(p) - (1 - p) \log(2^v |X|),$$

where $h(p)$ denotes the binary entropy. When $\delta_p \ll 1$, the bound states that the mutual information between the random variable and its estimate is bounded by $-pv \log(\delta_p)$.

Fig. 2. Schematic of $n$-use channel simulation. A client, equipped with a small quantum computer, executes a channel $B_{t,n}$, which enables a server to reproduce $n$ uses of a target channel $\mathcal{G}_t$. The part in yellow, i.e. the channels $\mathcal{E}$ and $\mathcal{D}$, are the encoder and the decoder of used by the server to interact with the client, while the part in blue, i.e. the channel $B_{t,n}$, is the channel performed by the client. The protocol is designed so that the overall transformation at the server’s end approximates the target channel $C_t^{\ominus \pi}$ for every choice of $t$ in a given parameter set $T$. 

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\[ \text{Δ} \]
The estimation of the channel $C$ from $n$ parallel uses is (approximately) related to the estimation of the channel $R_{\alpha}$ acting on lower-dimensional quantum systems. In particular, every protocol of estimating $C$ from $n$ parallel uses can be used as a protocol for estimating channel $R_{\alpha}$. The figure shows the general form of such a protocol, consisting in transforming the channel $R_{\alpha}$ into an approximation of channel $C^\otimes n$ using an encoder $E$, and a decoder $D$. The parameter $t$ is then estimated by preparing an input state $|\psi\rangle$ for (the approximation of) channel $C^\otimes n$, and and executing a measurement $\{M_t\}$ on the output.

Next, we look at the scenario of channel simulation from the point of view of quantum metrology. Consider the scheme of estimating $t$ as in Figure 3 where the protocol of the channel simulation protocol, characterized by the operations $P_{n,t}^{\text{sim}} := \{D, B_{n,t}, E\}$, is applied to an input state $|\psi\rangle$ and an estimate $\hat{t}$ is produced by performing a POVM $\{M_t\}$. The distribution for the outcome $\hat{t}$ is written by $\Pr[|\hat{t} - t| \leq \delta | \psi, P_{n,t}^{\text{sim}}, \{M_t\}]$. Notice that the input state $|\psi\rangle$ can be chosen to be pure without loss of generality. The inaccuracy of $\hat{t}$ in this estimation scheme is defined as

$$\delta_p(|\psi, \{P_{n,t}^{\text{sim}}\}, \{M_t\}) := \arg \min_\delta \max_t \Pr[|\hat{t} - t| \leq \delta | \psi, \{P_{n,t}^{\text{sim}}\}, \{M_t\}] = p.$$  \hspace{1cm} (9)

Next, we consider the inaccuracy of directly estimating $t$ from $n$ uses of $C$, as in the case of quantum metrology. Replacing the channel simulation operations $P_{n,t}^{\text{sim}}$ by $n$ uses of the true channel $C$ in the definition 9, we define $\delta_p(|\psi, \{C^\otimes n\}, \{M_t\})$. Since the simulation protocol $P_{n,t}^{\text{sim}}$ yields an $\varepsilon$-approximation of $C^\otimes n$, we have

$$\Pr[|\hat{t} - t| \leq \delta | \psi, P_{n,t}^{\text{sim}}, \{M_t\}] - \Pr[|\hat{t} - t| \leq \delta | \psi, C^\otimes n, \{M_t\}] < \varepsilon,$$  \hspace{1cm} (10)

which implies the continuity of the inaccuracy $\delta_p$:

$$\delta_p(|\psi, \{P_{n,t}^{\text{sim}}\}, \{M_t\}) \leq \delta_{p+\varepsilon}(|\psi, \{C^\otimes n\}, \{M_t\}).$$  \hspace{1cm} (11)

With the bounds 8 and 11, we are now ready to prove the main theorem:

**Proof of Theorem 2** As the assumption of Theorem 2 a set $\{C^\otimes n\}_{\alpha \in \mathcal{T}}$ of $n$-use quantum channels can be estimated to inaccuracy $O(n^{-\beta})$. More precisely, there exists an input $\psi$ and measurement $\{M_t\}$ such that $\max_t \text{Tr} V(\psi, C^\otimes n, \{M_t\}) \leq c \cdot n^{-\beta}$ for some constant $c > 0$, where $V(\psi, C^\otimes n, \{M_t\})$ is the covariance matrix with the input $\psi$, the measurement $\{M_t\}$, and $n$ use of $C$.

The generalized Chebyshev inequality implies $\delta_p(|\psi, \{C^\otimes n\}, \{M_t\})^2 (1-p) \leq \max_t \text{Tr} V(\psi, C^\otimes n, \{M_t\})$. The combination of these two inequalities implies

$$\min_{\psi, \{M_t\}} \delta_p(|\psi, \{C^\otimes n\}, \{M_t\}) \leq \sqrt{\frac{c}{(1-p)n^{\beta}}} \forall p \in (0,1).$$  \hspace{1cm} (12)

Now, combining 11 and 12, one has

$$\min_{\psi, \{M_t\}} \delta_p(|\psi, \{P_{n,t}^{\text{sim}}\}, \{M_t\}) \leq \min_{\psi, \{M_t\}} \delta_{p+\varepsilon}(\psi, \{C^\otimes n\}, \{M_t\}) \leq \sqrt{\frac{c}{(1-p-\varepsilon)n^{\beta}}}.$$

Suppose that the parameter $t$ is encoded into the channel $C^\otimes n$. For every pure input $\psi$, one has the following chain of (in)equalities (referring to Figure 3 for the labelling of Hilbert spaces):

$$N_{\text{max}} = \log d_a + \log d_b \geq H(4) + H(6) = H(3,5) + H(6) \geq H(3,5,6) \geq I(1 : 3,5,6) \geq I(1 : 8).$$  \hspace{1cm} (15)

The first inequality comes from the upper bound of entropy. The first equality comes from the observation that the encoder of the client, i.e. $E_{\alpha}$, can be assumed w.o.l.g. to be isometric. The second inequality holds since the mutual information is
always non-negative. The third inequality holds since the system labeled by 1 is a classical system. The forth inequality is the data processing inequality of the mutual information.

Substituting $\delta_p$ by $\delta_p(\psi, \{\mathcal{D}_{n,t}^{\text{sim}}\}, \{M_t\})$ in Eq. (15), one gets that

$$I(1 : 8) \geq -p v \log \left( \sqrt{\frac{1}{n}} \delta_p(\psi, \{\mathcal{D}_{n,t}^{\text{sim}}\}, \{M_t\}) \right) - (1 - p) \log(2^v |T|).$$  \hspace{1cm} (16)

Combining Eq. (16) and Eq. (15), one gets that

$$N_{\text{tot}} \geq -p v \log \left( \sqrt{\frac{1}{n}} \delta_p(\psi, \{\mathcal{D}_{n,t}^{\text{sim}}\}, \{M_t\}) \right) - (1 - p) \log(2^v |T|).$$  \hspace{1cm} (17)

Taking the same maximization in Eq. (17) and applying Eq. (14), one gets

$$N_{\text{tot}} \geq -v \log(\sqrt{c/(1 - p - \epsilon)n^p}) - (1 - p) \log(2^v |T|) = \frac{v p^2}{2} \log n + O(1)$$  \hspace{1cm} (18)

for any $p \in (0, 1 - \epsilon)$, which is equivalent to the statement in Theorem 1.

In the proof, we established a connection between two different criteria of quantum metrology. If a set of quantum channels is Heisenberg limited (standard quantum limited) in terms of the mean squared error, we showed that it is also Heisenberg limited (standard quantum limited) in terms of inaccuracy. In addition, the quantity on the left hand side of Eq. (16) is the mutual information between the true value $t$ and its estimate $\hat{t}$ in the setting of channel estimation, which amounts to the number of digits of $t$ that can be made precise in quantum metrology as interpreted in Ref. [49]. When the channel is Heisenberg limited, it is immediate from Eq. (16) that this quantity scales as $\log n$, which was defined in Ref. [49] as the information theoretic Heisenberg limit. In this sense, Eq. (16) establishes a general lower bound on the digitization of estimation precision, which extends the result in [48] from the setting of estimating ideal phase gates to the setting of multi-parameter and noisy metrology.

VI. PROOF OF THEOREM I

In this section, we show the proof of Theorem 1. It is clear that such a protocol demands $(1/2 + \alpha) \log n$ bits of communication at the leading order of $n$. What remains to be shown is the error rate.

First, we need to prepare a few concepts before our proof. The 2-Rényi divergence for two states $\rho$ and $\sigma$ is defined as $D_2(\rho||\sigma) = \log \text{Tr} \rho^2 \sigma^{-1}$ for $\text{supp}(\rho) \subset \text{supp}(\sigma)$. As a special case of [49] Definition 2.2, such a concept can be extended to a distance measure between quantum channels, defined as

$$D_2(\mathcal{A}||\mathcal{B}) := \sup_{\Psi \in \text{St}(\mathcal{H}_0 \otimes \mathcal{K})} D_2(\mathcal{A} \otimes I(\Psi)||\mathcal{B} \otimes I(\Psi)).$$  \hspace{1cm} (19)

The 2-Rényi divergence for quantum channels has an explicit form under the following condition:

(C3) The range of $\text{Choi}_\delta$ contains the range of $\text{Choi}_\delta^2$.

Otherwise, it is infinity.

Lemma 2. When Condition (C3) holds, the 2-Rényi divergence for two channels $\mathcal{A}$ and $\mathcal{B}$ has the following expression:

$$2^{D_2(\mathcal{A}||\mathcal{B})} = |\text{Tr}_{\text{out}} \text{Choi}_\delta \text{Choi}_\delta^{-1} \text{Choi}_\delta|,$$  \hspace{1cm} (20)

$$2^{D_2(\mathcal{A}||\mathcal{B})} - 1 = |\text{Tr}_{\text{out}} (\text{Choi}_\delta - \text{Choi}_\delta) \text{Choi}_\delta^{-1} (\text{Choi}_\delta - \text{Choi}_\delta)| = |(\text{Tr}_{\text{out}} \text{Choi}_\delta \text{Choi}_\delta^{-1} \text{Choi}_\delta) - I|,$$  \hspace{1cm} (21)

where $\text{Choi}_\delta$ is the Choi matrix of the channel $\mathcal{A}$.

Although Lemma 2 is given as a special case of [50] Item 2 of Theorem 3 with $\alpha = 2$, (21) follows from the maximum of the RLD Fisher information [36] Theorem 1 of the one-parameter family $\mathcal{A}_0 := \mathcal{A} + \theta(\mathcal{B} - \mathcal{A})$ at $\theta = 0$. This is because the RLD Fisher information of the family $\mathcal{A}_0 \otimes I(\Psi)$ at $\theta = 0$ equals $2^{D_2(\mathcal{A}_0 \otimes I(\Psi)||\mathcal{A}_0 \otimes I(\Psi))} - 1$. Then, (21) implies (20) because $(\text{Tr}_{\text{out}} \text{Choi}_\delta \text{Choi}_\delta^{-1} \text{Choi}_\delta) \geq I$ follows from $|\text{Tr}_{\text{out}} (\text{Choi}_\delta - \text{Choi}_\delta) \text{Choi}_\delta^{-1} (\text{Choi}_\delta - \text{Choi}_\delta)| = |(\text{Tr}_{\text{out}} \text{Choi}_\delta \text{Choi}_\delta^{-1} \text{Choi}_\delta) - I|$ for any $|\Psi\rangle$, where $|\Psi\rangle$ is the complex conjugate of $|\Psi\rangle$.

When channels $\mathcal{A}_1, \mathcal{A}_2, \mathcal{B}_1$, and $\mathcal{B}_2$ satisfy the range of $\text{Choi}_\delta$ contains the range of $\text{Choi}_\delta$ (i.e., $\delta_i = 1, 2$), as stated in [50] Item 3 of Theorem 3 with $\alpha = 2$, similar to Theorem 2 of [36], noticing that $\text{Choi}_{\mathcal{A} \otimes \mathcal{B}} = \text{Choi}_\delta \otimes \text{Choi}_\delta$, Eq. (20) immediately implies the additivity of 2-Rényi divergence for quantum channels:

$$D_2(\mathcal{A}_1 \otimes \mathcal{B}_1||\mathcal{A}_2 \otimes \mathcal{B}_2) = D_2(\mathcal{A}_1||\mathcal{A}_2) + D_2(\mathcal{B}_1||\mathcal{B}_2).$$  \hspace{1cm} (22)

Now we can show that the error scales like $n^{-\alpha}$ when the range of $\text{Choi}(|\xi\rangle)$ is independent of $t$. First, it is immediate from the assumed condition (C1) that Condition (C3) is satisfied for $\mathcal{A} = \mathcal{G}_t$ and $\mathcal{B} = \mathcal{G}_t$. Applying Lemma 1 one has

$$D_2(\mathcal{C}_t^{\alpha}|\mathcal{C}_t^{\alpha}) = n D_2(\mathcal{G}_t||\mathcal{G}_t).$$  \hspace{1cm} (23)
Define $\Psi_{\text{wc}}^t$ as the worst case input at $t$, i.e. $\Psi_{\text{wc}}^t := \arg\max_{\Psi} \| (C_{\text{wc}}^t - C_{\text{t}}^n) \otimes I(\Psi) \|_1$. By definition, $D_2$ for $\Psi_{\text{wc}}$ as an input to the channels is upper bounded by $D_2$ for the two channels, namely

$$D_2 \left( C_{\text{wc}}^t \otimes I(\Psi_{\text{wc}}^t) \right) \leq D_2 \left( C_{\text{wc}}^n \| C_{\text{t}}^n \right) . \tag{24}$$

By monotonicity of the $\alpha$-Rényi divergence with respect to $\alpha$, the quantity $D_2$ upper bounds the relative entropy. Further applying Pinsker’s inequality, one has

$$\epsilon_{\text{protocol}} \leq \sqrt{\frac{1}{2\log e} D \left( C_{\text{wc}}^t \otimes I(\Psi_{\text{wc}}^t) \| C_{\text{t}}^n \right) ,}$$

where $\epsilon_{\text{protocol}} := (1/2) \| (C_{\text{wc}}^t - C_{\text{t}}^n) \otimes I(\Psi_{\text{wc}}^t) \|_1$ is the error of the protocol. Substituting Eq. (23) and Eq. (24) into the above inequality, one gets

$$\epsilon_{\text{protocol}} \leq \sqrt{\frac{n}{2\log e} D_2 \left( C_{\text{wc}}^t \| C_{\text{t}}^n \right) .} \tag{25}$$

What remains is to find an expression of $D_2 \left( C_{\text{t}}^n \| C_{\text{t}}^n \right)$ that leads to the desired result. This can be done by Taylor expansion of this quantity, using the definition [19]:

$$D_2 \left( C_{\text{t}}^n \| C_{\text{t}}^n \right) \leq \| t - t^* \|^2 J_{\text{t}}^\text{R} + O (\| t - t^* \|^3) , \tag{26}$$

where $J_{\text{t}}^\text{R}$ is defined in Eq. (4). Derivation of Eq. (26) can be found in Appendix. Finally, by combining Eqs. (6), (25), and (26) one gets that $\epsilon_{\text{protocol}} = O(n^{-\alpha})$.

VII. CONCLUSION

We studied the cost of communicating $n$-use quantum channels. In the direct part, we proposed a protocol for sending the classical description of the channel. In the converse part, we derived a lower bound for the more general task of remote channel simulation, where a client, equipped with a small quantum computer, enables a server to execute a desired quantum channel on a large quantum system. The bound on remote channel simulation yields the desired bound for communicating quantum channels as a corollary. The bound is achieved by our concrete protocol for channels satisfying certain conditions. The bound captures the measurement sensitivity of quantum channels from an information-theoretic point of view and is therefore a step towards the unification of quantum metrology and quantum Shannon theory [48], [51], [46]. Potentially, the bound may have applications in various directions of delegate quantum computation [6], where a server is asked to execute a computation on the state held by a remote client. It can also, for example, be used to determine the bandwidth of a quantum sensor network [52] and to hint on how quantum programs can be conceived. These applications will become more desired as quantum devices are assembled into a network in the near future.

An interesting problem for future research is the compression of multiple-use channels, where the goal is to encode $n$ uses of an unknown quantum channel $C_{\text{t}}^n$ into another quantum channel $B_{\text{t},n}$ acting on a smaller system. This task is very similar to the simulation task considered in this paper, except that the parameter $t$ is now invisible. The counterpart of this task for states is the task of compressing multi-copy states, recently studied both theoretically [53], [54], [55], [56], [46] and experimentally [57]. The task of channel compression is more involved since the input of the channel is not necessary in the many-copy form, and these compression protocols for states cannot be applied directly. Our bound on remote channel simulation (cf. Theorem 2) applies also to compression, since it is harder. However, it remains open whether a concrete protocol achieving the bound exists.

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where $H(X)$ and $H(X|Y)$ denote the differential entropy and the conditional differential entropy, respectively, and the inequality holds since conditioning does not increase the entropy.

Now, what remains is to bound the differential entropy $H(X - \hat{X})$ under the constraint that

$$\Pr \left[ |X - \hat{X}| \leq \delta \right] = p.$$  

Recall that $X = \prod_{i=1}^{n} X_i$ with each $X_i = [x_{i,0}, x_{i,1}]$ being an interval. Then the domain of $X - \hat{X}$ is $X' = \prod_{i=1}^{n} [x_{i,0} - x_{i,1}, x_{i,1} - x_{i,0}]$, and we have $|X'| = 2^n |X|$. Using the Lagrangian multiplier approach, one can show that $H(X - \hat{X})$ is maximized by when $X - \hat{X}$ has the ladder-shape probability density function:

$$p(\hat{X} - X) = \begin{cases} \frac{p}{B_{\delta_p}} & |\hat{X} - X| \leq \delta_p \\ \frac{1-p}{2^n |X| - B_{\delta_p}} & |\hat{X} - X| > \delta_p, \end{cases}$$

where $B_{\delta_p} = \frac{(\sqrt{\pi} \delta_p)^n}{1/(e^2 + 1)}$ denotes the volume of a $v$-ball with radius $\delta_p$. Therefore, $H(X - \hat{X})$ can be bounded as

$$H(X - \hat{X}) \leq -p \log \left( \frac{B_{\delta_p}}{p} \right) + (1 - p) \log \left( \frac{2^n |X| - B_{\delta_p}}{1 - p} \right).$$

Combining Eq. (27) and Eq. (28) we get that Lemma 1.

B. Proof of Eq. (26)

First, it is straightforward that Condition (C1) for $\text{Choi}(G_t)$ implies that Condition (C3) holds for $\mathcal{A} = \mathcal{G}_t$ and $\mathcal{B} = \mathcal{G}_t$. Then, Lemma 2 implies

$$2^{D_2(G_t||G_r)} - 1 = \| \text{Tr}_{\mathcal{K}_{\text{out}}} \left( \text{Choi}(G_t) - \text{Choi}(G_r) \right) \text{Choi}^{-1}_t \left( \text{Choi}(G_t) - \text{Choi}(G_r) \right) \|.$$

By Taylor expansion of $\text{Choi}(G_t)$, for a normalized vector $s = (s_i)_{i=1,\ldots,v}$ and a small number $\varepsilon$, one obtains

$$2^{D_2(G_t+s||G_t)} - 1 = \| \text{Tr}_{\mathcal{K}_{\text{out}}} \left( \sum_i \frac{\partial \text{Choi}(G_t)}{\partial t_i} s_i + O(\varepsilon^2) \right) \text{Choi}^{-1}_t \left( \sum_i \frac{\partial \text{Choi}(G_t)}{\partial t_i} s_i + O(\varepsilon^2) \right) \|$$

$$= \varepsilon^2 \left\| \sum_{i,j} s_is_j \text{Tr}_{\mathcal{K}_{\text{out}}} \left( \frac{\partial \text{Choi}(G_t)}{\partial t_i} \text{Choi}^{-1}_t \frac{\partial \text{Choi}(G_t)}{\partial t_j} \right) \right\| + O(\varepsilon^3).$$

Notice that the value

$$\left(2^{D_2(G_t+s||G_t)} - 1 - \varepsilon^2 \left\| \sum_{i,j} s_is_j \text{Tr}_{\mathcal{K}_{\text{out}}} \left( \frac{\partial \text{Choi}(G_t)}{\partial t_i} \text{Choi}^{-1}_t \frac{\partial \text{Choi}(G_t)}{\partial t_j} \right) \right\| \right) / \varepsilon^3$$

can be uniformly bounded with respect to $s$.

Consider the set of quantum channels $\{\mathcal{G}_{t+s}\}_{s}$ parameterized by a single parameter $x$. The first term on the right hand side of Eq. (29) can be expressed as

$$\left\| \sum_{i,j} s_is_j \text{Tr}_{\mathcal{K}_{\text{out}}} \left( \frac{\partial \text{Choi}(G_t)}{\partial t_i} \text{Choi}^{-1}_t \frac{\partial \text{Choi}(G_t)}{\partial t_j} \right) \right\|_{(a)} = \left\| \max_{\Psi \in \mathcal{S}(\mathcal{K}_0 \otimes \mathcal{K})} \text{Tr}^R \left( \mathcal{G}_{t+s} \otimes \mathcal{I}_{\mathcal{K}}(\Psi) \right) \right\|$$

$$= \left\| \max_{\Psi \in \mathcal{S}(\mathcal{K}_0 \otimes \mathcal{K})} \sum_{i,j} s_is_j \text{Tr}^R \left( \mathcal{G}_{t+s} \otimes \mathcal{I}_{\mathcal{K}}(\Psi) \right) \right\|_{(b)}$$
where (a) follows from the application of [36] Theorem 1] to the set of quantum channels \( \{ \mathcal{C}_{t,s} \} \) with a single parameter \( x \), and \( J^R(\mathcal{C} \otimes I_R(\Psi)) \) on the right hand side of (b) denotes the RLD Fisher information matrix of the state family \( \{ \mathcal{C} \otimes I_R(\Psi) \}_t \). Next, replacing the fixed \( s \) by maximization over all unit vectors \( s' \), one gets

\[
\left\| \sum_{i,j} s_i s_j \text{Tr}_{\mathcal{H}_{\otimes}} \left( \frac{\partial}{\partial t_i} \text{Choi}(\mathcal{C}) \text{Choi}^{-1}(\mathcal{C}) \frac{\partial}{\partial t_j} \text{Choi}(\mathcal{C}) \right) \right\| \leq \max_{s':||s'||=1, \Psi \in \mathcal{S}(\mathcal{H}_{\otimes} \otimes \mathbb{R})} \sum_{i,j} s'_i s'_j J^R_{i,j}(\mathcal{C} \otimes I_R(\Psi)) + O(\varepsilon^3)
\]

\[
= \max_{s':||s'||=1, \Psi \in \mathcal{S}(\mathcal{H}_{\otimes} \otimes \mathbb{R})} \sum_{i,j} s'_i s'_j J^R_{i,j}(\mathcal{C} \otimes I_R(\Psi))
\]

\[
= \max_{s':||s'||=1} \left\| J^R(\mathcal{C} \otimes I_R(\Psi)) \right\| = J^R_{\mathcal{C}}.
\]  

(32)

Substituting it into Eq. (29), one gets

\[
2^{D^2(\mathcal{G}_{t+\varepsilon}||\mathcal{G})} - 1 \leq \varepsilon^2 J^R_{\mathcal{C}} + O(\varepsilon^3).
\]

(33)

Finally, substituting \( s = (t_r - t)/||t_r - t|| \) and \( \varepsilon = ||t_r - t|| \) into Eq. (33) and the uniform evaluation of (30), one gets \( 2^{D^2(\mathcal{G},||\mathcal{G})} - 1 \leq ||t_r - t||^2 J^R_{\mathcal{C}} + O(||t_r - t||^3) \), which implies Eq. (26).

In fact, one might make Taylor expansion of \( 2^{D^2(\mathcal{G}_{t+\varepsilon}||\mathcal{G} \otimes I_R(\Psi)) - 1} \) in the same way as (29) instead of \( 2^{D^2(\mathcal{G}_{t+\varepsilon}||\mathcal{G})} - 1 \). However, it is not so trivial to show that the error term like (30) can be uniformly bounded with respect to \( \Psi \) because the map \( \Psi \mapsto (\mathcal{G} \otimes I_R(\Psi))^{-1} \) is not necessarily bounded when \( \Psi \) is close to a pure state. Hence, we employ Taylor expansion of \( 2^{D^2(\mathcal{G}_{t+\varepsilon},||\mathcal{G})} - 1 \) with Choi matrix forms.