JET SCHEMES, LOG DISCREPANCIES AND INVERSION OF ADJUNCTION

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INTRODUCTION

Singularities play a key role in the Minimal Model Program. In this paper we show how some of the open problems in this area can be approached using jet schemes.

Let \((X,Y)\) be a pair, where \(X\) is a \(\mathbb{Q}\)-Gorenstein normal variety, and \(Y\) stands for a formal combination \(\sum_{i=1}^{k} q_i \cdot Y_i\), where \(q_i \in \mathbb{R}_+\) and \(Y_i \subset X\) are proper closed subschemes. Fix a closed subset \(\emptyset \neq W \subseteq X\). Using a suitable resolution of singularities for the pair \((X,Y)\) one can define numerical invariants \(\text{mld}(W;X,Y)\), called minimal log discrepancies. These invariants in turn can be used to define the classes of singularities which appear in Mori Theory.

We provide a way to compute minimal log discrepancies using arcs and jets. The \(m\)th jet scheme \(X_m\) of \(X\) is given set-theoretically as \(\text{Hom}(\text{Spec} \mathbb{C}[t]/(t^{m+1}), X)\). The limit of these schemes is the space of arcs \(X_\infty = \text{Hom}(\text{Spec} \mathbb{C}[[t]], X)\). This is an infinite dimensional space, but we may associate to \((X,Y)\) a family of subsets of \(X_\infty\) of finite codimension. Given \(W\), if we restrict these subsets over \(W\), then from their codimensions we can compute \(\text{mld}(W;X,Y)\). We stress that this characterization holds in complete generality. It extends the results in [Mu1], [Ya], and [ELM], where criteria were given for having non-negative log discrepancy, under certain hypotheses on the singularities of \(X\). The main ingredient in the proof of this characterization is the theory of motivic integration on singular varieties, developed by Denef and Loeser in [DL1].

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Note that our setting is slightly different than the standard one in Mori Theory. The usual setting is that of a pair $(X,D)$, where $X$ is a normal variety, and $D$ is a $\mathbb{Q}$-divisor such that $K_X + D$ is $\mathbb{Q}$-Cartier. We mention that our characterization of minimal log discrepancies has an analogue in this context (see Remark 2.8). However, for the approach via spaces of arcs, our setting seems more suggestive.

We leave the precise statement of our characterization for the main body of the paper (see Theorem 2.6) and describe the consequences. Our first application is a precise version of the Inversion of Adjunction Conjecture of Kollár and Shokurov, in the case when the ambient variety is smooth.

**Theorem 0.1.** Let $X$ be smooth, $Y = \sum_i q_i Y_i$ as above, and $D \subset X$ a normal effective divisor such that $D \not\subseteq \bigcup_i Y_i$. For every proper closed subset $W \subset D$, we have

$$\operatorname{mld}(W; X, D + Y) = \operatorname{mld}(W; D, Y|_D).$$

Kollár, Shokurov and Stevens proved special cases of Inversion of Adjunction: see [Kol], [Sh1] and [St]. The traditional approach to this problem involves applications of vanishing theorems. We refer to [Kol] for this part of the story. We mention also the result of Ambro [Am1] who proved Inversion of Adjunction in the case when $X = \mathbb{A}^n$ and $D$ is a hypersurface which is general with respect to its Newton polyhedron.

Together with our characterization of minimal log discrepancies, Theorem 0.1 can be used to characterize terminal hypersurface singularities. Recall that if $V$ is a locally complete intersection variety, it is proved in [Mu2] that $V$ has canonical singularities if and only if $V_m$ is irreducible for all $m$. At least in the case when $V$ is a hypersurface in a smooth variety, this follows also from the above results. Moreover, we get a similar characterization for the terminal case, which was suggested by Mirel Caibăr.

**Theorem 0.2.** Let $X$ be smooth and $D \subset X$ an irreducible and reduced divisor. $D$ has terminal singularities if and only if $D_m$ is normal for every $m$.

Our final application is towards a semicontinuity statement. Shokurov has given in [Sh2] a conjectural uniform bound for minimal log discrepancies. Ambro has made a stronger conjecture in [Am2] and he showed that this conjecture is equivalent to a semicontinuity statement about log discrepancies. We prove this conjecture in the case of an ambient smooth variety.

**Theorem 0.3.** Let $X$ be a smooth variety, and $Y = \sum_i q_i Y_i$ as above. The function $x \in X \mapsto \operatorname{mld}(x; X, Y)$ is lower semicontinuous.
A few words about the structure of the paper: in the first section we review the basic definitions and properties of minimal log discrepancies, while in the second section we prove our characterization of these invariants. In the next section we study the jet schemes of a hypersurface in a smooth variety, and as a result, we prove Theorems 0.1 and 0.2. In the last section we prove the above semicontinuity statement.

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1. Log discrepancies

All our varieties are defined over $\mathbb{C}$. In this section we review the definition and the basic properties of log discrepancies. For a detailed discussion and proofs we refer to [Am2]. Note that unlike in [Am2], in this paper we allow pairs of arbitrary codimension, but all the proofs can be reduced to the case of divisors.

We will always work in the following setting. Let $X$ be a normal, $\mathbb{Q}$-Gorenstein variety, and $Y$ a formal combination $Y = \sum_{i=1}^{k} q_i \cdot Y_i$, where $q_i \in \mathbb{R}$, and where $Y_i \subset X$ are proper closed subschemes. We will restrict later to the case when $q_i \geq 0$ for all $i$. A divisor $E$ over $X$ is a prime Weil divisor on $X'$, for some normal variety $X'$, proper and birational over $X$. We identify $E$ with the corresponding valuation of the function field of $X$. The center of this valuation on $X$ is denoted by $c_X(E)$.

Given a divisor $E$ over $X$, choose a proper, birational morphism $\pi : X' \to X$, with $X'$ normal and $\mathbb{Q}$-Gorenstein, such that $E$ is a Cartier divisor on $X'$, and such that all the scheme-theoretic inverse images $\pi^{-1}(Y_i)$ are Cartier divisors. We write $\pi^{-1}(Y) := \sum_i q_i \cdot \pi^{-1}(Y_i)$. The coefficient of $E$ in $K_{X'/X} - \pi^{-1}(Y)$ is $a(E; X, Y) - 1$. It is clear that $a(E; X, Y)$ does not depend on the particular model $X'$ we have chosen.

**Definition 1.1.** Let $W \subseteq X$ be a nonempty closed subset. The minimal log discrepancy of $(X, Y)$ on $W$ is defined by

$$\text{mld}(W; X, Y) := \inf_{c_X(E) \subseteq W} \{ a(E; X, Y) \}.$$
Definition 1.2. The pair \((X, Y)\) is called log canonical if we have \(\text{mld}(X; X, Y) \geq 0\).

We collect in the next proposition a few well-known facts about minimal log discrepancies.

Proposition 1.3. Let \((X, Y)\) and \(W \subseteq X\) be as above.

(i) If \(W_1, \ldots, W_r\) are the irreducible components of \(W\), then \(\text{mld}(W; X, Y) = \min_i \text{mld}(W_i; X, Y)\).

(ii) If \(U \subset X\) is an open subset such that both \(W \cap U\) and \(W \setminus U\) are nonempty, then

\[
\text{mld}(W; X, Y) = \min \{ \text{mld}(W \setminus U; X, Y), \text{mld}(W \cap U; U, Y|_U) \}.
\]

(iii) \(\text{mld}(W; X, Y) \geq 0\) if and only if there is an open subset \(U\) of \(X\), such that \(W \subseteq U\) and \((U, Y|_U)\) is log canonical. If \(\dim X \geq 2\), and if \(\text{mld}(W; X, Y) < 0\), then \(\text{mld}(W; X, Y) = -\infty\).

(iv) If \(\pi : X' \rightarrow X\) is a proper, birational morphism, with \(X'\) normal and \(\mathbb{Q}\)-Gorenstein, then

\[
\text{mld}(W; X, Y) = \text{mld}(\pi^{-1}(W); X', \pi^{-1}(Y) - K_{X'/X}),
\]

where \(\pi^{-1}(Y) = \sum_i q_i \cdot \pi^{-1}(Y_i)\).

The next proposition shows that minimal log discrepancies can be computed using log resolutions. Given \((X, Y)\) and \(W \subseteq X\), consider \(\pi : X' \rightarrow X\) proper, birational, with \(X'\) smooth, such that \(\pi^{-1}(Y) \cup \text{Ex}(\pi)\) is a divisor with simple normal crossings. Here \(\text{Ex}(\pi)\) denotes the exceptional locus of \(\pi\). In addition, if \(W \neq X\), we assume that \(\pi^{-1}(W) \cup \pi^{-1}(Y) \cup \text{Ex}(\pi)\) is also a divisor with simple normal crossings. Note that by [H], we can always find such a morphism. Let us write \(K_{X'/X} - \pi^{-1}(Y) = \sum_i (a_i - 1)E_i\).

Proposition 1.4. The pair \((X, Y)\) is log canonical if and only if \(a_i \geq 0\) for every \(i\). If \((X, Y)\) is log canonical on an open subset containing \(W\), then

\[
\text{mld}(W; X, Y) = \min_{\pi(E_i) \subseteq W} \{a_i\}.
\]

The following conjecture is a precise form of the Inversion of Adjunction Conjecture, due to Kollár and Shokurov.

Conjecture 1.5. Consider a pair \((X, Y)\) as above and a normal effective Cartier divisor \(D\) on \(X\), such that \(D \not\subseteq \bigcup_i Y_i\). For every nonempty, proper closed subset \(W \subseteq D\), we have

\[
\text{mld}(W; X, D + Y) = \text{mld}(W; D, Y|_D),
\]

where \(Y|_D = \sum_i q_i \cdot (Y_i \cap D)\).
We refer to [Kol] and [K+] for motivation and for a discussion of known results. Let us mention that there is a more general conjecture regarding Inversion of Adjunction for log canonical centers (see, for example, [Am3]). We will prove the following result in Section 3, as an application of our description of minimal log discrepancies in terms of jet schemes.

**Theorem 1.6.** The above conjecture is true if \( X \) is smooth and \( Y = \sum_i q_i \cdot Y_i \), where \( q_i \geq 0 \) for all \( i \).

**Remark 1.7.** The statement in Theorem 1.6 has been proved in [Am1] when \( X = \mathbb{A}^n \), \( D \) is a non-degenerate hypersurface, and \( W \) is the origin.

**Corollary 1.8.** Let \((X,Y)\) be a pair as in Theorem 1.6, with \( X \) smooth. If \( D \subset X \) is a normal divisor, such that \( D \not\subseteq \bigcup Y_i \), then \((X,D+Y)\) is log canonical around \( D \) if and only if \((D,Y|_D)\) is log canonical.

**Proof.** Apply Theorem 1.6 with \( W = D_{\text{sing}} \cup \bigcup_i (Y_i \cap D) \). Since we have
\[
\text{mld}(D; D,Y|_D) = \min\{\text{mld}(W; D,Y|_D), 1\},
\]
\[
\text{mld}(D; X, D+Y) = \min\{\text{mld}(W; X, D+Y), 0\},
\]
we are done since \((X,D+Y)\) is log canonical around \( D \) if and only if \( \text{mld}(D; X, D+Y) \geq 0 \).

We will consider also the following version of minimal log discrepancy.

**Definition 1.9.** With the notation in Definition 1.1, if \( W \subset X \) is a proper irreducible closed subset with generic point \( \eta_W \), then the minimal log discrepancy of \((X,Y)\) at \( \eta_W \) is
\[
\text{mld}(\eta_W; X,Y) := \inf_{E \in X} a(E; X,Y).
\]
If \( W = X \), then we put \( \text{mld}(\eta_W; X,Y) = 0 \).

We collect in the following proposition the basic properties of this invariant.

**Proposition 1.10.** Let \((X,Y)\) be a pair as above, and let \( W \subset X \) be a proper irreducible closed subset.

(i) \( \text{mld}(\eta_W; X,Y) \geq \text{mld}(W; X,Y) \).

(ii) If \( U \subset X \) is open, and \( U \cap W \neq \emptyset \), then \( \text{mld}(\eta_W; X,Y) = \text{mld}(\eta_W; U,Y|_U) \).

(iii) \( \text{mld}(\eta_W; X,Y) \geq 0 \) if and only if there is an open subset \( U \subset X \) with \( U \cap W \neq \emptyset \), and such that \((U,Y|_U)\) is log canonical. If \( \text{codim}(W,X) \geq 2 \), then \( \text{mld}(\eta_W; X,Y) \geq 0 \) if and only if \( \text{mld}(\eta_W; X,Y) \neq -\infty \).
(iv) If \((X,Y)\) is log canonical, and if \(\pi : X' \rightarrow X\) is a resolution as in Proposition 1.4, then
\[
\text{mld}(\eta; X,Y) = \inf_{\pi(E_i) = W} \{a_i\}.
\]

(v) There is an open subset \(U \subseteq X\) such that \(U \cap W \neq \emptyset\), and
\[
\text{mld}(\eta_W; X,Y) = \text{mld}(U \cap W; U, Y|_U).
\]

2. LOG DISCREPANCIES AND JET SCHEMES

For the basic definitions and properties of jet schemes, we refer to [DL1] (see also [Mu1] or [Mu2]). In particular, we will use freely the construction of motivic integrals from [DL1]. Recall our context: \(X\) is a normal, \(d\)-dimensional \(\mathbb{Q}\)-Gorenstein variety. Fix \(r \in \mathbb{N}^*\) such that \(rK_X\) is Cartier.

We denote the \(m\)th jet scheme of \(X\) by \(X_m\), and the space of arcs by \(X_{\infty}\). We have canonical morphisms \(\psi_m : X_{\infty} \rightarrow X_m\) and \(\phi_m : X_m \rightarrow X\). When the variety we consider is not obvious, we will write \(\psi_{X_m}\) and \(\phi_{X_m}\). For every \(m, j \in \mathbb{N}\), we put \(X_{m,j} = \text{Im}(X_m + j \rightarrow X_m)\) and similarly \(X_{m,\infty} = \text{Im}(\psi_m)\). It is a theorem of Greenberg from [Gr] that if \(j \gg 0\), then \(X_{m,\infty} = X_{m,j}\). In particular, \(X_{m,\infty}\) is constructible.

On each jet scheme \(X_m\) there is an \(\mathbb{A}^1\)-action \(\bullet\) such that the natural projections are compatible with these actions. Here \(\mathbb{A}^1\) is considered as a monoidal scheme under usual multiplication. Moreover, there are “zero-sections” \(\sigma_m : X \rightarrow X_m\) such that for every \(\gamma \in X_m\), we have \(0 \bullet \gamma = \sigma_m(\phi_m(\gamma))\). Note that if \(T \subseteq X_m\) is invariant under the \(\mathbb{A}^1\)-action, then \(\phi_m(T) = \sigma_m(T)\), hence it is closed in \(X\). For more details we refer to [Mu2].

We introduce now two subschemes of \(X\) which measure its singularities. Let \(i : U = X_{\text{reg}} \rightarrow X\) be the open immersion corresponding to the smooth part of \(X\). We have a canonical morphism
\[
(\Omega^d_X)^{\otimes r} \rightarrow i_*(\Omega^d_U)^{\otimes r} = \mathcal{O}_X(rK_X).
\]
This defines a closed subscheme \(Z \subset X\) of ideal \(\mathcal{I}_Z\), such that the image of the above morphism is \(\mathcal{I}_Z \otimes \mathcal{O}_X(rK_X)\).

We consider also the Jacobian subscheme \(Z'\) defined by the Jacobian ideal \(\mathcal{I}_{Z'} := \text{Fitt}_d(\Omega^1_X)\) (the \(d\)th Fitting ideal of \(\Omega^1_X\)). Working locally, we may assume that \(X \subseteq \mathbb{A}^N\) is defined by \((f_i)_i\). Then \(\mathcal{I}_{Z'}\) is generated by the restrictions to \(X\) of the \((N-d)\) minors of the Jacobian matrix \((\partial f_i/\partial X_j)_{i,j}\).

It is clear that we have \(\text{Supp}(Z) \subseteq X_{\text{sing}} = \text{Supp}(Z')\). Moreover, if \(X\) is locally complete intersection, then we may take \(r = 1\) and in this case \(\mathcal{I}_Z = \mathcal{I}_{Z'}\).
Recall that to every closed subscheme $T \hookrightarrow X$ we have an associated function $F_T : X_\infty \longrightarrow \mathbb{N} \cup \{\infty\}$ which measures the order of vanishing of an arc along $T$. For every $e \in \mathbb{N}$, let $X^{(e)}_\infty = F_{Z'}^{-1}(e) \subseteq X_\infty$. Similarly, if $m \geq e$, we denote by $X^{(e)}_m$ the set of jets in $X_m$ vanishing along $I_{Z'}$ with order exactly $e$. We put also $X^{(e)}_{m,j} = X^{(e)}_m \cap X_{m,j}$, and similarly for $X^{(e)}_{m,\infty}$.

It follows from Lemma 4.1 in [DL1] and its proof (see also [Lo]) that if $m \geq e$, then $X^{(e)}_m \cap X^{(e)}_{m,\infty}$ is locally trivial with fiber $A^d$. Moreover, $X^{(e)}_{m,\infty}$ is a locally closed subset of $X^{(e)}_m$.

Consider for all closed subschemes $T \subseteq X$ the corresponding subsets $\psi^{-1}_m(T_m)$, and let $A$ be an element in the algebra generated by all these subsets. We define $\text{codim}(A)$ as follows. Suppose first that $A \subseteq X^{(e)}_\infty$ for some $e$, and let us write $A = \psi^{-1}_m(B)$, where we may take $m \geq e$. We put $\text{codim}(A) := (m + 1)d - \dim(B \cap X^{(e)}_{m,\infty})$, and since $\dim(\psi_m(X_\infty)) = (m+1)d$ (see Lemma 4.3 in [DL1]), it follows that $\text{codim}(A)$ is a nonnegative integer. Moreover, the result mentioned in the above paragraph shows that the definition does not depend on which $m$ we have chosen. In general, we put $\text{codim}(A) := \min_{e \in \mathbb{N}} \text{codim}(A \cap X^{(e)}_\infty)$, with the convention that $\text{codim}(\emptyset) = \infty$. Note that $\text{codim}(A) = \infty$ if and only if $A \subseteq Z'_\infty$.

Let $\mu(A)$ be the Hodge realization of the motivic measure of $A$ (see [DL1]). $\mu(A)$ is a Laurent power series in two variables $u^{-1}$ and $v^{-1}$. If $\mu(A) \neq 0$, then there is only one monomial in $\mu(A)$ of maximal degree, namely $c(uv)^{-\text{codim}(A)}$, where $c$ is a positive integer. Moreover, $\mu(A) = 0$ if and only if $\text{codim}(A) = \infty$.

**Remark 2.1.** Suppose that $A$ is a finite intersection of sets of the form $F_{T_i}^{-1}(\geq m_i)$. One can show that in this case $\text{codim}(A) < \infty$ and, in fact, $\text{codim}(A) = \text{codim}(\psi_m(A), \psi_m(X_\infty))$, if $m \gg 0$. If $X$ is smooth, then one can also check that $\text{codim}(A)$ is the codimension of $A$ as a closed subset of $X_\infty$, but we do not know if this remains true in general.

We give now the version of the Change of Variable formula we will need (this is essentially the same version that was used in [Ya]). Let $X$ be as above, $Z$ the subscheme we have defined, and $W \subseteq X$ a closed subset. We consider $Y = \sum_{i=1}^k q_i \cdot Y_i$, where $q_i \in \mathbb{R}$, and $Y_i \subset X$ are proper closed subschemes. For $e \in \mathbb{N}$, $m = (m_i)_i \in \mathbb{N}^k$, we put $A = \cap_i F_{Y_i}^{-1}(m_i) \cap F_{Z'}^{-1}(e) \cap \psi_0^{-1}(W)$. 


Theorem 2.2. If $\pi : X' \to X$ is a proper, birational morphism, with $X'$ smooth, then
\[
\int_A (uv)^{(1/r)F_Z} = \int_{\pi_1^{-1}(A)} (uv)^{-F_{KX'/X}}.
\]

Proof. By the Change of Variable formula in [DL1], we have
\[
\int_A (uv)^{(1/r)F_Z} = \int_{\pi_1^{-1}(A)} (uv)^{(1/r)F_Z \circ \pi_\infty - F_{Z''}},
\]
where $Z''$ is the sheaf of ideals such that $\pi^* \Omega^1_X \to \Omega^1_{Z''} \otimes \Omega^1_X$, is an epimorphism. It follows from definition that
\[
\pi^{-1}(\mathcal{I}_Z) \cdot \mathcal{O}(-rK_{X'/X}) = \mathcal{I}_{Z''}.
\]
Since $F_Z \circ \pi_\infty = F_{\pi^{-1}(Z)}$, we deduce the formula in the statement.

The following is the main technical ingredient, which will allow us to connect log discrepancies and jet schemes. We first fix the notation. Let $(X, Y)$ and $Z \subset X$ be as above. We consider also a closed nonempty subset $W \subset X$. Fix a proper, birational morphism $\pi : X' \to X$, such that $X'$ is smooth, and such that $\pi^{-1}(Y) \cup \pi^{-1}(Z) \cup \text{Ex}(\pi)$ is a divisor with simple normal crossings. We first fix the notation. Let
\[
I = \sum_{j=1}^s q_j \pi^{-1}(Y_i).
\]
If $W \neq X$, then we put also the condition that $\pi^{-1}(W)$, together with the above union, is a divisor with simple normal crossings. We write $\pi^{-1}(Y_i) = \sum_{j=1} y_{i,j} D_j$, $\pi^{-1}(Z) = \sum_{j=1} z_j D_j$, and $K_{X'/X} = \sum_{j=1} k_j D_j$.

Theorem 2.3. With the above notation, for every $e \in \mathbb{N}$, $m \in \mathbb{N}^s$, we have
\[
\text{codim} \left( \bigcap_i F_{Y_i}^{-1}(m_i) \cap F_{Z}^{-1}(e) \cap \psi_0^{-1}(W) \right) = \frac{e}{r} + \min_{\nu \neq 0} \sum_{j=1}^s (k_j + 1)\nu_j.
\]
Here the infimum is over all $\nu \in \mathbb{N}^s$ with $\bigcap_{\nu_j \neq 0} D_j \neq \emptyset$, and such that $\sum_{j=1}^s \nu_j y_{i,j} = m_i$ for all $i$, and $\sum_{j=1}^s \nu_j z_j = e$. If $W \neq X$, then we have to add also the condition that there is $\nu_j \neq 0$ such that $\pi(D_j) \subset W$. By convention, the minimum over an empty set is $\infty$.

Proof. Let $A = \bigcap_i F_{Y_i}^{-1}(m_i) \cap F_{Z}^{-1}(e) \cap \psi_0^{-1}(W)$. By definition,
\[
\int_A (uv)^{(1/r)F_Z} = \mu(A)(uv)^{e/r}.
\]
We treat only the case $\mu(A) \neq 0$, the changes for the other case being obvious. Therefore the integral is given by a Laurent power series in $u^{-1}$ and $v^{-1}$ (with rational exponents) of degree $2((e/r) - \text{codim}(A))$. 
On the other hand, a direct computation shows that
\[
\int_{\pi_\infty^{-1}(A)} (uv)^{-F_{K'}} = \sum_\nu E(D_\nu)(uv - 1)^{|\nu|} (uv)^{-d - \sum_j (k_j + 1)\nu_j},
\]
where the sum is over all \( \nu \in \mathbb{N}^s \) such that \( \sum_i \nu_j y_{i,j} = m_i \), for all \( i \), and \( \sum_j \nu_j z_j = e \). We have put \( |\nu| = \text{Card}\{j | \nu_j > 0\} \). If \( W \neq X \), then we have to add also the condition that there is at least one \( \nu_j > 0 \) such that \( \pi(D_j) \subseteq W \). We have used the notation \( D_\nu = \bigcap_{\nu_j \neq 0} D_j \setminus \bigcup_{\nu_j = 0} D_j \).

The proof of this formula follows along the same lines as the proof of Theorem 2.15 in [Cr].

We deduce that the degree of the integral over \( \pi_\infty^{-1}(A) \) is equal to
\[-2\min_\nu \sum_j (k_j + 1) \nu_j, \]
where \( \nu \) runs over the set in the statement of the theorem. By Theorem 2.2, the two integrals are equal, and comparing their degrees we get the formula for \( \text{codim}(A) \).

**Remark 2.4.** When \( X \) is smooth, \( W = X \) and \( Y \) is a hypersurface, the formula in Theorem 2.3 follows also from the computation of motivic Igusa zeta function in [DL2]. Under the same assumption on \( X \) and \( W \), but for \( Y = q_1 \cdot Y_1 - q_2 \cdot Y_2 \), this is contained in [ELNM].

**Corollary 2.5.** With the notation in Theorem 2.3, we have
\[
\text{codim}\left( \bigcap_i F_{Y_i}^{-1}(\geq m_i) \cap F_{Z}^{-1}(\geq e) \cap \psi_0^{-1}(W) \right) = \min_\nu \sum_j \left( \frac{z_j}{r} + k_j + 1 \right) \nu_j.
\]
Here the infimum is over all \( \nu \in \mathbb{N}^s \) with \( \bigcap_{\nu_j \neq 0} D_j \neq \emptyset \), and such that \( \sum_j \nu_j y_{i,j} \geq m_i \) for all \( i \), and \( \sum_j \nu_j z_j \geq e \). If \( W \neq X \), then we have to add also the condition that there is \( \nu_j \neq 0 \) such that \( \pi(D_j) \subseteq W \).

**Proof.** If \( m \in \mathbb{N}^k \) and \( e \in \mathbb{N} \), then we put \( A_{m,e} \) for the set in Theorem 2.3 and \( A'_{m,e} \) for the set in the above statement. We put \( m' \geq m \) if \( m'_i \geq m_i \) for all \( i \). We have
\[
\bigcup_{m' \geq m, e' \geq e} A_{m', e'} \subseteq A'_{m, e},
\]
and the complement lies inside \( \bigcup_i (Y_i)_\infty \cup Z_\infty \). This gives
\[
\text{codim}(A'_{m, e}) = \min_{m', e'} \text{codim}(A_{m', e'}),
\]
and we conclude by Theorem 2.3.

We give now our characterization of minimal log discrepancies in terms of spaces of arcs. Using Proposition 1.3(ii), it is easy to see that for the computation of \( \text{mld}(W; X, Y) \) we may assume that \( W \neq X \).
Theorem 2.6. Let \((X,Y)\) be a pair as above, with \(Y = \sum_{i=1}^{k} q_i \cdot Y_i\). If \(W \subset X\) is a proper closed subset, and if \(\tau \in \mathbb{R}_+\), then the following are equivalent: 

(i) \(\text{mld}(W; X,Y) \geq \tau\). 

(ii) For every \(e \in \mathbb{N}\), \(m \in \mathbb{N}^k\), we have 

\[
(2) \quad \text{codim} \left( \bigcap_{i=1}^{k} F^{-1}_{Y_i}(m_i) \cap F^{-1}_Z(e) \cap \psi_0^{-1}(W) \right) \geq \frac{e}{r} + \sum_{i=1}^{k} q_i m_i + \tau.
\]

If \(q_i \geq 0\) for all \(i\), then the above conditions are also equivalent with 

(iii) For every \(e \in \mathbb{N}\), \(m \in \mathbb{N}^k\), we have 

\[
(3) \quad \text{codim} \left( \bigcap_{i=1}^{k} F^{-1}_{Y_i}(\geq m_i) \cap F^{-1}_Z(\geq e) \cap \psi_0^{-1}(W) \right) \geq \frac{e}{r} + \sum_{i=1}^{k} q_i m_i + \tau.
\]

Moreover, if \(\pi\) is a resolution as in Theorem 2.3, then in (ii) and (iii) above it is enough to put the conditions in (2) and (3), respectively, only for finitely many \(e\) and \(m\), depending on the numerical data of the resolution.

Proof. Let \(\pi\) be a resolution as in Theorem 2.3. We keep the notation in that theorem. By restricting to a suitable open neighbourhood of \(W\), we may assume that \(\pi(D_j) \cap W \neq \emptyset\), for all \(j\). In this case Proposition 1.4 shows that \(\text{mld}(W; X,Y) \geq \tau\) if and only if \(k_j + 1 - \sum_i q_i y_{i,j} \geq 0\) for all \(j\), and 

\[
k_j + 1 - \sum_i q_i y_{i,j} \geq \tau \quad \text{for all} \quad j \quad \text{such that} \quad \pi(D_j) \subset W.
\]

Suppose first that \(\text{mld}(W; X,Y) \geq \tau\). If \(\nu \in \mathbb{N}^s\) is such that \(\sum_j \nu_j y_{i,j} = m_i\), and such that \(\nu_l \geq 1\) for some \(l\) with \(\pi(D_l) \subset W\), then \(\sum_j (k_j + 1) \nu_j \geq \sum_i q_i m_i + \tau\), and we deduce (2) from the formula in Theorem 2.3. This proves (i) \(\Rightarrow\) (ii), and (i) \(\Rightarrow\) (iii) follows similarly, using Corollary 2.5.

We show now (ii) \(\Rightarrow\) (i). We take first \(j\) such that \(\pi(D_j) \subset W\). If \(k_j + 1 < \sum_i q_i y_{i,j} + \tau\), take \(m \in \mathbb{N}^k\) given by \(m_i = y_{i,j}\) for all \(i\), and let \(e = z_j\). By taking \(\nu_j = 1\), and \(\nu_{j'} = 0\) for \(j' \neq j\), the formula in Theorem 2.3 gives 

\[
\text{codim} \left( \bigcap_{i=1}^{k} F^{-1}_{Y_i}(m_i) \cap F^{-1}_Z(e) \cap \psi_0^{-1}(W) \right) < \frac{e}{r} + \sum_{i=1}^{k} q_i m_i + \tau,
\]

a contradiction with (ii).

We take now \(j\) such that \(\pi(D_j) \not\subset W\). Since \(\pi(D_j) \cap W \neq \emptyset\), there is \(j'\) such that \(\pi(D_{j'}) \subset W\), and \(D_j \cap D_{j'} \neq \emptyset\). For every \(\alpha \in \mathbb{N}\), we take \(\nu \in \mathbb{N}^s\) such that \(\nu_j = \alpha\), \(\nu_{j'} = 1\), and \(\nu_{j''} = 0\) if \(j'' \neq j, j'\). If \(m \in \mathbb{N}^k\) is such that \(m_i = y_{i,j} \alpha + y_{i,j'}\), and if \(e = z_j \alpha + z_{j'}\), then it follows from
and the formula in Theorem 2.3 that

\[(k_j + 1)\alpha + (k_{j'} + 1) \geq \sum_{i=1}^{k} q_i (y_{i,j}\alpha + y_{i,j'}) + \tau.\]

It is clear that there is a value for \(\alpha\), depending on the numerical data of the resolution, such that the above inequality implies \(k_j + 1 \geq \sum_i q_i y_{i,j}\).

We have thus shown that \(mld(W; X, Y) \geq \tau\).

Note that (iii) trivially implies (ii), as the codimension in (2) is always greater or equal to the codimension in (3). As the last assertion in the theorem follows from the above arguments, we are done.

**Remark 2.7.** We can deduce from the above theorem a condition for \((X, Y)\) to be log canonical, in terms of arcs. Namely, \((X, Y)\) is log canonical if and only if for every \(e \in \mathbb{N}\) and every \(m \in \mathbb{N}^k\), we have

\[\codim \left( \bigcap_{i=1}^{k} F_{Y_i}^{-1}(e) \cap F_{Z}^{-1}(e) \right) \geq \frac{e}{r} + \sum_{i=1}^{k} q_i m_i.\]

Moreover, if \(q_i \geq 0\) for all \(i\), then the above condition is equivalent with

\[\codim \left( \bigcap_{i=1}^{k} F_{Y_i}^{-1}(m_i) \cap F_{Z}^{-1}(e) \right) \geq \frac{e}{r} + \sum_{i=1}^{k} q_i m_i,\]

for every \(e \in \mathbb{N}\) and every \(m \in \mathbb{N}^k\). In order to see this, it is enough to apply the above theorem for \(W = \bigcup_i \text{Supp}(Y_i) \cup X_{\text{sing}}\).

Note that this characterization of log canonical singularities was proved in the case when \(\text{Supp}(Z) \subseteq \bigcup_i \text{Supp}(Y_i)\) in [Ya]. The above proof of Theorem 2.6 is inspired from his proof.

**Remark 2.8.** One can give an analogous description of minimal log discrepancies in the usual setting of Mori Theory. Suppose that \(X\) is a \(d\)-dimensional normal variety, and that \(D\) is a \(\mathbb{Q}\)-divisor on \(X\) such that \(r(K_X + D)\) is Cartier for some positive integer \(r\). For simplicity, we assume that \(D\) is effective, so we have a canonical morphism

\[(\wedge^d \Omega_X)^{\otimes r} \rightarrow (\wedge^d \Omega_X)^{\otimes r} \otimes \mathcal{O}_X(rD) \rightarrow \mathcal{O}_X(r(K_X + D)).\]

We have a closed subscheme \(T \subseteq X\) defined by the ideal \(\mathcal{I}_T\), such that the image of the above composition is \(\mathcal{I}_T \otimes \mathcal{O}_X(r(K_X + D))\). Note that in this case, this scheme depends also on \(D\).

The same arguments as above show, for example, that if \(\tau \in \mathbb{R}_+\), then \(mld(W; X, D) \geq \tau\) if and only if

\[\codim(F_T^{-1}(e) \cap \psi_0^{-1}(W)) \geq \frac{e}{r} + \tau\]
for every \( e \in \mathbb{N} \).

3. Inversion of Adjunction

In the case of a hypersurface, it is easy to understand the set of jets which can be lifted to the arc space. This will be enough to give a proof of Theorem 1.6.

Let us fix the notation for this section. We consider a smooth variety \( X \), with \( \dim X = d \), and a divisor \( D \subset X \) which is irreducible and reduced. Let \( Z \subset D \) be the jacobian subscheme of \( D \) defined by the ideal \( I_Z \). Recall that \( D_{m,e} = \text{Im}(D_{m+e} \to D_m) \), while \( D_{m,\infty} = \text{Im}(D_{\infty} \to D_m) \). In addition, if we restrict to those jets with order \( e \) along \( I_Z \), then we put \((e)\) as a superscript.

Lemma 3.1. Given \( D \) as above, and \( m, e \in \mathbb{N} \), with \( m \geq e \), we have \( D_{m,\infty}^{(e)} = D_{m,e}^{(e)} \). Moreover, if \( \eta : X_{m+e} \to X_m \) is the canonical projection, then \( \eta^{-1}(D_{m,e}^{(e)}) = D_{m+e}^{(e)} \).

Proof. We have to show that if \( u \in D_{m+e}^{(e)} \), then there is \( v \in D_\infty \) such that \( u \) and \( v \) have the same image in \( D_m \). In addition, if \( w \in X_{m+e} \) is such that \( \eta(u) = \eta(w) \), then \( w \in D_{m+e} \).

Let \( u_0 = \phi^D_{m}(u) \). By restricting to an open neighbourhood of \( u_0 \), we may assume that we have a regular system of parameters at \( u_0 \), denoted by \( x_1, \ldots, x_d \). We may also assume that \( D \) is defined by an equation \( f \). Note that the regular system of parameters induces an isomorphism \( \hat{\mathcal{O}}_{X,u_0} \simeq \mathbb{C}[[T_1, \ldots, T_d]] \), and we will identify \( f \) with a power series via this isomorphism.

For every \( p \), we have an isomorphism
\[
(\phi^X_p)^{-1}(u_0) \simeq (t\mathbb{C}[t]/(t^{p+1}))^d,
\]
which maps a morphism \( \gamma \) to \((\gamma_i)_i\), where \( \gamma_i = \gamma(x_i) \). Note that \( \gamma \in D_p \) if and only if \( f(\gamma_1, \ldots, \gamma_d) = 0 \). Similar considerations apply when \( p = \infty \).

In order to finish the proof, it is enough to prove the following assertions. Suppose that \( \gamma \in (t\mathbb{C}[[t]])^n \) is such that \( \text{ord}(f(\gamma)) \geq m + e + 1 \) and \( \text{ord}(\partial f/\partial T_1(\gamma), \ldots, \partial f/\partial T_n(\gamma)) = e \). Then there is \( \delta \in (\mathbb{C}[[t]])^n \), such that \( f(\gamma + t^{m+1} \delta) = 0 \). Moreover, if \( \gamma' \in (\mathbb{C}[[t]])^n \), then \( \text{ord}(f(\gamma + t^{m+1} \gamma')) \geq m + e + 1 \).

Consider the Taylor expansion:
\[
f(\gamma + t^{m+1} \gamma') = f(\gamma) + t^{m+1} \sum_{i=1}^d \frac{\partial f}{\partial T_i}(\gamma) \cdot \gamma_i' + t^{2m+2} \cdot (\ldots).
\]
Since \( \text{ord } f(\gamma) \geq m + e + 1 \), \( \text{ord } \frac{\partial f}{\partial t}(\gamma) \geq e \), and \( m \geq e \), we deduce \( \text{ord } f(\gamma + t^{m+1}\gamma') \geq m + e + 1 \). This gives the second of the above assertions. Moreover, it is easy to see from the above formula that there is \( \delta \) such that \( f(\gamma + t^{m+1}\delta) = 0 \). In fact, the terms of order zero in \( \delta \) are obtained solving a linear equation, while the higher terms can be deduced by a recursive argument. Hence we get the first of the above assertions. Alternatively, this statement can be deduced also from Newton’s Lemma (see [Gr]). \( \square \)

Consider now the following situation. Let \( m \in \mathbb{N} \), and let also \( R \subseteq X_m \) be an irreducible closed subset which is invariant under the \( \mathbb{A}^1 \)-action on \( X_m \). Suppose that \( a \leq m \) is such that \( \text{ord } \gamma(I_D) \geq a \) for every \( \gamma \in R \), where \( I_D \) is the ideal defining \( D \) in \( X \). We assume that \( \phi^X_m(R) \cap D \neq \emptyset \), and we put \( S = R \cap D_{m,\infty} \) and \( S = (\psi^R_m)^{-1}(S) \). Let

\[
eq \min\{\text{ord } \gamma(I_Z)|\gamma \in S\}.
\]

**Lemma 3.2.** With the above notation, we have

1. \( S \) is non-empty and \( e < \infty \).
2. If \( S^\circ := \{\gamma \in S|\text{ord } \gamma(I_Z) = e\} \) (which is open in \( S \)), then

\[
\text{codim}(S^\circ, D_\infty) \leq \text{codim}(R, X_m) + e - a.
\]

**Proof.** Let \( x \in \phi^X_m(R) \cap D \). We denote by \( x_m \) the image of \( x \) by the zero section to \( D_m \). Using the \( \mathbb{A}^1 \)-action on \( D_m \), we see that \( x_m \in S \).

The second assertion in (1) can be proved as follows. Let \( \mu : D' \longrightarrow D \) be a resolution of singularities for \( D \). The set \( f_{-1}^{-1}(S) \) is nonempty as it contains the zero section over any point in \( f^{-1}(x) \). Since it is the inverse image of a closed subset in \( D'_m \), and since \( D' \) is smooth, it can not be contained in \( f^{-1}(Z)_\infty \) (see, for example, Corollary 3.8 in [Mu2]). This shows that \( S \nsubseteq Z_\infty \).

Fix now \( p \geq \max\{m, e\} \). Let \( R := (\psi^X_m)^{-1}(R) \) We denote by \( R_{p+e} \) and \( R_p \) the projections of \( R \) to \( X_{p+e} \) and \( X_p \), respectively. Similarly, let \( S^\circ_p \) be the projection of \( S^\circ \) to \( X_p \). We denote by \( g : R_{p+e} \longrightarrow R_p \) the canonical projection.

Let \( T = R_{p+e} \cap D_{p+e} \). If \( \gamma \in T \) has order \( e' \leq e \) along \( I_Z \), then Lemma 3.1 shows that \( g(\gamma) \) lies over \( S \). Hence \( e' = e \). Therefore the set \( T^\circ := \{\gamma \in T|\text{ord } \gamma(I_Z) = e\} \) is an open subset of \( T \). Again, Lemma 3.1 implies that \( g \) induces a surjective map \( f : T^\circ \longrightarrow S^\circ_p \).

It is clear that all the fibers of \( f \) have dimension at most \( de \). On the other hand, \( T \) is cut out in \( R_{p+e} \) by \( p + e - a + 1 \) equations. If we put \( r = \text{codim}(R, X_m) \), then every irreducible component of \( T \) has dimension at least \( (p + 1)(d - 1) + de - (r + e - a) \). Therefore \( \dim S^\circ_p \geq (p+1)(d-1)-(r+e-a) \), hence \( \text{codim}(S^\circ, D_\infty) \leq r + e - a \). \( \square \)
We can now prove the case of Inversion of Adjunction which was stated in Section 1.

**Proof of Theorem 1.6** The inequality

\[ \text{mld}(W; X, D + Y) \leq \text{mld}(W; D, Y|_D) \]

is well-known in general and follows by adjunction (see, for example, the proof of Proposition 7.3.2 in [Kol]). We recall the argument for completeness.

Let \( \pi : X' \to X \) be proper, birational, such that \( X' \) is smooth, and \( \pi^{-1}(Y) \cup \pi^{-1}(D) \cup \pi^{-1}(W) \) is a divisor with simple normal crossings. Write

\[ K_{X'/X} - \pi^{-1}(Y) = \sum (a_i - 1) E_i, \]

and \( \pi^{-1}(D) = \tilde{D} + \sum_i b_i E_i. \) Note that by hypothesis, the strict transform \( \tilde{D} \) of \( D \) does not appear in \( K_{X'/X} - \pi^{-1}(Y) \).

The restriction \( \pi_0 : \tilde{D} \to D \) is a log resolution of \( (D, Y|_D \cup W) \), and the adjunction formula gives

\[ K_{\tilde{D}/D} - \pi_0^{-1}(Y|_D) = \sum (a_i - b_i - 1) E_i|_{\tilde{D}}. \]

Since \( \pi(\tilde{D}) \not\subseteq W \), we see that if \( \tilde{D} \cap E_i \neq \emptyset \), then \( \pi_0(\tilde{D} \cap E_i) \subseteq W \) if and only if \( \pi(E_i) \subseteq W \). This is enough to give the inequality we have claimed. The reverse inequality is not obvious, as some of the divisors \( E_i \) might not intersect \( \tilde{D} \).

We turn now to the proof of this reverse inequality. Suppose that \( \text{mld}(W; X, D + Y) < \tau \), for some \( \tau \in \mathbb{R}_+ \). It follows from Theorem 2.6 applied to the smooth variety \( X \) that there are \( m, a \in \mathbb{N}, b \in \mathbb{N}^k \), such that \( m \geq \max\{a, b_i\} \), and if

\[ A = \{ \gamma \in X_m | \text{ord} \gamma(I_D) \geq a, \text{ord} \gamma(I_{Y_i}) \geq b_i, \text{ord} \gamma(I_W) \geq 1 \}, \]

then there is an irreducible component \( R \) of \( A \), with \( \text{codim}(R, X_m) < a + \sum_i b_i q_i + \tau \).

It is clear that \( R \) satisfies the hypothesis of Lemma 3.2. With notation as in the lemma, we get the subset

\[ S^* \subseteq D_\infty \cap F_{Z}^{-1}(c) \cap \bigcap_i F_{Y_i}^{-1}(\geq b_i), \]

such that \( \text{codim}(S^*, D_\infty) < \sum_i q_i b_i + e + \tau \). Theorem 2.6 shows that \( \text{mld}(W; D, Y|_D) < \tau \). This completes the proof of the theorem.

We apply now Theorem 1.6 and 2.6 to deduce a characterization of terminal hypersurfaces. Fix a divisor \( D \subset X \), where \( X \) is smooth of
dimension $d$, and $D$ is reduced and irreducible. Recall that by Theorem 3.3 in [Mu2], $D$ has canonical (or equivalently, rational) singularities if and only if $D_m$ is irreducible for every $m$. Moreover, it is shown in [Mu2] that in this case $D_m$ is a locally complete intersection variety, of dimension $(m+1)(d-1)$. The following result similarly characterizes terminal singularities, giving a positive answer to a question of Mirel Caibăr.

**Theorem 3.3.** If $D \subset X$ is an irreducible and reduced divisor on a smooth variety $X$, then $D$ has terminal singularities if and only if $D_m$ is normal for every $m \in \mathbb{N}$.

**Proof.** By the results in [Mu2], we may assume that $D_m$ is a locally complete intersection variety of dimension $(m+1)(d-1)$. Therefore $D_m$ is normal if and only if $\dim(D_m)^{\text{sing}} \leq (m+1)(d-1)-2$.

Moreover, if $\phi_m : D_m \rightarrow D$ is the canonical projection, then $(D_m)^{\text{sing}} = \phi_m^{-1}(D^{\text{sing}})$. Indeed, the inclusion $\subseteq$ is trivial. To see the reverse inclusion, note that $D_m \subset X_m$ is defined by $(m+1)$ equations. Since $\dim D_m = \dim X_m - (m+1)$, if $u \in D_m$ is a smooth point, then these $(m+1)$ equations are part of a regular system of parameters at $u$. As the scheme defined by the first equation is locally isomorphic to $D \times \mathbb{A}^{md}$, $D$ has to be smooth at $\phi_m(u)$.

Therefore $D_m$ is normal for every $m$ if and only if $\dim \phi_m^{-1}(D^{\text{sing}}) \leq (m+1)(d-1)-2$ for every $m$. By Theorem 2.6, this is equivalent with $\mld(D^{\text{sing}}; X, D) \geq 2$. Since this minimal log discrepancy is an integer, this is further equivalent to $\mld(D^{\text{sing}}; X, D) > 1$. By Theorem 1.6, we have

$$\mld(D^{\text{sing}}; X, D) = \mld(D^{\text{sing}}; D).$$

As by definition $\mld(D^{\text{sing}}; D) > 1$ if and only if $D$ has terminal singularities, we are done. \qed

**Remark 3.4.** In fact, the above argument can be used to show that if $D$ is a normal divisor on a smooth $d$-dimensional variety $X$, then $D$ has log canonical singularities if and only if $D_m$ has pure dimension for every $m$. Indeed, $D_m$ has pure dimension if and only if $\dim D_m = (m+1)\dim D$. By Remark 2.7, this is true for every $m$ if and only if $(X, D)$ is log canonical. This is equivalent with $D$ being log canonical by Corollary 1.8.

Suppose now that $D$ is a normal divisor with log canonical singularities. The argument in the proof of Theorem 3.3 shows that $(D_m)^{\text{sing}} = \phi_m^{-1}(D^{\text{sing}})$. Moreover, we see that $\text{codim}((D_m)^{\text{sing}}, D_m) \geq$
mld(D_{\text{sing}}; D), for all m, and equality is achieved for some m. In particular, this implies Theorem 3.3 in [Mu2]: D has canonical singularities if and only if D_m is irreducible for every m.

4. SEMICONTINUITY OF MINIMAL LOG DISCREPANCIES

In this section we prove a semicontinuity statement for minimal log discrepancies in the case of a smooth ambient variety. Recall the following conjecture from [Am2].

Conjecture 4.1. If X is a normal, Q-Gorenstein variety, and if Y = \sum_i q_i \cdot Y_i, where q_i \in \mathbb{R}_+ and Y_i \subset X is a proper closed subscheme, for all i, then the function x \in X \rightarrow mld(x; X, Y) is lower semicontinuous.

It was shown in [Am2] that this conjecture is equivalent with the following one.

Conjecture 4.2. Let X and Y be as in Conjecture 4.1. For every two irreducible closed subsets V \subset W \subset X, we have

\[ mld(\eta_V; X, Y) \leq mld(\eta_W; X, Y) + \text{codim}(V, W). \]

Remark 4.3. In fact, in [Am2], Y is assumed to be a divisor. On the other hand, all the arguments can be extended to the case of an arbitrary subscheme.

One reason for conjecturing the above statements in [Am2] was to explain a conjecture of V. Shokurov from [Sh2], which was the particular case W = X in Conjecture 4.2.

We will show that Conjecture 4.2 is true if the ambient variety is smooth.

Theorem 4.4. Let X be a smooth variety, and Y = \sum_{i=1}^k q_i \cdot Y_i, where q_i \in \mathbb{R}_+ and Y_i \subset X is a proper closed subscheme, for all i. For every two irreducible closed subsets V \subset W, we have

\[ mld(\eta_V; X, Y) \leq mld(\eta_W; X, Y) + \text{codim}(V, W). \]

Proof. By taking a sequence of intermediate subvarieties, it is enough to consider the case when codim(V, W) = 1. If W = X, then V is a divisor. We clearly have mld(\eta_V; X, Y) \leq mld(\eta_X; X) = 1, which completes this case, as mld(\eta_X; X, q \cdot Y) = 0.

From now on, we suppose that W \neq X, so codim(V, X) \geq 2. If (X, Y) is not log canonical on any open subset meeting V, then mld(\eta_V; X, Y) = -\infty, and there is nothing to prove. If this is not the case, by restricting to a suitable open subvarieties, we may assume that (X, Y) is log canonical. Moreover, we may restrict to a suitable open
subset meeting $V$ in order to have $\text{mld}(V; X, q \cdot Y) = \text{mld}(\eta_V; X, q \cdot Y)$ (see Proposition 1.10(v)). Up to this point, the argument holds for arbitrary $X$.

Let $\tau = \text{mld}(\eta_V; X, Y)$. Since $X$ is smooth, by Theorem 2.6 there exists $m = (m_i)_i \in \mathbb{N}^k$ such that $\text{codim}(A, X_\infty) \leq \sum_i q_i m_i + \tau$, where $A = \bigcap_i F_{Y_i}^{-1}(\geq m_i) \cap \psi_0^{-1}(W)$. Fix $p \gg 0$ (depending on $m$), and let $B = \psi_p(A)$, so that $\dim B \geq (p + 1) \dim X - \sum_i q_i m_i - \tau$.

We claim that we can choose $m$ such that there is an irreducible component $T$ of $B$ with $\dim T \geq (p + 1) \dim X - \sum_i q_i m_i - \tau$, and such that $\phi_p(T) = W$. Indeed, note first that for every irreducible component $T$, $\phi_p(T)$ is a closed subset of $W$. This follows since $T$ is invariant under the $\mathbb{A}^1$-action on $X_m$. If we cannot find $T$ as claimed, then we may restrict to a suitable open subset meeting $W$ to deduce $\text{mld}(\eta_V; X, Y) > \tau$, a contradiction. For this we use the fact that by Theorem 2.6 in order to compute minimal log discrepancies it is enough to check finitely many jet schemes, depending on a log resolution of $(X, Y \cup W)$. Therefore we can find $T$ as claimed.

Let $\phi: T \to W$ be the restriction of $\phi_p$ to $T$. Since there is an irreducible component $S$ of $\phi^{-1}(V)$ with $\dim S \geq (\dim T - \dim W) + \dim V$, we deduce

$$\text{mld}(\eta_V; X, Y) = \text{mld}(V; X, Y) \leq \tau + \text{codim}(V, W)$$

via another application of Theorem 2.6. This concludes the proof. 

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**References**

[Am1] F. Ambro, Inversion of Adjunction for non-degenerate hypersurfaces, arXiv: math.AG/0108168.

[Am2] F. Ambro, On minimal log discrepancies, Math. Res. Letters 6 (1999), 573–580.

[Am3] F. Ambro, The adjunction conjecture and its applications, math.AG/9903060.

[Cr] A. Craw, An introduction to motivic integration, arXiv: math.AG/9911179.

[DL1] J. Denef and F. Loeser, Germs of arcs on singular varieties and motivic integration, Invent. Math. 135 (1999), 201–232.

[DL2] J. Denef and F. Loeser, Motivic Igusa zeta function, J. Alg. Geom. 7, 1998, 505–537.

[ELM] L. Ein, R. Lazarsfeld and M. Mustaţă, Contact loci in arc spaces, in preparation.

[Gr] M. Greenberg, Rational points in henselian discrete valuation rings, Publ. Math. I.H.E.S. 31 (1966), 59–64.

[Hi] H. Hironaka, Resolution of singularities of an algebraic variety over a field of characteristic zero, Ann. of Math. (2) 79. 1964, 109–326.

[Kol] J. Kollár, Singularities of pairs, in Algebraic Geometry, Santa Cruz 1995, volume 62 of Proc. Symp. Pure Math Amer. Math. Soc. 1997, 221–286.
[K+] J. Kollár (with 14 coauthors), Flips and abundance for algebraic threefolds, Astérisque 211, 1992.
[Lo] E. Looijenga, Motivic measures, Séminaire Bourbaki, Vol. 1999/2000, Astérisque 276 (2002), 267–297.
[Mu1] M. Mustaţă, Singularities of pairs via jet schemes, J. Amer. Math. Soc. 15 (2002), 599–615.
[Mu2] M. Mustaţă, Jet schemes of locally complete intersection canonical singularities, with an appendix by D. Eisenbud and E. Frenkel, Invent. Math. 145 (2001), 397–424.
[Sh1] V. Shokurov, 3-Fold log flips, Russian Acad. Sci. Izv. Math. 40, 1993, 95–202.
[Sh2] V. Shokurov, Problems about Fano varieties, in Birational geometry of algebraic varieties, Open Problems–Katata 1988, 30–32.
[St] J. Stevens, On canonical singularities as total spaces of deformations, Abh. Math. Sem. Univ. Hamburg 58, 275–278.
[Ya] T. Yasuda, Dimensions of jet schemes of log singularities, Amer. J. Math., to appear.

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