Gapped domain walls between 2+1D topologically ordered states

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The 2+1D topological order can be characterized by the mapping-class-group representations for Riemann surfaces of genus-1, genus-2, etc. In this paper, we use those representations to determine the possible gapped boundaries of a 2+1D topological order, as well as the domain walls between two topological orders. We find that mapping-class-group representations for both genus-1 and genus-2 surfaces are needed to determine the gapped domain walls and boundaries. Our systematic theory is based on the fixed-point partition functions for the walls (or the boundaries), which completely characterize the gapped domain walls (or the boundaries). The mapping-class-group representations give rise to conditions that must be satisfied by the fixed-point partition functions, which leads to a systematic theory. Such conditions can be viewed as bulk topological order determining the (non-invertible) gravitational anomaly at the domain wall, and our theory can be viewed as finding all types of the gapped domain wall given a (non-invertible) gravitational anomaly. We also developed a systematic theory of gapped domain walls (boundaries) based on the structure coefficients of condensable algebras.

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I. INTRODUCTION

Topological order is a new kind of order in gapped quantum states of matter beyond Landau symmetry breaking theory. In Ref. 3 and 4, it was conjectured that the non-Abelian geometric phases5 (both the \( U(1) \) part and the non-Abelian part) of degenerate ground states generated by the automorphism of Riemann surfaces can completely characterize different topological orders.3

The non-Abelian geometric phases contain an universal non-Abelian part\(^{3, 4}\) and a path dependent \( U(1) \) part\(^{3, 6}\). The non-Abelian part carries information about the projective representation of mapping class group (MCG) of the space manifold. For torus, the MCG is \( \Gamma_1 = SL(2, \mathbb{Z}) \), which is generated by a 90° rotation and a Dehn twist. The associated non-Abelian geometric phases for such two generators are denoted by \( S \) and \( T \), which are unitary matrices. \( S \) and \( T \) generate a projective representation of MCG \( SL(2, \mathbb{Z}) \) for torus.
The Abelian part of the non-Abelian geometric phases are also important: it is related to the gravitational Chern-Simons term in the partition function and carries information about the chiral central charge $c$ for the gapless edge excitations, The data $(S,T,c)$ is a quite complete description of 2+1D topological orders. However, to obtain a full description of 2+1D topological orders, the modular data for genus-1 surface is not enough, we must also use the non-Abelian geometric phases (i.e. the mapping class group representations) for genus-2 surfaces.

In this paper, we will study the boundary of topological orders, or more generally, the domain wall between two topological orders. We will see how the boundary properties are determined by the bulk topological orders. We would like to consider the following issues: What is the data that allow us to characterize different gapped domain walls between topologically ordered states? How to classify gapped domain walls? Ref. 16–19 studied those issues for the case of 2+1D Abelian topological orders, using condensable set of bosonic topological excitations. Ref. 20–26 considered this problem for the more general case of 2+1D non-Abelian topological orders, using boson condensation and/or condensable algebra. Some discussions for the boundaries of topological orders beyond 2+1D can be found in Ref. 9, 29, and 30. In particular, it was pointed out that the boundary effective theory of a bulk topological order has a gravitational anomaly. In fact, the bulk topological order gives an one-to-one classification of gravitational anomaly in one low dimension (realized by the boundary). Thus in some sense types of bosonic topological order = types of bosonic gravitational anomaly in one lower dimension. (Note that bosonic gravitational anomaly is a property of an effective theory that cannot be realized as a local bosonic system with finite cut-off. Two effective theories are said to have the same type of gravitational anomaly if one effective theory can change into another effective theory, possibly via phase transitions in the same dimension.)

In this paper, we will rederive the simple results based on the $(S,T,c)$ for the boundaries of general 2+1D non-Abelian topological orders introduced in Ref. 34. Our new approach also allows us to generalize the approach in Ref. 34 to high genus Riemann surface which may lead to a complete description of the gapped boundary of 2+1D topological orders. We try to address the following question: given a 2+1D topological order described by $(S,T,c)$ and the data from higher genus, how to describe and classify different gapped 1+1D boundary phases? If we find that there is no gapped 1+1D boundary for a type of 2+1D topological order, then such a type of topological order must have a gapless boundary. It can also be stated in the following way: given a type of gravitational anomaly determined, how to describe different gapped 1+1D phases? If we find that there is no gapped 1+1D phase for a type of gravitational anomaly, then such a type of gravitational anomaly will require (or ensure) the 1+1D phases to be gapless.

II. CHARACTERIZATION OF GAPPED DOMAIN WALLS

A. Dimensions of fusion spaces

To introduce data that can characterize different gapped domain walls, let us consider a 2D space $S^2$. Half of $S^2$ is occupied by the phase $A$ and the other half by the phase $B$. Let us assume the domain wall between the topological phases is gapped. We put a type-$a$ topological excitation in the phase $A$ and a type-$i$ topological excitation in the phase $B$. We denote such a configuration as $S^2_{BA;ia}$ (see Fig. 1). The ground state space for such a configuration is a fusion space. The dimension of the fusion space (i.e. the ground state degeneracy), denoted as $M^{ia}_{BA} \in \mathbb{N}$, is the data that characterizes the gapped domain wall between $A$ and $B$ phases. We may also view $M^{ia}_{BA}$ as the fusion coefficients for the fusion of type-$i$ particle, type-$a$ particle, and the domain wall $BA$.

B. Weighted wave function overlap

To obtain more data to characterize the domain walls, let us consider the degenerate ground states of the topologically ordered phase $A$, described by normalized wave function $|\psi_{IA}^A\rangle$, where the index $I_A$ label different ground states on a closed Riemann surface of genus $g$. Similarly, we have the degenerate ground states of the topologically ordered phase $B$: $|\psi_{IB}^B\rangle$. 

FIG. 1. Two topologically ordered phases $A$ and $B$, each occupies a hemisphere of a 2D space $S^2$. There is an excitation of type $a$ in the phase $A$ and an excitation of type $i$ in the phase $B$.

FIG. 2. (a): The ground state $|1\rangle$ on a torus that corresponds to the trivial particle $1$ can be represented by an empty solid torus. (b): The other ground state $|i\rangle$ that corresponds to a type-$i$ topological excitation can be represented by a solid torus with a world-line of the type-$i$ in the center.
of a quantum system, $H$ of excitation basis states. We will refer such a basis as the construction of degenerate ground states using world-lines give rise to a natural basis for the degenerate ground states. Also, the above construction of degenerate ground states using world-lines of those topological excitations. Also, the above construction of degenerate ground states using world-lines give rise to a natural basis for the degenerate ground states. We will refer such a basis as the excitation basis.

For example, on a genus-1 surface, the degenerate ground states are labeled by $I_B = i$ for the phase $B$ and $L_A = a$ for the phase $A$. Here $i = 1, \cdots, N_B$ label the types of the topological excitations in the phase $B$ and $a = 1, \cdots, N_A$ label the types of the topological excitations in the phase $A$. For genus-1 surface, those excitation-labeled ground states happen to be orthogonal. We see that the ground state degeneracies on genus-1 surface are given by $N_A$ for phase $A$ and $N_B$ for phase $B$.

As another example, on a genus-2 surface, the degenerate ground states are labeled by $I_B = (i, j, \mu, \nu)$ where $\mu = 1, \cdots, N_B^{ij}$ and $\nu = 1, \cdots, N_B^{ij}$ for the phase $B$. The degenerate ground states are labeled by $I_A = (a, b, c, \alpha, \beta)$ where $\alpha = 1, \cdots, N_A^{ab}$ and $\beta = 1, \cdots, N_A^{ab}$ for the phase $A$. The fusion coefficients of the topological excitations in the phase $A$, $N_B^{ij}$, are in general complex numbers, whereas $N_A^{ab}$, $N_A^{ac}$, and $N_A^{bc}$ are the fusion coefficients of the topological excitations in the phase $B$. We see that the ground state degeneracies on genus-2 surface are given by $\sum_{c=1}^{N_B^{ij}} N_B^{ij} N_B^{ij}$ for phase $B$ and $\sum_{c=1}^{N_A^{ab}} N_A^{ab} N_A^{bc}$ for phase $A$.

Motivated by the wave function overlap\cite{35–38} that can characterize different topological orders, here we will use the weighted wave function overlap to characterize different domain walls. We conjecture that [9, 39] the weighted overlap of the degenerate ground states on a closed genus-$g$ surface $\Sigma_g$ for topologically ordered phases $A$ and $B$ have the following form

$$\langle \psi_B^E | e^{-H_W} | \psi_A^A \rangle = e^{-\sigma_{BA} A_B A_A + o(\frac{1}{g})} W_{BA,g}^{IA}$$

(1)

where $H_W$ is local hermitian operator like a Hamiltonian of a quantum system, $A_{BA}$ is the area of the surface $\Sigma_g$ and $W_{BA,g}^{IA}$ is a topological invariant that characterize the domain between the phases $A$ and $B$.

In general, $W_{BA,g}^{IA}$ depends on the choices of $H_W$ which correspond to different choices of domain walls. A concrete calculation of the wave function overlap, $W_{BA,g}^{IA}$ for a simple topological order is presented in Section IV. We will show in next section that the wave function overlap can also be viewed as partition function of the domain wall.

When $B$ is a trivial product state, the index $I_B$ is always fixed to be 1, since $B$ has no ground state degeneracy. In this case we simplify

$$W_{BA,g}^{1A} = W_{A,g}^{IA}$$

(2)

by dropping index for trivial phase $B$ and $I_B$. Similarly, we simplify

$$M_{BA}^a = M_A^a$$

(3)

Those are data that describe a gapped boundary of topological order $A$.

We note that $W_{BA,g}^{IA}$ are in general complex numbers, whose phase can be changed by a change of phases for the ground states

$$|\psi_A^A \rangle \rightarrow e^{i\theta_{IA}} |\psi_A^A \rangle, \quad |\psi_B^B \rangle \rightarrow e^{i\theta_{IB}} |\psi_B^B \rangle.$$ (4)

Also $W_{BA,g}^{IA}$ may depend on some choices of basis that cannot be fixed. So $W_{BA,g}^{IA}$ by themselves are not a topological invariant, and they are not even physical. So we need to find a way mode out those phase and basis dependence. When we said $W_{BA,g}^{IA}$ are “topological invariant”, we mean that they are topological invariant up to those phase and basis choices.

For genus-1 surface $\Sigma_1$, we may choose the phases of $|\psi_1^A \rangle$ and $|\psi_1^B \rangle$ to make $W_{BA,1}^{11}$ real and positive. (Here 1 correspond to the trivial particle.) We then choose the phases of $|\psi_1^A \rangle$ to make $W_{BA,1}^{10}$ real and positive. Similarly, we choose the phases of $|\psi_1^B \rangle$ to make $W_{BA,1}^{11}$ real and positive. With such a choice, we find that $W_{BA,1}^{11} = M_{BA,1}^1$, which is the dimension of the fusion space defined in the last subsection (see Section III A for an explanation). For higher genus $g > 1$, $W_{BA,g}^{IA}$, after modding out some gauge redundancy, are new topological invariants. We hope $W_{BA,g}^{IA}$ carry enough information to fully characterize a gapped domain wall.

III. THE CONDITIONS ON THE DOMAIN-WALL DATA

In this section, we are going to derive some conditions on the data that characterize the gapped domain walls. For example we like to show that the dimension of the fusion state $M_{BA}^a$ (which are non-negative integers) and the weighted wave function overlap for torus $W_{BA,1}^{01}$ (which can be complex numbers) are actually equal to each other $M_{BA}^a = W_{BA,1}^{01}$. This is quite an amazing relation.

To derive those conditions, we need to introduce topological path integral, i.e. the path integral for triangle-
lated space-time whose value is re-triangulation invariant. This is done in Appendix A. We also need to introduce an algebraic (i.e., a categorical) approach to evaluate those topological path integrals, which is done in Appendix B. Using those results, we can derive the conditions on the domain-wall data.

A. Why $M_{BA}^{ia} = W_{BA;i}^{ia}$?

First, we like to show $M_{BA}^{ia} = W_{BA;i}^{ia}$. Let us consider the topological path integral on space-time $D^2_t \times S^1$, where $D^2_t$ is a disk with a puncture. Such a puncture corresponds to a world-line of a type-i topological excitation wrapping around $S^1$. The topological path integral on $D^2_t \times S^1$ only sum over the degrees of freedom in the bulk, and leave the degrees of freedom boundary fixed. Thus $Z^{top}(D^2_t \times S^1)$ corresponds to a wave function $|\psi_i\rangle$ on $S^1 \times S^1 = \partial(D^2_t \times S^1)$ (for details, see Ref. 39). First, we like to show that such a wave function $|\psi_i\rangle$ is automatically normalized. According to Ref. 39, $|\psi_i\rangle$ is given by $Z^{top}(S^2_{1,i} \times S^1)$ where $S^2_{1,i} \times S^1$ is obtained by gluing two $D^2_t \times S^1$ together along its boundary. $S^2_{1,i} \times S^1$ contains two world-lines of type-i and type-\(\bar{i}\).

Let us evaluate the above partition function on $S^2_{1,i} \times S^1$, which turns out to be $Z^{top}(S^2_{1,i} \times S^1) = 1$:  

\[
Z^{top}(S^2_{1,i} \times S^1) = \sum_{\alpha, \beta, k=1} Y^{\alpha \beta \bar{i}}_{i,\bar{k}} Z^{top}(S^2_{1,i} \times S^1) = \sum_{\alpha, \beta, k=1} Y^{\alpha \beta \bar{i}}_{i,\bar{k}} O^{\alpha \beta}_{\bar{k},i} Z^{top}(S^2 \times S^1) = 1,
\]

where we have used $N^{\bar{i}}_i = N^i_1 = 1$, so $\alpha = \beta = 1$ and the $\sum_{\alpha, \beta}$ only contain one term. $k$ is to be the trivial particle 1, since $k$ lives on a sphere $S^2$ alone. We have also used $Y^{\bar{i},i1}_{1,i1} O^{\bar{i},i1}_1 = 1$ (see appendix B for the definition of $Y$-move and $O$-move) and $Z^{top}(S^2 \times S^1) = 1$. So the topological path integral on $D^2_t \times S^1$ gives rise to normalized wave function $|\psi_i\rangle$.

Now, instead of a path integral on $S^2_{1,i} \times S^1$ which gives us $Z^{top}(S^2_{1,i} \times S^1) = 1$, let us consider a path integral on $S^2_{BA;ia} \times S^1$, where $S^2_{BA;ia}$ is described by Fig. 1. In space-time, the particle $a$ and $i$ correspond to world-lines wrapping around $S^1$ which is viewed as a space direction. The boundary between the two hemisphere in Fig. 1 is viewed as another space direction. The topological partition function for space-time $S^2_{BA;ia} \times S^1$ corresponds to the weighted wave function overlap $\langle \psi_i^B | e^{-H_W} | \psi_a^A \rangle$ with a fine tune choice of $H_W$, which has a form  

\[
\langle \psi_i^B | e^{-H_W} | \psi_a^A \rangle = Z^{top}(S^2_{BA;ia} \times S^1) = e^{-\sigma_{BA;ia} S_1} W_{BA;i}^{ia}.
\]

B. Invariance under the MCG action

Next, we divide the space-time $S^2_{BA;ia} \times S^1$ into three pieces $D^2_{A;i} \times S^1$ (see Fig. 4a), $D^2_{B;i} \times S^1$ (see Fig. 4c), and $S^2_{BA} \times S^1 \times S^1$ (see Fig. 4b), where $D^2_{A;i}$ is a disk occupied by the phase $A$ and the type-a particle, $D^2_{B;i}$ is a disk occupied by the phase $B$ and the type-i particle, and $S^2_{BA} \times S^1 \times S^1$ is a cylinder occupied by the $A$ and $B$ phases and the domain wall. We can use the three pieces, $D^2_{A;i} \times S^1$, $D^2_{B;j} \times S^1$, and $S^2_{BA} \times S^1 \times S^1$, to assemble the same closed space-time in two ways: (1) we glue $D^2_{A;i} \times S^1$ and $S^2_{BA} \times S^1 \times S^1$ directly along $S^1 \times S^1$ boundary, and glue $S^2_{BA} \times S^1 \times S^1$ and $D^2_{B;i} \times S^1$ with a twist $\hat{U} \in \Gamma_1 = SL(2,\mathbb{Z})$ for the boundary $S^1 \times S^1$; (2) we glue $D^2_{A;i} \times S^1$ and $S^2_{BA} \times S^1 \times S^1$ with a twist $\hat{U} \in \Gamma_1 = SL(2,\mathbb{Z})$ and glue $S^2_{BA} \times S^1 \times S^1$ and $D^2_{B;i} \times S^1$ directly.
The two ways to assemble the closed space-time only differ by a re-triangulation, and they must produce the same topological partition function (see Appendix A.4 and A.5). This leads to a matrix relation
\[
R_{B}^{ij}\omega_{B,A,1} = \omega_{B,A,1}R_{B}^{ij}.
\] (7)

We notice that the path integral on \(D_{A:a}^{2} \times S^{1}\) produce the basis state \(|a; A\) in the excitation basis for the phase \(A\) on the surface \(\partial(D_{A:a}^{2} \times S^{1})\). The action of the MCG transformation \(\hat{U}\) on \(|a; A\) is described by the unitary representation \(R_{B}^{ij}\) of \(\hat{U}\) for the phase \(A\). Similarly, the path integral on \(D_{B:i}^{2} \times S^{1}\) produce the basis state \(|i; B\) in the quasiparticle basis for the phase \(B\). The action of the MCG transformation \(\hat{U}\) on \(|i; B\) is described by the unitary representation \(R_{B}^{ij}\) of \(\hat{U}\) for the phase \(B\). This is how do we obtain eqn. (7).

\section{A consistent condition between the fusions in the bulk and on the wall}

In this section, we like to obtain some additional conditions. Let us consider the following fusion process: we bring a type-\(a\) excitation in phase \(A\) and a type-\(i\) excitation in phase \(B\) to the domain wall \(w_{B,A}\) between the two phase and fuse them into an excitation of type \(s\) on the domain wall. The corresponding fusion algebra is given by
\[
i \otimes_{w_{B,A}} a = M_{B,A,s}^{ia}s, \quad M_{B,A,s}^{ia} \in \mathbb{N}.
\] (8)

It turns out that
\[
M_{B,A}^{ia} = M_{B,A}^{ia,1}
\] (9)

where \(s = 1\) represents the trivial type of the excitations on the domain wall.

Now let us compute the dimension of the fusion space of \(S_{B,A}^{2}\) with excitations of types \(a\) and \(b\) in the phase \(A\) and excitations of types \(i\) and \(j\) in the phase \(B\). There are two ways to compute the dimension of the fusion space: (1) we may fuse \(a\) and \(b\) into \(c\) and fuse \(i\) and \(j\) into \(k\), or (2) we may first fuse \(a\) and \(i\) into \(s\) on the domain wall and fuse \(j\) and \(k\) into \(s\) on the domain, and then fuse \(s\) and \(s\) into the trivial excitation on the domain wall. The two fusion path should produce the same result, and we obtain
\[
\sum_{c,k} N_{A,c}^{ab} N_{B,k}^{ij} M_{B,A}^{kc} = \sum_{s} M_{B,A,s}^{ia} M_{B,A,s}^{jb}.
\] (10)

The two equations eqn. (9) and eqn. (10) imply the condition eqn. (14).

\section{The conditions on genus-1 data for a gapped domain wall}

Consider two topological orders, phases \(A\) and \(B\), described by \((S^{4}, T^{4}, c^{A})\) and \((S^{8}, T^{8}, c^{B})\). Suppose there are \(N^{A}\) and \(N^{B}\) types of topological excitations in the phase \(A\) and the phase \(B\), then the ranks of their modular matrices are \(N^{A}\) and \(N^{B}\) respectively. We find the following necessary conditions for the phase \(A\) and the phase \(B\) to be connected by a gapped domain wall:
\[
c^{A} = c^{B},
\] (11)

such that
\[
M_{B,A}^{ia} \in \mathbb{N},
\] (12)

\[
\sum_{b} S_{ij}^{AB} M_{B,A}^{ia} = \sum_{b} M_{B,A}^{jb} S_{ia}^{AB},
\] (13)

\[
\sum_{b} T_{ij}^{AB} M_{B,A}^{ia} = \sum_{b} M_{B,A}^{jb} T_{ia}^{AB},
\] (14)

\[
M_{B,A}^{ia} M_{B,A}^{jb} \leq \sum_{kc} N_{A,c}^{ab} M_{B,A}^{kc} N_{B,k}^{ij}.
\] (14)

Here \(\mathbb{N}\) denotes the set of non-negative integers. \(a, b, c, \ldots\) and \(i, j, k, \ldots\) are indices for the particle types in phases \(A\) and \(B\). \(N_{A,c}^{ab}\) and \(N_{B,k}^{ij}\) are fusion coefficient of phases \(A\) and \(B\). We may call Eq. (13) the commuting condition, and Eq. (14) the stable condition.

In fact the matrix \(M_{B,A}\) label a gapped domain wall between phases \(A\) and \(B\). Eqs. (11), (12), (13), and (14) is a set of necessary conditions a gapped domain wall \(M_{B,A}\) must satisfy, i.e., if there is a gapped domain wall, we will have a non-zero \(M_{B,A}\) that satisfies those conditions. This implies that if there is no non-zero solution of \(M_{B,A}\), the domain wall must be gapless. However, it is not clear if those condition are sufficient for a gapped domain wall to exist. In some simple examples, the solutions \(M_{B,A}\) are in one-to-one correspondence with gapped domain walls. However, for some complicated examples \([41]\), a \(M_{B,A}\) may correspond to more than one type of gapped domain wall. This indicates that some additional data are needed to completely characterize gapped domain walls.

\section{Wave function overlap on genus-2 Riemannian surfaces and topological path integral}

In the above, we have developed a theory on the domain wall based on the data \(W_{B,A}^{ia}\) of the wave function...
overlap on torus. However, such a data is insufficient to fully characterize the domain wall, i.e. there are known different domain walls to produce the same \( W_{BA} \). So in this section, we consider the overlap of normalized ground state wave functions, \( |\psi_A^1\rangle \) and \( |\psi_B^B\rangle \), on genus-2 surfaces, which gives rise to data \( W_{BA}^{1B} \) (see eqn. (1)).

To understand the structure of the data \( W_{BA}^{1B} \), let us generate the degenerate ground states on a genus-2 surface \( \Sigma \) via a path integral. We first consider a filled genus-2 surface denoted as \( \Sigma \_fill \); \( \Sigma = \partial \Sigma \_fill \). The path integral on \( \Sigma \_fill \) will generate a ground state on \( \Sigma \). To generate other ground states, we need to add three world-lines \( i, j, z \) in the interior of the filled genus-2 surface (see Fig. 3). Thus the linearly independent states on \( \Sigma \) are labeled by \( i, j, z, \mu, \nu \), where \( \mu = 1, \ldots, N^i, \nu = 1, \ldots, N^j \). Thus, for topological phase \( A \), the label \( L^A \) correspond to \( \Sigma^i \_fill \) \( \sim \) \( (iA, jA, zA, \muA, \nuA) \). For topological phase \( B \), the label \( L^B \) correspond to \( \Sigma^B \sim (iB, jB, zB, \muB, \nuB) \). So the wave function overlap on genus-2 surfaces is given by

\[
W_{BA}^{1B} = W_{BA}^{(i,j,\mu,\nu), (iA,jA,zA,\muA,\nuA)} \tag{15}
\]

The wave functions, \( |\Psi_{i,j,z,\mu,\nu}\rangle \), given by the path integral on \( \Sigma \_fill \) may not be normalized. In the following we will consider their normalization. First, we express the normalization via path integral

\[
\langle \Psi_{i,j,z,\mu,\nu} | \Psi_{i,j,z,\mu,\nu} \rangle = Z_{BA}^{top} \Sigma_{fill}^i \Sigma_{fill}^j \tag{16}
\]

where \( \Sigma_{fill}^i \) and \( \Sigma_{fill}^j \) are the closed 3-dimensional space-time obtained by gluing two \( \Sigma_{fill}^{i,j} \) along their \( \Sigma_2 \) boundary (see Fig. 6).

Now, we drop the three world lines in Fig. 6b. Next we deform the world-lines in Fig. 6a to those in Fig. 6b using eqn. (B6). This gives rise to a factor \( \sqrt{V_{i,11}^{i,j,11} V_{j,11}^{i,j,11}} \). Note that dash lines in Fig. 6b all go through \( S^2 \) and hence the dash lines must correspond to world lines of the trivial particles (i.e. type-1). We then drop the three dash lines in Fig. 6b. Next, we deform the two cluster of the world lines into two \( \Theta \) graphs, which give rise to a phase factor since the world lines may get twisted (see Fig. 16). Last we use eqn. (B5) to reduce the two \( \Theta \) graphs into two loops of the \( z \) world lines, which gives rise to a factor \( O_2^{\mu,\nu} O_2^{\mu,\nu} \) (here the repeated indices \( \mu \) and \( \nu \) are not summed). The two \( z \) loops produces \( d_2^2 \).

Collecting all those factors, we find that \( \langle \Psi_{i,j,z,\mu,\nu} | \Psi_{i,j,z,\mu,\nu} \rangle \) is given by (where \( \mu, \nu \) are not summed), up to a phase factor,

\[
Y_{1,11}^i Y_{1,11}^j Y_{1,11}^z O_2^{\mu,\nu} O_2^{\mu,\nu} d_2^2 Z_{BA}^{top} (\Sigma_{fill}^i \cup \Sigma_{fill}^j) \tag{17}
\]

\[
= \sqrt{d_1 d_1 d_2 d_2} \sqrt{d_3 d_3 d_4 d_4} \sqrt{d_5 d_5 d_6 d_6} \sqrt{d_7 d_7 d_8 d_8} Z_{BA}^{top} (\Sigma_{fill}^i \cup \Sigma_{fill}^j) = Z_{BA}^{top} (\Sigma_{fill}^i \cup \Sigma_{fill}^j),
\]

where \( Z_{BA}^{top} (\Sigma_{fill}^i \cup \Sigma_{fill}^j) \) is the topological partition function on \( \Sigma_{fill}^i \cup \Sigma_{fill}^j \) without worldlines.

Since \( \langle \Psi_{i,j,z,\mu,\nu} | \Psi_{i,j,z,\mu,\nu} \rangle \) is positive, the ambiguous phase factor must be 1. We see that the normalized ground state wave function is given by the topological path integral on \( \Sigma_2 \) with the world lines \( i, j, z \):

\[
|\psi_{i,j,z,\mu,\nu}\rangle = \frac{Z_{BA}^{top}(i, j, z, \mu, \nu)}{\sqrt{Z_{BA}^{top}(\Sigma_{fill}^i \cup \Sigma_{fill}^j)}}. \tag{18}
\]

This result allows us to express the wave function overlap in terms of path integral on \( \Sigma_{fill}^i \cup \Sigma_{fill}^j \):

\[
W_{BA}^{(i,j,\mu,\nu), (iA,jA,zA,\muA,\nuA)} = \frac{Z_{BA}^{top}(\Sigma_{fill}^i \cup \Sigma_{fill}^j)}{\sqrt{Z_{BA}^{top}(\Sigma_{fill}^i \cup \Sigma_{fill}^j)}} \tag{19}
\]

But now \( \Sigma_{fill}^{iA} \cup \Sigma_{fill}^{jA} \) is occupied by a topological phase \( A \) and the worldlines of \( i, j, z, \mu, \nu \) is occupied by a topological phase \( B \) and the worldlines of \( iB, jB, zB, \muB, \nuB \). Also, \( Z_{BA}^{top} \equiv Z_{BA}^{top}(\Sigma_{fill}^i \cup \Sigma_{fill}^j) \) is the topological partition function on space-time \( \Sigma_{fill}^i \cup \Sigma_{fill}^j \) filed with phase \( A \) without worldlines. Similarly, \( Z_{BA}^{top} \equiv Z_{BA}^{top}(\Sigma_{fill}^i \cup \Sigma_{fill}^j) \) is the topological partition function for phase \( B \).

Using the same reasoning as in eqn. (7), we find that the rectangular matrix \( W_{BA} \) satisfies

\[
R_{BA} W_{BA} = W_{BA} R_{BA}^T \tag{20}
\]

where \( R_{A,B} \) are the representations of genus-2 MCG \( \Gamma_2 \) for the phase \( A \) and \( B \).

The wave function overlap on genus-2 surface \( W_{BA} \) and the wave function overlap on genus-1 surface \( W_{BA,1} \) are related:

\[
W_{BA}^{(iA,jA,ZA,\muA,\nuA) \cup (iB,jB,ZB,\muB,\nuB)} = W_{BA,1}^{(iA,jA,ZA,\muA,\nuA)} \tag{21}
\]

Here \( W_{BA,1} \) is the topological partition function on the space-time obtained by gluing Fig. 7a and Fig. 7b together, and \( Z_{BA}^{top}(B^3(A) \cup B^3(B)) \) is the topological partition function on the space-time obtained by gluing Fig. 7c and Fig. 7d together.
wave function overlap via time obtained by gluing Fig. 8c and Fig. 8d together. Fig. 7b(right), and Fig. 7d together, we obtain a space-time obtained by gluing Fig. 8a and Fig. 7b(left), and Fig. 7c together along their surfaces, we obtain a space-time manifold in Fig. 8, after some cutting and gluing. To see this, we first cut Fig. 7a and Fig. 7b into left and right pieces. Then gluing Fig. 7a(left), Fig. 7b(left), and Fig. 7c together along their surfaces, we obtain a space-time obtained by gluing Fig. 8a and Fig. 8b together along their surfaces. Gluing Fig. 7a(right), Fig. 7b(right), and Fig. 7d together, we obtain a space-time obtained by gluing Fig. 8c and Fig. 8d together.

For convenience, we like to introduce a normalized wave function overlap via

$$\tilde{W}_{BA,g}^{i_A j_A} \equiv N_{BA,g} W_{BA,g}^{i_A j_A}$$

where we choose the normalization $N_{BA,g}$ such that $\tilde{W}_{BA,g}^{i_A j_A} = 1$ when $I,B, J,A$ correspond to trivial wordlines, i.e. for $g = 1,2$ we have

$$\tilde{W}_{BA,1}^{11} = 1, \quad \tilde{W}_{BA,2}^{(1,1,1,1,1),(1,1,1,1,1)} = 1.$$  \quad (23)

Thus we have

$$\tilde{W}_{BA,1}^{i_B j_B} = W_{BA,1}^{i_B j_B},$$

and

$$\tilde{W}_{BA,2}^{i_B j_B} = \sqrt{Z_{BA,1}^{top} Z_{BA,2}^{top} \tilde{Z}_{BA,2}^{top}} \tilde{Z}_{BA,2}^{top} (B^3 A \cup B^3 B)$$

$$\tilde{W}_{BA,2}^{i_B j_B} = \sqrt{Z_{BA,1}^{top} Z_{BA,2}^{top}} \tilde{Z}_{BA,2}^{top} (B^3 A) \cup \tilde{Z}_{BA,2}^{top} (B^3 B)$$

\quad (25)

The normalized wave function overlap for genus-1 and genus-2 have a simpler relation

$$\tilde{W}_{BA,2}^{i_B j_B} = \tilde{W}_{BA,1}^{i_B j_B} \tilde{W}_{BA,1}^{i_B j_B}$$

\quad (26)

IV. THE WAVE FUNCTION OVERLAP IN A LATTICE HAMILTONIAN MODEL

In this section we try to compute the wave function overlap in a concrete lattice Hamiltonian model. Above, in the path integral formulation in Euclidean spacetime, we learned the lesson that a gapped domain wall $W$ between phases $A,B$ (a defect along space direction), after a proper Wick-rotation, becomes an operator $e^{-H_W}$ (a defect along the time direction) that determines the weighted wave function overlap. However, in the Hamiltonian formulation there is no natural equivalence between space and time. To apply the previous results, we first try to analyze the physical picture in a lattice model.

Assume that there is a gapped domain wall at $x = 0$, phase $A$ in the region $x < 0$ a and phase $B$ in the region $x > 0$. Clearly the Hamiltonian near $x = 0$ can not be the same as those far from $x = 0$. For simplicity, we may assume that for some small positive number $\varepsilon$, the Hamiltonian is uniform in the regions $x < -\varepsilon$ and $x > \varepsilon$. On the other hand, in the region $-\varepsilon < x < \varepsilon$, the Hamiltonian is uniform along y and t directions, but changes with x. The Wick-rotated version of this picture, is that there are uniform phases $A$ at time $t < -\varepsilon$ and $B$ at time $t > \varepsilon$, and during time $-\varepsilon < t < \varepsilon$, the Hamiltonian evolves from that of $A$ to that of $B$. Thus $e^{-H_W}$ should correspond to the accumulated evolution operator during $-\varepsilon < t < \varepsilon$.

For the domain wall Hamiltonian in the region $-\varepsilon < x < \varepsilon$, the requirement is that it should keep the spectrum of the whole system gapped. However, we have no idea how this requirement is translated for $e^{-H_W}$. There is further another subtlety, that phases $A,B$ may not be defined on the same microscopic lattice, in particular, any generalized local unitary transformations can be inserted to deform $A$ or $B$. In the followings, we try to propose some reasonable assumptions that allows us to calculate the quantity $W_{BA,g}$.
• First, to remove the ambiguity of generalized local unitary transformations, we assume that proper generalized local unitary transformations have been applied such that $A, B$ are already on the same microscopic lattice and their Hamiltonians satisfy the following relation;

• We focus on the gapped domain walls that are directly induced by anyon condensations. Suppose $H^{ac}$ is the Hermitian operator that forces anyon condensation in $A$. (If only Abelian anyons are condensed, $H^{ac}$ is simply the sum of hopping string operators of the condensed anyons, that forces a total zero-momentum state of the condensed anyons. We will give more detail later in the toric code example.) We require the Hamiltonian $H_B$ of phase $B$ to be:

$$H_B = H_A + h H^{ac}, \quad h \rightarrow +\infty.$$  

where $H_A$ is the Hamiltonian of phase $A$.

• Then we assume that $e^{-H_W}$ acts on the ground state subspace of $H_A$ (or $H_B$) by multiplying a constant factor that may depend on the system size.

Now we look at the example of toric code model[42] which realize a $Z_2$ topological order:[43, 44]

$$H_A = -\sum_v \prod_{i \in \text{star}(v)} \sigma_z^i - \sum_p \prod_{i \in \partial p} \sigma_z^i. \quad (27)$$

The spins are on the links of the lattice. The first term sums over all vertices and $\text{star}(v)$ denotes the legs of the vertex $v$. The second term sums over all plaquettes and $\partial p$ denotes the boundary edges of the plaquette $p$. There are three types of nontrivial anyons $e, m, f$: $e$ is created by string operators of the form $\prod \sigma_z^i$ along links, $m$ is created by string operators of the form $\prod \sigma_z^i$ along links on the dual lattice, and $f$ is the fusion of $e$ and $m$. Both $e$ and $m$ can be condensed. The term that forces the condensation of $e$ is $H^{ac}_e = -\sum_i \sigma_z^i$, and that for $m$ is $H^{ac}_m = -\sum_i \sigma_z^i$. According to the above assumptions, we next calculate the overlap between the ground states of $H_A$ and $H_B = H_A + h H^{ac}_m$. It is easy to see that, under the $h \rightarrow +\infty$ limit, the ground state of $H_B$ is simply the spin polarized state $\otimes_i |0\rangle_i \pm |+\rangle_i$ or $\otimes_i |0\rangle_i$.

Now we write down the ground states of $A$ on a torus with the simplest lattice of only three links (see Fig. 9) according to the anyon flux along $y$ direction (measured by string operators winding around $x$ direction), the four ground states are

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|000\rangle + |101\rangle),$$

$$|\psi_e\rangle = \frac{1}{\sqrt{2}} (|110\rangle + |011\rangle),$$

$$|\psi_m\rangle = \frac{1}{\sqrt{2}} (|000\rangle - |101\rangle),$$

$$|\psi_f\rangle = \frac{1}{\sqrt{2}} (|110\rangle - |011\rangle). \quad (28)$$

FIG. 9. The simplest triangulated torus with three links and two vertices.

The corresponding ground states of $B$ are

$e$ condenses: $|\psi^{ac}_e\rangle = |++\rangle$, $m$ condenses: $|\psi^{ac}_m\rangle = |00\rangle$. \quad (29)

We obtain:

$$\langle \psi^{ac}_e | \psi_a \rangle = \frac{1}{2} (1, 1, 0, 0),$$

$$\langle \psi^{ac}_m | \psi_a \rangle = \frac{1}{\sqrt{2}} (1, 0, 1, 0), \quad a = 1, e, m, f. \quad (30)$$

which are indeed solutions to (13) and (14) after dropping the prefactors. One reason for the different prefactors of $e, m$ condensations is that the number of excitations the lattice can host is different. $e$ is hosted by vertices while $m$ is hosted by plaquettes and there are two vertices and only one plaquette in Fig. 9.

Let us generalize the above calculation to arbitrary system size and arbitrary lattice. We note that the four ground states on a torus for the toric-code model $H_A$ are sum of all closed strings formed by $|1\rangle$'s on the links, with fixed even-odd winding numbers in the two directions of torus, which are given by $|ee\rangle, |eo\rangle, |oe\rangle$, and $|oo\rangle$. The four ground states in the quasiparticle basis are given by

$$|\psi_1\rangle = \frac{1}{\sqrt{2}} (|ee\rangle + |oe\rangle),$$

$$|\psi_e\rangle = \frac{1}{\sqrt{2}} (|eo\rangle + |oo\rangle),$$

$$|\psi_m\rangle = \frac{1}{\sqrt{2}} (|ee\rangle - |oe\rangle),$$

$$|\psi_f\rangle = \frac{1}{\sqrt{2}} (|eo\rangle - |oo\rangle). \quad (31)$$

The ground states for the two possible $B$ phases are

$$|\psi^{ac}_e\rangle = \bigotimes_i |+\rangle_i, \quad |\psi^{ac}_m\rangle = \bigotimes_i |0\rangle_i. \quad (32)$$

To compute the overlap between $|\psi^{ac}_{e,m}\rangle$ with $|\psi_{1,e,m,f}\rangle$, let $N_v$ be the number of links, $N_v$ the number of vertices and $N_p$ the number of plaquettes of the triangulated torus (see Fig. 9). On the torus whose genus $g = 1$, they satisfy

$$N_f = N_v + N_p. \quad (33)$$
We note that each of $|ee\rangle$, $|eo\rangle$, $|oe\rangle$, and $|oo\rangle$ is an equal-weight superposition of $2^{N_p-1} = 2^{N_l-N_v-1}$ closed string configurations.

Since $|\psi_{ac}\rangle$ correspond to a single no-string configuration, it only overlaps with $|ee\rangle$ (which contains a no-string configuration). We find

$$\langle \psi_{ac}^{\text{ac}} | \psi_{1}\rangle = \frac{1}{\sqrt{2^N}} = 2^{-N/2},$$
$$\langle \psi_{ac}^{\text{ac}} | \psi_{e}\rangle = 0,$$
$$\langle \psi_{ac}^{\text{ac}} | \psi_{m}\rangle = \frac{1}{\sqrt{2^N}} = 2^{-N/2},$$
$$\langle \psi_{ac}^{\text{ac}} | \psi_{f}\rangle = 0.$$  \hfill (34)

After removing the area term $2^{-N/2}$, we see that the wave function overlaps $W_{ac,ac}$ are given by integers $W_{ac,ac} = (1,0,1,0)$, $a = 1,e,m,f$, when the $B$ phase is given by $m$-particle condensation.

Also $|\psi_{ac}\rangle$ have the same overlap $2^{-N_l/2}$ with any string configuration. We find

$$\langle \psi_{ac}^{\text{ac}} | \psi_{1}\rangle = \frac{2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2^{2N_p-1}}} = 2^{N_p-N_l/2} = 2^{-N/2},$$
$$\langle \psi_{ac}^{\text{ac}} | \psi_{e}\rangle = \frac{2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2^{2N_p-1}}} = 2^{N_p-N_l/2} = 2^{-N/2},$$
$$\langle \psi_{ac}^{\text{ac}} | \psi_{m}\rangle = \frac{2^{N_p-1} \times 2^{-N_l/2} - 2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2^{2N_p-1}}} = 0,$$
$$\langle \psi_{ac}^{\text{ac}} | \psi_{f}\rangle = \frac{2^{N_p-1} \times 2^{-N_l/2} - 2^{N_p-1} \times 2^{-N_l/2}}{\sqrt{2^{2N_p-1}}} = 0.$$  \hfill (35)

After removing the area term $2^{-N/2}$, we see that the wave function overlaps $W_{ac,ac}$ are given by integers $W_{ac,ac} = (1,1,0,0)$, $a = 1,e,m,f$, when the $B$ phase is given by $e$-particle condensation.

The above example supports our previous results that the area independent part of wave function overlap on torus is universal and are given by integers.

From a more experimental point of view, suppose there is a small physical system (say realized by a quantum simulator) where we can tune several parameters to force phase transitions, and the wave function overlaps can be measured (by interference for example). If quantization is observed in the wave function overlaps, it may relate to the universal integer part as above, and such quantization is a sign of topological order and anyon condensation.

V. GAPPED BOUNDARIES OF A 2+1D TOPOLOGICAL ORDER

We can use the results developed in this paper to study the gapped boundaries of 2+1D topological order, for example try to find out how many different gapped boundaries a topological order can have. In this section, we study some simple examples.

A. Boundary of $Z_2$ topological order

Let us choose phase $A$ to be $Z_2$ topological order (denoted by $Z_2$) and phase $B$ to be trivial. A domain wall between them is a boundary of $Z_2$ topological order. The modular matrices for the $Z_2$ topological order are given by

$$T_{Z_2} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, \quad S_{Z_2} = \frac{1}{2} \begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 1 & -1 & -1 \\ -1 & -1 & 1 & 1 \\ -1 & -1 & -1 & 1 \end{pmatrix}.$$  \hfill (36)

In the basis $(1,e,m,f)$. The modular matrices for the trivial topological order are $S_B = 1, T_B = 1$. From modular transformation for normalized wave function overlap

$$\tilde{W}_{Z_2,g=1} = \tilde{W}_{Z_2,g=1} S_{Z_2}, \quad \tilde{W}_{Z_2,g=1} = \tilde{W}_{Z_2,g=1} T_{Z_2}.$$  \hfill (37)

we find two types of gapped boundaries, characterized by two integer vector solutions of the above equation:

$$\tilde{W}_{Z_2,g=1} = (1,1,0,0), \quad \tilde{W}_{Z_2,g=1} = (1,0,1,0).$$  \hfill (38)

Next, let us consider the case of a genus-2 manifold $\Sigma_2$:

$$\begin{tikzpicture}[baseline=0pt]
\node (a) at (0,0) {$a_2$};
\node (b) at (1,0) {$a_1$};
\node (c) at (1,1) {$b_1$};
\node (d) at (0,1) {$b_2$};
\node (e) at (-0.5,0.5) {$\gamma$};
\node (f) at (-0.5,0) {$\gamma$};
\draw (a) .. controls (0.5,-0.5) and (1.5,0) .. (b);
\draw (b) .. controls (2.5,0.5) and (1.5,1.5) .. (c);
\draw (c) .. controls (0.5,0.5) and (-1,0) .. (d);
\draw (d) .. controls (-0.5,0) and (-1,1) .. (a);
\end{tikzpicture}$$

For $\text{MCG}(\Sigma_2)$, there are five generators which are Dehn twists along the closed curves $a_1$, $b_1$, $a_2$, $b_2$, and $\gamma$, as shown in (39). Denoting the projective representations of these five Dehn twists as $T_{a_1}, T_{b_1}, T_{a_2}, T_{b_2}$, and $T_{\gamma}$ respectively, we can use $T_{a_1}$ and $T_{b_2}$ to construct $S_i$ matrix that acts on the left (right) half of $\Sigma_2$, with $S_i = T_{b_2} \cdot T_{a_1}^{-1} \cdot T_{b_1} = T_{a_1}^{-1} \cdot T_{b_1} \cdot T_{a_1}^{-1}$. Then we have the following projective representations of the five generators of $\text{MCG}(\Sigma_2)$:

$$T_1, S_1, T_2, S_2, T_3.$$  \hfill (40)

where we have denoted $T_1 := T_{a_1}, T_2 := T_{a_2}$, and $T_3 := T_{\gamma}$. Then based on Eq.(20), we have

$$\tilde{W}_{Z_2,g=2} = \tilde{W}_{Z_2,g=2} \cdot R_{Z_2},$$  \hfill (41)

where $R_{Z_2} = S_1, T_1, S_2, T_2, T_3$. It is noted that for an Abelian theory, the basis in Fig.3 can be represented as

$$\begin{tikzpicture}[baseline=0pt]
\node (i) at (0,0) {$i$};
\node (j) at (1,0) {$j$};
\draw (i) .. controls (0.5,-0.5) and (1.5,0) .. (j);
\end{tikzpicture}$$

That is, the anyon $z$ in Fig.3 corresponds to the identity anyon $1$ now. Then the two anyon loops in (42)
are decoupled. With this basis, it is straightforward to check that for $R_{Z_2} = S_1, T_1, S_2$ and $T_2$, the solutions to Eq.(41) are simply tensor product of genus-1 solutions (see Eq.(26)):

\[
\begin{align*}
\hat{W}_{Z_2,g=2}^{(1)} &= \hat{W}_{Z_2,g=1}^e \otimes \hat{W}_{Z_2,g=1}^e, \\
\hat{W}_{Z_2,g=2}^{(2)} &= \hat{W}_{Z_2,g=1}^e \otimes \hat{W}_{Z_2,g=1}^m, \\
\hat{W}_{Z_2,g=2}^{(3)} &= \hat{W}_{Z_2,g=1}^m \otimes \hat{W}_{Z_2,g=1}^e, \\
\hat{W}_{Z_2,g=2}^{(4)} &= \hat{W}_{Z_2,g=1}^m \otimes \hat{W}_{Z_2,g=1}^m, \\
\end{align*}
\]

where $\hat{W}_{Z_2,g=1}^e$ and $\hat{W}_{Z_2,g=1}^m$ are the genus-1 results as expressed in Eq.(38). However, the solutions $\hat{W}_{Z_2,g=2}^{(2)}$ and $\hat{W}_{Z_2,g=2}^{(3)}$ in (43) are illegal, since both phase $Z_2$ and phase $T_2$ are homogeneous here and we do not consider the case that $e$-condensation and $m$-condensation boundaries coexist. In the following, we show that $\hat{W}_{Z_2,g=2}^{(2)}$ and $\hat{W}_{Z_2,g=2}^{(3)}$ are ruled out by considering $R_{Z_2} = T_5$ in Eq.(41).

For an Abelian theory, since the fusion result of $i \otimes j'$ is unique, the basis in (42) can be rewritten as

\[
\begin{array}{c}
i \\
\downarrow \\
\otimes \\

\downarrow \\
\\
\end{array}
\]

\[
\begin{array}{c}
\rightarrow j' \\
\leftarrow \\
\otimes \\
\leftarrow \\
\end{array}
\]

with $i \otimes j' = z'$. Then acting $T_5$ on the basis in (42) or equivalently (44) results in a phase $\theta z' = \theta z = e^{2\pi i z}$ (see Fig.16). In other words, the matrix $T_5$ in the basis (42) is a diagonal matrix with the diagonal elements being $\theta z'$. For $Z_2$ topological order, we have $\theta_1 = \theta_m = \theta_e = 1$, and $\theta_f = -1$. Given a vector $M_{Z_2,g=2}$, one can find that if the vector element corresponding to $\theta_f = -1$ is non-zero, then it cannot be the solution of $\hat{W}_{Z_2,g=2} = M_{Z_2,g=2} \cdot T_5$ in Eq.(41).

More explicitly, let us take the vector $M_{Z_2,g=2} = 2$ in Eq.(43) for example. Denoting the basis in (42) as

\[
\{ij\} = \{(1, e, m, f) \otimes (1, e, m, f) \}
\]

\[
\{(11, 1e, 1m, 1f; e1, ee, em, ef)\} \\
\{m1, me, mm, mf; f1, fe, fm, ff\}
\]

then $T_5$ is a diagonal matrix of the form

\[
\text{Diag}(T_5) = \begin{pmatrix}
1 & 1 & 1 & -1 \\
1 & 1 & -1 & 1 \\
1 & -1 & 1 & 1 \\
-1 & 1 & 1 & 1
\end{pmatrix}
\]

and

\[
\hat{W}_{Z_2,g=2}^{(2)} = \begin{pmatrix}
1 & 1 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}
\]

One can check explicitly that $\hat{W}_{Z_2,g=2}^{(2)} \neq \hat{W}_{Z_2,g=2} \cdot T_5$.

Similarly, one can find that $\hat{W}_{Z_2,g=2}^{(3)} \neq \hat{W}_{Z_2,g=2} \cdot T_5$, $\hat{W}_{Z_2,g=2}^{(1)} = \hat{W}_{Z_2,g=2} \cdot T_5$, and $\hat{W}_{Z_2,g=2}^{(4)} = \hat{W}_{Z_2,g=2} \cdot T_5$.

That is, only $\hat{W}_{Z_2,g=2}^{(1)}$ and $\hat{W}_{Z_2,g=2}^{(4)}$ in Eq.(43) are the true solutions of Eq.(41) by considering all the five generators $R_{Z_2} = T_1, S_1, T_2, S_2$ and $T_5$.

B. Boundary of $S_3$ topological order

$S_3$ topological order (described by quantum double of finite group $S_3$ with fusion rule given by Table I) is more interesting, since it is a non-Abelian theory. Let us choose phase $A$ to be $S_3$ topological order (denoted as $S_3$) and phase $B$ to be trivial (denoted as $I$). The modular matrices for the $S_3$ topological order are given by (with basis $(1, a^1, a^2, b, b^1, b^2, c, c^1)$)

\[
\text{Diag}(T) = \begin{pmatrix}
1 & 1 & 1 & 1 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2 \\
1 & 2 & 2 & 2
\end{pmatrix}
\]

\[
S = \begin{pmatrix}
1 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1 \\
0 & 1 & 1
\end{pmatrix}
\]

Eqn. (13) has five solutions with $M_{S_3}^{11} = 1$:

\[
\begin{array}{c}
\hat{W}_{S_3,g=1}^{(1)} = (1 0 0 1 0 0 1 0), \\
\hat{W}_{S_3,g=1}^{(2)} = (1 0 1 0 0 0 1 0), \\
\hat{W}_{S_3,g=1}^{(3)} = (1 1 0 0 0 0 0 1), \\
\hat{W}_{S_3,g=1}^{(4)} = (1 1 0 0 0 0 0 1), \\
\hat{W}_{S_3,g=1}^{(5)} = (1 1 1 0 0 0 0 1).
\end{array}
\]

It is found that the last solution does not satisfy the stable condition in eqn. (14), i.e.

\[
(\hat{W}_{S_3,g=1}^{(5)})^a (\hat{W}_{S_3,g=1}^{(5)})^b \preceq \sum_i N_i^a b_i (\hat{W}_{S_3,g=1}^{(5)})^i.
\]

So the $S_3$ topological order has only 4 types of gapped boundaries.

In fact, as discussed in the following, without resorting to the stable condition in (14), we show that the fake solution $\hat{W}_{S_3,g=1}^{(5)}$ can be ruled out by the genus-2 gapping boundary condition in Eq.(20), i.e.,

\[
\hat{W}_{S_3,g=2} = \hat{W}_{S_3,g=2} \cdot R_{S_3},
\]

which is a linear condition. Since the $S_3$ topological order is multiplicity-free (i.e., $N_k^a \leq 1$), we denote the component of $\hat{W}_{S_3,g=2}$ as $\hat{W}_{S_3,g=2}^{T_3}$. (See also Eq.(19)). Here the anyon types $i, j, z$ are used to label the basis vectors in the Hilbert space of degenerate ground states on a genus-2 manifold $\Sigma_2$ (see Fig.3). It is noted that both $\hat{W}_{S_3,g=2}^{T_3}$ and the projective representations $\hat{R}_{S_3}^{T_3}$ of $\text{MCG}(\Sigma_2)$ depend on the choice of basis vectors (see Appendix D).

In the following discussion, we use the basis vectors in Fig.3.

Similar to the previous subsection on $Z_2$ topological order, we check whether the genus-1 solutions in Eq.(50)
 TABLE I. Fusion rules $N$ab of 2+1D $S_3$ topological order. Here $b$ and $c$ correspond to pure flux excitations, $a^1$ and $a^2$ pure charge excitations, 1 the vacuum sector while $b^1$, $b^2$, and $c^1$ are charge-flux composites.

can be embedded in the genus-2 solutions of Eq. (52). Our logic is as follows: First, it is straightforward to find that

$$W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j} \quad (i, j = 1, 2, 3, 4)$$

each solutions $W_{S_3,g=2}^{i,j}$ of Eq. (52) with $z = 1$ if we simply consider $R_{S_3} = T_1, S_1, T_2, S_2$. Second, we need to check if $W_{S_3,g=1}^{i,j} \oplus W_{S_3,g=1}^{i,j}$ are solutions of Eq. (52) for $R_{S_3} = T_3$. In general, the operation $R_{S_3} = T_3$ will mix the components $W_{S_3,g=1}^{i,j} \oplus W_{S_3,g=1}^{i,j}$, and it is not apparent that $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$ are solutions of Eq. (52) for $R_{S_3} = T_3$. In this case, a careful study of Eq. (52) is needed. Third, we need to solve for the components $W_{S_3,g=2}^{i,j}$ with $z \neq 1$ in Eq. (52).

In the following, we solve all the components $W_{S_3,g=2}^{i,j}$ that are solutions of Eq. (52). The results can be mainly summarized as follows:

1. It is found there are in total 4 sets of independent solutions of the genus-2 condition in Eq. (52), which we denote as $W_{S_3,g=2}^{i,j}$, with $i = 1, 2, 3, 4$. They embed the genus-1 solutions of the form $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$ with $i = 1, 2, 3, 4$. Other pairings of $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$ (including $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$) are ruled out by Eq. (52), as summarized in Table II.

2. For the 4 sets of independent solutions $W_{S_3,g=2}^{i,j}$ with $i = 1, 2, 3, 4$, all the components can be uniquely determined as follows (see Appendix D for details):

- For $W_{S_3,g=2}^{i,j}$, which embeds the genus-1 solution of the form $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$, the condensed anyons are pure flux $1, b$, and $c$. With the genus-2 condition in Eq. (52), one can obtain the nonzero components of $W_{S_3,g=2}^{i,j,z}$ with $z \neq 1$ as

$$\begin{cases}
W_{S_3,g=2}^{i,j,z} = W_{S_3,g=2}^{i,j,z} = 1, \\
W_{S_3,g=2}^{i,j,z} = \sqrt{2}, \\
W_{S_3,g=2}^{i,j,z} = \frac{1}{\sqrt{2}}.
\end{cases}$$

The nonzero components $W_{S_3,g=2}^{i,j,z}$ with $z = 1$ can be expressed as the product of genus-1 results as

$$W_{S_3,g=2}^{i,j,z} = (W_{S_3,g=1}^{i,j})^i \cdot (W_{S_3,g=1}^{i,j})^j, \quad i, j \in \{1, 2, c\}.$$

For all the non-zero components $W_{S_3,g=2}^{i,j,z}$, one can find that $i, j, z \in \{1, 2, c\}$.

- For $W_{S_3,g=2}^{i,j}$, which embeds the genus-1 solution of the form $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$, the condensed anyons are $1, a^2$, and $c$. One can obtain the nonzero component of $W_{S_3,g=2}^{i,j}$ with $z \neq 1$ as

$$\begin{cases}
W_{S_3,g=2}^{i,j,z} = W_{S_3,g=2}^{i,j,z} = -1, \\
W_{S_3,g=2}^{i,j,z} = \sqrt{2}, \\
W_{S_3,g=2}^{i,j,z} = \frac{1}{\sqrt{2}}.
\end{cases}$$

The nonzero components $W_{S_3,g=2}^{i,j,z}$ with $z = 1$ can be expressed as the product of genus-1 results as

$$W_{S_3,g=2}^{i,j,z} = (W_{S_3,g=1}^{i,j})^i \cdot (W_{S_3,g=1}^{i,j})^j, \quad i, j \in \{1, 2, c\}.$$

For all the non-zero components $W_{S_3,g=2}^{i,j,z}$, one has $i, j, z \in \{1, a^2, c\}$.

- For $W_{S_3,g=2}^{i,j}$, which embeds the genus-1 solution of the form $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$, the condensed anyons are $1, a^1$, and $b$. As studied in Appendix D, all the components $W_{S_3,g=2}^{i,j,z}$ with $z \neq 1$ are zero. The only non-zero components can be considered as the product of genus-1 results as $W_{S_3,g=2}^{i,j,z} = (W_{S_3,g=1}^{i,j})^i \cdot (W_{S_3,g=1}^{i,j})^j, \quad i, j \in \{1, a^1, b\}$.

- For $W_{S_3,g=2}^{i,j}$, which embeds the genus-1 solution of the form $W_{S_3,g=1}^{i,j} \otimes W_{S_3,g=1}^{i,j}$, the condensed anyons are pure charges with $1, a^1$, and $a^2$. Similar
to the case of \( \tilde{W}^{(3)}_{S_3,g=2} \), all the components \( \tilde{W}^{i,j,z}_{S_3,g=2} \) with \( z \neq 1 \) are zero, and the only non-zero components correspond to the product of genus-1 results as \( \tilde{W}^{i,j,z}_{S_3,g=2} = (\tilde{W}^{(4)}_{S_3,g=1})^i \cdot (\tilde{W}^{(4)}_{S_3,g=1})^j \), where \( i, j \in \{1, a^1, a^2\} \).

1. Rule out the fake solution with genus-2 condition

As an illustration of how to determine the solutions of the genus-2 condition in Eq. (52), here we give an example on how \( \tilde{W}^{(5)}_{S_3,g=1} \otimes \tilde{W}^{(5)}_{S_3,g=1} \) is ruled out. The details of finding all the solutions of Eq. (52) can be found in Appendix D.

It is convenient to consider the following two choices of basis vectors:

\[
\begin{align*}
\text{basis I:} & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{basis1.png}
\end{array} \\
\text{basis II:} & \quad \begin{array}{c}
\includegraphics[width=0.2\textwidth]{basis2.png}
\end{array}
\end{align*}
\]

The concrete expression of Eq. (52) will depend on the choices of basis vectors. For example, choosing \( R = T_5 \) in Eq. (52), \( T_5 \) is a diagonal matrix in the basis II, but in general not diagonal in basis I. Denoting the wave function on a genus-2 manifold after anyon condensation as \( |\Psi_{ac}\rangle \), then the wavefunction overlap \( \tilde{W}^{i,j,z}_{S_3,g=2} \) in Eq. (52) depends on the choice of bases as follows

\[
\tilde{W}^{i,j,z}_{S_3,g=2} = \langle \Psi_{ac}\rangle |\Psi_{i,j,z}\rangle,
\]

(55)

Suppose \( \tilde{W}^{(5)}_{S_3,g=1} \otimes \tilde{W}^{(5)}_{S_3,g=1} \) is the solution of Eq. (52), then based on the expression in (50), we have \( \tilde{W}^{1,a^2,b,z}_{S_3,g=2} = 1 \). In the following, we will show that the genus-2 condition imposes that \( \tilde{W}^{1,a^2,b,z}_{S_3,g=2} = 0 \), and therefore \( \tilde{W}^{(5)}_{S_3,g=1} \otimes \tilde{W}^{(5)}_{S_3,g=1} \) cannot be the solution of Eq. (52).

To study the effect of \( T_5 \), it is convenient to consider the basis vectors II in (54). Let us focus on the components with \( i = a^2 \) and \( j = b \). (Recall that \( T_5 \) does not change \( i \) and \( j \).) Considering the fusion rule \( a^2 \otimes b = b_1 \oplus b_2 \), then in the basis vectors \( |\psi_{1,a^2,b,z}\rangle \), the anyon type \( z \) can only be chosen as \( b_1 \) or \( b_2 \), which have non-trivial topological spins [see (48)]. Then, considering \( R_{S_3} = T_5 \) in Eq. (52), we have

\[
\tilde{W}^{1,a^2,b,z}_{S_3,g=2} = \tilde{W}^{1,a^2,b,z}_{S_3,g=2} = 0.
\]

By inserting a complete set of basis vectors I in the expression (55), \( \tilde{W}^{1,a^2,b,z} \) can be expressed as

\[
\tilde{W}^{1,a^2,b,z} = \tilde{W}^{1,a^2,b,z}_{S_3,g=2} \cdot \tilde{W}^{1,a^2,b,z}_{S_3,g=2} = 0.
\]

Furthermore, since \( \langle \psi_{1,a^2,b,1} | \tilde{W}^{1,a^2,b} | \psi_{1,a^2,b} \rangle = \sqrt{d_{a^2} d_b d_{\bar{b}}}/D \) and \( \langle \psi_{1,a^2,b,1} | \tilde{W}^{1,a^2,b} | \psi_{1,a^2,b} \rangle = \sqrt{d_{a^2} d_b d_{\bar{b}}}/D \), which are nonzero. Then based on Eq. (57), one can immediately obtain

\[
\tilde{W}^{1,a^2,b,z}_{S_3,g=2} = 0.
\]

(58)

On the other hand, we know that if \( \tilde{W}^{(5)}_{S_3,g=1} \otimes \tilde{W}^{(5)}_{S_3,g=1} \) is the solution of Eq. (52), then we have \( \tilde{W}^{1,a^2,b,z}_{S_3,g=2} = 1 \), which contradicts with (58). Therefore, \( \tilde{W}^{(5)}_{S_3,g=1} \otimes \tilde{W}^{(5)}_{S_3,g=1} \) is ruled out as the solution of genus-2 condition in Eq. (52).

One can refer to Appendix D for all the solutions of Eq. (52) for \( S_3 \) topological order.

VI. SUMMARY

In this paper, we develop a systematic approach to the gapped domain walls between two topological orders \( \mathcal{A} \) and \( \mathcal{B} \). If \( \mathcal{B} \) is the trivial topological order, the domain wall becomes the boundary of topological order \( \mathcal{A} \). Our systematic approach is based on the topological partition function \( W_{\mathcal{A},\mathcal{B}}^{I_1A} \) of the domain wall \( \Sigma_g \), which is a Riemann surface of arbitrary genus \( g \), which is a multi-component partition function labeled by \( I_1, I_8 \). The multi-component partition function \( W_{\mathcal{A},\mathcal{B},g}^{I_1A} \) can also be viewed as the overlap of the degenerate ground states of \( \mathcal{A} \) and \( \mathcal{B} \) on \( \Sigma_g \), where \( I_4 \) labels the ground states of topological order \( \mathcal{A} \) on \( \Sigma_g \). This allows us to derive the following linear conditions on the data \( W_{\mathcal{A},\mathcal{B},g}^{I_1A} \) that characterized the domain walls:

\[
R_{\mathcal{A}}^{I_8, I_5} W_{\mathcal{A},\mathcal{B},g}^{I_1A} = W_{\mathcal{A},\mathcal{B},g}^{I_1A} R_{\mathcal{A}}^{I_8, I_5},
\]

(59)

where \( R_{\mathcal{A}} \) (\( R_{\mathcal{B}} \)) is the mapping-class-group representation for topological order \( \mathcal{A} \) (topological order \( \mathcal{B} \)) for
genus-$g$ Riemann surface. Eqn. (59) is a special case of a more general condition proposed in Ref. 40, for the partition function with non-invertible gravitational anomaly.

In this paper, through some simple examples, we try to demonstrate the validity of the condition (59) (and the condition in Ref. 40), by showing that the condition gives rise to a classification of gapped domain walls between two topological orders. In particular, we show that the condition in Ref. 40), by showing that the condition gives rise to a classification of gapped domain walls between two topological orders.

At moment, we do not known if the topological partition functions $W_{B_{A,g}}^{I_{a,b}I_{A}}$ for arbitrary genus-$g$ surface can fully characterize the gapped domain wall or not (although we think that, very likely, they do). In Appendix E, using the connection between anyon condensation and domain walls, we develop a classifying theory of gapped domain wall based on the structure coefficients $M_{k,v}^{c,u}$ that describe the condensable algebra in a topological order. We also give the conditions eqn. (E21), eqn. (E26), and eqn. (E28) on the structure coefficients $M_{k,v}^{c,u}$, so that they can described a gapped domain wall. However, those conditions are non-linear and is very hard to solve. Try to gain a deeper understanding of the two approaches may help us to find an easier way to fully classify gapped domain walls.

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Appendix A: Topological path integral on a space time with domain walls and world lines

1. Space-time lattice and branching structure

To find the conditions on the domain-wall data, we need to use extensively the space-time path integral. So we will first describe how to define a space-time path integral. We will first describe how to define a space-time path integral. The first approach is based on topological partition function $W_{B_{A,g}}^{I_{a,b}I_{A}}$ for genus-1 surface, plus the linear condition (59), is not enough.[34] We need to, at least, use the topological partition function $W_{B_{A,2}}^{I_{a,b}I_{A}}$ for genus-2 surface and its condition (59), to obtain a correct classification of gapped domain walls.

At moment, we do not known if the topological partition functions $W_{B_{A,g}}^{I_{a,b}I_{A}}$ for arbitrary genus-$g$ surface can fully characterize the gapped domain wall or not (although we think that, very likely, they do). In Appendix E, using the connection between anyon condensation and domain walls, we develop a classifying theory of gapped domain wall based on the structure coefficients $M_{k,v}^{c,u}$ that describe the condensable algebra in a topological order. We also give the conditions eqn. (E21), eqn. (E26), and eqn. (E28) on the structure coefficients $M_{k,v}^{c,u}$, so that they can described a gapped domain wall. However, those conditions are non-linear and is very hard to solve. Try to gain a deeper understanding of the two approaches may help us to find an easier way to fully classify gapped domain walls.

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FIG. 10. A 2-dimensional complex. The vertices (0-simplices) are labeled by $i$. The edges (1-simplices) are labeled by $\langle ij \rangle$. The faces (2-simplices) are labeled by $\langle ijk \rangle$. The degrees of freedoms may live on the vertices (labeled by $v_i$), on the edges (labeled by $e_{ij}$) and on the faces (labeled by $\phi_{ijk}$).

FIG. 11. (Color online) Two branched simplices with opposite orientations. (a) A branched simplex with positive orientation and (b) a branched simplex with negative orientation.
complex (see Fig. 10).

2. Discrete path integral

In this paper, we will only consider a type of 2+1D path integral that can be constructed from a tensor set \( T \) of two real and one complex tensors: \( T = (w_{01}, e_{01}, C_{v_0 v_1 v_2 v_3; v_0 v_1 v_2 v_3}) \). The complex tensor \( C_{v_0 v_1 v_2 v_3; v_0 v_1 v_2 v_3} \) can be associated with a tetrahedron, which has a branching structure (see Fig. 12). A branching structure is a choice of orientation of each edge in the complex so that there is no oriented loop on any triangle (see Fig. 12). Here the \( v_0 \) index is associated with the vertex-0, the \( e_01 \) index is associated with the edge-01, and the \( \phi_{012} \) index is associated with the triangle-012. They represent the degrees of freedom on the vertices, edges, and the triangles.

Using the tensors, we can define the path integral on any 3-complex that has no boundary:

\[
Z(M^3) = \sum_{001 \ldots} \prod_{v_01 \ldots} w_{01} \prod_{e_01 \ldots} e_{01} \prod_{\text{tetra}} \prod_{v_0123} C_{v_0 v_1 v_2 v_3; v_0 v_1 v_2 v_3}^{v_0123},
\]

where \( \sum_{001 \ldots} \) only sums over all the vertex indices, the edge indices, and the face indices, \( s_{0123} = 1 \) or \( s_{0123} = 0 \) depending on the orientation of tetrahedron (see Fig. 12). We believe such type of path integral can realize any 2+1D topological order.

3. Path integral on space-time with natural boundary

On the complex \( M^3 \) with boundary: \( B^2 = \partial M^3 \), the partition function is defined differently:

\[
Z(M^3) = \sum_{\{v_i; e_{ij}; \phi_{ijk}\}} \prod_{\text{vertex} \notin B^2} w_{01} \prod_{\text{edge} \notin B^2} d_{e_01} \prod_{\text{tetra}} \prod_{v_0123} C_{v_0 v_1 v_2 v_3; v_0 v_1 v_2 v_3}^{v_0123},
\]

where \( \sum_{v_i; e_{ij}; \phi_{ijk}} \) only sums over the vertex indices, the edge indices, and the face indices that are not on the boundary. The resulting \( Z(M^3) \) is actually a complex function of \( v_i \)'s, \( e_{ij} \)'s, and \( \phi_{ijk} \)'s on the boundary \( B^2 \): \( Z(M^3; \{v_i; e_{ij}; \phi_{ijk}\}) \). Such a function is a vector in the vector space \( V_{B^2} \). (The vector space \( V_{B^2} \) is the space of all complex function of the boundary indices on the boundary complex \( B^2 \): \( \Psi(\{v_i; e_{ij}; \phi_{ijk}\}) \). We denote such a vector as \( |\Psi(M^3)\rangle \) (boundary) are attached with the tensors \( w_{v_i} \) and \( d_{e_01} \). The boundary (A2) defined above is called a natural boundary of the path integral.

We also note that only the vertices and the edges in the bulk (i.e., not on the But when we glue two boundaries together, those tensors \( w_{v_i} \) and \( d_{e_01} \) are added back. For example, let \( M^3 \) and \( N^3 \) to have the same boundary (with opposite orientations)

\[
\partial M^3 = -\partial N^3 = B^2
\]

which give rise to wave function on the boundary \( |\Psi(M^3)\rangle \) and \( |\Psi(N^3)\rangle \) after the path integral in the bulk. Ghuing two boundaries together is like doing the inner product \( \langle \Psi(N^3)|\Psi(M^3)\rangle \). So the tensors \( w_{v_i} \) and \( d_{e_01} \) defines the inner product in the boundary Hilbert space \( V_{B^2} \). Therefore, we require \( w_{v_i} \) and \( d_{e_01} \) to satisfy the following unitary condition

\[
w_{v_i} > 0, \quad d_{e_01} > 0.
\]

4. Topological path integral

We notice that the above path integral is defined for any space-time lattice. The partition function \( Z(M^3) \) depends on the choices of of the space-time lattice. For example, \( Z(M^3) \) depends on the number of the cells in space-time, which give rise to the leading volume dependent term, in the large space-time limit (i.e. the thermodynamic limit)

\[
Z(M^3) = e^{-\epsilon V} Z_{\text{top}}(M^3)
\]

where \( V \) is the space-time volume, \( \epsilon \) is the energy density of the ground state, and \( Z_{\text{top}}(M^3) \) is the volume independent partition function. It was conjectured that the volume independent partition function \( Z_{\text{top}}(M^3) \) in the thermodynamic limit, as a function of closed space-time \( M^3 \), is a topological invariant that can fully characterize topological order.[9, 39] So it is very desirable to fine tune the path integral to make the energy density \( \epsilon = 0 \). This can be achieved by fine tuning the tensors \( w_{v_i} \) and \( d_{e_01} \). But we can also choose the tensor \( (w_{v_0}, d_{e_01}, C_{v_0 v_1 v_2 v_3; v_0 v_1 v_2 v_3}) \) to be the fixed-point tensor-set under the renormalization group flow of the tensor network.[48, 49] In this case, not only
the volume factor $e^{-\epsilon V}$ disappears, the volume independent partition function $Z_{\text{top}}(M^3)$ is also re-triangulation invariant, for any size of space-time lattice. In this case, we refer the path integral as a topological path integral, and denote the resulting partition function as $Z_{\text{top}}(M^3)$. $Z_{\text{top}}$ is also referred as the volume independent the partition function, which is a very important concept, since only volume independent the partition functions correspond to topological invariants. In particular, it was conjectured that such kind of topological path integrals describe all the topological order with gappable boundary. For details, see Ref. 9 and 39.

The invariance of partition function $Z$ under the re-triangulation in Fig. 13 and 14 requires that

$$
\sum_{\phi_{123}} \sum_{\phi_{012}} \sum_{\phi_{134}} \sum_{\phi_{234}} C_{\phi_{012}\phi_{123}\phi_{134}\phi_{234}} = \sum_{\phi_{012}} \sum_{\phi_{123}} \sum_{\phi_{013}} \sum_{\phi_{134}} \sum_{\phi_{234}} C_{\phi_{012}\phi_{123}\phi_{013}\phi_{134}\phi_{234}} \quad (A6)
$$

where $i,j,k,\cdots \in \{1,2,\cdots,N\}$ label the type of topological excitations, and $\alpha, \beta, \gamma$ label the fusion channels (i.e. different choice of actions at the junction of three world-lines). Again, the world line is defined via a different tensor set for the simplexes that touch the world line. We can choose the world line tensors to make the partition function with world line (after summing over the bulk and domain wall degrees of freedom) to be re-triangulation invariant (even for the re-triangulations that involve the domain wall). Therefore, the topological path integral can also be defined for spacetime with domain walls. Different choices of domain wall tensors give rise to different domain walls. Those domain walls can be characterized by the data introduced in Section II.

To find the conditions on those domain-wall data, we also need to use the space-time path integral with world-lines of topological excitations. We denote the resulting partition function as

$$
Z_{\text{top}} \left( \begin{array}{c} i \\ j \\ k \\ m \\ n \\ \alpha \\ \beta \\ \gamma \\ \epsilon \end{array} \right), \quad (A8)
$$

5. Topological path integral with domain walls and world lines

When the spacetime $M^3$ has a domain wall in it, we can have different tensor sets, $(w_{A,i}^j, d_A^{j_{01}^1}, C_{A,i^1j^2_kl}, d_{B,012}^{j_{01}^1}, C_{B,i^1j^2_kl}, d_{C,012}^{j_{01}^1})$, and $(w_{B,i}^j, d_B^{j_{01}^1}, C_{B,i^1j^2_kl}, d_{C,012}^{j_{01}^1}, C_{C,i^1j^2_kl}, d_{C,012}^{j_{01}^1})$, on the two sides of the domain wall. Here we will assume that the two tensor sets define topological path integrals in the bulk. The domain wall is defined via a different tensor set for the simplexes that touch the domain wall. Again we can choose the domain wall tensors to make the partition function with domain wall (after summing over the bulk and domain wall degrees of freedom) to be re-triangulation invariant (even for the re-triangulations that involve the domain wall). Therefore, the topological path integral can also be defined for spacetime with domain walls. Different choices of domain wall tensors give rise to different domain walls. Those domain walls can be characterized by the data introduced in Section II.

To find the conditions on those domain-wall data, we also need to use the space-time path integral with world-lines of topological excitations. We denote the resulting partition function as

$$
Z_{\text{top}} \left( \begin{array}{c} i \\ j \\ k \\ m \\ n \\ \alpha \\ \beta \\ \gamma \\ \epsilon \end{array} \right), \quad (A8)
$$

where $i,j,k,\cdots \in \{1,2,\cdots,N\}$ label the type of topological excitations, and $\alpha, \beta, \gamma$ label the fusion channels (i.e. different choice of actions at the junction of three world-lines). Again, the world line is defined via a different tensor set for the simplexes that touch the world line. We can choose the world line tensors to make the partition function with world line to be re-triangulation invariant (even for the re-triangulations that involve the world line). Therefore, the topological path integral can also be defined for spacetime with world lines. Different choices of world line tensors give rise to different world lines, which are labeled by the types of topological excitations. In this paper, we will only consider those topological path integrals with re-triangulation invariance.

Appendix B: Categorical approach to evaluate topological path integral with world lines

1. Planar world-lines and unitary m-fusion category

The topological path integrals with world lines (i.e. the re-triangulation invariant path integrals with world lines) $Z_{\text{top}}$ can be computed via an algebraic approach (or more precisely an categorical approach). In the following, we will give a brief introduction of such an approach. More details can be found in Ref. 51.
First, let us consider the partition functions with only planar world-line configurations. In this case, we may pretend the space-time to be 2-dimensional (or the space-time is actually 2-dimensional).

The partition functions with different world-line configurations can be related by some linear relations. For example

\[
Z^{\text{top}} \left( \begin{array}{ccc}
i & j & k \\ m & n & l \end{array} \right) = \sum_{\chi \delta} F_{kln, \chi \delta}^{ijm, \alpha \beta} Z^{\text{top}} \left( \begin{array}{ccc}
i & j & k \\ n & \chi & \delta \end{array} \right). \tag{B1}
\]

This is because the above partition functions describe the amplitude of fusion type-\(i, j, k\) topological excitations into degenerate type-\(l\) topological excitations. The two sides of the equation just correspond to different order of fusion which give rise to the same end product. Thus those amplitudes are related.

Let us consider the fusion of two topological excitations of type-\(i, j\). From far away, the two topological excitations may be viewed as spin-0 and spin-1 excitations which happen to have the same energy. So to describe the fusion type-\(i, j\) topological excitations produce \(N_{ij}^{\text{top}}\) type-\(i, j\) topological excitations that happen to have the same energy. For example, two spin-1/2 excitations can be viewed as spin-0 and spin-1 excitations which happen to have the same energy. Therefore, we have

\[
F_{ijm, \alpha \beta}^{kln, \gamma \delta} = 0 \quad \text{when} \quad N_{ij}^{kl} = 0 \text{ or } N_{ij}^{ln} = 0. \tag{B2}
\]

When \(N_{ij}^{kl} = 0\) or \(N_{ij}^{ln} = 0\), the left-hand-side of eqn. (B1) is always zero. Thus \(F_{ijm, \alpha \beta}^{kln, \gamma \delta} = 0\) when \(N_{ij}^{kl} = 0\) or \(N_{ij}^{ln} = 0\). When \(N_{ij}^{kl} = 0\) or \(N_{ij}^{ln} = 0\), amplitude on the right-hand-side of eqn. (B1) is always zero. So we can choose \(F_{ijm, \alpha \beta}^{kln, \gamma \delta} = 0\) when \(N_{ij}^{kl} = 0\) or \(N_{ij}^{ln} = 0\).

For fixed \(i, j, k, l\), the fusion type-\(i, j, k\) topological excitations produce \(N_{ij}^{kl}\) type-\(l\) topological excitations that happen to have the same energy. We have

\[
N_{ij}^{kl} = \sum_m N_{ij}^{lm} N_{mk} = \sum_n N_{in}^{ij} N_{nj}^{jk}. \tag{B3}
\]

Thus the matrix \(F_{ij}^{kl}\) with matrix elements \((F_{ij}^{kl})^{m, \alpha \beta}_{n, \chi \delta} = F_{kln, \chi \delta}^{ijm, \alpha \beta}\) is a matrix of dimension \(N_{ij}^{kl} \times N_{ij}^{lk}\). The matrix describe the relation between different orders of fusions same set of degenerate \(l\) particles. Thus the matrix \(F_{ij}^{kl}\) must be unitary:

\[
\sum_{n, \chi, \delta} F_{kln, \chi \delta}^{ijm', \alpha' \beta'} (F_{kln, \chi \delta}^{ijm, \alpha \beta})^* = \delta_{m', m} \delta_{\alpha', \alpha} \delta_{\beta', \beta}. \tag{B4}
\]

The second type of linear relations re-express the amplitude for \(l \rightarrow i, j, k\) in terms of the amplitude for \(i, j\):

\[
Z^{\text{top}} \left( \begin{array}{ccc}
i & j & k \\ m & n & l \end{array} \right) = O_{ij}^{k, \alpha \beta} \delta_{\mu \nu} Z^{\text{top}} \left( \begin{array}{ccc}
i & j & k \\ l & \mu & \nu \end{array} \right). \tag{B5}
\]

We call such local change of graph an O-move.

For fixed \(i, j\), \(l \rightarrow k\) for different \(\alpha \beta\) describe all the possible processes. So the amplitude for \(l \rightarrow i, j\) can be expressed in terms of the amplitudes of \(i, j \rightarrow l\):

\[
\sum_{k, \alpha \beta} Y_{ij}^{kl, \alpha \beta} Z^{\text{top}} \left( \begin{array}{ccc}
i & j & k \\ i & j & l \end{array} \right) = Z^{\text{top}} \left( \begin{array}{ccc}
i & j & l \\ i & j & l \end{array} \right). \tag{B6}
\]

We will call such a local change as a Y-move. We can choose

\[
Y_{ij}^{kl, \alpha \beta} = 0, \quad \text{if } N_{ij}^{kl} < 1. \tag{B7}
\]

We can adjust the action at the triple world-line junction to simplify \(O_{ij}^{k, \alpha \beta}\) and \(Y_{ij}^{kl, \alpha \beta}\). After the simplification,

\[
O_{ij}^{k, \alpha \beta} = \sqrt{\frac{d_i d_j}{d_k}} \delta_{i \beta}, \quad Y_{ij}^{kl, \alpha \beta} = \frac{d_k}{d_i d_j} \delta_{i \beta}, \quad d_i > 0, \tag{B8}
\]

where \(\delta_{i \beta} = 1\) for \(N_{ij}^{k} > 0\) and \(\delta_{i \beta} = 0\) for \(N_{ij}^{k} = 0\). Also \(d_i\) are the real and positive solutions from

\[
\sum_{ij} d_i d_j N_{ij}^{kl} = d_k D^2, \quad D \equiv \sqrt{\sum_i d_i^2}, \tag{B9}
\]

which are called the quantum dimensions of type-\(k\) topological excitation.

We see that the partition function \(A(X)\) for any world-line configurations can be characterized by tensor data \((N, N_{ij}^{kl}, F_{kln, \gamma \lambda})\). However, only certain tensor data \((N, N_{ij}^{kl}, F_{kln, \gamma \lambda})\), that satisfy some conditions can self-consistently describe partition function \(A(X)\). Those conditions form a set of non-linear equations whose variables are \(N_{ij}^{kl}, F_{kln, \gamma \lambda}, d_i\) (where \(d_i\) can be determined by \(N_{ij}^{kl}\) alone):

\[
\sum_{m=0}^N N_{ij}^{lm} N_{mk} = \sum_{n=0}^N N_{ij}^{nk} N_{ni}^{mn}. \quad \text{(B10)}
\]
\[
\sum_{jk} (N^{jk}_i)^2 \geq 1;
\]  
(B10)

\[
\sum_{nxy} F_{kln,\chi}^{ijm,\alpha \beta} (F_{kln,\chi}^{ijm,\alpha \beta})^* = \delta_{m,m'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'},
\]
\[
F_{kln,\chi}^{ijm,\alpha \beta} = 0 \text{ when } N^{ij}_m \leq 1 \text{ or } N^m_{kl} \leq 1 \text{ or } N^k_{ij} \leq 1 \text{ or } N^i_{ln} \leq 1,
\]
\[
\sum_i \sum_{\chi} F_{kl1,\chi}^{ijm,\alpha \beta} F_{kl1,\chi}^{ijm,\alpha \beta} = \sum_{\chi} F_{kl1,\chi}^{ijm,\alpha \beta} F_{kl1,\chi}^{ijm,\alpha \beta},
\]
\[
\sum_{i,j} d_i d_j N^{ij}_k = d_k D^2, \quad D = \sqrt{\sum_i d_i^2};
\]  
(B12)

\[
\sum_{nxy} F_{jnl,\beta}^{kmi,\alpha} (F_{jnl,\beta}^{kmi,\alpha})^* = \frac{d_i d_j}{d_m} \delta_{m,m'} \delta_{\alpha,\alpha'} \delta_{\beta,\beta'},
\]  
(B13)

We like to mention that, in the above we did not assume the existence of trivial particle, which fuse with other particles as an identity. We will call such kind of fusion as unitary m-fusion category.

As an application of the above algebraic structure – the unitary m-fusion category, let us consider two world-lines of type-i and type-j, wrapping around a torus \(S^1_2 \times S^1_2\) in the \(x\)-direction. Let \(W_i^x\) and \(W_j^x\) be the string operators that creates the world-lines. Applying the \(Y\)-move and then the \(O\)-move, and using eqn. (B8) (see Fig. 15), we find that

\[
W_i^x W_j^x = \sum_{k=1}^{N} \sum_{\alpha=1}^{N} Y_{k,\alpha}^{ij} O_{k,\alpha}^{ij} W_k^x
\]

We see that the algebra of the loop operator \(W_i^j\) forms a representation of fusion algebra \(i \otimes j = \sum_k N^{ij}_k W_k^x\).

2. Presence of trivial particle and unitary fusion category

Now let us assume such a trivial particle type to exist, and denoted it by \(1\), which satisfies the following fusion rule

\[
i \otimes 1 = 1 \otimes i = i.
\]  
(B15)

Thus \(N^{ij}_k\) satisfies

\[
N^{11}_j = N^{11}_j = \delta_{ij},
\]  
(B16)

We also requires that for every \(i\) there exists a unique \(\bar{i}\) such that

\[
i = i, \quad \bar{1} = 1, \quad \bar{N}^{ij}_k = \delta_{\bar{i} \bar{j}}
\]  
(B17)

We can represent a type-\(1\) string by a dash line. By examine the \(O\)-move with \(k = 1\):

\[
Z^{\text{top}} \left( \begin{array}{c} \bar{i} \\ \bar{1} \end{array} \right) = Z^{\text{top}} \left( \begin{array}{c} \bar{i} \\ \bar{1} \end{array} \right).
\]  
(B18)

We see that we can remove or add any vertex with dash line without changing \(Z^{\text{top}}\).

With the presence of trivial particle type, we can determine the amplitude for a loop of \(i\)-string. Using the rule of adding dash lines (the trivial strings) and \(O\)-move eqn. (B5), we find

\[
Z^{\text{top}} \left( \begin{array}{c} i \\ \bar{i} \end{array} \right) = Z^{\text{top}} \left( \begin{array}{c} i \end{array} \right) = O_i^{\bar{i}} Z^{\text{top}} \left( \begin{array}{c} \bar{i} \end{array} \right).
\]  
(B19)

Thus a loop of type-\(i\) world-line has an amplitude \(d_i\).

3. Non-planar diagram and braided fusion category

We have been considering planar graphs and the related fusion category theory. In this section we will consider non-planar graphs. Since the particles now live in 2-dimensional space (or higher), the fusion of the particles satisfies

\[
i \otimes j = j \otimes i,
\]  
(B20)
FIG. 16. (Color online) A “self-loop” with canonical framing corresponds to a twist by $2\pi$. A twist by $2\pi$ induces a phase $e^{i2\pi s_i}$ that defines the spin $s_i$ of the particle.

![Diagram](image)

FIG. 17. The two “self-loops” in (a) are “right-handed” and correspond to the same twist. The two “self-loops” in (b) are “left-handed” and also correspond to the same twist that is opposite to that in (a).

![Diagram](image)

and thus

$$N^{ij}_k = N^{ji}_k. \quad \text{(B21)}$$

So the fusion of 2D particles are commutative (while the fusion of 1D particles may not be commutative).

Here, we also like point out that a world-line of a particle are always framed (i.e., having a shadow world-line running parallel to it). When we draw a graph on a plane, there is canonical framing, obtain by shifting the graphs perpendicular to the plane (see Fig. 16). We have been using such a canonical framing in our previous discussion, and we have omitted drawing the framing. But if we do not use this canonical framing, then we need to draw the framing explicitly, as in Fig. 16.

Let us consider simple string configuration with crossing: a “self-loop” with the canonical framing (see Fig. 16). Such a “self-loop” corresponds to a straight line with a $2\pi$ twist, which is equal to a untwisted straight line with a phase $e^{i2\pi s_i}$. Here $s_i$ is the spin of the type-$i$ topological excitation, which is defined mod 1. We also note that the handedness of the “self-loop” determines the direction of the twist (see Fig. 17). As a result, a figure “8” of type-$i$ string has an amplitude $e^{i2\pi s_i} d_i$ (see Fig. 18).

![Diagram](image)

FIG. 18. A figure “8” of type-$i$ string has an amplitude $e^{i2\pi s_i} d_i$. \n
Appendix C: The gapped domain walls as 1+1D anomalous topological order

There is another ways to fully characterize the domain walls. We note that the gapped 1+1D domain walls can be viewed as 1+1D anomalous topological orders.[9] The 1+1D anomalous topological orders are characterized by unitary fusion categories (UFC) which are described by the following data (1) $N \in \mathbb{N}$: the number of types (including the trivial type) of topological excitations on the domain wall. We will use $i, j, k, \text{etc}$, to label the types of topological excitations and use 1 to label the trivial type. (2) $N^{ij}_k \in \mathbb{N}$: the fusion coefficients of the topological excitations. (3) $F^{ijm,\alpha\beta}_{kl,\gamma\delta}$: the unitary relation between different fusion spaces obtained via different fusion paths.

Those data $(N, N^{ij}_k, F^{ijm,\alpha\beta}_{kl,\gamma\delta})$ satisfy

$$\sum_{n=0}^{N} N^{ij}_m F^{mk}_n = \sum_{n=0}^{N} N^{jk}_n F^{ni}_l = \delta_{ij} \quad \text{for all } i,j,k \in \mathbb{N}. \quad \text{(C1)}$$

Where $F^{ijm,\alpha\beta}_{kl,\gamma\delta}$ is the fusion coefficient, $N^{ij}_k$ is the number of types (including the trivial type), $F^{ijm,\alpha\beta}_{kl,\gamma\delta}$ is the unitary relation between different fusion spaces, $\delta_{ij}$ is the Kronecker delta.

Here, we also like point out that a world-line of a particle are always framed (i.e., having a shadow world-line running parallel to it). When we draw a graph on a plane, there is canonical framing, obtain by shifting the graphs perpendicular to the plane (see Fig. 16). We have been using such a canonical framing in our previous discussion, and we have omitted drawing the framing. But if we do not use this canonical framing, then we need to draw the framing explicitly, as in Fig. 16.

Let us consider simple string configuration with crossing: a “self-loop” with the canonical framing (see Fig. 16). Such a “self-loop” corresponds to a straight line with a $2\pi$ twist, which is equal to a untwisted straight line with a phase $e^{i2\pi s_i}$. Here $s_i$ is the spin of the type-$i$ topological excitation, which is defined mod 1. We also note that the handedness of the “self-loop” determines the direction of the twist (see Fig. 17). As a result, a figure “8” of type-$i$ string has an amplitude $e^{i2\pi s_i} d_i$ (see Fig. 18).

![Diagram](image)

FIG. 18. A figure “8” of type-$i$ string has an amplitude $e^{i2\pi s_i} d_i$. \n
Appendix D: More details on the gapped boundary of $S_3$ topological order

1. Some preliminaries

In this section, we give the details of solving the gapped boundaries, using the genus-2 condition in \((52)\), for normalized wave function overlap $\tilde{W}_{S_3,g=2}$. For $g = 2$, it is
convenient to consider the following two choices of basis vectors:

\[ \text{basis I: } i \quad \nu \quad z \quad j \]  
\[ \text{(D1)} \]

and

\[ \text{basis II: } i \quad \mu \quad z \quad j \]  
\[ \text{(D2)} \]

In particular, ‘basis I’ will be useful in making a connection to the genus-1 solution by choosing \( z = 1 \), and ‘basis II’ will be useful in studying the effect of Dehn twist operator \( T_5 \) (or \( T_N \)) in Fig.39. Within basis II, \( T_5 \) will be a diagonal matrix with the diagonal elements corresponding to the topological spin \( \theta_z \) of any \( z \).

Before any concrete calculation, it is noted that for the solutions of gapped boundaries, if \( \bar{W}_{(3)}^{(I);i,j,z} \neq 0 \), then the topological spins of \( i, j, \) and \( z \) in \( D1 \) and \( D2 \) must be trivial. This can be understood by considering \( R_{S_3} = T_1, T_2, (S_1)^{-4}, (S_2)^{-4}, \) and \( T_5 \) in Eq.(41).

We denote basis I and basis II in \( D1 \) and \( D2 \) as

\[ |\psi^{1;ij,z};\mu,\nu \rangle, \]  
\[ \text{and} \]
\[ |\psi^{II;ij,z};\mu,\nu \rangle, \]  
\[ \text{(D3)} \]

respectively. Since the \( S_3 \) topological order is multiplicity free, we can write the above basis vectors as

\[ |\psi^{1;ij,z} \rangle, \]  
\[ \text{and} \]
\[ |\psi^{II;ij,z} \rangle. \]  
\[ \text{(D4)} \]

These two basis vectors are normalized as follows:

\[ \langle \psi^{1;ij,z} | \psi^{1';ij',z'} \rangle = \delta_{i,i'} \delta_{j,j'} \delta_{z,z'}, \]  
\[ \langle \psi^{II;ij,z} | \psi^{II';ij',z'} \rangle = \delta_{i,i'} \delta_{j,j'} \delta_{z,z'}. \]  
\[ \text{(D5)} \]

In addition, they are related to each other as,

\[ |\psi^{1;ij,z} \rangle = \sum_{z'} [F^{ij}_{ij}]_{(z,z')} \ |\psi^{II;ij,z'} \rangle, \]  
\[ \text{(D6)} \]

where \( [F^{ij}_{ij}]_{(z,z')} \) is defined by

\[ a \quad c \quad b \quad d \]  
\[ \frac{a}{c} = \sum_{f,\mu,\nu} \left[ F^{ab}_{cd} \right]_{(e,\alpha,\beta),(f,\mu,\nu)} \]  
\[ a \quad b \quad c \quad d \]  
\[ \text{(D7)} \]

Here \( [F^{ij}_{ij}]_{(z,z')} \) is related to the conventional \( F \)-matrix as defined in \( B1 \) through the following relation:

\[ F^{ab} = \left[ F^{cd} \right]_{(e,\alpha,\beta),(f,\mu,\nu)} \]  
\[ \text{(D8)} \]

It is convenient to study the effect of Dehn twist \( T_5 \) in basis II, which simply results in a phase factor, \( i.e., \)

\[ T_5 |\psi^{II;ij,z} \rangle = \theta_z |\psi^{II;ij,z} \rangle. \]  
\[ \text{(D9)} \]

Within basis I, one has

\[ T_5 |\psi^{1;ij,z} \rangle = \sum_{z'} [F^{ij}_{ij}]_{(z,z')} \theta_{z'} \ |\psi^{1;ij,z'} \rangle. \]  
\[ \text{(D10)} \]

Apparently, \( T_5 \) is in general not diagonal in basis I. However, if if the theory is abelian, then one has \( z = 1 \) and \( [F^{ij}_{ij}]_{1;z} = N_{ij}^z \). Then \( T_5 \) is a diagonal matrix with the matrix elements

\[ \langle \psi^{1;ij,z'} | T_5 |\psi^{1;ij,z} \rangle = \delta_{1,z} \delta_{1,z'} \theta_{z'} N_{ij}^z. \]  
\[ \text{(D11)} \]

Another interesting case is that if \( \theta_{z'} = 1 \) for all \( [F^{ij}_{ij}]_{(z,z')} \neq 0 \), then based on Eq.(D10), one has

\[ T_5 |\psi^{1;ij,z} \rangle = |\psi^{1;ij,z} \rangle, \]  
\[ \text{if } \theta_{z'} = 1 \text{ for all } [F^{ij}_{ij}]_{(z,z')} \neq 0. \]  
\[ \text{(D12)} \]

For \( S_3 \) topological order, all the \( F \) matrices have been obtained in Ref. 52 The so-called punctured \( S \) matrix

\[ S^{(z)}_{a,\mu;\nu;b,\nu} = \frac{1}{D} \cdot \frac{1}{\sqrt{\nu_z}} \]  
\[ \text{(D13)} \]

can be expressed in terms of \( F \)-matrix as (in the multiplicity-free case)

\[ S^{(z)}_{a,b} = \frac{1}{D} \cdot \frac{1}{\sqrt{\nu_z}} \sum_{f} \theta_{\alpha} \theta_{\beta} \sum_{f} F^{a_{\beta}f} \cdot \theta_f \cdot F^{a_{\alpha}f}_{b} \]  
\[ \text{(D14)} \]

based on which we can obtain the punctured \( S^{(z)} \) matrix. For our motivation of studying the gapped boundaries, we only need to consider \( S^{(z)} \) with \( \theta_z = 1 \) in \( D1 \). Based on Eq.(48) and the fusion rules of \( i \otimes i \) in Table I, we only need to check the cases of \( z = a^1, a^2, b \).

The results of punctures \( S^{(z)} \) and \( T^{(z)} \) matrices with \( z = a^1, a^2, b \) are summarized as follows:

- For \( z = a^1 \), one has (with the basis \( a^2, b, b^1, \) and \( b^2 \)):

\[ S^{(z=a^1)} = \frac{1}{3} (\omega \cdot \omega - \omega) \cdot \begin{pmatrix} 0 & 1 & 1 & -1 \\ 1 & 0 & 1 & 1 \\ 1 & -1 & -1 & 0 \\ -1 & 1 & 0 & 1 \end{pmatrix} \]  
\[ \text{(D15)} \]

and \( T^{(z=a^1)} = \text{diag}(1, 1, \omega, \omega^2) \), where \( \omega = e^{\frac{2\pi i}{3}} \).  

- For \( z = a^2 \), one has (with the basis \( c^1, c, \) and \( a^2 \)):

\[ S^{(z=a^2)} = \begin{pmatrix} 1 & 2 & 2 & \frac{1}{\sqrt{2}} \\ 1 & \frac{1}{2} & \frac{1}{2} & \frac{-1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \]  
\[ \text{(D16)} \]

and \( T^{(z=a^2)} = \text{diag}(-1, 1, 1) \).

- For \( z = b \), one can obtain (in the basis \( c^1, c, \) and \( b \)):

\[ S^{(z=b)} = \begin{pmatrix} 1 & -\frac{1}{2} & \frac{1}{2} & -\frac{1}{\sqrt{2}} \\ -\frac{1}{2} & 1 & \frac{1}{2} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{-1}{\sqrt{2}} & \frac{1}{2} & 0 \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} & 0 & \frac{1}{\sqrt{2}} \end{pmatrix}, \]  
\[ \text{(D17)} \]
and \( T(z=b) = \text{diag}(-1,1,1) \). For the punctured \( S(z) \) and \( T(z) \) presented above, one can check explicitly that they satisfied the so-called modular relation
\[
\left( S(z)T(z) \right)^3 = \left( S(z) \right)^2, \quad \left( S(z) \right)^4 = \theta_z^3. \tag{D18}
\]

Now we are ready to study the solutions of
\[
W = W \cdot R,
\]
with \( R = S(z) \) and \( T(z) \).

- For \( z = a^1 \), since \( T(z=a^1) \) is diagonal, one can find that \( W \) has the form \( W = (x,y,0) \). Then choosing \( R = S(z=a^1) \) in (D19), one can find that \( x = y = 0 \). That is, there is no non-zero solution of (D19) for \( R = S(z=a^1) \) and \( T(z=a^1) \). More explicitly, with the basis in (D1), we have
\[
\hat{W}_{S_3,g=2}^{1,i,j,z=a^1} = 0,
\]
where \( i, j \in \{a^2, b, b^1, b^2\} \).

- For \( z = a^2 \), since \( T(z=a^2) = \text{diag}(-1,1,1) \), we have \( W = (0,x,y) \). Then by choosing \( R = S(z=a^2) \) in (D19), one can find that \( x = -\sqrt{2}y \). That is,
\[
W = (0, -\sqrt{2}y, y),
\]
where \( y \) is be determined by other conditions. With the basis vectors chosen in (D1), Eq.(D21) indicates that
\[
\hat{W}_{S_3,g=2}^{1,i,c,z=a^2} = -\sqrt{2} \cdot \hat{W}_{S_3,g=2}^{1,a^2,i,z=a^2},
\]
where \( i \in \{a^2, c, c^1\} \).

- For \( z = b \), since \( T(z=b) = \text{diag}(-1,1,1) \), we have \( W = (0,x,y) \). Then by choosing \( R = S(z=b) \) in (D19), one can find that \( x = \sqrt{2}y \). That is,
\[
W = (0, \sqrt{2}y, y),
\]
where again \( y \) is be determined by other conditions. With the basis vectors chosen in (D1), Eq.(D23) indicates that
\[
\hat{W}_{S_3,g=2}^{1,i,c,z=b} = \sqrt{2} \cdot \hat{W}_{S_3,g=2}^{1,i,b,z=b},
\]
where \( i \in \{b, c, c^1\} \).

2. Solutions of genus-2 condition

In this subsection, we solve the genus-2 condition in Eq.(52), and find there are in total 4 sets of independent solutions, which embed the genus-1 solutions \( \hat{W}_{S_3,g=1}^{i} \) with \( i = 1, 2, 3, 4 \), respectively.

|   | \( \hat{W}_{S_3,g=1}^{(1)} \) | \( \hat{W}_{S_3,g=1}^{(2)} \) | \( \hat{W}_{S_3,g=1}^{(3)} \) | \( \hat{W}_{S_3,g=1}^{(4)} \) |
|---|---|---|---|---|
| \( \hat{W}_{S_3,g=1}^{(1)} \) | \( \checkmark \) | \( \times \) | \( \times \) | \( \times \) |
| \( \hat{W}_{S_3,g=1}^{(2)} \) | \( \checkmark \) | \( a^1 \odot c \) | \( a^2 \odot b \) | \( a^2 \odot b \) |
| \( \hat{W}_{S_3,g=1}^{(3)} \) | \( \checkmark \) | \( a^2 \odot b \) | \( a^2 \odot b \) | \( a^2 \odot b \) |
| \( \hat{W}_{S_3,g=1}^{(4)} \) | \( \checkmark \) | \( a^2 \odot b \) | \( a^2 \odot b \) | \( a^2 \odot b \) |

TABLE III. Only four sets of solutions are allowed by the genus-2 condition. Related fusion rules are \( a^1 \odot c = c^1 \), and \( a^2 \odot b = b^1 \oplus b^2 \).

Before solving the genus-2 condition in Eq.(52), it is first noted that the ground state degeneracy of a topological order on genus-2 closed manifold is
\[
\text{GSD}(S_2) = \sum_i \left( \frac{1}{\text{S}_i} \right)^2,
\]
where we sum over all the anyon types \( i \). One can check that for the \( S_3 \) topological order, we have \( \text{GSD}(S_2) = 116 \). To obtain the solutions of genus-2 condition in Eq.(41) means we need to obtain the 116 components of \( \hat{W}_{S_3,g=2}^{(1)I;j,i,z} \) if we choose the basis vectors \( I \) (II).

Let us start by considering different pairings of genus-1 solutions in Table III. First, it is noted that one common feature of pairings \( \hat{W}_{S_3,g=2}^{(1)I;j,i,z} \) if we choose the basis vectors \( I \) (II). Before solving the genus-2 condition in Eq.(52), it is
\[
\langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle = 0,
\]
which results from Eq.(52) with \( R_{S_3} = T_3 \). Now we insert a complete set of basis vectors I into the above equation, and obtain
\[
\langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle = \sum_{i,j,z} \langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle = 0.
\]

\[
(\langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle) = \sum_{i,j,z} \langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle = 0.
\]

\[
(\langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle) = \sum_{i,j,z} \langle \psi_{g=2}^{a^2,c,z} | \psi_{g=2}^{a^1,c^1,z} \rangle = 0.
\]
where we have considered the fusion rules \(a^1 \otimes a^1 = 1\), and \(c \otimes c = 1 \oplus a^2 \oplus b \oplus b^1 \oplus b^2\), and the only allowed component in \(|\psi_{\text{I}a^1,c^c}\rangle\) is for \(z = 1\). Since
\[
|\psi_{\text{I}a^1,c^c}\rangle = \sqrt{d_{a1}d_{a1}d_{a1}}/D
\]
which is nonzero in Eq. (D27), then based on Eqs. (D26) and (D27), we obtain
\[
W_{S_5,g=2}^{1,a^1,c^c} = 0. \tag{D28}
\]
This means the pairings \(W_{S_5,g=1}^{(p)} \otimes W_{S_5,g=1}^{(q)}\) with \((p, q) = (1, 3), (1, 4), (1, 5), (2, 3), (2, 4), \) and \((2, 5)\) cannot be the solutions of genus-2 condition in Eq. (52).

**Solutions of genus-2 condition:**

Now let us check the pairings of genus-1 solutions \(W_{S_5,g=1}^{(p)} \otimes W_{S_5,g=1}^{(q)}\) with \(p = 1, 2, 3, 4\) respectively in Table III.

\[
\begin{align*}
\tilde{W}_{S_5,g=1}^{(1)} \otimes \tilde{W}_{S_5,g=1}^{(1)} = 1,
\end{align*}
\]
and \(\tilde{W}_{S_5,g=2}^{1,i,j,z} = 0\), if there \(\exists i, j \notin \{1, b, c\}\).

In the following, we will show that \(W_{S_5,g=2}^{1,i,j,z} = 0\) are coupled to certain \(W_{S_5,g=2}^{1,i,j,z} = 0\) with \(\exists z \neq 1\) through the operation \(R_{S_5} = T_5\). We will frequently use the basis transformation in Eq. (D6). Let us start from \(i = b, j = c\). Then for \(|\psi_{\text{I}b^1,c^c}z\rangle = |\tilde{F}_{bc}\rangle_{zz'}|\psi_{\text{II}b^1,c^c}\rangle\), where \(z \in \{1, b\}\) and \(z' \in \{c, c_1\}\), one can find that
\[
|\tilde{F}_{bc}\rangle_{zz'} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}. \tag{D30}
\]

Since the action of \(T_5\) on \(|\psi_{\text{II}b^1,c^c}\rangle\) is simply \(T_5|\psi_{\text{II}b^1,c^c}\rangle = \theta_{\psi_{\text{II}b^1,c^c}}|\psi_{\text{II}b^1,c^c}\rangle\), then based on the basis transformation in Eqs. (D6) and (D30) one can find that \(T_5\) in the basis \(|\psi_{\text{I}b^1,c^1}\rangle, |\psi_{\text{I}b^1,c^b}\rangle\) has the expression:
\[
T_5 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{D31}
\]
By solving Eq. (52), one can obtain
\[
\tilde{W}_{S_5,g=2}^{1,b,c} = \tilde{W}_{S_5,g=2}^{1,b,c} = 0. \tag{D32}
\]
For the case of \(i = c\) and \(j = b\), the discussion is almost the same, and one can find that
\[
\tilde{W}_{S_5,g=2}^{1,c,b} = \tilde{W}_{S_5,g=2}^{1,c,b} = 0. \tag{D33}
\]

Next, let us consider the case \(i = j = c\). For \(|\psi_{\text{I}c^c}\rangle = |\tilde{F}_{cc}\rangle_{zz'}|\psi_{\text{II}c^c}\rangle\), where \(z, z' \in \{1, a^2, b, b^1, b^2\}\), one can find that
\[
[\tilde{F}_{cc}\rangle_{zz'} = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2 & -1 & -1 \\ \sqrt{2} & -1 & 2 & -1 \\ \sqrt{2} & -1 & -1 & 2 \end{pmatrix}. \tag{D34}
\]

Combining with \(T_5|\psi_{\text{II}c^c,c}\rangle = \theta_{\psi_{\text{II}c^c,c}}|\psi_{\text{II}c^c,c}\rangle\), one can find the expression of \(T_5\) in the basis \(|\psi_{\text{I}c^c,c^1}\rangle, |\psi_{\text{I}c^c,c^b}\rangle, |\psi_{\text{I}c^c,b^1}\rangle, |\psi_{\text{I}c^c,b^2}\rangle\) as follows
\[
T_5 = \frac{1}{3} \begin{pmatrix} 1 & \sqrt{2} & \sqrt{2} & \sqrt{2} \\ \sqrt{2} & 2 & -1 & -1 \\ \sqrt{2} & -1 & 2 & -1 \\ \sqrt{2} & -1 & -1 & 2 \end{pmatrix}. \tag{D35}
\]
By solving Eq. (52) with \(\tilde{W}_{S_5,g=2}^{1,c^c,c^1} = 1\) and \(\tilde{W}_{S_5,g=2}^{1,c^c,b^1} = 0\), one can obtain
\[
\tilde{W}_{S_5,g=2}^{1,c^c,c^1} + \tilde{W}_{S_5,g=2}^{1,c^c,b^1} = \sqrt{2}. \tag{D36}
\]
To further determine the concrete value of \(\tilde{W}_{S_5,g=2}^{1,c^c,c^1}\) and \(\tilde{W}_{S_5,g=2}^{1,c^c,b^1}\), we resort to Eqs. (D22) and (D24), based on which one can find that
\[
\tilde{W}_{S_5,g=2}^{1,c^c,c^1} = \tilde{W}_{S_5,g=2}^{1,c^c,b^1} = 0. \tag{D37}
\]
and
\[
\tilde{W}_{S_5,g=2}^{1,c^c,b^1} = 0. \tag{D38}
\]

Since we have obtained \(\tilde{W}_{S_5,g=2}^{1,c^c,b^1} = 1\) in Eq. (D33), then based on Eq. (D38) we have
\[
\tilde{W}_{S_5,g=2}^{1,c^c,b^1} = \sqrt{2}, \quad \tilde{W}_{S_5,g=2}^{1,c^c,b^1} = \tilde{W}_{S_5,g=2}^{1,c^c,b^1} = 1, \quad \tilde{W}_{S_5,g=2}^{1,c^c,b^1} = \frac{1}{\sqrt{2}}. \tag{D39}
\]
Then from Eq. (D36), one has \(\tilde{W}_{S_5,g=2}^{1,c^c,c^1} = 0\). Based on Eq. (D37), one has \(\tilde{W}_{S_5,g=2}^{1,c^c,c^1} = \tilde{W}_{S_5,g=2}^{1,c^c,c^1} = \tilde{W}_{S_5,g=2}^{1,c^c,c^1} = \tilde{W}_{S_5,g=2}^{1,c^c,b^1} = 0\).

Till now, we have obtained certain \(\tilde{W}_{S_5,g=2}^{1,i,j,z} \) that embeds \(\tilde{W}_{S_5,g=1}^{(1)} \otimes \tilde{W}_{S_5,g=1}^{(1)}\). In particular, the nonzero components as follows:
\[
\begin{align*}
\tilde{W}_{S_5,g=2}^{1,b,c,z} & = \tilde{W}_{S_5,g=2}^{1,b,c,z} = 1, \\
\tilde{W}_{S_5,g=2}^{1,c,b,z} & = \sqrt{2}, \\
\tilde{W}_{S_5,g=2}^{1,b,z} & = \frac{1}{\sqrt{2}}.
\end{align*}
\]

Furthermore, we claim that these are the only non-zero components of \(\tilde{W}_{S_5,g=2}^{1,i,j,z} \) that embeds \(\tilde{W}_{S_5,g=1}^{(1)} \otimes \tilde{W}_{S_5,g=1}^{(1)}\).
That is, for all the non-zero components $\tilde{W}_{i,j,z}^{a,c,b}$, one can find that $i,j,z \in \{1,b,c\}$. All the other $(116-13=103)$ components are zero. This can be checked explicitly as follows.

First, for $\tilde{W}_{i,j,z}^{a,c,b} = 1$ which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are $\tilde{W}_{S_{g}=g=2}$ such that there $\exists i,j,\not\in \{1,b,c\}$. Second, there are 18 components of $\tilde{W}_{i,j,z}^{a,c,b}$ with $g=2$, which are zero, with $\theta_{z} \neq 1$. They correspond to $\tilde{W}_{i,j,z}^{a,c,b}$ with $z= b^{1}, b^{2}$.

Next, let us consider the components with $i,j \in \{a^{2}, b^{1}, b^{2}\}$. There are in total 16 components.

Next, let us consider the components with $i \in \{a^{2}\}$, as we have analyzed above, all the 9 components $\tilde{W}_{i,j,z}^{a,c,b}$ with $a^{2}$ are zero.

Next, let us consider the components with $i= b$, we have $\tilde{W}_{i,j,z}^{a,c,b} = 0$, if there $\exists i,j = c^{1}$.

Finally, for $i= c$, this is not allowed by the fusion rules.

Altogether, for all the 116 genus-2 components $\tilde{W}_{i,j,z}^{a,c,b}$ that embed $\tilde{W}_{S_{g}=g=1}^{(1)} \otimes \tilde{W}_{S_{g}=g=1}^{(1)}$, one can find that there are 103 components that are zero, and all the 13 non-zero components are listed in Eqs.((D40) and (D41)).

Next, we consider the effect of punctured $S^{(2)}$ matrices. Based on Eq.(D42), we have

$$\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = -\sqrt{2} \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = -\sqrt{2} \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = 2 \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}}.$$ 

(D47)

Then we can obtain

$$\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = \sqrt{2} \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = -1, \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = 1 \sqrt{2}.$$ 

(D48)

Then we need to solve Eq.(52) with $T_{5}$ in (D35). One can find that $\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = 1$, $\tilde{W}_{S_{g}=g=2}^{a,c,b^{1}} = \tilde{W}_{S_{g}=g=2}^{a,c,b^{2}} = 0$, and $\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = \tilde{W}_{S_{g}=g=2}^{a,c,b^{1}} = \tilde{W}_{S_{g}=g=2}^{a,c,b^{2}} = 0$. Together with Eq.(D38), we have

$\tilde{W}_{S_{g}=g=2}^{a,c,b^{1}} = \tilde{W}_{S_{g}=g=2}^{a,c,b^{2}} = \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = 0.$ 

(D49)

Till now, we have obtained certain $\tilde{W}_{S_{g}=g=2}^{(2)}$ that embeds $\tilde{W}_{S_{g}=g=1}^{(2)} \otimes \tilde{W}_{S_{g}=g=1}^{(2)}$, with the non-zero components as follows:

$$\begin{align*}
\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} &= \tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} = -1, \\
\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} &= \sqrt{2}, \\
\tilde{W}_{S_{g}=g=2}^{a,c,a^{2}} &= 1 \sqrt{2}.
\end{align*}$$ 

(D50)

and

$$\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = 1, \quad \forall i,j \in \{1,a^{2},c\}. 
$$

(D51)

It is emphasized that for $\tilde{W}_{S_{g}=g=2}^{1,i,j,z}$ that embeds $\tilde{W}_{S_{g}=g=1}^{(2)} \otimes \tilde{W}_{S_{g}=g=1}^{(2)}$, the components in Eqs.(D50) and (D51) exhaust all the non-zero components. The discussion is almost the same as that for the case of $\tilde{W}_{S_{g}=g=1}^{(1)} \otimes \tilde{W}_{S_{g}=g=1}^{(1)}$, except for the following difference:

For $\tilde{W}_{S_{g}=g=2}^{1,i,j,z}$ which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are $\tilde{W}_{S_{g}=g=2}$ such that there $\exists i,j \not\in \{1,a^{2},c\}$. Relevant fusion rules are $a^{1} \otimes a^{2} = 1 \oplus b^{1} \oplus b^{2}$, $a^{2} \otimes c = c \oplus c^{1}$.) Based on the genus-1 solution, we have

$$\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = 1, \quad \forall i,j \in \{1,a^{2},c\}. 
$$

(D42)

and

$$\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = 0, \quad \forall i,j \not\in \{1,a^{2},c\}. 
$$

(D43)

Recalling that the action of $T_{5}$ on $|\psi_{1}^{(1,b,c,z')}\rangle$ is simply $T_{5}|\psi_{1}^{(1,b,c,z')}\rangle = \theta_{z'}|\psi_{1}^{(1,b,c,z')}\rangle$, based on the basis transformation in (D43), one can obtain the expression of $T_{5}$ in the basis $\{|\psi_{1}^{(1,a^{2},c)},\rangle,|\psi_{1}^{(1,a^{2},c')}\rangle\}$ as follows

$$T_{5} = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}. 
$$

(D44)

By solving Eq.(52), one can obtain

$$\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = -\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = -1. 
$$

(D45)

With the same procedure, one can find that

$$\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = -\tilde{W}_{S_{g}=g=2}^{1,i,j,z} = -1. 
$$

(D46)
One remarkable feature of \(|\psi^{1;i,j,z} = 1\)| with \(i, j \in \{1, a^1, b\}\) is that for the fusion results of \(i \times j = \sum \tilde{N}^z_{ij} z\) with \(\tilde{N}^z_{ij} > 0\), all the topological spins are trivial, i.e., \(\theta_z = 1\). In this case, based on Eq. (D12), one has

\[
T_5 |\psi^{1;i,j,z} = 1\rangle = |\psi^{1;i,j,z} = 1\rangle, \quad i, j \in \{1, a^1, b\}. \tag{D55}
\]

Then one can find that the solutions \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) in Eqs. (D52), (D53), and (D54) are decoupled from other components \(\tilde{W}_{S_{g=2}}^{1;i,j,z} \neq 1\) which are not. In the following, we will show that Eqs. (D52), (D53), and (D54) exhaust all the non-zero components of \(\tilde{W}_{S_{g=2}}^{1;i,j,z}\) that embed \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1 \in S_{g=1}^{1;1} \otimes S_{g=1}^{1;1} = 1\).

First, similar to previous discussions, for \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) which is the tensor product of genus-1 solutions, there are 55 components that are zero. They are \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) such that there \(\exists i, j \notin \{1, a^1, b\}\). Second, there are 18 components of \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) which are zero, with \(\theta_z = 1\). They correspond to \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) with \(z = b^1, b^2\).

The rest components with \(z \neq 1\) and \(\theta_z = 1\) can be checked case by case as follows.

- For \(z = a^2\), we have shown in Eq. (D20) that \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 0\), \(\forall i, j \in \{a^2, b, b', b^2\}\). There are in total 16 components.

- For \(z = a^2\) and \(b\), we need to solve Eq. (D52) with \(T_5\) in (D35). Different from the case of \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) and \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\), now we have \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) which is set by the genus-1 solutions \(\tilde{W}_{S_{g=1}}^{1;i,j,z} = 1\). Note also that \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 0\) because of the nontrivial topological spins of \(b^1\) and \(b^2\). Then by solving Eq. (D52) with \(T_5\) in (D35) we can get

\[
\tilde{W}_{S_{g=2}}^{1;i,j,z} = 0. \tag{D56}
\]

Then based on Eqs. (D22) and (D24), we can find that all the 18 components of \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) and \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) are zero.

In short, for \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\) that embed \(\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1 \otimes \tilde{W}_{S_{g=2}}^{1;i,j,z} = 1\), one can find there are in total 107 components that are zero, and 9 non-zero components which are listed in Eqs. (D52), (D53), and (D54).

\[
\tilde{W}_{S_{g=2}}^{1;i,j,z} = 1 \otimes \tilde{W}_{S_{g=2}}^{1;i,j,z} = 1. \tag{D57}
\]

The genus-1 solution \(\tilde{W}_{S_{g=1}}^{1;i,j,z} = 1\) in (50) corresponds to the condensation of anyons \(1, a^1, a^2\). Relevant fusion rules are \(a^1 \otimes a^1 = 1, a^1 \otimes a^2 = a^2, a^2 \otimes a^2 = 1 \otimes a^1 \otimes a^2\). Based on the genus-1 solution, we have

\[
\tilde{W}_{S_{g=1}}^{1;i,j,z} = 1 = \tilde{W}_{S_{g=2}}^{1;i,j,z} = 1. \tag{D58}
\]

Appendix E: Structure coefficients of condensable algebra and ground-state overlap

It is known that the gapped boundary is closely related to anyon condensation, whose data is fully encoded in the so-called condensable algebra [25] in the unitary modular tensor category (UMTC) that describes the anyons.

Let \((C, c)\) and \((D, c)\) be two 2d topological orders with the same chiral central charge \(c\), where \(C\) and \(D\) are two UMTC’s. We assume that they are Witten equivalent, in
other words, they can be connected by a gapped domain wall. There are two equivalent mathematical descriptions of such a gapped domain wall:

- A gapped domain wall between \((\mathcal{C}, c)\) and \((\mathcal{D}, c)\) can be described by a unitary modular tensor category \(\mathcal{M}\), which is equipped with a unitary braided monoidal equivalence

\[
\phi_{\mathcal{M}} : \mathcal{C} \boxtimes \mathcal{D} \rightarrow Z(\mathcal{M}).
\]

where \(\mathcal{C} \boxtimes \mathcal{D}\) is the UMTC describing the topological order obtained by stacking \(\mathcal{C}\) with the time-reversal of \(\mathcal{D}\) and \(Z(\mathcal{M})\) denotes the Drinfeld center of \(\mathcal{M}\).

- A gapped domain wall between \((\mathcal{C}, c)\) and \((\mathcal{D}, c)\) is uniquely determined by a Lagrangian condensable algebra \(A\) in \(\mathcal{C} \boxtimes \mathcal{D}\). Here Lagrangian means that \(A\) is “maximal” in the sense

\[
(dim A)^2 = \dim \mathcal{C} \dim \mathcal{D},
\]

In this case, the gapped domain wall is described by the UFC \((\mathcal{C} \boxtimes \mathcal{D})_A\), i.e., the category of right \(A\)-modules in \(\mathcal{C} \boxtimes \mathcal{D}\). A Lagrangian algebra \(A\), as an object in \(\mathcal{C} \boxtimes \mathcal{D}\), can be decomposed as follows:

\[
A = \sum_{i \in \text{Irr}(\mathcal{C}), j \in \text{Irr}(\mathcal{D})} (i \boxtimes j) \otimes W_{i j},
\]

where \(\text{Irr}(\mathcal{C})\) and \(\text{Irr}(\mathcal{D})\) denote the sets of isomorphic classes of simple objects.

In particular, when \(\mathcal{D}\) is the trivial topological order, the above describes a gapped boundary of \(\mathcal{C}\), and \(A\) is a Lagrangian algebra in \(\mathcal{C}\). The existence of a Lagrangian algebra, for example, in terms of the structure coefficients to be introduced in this section, is a necessary and sufficient condition for the existence of a gapped boundary. The approach presented in the main text, using the invariance of wave function overlaps under the modular transformations (or mapping class group transformations), however, only constitutes necessary conditions. The advantage of the modular invariance approach is that they are all linear equations and much easier to solve, as opposed to the non-linear equations for the structure coefficients such as the associativity condition. For the examples presented in this paper, each of our solutions does correspond to a Lagrangian algebra.

Physically, a condensable algebra is just a composite anyon that becomes the ground state after condensation (i.e., it is condensed). For this to be possible, it is necessary that this composite anyon has special algebraic structures, which can be thought as a generalization of usual commutative associative algebra. We know that after picking a basis, the multiplication of a usual associative algebra can be expressed by the structure coefficients. It is also the case for the condensable algebra. The structure coefficients encode all the “special algebraic structures” of the condensed composite anyon; in particular, they predict the form of the ground-state overlap.

To explain what a “basis” of a composite anyon means in a unitary modular tensor category, we begin by recalling the (orthonormal) basis of an \(n\)-dimensional Hilbert space \(H\), which is a set of state vectors \(|\alpha\rangle\) satisfying

\[
\langle \alpha | \beta \rangle = \delta_{\alpha \beta}, \quad \sum_{\alpha=1}^{n} |\alpha\rangle \langle \alpha | = \text{id}_H.
\]

To generalize this notion, note that an \(n\)-dimensional Hilbert space can be viewed as a composite anyon, that is composed of \(n\) trivial anyons, \(H = \oplus 1^{\otimes n}\). The trivial anyon is the tensor unit \(1\), namely the ground field \(\mathbb{C}\), in the tensor category. Moreover, a vector \(|\alpha\rangle\) can be identified with the embedding linear map \(q_\alpha : \mathbb{C} \rightarrow H\), by \(q_\alpha(\lambda) = \lambda|\alpha\rangle\). Similarly, \(\langle \alpha |\) can be identified with the projection \(p_\alpha : H \rightarrow \mathbb{C}\), by \(p_\alpha(|\psi\rangle) = \langle \alpha |\psi\rangle\).

Now, given a composite anyon

\[
A = \oplus 1^{\otimes N_i^A},
\]

where \(i\) labels the simple anyons, a basis of \(A\) is a set of morphisms (“linear maps” in generic tensor category) \(p_{i,\alpha}^A : A \rightarrow i, q_{i,\alpha}^A : i \rightarrow A, \alpha = 1, \ldots, N_i^A\)

\[
p_{i,\alpha}^A q_{j,\beta}^A = \delta_{ij} \delta_{\alpha \beta} \text{id}_i, \quad \sum_{\alpha=1}^{N_i^A} q_{i,\alpha}^A p_{i,\alpha}^A = \text{id}_A.
\]

We usually require that \(p_{i,\alpha}^A, q_{i,\alpha}^A\) are Hermitian conjugates:

\[
p_{i,\alpha}^A = (q_{i,\alpha}^A)^\dagger.
\]

It is intuitive to use the following graphs for the basis:

\[
\begin{align*}
\phi_{\alpha} &= p_{i,\alpha}^A, & \phi_{\alpha} &= q_{i,\alpha}^A. \\
\end{align*}
\]

Our choice of symbol is to remind the reader of the similarity to the basis of Hilbert spaces (rotate the graph by 90° anticlockwise). They satisfy similar orthonormal and complete conditions:

\[
\begin{align*}
\sum_{i} p_{i,\alpha}^A \sim |\alpha, i\rangle, & \quad \sum_{i} q_{i,\alpha}^A \sim |\alpha, i\rangle. \\
\end{align*}
\]

(8)

\[
\begin{align*}
\sum_{i} p_{i,\alpha}^A \sim |\alpha, i\rangle, & \quad \sum_{i} q_{i,\alpha}^A \sim |\alpha, i\rangle. \\
\end{align*}
\]

(9)

\[
\sum_{i} |\alpha, i\rangle \langle \alpha, i | \sim |\alpha, i\rangle |\alpha, i\rangle = \text{id}_A.
\]

(10)

The only subtle part is that the tensor product and braiding involving nontrivial \(i\) is different from usual intuitions from vector spaces, which will be explained below. First,
we need to take a basis of the tensor product of simple anyons \( i \otimes j \):

\[
\sum_{k} \sum_{\alpha=1}^{N^{ij}_{k}} \begin{array}{c} i \\ \alpha \end{array} \begin{array}{c} j \\ \beta \end{array} = \sum_{k} \sum_{\alpha=1}^{N^{ij}_{k}} \begin{array}{c} i \\ \alpha \end{array} \begin{array}{c} k \\ \gamma \end{array} \begin{array}{c} j \\ \beta \end{array} .
\]

(E11)

Choosing the orthonormal basis is equivalent to choosing \( Y \) and \( O \) to be identity matrix. The previous choice of vertex basis is in fact,

\[
\begin{array}{c} i \\ \alpha \end{array} \begin{array}{c} j \\ \beta \end{array} = \left( \frac{d_i d_j}{d_k} \right)^{1/4} \begin{array}{c} i \\ \alpha \end{array} \begin{array}{c} k \\ \gamma \end{array} \begin{array}{c} j \\ \beta \end{array} .
\]

(E12)

They may be referred to as the “rotatable basis”, since after the above rescaling, rotating a vertex (by bending the legs) leads to at most a phase factor or a unitary matrix, which cancels each other if we are considering a closed graph. Also, as explained in Sec. IIIE, in the rotatable basis, a closed graph representing a wave function is properly normalised to a constant (17) that does not depend on the anyon and vertex labels. However, in this section we prefer to use the orthonormal basis, indicated by the semicircle in the graph, which is more convenient for open graphs.

The associativity of tensor product is encoded in the \( F \)-matrix.

\[
\sum_{s,\xi,\delta} F_{kls,\xi,\delta}^{ijr,\alpha,\beta} = \sum_{\xi,\delta} F_{kls,\xi,\delta}^{ijr,\alpha,\beta} ,
\]

(E13)

where we used the definition of \( F \) matrix in (B1). The braiding can be similarly represented by the \( R \)-matrix:

\[
\begin{array}{c} j \\ \beta \end{array} = \sum_{b} R_{k,\beta,\alpha}^{ij} \begin{array}{c} i \\ \alpha \end{array} \begin{array}{c} j \\ \beta \end{array} .
\]

(E14)

We like to remark that, in general the tensors such as \( F, R \) matrices do depend on our choice of basis. However, it is easily verified that the two choices of basis (E12) differ by the same overall factor on both sides of (E13)(E14). Thus, \( F, R \) matrices remain the same under the change of basis (E12).

For an algebra \( A \) in a tensor category, there is a multiplication morphism \( m : A \otimes A \to A \). First take a basis of \( A \) as in (E5)

\[
\begin{array}{c} i \end{array} = \sum_{i} \sum_{\alpha=1}^{N^{i}_{\alpha}} \begin{array}{c} i \end{array} \begin{array}{c} \alpha \end{array} = \sum_{i} \sum_{\alpha=1}^{N^{i}_{\alpha}} p_{i,\alpha}^{A} .
\]

(E15)

The multiplication morphism \( m \) is then

\[
m = \sum_{ijk} F_{i,\alpha,j,\beta}^{k} \begin{array}{c} i \end{array} \begin{array}{c} \alpha \end{array} \begin{array}{c} j \end{array} = \sum_{i,\alpha,j,\beta} M_{i,\alpha,j,\beta}^{k} \begin{array}{c} i \end{array} \begin{array}{c} \alpha \end{array} \begin{array}{c} j \end{array} .
\]

(E16)

The central part can be expressed in terms of basis vertices \( p_{ij}^{k,\mu} \)

\[
\sum_{u} M_{i,\alpha,j,\beta}^{k} = \sum_{\mu} M_{i,\alpha,j,\beta}^{k,\mu} ,
\]

(E17)

Thus

\[
m = \sum_{ij} M_{i,\alpha,j,\beta}^{k} \begin{array}{c} i \end{array} \begin{array}{c} \alpha \end{array} \begin{array}{c} j \end{array} = \sum_{ij} \sum_{\xi,\delta} M_{i,\alpha,j,\beta}^{k,\mu} \begin{array}{c} i \end{array} \begin{array}{c} \alpha \end{array} \begin{array}{c} j \end{array} ,
\]

(E18)
$M_{i\alpha,j\beta}^{k\chi,\mu}$ is the “structure coefficients” of the algebra. In the category of vector spaces $\text{Vec}$, the object labels $i,j,k$ and the vertex label $\mu$ reduce to trivial, and $M_{i\alpha,j\beta}^{k\chi,\mu}$ reduces to structure coefficients of usual associative algebra, with $\alpha,\beta,\chi$ the labels of basis vectors. Again, similar to the usual associative algebra, the structure coefficients depend on the choice of basis, both the basis of $A$, $p_A^{i\alpha}, q_A^{j\beta}$ and the vertex basis $p_{ij}^{k\mu}$. It is easy to write down transformations of $M_{i\alpha,j\beta}^{k\chi,\mu}$ under a change of basis. For example, using the rotatable basis (E12), the corresponding structure coefficients is

$$M_{i\alpha,j\beta}^{k\chi,\mu} = \left(\frac{d_k}{d_{ij}}\right)^{1/4} M_{i\alpha,j\beta}^{k\chi,\mu}. \quad (E19)$$

Structure coefficients related by a change of basis are considered equivalent and describe the same algebra.

Now we are ready to define the condensable algebra $(A,M_{i\alpha,j\beta}^{k\chi,\mu})$, by listing the defining, i.e., sufficient and necessary conditions of $M_{i\alpha,j\beta}^{k\chi,\mu}$:

1. Associative:

$$\sum_{\omega} M_{i\alpha,j\beta}^{r\omega,\mu} M_{r\omega,k\chi}^{l\delta,\nu} = \sum_{s\xi} M_{k\chi,l\delta,k\chi}^{s\xi,\nu} M_{i\alpha,j\beta}^{l\delta,\nu}. \quad (E21)$$

2. Unital: There exists “the unit of the multiplication”, the unit morphism $\eta : 1 \rightarrow A$,

$$\eta \quad \eta = \eta, \quad (E22)$$

expressed in terms of embedding basis $\eta = \eta_A q_{1,\alpha}$, and

$$\sum_{\alpha=1}^{N_A^A} \eta_A M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_{\alpha=1}^{N_A^A} \eta_A M_{j\beta,\gamma\alpha}^{k\chi,1} = \delta_{j\beta} \eta_A. \quad (E23)$$

3. Connected: The range of index $\alpha$ in the above expression is $N_A^A = 1$, thus

$$M_{i1,j1}^{k\chi,1} = M_{j1,i1}^{k\chi,1} = \eta_1^{-1} \delta_{j\beta} \delta_{\gamma\chi}. \quad (E24)$$

4. Isometric:

$$\sum_{i\alpha,j\beta} M_{i\alpha,j\beta}^{k\chi,\mu} \left( M_{i\alpha,j\beta}^{k\chi,\mu} \right)^* = \delta_{kk'} \delta_{\chi\chi'}. \quad (E26)$$

5. Commutative:

$$M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_{\nu} R_{k\mu,i\alpha,j\beta}^{\nu} M_{j\beta,\nu}^{k\chi,\mu}. \quad (E28)$$

A unital connected isometric algebra in a unitary tensor category automatically satisfies the Frobenius condition [53]:

$$\sum_{i\alpha,j\beta} M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_{\nu} R_{k\mu,i\alpha,j\beta}^{\nu} M_{j\beta,\nu}^{k\chi,\mu}. \quad (E29)$$

As a direct corollary, $A$ is self dual and

$$|\eta_1|^2 = \text{dim} A = \sum_i N_i^A d_i, \quad (E30)$$

where $\eta_1$ is the from the unit morphism $\eta = \eta_A q_{1,1}^A : 1 \rightarrow A$. We choose the phase of $\eta_1$ and let

$$\eta_1 = \sqrt{\sum_i N_i^A d_i}. \quad (E31)$$

Also, by attaching counit $\eta^\dagger$ to top right $A$ in (E29), we obtain the following useful formula to exchange the upper and lower indices of $M_{i\alpha,j\beta}^{k\chi,\mu}$

$$M_{i\alpha,j\beta}^{k\chi,\mu} = \sum_{\gamma,\nu} \left( M_{i\alpha,j\beta}^{k\chi,\mu} \right)^* P_{j\beta,k\gamma}^{\nu} M_{j\gamma,\nu}^{k\chi,\mu} \eta_1^*, \quad (E32)$$
where \( F_{jk,i,i}^{k,j,i,\nu} \) comes from the following

\[
\sum_{i} F_{jk,i,i}^{k,j,i,\nu} = \sum_{i} \delta_{i,j} \delta_{i,k} \delta_{\nu,\mu} \delta_{i,\gamma} \delta_{i,\gamma}. \tag{E33}
\]

Similarly, attaching \( \eta^\dagger \) to topleft \( A \), we obtain

\[
M^{k,i}_{i,j,i} = \sum_{\gamma,\beta,\nu} \left( M^{\gamma,i}_{\beta,j} \right)^{\dagger} \left( M^{\nu,i}_{\mu,k} \right)^{\dagger} \left( M^{\mu,i}_{\alpha,j} \right)^{\dagger}. \tag{E34}
\]

Let us compute \( M^{k,i}_{i,j,i} \). From eqn. (E32), we obtain

\[
M^{k,i}_{i,j,i} = \sum_{\gamma} \left( M^{\gamma,i}_{\gamma,j} \right)^{\dagger} \left( M^{\gamma,i}_{\gamma,j} \right)^{\dagger} \left( M^{\gamma,i}_{\gamma,j} \right)^{\dagger}. \tag{E35}
\]

Using \( F_{i,j}^{i,j} = d_i^{-1} \), \( M^{k,i}_{i,j,i} = M^{k,i}_{i,j,i} \) and eqn. (E24), the above becomes

\[
\delta_{\beta,\gamma} = \sum_{j} \sum_{i} \left( M^{k,i}_{i,j,i} \right)^{\dagger} \left( M^{k,i}_{i,j,i} \right)^{\dagger} \delta_{\beta,\gamma}. \tag{E36}
\]

We see that \( M^{k,i}_{i,j,i} \) is proportional to a unitary matrix. By choosing a proper basis for eqn. (E8) we can make

\[
M^{k,i}_{i,j,i} = \frac{d_i \delta_{\alpha,\beta}}{\sqrt{\sum_{j} N^A_j d_j}}. \tag{E37}
\]

Together with (E28), one can permute any pair of \( i \alpha \) indices.

The structure coefficients can be used to compute the (topological universal part of) overlap between ground states before and after condensation. For simplicity, here we only discuss the case that the phase after condensation is topologically trivial and has a unique ground state. We start with the ground state on a torus. Before condensation, the loops of simple objects form a basis of the ground space on a torus:

\[
\text{After condensation, the ground state is, up to an overall factor that depends on the system size, the loop of the corresponding condensable algebra } A: \tag{E38}
\]

\[
A = \sum_{i} N_i^A. \tag{E39}
\]

Thus we can extract the multiplicity \( N_i^A \) from the ground-state overlap on a torus, which is exactly the universal part of wave function overlap (the labels in the trivial phase is omitted, see eqn. (2) and eqn. (3))

\[
\hat{W}^i_{C,1} = M_i^A = N_i^A, \tag{E40}
\]

On a closed two-dimensional surface with genus \( g = 2 \), one choice of the basis before condensation is given by

\[
\text{Still, after condensation, the ground state is} \tag{E41}
\]

Now, the structure coefficients kick in. After straightforward calculation, we find

\[
\text{Therefore, for genus 2, the wave function overlap, up to an overall factor, is given by (the labels in the trivial} \tag{E42}
\]

\[
\text{Therefore, for genus 2, the wave function overlap, up to an overall factor, is given by (the labels in the trivial} \tag{E43}
\]

\[
\text{Therefore, for genus 2, the wave function overlap, up to an overall factor, is given by (the labels in the trivial} \tag{E44}
\]
phase is omitted)
\[
\tilde{W}_{C:2}^{i,j,z,\mu,\nu} \propto \frac{1}{d_z} \sum_{\alpha,\beta} M_{i\alpha,z\chi}^{\mu,\nu} \left( M_{\chi,j\beta}^{\mu,\nu} \right)^* \tag{E44}
\]

Using (see eqn. (E24) and eqn. (26))
\[
M_{11,j\beta}^{k\chi} = M_{j\beta,11}^{k\chi} = \frac{1}{\sqrt{\sum_i N_i^A d_i}} \delta_{jk} \delta_{\chi}.
\]

we can fix the overall factor. Notice that (setting \( z = 1 \) in eqn. (E44))
\[
\sum_{\alpha,\beta} M_{i\alpha,11}^{\mu,\nu} \left( M_{\chi,j\beta}^{\mu,\nu} \right)^* = \frac{N_i^A N_j^A}{\sum_i N_i^A d_i} \tag{E46}
\]
where we have use the fact that \( \alpha = 1, \ldots, N_i^A \) and \( \beta = 1, \ldots, N_j^A \). Since \( \tilde{W}_{C:1}^{1} = N_i^A = 1 \), we find
\[
\tilde{W}_{C:2}^{i,j,z,\mu,\nu} = \sum_k N_k^A d_k \sum_{\alpha,\beta} M_{i\alpha,z\chi}^{\mu,\nu} \left( M_{\chi,j\beta}^{\mu,\nu} \right)^* \tag{E47}
\]

Another choice of basis is
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\]

whose difference from the first choice is given by the \( F \) matrix. Correspondingly,
\[
= \sum_{i,j,y,y' \mu,\nu} \left( \sum_{\alpha,\beta} M_{i\alpha,j\beta}^{y,y'} \left( M_{\alpha,\beta}^{y,y'} \right)^* \right)
\]

The compatibility between the two choices is guaranteed by the Frobenius condition (E29). It is straightforward to generalize to surfaces with any genus.

A Lagrangian algebra \( A \) is modular invariant, which was first proposed and proved in [54, Theorem 3.4] for a

special modular tensor category. But the proof automatically works for all modular tensor category. Below we explain the property in detail. Modular invariance means \( T \) and \( S \) invariance. First, the topological spin of \( A \) is trivial
\[
\theta_A = \frac{1}{\dim A} = 1, \tag{E50}
\]
which is a direct consequence of the commutative and Frobenius conditions. Thus a twist of string \( A \) leaves the graph invariant, which generates a subset a Dehn twists on the manifold. This property is the \( T \) invariance. Expressing in terms of simple objects,
\[
T_i N_i^A = N_i^A. \tag{E51}
\]

\( A \) also has the following invariance under the punctured \( S \) transformation, which, together with the twist of \( A \) above, generates all transformations in the mapping class group,
where we used the punctured $S$ matrix as in (D13). Therefore, the structure coefficients satisfy the (punctured) $S$ invariance:

$$\sum_{j,\nu} S^{(z)}_{j,\nu,\alpha,\mu} \sum_{\alpha} M_{j,\alpha,\nu}^{\alpha,\mu} = \sum_{\alpha} M_{\nu,\alpha,\mu}^{\alpha,\nu}. \quad (E54)$$

For a generic closed surface, as long as an appropriate graph basis such as (E41) is picked, part of the graph will look like (E52), and thus the wave function overlap is invariant under the punctured $S$ transformation. For example, on genus 2, taking the left half of (E41) and applying punctured $S$ transformation, we find that

$$\sum_{k_N} S^{(z)}_{k_N,\alpha,\mu} \tilde{W}_{c_1,2}^{(k_N,\alpha,\mu)} = \tilde{W}_{c_2,2}^{(k_N,\alpha,\mu)}. \quad (E55)$$

In particular, when $z = 1$, $\sum_{\alpha} M_{j,\alpha,\nu}^{\alpha,1} = \eta_1^{-1} N_i^A$, we have the $S$ invariance on torus

$$\sum_{j} S_{j,\nu} N_j^A = N_i^A. \quad (E56)$$

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