Research Article

Homology Groups in Warped Product Submanifolds in Hyperbolic Spaces

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In this paper, we show that if the Laplacian and gradient of the warping function of a compact warped product submanifold \( \Omega^{p+q} \) in the hyperbolic space \( H^m(−1) \) satisfy various extrinsic restrictions, then \( \Omega^{p+q} \) has no stable integral currents, and its homology groups are trivial. Also, we prove that the fundamental group \( \pi_1(\Omega^{p+q}) \) is trivial. These restrictions are also extended to the eigenvalues of the warped function, the integral Ricci curvature, and the Hessian tensor. The results obtained in the present paper can be considered as generalizations of the Fu–Xu theorem in the framework of the compact warped product submanifold which has the minimal base manifold in the corresponding ambient manifolds.

1. Introduction and Main Results

For any Riemannian manifold \( \Omega^n \), it is well known that any integral homology class in \( H_q(\Omega^n, \mathbb{Z}) \) which is nontrivial is correlated to integral stable currents. This result was initially proved by Federer and Fleming [1]. Utilizing the method of variational calculus for the geometric measure concept combining with the method of Federer and Fleming, Lawson and Simons [2] obtained the optimization for the second fundamental form, which leads to the vanishing homology in a range of intermediate dimensions and the nonexistence of stable currents in the submanifold in the simply connected space form, and obtained the key theorem of that paper.

\[ \sum_{a=1}^{p} \sum_{\beta=p+1}^{n} \left( 2\|\Pi(e_a, e_\beta)\|^2 - g(\Pi(e_a, e_a), \Pi(e_\beta, e_\beta)) \right) < p(n - \text{sgn}(c)p)c \]  

(1)

is satisfied, then \( \Omega^n \) has no stable \( p \)-currents with the vanished \( p \)th homology group, i.e.,

\[ H_p(\Omega^n, \mathbb{Z}) = H_q(\Omega^n, \mathbb{Z}) = 0, \]

(2)

where \( p + q = n \) and \( \text{sgn}(c) \) is the signature of constant curvature \( c \).

The geometric structure and topological properties of submanifolds in different spaces have been studied on a large scale during the past few years [4–17]. Many results showed that there is a closed relationship between stable currents which are nonexistent and the vanished homology groups of submanifolds in a different class of the ambient manifold.
obtained by imposing conditions on the second fundamental form (1). For example, as an application of the Ricci curvature and the ambient manifold is an Euclidean space, Vlachos [18] proved that a compact oriented submanifold \( \Omega^n \) of dimension \( n \) in Euclidean space \( \mathbb{E}^m \) of dimension \( m = n + k \) satisfies the pinching condition \( \text{Ric}(X) > \delta_1(n)g(A_HX,X) \), in which \( X \) is any unit vector, \( A_H \) is the shape operator regarding the mean curvature \( H \), and \( \delta_1(n) \) is a constant such that \( \delta_1(n) = n - 2 \) if \( n \) is even and \( \delta_1(n) = (n-1)(n-3)/(n-1) \) if \( n \) is odd. Therefore, \( \Omega^n \) has no stable currents. Moreover, \( \Omega^n \) is homeomorphic to \( \mathbb{S}^n \). Moreover, using Theorem 1 in [19], it was found that if a compact oriented submanifold of dimension \( n \Omega^n \) in space form \( \mathbb{E}^m \) satisfies the second fundamental form pinching condition \( S < a(n,k,|H|,c) \), for any integer \( k \) in which \( 0 < k < n \) and \( c \) is a constant sectional curvature, then the \( p \)th homology groups are vanishing, \( H_p(\Omega^n,\mathbb{Z}) = 0 \), for all \( p \in \{k, \ldots, n-k\} \), and if the fundamental group \( \pi_1(\Omega^n) \) is finite and simply connected, then \( \Omega^n \) is homeomorphic to \( \mathbb{S}^n \). Motivated by the nonexistence of stable submanifolds or stable currents, a number of topological properties have been studied by many authors [3, 4, 14, 15, 17, 19–24] inspired by Theorem 1.

Inspired by the aforementioned results, we want to obtain some similar results of warped product submanifold theory where the second fundamental form pinching condition shall be replaced by the warping function. Using Theorem 1, we now give the first main result of this note.

**Theorem 2.** Let \( \Psi: \Omega^{p,q} = N^p \times N^q \longrightarrow \mathbb{H}^m(-1) \) be an isometric embedding from a compact warped product submanifold \( \Omega^{p,q} \) into an \( m \)-dimensional hyperbolic space \( \mathbb{H}^m(-1) \) in which the base manifold \( N^q \) is minimal in \( \mathbb{H}^m(-1) \) and the following inequality

\[
3h\Delta h < 2\left(\|\nabla h\|^2 + \frac{p}{q} (2p+q)h^2 \right),
\]

hold, where \( \nabla h \) and \( \Delta h \) are the gradient and Laplacian of the warping function \( h \), respectively. Then, we have the following:

(a) The warped product submanifold \( \Omega^{p,q} \) does not have any stable integral \( p \)-currents.

(b) The \( i \)th integral homology groups of \( \Omega^{p,q} \) with integer coefficients are vanished, which means

\[
H_p(\Omega^{p,q},\mathbb{Z}) = \mathbb{H}_q(\Omega^{p,q},\mathbb{Z}) = 0.
\]

(c) If \( p = 1 \), then the fundamental group \( \pi_1(\Omega) \) is null, i.e., \( \pi_1(\Omega) = 0 \). Moreover, \( \Omega^{p,q} \) is a simply connected warped product manifold.

Motivated by the geometric rigidity (Theorem 2), the second goal of this approach is to prove a new vanishing theorem for compact warped product submanifolds in terms of the Ricci curvature and using the eigenvalue of Laplacian of the warping function. In particular, we can give the following theorem.

**Theorem 3.** Under the assumption of Theorem 2 and if the warping function \( h \) is an eigenfunction of Laplacian of \( \Omega^{p,q} \) corresponding to the first positive eigenvalue \( \lambda_1 \) and satisfying the strict inequality

\[
\|\nabla^2 h\|^2 + \text{Ric}(\nabla h,\nabla h)\frac{\lambda_1}{2q} \left(4\frac{p^2}{q} + 2p + 3\lambda_1 \right) h^2 > 0
\]

for integral Hessian tensor \( \nabla^2 h \) of the warping function \( h \) and the integral Ricci curvature along the gradient \( \nabla h \), then we have the following:

(a) The warped product submanifold \( \Omega^{p,q} \) does not have any stable integral \( p \)-currents.

(b) The \( i \)th integral homology groups of \( \Omega^{p,q} \) with integer coefficients are vanished; that is,

\[
H_p(\Omega^{p,q},\mathbb{Z}) = \mathbb{H}_q(\Omega^{p,q},\mathbb{Z}) = 0.
\]

(c) The fundamental group \( \pi_1(\Omega) \) is null, i.e., \( \pi_1(\Omega) = 0 \). Moreover, \( \Omega^{p,q} \) is a simply connected warped product manifold.

Now, we give a direct application of Theorem 3.

**Theorem 4.** Assume that \( \Psi: \Omega^{p,q} = N^p \times N^q \longrightarrow \mathbb{H}^m(-1) \) is an isometric embedding from a compact warped product submanifold \( \Omega^{p,q} \) into an \( m \)-dimensional hyperbolic space \( \mathbb{H}^m(-1) \) satisfying the following inequality:

\[
\int_{\Omega^{p,q}} \|\nabla^2 h\|^2 dV < \frac{\left(p - 1 - \lambda_1 \right)}{2q} \left(4\frac{p^2}{q} + 2p + 3\lambda_1 \right) \int_{\Omega^{p,q}} h^2 dV .
\]

Then, statements (a), (b), and (c) in Theorem 2 hold.

Another interesting result obtained from Theorem 4 is the following:

**Corollary 1.** Under the same assumption as Theorem 4 and if \( \nabla h \in \text{Ker}\,\Pi \) with the following holds,

\[
\int_{\Omega^{p,q}} \|\nabla^2 h\|^2 dV < \frac{\left(p - 1 - \lambda_1 \right)}{2q} \left(4\frac{p^2}{q} + 2p + 3\lambda_1 \right) \int_{\Omega^{p,q}} h^2 dV ,
\]

then, statements (a), (b), and (c) in Theorem 2 are satisfied.

**Remark 1.** Theorem 2 is the main vanishing homology theorem for a compact warped product submanifold with no
need for $\Omega^{p+q}$ to be simply connected. Moreover, our result is of significance due to involving the new pinching conditions in terms of the warping function, the integral of the squared norm of the Hessian tensor, the integral Ricci curvature, and the first nontrivial eigenvalue of the warped function.

2. Preliminaries

Let $\mathbb{H}^m(c)$ denote the hyperbolic space with dimension $(m)$ and constant sectional curvature $c = -1$. We use the fact that $\mathbb{H}^m(c)$ has a canonical isometric embedding in Lorentz–Minkowski space $L_1^{m+1}$ which is the vector space $\mathbb{R}^{m+1}$ with the metric
\[
\left< x, y \right> = \left( dx^1 \right)^2 + \cdots + \left( dx^{m+1} \right)^2,
\]
where $\{x^1, \ldots, x^{m+1}\}$ are canonical coordinates in $L_1^{m+1}$. Therefore, the hyperbolic space $\mathbb{H}^m$ with negative constant curvature $c$ is
\[
\mathbb{H}^m(c) = \left\{ x \in L_1^{m+1} : \left< x, x \right> = \frac{1}{c}, x^{m+1} > 0 \right\}.
\]
Thus,
\[
\tilde{\Omega}(X, Y, Z, W) = g(Y, W)g(X, Z) - g(X, W)g(Y, Z),
\]
\[
\forall X, Y, Z, W \in \Gamma(T\tilde{M}),
\]
where $T\tilde{M}$ is a tangent bundle of $\mathbb{H}^m$ and $\tilde{\Omega}$ is the Riemannian curvature tensor of $\mathbb{H}^m$. This means that $\mathbb{H}^m$ is a manifold of constant sectional curvature $-1$.

Assume that $\Omega^n$ admits isometric immersion into a Riemannian manifold $\hat{M}^m$ with induced metric $g$. Then, the formula
\[
\tilde{\Omega}(X, Y, Z, W) = R(X, Y, Z, W) + g(\tilde{\Omega}(X, Z), \tilde{\Omega}(Y, W)) - g(\tilde{\Omega}(X, W), \tilde{\Omega}(Y, Z))
\]
is the Gauss equation for $\Omega^n$ in which $X, Y, Z, W \in \mathfrak{X}(T\Omega)$, and $R$ and $\tilde{\Omega}$ are the curvature tensors on $\Omega^n$ and $\hat{M}$, respectively. The mean curvature vector $H$ for a local orthonormal basis $\{e_1, e_2, \ldots, e_n\}$ on $\Omega$ is defined by
\[
\|H\|^2 = \frac{1}{n} \sum_{a=1}^{n} \left( \sum_{a=1}^{n} \Pi_{aa} \right)^2.
\]
In addition, we set
\[
\Pi_{\alpha\beta} = g(\Pi(e_\beta, e_\alpha), e_r), \text{ and } \Pi(\Pi(e_\alpha, e_\beta), \Pi(e_\alpha, e_\beta))^2
\]
\[
= \sum_{a=1}^{n} g(\Pi(e_\alpha, e_\beta), \Pi(e_\alpha, e_\beta))^2.
\]
In this connection, we shall define the scalar curvature $\tau(T_x\Omega^n)$ of Riemannian submanifold $\Omega^n$, which is considered as a Riemannian intrinsic invariant, at some $x$ in $\Omega^n$ as follows:
\[
\tau(T_x\Omega^n) = \sum_{1 \leq \beta < \alpha \leq n} K_{\alpha\beta},
\]
where $K_{\alpha\beta} = K(e_\beta, e_\alpha)$. The first equality (15) is equivalent to the following equation which will be frequently used in subsequent proofs:
\[
2\tau(T_x\Omega^n) = \sum_{1 \leq \beta < \alpha \leq n} K_{\alpha\beta} + 1, \alpha \leq n.
\]
In a similar manner, the scalar curvature $\tau(L_x)$ of an $L$–plane is defined as
\[
\tau(L_x) = \sum_{1 \leq \beta < \alpha \leq n} K_{\alpha\beta}.
\]
If the plane sections are spanned by $e_\beta$ and $e_\alpha$, at $x$, we give denotations $\bar{K}_{\beta\alpha}$ and $K_{\beta\alpha}$, respectively, for the sectional curvature of the Riemannian manifold $M^m$ and submanifold $\Omega^n$, which are considered as the extrinsic and intrinsic sectional curvature of the span $\{e_\beta, e_\alpha\}$ at $x$. Using Gauss (12) and (15), we have
\[
\sum_{1 \leq \beta < \alpha \leq k + m} K_{\beta\alpha} = \sum_{1 \leq \beta < \alpha \leq k + m} \bar{K}_{\beta\alpha} + \sum_{r=m+1}^{\infty} \Pi_{\alpha\beta}^r \Pi_{\alpha\beta}^r - \left( \Pi_{\alpha\beta}^r \right)^2.
\]
On the contrary, the conception of warped product manifolds was originally introduced by Bishop and O’Neill [25] for manifolds of negative curvature. Assume that $N_{l}^p$ and $N_{l}^q$ are two Riemannian manifolds with Riemannian metrics $g_1$ and $g_2$, respectively. Assume that $h$ is a differentiable function on $N_{l}^p$. A warped product manifold is $\Omega^{p+q} = N_{l}^p \times_h N_{l}^q$ with $n = p + q$ and the Riemannian metric $g = g_1 + h^2 g_2$. Assume that $\Omega^{p+q} = N_{l}^p \times_h N_{l}^q$ is a warped product manifold. Hence, $\forall W_1 \in \Gamma(TN_{l}^p)$ and $W_2 \in \Gamma(TN_{l}^q)$, we attain that
\[
\nabla_{W_1}W_1 = \nabla_{W_1}W_2 = (W_1 \ln h)W_2.
\]
Furthermore, $\nabla (\ln h)$ is the gradient of $\ln h$, given by
\[
g(\nabla \ln h, W_1) = W_1 (\ln h).
\]
Thus, from [26], we have
\[
R(W_1, W_2)Y = \frac{H^h(W_1, Y)W_2}{h},
\]
where $H^h$ is a Hessian tensor of $h$. Let $\{e_1, \ldots, e_n\}$ be a local orthonormal basis of vector field $\Omega^{p+q}$; thus, the squared norm of the gradient of the positive differential function $h$ for an orthonormal basis $\{e_1, \ldots, e_n\}$ is
\[
\|\nabla h\|^2 = \sum_{i=1}^{n} (e_i(h))^2.
\]
Now, if we replace $p + q$ in (20) by $n$, we get
which leads to important tool in proving our result. In the first case, we 3.Proof of Main Results

also need to use the following method which is an

important tool in proving our result. In the first case, we

where

\[ g(W_1, W_2) = g(R(W_1, W_2)W_1, W_2) = \left( (\nabla W_1, W_1) \ln h g(W_2, W_2) \right) - g(\nabla W_1, (W_1 \ln h) W_2, W_2), \]

\[ = (\nabla W_1, W_1) \ln h g(W_2, W_2) - g(\nabla W_1, (W_1 \ln h) W_2) + (W_2 \ln h) \nabla W_1, W_2, W_2), \]

\[ = (\nabla W_1, W_1) \ln h g(W_2, W_2) - (W_1 \ln h)^2 - W_1 (W_1 \ln h). \]

Let us sum up over the vector fields such that

\[ \sum_{\beta=1}^{p} \sum_{a=1}^{q} K(e_{\beta} \wedge e_{a}) = \sum_{\beta=1}^{p} \sum_{a=1}^{q} \left( (\nabla e_{\beta}, e_{a}) \ln h - e_{\beta}(e_{a} \ln h) - (e_{\beta} \ln h)^2 \right). \]

Remark 2. It should be noted that we consider the opposite sign of Chen [27] of Laplacian of \( h \); that is, \( \Delta h = r \ln h \). Moreover, because unit vector fields \( W_1 \) and \( W_2 \) are tangent to \( N_1^r \) and \( N_2^q \), respectively, we get

\[ K(W_1 \wedge W_2) = g(R(W_1, W_2)W_1, W_2) = (\nabla W_1, W_1) \ln h g(W_2, W_2) - g(\nabla W_1, (W_1 \ln h) W_2), \]

which leads to

\[ \sum_{\beta=1}^{p} \sum_{a=1}^{q} K(e_{\beta} \wedge e_{a}) = -\frac{q \Delta h}{h}. \]

3. Proof of Main Results

We also need to use the following method which is an important tool in proving our result. In the first case, we

\[ \sum_{\beta=1}^{p} \sum_{a=1}^{q} \left\{ 2\left\| \Pi(e_{\beta}, e_{a}) \right\|^2 - g(\Pi(e_{\beta}, e_{a}), \Pi(e_{\beta}, e_{a})) \right\} \]

\[ = \sum_{\beta=1}^{p} \sum_{a=1}^{q} g(R(e_{\beta}, e_{a}) e_{\beta}, e_{a}) - \sum_{\beta=1}^{p} \sum_{a=1}^{q} g(R(e_{\beta}, e_{a}) e_{\beta}, e_{a}) + \sum_{r=1}^{m} \sum_{\beta=1}^{p} \sum_{a=1}^{q} (\Pi_{e_{a}})^2. \]

From \( R(e_{\beta}, e_{a}) e_{\beta} = (\mathcal{R}^h (e_{\beta}, e_{\beta})/h) e_{a} \) in (21), we derive

\[ \sum_{\beta=1}^{p} \sum_{a=1}^{q} \left\{ 2\left\| \Pi(e_{\beta}, e_{a}) \right\|^2 - g(\Pi(e_{\beta}, e_{a}), \Pi(e_{\beta}, e_{a})) \right\} \]

\[ = \sum_{\beta=1}^{p} \sum_{a=1}^{q} g(R(e_{\beta}, e_{a}) e_{\beta}, e_{a}) - \sum_{\beta=1}^{p} \sum_{a=1}^{q} g(R(e_{\beta}, e_{a}) e_{\beta}, e_{a}) + \sum_{r=1}^{m} \sum_{\beta=1}^{p} \sum_{a=1}^{q} (\Pi_{e_{a}})^2. \]
\[
\sum_{\beta=1}^{p} \sum_{\alpha=1}^{q} g(R(e_{\beta}, e_{\alpha})e_{\beta}, e_{\alpha}) = \frac{q}{h} \sum_{\beta=1}^{p} g(\nabla_{e_{\beta}} h, e_{\beta}). \quad (29)
\]

Thus, from (28) and (29), we derive

\[
\sum_{\beta=1}^{p} \sum_{\alpha=1}^{q} \left\{ 2\|\Pi(e_{\beta}, e_{\alpha})\|^2 - g(\Pi(e_{\alpha}, e_{\alpha}), \Pi(e_{\beta}, e_{\beta})) \right\}
\]

\[
= \frac{q}{h} \sum_{\beta=1}^{p} g(\nabla_{e_{\beta}} h, e_{\beta}) + \sum_{\beta=1}^{p} \sum_{\alpha=1}^{q} g(R(e_{\beta}, e_{\alpha})e_{\beta}, e_{\alpha}) \sum_{r=1}^{m} \sum_{s,l=1}^{q} (\Pi^{r}_{fs})^2.
\]  

(30)

Computing the Laplacian \(\Delta h\) on \(\Omega^{p,q}\), one obtains

\[
\Delta h = \sum_{\beta=1}^{p} g(\nabla_{e_{\beta}} \text{grad} h, e_{\beta}) + \sum_{i=1}^{n} g(\nabla_{e_{i}} \text{grad} h, e_{\beta}) = \sum_{\alpha=1}^{q} g(\nabla_{e_{\alpha}} \text{grad} h, e_{\alpha}).
\]  

(31)

Since \(N^{p}_{q}\) is totally geodesic in \(\Omega^{p,q}\), \(\text{grad} h \in \mathfrak{X}(TN_{1})\), and by utilizing the definition of the warped product, we obtain

\[
\Delta h = \frac{1}{h} \sum_{\alpha=1}^{q} g(e_{\alpha}, e_{\alpha})\|\nabla h\|^2 + \sum_{\beta=1}^{p} g(\nabla_{e_{\beta}} \text{grad} h, e_{\beta}).
\]  

(32)

Multiplying the above equation by \((1/h)\), we get

\[
\sum_{\beta=1}^{p} \sum_{\alpha=1}^{q} \left\{ 2\|\Pi(e_{\beta}, e_{\alpha})\|^2 - g(\Pi(e_{\alpha}, e_{\alpha}), \Pi(e_{\beta}, e_{\beta})) \right\}
\]

\[
= \sum_{r=m+1}^{m} \sum_{s,l=1}^{q} (\Pi^{r}_{fs})^2 + \frac{q\Delta h}{h} - q^2 \|\nabla \ln h\|^2 - \sum_{\beta=1}^{p} \sum_{\alpha=1}^{q} g(R(e_{\beta}, e_{\alpha})e_{\beta}, e_{\alpha}).
\]  

(35)

Next, using Gauss equation (12) for hyperbolic space \(H^m(-1)\), we find that

\[
n^2\|H\|^2 - n(n-1) = \|\Pi\|^2 + \sum_{1 \leq A < B \leq n} K(e_{A} \wedge e_{B}).
\]  

(36)

By rewriting the above equation for \(\Omega^{p,q}\) and utilizing (13) and (21), we attain

\[
\sum_{r=m+1}^{m} \left( \sum_{A=1}^{n} \Pi^{r}_{AA} \right)^2 + \sum_{r=m+1}^{m} \left( \Pi^{r}_{AA} \right)^2 + \sum_{r,m+1}^{m} \sum_{a,b=1}^{q} (\Pi^{r}_{ab})^2
\]

\[
+ 2 \sum_{r=m+1}^{m} \sum_{a=1}^{q} \sum_{\beta=1}^{p} (\Pi^{r}_{a\beta})^2 + \sum_{a=1}^{q} K(e_{\beta} \wedge e_{a}) + \sum_{1 \leq A < B \leq n} K(e_{A} \wedge e_{B}).
\]  

(37)

Using (18) and (27) in the above equation, we derive
After some rearrangements of the above equation, we get

\[
\sum_{t=n+1}^{m} \left( \sum_{A=1}^{n} \Pi_{AA}^t \right)^2 = \sum_{t=n+1}^{m} \sum_{i=1}^{p} (\Pi_{ii}^t)^2 + \sum_{i=1}^{p} (\Pi_{ii}^t)^2 + 2 \sum_{t=n+1}^{m} \sum_{a,b=1}^{q} (\Pi_{ab}^t)^2 - \frac{q \Delta h}{h} + n(n-1)
\]

\[
- p(p-1) - q(q-1) + 2 \sum_{t=n+1}^{m} \sum_{a,b=1}^{q} (\Pi_{ab}^t)^2 - \frac{q \Delta h}{h} - \sum_{t=n+1}^{m} \sum_{1 \leq i \leq p} (\Pi_{ii}^t)^2 - \sum_{1 \leq i \leq p} (\Pi_{ii}^t)^2 + \sum_{t=n+1}^{m} \Pi_{ii}^t \Pi_{ii}^t + n(n-1)
\]

\[+ \sum_{t=n+1}^{m} \left( \Pi_{11}^t \right)^2 + \cdots + \left( \Pi_{pp}^t \right)^2 - \sum_{i=1}^{m} \left( \Pi_{11}^t \right)^2 + \cdots + \left( \Pi_{pp}^t \right)^2
\]

\ [+ \sum_{t=n+1}^{m} \sum_{1 \leq a \leq b \leq q} \Pi_{ab}^t \Pi_{ab}^t - \sum_{t=n+1}^{m} \sum_{1 \leq a \leq b \leq q} (\Pi_{ab}^t)^2
\]

\[+ \sum_{t=n+1}^{m} \left( \Pi_{p+1,p+1}^t \right)^2 + \cdots + \left( \Pi_{mm}^t \right)^2 - \sum_{t=n+1}^{m} \left( \Pi_{p+1,p+1}^t \right)^2 + \cdots + \left( \Pi_{mm}^t \right)^2
\]

(38)

Thus, using the curvature equation and the sphere \( H^m(-1) \) and rearranging the last equation, we attain

\[
\sum_{t=n+1}^{m} \left( \sum_{A=1}^{n} \Pi_{AA}^t \right)^2 = \sum_{t=n+1}^{m} \sum_{i=1}^{p} (\Pi_{ii}^t)^2 + \sum_{i=1}^{p} (\Pi_{ii}^t)^2 + 2 \sum_{t=n+1}^{m} \sum_{a,b=1}^{q} (\Pi_{ab}^t)^2 - \frac{q \Delta h}{h} - 2pq
\]

(39)

After some rearrangements of the above equation, we get

\[
\sum_{t=n+1}^{m} \left( \sum_{A=1}^{n} \Pi_{AA}^t \right)^2 = \sum_{t=n+1}^{m} \sum_{i=1}^{p} (\Pi_{ii}^t)^2 + \sum_{i=1}^{p} (\Pi_{ii}^t)^2 + 2 \sum_{t=n+1}^{m} \sum_{a,b=1}^{q} (\Pi_{ab}^t)^2 - \frac{q \Delta h}{h}
\]

\[- 2pq + \sum_{t=n+1}^{m} \left\{ \sum_{1 \leq i \leq p} \Pi_{ii}^t \Pi_{ii}^t + \left( \Pi_{11}^t \right)^2 + \cdots + (\Pi_{pp}^t)^2 \right\}
\]

\[- \sum_{t=n+1}^{m} \left\{ \sum_{1 \leq i \leq p} (\Pi_{ii}^t)^2 + (\Pi_{11}^t)^2 + \cdots + (\Pi_{pp}^t)^2 \right\}
\]

\[+ \sum_{t=n+1}^{m} \left\{ \sum_{1 \leq a \leq b \leq q} \Pi_{ab}^t \Pi_{ab}^t + (\Pi_{p+1,p+1}^t)^2 + \cdots + (\Pi_{mm}^t)^2 \right\}
\]

\[- \sum_{t=n+1}^{m} \left\{ \sum_{1 \leq a \leq b \leq q} (\Pi_{ab}^t)^2 + (\Pi_{p+1,p+1}^t)^2 + \cdots + (\Pi_{mm}^t)^2 \right\}
\]

(40)
Utilizing the binomial theorem and the fact that the base
\( N^p \) of a warped product manifold \( N^p \times_h N^q \) is minimal, we

\[
\sum_{t=n+1}^{m} \left( \sum_{A=p+1}^{n} \Pi_{A,t}^t \right)^2 = -2pq + \sum_{t=n+1}^{m} \sum_{s,l=1}^{p} (\Pi_{s,l}^t)^2 + \sum_{t=n+1}^{m} \sum_{a,b=1}^{q} (\Pi_{a,b}^t)^2 \\
+ 2 \sum_{t=n+1}^{m} \sum_{s,l=1}^{p} (\Pi_{s,l}^t)^2 - \frac{q\Delta h}{h} \\
+ \sum_{t=n+1}^{m} \left( (\Pi_{11}^t)^2 + \cdots + (\Pi_{pp}^t)^2 \right) - \sum_{t=n+1}^{m} \sum_{s,l=1}^{p} (\Pi_{s,l}^t)^2 \\
- \sum_{t=n+1}^{m} \sum_{a,b=1}^{q} (\Pi_{a,b}^t)^2 + \sum_{t=n+1}^{m} \left( (\Pi_{p+1,p+1}^t)^2 + \cdots + (\Pi_{nn}^t)^2 \right)
\]

(41)

Utilizing the assumption of the theorem and since \( N^p \) is
minimal, the fifth term of the right-hand side (RHS) in
equation (41) is identically zero, and the first term of the left-
hand side is equal to the seventh term of the RHS. Hence, we
get

\[
\sum_{t=n+1}^{m} \sum_{s,l=1}^{p} (\Pi_{s,l}^t)^2 = \frac{q\Delta h}{h} + 2pq.
\]

(42)

From (36) and (42), we get

\[
\sum_{t=n+1}^{m} \sum_{s,l=1}^{p} (\Pi_{s,l}^t)^2 = \frac{q\Delta h}{h} + 2pq.
\]

(43)

Then, from equation (11), one obtains

\[
\sum_{t=n+1}^{m} \sum_{s,l=1}^{p} \sum_{a=1}^{q} \left( 2\|\Pi(e_{\beta}, e_a)\|^2 - g(\Pi, \Pi(e_{\alpha}, e_a)(e_{\beta}, e_{\beta})) \right) = \frac{q\Delta h}{h} - q^2\|\nabla\ln h\|^2 + \frac{q\Delta h}{2h} + pq
\]

(44)

This follows that

\[
\sum_{t=n+1}^{m} \sum_{s,l=1}^{p} \sum_{a=1}^{q} g(\tilde{R}(e_{\beta}, e_a)e_{\beta}, e_a).
\]

(45)

From our assumption (3) and (45), we obtain

\[
\sum_{t=n+1}^{m} \sum_{s,l=1}^{p} \left( 2\|\Pi(e_{\beta}, e_a)\|^2 - g(\Pi(e_{\alpha}, e_a), \Pi(e_{\beta}, e_{\beta})) \right) < \frac{3q\Delta h}{2h} - \frac{q^2}{h^2}\|\nabla\|^2.
\]

(46)
Applying Theorem 1 for constant holomorphic sectional curvature \( c = 1 \), we obtain that there are no stable \( p \)-currents in \( \Omega^{p,q} \) and \( H_p(\Omega^{p,q}, \mathbb{Z}) = H_q(\Omega^{p,q}, \mathbb{Z}) = 0 \), which completes the proof of (a) and (b) of the theorem. In the other part, from (45), substituting \( p = 1 \), we have

\[
\sum_{a=2}^{n} \left\{ 2\|\Pi(e_1, e_a)\|^2 - g(\Pi(e_a, e_a), \Pi(e_1, e_1)) \right\} = \frac{3q\Delta h}{2h} - \frac{q}{h^2}\|\nabla h\|^2. \tag{47}
\]

If the pinching condition (22) for \( p = 1 \) and \( q = n - 1 \) holds, then we get

\[
\sum_{a=2}^{n} \left\{ 2\|\Pi(e_1, e_a)\|^2 - g(\Pi(e_a, e_a), \Pi(e_1, e_1)) \right\} < (n + 1). \tag{48}
\]

Then, there are no stable 1-currents in \( \Omega^{1,q} \) and \( H_1(\Omega^{1,q}, \mathbb{Z}) = H_{n-1}(\Omega^{1,q}, \mathbb{Z}) = 0 \). Let us assume that \( \pi_{1}(\Omega) \neq 0 \). The compactness property of \( \Omega^{1,q} \); it follows from the classical theorem, using the results of Cartan and Hadamard, which states that there is a minimal closed geodesic in any nontrivial homotopy class in \( \pi_{1}(\Omega) \), and this leads to a contradiction. Therefore, \( \pi_{1}(\Omega) = 0 \). This is the third part of the theorem. If the finite fundamental group is null for any Riemannian manifold, this Riemannian manifold is simply connected. As a result, \( \Omega^{p,q} \) is simply connected.

3.2 Proof of Theorem 3. If \( \Delta h = -\lambda_1 h \), that is, \( h \) is the first eigenfunction of Laplacian \( \Delta h = \text{div} (\nabla h) \) of \( \Omega^{p,q} \) corresponding to the first nonzero eigenvalue \( \lambda_1 \) of \( h \). Recall the Bochner formula for a differentiable function \( h \) defined on a Riemannian manifold (see, e.g., [28]) which states that the following formula

\[
\frac{1}{2} \Delta \|\nabla h\|^2 = \|\nabla h\|^2 + \text{Ric}(\nabla h, \nabla h) + g(\nabla h, \nabla (\Delta h)), \tag{49}
\]

holds. Then, by integrating the above equation and using Stokes’ theorem, we get

\[
\int_{\Omega^{p,q}} \|\nabla^2 h\|^2 dV + \int_{\Omega^{p,q}} \text{Ric}(\nabla h, \nabla h) dV + \int_{\Omega^{p,q}} g(\nabla h, \nabla (\Delta h)) dV = 0. \tag{50}
\]

Now, using \( \Delta h = -\lambda_1 h \) and making some rearrangements in equation (50), we derive

\[
\int_{\Omega^{p,q}} \|\nabla^2 h\|^2 dV = \frac{1}{\lambda_1} \left( \int_{\Omega^{p,q}} \|\nabla^2 h\|^2 dV + \int_{\Omega^{p,q}} \text{Ric}(\nabla h, \nabla h) dV \right). \tag{51}
\]

By assumption of the theorem, (5) holds, then we have

\[
\int_{\Omega^{p,q}} \left\{ \|\nabla^2 h\|^2 + \text{Ric}(\nabla h, \nabla h) \right\} dV + \frac{\lambda_1}{2q} \left( \frac{4p^2}{q} + 2p + 3\lambda_1 \right) \int_{\Omega^{p,q}} h^2 dV > 0. \tag{52}
\]

Then, substituting equation (52) in (51), we get

\[
-\frac{\lambda_1}{2q} \left( \frac{4p^2}{q} + 2p + 3\lambda_1 \right) \int_{\Omega^{p,q}} h^2 dV < \lambda_1 \int_{\Omega^{p,q}} \|\nabla h\|^2 dV, \tag{53}
\]

which implies that

\[
-3\lambda_1 \int_{\Omega^{p,q}} h^2 dV < \frac{2p}{q} (2p + q) \int_{\Omega^{p,q}} h^2 dV + 2q \int_{\Omega^{p,q}} \|\nabla h\|^2 dV. \tag{54}
\]

Now, using \( \Delta = -\lambda_1 h \) in the left-hand side of equation (54), we arrive at

\[
\int_{\Omega^{p,q}} \left\{ 3h\Delta h - 2q\|\nabla h\|^2 - \frac{2p}{q} (2p + q)h^2 \right\} dV < 0. \tag{55}
\]

Then, the above equation gives us

\[
3h\Delta h < 2q\|\nabla h\|^2 + \frac{2p}{q} (2p + q)h^2. \tag{56}
\]
Then, using the above equation and also Theorem 2, we get the conclusion of our theorem.

3.3. Proof of Theorem 4. As we know that $\Omega^{p,q}$ is $N^p_1$, minimal compact warped product submanifold, from the Gauss equation, one obtains

$$R^l_{jkl} = -\left(\delta \delta_{jl} - \delta_d \delta_{jl}\right) + \sum_{r=1}^{m} \left(\Pi^r_i \Pi^r_{jl} - \Pi^r_i \Pi^r_{jk}\right),$$

(57)

which implies the following:

$$R^l_{jij} = -\left(\delta \delta_{ij} - \delta_d \delta_{ij}\right) + \sum_{r=1}^{m} \left(\Pi^r_i \Pi^r_{ij} - \Pi^r_i \Pi^r_{ij}\right).$$

(58)

Taking into account that $N^p_1$ is a minimal submanifold and using the argument of the Ricci curvature for the hyperbolic space, we get

$$\int_{\Omega^{p,q}} \sum_{i=1}^{p} \|\Pi (\nabla h, e_i)\|^2 dV + (p - 1 + \lambda_1) \int_{\Omega^{p,q}} \|\nabla h\|^2 dV = \int_{\Omega^{p,q}} \|\nabla h\|^2 dV.$$  
(62)

If our assumption (1.4) is satisfied, then

$$\int_{\Omega^{p,q}} \|\nabla^2 h\|^2 dV < \int_{\Omega^{p,q}} \sum_{i=1}^{p} \|\Pi (\nabla h, e_i)\|^2 dV + \frac{(p - 1 - \lambda_1)}{2q} \left(\frac{4p^2}{q} + 2p + 3\lambda_1\right) \int_{\Omega^{p,q}} h^2 dV.$$  
(63)

Rewrite the above equation using $\Delta h = -\lambda_1 h$:

$$\frac{3(p - 1 - \lambda_1)}{2q} \int_{\Omega^{p,q}} h \Delta h dV + \int_{\Omega^{p,q}} \|\nabla h\|^2 dV < \int_{\Omega^{p,q}} \sum_{i=1}^{p} \|\Pi (\nabla h, e_i)\|^2 dV + \frac{(p - 1 - \lambda_1)}{2q} \left(\frac{4p^2}{q} + 2p + 3\lambda_1\right) \int_{\Omega^{p,q}} h^2 dV.$$  
(64)

Thus, we get the conclusion of our theorem.

3.4. Proof of Corollary 1. Using the hypothesis of the corollary, $\forall h \in \text{Ker} \Omega$, we get $\Pi (\nabla h, e_i) = 0$. Using this condition in (7), we can easily obtain the required result.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare no conflicts of interest.
Authors’ Contributions

All authors contributed equally to this study and finalized the manuscript.

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