LYAPUNOV EXPONENTS OF DISCRETE QUASI-PERIODIC GEVREY SCHRÖDINGER EQUATIONS

WENMENG GENG AND KAI TAO*

College of Sciences, Hohai University, No.1 Xikang Road
Nanjing, Jiangsu, 210098, China

(Communicated by Xiaoying Han)

Abstract. In the study of the continuity of the Lyapunov exponent for the
discrete quasi-periodic Schrödinger operators, there is a pioneering result by
Wang-You [21] that the authors constructed examples whose Lyapunov ex-
ponent is discontinuous in the potential with the $C^0$ norm for non-analytic
potentials. In this paper, we consider this operators for some Gevrey poten-
tial, which is an analytic function having a Gevrey small perturbation, with
Diophantine frequency. We prove that in the large coupling regions, the Lya-
punov exponent is positive and jointly continuous in all parameters, such as the
energy, the frequency and the potential. Note that all analytic functions are
also Gevrey ones. Therefore, we also obtain that all of the large analytic poten-
tials are the non-perturbative weak Hölder continuous points of the Lyapunov
exponent in the Gevrey topology with $C^0$ norm. It is the first result about the
continuity in non-analytic potential with this norm and is complementary to
Wang-You’s result.

1. Introduction. In this paper, we consider the following discrete Schrödinger
operators on $l^2(\mathbb{Z}^+)$:

$$(S_{x, \lambda f + g, \omega} \phi)(n) = \phi(n + 1) + \phi(n - 1) + [\lambda f(x + n\omega) + g(x + n\omega)] \phi(n),$$

where $\lambda$ is called coupling number, $f(x)$ is a real analytic function on $\mathbb{T}$ and $g(x)$ is
a Gevrey function which is considered to be a perturbation in this paper. Here we
say a $C^\infty$ function $v$ is a Gevrey class $G^s(\mathbb{T})$ for some $s > 1$, if

$$\sup_{x \in \mathbb{T}} |\partial^m v(x)| \leq MK^m (m!)^s \quad \forall m \geq 0$$

for some constants $M, K \geq 0$. In [16], Klein showed that the condition (2) is
equivalent to the following weak exponential decay of the Fourier coefficients of $v$:

$$|\hat{v}(k)| \leq M \exp\left(-\rho |k|^{\frac{1}{s}}\right) \quad \forall k \in \mathbb{Z}$$

for some constants $M, \rho > 0$. Thus, we say $v \in G^{s, M, \rho}(\mathbb{T})$ when (3) holds. Obviously,

$$G^s(\mathbb{T}) = \bigcup_{\rho > 0, M > 0} G^{s, M, \rho}(\mathbb{T})$$

2020 Mathematics Subject Classification. Primary: 37C55, 37D25; Secondary: 30D60.
Key words and phrases. Lyapunov exponent, Schrödinger equations, Gevrey perturbation,
weak Hölder continuity, jointly continuous.

The second author was supported by the Fundamental Research Funds for the Central Universities(Grant B200202004) and China Postdoctoral Science Foundation (Grant 2019M650094).
* Corresponding author: Kai Tao.
and if \( s_1 < s_2, M_1 < M_2 \) and \( \rho_1 > \rho_2 \), then \( G^{s_1, M_1, \rho_1}(T) \subset G^{s_2, M_2, \rho_2}(T) \).

It is obvious that the characteristic equation \( S_{x, \lambda f + g, \omega} \phi = E \phi \) can be expressed as

\[
\begin{pmatrix}
\phi(n + 1) \\
\phi(n)
\end{pmatrix} = M_n(x, E, \lambda f + g, \omega) \begin{pmatrix}
\phi(1) \\
\phi(0)
\end{pmatrix},
\]

where

\[
M_n(x, E, \lambda f + g, \omega) = \prod_{j=n}^{1} \begin{pmatrix}
E - \lambda f(x + j\omega) - g(x + j\omega) & -1 \\
1 & 0
\end{pmatrix}
\]

is called transfer matrix of (1). Define

\[
L_n(E, \lambda f + g, \omega) = \frac{1}{n} \int_T \log \|M_n(x, E, \lambda f + g, \omega)\| dx,
\]

and the limit

\[
L(E, \lambda f + g, \omega) = \lim_{n \to \infty} L_n(E, \lambda f + g, \omega)
\]

exists by Kingman’s Subadditive Ergodic Theorem, which is called Lyapunov exponent of (1).

In this paper, we always assume \( \omega \) to be Diophantine \( D_{c,A} \). It means that \( \omega \) satisfies the Diophantine condition

\[
\|n\omega\| \geq \frac{c}{n^A} \quad \text{for all } n \neq 0,
\]

where \( A > 1 \). It is well known that almost every \( \omega \) satisfies (4).

Then we have the following theorem for the positivity and the weak Hölder continuity of Lyapunov exponents:

**Theorem 1.1.** Consider the Schrödinger operator (1) with some fixed analytic function \( f(x) \). There exists \( \lambda_0 = \lambda_0(f) \) such that for any \( \lambda > \lambda_0, \rho > 0, M > 0, s > 1, A > 1 \) and \( c > 0 \), there exists \( r = r(\lambda, c, A, \rho, s, M) \) such that for any \( g \in G^{s, M, \rho}(T) \) satisfying \( \|g\|_\infty < r \), \( \omega \in D_{c,A} \) and \( E \in \mathbb{R} \), it yields that

\[
L(E, \lambda f + g, \omega) > \frac{1}{2} \log \lambda,
\]

and

\[
|L(E, \lambda f, \omega) - L(E, \lambda f + g, \omega)| \leq \exp \left( - \log \|g\|_\infty \right)\frac{1}{2^d},
\]

where \( \|g\|_\infty = \max_{x \in T} |g(x)| \).

**Remark 1.** It shows that in the \( G^{s, M, \rho}(T) \) topology, there exists a neighborhood constructed by the centre \( \lambda f \) and the radius \( r \) in the \( C^0 \) norm, where the continuity of the Lyapunov exponent in the potential holds. Note that \( r \) depends on \( \lambda, c, A, \rho, s, M \). Therefore, \( \lambda f \) is always in the neighborhood for any \( G^{s, M, \rho}(T) \) and \( D_{c,A} \). And then, \( \lambda f \) is the non-perturbative continuity point of Lyapunov exponent in the \( G(T) = \bigcup_{s \geq 1, \rho > 0, M > 0} G^{s, M, \rho}(T) \) topology with \( C^0 \) norm for the \( D = \bigcup_{c > 0, A > 1} D_{c,A} \) frequency.

In the past long period of time, the study of the continuity of the Lyapunov exponent in parameters is a hot topics in our fields. In 2001, Goldstein-Schlag[11] proved that the Lyapunov exponent is Hölder continuous in \( E \) when the potential is analytic, which means that \( g \equiv 0 \) in (1). The frequency \( \omega \) in that paper is assumed to be strong Diophantine, which satisfies the strong Diophantine condition

\[
\|n\omega\| \geq \frac{C_\omega}{n ([\log n]^\alpha + 1)} \quad \text{for all } n \neq 0.
\]
It is thought as a very important breakthrough, as they developed two powerful techniques, the Large Deviation Theorem (LDT for short) and the Avalanche Principle. They are applied in the following many literatures, including ours, to prove the continuity of the Lyapunov exponent. For the same potential, Bourgain-Jitomirskaya [7] proved the continuity in $E$ with any irrational frequency, You-Zhang [27] obtained the Hölder continuity with weak Liouville frequency, and recently Han-Zhang [13] improved it to the finite Liouville frequency. Here the definitions of weak Liouville frequency and the finite Liouville one both come from the continued fraction approximates. For any irrational $\omega$, there exist its continued fraction approximates $\{p_s/q_s\}_{s=1}^{\infty}$, satisfying

$$\frac{1}{q_s(q_{s+1} + q_s)} < |\omega - p_s/q_s| < \frac{1}{q_s q_{s+1}}.$$ 

Define $\beta$ as the exponential growth exponent of $\{p_s/q_s\}_{s=1}^{\infty}$ as follows:

$$\beta(\omega) := \limsup_s \log \frac{q_{s+1}}{q_s} \in [0, \infty].$$

Obviously, if $\omega$ is strong Diophantine (5) or Diophantine (4), then $\beta(\omega) = 0$. We say $\omega$ is a weak Liouville number or finite Liouville number, if $\beta(\omega) < c$, where $c$ is a small constant, or $\beta(\omega) < \infty$. When the potential is analytic on $T^d$ with $d > 1$, Bourgain-Goldstein [6] proved the weak Hölder continuity in $E$ with the Diophantine multi-frequency, and Bourgain [5] obtained the continuity with the irrational multi-frequency. In [14], Jitomirskaya-Koslover-Schulteis showed the weak Hölder continuity on analytic quasi-periodic $GL(2, \mathbb{C})$ cocycles, which have the same determinant, with the strong Diophantine frequency. Then in [15], Jitomirskaya-Marx removed the restriction on determinants. In [19], our second author obtained the same result when the analytic cocycles are defined on $T^d$ with $d > 1$. For a class of Gevrey potentials satisfying (2) and the following called non-degeneracy condition:

$$\forall x \in T, \exists m \in \mathbb{Z}, s.t. \nu^{(m)}(x) \neq 0,$$

Klein [16] also proved the weak Hölder continuity in $E$ with the strong Diophantine frequency.

For the finitely smooth potential, the continuity may be not valid for general cocycles. The references [2, 3, 8] showed that discontinuity of Lyapunov exponent holds at every non-uniformly hyperbolic cocycles in $C^0$— topology. Wang-You [21, 22] gave examples that the Lyapunov exponents can be discontinuous in $C^r(1 < r \leq \infty)$ topology by the methods of Young [26]. Note that their result can be improved to degenerate Gevrey functions in $C^0$ norm easily. Then, Wang-Zhang [24] applied these methods to obtain the weak Hölder continuity in $E$ for the $C^2$ cos-type potential satisfying the non-degeneracy condition (6) with the strong Diophantine frequency. In [17], Liang-Wang-You improved it to be Hölder continuous in $E$ with the finite Liouville frequency and Xu-Ge-Wang [25] announced that $\frac{1}{2}$–Hölder continuity also holds for that operators with the strong Diophantine frequency.

If we review the above results, then we will find that when we want to study the continuity in the potential or the cocycles, Jitomirskaya et al obtained the affirmative answers for the analytic cocycles, but Wang and You gave the counterexamples for the $C^l(1 \leq l \leq \infty)$ cocycles. Therefore, there is a gap for the quasi-analytic cocycles. This is the reason why we consider our operators (1). Recall that Klein [16] had already studied the Lyapunov exponent of this discrete quasi-periodic Schrödinger operator with Gevrey potential. However, he needs the non-degeneracy condition...
(6) and proved the continuity in energy. In that reference, he applied this condition to obtain the Lojasiewicz-type inequality [18], which is always valid for analytic functions, for the Gevrey functions: there exist constants $C = C(v)$ and $\kappa = \kappa(v)$ such that for any $\delta > 0$,

$$\sup_{E \in \mathbb{R}} \text{mes} \{ x : |v(x) - E| < \delta \} < C \delta^\kappa. \tag{7}$$

It makes the large deviation theorem, which is the key lemma for the continuity of Lyapunov exponents, also to depend on $v$. However, it is very hard to express this dependence in $C^0$ norm. The reason is that to obtain (7), he first used the well-known fact that any open cover on a compact set must have a finite subcover to prove that the non-degeneracy condition (6) is equivalent to the following statement:

$$\exists m \geq 1, \exists c > 0, \text{such that } \forall x, \max_{1 \leq k \leq m} |v^{(k)}(x)| \geq c,$$

where $m = m(v)$ and $c = c(v)$. Therefore, it is not easy to calculate the limit of $m$ and the variation of the condition (6) when $v_n \to v$ in $C^0$ norm. In order to overcome this problem, we choose the analytic function, not the Lojasiewicz-type inequality (7), to help us obtain the large deviation theorem in this paper. So, we assume our potential to be a Gevrey perturbation of an analytic function and study the continuity of the Lyapunov exponent when this perturbation tends to 0. We do not think that our assumption has removed the non-degeneracy condition (6). We believe that it just gives a strong hint that the Gevrey functions may have a uniform non-degeneracy condition, when they are close to a large analytic function. Note that our Theorem 1.1 is not only not contradictory to [21], but also complementary to it.

Recently, Ge-Wang [9] announced a result that they can give the examples that the Lyapunov exponent is discontinuous in the Gevrey potential with the Gevrey norm:

$$\|g\|_{s, \rho} := \sum_{k \in \mathbb{Z}} |\hat{g}(k) \exp \left( \rho |k|^{\frac{s}{1}} \right)|.$$

They also said that they can give a very short proof to show that the non-degeneracy condition (6) will have a uniform control when $v_n \to v$ in this Gevrey norm and then the continuity of Lyapunov exponent in [16] will be extended easily to hold in the potential with this norm.

Obviously, compared to $C^0$ norm, it is harder to give the discontinuity example and easier to prove the continuity in the Gevrey norm. Therefore, the first highlight of our paper is that we obtain the continuity of Lyapunov exponent in Gevrey potential with a much worse norm. What’s more, our continuity at large analytic potentials is non-perturbative. The last highlight is that we improve the frequency from the strong Diophantine number to the Diophantine one.

At last, when considering the continuity in other parameters, we also obtain the following theorem which shows the joint continuity in all parameters:

**Theorem 1.2.** Consider the Schrödinger operator (1) with some fixed analytic function $f(x)$. There exists $\lambda_0 = \lambda_0(f)$ such that for any $\lambda > \lambda_0$, $\rho > 0$, $M > 0$, $A > 1$ and $c > 0$, there exists $r = r(\lambda, c, A, \rho, s)$ such that for any $g_1, g_2 \in G^{s,n,\rho}(\mathbb{T})$, $\omega_1, \omega_2 \in D_{c,A}$ and $E_1, E_2 \in \mathbb{R}$ satisfying $\|g_1\|_\infty, \|g_2\|_\infty < r$ and $\|g_1 - g_2\|_\infty + |E_1 - E_2| + |\omega_1 - \omega_2| < r$, 

$$\|g_1\|_{s, \rho} := \sum_{k \in \mathbb{Z}} |\hat{g}(k) \exp \left( \rho |k|^{\frac{s}{1}} \right)|.$$
\[ |L(E_1, \lambda f + g_1, \omega_1) - L(E_1, \lambda f + g_2, \omega_2)| \leq \exp \left( -|\log (\|g_1 - g_2\|_\infty + |E_1 - E_2| + |\omega_1 - \omega_2|)|^{\frac{1}{20}} \right). \]

This paper is organized as follows. In Section 2, we introduce many definitions and symbols, and prepare some important relations and lemmas, especially the distance between the cocycles defined by the Gevrey function and the ones by its truncation functions, and the called strong Birkhoff ergodic theorem with subharmonic functions defined on different complex strips. With their help, we apply the induction to obtain the key lemma, the large deviation theorem, in Section 3. Then in the last section, we give the proofs of Theorem 1.1 and 1.2. In the appendix, we calculate two important results both about the distance between different transfer matrices, whose proofs are independent of the main parts of our paper.

2. Preliminaries. For every positive integer \( n \), consider the truncation
\[ g_n(x) := \sum_{|k| \leq \bar{n}} \hat{g}(k)e^{ikx}, \]
where \( \bar{n} = \deg g_n \) will be determined later. Note that \( g_n(x) \) is an analytic function on \( \mathbb{T} \) and \( g_n(z) \) is a holomorphic on the strip \( \{ z : \Im z \leq \rho_n \} \), where
\[ \rho_n = \frac{\rho}{2\bar{n}^{\frac{1}{2}}}. \]
Indeed, if \( z = x + iy \) with \( |y| < \rho_n \), then
\[ |g_n(z)| \leq \sum_{|k| \leq \bar{n}} |\hat{g}(k)| e^{|k||y|} \leq 2M \sum_{k=0}^{\bar{n}} \exp \left( -\frac{|k|}{2} \right) e^{|k||y|} \]
\[ < 2M \sum_{k=0}^{\bar{n}} \exp \left( -\frac{\rho}{2} |k| \right) < C. \]

It is also obvious that for any \( n \),
\[ |g(x) - g_n(x)| \leq M \exp \left( -\frac{\rho}{2\bar{n}^{\frac{1}{2}}} \right). \]  
(8)
We choose \( \bar{n} = n^{2^s} \) to make the error in (8) super exponentially small:
\[ |g(x) - g_n(x)| \leq M \exp \left( -\frac{\rho}{2n^{2^s}} \right). \]  
(9)
Then, the width of holomorphicity of \( g_n \) becomes:
\[ \rho_n = \frac{\rho}{2n^{-2(s-1)}}. \]

Set
\[ M_n(x, E, \omega) = \prod_{j=n}^{1} \begin{pmatrix} E - \lambda v(x + j\omega) & -1 \\ 1 & 0 \end{pmatrix}, \]
\[ M_n^0(x, E, \omega) = \prod_{j=n}^{1} \begin{pmatrix} E - \lambda f(x + j\omega) & -1 \\ 1 & 0 \end{pmatrix}, \]
and
\[ M_n'(x, E, \omega) = \prod_{j=n}^{1} \begin{pmatrix} E - \lambda v_n(x + j\omega) & -1 \\ 1 & 0 \end{pmatrix}, \]
where \( v = f + \frac{1}{n}g \) and \( v_n = f + \frac{1}{n}g_n \). Then, define
\[
\begin{align*}
  u_n(x, E, \omega) &= \frac{1}{n} \log \|M_n(x, E, \omega)\|, L_n(E, \omega) = \int_T u_n(x, E, \omega) dx, \\
  u_n^a(x, E, \omega) &= \frac{1}{n} \log \|M_n^a(x, E, \omega)\|, L_n^a(E, \omega) = \int_T u_n^a(x, E, \omega) dx, \\
  u_n^t(x, E, \omega) &= \frac{1}{n} \log \|M_n^t(x, E, \omega)\| \text{ and } L_n^t(E, \omega) = \int_T u_n^t(x, E, \omega) dx.
\end{align*}
\]

We always assume that \( \|g\|_\infty < 1 \) and \( E \in [-\lambda\|f\|_\infty - 3, \lambda\|f\|_\infty + 3] \), or our question will be trivial. Now,
\[
1 \leq u_n(x, E, \omega), u_n^a(x, E, \omega), u_n^t(x, E, \omega) \leq \log(2\lambda\|f\|_\infty + 6).
\]

Therefore, for any \( 0 < \tau_1 < 1 \), there exists a constant \( \lambda_1(f, \tau_1) \) such that if \( \lambda > \lambda_1(f, \tau_1) \), then
\[
L_n(E, \omega) = S(\lambda, \tau_1) = (1 + \tau_1) \log \lambda.
\]

Due to (A.2), it has that
\[
|u_n(x, E, \omega) - u_n^a(x, E, \omega)| \leq \|g\|_\infty \exp((n - 1)S). \tag{11}
\]

Similarly, due to (A.1) and (9), it yields that there exists \( \bar{n}_0 = \bar{n}_0(S, \rho, M) \) such that for any \( n > \bar{n}_0 \),
\[
\|M_n^t(x, E, \omega)\| - \|M_n(x, E, \omega)\| \leq n \exp((n - 1)S) \exp\left(-\frac{\rho n^2}{2}\right) \leq e^{-n}.
\]

Therefore,
\[
|u_n(x, E, \omega) - u_n^t(x, E, \omega)| \leq e^{-n} \tag{12}
\]

and
\[
|L_n(E, \omega) - L_n^t(E, \omega)| < e^{-n}. \tag{13}
\]

In the second half of this section, we consider the strong Birkhoff Ergodic Theorem for the subharmonic functions with a Diophantine shift on the torus, which is a key lemma in this paper. We all know the famous Birkhoff Ergodic Theorem, which says that if \( T : X \to X \) is an ergodic transformation on a measure space \((X, \Sigma, m)\) and \( f \) is an \( m \)-integrable function, then the time average functions \( f_n(x) = \frac{1}{n} \sum_{k=0}^{n-1} f(T^k x) \) converge to the space average \( \langle f \rangle = \frac{1}{m(X)} \int_X f dm \) for almost every \( x \in X \). But it doesn’t tell us how fast do they converge. So, we call a theorem the strong Birkhoff Ergodic Theorem, if it gives the convergence rate.

At first, we consider the simplest subharmonic function \( \log|z| \) with the Diophantine shift on the strip \( Z = \{z : |\text{Im}z| \leq \rho\} \). Let \( \{x\} = x - \lfloor x \rfloor \). For any positive integer \( q \), complex number \( \zeta = \xi + i\eta \) and \( 0 \leq \chi < 1 \), define
\[
F_{q, \zeta}(x) = \sum_{0 \leq k < q} \log |\{x + k\omega\} - \zeta| \text{ and } I(\zeta) = \int_0^1 \log |y - \zeta| dy.
\]

Then, we have that

**Lemma 2.1** (Lemma 2.2 in [10]). There exist constants \( \tilde{c} = \tilde{c}(c, A) \) and \( \tilde{C} = \tilde{C}(c, A) \) such that for any \( n \geq 1, \zeta \in Z \) and \( 0 < \sigma \leq \tilde{c} \), we have
\[
\int_0^1 \exp(\sigma |F_{n, \zeta}(x) - nI(\zeta)|) dx \leq \exp \left( \tilde{C} \sigma n \delta_0^a \right),
\]
where \( \delta_0^a = (cn)^{-\frac{1}{4}} \log^2(cn) \).
Combining it with the following Riesz’s theorem proved in [12], we can start the proof of our strong Birkhoff Ergodic Theorem:

**Lemma 2.2.** Let $u : \Omega \to \mathbb{R}$ be a subharmonic function on a domain $\Omega \subset \mathbb{C}$. Suppose that $\partial \Omega$ consists of finitely many piece-wise $C^1$ curves. There exists a positive measure $\mu$ on $\Omega$ such that for any $\Omega_1 \subset \Omega$ (i.e., $\Omega_1$ is a compactly contained subregion of $\Omega$),

$$u(z) = \int_{\Omega_1} \log |z - \zeta| \, d\mu(\zeta) + h(z),$$

where $h$ is harmonic on $\Omega_1$ and $\mu$ is unique with this property. Moreover, $\mu$ and $h$ satisfy the bounds

$$\mu(\Omega_1) \leq C(\Omega, \Omega_1) (\sup_\Omega u - \sup_{\Omega_1} u),$$

$$\|h - \sup_{\Omega_1} u\|_{L^\infty(\Omega_2)} \leq C(\Omega, \Omega_1, \Omega_2) (\sup_\Omega u - \sup_{\Omega_1} u)$$

for any $\Omega_2 \subset \Omega_1$.

Assume that $u(z)$ is a subharmonic function on the strip $Z$ with $S = \max_{z \in Z} u(z)$. Choose $\Omega = Z$, $\Omega_1 = Z_1 := \{z : |\text{Im}z| \leq \rho/2\}$ and $\Omega_2 = Z_2 := \{z : |\text{Im}z| \leq \rho/4\}$ in Lemma 2.2. Then, applying the relationship between $\Omega$, $\Omega_1$ and $\Omega_2$ in the proof of this lemma in [12] and (15), we have that

$$\mu(\Omega_1) \leq \frac{CS}{\rho},$$

where $C$ is an absolute constant coming from the Green function with respect to the rectangle. Notice that the ergodic measure for the shift on the Torus is the Lebesgue measure and $m(T) = 1$. Then,

$$\langle u \rangle = \int_Z u(x) \, dx$$

and

$$\sum_{k=1}^n u(x + k\omega) - n < u > = \sum_{k=1}^n \int_{Z_1} \log |\{x + k\omega\} - \zeta| \, d\mu(\zeta) - n \int_{Z_1} I(\zeta) \, d\mu(\zeta)$$

$$+ \sum_{k=1}^n h(\{x + k\omega\}) - n \int_0^1 h(y) \, dy.$$

Recall that

$$\sum_{k=1}^n \int_{Z_1} \log |\{x + k\omega\} - \zeta| \, d\mu(\zeta) = \int_{Z_1} F_{n,\zeta}(x) \, d\mu(\zeta).$$

Then

$$\int_0^1 \exp \left(\sigma \sum_{k=1}^n u(x + k\omega) - n < u > \right) \, dx$$

$$\leq \left[ \int_0^1 \exp \left(2\sigma \left\{ \int_{Z_1} (F_{n,\zeta}(x) - n I(\zeta)) \, d\mu(\zeta) \right\} \right) \, dx \right]^{1/2}$$

$$\times \left[ \int_0^1 \exp \left(2\sigma \sum_{k=1}^n h(\{x + k\omega\}) - n \int_0^1 h(y) \, dy \right) \, dx \right]^{1/2}.$$
Since \( \exp(\sigma \cdot \cdot) \) is a convex function, the Jensen’s inequality implies that
\[
\int_0^1 \exp \left( \sigma \left| \int_{Z_1} (F_{n,\zeta}(x) - nI(\zeta)) d\mu(\zeta) \right| \right) dx 
\leq \int_0^1 \int_{Z_1} \exp (\sigma \mu(Z_1) | F_{n,\zeta}(x) - nI(\zeta)) | d\mu(\zeta) \mu(Z_1) dx 
= \int_{Z_1} \int_0^1 \exp (\sigma \mu(Z_1) | F_{n,\zeta}(x) - nI(\zeta)) | d\mu(\zeta) \mu(Z_1) dx 
\leq \int_{Z_1} \exp \left( \tilde{C} \sigma \mu(Z_1) n\delta_0^n \right) \frac{d\mu(\zeta)}{\mu(Z_1)} dx 
\leq \exp \left( \tilde{C} \sigma \mu(Z_1) n\delta_0^n \right).
\]

Note that \( h \) is a harmonic function, which implies that it has a much better estimation than \( \log |z| \). For the details, the readers can see Lemma 2.8 in [1]. Therefore, we have that for any \( 0 < \sigma \leq \frac{\tilde{c}}{2\mu(Z_1)} \),
\[
\int_0^1 \exp \left( \sigma \left| \sum_{k=1}^n u(x + k\omega) - n < u > \right| \right) dx < \exp \left( \tilde{C} \sigma \mu(Z_1) n\delta_0^n \right).
\]

Recall the Markov’s inequality: For any measurable extended real-valued function \( f(x) \) and \( \epsilon > 0 \), we have
\[
\text{mes} \left( \{ x \in X : |f(x)| \geq \epsilon \} \right) \leq \frac{1}{\epsilon} \int_X |f| dx.
\]
Let \( f(x) = \exp (\sigma \sum_{k=1}^n u(x + k\omega) - n < u >) \) and \( \epsilon = \exp(\sigma\delta n) \), then
\[
\text{mes} \left( \left\{ x \in X : \left| \sum_{k=1}^n u(x + k\omega) - n < u > \right| > \delta n \right\} \right) 
= \text{mes} \left( \left\{ x \in X : \exp \left( \sigma \left| \sum_{k=1}^n u(x + k\omega) - n < u > \right| \right) \geq \exp(\sigma\delta n) \right\} \right) 
\leq \exp \left( -\sigma\delta n + \tilde{C} \sigma \mu(Z_1) n\delta_0^n \right).
\]
Choose \( \sigma = \frac{\tilde{c}}{2\mu(Z_1)} \) and \( \delta > 2\tilde{C} \mu(Z_1) \delta_0^n \). Then, we obtain the strong Birkhoff ergodic theorem as follows:

**Theorem 2.3.** Let \( u \) be a subharmonic function defined in the strip \( Z \) with maximum \( S \) and \( \omega \) be Diophantine \( D_{c,A} \). Then for any positive integer \( k \) and \( \delta > \delta_0(\rho,k) := \frac{\tilde{C} S \log^2 \epsilon k}{\rho(ek)^{\frac{3}{2}}} \),
\[
\text{meas} \left( \left\{ x : \left| \sum_{j=1}^k u(x + j\omega) - k < u(\cdot) > \right| > \delta k \right\} \right) < \exp \left( -\frac{\tilde{c}\rho\delta k}{S} \right),
\]
where \( \tilde{c} = \tilde{c}(A,c) = \frac{\epsilon}{2C} \) and \( \tilde{C} = \tilde{C}(A,c) = 2\tilde{C} \).

3. **Large deviation Theorem.** In this section, we apply the induction to prove the following called Large Deviation Theorem.
Lemma 3.1. Consider the Schrödinger operator (1) with some fixed analytic function \( f(x) \). There exists \( \lambda_0 = \lambda_0(f) \) such that for any \( \lambda > \lambda_0, \rho > 0, M > 0, s > 1, A > 1 \) and \( c > 0 \), there exist \( n_0 = n_0(\lambda, A, c, \rho, s, M) \) and \( r_0 = r_0(\lambda, A, c, \rho, s, M) \) such that for any \( g \in G^{r, M, \rho}(\mathbb{T}) \) satisfying \( \|g\|_\infty < r_0, n > n_0, \omega \in D_{c, A} \) and \( E \in \mathbb{R} \), it yields that

\[
\text{meas} \left( \left\{ x \in \mathbb{T} : |u_n(x, E, \omega) - L_n(E, \omega)| > n^{-\frac{c}{2}} \right\} \right) < \exp \left( -cn^{\frac{a}{2}} \right).
\]

At first, we choose large \( \lambda \) such that the Lyapunov exponent of the analytic cocycles is close to \( \log \lambda \). This conclusion comes from [20]. In that paper, our second author considered the following quasi-periodic analytic Jacobi operators \( H_{x, \omega} \) on \( l^2(\mathbb{Z}) \):

\[
(H_{x, \omega})(n) = -\lambda a(x + (n + 1)\omega)\phi(n + 1) - \lambda a(x + n\omega)\phi(n - 1) + \lambda v(x + n\omega)\phi(n),
\]

where \( a \) and \( v \) are both analytic function on \( T \). Obviously, our operators (1) with \( g \equiv 0 \) is a special case of it. So, the lemma about the positivity of the Lyapunov exponent of Jacobi operators can be applied directly here:

Lemma 3.2. For any \( 0 < \tau_2 < 1 \), there exists \( \lambda_2 = \lambda_2(f, \tau_2) \) such that if \( \lambda > \lambda_2 \), then for any \( n \geq 1 \) and any irrational \( \omega \),

\[
L_n^\lambda(E, \omega) > (1 - \tau_2) \log \lambda.
\]

Let \( \tau_1 = \tau_2 = \frac{1}{100} \). Then, due to (10) and Lemma 3.2,

\[
\frac{99}{100} \log \lambda < L_n^\lambda(E, \omega) < \frac{101}{100} \log \lambda \quad \text{and} \quad L_n(E, \omega), L_n^\lambda(E, \omega) < \frac{101}{100} \log \lambda. \quad (17)
\]

Due to the convergence from the subadditivity,

\[
\frac{99}{100} \log \lambda < L^\lambda(E, \omega) < \frac{101}{100} \log \lambda \quad \text{and} \quad L(E, \omega), L^\lambda(E, \omega) < \frac{101}{100} \log \lambda.
\]

Secondly, assume that the real analytic function \( f(x) \) has an analytic extension on \( Z \). Choose \( \lambda > \lambda_0(f) = \max \{ \lambda_1(f, \frac{\lambda_0}{100}), \lambda_2(f, \frac{\lambda_0}{100}) \} \). Then, \( u_n^\lambda(z, E, \omega) \) is a subharmonic function on \( Z \) with the maximum \( \frac{101}{100} \log \lambda \) such that Theorem 2.3 can be applied to with \( n \geq \tilde{n}_0(c, A), k = n^{1 - \frac{1}{2\pi + \delta}} \) and \( \delta = \log^{-1} \lambda \cdot k^{-\frac{1}{2\pi}} = \log^{-1} \lambda \cdot n^{-\frac{1}{2\pi + \delta}} \):

\[
\text{meas} \left( \left\{ x : \left| \frac{1}{k} \sum_{j=1}^{k} u_n^\lambda(x + j\omega, E, \omega) - L_n^\lambda(E, \omega) \right| > \log \lambda \cdot n^{-\frac{1}{2\pi + \delta}} \right\} \right) < \exp \left( -cn \log^{-1} \lambda \cdot n^{\frac{2k - 1}{2\pi + \delta}} \right). \quad (18)
\]

Due to Lemma A.1,

\[
|u_n^\lambda(x, E, \omega) - u_n^\lambda(x + \omega, E, \omega)| < \frac{2S}{n}.
\]

Therefore,

\[
\left| \frac{1}{k} \sum_{j=1}^{k} u_n^\lambda(x + j\omega, E, \omega) - u_n^\lambda(x, E, \omega) \right| < \frac{(k + 1)S}{n} < 2 \log \lambda \cdot n^{-\frac{1}{2\pi + \delta}}.
\]
Combined it with (18), we obtain the large deviation theorem for $u_n^c$:

\[
\text{meas} \left( \left\{ x : |u_n^c(x, E, \omega) - L_n^c(E, \omega)| > 3 \log \lambda \cdot n^{-\frac{4}{4k+1}} \right\} \right) < \exp \left( -c \log^{-1} \lambda \cdot n^{\frac{2}{2k+1}} \right).
\]  

(19)

Thirdly, we want to get the LDT for $u_n$ at some $n$ by (19). To finish it, we choose $\|g\|_\infty$ small enough. Let $n_0 = n_0(\lambda, c, A, \rho, s, M) = \max\{\bar{n}_0, \tilde{n}_0, \check{n}_0\}$, where $\bar{n}_0 = \bar{n}_0(\lambda, c, A, s)$ satisfies that for any $n > \bar{n}_0$,

\[
n^{-\frac{4}{10}} < \frac{1}{100} n \log \lambda \quad \text{and} \quad \exp \left( -c \log^{-1} \lambda \cdot n^{\frac{2}{4k+1}} \right) < n^{-100B^4},
\]

where $B = 2As > 2$. Choose

\[
\|g\|_\infty \leq r_0(\lambda, c, A, \rho, s, M) := \log \lambda \cdot (2n_0)^{-\frac{4}{4k+1}} \exp \left( (1 - 2n_0)S \right) \ll 1.
\]

Then, due to (11), for any $n_0 \leq n \leq 2n_0$, it yields that

\[
|u_n(x, E, \omega) - u_n^c(x, E, \omega)| < \log \lambda \cdot n^{-\frac{4}{4k+1}}
\]

and

\[
|L_n(E, \omega) - L_n^c(E, \omega)| < \log \lambda \cdot n^{-\frac{4}{4k+1}}.
\]

(20)

Combined them with (19), we have

\[
\text{meas} \left( \left\{ x : |u_n(x, E, \omega) - L_n(E, \omega)| > 5 \log \lambda \cdot n^{-\frac{4}{4k+1}} \right\} \right) < \exp \left( -c \log^{-1} \lambda \cdot n^{\frac{2}{2k+1}} \right).
\]

And due to (20) and (17),

\[
\frac{49}{50} \lambda \leq L_{2n_0}(E, \omega) \leq L_{n_0}(E, \omega) \leq \frac{101}{100} \log \lambda.
\]

(21)

These conclusions are the initial step of our induction.

Now, we start the inductive step of LDT by the following lemma:

**Lemma 3.3.** Let $\lambda > \lambda_0(f)$. Assume

\[
\left( \frac{49}{50} - \frac{1}{3} \left( \sum_{k=0}^{i} \frac{1}{2^k} \right) \right) \log \lambda \leq L_{2n_i}(E, \omega) \leq L_{n_i}(E, \omega),
\]

(22)

\[
|L_{n_i}(E, \omega) - L_{2n_i}(E, \omega)| \leq \frac{1}{3 \cdot 2^{i+3}} \log \lambda,
\]

(23)

\[
\text{meas} \left( \left\{ x : |u_{n_i}(x, E, \omega) - L_{n_i}(E, \omega)| > n_i^{-\frac{4}{10}} \right\} \right) < n_i^{-100B^4},
\]

and

\[
\text{meas} \left( \left\{ x : |u_{2n_i}(x, E, \omega) - L_{n_i}(E, \omega)| > n_i^{-\frac{4}{10}} \right\} \right) < n_i^{-100B^4}.
\]

(24)

(25)

Then for any $n_i^{-2B} \leq n_{i+1} \leq n_i^{-6B^2}$, we have that

\[
\left( \frac{49}{50} - \frac{1}{3} \left( \sum_{k=0}^{i+1} \frac{1}{2^k} \right) \right) \log \lambda \leq L_{2n_{i+1}}(E, \omega) \leq L_{n_{i+1}}(E, \omega) \leq \frac{101}{100} \log \lambda,
\]

(26)

\[
|L_{n_{i+1}}(E, \omega) - L_{2n_{i+1}}(E, \omega)| \leq \frac{1}{3 \cdot 2^{i+4}} \log \lambda,
\]

(27)

\[
\text{meas} \left( \left\{ x : |u_{n_{i+1}}(x, E, \omega) - L_{n_{i+1}}(E, \omega)| > n_{i+1}^{-\frac{4}{10}} \right\} \right) < \exp \left( -cn_{i+1}^{\frac{4}{10}} \right),
\]

(28)
and

\[
\text{meas} \left( \left\{ x : |u_{2n_i+1}(x, E, \omega) - L_{2n_i+1}(E, \omega)| > n_i^{\frac{1}{10}} \right\} \right) < \exp \left( -cn_i^{\frac{1}{10}} \right). \tag{29}
\]

**Proof.** We shall fix \( \omega \) and \( E \) so that they can be suppressed from the notations.

Firstly, by the setting \( \lambda > \lambda_0 \) and (17), for any \( i \geq 0 \),

\[
L_{n_i+1}(E, \omega) \leq \frac{101}{100} \log \lambda.
\]

Secondly, assume that \( n_{i+1} \) is a multiple of \( n_i \), which means that there exists an integer \( m_i \) such that \( n_{i+1} = m_i n_i \). Let \( B_i(x) \) be the set of \( x \) which satisfies that

\[ |u_{n_i}(x) - L_{n_i}| > n_i^{\frac{1}{10}} \text{ and } |u_{2n_i}(x) - L_{2n_i}| > n_i^{\frac{1}{10}}. \]

Due to (24) and (25), we have \( \text{meas}(B) < 2n_i^{-100B^4} \). Thus, we can choose \( G_i = T \left( \bigcup_{j=0}^{m_i-1} B(x + jn_i \omega) \right) \) satisfying

\[
\text{meas}(T \setminus G_i) \leq 2m_i \cdot n_i^{-100B^4} \leq 2n_{i+1} \cdot n_i^{-100B^4} \leq 2n_i^{-95B^4}
\]
such that for any \( x \in G_i \) and \( j = 0, 1, \ldots, m_i - 1 \),

\[
||M_{n_i}(x + jn_i \omega)|| > \exp \left( n_i \left( L_{n_i} - n_i^{\frac{1}{10}} \right) \right) > \exp \left( \frac{1}{2} n_i \log \lambda \right),
\]

and

\[
\left| \log ||M_{n_i}(x + jn_i \omega)|| + \log ||M_{n_i}(x + (j + 1)n_i \omega)|| - \log ||M_{n_i}(x + (j + 1)n_i \omega)M_{n_i}(x + jn_i \omega)|| \right| < 2n_i^{\frac{1}{10}} + 2n_i|L_{n_i} - L_{2n_i}| \leq \frac{3}{25} n_i \log \lambda.
\]

Here we use the following relationship from (23):

\[
|L_{n_i} - L_{2n_i}| \leq \frac{3}{100} \log \lambda.
\]

Thus, the following avalanche principle can be applied for \( \mu = \exp \left( \frac{1}{2} n_i \log \lambda \right) \):

**Proposition 1** (Proposition 2.2 in [11]). Let \( A_1, \ldots, A_n \) be a sequence of \( 2 \times 2 \)-matrices whose determinants satisfy

\[
\max_{1 \leq j \leq n} |\det A_j| \leq 1. \tag{30}
\]

Suppose that

\[
\min_{1 \leq j \leq n} \|A_j\| \geq \mu > n \quad \text{and} \quad \max_{1 \leq j < n} |\log \|A_{j+1}\| + \log ||A_j|| - \log \|A_{j+1}A_j\||| < \frac{1}{2} \log \mu. \tag{31}
\]

Then

\[
\left| \log \|A_n \cdots A_1\| + \sum_{j=2}^{n-1} \log \|A_j\| - \sum_{j=1}^{n-1} \log \|A_{j+1}A_j\|| \right| < C \frac{n}{\mu} \tag{32}
\]

with some absolute constant \( C \).
Therefore, for any \( x \in G_i \),
\[
\left| u_{n_{i+1}}(x) + \frac{1}{m_i} \sum_{j=2}^{m_i-1} u_{n_i}(x + (j-1)n_i \omega) \right| \leq \exp \left( -\frac{1}{2} n_i \log \lambda \right).
\] (34)

Integrating it on \( G_i \) and applying the Hölder inequality on \( T \setminus G_i \), we get
\[
\left| L_{n_{i+1}} + \frac{m_i - 2}{m_i} L_{n_i} - \frac{m_i - 1}{m_i} L_{2n_i} \right| \leq \exp \left( -\frac{1}{2} n_i \log \lambda \right) + \log \lambda \cdot n_i^{-9B^2}.
\] (35)

Thus, due to the setting of \( m_i \), it yields that
\[
|L_{n_{i+1}} + L_{n_i} - 2L_{2n_i}| \leq 10 \log \lambda \cdot m_i^{-1},
\] (36)

and
\[
L_{n_{i+1}} \geq L_{n_i} - 2|L_{n_i} - L_{2n_i}| - 10 \log \lambda \cdot m_i^{-1}
\]

\[
\geq \left( \frac{49}{50} - \frac{1}{3} \left( \sum_{k=0}^{i} \frac{1}{2^k} \right) \right) \log \lambda - 2 \cdot \frac{1}{3} \cdot \frac{1}{2^{i+3}} \log \lambda - 10 \log \lambda \cdot m_i^{-1}
\]

\[
\geq \left( \frac{49}{50} - \frac{1}{3} \left( \sum_{k=0}^{i+1} \frac{1}{2^k} \right) \right) \log \lambda.
\]

Similarly, repeating the same argument with a block of size \( 2m_i \), we obtain that
\[
|L_{2n_{i+1}} + L_{n_i} - 2L_{2n_i}| \leq 10 \log \lambda \cdot m_i^{-1},
\] (37)

and
\[
L_{2n_{i+1}} \geq \left( \frac{49}{50} - \frac{1}{3} \left( \sum_{k=0}^{i+1} \frac{1}{2^k} \right) \right).
\]

Combined (36), (37) and the setting that \( n_{i+1} \geq n_i^{2B} \), we have that
\[
|L_{2n_{i+1}} - L_{n_{i+1}}| \leq 20 \log \lambda \cdot n_i^{-2B} \ll \frac{1}{3 \cdot 2^{i+4}} \log \lambda.
\]

Thirdly, we use the strong Birkhoff Ergodic Theorem, Theorem 2.3, to obtain the LDT for \( u_{n_{i+1}} \) and \( u_{2n_{i+1}} \). Replacing \( x \) by \( x, x + \omega, \cdots, x + (n_i - 1)\omega \) in (34) and averaging them, we have
\[
\left| \frac{1}{n_i} \sum_{j=0}^{n_i-1} u_{n_{i+1}}(x + j \omega) + \frac{1}{n_i+1} \sum_{j=n_i}^{n_i+n_i-1} u_{n_i}(x + j \omega) \right| - \frac{2}{n_{i+1}} \sum_{j=0}^{n_i-1} u_{2n_i}(x + j \omega) \right| \leq \exp \left( -\frac{9}{10} n_i \log \lambda \right)
\]

up to a set not exceeding \( n_i^{-90B^4} \) in measure. Due to Remark 3, it yields that
\[
\left| u_{n_{i+1}}(x) + \frac{1}{n_{i+1}} \sum_{j=n_i}^{n_i+n_i-1} u_{n_i}(x + j \omega) \right| - \frac{2}{n_{i+1}} \sum_{j=0}^{n_i-1} u_{2n_i}(x + j \omega) \right| \leq \exp \left( -\frac{9}{10} n_i \log \lambda \right) + 2 \frac{101}{100} \log \lambda \cdot m_i^{-1} \leq \frac{5}{2} \log \lambda \cdot m_i^{-1}
\] (38)
up to this set. Now we have gotten the Dirichlet sum of $u_{n_i}$ and $u_{2n_i}$ in this formula. However, the functions $u_{n_i}$ and $u_{2n_i}$ are not subharmonic. Therefore, we apply (12) to (38) by changing $u_{n_i}$ and $u_{2n_i}$ to $u_{n_i}^t$ and $u_{2n_i}^t$ and obtain

$$
\left| u_{n_{i+1}}(x) + \frac{1}{n_{i+1}} \sum_{j=n_i}^{n_{i+1}-1} u_{n_i}^t(x+j\omega) - \frac{2}{n_{i+1}} \sum_{j=0}^{n_{i+1}-n_i-1} u_{2n_i}^t(x+j\omega) \right| \leq \frac{5}{2} \log \lambda \cdot m_i^{-1} + \exp(-n_i) \leq 3 \log \lambda \cdot m_i^{-1}. 
$$

(39)

Note that $u_{n_i}^t$ and $u_{2n_i}^t$ have subharmonic extensions on the strip $\mathcal{Z}_i$, where $\mathcal{Z}_i := \{ z : |\text{Im}z| \leq \frac{\epsilon}{2} (2n_i)^{-2(s-1)} \}$. Then, we apply Theorem 2.3 for $u_{n_i}^t$ and $u_{2n_i}^t$ with

$k = n_{i+1} - 2n_i$ and $n_{i+1} - n_i$ and $\delta = n_{i+1}^{-1} > \delta_0(\rho_n, n_{i+1}/2)$:

$$
\text{meas} \left( \left\{ x : \frac{1}{n_{i+1}} \sum_{j=n_i}^{n_{i+1}-1} u_{n_i}^t(x+j\omega) - \frac{m_i - 2}{m_i} < u_{n_i}^t(\cdot) > > n_{i+1}^{-1} \right\} \right) < \exp(-n_{i+1}^{-1} - \frac{\delta}{m_i}),
$$

$$
\text{meas} \left( \left\{ x : \frac{2}{n_{i+1}} \sum_{j=0}^{n_{i+1}-n_i-1} u_{2n_i}^t(x+j\omega) - \frac{2m_i - 1}{m_i} < u_{2n_i}^t(\cdot) > > n_{i+1}^{-1} \right\} \right) < \exp(-n_{i+1}^{-1} - \frac{\delta}{m_i}).
$$

Combined them with (13) and (39), we have

$$
\left| u_{n_{i+1}}(x) + \frac{m_i - 2}{m_i} L_n - \frac{2m_i - 1}{m_i} L_{2n} \right| \leq 3 \log \lambda \cdot m_i^{-1} + 2n_{i+1}^{-1} \leq 3n_{i+1}^{-1}
$$

up to a set not exceeding $n_{i+1}^{-1}70\log^4$ in measure. By (35), it yields that

$$
\left| u_{n_{i+1}}(x) - L_{n_{i+1}} \right| \leq 3n_{i+1}^{-1} + 2 \log \lambda \cdot m_i^{-1} \leq 4n_{i+1}^{-1}
$$

(40)

up to this set. Notice that the exception measure of this LDT (40) is not good enough for us. We will apply the following lemma, which is Corollary 4.10 in [4], to boost it:

**Lemma 3.4.** Assume $u = u(x) : \mathbb{T} \to \mathbb{R}$ has a subharmonic extension $u(z)$ to the strip $\{ z : |\text{Im}z| \leq \rho \}$, $\rho > 0$, such that $|u(z)| \leq S$ for all $z$. If

$$
\text{meas} \left( \left\{ x \in \mathbb{T} : \left| u(x) - \int_\mathbb{T} u(x) \, dx \right| > \epsilon_0 \right\} \right) < \epsilon_1,
$$

(41)

then, for an absolute constant $c > 0$,

$$
\text{meas} \left( \left\{ x \in \mathbb{T} : \left| u(x) - \int_\mathbb{T} u(x) \, dx \right| > \sqrt{\epsilon_0} \right\} \right) < \exp \left( -c \left( \sqrt{\epsilon_0} + \sqrt{\frac{\epsilon_1 \cdot S}{\epsilon_0 \cdot \rho}} \right)^{-1} \right).
$$

(42)

Again, this lemma can be used only for the subharmonic functions. We need to use (12) and (13) to exchange $u_{n_{i+1}}$ and $L_{n_{i+1}}$ by the truncation function $u_{n_{i+1}}^t$ and
its average $L_{n_{i+1}}^t$ and yield that
\[
\left| u_{n_{i+1}}^t(x) - L_{n_{i+1}}^t \right| \leq 4n_{i+1}^{-\frac{1}{2}} + 2\exp(-n_{i+1}) \leq n_{i+1}^{-\frac{1}{2}}
\]
up to a set not exceeding $n_{i+1}^{-70B^4}$ in measure. Therefore, Lemma 3.4 is applied for $u_{n_{i+1}}^t$ with $\rho_{n_{i+1}} = \frac{6}{2}n_{i+1}^{2(s-1)}$, $\epsilon_0 = n_{i+1}^{-\frac{1}{2}}$ and $\epsilon_1 = n_{i+1}^{-14B^4} > n_{i+1}^{-70B^4}$:
\[
\text{meas} \left( \left\{ x \in \mathbb{T} : \left| u_{n_{i+1}}^t(x) - L_{n_{i+1}}^t \right| > n_{i+1}^{-\frac{1}{2}} \right\} \right) < \exp\left(-cn_{i+1}^{\frac{1}{2}}\right).
\]
Applying (12) and (13) again, we obtain the desired LDT for $u_{n_{i+1}}^t$:
\[
\text{meas} \left( \left\{ x \in \mathbb{T} : \left| u_{n_{i+1}}^t(x) - L_{n_{i+1}}^t \right| > n_{i+1}^{-\frac{1}{2}} \right\} \right) < \exp\left(-cn_{i+1}^{\frac{1}{2}}\right).
\]
Obviously, the above process also holds for $u_{2n_{i+1}}$.

At last, we begin to consider the condition that $n_{i+1}$ is not a multiple of $n_i$, which means that $n_{i+1} = m_in_i + r_i$ where $0 < r_i < n_i$. Then,
\[
\left| u_{n_{i+1}}^t(x) - u_{m_in_i}(x) \right| \leq 10\log \lambda \cdot m_i^{-1}
\]
and (40) also holds for any $n_i^{2B} \leq n_{i+1} \leq n_i^{5B^2}$. Indeed, denote $M_{n_{i+1}}(x) = R_i(x) \cdot M_{m_in_i}(x)$ where
\[
R_i(x) = \prod_{j=m_i+1}^{m_in_i+1} M(x + j\omega) = \prod_{j=m_i+r_0+1}^{m_in_i+1} M(x + j\omega).
\]
Obviously, $R_i(x)$ is a unimodular matrix and for any $x$
\[
1 \leq \|R(x)\| = \|R^{-1}(x)\| \leq \exp(rS) \leq \exp(n_iS).
\]
Therefore,
\[
\left| u_{n_{i+1}}^t(x) - u_{m_in_i}(x) \right| = \left| \frac{1}{n_{i+1}} \log \|M_{n_{i+1}}(x)\| - \frac{1}{m_in_i} \log \|M_{m_in_i}(x)\| \right|
\leq \left| \frac{1}{n_{i+1}} \log \|M_{n_{i+1}}(x)\| - \frac{1}{n_{i+1}} \log \|M_{m_in_i}(x)\| \right|
+ \left| \frac{1}{n_{i+1}} \log \|M_{m_in_i}(x)\| - \frac{1}{m_in_i} \log \|M_{m_in_i}(x)\| \right|
\leq \left| \frac{1}{n_{i+1}} \log \|R_i(x)\| \right| + \frac{r_i}{n_{i+1}m_i} \log \|M_{m_in_i}(x)\| \leq 10\log \lambda \cdot m_i^{-1}.
\]

Then, Lemma 3.4 can be applied no matter $n_{i+1}$ is a multiple of $n_i$ or not. \qed

Now, we start the proof of Lemma 3.1.

Proof. Recall that for any $\lambda > \lambda_0$, $\rho > 0$, $M > 0$, $s > 1$, $A > 1$ and $c > 0$, there exist $n_0 = n_0(\lambda, A, c, \rho, s, M)$ and $r_0 = r_0(\lambda, A, c, \rho, s, M)$ such that for any $g \in G_{s,M,\rho}(\mathbb{T})$ satisfying $\|g\|_{\infty} < r_0$, $\omega \in \mathcal{D}_{c,A}$ and $E \in \mathbb{R}$,
\[
\frac{49}{50} \log \lambda \leq L_{2n_0}(E,\omega) \leq L_{n_0}(E,\omega) \leq \frac{101}{100} \log \lambda,
\]
and for any $n_0 \leq n \leq 2n_0$
\[
\text{meas} \left( \left\{ x : |u_n(x,E,\omega) - L_n(E,\omega)| > n^{-\frac{1}{2}} \right\} \right) < n^{-100B^4}.
\]
Then, the assumptions of Lemma 3.3 hold for $n_0$. Therefore, (26), (27), (28) and (29) hold for any $n \in [n_0^{2B}, n_0^{5B^2}]$. Choose $n_1 = n_0^{2B}$. Then, the assumptions of
Lemma 3.3 hold for \( n_1 \), and (26), (27), (28) and (29) hold for any \( n \in [n_1^{2B}, n_1^{5B^2}] \).

From now on, we choose \( n_{i+1} = n_i^{2B} \) for any \( i \geq 1 \). Then, due to Lemma 3.3 and the induction, we obtain that (26), (27), (28) and (29) hold for any \( n \in \bigcup_{i=0}^{\infty} [n_i^{2B}, n_i^{5B^2}] \).

It is obvious that \( n_i^{5B^2} > (n_i^{2B})^{2B} = n_{i+1} \). Thus, (26), (27), (28) and (29) hold for any \( n > n_1 \). At last, choosing \( n_0 = n_1 \), we have finished this proof. \( \square \)

**Remark 2.** The proof shows that \( n_0 \) depends continuously on \( \lambda, A, c, \rho, M \) and \( s \).

4. **Proofs of Theorem 1.1 and Theorem 1.2.** Now we can combine Lemma 3.1 with the avalanche principle to obtain the following lemma.

**Lemma 4.1.** There exists \( \lambda_0 = \lambda_0(f) \) such that for any \( \lambda > \lambda_0, \omega \in \mathcal{D}_{c, A} \) and \( g \in G^{s, M, \rho}(T) \), there exist \( n_0 = n_0(\lambda, A, c, \rho, s, M) \) and \( r_0 = r_0(\lambda, A, c, \rho, s, M) \) such that for any \( n \geq n_0 \), \( \| g \|_\infty \leq r_0 \),

\[
|L(E, \lambda f + g, \omega) + L_n(E, \lambda f + g, \omega) - 2L_{2n}(E, \lambda f + g, \omega)| \leq \exp \left(-cn_0^{5\sigma} \right). \tag{43}
\]

**Proof.** We suppress \( \omega, E \) and \( \lambda f + g \) from the notations for ease. Recalling the proof of Lemma 3.1, we have that for any \( n > n_0 \), where \( n_0 \) comes from Lemma 3.1,

\[
|L_n - L_{2n}| \leq \frac{1}{3 \cdot 2^4} \log \lambda \leq \frac{1}{40} \log \lambda, \tag{44}
\]

and

\[
L_n > \frac{1}{2} \log \lambda.
\]

Choose \( n_1 = n \), and \( n_{i+1} \sim \exp \left(\frac{2}{5}n_i^{5\sigma} \right) \), which is a multiple of \( n_i \). Define \( m_i = \frac{n_{i+1}}{n_i} > \exp \left(\frac{2}{5}n_i^{5\sigma} \right) \). Then Lemma 3.1 permits us to ensure that there exist a set \( \Omega_i \), satisfying that expect for \( x \in \Omega_i \subset T \)

\[
\text{meas}(\Omega_i) < 4n_i \exp \left(-cn_i^{5\sigma} \right) < 4n_i \exp \left(-cn_i^{5\sigma} \right) < \exp \left(-\frac{c}{2}n_i^{5\sigma} \right),
\]

such that for any \( x \in T \setminus \Omega_i \),

\[
\max_{1 \leq j \leq m_i} \left| \frac{1}{n_i} \log \| M_{n_i}(x + j\omega) \| - L_i \right| < n_i^{5\sigma} < \frac{1}{100} \log \lambda < \frac{1}{50} L_n. \tag{45}
\]

and

\[
\max_{1 \leq j \leq m_i} \left| \frac{1}{2n_i} \log \| M_{2n_i}(x + j\omega) \| - L_{2n_i} \right| < (2n_i)^{5\sigma} < \frac{1}{100} \log \lambda < \frac{1}{50} L_{2n_i}. \tag{46}
\]

Now we can verify the conditions of Proposition 1 for

\[
A_k = M_{n_i}(x + (k-1)n_i\omega) \in SL_2(R).
\]

By (45)

\[
\|A_k\| > \exp \left(\frac{49}{50}n_iL_n \right) = \mu > m_i,
\]

and by (44), (45) and (46),

\[
\|\log \|A_k\| + \log \|A_{k+1}\| - \log \|A_{k+1}A_k\|| < \left|2n_iL_n - 2n_iL_{2n_i}\right| + \frac{1}{10} n_iL_n,
\]

\[
< \frac{1}{5} n_iL_n < \frac{1}{2} \log \mu.
\]
Thus, the avalanche principle implies that
\[
\left| \log \|A_{m_1} \cdots A_i\| + \sum_{k=2}^{m_i-1} \log \|A_k\| - \sum_{k=1}^{m_i-1} \log \|A_{k+1}A_k\| \right| < C \frac{m_i}{\mu}.
\]
Hence, for any \( x \in T \setminus \Omega \)
\[
\left| \log \|M_{n_{i+1}}(x)\| + \sum_{k=2}^{m_i-1} \log \|M_{n_i}(x+(k-1)n_i\omega)\| - \sum_{k=1}^{m_i-1} \log \|M_{2n_i}(x+(k-1)n_i\omega)\| \right| < \exp \left( -\frac{49}{50} n_i L_{n_i} \right) < \exp \left( -\frac{1}{4} n_i \log \lambda \right).
\]
Integration in \( x \) implies that
\[
\left| L_{n_{i+1}} + \frac{m_i - 2}{m_i} L_{n_i} - 2 \frac{m_i - 1}{m_i} 2 L_{2n_i} \right| < \exp \left( -\frac{1}{4} n_i \log \lambda \right) + 10 \log \lambda \cdot \text{meas}(\Omega)
\]
\[
< 20 \log \lambda \exp \left( -\frac{c}{2} n_i^{\frac{1}{M}} \right),
\]
and
\[
\left| L_{n_{i+1}} + L_{n_i} - 2L_{2n_i} \right| < \frac{2L_{n_i} + 2L_{2n_i}}{m_i} + 20 \log \lambda \exp \left( -\frac{c}{2} n_i^{\frac{1}{M}} \right)
\]
\[
< 5 \log \lambda \exp \left( -\frac{c}{4} n_i^{\frac{1}{M}} \right) + 20 \log \lambda \exp \left( -\frac{c}{2} n_i^{\frac{1}{M}} \right)
\]
\[
< \exp \left( -\frac{c}{5} n_i^{\frac{1}{M}} \right).
\]
Writing the same inequality with \( n_{i+1} \) replaced by \( 2n_{i+1} \) gives
\[
\left| L_{2n_{i+1}} + L_{n_i} - 2L_{2n_i} \right| < \exp \left( -\frac{c}{5} n_i^{\frac{1}{M}} \right).
\]
Thus, we have
\[
\left| L_{2n_{i+1}} - L_{n_{i+1}} \right| < 2 \exp \left( -\frac{c}{5} n_i^{\frac{1}{M}} \right).
\]
Note that the upper estimations in this proof hold for any \( i \geq 1 \). Then it yields that
\[
\left| L_{n_{i+1}} - L_{n_i} \right| < 5 \exp \left( -\frac{c}{5} n_i^{\frac{1}{M}} \right).
\]
Then
\[
\left| L + L_{n_1} - 2L_{2n_1} \right| < \sum_{i \geq 2} \left| L_{n_{i+1}} - L_{n_i} \right| + \left| L_{n_2} + L_{n_1} - 2L_{2n_1} \right|
\]
\[
< \sum_{i \geq 2} 5 \exp \left( -\frac{c}{5} n_i^{\frac{1}{M}} \right) + \exp \left( -\frac{c}{5} n_1^{\frac{1}{M}} \right)
\]
\[
< \exp \left( -c n_1^{\frac{1}{M}} \right).
\]
This proves (43) by recalling the setting \( n_1 = n \). \( \square \)

Therefore, due to Remark 2 and Lemma 4.1, it yields that for any \( n \geq n_0 \),
\[
|L(E, \lambda f + g, \omega) - L(E, \lambda f, \omega)| \leq 2 \exp \left( -c n_1^{\frac{1}{M}} \right) + |L_n(E, \lambda f + g, \omega) - L_n(E, \lambda f, \omega)|
\]
\[
+ 2 |L_{2n}(E, \lambda f + g, \omega) - L_{2n}(E, \lambda f, \omega)|.
\]
Note that if det \( C \), we have that
\[
|L_n(E, \lambda f + g, \omega) - L_n(E, \lambda f, \omega)| < \|g\|_\infty \exp((n - 1)S)
\]
and
\[
|L_{2n}(E, \lambda f + g, \omega) - L_{2n}(E, \lambda f, \omega)| < \|g\|_\infty \exp((2n - 1)S).
\]
Thus,
\[
|L(E, \lambda f + \kappa g, \omega) - L(E, \lambda f, \omega)| \leq 2 \exp \left(-cn^{\frac{1}{10}}\right) + 3\|g\|_\infty \exp((2n - 1)S).
\]
For any \( \|g\|_\infty \leq r = \exp \left(-cn^{\frac{1}{10}}\right) \exp((1 - 2n_0)S) < r_0 \), there exists \( n \geq n_0 \) such that
\[
\|g\|_\infty \sim \exp \left(-cn^{\frac{1}{10}}\right) \exp((1 - 2n)S)
\]
and then
\[
|L(E, \lambda f + g, \omega) - L(E, \lambda f, \omega)| \leq 5 \exp \left(-cn^{\frac{1}{10}}\right) < \exp \left(-\log \|g\|_\infty \frac{1}{10} \right).
\]
This finishes the proof of Theorem 1.1.

Note that Lemma 4.1 also holds for any \( g_1, g_2 \in G^{s,M,\omega} (T) \), \( \omega_1, \omega_2 \in D_{c,A} \) and \( \|g_1\|_\infty, \|g_2\|_\infty < r \). Then, if \( |E_1 - E_2| + |\omega_1 - \omega_2| + \|g_1 - g_2\|_\infty < r \), then we can also find some \( n > n_0 \) such that
\[
|E_1 - E_2| + |\omega_1 - \omega_2| + \|g_1 - g_2\|_\infty \sim \exp \left(-cn^{\frac{1}{10}}\right) \exp((1 - 2n)S).
\]
Therefore, we finish the proof of Theorem 1.2, since the other part is trivial obviously.

Appendix a: Some distances between the multiplications of matrices. In this appendix, we study the distances between the functions which are related to the multiplications of matrices. The two lists of matrices in the first lemma come from the same matrix function. Then, they will be different in the next lemma.

Lemma A.1. Let \( A : T \rightarrow SL(2, \mathbb{R}) \) and \( \max_{x \in T} \|A(x)\| \leq \exp(S) \). Then for any \( x \in T \), \( n \in \mathbb{Z}^+ \) and \( \omega \),
\[
\left| \frac{1}{n} \log \left| \prod_{j=0}^{n-1} A(x + j\omega) \right| - \frac{1}{n} \log \left| \prod_{j=1}^{n} A(x + j\omega) \right| \right| \leq \frac{2S}{n}
\]

Proof. Note that if \( \det C = 1 \), then \( \|C\| = \|C^{-1}\| \geq 1 \). Thus,
\[
\left| \frac{1}{n} \log \left| \prod_{j=0}^{n-1} A(x + j\omega) \right| - \frac{1}{n} \log \left| \prod_{j=1}^{n} A(x + j\omega) \right| \right|
= \frac{1}{n} \log \left| \frac{\prod_{j=0}^{n-1} A(x + j\omega)}{\prod_{j=1}^{n} A(x + j\omega)} \right|
\leq \frac{1}{n} \max \left\{ \log \|A(x)A^{-1}(x + n\omega)\|, \log \|A^{-1}(x)A(x + n\omega)\| \right\}
\leq \frac{2S}{n}
\]
\( \square \)
Remark 3. Obviously, for any $1 \leq k \leq n$,
\[
\frac{1}{n} \log \left\| \prod_{j=0}^{n-1} A(x + j \omega) \right\| - \frac{1}{n} \log \left\| \prod_{j=k}^{n-1} A(x + j \omega) \right\| \leq \frac{2Sk}{n}.
\]

Lemma A.2. Let $A, B : \mathbb{T} \to SL(2, \mathbb{R})$, $\max_{x \in \mathbb{T}} \|A(x)\|$ and $\max_{x \in \mathbb{T}} \|B(x)\| \leq \exp(S)$ and $\max_{x \in \mathbb{T}} \|A(x) - B(x)\| < \kappa$. Then, for any $x \in \mathbb{T}$, $n \in \mathbb{Z}^+$ and $\omega$,
\[
\left\| \prod_{j=0}^{n-1} A(x + j \omega) \right\| - \left\| \prod_{j=0}^{n-1} B(x + j \omega) \right\| \leq n \kappa \exp((n - 1)S), \tag{A.1}
\]
and
\[
\left\| \prod_{j=0}^{n-1} A(x + j \omega) \right\| - \left\| \prod_{j=0}^{n-1} B(x + j \omega) \right\| \leq \kappa \exp((n - 1)S). \tag{A.2}
\]

Proof.
\[
\left\| \prod_{j=0}^{n-1} A(x + j \omega) \right\| - \left\| \prod_{j=0}^{n-1} B(x + j \omega) \right\| \leq \left\| \prod_{j=0}^{n-1} A(x + j \omega) \right\| - \left\| \prod_{j=0}^{n-1} B(x + j \omega) \right\|
\]
\[
\leq \left\| \prod_{j=0}^{n-1} A(x + j \omega) - \prod_{j=0}^{n-2} A(x + j \omega)B(x + (n - 1)\omega) \right\|
\]
\[
+ \left\| A(x)A(x + \omega) \prod_{j=0}^{n-1} B(x + j \omega) - A(x) \prod_{j=1}^{n-1} B(x + j \omega) \right\|
\]
\[
+ \left\| A(x) \prod_{j=1}^{n-1} B(x + j \omega) - \prod_{j=0}^{n-1} B(x + j \omega) \right\|
\]
\[
\leq n \kappa \exp((n - 1)S).
\]

Therefore,
\[
\frac{1}{n} \log \left\| \prod_{j=0}^{n-1} A(x + j \omega) \right\| - \frac{1}{n} \log \left\| \prod_{j=0}^{n-1} B(x + j \omega) \right\|
\]
\[
= \frac{1}{n} \log \left\| \frac{\prod_{j=0}^{n-1} A(x + j \omega)}{\prod_{j=0}^{n-1} B(x + j \omega)} \right\|
\]
\[
= \frac{1}{n} \log \left( 1 + \left\| \frac{\prod_{j=0}^{n-1} A(x + j \omega) - \prod_{j=0}^{n-1} B(x + j \omega)}{\prod_{j=0}^{n-1} B(x + j \omega)} \right| \right)
\]
\[
\leq \frac{1}{n} \log \left( 1 + \left\| \frac{\prod_{j=0}^{n-1} A(x + j \omega) - \prod_{j=0}^{n-1} B(x + j \omega)}{\prod_{j=0}^{n-1} B(x + j \omega)} \right| \right)
\]
\[
\leq \frac{1}{n} \left\| \prod_{j=0}^{n-1} A(x + j\omega) \right\| - \left\| \prod_{j=0}^{n-1} B(x + j\omega) \right\| = \kappa \exp((n - 1)S).
\]

Acknowledgments. We would like to thank the referees very much for their valuable comments and suggestions.

REFERENCES

[1] A. Avila, S. Jitomirskaya and C. A. Marx, Spectral theory of extended Harper’s model and a question by Erdős and Szekeres, Inv. Math., 210 (2017), 283–339.

[2] J. Bochi, Genericity of zero Lyapunov exponents, Ergod. Theory Dyn. Syst., 22 (2002), 1667–1696.

[3] J. Bochi and M. Viana, The Lyapunov exponents of generic volume perserving and symplectic maps, Ann. of Math., 161 (2005), 1423–1485.

[4] J. Bourgain, Green’s Function Estimates for Lattice Schrödinger Operators and Applications, Annals of Mathematics Studies, 158. Princeton University Press, Princeton, NJ, 2005.

[5] J. Bourgain, Positivity and continuity of the Lyapunov exponent for shifts on \( T^d \) with arbitrary frequency vector and real analytic potential, J. Anal. Math., 96 (2005), 313–355.

[6] J. Bourgain and M. Goldstein, On nonperturbative localization with quasi-periodic potential, Ann. of Math., 152 (2000), 835–879.

[7] J. Bourgain and S. Jitomirskaya, Continuity of the Lyapunov exponent for quasiperiodic operators with analytic potential, J. Stat. Phys., 108 (2002), 1028–1218.

[8] A. Furman, On the multiplicative ergodic theorem for uniquely ergodic systems, Ann. Inst. H. Poincaré Probab. Statist., 33 (1997), 797–815.

[9] L. Ge and Y. Wang, work in progress.

[10] M. Geng and K. Tao, Large deviation theorems for Dirichlet determinants of analytic quasi-periodic Jacobi operators with Brjuno-Rüssmann frequency, preprint, arXiv:1906.11136.

[11] M. Goldstein and W. Schlag, Hölder continuity of the integrated density of states for quasiperiodic Schrödinger equations and averages of shifts of subharmonic functions, Ann. of Math., 154 (2001), 155–203.

[12] M. Goldstein and W. Schlag, Fine properties of the integrated density of states and a quantitative separation property of the Dirichlet eigenvalues, Geom. Funct. Analysis, 18 (2008), 755–869.

[13] R. Han and S. Zhang, Optimal large deviation estimates and Hölder Regularity of the Lyapunov exponents for quasi-periodic Schrödinger cocycles, preprint, arXiv:1803.02035v1.

[14] S. Jitomirskaya, D. A. Koslover and M. S. Schulteis, Continuity of the Lyapunov exponent for analytic quasi-periodic cocycles, Ergod. Theory Dyn. Syst., 29 (2009), 1881–1905.

[15] S. Jitomirskaya and C. A. Marx, Continuity of the Lyapunov Exponent for analytic quasi-periodic cocycles with singularities, Journal of Fixed Point Theory and Applications, 10 (2011), 129–146.

[16] S. Klein, Anderson localization for the discrete one-dimensional quasi-periodic Schrödinger operator with potential defined by a Gevrey-class function, J. Funct. Anal., 218 (2005), 255–292.

[17] J. Liang, Y. Wang and J. You, Hölder continuity of Lyapunov exponent for a family of smooth Schrödinger cocycles, preprint, arXiv:1806.03284.

[18] S. Lojasiewicz, Sur le problème de la division, Studia Math., 18 (1959), 87–136.

[19] K. Tao, Continuity of Lyapunov exponent for analytic quasi-periodic cocycles on higher-dimensional torus, Front. Math. China, 7 (2012), 521–542.

[20] K. Tao, Strong Birkhoff Ergodic Theorem for subharmonic functions with irrational shift and its application to analytic quasi-periodic cocycles, preprint, arXiv:1805.00431.

[21] Y. Wang and J. You, Examples of discontinuity of Lyapunov exponent in smooth quasi-periodic cocycles, Duke Math., 162 (2013), 2363–2412.

[22] Y. Wang and J. You, The set of smooth quasi-periodic Schrödinger cocycles with positive Lyapunov exponent is not open, Commun. Math. Phys., 362 (2018), 801–826.
[23] Y. Wang and Z. Zhang, Uniform positivity and continuity of Lyapunov exponents for a class of $C^2$ quasiperiodic Schrödinger cocycles, *J. Funct. Anal.*, 268 (2015), 2525–2585.

[24] J. Xu, L. Ge and Y. Wang, work in progress.

[25] L.-S. Young, Lyapunov exponents for some quasi-periodic cocycles, *Ergod. Theory Dyn. Syst.*, 17 (1997), 483–504.

[26] J. You and S. Zhang, Hölder continuity of the Lyapunov exponent for analytic quasiperiodic Schrödinger cocycles with weak Liouville frequency, *Ergod. Theory Dyn. Syst.*, 34 (2014), 1395–1408.

Received October 2019; revised March 2020.

_E-mail address: Wennmeng_geng@163.com_
_E-mail address: ktao@hhu.edu.cn_