Existence of positive solutions to quasi-linear elliptic equations with exponential growth in the whole Euclidean space

Yunyan Yang

Department of Mathematics, Renmin University of China, Beijing 100872, PR China

Received 28 June 2011; accepted 14 November 2011
Available online 21 November 2011
Communicated by H. Brezis

Abstract

In this paper a quasi-linear elliptic equation in the whole Euclidean space is considered. The nonlinearity of the equation is assumed to have exponential growth or have critical growth in view of Trudinger–Moser type inequality. Under some assumptions on the potential and the nonlinearity, it is proved that there is a nontrivial positive weak solution to this equation. Also it is shown that there are two distinct positive weak solutions to a perturbation of the equation. The method of proving these results is combining Trudinger–Moser type inequality, Mountain-pass theorem and Ekeland’s variational principle.

© 2011 Elsevier Inc. All rights reserved.

Keywords: Trudinger–Moser inequality; Singular Trudinger–Moser inequality; N-Laplace equation; Exponential growth

1. Introduction

Let $\Omega \subset \mathbb{R}^N$ be a bounded smooth domain. There are fruitful results on the following problem

\[
\begin{cases}
-\Delta_p u = f(x, u) \quad \text{in } \Omega, \\
u \in W_0^{1,p}(\Omega),
\end{cases}
\]

(E-mail address: yunyanyang@ruc.edu.cn.)
where \( \Delta_p u = \text{div}(|\nabla u|^{p-2} \nabla u) \). When \( p = 2 \) and \( |f(x, u)| \leq c(|u| + |u|^{q-1}) \), \( 1 < q \leq 2^* = 2N/(N - 2) \), \( N \geq 3 \). Among pioneer works we mention Brézis [9], Brézis and Nirenberg [11], Bartsch and Willem [8] and Capozzi, Fortunato and Palmieri [13]. For \( p \leq N \) and \( p^2 \leq N \), Garcia and Alonso [23] generalized Brézis–Nirenberg’s existence and nonexistence results to \( p \)-Laplace equation. When \( \Omega = \mathbb{R}^N \) and \( p = 2 \), one may consider the semilinear Schrödinger equation instead of (1.1):

\[
\begin{cases}
-\Delta u + V(x)u = f(x, u) & \text{in } \mathbb{R}^N, \\
u \in W^{1,N}(\mathbb{R}^N),
\end{cases}
\]

where again \( |f(x, u)| \leq c(|u| + |u|^{q-1}) \), \( 1 < q \leq 2^* = \frac{2N}{N - 2} \). Many papers are devoted to (1.2), we refer the reader to Kryszewski and Szulkin [25], Alama and Li [6], Ding and Ni [17] and Jeanjean [24]. Sobolev embedding theorems and critical point theory, in particular the mountain-pass theorem would play an important role in studying problems (1.1) and (1.2) since both of them have variational structure. When \( p = N \) and \( f(x, u) \) behaves like \( e^{\alpha |x|^{N/(N - 1)}} \) as \( |x| \to \infty \), problem (1.1) was studied by Adimurthi [2], Adimurthi and Yadava [4], Ruf et al. [15,16], J.M. do Ó [18], Panda [31] and the references therein. To the author’s knowledge, all these results are based on Trudinger–Moser inequality [29,32,35] and critical point theory.

In this paper we consider the existence of positive solutions of the quasi-linear equation

\[
-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x, u)}{|x|^{\beta}}, \quad x \in \mathbb{R}^N (N \geq 2),
\]

where \( \Delta_N u = \text{div}(|\nabla u|^{N-2} \nabla u) \), \( V : \mathbb{R}^N \to \mathbb{R} \) is a continuous function, \( f(x, s) \) is continuous in \( \mathbb{R}^N \times \mathbb{R} \) and behaves like \( e^{\alpha_N |s|^{N/(N-1)}} \) as \( s \to +\infty \), and \( 0 \leq \beta < N \). Problem (1.3) can be compared with (1.2) in this way: Sobolev embedding theorem can be applied to (1.2), while Trudinger–Moser embedding theorem cannot be applied to (1.3). When \( \beta = 0 \), problem (1.3) was studied by D. Cao [12] in the case \( N = 2 \), by Panda [30], J.M. do Ó [19] and Alves and Figueiredo [7] in general dimensional case. Denote \( \alpha_N = N\omega_{N-1}^{1/(N-1)} \), where \( \omega_{N-1} \) is the measure of unit sphere in \( \mathbb{R}^N \). When \( 0 < \beta < N \), problem (1.3) is closely related to a singular Trudinger–Moser type inequality, namely

**Theorem A. (See [5].)** For all \( \alpha > 0 \), \( 0 \leq \beta < N \), and \( u \in W^{1,N}(\mathbb{R}^N) (N \geq 2) \), there holds

\[
\int_{\mathbb{R}^N} e^{\alpha |x|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{kN/(N-1)}}{k!} \frac{dx}{|x|^{\beta}} < \infty.
\]

Furthermore, we have for all \( \alpha \leq (1 - \frac{\beta}{N})\alpha_N \) and \( \tau > 0 \),

\[
\sup_{f: \mathbb{R}^N \to [0, \infty]} \int_{\mathbb{R}^N} e^{\alpha |x|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{kN/(N-1)}}{k!} \frac{dx}{|x|^{\beta}} \leq \int_{\mathbb{R}^N} e^{\alpha |x|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha^k |u|^{kN/(N-1)}}{k!} \frac{dx}{|x|^{\beta}} < \infty.
\]

This inequality is sharp: for any \( \alpha > (1 - \frac{\beta}{N})\alpha_N \), the supremum is infinity.
This theorem extends a result of Adimurthi and Sandeep [3] on a bounded smooth domain.
When $\beta = 0$ and $\tau = 1$, (1.5) was proved by B. Ruf [34] in the case $N = 2$ via symmetrization method and by Li and Ruf [26] in general dimensional case via the method of blow-up analysis. When $\beta = 0$ and $\alpha < \alpha_N$, (1.5) was first proved by Cao [12] in the case $N = 2$, and then by Panda [30], J.M. do Ó [19] in general dimensional case. A similar but different type inequality was obtained by Adachi and Tanaka [1].

We assume the following two conditions on the potential $V(x)$:

$(V_1)$ $V(x) \geq V_0 > 0$ in $\mathbb{R}^N$ for some $V_0 > 0$;
$(V_2)$ The function $\frac{1}{V(x)}$ belongs to $L^{\frac{1}{N-1}}(\mathbb{R}^N)$.

As for the nonlinearity $f(x, s)$ we suppose the following:

$(H_1)$ There exist constants $\alpha_0, b_1, b_2 > 0$ such that for all $(x, s) \in \mathbb{R}^N \times \mathbb{R}^+$,
$$|f(x, s)| \leq b_1 s^{N-1} + b_2 \left\{ e^{\alpha_0 |s|^{N/(N-1)}} - \sum_{k=0}^{N-2} \frac{\alpha_0^k s^{Nk/(N-1)}}{k!} \right\};$$

$(H_2)$ There exists $\mu > N$ such that for all $x \in \mathbb{R}^N$ and $s > 0$,
$$0 < \mu F(x, s) \equiv \mu \int_0^s f(x, t) \, dt \leq s f(x, s);$$

$(H_3)$ There exist constants $R_0, M_0 > 0$ such that for all $x \in \mathbb{R}^N$ and $s \geq R_0$,
$$F(x, s) \leq M_0 f(x, s).$$

Define a function space
$$E = \left\{ u \in W^{1,N}(\mathbb{R}^N) : \int_{\mathbb{R}^N} V(x)|u|^N \, dx < \infty \right\}. \tag{1.6}$$

We say that $u \in E$ is a weak solution of problem (1.3) if for all $\varphi \in E$ we have
$$\int_{\mathbb{R}^N} \left( |\nabla u|^{N-2} \nabla u \nabla \varphi + V(x)|u|^{N-2} u \varphi \right) \, dx = \int_{\mathbb{R}^N} \frac{f(x, u)}{|x|^{\beta}} \varphi \, dx.$$

The assumption $(V_1)$ implies that $E$ is a reflexive Banach space when equipped with the norm
$$\|u\|_E = \left\{ \int_{\mathbb{R}^N} \left( |\nabla u|^N + V(x)|u|^N \right) \, dx \right\}^{\frac{1}{N}}, \tag{1.7}$$

and for any $q \geq N$, the embedding
is continuous. However $(V_2)$ together with $(V_1)$ implies that $E \hookrightarrow L^q(\mathbb{R}^N)$ is compact for all $q \geq 1$ (see Lemma 2.4 below). Surprisingly the assumption $(V_2)$ is much better than

$$(V_2') \ V(x) \to +\infty \text{ as } |x| \to +\infty,$$

since $(V_1)$ together with $(V_2')$ only leads to the compact embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \geq N$ (see for example Costa [14] for details). However in this paper our argument of proving the main results depends crucially on the compact embedding $E \hookrightarrow L^q(\mathbb{R}^N)$ for all $q \geq 1$.

For any $\beta: 0 \leq \beta < N$, we define a singular eigenvalue for the $N$-Laplace operator by

$$\lambda_\beta = \inf_{u \in E, u \neq 0} \frac{\|u\|_E^N}{\int_{\mathbb{R}^N} \frac{|u|^N}{|x|^\beta} \ dx}.$$  \hspace{1cm} (1.8)

It is easy to see that $\lambda_\beta > 0$. Write $m(r) = \sup_{|x| \leq r} V(x)$ and

$$\mathcal{M} = \inf_{r > 0} \frac{(N - \beta)^N}{\alpha_0^{N-1} r^{N-\beta}} e^{(N-\beta)m(r) \frac{(N-2)N}{N^2} r^N},$$

where $\alpha_0$ is given by $(H_1)$. If $V(x)$ is continuous and $(V_1)$ is satisfied, then $m(r)$ is a positive continuous function and $\mathcal{M}$ can be attained by some $r > 0$.

One of our main results can be stated as follows:

**Theorem 1.1.** Assume that $V(x)$ is a continuous function satisfying $(V_1)$ and $(V_2)$, $f: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R}$ is a continuous function and the hypotheses $(H_1)$, $(H_2)$ and $(H_3)$ hold. Furthermore we assume

$$(H_4) \ \limsup_{s \to 0^+} \frac{N|F(x,s)|}{s^N} < \lambda_\beta \text{ uniformly with respect to } x \in \mathbb{R}^N;$$

$$(H_5) \ \liminf_{s \to +\infty} sf(x,s) e^{-\alpha_0 s^{N-1}} = \beta_0 > \mathcal{M} \text{ uniformly with respect to } x \in \mathbb{R}^N.$$  \hspace{1cm} (1.9)

Then Eq. (1.3) has a nontrivial positive mountain-pass type weak solution.

Here and throughout this paper, we say that a weak solution $u$ is positive if $u(x) \geq 0$ for almost every $x \in \mathbb{R}^N$. It should be pointed out that $\mathcal{M}$ is not the best constant in $(H_5)$. It would be interesting if one can find an explicit smaller number replacing $\mathcal{M}$.

In [5], Theorem A has been employed to study a perturbation of Eq. (1.3), namely

$$-\Delta_N u + V(x)|u|^{N-2}u = \frac{f(x,u)}{|x|^{\beta}} + \epsilon h, \quad x \in \mathbb{R}^N \ (N \geq 2),$$  \hspace{1cm} (1.10)

where $\epsilon > 0$ is a constant and $h: \mathbb{R}^N \to \mathbb{R}$ is a function belonging to $E^*$, the dual space of $E$. If $V(x)$ satisfies $(V_1)$, $(V_2)$, and $f(x,s)$ satisfies $(H_1)$–$(H_4)$, then it was shown in [5] that when $\epsilon > 0$ is sufficiently small and $h \neq 0$, the problem (1.10) has two weak solutions: one is of
mountain-pass type and the other is of negative energy. But we cannot conclude that the two solutions are distinct. In this paper, replacing \((V_2')\) by \((V_2)\) and imposing additional condition \((H_5)\), we can prove that the above two solutions are distinct, namely

**Theorem 1.2.** Suppose that \(f(x,s)\) is continuous in \(\mathbb{R}^N \times \mathbb{R}\) and \((H_1)-(H_5)\) hold. \(V(x)\) is continuous in \(\mathbb{R}^N\) satisfying \((V_1)\) and \((V_2)\), \(h\) belongs to \(E^*\), the dual space of \(E\), with \(h \geq 0\) and \(h \neq 0\). Then there exists \(\epsilon_0 > 0\) such that if \(0 < \epsilon < \epsilon_0\), then the problem \((1.10)\) has two distinct positive weak solutions.

The proof of Theorem 1.1 and Theorem 1.2 is based on Theorem A, the mountain-pass theorem without the Palais–Smale condition [33] and the Ekeland’s variational principle [36], which were also used in [5,21]. Let us make some reduction on problems \((1.3)\) and \((1.10)\). Set

\[
\tilde{f}(x,s) = \begin{cases} 
0, & f(x,s) < 0, \\
 f(x,s), & f(x,s) \geq 0.
\end{cases}
\]

Assume \(u \in E\) is a weak solution of

\[
-\Delta_N u + V(x)|u|^{N-2}u = \frac{\tilde{f}(x,u)}{|x|^{\beta}} + \epsilon h,
\]

where \(h \geq 0\) and \(\epsilon > 0\), then the negative part of \(u\), namely

\[
u_-(x) = \begin{cases} 
0, & u(x) > 0, \\
 u(x), & u(x) \leq 0.
\end{cases}
\]

belongs to the function space \(E\) and satisfies

\[
\int_{\mathbb{R}^N} \left( |\nabla u_-|^N + V(x)|u_-|^N \right) dx = \int_{\mathbb{R}^N} \frac{\tilde{f}(x,u)}{|x|^{\beta}} u_- dx + \epsilon \int_{\mathbb{R}^N} hu_- dx \\
= \epsilon \int_{\mathbb{R}^N} hu_- dx \leq 0.
\]

Hence \(u_-(x) = 0\) for almost every \(x \in \mathbb{R}^N\) and thus \(u\) is a positive weak solution of \((1.11)\). This together with \((H_2)\) implies \(f(x,u) \geq 0\). It follows that \(\tilde{f}(x,u) = f(x,u)\). Therefore \(u\) is also a positive weak solution of \((1.10)\). When \(h = 0\), \((1.10)\) becomes \((1.3)\). Based on this, to prove Theorems 1.1 and 1.2, it suffices to find weak solutions of \((1.3)\) and \((1.10)\) with \(f\) replaced by \(\tilde{f}\) respectively. So throughout this paper, we can assume without loss of generality

\[
f(x,s) \equiv 0, \quad \forall s < 0.
\]

Before ending this introduction, we would like to mention that results similar to Theorem 1.2 in two-dimensional case, i.e. \(N = 2\), were obtained by J.M. do Ó [21]. Similar problems for bi-Laplace equation in \(\mathbb{R}^4\) was considered by the author in [37]. For compact Riemannian manifold case, we refer the reader to [22,38]. Also it should be remarked that results obtained in [5]
and in the present paper still hold if there is only the subcritical case of (1.5), namely for any $\alpha < (1 - \beta/N)\alpha_N$ and $\tau > 0$,

$$
\sup_{f \in \mathbb{R}^N} \left( |\nabla u|^N + \tau |u|^N \right) dx \leq 1.
$$

In fact, in [7,12,19,30], all the contributors only used the above subcritical inequality.

The remaining part of this paper is organized as follows: In Section 2, we display several key estimates in later compactness analysis. In Section 3, we consider the functionals related to problems (1.3) and (1.10). Finally Theorem 1.1 is proved in Section 4 and Theorem 1.2 is proved in Section 5.

2. Key estimates for the subsequent compactness analysis

In this section we will derive several technical lemmas for our use later. For any integer $N \geq 2$ and real number $s$, we define a function $\zeta : \mathbb{N} \times \mathbb{R} \to \mathbb{R}$ by

$$
\zeta(N, s) = e^s - \sum_{k=0}^{N-2} \frac{s^k}{k!} = \sum_{k=N-1}^{\infty} \frac{s^k}{k!}.
$$

(2.1)

**Lemma 2.1.** Let $s \geq 0$, $p \geq 1$ be real numbers and $N \geq 2$ be an integer. Then there holds

$$
(\zeta(N, s))^p \leq \zeta(N, ps).
$$

(2.2)

**Proof.** We prove (2.2) by induction with respect to $N$. Define a function $\phi(s) = (e^s - 1)^p - (e^{ps} - 1)$. It is easy to see that for $s \geq 0$ and $p \geq 1$,

$$
\phi'(s) = p(e^s - 1)^{p-1} - pe^{ps} \leq 0.
$$

Hence $\phi(s) \leq \phi(0) = 0$ and thus (2.2) holds for $N = 2$. Suppose (2.2) holds for $N \geq 2$, we only need to prove that

$$
(\zeta(N + 1, s))^p \leq \zeta(N + 1, ps).
$$

(2.3)

For this purpose we set

$$
\psi(s) = \left( e^s - \sum_{k=0}^{N-1} \frac{s^k}{k!} \right)^p - \left( e^{ps} - \sum_{k=0}^{N-1} \frac{(ps)^k}{k!} \right).
$$

A straightforward calculation shows
\[
\begin{align*}
\psi'(s) &= p \left( e^s - \sum_{k=0}^{N-1} \frac{s^k}{k!} \right)^{p-1} \left( e^s - \sum_{k=1}^{N-1} \frac{s^{k-1}}{(k-1)!} \right) - \left( pe^{ps} - p \sum_{k=1}^{N-1} \frac{(ps)^{k-1}}{(k-1)!} \right) \\
&\leq p \left\{ \left( e^s - \sum_{k=1}^{N-1} \frac{s^{k-1}}{(k-1)!} \right)^p - \left( e^{ps} - \sum_{k=1}^{N-1} \frac{(ps)^{k-1}}{(k-1)!} \right) \right\} \\
&= p \left\{ \left( e^s - \sum_{k=0}^{N-2} \frac{s^k}{k!} \right)^p - \left( e^{ps} - \sum_{k=1}^{N-2} \frac{(ps)^k}{k!} \right) \right\} \leq 0.
\end{align*}
\]

Here we have used the induction assumption \( (\zeta(N,s))^p \leq \zeta(N,ps) \). Thus \( \psi(s) \leq \psi(0) = 0 \) for \( s \geq 0 \), and whence (2.3) holds. Therefore (2.2) holds for any integer \( N \geq 2 \).

Lemma 2.2. For all \( N \geq 2, s \geq 0, t \geq 0, \mu > 1 \) and \( v > 1 \) with \( 1/\mu + 1/v = 1 \), there holds

\[
\zeta(N, s + t) \leq \frac{1}{\mu} \zeta(N, \mu s) + \frac{1}{v} \zeta(N, vt).
\]

Proof. Observing that

\[
\frac{\partial^2}{\partial s^2} \zeta(2, s) = e^s \geq 0, \quad \frac{\partial^2}{\partial s^2} \zeta(3, s) = e^s \geq 0
\]

and when \( N \geq 4, \)

\[
\frac{\partial^2}{\partial s^2} \zeta(N, s) = e^s - \sum_{k=2}^{N-2} \frac{s^{k-2}}{(k-2)!} = e^s - \sum_{k=0}^{N-4} \frac{s^k}{k!} \geq 0,
\]

we conclude that \( \zeta(N, s) \) is convex with respect to \( s \) for all \( N \geq 2 \). Hence

\[
\zeta(N, s + t) = \zeta \left( N, \frac{1}{\mu} \mu s + \frac{1}{v} vt \right) \leq \frac{1}{\mu} \zeta(N, \mu s) + \frac{1}{v} \zeta(N, vt).
\]

This concludes the lemma.

Lemma 2.3. Let \( (w_n) \) be a sequence in \( E \). Suppose \( \|w_n\|_E = 1 \), \( w_n \rightharpoonup w_0 \) weakly in \( E \), \( w_n(x) \to w_0(x) \) and \( \nabla w_n(x) \to \nabla w_0(x) \) for almost every \( x \in \mathbb{R}^N \). Then for any \( p: 0 < p < \frac{1}{(1 - \|w_0\|_E^2)^{1/(N-1)}} \)

\[
\sup_n \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_N(1 - \beta/N)p |w_n|^\frac{N}{N-1})}{|x|^\beta} dx < \infty. \tag{2.4}
\]

Proof. We follow the lines of Lions [28]. Noticing that

\[
|w_n|^{\frac{N}{N-1}} = |w_n - w_0 + w_0|^{\frac{N}{N-1}} \leq (1 + \epsilon) |w_n - w_0|^{\frac{N}{N-1}} + c(\epsilon) |w_0|^{\frac{N}{N-1}},
\]
we have by using Lemma 2.2
\[
\zeta(N, \alpha_N (1 - \beta/N) p |w_n|^N) \leq \frac{1}{\mu} \zeta(N, \mu (1 + \epsilon) \alpha_N (1 - \beta/N) p |w_n - w_0|^N) + \frac{1}{v} \zeta(N, v c(\epsilon) \alpha_N (1 - \beta/N) p |w_0|^N) \\
\leq \zeta(N, (1 + \epsilon) \alpha_N (1 - \beta/N) p \|w_n - w_0\|_{E}^{N} \left( \frac{|w_n - w_0|}{\|w_n - w_0\|_{E}} \right)^{N} \\
+ \zeta(N, v c(\epsilon) \alpha_N (1 - \beta/N) p |w_0|^N),
\]
where \( \mu > 1, v > 1 \) and \( 1/\mu + 1/v = 1 \). By Brézis–Lieb’s lemma [10],
\[
\|w_n - w_0\|_{E}^{N} = 1 - \|w_0\|_{E}^{N} + o_n(1),
\]
where \( o_n(1) \to 0 \) as \( n \to \infty \). Hence for any \( p: 0 < p < \frac{1}{(1 - \|w_0\|_{E}^{N})^{1/(N-1)}} \), one can choose \( \epsilon > 0 \) sufficiently small and \( \mu > 1 \) sufficiently close to 1 such that
\[
\mu (1 + \epsilon) \alpha_N (1 - \beta/N) p \|w_n - w_0\|_{E}^{N} < \alpha_N (1 - \beta/N).
\]
Now (2.4) follows from Theorem A immediately. \( \square \)

**Lemma 2.4.** Assume \( V: \mathbb{R}^N \times \mathbb{R} \to \mathbb{R} \) is continuous and \((V_1), (V_2)\) hold. Then \( E \) is compactly embedded in \( L^q(\mathbb{R}^N) \) for all \( q \geq 1 \).

**Proof.** By \((V_1)\), the standard Sobolev embedding theorem implies that the following embedding is continuous
\[
E \hookrightarrow W^{1,N}(\mathbb{R}^N) \hookrightarrow L^q(\mathbb{R}^N) \quad \text{for all } N \leq q < \infty.
\]
It follows from the Hölder inequality and \((V_2)\) that
\[
\int_{\mathbb{R}^N} |u|^{\gamma} \, dx \leq \left( \int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} \, dx \right)^{1-1/N} \left( \int_{\mathbb{R}^N} V |u|^N \, dx \right)^{1/N} \leq \left( \int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} \, dx \right)^{1-1/N} \|u\|_E.
\]
For any \( 1 < \gamma < N \), there holds
\[
\int_{\mathbb{R}^N} |u|^\gamma \, dx \leq \int_{\mathbb{R}^N} (|u| + |u|^N) \, dx \leq \left( \int_{\mathbb{R}^N} \frac{1}{V^{\frac{1}{N-1}}} \, dx \right)^{1-1/N} \|u\|_E + \frac{1}{V_0} \|u\|_E^N,
\]
where \( V_0 \) is given by \((V_1)\). Thus we get continuous embedding \( E \hookrightarrow L^q(\mathbb{R}^N) \) for all \( q \geq 1 \).

To prove that the above embedding is also compact, take a sequence of functions \((u_k) \subset E\) such that \( \|u_k\|_E \leq C \) for all \( k \), we must prove that up to a subsequence there exists some \( u \in E \)
such that $u_k$ convergent to $u$ strongly in $L^q(\mathbb{R}^N)$ for all $q \geq 1$. Without loss of generality we may assume

$$\begin{cases}
u_k \rightarrow u \quad \text{weakly in } E, \\
u_k \rightarrow u \quad \text{strongly in } L^q_{\text{loc}}(\mathbb{R}^N), \forall q \geq 1, \\
u_k \rightarrow u \quad \text{almost everywhere in } \mathbb{R}^N.
\end{cases} \quad (2.5)$$

In view of $(V_2)$, for any $\epsilon > 0$, there exists $R > 0$ such that

$$\left( \int_{|x| > R} \frac{1}{V^{1/N} dx} \right)^{1-1/N} < \epsilon.$$  

Hence

$$\int_{|x| > R} |u_k - u| dx \leq \left( \int_{\mathbb{R}^N} \frac{1}{V^{1/N} dx} \right)^{1-1/N} \left( \int_{\mathbb{R}^N} V |u|^N dx \right)^{1/N} \leq \epsilon \|u_k - u\|_E \leq C \epsilon. \quad (2.6)$$

Here and in the sequel we often denote various constants by the same $C$. On the other hand, it follows from (2.5) that $u_k \rightarrow u$ strongly in $L^1(B_R(0))$, where $B_R(0) \subset \mathbb{R}^N$ is the ball centered at 0 with radius $R$. This together with (2.6) leads to

$$\limsup_{k \rightarrow \infty} \int_{\mathbb{R}^N} |u_k - u| dx \leq C \epsilon.$$

Since $\epsilon$ is arbitrary, we obtain

$$\lim_{k \rightarrow \infty} \int_{\mathbb{R}^N} |u_k - u| dx = 0.$$  

For $q > 1$, it follows from the continuous embedding $E \hookrightarrow L^s(\mathbb{R}^N)$ ($s \geq 1$) that

$$\int_{\mathbb{R}^N} |u_k - u|^q dx = \int_{\mathbb{R}^N} |u_k - u|^\frac{1}{2} |u_k - u|^{(q-\frac{1}{2})} dx$$

$$\leq \left( \int_{\mathbb{R}^N} |u_k - u| dx \right)^{1/2} \left( \int_{\mathbb{R}^N} |u_k - u|^{2q-1} dx \right)^{1/2}$$

$$\leq C \left( \int_{\mathbb{R}^N} |u_k - u| dx \right)^{1/2} \rightarrow 0$$

as $k \rightarrow \infty$. This concludes the lemma. \qed
3. Functionals and compactness analysis

3.1. The functionals and their profiles

As we mentioned in the introduction, problems (1.3) and (1.10) have variational structure. To apply the critical point theory, we define the functional $J_{\beta,\epsilon} : E \to \mathbb{R}$ by

$$J_{\beta,\epsilon}(u) = \frac{1}{N} \|u\|_E^N - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta} \, dx - \epsilon \int_{\mathbb{R}^N} hu \, dx,$$

where $\epsilon \geq 0$, $0 \leq \beta < N$, $\|u\|_E$ is the norm of $u \in E$ defined by (1.7) and $F(x,s) = \int_0^s f(x,t) \, dt$ is the primitive of $f(x,s)$. Assume $f$ satisfies the hypothesis $(H_1)$. Then there exist some positive constants $\alpha_1 > \alpha_0$ and $b_3$ such that for all $(x,s) \in \mathbb{R}^N \times \mathbb{R}$, $F(x,s) \leq b_3 \zeta(N,\alpha_1|s|^{N/(N-1)})$. Thus $J_{\beta,\epsilon}$ is well defined thanks to Theorem A. In the case $\epsilon = 0$, we denote $J_{\beta,0}$ for simplicity by

$$J(u) = \frac{1}{N} \|u\|_E^N - \int_{\mathbb{R}^N} \frac{F(x,u)}{|x|^\beta} \, dx.$$

The profiles of the functionals $J_{\beta,\epsilon}$ and $J(u)$ are well described in the following lemma.

**Lemma 3.1.** Assume $(V_1)$, $(H_1)$, $(H_2)$, $(H_3)$ and $(H_4)$ are satisfied. Then (i) for any nonnegative, compactly supported function $u \in W^{1,N} (\mathbb{R}^N \setminus \{0\})$, there holds $J_{\beta,\epsilon}(u) \to -\infty$ as $t \to +\infty$; (ii) there exists $\epsilon_1 > 0$ such that when $0 < \epsilon < \epsilon_1$, one can find $r_\epsilon$, $\vartheta_\epsilon > 0$ such that $J_{\beta,\epsilon}(u) \geq \vartheta_\epsilon$ for all $u$ with $\|u\|_E = r_\epsilon$, where $r_\epsilon$ can be further chosen such that $r_\epsilon \to 0$ as $\epsilon \to 0$. When $\epsilon = 0$, there exist $\delta > 0$ and $r > 0$ such that $J(u) \geq \delta$ for all $\|u\|_E = r$; (iii) assume $\epsilon > 0$ and $h \not\equiv 0$, there exists a constant $\tau > 0$ such that if $0 < t < \tau$, then $\inf_{\|u\|_E \leq t} J_{\beta,\epsilon}(u) < 0$.

**Proof.** We refer the reader to [5, Lemmas 4.1, 4.2 and 4.3] for details. It is remarkable that we can also apply Lemma 2.1 and Lemma 2.4 instead of decreasing rearrangement argument in the proof of [5, Lemma 4.2] and thus simplify it.

To use the critical point theory, we need some regularity of the functionals $J_{\beta,\epsilon}$ and $J$. In fact, by Proposition 1 in [21] and standard arguments (see for example [33]), one can see that both $J_{\beta,\epsilon}$ and $J$ belong to $C^1(E, \mathbb{R})$. A straightforward calculation shows

$$\langle J'(u), \phi \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{N-2} \nabla u \nabla \phi + V |u|^{N-2} u \phi) \, dx - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \phi \, dx,$$

$$\langle J'_{\beta,\epsilon}(u), \phi \rangle = \int_{\mathbb{R}^N} (|\nabla u|^{N-2} \nabla u \nabla \phi + V |u|^{N-2} u \phi) \, dx - \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \phi \, dx - \epsilon \int_{\mathbb{R}^N} h \phi \, dx$$

(3.2)
for all $\phi \in E$. Hence weak solutions of (1.3) and (1.10) are critical points of $J$ and $J_{\beta,\epsilon}$ respectively.

### 3.2. Min–max value

In this subsection, we prepare for estimating the min–max value of the functionals $J$ and $J_{\beta,\epsilon}$. The idea is to construct a sequence of functions $M_n \in E$ and estimate $\max_{t \geq 0} J(tM_n)$ and $\max_{t \geq 0} J_{\beta,\epsilon}(tM_n)$. Recall Moser’s function sequence

$$\tilde{M}_n(x, r) = \frac{1}{\omega_N^{1/N-1}} \begin{cases} \left(\log n\right)^{1-1/N}, & |x| \leq r/n, \\ \frac{\log r}{(\log n)^{1/N}}, & r/n < |x| \leq r, \\ 0, & |x| > r. \end{cases}$$

Let $M_n(x, r) = \frac{1}{\|\tilde{M}_n\|_E} \tilde{M}_n(x, r)$. Then $M_n$ belongs to $E$ with its support in $B_r(0)$ and $\|M_n\|_E = 1$.

**Lemma 3.2.** Assume $V(x)$ is continuous and $(V_1)$ is satisfied. Then there holds

$$\|\tilde{M}_n\|_E^N \leq 1 + \frac{m(r)}{\log n} \left( \frac{(N-1)!}{N^N} r^N + o_n(1) \right), \quad (3.3)$$

where $m(r) = \max_{|x| \leq r} V(x)$ and $o_n(1) \to 0$ as $n \to \infty$.

**Proof.** It is easy to calculate

$$\int_{\mathbb{R}^N} |\nabla \tilde{M}_n|^N dx = \frac{1}{\omega_N} \int_{\frac{r}{n} \leq |x| \leq r} \frac{1}{|x|^N \log n} dx = 1.$$

Integration by parts gives

$$\int_{\frac{r}{n} \leq |x| \leq r} \left( \log \frac{r}{|x|} \right)^N dx$$

$$= \omega_N \int_{\frac{r}{n}}^r s^{N-1} \left( \log \frac{r}{s} \right)^N ds$$

$$= -\omega_N \left( \frac{r}{n} \right)^N (\log n)^N + \omega_N \int_{\frac{r}{n}}^r s^{N-1} \left( \log \frac{r}{s} \right)^{N-1} ds$$

$$= -\omega_N \left( \frac{r}{n} \right)^N \left\{ \frac{1}{N} (\log n)^N + \frac{1}{N} (\log n)^{N-1} + \frac{N-1}{N^2} (\log n)^{N-2} \right\}$$
\[ + \cdots + \frac{(N-1)(N-2)\cdots 3}{N^{N-2}}(\log n)^2 \Bigg] + \omega_{N-1} \left( \frac{(N-1)!}{N^{N-2}} \right) \int_{\frac{r}{N}}^{r} s^{N-1} \log \frac{r}{s} \, ds \]

\[ = \omega_{N-1} \frac{(N-1)!}{N^N} r^N + o_n(1). \]

Hence

\[ \int_{\mathbb{R}^N} |\tilde{M}_n|^N \, dx = \frac{1}{\omega_{N-1}} \int_{|x| \leq r/n} (\log n)^{N-1} \, dx + \frac{1}{\omega_{N-1}} \int_{\frac{r}{n} \leq |x| \leq r} \frac{(\log \frac{r}{|x|})^N}{\log n} \, dx \]

\[ = \left( \frac{r}{n} \right)^N \frac{(\log n)^{N-1}}{N} + \frac{1}{\omega_{N-1} \log n} \int_{\frac{r}{n} \leq |x| \leq r} \left( \log \frac{r}{|x|} \right)^N \, dx \]

\[ = \frac{1}{\log n} \left( \frac{(N-1)!}{N^N} r^N + o_n(1) \right), \]

and thus

\[ \| \tilde{M}_n \|_E^N = \int_{\mathbb{R}^N} |\nabla \tilde{M}_n|^N \, dx + \int_{\mathbb{R}^N} V(x)|\tilde{M}_n|^N \, dx \]

\[ \leq 1 + m(r) \int_{\mathbb{R}^N} |\tilde{M}_n|^N \, dx \]

\[ = 1 + m(r) \left( \frac{(N-1)!}{N^N} r^N + o_n(1) \right). \]

This is exactly (3.3). \( \square \)

**Lemma 3.3.** Assume \((V_1), (H_1), (H_2), (H_3)\) and \((H_5)\). There exists some \( n \in \mathbb{N} \) such that

\[ \max_{t \geq 0} J(tM_n) < \frac{1}{N} \left( \frac{N - \beta}{N - \alpha_0} \right)^{N-1}. \]  

(3.4)

Furthermore for the above \( n \) there exist some \( \epsilon^* > 0 \) and \( \delta^* > 0 \) such that if \( 0 \leq \epsilon < \epsilon^* \), then

\[ \max_{t \geq 0} J_{\beta, \epsilon}(tM_n) < \frac{1}{N} \left( \frac{N - \beta}{N - \alpha_0} \right)^{N-1} - \delta^*. \]  

(3.5)

**Proof.** We first prove (3.4). By \((H_5)\) and (1.9) (the definition of \( \mathcal{M} \)), there exists some \( r > 0 \) such that

\[ \beta_0 > \frac{(N-\beta)^N}{\alpha_0^{N-1} r^{N-\beta}} e^{(N-\beta)m(r)} \left( \frac{N-2\beta}{N\beta} \right) r^N. \]  

(3.6)
Suppose by contradiction that for all \( n \in \mathbb{N} \)

\[
\max_{t \geq 0} J(tM_n) \geq \frac{1}{N} \left( \frac{N - \beta \alpha_N}{N \alpha_0} \right)^{N-1}.
\] (3.7)

By (i) of Lemma 3.1, \( \forall n \in \mathbb{N} \), there exists \( t_n > 0 \) such that

\[
J(t_n M_n) = \max_{t \geq 0} J(t M_n).
\]

Thus (3.7) gives

\[
J(t_n M_n) = \frac{t_n^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t_n M_n)}{|x|^\beta} \, dx \geq \frac{1}{N} \left( \frac{N - \beta \alpha_N}{N \alpha_0} \right)^{N-1}.
\]

Noticing that \( F(x, \cdot) \geq 0 \), we have

\[
t_n^N \geq \left( \frac{N - \beta \alpha_N}{N \alpha_0} \right)^{N-1}.
\] (3.8)

It is easy to see that at \( t = t_n \),

\[
\frac{d}{dt} \left( \frac{t_n^N}{N} - \int_{\mathbb{R}^N} \frac{F(x, t M_n)}{|x|^\beta} \, dx \right) = 0,
\]

or equivalently

\[
t_n^N = \int_{\mathbb{R}^N} \frac{t_n M_n f(x, t_n M_n)}{|x|^\beta} \, dx.
\] (3.9)

By \((H_5)\), \( \forall \eta > 0, \exists R_\eta > 0 \) such that for all \( x \in \mathbb{R}^N \) and \( u \geq R_\eta \)

\[
uf(x, u) \geq (\beta_0 - \eta)e^{\alpha_0 |u|^{\frac{N}{N-1}}}.\] (3.10)

By Lemma 3.2, when \( |x| \leq \frac{r}{\eta} \), we have

\[
M_{\frac{N}{N-1}}^N(x, r) \geq \frac{1}{\omega_{N-1}^{\frac{1}{N-1}}} \frac{\log n}{1 + \frac{1}{N-1} \log n \left( \frac{(N-1)!}{N^N} r^N + o_n(1) \right)}
\]

\[
= \omega_{N-1}^{-\frac{1}{N-1}} \log n - \omega_{N-1}^{-\frac{1}{N-1}} m(r) \left( \frac{N-2)!}{N^N} r^N + o_n(1) \right).\] (3.11)

Hence we have by combining (3.9) and (3.10) that
\[ t_n^N \geq (\beta_0 - \eta) \int_{|x| \leq r_n} e^{\alpha_0 |t_n M_n|^N N^{-1}} |x|^{-\beta} \, dx \]

\[ = (\beta_0 - \eta) \int_{|x| \leq r_n} e^{\alpha_0 \alpha_{N-1}^{-1} t_n^{N^{-1}}} (\log n - m(r))^{(N-2)N^{-1} r_N + o_n(1))} e^{\alpha_0 |t_n M_n|^N N^{-1}} |x|^{-\beta} \, dx \]

\[ = (\beta_0 - \eta) \frac{\alpha_{N-1}}{N - \beta} \left( \frac{r}{n} \right)^{N-\beta} e^{\alpha_0 \alpha_{N-1}^{-1} t_n^{N^{-1}}} (\log n - m(r))^{(N-2)N^{-1} r_N + o_n(1)} \] \hspace{1cm} (3.12)

This yields that \( t_n \) is a bounded sequence. In view of (3.8), we can also see from (3.12) that

\[ \lim_{n \to \infty} t_n^N = \left( \frac{N - \beta}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}. \] \hspace{1cm} (3.13)

For otherwise there exists some \( \delta > 0 \) such that for sufficiently large \( n \)

\[ t_n^N \geq \left( \delta + \frac{N - \beta}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}. \]

Thus

\[ \alpha_0 \alpha_{N-1}^{-1} t_n^{N^{-1}} \geq N - \beta + \alpha_0 \alpha_{N-1}^{-1} \delta \]

and whence the right hand of (3.12) tends to infinity which contradicts the bounded-ness of \( t_n \).

Now we estimate \( \beta_0 \) from above. It follows from (3.9) and (3.10) that

\[ t_n^N \geq (\beta_0 - \eta) \int_{|x| \leq r_n} e^{\alpha_0 |t_n M_n|^N N^{-1}} |x|^{-\beta} \, dx + \int_{t_n M_n < R_\eta} t_n M_n f(x, t_n M_n) \frac{1}{|x|^{-\beta}} \, dx \]

\[ - (\beta_0 - \eta) \int_{t_n M_n < R_\eta} e^{\alpha_0 |t_n M_n|^N N^{-1}} |x|^{-\beta} \, dx. \] \hspace{1cm} (3.14)

Since \( M_n \to 0 \) almost everywhere in \( \mathbb{R}^N \), we have by using the Lebesgue’s dominated convergence theorem

\[ \lim_{n \to \infty} \int_{t_n M_n < R_\eta} t_n M_n f(x, t_n M_n) \frac{1}{|x|^{-\beta}} \, dx = 0, \] \hspace{1cm} (3.15)

\[ \lim_{n \to \infty} \int_{t_n M_n < R_\eta} e^{\alpha_0 |t_n M_n|^N N^{-1}} |x|^{-\beta} \, dx = \int_{|x| \leq r} \frac{1}{|x|^{-\beta}} \, dx = \frac{\alpha_{N-1} r^{N-\beta}}{N - \beta}. \] \hspace{1cm} (3.16)
Using (3.8),
\[
\int_{|x| \leq r} e^{\alpha_0 |x|} \frac{N}{|x|^\beta} \, dx \geq \int_{|x| \leq r} e^{\alpha N (1-\beta/N) M_n^{N-1}} \frac{N}{|x|^\beta} \, dx \\
+ \int_{\frac{r}{N} \leq |x| \leq r} e^{\alpha N (1-\beta/N) M_n^{N-1}} \frac{N}{|x|^\beta} \, dx.
\]
(3.17)

On one hand we have by (3.11)
\[
\int_{|x| \leq r} e^{\alpha N (1-\beta/N) M_n^{N-1}} \frac{N}{|x|^\beta} \, dx \geq e^{\alpha N (1-\beta/N) \omega_{N-1} \frac{1}{N} \log n - \omega_{N-1} \frac{1}{N} m(r) (N-2)^N r^N + o_n(1)} \int_{|x| \leq r} \frac{1}{|x|^\beta} \, dx \\
= \frac{\omega_{N-1}}{N-\beta} \left( \frac{r}{n} \right)^{N-\beta} e^{(N-\beta)\log n - (N-\beta)m(r) (N-2)^N r^N + o_n(1)} \\
= \frac{\omega_{N-1} r^{N-\beta}}{N-\beta} e^{-(N-\beta)m(r) (N-2)^N r^N + o_n(1)}.
\]

On the other hand, by definition of $M_n$,
\[
\int_{\frac{r}{N} \leq |x| \leq r} e^{\alpha N (1-\beta/N) M_n^{N-1}} \frac{N}{|x|^\beta} \, dx \\
= \int_{\frac{r}{N} \leq |x| \leq r} e^{(N-\beta)((\log n)^{-1/N} \| \tilde{M}_n \|_{E}^{-1} \log \frac{r}{n})^{N-1}} \frac{N}{|x|^\beta} \, dx \\
= \omega_{N-1} \frac{r^{N-\beta}}{N-\beta} \int_{\frac{r}{n}}^r t^{N-\beta-1} \left( (\log n)^{-1/N} \| \tilde{M}_n \|_{E}^{-1} \log \frac{t}{n} \right)^{N-1} \, dt \\
\geq \omega_{N-1} r^{N-\beta} \int_0^r \left( (\log n)^{1/N} \| \tilde{M}_n \|_{E}^{-1} \right)^{N-1-(N-\beta)} \| \tilde{M}_n \|_{E} e^{-(N-\beta)\| \tilde{M}_n \|_{E} (\log n)^{1/N} s} \, ds \\
= \omega_{N-1} r^{N-\beta} \left( 1 - e^{-(N-\beta)\log n} \right).
Here we have used the change of variable $t = re^{-\|\tilde{M}_n\|_E(\log n)^{1/N}s}$ in the third equality. Hence we obtain by passing to the limit $n \to \infty$ in (3.17)

$$\liminf_{n \to \infty} \int_{|x| \leq r} e^{\omega_0|\alpha|nM_e} \frac{N}{|x|^\beta} \frac{N}{N-1} = \frac{\omega_{N-1}r^{N-\beta}}{N-\beta} \left(1 + e^{-(N-\beta)m(r)(N-2)^rN}\right).$$

This together with (3.13)–(3.16) implies

$$\left(\frac{N-\beta}{N} \frac{\alpha_0}{\alpha_0}\right)^{N-1} \geq (\beta_0 - \eta) \frac{\omega_{N-1}r^{N-\beta}}{N-\beta} e^{-(N-\beta)m(r)(N-2)^rN}.$$

Since $\eta > 0$ is arbitrary, we have

$$\beta_0 \leq \frac{(N-\beta)}{\alpha_0^{N-1}} e^{(N-\beta)m(r)(N-2)^rN}.$$

This contradicts (3.6) and ends the proof of (3.4).

Secondly it follows from (3.4) and the definition of $J_{\beta,\epsilon}$ that (3.5) holds.

3.3. Palais–Smale sequence

In this subsection, we will show that the weak limit of a Palais–Smale sequence for $J_{\beta,\epsilon}$ is the weak solution of (1.10). (Respectively the weak limit of a Palais–Smale sequence for $J$ is also the weak solution of (1.3).)

**Lemma 3.4.** Assume that $(V_1), (V_2), (H_1), (H_2)$ and $(H_3)$ are satisfied. Let $(u_n) \subset E$ be an arbitrary Palais–Smale sequence of $J_{\beta,\epsilon}$, i.e.,

$$J_{\beta,\epsilon}(u_n) \to c, \quad J'_{\beta,\epsilon}(u_n) \to 0 \quad \text{in } E^* \text{ as } n \to \infty,$$

(3.18)

where $E^*$ denotes the dual space of $E$. Then there exist a subsequence of $(u_n)$ (still denoted by $(u_n)$) and $u \in E$ such that $u_n \to u$ weakly in $E$, $u_n \to u$ strongly in $L^q(\mathbb{R}^N)$ for all $q \geq 1$, and

$$\begin{cases}
\nabla u_n(x) \to \nabla u(x) \quad &\text{a.e. in } \mathbb{R}^N, \\
\frac{f(x,u_n)}{|x|^\beta} \to \frac{f(x,u)}{|x|^\beta} \quad &\text{strongly in } L^1(\mathbb{R}^N), \\
\frac{F(x,u_n)}{|x|^\beta} \to \frac{F(x,u)}{|x|^\beta} \quad &\text{strongly in } L^1(\mathbb{R}^N).
\end{cases}$$

Furthermore $u$ is a weak solution of (1.10). The same conclusion holds when $\epsilon = 0$.

**Proof.** Assume $(u_n)$ is a Palais–Smale sequence of $J_{\beta,\epsilon}$. By (3.18), we have

$$\frac{1}{N} \|u_n\|_E^N - \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^\beta} \, dx - \epsilon \int_{\mathbb{R}^N} hu_n \, dx \to c \quad \text{as } n \to \infty,$$

(3.19)
\[
\left| \int_{\mathbb{R}^N} \left( |\nabla u_n|^{N-2} \nabla u_n \nabla \psi + V |u_n|^{N-2} u_n \psi \right) dx - \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} \psi dx - \epsilon \int h \psi \, dx \right| \\
\leq \tau_n \| \psi \|_E
\] (3.20)

for all \( \psi \in E \), where \( \tau_n \to 0 \) as \( n \to \infty \). Noticing that (1.12), we have by \( (H_2) \) that \( 0 \leq \mu F(x,u_n) \leq u_n f(x,u_n) \) for some \( \mu > N \). Taking \( \psi = u_n \) in (3.20) and multiplying (3.19) by \( \mu \), we have

\[
\left( \frac{\mu}{N} - 1 \right) \| u_n \|_E \leq \left( \frac{\mu}{N} - 1 \right) \| u_n \|_E - \int_{\mathbb{R}^N} \frac{\mu F(x,u_n) - f(x,u_n) u_n}{|x|^\beta} \, dx \\
\leq \mu |c| + \tau_n \| u_n \|_E + (\mu + 1) \epsilon \| h \|_{E^*} \| u_n \|_E.
\]

Therefore \( \| u_n \|_E \) is bounded. It then follows from (3.19), (3.20) that

\[
\int_{\mathbb{R}^N} \frac{f(x,u_n) u_n}{|x|^\beta} \, dx \leq C, \quad \int_{\mathbb{R}^N} \frac{F(x,u_n)}{|x|^\beta} \, dx \leq C
\] (3.21)

for some constant \( C \) depending only on \( \mu, N \) and \( \| h \|_{E^*} \). By Lemma 2.4, up to a subsequence, \( u_n \to u \) strongly in \( L^q(\mathbb{R}^N) \) for some \( u \in E \), \( \forall q \geq 1 \). This immediately leads to \( u_n \to u \) almost everywhere in \( \mathbb{R}^N \). Now we claim that up to a subsequence

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|f(x,u_n) - f(x,u)|}{|x|^\beta} \, dx = 0.
\] (3.22)

In fact, since \( f(x,\cdot) \geq 0 \), it suffices to prove that up to a subsequence

\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} \, dx = \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{f(x,u)}{|x|^\beta} \, dx.
\] (3.23)

Since \( u, \frac{f(x,u)}{|x|^\beta} \in L^1(\mathbb{R}^N) \), we have

\[
\lim_{\eta \to +\infty} \int_{|x| \geq \eta} \frac{f(x,u)}{|x|^\beta} \, dx = 0.
\]

Let \( C \) be the constant in (3.21). Given any \( \delta > 0 \), one can select some \( M > C/\delta \) such that

\[
\int_{|x| \geq M} \frac{f(x,u)}{|x|^\beta} \, dx < \delta.
\] (3.24)
It follows from (3.21) that
\[
\int_{|u_n| \geq M} \frac{f(x, u_n)}{|x|^{\beta}} \, dx \leq \frac{1}{M} \int_{|u_n| \geq M} \frac{f(x, u_n)u_n}{|x|^{\beta}} \, dx < \delta.
\] (3.25)

For all \( x \in \{ x \in \mathbb{R}^N : |u_n| < M \} \), by our assumption \((H_1)\), there exists a constant \( C_1 \) depending only on \( M \) such that \( |f(x, u_n(x))| \leq C_1 |u_n(x)|^{N-1} \). Notice that \( |x|^{-\beta} |u_n|^{N-1} \to |x|^{-\beta} |u|^{N-1} \) strongly in \( L^1(\mathbb{R}^N) \) and \( u_n \to u \) almost everywhere in \( \mathbb{R}^N \). By the generalized Lebesgue’s dominated convergence theorem, we obtain
\[
\lim_{n \to \infty} \int_{|u_n| < M} \frac{f(x, u_n)}{|x|^{\beta}} \, dx = \int_{|u| < M} \frac{f(x, u)}{|x|^{\beta}} \, dx.
\] (3.26)

Combining (3.24), (3.25) and (3.26), we can find some \( K > 0 \) such that when \( n > K \),
\[
\left| \int_{\mathbb{R}^N} \frac{f(x, u_n)}{|x|^{\beta}} \, dx - \int_{\mathbb{R}^N} \frac{f(x, u)}{|x|^{\beta}} \, dx \right| < 3\delta.
\]

Hence (3.23) holds and thus our claim (3.22) holds. By \((H_1)\) and \((H_3)\), there exist constants \( c_1, c_2 > 0 \) such that
\[
F(x, u_n) \leq c_1 |u_n|^N + c_2 f(x, u_n).
\]

In view of (3.22) and Lemma 2.4, it follows from the generalized Lebesgue’s dominated convergence theorem
\[
\lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{|F(x, u_n) - F(x, u)|}{|x|^{\beta}} \, dx = 0.
\]

Using the argument of proving (4.26) in [5], we have \( \nabla u_n(x) \to \nabla u(x) \) a.e. in \( \mathbb{R}^N \) and
\[
|\nabla u_n|^{N-2} \nabla u_n \to |\nabla u|^{N-2} \nabla u \quad \text{weakly in } (L^{N/(N-2)}(\mathbb{R}^N))^N.
\]

Finally passing to the limit \( n \to \infty \) in (3.20), we have
\[
\int_{\mathbb{R}^N} \left( |\nabla u|^{N-2} \nabla u \nabla \psi + V |u|^{N-2} u \psi \right) \, dx - \int_{\mathbb{R}^N} \frac{f(x, u)}{|x|^{\beta}} \psi \, dx - \epsilon \int_{\mathbb{R}^4} h \psi \, dx = 0
\]
for all \( \psi \in C_0^\infty(\mathbb{R}^N) \), which is dense in \( E \). Hence \( u \) is a weak solution of (1.10). After checking the above argument, \( \epsilon \) need not to be nonzero, i.e. the same conclusion holds for \( J \).

\textbf{Remark 3.5.} Similar results of Lemma 3.4 was also established by J.M. do Ó in two-dimensional case [20] and by the author for bi-Laplace equation in four-dimensional Euclidean space [37].
4. Nontrivial positive solution

In this section, we will prove Theorem 1.1. It suffices to look for nontrivial critical points of the functional $J$ in the function space $E$.

**Proof of Theorem 1.1.** By (i) and (ii) of Lemma 3.1, $J$ satisfies all the hypothesis of the mountain-pass theorem except for the Palais–Smale condition: $J \in C^1(E, \mathbb{R})$; $J(0) = 0$; $J(u) \geq \delta > 0$ when $\|u\|_E = r$; $J(e) < 0$ for some $e \in E$ with $\|e\|_E > r$. Then using the mountain-pass theorem without the Palais–Smale condition [33], we can find a sequence $(u_n)$ in $E$ such that

$$J(u_n) \to c > 0, \quad J'(u_n) \to 0 \quad \text{in } E^*,$$

where

$$c = \min \max_{\gamma \in \Gamma, u \in \gamma} J(u) \geq \delta$$

is the min–max value of $J$, where $\Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \gamma(1) = e \}$. By (3.1), this is equivalent to saying

$$\frac{1}{N} \|u_n\|_E^N - \int_{\mathbb{R}^N} \frac{F(x, u_n)}{|x|^{\beta}} \, dx \to c \quad \text{as } n \to \infty,$$

$$\left| \int_{\mathbb{R}^N} \left( |\nabla u_n|^{N-2} \nabla u_n \nabla \psi + V |u_n|^{N-2} u_n \psi \right) \, dx - \int_{\mathbb{R}^N} \frac{f(x, u_n)}{|x|^{\beta}} \psi \, dx \right| \leq \tau_n \|\psi\|_E$$

for all $\psi \in E$, where $\tau_n \to 0$ as $n \to \infty$. By Lemma 3.4, up to a subsequence, there holds

$$\begin{cases}
  u_n \rightharpoonup u \quad \text{weakly in } E, \\
  u_n \to u \quad \text{strongly in } L^q(\mathbb{R}^N), \quad \forall q \geq 1, \\
  \lim_{n \to \infty} \int_{\mathbb{R}^N} \frac{F(x, u_n)}{|x|^{\beta}} \, dx = \int_{\mathbb{R}^N} \frac{F(x, u)}{|x|^{\beta}} \, dx, \\
  u \text{ is a weak solution of (1.3)}.
\end{cases}$$

Now suppose by contradiction $u \equiv 0$. Since $F(x, 0) = 0$ for all $x \in \mathbb{R}^N$, it follows from (4.1) and (4.3) that

$$\lim_{n \to \infty} \|u_n\|_E^N = Nc > 0.$$ 

Thanks to the hypothesis $(H_5)$, we have $0 < c < \frac{1}{N} \left( \frac{N-\beta}{N \alpha_0} \right)^{N-1}$ by applying Lemma 3.3. Thus there exist some $\eta_0 > 0$ and $K > 0$ such that $\|u_n\|_E^N \leq \left( \frac{N-\beta}{N \alpha_0} - \eta_0 \right)^{N-1}$ for all $n > K$. Choose $q > 1$ sufficiently close to 1 such that $q \alpha_0 \|u_n\|_E^{\alpha_0} \leq (1 - \beta/N) \alpha_N - \alpha_0 \eta_0/2$ for all $n > N$. By $(H_1)$,
\[ |f(x, u_n)u_n| \leq b_1|u_n|^N + b_2|u_n|\zeta(N, \alpha_0|u_n|^{N-1}), \]

where the function \(\zeta(\cdot, \cdot)\) is defined by (2.1). It follows from the Hölder inequality, Lemma 2.1 and Theorem A that

\[
\int_{\mathbb{R}^N} \frac{|f(x, u_n)u_n|}{|x|^\beta} \, dx \\
\leq b_1 \int_{\mathbb{R}^N} \frac{|u_n|^N}{|x|^\beta} \, dx + b_2 \int_{\mathbb{R}^N} \frac{|u_n|\zeta(N, \alpha_0|u_n|^{N-1})}{|x|^\beta} \, dx \\
\leq b_1 \int_{\mathbb{R}^N} \frac{|u_n|^N}{|x|^\beta} \, dx + b_2 \left( \int_{\mathbb{R}^N} \frac{|u_n|^{q'}}{|x|^\beta} \, dx \right)^{1/q'} \left( \int_{\mathbb{R}^N} \frac{\zeta(N, q\alpha_0|u_n|^{N-1})}{|x|^\beta} \, dx \right)^{1/q} \\
\leq b_1 \int_{\mathbb{R}^N} \frac{|u_n|^N}{|x|^\beta} \, dx + C \left( \int_{\mathbb{R}^N} \frac{|u_n|^{q'}}{|x|^\beta} \, dx \right)^{1/q'} \rightarrow 0 \quad \text{as } n \rightarrow \infty.
\]

Here we used (4.3) again (precisely \(u_n \rightarrow u\) in \(L^s(\mathbb{R}^N)\) for all \(s \geq 1\)) in the last step of the above estimates. Inserting this into (4.2) with \(\psi = u_n\), we have

\[ \|u_n\|_E \rightarrow 0 \quad \text{as } n \rightarrow \infty, \]

which contradicts (4.4). Therefore \(u \neq 0\) and we obtain a nontrivial weak solution of (1.3).

5. Multiplicity results

In this section we will prove Theorem 1.2. The proof is divided into three steps, namely

**Step 1.** Let \(\epsilon_1\) be given by (ii) of Lemma 3.1, and \(\epsilon^*, \delta^*\) be given by Lemma 3.3. Then when \(0 < \epsilon < \epsilon_1\), there exists a sequence \((v_n) \subset E\) such that

\[ J_{\beta, \epsilon}(v_n) \rightarrow c_M, \quad J'_{\beta, \epsilon}(v_n) \rightarrow 0, \quad \text{as } n \rightarrow \infty, \quad (5.1) \]

where \(c_M\) is a min–max value of \(J_{\beta, \epsilon}\). Let \(\epsilon_2 = \min\{\epsilon_1, \epsilon^*\}\). Then when \(0 < \epsilon < \epsilon_2\), we can take \(c_M\) such that

\[ 0 < c_M < \frac{1}{N} \left( \frac{N - \beta \alpha_N}{N \alpha_0} \right)^{N-1} - \delta^*. \quad (5.2) \]

In addition, up to a subsequence, there holds \(v_n \rightharpoonup u_M\) weakly in \(E\), and \(u_M\) is a weak solution of (1.10).

**Proof.** By (i) and (ii) of Lemma 3.1, when \(0 < \epsilon < \epsilon_1\), \(J_{\beta, \epsilon}\) satisfies the following condition: \(J_{\beta, \epsilon} \in C^1(E, \mathbb{R})\); \(J_{\beta, \epsilon}(0) = 0\); \(J_{\beta, \epsilon}(u) \geq \vartheta_\epsilon > 0\) when \(\|u\|_E = r_\epsilon\); \(J_{\beta, \epsilon}(e) < 0\) for some \(e \in E\).
with \( \|e\| > \max\{r_\epsilon, 1\} \). Then using the mountain-pass theorem without the Palais–Smale condition [33], we can find a sequence \( (v_n) \) in \( E \) such that

\[
J_{\beta, \epsilon}(v_n) \to c_M > 0, \quad J'_{\beta, \epsilon}(v_n) \to 0 \text{ in } E^*,
\]

where

\[
c_M = \min_{\gamma \in \Gamma} \max_{u \in \gamma} J_{\beta, \epsilon}(u) \geq \vartheta_\epsilon
\]

is a min–max value of \( J_{\beta, \epsilon} \), where \( \Gamma = \{ \gamma \in C([0, 1], E) : \gamma(0) = 0, \ \gamma(1) = e \} \). Clearly (5.2) follows from Lemma 3.3. The last assertion follows from Lemma 3.4 immediately. \( \square \)

**Step 2.** Let \( r_\epsilon \) be given by (ii) of Lemma 3.1 such that \( r_\epsilon \to 0 \) as \( \epsilon \to 0 \). There exists \( \epsilon_3 : 0 < \epsilon_3 < \epsilon_2 \) such that if \( 0 < \epsilon < \epsilon_3 \), then there exists a sequence \( (u_n) \subset E \) such that

\[
J_{\beta, \epsilon}(u_n) \to c_\epsilon := \inf_{\|u\|_E \leq r_\epsilon} J_{\beta, \epsilon}(u) \quad (5.3)
\]

and

\[
J'_{\beta, \epsilon}(u_n) \to 0 \text{ in } E^* \text{ as } n \to \infty, \quad (5.4)
\]

where \( c_\epsilon < 0 \) and \( c_\epsilon \to 0 \) as \( \epsilon \to 0 \). In addition, up to a subsequence, there holds \( u_n \to u_0 \) strongly in \( E \), and \( u_0 \) is a weak solution of (1.10) with \( J_{\beta, \epsilon}(u_0) = c_\epsilon \).

**Proof.** Let \( r_\epsilon \) be given by (ii) of Lemma 3.1, i.e. \( J_{\beta, \epsilon}(u) > \vartheta_\epsilon > 0 \) for all \( u \) with \( \|u\|_E = r_\epsilon \). Since \( r_\epsilon \to 0 \) as \( \epsilon \to 0 \), one can choose \( \epsilon_3 : 0 < \epsilon_3 < \epsilon_2 \) such that when \( 0 < \epsilon < \epsilon_3 \),

\[
r_\epsilon < \left( \frac{N - \beta}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}. \quad (5.5)
\]

By \((H1)\) and \((H2)\), we have

\[
F(x, u) \leq b_1|u|^N + b_2|u|\xi \left( N, \alpha_0 \right) \left( \|u\|_E \right)^{N/(N-1)} \left( \|u\|_E \right)^{N/(N-1)}. \quad (5.6)
\]

Here again \( \xi(\cdot, \cdot) \) is defined by (2.1). When \( \|u\|_E \leq r_\epsilon \), we have \( \alpha_0 \|u\|_E^{N/(N-1)} < (1 - \beta/N) \alpha_N \). It then follows from Lemma 2.1 and Theorem A that \( F(x, u)/|x|^{\beta} \) is bounded in \( L^p(\mathbb{R}^N) \cap L^1(\mathbb{R}^N) \) for some \( p > 1 \) when \( \|u\|_E \leq r_\epsilon \). Hence \( J_{\beta, \epsilon} \) has lower bound on the ball \( B_{r_\epsilon} = \{ u \in E : \|u\|_E \leq r_\epsilon \} \).

The closure of \( B_{r_\epsilon}, \overline{B}_{r_\epsilon} \subset E \) is a complete metric space with the metric given by the norm of \( E \), convex and \( J_{\beta, \epsilon} \) is of class \( C^1 \) and has lower bound on \( \overline{B}_{r_\epsilon} \). By the Ekeland’s variational principle [36], there exists a sequence \( (u_n) \subset \overline{B}_{r_\epsilon} \) such that (5.3) and (5.4) hold.

By (iii) of Lemma 3.1, \( c_\epsilon < 0 \). Since \( r_\epsilon \to 0 \) as \( \epsilon \to 0 \), noticing (5.6), we have by using the Hölder inequality and Lemma 2.4
\[
\sup_{\|u\| \leq r \epsilon} \int_{\mathbb{R}^N} F(x,u) \frac{dx}{|x|^\beta} \to 0, \quad \sup_{\|u\| \leq r \epsilon} \int_{\mathbb{R}^N} hu \, dx \to 0
\]
as \(\epsilon \to 0\). This implies \(c_\epsilon \to 0\) as \(\epsilon \to 0\).

Now we are proving the last assertion. Assume \(u_n \rightharpoonup u_0\) weakly in \(E\). (5.4) is equivalent to
\[
\left| \langle J'_{\beta,\epsilon}(u_n), \phi \rangle \right| \leq \tau_n \|\phi\|_E, \quad \forall \phi \in E, \tag{5.7}
\]
where \(\tau_n \to 0\) as \(n \to \infty\). Recalling (3.2) and choosing \(\phi = u_n - u_0\) in (5.7), we have
\[
\int_{\mathbb{R}^N} \left( |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u_0) + V(x)|u_n|^{N-2}u_n(u_n - u_0) \right) \, dx
\]
\[
- \int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} (u_n - u_0) \, dx - \epsilon \int_{\mathbb{R}^N} h(u_n - u_0) \, dx = o_n(1),
\]
where \(o_n(1) \to 0\) as \(n \to \infty\). Hölder inequality together with (5.5), Theorem A and Lemma 2.4 implies that
\[
\int_{\mathbb{R}^N} \frac{f(x,u_n)}{|x|^\beta} (u_n - u_0) \, dx = o_n(1), \quad \epsilon \int_{\mathbb{R}^N} h(u_n - u_0) \, dx = o_n(1).
\]
Hence
\[
\int_{\mathbb{R}^N} \left( |\nabla u_n|^{N-2} \nabla u_n \nabla (u_n - u_0) + V(x)|u_n|^{N-2}u_n(u_n - u_0) \right) \, dx = o_n(1). \tag{5.8}
\]
On the other hand, since \(u_n \rightharpoonup u_0\) weakly in \(E\), we obtain
\[
\int_{\mathbb{R}^N} \left( |\nabla u_0|^{N-2} \nabla u_0 \nabla (u_0 - u_0) + V(x)|u_0|^{N-2}u_0(u_0 - u_0) \right) \, dx = o_n(1). \tag{5.9}
\]
Subtracting (5.9) from (5.8), using a well-known inequality (see for example Chapter 10 of [27])
\[
2^{2-N} |b-a|^N \leq \left( |b|^{N-2}b - |a|^{N-2}a, b-a \right), \quad \forall a, b \in \mathbb{R}^N, \tag{5.10}
\]
we obtain \(\|u_n - u_0\|^N_E \to 0\) and thus \(u_n \to u_0\) strongly in \(E\) as \(n \to \infty\). Since \(J'_{\beta,\epsilon} \in C^1(E, \mathbb{R})\), there hold \(J_{\beta,\epsilon}(u_0) = c_\epsilon\) and \(J'_{\beta,\epsilon}(u_0) = 0\), i.e. \(u_0\) is a weak solution of (1.10). \(\square\)

**Step 3.** There exists \(\epsilon_0: 0 < \epsilon_0 < \epsilon_3\) such that if \(0 < \epsilon < \epsilon_0\), then \(u_M \neq u_0\).

**Proof.** Suppose by contradiction that \(u_M \equiv u_0\). Then \(v_n \rightharpoonup u_0\) weakly in \(E\). By (5.1),
\[
J_{\beta,\epsilon}(v_n) \to c_M > 0, \quad \left| \langle J'_{\beta,\epsilon}(v_n), \phi \rangle \right| \leq \gamma_n \|\phi\|_E \tag{5.11}
\]
with $\gamma_n \to 0$ as $n \to \infty$. On one hand, by Lemma 3.4, we have

$$\int_{\mathbb{R}^N} \frac{F(x, v_n)}{|x|^{\beta}} \, dx \to \int_{\mathbb{R}^N} \frac{F(x, u_0)}{|x|^{\beta}} \, dx \text{ as } n \to \infty.$$  \hspace{1cm} (5.12)

Here and in the sequel, we do not distinguish sequence and subsequence. On the other hand, since $v_n \rightharpoonup u_0$ weakly in $E$, it follows from the Hölder inequality and Lemma 2.4 that

$$\int_{\mathbb{R}^N} h v_n \, dx \to \int_{\mathbb{R}^N} h u_0 \, dx \text{ as } n \to \infty.$$ \hspace{1cm} (5.13)

Inserting (5.12) and (5.13) into (5.11), we obtain

$$\frac{1}{N} \|v_n\|^N_E = c_M + \int_{\mathbb{R}^N} \frac{F(x, u_0)}{|x|^{\beta}} \, dx + \epsilon \int_{\mathbb{R}^N} h u_0 \, dx + o_n(1).$$ \hspace{1cm} (5.14)

In the same way, one can derive

$$\frac{1}{N} \|u_n\|^N_E = c_\epsilon + \int_{\mathbb{R}^N} \frac{F(x, u_0)}{|x|^{\beta}} \, dx + \epsilon \int_{\mathbb{R}^N} h u_0 \, dx + o_n(1).$$ \hspace{1cm} (5.15)

Combining (5.14) and (5.15), we have

$$\|v_n\|^N_E - \|u_0\|^N_E = N(c_M - c_\epsilon + o_n(1)).$$ \hspace{1cm} (5.16)

From Step 2, we know that $c_\epsilon \to 0$ as $\epsilon \to 0$. This together with (5.2) leads to the existence of $\epsilon_0$: $0 < \epsilon_0 < \epsilon_3$ such that if $0 < \epsilon < \epsilon_0$, then

$$0 < c_M - c_\epsilon < \frac{1}{N} \left( \frac{N - \beta}{N} \frac{\alpha_N}{\alpha_0} \right)^{N-1}.$$ \hspace{1cm} (5.17)

Write

$$w_n = \frac{v_n}{\|v_n\|_E}, \quad w_0 = \frac{u_0}{(\|u_0\|^N_E + N(c_M - c_\epsilon))^{1/N}}.$$

It follows from (5.16) and $v_n \rightharpoonup u_0$ weakly in $E$ that $w_n \rightharpoonup w_0$ weakly in $E$. Notice that

$$\int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_0 |v_n|^{N/(N-1)})}{|x|^{\beta}} \, dx = \int_{\mathbb{R}^N} \frac{\zeta(N, \alpha_0 \|v_n\|_E^{N/(N-1)} |w_n|^{N/(N-1)})}{|x|^{\beta}} \, dx.$$

By (5.16) and (5.17), a straightforward calculation shows

$$\lim_{n \to \infty} \alpha_0 \|v_n\|_E^{N/(N-1)} (1 - \|w_0\|_E^{N/(N-1)})^{1/(N-1)} < \left( 1 - \frac{\beta}{N} \right) \alpha_N.$$
Whence Lemma 2.3 together with Lemma 2.1 implies that
\[ \zeta(N, \alpha_0 |v_n|^{N/(N-1)}) / |x|^\beta \]
is bounded in \( L^q(\mathbb{R}^N) \) for some \( q > 1 \). By \((H_1)\),
\[ |f(x, v_n)| \leq b_1 |v_n|^{N-1} + b_2 \zeta(N, \alpha_0 |v_n|^{N/(N-1)}). \]

Then it follows from the continuous embedding \( E \hookrightarrow L^p(\mathbb{R}^N) \) for all \( p \geq 1 \) that \( f(x, v_n) / |x|^\beta \) is bounded in \( L^q(\mathbb{R}^N) \) for some \( q_1 : 1 < q_1 < q \). This together with Lemma 2.4 and the Hölder inequality gives
\[ \left| \int_{\mathbb{R}^N} \frac{f(x, v_n)(v_n - u_0)}{|x|^\beta} \, dx \right| \leq \left\| \frac{f(x, v_n)}{|x|^\beta} \right\|_{L^{q_1}(\mathbb{R}^N)} \|v_n - u_0\|_{L^{q_1'}(\mathbb{R}^N)} \to 0, \quad (5.18) \]
where \( 1/q_1 + 1/q_1' = 1 \).

Taking \( \phi = v_n - u_0 \) in (5.11), we have by using (5.13) and (5.18) that
\[ \int_{\mathbb{R}^N} \left( |\nabla v_n|^{N-2} \nabla v_n \nabla (v_n - u_0) + V(x) |v_n|^{N-2} v_n (v_n - u_0) \right) \, dx \to 0. \quad (5.19) \]

However the fact \( v_n \rightharpoonup u_0 \) weakly in \( E \) leads to
\[ \int_{\mathbb{R}^N} \left( |\nabla u_0|^{N-2} \nabla u_0 \nabla (v_n - u_0) + V(x) |u_0|^{N-2} u_0 (v_n - u_0) \right) \, dx \to 0. \quad (5.20) \]

Subtracting (5.20) from (5.19), using the inequality (5.10), we have
\[ \|v_n - u_0\|_E \to 0. \]

This together with (5.16) implies that
\[ c_M = c_\epsilon, \]
which is absurd since \( c_M > 0 \) and \( c_\epsilon < 0 \). Therefore we end Step 3 and complete the proof of Theorem 1.2. \( \Box \)

Acknowledgments

This work was partly supported by the NSFC 11171347 and the NCET program 2008–2011. The author thanks the referee for his valuable comments and suggestions on this paper.

References

[1] S. Adachi, K. Tanaka, Trudinger type inequalities in \( \mathbb{R}^N \) and their best exponents, Proc. Amer. Math. Soc. 128 (2000) 2051–2057.
[2] Adimurthi, Existence of positive solutions of the semilinear Dirichlet Problem with critical growth for the \( N \)-Laplacian, Ann. Sc. Norm. Super. Pisa XVII (1990) 393–413.
[3] Adimurthi, K. Sandeep, A singular Moser–Trudinger embedding and its applications, NoDEA Nonlinear Differential Equations Appl. 13 (2007) 585–603.
[4] Adimurthi, S.L. Yadava, Multiplicity results for semilinear elliptic equations in a bounded domain of $\mathbb{R}^2$ involving critical exponent, Ann. Sc. Norm. Super. Pisa XVII (1990) 481–504.
[5] Adimurthi, Y. Yang, An interpolation of Hardy inequality and Trudinger–Moser inequality in $\mathbb{R}^N$ and its applications, Int. Math. Res. Not. 13 (2010) 2394–2426.
[6] S. Alama, Y.Y. Li, Existence of solutions for semilinear elliptic equations with indefinite linear part, J. Differential Equations 96 (1992) 89–115.
[7] C.O. Alves, G.M. Figueiredo, On multiplicity and concentration of positive solutions for a class of quasilinear problems with critical exponential growth in $\mathbb{R}^N$, J. Differential Equations 246 (2009) 1288–1311.
[8] T. Bartsch, M. Willem, On an elliptic equation with concave and convex nonlinearities, Proc. Amer. Math. Soc. 123 (1995) 3555–3561.
[9] H. Brézis, Elliptic equations with limiting Sobolev exponents, Comm. Pure Appl. Math. 39 (1986) 517–539.
[10] H. Brézis, E. Lieb, A relation between pointwise convergence of functions and convergence of functionals, Proc. Amer. Math. Soc. 88 (1983) 486–490.
[11] H. Brézis, L. Nirenberg, Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents, Comm. Pure Appl. Math. 36 (1983) 437–477.
[12] D. Cao, Nontrivial solution of semilinear elliptic equations with critical exponent in $\mathbb{R}^2$, Comm. Partial Differential Equations 17 (1992) 407–435.
[13] A. Capozzi, D. Fortunato, G. Palmieri, An existence result for nonlinear elliptic problems involving critical Sobolev exponent, Ann. Inst. H. Poincaré Anal. Non Linéaire 2 (1985) 463–470.
[14] D. Costa, On a class of elliptic systems in $\mathbb{R}^N$, Electron. J. Differential Equations 7 (1994) 1–14.
[15] D.G. de Figueiredo, J.M. do Ó, B. Ruf, On an inequality by N. Trudinger and J. Moser and related elliptic equations, Comm. Pure Appl. Math. LV (2002) 135–152.
[16] D.G. de Figueiredo, O.H. Miyagaki, B. Ruf, Elliptic equations in $\mathbb{R}^2$ with nonlinearities in the critical growth range, Calc. Var. 3 (1995) 139–153.
[17] W.Y. Ding, W.M. Ni, On the existence of positive entire solutions of a semilinear elliptic equation, Arch. Ration. Math. Anal. 31 (1986) 283–308.
[18] J.M. do Ó, Semilinear Dirichlet problems for the $N$-Laplacian in $\mathbb{R}^N$ with nonlinearities in the critical growth range, Differential Integral Equations 9 (1996) 967–979.
[19] J.M. do Ó, $N$-Laplacian equations in $\mathbb{R}^N$ with critical growth, Abstr. Appl. Anal. 2 (1997) 301–315.
[20] J.M. do Ó, M. de Souza, On a class of singular Trudinger–Moser type inequalities and its applications, Math. Nachr. 284 (2011) 1754–1776.
[21] J.M. do Ó, E. Medeiros, U. Severo, On a quasilinear nonhomogeneous elliptic equation with critical growth in $\mathbb{R}^N$, J. Differential Equations 246 (2009) 1363–1386.
[22] J.M. do Ó, Y. Yang, A quasi-linear elliptic equation with critical growth on compact Riemannian manifold without boundary, Ann. Global Anal. Geom. 38 (2010) 317–334.
[23] A. Garcia, P. Alonso, Existence and non-uniqueness for the $p$-Laplacian, Comm. Partial Differential Equations 12 (1987) 1389–1430.
[24] L. Jeanjean, Solutions in spectral gaps for a nonlinear equation of Schrödinger type, J. Differential Equations 112 (1994) 53–80.
[25] W. Kryszewski, A. Szulkin, Generalized linking theorem with an application to semilinear Schrödinger equation, Adv. Difference Equ. 3 (1998) 441–470.
[26] Y. Li, B. Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^N$, Indiana Univ. Math. J. 57 (2008) 451–480.
[27] P. Lindqvist, Notes on the $p$-Laplace equation, preprint.
[28] P.L. Lions, The concentration-compactness principle in the calculus of variation, the limit case, part I, Rev. Mat. Iberoamericana 1 (1985) 145–201.
[29] J. Moser, A sharp form of an inequality by N. Trudinger, Indiana Univ. Math. J. 20 (1971) 1077–1091.
[30] R. Panda, Nontrivial solution of a quasilinear elliptic equation with critical growth in $\mathbb{R}^n$, Proc. Indian Acad. Sci. Math. Sci. 105 (1995) 425–444.
[31] R. Panda, On semilinear Neumann problems with critical growth for the $N$-Laplacian, Nonlinear Anal. 26 (1996) 1347–1366.
[32] S. Pohozaev, The Sobolev embedding in the special case $pl = n$, in: Proceedings of the Technical Scientific Conference on Advances of Scientific Research 1964–1965, Mathematics Sections, Moscow. Energet. Inst., Moscow, 1965, pp. 158–170.
[33] P.H. Rabinowitz, On a class of nonlinear Schrödinger equations, Z. Angew. Math. Phys. 43 (1992) 270–291.
[34] B. Ruf, A sharp Trudinger–Moser type inequality for unbounded domains in $\mathbb{R}^2$, J. Funct. Anal. 219 (2005) 340–367.
[35] N.S. Trudinger, On embeddings into Orlicz spaces and some applications, J. Math. Mech. 17 (1967) 473–484.
[36] M. Willem, Minimax Theorems, Birkhäuser, 1996.
[37] Y. Yang, Adams type inequalities and related elliptic partial differential equations in dimension four, J. Differential Equations 252 (2012) 2266–2295.
[38] Y. Yang, L. Zhao, A class of Adams–Fontana type inequalities and related functionals on manifolds, NoDEA Nonlinear Differential Equations Appl. 17 (2010) 119–135.