Nonlocal transformations of the Generalized Liénard type equations and dissipative Ermakov-Milne-Pinney systems

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Abstract

We employ the method of nonlocal generalized Sundman transformations to formulate the linearization problem for equations of the generalized Liénard type and show that they may be mapped to equations of the dissipative Ermakov-Milne-Pinney type. We obtain the corresponding new first integrals of these derived equations, this method yields a natural generalization of the construction of Ermakov-Lewis invariant for a time dependent oscillator to (coupled) Liénard and Liénard type equations. We also study the linearization problem for the coupled Liénard equation using nonlocal transformations and derive coupled dissipative Ermakov-Milne-Pinney equation. As an offshoot of this nonlocal transformation method when the standard Liénard equation, \( \ddot{x} + f(x)\dot{x} + g(x) = 0 \), is mapped to that of the linear harmonic oscillator equation we obtain a relation between the functions \( f(x) \) and \( g(x) \) which is exactly similar to the condition derived in the context of isochronicity of the Liénard equation.

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1 Introduction

The linearization of a nonlinear ordinary differential equation (ODE) has been an object of immense interest for many years. The most commonly employed method is to seek a point transformation such that the transformed ODE becomes linear and hence may be solved by some known method.

Of late however, a number of attempts have been successfully made to tackle this problem by using Sundman transformations [30], which are nonlocal in character. Besides, second-order ODEs for which the linearization problem is well studied, Euler et al [11, 12, 13] have also extended their procedure to deal with higher-order (mainly third-order) ODEs. The most general type of a nonlocal transformation (see for example, [17] ) that may be considered is of the form

\[
dX = A(x, t)dx + B(x, t)dt, \quad dT = C(x, t)dx + D(x, t)dt.
\]  

(1.1)

The usual case of a point transformation corresponds to the situation where \( A_t(x, t) = B_x(x, t) \) and \( C_t(x, t) = D_x(x, t) \) so that \( (X(x, t), T(x, t)) \rightarrow (F(x, t), G(x, t)) \), such that \( X(x, t) = F(x, t) \) and \( T(x, t) = G(x, t) \). In [11] as also in [9] it was assumed that \( X(x, t) = F(x, t) \) but that \( C_t(x, t) \neq D_x(x, t) \), so that the temporal part is nonlocal. In fact they took \( C(x, t) = 0 \) so that \( dT = D(x, t)dt \) assuming that \( D_x \neq 0 \). Such a nonlocal transformation is commonly referred to as a Sundman transformation [1].

The Liénard equation [22]

\[
\ddot{x} + f(x)\dot{x} + g(x) = 0,
\]  

(1.2)

has been extensively studied owing to its diverse physical applications and its appearance in the context of limit cycles of the Van der Pol equation [19]. The higher dimensional Liénard equation presents substantial additional difficulties which prevent straightforward extensions of planar results. This is particularly true for stability properties of equilibria, which are essential in studying the dynamics of perturbed systems. In a recent paper Briata and Sabatini [2] proved the asymptotic stability of the equilibrium solution of a class of vector Liénard equations by means of the LaSalle invariance principle.

Recently Chandrasekar et al [5] have described a method for the linearization of coupled systems which may be briefly summarized as follows. Given a system of coupled equations

\[
\ddot{x} = \phi_1(x, y, \dot{x}, \dot{y}, t), \quad \ddot{y} = \phi_2(x, y, \dot{x}, \dot{y}, t),
\]  

(1.3)

one looks for a transformation

\[
\omega_i = f_i(t, x, y), \quad z_i = \int f_{i+2}(t, x, y) \, dt \quad i = 1, 2
\]  

(1.4)

such that \( \frac{d^2\omega_1}{dz_1^2} = 0 \) and \( \frac{d^2\omega_2}{dz_2^2} = 0 \). In the event that such a transformation exists the coupled system is said to be linearizable. The procedure described by them depends on the existence of two first integrals

\[
I_1 = F(t, x, y, \dot{x}, \dot{y}) \quad I_2 = G(t, x, y, \dot{x}, \dot{y}).
\]  

(1.5)
In fact they have shown that if the coupled system is linearizable then the above transformation is completely determined by these first integrals which must necessarily be of the form \( I_1 = \frac{1}{f_3} \frac{df_1}{dt} \) and \( I_2 = \frac{1}{f_4} \frac{df_2}{dt} \). This can be checked easily as follows. Define \( z_1 \) and \( z_2 \) as \( \frac{dz_1}{dt} = f_3 \) and \( \frac{dz_2}{dt} = f_4 \). This immediately yields

\[
I_1 = \frac{1}{dz_1} \frac{df_1}{dt} = \frac{df_1}{dz_1} dt, \quad I_2 = \frac{1}{dz_2} \frac{df_2}{dt} = \frac{df_2}{dz_2} dt.
\]

If we identify \( \omega_1 \equiv f_1 \) and \( \omega_2 \equiv f_2 \) then \( I_1 = \frac{d\omega_1}{dz_1} \) and

\[
\frac{d}{dz_1} \left( \frac{d\omega_1}{dz_1} \right) = \frac{dI_1}{dt} / \frac{dz_1}{dt} = 0.
\]

The major shortcoming of this method is that it requires explicit knowledge of the first integrals which in itself is a non trivial problem.

A pioneering contribution towards the linearization of the equations of motion occurring in celestial mechanics was made by Sundman [30] who introduced the transformation \( dt = rd\tau \) in his study of the 3-body problem, where \( r \) is the dependent variable (radial component). About a quarter of a century ago Sundman’s method was revitalized by Szébehely and Bond [32], who considered a transformation of the dependent variable \( r = F(\rho) \). The theoretical importance of the generalized Sundman transformations stems from their occurrence in various areas of mechanics and dynamical systems. In particular transformations of the Sundman type which are also referred to as non-point transformations by some authors [6] are especially effective for obtaining solutions of many nonlinear ODEs.

In [9] the authors derived the most general condition under which a second-order ordinary differential equation is transformable to the linear equation \( X''(T) = 0 \), (here \( X' = \frac{dX}{dT} \)) under a generalized Sundman transformation. In this communication we derive a systematic procedure to find the first integral for SODE which are transformed to \( X''(T) + \omega^2 X = 0 \) under generalized Sundman transformation.

**Prelude, motivation and result:** This equation falls in the class of the so-called projective connections and closely connected with different geometric problems, i.e., SODE of the form

\[
\ddot{x} + A_3(t; x) \dot{x}^3 + A_2(t; x) \dot{x}^2 + A_1(t; x) \dot{x} + A(t; x) = 0.
\]  

(1.6)

Lie was the first to study the linearization problem of SODE, he showed that every linearizable SODE can be recasted to the above form of equation (1.6) and the coefficients satisfy the conditions (see for example, [29])

\[
3A_{3tt} - 2A_{2tx} + A_{1xx} = (3A_1A_3 - A_2^2)t - 3(AA_3)_x - 3A_3A_x + A_2A_{1x}, \quad (1.7)
\]

\[
3A_{xx} - 2A_{1tx} + A_{2tt} = 3(AA_3)_t + (A_1^2 - 3AA_2)_x + 3AA_{3t} - A_1A_2. \quad (1.8)
\]

Such equations were studied by Lie [21], Tresse [33], Cartan [1], Liouville [23], etc. (see, for example, [23]). The class of equations (1.6) is closed under generic point transformations. It means that the transformed equation is again given by (1.6) but with some other coefficients.
The problem of existence of the change of variables that transforms equation (1.6) into other with different coefficient is called the Equivalence Problem.

If an equation of this type admits an integral of the form $A(t,x)\dot{x} + B(t,x)$ then it must be of the form

$$\ddot{x} + A_2(t,x)\dot{x}^2 + A_1(t,x)\dot{x} + A_0(t,x) = 0.$$ 

This result is contained in [26]. A transparent motivation of this result in terms of $\lambda$-symmetries is contained, for instance, in the paper [25], which is coauthored by the authors the present paper. More motivations in terms of projective structures can be also found in [3].

Another motivation for the present article stems from a recent paper of Padmanabhan [27] in which he pointed out the physical basis for the Ermakov-Lewis invariant. A generalization of Padmanabhan’s original Lagrangian has been made by the authors of [14] by including an additional potential term. Recently we reported the results of a further modification of the transformation used in [27] to derive generalizations of the time-dependent oscillator equation and its associated partner, namely the Ermakov-Pinney equation. There is a hidden nonlocal transformation embedded in this transformation. A few years ago we examined the connection between a time-dependent second-order ODE and the Ermakov-Pinney system [15], where it was shown that by a simple rational transformation of the dependent variable one could easily extract the well known Ermakov-Lewis invariant. In spite of such a large number of applications in physics (for example, see [20] [8] for exhaustive references and historical background [20]), the Ermakov-Pinney equation in itself does not have any dissipation term, but the physical system demands a natural generalization of the model by inclusion of the damping mechanism. It is known from [18] that damped Ermakov-Pinney equations arise in quantum mechanical models with dissipation.

It is clear from the work of Padmanabhan [27] [16] that there is a hidden nonlocal transformation embedded in his construction and this provides the motivation to explore the connection between equations of the Liénard type and the dissipative Ermakov-Pinney equations. In the second part of the paper we introduce the damped Pinney equation considered in [18] was defined as the model arising when a damping term, linear in the velocity, is included in the Pinney equation. We generalize our results to coupled Liénard equations and consider the mapping to a coupled dissipative Ermakov-Milne-Pinney equation. As an offshoot of our program we obtain the isochronous conditions stated by Sabatini on $f(x)$ and $g(x)$ of the standard Liénard equation [31] [7].

The article is organized as follows. In Section 2 we introduce a nonlocal transformation and demonstrate the mapping of the the Liénard type equation to the Ermakov-Pinney type equation. In Section 3 we study the mapping between the generalized Liénard equation and generalized dissipative Ermakov-Milne-Pinney equation. Section 4 is devoted to the coupled Liénard equation and the coupled dissipative Ermakov-Milne-Pinney equation. In Section 5 we study the linearization of the standard Liénard equation, $\ddot{x} + f(x)\dot{x} + g(x) = 0$, using a nonlocal transformation and demonstrate that the isochronous conditions stated by Sabatini on $f(x)$ and $g(x)$ of the standard Liénard equation follows almost naturally.
2 Linearization via Nonlocal transformations

In addressing the broader issue of linearizing a given second-order nonlinear ODE, it is useful if we enlarge the class of transformations usually considered to beyond point transformations and look for a more general class of transformations which are of a nonlocal character. The theory of nonlocal transformations can be traced back to the original works of Sundman (1912). Euler et al. [11, 12] have considered the class of Sundman transformations and have profitably used them to not only linearize nonlinear ODEs but also to identify the so-called generalized Sundman symmetries of such equations. The present authors [17] have further generalized the Sundman transformations and have applied them to deduce first integrals of time-dependent second-order ODEs. Duarte et al. [9] have used nonlocal transformations for determining when a given ODE is equivalent to a linear differential equation. In the notation of [9] it is usual to begin with a transformation of the form

\[ X = F(x, t), \quad dT = G(x, t)dt, \]  

(2.1)

and to determine the functions \( F \) and \( G \) such that the original second-order ODE \( \ddot{x} = F(x, \dot{x}) \) is mapped to a linear ODE, in particular to the free particle equation \( \ddot{X} + \omega^2 X = 0 \). Here we consider a modification of the above form and assume that the nonlocal transformation is defined, in general, by

\[ dh(X) = A(x, t)dx + B(x, t)dt, \]  

(2.2)

where \( h(X) \) is some suitable function to be chosen while \( G(x, t) = 1 \) so that \( T = t \).

2.1 Illustration: linearization of Liénard type equations

As an illustration we consider a second-order ordinary differential equation (SODE) of the Liénard type having the form

\[ \ddot{x} + f(x)\dot{x}^2 + g(x) = 0, \]  

(2.3)

and search for a nonlocal transformation such that it is mapped to the simple harmonic equation

\[ \ddot{X} + \omega^2 X = 0, \]  

(2.4)

(here \( \ddot{X} = \frac{dX}{dt} \)) by the transformation

\[ \frac{dX}{X} = A(x, t)dx + B(x, t)dt. \]  

(2.5)

It is a matter of straightforward computation to show that (2.3) is mapped to (2.4) provided its coefficients satisfy the following conditions:

\[ A^2 + A_x - A(x)f(x) = 0, \]  

(2.6)

\[ B_x + 2A(x)B(x) = 0, \]  

(2.7)

\[ A(x)g(x) - B^2 = \omega^2. \]  

(2.8)
Suppose $A = \frac{v_x}{v}$, then after solving the first and second equations stated above we get

$$v_x = \exp\left(\int f(s)ds\right), \quad B(x) = \frac{1}{v^4}.$$  

The final equation then becomes

$$\frac{v_x}{v} g(x) - \frac{1}{v^4} = \omega^2.$$  

Let us now set $f(x) = \frac{\omega}{x}$, which readily yields $v = \frac{x^{\alpha+1}}{\alpha+1}$. Thus $A$ and $B$ are given by

$$A(x) = \frac{\alpha + 1}{x}, \quad B(x) = \frac{(\alpha + 1)^2}{x^{2(\alpha+1)}}$$  \hspace{1cm} (2.9)

and

$$g(x) = \frac{(\alpha + 1)^3}{x^{4\alpha+3}} + \omega^2 \frac{x}{(\alpha + 1)}.$$  \hspace{1cm} (2.10)

Therefore, the singular Sundman type transformation has the appearance

$$X = x^{\alpha+1} \exp\left((\alpha + 1)^2 \int \frac{dt}{x^{2(\alpha+1)}}\right) \quad T = t,$$  \hspace{1cm} (2.11)

and corresponds to the equation

$$\ddot{x} + \frac{\alpha}{x} \dot{x}^2 + \frac{1}{\alpha + 1} \omega^2 \dot{x} + \frac{(\alpha + 1)^3}{x^{4\alpha+3}} = 0.$$  \hspace{1cm} (2.12)

On the other hand by integrating (2.4) we obtain

$$\left(\frac{dX}{dT}\right)^2 + \omega^2 X^2 = I(t, x, \dot{x}),$$  \hspace{1cm} (2.13)

where $I(t, x, \dot{x})$ is the first integral. Using this recipe the first integral of (2.12) reads

$$I = (\alpha + 1)^2 \left(x^\alpha \dot{x} + \frac{\alpha + 1}{x^{\alpha+1}}\right)^2 + \omega^2 \left(x^{\alpha+1} \exp((\alpha + 1)^2 \int \frac{dt}{x^{2(\alpha+1)}})\right)^2.$$  

**Remark** For $\alpha = 0$, equation (2.12) boils down to the Ermakov-Pinney equation while for $\alpha = -\frac{1}{2}$ it corresponds to a reduced version of an equation of the Gambier type, namely

$$\ddot{x} - \frac{1}{2x} \dot{x}^2 + 2\omega^2 x - \frac{1}{8x} = 0,$$  \hspace{1cm} (2.14)

which incidentally has the additional feature of exhibiting the property of isochronicity.
3 Nonlocal Sundman transformation of the generalized Liénard equation and the generalized dissipative Ermakov-Milne-Pinney equations

In this section we consider a general second-order differential equation (SODE) of the type
\[ \ddot{x} + A_2(t; x)\dot{x}^2 + A_1(t; x)\dot{x} + A_0(t; x) = 0 \] (3.1)
and look for a generalized Sundman transformation such that it is mapped to the following equation
\[ X''(T) + \omega^2 X = 0. \] (3.2)
Formally we define a generalized Sundman transformation for as follows.

**Definition 3.1 (Sundman transformation)** A coordinate transformation of the form
\[ X(T) = F(t, x), \quad dT = G(t, x)dt, \quad \frac{\partial F}{\partial x} \neq 0, \quad G \neq 0 \] (3.3)
is said to be a generalized Sundman transformation if differentiable functions \( F \) and \( G \) are determined such that an \( n \)-th-order ordinary differential equation
\[ x^{(n)} = w(t, x, \dot{x}, \ddot{x}, \ldots, x^{(n-1)}), \quad x^{(k)} = \frac{d^k x}{dt^k}, \]
is transformed to the autonomous equation
\[ X^{(n)} = w_0(X, X', \ldots, X^{(n-1)}), \] (3.4)
where \( X' = dX/dT \) etc.

Straightforward computation then shows that (3.1) is mapped to (3.2) provided its coefficients satisfy the following conditions:
\[ \frac{F_{xx}}{F_x} - \frac{G_x}{G} = A_2(t, x) \] (3.5)
\[ 2\frac{F_{xt}}{F_x} - \frac{G_x}{G}\frac{F_t}{F_x} - \frac{G_t}{G} = A_1(t, x) \] (3.6)
\[ \frac{F_{tt}}{F_x} - \frac{G_t}{G}\frac{F_t}{F_x} + \omega^2 F^2 = A_0(t, x). \] (3.7)

Therefore given a SODE, so that the explicit form of the coefficients \( A_i(t, x) \)'s are known, by solving the set of equations (3.5) to (3.7) if one can deduce the functions \( F \) and \( G \) then the linearizing transformation (3.3) may be obtained and consequently equation (3.1) may be linearized to the equation of a linear harmonic oscillator (3.2).

Integrating (3.2), we get
\[ \left( \frac{dX}{dT} \right)^2 + \omega^2 X^2 = \left( \frac{F_x}{G}\frac{\dot{X}}{G} + \frac{F_t}{G} \right)^2 + \omega^2 F^2 = I(t, x, \dot{x}) = \text{constant}, \] (3.8)
where \( I(t, x, \dot{x}) \) is the first integral. Having explained the general idea behind construction of the linearizing transformation and a first integral for a given equation of the type considered in (3.1), let us pass on to a description of the actual details of their construction.

Integrating (3.5) w.r.t. \( x \), we obtain

\[
G = b(t, x)F_x, \tag{3.9}
\]

where

\[
b(t, x) = a(t) \exp \left( - \int A_2(t, x)dx \right). \tag{3.10}
\]

Here \( a(t) \) is an arbitrary function of \( t \). From equations (3.6) and (3.7), we have

\[
S_x - \frac{b_x}{b} S = A_1(t, x) + \frac{b_t}{b}, \tag{3.11}
\]

\[
S_t - \frac{b_t}{b} S + \omega^2 b^2 F F_x = A_0(t, x), \tag{3.12}
\]

where

\[
S = \frac{F_t}{F_x}. \tag{3.13}
\]

Solving equation (3.11), we find that

\[
S = c(t) b(x, t) + b(x, t) \int \frac{A_1(t, x) + \frac{b_t}{b}}{b(x, t)} dx, \tag{3.14}
\]

where \( c(t) \) is an arbitrary function of \( t \). The explicit form of \( F \) can now be determined by substituting the expression for \( S \) into (3.13) and solving the resultant first-order partial differential equation for \( F \) namely,

\[
F_t - SF_x = 0. \tag{3.15}
\]

Once \( F \) is known \( G \) can be found from (3.9) and (3.10) which in turn provide us the GST as given in (3.3).

Note that when the expressions for \( S \) and \( F \) are substituted into (3.12) then the latter must be identically satisfied.

We illustrate the procedure described above with a few simple but nontrivial examples. All these examples are related to the parametric extensions of the Gambier equation.

\[
\ddot{x} = \left( 1 - \frac{1}{n} \right) \frac{\dot{x}^2}{x} + a \frac{n+2}{n} x \ddot{x} + b \dot{x} - \left( 1 - \frac{2}{n} \right) s \frac{\dot{x}}{x} - \frac{a^2}{n} x^3 + (\dot{a} - ab) x^2 + \left( cn - \frac{2as}{n} \right) x - bs - \frac{s^2}{nx}, \tag{3.16}
\]

In our illustration we assume all the coefficients are functions of the independent variable \( t \).
Proposition 3.1 A time dependent first integral of the second order equation of the form

\[ \ddot{x} + \frac{\alpha}{x} \dot{x}^2 + \frac{\beta}{t} \dot{x} + A_0(x, t) = 0 \]

is given by the function

\[ I(t, x, \dot{x}) = \left( \frac{x^\alpha \dot{x} + x^{\alpha+1}(\beta a + t\dot{a})}{t(\alpha + 1)a^2} \right)^2 + \omega^2 \left( (x^{\alpha+1} t^\beta a)^2 \right), \]

where

\[ A_0(x, t) = \left( \frac{\ddot{a}}{a} - 2 \dot{a}^2 + \frac{\dot{\beta}}{t} - \frac{\beta}{t^2} - \frac{\ddot{a} \beta}{at} \right) \frac{x}{\alpha + 1} + \omega^2 \lambda (\alpha + 1) a^{2\lambda + 2} t^{2\lambda \beta} x^{2\lambda a + 2\lambda - 2\alpha - 1}. \]

\( \alpha, \beta \) are constants, \( \lambda \) is an integer and \( a(t) \) is an arbitrary function of \( t \).

Proof: In the above proposition \( A_2(x, t) = \frac{\alpha}{x} \) and \( A_1(x, t) = \frac{\beta}{t} \). Our main aim is to find \( F \) and \( G \). From (3.10) we find that

\[ b(x, t) = \frac{a}{x^\alpha}. \tag{3.17} \]

Substituting \( A_1(x, t) \) and \( b(x, t) \) in (3.14), we have

\[ S(x, t) = \left( \frac{\dot{a}}{a} + \frac{\beta}{t} \right) \frac{x}{\alpha + 1}, \tag{3.18} \]

where we have set \( c(t) = 0 \). Now, substituting \( S \) in (3.13) we have the first order partial differential equation

\[ \frac{F_t}{\frac{a}{a} + \frac{\beta}{t}} - \frac{F_x}{\frac{x}{\alpha + 1}} = 0. \tag{3.19} \]

By using the method of characteristics we obtain the general of \( F(x, t) \) in the form

\[ F(x, t) = J(a x^{\alpha+1} t^\beta), \tag{3.20} \]

where \( J(\xi) \) is any arbitrary function of the characteristic coordinate \( \xi = a x^{\alpha+1} t^\beta \). Assuming \( F = (a x^{\alpha+1} t^\beta)^\lambda \), we have from (3.9) the following expression for \( G \),

\[ G = \lambda (\alpha + 1) a^{\lambda + 1} t^{\lambda \beta} x^{(\alpha + 1)(\lambda - 1)}. \tag{3.21} \]

Therefore, the GST is given by;

\[ X = a x^{\alpha+1} t^\beta \lambda, \]

\[ dT = \lambda (\alpha + 1) a^{\lambda + 1} t^{\lambda \beta} x^{(\alpha + 1)(\lambda - 1)} dt. \]

Note that the expressions for \( S \) and \( F \) as given in (3.18) and (3.20) respectively when substituted in (3.12) indeed give \( A_0(x, t) \). The expression for the first integral is obtained from (3.8) after substituting \( F \) and \( G \) from (3.20) and (3.21) respectively.
Proposition 3.2  A time dependent first integral of a second-order equation of the form

\[ \ddot{x} + \frac{\alpha}{x} \dot{x}^2 + \beta(t)x^\lambda \dot{x} + A_0(x, t) = 0 \]  

(3.22)

is given by the function

\[ I(t, x, \dot{x}) = \left( \frac{x^\alpha \dot{x}}{a} + \frac{\beta(t)x^{\lambda + \alpha + 1}}{\lambda + \alpha + 1} \right)^2 + \omega^2 \left( \beta_1(t) - \frac{\lambda + \alpha + 1}{\lambda} x^{-\lambda} \right)^2 \]

where

\[ A_0(x, t) = \frac{\dot{\beta}(t)x^{\lambda + 1}}{\lambda + \alpha + 1} + \omega^2 (\lambda + \alpha + 1) \beta_1(t)x^{-2\alpha - \lambda - 1} - \omega^2 (\lambda + \alpha + 1)^2 \frac{x^{-2\alpha - 2\lambda - 1}}{\lambda}, \]

\[ \beta_1(t) = \int \beta(t) dt. \]

Proof:  In the above proposition \( A_2(x, t) = \frac{a}{x} \) and \( A_1(x, t) = \beta(t)x^\lambda \). Once again our main aim is to find \( F \) and \( G \) under the transformation \( X = F(t; x) \) and \( dT = G(t; x) dt \). For this, first of all we evaluate \( b(x, t) \). From \((3.10)\) we have

\[ b(x, t) = \frac{a}{x^\alpha}. \]  

(3.23)

Substituting \( b(x, t) \) in \((3.11)\), we have

\[ S_x + \frac{\alpha}{x} S = \frac{\dot{a}}{a} + \beta(t)x^\lambda. \]  

(3.24)

A particular solution of \((3.24)\) is clearly given by

\[ S(x, t) = \frac{\dot{a}}{a} x + \beta(t)x^\lambda, \]  

(3.25)

where we have set the constant of integration to be zero. Setting the arbitrary function \( a(t) = \) constant \( = 1 \), we have

\[ S = \beta(t) \frac{x^{\lambda + \alpha + 1}}{\lambda + \alpha + 1}. \]  

(3.26)

Now, substituting \( S \) in \((3.13)\) we have the first order partial differential equation

\[ \frac{x^{\lambda + \alpha + 1}}{\lambda + \alpha + 1} F_x - \frac{F_t}{\beta(t)}. \]

By using the method of characteristics we obtain the general of \( F(x, t) \) in the form

\[ F = J(\beta_1(t) - \frac{\lambda + \alpha + 1}{\lambda} x^{-\lambda}), \]  

(3.27)
where \( J(\xi) \) is any arbitrary function of the characteristic coordinate \( \xi = \beta_1(t) - \frac{\lambda + \alpha + 1}{\lambda} x^{-\lambda} \). Setting \( F = \beta_1(t) - \frac{\lambda + \alpha + 1}{\lambda} x^{-\lambda} \), we have from (3.9) and using (3.10), \( G = \frac{\lambda + \alpha + 1}{x^{\lambda + \alpha + 1}} \). Therefore, the GST looks like,

\[
X = \beta_1(t) - \frac{\lambda + \alpha + 1}{\lambda} x^{-\lambda},
\]

\[
dT = \frac{\lambda + \alpha + 1}{x^{\lambda + \alpha + 1}} dt.
\]

It is easy to verify that these expression for \( S \) and \( F \) gives the required expression for \( A_0(x,t) \) when substituted in (3.12). Again, we find the required expression for the first integral after substituting \( F \) and \( G \) in (3.8).

**Corollary 3.1** For \( \alpha = -3, \lambda = 4, \beta(t) = 1/2 \) equation (3.22) reduces to

\[
\ddot{x} - \frac{3}{x} \dot{x}^2 + \frac{1}{2} x^4 \dot{x}^2 + \omega^2 x = \frac{\omega^2}{x^3},
\]

where “derivative free” terms coincide with the Ermakov-Pinney equation.

**Proposition 3.3** A time dependent first integral of the second order equation of the form

\[
\ddot{x} + 3\alpha(t) x \dot{x} + \frac{3}{2} \dot{\alpha}(t) x^2 + \frac{2\omega^2}{3x^2} \alpha_1(t) - \frac{4\omega^2}{9x^3} = 0
\]

is given by the function

\[
I = (t, x, \dot{x}) = \left( \dot{x} + \frac{3\alpha(t) x^2}{2} \right)^2 + \omega^2 \left( \alpha_1(t) - \frac{2}{3x} \right)^2,
\]

where \( \alpha_1(t) = \int \alpha(t) dt \).

**Proof:** In the above proposition

\[
A_2(x,t) = 0, \quad A_1(x,t) = 3\alpha(t)x, \quad \text{and} \quad A_0(x,t) = \frac{3}{2} \dot{\alpha}(t) x^2 + \frac{2\omega^2}{3x^2} \alpha_1(t) - \frac{4\omega^2}{9x^3}.
\]

Our main aim is to find \( F \) and \( G \). For this, first of all we evaluate \( b(x,t) \). From (3.10) we have

\[
b(x,t) = a(t).
\]

(3.28)

Again, from (3.9), \( G \) can be written as

\[
G = a(t) F_x.
\]

(3.29)

Therefore, the equations (3.11) and (3.12) can be written as

\[
S_x = 3\alpha(t)x \dot{x} + \frac{\dot{\alpha}}{a},
\]

(3.30)
\[ S_t - \frac{\dot{a}}{a} S + \omega^2 a^2 F F_x = \frac{3}{2} \dot{\alpha}(t)x^2 + \frac{2 \omega^2}{3x^2} \alpha_1(t) - \frac{4 \omega^2}{9x^3}. \] (3.31)

A particular solution of (3.30) is clearly given by
\[ S = \frac{3}{2} \alpha(t)x^2 + \frac{\dot{a}}{a} x, \] (3.32)
where we have set the constant of integration to be zero. Setting the arbitrary function \( a(t) = \text{constant} = 1 \), we have
\[ S = \frac{3}{2} \alpha(t)x^2. \] (3.33)

Now, substituting \( S \) in (3.13) we have the first order partial differential equation
\[ \frac{3}{2} x^2 F_x - \frac{1}{\alpha(t)} F_t = 0 \]
By using the method of characteristics we obtain the general of \( F(x, t) \) in the form
\[ F = J(\alpha_1(t) - \frac{2}{3x}), \] (3.34)
where \( J(\xi) \) is any arbitrary function of the characteristic coordinate \( \xi = \alpha_1(t) - \frac{2}{3x} \). Setting \( F = \alpha_1(t) - \frac{2}{3x} \), we have from (3.29), \( G = \frac{2}{3x^2} \). Therefore, the GST looks like,
\[ X = \alpha_1(t) - \frac{2}{3x}, \]
\[ dT = \frac{2}{3x^2} dt. \]

It is easy to verify that the equation (3.31) is identically satisfied for these \( S \) and \( F \). Again, we find the required expression for the first integral after substituting \( F \) and \( G \) in (3.8).

**Proposition 3.4** A time dependent first integral of the second order equation of the form
\[ \ddot{x} + \frac{\alpha}{x} \dot{x}^2 + \left( \frac{\ddot{a}}{a} - 2 \frac{\dot{a}^2}{a^2} \right) \frac{x}{\alpha + 1} + \lambda \omega^2 (\alpha + 1) a^{2\lambda+1} x^{2\lambda+2\lambda-2\alpha-1} = 0 \] (3.35)
is given by the function
\[ I(t, x, \dot{x}) = \left( \frac{x^\alpha}{a} \dot{x} + \frac{\dot{a} x^{\alpha+1}}{a^2(\alpha + 1)} \right)^2 + \omega^2 (ax^{\alpha+1})^{2\lambda} \]
where \( \alpha, \lambda \) is constant and \( a(t) \) is an arbitrary function of \( t \).

**Proof:** In the above proposition \( A_2(x, t) = 0, A_1(x, t) = \frac{\alpha}{x} \) and
\[ A_0(x, t) = \left( \frac{\ddot{a}}{a} - 2 \frac{\dot{a}^2}{a^2} \right) \frac{x}{\alpha + 1} + \lambda \omega^2 (\alpha + 1) a^{2\lambda+2} x^{2\lambda+2\lambda-2\alpha-1}. \]
The proof is similar to proposition (3.1) since this proposition is a particular case of Proposition (3.3) with \( \beta(t) = 0. \) \( \Box \)

It should be noted that equation (3.35) is the master equation of many Ermakov-Pinney type equation. We give few of them.
Corollary 3.2 (a) If we set \( \alpha = 0 \) and \( \lambda = -1 \), then the equation (3.35) reduces to an equation of the Ermakov-Pinney type, viz
\[
\ddot{x} + \left( \frac{\dot{a}}{a} - 2\frac{\ddot{a}^2}{a^2} \right) x = \frac{\omega^2}{ax^3}, \tag{3.36}
\]
and the corresponding first integral is given by
\[
I_{gEP}(t, x, \dot{x}) = \left( \frac{\dot{x}}{a} + \frac{\dot{a}x}{a^2} \right)^2 + \omega^2(ax)^{-2}.
\]

(b) For \( \alpha = -\frac{3}{2}, \lambda = -1 \), we obtain the (parametric) Kummer-Schwarz equation, viz
\[
\ddot{x} - \frac{3}{2} \frac{x^2}{2} - 2b(t)x + \frac{\omega^2}{2a} x^3 = 0, \tag{3.37}
\]
where \( b(t) = (\frac{\ddot{a}}{a} - 2\frac{\ddot{a}^2}{a^2}) \).

Corollary 3.3 If we set \( \alpha = -\frac{1}{2}, \ a = 1, \ \lambda = 1 \) and shifting \( \omega^2 \mapsto 2\omega^2 \), then equation (3.35) reduces to
\[
\ddot{x} - \frac{\dot{x}^2}{x} + \omega^2 x = 0, \tag{3.38}
\]
whose first integral is given by \( I = \frac{\dot{x}^2}{x} + 2\omega^2 x \).

4 Mapping of the coupled Liénard equation to a coupled dissipative Ermakov-Milne-Pinney equation via nonlocal transformation

We consider the following coupled Liénard equation
\[
\ddot{x} + f_1(x, y)\dot{x} + g_1(x, y) = 0, \quad \ddot{y} + f_2(x, y)\dot{y} + g_2(x, y) = 0, \tag{4.1}
\]
where \( f \) is defined on an open connected subset \( \Omega \) of \( \mathbb{R}^2 \). Briata and Sabatini [2] studied a coupled equation where the coupling is entirely due to the dissipative terms and \( g(x) = (g_1(x), g_2(y)) \).

Proposition 4.1 The coupled Liénard equation \( \ddot{x} + f_1(x, y)\dot{x} + g_1(x, y) = 0 \) and \( \ddot{y} + f_2(x, y)\dot{y} + g_2(x, y) = 0 \) is mapped to the equations of two linear harmonic oscillator \( \ddot{X} + \omega^2 X = 0 \) and \( \ddot{Y} + \omega^2 Y = 0 \), under the following nonlocal transformation
\[
\frac{dX}{X} = A_1(x, y)dx + B_1(x, y)dy + C_1(x, y)dt, \quad T = t \tag{4.2}
\]
\[
\frac{dY}{Y} = A_2(x, y)dx + B_2(x, y)dy + C_2(x, y)dt, \quad T = t, \tag{4.3}
\]
where
\[ A_i = B_i = \frac{1}{x + y}, \quad (x+y)^2 C_1(x,y) = \int_0^x (s+y)f_1(s,y)ds, \quad (x+y)^2 C_2(x,y) = \int_0^y (x+s)f_2(x,s)ds \]
provided
\[ g_1 + g_2 = \omega^2 (x+y) + \frac{1}{(x+y)^3} \left( \int_0^x (s+y)f_1(s,y)ds \right) = \omega^2 (x+y) + \frac{1}{(x+y)^3} \left( \int_0^y (x+s)f_2(x,s)ds \right). \]
The consistency condition implies \((f_1 - f_2) = (x+y)(f_{2x} - f_{1y})\).

**Proof:** From \(dX/X = A_1(x,y)dx + B_1(x,y)dy + C_1(x,y)dt\) we have
\[ \frac{\dot{X}}{X} = A_1(x,y)\dot{x} + B_1(x,y)\dot{y} + C_1(x,y), \]
which after further differentiation we obtain
\[ \ddot{X} = \left[ A_{1x}\dot{x}^2 + (A_{1y} + B_{1x})\dot{x}\dot{y} + B_{1y}\dot{y}^2 + (C_{1x} - A_1f_1)\dot{x} + (C_{1y} - B_1f_2)\dot{y} - A_1g_1 - B_1g_2 \right] X \]
\[ + \left[ A_1\dot{x} + B_1\dot{y} + C_1 \right]^2 X. \]

Now we set
\[ A^2 + A_{1x} = 0 \]
\[ B_{1x} + 2A_1B_1 + A_{1y} = 0 \]
\[ C_{1x} + 2A_1C_1 - A_1f_1 = 0 \]
\[ C_{1y} + 2B_1C_1 - B_1f_2 = 0 \]
\[ \omega^2 = A_1g_1 + B_1g_2 - C_1^2. \]
A particular solution of the first two equations is obvious, given by \(A_1 = 1/(x+y) = B_1\). Here we have chosen the constant of integration to be zero. Inserting these values of \(A_1\) and \(B_1\) we obtain the expression of \(C_1\). The last equation yields the value of \(g_1 + g_2\). From the consistency condition of
\[ \frac{\partial}{\partial x}((x+y)^2C_1) = (x+y)f_1(x,y) \quad \frac{\partial}{\partial y}((x+y)^2C_1) = (x+y)f_2(x,y) \]
we obtain \((f_1 - f_2) = (x+y)(f_{2x} - f_{1y})\). □

Thus a sufficient condition is proposed for the simultaneous linearization of the coupled equations. Once again we assume our transformation is defined everywhere except at the singular points.

**Illustration:**
Let us take
\[ f_1(x,y) = \frac{1}{x^3y} \quad \text{and} \quad f_2(x,y) = \frac{1}{xy^3}. \]
This immediately yields \( C_1 = -\frac{1}{2}x^2y^2 \) and
\[
g_1 + g_2 = \left( \omega^2 + \frac{1}{4x^4y^4} \right) (x + y).
\]
The natural choice of \( g_1 \) and \( g_2 \) are
\[
g_1(x, y) = \omega^2 x + \frac{1}{4x^3y^4} \quad \text{and} \quad g_2(x, y) = \omega^2 y + \frac{1}{4x^4y^3}.
\]
Thus we obtain a coupled version
\[
\ddot{x} + \frac{\dot{x}}{x^3y} + \omega^2 x + \frac{1}{4x^3y^4} = 0
\]
\[
\ddot{y} + \frac{\dot{y}}{xy^3} + \omega^2 y + \frac{1}{4x^4y^3} = 0
\]
of the Ermakov-Milne-Pinney equation. The first integral of this coupled equation is given by
\[
I = (\dot{x}y - x\dot{y}) - \frac{1}{2} \left( \frac{1}{x^2} - \frac{1}{y^2} \right).
\]

5 Linearization and isochronous conditions for the Liénard equation

The Liénard equation \( \ddot{x} + f(x)\dot{x} + g(x) = 0 \) has been studied extensively owing to its applications and associated mathematical properties such as the existence of limit cycles for suitable choices of the functions \( f(x) \) and \( g(x) \). Furthermore this equation also displays isochronous behavior when \( f(x) \) and \( g(x) \) bear a specific relationship together with certain conditions on their character as will be stated in the sequel.

However, we will first focus on its linearization via a nonlocal transformation, which is defined everywhere except at the singular points, to the equation of a linear harmonic oscillator.

**Proposition 5.1** The Liénard equation \( \ddot{x} + f(x)\dot{x} + g(x) = 0 \) is mapped to the equation of a linear harmonic oscillator, \( \ddot{X} + \omega^2 X = 0 \), under the following nonlocal transformation
\[
\frac{dX}{X} = A(x)dx + B(x)dt, \quad T = t
\]
where \( A(x) = 1/x \) and \( x^2B(x) = \int_0^x sf(s)ds \) provided
\[
g(x) = \omega^2 x + \frac{1}{x^3} \left( \int_0^x sf(s)ds \right)^2.
\]
**Proof:** From \( dX/X = A(x)dx + B(x)dt \) we have

\[
\frac{\dot{X}}{X} = A(x)\dot{x} + B(x)
\]

which implies upon further differentiation (note that since \( T = t \) we continue denoting the derivatives with overdots)

\[
\ddot{X} = [(A_x + A^2)\dot{x}^2 + (B_x + 2A(x)B(x) - A(x)f(x))\dot{x} + (B^2(x) - A(x)g(x))]X.
\]

Next we set

\[
\begin{align*}
A_x + A^2 &= 0 \quad (5.1) \\
B_x + 2A(x)B(x) - A(x)f(x) &= 0 \quad (5.2) \\
B^2(x) - A(x)g(x) &= -\omega^2. \quad (5.3)
\end{align*}
\]

A particular solution of the first of these equations is obviously \( A(x) = 1/x \) where we have chosen the constant of integration (which can be an arbitrary function of \( t \)) to be zero. Inserting this expression for \( A(x) \) into the second equation we get after integration

\[
B(x) = \frac{1}{x^2} \int_0^x sf(s)ds.
\]

Finally it follows from the last equation that \( g(x) = (B^2 + \omega^2)/A(x) = x(B^2 + \omega^2) \), since \( A(x) = 1/x \), so that

\[
g(x) = \omega^2 x + \frac{1}{x^3} \left( \int_0^x sf(s)ds \right)^2. \quad (5.4)
\]

\( \square \)

**Remark about isochronicity condition:** Here we dwell on the aspect of isochronicity of such an equation. In [31] the author has studied the monotonicity properties of the period function of \([1,2]\). In particular it is shown that if the functions \( f \) and \( g \) be analytic, \( g \) odd, \( f(0) = g(0) = 0 \) and \( g'(0) > 0 \) then the origin is an isochronous center if and only if \( f \) is odd and

\[
\tau(x) := \left( \int_0^x sf(s)ds \right)^2 - x^3(g(x) - g'(0)x) \equiv 0. \quad (5.5)
\]

Christopher and Devlin [7] is seemingly re-proved by using a different technique.

In all our subsequent calculations we will take \( \omega = 1 \) without loss of generality. This is in fact identical to the condition derived by Sabatini [31] given by (5.5).

It is evident that here the function \( h(X) = \log X \) while

\[
X = F(x,t) = x \exp(\int B(x)dt), \quad T = t.
\]

(5.6)

It is clear that when \( \omega = 0 \), then we can linearize the Liénard equation to \( \ddot{X} = 0 \), by means of the above nonlocal transformation provided

\[
g(x) = \frac{1}{x^3} \left( \int_0^x sf(s)ds \right)^2.
\]
6 Conclusion

We have studied several nonlinear differential equations of Generalized Liénard type using nonlocal transformations. In particular, we have specifically chosen to map the initial nonlinear equation to that of a linear harmonic oscillator under a nonlocal/Sundam transformation and have even extended the procedure to a system of coupled nonlinear ODEs. As illustrated by our examples they can be used profitably to obtain the first integrals of several nonlinear ODEs displaying different types of damping /velocity dependance, and nature of these equations are damped Ermakov-Pinney type. Thus we have generalized the construction of Ermakov-Lewis invariant for a time dependent oscillator to (coupled) Liénard and Liénard type equations, in fact, it was one of our motivation to write this paper. As the preceding section illustrates nonlocal transformations are quite useful for deriving the conditions to be obeyed by the functions $f$ and $g$ of the standard Liénard equation to allow for isochronous motions. This utility can be extended to a much wider context, by treating such transformations as legitimate entities.

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