ALGEBRAIC DE RHAM COHOMOLOGY OF LOG-RIEMANN SURFACES OF FINITE TYPE

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Abstract. Log-Riemann surfaces of finite type are certain branched coverings, in a generalised sense, of \( \mathbb{C} \). They have finitely generated fundamental group and finitely many ramification points, but ramification points of infinite order are allowed (in which case the map has infinite degree). Biswas and Perez-Marco showed that any such log-Riemann surface is given by a meromorphic function \( \pi \) on a punctured Riemann surface \( S' \) (i.e. a compact Riemann surface \( S \) minus finitely many punctures) such that the differential \( d\pi \) has exponential singularities at the punctures (for example in the case of genus zero with one puncture these are given by functions of the form \( \pi = \int R(z) e^{P(z)} \, dz \), where \( R \) is a rational function, and \( P \) is a polynomial of degree equal to the number of infinite order ramification points). Let \( R \) be the set of infinite order ramification points and let \( S^* = S' \cup R \). We define an algebraic de Rham cohomology group \( H^1_{dR}(S^*) \) (given by certain differentials with exponential singularities modulo differentials of certain functions with exponential singularities) for which we show that integration along curves gives a nondegenerate pairing with the relative homology group \( H_1(S^*, R; \mathbb{C}) \). In particular the dimension of \( H^1_{dR}(S^*) \) is \( 2g + \# R + (n-2) \), where \( g \) is the genus of \( S \) and \( n \) the number of punctures. The periods of these differentials along curves joining the infinite order ramification points can be written in the form \( \int_{\gamma} R_1(z, w) e^{\int_{\gamma} R_2(z,w) \, dz} \), where \( z, w \) are meromorphic functions generating the function field of \( S \), \( R_1, R_2 \) are rational functions of \( z, w \), and \( \gamma \) is a curve such that \( z \to \infty \) along certain directions at the two endpoints of \( \gamma \).

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1. Introduction

We recall that for a nonsingular projective algebraic curve \( S \) over \( \mathbb{C} \) (Riemann surface), the algebraic de Rham cohomology group \( H^1_{dR}(S) \) may be defined as the quotient of the space of differentials of the second kind (meromorphic 1-forms with
all residues equal to zero) by the subspace of exact differentials (differentials of meromorphic functions). Integration along closed curves gives a nondegenerate pairing of $H^1_{\text{R}}(S)$ with the homology $H_1(S, \mathbb{C})$ of $S$, so that the dimension of $H^1_{\text{R}}(S)$ is $2g$, where $g$ is the genus of $S$.

A choice of nonconstant meromorphic function $z$ on $S$ realizes $S$ as a finite sheeted branched covering of the Riemann sphere $\mathbb{C}$. In this article we define an algebraic de Rham cohomology group and a period pairing for certain branched coverings, in a generalized sense, of $\mathbb{C}$, namely log-Riemann surfaces of finite type, which are given by transcendental functions of infinite degree. A log-Riemann surface consists of a Riemann surface together with a local holomorphic diffeomorphism $\pi$ from the surface to $\mathbb{C}$ such that the set of points $R$ added to the surface, when completing it with respect to the path-metric induced by the flat metric $d\pi$, is discrete. Log-Riemann surfaces were defined and studied in [BPM15a] (see also [BPM06]), where it was shown that the map $\pi$ restricted to any small enough punctured metric neighbourhood of a point $w^*$ in $R$ gives a covering of a punctured disc in $\mathbb{C}$, and is thus equivalent to either $(z \mapsto z^n)$ restricted to a punctured disc $\{0 < |z| < \epsilon\}$ (in which case we say $w^*$ is a ramification point of order $n$) or to $(z \mapsto \epsilon^z)$ restricted to a half-plane $\{\Re z < C\}$ (in which case we say $w^*$ is a ramification point of infinite order).

A log-Riemann surface is said to be of finite type if it has finitely many ramification points and finitely generated fundamental group. We will only consider those for which the set of infinite order ramification points is nonempty (otherwise the map $\pi$ has finite degree and is given by a meromorphic function on a compact Riemann surface). In [BPM15a], [BPM13], uniformization theorems were proved for log-Riemann surfaces of finite type, which imply that a log-Riemann surface of finite type is given by a pair $(S' = S - \{p_1, \ldots, p_n\}, \pi)$, where $S$ is a compact Riemann surface, and $\pi$ is a meromorphic function on the punctured surface $S'$ such that the differential $d\pi$ has essential singularities at the punctures of a specific type, namely exponential singularities.

Given a germ of meromorphic function $h$ at a point $p$ of a Riemann surface, a function $f$ with an isolated singularity at $p$ is said to have an exponential singularity of type $h$ at $p$ if locally $f = ge^h$ for some germ of meromorphic function $g$ at $p$, while a 1-form $\omega$ is said to have an exponential singularity of type $h$ at $p$ if locally $\omega = \alpha e^h$ for some germ of meromorphic 1-form $\alpha$ at $p$. Note that the spaces of germs of functions and 1-forms with exponential singularity of type $h$ at $p$ only depend on the equivalence class $[h]$ in the space $M_p/O_p$ of germs of meromorphic functions at $p$ modulo germs of holomorphic functions at $p$.

Thus the uniformization theorems of [BPM15a], [BPM13] give us $n$ germs of meromorphic functions $h_1, \ldots, h_n$ at the punctures $p_1, \ldots, p_n$, with poles of orders $d_1, \ldots, d_n \geq 1$ say, such that near a puncture $p_j$ the map $\pi$ is of the form $\int g_j e^{h_j} dz$, where $g_j$ is a germ of meromorphic function near $p_j$ and $z$ a local coordinate near $p_j$. The punctures correspond to ends of the log-Riemann surface, where at each puncture $p_j$, $d_j$ infinite order ramification points are added in the metric completion, so that the total number of infinite order ramification points is $\sum_j d_j$. The $d_j$ infinite order ramification points added at a puncture $p_j$ correspond to the $d_j$ directions of approach to the puncture along which $\Re h_j \to -\infty$ so that $e^{h_j}$ decays exponentially and $\int \sum_j g_j e^{h_j} dz$ converges. In the case of genus zero with one puncture
for example, which is considered in [BPM15b], $\pi$ must have the form $\int R(z)e^{P(z)}dz$ where $R$ is a rational function and $P$ is a polynomial of degree equal to the number of infinite order ramification points.

For a log-Riemann surface of finite type we would like to define a period pairing between differentials and curves on the surface which takes into account the extra structure coming from the infinite order ramification points. It is natural to consider then, in addition to closed curves, also curves joining the infinite order ramification points. As the infinite order ramification points correspond to directions of approach to the punctures, we need to consider differentials whose integrals along curves asymptotic to these directions converge.

Let $S^* := S' \cup \mathcal{R}$ be the space obtained by adding the infinite order ramification points to the punctured surface $S'$. Let $\Omega(S^*)$ be the space of 1-forms meromorphic on $S'$ and with exponential singularities at the punctures $p_1,\ldots,p_n$ of types $h_1,\ldots,h_n$, and let $\Omega_{11}(S^*)$ be the subspace of such 1-forms whose residues at all points of $S'$ vanish (this will be the analogue of the differentials of the second kind). Let $\mathcal{M}(S^*)$ be the space of functions $f$ which are meromorphic on $S'$ with exponential singularities at the punctures $p_1,\ldots,p_n$ of types $h_1,\ldots,h_n$. For $f$ in $\mathcal{M}(S^*)$ the differential $df$ lies in $\Omega_{11}(S^*)$, and $f$ vanishes at all the infinite order ramification points, thus so do the integrals of $df$ over curves joining the infinite order ramification points. We let $d\mathcal{M}(S^*) \subset \Omega_{11}(S^*)$ be the space of such differentials, and define the algebraic de Rham cohomology group of the log-Riemann surface to be the quotient space

$$H^1_{dR}(S^*) := \Omega_{11}(S^*) / d\mathcal{M}(S^*)$$

The integrals of cohomology classes along closed curves in $S$ and along curves joining infinite order ramification points are then well-defined, giving a bilinear pairing between $H^1_{dR}(S^*)$ and the relative homology $H_1(S^*,\mathcal{R};\mathbb{C})$.

**Theorem 1.1.** The period pairing $H^1_{dR}(S^*) \times H_1(S^*,\mathcal{R};\mathbb{C}) \rightarrow \mathbb{C}$ is nondegenerate. In particular

$$\dim H^1_{dR}(S^*) = \dim H_1(S^*,\mathcal{R};\mathbb{C}) = 2g + (n-1) + \sum_j (d_j-1) + (n-1) = 2g + #\mathcal{R} + (n-2)$$

We can also consider the subspaces $\mathcal{O}(S^*) \subset \mathcal{M}(S^*)$ and $\Omega^0(S^*) \subset \Omega_{11}(S^*)$ of functions and 1-forms respectively in these spaces which are holomorphic on $S'$. The differential of any $f$ in $\mathcal{O}(S^*)$ lies in $\Omega^0(S^*)$ and we can define another cohomology group $H^1_{dR,0}(S^*) := \Omega^0(S^*) / d\mathcal{O}(S^*)$.

**Theorem 1.2.** The period pairing $H^1_{dR,0}(S^*) \times H_1(S^*,\mathcal{R};\mathbb{C}) \rightarrow \mathbb{C}$ is nondegenerate. In particular

$$\dim H^1_{dR,0}(S^*) = \dim H_1(S^*,\mathcal{R};\mathbb{C}) = 2g + #\mathcal{R} + (n-2)$$

The proofs of the above theorems will be given in section 5.

We note that all the spaces of functions and differentials defined above only depend on the types $h_1,\ldots,h_n$ of the exponential singularities of $d\pi$, which are unaltered if the differential $d\pi$ is multiplied by a non-zero meromorphic function. It is natural then to consider a less rigid structure than that of a log-Riemann surface. Namely we declare two maps $\pi_1,\pi_2$ inducing log-Riemann surface structures of finite
type on a surface $S$ to be equivalent if the function $d\pi_1/d\pi_2$ is meromorphic on $S$, and define an exp-algebraic curve to be an equivalence class of such finite type log-Riemann surfaces. Then exp-algebraic curves correspond precisely to the data of a compact Riemann surface $S$ together with a Mittag-Leffler distribution of principal parts $(h_1, \ldots, h_n)$ of meromorphic functions at punctures $p_1, \ldots, p_n$.

Periods of differentials with exponential singularities have also appeared in the work of Bloch-Esnault ([BE04]), who define homology and cohomology groups associated to an irregular connection on a Riemann surface, and show that an associated pairing is nondegenerate. It is not clear however how the groups defined in [BE04] are related to the ones defined in this article.

Certain functions with exponential singularities, namely the $n$-point Baker-Akhiezer functions ([Bak28], [Akh61]), have been used in the algebro-geometric integration of integrable systems (see, for example, [Kri76], [Kri77a] and the surveys [Kri77b], [Dub81], [KN80], [DKN85]). Given a divisor $D$ on $S'$, an $n$-point Baker-Akhiezer function (with respect to the data $(\{p_j\}, \{h_j\}, D)$) is a function $f$ in the space $M(S')$ satisfying the additional properties that the divisor $(f)$ of zeroes and poles of $f$ on $S'$ satisfies $(f) + D \geq 0$, and that $f \cdot e^{-h_j}$ is holomorphic at $p_j$ for all $j$. For $D$ a non-special divisor of degree at least $g$, the space of such Baker-Akhiezer functions is known to have dimension $\deg D - g + 1$.

In section 3 we show how to naturally associate to the data $\mathcal{H} = \{h_j\}$ a degree zero line bundle $L_\mathcal{H}$ together with a meromorphic connection $\nabla_\mathcal{H}$ and a single-valued horizontal section $s_\mathcal{H}$. These can be used to construct a non-zero meromorphic function $f_0$ on $S'$ with the prescribed types of exponential singularities $h_1, \ldots, h_n$ at the punctures $p_1, \ldots, p_n$, so that all the spaces of functions and 1-forms defined above associated to the data $\{h_j\}$ are non-trivial.

Finally we remark that functions and differentials with exponential singularities on compact Riemann surfaces have also been studied by Cutillas ([Cut84], [Cut89], [Cut90]), where they arise naturally in the solution of the Weierstrass problem of realizing arbitrary divisors on compact Riemann surfaces, and by Taniguchi ([Tan01], [Tan02]), where entire functions satisfying certain topological conditions (called "structural finiteness") are shown to be precisely those entire functions whose derivatives have an exponential singularity at $\infty$, namely functions of the form $\int Q(z)e^{P(z)} \, dz$, where $P, Q$ are polynomials.

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2. LOG-RIEMANN SURFACES OF FINITE TYPE AND EXP-ALGEBRAIC CURVES

We recall some basic definitions and facts from [BPM15a], [BPM15b], [BPM13].

**Definition 2.1.** A log-Riemann surface is a pair $(S, \pi)$ where $S$ is a Riemann surface and $\pi : S \to \mathbb{C}$ is a local holomorphic diffeomorphism such that the set of points $\mathcal{R}$ added to $S$ in the completion $S^* := S \sqcup \mathcal{R}$ with respect to the path metric induced by the flat metric $|d\pi|$ is discrete.
The map $\pi$ extends to the metric completion $S^*$ as a 1-Lipschitz map. In [BPM15a] it is shown that the map $\pi$ restricted to a sufficiently small punctured metric neighbourhood $B(w^*, r) - \{w^*\}$ of a ramification point is a covering of a punctured disc $B(\pi(w^*), r) - \{\pi(w^*)\}$ in $\mathbb{C}$, and so has a well-defined degree $1 \leq n \leq +\infty$, called the order of the ramification point (we assume that the order is always at least 2, since order one points can always be added to $S$ and $\pi$ extended to these points in order to obtain a log-Riemann surface).

**Definition 2.2.** A log-Riemann surface is of finite type if it has finitely many ramification points and finitely generated fundamental group.

For example, the log-Riemann surface given by $(\mathbb{C}, \pi = e^z)$ is of finite type (with the metric $|d\pi|$ it is isometric to the Riemann surface of the logarithm, which has a single ramification point of infinite order), as is the log-Riemann surface given by the Gaussian integral $(\mathbb{C}, \pi = \int e^z^2 dz)$, which has two ramification points, both of infinite order, as in the figure below:

![Log-Riemann surface of the Gaussian integral](image)

In [BPM13], it is shown that a log-Riemann surface of finite type (which has at least one infinite order ramification point) is of the form $(S', \pi)$, where $S'$ is a punctured compact Riemann surface $S' = S - \{p_1, \ldots, p_n\}$ and $\pi$ is meromorphic on $S'$ and $d\pi$ has exponential singularities at the punctures $p_1, \ldots, p_n$. Let $h_1, \ldots, h_n$ be the types of the exponential singularities of $d\pi$ at the punctures $p_1, \ldots, p_n$. As described in [BPM13], each puncture $p_j$ corresponds to an end of the log-Riemann surface where $d_j$ infinite order ramification points are added, $d_j$ being the order of the pole of $h_j$ at $p_j$.

Let $w^*$ be an infinite order ramification point associated to a puncture $p_j$. An $\epsilon$-ball $B_\epsilon$ around $w^*$ is isometric to the $\epsilon$-ball around the infinite order ramification point of the Riemann surface of the logarithm (given by cutting and pasting infinitely many discs together), and there is an argument function $\arg_{w^*} : B_\epsilon^* \rightarrow \mathbb{R}$ defined on the punctured ball $B_\epsilon^*$. While the function $\pi$, which is of the form
\[ \pi = \int e^{h_j \alpha_j} \text{ in a punctured neighbourhood of } p_j \text{ for some meromorphic 1-form } \alpha_j, \]
extends continuously to \( w^* \) for the metric topology on \( S^* \), in general functions of the form \( f = \int e^{h_j \alpha} \) (where \( \alpha \) is a 1-form meromorphic near \( p_j \)) do not extend continuously to \( w^* \) for the metric topology \([\text{BPM08}]\). Limits of these functions in sectors \( \{ p \in B_* : c_1 < \arg w^*(p) < c_2 \} \) do exist however and are independent of the sector; we say that the function is Stolz continuous at points of \( \mathcal{R} \).

**Definition 2.3.** Define spaces of functions and 1-forms on \( S^* \): 

\[
\mathcal{M}(S^*) := \{ f \text{ meromorphic function on } S' : f \text{ has exponential singularities at } p_1, \ldots, p_n \text{ of types } h_1, \ldots, h_n \}
\]

\[
\mathcal{O}(S^*) := \{ f \in \mathcal{M}(S^*) : f \text{ holomorphic on } S' \}
\]

\[
\Omega(S^*) := \{ \omega \text{ meromorphic 1-form on } S' : \omega \text{ has exponential singularities at } p_1, \ldots, p_n \text{ of types } h_1, \ldots, h_n \}
\]

\[
\Omega_{11}(S^*) := \{ \omega \in \Omega(S^*) : \text{Residues of } \omega \text{ at all points of } S' \text{ vanish} \}
\]

\[
\Omega^0(S^*) := \{ \omega \in \Omega(S^*) : \omega \text{ holomorphic on } S' \}
\]

Functions in \( \mathcal{M}(S^*) \) are Stolz continuous at points of \( \mathcal{R} \) taking the value 0 there. The integrals of 1-forms \( \omega \) in \( \Omega_{11}(S^*) \) over curves \( \gamma : [a, b] \to S^* \) joining points \( \omega^* \), \( \omega^*_2 \) of \( \mathcal{R} \) converge if \( \gamma \) is disjoint from the poles of \( \omega \) and tends to these points through sectors \( \{ p \in B_* : c_1 < \arg \omega^*(p) < c_2 \}, \{ p \in B_* : c_1 < \arg \omega^*_2(p) < c_2 \} \) (since any primitive of \( \omega \) on a sector is Stolz continuous).

The definitions of the above spaces only depend on the types \( \{ [h_i] \in \mathcal{M}_{p_i}/\mathcal{O}_{p_i} \} \) of the exponential singularities of the 1-form \( d\pi \), which do not change if \( d\pi \) is multiplied by a meromorphic function. It is natural to define then a structure less rigid than that of a log-Riemann surface of finite type.

**Definition 2.4 (Exp-algebraic curve).** Given a punctured compact Riemann surface \( S' = S - \{ p_1, \ldots, p_n \} \), two meromorphic functions \( \pi_1, \pi_2 \) on \( S' \) inducing log-Riemann surface structures of finite type are considered equivalent if \( d\pi_1/d\pi_2 \) is meromorphic on the compact surface \( S \). An exp-algebraic curve is an equivalence class of such log-Riemann surface structures of finite type.

It follows from the uniformization theorem of \([\text{BPM13}]\) that an exp-algebraic curve is given by the data of a punctured compact Riemann surface and \( n \) (equivalence classes of) germs of meromorphic functions \( \mathcal{H} = \{ [h_i] \in \mathcal{M}_{p_i}/\mathcal{O}_{p_i} \} \) with poles at the punctures.

We can associate a topological space \( \hat{S} \) to an exp-algebraic curve, given as a set by \( \hat{S} = S' \cup \mathcal{R} \), where \( \mathcal{R} \) is the set of infinite ramification points added with respect to any map \( \pi \) in the equivalence class of log-Riemann surfaces of finite type, and the topology is the weakest topology such that all maps \( \hat{\pi} \) in the equivalence class extend continuously to \( \hat{S} \).

Finally, for a meromorphic function \( f \) on \( S' \) (respectively meromorphic 1-form \( \omega \) on \( S' \)) with exponential singularities of types \( h_1, \ldots, h_n \) at points \( p_1, \ldots, p_n \) we can define a divisor \( (f) = \sum_{p \in S'} n_p \cdot p \) (respectively \( (\omega) = \sum_{p \in S} m_p \cdot p \) by \( n_p = ord_p(f) \) if \( p \in S' \) and \( n_p = ord_{p_i}(g) \) if \( p = p_i \), where \( g \) is a germ of meromorphic function.
at \( p_i \) such that \( f = ge^{h_i} \) (respectively \( m_p = \text{ord}_p(\omega) \) if \( p \in S' \) and \( n_p = \text{ord}_p(\alpha) \) if \( p = p_i \), where \( \alpha \) is a germ of meromorphic 1-form at \( p_i \) such that \( \omega = \alpha e^{h_i} \)).

Note that the divisor (\( f \)) can also be defined by \( n_p = \text{Res}(df/f, p) \), so it follows from the Residue Theorem applied to the meromorphic 1-form \( df/f \) that the divisor (\( f \)) has degree zero.

3. EXP-ALGEBRAIC CURVES AND LINE BUNDLES WITH MEROMORPHIC CONNECTIONS

Let \((S, \mathcal{H} = \{[h_1], \ldots, [h_n]\})\) be an exp-algebraic curve, where \( S \) is a compact Riemann surface of genus \( g \) and \( h_1, \ldots, h_n \) are germs of meromorphic functions at points \( p_1, \ldots, p_n \). Let \( \Omega(S) \) be the space of holomorphic 1-forms on \( S \). The data \( \mathcal{H} \) defines a degree zero line bundle \( \mathcal{L}_\mathcal{H} \) together with a transcendental section \( s_\mathcal{H} \) of this line bundle which is non-zero on the punctured surface \( S' \) as follows:

Solving the Mittag-Leffler problem locally for the distribution \( \{h_1, \ldots, h_n\} \) gives meromorphic functions on an open cover such that the differences are holomorphic on intersections, and hence gives an element of \( H^1(S, \mathcal{O}) \). Under the exponential this gives a degree zero line bundle as an element of \( H^1(S, \mathcal{O}^*) \). Explicitly this is constructed as follows:

Let \( B_1, \ldots, B_n \) be pairwise disjoint coordinate disks around the punctures \( p_1, \ldots, p_n \) and let \( V \) be an open subset of \( S' \) intersecting each disk \( B_i \) in an annulus \( U_i = V \cap B_i \) around \( p_i \) such that \( \{B_1, \ldots, B_n, V\} \) is an open cover of \( S \). Define a line bundle \( \mathcal{L}_\mathcal{H} \) by taking the functions \( e^{-h_i} \) to be the transition functions for the line bundle on the intersections \( U_i \). Define a holomorphic non-vanishing section of \( \mathcal{L}_\mathcal{H} \) on \( S' \) by:

\[
s_\mathcal{H} := \begin{cases} 1 & \text{on } V \\ e^{-h_i} & \text{on } B_i - \{p_i\} \end{cases}
\]

Define a connection \( \nabla_\mathcal{H} \) on \( \mathcal{L}_\mathcal{H} \) by declaring that \( \nabla_\mathcal{H}(s_\mathcal{H}) = 0 \). Then for any holomorphic section \( s \) on \( V \), \( s = fs_\mathcal{H} \) for some holomorphic function \( f \), and \( \nabla_\mathcal{H}(s) = df s_\mathcal{H} \), so \( \nabla_\mathcal{H} \) is holomorphic on \( V \). On each disk \( B_i \), letting \( s_i \) be the section which is constant equal to 1 on \( B_i \) (with respect to the trivialization on \( B_i \)), for any holomorphic section \( s \) on \( B_i \), \( s = fs_i \) for some holomorphic function \( f \), and \( s_i = e^{h_i} s_\mathcal{H} \), so

\[
\nabla_\mathcal{H}(s) = \nabla_\mathcal{H}(fs_i) \\
= \nabla_\mathcal{H}(fe^{h_i} s_\mathcal{H}) \\
= (df + f dh_i)e^{h_i} s_\mathcal{H} \\
= (df + f dh_i)s_i
\]

thus the connection 1-form of \( \nabla_\mathcal{H} \) with respect to \( s_i \) is given by \( dh_i \), so \( \nabla_\mathcal{H} \) is meromorphic on \( B_i \) with a single pole at \( p_i \) of order \( d_i + 1 \geq 2 \).

Let \( s_\mathcal{H}^* \) be the unique section of the dual bundle \( \mathcal{L}_\mathcal{H}^* \) on \( S' \) such that \( s_\mathcal{H} \otimes s_\mathcal{H}^* = 1 \) on \( S' \). Then for any non-zero meromorphic section \( s \) of \( \mathcal{L}_\mathcal{H} \), the function \( f := s \otimes s_\mathcal{H}^* \) is meromorphic on \( S' \) with exponential singularities at \( p_1, \ldots, p_n \) of types \( h_1, \ldots, h_n \), and the divisors of \( s \) and \( f \) coincide. Thus the line bundle \( \mathcal{L}_\mathcal{H} \) has degree zero. In summary we have:
Theorem 3.1. For any log-Riemann surface of finite type \( S^* \), the line bundle \( L_H \) has degree zero and the maps \( s \mapsto s \otimes s_H^*, f \mapsto f \cdot s_H^* \) (respectively \( \alpha \mapsto \alpha \otimes s_H^*, \omega \mapsto \omega \cdot s_H^* \)) are mutually inverse isomorphisms between the spaces of meromorphic sections of \( L_H \) and \( \mathcal{M}(S^*) \) (respectively the spaces of meromorphic \( L_H \)-valued 1-forms and \( \Omega(S^*) \)) preserving divisors.

In particular the vector spaces \( \mathcal{M}(S^*), \Omega(S^*), \Omega(S^*), \Omega_M(S^*), \Omega^0(S^*) \) are non-zero.

Proof: Since the isomorphisms above preserve divisors, the spaces \( \mathcal{O}(S^*), \Omega^0(S^*) \) correspond to the spaces of meromorphic sections of \( L_H \) and meromorphic \( L_H \)-valued 1-forms which are holomorphic on \( S^* \), both of which are non-empty. \( \diamond \)

Proposition 3.2. The correspondence \( H \mapsto (L_H, \nabla_H) \) gives a one-to-one correspondence between exp-algebraic structures on \( S \) and degree zero line bundles on \( S \) with meromorphic connections with all poles of order at least two, zero residues, and trivial monodromy.

Proof: Since the connection 1-form of \( \nabla_H \) is given by \( dh_i \) on \( B_i \), all residues of \( \nabla_H \) are equal to zero, while the monodromy of \( \nabla_H \) is trivial since \( s_H \) is a single-valued horizontal section.

Conversely, given such a meromorphic connection \( \nabla \) on a degree zero line bundle \( L \), if \( p_1, \ldots, p_n \) are the poles of \( \nabla \) and \( \omega_1, \ldots, \omega_n \) are the connection 1-forms of \( \nabla \) with respect to trivializations near \( p_1, \ldots, p_n \), then each \( \omega_i \) has zero residue at \( p_i \) and pole order at least two, hence there exist meromorphic germs \( h_1, \ldots, h_n \) near \( p_1, \ldots, p_n \) such that \( \omega_i = dh_i \). We obtain an exp-algebraic curve \( (S, H, \nabla) \).

It is clear for an exp-algebraic curve \( (S, H) \) that \( H(L_H, \nabla_H) = H \), so the correspondences are inverses of each other. \( \diamond \)

Finally we remark that by Serre Duality, the degree zero line bundle \( L_H \), given as an element of \( H^1(S, \mathcal{O}) \), can also be described as an element of \( H^0(S, \Omega)^* = \Omega(S)^* \) using residues, as the linear functional

\[
\text{Res}_H : \Omega(S) \to \mathbb{C} \\
\xi \mapsto \sum_i \text{Res}(\xi \cdot h_i, p_i)
\]

4. The Local Period Pairing

Let \( p = p_i \) be one of the punctures of the exp-algebraic curve \( (S, H) \), and let \( (U, w = 1/z) \) be a coordinate disk near \( p \) such that \( z(p) = \infty \) and \( h = h_i \) is given by \( h = z^d \), where \( d = d_i \) is the order of the pole of \( h_i \) at \( p_i \). The completion \( U^* \) of the punctured disk \( U' = U - \{ p \} \) with respect to the metric \( |e^{z^d}||dz| \) is given by adding \( d \) infinite order ramification points \( w_0, \ldots, w_{d-1} \) to \( U' \), \( U^* = U' \cup \{ w_1, \ldots, w_d \} \), such that \( z \in U' \) converges to \( w_j \) if \( z \to \infty \) along any ray \( \arg z = \theta \) with \( |\theta - (2j-1)\pi/d| < \pi/2d \).

Definition 4.1. Let \( \mathcal{O}_p([h]), \Omega_p([h]) \) be the spaces of holomorphic functions and 1-forms respectively on \( U' \) with exponential singularities of type \( h \) at \( p \).
Let $\gamma \subset U'$ be a small circle around $p$ traversed anticlockwise, and let $\gamma_1, \ldots, \gamma_{d-1}: \mathbb{R} \to U'$ be curves such that $\gamma_k(t) \to w^*_0$ as $t \to -\infty$, $\gamma_k(t) \to w^*_k$ as $t \to +\infty$ (so that $\gamma_k$ is a curve joining the infinite order ramification points $w^*_0$ and $w^*_k$). Then the curves $\gamma, \gamma_1, \ldots, \gamma_{d-1}$ form a basis for $H_1(U^*, \{w^*_0, \ldots, w^*_{d-1}\}; \mathbb{C})$, and integration along curves gives a pairing of this space with the space $\Omega_p([h])/d\mathcal{O}_p([h])$, the 'local period pairing' at $p$. We show in the following sections that this local period pairing is nondegenerate.

First we show:

**Theorem 4.2.**

$$\dim \Omega_p([h])/d\mathcal{O}_p([h]) = d$$

### 4.1. Differentials of the form $\omega = z^ke^zdz$.

Let $g(z)e^zdz$ be a differential in $\Omega_p([h])$ with $g$ meromorphic at infinity. In the case when $g$ is a polynomial we have:

**Proposition 4.3.** (1) For $d = 1$,

$$\mathbb{C}[z]e^zdz = d(\mathbb{C}[z]e^z)$$

(2) For $d \geq 2$,

$$\mathbb{C}[z]e^zdz = d \left( \mathbb{C}[z]e^z \right)^{d-2} \bigoplus \mathbb{C} \cdot z^j e^zdz$$

**Proof:** For $d = 1$, $e^zdz = d(e^z)$ and for $k \geq 0$, $z^{k+1}e^zdz = d(z^{k+1}e^z) - (k+1)z^ke^zdz$ (integration by parts), so the result follows by induction.

Let $d \geq 2$. For $0 \leq k \leq d-2$ there is nothing to prove, for $k = d-1$ we have $z^{d-1}e^zdz = (1/d)d(e^z)$, and for $k \geq d-1$, $z^{k+1} = z^{d-1}z^k - d^2$ so

$$z^{k+1}e^zdz = \frac{1}{d} \left( z^{k-d+2}e^zd - \frac{k-d+2}{d} z^{k-d+1}e^zd \right)$$

(integration by parts) so the result follows by induction. \hfill \Box

For $g$ a Laurent polynomial we have:

**Proposition 4.4.** For $d \geq 1$,

$$\mathbb{C}[z, 1/z]e^zdz = d \left( \mathbb{C}[z, 1/z]e^z \right)^{d-2} \bigoplus \mathbb{C} \cdot z^j e^zdz$$

**Proof:** Let $1 \leq k \leq d$, consider $(1/z^k)e^zdz$. If $k = 1$ there is nothing to prove. If $1 < k \leq d$, then

$$\frac{e^zd}{z^k}dz = d \left( \frac{-e^zd}{(k-1)z^{k-1}} \right) + \frac{d}{k-1} z^{d-k}e^zd$$

(integration by parts) and we are done by the previous Proposition since $d-k \geq 0$. For $k > d$ the result follows by induction since $k-d < k$. \hfill \Box

Let $\mathcal{M}_p$ be the space of germs of meromorphic functions at $p$. Theorem 4.2 follows immediately from the following proposition:
Proposition 4.5. For \( d \geq 1 \),
\[
\mathcal{M}_p \cdot e^zdz = d(\mathcal{M}_p \cdot e^zd) \bigoplus_{j=-1}^{d-2} \mathbb{C} \cdot z^je^zd
\]

Proof: A straightforward induction using the proof of Proposition 4.4 above shows that for \( k > 1 \),
\[
e^{z^kd}dz = \sum_{j=1}^{k-1} \alpha_{j,k}d \left( \frac{e^{z^j}}{z^j} \right) + \sum_{j=-1}^{d-2} \beta_{j,k}z^je^{z^kd}dz
\]
where the constants \( \alpha_{j,k} \) and \( \beta_{j,k} \) are given by
\[
\alpha_{j,(j+k)d+1} = \left( -\frac{1}{j} \right) \left( \frac{d}{j+d} \right) \left( \frac{d}{j+2d} \right) \cdots \left( \frac{d}{j+kd} \right), \quad j \geq 1, k \geq 0,
\]
\[
\beta_{d-j,d+kj} = \left( \frac{d}{j-1} \right) \left( \frac{d}{j-1+d} \right) \left( \frac{d}{j-1+2d} \right) \cdots \left( \frac{d}{j-1+kd} \right), \quad 2 \leq j \leq d+1, k \geq 0
\]
and \( \alpha_{j,k} = \beta_{j,k} = 0 \) otherwise.

Given \( \omega = g(z)e^{z^d}dz \in \Omega_p([h]) \) with \( g \) a convergent Laurent series \( g(z) = g_nz^n + g_{n-1}z^{n-1} + \cdots + g_0 + g_{-1}z^{-1} + \cdots \), by Proposition 4.3
\[
(g_nz^n + \cdots g_0 + g_{-1}z^{-1})e^{z^d}dz \in d(\mathcal{O}_p) \bigoplus_{j=-1}^{d-2} z^je^{z^d}dz
\]
while for the remaining terms, using the above formula for \( z^{-k}e^{z^d}dz \), we have an equality of formal power series
\[
\left( \sum_{k=2}^{\infty} \frac{g_k}{z^k} \right) e^{z^d}dz = d \left( \sum_{j=1}^{\infty} \left( \sum_{k=j+1}^{\infty} \alpha_{j,k}g_{-k} \right) \frac{1}{z^j} \right) e^{z^d}dz + \sum_{j=-1}^{d-2} \left( \sum_{k=2}^{\infty} \beta_{j,k}g_{-k} \right) z^je^{z^d}dz
\]
It suffices to show that the series on the right above are convergent:

Let \( C, R > 0 \) such that \( |g_{-k}| \leq CR^k \). Choose \( k_0 - 1 > d \cdot (2R)^d \). Then it is easy to see from the above formulae that
\[
|\alpha_{j,k}| \leq \left( \frac{1}{j} \right) \left( \frac{1}{(2R)^d} \right)^{(k-k_0)/d}, \quad j \geq 1, k \geq k_0
\]
\[
|\beta_{d-j,k}| \leq \left( \frac{1}{(2R)^d} \right)^{(k-k_0)/d}, \quad 2 \leq j \leq d+1, k \geq k_0
\]
so
\[
\sum_{k \geq j+1} |\alpha_{j,k}g_{-k}| < +\infty
\]
\[
\sum_{k \geq 2} |\beta_{d-j,k}g_{-k}| < +\infty
\]
and moreover for \( j > k_0 \), \( \sum_{k \geq j+1} |\alpha_{j,k}g_{-k}| < C'/j \) (for some \( C' > 0 \)), so the power series with coefficient of \( z^j \) equal to \( \sum_{k \geq j+1} \alpha_{j,k}g_{-k} \) has a positive radius of convergence. The proposition follows. ∎
4.2. **Nondegeneracy of the local period pairing.** Define a "local period mapping" by

\[ \Phi : \Omega^p([h]) \to \mathbb{C}^d \]

\[ \omega \mapsto \left( \int_{\gamma} \omega, \int_{\gamma_1} \omega, \ldots, \int_{\gamma_{d-1}} \omega \right) \]

The nondegeneracy of the local period pairing follows from the following Theorem:

**Theorem 4.6.** For \( \omega \in \Omega^p([h]) \), \( \Phi(\omega) = 0 \) if and only if \( \omega \in dO^p([h]) \).

For the computations which follow, it will be convenient to take the local coordinate \( w = 1/z \) for this section only such that \( h(z) = -z^d \). As before we let \( \gamma \) be a small circle going anticlockwise around \( p \), and let \( \gamma_1, \ldots, \gamma_{d-1} \) be curves joining the ramification point \( w^*_0 \) to \( w^*_1, \ldots, w^*_{d-1} \) respectively.

By Proposition 4.5, it suffices to consider the periods of the differentials \( z^j e^{-z^d} dz, j = -1, \ldots, d - 2 \). Let \( \Pi \) be the "period matrix"

\[
\Pi = \begin{pmatrix}
\int_{\gamma} z^{-1} e^{-z^d} dz & \int_{\gamma} e^{-z^d} dz & \cdots & \int_{\gamma} z^{d-2} e^{-z^d} dz \\
\int_{\gamma_1} z^{-1} e^{-z^d} dz & \int_{\gamma_1} e^{-z^d} dz & \cdots & \int_{\gamma_1} z^{d-2} e^{-z^d} dz \\
\vdots & \vdots & \ddots & \vdots \\
\int_{\gamma_{d-1}} z^{-1} e^{-z^d} dz & \int_{\gamma_{d-1}} e^{-z^d} dz & \cdots & \int_{\gamma_{d-1}} z^{d-2} e^{-z^d} dz
\end{pmatrix}
\]

Then Theorem 4.6 follows from:

**Proposition 4.7.** \( \Pi \) is nonsingular.

**Proof:** The first row of \( \Pi \) is \( (2\pi i, 0, 0, \ldots, 0) \). It suffices to show then that the \((d-1)\)-by-\((d-1)\) minor

\[
\Pi_1 = \begin{pmatrix}
\int_{\gamma_1} e^{-z^d} dz & \cdots & \int_{\gamma_1} z^{d-2} e^{-z^d} dz \\
\vdots & \ddots & \vdots \\
\int_{\gamma_{d-1}} e^{-z^d} dz & \cdots & \int_{\gamma_{d-1}} z^{d-2} e^{-z^d} dz
\end{pmatrix}
\]

is nonsingular. Let \( \omega_k = e^{2\pi i k/d} \) and \( v_k \) be the row vector

\[ v_k = (\int_0^{\omega_k^{-\infty}} e^{-z^d} dz, \ldots, \int_0^{\omega_k^{-\infty}} z^{d-2} e^{-z^d} dz) \]

Then the rows of \( \Pi_1 \) are \( v_1 - v_0, v_2 - v_0, \ldots, v_{d-1} - v_0 \).

A change of variables \( z = \omega_k t^{1/d} \) gives

\[
\int_0^{\omega_k^{-\infty}} z^j e^{-z^d} dz = \frac{1}{d} \omega_k^{j+1} \Gamma \left( \frac{j+1}{d} \right)
\]
for \(0 \leq k \leq d - 1, j \geq 0\). Let \(\Pi_2\) be the matrix with rows \(v_1, \ldots, v_{d-1}\), then
\[
\det \Pi_2 = \begin{vmatrix}
\frac{1}{d} \Gamma\left(\frac{1}{d}\right) \omega_1 & \frac{1}{d} \Gamma\left(\frac{2}{d}\right) \omega_2 & \ldots & \frac{1}{d} \Gamma\left(\frac{d-1}{d}\right) \omega_{d-1} \\
\frac{1}{d} \Gamma\left(\frac{1}{d}\right) \omega_2 & \frac{1}{d} \Gamma\left(\frac{2}{d}\right) \omega_2 & \ldots & \frac{1}{d} \Gamma\left(\frac{d-1}{d}\right) \omega_{d-1} \\
\vdots & \vdots & \ddots & \vdots \\
\frac{1}{d} \Gamma\left(\frac{1}{d}\right) \omega_{d-1} & \frac{1}{d} \Gamma\left(\frac{2}{d}\right) \omega_{d-1} & \ldots & \frac{1}{d} \Gamma\left(\frac{d-1}{d}\right) \omega_{d-1}
\end{vmatrix}
= \left(\frac{1}{d}\right)^{(d-1)^2} \Gamma\left(\frac{1}{d}\right) \Gamma\left(\frac{2}{d}\right) \ldots \Gamma\left(\frac{d-1}{d}\right)
\begin{vmatrix}
\omega_1^2 & \ldots & \omega_1^{d-1} \\
\omega_2^2 & \ldots & \omega_2^{d-1} \\
\vdots & \ddots & \vdots \\
\omega_{d-1}^2 & \ldots & \omega_{d-1}^{d-1}
\end{vmatrix}
= \left(\frac{1}{d}\right)^{(d-1)^2} \Gamma\left(\frac{1}{d}\right) \Gamma\left(\frac{2}{d}\right) \ldots \Gamma\left(\frac{d-1}{d}\right) (\omega_1 \ldots \omega_{d-1}) \Pi_{1 \leq i < j \leq d-1} (\omega_i - \omega_j)
\neq 0
\]

(using the formula for the Vandermonde determinant). It is easy to see that \((-v_0) = v_1 + v_2 + \cdots + v_{d-1}\), giving
\[
\det \Pi_1 = \det(v_1 - v_0, \ldots, v_{d-1} - v_0)
= \det(v_1, v_2, \ldots, v_{d-1}) + \det(-v_0, v_2, \ldots, v_{d-1}) + \det(v_1, -v_0, v_3, \ldots, v_{d-1}) + \cdots + \det(v_1, \ldots, v_{d-2}, -v_0)
= d \det \Pi_2
\]
so \(\det \Pi = 2\pi i \det \Pi_1 = 2\pi i d \det \Pi_2 \neq 0\).

4.3. Local periods as residues of Borel transforms. In this section we show that the local periods of a differential \(\omega \in \Omega_p([h])\) can also be represented as residues of the Borel transform \(B_d f\) of order \(2\) of a certain function \(f = f_\omega\) associated with \(\omega\) (where \(d\) is the order of the pole of \(h\) at \(p\)), and use it to give another proof of the fact that \(\omega\) is 'exact' (\(\omega \in d\Omega_p([h])\)) if all its local periods vanish.

We recall the definitions of Borel and Laplace transforms of order \(d\) ([Ba100], §5.1, §5.2):

**Definition 4.8** (Borel transform of order \(d\) in the direction \(\theta\). Given \(d \geq 1\) and \(f\) holomorphic in a small sector \(V = \{0 < |w| < \rho, \arg w - \theta < \alpha/2\}\) centered around the direction \(\theta \in \mathbb{R}\) of opening \(\alpha > \pi/d\) which is bounded near the origin, let \(\gamma(\theta)\) denote a closed curve starting out from the origin along a ray \(\arg w = \theta + (\pi + \epsilon)/2d\) going up to a point \(w_0\) with \(|w_0| < \rho\), then going clockwise along the circle \(|w| = |w_0|\) till a ray \(\arg w = \theta - (\pi - \epsilon)/2d\), and then going back to the origin along this ray, where \(\epsilon\) is chosen small enough so that \(\gamma(\theta)\) is contained in the sector \(V\).

The Borel transform of order \(d\) of \(f\) in the direction \(\theta\) is defined by
\[
(B_d f)(\xi) = \frac{1}{2\pi i} \int_{\gamma(\theta)} w^d f(z) \exp((\xi/w)^d) d(w^{-d})
\]
for $\xi$ in an infinite sector $W = \{\xi \neq 0, |\arg \xi - \theta| < \epsilon/2d\}$ centered around the direction $\theta$.

Note that for $\xi$ in $W$ the integral converges absolutely and uniformly on compacts (the exponential term decays rapidly along the radial parts of $\gamma(\theta)$ for $\xi$ in $W$), and so $B_d f$ is holomorphic in the sector $W$. Moreover by Cauchy’s Theorem the integral is independent of the choices of $w_1, \epsilon$.

**Definition 4.9** (Laplace transform of order $d$ in the direction $\theta$). Given $d \geq 1$ and a function $g$ holomorphic in an narrow infinite sector $W = \{\xi \neq 0, |\arg \xi - \theta| < \epsilon/2d\}$ centered around the direction $\theta$ of opening $\epsilon/d < 2\pi$, such that $g$ has exponential growth of order at most $d$ near $\infty$ ($|g(\xi)| = O(\exp(k|\xi|^d))$ for some constant $k > 0$), the Laplace transform of order $d$ of $g$ in the direction $\theta$ is defined by

$$(L_d g)(w) := w^{-d} \int_0^{1/\alpha} g(\xi) \exp(-(\xi/w)^d) d(\xi^d)$$

for $w$ in a small sector $V = \{0 < |w| < \rho, |\arg w - \theta| < \alpha/2\}$. Note that the integral converges absolutely and uniformly on compacts (the exponential term decays rapidly enough along the ray $[0, \theta \cdot \infty]$ for $w$ in $V$ if $\rho$ is small enough), so $L_d g$ is holomorphic in the sector $V$.

Then the following inversion theorem holds ([Bal00], §5.3):

**Theorem 4.10.** Let $f$ be holomorphic and bounded near the origin in a small sector $V$ centered around the direction $\theta$ of opening $\alpha > \pi/d$. Then

$$g = B_d f$$

is holomorphic and of exponential growth at most $d$ in a narrow sector $W$ centered around the direction $\theta$, and

$$f = L_d g$$

Let $\omega \in \Omega_p([h])$. As before, let $w = 1/z$ be a local coordinate near $p$ such that $z(p) = \infty$ and $h(z) = z^d$, and let $w^*_0, \ldots, w^*_{d-1}$ be the infinite order ramification points added when completing a punctured neighbourhood of $p$ with respect to the metric $|ez^d dz|$.

Note that in any small sector $V$ with vertex at $w = 0$ which contains a ray towards the origin along which $w$ tends to a ramification point $w^*_j$, we can always write $\omega$ in the form

$$\omega = d (f \cdot u^d e^h)$$

where $f = f_\omega$ is holomorphic in $V$ and given by

$$f = \frac{1}{w^d} \cdot \frac{1}{e^h} \cdot \int_{w_j}^{w} \omega = z^d \cdot \frac{1}{e^h} \cdot \int_{w_j}^{z} \omega$$

Then $\omega \in d\mathcal{O}_p([h])$ if and only if $f$ extends to a meromorphic function in a neighbourhood of $w = 0$.

In what follows, it will be convenient to work with $f$ as a function of the $z$-variable, in which case the sector $V$ will be a small sector with vertex at $z = \infty$. In this variable, if we take the sector $V$ centered around the direction $\arg z = 0$, then the integral defining the Borel transform is over a curve $\gamma$ going from $z = \infty$ to
some point $z_1$ (with $|z_1|$ large) along a ray $\arg z = -\pi/2d - \epsilon$, then going clockwise along the circle $\{ |z| = |z_1| \}$ up to the ray $\arg z = \pi/2d + \epsilon$ and then going back to $z = \infty$ along this ray. Note that the curve $\gamma$ is a curve joining $w^*_0$ to $w^*_1$. In this sector, we take $f$ to be given by

$$f = z^d \cdot \frac{1}{e^{z^d}} \cdot \int_{w^*_1}^z \omega$$

**Theorem 4.11.** Let $\omega = e^{z^d} \phi(z) \ dz \in \Omega_p([-h])$ with $\phi(z) \ dz$ holomorphic near $z = \infty$. Then:

1. The Borel transform $B_d f$ of order $d$ in the direction $\theta = 0$ of $f \omega$ extends to a meromorphic function on $\mathbb{C}$ which is holomorphic outside the finite set of $d$th roots of unity $\xi = e^{2\pi ik/d}$, and has exponential growth of order at most $d$ near $\xi = \infty$.

2. The Borel transform $B_d f$ has at most simple poles at the $d$th roots of unity, and the residues of $B_d f$ at these points are given in terms of periods of $\omega$:

   $$\text{Res}(B_d f, \xi = e^{2\pi ik/d}) = \left( \frac{1}{de^{2\pi ik(d-1)/d}} \right) \left( \frac{-1}{2\pi i} \right) \int_{w^*_k}^{w^*_{k+1}} \omega$$

**Proof:** We compute:

$$\text{(B}_df)(\xi) = \frac{1}{2\pi i} \int_{\gamma} \int_{\gamma}^z \frac{f(z)e^{(\xi z)^d}}{z^d} \ d(z^d)$$

$$= \frac{1}{2\pi i} \int_{\gamma} \left( \int_{w^*_1}^z \phi(t)e^{\xi t^d} \ dt \right) e^{-z^d} e^{(\xi z)^d} \ d(z^d)$$

$$= \frac{-1}{2\pi i} \int_{\gamma} \left( \int_{z}^{w^*_1} \phi(t)e^{\xi t^d} \ dt \right) e^{(\xi^d - 1)z^d} \ d(z^d)$$

At this point we note that the inner integral from $z$ to $w^*_1$ can be taken along the curve $\gamma$ as well, so that the integral above becomes an integral over a triangle $\{ 0 \leq x \leq y \leq 1 \} \subset [0, 1] \times [0, 1]$, taking $[0, 1] \times [0, 1]$ to be the domain of $\gamma \times \gamma : (x, y) \mapsto (\gamma(x), \gamma(y)) \in U^*$. Applying Fubini’s Theorem gives

$$\text{(B}_df)(\xi) = \frac{-1}{2\pi i} \int_{\gamma} \left( \int_{w^*_0}^{t} e^{(\xi^{-1})z^d} d(z^d) \right) \phi(t)e^{t^d} \ dt$$

$$= \frac{-1}{2\pi i} \int_{\gamma} \frac{1}{\xi^d - 1} e^{(\xi^{-1})t^d} \phi(t)e^{t^d} \ dt$$

$$= \left( \frac{-1}{2\pi i} \int_{\gamma} \phi(t)e^{t^d} \ dt \right) \cdot \frac{1}{\xi^d - 1}$$

$$= \frac{A(\xi)}{\xi^d - 1}$$

where

$$A(\xi) = \frac{-1}{2\pi i} \int_{\gamma} \phi(t)e^{t^d} \ dt$$


Now the function $A$ is holomorphic in an infinite sector centered around the positive real axis in the $\xi$-plane. However it can be analytically continued to all of $\mathbb{C}^*$ by the standard device of ‘rotating’ the contour $\gamma$. Namely, if $\lambda \cdot \gamma$ denotes the curve $\gamma$ multiplied by $\lambda \in S^1$, then we can extend $A$ to all of $\mathbb{C}^*$ by defining

$$A(\xi) := \frac{-1}{2\pi i} \int_{\xi/|\xi|\gamma} \phi(t)e^{\xi t/d} dt$$

for all $\xi \in \mathbb{C}^*$. Moreover the isolated singularity of $A$ at $\xi = 0$ is removable since $A(\xi) \to 0$ as $\xi \to 0$, because the integrand converges to $\phi(t)dt$ and $\phi(t)dt$ is holomorphic near $t = \infty$. Thus $A$ is entire, and it is not hard to show from the above equation that $A$ is of exponential growth of order at most $d$ near $\xi = \infty$. This proves (1). For (2), we have

$$\text{Res}(B_d f, \xi = e^{2\pi i k/d}) = \frac{A(e^{2\pi i k/d})}{d \cdot (e^{2\pi i k/d})^{d-1}}$$

$$= \left(\frac{1}{de^{2\pi i k(d-1)/d}}\right) \left(\frac{-1}{2\pi i}\right) \int_{\gamma_{k+1}} \omega$$

$\diamond$

As a corollary we obtain another proof of Theorem 4.6.

**Proof of Theorem 4.6**: Clearly all periods vanish if $\omega \in d\mathcal{O}_p([h])$. For the converse, assume all periods of $\omega$ vanish. By Proposition 4.5, $\omega \equiv e^{z^d}P(z)dz (\text{mod } d\mathcal{O}_p([h]))$ for some polynomial $P$ of degree at most $d-2$, and from the proof of Proposition 4.4 and the fact that $\text{Res}(\omega, p) = 0$ it follows that $e^{z^d}P(z)dz \equiv \omega_1 = e^{z^d}\phi(z)dz (\text{mod } d\mathcal{O}_p([h]))$ for some 1-form $\phi(z)dz$ which is holomorphic near $z = \infty$. Letting $f = f_{\omega_1}$, by the previous Theorem $B_d f$ is entire (since its residues are multiples of the periods of $\omega_1$ which vanish by hypothesis) of exponential growth at most $d$, so by the Inversion Theorem for Borel-Laplace transforms, $f = \mathcal{L}_d(B_d f)$ defines a holomorphic germ near $z = \infty$, thus $\omega \equiv \omega_1 \equiv 0 (\text{mod } d\mathcal{O}_p([h]))$. $\diamond$

5. The Global Period Pairing

Let $(S, \mathcal{H} = \{h_j\})$ be an exp-algebraic curve, where $h_1, \ldots, h_n$ are germs of meromorphic functions with poles of order $d_1, \ldots, d_n \geq 1$ at points $p_1, \ldots, p_n$ and $S$ is a compact Riemann surface of genus $g$. Let $S^* = S \cup \mathcal{R}$ be the completion of $S' = S - \{p_1, \ldots, p_n\}$ with respect to some log-Riemann surface structure of finite type given by a meromorphic map $\pi : S' \to \hat{\mathbb{C}}$ such that $d\pi$ has exponential singularities at $p_1, \ldots, p_n$ of types $h_1, \ldots, h_n$.

For each puncture $p_i$, let $w_j^{*i}, 0 \leq j \leq d_i - 1$ be the infinite order ramification points associated to the puncture $p_i$, so that $\mathcal{R} = \{w_j^{*i}, 0 \leq j \leq d_i - 1, 1 \leq i \leq n\}$. Let $\gamma^{(i)}$ be a small circle around $p_i$, and let $\{\gamma^{(i)}_j : 1 \leq j \leq d_i - 1\}$ be curves, as in the previous section, joining the ramification point $w_0^{*i}$ to the ramification points $w_j^{*i}, 1 \leq j \leq d_i - 1$ and disjoint from $\gamma^{(i)}$.

We fix a standard homology basis $a_1, \ldots, a_g, b_1, \ldots, b_g$ of $S$, choosing these closed curves so that they are disjoint from the punctures $p_1, \ldots, p_n$ and intersect at only
Proof of Proposition 5.1:

Given a vanish, in particular the integrals over since the integrals of primitive the period mapping Proposition 5.2. The period mapping Proposition 5.1. has a Stolz continuous extension to $S$. Theorem 4.6 there is a germ $\in M_{F}$ in a punctured neighbourhood of $p$. Theorem 5.3. Let $\vdash$ vanish, $F$ theorem $(\omega$ sending a 1-form $\omega$ over the curves $\beta_{1}$ vanishing for each $\omega$ has a primitive $F$ on $S'$ which is meromorphic on $S'$. Moreover since $dF = \omega \in \Omega(S')$, $F$ has a Stolz continuous extension to $S'$ which we may assume satisfies $F(w_{0}^{(1)}) = 0$. Since the integrals of $\omega$ over the curves $\{\gamma_{j}^{(i)}\}$ vanish, for each $i = 1, \ldots, n$, by Theorem 4.6 there is a germ $f_{i} \in O_{p_{i}}([h_{i}])$ and a constant $c_{i}$ such that $F = f_{i} + c_{i}$ in a punctured neighbourhood of $p_{i}$. Since the integrals of $\omega$ over the curves $\beta_{i}$ vanish, $F(w_{0}^{(i)}) = F(w_{n}^{(i)}) = 0$ for all $i$, hence $c_{i} = F(w_{0}^{(i)}) = 0$ for all $i$. Thus $F \in \mathcal{M}(S')$ and $[\omega] = 0$. 

For the proof of Proposition 5.2, we will make use of the following Mergelyan type Theorem for compact Riemann surfaces due to Gusman (Gus60):

Theorem 5.3. Let $S$ be a compact Riemann surface and $E \subset S$ a closed subset such that $S - E$ has finitely many connected components $V_{1}, \ldots, V_{m}$, and for each $i$ let $q_{i}$ be a point of $V_{i}$. Then any continuous function $f$ on $E$ which is holomorphic in the interior of $E$ can be uniformly approximated on $E$ by functions meromorphic on $S$ with poles only in the set $\{q_{1}, \ldots, q_{m}\}$.
We fix an auxiliary point $q$ in the interior of the polygon $P$ and a small circle $\gamma$ around $q$ such that $q, \gamma$ are disjoint from all curves in $\mathcal{C}$. We let $\mathcal{C}' = \mathcal{C} \cup \{ \gamma \}$, and let $\Omega_{II,q}(S^*)$ be the space of all 1-forms in $\Omega(S^*)$ whose residues are zero at all points of $S'' := S \setminus \{ p_1, \ldots, p_n, q \}$. We define another period mapping

$$\Psi : \Omega_{II,q}(S^*) \rightarrow \mathbb{C}^{N+1}$$

taking a 1-form $\omega$ to the vector of its integrals over all curves in $\mathcal{C}'$.

From Theorem 3.3 we can choose a function $f_0 \in \mathcal{O}(S^*) - \{ 0 \}$. Multiplying $f_0$ by a meromorphic function $g$ which is holomorphic on $S'$ if necessary, we can assume that near each $p_i$ the function $f_0$ is of the form $f_0 = g_i e^{h_i}$ where $g_i$ is meromorphic near $p_i$ and has a pole at $p_i$ (existence of such a meromorphic function $g$ holomorphic on $S'$ with poles of high enough order at the points $p_i$ follows from the Riemann-Roch Theorem). We can also assume that the (finitely many) zeroes and critical points of $f_0$ on $S'$ are disjoint from all curves in $\mathcal{C}'$ and from $\{ q \}$. Then the differential $\omega_0 := df_0$ lies in $\Omega^0(S^*)$, is holomorphic and non-zero on all curves in $\mathcal{C}'$ and $q$, and near each $p_i$ takes the form $\omega_0 = k_i e^{h_i} dw$ where $w$ is a local coordinate near $p_i$ and $k_i = dg_i/dw + g_i dh_i/dw$ is a meromorphic function with a pole of order at least $d_i + 2$ at $p_i$.

We fix a compact subset $E$ given by the union of the curves in $\mathcal{C}_1, \mathcal{C}_2$ and small pairwise disjoint closed disks $U_i$ around the points $p_i$ such that $U_i$ contains the curves $\{ \gamma^{(i)} : 1 \leq j \leq d_i - 1 \}$ and $\gamma^{(i)}$. Then $S - E$ is connected and contains the point $q$.

**Lemma 5.4.** Given $c \in \mathcal{C}_1$ and $\epsilon > 0$, there exists $\omega \in \Omega_{II,q}(S^*)$ such that $|\int_c \omega - 1| < \epsilon$ and $|\int_{c'} \omega| < \epsilon$ for all $c' \in \mathcal{C}' - \{ c \}$.

**Proof:** Let $f$ be a continuous function on $E$ such that $f$ is supported in a small arc in the interior of the curve $c \in \mathcal{C}_1$, $\int_c f \omega_0 = 1$, and $f$ is zero elsewhere on $E$. Then by Theorem 5.3 we can find a meromorphic function $R$ on $S$ holomorphic on $S - \{ q \}$ such that the 1-form $\omega := R \omega_0 \in \Omega_{II,q}(S^*)$ satisfies $|\int_c \omega - 1| < \epsilon$ and $|\int_{c'} \omega| < \epsilon/n$ for all $c' \in \mathcal{C} - \{ c \}$. It follows that $|\int_\gamma \omega| < \epsilon$ since the sum of residues of $\omega$ at $p_1, \ldots, p_n, q$ is zero. $\diamond$

**Lemma 5.5.** Given $c \in \mathcal{C}_2$ and $\epsilon > 0$, there exists $\omega \in \Omega_{II,q}(S^*)$ such that $|\int_c \omega - 1| < \epsilon$ and $|\int_{c'} \omega| < \epsilon$ for all $c' \in \mathcal{C}' - \{ c \}$.

**Proof:** Let $f$ be a continuous function on $E$ which is supported on a small arc of $c$ disjoint from the disks $U_1, \ldots, U_n$, such that $\int_c f \omega_0 = 1$. Again by Theorem 5.3 we can find a meromorphic function $R$ on $S$ holomorphic on $S - \{ q \}$ such that the 1-form $\omega := R \omega_0 \in \Omega_{II,q}(S^*)$ satisfies $|\int_c \omega - 1| < \epsilon$ and $|\int_{c'} \omega| < \epsilon/n$ for all $c' \in \mathcal{C} - \{ c \}$. It follows that $|\int_\gamma \omega| < \epsilon$ since the sum of residues of $\omega$ at $p_1, \ldots, p_n, q$ is zero. $\diamond$

**Lemma 5.6.** Given $c \in \mathcal{C}_3$ and $\epsilon > 0$, there exists $\omega \in \Omega_{II,q}(S^*)$ such that $|\int_c \omega - 1| < \epsilon$ and $|\int_{c'} \omega| < \epsilon$ for all $c' \in \mathcal{C}' - \{ c \}$.

**Proof:** Let $c = \gamma^{(i)}$ for some $1 \leq i \leq n, 1 \leq j \leq d_i - 1$. By Proposition 5.3 we can choose a 1-form $\omega_i \in \Omega_p([h_i])$ such that $\omega_i = \phi_i e^{h_i} dw$ for some meromorphic
function \( \phi_i \) on \( U_i \) with only pole on \( U_i \) at \( p_i \) of order between 1 and \( d_i \) such that \( \int_{j, i} \omega_i = 1, \int_{j', i} \omega_i = \int_{\gamma, i} \omega_i = 0 \) for all \( j' \neq j \). We then take \( f \) to be a continuous function on \( E \) such that \( f \omega_0 = \omega_i \) on \( U_i \), \( f = 0 \) on all other disks \( U_{i'}, i' \neq i \), \( f = 0 \) on all curves in \( C_1 \), and such that the integral of \( f \omega_0 \) over all curves in \( C_2 \) is zero. Note that \( f \) is holomorphic on the disc \( U_i \) since \( \phi_i \) has a pole of order at most \( d_i \) at \( p_i \) and \( k_i \) has a pole of order at least \( d_i + 1 \) at \( p_i \).

Again by Theorem 5.3, we can find a meromorphic function \( R \) on \( S \) holomorphic on \( S \setminus \{ q \} \) such that the 1-form \( \omega := R \omega_0 \in \Omega_{I, g}(S^*) \) satisfies \( |\int_c \omega - 1| < \epsilon \) and \( |\int_c \omega| < \epsilon/n \) for all \( c' \in C \setminus \{ c \} \). It follows that \( |\int_q \omega| < \epsilon \) since the sum of residues of \( \omega \) at \( p_1, \ldots, p_n, q \) is zero. \( \diamond \)

**Lemma 5.7.** Given \( y_1, \ldots, y_n \in \mathbb{C} \) such that \( y_1 + \cdots + y_n = 0 \) and \( \epsilon > 0 \), there exists \( \omega \in \Omega_{I, g}(S^*) \) such that \( |\int_{y_i} \omega - y_i| < \epsilon \) for \( i = 1, \ldots, n \), \( |\int_q \omega| < \epsilon \) and \( |\int_c \omega| < \epsilon \) for all other curves in \( C' \).

**Proof:** As in the proof of the previous Lemma, for each \( i \) we choose a 1-form \( \omega_i \omega e^{\bar{w}} dw_i \in \Omega_{p_i}([w_i]) \) (where \( w_i \) is a local coordinate at \( p_i \)) such that the integrals of \( \omega_i \) over the curves \( \gamma_j \) vanish while the integral of \( \omega_i \) over the circle \( \gamma_j \) equals \( y_i \). Let \( f \) be a continuous function on \( E \) such that \( f \omega_0 = \omega_i \) on \( U_i \) (then as before \( f \) is holomorphic on \( U_i \)), \( f = 0 \) on all curves in \( C_1 \), and the integrals of \( f \) over all curves in \( C_2 \) vanish.

Again by Theorem 5.3, we can find a meromorphic function \( R \) on \( S \) holomorphic on \( S \setminus \{ q \} \) such that the 1-form \( \omega := R \omega_0 \in \Omega_{I, g}(S^*) \) satisfies \( |\int_{y_i} \omega - y_i| < \epsilon/n \) and \( |\int_q \omega| < \epsilon/n \) for all other curves \( c \in C \). It follows that \( |\int_q \omega| < \epsilon \) since the sum of residues of \( \omega \) at \( p_1, \ldots, p_n, q \) is zero. \( \diamond \)

**Proof of Proposition 5.2.** It follows from Lemmas 5.4-5.7 that the image of the period mapping \( \Psi \) is dense in the codimension two subspace \( W \) of \( \mathbb{C}^{N+1} \) given by \( W' := \{(x_1, \ldots, x_{N+1}) \in \mathbb{C}^{N+1} | x_{(n+1)-(n-1)} + \cdots + x_{(N+1)-1} = 0, x_{N+1} = 0 \} \), hence \( W' \) is contained in the image of \( \Psi \). Let \( V := \Psi^{-1}(W') \subset \Omega_{I, g}(S^*) \), then it follows from the second equation defining \( W' \) that the residue of any \( \omega \) in \( V \) at the point \( q \) vanishes, thus \( V \subset \Omega_{I, g}(S^*) \) and \( \Psi(V) = W' \). It follows that \( \Phi(V) = W \subset \mathbb{C}^N \). \( \diamond \)

Theorem 1.1 follows immediately from Proposition 5.1 and 5.2.

For the proof of Theorem 1.2 we consider a similar period mapping (keeping the same notation as before)

\[
\Phi : H^1_{dR, \Theta}(S^*) \rightarrow \mathbb{C}^N.
\]

As before it suffices to show that \( \Phi \) is injective and that \( \Phi \) surjects onto the codimension one subspace \( W \). The proof of injectivity is the same as before, observing that if \( F \in \mathcal{M}(S^*) \) is the primitive of a 1-form \( \omega \in \Omega^0(S^*) \) then \( F \in \mathcal{O}(S^*) \). We proceed to the proof of surjectivity.

As before we fix the point \( q \) and the function \( f_0 \) and 1-form \( \omega_0 = df_0 \). The following lemma is a straightforward consequence of the Riemann-Roch Theorem:

**Lemma 5.8.** Given an integer \( l \) and a point \( p \neq q \), there exists a meromorphic function \( R \) on \( S \) such that \( R \) is holomorphic on \( S \setminus \{ p, q \} \) and the order of \( R \) at \( q \) is equal to \(-l\).
Proof: For $N$ a large enough positive integer, there exists $R \in \mathcal{O}_D - \mathcal{O}_{D'}$, where $D = p^N q^{s+1}$, $D' = p^N q^s$. 

Lemma 5.9. Given $\omega \in \Omega(S^*)$ such that $\omega$ is holomorphic on $S' = \{p_1, \ldots, p_n, q\}$, there exists $f \in \mathcal{M}(S^*)$ such that $f$ is holomorphic on $S''$ and $\omega + df$ has at most a simple pole at $q$.

Proof: Let $\alpha$ be the logarithmic differential $d\log f / f_0$, which is a meromorphic 1-form on $S'$, and fix a local coordinate $z$ at the point $q$ such that $z(q) = 0$. We may assume that $\omega$ has a pole at $q$ of order $m \geq 2$ (otherwise we can take $f = 0$ and we are done). Then, $f_0$ is non-zero at $q$, so near $q$ we have

$$\omega = \left(\frac{b_{m-1}}{z^{m-1}} + \cdots + \frac{b_1}{z} + O(1)\right) \cdot f_0 \cdot dz$$

for some constants $\{b_k\}$. By Lemma 5.8 we can choose a function $R$ meromorphic on $S - \{p, q\}$ such that $R$ has a pole of order $(m - 1)$ at $q$, and multiplying by a constant if necessary we may assume that

$$R(z) = \frac{c_{-(m-1)}}{z^{m-1}} + O\left(\frac{1}{z^{m-2}}\right)$$

near $q$, where $c_{-(m-1)} = -b_{m-1}/(m-1)$. Then $R \cdot f_0 \in \mathcal{M}(S^*)$ is holomorphic on $S''$, and $d(Rf_0) = (dR + R)\alpha f_0$, where $\alpha$ is holomorphic near $q$, so

$$\omega + d(Rf_0) = \left[\left(\frac{b_{m-1}}{z^{m-1}} + O\left(\frac{1}{z^{m-2}}\right)\right) + \left(-\frac{(m-1)c_{-(m-1)}}{z^m} + O\left(\frac{1}{z^{m-1}}\right)\right) + O\left(\frac{1}{z^{m-1}}\right) \cdot O(1)\right] \cdot f_0 \cdot dz$$

Thus $\omega + d(Rf_0)$ has at most a pole of order $(m - 1)$ at $q$, so we are done by induction on $m$. 

Proof of 1.2. Let $\Psi : \Omega_{H,q}(S^*) \to \mathbb{C}^{N+1}$ be as before.

It follows from the proof of Proposition 1.2 that for any vector $\vec{v} \in W'$ and any $\epsilon > 0$ there is an $\omega \in \Omega(S^*)$ which is holomorphic on $S''$ such that $||\Psi(\omega) - \vec{v}|| < \epsilon$. By the previous lemma, we can find a 1-form $\omega' = \omega + df$ such that $\omega' \in \Omega(S^*)$, $\omega'$ is holomorphic on $S''$, has at most a simple pole at $q$, and $\Psi(\omega') = \Psi(\omega)$. Letting $V_q$ denote the subspace of $\Omega(S^*)$ consisting of 1-forms which are holomorphic on $S''$ with at most a simple pole at $q$, it follows that $\Psi(V_q)$ is dense in, and hence contains, $W'$. Letting $V' \subset V_q$ be the subspace of $V_q$ such that $\Psi(V') = W'$, we have $V' \subset \Omega(S^*)$ (since 1-forms in $V'$ have at most a simple pole at $q$ and their residue at $q$ vanishes) and $\Phi(V') = W$, so $\Phi : H^1_{dR,0}(S^*) \to W$ is surjective. 

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