The b-Chromatic Number of Regular Graphs via The Edge Connectivity

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Abstract

The b-chromatic number of a graph \( G \), denoted by \( \varphi(G) \), is the largest integer \( k \) that \( G \) admits a proper coloring by \( k \) colors, such that each color class has a vertex that is adjacent to at least one vertex in each of the other color classes. El Sahili and Kouider [About b-colorings of regular graphs, Res. Rep. 1432, LRI, Univ. Orsay, France, 2006] asked whether it is true that every \( d \)-regular graph \( G \) of girth at least 5 satisfies \( \varphi(G) = d + 1 \). Blidia, Maffray, and Zemir [On b-colorings in regular graphs, Discrete Appl. Math. 157 (2009), 1787-1793] showed that the Petersen graph provides a negative answer to this question, and then conjectured that the Petersen graph is the only exception. In this paper, we investigate a strengthened form of the question.

The edge connectivity of a graph \( G \), denoted by \( \lambda(G) \), is the minimum cardinality of a subset \( U \) of \( E(G) \) such that \( G \setminus U \) is either disconnected or a graph with only one vertex. A \( d \)-regular graph \( G \) is called super-edge-connected if every minimum edge-cut is the set of all edges incident with a vertex in \( G \), i.e., \( \lambda(G) = d \) and every minimum edge-cut of \( G \) isolates a vertex. We show that if \( G \) is a \( d \)-regular graph that contains no 4-cycle, then \( \varphi(G) = d + 1 \) whenever \( G \) is not super-edge-connected.

Keywords: b-chromatic number, edge connectivity, super-edge-connected.
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1 Introduction

All graphs considered in this paper are finite and simple (undirected, loopless, and without multiple edges). Let \( G = (V, E) \) be a graph. A (proper vertex) coloring of \( G \), is a function \( f_G : V(G) \rightarrow C \) such that for each \( \{u, v\} \in E(G) \), \( f_G(u) \neq f_G(v) \). Each \( c \) in \( C \) is called a color. Also, for each \( c \) in \( C \), \( f_G^{-1}(c) \) is called a color class of \( f_G \). We say \( v \) is colored by \( c \) if \( f_G(v) = c \). We mean by \( \chi(G) \), the minimum cardinality of a set \( C \) that a coloring \( f_G : V(G) \rightarrow C \) exists. A b-coloring of the graph \( G \) is a coloring \( f_G : V(G) \rightarrow C \) such that for each \( c \) in \( C \), there exists some vertex \( v \) in \( V(G) \) such that \( f_G(v) = c \) and \( f_G(N_G(v)) = C \setminus \{c\} \), where \( N_G(v) := \{w : \{v, w\} \in E(G)\} \). In other words, a coloring of \( G \) is called a b-coloring, if each color class contains a vertex that is adjacent to at least one vertex in each of the other color classes. Obviously, each coloring of \( G \) with \( \chi(G) \) colors, is a b-coloring. Also, if \( f_G : V(G) \rightarrow C \) is a b-coloring, then \( |C| \leq \Delta(G) + 1 \), where \( \Delta(G) \) denotes the maximum degree of \( G \). The b-chromatic number of \( G \), denoted by \( \varphi(G) \), is the maximum cardinality of a set \( C \) such that a b-coloring \( f_G : V(G) \rightarrow C \) exists. The concept of b-coloring of graphs introduced by Irving and Manlove in 1999 in [4], and has received attention.
Given a graph $G$ and a coloring $f_G : V(G) \to C$, a vertex $v$ in $V(G)$ is called a \textit{color-dominating} vertex (with respect to $f_G$) if $f_G(N_G(v)) = C \setminus \{f_G(v)\}$. In other words, $v$ is a color-dominating vertex, if it is adjacent to at least one vertex in each of the other color classes. We say that the color $c$ realizes, if there exists a color-dominating vertex which is colored by $c$.

As mentioned, for each graph $G$ with maximum degree $\Delta(G)$, $\varphi(G) \leq \Delta(G) + 1$. Therefore, for each $d$-regular graph $G$, $\varphi(G) \leq d + 1$. Since $d + 1$ is the maximum possible b-chromatic number for $d$-regular graphs, determining necessary or sufficient conditions to achieve this bound is of interest. Kratochvíl, Tuza, and Voigt \cite{7} proved that every $d$-regular graph $G$ with at least $d^4$ vertices satisfies $\varphi(G) = d + 1$. In \cite{2}, Cabello and Jakovac lowered $d^4$ to $2d^3 - d^2 + d$. These amazing bounds confirm that for each natural number $d$, there are only a finite number of $d$-regular graphs (up to isomorphism) that their b-chromatic numbers are not $d + 1$. El Sahili and Kouider \cite{3} asked whether it is true that every $d$-regular graph $G$ of girth at least 5 satisfies $\varphi(G) = d + 1$. In this regard, Blidia, Maffray, and Zemir \cite{1} showed that the Petersen graph provides a negative answer to this question. They proved that the b-chromatic number of the Petersen graph is 3, and then conjectured that the Petersen graph is the only exception. They also proved this conjecture for $d \leq 6$. In \cite{6}, Kouider proved that the b-chromatic number of any $d$-regular graph of girth at least 6 is $d + 1$. El Sahili and Kouider \cite{3} showed that the b-chromatic number of any $d$-regular graph of girth 5 that contains no 6-cycle is $d + 1$. In \cite{2}, Cabello and Jakovac proved a celebrated theorem for the b-chromatic number of regular graphs of girth 5, which guarantees that the b-chromatic number of each $d$-regular graph with girth 5, is bounded below by a linear function of $d$. They proved that a $d$-regular graph with girth at least 5, has b-chromatic number at least $\left\lceil \frac{d+1}{2} \right\rceil$. Also, they proved that for except small values of $d$, every connected $d$-regular graph that contains no 4-cycle and its diameter is at least $d$, has b-chromatic number $d + 1$. It is shown in \cite{3} that if $G$ is a $d$-regular graph that contains no 4-cycle, then $\varphi(G) \geq \left\lceil \frac{d+4}{2} \right\rceil$. This lower bound, is sharp for the Petersen graph. Besides, If $G$ has a triangle, then $\varphi(G) \geq \left\lceil \frac{d+4}{2} \right\rceil$. Also, if $G$ is a $d$-regular graph that contains no 4-cycle and $\text{diam}(G) \geq 6$, then $\varphi(G) = d + 1$.

The \textit{vertex connectivity} of a graph $G$, denoted by $\kappa(G)$, is the minimum cardinality of a subset $U$ of $V(G)$ such that $G \setminus U$ is either disconnected or a graph with only one vertex. Also, the \textit{edge connectivity} of a graph $G$, denoted by $\lambda(G)$, is the minimum cardinality of a subset $U$ of $E(G)$ such that $G \setminus U$ is either disconnected or a graph with only one vertex. It is well-known that for each graph $G$, $\kappa(G) \leq \lambda(G) \leq \delta(G)$, where $\delta(G)$ denotes the minimum degree of $G$. By an \textit{edge-cut} of $G$, we mean a subset of $E(G)$ such that deleting all of its elements from $G$, yields a disconnected graph. Therefore, for each graph $G$ with at least two vertices, $\lambda(G)$ is the minimum cardinality of all edge-cuts of $G$. For every edge-cut $T$ of $G$, $\text{sat}(T)$ stands for the set of all vertices of $G$ that are saturated by some edges in $T$, i.e., $\text{sat}(T) := \{v \mid v \in V(G), \text{there exists some} w \in V(G) \text{for which} \{v, w\} \in T\}$. We mean by a minimum edge-cut of $G$, an edge-cut of $G$ with cardinality $\lambda(G)$. A $d$-regular graph $G$, is called \textit{super-edge-connected}, if every minimum edge-cut of $G$, is the set of all edges incident with a vertex in $G$, i.e., $\lambda(G) = d$ and deleting each minimum edge-cut of $G$ from $G$, yields a graph which has an isolated vertex.
An edge-cut of a graph $G$ is called trivial whenever it is equal to the set of all edges incident with a vertex of $G$. With this terminology, a $d$-regular graph $G$ is super-edge-connected if and only if every minimum edge-cut of $G$ is trivial.

It has been proved in [8] that for any $d$-regular graph $G$ that contains no 4-cycle, if $\kappa(G) \leq \frac{d+1}{2}$, then $\varphi(G) = d + 1$. This upper bound is sharp in the sense that the vertex connectivity of the Petersen graph is $\frac{d+1}{2} + 1$; nevertheless, its b-chromatic number is not $d + 1$. Also, if $\kappa(G) < \frac{3d-3}{4}$, then $\min\{2(d - \kappa(G) + 1), d + 1\} \leq \varphi(G) \leq d + 1$. Furthermore, if there exists a subset $U$ of $V(G)$ such that $|U| = \kappa(G)$ and $G \setminus U$ has at least four connected components, then $\varphi(G) = d + 1$. Moreover, if $\kappa(G) < \frac{2d-1}{3}$ and there exists a subset $U$ of $V(G)$ such that $|U| = \kappa(G)$ and $G \setminus U$ has at least three connected components, then $\varphi(G) = d + 1$. If the girth of $G$ is 5, $\frac{3d-3}{4}$ and $\frac{2d-1}{3}$ can be replaced by $\frac{3d}{4}$ and $\frac{2d+1}{3}$, respectively.

In this paper, we investigate the b-chromatic number of $d$-regular graphs with no 4-cycles. We show that if $G$ is a $d$-regular graph that contains no 4-cycle, then $\varphi(G) = d + 1$ whenever $G$ is not super-edge-connected. Throughout the paper, for each nonnegative integer $n$, the symbol $[n]$ stands for the set $\{i \mid i \in \mathbb{N}, 1 \leq i \leq n\}$.

## 2 The Main Result

This section concerns a relation between the b-chromatic number and the edge-connectivity in $d$-regular graphs that do not contain 4-cycles. We show that every $d$-regular graph that does not contain $C_4$ as a subgraph, achieves the maximum b-chromatic number $d + 1$, unless it is super-edge-connected. In this regard, first we mention Lemma 1 and Lemma 2; the former is related to the super-edge-connected graphs without 4-cycle, and the latter presents a sufficient condition for bipartite graphs to have a perfect matching.

**Lemma 1.** Let $d \geq 4$ and $G$ be a $d$-regular graph that contains no 4-cycle, and $T$ be a minimum edge-cut of $G$ which is not trivial. Suppose that $G_1, \ldots, G_l$ are connected components of $G \setminus T$. Then for each $i$ in $\{l\}$, there exists some $a_i$ in $V(G_i) \setminus \text{sat}(T)$ such that $|N_G(a_i) \cap \text{sat}(T)| \leq 2$.

**Proof.** Let us regard an arbitrary $i$ in $\{l\}$ as fixed. Set $A_i := V(G_i) \cap \text{sat}(T)$ and $s := |V(G_i)|$. Since $T$ is not trivial, $s \geq 2$. Obviously, $s \neq 2$; otherwise, the number of edges between $V(G_i)$ and $V(G) \setminus V(G_i)$ in the graph $G$ is at least $2(d - 1)$. So $|T| \geq 2(d - 1) \geq d + 2$, a contradiction. Hence, $s \geq 3$. It is well-known that the number of edges of a graph with $n$ vertices which contains no 4-cycle is at most $\frac{s}{2}(1 + \sqrt{4n - 3})$. Since the graph $G_i$ does not contain any 4-cycles, $sd = \sum_{x \in V(G_i)} \deg_G(x) \leq \frac{s}{2}(1 + \sqrt{4s - 3}) + |T| \leq \frac{s}{2}(1 + \sqrt{4s - 3}) + d$. Hence,

\[(s - 1)d \leq \frac{s}{2}(1 + \sqrt{4s - 3}) \implies 2(s - 1)d - s \leq s\sqrt{4s - 3} \implies (2d - 1)(s - 1) - 1 \leq s\sqrt{4s - 3} \implies (2d - 2)(s - 1) < s\sqrt{4s - 3} \text{ (since } 2 < s) \implies d - 1 < \frac{s\sqrt{4s - 3}}{2(s - 1)}\sqrt{4s - 3}.

One can easily observe that the derivative of the function $f(x) = \frac{x}{2(x-1)}\sqrt{4x - 3} - (d - 1)$ is positive for $x \in [3, +\infty)$. So $f$ is strictly increasing in the interval $[3, +\infty)$.
Thus, proving \( f(d + 3) < 0 \), implies \( s \geq d + 4 \). Since the derivative of the function
\[
g(y) = \frac{y+3}{2y^2} + \sqrt{y} - (y - 1)
\]
is negative for \( y \in (0, +\infty) \) and \( g(4) = \frac{1}{3} < 0 \), we obtain that for each natural number \( d \) which \( d \geq 4 \), \( g(d) < 0 \). Hence, \( f(d + 3) = g(d) < 0 \); and therefore, \( s \geq d + 4 \). Since \( |A_i| \leq |T| \leq d, |V(G_i) \setminus A_i| \geq 4 \). Now, consider four elements \( x_1, x_2, x_3, x_4 \) in \( V(G_i) \setminus A_i \). Since \( G \) does not have any 4-cycles, there exists some \( j \in \{1, 2, 3, 4\} \) such that \( |N_G(x_j) \setminus A_i| < d \). So \( |N_G(x_j) \setminus A_i| > 0 \). Set \( N_G(x_j) \setminus A_i = \{y_k|1 \leq k \leq |N_G(x_j) \setminus A_i|\} \). Obviously, there exists some \( k \) in \( |N_G(x_j) \setminus A_i| \) such that \( |N_G(y_k) \setminus (A_i \setminus N_G(x_j))| \leq 1 \); otherwise, \( d \geq |A_i| \geq |N_G(x_j) \setminus A_i| + 2|N_G(x_j) \setminus A_i| = d + |N_G(x_j) \setminus A_i| > d \), which is impossible. We conclude that there exists an element \( y_k \) in \( N_G(x_j) \setminus A_i \) for which \( |N_G(y_k) \setminus (A_i \setminus N_G(x_j))| \leq 1 \). Since \( y_k \) has at most one neighbor in \( N_G(x_j) \setminus A_i \); hence, \( y_k \) has at most two neighbors in \( A_i \). Therefore, \( a_i := y_k \) is a desired vertex.

Lemma 2. \([2]\) Let \( H \) be a bipartite graph with parts \( U \) and \( V \) such that \( |U| = |V| \). Let \( u^* \in U \) and \( v^* \in V \). If for each vertex \( x \) in \( V(H) \setminus \{u^*, v^*\} \), \( \deg_H(x) \geq \frac{|V|}{2} \), \( \deg_H(u^*) > 0 \), and \( \deg_H(v^*) > 0 \), then \( H \) has a perfect matching.

We are now in a position to prove the main result of the paper.

Theorem 1. Let \( G \) be a \( d \)-regular graph that contains no 4-cycle. If \( G \) is not super-edge-connected, then \( \varphi(G) = d + 1 \).

Proof. There is nothing to prove when \( d \in \{0, 1, 2\} \). Also, Jakovac and Klavžar, in \([5]\), showed that the only cubic graph that contains no 4-cycle and its b-chromatic number is not equal to 4, is the Petersen graph. Since the Petersen graph is super-edge-connected, the proof is completed for \( d = 3 \). So we suppose that \( d \geq 4 \). Since \( G \) is not super-edge-connected, there exists a minimum edge-cut \( T \) of \( G \) which is not trivial. Suppose that \( G_1, \ldots, G_l \) are connected components of \( G \setminus T \). For each \( i \) in \([l]\), define \( A_i := \{x \mid x \in V(G_i), \exists y \in V(G) \setminus V(G_i) \text{ such that } (x, y) \in T\} \). According to the Lemma 1 for each \( i \) in \([l]\), there exists some \( a_i \) in \( V(G_i) \setminus A_i \) for which \( |N_G(a_i) \setminus A_i| \leq 2 \).

Let \( x_1, \ldots, x_{d - |N_G(a_1) \setminus A_1|} \) be an arbitrary ordering of all elements of \( N_G(a_1) \setminus A_1 \). Color the vertex \( a_1 \) by color \( 1 \), and for each \( i \) in \([|N_G(a_1) \setminus A_1]|\), assign the color \( i + 1 \) to the vertex \( x_i \). Also, color all vertices that are in the set \( N_G(a_1) \cap A_1 \) by all colors that are in the set \( \{d + 1\} \setminus |N_G(a_1) \setminus A_1| + 1 \} \) injectively. Then, color all vertices that are in the set \( N_G(a_2) \cap A_2 \) by some colors that are in the set \( \{1, \lfloor \frac{d+2}{2} \rfloor \} \) injectively.

The vertex \( a_1 \) is a color-dominating vertex with color \( 1 \). Now, our task is to color all the vertices in \( \bigcup_{i=1}^{\lfloor \frac{d}{2} \rfloor} N_G(x_i) \setminus \{a_1 \cup N_G(a_1)\} \) by colors in \( \{d + 1 \} \) such that for each \( i \) in \([\lfloor \frac{d}{2} \rfloor]\), all colors that are in the set \( \{d + 1 \} \setminus i + 1 \} \), appear on \( N_G(x_i) \).

For each \( i \) in \([\lfloor \frac{d}{2} \rfloor]\), set \( V_i, S_i, \) and \( C_i, \) as follows:

- \( V_i := N_G(x_i) \setminus \{a_1 \cup N_G(a_1)\} \);
- \( S_i := \{a_1 \cup N_G(a_1) \cup \bigcup_{j=1}^{i} V_j \} \cup (N_G(a_2) \cap A_2) \);
- \( C_i := \{d + 1 \} \setminus \{1\} \setminus \)the set of colors that were appeared on \( \{x_i \} \cup (N_G(x_i) \cap N_G(a_1)) \).
Since $G$ contains no 4-cycle, for any two distinct natural numbers $i$ and $j$ in $[\lfloor \frac{d}{2} \rfloor]$, \( V_i \cap V_j = \emptyset \). Also, since $G$ contains no 4-cycle, the maximum degree of the induced subgraph of $G$ on $N_G(a_1)$ is at most one. So $|V_i| = |C_i| = d - 1$ or $|V_i| = |C_i| = d - 2$. Moreover, $|V_i| = |C_i| = d - 2$ if and only if $|N_G(x_i) \cap N_G(a_1)| = 1$. Now, we follow $[\frac{d}{2}]$ steps inductively. For each $i$ in $[\lfloor \frac{d}{2} \rfloor]$, at $i$-th step, we only color all vertices that are in $V_i$ by all colors that are in $C_i$ injectively. Suppose by induction that $1 \leq i \leq [\frac{d}{2}]$ and for each $k$ in $[i - 1]$, at $k$-th step, we have only colored all vertices that are in $V_k$ by all colors that are in $C_k$ injectively in such a way that the resulting partial coloring on $S_k$ is a proper coloring. Now, at $i$-th step, we want to color only all vertices that are in $V_i$ by all colors that are in $C_i$ injectively such that the resulting partial coloring on $S_i$ be a proper partial coloring. Consider a bipartite graph $H_i$ with one part $V_i$ and the other part $C_i$, which a vertex $v$ in $V_i$ is adjacent to a color $c$ in $C_i$ in the graph $H_i$ if and only if (in the graph $G$) $v$ does not have any neighbors in $S_i$ already colored by $c$. Such a coloring of all vertices that are in $V_i$ by all colors that are in $C_i$ (as mentioned) exists if and only if $H_i$ has a perfect matching.

Let $v$ be an arbitrary element of $V_i$. The set of neighbors of $v$ in the graph $G$ that were already colored, is a subset of $\{x_i\} \cup \bigcup_{j=1}^{i-1} V_j \cup (N_G(a_2) \cap A_2)$. Since $G$ contains no 4-cycle, for each $j$ in $[i - 1]$, $v$ has at most one neighbor in $V_j$. Also, $v$ has at most one neighbor in $N_G(a_2) \cap A_2$. However, the color of the vertex $x_i$ does not belong to $C_i$. Therefore, $deg_{H_i}(v) \geq |C_i| - i$. Also, since for each $j$ in $[i - 1]$, $v_j$ sees all colors of $[d + 1]$ on its closed neighborhood, each color of $[d + 1]$ appears at most once on $V_j$. Besides, each color of $[d + 1]$ appears at most once on $N_G(a_2) \cap A_2$. Therefore, for each $c$ in $C_i$, $deg_{H_i}(c) \geq |V_i| - i$. Hence, for each $v$ in $V(H_i)$, $deg_{H_i}(v) \geq |V_i| - i$.

Since $|V_i| \geq d - 2$, if $1 \leq i \leq [\frac{d - 2}{2}]$, then $deg_{H_i}(v) \geq |V_i| - i \geq \frac{V_i}{2}$. In the case $i = [\frac{d}{2}]$, since the color of each vertex in $N_G(a_2) \cap A_2$ belongs to the set $\{1, [\frac{d - 2}{2}]\}$ and also $C_i \cap \{1, [\frac{d - 2}{2}]\} = \emptyset$, for each $v$ in $V(H_i)$, $deg_{H_i}(v) \geq |V_i| - (i - 1) \geq \frac{|V_i|}{2}$.

So Lemma 2 implies that $H_i$ has a perfect matching and we are done. We conclude that there exists a partial coloring on $S_{[\frac{d}{2}]}$ by all colors of the set $[d + 1]$, such that $a_1, x_1, \ldots, x_{[\frac{d}{2}]}$ are color-dominating vertices whose colors are $1, 2, \ldots, [\frac{d + 2}{2}]$, respectively.

The next task is to color some uncoded vertices from $V(G_2)$ in such a way that all colors in $[d + 1] \setminus [\lfloor \frac{d + 2}{2} \rfloor]$ realize. The procedure is to find $[\frac{d - 1}{2}]$ suitable vertices $z_{1}, \ldots, z_{[\frac{d - 1}{2}]}$ in $N_G(a_2) \setminus A_2$ in order to make them along $a_2$ color-dominating.

Let $N_G(a_2) \setminus A_2 = \{y_i \mid 1 \leq i \leq |N_G(a_2) \setminus A_2|\}$. For each $i$ in $[|N_G(a_2) \setminus A_2|]$, set $W_i$ and $e_{yi}$ as follows:

- $W_i := N_G(y_i) \setminus (\{a_2\} \cup N_G(a_2))$;
- $e_{yi} := \{(s, t) \mid (s, t) \in E(G), s \in W_i, t \in A_1\}$.

Since $G$ does not contain any 4-cycles, for any two different natural numbers $i$ and $j$ in $[|N_G(a_2) \setminus A_2|]$, $W_i \cap W_j = \emptyset$. Without loss of generality, we can assume that the sequence $\{e_{yi}\}_{i = 1}^{|N_G(a_2) \setminus A_2|}$ is decreasing, i.e., $e_{y_1} \geq \cdots \geq e_{y_{|N_G(a_2) \setminus A_2|}}$. We show that for each natural number $i$, $|N_G(a_2) \setminus A_2| - \lfloor \frac{d - 1}{2} \rfloor + 1 \leq i \leq |N_G(a_2) \setminus A_2|$, $e_{yi} \leq 1$. 

\[5\]
Thus, there exists at most one element in $e_i$ if and only if $d_{i} \leq |H|$ perfect matching in $G$. Suppose, on the contrary, that for some natural number $i$, $|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor + 1 \leq i \leq |N_{G}(a_{2}) \setminus A_{2}|$, $e_{y_{i}} > 1$. Therefore, for each $j$ in $[|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor + 1]$, $e_{y_{j}} \geq 2$. Since each vertex in $N_{G}(a_{2}) \cap A_{2}$ is incident with an edge of $T$, so

$$d \geq |T| \geq |N_{G}(a_{2}) \cap A_{2}| + 2(|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor + 1) =$$

$$|N_{G}(a_{2}) \cap A_{2}| + 2(d - |N_{G}(a_{2}) \cap A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor) \geq$$

$$d + 3 - |N_{G}(a_{2}) \cap A_{2}| \geq d + 3 - 2 = d + 1,$$

which is impossible. Accordingly, for each natural number $i$, $|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor + 1 \leq i \leq |N_{G}(a_{2}) \setminus A_{2}|$, $e_{y_{i}} \leq 1$. For each $i$ in $[|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor + 1]$, put $z_{i} := y_{i}[N_{G}(a_{2}) \setminus A_{2}] - \lfloor \frac{d_{i}}{2} \rfloor + i$; therefore, $e_{z_{i}} \leq 1$.

Now, color the vertex $a_{2}$ by color $d + 1$ and for each $i$, $1 \leq i \leq \lfloor \frac{d_{i}}{2} \rfloor$, assign the color $\lfloor \frac{d_{i} + 2}{2} \rfloor + i$ to the vertex $z_{i}$. Also, color all vertices that are in the set $N_{G}(a_{2}) \setminus (A_{2} \cup \{z_{i} | 1 \leq i \leq \lfloor \frac{d_{i}}{2} \rfloor\})$ by some colors of $[d + 1]$ injectively in such a way that all colors of the set $[d + 1]$ appear on the closed neighborhood of $a_{2}$.

For each $i$ in $[|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i}}{2} \rfloor]$, define $V_{i}^{\prime}$, $S_{i}^{\prime}$, and $C_{i}^{\prime}$, as follows:

- $V_{i}^{\prime} := N_{G}(z_{i}) \setminus (\{a_{2}\} \cup N_{G}(a_{2}))$;
- $S_{i}^{\prime} := \{a_{2}\} \cup N_{G}(a_{2}) \cup (\bigcup_{j=1}^{i} V_{j}^{\prime}) \cup S_{\{4\}}^{\prime}$;
- $C_{i}^{\prime} := [d] \setminus (\text{the set of colors that were appeared on } \{z_{i}\} \cup (N_{G}(z_{i}) \cap N_{G}(a_{2})))$.

The maximum degree of the induced subgraph of $G$ on $N_{G}(a_{2})$ is at most one. So $|V_{i}^{\prime}| = |C_{i}^{\prime}| = d - 1$ or $|V_{i}^{\prime}| = |C_{i}^{\prime}| = d - 2$. Furthermore, $|V_{i}^{\prime}| = |C_{i}^{\prime}| = d - 2$ if and only if $|N_{G}(z_{i}) \cap N_{G}(a_{2})| = 1$. Now, we follow $\lfloor \frac{d_{i} - 1}{2} \rfloor$ steps inductively. For each $i$ in $[|N_{G}(a_{2}) \setminus A_{2}| - \lfloor \frac{d_{i} - 1}{2} \rfloor]$, at $i$-th step, we only color all vertices that are in $V_{i}^{\prime}$ by all colors that are in $C_{i}^{\prime}$ injectively. Suppose by induction that $1 \leq i \leq \lfloor \frac{d_{i} - 1}{2} \rfloor$ and for each $k$ in $[i - 1]$, at $k$-th step, we have only colored all vertices that are in $V_{k}^{\prime}$ by all colors in $C_{k}^{\prime}$ injectively in such a way that the resulting partial coloring on $S_{k}^{\prime}$ is a proper coloring. Now, at $i$-th step, we aim to color only all vertices that are in $V_{i}^{\prime}$ by all colors that are in $C_{i}^{\prime}$ injectively such that the resulting partial coloring on $S_{i}^{\prime}$ be a proper partial coloring. Consider a bipartite graph $H_{i}^{\prime}$ with one part $V_{i}^{\prime}$ and the other part $C_{i}^{\prime}$, that a vertex $v$ in $V_{i}^{\prime}$ is adjacent to a color $c$ in $C_{i}^{\prime}$ in the graph $H_{i}^{\prime}$, if and only if (in the graph $G$) $v$ does not have any neighbors in $S_{i}^{\prime}$ already colored by $c$. Such a coloring of all vertices that are in $V_{i}^{\prime}$ by all colors that are in $C_{i}^{\prime}$ exists if and only if $H_{i}^{\prime}$ has a perfect matching. So our goal is to prove the existence of a perfect matching in $H_{i}^{\prime}$. In this regard, we again apply Lemma \(\mathbb{2}\).

Since $e_{z_{i}} \leq 1$, there exists at most one edge between $V_{i}^{\prime}$ and $A_{1}$ in the graph $G$. Thus, there exists at most one element in $V_{i}^{\prime}$ that has a neighbors in $V(G_{1})$ (in the graph $G$). If such a vertex exists, call it $v^{\ast}$. Therefore, in the graph $H_{i}^{\prime}$, the degree of each vertex in $V_{i}^{\prime} \setminus \{v^{\ast}\}$ is at least $|C_{i}^{\prime}| - (i - 1)$. Similarly, there exists at most one color in $C_{i}^{\prime}$ for which there is an element in $V_{i}^{\prime}$ that has a neighbor in $V(G_{1})$ with this color. If such a color exists, call it $c^{\ast}$. Hence, in the graph $H_{i}^{\prime}$, the degree of each vertex in $C_{i}^{\prime} \setminus \{c^{\ast}\}$ is at least $|V_{i}^{\prime}| - (i - 1)$. We conclude that for each vertex $x$ in $V_{H_{i}^{\prime}} \setminus \{v^{\ast}, c^{\ast}\}$, $deg_{H_{i}^{\prime}}(x) \geq |V_{i}^{\prime}| - (i - 1)$; and since $i \leq \lfloor \frac{d_{i} - 1}{2} \rfloor$ and $|V_{i}^{\prime}| \geq d - 2$,
\[ \text{deg}_{H_i'}(x) \geq \frac{|V_i'|}{2}. \] Also, the degree of each vertex of \( H_i' \) is at least \(|V_i'| - i\) which is positive.

We conclude that \(|V_i'| = |C_i'|\) and the degree of each vertex in \( V_{H_i'} \setminus \{v^*, c^*\} \) is at least \(|V_i'|/2\). Also, the degree of each vertex of \( H_i' \) is positive. Accordingly, the Lemma \( \sqrt{2} \) implies that \( H_i' \) has a perfect matching. Therefore, there exists a partial coloring on \( S_{\lceil d-1/2 \rceil} \) by all colors of the set \([d + 1]\) such that all colors realize. This partial coloring can be extended to a coloring of the graph \( G \) greedily. So \( \varphi(G) = d + 1 \). 

\[ \blacksquare \]

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