On multivariate Newton-like inequalities

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Abstract

We study multivariate entire functions and polynomials with non-negative coefficients. A class of Strongly Log-Concave entire functions, generalizing Minkowski volume polynomials, is introduced: an entire function \( f \) in \( m \) variables is called Strongly Log-Concave if the function \( (\partial x_1)^{c_1} \cdots (\partial x_m)^{c_m} f \) is either zero or

\[
\log((\partial x_1)^{c_1} \cdots (\partial x_m)^{c_m} f)
\]

is concave on \( \mathbb{R}^m_+ \). We start with yet another point of view (of propagation) on the standard univariate (or homogeneous bivariate) Newton Inequalities. We prove analogues of the Newton Inequalities in the multivariate Strongly Log-Concave case. One of the corollaries of our new Newton-like inequalities is the fact that the support \( \text{supp}(f) \) of a Strongly Log-Concave entire function \( f \) is discretely convex (\( D \)-convex in our notation). The proofs are based on a natural convex relaxation of the derivatives \( \text{Der}_f(r_1, ..., r_m) \) of \( f \) at zero and on the lower bounds on \( \text{Der}_f(r_1, ..., r_m) \), which generalize the Van Der Waerden-Falikman-Egorychev inequality for the permanent of doubly-stochastic matrices. A few open questions are posed in the final section.

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1 Introduction

This paper is concerned with multivariate polynomials and entire functions with nonnegative real coefficients. (All Taylor’s series in this paper are taken at zero.) We continue the research, initiated in the recent papers [9], [10], [11], [7], [12] by the present author, on “combinatorics and combinatorial applications hidden in certain homogeneous polynomials with non-negative coefficients.” Essentially, the main goal here is understanding how far one can push the approach from the above mentioned papers. The following definition introduces the main notation of the paper.

Definition 1.1:

1. We denote by $\text{Sim}(n)$ the standard simplex in $\mathbb{R}^n$:
   \[ \text{Sim}(n) = \{(a_1, ..., a_n) : a_i \geq 0, 1 \leq i \leq n; \sum_{1 \leq i \leq n} a_i = 1. \}
   
2. We denote by $\text{Pol}_+(m, n)$ the convex cone of polynomials with nonnegative coefficients in $m$ variables of total degree $n$; the corresponding convex cone of homogeneous polynomials is denoted as $\text{Hom}_+(m, n)$.
   
   We denote by $\text{Ent}_+(m)$ the convex cone of entire functions on $\mathbb{C}^m$ with nonnegative Taylor’s series.

3. An entire function $f \in \text{Ent}_+(m)$ is called Strongly Log-Concave if for all integer vectors $(c_1, ..., c_m) \in \mathbb{Z}^m$ the function $(\partial x_1)^{c_1}...(\partial x_m)^{c_m} f$ is either zero or $\log((\partial x_1)^{c_1}...(\partial x_m)^{c_m} f)$ is concave on $\mathbb{R}^m$. A set of Strongly Log-Concave polynomials $p \in \text{Pol}_+(m, n)$ is denoted as $\text{SLC}(m, n)$ and a set of Strongly Log-Concave entire functions $f \in \text{Ent}_+(m)$ is denoted as $\text{SLC}(m)$.

4. A (discrete) subset $S \subset \mathbb{Z}^m$ is called D-convex if
   \[ \text{Conv}(S) \cap \mathbb{Z}^m = S, \]
   
   where $\text{Conv}(S)$ is the convex hull of $S$ and $\mathbb{Z}^m$ is the $m$-dimensional integer lattice.
   
   A map $G : \mathbb{Z}^m \to [-\infty, +\infty]$ is called D-concave if
   \[ G(\sum_{1 \leq i \leq k < \infty} a_i Y_i) \geq \sum_{1 \leq i \leq k < \infty} a_i G(Y_i) \]
   
   for all sequences $(a_1, ..., a_k) \in \text{Sym}(k)$ and all vectors $Y_1, ..., Y_k \in \mathbb{Z}^m$ such that $\sum_{1 \leq i \leq k < \infty} a_i Y_i \in \mathbb{Z}^m$.

   Our notion of D-convexity coincides with the notion of pseudo-convexity from [3]. As the term “pseudo-convex” is already occupied in the complex analysis, we think that the term D-convexity is more appropriate (and informative).
5. The support of an entire function

\[ f(x_1, \ldots, x_m) = \sum_{(r_1, \ldots, r_m) \in \mathbb{Z}_+^m} a_{r_1, \ldots, r_m} \prod_{1 \leq i \leq m} x_i^{r_i} \]  

is defined as \( \text{supp}(f) = \{(r_1, \ldots, r_m) : a_{r_1, \ldots, r_m} \neq 0\} \).

6. For an entire function \( f \in \text{Ent}_+(m) \) and an integer vector \( R = (r_1, \ldots, r_m) \in \mathbb{Z}_+^m \) we define \( \text{Der}_f(R) = (\partial x_1)^{r_1} \cdots (\partial x_m)^{r_m}f(0) \). In the notation of (1), \( \text{Der}_f(R) = a_{r_1, \ldots, r_m} \prod_{1 \leq i \leq m} r_i! \).

**Example 1.2:**

1. First, we note that a homogeneous polynomial \( p \in \text{Hom}_+(m, n) \) is log-concave on \( R_+^m \) if and only if the function \( p^\frac{1}{n} \) is concave on \( R_+^n \).

2. A natural class of **Strongly Log-Concave** homogeneous polynomials in \( \text{Hom}_+(m, n) \) consists of \( \text{H-Stable} \) polynomials: a polynomial \( p \in \text{Hom}_C(m, n) \) is called **H-Stable** if \( p(Z) \neq 0 \) provided \( \text{Re}(Z) > 0 \). It is easy to show and is well known that if \( p \in \text{Hom}_C(m, n) \) is \( \text{H-Stable} \) then the polynomial \( \frac{p}{p(x_1, \ldots, x_m)} \in \text{Hom}_+(m, n) \) for any positive real vector \( (x_1, \ldots, x_m) \) and \( \langle \partial x_i \rangle p \) is either zero or \( \text{H-Stable} \). Consider an univariate polynomial \( R(t) = \sum_{0 \leq i \leq k} a_i t^i, a_k \neq 0 \) and the associated homogeneous polynomial \( p \in \text{Hom}_+(2, n) \). \( p(x, y) = \sum_{0 \leq i \leq k} a_i x^i y^{n-i} \).

Then \( p \) is **H-Stable** iff the roots of \( R \) are non-positive real numbers, which shows that \( \text{H-Stable} \) polynomials are **Strongly Log-Concave**.

3. Another, different from \( \text{H-Stable} \), class of **Strongly Log-Concave** homogeneous polynomials in \( \text{Hom}_+(m, n) \) consists of Minkowski polynomials \( \text{Vol}_n(\sum_{1 \leq i \leq m} x_i K_i) \), where \( \text{Vol}_n \) stands for the standard volume in \( R^n \) and \( K_1, \ldots, K_m \) are convex compact subsets of \( R^n \). The **Strong Log-Concavity** of Minkowski polynomials is essentially equivalent to the famous **Alexandrov-Fenchel** inequalities \( \prod \) for the mixed volumes.

**Remark 1.3:** \( \text{H-Stable} \) and Minkowski polynomials satisfy a seemingly stronger property: they are invariant with respect to the changes of variables \( Y = AX \), where \( A \) is a rectangular matrix with **non-negative** entries and without zero rows. We don’t know whether such invariance holds in the general **Strongly Log-Concave** case.
Spelling out the definition of $D$-concavity gives us a reformulation of the famous Newton’s inequalities.

In the case of **Strongly Log-Concave** multivariate entire functions, the map $\log(Der f)$ is not necessary $D$-concave.

We introduce the following map

$$C_f(r_1, ..., r_m) = \inf_{x_i > 0} \frac{f(x_1, ..., x_m)}{\prod_{1 \leq i \leq m} (\frac{x_i}{r_i})^{r_i}}, (r_1, ..., r_m) \in Z_+^m$$  \hspace{1cm} (2)

It is easy to show that if $f \in \text{Ent}_+(m)$ and $\log(f)$ is concave on $R^m_+$ then $\log(C_f)$ is $D$-concave. Therefore, the $D$-convexity of the support $\text{supp}(f)$ would follow from the property

$$C_f(R) > 0 \iff Der f(R) > 0. \hspace{1cm} (3)$$

We prove in this paper a sharp quantitative version of (3):

$$\prod_{1 \leq i \leq m} \frac{r_i}{r_i} C_f(R) \geq Der f(R) \geq \exp \left( - \left( \sum_{1 \leq i \leq m} r_i \right) \right) C_f(R). \hspace{1cm} (4)$$

The inequalities (4) (and their more refined versions) generalize the Van Der Waerden-Falikman-Egorychev lower bound on the permanent of doubly-stochastic matrices [6], [5] and used in this paper to prove Newton-like inequalities for **Strongly Log-Concave** entire functions.

### 2 Univariate Newton-like Inequalities

#### 2.1 Propagatable sequences (weights)

**Definition 2.1:** Let us define the following closed subset of $R^{n+1}$ of log-concave sequences:

$$LC = \{(d_0, ..., d_n) : d_i \geq 0, 0 \leq i \leq n; d_i^2 \geq d_{i-1}d_{i+1}, 1 \leq i \leq n-1\}.$$  

We also associate with a given positive vector $(c_0, ..., c_n)$ the weighted shift operator $Shift_c : R^{n+1} \rightarrow R^{n+1},$

$$Shift_c((x_0, ..., x_n)^T) = (c_0x_1, ..., c_{n-1}x_n, 0)^T.$$  

If $c$ is the vector of all ones, then $Shift_c =: Shift.$

A positive finite sequence $(b_0, ..., b_n)$ is called **propagatable** if the following implication holds:

$$(p^{(0)}(0)b_0, ..., p^{(n)}(0)b_n) \in LC \implies (p^{(0)}(t)b_0, ..., p^{(n)}(t)b_n) \in LC, t \geq 0, \hspace{1cm} (5)$$

where $p$ is a polynomial of degree at most $n.$  

Analogously, we define infinite **propagatable** sequences by considering infinite log-concave sequences and entire functions in (5).
Proposition 2.2: Let $c_0, \ldots, c_{n-1}$ be a nonnegative sequence. Then $\exp(t(\text{Shift}_c))(L) \subset L$ for all $t \geq 0$ if and only if

$$2c_i \geq c_{i+1} + c_{i-1}, 1 \leq i \leq n-2; 2c_{n-1} \geq c_{n-2}.$$  

(In other words, the infinite sequence $(c_0, \ldots, c_{n-1}, 0, \ldots)$ is concave).

Proof:

1. The "only if" part: Consider the linear system of differential equations:

$$X'(t) = \text{Shift}_c X(t) : X(0) = (1, 1, \ldots, 1), X(t) = (X_0(t), \ldots, X_n(t)).$$

Suppose that $\exp(t(\text{Shift}_c))(L) \subset L, t \geq 0$, i.e $X(t) \in L : t \geq 0$.

Define the following smooth functions:

$$r_i(t) = (X_i(t))^2 - X_{i+1}(t)X_{i-1}(t), 1 \leq i \leq n - 1.$$  

It follows that $r_0(0) = 0$ and $r_i(t) \geq 0, t \geq 0$. Therefore $r'_i(0) \geq 0$. Thus

$$0 \leq r'_i(0) = 2c_i - c_{i+1} - c_{i-1}, 1 \leq i \leq n - 2; 0 \leq r'_{n-1}(0) = 2c_{n-1} - c_{n-2}.$$  

2. The "if" part: As $\exp(A) = \lim_{n \to \infty} (I + \frac{A}{n})^n$, thus it is sufficient to prove that $(I + t(\text{Shift}_c))(L) \subset L$ for all $t \geq 0$, which is done by straightforward derivations.

Remark 2.3: The observation that $(I + \text{Shift})(L) \subset L$ is probably well known; we have learned it from Julius Borcea.

Theorem 2.4: Let $(b_0, \ldots, b_k)$ be a positive sequence. Define $c_i = \frac{b_i}{b_{i+1}}, 0 \leq i \leq k - 1$. The sequence $(b_0, \ldots, b_k)$ is propagatable iff the infinite sequence $(c_0, \ldots, c_{k-1}, 0, \ldots)$ is concave.

Proof: Define a vector function $\text{Mom}_b(t) = (b_0p^{(0)}(t), \ldots, b_np^{(n)}(t))$. Clearly, $\text{Mom}_b(t)$ solves the following system of linear differential equations:

$$\text{Mom}_b(t)' = \text{Shift}_c(\text{Mom}_b(t)).$$

Therefore $(b_0, \ldots, b_n)$ is propagatable iff $\exp(t(\text{Shift}_c))(L) \subset L$ for all $t \geq 0$. The result now follows from Proposition (2.2).

The following result follows fairly directly from Theorem (2.4).

Corollary 2.5: Let $(b_0, \ldots, b_k, \ldots)$ be a positive infinite sequence. Define $c_i = \frac{b_i}{b_{i+1}}, 0 \leq i < \infty$. The sequence $(b_0, \ldots, b_k, \ldots)$ is propagatable iff the infinite sequence $(c_0, \ldots, c_{k-1}, \ldots)$ is concave.
Example 2.6: A polynomial $p(t) = \sum_{0 \leq i \leq k} a_i t^i$ with nonnegative coefficients is called $n$-Newton for $n \geq k$ if
\[ d_i^2 \geq d_{i-1} d_{i+1} : 1 \leq i \leq k-1, d_i = \frac{a_i}{(n)_i}. \] (6)

Or, in other words, the vector $(p^{(0)}(0)b_0, \ldots, p^{(k)}(0)b_k) \in LC$, where $b_i = (n-i)!$.

As $c_i = \frac{b_i}{b_{i+1}} = n - i : 0 \leq i \leq k - 1$ hence it follows from Theorem (2.4) that $(p^{(0)}(t)b_0, \ldots, p^{(k)}(t)b_k) \in LC : t \geq 0$. Equivalently,
\[ (p^{(i+1)}(t))^2 \geq \frac{n-i}{n-i-1} p(i)(t)p^{(i+2)}(t) : t \geq 0, i \leq k - 2, \] (7)

which means that the functions $n^{-i}/p^{(i)} : 0 \leq i \leq k$ are concave on $R_+$. 

Let $f \in Ent_+(1)$ be entire univariate function, $f(t) = \sum_{0 \leq i < \infty} a_i t^i$.
A natural generalization of the $n$-Newton property, i.e. when $n \to \infty$, is the log-concavity of the infinite sequence $f^{(0)}(0), \ldots, f^{(k)}(0), \ldots$. Corollary (2.5) proves that this property is equivalent to Strong Log-Convexity of $f$.

We collect the above observations in the following proposition.

Proposition 2.7:
1. A polynomial $p$ with nonnegative coefficients is $n$-Newton, where $n \geq \deg(p)$, iff the functions $n^{-i}/p^{(i)} : 0 \leq i \leq k$ are concave on $R_+$.

Let us $n$-homogenize the univariate polynomial $p$, i.e. put $R(x, y) = y^n p\left(\frac{x}{y}\right)$. Then, $R \in Hom_+(2, n)$ and the functions $n^{-i}/p^{(i)} : 0 \leq i \leq k$ are concave on $R_+$ if and only if the polynomial $R$ is Strongly Log-Convex.

2. An entire function $f \in Ent_+(1)$ is Strongly Log-Convex iff the infinite sequence $f^{(0)}(0), \ldots, f^{(k)}(0), \ldots$ is log-concave.

Remark 2.8: The standard Newton Inequalities correspond to the case $n = \deg(p)$ and hold if, for instance, the roots of $p$ are real. It was proved by G. C. Shephard in [23] that a polynomial $p$ is $n$-Newton iff $p(t) = \text{Vol}_n(tK_1 + K_2)$ for some convex compact subsets(simplices) $K_1, K_2 \subset R^n$. This remarkable result can be used (see [14] and [15]) for alternative short proofs of Proposition (2.7) and Liggett’s convolution theorem, which states that $pq$ is $m + n$-Newton provided that $p$ is $n$-Newton and $p$ is $m$-Newton.

The literature on univariate Newton Inequalities is vast, we refer the reader to the recent survey [20]. But the results presented here seem to be new, nothing of the kind is mentioned in [20].
3 Multivariate Case

The main upshot of Proposition 2.7 is that in the univariate case as well in the bivariate homogeneous case the following equivalence holds:

“$f$ is Strongly Log-Concave” $\iff$ “the map $\log(Der_f)$ is $D$-concave”.

In the general multivariate case both implication fail.

Example 3.1:

1. Consider the polynomial $p(x_1, ..., x_{2n}) = (x_1 + x_2)(x_2 + x_3)...(x_{2n-1} + x_{2n})(x_{2n} + x_1)$. Clearly, it is H-Stable. Consider three vectors: $R_0 = (1, ..., 1)$, $R_1 = (2, 0, 2, ..., 0, 2)$, $R_2 = (0, 2, ..., 0, 2)$. By direct inspection, $Der_p(R_0) = 2$, $Der_p(R_1) = Der_p(R_2) = 2^n$. Which gives

$$\log(Der_p(\frac{1}{2} (R_1 + R_2))) = \frac{1}{2} (\log(Der_p(R_1)) + \log(Der_p(R_2))) - (n-1) \log(2). \quad (8)$$

2. Alexandrov-Fenchel Inequalities.

Consider a homogeneous Strongly Log-Concave polynomial $p \in Hom_+(m, n)$ and fix a non-negative integer vector $R = (r_1, r_2, ..., r_m), \sum_{1 \leq i \leq m} r_i = m$. Define the following polynomial $q \in Hom_+(2, n - \sum_{3 \leq i \leq m} r_i)$,

$$q(x_1, x_2) = (\partial x_3)^{r_3}...(\partial x_m)^{r_m}p(x_1, x_2, 0, ..., 0).$$

Then $q$ is either zero or Strongly Log-Concave. This observation leads to the following inequalities: if both vectors

$$R_1 = (r_1 + 1, r_2 - 1, r_3, ..., r_m), R_2 = (r_1 - 1, r_2 + 1, r_3, ..., r_m)$$

are non-negative then

$$Der_p(R) = Der_p(\frac{1}{2}(R_1 + R_2)) \geq (Der_p(R_1))^\frac{1}{2}(Der_p(R_2))^\frac{1}{2} \quad (9)$$

3. Consider $p \in Hom_+(4, 4), p(x_1, x_2, x_3, x_4) = x_1x_2x_3x_4 + \frac{1}{4}(x_1x_2)^2 + (x_3x_4)^2)$. Here the map $\log(Der_f)$ is $D$-concave but the polynomial $p$ is not log-concave on $R^4_+$. We prove in this paper that in the general multivariate case if $f$ is Strongly Log-Concave then the map $\log(Der_f)$ is “almost” $D$-concave.
3.1 Generalized Van Der Waerden-Falikman-Egorychev lower bounds

This section follows the recent inductive approach by the author [10].

**Definition 3.2:** For an entire function \( f \in \text{Ent}_+(n) \) we define its **Capacity** as

\[
\text{Cap}(f) = \inf_{x_i > 0} \frac{p(x_1, \ldots, x_n)}{\prod_{1 \leq i \leq n} x_i}
\]

We need the following elementary result:

**Lemma 3.3:** Consider a function \( f : \mathbb{R}_+ \to \mathbb{R}_+ \) such that the derivative \( f'(0) \) exists.

1. If \( f^{\frac{1}{k}} \) is concave on \( \mathbb{R}_+ \) for \( k > 1 \) then \( f'(0) \geq (\frac{k-1}{k})^{k-1} \inf_{t > 0} \frac{f(t)}{t} \).

2. If \( f \) is log-concave on \( \mathbb{R}_+ \) then \( f'(0) \geq \frac{1}{e} \inf_{t > 0} \frac{f(t)}{t} \).

   If, additionally, the function \( f \) is analytic and \( f'(0) = \frac{1}{e} \inf_{t > 0} \frac{f(t)}{t} \) then \( f(t) = \exp(at) \), \( a > 0 \).

3. Let \( R(t) = a_0 + \ldots + a_n t^n \) be a strongly log-concave on \( \mathbb{R}_+ \) univariate polynomial with nonnegative coefficients:

   \[ G(i)^2 \geq G(i-1)G(i+1) : 1 \leq i \leq n - 1, G(i) = a_i! \]

   Then \( f'(0) \geq L(n) \inf_{t > 0} \frac{f(t)}{t} \), where \( L(n) = (\inf_{t > 0} \frac{\exp_n(t)}{t})^{-1} \) and the truncated exponential is defined as \( \exp_n(t) = 1 + \ldots + \frac{1}{n!} t^n \). (Note that \( \exp_n \) is strongly log-concave on \( \mathbb{R}_+ \).

**Proof:**

1. If \( f(0) = 0 \) then, obviously, \( f'(0) \geq \inf_{t > 0} \frac{f(t)}{t} \). Therefore, we can assume that \( f(0) = 1. \)

   As \( f^\frac{1}{k} \) is concave and non-negative on \( \mathbb{R}_+ \) thus \( f(t) \leq (1 + \frac{f'(0)}{k} t)^k, t \geq 0. \)

   The standard calculus gives us for \( l(t) = (1 + \frac{f'(0)}{k} t)^k \) that

   \[
   \inf_{t > 0} \frac{l(t)}{t} = f'(0)(g(k))^{-1}, g(k) = \left(\frac{k-1}{k}\right)^{k-1}.
   \]

   As \( \inf_{t > 0} \frac{f(t)}{t} \leq \inf_{t > 0} \frac{l(t)}{t} \), we deduce that \( f'(0) \geq g(k) \inf_{t > 0} \frac{f(t)}{t} \).

2. As in the proof above, we can assume that \( f(0) = 1. \) It follows from the log-concavity that \( f(t) \leq \exp(f'(0)t), t \geq 0. \) It is easy to see that

   \[
   \inf_{t > 0} \frac{f(t)}{t} \leq \inf_{t > 0} \frac{\exp(f'(0)t)}{t} = f'(0)\exp(1) = \frac{\exp(f'(0)s)}{s} \rightarrow f'(0)^{-1}.
   \]

   Therefore, \( f'(0) \geq \frac{1}{e} \inf_{t > 0} \frac{f(t)}{t} \).

   If \( f'(0) = \frac{1}{e} \inf_{t > 0} \frac{f(t)}{t} \) then, using the log-concavity again, we get that \( f(t) = \exp(f'(0)t), 0 \leq t \leq s. \) If \( f \) is analytic then \( f(z) = \exp(az), z \in C, a = f'(0) > 0. \)
3. Again, assume WLOG that \( R(0) = 1 \). It follows then from the strong log-concavity that
\[
R(t) \leq 1 + \ldots + \frac{1}{n!} t^n = \exp_n(t), t \geq 0.
\]

The rest of the proof is now as above.

**Corollary 3.4:** Let \( f \in \text{Ent}_+(n+1) \) and \( g_n(x_1, \ldots, x_n) = (\partial x_{n+1}) p(x_1, \ldots, x_n, 0) \).

If \( f \) is log-concave on \( R^{n+1}_+ \) then
\[
\text{Cap}(q_n) \geq \frac{1}{e} \text{Cap}(f).
\]  

(11)

If \( p \in \text{Hom}_+(n+1, n+1) \) is log-concave on \( R^{n+1}_+ \) then
\[
\text{Cap}(q_n) \geq g(n+1) \text{Cap}(p), \text{ where } g(k) = \left( \frac{k-1}{k} \right)^{k-1}.
\]  

(12)

**Proof:** We need to prove that \( (\partial x_{n+1}) p(x_1, \ldots, x_n, 0) \geq \frac{1}{e} \text{Cap}(p) \prod_{1 \leq i \leq n} x_i \). Define an univariate log-concave entire function \( R(t) = f(x_1, \ldots, x_n, t) \).

Then \( R(t) \geq \text{Cap}(p) t \prod_{1 \leq i \leq n} x_i : t \geq 0 \) and \( R'(0) = (\partial x_{n+1}) f(x_1, \ldots, x_n, 0) \).


It follows from the second item in Lemma (3.3) that
\[
(\partial x_{n+1}) p(x_1, \ldots, x_n, 0) \geq \frac{1}{e} \text{Cap}(p) \prod_{1 \leq i \leq n} x_i.
\]

The inequality (12) is proved in the very same way, using the first item in Lemma (3.3) and the fact that if \( p \in \text{Hom}_+(n+1, n+1) \) is log-concave on \( R^{n+1}_+ \) then also \( p^{\frac{1}{n+1}} \) is concave on \( R^{n+1}_+ \).

We use below the following notation:

\[
vdw(n) = \frac{n!}{n^n}.
\]

**Theorem 3.5:**

1. Let \( f \in \text{Ent}_+(n) \) be **Strongly Log-Concave** entire function in \( n \) variables. Then the following inequality holds:
\[
\text{Cap}(f) \geq \frac{\partial^n}{\partial x_1 \ldots \partial x_n} f(0) \geq \frac{1}{e^n} \text{Cap}(f)
\]  

(13)

Note that the right inequality in (13) becomes equality if \( f = \exp(\sum_{1 \leq i \leq n} a_i x_i) \) where \( a_i > 0, 1 \leq i \leq n \).
2. Let a homogeneous polynomial \( p \in \text{Hom}_+(n, n) \) be **Strongly Log-Concave**. Then the next inequality holds:

\[
\text{Cap}(f) \geq \frac{\partial^n}{\partial x_1...\partial x_n} f(0) \geq vdw(n)\text{Cap}(p)
\]  

(14)

Note that the right inequality in (14) becomes equality if \( p = (\sum_{1 \leq i \leq n} a_i x_i)^n \) where \( a_i > 0, 1 \leq i \leq n \).

3. Let a polynomial \( p \in \text{Pol}_+(n, n) \) be **Strongly Log-Concave**. Then the next inequality holds:

\[
\text{Cap}(f) \geq \frac{\partial^n}{\partial x_1...\partial x_n} f(0) \geq \prod_{1 \leq i \leq n} L(i)\text{Cap}(p),
\]  

where \( L(n) = (\inf_{t > 0} \frac{\exp(n(t))}{t})^{-1}. \)

(Note that \( L(1) = 1, L(2) = (1 + \sqrt{2})^{-1} \) and \( L(n) > e^{-1}, n \geq 1. \))

**Proof:**

1. Define the following entire functions \( q_i \in \text{Ent}_+(i) \):

\[
q_n = f, \quad q_i(x_1, ... , x_i) = \frac{\partial^n}{\partial x_{i+1}...\partial x_n} f(x_1, ... , x_i, 0, ..., 0).
\]

Notice that \( q'_1(0) = \frac{\partial^n}{\partial x_1...\partial x_n} f(0) \).

By the definition of **Strongly Log-Concavity**, these entire functions are either log-concave or zero. Using the inequality (11), we get that

\[
\text{Cap}(q_i) \geq \frac{1}{e} \text{Cap}(q_{i+1}), 1 \leq i \leq n - 1.
\]

Therefore

\[
\inf_{t > 0} \frac{q_1(t)}{t} = \text{Cap}(q_1) \geq (\frac{1}{e})^{n-1} \text{Cap}(f).
\]

Finally, using Lemma (3.3), we get that

\[
\frac{\partial^n}{\partial x_1...\partial x_n} f(0) = q'_1(0) \geq \frac{1}{e} \inf_{t > 0} \frac{q_1(t)}{t} \geq \frac{1}{e^n} \text{Cap}(f).
\]

2. If a homogeneous polynomial \( p \in \text{Hom}_+(n, n) \) is **Strongly Log-Concave** then the polynomials \( q_i \in \text{Hom}_+(i, i) \), \( \frac{\partial^n}{\partial x_1...\partial x_n} p(0) = \text{Cap}(q_1) \) and \( q_i^\downarrow \) is concave on \( \mathbb{R}^+_i \), \( 1 \leq i \leq n \).

It follows from the inequality (12) that

\[
\frac{\partial^n}{\partial x_1...\partial x_n} p(0) = \text{Cap}(q_1) \geq \prod_{2 \leq k \leq n} g(k)\text{Cap}(p) = \frac{n!}{n^n} \text{Cap}(p).
\]
3.2 General monomials

Consider an entire function \( f \in \text{Ent}^+(m) \) and an integer non-negative vector \( R = (r_1, \ldots, r_m) \). Assume WLOG that \( R = (r_1, \ldots, r_k, 0, \ldots, 0) : r_i > 0, 1 \leq i \leq k; k \leq n \). Let us define the entire function \( f_R \in \text{Ent}^+(|R|_1) \), where \(|R|_1 = r_1 + \ldots + r_k\).

\[
f_R(y_1, \ldots, y_{|R|_1}) = f(e_1(y_1 + \ldots + y_{r_1}) + \ldots + e_k(y_{r_1+\ldots+r_{k-1}+1} + \ldots + y_{r_1+\ldots+r_k}))
\]

where \( \{e_1, \ldots, e_m\} \) is the standard basis in \( C^m \). The following identity is obvious:

\[
(\partial x_1)^{r_1} (\partial x_m)^{r_m} f(0) = (\partial y_1)(\partial y_{|R|_1}) f_R(0).
\]

Note that if the original entire function (homogeneous polynomial) \( f \) is Strongly Log-Concave (H-Stable) then the same holds for the entire function (homogeneous polynomial) \( f_R \).

It easily follows from the arithmetic-geometric means inequality that

\[
\text{Cap}(f_R) = C_f(r_1, \ldots, r_m) = \inf_{x_i > 0} \frac{f(x_1, \ldots, x_m)}{\prod_{1 \leq i \leq m} (\frac{x_i}{r_i})^{r_i}} \quad (16)
\]

As we deal only with entire functions with the non-negative coefficients hence the following inequality holds:

\[
\left(\prod_{1 \leq i \leq m} vdw(r_i)\right) C_f(r_1, \ldots, r_m) \geq (\partial x_1)^{r_1} (\partial x_m)^{r_m} f(0) \quad (17)
\]

Putting these observations together, we get the Corollary to Theorem 3.5.

**Corollary 3.6:**

1. Let \( f \in \text{Ent}^+(m) \) be Strongly Log-Concave entire function in \( m \) variables. Then for all integer vectors \( R = (r_1, \ldots, r_m) \in \mathbb{Z}_+^m \) the next inequalities hold:

\[
\left(\prod_{1 \leq i \leq m} vdw(r_i)\right) C_f(r_1, \ldots, r_m) \geq (\partial x_1)^{r_1} (\partial x_m)^{r_m} f(0) = \exp(-|R|_1) C_f(r_1, \ldots, r_m) \quad (18)
\]

2. Let a homogeneous polynomial \( p \in \text{Hom}^+(m, n) \) be Strongly Log-Concave. Then for all integer vectors \( R = (r_1, \ldots, r_m) \in \mathbb{Z}_+^m, \sum_{1 \leq i \leq m} r_i = n \) the next inequalities hold:

\[
\left(\prod_{1 \leq i \leq m} vdw(r_i)\right) C_p(r_1, \ldots, r_m) \geq (\partial x_1)^{r_1} (\partial x_m)^{r_m} p(0) \geq vdw(n) C_p(r_1, \ldots, r_m) \quad (19)
\]

Let us recall the generalized Schrijver’s inequality from [10].
**Theorem 3.7:** Let $p \in \text{Hom}_+(n, n)$ be H-Stable. Let us denote the degree of variable $x_i$ in the polynomial $p$ as $\text{deg}_p(i)$.

If $\text{deg}_p(i) \leq k \leq n, 1 \leq i \leq n$, Then the next inequality holds:

$$\text{Cap}(p) \geq \frac{\partial^n}{\partial x_1 \cdots \partial x_n} p(0) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \text{vdw}(k) \text{Cap}(p)$$ \quad (20)

Combining Theorem 3.7 and observations (16), (17) we get the following Corollary.

**Corollary 3.8:**

Let $p \in \text{Hom}_+(n, n)$ be H-Stable. Assume that the degree of variable $x_i$ in the polynomial $p$, $\text{deg}_p(i) \leq k \leq n, 1 \leq i \leq n$. Then the following inequalities hold:

$$\left( \prod_{1 \leq i \leq m} \text{vdw}(r_i) \right) C_p(r_1, \ldots, r_m) \geq (\partial x_1)^{r_1} \cdots (\partial x_m)^{r_m} p(0) \geq (\frac{k-1}{k})^{(k-1)(n-k)} \text{vdw}(k) C_p(r_1, \ldots, r_m)$$ \quad (21)

### 3.3 A lower bound on the inner products of H-Stable polynomials

**Theorem 3.9:** Let us consider two H-Stable polynomials $p, q \in \text{Hom}_+(m, n)$:

$$p(x_1, \ldots, x_m) = \sum_{r_1 + \cdots + r_m = n} a_{r_1, \ldots, r_m} \prod_{1 \leq i \leq m} x_i^{r_i}, q(x_1, \ldots, x_m) = \sum_{r_1 + \cdots + r_m = n} b_{r_1, \ldots, r_m} \prod_{1 \leq i \leq m} x_i^{r_i},$$

and a nonnegative vector $(l_1, \ldots, l_m)$ such that $\sum_{1 \leq i \leq m} l_i = n$.

Let us assume that

$$\inf_{x_i > 0, 1 \leq i \leq m} \frac{p(x_1, \ldots, x_m)}{\prod_{1 \leq i \leq m} x_i^{l_i}} =: A > 0, \inf_{x_i > 0, 1 \leq i \leq m} \frac{q(x_1, \ldots, x_m)}{\prod_{1 \leq i \leq m} x_i^{l_i}} =: B > 0.$$ \quad (22)

Then the following inequality holds:

$$< p, q >=: \sum_{r_1 + \cdots + r_m = n} a_{r_1, \ldots, r_m} b_{r_1, \ldots, r_m} \geq AB \frac{\text{vdw}(nm)}{\text{vdw}(n)^m}$$ \quad (23)

**Proof:** Let us consider a rational function $F = \prod_{1 \leq i \leq m} x_i^{l_i} p(x_1, \ldots, x_m) q(\frac{1}{x_1}, \ldots, \frac{1}{x_m})$. It is clear that, in fact, $F \in \text{Hom}_+(m, nm)$ and $F$ is H-Stable. Note that

$$(n!)^m \sum_{r_1 + \cdots + r_m = n} a_{r_1, \ldots, r_m} b_{r_1, \ldots, r_m} = (\partial x_1)^n \cdots (\partial x_m)^n F(0).$$

It follows from (22) that $C_F(n, \ldots, n) \geq ABn^{nm}$. Using the right inequality in (19), we get that

$$\sum_{r_1 + \cdots + r_m = n} a_{r_1, \ldots, r_m} b_{r_1, \ldots, r_m} \geq AB \frac{\text{vdw}(nm)}{\text{vdw}(n)^m}.$$
Remark 3.10:

1. It is easy to see that the inequalities (22) holds for some vector \((l_1, \ldots, l_m)\) if and only if the Newton polytopes, \(\text{Newt}(p)\) and \(\text{Newt}(q)\), have non-empty intersection. (Recall that the Newton polytope \(\text{Newt}(p)\) is the convex hull of the support \(\text{supp}(p)\).)

   One of the corollaries of Theorem (3.9) is the fact that the intersection \(\text{Newt}(p) \cap \text{Newt}(q)\) is not empty iff the intersection \(\text{supp}(p) \cap \text{supp}(q)\) is not empty. There is alternative (and harder) way to prove this fact. It was proved in [9] and [11] that if \(p\) is a \textbf{H-Stable} polynomial then the Newton polytope \(\text{Newt}(p)\) is the polymatroid, based on some integer valued submodular function. It follows from the celebrated Edmonds’ result [4] that all the vertices of \(\text{Newt}(p) \cap \text{Newt}(q)\) are integer. Therefore, if \(\text{Newt}(p) \cap \text{Newt}(q)\) is not empty then the exists an integer vector \((r_1, \ldots, r_m)\) \(\in \text{Newt}(p) \cap \text{Newt}(q)\). But all integer vectors in \(\text{Newt}(p) \cap \text{Newt}(q)\) belong to the support \(\text{supp}(p) \cap \text{supp}(q)\).

   The inequality (23) is unlikely sharp. We conjecture here a sharp version:

   \[
   \sum_{r_1+\ldots+r_m=n} a_{r_1,\ldots,r_m} b_{r_1,\ldots,r_m} \prod_{1\leq i\leq m} (r_i)! \geq AB \frac{n!}{m^n},
   \]

2. If \textbf{H-Stable} polynomials \(p, q \in \text{Hom}_+(m, n)\) are both multilinear, i.e. \(\deg_p(i), \deg_q(i) \leq 1, 1 \leq i \leq m\), then

   \[
   \langle p, q \rangle = (\partial x_1) \ldots (\partial x_m) G(0), \text{ where } G(x_1, \ldots, x_m) = \left( \prod_{1 \leq i \leq m} x_i \right) p(x_1, \ldots, x_m) q\left(\frac{1}{x_1}, \ldots, \frac{1}{x_m}\right).
   \]

   Note that the polynomial \(G \in \text{Hom}_+(m, m)\) is \textbf{H-Stable}, \(\text{Cap}(G) \geq AB\) and \(\deg_G(i) \leq 2, 1 \leq i \leq m\). Using Theorem (5.1), we get the following inequality:

   \[
   \langle p, q \rangle \geq AB 2^{-m+1}
   \]

   The inequality (24) is sharp for \(m = 2n\).

4 Multivariate Newton Inequalities

We start with the following simple fact.

**Fact 4.1:** If an entire function \(f \in \text{Ent}_+(m)\) is log-concave on \(R_+^m\) then the map \(C_f\), defined as

\[
C_f(y_1, \ldots, y_m) = \inf_{x_i > 0} \frac{f(x_1, \ldots, x_m)}{\prod_{1 \leq i \leq m} (\frac{x_i}{y_i})^{w_i}} \geq 0
\]

is log-concave on \(R_+^m\).
Proof: Assume WLOG that \( y_i > 0, 1 \leq i \leq k \leq m \) and \( y_j = 0, k + 1 \leq j \leq m \). It follows from the monotonicity of \( f \) that

\[
C_f(x_1, \ldots, x_m) = \inf_{x_i > 0, 1 \leq i \leq k} \frac{f(x_1, \ldots, x_k, 0, \ldots, 0)}{\prod_{1 \leq i \leq k} (x_i/y_i)}.
\]

Therefore \( C_f(y_1, \ldots, y_m) \geq a \) iff \( \log(f(x_1y_1, \ldots, x_my_m)) \geq \log(a) + \sum_{1 \leq i \leq m} y_i \log(x_i) \) for all positive vectors \((x_1, \ldots, x_m)\). The desired log-concavity follows now from the log-concavity of the function \( f \) and of the logarithm.

Let \( Y = (r_1, \ldots, r_m) \in Z_+^m \) be an integer vector. We use below the following notations:

\[
VDW(Y) = \prod_{1 \leq i \leq m} vdw(r_i), \quad \text{where} \quad vdw(r) = \frac{r!}{r^r}.
\]

**Theorem 4.2:** Let us consider integer vectors \( Y_0, Y_1, \ldots, Y_k \in Z_+^m \) such that

\[
Y_0 = \sum_{1 \leq i \leq k} a_i Y_i; a_i \geq 0, \sum_{1 \leq i \leq k} a_i = 1.
\]

1. Suppose that the entire function \( f \in \text{Ent}_+(m) \) is **Strogy Log-Concave**. Then

\[
\text{Der}_f(Y_0) \geq \left( \exp(-|Y_0|_1) \prod_{1 \leq i \leq k} (VDW(Y_i))^{-a_i} \right) \prod_{1 \leq i \leq k} (\text{Der}_f(Y_i))^{a_i} \quad (25)
\]

2. If \( p \in \text{Hom}_+(m,n) \) is **Strogy Log-Concave** then

\[
\text{Der}_f(Y_0) \geq \left( vdw(n) \prod_{1 \leq i \leq k} (VDW(Y_i))^{-a_i} \right) \prod_{1 \leq i \leq k} (\text{Der}_f(Y_i))^{a_i} \quad (26)
\]

3. If \( p \in \text{Hom}_+(m,n) \) is **H-Stable** and \( \text{deg}_p(i) \leq k \leq n \) for all \( 1 \leq i \leq m \) then

\[
\text{Der}_f(Y_0) \geq \left( (k - 1) \frac{(k - 1)(n-k)}{k} vdwn(k) \prod_{1 \leq i \leq k} (VDW(Y_i))^{-a_i} \right) \prod_{1 \leq i \leq k} (\text{Der}_f(Y_i))^{a_i} \quad (27)
\]

**Proof:** We will prove only the inequality \( (25) \) as the other ones are proved in the same way. Using the the right inequality in \( (18) \), we get that

\[
\text{Der}_f(Y_0) \geq \exp(-|Y_0|_1) C_f(Y_0).
\]

Since the map \( C_f \) is log-concave hence

\[
C_f(Y_0) \geq \prod_{1 \leq i \leq k} (C_f(Y_i))^{a_i}.
\]

Finally, we use the left inequality in \( (18) \):

\[
C_f(Y_i) \geq (VDW(Y_i))^{-1} \text{Der}_f(Y_i).
\]

\[\square\]
Corollary 4.3: The support \( \text{supp}(f) \) of Strogly Log-Concave entire function \( f \in \text{Ent}_+(m) \) is \( D \)-convex.

Example 4.4:

1. Let us consider the following vectors in \( \mathbb{Z}_+^n \):

\[
Y_0 = (1, 1, ..., 1); Y_1 = (n, 0, ..., 0), ..., Y_n = (0, 0, ..., n).
\]

Note that \( Y_0 = \sum_{1 \leq i \leq n} \frac{1}{n} Y_i \). If \( p \in \text{Hom}_+(n, n) \) is Strogly Log-Concave then \( (26) \) gives the next inequality

\[
\text{Der}_p(Y_0) \geq \prod_{1 \leq i \leq k} \left( \text{Der}_f(Y_i) \right)^{\frac{1}{n}},
\]

which is attained on \( p(x_1, ..., x_n) = (x_1 + ... + x_n)^n \).

2. Consider three vectors in \( \mathbb{Z}_+^{2n} \):

\[
Y_0 = (1, 1, ..., 1); Y_1 = (2, ..., 2, 0, ..., 0), Y_2 = (0, ..., 0, 2, ..., 2), |Y_1|_1 = |Y_2|_1 = 2n.
\]

If \( p \in \text{Hom}_+(2n, 2n) \) is \( \text{H-Stable} \) and \( \text{deg}_p(i) \leq 2 \leq 2n \) for all \( 1 \leq i \leq 2n \) then it follows from \( (27) \) that

\[
\text{Der}_p(Y_0) \geq 2^{-n+1} \prod_{1 \leq i \leq 2} \left( \text{Der}_p(Y_i) \right)^{\frac{1}{2}}. \tag{28}
\]

The inequality \( (28) \) is attained on the polynomial \( p(x_1, ..., x_{2n}) = (x_1+x_2)(x_2+x_3)\cdots(x_{2n-1}+x_{2n})(x_{2n}+x_1) \).

5 Comments and Open problems

1. The inequality \( (14) \) is a far going generalization of the famous Van der Waerden conjecture on the permanent of doubly-stochastic matrices\( (19), (6), (5) \) and the Bapat’s conjecture \( (2) \). See more on this combinatorial connection in \( (11), (13), (7) \). The Van der Waerden conjecture conjecture corresponds to \( \text{H-Stable} \) polynomials

\[
\text{Prod}_A(x_1, ..., x_n) = \prod_{1 \leq i \leq n} \prod_{1 \leq j \leq n} A(i, j)x_j,
\]

where \( n \times n \) matrix is non-negative entry-wise and has no zero rows. If such a matrix is doubly-stochastic, i.e. all its rows and columns sum to 1, then \( \text{Cap}(\text{Prod}_A) = 1 \).

The convex relaxation approach to Newton-like inequalities in Theorem\( (4.2) \) was introduced by the author in \( (12) \) for the determinantal polynomials \( \text{det}(\sum_{1 \leq i \leq m} x_i A_i) \), where \( A_1, ..., A_n \) are \( n \times n \) hermitian PSD matrices. The corresponding inequalities in \( (12) \) are weaker than in the present paper.
2. Just the log-concavity of $f$ is not sufficient for $D$-convexity of its support $\text{supp}(f)$ even for univariate polynomials with non-negative coefficients. Indeed, consider $p(t) = t + t^3$. The fourth root $\sqrt[4]{p(t)}$ is concave on $R_+$:
\[
(p^{(1)}(t))^3 - \frac{4}{3}p(t)p^{(2)}(t) = (1 + 3t^2)^2 - \frac{4}{3}(t + t^3)6t = (t^2 - 1)^2 \geq 0.
\]
This example can be "lifted" to a "bad" log-concave homogeneous polynomial $q \in \text{Hom}_+(4,4)$:
\[
q(x, y, v, w) = (x + y)^3(v + w) + (v + w)^3(x + y).
\]
It is easy to see that $\text{Cap}(q) = 2^5$ but\[
\frac{\partial^4}{\partial x \partial y \partial v \partial w} q(0) = 0.
\]

3. In the case of $\mathbf{H}$-Stable polynomials, Corollary (4.3) can be made much more precise: Define, for a subset $S \subset \{1, ..., m\}$ and a polynomial $p \in \text{Hom}_+(m, n)$, the integer number $\deg_p(S)$ equal to the maximum total degree attained on variables in $S$. Then the following relation holds:
\[
ar_{1, \ldots, r_m} > 0 \iff \sum_{j \in S} r_j \leq \deg_p(S) : S \subset \{1, ..., m\}, p \in \text{Hom}_+(m, n).
\]
(29)

Additionaly, the integer valued map $\deg_p : 2^{\{1, \ldots, m\}} \to \{0, \ldots, n\}$ is submodular.

The characterization (29), proved in [9], is a far going generalization of the Hall-Rado theorems on the existence of perfect matchings. The paper [11] provides algorithmic applications of this result: strongly polynomial deterministic algorithms for the membership problem as for the support as well for the Newton polytope of $\mathbf{H}$-Stable polynomials $p \in \text{Hom}_+(m, n)$, given as oracles.

We don’t know whether (29) works for Strongly Log-Concave homogeneous polynomials. But it would follow from the following conjecture/question:

**Conjecture 5.1:** Let $p \in \text{Hom}_+(3, n)$ be Strongly Log-Concave. Then there exist convex compact subsets $K_1, K_2, K_3 \subset R^n$ such that
\[
p(x_1, x_2, x_3) = \text{Vol}_n(x_1K_1 + x_2K_2 + x_3K_3) : x_1, x_2, x_3 \geq 0
\]
(30)

Or put more modestly:

**Question 5.2:** Which Strongly Log-Concave polynomials $p \in \text{Hom}_+(3, n)$ allow the representation (30)?

The Minkowski polynomials $\text{Vol}_n \in \text{Hom}_+(3, n), \text{Vol}_n(x_1K_1 + x_2K_2 + x_3K_3)$ actually have seemingly stronger, than Strong Log-Concavity, property: the polynomials $\prod_{1 \leq i \leq r \leq n}(\sum_{1 \leq i \leq 3}(a_{i,j} \partial x_i)\text{Vol}_n$ are either zero or log-concave on $R^3_+$ provided that $a_{i,j} \geq 0$. 

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4. **Question 5.3:** Is the set of **Strongly Log-Concave** entire functions closed under the multiplications? If true it would imply that the Minkowski sum $\text{supp}(f) + \text{supp}(g) = \text{supp}(fg)$ is $D$-convex if $f, g$ are **Strongly Log-Concave.**

We note that the problem of $D$-convexity of Minkowski sums of $D$-convex sets was studied from a combinatorial point of view in [3].

5. **Question 5.4:** What are the asymptotically exact constants in Theorem(4.2)? Are the cyclic polynomials in Example(3.1) extremal?

6. Can recently refuted Okounkov’s conjecture [21], in the representation theory, on log-concavity of multiplicities be fixed/generalized in the way similar to Theorem(4.2)?

7. Stable multivariate polynomials form a backbone of linear multivariate control. If $p \in HSP_+(n, n)$ then $\text{Cap}(p) = \inf_{\text{Re}(z_i)>0} \left| \prod_{1 \leq i \leq n} \frac{\text{Re}(z_i)}{\text{Re}(z_i)} \right|$. In other words, the capacity can be viewed as a measure of stability. What is a meaning of capacity if terms of control/dynamics or in terms of the corresponding hyperbolic PDE?

8. Can our results be reasonably generalized to the fractional derivatives?

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