Emergent Spacetime and Cosmic Inflation

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ABSTRACT

We propose a background-independent formulation of cosmic inflation. The inflation in this picture corresponds to a dynamical process to generate space and time while the conventional inflation is simply an (exponential) expansion of a preexisting spacetime owing to the vacuum energy carried by an inflaton field. We observe that the cosmic inflation is triggered by the condensate of Planck energy into vacuum responsible for the generation of spacetime and must be a single event according to the exclusion principle of noncommutative spacetime caused by the Planck energy condensate in vacuum. The emergent spacetime picture admits a background-independent formulation so that the inflation can be described by a conformal Hamiltonian system characterized by an exponential phase space expansion without introducing any inflaton field as well as an *ad hoc* inflation potential. This implies that the emergent spacetime may incapacitate all the rationales to introduce the multiverse hypothesis.

Keywords: Emergent spacetime, Cosmic inflation, Quantum gravity

March 6, 2015
1 Introduction

History is a mirror to the future. If we do not learn from the mistakes of history, we are doomed to repeat them. In the middle of the 19th century, Maxwell’s equations for electromagnetic phenomena predicted the existence of an absolute speed, \( c = 2.998 \times 10^8 \text{ m/sec}, \) which apparently contradicted the Galilean relativity, a cornerstone on which the Newtonian model of space and time rested. Since most physicists, by then, had developed deep trust in the Newtonian model, they concluded that Maxwell’s equations can only hold in a specific reference frame, called the ether. However, by doing so, they reverted back to the Aristotelian view that Nature specifies an absolute rest frame. It was Einstein to realize the true implication of this quandary: It was asking us to abolish Newton’s absolute time as well as absolute space. The ether was removed by the Einstein’s special relativity by radically modifying the concept of space and time in the Newtonian dynamics. Time lost its absolute standing and the notion of absolute simultaneity was physically untenable. Only the four-dimensional spacetime has an absolute meaning. The new paradigm of spacetime has completely changed the Newtonian world with dramatic consequences.

The physics of the last century had devoted to the study of two pillars: general relativity and quantum field theory. And the two cornerstones of modern physics can be merged into beautiful equations, the so-called Einstein equations given by

\[
R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} R = 8\pi G T_{\mu\nu},
\]

where the right-hand side is the energy-momentum tensor whose contents are described by (quantum) field theories. Although the revolutionary theories of relativity and quantum mechanics have utterly changed the way we think about Nature and the Universe, new open problems have emerged which have not yet been resolved within the paradigm of the 20th century physics. For example, a short list of them is the cosmological constant problem, the hierarchy problem, dark energy, dark matter, cosmic inflation and quantum gravity. In particular, recent developments in cosmology, particle physics and string theory have led to a radical proposal that there could be an ensemble of universes that might be completely disconnected from ours. Of course, it would be perverse to claim that nothing exists beyond the horizon of our observable universe. The observable universe is one causal patch of a much larger unobservable universe. However, a painful direction is to use the string landscape or multiverse to explain some notorious problems in theoretical physics based on the anthropic argument. “And it’s pretty unsatisfactory to use the multiverse hypothesis to explain only things we don’t understand.”

Taking history as a mirror, this situation is very reminiscent of the hypothetical luminiferous ether in the late 19th century. Looking forward to the future, we may need another turn of the spacetime picture to defend the integrity of physics.

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1George Santayana.
2Graham Ross in Quanta magazine “At multiverse impasse, a new theory of scale” (August 18, 2014) and Wired.com “Radical new theory could kill the multiverse hypothesis.”
In physical cosmology, cosmic inflation is the exponential expansion of space in the early universe. Suppose that spacetime evolution is determined by a single scale factor $a(t)$ and its Hubble expansion rate $H \equiv \frac{\dot{a}}{a}$ according to the cosmological principle and driven by the dynamics of a scalar field $\phi$, called the inflaton \[3, 4\]. Then the Einstein equation (1.1) reduces to the Friedmann equation

$$H^2 = \frac{8\pi G_N}{3} \left( \frac{1}{2} \dot{\phi}^2 + V(\phi) \right).$$

(1.2)

The evolution equation of the inflaton in the Friedmann universe is described by

$$\ddot{\phi} + 3H\dot{\phi} + \frac{\delta V}{\delta \phi} = 0.$$  

(1.3)

The Friedmann equation (1.2) tells us that in the early universe with $V(\phi) \approx V_0$ and $\dot{\phi} \approx 0$, there was an inflationary epoch of the exponential expansion of space, i.e., $a(t) \propto e^{Ht}$ where $H = \sqrt{\frac{8\pi G_N V_0}{3}}$ is called the Hubble constant. In order to successfully fit to data, one finds

$$V_0 \geq (2 \times 10^{16}\text{GeV})^4 \approx (10^{-2} M_P)^4$$

(1.4)

where $M_P = 1/\sqrt{8\pi G_N}$ is the Planck mass.

Let us contemplate the inflationary scenario with a critical eye. According to this scenario \[3, 4\], inflation is described by the exponential expansion of the universe in a supercooled false vacuum state that is a metastable state without any fields or particles but with a large energy density. It should be emphasized that the inflation scenario so far has been formulated in the context of effective field theory coupled to general relativity. Thus, in this scenario, the existence of space and time is \textit{a priori} assumed from the beginning although the evolution of spacetime is determined by Eq. (1.1). In other words, the inflationary scenario does not describe any generation (or creation) of spacetime but simply characterizes an expansion of a preexisting spacetime. It never addresses the (dynamical) origin of spacetime. However, there has to be a definite beginning to an inflationary universe \[5\]. This means that the inflation is incomplete to describe the very beginning of our universe and some new physics is needed to probe the past boundary of the inflating regions. The Friedmann equation (1.2) shows that the cosmic inflation is triggered by the potential energy carried by inflaton whose energy scale is near the Planck energy over which quantum gravity effects become strong and effective field theory description may break down. Although an inflating false vacuum is metastable, essentially all models of inflation lead to eternal inflation to the future since expansion rate is much greater than decay rate \[3\]. Once inflation starts, it never stops. If one identifies the slowly varying inflaton field $\phi(t)$ with a particle trajectory $x(t) = \phi(t)$ and $\dot{\phi}(t)$ with its velocity $v(t) = \dot{x}(t)$, the evolution equation (1.3) tells us that the frictional force, $3Hv(t)$, caused by the inflating spacetime is (almost) balanced with an external force $F(x) = -\frac{dV}{dx}$, i.e.,

$$\dot{x}(t) \approx \frac{F(x)}{3H},$$

(1.5)
because $\ddot{x} \approx 0$ during inflation. This implies that the cosmic inflation as a dynamical system corresponds to a non-Hamiltonian system.\(^\text{3}\)

Recent developments in string theory have revealed a remarkable and radical new picture about gravity. For example, the AdS/CFT correspondence illustrates a surprising picture that an $SU(N)$ gauge theory in lower dimensions defines a nonperturbative formulation of quantum gravity in higher dimensions.\(^5\) In particular, the AdS/CFT duality shows a typical example of emergent gravity and emergent space because gravity in higher dimensions is defined by a gravityless field theory in lower dimensions. Now we have many examples from string theory in which spacetime is not fundamental but only emerges as a large distance, classical approximation.\(^7\) Therefore, the rule of the game in quantum gravity is that space and time are an emergent concept. Since the emergent spacetime, as quantum gravity, is a significant new paradigm, we want to apply the emergent spacetime picture to cosmic inflation. We will propose a background-independent formulation of the cosmic inflation. This means that we do not assume the prior existence of spacetime but define a spacetime structure as a solution of an underlying background-independent theory such as matrix models. The inflation in this picture corresponds to a dynamical process to generate space and time which is very different from the standard inflation simply describing an (exponential) expansion of a preexisting spacetime. Spacetime is emergent from the Planck energy condensate in vacuum that generates an extremely large Universe in which our observable patch within cosmic horizon is a very tiny part $\sim 10^{-60}$ of the entire spacetime. Originally the multiverse hypothesis has been motivated by an attempt to explain the anthropic fine-tuning such as the cosmological constant problem and boosted by the chaotic and eternal inflation scenarios and the string landscape derived from the Kaluza-Klein compactification of string theory which are all based on the traditional spacetime picture. Since emergent spacetime is radically different from any previous physical theories, all of which describe what happens in a given spacetime, the multiverse picture must be reexamined from the standpoint of emergent spacetime and a background-independent theory. The background-independent formulation of cosmic inflation will certainly open a new prospect that may cripple all the rationales to introduce the multiverse picture.

Since the concept of the multiverse raises deep conceptual issues even to require to change our view of science itself, it should be important to ponder on the real status of the multiverse whether it is simply a mirage developed from an incomplete physics like the ether in the late 19th century or it is of vital importance even in more complete theories. The main purpose of this paper is to illuminate how the emergent spacetime picture brings about radical changes of physics, especially, regarding to physical cosmology. In particular, a background-independent theory such as matrix models provides a concrete realization of the idea of emergent spacetime which has a sufficiently elegant and explanatory power to defend the integrity of physics against the multiverse hypothesis.

\(^3\)Nonetheless, the friction term does not lead to dissipative energy production. This fact can be seen by observing that Eq. 1.3 can be derived from the first law of thermodynamics, $dE + pdV = Vd\rho + (\rho + p)dV = 0$, where $\rho + p = \dot{\phi}^2$ and $\dot{\rho} = \left(\ddot{\phi} + \frac{\delta V}{\delta \phi}\right)\dot{\phi}$. 

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The emergent spacetime is a completely new paradigm that the multiverse debate in physics circles has to seriously take into account. This paper is intended to be self-contained as much as possible and mathematical backgrounds underlying our arguments are also briefly reviewed in Appendices.

This paper is organized as follows. In Sec. 2, we compactly review the background-independent formulation of emergent gravity and emergent spacetime in terms of matrix models [11] [12] [13] [14] [15]. See also [16, 17]. The pith and marrow of the underlying argument is the realization that NC spaces arise as a solution in a particular Coulomb branch of a large $N$ matrix model and this vacuum in the Coulomb branch admits a separable Hilbert space as quantum mechanics [18]. General solutions are generated by considering generic deformations of a primitive vacuum, e.g., the Moyal NC space obeying the Heisenberg algebra. These deformations are intrinsically dynamical and described by general inner automorphisms of an underlying NC algebra $A_\theta$. The (emergent) time is defined through the Hamiltonian description of the dynamical system like quantum mechanics. The emergent geometry is simply derived from the nontrivial inner automorphism of the NC algebra $A_\theta$, in which the NC nature is crucial to realize the emergent gravity [13]. An important point is that the matrix model does not presuppose any spacetime background on which fundamental processes develop. Rather the background-independent theory provides a mechanism of spacetime generation such that every spacetime structure including flat spacetime arises as a solution of the theory itself.

In Sec. 3, we observe that a spacetime background arises as a vacuum solution that is the NC space in a Coulomb branch of the matrix model. The vacuum configuration is the Planck energy condensate responsible for the generation of spacetime and results in extremely large spacetime. We demonstrate why the emergent gravity clearly resolves the notorious cosmological constant problem [11] [12] [19]. A principal reason is that the huge vacuum energy being a perplexing cosmological constant in general relativity was simply used to generate flat spacetime and thus does not gravitate. The emergent gravity is in stark contrast to general relativity since it does not allow the coupling of the cosmological constant thanks to a general property. Because the Planck energy condensate into vacuum must be a dynamical process, we explore the dynamical mechanism for the instantaneous condensation of vacuum energy to enormously spread out spacetime. We find that the cosmic inflation as a dynamical system can be described by a generalized phase space of time-dependent Hamiltonian system. We show that a locally conformal (co)symplectic manifold (see Appendix A for the definition) is a natural phase space describing the cosmic inflation of our universe. Since the generalized symplectic manifold admits a rich variety of vector fields, in particular, Liouville vector fields that generate an exponential phase space expansion, the inflation can be described by the so-called conformal Hamiltonian system [20] without introducing any inflaton field as well as an ad hoc inflation potential. It is remarkable to see that an inflationary vacuum describing the creation of spacetime also arises as a solution of time-dependent matrix model although the corresponding temporal gauge field must be nonlocal. Our work is not the first to address physical cosmology using matrix models. There have been interesting earlier attempts [21]. In particular, the cosmic inflation was addressed in very interesting works [22] using the Monte Carlo analysis of the type IIB matrix model in Lorentzian
signature and it was found that three out of nine spatial directions start to expand at some critical time after which exactly (3+1)-dimensions dynamically become macroscopic.

In Sec. 4, we emphasize that NC spacetime necessarily implies emergent spacetime if spacetime should be viewed as NC. Although spacetime at microscopic scales is intrinsically NC, we understand the NC spacetime through the quantization of a symplectic manifold. Since the most natural object to explore the symplectic geometry is a string rather than a particle [13] or a pseudoholomorphic curve which is a stringy generalization of a geodesic worldline in Riemannian geometry [23], we need a mathematically precise framework for describing strings in a background independent way to make sense of the emergent spacetime proposal. We show that the pseudoholomorphic curves can be lifted to NC spacetimes by the matrix string theory [24, 25]. We argue that NC spacetimes must be viewed as second-quantized strings for the background-independent formulation of quantum gravity, which is still elusive in contemporary string theory. Hence we need to read old literatures with the new perspective.

In Sec. 5, we summarize why the emergent spacetime picture may incapacitate all the rationales to introduce the multiverse hypothesis. Since the emergent spacetime picture is radically different from the conventional picture in general relativity so that they are exclusive and irreconcilable each other, we reconsider from the standpoint of the emergent spacetime and background independentness main sources to introduce the multiverse hypothesis: (A) cosmological constant problem, (B) chaotic and eternal inflation scenarios, (C) string landscape. We argue that the emergent spacetime certainly opens a new perspective that may cripple all the rationales to introduce the multiverse hypothesis. We also discuss a speculative mechanism to end the inflation by some nonlinear damping through interactions between the inflating background and ubiquitous local fluctuations. Finally we discuss possible ways to understand our real world $\mathbb{R}^{1,3}$ that is unfortunately beyond our current approach because $\mathbb{R}^{1,3}$ does not belong to the family of (almost) symplectic manifolds.

In the first appendix we briefly review the mathematical foundation of locally conformal cosymplectic (LCC) manifolds that correspond to a natural phase space describing the cosmic inflation of our universe. In the second appendix we give a brief exposition of harmonic oscillator with time-dependent mass to illustrate how nonconservative dynamical systems with friction can be formulated by time-dependent Hamiltonian systems which may be useful to understand the cosmic inflation as a dynamical system.

2 Emergent spacetime from matrix model

Let us start with a zero-dimensional matrix model with a bunch of $N \times N$ Hermitian matrices, $\{\phi_a \in \mathcal{A}_N | a = 1, \ldots, 2n\}$, whose action is given by [26]

$$S = -\frac{1}{4} \sum_{a,b=1}^{2n} \text{Tr} [\phi_a, \phi_b]^2.$$ (2.1)
In particular, we are interested in the matrix algebra $A_N$ in the limit $N \to \infty$. We require that the matrix algebra $A_N$ is associative, from which we get the Jacobi identity

$$[\phi_a, [\phi_b, \phi_c]] + [\phi_b, [\phi_c, \phi_a]] + [\phi_c, [\phi_a, \phi_b]] = 0.$$  

(2.2)

We also assume the action principle, from which we yield the equations of motion:

$$\sum_{b=1}^{2n} [\phi_b, [\phi_a, \phi_b]] = 0.$$  

(2.3)

We emphasize that we have not introduced any spacetime structure to define the action (2.1). It is enough to suppose the matrix algebra $A_N$ consisted of a bunch of matrices which are subject to a few relationships given by Eqs. (2.2) and (2.3).

First suppose that the vacuum configuration of $A_N$ is given by

$$\langle \phi_a \rangle_{\text{vac}} = p_a \in A_N,$$  

(2.4)

which must be a solution of Eqs. (2.2) and (2.3). An obvious solution in the limit $N \to \infty$ is given by the Moyal-Heisenberg algebra

$$[p_a, p_b] = -iB_{ab},$$  

(2.5)

where $(B_{ab}) = -L_P^{-2}(1_n \otimes i\sigma^2)$ is a $2n \times 2n$ constant symplectic matrix and $L_P$ is a typical length scale set by the vacuum. A general solution will be generated by considering all possible deformations of the Moyal-Heisenberg algebra (2.5). It is assumed to take the form

$$\phi_a = p_a + \hat{A}_a \in A_N,$$  

(2.6)

obeying the deformed algebra given by

$$[\phi_a, \phi_b] = -i(B_{ab} - \hat{F}_{ab}),$$  

(2.7)

where

$$\hat{F}_{ab} = \partial_a \hat{A}_b - \partial_b \hat{A}_a - i[\hat{A}_a, \hat{A}_b] \in A_N$$  

(2.8)

with the definition $\partial_a \equiv -i\text{ad}_{p_a} = -i[p_a, \cdot]$. For the general matrix $\phi_a \in A_N$ to be a solution of Eqs. (2.2) and (2.3), the set of matrices $\hat{F}_{ab} \in A_N$, called the field strengths of NC $U(1)$ gauge fields $\hat{A}_a \in A_N$, must obey the following equations

$$\hat{D}_a \hat{F}_{bc} + \hat{D}_b \hat{F}_{ca} + \hat{D}_c \hat{F}_{ab} = 0,$$  

(2.9)

$$\sum_{b=1}^{2n} \hat{D}_b \hat{F}_{ab} = 0,$$  

(2.10)

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4The conventional choice of vacuum in Coulomb branch is given by $[\phi_a, \phi_b]_{\text{vac}} = 0$ and so $\langle \phi_a \rangle_{\text{vac}} = \text{diag}(\alpha_a)_1, (\alpha_a)_2, \cdots, (\alpha_a)_N)$. However, it turns out (see Section III.C in [12]) that, in order to describe a classical geometry from a background-independent theory, it is necessary to have a nontrivial vacuum defined by a coherent condensation such as the vacuum (2.5). For this reason, we will choose the Moyal-Heisenberg vacuum instead of the conventional vacuum. A similar reasoning was also advocated in footnote 2 in Ref. [18].
\[
\tilde{D}_a \tilde{F}_{bc} \equiv -i \text{ad}_{\phi_a} \tilde{F}_{bc} = -i [\phi_a, \tilde{F}_{bc}] = -[\phi_a, [\phi_b, \phi_c]].
\] (2.11)

The algebra \( \mathcal{A}_N \) admits a large amount of inner automorphisms \( \text{Inn}(\mathcal{A}_N) \). Note that any automorphism of the matrix algebra \( \mathcal{A}_N \) is inner. Suppose that \( \mathcal{A}'_N = \{ \phi'_a | a = 1, \ldots, m \} \) is another matrix algebra composed of elements of \( \tilde{N} \times \tilde{N} \) Hermitian matrices. We will identify two matrix algebras, i.e. \( \mathcal{A}_N \cong \mathcal{A}'_N \) if \( m = 2n \) and \( \tilde{N} = N \) and there exists a unitary matrix \( U \in \text{Inn}(\mathcal{A}_N) \) such that \( \phi'_a = U \phi_a U^{-1}, \forall a = 1, \ldots, 2n \). It is important to recall that the NC algebra \( \mathcal{A}_N \) generated by the vacuum operators \( p_a \) admits an infinite-dimensional separable Hilbert space

\[
\mathcal{H} = \{|n\rangle | n = 1, \ldots, N \to \infty \}
\] (2.12)

that is the Fock space of the Moyal-Heisenberg algebra (2.5). As is well-known from quantum mechanics [27], there is a one-to-one correspondence between the operators in \( \mathcal{H} \) and the set of \( N \times N \) matrices over \( \mathbb{C} \) where \( V \) is an \( N \)-dimensional complex vector space. In our case, \( V = \mathcal{H} \) is a Hilbert space and \( N = \text{dim}(\mathcal{H}) \to \infty \). Then the matrix algebra \( \mathcal{A}_N \) can be realized as the NC \(*\)-algebra

\[
\mathcal{A}_\theta = \{ \hat{\phi}_a(y) \in \text{Hom}(\mathcal{H}) | a = 1, \ldots, 2n \},
\] (2.13)

which is generated by the set of coordinate generators obeying the commutation relation

\[
[y^a, y^b]_\star = i \theta^{ab}.
\] (2.14)

The \(*\)-commutator (2.14) is related to the Moyal-Heisenberg algebra (2.5) by \( \theta^{ab} = (B^{-1})^{ab} \) and \( p_a = B_{ab}y^b \). To be specific, given a Hermitian operator \( \hat{\phi}_a(y) \in \mathcal{A}_\theta \), we have a matrix representation in \( \mathcal{H} \) as follows:

\[
\hat{\phi}_a(y) = \sum_{n,m=1}^{\infty} |n\rangle \langle n| \hat{\phi}_a(y)|m\rangle \langle m| = \sum_{n,m=1}^{\infty} (\phi_a)_{nm} |n\rangle \langle m|
\] (2.15)

using the completeness of \( \mathcal{H} \), i.e. \( \sum_{n=1}^{\infty} |n\rangle \langle n| = 1_\mathcal{H} \). The unitary equivalence between two matrix algebras, \( \mathcal{A}_N \cong \mathcal{A}'_N \), can then be understood as the unitary equivalence between two Hilbert spaces

\[
\mathcal{H}'_N = U \mathcal{H}_N
\] (2.16)

where \( U : \mathcal{H}_N \to \mathcal{H}_N \) is a unitary operator acting on the \( N \)-dimensional Hilbert space \( \mathcal{H}_N \).

As a result, the inner automorphism \( \text{Inn}(\mathcal{A}_N) \) of the matrix algebra \( \mathcal{A}_N \) is translated into that of the NC \(*\)-algebra \( \mathcal{A}_\theta \), denoted by \( \text{Inn}(\mathcal{A}_\theta) \). Its infinitesimal generators consist of an inner derivation \( \mathcal{D} \) defined by the map [11][12][13]

\[
\mathcal{A}_\theta \to \mathcal{D} : \mathcal{O} \mapsto \text{ad}_\mathcal{O} = -i[\mathcal{O}, \cdot]_\star
\] (2.17)

for any \( \mathcal{O} \in \mathcal{A}_\theta \). Using the Jacobi identity of the NC \(*\)-algebra \( \mathcal{A}_\theta \), one can easily verify the Lie algebra homomorphism:

\[
[\text{ad}_\mathcal{O}_1, \text{ad}_\mathcal{O}_2] = -iad[\mathcal{O}_1, \mathcal{O}_2]_\star
\] (2.18)
for any $O_1, O_2 \in \mathcal{A}_\theta$. In particular, we consider the set of derivations determined by NC gauge fields in Eq. (2.13):

$$\{ \hat{V}_a \equiv \text{ad}\hat{\phi}_a \in \mathcal{D} | \hat{\phi}_a(y) = p_a + \hat{A}_a(y) \in \mathcal{A}_\theta, \quad a = 1, \cdots, 2n \}. \quad (2.19)$$

In a large-distance limit, i.e. $|\theta| \to 0$, one can expand the NC vector fields $\hat{V}_a$ using the explicit form of the Moyal $\star$-product. The result takes the form

$$\hat{V}_a = V_a^\mu(y) \frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} V_{a \mu_2 \cdots \mu_p}(y) \frac{\partial}{\partial y^\mu_2} \cdots \frac{\partial}{\partial y^\mu_p} \in \mathcal{D}. \quad (2.20)$$

Thus the Taylor expansion of NC vector fields in $\mathcal{D}$ generates an infinite tower of the so-called polyvector fields [13]. Note that the leading term gives rise to the ordinary vector fields that will be identified with a frame basis associated to the tangent bundle $T\mathcal{M}$ of an emergent manifold $\mathcal{M}$.

Since we have started with a large $N$ matrix model, it is natural to expect that the IKKT-type matrix model (2.1) is dual to a higher-dimensional gravity or string theory according to the large $N$ duality or gauge/gravity duality [28]. The emergent gravity is realized via the gauge/gravity duality as follows [13]:

$$\mathcal{A}_N \quad \Rightarrow \quad \mathcal{A}_\theta \quad \Rightarrow \quad \mathcal{D}. \quad (2.21)$$

The gauge theory side of the duality is described by the set of large $N$ matrices that is an associative, but NC, algebra $\mathcal{A}_N$. By choosing a proper vacuum such as Eq. (2.4), the matrix algebra $\mathcal{A}_N$ is realized as a representation on a separable Hilbert space $\mathcal{H}$ so that a matrix in $\mathcal{A}_N$ is regarded as a linear operator acting on the Hilbert space $\mathcal{H}$, i.e., $\mathcal{A}_N \cong \text{End}(\mathcal{H})$. In the end the algebra $\mathcal{A}_N$ is isomorphically mapped to the NC $\star$-algebra $\mathcal{A}_\theta$, as Eq. (2.15) has clearly illustrated. The noncommutativity of an underlying algebra is crucial to realize the emergent gravity. The gravity side of the duality is defined by associating the derivation $\mathcal{D}$ of the algebra $\mathcal{A}_\theta$ with a quantized frame bundle $\hat{\mathcal{X}}(\mathcal{M})$ of an emergent spacetime manifold $\mathcal{M}$. After all, to describe a quantum geometry mathematically, we need to find a right NC algebra.

It is important to perceive that the realization of emergent geometry through the duality chain in Eq. (2.21) is intrinsically local. Therefore it is necessary to consider patching or gluing together the local constructions to form a set of global quantities. Let us explain this feature briefly since its extensive exposition was already given in Ref. [13]. Its characteristic feature becomes transparent

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5 As we discussed in footnote 4 this is the reason why we need the Moyal-Heisenberg vacuum (2.5) instead of the conventional Coulomb branch vacuum. If we choose the conventional vacuum, we will fail to realize the isomorphism between $\mathcal{A}_N$ and $\mathcal{A}_\theta$.

6 The explicit realization of the duality chain (2.21) depends on the data of the matrix algebra $\mathcal{A}_N$. In particular, the data of $\mathcal{A}_N$ may include choosing $N$ to be finite or infinite and the number of linearly independent matrices. Then the vacuum will be specified by choosing the most primitive one so that more general solutions are generated by deformations of the primitive vacuum as we already implemented in Eq. (2.6). For instance, for our particular choice given by $N \to \infty$ and even number of matrices, the Moyal-Heisenberg algebra (2.5) is the most primitive vacuum for emergent geometry. This statement may be regarded as a quantum version of the Darboux theorem in symplectic geometry.
when the commutative limit, i.e. $|\theta| \to 0$, is taken into account. In this limit, the NC $\ast$-algebra $\mathcal{A}_\theta$ reduces to a Poisson algebra $\mathfrak{P}^{(i)} = (C^\infty(U_i), \{-, -\}_\theta)$ defined on a local patch $U_i \subset M$ in an open covering $M = \bigcup_{i \in I} U_i$. The Poisson algebra $\mathfrak{P}^{(i)}$ arises as follows. Let $L \to M$ be a line bundle over $M$. We assume that the curvature $\mathcal{F}$ of the line bundle $L$ is a nondegenerate, closed two-form. Therefore we identify the curvature two-form $\mathcal{F} = dA$ with a symplectic structure of $M$. On an open neighborhood $U_i \subset M$, it is possible to represent $\mathcal{F}^{(i)} = B + F^{(i)}$ where $F^{(i)} = dA^{(i)}$ and $B$ is the constant symplectic two-form already introduced in Eq. (2.5). Consider a chart $(U_i, \phi^{(i)})$ where $\phi^{(i)} \in \text{Diff}(U_i)$ is a local trivialization of the line bundle $L$ over an open subset $U_i$ obeying $\phi^{\ast}(\mathcal{F}^{(i)}) = B$. Such a local chart always exists owing to the Darboux theorem or the Moser lemma in symplectic geometry [29] and the local coordinate chart obeying $\phi^{\ast}(\mathcal{F}^{(i)}) = B$ is called Darboux coordinates. Then the line bundle $L \to M$ corresponds to a dynamical symplectic manifold $(M, \mathcal{F})$ where $\mathcal{F} = B + dA$. The dynamical system is locally described by the Poisson algebra $\mathfrak{P}^{(i)} = (C^\infty(U_i), \{-, -\}_\theta)$ in which the vector space $C^\infty(U_i)$ is formed by the set of Darboux transformations $\phi^{(i)} \in \text{Diff}(U_i)$ equipped with the Poisson bracket defined by the Poisson bivector $\theta = B^{-1} \in \Gamma(L^2 TM)$.

Consider a collection of local charts to make an atlas $\{(U_i, \phi^{(i)})\}$ on $M = \bigcup_{i \in I} U_i$ and complete the atlas by gluing these charts on their overlap. To be precise, suppose that $(U_i, \phi^{(i)})$ and $(U_j, \phi^{(j)})$ are two coordinate charts and $F^{(i)} = dA^{(i)}$ and $F^{(j)} = dA^{(j)}$ are local curvature two-forms on $U_i$ and $U_j$, respectively. We choose the coordinate maps $\phi^{(i)} \in \text{Diff}(U_i)$ and $\phi^{(j)} \in \text{Diff}(U_j)$ such that $\phi^{(i)}_\ast(B + F^{(i)}) = B$ and $\phi^{(j)}_\ast(B + F^{(j)}) = B$. On an intersection $U_i \cap U_j$, the local data $(A^{(i)}, \phi^{(i)})$ and $(A^{(j)}, \phi^{(j)})$ on Darboux charts $(U_i, \phi^{(i)})$ and $(U_j, \phi^{(j)})$, respectively, are patched or glued together by \[ A^{(j)} = A^{(i)} + d\lambda^{(ji)}, \tag{2.22} \]
\[ \phi^{(ji)} = \phi^{(j)} \circ \phi^{-1}^{(i)}, \tag{2.23} \]
where $\phi^{(ji)} \in \text{Diff}(U_i \cap U_j)$ is a symplectomorphism on $U_i \cap U_j$ generated by a Hamiltonian vector field $X_{\lambda^{(ji)}}$ satisfying $\iota(X_{\lambda^{(ji)}}) B + d\lambda^{(ji)} = 0$. We sometimes denote the interior product $\iota_X$ by $\iota(X)$ for a notational convenience. Similarly, we can glue the local Poisson algebras $\mathfrak{P}^{(i)}$ to form a globally defined Poisson algebra $\mathfrak{P} = \bigcup_{i \in I} \mathfrak{P}^{(i)}$. This globalization can also be applied to the derivation $\mathfrak{D}$ in Eq. (2.20) to yield global vector fields $V_a = V_a^\mu(y) \frac{\partial}{\partial y^\mu} \in \Gamma(TM)$, $a = 1, \cdots, 2n$, which form a linearly independent basis of the tangent bundle $TM$ of a $2n$-dimensional emergent manifold $M$. As a consequence, the set of global vector fields $\mathfrak{X}(M) = \{V_a | a = 1, \cdots, 2n\}$ results from the globally defined Poisson algebra $\mathfrak{P}$ [13].

The vector fields $V_a \in \mathfrak{X}(M)$ are related to an orthonormal frame, the so-called vielbeins $E_a \in \Gamma(TM)$, in general relativity by the relation \[ V_a = \lambda E_a, \quad a = 1, \cdots, 2n. \tag{2.24} \]

The conformal factor $\lambda \in C^\infty(M)$ is determined by imposing the condition that the vector fields $V_a$
preserve a volume form

\[ \nu = \lambda^2 v^1 \wedge \cdots \wedge v^{2n}, \]

(2.25)

where \( v^a = v^a_\mu(y)dy^\mu \in \Gamma(T^*\mathcal{M}) \) are coframes dual to \( V_a \), i.e., \( \langle v^a, V_b \rangle = \delta^a_b \). This means that the vector fields \( V_a \) obey the conditions

\[ \mathcal{L}_{V_a} \nu = (\nabla \cdot V_a + (2 - 2n)V_a \ln \lambda) \nu = 0, \quad \forall a = 1, \cdots, 2n. \]

(2.26)

Note that a symplectic manifold always admits such volume-preserving vector fields. After implementing the volume-preserving condition (2.26), the relation (2.24) completely determines a 2\( n \)-dimensional Riemannian manifold \( \mathcal{M} \) whose metric is given by [11, 12, 13]

\[ ds^2 = g_{\mu\nu}(x)dx^\mu \otimes dx^\nu = e^a \otimes e^a \]

(2.27)

where \( e^a = e^a_\mu(x)dx^\mu = \lambda v^a \in \Gamma(T^*\mathcal{M}) \) are orthonormal one-forms on \( \mathcal{M} \). After all, the 2\( n \)-dimensional Riemannian manifold \( \mathcal{M} \) is emergent from the commutative limit of NC \( U(1) \) gauge fields which are mapped to polyvector fields \( \hat{V}_a = V_a + O(\theta^2) \in \mathcal{D} \).

So far we have discussed the emergence of spaces only. However, the theory of relativity dictates that space and time must be coalesced into the form of Minkowski spacetime in a (locally) inertial frame. Hence, if general relativity is realized from a NC \( \star \)-algebra \( \mathcal{A}_\theta \), it is necessary to put space and time on an equal footing in the NC \( \star \)-algebra \( \mathcal{A}_\theta \). If a space is emergent, so should time. Thus, an important problem is how to realize the emergence of “time.” Quantum mechanics offers us a lesson that the definition of (particle) time is strictly connected with the problem of dynamics. In quantum mechanics, the time evolution of a dynamical system is defined as an inner automorphism of NC algebra \( \mathcal{A}_\hbar \) generated by the NC phase space

\[ [x^i, x^j] = 0, \quad [x^i, p_j] = i\hbar \delta^i_j, \quad i, j = 1, \cdots, n. \]

(2.28)

The time evolution for an observable \( f \in \mathcal{A}_\hbar \) is simply an inner derivation of \( \mathcal{A}_\hbar \) given by

\[ \frac{df}{dt} = \frac{i}{\hbar} [H, f]. \]

(2.29)

A remarkable picture, as observed by Feynman [31], Souriau, and Sternberg [32], is that the physical forces such as the electromagnetic, weak and strong forces, can be realized as the deformations of an underlying vacuum algebra such as Eq. (2.28). For example, the most general deformation of the Heisenberg algebra (2.28) within the associative algebra \( \mathcal{A}_\hbar \) is given by

\[ x^i \rightarrow x^i, \quad p_i \rightarrow p_i + A_i(x,t), \quad H \rightarrow H + A_0(x,t), \]

(2.30)

where \( (A_0, A_i)(x,t) \) must be electromagnetic gauge fields. Then the time evolution of a particle system under a time-dependent external force is given by

\[ \frac{df}{dt} = \frac{\partial f}{\partial t} + \frac{i}{\hbar} [H, f]. \]

(2.31)
Note that the construction of the NC algebra $\mathcal{A}_N$ (or $\mathcal{A}_\theta$) bears a close parallel to quantum mechanics. The former is based on the NC space (2.5) or (2.14) while the latter is based on the NC phase space (2.23). The NC $U(1)$ gauge fields in Eq. (2.6) act as a deformation of the vacuum algebra (2.5) in the matrix algebra $\mathcal{A}_N$, similarly to Eq. (2.30) in the quantum algebra $\mathcal{A}_\theta$. Therefore we can apply the same philosophy to the NC algebra $\mathcal{A}_N$ (or $\mathcal{A}_\theta$) to define a dynamical system based on the Moyal-Heisenberg algebra (2.5) or (2.14). In other words, we can introduce an evolution equation for the dynamical system characterized by the deformations (2.6) and (2.7). For this purpose, we extend the NC algebra $\mathcal{A}_\theta$ to $\mathcal{A}_\theta^\perp \equiv \mathcal{A}_\theta \left( C^\infty(\mathbb{R}) \right) = C^\infty(\mathbb{R}) \otimes \mathcal{A}_\theta$ whose generic element takes the form

$$\widehat{f}(t, y) \in \mathcal{A}_\theta^\perp,$$

(2.32)

The matrix representation (2.15) is then replaced by

$$\widehat{f}(t, y) = \sum_{n,m=1}^{\infty} |n\rangle \langle n| \widehat{f}(t, y) |m\rangle \langle m| = \sum_{n,m=1}^{\infty} f_{nm}(t) |n\rangle \langle m|$$

(2.33)

where $f_{nm}(t) := [f(t)]_{nm}$ are elements of a matrix $f(t)$ in $\mathcal{A}_N^1 \equiv \mathcal{A}_N \left( C^\infty(\mathbb{R}) \right) = C^\infty(\mathbb{R}) \otimes \mathcal{A}_N$ as a representation of the observable (2.32) on the Hilbert space (2.12). As the Heisenberg equation (2.31) in quantum mechanics suggests, the evolution equation for an observable $\widehat{f}(t, y) \in \mathcal{A}_\theta^\perp$ in the Heisenberg picture is defined by

$$\frac{d\widehat{f}(t, y)}{dt} = \frac{\partial \widehat{f}(t, y)}{\partial t} - i[\widehat{A}_0(t, y), \widehat{f}(t, y)]_\ast \equiv \widehat{D}_0 \widehat{f}(t, y)$$

(2.34)

where we denoted the local Hamiltonian density by

$$\widehat{H}(t, y) \equiv -\widehat{A}_0(t, y) \in \mathcal{A}_\theta^\perp.$$

(2.35)

The definition (2.34) is intended for the following reason. Note that

$$-i[\phi_a, \widehat{f}(t)] = \partial_a \widehat{f}(t, y) - i[\widehat{A}_a(t, y), \widehat{f}(t, y)]_\ast \equiv \widehat{D}_a \widehat{f}(t, y),$$

(2.36)

where the representation (2.33) has been employed. Then one can see that the inner automorphism $\text{Inn}(\mathcal{A}_\theta)$ of $\mathcal{A}_\theta$ can be lifted to the automorphism of $\mathcal{A}_\theta^\perp$ given by

$$\widehat{A}_0(t, y) \rightarrow \widehat{U}(t, y) \star \frac{\partial \widehat{U}^{-1}(t, y)}{\partial t} + \widehat{U}(t, y) \star \widehat{A}_0(t, y) \star \widehat{U}^{-1}(t, y),$$

(2.37)

$$\widehat{A}_a(t, y) \rightarrow \widehat{U}(t, y) \star \frac{\partial \widehat{U}^{-1}(t, y)}{\partial y^a} + \widehat{U}(t, y) \star \widehat{A}_a(t, y) \star \widehat{U}^{-1}(t, y),$$

(2.38)

where $\widehat{U}(t, y) = e^{i\widehat{\lambda}(t, y)}$ with $\widehat{\lambda}(t, y) \in \mathcal{A}_\theta^\perp$. It is obvious that the above automorphism is nothing but gauge transformations for NC $U(1)$ gauge fields in $(2n + 1)$-dimensions [53].

Our leitmotif is that a consistent theory of quantum gravity should be background-independent, so that it should not presuppose any spacetime background on which fundamental processes develop.
Hence the background-independent theory must provide a mechanism of spacetime generation such
that every spacetime structure including the flat spacetime arises as a solution of the theory itself.
One of the most natural candidates for such a background-independent theory is a zero-dimensional
matrix model such as Eq. (2.1) because it is not necessary to assume the prior existence of spacetime
to define the theory. We emphasized that the NC nature of a vacuum solution, e.g. Eq. (2.5), is
essential to realize the large $N$ duality via the duality chain (2.21). A profound feature is that the
background-independent theory is intrinsically dynamical because the space of all possible solutions
is extremely large, typically infinite-dimensional and generic deformations such as Eq. (2.5) will span
a large subspace in the solution space [13]. We argued that the dynamics under the Moyal-Heisenberg
vacuum (2.4) is described by the NC algebra
$$\mathcal{A}_N^1 = \mathcal{A}_N(C^\infty(\mathbb{R})) = C^\infty(\mathbb{R}) \otimes \mathcal{A}_N.$$ One may regard
$\mathcal{A}_N^1$ as a one-parameter family of deformations of the algebra $\mathcal{A}_N$. In this case we can generalize the
duality chain (2.21) to realize the “time-dependent” gauge/gravity duality as follows:
$$\mathcal{A}_N^1 \Rightarrow \mathcal{A}_b^1 \Rightarrow \mathcal{D}^1. \quad (2.39)$$
It is well-known [34] that in the case of $\mathcal{A}_N^1$ (or $\mathcal{A}_b^1$), the module of its derivations can be written as a
direct sum of the submodules of horizontal and inner derivations:
$$\mathcal{D}^1 = \text{Hor}(\mathcal{A}_N^1) \oplus \mathcal{D}(\mathcal{A}_N^1) \cong \text{Hor}(\mathcal{A}_b^1) \oplus \mathcal{D}(\mathcal{A}_b^1) \quad (2.40)$$
where horizontal derivations are liftings of smooth vector fields on $\mathbb{R}$ onto $\mathcal{A}_N^1$ (or $\mathcal{A}_b^1$) and are locally
generated by a vector field
$$g(t,y)\frac{\partial}{\partial t} \in \text{Hor}(\mathcal{A}_b^1). \quad (2.41)$$
The inner derivation $\mathcal{D}(\mathcal{A}_b^1)$ is defined by lifting the NC vector fields in Eq. (2.19) onto $\mathcal{A}_b^1$ and
generated by
$$\{\hat{V}_a(t) \equiv \text{ad}_\hat{\varphi}_a \in \mathcal{D}(\mathcal{A}_b^1)\} \hat{\phi}_a(t,y) = p_a + \hat{A}_a(t,y) \in \mathcal{A}_b^1, \; a = 1, \ldots, 2n \} \quad (2.42)$$
and
$$\{\hat{V}_0(t) - \frac{\partial}{\partial t} \equiv \text{ad}_{\hat{A}_0} \in \mathcal{D}(\mathcal{A}_b^1)\}.$$ \quad (2.43)
It might be remarked that the definition of the time-like vector field $\hat{V}_0(t)$ is motivated by the quantum
Hamilton’s equation (2.34), i.e.,
$$\hat{V}_0(t) := \frac{d}{dt}, \quad (2.44)$$
Consequently, the module of the derivations of the NC algebra $\mathcal{A}_b^1$ is generated by
$$\mathcal{D}^1 = \{\hat{V}_A(t) = (\hat{V}_0, \hat{V}_a)(t) = \frac{\partial}{\partial t} + \text{ad}_{\hat{A}_0}, \; \hat{V}_a(t) = \text{ad}_\hat{\varphi}_a, \; A = 0, 1, \ldots, 2n\}. \quad (2.45)$$
In the commutative limit, $|\theta| \to 0$, the time-dependent polyvector fields $\hat{V}_A(t)$ in $\mathcal{D}^1$ will take the following form

$$\hat{V}_0(t) = \frac{\partial}{\partial t} + A_0^\mu(t,y) \frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} A_0^{\mu_2 \ldots \mu_p}(t,y) \frac{\partial}{\partial y^{\mu_2}} \cdots \frac{\partial}{\partial y^{\mu_p}},$$

(2.46)

$$\hat{V}_a(t) = V_a^\mu(t,y) \frac{\partial}{\partial y^\mu} + \sum_{p=2}^{\infty} V_a^{\mu_2 \ldots \mu_p}(t,y) \frac{\partial}{\partial y^{\mu_2}} \cdots \frac{\partial}{\partial y^{\mu_p}}.$$  

(2.47)

Let us truncate the above polyvector fields to ordinary vector fields given by

$$\mathfrak{X}(\mathcal{M}) = \left\{ V_A = V_A^M(t,y) \frac{\partial}{\partial y^M} | A, M = 0, 1, \ldots, 2n \right\}$$

(2.48)

where $V_A^0 = \delta_A^0$ and $y^M = (t, y^\mu)$ are local coordinates on an emergent *Lorentzian* manifold $\mathcal{M}$ of $(2n + 1)$-dimensions. The orthonormal vielbeins on $T\mathcal{M}$ are then defined by the relation

$$V_A = \lambda E_A \in \Gamma(T\mathcal{M})$$

(2.49)

or on $T^*\mathcal{M}$

$$v^A = \lambda^{-1} e^A \in \Gamma(T^*\mathcal{M}).$$

(2.50)

The conformal factor $\lambda \in C^\infty(\mathcal{M})$ is similarly determined by the volume preserving conditions

$$\mathcal{L}_{V_A} \nu_t = \left( \nabla \cdot V_A + (1 - 2n) V_A \ln \lambda \right) \nu_t = 0, \quad \forall A = 0, 1, \ldots, 2n,$$

(2.51)

where

$$\nu_t \equiv dt \wedge \nu = \lambda^2 dt \wedge v^1 \wedge \cdots \wedge v^{2n}$$

(2.52)

is a $(2n + 1)$-dimensional volume form on $\mathcal{M}$. In the end, the Lorentzian metric on a $(2n + 1)$-dimensional spacetime manifold $\mathcal{M}$ is given by

$$ds^2 = G_{MN}(x)dx^M \otimes dx^N = e^A \otimes e^A$$

$$= \lambda^2 v^A \otimes v^A = \lambda^2 \left( - dt^2 + v^a_\mu v^a_\nu (dy^\mu - A^\mu)(dy^\nu - A^\nu) \right)$$

(2.53)

where $A^\mu := A_0^\mu(t,y)dt$.

It should be noted that the time evolution (2.44) for a general time-dependent system is not completely generated by an inner automorphism since $\text{Hor}(A_0^\theta)$ is not an inner but outer derivation. This happens since the time variable $t$ is a bach. Thus one may extend the phase space by introducing a conjugate variable $H$ of $t$ so that the extended phase space becomes a symplectic manifold. Then it is well-known [29] that the time evolution of a time-dependent system can be defined by the inner automorphism of the extended phase space whose extended Poisson bivector is given by

$$\vartheta = \theta + \frac{\partial}{\partial t} \wedge \frac{\partial}{\partial H}$$

(2.54)
where
\[ \theta = \frac{1}{2} \theta^{\mu\nu} \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu} \] (2.55)
is the original Poisson bivector responsible for the NC space (2.14). As a result, one can see [12] that the temporal vector field (2.44) is realized as a generalized Hamiltonian vector field defined by
\[ V_0 = \mathcal{X}_H = \partial (dH) = \frac{\partial}{\partial t} + X_H \] (2.56)
where \( X_H = \theta (dA_0) \) is the original Hamiltonian vector field realized as an inner derivation \( \text{ad}_{\hat{A}_0} = X_H + \mathcal{O}(\theta^2) \in \mathcal{D} \). But we have to pay the price for the extension of phase space. The extended Poisson structure (2.54) raises a serious issue whether the time variable for a general time-dependent system might also be quantized; in other words, time also becomes an operator obeying the commutation relation \([t, H] = i\). We want to be modest not to address this issue since it is a challenging open problem even in quantum mechanics.

We figure out the time issue in a less ambitious way. Suppose that \((M, B \equiv \theta^{-1})\) is the original symplectic manifold responsible for the emergence of spaces. Now we consider a contact manifold \((\mathbb{R} \times M, \mathcal{B})\) where \( \mathcal{B} = \tilde{B} + dH \wedge dt \) and \( \tilde{B} = \pi_2^* B \) is defined by the projection \( \pi_2 : \mathbb{R} \times M \to M, \pi_2(t, x) = x \) [29]. We define the concept of (space)time in emergent gravity through the contact manifold \((\mathbb{R} \times M, \mathcal{B})\) in the sense that the derivations in Eq. (2.45) can be obtained by quantizing the contact manifold \((\mathbb{R} \times M, \mathcal{B})\). Indeed it is shown in Appendix A that the time-like vector field \( V_0 \) in Eq. (2.56) arises as a Hamiltonian vector field of a cosymplectic manifold whose particular class is a contact manifold. Note that the emergent geometry described by the metric (2.53) respects the (local) Lorentz symmetry. If one gazes at the the metric (2.53), one can see that the Lorentzian manifold \( \mathcal{M} \) becomes a flat Minkowski spacetime on a local Darboux chart in which all fluctuations die out, i.e., \( v^a_\mu \to \delta^a_\mu, A^\mu \to 0 \), so \( \lambda \to 1 \). We have to emphasize (see footnote 7) that the vacuum algebra responsible for the emergence of the flat Minkowski spacetime is the Moyal-Heisenberg algebra (2.5). Many surprising results will immediately come out from this dynamical origin of the flat spacetime [11, 19], which is absent in general relativity.

We close this section by observing that the quantized version of the contact manifold \((\mathbb{R} \times M, \mathcal{B})\) is described by the BFSS-like matrix model whose action is given by
\[ S = -\frac{1}{g^2} \int dt \text{Tr} \left( \frac{1}{2} (D_0 \phi_a)^2 - \frac{1}{4} [\phi_a, \phi_b]^2 \right), \] (2.57)
where \( D_0 \phi_a = \frac{\partial \phi_a}{\partial t} - i [A_0, \phi_a] \). We interpret the BFSS matrix model (2.57) as a Hamiltonian system of the IKKT matrix model whose action is given by Eq. (2.1). Note that the original BFSS matrix model [35] contains 9 adjoint scalar fields while the action (2.57) has even number of adjoint scalar fields. For the former case, on the one hand, we have no idea how to realize the adjoint scalar fields as a matrix representation on a Hilbert space like as (2.32). Even it may be nontrivial to construct the Hilbert space because the M-theory is involved with a 3-form instead of symplectic 2-form. For the
latter case, on the other hand, the previous Moyal-Heisenberg vacuum (2.4) is naturally extended to the vacuum configuration of $A^1_N$ given by

$$\langle \phi_a \rangle_{\text{vac}} = p_a, \quad \langle \hat{A}_0 \rangle_{\text{vac}} = 0,$$

(2.58)

where the vacuum moduli $p_a \in A^1_N$ satisfy the commutation relation (2.5). We consider all possible deformations of the vacuum (2.58) and parameterize them as

$$\hat{\phi}_A(t, y) = p_A + \hat{A}_A(t, y) \in A^1_\theta,$$

(2.59)

where the isomorphism (2.33) between $A^1_N$ and $A^1_\theta$ was used. Note that

$$[\hat{\phi}_A, \hat{\phi}_B]_* = -i (B_{AB} - \hat{F}_{AB}),$$

(2.60)

where

$$\hat{F}_{AB} = \partial_A \hat{A}_B - \partial_B \hat{A}_A - i [\hat{A}_A, \hat{A}_B]_* \in A^1_\theta$$

(2.61)

and

$$B_{AB} = \begin{pmatrix} 0 & 0 \\ 0 & B_{ab} \end{pmatrix}.$$

Plugging the fluctuations (2.59) into the action (2.57) leads to a $(2n + 1)$-dimensional NC $U(1)$ gauge theory whose action is given by

$$S = -\frac{1}{g^2} \int dt \text{Tr} \left( \frac{1}{2} (D_0 \phi_0)^2 - \frac{1}{4} [\phi_0, \phi_0]^2 \right)$$

$$= -\frac{1}{g^2_{YM}} \int d^{2n+1}y \frac{1}{4} (\hat{F}_{AB} - B_{AB})^2,$$

(2.62)

where $g^2_{YM} = (2\pi)^n |\text{Pf} \theta| g^2$ is the $(2n + 1)$-dimensional gauge coupling constant. By applying the time-dependent duality chain (2.39) to time-dependent matrices in $A^1_N$, it is straightforward to derive the module $\mathcal{D}^1$ in Eq. (2.45) from the large $N$ matrices or NC $U(1)$ gauge fields in the action (2.62).

A Lorentzian geometry described by the metric (2.53) corresponds to a classical spacetime emergent from a quantum geometry derived from the NC module $\mathcal{D}^1$ [13].

\footnote{It is worth remarking that the matrix quantum mechanics (2.57) can be written in a more symmetrical way by defining $\phi_0 \equiv \frac{\partial}{\partial t} - i A_0(t)$. With this definition, the action can be succinctly written as

$$S = \frac{1}{4g^2} \int dt \text{Tr} [\phi_A, \phi_B]^2$$

where $\phi_A(t) = (\phi_0, \phi_a)(t)$. Then the vacuum configuration (2.58) can be stated as $\langle \phi_A \rangle_{\text{vac}} = p_A = (\hat{\phi}_0, p_a)$. This notation makes a merit of the emphasis that the temporal differential operator in $\phi_0$ must be regarded as a timelike background on an equal footing with the spatial vacuum moduli $p_a$. It is important to note that the vacuum (2.58) gives rise to the flat Minkowski spacetime because the vector fields in Eq. (2.49) on this vacuum are given by $\langle V_A \rangle_{\text{vac}} = \delta^M_A \frac{\partial}{\partial y^M}$ and they determine a vacuum geometry for the general Lorentzian metric (2.53).}
3 Cosmic inflation as a time-dependent Hamiltonian system

Let us recapitulate how we have obtained the \((2n + 1)\)-dimensional Lorentzian spacetime \( \mathcal{M} \) described by the nontrivial metric \( (2.53) \). At the outset, we have considered a background independent theory in which any existence of spacetime is not assumed but defined by the theory. Of course, the background-independent theory does not mean that the physics is independent of the background. Background independence here means that, although a physical phenomenon occurs in a particular background with a specific initial condition, an underlying theory itself describing such a physical event should presuppose neither any kind of spacetime nor material backgrounds. Therefore the background itself should arise from a vacuum solution of the underlying theory. In particular, the background-independent theory has to make no distinction between geometry and matter since it has no predetermined background. We have prescribed a most primitive vacuum such that it generates the most simplest spacetime structure. General and more complicated structures of spacetime are obtained by deforming the primitive vacuum in all possible ways. These deformations correspond to physical processes that happen upon a particular (spacetime) background. Motivated by a close analogy with quantum mechanics, we argued that the evolution of spacetime structure supported on a vacuum solution must be understood as a dynamical system. We have formulated the dynamical system described by the fundamental action \((2.62)\), from which an emergent \((2n + 1)\)-dimensional Lorentzian spacetime \( \mathcal{M} \) with the metric \( (2.53) \) is derived.

Let us look at the vacuum geometry for the metric \( (2.53) \) in which \( v^\mu \rightarrow \delta^\mu, \quad A^\mu \rightarrow 0 \). In this case, \( V_A^{(0)} \equiv \langle V_A \rangle_{\text{vac}} = \delta^A_M \frac{\partial}{\partial y^T}, \) so \( \lambda^2 \rightarrow 1 \) because \( 2V_A \ln \lambda = g_{BAB} \) if the structure equations of vector fields \( V_A = \lambda E_A \in \Gamma(T\mathcal{M}) \) are defined by \([11, 12]\)

\[
[V_A, V_B] = -g_{AB}^C V_C. \tag{3.1}
\]

Since \( \langle \phi_A \rangle_{\text{vac}} = p_A = (\frac{\partial}{\partial t}, p_a) \) and the corresponding vector fields are given by \( V_A^{(0)} = E_A^{(0)} = (\frac{\partial}{\partial t}, \text{ad}_{p_a}) = \delta^M_A \frac{\partial}{\partial y^T}, \) we see that the flat Minkowski spacetime was originated from the vacuum configuration \((2.58)\). See also footnote 7. We can calculate the vacuum energy density caused by the condensate \((2.58)\) using the action \((2.62)\):

\[
\rho_{\text{vac}} = \frac{1}{4g_{YM}^2} |B_{ab}|^2. \tag{3.2}
\]

A striking fact is that the vacuum \((2.58)\) responsible for the generation of flat spacetime is not empty. Rather the flat spacetime had been originated from the uniform vacuum energy \((3.2)\) known as the cosmological constant in general relativity. This is a tangible difference from Einstein gravity, in which \( T_{\mu \nu} = 0 \) in flat spacetime as one can see from Eq. \((1.1)\). In the end, the emergent gravity reveals a remarkable picture that a uniform vacuum energy such as Eq. \((3.2)\) does not gravitate. As a result, it turns out that the emergent gravity does not contain the coupling of cosmological constant like \( \int d^{2n+1}x \sqrt{-G} \Lambda \), so it presents a striking contrast to general relativity. This important conclusion may be strengthened by applying the Lie algebra homomorphism \((2.18)\) to the commutators in Eq.
ad_{i[\hat{\phi}_A,\hat{\phi}_B]} = [\hat{V}_A,\hat{V}_B] \equiv \hat{V}_{F_{AB}} - B_{AB} \equiv \hat{V}_{F_{AB}} \in \mathcal{O}^1 \tag{3.3}

for a constant field strength $B_{AB}$. To emphasize clearly, the gravitational fields emergent from NC $U(1)$ gauge fields must be insensitive to the constant vacuum energy such as Eq. (3.2). In consequence, the emergent gravity clearly dismisses the notorious cosmological constant problem \[11, 12, 19\].

In order to estimate the dynamical energy scale for the vacuum condensate (2.58), note that the Newton constant $G_N$ in emergent gravity has to be determined by field theory parameters only such as the gauge coupling constant $g_{YM}$ and $\theta = B^{-1}$ defining the NC $U(1)$ gauge theory (2.62). A simple dimensional analysis leads to the result \[12, 19\].

$$G_N \hbar^2 c^2 \sim g_{YM}^2 |\theta|, \tag{3.4}$$

where $|\theta| := |\text{Pf} \theta|^{1/2}$. To be specific, when considering the four-dimensional case in which $M_P = (8\pi G_N)^{-1/2} \sim 10^{18}$ GeV and $g_{YM}^2 \sim \frac{1}{137}$, the vacuum energy (3.2) due to the condensate (2.58) is at a moderate estimate given by

$$\rho_{\text{vac}} = \frac{1}{4g_{YM}^2} |B_{ab}|^2 \sim g_{YM}^2 M_P^4 \sim 10^{-2} M_P^4. \tag{3.5}$$

Amusingly emergent gravity discloses that the perverse vacuum energy $\rho_{\text{vac}} \sim M_P^4$ was actually the origin of flat spacetime. It is worthwhile to remark that the gravitational constant $G_N \sim M_P^{-2}$ naturally sets a dynamical scale for the emergence of gravity and spacetime if quantum gravity should be formulated in a background-independent way so that the spacetime geometry emerges from a vacuum configuration of some fundamental ingredients in the underlying theory. Therefore it may be not a surprising result but rather an inevitable consequence that the Planck energy density (3.5) in vacuum was the genetic origin of spacetime.

Note that the Planck energy condensate in vacuum resulted in an extremely extended spacetime as the metric (2.53) clearly indicates. However the spacetime was not existent at the beginning but simply emergent from the vacuum condensate (2.58). Therefore the Planck energy condensation into vacuum must be regarded as a dynamical process. Since the dynamical scale for the vacuum condensate is about of the Planck energy, the time scale for the condensation will be roughly of the Planck time $\sim 10^{-44}$ sec. Inflation scenario asserts that our Universe at the beginning had undergone an explosive inflation era lasted roughly $\sim 10^{-33}$ seconds. Thus it is natural to consider the cosmic inflation as a dynamical process for the instantaneous condensation of vacuum energy $\rho_{\text{vac}} \sim M_P^4$ to enormously spread out spacetime \[19\]. Now we will explore how the cosmic inflation is triggered by the condensate of Planck energy in vacuum responsible for the emergence of spacetime.

First let us understand intuitively Eqs. (1.2) and (1.3) to get some dear insight from the old wisdom. Suppose that a test particle with mass $m$ is placed in the condensate (3.5). Consider a ball of
radius $r(t)$ and the test particle placed on its surface. According to the Gauss’s law, the particle will be subject to the gravitational potential energy $V(r) = -\frac{G_N m}{r}$ caused by the condensate (3.5), where $M(r) = \frac{4\pi r^3}{3} \rho_{\text{vac}}$ is the total mass inside the ball. In order to preserve the total energy $E$ of the particle, the ball has to expand so that the kinetic energy $K(r) = \frac{1}{2}m\dot{r}(t)^2$ generated by the expansion compensates the negative potential energy. That is, the energy conservation implies the following relation

$$H^2 = \frac{8\pi G_N \rho_{\text{vac}}}{3} - \frac{k}{r(t)^2},$$

(3.6)

where $H = \frac{\dot{r}(t)}{r(t)}$ is the expansion rate and $k \equiv -\frac{2E}{m}$. By comparing the above equation with the Friedmann equation (1.2) after the identification $r(t) = Ra(t)$, we see that Eq. (3.6) corresponds to $\rho_{\text{vac}} = V(\phi) \approx V_0$ and $\dot{\phi} \approx 0$ with $k = 0$. At the outset we actually assumed the spatially flat universe, $k = 0$, for the Friedmann equation (1.2). In our approach with a background-independent theory, the condition $k = 0$ is automatic since the very beginning should be absolutely nothing! This conclusion is consistent with the metric (2.53) which describes a final state of cosmic inflation. Hence we may moderately claim that the background-independent theory for cosmic inflation predicts a spatially flat universe, in which the constant $k$ must be exactly zero.

From the above simple argument, we see that the size of the ball exponentially expands, i.e.,

$$a(t) = a_0 e^{Ht}$$

(3.7)

where

$$H = \sqrt{\frac{8\pi G_N \rho_{\text{vac}}}{3}}$$

(3.8)

is a constant. Let us introduce fluctuations around the inflating solution (3.7) by considering $\rho_{\text{vac}} \to \rho_{\text{vac}} + \delta \rho$ and $\dot{\phi} \neq 0$, where $\delta \rho$ is the mechanical energy due to the fluctuations of the inflaton $\phi(t)$. Then the evolution equation (3.6) is replaced by

$$H^2 = \frac{8\pi G_N}{3}(\rho_{\text{vac}} + \delta \rho),$$

(3.9)

and the dynamics of the inflaton is described by Eq. (1.3). As we already remarked in Eq. (1.5), the dynamics of the inflaton must be described by a non-Hamiltonian system, whose mathematical basis will be reviewed in Appendix A. Therefore, in order to describe the cosmic inflation in the context of emergent gravity, we need to extend the module $D^1$ of differential operators in Eq. (2.39) so that the exponential behavior (3.7) is derived from it. In classical limit, such vector fields are known as conformal vector fields whose flow preserves a symplectic form up to a constant, so they appear in the conformal Hamiltonian dynamics such as simple mechanical systems with friction [20].

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8It might be remarked that this experiment is a simple twist of the well-known solution of Gauss’s law for gravity inside the earth, in which the minus sign in the gravitational potential energy presupposes a repulsive force rather than the usual attractive force. Moreover the repulsive force is given by $F = k_g r = -\nabla V(r)$ where $k_g = \frac{4\pi G_N m \rho_{\text{vac}}}{3}$ and $V(r) = -\frac{G_N M(r)m}{2r}$ is the gravitational potential energy in Newtonian gravity. The factor 2 enhancement is due to the general relativity effect.
However, the vector field defined by Eq. (3.12) is more convenient for our case. The equations of motion for the Hamiltonian $H$ are given by

$$ \dot{\boldsymbol{q}} - \kappa \dot{\boldsymbol{q}} + \frac{\partial V}{\partial \boldsymbol{q}} = 0, $$

where $V(q) = U(q) + \frac{\kappa^2}{8} q^2$. To be specific, the integral curves for $U(q) = \frac{1}{2} \omega^2 q^2$ are given by

$$ q^i(t) = e^{\frac{\kappa}{2} t} q^i(\kappa = 0; t), \quad p_i(t) = e^{\frac{\kappa}{2} t} p_i(\kappa = 0; t), $$

where $q^i(\kappa = 0; t) = A^i \sin(\omega t + \theta)$ and $p_i(\kappa = 0; t) = B_i \cos(\omega t + \theta)$ describe the usual harmonic oscillator with a closed orbit when $\kappa = 0$. Therefore we see that the flow generated by a conformal vector field has the property

$$ \phi^* \omega = e^{\kappa t} \omega, $$

which may be directly obtained by integrating Eq. (3.11). This means that the volume of phase space exponentially expands (contracts) if $\kappa > 0$ ($\kappa < 0$). When we compare the equations of motion

---

Note that $a = b + d \lambda$ where $b = -p_i dq^i$ and $\lambda = \frac{1}{2} q^i d p_i$. Thus one can also define the conformal vector field $X$ by $\iota_X \omega = kb + d H'$ where $H' = H + \kappa \lambda$. In this case $X = \kappa p_i \frac{\partial}{\partial p_i} + X_H$, and the equations of motion are given by $\frac{dq^i}{dt} = \frac{\partial H'}{\partial q^i}$ and $\frac{dp_i}{dt} = \kappa p_i - \frac{\partial H'}{\partial q^i}$. For $H' = \frac{1}{2} (p_i^2 + \omega^2 q_i^2)$, the general solution is given by $q^i(t) = A^i e^{\frac{\kappa}{2} t} \sin \left( \sqrt{\omega^2 - \frac{\kappa^2}{4}} t + \theta \right)$. However, the vector field defined by Eq. (3.12) is more convenient for our case.

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The proof goes as follows. Let $\phi_t$ denote the flow of $X$. By the Lie derivative theorem, we have $\frac{d}{dt}(\phi_t^* \omega) = \phi_t^* \mathcal{L}_X \omega = \kappa \phi_t^* \omega$, which has the unique solution (3.17).
(3.15) with Eq. (1.3), we see that \( \kappa = -3H \), so the expansion of spacetime, i.e. \( H > 0 \), leads to a damping of inflaton fluctuations.

The mathematical parallelism between quantum mechanics and NC spacetime suggests how to formulate the cosmic inflation as a dynamical system. First note that the NC space (2.14) in commutative limit becomes a phase space with the symplectic form

\[
B = \frac{1}{2} B_{ab} dy^a \wedge dy^b.
\]

(3.18)

The dynamics of Hamiltonian systems is characterized by the invariance of phase space volume under time evolution and the conservation of phase space volume for divergenceless Hamiltonian flows is known as the Liouville theorem [29]. However, the cosmic inflation means that the volume of spacetime phase space has to exponentially expand as Eq. (3.7) clearly suggests. Hence the cosmic inflation as a dynamical system has to be regarded as a non-Hamiltonian system[11] and a generalized Liouville theorem is necessary to describe the exponential expansion of spacetime. We have already explained above how such a non-Hamiltonian dynamics can be formulated in terms of a conformal Hamiltonian system characterized by the (local) flow obeying Eq. (3.11). See Appendix A for a mathematical exposition of more general time-dependent nonconservative dynamical systems.

Let us apply the conformal Hamiltonian dynamics to the cosmic inflation. Recall that we have considered an atlas \( \{(U_i, \phi_{(i)})\} \) on \( M = \bigcup_{i \in I} U_i \) as a collection of local Darboux charts and complete it by gluing these local charts on their overlap. On each local chart, we have a local symplectic structure \( \Omega_i = \frac{1}{2} B_{ab} dy^a_{(i)} \wedge dy^b_{(i)} \) where \( \{y^a_{(i)}\} \) are Darboux coordinates on a local patch \( U_i \subset M \). As was explained in Refs. [40, 41] and reproduced in Appendix A, the phase space coordinates \( \{y^a_{(i)}\}_{U_i} \) of a conformal Hamiltonian system undergo a nontrivial time evolution even in a local Darboux frame. For example, look at the equations of motion (3.13) and (3.14) to recognize such a nontrivial time evolution even when \( H = 0 \). The dynamics in this case consists of the orbits of a conformal vector field \( X \) obeying the condition (A.29). The result is essentially the same as the previous mechanical system with negative-friction. To be specific, write \( \Omega_i = da_{(i)} \) on a local patch \( U_i \subset M \) where

\[
a_{(i)} = -\frac{1}{2} p_a^{(i)} dy^a_{(i)} \text{ with } p_a^{(i)} = B_{ab} y^b_{(i)}
\]

and consider a conformal vector field \( X \) defined by

\[
\iota_X \Omega_i = \kappa a_{(i)} + dH_i,
\]

(3.19)

where \( H_i : U_i \to \mathbb{R} \) is a local Hamiltonian and \( \kappa \) is a positive constant. Using the fact that \( d\Omega_i = 0 \), it is easy to derive the condition (A.29) from Eq. (3.19), i.e.,

\[
\mathcal{L}_X \Omega_i = \kappa \Omega_i.
\]

(3.20)

[11] We want to remark that such systems ubiquitously arise in, e.g., dynamical systems with friction and nonequilibrium statistical mechanics. Recently the statistical mechanics of non-Hamiltonian systems has been formulated using a generalized Liouville measure to study the simulations of molecular dynamics. See, for example, [36, 37, 38, 39]. We think that their formulation may be useful to understand the evolution of our early universe, especially, regarding to the issue of the cosmic Landau damping discussed in the last section.
The vector field $X$ obeying Eq. (3.19) is given by

$$X = \frac{\kappa}{2} y_a^{(i)} \frac{\partial}{\partial y_a^{(i)}} + X_{H_i},$$

(3.21)

where $X_{H_i}$ is the ordinary Hamiltonian vector field satisfying $\iota(X_{H_i})\Omega_i = dH_i$. We will set $H_i = 0$ for simplicity. Later we will explain why we could simplify the problem with $H_i = 0$. The time evolution of local Darboux coordinates is then determined by the equations

$$\frac{dy_a^{(i)}}{dt} = X(y_a^{(i)}) = \frac{\kappa}{2} y_a^{(i)}.$$

(3.22)

The solution is given by

$$y_a^{(i)}(t) = e^{\frac{\kappa}{2}t} y_a^{(i)}(0).$$

(3.23)

We may glue the local solutions (3.23) to have a global form

$$p_a(t) = B_{ab} y^b(t) = e^{\frac{\kappa}{2}t} p_a.$$

(3.24)

Then the time-dependent canonical one-form is given by

$$a(t) = -\frac{1}{2} p_a(t) dy_a(t) = -e^{\kappa t} p_a dy_a$$

(3.25)

and thus

$$\Omega(t) = da(t) = e^{\kappa t} B.$$

(3.26)

The exterior derivative above acts only on $\mathbb{R}^{2n}$. One can show using the proof in footnote [10] that the result (3.26) is the integral form of Eq. (3.20). More generally, the result (3.26) is a particular case of the general Moser flow $\phi_t$ generated by a time-dependent vector field $X_t$ for locally conformal symplectic manifolds (see Appendix A for the definition) which is given by [42]

$$\phi_t^* \Omega_t = \exp \left( \int_0^t \phi_s^* (b_s(X_s)) ds \right) \cdot \Omega,$$

(3.27)

where the one-form $b$ is the Lee form of $\Omega$. The above result (3.26) is simply obtained from Eq. (3.27) when $b(X)$ is a constant $\kappa$.

We have motivated the cosmic inflation with the idea that the vacuum configuration (2.58) is a final state accumulating the vacuum energy (3.2). In other words, there must be a dynamical system describing the mutation from the initial state, so-called “absolutely nothing,” to the final state. For this purpose, let us consider a symplectic manifold $(M, \Omega(t))$ with the symplectic form given by Eq. (3.26). It is natural to consider this symplectic manifold as a time-dependent vacuum configuration

$$\langle \phi_a(t) \rangle_{\text{vac}} = p_a(t) = e^{\frac{\kappa}{2}t} p_a, \quad \langle \hat{A}_0(t) \rangle_{\text{vac}} = 0.$$

(3.28)

The second condition of Eq. (3.28) corresponds to our previous setting $H_i = 0$ according to the identification (2.35). Since the vacuum (3.28) is in highly nonequilibrium, it is expected that it will
eventually evolve to the final state \([2.58]\) through the so-called reheating process. However we do not know a possible mechanism for the reheating. We will speculate in Sec. 5 a plausible picture for the reheating mechanism. Since

\[
\langle \left[ \phi_a(t), \phi_b(t) \right] \rangle_{\text{vac}} = -ie^{\kappa t} B_{ab} = -i\Omega_{ab}(t),
\]

we will regard \(\Omega(t)\) as the symplectic structure of the inflating vacuum \([3.28]\). To appreciate the physical picture of the vacuum configuration \([3.28]\), let us specify a typical length scale at \(t = 0\) as \(|y^a(t = 0)| \sim L_P\). It should be reasonable to identify \(L_P\) with the Planck length. According to the standard inflation scenario, inflation needed to expand the universe by at least a factor of \(e^{60}\). If we simply assume that the radius of the universe at the beginning of inflation is about \(L_P\), 60 e-foldings at \(t = t_{\text{end}} = 10^{-36} \sim 10^{-33} \text{ sec} \) mean that \(\kappa t_{\text{end}} \sim 120\) since \(|y^a(t = t_{\text{end}})| = e^{\frac{\kappa}{2} t_{\text{end}}} |y^a(t = 0)| \sim e^{\frac{\kappa}{2} t_{\text{end}}} L_P\). This implies that \(\kappa = 10^{11} \sim 10^{14} \text{ GeV} \) because \(1 \text{ eV} = (6.6 \times 10^{-16} \text{ sec})^{-1} \). It turns out that \(\kappa\) is identified with the inflationary Hubble constant \(H\) in Eq. \([3.7]\), and this informs us of the inflationary energy scale given by \([3.30]\)

\[
\kappa = 2H = 10^{11} \sim 10^{14} \text{ GeV}.
\]

Let us first determine the vacuum geometry emergent from the vacuum configuration \([3.28]\). In this case it is not necessary to glue Darboux charts because we have not yet introduced local fluctuations, so the Darboux coordinates in \([3.28]\) are globally defined. The underlying Poisson algebra is defined by the time-dependent bivector \(\theta(t) \equiv \Omega(t)^{-1} = e^{-\kappa t} \theta(0) = \frac{1}{2} e^{-\kappa t} \theta_{\mu\nu} \frac{\partial}{\partial y^\mu} \wedge \frac{\partial}{\partial y^\nu}\). According to the definition \([A.11]\), we get (omitting the symbol indicating the vacuum for a notational simplicity)

\[
V_a(t) = \theta(t)(dp_a(t)) = e^{-\frac{\kappa}{2} t} V_a(0)
\]

where \(V_a(0) = \delta_a^\mu \frac{\partial}{\partial y^\mu}\). Similarly,

\[
V_0(t) = \frac{\partial}{\partial t} - \theta(t)(d\hat{A}_0(t, y)) = \frac{\partial}{\partial t}
\]

since \(\hat{A}_0(t, y) = 0\) according to Eq. \([3.28]\). Thus the dual one-forms are given by

\[
v^0(t) = dt, \quad v^a(t) = e^{\frac{\kappa}{2} t} v^a(0)
\]

where \(v^a(0) = \delta^a_\mu dy^\mu\). This coframe basis leads to the volume-form of the vacuum manifold:

\[
\tilde{v}_t = \lambda^2 v^0(t) \wedge v^1(t) \wedge \cdots \wedge v^{2n}(t) = e^{nkt} \lambda^2 dt \wedge dy^1 \wedge \cdots \wedge dy^{2n}.
\]

Then the conformal factor \(\lambda^2\) is given by the formula

\[
\lambda^2 = \tilde{v}_t(V_0(t), V_1(t), \cdots, V_{2n}(t)) = 1.
\]
By applying the results (3.33) and (3.35) to the metric (2.53), we can finally determine the vacuum geometry for the vacuum configuration (3.28) given by

\[ ds^2 = -dt^2 + e^{\kappa t} dy^\mu dy^\nu. \]  

(3.36)

This is the de Sitter space in flat coordinates which covers half of the de Sitter manifold. Therefore we see that the constant \( \kappa \) defining the conformal vector fields in Eq. (3.20) is equal to the inflationary Hubble constant \( H \) as we already identified in (3.30).

Important remarks are in order. First we see that the cosmic inflation is a typical example of a locally conformal symplectic (LCS) manifold. The LCS manifold has a disparate property compared to symplectic manifolds. First of all, the conformal vectorfield defined by Eq. (3.20) (or more generally Eq. (A.25)) is nontrivial even if a corresponding Hamiltonian function identically vanishes. The so-called Liouville vector field \( Z = \frac{1}{2} y^a \frac{\partial}{\partial y^a} \) is still nontrivial [43] and it generates the exponential expansion of spacetime described by the metric (3.36).\(^\text{12}\)

It might be emphasized that the inflation metric (3.36) is emergent from the vacuum (3.28) in nonequilibrium which incorporates no local fluctuations yet. Later we will consider these local fluctuations around the time-dependent vacuum (3.28). If \( \kappa a = -H b \) in Eq. (3.19) where \( b \) is a closed one-form, \( X \) is called a Hamiltonian vector field of an LCS manifold. See the definition (A.10). Even in this case the Hamiltonian vector field shows a peculiar property different from the symplectic case. If \( b \) is not exact, \( X = 0 \) only if \( H = 0 \). Therefore we see that the vector fields of an LCS manifold is in stark contrast to those of symplectic manifolds, in which \( X_f = 0 \) implies \( f = \text{constant} \) only. Remarkably, the last property of Eq. (3.3) does not hold during the inflation.

It may be instructive to compare the above situation with the equilibrium case described by the metric (2.53). First note that the invariant volume form (2.52) can be written as

\[ \nu_t = \lambda^{1-2n} \nu_g, \]  

(3.37)

where \( \nu_g = e^0 \wedge \cdots \wedge e^{2n} = \sqrt{-G} d^{2n+1}x \) is the Riemannian volume form. Therefore, the vector fields \( V_A \) do not necessarily preserve the Riemannian volume form \( \nu_g \) although they preserve the volume form \( \nu_t \). However, since \( \lambda^2 \to 1 \) at spatial infinity as was explained above Eq. (3.1), they do preserve both \( \nu_t|_\infty \) and \( \nu_g|_\infty \). In other words, the flow generated by \( V_A \) leads to only local changes of the spacetime volume while it preserves the volume element at asymptotic regions. On the contrary, the conformal vector fields change the spacetime volume everywhere. Accordingly they definitely give rise to the exponential expansion of the spacetime volume. After all, we see that a natural phase space

\(^{12}\text{It would be worthwhile remarking that it is not possible to realize the Liouville vector field in terms of a Hamiltonian function. Thus the inflation is a dynamical system without any Hamiltonian. It may explain why even string theory faces many difficulties to realize the cosmic inflation. Probably this situation becomes more transparent by the mechanical analogue described by Eq. (3.15). However we show in Appendix B that this situation can be cured by introducing a time-dependent Hamiltonian. Combining this fact with the gedanken experiment regarding to Eq. (3.6) leads to a tempting picture that the universe might have been created from nothing by a tunneling event.}\)
for the cosmic inflation has to contain an LCS manifold replacing standard symplectic manifolds. Including time, it becomes an LCC manifold (whose definition can be found in Appendix A).

As we have advocated the vitality of the background-independent formulation of emergent space-time, we need to realize the inflating background \( (3.28) \) as a solution of the matrix model \( (2.57) \). Now we will examine whether the inflating behavior in Eq. \( (3.28) \) arises as the solution. The matrix action \( (2.57) \) leads to the equations of motion given by \( [28] \)

\[
D_0^2 \phi_a + [\phi_b, [\phi_a, \phi_b]] = 0,
\]

which must be supplemented with the Gauss constraint

\[
[\phi_a, D_0 \phi_a] = 0.
\]

Note that the second term in Eq. \( (3.38) \) identically vanishes for the background \( (3.28) \). Therefore it is necessary to impose the condition

\[
D_0 \phi_a = e^{\frac{\kappa}{2}} (\frac{\kappa}{2} p_a - i [\hat{A}_0, p_a]) = 0
\]

(3.40)

to satisfy both \( (3.38) \) and \( (3.39) \). Thereby, we see that it is required to turn on the temporal gauge field in vacuum, i.e.,

\[
\langle \hat{A}_0(t) \rangle_{\text{vac}} \equiv \hat{a}_0(t, y) \neq 0
\]

(3.41)

and the background metric \( (3.36) \) has also to take its influence into account. To solve the above equation, it may be more convenient to map the matrix algebra \( A^1_N \) to the NC \( \star \)-algebra \( A^1_\theta \) by assuming the matrix representation \( (2.33) \). In terms of the NC \( \star \)-algebra \( A^1_\theta \), Eq. \( (3.40) \) reads as

\[
\frac{\partial \hat{a}_0(t, y)}{\partial y^a} = \frac{\kappa}{2} p_a.
\]

(3.42)

The solution is given by an open Wilson line \( [44] \)

\[
\hat{a}_0(y) = \frac{\kappa}{2} \int_0^1 d\sigma \frac{dy^\mu(\sigma)}{d\sigma} p_\mu(\sigma)
\]

(3.43)

along a path parameterized by the curve

\[
y^\mu(\sigma) = y_0^\mu + \zeta^\mu(\sigma)
\]

(3.44)

where \( \zeta^\mu(\sigma) = \theta^{\mu\nu} k_\nu \sigma \) with \( 0 \leq \sigma \leq 1 \) and \( y^\mu(\sigma = 0) \equiv y_0^\mu \) and \( y^\mu(\sigma = 1) \equiv y^\mu \). One can easily check Eq. \( (3.42) \) using the formula

\[
\frac{\partial}{\partial y^\mu} \int_0^1 d\sigma \frac{dy^\nu(\sigma)}{d\sigma} K(y(\sigma)) = \delta^\nu_\mu K(y)
\]

(3.45)

for some differentiable function \( K(y) \).
Before correcting the metric (3.36) due to the background gauge field (3.41), we want to discuss some physical significance of the nonlocal term (3.43). First note that the temporal gauge field (3.43) is a background Hamiltonian density according to the identification (2.35) and time-independent. We will see soon that the gravitational metric including the effect of the nonlocal term (3.43) is still local as it should be. It was already noticed in [45] that nonlocal observables in emergent gravity are in general necessary to describe some gravitational metric that is nonetheless local. Moreover the appearance of such nonlocal terms should not be surprising in NC gauge theories, in which there exist no local gauge invariant observables. Indeed it was shown in [44] that nonlocal observables are the NC generalization of gauge invariant operators in NC gauge theories.

Now let us include the nonlocal term (3.43) and take into account its influence on the metric (3.36). It causes the timelike vector field (3.32) to change into the following form

\[ V_0(t) = \frac{\partial}{\partial t} + \kappa \frac{e^{-\kappa t}}{2} y^\mu \frac{\partial}{\partial y^\mu} \]

(3.46)

Note that the vector field takes the local form again as the result of applying the formula (3.45). Then the dual one-forms are instead given by

\[ v^0(t) = dt, \quad v^\mu(t) = e^{\frac{\kappa}{2} t} (dy^\mu - a^\mu), \]

(3.47)

where

\[ a^\mu = e^{-\kappa t} y^\mu dt. \]

(3.48)

It is straightforward to see that the conformal factor defined by Eq. (3.35) is not affected by the background (3.41), i.e., \( \lambda^2 = 1 \). In the end, the de Sitter metric (3.36) is replaced by the metric

\[ ds^2 = -dt^2 + e^{\kappa t} (dy^\mu - a^\mu)(dy^\mu - a^\mu) \]

(3.49)

after including the background gauge field (3.43). It may be worthwhile to point out that the temporal one-form (3.48) rapidly decays so that the previous background (3.28) was a sufficiently good approximation at late times although the temporal gauge field was crucial to be compatible with Eqs. (3.38) and (3.39).

So far we have focused on the inflating background defined by

\[ \langle \phi_a(t) \rangle_{\text{vac}} = p_a(t) = e^{\frac{\kappa}{2} t} p_a, \quad \langle \hat{A}_0(t) \rangle_{\text{vac}} = \hat{a}_0(t, y). \]

(3.50)

Hence it is necessary to consider fluctuations around the background (3.50). This can be easily done by putting the fluctuations on the inflating background:\footnote{One may wonder why the time direction is not inflating. This is due to our choice of a coordinate frame to describe the dynamical system. The time evolution operator (3.32) is defined in the so-called comoving frame. In general, one can choose an arbitrary frame in which the time evolution is described by (2.41). A particularly interesting frame is the conformal coordinates with which the metric is given by \( ds^2 = a(\eta)^2 (-d\eta^2 + dx \cdot dx) \) where \( a(\eta) = \frac{2}{\kappa \eta} \) and \(-\infty < \eta < 0 \). The conformal coordinates can be easily transformed to the comoving coordinates by \( a(\eta) d\eta = dt \).}

\[ \langle \hat{A}_a(t, y) \rangle \equiv \left\{ \hat{\phi}_0(t, y) = \frac{\partial}{\partial t} - i \hat{A}_0(t, y), \quad \hat{\phi}_a(t, y) = e^{\frac{\kappa}{2} t} (p_a + \hat{A}_a(t, y)) \right\}, \]

(3.51)
where \( \tilde{A}_0(t, y) = \tilde{a}_0(y) + \delta \tilde{a}_0(t, y) \). The pathological behavior of dynamical variables at \( t \to \infty \) poses no threat whatsoever since we have assumed the finite duration of inflation, i.e., \( 0 \leq t \leq t_{\text{end}} \sim 10^{-33} \) sec. If we apply the matrix representation (2.33) to the above dynamical variables in the NC \( \ast \)-algebra \( \mathcal{A}_0^\dagger \), we get a bunch of time-dependent matrices whose set is denoted by \( \mathcal{A}_N^1 \). In consequence, the cosmic inflation is described by extending the duality chain (2.39) to the time-dependent background (3.50) as follows:

\[
\mathcal{A}_N^1 \implies \mathcal{A}_0^\dagger \implies \mathcal{D}^1. \tag{3.52}
\]

The module \( \mathcal{D}^1 \) of the derivations of the NC algebra \( \mathcal{A}_0^\dagger \) is given by

\[
\mathcal{D}^1 = \left\{ \tilde{V}_A(t) = (\tilde{V}_0, \tilde{V}_a)(t) | \frac{\partial}{\partial t} + e^{-\kappa t} \text{ad}_{\tilde{A}_0}, \ \tilde{V}_a(t) = e^{-\frac{\kappa}{2} t} \text{ad}_{\delta \tilde{a}_a} \right\}, \tag{3.53}
\]

where \( A = 0, 1, \cdots, 2n \) and the adjoint operations are defined by Eqs. (2.42) and (2.43). In the end the general Lorentzian metric describing \((2n + 1)\)-dimensional inflating spacetime is given by

\[
d s^2 = \lambda^2 v^A \otimes v^A = \lambda^2 (- dt^2 + e^{\kappa t} v_\mu^a v_\nu^a (dy^\mu - A^\mu)(dy^\nu - A^\nu)), \tag{3.54}
\]

where \( A^\mu := a^\mu + e^{-\kappa t} \delta a_0^\mu(t, y) dt \) and \( \lambda^2 \) is determined by Eq. (3.35) that is, of course, no longer 1. If all fluctuations are turned off for which \( v_\mu^a = \delta_\mu^a \), \( A^\mu = a^\mu \) and \( \lambda^2 = 1 \), we recover the inflation metric (3.49).

### 4 NC spacetime as a second-quantized string

We know that quantum mechanics is the more fundamental description of nature than classical physics. The microscopic world is already quantum. Nevertheless, the quantization is necessary to find a quantum theoretical description of nature since we have understood our world starting with the classical description which we understand better. After quantization, the quantum theory is described by a fundamental NC algebra \( \mathcal{A}_h \) such as Eq. (2.28). A striking feature is that every point in \( \mathbb{R}^n \) is unitarily equivalent because translations in \( \mathbb{R}^n \) are generated by an inner automorphism of \( \mathcal{A}_h \), i.e., \( f(x+a) = U(a) f(x) U(a)^\dagger \) where \( f(x) \in \mathcal{A}_h \) and \( U(a) = e^{i p_i a_i / h} \in \text{Inn}(\mathcal{A}_h) \). Therefore, through the quantization, the concept of (phase) space is doomed. Instead the (phase) space is replaced by a state in a Hilbert space \( \mathcal{H} \) and dynamical variables become operators acting on \( \mathcal{H} \). Only in the classical limit, a phase space with the symplectic structure \( \omega = dx^i \wedge dp_i \) is emergent from the quantum algebra \( \mathcal{A}_h \) such as (2.28).

Recall that the mathematical structure of NC spacetime is basically equivalent to the NC phase space in quantum mechanics [17]. Therefore essential features in quantum mechanics must be applied to the NC spacetime too. In particular, NC algebras such as the NC space (2.14) also play a fundamental role and every points in the NC space \( \mathcal{A}_0 \) are indistinguishable, i.e., unitarily equivalent because any two points are connected by an inner automorphism of \( \mathcal{A}_0 \). In other words, there is no space(time)
for the same reason as quantum mechanics and a classical spacetime must be derived from the NC algebra \( A_\theta \). After all, an important lesson is that NC spacetime implies emergent spacetime.

Although spacetime at a microscopic scale, e.g. \( L_P \), is intrinsically NC, we understand the NC spacetime through the quantization of a symplectic (or more generally Poisson) manifold. Let \((M, B)\) be a symplectic manifold. On one hand, the basic concept in symplectic geometry is an area defined by the symplectic two-form \( B \) which is a nondegenerate, closed two-form. On the other hand, the basic concept in Riemannian geometry determined by a pair \((M, g)\) is a distance defined by the metric tensor \( g \) which is a nondegenerate, symmetric bilinear form. One may identify this distance with a geodesic worldline of a “particle” moving in \( M \). On the contrary, the area in symplectic geometry may be regarded as a minimal worldsheet swept by a “string” moving in \( M \). In this picture, the wiggly string, so a fluctuating worldsheet, corresponds to a deformation of symplectic structure in \( M \). This picture becomes more transparent by the so-called pseudoholomorphic or \( J \)-holomorphic curve introduced by Gromov [46].

Let \((M, J)\) be an almost complex manifold and \((\Sigma, j)\) be a Riemann surface. By the compatibility of \( J \) to \( B \), we have the relation \( g(X, Y) = B(X, JY) \) for any vector fields \( X, Y \in \mathfrak{X}(M) \). Let us also fix a Hermitian metric \( h \) of \((\Sigma, j)\). A smooth map \( f : \Sigma \to M \) is called pseudoholomorphic if the differential \( df : T\Sigma \to TM \) is a complex linear map with respect to \( j \) and \( J \):

\[
df \circ j = J \circ df.
\]

This condition corresponds to the commutativity of the following diagram

\[
\begin{array}{ccc}
T\Sigma & \xrightarrow{j} & T\Sigma \\
\downarrow df & & \downarrow df \\
TM & \xrightarrow{J} & TM
\end{array}
\]

Since \( J^{-1} = -J \), it is also equivalent to \( \overline{\partial}_J f = 0 \) where \( \overline{\partial}_J f := \frac{1}{2}(df + J \circ df \circ j) \). For example, suppose that the Riemann surface is \((\Sigma, i)\) where \( i \) is the standard complex structure. We can work in a chart \( u_\alpha : U_\alpha \to \mathbb{C} \) with local coordinates \( z = \tau + i\sigma \) where \( U_\alpha \subset \Sigma \) is an open neighborhood. Define \( f_\alpha = f \circ u_\alpha^{-1} \). In this case, we have

\[
\overline{\partial}_J f = \frac{1}{2} \left[ \left( \frac{\partial f_\alpha}{\partial \tau} + J(f_\alpha) \frac{\partial f_\alpha}{\partial \sigma} \right) d\tau + \left( \frac{\partial f_\alpha}{\partial \sigma} - J(f_\alpha) \frac{\partial f_\alpha}{\partial \tau} \right) d\sigma \right].
\]

Thus we see that \( \overline{\partial}_J f = 0 \) if

\[
\frac{\partial f_\alpha}{\partial \tau} + J(f_\alpha) \frac{\partial f_\alpha}{\partial \sigma} = 0.
\]

Since \( J \) is \( B \)-compatible, every smooth map \( f : \Sigma \to M \) satisfies

\[
\frac{1}{2} \int_\Sigma ||df||_g^2 d\text{vol}_\Sigma = \int_\Sigma ||\overline{\partial}_J f||_g^2 d\text{vol}_\Sigma + \int_\Sigma f^* B,
\]

27
where the norms are taken with respect to the metric $g$ and $d \text{vol}_\Sigma$ is a volume form on $\Sigma$. In terms of local coordinates, $(\sigma^1, \sigma^2)$ on $\Sigma$ and $f(\sigma) = (x^1, \cdots, x^{2n})$ on $M$,

$$||df||_g^2 = g_{\mu\nu}(f(\sigma)) \frac{\partial x^\mu}{\partial \sigma^a} \frac{\partial x^\nu}{\partial \sigma^b} h^{ab}(\sigma)$$

(4.5)

and $d \text{vol}_\Sigma = \sqrt{h} d^2 \sigma$. Therefore, the left-hand side of Eq. (4.4) is nothing but the Polyakov action in string theory. For a pseudoholomorphic curve $f : \Sigma \to M$ that obeys $\overline{\partial}_f f = 0$, we thus have the identity

$$S_P(f) = \frac{1}{2} \int_{\Sigma} ||df||_g^2 \ d \text{vol}_\Sigma = \int_{\Sigma} f^* B.$$  

(4.6)

This means that any pseudoholomorphic curves minimize the “harmonic energy” $S_P(f)$ in a fixed homology class and so are harmonic maps. In other words, their symplectic areas coincide with surface areas. Therefore, any pseudoholomorphic curve is a solution of the worldsheet Polyakov action $S_P(f)$. For instance, if $M = \mathbb{C}^n$ with complex coordinates $\phi^i = x^{2i-1} + \sqrt{-1} x^{2i}$ ($i = 1, \ldots, n$) and $f_\alpha(z, \bar{z}) \equiv \phi^i(z, \bar{z})$, Eq. (4.3) becomes

$$1 \over 2 \left( \frac{\partial}{\partial \tau} + \sqrt{-1} \frac{\partial}{\partial \sigma} \right) \phi^i(z, \bar{z}) = \partial_z \phi^i(z, \bar{z}) = 0.$$  

(4.7)

In this case, pseudoholomorphic curves coincide with holomorphic curves. Moreover such curves are harmonic and minimal surfaces.$^{14}$

The pseudoholomorphic curve also provides us a useful tool to understand the emergent gravity picture. To demonstrate this aspect, let us include a boundary interaction in the sigma model (4.4) such that the open string action is given by

$$S_A(f) = \frac{1}{2} \int_{\Sigma} ||df||_g^2 \ d \text{vol}_\Sigma + \int_{\partial \Sigma} f^* A,$$  

(4.8)

where the one-form $A$ is the connection of a line bundle $L \to M$. Using the Stokes’ theorem, the second term can be written as

$$\int_{\partial \Sigma} f^* A = \int_{\Sigma} f^* dA.$$  

(4.9)

After combining the identities (4.4) and (4.9) together, we write the action

$$S_A(f) = \int_{\Sigma} ||\overline{\partial}_f f||_g^2 \ d \text{vol}_\Sigma + \int_{\Sigma} f^* \mathcal{F},$$  

(4.10)

where $\mathcal{F} = B + F$ and $F = dA$. If one recalls the derivation of Eq. (4.4), one may immediately realize that the action $S_A(f)$ can equivalently be written as the form of the Polyakov action

$$S_P(\psi) = \frac{1}{2} \int_{\Sigma} ||d\psi||_G^2 \ d \text{vol}_\Sigma,$$  

(4.11)

$^{14}$In the topological A-model that is concerned with pseudoholomorphic maps from $\Sigma$ to $M = T^* N$, there is a vanishing theorem$^{49}$ stating that $\int_{\Sigma} f^* B = 0$. In particular, the mappings from $\partial \Sigma$ to $N$ are necessarily constant.
where the differential \(d\psi\) for a smooth map \(\psi : \Sigma \to M\) has the norm taken with respect to some metric \(G\). For this purpose, let us assume that the almost complex structure \(J\) is also compatible with the deformed symplectic structure \(F\), i.e.,

\[
G(X, Y) = F(X, JY), \quad \forall X, Y \in \mathfrak{X}(M)
\]

is a Riemannian metric on \(M\). An explicit representation of the Polyakov action (4.11) can be made by introducing local coordinates \(\psi(\sigma) = (X^1, \cdots, X^{2n})\) on an open set \(U_i \subset M\) so that

\[
||d\psi||_G^2 = G_{\mu\nu}(\psi(\sigma)) \frac{\partial X^\mu}{\partial \sigma^a} \frac{\partial X^\nu}{\partial \sigma^b} h^{ab}(\sigma).
\]

One can then apply the same derivation of Eq. (4.4) to the action (4.11) to derive the identity

\[
\frac{1}{2} \int_{\Sigma} ||d\psi||_G^2 \, d\text{vol}_\Sigma = \int_{\Sigma} ||\bar{\partial}_J \psi||_G^2 \, d\text{vol}_\Sigma + \int_{\Sigma} \psi^* F.
\]

For pseudoholomorphic curves \(\psi : \Sigma \to M\) satisfying \(\bar{\partial}_J \psi = 0\), we finally get the result

\[
S_P(\psi) = \frac{1}{2} \int_{\Sigma} ||d\psi||_G^2 \, d\text{vol}_\Sigma = \int_{\Sigma} \psi^* F.
\]

The above argument reveals a nice picture that dynamical \(U(1)\) gauge fields in a line bundle \(L\) over \(M\) embody the deformation of an underlying symplectic structure \((M, B)\) and this deformation is transformed into the dynamics of gravity [13]. This is a reincarnation of the duality chain (2.21) indicating the gauge/gravity duality. As we observed before, the symplectic geometry is probed by strings while the Riemannian geometry is probed by particles. In particular, the impetus of pseudoholomorphic or \(J\)-holomorphic curves in symplectic geometry uncover why string theory can be regarded as a stringy realization of symplectic geometry. It is in general known [23] that there is a nonlinear Fredholm theory which describes the deformations of a given pseudoholomorphic curve \(f : \Sigma \to (M, J)\) and the deformations are parameterized by a finite-dimensional moduli space. (This moduli space may be greatly enriched by considering pseudoholomorphic curves in the context of LCS manifolds we have reviewed in Appendix A.) When symplectic manifolds are probed with strings or pseudoholomorphic curves, the notion of wiggly strings in this probe picture corresponds to the deformation of symplectic structures. Hence the emergence of gravity from symplectic geometry or more precisely NC \(U(1)\) gauge fields may not be surprising because we know from string theory that a Riemannian geometry (or general relativity) is emergent from the wiggly strings.

The pseudoholomorphic curve makes a geometrical and physical picture of NC spacetime become clear. The NC spacetime is defined by the quantization of a symplectic manifold \((M, B)\). We can think of the integral \(A(f) = \int_{\Sigma} f^* B\) in two ways if \(f\) is a pseudoholomorphic curve. On the one hand, the pointwise compatibility between the structures \((B, J)\) means that \(A(f)\) is essentially the area of the image of \(f\), measured in the Riemannian metric \(g\). On the other hand, the condition that \(B\) is closed means that \(A(f)\) is a topological (homotopy) invariant of the map \(f\). Hence we can
use the curves in two main ways \[23\]. The first way is as geometrical probes to explore symplectic manifolds, as we advocated above. The second way is as the source of numerical invariants known as the Gromov-Witten invariants. Using the pseudoholomorphic curves, Gromov proved a surprising non-squeezing theorem \[46, 47, 48\] stating that, if 
\[
\iota : B_{2n}(r) \subset \mathbb{C}^n \to (M, B)
\]

is a symplectic embedding of the ball \(B_{2n}(r)\) of radius \(r\) in \((M, B)\) with the standard symplectic form \(B\) into the cylinder \(Z_{2n}(R) = B_2(R) \times \mathbb{R}^{2n-2}\), then \(R \geq r\). A further generalization is to replace \(\mathbb{R}^{2n-2}\) by a \((2n - 2)\)-dimensional compact symplectic manifold \(V\) with \(\pi_2(V) = 0\).

One may try to lift the notion of pseudoholomorphic curves to quantized symplectic manifolds, namely, NC spaces such as Eq. (2.14). The quantization of a symplectic manifold leads to a radical change of classical concepts such as spaces and observables. The classical space is replaced by a Hilbert space and dynamical observables become operators acting on the Hilbert space. Then, as we have constantly argued, the NC spacetime provides a more elegant framework for the background-independent formulation of quantum gravity in terms of matrix models, which is still elusive in string theory. We showed in Sec. 3 that the dynamical Lorentzian spacetime (2.53) emerges from a classical limit of the matrix model (2.57). Remarkably, the cosmic inflation described by the metric (3.49) also arises as a vacuum solution of the time-dependent matrix model as was shown in Eq. (3.40). In order to grasp what a pseudoholomorphic curve looks like in NC spacetime, let us consider the simplest case in Eq. (4.7). After quantization, coordinates of \(\mathbb{C}^n\) denoted by \(\phi^i(z, \bar{z})\) become operators in a NC \(\star\)-algebra \(A_\theta\equiv A_\theta(\mathbb{C}_\infty(\mathbb{R}^2)) = \mathbb{C}_\infty(\mathbb{R}^2) \otimes A_\theta\), i.e., \(\phi^i(z, \bar{z}) \to \hat{\phi}^i(z, \bar{z}) \in A_\theta^2\). The worldsheet \(\mathbb{R}^2\) may be replaced by \(T^2\) or \(\mathbb{R} \times S^1\). Let us clarify the notation \(A_\theta^2\) after the Wick rotation of the worldsheet coordinate \(\tau = it\), so \(\mathbb{R}^2 \to \mathbb{R}^{1,1}\). Consider a generic element in the NC \(\star\)-algebra \(A_\theta^2\) given by

\[
\hat{f}(t, \sigma, y) \in A_\theta^2. \tag{4.16}
\]

The matrix representation (2.33) is now generalized to

\[
\hat{f}(t, \sigma, y) = \sum_{n,m=1}^{\infty} |n\rangle \langle n| \hat{f}(t, \sigma, y) |m\rangle \langle m| = \sum_{n,m=1}^{\infty} f_{nm}(t, \sigma) |n\rangle \langle m| \tag{4.17}
\]

where the coefficients \(f_{nm}(t, \sigma) := [f(t, \sigma)]_{nm}\) are elements of a matrix \(f(t, \sigma)\) in \(A_N^2 \equiv A_N(\mathbb{C}_\infty(\mathbb{R}^{1,1})) = C^\infty(\mathbb{R}^{1,1}) \otimes A_N\) as a representation of the observable (4.16) on the Hilbert space (2.12). Then we have an obvious generalization of the duality chain (2.39) as follows:

\[
A_N^2 \implies A_\theta^2 \implies \mathcal{D}^2. \tag{4.18}
\]

The module of derivations is similarly a direct sum of the submodules of horizontal and inner derivations [34]:

\[
\mathcal{D}^2 = \text{Hor}(A_N^2) \oplus \mathcal{D}(A_N^2) \cong \text{Hor}(A_\theta^2) \oplus \mathcal{D}(A_\theta^2), \tag{4.19}
\]

where horizontal derivations are locally generated by a vector field

\[
k(t, \sigma, y) \frac{\partial}{\partial t} + l(t, \sigma, y) \frac{\partial}{\partial \sigma} \in \text{Hor}(A_\theta^2). \tag{4.20}
\]
It can be shown [12, 13] that the matrix model for the duality chain (4.18) is given by

\[ S = -\frac{1}{g_s} \int d^2 \sigma \text{Tr} \left( \frac{1}{4} F_{\alpha \beta}^2 + \frac{1}{2} (D_\alpha \phi_a)^2 - \frac{1}{4} [\phi_a, \phi_b]^2 \right), \quad (4.21) \]

where \( a = 2, \ldots, 2n + 1 \) and \( \sigma^\alpha = (t, \sigma) \), \( \alpha = 0, 1 \) and \( F_{\alpha \beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha - i [A_\alpha, A_\beta] \). The \( n = 4 \) case is known as the matrix string theory which is supposed to describe a nonperturbative type IIA string theory in light-cone gauge [25]. The matrix string theory can be obtained from the BFSS matrix model via compactification on a circle [28]. To achieve this goal, the BFSS matrix model has to have 9 adjoint scalar fields, \( \phi_a(t) \) \( (a = 1, \cdots, 9) \), unlike the action (2.57) with even number of adjoint scalar fields. The reason why we consider only even number of adjoint scalar fields is to realize the equivalence (2.62). In this case the action (2.57) can be understood as a Hilbert space representation of certain NC gauge theory under a symplectic vacuum such as (2.5) with rank \( (B) = 2n \). We do not know a corresponding NC gauge theory whose Hilbert space representation precisely reproduces the BFSS matrix model. We will further comment on this issue later. However the matrix string theory (4.21) has 8 adjoint scalar fields for \( n = 4 \). Thus it is possible to realize it as the Hilbert space representation of (9 + 1)-dimensional NC \( U(1) \) gauge theory with rank \( (B) = 8 \) [12, 13]. Therefore it will be interesting to understand how to derive the matrix string theory (4.21) from the BFSS-like matrix model (2.57).

The basic idea is similar to the previous scheme to construct the one-dimensional matrix model (2.57) through the contact structure of zero-dimensional matrices. A difference is that we start with the one-dimensional matrix model (2.57) and introduce an additional contact structure along a spatial direction whose coordinate is called \( \sigma \) in our case. Ultimately, the matrix string theory (4.21) can be realized as the quantization of a regular 2-contact manifold. See Ref. [40] for a general \( k \)-contact manifold. First let us consider the projection \( \pi_2 : \mathbb{R}^2 \times M \rightarrow M, \pi_2(\sigma^\alpha, x) = x \) where \( M \) is a symplectic manifold with the symplectic form \( B \). The regular 2-contact \((2n + 2)\)-dimensional manifold is defined by a quartet \((\mathbb{R}^2 \times M, B, \eta^\alpha)\), \( \alpha = 0, 1 \), where \( B = \pi_2^* B \), such that

\[ \eta^1 \wedge \eta^2 \wedge B^\alpha \neq 0 \quad (4.22) \]

everywhere and \( d\eta^\alpha = \gamma^\alpha B \) with constants \( \gamma^\alpha \) and \( dB = 0 \). Moreover there are uniquely defined two Reeb vectors \( R_\alpha \) \( (\alpha = 0, 1) \) given by

\[ \iota_{R_\alpha} \eta^\beta = \delta^\beta_\alpha, \quad \iota_{R_\alpha} B = 0, \quad \alpha, \beta = 0, 1. \quad (4.23) \]

The above relations imply

\[ \mathcal{L}_{R_\alpha} \eta^\beta = 0, \quad \mathcal{L}_{R_\alpha} B = 0, \quad [R_1, R_2] = 0. \quad (4.24) \]

\[ \text{It is possible to replace } \mathbb{R}^2 \times M \text{ by a general } (2n + 2)\text{-dimensional manifold} N \text{ as far as there is a well-defined two-dimensional foliation } \mathcal{V} \text{ such that the corresponding space of leaves } N/\mathcal{V} = M \text{ is a Hausdorff differentiable manifold} \text{ [40]. See (A.24) for a relevant discussion. We will keep the maximal simplicity for a plain argument.} \]
For example, the contact forms for the matrix string theory (4.21) are given by
\[ \eta^1 = dt - \frac{1}{2} p_a dy^a, \quad \eta^2 = d\sigma - \frac{1}{2} p_a dy^a, \]
which determines the corresponding Reeb vectors
\[ R_1 = \frac{\partial}{\partial t}, \quad R_2 = \frac{\partial}{\partial \sigma}. \]
These Reeb vectors span the space of horizontal derivations in Eq. (4.20).

Since there are two independent contact structures, each contact structure generates its own Hamiltonian vector field defined by (A.42). For the contact structures in Eq. (4.25), they are given by
\[ V_\alpha = \frac{\partial}{\partial \sigma^\alpha} + A^\mu_\alpha(t, \sigma, y) \frac{\partial}{\partial y^\mu}. \]

The quantization of the 2-contact manifold \((\mathbb{R}^2 \times M, \tilde{B}, \eta^\alpha)\) is particularly simple because it is performed using the Darboux coordinates \((\sigma^\alpha, y^a)\). It is basically defined by the quantization of the symplectic manifold \((M, B)\) in which \(\sigma^\alpha\) are regarded as classical variables like the time coordinate in the algebra \(A_\theta\). After quantization, a generic element of the NC \(*\)-algebra \(A_\theta^2\) takes the form (4.16). Then the module \(\mathcal{D}^2\) in Eq. (4.19) is generated by
\[ \mathcal{D}^2 = \left\{ \hat{V}_A(t, \sigma) = (\hat{V}_A, \hat{V}_a)(t, \sigma) | \hat{V}_A(t, \sigma) = \frac{\partial}{\partial \sigma^\alpha} + \text{ad}_{\hat{A}_\alpha}, \hat{V}_a(t, \sigma) = \text{ad}_{\hat{A}_a} \right\}, \]
where \(A = 0, 1, \ldots, 2n + 1\) and the adjoint operations are inner derivations of \(A_\theta^2\). Finally the corresponding Lorentzian metric dual to the matrix string theory (4.21) is given by [12, 13]
\[ ds^2 = \lambda^2 v^A \otimes v^A = \lambda^2 \left( - \eta_{\alpha\beta} d\sigma^\alpha d\sigma^\beta + v^\alpha v^\beta (dy^\mu - A^\mu_y(dy^\nu - A^\nu)) \right), \]
where \(A^\mu := A^\mu_\alpha(t, \sigma, y) d\sigma^\alpha\) and \(\lambda^2 = \nu(t, \sigma)(V_0, V_1, \ldots, V_{2n+1})\) is determined by the volume preserving condition, \(L_{V_A} \nu(t, \sigma) = 0\), with respect to a given volume form
\[ \nu(t, \sigma) = dt \wedge d\sigma \wedge \nu = \lambda^2 dt \wedge d\sigma \wedge v^1 \wedge \cdots v^{2n}. \]

Let us come back to our previous question about the generalization of pseudoholomorphic curves to a quantized spacetime. In order to address this issue, let us return to the Euclidean space again by the Wick rotation \(t = -i\tau\). If the quantum version of pseudoholomorphic curves exists, Eq. (4.3) suggests that it will also obey the first-order partial differential equations. It is well-known [50] that the matrix action (4.21) admits such a first-order system. For simplicity, assume that adjoint scalar fields mostly vanish except \((\phi_2, \phi_3) \neq 0\). It is convenient to use the complex variables
\[ \phi = \frac{1}{2}(\phi_2 - i\phi_3), \quad \phi^\dagger = \frac{1}{2}(\phi_2 + i\phi_3). \]
It is not difficult to show that the Euclidean action with $\phi_a = 0$ for $a = 4, \cdots, 9$ can be written as the Bogomol’nyi-type

$$S = \frac{1}{g_s^2} \int d^2\sigma \text{Tr} \left( \frac{1}{4} F^2_{\alpha\beta} + \frac{1}{2}(D_{\alpha}\phi_a)^2 - \frac{1}{4}[\phi_a,\phi_b]^2 \right)$$

$$= \frac{2}{g_s^2} \int d^2\sigma \text{Tr} \left( (iF_{z\bar{z}} - [\phi,\phi^\dagger])^2 + |D_{\bar{z}}\phi|^2 - i\partial_{\alpha}(\varepsilon^{\alpha\beta}\phi^\dagger D_{\beta}\phi) \right).$$

(4.32)

Since the last term is a topological number, the minimum of the action is achieved in the configurations obeying

$$F_{z\bar{z}} + i[\phi,\phi^\dagger] = 0, \quad D_{\bar{z}}\phi = 0.$$  

(4.33)

Note that the above equations recover Eq. (4.7) in a very commutative limit where $[\phi^\dagger,\phi] = 0$. Therefore it is reasonable to identify Eq. (4.33) with the quantum version of pseudoholomorphic curves.

Mathematically Eq. (4.33) is equivalent to the Hitchin equations describing a Higgs bundle [51]. A Higgs bundle is a system composed of a connection $A$ on a principal $G$-bundle or simply a vector bundle $E$ over a Riemann surface $\Sigma$ and a holomorphic endomorphism $\phi$ of $E$ satisfying Eq. (4.33). The Hitchin equations describe four-dimensional Yang-Mills instantons on $\mathbb{R}^4$ which are invariant with respect to the translation group $\mathbb{R}^2$. (This $\mathbb{R}^2$ is transverse to the Riemann surface, so independent of the worldsheet $\mathbb{R}^2$.) Using the translation invariance, the Yang-Mills instantons can be dimensionally reduced to the Riemann surface $\Sigma$ in which Yang-Mills gauge fields along the isometry directions become an adjoint Higgs field $\phi$. In our case the gauge group $G$ is $U(N)$. In particular, we are interested in the large $N$ limit, i.e., $N \to \infty$. In this limit, the action (4.32) can be mapped to four-dimensional NC $U(1)$ gauge theory under a Coulomb branch vacuum $\langle \phi_a \rangle_{\text{vac}} = p_a$, $a = 2, 3$ obeying the commutation relation $[p_2, p_3] = -iB_{23}$ and the Hitchin equations (4.33) precisely become the self-duality equation for NC $U(1)$ instantons on $\mathbb{R}^2(\text{or} \Sigma) \times \mathbb{R}_\theta$ [52, 53]. The corresponding gravitational metric for the case $n = 1$ was already identified in Eq. (4.29) with the analytic continuation $t = -i\tau$. It was shown in [45, 54, 55] that the solutions of the Hitchin equations (4.33) are dual to four-dimensional gravitational instantons which are hyper-Kähler manifolds. In particular, the real heaven is governed by the $su(\infty)$ Toda equation and the self-duality equation for the real heaven exactly reduces to the commutative limit of the Hitchin equations (4.33). See eq. (4.31) in Ref. [45]. Thus the Hitchin system with the gauge group $G = U(N \to \infty)$ may be closely related to the Toda field theory. Indeed this interesting connection was already analyzed in [56]. In sum, Hitchin equations, NC $U(1)$ instantons, gravitational instantons and pseudoholomorphic curves may be small islands in the matrix string theory (4.21) that have barely showed themselves.

Let us conclude this section by drawing an invaluable insight. So far we have understood NC spacetimes too easily. However the NC spacetime is much more radical and mysterious than we thought. It is fair to say that we have not yet fully understood the mathematically precise sense in which spacetime should be NC. Indeed we have observed at the outset of this section that NC spacetime implies emergent spacetime if spacetime should be viewed as NC. This means that spacetime
is somehow a derived concept. Since we form our picture of the world by recognizing the NC spacetime as a small deformation of classical symplectic or Poisson manifolds, we need an efficient tool to explore the symplectic geometry. The most natural object to probe symplectic manifolds is a pseudoholomorphic curve which is a stringy generalization of a geodesic worldline in Riemannian geometry [23]. In other words, the pseudoholomorphic curve is basically a minimal surface or a string worldsheet embedded into spacetime. However, to make sense of the emergent spacetime proposal, we need a mathematically precise framework for describing strings in a background-independent way. If it is so, the background-independent theory does not have to assume from the outset that strings are vibrating in a preexisting spacetime. In this section we have taken aim at clarifying how the pseudoholomorphic curves can be lifted to NC spacetimes by the matrix string theory. The matrix string theory naturally extends the first-quantized string theory so that it also describes the perturbative interactions of splitting and joining of strings, producing surfaces with nontrivial topology [25]. That is, the matrix string theory is a second-quantized theory in which spacetime emerges from the collective behavior of matrix strings. In the end, the NC spacetime may be viewed as a second-quantized string for the background-independent formulation of quantum gravity, which is still elusive in ordinary string theory.

5 Discussion

In string theory, there are two exclusive spacetime pictures based on the Kaluza-Klein (KK) theory vs. emergent gravity although they are conceptually in deep discord with each other. On the one hand, the KK gravity is defined in higher dimensions as a more superordinate theory and gauge theories in lower dimensions are derived from the KK theory via compactification. Since the KK theory is just the Einstein gravity in higher dimensions, the prior existence of spacetime is \textit{a priori} assumed. On the other hand, in emergent gravity picture, gravity in higher dimensions is not a fundamental force but a collective phenomenon emergent from more fundamental ingredients defined in lower dimensions. In emergent gravity approach, the existence of spacetime is not \textit{a priori} assumed but the spacetime structure is defined by the theory. This picture leads to the concept of emergent spacetime. In some sense, emergent gravity is the inverse of KK paradigm, schematically summarized by

\[(1 \otimes 1)_S \rightleftharpoons 2 \oplus 0 \quad (5.1)\]

where $\rightarrow$ means the emergent gravity picture while $\leftarrow$ indicates the KK picture.

Recent developments in string theory have revealed growing evidences for the emergent gravity and emergent spacetime. The AdS/CFT correspondence and matrix models are typical examples supporting the emergent gravity and emergent spacetime [7]. Since the emergent spacetime is a

\footnote{This prospect has been recently advocated by Moore in (especially, Sec. 9) “Physical mathematics and the future” (available at \url{http://www.physics.rutgers.edu/~gmoore/}). See also Segal in “Space and spaces” (available at \url{http://www.lms.ac.uk/sites/lms.ac.uk/files/files/AboutUs/AGM-talk.pdf}) as cited by the former and [57].}
new fundamental paradigm for quantum gravity and radically different from any previous physical theories, all of which describe what happens in a given spacetime, it is required to seriously reexamine all the rationales to introduce the multiverse hypothesis from the perspective of emergent spacetime. However, we do not intend to make an objection to the existence of more diverse subregions in the Universe. The Universe is rather likely much larger than we previously thought. Actually the emergent spacetime picture implies that our observable patch within cosmic horizon is a very tiny part $\sim 10^{-60}$ of the entire spacetime, as we will discuss soon. Instead we will pose the issue whether the existence of more diverse subregions besides ours means that the laws of physics are ambiguous or all these subregions follow the same laws of physics and the physical laws of our causal patch in the Universe can be understood as accurately as possible without reference to the existence of other subregions.

First let us summarize the main (not exhausting) sources of the multiverse idea [1]:

A. Cosmological constant problem.

B. Chaotic and eternal inflation scenarios.

C. String landscape.

First of all, we have to point out that these are all based on the traditional spacetime picture. The cosmological constant problem (A) is the problem in all traditional gravity theories such as Einstein gravity and modified gravities. So far any such a theory has not succeeded to resolve the problem A. The inflation scenarios (B) are also based on the traditional gravity theory coupled to an effective field theory for inflaton(s). Thus, in these scenarios, the prior existence of spacetime is simply assumed. The string landscape (C) also arises from the conventional KK compactification of string theory although the string theory is liberal enough to allow two exclusive spacetime pictures, as we already remarked above. Since superstring theories can consistently be defined only in ten-dimensions, extra six-dimensional internal spaces need to be compactified to explain our four-dimensional world. Moreover it is important to determine the shapes and topology of internal spaces to make contact with a low-energy phenomenology in four-dimensions because the internal geometry of string theory determines a detailed structure of the multiplets for elementary particles and gauge fields via the KK compactification. The string landscape (C) means that the huge variety of compactified internal geometries exist, typically, in the range of $10^{500}$ and almost the same number of four-dimensional worlds with different low-energy phenomenologies accordingly survive [9] [10].

We have to stress again that the emergent spacetime picture is radically different from the conventional picture in general relativity so that they are exclusive and irreconcilable each other. Therefore, if the emergent spacetime picture is correct to explain our Universe, we have to give up the traditional spacetime picture and KK paradigm. For this reason, we will reconsider all the rationales (A,B,C) from the standpoint of emergent spacetime and the background independentness.
We already justified at the beginning of Sec. 3 why emergent gravity definitely dismisses the problem A. See also Refs. [11, 12, 19] for more extensive discussion of this issue. There is no cosmological constant problem in emergent gravity approach founded on the emergent spacetime. The foremost reason is that the huge vacuum energy (3.2) or (3.5) that is a cosmological constant in general relativity was simply used to generate the flat spacetime and thus it does not gravitate any more. The emergent gravity does not allow the coupling of the cosmological constant thanks to the general property (3.3), which is a tangible difference from general relativity. Consequently there is no demanding reason to rely on the anthropic fine-tuning to explain the tiny value of current dark energy. We will also discuss later what dark energy is from the emergent gravity picture following the observation in Refs. [11, 12, 19].

The multiverse picture arises in inflationary cosmology (B) as follows [3, 4]. In theories of inflationary model, even though false vacua are decaying, the rate of exponential expansion is always much faster than the rate of exponential decay. Once inflation starts, the total volume of the false vacuum continues to grow exponentially with time. The chaotic inflation is also eternal, in which large quantum fluctuations during inflation can significantly increase the value of the energy density in some parts of the universe. These regions expand at a greater rate than their parent domains, and quantum fluctuations inside them lead to production of new inflationary domains which expand even faster. Jumps of the inflaton field due to quantum fluctuations lead to a process of eternal self-production of inflationary universe. In most inflationary models, once inflation happens, it produces not just one universe, but an infinite number of universes.

Now an important question is whether the emergent spacetime picture can also lead to the eternal inflation. The answer is certainly no. The reason is the following. We showed that the inflationary vacuum (3.50) arises as a solution of the (BFSS-like) matrix model (2.57). In order to define the matrix model (2.57), however, we have not introduced any spacetime structure. The vacuum (3.50) corresponds to the creation of spacetime unlike the traditional inflationary models that describe just the exponential expansion of a preexisting spacetime. Moreover, the inflationary vacuum (3.50) describes a dynamical process of the Planck energy condensate responsible for the emergence of spacetime. It is not an empty space but full of the Planck energy as Eq. (3.5) clearly indicates. An important point is that the Planck energy condensate results in a highly coherent vacuum called the NC space (2.5). As the NC phase space in quantum mechanics necessarily brings about the Heisenberg’s uncertainty relation, \( \Delta x \Delta p \geq \frac{\hbar}{2} \), the NC space (2.5) also leads to the spacetime uncertainty relation. Therefore any further accumulation of energy over the vacuum (3.50) must be subject to the exclusion principle known as the UV/IR mixing [58]. The Gromov’s non-squeezing theorem discussed in Sec. 4 is arguably the classical analogue of the exclusion principle due to the NC space [13]. Consequently, it is not possible to further accumulate the Planck energy density \( \delta \rho \sim M_P^4 \) over the inflationary vacuum (3.50). This means that it is impossible to superpose a new inflating subregion over the inflationary vacuum (3.50). In other words, the cosmic inflation triggered by the Planck energy condensate into vacuum must be a single event [19]. In the end we have a beautiful picture: The NC spacetime is
necessary for the emergence of spacetime and the exclusion principle of NC spacetime guarantees the stability of spacetime. In conclusion, the emergent spacetime does not allow the pocket universes appearing in the eternal inflation.

The above argument suggests an intriguing picture for the dark energy too. Suppose that the inflation ended. This means that the inflationary vacuum (3.50) in nonequilibrium makes a (first-order) phase transition to the vacuum (2.4) in equilibrium in some way. We do not know how to do it. We will discuss a possible scenario later. Since the vacuum (2.4) satisfies the commutation relation (2.5), any local fluctuations over the vacuum (2.4) must also be subject to the spacetime uncertainty relation or UV/IR mixing. This implies that any UV fluctuations are paired with corresponding IR fluctuations. For example, the most typical UV fluctuations are characterized by the Planck mass $M_P$ and these will be paired with the most typical IR fluctuations with the largest possible wavelength denoted by $L_H = M_H^{-1}$. This means that these UV/IR fluctuations are extended up to the scale $L_H$ which may be identified with the current size of cosmic horizon. By a simple dimensional analysis one can estimate the energy density of these fluctuations:

$$\delta \rho \sim M_P^2 M_H^2 = \frac{1}{L_P^2 L_H^2}. \quad (5.2)$$

It may be emphasized that, if the microscopic spacetime is NC, then the UV/IR mixing is inevitable and the extended (nonlocal) energy (5.2) is necessarily induced [19]. If we identify $L_H$ with the cosmic horizon of our observable universe, $L_H \sim 1.3 \times 10^{26}$ meter, $\delta \rho$ is roughly equal to the current dark energy, i.e.,

$$\delta \rho = M_{DE}^4 \sim (10^{-3} \text{eV})^4. \quad (5.3)$$

Thus the emergent gravity predicts the existence of dark energy whose scale is characterized by the size of our visible universe. Since the characteristic scale of entire spacetime is set by the Planck mass $M_P$, this implies that our observable universe is one causal patch out of much larger unobservable patches, $M_P/M_H = M_P^2/M_{DE}^2 \sim 10^{60}$. However we have to concede that the result (5.2) was obtained in [11] from the Euclidean metric (2.27) dual to the IKKT matrix model (2.1) with $n = 2$ after the Wick rotation since we do not know a NC field theory obtained from the BFSS matrix model with $(2n - 1)$ adjoint scalar fields.

The gauge/gravity duality such as the AdS/CFT correspondence has clarified how a higher dimensional gravity can emerge from a lower dimensional gauge theory. A mysterious point is that the emergence of gravity requires the emergence of spacetime too. If spacetime is emergent, everything supported on the spacetime should be emergent too for an internal consistency of the theory. In particular, matters cannot exist without spacetime and thus must be emergent together with the spacetime. Eventually, the background-independent theory has to make no distinction between geometry and matter [13]. This is the reason why the emergent spacetime picture cannot coexist peacefully with the KK paradigm. As we pointed out before, the string landscape has been derived from the KK compactification of string theory. Therefore, if the emergent spacetime picture is correct, we need to carefully reexamine the string landscape (C) from that point of view. The emergent spacetime picture
may endow the string landscape with a completely new interpretation since reversing the arrow in (5.1) accompanies a radical change of physics. For example, a geometry is now derived from a gauge theory while previously the gauge theory was derived from the geometry.

The KK compactification of string theory advocates that the Standard Model in four dimensions is determined by a six-dimensional internal geometry, e.g., a Calabi-Yau manifold. Thus different internal geometries mean different physical laws in four dimensions, so different universes governed by the different Standard Models. However, the emergent gravity reverses the arrow in (5.1). Rather internal geometries are determined by microscopic configurations of gauge fields and matter fields in four dimensions. As a consequence, different internal geometries mean different microscopic configurations of four-dimensional particles and nonperturbative objects such as solitons and instantons. This picture may be more strengthened by the fact [59] that Calabi-Yau manifolds are emergent from six-dimensional NC $U(1)$ instantons and thus the origin of Calabi-Yau manifolds is actually a gauge theory. If the microscopic configuration changes by interactions, then the corresponding change of the internal geometry will also be induced by the interactions. If so, the huge variety of internal geometries may correspond to the ensemble of microscopic configurations in four dimensions and $10^{500}$ would be the Avogadro number for the microscopic ensemble. Recall that NC geometry begins from the rough correspondence–contravariant functor–between the category of topological spaces and the category of commutative algebras over $\mathbb{C}$ and then changes the commutative algebras by NC algebras to define corresponding NC spaces. In this correspondence, different internal geometries correspond to choosing different NC algebras. We have observed that the latter allows a background-independent formulation which does not require a background geometry and a large amount (possibly infinitely many) of spacetime geometries can be described by generic deformations of a vacuum algebra in a master theory. Hence a background-independent quantum gravity seems to bring a new perspective that cripples all the rationales to introduce the multiverse hypothesis.

We certainly live in the universe where the inflationary epoch had lasted only for a very tiny period at very early times although it is currently in an accelerating phase derived by the dark energy. Therefore there should be some relaxation mechanism for the (first-order) phase transition from the universe in nonequilibrium to the stable universe in equilibrium. We showed that the former is described by the metric (3.54) whereas the latter is described by (2.53) and both arise as solutions of the background-independent matrix model (2.57). In scalar field inflation theories the relaxation mechanism is known as the reheating in which the scalar field switches from being overdamped to being underdamped and begins to oscillate at the bottom of the potential to transfer it’s energy to a radiation dominated plasma at a temperature sufficient to allow standard nucleosynthesis [3, 4]. For this purpose, it is necessary to introduce a very ad hoc potential for the inflaton. In our case, however, we have introduced neither an inflaton field nor an inflation potential. Nevertheless, the inflation was possible since an LCS manifold admits a rich variety of vector fields, in particular, Liouville vector fields which generate the inflation. Thus an urgent question is: how to end the inflation in emergent gravity?
We do not know the answer yet for the ending mechanism. Thereby we will briefly speculate a plausible scenario only. Let us start with a naive observation. The Lorentzian metric (3.54) describes general scalar-tensor perturbations on the inflating spacetime (3.36). Since the fluctuations have been superposed on the inflating background, we suspect that there may be some nonlinear damping mechanism through the interactions between the background and the density fluctuations. To be precise, there may be a cosmic analogue of the Landau damping in plasma physics originally applied to longitudinal oscillations of an electron plasma. The Landau damping in a plasma occurs due to the energy exchange between an electromagnetic wave and particles in the plasma with velocity approximately equal to phase velocity of the wave and leads to exponentially decaying collective oscillations.\footnote{There is a nice exposition on the Landau damping by Werner Herr, “Physics of Landau Damping: An introduction (to a mysterious topic),” available at https://indico.cern.ch/event/216963/contribution/41/material/slides/0.pdf. Recently the Landau damping has been mathematically established even at the non-linear level [60].} The Landau damping may be intuitively understood by considering how a surfer gains energy from the sea wave. If the suffer is slightly slower than the wave mode, the mode loses energy to the surfer. For the wave to be damped, the wave velocity and the surfer velocity must be similar and then the surfer is trapped by the wave. A similar situation may happen in the inflating spacetime (3.54). Local fluctuations (suffers) on the inflating spacetime (the wave mode) are given by Eq. (3.51). Note that these local fluctuations carry an additional localized energy and this local energy will cause a slight delay of the drift of local lumps compared to the inflating background. Moreover these drift delays will occur everywhere since (quantum) fluctuations are everywhere. Then this is precisely the condition for the Landau damping to occur. If this is true, the inflating mode will transfer its kinetic energy (see Eq. (3.6)) to ubiquitous local fluctuations, ending the inflation through an exponential damping and entering to a radiation dominated era via the reheating at a sufficiently high temperature for standard Big Bang.

The above speculation may be too good to be true. However, it may not be so absurd, considering the fact that the underlying theory for emergent gravity is the Maxwell’s electromagnetism on NC spacetime. Furthermore it seems to be a reasonable clue since the Landau damping can be realized even at a nonlinear level [60]. Therefore it will be important to verify whether the innocent idea can work or not.

Our real world, \(\mathbb{R}^{1,3} \cong \mathbb{R} \times \mathbb{R}^3\), is mystic as ever because the spatial 3-manifold \(\mathbb{R}^3\) does not belong to the family of (almost) symplectic manifolds. We thus finally want to list possible stairways to our real world, the four-dimensional Lorentzian spacetime \(\mathcal{M}\):

A. Analytic continuation or Wick rotation from four-manifolds.

B. KK compactification \(\mathcal{M} \times S^1\).

C. Constact manifold \((\mathbb{R}^3, \eta)\).

D. Nambu structure \((\mathbb{R}^3, C)\).
Here $\eta$ is a contact form on $\mathbb{R}^3$ and $C = \frac{1}{3!} C_{\mu\nu\lambda} dx^\mu \wedge dx^\nu \wedge dx^\lambda$ is a nondegenerate, closed three-form on $\mathbb{R}^3$. In the case (A), the Lorentzian metric is obtained from Eq. (2.27) by the Wick rotation $y^4 = iy^0$. We used this boring method to evaluate the dark energy (5.2). It is also straightforward to compactify the $(4 + 1)$-dimensional Lorentzian metric (2.53) onto $S^1$ to get the result (B). Since the time is also defined as a contact structure, the case (C) has two contact structures as the matrix string theory discussed in Sec. 4. It may be interesting to briefly explore some clue for the cosmic inflation in the context (C). Let $N = \mathbb{R} \times \mathbb{R}^3$ and $t \in \mathbb{R}$ be the time coordinate and $f_t = f(t)$ be a positive monotonic function. Define a time-dependent closed two-form on $N$ by

$$B_t = d\lambda_t = f_t(dT \wedge \eta + d\eta)$$

(5.4)

where $\lambda_t = f_t \eta$ and $T = \ln f_t$. Since $B_t^2 = e^{2T} dT \wedge \eta \wedge d\eta$ is nowhere vanishing, $B_t$ is a symplectic form on $N$. Consider a time-dependent Hamiltonian $H : N \to \mathbb{R}$ such that $dH = -e^T dT$ and denote the Hamiltonian vector field of $H$ by $X_H$. Let $R$ be the Reeb vector field associated with the contact form $\eta$. Then it is easy to show that

$$\iota_R B_t = dH,$$

(5.5)

that is, $R = X_H$. A very interesting property is that

$$Z = \frac{\partial}{\partial T}$$

(5.6)

is the Liouville vector field of the symplectic form $B_t$, i.e., $\mathcal{L}_Z B_t = B_t$ or $\iota_Z B_t = \lambda_t$. This condition can be written as $\mathcal{L}_Z \lambda_t = \lambda_t$. One can regard the Liouville vector field $Z$ as the Reeb vector field associated with the contact form $dT$. Since $\iota_Z (B_t^2) = e^{2T} \eta \wedge d\eta$, the one-form $\lambda_t$ gives rise to a contact form on every three-dimensional submanifold $M \subset N$ transverse to $Z$. Thus we expect that the conformal vector field $Z$ will generate an inflationary metric given by

$$ds^2 = -dT^2 + e^{2T} dx \cdot dx,$$

(5.7)

although we do not know its microscopic formulation in terms of matrix models. The approach in [22] may be useful for this case. Since we have no idea how to formulate emergent gravity based on the Nambu structure (D), the last case would remain to be our dream. It may be of M-theory origin because it is involved with the 3-form $C$ instead of symplectic 2-form $B$.

Acknowledgments

We would like to thank Seokcheon Lee for helpful discussions. We also thank Jungjai Lee for insightful discussions on the Landau damping. This work was supported by the National Research Foundation of Korea (NRF) grant funded by the Korea government (MOE) (No. 2011-0010597) and in part by (MSIP) (No. NRF-2012R1A1A2009117).
In this Appendix we briefly review the mathematical foundation of locally conformal cosymplectic (LCC) manifolds. It was shown in [41] that an LCC manifold can be seen as a generalized phase space of time-dependent Hamiltonian system. Thus we argue that the LCC manifold is also a natural phase space describing the cosmic inflation of our universe. Our argument for that is a direct application of the results in Refs. [40, 41] to emergent gravity.

First let us consider locally conformal symplectic (LCS) manifolds. An LCS manifold is a triple $(M, \Omega, b)$ where $b$ is a closed one-form and $\Omega$ is a nondegenerate (but not closed) two-form satisfying $d\Omega - b \wedge \Omega = 0$. (A.1)
The dimension of $M$ will be assumed to be at least 4 and the one-form $b$ is called the Lee form. If the Lee form $b$ is exact, the manifold is globally conformal symplectic (GCS). A symplectic manifold corresponds to the case with $b = 0$. Locally by choosing $b = d\lambda(\alpha)$ for a local function $\lambda(\alpha)$: $U_\alpha \rightarrow \mathbb{R}$ on an open neighborhood $U_\alpha$, Eq. (A.1) is equivalent to $d(e^{-\lambda(\alpha)} \Omega) = 0$, so the local geometry of LCS manifolds is exactly the same as that of symplectic manifolds. Thus an LCS form on a manifold $M$ is a non-degenerate two-form $B$ that is locally conformal to a symplectic form. In other words, on an LCS manifold $(M, \Omega, b)$, there exists an open covering $\{U_\alpha\}$ of $M$ and a smooth positive function $f_\alpha$ on each $U_\alpha$ such that $f_\alpha \Omega|_{U_\alpha}$ is symplectic on $U_\alpha$. Two LCS forms $\Omega$ and $\Omega'$ are said to be (conformally) equivalent if there exists some positive function $f$ such that $\Omega' = f\Omega$, where the Lee form of $\Omega'$ is just $b' = b + d\ln f$. An interesting example [61] is provided by the Hopf manifolds that are diffeomorphic to $S^1 \times S^{2n-1}$ and have a locally conformal Kähler metric while they admit no Kähler metric.

An LCS manifold can be seen as a generalized phase space of Hamiltonian dynamical systems since the form of the Hamilton’s equations is preserved by homothetic canonical transformations. Let us recapitulate how the LCS manifolds naturally arise from the Hamiltonian dynamics of particles. Consider a dynamical system with $n$ degrees of freedom so that its phase space is a $2n$-dimensional differentiable manifold $M$ endowed with an open covering of coordinate neighborhoods $\{U_\alpha\}_{\alpha \in I}$ with local coordinates $(q^i(\alpha), p^i(\alpha))$, $i = 1, \cdots, n$. Then we know that the dynamics consists of the orbits of a Hamiltonian vector field $X_H$. Every point of $M$ has an open neighborhood $U_\alpha$ with the local Darboux coordinates $(q^i(\alpha), p^i(\alpha))$. One can restrict the Hamiltonian $H$ and a nondegenerate two-form $\omega$ to each $U_\alpha$ to have a local Hamiltonian $H_\alpha = H_\alpha(q^i(\alpha), p^i(\alpha))$ and a symplectic structure $\omega_\alpha = dq^i(\alpha) \wedge dp^i(\alpha)$. Similarly the globally defined Hamiltonian vector field $X_H$ restricted to $U_\alpha$ is precisely given by $X_{H_\alpha}$. Then the orbits are defined by the Hamilton’s equations

$$\frac{dq^i(\alpha)}{dt} = \frac{\partial H_\alpha}{\partial p^i(\alpha)}, \quad \frac{dp^i(\alpha)}{dt} = -\frac{\partial H_\alpha}{\partial q^i(\alpha)}.$$ (A.2)

When one takes the coordinate chart definition of symplectic manifolds, there is no compulsory reason why one should require the two-form $\omega$ to be closed. Indeed, the Hamiltonian formulation of
particle dynamics consists in asking the local forms $\omega_\alpha$ and local functions $H_\alpha$ to glue up to a global symplectic form $\omega$ and a global Hamiltonian $H$. However, since the dynamical information is given by a global vector field, it is more natural to only require that the transition functions

\[ q^i_\beta = q^i_\alpha \left( q^i_\alpha, p^i_\alpha \right), \quad p^i_\beta = p^i_\alpha \left( q^i_\alpha, p^i_\alpha \right) \]  

(A.3)
on an overlap $U_\alpha \cap U_\beta \neq \emptyset$ preserve the form of the Hamilton’s equations (A.2). This happens not only if Eq. (A.3) implies

\[ \omega_\beta = dq^i_\beta \wedge dp^i_\beta = dq^i_\alpha \wedge dp^i_\alpha = \omega_\alpha, \quad H_\beta = H_\alpha, \]  

(A.4)
where $H_\alpha : U_\alpha \to \mathbb{R}, \alpha \in I$, but also if it implies

\[ \omega_\beta = \lambda_{\beta \alpha} \omega_\alpha, \quad H_\beta = \lambda_{\beta \alpha} H_\alpha, \]  

(A.5)
where $\lambda_{\beta \alpha} = \text{constant} \neq 0$. Since $\iota(X_{H_\alpha})\omega_\alpha = dH_\alpha$, from Eq. (A.5) we obtain

\[ X_{H_\alpha} = X_{H_\beta}, \]  

(A.6)
so the integral curves of $X_{H_\alpha}$ and $X_{H_\beta}$ are the same. Furthermore, Eq. (A.5) implies the cocycle condition

\[ \lambda_{\gamma \beta} \lambda_{\beta \alpha} = \lambda_{\gamma \alpha} \]  

(A.7)
as the gluing condition. We know that the cocycle condition (A.7) implies the existence of the local functions $\sigma_\alpha : U_\alpha \to \mathbb{R}$ satisfying

\[ \lambda_{\beta \alpha} = \frac{e^{\sigma_\alpha}}{e^{\sigma_\beta}}. \]  

(A.8)
Thus Eq. (A.5) shows that

\[ \omega = e^{\sigma_\alpha} \omega_\alpha, \quad H = e^{\sigma_\alpha} H_\alpha \]  

(A.9)
are globally defined on $M$. Moreover a Hamiltonian vector field is globally defined, i.e. $X_H = X_{H_\alpha}$, as was indicated in Eq. (A.6). Hence we have a basic line bundle $L$ over $M$ and a Hamiltonian $H$ as a cross-section of $L$ (a “twisted Hamiltonian”) instead of a simple function. Therefore $(M, \omega)$ is an LCS manifold that can be considered as a natural phase space of Hamiltonian dynamical systems, more general than the symplectic manifolds.

As we explained in Sec. 2, the realization of emergent geometry is intrinsically local too. The emergent geometry is constructed by gluing local Darboux charts and their local Poisson algebras. Therefore the construction of LCS manifolds as natural generalized phase spaces for particle dynamics should also be applied to the emergent geometry. Therefore we briefly review infinitesimal automorphisms of an LCS manifold $(M, \Omega, b)$. The infinitesimal automorphism (IA) will be denoted by $\mathfrak{A}_f$. Let $C^\infty(M)$ denote the associative algebra of smooth functions on $M$ and $f : M \to \mathbb{R}$ be such a globally defined function. The Hamiltonian vector field $X_f$ of $f \in C^\infty(M)$ with respect to the LCS form $\Omega$ is defined by

\[ \iota(X_f)\Omega = df = fb. \]  

(A.10)
As we observed above, there is a well-defined line bundle $L$ over $M$ in which local functions $f_\alpha \equiv e^{-\sigma_\alpha} f$ on a patch $U_\alpha \subset M$ correspond to sections of $L \to U_\alpha$. If we take the Lee form on $U_\alpha$ as $b|_{U_\alpha} = d\sigma_\alpha$, Eq. (A.10) refers to the usual (local) Hamiltonian vector field $X_{f_\alpha} = X_f$ defined by

$$\iota(X_{f_\alpha})\Omega_\alpha = df_\alpha$$

(A.11)

where $\Omega_\alpha = e^{-\sigma_\alpha}\Omega$. Using the Cartan formula for the Lie derivative

$$\mathcal{L}_X = dt_X + \iota_X d,$$

(A.12)

one can immediately deduce from Eqs. (A.1) and (A.10) that

$$\mathcal{L}_X f_\alpha \Omega = b(X_f)\Omega,$$

(A.13)

$$\mathcal{L}_X b = db(X_f).$$

(A.14)

Therefore, unlike the symplectic case, the Hamiltonian vector field $X_f$ is in general not an IA of LCS manifolds.

Using the Hamiltonian vector fields defined by Eq. (A.10), we define the Poisson bracket

$$\{f, g\} = \iota(X_f)\iota(X_g)\Omega = -\Omega(X_f, X_g) = e^{\sigma_\alpha} \iota(X_{f_\alpha})\iota(X_{g_\alpha})\Omega_\alpha = e^{\sigma_\alpha} \{f_\alpha, g_\alpha\}_{\Omega_\alpha}.$$  

(A.15)

Then we can calculate the double Poisson bracket

$$\{\{f, g\} \Omega, h\} = X_h(\Omega(X_f, X_g)) - b(X_h)\Omega(X_f, X_g).$$

(A.16)

Using this result, it is easy to check the Jacobi identity of the Poisson bracket:

$$\{\{f, g\} \Omega, h\} + \{\{g, h\} \Omega, f\} + \{\{h, f\} \Omega, g\} = (d\Omega - b \wedge \Omega)(X_f, X_g, X_h) = 0.$$  

(A.17)

Let $\mathfrak{P} = (C^\infty(M), \{-, -\}_{\Omega})$ be the Poisson-Lie algebra of $(M, \Omega)$ and $\mathfrak{X}(M)$ the Lie algebra of vector fields of $M$. The result (A.15) shows that the mapping $\mathfrak{H} : \mathfrak{P} \to \mathfrak{X}(M)$ given by $f \mapsto X_f$ is a Lie algebra homomorphism because one can derive the relation

$$X_{\{f, g\} \Omega} = [X_f, X_g]$$

(A.18)

from the Jacobi identity (A.17). However, if $(M, \Omega)$ is a (connected) LCS manifold that is not GCS, then $\mathfrak{H}$ must be a monomorphism, i.e., an injective homomorphism. See the Proposition 2.1 in [40] for the proof. This means that $X_f = 0$ implies $f = 0$. This is in stark contrast to symplectic manifolds, in which $X_f = 0$ just implies $f = \text{constant}$. Since we argue that the phase space for the cosmic inflation is a locally conformal (co)symplectic manifold, this in turn implies that the last property of Eq. (3.3) does not hold during the inflation. However, it does not mean that the cosmological constant problem threatens the emergent gravity too because physical quantities during inflation are not constant but time-dependent.

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Let us denote the IA of \((M, \Omega)\) by \(\mathfrak{X}_\Omega(M)\) whose elements obey \(\mathcal{L}_X \Omega = 0\). Then we have \(\mathcal{L}_X b = 0\) by Eq. (A.1) which implies the condition \(b(X) = \text{constant}\). In particular, if \(X, Y \in \mathfrak{X}_\Omega(M)\), then \(b(X) = \text{constant}, \ b(Y) = \text{constant and } db(X, Y) = 0\) yields \(b([X, Y]) = 0\) using the formula
\[
db(X, Y) = X(b(Y)) - Y(b(X)) - b([X, Y]).
\]
Hence, the application \(l : \mathfrak{X}_\Omega(M) \to \mathbb{R}\) defined by \(l(X) = b(X)\) is a Lie algebra homomorphism, called the Lee homomorphism of \(\mathfrak{X}_\Omega(M)\). The kernel \(\ker(l)\) is the Lie algebra of the horizontal elements of \(\mathfrak{X}_\Omega(M)\), denoted by \(\mathfrak{X}^\text{hor}_\Omega(M)\). The IA \(X \in \mathfrak{X}_\Omega(M)\) with \(l(X) \neq 0\) is called transversal IA and an LCS manifold \(M\) is called the first kind if it has a transversal IA. Otherwise, \(M\) is of the second kind and the Lee homomorphism is trivial. Note that, if \((M, \Omega)\) is of the first kind and \(f : M \to \mathbb{R}\) is a function such that \(df|_{x_0} = b(x_0)\), then \((M, e^{-f}\Omega)\) has the Lee form \(b - df\) with a vanishing point, so it becomes an LCS manifold of the second kind.

There is a special vector field \(A\) defined by \(\iota_A \Omega = b\). Then it is easy to get
\[
\iota_A b = 0, \quad \mathcal{L}_A b = 0, \quad \mathcal{L}_A \Omega = 0.
\]
We do have \(X_f \in \mathfrak{X}_\Omega(M)\) if and only if \(b(X_f) = 0\) according to Eq. (A.13) or equivalently \(b(X_f) = \iota_{X_f} \iota_A \Omega = -\iota_A (df - f b) = - A(f) = 0\). Let us fix an element \(B \in \mathfrak{L}^{-1}(1) \subset \mathfrak{X}_\Omega(M)\). Then every element \(Y\) in \(\mathfrak{X}_\Omega(M)\) has a unique decomposition
\[
Y = X + l(Y) B, \quad X \in \mathfrak{X}^\text{hor}_\Omega(M).
\]
Now, put \(a \equiv -\iota_B \Omega\), so \(a(B) = 0\) and \(a(A) = \iota_B \iota_A \Omega = b(B) = 1\). Since \(\mathcal{L}_B \Omega = (\iota_B d + d \iota_B) \Omega = 0\), this yields a particular expression for \(\Omega\) given by
\[
\Omega = da - b \wedge \alpha = d_b a,
\]
where \(d_b\) is the Lichnerowicz differential defined by \(d_b \beta = d \beta - b \wedge \beta\) for any \(k\)-form \(\beta\) and satisfies \(d_b^2 = 0\). Furthermore, using the formula \([\mathcal{L}_X, \iota_Y] = \iota_{[X,Y]}\) for vector fields \(X, Y\), we have \(\mathcal{L}_B a = 0\), hence \(\iota_B da = 0\) that means rank \(da < 2n\). Since \(\Omega^n \neq 0\), one can deduce from Eq. (A.22) the condition
\[
b \wedge \alpha \wedge (da)^{n-1} \neq 0
\]
everywhere. This yields the Proposition 2.2 in Ref. [40] that a manifold \(M\) of dimension \(2n\) admits an LCS structure of the first kind if and only if it admits two one-forms \(a, b\) such that \(db = 0\), rank \(da < 2n\) and Eq. (A.23) holds at every point of \(M\). Note also that \(\iota_A da = \iota_A (\Omega + b \wedge \alpha) = b - a(A) b = 0\). This means that \([A, B] = 0\) because \(\iota_A da = \mathcal{L}_A a = -\mathcal{L}_A \iota_B \Omega = -\iota_{[A,B]} \Omega = 0\). In sum, there exist particular vector fields \(A\) and \(B\) in \(\mathfrak{X}_\Omega(M)\) that obey
\[
[A, B] = 0, \quad a(A) = b(B) = 1, \quad a(B) = b(A) = 0.
\]
Thus one can obtain on \(M\) the vertical foliation \(V = \text{span}\{A, B\}\), whose leaves are the orbits of a natural action of \(\mathbb{R}^2\).
Suppose that \((M, \Omega)\) is an LCS manifold of the first kind and \(B\) is a basic transversal IA. Let \(\mathfrak{X}^\text{hor}_\Omega(M, B)\) be the Lie subalgebra of \(\mathfrak{X}^\text{hor}_\Omega(M)\) whose automorphisms also preserve \(B\). It turns out that \(X \in \mathfrak{X}^\text{hor}_\Omega(M, B)\) if and only if \(\mathcal{L}_X \Omega = 0\), \(b(X) = 0\) and \([X, B] = 0\). Similarly consider the subset of \(C^\infty(M)\) that consists of functions satisfying \(A(f) = B(f) = 0\) and is denoted by \(C^\infty_V(M)\). Then one can show that \(\mathcal{P}_V = (C^\infty_V(M), \{-, -\}_\Omega)\) is a Poisson-Lie subalgebra of \(\mathfrak{g}\) and \(\mathcal{P}_V : \mathfrak{g} \rightarrow \mathfrak{X}^\text{hor}_\Omega(M, B)\) is an isomorphism. A striking fact is that a semi-simple Lie group \(G\) cannot act transitively on a nonsymplectic LCS manifold.

The formula (A.13) proves that a Hamiltonian vector field is a conformal infinitesimal transformation (CIT) of \((M, \Omega)\). In general, a vector field \(X\) is a CIT if

\[
\mathcal{L}_X \Omega = \alpha_X \Omega
\]

where \(\alpha_X\) is a function on \(M\). The CIT forms a Lie algebra denoted by \(\mathfrak{X}^c_\Omega(M)\). By differentiating Eq. (A.25), one can derive that \(\mathcal{L}_X b = d\alpha_X\), which implies

\[
\alpha_X = b(X) + \kappa, \quad \kappa = \text{constant}.
\]

One can rewrite Eq. (A.25) as

\[
\kappa \Omega = d_b(\iota_X \Omega).
\]

Thus an LCS form \(\Omega\) is \(d_b\)-exact if there is a CIT \(X\). Or it can be written in terms of a local symplectic form \(\Omega_\alpha = e^{-\sigma_\alpha} \Omega\) as

\[
\mathcal{L}_X \Omega_\alpha = (\alpha_X - b(X)) \Omega_\alpha.
\]

That is, the local form of the CIT is given by

\[
\mathcal{L}_X \Omega_\alpha = \kappa \Omega_\alpha.
\]

If we write \(\Omega_\alpha = dA_{(\alpha)}\) on an open neighborhood \(U_\alpha\) according to the Poincaré lemma, Eq. (A.29) can be written as the form

\[
\iota_X \Omega_\alpha = \kappa A_{(\alpha)} + df_\alpha,
\]

where \(f_\alpha : U_\alpha \rightarrow \mathbb{R}\) is a smooth function on \(U_\alpha\). If the conditions (A.29) and (A.30) hold either locally or globally, we will call \(X\) a conformal vector field which will play an important role in our discussion. If \(H^1(M) = 0\), the conformal vector field \(X\) has a unique decomposition given by

\[
X = \kappa Z + X_f,
\]

where \(\iota_Z \Omega = A\) and \(\iota_{X_f} \Omega = df\). The vector field \(Z\) is called the Liouville vector field in [43]. Note that, even though \(f = 0\) identically, the conformal vector field \(X = \kappa Z\) is nontrivial. We argued in Sec. 3 that this remarkable property leads to a desirable consequence for the cosmic inflation.

We can extend the Lee homomorphism to \(l : \mathfrak{X}^c_\Omega(M) \rightarrow \mathbb{R}\) by defining \(l(X) = b(X) - \alpha_X = -\kappa\). If \(X, Y \in \mathfrak{X}^c_\Omega(M)\), we get \(\alpha_{[X, Y]} = X(b(Y)) - Y(b(X))\) from \(\mathcal{L}_{[X, Y]} \Omega = \alpha_{[X, Y]} \Omega\) and so
\( l([X,Y]) = b([X,Y]) - \alpha_{[X,Y]} = -db(X,Y) = 0 \) using the formula (A.19). Hence the extended \( l \) is also a Lie algebra homomorphism. Its kernel is denoted by \( \ker l = \mathfrak{x}_{\text{Ham}}^l(M) \) and consists of vector fields \( X \) such that \( \mathcal{L}_X \Omega_\alpha = 0 \), i.e., of locally Hamiltonian vector fields. Note that \( \tilde{l}(X) \) for \( \tilde{\Omega} = e^{\varphi} \Omega \) is equal to \( l(X) \) for \( \Omega \). Thus the Lee homomorphism \( l \) is conformally invariant. If we fix an element \( C \in l^{-1}(1) \), we can get for every \( Y \in \mathfrak{x}_\tilde{\Omega}(M) \) the unique decomposition

\[
Y = X + l(Y)C, \quad X \in \mathfrak{x}_\Omega^c(M). \tag{A.32}
\]

Then, if \( c = -\iota_C \Omega \), we can solve \( \mathcal{L}_C \Omega = (\iota_C d + d\iota_C)\Omega = \alpha_C \Omega \) to get a particular expression for \( \Omega \) given by

\[
\Omega = dc - b \wedge c = d\iota_c. \tag{A.33}
\]

In a conservative dynamical system described by a Hamiltonian vector field, time coordinate \( t \) is not a phase space coordinate but an affine parameter on particle trajectories. But, for a general time-dependent system, it is necessary to include the time coordinate as an extra phase space coordinate. The corresponding \((2n + 1)\)-dimensional manifold is known as an almost cosymplectic manifold which is a triple \((M, \Omega, \eta)\) where \( \Omega \) and \( \eta \) are a two-form and a one-form on \( M \) such that \( \eta \wedge \Omega^n \neq 0 \). If \( \Omega \) and \( \eta \) are closed, i.e., \( d\Omega = d\eta = 0 \), then \( M \) is said to be a cosymplectic manifold. Thus an odd-dimensional counterpart of a symplectic manifold is given by a cosymplectic manifold, which is locally a product of a symplectic manifold with a circle or a line. A contact manifold constitutes a subclass of cosymplectic manifolds with \( \Omega = d\eta \). Then the one-form \( \eta \) is a contact structure or a contact one-form. Given a contact one-form \( \eta \), there is a unique vector field \( R \) such that \( \iota_R \eta = 1 \) and \( \iota_R \Omega = 0 \). This vector field \( R \) is known as the Reeb vector field of the contact form \( \eta \). Two contact forms \( \eta \) and \( \eta' \) on \( M \) are equivalent if there is a smooth positive function \( \rho \) on \( M \) such that \( \eta' = \rho \eta \), since \( \eta' \wedge (d\eta')^n = \rho^{n+1} \eta \wedge (d\eta)^n \neq 0 \). The contact structure \( C(\eta) \) determined by \( \eta \) is the equivalence class of \( \eta \).

The Darboux theorem for a contact manifold \((M, \eta)\) states that, in an open neighborhood of each point of \( M \), it is always possible to find a set of local (Darboux) coordinates \((x^1, \ldots, x^n, y_1, \ldots, y_n, z)\) such that the one-form \( \eta \) can be written as

\[
\eta = dz - \sum_{i=1}^n y_i dx^i \tag{A.34}
\]

and the Reeb vector field is given by

\[
R = \frac{\partial}{\partial z}. \tag{A.35}
\]

To understand the condition on the contact one-form \( \eta \), first let us denote by \( \mathcal{D} \) the contact distribution or subbundle defined by the kernel of \( \eta \). If \( X, Y \) are (local) vector fields in \( \mathcal{D} \), we have

\[
d\eta(X,Y) = X(\eta(Y)) - Y(\eta(X)) - \eta([X,Y]) = -\eta([X,Y]). \tag{A.36}
\]

This says that the distribution is integrable if and only if \( d\eta \) is zero on \( \mathcal{D} \). However the condition \( \eta \wedge (d\eta)^n \neq 0 \) means that the kernel of \( d\eta \) is one-dimensional and everywhere transverse to \( \mathcal{D} \).
Consequently, $d\eta$ is a linear symplectic form on $D$ and the largest integral submanifolds of $D$ are $n$-dimensional, so maximally non-integrable. In other words, a contact structure is nowhere integrable.

In the above Darboux coordinate system, the contact subbundle $D$ is spanned by

$$X_i = \frac{\partial}{\partial x^i} + y_i \frac{\partial}{\partial z}, \quad Y^i = \frac{\partial}{\partial y^i}, \quad i = 1, \ldots, n,$$

so they obey the bracket relations

$$[X_i, Y^j] = -\delta^j_i R, \quad [X_i, R] = [Y^i, R] = 0.$$  \hspace{1cm} (A.38)

Since $d\eta = \sum_{i=1}^{n} dx^i \wedge dy^i$ is a symplectic form with rank $2n$, the kernel of $d\eta$ is one-dimensional and generated by the Reeb vector $R$. Therefore every vector field $X$ on $M$ can be uniquely written as $X = fR + Y$ where $f \in C^\infty(M)$ and $Y$ is a section of $D$. A contact structure is regular if $R$ is regular as a vector field, that is, every point of the manifold has a neighborhood such that any integral curve of the vector field passing through the neighborhood passes through only once.

Given a $(2n-1)$-dimensional contact manifold $M$ with a contact form $a$, i.e. $a \wedge (da)^{n-1} \neq 0$, one can construct an LCS manifold by considering a principal bundle $p : V \to M$ with group $\mathbb{S}^1$ over $M$. Consider $V = \mathbb{S}^1 \times M$ endowed with the form $\Omega = da - b \wedge a = d_\theta a$, where $b$ is the canonical one-form on $\mathbb{S}^1$. Clearly, $\Omega$ is nondegenerate and $b$ is closed but not exact. And it obeys $d\Omega - b \wedge \Omega = d_b \Omega = d_\theta^2 a = 0$. Hence, $(V, \Omega)$ is an LCS manifold having $b$ as its Lee form but it is not GCS. More generally, let $p : V \to M$ be an arbitrary principal bundle with group $\mathbb{S}^1$ over a $(2n-1)$-dimensional manifold $M$. And let $a$ be the connection one-form on this principal bundle and $F = da$ be the corresponding curvature two-form. Then, if $b \wedge a \wedge F^{n-1} \neq 0$, the form $\Omega = F - b \wedge a$ defines an LCS structure on $V$ which is not GCS.

Let $\mathfrak{X}(M), \Lambda^1(M)$ be the $C^\infty(M)$-modules of differentiable vector fields and one-forms on $M$, respectively. If $(M, \Omega, \eta)$ is a cosymplectic manifold, then there exists an isomorphism of $C^\infty(M)$-modules

$$\Upsilon : \mathfrak{X}(M) \to \Lambda^1(M)$$

defined by

$$\Upsilon(X) = \iota_X \Omega + \eta(X) \eta.$$  \hspace{1cm} (A.39)

The Reeb vector field is given by $R = \Upsilon^{-1}(\eta)$. Let $f : M \to \mathbb{R}$ be a smooth function on $M$. The Hamiltonian vector field $X_f$ is then defined by

$$\Upsilon(X_f) = df - R(f) \eta + \eta.$$  \hspace{1cm} (A.41)

In other words, $X_f$ is the vector field characterized by the identities

$$\iota(X_f) \Omega = df - R(f) \eta, \quad \eta(X_f) = 1.$$  \hspace{1cm} (A.42)

Then one can check that the time-like vector field $V_0$ in Eq. (2.56) is a Hamiltonian vector field for a cosymplectic manifold $(\mathbb{R} \times M, \pi_2^* B, dt)$ where $\pi_2 : \mathbb{R} \times M \to M$ and $(M, B)$ is a symplectic manifold.
An almost cosymplectic manifold \((M, \Omega, \eta)\) is said to be LCC, if there exist an open covering \(\{U_\alpha\}_{\alpha \in I}\) and local functions \(\sigma_\alpha : U_\alpha \to \mathbb{R}\) such that

\[
d(e^{-\sigma_\alpha} \Omega) = 0, \quad d(e^{-\sigma_\alpha} \eta) = 0. \tag{A.43}
\]

The local one-forms \(d\sigma_\alpha\) glue up to a closed one-form \(b\) satisfying

\[
d\Omega - b \wedge \Omega = d\eta - b \wedge \eta = 0. \tag{A.44}
\]

Two LCC structures \((\Omega', \eta')\) and \((\Omega, \eta)\) are equivalent if \(\Omega' = f \Omega\) and \(\eta' = f \eta\) for a positive function \(f\) on \(M\) where the Lee form of \(\Omega'\) is given by \(b' = b + d \ln f\). An LCC manifold reduces to a cosymplectic manifold if the Lee form \(b\) vanishes while it becomes an LCS manifold if \(\eta = 0\) identically. The isomorphism \([A.40]\) can be generalized to LCC manifolds and the corresponding Hamiltonian vector field is defined by

\[
X_f = \Upsilon^{-1}(df - R(f)\eta + \eta) + fS \tag{A.45}
\]

where \(S\) is called the canonical vector field defined by

\[
\Upsilon(S) = b(R)\eta - b. \tag{A.46}
\]

That is, \(X_f\) is characterized by the identities

\[
\iota(X_f)\Omega = df - R(f)\eta + f(b(R)\eta - b), \quad \eta(X_f) = 1. \tag{A.47}
\]

It was shown in [41] that LCC manifolds may be seen as generalized phase space of time-dependent Hamiltonian systems. We argued that an LCC manifold plays a role of a generalized phase space for the cosmic inflation of our Universe in the context of emergent spacetime, in particular, in a background-independent theory.

### B Harmonic oscillator with time-dependent mass

We observed that NC spacetime \(\mathbb{R}^{2n}_q\) in equilibrium is described by the Hilbert space of an \(n\)-dimensional harmonic oscillator while the inflating spacetime in nonequilibrium is described by the \(n\)-dimensional harmonic oscillator with a negative friction. The corresponding harmonic oscillator of constant frequency \(\omega\) and friction coefficient \(\alpha\) is given by

\[
\dddot{q}_i + 2\alpha \dot{q}_i + \omega^2 q_i = 0, \quad i = 1, \ldots, n. \tag{B.1}
\]

The inflationary coordinates \([3, 23]\) correspond to the case \(\alpha = -\frac{\kappa}{2} < 0\). It is known that the above second-order equation of motion cannot be directly derived from the Euler-Lagrange equation of any Lagrangian. However, there is an equivalent second-order equation

\[
e^{2\alpha t}(\dddot{q}_i + 2\alpha \dot{q}_i + \omega^2 q_i) = 0 \tag{B.2}
\]
for which a variational principle can be found [62]. Although Eq. (B.1) is traditionally considered to be non-Lagrangian, there exists an action principle for the equation of motion (B.2) in terms of the Lagrangian

\[ L = \frac{1}{2} m(q^2 - \omega^2 q^2)e^{2\alpha t}. \]  

(B.3)

The corresponding Hamiltonian is given by

\[ H = \frac{1}{2} \left( e^{-2\alpha t} p^2 + e^{2\alpha t} m^2 \omega^2 q^2 \right) \]

(B.4)

where \( p_i = m\dot{q}_i e^{2\alpha t} \).

It is interesting to notice that the equation of motion (B.2) can be derived from an \( n \)-dimensional harmonic oscillator with a time-dependent mass \( m(t) \) whose action is given by

\[ S = \frac{1}{2} \int dt (m(t)\dot{q}^2 - k(t)q^2) \]

(B.5)

where \( k(t) = m(t)\omega^2 \) with constant frequency \( \omega \). The variational principle, \( \delta S = 0 \), with respect to arbitrary variations \( \delta q^i \) leads to the equation of motion

\[ m(t)\left( \ddot{q}^i + \frac{\dot{m}(t)}{m(t)} \dot{q}^i + \omega^2 q^i \right) = 0. \]

(B.6)

The second-order equation (B.2) corresponds to the case

\[ \frac{\dot{m}(t)}{m(t)} = 2\alpha \quad \Rightarrow \quad m(t) = m_0 e^{2\alpha t}. \]

(B.7)

Note that the equation of motion (1.3) for the inflaton corresponds to the case with the time-dependent mass \( m(t) = m_0 e^{3Ht} \).

There is also the first-order formalism for the dynamical system (B.5). The action has the form

\[ S = \frac{1}{2} \int dt \left( y\dot{x} - x\dot{y} - (y^2 + 2\alpha xy + \omega^2 x^2) \right) e^{2\alpha t}. \]

(B.8)

The equations of motion derived from the action (B.8) are given by

\[ (\dot{y} + 2\alpha xy + \omega^2 x)e^{2\alpha t} = 0, \quad (\dot{x} - y)e^{2\alpha t} = 0. \]

(B.9)

The above action (B.8) describes a singular system with second-class constraints

\[ \phi_x = p_x - \frac{1}{2} ye^{2\alpha t}, \quad \phi_y = p_y + \frac{1}{2} xe^{2\alpha t} \]

(B.10)

with the Hamiltonian

\[ H(x, y, t) = \frac{1}{2} (y^2 + 2\alpha xy + \omega^2 x^2)e^{2\alpha t}. \]

(B.11)

Even though the constraints are explicitly time-dependent, it is still possible to apply the Hamiltonian formalism with the help of Dirac brackets and perform the canonical quantization of the system. It was shown in [62] that the classical and quantum description of the harmonic oscillator described by the action (B.5) is equivalent to the first-order approach given in terms of the constraint system described by the action (B.8). Furthermore it can be proved that the dynamical system described by Eq. (B.2) is locally (i.e., \( |t| < \infty \)) equivalent to the system with the equation of motion (B.1).
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