Gradient Recovery for the BEM-based FEM and VEM

Daniel Seibel1,* and Steffen Weißer1,**

1 Department of Mathematics, Saarland University, 66041 Saarbrücken, Germany

In this article, we propose new gradient recovery schemes for the BEM-based Finite Element Method (BEM-based FEM) and Virtual Element Method (VEM). Supporting general polytopal meshes, the BEM-based FEM and VEM are highly flexible and efficient tools for the numerical solution of boundary value problems in two and three dimensions. We construct the recovered gradient from the gradient of the finite element approximation via local averaging. For the BEM-based FEM, we show that, under certain requirements on the mesh, superconvergence of the recovered gradient is achieved, which means that it converges to the true gradient at a higher rate than the untreated gradient. Moreover, we propose a simple and very efficient a posteriori error estimator, which measures the difference between the unprocessed and recovered gradient as an error indicator. Since the BEM-based FEM and VEM are specifically suited for adaptive refinement, the resulting adaptive algorithms perform very well in numerical examples.

© 2019 The Authors Proceedings in Applied Mathematics & Mechanics published by Wiley-VCH Verlag GmbH & Co. KGaA Weinheim

1 Introduction

Gradient or stress recovery has a long tradition in the context of Finite Element Methods (FEM) [1–4]. In essence, gradient recovery is a post-processing applied to the gradient of the finite element solution and it has two major applications, namely the construction of superconvergent solutions [5] and a posterior error estimation [6]. In the past decades, it has been extensively studied and several different post-processing strategies have been proposed, for example the L2-recovery [7] or the popular patch recovery by Zienkiewicz and Zhu [8].

Yet, the application of gradient recovery is almost limited to standard FEM on triangular or quadrilateral discretisations. In [9], Guo, Xie and Zhao propose a Zienkiewicz-Zhu-type recovery scheme for Virtual Element Method (VEM), which is a new FEM-like method for the numerical solution of partial differential equations [10, 11]. The VEM belongs to the family of Galerkin methods based on polytopal grids and, as such, features a great flexibility with handling complex geometries and allows for easy adaptive refinement and coarsening. Despite its newness, it has already been applied to a wide range of problems [12–14]. Besides the VEM, the hybridised discontinuous Galerkin method [15], the mimetic finite difference method [16] and the BEM-based Finite Element Method (BEM-based FEM) [17] are prominent examples for numerical methods on polytopal grids. The latter has been introduced in [18] and has been studied, in particular, for adaptive FEM strategies involving residual [19, 20] and goal-oriented error estimators [21]. Other fields of application of the BEM-based FEM include, but are not limited to, convection dominated problems [22], anisotropic discretisations [23] and Nyström-based formulations [24].

In this work, we formulate gradient recovery schemes by averaging for the lowest order BEM-based FEM and VEM. In Section 3, we show that for the BEM-based FEM the centroids of regular k-gons are points of extraordinary accuracy, which we use to construct superconvergent recovered gradients in Section 4. Thereafter, we propose an a posteriori error estimator based on this recovery scheme in Section 5.

2 Preliminaries

Let $\Omega \subset \mathbb{R}^2$ be a bounded polygonal domain that admits a finite decomposition $\mathcal{T}_h$ into open non-overlapping polygonal elements $E$ with maximal diameter $h > 0$ such that $\overline{\Omega} = \bigcup_{E \in \mathcal{T}_h} \overline{E}$. The boundary $\partial E$ of each element $E$ is assumed to be not self-intersecting. Moreover, we denote by $x_i$, $i=1, \ldots, N$, the set of interior nodes of $\mathcal{T}_h$.

We denote by $L^2(\Omega)$ the space of square-integrable functions and by $H^k(\Omega)$ the Sobolev space of order $k \in \mathbb{N}$ with corresponding norms $\|\cdot\|_{L^2(\Omega)}$ and $\|\cdot\|_{H^k(\Omega)}$. Furthermore, we define $H^k_0(\Omega)$ to be the closure of $C_0^\infty(\Omega)$ in $H^k(\Omega)$.

For simplicity, we consider the Poisson equation with right-hand side $f \in L^2(\Omega)$ and zero Dirichlet conditions, i.e., find $u \in V = H^1_0(\Omega)$ such that

$$\int_{\Omega} \nabla u \cdot \nabla v \, dx = \int_{\Omega} f v \, dx, \quad \forall v \in V.$$  \hspace{1cm} (1)
In the following, we introduce the shape functions of lowest order used by both methods. Let \( \mathcal{P}_p(E) \) and \( \mathcal{P}_h(e) \) be the spaces of polynomials of degree \( p \) on the element \( E \) and edge \( e \) respectively. We define the local space of shape functions by
\[
V_h^E = \left\{ v_h \in H^1(E) \mid \Delta v_h = 0 \text{ on } E \text{ and } (v_h)_e \in \mathcal{P}_1(e) \text{ for every edge } e \text{ of } E \right\}.
\]
Consequently, the global finite element space and degrees of freedoms are given by
\[
V_h = \left\{ v_h \in V \mid (v_h)_e \in V_h^E, \forall E \in T_h \right\} \cap C^0(\Omega) \text{ and } N_i(v_h) = v_h(x_i), \quad i = 1, \ldots, N.
\]
The two methods depart in the discretisation of (1). The BEM-based FEM uses the theory of boundary integral operators, whereas the VEM reduces the problem to the polynomial component \( \mathcal{P}_1(E) \) of \( V_h^E \). We refer to [10, 17] for details. Nonetheless, in both cases, we end up with a discrete formulation of the form: find \( u_h \in V_h \) such that
\[
a_h(u_h, v_h) = b_h(v_h), \quad \forall v_h \in V_h, \quad \text{with } a_h(u_h, v_h) = \sum_{E \in T_h} a_h^E(u_h, v_h), \quad b_h(v_h) = \sum_{E \in T_h} b_h^E(v_h).
\]
By introducing the Lagrangian basis
\[
u_h = \sum_{i=1}^{N} c_i \varphi_i, \quad c_i \in \mathbb{R}, \quad N_i(\varphi_j) = \begin{cases} 1, & i = j, \\ 0, & i \neq j, \end{cases} \quad \text{for } i, j = 1, \ldots, N,
\]
we can re-write the discrete formulation (2) as a system of linear equations
\[
A c = b, \quad A_{ij} = a_h(\varphi_i, \varphi_j), \quad b_i = b_h(\varphi_i), \quad i, j = 1, \ldots, N.
\]

### 3 Superconvergent Points

Originally, the term “gradient superconvergence” describes the phenomenon that for certain types of elements there exist points at which the gradient of the finite element solution converges to the true solution at a higher rate than that encountered globally. Such points of extraordinary accuracy are known as stress points and were first discovered by Barlow [1]. To the best of our knowledge, there are no results on superconvergent points for the FEM on polygonal elements.

In the following, we extend the strategy of locating stress points by Strang and Fix [2] to the BEM-based FEM on regular polygonal elements. To this end, let \( \Omega \subset \mathbb{R}^2 \) be a regular convex or star polygon with \( k \) vertices. Note that the number of degrees of freedom equals \( k \) for convex and \( 2k \) for star \( k \)-gons. Given a quadratic harmonic polynomial \( m \in \mathcal{P}^2(\Omega) \), we define \( \hat{m} \) to be the BEM-interpolant in terms of the shape functions \( \varphi_i \). Then, we may characterise stress points as those points, where \( \hat{m} \) coincides with \( m \) up to a BEM error. We check if the centroid \( x_c \) of \( \Omega \) is superconvergent, i.e., if the interpolant \( \hat{m} \) is almost exact at \( x_c \).

In our example, \( \Omega \) is always centred at the origin with diameter less than one and we choose \( m = (x - 1)^2 - (y + 0.05)^2 \). In Table 1 the relative error measured in the Euclidean norm is listed and we observe that the centroid is a stress point in almost all cases.

**Table 1**: Relative point-wise error between \( m \) and \( \hat{m} \) at the centroid \( x_c \) for regular convex and star \( k \)-gons.

| \( k \) | convex | star |
|-------|-------|------|
| 3     | \( 4.99 \cdot 10^{-2} \) | – |
| 4     | \( 3.04 \cdot 10^{-10} \) | – |
| 5     | \( 4.53 \cdot 10^{-11} \) | \( 9.31 \cdot 10^{-7} \) |
| 6     | \( 1.41 \cdot 10^{-11} \) | \( 9.98 \cdot 10^{-9} \) |
| 7     | \( 6.34 \cdot 10^{-12} \) | \( 8.53 \cdot 10^{-10} \) |
| 8     | \( 3.51 \cdot 10^{-12} \) | \( 1.74 \cdot 10^{-10} \) |
| 9     | \( 2.22 \cdot 10^{-12} \) | \( 5.60 \cdot 10^{-11} \) |
| 10    | \( 1.54 \cdot 10^{-12} \) | \( 2.37 \cdot 10^{-11} \) |

**Fig. 1**: The error against the number of degrees of freedom in a log-log plot.

### 4 Superconvergent Gradient Recovery

The primary use of superconvergent points lies in superconvergent gradient recovery. To be more precise, we apply a post-processing technique that incorporates superconvergent points, with the intention that the recovered gradient is superconvergent not only at certain points but throughout subdomains or even the whole domain.
In the following, we assume the mesh to be made of regular hexagons. We define the recovery operator \( G : V_h \to V_h^2 \) by
\[
N_i(Gu_h) = \frac{1}{\#E(x_i)} \sum_{E \in E(x_i)} \nabla u_h(x_c(E)) \quad \text{with} \quad E(x_i) = \{ E \in T_h | x_i \in E \}, \quad i = 1, \ldots, N. \tag{3}
\]
Since this translates to local averaging, the recovery process is fairly simple and localised and practically does not increase numerical costs. Note that \( \nabla u_h \) is only given implicitly and therefore approximated by means of the BEM. Moreover, we observe superconvergence of the recovered gradient in numerical experiments. To demonstrate this, we solve the Laplace problem with Dirichlet conditions \( g(x, y) = \exp(2\pi(x - 0.3)) \cos(2\pi(y - 0.3)) \) on \( \Omega = (-1, 1)^2 \) for a series of meshes made of regular hexagons and apply the averaging technique (3) on each level.

The error against the number of degrees of freedom for this example is depicted in Figure 1. We observe superconvergence, which is highlighted by an increase in the order of convergence, i.e., from \( O(N^{-1/2}) \) for \( \nabla u_h \) to approximately \( O(N^{-3/4}) \) for \( \mathcal{G}u_h \) with \( N \) being the number of degrees freedom in the mesh.

5 Recovery-based error estimators

Another significant application of gradient recovery lies in a posteriori error estimation with the aim of adaptive mesh refinement techniques. In order to take general meshes into account, we modify our averaging scheme as follows
\[
N_i(\mathcal{G}u_h) = \frac{1}{|E(x_i)|} \sum_{E \in E(x_i)} |E| \nabla u_h(x_c(E)), \quad i = 1, \ldots, N. \tag{4}
\]
Since \( u_h \) is only given implicitly in the VEM, we apply the scheme to its projection \( \Pi_1^V u_h \) instead, compare [25].
Algorithm 1 We follow the basic concept of an adaptive FEM algorithm of the form

1. **Solve:** We solve on the current mesh level and compute $\nabla u_h$ (BEM-based FEM) or $\nabla \Pi_1 u_h$ (VEM).

2. **Recovery:** Then, we apply the post-processing by averaging (4) and obtain the recovered gradient $\hat{G} u_h$ (BEM-based FEM) or $\hat{G} \Pi_1 u_h$ (VEM).

3. **Estimate:** Subsequently, we compute the estimates $\eta^2 = \sum_E \eta^2_E$ with $\eta^2_E = \| \hat{G} u_h - \nabla u_h \|_{L^2(E)}^2$ (BEM-based FEM) or $\eta^2 = \| \Pi_1 \hat{G} \Pi_1 u_h - \Pi_1 \nabla u_h \|_{L^2(E)}^2$ (VEM).

4. **Mark:** Afterwards, we mark the elements for refinement. Here, we apply the Dörfler marking strategy \[26\].

5. **Refine:** Finally, we refine the marked elements and start again. Here, we use the bisection algorithm introduced in [19].

We stop the algorithm if the maximum mesh level is reached or $\eta$ is sufficiently small.

In the following, we test the performance of the recovery-based estimators for both methods. To this end, we consider the Laplace problem on the L-shaped domain $\Omega = (-1,1)^2 \setminus [0,1]^2$ with Dirichlet conditions $g(r, \varphi) = r^{2/3} \sin(2(\varphi - \pi/2)/3)$ in polar coordinates $(r, \varphi)$. This example is a popular benchmark for adaptive algorithms, since the rate of convergence of the error in the energy norm is limited to $O(N^{-1/3})$ for uniform refinement. As depicted in Figure 2, the recovery-based estimators recover the optimal convergence rate of $O(N^{-1/2})$ and produce similar results compared to residual-based estimators \[20, 27\]. In addition, we study the efficiency $\Psi$ of the estimators, which is the quotient of estimated and true error. We see in Figure 3 that the recovery-based estimators operate nearly optimally and also outperform the residual-based ones.

Overall, both estimators perform very well in this particular example. Due to the fact that the recovery-based scheme achieves slightly better results and estimates the error more accurately, we prefer it over the residual-based estimator.

References

[1] J. Barlow, Int. J. Numer. Meth. Eng. 10(2), 243–251 (1976).
[2] G. Strang and G. J. Fix, An analysis of the finite element method (Prentice-Hall, Inc., Englewood Cliffs, N. J., 1973), Prentice-Hall Series in Automatic Computation.
[3] M. Zlámal, Math. Comp. 32(143), 663–685 (1978).
[4] O. C. Zienkiewicz and J. Z. Zhu, Internat. J. Numer. Methods Engrg. 24(3), 337–357 (1987).
[5] M. Zlámal, Some superconvergence results in the finite element method, Lecture Notes in Mathematics, Vol. 606 (Springer, Berlin, 1977).
[6] M. Ainsworth and J. T. Oden, A posteriori error estimation in finite element analysis, Pure and Applied Mathematics (New York) (Wiley-Interscience [John Wiley & Sons, New York, 2000).
[7] J. T. Oden and H. J. Brauchli, Int. J. Numer. Meth. Eng. 3(3), 317–325 (1971).
[8] O. C. Zienkiewicz and J. Z. Zhu, Internat. J. Numer. Methods Engrg. 33(7), 1331–1364 (1992).
[9] H. Guo, C. Xie, and R. Zhao, arXiv e-prints(April), arXiv:1804.10194 (2018).
[10] L. Beirão da Veiga, F. Brezzi, A. Cangiani, G. Manzini, L. D. Marini, and A. Russo, Math. Models Methods Appl. Sci. 23(1), 199–214 (2013).
[11] L. Beirão da Veiga, F. Brezzi, L. D. Marini, and A. Russo, Math. Models Methods Appl. Sci. 24(8), 1541–1573 (2014).
[12] L. Mascotto, I. Perugia, and A. Pichler, Comput. Methods Appl. Mech. Engrg. 347, 445–476 (2019).
[13] A. Cangiani, V. Gryrya, and G. Manzini, SIAM J. Numer. Anal. 54(6), 3411–3435 (2016).
[14] P.F. Antonietti, S. Berrone, M. Verani, and S. Weißer, The virtual element method on anisotropic polygonal discretizations, in: Numerical Mathematics and Advanced Applications - ENUMATH 2017, edited by F. Radu, K. Kumer, I. Berre, J. Nordbotten, and I. Pop, Lect. Notes Comput. Sci. Eng. Vol. 126 (Springer International Publishing, 2018).
[15] B. Cockburn, J. Gopalakrishnan, and R. Lazarov, SIAM J. Numer. Anal. 47(2), 1319–1365 (2009).
[16] F. Brezzi, A. Buffa, and K. Lipnikov, M2AN Math. Model. Numer. Anal. 43(2), 277–295 (2009).
[17] S. Rjasanow and S. Weißer, SIAM J. Numer. Anal. 50(5), 2357–2378 (2012).
[18] D. Copeland, U. Langer, and D. Pusch, From the boundary element domain decomposition methods to local Trefftz finite element methods on polyhedral meshes, in: Domain decomposition methods in science and engineering XVIII, , Lect. Notes Comput. Sci. Eng., Vol. 70 (Springer, Berlin, 2009), pp. 315–322.
[19] S. Weißer, Numer. Math. 118(4), 765–788 (2011).
[20] S. Weißer, Comput. Math. Appl. 73(2), 187–202 (2017).
[21] S. Weißer and T. Wick, Comput. Methods Appl. Math. (2017).
[22] C. Hofreither, U. Langer, and S. Weißer, ZAMM Z. Angew. Math. Mech. 96(12), 1467–1481 (2016).
[23] S. Weißer, ESAIM: M2AN 53(2), 475–501 (2019).
[24] A. Anand, J. S. Ovall, and S. Weißer, Comput. Math. Appl. 75(11), 3971–3986 (2018).
[25] B. Ahmad, A. Alsaedi, F. Brezzi, L. D. Marini, and A. Russo, Comput. Math. Appl. 66(3), 376–391 (2013).
[26] W. Dörfler, SIAM J. Numer. Anal. 33(3), 1106–1124 (1996).
[27] A. Cangiani, E. H. Georgoulis, T. Pryer, and O. J. Sutton, Numer. Math. 137(4), 857–893 (2017).