Coxeter decompositions of hyperbolic simplices.
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1 Introduction

Definition 1. A convex polyhedron in a space of constant curvature is called a Coxeter polyhedron if all dihedral angles of this polyhedron are the integer parts of $\pi$.

Definition 2. A Coxeter decomposition of a convex polyhedron $P$ is a decomposition of $P$ into finitely many tiles such that each tile is a Coxeter polyhedron and any two tiles having a common facet are symmetric with respect to this facet.

Coxeter decompositions of hyperbolic triangles were studied in [10], [11], [12], [13] and [2]. Coxeter decompositions of hyperbolic tetrahedra are listed in [4]. In this paper, we classify Coxeter decompositions of simplices in hyperbolic spaces $H^n$, where $n \geq 4$. The paper completes the classification of the Coxeter decompositions of hyperbolic simplices.

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Basic definitions

The tiles in Definition 2 are called fundamental polyhedra. Clearly, any two fundamental polyhedra are congruent to each other. A hyperplane $\alpha$ containing a facet of a fundamental polyhedron is called a mirror if $\alpha$ contains no facet of $P$.

Definition 3. Given a Coxeter decomposition of a polyhedron $P$, a dihedral angle of $P$ formed up by facets $\alpha$ and $\beta$ is called fundamental if no mirror contains $\alpha \cap \beta$.

From now on we consider only the polyhedra bounded by the mirrors of some Coxeter decomposition.

Notation.
$P$ is a simplex equipped with a Coxeter decomposition; $F$ is a fundamental polyhedron considered up to an isometry of $H^n$; $\Sigma(T)$ is a Coxeter diagram of a Coxeter simplex $T$;
$N$ is a number of the fundamental polyhedra inside $P$.

A decomposition is called **non-trivial** if $N > 1$.

We use the standard notation for the Euclidean and spherical Coxeter simplices: $A_n, \tilde{A}_n, B_n, \tilde{B}_n, C_n, D_n, \tilde{D}_n, \ldots$. The notation for the hyperbolic Coxeter simplices is introduced in Table 2.

## 2 Properties of Coxeter decompositions of simplices

### 2.1 Fundamental polyhedron

**Theorem 1.** If $P$ is a hyperbolic simplex then $F$ is a simplex too.

**Proof.** Since the decomposition contains finitely many tiles, there is a **minimal** simplex inside $P$ containing no proper simplices bounded by mirrors and faces of $P$. We are aimed to prove that the minimal simplex is a fundamental polyhedron of the decomposition. Without loss of generality, we assume that $P$ is minimal itself.

Suppose that $P$ has a non-fundamental dihedral angle. Let $\Pi$ be a mirror decomposing the dihedral angle of $P$. Then $\Pi$ decomposes $P$ into two smaller simplices. The contradiction to the minimal property of $P$ shows that any dihedral angle of $P$ is fundamental.

Suppose that $F$ is not a simplex. Consider the set $S$ of polyhedra which could be cut from $P$ by a single mirror. The number of mirrors is finite. Thus, $S$ contains a **minimal** polyhedron $M$ such that no polyhedron contained in $M$ belongs to $S$. Let $\Pi$ be a mirror which cuts $M$ from $P$.

Consider the dihedral angles of $M$. Any dihedral angle $\alpha$ of $M$ formed up by $\Pi$ and a facet of $P$ is fundamental (otherwise $M$ contains a polyhedron which belongs to $S$ in contradiction to the minimal property of $M$). The rest dihedral angles of $M$ are the dihedral angles of $P$. Thus, all dihedral angles of $M$ are fundamental, and $M$ is a Coxeter polyhedron.

Let $A_0, \ldots, A_k$ be the vertices of $P$ contained in $M$, let $A_{k+1}, \ldots, A_n$ be the rest vertices of $P$. If $k = 0$ then $M$ is a simplex in contradiction to the minimal property of $M$. Therefore, $k \neq 0$. Consider the lines $A_0A_n$ and $A_1A_n$. These lines contain the non-adjacent edges of $M$. But these lines have a common point $A_n$. This is impossible, since $M$ is an acute angled polyhedron (see [1]). The contradiction shows that $F$ is a simplex.

\[\square\]
Lemma 1. Let $F$ be a hyperbolic Coxeter simplex. The number of simplices admitting a Coxeter decomposition with the fundamental polyhedron $F$ is finite.

Proof. The dihedral angles completely determine a non-Euclidean simplex. The dihedral angles of a simplex $P$ admitting a Coxeter decomposition are the multiples of the dihedral angles of the fundamental polyhedron. Thus, for any fundamental simplex $F$ there exist finitely many ways to prescribe the dihedral angles of the simplex $P$.

Lemma 2 (Volume property). \( \frac{\text{Vol}(P)}{\text{Vol}(F)} \in \mathbb{Z} \), where $\text{Vol}(T)$ is a volume of a simplex $T$.

The lemma is evident. The volumes of the hyperbolic Coxeter simplices are computed in [8]. In Table 2, we list the volumes of the Coxeter simplices in $\mathbb{H}^n$, where $n \geq 4$.

2.2 Three types of Coxeter decompositions of simplices

Consider a Coxeter decomposition of a simplex $P$. Suppose that $P$ has a non-fundamental dihedral angle. Suppose in addition, that any non-fundamental simplex inside $P$ has a non-fundamental dihedral angle. Let $\Pi$ be a mirror decomposing the dihedral angle of $P$. Then $\Pi$ decomposes $P$ into two smaller simplices $P_1$ and $P_2$. Given the decompositions of $P_1$ and $P_2$, it is possible to find the decomposition of $P$. By the assumption each of $P_1$ and $P_2$ is either fundamental or has a non-fundamental dihedral angle. In the latter case, the small simplex is decomposed into two smaller ones. Thus, we decompose the simplex into smaller and smaller simplices. The process stops only when $P$ is decomposed into fundamental simplices. Since $P$ contains finitely many fundamental simplices, the process stops after finitely many steps.

Invert the process. We obtain the following inductive algorithm for constructing the decomposition of the big simplex from the decompositions of the smaller simplices:

- Step 0. Let $F$ be a fundamental simplex. Set $P_0 = F$.
- Step 1. Take two copies of $P_0$. Glue these copies together to construct a simplex admitting a Coxeter decomposition with $N = 2$. Use all the ways to glue these copies together and obtain all the simplices $P_1, ..., P_k$ admitting the decomposition with this fundamental polyhedron and $N = 2$.
- Step 2. Find all simplices $P_{k+1}, ..., P_l$ which consist either of $P_1$ and
$P_0$ or of two copies of $P_1$ (in the former case we take one copy of each simplex).

- Step m. Suppose that after the step $(m - 1)$ we have the simplices $P_0, P_1, ..., P_n$. At the step $m$ compose all the simplices $P_{n+1}, ..., P_s$ from one copy of $P_{m-1}$ and one copy of $P_i$ for each $i \leq m - 1$ (if the simplex obtained is already in the list $P_0, P_1, ..., P_n$, one should not add this simplex to the list).

Suppose that $P$ admits a decomposition with the fundamental simplex $F$. Suppose that any non-fundamental simplex inside $P$ has a non-fundamental dihedral angle. Then the decomposition of $P$ will be constructed after finitely many steps of the inductive algorithm. In general, a simplex constructed by the algorithm contains some non-fundamental simplices with fundamental dihedral angles. But any simplex obtained by the algorithm has a non-fundamental dihedral angle.

**Definition 4.** Let $\Theta(P)$ be a Coxeter decomposition of a simplex $P$.

- $\Theta(P)$ is called a decomposition of the first type if the decomposition can be obtained by the inductive algorithm.
- $\Theta(P)$ is called a decomposition of the second type if all the dihedral angles of $P$ are fundamental.
- $\Theta(P)$ is called a decomposition of the third type if $\Theta(P)$ is neither of the first type nor of the second.

To find the decompositions of the first type, we use the inductive algorithm. To classify the decompositions of the second type is more difficult, since there is no algorithm for the classification. It turns out that any decompositions of the third type is a superposition of the decompositions of the first and the second types.

### 3 Classification of the decompositions

#### 3.1 Decompositions of the first type

Consider the inductive algorithm for the fundamental simplex $F$. The number of simplices admitting a decomposition with the fundamental simplex $F$ is finite. Each decomposition can be obtained by finitely many ways. Thus, the inductive algorithm terminates after finitely many steps. The result of the algorithm is the complete set of the decompositions of the first type with the fundamental simplex $F$. 

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There are finitely many hyperbolic Coxeter simplices in the dimensions greater than 2. Thus, we can use the inductive algorithm for one Coxeter simplex after another. We can use a computer for the computations.

**Definition 5.** Let $\Theta(F, P)$ be a Coxeter decomposition of the simplex $P$ with fundamental simplex $F$. Let $\Theta_1(T, P)$ be a Coxeter decomposition of the simplex $P$ with fundamental simplex $T$. Suppose that each mirror of $\Theta_1(T, P)$ is a mirror of $\Theta(F, P)$. Then $\Theta(F, P)$ is called a superposition of the decompositions $\Theta_1(T, P)$ and $\Theta(F, T)$, where $\Theta(F, T)$ is a restriction of $\Theta(F, P)$ to the simplex $T$. A decomposition is called simple if it is not a superposition of the non-trivial decompositions.

Clearly, it is sufficient to list the simple decompositions. Tables 3 and 4 contain the list of the simple decompositions of simplices of the first type.

### 3.2 Decompositions of the second type

Let $P$ be a simplex admitting a Coxeter decomposition of the second type. Clearly, $P$ is a Coxeter simplex.

**Lemma 3 (Subdiagram property).** Let $P$ be a simplex admitting a Coxeter decomposition of the second type with fundamental simplex $F$. Let $\Sigma(P)$ be a Coxeter diagram of $P$ and $\Sigma(F)$ be a Coxeter diagram of $F$. Let $v$ be a node of $\Sigma(P)$. There exists a node $w$ of $\Sigma(F)$ such that either $\Sigma(P) \setminus v = \Sigma(F) \setminus w$ or the simplex $p$ determined by the subdiagram $\Sigma(P) \setminus v$ admits a Coxeter decomposition of the second type with fundamental simplex $f$ determined by $\Sigma(F) \setminus w$.

**Proof.** The node $v$ of $\Sigma(P)$ corresponds to a facet $\alpha$ of $P$. Let $A$ be a vertex of $P$ opposite to the facet $\alpha$.

Suppose that the fundamental simplex $F_A$ containing $A$ is unique. Then all but one facets of $F_A$ are the facets of $P$. Let $w$ be a node of $\Sigma(F)$ correspondent to the rest facet of $F_A$. Then $\Sigma(P) \setminus v = \Sigma(F) \setminus w$.

Suppose that $A$ belongs to several fundamental simplices. Consider a small sphere $s$ centered in $A$ (if the vertex $A$ is ideal, consider the horosphere $s$ centered in $A$). Let $p = P \cap s$. The Coxeter decomposition of $P$ restricted to $p$ is a Coxeter decomposition of a spherical (Euclidean) simplex $p$ with some fundamental simplex $f$. Clearly, a Coxeter diagram of $p$ is $\Sigma(P) \setminus v$, and a Coxeter diagram of $f$ is $\Sigma(F) \setminus w$ for some face $w$ of $F$. Since the dihedral angles of $P$ are fundamental, the dihedral angles of $p$ are fundamental too. 

\[\square\]
To use the Subdiagram property we need the classification of the Coxeter decompositions of spherical simplices of the second type (see [3]).

**Definition 6.** A Coxeter decomposition of a spherical polyhedron is called **indecomposable** if the Coxeter diagram of the fundamental simplex is connected.

In [3], it is proved that any Coxeter decomposition of a spherical simplex is a direct product of the indecomposable decompositions. Moreover, a decomposition of the second type is a product of the indecomposable decompositions of the second type.

**Theorem 2.** An indecomposable decomposition of a spherical simplex of the second type is uniquely determined by the pair \((F, P)\). The possibilities for the pairs \((F, P)\) are listed in Table 1.

| \(F\) | \(P\) |
|---|---|
| \(H_3\) | \(3A_1\) |
| \(F_4\) | \(2A_2\) |
| \(H_4\) | \(A_4, 2G_2^{(5)}, 2A_2, H_3 + A_1, D_4, 4A_1\) |
| \(D_n\) | \(D_{m_1} + \ldots + D_{m_r} (m_1 + \ldots + m_r = n), m_1 \geq m_2 \geq \ldots \geq m_r > 1, \) where \(D_2 = 2A_1, D_3 = A_3\) |
| \(E_6\) | \(A_5 + A_1, 3A_2\) |
| \(E_7\) | \(D_6 + A_1, A_5 + A_2, 2A_3 + A_1, A_7, D_4 + 3A_1, 7A_1\) |
| \(E_8\) | \(A_8, A_7 + A_1, A_5 + A_2 + A_1, 2A_4, 4A_2, A_6 + A_2, E_7 + A_1, D_8, D_6 + 2A_1, D_5 + A_3, 2D_4, D_4 + 4A_1, 2A_3 + 2A_1, 8A_1\) |

If \(F \neq H_3, H_4, F_4, B_n\), the classification follows immediately from [7]. The rest cases of Theorem 2 are proved in [3].

**Lemma 4.** Let \(P\) be an \(n\)-dimensional simplex and \(\Theta(P)\) be a non-trivial Coxeter decomposition such that for any vertex \(v\) of \(P\) a fundamental simplex \(F_v\) containing \(v\) is unique. Then \(v\) is the only vertex of \(P\) contained in \(F_v\).

**Proof.** Let \(\alpha_v\) be a facet of \(F_v\) opposite to \(v\). The rest facets of \(F_v\) are the facets of \(P\). Suppose that \(F_v\) contains a vertex \(w \neq v\) of \(P\). Then \(\alpha_v\) contains \(w\). Since the decomposition is non-trivial, \(\alpha_v\) is not a facet of \(P\).
Thus, \( w \) belongs to at least two fundamental simplices, that is impossible by the hypothesis of the lemma.

\[ \square \]

**Lemma 5.** Let \( P \) be an \( n \)-dimensional simplex and \( \Theta(P) \) be a Coxeter decomposition such that for any vertex \( v \) of \( P \) a fundamental simplex \( F_v \) containing \( v \) is unique. Then \( N \geq 2(n + 1) \).

**Proof.** Let \( v \) be a vertex of \( P \) and \( F_v \) be the fundamental simplex containing \( v \). Let \( F^v \) be a fundamental simplex having a common facet with \( F_v \) (the fundamental simplex with this property is unique). Let \( F_w \) and \( F^w \) be the analogous simplices for the vertex \( w \neq v \). To prove the lemma, it is sufficient to show that the simplices \( F_v, F^v, F_w \) and \( F^w \) are distinct.

By the hypothesis of the lemma \( F_v \neq F^v \) and \( F_w \neq F^w \). Suppose that either \( F_v = F_w \), or \( F^v = F^w \), or \( F^v = F_w \), or \( F^v = F^w \). Then the simplices \( F_v \cap F^v \) and \( F_w \cap F^w \) are the facets of the same \( n \)-dimensional simplex \( S \). By Lemma 4, \( F_v \cap F^v \) has a vertex in the inner part of each edge of \( P \) incident to \( v \). The simplex \( F_w \cap F^w \) has a vertex in the inner part of each edge incident to \( w \). Thus, \( S \) has at least \( 2n - 1 \) vertices. Since \( n > 2 \), this is impossible.

\[ \square \]

**Lemma 6.** Let \( \Theta(P) \) be a Coxeter decomposition containing exactly two fundamental simplices. Then \( P \) has a non-fundamental dihedral angle.

The lemma is evident.

**Lemma 7.** Let \( P \) be a simplex in \( \mathbb{E}^n \) admitting a non-trivial Coxeter decomposition. Suppose that \( F \) is similar to \( P \). Suppose that the Coxeter diagram \( \Sigma(F) \) differs from \( \tilde{\Sigma}_2 \) and \( \tilde{G}_2^{(5)} \). Then \( N \geq 2^n \).

**Proof.** Let \( \Sigma(P) \) be the affine Dynkin diagram and \( \bar{v} \) be the node of \( \Sigma(P) \) correspondent to the lowest root. Let \( v \) be the vertex of \( P \) opposite to the facet that is represented by \( \bar{v} \). Let \( \bar{W} \) be the affine Weyl group. The stabilizer of \( v \) in \( \bar{W} \) is the set of all linear parts of the isometries contained in \( \bar{W} \). Thus, the fundamental simplex \( F \) containing \( v \) is unique. Moreover, if \( \Sigma(F) \) differs from \( \tilde{\Sigma}_2 \) and \( \tilde{G}_2^{(5)} \) then \( F \) is homothetic to \( P \).

Suppose that \( \Sigma(P) \neq \tilde{A}_n \). Let \( \alpha \) be the facet of \( F \) opposite to \( v \). Consider an edge \( r_F \) of \( F \) incident to \( v \) and orthogonal to \( \alpha \). Let \( r_F \) be an edge of \( P \) containing \( r_F \). The reflection with respect to \( \alpha \) preserves the line containing \( r_F \). Hence, \( r_F \) is at least to times longer than \( r_F \), and the lemma is proved.

Suppose that \( \Sigma(P) = \tilde{A}_n \). To show that in this case the coefficient of the homothety is at least 2, consider a set \( R_{\min} \) of the shortest edges of \( F_v \). The symmetry group of \( F_v \) acts transitively on the set of vertices of \( F_v \). Thus,
Let \( R_{\text{min}} \) contain at least one edge \( r_v \) incident to \( v \). Let \( t_v \) be the edge of \( P \) containing \( r_v \). The fundamental simplex contains no edge shorter than \( r_v \). Hence, \( t_v \) is at least \( t \) times longer than \( r_v \), and \( \frac{\text{Vol} P}{\text{Vol} F} \geq 2^n \).

Let \( (F, P) \) be a pair of simplices in \( IH^n \), \( n \geq 4 \). Suppose that \( (F, P) \) satisfies Volume Property, Subdiagram Property, Lemma 5 and Lemma 6. It turns out that in this case \( (F, P) \) is one of the following pairs:

- \( (H^4_3, H^4_9) \), \( N = 10 \);
- \( (H^5_5, H^5_{12}) \), \( N = 16 \);
- \( (H^5_9, H^5_{11}) \), \( N = 6 \);
- \( (H^5_3, H^5_{11}) \), \( N = 20 \);
- \( (H^5_5, H^5_{5}) \), \( N = 272 \);
- \( (H^6_9, H^6_{3}) \), \( N = 527 \)

(see Table 2 for the notation of the hyperbolic Coxeter simplices).

Consider these five cases to figure out if there exist decompositions corresponding to these pairs \( (F, P) \). We call \( (F, P) \), the symbol of the decomposition. Sometimes we write the number \( N \) near the pair \( (F, P) \).

### 3.2.1 There exists a unique decomposition with symbol \( (H^4_3, H^4_9) \) and \( N = 10 \).

The fundamental simplex \( H^4_3 \) has a unique ideal vertex \( v \). The section of \( H^4_3 \) by the horosphere centered in \( v \) is determined by the parabolic diagram \( \tilde{A}_3 \) (we call \( v \) the vertex of the type \( \tilde{A}_3 \)). The simplex \( H^4_9 \) has three ideal vertices of the type \( \tilde{A}_3 \). By Lemma 7, for any decomposition with symbol \( (\tilde{A}_3, \tilde{A}_3) \) we have \( N = 1 \) or \( N \geq 8 \). Since 10 fundamental simplices have 10 ideal vertices, one of the ideal vertices of \( H^4_9 \) (say, \( v_1 \)) belongs to 8 fundamental simplices, the rest two ideal vertices of \( H^4_9 \) belongs to one fundamental simplex each.

The only decomposition with symbol \( (\tilde{A}_3, \tilde{A}_3) \), \( N = 8 \) is a decomposition shown in Fig. 1. This is the decomposition of the section by horosphere centered in \( v_1 \). Let \( F_1, \ldots, F_8 \) be the fundamental simplices containing \( v_1 \), let \( f_1, \ldots, f_8 \) be the facets of \( F_1, \ldots, F_8 \) opposite to \( v_1 \). Let \( S \) be the union of \( f_1, \ldots, f_8 \). The combinatorics of \( S \) coincides with the decomposition shown in Fig. 1. But the facets \( f_1, \ldots, f_8 \) belong to different hyperplanes. Indeed, \( f_i \) is orthogonal to all but one facets of \( F_i \). The rest facet intersects \( f_1 \) with the dihedral angle \( \frac{\pi}{2} \). A 2-dimensional face of the fundamental simplex \( H^4_3 \) assigned with the dihedral angle \( \frac{\pi}{n} \) will be called an \( n \)-face. Let \( t \) be a 2-dimensional face of the fundamental simplex \( \tilde{A}_3 \) in the decomposition with symbol \( (\tilde{A}_3, \tilde{A}_3) \). We call \( t \) an \( n \)-face if \( t \) corresponds to the \( n \)-face of \( H^4_3 \).

It is not evident which face of the tetrahedron \( AEFG \) is a 3-face (see Fig. 1).

**Proposition.** Exactly two of 3-faces cut \( ABCD \). These facets are \( LMF \) and \( LEK \).
Proof. The 3-faces cut the set $S$ into parts contained in different hyperplanes. Only one of these hyperplanes (say, $\Pi$) contains a facet of $H^4_9$. Let $f_i \notin \Pi$. Then $f_i$ belongs to exactly two fundamental simplices. Since the decomposition contains 10 fundamental simplices, exactly two of the faces $f_1,...,f_8$ are not in $\Pi$. Thus, at most two 3-faces cut $S$. Hence, at most two 3-faces correspond to the triangles in the inner part of the tetrahedron $ABCD$ (see Fig. 1). If a face is symmetric to another face with respect to some mirror, then these faces are either 2-faces both or 3-faces both. These conditions are satisfied only if $KLM$, $LMF$, $LEF$, $LEK$, $KMD$ and $AEF$ are 3-faces, and the rest triangles are 2-faces. The 3-faces inside $ABCD$ are $LMF$ and $LEK$.

Consider a union of $F_1,...,F_8$. Attach to the union of $F_1,...,F_8$ two additional simplices, glue these simplices to the facets correspondent to the tetrahedra $LMGC$ and $BKEL$. We obtain a simplex $H^4_9$ and the decomposition with symbol $(H^4_3, H^4_9)$, $N = 10$. Clearly, the decomposition with this symbol is unique.
3.2.2 There is no decomposition with symbol \((H_5^5, H_{12}^5)\), \(N = 16\).

The Coxeter diagram of \(H_{12}^5\) has two parabolic subdiagrams of the type \(\tilde{C}_4\) and four parabolic subdiagrams of the type \(\tilde{F}_4\). The Coxeter diagram of \(H_5^5\) has one subdiagram of the type \(\tilde{C}_4\) and one subdiagram of the type \(\tilde{F}_4\). A diagram \(\tilde{F}_4\) contains no subdiagram \(2B_2\) and a diagram \(\tilde{C}_4\) contains no subdiagram \(2A_2\). Moreover, \(2A_2\) and \(2B_2\) admit no decomposition of the second type with \(N < 32\). Therefore, the vertex of \(P\) of the type \(\tilde{C}_4\) is tiled by sixteen vertices of the type \(\tilde{C}_4\); the vertex of the type \(\tilde{F}_4\) is tiled by sixteen vertices of the type \(\tilde{F}_4\). By Lemma \(\text{[4]}\), a non-trivial decomposition with symbol \((\tilde{F}_4, \tilde{F}_4)\) contains at least 16 fundamental simplices. Therefore, each of the four vertices of \(H_{12}^5\) of the type \(\tilde{F}_4\) belongs to either one or to at least 16 fundamental simplices. But 16 fundamental simplices have 16 vertices of the type \(\tilde{F}_4\). This is impossible.

3.2.3 There is no decomposition with symbol \((H_7^5, H_{11}^5)\), \(N = 6\).

Suppose that there exists a decomposition with symbol \((H_7^5, H_{11}^5)\) and \(N = 6\). Then 6 fundamental simplices contain \(3 \cdot 6 = 18\) ideal vertices. These 18 vertices coincide with 5 ideal vertices of \(P\). An ideal vertex of \(P\) belongs to at least two fundamental simplices, since the diagram of \(F\) contains no subdiagram \(\tilde{D}_4\). No vertex of \(P\) belongs to exactly two fundamental simplices, otherwise by Lemma \(\text{[4]}\), \(P\) has a non-fundamental dihedral angle.

Suppose that an ideal vertex of \(P\) belongs to exactly 3 fundamental simplices. Then there exists a decomposition of the second type with \(N = 3\) and either \((F, P) = (\tilde{B}_4, \tilde{D}_4)\) or \((F, P) = (\tilde{F}_4, \tilde{D}_4)\). But \(\tilde{D}_4\) contains a subdiagram \(4A_1\), and none of the diagrams \(\tilde{B}_4\) and \(\tilde{F}_4\) contains this subdiagram. Moreover, the simplex \(4A_1\) admits no decomposition of the second type with \(N = 3\). Therefore, there is no decomposition with \(N = 3\) and with the symbols \((\tilde{B}_4, \tilde{D}_4)\) and \((\tilde{F}_4, \tilde{D}_4)\). Hence, no ideal vertex of \(P\) belongs to exactly 3 fundamental simplices.

Thus, each of five ideal vertices of \(P\) belongs to at least four fundamental simplices. This contradicts to the fact that six fundamental simplices have 18 ideal vertices.

3.2.4 There exists a unique decomposition with symbol \((H_4^5, H_{11}^5)\) and \(N = 20\).

By Lemma \(\text{[4]}\), the ideal vertex of \(P\) belongs to either exactly one fundamental simplex or to at least 16 fundamental simplices. The simplex \(P = H_{11}^5\) has 5 ideal vertices. Since \(N = 20\), four of the ideal vertices of \(P\) belong to exactly
one fundamental simplex each, the rest vertex belongs to 16 fundamental simplices.

**Lemma 8.** There exists at most one decomposition with symbol \((H_4^5, H_{11}^5)\).

**Proof.** Let \(v\) be a vertex of \(P\) which belongs to exactly one fundamental simplex \(F_1\). Let \(\alpha\) be a facet of \(F_1\) opposite to \(v\). The position of \(F_1\) with respect to \(P\) determines the decomposition. Hence, it is sufficient to show that the way to put the facet \(\alpha\) is unique. Let \(\beta_1,\ldots,\beta_5\) be the facets of \(P\) containing the vertex \(v\). Clearly, \(\alpha\) intersects all but one of \(\beta_1,\ldots,\beta_5\) perpendicularly. The rest facet can be chosen arbitrary in the set \(\{\beta_1,\ldots,\beta_5\}\) (the change of this facet leads to a symmetry of the decomposition and does not determine new decompositions).

\[\square\]

**Lemma 9.** There exists a decomposition with symbol \((H_4^5, H_{11}^5), N = 20\).

**Proof.** The proof is a straightforward calculation in the linear model of the hyperbolic space.

\[\square\]

Now we are aimed to describe the decomposition. The decomposition is very similar to the decomposition with symbol \((H_3^4, H_4^4), N = 10\). We can visualize the decomposition as follows.

First, we describe the decomposition in the neighborhood of the vertex \(v\) which is incident to sixteen fundamental simplices \(F_1, \ldots, F_{16}\). Let \(s\) be a horosphere centered in \(v\). Then \(s \cap P\) is a decomposition with symbol \((\bar{D}_4, \bar{D}_4), N^4 = 16\). The simplex \(\bar{D}_4\) admitting a decomposition can be explained as \(\{P^4 = x_1 + x_2 + x_3 + x_4 \leq 2, | x_i \geq 0\}\) in Cartesian coordinates. Then the vertex \((0, 0, 0, 0)\) belongs to twelve fundamental simplices \(f_1, \ldots, f_{12}\). The decomposition in the neighborhood of \((0, 0, 0, 0)\) coincides with the decomposition with symbol \((\bar{D}_3, A_1 + A_1 + A_1 + A_1), N^3 = 12\). The rest four fundamental simplices \(f_{13}, \ldots, f_{16}\) are cut from \(P^4\) by the 3-dimensional faces \(x_i = 1, i = 1, \ldots, 4\).

Let \(\bar{f}_i\) be a facet of \(F_i\) \((i = 1, \ldots, 16)\) opposite to the vertex \(v\). Then \(\bar{f}_1, \ldots, \bar{f}_{12}\) belong to the same hyperplane \(\Pi\). The facets \(\bar{f}_{13}, \ldots, \bar{f}_{16}\) belong to four different hyperplanes, which intersect \(\Pi\) with dihedral angles \(\pi/7\). Reflect the simplices \(F_{13}, \ldots, F_{16}\) with respect to \(\bar{f}_{13}, \ldots, \bar{f}_{16}\) respectively to obtain the rest four fundamental simplices \(F_{17}, \ldots, F_{20}\).
3.2.5 There exists a unique decomposition with symbol \((H_1^8, H_4^3)\) and \(N = 272\).

**Lemma 10.** There exists a unique decomposition with symbol \((H_1^8, H_4^3)\) and \(N = 272\).

**Proof.** The existence is proved in \([9]\). Here we reproduce the sketch of the proof.

Denote the facets of \(F = H_1^8\) as shown in Fig. 3.2.5.a. Let \(v_i\) be a unit outward normal for the \(i\)-th facet of \(F\). Set \(v_9 = 2v_0 + 2v_2 + 3v_4 + 2v_5 + v_6\). It is easy to check that the vectors \(v_1, v_2, ..., v_9\) are the outward normals for the simplex \(P = H_4^3\) (see Fig. 3.2.5.b). As it is shown in \([9]\), \(v_9\) can be obtained from \(v_2\) by the sequence of the reflections with respect to the facets orthogonal to \(v_i, i = 0, ..., 8\).

To prove the uniqueness, consider the Coxeter diagram of \(P = H_4^3\). This diagram has a subdiagram \(A_8\) (numbered by 1, 2, ..., 8 in Fig. 3.2.5.b). The spherical simplex \(A_8\) can be decomposed either into one copy of \(A_8\) or into \(2^{14} \cdot 3^5 \cdot 5^2 > 272\) copies of \(E_8\). The latter is impossible, since \(N = 272\). Hence, \(A_8\) is tiled by one copy of \(A_8\) and we have two possibilities: the facets of \(P\) are numbered either as shown in in Fig. 3.2.5.b or as shown in in Fig. 3.2.5.

Suppose that there exists a decomposition corresponding to the letter numbering. Let \(u\) be a unit vector orthogonal to the 9-th facet. It is easy to check that \(u\) does not belong to the lattice \(\sum_{i=0}^{8} \mathbb{Z}v_i\). Thus, \(u\) can not be obtained form \(v_i\) \((i = 0, ..., 8)\) by a sequence of the reflections with respect to the facets of \(F\). Hence, in this case the decomposition is impossible.
The proof of the lemma shows that the facets of $P$ are numbered as it is shown in Fig. 3.2.5. Let $A_i$ be a vertex opposite to the $i$-th facet of $P$. $P$ has two ideal vertices: $A_3$ and $A_8$. The fundamental simplex has one ideal vertex. It is easy to check that $A_3$ belongs to 200 fundamental simplices and $A_8$ belongs to 72 fundamental simplices.

3.2.6 There exists a unique decomposition with symbol $(H_1^9, H_3^9)$ and $N = 527$.

Denote the facets of $P = H_3^9$ as shown in Fig. 4. Let $v$ be a vertex of $P$ opposite to the fourth facet. It follows from Subdiagram Property that a fundamental simplex $F_1$ containing $v$ is unique. There is a unique way to put the simplex $F_1$ inside $P$. Hence, the decomposition with symbol $(H_1^9, H_3^9)$ is unique. To check that there exists the decomposition with this symbol, consider the simplex $F$ in the linear model of $H^9$. It is not difficult to check using a computer that after a number of reflections we obtain the facet of $P$ opposite to $v$. It turns out that the ideal vertex opposite to the first facet belongs to 270 fundamental simplices, the ideal vertex opposite to the ninth facet belongs to 256 fundamental simplices and the ideal vertex opposite to the tenth facet belongs to a unique fundamental simplex.

Thus, the following Theorem is proved:

**Theorem 3.** A Coxeter decomposition of a hyperbolic simplex of the second type is uniquely determined by the pair $(F, P)$. The possibilities for the pairs $(F, P)$ are listed in Table 5.

3.3 Decompositions of the third type

Recall that a Coxeter decomposition of the simplex $P$ is called a decomposition of the third type if $P$ has a non-fundamental dihedral angle and the decomposition can not be obtained by the inductive algorithm (see section 2.2).

Let $\Theta_1(T, P)$ be a decomposition of the simplex $P$ with fundamental simplex $T$ and $\Theta_2(F, T)$ be a decomposition of $T$ with fundamental simplex
Clearly, \( P \) admits a decomposition \( \Theta(F, P) \) with fundamental simplex \( F \), where \( \Theta(F, P) \) is a superposition of \( \Theta_1(T, P) \) and \( \Theta_2(F, T) \).

Denote by \( \Theta(F, T) \) the restriction of \( \Theta(F, P) \) to the simplex \( T \).

**Lemma 11.** Let \( \Theta(F, P) \) be a Coxeter decomposition of \( P \) with fundamental simplex \( F \). Suppose that \( P \) is decomposed by a mirror of the decomposition into two simplices \( P_1 \) and \( P_2 \). Suppose that there exists a simplex \( T \subseteq P_1 \) such that \( \Theta(F, P_1) \) is a superposition of the decompositions \( \Theta(F, T) \) and \( \Theta_1(T, P_1) \), where \( \Theta_1(T, P_1) \) is a decomposition of \( P_1 \) with fundamental simplex \( T \). Suppose that the dihedral angles in the decomposition \( \Theta(F, T) \) are fundamental. Then the decomposition \( \Theta(F, P) \) is a superposition of \( \Theta(F, T) \) and \( \Theta_1(T, P) \), where \( \Theta_1(T, P) \) is a decomposition of \( P \) with fundamental simplex \( T \).

**Proof.** Let \( P = A_0A_1A_2...A_n, P_1 = A_0A_1...A_{n-1}B, P_2 = BA_1A_2...A_{n-1}A_n \) \((B \in A_0A_n)\). Let \( \alpha = A_1A_2...A_{n-1}A_n \). Extend the decomposition \( \Theta_1(T, P_1) \) to the decomposition \( \Theta_1(T) \) of the hyperbolic space \( \mathbb{H}^n \). Clearly, any facet of \( P \) except \( \alpha \) belongs to a mirror of the decomposition \( \Theta_1(T) \). To prove the lemma, it is sufficient to show that \( \alpha \) belongs to a mirror of the decomposition \( \Theta_1(T) \).

Let \( S \) be a set of mirrors of \( \Theta_1(T) \) containing the \((n-2)\)-dimensional face \( A_1...A_{n-1} \). Suppose that \( \alpha \) does not belong to \( S \). Then \( \alpha \) decomposes a dihedral angle of some fundamental simplex \( T_1 \) of the decomposition \( \Theta_1(T) \). This contradicts to the assumption that the dihedral angles in the decomposition \( \Theta(F, T) \) are fundamental.

\[ \square \]

**Theorem 4.** Any decomposition of the third type is a superposition of some non-trivial decompositions.

**Proof.** Let \( \Theta(F, P) \) be a decomposition of the third type of a simplex \( P \) with fundamental simplex \( F \). Then by the definition of the decomposition of the third type there exists a mirror decomposing \( P \) into two simplices \( P_1, P_2 \). Let \( \Theta_1 \) and \( \Theta_2 \) be the restrictions of \( \Theta(F, P) \) to the simplices \( P_1 \) and \( P_2 \). At least one of \( \Theta_1 \) and \( \Theta_2 \) is a decomposition either of the second or of the third type (otherwise, \( \Theta(F, P) \) could be obtained by the inductive algorithm). Suppose that \( \Theta_1 \) is a decomposition of the second type. Then, applying Lemma 11 to the simplex \( T = P_1 \), we obtain that \( \Theta(F, P) \) is a superposition of \( \Theta_1 \) and \( \Theta(P_1, P) \), where \( \Theta(P_1, P) \) is a decomposition of \( P \) with fundamental simplex \( P_1 \).

Suppose that \( \Theta_1 \) is a decomposition of the third type. Then \( P_1 \) is decomposed by a mirror into two smaller simplices. If the decompositions of these simplices are of the third type, then these simplices are decomposed into
some smaller simplices. Since $\Theta(F, P)$ is a decomposition of the third type, the finite number $k$ of the steps leads to a simplex with the decomposition of the second type. Then by Lemma 11 the decompositions of the simplices at the step $(k - 1)$ is a superposition of some decompositions. Use Lemma 11 again to show that the decomposition at the step $(k - 2)$ is a superposition too. Applying the Lemma 11 for $k$ times, we obtain the statement of the theorem.

Thus, any simple decomposition of a hyperbolic simplex is a decomposition either of the first or of the second type. Hence, all the simple decompositions are listed in Table 3 and Table 4.
## 4 Tables

Table 2: Hyperbolic Coxeter simplices in $H^n$, $\geq 4$.  

| Coxeter diagram | Notation | Volum |
|-----------------|----------|-------|
| ![Diagram](image1.png) | $H_1^{(4)}$ | 0.00091385226 |
| ![Diagram](image2.png) | $H_2^{(4)}$ | 0.00776774420 |
| ![Diagram](image3.png) | $H_3^{(4)}$ | 0.01553548841 |
| ![Diagram](image4.png) | $H_4^{(4)}$ | 0.02376015874 |
| ![Diagram](image5.png) | $H_5^{(4)}$ | 0.02513093713 |
| ![Diagram](image6.png) | $H_6^{(4)}$ | 0.00685389195 |
| ![Diagram](image7.png) | $H_7^{(4)}$ | 0.01142315324 |
| ![Diagram](image8.png) | $H_8^{(4)}$ | 0.01370778389 |
| ![Diagram](image9.png) | $H_9^{(4)}$ | 0.02284630648 |
| ![Diagram](image10.png) | $H_{10}^{(4)}$ | 0.03426945973 |
| ![Diagram](image11.png) | $H_{11}^{(4)}$ | 0.06853891945 |
| ![Diagram](image12.png) | $H_{12}^{(4)}$ | 0.06853891945 |
| ![Diagram](image13.png) | $H_{13}^{(4)}$ | 0.09138522594 |
| ![Diagram](image14.png) | $H_{14}^{(4)}$ | 0.13707783890 |

The volumes are rewritten from [8].
Table 2, Continued.

| Coxeter diagram | Notation | Volum                  |
|-----------------|----------|------------------------|
| ![Diagram](image) | $H_1^6$  | $0.3987432701 \times 10^{-4}$ |
|                 | $H_2^6$  | $0.7974865401 \times 10^{-4}$ |
|                 | $H_3^6$  | $2.9620928633 \times 10^{-4}$ |

4.1 Decompositions of the first type

Notation

Simplices with the dihedral angles $\pi \frac{k}{q}$ are represented by the diagrams similar to the Coxeter diagrams. A dihedral angle equal to $\pi \frac{k}{q}$ is represented by a $(q - 2)$-fold edge decomposed into $k$ parts.

The list of the decompositions of the first type was obtained by the inductive algorithm. In the tables, the Coxeter diagram of the fundamental simplex is followed by the non-trivial decompositions with this fundamental simplex. We omit Coxeter simplices that are not fundamental for non-trivial decompositions. The non-trivial decompositions are listed in the order their were obtained in the inductive algorithm. The tables contain the simple
decompositions only (see section 3.1 for the definition of the simple decomposition). The decompositions of the Coxeter simplices are listed in Table 3.

The numbers \((N, s ; k, l, i, j)\) shown under the diagram of the decomposition describe the decomposition:

- \(N\) is a number of the fundamental simplices in the decomposition
- \(s\) is a number of gluing necessary to obtain the decomposition (if the decomposition is obtained by the gluing together of the simplices with the gluing numbers \(s_1\) and \(s_2\), then \(s = 1 + \max\{s_1, s_2\}\)),
- \(k\) and \(l\) are the numbers of the simplices glued together to obtain the decomposition
- \(i\) and \(j\) are the numbers of the facets of the simplices \(k\) and \(l\) which should be glued together (the nodes of the diagrams are numbered from the left to the right by the numbers 0,1,...,\(n\)).

Table 3: Simple decompositions of the Coxeter simplices of the first type.

| \(F\) | \(P\) | \(N\) | \(s\) | \((k, l, i, j)\) |
|-------|-------|-------|-------|----------------|
| \(H_2^{(3)}\) | \(H_3^{(4)}\) | 2 | 1 | (0,0,4,4) |
| \(H_6\) | \(H_9\) | 2 | 1 | (0,0,4,4) |
| \(H_8\) | \(H_{10}\) | 2 | 1 | (0,0,1,1) |
| \(H_5\) | \(H_6\) | 2 | 1 | (0,0,4,4) |
| \(H_7\) | \(H_8\) | 2 | 1 | (0,0,0,0) |
| \(H_4\) | \(H_7\) | 3 | 2 | (0,0,3,3) |
| \(H_5\) | \(H_8\) | 4 | 2 | (1,0,0,1) |
| \(H_6\) | \(H_8\) | 2 | 1 | (2,2,0,0) |
| \(H_5\) | \(H_6\) | 3 | 2 | (1,0,0,3) |
| \(H_6\) | \(H_7\) | 2 | 1 | (0,0,3,3) |
| \(H_1\) | \(H_5\) | 5 | 3 | (3,2,0,0) |

Table 3: Simple decompositions of the Coxeter simplices of the first type.
Table 4: Simple decompositions of the non-Coxeter simplices of the first type.

Decomposition of the bounded simplices in $H^4$.

1. $F = H^4_1$

|   | 1       | 2       | 3       | 4       | 5       |
|---|---------|---------|---------|---------|---------|
| 0 | ![Diagram](image1) | ![Diagram](image2) | ![Diagram](image3) | ![Diagram](image4) | ![Diagram](image5) |
| 6 | (2.1 ; 0.0,0,0) | (2.1 ; 0.0,4,4) | (4.2 ; 1.1,3,3) | (3.2 ; 2.0,0,3) | (6.3 ; 3.1,0,2) |
| 12 | (4.3 ; 4.0,0,2) | (6.3 ; 4.4,2,2) | (6.3 ; 4.4,4,4) | (12.4 ; 5.4,1,1) | (5.4 ; 6.0,1,1) |
| 18 | (8.4 ; 6.6,1,1) | (10.4 ; 8.6,0,0) | (12.4 ; 8.8,1,1) | (10.5 ; 10.1,0,0) | (10.5 ; 10.1,0,3) |
| 24 | (11.5 ; 11.1,0,4) | (12.5 ; 11.1,1,1) | (20.5 ; 13.1,1,1) | (20.5 ; 13.1,2,3) | (15.5 ; 14.4,0,1) |
| 30 | (24.5 ; 14.1,1,3) | (14.6 ; 15.6,0,2) | (20.6 ; 15.1,0,0) | (20.6 ; 15.1,1,1) | (16.6 ; 17.1,1,1) |
| 36 | (22.6 ; 18.1,8,2,2) | (22.6 ; 18.1,8,3,3) | (26.6 ; 20.1,1,0) | (40.6 ; 20.2,0,1) | (30.6 ; 21.1,1,3,3) |
| 42 | (30.6 ; 22.1,2,3) | (36.6 ; 23.1,3,3) | (32.7 ; 25.2,0,0) | (28.7 ; 25.2,5,1,1) | (30.7 ; 26.13,0,4) |
| 48 | (40.7 ; 26.2,6,2,2) | (44.7 ; 30.3,3,3) | (48.7 ; 32.3,0,0) | (52.7 ; 32.3,2,1) | (52.7 ; 32.3,4,4) |
| 54 | (50.7 ; 34.26,0,0) | (60.7 ; 35.35,5,0) | (60.7 ; 35.35,1,1) | (60.8 ; 40.40,0,3) | (80.8 ; 41.41,2,2) |

2. $F = H^4_2$

|   | 1       |
|---|---------|
| 0 | ![Diagram](image6) |
| 6 | (2.1 ; 0.0,0,0) |

3. $F = H^4_3$

|   | 1       | 2       | 3       | 4       | 5       |
|---|---------|---------|---------|---------|---------|
| 0 | ![Diagram](image7) | ![Diagram](image8) | ![Diagram](image9) | ![Diagram](image10) | ![Diagram](image11) |
| 6 | (2.1 ; 0.0,2,2) | (2.1 ; 0.0,4,4) | (4.2 ; 1.1,3,3) | (4.2 ; 1.1,3,0) | (8.3 ; 3.3,0,0) |

4. $F = H^4_4$

|   | 1       | 2       |
|---|---------|---------|
| 0 | ![Diagram](image12) | ![Diagram](image13) |
| 6 | (2.1 ; 0.0,0,0) | (4.2 ; 1.1,3,3) |
Decomposition of the unbounded simplices in $\mathcal{H}^4$.

1. $F = H_1^4$
   1
   (2,1 ; 0,0,0,0)
   2
   (2,1 ; 0,0,4,4)
   3
   (3,2 ; 1,0,0,1)
   4
   (4,2 ; 1,1,3,3)
   5
   (6,3 ; 3,3,1,1)

2. $F = H_2^4$
   0
   (2,1 ; 0,0,0,0)
   1
   (2,1 ; 0,0,4,4)

3. $F = H_3^4$
   0
   (2,1 ; 0,0,0,0)
   1
   (2,1 ; 0,0,4,4)

4. $F = H_4^4$
   0
   (2,1 ; 0,0,0,0)
   1
   (2,1 ; 0,0,4,4)
   2
   (2,1 ; 0,3,3,3)
   3
   (4,2 ; 1,1,2,2)

5. $F = H_5^4$
   0
   (2,1 ; 0,0,0,0)
   1
   (2,1 ; 0,0,4,4)

6. $F = H_6^4$
   0
   (2,1 ; 0,0,2,2)
   1
   (2,1 ; 0,0,4,4)

Decomposition of the simplices in $\mathcal{H}^5$.

1. $F = H_1^5$
   0
   (2,1 ; 0,0,0,0)
   1
   (2,1 ; 0,0,5,5)
   2
   (3,2 ; 1,0,0,1)
   3
   (4,2 ; 1,1,4,4)
   4
   (6,3 ; 5,5,0,0)
   5
   (6,3 ; 5,5,3,3)
   6
   (8,4 ; 7,7,1,1)
   7
   (12,4 ; 6,6,3,3)
   8
   (16,5 ; 10,10,2,2)
   9
   (24,5 ; 11,11,1,1)
   10
   (8,4 ; 7,7,1,1)
   11
   (12,4 ; 6,6,0,0)
   12
   (12,4 ; 6,6,3,3)
   13
   (7,4 ; 7,3,5,0)
   14
   (6,3 ; 5,5,3,3)
   15
   (14,5 ; 13,13,0,0)
   16
   (14,5 ; 13,13,3,3)
   17
   (24,5 ; 15,15,3,3)
   18
   (48,6 ; 17,17,4,4)
   19
   (24,6 ; 20,20,4,4)
   20
   (21,5 ; 15,10,0,3)
   21
   (24,5 ; 15,15,3,3)
   22
   (28,6 ; 18,18,3,3)
   23
   (42,6 ; 20,20,4,4)
   24
   (21,5 ; 15,10,0,3)

2. $F = H_2^5$
   0
   (2,1 ; 0,0,4,4)
   1
   (2,1 ; 0,0,4,4)
   2
   (2,1 ; 0,0,5,5)
   3
   (3,2 ; 1,0,0,3)
   4
   (4,2 ; 1,1,1,1)
   5
   (5,3 ; 3,2,0,0)
   6
   (6,3 ; 3,3,3,3)
   7
   (10,4 ; 5,5,2,2)
3. $F = H_3^5$

5. $F = H_5^5$

6. $F = H_6^5$

7. $F = H_7^5$

8. $F = H_8^5$

9. $F = H_{10}^5$

10. $F = H_{12}^5$

Decomposition of the simplices in $H^6$. 

1. $F = H_1^6$
2. \( F = H_2^6 \)

\[
\begin{array}{cccc}
5 & 6 & 7 & 8 \\
(5,3 ; 3,2,0,0) & (6,3 ; 3,3,1,1) & (10,4 ; 5,5,0,0) & (10,4 ; 5,5,4,4) & (20,5 ; 7,7,4,4) \\
4 & 3 & 2 & 1 \\
(2,1 ; 0,0,0,0) & (2,1 ; 0,0,4,4) & (2,1 ; 0,0,6,6) & (3,2 ; 1,0,0,1) \\
(4,2 ; 1,1,3,3) & (4,2 ; 1,1,5,5) & (4,2 ; 2,0,0,0) & (4,2 ; 2,2,1,1) & (5,3 ; 4,2,0,0) \\
(6,3 ; 4,4,1,1) & (6,3 ; 4,4,3,3) & (8,3 ; 5,5,0,0) & (8,3 ; 5,5,4,4) & (10,4 ; 9,9,4,4) \\
(12,4 ; 10,0,3,3) \\
\end{array}
\]

3. \( F = H_3^6 \)

\[
\begin{array}{c}
1 \\
(2,1 ; 0,0,6,6) \\
\end{array}
\]

Decomposition of the simplices in \( H^7 \).

1. \( F = H_1^7 \)

\[
\begin{array}{ccc}
1 & 2 & 3 \\
(2,1 ; 0,0,0,0) & (2,1 ; 0,0,5,5) & (3,2 ; 1,0,0,1) \\
(4,2 ; 1,1,4,4) & (4,2 ; 1,1,6,6) & (3,2 ; 1,0,0,4) & (6,3 ; 3,3,0,0) \\
(6,3 ; 3,3,1,1) & (6,3 ; 3,3,4,4) & (6,3 ; 3,3,6,6) & (6,3 ; 4,1,0,3) \\
(8,3 ; 4,4,5,5) & (4,3 ; 6,0,7,3) & (6,3 ; 6,6,0,0) & (6,3 ; 6,6,3,3) \\
(12,4 ; 7,7,1,1) & (12,4 ; 7,7,4,4) & (12,4 ; 8,8,1,1) & (12,4 ; 8,8,3,3) \\
\end{array}
\]
2. \( F = H_2^7 \)

3. \( F = H_3^7 \)
4. \( F = H_4^7 \)

(20,5 ; 17,17,3,3)

Decomposition of the simplices in \( H^8 \).

1. \( F = H_1^8 \)

(20,5 ; 19,19,0,0)

(2,1 ; 0,7,7)

(2,1 ; 0,0,7,7)

(2,1 ; 0,0,8,8)

(4,2 ; 1,1,7,7)

(4,2 ; 1,1,1,1)

(6,3 ; 4,4,7,7)

(8,4 ; 8,8,7,7)

(12,4 ; 9,9,7,7)

(12,4 ; 10,10,2,2)

(12,4 ; 12,6,5,4)

(12,4 ; 13,6,1,1)

(12,5 ; 14,8,0,3)

(16,5 ; 14,14,2,2)

(18,5 ; 18,11,2,4)

(16,5 ; 19,6,8,5)

(24,5 ; 19,19,1,1)

(16,5 ; 20,20,4,4)

(24,6 ; 22,22,2,2)

(24,6 ; 22,22,3,3)

(24,6 ; 23,14,0,3)
2. \( F = H_2^8 \)

3. \( F = H_3^8 \)
Decomposition of the simplices in $H^9$.

1. $F = H^9$

4. $F = H^8$

...
4.2 Decompositions of the second type

Table 5: Decompositions of the second type.

| $F$  | $P$  | $N$ | Description     |
|------|------|-----|-----------------|
| $H_3^4$ | $H_9^4$ | 10  | see section 3.2.2 |
| $H_4^5$ | $H_{11}^5$ | 20  | see section 3.2.4 |
| $H_1^8$ | $H_4^8$ | 272 | see section 3.2.5 |
| $H_1^9$ | $H_3^9$ | 527 | see section 3.2.6 |

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