CONVERGENCE OF GAUSS CURVATURE FLOWS TO TRANSLATING SOLITONS

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Abstract. We address the asymptotic behavior of the $\alpha$-Gauss curvature flow, for $\alpha > 1/2$, with a complete non-compact convex initial hypersurface which is contained in a cylinder of a bounded cross section. We show that the flow converges, as $t \to +\infty$, locally smoothly to a translating soliton which is uniquely determined by the asymptotic cylinder of the initial hypersurface.

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1. Introduction

Given $\alpha > 0$, the $\alpha$-Gauss curvature flow ($\alpha$-GCF in abbreviation) is a one-parameter family of embeddings $F : M^n \times [0,T) \to \mathbb{R}^{n+1}$ such that for each $t \in [0,T)$, $F(M^n, t) = \Sigma_t$ is a complete convex hypersurface in $\mathbb{R}^{n+1}$, and $F(\cdot, t)$ satisfies

$$\frac{\partial}{\partial t} F(p, t) = -K^\alpha(p, t) \nu(p, t).$$

Here, $K(p, t)$ is the Gauss curvature of $\Sigma_t$ at $F(p, t)$, and $\nu(p, t)$ is the unit normal vector of $\Sigma_t$ at $F(p, t)$ pointing outward of the convex hull of $\Sigma_t$.

The classical Gauss curvature flow (GCF), the $\alpha = 1$ case, was first introduced by W. Firey [23] to describe the shape of worn stones and the asymptotic behavior when it disappears. In [23], W. Firey proved that if a closed strictly convex solution to the GCF in $\mathbb{R}^3$ has the central symmetry, then it converges to a round sphere after rescaling. Later, B. Andrews [3] removed the central symmetry condition. In higher dimensions $n \geq 3$, P. Guan and L. Ni [24] obtained the convergence to a self-shrinking soliton after rescaling, and K. Choi and P. Daskalopoulos [16] showed the uniqueness of self-shrinking solitons. Namely, a closed strictly convex solution to the GCF in $\mathbb{R}^{n+1}$ converges to a round sphere after rescaling.

In addition to the classical case $\alpha = 1$, the asymptotic behavior of the $\alpha$-GCF also has been widely studied. In particular, in the $\alpha = \frac{1}{n+2}$ case, an affine transform of a solution remains as a solution, and thus we call the $\frac{1}{n+2}$-GCF as the affine normal flow. E. Calabi [11] showed that a self-shrinking soliton to the affine normal flow is an ellipsoid. (See also [9] for an alternative proof.) B. Andrews [2] obtained the convergence of the closed affine normal flow to an ellipsoid after rescaling.

In the range of $\alpha > \frac{1}{n+2}$, the convergence of the closed $\alpha$-GCF to a round sphere after rescaling has been shown by B. Chow [19] for $\alpha = \frac{1}{n}$, and by B. Andrews and X. Chen [6] for $\frac{1}{2} \leq \alpha \leq 1$ and $n = 2$. Later,
for the all $\alpha > \frac{1}{n+2}$. B. Andrews, P. Guan and L. Ni [8] showed the convergence to a self-similar soliton after rescaling. Moreover S. Brendle, K. Choi, and P. Daskalopoulos [9] proved the uniqueness of self-shrinking solitons. Namely, for $\alpha > \frac{1}{n+2}$, a closed strictly convex solution to the $\alpha$-GCF in $\mathbb{R}^{n+1}$ converges to a round sphere after rescaling.

In the range of small powers $\alpha \in (0, \frac{1}{n+2})$, the asymptotic behavior remains as an open problem. B. Andrews classified closed self-shrinking solitons in the curve case $n = 1$ [5], and showed the existence of non-trivial closed self-shrinking solitons in higher dimensions [4].

Regarding the non-compact case, the translating solitons to the $\alpha$-GCF have been classified for $\alpha = \frac{1}{n+2}$ and $\alpha > \frac{1}{2}$. In the affine normal case $\alpha = \frac{1}{n+2}$, the translating solitons are paraboloids. The $n = 2$ case showed first by K. Jörgens [28], and later by J.C.C. Nitsche [29] with another proof by using the complex analysis. E. Calabi [10] extended the result for $n \leq 5$, and A.V. Pogorelov [30] proved for all dimensions. S.Y. Cheng and S.T. Yau [12] provided an alternative proof by using the affine geometry. See also the recent classification result [15] of K.Choi, B.Choi and S.Kim for the case $n = 2$ and $\alpha < \frac{1}{2}$.

In [32, 33], J. Urbas showed that every translating soliton for $\alpha > \frac{1}{2}$ is contained in a bounded cylinder $\overline{\Omega} \times \mathbb{R}$, namely $\Omega \subset \mathbb{R}^n$ is bounded. Moreover, if $\alpha > \frac{1}{2}$ then given a bounded convex body $\overline{\Omega} \subset \mathbb{R}^n$ there exists a translating soliton asymptotic to $\partial \Omega \times \mathbb{R}$. Furthermore, for each bounded convex body $\overline{\Omega}$, the translating soliton is unique up to translations. One the other hand, for small powers $\alpha \in (0, \frac{1}{2})$, H. Jian and X.J. Wang [27] showed the existence of infinitely many entire translating solitons.

Recently the authors [13] showed the convergence to a translating soliton for $n = 1$ and $\alpha > \frac{1}{2}$. In this paper, we establish its higher dimensional result for $n \geq 2$ as follows.

**Theorem 1.1.** Let $\mathcal{K}_0 \subset \mathbb{R}^{n+1}$ be an unbounded convex body asymptotic to a convex cylinder $\overline{\Omega} \times \mathbb{R}(\neq \mathcal{K}_0)$ with the bounded section $\Omega \subset \mathbb{R}^n$. Then, given $\alpha \geq 1$, the viscosity solution$^1$ to the $\alpha$-Gauss curvature flow from the initial hypersurface $\Sigma_0 = \partial \mathcal{K}_0$ locally smoothly converges to the translating soliton asymptotic to $\partial \Omega \times \mathbb{R}$ as $t \to +\infty$.

**Local convergence:** The viscosity flow $\Sigma_t$ is asymptotic to the initial asymptotic cylinder, say $\partial \Omega \times \mathbb{R}$, for all time by Theorem 2.7 and thus $\Sigma_t$ can be written as convex graphs on a fixed domain

$$\Sigma_t = \partial \{ x_{n+1} > u(x, t) : x \in \Omega \}.$$

The local smooth convergence in the statement of the above theorem implies the $C_{loc}^\infty(\Omega)$ convergence of the functions $u(\cdot, t) = \inf_{x \in \text{Int}(\Omega)} u(x, t)$ to $u_\Omega \in C^\infty(\Omega)$, which represents the translating soliton asymptotic to $\partial \Omega \times \mathbb{R}$. If the weakly convex domain $\Omega$ is not strictly convex, then the corresponding translating soliton may touch the boundary of the cylinder and have flat sides. (See the work by K. Choi, P. Daskalopoulos, and K.A. Lee in [17].) Therefore, the smooth convergence up to boundary is not expected.

**Viscosity solution:** We introduce the notion of the viscosity solutions to $\alpha$-GCF in Definition 2.6 to state and prove the convergence of flows from weakly convex non-smooth initial hypersurfaces. The existence and uniqueness of the viscosity flow is shown in Theorem 2.7. Note that if $\Sigma_0$ is weakly convex and has flat sides, the solution $\Sigma_t$ preserves the flat sides for a certain amount of time by the result of R. Hamilton [24]. See also the optimal regularity of an evolving flat side for short time [21] and for long time [22]. Regardless of the regularity of $\Sigma_0$, for each $\Omega \subset \subset \Omega$ we show that the flow $\Sigma_t$ becomes smooth and strictly convex in $\Omega' \times \mathbb{R}$ for large time $t \gg 1$ and smoothly converges to the translating soliton. In our subsequent work [14], we show the uniqueness of ancient solutions which are asymptotic to a convex cylinder $\partial \Omega \times \mathbb{R}$ and we

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$^1$In this paper, $\Omega \subset \mathbb{R}^n$ denotes an open set.

$^2$In this paper, we say that $\mathcal{K}$ is a bounded convex body if it is a compact convex set with non-empty interior. In addition, an unbounded convex body means an unbounded closed convex set with non-empty interior.

$^3$See Definition 2.6.
use Theorem 1.1 with the notion of viscosity solution, in a crucial way. Indeed, some ancient solutions are not of class $C^2$ [17, 14].

**Additional steps in higher dimensions:** Compared to the $n=1$ case [13], the entropy formulas here become more involved so we provide them in Appendix. Moreover, for the local smooth convergence, one needs to establish local upper and lower bounds on the principal curvatures, which are independent of the regularity of the initial data $\Sigma_0 = \partial K_0$. Since the linearized operator highly degenerates along horizontal directions, in Section 4 we introduce some geometric ideas and establish new estimates.

For small $\alpha \in (\frac{1}{2},1)$ the same result holds under the technical assumption that $\Sigma_0$ can be approximated by closed hypersurfaces with uniform bounds for $\int K^\alpha dg$ and $(\alpha - 1) \int PK^\alpha dg$, where $P$ is defined at (2.1).

Notice that $\int K^\alpha dg$ and $(\alpha - 1) \int PK^\alpha dg$ denote the total speed and total acceleration, respectively. See Lemma A.1. Our result for $\alpha \in (\frac{1}{2},1)$ states as follows:

**Theorem 1.2.** Let $K_0 \subset \mathbb{R}^{n+1}$ be an unbounded convex body asymptotic to a convex cylinder $\overline{\Omega} \times \mathbb{R}(\neq K_0)$ with bounded section $\overline{\Omega} \subset \mathbb{R}^n$. Suppose that given $\alpha \in (\frac{1}{2},1)$, there is a sequence of bounded strictly convex bodies $K_i$ with smooth boundaries $\Sigma_i = \partial K_i$ which increases to $\Sigma_\infty = \partial K_\infty$ (i.e. $K_i \subset K_{i+1}$ and $\partial(\cup_i K_i) = \partial K_\infty$) with uniform upper bounds for $\int_{\Sigma_i} K^\alpha dg$ and $(\alpha - 1) \int_{\Sigma_i} PK^\alpha dg$. Then, the viscosity solution to the $\alpha$-Gauss curvature flow converges locally smoothly to the translating soliton asymptotic to $\partial \Omega \times \mathbb{R}$ as $t \to +\infty$.

**Convergence with small $\alpha \in (\frac{1}{2},1)$:** Since we have $K dg = dvol_{\Sigma_0}$ under the Gauss map, upper bounds for $\int_{\Sigma_i} K^\alpha d\Sigma = \int_{S_{\infty}} K^{\alpha-1} dvol_{\Sigma_0}$ with $\alpha < 1$ are related to local lower bounds for $K$. In the one-dimensional case [13], the local lower bounds for the curvature $\kappa$ were obtained by considering the evolution equation of $\kappa$ as a fast diffusion equation. However, in higher dimensions the Gauss curvature $K$ is not a solution to a porous medium equation any more, and thus it is hard to derive lower bounds for $K$. The convergence for $\alpha \in (\frac{1}{2},1)$ without the technical assumption of the bounded total speed and acceleration poses an interesting question that remained to be addressed.

Let us remark that in order to converge to a translating soliton, the initial hypersurface $\Sigma_0$ must be contained in a bounded cylinder. Jointly with L. Kim and K.A. Lee, the second and third authors in [18] showed by a barrier argument that if $\Sigma_0$ is a graph over a (possibly non-compact) domain $\Omega_0 \subset \mathbb{R}^n$, then any solution $\Sigma_i$ running from $\Sigma_0$ must remain as a graph over the same domain $\Omega_0$. On the other hand, every translating soliton for the $\alpha$-GCF with $\alpha > 1/2$ is asymptotic to a cylinder of a bounded cross section by (32, 33). Hence, it is necessary to assume that $\Sigma_0$ is contained in a bounded cylinder.

The following monotonicity formula will be used to identify the limit as a soliton. The technical assumptions in Theorem 1.2 were made so that this inequality can be applied.

**Theorem 1.3.** Given $\alpha \geq \frac{1}{n}$, compact strictly convex smooth solution $\Sigma_i$ to the $\alpha$-GCF satisfies

$$\frac{d}{dt} \int_{\Sigma_i} PK^\alpha dg \geq (n^{-1} + 2\alpha - 1) \int_{\Sigma_i} P^2 K^\alpha dg \geq 0.$$ 

We notice that B. Chow [20] obtained the above monotonicity formula for the GCF ($\alpha = 1$); (see the proof of Lemma 4.3 in [20]). In the same paper, B. Chow also obtained a monotonicity formula (Lemma 5.2 in [20]) for the rescaled GCF. In [11] B. Andrews generalized the monotonicity formula for the rescaled $\alpha$-GCF. Although Theorem 1.3 is a straightforward generalization of [20], the formula seems not to be shown or used before, so we prove it in Appendix.
2. Preliminaries

Definition 2.1. (i) $\Sigma \subset \mathbb{R}^{n+1}$ is a convex hypersurface if it is the boundary of a convex body $\mathcal{K}$, which is either bounded or unbounded. Notice that the convex hypersurface $\Sigma = \partial \mathcal{K}$ is complete and embedded.

(ii) For a $C^2$ convex hypersurface $\Sigma = \partial \mathcal{K}$, we say it is strictly convex at $p \in \Sigma$ if the second fundamental form with respect to the inner normal is positive definite.

Throughout this paper, $h_{ij}$ denotes the second fundamental form. For a strictly convex solution, one may consider the inverse $b^{ij}$ of the second fundamental form $h_{ij}$, which satisfies $b^{ik}h_{kj} = \delta^i_j$. We also denote by $dg := \sqrt{\det g} dx$ the volume form induced from the ambient Euclidean metric. Let $S := \langle F, \nu \rangle$ and $S_{x_0} := \langle F - x_0, \nu \rangle$ denote the support functions with respect to the origin and $x_0 \in \mathbb{R}^{n+1}$, respectively. Moreover, we recall the following tensor $P_{ij}$ and the quantity $P$ defined by B. Chow in [20]:

\begin{equation}
P_{ij} := \nabla^2 K^\alpha - b^{mn}\nabla_m h_{ij} \nabla_n K^\alpha + K^\alpha h^k_i h_{kj} \quad \text{and} \quad P := b^{ij} P_{ij}.
\end{equation}

Note that, for solutions to the $\alpha$-GCF, (2.15) implies

\begin{equation}
P = \frac{1}{\alpha K^\alpha} (\partial_t K^\alpha - b^{ij} \nabla_i K^\alpha \nabla_j K^\alpha).
\end{equation}

Let us recall the unique existence of translating solitons by J. Urbas and state the result in the way we will use in work.

Definition 2.2 (Theorem of J. Urbas [32, 33]). For $\alpha > 1/2$ and a given bounded convex domain $\Omega \subset \mathbb{R}^n$, let $u_\Omega : \Omega \rightarrow \mathbb{R}$ denote the graph function of the unique translating soliton which is asymptotic to $\partial \Omega \times \mathbb{R}$, it moves in the positive $e_{n+1}$ direction, and satisfies $\inf u_\Omega = 0$. In other words, the hypersurface given by $\partial \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > u_\Omega(x')\}$ defines the translating soliton.

Remark 2.3 (The result by Urbas in [33]). In the case where $\Omega$ is not a strictly convex domain, it is possible that $\limsup_{x \to x_0} u_\Omega(x') < \infty$, for some $x_0 \in \partial \Omega$. Hence the hypersurface $\{x_{n+1} = u_\Omega(x')\}$ is not necessarily complete. This is the reason why in the definition above we defined the translating soliton as $\partial \{(x', x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > u_\Omega(x')\}$. Urbas [33] showed the existence of such solitons and their uniqueness among solutions realized in a certain generalized sense. To be more specific, Urbas [33] showed that if a convex function $u(x')$ defined on $\Omega$ satisfies the translating soliton equation

\begin{equation}
det D^2 u = \beta \left(1 + |Du|^2\right)^{\frac{n+2}{2n+2}}
\end{equation}

for some $\beta > 0$ in the sense of Alexandrov, and $|\mathbb{R}^n - Du(\Omega)| = 0$, then $u = u_\Omega + C$, for some constant $C$. We will use this characterization of solitons in the proof of Theorem 1.1.

Definition 2.4. For $\alpha > 1/2$ and a given convex bounded domain $\Omega \subset \mathbb{R}^n$, let us note the speed of the associated translating soliton by

\begin{equation}
\lambda := \frac{1}{|\Omega|^\alpha} \left[ \int_{\mathbb{R}^n} \frac{1}{(1 + |p|^2)^{n+2-\frac{\alpha}{2}}} dp \right]^\alpha.
\end{equation}

(The derivation of this formula follows from (5.7) and $Du(\Omega) = \mathbb{R}^n$). Moreover, note that when $\alpha = 1$,

\begin{equation}
\lambda := \frac{1}{|\Omega|} \left[ \int_{\mathbb{R}^n} \frac{1}{(1 + |p|^2)^{n+1}} dp \right] = \frac{\omega_n}{2|\Omega|}
\end{equation}

holds, where $\omega_n = |\mathbb{S}^n|$.

We derive the evolution equations of basic geometric quantities.
Proposition 2.5. For strictly convex hypersurfaces, we have

\begin{equation}
\nabla_m K = K b^{ij} \nabla_m h_{ij}
\end{equation}

\begin{equation}
\nabla_i (b^{ij} K) = 0
\end{equation}

\begin{equation}
\nabla b^{ij} = -b^{ij} \nabla_i h_{pq} b^{pq}.
\end{equation}

For smooth strictly convex solutions to the $\alpha$-GCF, we have

\begin{equation}
\partial_t g_{ij} = -2K^\alpha h_{ij}
\end{equation}

\begin{equation}
\partial_t dg = -K^\alpha H dg
\end{equation}

\begin{equation}
\partial_t \nu = \nabla K^\alpha = \nabla_i K^\alpha \nabla^i F
\end{equation}

\begin{equation}
\partial_t h_{ij} = \nabla^2_i K^\alpha - K^\alpha h_{ik} h^k_i
\end{equation}

\begin{equation}
\partial_t b^{pq} = \alpha K^\alpha b^{ij} \nabla^2_i h_{pq} - \alpha K^\alpha b^{ij} b^{pq} \nu - \alpha K^\alpha \beta^{ij} b^{pq} (\alpha b^{kl} b^{mn} - b^{km} b^{ln}) \nabla_i h_{mn} \nabla_j h_{kl} + \alpha K^\alpha H h_{ij} - (1 + \alpha) K^\alpha t 
\end{equation}

\begin{equation}
\partial_t K^\alpha = \alpha K^\alpha b^{ij} \nabla^2_i K^\alpha + \alpha H K^{2\alpha}
\end{equation}

\begin{equation}
\partial_t |F|^2 = 2\alpha K^\alpha b^{ij} \nabla^2_i |F|^2 + 2(\alpha - 1) K^\alpha S - 2K^\alpha b^{ij} g_{ij}
\end{equation}

\begin{equation}
\partial_t S = \alpha K^\alpha b^{ij} \nabla^2_i S + \alpha K^\alpha HS - (1 + \alpha) K^\alpha.
\end{equation}

Proof. By $K = (\det g^{ij})(\det h_{ij})$

\begin{equation}
\nabla_m K = K \nabla_m \log K = K \nabla_m \log(\det h_{ij}) = K b^{ij} \nabla_m h_{ij}.
\end{equation}

Next,

\begin{equation}
\nabla_i (b^{ij} K) = (\nabla_i b^{ij}) K + b^{ij} \nabla_i K = -b^{ik} b^{ij} (\nabla_i h_{kl}) K + b^{ij} K b^{kl} (\nabla_i h_{kl}) = 0.
\end{equation}

The identity follows from taking a derivative on $b^{ij} h_{jk} = \delta_{ik}$. The evolution equations are shown in Proposition 2.1. Note that

\begin{equation}
(\partial_t - \alpha K^\alpha b^{ij} \nabla^2_i) F = -K^\alpha \nu - \alpha K^\alpha b^{ij} h_{ij} (-\nu)
\end{equation}

\begin{equation}
= (\alpha - 1) K^\alpha \nu.
\end{equation}

Thus, we have

\begin{equation}
(\partial_t - \alpha K^\alpha b^{ij} \langle F, F \rangle) = 2\langle F, (\alpha - 1) \nu \rangle - 2\alpha K^\alpha b^{ij} \langle \nabla_i F, \nabla_j F \rangle
\end{equation}

\begin{equation}
= 2(\alpha - 1) S - 2K^\alpha b^{ij} g_{ij}
\end{equation}

and, using $\nabla^2_i \nu = \nabla_i (h_{jk} h^{jk} F) = -h_{jk} h^{jk} \nu + \nabla_k h_{ij} \nabla^k F$, we obtain

\begin{equation}
(\partial_t - \alpha K^\alpha b^{ij} \nabla^2_i) \langle F, \nu \rangle = (\alpha - 1) K^\alpha + \langle F, (\partial_t - \alpha K^\alpha b^{ij} \nabla^2_i) \nu \rangle - 2\alpha K^\alpha b^{ij} \langle \nabla_i F, \nabla_j \nu \rangle
\end{equation}

\begin{equation}
= (\alpha - 1) K^\alpha + \langle F, \nabla K^\alpha - \alpha K^\alpha b^{ij} \nabla_k h_{ij} \nabla^k F + \alpha K^\alpha H \nu \rangle - 2nK^\alpha
\end{equation}

\begin{equation}
= -(\alpha - 1) K^\alpha + \alpha K^\alpha HS.
\end{equation}

\square

Let us next introduce the following definition of viscosity solutions that we will employ throughout this work. Similar definitions have been frequently used in the literature, for instance in [4, 7].

Definition 2.6 (viscosity solution). Let $K_t \subset \mathbb{R}^{n+1}, t \in [0, T]$, be a continuous one-parameter family of convex bodies which are either bounded or unbounded. $\Sigma_t = \partial K_t \subset \mathbb{R}^{n+1}, t \in [0, T]$, is a viscosity subsolution to the $\alpha$-GCF if the following holds for every $t_0 \in [0, T]$: for any smooth strictly convex solution to the $\alpha$-GCF $\Sigma_t = \partial K_t'$ with $K_{t_0}' \subset K_{t_0}$, the comparison $K_t' \subset K_t$ holds for all $t \in [t_0, T]$. Similarly, $\Sigma_t = \partial K_t$ is a viscosity supersolution to the $\alpha$-GCF if the following holds for every $t_0 \in [0, T]$: for any smooth strictly
convex solution to the $\alpha$-GCF $\Sigma_i' = \partial K_i'$ with $K_{i,0} \subset K_i'$, the comparison $\Sigma_i \subset K_i'$ holds for all $t \in [t_0, T)$. $\Sigma_i$ is a viscosity solution if it is both a viscosity subsolution and a viscosity supersolution.

We state the uniqueness and existence of a viscosity solution starting at any convex hypersurface $\Sigma_0 = \partial K_0 \subset \mathbb{R}^{n+1}$, compact or non-compact, and asymptotic to a cylinder. Its proof is rather a straightforward application of standard smooth approximations and the comparison principle.

**Theorem 2.7.** Let $\Sigma_0 = \partial K_0 \subset \mathbb{R}^{n+1}$ be a convex hypersurface. If $\Sigma_0$ is compact, then there is a unique viscosity solution $\Sigma_i$ to the $\alpha$-GCF running from $\Sigma_0$ and defined over $t \in [0, T)$ for some $T < +\infty$. If $\Sigma_0$ is non-compact and asymptotic to a cylinder $\partial \Omega \times \mathbb{R}$ then there is a unique viscosity solution $\Sigma_i$ to the $\alpha$-GCF running from $\Sigma_0$ defined for all $t \in [0, +\infty)$. Moreover, $\Sigma_i$ is non-compact and asymptotic to $\Omega \times \mathbb{R}$ for all $t \in [0, \infty)$. 

**Proof.** Consider the first case that $\Sigma_0$ is compact. Choose an increasing sequence of convex bodies $K_{i,0}$ with smooth strictly convex boundaries $\Sigma_{i,0}$ (see Ch3.4 [11] for an approximation by smooth strictly convex hypersurfaces) satisfying $\cup_i K_{i,0} = \text{int}(K_0)$. Let $\Sigma_{i,t}$ be the unique smooth solution to the $\alpha$-GCF starting from $\Sigma_{i,0}$ (see in [19]). By the comparison principle the sequence $\Sigma_{i,t}$ is increasing in $i$, and hence the limits $K_t := \cup_i K_{i,t}$ and $\Sigma_t = \partial K_t$ exist. We claim that $\Sigma_i$ is a viscosity solution with initial data $\Sigma_0$. By the construction, $\Sigma_t$ is a viscosity supersolution.

Let us next show that $\Sigma_i$ is a viscosity subsolution as well. Assume without loss of generality that $0 \in K_0$ and $t_0 = 0$. Let $\Sigma'_i = \partial K_i$ be a smooth strictly convex $\alpha$-GCF flow with $K_0' \subset K_0$. Given a small $\epsilon > 0$, we consider the rescaled solution $\Sigma'_i, \tau = (1 - \epsilon)^{(1+n\alpha)} t$ starting at $(1 - \epsilon) \Sigma_i'$. If $i_0$ is sufficiently large, $(1 - \epsilon) K_{i,0}' \subset K_{i,0}$ holds if $i \geq i_0$. Thus, the comparison principle guarantees $(1 - \epsilon) K_{i,t}' \subset K_{i,t}$. Taking the limit $i \to \infty$ and passing $\epsilon \to 0$ yield the inclusion $K_i' \subset K_t$. This proves that $\Sigma_i$ is a viscosity subsolution. We then conclude that $\Sigma_t$ is a viscosity solution.

For the uniqueness assertion, let us assume that we have another viscosity solution $\Sigma_i''$ starting at $\Sigma_0$.

Then, the same argument as above, shows that each small $\epsilon > 0$, there is $i_0 \gg 1$ such that

$$(1 - \epsilon) K_{i,(1-\epsilon)^{-(1+n\alpha)} t}'' \subset K_t'' \subset (1 + \epsilon) K_{i,(1+\epsilon)^{-(1+n\alpha)} t},$$

for $i \geq i_0$.

Taking the limit $i \to \infty$ and passing $\epsilon \to 0$, we conclude that $\Sigma_i'' = \Sigma_i''$. The finiteness of $T$ follows by comparing the solution with a huge spherical solution containing it.

Consider the next case that $\Sigma_0$ is non-compact and asymptotic to $\Omega \times \mathbb{R}$. Choose a sequence of increasing compact sets $K_{i,0}$ with smooth strictly convex boundaries $\Sigma_{i,0}$ such that $\cup_i K_{i,0} = K_0$. Let $\Sigma_{i,t}$, $t \in [0, T_i)$, be the unique smooth strictly convex solutions to the $\alpha$-GCF and define $K_i := \cup_i K_{i,t}$ and $\Sigma_{i,t} = \partial K_{i,t}$ as before. Note $\Sigma_{i,t}$ exists for $t \in [0, T_i)$, where $T_i \geq \lim_{i \to \infty} T_i = (0, \infty)$. By the construction, $\Sigma_i$ is already a viscosity supersolution. Let $\Sigma'_i = \partial K'_i$ be a smooth strictly convex $\alpha$-GCF with $K'_i \subset K_i$. When $K'_i$ is compact, one can use the same argument as before to show $K'_i \subset K_i$. Let us assume $K'_i$ be non-compact. Then $K'_i$ has to be asymptotic to a cylinder $\Omega' \times \mathbb{R}$ with $\Omega' \subset \Omega$. By the same scaling and limiting argument, we may assume $\Omega' \subset \Omega$. For such a $\Sigma'_i$, [18] shows that there exists a unique smooth solution (thus it is equal to $\Sigma'_i$ by the uniqueness) for all $t \in (0, \infty)$ and the solution is written on the fixed domain $\Omega'$. Moreover, the construction in [18] shows $K'_i$ can be approximated by an increasing sequence of compact smooth strictly convex $\alpha$-GCFs $\Sigma'_{i,t} = \partial K'_{i,t}$. For each $K'_{i,t}$, there is $j_i$ such that $K'_{j_i,t} \subset K_{i,t}$. This implies $K'_{j_i,t} \subset K_{j_i,t} \subset K_t$ and proves $K'_i \subset K_t$, i.e. $\Sigma_i$ is a viscosity solution. This also shows $T = \infty$ since we may put a non-compact rotationally strictly convex hypersurface which is asymptotic to a round cylinder in the inside of $K_0$ and apply the comparison principle. Finally, the cylinder asymptotic to $K_t$ does not shrink along the flow since we can insert such a barrier arbitrarily close to the boundary of $\Omega \times \mathbb{R}$ at the initial time $t = 0$. □

**Corollary 2.8.** Let $K_0$ be either a bounded convex body or an unbounded convex body which is contained in a bounded cylinder. If $K_{i,0}$ is an increasing sequence of convex bodies such that $\partial (\cup_i K_{i,0}) = \partial K_0$, then $\partial (\cup_i K_{i,t}) = \partial K_t$. Here, $\partial K_{i,t}$ and $\partial K_t$ are the viscosity $\alpha$-GCFs running from $\partial K_{i,0}$ and $\partial K_0$, respectively.
In this paper, when $\Sigma_{i,t} = \partial K_{i,t}$ is referred, it means approximating smooth compact strictly convex solutions of $\Sigma_t$ from inside unless otherwise stated.

3. LOCAL SPEED ESTIMATE

We review the following Harnack estimate which was shown by B. Chow in [20].

**Theorem 3.1** (B. Chow [20]). For a smooth compact strictly convex solution to the $\alpha$-GCF with $\alpha > 0$, there holds

\[
\frac{1}{K^{\alpha}} (\partial_t K^{\alpha} - b^{ij} \nabla_i K^{\alpha} \nabla_j K^{\alpha}) \geq -\frac{n\alpha}{1+n\alpha} \frac{1}{t}.
\]

This has the following consequence:

**Proposition 3.2.** Let $x_{n+1} = u(x', t)$ be a smooth strictly convex graphical solution to the $\alpha$-GCF with $\alpha > 0$ over some domain $\Omega' \subset \mathbb{R}^n$ and assume it is part of a compact smooth solution or a smooth limit of such solutions. Then,

\[
\frac{u_{tt}}{t} \geq -\frac{\frac{n\alpha}{1+n\alpha}}{t} u_t,
\]

and hence, for $t_2 \geq t_1 > 0$,

\[
u_t(t, t_2) \geq \left( \frac{t_1}{t_2} \right)^{\frac{n\alpha}{1+n\alpha}} \nu_t(t, t_1).
\]

**Proof.** For any 1-form $V_i$, $\alpha^{ij} b^{ij} (V_i + \nabla_i \log K^\alpha)(V_j + \nabla_j \log K^\alpha) \geq 0$ and the Harnack imply

\[
\partial_t \log K^\alpha + 2\alpha^{ij} b^{ij} V_i \nabla_j \log K^\alpha + K^{\alpha} b^{ij} V_i V_j \geq -\frac{n\alpha}{1+n\alpha} \frac{1}{t}.
\]

In other words, for any vector field $U^i = K^{\alpha} b^{ij} V_j$,

\[
\partial_t K^\alpha + 2U^i \nabla_i K^\alpha + h_{ij} U^i U^j \geq -\frac{n\alpha}{1+n\alpha} \frac{K^\alpha}{t}.
\]

For a graphical solution of $\alpha$-GCF, $x_{n+1} = u(x', t)$, note that $\partial_t u = \frac{K^\alpha}{\langle -\nu, e_{n+1} \rangle}$ and

\[
\partial_{tt} u(x', t) = (\partial_t + W^i \nabla_i) \left( \frac{K^\alpha}{\langle -\nu, e_{n+1} \rangle} \right) \text{ with } W = \frac{K^\alpha}{\langle -\nu, e_{n+1} \rangle} e_{n+1}^{\tan}.
\]

Here $e_{n+1}^{\tan} = e_{n+1} - \langle e_{n+1}, \nu \rangle \nu$. Using this and $\partial_t \nu = \nabla K^\alpha$, we check

\[
(\partial_t + W^i \nabla_i) \left( \frac{K^\alpha}{\langle -\nu, e_{n+1} \rangle} \right) = \frac{1}{\langle -\nu, e_{n+1} \rangle} \left( \partial_t K^\alpha + W^i \nabla_i K^\alpha \right) + K^\alpha \left( \partial_t + W^i \nabla_i \right) \frac{1}{\langle -\nu, e_{n+1} \rangle} \frac{K^\alpha}{\langle -\nu, e_{n+1} \rangle}
\]

\[
= \frac{1}{\langle -\nu, e_{n+1} \rangle} \left( \partial_t K^\alpha + 2W^i \nabla_i K^\alpha + h_{ij} W^i W^j \right) \geq -\frac{n\alpha}{n + \alpha} \frac{K^\alpha}{t}\langle -\nu, e_{n+1} \rangle.
\]

□

Suppose that $\Sigma_t$ is a non-compact viscosity $\alpha$-GCF asymptotic to $\partial \Omega \times \mathbb{R}$ and let $\{x_{n+1} = u_i(x', t)\}$ be the graph representation of the lower parts of the approximating compact smooth strictly convex solutions $\Sigma_{i,t}$. Let us denote by $\Omega_{i,t}$ the spatial domain of $u_i(\cdot, t)$, namely $\Omega_{i,t}$ is the projection of $\Sigma_{i,t} \subset \mathbb{R}^{n+1}$ to the hyperplane $\{x_{n+1} = 0\}$. 
Proposition 3.3. For each $\Omega' \subset \subset \Omega$ and $t_0 > 0$ there is $L > 0$ with the following significance: for all $T > t_0$ there is $i_0$ so that

$$(3.4) \quad \partial_t u_i(x', t) = \frac{K^\alpha}{\langle -\nu, \epsilon_{n+1} \rangle} \leq L \quad \text{for } (x', t) \in \Omega' \times [t_0, T] \text{ and } i > i_0.$$ 

Moreover, for each $\Omega' \subset \subset \Omega$ there are positive constants $t_0$, $\delta$, $L$ with the following significance: for all $T > t_0$ there is $i_0$ so that

$$(3.5) \quad 0 < \delta \leq \partial_t u_i(x', t) = \frac{K^\alpha}{\langle -\nu, \epsilon_{n+1} \rangle} \quad \text{for } (x', t) \in \Omega' \times [t_0, T] \text{ and } i > i_0.$$ 

Proof. Let us assume, without loss of generality, that $\Omega$ contains the origin and that the speed of the translating soliton defined on $\Omega$, call it $u_0$, is $\lambda$. Fix a small $\epsilon_0 \in (0, 1/6)$ so that $\Omega' \subset (1 + \epsilon_0)^{1/n} \Omega$. Since $\mathcal{K}_t$ is asymptotic to $\Omega \times \mathbb{R}$ for all $t \geq 0$ and $\cup_i \mathcal{K}_{i,t} = \mathcal{K}_t$, given $T'$ there is $i_0$ such that if $i > i_0$ 

$$(3.6) \quad (1 + \epsilon_0)^{-1/n} \Omega \subset \subset \Omega_{i,t} \quad \text{for all } t \in [0, T'].$$

$T'$ is some number which will be chosen later.

By rescaling the flow, if we define 

$$\tilde{u}(x') := (1 + \epsilon_0)^{-1/n} u_{1+\epsilon_0}(1 + \epsilon_0)^{1/n}(x'),$$

then $\tilde{u}$ is the translating soliton on $(1 + \epsilon_0)^{-1/n} \Omega$ which has the speed $(1 + \epsilon_0)\lambda$. Similarly, we define the translating soliton 

$$\hat{u}(x') := (1 - \epsilon_0)^{-1/n} u_{1-\epsilon_0}(1 - \epsilon_0)^{1/n}(x')$$

on $(1 - \epsilon_0)^{-1/n} \Omega$ which has the speed $(1 - \epsilon_0)\lambda$. Depending on $\Sigma_0$, we may find a large $L > 0$ such that 

$$\hat{u}(x') - \frac{L}{2} \leq u(x', 0) \quad \text{on } \Omega$$

and 

$$u(x', 0) \leq \hat{u}(x') + \frac{L}{2} \quad \text{on } (1 + \epsilon_0)^{-1/n} \Omega.$$ 

It follows that there is an $i_0$ such that for $i > i_0$, then 

$$\tilde{u}(x') - \frac{L}{2} \leq u_i(x', 0) \quad \text{on } \Omega_{i,0} \quad \text{and} \quad u_i(x', 0) \leq \hat{u}(x') + L \quad \text{on } (1 + \epsilon_0)^{-1/n} \Omega.$$ 

Furthermore, by (3.6) and (3.7), one can apply the comparison principle between $x_{n+1} = u_i(x', t)$ and two barriers so that we obtain, for $i > i_0$, 

$$\hat{u}(x') - L + (1 - \epsilon_0)\lambda t \leq u_i(x', t) < \tilde{u}(x') + L + (1 + \epsilon_0)\lambda t \quad \text{on } \Omega' \times [0, T'].$$ 

In particular, we have for $t \in [0, T']$ and $i > i_0$ 

$$0 \leq f(x', t) := (\hat{u}(x') + L + (1 + \epsilon_0)\lambda t) - u_i(x', t) \leq 2(L + \epsilon_0\lambda t)$$

and 

$$0 \leq g(x', t) := u_i(x', t) - (\tilde{u}(x') - L + (1 - \epsilon_0)\lambda t) \leq 2(L + \epsilon_0\lambda t).$$

We first prove the upper bound (3.4). Choose $T'$ in (3.6) by $T' = 2T$. Suppose $\partial_t u_i(x_0, t_1) = C$ at some $x_0 \in \Omega'$ and $t_1 \in [t_0, T]$. Then by (3.3), $\partial_t u_i(x_0, t) \geq C \eta$ for some $\eta = \eta(\alpha, n) \in (0, 1)$ and all $t \in [t_1, 2t_1]$. We have 

$$0 \leq f(x_0, 2t_1) = f(x_0, t_1) + \int_{t_1}^{2t_1} \partial_t f \leq 2(L + \epsilon_0\lambda t_1) + [(1 + \epsilon_0)\lambda t_1 - C\eta t_1]$$

and hence $\partial_t u_i(x_0, t_0) = C \leq \frac{(1 + 3\epsilon_0)\lambda}{\eta} + \frac{2L}{\eta t_1}$, proving that the bound from above in (3.4) holds for any $t_1 \in [t_0, T]$ and $t_0$ fixed.
Next, we prove the lower bound \((3.5)\). To this end, suppose \(\partial_t u_i(x_0, t_0) = c\) at some \(x_0 \in \Omega'\) and \(t_0 > 0\). Provided \(T' > t_0\) and \(i > i_0\), \((3.3)\) implies that \(\partial_t u_i(x_0, t) \leq \gamma c\) for any \(t \in [t_0/2, t_0]\) and some \(\gamma = \gamma(\alpha, n) > 1\). Thus,

\[
0 \leq g(x_0, t_0) = g(x_0, t_0/2) + \int_{t_0/2}^{t_0} \partial_t g \leq 2 \left(L + \epsilon_0 \lambda \frac{t_0}{2}\right) + \left[\gamma c \frac{t_0}{2} - (1 - \epsilon_0) \lambda \frac{t_0}{2}\right]
\]

implies that for any \(\epsilon_0 < 1/6\) we have

\[
c \geq \frac{1 - 3\epsilon_0}{\gamma} \lambda - \frac{4L}{\gamma t_0} \geq \frac{\lambda}{2\gamma} - \frac{4L}{\gamma t_0}.
\]

Hence, \(\partial_t u_i(x_0, t_0) = c \geq \frac{\lambda}{2\gamma}\) if \(t_0 \geq \frac{16L}{\lambda}\). Let us choose \(t_0 := 16L/\lambda > 0\). For every \(T \geq t_0\), if we choose \(T' = T\) the previous yields the lower bound \((3.5)\) for \(i > i_0\).

On a strictly convex smooth solution \(\Sigma_t\), we may define the Gaussian curvature \(K\) as a function of the normal vector \(\nu\) at a point \(p\), i.e. we define \(\bar{K}(\nu, t) := K(p(\nu, t), t)\) where \(p = p(\nu, t)\) is the unique point with \(\nu(p) = \nu\). By the evolution of \(\nu\) in \((2.11)\), \(\partial_t \bar{K}^\alpha = \partial_t K^\alpha - b \nabla_i K^\alpha \nabla_j K^\alpha\). Hence Chow’s Harnack inequality \((3.1)\) implies

\[
(3.8) \quad \partial_t \bar{K}^\alpha \geq - \frac{n\alpha}{1 + n\alpha} \frac{\bar{K}^\alpha}{t}
\]

which, after integrated in time \(t \in [t_1, t_2]\), gives

\[
(3.9) \quad \bar{K}^\alpha(\cdot, t_2) \geq \left(\frac{t_1}{t_2}\right)^{\frac{n\alpha}{1 + n\alpha}} \bar{K}^\alpha(\cdot, t_1).
\]

A similar argument of Proposition \((3.3)\) applied to the support function \(S(\cdot, t)\) instead of the height function \(u(\cdot, t)\), was actually used by the authors in \([13\text{ Section 2}]\). We will need this result for the current problem as well. Following similar arguments as in Proposition \((3.3)\) and \([13]\), we obtain the following:

**Proposition 3.4.** Let \(\Sigma_{i, t}\) be a sequence of compact smooth strictly convex solutions which approximate the non-compact viscosity solution \(\Sigma_t\) asymptotic to the cylinder \(\partial \Omega \times \mathbb{R}\) of the bounded section \(\Omega\). For any small \(\mu > 0\), there are positive constants \(t_0, \delta\) depending on \(\Sigma_0\) and \(\mu\) with the following significance: for all \(T > 0\) there is \(i_0\) such that, for \(\Sigma_{i, t}\) with \(i > i_0\),

\[
\delta \leq K^\alpha(p, t) \quad \text{if} \quad t_0 \leq t \leq T, \quad \text{and} \quad \langle -\nu(p, t), e_{n+1} \rangle \geq \mu.
\]

For given \(t_0\), there is \(M\) depending on \(\Sigma_0\) and \(t_0\) with the following significance: for all \(T > 0\) there is \(i_0\) such that, for \(\Sigma_{i, t}\) with \(i > i_0\),

\[
K^\alpha(p, t) \leq M \quad \text{if} \quad t_0 \leq t \leq T, \quad \text{and} \quad \langle -\nu(p, t), e_{n+1} \rangle \geq 0.
\]

**Proof.** Assume \(\Omega \subset \mathbb{R}^n\) contains the origin. Define the support function \(\bar{S}(\nu(t)) = \sup_{x \in \Sigma_0} \langle x, \nu(t) \rangle\). Let \(\epsilon_0 > 0\). As in Proposition \((3.3)\) let us consider a translator on a slightly larger domain whose translator speed is \((1 - \epsilon_0)\lambda\). Here \(\lambda\) is the speed of the translator \(u_0\) on \(\Omega\). We can make that this translator contains our initial surface \(\Sigma_0\) (and hence all \(\Sigma_{i, t}\)) by translating the translator in \(-\epsilon_{n+1}\) direction. If \(\bar{S}^+(\nu, t)\) denotes the support function of this translator outside, then the comparison principle between support functions \([13\text{ Lemma 2.6}]\) yields

\[
\bar{S}(\nu, t) \leq \bar{S}^+(\nu, t) = C + (1 - \epsilon_0) \lambda t \langle \nu, e_{n+1} \rangle \quad \text{on} \quad \Sigma_{i, t} \cap \{\langle -\nu, e_{n+1} \rangle \geq 0\} \quad \text{by some} \quad C(\epsilon_0, \Sigma_0, \alpha, n) > 0.
\]
On the other hand, by inserting a translating soliton of the speed $(1 + \epsilon_0)\lambda$ inside, we know that the point $(L + (1 + \epsilon_0)\lambda t)e_{n+1}$ for some $L > 0$ is located inside of $\Sigma_t$. Thus, $\langle F - (L + (1 + \epsilon_0)\lambda t)e_{n+1}, \nu \rangle \geq 0$ and hence, in terms of approximating solutions, for each $T' > 0$ there is $i_0$ with

$$-C + (1 + \epsilon_0)\lambda t \langle \nu, e_{n+1} \rangle \leq \bar{S}(\nu, t) \quad \text{on } \Sigma_{i,t} \quad \text{if } i > i_0 \quad \text{by some } C(\epsilon_0, \Sigma_0, \alpha, n) > 0.$$  

In particular, if $i > i_0$, we have

$$0 \leq f(\nu, t) := \bar{S}^+(\nu, t) - \bar{S}(\nu, t) \leq 2\left(C - \epsilon_0\lambda t \langle \nu, e_{n+1} \rangle\right) \quad \text{for } t \in [0, T']$$

and

$$0 \leq g(\nu, t) := \bar{S}(\nu, t) - C + (1 + \epsilon_0)\lambda t \langle \nu, e_{n+1} \rangle \leq 2\left(C - \epsilon_0\lambda t \langle \nu, e_{n+1} \rangle\right) \quad \text{for } t \in [0, T'].$$

In the meantime, note that $\partial_t \bar{S}(\nu, t) = \bar{K}^\alpha(\nu, t)$. In the estimates below, we assume $i > i_0 = i_0(T')$. Let us prove the upper bound. Given $t_0 > 0$, suppose that $\bar{K}^\alpha(\nu_0, t_0) = a$ at some $\nu_0 \in S^n := S^n \cap \{x_{n+1} \leq 0\}$ and $0 < t_0 < T'/2$. Then (3.38) implies that $\bar{K}^\alpha(\nu_0, t) \geq \eta a$ for $t \in [t_0, 2t_0]$ by some $\eta = \eta(\alpha, n) \in (0, 1)$. Therefore

$$0 \leq f(\nu_0, 2t_0) = f(\nu_0, t_0) + \int_{t_0}^{2t_0} \partial_t f \leq 2\left(C + \epsilon_0\lambda t_0\right) + \left[1 + \epsilon_0\lambda t_0 - \eta a t_0\right]$$

implies that the upper bound $\bar{K}^\alpha(\nu_0, t) \leq \frac{(1 + 3\epsilon_0)\lambda}{\eta} + 2C$ for some $M$ depends on $\Sigma_0$ and $t_0$. This proves the upper bound.

Let us prove the lower bound. Given $\mu > 0$, suppose $\bar{K}^\alpha(\nu_0, t_0) = a$ at some $(\nu_0, t_0)$ with $\langle - \nu_0, e_{n+1} \rangle \geq \mu > 0$ and $0 < t_0 < T'$. Then by (3.39), $\bar{K}^\alpha(\nu_0, t) \leq \gamma c$ for $t \in [t_0/2, t_0]$ and some $\gamma = \gamma(\alpha, n) > 1$. Hence, for $\epsilon_0 \in (0, 1)$ to be chosen later, there is $C = C(\epsilon_0, \Sigma_0, \alpha, n)$ such that

$$0 \leq g(\nu_0, t_0) = g(\nu_0, t_0/2) + \int_{t_0/2}^{t_0} \partial_t g \leq 2\left(C + \epsilon_0\lambda t_0/2\right) + \left(\gamma a t_0/2 - (1 - \epsilon_0)\mu \lambda t_0/2\right)$$

implying that

$$a \geq \frac{(1 - \epsilon_0)\mu - 2\epsilon_0\lambda}{\gamma} - \frac{4C}{\gamma t_0}.$$  

Now by choosing $\epsilon_0 := \frac{\mu}{3 + \mu}$ (hence $(1 - \epsilon_0)\mu = 3\epsilon_0$) we have

$$a \geq \frac{\epsilon_0}{\gamma} - \frac{4C}{\gamma t_0} = \frac{\mu}{3 + \mu} - \frac{4C}{\gamma t_0} \quad \text{for some } C = C(\mu, \Sigma_0, \alpha, n).$$

Therefore

$$a = \bar{K}^\alpha(\nu_0, t_0) \geq \frac{\mu}{3 + \mu} \frac{\lambda}{2\gamma} \quad \text{if } t_0 \geq 8C \frac{3 + \mu}{\mu \lambda}.$$  

In summary, given $\mu \in (0, 1)$, there is $t_0 = 8C \frac{3 + \mu}{\mu \lambda}$ such that if $t_0 \leq t \leq T'$ and $\langle - \nu, e_{n+1} \rangle \geq \mu$ then $\bar{K}^\alpha(\nu_0, t_0) \geq \delta$ holds on $\Sigma_{i,t}$ with $t > i_0$, where $\delta > 0$ is some constant depending on $\mu$, $\Sigma_0$, $\alpha$, and $n$. □

4. Local convexity estimate

This section, we prove estimates which give local bounds from below on the minimum principal curvature $\lambda_{\min}$ of our solution $\Sigma_t$ in terms of upper and lower bounds of the speed $\bar{K}^\alpha$. The estimates allow us to pass to the limit of solutions and it is important later in the proof of the main theorem. We need some preliminary results and we begin with simple observations on convex graphs.
Lemma 4.1. Let $x_{n+1} = u(x')$ be a $C^2$ convex graph on $\{|x'| \leq 2r\}$ and assume there is $\delta > 0$ such that
$$\frac{K}{\langle -\nu, e_{n+1} \rangle} > \delta,$$
where $\nu = \frac{(Du, -1)}{\sqrt{1 + |Du|^2}}$ denotes a unit normal vector to the graph. Then there is $C = C(\delta, r^{-n}, n)$ such that
$$\sup_{|x'| \leq r} u - \inf_{|x'| \leq r} u \leq Cr.$$ 

**Proof.** We may assume without loss of generality that $r = 1$ and that
$$\inf_{|x'| \leq 1} u = u(x'_1) = 0 \quad \text{and} \quad L := \sup_{|x'| \leq 1} u = \sup_{|x'| = 1} u = u(x'_2) > 0$$
with $|x'_1| \leq 1$ and $|x'_2| = 1$. Since $u$ is convex, the set $A := \{x' : u(x') \leq L \text{ with } |x'| < 2\}$ is convex, $\{|x'| \leq 1\} \subset A$ and $x'_2 \in \partial A$. This implies that $u \geq L$ on $B := \{x' : \langle x', x'_2 \rangle > 1 \text{ and } |x'| < 2\}$. Also, the convexity of $u$ implies that, for every $x' \in B$,
$$|Du(x')| \geq \frac{u(x') - u(x'_1)}{|x' - x'_1|} \geq \frac{L}{4}.$$ 
It follows that the normal vectors $\nu = \frac{(Du, -1)}{\sqrt{1 + |Du|^2}}$, are contained in
$$C := \{v \in \mathbb{S}^n : 0 \leq \langle v, -e_{n+1} \rangle \leq \frac{1}{\sqrt{1 + (L/4)^2}}\}.$$ 
One can roughly bound $|C| \leq c_nL^{-1}$. On the other hand, note $|B| = c_n > 0$ and hence our assumption yields $|\nu[B]| = \int_B \frac{K}{\langle -\nu, e_{n+1} \rangle} dx' \geq c_n\delta$. Since $\nu[B] \subset C$, we conclude $c_n\delta \leq c_n L^{-1}$ or $L \leq c_n \delta^{-1}$. Recalling $L := \sup_{|x'| \leq 1} u$ and $\inf_{|x'| \leq 1} u = 0$, this finishes the proof of the lemma. \hfill $\square$

Lemma 4.2. Let $\Sigma = \partial K$ be a complete $C^2$ convex hypersurface in $\mathbb{R}^{n+1}$. Suppose $0 \in \Sigma$, $\langle -\nu(0), e_{n+1} \rangle > 0$, and that, around the origin, $\Sigma$ can be represented as a convex graph over a disk $D_{4\rho} := \{x' \in \mathbb{R}^n : |x'| \leq 4\rho\}$, for some $\rho > 0$. i.e. there is a convex function $u : D_{4\rho} \to \mathbb{R}$ such that
$$\{(x', u(x')) : x' \in D_{4\rho}\} = \{(x', x_{n+1}) \in \Sigma : \langle -\nu(x), e_{n+1} \rangle > 0 \text{ and } x' \in D_{4\rho}\} := \Gamma.$$ 
If we further assume that
$$\frac{K}{\langle -\nu, e_{n+1} \rangle} \geq \delta \text{ on } \Gamma \text{ for some } \delta > 0,$$
then there is $C = C(\delta, \rho, n)$ such that
$$\langle -\nu(x), e_{n+1} \rangle^{-1} \leq C \quad \text{on } \{x \in \Sigma : \langle x, \nu(x) \rangle \leq \rho \text{ and } \langle -\nu, e_{n+1} \rangle \geq 0\}.$$ 

**Proof.** We may assume that $u(0) = 0$. By Lemma 4.1, $u(x') = u(x') - u(0) \leq C'\rho$ on $\{|x'| \leq 2\rho\}$, for some $C'(\delta, \rho, n)$. Therefore, the ball $B_{2\rho}((C' + 2)\rho e_{n+1})$ is located above to $\Sigma \cap \{\langle -\nu, e_{n+1} \rangle \geq 0\}$. Hence around this center point $x_1 := (C' + 2)\rho e_{n+1}$, we have $\langle x - x_1, \nu(x) \rangle \geq 2\rho$, for all $x \in \Sigma \cap \{\langle -\nu, e_{n+1} \rangle \geq 0\}$. It follows that for all $x \in \Sigma \cap \{\langle -\nu, e_{n+1} \rangle \geq 0\}$ satisfying $\langle x, \nu(x) \rangle \leq \rho$, we have
$$2\rho \leq \langle x - x_1, \nu(x) \rangle = \langle x, \nu(x) \rangle - \langle x_1, \nu(x) \rangle \leq \rho - \rho(C' + 2)\langle \nu(x), e_{n+1} \rangle$$
which implies the desired bound $\frac{1}{C' + 2} \leq \langle -\nu(x), e_{n+1} \rangle$. \hfill $\square$

The following proposition is obtained by combining the results above.

**Proposition 4.3.** Let $\Sigma = \partial K \subset \mathbb{R}^{n+1}$ be a $C^2$ convex hypersurface a part of which is a convex graph $x_{n+1} = u(x')$ on convex domain $\Omega \subset \mathbb{R}^n$. For given $x_0 = (x'_0, u(x'_0)) \in \Sigma$ with $x'_0 \in \Omega$, suppose that
$$d(x'_0, \partial \Omega) := 4\epsilon \text{ and } \frac{K}{\langle -\nu, e_{n+1} \rangle} \geq \delta > 0 \text{ on } \{(x', u(x')) : |x' - x'_0| \leq 2\epsilon\}. \text{ Then }$$
$$\{x \in \Sigma : \langle x - x_0, \nu(x) \rangle \leq \epsilon, \langle -\nu(x), e_{n+1} \rangle \geq 0\}$$
is compact and, on this set, there is $C = C(\delta, \epsilon, n)$ such that

$$(-\nu(x), e_{n+1})^{-1} \leq C \quad \text{and} \quad |x - x_0| \leq C \text{diam}(\Omega).$$

**Proof.** The first gradient bound follows directly from Lemma 4.1 and 4.2. The second is a consequence the gradient bound. □

Next, we show our convexity estimates. The proof is independent of previous propositions, but they will be combined in Corollary 4.5 to give the regularity estimates for the viscosity solutions asymptotic to a cylinder.

**Theorem 4.4.** For $\alpha > 0$, let $\Sigma = F(\cdot, t)(\Sigma^n)$ be a complete smooth strictly convex solution to the $\alpha$-GCF. For $F_0 := F(p_0, t_0) \in \Sigma_{t_0}$, suppose there exist constants $\epsilon, \delta, L > 0$ such that

$$\delta \leq K^\alpha(p, t) \leq L \quad \text{and} \quad |F(p, t) - F_0| \leq L$$
on the $(p, t) \in [0, t_0] : (F(p, t) - F_0, \nu(p, t)) \leq \epsilon$. Then there is $C = C(\epsilon, \delta, L, \alpha, n)$ such that

$$\lambda_{\min}^{-1}(p_0, t_0) \leq C (1 + t_0^{-1}).$$

**Proof.** We may assume $F_0 = F(p_0, t_0) = 0$. Let $S := \langle F, \nu \rangle$ be the support function. Under the $\alpha$-GCF, by (2.9) and (2.14) we have

$$(\partial_t - Kb^\alpha \nabla^2) b_1^1 = -\alpha K^\alpha b^1 b^1(\alpha b^{kl} b^{mn} + b^{km} b^{ln}) \nabla_i h_{kl} \nabla_j h_{mn} - \alpha K^\alpha H b_i^1 + (1 + n\alpha) K^\alpha - 2K^\alpha.$$

Define the cut off function

$$\eta := (\epsilon - S)_+,$$

and compute that

$$(\partial_t - Kb^j \nabla_i) \ln \eta = \frac{(n\alpha + 1)K^\alpha}{\eta} - \frac{\alpha K^\alpha HS}{\eta} + \frac{\alpha K^\alpha b^j \nabla_i \eta \nabla j \eta}{\eta^2}.$$

For some $\gamma > 0$ to be chosen later, let us consider the auxiliary test function

$$w := \eta^2 e^{\gamma \int \eta^2 t}$$

and apply the maximum principle to bound the maximum of $\eta^2 \lambda_{\min}^{-1} e^{\gamma \int \eta^2 t}$ on $\Sigma \times (0, t_0)$ is obtained at $(p', t')$. At this point, choose local coordinates such that $b^{ij} = \lambda_{ij}^{-1} \delta^{ij}$, $\lambda_1 = \lambda_{\min}$, and $g_{ij} = \delta_{ij}$ at $(p', t')$. A direct calculation using (2.17) and (2.10) shows that at the maximum point $(p', t')$ we have

$$0 \leq (\partial_t - \alpha K^\alpha b^j \nabla_i \eta^2) \ln w \leq 2 \left[ \frac{(n\alpha + 1)K^\alpha}{\eta} - \frac{\alpha K^\alpha HS}{\eta} + \frac{\alpha K^\alpha b^j \nabla_i \eta \nabla j \eta}{\eta^2} \right]$$

$$- \frac{1}{b^j} \left[ \alpha K^\alpha b^1 b_1^1(\alpha b^{kl} b^{mn} + b^{km} b^{ln}) \nabla_i h_{kl} \nabla_j h_{mn} \right] + \frac{\alpha K^\alpha b^j \nabla_i \eta b^{11} \nabla j b^{11}}{\eta^2} + 2(n\alpha - 1)\gamma K^\alpha S - 2\gamma \alpha K^\alpha g_{ij} + \frac{1}{\eta}.$$

Notice that $S \geq 0$ for $t \leq t_0$ since $0 \in \Sigma_{t_0}$ and also $S \leq \epsilon$ on the support of $\eta$. Therefore we may bound three terms in the inequality above as

$$2 \frac{(n\alpha + 1)K^\alpha}{\eta} - 2 \frac{\alpha K^\alpha HS}{\eta} + 2(n\alpha - 1)\gamma K^\alpha S \leq C \left( \frac{1}{\eta} + \epsilon \gamma \right).$$
for some $C = C(L, \delta, n, \alpha)$.

On the other hand, at this maximum point we have

$$\nabla \ln w = 2 \frac{\nabla \eta}{\eta} + \frac{\nabla b_{11}^{11}}{b_{11}^{11}} + \gamma \nabla |F|^2 = 0$$

and therefore for fixed $i$ (we are not summing over $i$)

$$2 \frac{\alpha K^\alpha b_{11}^{ii} \nabla_i \eta \nabla_i \eta}{\eta^2} = \frac{1}{2} \alpha K^\alpha b_{11}^{ii} \left( \frac{\nabla_i b_{11}^{11}}{b_{11}^{11}} + \gamma \nabla_i |F|^2 \right) \left( \frac{\nabla_i b_{11}^{11}}{b_{11}^{11}} + \gamma \nabla_i |F|^2 \right)$$

(4.2)

$$\leq \frac{\alpha K^\alpha b_{11}^{ii} \nabla_i b_{11}^{11} \nabla_i b_{11}^{11}}{(b_{11}^{11})^2} + \gamma^2 \alpha K^\alpha b_{11}^{ii} |F|^2 \nabla_i |F|^2$$

$$\leq \frac{\alpha K^\alpha b_{11}^{ii} \nabla_i b_{11}^{11} \nabla_i b_{11}^{11}}{(b_{11}^{11})^2} + 4(\sup |F|^2) \gamma^2 \alpha K^\alpha b_{11}^{ii}.$$

We use (4.2) for all $i \neq 1$ and plug them into (4.1). Then, there exists $C = C(L, \delta, \alpha, n) > 0$ such that

$$0 \leq 2 \frac{\alpha K^\alpha b_{11}^{ii} \nabla_i \eta \nabla_i \eta}{\eta^2} + \left[ -\alpha K^\alpha b_{11}^{ii} (\alpha b_{kl} b_{mn} + b_{km} b_{ln}) \nabla_i h_{kl} \nabla_i h_{mn} \right]$$

$$+ \alpha K^\alpha (b_{11}^{11})^3 \nabla_i h_{11} \nabla_i h_{11} + \sum_{i \neq 1} 2 \alpha K^\alpha b_{11}^{ii} (b_{11}^{11})^2 \nabla_i h_{11} \nabla_i h_{11}$$

$$+ C \left( \frac{1}{\eta} + \epsilon \gamma \right) - (2 \gamma - 4(\sup |F|^2) \gamma^2) \alpha K^\alpha b_{11}^{ii} g_{ii} + \frac{1}{b'}.$$  

Here, a crucial observation is the cancellation among the third order derivatives

(4.4)  

$$- \alpha K^\alpha b_{11}^{11} b_{km} b_{ln} \nabla_i h_{kl} \nabla_i h_{mn} + \alpha K^\alpha (b_{11}^{11})^3 \nabla_i h_{11} \nabla_i h_{11} + \sum_{i \neq 1} 2 \alpha K^\alpha b_{11}^{ii} (b_{11}^{11})^2 \nabla_i h_{11} \nabla_i h_{11} \leq 0.$$

Let us choose $\gamma = \frac{1}{4(\sup |F|^2)}$. Plugging $(\nabla_i \eta)^2 = (h_{11})^2 (F, \nabla_i F)^2$ and (4.4) into (4.3), we obtain

$$\gamma \alpha K^\alpha b_{11}^{ii} g_{ii} \leq C \left( \frac{1}{\eta} + \epsilon \gamma \right) + \frac{1}{b'} + 2 \alpha K^\alpha (\sup |F|^2) (b_{11}^{11} \eta^2)^{-1}.$$

Combining this inequality with the bound $K^\alpha \geq \delta$, we conclude that there is $C = C(\epsilon, \delta, L, \sup |F|^2, \alpha, n)$ such that

$$b_{11}^{11} \leq C \left( 1 + \frac{1}{b'} + \frac{1}{\eta} + (b_{11}^{11} \eta^2)^{-1} \right).$$

Note that $0 < t' \leq t_0$, $\eta \leq \epsilon$ and $1 \leq e^{\gamma |F|^2} \leq e^{1/4}$. Hence the last bound yields

$$w(p', t') = \eta^2 b_{11}^{11} e^{3|z|^2} t' \leq C \left( 1 + t_0 + \frac{t_0}{w(p', t')} \right)$$

from which we conclude the bound

$$w(p', t') \leq C t_0 \left( 1 + \frac{1}{t_0} \right).$$

The theorem readily follows from

$$w(p_0, t_0) := e^{2b_{11}^{11}(p_0, t_0)} t_0 \leq w(p', t').$$

□
Corollary 4.5. Let \( \Sigma_i \subset \mathbb{R}^{n+1} \) be a non-compact viscosity \( \alpha \)-GCF asymptotic to \( \partial \Omega \times \mathbb{R} \) and \( x_{n+1} = u(x', t) \), be the graphical representation of \( \Sigma_i \) on \( \Omega \). Then, for any \( \Omega' \subset \subset \Omega \) there exists \( t_0 > 0 \) and \( L > 0 \) such that
\[
|u_i - u| < C \infty (\Omega' \times [t_0, \infty)), \quad \text{and} \quad \frac{1}{(-\nu, e_{n+1})}, \lambda_{\min}, \lambda_{\max} \leq L \quad \text{on} \quad (x', t) \in \Omega' \times [t_0, \infty).
\]

Proof. Let us denote \( 4 \epsilon := d(\Omega', \partial \Omega) > 0 \). We also fix an approximating sequence \( \Sigma_{i, t} \) and denote the graph representation of \( \Sigma_{i, t} \cap \{(-\nu, e_{n+1}) \geq 0 \} \) by \( x_{n+1} = u_i(x', t) \). By Proposition 4.3 and Proposition 4.3 we obtain \( T_0 = T_0(\Sigma_0, \Omega', \alpha, n) \) with the following: for all \( T > T_0 \) there is \( i_0 \) so that for every \( x_i = (x_i', u_i(x_i', t_0)) \in \Sigma_{i, t_0} \) with \( i > i_0 \), \( x_i' \in \Omega' \) and \( T_0 \leq t_0 \leq T \), there hold
\[
(4.5) \quad \frac{1}{(-\nu(x), e_{n+1})} \leq C \quad \text{and} \quad |x - x_i| \leq C \text{diam}(\Omega) \quad \text{on} \quad \{S_{x_i}(x) \leq \epsilon \}
\]
for some \( C = C(\Sigma_0, \epsilon, \alpha, n) \). Meanwhile, Proposition 4.4 gives upper and lower bounds of \( K^\alpha \) on the region \( \Sigma_{i, t} \cap \{(-\nu, e_{n+1})^{-1} \leq C \} \) for \( T_0 \leq t \leq T \). i.e. we have two-sided bounds of \( K^\alpha \) on \( \{S_{x_i}(x) \leq \epsilon \} \) for \( t \in [T_0, T] \) for large \( i > i_0 \). Consequently, Theorem 4.4 gives a bound of \( \lambda_{\min} \) at \( x_i \in \Sigma_{i, t_0} \) when \( x_i' \in \Omega' \) and \( T_0 + 1/2 \leq t_0 \leq T \). The bound on \( \lambda_{\max} \) follows from the bounds on \( \lambda_{\min} \) and \( K^\alpha \).

To summarize, for \( i > i_0 \), the solutions \( x_{n+1} = u_i(x', t) \) on \( \Omega' \times [T_0 + 1/2, T] \) have uniform bounds on \( (1 + |Du_i|^2)^{1/2}, \lambda_1(x'), \) and \( \lambda_n(x') \). One can use standard regularity estimates of uniformly parabolic equations to deduce that \( u_i \) converges to \( u \) in \( C^\infty \) sense on the specified domain.

5. Convergence to translating soliton

In this section we give the proof of our main convergence result Theorem 1.1. It will be based on the following monotonicity formula which holds on compact solutions and is shown in Corollary A.4 in Appendix. Recall the definition of \( P_{ij} \) given in (2.1) and \( P := b^i P_{ij} \).

Theorem 5.1. Let \( \Sigma_i \) be a smooth compact closed strictly convex solution of the \( \alpha \)-GCF with \( \alpha > 0 \). Then
\[
(5.1) \quad \frac{d}{dt} \int P K^\alpha \, dg = \int (P_{ij} P_{kl} b^k b^l + (2\alpha - 1) P^2) K^\alpha \, dg \geq (n^{-1} + 2\alpha - 1)(\int P K^\alpha \, dg)^2 \int K^\alpha \, dg.
\]
In particular, when \( \alpha = 1 \) the last term is \( \frac{\omega_n}{\omega_n} \left( \int P K \, dg \right)^2 \) where \( \omega_n = |S^n| = \int K \, dg \).

Proof. Shown in Corollary A.4 in Appendix.

Proposition 5.2. For \( \alpha \geq 1 \), let \( \Sigma_i = \partial \{x_{n+1} = u(x', t)\} \) be a non-compact viscosity \( \alpha \)-GCF asymptotic to \( \Omega \times \mathbb{R} \) for some bounded convex domain \( \Omega \). Then for every \( \tau > 0 \) and \( U \subset \subset \Omega \),
\[
\lim_{t \to \infty} \int_{t-t}^{t+\tau} \int_{(x', u(x', s)) : x' \in U} P^2 K^\alpha \, dg \, ds = \lim_{t \to \infty} \int_{t-t}^{t+\tau} \int_{\bar{\Omega} \cap (x' \in U)} P^2 K^\alpha \, dg \, ds = 0.
\]

Proof. Let us consider an approximating sequence of smooth compact strictly convex solutions (from inside) \( \Sigma_{i, t} \) with an additional assumption that \( \Sigma_{i, 0} \) has the reflection symmetry about \( \{x_{n+1} = i\} \). By Corollary 4.3 \( \Sigma_{i, t} \) converges locally smoothly to \( \Sigma_t \) when their lower parts are viewed as graphs.

The approximation of \( \Sigma_t \) by \( \Sigma_{i, t} \) shown above and the positivity of \( P^2 K^\alpha \), imply that it suffices to show the following statement: for given \( \tau > 0 \) and \( \epsilon > 0 \), there is \( t_0 \) such that for each \( t \geq t_0 \), we have
\[
(5.2) \quad \limsup_{i \to \infty} \int_{t-t}^{t+\tau} \int_{\Sigma_{i, t}} P^2 K^\alpha \, dg \, ds \leq \epsilon.
\]

Claim 5.1. For any fixed finite time interval \( [1, T] \), there is some large \( i_0 \) such that \( K^{\alpha-1} \leq C < \infty \) on \( \Sigma_{i, t} \) for \( i \geq i_0, t \in [1, T] \). The constant \( C \) only depends on \( \Sigma_0 \).
Proof of Claim. This is by Proposition 5.2 and the symmetry of \( \Sigma_{i,t} \) with respect to \( \{x_{n+1} = j\} \). 

By shifting \( t = 1 \) as the initial time we may assume the claim holds from time \( t = 0 \). Let us continue to show 5.2. The Harnack inequality (3.1) and Claim 5.1 yield that, for any \( T > 0 \), there holds

\[
J^{(i)}(s) := \int_{\Sigma_{i,s}} PK^{\alpha} dg \geq -\frac{n}{1+n\alpha} \int_{\Sigma_{i,s}} K^{\alpha} dg \geq -\frac{n\omega_n}{1+n\alpha} C
\]

for all \( i \geq i_0 = i_0(T) \) and \( s \in [0, T] \).

Let us choose \( t_0 := \frac{n\omega_n}{1+n\alpha} \frac{2}{2\alpha-1} \). We have \( J^{(i)}(s) \geq -\frac{2\alpha-1}{2} \epsilon \), for all \( t_0 \leq s \leq t \) and \( i \geq i_0 = i_0(T) \). The monotonicity formula (5.1) gives that \( \int_{\Sigma_{i,t}} P^{2}K^{\alpha} dg \), for all \( t > 0 \).

If there are \( t \geq t_0 \) and \( l' > i_0(T) \) \( (T > t + \tau \) will be determined later) such that \( \int_{t}^{t+\tau} \int_{\Sigma_{i',t'}} P^{2}K^{\alpha} dg ds > \epsilon \), then

\[
J^{(i')}(t+\tau) = J^{(i')}(t) + \int_{t}^{t+\tau} \partial_s J^{(i')}(s) ds \geq \frac{2\alpha-1}{2} \epsilon + (2\alpha-1) \int_{t}^{t+\tau} \int_{\Sigma_{i',t'}} P^{2}K^{\alpha} dg ds \geq \frac{2\alpha-1}{2} \epsilon.
\]

From (5.1), we have

\[
\partial_s J^{(i')}(s) \geq (2\alpha-1) \frac{[J^{(i')}(s)]^2}{\Sigma_{i',t'}} K^{\alpha} dg \geq \frac{2\alpha-1}{\omega_n} \sup_{\Sigma_{i',t'}} \frac{[J^{(i')}(s)]^2}{K^{\alpha-1}}.
\]

Under the assumption that \( K^{\alpha-1} \leq C \) and \( J^{(i')}(t+\tau) \geq \frac{2\alpha-1}{2} \epsilon \), this ODE inequality blows up before finite time \( T = T(\epsilon, \alpha, C, n, t+\tau) \). If we choose this \( T \) and then the argument shows there is no such \( l' > i_0 = i_0(T) \).

When \( \alpha = 1 \), we do not need Claim 5.1 and the previous proof shows the following slightly general version. This result will be used our subsequential research [14].

**Proposition 5.3.** For any \( \tau > 0 \) and \( \epsilon > 0 \), there is \( T(\tau, \epsilon, n) > 0 \) such that the following holds: if \( x_{n+1} = u(x,t) \) on \( (x,t) \in \bar{U} \times [-T,T] \) for some bounded \( \bar{U} \subset \mathbb{R}^n \) is a smooth graphical convex solution to the classical GCF (possibly incomplete) which is a smooth limit of (parts of) smooth strictly convex closed solutions, then

\[
\int_{-\tau}^{T} \int \{(x,u(x,s)) : x \in U\} P^{2}K dg ds \leq \epsilon.
\]

Next, we will show that the result of Proposition 5.2 also holds for \( \alpha \in (1/2,1) \). In this range of exponents we need to impose additional assumptions on the initial data \( \Sigma_0 \).

**Proposition 5.4.** For \( \alpha \in (1/2,1) \), suppose that \( \Sigma_0 \) satisfies the assumptions of Theorem 1.2 Then the conclusion of Proposition 5.3 holds.

**Proof.** By the assumptions, we have approximating compact hypersurfaces \( \Sigma_{i,0} \) such that \( N^{(i)}(0) \leq C \) and \( J^{(i)}(0) \geq -C \). Since \( (N^{(i)}(t)) \theta^{-\alpha} \) is concave in time (by Corollary A.3) and \( \partial_t N^{(i)}(t) = (\alpha - 1)J^{(i)}(t) \) (by Lemma A.1), we conclude that

\[
(N^{(i)}(t)) \theta^{-\alpha} \leq M + Mt
\]

for some \( M = M(C,\alpha) > 0 \). Since \( \frac{1}{\alpha} < 1 \), it follows that \( N^{(i)}(t) \leq (M + Mt)^{\frac{1}{\alpha}} \), that is the function \( N^{(i)}(t) \) has the sublinear growth rate.

By an argument similar to (5.3),

\[
J^{(i)}(t) \geq \frac{-\lambda}{1+n\alpha} \int_{\Sigma_{i,t}} K^{\alpha} dg \geq -\frac{n}{1+n\alpha} \frac{(M + Mt)^{\frac{1}{\alpha}}}{t}.
\]


Hence there is \( t_0 = t_0(n, \alpha, \epsilon, M) \) such that \( J^{(i)}(t) \geq -\frac{2\alpha - 1}{2} \epsilon \) for all \( t \geq t_0 \).

If there exist \( t \geq t_0 \) and \( i \) for which

\[
(5.6) \quad \int_{t}^{t+\tau} \int_{\Sigma_{t,s}} P^2 K^\alpha \, dg \, ds > \epsilon,
\]

then by the argument of (5.3) we obtain \( J^{(i)}(t + \tau) \geq \frac{2\alpha - 1}{2} \epsilon \).

From (5.5), we derive the following ODE inequality

\[
\frac{\partial}{\partial s} J^{(i)}(s) \geq (2\alpha - 1) \frac{[J^{(i)}(s)]^2}{(M + Ms)^{\frac{\alpha}{2}}},
\]

By the sublinear growth of the denominator, it can be checked that the ODE blows up in the finite time

\[
T = T(\epsilon, \alpha, M, t + \tau) \quad \text{if} \quad J^{(i)}(t + \tau) \geq \frac{2\alpha - 1}{2} \epsilon > 0.
\]

Therefore we have the opposite inequality of (5.6) if \( i \) is sufficiently large so that the maximum existence time \( T_i \) of \( \Sigma_{i,t} \) satisfies \( T_i \geq T \).

Next lemma shows that an \( \alpha \)-GCF satisfying \( P \equiv 0 \) is a translating soliton as like in the result by R. Hamilton \[20\] for the mean curvature flow.

**Lemma 5.5.** For a manifold \( M^n \), let \( F : M^n \times [-\epsilon, \epsilon] \to \mathbb{R}^{n+1} \) be a strictly convex smooth immersion which satisfies the \( \alpha \)-GCF and

\[
P = \frac{1}{\alpha K^\alpha} (\partial_t K^\alpha - b^ij \nabla_i K^\alpha \nabla_j K^\alpha) \equiv 0.
\]

Then \( F(M^n, 0) \) has to be a (possibly incomplete) translating soliton.

**Proof.** First, observe that for such a solution the evolution of \( P \) in (A.7) implies that \( P_{ij} \equiv 0. \) Let us define

\[
T := b^ij \nabla_i K^\alpha \nabla_j F + K^\alpha \nu.
\]

Then

\[
\nabla_m T = \nabla_m b^ij \nabla_i K^\alpha \nabla_j F + b^ij \nabla^2_{im} K^\alpha \nabla_j F + b^ij \nabla_i K^\alpha (\nu_{jm}) + \nabla_m K^\alpha \nu + K^\alpha h_{mj} \nabla^j F.
\]

Using

\[
0 = P_{im} = \nabla^2_{im} K^\alpha - b^kl \nabla_k h_{im} \nabla_l K^\alpha + K^\alpha h_{ik} h^m_k,
\]

we obtain

\[
\nabla_m T = -b^ik b^jl \nabla_m h_{kl} \nabla_i K^\alpha \nabla_j F + b^ij \nabla_j F (b^kl \nabla_k h_{im} \nabla_l K^\alpha - K^\alpha h_{ik} h^m_k) + K^\alpha h_{mj} \nabla^j F = 0.
\]

Namely, \( T \) is a constant vector. Note that \( \langle T, \nu \rangle = K^\alpha \) and this shows \( F(M^n, 0) \) is a translating soliton with a velocity \(-T\).

\( \square \)

**Proof of Theorem 1.1.** In view of Corollary 4.5 and the standard parabolic regularity theory, for any given \( \tau_1 \to \infty \), we may take a further subsequence (which we still denote by \( \tau_i \)) so that

\[
u(x', t + \tau_i) - \inf_{\Omega} u(x', \tau_i) \to u_\infty(x', t) \quad \text{in} \quad C^\infty_{\text{loc}}(\Omega \times (-\infty, \infty)).
\]

By Proposition 5.2 and Lemma 5.3, \( x_{n+1} = u_\infty(x', t) \) on \( \Omega \times (-\infty, \infty) \) has to be a (possibly incomplete) translating soliton. It suffices to show this is actually the unique translating soliton defined on \( \Omega \). i.e. \( u_\infty(x', 0) \equiv u_\Omega(x') \).

Let us denote \( u_{\infty, 0} := u_\infty(\cdot, 0) \), and the velocity of this possibly incomplete translating soliton by \( \lambda \epsilon_{n+1} \).

i.e.

\[
K^\alpha = \lambda \langle \nu, \epsilon_{n+1} \rangle \iff \left[ \frac{\det D^2 u_{\infty, 0}}{(1 + |Du_{\infty, 0}|^2)^{\frac{\alpha}{2}}} \right] = \lambda (1 + |Du_{\infty, 0}|^2)^{-1/2} \text{ on } \Omega.
\]
This implies
\[
\lambda^{1/\alpha} |\Omega| = \int_{\Omega} \frac{\det D^2 u_{\infty,0}}{\left(1 + |Du_{\infty,0}|^2\right)^{n+2+\frac{4}{\alpha}}} = \int_{Du_{\infty,0}(\Omega)} \frac{1}{\left(1 + |p|^2\right)^{n+2+\frac{4}{\alpha}}} \leq \int_{\mathbb{R}^n} \frac{1}{\left(1 + |p|^2\right)^{n+2+\frac{4}{\alpha}}} =: \Lambda(n, \alpha) < \infty \quad \text{provided} \quad \alpha > \frac{1}{2}.
\]

Note the equality holds if and only if \(|\Omega| = 0\), i.e., when \(u_{\infty,0} = u_{\Omega}\); (see the characterization of \(u_{\Omega}\) which is discussed after Definition 2.2).

Assume without loss of generality that \(\Omega\) contains the origin. Since we can apply the previous argument for every subsequence of the sequence \(\tau_i\), this implies
\[
\limsup_{s \to \infty} u_i(0, s) \leq \left(\frac{\Lambda(n, \alpha)}{|\Omega|}\right)^{\alpha} =: \lambda_\Omega.
\]

In view of the argument in the first paragraph, we can always find a converging subsequence. Thus it suffices to show
\[
\liminf_{s \to \infty} u_i(0, s) \geq \lambda_\Omega.
\]

On the contrary, suppose there is a sequence of time \(\tau_i \to \infty\) such that \(u_i(0, \tau_i) \leq \lambda_\Omega(1 - 8\delta)\) for some \(\delta > 0\). Due to Proposition 5.2, there is a small \(c > 0\) such that \(u_i(0, s) \leq \lambda_\Omega(1 - 4\delta)\) on \(s \in [(1 - c)\tau_i, \tau_i]\). By (5.8), for every fixed \(\epsilon > 0\), \(u_i(0, (1 - c)\tau_i) \leq (1 + \epsilon)\lambda_\Omega(1 - c)\tau_i + O(1)\) as \(i \to \infty\). Thus
\[
u_i(0, \tau_i) \leq u_i(0, (1 - c)\tau_i) + \lambda_\Omega(1 - 4\delta) c\tau_i \leq (1 + (\epsilon(1 - c) - 4\delta c))\lambda_\Omega \tau_i + O(1).
\]
Choosing \(\epsilon := \frac{2\delta}{1 - c}\), we get
\[
u_i(0, \tau_i) \leq (1 - 2\delta c)\lambda_\Omega \tau_i + O(1) \quad \text{as} \quad i \to \infty.
\]
i.e. the average speed evaluated along the sequence \(\tau_i\) is strictly less than \(\lambda_\Omega\).

On the other hand, by considering slightly slower translating soliton defined on larger domain
\[
u(x') := (1 - \delta c)^{-\frac{1}{\alpha}} u_i((1 - \delta c)\frac{t}{\alpha}) = u_i(x') \quad \text{for some} \quad x' \in \Omega' \quad \text{as} \quad t \to \infty.
\]
and use \(\nu(-t) - L\) for large \(L\) as an initial barrier as we did in the proof of Proposition 5.2, we obtain
\[
u(0, t) \geq (1 - \delta c) t - O(1) \quad \text{as} \quad t \to \infty.
\]
This is a contradiction and finishes the proof.

\[
\square
\]

\textbf{Proof of Theorem 1.2.} In this case, we assume that
\[
\mathcal{N}(0) := \int_{\Sigma_0} K^\alpha \, dg \leq C
\]
and also that
\[
\mathcal{J}(0) := \int_{\Sigma_t} P_{ij} b^{ij} K^\alpha \, dg \geq -C
\]
for some \(C < \infty\), with \(P_{ij} := \nabla_i^2 K^\alpha - b^{m[m} \nabla_{m} h_{ij} \nabla_{n} K^\alpha + K^\alpha h_{i}^{k} h_{k} j\). Note that for compact solutions Lemma 4.1 gives \(\partial_t \mathcal{N} = (\alpha - 1) \mathcal{J}\) and hence this assumption corresponds to giving upper bounds on \(\mathcal{N}\) and its first time derivative at \(t = 0\). Then the proof goes same as the proof of Theorem 1.1 except that we use Proposition 5.4 instead of Proposition 5.2. \(\square\)
Appendix A. Monotonicity Formula

Let $F : M^n \times [0, T] \to \mathbb{R}^{n+1}$ be a parametrization of a smooth strictly convex closed solution $\Sigma_t$ of the $\alpha$-GCF. We define the entropies

(A.1) \[ \mathcal{N}(t) := \int_{\Sigma_t} K^\alpha \, dg \]
and

(A.2) \[ \mathcal{J}(t) := \int_{\Sigma_t} P_{ij} b^{ij} K^\alpha \, dg \]
where

(A.3) \[ P_{ij} := \nabla^2_{ij} K^\alpha - b^{mn} \nabla_m h_{ij} \nabla_n K^\alpha + K^\alpha h^k_i h^j_k. \]

Here, $dg := \sqrt{\det \ g \, dx}$ is the intrinsic volume form inherited from the imbedding $F$. In this section we will summarize and prove certain entropy identities and inequalities which are used in this work.

Lemma A.1.

(A.4) \[ \frac{d}{dt} \mathcal{N}(t) = (\alpha - 1) \mathcal{J}(t). \]

Proof. By equation (2.15) and (2.10),

(A.5) \[ \frac{d}{dt} (K^\alpha \, dg) = \frac{d}{dt} (K^\alpha) \, dg + K^\alpha \frac{d}{dt} (K^\alpha) \, dg = \left( \alpha K^\alpha b^{ij} \nabla^2_{ij} K^\alpha + \alpha H K^{2\alpha} - H K^{2\alpha} \right) \, dg \]

Hence

\[ \frac{d}{dt} \mathcal{N} = \int_{\Sigma_t} \left( \alpha b^{ij} \nabla^2_{ij} K^\alpha - (\alpha - 1) H K^\alpha \right) K^\alpha \, dg. \]

Using the following integration by parts

\[ \int K^\alpha b^{ij} \nabla^2_{ij} K^\alpha \, dg = \int K^{\alpha - 1} b^{ij} \nabla^2_{ij} K^\alpha \, dg \]
\[ = - \int b^{ij} K \nabla_i K^{\alpha - 1} \nabla_j K^\alpha \, dg \quad \text{(by eq (2.7))}, \]
\[ = - \int \frac{\alpha - 1}{\alpha} b^{ij} \nabla_i K^\alpha \nabla_j K^\alpha \, dg \]

we conclude the desired identity

\[ \frac{d}{dt} \mathcal{N}(t) = (\alpha - 1) \int_{\Sigma_t} \left( b^{ij} \nabla^2_{ij} K^\alpha - \frac{b^{ij}}{\alpha K^\alpha} \nabla_i K^\alpha \nabla_j K^\alpha + H K^\alpha \right) K^\alpha \, dg \]
\[ = (\alpha - 1) \mathcal{J}(t). \]

\[ \square \]

Theorem A.2.

\[ \frac{d}{dt} \mathcal{J}(t) = \int_{\Sigma_t} \left( b^{ik} b^{ij} P_{ij} + (2\alpha - 1)(b^{ij} P_{ij})^2 \right) K^\alpha \, dg. \]

Remark A.3. Note that

(A.6) \[ b^{ij} P_{ij} = b^{ij} \nabla^2_{ij} K^\alpha - \frac{b^{ij}}{\alpha K^\alpha} \nabla_i K^\alpha \nabla_j K^\alpha + K^\alpha H \]

which follows from (2.6).
Proof of Theorem. The evolution of $b^{kl} P_{kl}$, shown in Theorem 3.2 [20], is given by

\begin{equation}
\frac{d}{dt} (b^{kl} P_{kl}) = \alpha K^{\alpha} b^{ij} \nabla_{ij}^2 (b^{kl} P_{kl}) + 2ab^{ij} \nabla_i K^{\alpha} \nabla_j (b^{kl} P_{kl}) + b^{ik} b^{jl} P_{ij} P_{kl} + \alpha (b^{kl} P_{kl})^2.
\end{equation}

By (A.7) and (A.5), we obtain

\begin{equation}
\frac{d}{dt} \mathcal{J} = \int \frac{d}{dt} (b^{kl} P_{kl}) K^\alpha 
\end{equation}

where

\begin{equation}
I := \int \left( \alpha K^{\alpha} b^{ij} \nabla_{ij}^2 (b^{kl} P_{kl}) + 2ab^{ij} \nabla_i K^{\alpha} \nabla_j (b^{kl} P_{kl}) \right.
\end{equation}

To finish the proof of the theorem it suffices to show that $I = (\alpha - 1) \int (b^{kl} P_{kl})^2 K^\alpha \, dg$. Note that for any two functions $F$ and $G$ we have the following integration by parts formula:

\begin{equation}
\int (\nabla_{ij}^2 G) (b^{ij} F K^\alpha) \, dg = - \int \nabla_j G \nabla_i (b^{ij} F K^\alpha) \, dg
\end{equation}

Applying formula (A.10) with $F := \alpha K^{\alpha}$ and $G := b^{kl} P_{kl}$ we obtain

\begin{equation}
\int \nabla_{ij}^2 (b^{kl} P_{kl}) (\alpha K^{\alpha} b^{ij}) K^\alpha \, dg = (-2 + 1) \int b^{ij} \nabla_i K^{\alpha} \nabla_j (b^{kl} P_{kl}) K^\alpha \, dg.
\end{equation}

Hence,

\begin{equation}
\int (\alpha K^{\alpha} b^{ij} \nabla_{ij}^2 (b^{kl} P_{kl}) + 2ab^{ij} \nabla_i K^{\alpha} \nabla_j (b^{kl} P_{kl})) K^\alpha \, dg
\end{equation}

Plugging this into (A.9) yields

\begin{equation}
I = \int \left( \alpha K^{\alpha} b^{ij} \nabla_{ij}^2 (b^{kl} P_{kl}) + 2ab^{ij} \nabla_i K^{\alpha} \nabla_j (b^{kl} P_{kl}) \right.
\end{equation}

This finishes the proof of the theorem. \qed
Corollary A.4. For $\alpha \geq \frac{n-1}{2n}$, we have

$$\frac{d}{dt} \int (b^{ij} P_{ij}) K^\alpha \, dg \geq \left( \frac{1}{n} + 2\alpha - 1 \right) \int (b^{ij} P_{ij})^2 K^\alpha \, dg \geq \left( \frac{1}{n} + 2\alpha - 1 \right) \frac{\left( \int (b^{ij} P_{ij}) K^\alpha \, dg \right)^2}{\int K^\alpha \, dg} \geq 0.$$ 

Proof. The $\alpha = 1$ case is proven in Lemma 4.3 [20]. In the more general case, the result readily follows by the previous Theorem, the inequality

$$b^{ik} b^{jl} P_{ij} P_{kl} \geq \frac{1}{n} (b^{ij} P_{ij})^2$$

and the Hölder inequality. 

Corollary A.5. For $\alpha > 0$ with $\alpha \neq 1$, we have

$$\frac{d^2}{dt^2} N^{\frac{n-\alpha}{n}} \leq 0.$$ 

Proof. Recall that \( \frac{d}{dt} N = (\alpha - 1) \int (b^{ij} P_{ij}) K^\alpha \, dg \), by Lemma [A.1]. Hence

$$\frac{d^2}{dt^2} N^{\frac{n-\alpha}{n}} = \frac{d}{dt} \left( \frac{\alpha}{\alpha - 1} N \int N^{\frac{2\alpha - 1}{\alpha - 1}} \right)$$

$$= \frac{d}{dt} \left( -\alpha N^{\frac{2\alpha - 1}{\alpha - 1}} \right)$$

$$= -\alpha \left( \frac{d}{dt} J N^{\frac{2\alpha - 1}{\alpha - 1}} - (2\alpha - 1) J^2 N^{\frac{3\alpha - 2}{\alpha - 2}} \right)$$

$$\leq 0 \quad \text{by Corollary A.4.}$$

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