A dynamical model of Sayano-Shushenskaya hydropower plant: stability, oscillations, and accident

Dynamical model of Sayano-Shushenskaya hydropower plant

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Abstract    This work is devoted to the construction and study of a mathematical model of hydropower unit, consisting of synchronous generator, hydraulic turbine, and speed governor. It is motivated by the accident happened on the Sayano-Shushenskaya hydropower plant in 2009 year. Parameters of the Sayano-Shushenskaya hydropower plant were used for modeling the system. Oscillations in zones, which were not recommended for operation, were found. The obtained results are consistent with the full-scale test results carried out for hydropower units of the Sayano-Shushenskaya hydropower plant in 1988.

Keywords    Sayano-Shushenskaya hydropower plant, hydropower unit, synchronous generator, hydraulic turbine, speed governor, oscillations

1 Introduction

Nowadays one of the most important source of electricity is the electric energy produced by hydroelectric facilities. According to [11], Norway gets 99% of its electric power from water, Brazil 84%, Austria 59%, Canada 58% and Russia 18%. Failures of hydropower plants cause loss of lives, major damage in the surrounding area, and serious economic consequences. In recent years, accidents at the hydropower plants have become frequent (see, e.g., Biudron Hydroelectric Power Station (Switzerland, 2000), Taum Sauk Hydroelectric Power Station (Missouri, USA, 2005), Sayano–Shushenskaya Dam (Russia, 2009), Itaipu Dam (Brazil, 2009), Srisailam Dam (India, 2013), Dhauliganga hydro electric station (India, 2013)). In order to prevent such accidents, it is necessary to investigate their causes.

This work is motivated by the accident happened on the Sayano-Shushenskaya hydropower plant in 2009 year. According to the act of special commission of the Russian Federal Environmental, Industrial and Nuclear Supervision Service, immediately before the accident the power of the second hydropower unit was 475 MW at a head of 212 meters [36], i.e., it worked in the not recommended zone II (Fig. 1). Zone II of hydropower unit work is characterized by strong hydraulic turbine blows in flowing part and vibrations. For the normal operation it is recommended power range, corresponding to the zone III, in which the efficiency of turbines has a maximum value. Also operation is allowed in the zone I, in which the dynamics is allowed, but the level of efficiency of the turbines are low. Operation in zone IV is not allowed. These work zones of hydropower unit of the Sayano-Shushenskaya hydropower plant were obtained by the full-scale test of hydropower units in the late 80s of 20th century and were published in the technical report “Full-scale testing of turbines of Sayano-Shushenskaya hydropower plant with standard runner” No. 1008 [10].

It is known [36], that the accident was caused by vibrations in the hydropower unit. Various possible reasons of occurrence of such vibrations were discussed in [36,15]. However main attention in these works was given to the dynamical processes in the turbines and the full mathematical model turbine–speed governor–synchronous generator was not considered.

In this paper an adequate mathematical model turbine–speed governor–synchronous generator is suggested. This model joins together differential equations of turbine (Pervozvanskiy’s model) [35], differential equations of
synchronous generator (Park-Gorev model) [15], and differential equations of speed governor [33]. Parameters of the model are taken from [13]. As a result of using numerical methods the periodic solutions of differential equations of the model were found during the analysis of transient processes. These solutions correspond to hydropower unit vibrations observed for some operation modes at the Sayano-Shushenskaya hydropower plant.

Remark, that we considered also some other more simple models of hydropower unit (based on more simple models of turbines (see [33]), synchronous generators (the Tricomi equation [40,25], the equations of synchronous generator with parallel connection in feed system [25]). However, for these models the above effects have not been found.

2 Components of hydropower unit

Following [30], let us consider the main constructive elements of hydropower unit. The hydropower unit in Fig. 2 consists of synchronous generator and radial-axial hydraulic turbine. In this work a nonregulated synchronous generator is considered. The rotor of generator and the runner are connected together by a rigid shaft.

The dam creates a difference in water level between the upper reservoir and lower reservoir. The flow of water is delivered from upper reservoir to turbine by penstock through the spiral casing. Water jets impact on the blades of the turbine producing torque applied to the rotating shaft. Since the turbine shaft is rigidly connected with the generator rotor, the rotor starts to rotate and to produce electricity, which is transferred to the grid. The water flow is controlled by means of guide vanes.

For the safety of the power network, the frequency should remain almost constant. This is reached by keeping the same speed of the synchronous generator. The rotational speed is controlled by the speed governor.

The main structural elements of hydropower unit are presented in Fig. 3. Introduce the following notations: \( s \) is a signal of angular (rotational) speed deviation, \( \mu \) is a position of guide vanes, \( M_T \) is a turbine torque, \( M_G \) is a generator torque, \( U \) is a voltage required by the power network. The power network represents a set of energy suppliers and consumers.

Thus, the hydropower unit represents a system consisting of hydraulic turbine, which produces a mechanical torque, generator, which converts mechanical energy into electrical energy, and automatic speed governor, which regulates the rotation speed of hydraulic turbine.

In order to develop a mathematical model of hydropower unit it is necessary to describe each structural element of hydropower unit, presented in Fig. 3. Let us consider each elements in details.

3 Mathematical model of synchronous generator

The main constructive elements of synchronous generator are stationary stator (Fig. 3 a) and rotating rotor (Fig. 3 b). The windings are placed in the stator and rotor slots. Stator windings are arranged in such a way
that in the case of alternate current they generates a rotating magnetic field.

Using Park’s transformation, the three-phase windings of stator can be substituted by two equivalent short-circuited windings, and the rotor can be described as two equivalent short-circuited damper windings and one field winding. Let us choose the rotating coordinate system \((d, q)\), connected with the rotor.

\[
\begin{align*}
\dot{\psi}_d &= -\omega_0 (1 + s) \psi_q - \omega_r i_d - \omega_0 u_d, \\
\dot{\psi}_q &= \omega_0 (1 + s) \psi_d - \omega_r i_q - \omega_0 u_q, \\
\dot{\psi}_r &= \frac{1}{T_r} (E_r - E_q), \\
\dot{\psi}_{rd} &= -\frac{1}{T_{rd}} E_{rq}, \\
\dot{\psi}_{rq} &= \frac{1}{T_{rq}} E_{rd},
\end{align*}
\]

where flux linkages are determined as follows

\[
\begin{align*}
\psi_d &= x_d i_d + E_q + E_{rq}, \\
\psi_q &= x_q i_q - E_{rd}, \\
\psi_r &= \frac{x^2}{x_r} x_d i_d + \frac{x_{ad}}{x_r} E_{rq}, \\
\psi_{rd} &= \frac{x^2}{x_{rd}} x_d i_d + E_q + \frac{x_{ad}}{x_{rd}} (E_q + E_{rq}), \\
\psi_{rq} &= \frac{x^2}{x_{rq}} x_q i_q - E_{rd} - \frac{x_{aq}}{x_{rq}} E_{rd}.
\end{align*}
\]

Here the following variables and coefficients are relative to the corresponding base values (voltage, current, flux linkage, impedance, inductance, power): \(r\) is a stator resistance, \(i_d, i_q\) are currents in stator windings, \(u_d, u_q\) are stator voltages, \(\psi_d, \psi_q, \psi_r, \psi_{rd}, \psi_{rq}\) are flux linkages, \(E_r\) is a field voltage, \(E_q, E_{rd}, E_{rq}\) are electromotive forces, induced in the stator by the magnetic field of rotor winding currents for synchronous rotor speed, \(x_d, x_q\) are synchronous inductances (reactances) along the axes \(d\) and \(q\), \(x_r, x_{rd}, x_{rq}\) are impedances of field winding, damper windings along the axes \(d\) and \(q\). The following coefficients are time constants: \(T_r\) is a field-winding time constant with open stator and damper windings \([s]\), \(T_{rd}, T_{rq}\) are damper winding time constants with open stator and field windings \([s]\).
The stator voltages $u_d$ and $u_q$ along the d- and q-axes in per unit values are determined according to the laws

$$u_d = -U \sin(\theta_0 + \theta_\Delta), \quad u_q = U \cos(\theta_0 + \theta_\Delta),$$

where $U = \frac{\dot{V}}{U_b}$ is a voltage relative to the base voltage [\text{p.u.}], $\dot{V}$ is a voltage in power network [V], $U_b$ is the base voltage [V].

The motion of synchronous generator rotor about shaft is described by the torque equation in physical values [11] pp.132-149], [5] pp.5.15-5.17:

$$\dot{\omega} = \frac{J}{M} \left( \theta_\Delta - \left( \hat{\Psi}_d \hat{i}_q - \hat{\Psi}_q \hat{i}_d \right) \right),$$

(3)

Here $J$ is a moment of inertia of rotor (is a moment of inertia of hydropower unit) [kg \cdot m$^2$], $\hat{\Psi}_d$, $\hat{\Psi}_q$ are flux linkages in physical unit values [Wb = $\frac{kg \cdot m^2}{\sqrt{2}}$], $\hat{i}_d$, $\hat{i}_q$ are currents in stator windings in physical unit values [A], $M$ is a torque relation [\text{kg} \cdot \text{m}]. Since the shaft of turbine is rigidly connected with the rotor of generator, a moment of inertia of rotor coincides with a moment of inertia of hydropower unit and an angular rotor speed coincides with an angular turbine speed. In our case the rotation torque $M_T$ is a turbine torque, which is created by the pressure of water on the runner.

Rewrite system of equations (3) in terms of base variables. For this purpose the second equation is divided twice by the base voltage:

$$U_b = \omega_0 \Psi_b = Z_b I_b,$$

where $\Psi_b$ is the base flux linkage [Wb = $\frac{kg \cdot m^2}{\sqrt{2}}$], $Z_b$ is the base impedance (valid also for resistances and reactances)[Ohms], $I_b$ is the base current [A]. Then

$$\frac{J}{\Psi_b Z_b I_b} \left( \frac{\dot{\omega}}{\omega_0} \right) = \frac{M_T}{\omega_0 \Psi_b Z_b I_b} - \frac{1}{\omega_0 Z_b} \left( \Psi_d \hat{i}_q - \hat{\Psi}_d \hat{i}_d \right),$$

$$\frac{J \omega_0^2}{U_b I_b} \dot{s} = \frac{M_T}{\Psi_b I_b} - \left( \hat{\Psi}_d i_q - \hat{\Psi}_q i_d \right).$$

Thus, as a result of given transformation one obtains system of equations (3) in per unit values:

$$\dot{\theta}_\Delta = \omega_0 s,$$

$$\dot{s} = \frac{1}{T_J} \left( \frac{M_T}{\Psi_b I_b} - \left( \hat{\Psi}_d i_q - \hat{\Psi}_q i_d \right) \right),$$

(4)

where $T_J = J \frac{\omega_0}{\Psi_b I_b}$ is inertial constant of hydropower unit [s], $S_b = U_b I_b$ is the base power [W = $\frac{kg \cdot m^2}{\sqrt{2}}$].

4 Mathematical model of hydraulic turbine

Hydraulic turbines can be classified by their type of construction, the most important ones being the Francis, Pelton and Kaplan or Propeller turbines [11]. They are distinguished by construction of runner and control methods of the rotation speed of the turbine. However, regardless of the turbine structure, the rotation equation of turbine shaft in earth-fixed coordinate system has the following form [17,23]

$$J \dot{\omega} = M_T - M_G,$$

where $M_G$ is a resistance torque, which is created by the generator.

The rotational torque is defined by the relation

$$M_T = \frac{3}{\omega(t)} P_T(t),$$

where $P_T$ is the turbine mechanical power [W = $\frac{kg \cdot m^2}{\sqrt{2}}$], $\frac{3}{\omega(t)}$ is a constant depending on the construction of turbine [\text{p.u.}].

By virtue of the laws of hydraulics, the turbine power $P_T$ is calculated by the formula:

$$P_T(t) = h(t) Q(t),$$

where $h$ is a pressure drop [Pa = $\frac{kg}{m^2}$], $Q$ is the water flow through the turbine [$\frac{m^3}{s}$].

The water flow through the turbine is related with $h$ and the opening (closing) guide vanes in the following way:

$$Q(t) = C \mu(t) \sqrt{h(t)},$$

$$\mu(t) = \mu_0 + \mu_\Delta(t),$$

where $\mu$ is a position of guide vanes [\text{p.u.}], $\mu_0$ is a given position of guide vanes [\text{p.u.}], $\mu_\Delta$ is the deviation of given position of guide vanes [\text{p.u.}], $C = S/\sqrt{\rho}$ is a constant depending on the construction of penstock [$\frac{m^3}{s \cdot \sqrt{\rho}}$], $S$ is a sectional area of water conduit [m$^2$], $\rho$ is the density of water [$\frac{kg}{m^3}$].

The differential equation, describing the change of water flow through the turbine, can be written according to the Newton equation for the liquid column enclosed in the penstock. Then, taking into account that $S$ is a sectional area of penstock [m$^2$], $l$ is a length of penstock [m], $\rho$ is a density of water [$\frac{kg}{m^3}$], $p_u$ is a constant pressure on the upper end of penstock [Pa], $p_l$ is a constant pressure on the lower end of penstock (after turbine) [Pa], one obtains

$$\dot{Q} = \frac{S}{l \rho} (p_u - p_l - h(t)).$$
Thus, the dynamics of hydraulic turbine is described by the system of differential equations
\[
\begin{align*}
J \ddot{\omega} &= \frac{k}{C^2(\mu_0 + \mu_z)^2} \sigma^{3/2} \omega - M_G, \\
\dot{Q} &= \frac{S}{\lambda_p} \left( p_i - p_t - \frac{Q^2}{C^2(\mu_0 + \mu_z)^2} \right),
\end{align*}
\]
where \( M_G \) is a generator torque in the physical values.

In order to pass to the rotating coordinates \((d,q)\), it is necessary to make the following transformation
\[
\omega(t) = \omega(t) - \omega_0.
\]

The equation of servomotor motion is as follows
\[
\dot{s} = \frac{1}{T_f} \left( \frac{k}{C^2(\mu_0 + \mu_z)^2} \sigma^{3/2} \omega_0 (1 + s) - M_G \right),
\]
where
\[
k = \frac{k}{\psi_d I_f}
\]
and in Section 3 there was defined the generator torque in per unit values
\[
\tilde{M}_G = \psi_d i_q - \psi_q i_d.
\]

Note that the obtained system contains one control signal \( \mu_{\Delta} \), which corresponds to automatic speed governor of turbine.

5 Mathematical model of speed governor

Let us consider the simplified scheme of automatic speed governor of turbine \([33, \text{pp.153-157}]\), presented in Fig. 6. In this work parameters of hydropower unit are chosen so that the speed governor does not get to saturation. To simulate the case of saturation it is necessary to consider more complex models of speed governors (see, e.g., \([6,31,37,38,39]\)).

The governor has the deadband \( z \), which is specified by the technical conditions. The deadband of the hydraulic turbine is \( 30 \text{ mHz} \) \([33]\). The deadband in the per unit values is \( z = 0.002 \).

The input signal is a relative deviation of rated angular speed \( s \). After the input signal passes through the deadband, one obtains the signal \( \eta_s \), which corresponds to a signal of measuring device:
\[
\eta_s = \sigma \chi_s(s) = \begin{cases} \sigma (s - z/2), & s \geq z/2, \\ \sigma (s + z/2), & s \leq -z/2, \\ 0, & |s| < z/2, \end{cases}
\]

where \( \sigma \) is a transmission coefficient of open-cycle control system.

The control signal \( \eta \) is formed by the formula
\[
\eta = -\eta_s - \tilde{\mu}_\Delta,
\]
where \( \tilde{\mu}_\Delta \) is a signal of rigid negative feedback.

Then the control signal is cut off because of restriction on the velocity of change of guide vanes position:
\[
\rho = \chi_\rho(\eta) = \begin{cases} \eta, & \rho_0 \leq \eta \leq \rho_c, \\ \rho_0, & \eta < \rho_0, \\ \rho_c, & \rho_0 < \eta, \end{cases}
\]
where \( \rho_0, \rho_c \) are the maximum velocities of opening and closing the vanes.

A servomotor is presented by the integrator with time constant \( T_c \) [s]. The output value of servomotor is the relative displacement of guide vanes \( \tilde{\mu}_\Delta \). The stroke of servomotor has limit stops \( \mu_{\min}, \mu_{\max} \), which correspond to the minimum and maximum power of the turbine \([33]\). If guide vanes reach limit stops, at which further displacement of vanes in the same direction is not possible, then displacement of vanes is stopped. In some research works it is recommended to model such saturation through direct feedback \([9,35,39,33]\).

Consequently, the control signal takes the following form
\[
\xi = \rho - \xi_\rho(\mu_\Delta, \mu_0),
\]
where
\[
\xi_\rho(\mu_\Delta, \mu_0) = \begin{cases} \rho, & \mu_\Delta + \mu_0 < \mu_{\min} \text{ and } \rho < 0, \\ \rho, & \mu_\Delta + \mu_0 > \mu_{\max} \text{ and } \rho > 0, \\ 0, & \text{otherwise}. \end{cases}
\]

The equation of servomotor motion is as follows
\[
\dot{\tilde{\mu}}_\Delta = \frac{\rho - \xi_\rho(\mu_\Delta, \mu_0)}{T_c},
\]
where
\[
\tilde{\mu}_\Delta = \chi_\mu(\mu_\Delta, \mu_0) = \begin{cases} \mu_\Delta, & \mu_{\min} - \mu_0 \leq \mu_\Delta \leq \mu_{\max} - \mu_0, \\ \mu_{\min} - \mu_0, & \mu_\Delta < \mu_{\min}, \\ \mu_{\max} - \mu_0, & \mu_\Delta > \mu_{\max} - \mu_0. \end{cases}
\]
In other words, the signal $\mu_\Delta$ is the signal $\mu_\Delta$, passed through the saturation. Consequently, equation (6) can be rewritten in the form

$$\dot{\mu}_\Delta = \frac{\rho - \xi_\rho(\mu_\Delta, \mu_0)}{T_c},$$

$$\mu_{\min} - \mu_0 \leq \mu_\Delta \leq \mu_{\max} - \mu_0.$$  

Thus, the automatic speed governor of the turbine can be described by the following differential equation

$$\dot{\mu}_\Delta = \frac{\chi_\rho(-\sigma \chi_\sigma(s) - \chi_\mu(\mu_\Delta, \mu_0) - \xi_\rho(\mu_\Delta, \mu_0))}{T_c}.$$  

6 Mathematical model of hydropower unit. 

Operating modes

Using equations of each structural element of hydropower unit, one writes the equations of hydropower unit with automatic speed governor in per unit values:

$$\dot{\theta}_\Delta = \omega_0 s,$$

$$\dot{s} = \frac{k}{T_f} \left( \frac{Q^2}{C^2(\mu_0 + \mu_\Delta)^2} \omega_0^2 (1 + s) - \Psi_q i_q + \Psi_i q_i d \right),$$

$$\dot{Q} = \frac{S}{I_p} \left( (p_u - p_t) - \frac{Q^2}{C^2(\mu_0 + \mu_\Delta)^2} \right),$$

$$\dot{\Psi}_d = -\omega_0 (1 + s) \Psi_q - \omega_0 r i_d + \omega_0 U \sin(\theta_0 + \theta_\Delta),$$

$$\dot{\Psi}_q = \omega_0 (1 + s) \Psi_d - \omega_0 r i_q - \omega_0 U \cos(\theta_0 + \theta_\Delta),$$

$$\dot{T}_r \dot{\Psi}_E = E_r - E_q,$$

$$T_{rd} \dot{\Psi}_i d = -E_{rq},$$

$$T_{rq} \dot{\Psi}_r q = E_{rd},$$

$$\dot{\mu}_\Delta = \frac{\chi_\rho(-\sigma \chi_\sigma(s) - \chi_\mu(\mu_\Delta, \mu_0) - \xi_\rho(\mu_\Delta, \mu_0))}{T_c},$$

$$\dot{\psi}_d = x_{rd} i_d + E_q + E_{rq},$$

$$\dot{\psi}_q = x_q i_q - E_{rd} + E_{rq},$$

$$\dot{\psi}_r = x_{rd} i_d + x_r E_q + x_{rd} \dot{E}_r,$$

$$\dot{\psi}_{rd} = x_{rd} i_d + x_r \dot{E}_q + x_{rd} \dot{E}_r + (E_q + E_{rq}),$$

$$\dot{\psi}_{rq} = x_{r}^2 i_q - E_{rd} - x_{r} \dot{E}_q.$$  

Dynamical stability of hydropower units is considered in terms of maintaining a given mode. This means that if sudden, significant changes of network mode arises, then after the transient processes the output power of hydropower unit must correspond to the required power. For example, after short circuits in one or more power lines, blackouts, changes of the external load, etc. Note that the hydropower unit in the considered processes can not be represented as a source of current or electromotive force since dynamical processes in this case have a significant effect both on a hydropower unit, and on the network mode.

An operating mode of hydropower unit corresponds to the asymptotically stable equilibrium point of system (7). The equilibrium points are the following

$$s^* = 0 \quad \text{(i.e.} \quad \omega_\Delta^* = 0), \quad \mu^* = 0,$$

$$\theta^*_\Delta = \text{const}, \quad Q^* = \frac{C \mu_0 \sqrt{p_u - p_t}}{\omega_0},$$

$$\rho^* = 0, \quad E^*_{rq} = 0, \quad E^*_{rd} = 0,$$

$$i^*_d = -\frac{x_q}{r^2 + x_d x_q}(-\frac{r}{x_q} U \sin \theta - U \cos \theta + E_r),$$

$$i^*_q = -\frac{r}{r^2 + x_d x_q}(-\frac{x_d}{x_q} U \sin \theta - U \cos \theta - E_r),$$

$$\psi^*_d = x_q i_d + E_r, \quad \psi^*_q = x_{rd} i_q, \quad \psi^*_r = x_{rd} i_q.$$  

The position of guide vanes $\mu_0$ is defined from the following equation, which will be called the balance equation between the turbine and the generator torques:

$$M_T(\mu_0, \omega_0) = M_G(U, \theta_0),$$

where

$$M_T(\mu_0, \omega_0) = \frac{k C (p_u - p_t)^\frac{1}{2}}{\omega_0^3} \mu_0,$$

$$M_G(U, \theta) = \frac{r (x_d - x_q)}{r^2 + x_d x_q} (-U^2 x_d \sin^2 \theta + U^2 x_q \cos^2 \theta - \frac{r^2 - x_d x_q}{r} U^2 \sin \theta \cos \theta - \frac{r^2}{x_d x_q} U^2 \sin \theta \cos \theta - \frac{r x_d x_q}{r} U \sin \theta + 2 x_q E_r U \cos \theta - \frac{E_r}{r} (-x_d U \sin \theta + r U \cos \theta) + \frac{r x_q E_r (x_d - x_q)}{(r^2 + x_d x_q)^2} - \frac{r E_r^2}{(r^2 + x_d x_q)^2}) \frac{1}{r^2 + x_d x_q}.$$  

Graphical solutions of the balance equation depending on voltage change in the power network are presented in Fig. 7. The parameter corresponding to the voltage is represented as $U = \gamma U_{nom}$, where $\gamma > 0$.

Equation (8) contains the input parameter $U$ and the variable $\mu_0$. The equality of turbine and generator torques is attained due to the variable $\mu_0$. Recall that $\omega_0 = 14.954 \text{ rad/s}, \theta_0 = \arccos(0.9)$. Then $\mu_0$, depending on the voltage $U$, is found from the balance
The instantaneous power is determined by the formula
\[ P(U, \theta(t)) = -\frac{3}{2} (i_d(t)U \sin(\theta(t)) + i_q(t)U \cos(\theta(t))) . \]
Represent the instantaneous power in the following form
\[ P(U, \theta(t)) = P_0 + P_\Delta(U, \theta(t)), \]
where \( P_0 \) is the required (nominal) power, which is determined by the formula \( P_0 = P(U, \theta_0) \), \( P_\Delta \) is a deviation of the required power.

The change of \( U \) leads to the change of power. A plot of \( P \) against \( U \) is shown in Fig. 10.

7 Calculation of generator and turbine parameters for the Sayano-Shushenskaya hydropower plant

The radial-axial vertical hydraulic turbines RO-230/833-B-677, connected with synchronous generator on the umbrella type SVF-1285/275-42 UHL4, are installed at the Sayano-Shushenskaya hydropower plant.

The rest of the system parameters are determined as follows:
Thus, for modeling the parameters of hydropower unit of the Sayano-Shushenskaya hydropower plant were used (13):

\[
\omega_0 = 2\pi 142.8/60 \text{ [rad/s]}, \quad r_0 = 0.0034 \text{ [p.u.],}
\]

Thus, for modeling the parameters of hydropower unit of the Sayano-Shushenskaya hydropower plant were used (13):

\[
\omega_0 = 2\pi 142.8/60 \text{ [rad/s]}, \quad r_0 = 0.0034 \text{ [p.u.],}
\]

\[
x_d = 1.58 \text{ [p.u.], } x_q = 0.97 \text{ [p.u.], } T_r = 8.21 \text{ [s], } J = 25.5 \cdot 10^6 \text{ [kg} \cdot \text{m}^2], \quad E_r = 530 \text{ [p.u.], } C = 0.27 \text{ [m}^3/\text{m}^3/\sqrt{\text{kg}]},
\]

\[
x_{ad} = 1.396 \text{ [p.u.], } x_{aq} = 0.786 \text{ [p.u.], } x_r = 1.6946 \text{ [p.u.],}
\]

\[
x_{rd} = 1.6155 \text{ [p.u.], } x_{rq} = 0.9361 \text{ [p.u.], } T_{rd} = 0.8666 \text{ [s], } T_{rq} = 0.7604 \text{ [s], } S = \pi/4 \cdot 7.5^2 \text{ [m}^2], \quad l = 192 \text{ [m],}
\]

\[
\rho = 0.98 \cdot 10^3 \text{[kg/m}^3], \quad p_u = 2.7 \cdot 10^6 \text{ [Pa], } p_l = 0.35 \cdot 10^6 \text{ [Pa], } k = 40 \text{ [kg} \cdot \text{m}^2/\text{s}^2], \quad Q_{\text{max}} = 358 \text{ [m}^3/\text{s}.]
\]

8 Local analysis

Let us study the local stability of equilibrium points of system (7). It is enough to carry out the analysis of stability on the interval [0, 2\pi] since the solutions of the system are 2\pi-periodic. The equilibrium points with respect to \( \theta \) are defined from balance equation (8).

System (7) may have 0, 1, 2, 3 or 4 equilibrium points on the interval (0, 2\pi).

Let us find the Jacobian matrix of the right-hand side of system (7). For this purpose algebraic system of equations (2) is solved for \( i_d, i_q, E_q, E_r, E_{rd}, E_{rq} \):

\[
i_d = X_d \psi_d - X_r \psi_r + X_{rd} \psi_{rd}, \quad i_q = Y_q \psi_q - Y_{rq} \psi_{rq},
\]

\[
E_q = Z_d \psi_d + Z_r \psi_r - Z_{rd} \psi_{rd}, \quad E_{rd} = P_q \psi_q - P_{rq} \psi_{rq},
\]

\[
E_{rq} = -Q_d \psi_d + Q_r \psi_r + Q_{rd} \psi_{rd},
\]

where

\[
X_d = a_4 a_6 - 1/b_1, \quad X_r = a_6 - 1/b_1, \quad X_{rd} = 1 - a_4/b_1,
\]

\[
Y_q = 1/a_2 - a_7, \quad Y_{rq} = 1/a_2 - a_7,
\]

\[
Z_d = (b_1 - (a_1 - a_3) (a_4 a_6 - 1))/b_1 (1 - a_6), \quad Z_r = a_5 - a_1/b_1,
\]

\[
Z_{rd} = b_1 + (a_1 - a_3) (1 - a_4)/b_1 (1 - a_6),
\]

\[
P_q = a_7,a_2 - a_7, \quad P_{rq} = a_2/a_2 - a_7,
\]

\[
Q_d = (a_1 - a_3) (a_4 a_6 - 1) - b_1/b_1 (1 - a_4),
\]

\[
Q_r = (a_1 - a_3) (a_4 a_6 - 1) - b_1/b_1 (1 - a_4), \quad Q_{rd} = a_3 - a_1/b_1,
\]

\[
a_1 = x_d, \quad a_2 = x_q, \quad a_3 = x_d^2/x_r, \quad a_4 = x_{ad}/x_r,
\]

\[
a_5 = x_{ad}/x_{rd}, \quad a_6 = x_{ad}/x_{rd}, \quad a_7 = x_{ad}/x_{rd},
\]

\[
b_1 = (a_1 - a_3) (a_4 a_6 - 1) - (a_1 a_3 - a_3) (1 - a_6).
\]

Then nonzero elements of the Jacobi matrix

\[
J = \{j_i,k\}_{k=1...9}
\]
of the right-hand side of system (7) in stationary point are defined by the formulas:

\[ j_{1,2} = \omega_0, \quad j_{2,2} = \frac{-k (Q^{st})^3}{T_J C_2^2 \omega_0^2 \mu_0^2}, \quad j_{2,3} = \frac{3 k (Q^{st})^2}{T_J C_2^2 \omega_0^3 \mu_0^3}, \]

\[ j_{2,4} = \frac{(X_d - Y_q) \Psi_d + Y_{rq} \Psi_{rq}}{T_J}, \]

\[ j_{2,5} = -Y_q \Psi_d + X_d \Psi_d - X_r \Psi_r + X_{rd} \Psi_{rd}, \]

\[ j_{2,6} = -X_r \Psi_d, \quad j_{2,7} = \frac{X_{rd} \Psi_d}{T_J}, \quad j_{2,8} = \frac{Y_{rq} \Psi_d}{T_J}, \]

\[ j_{2,9} = \frac{2 k (Q^{st})^3}{T_J C_2^2 \mu_0^2 \omega_0^2}, \quad j_{3,3} = -\frac{2 S Q^{st}}{l \rho C^2 \mu_0^2}, \]

\[ j_{3,9} = -\frac{2 S (Q^{st})^2}{l \rho C^2 \mu_0^2}, \quad j_{4,1} = \frac{\omega_0 U \cos(\theta_0 + \theta_0^2)}{T_J}, \]

\[ j_{4,2} = -\omega_0 \Psi_d, \quad j_{4,4} = -\omega_0 r X_d, \quad j_{4,5} = -\omega_0 j_{4,6} = \omega_0 r X_r, \quad j_{4,7} = -\omega_0 r X_{rd}, \]

\[ j_{5,1} = \omega_0 U \sin(\theta_0 + \theta_0^2), \quad j_{5,2} = \omega_0 \Psi_d, \]

\[ j_{5,4} = \omega_0, \quad j_{5,5} = -\omega_0 r Y_q, \quad j_{5,8} = \omega_0 r Y_{rq}, \]

\[ j_{6,4} = \frac{Z_d}{T_r}, \quad j_{6,6} = -\frac{Z_r}{T_r}, \quad j_{6,7} = \frac{Z_{rd}}{T_r}, \]

\[ j_{7,4} = \frac{Q_d}{T_{rd}}, \quad j_{7,6} = -\frac{Q_r}{T_{rd}}, \quad j_{7,7} = -\frac{Q_{rd}}{T_{rd}}, \]

\[ j_{8,5} = \frac{P_q}{T_{rq}}, \quad j_{8,8} = -\frac{P_{rq}}{T_{rq}}, \quad j_{9,2} = \frac{1}{T_c}, \quad j_{9,9} = -\frac{1}{T_c}. \]

In order to study the local stability of the equilibrium states of system (7) relative to the voltage, the standard function leqnonLin of the application package MathLab was used [1]. This function is based on the least-squares method, i.e. on an iterative approximation to equilibrium state, that allows one to reduce the computational error and to define more exactly the interval of instability of the system. The initial data for this method is putative equilibrium state. Using this method the Jacobi matrix calculated at an equilibrium is found, and then the Routh–Hurwitz stability criterion is applied. Results are presented in Fig. 11 the equilibrium state \( \theta = \theta_0 \) is unstable for \( \gamma \in [\gamma_1 \approx 0.86; \gamma_2 \approx 0.9] \) and stable for other \( \gamma \), the rest three equilibria states are always unstable.

1 First, the study of local stability of equilibria was carried out via eigenvalues of the Jacobian matrix and the standard function eig from MathLab. It turned out that all equilibrium states are unstable for all \( \gamma \). Further, the stability criterion of Routh–Hurwitz for the Jacobian matrix was applied. It showed that the equilibrium state \( \theta = \theta_0 \) is unstable for \( \gamma \in [\gamma_1 \approx 0.84; \gamma_2 \approx 1.17] \) and stable for other \( \gamma \). The obtained results of these methods can be explained by the fact that all calculations are made with some numerical error and because of the substantial difference in magnitude.

2 An oscillation can generally be easily numerically localized if the initial data from its open neighborhood in the phase

**Fig. 11** Stability of equilibrium states of system (7), defined by the least-squares method: blue pluses are stable equilibrium states, red crosses are unstable equilibrium states

### 9 Analysis of transient processes

During the operation of hydropower unit the transient processes related to sudden changes of the work parameters of the hydraulic unit often occur. As a result, the following problem arises: to find parameters, under which the hydropower unit pulls in the new operating mode after transient processes. This problem is closely related to the limit (ultimate) load problem, which arises in practice of operation of electrical motors after sudden change of load torque on the shaft [3 12 13 23 24]. For its solution the equal-area method is widely used in engineering practice. This method was used for some models in the works of A.A. Yanko–Trinitskii [12]. In our work modern methods of numerical integration of the system (Runge–Kutta method) are combined with the analysis in the spirit of the classical ideas of Yanko–Trinitskii.

First the numerical analysis of transient processes was carried out with the initial data taken from a small neighborhood of the equilibrium state. It is verified whether the trajectory goes out from this neighborhood after a long integration time (1000 s) or not. As a result of the study it was obtained that the equilibrium state \( \theta = \theta_0 \), corresponding to the operating mode, is unstable for \( \gamma \in [\gamma_1 \approx 0.85; \gamma_2 \approx 0.91] \). The received interval of instability is consistent with the local analysis and corresponds to the interval \([S_1, S_2]\) in Fig. 11. It can be defined more exactly due to coefficient \( k \), corresponding the turbine used at the Sayano-Shushenskaya hydropower plant.

Further, the numerical analysis of transient processes was carried out with various initial data. For values \( \gamma \) corresponding the local stability, hidden oscillations
are not found numerically and all simulated trajectories attract to the equilibrium states. For values $\gamma$, corresponding to the local instability, simulated trajectories attract to self-excited periodic solutions. The local bifurcation, in which an equilibrium loses stability and a small stable limit cycle is born, occurs in considered multidimensional system (this bifurcation is an analog of Andronov-Hopf bifurcation [21 22 23]).

According to [26], immediately before the accident the power of the second hydropower unit was 475 MW at a head of 212 meters (i.e., it worked in the not recommended zone II (Fig. 1)). On the day of the accident the power of the second hydropower unit was reduced in accordance with the commands of the group controller of active and reactive power.

Below three cases are modeled:

1. operation of hydropower unit at the rated voltage (Fig. 1 point A) with initial data $(\theta_\Delta, s, Q, \varphi_d, \varphi_q, \varphi_t, \varphi_{rd}, \varphi_{rq}, \varphi_D) = (0, 1, 0, 0, 0, 0, 0, 0, 0)$,
2. reducing the power of hydropower unit, that corresponds to reducing voltage to 0.89 of the rated voltage (Fig. 1 point B),
3. reducing power of hydropower unit, that corresponds to reducing voltage to 0.7 of the rated voltage (Fig. 1 point C).

The results of modeling have shown that at the rated voltage the trajectory of the system after transient processes is attracted to the equilibrium state, which corresponds to operating mode of hydropower unit (Figs. 12 13). Further at some instant the voltage is reduced to 0.89 of the rated voltage. In this case space lead to a long-term behavior that approaches the oscillation. Therefore, from a computational perspective, it is natural to suggest the following classification of attractors [21 31 32 28 19], which is based on the simplicity of finding their basins of attraction in the phase space: An attractor is called a self-excited attractor if its basin of attraction intersects with any open neighborhood of an equilibrium, otherwise it is called a hidden attractor. For a self-excited attractor its basin of attraction is connected with an unstable equilibrium and, therefore, (standard computational procedure) self-excited attractors can be localized numerically by the standard computational procedure: by constructing a solution using initial data from an unstable manifold in a neighborhood of an unstable equilibrium, and observing how it is attracted, and visualizing the oscillation. In contrast, the basin of attraction for a hidden attractor is not connected with any equilibrium. For example, hidden attractors are attractors in systems with no equilibria or with only one stable equilibrium (a special case of the multistability: coexistence of attractors in multistable systems). Well known examples of the hidden oscillations are nested limit cycles in 16th Hilbert problem (see, e.g., [20 28]) and counterexamples to the Aizerman and Kalman conjectures on the absolute stability of nonlinear control systems [23 24 27 28]. Hidden oscillations in the models of electrical machines are discussed, e.g., in [10 11 25 26 14].

The trajectory of the system after transient processes is attracted to the stable limit cycle (Figs. 14 15), i.e., vibrations are arisen in the hydropower units. Then the voltage is reduced to 0.7 of the rated voltage. The trajectory of the system after transient processes is attracted to equilibrium state (Figs. 16 17).

![Fig. 12 Stable equilibrium in the mathematical model of hydropower unit, $U = 15.75 \cdot 10^3$ [V]](image1)

![Fig. 13 Stable equilibrium in the mathematical model of hydropower unit – projection onto ($\theta_\Delta, \mu_\Delta, s$), $U = 15.75 \cdot 10^3$ [V]](image2)
From Fig. 9 it is clear that the maximum value of the oscillation amplitude is reached at $U = 0.89U_{nom}$.

Thus, the results of modeling are sufficiently consistent with the full-scale tests carried out for hydropower units of the Sayano-Shushenskaya hydropower plant.

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