Compactification and signature transition in Kaluza-Klein spinor cosmology

B. Vakili∗, S. Jalalzadeh† and H. R. Sepangi‡
Department of Physics, Shahid Beheshti University, Evin, Tehran 19839, Iran

November 1, 2021

Abstract

We study the classical and quantum cosmology of a 4 + 1-dimensional space-time with a non-zero cosmological constant coupled to a self interacting massive spinor field. We consider a spatially flat Robertson-Walker universe with the usual scale factor $R(t)$ and an internal scale factor $a(t)$ associated with the extra dimension. For a free spinor field the resulting equations admit exact solutions, whereas for a self interacting spinor field one should resort to a numerical method for exhibiting their behavior. These solutions give rise to a degenerate metric and exhibit signature transition from a Euclidean to a Lorentzian domain. Such transitions suggest a compactification mechanism for the internal and external scale factors such that $a \sim R^{-1}$ in the Lorentzian region. The corresponding quantum cosmology and the ensuing Wheeler-DeWitt equation have exact solutions in the mini-superspace when the spinor field is free, leading to wavepackets undergoing signature change. The question of stabilization of the extra dimension is also discussed.

PACS umbers: 04.20.-q, 04.50.+h, 04.60.-m

∗email: b-vakili@cc.sbu.ac.ir
†email: s-jalalzadeh@cc.sbu.ac.ir
‡email: hr-sepangi@cc.sbu.ac.ir
1 Introduction

Higher-dimensional cosmology resulting from the solutions of higher-dimensional Einstein field equations begun to develop with the works of Kaluza and Klein when they tried to geometrically unify gravitational and electromagnetic interactions by introducing an extra dimension in the space-time metric. Over a long period of time this work has been the focus of an active area of research [1]. The extra dimension in Kaluza-Klein approach has compact topology with compactification scale $l$ so that at scales much larger than $l$ extra dimensions are not observable. However, if one relaxes this condition by allowing the extra dimensions to assume large sizes, interesting physical effects appear. The birth of brane cosmology some few years ago has been the focus of much attention in this regard [2, 3]. The assumption that our observable universe may be embedded in a higher dimensional bulk has opened a new window in cosmology with different approaches adopted in utilizing the size of extra dimensions involved [4]. A rather similar approach to models with large extra dimensions comes from the space-time-matter theory where our 4D world is embedded in a 5D bulk devoid of matter [5]. This theory was later shown to be basically equivalent to the brane approach [6].

A question of interest in the classes of problems dealing with higher dimensional cosmology is the mechanisms through which compactification of extra dimensions can be achieved. One of the common methods in this regard is to use mechanisms based on signature transition in classical and quantum cosmology, first addressed in the works of Hartle and Hawking [7]. They argued that quantum cosmology amplitudes should be expressed as the sum of all compact Riemannian manifolds whose boundaries are located at the signature changing hypersurface. This phenomenon has been studied at the classical and quantum cosmology level by other authors, see for example [8], [9] and [10]. There are a number of works which support the idea that signature transition would provide a mechanism for compactification in higher dimensional cosmology [11, 12, 13, 14]. In [11] a higher dimensional FRW model with a positive cosmological constant was considered with topology $S^3 \times S^6$ as the spatial section and two scale factors $a_1$ and $a_2$. It was shown that classical signature change induces compactification, that is, drags the size of $S^6$ down and gives rise to a long-time stability at an unobservable small scale. In [12] the compactification mechanism is studied for a $D + 1$-dimensional compact Kaluza-Klein cosmology with a negative cosmological constant in which the matter field is either dust or coherent excitations of a dilaton field. Another effort in this direction can be found in [13] where the authors consider an empty 4 + 1-dimensional Kaluza-Klein cosmology with a negative cosmological constant in which the external space is an open FRW metric and the internal space contains a compact scale factor $a$. It is then shown that signature transition induces compactification on the scale factor $a$ in the Lorentzian domain, dragging it to a small size of order $|\Lambda|^{1/2}$. These efforts have mainly been made on the assumption that either the space is empty with a cosmological constant or filled with a scalar field. The cases in which the matter source is a spinor field have seldom been considered in the literature and it would thus be of interest to employ such fields in this study.

In general, theories studying spinor fields coupled to gravity result in Einstein-Dirac system which are somewhat difficult to solve. The cosmological solutions of such systems have been studied in a few cases by a number of authors [15]. A general discussion on the possibility that classical homogeneous spinor fields can play the role of matter in cosmology can be found in [16]. The results of such a spinor cosmology are notable relative to the scalar field driven cosmology, particularly in inflation scenarios [17] and the prediction of cyclic universes [18]. In quantum cosmology, the Wheeler-DeWitt (WD) equation for a universe filled with a scalar field is a Schrödinger like equation with a potential barrier [19], suggesting a tunnelling mechanism for creation of the universe from nothing. In contrast, in spinor quantum cosmology, the WD equation becomes a Schrödinger like equation with a potential well and thus the creation of the universe can not be described by such tunnelling procedures, for a discussion on this issue see [10].

In this paper we consider a 4 + 1-dimensional Kaluza-Klein cosmology with a spatially flat Robertson-Walker metric, having two dynamical variables, the usual external scale factor $R$ and
the internal scale factor $a$. The matter source in our model is a Dirac spinor field which can be free or self-interacting. The solutions of the resulting field equations show that they are continuous functions passing smoothly from a Euclidean to a Lorentzian domain through the signature changing hypersurface. They also provide a mechanism for compactification of the scale factors with different compactification scales, that is, of the order $|\Lambda|^{-1/2}$ for $R$ and $|\Lambda|^{1/2}$ for $a$. These results are in agreement with the results of [13]. We also touch upon the interesting question of stabilization of the extra dimension in the case where the spinor field is self interacting. Finally, the quantum cosmology of the model is studied by presenting exact solutions to the WD equation in the case of free spinor fields.

2 The action

We start by considering a cosmological model in which the spacetime is assumed to be of spatially flat Robertson-Walker type with one compact extra dimension as the internal space such that the total bulk manifold has warped product topology

$$M = M_{1+3} \times S^1. \quad (1)$$

Adopting the chart $\{\beta, x, y, z, \rho\}$ where $\beta$ represents the lapse function, $x, y, z$ denote the external space coordinates and $\rho$ is the internal space coordinate, the metric can be written as

$$ds^2 = -\beta d\beta^2 + R^2(\beta) \left( dx^2 + dy^2 + dz^2 \right) + a^2(\beta)d\rho^2, \quad (2)$$

where $R(\beta)$ and $a(\beta)$ are the scale factor of the universe and the radius of the internal space respectively. From (2) it is clear that the sign of $\beta$ determines the geometry, being Lorentzian if $\beta > 0$ and Euclidean if $\beta < 0$. For $\beta > 0$ the traditional cosmic time can be recovered by the substitution $t = \frac{2}{3} \beta^{3/2}$ and metric (2) becomes

$$ds^2 = -dt^2 + R^2(t) \left( dx^2 + dy^2 + dz^2 \right) + a^2(t)d\rho^2, \quad (3)$$

where $R(t) = R(\beta(t))$ and $a(t) = a(\beta(t))$ in the $\{t, x, y, z, \rho\}$ chart. It should be noted that what we have called the lapse function here is slightly different than the much more popular ADM lapse function since this one enters the metric as beta instead of $\beta^2$. As in [8] we formulate our equations in a region that does not include $\beta = 0$ and seek real solutions for $R$ and $a$ passing smoothly through the $\beta = 0$ hypersurface. The Ricci scalar corresponding to metric (3) is

$$\mathcal{R} = 6 \left[ \frac{\ddot{R}}{R} + \frac{\dot{R}}{R} \right] + 2 \frac{\ddot{a}}{a} + 6 \frac{\ddot{R}}{R} \frac{\dot{a}}{a}, \quad (4)$$

where a dot represents differentiation with respect to $t$.

With this preliminary setup we write the action functional as

$$S = \int \sqrt{-g}L dt d^3x d\rho, \quad (5)$$

where

$$L = L_{grav} + L_{matt}. \quad (6)$$

Here, $L_{grav}$ represents the Einstein-Hilbert Lagrangian for the gravitational field with a cosmological constant $\Lambda$

$$L_{grav} = \mathcal{R} - \Lambda. \quad (7)$$

The term $L_{matt}$ in (6) denotes the Lagrangian for the matter source, which we assume is a Dirac spinor field $\psi$. As is well known the spinor field Lagrangian in curved spacetime is given by

$$L_{matt} = \frac{1}{2} \left[ \bar{\psi} \gamma^\mu (\partial_\mu + \Gamma_\mu) \psi - \bar{\psi} (\partial_\mu - \Gamma_\mu) \gamma^\mu \psi \right] - V(\bar{\psi}, \psi), \quad (8)$$
Also, the spin connections satisfy the relation
\[ \gamma^\mu = e^\mu_a \gamma^a, \quad \gamma_\mu = e^a_\mu \gamma^a. \]  
For metric (3) the vielbeins can be easily obtained from their definition, that is, \( g_{\mu\nu} = e^a_\mu e^b_\nu \eta_{ab} \) leading to
\[ e^a_\mu = \text{diag}(1, R, R, R, a), \quad e^a_\mu = \text{diag}(1, 1/R, 1/R, 1/R, 1/a). \]  
Also, the spin connections satisfy the relation
\[ \Gamma_\mu = \frac{1}{4} g_{\nu\lambda} \left( \partial_\mu e^\lambda_a + \Gamma^\lambda_\sigma_\mu e^\sigma_a \right) \gamma^\nu \gamma^a. \]  
Thus for line element (3), use of (9) and (10) yields
\[ \Gamma_0 = 0, \quad \Gamma_i = -\frac{\dot{R}}{2} \gamma^0 \gamma^i, \quad i = 1, 2, 3, \quad \text{and} \quad \Gamma_4 = -\frac{\dot{a}}{2} \gamma^0 \gamma^4, \]  
where \( \gamma^0, \gamma^i \) and \( \gamma^4 \) are the Dirac matrices in the five dimensional Minkowski spacetime and we may always adopt a suitable representation in which \( \gamma^0 \) is diagonal with \( (\gamma^0)^2 = -1 \). The final remark about the Lagrangian (8) is that for consistency of Einstein field equations with a spinor field as the matter source in the background metric (3), the spinor field \( \psi \) must depend on \( t \) only, that is \( \psi = \psi(t) \) [10].

Now, substituting (7) and (8) into (5) and integrating over spatial dimensions results in an effective Lagrangian \( \mathcal{L} \) in the mini-superspace \( \{ R, a, \psi, \bar{\psi} \} \), that is
\[ \mathcal{L} = \frac{1}{2} R a \dot{R}^2 + \frac{1}{2} R^2 \dot{R} \dot{a} + \frac{1}{6} \Lambda R^3 a + \frac{1}{12} R^3 a \left[ \bar{\psi} \gamma^0 \psi - \bar{\psi} \gamma^0 \psi - 2V(\bar{\psi}, \psi) \right]. \]  

### 3 Field equations

Variation of Lagrangian (13) with respect to \( \bar{\psi}, \psi, R \) and \( a \) yields the equation of motion of the spinor field and the Einstein field equations
\[ \dot{\psi} + \left( 3 \frac{\dot{R}}{R} + \frac{\dot{a}}{2a} \right) \psi + \gamma^0 \frac{\partial V}{\partial \psi} = 0, \]  
\[ \dot{\bar{\psi}} + \left( 3 \frac{\dot{R}}{R} + \frac{\dot{a}}{2a} \right) \bar{\psi} - \frac{\partial V}{\partial \bar{\psi}} \gamma^0 = 0, \]  
\[ 2 \frac{\dot{R}}{R} + \frac{\dot{R}^2}{R^2} + \frac{\dot{R} \dot{a}}{R a} - \Lambda = \frac{1}{2} \left( \bar{\psi} \frac{\partial V}{\partial \psi} + \frac{\partial V}{\partial \bar{\psi}} \psi \right) - V(\bar{\psi}, \psi), \]  
\[ 3 \frac{\dot{R}^2}{R^2} + 3 \frac{\dot{R}}{R} - \Lambda = \frac{1}{2} \left( \bar{\psi} \frac{\partial V}{\partial \psi} + \frac{\partial V}{\partial \bar{\psi}} \psi \right) - V(\bar{\psi}, \psi). \]  
The “zero energy condition” also reads
\[ \mathcal{H} = \frac{\partial \mathcal{L}}{\partial \dot{R}} \dot{R} + \frac{\partial \mathcal{L}}{\partial \dot{a}} \dot{a} + \frac{\partial \mathcal{L}}{\partial \bar{\psi}} \dot{\bar{\psi}} + \bar{\psi} \frac{\partial \mathcal{L}}{\partial \psi} - \mathcal{L} = 0, \]  
yielding the constraint equation as
\[ 3 \frac{\dot{R}^2}{R^2} + 3 \frac{\dot{R}}{R a} - \Lambda = -V(\bar{\psi}, \psi). \]
It is clear that the right hand sides of equations (16), (17) and (19) represent the components of the energy-momentum tensor where the energy density of the spin or field is given by

$$\rho = T_{00} = -V(\bar{\psi}, \psi).$$

To make Lagrangian (13) manageable, consider the following change of variables

$$R = (u + v)^{1/2}, \quad a = (u + v)^{-1/2} (u - v).$$

In terms of the new variables \((u, v)\), Lagrangian (13) takes the form

$$L = \frac{1}{4} (\dot{u}^2 - \dot{v}^2) + \frac{1}{6} \Lambda \left( u^2 - v^2 \right) + \frac{1}{12} \left( u^2 - v^2 \right) \left[ \bar{\psi} \gamma^0 \psi - \bar{\psi} \gamma^0 \psi - 2V(\bar{\psi}, \psi) \right],$$

with the energy constraint (19) becoming

$$H = \frac{1}{4} (\dot{u}^2 - \dot{v}^2) - \frac{1}{6} \Lambda \left( u^2 - v^2 \right) + \frac{1}{6} \left( u^2 - v^2 \right) V(\bar{\psi}, \psi) = 0.$$

In terms of the new variables, the field equations (14)-(17) can be obtained by the variation of Lagrangian (22) with respect to its dynamical variables \((\bar{\psi}, \psi, u, v)\) with the result

$$\dot{\psi} + \frac{uu - vv}{u^2 - v^2} \psi + \gamma^0 \frac{\partial V}{\partial \bar{\psi}} = 0,$$

$$\dot{\bar{\psi}} + \frac{uu - vv}{u^2 - v^2} \bar{\psi} - \gamma^0 \frac{\partial V}{\partial \psi} = 0,$$

$$\ddot{u} - \frac{2}{3} \Lambda u - \frac{1}{3} \left( \frac{\dot{\psi}}{\psi} + \frac{\dot{\bar{\psi}}}{\bar{\psi}} \right) \bar{\psi} \gamma^0 \psi - 2V(\bar{\psi}, \psi) = 0,$$

$$\ddot{v} - \frac{2}{3} \Lambda v - \frac{1}{3} \left( \frac{\dot{\psi}}{\psi} + \frac{\dot{\bar{\psi}}}{\bar{\psi}} \right) \bar{\psi} \gamma^0 \psi - 2V(\bar{\psi}, \psi) = 0.$$

Our goal would now be to concentrate on the solutions of these equations for various forms of the potential \(V\).  

### 4 Solutions

Up to this point the cosmological model has been rather general. However, motivated by the desire to find suitable smooth functions \(R(t), a(t), \psi(t)\) and in particular to stabilize the internal degree of freedom, a suitable potential \(V(\bar{\psi}, \psi)\) should be chosen. First, we assume that this potential has the property

$$\bar{\psi} \gamma^0 \frac{\partial V}{\partial \psi} - \frac{\partial V}{\partial \bar{\psi}} \gamma^0 \psi = 0.$$  

This is not a severe restriction on the form of the potentials as most of the potentials in use would satisfy the above condition. With this property in mind, we can integrate equations (24) and (25) to obtain

$$\ddot{\psi} = \frac{C}{u^2 - v^2} = \frac{C}{R^3 a},$$  

where \(C\) is an integrating constant and we have used transformations (21).

In general the potential \(V(\bar{\psi}, \psi)\) should describe a physical self-interacting spinor field and is usually an invariant function constructed from the spinor \(\psi\) and its adjoint \(\bar{\psi}\). Some of the common forms for \(V\) are: \(V(\psi, \bar{\psi}) = m \bar{\psi} \psi\) representing a free spinor field of mass \(m\), \(V(\bar{\psi}, \psi) = m \bar{\psi} \psi + J^\mu J_\mu\) where \(J^\mu = \bar{\psi} \gamma^\mu \psi\) and known as the Thirring model, \(V(\bar{\psi}, \psi) = m \bar{\psi} \psi + \lambda (\bar{\psi} \psi)^n\) and so on. In what follows we focus attention on two cases, the free and self-interacting spinor field potentials with the latter having a term of the form \((\bar{\psi} \psi)^n\).  

5
4.1 Free spinor field

This is the simplest case in which the potential has the form \( V(\bar{\psi}, \psi) = m\bar{\psi}\psi \). Use of equations (20) and (29) results in obtaining the energy density of the spinor field

\[
\rho = -\frac{mC}{R^3a}. \tag{30}
\]

The condition \( \rho > 0 \) demands that the integrating constant \( C \) be negative and as such, we choose it to be \(-1\). For the initial conditions we take \( \dot{a}(0) = \dot{v}(0) = 0 \), whose relevance will be discussed in the next section. Integrating equations (26) and (27), one is led to

\[
\begin{align*}
  u(t) &= \mathcal{A} \cos \left( \sqrt{-\frac{2}{3}\Lambda} t \right), \\
  v(t) &= \mathcal{B} \cos \left( \sqrt{-\frac{2}{3}\Lambda} t \right),
\end{align*} \tag{31}
\]

where \( \mathcal{A} \) and \( \mathcal{B} \) are two integrating constants. Now, these solutions must satisfy the “zero energy condition,” equation (23). Thus, substitution of equations (31) into (23) gives a relation between \( \mathcal{A} \) and \( \mathcal{B} \)

\[
\mathcal{A}^2 - \mathcal{B}^2 = \frac{m}{-\Lambda}. \tag{32}
\]

If one chooses the constants \( \mathcal{A} \) and \( \mathcal{B} \) such that \( \mathcal{A} - \mathcal{B} = 1 \), then one finds the following solutions in terms of \( \beta \)

\[
\begin{align*}
  R(\beta) &= \left( \frac{m}{-\Lambda} \right)^{1/2} \cos^{1/2} \left( \frac{2}{3} \sqrt{-\frac{2}{3}\Lambda}\beta^{3/2} \right), \\
  a(\beta) &= \left( \frac{-\Lambda}{m} \right)^{1/2} \cos^{1/2} \left( \frac{2}{3} \sqrt{-\frac{2}{3}\Lambda}\beta^{3/2} \right). \tag{33}
\end{align*}
\]

Also, the energy density and Ricci scalar are obtained as

\[
\begin{align*}
  \rho &= \frac{-\Lambda}{\cos^2 \left( \frac{2}{3} \sqrt{-\frac{2}{3}\Lambda}\beta^{3/2} \right)}, \\
  \mathcal{R} &= -\Lambda \left[ \tan^2 \left( \frac{2}{3} \sqrt{-\frac{2}{3}\Lambda}\beta^{3/2} \right) - 3 \right]. \tag{35}
\end{align*}
\]

The classical solutions (33) and (34) describe a Kaluza-Klein universe with a cosmological constant filled with a free spinor field. When \( \Lambda < 0 \), both scale factors \( R \) and \( a \) are unbounded in the Euclidean region \( \beta < 0 \), passing continuously through \( \beta = 0 \) and exhibiting bounded oscillations in the Lorentzian region \( \beta > 0 \). For \( |\Lambda| \gg 0 \), these solutions give rise to a small scale factor \( R \) passing smoothly from Euclidean to Lorentzian regions. The scale factor \( a \) will be very large in both regions compared to the scale factor \( R \) and passes through \( \beta = 0 \) smoothly. The scale factor \( R \) is thus compactified to a small size of order \( |\Lambda|^{-1/2} \) in the Lorentzian domain. Taking \( |\Lambda| \simeq 0 \), the scale factor \( a(\beta) \) becomes very small in comparison to \( R(\beta) \). As it is clear, it passes through \( \beta = 0 \) continuously with a very small value \( a(0) = \left( \frac{|\Lambda|}{m} \right)^{1/2} \) and oscillates for \( \beta > 0 \) with amplitude \( \left( \frac{|\Lambda|}{m} \right)^{1/2} \).

One of the interesting features in this case is that signature transition can induce compactification on the scale factor \( a(\beta) \) in the Lorentzian region, dragging it to a small size of order \( |\Lambda|^{1/2} \). The first zero of the oscillatory functions appearing in (33) and (34) in the \( \beta > 0 \) region occurs at

\[
\beta_0 = \left( \frac{3}{4\pi} \sqrt{\frac{3}{-2\Lambda}} \right)^{2/3}. \tag{36}
\]

From (36) it is seen that for a small value of the cosmological constant (large \( \beta_0 \)) we have an extended Lorentzian region \( 0 \leq \beta \leq \beta_0 \) which would correspond to a Kaluza-Klein cosmology with a large scale factor \( R \) and a stable compactified internal scale factor \( a \). Note that for \( |\Lambda| \sim 10^{-56} \text{cm}^{-2} \) we have \( \beta_0 \geq \text{present age of the universe} \). These results are completely in agreement with [13] in which an
empty Kaluza-Klein universe with negative curvature \((k = -1)\) was investigated. However, it is worth noting that for \(k = 0\) this model becomes singular due to the appearance of \(k\) in the denominator of the scale factor \(R\). In the present discussion our model is free of such singularities for \(k = 0\).

When \(\Lambda > 0\), both scale factors \(R(\beta)\) and \(a(\beta)\) have oscillatory behavior in the Euclidean region passing smoothly through \(\beta = 0\) and are unbounded in the Lorentzian domain. Since signature transition does not lead to compactification and the energy density of the spinor field becomes negative, this case is not of interest in our present investigation.

### 4.2 Self-interacting spinor field

In this section we consider a self interacting potential which we have chosen to be of the following form

\[
V(\bar{\psi}, \psi) = m\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^n,
\]

where \(\lambda\) is a coupling constant and \(n > 0\). Since this potential has the property represented by equation (28), integrating equations (24) and (25) again yields

\[
\ddot{\psi} = \frac{D}{u^2 - v^2},
\]

where \(D\) is a constant which is not necessarily equal to \(C\) in (30). Using this result in equations (26) and (27) one obtains the following dynamical equations for \(u\) and \(v\) in terms of the evolution variable \(\beta\)

\[
\begin{align*}
    u'' &= \frac{1}{2\beta} u - \frac{3}{4} k\beta u(u^2 - v^2)^{-n} + \frac{2}{3} \Lambda \beta u, \\
    v'' &= \frac{1}{2\beta} v - \frac{3}{4} k\beta v(u^2 - v^2)^{-n} + \frac{2}{3} \Lambda \beta v,
\end{align*}
\]

where \(\frac{3}{4}k = \frac{2}{9}(1 - n)\lambda D^n\) and a prime represents differentiation with respect to \(\beta\). Note that \(k\) should not be confused with curvature here. Also the zero energy constraint (23) takes the form

\[
-4\mathcal{H} = \frac{1}{\beta}(-u'u'' + v'v'') - \frac{9}{4}k(u^2 - v^2)^{1-n} + \frac{2}{3}\Lambda(u^2 - v^2) - \frac{2}{3}mD = 0.
\]

The coupled equations (39) and (40) should now be solved and since their solutions must automatically satisfy the energy constraint, equation (41) becomes only a restriction on the initial conditions. Note that equations (39) and (40) are a system of coupled non-linear differential equations with moving singularities [21] at the critical values \(\beta_c\) at which \(u(\beta_c) = \pm v(\beta_c)\). In terms of the original variables we have \(a(\beta_c) = 0, R(\beta_c) = 0\) and \(a(\beta_c) = \infty\). These equations do not have closed form solutions. However, such system of equations have been investigated in [9] and [21] where analytic solutions near \(\beta = 0\) are obtained and a numerical method to find the solutions for the full range of \(\beta\) is presented.

For the problem at hand, in order to have well behaved solutions close to \(\beta = 0\), the first term in equation (41) imposes that \(u'(\beta) \sim \beta^r\) and \(v'(\beta) \sim \beta^p\), where \(r, p \geq \frac{1}{2}\) or \(|u'(0)| = |v'(0)|\). On the other hand the first terms on the right hand side of equations (39) and (40) imply that near \(\beta = 0\) these equations admit solutions of the form \(u'(\beta) \sim \beta^{1/2}\) and \(v'(\beta) \sim \beta^{1/2}\) which impose a more severe restriction on the initial conditions. Also, they are not real or \(C^2\) across \(\beta = 0\). It is easy to show that regular solutions close to \(\beta = 0\) are of the form [9]

\[
\begin{align*}
    u(\beta) &= A\beta^3 + u_0 \quad \text{where} \quad A = -\frac{2}{9} \left[\frac{3}{4} \frac{k v_0}{(u_0^2 - v_0^2)^n} - \frac{2}{3} \Lambda u_0\right], \\
    v(\beta) &= B\beta^3 + v_0 \quad \text{where} \quad B = -\frac{2}{9} \left[\frac{3}{4} \frac{k v_0}{(u_0^2 - v_0^2)^n} - \frac{2}{3} \Lambda v_0\right],
\end{align*}
\]

where \(u_0 = u(0)\) and \(v_0 = v(0)\). Thus the following initial condition must be satisfied

\[
u'(0) = v'(0) = 0 \quad \text{and} \quad u''(0) = v''(0) = 0.
\]
The above initial conditions are those we used for finding solutions (31). One should note that since we have a system of coupled second order differential equations, the relations $u''(0) = v''(0) = 0$ are not initial conditions but rather consistency checks. Therefore the initial values for functions $u$ and $v$ should now satisfy equation (41)

$$\frac{9}{4}k(u_0^2 - v_0^2)^{1-n} - \frac{2}{3}\Lambda(u_0^2 - v_0^2) + \frac{2}{3}m_D = 0. \hspace{1cm} (45)$$

Equation (45) shows an implicit relation from which one can find the possible initial values for $u$ and $v$. What remains is the solution of the system of equations (39) and (40) with the initial conditions (45). As mentioned before this system cannot not be solved analytically and one may use the same numerical method as described in [9, 21]. The results for the scale factors $R$ and $a$ are shown in figure 1 for typical values of the parameters. As it is clear from the figure, both external scale factor $R$ and internal scale factor $a$ are unbounded functions in the Euclidean region $\beta < 0$ passing smoothly through the signature changing hypersurface $\beta = 0$ and are bounded in the Lorentzian region $\beta > 0$. Note that as in the free spinor field case, signature transition induces compactification on the scale factors but the compactification scale for the internal space is much smaller than that for the external space. Indeed equations (33), (34) and figure 1 show that in the presence of a free or self interacting spinor field the corresponding Kaluza-Klein cosmology has a large external scale factor in the Lorentzian domain with a size of order $|\Lambda|^{-1/2}$ together with a small compact internal space having a radius of order $|\Lambda|^{1/2}$.

## 5 Stabilization

A problem of great importance in theories dealing with extradimensions is the question of their stabilization which we are now going to address. In general, as it was shown in [22], stabilization of the extra dimension is related to the properties of the potential appearing in the theory. From Lagrangian (13) we can write the potential as

$$U = -\frac{1}{6}\Lambda R^3a + \frac{1}{6}R^3aV(\bar{\psi}, \psi). \hspace{1cm} (46)$$

Taking $V(\bar{\psi}, \psi) = m\bar{\psi}\psi + \lambda(\bar{\psi}\psi)^n$ and writing $\bar{\psi}\psi = D/R^3a$ from equation (38), the potential takes the form

$$U = -\frac{1}{6}\Lambda R^3a + \frac{1}{6}\lambda D^n(R^3a)^{1-n} + \frac{1}{6}m_D, \hspace{1cm} (47)$$

whose minimum occurs when $\frac{\partial U}{\partial R}|_{R_0} = 0$ and $\frac{\partial U}{\partial a}|_{a_0} = 0$, yielding

$$a_0 \sim \left[\lambda D^n(n - 1)|\Lambda|^{3n/2-1}\right]^{1/n}, \hspace{1cm} (48)$$
where we have used the approximation \( R_0 \sim |\Lambda|^{-1/2} \) resulting from our classical solution (33). Also, the second derivative of \( U \) with respect to \( a \) becomes

\[
U''(a_0) = n(n-1)\lambda D^n R_0^{3-3n}a_0^{-n-1},
\]

which for \( n > 1 \) renders the potential minimum at \( a_0 \). Let us now expand Lagrangian (13) around this minimum as \( a(t) = a_0 + \delta a \). Defining \( \delta = \delta a / a_0 \), it is easy to show that \( \delta = \delta_0 e^{i\omega t} \) where \( \omega^2 \propto U''(a_0) > 0 \) [22]. One should note that if the spinor field is free, \( \lambda = 0 \), then the potential does not have a minimum and the internal degree of freedom \( a(t) \) is not stable. This is also clear from the classical solution (34) which also shows that the internal scale factor oscillates between zero and a finite size. In summary, having a stable internal space is only possible in this model if the spinor field is self-interacting with \( n > 1 \) in equation (37).

6 Quantum cosmology

We now focus attention on the study of the quantum cosmology of the model described above. We start by writing the Wheeler-DeWitt equation from Hamiltonian (23). As we saw earlier, a self-interacting spinor field results in equations which are complicated and difficult to solve. One would therefore expect the corresponding quantum cosmology to become equally complicated. However, this is not the case when we have a free spinor field. This motivates us to concentrate on the quantum cosmology corresponding to the classical solutions represented by equations (33) and (34) which can be cast into an oscillator-ghost-oscillator system whose solutions are well known. The WD equation resulting from Hamiltonian (23) with \( V(\bar{\psi}, \psi) = m\bar{\psi}\psi \) and equation (29) can be written as

\[
\mathcal{H}\Psi(u, v) = \left[ \frac{\partial^2}{\partial u^2} - \frac{\partial^2}{\partial v^2} - (u^2 - v^2)\omega^2 - \frac{1}{6}m \right] \Psi(u, v) = 0,
\]

where \( \omega^2 = -\frac{1}{8} \Lambda \). This equation is separable in terms of the mini-superspace variables and a solution can be written as

\[
\Phi_{n_1, n_2}(u, v) = \alpha_{n_1}(u)\beta_{n_2}(v) \quad n_1, n_2 = 0, 1, 2, ...
\]

where

\[
\alpha_n(u) = \left( \frac{\omega}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\omega u^2/2} H_n(\sqrt{\omega}u),
\]

\[
\beta_n(v) = \left( \frac{\omega}{\pi} \right)^{1/4} \frac{1}{\sqrt{2^n n!}} e^{-\omega v^2/2} H_n(\sqrt{\omega}v),
\]

and \( H_n(x) \) is a Hermite polynomial. The zero energy constraint \( \mathcal{H} = 0 \) then yields

\[
n_2 - n_1 = \frac{1}{12} \frac{m}{\omega}.
\]

Note that since equation (30) shows \( m \) as the total energy of the spinor field, this equation represents a quantization condition of the spinor field energy. The set \( \{ \Phi_{n_1, n_2}(u, v) \} \) is a base for the Hilbert space of the measurable square-integrable functions on \( R^2 \) with the usual inner product

\[
\int \Phi_{n_1, n_2}(u, v)\Phi_{n'_1, n'_2}(u, v) du dv = \delta_{n_1, n'_1}\delta_{n_2, n'_2},
\]

which is the result of the orthogonality and completeness properties of the Hermite polynomials. Now, the general solution of equation (50) can be written in the form

\[
\Psi(u, v) = \sum_{n_1, n_2} A_{n_1, n_2} \Phi_{n_1, n_2}(u, v),
\]
where the prime on the sum indicates summation over all values of \( n_1 \) and \( n_2 \) satisfying constraint (54). The coefficients \( A_{n_1,n_2} \) are given by

\[
A_{n_1,n_2} = \frac{\pi^{1/4} (n_2/2)! c_{n_1}}{\sqrt{2^m n_2!} (-1)^{n_2/2} n_2!},
\]

for even values of \( n_2 \) and are arbitrary for odd values of \( n_2 \). In equation (57), \( c_n \) is defined as

\[
c_n = e^{-1/4|\chi_0|^2} \frac{\chi_0^n}{\sqrt{2^m n!}},
\]

where \( \chi_0 \) is an arbitrary complex number. For a more detailed discussion on the forms of the solutions and their various graphical representations and different boundary conditions the reader is referred to [20].

To understand the predictions of the quantum cosmology of the model, we note that equation (50) is a Schrödinger-like equation for a two dimensional oscillator-ghost-oscillator with zero energy moving in the potential

\[
U(u,v) = v^2 - u^2 - \frac{1}{6} \frac{m}{\omega^2}.
\]

This potential has a saddle point at \( u = v = 0 \), making it possible to divide the mini-superspace into two regions characterized by \( U > 0 \) for the classically forbidden or Euclidean and \( U < 0 \) for the classically allowed or Lorentzian region. The boundary of the two regions is given by \( U = 0 \), that is

\[
v^2 - u^2 = \frac{m}{6\omega^2} = \frac{m}{-\Lambda}.
\]

Comparing the above values of \( u \) and \( v \) with the classical solutions (31) and (32) shows that they correspond to \( \beta = 0 \), thus the same boundary separates the Euclidean and Lorentzian regions in both classical and quantum cosmology. The behavior of the wave function is exponential in the Euclidean domain and oscillatory in the Lorentzian region. Therefore, the smallest allowed values for \( n_2 \) and \( n_1 \) in (56) lead to a wave function defined in the Euclidean region. For other values and with more terms, equation (56) gives the wave function in the Lorentzian domain. As in the theory of oscillatory motion in quantum mechanics where the square of the wave function has its largest value at the turning points, the square of our wave function in the allowed (Lorentzian) region has also its largest value corresponding to the values of \( u \) and \( v \) satisfying equation (60), i.e. at the signature changing hypersurface in whose vicinity the classical solutions are defined. In general, such wavepackets represent a universe with no singularity at the signature changing hypersurface, that is, the universe appears from a Euclidean region without tunnelling.

7 Conclusions

In this paper we have studied the solutions of Einstein field equations coupled to a Dirac spinor field in a 4 + 1-dimensional Kaluza-Klein cosmology with a Robertson-Walker type metric and a negative cosmological constant. We have also explored the possibility of having solutions that are described by degenerate metrics signifying transition from a Euclidean to a Lorentzian domain. The spinor fields representing the matter source in this study have been taken to be either free or self interacting. In the case of a free spinor field we obtained exact solutions for the resulting Einstein-Dirac system. However, when the matter field is self interacting, the resulting equations are not amenable to exact solutions and possess the property of having moving singularities. A numerical method for this class of differential equations has been developed previously and were used to find the solutions for this case. The behavior of the solutions shows that the role of the coupling constant in the self-interacting potential is the same as a perturbation on the free spinor solutions. Both of these solutions predict signature transition from a Euclidean to a Lorentzian region. We have shown that for a very small
cosmological constant, signature change can induce compactification either on the external or internal scale factors in the Lorentzian region, but with different compactification scales, that is, \( \sim |\Lambda|^{-1/2} \) for \( R \) and \( \sim |\Lambda|^{1/2} \) for \( a \). An interesting property of this model is that the extra dimension turns out to be stable in the case of self-interacting spinor fields. The quantum cosmology of the model for the case when the spinor field is free was also studied. The corresponding WD equation was found from the Hamiltonian describing an oscillator-ghost-oscillator system. The exact solutions of the WD equation show wavepackets undergoing signature transition and lead to a quantization condition for the total energy of the system.

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