BIG POLYGON SPACES

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ABSTRACT. We study a new class of compact orientable manifolds, called big polygon spaces. They are intersections of real quadrics and related to polygon spaces, which appear as their fixed point set under a canonical torus action.

What makes big polygon spaces interesting is that they exhibit remarkable new features in equivariant cohomology: The Chang–Skjelbred sequence can be exact for them and the equivariant Poincaré pairing perfect although their equivariant cohomology is never free as a module over the cohomology ring $BT$. More generally, big polygon spaces show that a certain bound on the syzygy order of the equivariant cohomology of compact orientable $T$-manifolds obtained by Allday, Puppe and the author is sharp.

1. INTRODUCTION

Let $R = \mathbb{Q}[t_1, \ldots, t_r]$ be a polynomial ring. A finitely generated $R$-module $M$ is called an $m$-th syzygy if there is an exact sequence

\[ 0 \to M \to F_1 \to \cdots \to F_m \]

with finitely generated free modules $F_1, \ldots, F_m$. The first syzygies are exactly the torsion-free modules and the $r$-th syzygies the free ones. In this sense, syzygies interpolate between torsion-freeness and freeness. We also call any $M$ a zeroeth syzygy. If $M$ is an $m$-th syzygy, but not one of order $m + 1$ (or if $m = r$), then we say that it is of order exactly $m$, and we write $\text{syzord } M = m$.

Allday, Puppe and the author have initiated the study of syzygies in the context of torus-equivariant cohomology [1], [2]. Let $T = (S^1)^r$ be a torus, and let $X$ be a $T$-manifold such that its rational cohomology $H^*(X)$ is finite-dimensional. Then $R = H^*(BT)$, the cohomology of the classifying space of $T$, is of the form described above with generators of degree 2, and the (Borel) equivariant cohomology $H^*_T(X)$ is a module, even an algebra, over $R$. Many authors have investigated the cases where $H^*_T(X)$ is torsion-free or free. One insight of [1] was that reflexive $R$-modules (the second syzygies) are equally important: $H^*_T(X)$ is reflexive if and only if the Chang–Skjelbred sequence

\[ 0 \to H^*_T(X) \to H^*_T(X^T) \to H^*_{T^+}(X_1, X^T) \]

is exact, where $X_1$ denotes the union of the fixed point set $X^T$ and the 1-dimensional orbits [1 Thm. 1.1]. This often permits an efficient computation of $H^*_T(X)$. Moreover, if $X$ is compact oriented with equivariant orientation $[X]_T$, then the equivariant Poincaré pairing

\[ H^*_T(X) \times H^*_T(X) \to R, \quad (\alpha, \beta) \mapsto \langle \alpha \cup \beta, [X]_T \rangle, \]

is perfect if and only if $H^*_T(X)$ is reflexive [1 Cor. 1.3].

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Syzygies of any order can appear as the equivariant cohomology of $T$-manifolds. Assume for example that $X$ is a (compact) toric manifold; it is well-known that $H^*_T(X)$ is free over $R$ in this case. By removing two fixed points from $X$, one can obtain, for any $m < r$, syzygies of order exactly $m$ [1 Sec. 6.1], [2 Sec. 6.1]. The situation changes dramatically in the presence of Poincaré duality [1 Cor. 1.4]:

**Theorem 1.1** (Allday–Franz–Puppe). *Let $X$ be a compact orientable $T$-manifold. If $H^*_T(X)$ is a syzygy of order $m \geq r/2$, then it is free over $R$.*

The aim of this note is to show that this bound is sharp. For $r \in \{3, 5, 9\}$, Puppe and the author [5] have previously constructed compact orientable $T$-manifolds $X$ with syzord $H^*_T(X) = 1$. A modest generalization of the construction appeared in [7] Sec. 6.2]. However, no examples were known so far of rational Poincaré duality spaces $X$ such that $H^*_T(X)$ is reflexive, but not free. By providing these, we now in particular give the first examples of rational Poincaré duality spaces $X$ such that $H^*_T(X)$ is not free over $R$, but such that the Chang–Skjelbred sequence (1.2) is exact and the equivariant Poincaré pairing (1.3) perfect.

A vector $\ell \in \mathbb{R}^r$ is called *generic* if one cannot split up its components into two groups of equal sum. For generic $\ell$ and $a, b \geq 1$ consider the real algebraic variety $X_{a,b}(\ell) \subset \mathbb{C}^{r(a+b)}$ defined by the equations

$$
\|u_j\|^2 + \|z_j\|^2 = 1 \quad (1 \leq j \leq r),
$$

$$
\ell_1 u_1 + \cdots + \ell_r u_r = 0,
$$

where $u_1, \ldots, u_r \in \mathbb{C}^a$ and $z_1, \ldots, z_r \in \mathbb{C}^b$. We call $X_{a,b}(\ell)$ a *big polygon space*. The torus $T = (S^1)^r$ acts on it by scalar multiplication on the variables $z_j$,

$$
(g_1, \ldots, g_r) \cdot (u_1, \ldots, u_r, z_1, \ldots, z_r) = (u_1, \ldots, u_r, g_1 z_1, \ldots, g_r z_r).
$$

Given that $\ell$ is generic, $X_{a,b}(\ell)$ is an orientable compact connected $T$-manifold. Permuting the coordinates of $\ell$ or changing their signs does not produce new equivariant diffeomorphism types, so one can always assume the components of $\ell$ to be non-negative and weakly increasing (Lemma 2.1).

If $\ell$ has positive components, one can think of $X_{a,b}(\ell)$ as the set of all $r$-tuples of vectors in $\mathbb{C}^{a+b}$ of lengths $\ell_1, \ldots, \ell_r$ whose sum lies on a fixed $b$-dimensional complex subspace. The set of $T$-fixed points corresponds to setting $z = 0$, which gives all $r$-tuples of vectors in $\mathbb{C}^a$ of lengths $\ell_1, \ldots, \ell_r$ which form a polygon in the sense that they add up to 0. This is an example of a “space of polygons”. Various kinds of polygon spaces have been studied by Walker, Hausmann, Klyachko, Kapovich, Millson, Knutson, Farber, Schütz, Fromm and others, see [5], [6], [12], [13] §10.3 and the references given therein.

Our main result says that for any choice of $a$, $b$ and $r$ there is a big polygon space $X_{a,b}(\ell)$ that produces a maximal non-free syzygy in equivariant cohomology, and it is essentially unique.

**Theorem 1.2.** *Let $a, b, r \geq 1$, and let $\ell \in \mathbb{R}^r$ be generic with $0 \leq \ell_1 \leq \cdots \leq \ell_r$. 

1. Assume $r = 2m + 1$. Then syzord $H^*_T(X_{a,b}(\ell)) = m$ if and only if $X_{a,b}(\ell)$ is equivariantly diffeomorphic to $X_{a,b}(1, \ldots, 1)$.

2. Assume $r = 2m + 2$. Then syzord $H^*_T(X_{a,b}(\ell)) = m$ if and only if $X_{a,b}(\ell)$ is equivariantly diffeomorphic to $X_{a,b}(0, 1, \ldots, 1)$.***
This implies that all syzygy orders less than \( r/2 \) can be realized via big polygon spaces (Corollary 5.3).

The proof of Theorem 1.2 appears in Sections 5 and 6. Before, we discuss generalities of big polygon spaces (Section 2) and their cohomology, first non-equivariant (Section 3, including an analogue of Walker’s conjecture) and then equivariant (Section 4). We conclude with several additional comments in Section 7.

Unless specified otherwise, all (co)homology in this paper is taken with coefficients in a field \( k \) of characteristic 0, and all tensor products are over \( k \). All manifolds we consider are assumed to be smooth and to have finite-dimensional cohomology. Products of oriented manifolds are oriented in the canonical way according to the order of the factors. We orient the unit sphere \( S \subset \mathbb{C}^n \) such that the induced orientation on \((0, \infty) \times S\) agrees with the canonical orientation on \( \mathbb{C}^n \).

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### 2. First properties

Let \( r \geq 1 \). We use the abbreviation \([r] = \{1, \ldots, r\}\), and for a subset \( J \subset [r] \) we write \(|J|\) for the size of \( J \), \( J^c = [r] \setminus J \) and \( J \cup j \) instead of \( J \cup \{j\} \) etc. for \( j \in [r] \). Moreover, given two disjoint subsets \( J, K \subset [r] \), we denote the sign of the shuffle \((J, K)\) by

\[
(-1)^{|(J,K)|} = (-1)^{|\{(j,k) \in J \times K \mid j > k\}|}.
\]

For a vector \( \ell \in \mathbb{R}^r \), called length vector in this context, and \( J \subset [r] \) we define

\[
\ell(J) = \sum_{j \in J} \ell_j.
\]

One says that \( \ell \) is generic if

\[
\forall J \subset [r] \quad \ell(J) \neq \ell(J^c).
\]

In this case \( J \) is called \( \ell \)-long or \( \ell \)-short, depending on whether the left or the right hand side dominates in (2.3). If \( \ell \) is clear from the context, we just say ‘long’ or ‘short’. The non-generic length vectors lie on hyperplanes given by normal vectors with coordinates equal to \( \pm 1 \). The connected components of the complement of this hyperplane arrangement are called chambers. Two length vectors \( \ell \) and \( \ell' \) lie in the same chamber if and only if they induce the same notion of ‘long’ and ‘short’; we write \( \ell \sim \ell' \) in this case. Permuting components of \( \ell \) or changing their signs does not affect genericity.
The big polygon space $X_{a,b}(\ell)$ for $a, b, r \geq 1$ and generic $\ell \in \mathbb{R}^r$ as well as the action of $T = (S^1)^r$ on it have been defined in the introduction. Note that unlike polygon spaces, big polygon spaces are non-empty for any $\ell$. We write points of $X_{a,b}(\ell)$ or the ambient space $C^{r(a+b)}$ in the form $(u, z)$. The fixed point set of $X_{a,b}(\ell)$ is the space of polygons $E_{2a}(\ell)$ studied in \cite{6, 9, 5} and \cite{13} §10.3. Note that if $\ell$ has a zero coordinate, say $\ell = (0, \ell')$, then $\ell'$ is again generic, and there is an equivariant diffeomorphism

$$ (2.4) \quad X_{a,b}(\ell) \cong S^{2a+2b-1} \times X_{a,b}(\ell') $$

where the $S^1$-action on $S^{2a+2b-1}$ comes from scalar multiplication in $\mathbb{C}^b$.

**Lemma 2.1.** Let $\ell$ and $\ell'$ be generic length vectors in $\mathbb{R}^r$.

1. $X_{a,b}(\ell)$ is an orientable compact connected $T$-manifold; its dimension is $(2a + 2b - 1)r - 2a$.

2. If $\ell'$ is obtained from $\ell$ by changing the sign of some components and/or by permuting them, then $X_{a,b}(\ell)$ and $X_{a,b}(\ell')$ are diffeomorphic, equivariantly with respect to the corresponding permutation of the components of $T$.

3. If $\ell \sim \ell'$, then $X_{a,b}(\ell)$ and $X_{a,b}(\ell')$ are equivariantly diffeomorphic.

**Proof.** It is clear that $X_{a,b}(\ell)$ is compact. Also, by scaling $u$ by $\lambda \in [0, 1]$ and suitably increasing the variables $z_j$, one can connect any point $(u, z) \in X_{a,b}(\ell)$ to the subset $X_{a,b}(\ell) \cap \{u = 0\} \cong T$. Hence $X_{a,b}(\ell)$ is path-connected.

To complete the proof of (1), we show that $X_{a,b}(\ell)$ is an orientable $T$-submanifold of $C^{r(a+b)}$; for this it suffices to verify that 0 is a regular value of the $T$-invariant function

$$ F_\ell: C^{r(a+b)} \to \mathbb{R}^r \times \mathbb{C}^a, $$

$$ (u, z) \mapsto (\|u_1\|^2 + \|z_1\|^2 - 1, \ldots, \|u_r\|^2 + \|z_r\|^2 - 1, \sum_{j=1}^r \ell_j u_j). $$

We start with the case where all components of $\ell$ are non-negative. Because the function $C^{r(a+b)} \supset S^{2a+2b-1} \to \mathbb{R}^r$, $(u_j, z_j) \mapsto \|u_j\|^2 + \|z_j\|^2 - 1$ is submersive, we may by induction assume that all components of $\ell$ are actually positive. In this case it is easy to see that the differential $DF_\ell(u, z)$ is surjective for all $(u, z) \in X_{a,b}(\ell)$ such that $z \neq 0$. If $z = 0$, then we are inside the space of polygons $E_{2a}(\ell)$, and the argument in \cite{12} Thm. 3.1) (or \cite{5} Prop. 3.1) applies.

If $\ell'$ is obtained from $\ell$ as in (2), then changing the sign of some $u_j$ and/or permuting the pairs $(u_j, z_j)$ defines an automorphism of $C^{r(a+b)}$ that carries $X_{a,b}(\ell)$ to $X_{a,b}(\ell')$; this automorphism is equivariant with respect to the corresponding permutation of the components of $T$. This proves (1) for general $\ell$ and also part (2).

The last claim follows from an equivariant version of the Ehresmann fibration theorem: Let $C$ be a chamber and consider the map

$$ (2.6) \quad F: C^{r(a+b)} \times C \to \mathbb{R}^r \times \mathbb{C}^a \times C, \quad (u, z, \ell) \mapsto (F_\ell(u, z), \ell). $$

The first part implies that $F$ is again a submersion, hence so is the restriction $F: F^{-1}(0, C) \to C$, whose fibre over $\ell \in C$ is $X_{a,b}(\ell)$. Now take a non-vanishing vector field $\xi$ on the line segment $L \subset C$ connecting $\ell$ and $\ell'$ and lift it to a vector field $\tilde{\xi}$ on $F^{-1}(L)$ which is perpendicular to $\ker T\tilde{F}$ with respect to some $T$-invariant metric. From the flow of $\tilde{\xi}$ we get a $T$-equivariant diffeomorphism between $X_{a,b}(\ell)$ and $X_{a,b}(\ell')$.  \qed
Assumption 2.2. We assume for the rest of this paper that all length vectors \( \ell \) we consider are generic and of the form \( 0 \leq \ell_1 \leq \cdots \leq \ell_r \).

This is justified by part (2) of Lemma 2.1 given that we are only concerned with cohomological features of the big polygon spaces. Because the chambers are open in \( \mathbb{R}^r \), we may by (3) even assume the \( \ell_j \) to be positive and strictly increasing. This will sometimes be convenient.

3. Non-equivariant cohomology

We now compute the non-equivariant cohomology of \( X_{a,b}(\ell) \); this will also serve as a warm-up for the equivariant situation in the next section. Our general approach is modelled on that of Farber–Fromm [9], [5]. The (equivariant) perfection of the Morse–Bott function however will follow from a simple symmetry argument.

Let \( a, b, r \geq 1 \). We write \( X = X_{a,b}(\ell) \) and introduce the abbreviations

\[
V = S^{2a+2b-1} \subset \mathbb{C}^a \times \mathbb{C}^b, \quad d = \dim V = 2a + 2b - 1,
\]

\[
\bar{V} = V \cap (\mathbb{C}^a \times 0) = S^{2a-1}, \quad \bar{d} = \dim \bar{V} = 2a - 1.
\]

We choose a base point \(* \in \bar{V} \subset V \) and define for \( J \subset [r] \) the Cartesian product

\[
V_J = \{(u, z) \in V^r \mid \forall j \notin J \quad (u_j, z_j) = *\}
\]

and for short \( J \) also

\[
W_J = \{(u, z) \in V^r \mid \forall i, j \notin J \quad u_i = u_j, \quad z_i = z_j = 0\} \cong V_J \times \bar{V},
\]

\[
P_J = \{u \in \bar{V}^r \mid \forall j \in J, \; i, k \notin J \quad u_j = -u_i = -u_k\} \cong \bar{V}.
\]

Then \( V_J \subset W_J \subset V^r \setminus X \) and \( P_J \subset (W_J)^\circ \), moreover

\[
\dim V_J = |J|d, \quad \dim W_J = |J|d + \bar{d}.
\]

(Our \( V_J \) and \( W_J \) correspond to \( V_{J^c} \) and \( W_{J^c} \) in the notation of [5, p. 3105].)

We orient the manifolds \( V_J \) and \( W_J \) as in [9, p. 71]: Let \( J = \{j_1 < \cdots < j_k\} \). The orientations of \( V \) and \( \bar{V} \) give canonical orientations of \( V^k \) and \( V^k \times \bar{V} \) according to the order of the factors. We transport these orientations to \( V_J \) and \( W_J \) via the diffeomorphisms

\[
V_J \to V^k, \quad v \mapsto (v_{j_1}, \ldots, v_{j_k}),
\]

\[
W_J \to V^k \times \bar{V}, \quad v \mapsto (v_{j_1}, \ldots, v_{j_k}, v_i),
\]

where \( i \) is some index not in \( J \).

Define

\[
f : V^r \setminus X \to \mathbb{R}, \quad (u, z) \mapsto -\|u_1 + \cdots + u_r\|^2.
\]

Lemma 3.1. This \( f \) is a Morse–Bott function. Its critical submanifolds are the \( P_J \) for short \( J \). The negative normal bundle of \( P_J \) in \( V^r \setminus X \) is its normal bundle in \( W_J \), and the index of \( P_J \) is \( |J|d \).

Proof. Let \( (v, w) \in \mathbb{C}^{a+b} \) be a tangent vector at \((u, z) \in V^r \setminus X\). Then

\[
Df(u, z) \cdot (v, w) = 2(u_1 + \cdots + u_r, v_1 + \cdots + v_r).
\]

If \( z_j \neq 0 \) for some \( j \), then the map \( (v, w) \mapsto v_1 + \cdots + v_r \) is surjective. Since \( u_1 + \cdots + u_r \neq 0 \), this implies that \((u, z)\) cannot be critical. Hence all critical points satisfy \( z = 0 \). Formula (3.10) shows that the critical points there are those of the
discussion of the $G$-basis of these relative homology groups. (Recall that each $D$ and let $W$. By induction, the $[V_J]$ and $[W_J]$ for short $J$.)

**Proof.** Let $g_1, \ldots, g_r$ be generators of the group $G = (\mathbb{Z}_2)^r$. By letting $g_j$ act as complex conjugation on some coordinate of $z_j$, we get a $G$-action on $Y = V^r \setminus X$ such that $f$ is $G$-invariant. For any short $J$ we have

\[
(3.11) \quad g_j[V_J] = \begin{cases} -[V_J] & \text{if } j \in J, \\ +[V_J] & \text{if } j \notin J, \end{cases}
\]

and analogously for $[W_J]$.

Let $c_1 < \cdots < c_m < 0$ be the critical values of $f$, and for some small $\varepsilon > 0$ set $Z_k = f^{-1}(((-\infty, c_k + \varepsilon)) \subset V^r \setminus X$. Also let $Z_{-1} = \emptyset$. We prove by induction on $k$ that a basis of $H_*(Z_k)$ is given by the short $[V_J]$ and $[W_J]$ such that $f(P_J) \leq c_k$. (In this proof, all homology is with integer coefficients.) For $k = m$ this is the desired result.

Consider the long exact sequence

\[
(3.12) \quad \cdots \to H_*(Z_{k+1}) \to H_*(Z_k) \to H_*(Z_k, Z_{k-1}) \xrightarrow{\delta} H_{*-1}(Z_{k-1}) \to \cdots.
\]

By induction, the $[V_J]$ and $[W_J]$ such that $f(P_J) < c_k$ form a basis for $H_*(Z_{k-1})$. Let $D_-$ be union of the negative normal bundles to the $P_J$ with $f(P_J) = c_k$, and let $S_-$ be the union of the associated sphere bundles. Then $H_*(Z_k, Z_{k-1}) \cong H_*(D_-, S_-)$. By Lemma 3.1 the images of the $[V_J]$ and $[W_J]$ with $f(P_J) = c_k$ form a basis of these relative homology groups. (Recall that each $P_J$ is a sphere.) The discussion of the $G$-action above implies that $H_*(Z_k, Z_{k-1})$ and $H_*(Z_{k-1})$ have no $G$-characters in common. The map $\delta$ in (3.12) therefore is trivial, and the sequence splits. This completes the inductive step. \qed

**Proposition 3.3.** $H^*(X_{a,b}(\ell); \mathbb{Z})$ is free, and its Poincaré polynomial is given by

\[
P(X_{a,b}(\ell), x) = \sum_{J \text{ short}} x^{|J|d} + \sum_{J \text{ long}} x^{|J|d-d-1}.
\]

In particular, the Betti sum of $X_{a,b}(\ell)$ is $2^r$.

**Proof.** We have $H^*(X; \mathbb{Z}) \cong H_{rd-\ell}(V^r, V^r \setminus X; \mathbb{Z})$ by Poincaré–Alexander–Lefschetz duality. Thus, it is enough to verify that this relative homology is free and with Poincaré polynomial

\[
\sum_{J \text{ long}} x^{|J|d} + \sum_{J \text{ short}} x^{|J|d+d+1}.
\]
Proposition 3.6. There is a basis of $E$ the length vector: The Betti sum of $\mathcal{X}$ is realized if and only if $\ell$ if $\alpha$ that $(3.16)$

$$\dim H^*(E_{2a}(\ell)) \leq 2^r - 2\left(\frac{2m}{m}\right)$$

is the fixed point set of the involution of $X_{1,b}(\ell)$ induced by the complex conjugation on $\mathbb{C}^{1+b}$, hence a sort of “real locus” of $X_{1,b}(\ell)$. We note that the mod 2 Betti numbers of $\mathcal{B}C_{b+1}^r(\ell, 0)$ computed in $[13, \text{Thm. 10.3.16}]$ are the same as those of $X_{1,b}(\ell)$, up to degree shifts.

Remark 3.4. The manifold $\mathcal{B}C_{b+1}^r(\ell, 0)$ defined in $[13] \S 10.3.1\footnote{Strictly speaking, the definition of a big chain space in [13] requires all edges to have positive length. However, as in the proof of Lemma 2.1 (3) one can see that $\mathcal{B}C_{b+1}^r(\ell, 0)$ is diffeomorphic to the big chain space $\mathcal{B}C_{b+1}^r(\ell, \varepsilon)$ for small $\varepsilon.$}$ is the product structure of $H^*(X; \mathbb{Z})$.

**Proposition 3.6.** There is a basis of $H^*(X_{a,b}(\ell); \mathbb{Z})$ consisting of elements $\alpha_J$ of degree $|J|d$ (J short) and elements $\beta_J$ of degree $|J|d - d - 1$ (J long) such that $\alpha_{\emptyset} = 1$ and

$$\alpha_J \cup \alpha_K = \begin{cases} (-1)^{|J|K} \alpha_{J \cup K} & \text{if } J \cap K = \emptyset \text{ and } J \cup K \text{ short,} \\ 0 & \text{otherwise,} \end{cases}$$

$$\alpha_J \cup \beta_K = \begin{cases} (-1)^{|J|K} \beta_{J \cup K} & \text{if } J \cap K = \emptyset, \\ 0 & \text{otherwise,} \end{cases}$$

$$\beta_J \cup \beta_K = 0.$$
Proof. We combine Fromm’s approach to the ring structure of the cohomology of spaces of polygons [9, Prop. A.2.4] with representation theory. As in the proof of Lemma 3.2, we use the action of $G = (\mathbb{Z}_2)^r$ on the various spaces. The characters of $G$ are canonically indexed by the subsets of $[r]$ with $\emptyset$ corresponding to the trivial character.

Each $[V_J]$ and $[W_J]$ transforms according to the character $J$. From the proof of Proposition 3.3 we see that each character $J \subset [r]$ occurs in $H_*(V^r, V^r \setminus X; \mathbb{Z})$ with multiplicity 1; the corresponding isotypical component is spanned by the image of $[V_J]$ if $J$ is long and by a preimage of $[W_J]$ if $J$ is short.

From the naturality of Poincaré–Alexander–Lefschetz duality we get the following commutative diagram:

\[
\begin{array}{ccc}
H^*(V^r; \mathbb{Z}) & \longrightarrow & H^*(X; \mathbb{Z}) \\
\downarrow \cong & & \downarrow \cong \\
H_{rd-\ast}(V^r; \mathbb{Z}) & \longrightarrow & H_{rd-\ast}(V^r, V^r \setminus X; \mathbb{Z})
\end{array}
\]

(3.17)

The vertical isomorphisms are induced by the cap product with the fundamental class $[V_r]$ of $V^r$. This implies that they interchange the isotypical components corresponding to $J$ and $J^c$.

We define the $\alpha_J$’s as the duals of the $V_J$. In particular, $\alpha_\emptyset = 1$. Since the $\alpha_J$’s are images of the corresponding elements in $H^*(V^r)$, we get their multiplication rule as well as the last claim. The $\beta_J$’s are duals of the preimages of the $[W_J]$’s. By Schur’s lemma and Poincaré duality, we have $\alpha_{J^c} \cup \beta_J = \pm \beta_{[r]}$; we choose $\beta_J$ such that the sign equals $\sigma_{J^c,J}$.

Now assume that $J$ is short and $K$ long. If $J$ and $K$ are disjoint, then $\alpha_J \cup \beta_K$ must be a multiple of $\beta_{J \cup K}$, again by Schur’s lemma. To see that the scalar is as claimed, we set $L = (J \cup K)^c$ and compute

\[
\begin{align*}
\alpha_L \cup (\alpha_J \cup \beta_K) &= \sigma_{L,J} \alpha_{L \cup J} \cup \beta_K = \sigma_{L,J} \sigma_{L \cup J,K} \beta_{[r]} \\
&= \sigma_{L,J \cup K} (-1)^{(J,K)} \beta_{[r]} = \alpha_L \cup (-1)^{(J,K)} \beta_{J \cup K}.
\end{align*}
\]

(3.18)\hspace{1cm} (3.19)

If $J$ and $K$ are not disjoint, then $\alpha_J \cup \beta_K$ is a multiple of $\alpha_{J \Delta K}$ or $\beta_{J \Delta K}$, depending on whether the symmetric difference $J \Delta K$ is short or long. But the degree of either candidate is strictly smaller than the sum of the degrees of $\alpha_J$ and $\beta_K$. Hence the product vanishes.

The degree of an $\alpha_J$ is congruent to 0 modulo $d$ and that of a $\beta_J$ congruent to $-d-1 \equiv 2b$. Hence the degree of a product $\beta_J \cup \beta_K$ is congruent to $4b$ modulo $d$. Since $d$ is odd and $b \not\equiv 0$, $4b$ is neither congruent to 0 nor to $2b$, which implies that such a product vanishes, too. \hfill \Box

The Walker conjecture (1985) asserted that two generic length vectors $\ell$, $\ell'$ are equivalent if $E_2(\ell)/SO(2)$ and $E_2(\ell')/SO(2)$ have isomorphic integral cohomology rings. This was finally proven by Schütz in 2010, based on work of Farber–Hausmann–Schütz; the analogous question for the spaces of polygons $E_2a(\ell)$ was resolved by Farber–Fromm, see [5] and [13, §10.3.4]. Using Proposition 3.4 we can easily obtain a version for big polygon spaces.

**Proposition 3.7.** Let $\ell$ and $\ell'$ be two generic length vectors. Then $\ell \sim \ell'$ if and only if $H^*(X_{a,b}(\ell); \mathbb{Z}_2)$ and $H^*(X_{a,b}(\ell'); \mathbb{Z}_2)$ are isomorphic as graded rings.

Proof. We only have to do the ‘if’ part; the ‘only if’ is Lemma 2.1.[3].
Since $H^*(X_a,\hat{\ell};\mathbb{Z})$ is torsion-free by Proposition 5.3 (or Proposition 3.3), we have $H^d(X(\ell);\mathbb{Z}) = H^d(X_a,\hat{\ell};\mathbb{Z}) \otimes \mathbb{Z} \mathbb{Z}$; we denote this ring by $A^*(\ell)$. By assumption, $A^*(\ell) \cong A^*(\ell')$ as graded rings.

The $\mathbb{Z}_2$-dimension of $A^1(\ell)$ is the number of $\ell$-short singleton sets. It is either $r$ or $r-1$ as two long sets always intersect. Given that we assume length vectors to be weakly increasing, the latter case implies that any set containing $r$ is long. These are already half of all subsets, so all other subsets are short. Hence $\ell \sim (0,\ldots,0,1)$ in this case. Because the same applies to $\ell'$, we may assume all singleton sets to be $\ell$-short and $\ell'$-short.

From Proposition 3.6 we see that
\[(3.20) \quad A^*(\ell) = H^d(V^r;\mathbb{Z}_2) / \langle \alpha_J \mid J \subset \{r\} \ell\text{-long} \rangle,\]
and analogously for $A^*(\ell')$. From a result of Gubeladze on isomorphic monoid rings (cf. [13, Thm. 4.7.53]) it now follows that $\ell$ and $\ell'$ define the same notion of 'long' and 'short'. (We have excluded the case of long singleton sets to ensure that the abstract simplicial complexes on $\{r\}$ defined by the $\ell$-short and $\ell'$-short subsets indeed have $r$ vertices, as required by the definition in [13, §2.1].) \hfill \Box

4. Equivariant cohomology

It will be convenient to consider equivariant homology along with equivariant cohomology. We therefore start with some general remarks about the former; details can be found in [11, Sec. 3] or [12, Sec. 2]. We remind the reader that this equivariant homology is not the homology of the Borel construction.

Let $X$ be a $T$-manifold of dimension $n$. The equivariant homology $H^T_*(X)$ of $X$ (with compact supports) as well as the equivariant homology $H^T_{s,c}(X)$ with closed supports are modules over the polynomial ring $R = k[t_1,\ldots,t_r]$; for compact $X$ they coincide.

There is a canonical restriction map
\[(4.1) \quad H^T_{s,c}(X) \to H^c_{s}(X) = \text{Hom}(H^s_{s}(X),k);\]
it is the edge homomorphism of a spectral sequence with $E_2 = H^c_{s}(X) \otimes R$ and converging to $H^T_{s,c}(X)$, see [12, Prop. 2.3]. Under this map, any orientation $o \in H^c_{n}(X)$ of $X$ lifts uniquely to an equivariant orientation $o_T \in H^T_{n,c}(X)$ [12, Prop. 3.2].

The equivariant Poincaré duality isomorphism $PD_X : H^s_T(X) \to H^{T,c}_{n-s}(X)$ is the cap product with $o_T$.

Suppose that $X$ is a $T$-stable closed submanifold of a $T$-manifold $Y$ and let $\nu^T_r : H^T_{s,c}(X) \to H^T_{s,c}(Y)$ be the map induced by the inclusion. The orientation of $X$ being understood, we write $[X]_T \in H^T_{s,c}(Y)$ for its image under $\nu^T_r$.

**Proposition 4.1.** Let $T = K \times L$ be a decomposition into subtori, inducing a decomposition $R = R_K \otimes R_L$ (with the obvious meaning). For a $K$-manifold $X$ and an $L$-manifold $Y$ there is an isomorphism of $R$-modules
\[\times : H^K_{s,c}(X) \otimes H^L_{s,c}(Y) \to H^T_{s,c}(X \times Y).\]

For any $K$-stable oriented closed submanifold $M \subset X$ and any $L$-stable oriented closed submanifold $N \subset Y$ one has $[M]_K \times [N]_L = [M \times N]_T$.

**Proof.** We denote by $C^T_{s,c}(-)$ the singular Cartan model for equivariant cohomology with compact supports; its $R$-dual $C^{T,c}_{s}(-) = \text{Hom}_R(C^T_{s,c}(-), R)$ gives rise to
equivariant homology with closed supports, see \cite{2} Sec. 2.3. Also, let $\pi_X$ and $\pi_Y$ be the projections of $X \times Y$ onto $X$ and $Y$, respectively.

The well-known cross product isomorphism in equivariant cohomology (here with compact supports) is induced by the quasi-isomorphism of $R$-algebras

\begin{equation}
C_{K,H}(X) \otimes C_{L,H}(X) \to C_{T,H}(X \times Y), \quad \alpha \otimes \beta \mapsto \pi_X(\alpha) \cup \pi_Y(\beta).
\end{equation}

Moreover, because we assume $H^*_c(X) \cong H_*(X)$ and $H^*_c(Y) \cong H_*(Y)$ to be finite-dimensional, the map

\begin{equation}
\text{Hom}_k(C^*_c(X), k) \otimes \text{Hom}_k(C^*_c(Y), k) \to \text{Hom}_k(C^*_c(X) \otimes C^*_c(Y), k)
\end{equation}

is a quasi-isomorphism of complexes. A spectral sequence argument as in \cite[Rem. 3.3]{1} shows that its equivariant extension

\begin{equation}
C^{K,H}_*(X) \otimes C^{L,H}_*(Y) \to \text{Hom}_R(C^{K,H}_*(X) \otimes C^{L,H}_*(Y), R)
\end{equation}

is a quasi-isomorphism of $R$-modules. Combining it with the Künneth formula and the quasi-isomorphism dual to \eqref{4.2} establishes the isomorphism in equivariant homology.

The last claim follows by verifying that both $[M]_K \otimes [N]_L$ and $[M \times N]_T$ restrict to $[M] \times [N] = [M \times N] \in H^*_c(M \times N)$ according to the way we orient products.

Let $X$ be a closed $T$-stable submanifold of a $T$-manifold $Y$ with both $X$ and $Y$ oriented. The equivariant Euler class of $X \subset Y$ then is $e_T(X \subset Y) = \nu^T_1(1)$, where the push-forward map $\nu^T: H^*_T(X) \to H^*_T(Y)$ is defined as the composition $\nu^T = PD_{\nu}^{-1}\nu^T_1PD_X$. If $G = S^1$ acts by scalar multiplication on $\mathbb{C}$ (with the canonical orientation), then $e_G(\ast \subset \mathbb{C}) = t \in \mathbb{k}[t] = H^*(BG)$. We will need a related case.

**Lemma 4.2.** With the above notation, let $G$ act trivially on $\mathbb{C}^a$ and by scalar multiplication on $\mathbb{C}^b$, and let $S \subset \mathbb{C}^a \times \mathbb{C}^b$ be the unit sphere. Then

$$e_G(S^G \subset S) = t^b.$$ 

Equivalently, $[S^G]_G = t^b \cdot [S]_G \in H^*_G(S)$.

**Proof.** By naturality, we can replace $S$ by the normal bundle $W$ of $S^G$ in $S$. This bundle is trivial, $W \cong S^G \times \mathbb{C}^b$, where $G$ acts by scalar multiplication on $\mathbb{C}^b$, and the product orientation on $W$ coincides with the one inherited from $S$. This implies

\begin{equation}
e_G(S^G \subset S) = e_G(\ast \subset \mathbb{C}^b) = (e_G(\ast \subset \mathbb{C}))^b = t^b.
\end{equation}

Clearly, $t^b \cdot [S]_G$ is Poincaré dual to $t^b \in H^*_G(S)$. So the homological formulation follows once we observe that the restriction map $H^*_G(S) \to H^*_G(S^G)$ is injective. This can be seen by a direct computation, or as follows: Since $S$ and $S^G$ have the same Betti sum, $H^*_G(S)$ is free over $\mathbb{k}[t]$, cf. \cite{3} Thm. 3.10.4. Hence restriction to the fixed point set is injective, for example because of the Chang–Skjelbred sequence \eqref{4.2}.

We are now ready to look at the equivariant cohomology of the big polygon space $X = X_{a,b}(\ell)$. Before starting in earnest, we make a simple observation. It will be sharpened in Proposition \eqref{5.3} that time without appealing to \eqref{6}.

**Lemma 4.3.** $H^*_T(X)$ is not free over $R$. In fact, syzord $H^*_T(X) < r/2$. 


Lemma 4.5. Let $\binom{r}{\ell} = \Delta_{\ell}$ be the equivariant analogue of $\binom{r}{n} = \Delta_{n}$ for short.

Proof. By comparing the Farber–Schütz bound \cite{FarberSchuetz} with Proposition \cite{FarberLaures}, we see that the Betti sum of $X$ is always larger than that of its fixed point set. Similar to the preceding proof, this implies that $H^\ell_{\ast}(X)$ is not free over $R$. The latter claim now follows from Theorem \cite{FarberLaures}.

Let $i^T_\ast: H^\ast_{\ast}(V^r \times X) \to H^\ast_{\ast}(V^r)$ be the equivariant analogue of $i_\ast$.

Lemma 4.4. There is a short exact sequence\footnote{By an argument due to Puppe \cite{Puppe}, one can show that this sequence splits.}

\[ 0 \to (\ker i^T_\ast)[rd] \to H^\ast_{\ast}(X) \to (\ker i^T_\ast)[rd - 1] \to 0. \]

Note that here and throughout we use a cohomological grading. For example, an element $c \in H^k_{\ast}(V^r)$ has degree $-k$, and degree $rd - k$ in $H^\ast_{\ast}(V^r)[rd]$.

Proof. Set $n = \dim V^r = rd$. Because of the naturality of equivariant Poincaré–Alexander–Lefschetz duality \cite[Thm. 3.4]{FarberLaures}, the following diagram is commutative:

\[
\begin{array}{cccccc}
H^{\ast+s-1}(X) & \longrightarrow & H^{\ast+s}(V^r, X) & \longrightarrow & H^{\ast+s}(V^r) & \longrightarrow & H^{\ast+s}(X) \\
\downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
H^T_{\ast+1}(V^r, V^r \times X) & \longrightarrow & H^T_{\ast}(V^r \times X) & \longrightarrow & H^T_{\ast}(V^r) & \longrightarrow & H^T_{\ast}(V^r, V^r \times X).
\end{array}
\]

Lemma 4.5.

1. $H^\ast_{\ast}(V^r)$ is a free $R$-module with basis $[V_j]_T$, $J \subset [r]$.

2. $H^\ast_{\ast}(V^r \times X)$ is a free $R$-module with basis $[V_j]_T$ and $[W_j]_T$, $J$ short.

Proof. The $[V_j]_T$ are preimages of the $[V_j]$ under the restriction map \cite{FarberLaures}. Since the latter form a basis of $H_\ast(V^r)$, this implies, as in the Leray–Hirsch theorem, that the spectral sequence $E_2 = H^\ast_\ast(V^r) \otimes R \Rightarrow H^\ast_{\ast}(V^r)$ collapses and that the $[V_j]_T$ form a basis of $H^\ast_{\ast}(V^r)$ over $R$.

By Lemma \cite{FarberLaures} a basis for $H_\ast(V^r \times X)$ is given by the $[V_j]$ and $[W_j]$ for short $J$. Hence the same proof works for $H^\ast_{\ast}(V^r \times X)$.

Proposition 4.6. For $J$ short,

\[
i^T_\ast[V_j]_T = [V_j]_T; \quad i^T_\ast[W_j]_T = \sum_{j \in J} (-1)^{|J|} t_j^0 \cdot [V_{j \in J}]_T.
\]

Proof. The claim is clear for $V_j$. For $W_j$, consider first the case $J = \emptyset$, where $W_\emptyset = \Delta_J$ is the diagonal of $V^r$. Then

\[
[\Delta_J] = \sum_{j \in [r]} [V_{(j)}] \times \{(*)^{[r] \setminus j}]
\]

in $H_\ast(V^r)$, hence also in $H^\ast_{\ast}(V^r) = H_\ast(V^r) \otimes R$. By naturality and Lemma \cite{FarberLaures} we get

\[
i^T_\ast[W_\emptyset]_T = \sum_{j \in [r]} t_j^0 \cdot [V_{(j)}]_T \in H^\ast_{\ast}(V^r).
\]
The general case now follows from Proposition 4.1: Define the subtori $K = (S^1)^J$ and $L = (S^1)^I$ of $T = (S^1)^r$, where $I = J^c$, and let $\Delta_I \subset V^I$ be the diagonal and $s$ the sign of the shuffle $(J, I)$. Then

$$\iota^*_j[W_J]|_T = s \iota^*_j([V_J]|_K \times [\Delta_I]|_L)$$

$$= s \sum_{j \in J} [V_J]|_K \times (t^b_j \cdot [V_{\{j\}}]|_L) = \sum_{j \not\in J} (-1)^{(J, j)} t^b_j \cdot [V_{J \cup \{j\}}]|_T$$

because of the way the orientation of each $V_{J \cup \{j\}}$ is defined.

5. The equilateral case

We now consider the equilateral case given by $\ell = (1, \ldots, 1) \in \mathbb{R}^r$. This length vector is generic if and only if $r = 2m + 1$ is odd. In this case, a subset $J \subset [r]$ is short if and only if $|J| \leq m$. We are going to identify $H^*_T(X_{a, b}(\ell))$ with the syzygies appearing in the Koszul resolution of $R/(t_{a, 1}^b, \ldots, t_{a, r}^b)$, which we review first.

Let $N$ be an $r$-dimensional $k$-vector space, concentrated in degree $2b$, and let $(e_1, \ldots, e_r)$ be a basis of the the $k$-dual $\bar{N}$ of $N$. (Recall that the generators of $R = \mathbb{k}[t_1, \ldots, t_r]$ have degree 2.) We write $N^{\wedge k}$ for the $k$-th exterior power of $N$. The Koszul resolution of $M = R/(t_{a, 1}^b, \ldots, t_{a, r}^b)$ over $R$ is

$$0 \xrightarrow{\delta_{r-1}} R \otimes N^{\wedge r} \xrightarrow{\delta_r} R \otimes N^{\wedge (r-1)} \xrightarrow{\delta_{r-2}} \cdots$$

$$\xrightarrow{\delta_3} R \otimes N^{\wedge 2} \xrightarrow{\delta_2} R \otimes N \xrightarrow{\delta_1} R \xrightarrow{\delta_0} M \longrightarrow 0$$

with connecting homomorphisms

$$\delta_k : R \otimes N^{\wedge k} \rightarrow R \otimes N^{\wedge (k-1)} \quad f \otimes \alpha \mapsto \sum_{j=1}^r f t^b_j \otimes e_j \lrcorner \alpha$$

for $k > 0$; here $e_j \lrcorner \alpha$ denotes the contraction of $\alpha$ with $e_j$. The map $\delta_0$ is the canonical projection.

For $0 \leq k \leq r + 1$, we define the $k$-th Koszul syzygy to be

$$K_{b, k} = \text{im} \delta_k[-2bk],$$

cf. [1] Sec. 2.4; the degree shift ensures that each $K_{b, k}$ is generated in degree 0.

For example, $K_{b, 0} = M$, $K_{b, 1}[2b] = (t_{a, 1}^b, \ldots, t_{a, r}^b) \triangleleft R$ (which for $b = 1$ is the maximal homogeneous ideal), $K_{b, r} = R$ and $K_{b, r+1} = 0$. It is clear from the definition that $K_{b, k}$ is a $k$-th syzygy. In fact, for $k \leq r$ we have

$$\text{syzord} K_{b, k} = k$$

because otherwise Hilbert’s syzygy theorem would imply that the homological dimension of $K_{b, k}$ is less than $r - k$ and therefore that of $M$ less than $r$. But this is impossible as setting all $t_j = 0$ in the resolution $[5.1]$ gives $\text{Tor}^R_j(M, k) = \mathbb{k}[2br]$.

We write the basis of $N^{\wedge k}$ induced by the chosen basis of $\bar{N}$ as

$$e_J = e_{j_1} \wedge \cdots \wedge e_{j_k} \quad \text{for} \quad J = \{j_1 < \cdots < j_k\} \subset [r].$$

Note that $e_J$ is of degree $-2b|J|$. Because of the self-duality of the resolution $[5.1]$, $K_{b, k+1}[-2b(r - k - 1)]$ and $K_{b, k}[-2b(r - k)]$ are for $1 \leq k \leq r$ the kernel and image,
(5.6) \[ \delta_{r-k} : R \otimes \tilde{N}^{(r-k)} \to R \otimes \tilde{N}^{(r-k+1)}, \]

(5.7) \[ f \otimes e_J \mapsto \sum_{j=1}^{r} f_{j}^b \otimes e_J \wedge e_j = \sum_{j \notin J} (-1)^{|J,j|} f_{j}^b \otimes e_{J,J} \]

**Proposition 5.1.** Let \( r = 2m + 1 \) and \( \ell = (1, \ldots, 1) \in \mathbb{R}^r \). Then

\[ H^*_T (X_{a,b} (\ell)) \cong \bigoplus_{|J| < m} R[|J|d] \oplus K_{b,m}[md] \]

\[ \oplus K_{b,m+2} [(m+2)d - 2\ell - 1] \oplus \bigoplus_{|J| > m+1} R[|J|d - \ell - 1]. \]

In particular, \( \text{syzord} H^*_T (X_{a,b} (\ell)) = m. \)

**Proof.** We start by computing the kernel and cokernel of \( i^*_T \) in the short exact sequence

(5.8) \[ 0 \to (\ker i^*_T)[rd] \to H^*_T (X) \to (\ker i^*_T)[rd - 1] \to 0 \]

from Lemma 4.4. It follows from Proposition 4.6 that for any \( 0 \leq j \leq m \) the restriction

(5.9) \[ i^*_T : \bigoplus_{|J| = j} R[|J|d] \to \bigoplus_{|J| = j+1} R[|J|d] \]

is essentially the map \( \tilde{\delta}_j \) from (5.7), up to a degree shift by \( |W_J| - |e_J| = -(j+1)\ell \), compare (5.6). (Note that we grade homology negatively.) Set \( \tilde{s} = -(m+1)\ell \) and \( s = \tilde{s} - 2b(m+1) = -(m+1)\ell. \) The kernel of \( i^*_T \) is spanned by the elements

(5.10) \[ [W_J]_T - \sum_{j \notin J} (-1)^{|J,j|} \, f_{j}^b : [V_{J,J}]_T \]

for \( |J| < m \) plus the kernel \( K_{b,m+2} [\tilde{s} - 2b(m-1)] = K_{b,m+2} [-(m+1)d - 2\ell] \) of \( \tilde{\delta}_m[\tilde{s}] = \delta_{r-m-1}[\tilde{s}], \) and the cokernel of \( i^*_T \) is spanned by the \( [V_J] \) with \( |J| > m + 1 \) plus the cokernel of \( \delta_{r-m-1}[\tilde{s}], \) which is the image \( K_{b,m} [\tilde{s} - 2b(m+1)] \) of \( \delta_{r-m}[\tilde{s}]. \)

Hence

(5.11) \[ \ker i^*_T \cong \bigoplus_{|J| < m} R[|J|d - \ell] \oplus K_{b,m+2} [-(m+1)d - 2\ell], \]

(5.12) \[ \text{coker } i^*_T \cong K_{b,m} [-(m+1)d] \oplus \bigoplus_{|J| > m+1} R[|J|d]. \]

Next we show that the extension (5.8) is trivial. The free summands of \( \ker i^*_T \) clearly pose no problem. Because \( K_{b,m} \) and \( K_{b,m+2} \) both live in even degrees and their degree shifts in (5.8) differ by an odd number, there is no problem, either. Finally, extensions of the form

(5.13) \[ 0 \to R[l] \to M \to K_{b,l}[l'] \to 0 \]

are always trivial if \( l - l' \neq 2b. \) For \( b = 1 \) this has been shown in [1] Lemma 2.4]; the general case is analogous. The sequence (5.8) thus splits.

Hence \( H^*_T (X) \) is a direct sum of \( m \)-th syzygies by (5.13), so it is an \( m \)-th syzygy itself. This is the maximum possible by Lemma 4.3. \( \square \)
Lemma 5.2. Let $K$, $L$ be tori and set $T = K \times L$. Let $Y$ be a $K$-manifold such that $H^*_K(Y)$ is not free over $R_K$, and let $Z$ be an $L$-manifold such that $H^*_L(Z)$ is free over $R_L$. Then

\[ \text{syzord}_K H^*_T(Y \times Z) = \text{syzord}_{R_K} H^*_K(Y). \]

Proof. Let $j = \text{syzord}_K H^*_K(Y) < r$. Tensoring an exact sequence of the form (1.1) for $H^*_K(Y)$ with $H^*_L(Z)$ over $k$ shows that $H^*_T(Y \times Z) = H^*_K(Y) \otimes H^*_L(Z)$ is again a $j$-th syzygy. That it cannot be a higher syzygy follows from the characterization of syzygies in terms of regular sequences, cf. \[4, \text{App. E}\]: Since $H^*_K(Y)$ is not a syzygy of order $j$, there exists a regular sequence $f_1, \ldots, f_{j+1}$ in $R_K \subset R$ which is not regular for $H^*_K(Y)$. Hence the same sequence cannot be regular for $H^*_K(Y) \otimes H^*_L(Z)$, so $H^*_T(Y \times Z)$ is not a syzygy of order $j + 1$. \hfill \Box

We have seen in Lemma 4.3 that $\text{syzord}_H^*_T(X_{a,b}(\ell))$ is less than $r/2$. Now we can deduce that big polygon spaces actually realize all syzygy orders less than $r/2$. In particular, the bound given by Theorem 1.1 is sharp for any $r$.

Corollary 5.3. Let $m \geq 0$ and $r \geq 2m + 1$ and set

\[ \ell = (0, \ldots, 0, 1, \ldots, 1) \subseteq \mathbb{R}^r. \]

Then $\text{syzord}_H^*_T(X_{a,b}(\ell)) = m$.

Proof. Write $k = r - (2m + 1)$. We have $X_{a,b}(\ell) = V^k \times X_{a,b}(1, \ldots, 1)$ by (2.4). $H^*_K(V^k)$ is free over $R_K$ for the induced action of $K = (S^1)^k$, for example because the fixed point set is again a product of $k$ spheres. Now apply Lemma 5.2 \hfill \Box

6. The general case

In this section $\ell \in \mathbb{R}^r$ is again a generic length vector, which we assume to be non-negative and weakly increasing.

Lemma 6.1. Consider the short exact sequence of finitely generated $R$-modules

\[ 0 \to M \to M' \to M'' \to 0. \]

If $\text{syzord } M'' > \text{syzord } M$, then $\text{syzord } M' = \text{syzord } M$.

Proof. Recall that a finitely generated $R$-module $N$ is a $k$-th syzygy if and only if

\[ \text{depth } N_p \geq \min(k, \text{depth } R_p) \]

for any prime ideal $p \lhd R$ [4, App. E]. Here ‘depth $N_p$’ refers to the depth over $R_p$.

Localizing the given short exact sequence at $p$, we get the short exact sequence

\[ 0 \to M_p \to M'_p \to M''_p \to 0. \]

Together with (6.1), the usual bounds for the depth of modules [4, Prop. 16.14]

\[ \text{depth } M_p \geq \min(\text{depth } M'_p, \text{depth } M''_p + 1), \]

\[ \text{depth } M'_p \geq \min(\text{depth } M_p, \text{depth } M''_p) \]

imply depth $M_p = \text{depth } M'_p$, which proves the claim. \hfill \Box

Lemma 6.2. $\text{syzord}_H^*_T(X_{a,b}(\ell)) = \text{syzord coker } i^*_T$. 

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Proof. Consider the exact sequence
\begin{equation}
0 \to \ker \iota^T \to H^T_*(V^T \setminus X_{a,b}^T) \xrightarrow{\iota^T_*} H^T_*(V^T) \to \coker \iota^T_* \to 0,
\end{equation}
whose two middle terms are finitely generated free \( R \)-modules by Lemma 4.5. By splicing \( \mu \) together with the exact sequence (1.1) for \( M = \coker \iota^T_* \), we see that \( \sigma_{\ell}(J) = \min\{ \sigma_{\ell}(J) \mid J \text{ \( \ell \)-long and } \sigma_{\ell}(J) > 0 \} \).

Note that there is always a long \( J \) such that \( \sigma_{\ell}(J) > 0 \). For instance, if \( J \) is a long subset of minimal size, then \( J \neq \emptyset \) and \( \sigma_{\ell}(J) = |J| > 0 \).

**Proposition 6.3.** \( \text{syzord} H^T_*(X_{a,b}^T) \leq \mu(\ell) - 1 < r/2 \).

Proof. By Lemma 2.1 the syzygy order of \( H^T_*(X_{a,b}^T) \) depends only on the chamber containing \( \ell \), and the same holds for \( \mu(\ell) \). So we may assume \( \ell \) to be positive. By Lemma 6.2 to show the first inequality it is enough to verify that \( M = \coker \iota^T_* \) is not a syzygy of order \( k = \mu(\ell) \).

From Proposition 4.6 we know that \( M \) is generated by the \( [V_j]^T \) for long \( J \), subject to the relations
\begin{equation}
\sum_{j \not\in J} (-1)^{(j,j)} t_j \cdot [V_{j\cup j}]^T = 0
\end{equation}
for short \( J \). Because the short sets are the complements of the long ones, there is a short \( J \) such that \( J \cup j \) is long only for \( k \) values \( j = j_1, \ldots, j_k \). We claim that the regular sequence \( t_{j_1}, \ldots, t_{j_k} \in R \) is not \( M \)-regular: The image of \( [V_{j_k}]^T \) in the quotient \( M/[t_{j_1}, \ldots, t_{j_k-1}]M \) is non-zero, but \( t_{j_k}[V_{j_k}]^T = 0 \) there because of (1.8). Hence multiplication by \( t_{j_k} \) is not injective. Since there is a regular sequence in \( R \) of length \( k \) which is not \( M \)-regular, \( M \) is not a \( k \)-th syzygy.

The second inequality, which together with the first reproves Lemma 4.3, follows by looking at long subsets of minimal size, keeping in mind that half of all subsets are long.

**Corollary 6.4.**

1. Assume that \( r = 2m+1 \) is odd. Then \( \text{syzord} H^T_*(X_{a,b}^T) = m \) if and only if \( \ell \sim (1,\ldots,1) \).
2. Assume that \( r = 2m+2 \) is even. Then \( \text{syzord} H^T_*(X_{a,b}^T) = m \) if and only if \( \ell \sim (0,1,\ldots,1) \).

Together with Lemma 2.1, 4.3 this proves Theorem 1.2.

Proof. We have seen in Section 5 that the length vectors \((1,\ldots,1) \in \mathbb{R}^{2m+1} \) and \((0,1,\ldots,1) \in \mathbb{R}^{2m+2} \) produce syzygies of order exactly \( m \).

For the converse, we start with the case of odd \( r \). By Proposition 6.3 all \( \ell \)-long subsets have size at least \( m+1 \). This holds for half of all subsets of \( [r] \). At the same time, half of all subsets are long. So we see that the \( \ell \)-long subsets are exactly those with at least \( m+1 \) elements. Hence \( \ell \sim (1,\ldots,1) \).
We now turn to the even case; by Lemma 2.3 we may assume $\ell$ to be strictly increasing. The long subsets have again size at least $m+1$. Among all long subsets $J$ of size $m+1$, pick one with minimal $\ell(J)$. (Since half of all subsets are long, it is impossible that there is no long subset of this size.) Let $j_{\min}$ and $j_{\max}$ be the minimal and maximal element of $J$, respectively. Set $I = J \setminus \{j_{\max}\}$; it is short. There must be at least $m+1$ elements $j \notin I$ such that $I \cup j$ is long for otherwise $\mu(\ell) \leq \sigma(\ell^c) < m + 1$. Since we have chosen a $J$ with minimal $\ell(J)$, this can only happen if $\ell_j \geq \ell_{\max}$, i.e., if $j \geq j_{\max}$. Hence there are at least $m$ elements $j$ greater than $j_{\max}$. If $j_{\min} = 1$, then $\ell(J) \leq \ell(J^c)$, so $J$ would not be long. Thus, $J = \{2, 3, \ldots, m+2\}$. Since $J^c = \{1, m+3, \ldots, r\}$ is short, so must be all subsets possibly containing 1 and up to $m$ other elements. But these are already half of all subsets. We conclude that the short subsets are exactly those which contain at most $m$ elements greater than 1. This is the same notion of ‘short’ as given by the length vector $(0, 1, \ldots, 1)$. \hfill $\Box$

**Remark 6.5.** Comparing Corollary 6.4 with Remark 6.3, we see that $\text{syzord} H^*_T(X)$ is largest exactly for those big polygon spaces which maximize the Betti sum of the fixed point set. This reminds of the general fact, cf. Section 4, that the equivariant cohomology of a $T$-manifold $Y$ is free (a syzygy of order $r$) if and only if the Betti sum of $Y^r$ is as big as possible, namely equal to the Betti sum of $Y$.

However, there seems to be no relation between $\text{syzord} H^*_T(X)$ and $\dim H^*(X^r)$ for big polygon spaces in general. For example, we have

\begin{align}
(6.9) & \quad \text{syzord} H^*_T(X_{a,b}(0,0,0,1,1,1)) = 1 > 0 = \text{syzord} H^*_T(X_{a,b}(1,2,2,2,3,3)) \\
(6.10) & \quad \dim H^*(E_{2a}(0,0,0,1,1,1)) = 32 < 36 = \dim H^*(E_{2a}(1,2,2,2,3,3)).
\end{align}

**Conjecture 6.6.** $\text{syzord} H^*_T(X_{a,b}(\ell)) = \mu(\ell) - 1$.

Using Macaulay2 [11] and the lists of chambers computed by Hausmann–Rodriguez and Wang [14], this has been verified for all chambers in dimensions $r \leq 9$.

7. Comments

7.1. The mutants. In [8, Sec. 4] Puppe and the author constructed three examples of compact orientable $T$-manifolds such that $\text{syzord} H^*_T(X) = 1$. These spaces were called “mutants of compactified representations”. We sketch a proof that $Z_3$, the smallest of those examples, is equivariantly homeomorphic to the big polygon space $X = X_{1,1}(1,1,1)$. The mutants for $r = 5$ and $r = 9$ are not big polygon spaces, however: While they have the same dimension and the same Betti sum as $X_{1,1}(\ell)$ for $\ell \in \mathbb{R}^r$, the individual Betti numbers differ.

We start by observing that the quotient $X/T$ can be identified with the subspace $X_+$ of $X$ where all coordinates $z_j$ are non-negative real numbers. Moreover, the restriction $T \times X_+ \to X$ of the action displays $X$ as an identification space,

\begin{equation}
X = \left( T \times X_+ \right) / \sim.
\end{equation}

Here two points $(g,u,z)$, $(g',u',z') \in T \times X_+ \subset T \times \mathbb{C}^{ra} \times \mathbb{C}^{rb}$ are identified if $(u,z) = (u',z')$ and if $g^{-1}g'$ lies in the coordinate subtorus

\begin{equation}
\{ g \in T = (S^1)^3 \mid g_j = 1 \text{ if } z_j \neq 0 \} \subset T.
\end{equation}
The mutant $Z_2$ is an identification space by construction,

$$Z_2 = (T \times Q) / \sim.$$  

Here $Q$ is a 4-ball, and the non-trivial identifications happen over the 3-sphere $\partial Q$ in the following way: Take a 2-sphere and divide it into three spherical digons. (This is the boundary of the orbit space of the compactified standard representation of $T$ on $\mathbb{C}^3$ with its partition into orbit types.) Lift this partition along the Hopf fibration $\partial Q \approx S^3 \to S^2$. The subtori needed for the identification space (7.3) then are the isotropy groups occurring in $\mathbb{C}^3$, which are again the coordinate subtori of $T$.

Hence it is enough to find a homeomorphism between the two orbit spaces $X_+$ and $Q$ that respects the partitions used for the identifications. Let $D \subset \mathbb{C}^3$ be the unit ball with respect to the maximum norm $\|u\|_\infty = \max(\|u_1\|, \|u_2\|, \|u_3\|)$. Since in $X_+$ all coordinates $z_j$ are non-negative real numbers, the projection

$$X_+ \to \mathbb{C}^3, \quad (u, z) \mapsto u$$

is a homeomorphism onto the intersection of $D$ with the subspace $u_1 + u_2 + u_3 = 0$; call it $P$. (This is the configuration space of all triangles, possibly degenerate, with sides of length at most 1.) Note that $P$ is homeomorphic to a 4-ball, and that the non-trivial identifications in (7.1) happen exactly over its boundary $\partial P \approx S^3$.

Now consider the map

$$p: \partial P \to \mathbb{C} \times \mathbb{R}, \quad u \mapsto \left( \sum_{j=1}^3 (1 - |u_j|) \lambda_j, A(u) \right)$$

where $A(u)$ is the oriented area of the polygon with sides $u_1$, $u_2$, $u_3$, and the $\lambda_j$’s are the cubic roots of unity.

**Lemma 7.1.** The image $B$ of $p$ is homeomorphic to a 2-sphere, and $p: \partial P \to B$ is the Hopf fibration.

**Proof.** We start by showing that the image of the map

$$\bar{p}: \partial P \to \mathbb{C}, \quad u \mapsto \sum_{j=1}^3 (1 - |u_j|) \lambda_j$$

is a triangle. Since $u \in \partial P$, at least one the $u_j$’s has length 1, say $u_3$. Then $\bar{p}(u)$ lies inside the triangle with vertices $\lambda_1$, $\lambda_2$ and the origin. In fact, it is the whole triangle because the only restriction on the lengths $|u_j| \in [0, 1]$ is the triangle inequality, which under the assumption $|u_3| = 1$ translates into the equation

$$(1 - |u_1|) + (1 - |u_3|) \leq 1 + (1 - |u_3|) = 1.$$

Thus, the image of $\bar{p}$ is the triangle spanned by the $\lambda_j$’s. The argument also shows that one can recover the lengths $|u_j|$ from $\bar{p}(u)$.

If the lengths are known, there are only two choices for the oriented area $A(u)$, except when all sides are parallel, in which case there is only one. The latter case corresponds to the boundary of the triangle $\bar{p}(\partial P)$. Hence the image of $p$ consists of two triangles, glued together at their boundaries. This gives a 2-sphere. It is obvious that $p$ is invariant under rotations of the complex plane. This action is free since at most one $u_j$ can be zero. Moreover, the side lengths and the oriented area determine the triangle up to rotation. So $p$ is a principal $S^1$-fibration. Since domain and codomain are spheres, it must be the Hopf fibration. \qed
One readily checks that the partition of $X_+$ by orbit type corresponds to a partition of $B$ into three spherical digons, each containing both poles and one edge of the triangle forming the equator. Because this is the same partition as for the mutant, this proves that $X$ and $Z$ are $T$-equivariantly homeomorphic.

7.2. Connected sums of products of spheres. Assume $r = 3$ and consider $u_1$ and $u_2$ as elements of $\mathbb{R}^{2a}$. Introduce new variables $\tilde{u} \in \mathbb{C}^{2a}$ and $\tilde{z}_1, \tilde{z}_2, \tilde{z}_3 \in \mathbb{C}$ via

$$
(7.8) \quad \tilde{z}_k = \frac{z_k}{\sqrt{3}}, \quad \tilde{u} = \frac{u_1 + u_2}{\sqrt{2}} + i \frac{u_1 - u_2}{\sqrt{6}}.
$$

It is elementary, but somewhat tedious to verify that $X_{a,b}(1,1,1)$ can be defined by the equations

$$
(7.9) \quad \lambda_1 \|\tilde{z}_1\|^2 + \lambda_2 \|\tilde{z}_2\|^2 + \lambda_3 \|\tilde{z}_3\|^2 + \sum_{l=1}^{2a} \tilde{u}_l^2 = 0,
$$

$$
(7.10) \quad \|\tilde{z}_1\|^2 + \|\tilde{z}_2\|^2 + \|\tilde{z}_3\|^2 + \|\tilde{u}\|^2 = 1,
$$

where $\lambda_k = 2 e^{2\pi ki/3}$ are the cubic roots of $8$. The intersection of the two real quadrics (7.9) has an isolated singularity at the origin, and equation (7.10) exhibits $X_{a,b}(1,1,1)$ as its link.

Because the origin is in the interior of the triangle spanned by the $\lambda_k$'s, it follows from a result of Gómez Gutiérrez and López de Medrano [10, Main Thm.] that this manifold is diffeomorphic to a connected sum of products of spheres,

$$
(7.11) \quad X_{a,b}(1,1,1) \cong \#_3 S^{2a+4b-1} \times S^{2a+4b-2};
$$

for $a = b = 1$ a homeomorphism of this form was already established in [8 Sec. 7]. This is essentially the only case where this happens, apart from the trivial case with a single summand, cf. (2.3).

$$
(7.12) \quad X_{a,b}(0,\ldots,0,1) = (S^{2a+2b-1})^{r-1} \times S^{2b-1}.
$$

Proposition 7.2. If $X_{a,b}(\ell)$ has the cohomology algebra of a connected sum of products of spheres, then either $\ell \sim (0,\ldots,0,1)$ or $\ell \sim (1,1,1)$.

Proof. Recall first that the chamber given by $\ell = (0,\ldots,0,1)$ is the only one for $r \leq 2$ (assuming that $\ell$ is non-negative and weakly-increasing); for $r = 3$ there is exactly one more, given by $\ell = (1,1,1)$, cf. [13] p. 448.

Write $X = X_{a,b}(\ell)$. By (7.12) we can assume $\ell \not\sim (0,\ldots,0,1)$. Then $r \geq 3$, and all singleton sets are short. Assume

$$
(7.13) \quad H^*(X) \cong H^*(Y_1 \# \cdots \# Y_k),
$$

where each $Y_i$ is a product of at least two spheres. By Proposition [3.3] the dimension of each sphere must be at least $d$, and if $r_i$ denotes the number of $d$-spheres in $Y_i$, then $r_1 + \cdots + r_k = \dim H^d(X) = r$.

As remarked after Proposition [6.6] $H^d(X)$ generates a subalgebra of dimension $2^{r-1}$. Since each $Y_i$ is a product of spheres, the subalgebra generated by $H^d(Y_i)$ has dimension $2^{r_i}$ (and vanishes in degree $\dim X$). Hence (7.13) implies

$$
(7.14) \quad (2^{r_1} - 1) + \cdots + (2^{r_k} - 1) = 2^{r-1} - 1.
$$

Clearly, one solution is $k = 3$ and $r_1 = r_2 = r_3 = 1$. In this case we have $\ell \sim (1,1,1)$ by the remark made at the beginning. We claim that there is no other solution.
This claim is obviously true for \( k = 1 \). For \( k = 2 \) the equation (7.14) is not satisfied if \( r_1 = 1 \) or \( r_2 = 1 \) or \( r_1 = r_2 = 2 \). In the latter case, the right-hand side dominates, as it does for \( k \geq 4 \) and \( r_1 = \cdots = r_k = 1 \). To finish the proof, it suffices to observe that whenever one has the inequality “\( \leq \)” in (7.14) and \( r_i < r - 1 \) for some \( i \), then increasing \( r_i \) makes the right-hand side dominate strictly.

**Remark 7.3.** One can write any \( X_{a,b}(\ell) \) as the link of an intersection of \( r-1 \) homogeneous quadrics: The sum of all \( r \) equations (1.4) defines a sphere \( S \) in \( \mathbb{C}^{r(a+b)} \), and subtracting a multiple of this equation from the other ones makes them homogeneous. Eliminating some variables disposes of (1.5).

Let \( Y \) be the real algebraic variety defined by \( r-1 \) of the homogeneous quadrics thus obtained. It is smooth at the points lying on the sphere \( S \) because together with the equation for \( S \) these quadrics define the manifold \( X_{a,b}(\ell) \). By homogeneity, this implies that \( Y \) has at most an isolated singularity at the origin, with link \( X_{a,b}(\ell) \).

**7.3. Minimal dimension.** We have seen that for any \( m \geq 0 \) there are compact orientable \( T \)-manifolds whose \( T \)-equivariant cohomology is not free and a syzygy of order exactly \( m \), namely the equilateral big polygon spaces \( X_{a,b}(1, \ldots, 1) \). Here the torus rank is \( r = 2m + 1 \), which is minimal by Theorem 1.1. The dimension of \( X_{a,b}(1, \ldots, 1) \) is at least \( n = 6m + 1 \); this value is realized for \( a = b = 1 \). For \( m = 0 \) this is clearly the minimum dimension possible as any torus action on a discrete space is trivial. More surprisingly, it is also minimal for \( m = 1 \). This follows from the bound on the torus rank together with the following consequence of the quotient criterion for syzygies in equivariant cohomology [7, Sec. 7.2]:

**Proposition 7.4.** Let \( X \) be a compact orientable \( T \)-manifold such that \( H_T^*(X) \) is torsion-free, but not free over \( R \). Then \( \dim X \geq 2r + 1 \).

**Question 7.5.** For \( m \geq 2 \), do examples of maximal syzygies exist in dimension smaller than \( 6m + 1 \)?

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