An H-theorem for incompressible fluids

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Abstract

A basic aspect of the kinetic descriptions of incompressible fluids based on an inverse kinetic approaches is the possibility of satisfying an H-theorem. This property is in fact related to the identification of the kinetic distribution function with a probability density in an appropriate phase space. Goal of this investigation is to analyze the conditions of validity of an H-theorem for the inverse kinetic theory recently proposed by Ellero and Tessarotto [2004, 2005]. It is found that the time-dependent contribution to the kinetic pressure, characteristic of such a kinetic model, can always be uniquely defined in such a way to warrant the constancy of the entropy.

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Incompressible Navier-Stokes equations: kinetic theory; H-theorem.

1 Introduction

The possibility of defining an inverse kinetic theory for the 3D incompressible Navier-Stokes equations (INSE; see Appendix A), recently pointed out\textsuperscript{1, 2, 3}, raises the interesting question whether the relevant kinetic distribution function satisfies an H-theorem, namely the related (Shannon) kinetic entropy can be specified in such a way to result monotonically non-decreasing in time for arbitrary fluid fields defined in a internal domain, to be identified with a bounded three-dimensional domain $\Omega \subseteq \mathbb{R}^3$ (fluid domain). The validity of such a theorem is in fact a sufficient condition of strict positivity of the

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distribution function. Therefore, this result is important in order to establish also the consequent interpretation of the kinetic distribution function, to be suitably normalized, in terms of a probability density on an appropriate phase space $\Gamma$.

Purpose of this Note is to evaluate the kinetic entropy and the related entropy production rate for strictly positive, suitably smooth, but otherwise arbitrary, distribution functions $f(x,t)$ which correspond to an arbitrary strong solution of the initial-boundary value problem of INSE (see Appendix A and Refs. [3, 4]), defined for an internal domain of $\mathbb{R}^3$, and for kinetic distributions functions which are not necessarily Maxwellian. We intend to prove that, under a suitable assumptions, which involve the specification of the only non-observable free parameter of the theory, a strictly positive time-dependent additive contribution to the kinetic pressure, the kinetic entropy results identically conserved, thus yielding an H-theorem for the kinetic distribution function. The conclusion is obtained invoking mild assumptions both on the initial kinetic distribution function and on the fluid fields $\{\rho(r,t) = \rho_o > 0, V(r,t), p(r,t)\}$.

In the sequel we intend to show that the proof of this statement, and hence of the positivity of the kinetic distribution function, relies essentially only on the following assumptions:

a) the strict positivity of the initial distribution function $f(x,t_o)$ in the whole phase space $\Gamma$;

b) the requirement that $f(x,t_o)$ belongs to the functional class

$$f(x,t_o) \in C^{(1, 1)}(\Omega \times I), \quad (1)$$

and results suitably summable in $\Gamma$ and smooth in $\Gamma \times I$;

c) no-slip, Dirichlet boundary conditions are imposed on the fluid fields (see Appendix A);

d) the total mass of the fluid is conserved, i.e.,

$$\int_{\Omega} d^3r \rho(r,t) = M = const. \quad (2)$$

e) the fluid is subject to a volume force density $f(r,t)$ which is assumed suitably smooth, precisely at least

$$f(r,t) \in C^{(1, 0)}(\Omega \times I), \quad (3)$$

$$f(r, v, t) \in C^{(0)}(\Omega \times I), \quad (4)$$

where $C^{(i,j)}(\Omega \times I) \equiv C^{(i)}(\Omega) \times C^{(j)}(I)$, with $i, j \in \mathbb{N}$;
f) the existence of a strong solution of INSE exists in $\Omega \times I$ which belongs to the functional class:

\[
\begin{cases}
V(r,t), p(r,t) \in C^{(0)}(\Omega \times I), \\
V(r,t), p(r,t) \in C^{(2,1)}(\Omega \times I),
\end{cases}
\]

(5)

with $I$ denoting the time axis, generally to be identified with a finite subset of $\mathbb{R}$;

g) the determination of the time-dependent term of the kinetic pressure ($p_1$). It is found that $p_1$ can always be defined in the same time interval $I$, up to an arbitrary positive constant, in such a way that the entropy production rate vanishes identically.

The basic result can be summarized as follows.

**Theorem 1 - H-theorem (entropy conservation for the inverse kinetic theory of INSE)**

Let us denote by $f(x(t),t)$ the kinetic distribution function solution, assumed to exist and result suitably regular, of the inverse kinetic equation represented in the form

\[ f(x(t),t) = T_{t,t_0}f(x_{o},t_{o}), \]

(6)

$x = (r, v)$ being a state vector belonging to the phase space $\Gamma = \Omega \times U$ and $U = \mathbb{R}^3$ the velocity space. $f(x_{o},t_{o})$, the initial kinetic distribution function, is assumed to be suitably smooth in the sense of Eq.(1), strictly positive and summable in $\Gamma$ for appropriate weight functions. Moreover, $T_{t,t_0}$ is a suitably-defined diffeomorphism, denoted as Navier-Stokes evolution operator, such that $\forall t \in I$, including the initial time $t_{o} \in I$, the fluid fields at any time $t \in I$, $\{\rho_o, V(r,t), p(r,t)\}$ are uniquely defined by the following velocity moments of the kinetic distribution function:

\[ \rho_o = \int d^3v f(x,t) = \rho_o = \int d^3v T_{t,t_0}f(x_{o},t_{o}), \]

(7)

\[ V(r,t) = \frac{1}{\rho_o} \int d^3v vf(x,t) = \frac{1}{\rho_o} \int d^3v T_{t,t_0}f(x_{o},t_{o}), \]

(8)

\[ p(r,t) = p_1(r,t) - P_o, \]

(9)

where

\[ p_1(r,t) = \int d\mathbf{v} E f(x,t) = \int d\mathbf{v} ET_{t,t_0}f(x_{o},t_{o}). \]

(10)

Here $E \equiv \frac{1}{2}u^2$, with $u = v - V$, while $P_o = P_o(t)$ is a smooth strictly positive function. Then, the following results follow:
1) provided the initial distribution function \( f(x_o, t_o) \) results strictly positive and suitably smooth in the sense of (1), it follows that the velocity moments 
\[
F_G(r, t) = \int \nu G(x, t) f(x, t) \mathrm{d}^3v,
\]
where \( G(x, t) = 1, \nu, E \equiv \frac{1}{3}u^2, \nu u, uE \) and \( u = v - V \), exist, are continuous \( \Omega \times I \) and suitably smooth in \( \Omega \times I \) in the sense of the settings (2),(3),(4) and (5). In addition, the kinetic entropy
\[
S(t) = -\int_G \mathrm{d}x f(x, t) \ln f(x, t)
\]
exists and is suitably smooth in \( I \);

2) provided the kinetic pressure \( p(r, t) = \int \nu E f(x, t) \) is suitably prescribed, there results identically in \( I \)
\[
S(t) = S(t_o)
\]
(law of entropy conservation).

For greater clarity, in Sec.2 the relevant aspects of the inverse kinetic theory previously developed are recalled [3]. This is useful to introduce the Navier-Stokes evolution operator \( T_{t,t_o} \) and define the related probability density in the phase space \( G \). As a consequence, it is immediate to prove that \( T_{t,t_o} \) conserves probability in the same space. Subsequently, in Sec. 3, the kinetic entropy the \( S(t) \) and its time derivative \( \partial S(t)/\partial t \) are evaluated. It is found that by suitably defining the non-negative kinetic pressure \( (p_1) \) the entropy production rate can always be set equal to zero in the whole time interval \( I \) in which by assumption a strong solution of INSE exists. Implications of the result are pointed out.

2 The Navier-Stokes evolution operator

In this Section we briefly recall the formulation of the inverse kinetic theory developed in [3]. This is useful in order to identify the Navier-Stokes evolution operator \( T_{t,t_o} \) which determines the evolution of the relevant kinetic distribution function \( f(x, t) \) in the extended phase space \( G \times I \) ( where \( G \equiv \Omega \times U \), with \( U \equiv \mathbb{R}^3 \) denoting a suitable "velocity" space).

The result is obtained by requiring the \( f(x, t) \) obeys a Vlasov kinetic equation of the form
\[
L(F)f = 0,
\]
where \( x \equiv (r, v) \) is a state vector spanning the phase space \( G \), \( L \) the streaming operator
\[
L(F) = \frac{\partial}{\partial t} + \frac{\partial}{\partial x} \cdot \{X\}
\]
and $X$ a vector field of the form $X(x,t) = \{v, F(x,t)\}$. This equation can formally be written in integral form by introducing the initial kinetic distribution function $f(x_0, t_0) \equiv f_0(x_0)$, defined at the initial time $t_0 \in I$, and the flow, i.e., the the diffeomorphism $x_0 \rightarrow x(t) = T_{t,t_0}x_0$ generated by the vector field $X$, via the initial-value problem

$$
\begin{cases}
\frac{dx}{dt} = X(x,t) \\
x(t_0) = x_0
\end{cases}
$$

(15)

and its related evolution operator $T_{t,t_0}$ (Navier-Stokes evolution operator). This implies that provided the solution of the initial-value problem exists, is unique and suitably smooth, the Jacobian of the flow $x_0 \rightarrow x(t)$, $J(x(t),t) \equiv \left| \frac{\partial x(t)}{\partial x_0} \right|$ is non singular and reads

$$J(x(t),t) = \exp \left\{ \int_{t_0}^{t} dt' \frac{\partial}{\partial v(t')} \cdot F(x(t'),t') \right\}.
$$

(16)

Thus, the evolution operator, acting on the kinetic distribution function $f_0(x_0)$ results

$$f(x(t),t) = T_{t,t_0}f_0(x_0) \equiv f_0(x_0) \exp \left\{ -\int_{t_0}^{t} dt' \frac{\partial}{\partial v(t')} \cdot F(x(t'),t') \right\}.
$$

(17)

Previously it has been shown [4] that the functional form of the vector field $X$ and of the "mean-field force" $F(x,t)$ yielding and inverse kinetic theory for INSE can be uniquely determined under suitable prescriptions. These include, in particular, the requirements that:

1) the local Maxwellian kinetic distribution function

$$f_M(x,t; V,p_1) = \frac{\rho_o}{(\pi)^{\frac{d}{2}} v_{th}^d} \exp \left\{ -X^2 \right\},
$$

(18)

[where $X^2 = \frac{u^2}{v_{th}^2}, v_{th}^2 = 2p_1/\rho_o$ and $u$ is the relative velocity $u \equiv v - V(r,t)$] results a particular solution of the inverse kinetic equation if and only if $\{\rho, V, p\}$ satisfy INSE;

2) suitable bounce-back boundary conditions are imposed for the kinetic distribution function on the boundary $\delta \Omega$ [3];

3) the moment equations corresponding to the velocity moments $M_G(r,t) = \int d^3v G(x,t)f(x,t)$ for $G(x,t) = 1, v/\rho_o, E = \frac{1}{3} u^2$ coincide with the differential equations of INSE;
4) the fluid fields $\rho_o$ and $\mathbf{V}(\mathbf{r}, t)$ are identified respectively with the velocity moments for $G(x, t) = 1, v/\rho_o$; similarly, the fluid pressure $p(\mathbf{r}, t)$ is defined in terms of the kinetic pressure $p_1(\mathbf{r}, t)$ [see Eq.(9)] by requiring

$$p(\mathbf{r}, t) = p_1(\mathbf{r}, t) - P_o.$$  \hspace{1cm} (19)

It is obvious, in order that $\nabla p = \nabla p_1$, that $P_o$ can be in principle an arbitrary strictly positive function independent of $\mathbf{r}$. Thus it can always be assumed to be $\forall t \in I$ a smooth function of $t$. The resulting form of $\mathbf{F}(\mathbf{r}, \mathbf{v}, t)$ is recalled in Appendix B. It implies

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{x}, t) = \frac{3}{2p_1} \left\{ \frac{D}{Dt} p_1 + \nabla \cdot \mathbf{Q} + \frac{1}{p_1} \mathbf{u} \cdot \nabla \Pi \right\} + \frac{1}{p_1} \mathbf{u} \cdot \nabla p,$$  \hspace{1cm} (20)

where $\frac{D}{Dt} \equiv \frac{\partial}{\partial t} + \mathbf{V} \cdot \frac{\partial}{\partial \mathbf{r}}$ is the Lagrangian (or convective) derivative. For $f \equiv f_M$ it becomes

$$\frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{x}, t) = \frac{3}{2p_1} \frac{D}{Dt} p_1 + \frac{1}{p_1} \mathbf{u} \cdot \nabla p,$$  \hspace{1cm} (21)

which implies that if $f \equiv f_M$ at a given time. These expressions, in particular (20) and (21), permit to determine uniquely the Jacobian $J(x(t), t)$ and the evolution operator $T_{t,t_o}$. Thus, introducing the normalized kinetic distribution function

$$\hat{f}(\mathbf{x}, t) = \frac{1}{\rho_o} f(\mathbf{x}, t)$$  \hspace{1cm} (22)

and requiring that the initial kinetic distribution function $\hat{f}(\mathbf{x}_o, t_o) \equiv \hat{f}_o(\mathbf{x}_o)$ results at least of class $C^1(\Gamma \times I)$ and summable in $\Gamma$ it follows

$$d\mathbf{x}(t) \hat{f}(\mathbf{x}(t), t) = d\mathbf{x}(t_o) \hat{f}(\mathbf{x}(t_o), t_o) \equiv d\mathbf{x}_o \hat{f}_o(\mathbf{x}_o),$$  \hspace{1cm} (23)

and in particular

$$\int_{\Gamma} d\mathbf{x}(t) \hat{f}(\mathbf{x}(t), t) = \int_{\Gamma} d\mathbf{x}_o \hat{f}_o(\mathbf{x}_o) = 1.$$  \hspace{1cm} (24)

In order to prove that $\hat{f}(\mathbf{x}(t), t)$ can be interpreted as probability density in the next section we intend to establish an H-theorem.

3 Shannon kinetic entropy

In terms of the Navier-Stokes evolution operator $T_{t,t_o}$ and Eq.(20) [or (21) in the case in which $f(\mathbf{x}, t)$ coincides with a local Maxwellian distribution (18)]
it is now immediate to evaluate the Shannon kinetic entropy associated to the kinetic distribution function \( f(x,t) \), namely \( S(t) = -\int_{\Gamma} dx f(x,t) \ln f(x,t) \). Let us assume, for this purpose that the initial kinetic distribution function \( f(x_0, t_0) \equiv f_0(x_0) \) be defined in such a way that it results strictly positive in \( \Gamma \), at least of class \( C^{(1)}(\Gamma \times I) \) and summable in \( \Gamma \) so that the Shannon kinetic entropy \( S(t_0) = -\int_{\Gamma} dx f_0(x_0, t_0) \ln f(x_0, t_0) \) results defined and at least of class \( C^{(1)}(I) \). Thanks to the integral kinetic equation (17) and the condition of conservation (23) it follows that \( S(t) \) and \( S(t_0) \) are related by means of the equation:

\[
S(t) = S(t_0) + \int_{\Gamma} dx_0 f_0(x_0) \int_{t_0}^t dt' \frac{\partial}{\partial \mathbf{v}(t')} \cdot \mathbf{F}(x(t'), t').
\] (25)

Therefore, the entropy production rate \( \frac{\partial}{\partial t} S(t) \) results

\[
\frac{\partial}{\partial t} S(t) = \int_{\Gamma} d\mathbf{x} f(\mathbf{x}, t) \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{x}, t),
\] (26)

where \( \frac{\partial}{\partial \mathbf{v}} \cdot \mathbf{F}(\mathbf{x}, t) \) is given or an arbitrary kinetic distribution function by Eq.(20) [or (21) for the Maxwellian case]. It follows

\[
\frac{\partial}{\partial t} S(t) = \frac{3}{2} \int_{\Gamma} d\mathbf{x} \frac{1}{P_o + p(r,t)} f(\mathbf{x}, t) \left\{ \frac{\partial}{\partial t} P_o(t) + \frac{D}{D_t} p + \nabla \cdot \mathbf{Q} + \frac{1}{2p_1} \left[ \nabla \cdot \Pi \right] \cdot \mathbf{Q} \right\}
\] (27)

Since \( P_o + p(r,t) \) and \( f(\mathbf{x}, t) \) are strictly positive, we can always define \( P_o(t) \) so that in the finite time interval \( I \) there results identically

\[
\frac{\partial}{\partial t} P_o(t) = -\int_{\Omega} d\mathbf{r} \frac{1}{P_o(t) + p(r,t)} \left[ \frac{D}{D_t} p + \nabla \cdot \mathbf{Q} + \frac{1}{2p_1} \left[ \nabla \cdot \Pi \right] \cdot \mathbf{Q} \right].
\] (29)

In the case in which the initial condition \( f_0(\mathbf{x}_0) \) coincides identically with \( f_M \) in the whole phase space \( \Gamma \) it follows in particular

\[
\frac{\partial}{\partial t} P_o(t) = -\int_{\Omega} d\mathbf{r} \frac{1}{P_o(t) + p(r,t)} \frac{D}{D_t} p.
\] (30)

Therefore, condition (29) [or (30) in the Maxwellian case] implies that in the same time interval \( I \) the entropy production rate must vanish identically, i.e.,

\[
\frac{\partial}{\partial t} S(t) \equiv 0.
\] (31)

We stress that Eq.(31) holds, in principle, for an arbitrary initial condition \( P_o(t_0) = P_{oo} > 0 \) with \( P_{oo} \) suitably large. Therefore, the kinetic pressure \( p_1 \),
given by Eqs.(9) and (10), remains still non-unique since it is determined in terms of Eq.(29) only up to an arbitrary positive constant $P_{oo}$.

It follows that the Shannon entropy for the kinetic distribution function $f(x,t)$ results always conserved by imposing a suitable prescription on the the time-dependent part of the kinetic pressure $P_o(t)$. Since the latter is unrelated to the physical observables (i.e., the fluid fields) the constraint condition imposed on the kinetic pressure [respectively (29) or (30)] can always be satisfied. As a consequence, with such prescriptions the normalized kinetic distribution function $\hat{f}(x,t)$ can be interpreted as probability density.

3.1 Conclusions

In this paper the condition of positivity of the kinetic distribution function $f(x,t)$ which characterizes the inverse kinetic theory recently developed for the incompressible Navier-Stokes equations has been investigated [3, 4]. We have proven that the Shannon entropy is exactly conserved for arbitrary kinetic distribution function, provided the kinetic pressure is suitably defined and the initial kinetic distribution function results positive definite and suitably regular. As indicated, these conditions can always be satisfied without imposing any constraint on the physical observables, here represented by the fluid fields $\{\rho_0, V, p\}$.

The conclusion applies in principle to arbitrary, suitable smooth in the sense (5), strong solutions of INSE which are defined in three dimensional internal domains of $\mathbb{R}^3$. Assuming, mass conservation and no-slip boundary conditions (i.e., Dirichlet boundary conditions) on the boundary $\delta\Omega$, the same result holds also for non-isolated systems characterized by moving boundaries. In addition, arbitrary volume forces which satisfy (3),(4) or analogous surface forces obtained by applying a non-uniform pressure on the boundary $\delta\Omega$, can be included.

An immediate consequence of the H-theorem here obtained is the possibility of imposing the maximum entropy principle in order to determine the initial kinetic distribution function $f_0(x)$, i.e., requiring the variational equation $\delta S(f_0) = 0$ subject to suitable constraint equations. Thus, for example, the local Maxwellian distribution (18) is obtained by imposing solely the constraints provided by the moments (7),(8),(9) and (10), to be considered as prescribed. However, in principle, the variational principle can also be used to determine non-Maxwellian initial distributions [5].

These results appear significant both from the mathematical viewpoint and the physical interpretation of the theory.
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4 Appendix A: INSE

The incompressible Navier-Stokes equations (INSE) are defined by the following set of PDE’s and inequalities for the fluid fields \( \{ \rho, \mathbf{V}, p \} \)

\[
\frac{\partial}{\partial t} \rho + \nabla \cdot (\rho \mathbf{V}) = 0, \quad (32)
\]

\[
\rho \frac{D}{Dt} \mathbf{V} + \nabla p + \mathbf{f} - \mu \nabla^2 \mathbf{V} = 0, \quad (33)
\]

\[
\nabla \cdot \mathbf{V} = 0, \quad (34)
\]

\[
\rho(r,t) > 0, \quad (35)
\]

\[
p(r,t) \geq 0, \quad (36)
\]

\[
\rho(r,t) = \rho_o > 0. \quad (37)
\]

The first three equations (32),(33) and (34), denoting respectively the continuity, forced Navier-Stokes and isochoricity equations, are assumed to be satisfied in the open three-dimensional set \( \Omega \subseteq \mathbb{R}^3 \) (fluid domain) and in a possibly bounded time interval \( I \subset \mathbb{R} \), while the last three inequalities, (35)-(37) apply also in the closure of the fluid domain \( \overline{\Omega} \equiv \Omega \cup \delta \Omega \). Here the notation is standard\[3\]. Hence \( \frac{D}{Dt} = \frac{\partial}{\partial t} + \mathbf{V} \cdot \nabla \) and \( \mu \equiv \nu \rho_o > 0 \) is the constant fluid viscosity, with \( \nu \) the related kinematic viscosity. The volume force density \( \mathbf{f}(r,t) \) acting on the fluid element by assumption is taken in the functional setting (3),(4) and (5). Consequently, the fluid fields \( \{ \mathbf{V}(r,t), p(r,t) \} \) are required to satisfy the regularity conditions (5). The initial-boundary value problem for INSE is defined as follows. The initial condition is defined by imposing

\[
\rho(r,t_o) = \rho_o > 0 \quad (38)
\]

\[
p(r,t_o) = p_o(r), \quad (39)
\]

\[
\mathbf{V}(r,t_o) = \mathbf{V}_o(r), \quad (40)
\]

where \( \{ \mathbf{V}_o(r), p_o(r) \} \) belong to the functional class

\[
\begin{cases}
\mathbf{V}_o(r), p_o(r) \in C^0(\overline{\Omega}), \\
\mathbf{V}_o(r), p_o(r) \in C^2(\Omega),
\end{cases}
\]

(41)
and moreover satisfy respectively the isochoricity condition (34) and the Poisson equation
\[ \nabla^2 p_o(r) = -\nabla \cdot \{ \rho_o \nabla V_o \cdot \nabla V_o + f(r,t_o) \}. \quad (42) \]
The boundary conditions can be specified, for example, by means of the Dirichlet boundary conditions (which for the velocity are usually denoted as no-slip boundary conditions), i.e., letting \( \forall t \in I \) and imposing in each point \( r_W \) of the boundary \( \delta \Omega \)
\[ \rho(\cdot,t) = \rho_o > 0, \quad (43) \]
\[ p(\cdot,t) = p_W(\cdot,t), \quad (44) \]
\[ V(\cdot,t) = V_W(\cdot,t). \quad (45) \]
Here \( \{ V_W(r,t), p_W(r,t) \} \) denote respectively the velocity and the pressure at an arbitrary point \( r_W \) belonging to the boundary \( \delta \Omega \). Fields, both required to belong to the same functional class (5).

5 Appendix B: mean-field force

For a generic (i.e., non-Maxwellian) distribution function \( f(x,t) \), the mean-field force \( F \) reads \( F(x,t) = F_0(x,t) + F_1(x,t) \), where \( F_0 \) and \( F_1 \) are the vector fields:
\[ F_0(x,t) = \frac{1}{\rho_o} \left[ \nabla \cdot \Pi - \nabla p_1 - f \right] + a + \nu \nabla^2 V, \quad (46) \]
\[ F_1(x,t) = \frac{1}{2p_1} u \left\{ \frac{D}{Dt} p_1 + \nabla \cdot Q + \frac{1}{2p_1} \left[ \nabla \cdot \Pi \right] \cdot Q \right\} + \frac{v^2_{th}}{2p_1} \nabla \cdot \Pi \left\{ X^2 - \frac{3}{2} \right\}, \quad (47) \]
where \( X^2 = \frac{u^2}{v^2_{th}} \) and \( v^2_{th} = 2p_1/\rho_o \). Here \( Q \) and \( \Pi \) are the velocity-moments \( Q = \int d^3v u \frac{u^2}{3} f, \Pi = \int d^3v uu \), while \( f \) denotes the volume force density acting on the fluid element and finally \( \nu > 0 \) is the constant kinematic viscosity. In particular, for the Maxwellian kinetic equilibrium (18) there results \( \Pi = p_1 \frac{1}{2} \Pi = 0 \). Moreover, \( a \) is the convective term which according to Ref. [4] is uniquely defined and reads \( a = \frac{1}{2} u \cdot \nabla V + \frac{1}{2} \nabla \cdot V \cdot u \).

References

[1] M. Ellero and M. Tessarotto, Bull. Am Phys. Soc. 45(9), 40 (2000).
[2] M. Tessarotto and M. Ellero, RGD24 (Italy, July 10-16, 2004), AIP Conf. Proceedings 762, 108 (2005).

[3] M. Ellero and M. Tessarotto, Physica A, An inverse kinetic theory for the incompressible Navier-Stokes equations, Physica A, doi:10.1016/j.physa.2005.03.021 (2005).

[4] M. Tessarotto and M. Ellero, A unique representation of inverse-kinetic theory for incompressible Navier-Stokes equations, submitted (2006).

[5] E.T. Jaynes, Phys. Rev. 106, 620 (1957).