Bike-sharing systems with a dual selection mechanism and a dynamic double-threshold repositioning policy

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Abstract
Bike-sharing systems have gained considerable attention with the tide of sharing economy. This paper proposes a dual selection mechanism and a dynamic double-threshold repositioning policy to reduce problematic stations. The selection mechanism is a preventive self-balancing measure, which is carried out before problematic stations appear. To prove the effectiveness of this selection mechanism, a system of differential equations is set up, the steady-state probabilities of the bike-sharing system are computed, and numerical experiments to verify the applicability and computability of the computational method are provided. The method provid in this study contributes to improving the efficiency of the bike-sharing by self-balancing measure.

1 INTRODUCTION

Bike sharing systems (BSSs), adopted as sustainable transportation systems, have been launched in more than 2900 cities in the world [1]. BSSs increase customers’ commuting flexibility by enabling them to pick up or drop off a bike at any station. The increased flexibility comes with the challenge of unpredictable and fluctuating demand for bikes and lockers. As a result, imbalanced problems occur, such as no bike or no locker at a station (i.e. a problematic station). It is the main operations management problem to reduce problematic stations. Now, a common solution is to employ staffs to manually reposition bikes by trucks. But truck scheduling leads to considerable operational cost and produces extra carbon emission, which goes against the green concept of BSSs. Manually repositioning process is a passive procedure, which motivates me to seek an active way.

It is an active repositioning method to incentivize customers to reposition bikes, and some incentive mechanisms have been implemented in actual BSSs. The Citi Bike in New York carries out a ‘Bike Angels’ program, which rewards customers who move bikes out from full stations or return bikes to empty stations. Operators mark some stations with different pick-up points or drop-off points in the BSS APP. The points of stations are updated every 15 min. Customers will receive exclusive gifts depending on the points they get [2]. There is a similar incentiv mechanism in China, called ‘Red Packet Bikes’. Operators set bikes that are unused for 24 h as red packet bikes in the BSS APP. If a customer rents a red packet bike and rides it for more than 10 min, he can ride the bike for free in 2 h and will get 1 to 500 RMB randomly as a reward. And if a customer returns it to a specified region pointed in the APP, he can also get a pecuniary reward. This program aims to encourage customers to move the long-time unused bikes or return bikes to a place lacked bikes [3, 4].

To implement an incentive mechanism, we first analyze the customer behaviours to recognize who will accept it. The prerequisite for applying the pecuniary incentive mechanism is to have heterogeneity for customers in degrees of price sensitivity (PS) and value of time (VOT), which makes it possible to guide different types of customers to choose different policies (for a detailed analysis, see Tan et al. [5]). Customers of BSSs can be divided into two categories with heterogeneity: commuters and leisure travellers. Based on the data of BSSs in three famous cities in China, the customers consist of 73% commuters and 27% leisure travellers in Guangzhou [6], 42% commuters and 58% leisure travellers in Hangzhou [7] and 55.2% commuters and 44.8% leisure travellers in Nanjing [8]. A few commuters may change their routes, and many leisure travellers will change their routes in an acceptable distance for either walk or ride with pecuniary rewards. Thus incentive mechanisms make sense.
Note that almost present incentive mechanisms are remedial measures of imbalanced situation. This paper proposes a preventive incentive mechanism to encourage customers to self-balance bikes before problematic stations appear. When problematic stations come out, some existing incentive mechanisms motivate customers to self-rebalance bikes. Nevertheless, once problematic stations occur, the service of quality and customer satisfaction may drop. Why do not we encourage customers to suitable stations before problematic stations appear? Thus we propose a dual selection mechanism to motivate customers to pick up a bike from a recommended station and return the bike to a suggested station in exchange for rewards. This selection mechanism is a policy for operators to design options in the BSS APP as a preemptive measure to encourage customers to self-balance bikes. When a customer intends to rent (or return) a bike and sets his starting place (or destination) in the APP, the APP will select a recommended station for him. That is the reason why the selection mechanism is called a ‘dual’ selection mechanism. If the customer chooses the recommended station, he will get a pecuniary reward. If he does not select the recommended station, he can rent or return a bike normally but without a reward. Nowadays, almost all the BSSs have their smart phone APPs. With fast development of communication technology, the BSS APPs can provide customers real-time information of the number of bikes in stations, which make the incentive mechanism executable.

Since not all the customers will participate in the dual selection mechanism, and in rush hours, even though the incentive mechanism is performed well, there still may exist problematic stations. So besides the dual selection mechanism, we propose a dynamic double-threshold repositioning policy as a supplementary measure, under which staffs remove bikes to rebalance stations manually by trucks.

The main contributions of this paper are threefold. The first one is to propose a dual selection mechanism as a preventive measure to ease imbalanced situations. The second one is to provide a dynamic double-threshold repositioning policy, under which bikes can be repositioned in batch dynamically. The third one is to obtain the steady-state probabilities of problematic stations, verify the effects of the selection mechanism, and provide guidance for operators on how to control the probabilities of problematic stations by regulating system parameters, selection mechanism parameters, and customers’ participation proportions.

The remainder of this paper is structured as follows: Section 2 provides the literature review of BSSs. Section 3 describes a BSS with a dual selection mechanism and a dynamic double-threshold repositioning policy. Section 4 sets up a system of differential equations. In Section 5, we use operator semi-group to provide the limit for the Markov process and compute the system of differential equations. Section 6 provides numerical experiments to verify the applicability and computability of the computational method and analyze the effect of main parameters on the performance. In Section 7, we conclude our results and give an outlook for future research. Finally, two appendices are provided.

2 | LITERATURE REVIEW

The literature of BSSs related to this paper can be mainly classified into three aspects. The first thread of research focuses on system design, including stations’ number, capacities, and locations. Li and Yang [9] proposed a model to determine the number and locations of stations, the network structure of bike paths connected between stations, and travel paths for users between each pair of origins and destinations. Lin et al. [10] addressed a strategic design problem with bike stock consideration by a hub location inventory model. Nair and Miller-Hooks [11] formulated an equilibrium network design model to determine locations of stations. These papers prove that a better system design of fleet size, station location and capacity is beneficial for improving system performance.

The second thread of research focuses on the repositioning problem. The repositioning strategies are divided into two different classes: static repositioning and dynamic repositioning. For the static repositioning, Schuijbroek et al. [12] provided cluster-based models to analyze the static repositioning problem and showed that they can reduce the length of the rebalancing. Cruz et al. [13] proposed an iterated local search based heuristic to find a least-cost route that the met the demand of all stations and did not violate the minimum and maximum load limits along the tour. Liu et al. [14] developed a Meteorology Similarity Weighted K-Nearest-Neighbour regressor to predict the station pick-up demand based on large-scale historic trip records and Adaptive Capacity Constrained K-centre Clustering based bike repositioning optimization model. Bulhoes et al. [15] proposed a branch-and-cut algorithm over an extended network-based mathematical formulation to analyze the static repositioning problem. Jia et al. [16] proposed an approach integrating multi-objective optimization and a weighting factor based on the shortage event types of each station to solve the repositioning problem during rush hours. They used practical instances to verify that the proposed approach can reduce the number of occurred shortage events and the total cost time of repositioning operations. Static repositioning neglects the movements of bikes during the repositioning period and focuses on the problem of finding routes for a fixed set of trucks to reposition bikes at the end of the day. Static repositioning typically cannot adequately capture the surges in customer demand even if they are predictable. Thus dynamic repositioning needs to be studied sufficiently.

For the dynamic repositioning, Shu et al. [17] predicted the stochastic demand from customer trip data of the Singapore metro system. They proposed a dynamic repositioning model to minimize the number of unsatisfied customers and they found that repositioning operations can lead to an additional 15–20% of trips. Caggiani et al. [18] proposed a new comprehensive dynamic bike repositioning method that started from the prediction of the numbers and positions of bikes over a system operating area and ended with a relocation decision support system. Legros [19] used a Markov decision process approach to develop an implementation decision-support tool. He developed a one-step policy improvement, which solved the
problem referred to as ‘the curse of dimensionality’. To analyze large BSS, they further developed a decomposition method based on proper station grouping.

The third thread of research focuses on the design and analysis of incentive mechanisms. Pfommer et al. [20] proposed a dynamic incentive scheme where customers were encouraged to change their target stations in exchange for payments. They applied the incentive policy to London’s Barclays Cycle Hire scheme, simulated it based on historical data and found that it was possible to trade off reward payout to customers against the cost of hiring staffs to reposition bikes. Singla et al. [21] proposed a crowdsourcing mechanism to encourage customers to change their travel behaviours following the bike repositioning requirement. They deployed the incentive mechanism on a real-world BSS for 30 days in Mainz, Germany, and found that the acceptance rate of the incentive mechanism in overall participants was about 60%. Ban and Hyun [22] defined three major parameters to simulate the incentive mechanism for customers depending on the city’s budget: The maximum walking distance, the incentive threshold for walking motivation, and the user participation rate. They found that the extra walking distance is the most influential element. Wu et al. [23] adopted the ranking and selection approach to select the optimal incentive plan. They analyzed a profit maximization problem with a constraint on the customer service level under the incentive plan and proved that the profit and service level can be improved significantly compared with the scenario without incentive provision. Wu [24] respectively analyzed the incentive methods for individual workers and for a group of workers. Pan et al. [25] provided a novel spatial temporal bike repositioning framework and proposed a hierarchical reinforcement pricing (HRP) algorithm to decide the amount of reward for users at each time. Duan and Wu [26] proposed a deep reinforcement learning architecture designed for source incentive to optimize the incentive scheme for considering destination incentive. They integrated source and destination incentives by adjusting the detour level at the source and destination. They used experiment to prove that the adapted learning algorithm outperforms the one that only considers source incentive in maximizing the long-term number of satisfied users. Based on the above literature, it is obvious that the incentive mechanism is a feasible method to deal with congestion and imbalanced situation by encouraging customers to change their routes slightly. But all of these incentive mechanisms are redeeming mechanisms.

3 | MODEL DESCRIPTION

In this section, we first describe a BSS with a dual selection mechanism and a dynamic double-threshold repositioning policy. Then we introduce a sequence of fraction vectors, which are used to express the BSS as a Markov process.

We propose a dual selection mechanism to incentive customers to change their travel routes slightly. When a customer opens the BSS APP and sets his starting position, the APP will select a station with the most bikes from several stations around his starting position. If the customer selects the recommended station to rent a bike, he will get a pecuniary reward. If he selects another station without using the APP, he can rent a bike without a reward. Similarly, when a customer needs to return a bike and sets his destination in the APP, the APP will select a station with the fewest bikes among several stations around his destination. If the customer selects the suggested station to return the bike, he will get a pecuniary reward; Otherwise, he can return the bike without a reward.

The model notations, operation mechanism, and the repositioning policy are described as follows:

(1) Stations: We assume the BSS consists of \( N \) identical stations, each of which contains \( S \) bikes at initial time \( t = 0 \) and has capability \( C \) for \( 1 \leq S < C < +\infty \). This paper studies a large-scale BSS and analyzes how the selection mechanism affects the BSS. For simplification of analysis, we do not consider the differences among stations and among preferences of stations.

(2) Arrival process: We assume that customers arrive at the BSS as a Poisson process with arrival rate \( \lambda \) for \( \lambda > 0 \).

Arrival process selection mechanism: When a customer intends to rent a bike and sets his starting place in the BSS APP, the APP will select a station with the most bikes from \( d_1 \) (\( 1 \leq d_1 \leq N \)) stations around his starting place. If there is a tie, the customer can select one randomly. If there is at least one bike in the recommended station, the customer enters the station, rents a bike and starts his trip. If there is no bike in the recommended station (i.e., all the \( d_1 \) stations are empty), the customer will leave the system immediately. We assume that only a fraction \( \alpha_1 \) of customers follow the selection mechanism for the arrival process for \( 0 \leq \alpha_1 \leq 1 \).

(3) Riding-bike time: The riding-bike times on the road are i.i.d. and exponential with parameter \( 1/\mu \).

(4) Return processes: When a customer is going to finish his riding-bike trip, he needs to return the bike to a station according to the following two cases:

The first return process selection mechanism: When a customer wants to finish his riding-bike trip and sets his destination in the APP, the APP will select a station with the fewest bikes from \( d_2 \) (\( 1 \leq d_2 \leq N \)) stations around his destination. If there is a tie, he can select one randomly. If there is at least one vacant locker in the recommended station, the customer returns the bike in the station and leaves the system. If the recommended station is full, the customer cannot return the bike at the station, and he has to start a further return process.

Further return process selection mechanism: When all the \( d_2 \) stations around the customer’s destination are full, the APP will provide another recommended station with the fewest bikes selected from other \( d_2 \) stations around his destination. If there is at least one bike in the recommended station, the customer returns the bike at the station and leaves the system immediately. If the recommended station is full, the customer has to start another return process. Based on such a return mode, the customer has to follow the further return process until he finds a non-full station and returns the bike successfully. We assume that only a fraction \( \alpha_2 \) of customers follow the selection mechanism for the return process for \( 0 \leq \alpha_2 \leq 1 \).

(5) Dynamic double-threshold repositioning policy: Since bikes are not legitimately distributed in the \( N \) stations,
trucks are used to reposition them according to a dynamic double-threshold repositioning policy. Interarrival times of trucks to each station are i.i.d. and are exponential with arrival rate $\beta > 0$. The dynamic double-threshold repositioning policy is described as follows:

If a truck arrives at one station which has $n$ bikes for $L \leq n \leq C (L > 5)$, the truck takes away $n - S$ bikes which will be repositioned to other stations. If a truck arrives at one station which has $n$ bikes for $0 \leq n \leq M (M < R < S)$, the truck lays down $R - n$ bikes in this station which will have $R$ bikes due to the new adding bikes. If a truck arrives at a station with $n$ bikes for $M + 1 \leq n \leq L - 1$, no bike will be taken away or laid down.

(6) Leaving principle: The customer departure process has two different cases: (a) A customer directly leaves the BSS if the recommended station is empty, and (b) if a customer completes his trip and returns the bike to a station successfully, he immediately leaves the BSS.

The notation used in this paper is summarized as follows:

- $N$: Number of stations;
- $C$: Capacity of a station;
- $S$: Number of bikes at initial time in a station, and the number of removing bikes to;
- $\mu$: Riding rate of bike-riding on road;
- $\lambda$: Customer arrival rate of a station;
- $d_1$: Number of stations from which to select the recommended station to rent bikes;
- $d_2$: Number of stations from which to select the recommended station to return bikes;
- $\alpha_1$: Proportion of customers following the arrival process selection mechanism;
- $\alpha_2$: Proportion of customers following the return process selection mechanism;
- $\beta$: Arrival rate of trucks;
- $M$: Threshold of supplementing bikes;
- $L$: Threshold of removing bikes;
- $R$: Number of supplementing bikes to.

**Remark 1.** There are some limitations of the model assumptions. First, we study a homogeneous BBS with identical stations. In an actual situation, the arrival rates and service rates are different, but for a large-scale system, it is a measure to treat stations identical [as Fricker and Gast [27]]. Second, we assume the arrangement of trucks can always meet the requirement of bikes without considering the limit of the capacity of trucks. The limit of the capacity of trucks will delay the repositioning process. A suitable arrangement of trucks needs to be designed in the future.

**Remark 2.** For the dynamic double-threshold repositioning policy, the number of bikes in a station is supplemented to $R$ with $0 < R \leq S$, because some bikes are ridden on the road. Thus it is not able to supplement the number of bikes up to $S$ in each station.

In the remainder of this section, we introduce a fraction vector of stations with at least $k$ bikes to construct an empirical measure Markov process which expresses the states of the BSS. We denote by $n_k(t)$ the number of stations with at least $k$ bikes at time $t \geq 0$ for $k \geq 0$. Clearly, $n_0(t) = N$, and $0 \leq n_k(t) \leq N$ for $k \geq 1$. We write that for $k \geq 0$

$$U_k^{(N)}(t) = \frac{n_k(t)}{N},$$

which is the fraction of stations with at least $k$ bikes at time $t \geq 0$. Hence $U_0^{(N)}(t) \equiv 1$ for $t \geq 0$. Let $U^{(N)}(t) = (U_0^{(N)}(t), U_1^{(N)}(t), \ldots, U_C^{(N)}(t))$. Then the states of the BSS are described as a stochastic process $\{U^{(N)}(t) : t \geq 0\}$. Since the arrival processes of customers and trucks are Poisson processes, and the travel times are exponential, $\{U^{(N)}(t) : t \geq 0\}$ is a Markov process whose state space $E_N$ is given by

$$E_N = \left\{ (i_0^{(N)}, i_1^{(N)}, \ldots, i_C^{(N)}) : 1 = i_0^{(N)} \geq i_1^{(N)} \geq \cdots \geq i_C^{(N)} \geq 0, i_k^{(N)} \right\}$$

is a non-negative integer for $0 \leq k \leq C$.

It is seen from the stochastic order that $U_k^{(N)}(t) \geq U_{k+1}^{(N)}(t)$ for $0 \leq k \leq C - 1$, hence we get that $1 = U_0^{(N)}(t) \geq U_1^{(N)}(t) \geq U_2^{(N)}(t) \geq \cdots \geq U_C^{(N)}(t) \geq 0$.

To study the Markov process $\{U^{(N)}(t) : t \geq 0\}$, we write the expected fractions as follows:

$$\nu_k^{(N)}(t) = E\left[U_k^{(N)}(t)\right], \quad 0 \leq k \leq C,$$

and $\mathbf{u}^{(N)}(t) \equiv (\nu_0^{(N)}(t), \nu_1^{(N)}(t), \ldots, \nu_C^{(N)}(t))$. It is clear that $\nu_0^{(N)}(t) \equiv 1$ and $\nu_0^{(N)}(t) \geq \nu_1^{(N)}(t) \geq \nu_2^{(N)}(t) \geq \cdots \geq \nu_C^{(N)}(t)$.

## 4 THE SYSTEM OF DIFFERENTIAL EQUATIONS

In this section, we provide a probability analysis for both the arrival process and the return process. Based on this, we set up a system of differential equations satisfied by the expected fraction vectors. We need to determine the expected change of the number of stations with at least $k$ customers over a small period $[0, dt)$.

(a) Arrival process

For the arrival process selection mechanism, the recommended station with the most bikes is selected from $d_1$ stations, that is, the number of bikes in the recommended station is $k \geq 1$ (when $k = 0$, the customer leaves the BSS), while the numbers of bikes in the remaining stations are all not more than $k - 1$. When customers follow the selection mechanism, the arrival
rate during a small period \([0, \, dt]\) is given by

\[
\lambda N \left[ \alpha_1 \sum_{m=1}^{d_1} C_{d_1} \left( \alpha_2 \right)^m \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)^{d_1-m} \right] \times \left( 1 - \lambda u_k^{(N)}(t) \right) + (1 - \alpha_1) \
\left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right) \right] dt,
\]

\[1 \leq k \leq C - 1, \quad (1)\]

and

\[
\lambda N \left[ \alpha_1 \sum_{m=1}^{d_1} C_{d_1} \left( \alpha_2 \right)^m \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)^{d_1-m} \right] \times \left( 1 - \lambda u_k^{(N)}(t) \right) + (1 - \alpha_1) \lambda u_k^{(N)}(t) \right] dt.
\]

\[2\]

Note that \((\lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t))^{d_1-m}\) is the probability that there are \(m\) stations with \(k\) bikes, while \((1 - \lambda u_k^{(N)}(t))^{d_1-m}\) is the probability that the other \(d_1 - m\) stations whose bike numbers are all less than \(\lambda k\).

(b) Return process

When the trip is finished, the customer must return the bike to a station. There are two cases of the return process. For the return process selection mechanism, the recommended station with the fewest bikes is selected from \(d_2\) stations around his destination, that is the bike number in the recommend station is \(k\) \((0 \leq k \leq C - 1)\), and the numbers of bikes in the other \(d_2 - 1\) stations are all not less than \(k + 1\). The actual service rate during a short time \([0, \, dt]\) under the selection mechanism is given by

\[
\nu_k^{(N)}(t) = \sum_{m=1}^{d_2} C_{d_2} \left( \alpha_2 \right)^m \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)^{d_2-m} \left( 1 - \lambda u_k^{(N)}(t) \right)^{d_2-m} \right] dt,
\]

\[3\]

and

\[
\nu_k^{(N)}(t) = \sum_{m=1}^{d_2} C_{d_2} \left( \alpha_2 \right)^m \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)^{d_2-m} \left( 1 - \lambda u_k^{(N)}(t) \right)^{d_2-m} \right] dt.
\]

\[4\]

Then formulas (1)-(3) can be simplified as

\[
\lambda N \left[ \alpha_1 \nu_k^{(N)}(t) + 1 - \alpha_1 \right] \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right),
\]

\[1 \leq k \leq C - 1,
\]

\[
\lambda N \left[ \alpha_1 \nu_k^{(N)}(t) + 1 - \alpha_1 \right] \lambda u_k^{(N)}(t),
\]

\[5\]

and

\[
\mu \left\{ C - l + (N - 1) \left[ C - \sum_{k=1}^{C} \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right) \right] \right.
\]

\[\left( 1 - \lambda u_k^{(N)}(t) \right)^{d_1-m} \right] \times \left[ \alpha_2 \sum_{m=1}^{d_3} C_{d_3} \left( \alpha_2 \right)^m \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)^{d_3-m} \right] \times \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)\right\] \[\left( 1 - \lambda u_k^{(N)}(t) \right)^{d_2-m} \right) + (1 - \alpha_2) \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right) \right] dt,
\]

\[0 \leq k \leq C - 1, \quad (3)\]

To simplify formulas (1)-(3), we write

\[
\xi_k^{(N)}(t) = \lambda N \left[ \alpha_1 \nu_k^{(N)}(t) + 1 - \alpha_1 \right] \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right), \quad 1 \leq k \leq C - 1,
\]

\[7\]

\[
\xi_k^{(N)}(t) = \lambda N \left[ \alpha_1 \nu_k^{(N)}(t) + 1 - \alpha_1 \right] \lambda u_k^{(N)}(t),
\]

\[8\]

We let

\[
\eta_k^{(N)}(t) = \mu \left\{ C - l + (N - 1) \left[ C - \sum_{k=1}^{C} \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right) \right] \right.
\]

\[\left( 1 - \lambda u_k^{(N)}(t) \right)^{d_1-m} \right] \times \left[ \alpha_2 \sum_{m=1}^{d_3} C_{d_3} \left( \alpha_2 \right)^m \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)^{d_3-m} \right] \times \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right)\right\] \[\left( 1 - \lambda u_k^{(N)}(t) \right)^{d_2-m} \right) + (1 - \alpha_2) \left( \lambda u_k^{(N)}(t) - \lambda u_k^{(N)}(t) \right) \right] dt,
\]

\[9\]

Based on the above analysis, the Markov process belongs to the family of density-dependent population processes, defined by Kurtz [28]. We obtain a system of differential equations as follows:

for \(k = 0\)

\[
\frac{d u_k^{(N)}(t)}{d t} = -\left( \xi_k^{(N)}(t) + \beta R \right) \left( u_k^{(N)}(t) - u_k^{(N)}(t) \right)
\]

\[\eta_k^{(N)}(t) \right) \left( u_k^{(N)}(t) - u_k^{(N)}(t) \right),
\]

\[10\]
for $1 \leq k \leq M$

$$\frac{d\xi_k^{(N)}(t)}{dt} = \xi_{k-1}^{(N)}(t)\left(u_{k-1}^{(N)}(t) - u_k^{(N)}(t)\right)$$

$$- \left[\xi_k^{(N)}(t) + \eta_k^{(N)}(t) + \beta(R-k)\right]$$

$$\times \left(u_k^{(N)}(t) - u_{k+1}^{(N)}(t)\right)$$

$$+ \eta_{k+1}^{(N)}(t)\left(u_{k+1}^{(N)}(t) - u_{k+2}^{(N)}(t)\right),$$

(11)

for $k = R$

$$\frac{d\xi_k^{(N)}(t)}{dt} = \xi_{k-1}^{(N)}(t)\left(u_{k-1}^{(N)}(t) - u_k^{(N)}(t)\right)$$

$$- \left[\xi_k^{(N)}(t) + \eta_k^{(N)}(t)\right]\left(u_k^{(N)}(t) - u_{k+1}^{(N)}(t)\right)$$

$$+ \eta_{k+1}^{(N)}(t)\left(u_{k+1}^{(N)}(t) - u_{k+2}^{(N)}(t)\right)$$

$$+ \beta \sum_{l=0}^{M} (R-l)\left(u_l^{(N)}(t) - u_{l+1}^{(N)}(t)\right),$$

(12)

for $k = S$

$$\frac{d\xi_k^{(N)}(t)}{dt} = \xi_{k-1}^{(N)}(t)\left(u_{k-1}^{(N)}(t) - u_k^{(N)}(t)\right)$$

$$- \left[\xi_k^{(N)}(t) + \eta_k^{(N)}(t)\right]\left(u_k^{(N)}(t) - u_{k+1}^{(N)}(t)\right)$$

$$+ \eta_{k+1}^{(N)}(t)\left(u_{k+1}^{(N)}(t) - u_{k+2}^{(N)}(t)\right)$$

$$+ \beta \sum_{l=L}^{C} (l - S)\left(u_l^{(N)}(t) - u_{l+1}^{(N)}(t)\right),$$

(13)

for $L \leq k \leq C - 1$

$$\frac{d\xi_k^{(N)}(t)}{dt} = \xi_{k-1}^{(N)}(t)\left(u_{k-1}^{(N)}(t) - u_k^{(N)}(t)\right)$$

$$- \left[\xi_k^{(N)}(t) + \eta_k^{(N)}(t) + \beta(k-S)\right]$$

$$\times \left(u_k^{(N)}(t) - u_{k+1}^{(N)}(t)\right)$$

$$+ \eta_{k+1}^{(N)}(t)\left(u_{k+1}^{(N)}(t) - u_{k+2}^{(N)}(t)\right),$$

(14)

for $k = C$

$$\frac{d\xi_C^{(N)}(t)}{dt} = \xi_{C-1}^{(N)}(t)\left(u_{C-1}^{(N)}(t) - u_C^{(N)}(t)\right)$$

$$- \left[\eta_C^{(N)}(t) + \beta(k-S)\right]\left(u_C^{(N)}(t) - u_{C+1}^{(N)}(t)\right),$$

(15)

for either $M + 1 \leq k \leq R - 1$, $R + 1 \leq k \leq S - 1$ or $S + 1 \leq k \leq L - 1$,

$$\frac{d\xi_k^{(N)}(t)}{dt} = \xi_{k-1}^{(N)}(t)\left(u_{k-1}^{(N)}(t) - u_k^{(N)}(t)\right)$$

$$- \left[\xi_k^{(N)}(t) + \eta_k^{(N)}(t)\right]\left(u_k^{(N)}(t) - u_{k+1}^{(N)}(t)\right)$$

$$+ \eta_{k+1}^{(N)}(t)\left(u_{k+1}^{(N)}(t) - u_{k+2}^{(N)}(t)\right),$$

(16)

with the boundary condition

$$\xi_0^{(N)}(t) = 1, \quad t \geq 0,$$

(17)

and the initial conditions

$$\xi_k^{(N)}(0) = g_k, \quad k \geq 0.$$

(18)

5 The Martingale Limit

In this section, we use the operator semi-group to provide a limit for the sequence $\{U^{(N)}(t) : t \geq 0\}$ of Markov process. For the vector $g = (g_0, g_1, g_2, \ldots, g_C)$, the vector space $\Omega_N$ is given by $\Omega_N = \{g : g \geq 0, g_C = 1\}$, in which $g_k$ is a non-negative number for $0 \leq k \leq C$. We take a metric

$$\rho(g, g') = \max_{0 \leq k \leq C} \{|g_k - g'_k|\}, \quad g, g' \in \Omega_N.$$  

Note that under the metric $\rho(g, g')$, the state space $\Omega_N$ is separable and compact. We denote by $T_N(t)$ the operator semi-group of the Markov process $\{U^{(N)}(t), t \geq 0\}$. Note that the operator semi-group of the Markov process is defined and analyzed in Ethier and Kurtz [29] for more details.

If $f : \Omega_N \to C^1$, then for $g \in \Omega_N$ and $t \geq 0$, $T_N(t)f(g) = E[f(U_N(t)) | U_N(0) = g]$. Let $A_N$ be the generating operator of the operator semi-group $T_N(t)$, and it is easy to see that $T_N(t) = \exp\{A_Nt\}$ for $t \geq 0$. Then for every function $f : \Omega_N \to C^1$,

$$\frac{d}{dt}(U^{(N)}(t)) = A_N f(U^{(N)}(t)),$$

where $A_N$ acting on functions $f : \Omega_N \to R$ is the generating operator of the Markov process $\{U^{(N)}_0(t) : t \geq 0\}$,

$$A_N = A_N^{\text{Arrival}} + A_N^{\text{Return}} + A_N^{\text{Track-In}} + A_N^{\text{Track-Out}},$$

(19)

for $g = (g_0, g_1, g_2, \ldots, g_C) \in \Omega_N$. 

Substitute Equations (4) and (5) into Equations (1) and (2), we obtain

\[
\lambda N \left\{ \alpha_1 \left[ 1 - n_k^{(N)}(t) \right] + (1 - \alpha_2) \left\{ 1 - n_k^{(N)}(t) \right\} \right\}, \quad 1 \leq k \leq C - 1, \tag{20}
\]

and

\[
\lambda \left\{ \alpha_1 \left[ 1 - n_k^{(N)}(t) \right] + (1 - \alpha_2) n_C^{(N)}(t) \right\}. \tag{21}
\]

Similarly, substitute Equation (6) into Equation (3), we get

\[
\mu \left\{ C - l + (N - 1) \left[ C - \sum_{k=1}^{C} k \left( n_k^{(N)}(t) - n_k^{(N+1)}(t) \right) \right. \right. \\
+ \frac{n_C^{(N)}(t)}{1 - n_C^{(N)}(t)} \left\{ \alpha_2 \left[ n_k^{(N)}(t) \right] + (1 - \alpha_2) n_C^{(N)}(t) \right\} \right\}, \quad 0 \leq k \leq C - 1, \tag{22}
\]

The proof of Equations (20) to (22) are given in Appendix A. Based on Equations (20) to (22), we write

\[
\mathbf{A}_N^{\text{Arrival}} \ f(g) = \lambda N \sum_{k=1}^{C-1} \left\{ \alpha_1 \left[ 1 - g_{k+1} \right] - (1 - g_k) \right\} \\
+ (1 - \alpha_1) \left[ g_k - g_{k+1} \right] \left[ f(g - \frac{e_k}{N}) - f(g) \right] \\
+ \lambda N \left\{ \alpha_1 \left[ 1 - g_C \right] + (1 - \alpha_1) g_C \right\} \times \left[ f(g - \frac{e_C}{N}) - f(g) \right], \tag{23}
\]

\[
\mathbf{A}_N^{\text{Return}} \ f(g) = \mu \left\{ C - l + (N - 1) \left[ C - \sum_{k=1}^{C} k (g_k - g_{k+1}) \right. \right. \\
+ \frac{g_C}{1 - g_C} \left\{ \alpha_2 \left[ g_k \right] + (1 - \alpha_2) \right\} \right\} \sum_{k=1}^{C} \left\{ \alpha_2 \left[ g_k \right] - (1 - g_{k+1}) \right\} \left[ f(g + \frac{e_k}{N}) - f(g) \right]. \tag{24}
\]

\[
\mathbf{A}_N^{\text{Track-In}} \ f(g) = \beta N \sum_{k=0}^{M} \left[ g_k - g \right] \left[ f(g + \frac{(R - k)e_k}{N}) - f(g) \right], \tag{25}
\]

and

\[
\mathbf{A}_N^{\text{Track-Out}} \ f(g) = \beta N \sum_{k=1}^{C} \left[ g_k - g \right] \left[ f(g + \frac{(k - S)e_k}{N}) - f(g) \right], \tag{26}
\]

in which \(e_k\) is a row vector with the \(k\)th entry one and all others zeroes. Thus it follows from Equations (19) to (26) that

\[
\mathbf{A}_N \ f(g) = \lambda N \sum_{k=1}^{C-1} \left\{ \alpha_1 \left[ 1 - g_{k+1} \right] - (1 - g_k) \right\} \\
+ (1 - \alpha_1) \left[ g_k - g_{k+1} \right] \left[ f(g - \frac{e_k}{N}) - f(g) \right] \\
+ \lambda N \left\{ \alpha_1 \left[ 1 - g_C \right] + (1 - \alpha_1) g_C \right\} \times \left[ f(g - \frac{e_C}{N}) - f(g) \right] + \mu \left[ C - l + (N - 1) \right. \\
\times \left[ C - \sum_{k=1}^{C} k (g_k - g_{k+1}) + \frac{g_C}{1 - g_C} \right] \right\} \\
\times \sum_{k=1}^{C} \left\{ \alpha_2 \left[ g_k \right] - (1 - g_{k+1}) \right\} + (1 - \alpha_2) \left[ g_k - g_{k+1} \right] \times \left[ f(g + \frac{e_k}{N}) - f(g) \right] \\
+ \beta N \sum_{k=0}^{M} \left[ g_k - g \right] \left[ f(g + \frac{(R - k)e_k}{N}) - f(g) \right] \tag{27}
\]

Let \(U(t) = \lim_{N \to \infty} U^{(N)}(t)\) for \(t \geq 0\). Then two formal limits for the sequence \(\{A_N\}\) of generating operators and for the sequence \(\{T_N(t)\}\) of semi-groups are expressed as \(A = \lim_{N \to \infty} A_N\) and \(T(t) = \lim_{N \to \infty} T_N(t)\) for \(t \geq 0\), respectively. Note that as \(N \to \infty\)

\[
\frac{N}{f(g - \frac{e_k}{N}) - f(g)} \to -\frac{\partial}{\partial g_k} f(g), \tag{28}
\]

\[
\frac{N}{f(g + \frac{e_k}{N}) - f(g)} \to \frac{\partial}{\partial g_k} f(g). \tag{29}
\]

Thus when \(N \to \infty\), Equation (27) can be written as

\[
A f(g) = -\lambda \sum_{k=1}^{C-1} \left\{ \alpha_1 \left[ 1 - g_{k+1} \right] - (1 - g_k) \right\} \\
+ (1 - \alpha_1) \left[ g_k - g_{k+1} \right] \frac{\partial}{\partial g_k} f(g) \\
- \lambda \left\{ \alpha_1 \left[ 1 - g_C \right] + (1 - \alpha_1) g_C \right\} \frac{\partial}{\partial g_C} f(g) \tag{30}
\]
Using the operator semi-group and density-dependent population process, we know that the inhomogeneous Markov process \( \{ U^{(N)}(t) : t \geq 0 \} \) exist quadratic limit. For detailed derivative process, readers can refer to Li et al. [30] and Fricker and Gast [27].

Let \( y_k = \lim_{N \to \infty, t \to +\infty} u_{k}^{(N)}(t) \) for \( 0 \leq k \leq C \) and \( y = (y_0, y_1, y_2, \ldots, y_C) \). It is well known that if \( y \) is a fixed point of the expect fraction vector \( u^{(N)}(t) \), then

\[
\lim_{N \to +\infty} \left[ \frac{d}{dt} u^{(N)}(t) \right] = 0.
\]

Taking \( N \to \infty, t \to +\infty \) in both sides of Equations (4)–(9), we get

\[
\xi_k = \lambda [\alpha_1 W_k + (1 - \alpha_1)], \quad 1 \leq k \leq C - 1,
\]

\[
\xi_C = \lambda [\alpha_1 W_C + (1 - \alpha_1)],
\]

and

\[
\eta_l = \mu \left[ C - \sum_{k=1}^{C} k (y_k - y_{k+1}) + \frac{y_C}{(1 - y_C)^2} \right] \\
\times [\alpha_2 \nu_l + (1 - \alpha_2)], \quad 0 \leq l \leq C - 1,
\]

in which

\[
W_k = \sum_{m=1}^{d_1} C_{d_1}^{(m)} (y_k - y_{k+1})^{m-1} (1 - y_k)^{d_1-m},
\]

\( 1 \leq k \leq C - 1, \)

\[
W_C^{(N)}(t) = \sum_{m=1}^{d_1} C_{d_1}^{(m)} (y_C)^{m-1} (1 - y_C)^{d_1-m},
\]

and

\[
V_k = \sum_{m=1}^{d_2} C_{d_2}^{(m)} (y_k - y_{k+1})^{m-1} (y_{k+1})^{d_2-m}, \quad 0 \leq k \leq C - 1.
\]

for \( k = 0 \)

\[
-(\xi_k + \beta R)(y_k - y_{k+1}) + \eta_{k+1}(y_{k+1} - y_{k+2}) = 0,
\]

for \( 1 \leq k \leq M \)

\[
\xi_k - (y_k - y_{k-1}) - [\xi_k + \eta_k + \beta (R - k)](y_k - y_{k+1}) + \eta_k(y_{k+1} - y_{k+2}) = 0,
\]

for \( k = R \)

\[
\xi_k - (y_k - y_{k-1}) - [\xi_k + \eta_k + \beta (R - k)](y_k - y_{k+1}) + \eta_k(y_{k+1} - y_{k+2}) = 0,
\]

for \( k = S \)

\[
\xi_k - (y_k - y_{k-1}) - [\xi_k + \eta_k + \beta (S - k)](y_k - y_{k+1}) + \eta_k(y_{k+1} - y_{k+2}) = 0,
\]

for \( L \leq k \leq C - 1 \)

\[
\xi_k - (y_k - y_{k-1}) - [\xi_k + \eta_k + \beta (S - k)](y_k - y_{k+1}) + \eta_k(y_{k+1} - y_{k+2}) = 0,
\]

for \( k = C \)

\[
\xi_k - (y_k - y_{k-1}) - [\eta_k + \beta (k - S)](y_k - y_{k+1}) + \eta_{k+1}(y_{k+1} - y_{k+2}) = 0,
\]

for either \( M + 1 \leq k \leq R - 1, R + 1 \leq k \leq S - 1 \) or \( S + 1 \leq k \leq L - 1, \)

\[
\xi_k - (y_k - y_{k-1}) - [\xi_k + \eta_k + \beta (S - k)](y_k - y_{k+1}) + \eta_k(y_{k+1} - y_{k+2}) = 0,
\]

with the boundary condition

\[
y_0 = 1.
\]

We denote \( \pi_k = y_k - y_{k+1} \) for \( 0 \leq k \leq C - 1 \), and \( \pi_C = y_C \). And then \( B = (\pi_0, \pi_1, \pi_2, \ldots, \pi_C) \). Then we can get

\[
B V = 0,
\]

and

\[
B e = 1,
\]
in which

\[ \mathbf{V}_\beta = \begin{pmatrix} B_{1,1} & B_{1,2} \\ B_{2,1} & B_{2,2} \\ B_{3,2} & B_{3,3} \\ B_{4,3} & B_{4,4} \\ B_{5,4} & B_{5,5} \end{pmatrix}, \]  

(38)

where the elements of the matrix \( \mathbf{V}_\beta \) are given in Appendix B.

Let

\[ \begin{align*}
\mathbf{\beta} &= (\pi^{(1)}, \pi^{(2)}, \pi^{(3)}, \pi^{(4)}, \pi^{(5)}) , \\
\pi^{(1)} &= (\pi_0, \pi_1, \pi_2, \ldots, \pi_{M-1}, \pi_M) , \\
\pi^{(2)} &= (\pi_{M+1}, \pi_{M+2}, \pi_{M+3}, \ldots, \pi_{R-1}, \pi_R) , \\
\pi^{(3)} &= (\pi_{R+1}, \pi_{R+2}, \pi_{R+3}, \ldots, \pi_{P-2}, \pi_{P-1}) , \\
\pi^{(4)} &= (\pi_3, \pi_4, \pi_5, \pi_6, \pi_7) , \\
\pi^{(5)} &= (\pi_{3,4}, \pi_{4,5}, \pi_{5,6}, \pi_{6,7}, \pi_{C-1}, \pi_C) .
\end{align*} \]

Based on Equations (36) and (37), we get

\[ \begin{align*}
\pi^{(1)} B_{1,1} + \pi^{(2)} B_{2,1} &= 0 , \\
\pi^{(1)} B_{1,2} + \pi^{(2)} B_{2,2} + \pi^{(3)} B_{3,2} &= 0 , \\
\pi^{(2)} B_{2,3} + \pi^{(3)} B_{3,3} + \pi^{(4)} B_{4,3} &= 0 , \\
\pi^{(3)} B_{3,4} + \pi^{(4)} B_{4,4} + \pi^{(5)} B_{5,4} &= 0 , \\
\pi^{(4)} B_{4,5} + \pi^{(5)} B_{5,5} &= 0 , \\
\pi^{(1)} e + \pi^{(2)} e + \pi^{(3)} e + \pi^{(4)} e + \pi^{(5)} e &= 1.
\]  

(39)

To get the probability vector \( \mathbf{\beta} \), we need to solve this block-structured system. Using the LU-type RG-factorization in Li [31], we write

\[ \begin{align*}
U_1 &= B_{1,1} , \\
U_2 &= B_{2,2} + B_{2,1} (-U_1)^{-1} B_{1,2} , \\
U_3 &= B_{3,3} + B_{3,2} (-U_2)^{-1} B_{2,3} , \\
U_4 &= B_{4,4} + B_{4,3} (-U_3)^{-1} B_{3,4} , \\
U_5 &= B_{5,5} + B_{5,4} (-U_4)^{-1} B_{4,5} ,
\end{align*} \]

and

\[ R_{k+1} = B_{k+1,k} (-U_k)^{-1}, \ k = 1, 2, 3, 4. \]

We get the probability vector \( \mathbf{\beta} \) as follows:

\[ \begin{align*}
\pi^{(1)} &= \tau \pi_5 R_5 R_4 R_3 R_2 , \\
\pi^{(2)} &= \tau \pi_5 R_5 R_4 R_3 , \\
\pi^{(3)} &= \tau \pi_5 R_5 R_3 , \\
\pi^{(4)} &= \tau \pi_5 R_3 , \\
\pi^{(5)} &= \tau \pi_5 ,
\end{align*} \]

where \( \pi_5 \) is the steady-state probability vector of Markov chain \( U_5 = B_{5,5} + B_{5,4} (-U_4)^{-1} B_{4,5} \) and \( \tau = 1/(\tau_5 e + \tau_5 R_5 e + \tau_5 R_5 R_4 e + \tau_5 R_5 R_4 R_3 e + \tau_5 R_5 R_4 R_3 R_2 e) \).

Since \( \tau_5 \) and \( \tau_k \) all depend on \( \beta \) for \( k = 2, 3, 4, 5 \), based on Equation (39) and the Brouwer fixed point theorem (see Smart [32]), we obtain that probability vector \( \mathbf{\beta} \) satisfies a non-linear equation

\[ \mathbf{\beta} = (\tau_5 \tau R_5 \tau_5 R_5 R_4, \tau_5 \tau R_5 R_4 R_3, \tau_5 \tau R_5 R_4 R_3 R_2) . \]

6 | NUMERICAL ANALYSIS

In this section, we provide a performance analysis of the BSS with the dual selection mechanism and dynamic double-threshold repositioning policy. Obviously, the steady-state probability of problematic stations plays a key role in measuring the quality of service of the BSS. As seen in Section 5, \( \pi_0 \) is the empty probability that there is no bike in a station, and \( \pi_C \) is the full probability that there is no vacant locker in a station. In what follows, we provide numerical experiments to focus on two key points: empty probability \( \pi_0 \) and full probability \( \pi_C \). Note that such a numerical analysis is useful and helpful in design and operations management of the BSSs through reducing the probability of problematic stations, so that the quality of service of the BSS can be improved sufficiently.

We analyze a practical BSS in New York, called Citi Bike, which is operated by NYC Bike Share, LLC, a subsidiary of Lyft, Inc. The data of this BSS is published on the website 'Citi Bike System Data [33]'. We collected and analyzed the data in August 2019. There were 781 active stations and the bike fleet was 13,144. There were 2,393,606 trips in that month, with an average of 77,213 trips per day. The Service Delivery Department used box trucks, vans, contracted trikes, articulated trikes, and member incentives ('Bike Angels') to redistribute bikes and Citi Bike staffs rebalanced 196,008 bikes in August.

We select the Grove St PATH Station to analyze the operational situation. The capacity of this station is 40. This station has 5463 times bike-renting and 6680 times bike-returning in August. We focus on 20 August 2019 which is a Tuesday. Figure 1 shows the times of bike-renting and bike-returning on August 20. It is observed that the renting and returning processes are asymmetrical, which leads to the imbalanced situation obviously. Especially, during 9:00-10:00, there are 4 bikes are rented, while 68 bikes are returned. And during 19:00-20:00, there are 71 bikes are rented, while 19 bikes are returned. Thus repositioning process is indispensable. Figure 2 characterizes the bike-riding duration of the Grove St PATH Station on 20 August 2019. We find that the riding duration varies from 1 to 60 min and concentrates on 3-5 min.

We set some basic parameters in the following numerical experiments according to the data of the Grove St PATH Station of Citi Bike. As seen from Figure 1, we obtain that the times of bike-renting are 0-68 per hour, so that we set \( \lambda \in (0, 70) \). From Figure 2, it is seen that the duration of bike-renting varies from 1 to 57 min, so we set \( \mu \in (1, 60) \). We compute five
different values of \(d_1\) and \(d_2\) from 1 to 5. When \(d_1 = 1\) \((d_2 = 1)\), it means that the selection mechanism in arrival (return) process is not adopted. Let \(C = 40\) and we take the parameters of the double-threshold repositioning policy as \(M = 5, R = 15, S = 25, L = 30\).

6.1 Analysis of the empty probability \(\pi_0\)

We hope that the value of the empty probability \(\pi_0\) is as small as possible, thus more customers can rent bikes successfully. We first analyze the relation between \(\pi_0\) and \(\lambda\) without applying the selection mechanism, that is \(d_1 = d_2 = 1, \alpha_1 = \alpha_2 = 0\).

Figure 3 characterises how \(\lambda\) affects \(\pi_0\). It shows that \(\pi_0\) increases as \(\lambda\) increases. This result can intuitively be understood as follows: If \(\lambda\) increases, more customers arrive at the system, so that more bikes will be rented from stations. This leads that the empty probability of stations increases.

When following the arrival process selection mechanism but without applying the return process selection mechanism, we analyze the effect of \(d_1\) and \(\alpha_1\) on \(\pi_0\), respectively. The left of Figure 4 indicates how \(d_1\) affects \(\pi_0\). We let \(d_2 = 1, \alpha_2 = 0\), and \(\alpha_1 = 0.8\). It is observed that \(\pi_0\) decreases as \(d_1\) increases. This result can be intuitively understood: If \(d_1\) increases, it means that the recommended station is the one with most bikes selected.
from more stations, so that the recommended station may have more bikes. Thus the empty probability of stations decreases.

Since not all the customers accept the selection mechanism, we analyze how the participation proportion $\alpha_1$ affects $\pi_0$. In the right of Figure 4, we set $d_2 = 1, \alpha_2 = 0$, and $d_1 = 3$. It is seen that $\pi_0$ decreases as $\alpha_1$ increases. It means that when more customers adopt the arrival process selection mechanism, the empty probability $\pi_0$ decreases.

![Figure 5](image1.png)  
**FIGURE 5** $\pi_C$ vs. $\mu$

6.2 | Analysis of the full probability $\pi_C$

Our aim in the numerical experiments is to reduce the full probability $\pi_C$, so that more customers can return bikes successfully. We consider the relation between $\pi_C$ and $\mu$ without applying the return process selection mechanism. To this end, we set $d_1 = d_2 = 1, \alpha_1 = \alpha_2 = 0$.

Figure 5 characterises the effect of $\mu$ on $\pi_C$. It is easy to see that $\pi_C$ increases as $\mu$ increases. We can intuitively understand the increasing observation. If $\mu$ increases, then the bike-riding duration decreases, thus more bikes will be returned to stations, so that the probability of full stations increases.

When applying the return process selection mechanism, we analyze the effect of $d_2$ and $\alpha_2$ on $\pi_C$, respectively. In the left of Figure 6, we assume $d_1 = 1, \alpha_1 = 0$ and $\alpha_2 = 0.8$. It is seen that $\pi_C$ decreases as $d_2$ increases. This result can be understood as follows: If $d_2$ increases, then the recommended station with the fewest bikes is selected from more stations, thus the recommended station may have fewer bikes, so that the full probability $\pi_C$ decreases.

In the right of Figure 6, we analyze how the participation proportion $\alpha_2$ affects $\pi_C$. We set $d_2 = 3, d_1 = 1, \alpha_1 = 0$. It is seen that $\pi_C$ decreases as $\alpha_2$ increases. This shows that when more customers adopt the return process selection mechanism, the full probability $\pi_C$ decreases as $\alpha_2$ increases. What’s more, when all the customers participate in the return process selection mechanism, it can realize that the full probability is nearly zero.

Note that the numerical experiments are to validate the dual selection mechanism and the model computation. We analyze the effect of main parameters $\lambda, \mu, d_1, d_2, \alpha_1$ and $\alpha_2$ on the probabilities of problematic stations. These results verify the balancing effort of the dual selection mechanism. Operators can design the parameters of dual selection mechanism based on the characteristics of cities (city size, tourist or industrial) to control the probabilities of problematic stations under an expected level. Also, operators can align the incentives of customers by regulating the participation proportions to control the probabilities of problematic stations.

![Figure 6](image2.png)  
**FIGURE 6** $\pi_C$ vs. $d_2$ and $\alpha_2$

7 | CONCLUSION

This paper combines a dual selection mechanism with a dynamic double-threshold repositioning policy to balance the bike sharing system. The dual selection mechanism is an active and preventive incentive mechanism, which aims to balance the BBS before problematic stations occur. We compute the steady-state probability of problematic stations of the BSS and provide analysis on how to reduce the probability by regulating relevant parameters. We verify that the dual selection mechanism can significantly reduce the steady-state probability of problematic stations. Owing to the GPS built-in, some accurate and real-time information on bikes in the stations has been posted on the smart phone APP, and the data can be used to guide customer behaviour for renting bikes in a suitable station. Many redeeming incentive mechanisms have been deployed when repositioning requests are needed. All of the mechanisms point out the rewarded stations following the existing repositioning requirement. This paper gives a computational method to determine a
recommended station guiding against the problematic stations. This preventive incentive mechanism is needed to be carried out to incentive customers to select the recommended stations before problematic stations come out. In addition, this paper also considers the participation proportions of customers who follow the selection mechanisms, which makes the analysis of the incentive mechanism more general.

This paper provides a complete picture of how to use Markov processes and martingale limits to analyze and compute the steady-state probability of problematic stations of large-scale BSSs. The methodology and results of this paper give a new highlight on understanding the influence of system parameters on performance measures of BSSs. The selection mechanism and computation of this paper can easily extend to free floating bike sharing systems. And we hope our methodology developed in this paper can be applicable to car sharing systems, rerouting customers on over-booked flights and so forth. On such a research line, there are several interesting directions for potential future research for BSSs as follows: (1) Introducing a combination of this preventive selection mechanism with redeeming incentive mechanisms, and (2) studying the optimal arrangement of pecuniary incentive design and manually repositioning.

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REFERENCES
1. Meddin, R., Demaio, P.: The bike-sharing world map. (2020). Accessed 4 Aug 2020
2. Bike Angel Program. https://bikeangels.citibikenyc.com (2019). Accessed 1 Aug 2019
3. Mobike. https://mobike.com/cn/made-by-mobike/ (2020). Accessed 1 Aug 2020
4. Lan, J., et al.: Enabling value co-creation in the sharing economy: the case of Mobike. Sustainability, 9, 1504 (2017)
5. Tan, Z., Yang, H., Guo, R.Y.: Dynamic congestion pricing with day-to-day flow evolution and user heterogeneity. Transp. Res. Pt. C-Emerg. Technol. 61, 87–105 (2015)
6. Guangzhou Investigation Report, Investigation report on the use of public bicycles in guangzhou. http://www.docin.com/p-500058471.html. (2017)
7. Hangzhou Investigation Report. Investigation and analysis on the operation status of bicycle sharing system in Hangzhou, https://wenku.baidu.com/view/3d235b706e9951e796892753.html (2012). Accessed 12 May 2012
8. Nanjing Investigation Report, China. Nanjing Network Information Center. Investigation report on the use of public bicycles in Nanjing. http://www.nanjing.gov.cn/hdl/zjke/wdsc/dhgc/201709/t20170925_360504.html. (2017). Accessed 25 September 2017
9. Lin, J.R., Yang, T.H.: Strategic design of public bicycle sharing systems with service level constraints. Transp. Res. Pr. e-Logist. Transp. Rev. 47(2), 284–294 (2011)
10. Lin, J.R., Yang, T.H., Chang, Y.C.: A hub location inventory model for bicycle sharing system design: formulation and solution. Comput. Ind. Eng. 65(1), 77–86 (2013)
11. Nair, R., Miller-Hooks, E.: Equilibrium design of bicycle sharing systems: the case of Washington DC. Comput. Ind. Eng. 5(3), 321–344 (2016)
12. Schuijbroek, J., Hampshire, R.C., Van Hoeve, W.J.: Inventory rebalancing and vehicle routing in bike sharing systems. Eur. J. Oper. Res. 257(3), 992–1004 (2017)
13. Cruz, F., et al.: A heuristic algorithm for a single vehicle static bike sharing rebalancing problem. Comput. Oper. Res. 79, 19–33 (2017)
14. Liu, J., et al.: Rebalancing bike sharing systems: A multi-source data smart optimization. In: Proceedings of International Conference on Knowledge Discovery and Data Mining, San Francisco, (2016)
15. Bulhoes, T., et al.: The static bike relocation problem with multiple vehicles and visits. Eur. J. Oper. Res. 264(2), 508–523 (2018)
16. Jia, H., et al.: Multiobjective bike repositioning in bike-sharing systems via a modified artificial bee colony algorithm. IEEE Trans. Autom. Sci. Eng. 17(2), 909–920 (2020)
17. Shu, J., et al.: Models for effective deployment and redistribution of bicycles within public bicycle-sharing systems. Oper. Res. 61(6), 1346–1359 (2013)
18. Caggiani, L., et al.: A modeling framework for the dynamic management of free-floating bike-sharing system. Transp. Res. Pr. C-Emerg. Technol. 87, 159–182 (2018)
19. Legros, B.: Dynamic repositioning strategy in a bike-sharing system; how to prioritize and how to rebalance a bike station. Eur. J. Oper. Res. 272(2), 740–753 (2019)
20. Pfrommer, J., et al.: Dynamic vehicle redistribution and online price incentives in shared mobility systems. IEEE Trans. Intell. Transp. Syst. 15(4), 1567–1578 (2014)
21. Singla, A., Santoni, M., Bartók, G.: Incentivizing users for balancing bike sharing systems. In: Proceedings of the 29th AAAI Conference on Artificial Intelligence, Austin (2015)
22. Ban, S., Hyun, K.H.: Designing a user-participation-based bike rebalancing service. Sustainability 11(8), 2396 (2019)
23. Wu, R., Liu, S., Shi, Z.: Customer incentive rebalancing plan in free-float bike-sharing system with limited information. Sustainability 11(11), 3088 (2019)
24. Wu, J.: Challenges and opportunities in algorithmic solutions for rebalancing in bike sharing systems. Tsinghua Sci. Technol. 25(6), 721–733 (2020)
25. Pan, Q., Cai, Z., Fang, P.: Tang: A deep reinforcement learning framework for rebalancing dockless bike sharing systems. In: Proceedings of the 2019 AAAI Conference on Artificial Intelligence, Honolulu (2019)
26. Duan, Y., Wu, J.: Optimizing rebalance scheme for dock-less bike sharing systems with adaptive user incentive. In: Proceedings of the 20th IEEE International Conference on Mobile Data Management, Hong Kong (2019)
27. Fricker, C., Gast, N.: Incentives and redistribution in homogeneous bike-sharing systems with finite capacity. Euro Journal on Transportation and Logistics 5(3), 261–291 (2016)
28. Kurtz, T.G.: Approximation of Population Processes. Society for Industrial and Applied Mathematics, Philadelphia (1981)
29. Ethier, S.N., Kurtz, T.G.: Markov processes: characterization and convergence. John Wiley & Sons, New Jersey (2009)
30. Li, Q.L.: Constructive Computation in Stochastic Models with Applications. Springer, Science & Business Media Press, Singapore (2010)
31. Li, Q.L., et al.: The mean-field computation in a supermarket model with returns and a dynamic double-threshold repositioning policy. IET Intell Transp Syst. 15:712–725, https://doi.org/10.1049/itrs.2012.12056
32. Smart, D.R.: Fixed Point Theorems. Cambridge University Press, Cambridge (1980)
APPENDIX A: THE PROOF OF EQUATIONS (20) TO (22)

We only prove Equations (20) and (21), while Equation (22) can be proved similarly. Based on Equation (4), we get that for $1 \leq k \leq C - 1$

$$W_k^{(N)}(t)(1 - u_{k+1}^{(N)}(t)) = \sum_{m=1}^{d_1} C_{d_1}^{m} \left( u_k^{(N)}(t) - u_{k+1}^{(N)}(t) \right)^m \left( 1 - u_k^{(N)}(t) \right)^{d_1-m}$$
$$= \sum_{m=0}^{d_1} C_{d_1}^{m} \left( u_k^{(N)}(t) - u_{k+1}^{(N)}(t) \right)^m \left( 1 - u_k^{(N)}(t) \right)^{d_1-m} - \left( 1 - u_k^{(N)}(t) \right)^{d_1}$$

then

$$\lambda_N \left[ \alpha_1 \sum_{m=1}^{d_1} C_{d_1}^{m} \left( u_k^{(N)}(t) - u_{k+1}^{(N)}(t) \right)^m \left( 1 - u_k^{(N)}(t) \right)^{d_1-m} + (1 - \alpha_1) \left( u_k^{(N)}(t) - u_{k+1}^{(N)}(t) \right)^{d_1} \right] = \lambda_N \left\{ \left[ 1 - u_{k+1}^{(N)}(t) \right] - \left( 1 - u_k^{(N)}(t) \right)^{d_1} \right\}$$

Based on Equation (5), we get that

$$W_C^{(N)}(t)u_C^{(N)}(t) = \sum_{m=1}^{d_1} C_{d_1}^{m} \left( u_C^{(N)}(t) \right)^m \left( 1 - u_C^{(N)}(t) \right)^{d_1-m}$$
$$= \sum_{m=0}^{d_1} C_{d_1}^{m} \left( u_C^{(N)}(t) \right)^m \left( 1 - u_C^{(N)}(t) \right)^{d_1-m} - \left( 1 - u_C^{(N)}(t) \right)^{d_1}$$

then

$$\lambda N \left[ \alpha_1 \sum_{m=1}^{d_1} C_{d_1}^{m} \left( u_C^{(N)}(t) \right)^m \left( 1 - u_C^{(N)}(t) \right)^{d_1-m} + (1 - \alpha_1) \left( u_C^{(N)}(t) \right)^{d_1} \right] = \lambda \left\{ \left[ 1 - (1 - u_k^{(N)}(t)) \right] - \left( 1 - u_C^{(N)}(t) \right)^{d_1} \right\}$$

This completes the proof.

APPENDIX B: THE ELEMENTS OF MATRIX $V_m$ IN EQUATION (38)

Let $\omega_k = \xi_k + \eta_k + \beta (R - k)$ for $1 \leq k \leq M$, and $\chi_k = \xi_k + \eta_k$ for $M + 1 \leq k \leq C$.

$$B_{1,1} = \begin{pmatrix} - (\xi_0 + \beta R) & \xi_0 \\ \eta_1 & -\omega_1 & \xi_1 \\ & \ddots & \ddots & \ddots \\ & & \eta_M & -\omega_M \\ \end{pmatrix}_{(M+1) \times (M+1)}$$

$$B_{1,2} = \begin{pmatrix} \beta R \\ \beta (R - 1) \\ \vdots \\ \beta (R - M) \\ \end{pmatrix}_{(M+1) \times (R - M + 1)}$$

$$B_{2,1} = \begin{pmatrix} \xi_{M+1} \\ \vdots \\ \xi_C \\ \end{pmatrix}_{(R - M) \times (M+1)}$$

$$B_{2,2} = \begin{pmatrix} -\chi_{M+1} & \xi_{M+1} \\ -\eta_{M+1} - \chi_{M+2} - \xi_{M+2} & \xi_{M+2} \\ \vdots & \ddots & \ddots \\ \eta_R - \chi_R & \xi_R - \chi_R \\ \end{pmatrix}_{(R - M) \times ((R - M) - 1)}$$

$$B_{3,3} = \begin{pmatrix} -\chi_{R+1} & \xi_{R+1} \\ -\eta_{R+2} - \chi_{R+2} - \xi_{R+2} & \xi_{R+2} \\ \vdots & \ddots & \ddots \\ \eta_{R+1} - \chi_{R+1} & \xi_{R+1} \\ \end{pmatrix}_{(R - M) \times ((R - M) - 1)}$$

$$B_{3,4} = \begin{pmatrix} \xi_{R+1} \\ \vdots \\ \xi_{C - 1} \\ \end{pmatrix}_{(R - M) \times (L - R - 1)}$$

$$B_{4,3} = \begin{pmatrix} \eta_{L} \\ \vdots \\ \eta_{C - 1} \\ \end{pmatrix}_{(L - 1) \times (C - 1)}$$
\[ B_{4,4} = \begin{pmatrix}
\xi_S & \xi_{S+1} & \cdots & \xi_{L-1} \\
\eta_{S+1} - \chi_{S+1} & \eta_{S+2} - \chi_{S+2} & \cdots & \eta_{L-2} - \chi_{L-2} \\
\vdots & \vdots & \ddots & \vdots \\
\eta_{L-1} - \chi_{L-1} & \cdots & \cdots & \eta_{L-1} - \chi_{L-1}
\end{pmatrix}_{(L-S) \times (L-S)}, \]

\[ B_{5,4} = \begin{pmatrix}
\beta (L-S) \\
\beta (L+1-S) \\
\vdots \\
\beta (C-S)
\end{pmatrix}_{(C-L+1) \times (L-S+1)}, \]

\[ B_{4,5} = \begin{pmatrix}
\xi_S & \xi_{S+1} & \cdots & \xi_{L-1} \\
\eta_{L+1} - \xi_{L+1} & \eta_{L+2} - \xi_{L+2} & \cdots & \eta_{C} - \xi_{C} \\
\vdots & \vdots & \ddots & \vdots \\
\xi_{L-1} & \cdots & \cdots & \xi_{L-1}
\end{pmatrix}_{(L-L+1) \times (C-L+1)}, \]

\[ B_{5,5} = \begin{pmatrix}
\beta (L-S) \\
\beta (L+1-S) \\
\vdots \\
\beta (C-S)
\end{pmatrix}_{(C-L+1) \times (C-L+1)}.