RESONANCE INTERACTIONS OF MULTI-PARTICLE SYSTEMS

SIMON BARTH, ANDREAS BITTER AND SEMJON VUGALTER

Abstract. We consider $N$-body Schrödinger operators with a virtual level at the threshold of the essential spectrum. We show that in the case of $N \geq 3$ particles in dimension $d \geq 3$ virtual levels turn into simple eigenvalues of the system and we obtain decay rates of the corresponding eigenfunctions in dependence on the dimension and the number of particles. We prove that in dimension $d \geq 3$ the Hamiltonian of $N \geq 4$ particles interacting via short-range potentials admits only a finite number of negative eigenvalues. We extend our results to dimension $d = 1$ and $d = 2$ in case of $N \geq 4$ fermions.

1. Introduction

A remarkable physical phenomenon in three-body quantum systems is the so-called Efimov effect, which was first discovered by the physicist V. Efimov in 1970 [5]. It reads as follows: The three-body Schrödinger operator of three-dimensional particles interacting via short-range potentials has an infinite number of negative eigenvalues if every two-body subsystem has non-negative spectrum and at least two of them have a resonance at zero. As was predicted by V. Efimov these three-body bound states should have very unusual properties. In particular, they accumulate logarithmically at zero with accumulation rate depending on the masses of the particles but not on the shapes of the potentials.

It became an outstanding challenge to understand this phenomenon, both from the physical and the mathematical point of view. The first mathematical proof of the Efimov effect was given by D. R. Yafaev in 1974 [32], where he studied a symmetrized form of the Faddeev equations for the eigenvalues of the three-particle Schrödinger operator together with the low-energy behaviour of the resolvents of the two-body Hamiltonians. This proof constituted a major step forward in the understanding of this problem. Later he also proved that such an effect cannot occur if at least two of the two-body Hamiltonians do not have any resonances [33]. By the middle of 1990’s a large number of physical and mathematical results were obtained on this topic, e.g. [24, 26, 25, 20, 18, 17, 29, 30, 31, 28, 27].

A new wave of interest for the Efimov effect came at the beginning of the 21st century with the experimental discovery of this effect in an ultracold gas of caesium atoms [14] (for a detailed review of experimental works see [6]). In 2013 the physicists Y. Nishida, S. Moroz and D. T. Son discovered the so-called super Efimov effect [16], which states that in the case of three spinless fermions in dimension two the system has infinitely many negative bound states, provided every two-body subsystem admits a $p$-wave resonance at zero. Later this was mathematically proved by D. K. Gridnev [11], where similar techniques to D. Yafaev’s original proof of the Efimov effect have been used.

It is a fundamental question to ask whether the Efimov effect can be extended to multi-particle systems with more than three particles. In 1973, the physicists R. D. Amado and F. C. Greenwood [3] concluded that in the case of $N \geq 4$ bosons in dimension three, the effect cannot emerge if only $(N - 1)$-particle subsystems have resonances. The reason for this conclusion is that a three-particle virtual level should be an eigenvalue at the edge of the essential spectrum, in contrast to the two-particle case where it is a resonance. As was mathematically rigorous proved
in [30] the existence of eigenvalues at the edge of the essential spectrum of the two-particle subsystems can not lead to an Efimov-type effect.

In [19] Y. Nishida and S. Tan predicted that universal effects similar to the Efimov effect can be found in several types of N-particle systems with $N \geq 4$ in different dimensions. In 2017, Y. Nishida also predicted on a physical level of rigoroussness that a similar effect is possible in case of four two-dimensional bosons [15]. Here, the three-body resonances lead to the infiniteness of the discrete spectrum of the four-body Hamiltonian.

The first attempt to give a mathematically rigorous proof of the result by Amado and Greenwood was made in 2013 by D. K. Gridnev [10]. His approach is based on similar ideas as in the articles of D. R. Yafaev [32] and A. V. Sobolev [24], extended to the case of $N \geq 4$ particles. As usual for $N$-particle systems with $N \geq 4$ this method requires a lot of technical estimates and strong restrictions on the potentials $V_{ij}$; in particular, in [10] it is assumed that $V_{ij} \in L^1(\mathbb{R}^3) \cap L^3(\mathbb{R}^3)$. The most difficult part of the proof is to show that in the case of $N \geq 3$ particles the bottom of the essential spectrum of the corresponding Hamiltonian cannot be a resonance, provided every subsystem has no negative bound states or resonances at the threshold of the essential spectrum. The proof of this statement is stretched over two articles. Firstly, it was proved in the case $N = 3$ [8], where it was shown that the three-body system, which is at the three-body coupling constant threshold, has a square integrable zero energy solution if none of the two-body subsystems is bound or has a zero-energy resonance. Here, it was assumed that the pair interactions $V_{ij}$ are non-negative. Later, this result was generalized to the case of $N \geq 4$ particles and where the potentials $V_{ij}$ are allowed to change signs [9].

In the work at hand we present a different and very transparent approach to the study of decay properties of zero-resonances and eigenfunctions of multi-particle Schrödinger operators at the edge of the essential spectrum. This approach is a further development of Agmon’s method of proving the exponential decay of eigenfunctions [2]. In particular, we establish connections between the rate of decay of a virtual level at zero and Hardy’s constant in the corresponding space. Since our method is purely variational it allows us to work with very weak restrictions on the potentials. In addition, as it is usual for variational methods for multi-particle Schrödinger operators our approach allows us to work on subspaces with fixed permutational symmetry. Combining our results on the decay of virtual levels with the ideas of [30] we give a purely variational proof of the absence of the Efimov effect for $N \geq 4$ particles in all dimensions $n \geq 3$. We extend this result to systems of $N \geq 4$ identical fermions on the subspace of antisymmetric functions in dimension $n = 1$ and $n = 2$.

The paper is organized as follows. In Section 2, we introduce our notation and give sufficient conditions for the existence of solutions in the space $\dot{H}^1(\mathbb{R}^n)$ of the equation

$$(-\Delta + V(x))\psi = 0, \quad x \in \mathbb{R}^n, \quad n \geq 3,$$

without assuming that the potential $V(x)$ decays as $|x| \to \infty$. We then prove estimates on the rate of decay of such solutions. The conditions on the potential $V(x)$ are chosen in such a way that this result can be applied to multi-particle systems. In Section 3, we extend this result to Schrödinger operators considered on subspaces of states with fixed symmetries. Section 4 is devoted to the applications of the results obtained in Section 1 and Section 2. In particular, in this section we prove that for $N \geq 4$ in dimension $n \geq 3$ the virtual level is an eigenfunction. We give estimates on the rate of decay of these eigenfunctions in dependence on the number of particles and the corresponding dimension. In Section 5 we prove the absence of the Efimov effect for $N \geq 4$ particles in dimension $n \geq 3$. In Section 6 we extend the result of Section 4 and Section 5 to the case of $N \geq 4$ one- and two-dimensional fermions. In the Appendix we prove several technical results. Some of these results were known before and are given for the convenience of the reader only.
2. Decay properties of zero-energy solutions of the Schrödinger equation

In the following we consider the Schrödinger operator

$$h_0 = -\Delta + V$$  \hspace{1cm} (2.1)

in $L^2(\mathbb{R}^d)$, where $d \geq 3$. We assume that the potential $V$ is relatively bounded with bound zero, i.e. for every $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that

$$\langle |V| \psi, \psi \rangle \leq \varepsilon \|\nabla \psi\|^2 + C(\varepsilon) \|\psi\|^2$$  \hspace{1cm} (2.2)

holds for any function $\psi \in C_0^\infty(\mathbb{R}^d)$. According to the KLMN-Theorem (see [21]) assumption (2.2) implies that $h_0$ is a self-adjoint operator in $L^2(\mathbb{R}^d)$, corresponding to the quadratic form

$$L[\varphi] = \|\nabla \varphi\|^2 + \langle V \varphi, \varphi \rangle$$  \hspace{1cm} (2.3)

with form domain $H^1(\mathbb{R}^d)$. For any $\varepsilon \in (0, 1)$ we denote

$$h_\varepsilon = h_0 + \varepsilon \Delta.$$  \hspace{1cm} (2.4)

Let $\hat{H}^1(\mathbb{R}^d)$ be the closure of $C_0^\infty(\mathbb{R}^d)$ with respect to the gradient-seminorm

$$\left( \int_{\mathbb{R}^d} |\nabla \psi|^2 \, dx \right)^{\frac{1}{2}}.$$  \hspace{1cm} (2.5)

For any self-adjoint operator $A$ we denote by $\mathcal{S}(A)$, $\mathcal{S}_{\text{ess}}(A)$ and $\mathcal{S}_{\text{disc}}(A)$ the spectrum, the essential spectrum and the discrete spectrum of $A$, respectively. The main results of this section is the following

**Theorem 2.1.** Suppose that $V$ satisfies (2.2). Further, assume that

$$h_0 \geq 0 \quad \text{and} \quad \inf \mathcal{S}(h_\varepsilon) < 0$$  \hspace{1cm} (2.6)

holds for any $\varepsilon \in (0, 1)$. If there exist constants $\alpha_0 > 0$, $b > 0$ and $\gamma_0 \in (0, 1)$, such that for any function $\psi \in H^1(\mathbb{R}^d)$ with $\text{supp} \psi \subset \{ x \in \mathbb{R}^d : |x| \geq b \}$ we have

$$\langle h_\varepsilon \psi, \psi \rangle - \gamma_0 \|\nabla \psi\|^2 - (\alpha_0^2 |x|^{-2} \psi, \psi) \geq 0,$$  \hspace{1cm} (2.7)

then the following assertions hold:

(i) If $\alpha_0 > 1$, then zero is a simple eigenvalue of $h_0$ and the corresponding eigenfunction $\varphi_0$ satisfies $(1 + |x|)^{\alpha_0 - 1} \varphi_0 \in L^2(\mathbb{R}^d)$. Moreover, there exists a constant $\delta_0 > 0$, such that for any function $\psi \in H^1(\mathbb{R}^d)$ with $\langle \nabla \psi, \nabla \varphi_0 \rangle = 0$ it holds

$$\langle h_0 \psi, \psi \rangle \geq \delta_0 \|\nabla \psi\|^2.$$  \hspace{1cm} (2.8)

(ii) If $\alpha_0 \in (0, 1)$ and in addition

$$\langle |V| \psi, \psi \rangle \leq C \|\nabla \psi\|^2$$  \hspace{1cm} (2.9)

holds for any function $\psi \in \hat{H}^1(\mathbb{R}^d)$ and some constant $C > 0$, then there exists a function $\varphi_1 \in \hat{H}^1(\mathbb{R}^d)$ satisfying

$$\|\nabla \varphi_1\|^2 + \langle V \varphi_1, \varphi_1 \rangle = 0$$  \hspace{1cm} (2.10)

and $(1 + |x|)^{\alpha_0 - 1} \varphi_1 \in L^2(\mathbb{R}^d)$. If we assume that for some $C > 0$

$$\|V \psi\|^2 \leq C \left( \|\nabla \psi\|^2 + \|\psi\|^2 \right)$$  \hspace{1cm} (2.11)

holds for every function $\psi \in C_0^\infty(\mathbb{R}^d)$, then the solution $\varphi_1 \in \hat{H}^1(\mathbb{R}^d)$ of (2.10) is unique. Moreover, there exists a constant $\delta_1 > 0$, such that for any function $\psi \in \hat{H}^1(\mathbb{R}^d)$ with $\langle \nabla \psi, \nabla \varphi_1 \rangle = 0$ it holds

$$\langle h_0 \psi, \psi \rangle \geq \delta_1 \|\nabla \psi\|^2.$$  \hspace{1cm} (2.12)
If instead of \((2.7)\) a stronger inequality
\[
\langle h_0 \psi, \psi \rangle - \gamma_0 \| \nabla \psi \|^2 - \langle \alpha_0^2 |x|^{-\beta} \psi, \psi \rangle \geq 0
\]
holds for some constants \(\alpha_0, \gamma_0 > 0\) and \(\beta \in (0, 2)\), then the function \(\varphi_0\) in part (i) of the theorem satisfies
\[
\exp\left(\alpha_0 \kappa^{-1} |x|^\kappa\right) \varphi_0 \in L^2(\mathbb{R}^d), \quad \text{where} \quad \kappa = 1 - 2^{-1} \beta.
\]

**Remark.** (i) Note that assumption \((2.7)\) implies that for any \(0 < \varepsilon < \gamma_0\) the essential spectrum of the operator \(h_\varepsilon\) is non-negative. Hence \((2.6)\) implies that for any sufficiently small \(\varepsilon > 0\) the operator \(h_\varepsilon\) has a discrete eigenvalue.

(ii) We assume \((2.11)\) to be able to apply the results by M. Schechter and B. Simon [22] on the unique continuation theorem, which allows us to prove the uniqueness of \(\varphi_1\). Without this assumption the subspace of functions in \(\dot{H}^1(\mathbb{R}^d)\) satisfying \((2.10)\) is at most finite (see Lemma A.1 in Appendix).

(iii) As mentioned in the first remark the operator \(h_\varepsilon\) has negative eigenvalues for small \(\varepsilon > 0\). We should not expect that a sequence of the corresponding eigenfunctions \(\varphi_\varepsilon\) always converges in \(L^2(\mathbb{R}^d)\) as \(\varepsilon \to 0\), because we know that for one-particle Schrödinger operators with short-range potentials in \(\mathbb{R}^3\) this is not the case. However, if we normalize the sequence \(\varphi_\varepsilon\) with the seminorm \((2.5)\), condition \((2.7)\) will make it energetically disadvantageous for \(\varphi_\varepsilon\) to leave all compact regions. This allows us to prove that the quadratic form of \(h_0\) has a minimizer in \(\dot{H}^1(\mathbb{R}^d)\).

In the proof of Theorem 2.1 we will apply the following localization error estimate, which is a straightforward modification of Lemma 5.1. in [30]. For the sake of completeness we will give the corresponding proof in the Appendix.

**Lemma 2.2.** For every \(\varepsilon > 0\) and every fixed \(b > 0\) one can find \(\bar{b} > b\) and smooth functions \(\chi_1, \chi_2 : \mathbb{R}^d \to \mathbb{R}\), such that
\[
\chi_1^2 + \chi_2^2 = 1, \quad \chi_1(x) = \begin{cases} 1, & |x| \leq \bar{b} \\ 0, & |x| > \bar{b} \end{cases}
\]
and
\[
|\nabla \chi_1|^2 + |\nabla \chi_2|^2 \leq \varepsilon |x|^{-2}.
\]

**Remark.** Note that by \((2.16)\) and Hardy’s inequality
\[
\int |\nabla \chi_i|^2 |\psi|^2 \, dx \leq \varepsilon \| \nabla \psi \|^2
\]
holds for every \(\psi \in \dot{H}^1(\mathbb{R}^d)\), where \(d \geq 3\) and \(i = 1, 2\). This estimate shows that if the constant \(\bar{b}\) is chosen much larger than \(b\), then the localization error can be compensated with an \(\varepsilon\)-part of \(\| \nabla \psi \|^2\).

**Proof of statement (i) of Theorem 2.1.** By Lemma A.1 there exists a sequence of eigenfunctions \(\psi_n \in H^1(\mathbb{R}^d)\), corresponding to eigenvalues \(E_n < 0\) of the operator \(h_{n-1}\), i.e. it holds
\[
-(1 - n^{-1}) \Delta \psi_n + V \psi_n = E_n \psi_n.
\]
We normalize the sequence \((\psi_n)_{n \in \mathbb{N}}\) by \(\| \nabla \psi_n \| = 1\) and take a weakly convergent subsequence (also denoted by \((\psi_n)_{n \in \mathbb{N}}\)), which has a weak limit \(\varphi_0 \in \dot{H}^1(\mathbb{R}^d)\). Note that by the Rellich–Kondrachov theorem \((\psi_n)_{n \in \mathbb{N}}\) converges to \(\varphi_0\) in \(L^2_{\text{loc}}(\mathbb{R}^d)\). We will prove statement (i) of Theorem 2.1 successively by the following Lemmas.

**Lemma 2.3.** The weak limit \(\varphi_0 \in \dot{H}^1(\mathbb{R}^d)\) of the sequence \((\psi_n)_{n \in \mathbb{N}}\) is not identically zero.
such that for any eigenfunction $\psi$ operator $h$ for every $H$.

Hence, (Lemma 2.4.

Assume that inequality proves the Lemma.

□

Proof. We consider the functional

$$L[\psi, \varepsilon] := (1 - \varepsilon)\|\nabla \psi\|^2 + \langle V \psi, \psi \rangle,$$

(2.19)

where $\psi \in H^1(\mathbb{R}^d)$ and $\varepsilon > 0$. We fix constants $\varepsilon_1 > 0$ and $b > 0$, such that (2.7) holds and construct functions $\chi_1, \chi_2$ in accordance with Lemma 2.2, which implies

$$L[\psi, \varepsilon] \geq L[\psi \chi_1, \varepsilon + \varepsilon_1] + L[\psi \chi_2, \varepsilon + \varepsilon_1]$$

(2.20)

for every $\psi \in H^1(\mathbb{R}^d)$ independently of $\varepsilon$. Since the operator $h_0$ is non-negative we have

$$L[\psi \chi_1, \varepsilon + \varepsilon_1] = (1 - \varepsilon - \varepsilon_1)\|\nabla (\psi \chi_1)\|^2 + \langle V \psi \chi_1, \psi \chi_1 \rangle$$

$$\geq - (\varepsilon + \varepsilon_1)\|\nabla (\psi \chi_1)\|^2.$$

(2.21)

In addition, since $\text{supp} (\psi \chi_2) \subset \{x \in \mathbb{R}^d : |x| \geq b\}$ we conclude by (2.7) that

$$L[\psi \chi_2, \varepsilon + \varepsilon_1] = (1 - \varepsilon - \varepsilon_1)\|\nabla (\psi \chi_2)\|^2 + \langle V \psi \chi_2, \psi \chi_2 \rangle$$

$$= (1 - \gamma_0)\|\nabla (\psi \chi_2)\|^2 + \langle V \psi \chi_2, \psi \chi_2 \rangle + (\gamma_0 - \varepsilon - \varepsilon_1)\|\nabla (\psi \chi_2)\|^2$$

$$\geq (\gamma_0 - \varepsilon - \varepsilon_1)\|\nabla (\psi \chi_2)\|^2.$$

(2.22)

Hence, (2.21) and (2.22) imply

$$L[\psi, \varepsilon] \geq -(\varepsilon + \varepsilon_1)\|\nabla \psi \chi_1\|^2 + (\gamma_0 - \varepsilon - \varepsilon_1)\|\nabla (\psi \chi_2)\|^2.$$

(2.23)

For $\psi = \psi_n$ and $\varepsilon = n^{-1}$, estimate (2.23) yields

$$-(\varepsilon_1 + n^{-1})\|\nabla (\psi_n)\chi_1\|^2 + (\gamma_0 - \varepsilon_1 - n^{-1})\|\nabla (\psi_n)\chi_2\|^2 < 0,$$

(2.24)

which implies

$$(\gamma_0 - \varepsilon_1 - n^{-1}) (\|\nabla (\psi_n)\chi_1\|^2 + \|\nabla (\psi_n)\chi_2\|^2) < \gamma_0\|\nabla (\psi_n)\chi_1\|^2.$$

(2.25)

By the IMS localization formula we have

$$\|\nabla (\psi_n)\chi_1\|^2 + \|\nabla (\psi_n)\chi_2\|^2 \geq \|\nabla \psi_n\|^2 = 1$$

(2.26)

for every $n \in \mathbb{N}$. Hence, by (2.25) we obtain

$$\|\nabla (\psi_n)\chi_1\|^2 \geq \frac{\gamma_0 - \varepsilon_1 - n^{-1}}{\gamma_0} \geq 1 - \varepsilon_2,$$

(2.27)

where $\varepsilon_2 > 0$ can be chosen arbitrarily small by choosing $\varepsilon_1 > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large. Due to (2.22) we have $L[\psi \chi_2, n^{-1} + \varepsilon_1] > 0$. This, together with (2.20) and $L[\psi_n, n^{-1}] < 0$ implies

$$0 > L[\psi_n \chi_1, n^{-1} + \varepsilon_1] = (1 - n^{-1} - \varepsilon_1)\|\nabla (\psi_n)\chi_1\|^2 + \langle V \psi_n \chi_1, \psi_n \chi_1 \rangle$$

$$\geq (1 - n^{-1} - 2\varepsilon_1)\|\nabla (\psi_n)\chi_1\|^2 - C(\varepsilon_1)\|\psi_n \chi_1\|^2,$$

(2.28)

where in the last inequality we used (2.2). By combining (2.28) and (2.27) we arrive at

$$\|\psi_n \chi_1\|^2 \geq \frac{(1 - n^{-1} - 2\varepsilon_1)(1 - \varepsilon_2)}{C(\varepsilon_1)}.$$

(2.29)

Since $|\chi_1| \leq 1$, $\chi_1$ is compactly supported and $(\psi_n)_{n \in \mathbb{N}}$ converges to $\varphi_0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$, the last inequality proves the Lemma.

□

Lemma 2.4. Assume that (2.6) and (2.7) hold for $\alpha_0 > 1$. Then there exists a constant $C_1 > 0$, such that for any eigenfunction $\psi_n \in H^1(\mathbb{R}^d)$ corresponding to a negative eigenvalue of the operator $h_{n-1}$, normalized by $\|\nabla \psi_n\| = 1$, we have $\|(1 + |x|)^{\alpha_0 - 1} \psi_n\| \leq C_1$. 

Remark. Recall that eigenfunctions $\psi_n$ of the operators $h_n^{-1}$ decay exponentially with powers depending on the distances from the corresponding eigenvalues to zero. Since for $n \to \infty$ the negative eigenvalues of $h_n^{-1}$ converge to zero, these exponential estimates are not uniform in $n \in \mathbb{N}$. However, Lemma 2.4 shows that if condition (2.7) holds for functions supported far from the origin, a uniform estimate on the rate of decay of eigenfunctions of $h_n^{-1}$ exists. This estimate is of the polynomial type and the corresponding power depends on the parameter $\alpha_0$ in (2.7) only.

Proof. For any $\varepsilon > 0$ we define the function

$$G_\varepsilon(x) = \frac{|x|^{\alpha_0}}{1 + \varepsilon|x|^{\alpha_0}} \chi_R(x),$$

where $\chi_R$ is a $C^\infty$ cutoff function, such that

$$\chi_R(x) = \begin{cases} 0, & |x| \leq R \\ 1, & |x| \geq 2R \end{cases}$$

Since for eigenfunctions $\psi_n$ we have

$$-(1 - n^{-1}) \Delta \psi_n + V \psi_n = E_n \psi_n$$

with $E_n < 0$ and each $\psi_n$ decays exponentially, we can multiply (2.32) with $G_\varepsilon^2 \psi_n$ and integrate by parts to obtain

$$(1 - n^{-1}) \langle \nabla \psi_n, \nabla (G_\varepsilon^2 \psi_n) \rangle + \langle V \psi_n, G_\varepsilon^2 \psi_n \rangle = E_n \|G_\varepsilon \psi_n\|^2 < 0.$$  (2.33)

Note that

$$\text{Re} \langle \nabla \psi_n, \nabla (G_\varepsilon^2 \psi_n) \rangle = \text{Re} \langle \nabla \psi_n, G_\varepsilon \psi_n \nabla G_\varepsilon \rangle + \text{Re} \langle (\nabla \psi_n)G_\varepsilon, \nabla (G_\varepsilon \psi_n) \rangle$$

$$= \text{Re} \langle \nabla (\psi_n G_\varepsilon), \psi_n \nabla G_\varepsilon \rangle - \text{Re} \langle \psi_n \nabla G_\varepsilon, \psi_n \nabla G_\varepsilon \rangle$$

$$+ \text{Re} \langle \nabla (\psi_n G_\varepsilon), \nabla (\psi_n G_\varepsilon) \rangle - \text{Re} \langle \psi_n \nabla G_\varepsilon, \nabla (\psi_n G_\varepsilon) \rangle$$

$$= \text{Re} \langle \nabla (\psi_n G_\varepsilon), \nabla (\psi_n G_\varepsilon) \rangle - \text{Re} \langle \psi_n \nabla G_\varepsilon, \psi_n \nabla G_\varepsilon \rangle.$$  (2.34)

Hence, we have

$$\langle \nabla \psi_n, \nabla (G_\varepsilon^2 \psi_n) \rangle = \|\nabla (\psi_n G_\varepsilon)\|^2 - \|\psi_n \nabla G_\varepsilon\|^2,$$  (2.35)

which together with (2.33) implies

$$\left(1 - \frac{1}{n}\right) \left(\|\nabla (\psi_n G_\varepsilon)\|^2 - \int |\psi_n|^2 |\nabla G_\varepsilon|^2 \, dx \right) + \int V|\psi_n G_\varepsilon|^2 \, dx < 0.$$  (2.36)

For $|x| > 2R$ we can estimate

$$|\nabla G_\varepsilon| = \frac{\alpha_0 |x|^{\alpha_0 - 1}}{(1 + \varepsilon|x|^{\alpha_0})^2} \leq \alpha_0 |x|^{-1} |G_\varepsilon|.$$  (2.37)

For $|x| \in [R, 2R]$ the function $|\nabla G_\varepsilon|$ is uniformly bounded in $\varepsilon$, which together with Hardy’s inequality implies

$$\int_{|x| \leq 2R} |G_\varepsilon|^2 |\psi_n|^2 \, dx \leq C \int_{|x| \leq 2R} |\psi_n|^2 \, dx \leq 4CR^2 \int |\nabla \psi_n|^2 \, dx =: C_0.$$  (2.38)

Substituting (2.37) and (2.38) into (2.36) we obtain

$$(1 - n^{-1}) \|\nabla (\psi_n G_\varepsilon)\|^2 + \langle VG_\varepsilon \psi_n, G_\varepsilon \psi_n \rangle - \alpha_0^2 \int_{|x| > 2R} \frac{|G_\varepsilon \psi_n|^2}{|x|^2} \, dx \leq C_1,$$  (2.39)

where $C_1 > 0$ does not depend on $n \in \mathbb{N}$ or $\varepsilon > 0$. Note that the function $G_\varepsilon \psi_n$ is supported outside the ball with radius $R > 0$. By choosing $R > b$ it satisfies (2.7), i.e. it holds

$$(1 - \gamma_0) \|\nabla (G_\varepsilon \psi_n)\|^2 + \langle VG_\varepsilon \psi_n, G_\varepsilon \psi_n \rangle - \alpha_0^2 |x|^{-2} G_\varepsilon \psi_n, G_\varepsilon \psi_n \rangle \geq 0.$$  (2.40)
For $n > 2\gamma_0^{-1}$ estimates (2.39) and (2.38) imply
\[
\frac{2}{\gamma_0} \| \nabla (G_\varepsilon \psi_n) \|^2 \leq C_1.
\] (2.41)
Taking $\varepsilon \to 0$ yields $\| \nabla (|x|^{\alpha_0} \psi_n) \| \leq C$, which together with Hardy’s inequality completes the proof.

\textbf{Lemma 2.5.} Assume that (2.6) and (2.7) hold for $\alpha_0 > 1$, then zero is an eigenvalue of $h_0$ and the corresponding eigenfunction $\varphi_0$ satisfies
\[
(1 + |x|)^{\alpha_0 - 1} \varphi_0 \in L^2(\mathbb{R}^d).
\] (2.42)
Proof. We take a sequence of eigenfunctions $\psi_n$ of $h_{n-1}$ normalized by $\| \nabla \psi_n \| = 1$. This sequence has a subsequence (also denoted by $(\psi_n)_{n \in \mathbb{N}}$) with a weak limit $\varphi_0 \in H^1(\mathbb{R}^d)$. According to Lemma 2.3 we have $\varphi_0 \not\equiv 0$. Since $(\psi_n)_{n \in \mathbb{N}}$ converges to $\varphi_0$ in $L^2_{\text{loc}}(\mathbb{R}^d)$ and by Lemma 2.4 we have $\| (1 + |x|)^\gamma \psi_n \| \leq C$ with $\gamma > 0$ and $C$ independent of $n \in \mathbb{N}$, we conclude that (2.42) holds. This also shows that $\langle V \psi_0, \psi_0 \rangle$ is well defined. Our next goal is to prove $\langle V \psi_0, \varphi_0 \rangle = -1$. We write
\[
\langle V \varphi_0, \varphi_0 \rangle = \langle V \varphi_0, \varphi_0 - \psi_n \rangle + \langle V \varphi_0, \psi_n \rangle
\]
Due to (2.2) the first term on the r.h.s. of (2.43) can be estimated by
\[
\| (V \varphi_0, \varphi_0 - \psi_n) \| \leq \left( \| \nabla \varphi_0 \|^2 + C(1) \| \varphi_0 \|^2 \right)^{\frac{1}{2}} \left( \epsilon \| \nabla (\varphi_0 - \psi_n) \|^2 + C(\epsilon) \| \varphi_0 - \psi_n \|^2 \right)^{\frac{1}{2}}
\]
\[
\leq C \left( \epsilon \| \nabla (\varphi_0 - \psi_n) \|^2 + C(\epsilon) \| \varphi_0 - \psi_n \|^2 \right)^{\frac{1}{2}}.
\] (2.44)
Since $\| \psi_n - \varphi_0 \| \to 0$ as $n \to \infty$, choosing $\epsilon > 0$ sufficiently small and $n \in \mathbb{N}$ sufficiently large shows that (2.44) can be done arbitrarily small. Similar arguments show that the second term on the r.h.s. of (2.43) can be done arbitrarily small as well. Consequently, we have $\langle V \psi_n, \psi_n \rangle \to \langle V \varphi_0, \varphi_0 \rangle$ as $n \to \infty$. By
\[
(1 - n^{-1}) \| \nabla \psi_n \|^2 + \langle V \psi_n, \psi_n \rangle \leq 0 \quad \text{and} \quad \| \nabla \psi_n \| = 1
\] (2.45)
we conclude $\langle V \varphi_0, \varphi_0 \rangle = -1$. By the semi-continuity of the norm we have $\| \nabla \varphi_0 \| \leq 1$. This yields
\[
\| \nabla \varphi_0 \|^2 + \langle V \varphi_0, \varphi_0 \rangle \leq 0,
\] (2.46)
which is only true for $\| \nabla \varphi_0 \| = 1$. Hence, $\varphi_0$ is a minimizer of the quadratic form of $h_0$. Since $\varphi_0 \in L^2(\mathbb{R}^d)$, we conclude that it is an eigenfunction of $h_0$, corresponding to the eigenvalue zero. \hfill \square

Our next goal is to prove inequality (2.8) and the nondegeneracy of $\varphi_0$. We will do it in the following Lemmas 2.6 - 2.8.

\textbf{Lemma 2.6.} For every $\varepsilon > 0$ one can find $n_0 \in \mathbb{N}$, such that for any $n \geq n_0$ and any eigenfunction $\psi_n$ with $\| \nabla \psi_n \| = 1$, corresponding to some negative eigenvalue of the operator $h_{n-1}$, it holds $\| \psi_n - \varphi_0 \| < \varepsilon$.

Proof. Assume that we have a sequence $a(n) \in (0, 1)$ with $a(n) \to 0$ as $n \to \infty$ and $\| \psi_{a(n)} - \varphi_0 \| \geq C > 0$ for every $n \in \mathbb{N}$. Proceeding as in the proof of Lemma 2.3 and Lemma 2.5 we can find a subsequence (also denoted by $a(n)$), such that $(\psi_{a(n)})_{n \in \mathbb{N}}$ converges to some function $\varphi_0 \in L^2(\mathbb{R}^d)$ with $\varphi_0 \not\equiv 0$, $\| \nabla \varphi_0 \| = 1$ and
\[
\| \nabla \varphi_0 \|^2 + \langle V \varphi_0, \varphi_0 \rangle = 0.
\] (2.47)
By $\|\nabla \varphi_0\| = \|\nabla \tilde{\varphi}_0\| = 1$ and $\|\psi_{a(n)} - \varphi_0\| \geq C > 0$ we conclude that $\varphi_0$ and $\tilde{\varphi}_0$ are linearly independent. According to [7] the eigenvalue of a Schrödinger operator coinciding with the bottom of the spectrum cannot be degenerate. Consequently, $\varphi_0$ and $\tilde{\varphi}_0$ cannot be linearly independent. \hfill $\Box$

**Lemma 2.7.** For any sufficiently small $\varepsilon > 0$ the operator $h_\varepsilon$ has only one negative eigenvalue, which is non-degenerate.

**Proof.** Assume there is a sequence $a(n) \in (0,1)$ with $a(n) \to 0$ as $n \to \infty$, such that for any $n \in \mathbb{N}$ the operator $h_{a(n)}$ has at least two eigenvalues. Recall that the lowest eigenvalue of $h_{a(n)}$ is non-degenerate. We consider two eigenfunctions $\psi_{a(n)}^{(1)}$ and $\psi_{a(n)}^{(2)}$, normalized by $\|\psi_{a(n)}^{(1)}\| = \|\psi_{a(n)}^{(2)}\| = 1$, where $\psi_{a(n)}^{(1)}$ corresponds to the lowest eigenvalue. Now $\psi_{a(n)}^{(1)}$ and $\psi_{a(n)}^{(2)}$ are orthogonal in $L^2(\mathbb{R}^d)$ and by Lemma 2.6 $\psi_{a(n)}^{(1)}$ and $\psi_{a(n)}^{(2)}$ both converge to $\varphi_0 \in L^2(\mathbb{R}^d)$, which is a contradiction. \hfill $\Box$

**Lemma 2.8.** There exists a constant $\delta_0 > 0$, such that for every function $\psi \in H^1(\mathbb{R}^d)$ with $\langle \nabla \psi, \nabla \varphi_0 \rangle = 0$ it holds

\begin{equation}
(1 - \delta_0)\|\nabla \psi\|^2 + \langle V\psi, \psi \rangle \geq 0.
\end{equation}

**Proof.** We prove the Lemma by contradiction. Assume that there is no such constant $\delta_0 > 0$. Then there exists a sequence of functions $g_n \in H^1(\mathbb{R}^d)$ with

\begin{equation}
\langle \nabla g_n, \nabla \varphi_0 \rangle = 0 \quad \text{and} \quad \langle h_{n^{-1}}g_n, g_n \rangle < 0.
\end{equation}

Note that for all $c_1, c_2 \in \mathbb{R}$ we have

\begin{equation}
\langle h_{n^{-1}}(c_1g_n + c_2\varphi_0), (c_1g_n + c_2\varphi_0) \rangle = c_1^2\langle h_{n^{-1}}g_n, g_n \rangle + c_2^2\langle h_{n^{-1}}\varphi_0, \varphi_0 \rangle + 2\text{Re}c_1c_2\langle h_{n^{-1}}g_n, \varphi_0 \rangle.
\end{equation}

Further, it is easy to see that

\begin{equation}
\langle h_{n^{-1}}g_n, \varphi_0 \rangle = \langle g_n, h_0\varphi_0 \rangle - n^{-1}\text{Re}\langle \nabla g_n, \nabla \varphi_0 \rangle = 0
\end{equation}

and

\begin{equation}
\langle h_{n^{-1}}\varphi_0, \varphi_0 \rangle = \langle h_0\varphi_0, \varphi_0 \rangle - n^{-1}\|\nabla \varphi_0\|^2 = -n^{-1}
\end{equation}

hold for every $n \in \mathbb{N}$. Hence, we conclude that for any linear combination $c_1g_n + c_2\varphi_0$ we have

\begin{equation}
\langle h_{n^{-1}}(c_1g_n + c_2\varphi_0), (c_1g_n + c_2\varphi_0) \rangle < 0.
\end{equation}

Since by (2.49) functions $\varphi_0$ and $g_n$ are linearly independent, for any $n \in \mathbb{N}$ we can find a linear combination $f_n$ of $\varphi_0$ and $g_n$, such that $f_n$ is orthogonal to the ground state of $h_{n^{-1}}$. Since by Lemma 2.7 for sufficiently large $n \in \mathbb{N}$ the operator $h_{n^{-1}}$ has only one negative eigenvalue, we have $\langle h_{n^{-1}}f_n, f_n \rangle \geq 0$. This is a contradiction to (2.53). \hfill $\Box$

Combining Lemma 2.5 and Lemma 2.8 proves statement (i) of Theorem 2.1.

**Proof of statements (ii) and (iii) of Theorem 2.1.** Note that in case of $\alpha_0 \in (0,1)$ the sequence of eigenfunctions $\psi_{\alpha}$ of the operators $h_{\alpha^{-1}}$, normalized by $\|\nabla \psi_{\alpha}\| = 1$, does not necessarily converge in $L^2(\mathbb{R}^d)$, as for example happens in the case of a one-particle Schrödinger operator in $\mathbb{R}^3$. To ensure that the functional $\|\nabla \psi\|^2 + \langle V\psi, \psi \rangle$ is well defined for the weak limit $\varphi_0 \in H^1(\mathbb{R}^d)$ and that $\langle V\psi_{\alpha}, \psi_{\alpha} \rangle$ converges to $\langle V\varphi_0, \varphi_0 \rangle$ as $n \to \infty$, we assume (2.9). We will prove part (ii) of Theorem 2.1 in two steps. In Lemma 2.9 we prove the existence of the function $\varphi_1$ satisfying (2.10). Then, in Lemma 2.10 we prove the uniqueness of $\varphi_1$ and the inequality (2.12).
Lemma 2.9. Assume that (2.6) and (2.7) hold for \( \alpha_0 \in (0,1) \) and in addition
\[
\langle |V| \psi, \psi \rangle \leq C \| \nabla \psi \|^2 \tag{2.54}
\]
holds for any function \( \psi \in \dot{H}^1(\mathbb{R}^d) \) and some constant \( C > 0 \). Then, there exists a function \( \varphi_1 \in \dot{H}^1(\mathbb{R}^d) \) with
\[
\| \nabla \varphi_1 \|^2 + (V \varphi_1, \varphi_1) = 0. \tag{2.55}
\]
Moreover, \( \varphi_1 \) satisfies \( (1 + |x|)^{\alpha_0-1} \varphi_1 \in L^2(\mathbb{R}^d) \).

Proof. By assumption (2.6) there exists a sequence of functions \( \psi_n \in \dot{H}^1(\mathbb{R}^d) \) satisfying
\[
(1 - n^{-1}) \| \nabla \psi_n \|^2 + \langle V \psi_n, \psi_n \rangle < 0 \quad \text{and} \quad \| \nabla \psi_n \| = 1. \tag{2.56}
\]
Repeating the same arguments as in Lemma 2.3 shows that a subsequence (also denoted by \( (\psi_n)_{n \in \mathbb{N}} \)) converges in \( L^2_{\text{loc}}(\mathbb{R}^d) \) to some function \( \varphi_1 \in \dot{H}^1(\mathbb{R}^d) \) with \( \varphi_1 \neq 0 \). Let us prove that \( \varphi_1 \) is a minimizer of the quadratic form of \( h_0 \) in \( \dot{H}^1(\mathbb{R}^d) \) by showing \( (V \varphi_1, \varphi_1) = -1 \). We fix the constant \( b > 0 \) and construct functions \( \chi_1, \chi_2 \) according to Lemma 2.2. Since \( \chi_1^2 + \chi_2^2 = 1 \) we have
\[
(V \varphi_1, \varphi_1) = (V \varphi_1, \varphi_1 \chi_1^2) + (V \varphi_1, \varphi_1 \chi_2^2). \tag{2.57}
\]
Note that
\[
(V \varphi_1, \varphi_1 \chi_1^2) = (V \varphi_1 - \psi_n, \varphi_1 \chi_1^2) + (V \psi_n, \varphi_1 \chi_1^2)
= (V \varphi_1 - \psi_n, \varphi_1 \chi_1^2) + (V \psi_n, \varphi_1 \chi_1^2) + (V \psi_n, (\varphi_1 - \psi_n) \chi_1^2). \tag{2.58}
\]
At first we estimate the first term on the r.h.s. of (2.58). By (2.16) and \( \| \nabla \varphi_1 \| \leq 1 \) we have
\[
(V \varphi_1 \chi_1, \varphi_1 \chi_2) \leq 2 C \| \nabla \varphi_1 \| \leq 2 \| \nabla \varphi_1 \|^2 + 2 \leq C. \tag{2.59}
\]
Hence, by (2.2) and (2.59) we conclude
\[
\| (V \varphi_1 - \psi_n, \varphi_1 \chi_1^2) \| \leq (\| (V \varphi_1 \chi_1, \varphi_1 \chi_2) \|)^{\frac{1}{2}} (\| (V \varphi_1 - \psi_n, \varphi_1 - \psi_n) \|)^{\frac{1}{2}}
\leq C \| \nabla ((V \varphi_1 - \psi_n, \varphi_1 - \psi_n) \|)^{\frac{1}{2}}. \tag{2.60}
\]
Since \( \chi_1 \) is compactly supported and due to \( \psi_n \to \varphi_1 \) in \( L^2_{\text{loc}}(\mathbb{R}^d) \), the second term on the r.h.s. of (2.60) tends to zero as \( n \to \infty \). Hence, by \( \| \nabla \varphi_1 \|, \| \nabla \psi_n \| \leq 1 \) and since \( \| \nabla \chi_1 \| \) is bounded, the r.h.s. of (2.60) can be done arbitrarily small by choosing \( \varepsilon > 0 \) sufficiently small and \( n \in \mathbb{N} \) sufficiently large. Applying similar arguments to the last term of (2.58) yields
\[
(V \varphi_1 \chi_1, \varphi_1 \chi_1) \leq (V \psi_n \chi_1, \psi_n \chi_1) + \varepsilon. \tag{2.61}
\]
Further, by (2.54) we have
\[
(V \varphi_1 \chi_2, \varphi_1 \chi_2) \leq C \| \nabla (\varphi_1 \chi_2) \| \leq 2 C \| (\nabla \varphi_1) \chi_2 \| + 2 C \| (\nabla \chi_2) \varphi_1 \|. \tag{2.62}
\]
Since \( \varphi_1 \in \dot{H}^1(\mathbb{R}^d) \) and \( \chi_2 \) is bounded and supported in the region \( \{ x \in \mathbb{R}^d : |x| \leq b \} \), the first term on the r.h.s. of (2.62) is arbitrary small if \( b \) is sufficiently large. Due to (2.16) it holds
\[
\| (\nabla \chi_2) \varphi_1 \| \leq \varepsilon \| \nabla \varphi_1 \| = \varepsilon, \tag{2.63}
\]
which shows that the second term of (2.62) can be done arbitrarily small by choosing \( \varepsilon > 0 \) sufficiently large. Hence, we obtain
\[
(V \varphi_1 \chi_2, \varphi_1 \chi_2) \leq \varepsilon. \tag{2.64}
\]
Collecting estimates (2.61) and (2.64) yields
\[
(V \varphi_1, \varphi_1) \leq (V \psi_n \chi_1, \psi_n \chi_1) + \varepsilon, \tag{2.65}
\]
where \( n \in \mathbb{N} \) is sufficiently large. Let us estimate the r.h.s. of (2.65). Assumption (2.54) implies
\[
(V \psi_n \chi_1, \psi_n \chi_1) = (V \psi_n, \psi_n) - (V \psi_n \chi_2, \psi_n \chi_2) \leq \| V \psi_n \| \psi_n + C \| (\nabla \psi_n \chi_2) \|. \tag{2.66}
\]
By the use of (2.7) and similar arguments as in the proof of Lemma 2.3, for any fixed \( \varepsilon > 0 \) we can choose \( \tilde{b} > 0 \) and \( n \in \mathbb{N} \) large enough, such that \( \| \nabla (\psi_n \chi_2) \| < \varepsilon \). Hence, by the use of (2.75)
\[
|V\psi_n, \psi_n| \leq (1 - n^{-1}) \| \nabla \psi_n \|^2 \leq n^{-1} - 1, \tag{2.67}
\]

and together with (2.65) we conclude \( \langle V\varphi_1, \varphi_1 \rangle = -1 \), i.e. it holds
\[
\| \nabla \varphi_1 \|^2 + \langle V\varphi_1, \varphi_1 \rangle = 0. \tag{2.68}
\]

Now we prove that \( (1 + |x|)^{a_{n-1}}\varphi_1 \in L^2(\mathbb{R}^d) \). Let \( G_\varepsilon \) be the function defined by (2.30). Since \( \varphi_1 \) is a minimizer of the quadratic form of \( h_0 \), it satisfies the Euler-Lagrange equation in a generalized sense, i.e. it holds
\[
\langle \nabla \varphi_1, \nabla \psi \rangle + \langle V\varphi_1, \psi \rangle = 0 \tag{2.69}
\]

for every function \( \psi \in \dot{H}^1(\mathbb{R}^d) \). By setting \( \psi = G_\varepsilon^2\varphi_1 \) we obtain
\[
\langle \nabla \varphi_1, \nabla (G_\varepsilon^2\varphi_1) \rangle + \langle V\varphi_1, G_\varepsilon^2\varphi_1 \rangle = 0. \tag{2.70}
\]

Similar computation to (2.34) yields
\[
\| \nabla (\varphi_1 G_\varepsilon) \|^2 - \int |\varphi_1|^2 |\nabla G_\varepsilon|^2 \, dx + \int V|\varphi_1 G_\varepsilon|^2 \, dx = 0. \tag{2.71}
\]

By the use of (2.37) we can rewrite (2.71) as
\[
\| \nabla (\varphi_1 G_\varepsilon) \|^2 + \langle V \varphi_1 G_\varepsilon, \varphi_1 G_\varepsilon \rangle - \alpha_0 \int \frac{|G_\varepsilon^2 \varphi_1|^2}{|x|^2} \, dx \leq \int \frac{|\varphi_1|^2 |\nabla G_\varepsilon|^2}{|x|^2} \, dx. \tag{2.72}
\]

Since the function \( |\nabla G_\varepsilon| \) is uniformly bounded in \( \varepsilon \) for \( |x| \in [R, 2R] \), we have
\[
\int_{\{R \leq |x| \leq 2R\}} |\varphi_1|^2 |\nabla G_\varepsilon|^2 \, dx \leq C \int_{\{R \leq |x| \leq 2R\}} |\varphi_1|^2 \, dx \leq 4CR^2 \int \frac{|\varphi_1|^2}{(2|x|)^2} \, dx \leq C_1 \int |\nabla \varphi_1|^2 \, dx \leq C_1, \tag{2.73}
\]

where the constant \( C_1 > 0 \) does not depend on \( \varepsilon > 0 \). Similar to the proof of Lemma 2.4, assumption (2.7) implies
\[
\| \nabla (\varphi_1 G_\varepsilon) \| \leq C.
\]

Taking \( \varepsilon \to 0 \) yields \( \| \nabla (|x|^{a_{n-1}} \varphi_1) \| < \infty \), which together with Hardy’s inequality implies
\[
(1 + |x|)^{a_{n-1}} \varphi_1 \in L^2(\mathbb{R}^d). \tag{2.74}
\]

This completes the proof. \( \square \)

**Lemma 2.10.** Assume that
\[
\| V\psi \|^2 \leq C (\| \nabla \psi \|^2 + \| \psi \|^2) \tag{2.75}
\]

holds for some \( C > 0 \) and every function \( \psi \in C_0^\infty(\mathbb{R}^d) \), then the solution \( \varphi_1 \in \dot{H}^1(\mathbb{R}^d) \) in Lemma 2.9 is unique. Moreover, there exists a constant \( \delta_1 > 0 \), such that for any function \( \psi \in \dot{H}^1(\mathbb{R}^d) \) with \( \langle \nabla \psi, \nabla \varphi_1 \rangle = 0 \) it holds
\[
\langle h_0 \psi, \psi \rangle \geq \delta_1 \| \nabla \psi \|^2. \tag{2.76}
\]

**Proof.** We will prove the Lemma by contradiction. Assume that there is no such constant \( \delta_1 > 0 \), then there exists a sequence of functions \( \psi_{n(1)} \) in \( \dot{H}^1(\mathbb{R}^d) \), such that
\[
\| \nabla \psi_{n(1)} \|^2 = 1, \quad \langle \nabla \psi_{n(1)}, \nabla \varphi_1 \rangle = 0, \quad (1 - n^{-1}) \| \nabla \psi_{n(1)} \|^2 + \langle V\psi_{n(1)}, \psi_{n(1)} \rangle < 0. \tag{2.77}
\]

Moreover, there exists a subsequence (which by abuse of notation is denoted by \( \psi_{n(1)} \) again) and a function \( \tilde{\varphi}_1 \in \dot{H}^1(\mathbb{R}^d) \), such that \( \psi_{n(1)} \rightharpoonup \tilde{\varphi}_1 \) in \( \dot{H}^1(\mathbb{R}^d) \) and therefore \( \psi_{n(1)} \rightharpoonup \tilde{\varphi}_1 \) in \( L^2_{\text{loc}}(\mathbb{R}^d) \). Obviously, \( \varphi_1 \) and \( \tilde{\varphi}_1 \) are linearly independent and \( \tilde{\varphi}_1 \) is a minimizer of the quadratic
form of $h_0$ as well. Since (2.69) holds for $\psi = \tilde{\psi}_1$, any linear combination of $\varphi_1$ and $\tilde{\varphi}_1$ is also a minimizer of the quadratic form of $h_0$. By Hardy’s inequality both functions $\varphi_1$ and $\tilde{\varphi}_1$ belong to the weighted $L^2$-space with weight $(1 + |x|)^{-2}$. Since the subspace of linear combinations of $\varphi_1$ and $\tilde{\varphi}_1$ is two-dimensional, it contains two orthogonal functions with respect to the weighted scalar product. At least one of these functions, say $f$, has a nontrivial positive part $f_+$ and a nontrivial negative part $f_-$, which are also minimizers of the quadratic form of the operator $h_0$ and satisfy the corresponding Schrödinger equation. Functions $f_+$ and $f_-$ are zero on some open sets. Since $V$ satisfies (2.75), the unique continuation Theorem yields $f_+ = f_- = 0$ (see Theorem 2.1 [22]). The obtained contradiction proves (2.76), which implies the uniqueness of $\varphi_1$ in particular. Statement (ii) of Theorem 2.1 is proved.

Statement (iii) follows easily from Lemma 2.4 and Lemma 2.5 by replacing the function $G_\varepsilon$ in (2.30) by the function

$$J_\varepsilon = \exp \left( \frac{\tilde{a}|x|^\alpha}{1 + \varepsilon |x|^\alpha} \right) \chi_R(|x|)$$

(2.78)

for $\alpha, \tilde{a}, R > 0$. This completes the proof of Theorem 2.1.

\square

3. RESONANCES AND EIGENFUNCTIONS ON SUBSPACES WITH FIXED SYMMETRIES

Let $h_0 = -\Delta + V$ be invariant under action of some symmetry group $G$ and let $\sigma$ be a type of irreducible representation of $G$. Denote by $P^\sigma$ the projection in $L^2(\mathbb{R}^d)$ onto the subspace of functions transformed according to the representation $\sigma$. In the following we assume that for every function $\psi \in L^2(\mathbb{R}^d)$ and $\chi \in C_0(\mathbb{R}^d)$ with $\chi(x) = \chi(|x|)$ the condition $P^\sigma \psi = \psi$ implies $P^\sigma \chi \psi = \chi \psi$. We denote $h_0^\sigma = P^\sigma h_0$, $h_0^\sigma = P^\sigma h_0$, $\mathcal{H}^\sigma = P^\sigma H^1(\mathbb{R}^d)$ and $\mathcal{H}^\sigma = P^\sigma H^1(\mathbb{R}^d)$.

**Theorem 3.1.** Suppose that $V$ satisfies (2.2). Further, assume that

$$h_0^\sigma \geq 0 \quad \text{and} \quad \inf \mathcal{S}(h_0^\sigma) < 0 \quad (3.1)$$

holds for any $\varepsilon \in (0, 1)$. If there exist constants $\alpha_0 > 0$, $b > 0$ and $\gamma_0 \in (0, 1)$, such that for any function $\psi \in \mathcal{H}^\sigma$ with supp $\psi \subset \{x \in \mathbb{R}^d : |x| \geq b\}$ we have

$$\langle h_0^\sigma \psi, \psi \rangle - \gamma_0 \|
abla \psi\|^2 - (\alpha_0^2 |x|^{-2} \psi, \psi) \geq 0, \quad (3.2)$$

then the following assertions hold:

(i) If $\alpha_0 > 1$, then zero is an eigenvalue of $h_0^\sigma$ with finite degeneracy. Let $\mathcal{W}_0$ be the corresponding eigenspace, then for any $\varphi_0 \in \mathcal{W}_0$ we have $(1 + |x|)^{\alpha_0-1} \varphi_0 \in L^2(\mathbb{R}^d)$. Moreover, there exists a constant $\delta_0 > 0$, such that for any function $\psi \in \mathcal{H}^\sigma$ with $\langle \nabla \psi, \nabla \varphi_0 \rangle = 0$ for all $\varphi_0 \in \mathcal{W}_0$ it holds

$$\langle h_0^\sigma \psi, \psi \rangle \geq \delta_0 \|
abla \psi\|^2. \quad (3.3)$$

(ii) If $\alpha_0 \in (0, 1)$ and in addition

$$\langle |V| \psi, \psi \rangle \leq C \|
abla \psi\|^2 \quad (3.4)$$

holds for any function $\psi \in \mathcal{H}^\sigma$ and some constant $C > 0$, then there exists a finite-dimensional subspace $\mathcal{W}_1 \subset \mathcal{H}^\sigma$, such that for any function $\varphi_1 \in \mathcal{W}_1$ it holds

$$\|
abla \varphi_1\|^2 + \langle V \varphi_1, \varphi_1 \rangle = 0. \quad (3.5)$$

Moreover, any $\varphi_1 \in \mathcal{W}_1$ satisfies $(1 + |x|)^{\alpha_0-1} \varphi_1 \in L^2(\mathbb{R}^d)$ and there exists a constant $\delta_1 > 0$, such that for any function $\psi \in \mathcal{H}^\sigma$ with $\langle \nabla \psi, \nabla \varphi_1 \rangle = 0$ for all $\varphi_1 \in \mathcal{W}_1$ it holds

$$\langle h_0^\sigma \psi, \psi \rangle \geq \delta_1 \|
abla \psi\|^2. \quad (3.6)$$
(iii) If instead of (3.2) a stronger inequality
\begin{equation}
\langle h_0^* \psi, \psi \rangle - \gamma_0 \| \nabla \psi \|^2 - (\alpha_0^* |x|^{-\beta} \psi, \psi) \geq 0
\end{equation}
holds for some constant \( \alpha_0 > 0 \) and \( \beta \in (0, 2) \), then each function \( \varphi_0 \in \mathcal{W}_0 \) in part (i) of the theorem satisfies
\begin{equation}
\exp \left( \alpha_0 \kappa^{-1} |x|^{-\kappa} \right) \varphi_0 \in L^2(\mathbb{R}^d), \quad \text{where} \quad \kappa = 1 - 2^{-1} \beta.
\end{equation}

Proof. The proof of Theorem 3.1 is a straightforward generalization of the proof of Theorem 2.1. The main difference between these two theorems is that in Theorem 2.1 we have non-degenerate minimizers \( \varphi_0 \) and \( \varphi_1 \) of the quadratic form of the operator \( h_0 \) in the spaces \( H^1(\mathbb{R}^d) \) and \( \dot{H}^1(\mathbb{R}^d) \), respectively. In Theorem 3.1 the corresponding subspaces \( \mathcal{W}_0 \) and \( \mathcal{W}_1 \) are not necessarily one-dimensional. However, due to Lemma A.1 they are always finite-dimensional. \( \Box \)

Remark. Theorem 2.1 and Theorem 3.1 require \( d \geq 3 \). We used this condition twice. At first, we used Hardy’s inequality to compensate the localization error \( \varepsilon |x|^{-2} \) with a part of the kinetic energy in Lemma 2.2. Secondly, we used the Rellich–Kondrachov theorem in the proof of Theorem 2.1 to obtain convergence of the constructed subsequence in \( L^2_{\text{loc}}(\mathbb{R}^d) \). If the dimension is one or two, but the operator \( h_0 \) is considered on a subspace with a fixed symmetry \( \sigma \), such that Hardy’s inequality
\begin{equation}
\| \nabla \psi \|^2 \geq C \| \psi |x|^{-1} \|^2
\end{equation}
holds for some \( C > 0 \), the statement of Theorem 3.1 remains true.

4. Applications

4.1. Virtual levels of one-body Schrödinger operators. The main goal of our paper is to study decay properties of virtual levels of multiparticle Schrödinger operators. However, in order to show how effective Theorem 2.1 is we start with the easiest case of one-particle Schrödinger operators. Some of the results below are already known.

Let
\begin{equation}
h_0 = -\Delta + V \quad \text{and} \quad h_\varepsilon = h_0 + \varepsilon \Delta
\end{equation}
in \( L^2(\mathbb{R}^d) \), where \( d \geq 3 \) and \( \varepsilon > 0 \). We assume that \( V = V_1 + V_2 \), such that \( V_1 \in L^2(\mathbb{R}^3) \) for \( d = 3 \), \( V_1 \in L^2(\mathbb{R}^4) \cap L^{2+\gamma}(\mathbb{R}^4) \) for \( d = 4 \) and some \( \gamma > 0 \), and \( V_1 \in L^d(\mathbb{R}^d) \) for \( d \geq 5 \). Further, let \( V_2 \geq 0 \) be bounded and \( V_2(x) \to 0 \) as \( |x| \to 0 \). According to Theorem X.19 and Theorem X.20 [21], these assumptions imply that \( V \) is relatively bounded with bound zero, i.e. it holds (2.2). We assume that \( h_0 \geq 0 \). It is easy to see that for \( \varepsilon \in (0, 1) \) we have \( \mathcal{S}_{\text{ess}}(h_\varepsilon) = [0, \infty) \) and only discrete eigenvalues of \( h_\varepsilon \) can appear below zero.

Definition 4.1. If \( h_0 \geq 0 \) and for any \( \varepsilon \in (0, 1) \) \( \inf \mathcal{S}(h_\varepsilon) < 0 \), we say that the operator \( h_0 \) has a virtual level at zero.

Theorem 4.2. Consider \( h_0 = -\Delta + V \), where \( V \) satisfies the assumptions mentioned at the beginning of this section. Assume that \( h_0 \) has a virtual level. Then there exists a function \( \varphi_0 \in H^1(\mathbb{R}^d) \), such that
\begin{equation}
\| \nabla \varphi_0 \|^2 + \langle V \varphi_0, \varphi_0 \rangle = 0.
\end{equation}

Definition 4.3. If the function \( \varphi_0 \) in Theorem 4.2 is not an eigenfunction of \( h_0 \), then it is called a zero resonance.

Theorem 4.4. Let \( \varphi_0 \in H^1(\mathbb{R}^d) \) be the function from Theorem 4.2, then
(i) \( \varphi_0 \) satisfies
\begin{equation}
(1 + |x|)^{\alpha} \varphi_0 \in L^2(\mathbb{R}^d)
\end{equation}
for every \( \alpha < 2^{-1}(4 - d) \). In particular, for \( d \geq 5 \) zero is an eigenvalue of \( h_0 \).
(ii) If $V_2(x) \geq \alpha_1|x|^{-2}$ holds for some constant $\alpha_1 > 0$, then (4.3) holds for any $\alpha < \sqrt{\alpha_1 + 4^{-1}(d - 2)^2} - 1$. In particular, if $d = 3$ and $\alpha_1 > \frac{4}{3}$, then zero is an eigenvalue of $h_0$. If $d = 4$, then zero is an eigenvalue of $h_0$ for any $\alpha > 0$.

(iii) If $V_2(x) \geq \alpha_2|x|^{-\beta}$ holds for some constants $\alpha_2 > 0$ and $\beta \in (0, 1)$, then the eigenfunction $\varphi_0$ satisfies $\exp(\alpha|x|^{\kappa})\varphi_0 \in L^2(\mathbb{R}^d)$ for any $\alpha < \alpha_2\kappa^{-1}$, where $\kappa = 1 - \frac{d}{2}$.

**Remark.** One can also obtain decay rates of resonances and eigenfunctions by the use of Green’s function, see for example [12]. However, our method requires very mild conditions on the singularities of the potential and allows its positive part to decay very slowly.

**Proof of Theorem 4.2 and 4.4.** We only need to prove that
\[(1 - \gamma_0)\|\nabla \psi\|^2 + \langle V\psi, \psi \rangle - \alpha_0^2\|x^{-1}\psi\|^2 \geq 0 \tag{4.4}\]
holds for every function $\psi$ with $\text{supp} (\psi) \subset \{ x \in \mathbb{R}^d : |x| \geq b \}$. Note that
\[
\langle V\psi, \psi \rangle \geq -\langle |V|\frac{\psi}{|x|} \rangle \geq \left( \int_{\{|x|>b\}} |V|\frac{\psi}{|x|} \, dx \right)^\frac{1}{2} \left( \int \frac{|\psi|^2}{|x|^2} \, dx \right)^\frac{1}{2} \geq -\varepsilon \|\nabla \psi\|^2 \tag{4.5}
\]
holds due to the Sobolev inequality for $b > 0$ sufficiently large. Hence, (4.4) follows for sufficiently small $\gamma_0 > 0$ with $\alpha_0 > 0$, such that $\alpha_0^2$ is smaller than the Hardy constant $\frac{(d-2)^2}{4}$.

### 4.2 Virtual levels of N-body Schrödinger operators.

Now we consider a system of $N \geq 3$ quantum particles in dimension $n \geq 3$ with masses $m_i > 0$, $i = 1, \ldots, N$, and position vectors $x_i \in \mathbb{R}^n$, $i = 1, \ldots, N$. Such a system, denoted by $Z_1$, is described by the Hamiltonian $H_N$, acting on $L^2(\mathbb{R}^{nN})$, which is given by
\[
H_N = -\sum_{i=1}^N \frac{1}{m_i} \Delta x_i + \frac{1}{2} \sum_{i,j=1,i\neq j}^N V_{ij}(x_{ij}), \ x_{ij} = x_i - x_j, \tag{4.6}
\]
where the potentials $V_{ij}$ describe the particle pair interaction. In the following we assume that $V_{ij} = V_{ij}^{(1)} + V_{ij}^{(2)}$, such that
\[
V_{ij}^{(1)} \in L^\infty_{\text{loc}}(\mathbb{R}^n) \quad \text{and} \quad |V_{ij}^{(1)}(x)| \leq C|x|^{-2-\nu}, \text{ if } |x| \geq A \tag{4.7}
\]
holds for some constants $A > 0$ and $\nu > 0$. Further, we assume that
\[
V_{ij}^{(2)} \geq 0 \quad \text{is bounded and } V_{ij}^{(2)}(x) \to 0 \quad \text{as } |x| \to \infty. \tag{4.8}
\]
We will consider the operator $H_N$ in the center-of-mass frame. Following [23] we introduce the scalar product $\langle \cdot, \cdot \rangle_1$ on $\mathbb{R}^{nN}$ by
\[
\langle x, \tilde{x} \rangle_1 = \sum_{i=1}^N m_i (x_i, \tilde{x}_i), \quad |x|^2_1 = \langle x, x \rangle_1, \quad x, \tilde{x} \in \mathbb{R}^{nN}. \tag{4.9}
\]
Here we denote by $\langle \cdot, \cdot \rangle$ the standard scalar product on $\mathbb{R}^n$. Denote by
\[
R_0 = \left\{ x \in \mathbb{R}^{nN} : \sum_{i=1}^N m_i x_i = 0 \right\} \tag{4.10}
\]
the space of relative motion of the system. We define the Laplacian $\Delta_0$ acting in $L^2(R_0)$ and $V = \frac{1}{2} \sum_{i,j=1, i \neq j}^N V_{ij}(x_{ij})$. The operator $H_0$ is given by
\[
H_0 = -\Delta_0 + V. \tag{4.11}
\]
For an arbitrary subsystem \( C \subseteq Z_1 \) let
\[
R_0[C] = \left\{ x \in \mathbb{R}^{nN} : \sum_{i \in C} m_i x_i = 0, \ x_j = 0, \ j \notin C \right\}
\]  
be the space of the relative motion of the subsystem \( C \). Denote by \( \Delta_0[C] \) the Laplacian on \( R_0[C] \) and let \( V[C] = \frac{1}{2} \sum_{i,j \in C, i \neq j} V_{ij} \). The Hamiltonian of the subsystem \( C \) is given by
\[
H_0[C] = -\Delta_0[C] + V[C].
\]  
(4.13)

In the following all Hamiltonians are considered in the sense of quadratic forms. We say that \( Z_p = (C_1, \ldots, C_p) \) is a breaking of the system \( Z_1 \) (of order \( |Z_p| = p \)), if
\[
\emptyset \neq C_i \subseteq Z_1, \quad C_i \cap C_j = \emptyset, \quad \bigcup_{j=1}^p C_j = Z_1
\]  
holds for all \( 1 \leq i \neq j \leq p \). Let
\[
R_0(Z_p) = \bigoplus_{C_k \in Z_p} R_0[C_k], \quad R_c(Z_p) = R_0 \ominus R_0(Z_p).
\]  
(4.15)
The Hamiltonian of the breaking \( Z_p \) is given by
\[
H_0(Z_p) = \sum_{C_k \in Z_p} H_0[C_k].
\]  
(4.16)

We define the operator \( I(Z_p) \) of inter-cluster interactions by
\[
I(Z_p) = V - \sum_{C_k \in Z_p} V[C_k].
\]  
(4.17)

Further, we introduce the projections \( P_0(Z_p) \) and \( P_c(Z_p) \) in \( R_0 \) on \( R_0(Z_p) \) and \( R_c(Z_p) \), respectively. For \( x \in R_0 \) let
\[
q(Z_p) = P_0(Z_p)x, \quad \xi(Z_p) = P_c(Z_p)x
\]  
(4.18)

the corresponding invariant coordinates. For \( \kappa, R > 0 \) we define the regions
\[
S(R) = \{ x \in R_0 : |x|_1 \leq R \},
\]
\[
K(Z_p, \kappa) = \{ x \in R_0 : |q(Z_p)|_1 \leq \kappa |\xi(Z_p)|_1 \}
\]  
(4.19)

**Definition 4.5.** For an arbitrary subsystem \( C \subseteq Z_1 \) we say that the corresponding operator \( H_0[C] = -\Delta_0[C] + V[C] \) has a virtual level at zero, if \( H_0[C] \geq 0 \) and for all sufficiently small \( \varepsilon > 0 \) it holds
\[
S_{\text{ess}}(-(1-\varepsilon)\Delta_0[C] + V[C]) = [0, \infty), \quad S_{\text{disc}}(-(1-\varepsilon)\Delta_0[C] + V[C]) \neq \emptyset.
\]  
(4.20)

The main result of this section is the following

**Theorem 4.6.** Let \( Z_1 \) be a system of \( N \geq 3 \) particles in dimension \( n \geq 3 \). Suppose that the potentials \( V_{ij} \) satisfy the assumptions (4.7) and (4.8). Assume that \( H_0 \) has a virtual level at zero and for each subsystem \( C \subseteq Z_1 \) it holds
\[
S((1-\varepsilon)\Delta_0[C] + V[C]) = [0, \infty)
\]  
(4.21)

for sufficiently small \( \varepsilon > 0 \). Then

(i) zero is an eigenvalue of \( H_0 \) and the corresponding eigenfunction \( \varphi_0 \) satisfies
\[
(1 + |x|_1)^{\alpha_0-1}\varphi_0 \in L^2(R_0), \quad \alpha_0 < \frac{n(N-1)-2}{2}.
\]  
(4.22)
(ii) There exists a constant $\delta_0 > 0$, such that for every function $\psi \in H^1(R_0)$ satisfying 
\[ (\nabla_0 \psi, \nabla_0 \varphi_0) = 0 \] it holds 
\[ (1 - \delta_0)\|\nabla_0 \psi\|^2 + \langle V \psi, \psi \rangle \geq 0. \] 
(4.23) If for $V^{(2)}_0$ it holds $V^{(2)}_{ij}(x) \geq \alpha_{ij}|x|^{-\beta}$ for some constants $\alpha_{ij} > 0$ and $\beta \in (0, 2)$, then
zero is an eigenvalue of $H_0$ and the corresponding eigenfunction $\varphi_0$ satisfies
\[ e^{\kappa|x|^2} \varphi_0 \in L^2(R_0), \] 
where $\kappa = 1 - \frac{\beta}{2}$ and $\mu > 0$ depends only on the coefficients $\alpha_{ij}$ and on the masses of the particles.
Remark. (i) Theorem 4.6 tells us that for $n$-dimensional particles with $n \geq 3$ only two-particle systems virtual levels may be resonances. This is the reason why the Efimov effect does not occur for $n \geq 3$ and $N \geq 3$.
(ii) Part (iii) of Theorem 4.6 shows that if the interactions of particles for large distances are long-range and positive, an Agmon-type method can be used to prove the sub-exponential decay of eigenfunctions at the bottom of the essential spectrum. This idea was privately communicated to one of the authors by Dirk Hundertmark, who used it in a different context.

Before proving the theorem, we will generalize it in two directions: We will give an analogue of this theorem for systems including particles with infinite masses (Theorem 4.7) and we will consider systems with symmetry restrictions (Theorem 4.9).

4.2.1. Systems of particles with infinite masses. Let $Z_0 = \{0, 1, \ldots, N - 1\}$ be an $N$-particle system with particle 0 having infinite mass. We assume that this particle is located at the origin and define the $n(N - 1)$-dimensional space of relative motion of particles $\{1, \ldots, N - 1\}$ as
\[ R_0 = \{ x = (x_0, \ldots, x_{N-1}) \in \mathbb{R}^{nN} : x_0 = 0 \}, \quad n \geq 3. \] 
(4.25)
On $R_0$ we introduce the scalar product
\[ \langle x, \bar{x} \rangle_1 = \sum_{i=1}^{N-1} m_i \langle x_i, \bar{x}_i \rangle, \quad |x|^2 = \langle x, x \rangle_1, \quad x, \bar{x} \in R_0. \] 
(4.26)
Let $C \subseteq Z_0$ be an arbitrary subsystem of $Z_0$. If $0 \notin C$ we set
\[ R_0[C] = \left\{ x \in \mathbb{R}^{nN} : \sum_{i \in C} m_i x_i = 0, \quad x_j = 0, \quad j \notin C \right\} \] 
(4.27)
and if $0 \in C$ we define
\[ R_0[C] = \{ x \in \mathbb{R}^{nN} : x_0 = 0, \quad x_j = 0, \quad j \notin C \}. \] 
(4.28)
In abuse of notation we use $R_0$ and $R_0[C]$ as in the case of particles with finite masses. Let $\Delta_0[C]$ be the Laplace operator on $L_2(R_0[C])$ and the Hamiltonian $H_0[C]$ of the subsystem $C$ is given by
\[ H_0[C] = -\Delta_0 + V[C], \quad V[C] = \frac{1}{2} \sum_{i,j \in C, \ i \neq j} V_{ij}. \] 
(4.29)
For a breaking $Z_p$ of $Z_0$ into $p$ clusters let
\[ R_0(Z_p) = \bigoplus_{C_k \in Z_p} R_0[C_k], \quad R_e(Z_p) = R_0 \ominus R_0(Z_p). \] 
(4.30)
The Hamiltonian of the breaking $Z_p$ is given by

$$H_0(Z_p) = \sum_{C_k \in Z_p} H_0[C_k].$$

\hspace{1cm} (4.31)

**Theorem 4.7.** Let $N \geq 3$ and $Z_0 = \{0, 1, \ldots, N - 1\}$ be a system of $N$ particles, where particle 0 has infinite mass. Then assertions (i)-(iii) of Theorem 4.6 hold for $Z_1$ replaced by $Z_0$.

4.2.2. Systems with permutational symmetry. Assume now that $Z_1$ is a system of several identical particles, where every particle has a finite mass. Let $S$ be the group of permutaions of identical particles in $Z_1$ and $\sigma$ be a type of irreducible representation of this group. Let $P^\sigma$ be the corresponding projection on the subspace of functions transformed according to the representation $\sigma$. For any fixed breaking $Z_p = (C_1, \ldots, C_p)$, $2 \leq p \leq N - 1$, we define $S(Z_p)$ as a group, which permutes identical particles within the subsystem $C_k \subset Z_p$ and permutes identical subsystems if such subsystems exist in $Z_p$. Obviously $S(Z_p)$ is a subgroup of $S$. Denote by $\sigma'(Z_p)$ types of irreducible representations of $S(Z_p)$. We say that the representation $\sigma'(Z_p)$ of the group $S(Z_p)$ is induced by the representation $\sigma$ of the group $S$ and write $\sigma'(Z_p) \prec \sigma$, if $\sigma'(Z_p)$ is contained in $\sigma$ restricted to $S(Z_p)$.

**Definition 4.8.** We say that $H_0^\sigma := P^\sigma H_0$ has a virtual level of symmetry $\sigma$, if $H_0^\sigma \geq 0$ and for all sufficiently small $\varepsilon > 0$ it holds

$$\mathcal{S}_{\text{ess}}(P^\sigma(H_0 + \varepsilon \Delta_0)) = [0, \infty), \quad \mathcal{S}_{\text{disc}}(P^\sigma(H_0 + \varepsilon \Delta_0)) \neq \emptyset$$

\hspace{1cm} (4.32)

**Theorem 4.9.** Suppose that $N \geq 3$ and consider the operator $H_0^\sigma$, where the potentials $V_{ij}$ satisfy (4.7) and (4.8). Assume that for any breaking $Z_p$ and any type of irreducible representation $\sigma'(Z_p) \prec \sigma$ it holds

$$P^\sigma(Z_p) (H_0(Z_p) + \varepsilon \Delta_0(Z_p)) \geq 0$$

\hspace{1cm} (4.33)

for any sufficiently small $\varepsilon > 0$. Further, assume that $H_0^\sigma$ has a virtual level of symmetry $\sigma$. Then

(i) zero is an eigenvalue of $H_0^\sigma$ with finite degeneracy. Let $W_0$ be the corresponding eigenspace, then for any $\varphi_0 \in W_0$ we have

$$(1 + |x|)^{\alpha_0 - 1} \varphi_0 \in L^2(R_0), \quad \alpha_0 < \frac{n(N - 1) - 2}{2}.$$ 

\hspace{1cm} (4.34)

(ii) There exists a constant $\delta_0 > 0$, such that for any function $\psi \in P^\sigma H^1(R_0)$ with $\langle \nabla_0 \psi, \nabla_0 \varphi_0 \rangle = 0$ for all $\varphi_0 \in W_0$ it holds

$$(1 - \delta_0)\|\nabla_0 \psi\|^2 + \langle V \psi, \psi \rangle \geq 0.$$ 

\hspace{1cm} (4.35)

(iii) If for $V_{ij}^{(2)}$ it holds $V_{ij}^{(2)}(x) \geq \alpha_{ij} |x|^{-\beta}$ for some constants $\alpha_{ij} > 0$ and $\beta \in (0, 2)$, then every function $\varphi_0 \in W_0$ satisfies

$$e^{\mu |x|^{\kappa}} \varphi_0 \in L^2(R_0),$$

\hspace{1cm} (4.36)

where $\kappa = 1 - \frac{\beta}{2}$ and $\mu > 0$ depends only on the coefficients $\alpha_{ij}$ and on the masses of the particles.

**Remark.** The rate of decay of the eigenfunctions $\varphi_0 \in W_0$ depends on the corresponding Hardy constant $c_\mu$, which on the whole space $L^2(R_0)$ is given by $c_\mu = \left(\frac{\dim R_0 - 2}{4}\right)^{\frac{\beta}{2}}$. However, if $\sigma$ is different from the representation symmetric with respect to permutations of each pair of particles the Hardy constant can become larger. This can result in a stronger rate of decay of the eigenfunctions.
Proof of Theorem 4.6

To explain the main ideas of the proof we start with $N = 3$ and extend the strategy to the case $N \geq 4$ afterwards. We will use the following two Lemmas proved earlier in [34] and [30], respectively.

Lemma 4.10. [34, Lemma 2.1] Suppose that $Z_2 = (C_1, C_2)$ is an arbitrary breaking of the system $Z_1$ into two clusters and $K(Z_2, \kappa)$ are the regions defined in (4.19). Then there exists $\kappa_0 > 0$, such that for all $0 < \kappa < \kappa_0$ we have

$$K(Z_2, \kappa) \cap K(Z_2, \kappa) \subset S(R). \quad (4.37)$$

The following estimate for the localization error, originally proved in [30], plays a crucial role in the proof of Theorems 4.6, 4.7 and 4.9. For the convenience of the reader a complete proof of this estimate is given in the Appendix.

Lemma 4.11. [30, Lemma 5.1] Given $\varepsilon > 0$ and $\kappa > 0$, for each breaking $Z_p$ one can find $0 < \kappa' < \kappa$ and functions $u_{Z_p}, v_{Z_p} : R_0 \to \mathbb{R}$, such that

$$u_{Z_p}^2 + v_{Z_p}^2 = 1, \quad u_{Z_p}(x) = \begin{cases} 1, & x \in K(Z_p, \kappa') \\ 0, & x \notin K(Z_p, \kappa) \end{cases} \quad (4.38)$$

and

$$|\nabla_0 u_{Z_p}|^2 + |\nabla_0 v_{Z_p}|^2 < \varepsilon \left[ |v_{Z_p}|^2|x|^{-2} + |u_{Z_p}|^2|q(Z_p)| |x|^{-2} \right] \quad (4.39)$$

for $x \in K(Z_p, \kappa) \cap K(Z_p, \kappa')$.

Proof of Theorem 4.6 for $N = 3$ particles and $n = 3$. Note that in this case the constant $\alpha_0$ in (4.22) should be strictly less than two. We will prove that all conditions of statement (i) of Theorem 2.1 are fulfilled. We will also show that if in addition $V^{(2)}_ij(x) \geq \alpha_{ij}|x|^{-\beta}$ holds for some constants $\alpha_{ij} > 0$ and $\beta \in (0, 2)$, then (4.36) follows from statement (iii) of Theorem 2.1. Since $V_{ij} \in L^2_{\inf}(\mathbb{R}^3)$ and it decays at infinity, according to [21] for any $\varepsilon > 0$ there exists a constant $C(\varepsilon) > 0$, such that

$$\langle V_{ij}|\varphi, \varphi \rangle \leq \varepsilon \|\nabla_{ij}\varphi\|^2 + C(\varepsilon)\|\varphi\|^2 \quad (4.40)$$

holds for any function $\varphi \in C_0^\infty(R_0)$, which obviously implies (2.2) for $V = \frac{1}{2} \sum_{i,j,i\neq j} V_{ij}$. It remains to prove that

$$L[\psi] := (1 - \gamma_0) \|\nabla_0 \psi\|^2 + \langle V\psi, \psi \rangle - \|\alpha_0|x|^{-1}\psi\|^2 \geq 0 \quad (4.41)$$

holds for some $\gamma_0 > 0$, $\alpha_0 \in (1, 2)$ and any function $\psi \in H^1(R_0)$ with supp($\psi$) $\subset \{x \in R_0 : |x| \geq R\}$ for some sufficiently large $R > 0$.

The proof of (4.41) follows the ideas of the estimate from below of the quadratic form of the multi-particle Schrödinger operator in [31] in the easiest case when the subsystems do not have bound states or virtual levels. The difference between (4.41) and a similar inequality proved in [31] is that for the purposes of [31] it was sufficient to prove this inequality with an arbitrary small $\alpha > 0$. Now we need to prove (4.41) with $\alpha \in (1, 2)$. Following [29] we will make a partition of the unity of the configuration space of the system, separating regions $K(Z_2, \kappa)$, corresponding to different breakings of the system into two clusters. We will choose $\kappa$ very small to be able to compensate the term $-\alpha|x|^{-1}$ with a small part of the kinetic energy.

Let $u_{Z_2}$ be the localization functions defined by (4.38). Recall that $u_{Z_2}$ is supported in the cone in the configuration space, where two particles belonging to the same cluster in $Z_2$ are close one to another and the third particle is very from this cluster. Applying Lemma 4.10 and Lemma 4.11 yields

$$L[\varphi] \geq \sum_{Z_2} L_1[\varphi u_{Z_2}] + L_2[\varphi V], \quad (4.42)$$
where $\mathcal{V} = \sqrt{1 - \sum_{j=1}^{2} u_{j}^{2}}$ and the functionals $L_{1}, L_{2} : H^{1}(R_{0}) \to \mathbb{R}$ are defined by

\begin{equation}
L_{1}[\psi] := (1 - \gamma_{0}) \| \nabla \psi \|^{2} + (V \psi, \psi) - \| \alpha_{0} |x|^{-1} \psi \|^{2} - \varepsilon \| \eta_{0} |x|^{-1} \psi \|^{2},
\end{equation}

\begin{equation}
L_{2}[\psi] := (1 - \gamma_{0}) \| \nabla \psi \|^{2} + (V \psi, \psi) - \| \alpha_{0} |x|^{-1} \psi \|^{2} - \varepsilon \| |x|^{-1} \psi \|^{2}.
\end{equation}

We will prove that $L_{1}[\varphi u_{Z_{2}}] \geq 0$ and $L_{2}[\varphi \psi] \geq 0$, if $\varepsilon, \gamma_{0} > 0$ and $\kappa > 0$ are sufficiently small and $R > 0$ is sufficiently large. Here, $\kappa$ is the parameter in the definition of the cone $K(Z_{2}, \kappa)$.

At first we estimate $L_{1}[\varphi u_{Z_{2}}]$ for an arbitrary breaking $Z_{2} = (C_{1}, C_{2})$. Note that

\begin{align*}
L_{1}[\varphi u_{Z_{2}}] &= \langle H_{0}(Z_{2}) \varphi u_{Z_{2}}, \varphi u_{Z_{2}} \rangle - \gamma_{0} \| \nabla_{q}(Z_{2}) (\varphi u_{Z_{2}}) \|^{2} \\
&\quad + (1 - \gamma_{0}) \| \nabla \xi(Z_{2}) (\varphi u_{Z_{2}}) \|^{2} + \langle I(Z_{2}) \varphi u_{Z_{2}}, \varphi u_{Z_{2}} \rangle \\
&\quad - \| \alpha_{0} |x|^{-1} \varphi u_{Z_{2}} \|^{2} - \varepsilon \| \eta_{0} |x|^{-1} \varphi u_{Z_{2}} \|^{2}.
\end{align*}

Without loss of generality we assume that in $Z_{2} = (C_{1}, C_{2})$ the cluster $C_{1}$ has two particles and $C_{2}$ has only one particle. Since the operators $H_{0}[C_{1}]$ do not have virtual levels and $H_{0}(Z_{2}) = H_{0}(C_{1})$, it holds

\begin{equation}
\langle H_{0}(Z_{2}) \varphi u_{Z_{2}}, \varphi u_{Z_{2}} \rangle \geq \mu_{0} \| \nabla_{q}(Z_{2}) (\varphi u_{Z_{2}}) \|^{2}
\end{equation}

together with Hardy’s inequality implies

\begin{equation}
\langle H_{0}(Z_{2}) \varphi u_{Z_{2}}, \varphi u_{Z_{2}} \rangle - \gamma_{0} \| \nabla_{q}(Z_{2}) (\varphi u_{Z_{2}}) \|^{2} - \varepsilon \| \eta_{0} |x|^{-1} \varphi u_{Z_{2}} \|^{2} \geq \frac{\mu_{0}}{2} \| \nabla_{q}(Z_{2}) (\varphi u_{Z_{2}}) \|^{2}.
\end{equation}

Therefore, we arrive at

\begin{equation}
L_{1}[\varphi u_{Z_{2}}] \geq \frac{\mu_{0}}{2} \| \nabla_{q}(Z_{2}) (\varphi u_{Z_{2}}) \|^{2} + (1 - \gamma_{0}) \| \nabla \xi(Z_{2}) (\varphi u_{Z_{2}}) \|^{2} \\
\quad + \langle I(Z_{2}) \varphi u_{Z_{2}}, \varphi u_{Z_{2}} \rangle - \| \alpha_{0} |x|^{-1} \varphi u_{Z_{2}} \|^{2}.
\end{equation}

On the support of $u_{Z_{2}}$ we have $|q(Z_{2})|_{1} \leq \kappa |\xi(Z_{2})|_{1}$, which by the Poincaré-Friedrichs’s inequality (Theorem 6.30, [1]) implies

\begin{equation}
\frac{\mu_{0}}{2} \| \nabla_{q}(Z_{2}) (u_{Z_{2}} \varphi) \|^{2} \geq \frac{\mu_{0}}{8\kappa^{2}} \| \xi(Z_{2}) |^{-1} | \varphi u_{Z_{2}} \|^{2}.
\end{equation}

Since supp $(\varphi u_{Z_{2}}) \subset K(Z_{2}, \kappa) \setminus S(R)$ it holds $|x_{ij}| \geq C |\xi(Z_{2})|_{1}$ for $i \in C_{1}, j \in C_{2}$ and some $C > 0$. Therefore, by $V_{ij} \geq V_{ij}^{(1)}$ and $|V_{ij}(x_{ij})| \leq C |\xi(Z_{2})|_{1}^{-2^{\nu}}$, for sufficiently small $\kappa > 0$ we have

\begin{equation}
\frac{\mu_{0}}{2} \| \nabla_{q}(Z_{2}) (u_{Z_{2}} \varphi) \|^{2} + \langle I(Z_{2}) \varphi u_{Z_{2}}, \varphi u_{Z_{2}} \rangle - \| \alpha_{0} |x|^{-1} \varphi u_{Z_{2}} \|^{2} \geq 0.
\end{equation}

Combining (4.46) and (4.50) yields $L_{1}[\varphi u_{Z_{2}}] \geq 0$.

To prove part (i) and part (ii) of the Theorem in the case of $N = 3$ it suffices to show $L_{2}[\varphi \psi] \geq 0$. Note that on the support of $\mathcal{V}$ all the distances between the particles are large. Since $V_{ij} \geq V_{ij}^{(1)}$ and on the support of $\mathcal{V} \varphi$ we have

\begin{equation}
|V_{ij}^{(1)}(x_{ij})| \leq C |x_{ij}|^{-2^{\nu}} \leq \varepsilon |x_{ij}|^{-2}, \quad i, j = 1, 2, 3, i \neq j,
\end{equation}

where $\varepsilon > 0$ can be chosen arbitrarily small by choosing $R > 0$ sufficiently large, we conclude

\begin{equation}
L_{2}[\mathcal{V} \varphi] \geq (1 - \gamma_{0}) \| \nabla_{0} \mathcal{V} \varphi \|^{2} - (\alpha_{0}^{2} - 2\varepsilon) \| \mathcal{V} \varphi \|^{2}.
\end{equation}

Note that dim $R_{0} = 6$, which together with Hardy’s inequality implies

\begin{equation}
\| \nabla_{0} \mathcal{V} \varphi \|^{2} \geq 4 \| x_{1}^{-1} \mathcal{V} \varphi \|^{2}.
\end{equation}

Since $\alpha_{0} < 2$ we can choose $0 < \varepsilon < \frac{4 - \alpha_{0}^{2}}{4}$ and $\gamma_{0} > 0$ sufficiently small, such that $L_{2}[\mathcal{V} \varphi] \geq 0$, which completes the proof of statement (i) and (ii) for $d = 3$ and $N = 3$. To prove statement
(iii) it suffices to note that for $\beta \in (0, 2)$ and $\alpha_{ij} > 0$ we have $\sum_{i,j} V_{ij}^{(2)}(x_{ij}) \geq C|x_1|^{-\beta}$. Applying statement (iii) of Theorem 2.1 completes the proof for $N = 3$. \hfill $\square$

To prove Theorem 4.6 for $n = 3$ and $N \geq 4$ we use the following Lemma, which is similar to (Lemma 3.5, [31]).

**Lemma 4.12.** Let $3 \leq m \leq N-1$ and $\kappa'(2), \ldots, \kappa'(m-1) > 0$ and $R > 0$. Further, let $Z_m$, $Z'_m$ be breakings of $Z_1$ with $|Z_m| = |Z'_m| = m$ and $Z_m \neq Z'_m$. Then we can find $\kappa(m) > 0$, such that

$$K(Z_m, \kappa(m)) \cap K(Z'_m, \kappa(m)) \subset \bigcup_{Z_n: n < m} K(Z_n, \kappa(n)) \cup S(R).$$

**(4.54)**

**Proof of Theorem 4.6 for $n = 3$ and $N \geq 4$.** Without loss of generality we can assume that $V_{ij}^{(2)} \equiv 0$ holds for $i, j = 1, \ldots, N$, $i \neq j$. Let $L[\varphi]$ be the functional defined in (4.41). We will show that $L[\varphi] \geq 0$ holds for every $0 \leq \alpha_0 < \frac{\kappa'}{\kappa} \alpha$ and every $\varphi \in H^1(R_0)$ with $\text{supp}(\varphi) \subset R_0 \setminus S(R)$, where $R > 0$ is sufficiently large. Analogously to the case $N = 3$ we get

$$L[\varphi] \geq \sum_{Z_2} L_1[\varphi u_{Z_2}] + L_2[\varphi v_2],$$

where the functionals $L_1, L_2$ are defined in (4.43) and $V_2 = \sqrt{1 - \sum_{Z_2} u_{Z_2}^2}$. By repeating the same arguments as in the case $N = 3$, one can easily show that $L_1[\varphi u_{Z_2}] \geq 0$ holds for all two-cluster decompositions $Z_2$. We only need to prove $L_2[\varphi v_2] \geq 0$. By applying Lemma 4.12 we can find $\kappa(3) > 0$, such that on the support of $V_2 \varphi$ the cones $K(Z_3, \kappa(3))$ and $K(Z'_3, \kappa(3))$ do not overlap for $Z_3 \neq Z'_3$. Applying Lemma 4.11 yields

$$L_2[\varphi v_2] \geq \sum_{Z_3} L_1[\varphi u_{Z_3}, V_2 \varphi] + L_2[\varphi v_3, V_2 \varphi],$$

where $V_3 = \sqrt{1 - \sum_{Z_3} u_{Z_3}^2}$ on the support of $V_2 \varphi$ and

$$L_1'[\varphi] = \langle (H_0(Z_3) \varphi, \varphi) - \alpha_0 \left[ \nabla q(Z_3) \varphi \right]^2 + (1 - \gamma_0) \left[ \nabla q(Z_3) \varphi \right]^2 + (I(Z_3) \varphi, \varphi) - \alpha_0^2 \left[ \nabla q(Z_3) \varphi \right]^2 - \gamma \left[ \varphi \right]^2 \rangle,$$

$$L_2'[\varphi] = (1 - \gamma_0) \left[ \nabla q \varphi \right]^2 + \langle V \varphi, \varphi \rangle - ||| \alpha_0 |x_1|^{-1} \varphi \rangle - 2 \gamma \left[ \varphi \right]^2.$$  

(4.57)

Since for each subsystem $C_j$ in the breaking $Z_3$ the corresponding operator $H_0|C_j$ does not have a virtual level we have

$$\langle (H_0(Z_3) \varphi, \varphi) \rangle \geq \mu_0 \left[ \nabla q(Z_3) \varphi \right]^2$$

(4.58)

for some $\mu_0 > 0$, independent of $\varphi$. In addition, on the support of $u_{Z_3} V_3$ it holds $|V_{ij} (x_{ij})| \leq c||| \xi(Z_3) |x_1|^{-2-\nu}$ for $i, j$ belonging to different clusters in $Z_3$. Consequently, by the same arguments as in the estimate of $L_1[\varphi u_{Z_3}, V_2 \varphi]$ we get $L_1'[\varphi u_{Z_3}, V_2 \varphi] \geq 0$. Repeating this process, we see that to prove the theorem it suffices to show

$$L_3[\varphi] := (1 - \gamma_0) \left[ \nabla q \varphi \right]^2 + \langle V \varphi, \varphi \rangle - ||| \alpha_0 |x_1|^{-1} \varphi \rangle - \epsilon \left[ \varphi \right]^2 \geq 0$$

(4.59)

for small $\epsilon, \gamma_0 > 0$ and for functions $\varphi \in H^1(R_0)$, which are supported in the region where

$$|V_{ij} (x_{ij})| \leq c|x_1|^{-2-\nu}$$

(4.60)

holds for all $i, j = 1, \ldots, N$, $i \neq j$. We choose $0 < \epsilon < \frac{(3(N-1)-2)^2}{4} \alpha_0^2$ and $R > 0$ sufficiently large, such that by Hardy’s inequality in dimension $3(N-1)$ it holds (4.59). Now we can apply Theorem 2.1 and conclude that zero is a simple eigenvalue of $H_0$ and the corresponding eigenfunction $\varphi_0$ satisfies

$$(1 + |x_1|)^{\alpha_0-1} \varphi_0 \in L^2(R_0)$$

(4.61)
for every $\alpha_0 < \frac{3N-5}{2}$. This completes the proof of statement (i) and (ii) of Theorem 4.6 in the case $n = 3$ and $N \geq 4$. Finally, since Hardy’s inequality holds for every $n \geq 3$, the proof of the theorem can trivially be adapted to the case $n \geq 4$ by replacing the Hardy constant in the corresponding dimension. Statement (iii) of the theorem follows from statement (iii) of Theorem 2.1 similar to the case of $N = 3$. \hfill $\square$

Theorem 4.7 can be proved by similar arguments. Theorem 4.9 can be proved similar to Theorem 4.6 by using Theorem 3.1 instead of Theorem 2.1.

5. Absence of the Efimov effect in $N$-particle systems with $N \geq 4$

In this section we prove that the Efimov effect does not occur in the case of more than three particles in any dimension $n \geq 3$. The main reason for this is that for such systems the virtual level is always an eigenvalue, see Theorem 4.6. Our proof is based on the ideas of [30], where it was shown that in case of three particles, restricted to certain symmetries, the Efimov effect does not occur as well. We will adapt this technique to arbitrary $N$-body systems.

**Theorem 5.1.** Consider the operator $H_0$ with $n \geq 3$ and $N \geq 4$ particles, where the potentials $V_{ij}$ satisfy (4.7) and (4.8). Assume that for all subsystems $C \subseteq Z_1$ it holds $H_0[C] \geq 0$ and for all $C$ with $|C| \leq N-2$ the operators $H_0[C]$ do not have virtual levels. Then the discrete spectrum of $H_0$ is finite.

**Remark.** (i) We emphasize that in Theorem 5.1 the operator $H_0[C]$ with $|C| = N - 1$ may have a virtual level.

(ii) Theorem 5.1 can be easily generalized to the case when one of the particles has infinite mass.

(iii) The results of Theorem 5.1 can be easily generalized to the case when the operator $H_0$ is considered on a subspace of functions with fixed permutational symmetry. Namely, the following theorem holds.

**Theorem 5.2.** Consider the operator $H_0^n$ with $n \geq 3$ and $N \geq 4$ particles, where the potentials $V_{ij}$ satisfy (4.7) and (4.8). Let the operators $H_0(Z_p)$, the group $S(Z_p)$ and the inducing of the symmetry $\sigma(Z_p)$ $\rho \sigma$ be defined as in section 4.2.2. Assume that for any breaking $Z_p = (C_1, \ldots, C_p)$ with $p \geq 3$ or $p = 2$ and $|C_1|, |C_2| < N - 1$ and any type of irreducible representation $\sigma'(Z_p) \sim \sigma$ it holds

$$P^{\sigma'(Z_p)}(H_0(Z_p) + \varepsilon \Delta_0(Z_p)) \geq 0$$

for sufficiently small $\varepsilon > 0$. Then the discrete spectrum of $H_0^n$ is finite.

**Proof of Theorem 5.1.** Let for $\varphi \in H^1(\mathbb{R}^d)$

$$L_1[\varphi] := \langle H_0 \varphi, \varphi \rangle - \varepsilon \|x\|^{-1} \varphi\|^2.$$  

(5.2)

Due to Lemma A.1 to prove the theorem it suffices to show that there exist constants $\varepsilon > 0$ and $b > 0$, such that $L_1[\varphi] \geq 0$ holds for all functions $\varphi \in H^1(\mathbb{R}^d)$ with supp $\varphi \subset \{x \in \mathbb{R}^d, |x| \geq b\}$. Applying Lemma 4.11 yields

$$L_1[\varphi] \geq \sum_{Z_2} L_2[\varphi Z_2] + L_3[\varphi]$$

(5.3)

where $V = \sqrt{1 - \sum_{Z_2} u_x^2}$ and the functionals $L_2, L_3 : H^1(R_0) \rightarrow \mathbb{R}$ are defined by

$$L_2[\psi] := \langle H_0 \psi, \psi \rangle - \varepsilon \|x\|^{-1} \psi\|^2 - \varepsilon_1 \|q(Z_2)\|^{-1} \psi\|^2,$$

(5.4)

$$L_3[\psi] := \langle H_0 \psi, \psi \rangle - (\varepsilon + \varepsilon_1) \|x\|^{-1} \psi\|^2,$$

(5.5)
where
\[ \Omega(Z_2) \subset \{ x \in R^0 : |x| \geq b, \kappa'|\xi(Z_2)| \leq |q(Z_2)| \leq \kappa|\xi(Z_2)| \}. \] (5.6)
The constants \( \varepsilon_1 \) > 0 and \( \kappa > 0 \) can be chosen arbitrarily small and \( \kappa' > 0 \) depends on \( \varepsilon_1 \) and \( \kappa \). At first we prove that \( L_2[\varphi u_{Z_2}] \geq 0 \). We need to distinguish between two different types of breakings \( Z_2 = (C_1, C_2) \):

(i) \( |C_1| < N - 1 \) and \( |C_2| < N - 1 \),

(ii) \( |C_1| = N - 1 \) or \( |C_2| = N - 1 \).

In case (i) by the assumption of the theorem the operators \( H_0[C_1] \) and \( H_0[C_2] \) do not have virtual levels, i.e. there exists a constant \( \mu_0 > 0 \), such that
\[ \langle H_0(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2} \rangle \geq \mu_0\|\nabla_0(\varphi u_{Z_2})\|^2 \] (5.7)
holds for any \( \varphi \in H^1(R_0) \). In this case we can make use of similar arguments as in the proof of Theorem 4.6 to conclude \( L_2[\varphi u_{Z_2}] \geq 0 \).

We turn to case (ii), where the Hamiltonians of the subsystems may have virtual levels. Suppose that \( |C_1| = N - 1 \) and that \( H_0[C_1] \) has a virtual level. Then, according to Theorem 4.6 zero is a simple eigenvalue of \( H_0[C_1] \). Let \( \varphi_0 \) be the corresponding eigenfunction. We estimate \( L_2[\varphi u_{Z_2}] \) by adapting the strategy of [30]. We write
\[ \varphi u_{Z_2}(q(Z_2), \xi(Z_2)) = \varphi_0(q(Z_2))f(\xi(Z_2)) + g(q(Z_2), \xi(Z_2)), \] (5.8)
where \( \|\varphi_0\| = 1 \) and
\[ \langle \nabla_q(Z_2)g(\cdot, \xi(Z_2)), \nabla_q(Z_2)\varphi_0 \rangle = 0 \] (5.9)
holds for almost every \( \xi_{Z_2} \). Note that
\[ L_2[\varphi u_{Z_2}] = \langle H_0[C_1] g, g \rangle + \langle H_0[C_1] \varphi_0 f, \varphi_0 f \rangle + 2 \Re(H_0[C_1] g, \varphi_0 f) \]
\[ + \|\nabla_0(Z_2)\varphi u_{Z_2}\|^2 + \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2} \rangle \]
\[ - \varepsilon_1\|\varphi u_{Z_2}\|^2 - \varepsilon_1\|q(Z_2)|^{-1}\varphi u_{Z_2}\|^2. \] (5.10)
Since \( H_0[C_1]\varphi_0 = 0 \) the second term and the third term on the r.h.s. of (5.10) are zero. Due to the orthogonality condition (5.9) Theorem 4.6 yields
\[ \langle H_0[C_1] g, g \rangle \geq \delta_0\|\nabla_q(Z_2)g\|^2 \] (5.11)
for some \( \delta_0 > 0 \). We arrive at
\[ L_2[\varphi u_{Z_2}] \geq \delta_0\|\nabla_q(Z_2)g\|^2 + \|\nabla_0(Z_2)\varphi u_{Z_2}\|^2 + \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2} \rangle \]
\[ - \varepsilon_1\|\varphi u_{Z_2}\|^2 - \varepsilon_1\|q(Z_2)|^{-1}\varphi u_{Z_2}\|^2. \] (5.12)
Further, since \( V_{ij} \geq V_{ij}^{(1)} \) we have
\[ \langle I(Z_2)\varphi u_{Z_2}, \varphi u_{Z_2} \rangle \geq \sum_{i \in C_1, j \in C_2} \langle V_{ij}^{(1)} \varphi u_{Z_2}, \varphi u_{Z_2} \rangle \geq - \sum_{i \in C_1, j \in C_2} \langle V_{ij}^{(1)} \varphi u_{Z_2}, \varphi u_{Z_2} \rangle \]
\[ \geq - C\|\xi(Z_2)|^{-1}\varphi u_{Z_2}\|^2 \geq - \varepsilon_1\|\nabla_0(Z_2)\varphi u_{Z_2}\|^2, \] (5.13)
where \( \varepsilon > 0 \) can be chosen arbitrarily small by choosing \( b \) sufficiently large. Here we used that on supp \( (\varphi u_{Z_2}) \) we have \( |V_{ij}^{(1)}(x_{ij})| \leq C|\xi(Z_2)|^{-2} \leq \varepsilon|\xi(Z_2)|^{-2} \) for \( i, j \) belonging to different clusters. Moreover, since on the support of \( \varphi u_{Z_2} \) we also have \( |x_{ij}|\geq (1 + \kappa^2)^{-\varepsilon} \xi(Z_2)|^{-1} \), we arrive at
\[ L_2[\varphi u_{Z_2}] \geq \delta_0\|\nabla_q(Z_2)g\|^2 + (1 - \varepsilon_1)\|\nabla_0(Z_2)\varphi u_{Z_2}\|^2 - \varepsilon_1\|q(Z_2)|^{-1}\varphi u_{Z_2}\|^2. \] (5.14)
Since
\[ \|q(Z_2)|^{-1}\varphi u_{Z_2}\|^2 \leq 2\|q(Z_2)|^{-1}\varphi_0 f\|^2 + 2\|g\|q(Z_2)|^{-1}\varphi u_{Z_2}|^2, \] (5.15)
Now we estimate the last term on the r.h.s. of (5.16). Note that for $\kappa > 0$ sufficiently small it holds

$$
\|q(Z_2)\|_{1}^{-1} f \varphi_0 \|_{L^2(\Omega(Z_2))}^2 \leq \int_{\{\varphi_0 |_{\Omega(Z_2)} > \frac{\kappa}{2}\}} |f|^2 d\xi(Z_2) \int_{\Omega(Z_2)} |\varphi_0|^2 q(Z_2)_{1}^{-2} dq(Z_2)
$$

$$
\leq (\kappa')^{-2} \int_{\{\varphi_0 |_{\Omega(Z_2)} \geq \frac{\kappa}{2}\}} \Phi |f|^2 |\xi(Z_2)|_{1}^{-2} d\xi(Z_2),
$$

where $\tilde{\Omega}(Z_2) = \{q(Z_2) : \kappa' |\xi(Z_2)|_{1} \leq |q(Z_2)|_{1} \leq \kappa |\xi(Z_2)|_{1}\}$ and

$$
\Phi(\xi(Z_2)) = \int_{\Omega(Z_2)} |\varphi_0(q(Z_2))|^2 dq(Z_2).
$$

(5.17)

Since $\varphi_0$ is square-integrable in $q(Z_2)$, for any $\delta > 0$ one can find $b > 0$, such that $\Phi(\xi(Z_2)) < \delta$ holds uniformly in $|\xi(Z_2)|_{1} \geq \frac{b}{2}$. Hence, for any fixed $\tilde{\varepsilon} > 0$ we can choose $b > 0$ sufficiently large, such that

$$
\|q(Z_2)\|_{1}^{-1} f \varphi_0 \|_{L^2(\Omega(Z_2))}^2 \leq \tilde{\varepsilon} \int_{\{\kappa' |\xi(Z_2)|_{1} \geq \frac{b}{2}\}} |\xi(Z_2)|_{1}^{-2} |f(\xi(Z_2))|^2 d\xi(Z_2).
$$

(5.19)

Due to Lemma 5.3. in [30] there exists a constant $\gamma > 0$, depending on $\|\varphi\|, \|\nabla \varphi\|$ and $\|\Delta \varphi\|$ only, such that

$$
\|\nabla \xi(Z_2) \varphi u_{Z_2}\|^2 \geq \gamma (\|\nabla \xi(Z_2) \varphi f\|^2 + \|\nabla \xi(Z_2) g\|^2),
$$

(5.20)

which together with (5.16) and (5.19) yields

$$
L_2[\varphi u_{Z_2}] \geq (1 - \varepsilon) \gamma \|\nabla \xi(Z_2) f\|^2 - 2\varepsilon_1 \tilde{\varepsilon} \|\xi(Z_2)|_{1}^{-1} f\|^2 \geq 0.
$$

(5.21)

Thus, it remains to prove that $L_3[\psi V] \geq 0$ holds for every function $\varphi \in H^1(R_0)$ with $\text{supp} \varphi \subset \{x \in \mathbb{R}^d : |x|_{1} \geq b\}$. For any breaking $Z_p = (C_1, \ldots, C_p)$ with $p \geq 3$ the corresponding operators $H[C_i]$ do not have virtual levels. Therefore, we can estimate the functional $L_3[\psi V]$ in cones corresponding to breakings $Z_p$ into $3 \leq p \leq N - 1$ clusters, similarly to the proof of Theorem 4.6. In the region, which remains after the separation of cones corresponding to all $Z_p$ with $p \leq N - 1$ it holds $|V^{(1)}_{ij}(x_{ij})| \leq c|x|_{1}^{-2-\nu}$ for all $i \neq j$. Applying Hardy’s inequality completes the proof.

6. Systems of $N \geq 4$ fermions in dimension $n = 1$ or $n = 2$

We consider a system $Z_1$ of $N \geq 3$ one- or two-dimensional particles and the corresponding Hamiltonian given in (4.6), where the potentials $V_{ij}$ satisfy

$$
V_{ij} \in L^2_{\text{loc}}(\mathbb{R}^n), \quad |V_{ij}(x)| \leq C|x|^{-2-\nu}, \text{ if } |x| \geq A
$$

(6.1)

for some constants $A > 0$ and $\nu > 0$. Further, we assume that all particles are identical, i.e. $m_i = m_j$ for $1 \leq i, j \leq N$ and for $i \neq j$, $k \neq l$ we have

$$
V_{ij}(x) = V_{ij}(-x), \quad V_{ij}(x) = V_{kl}(x), \quad x \in \mathbb{R}^n, \quad n = 1, 2.
$$

(6.2)

We reduce the center of mass by defining the space $R_0$ and the operator $H_0$ according to (4.10) and (4.29), respectively. Since the particles are identical, the operator $H_0$ is invariant under action of the group $S_N$ of permutation of particles. Let $\sigma_{0w}$ be the irreducible representation of $S_N$, antisymmetric with respect to permutation of each pair of particles. Let $P^{\sigma_{0w}}$ be the
projection in $R_0$ onto the subspace of the $\sigma_{as}$. We will consider the operator $H_0$ on the subspace $P_{\sigma_{as}} L^2 (R_0)$ and define $H_0^{\sigma_{as}} = P_{\sigma_{as}} H_0$. Given a subsystem $C \subset Z_1$, let $S[C]$ be the subgroup of $S_N$ corresponding to permutations of particles in the subsystem $C$. We denote by $\sigma_{as}[C]$ the irreducible representation of $S[C]$, antisymmetric with respect to permutation of each pair of particles in $C$. Let $H_0^{\sigma_{as}}[C] = P_{\sigma_{as}}[C] H_0[C]$.

**Definition 6.1.** For an arbitrary subsystem $C \subset Z_1$ we say that the corresponding operator $H_0^{\sigma_{as}}[C]$ has a virtual level at zero, if $H_0^{\sigma_{as}}[C] \geq 0$ and for sufficiently small $\varepsilon > 0$ it holds

$$S_{\text{ess}} \left( P_{\sigma_{as}}[C] \left[ - (1 - \varepsilon) \Delta_0[C] + V[C] \right] \right) = [0, \infty)$$

and

$$S_{\text{disc}} \left( P_{\sigma_{as}}[C] \left[ - (1 - \varepsilon) \Delta_0[C] + V[C] \right] \right) \neq \emptyset.$$

We are now ready to state the main theorem of this section.

**Theorem 6.2.** Let $n = 1$ or $n = 2$ and let $Z_1$ be a system of $N \geq 3$ particles. Assume that the potentials $V_{ij}$ satisfy (6.1) and (6.2). Further, assume that $H_0^{\sigma_{as}}$ has a virtual level at zero and for each subsystem $C \subset Z_1$ and sufficiently small $\varepsilon > 0$ it holds

$$S \left( P_{\sigma_{as}}[C] \left[ - (1 - \varepsilon) \Delta_0[C] + V[C] \right] \right) = [0, \infty).$$

Then, zero is an eigenvalue of $H_0^{\sigma_{as}}$.

**Proof.** According to Theorem 3.1, it suffices to show that there exist $R > 0$, $\gamma_0 > 0$ and $\alpha_0 > 1$, such that for any function $\varphi \in P_{\sigma_{as}} H^1 (R_0)$ with $\text{supp}(\varphi) \subset R_0 \setminus S(R)$ we have

$$L[\varphi] := \langle H_{\sigma_{as}} \varphi, \varphi \rangle - \gamma_0 \|\nabla \varphi\|^2 - \alpha_0 \|x^{-1} \varphi\|^2 \geq 0.$$  \hfill (6.6)

Note that in dimension $n = 1$ and $n = 2$ Hardy’s inequality holds for antisymmetric functions [4]. If $n = 2$ and $N \geq 4$ or $n = 1$ and $N \geq 6$ we can repeat the same arguments as in Theorem 4.6 for $0 < \alpha_0 < \frac{2(N-1)}{N}$, if $n = 2$ and $0 < \alpha_0 < \frac{N-3}{N-2}$, if $n = 1$, respectively.

We only need to consider the cases $n = 2$, $N = 3$ and $n = 1, N = 3, 4, 5$. We start with the case $n = 2, N = 3$. By the same arguments as in the proof of Theorem 4.6, it suffices to show that $L_2[\varphi \varphi] \geq 0$ holds for $\varphi \in P_{\sigma_{as}} H^1 (R_0)$, where $L_2[\varphi \varphi]$ and $\varphi \varphi$ are defined in (4.43). Since the multiplication with $\varphi$ does not change the symmetry property of $\varphi$, the function $\varphi \varphi$ is antisymmetric with respect to permutations of particles. Hence, it is orthogonal to all functions depending on $|x|_1$ only. Therefore, for $\varphi \varphi$ we have (see for example [12] p. 254)

$$\|\nabla (\varphi \varphi)\|^2 \geq L(L+1) \|x^{-1} \varphi \varphi\|^2, \quad L = l + \frac{1}{2} (\dim R_0 - 3)$$ \hfill (6.7)

with $l = 1$ and $\dim R_0 = 4$. Substituting this inequality in $L_2[\varphi \varphi]$ gives the desired estimate for $n = 2$ and $N \geq 3$.

Now we turn to the case $n = 1$ and $N = 3, 4, 5$. Let $N = 4$ or $N = 5$. In this case we have $\dim R_0 = 3$ or $\dim R_0 = 4$, respectively. By the same argument as in the case of $n = 2, N = 3$, we only have to consider the functional $L_2[\varphi \varphi]$. Since $\varphi \varphi$ is orthogonal to all functions with $l = 0$, applying the Hardy-type inequality (6.7) with $\dim R_0 = 4$ and $\dim R_0 = 5$, respectively, yields the result for $n = 1$ and $N = 4, N = 5$. To complete the proof it remains to consider the case of $n = 1$ and $N = 3$. In this case we will prove the following

**Lemma 6.3.** Let $R_0$ be the space defined in (4.10) with $n = 1$ and $N = 3$ and let $\psi \in C_0^1 (R_0)$ be antisymmetric with respect to exchange of each pair of coordinates $(x_i, x_j)$. Then we have

$$\|\nabla \psi\|^2 \geq 9 \|\psi x_1^{-1}\|^2.$$  \hfill (6.8)
Remark. Combining the arguments of the proof of Theorem 6.2 with the estimate (6.3) one can easily obtain an estimate on the rate of decay of virtual levels in this system. In particular, it is easy to see that a zero energy eigenfunction $\varphi_0$ for a system of three one-dimensional fermions on the subspace of functions antisymmetric with respect to permutations of coordinates of particles satisfies $(1 + |x|)^{2-\varepsilon} \varphi_0 \in L^2(\mathbb{R}^3)$ for any $\varepsilon > 0$.

Let us use Lemma 6.3, which will complete the proof of Theorem 6.2. Note that for $n = 1$ and $N = 3$ we have $\dim R_0 = 2$. On the plane $R_0$ we introduce the polar coordinates $\psi = \psi(\rho, \theta)$, where $\rho = \sqrt{\sum_{i=1}^{3} x_i^2}$ and $\theta$ is the angle between $x$ and $\frac{1}{\sqrt{2}}(1, -1, 0)$. Obviously, the axes $x_1 = x_2$, $x_2 = x_3$, $x_1 = x_3$ cut $R_0$ into six sectors, each with angle $\frac{\pi}{3}$. Since $\psi$ is antisymmetric with reflection on these axes we conclude that $\psi$ is a periodic function in $\theta$ with period $\frac{\pi}{3}$ and $\psi(\rho, 0) = 0$. We represent $\psi$ as a Fourier series, i.e. we write for almost all $\rho$

$$\psi(\rho, \theta) = \sum_{n=1}^{\infty} a_n(\rho) \sin(3n\theta). \quad (6.9)$$

Differentiating (6.9), we get

$$\|\nabla \psi\|^2 \geq \left\| \frac{1}{\rho} \frac{\partial}{\partial \theta} \psi \right\|^2 \geq 9\|\rho^{-1}\|^2. \quad (6.10)$$

This completes the proof. \qed

For the absence of the Efimov effect in systems of $N \geq 4$ one- or two-dimensional particles we get now the following result.

Theorem 6.4. Let $n = 1$ or $n = 2$ and let $Z_1$ be a system of $N \geq 4$ particles. Assume that the potentials $V_i$ satisfy (6.1) and (6.2). Further assume that for each subsystem $C \subset Z_1$ we have $H_0^{\text{vac}}[C] \geq 0$ and if $|C| \leq N - 1$ the operator $H_0^{\text{vac}}[C]$ does not have a virtual level at zero. Then the discrete spectrum of $H_0^{\text{vac}}$ is finite.

Proof. The proof of Theorem 6.4 goes along the same line as that of Theorem 5.1. The only difference is that if for a subsystem $C$ with $|C| = N - 1$ the operator $H_0^{\text{vac}}[C]$ has a virtual level, zero might be a degenerate eigenvalue of finite multiplicity. However, in this case we can find a decomposition similar to that in (5.8) with a function $g$ which is orthogonal to the corresponding eigenspace. Repeating the arguments of the proof of Theorem 5.1 proves Theorem 6.4. \qed

Appendix A.

Proof of Lemma 2.2. Let $\varepsilon > 0$ and $b > 0$ be fixed. Let $b > b$ and $u \in C^1(\mathbb{R}^+)$, such that $u(t) = 1$ if $t \leq b$ and $u$ is non-increasing on $[b, \infty)$. Moreover, for $t \to b$ let $u'(t) (1 - u^2(t))^{-\frac{1}{2}} \to 0$. We define $v := \sqrt{1 - u^2}$, $\chi_1(x) := u(|x|)$ and $\chi_2(x) := v(|x|)$.\quad (A.1)

Then, since $\chi_1^2 + \chi_2^2 = 1$ holds we have

$$|\nabla \chi_1|^2 + |\nabla \chi_2|^2 = \frac{|\nabla \chi_1|^2}{1 - \chi_1^2} = \frac{u'(|x|)^2}{1 - u(|x|)^2}. \quad (A.2)$$

Now since $u'(|x|) (1 - u^2(|x|))^{-\frac{1}{2}} \to 0$ as $|x| \to b$, we can take $b' > b$ so close to $b$ that

$$\frac{u'(|x|)^2}{1 - u(|x|)^2} \leq \varepsilon |x|^{-2}, \quad |x| \in [b, b']. \quad (A.3)$$

This together with (A.2) implies

$$|\nabla \chi_1|^2 + |\nabla \chi_2|^2 \leq \varepsilon |x|^{-2}, \quad |x| \in [b, b']. \quad (A.4)$$
Now we define the function $u$ for $t \geq b'$ as
\[ u(t) = u(b') \ln \left( \frac{b'}{b} \right) \left( \ln \left( \frac{b'}{b} \right) \right)^{-1}, \quad t \in [b', \bar{b}] \text{ and } u(t) = 0, \quad t \geq \bar{b}. \quad (A.5) \]

Note that $u(b')$ is close to 1, but it is strictly less than 1. As before we set
\[ \chi_1(x) = u(|x|), \quad \chi_2(x) = v(|x|), \quad |x| \geq b'. \quad (A.6) \]

We have for $|x| \geq b'$
\[ |\nabla \chi_1|^2 + |\nabla \chi_2|^2 \leq \frac{u^2(b')}{1 - u^2(b')} \left( \frac{b'}{b} \right)^{-2} |x|^{-2}. \quad (A.7) \]

Since $b'$ is close to $b$ and $\bar{b}$ can be done arbitrarily large the r.h.s. of (A.7) can be estimated as $\varepsilon |x|^{-2}$. \qed

**Proof of Lemma 4.11.** Let $\kappa > 0$ and let $Z_p$ be an arbitrary breaking into $p$ clusters. For the sake of brevity we write $q$ and $\xi$ instead of $q(Z_p)$ and $\xi(Z_p)$, respectively.

Let $v_1 \in C^1(\mathbb{R}_+)$, such that $v_1(t) = 1$, if $t \geq \kappa$ and $v_1$ is non-decreasing on $[0, \kappa]$. We assume $v'_1(t) (1 - v_1^2(t))^{-\frac{1}{2}} \to 0$ as $t \to \kappa$ and define $u_1(t) := (1 - v_1^2(t))^{\frac{1}{2}}$. For $0 < \kappa'' < \kappa$ and $x = (q, \xi) \in K(Z_p, \kappa) \setminus K(Z_p, \kappa'')$ let
\[ u(x) = u_1 \left( \frac{|q|_1}{|\xi|_1} \right), \quad v(x) = v_1 \left( \frac{|q|_1}{|\xi|_1} \right). \quad (A.8) \]

Then we have
\[ |\nabla u|^2 + |\nabla v|^2 = (1 - v_1^2(t))^{-1} (1 + |q|_1^2 |\xi|_1^{-2}) |\xi|_1^{-2} (v'_1(t))^2, \quad (A.9) \]

where $t = |q|_1 |\xi|_1^{-1}$. Since $\kappa'' \leq |q|_1 |\xi|_1^{-1} \leq \kappa$ and $|x|^2 = |q|^2 + |\xi|^2$ we have $|\xi|_1^{-2} \leq (1 + \kappa^2) |x|^{-2}$. Hence, (A.9) yields
\[ |\nabla u|^2 + |\nabla v|^2 \leq (v'_1(t))^2 (1 - v_1(t)^2)^{-1} (1 + \kappa^2) |x|^{-2}. \quad (A.10) \]

Since $v'_1(t) (1 - v_1^2(t))^{-\frac{1}{2}} \to 0$ as $t \to \kappa$ we can choose $\kappa''$ so close to $\kappa$, such that for $x \in K(Z_p, \kappa) \setminus K(Z_p, \kappa'')$ and $t = |q|_1 |\xi|_1^{-1}$ we have
\[ (v'_1(t))^2 (1 - v_1(t)^2)^{-1} (1 + \kappa^2)^2 |x|^{-2} < \varepsilon |x|^{-2}. \quad (A.11) \]

Now we define $u$ and $v$ for $x \in K(Z_p, \kappa'')$. Let $0 < \kappa' < \kappa''$ and set
\[ v_1(t) = v_1(\kappa'') \ln(\kappa''/\kappa')^{-1} \ln(t/\kappa'), \quad t \leq \kappa''. \quad (A.12) \]

Let
\[ v(x) = v_1 \left( \frac{|q|_1}{|\xi|_1} \right), \quad x \in K(Z_p, \kappa'') \setminus K(Z_p, \kappa') \text{ and } v(x) = 0, \quad x \in K(Z_p, \kappa'). \quad (A.13) \]

Since $v_1(t) < v_1(\kappa'') < 1$, if $t < \kappa''$ we have
\[ (|\nabla u|^2 + |\nabla v|^2) |u|^{-2} = |\nabla v|^2 (1 - v_1^2)^{-1} |u|^{-2} < |\nabla v|^2 (1 - v_1^2(\kappa''))^{-2}. \quad (A.14) \]

For $t = |q|_1 |\xi|_1^{-1} \leq \kappa''$ we get
\[ |\nabla v|^2 = (v'_1(t))^2 (1 + |q|_1^2 |\xi|_1^{-2}) |\xi|_1^{-2} \leq (v'_1(t))^2 (1 + (\kappa'')^2) |\xi|_1^{-2}. \quad (A.15) \]

Note that
\[ v'_1(t) = v_1(\kappa'') (\ln(\kappa''/\kappa'))^{-1} t^{-1}. \quad (A.16) \]
Hence, by combining (A.14), (A.15), (A.16), substituting \( t = |q(Z_2)|_1 |\xi(Z_2)|^{-1} \) and multiplying both sides of (A.14) with \(|u|^2\) we conclude

\[
|\nabla_0 u|^2 + |\nabla_0 v|^2 < \varepsilon |q|^{-2} |u|^2,
\]
(A.17)

for \(|q| < \kappa''|\xi|\) if \(\kappa' > 0\) is chosen sufficiently small. This, together with (A.10) completes the proof. \(\square\)

**Lemma A.1.** Let \( h_0 = -\Delta + V \) in \( L^2(\mathbb{R}^d) \), \( d \geq 3 \) with \( V \) satisfying (2.2). Assume there exist \( \varepsilon > 0 \) and \( b > 0 \), such that

\[
\langle h_0 \psi, \psi \rangle - \varepsilon \langle |x|^{-2} \psi, \psi \rangle \geq 0
\]
(A.18)

holds for any \( \psi \in H^1(\mathbb{R}^d) \) with \( \text{supp} \psi \subset \{ x \in \mathbb{R}^d, |x| \geq b \} \). Then the following assertions hold.

(i) \( \inf S_{\text{ess}}(h_0) \geq 0 \).

(ii) Zero is not an infinitely degenerate eigenvalue of \( h_0 \).

(iii) If in addition (2.9) holds, then the subspace of functions in \( \tilde{H}^1(\mathbb{R}^d) \) satisfying

\[
- \Delta \psi + V \psi = 0
\]
(A.19)

is at most finite-dimensional.

**Remark.**

(i) The Lemma is a slightly modified variant of a part of the proof of the main Theorem in [34].

(ii) This result can be easily extended to the case where the operator \( h_0 \) is invariant under action of a symmetry group \( G \) and we consider this operator on some symmetry space \( P^\sigma L^2(\mathbb{R}^d) \), here \( \sigma \) is a type of irreducible representation of \( G \).

**Proof.** We construct a finite-dimensional subspace \( M \subset L^2(\mathbb{R}^d) \), such that \( \langle h_0 \psi, \psi \rangle > 0 \) holds for any \( \psi \in H^1(\mathbb{R}^d) \) \( \left( \tilde{H}^1(\mathbb{R}^d) \right) \) orthogonal to \( M \). Due to Lemma 2.2 we have

\[
\langle h_0 \psi, \psi \rangle \geq L[\psi \chi_1] + L[\psi \chi_2],
\]
(A.20)

where \( \chi_1, \chi_2 \) are defined in Lemma 2.2 and the functional \( L \) is given by

\[
L[\psi] = \langle h_0 \psi, \psi \rangle - \varepsilon \langle |x|^{-2} \psi, \psi \rangle.
\]
(A.21)

Since \( \psi \chi_2 \) is supported outside the ball of radius \( b \), condition (A.18) implies \( L[\psi \chi_2] \geq 0 \). To prove the Lemma it suffices to show that \( L[\psi \chi_1] > 0 \) holds for any \( \psi \perp M \) for some finite-dimensional space \( M \). By Hardy’s inequality and (2.2) it holds

\[
L[\psi \chi_1] \geq (1 - 5\varepsilon) \|
abla (\chi_1 \psi)\|^2 - C(\varepsilon) \|\chi_1 \psi\|^2.
\]
(A.22)

Let

\[
M_k := \{ \varphi_1 \chi_1, \ldots, \varphi_k \chi_1 \},
\]
(A.23)

where \( \{ \varphi_1, \ldots, \varphi_k \} \) is an orthonormal set of eigenfunctions corresponding to the lowest eigenvalues of the Laplacian, acting on \( L^2(S(b_1)) \) with Dirichlet boundary conditions. For \( \psi \perp M_k \) we have \( \psi \chi_1 \perp \psi_1, \ldots, \psi_k \), which for sufficiently large \( k \) implies

\[
\| \nabla (\psi \chi_1) \|^2 \geq 2 (1 - \varepsilon)^{-1} C(\varepsilon) \| \psi \chi_1 \|^2.
\]
(A.24)

Therefore, we conclude \( L[\psi \chi_1] > 0 \). \(\square\)
Acknowledgements

Simon Barth and Andreas Bitter are deeply grateful to Timo Weidl for his support and for the useful discussions. Their work was supported by the Deutsche Forschungsgemeinschaft (DFG) through the Research Training Group 1838: Spectral Theory and Quantum Systems. Semjon Vugalter gratefully acknowledges the funding by the Deutsche Forschungsgemeinschaft (DFG), German Research Foundation Project ID 258734477 - SFB 1173. Semjon Vugalter is grateful to the University of Toulon for the hospitality during his stay there.

The authors thank the Mittag-Leffler Institute, where a part of the work was done during the semester program Spectral Methods in Mathematical Physics.

References

[1] R. Adams and J. Fournier. Sobolev Spaces. Pure and Applied Mathematics. Elsevier Science, 2003.
[2] S. Agmon. Lectures on Exponential Decay of Solutions of Second-Order Elliptic Equations: Bounds on Eigenfunctions of N-Body Schrodinger Operations. (MN-29). Princeton University Press, 1982.
[3] R. D. Amado and F. C. Greenwood. There is no Efimov effect for four or more particles. Phys. Rev. D, 7:2517–2519, Apr 1973.
[4] M. S. Birman. On the spectrum of singular boundary-value problems. Mat. Sb. (N.S.), 55 (97):125–174, 1961.
[5] V. Efimov. Weakly bound states of three resonantly interacting particles. Yadern. Fiz., 12:1080–1091, 1970.
[6] E. Ferlaino and R. Grimm. Forty years of Efimov physics: How a bizarre prediction turned into a hot topic. Physics, 3, 01 2010.
[7] H.-W. Goelden. On non-degeneracy of the ground state of Schrödinger operators. Math. Z., 155(3):239–247, 1977.
[8] D. K. Gridnev. Zero-energy bound states and resonances in three-particle systems. Journal of Physics A: Mathematical and Theoretical, 45(17):175203, apr 2012.
[9] D. K. Gridnev. Zero energy bound states in many–particle systems. Journal of Physics A: Mathematical and Theoretical, 45(39):395302, sep 2012.
[10] D. K. Gridnev. Why there is no Efimov effect for four bosons and related results on the finiteness of the discrete spectrum. J. Math. Phys., 54(4):042105, 41, 2013.
[11] D. K. Gridnev. Three resonating fermions in flatland: proof of the super Efimov effect and the exact discrete spectrum asymptotics. J. Phys. A, 47(50):505204, 25, 2014.
[12] D. K. Gridnev and M. E. Garcia. Rigorous conditions for the existence of bound states at the threshold in the two-particle case. Journal of Physics A: Mathematical and Theoretical, 40(30):905302, 25, 2007.
[13] M. Klaus and B. Simon. Coupling constant thresholds in nonrelativistic quantum mechanics. i. short-range two-body case. Annals of Physics, 130(2):251 – 281, 1980.
[14] T. Kraemer, M. Mark, P. Waldburger, J. G Danzl, C. Chin, B. Engeser, A. Lange, K. Pich, A. Jaakkola, H.-C. Nägerl, and R. Grimm. Evidence for Efimov quantum states in an ultracold gas of caesium atoms. Nature, 440:315–8, 04 2006.
[15] Y. Nishida. Semisuper efimov effect of two-dimensional bosons at a three-body resonance. Physical Review Letters, 118, 02 2017.
[16] Y. Nishida, S. Moroz, and D. T. Son. Super efimov effect of resonantly interacting fermions in two dimensions. Phys. Rev. Lett., 110:235301, Jun 2013.
[17] Y. Nishida, S. Moroz, and S. Thanh. Super Efimov effect of resonantly interacting fermions in two dimensions. Physical Review Letters, 110, 01 2013.
[18] Y. Nishida and S. Tan. Liberating Efimov physics from three dimensions. Few-Body Systems, 51(2):191, Jul 2011.
[19] Y. Nishida and S. Tan. Liberating Efimov physics from three dimensions. Few-Body Systems, 51(2):191, Jul 2011.
[20] Y. N. Ovchinnikov and I. M. Sigal. Number of bound states of three-body systems and Efimov’s effect. Ann. Physics, 123(2):274–295, 1979.
[21] M. Reed and B. Simon. II: Fourier Analysis, Self-Adjointness. Methods of Modern Mathematical Physics. Elsevier Science, 1975.
[22] M. Schechter and B. Simon. Unique continuation for schrodinger operators with unbounded potentials. Journal of Mathematical Analysis and Applications, 77(2):482 – 492, 1980.
[23] A. G. Sigalov and I. M. Sigal. Description of the spectrum of the energy operator of quantum-mechanical systems that is invariant with respect to permutations of identical particles. Theoretical and Mathematical Physics, 5(1):990–1005, Oct 1970.
28

[24] A. V. Sobolev. The Efimov effect. Discrete spectrum asymptotics. Comm. Math. Phys., 156(1):101–126, 1993.
[25] H. Tamura. The Efimov effect of three-body Schrödinger operators. Journal of Functional Analysis, 95(2):433 – 459, 1991.
[26] H. Tamura. The Efimov effect of three-body Schrödinger operators: asymptotics for the number of negative eigenvalues. Nagoya Math. J., 130:55–83, 1993.
[27] S. Vugalter. Absence of the efimov effect in a homogeneous magnetic field. Letters in Mathematical Physics, 37(1):79–94, May 1996.
[28] S. Vugalter. Discrete spectrum of a three-particle schrödinger operator with a homogeneous magnetic field. Journal of the London Mathematical Society, 58(2):497–512, 1998.
[29] S. A. Vugal’ter and G. M. Zhislin. On the finiteness of the discrete spectrum of energy operators of many-atom molecules. Teoret. Mat. Fiz., 55(1):66–77, 1983.
[30] S. A. Vugal’ter and G. M. Zhislin. The symmetry and efimov’s effect in systems of three-quantum particles. Communications in Mathematical Physics, 87, 01 1983.
[31] S. A. Vugal’ter and G. M. Zhislin. On the finiteness of discrete spectrum in the n-particle problem. Rep. Math. Phys., 19(1):39–90, 1984.
[32] D. R. Yafaev. On the theory of the discrete spectrum of the three-particle Schrödinger operator. Mat. Sb. (N.S.), 94(136):567–593, 655–656, 1974.
[33] D. R. Yafaev. On the point spectrum in the quantum-mechanical many-body problem. Mathematics of the USSR-Izvestiya, 10(4):861–896, aug 1976.
[34] G. M. Zhislin. Finiteness of the discrete spectrum in the quantum problem of n particles. Teoret. Mat. Fiz., 21:60–73, 1974.

Simon Barth, Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: simon.barth@mathematik.uni-stuttgart.de

Andreas Bitter, Institut für Analysis, Dynamik und Modellierung, Universität Stuttgart, Pfaffenwaldring 57, D-70569 Stuttgart, Germany
E-mail address: andreas.bitter@mathematik.uni-stuttgart.de

Semjon Wugalter, Institute for Analysis, Karlsruhe Institute of Technology (KIT), Englerstrasse 2, 76131 Karlsruhe, Germany
E-mail address: semjon.wugalter@kit.edu