Ionization of Atoms by Intense Laser Pulses

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Dedicated to our friend and colleague Robert Schrader on the occasion of his 70th birthday

Abstract. The process of ionization of a hydrogen atom by a short infrared laser pulse is studied in the regime of very large pulse intensity, in the dipole approximation. Let $A$ denote the integral of the electric field of the pulse over time at the location of the atomic nucleus. It is shown that, in the limit where $|A| \to \infty$, the ionization probability approaches unity and the electron is ejected into a cone opening in the direction of $-A$ and of arbitrarily small opening angle. Asymptotics of various physical quantities in $|A|^{-1}$ is studied carefully. Our results are in qualitative agreement with experimental data reported in Eckle et al. (Science 322, 1525–1529; 2008, Nature (physics) 4, 565–570 2008).

1. Experimental Findings and Preliminary Theoretical Considerations

In recent experimental work [1,2], Eckle et al. have investigated the ionization of Helium atoms by highly intense elliptically polarized infrared laser pulses of short duration. One of the purposes of their work has been to perform an (indirect) measurement of the tunneling delay time in strong-field ionization of Helium atoms. The experimental parameters in their work have been chosen as follows: the pulse duration, $T$, is around 5.5 femtoseconds; the peak intensity, $I_0$, is between $2.3 \times 10^{14}$ and $3.5 \times 10^{14} \text{ W/cm}^2$, and the center wavelength is around 725 nm. The ionization potential, $I_p$, of a Helium atom in its groundstate is known to be $I_p \approx 24.6 \text{ eV}$. These parameter values yield a Keldysh parameter, $\gamma$, for circularly polarized light ranging from 1.17 to 1.45. The Keldysh parameter for circular polarization is given by

$$\gamma \approx 0.33 \sqrt{\frac{I_p(\text{eV})}{I_0(10^{14}\text{W/cm}^2)[\mathcal{L} (\mu m)]^2}}. \quad (1.1)$$

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If $\gamma \gg 1$, i.e., for short wave lengths, $\mathcal{L}$, and low intensity, $I_0$, the ionization process can be described in terms of multi-photon absorption, and one may attempt to treat the ionization problem perturbatively (for a theoretical analysis of a related problem, see, e.g., [3]).

If $\gamma \ll 1$, i.e., for high intensities and long wave lengths, a regime is approached where the electromagnetic field can be treated classically. However, due to the high intensity of the pulse, the theoretical analysis of the ionization process is intrinsically non-perturbative in the coupling of the electrons to the electromagnetic field. This is the regime we study in this paper.

For the values of $\gamma$ between 1.17 and 1.45 realized in the experiments described in [1,2], reliable analytical calculations of the ionization process appear to be very difficult to come by, and it is advisable to perform numerical studies; see [4]. We find, however, that our analytical results are in good qualitative agreement with the experimental findings in [1,2]. One key point of these findings is that the ionization process of a Helium atom by a short, intense near-infrared laser pulse is essentially instantaneous, in contrast to theoretical predictions based on an approximate theoretical picture taken from [5,6]: experimentally, an upper bound on the time it takes to ionize a Helium atom (with experimental parameters chosen as discussed above) appears to lie between 12 and 34 attoseconds, while a theoretical prediction relying on [5,6] yields an ionization (or “barrier traversal”) time of 450–560 attoseconds. Obviously, there is a problem with either the interpretation of the experimental findings in terms of an “ionization time” or with the approximate theory of the ionization process based on [5,6]; but most likely with both. The purpose of our paper is to provide a qualitative theoretical interpretation of the data gathered in the experiments described in [1,2].

We start with a brief sketch of the picture on which the theoretical interpretation of the experimental results is based that the authors of [1] have advocated implicitly. We then describe our own approach and state our main results.

Without harm, we may simplify our discussion by considering the ionization of Hydrogen atoms or Helium$^+$ ions by elliptically polarized laser pulses. The direction of propagation of the pulses through a very dilute, cold gas of atoms or ions is chosen to be our $z$ axis. The electric and magnetic field of the pulse are then parallel to the $x−y$ plane. If $\mathcal{E}_0$ denotes the peak electric field of the pulse at the location of an atom or ion and $T$ denotes the duration of the pulse then the field of the pulse is assumed to be homogeneous over a region of the $x−y$ plane of large diameter, $d$, as compared to $\mathcal{E}_0 T^2$, centered at the location of the atom or ion. Note that $\mathcal{E}_0 T^2$ has the dimension of length. This assumption partially justifies to use the dipole approximation.

The Hamiltonian generating the time evolution of the electron in the atom or ion then only depends on the electric field, $E(t)$, at the location of the atomic (or ionic) nucleus; ($t$ denotes time). The vector $E(t)$ can be chosen to have the form

$$E(t) = \mathcal{E}_0(t) \left( \cos \left[ \omega \left( t - \frac{T}{2} \right) \right], \epsilon \sin \left[ \omega \left( t - \frac{T}{2} \right) \right], 0 \right)$$

(1.2)
where $E_0(t)$ is a smooth envelope function with support in the interval $[0, T]$, $\omega = 2\pi c/\mathcal{L}$ is the angular frequency of the pulse (with $\mathcal{L} \ll c T$), and $\epsilon$ is a parameter describing the elliptical polarization of the pulse. To be concrete, we choose $E_0(t)$ to be non-negative, symmetric-decreasing about $t = T/2$, with a maximum, $E_0(T/2) =: E_0$, at $t = T/2$.

Apparently, the pulse arrives at the location of the nucleus at time $t = 0$ and lasts until time $t = T$. An important quantity is the vector potential

$$A(t) = \int_0^t d\tau E(\tau). \quad (1.3)$$

Clearly, $A(t) = 0$, for $t \leq 0$, and $A(t) \equiv A(T)$, for $t \geq T$. For our choice of the envelope function $E_0(t)$,

$$A(T) = \text{const} \cdot E_0(1, 0, 0), \quad (1.4)$$

where the constant depends on $\omega$ and on $E_0(t)$; it tends to 0 rapidly, as $\omega \to \infty$, i.e., in the ultraviolet. In this regime, the Keldysh parameter $\gamma$ becomes very large, and the analysis presented in our paper is not applicable. It does, however, apply to the situation where $\text{const} \cdot E_0$, in Eq. (1.4), becomes large, meaning that $\gamma$ becomes small.

To anticipate our main result, we will show that, for a laser pulse of the form in Eq. (1.2),

(i) the ionization probability approaches unity, as $E_0 \to \infty$ (with a rate that will be estimated explicitly), and

(ii) the electron is ejected by the pulse into a cone with axis parallel to $A(T)$ and a small opening angle $\Theta = \Theta(E_0)$; its average velocity $v = v(E_0)$ is approximately parallel to $A(T)$. Moreover,

$$\Theta(E_0) \to 0, \text{ as } E_0 \to \infty, \quad (1.5)$$

(with a rate that will be estimated), and

$$v(E_0) \parallel A(T), \text{ as } E_0 \to \infty, \quad (1.6)$$

with $|v(E_0)| \propto E_0$.

These theoretical results are in good qualitative agreement with the experimental findings described in [1,2]. In the experiments, the motion of the ions after ionization is measured. However, by momentum conservation, such measurements also determine the motion of the electron.

In [1], data compatible with Eqs. (1.5) and (1.6) are interpreted as saying that the ionization process is nearly instantaneous. This interpretation is based, implicitly, on arguments that rely on the “Ritz Hamiltonian” for the motion of the electron:

$$H_{\text{Ritz}}(t) = -\Delta - \frac{Z}{|x|} - E(t) \cdot x. \quad (1.7)$$

$\Delta$ is the Laplacian, $Z$ is the charge of the nucleus, and $E(t)$ is the electric field of the laser pulse at the location of the nucleus, see Eq. (1.2). Here we work in units such that $\hbar = 1, m_{el} = 1/2$ and $e = 1$, where $m_{el}$ is the mass of an
electron and \( e \) is the elementary electric charge. Therefore, in our units, the numerical value of the speed of light, \( c \), is around 137. Hereafter, we follow the convention that the dimension of a physical quantity is a function of the length only, namely: \([\text{length}] = \text{length} \); \([\text{mass}] = \text{length}^{-1} \); \([\text{time}] = \text{length} \); the electric charge is dimensionless.

At a fixed moment, \( t = t_0 \), of time, the potential

\[ U_{t_0}(x) := -\frac{Z}{|x|} - E(t_0) \cdot x \]  

(1.8)

has a shape indicated in Fig. 1.

Initially, the electron is localized near the nucleus placed at the origin, \( O \), of our coordinate system and treated as static for the duration of the tunneling process. If \( E(t) \) depends slowly on time \( t \), i.e., for rather large pulse duration \( T \) and long wave lengths, one may expect that an adiabatic approximation for the description of the tunneling process of the electron through the barrier of the potential \( U_{t_0}(x) \) to the point \( x_T \) (see Fig. 1) is appropriate. If \( \Delta t_T \) denotes the barrier traversal time, the electric field acting on the (nearly free) electron, after it has traversed the barrier, is given by \( E(t) \), with \( t \geq t_0 + \Delta t_T \). If we interpret \( t_0 = 0 \) as the time of onset of barrier traversal then the electron, after barrier traversal will be ejected in a direction roughly parallel to the vector

\[ X := \int_{t_0 + \Delta t_T}^{T} d\tau E(\tau). \]  

(1.9)

For a pulse described by Eq. (1.2) and a strictly positive barrier traversal time, \( \Delta t_T \), the direction of \( X \) in which the electron is ejected is not parallel to the direction of \( A(T) \) (parallel to the \( x \)-axis, for our concrete choice of an envelope function \( E_0(t) \)). By tuning the direction of \( A(T) \) and measuring the direction in which the electrons are ejected, one can determine the angle, \( \phi \), between \( X \) and \( A(T) \). This angle then provides information on the barrier traversal time \( \Delta t_T \). Experimentally, \( \phi \) is very small, so that \( \Delta t_T \) is argued to be very short.

![Figure 1. The potential \( U_{t_0}(x) \)](image-url)
The analysis presented in this paper shows that, for large $E_0$, $\phi$ is small. We have found the Ritz Hamiltonians in Eq. (1.7) to be rather inconvenient for an analysis of ionization processes. It is advantageous to, instead, consider the “Kramers Hamiltonians”

$$H(t) = (p - A(t))^2 - \frac{Z}{|x|},$$

where $p = -i\nabla$ is the usual electron momentum operator and $A(t)$ is the vector potential at the location of the nucleus given in Eq. (1.3). The evolutions generated by $H_{Ritz}(t)$ see (1.7) and $H(t)$, as in (1.10), are related to each other by a time-dependent gauge transformation given by

$$\Lambda(x, t) := A(t) \cdot x.$$  

(1.11)

If $(E(T) \cdot x, 0)$ denotes the 4-vector potential before the gauge transformation (1.11) is made then, after this gauge transformation, it is given by $(0, A(t))$. Quantum mechanically, the gauge equivalence of the time evolutions generated by the Ritz Hamiltonians, Eq. (1.7), and the Kramers Hamiltonians, Eq. (1.10), can easily be verified using the Trotter product formula (see, e.g., [7]) for the propagators and the identity

$$e^{-i\Lambda(x, t)H(t)}e^{i\Lambda(x, t)} = p^2 - \frac{Z}{|x|},$$

(1.12)

with $H(t)$ as in (1.10).

Next, we sketch some key ideas in our analysis of the time evolution generated by the Kramers Hamiltonians. As an initial condition, $\psi_0$, for the electron we choose a bound state wave function, typically the atomic groundstate. In our units, it has a spatial spread of order $O(Z^{-1})$. The quantum-mechanical propagator generated by the Kramers Hamiltonians $H(t)$, defined in Eq. (1.10), is denoted by $U(t, t_0) = U(t, t_0; Z)$. It evolves an electronic wave function from time $t_0$ to time $t$ and solves the equation

$$i\partial_t U(t, t_0; Z) = H(t)U(t, t_0; Z),$$

(1.13)

with $U(t_0, t_0; Z) = 1$, for any arbitrary $t_0$; see [8]. We note that the propagator $U_0(t, t_0) \equiv U(t, t_0; Z = 0)$ can be calculated explicitly:

$$U_0(t, t_0) = \exp \left[ -i \int_{t_0}^{t} (p - A(\tau))^2 d\tau \right]$$

(1.14)

$$= e^{i\phi(t, t_0)} e^{-i(t-t_0)p^2} \exp \left[ 2ip \cdot \int_{t_0}^{t} A(\tau) d\tau \right].$$

(1.15)

The first factor on the R. S. of (1.15) is a pure phase factor (with $\phi(t, t_0) = -\int_{t_0}^{t} A(\tau)^2 d\tau$), the second factor is the free time evolution, and the third factor is a space translation by the vector $2\int_{t_0}^{t} A(\tau) d\tau$.

As our initial time, we choose $t_0 = 0$, and the initial condition at $t = 0$ is chosen to be $\psi_0$, as described above. The laser pulse hits the atom at time $t = 0$ and lasts up to time $T$. Because of the space translation,
in the free propagator (1.15), which moves the initial wave function, \( \psi_0 \), far out of the potential well (described by \(-Z/|x|\)), provided \( \mathcal{E}_0 \) (the peak electric field) is large, one expects that

\[
U(T, 0; Z) \psi_0 \approx U_0(T, 0) \psi_0, \tag{1.17}
\]

with an error term that tends to 0, as \( \mathcal{E}_0 \to \infty \). Results of this type have first been proven by Fring, Kostrykin and Schrader in [9]. We will reproduce their results in Sect. 2, below.

As noted in (1.3),

\[
A(t) = A(T), \text{ for } t \geq T, \tag{1.18}
\]

i.e., the vector potential is constant when the pulse has passed. We may therefore use a gauge transformation to remove it:

\[
e^{-i\Lambda(x,T)} U(t, T; Z) e^{i\Lambda(x,T)} = U_C(t, T), \text{ for all } t \geq T, \tag{1.19}
\]

where \( U_C(t, T) = \exp[-i(t - T)H_C] \), and

\[
H_C := p^2 - \frac{Z}{|x|} \tag{1.20}
\]

is the Coulomb Hamiltonian.

Next we note that, by Eq. (1.11),

\[
e^{-i\Lambda(x,T)} = e^{-iA(T) \cdot x}, \tag{1.21}
\]

i.e., \( e^{-i\Lambda(x,T)} \) is a translation in momentum space: it translates \( \widehat{\psi}_T(p) \) to

\[
\widehat{\psi}_{A(T)}(p) := \widehat{\psi}_T(p + A(T)), \tag{1.22}
\]

where

\[
\psi_T(x) = (U(T, 0; Z) \psi_0)(x), \tag{1.23}
\]

and \( \widehat{\psi}_T \) is the Fourier transform of \( \psi_T \). An electron in the state given by \( \psi_{A(T)} \), see Eq. (1.22), has a mean distance from the nucleus of order \( O(\sqrt{TZ}) \) and a mean velocity in the direction of \( A(T) \) of magnitude \( |A(T)| \). Thus, the mean distance of \( \psi_{A(t)} \) from the nucleus and the mean velocity of the electron, parallel to \( A(T) \), diverge, as the peak electric field, \( \mathcal{E}_0 \), of the pulse tends to \( \infty \). However, by Eqs. (1.17) and (1.15), the spread of the wave function \( \psi_{A(t)} \) in \( x \)-space around its mean position is of order \( O(TZ) \), which is independent of \( \mathcal{E}_0 \). It is then almost obvious that, for \( t \geq T \),

\[
U(t, 0; Z) \psi_0 = U(t, T; Z) \psi_T
= e^{i\Lambda(x,T)} U_C(t, T) \psi_{A(T)}
\approx e^{i\Lambda(x,T)} e^{-i(t - T)p \cdot \psi_{A(T)}},
\]

with an error term that tends to 0, as \( \mathcal{E}_0 \to \infty \), uniformly in \( t \geq T \). This will be proven mathematically in Sect. 2.2., below. The phase factor, \( e^{i\Lambda(x,T)} \), on
the R.S. of (1.26) is unimportant. Moreover, \( \exp \left[ -i (t - T)p^2 \right] \psi_{A(T)} \) is the free time evolution of an electron wave function initially located at a distance of order \( \mathcal{O}(|\int_0^T A(\tau) d\tau|) \) from the nucleus and with a mean velocity parallel to \( A(T) \) and of magnitude \( |A(T)| \). Its spread in the direction perpendicular to \( A(T) \) is of order \( \mathcal{O}(tZ) \), which is independent of \( \mathcal{E}_0 \). Thus, the state \( U(t, 0; Z)\psi_0 \) propagates into a cone with axis parallel to \( A(T) \) and with an opening angle of order \( \mathcal{O}(Z/|A(T)|) \), which tends to 0, as \( \mathcal{E}_0 \to \infty \).

In the technical sections of this paper, these claims are verified mathematically, and the asymptotics in \( 1/\mathcal{E}_0 \) is estimated quite carefully. This is crucial, because the Kramers Hamiltonians \( H(t) \) of Eq. (1.10) do not capture the physics of the ionization process correctly for very large values of \( \mathcal{E}_0 \), for the following reasons:

1. Non-relativistic kinematics for the electron is justified in our study of the ionization process only if the (mean) electron speed after ionization, \( |A(T)| \), is small compared to the speed of light, \( c \) (with \( c \approx 137 \), in our units). If this condition is violated relativistic kinematics would have to be employed, and electron-positron pair creation by the laser pulse in the Coulomb field of the nucleus would have to be incorporated in our analysis, i.e., the whole process would have to be studied by using methods of relativistic QED.

2. The dipole approximation used in the Hamiltonians defined in Eqs. (1.7) and (1.10) can only be justified under the following conditions:

   (i) The wave length \( \mathcal{L} \) and the spatial extension, \( Tc \), of the laser pulse in the propagation direction (here the \( z \)-axis) must be large, as compared to the spatial spread in the \( z \)-direction of the electron wave function at time \( t = T \), which is of order \( \mathcal{O}(TZ) \). It follows right away that \( Z \ll 137 \), i.e., our analysis only applies to light atoms, such as Hydrogen or Helium, which, of course, was to be expected. Thus, we must impose that

   \[ TZ \ll \mathcal{L} \ll Tc. \tag{1.27} \]

   (ii) In order to justify neglecting the spatial dependence of the vector potential, \( A(x, t) \), of the laser pulse in the Pauli-Fierz Hamiltonian

   \[ H_{PF}(t) := (p - A((x, t))^2 - \frac{Z}{|x|} \tag{1.28} \]

   that should be used in our analysis, instead of the Kramers Hamiltonian, Eq. (1.10), the laser pulse must be spatially homogenous in the \( x \)- and the \( y \)-directions up to a distance \( d \) from the nucleus large compared to the mean distance of the electron from the nucleus at time \( T \), which is given by \( 2|\int_0^T A(\tau) d\tau| \).

   (iii) Finally, terms like \( |A(x, t)^2 - A(0, t)^2| \) should be small in the tales of the electron wave function, \( \psi_t \), for all times. These conditions are satisfied if \( Z \ll 137 \) and if \( \mathcal{E}_0 \) is fairly small compared to \( Z \); e.g., \( Z \) and \( \mathcal{E}_0 \) of order 1.
Since our analytical methods only yield asymptotics in $1/\mathcal{E}_0$, we would be lucky if our results gave reliable information about the ionization process for $\mathcal{E}_0$ of order 1, (i.e., $\gamma \approx 1$), corresponding to the experimental situation and needed to justify the dipole approximation. More precise quantitative information can presumably only be obtained from extensive numerical simulations.

Yet, it is gratifying to note that our results are in good qualitative agreement with the experimental findings. Moreover, our analysis, which is based on the Kramers Hamiltonian in Eq. (1.10), suggests that naive calculations of “barrier traversal times” based on an adiabatic approximation to the Ritz Hamiltonians, Eq. (1.7), may not yield reliable results.

2. Description of the Theoretical Setup

We consider an electron bound to a nucleus by a static potential $V(x)$ and under the influence of a laser pulse described, in the Coulomb gauge, by the time dependent vector potential $A(t)$, which we assume to be independent of $x$. The Hamiltonian is given by

$$H(t) = (p - A(t))^2 + V(x)$$

and acts on the Hilbert space $L^2(\mathbb{R}^3)$. Here $p = -i\nabla$ is the momentum operator. We denote by $U(t,s)$ the propagator generated by the time-dependent Hamiltonian $H(t)$, that is

$$i\partial_t U(t,s) = H(t)U(t,s), \quad \text{with} \quad U(s,s) = 1 \quad \text{for all } s \in \mathbb{R}. \quad (2.1)$$

2.1. The Pulse

We consider a pulse with amplitude $\lambda$ lasting for a time $T > 0$. We will be interested here in fixed $T$ and large $\lambda$.

The electric component of the pulse is given by

$$E(t) = \frac{\lambda}{T} f(t/T)$$

for a vector valued function $f : \mathbb{R} \to \mathbb{R}^3$, with $\text{supp } f \subset [0,1]$. (In Sec. 1, we have used the notation $\mathcal{E}_0 \simeq \lambda/T$). The vector potential $A(t)$ is then given by

$$A(t) = \int_{-\infty}^{t} ds \, E(s) = \lambda F(t/T)$$

with

$$F(s) = \int_{-\infty}^{s} d\tau f(\tau).$$

By definition $F(s) = 0$, for all $s < 0$, and $F(s) = F(1)$, for all $s \geq 1$. 
The time integral of the vector potential will also play an important role in our analysis. We set

$$G(s) = \int_{-\infty}^{s} d\tau F(\tau).$$

Then

$$\int_{-\infty}^{t} A(s)ds = \lambda T G(t/T).$$

By definition $G(s) = 0$, for all $s < 0$, and $G(s) = G(1) + (s - 1)F(1)$, for all $s > 1$.

**Assumptions on pulse.** We assume that

$$|G(s)|^{-1} \in L^1((s_0, 1)), \quad \text{for all } 0 < s_0 < 1. \quad (2.2)$$

Moreover, we assume that

$$F(1) \neq 0 \quad (2.3)$$

and that

$$|G(s)| \geq C_s, \quad \text{for all } s \geq 1. \quad (2.4)$$

Assuming that $F(1) \neq 0$, this last condition is satisfied if $F(1) \cdot G(1) \geq 0$; in other words, if the angle between $F(1)$ and $G(1)$ is less or equal to $\pi$. In fact, for arbitrary $s \geq 1$,

$$|G(s)|^2 = |G(1) + (s - 1)F(1)|^2$$

$$= |G(1)|^2 + (s - 1)^2|F(1)|^2 + 2(s - 1)G(1) \cdot F(1)$$

$$\geq |G(1)|^2 + (s - 1)^2|F(1)|^2$$

$$\geq \frac{\min(|G(1)|^2, |F(1)|^2)}{2} s^2.$$ 

**Examples.** A simple example of a pulse satisfying the assumptions (2.3), (2.4) is obtained by setting

$$f(s) = \varepsilon 1(0 \leq s \leq 1)$$

for a fixed polarization vector $\varepsilon \in \mathbb{R}^3$ (pulse with linear polarization). Then $F(s) = 0$, for $s \leq 0$, $F(s) = \varepsilon s$, for $s \in [0, 1]$, and $F(s) = \varepsilon$ for $s \geq 1$. This gives $G(s) = 0$ for $s \leq 0$, $G(s) = (s^2/2)\varepsilon$ for $s \in [0, 1]$, $G(s) = (s - 1/2)\varepsilon$ for $s \geq 1$. Another example is a pulse with modulated circular polarization. If the polarization is perpendicular to the $z$-axis, such a pulse is described by

$$f(s) = h(s)(\cos(\omega(s - 1/2)), \sin(\omega(s - 1/2)), 0)$$

where $h(s) \geq 0$ is symmetric decreasing about $s = 1/2$, with supp $h \subset [0, 1]$. If the effect of the pulse does not average out to zero, it is simple to check that, in this case, too, the conditions (2.3) and (2.4) are satisfied; see Sect. 1.
2.2. The Potential

To describe the coupling of the electron to the nucleus, we consider a static potential \( V(x) \). We distinguish two sets of assumptions on the potential \( V \).

**Short range potential.** We assume that there is a constant \( V_0 \) (with \( |V_0| = \text{length}^{-1} \)), a length scale \( D > 0 \), and an \( \alpha > 0 \) such that

\[
|V(x)| \leq V_0D \frac{1}{|x|} \frac{1}{(1 + (x/D)^2)^{\alpha/2}}. \tag{2.5}
\]

From the physical point of view, it is important to also cover an attractive Coulomb potential.

**Coulomb potential.**

\[
V(x) = -\frac{Z}{|x|}, \quad Z > 0. \tag{2.6}
\]

2.3. The Initial Wave Function

We require exponential decay of the wave function \( \psi \) and of its first and second derivatives. In other words, we assume that

\[
|\psi(x)| \leq CR^{-3/2}e^{-|x|/R}, \quad |\nabla \psi(x)| \leq CR^{-5/2}e^{-|x|/R},
\]

\[
|\Delta \psi(x)| \leq CR^{-5/2}e^{-|x|/R}(R^{-1} + |x|^{-1}) \tag{2.7}
\]

for some \( R > 0 \) and some dimensionless constants \( C \).

Moreover, we will also need decay in momentum space. We assume that

\[
|\hat{\psi}(p)| \leq CR^{3/2} \frac{1}{(1 + (Rp)^2)^{\gamma/2}} \tag{2.8}
\]

for a dimensionless constant \( C \), and for some \( \gamma > 3/2 \).

2.4. The Observable

For fixed \( \delta, \theta > 0 \), we are interested in the probability that the electron is ejected in the direction \( G(t/T) \) of the pulse (with \( G(t/T) \to F(1) \), as \( t/T \to \infty \)). To this end, we propose to estimate the norm

\[
N(t) = \|\chi_{\delta,\theta}(t)U(t,0)\psi\|,
\]

where the propagator \( U(t,0) \) is defined in (2.1), and

\[
\chi_{\delta,\theta}(t) = 1(|x| \geq \delta t) 1(x \cdot G(t/T) \geq |x||G(t/T)| \cos \theta)
\]

for some fixed positive \( \delta, \theta \), with \( \theta > 0 \) arbitrarily small. We will prove that if the dimensionless quantity \( R\lambda \) is sufficiently large the norm \( N(t) \) can be made arbitrarily close to one. Note that our results are uniform in time \( t \). In particular, they hold in the limit of large \( t/T \). We observe that, for large \( t/T \), the direction of \( G(t/T) \) approaches the direction of \( F(1) \); in other words, the vector \( F(1) \) determines the direction in which the electron propagates asymptotically, after ionization, in the limit of large \( R\lambda \).
3. Results and Proofs

3.1. Short Range Potentials

We begin our analysis by considering an interaction potential decaying faster than Coulomb. That is, we assume, in this subsection, that \( V \) satisfies condition (2.5), for some \( \alpha > 0 \).

**Notation.** Throughout the paper, \( C \) will denote a universal constant, independent of the parameters \( \lambda, T, R, D, V_0 \) characterizing the pulse, the initial wave function, and the interaction potential.

**Remark.** Note that, with our conventions, \([T] = [D] = [R] = [\lambda] = \text{length}^{-1}\), and \( Z = Ze^2 \) is dimensionless. We have chosen the numerical value \( m_{el} = 1/2 \) for the electron mass. Therefore, in the formulae below, \( t \) stands for \( t/2m_{el} \), \( T \) stands for \( T/2m_{el} \), \( \delta \) stands for \( 2\delta m_{el} \), \( V_0 \) stands for \( 2V_0m_{el} \), and (in Section 3.2) \( Z \) stands for \( 2Zm_{el} \).

**Theorem 3.1.** Assume that conditions (2.2), (2.3), (2.4), (2.5) for some \( \alpha > 0 \), (2.7), (2.8) for some \( \gamma > 5/2 \), are satisfied. Then we have that, uniformly in \( t \geq T \),

\[
\| \chi_{\delta, \theta}(t)U(t, 0)\psi \| \geq 1 - C \left[ \frac{1}{R(C\lambda - \delta)} + \frac{1}{R\lambda \tan \theta} \right] \left[ 1 + \frac{R^2}{t} \right] - \frac{CV_0T}{\alpha(\lambda T/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right] - CV_0DR \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda
\]

where the dimensionless quantity \( \kappa_\lambda \) is given by

\[
\kappa_\lambda = \inf_{0 < s_0 < 1} \left\{ \frac{T}{R^2 s_0} + \frac{1}{R\lambda} \int_{s_0}^{1} \frac{d\tau}{G(\tau)} \left[ 1 + \frac{1}{\tau^2} \right] \right\}.
\]

**Remark.** It follows from assumption (2.2) that \( \kappa_\lambda \to 0 \), as \( R\lambda \to \infty \). In fact, it follows from (2.2) that the function \( K(s_0) = \int_{s_0}^{1} d\tau |G(\tau)|^{-1}(1 + \tau^{-2}) \) is finite, for all \( s_0 > 0 \). Clearly, \( K(s_0) \) is monotonically decreasing in \( s_0 \) and can therefore be inverted. Typically \( K(s_0) \to \infty \), as \( s_0 \to 0 \). Thus, for \( R\lambda \) large enough, we can choose \( s_0 = K^{-1}((R\lambda)^{1/2}) \). Then \( s_0 \to 0 \) and \( K(s_0)(\lambda R)^{-1} \to 0 \), as \( R\lambda \to \infty \). To have more precise information about how fast \( \kappa_\lambda \) tends to zero, as \( R\lambda \to \infty \), one needs more information about the pulse.

**Example.** If \( f(s) = \varepsilon \), for a fixed \( \varepsilon \in \mathbb{R}^3 \), for all \( s \in [0, 1] \), and \( f(s) = 0 \) for \( s \notin [0, 1] \), it follows that \( F(s) = s \varepsilon \) and \( G(s) = (s^2/2)\varepsilon \), for all \( s \in [0, 1] \). Then we find that

\[
\kappa_\lambda = \inf_{s_0 \in (0, 1)} \left\{ \frac{T}{R^2 s_0} + \frac{2}{R\lambda s_0} \frac{1}{\tau} \left[ 1 + \frac{1}{\tau^2} \right] \right\} \leq \inf_{s_0 \in (0, 1)} \left\{ \frac{T}{R^2 s_0} + \frac{2}{R\lambda s_0} + \frac{2}{3R\lambda s_0^3} \right\}.
\]
It is easy to check that the infimum is attained at
$$s_0^2 = (R/T\lambda)(1 + \sqrt{1 + 2T\lambda/R}).$$
For $R\lambda \gg 1$ (and $R^2/T \simeq 1$), the infimum is attained at $t_0 \simeq (2TR^5/\lambda)^{1/4}$ and is given by
$$\kappa_\lambda \simeq \frac{4}{3} \left( \frac{2T^3}{R^7\lambda} \right)^{1/4}.$$

Remark. It follows from Theorem 3.1, that, as $t \to \infty$, the electron will propagate, with probability approaching one, as $R\lambda \to \infty$, into the cone with an opening angle smaller than an arbitrary $\theta > 0$ around the direction of $F(1)$. In other words, with $\tilde{\chi}_{\delta, \theta}(t) = 1(|x| \geq \delta t) 1(x \cdot F(1) \geq |x||F(1)|\cos \theta)$, we find
$$\lim \inf_{t \to \infty} \| \tilde{\chi}_{\delta, \theta}(t) U(t, 0) \psi \| \geq 1 - C \left[ \frac{1}{R(C\lambda - \delta)} + \frac{1}{R\lambda\tan \theta} \right]$$
$$- \frac{CV_0T}{(\lambda T/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right] - CV_0DR \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda. \quad (3.2)$$

To prove (3.2), observe that $\| 1(x \cdot G(t/T) \geq |x||G(t/T)|\cos \theta) \psi \| \geq \| 1(x \cdot F(1) \geq |x||F(1)|\cos (\theta/2)) \psi \|$, if the angle between $G(t/T)$ and $F(1)$ is smaller than $\theta/2$. Since $G(t/T) = G(1) + (t/T - 1)F(1)$, the angle between $G(t/T)$ and $F(1)$ is certainly smaller than $\theta/2$, for sufficiently large $t/T \gg 1$.

To prove Theorem 3.1, we first show how the evolution up to time $T$ can be approximated by the evolution generated by the time dependent Kramers Hamiltonian without potential. The next lemma is due to Fring, Kostrykin and Schrader; see [9].

Lemma 3.2. Let $U_0(t, s) = e^{-i \int_s^t d\tau (p - A(\tau))^2}$. Assume that conditions (2.3), (2.4), (2.7), (2.8), and (2.5), for some $\alpha \geq 0$, are satisfied; (for $\alpha = 0$ we recover the Coulomb potential (2.6) by taking $V_0D = Z$). Then there exists a constant $C$ such that
$$\| (U(T, 0) - U_0(T, 0)) \psi \| \leq CV_0DR \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda$$
with $\kappa_\lambda$ defined in (3.1).

Proof. We define the new propagator
$$\tilde{U}(s, 0) = e^{-2i p \cdot \int_s^0 d\tau A(\tau)} e^{i \int_s^0 d\tau A^2(\tau)} U(s, 0).$$
Then
$$i \frac{d}{ds} \tilde{U}(s, 0) = \tilde{H}(s) \tilde{U}(s, 0)$$
with
$$\tilde{H}(s) = p^2 + V(x - 2\lambda TG(s/T)).$$
Since
\[ U(T, 0) - U_0(T, 0) = e^{2i\lambda T p \cdot G(1)} e^{-i \int_0^T d\tau A^2(\tau)} \left( \tilde{U}(T, 0) - e^{-i T p^2} \right) \]
we find that
\[
\| (U(T, 0) - U_0(T, 0)) \psi \| = \left\| \left( \tilde{U}(T, 0) - e^{-i T p^2} \right) \psi \right\|
\leq \int_0^T ds \left\| V(x - 2\lambda T G(s/T))e^{-isp^2} \psi \right\|. \tag{3.3}
\]
Now, we observe that, on one hand, by (2.5),
\[
\left\| V(x - 2\lambda T G(s/T))e^{-isp^2} \psi \right\| \leq 2V_0 D \int dx |\nabla \psi(x)|^2 \leq C \frac{V_0 D}{R}. \tag{3.4}
\]
On the other hand, by Lemma 3.3 (see (3.12) below), we have that
\[
\left\| V(x - 2\lambda T G(s/T))e^{-isp^2} \psi \right\| \leq \frac{CV_0 D}{\lambda T |G(s/T)|} \left[ 1 + \frac{R^4}{s^2} \right].
\]
(We are neglecting here the factor \((1 + \lambda^2 T^2 |G(s/T)|^2/D^2)^{-\alpha/2}\) on the r.h.s. of (3.12). This factor will play an important role for large times; here it would just give a faster decay in \(\lambda\).) Thus
\[
\| (U(T, 0) - U_0(T, 0)) \psi \| \leq \int_0^{t_0} ds \frac{C \frac{V_0 D}{R}}{\lambda T |G(s/T)|} \left[ 1 + \frac{R^4}{s^2} \right] + \int_{t_0}^T ds \frac{CV_0 D}{\lambda T |G(s/T)|} \left[ 1 + \frac{R^4}{s^2} \right], \tag{3.5}
\]
for arbitrary \(t_0 \in [0, T]\), and hence
\[
\| (U(T, 0) - U_0(T, 0)) \psi \| \leq C V_0 D \left[ 1 + \frac{R^4}{T^2} \right]
\times \inf_{0 < t_0 < T} \left\{ \frac{t_0}{R} + \frac{1}{\lambda} \int_{t_0/T}^{1} \frac{d\tau}{|G(\tau)|} \left[ 1 + \frac{1}{\tau^2} \right] \right\}. \]

Proof of Theorem 3.1. We begin by writing
\[
\chi_{\delta, \theta}(t) U(t, 0) \psi = \chi_{\delta, \theta}(t) U(t, T) U(T, 0) \psi
\]
\[
= \chi_{\delta, \theta}(t) U(t, T) \left( U(T, 0) - e^{-i \int_0^T d\tau (p - A(\tau))^2} \right) \psi
\]
\[
+ \chi_{\delta, \theta}(t) U(t, T) e^{-i \int_0^T d\tau (p - A(\tau))^2} \psi.
\]
Therefore, by Lemma 3.2,
\[
\|\chi_{\delta,\theta}(t)U(t,0)\psi\| \geq \|\chi_{\delta,\theta}(t)U(t,T) e^{-i 0^T \tau(p-A(\tau))^2} \psi\| \\
- \|(U(T,0) - e^{-i 0^T \tau(p-A(\tau))^2}) \psi\| \\
\geq \|\chi_{\delta,\theta}(t)U(t,T) e^{-i 0^T \tau(p-A(\tau))^2} \psi\| \\
- CV_0 DR \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda,
\]
with \( \kappa_\lambda \) defined in (3.1). Since \( A(t) = A(T) \), for all \( t > T \), we obtain that
\[
\|\chi_{\delta,\theta}(t)U(t,0)\psi\| \geq \|\chi_{\delta,\theta}(t) e^{-(t-T)\left[(p-A(T))^2 + V(x)\right]} e^{-i 0^T \tau(p-A(\tau))^2} \psi\| \\
- CV_0 DR \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda. \quad (3.6)
\]
Next, we notice that
\[
\|\chi_{\delta,\theta}(t) e^{-i(t-T)\left[(p-A(T))^2 + V(x)\right]} e^{-i 0^T \tau(p-A(\tau))^2} \psi\| \\
\geq \|\chi_{\delta,\theta}(t) e^{-i 0^T \tau(p-A(\tau))^2} \psi\| \\
- \left\| e^{-i(t-T)\left[(p-A(T))^2 + V(x)\right]} - e^{-i(t-T)\left[(p-A(T))^2\right]} \right\| e^{-i 0^T \tau(p-A(\tau))^2} \psi\| \\
\geq \|\chi_{\delta,\theta}(t) e^{-i 0^T \tau(p-A(\tau))^2} \psi\| - \int_0^{t-T} ds \left\| V(x) e^{-i 0^T + s \tau(p-A(\tau))^2} \psi\right\|. \quad (3.7)
\]
We then use
\[
\int_0^{t-T} ds \left\| V(x) e^{-i 0^T + s \tau(p-A(\tau))^2} \psi\right\| \\
= \int_0^{t-T} ds \left\| V(x) e^{-i (T+s)p^2 } e^{2i \lambda T p G(1+s/T)} \psi\right\| \\
= \int_0^{t-T} ds \left\| V(x - 2\lambda T G(1+s/T)) e^{-i (T+s)p^2 } \psi\right\|. \quad (3.8)
\]
To bound the integrand, we observe that, by Lemma 3.3 (see (3.12) below),
\[
\left\| V(x - 2\lambda T G(1+s/T)) e^{-i (T+s)p^2 } \psi\right\| \leq \frac{CV_0}{(\lambda T |G((1+s/T))|/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right]
\]
for all $s \geq 0$. Hence, by (2.4),
\[
\int_0^{t-T} ds \left\| V(x - 2\lambda TG(1 + s/T))e^{-i(T+s)p^2} \psi \right\|
\leq \frac{CV_0}{(\lambda T/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right] \int_0^{t-T} ds \frac{1}{|G(1 + s/T)|^{1+\alpha}}.
\]
\[
\leq \frac{CV_0 T}{(\lambda T/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right] \int_0^\infty d\tau \frac{1}{(1 + \tau)^{1+\alpha}}.
\]
\[
\leq \frac{CV_0 T}{\alpha(\lambda T/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right].
\]

Therefore, from (3.7),
\[
\left\| \chi_{s,\theta}(t)e^{-i(t-T)|[p-A(T)]^2 + V(x)|}e^{-i \int_0^T d\tau (p-A(\tau))^2} \psi \right\|
\geq \left\| \chi_{s,\theta}(t)e^{-i \int_0^T d\tau (p-A(\tau))^2} \psi \right\| - \frac{CV_0 T}{\alpha(\lambda T/D)^{1+\alpha}} \left[ 1 + \frac{R^4}{T^2} \right]. \tag{3.9}
\]

The first term on the right hand side of (3.9) can be bounded by
\[
\left\| \chi_{s,\theta}(t)e^{-i \int_0^T d\tau (p-A(\tau))^2} \psi \right\|
\geq 1 - \left\| 1(|x| \leq \delta t)e^{2\lambda TpG(t/T)}e^{-itp^2} \psi \right\|
- \left\| 1(|x - G(t/T)| \leq |x||G(t/T)| \cos \theta)e^{2\lambda TpG(t/T)}e^{-itp^2} \psi \right\|
\geq 1 - \left\| 1(|x - 2\lambda TG(t/T)| \leq \delta t)e^{-itp^2} \psi \right\|
- \left\| 1(|x - 2\lambda TG(t/T)| \leq |x - 2\lambda TG(t/T)||G(t/T)| \cos \theta) \right. 
\times e^{-itp^2} \psi \left. \right\|. \tag{3.10}
\]

Since, by (2.4), $|G(s)| \geq Cs$ for all $s \geq 1$, we find that
\[
\left\| \chi_{s,\theta}(t)e^{-i \int_0^T d\tau (p-A(\tau))^2} \psi \right\| \geq 1 - \left\| 1(|x| \geq (C\lambda - \delta)t)e^{-itp^2} \psi \right\|
- \left\| 1(|x| \geq C\lambda t \tan \theta)e^{-itp^2} \psi \right\|. \tag{3.11}
\]

To conclude, we observe that
\[
\left\| 1(|x| \geq Kt)e^{-itp^2} \psi \right\|^2 \leq \frac{1}{(Kt)^2} \left\langle e^{-itp^2} \psi, x^2e^{-itp^2} \psi \right\rangle
\leq \frac{1}{(Kt)^2} \left\langle \psi, (x + 2tp)^2 \psi \right\rangle
\leq \frac{2}{(Kt)^2} \left\langle \psi, (x^2 + 4t^2p^2) \psi \right\rangle
\leq \frac{C}{(Kt)^2} (R^2 + t^2 R^{-2}) \leq \frac{C}{(KR)^2} \left( 1 + \frac{R^4}{t^2} \right)
\]
using (2.7), and (2.8) for some $\gamma > 5/2$. Hence, (3.11) yields
\[
\|\chi_{\delta,\theta}(t) e^{-i \int_0^t d\tau (p-A(\tau))^2} \psi\| \geq 1 - C \left[ \frac{1}{R(C\lambda - \delta)} + \frac{1}{R\lambda \tan \theta} \right] \left[ 1 + \frac{R^2}{t} \right].
\]
Together with (3.6) and (3.9), this concludes the proof of the theorem. \hfill \Box

**Lemma 3.3.** Assume (2.5), for some $\alpha > 0$, and (2.7), (2.8), for some $\gamma > 3/2$. Then
\[
\|V(x - 2\lambda T G(t/T)) e^{-i tp^2} \psi\| \leq \frac{C V_0 D}{\lambda T |G(t/T)|} \left( 1 + \frac{\lambda^2 T^2 |G(t/T)|^2}{D^2} \right)^{\alpha/2} \left[ 1 + \frac{R^4}{t^2} \right].
\]  

**Proof.** We notice that
\[
(e^{-i tp^2} \psi)(x) = \frac{e^{ix^2/4t}}{(4\pi it)^{3/2}} \int dy e^{-iy \cdot x/2t} e^{iy^2/4t} \psi(y)
\]
\[
= \frac{e^{ix^2/4t}}{(4\pi it)^{3/2}} \hat{\psi}(x/2t)
\]
\[
+ \frac{e^{ix^2/4t}}{(4\pi it)^{3/2}} \int dy e^{-iy \cdot x/2t} \left( e^{iy^2/4t} - 1 \right) \psi(y).
\]
In the second term, we perform integration by parts to obtain decay in the $x$-variable.
\[
\int dy e^{-iy \cdot x/2t} \left( e^{iy^2/4t} - 1 \right) \psi(y)
\]
\[
= -\int dy \frac{\Delta_y e^{-iy \cdot x/2t}}{(x/2t)^2} \left( e^{iy^2/4t} - 1 \right) \psi(y)
\]
\[
= -\frac{1}{(x/2t)^2} \int dy e^{-iy \cdot x/2t}
\]
\[
\times \left[ (\Delta \psi)(y) \left( e^{iy^2/4t} - 1 \right) + i(\nabla \psi)(y) \cdot \frac{y}{2t} e^{iy^2/4t} + \psi(y) \left( -\frac{y^2}{4t^2} + \frac{3i}{2t} \right) e^{iy^2/4t} \right].
\]
Therefore, we obtain that
\[
\left| \int dy e^{-iy \cdot x/2t} \left( e^{iy^2/4t} - 1 \right) \psi(y) \right|
\]
\[
\leq \frac{1}{(x/2t)^2} \int dy \left\{ |\Delta \psi(y)| \frac{|y|^2}{4t} + |\nabla \psi(y)| \frac{|y|}{t} + |\psi(y)| \left( \frac{3}{2t} + \frac{|y|^2}{4t^2} \right) \right\}
\]
\[
\leq C \frac{R^{3/2}}{t} \frac{1}{(x/t)^2} \left( 1 + \frac{R^2}{t} \right).
\]
Since, on the other hand,
\[
\left| \int dy e^{-iy \cdot x/2t} \left( e^{iy^2/4t} - 1 \right) \psi(y) \right| \leq CR^{3/2} \frac{R^2}{t},
\]
it follows that
\[
\left| \int \mathrm{d}y e^{-iyx/2t} \left( e^{iy^2/4t} - 1 \right) \psi(y) \right| \leq CR^{3/2} \frac{R^2}{t} \frac{1}{1 + (Rx/t)^2} \left( 1 + \frac{R^2}{t} \right).
\]

Hence, using (2.5) and (2.8),
\[
\left\| V(x - 2\lambda TG(t/T)) e^{-itp^2} \psi \right\|^2 \leq CV^2 D^2 \left( 1 + \frac{R^4}{t^2} \right)^2 \left\| \frac{\hat{\psi}(x/2t)}{t^3} \right\|^2
\]
\[
+ CR^3 \frac{R^4}{t^2} \left( 1 + \frac{R^2}{t} \right)^2 \int \mathrm{d}x \frac{V^2(x - 2\lambda TG(t/T))}{(1 + (Rx/t)^2)^2} \psi(x) \psi(x)
\]
\[
\leq CV^2 D^2 \int \frac{\mathrm{d}x}{(1 + (Rx)^2)^{-2}} \frac{1}{|G(t/T)|^2} \left( 1 + 4t^2 D^2 \right) \left| x - \lambda(T/t)G(t/T) \right|^2 \alpha
\]
\[
+ \frac{CV^2 D^2}{t^2} \frac{R^3}{t^2} \left( 1 + \frac{R^2}{t} \right)^2 \left[ 1 + \frac{R^4}{t^2} \left( 1 + \frac{R^2}{t} \right)^2 \right] \left| x - 2R \lambda(T/t)G(t/T) \right|^2 \alpha
\]
\[
\times \left[ 1 + (1 + x^2)^{-\beta} \right] \left[ 1 + \frac{1 + x^2}{t^2} \right] \left[ 1 + \frac{1 + x^2}{t^2} \right] \left| G(t/T) \right|^2 \alpha
\]
where \( \beta = \min(\gamma, 2) > 3/2 \). It follows that
\[
\left\| V(x - 2\lambda TG(t/T)) e^{-itp^2} \psi \right\|^2 \leq \frac{CV^2 D^2}{\lambda^2 T^2 |G(t/T)|^2} \left( 1 + \frac{\lambda^2 T^2}{D^2} |G(t/T)|^2 \right) \left[ 1 + \frac{R^4}{t^2} \right]^2.
\]

\[\square\]

3.2. Coulomb Potentials

In this section we consider the physically more interesting case of a Coulomb interaction. The long range of the Coulomb potential requires some modification of the argument used in the previous section; in particular, to obtain results uniform in time, we need to approximate the long time evolution by a “Dollard-modified” free dynamics (see [10]).

As initial data we consider here the ground state of the Schrödinger operator with an attractive Coulomb interaction, which satisfies the assumptions (2.7), and (2.8), with \( \gamma = 4 \). (In the following theorem we will therefore assume (2.8) with \( \gamma = 4 \); but, of course, other values of \( \gamma \) can also be considered.)

**Theorem 3.4.** Assume that conditions (2.3), (2.4), (2.6), and (2.8), for \( \gamma = 4 \), are satisfied. Suppose that there exists a constant \( C \) such that \( C^{-1} \leq R^2/T \leq \)
$C$, that $Z \leq \lambda$, and that $\lambda R \geq 1$ is large enough. Then we have that, uniformly in $t \geq T$,

$$
\|\chi_{\delta, \theta}(t) U(t, 0) \psi\| \geq 1 - C \left( \frac{1}{R \lambda \tan \theta} + \frac{1}{R(C\lambda - \delta)} \right) \left( 1 + \frac{R^2}{t} \right) - Z R \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda - \frac{C}{(R \lambda)^{1/7}} \left( \frac{Z T^{3/2}}{R^2} \right)^{4/7}
$$

where the dimensionless quantity $\kappa_\lambda$ was defined in (3.1). Since, by assumption (2.2), $\kappa_\lambda \to 0$, as $(\lambda R) \to \infty$, it follows in particular that

$$
\lim_{\lambda R \to \infty} \|\chi_{\delta, \theta}(t) U(t, 0) \psi\| = 1
$$

uniformly in $t \geq T$.

**Remark.** Just like Theorem 3.1, Theorem 3.4 implies that

$$
\lim_{t \to \infty} \|\tilde{\chi}_{\delta, \theta}(t) U(t, 0) \psi\| \geq 1 - C \left( \frac{1}{R \lambda \tan \theta} + \frac{1}{R(C\lambda - \delta)} \right) \left( 1 + \frac{R^2}{t} \right) - Z R \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda - \frac{C}{(R \lambda)^{1/7}} \left( \frac{Z T^{3/2}}{R^2} \right)^{4/7}
$$

where $\tilde{\chi}_{\delta, \theta}(t) = 1(|x| \geq t\delta)\mathbf{1}(x \cdot F(1) \geq |x||F(1)| \cos \theta)$. In other words, it is the vector $F(1)$ that determines, with probability approaching one, as $R \lambda \to \infty$, the direction in which the electron propagates asymptotically.

**Proof.** By Lemma 3.2, we have that

$$
\|\chi_{\delta, \theta}(t) U(t, 0) \psi\| \geq \|\chi_{\delta, \theta}(t) U(t, T) e^{-i \int_0^T d\tau (p - A(\tau))^2} \psi\| - C Z R \left[ 1 + \frac{R^4}{T^2} \right] \kappa_\lambda.
$$

(3.13)

To replace the unitary evolution $U(t, T)$ by a free evolution, we introduce, first of all, a cutoff in momentum space. We choose a smooth function $\chi \in C_0^\infty(\mathbb{R}^3)$, with $\chi(x) = 0$ for all $|x| \geq 1$ and $\chi(x) = 1$ for all $|x| \leq 1/2$. We define $\bar{\chi} = 1 - \chi$. Then we have

$$
\|\chi_{\delta, \theta}(t) U(t, T) e^{-i \int_0^T d\tau (p - A(\tau))^2} \psi\| \geq \|\chi_{\delta, \theta}(t) U(t, T) e^{-i \int_0^T d\tau (p - A(\tau))^2} \chi(p/K_0) \psi\| - \|\bar{\chi}(p/K_0) \psi\| - \frac{C}{(R K_0)^{\gamma - 3/2}} (3.14)
$$
for arbitrary $K_0 > 0$; we will later optimize the choice of $K_0$. Next, we let

$$\psi_T = e^{-i T p^2} \chi(p/K_0) \psi,$$

and we observe that

$$\chi \delta \theta (t) \mathcal{U}(t, T) e^{-i \int_0^T ds (p - A(s))} \chi(p/K_0) \psi$$

$$= \chi \delta \theta (t) e^{-i (t-T)} [p - A(T)]^2 Z/|x| e^{2i \lambda T p \cdot G(1)} e^{-i \int_0^T ds A^2(s)} \psi_T$$

$$= e^{i x \cdot A(T)} \chi \delta \theta (t) e^{-i (t-T)} [p^2 - Z/|x|] e^{-i x \cdot A(T)} e^{2i \lambda T p \cdot G(1)} e^{-i \int_0^T ds A^2(s)} \psi_T$$

$$= e^{-i \int_0^T ds A^2(s)} e^{2i \lambda T \mathcal{G}(1) \cdot A(1)} e^{i x \cdot A(T)} \chi \delta \theta (t) e^{2i \lambda T p \cdot G(1)}$$

$$\times e^{-i (t-T)} [p^2 - Z/|x-2 \lambda T \mathcal{G}(1)|] e^{-i x \cdot A(T)} \psi_T$$

and we write

$$e^{-i (t-T) [p^2 - Z/|x-2 \lambda T \mathcal{G}(1)|]} e^{-i x \cdot A(T)} \psi_T$$

$$= e^{-i (t-T) p^2} e^{i Z \int_0^T \mathrm{d} \tau} e^{-i x \cdot A(T)} \psi_T$$

$$+ \left[ e^{-i (t-T) [p^2 - Z/|x-2 \lambda T \mathcal{G}(1)|]} - e^{-i (t-T) p^2} e^{i Z \int_0^T \mathrm{d} \tau} \right]$$

$$\times e^{-i x \cdot A(T)} \psi_T.$$

(3.15)

Observe here that $\psi_T = \chi(p/K_0) e^{-i T p^2} \psi$ is supported, in momentum space, in the ball of radius $K_0$ around the origin. This implies that $|p + \lambda F(1)| \leq K_0$ for all $p$ in the support of the Fourier transform of $e^{-i x \cdot A(T)} \psi_T$. Therefore $|2 \tau p - 2 \lambda T \mathcal{G}(1)| \geq 2 \lambda T |G(1 + \tau/T)| - 2 \tau K_0 \leq C \lambda T + (C \lambda - K_0) \tau$ for all $\tau \in [0, T]$. In particular, if we require that $K_0 \leq C \lambda/2$, the integral $\int_0^T \mathrm{d} \tau |2 \tau p - 2 \lambda T \mathcal{G}(1)|^{-1}$ is well defined (at the end, we will choose $K_0 R \simeq (\lambda R)^{2/35}$, and therefore the condition $K_0 \leq C \lambda/2$ is certainly satisfied for sufficiently large values of $(\lambda R)$). It follows that

$$\| \chi \delta \theta (t) \mathcal{U}(t, T) e^{i \int_0^T ds (p - A(s))} \chi(p/K_0) \psi \|$$

$$\geq \| \chi \delta \theta (t) e^{2i \lambda T p \cdot G(1)} e^{-i (t-T) p^2} e^{i Z \int_0^T \mathrm{d} \tau} e^{-i x \cdot A(T)} \psi_T \|$$

$$- \left\| \left[ e^{-i (t-T) [p^2 - Z/|x-2 \lambda T \mathcal{G}(1)|]} - e^{-i (t-T) p^2} e^{i Z \int_0^T \mathrm{d} \tau} \right]$$

$$\times e^{-i x \cdot A(T)} \psi_T \right\|.$$

(3.17)

To bound the first term, we observe that

$$\| \chi \delta \theta (t) e^{2i \lambda T p \cdot G(1)} e^{-i (t-T) p^2} e^{i Z \int_0^T \mathrm{d} \tau} e^{-i x \cdot A(T)} \psi_T \|$$

$$= \| \chi \delta \theta (t) e^{2i \lambda T (p - A(T))} e^{-i (t-T) (p-A(T))^2} e^{i Z \int_0^T \mathrm{d} \tau} e^{-i (t-T) p^2} e^{i Z \int_0^T \mathrm{d} \tau} \psi_T \|$$

$$= \| \chi \delta \theta (t) e^{2i \lambda T p \cdot G(t/T)} e^{-i T p^2} e^{i Z \int_0^T \mathrm{d} \tau} \chi(p/K_0) \psi \|.$$  

(3.18)
From (3.18), we obtain

\[
\left\| \chi_{\delta, \theta}(t)e^{2i\lambda T_{p}G(1)}e^{-i(t-T)p^{2}}e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_{T} \right\| \\
= \left\| 1((x - 2\lambda T G(t/T)) \cdot G(t/T) \geq |x - 2\lambda T G(t/T)| |G(t/T)| \cos \theta \right\| \\
\times 1(|x - 2\lambda T G(t/T)| \geq \delta t) e^{-itp^{2}} e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1 + \tau / T)|}} \chi(p/K_{0}) \psi \right\| \\
\geq \left\| \chi(p/K_{0}) \psi \right\| \\
- \left\| 1(|x - 2\lambda T G(t/T)| \leq \delta t) e^{-itp^{2}} e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1 + \tau / T)|}} \chi(p/K_{0}) \psi \right\| \\
- \left\| 1(|x - 2\lambda T G(t/T)) \cdot G(t/T) \leq |x - 2\lambda G(t/T)||G(t/T)| \cos \theta \right\| \\
\times e^{-itp^{2}} e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1 + \tau / T)|}} \chi(p/K_{0}) \psi \right\| \\
\geq 1 - (RK_{0})^{-5/2} \\
- \left\| 1(|x| \geq (C\lambda - \delta)t) e^{-itp^{2}} e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1)|}} \chi(p/K_{0}) \psi \right\| \\
- \left\| 1(|x| \geq C\lambda t \tan \theta) e^{-itp^{2}} e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1 + \tau / T)|}} \chi(p/K_{0}) \psi \right\|.
\]

From Lemma 3.5, below, we find that

\[
\left\| \chi_{\delta, \theta}(t)e^{2i\lambda T_{p}G(1)}e^{-i(t-T)p^{2}}e^{iZ \int_{0}^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_{T} \right\| \\
\geq 1 - (RK_{0})^{-5/2} - C \left( \frac{1}{R\lambda \tan \theta} + \frac{1}{R(C\lambda - \delta)} \right) \left( 1 + \frac{R^{2}}{t} \right). \tag{3.19}
\]

As for the second term on the r.h.s. of (3.17), we use the bound

\[
\left\| e^{-i(t-T)p^{2}} e^{iZ \int_{0}^{t} \frac{dt}{|2\tau p - 2\lambda T G(1)|}} \right\| \\
\leq Z \int_{0}^{t-T} ds \left\| \frac{1}{|x - 2\lambda T G(1)|} - \frac{1}{|2sp - 2\lambda TG(1)|} \right\| \\
\times e^{-isp^{2}} e^{iZ \int_{0}^{s} \frac{dt}{|2\tau p - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_{T} \right\|. \tag{3.20}
\]

We first handle small values of \(s \in [0, t - T]\). To this end, we observe that

\[
\left\| \frac{1}{|x - 2\lambda T G(1)|} e^{-isp^{2}} e^{iZ \int_{0}^{s} \frac{dt}{|2\tau p - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_{T} \right\| ^{2} \\
= \left\| \frac{1}{|x - 2\lambda T G(1)|} e^{-ix \cdot A(T)} e^{-is(p - A(T))^{2}} e^{iZ \int_{0}^{s} \frac{dt}{|2\tau p - 2\lambda T G(1 + \tau / T)|}} \psi_{T} \right\| ^{2} \\
\leq \int \frac{dx}{|x - 2\lambda T G(1)| ^{2}} \left\| e^{-is(p - A(T))^{2}} e^{iZ \int_{0}^{s} \frac{dt}{|2\tau p - 2\lambda T G(1 + \tau / T)|}} \psi_{T} (x) \right\| ^{2} \\
\leq 4 \int_{|p| \leq K_{0}} dp |p|^{2} \left\| \hat{\psi}(p) \right\| ^{2} \leq CR^{-2}
\]
using (2.8), with $\gamma = 4$. On the other hand, we have that
\[
\left\| \frac{1}{|2sp - 2\lambda T G(1)|} e^{-isp^2} e^{iZ \int_0^s \frac{dx}{|2sp - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_T \right\|
\]
\[
= \left\| \frac{1}{|2sp - 2\lambda T G(1 + s/T)|} \chi(p/K_0) \psi \right\|
\]
\[
\leq \frac{1}{2\lambda T |G(1 + s/T)| - sK_0} \leq \frac{1}{C\lambda T}
\]
for all $s \in [0, t]$, if $K_0 < C\lambda /2$; here we used the assumption (2.4). Therefore
\[
\left\| \left[ \frac{1}{|x - 2\lambda G(T)|} - \frac{1}{|2sp - 2\lambda T G(1)|} \right] e^{-isp^2} e^{iZ \int_0^s \frac{dx}{|2sp - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_T \right\|
\]
\[
\leq C \left( \frac{1}{R} + \frac{1}{\lambda T} \right) \leq CR^{-1}
\]
assuming $\lambda T \geq R$. In conclusion
\[
\int_0^{t-T} ds \left[ \frac{1}{|x - 2\lambda T G(1)|} - \frac{1}{|2sp - 2\lambda T G(1)|} \right]
\]
\[
\times e^{-isp^2} e^{iZ \int_0^s \frac{dx}{|2sp - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_T \right\|
\]
\[
\leq Ct_0 \frac{1}{R} + \int_{t_0}^{t-T} ds \left[ \frac{1}{|x - 2\lambda T G(1)|} - \frac{1}{|2sp - 2\lambda T G(1)|} \right]
\]
\[
\times e^{-isp^2} e^{iZ \int_0^s \frac{dx}{|2sp - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_T \right\|
\]
(3.21)
for an arbitrary $t_0 > 0$.

To estimate the second term, we use the kernel representation
\[
(e^{-isp^2} \psi)(x) = \frac{1}{(4\pi is)^{3/2}} \int dy e^{i(x-y)^2/4s} \psi(y)
\]
implying that
\[
\frac{1}{|x - 2\lambda T G(1)|} \left( e^{-isp^2} e^{iZ \int_0^s \frac{dx}{|2sp - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_T \right)(x)
\]
\[
= \frac{e^{ix^2/4s}}{(4\pi is)^{3/2}|x - 2\lambda T G(1)|} e^{iZ \int_0^s \frac{dx}{|x/s - 2\lambda T G(1)|}} \hat{\psi}_T(x/2s + \lambda F(1))
\]
\[
+ \tilde{R}_\lambda^{(1)}(s, x),
\]
with
\[
\tilde{R}_\lambda^{(1)}(s, x) = \frac{e^{ix^2/4s}}{(4\pi is)^{3/2}|x - 2\lambda T G(1)|}
\]
\[
\times \int dy e^{-iy \cdot x/2s} \left( e^{iy^2/4s} - 1 \right) \left( e^{iZ \int_0^s \frac{dx}{|2sp - 2\lambda T G(1)|}} e^{-ix \cdot A(T)} \psi_T \right)(y).
\]
Similarly, we notice that
\[
\frac{1}{2sp - 2\lambda TG(1)} e^{-isp^2} e^{iZ \int_{T_0}^x e^{is(T)} e^{-ix \cdot A(T)} \psi_T(x)}
\]
\[
= \frac{e^{ix^2/4s}}{(4\pi is)^{3/2}} \left| x - 2\lambda TG(1) \right| e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} \psi_T(x/2s + \lambda F(1))
\]
\[+ R^{(2)}_\lambda(s, x),
\]
with
\[
R^{(2)}_\lambda(s, x) = \frac{e^{ix^2/4s}}{(4\pi is)^{3/2}} \int dy e^{-iy \cdot x/2s} \left( e^{iy^2/4s} - 1 \right)
\]
\[
\times \left( \frac{1}{2sp - 2\lambda TG(1)} e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} e^{-ix \cdot A(T)} \psi_T \right)(y).
\]
From (3.20), we find that
\[
\int_{t_0}^{t-T} ds \left\| \left| \frac{1}{|x - 2\lambda TG(1)|} - \frac{1}{sp - 2\lambda TG(1)} \right| \right. \]
\[
\times e^{-isp^2} e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} e^{-ix \cdot A(T)} \psi_T \left\| \right.
\]
\[
\leq \int_{t_0}^{t-T} ds \left( \| R^{(1)}_\lambda(s, x) \| + \| R^{(2)}_\lambda(s, x) \| \right).
\]
To control the first remainder term, we compute
\[
\left( e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} e^{-ix \cdot A(T)} \psi_T \right)(y)
\]
\[
= \frac{1}{(2\pi)^{3/2}} \int dke^{ik \cdot y} e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} \psi_T(k + \lambda F(1))
\]
\[
= \frac{e^{-i\lambda F(1) \cdot y}}{(2\pi)^{3/2}} \int dke^{ik \cdot y} e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} \psi_T(k).
\]
Hence
\[
R^{(1)}_\lambda(s, x) = \frac{e^{ix^2/4s}}{(8\pi^2 is)^{3/2} |x - 2\lambda TG(1)|}
\]
\[
\times \int dy e^{-iy \cdot (x/2s + \lambda F(1))} \left( e^{iy^2/4s} - 1 \right) h_\lambda(s, y)
\]
with
\[
h_\lambda(s, y) = \int dke^{ik \cdot y} e^{iZ \int_{T_0}^x \frac{dx^2}{2s - 2\lambda TG(1)}} \psi_T(k).
\]
In Lemma 3.6, below, we show that, for every multi-index \( \beta \in \mathbb{N}^3 \),
\[
|D^\beta_x h_\lambda(s, x)| \leq CR^{-3/2} K_0^{1/|\beta|} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2(1 + s/T) \right] .
\]
Therefore, on the one hand,

\[
|R^{(1)}(s, x)| \leq \frac{C}{s^{5/2}|x - 2\lambda G(T)|} \int dy |y|^2 |h_\lambda(s, y)|
\]

\[
\leq \frac{CR^{7/2}}{s^{5/2}|x - 2\lambda G(T)|} \left( \frac{T K_0}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^{2n}(1 + s/T) \right]
\]

(3.23)

for all \( n > 5/2 \). On the other hand, from

\[
R^{(1)}(s, x) = \frac{e^{ix^2/4s}}{(8\pi^2i s)^{3/2}|x - 2\lambda TG(1)|}
\]

\[
\times \int dy \left[ \Delta_y e^{-iy \cdot (x / 2s + \lambda F(1))} \right] \left[ \frac{e^{iy^2/4s}}{2m} - 1 \right] h_\lambda(s, y)
\]

we find by integrating by parts that

\[
|R^{(1)}(s, x)| \leq \frac{C}{s^{3/2}|x - 2\lambda TG(1)||x / 2s + \lambda F(1)|^{2m}}
\]

\[
\times \sum_{|\alpha| + |\beta| = 2m} \int dy \left| D^\alpha \left( e^{iy^2/4s} - 1 \right) \right| \left| D^\beta h_\lambda(s, y) \right|
\]

Using that

\[
|D^\alpha(e^{iy^2/4s} - 1)| \leq \frac{C}{s^r} \left( 1 + \frac{|y|^{2r}}{s^r} \right)
\]

(3.24)

if \(|\alpha| = 2r, r \geq 1\), and that

\[
|D^\alpha(e^{iy^2/4s} - 1)| \leq \frac{C|y|}{s^r} \left( 1 + \frac{|y|^{2(r-1)}}{s^{r-1}} \right)
\]

(3.25)

if \(|\alpha| = 2r - 1, r \geq 1\), we arrive at

\[
|R^{(1)}(s, x)|
\]

\[
\leq \frac{CR^{-3/2}K_0^{2m}}{s^{3/2}|x - 2\lambda TG(1)||x / 2s + \lambda F(1)|^{2m}} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^{2n}(1 + s/T) \right]
\]

\[
\times \left\{ \int dy \frac{|y|^2}{s} \frac{1}{1 + (|y|/R)^{2n}}
\right.
\]

\[
+ \sum_{r=1}^{m} \frac{1}{(K_0^2 s)^r} \int dy \left( 1 + \frac{|y|^{2r}}{s^r} \right) \frac{1}{1 + (|y|/R)^{2n}}
\]

\[
+ \sum_{r=1}^{m} \frac{1}{(K_0^2 s)^r} \int dy (K_0 |y|) \left( 1 + \frac{|y|^{2(r-1)}}{s^{r-1}} \right) \frac{1}{1 + (|y|/R)^{2n}} \right\},
\]
where the first term in the parenthesis corresponds to $|\alpha| = 0$, the second to $|\alpha| = 2r$ and the third to $|\alpha| = 2r - 1$. It follows that

$$
|R^{(1)}_{\lambda}(s, x)| \leq \frac{C R^{3/2} K_0^{2m}}{s^{3/2}|x - 2\lambda T G(1)||x/2s + \lambda F(1)|^{2m}} \left( \frac{K_0 T}{R} \right)^{2n} \\
\times \left[ 1 + \frac{Z}{\lambda} \log^n(1 + s/T) \right] \\
\times \left\{ \frac{R^2}{s} + \frac{R^2}{s} \sum_{r=1}^{m} \frac{1}{(RK_0)^{2r-1}} \left( 1 + \left( \frac{R^2}{s} \right)^{2r} \right) \right\} (3.26)
$$

for all $n > m + 3/2$, and all $m \geq 1$. Combining this bound with (3.23), we find that

$$
|R^{(1)}_{\lambda}(s, x)| \leq \frac{C R^{7/2}}{s^{5/2}|x - 2\lambda T G(1)|(1 + (|x/2s + \lambda F(1)|/K_0)^{2m})} \left( \frac{K_0 T}{R} \right)^{2n} \\
\times \left[ 1 + \frac{Z}{\lambda} \log^n(1 + s/T) \right] \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right)
$$

for all $n > m + 3/2$, and all $m \geq 1$. We thus conclude that

$$
\|R^{(1)}_{\lambda}(s, x)\| \leq \frac{C R^{7/2}}{s^{5/2}} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^n(1 + s/T) \right] \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right) \\
\times \left( \int \frac{dx}{|x - 2\lambda T G(1)|^{2(1 + (|x/2s + \lambda F(1)|/K_0)^{2m})^2}} \right)^{1/2}. (3.27)
$$

Since

$$
\int \frac{dx}{|x - 2\lambda T G(1)|^2 (1 + (|x/2s + \lambda F(1)|/K_0)^{2m})^2} = 2sK_0 \int \frac{dx}{|x - \frac{\lambda T G(1+s/T)}{sK_0}|^2(1 + x^{2m})^2} \leq \frac{C(sK_0)^3}{\lambda^2 T^2 |G(1 + s/T)|^2}
$$

we find that

$$
\|R^{(1)}_{\lambda}(s, x)\| \leq \frac{C R^{7/2} K_0^{3/2}}{\lambda s T |G(1 + s/T)|} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^n(1 + s/T) \right] \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right) (3.28)
$$
for any $m \geq 1$. Since $|G(1 + s/2T)| \geq C(1 + s/2T)$, we find

$$
\int_{t_0}^{t_\cdot T} ds \| R^{(1)}_\lambda (s, x) \| \leq \frac{CR^{7/2} K^{3/2}_0}{\lambda T} \left( \frac{K_0 T}{R} \right)^{2n} \times \int_{t_0}^\infty \frac{ds}{s(1 + s)} \left[ 1 + \frac{Z}{\lambda} \log^2 (1 + s) \right] \left( 1 + \left( \frac{R^2}{T} \right)^{2m} \frac{1}{s^{2m}} \right)
$$

(3.29)

for any $m \geq 1$ and $n > m + 3/2$, and any $\varepsilon > 0$.

To control the second remainder term on the r.h.s. of (3.22), we write

$$
\left( \frac{1}{|2sp - 2\lambda T G(1)|} \right)^{e^{\mu} f_0^{\mu} (\mu T - \Delta T G(1))} e^{-ix \cdot A(T) \psi_T} (y) = \frac{1}{(2\pi)^3/2} \int \frac{dk}{|2sk - 2\lambda T G(1)|} e^{ik \cdot y} e^{iZ f_0^{\mu} (\mu T - \Delta T G(1))} \hat{\psi}_T (k + \lambda F(1)) = e^{-i\lambda F(1) \cdot y} \int \frac{dk}{(2\pi)^3/2} \frac{e^{ik \cdot y} e^{iZ f_0^{\mu} (\mu T - \Delta T G(1))}}{|2sk - 2\lambda T G(1 + s/T)|} \hat{\psi}_T (k).
$$

Hence

$$
R^{(2)}_\lambda (s, x) = \frac{e^{ix^2/4s}}{(8\pi^2 is)^{3/2}} \int dy \ e^{-iy \cdot (x/2 + \lambda F(1))(e^{iy^2/4s} - 1)} g_\lambda (s, y)
$$

with

$$
g_\lambda (s, y) = \int \frac{dk}{|2sk - 2\lambda T G(1)|} e^{ik \cdot y} e^{iZ f_0^{\mu} (\mu T - \Delta T G(1 + s/T))} \hat{\psi}_T (k).
$$

In Lemma 3.7, we show that for every multi-index $\beta \in \mathbb{N}^3$,

$$
|D^\beta_x g_\lambda (s, x)| \leq \frac{C}{\lambda (T + s)} \left( \frac{TK_0}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2 (1 + s/T) \right].
$$

Therefore, on the one hand

$$
|R^{(2)}_\lambda (s, x)| \leq \frac{C}{s^{5/2}} \int dy |y|^2 |g_\lambda (s, y)| \leq \frac{CR^{7/2}}{\lambda s^{5/2}(s + T)} \left( \frac{TK_0}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2 (1 + s/T) \right]
$$

(3.30)

for all $n > 3/2$. On the other hand, from

$$
R^{(2)}_\lambda (s, x) = \frac{e^{ix^2/4s}}{(8\pi^2 is)^{3/2}} \int dy \frac{\Delta^m_y e^{-iy \cdot (x/2 + \lambda F(1))}}{(-1)^m x/2s + \lambda F(1)|2m(e^{iy^2/4s} - 1)} g_\lambda (s, y)
$$
we find by integrating by parts that

\[
|R_\lambda^{(2)}(s, x)| \leq \frac{C}{s^{3/2}|x/2s + \lambda F(1)|^{2m}} \times \sum_{|\alpha| + |\beta| = 2m} \left\| D^\alpha (e^{iy^2/4s} - 1) \right\| D^\beta g_\lambda(s, y).
\]

Using the bounds (3.24), (3.25), we obtain, similarly to (3.26), the bound

\[
|R_\lambda^{(2)}(s, x)| \leq CR^{7/2}K_0^{2m} \frac{\lambda s^{5/2}(s + T)}{\lambda s^{5/2}(s + T)(1 + (|x/2s + \lambda F(1)|/K_0)^{2m})} \left( \frac{K_0 T}{R} \right)^{2n} \times \left[ 1 + \frac{Z}{\lambda} \log^2(1 + s/T) \right] \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right).
\]

for all \( n > m + 3/2 \), and all \( m \geq 1 \). Combining the last bound with (3.30), we conclude that

\[
|R_\lambda^{(2)}(s, x)| \leq CR^{7/2}K_0^{2m} \frac{\lambda s^{5/2}(s + T)}{\lambda s^{5/2}(s + T)(1 + (|x/2s + \lambda F(1)|/K_0)^{2m})} \left( \frac{K_0 T}{R} \right)^{2n} \times \left[ 1 + \frac{Z}{\lambda} \log^2(1 + s/T) \right] \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right).
\]

Hence we have

\[
\|R_\lambda^{(2)}(s, x)\| \leq CR^{7/2}K_0^{3/2} \frac{\lambda s(T + s)}{\lambda s(T + s)} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2(1 + s/T) \right] \times \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right) \left( \frac{R^2}{t_0} \right)^{1/2} \left( \frac{d\lambda}{(1 + (|x/2s + \lambda F(1)|/K_0)^{2m})^2} \right)^{1/2}
\]

\[
\leq CR^{7/2}K_0^{3/2} \frac{\lambda s(T + s)}{\lambda s(T + s)} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2(1 + s/T) \right] \times \left( 1 + \left( \frac{R^2}{s} \right)^{2m} \right) \left( \frac{R^2}{t_0} \right)^{1/2}
\]

for all \( m \geq 1 \). Similarly to (3.29), this implies that

\[
\int_{t_0}^{t-T} ds \|R_\lambda^{(2)}(s, x)\| \leq C \frac{(K_0 R)^{3/2}}{\lambda} \left( \frac{K_0 T}{R} \right)^{2n} \frac{R^2}{t_0} \times \left( 1 + \left( \frac{R^2}{t_0} \right)^{2m} \right) \left( 1 + \frac{Z}{\lambda} \left( \frac{t_0}{T} \right)^{\varepsilon} \right)
\]

for any \( m \geq 1, n > m + 3/2 \), and \( \varepsilon > 0 \).
From (3.20), (3.21), (3.29), (3.32), we find that
\[
\left\| e^{-i(t-T)|p^2-Z|/x-2\lambda T G(1)|} - e^{-i(t-T)p^2 e^{i T f_{0}^{t-T} |\tau p-2\lambda T G(1)|}} e^{-ix \cdot A(T) \psi_{T}} \right\|
\leq \frac{C Z t_{0}}{R} + C \frac{Z}{\lambda} (K_{0} R)^{3/2} \left( \frac{K_{0} T}{R} \right)^{2n} \frac{R^{2}}{t_{0}} \left( 1 + \frac{R^{2}}{t_{0}} \right) \left( 1 + \frac{Z}{\lambda} \frac{(t_{0})}{T} \right)
\]
for all \( m \geq 1 \), \( n > m + 3/2 \), and \( \varepsilon > 0 \). We now choose \( m = 1 \) and \( n = 3 \), and we set
\[
t_{0} \frac{R}{2} = \left( \frac{(K_{0} R)^{3/2} (K_{0} T)^{6}}{R \lambda} \right)^{1/4}.
\]
We will choose \( K_{0} \) so that \( K_{0} R \) and \( K_{0} T / R \) are large in the limit of large \( (R \lambda) \) so that we may assume that \( t_{0} / R^{2} \leq 1 \), \( (t_{0} / T) \leq \lambda / (Z) \). Then
\[
\left\| e^{-i(t-T)|p^2-Z|/x-2\lambda T G(1)|} - e^{-i(t-T)p^2 e^{i T f_{0}^{t-T} |\tau p-2\lambda T G(1)|}} e^{-ix \cdot A(T) \psi_{T}} \right\|
\leq C Z R \left( \frac{(K_{0} R)^{3/2} (K_{0} T)^{6}}{R \lambda} \right)^{1/4}.
\]
This last bound, together with (3.19), implies that
\[
\| \chi_{\delta,\theta}(t) \mathcal{U}(t, T) e^{-i \int_{0}^{T} d \tau (p-A(\tau))^{2} \psi} \| \geq 1 - \frac{C}{R \lambda} \left( 1 + \frac{R^{2}}{t} \right) - C (R K_{0})^{-5/2}
- C Z R \left( \frac{(K_{0} R)^{3/2} (K_{0} T)^{6}}{R \lambda} \right)^{1/4}.
\]
We finally optimize our choice of \( K_{0} \) by setting \( (K_{0} T / R) = C_{0} (R \lambda)^{2/35} \), for an appropriate constant \( C_{0} \). This yields
\[
\| \chi_{\delta,\theta}(t) \mathcal{U}(t, T) e^{-i \int_{0}^{T} d \tau (p-A(\tau))^{2} \psi} \| \geq 1 - \frac{C}{R \lambda} \left( 1 + \frac{R^{2}}{t} \right)
- \frac{C}{(R \lambda)^{1/7}} \left( \frac{Z T^{3/2}}{R^{2}} \right)^{4/7},
\]
which, combined with (3.13), concludes the proof of the theorem. \( \square \)

**Lemma 3.5.** Suppose that \( \chi \in C^{\infty}_{0}(\mathbb{R}^{3}) \), with \( \chi(x) = 0 \) for all \( |x| \geq 1 \) and \( \chi(x) = 1 \) for all \( |x| \leq 1/2 \). Assume that \( C^{-1} \leq R^{2} / T \leq C \), \( Z \leq \lambda \), \( K_{0} \leq C \lambda \) for an appropriate constant \( C \), and that \( \lambda R \geq 1 \) is large enough (at the end \( (K_{0} R) \simeq (R \lambda)^{2/35} \), and therefore the condition \( K_{0} \leq C \lambda \) is satisfied for sufficiently large \( (R \lambda) \)). Then, for every \( t \geq T \), and for every constant \( D > 0 \), we have that
\[
\| 1(|x| \geq D t) e^{-it p^2} e^{i T f_{0}^{t-T} |\tau p-2\lambda T G(1)|} \chi(p / K_{0}) \psi \| \leq \frac{C}{D R} \left( 1 + \frac{R^{2}}{t} \right).
\]
(3.33)
Proof. We notice that
\[
\left\Vert 1(|x| \geq Dt) e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi \right\Vert^2 \leq \frac{1}{(Dt)^2} W^2(t)
\]
(3.34)

where
\[
W^2(t) := \left( e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi, x^2 e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi \right).
\]

Next we compute
\[
\frac{d}{dt} W^2(t) = \left\langle e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi, i \left[ p^2 + \frac{Z}{2(t-T)p - 2\lambda T G(t/T)}, x^2 \right] \right\rangle
\]
\[
\times e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi \right\rangle
\]
\[
= 2\text{Im} \left\langle e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi, x \cdot \left( 2p + 2Z(t-T) \frac{2(t-T)p - 2\lambda T G(t/T)}{|2(t-T)p - 2\lambda T G(t/T)|} \right) \right\rangle
\]
\[
\times e^{-itp^2} e^{iZ f_0^{t-T} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \chi(p/K_0) \psi \right\rangle
\]
which, using \(|2(t-T)p - 2\lambda T G(t/T)| \geq (C \lambda - K_0)t\), implies that
\[
\left| \frac{d}{dt} W^2(t) \right| \leq CW(t) \left( \frac{Z}{t \lambda^2} + ||p|\psi| \right) \leq CR^{-1} W(t)
\]
(3.35)
for all \( t > T \) (because \( \lambda R \geq 1 \), and \( Z/\lambda \leq 1 \), \( \lambda T/R \geq 1 \)). By Gronwall’s Lemma we find that
\[
W(t) \leq C(t-T)R^{-1} + W(T)
\]
(3.36)
where
\[
W^2(T) = \langle e^{-iTp^2} \chi(p/K_0) \psi, x^2 e^{-iTp^2} \chi(p/K_0) \psi \rangle.
\]

Similarly to (3.35), we find that
\[
\left| \frac{d}{dT} W^2(T) \right| \leq 2W(T)||p|\psi| \leq CR^{-1} W(T)
\]
which implies that
\[
W(T) \leq CTR^{-1} + W(0) \leq C (TR^{-1} + R)
\]
and thus, combining the last equation with (3.34) and (3.36), we obtain (3.33). □

Lemma 3.6. Let
\[
h_\lambda(s, x) = \int dk e^{ik \cdot y} e^{iZ f_0^{s} \frac{dt}{|2\tau p - 2\lambda T G(1+\tau/T)|}} \widehat{\psi}_T(k)
\]
with \( \hat{\psi}_T(k) = e^{-iTk^2}\chi(k/K_0)\psi \), with \( \chi \in C_0^\infty(\mathbb{R}^3) \) such that \( \chi(y) = 1 \) for \( |y| \leq 1/2 \) and \( \chi(y) = 0 \) for \( |y| \geq 1 \). Assume that \( R^{-1} + RT^{-1} \leq K_0 \leq C\lambda \) for an appropriate constant \( C \) (at the end, we will choose \( K_0 R \approx (R\lambda)^{2/35} \), and therefore these conditions are satisfied for large enough \( \lambda R \)). Assume also \( Z \leq \lambda \). Then, for every \( \beta \in \mathbb{N}^3 \), we have
\[
|D_x^\beta h_\lambda(s, x)| \leq \frac{CR^{-3/2}K_0^{\beta}|\beta|}{1 + (x/R)^{2n}} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^n(1 + s/T) \right].
\]

Proof. We have
\[
D_x^\beta h_\lambda(s, x) = \int dk (ik)^\beta e^{ik \cdot x} d\int_0^s e^{i2\pi x_\beta} (\tau \lambda T)^n \hat{\psi}_T(k).
\]

Hence
\[
|D_x^\beta h_\lambda(s, x)| \leq \int dk |k|^\beta \chi(k/K_0)|\hat{\psi}(k)| \leq R^{-3/2-|\beta|} \int dk |k|^\beta \chi(k/RK_0) \frac{1}{(1 + k^2)^2} \leq CR^{-3/2-|\beta|} \int_{|k| \leq 2RK_0} \frac{1}{(1 + |k|)^4-|\beta|} \leq CR^{-3/2}K_0^{\beta}\chi(k/K_0)|\hat{\psi}(k)|.
\]

Integrating by parts in (3.37), we arrive at
\[
D_x^\beta h_\lambda(s, x) = \int dk (ik)^\beta \frac{\Delta_n e^{ik \cdot x}}{(-1)^n |x|^{2n}} d\int_0^s e^{i2\pi x_\beta} (\tau \lambda T)^n \hat{\psi}_T(k) = \int dk \frac{e^{ik \cdot x}}{(-1)^n |x|^{2n}} \Delta_n [ (ik)^\beta e^{i2\pi x_\beta} (\tau \lambda T)^n \hat{\psi}_T(k) ]
\]
and therefore
\[
|D_x^\beta h_\lambda(s, x)| \leq \frac{C}{|x|^{2n}} \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| = 2n} \int dk |k|^{\beta-|\alpha_1|} \times |D^{\alpha_2} e^{i2\pi x_\beta} f_0^{s} |d\tau T|^{\alpha_3}| D^{\alpha_3} \hat{\psi}_T(k) |.
\]

Observe that, for all \( |\alpha_2| \geq 1 \),
\[
|D^{\alpha_2} e^{i2\pi x_\beta} f_0^{s} |d\tau T|^{\alpha_3}| \leq C \sum_{|\alpha_2|} \sum_{m=1}^{\left|\alpha_2\right|} \int_{j_1,...,j_m \geq 1:j_1+...+j_m = |\alpha_2|} \prod_{i=1}^m \int_0^s \frac{d\tau T}{2\tau k - 2\lambda T G(1 + \tau/T)|j_i+1|}.
\]

Using the fact that \( |k| \leq K_0 \) on the support of \( \hat{\psi}_T \), we find \( |2\tau k - 2\lambda T G(1 + \tau/T)| \geq 2\lambda T G(1 + \tau/T)| - 2K_0 \tau \geq C\lambda T + (C\lambda - K_0) \tau \). Therefore, assuming
that $K_0 < C\lambda/2$, and that $Z < \lambda$,

\[
|D^{\alpha_2} e^{iZ} \int_0^s \frac{d\tau}{(s^2 + 2\tau T \log (1 + \tau/T))^{\frac{3}{2}}}|
\]

\[
\leq C \sum_{m=1}^{\alpha_2} \sum_{j_1, \ldots, j_m \geq 1: j_1 + \ldots + j_m = |\alpha_2|} \frac{Z}{(s^2 + 2\tau T \log (1 + \tau/T))^{\frac{3}{2}}}
\]

\[
\leq C \frac{Z}{(s^2 + 2\tau T \log (1 + \tau/T))^{\frac{3}{2}}}
\]

\[
\leq C \frac{Z}{(s^2 + 2\tau T \log (1 + \tau/T))^{\frac{3}{2}}}
\]

(3.40)

On the other hand, by a simple computation, we have

\[
|D^{\alpha_3} \tilde{\psi}_T(k)| \leq C \chi(k/K_0) \frac{R^{3/2 + |\alpha_3|}}{(1 + (RK)^2)^2}
\]

\[
\times \left( \frac{T^{1/2}}{R} (1 + |k|T^{1/2}) + \frac{1}{RK} + \frac{1}{\sqrt{1 + (RK)^2}} \right)^{|\alpha_3|}
\]

\[
\leq C \chi(k/K_0) \frac{R^{3/2}}{(1 + (RK)^2)^2} (K_0 T)^{|\alpha_3|}
\]

(3.41)

assuming that $K_0 R \geq 1$ and $K_0 T/R \geq 1$.

From (3.39), it follows that

\[
|x|^{2n} |D^\beta h_\lambda(s, x)|
\]

\[
\leq C \sum_{|\alpha_1| + |\alpha_3| = 2n} \int dk |k|^{|\beta| - |\alpha_1|} (K_0 T)^{|\alpha_3|} R^{3/2} \frac{(1 + (RK)^2)^2}{(1 + (Rk)^2)^2}
\]

\[
+ C \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| = 2n, |\alpha_2| \geq 1} \frac{Z}{(s^2 + 2\tau T \log (1 + \tau/T))^{\frac{3}{2}}}
\]

\[
\times \int_{|k| \leq K_0} dk |k|^{|\beta| - |\alpha_1|} (K_0 T)^{|\alpha_3|} R^{3/2} \frac{(1 + (RK)^2)^2}{(1 + (Rk)^2)^2}
\]

\[
\leq C \sum_{|\alpha_1| + |\alpha_3| = 2n} \frac{R^{-3/2 - |\beta| + |\alpha_1|}}{(1 + |k|^4)^4} \frac{(K_0 T)^{|\alpha_3|}}{\lambda^{|\alpha_2|}} \int_{|k| \leq RK_0} \frac{dk}{(1 + |k|^4)^4 - |\beta| + |\alpha_1|}
\]

\[
+ C \frac{Z}{\lambda} (1 + \log 2n (1 + s/T))
\]

\[
\times \sum_{|\alpha_1| + |\alpha_2| + |\alpha_3| = 2n, |\alpha_2| \geq 1} \frac{R^{-3/2 - |\beta| + |\alpha_1|}}{\lambda^{|\alpha_2|}} (K_0 T)^{|\alpha_3|} \int_{|k| \leq RK_0} \frac{dk}{(1 + |k|^4)^4 - |\beta| + |\alpha_1|}
\]
and therefore
\[ |x|^{2n} |D_x^2 h_\lambda(s, x)| \]
\[ \leq CR^{-3/2+2n-|\beta|} \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=2n} \left( \frac{K_0 T}{R} \right)^{|\alpha_3|} (1 + (K_0 R)^{1+|\beta|-|\alpha_1|+\varepsilon}) \]
\[ + C \frac{Z R^{-3/2+2n-|\beta|}}{\lambda} (1 + \log^2 n (1 + s/T)) \]
\[ \times \sum_{|\alpha_1|+|\alpha_2|+|\alpha_3|=2n} \frac{1}{(R\lambda)^{|\alpha_2|}} \left( \frac{K_0 T}{R} \right)^{|\alpha_3|} (1 + (K_0 R)^{1+|\beta|-|\alpha_1|+\varepsilon}) \]
\[ \leq CR^{-3/2+2n} K_0^{|\beta|} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2 n (1 + s/T) \right]. \]

Combining this bound with (3.38), we find, for arbitrary \( n \geq 0 \),
\[ |D_x^2 h_\lambda(s, x)| \leq \frac{CR^{-3/2} K_0^{|\beta|}}{1 + (x/R)^{2n}} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2 n (1 + s/T) \right]. \]

\[ \square \]

**Lemma 3.7.** Let
\[ g_\lambda(s, x) = \int dk e^{ik\cdot x} \frac{e^{iZ \int_0^s \frac{d\tau}{2\pi} \chi(\tau) - 2\lambda T G(1 + s/T)}}{2sk - 2\lambda T G(1 + s/T)} \tilde{\psi}_T(k) \]
with \( \tilde{\psi}_T(k) = e^{-itk^2} \chi(k/K_0) \psi \), with \( \chi \in C_0^\infty(\mathbb{R}^3) \) such that \( \chi(y) = 1 \) for \( |y| \leq 1/2 \) and \( \chi(y) = 0 \) for \( |y| \geq 1 \). Assume that \( R^{-1} + RT^{-1} \leq K_0 \leq CA \) for an appropriate constant \( C \) (at the end, we will choose \( K_0 R \approx (R\lambda)^{2/35} \), and therefore these conditions are satisfied for large enough \( \lambda R \)). Assume also \( Z \leq \lambda \). Then, for every \( \beta \in \mathbb{N}^3 \), we have
\[ |D_x^2 g_\lambda(s, x)| \leq \frac{C}{\lambda(T+s)} \frac{R^{-3/2} K_0^{|\beta|}}{1 + (x/R)^{2n}} \left( \frac{K_0 T}{R} \right)^{2n} \left[ 1 + \frac{Z}{\lambda} \log^2 n (1 + s/T) \right]. \]

**Proof.** We have
\[ D_x^2 g_\lambda(s, x) = \int dk (ik)^\beta e^{ik\cdot x} \frac{e^{iZ \int_0^s \frac{d\tau}{2\pi} \chi(\tau) - 2\lambda T G(1 + s/T)}}{2sk - 2\lambda T G(1 + s/T)} \tilde{\psi}_T(k). \]
(3.42)

Since, for \( |k| \leq K_0 \), we have \( |sk - 2\lambda T G(1 + s/2T)| \geq C\lambda(T+s) - sK_0 \geq C\lambda T + s(C\lambda - K_0) \leq C\lambda(T+s) \), we find
\[ |D_x^2 g_\lambda(s, x)| \leq \frac{C}{\lambda(T+s)} \int dk |k|^\beta \chi(k/K_0) |\tilde{\psi}(k)| \]
\[ \leq \frac{CR^{-3/2-|\beta|}}{\lambda(T+s)} \int dk |k|^\beta \chi(k/RK_0) \frac{1}{(1 + k^2)^2} \]
\[ \leq \frac{CR^{-3/2} K_0^{|\beta|}}{\lambda(T+s)}. \]
(3.43)
Integrating by parts in (3.42), we arrive at
\[ D_\beta x g_\lambda(s, x) = \int dk (ik)^\beta \Delta_n e^{ikx} \frac{\Delta_n e^{ikx}}{(1)^n|x|^{2n}} e^{iZ f_0^{\alpha_2}} e^{iZ f_0^{\alpha_3}} e^{iZ f_0^{\alpha_4}} \psi_T(k) \]
and therefore
\[ |D_\beta x h_\lambda(s, x)| \leq C|x|^{2n} \]
\[ \sum_{|\alpha_1|+\cdots+|\alpha_4|=2n} \int dk|k||\beta|-|\alpha_1||\Delta_n e^{ikx} \frac{\Delta_n e^{ikx}}{(1)^n|x|^{2n}} e^{iZ f_0^{\alpha_2}} e^{iZ f_0^{\alpha_3}} e^{iZ f_0^{\alpha_4}} \psi_T(k) | \]
\[ 2sk - 2\lambda T G(1 + s/T) ||\alpha_3|+1|D_\alpha \tilde{\psi}_T(k)|. \] (3.44)

From (3.40), (3.41), and since \(|sk - \lambda T G(1 + s/T)| \geq C\lambda(T + s)|

\[ |D_\beta x h_\lambda(s, x)| \leq \frac{CR^{3/2}}{|x|^{2n}} \]
\[ \sum_{|\alpha_1|+\cdots+|\alpha_4|=2n} \frac{(K_0T)^{|\alpha_4|}}{\lambda^{(|\alpha_3|+1)(T + s)}} \int \frac{|k||\beta|-|\alpha_1||}{(1 + (Rk)^2)^2} \]
\[ + \frac{CR^{3/2} Z}{\lambda} (1 + \log^{2n}(1 + s/T)) \]
\[ \sum_{|\alpha_1|+\cdots+|\alpha_4|=2n} \frac{\lambda^{(|\alpha_2|+|\alpha_3|+1)(T + s)}}{|\alpha_2| \geq 1} \int \frac{|k||\beta|-|\alpha_1||}{(1 + (Rk)^2)^2} \]
\[ \leq \frac{CR^{-3/2} K_0^{[\beta]}}{\lambda(T + s)(|x|/R)^{2n}} \left[ 1 + \frac{Z}{\lambda} \log^{2n}(1 + s/T) \right] \left( \frac{K_0T}{R} \right)^{2n} \]

where we used \(K_0 T/R \geq 1, R\lambda > 1, Z < \lambda\). Combining the last equation with (3.43), we conclude the proof of the lemma.

\[ \Box \]

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