Sparsest Cut on Bounded Treewidth Graphs: Algorithms and Hardness Results

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Abstract

We give a 2-approximation algorithm for Non-Uniform Sparsest Cut that runs in time \( n^{O(k)} \), where \( k \) is the treewidth of the graph. This improves on the previous \( 2^{2^k} \)-approximation in time \( \text{poly}(n) 2^{O(k)} \) due to Chlamtác et al. [CKR10].

To complement this algorithm, we show the following hardness results: If the Non-Uniform Sparsest Cut problem has a \( \rho \)-approximation for series-parallel graphs (where \( \rho \geq 1 \)), then the MaxCut problem has an algorithm with approximation factor arbitrarily close to \( 1/\rho \). Hence, even for such restricted graphs (which have treewidth 2), the Sparsest Cut problem is NP-hard to approximate better than \( 17/16 - \varepsilon \) for \( \varepsilon > 0 \); assuming the Unique Games Conjecture the hardness becomes \( 1/\alpha_{GW} - \varepsilon \). For graphs with large (but constant) treewidth, we show a hardness result of \( 2 - \varepsilon \) assuming the Unique Games Conjecture.

Our algorithm rounds a linear program based on (a subset of) the Sherali-Adams lift of the standard Sparsest Cut LP. We show that even for treewidth-2 graphs, the LP has an integrality gap close to 2 even after polynomially many rounds of Sherali-Adams. Hence our approach cannot be improved even on such restricted graphs without using a stronger relaxation.

1 Introduction

The Sparsest Cut problem takes as input a “supply” graph \( G = (V, E_G) \) with positive edge capacities \( \{\text{cap}_e\}_{e \in E_G} \), and a “demand” graph \( D = (V, E_D) \) (on the same set of vertices \( V \)) with demand values \( \{\text{dem}_e\}_{e \in E_D} \), and aims to determine

\[
\Phi_{G,D} := \min_{S \subseteq V} \frac{\sum_{e \in \partial_G(S)} \text{cap}_e}{\sum_{e \in \partial_D(S)} \text{dem}_e},
\]

where \( \partial_G(S) \) denotes the edges crossing the cut \((S, V \setminus S)\) in graph \( G \). When \( E_D = (V) \) with \( \text{dem}_e = 1 \), the problem is called Uniform Demands Sparsest Cut, or simply Uniform Sparsest Cut. Our results all hold for the non-uniform demands case.

The Sparsest Cut problem is known to be NP-hard due to a result of Matula and Shahrokhi [MS90], even for unit capacity edges and uniform demands. The best algorithm for Uniform Sparsest Cut on general graphs is an \( O(\sqrt{\log n}) \)-approximation due to Arora, Rao, and Vazirani [ARV09]; for Non-Uniform Sparsest Cut the best factor is \( O(\sqrt{\log n \log \log n}) \) due to Arora, Lee and Naor [ALN08]. An older \( O(\sqrt{\log n}) \)-approximation for Non-Uniform Sparsest Cut is known for all excluded-minor families of graphs [Rao99], and constant-factor approximations exist for more restricted classes of graphs [GNRS04, CGN+06, CJLV08, LR10, LS09, CSW10]. Constant-factor approximations are known for Uniform

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Sparsest Cut for all excluded-minor families of graphs [KPR93, Rab03]. [GS13] give a \((1 + \varepsilon)\)-approximation algorithm for non-uniform Sparsest Cut that runs in time depending on generalized spectrum of the graphs \((G, D)\). All above results, except [GS13], consider either the standard linear or SDP relaxations. The integrality gaps of convex relaxations of Sparsest Cut are intimately related to questions of embeddability of finite metric spaces into \(\ell_1\); see, e.g., [LLR95, GNRS04, KV05, KR09, LN06, CKN09, LS11, CKN11] and the many references therein. Integrality gaps for LPs/SDPs obtained from lift-and-project techniques appear in [CMM09, KS09, RS09, GSZ12]. [GNRS04] conjectured that metrics supported on graphs excluding a fixed minor embed into \(\ell_1\) with distortion \(O(1)\) (depending on the excluded minor, but independent of the graph size); this would imply \(O(1)\)-approximations to Non-Uniform Sparsest Cut on instances \((G, D)\) where \(G\) excludes a fixed minor. This conjecture has been verified for several classes of graphs, but remains open (see, e.g., [LS09] and references therein).

The starting point of this work is the paper of Chlamtác et al. [CKR10], who consider non-uniform Sparsest Cut on graphs of treewidth \(k\). They ask if one can obtain good algorithms for such graphs without answering the [GNRS04] conjecture; in particular, they look at the Sherali-Adams hierarchy. In their paper, they give a \(2^{2^k}\)-approximation in time \(\text{poly}(n) \cdot 2^{O(k)}\) by solving the \(k\)-round Sherali-Adams linear program and ask whether one can achieve an algorithm whose approximation ratio is independent of the treewidth \(k\). We answer this question in the affirmative.

**Theorem 1.1 (Easiness)** There is an algorithm for the Non-Uniform Sparsest Cut problem that, given any instance \((G, D)\) where \(G\) has treewidth \(k\), outputs a \(2\)-approximation in time \(n^{O(k)}\).

Graphs that exclude some planar graph as a minor have bounded treewidth, and \(H\)-minor-free graphs have treewidth \(O(|H|^{3/2} \sqrt{n})\). This implies a \(2\)-approximation for planar-minor-free graphs in polytime, and for general minor-free graphs in time \(2^{O(\sqrt{n})}\). In fact, we only need \(G\) has a recursive vertex separator decomposition where each separator has \(k\) vertices for the above theorem to apply.

Our algorithm is also based on solving an LP relaxation, one whose constraints form a subset of the \(O(k \log n)\)-round Sherali-Adams lift of the standard LP, and then rounding it via a natural propagation rounding procedure. We show that further applications of the Sherali-Adams operator (even for a polynomial number of rounds) cannot do better:

**Theorem 1.2 (Tight Integrality Gap)** For every \(\varepsilon > 0\), there are instances \((G, D)\) of the Non-Uniform Sparsest Cut problem with \(G\) having treewidth 2 (a.k.a. series-parallel graphs) for which the integrality gap after applying \(r\) rounds of the Sherali-Adams hierarchy still remains \(2 - \varepsilon\), even when \(r = n^\delta\) for some constant \(\delta = \delta(\varepsilon) > 0\).

This result extends the integrality gap lower bound for the basic LP on series-parallel graphs shown by Lee and Raghavendra [LR10], for which Chekuri, Shepherd and Weibel gave a different proof [CSW10].

On the hardness side, Ambühl et al. [AMS11] showed that if Uniform Sparsest Cut admits a PTAS, then SAT has a randomized sub-exponential time algorithm. Chawla et al. [CKK+06] and Khot and Vishnoi [KV05] showed that Non-Uniform Sparsest Cut is hard to approximate to any constant factor, assuming the Unique Games Conjecture. The only \(\text{APX}\)-hardness result (based on \(P \neq \text{NP}\)) for Non-Uniform Sparsest Cut is recent, due to Chuzhoy and Khanna [CK09, Theorem 1.4]. Their reduction from \(\text{MaxCut}\) shows that the problem is \(\text{APX}\)-hard even when \(G\) is \(K_{2,n}\), and hence of treewidth or even pathwidth 2. (This reduction was rediscovered by Chlamtác, Krauthgamer, and Raghavendra [CKR10].) We extend their reduction to show the following hardness result for the Non-Uniform Sparsest Cut problem:

**Theorem 1.3 (Improved NP-Hardness)** For every constant \(\varepsilon > 0\), the Non-Uniform Sparsest Cut problem is hard to approximate better than \(\frac{17}{16} - \varepsilon\) unless \(P = \text{NP}\) and hard to approximate better than

\[^1\text{We emphasize that only the supply graph } G \text{ has bounded treewidth; the demand graphs } D \text{ are unrestricted.}\]
\(1/\alpha_{GW} - \varepsilon\) assuming the Unique Games Conjecture, even on graphs with treewidth 2 (series-parallel graphs).

Our proof of this result gives us a hardness-of-approximation that is essentially the same as that for MAXCUT (up to an additive \(\varepsilon\) loss). Hence, improvements in the NP-hardness for MAXCUT would translate into better NP-hardness for Non-Uniform Sparsest Cut as well.

If we allow instances of larger treewidth, we get a Unique Games-based hardness that matches our algorithmic guarantee:

**Theorem 1.4 (Tight UG Hardness)** For every constant \(\varepsilon > 0\), it is UG-hard to approximate Non-Uniform Sparsest Cut on bounded treewidth graphs better than \(2 - \varepsilon\). I.e., the existence of a family of algorithms, one for each treewidth \(k\), that run in time \(n^{f(k)}\) and give \((2 - \varepsilon)\)-approximations for Non-Uniform Sparsest Cut would disprove the Unique Games Conjecture.

### 1.1 Other Related Work

There is much work on algorithms for bounded treewidth graphs: many NP-hard problems can be solved exactly on such graphs in polynomial time (see, e.g., [RS86]). Bienstock and Ozbay [BO04] show, e.g., that the stable set polytope on treewidth-\(k\) graphs is integral after \(k\) levels of Sherali-Adams; Magen and Moharrami [MM09] use their result to show that \(O(1/\varepsilon)\) rounds of Sherali-Adams are enough to \((1 + \varepsilon)\)-approximate stable set and vertex cover on minor-free graphs. Wainwright and Jordan [WJ04] show conditions under which Sherali-Adams and Lasserre relaxations are integral for combinatorial problems based on the treewidth of certain hypergraphs. In contrast, our lower bounds show that the Sparsest Cut problem is \(\text{APX}\)-hard even on treewidth-2 supply graphs, and the integrality gap stays close to 2 even after a polynomial number of rounds of Sherali-Adams.

## 2 Preliminaries and Notation

We use \([n]\) to denote the set \(\{1, 2, \ldots, n\}\). For a set \(A\) and element \(i\), we use \(A + i\) to denote \(A \cup \{i\}\).

### 2.1 Cuts and MaxCut Problem

All the graphs we consider are undirected. For a graph \(G = (V, E)\) and set \(S \subseteq V\), let \(\partial_G(S)\) be the edges with exactly one endpoint in \(S\); we drop the subscript when \(G\) is clear from context. Given vertices \(V\) and special vertices \(s, t\), a cut \((A, V \setminus A)\) is \(s\)-\(t\)-separating if \(|A \cap \{s, t\}| = 1\).

In the (unweighted) MAXCUT problem, we are given a graph \(G = (V, E)\) and want to find a set \(S \subseteq V\) that maximizes \(|\partial_G(S)|\); the weighted version has weights on edges and seeks to maximize the weight on the crossing edges. The approximability of weighted and unweighted versions of MAXCUT differ only by an \((1 + o(1))\)-factor [CST01], and henceforth we only consider the unweighted case.

### 2.2 Tree Decompositions and Treewidth

Given a graph \(G = (V, E_G)\), a tree decomposition consists of a tree \(T = (X, E_X)\) and a collection of node subsets \(\{U_i \subseteq V\}_{i \in X}\) called “bags” such that the bags containing any node \(v \in V\) form a connected component in \(T\) and each edge in \(E_G\) lies within some bag in the collection. The width of such a tree decomposition is \(\max_{i \in X} |U_i| - 1\), and the treewidth of \(G\) is the smallest width of any tree-decomposition for \(G\). See, e.g., [Die00, Bod98] for more details and references.

The notion of treewidth is intimately connected to the underlying graph \(G\) having small vertex separators. Indeed, say graph \(G = (V, E)\) admits (weighted) vertex separators of size \(K\) if for every assignment of positive weights to the vertices \(V\), there is a set \(X \subseteq V\) of size at most \(K\) such that no component of \(G - X\) contains more than \(\frac{2}{3}\) of the total weight \(\sum_{v \in V} w_v\). For example, planar graphs admit weighted
vertex separators of size at most $\sqrt{n}$. It is known (see, e.g., [Ree92, Theorem 1]) that if $G$ has treewidth $k$ then $G$ admits weighted vertex separators of size at most $k + 1$; conversely, if $G$ admits weighted vertex separators of size at most $K$ then $G$ has treewidth at most $4K$. (The former statement is easy. A easy weaker version of the latter implication with treewidth $O(K \log n)$ is obtained as follows. Find an unweighted vertex separator $X \subseteq V$ of size $K$ to get subgraphs $G_1, G_2, \ldots, G_t$ each with at most $2/3$ of the nodes. Recurse on the subgraphs $G_i \cup X$ to get decomposition trees $T_1, \ldots, T_t$. Attach a new empty bag $U$ and connecting $U$ to the “root” bag in each $T_i$ to get the decomposition tree $T$, add the vertices of $X$ to all the bags in $T$, and designate $U$ as its root. Note that $T$ has height $O(\log n)$ and width $O(K \log n)$. In fact, this tree decomposition can be used instead of the one from Theorem 3.1 for our algorithm in Section 3 to get the same asymptotic guarantees.)

### 2.3 The Sherali-Adams Operator

For a graph with $|V| = n$, we now define the Sherali-Adams polytope. We can strengthen an LP by adding all variables $x(S,T)$ such that $|S| \leq r$ and $T \subseteq S$. The variable $x(S,T)$ has the “intended solution” that the chosen cut $(A, \overline{A})$ satisfies $A \cap S = T$. We can then define the $r$-round Sherali-Adams polytope (starting with the trivial LP), denoted $\text{SA}_r(n)$, to be the set of all vectors $(y_{uv})_{u,v \in V} \in \mathbb{R}^{\binom{n}{2}}$ satisfying the following constraints:

\[
y_{uv} = x(\{u, v\}, \{u\}) + x(\{u, v\}, \{v\}) \quad \forall u, v \in V \tag{2.1}
\]

\[
\sum_{T \subseteq S} x(S,T) = 1 \quad \forall S \subseteq V \text{ s.t. } |S| \leq r \tag{2.2}
\]

\[
x(S,T) = x(S + u, T) + x(S + u, T + u) \quad \forall S \subseteq V \text{ s.t. } |S| \leq r - 1, T \subseteq S, u \notin S \tag{2.3}
\]

\[
x(S,T) \geq 0 \quad \forall S \subseteq V \text{ s.t. } |S| \leq r, T \subseteq S \tag{2.4}
\]

We will refer to (2.3) as consistency constraints. These constraints immediately imply that the $x(S,T)$ variables satisfy the following useful property:

**Lemma 2.1** For every pair of disjoint sets $S, S' \subseteq V$ such that $|S \cup S'| \leq r$ and for any $T \subseteq S$, we have:

\[
x(S,T) = \sum_{T' \subseteq S'} x(S \cup S', T \cup T')
\]

**Proof.** This follows by repeated use of (2.3). 

We can now use $\text{SA}_r(n)$ to write an LP relaxation for an instance $G = (V,E)$ of MaxCut:

\[
\max \sum_{(u,v) \in E} y_{uv} \tag{2.5}
\]

s.t. $y_{uv} \in \text{SA}_r(n) \quad \forall u, v \in V$

We can also define an LP relaxation for an instance $(G,D)$ of Non-Uniform Sparsest Cut:

\[
\min \sum_{(u,v) \in E_G} \frac{\text{cap}_{uv} y_{uv}}{\sum_{(u,v) \in E_D} \text{dem}_{uv} y_{uv}} \tag{2.6}
\]

s.t. $y_{uv} \in \text{SA}_r(n) \quad \forall u, v \in V$

Note that the Sparsest Cut objective function is a ratio, so this is not actually an LP as stated. Instead, we could add the constraint $\sum_{(u,v) \in E_D} \text{dem}_{uv} y_{uv} \geq \alpha$, minimize $\sum_{(u,v) \in E_G} \text{cap}_{uv} y_{uv}$, and use binary search to find the correct value of $\alpha$. In Section 3, we will use (a slight weakening of) this relaxation in our approximation algorithm for Sparsest Cut on bounded-treewidth graphs, and in Section 6 we will show that Sherali-Adams integrality gaps for the MaxCut LP (2.5) can be translated into integrality gaps for the Sparsest Cut LP (2.6).

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2In some uses of Sherali-Adams, variables $x_{S,T}$ are intended to mean that $A \cap (S \cup T) = S$—this is not the case here.
3 An Algorithm for Bounded Treewidth Graphs

In this section, we present a 2-approximation algorithm for Sparsest Cut that runs in time \( n^{O(\text{treewidth})} \).

Consider an instance \((G, D)\) of Sparsest Cut, where \(G\) has treewidth \(k'\), but there are no constraints on the demand graph \(D\). We assume that we are also given an initial tree-decomposition \((T' = (X', E_{X'}); \{U_i' \subseteq V | i \in X'\})\) for \(G\). This is without loss of generality, since such an tree-decomposition \(T'\) can be found, e.g., in time \( O(n^{k'+2}) \) [ACP87] or time \( O(n)\exp(poly(k')) \) [Bod96]; a tree-decomposition of width \( O(k' \log k') \) can be found in \(\text{poly}(n)\) time [Ami10].

3.1 Balanced Tree Decompositions and the Linear Program

We start with a result of Bodlaender [Bod89, Theorem 4.2] which converts the initial tree decomposition into a “nice” one, while increasing the width only by a constant factor:

**Theorem 3.1 (Balanced Tree Decomp.)** Given graph \(G = (V, E_G)\) and a tree decomposition \((T' = (X', E_{X'}); \{U_i' \subseteq V | i \in X'\})\) for \(G\) with width at most \(k'\), there is a tree decomposition \((T = (X, E_X); \{U_i \subseteq V | i \in X\})\) for \(G\) such that

(a) \(T\) is a binary tree of depth at most \(\lambda := 2[\log_{5/4}(2n)]\), and

(b) \(\max_{i \in X} |U_i| \) is at most \(k := 3k' + 3\), and hence the width is at most \(k - 1\).

Moreover, given \(G\) and \(T'\), such a decomposition \(T\) can be found in \(\text{time} O(n)\).

From this point on, we will work with the balanced tree decomposition \(T = (X, E_X)\), whose root node is denoted by \(r \in X\). Let \(P_{ra}\) denote the set of nodes on the tree path in \(T\) between nodes \(a, r \in X\) (inclusive), and let \(V_a = \bigcup_{b \in P_{ra}} U_b\) be the union of the bags \(U_b\)'s along this \(r-a\) tree path. Note that \(|V_a| \leq k \cdot \lambda\).

Recall the Sherali-Adams linear program (2.6), with variables \(x(S, T)\) for \(T \subseteq S\) having the intended meaning that the cut \((A, \overline{A})\) satisfies \(A \cap S = T\). We want to use this LP with the number of rounds \(r\) being \(\max_{a \in X} 2|V_a|\), but solving this LP would require time \(n^{O(k \log n)}\), which is undesirable. Hence, we write an LP that uses only some of the variables from (2.6). Let \(S_a\) denote the power set of \(V_a\). Let \(S_{ab}\) be the power set of \(V_a \cup V_b\) and let \(S := \bigcup_{a,b \in X} S_{ab}\). For every set \(S \in S\), and every subset \(T \subseteq S\), we retain the variable \(x(S, T)\) in the LP, and drop all the others. There are at most \(\text{poly}(n)\) nodes in \(X\), and hence \(\text{poly}(n)\) sets \(S_{ab}\), each of these has at most \(2^{2k \lambda} = n^{O(k)}\) many sets. This results in an LP with \(n^{O(k)}\) variables and a similar number of constraints.

Finally, as mentioned above, to take care of the non-linear objective function in (2.6), we guess the optimal value \(\alpha^* > 0\) of the denominator, and add the constraint

\[
\sum_{(u,v) \in E_G} \text{dem}_{uv} y_{uv} \geq \alpha^*
\]

as an additional constraint to the LP, thereby just minimizing \(\sum_{(u,v) \in E_G} \text{cap}_{uv} y_{uv}\). For the rest of the discussion, let \((x, y)\) be an optimal solution to the resulting LP.

3.2 The Rounding Algorithm

The rounding algorithm is a very natural top-down propagation rounding procedure. We start with the root \(r \in X\); note that \(V_r = U_r\) in this case. Since \(\sum_{S \subseteq V_r} x(V_r, T) = 1\) by the constraints (2.2) of the LP, the \(x\) variables define a probability distribution over subsets of \(V_r\). We sample a subset \(A_r\) from this distribution.

In general, for any node \(a \in X\) with parent \(b\), suppose we have already sampled a subset for each of its ancestor nodes \(b, \cdot \cdot \cdot, r\), and the union of these sampled sets is \(A_b \subseteq V_b\). Now, let \(B_a = \{A' \subseteq V_a |\)
$A' \cap V_b = A_b$}; i.e., the family of subsets of $V_a$ whose intersection with $V_b$ is precisely $A_b$. By Lemma 2.1, we have

$$x(V_b, A_b) = \sum_{A' \in B_a} x(V_a, A').$$

Thus the values $x(V_a, A')/x(V_b, A_b)$ define a probability distribution over $B_a$. We now sample a set $A_a$ from this distribution. Note that this rounding only uses sets we retained in our pared-down LP, so we can indeed implement this rounding. Moreover, this set $A_a \supseteq A_b$. Finally, we take the union of all the sets

$$A := \cup_{a \in X} A_a,$$

and output the cut $(A, \overline{A})$. The following lemma is immediate:

**Lemma 3.2** For any $a \in X$ and any $S \in S_a$, we get $\Pr[(A \cap S) = T] = x(S, T)$ for all $T \subseteq S$.

**Proof.** First, we claim that $\Pr[A_a = T] = x(V_a, T)$ for all $a \in X$. This is a simple induction on the depth of $a$: the base case is directly from the algorithm. For $a \in X$ with parent node $b$,

$$\Pr[A_a = T] = \Pr[A_b = T \cap V_b] \cdot \Pr[A_a = T \mid A_b = T \cap V_b] = x(V_b, T \cap V_b) \cdot \frac{x(V_a, T)}{x(V_b, T \cap V_b)} = x(V_a, T),$$

as claimed. Now we prove the statement of the lemma: Since $S \subseteq V_a$, we know that $\Pr[A \cap S = T] = \Pr[A_a \cap S = T]$, because none of the future steps can add any other vertices from $V_a$ to $A$. Moreover,

$$\Pr[A_a \cap S = T] = \sum_{T' \subseteq V_a \setminus S} \Pr[A_a = T \cup T'] = \sum_{T' \subseteq V_a \setminus S} x(V_a, T \cup T'),$$

the last equality using the claim above. Defining $S' := V_a \setminus S$, this equals $\sum_{T' \subseteq S'} x(S \cup S', T \cup T')$, which by Lemma 2.1 equals $x(S, T)$ as desired.  

**Lemma 3.3** The probability of an edge $(u, v) \in E_G$ being cut by $(A, \overline{A})$ equals $y_{uv}$.

**Proof.** By the properties of tree-decompositions, each edge $(u, v) \in E_G$ lies within $U_a$ for some $a \in X$, and $\{u, v\} \subseteq S_a$. The probability of the edge being cut is

$$\Pr[A \cap \{u, v\} = \{u\}] + \Pr[A \cap \{u, v\} = \{v\}] = x(\{u, v\}, \{u\}) + x(\{u, v\}, \{v\}) = y_{uv}.$$

The first equality above follows from Lemma 3.2, and the second from the definition of $y_{uv}$ in (2.1).

Thus the expected number of edges in the cut $(A, \overline{A})$ equals the numerator of the objective function.

**Lemma 3.4** The probability of a demand pair $(s, t) \in E_D$ being cut by $(A, \overline{A})$ is at least $y_{st}/2$.

**Proof.** Let $a, b \in X$ denote the (least depth) nodes in $T$ such that $s \in U_a$ and $t \in U_b$ respectively; for simplicity, assume that the least common ancestor of $a$ and $b$ is $r$. (An identical argument works when the least common ancestor is not the root.) We can assume that $r \notin \{a, b\}$, or else we can use Lemma 3.2 to claim that the probability $s, t$ are separated is exactly $y_{st}$.

Consider the set $V_a \cup V_b$, and consider the set-valued random variable $W$ (taking on values from the power set of $V_a \cup V_b$) defined by $W[T] := x(V_a \cup V_b, T)$. Denote the distribution by $\mathcal{D}_{ab}$, and note that this is just the distribution specified by the Sherali-Adams LP restricted to $V_a \cup V_b$. Let $X_s$ and $X_t$ denote the indicator random variables of the events $\{s \in W\}$ and $\{t \in W\}$ respectively; these variables are dependent in general. For a set $T \subseteq V_r$, let $X_{s|T}$ and $X_{t|T}$ be indicators for the corresponding events conditioned on $W \cap V_r = T$. Then by definition,

$$y_{st} = \Pr_{\mathcal{D}_{ab}}[X_s \neq X_t] = \mathbb{E}_T \Pr_{\mathcal{D}_{ab}}[X_s \neq X_t|T]$$

(3.7)
where the expectation is taken over outcomes of \( T = W \cap V_r \).

Let \( \mathcal{D} \) denote the distribution on cuts defined by the algorithm. Let \( Y_s \) and \( Y_t \) denote events that \( \{ s \in A \} \) and \( \{ t \in A \} \) respectively, and let \( Y_{s|T} \) and \( Y_{t|T} \) denote these events conditioned on \( A \cap V_r = T \). Thus the probability that \( s \) and \( t \) are separated by the algorithm is

\[
\text{alg}(s, t) = \Pr_\mathcal{D}[Y_s \neq Y_t] = \mathbb{E}_T \Pr_\mathcal{D}[Y_{s|T} \neq Y_{t|T}]
\]  

(3.8)

where the expectation is taken over the distribution of \( T = A \cap V_r \); by Lemma 3.2 this distribution is the same as that for \( W \cap V_r \).

It thus suffices to prove that for any \( T \),

\[
\Pr_\mathcal{D_a}[X_{s|T} \neq X_{t|T}] \leq 2 \Pr_\mathcal{D_a}[Y_{s|T} \neq Y_{t|T}].
\]

(3.9)

Now observe that \( Y_{s|T} \) is distributed identically to \( X_{s|T} \) (with both being 1 with probability \( \frac{x_{(V_s \cup \{s\}, T \cup A)}}{x_{(V_r, T)}} \)), and similarly for \( Y_{t|T} \) and \( X_{t|T} \). However, since \( s \) and \( t \) lie in different subtrees, \( Y_{s|T} \) and \( Y_{t|T} \) are independent, whereas \( X_{s|T} \) and \( X_{t|T} \) are dependent in general.

We can assume that at least one of \( \mathbb{E}_{\mathcal{D_a}}[X_{s|T}], \mathbb{E}_{\mathcal{D_a}}[X_{t|T}] \) is at most 1/2; if not, we can do the following analysis with the complementary events \( \mathbb{E}_{\mathcal{D_a}}[\bar{X}_{s|T}], \mathbb{E}_{\mathcal{D_a}}[\bar{X}_{t|T}] \), since (3.9) depends only on random variables being unequal. Moreover, suppose

\[
\mathbb{E}_{\mathcal{D_a}}[X_{t|T}] \leq \mathbb{E}_{\mathcal{D_a}}[X_{s|T}]
\]

(else we can interchange \( s, t \) in the following argument). Define the distribution \( \mathcal{D}' \) where we draw \( X_{s|T}, X_{t|T} \) from \( \mathcal{D_a} \), set \( Y_{s|T} \) equal to \( X_{s|T} \), and draw \( Y_{t|T} \) independently from \( \mathcal{D} \). By construction, the distributions of \( X_{s|T}, X_{t|T} \) in \( \mathcal{D}_{st} \) and \( \mathcal{D}' \) are identical, as are the distributions of \( Y_{s|T}, Y_{t|T} \) in \( \mathcal{D} \) and \( \mathcal{D}' \).

We claim that

\[
\mathbb{E}_{\mathcal{D}'}[X_{t|T} \neq Y_{t|T}] \leq \mathbb{E}_{\mathcal{D}'}[X_{s|T} \neq Y_{t|T}].
\]

(3.10)

Indeed, if \( \mathbb{E}_{\mathcal{D}'}[X_{s|T}] = a \) and \( \mathbb{E}_{\mathcal{D}'}[X_{t|T}] = b \), then \( \mathbb{E}_{\mathcal{D}'}[Y_{t|T}] = b \) as well, with \( b \leq a \) and \( b \leq 1/2 \). Thus, (3.10) claims that \( 2b(1-b) \leq a(1-b) + b(1-a) \) (recall here that \( Y_{t|T} \) is chosen independently of the other variables). This holds if \( b(1-2b) \leq a(1-2b) \), which follows from our assumptions on \( a, b \) above. Finally,

\[
\Pr_{\mathcal{D}'}[X_{s|T} \neq X_{t|T}] \leq \Pr_{\mathcal{D}'}[X_{s|T} \neq Y_{t|T}] + \Pr_{\mathcal{D}'}[X_{t|T} \neq Y_{t|T}].
\]

(3.11)

Combining (3.10) and (3.11) and observing that \( X_{s|T} = Y_{s|T} \) in our construction, the claim follows. \( \Box \)

By Lemmas 3.3 and 3.4, a random cut \((A, \overline{A})\) chosen by our algorithm cuts an expected capacity of exactly \( \sum_{uv \in E_G} \text{cap}_{uv} y_{uv} \); whereas the expected demand cut is at least \( \frac{1}{2} \sum_{st \in E_D} \text{dem}_{st} y_{st} \). This shows the existence of a cut in the distribution whose sparsity is within a factor of two of the LP value. Such a cut can be found using the method of conditional expectations; we defer the details to the next section. Moreover, the analysis of the integrality gap is tight: Section 6 shows that for any constant \( \gamma > 0 \), the Sherali-Adams LP for Sparsest Cut has an integrality gap of at least \( 2 - \varepsilon(\gamma) \), even after \( n^\gamma \) rounds.

### 3.3 Derandomization

In this section, we use the method of conditional expectations to derandomize our rounding algorithm, which allows us to efficiently find a cut \((A, \overline{A})\) with sparsity at most twice the LP value. We will think of the set \( A \) as being a \( \{0,1\}\)-assignment/labeling for the nodes in \( V \), where \( i \in A \iff A(i) = 1 \).
In the above randomized process, let $Y_{ij}$ be the indicator random variable for whether the pair $(i, j)$ is separated. We showed that for $(i, j) \in E_G$, $\mathbb{E}[Y_{ij}] = y_{ij}$, and so for all other $(i, j) \in \binom{V}{2}$, $\mathbb{E}[Y_{ij}] \geq y_{ij}/2$. Now if we let $Z = \sum_e \text{cap}_e Y_e$ be the r.v. denoting the edge capacity cut by the process and $Z' = \sum_{st} \text{dem}_{st} Y_{st}$ be the r.v. denoting the demand separated, then the analysis of the previous section shows that

$$\frac{\mathbb{E}[Z]}{\mathbb{E}[Z']} \leq 2 \cdot \frac{\sum_e \text{cap}_e y_e}{\alpha}.$$  

(Recall that $\alpha$ was the “guessed” value of the total demand separated by the actual sparsest cut.) Equivalently, defining $\text{LP}^* := \sum_e \text{cap}_e y_e$, and

$$W := \frac{Z}{\text{LP}^*} - \frac{2 Z'}{\alpha},$$

we know that $\mathbb{E}[W] \leq 0$.

The algorithm is the natural one: for the root $r$, enumerate over all $2^k$ assignments for the bag $V_r$, and choose the assignment $A_r$ minimizing $\mathbb{E}[W | A_r]$. Since $\mathbb{E}[W] \leq 0$, it must be the case that $\mathbb{E}[W | A_r] \leq 0$ by averaging. Similarly, given the choices for nodes $X' \subseteq X$ such that $T[X']$ induces a connected tree and $\mathbb{E}[W | \{ A_x \}_{x \in X'}] \leq 0$, choose any $a \in X$ whose parent $b \in X'$, and choose an assignment $A_a$ for the nodes in $V_a \setminus V_b$ so that the new $\mathbb{E}[W | \{ A_x \}_{x \in X'} \cup \{ A_a \}] \leq 0$. The final assignment $A$ will satisfy $\mathbb{E}[W | \{ A_a \}_{a \in X}] \leq 0$, which would give us a cut with sparsity at most $2\text{LP}^*/\alpha$, as desired.

It remains to show that we can compute $\mathbb{E}[W | \{ A_x \}_{x \in X'}]$ for any subset $X' \subseteq X$ containing the root $r$, such that $T[X']$ is connected. Let $V' = \cup_{x \in X'} \bar{U}_x$ be the set of nodes already labeled. For any vertex $v \in V$, let $b(v) \in X$ be the highest node in $T$ such that $v \in U_{b(v)}$. If $v$ is yet unlabeled, then $b(v) \notin X'$, and hence let $\ell(v)$ be the lowest ancestor of $b(v)$ in $X'$. In other words, we have chosen an assignment $A_{\ell(v)}$ for the bag $V_{\ell(v)}$. By the properties of our algorithm, we know that

$$\Pr[v \in A | A_{\ell(v)}] = \frac{x(V_{\ell(v)} \cup \{ v \}, A_{\ell(v)} \cup \{ v \})}{x(V_{\ell(v)}, A_{\ell(v)})}.$$  

(3.12)

Moreover, if $u, v$ are both unlabeled such that their highest bags $b(u), b(v)$ share a root-leaf path in $T$, then

$$\Pr[u, v \text{ separated } | A_{\ell(v)}] = \frac{x(V_{\ell(v)} \cup \{ u, v \}, A_{\ell(v)} \cup \{ u \}) + x(V_{\ell(u)} \cup \{ u, v \}, A_{\ell(u)} \cup \{ v \})}{x(V_{\ell(v)}, A_{\ell(v)})},$$  

(3.13)

where $\ell(v) = \ell(u)$ is the lowest ancestor of $b(u), b(v)$ that has been labeled. If $u, v$ are yet unlabeled, but we have chosen an assignment for $a = \text{lca}(b(u), b(v))$, then $u, v$ will be labeled independently using (3.12). Finally, if $u, v$ are unlabeled, and we have not yet chosen an assignment for $a = \text{lca}(b(u), b(v))$, then the probability of $u, v$ being cut is precisely

$$\sum_{U \subseteq \bar{V}_a \setminus V_{\ell(v)}} \frac{x(V_a, A_{\ell(v)} \cup U)}{x(V_{\ell(v)}, A_{\ell(v)})} \cdot \Pr[(u, v) \text{ separated } | V_a \text{ labeled } A_{\ell(v)} \cup U],$$

where the probability can be computed using (3.12), since $u, v$ will be labeled independently after conditioning on a labeling for $V_a$. There are at most $n^{O(k)}$ terms in the sum, and hence we can compute this in the claimed time bound. Now, we can compute $\mathbb{E}[W | \{ A_x \}_{x \in X'}]$ using the above expressions in time $n^{O(k)}$, which completes the proof.

### 3.3.1 Embedding into $\ell_1$

Our algorithm and analysis also implies a 2-approximation to the minimum distortion $\ell_1$ embedding of a treewidth $k$ graph in time $n^{O(k)}$. We will describe an algorithm that, given $D$, either finds an
embedding with distortion $2D$ or certifies that any $\ell_1$ embedding of $G$ requires distortion more than $D$. It is easy to use such a subroutine to get a $2 + o(1)$-approximation to the minimum distortion $\ell_1$ embedding problem.

Towards this end, we write a relaxation for the distortion $D$ embedding problem as follows. Given $G$ with treewidth $k$, we start with the $r$-round Sherali-Adams polytope $SA_r(n)$ with $r = O(k \log n)$. We add the additional set of constraints $C \cdot d(u, v) \leq y_{uv} \leq D \cdot C \cdot d(u, v)$, for every pair of vertices $u, v \in V$. The cut characterization of $\ell_1$ implies that this linear program is feasible whenever there is a distortion $D$ embedding. Given a solution to the linear program, we round it using the rounding algorithm of the last section. It is immediate from our analysis that a random cut sampled by the algorithm satisfies $Pr[(u, v) \text{ separated}] \in [y_{uv}/2, y_{uv}]$.

Moreover, since the analysis of the rounding algorithm only uses $n^{O(k)}$ equality constraints on the expectations of random variables, we can use the approach of Karger and Koller [KK97] to get an explicit sample space $\Omega$ of size $|\Omega| = n^{O(k)}$ that satisfies all these constraints. Indeed, each of the points $\omega \in \Omega$ of this sample space gives us a $\{0, 1\}$-embedding of the vertices of the graph. We can concatenate all these embeddings and scale down suitably in time $|\Omega| \cdot \poly(n)$ to get an $\ell_1$-embedding $f : V \rightarrow \mathbb{R}^{[2]}$ with the properties that (a) $\|f(u) - f(v)\|_1 = y_{uv}$ for all $(u, v) \in E_G$, and (b) $\|f(u) - f(v)\|_1 \geq y_{uv}/2$ for $(u, v) \in \binom{V}{2}$. Scaling $f$ by a factor of $C$ gives an embedding with distortion $2D$.

## 4 The Hardness Result

In this section, we prove the $\text{APX}$-hardness claimed in Theorem 1.3. In particular, we show the following reduction from the $\text{MAXCUT}$ problem to the Non-Uniform Sparsest Cut problem.

**Theorem 4.1** For any $\varepsilon > 0$, a $\rho$-approximation algorithm for Non-Uniform Sparsest Cut on series-parallel graphs (with arbitrary demand graphs) that runs in time $T(n)$ implies a $(\frac{1}{\rho} - \varepsilon)$-approximation to $\text{MAXCUT}$ on general graphs running in time $T(n^{O(1/\varepsilon)})$.

The current best hardness-of-approximation results for $\text{MAXCUT}$ are: (a) the $(\frac{16}{17} + \varepsilon)$-factor hardness (assuming $P \neq \text{NP}$) due to Håstad [Hås01] (using the gadgets from Trevisan et al. [TSSW00]) and (b) the $(\alpha_{GW} - \varepsilon)$-factor hardness (assuming the Unique Games Conjecture) due to Khot et al. [KKMO07, MOO10], where $\alpha_{GW} = 0.87856 \ldots$ is the constant obtained in the hyperplane rounding for the $\text{MAXCUT}$ SDP. Combined with Theorem 4.1, these imply hardness results of $(\frac{17}{16} - \varepsilon)$ and $(1.138 - \varepsilon)$ respectively for Non-Uniform Sparsest Cut and prove Theorem 1.3.

The proof of Theorem 4.1 proceeds by taking the hard $\text{MAXCUT}$ instances and using them to construct the demand graphs in a Sparsest Cut instance, where the supply graph is the familiar fractal obtained from the graph $K_{2, n}$.

The base case of this recursive construction is in Section 4.1, and the full construction is in Section 4.2. The analysis of the latter is based on a generic powering lemma, which will be useful for showing tight Unique Games hardness for bounded treewidth graphs in Section 5 and the Sherali-Adams integrality gap in Section 6.

### 4.1 The Basic Building Block

Given a connected (unweighted) $\text{MAXCUT}$ instance $H = ([n], E_H)$, let $m = |E_H|$, and let $mc(H) := \max_{A \subseteq [n]} |\partial_H(A)|$. Let the supply graph be $G'_1 = (V_1, E_1)$, with vertices $V_1 = \{s, t\} \cup [n]$ and edges $E_1 = \cup_{i \in [n]} \{(s, i), \{t, i\}\}$. Define the capacities $\text{cap}_s,i = \text{cap}_t,i = \deg_H(i)/2m$. Define the demands thus: $\text{dem}_{s,i} = 1$, and for $i, j \in [n]$, let $\text{dem}_{i,j} = 1_{\{i,j\} \in E_H}/m$ (i.e., $i, j$ have $1/m$ demand between them if $\{i, j\}$ is an edge in $H$, and zero otherwise). Let this setting of demands be denoted $D'_1$. (The hardness

\footnote{The fractal for $K_{2, 2}$ has been used for lower bounds on the distortion incurred by tree embeddings [GNRS04], Euclidean embeddings [NR03], and low-dimensional embeddings in $\ell_1$ [BC05, LN04, Reg12]. Moreover, the fractal for $K_{2, n}$ shows the integrality gap for the natural metric relaxation for Sparsest Cut [LR10, CSW10].}
results in Chuzhoy and Khanna [CK09] and Chlamtác et al. [CKR10] used the same graph $G_1$, but with a different choice of capacities and demands.)

**Claim 4.2** The sparsest cuts in $G'_1$ are s-t-separating, and have sparsity $m/(m + mc(H))$.

**Proof.** For $A \subseteq [n]$, the cut $(A + s, \overline{A} + t)$ has sparsity
\[
\frac{\sum_{i \in A} \deg_H(i) + \deg_H(s)/2m}{|\partial_H(A)|/m + 1} = \frac{m}{|\partial_H(A)| + m} < 1,
\]
since $\frac{1}{2} \sum_i \deg_H(i) = m$. The cut $(A, \overline{A} + s + t)$ has sparsity
\[
\frac{2 \sum_{i \in A} \deg_H(i) + \deg_H(s)/2m}{|\partial_H(A)| + m} \geq \frac{2 \sum_{i \in A} \deg_H(i) + \deg_H(s)/2m}{\sum_{i \in A} \deg_H(i)} \geq 1,
\]
which is strictly worse than any s-t-separating cut. Hence the sparsest cut is the cut $(A + s, \overline{A} + t)$ that maximizes $|\partial_H(A)|$.

Given a $cm$-vs-$sm$ hardness result for MAXCut, this gives us a $(1 + c)$-vs-$(1 + s)$ hardness for Sparsest Cut. However, we can do better using a recursive “fractal” construction, as we show next. Before we proceed further, we remark that if we remove the s-t demand from the instance $G'_1$, we obtain an instance $G_1$ with the following properties.

**Lemma 4.3** The instance $G_1$ constructed by removing $\text{dem}_{s,t}$ from $G'_1$ satisfies:

- If $H$ has a cut of size $cm$, then there is an s-t separating cut of capacity 1 that separates $c$ demand.
- Any s-t separating cut has capacity at least 1.
- If the maximum cut in $H$ has size $sm$, then every s-t separating cut has sparsity at least $s^{-1}$.
- Any cut that does not separate $s$ and $t$ has sparsity at least 1.

While $G_1$ by itself is not a hard instance of Sparsest Cut, the above properties will make it a useful building block in the powering operation below.

![Figure 4.1: Base case of the construction $G'_1$ for $n = 5$.](image)

### 4.2 An Instance Powering Operation

In this section, we describe a powering operation on Sparsest Cut instances that we use to boost the hardness result. This is the natural fractal construction. We start with an instance $G_1 = (V_1 = \{s, t\} \cup [n], \text{cap}_e, \text{dem}_e)$ of the sparsest cut problem. In other words, we have a Sparsest Cut instance with two designated vertices $s$ and $t$. (For concreteness, think of the $G_1$ from the previous section, but any graph $G_1$ would do.)

For $\ell \geq 2$, consider the graph $G_\ell$ obtained by taking $G_1$ and replacing each capacity edge $e = (u, v)$ in $G_1$ with a copy of $G_{\ell-1}$ in the natural way. In other words, for every $e = (u, v)$, we create a copy $G_e^\ell$ of $G_{\ell-1}$, and identify its vertex $s$ with $u$ and its $t$ with $v$. Moreover, $G_e^\ell_{\ell-1}$ is scaled down by $\text{cap}_e$. 


Thus if edge $f \in E_{\ell-1}$ has capacity $\text{cap}_f$ in $G_{\ell-1}$, then the corresponding edge in $G^v_{\ell-1}$ has capacity $\text{cap}_e \cdot \text{cap}_f$; the demands in $G^v_{\ell-1}$ are also scaled by the same factor. In addition to the scaled demands from copies of $G_{\ell-1}$, $G_\ell$ contains new level-$\ell$ demands $\text{dem}_{i,j}$ from the base graph $G_1$. Note that this instance contains vertices of $V_1$ in its vertex set and will have $s$ and $t$ as its designated vertices.

The following properties are immediate.

**Observation 4.4** If $G_1$ has $n$ vertices and $m$ capacity edges, then $G_\ell$ has $m^{\ell-1}n$ vertices and $m^\ell$ capacity edges. Moreover, if the supply graph in $G_1$ has treewidth $k$, then the supply graph of $G_\ell$ also has treewidth $k$.

We next argue “completeness” and “soundness” properties of this operation. We will distinguish between cuts that separate $s$ and $t$, and those that do not. We call the former cuts **admissible** and the latter **inadmissible**.

**Lemma 4.5** If $G_1$ has an admissible cut $(A, \overline{A})$ that cuts $\text{cap}(A, \overline{A})$ capacity and $\text{dem}(A, \overline{A})$ demand, then there exists an admissible cut in $G_\ell$ of capacity $(\text{cap}(A, \overline{A}))^\ell$ that cuts $\text{dem}(A, \overline{A}) \cdot (\sum_{i=0}^{\ell-1} \text{cap}(A, \overline{A})^i)$ demand.

**Proof.** The proof is by induction on $\ell$. The base case $\ell = 1$ is an assumption of the lemma. Assume the claim holds for $G_{\ell-1}$. Let $(A_{\ell-1}, \overline{A}_{\ell-1})$ denote the admissible cut satisfying the induction hypothesis and let $s \in A_{\ell-1}$. Recall that $G_\ell$ is created by replacing the edges of $G_1$ by copies of $G_{\ell-1}$. Define the cut $A_\ell$ in the natural way: Start with $A_\ell = A$. Then for each $e = (u, v) \in G_1$ such that $u, v \in A$, we place all of $G^e_{\ell-1}$ in $A_\ell$; similarly if $u, v \in \overline{A}$ then place all of $G^e_{\ell-1}$ in $A_\ell$. For $(u, v) \in G_1$ such that $u \in A, v \in \overline{A}$, we cut $G^e_{\ell-1}$ according to $(A_{\ell-1}, \overline{A}_{\ell-1})$: i.e., the copy of a vertex $x \in A_{\ell-1}$ is placed in $A_\ell$. Similarly, if $u \in \overline{A}, v \in A$, we put the copy of $x$ in $A_\ell$ if $x \in \overline{A}_{\ell-1}$. This defines the cut $(A_\ell, \overline{A}_\ell)$.

The capacity of the cut can be computed as follows: For each edge of $G_1$ cut by $A$, the corresponding copy of $G_{\ell-1}$ contributes $\text{cap}_e \cdot \text{cap}(A_{\ell-1}) = \text{cap}_e \cdot (\text{cap}(A, \overline{A})^{\ell-1})$ to the cut, where we used the inductive hypothesis for $A_{\ell-1}$. For edges not cut by $(A, \overline{A})$, the corresponding $G^e_{\ell-1}$ is uncut and contributes 0. Thus

$$\text{cap}(A_\ell, \overline{A}_\ell) = \sum_{e \in (A, \overline{A})} \text{cap}_e \cdot (\text{cap}(A, \overline{A})^{\ell-1}) = \text{cap}(A, \overline{A})^\ell.$$  

Similarly, the demand from copies of $G_{\ell-1}$ cut by $A_\ell$ is exactly

$$\sum_{e \in (A, \overline{A})} \text{cap}_e \cdot \text{dem}(A_{\ell-1}, \overline{A}_{\ell-1}) = \text{cap}(A, \overline{A}) \cdot \text{dem}(A_{\ell-1}, \overline{A}_{\ell-1})$$

$$= \text{cap}(A, \overline{A}) \cdot \text{dem}(A, \overline{A}) \cdot (\sum_{i=0}^{\ell-2} \text{cap}(A, \overline{A})^i)$$

$$= \text{dem}(A, \overline{A}) \cdot (\sum_{i=1}^{\ell-1} \text{cap}(A, \overline{A})^i).$$

(The second equality is from the induction hypothesis.) Additionally, $A_\ell$ cuts exactly $\text{dem}(A, \overline{A})$ units of the level-$\ell$ demands. The claim follows by the summing the two.  

Note that if $G_1$ has an admissible cut $(A, \overline{A})$ of capacity 1 that cuts $\text{dem}(A, \overline{A})$ units of demand, then the above lemma gives us a cut of capacity 1 that cuts $\ell \text{dem}(A, \overline{A})$ units of demand.

Now, for soundness analysis, we argue that if $G_1$ has no “good” cuts, then neither does $G_\ell$. It will be convenient to separately argue about the admissible and inadmissible cuts.

We will need the notion of “connected” cuts. Given a graph $G = (V, E)$, call a cut $(X, V \setminus X)$ **connected** if the resulting components $G[X]$ and $G[V \setminus X]$ are both connected graphs. Observe that for a connected admissible cut $(A + s, \overline{A} + t)$ in $G_\ell$, along any $s$-$t$ shortest path $P$, the vertices in $P \cap (A + s)$ forms some prefix of $P$—this path is cut exactly once.
Lemma 4.6 ([OS81], Lemma 2.1(ii)) For any connected Sparsest Cut instance \((G, D)\), there exists a sparsest cut that is connected.

**Proof.** Let \(S\) be the sparsest cut in \(G\) such that \(\sum_{e \in \partial_G(S)} \text{cap}_e\) is as small as possible. We claim that \(S\) must be connected. Suppose not, and let \(S_1, \ldots, S_k\) be the partition of \(S\) into connected components with \(k \geq 3\). Let \(\text{cap}_e\) be the capacity on edge \(e \in E_G\) and let \(\text{dem}_e\) be the demand on edge \(e \in E_D\). Let \(\Phi_{G, D}\) be the value of the Sparsest Cut instance. Then

\[
\Phi_{G, D} = \frac{\sum_{e \in \partial_G(S)} \text{cap}_e}{\sum_{e \in \partial_D(S)} \text{dem}_e} \geq \frac{\sum_{i=1}^k \sum_{e \in \partial_G(S_i)} \text{cap}_e}{\sum_{i=1}^k \sum_{e \in \partial_D(S_i)} \text{dem}_e}.
\]

Since \(S\) is a sparsest cut, each \(S_i\) has sparsity at least \(\Phi_{G, D}\). It follows that in fact all \(S_i\)'s have the same sparsity. Thus for each \(i\),

\[
\Phi_{G, D} = \frac{\sum_{e \in \partial_G(S_i)} \text{cap}_e}{\sum_{e \in \partial_D(S_i)} \text{dem}_e}.
\]

Since \(G\) is connected, each of the \(\sum_{e \in \partial_G(S_i)} \text{cap}_e\) quantities are positive. Moreover, since \(k \geq 3\), there must be an \(i\) such that \(\sum_{e \in \partial_G(S_i)} \text{cap}_e\) is strictly smaller than \(\sum_{e \in \partial_G(S)} \text{cap}_e\). This, however, contradicts the definition of \(S\), and the claim follows.

We now proceed to main technical result of this section.

**Lemma 4.7** Suppose that for some constant \(\gamma\), \(G_1\) satisfies:

- Any admissible cut \((A, \overline{A})\) has capacity \(\text{cap}(A, \overline{A})\) at least 1.
- Any admissible cut \((A, \overline{A})\) cuts at most \(\gamma \cdot \text{cap}(A, \overline{A})\) demand.
- Any inadmissible cut \((A, \overline{A})\) cuts at most \(\text{cap}(A, \overline{A})\) demand.

Then \((G_{\ell}, D_{\ell})\) satisfies:

- Any admissible cut \((A, \overline{A})\) has capacity \(\text{cap}(A, \overline{A})\) at least 1.
- Any admissible cut \((A, \overline{A})\) cuts at most \(\ell \gamma \cdot \text{cap}(A, \overline{A})\) demand.
- Any inadmissible cut \((A, \overline{A})\) cuts at most \((\ell - 1) \gamma + 1 \cdot \text{cap}(A, \overline{A})\) demand.

**Proof.** The proof is by induction on \(\ell\). The base case \(\ell = 1\) is the assumption of the lemma. Suppose that the claim holds for \(G_{\ell-1}\).

First, let \((A_{\ell}, \overline{A}_{\ell})\) be an admissible cut. Let \((A_1, \overline{A}_1)\) denote the projection of this cut onto \(\{s, t\} \cup [n]\), i.e., \(A_1 = A_{\ell} \cap (\{n\} \cup \{s, t\})\). For each edge \(e \in (A_1, \overline{A}_1)\), the cut \(A_{\ell}\) induces an admissible cut on \(G^e_{\ell-1}\). This contributes at least unit capacity to the corresponding level-(\(\ell - 1\)) cut (by the induction hypothesis), and thus \(\text{cap}_e \cdot 1\) to the cut \((A_{\ell}, \overline{A}_{\ell})\) because of the scaling-down in the construction of \(G_{\ell}\). Summing over all edges \(e \in (A_1, \overline{A}_1)\), we conclude that \(\text{cap}(A_{\ell}, \overline{A}_{\ell})\) is at least the capacity of \((A_1, \overline{A}_1)\) in \(G_1\). Using the fact that all admissible cuts in \(G_1\) have capacity at least 1, the first part of the claim follows.

Next, we estimate the demand cut by \((A_{\ell}, \overline{A}_{\ell})\). The total level \(\ell\) demand cut is at most \(\gamma\) times the capacity of \((A_1, \overline{A}_1)\) in \(G_1\), and hence by the argument above, contributes at most \(\gamma \cdot \text{cap}(A_{\ell}, \overline{A}_{\ell})\). Moreover, if \(\text{dem}_{\ell-1}^e\) and \(\text{cap}_{\ell-1}^e\) denotes the total demand and capacity cut by \(A_{\ell}\) inside \(G^e_{\ell-1}\), then by the induction hypothesis, we have \(\text{dem}_{\ell-1}^e \leq (\ell - 1) \cdot \gamma \cdot \text{cap}_{\ell-1}^e\). Since these demands and capacities are coming from disjoint sets of edges, we can add these inequalities to conclude that \(\text{dem}(A_{\ell}, \overline{A}_{\ell})\) is at most \(\gamma \cdot \text{cap}(A_{\ell}, \overline{A}_{\ell}) + \sum_e \text{dem}^e_{\ell-1} \leq (\gamma + (\ell - 1) \cdot \gamma) \cdot \text{cap}(A_{\ell}, \overline{A}_{\ell})\). The second part of the claim follows.

Finally, let \((A_{\ell}, \overline{A}_{\ell})\) be an inadmissible cut with \(s, t \notin A_{\ell}\) by Lemma 4.6, we can assume it is connected. Let \((A_1, \overline{A}_1)\) denote the projection of \((A_{\ell}, \overline{A}_{\ell})\) onto \(G_1\). Our construction guarantees that \(A_1\) is either empty, or induces a connected cut in \(G_1\). In the former case, \(A_{\ell}\) induces an inadmissible cut in some
Given a bipartite graph \( B \), one standard form of the Unique Label Cover problem is the following. We are given a bipartite graph \( B = (U, V, E_B) \). There is a label set with \( d \) labels. Each edge \((u, v) \in E_B\) has an associated bijective map \( \sigma_{u,v} : [d] \to [d] \). A labeling is a map from \( U \cup V \) to \([d]\), and satisfies an edge \((u, v) \in E_B\) if \( \sigma_{u,v}(\text{label}(u)) = \text{label}(v) \).

The optimum of the Unique Label Cover problem is the maximum fraction of edges satisfied by any labeling.

**Conjecture 5.1 (Unique Games Conjecture)** For any \( \eta, \gamma > 0 \), there is a large enough constant \( d = d(\eta, \gamma) \) such that it is NP-hard to distinguish whether a Unique Label Cover instance with label size \( G_{t-1} \); the demand and capacity cut by \( A_t \) equals those in this inadmissible cut of \( G_{t-1} \), so we can use the inductive hypothesis. Since both the demands and capacities are scaled by the same amount, this gives us the proof for the first case. In the latter case, for every edge \( e \in G_1 \) cut by \((A_1, \overline{A_1})\), the cut \((A_t, \overline{A_t})\) induces an admissible cut in \( G_{t-1} \). Let \( \text{cap}^G_{t-1} \) and \( \text{dem}^G_{t-1} \) denote the capacity and demand cut by \( A_t \) inside \( G_{t-1} \). By the inductive hypothesis, each of these admissible cuts has at least unit capacity in the unscaled version of \( G_{t-1} \), so the scaled-down capacity \( \text{cap}^G_{t-1} \geq \text{cap}^{G^1}_{t-1} \). Summing over all cut edges, \( \text{cap}(A_t, \overline{A_t}) \geq \text{cap}^{G^1}(A_1, \overline{A_1}) \). The level-\( \ell \) demand cut by \( A_t \) is equal to the demand cut in \( G_1 \) by \( A_1 \); by the last assumption this is at most \( \text{cap}^{G^1}(A_1, \overline{A_1}) \leq \text{cap}(A_t, \overline{A_t}) \). Moreover, using the second part of the induction hypothesis, we conclude that for any \( e \), \( \text{dem}^G_{t-1} \leq (\ell - 1) \gamma \text{cap}^G_{t-1} \). Since the demands and capacities contribution to \( \text{dem}^G_{t-1} \) and \( \text{cap}^G_{t-1} \) are disjoint for different edges \( e \), we can add these inequalities to conclude that the total demand cut is at most \( \text{cap}(A_t, \overline{A_t}) + \sum_e \text{dem}^G_{t-1} \leq \text{cap}(A_t, \overline{A_t})(1 + (\ell - 1)\gamma) \), which completes the proof.

### 4.2.1 Putting It Together

Let \( G_1 \) be the instance defined in Section 4.1 and \( G_\ell \) be obtained by the powering operation starting with \( G_1 \). Lemmas 4.3 and 4.5 imply that if \( H \) has a cut of size \( cm \), then \( G_\ell \) has a cut of sparsity \( \frac{1}{\ell c} \). Moreover, using Lemma 4.7 along with Lemma 4.3 shows that if \( H \) has max cut size at most \( sm \), then the sparsest cut in \( G_\ell \) has sparsity at least \( \frac{1}{1+((\ell-1)s)} \). Hence, a \( cm \)-vs-\( sm \) hardness for MaxCut translates to a \( \frac{\ell c}{1+((\ell-1)s)} \geq \frac{c}{s}(1 - \frac{1}{\ell s}) \) hardness for Sparsest Cut. Taking \( \ell = \frac{1-s}{\varepsilon s} = \Omega(1/\varepsilon) \) gives us a hardness of \( \frac{c}{s}(1 - \varepsilon) \).

Note that Lee and Raghavendra [LR10] show the integrality gap of the natural LP relaxation for Non-Uniform Sparsest Cut is 2 for series-parallel graphs; Chekuri, Shepherd, and Weibel [CSW10] give a different analysis of the integrality gap lower bound. Their instances are the graphs \( G_\ell \) above, but with \( K_n \) as the MaxCut instance \( H \). In hindsight, their gaps follow from the fact that the integrality gap of the LP relaxation of MaxCut on \( K_n \) is 2. This theme will be revisited when we show an integrality gap for the Sherali-Adams LP using the Sherali-Adams integrality gaps for MaxCut.

### 5 A Tight Unique Games Hardness

In this section, we show that, assuming the Unique Games Conjecture, the Sparsest Cut problem is hard to approximate better than a factor of 2, even on bounded treewidth graphs. Specifically, for every constant \( \varepsilon > 0 \), having a polynomial-time algorithm for every fixed value of treewidth that gave a \((2 - \varepsilon)\)-approximation to Sparsest Cut would violate the Unique Games Conjecture.

The proof in this section first abstracts out a useful form of the Unique Games problem and builds a basic instance from it that shows a hardness of \( 3/2 - \varepsilon \). Then we use the powering ("fractalization") operation from Section 4.2 to boost the hardness to \( 2 - \varepsilon \).

#### 5.1 A Convenient Form of Unique Games

One standard form of the Unique Label Cover (a.k.a. Unique Games) problem is the following. We are given a bipartite graph \( B = (U, V, E_B) \). There is a label set with \( d \) labels. Each edge \((u, v) \in E_B\) has an associated bijective map \( \sigma_{u,v} : [d] \to [d] \). A labeling is a map from \( U \cup V \) to \([d]\), and satisfies an edge \((u, v) \in E_B\) if \( \sigma_{u,v}(\text{label}(u)) = \text{label}(v) \).

The optimum of the Unique Label Cover problem is the maximum fraction of edges satisfied by any labeling.
\(d\) has optimum at least \(1 - \eta\) or at most \(\gamma\).

It will be most convenient for us to consider non-bipartite versions of Unique Label Cover, where there is a general (multi-)graph \(H = (V_H, E_H)\). For each \(e = (v, w) \in E_H\) there is again a bijective map \(\sigma_e : [d] \to [d]\) and the goal is to find a labeling maximizing the number of satisfied edges. We call such a multigraph \(H\) a union of cliques if there exists a partition of \(E_H\) into (edge-disjoint) cliques \(C_1, C_2, \ldots, C_r\) for some \(r\), i.e., each \(C_i\) is a complete graph on some subset \(S_i \subseteq V_H\). (Recall that \(H\) is a multigraph, so these sets \(S_i\) may have more than single vertices in common, resulting in parallel edges.) We call a Unique Label Cover instance \((H, \{\sigma_e\}_{e \in E_H}, d)\) \(\Delta\)-nice if

- The edge set \(E_H\) is an (edge-disjoint) union of cliques \(C_1, C_2, \ldots, C_n\) where \(n := |V_H|\).
- Each clique \(C_i\) is over some subset \(S_i\) of size \(\Delta\).
- Each vertex \(v \in V_H\) lies in exactly \(\Delta\) cliques.

Note these properties mean that the degree of each vertex is exactly \(\Delta(\Delta - 1)\), where we count parallel edges. Moreover, the total number of edges in \(H\) is \(n\left(\frac{\Delta}{2}\right)\). A Unique Label Cover instance is nice if it is \(\Delta\)-nice for some \(\Delta\). (The use of such a Unique Label Cover instance is also implicit, e.g., in \([KKMO07]\).)

**Lemma 5.2** Assuming the UGC, for any \(\eta > 0\), there is a large enough constant \(d = d(\eta)\) such that it is \(NP\)-hard to distinguish whether a nice Unique Label Cover instance with label size \(d\) has optimum at least \(1 - \eta\), or at most \(\eta\). Moreover, this holds for \(\Delta\)-nice instances where \(\eta < 1/2\Delta\).

**Proof.** We can assume we have instances \(B = (U, V, E_B)\) of bipartite Unique Label Cover which are \(\Delta\)-regular—all vertices in \(U \cup V\) have degree \(\Delta\) in \(B\)—where we cannot distinguish between \((1 - \delta)\)-versus-\(\delta\) fraction of satisfiable constraints. We define a non-bipartite Unique Label Cover instance \(H = (V, E_H)\) on the set \(V\) as follows: For each \(u \in U\), add edges in \(E_H\) between all its neighbors. (Hence, the number of edges added between \(v, w \in V\) equals the number of common neighbors they have.) Since each vertex \(u \in U\) results in a clique being added among its \(\Delta\) neighbors, the new instance is a union of \(n\) edge-disjoint \(\Delta\)-cliques. Moreover, each vertex in \(V\) belongs to \(\Delta\) cliques, equal to its degree in \(B\). So the instance is \(\Delta\)-nice. Finally, the bijection constraints are \(\sigma'_uv := \sigma_{uw} \circ \sigma_{wv}^{-1}\), with label set \([d]\).

Suppose the bipartite instance \(B\) was \(1 - \delta\) satisfiable. Let \(\text{label} : U \cup V \to [d]\) denote a labeling that satisfies \((1 - \delta)\) fraction of the constraints. Let \(\delta_u\) denote the fraction of constraints incident on \(u\) that are violated by \(\text{label}\) so that \(\sum_u \delta_u \leq n\delta\). In the clique corresponding to \(u\), this labeling violates at most \((\delta_u\Delta) \cdot (\Delta - 1)\) constraints. Thus the total number of violated constraints is at most \(\sum_u \delta_u \Delta(\Delta - 1) \leq n\delta\Delta(\Delta - 1) = 2\delta n\left(\frac{\Delta}{2}\right)\). Thus at least \(1 - 2\delta\) of the constraints in \(H\) are satisfiable.

Conversely, suppose a \(\eta\) fraction of constraints in the union of cliques instance \(H\) was satisfied by a labeling \(\chi\). We would like to extend this coloring to the vertices of \(U\) so that at least an \(\eta\) fraction of the constraints in \(B\) are satisfied. To this end, consider any \(u \in U\) and restrict attention to the clique of edges added among the neighbors of \(u\). Suppose \(\eta_u\) fraction of the \(\binom{\Delta}{2}\) edges in this clique were satisfied in \(H\) by \(\chi\). Note that \(E_u[\eta_u] = \eta\). Suppose these \(\eta_u\left(\frac{\Delta}{2}\right)\) satisfied edges form \(t\) connected components, with sizes \(k_1 \geq k_2 \geq \ldots \geq k_t\). The maximum number of satisfied edges within these components is \(\sum_{i=1}^t \binom{k_i}{2}\), which must be at least \(\eta_u \left(\frac{\Delta}{2}\right)\). This implies that \(\frac{1}{k_i} \binom{k_i}{2} \geq \eta_u \left(\frac{\Delta}{2}\right)\), and in turn, \(k_i \geq \eta_u \cdot \Delta\). In other words, there exists a connected component of satisfied edges with \(\eta_u \Delta\) vertices. Pick any vertex \(v\) in this component and set \(\chi(u) := \sigma_{wv}^{-1}(\chi(v))\). Note that this label for \(u\) satisfies not only the edge \((u, v) \in E_B\), but also the edges \((u, w)\) for \(w\) lying in this connected component. (This follows by the uniqueness of the constraints.) Do this independently for each \(u \in U\). The total fraction of constraints satisfied by this extension of \(\chi\) to a labeling of \(U \cup V\) is now \(E_u[\eta_u] = \eta\), which completes the proof.

Finally, observe that in any \(\Delta\)-nice instance, we can find a matching of size at least \(1/2\Delta\) and satisfy all these edges; hence the parameter \(\eta\) must be below \(1/2\Delta\).

In the next section, we give a reduction from such \(\Delta\)-nice instances of Unique Label Cover to Sparsest Cut on constant treewidth graphs. A little notation will be useful: for a vertex \(x = (x_1, x_2, \ldots, x_d) \in \{-1, 1\}^d\)
and a permutation \(\sigma\), define \(\sigma(x)\) to be the vector
\[
\left(x_{\sigma^{-1}(1)}, x_{\sigma^{-1}(2)}, \ldots, x_{\sigma^{-1}(d)}\right)
\]
and define \(\bar{x} = (-x_1, -x_2, \ldots, -x_d)\).

5.2 The Basic Instance

Consider a \(\Delta\)-nice Unique Label Cover instance \(H = (V_H, E_H)\) with bijections \(\{\sigma_{u,v}\}_{(u,v) \in E_H}\) and label set \(d\). Let \(n := |V_H|\) be the number of vertices in \(H\) and \(m := |E_H| = n(\frac{\Delta}{2})\) be the number of edges in \(H\). Let \(N := n \cdot 2^d\) and \(M := m \cdot 2^d\); hence \(M/N = m/n = (\frac{\Delta}{2})\).

The Sparsest Cut instance \((G, D)\) on the new set of nodes \(V\) is the following:

- **Nodes.** Take a cube \(Q_v = \{-1,1\}^d\) for each \(v \in V_H\). Add two new nodes \(s, t\). This is the new set of nodes \(V\) of size \(n \cdot 2^d + 2 = N + 2\). We will use \(x \in \{-1,1\}^d\) to denote cube nodes (i.e., those in \(V \setminus \{s, t\}\)), and will use the notation \(x^v\) to indicate that \(x\) is a cube node in cube \(Q_v\).

- **Supply Edges.** Add edges of capacity \(1/N\) from \(s\) to each node in \(\cup_{v \in V_H} Q_v\), and the same from \(t\) to these nodes. There are \(2N\) such edges, which we call star edges.

Add all the hypercube edges, but with capacity \(\alpha/N\) for some parameter \(\alpha > 0\). There are \(n \cdot d^{2d-1} = dN/2\) cube edges, which have total capacity \(\alpha d/2\). (Think of \(\alpha\) as a small quantity.)

- **Demand Edges.** For each edge \(e = (v, w) \in E_H\) and for each \(x \in \{-1,1\}^d\), add a demand edge with demand \(1/M\) between \(x \in Q_v\) and \(\sigma_{vw}(x) \in Q_w\). (If \(\sigma_{vw}\) were the identity permutation, then we would add a demand edge between each node in \(Q_v\) and its antipodal node in \(Q_w\).) Thus there is a total demand of \(2d/M\) for each edge \(e \in E_H\) and hence a total of \(m \cdot 2^d/M = 1\) of such demand. Observe that there are \(\Delta(\Delta - 1)\) demand edges \((x^v, \cdot)\) incident to each node \(x^v \in Q_v\) for each \(v \in V_H\).

This instance will be a good starting point for the powering operation of Section 4.2. Recall that a cut is admissible if it separates \((s, t)\), and is inadmissible otherwise. We will show that in the good case, there is a sparse admissible cut, whereas in the bad case, all admissible cuts have much higher sparsity. Additionally, we will prove a (weaker) lower bound on the sparsity of inadmissible cuts.

Intuitively, the factor of two comes from the following facts: if we connect two hypercubes with demands between \(x\) in the first hypercube to \(\bar{x}\) in the second, choosing the same dictator cut on both hypercubes cuts all demand pairs, while choosing different dictator cuts on the two hypercubes cuts only half the demand pairs. Thus restricted to dictator cuts, we get a gap of two. How do we exclude cuts that are far from dictators\(^4\)? Here we use the fact that sparse cuts should cut few supply edges within each hypercube. Freidgut’s junta theorem tells us that cuts that do not cut much more than a dictator cut in a hypercube, are close to juntas, which is sufficient to be able to “decode” a sparse cut into a good assignment to the unique games instance.

We start by recording some basic properties of \(G\), which will prove useful for the powering operation.

**Observation 5.3** Let \(G\) be the network defined above.

- The total capacity in the network is \((1 + \alpha d/2)\), and the total demand is 1.
- The treewidth of the supply graph is at most \(2^d\), the size of each hypercube.
- Any admissible cut has capacity at least 1.

\(^4\)This is where we are able to improve on the MaxCut UG-hardness of [KKMO07, MOO10], where the noise operator is applied to such a construction, and majority-is-stablest is used to exclude cuts far from dictators.
5.2.1 Completeness

Lemma 5.4 If there exists a labeling \( f : V \rightarrow [d] \) satisfying at least \( 1 - \eta \) fraction of the Unique Label Cover instance, then there exists a cut in \((G,D)\) of capacity at most \((1+\alpha/2)\) that cuts at least \((1-\eta)\) demand.

Proof. Consider the set \( A \) that contains \( s \) and, for each \( v \in V_H \), contains \( \{x \in Q_v \mid x_{f(v)} = 1\} \). This cut separates each node in \( V \setminus \{s,t\} \) from either \( s \) or \( t \), and hence separates \( N \) of the star edges to cut total capacity \( N \cdot 1/N = 1 \). Moreover, it cuts exactly \( 2^{d-1} \) edges from each hypercube, and hence \( \alpha/N : n2^{d-1} = \alpha/2 \) capacity in the cube edges. This gives a total of \((1+\alpha/2)\) capacity cut by \((A,\overline{A})\).

There are at least \((1-\eta)m \) edges \((v,w) \in E_H \) with \( \sigma_{w,v}(f(v)) = f(w) \). Consider one such edge \((v,w) \) and some \( x \in Q_v; \) say \( x_{f(v)} = b \in \{-1,1\} \). Then its demand edge corresponding to \((v,w) \) goes to \( \sigma_{w,v}(x) \in Q_w \), whose \( f(w) \) th coordinate contains

\[-x_{\sigma^{-1}_w(f(w))} = -x_{f(v)} = -b.\]

Hence, \( A \) cannot contain both \( x \in Q_v \) and \( \sigma_{w,v}(x) \in Q_w \) and thus cuts all \( 2^d \) demands corresponding to \((v,w) \in E_H \). The total demand cut is therefore at least

\[(1-\eta)m \cdot 2^d/M = (1-\eta).\]

This completes the proof.

5.2.2 Soundness

Theorem 5.5 For \( \varepsilon > 0 \), let \( \alpha = \varepsilon^2 \). If the resulting Sparsest Cut instance has an admissible cut \((A,\overline{A})\) in \((G,D)\) of sparsity less than \((2-4\varepsilon)\), then there exists a labeling for the Unique Label Cover instance that satisfies at least \( \eta' := (\varepsilon/2 - 1/\Delta)^{2(\varepsilon^{-4})} \) fraction of the edges in \( H \).

Proof. A road-map of the proof: We first show that a large fraction of the cubes are “good”, i.e., for most of the cubes, very few of the supply edges are cut. Using Friedgut’s Junta Theorem, these good cubes are close to being “juntas”, i.e., each contains a small number of influential coordinates. Finally, for a large amount of the demand to be separated using these juntas, there is a non-trivial number of edges in \( H \) that have end-points whose juntas share a common coordinate, so a randomized rounding procedure gives the claimed good labeling for \( H \).

Claim 5.6 The cut \((A,\overline{A})\) must separate more than \((\frac{1}{2} + \varepsilon)\) demand.

Proof. By Observation 5.3, \((A,\overline{A})\) has capacity at least 1. For the sparsity to be less than \((2-4\varepsilon)\), \((A,\overline{A})\) must cut at least \(1/(2-4\varepsilon) > (\frac{1}{2} + \varepsilon)\) units of demand.

Call a vertex \( v \in V_H \) good if the number of cube edges from \( Q_v \) cut by \((A,\overline{A})\) is at most \( \frac{32\alpha}{\varepsilon^2} 2^{d-1} \); we also say such a cube is good. (Note that a dimension cut—a.k.a. a dictator—would cut exactly \( 2^{d-1} \) edges, so good cubes do not have “too many more” cut edges than a dictator.)

Claim 5.7 There are at most \((\varepsilon/8)n \) bad vertices in \( V_H \).

Proof. Suppose not: Then the number of cube edges cut is strictly more than \((\varepsilon/8)n \cdot (32/\alpha \varepsilon) 2^{d-1} \), each of capacity \( \alpha/N \). This gives a total capacity of more than 2. But the total demand in the instance is 1, which would imply the sparsity of \((A,\overline{A})\) is more than 2, a contradiction.

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The rest of the argument shows that a cut separating \((1/2 + \varepsilon)\) units of demand (by Claim 5.6) that is good on most cubes (by Claim 5.7) can be decoded to a labeling for the Unique Label Cover instance. (Henceforth, we will not worry about the supply edges, etc., and will just consider the cubes and the demands.)

Let us view the cut \((A, \overline{A})\) as a function \(f'' : \bigcup_{v \in V_H} Q_v \to \{-1, 1\}\). We now perform two small changes to this function. Firstly, for any bad vertex \(v \in V_H\), let us change \(f''\) on the nodes of \(Q_v\) to be identically equal to 1. Note that the new function (which we call \(f'\)) might separate less demand, but since we changed the function value on an \(\varepsilon/8\) fraction of the cube nodes and the instance is regular, the demand separated must still be \((1/2 + \varepsilon - 2 \cdot \varepsilon/8) = (1/2 + 3\varepsilon/4)\).

Secondly, we use the following rephrasing of Freidgut’s Junta Theorem [Fri98] that has been used in the context of hardness for cut-related problems (see, e.g., [CKK+06]):

**Theorem 5.8 (Freidgut’s Junta Theorem)** If a function \(f' : \{-1, 1\}^d \to \{-1, 1\}\) cuts at most \(c 2^{d-1}\) edges of the cube \([-1, 1]^d\), then it is \(\varepsilon\)-close (i.e., differs in at most \(\varepsilon 2^d\) points) from a function \(f : \{-1, 1\}^d \to \{-1, 1\}\) that is a \(\exp\{O(c/\varepsilon)\}\)-junta.

(Recall that the function \(f\) is a \(K\)-junta if there is a set \(S \subseteq [d]\) of size at most \(K\), such that for any \(x \in \{-1, 1\}^d\), specifying the coordinates of \(x\) that lie within \(S\) determines the value \(f(x)\). I.e., \(f\) is constant on the subcubes obtained by fixing the bits in \(S\) and running over all the other bits.)

Using Theorem 5.8 for each good cube, we replace the function \(f'\) restricted to that cube by its \(\varepsilon/8\)-close \(K\)-junta, where \(K := \exp\{O(1/\varepsilon^2)\} = \exp\{O(\varepsilon^{-4})\}\). Here we used the fact that the number of edges cut in a good cube is at most \(32 \alpha^4 2^{d-1}\) the assumption \(\alpha = \varepsilon^2\). This redefined function, which we call \(f\), differs from \(f'\) on at most \(32 \alpha^4 2^{d-1}\cdot \varepsilon/8\) fraction of all the \(n \cdot 2^d\) cube nodes, which might result in at most \(2 \cdot \varepsilon/8 = \varepsilon/4\) fraction of the demand no longer being separated. This still leaves at least \(1/2 + 3\varepsilon/4 - \varepsilon/4 = 1/2 + \varepsilon/2\) demand separated. Moreover, now the newly redefined function \(f\) is a \(K\)-junta on each of the cubes—on the bad cubes by the first redefinition and on the good cubes by the second—and \(f\) still separates \((1/2 + \varepsilon/2)\) units of demand.

For each \(v \in V_H\), define the set \(J_v \subseteq [d]\) to be the set of \(K\) most influential coordinates of \(f\) restricted to the cube \(Q_v\): since \(f|_{Q_v}\) is a \(K\)-junta, specifying \(\{x_i^v\}_{i \in S}\) fixes the value \(f(x^v)\). Secondly, call an edge \((v, w) \in E_H\) compatible if \(\sigma_{vw}(J_v) \cap J_w \neq \emptyset\), i.e., if \(f|_{Q_v}\) and \(f|_{Q_w}\) “share” an influential coordinate (after applying the right permutation). Observe that if an edge is compatible, then assigning each vertex \(v\) a label randomly chosen from its set \(J_v\) would satisfy each compatible edge with probability at least \(1/2K^2\). The following lemma is now the final piece in the argument.

**Lemma 5.9** The number of compatible edges \((v, w) \in E_H\) is at least \((\varepsilon/2 - 1/\Delta)m\).

**Proof.** Consider all the cubes \(Q_v\). Recall that we don’t care about the supply edges for this argument, only the demand edges. For each \((v, w) \in E_H\), these demand edges give a matching between the nodes of \(Q_v\) and \(Q_w\); the union of all these matchings gives us all the demand edges. Finally, \(f\) gives us a \([−1, 1]\)-coloring of the nodes of the cubes and separates \((1/2 + \varepsilon/2)\) fraction of these demand edges.

For each cube \(Q_v\), collapse all the nodes \(x^v\) whose values on the coordinates in \(J_v\) are the same to get a cube \(\tilde{Q}_v\) isomorphic to \([-1, 1]^K\). (Each new node \(\tilde{x}^v\) in \(\tilde{Q}_v\) comprises of \(2^{d-K}\) nodes collapsed together.) Moreover, since any two nodes collapsed together agree on their \(f\)-value, the number of separated demand edges in the resulting multigraph remains unchanged.

Suppose \((v, w)\) is not compatible. We claim that the \(2^d\) demand edges between \(Q_v\) and \(Q_w\) now form a complete bipartite multigraph between \(\tilde{Q}_v\) and \(\tilde{Q}_w\), with \(2^{d-K}\) demand edges going between each \(\tilde{x}^v \in \tilde{Q}_v\) and \(\tilde{x}^w \in \tilde{Q}_w\). For simplicity, assume that \(\sigma_{vw}\) is the identity map, so incompatibility means that \(J_v \cap J_w = \emptyset\), and imagine that \(J_v = \{1, 2, \ldots, K\}\) and \(J_w = \{K + 1, K + 2, \ldots, 2K\}\). A demand edge that used to go between \((b_1, b_2, \ldots, b_d) \in Q_v\) and \((-b_1, -b_2, \ldots, -b_d) \in Q_w\) now goes between
There are \( 2^{d-2K} \) of these edges, and they are the only ones. This proves the claim.

Now suppose there were no compatible edges at all. Then for each edge \((v,w)\) in \( H \), we would add a complete bipartite multigraph between the \( 2^K \) nodes in \( \hat{Q}_v \) and \( \hat{Q}_w \). This would be exactly the multigraph \( H \times I_\ell^K \) (where \( I_\ell \) is a graph on \( \ell \) vertices and no edges at all), but with each edge replicated some number \( 2^{d-2K} \) number of times. It is easy to show that, because \( H \) is a union of \( \Delta \)-cliques, \( f \) can cut at most \( 1/2 + 1/\Delta \) fraction of the demand edges; note this is precisely where we use the union-of-cliques structure of \( H \). The simple proof is in Claim 5.11.

However, we claimed that \( f \) cut \( 1/2 + \varepsilon/2 \) fraction of the edges, so there must be at least one compatible edge. Observe that the number of cut edges behaves in a Lipschitz fashion as we change the number of compatible edges: Each compatible edge \((v,w)\) ∈ \( E_H \) means the cut may be higher by at most an additive \( 2^d \) amount (since this is the total number of demand edges corresponding to an \( H \)-edge), which is a \( 1/m \) fraction of the total demand. If we want the fraction of demand edges cut to increase by \( \varepsilon/2 - 1/\Delta \), and each compatible edge can increase this additively by at most \( (\varepsilon/2 - 1/\Delta)m \) compatible edges.

Choosing a label for each \( v \in V_H \) randomly from \( J_v \) and using Lemma 5.9, the expected fraction of satisfied edges in \( E_H \) is at least \( (\varepsilon/2 - 1/\Delta)/K^2 \). Noting that \( K = \exp\{O(\varepsilon^{-4})\} \) completes the proof of Theorem 5.5.

Finally, we show that inadmissible cuts cannot have sparsity better than 1

**Claim 5.10** Any inadmissible cut \((A, \overline{A})\) satisfies \( \dem(A, \overline{A}) \leq \cap(A, \overline{A}) \).

**Proof.** By Lemma 4.6, we can assume \((A, \overline{A})\) is connected and assume w.l.o.g. that \( s, t \notin A \). Then \( A \) is a connected subset of one cube \( Q_v \). This means \( \partial A \) contains all star edges adjacent to \( A \), so \( \cap(A, \overline{A}) \geq 2|A| \cdot 1/N \). Moreover, for each \( x^v \in A \), all the \( \Delta(\Delta - 1) \) demand edges incident to \( x^v \) are cut, so the total demand cut is \( \Delta(\Delta - 1) \cdot |A| \cdot 1/M = 2|A|/N \).

### 5.2.3 MaxCut on Lifts of \( K_n \)

The following simple lemma was useful in the analysis of Theorem 5.5.

**Claim 5.11** Let \( H \) be a multigraph that is a union of cliques, each on exactly \( \Delta \geq 2 \) vertices. Let \( I_\ell = (\{1,2,\ldots,\ell\},\emptyset) \) be a graph of \( \ell \) vertices with no edges. For each \( \ell \in \mathbb{Z}_{\geq 1} \), every cut in the graph \( H \times I_\ell \) contains at most \( \left\lfloor \frac{\Delta}{2} \right\rfloor \left\lfloor \frac{\Delta}{2} \right\rfloor \leq \left( \frac{\Delta}{2} \right) \left( \frac{1}{2} + \frac{1}{2(\Delta-1)} \right) \) edges. Hence, any cut separates at most the claimed fraction of edges in each clique, and the fact that \( H \) is a union of (edge-disjoint) cliques gives the result for \( \ell = 1 \).

For general \( \ell \) case: For each \( v \), let \( b_v \) be the fraction of its \( \ell \) copies colored blue. Color \( v \) blue with probability \( b_v \) and red with probability \( 1-b_v \). This gives a coloring (i.e., cut) of the vertices of \( H \). For any edge \((u,v)\) in \( H \), this coloring cuts \( \ell^2 \cdot (b_u(1-b_v) + (1-b_u)b_v) \) edges in \( H \times I_\ell \), which is exactly \( \ell^2 \) times the probability of \((u,v)\) being cut by the random coloring in \( H \). Hence the fraction of edges cut in \( H \times I_\ell \) is exactly the expected fraction of edges in \( H \) cut by the random coloring. But the argument above implies that each 2-coloring of \( H \) cuts at most \( 1/2 + 1/2(\Delta - 1) \) fraction of its edges, which lower bounds the expectation and proves the claim.
5.3 Putting It Together

To boost the hardness result from Section 5.2, we will use the powering operation on the instance defined above. Intuitively, our proof gave us a $2 - \varepsilon$ gap as we long as we could ignore the inadmissible cuts. The powering operation decreases the sparsity in both the good and the bad cases. However, it ensures that even inadmissible cuts induce admissible cuts at lower levels and thus cannot have much better sparsity.

Let $G_1$ be the instance from Section 5.2. Then Observation 5.3, Theorem 5.5, and Claim 5.10 can be summarized as:

**Lemma 5.12** Suppose that there is no labeling $f : V \to [d]$ satisfying an $\eta$ fraction of the unique label cover instance. Then there exists $\eta' := \eta'(\varepsilon, \eta)$ such that the graph $G_1$ satisfies:

- Any admissible cut $(A, \overline{A})$ has capacity $\text{cap}(A, \overline{A})$ at least 1.
- Any admissible cut $(A, \overline{A})$ cuts at most $(\frac{1}{2} + \eta') \cdot \text{cap}(A, \overline{A})$ demand.
- Any inadmissible cut $(A, \overline{A})$ cuts at most $\text{cap}(A, \overline{A})$ demand.

Let $G_\ell$ be the instance applying the powering operation to the instance $G_1$, and let $\ell = \frac{4}{\varepsilon}$ and $\alpha = \varepsilon^2$. Using Lemma 5.4 and Lemma 4.5, a straightforward calculation shows:

**Lemma 5.13** If there exists a labeling $f : V \to [d]$ satisfying at least $1 - \eta$ fraction of the unique label cover instance, then there exists a cut $(A_\ell, \overline{A}_\ell)$ in $G_\ell$ such that $\text{cap}(A, \overline{A}) \leq (1 + \frac{\varepsilon}{2})^\ell \leq 1 + 4\varepsilon$ and $\text{dem}(A, \overline{A}) \geq \ell(1 - \eta)$.

Hence, in the “yes” case, the sparsity is at most

$$\frac{(1 + 4\varepsilon)}{\ell(1 - \eta)}.$$  \hfill (5.14)

On the other hand, Lemma 4.7 implies that:

**Lemma 5.14** Suppose that there is no labeling $f : V \to [d]$ satisfying an $\eta$ fraction of the unique label cover instance. Then the graph $G_\ell$ satisfies:

- Any admissible cut $(A, \overline{A})$ cuts at most $\ell(\frac{1}{2} + \eta') \cdot \text{cap}(A, \overline{A})$ demand.
- Any inadmissible cut $(A, \overline{A})$ cuts at most $(\ell(\frac{1}{2} + \eta') + \frac{1}{2})\text{cap}(A, \overline{A})$ demand.

In both these cases, the sparsity is at least

$$\frac{1}{\ell(\frac{1}{2} + \eta') + \frac{1}{2}} \geq \frac{1}{\ell(\frac{1}{2} + \eta' + \varepsilon/2)}.$$  \hfill (5.15)

From (5.14) and (5.15), we conclude that the hardness is at least $\frac{2(1-\eta)}{(1+4\varepsilon)(1+2\eta'+\varepsilon)} \geq 2(1 - 5\varepsilon - \eta - 2\eta')$. Thus for any $\varepsilon' > 0$, we can pick $\varepsilon$ small enough to get a $2 - \varepsilon'$ hardness.

6 A $2 - \varepsilon$ Integrality Gap for Sherali-Adams

In this section, we show that how to translate Sherali-Adams integrality gaps for the MaxCut problem into corresponding Sherali-Adams integrality gaps for Non-Uniform Sparsest Cut.

6.1 The Sherali-Adams LP and Consistent Local Distributions

We begin by recording a standard result stating that an $r$-round Sherali-Adams solution is essentially equivalent to the existence of a collection of consistent “local” distributions $\{D_S\}_{|S| \leq r}$.
Theorem 6.1 There exists a \((x,y) \in S_{A_\epsilon}(n)\) if and only if for every \(S \subseteq V\) of size at most \(r\), there exists a distribution \(D_S\) over subsets of \(S\) and these distributions satisfy the following property: For any \(Q \subseteq S\) and for every \(A \subseteq Q\),
\[
D_Q(\{A\}) = D_S(\{B : B \cap Q = A\}).
\] (6.16)

**Proof.** Assume we have \((x,y) \in S_{A_\epsilon}(n)\). For a set \(S \subseteq V\) such that \(|S| \leq r\), we define \(D_S\) such that \(D_S(T) = x(S,T)\). By (2.4) and (2.2), all \(x(S,T) \geq 0\) and \(\sum_{S \subseteq T} x(S,T) = 1\), so this is a well-defined distribution. We now need to show that these distributions satisfy (6.16), i.e., we need to show that
\[
x(Q,A) = \sum_{B \subseteq S, B' \cap Q = A} x(S,B).
\]
This follows directly from Lemma 2.1:
\[
\sum_{B \subseteq S, B' \cap Q = A} x(S,B) = \sum_{B' \subseteq S \setminus Q} x(Q \cup (S \setminus Q), B' \cup A)
= x(Q,A).
\]

On the other hand, assume we have distributions \(D_S\) for all \(S \subseteq V\) such that \(|S| \leq r\). Set \(x(S,T) = D_S(T)\). The fact that \(D_S\) is a distribution implies that \(\sum_{T \subseteq S} D_S(T) = 1\) and the non-negativity constraints are satisfied. Finally, (6.16) implies that the consistency constraint
\[
D_S(T) = D_{S+u}(T) + D_{S+u}(T + u)
\]
is satisfied.

6.2 The Integrality Gap Instance

In this section, we show that the integrality gap of the LP for Sparsest Cut remains \(2 - \epsilon\), even after polynomially many rounds of Sherali-Adams. Recall the construction from Section 4: Given a connected unweighted instance \(H = (V,E_H)\) of MAXCUT, it produces an instance \(G_\ell\) with vertices \(V_\ell\) and supply edges \(E_\ell\), such that the sparsest cut in \(G_\ell\) is related to the max cut in \(H\). We use the same construction here, but instead of starting with a hard instance \(H\) of MAXCUT, we start with a MAXCUT instance exhibiting an integrality gap for \(r\) rounds of Sherali-Adams.

We first show that there exist “local” distributions satisfying the conditions of Theorem 6.1 for all subsets of the vertices of \(G_\ell\) of size at most \(O(r)\). This implies the existence of an \(O(r)\)-round Sherali-Adams solution for the Sparsest Cut instance. We then calculate the value of this fractional solution for \(G_\ell\) and relate it to the integral optimum. This result is a natural extension of those of [LR10, CSW10], who showed a gap of 2 for the basic LP also on the fractal for \(K_{2,n}\) using the fact that the complete graph \(K_n\) exhibits an integrality gap for the basic MAXCUT LP.

**Lemma 6.2** Consider an unweighted MAXCUT instance \(H\) for which we have an \(r\)-round Sherali-Adams solution and \(G_\ell\) constructed from \(H\). For any \(T \subseteq V_\ell\) of size at most \(r\), there exists a distribution \(D_T\) over subsets of \(T\) such that for any \(Q \subseteq T\) and for every \(A \subseteq Q\), (6.16) holds.

**Proof.** Let us first define these local distributions. By Theorem 6.1, an \(r\)-round Sherali-Adams solution for the MAXCUT instance \(H\) gives, for each subset \(R \subseteq V_H\) of size at most \(r\), a probability distribution \(F_R\) over subsets of \(R\). Moreover, these local distributions \(F_R\) satisfy the consistency constraints (6.16).

We will use this set of distributions \(\{F_R\}_{R \subseteq V_H, |R| \leq r}\) to define another set of distributions \(\{D_T\}_{T \subseteq V_\ell, |T| \leq r}\) which also satisfy the consistency constraints (6.16).

Recall that \(G_\ell\) is obtained by taking a copy of \(G_1 = (V_1,E_1)\) and then replacing each edge \(e \in G_1\) by a copy of instance \(G_{\ell-1}\) (which we call \(G_{\ell-1}^e\)). To define the distribution for \(T \subseteq V_\ell\) with \(|T| \leq r\), we first extend \(T\) to a set \(T'\) in the following way:
To show that the size of $T$ some notation: For sets $A$ and $D$ of the local distributions been added due to some $y$. Since we make draws from the probability distributions $F$, the distribution $D$ Let $T_1'$ be the vertices of $T'$ that are in $G_1$, i.e., $T_1' = T' \cap V_1$, and let $T_\ell$ be all vertices of $T'$ in the instance of $G_{\ell-1}$ corresponding to edge $e$ in $G_1$.

The distribution $D_T$ is given by the following recursive process that takes an instance $G_\ell$ and a set $T \in V_\ell$ and outputs a random subset $X \subseteq T$:

(i) Put $s$ in $X$; $t$ will never be in $X$.
(ii) Draw a subset $Y$ from $F_{T_1'}$. With probability 1/2, set $X \leftarrow X \cup Y$. With probability 1/2, set $X \leftarrow X \cup (T_1' \setminus Y)$.
(iii) For each vertex $v \in V_1$, do the following:
   (a) If $v \in X$, set $X \leftarrow X \cup T_{st}$ and recurse on $G_{\ell-1}', T_{st}'$.
   (b) If $v \notin X$, set $X \leftarrow X \cup T_{vt}$ and recurse on $G_{\ell-1}', T_{vt}'$.
(iv) Note that $X \subseteq T'$, so output $X \cap T$.

Since we make draws from the probability distributions $F_{T_1'}$, we should ensure that these distributions are well-defined. In particular, since $T_1'$ is a subset of $T'$ which could be larger than $|T| = r$, we need to show that the size of $T_1'$ is at most $r$. Indeed, if $T_1'$ contains some vertex $x$ not in $T$, this vertex has been added due to some $y \in T$ in $G_{\ell-1}'$ and hence can be charged to $y \in T$. This finishes the definition of the local distributions $D_T$.

It remains to show that for any $A \subseteq Q \subseteq T$, the consistency condition (6.16) holds. Let us introduce some notation: For sets $Y \subseteq X$, let $\Pr_X(pick Y)$ indicate the probability that $Y$ is chosen from the distribution $D_X$. Using this notation, (6.16) is the same as showing that for $T \subseteq V_\ell$ of size at most $r$ and any $A \subseteq Q \subseteq T$, $$\Pr_Q(pick A) = \sum_{B \subseteq T : B \cap Q = A} \Pr_T(pick B).$$ (6.17)

It is easy to see that it suffices to prove this for the case where $T = Q + q$ for some $q \in V_\ell \setminus Q$. In this case, things simplify to showing that for every $A \subseteq Q$, $$\Pr_Q(pick A) = \Pr_T(pick A) + \Pr_T(pick (A + q)).$$ (6.18)

To prove (6.18), it then suffices to show $$\Pr_{Q'}(pick A) = \Pr_{T'}(pick A) + \Pr_{T'}(pick (A + q))$$ (6.19) for any $A \subseteq Q'$. Recall that even though $|T \setminus Q| = 1$, the extended sets $T'$ and $Q'$ may differ in more vertices.

We proceed by induction on $\ell$. In the base case, consider $G_1$. If $q$ is $s$ or $t$, this is trivial. Otherwise, the claim holds by the consistency properties of the $\{F\}$ distributions. In the inductive case, we assume (6.19) holds for $G_{\ell-1}$ and want to show that this claim holds for $G_{\ell}$. There are two subcases: $q \in V_1$ and $q \notin V_1$. If $q \in V_1$, the sets $Q'$ and $T'$ only differ on vertices in $V_1$ and the claim holds by the consistency of the $\{F\}$ distributions.

Otherwise, $q \notin V_1$. Let $Q_1' = Q' \cap V_1$ and $A_1 = A \cap V_1$. We denote by $V_{\ell-1}'$ the vertices of $G_{\ell-1}'$. Let $A_e$ be $A \cap V_{\ell-1}'$ and let $Q_e' = Q' \cap V_{\ell-1}'$ for any edge $e$ in $G_1$. Because our selection process is independent on each of the $V_{\ell-1}'$'s given its choice in $V_1$, we have that $$\Pr_{Q'}(pick A) = \Pr_{Q_1'}(pick A_1) \prod_{e \in G_1} \Pr_{Q_e'}(pick A_e | pick A_1)$$
and
\[ \Pr_T(\text{pick } A) = \Pr_T(\text{pick } A_1) \prod_{e \in G_1} \Pr_T(\text{pick } A_e \mid \text{pick } A_1). \]

Since \( q \notin V_1 \), \( Q'_1 = T'_1 \). Also, since \( T = Q + q \), observe that \( Q'_e \) and \( T'_e \) must be the same for all edges \( e \) of \( G_1 \) except for one. Call this edge \( e^* \). These two facts imply that
\[ \Pr_T(\text{pick } A) = \Pr_T(\text{pick } A_1) \Pr_{Q'_1}(\text{pick } A_{e^*} \mid \text{pick } A_1) \prod_{e \neq e^*} \Pr_{Q'_e}(\text{pick } A_e \mid \text{pick } A_1). \tag{6.20} \]

Similarly, we know that
\[ \Pr_T(\text{pick } (A + q)) = \Pr_T(\text{pick } A_1) \Pr_{Q'_1}(\text{pick } (A_{e^*} + q) \mid \text{pick } A_1) \prod_{e \neq e^*} \Pr_{Q'_e}(\text{pick } A_e \mid \text{pick } A_1). \tag{6.21} \]

By the inductive assumption, we know that
\[ \Pr_{Q'_e}(\text{pick } A_{e^*}) = \Pr_{T'_e}(\text{pick } A_{e^*} \mid \text{pick } A_1) + \Pr_{T'_e}(\text{pick } (A_{e^*} + q) \mid \text{pick } A_1). \]

Adding (6.20) and (6.21) therefore gives us
\[ \Pr_T(\text{pick } A) + \Pr_T(\text{pick } (A + q)) = \Pr_T(\text{pick } A_1) \prod_{e \in G_1} \Pr_{Q'_e}(\text{pick } A_e \mid \text{pick } A_1) = \Pr_T(\text{pick } A). \]

Now, applying Theorem 6.1 to the MaxCut instance from Lemma 6.2, we get an \( r \)-round Sherali-Adams solution \( y \) for Sparsest Cut. Recall that we set \( y_e = D_{\{i,j\}}(\{\{i\}, \{j\}\}) \), which is the probability over the distribution \( D_{\{i,j\}} \) that exactly one of the endpoints is chosen. We analyze the Sparsest Cut objective function value of this fractional solution \( y \) next.

**Lemma 6.3** Let the value of the \( r \)-round Sherali-Adams relaxation solution \( z \) for the MaxCut instance on \( H \) implied by the distributions \( \{F_S\} \) be \( \sum_{e \in E_H} z_e = c.m. \) Then the sparsity of the \( r \)-round solution \( y \) is \( \frac{1}{\ell c} \).

**Proof.** Let \( \text{cap}_{ij} \) be the capacity of edge \( \{i, j\} \) and \( \text{dem}_{ij} \) be the demand on edge \( \{i, j\} \) in the instance \( G_t \). Let \( E_t^c \) be its capacity edges and \( E_t^d \) its demand edges.

We begin by proving that \( \sum_{e \in E_t^c} \text{cap}_e y_e = 1 \). First, we claim that \( y_e = 2^{-\ell} \) for any capacity edge \( e \). By the symmetry introduced in Step ii(c) of the definition of \( D_T \), a capacity edge of \( G_1 \) is cut with probability \( 1/2 \), so \( y_e = 1/2 \) for any such edge \( e \). Using this fact, a simple induction then suffices. Second, we claim that \( \sum_{e \in E_t^c} \text{cap}_e = 2^\ell \). Since \( \sum_{e \in E_t^c} \text{cap}_e = 2^\ell \), a simple induction again gives us the desired result. Combining these facts, we get \( \sum_{e \in E_t^c} \text{cap}_e y_e = 1 \).

Next, we show that \( \sum_{e \in E_t^d} \text{dem}_e y_e = \ell c \). The proof is again by induction on \( \ell \). The base case is \( G_1 \), where the value is \( c \) because \( y_e = z_e \) for \( e \in E_H \) and all demands are \( 1/m \). For the induction step, assume \( \sum_{e \in E_{\ell-1}^d} \text{dem}_e y_e = (\ell - 1)c \). For each \( e \in E_1^c \), the corresponding \( G_{\ell - 1}^e \) contributes \( \frac{1}{2}(\ell - 1)c \cdot \text{cap}_e \) by the inductive hypothesis and the fact that \( y_{e'} = \frac{1}{2}y_e \) for all \( e' \in E_1^c \) and \( e'' \in E_{\ell - 1}^c \). Summing over all \( e \in E_1^c \) gives \( (\ell - 1)c \). By the base case, level \( \ell \) demands contribute \( c \), giving a total of \( \ell c \) demand cut.

Given the above construction, we can now prove the main theorem of this section, which allows us to convert Sherali-Adams integrality gaps for MaxCut to Sherali-Adams integrality gaps for Sparsest Cut.
Theorem 6.4  Given an unweighted MaxCut instance $H$ on $n$ nodes and $m$ edges with a max cut of size $sm$ and an $r$-round Sherali-Adams solution of value at least $cm$, for any constant $\varepsilon > 0$ there exists a Sparsest Cut instance $G$ with an $r$-round Sherali-Adams integrality gap of $(c/s) - \varepsilon$, such that the size of $G$ is $n^{O(c/s^2\varepsilon)}$.

Proof. We will set $G$ to be the graph $G_\ell$ constructed from base instance $H$ for some value of $\ell$ to be chosen later. By Lemma 6.2, there exists an $r$-round Sherali-Adams solution $y$ whose sparsity is $\frac{1}{\ell}$ by Lemma 6.3. By calculations as in Section 4.2.1, the actual sparsest cut value for $G_\ell$ is at least $\frac{c}{s} - \varepsilon$ for $\ell = \frac{c}{s^2\varepsilon}$. This gives us an $r$-round Sherali-Adams integrality gap of $\frac{\ell c}{1+(\ell-1)s} \geq \frac{c}{s} - \varepsilon$ for $\ell = \frac{c}{s^2\varepsilon}$. Plugging in the Sherali-Adams integrality gap instance for MaxCut due to Charikar et al. [CMM09] gives us the following corollary:

Corollary 6.5 For every $\varepsilon > 0$, there exists $\gamma > 0$ such that the integrality gap of the Sparsest Cut relaxation is $2 - \varepsilon$ even after $n^\gamma$ rounds of Sherali-Adams.

Proof. From [CMM09, Theorem 5.3], we have that for any $\varepsilon' > 0$, there exists $\gamma' > 0$ and an unweighted MaxCut instance with $M$ edges and $N$ nodes such that the optimal integral solution is at most $M(1/2 + \varepsilon'/6)$ and the LP value after $N^{\gamma'}$ rounds of Sherali-Adams is at least $N(1 - \varepsilon'/6)$. From this, we get $c/s > 2 - \varepsilon'$. Now Theorem 6.4 allows us, for any $\varepsilon'' > 0$, to obtain a Sparsest Cut instance with integrality gap $2 - \varepsilon' = \varepsilon''$ after $N^{\gamma'}$ rounds of Sherali-Adams. Setting $\varepsilon' = \varepsilon'' = \varepsilon/2$ gives us the integrality gap of $2 - \varepsilon$. Moreover, the size of the new instance is $n = N^{O(c/s^2\varepsilon)}$, so setting $\gamma = O(s^2\varepsilon\gamma'/c)$ completes the proof.

7 Conclusions

We show how to use the Sherali-Adams hierarchy to get a factor-2 approximation for the Non-Uniform Sparsest Cut problem on treewidth-$k$ graphs in time $n^{O(k)}$. (This also gives $2^{O(\sqrt{n})}$-time 2-approximation algorithms for Sparsest Cut on minor-free graphs.) We also show that the Non-Uniform Sparsest Cut problem is as hard as the MAXCut problem, even for treewidth-2 graphs, which gives us the best NP-hardness known (even for the unconstrained problem). Assuming the UGC, this gives a hardness of $1/0.878 - \varepsilon$ for these series-parallel graphs. For graphs of large constant treewidth, we show a Unique Games hardness of $2 - \varepsilon$, which matches our algorithm. Finally, we demonstrate an integrality gap of $2 - \varepsilon$ for Sherali-Adams relaxations after a polynomial number of rounds, even for treewidth-2 graphs.

Many research directions remain open. Among them are getting better hardness results for Non-Uniform Sparsest Cut, both for restricted graph classes and for the general problem, getting polynomial-time $O(1)$-approximation algorithms for planar or minor-closed families (using LP/SDP hierarchies or otherwise), and making progress on the embeddability conjecture from [GNRS04].

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