Dynamical chiral symmetry breaking in $SU(N_c)$ gauge theories with large number of fermion flavors

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In this paper we examine a phase transition in $SU(N_c)$ gauge theories governed by the existence of an infrared fixed point of the renormalization group $\beta$ function. The nonlinear integral Schwinger-Dyson equation for a mass function of massless fermions is solved numerically using the exact expression of the running coupling in two-loop approximation for an $SU(3)$ gauge theory. Based on the obtained solution of the Schwinger-Dyson equation, the value of the chiral condensate, $(\bar{q}q)$, and the decay constant, $f$, of bound states are calculated for several values of fermion flavors $N_f$. We show that this kind of phase transition is a transition of finite order.

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I. INTRODUCTION

Gauge field theories with a large number of massless fermions are becoming an attractive topic for theoretical research. A recent discovery of a phase transition in such theories [1], [2] has led to additional interest. Depending on the number of fermions, there are two possible phases. The first phase is a phase with broken chiral symmetry and confinement which occurs when the number of fermions is less than some critical value $(N_f < N^c_f)$, where $N^c_f$ is a critical value of the number of fermions. The second phase, occurs when $(N_f > N^c_f)$, is a phase with strict chiral symmetry and absence of confinement. This type of phase transition occurs for example in minimal supersymmetrical QCD [3]. The dynamics of these two phases is well studied in the approximation discussed below.

The reason for the phase transition is the existence of an infrared fixed point (IFP), with coupling constant $\alpha_c = \alpha(0)$, in two loop approximation for the renormalization group $\beta$ function. Also the mathematical aspect of the phase transition becomes clear from the analysis of the integral Schwinger-Dyson equation (SDE) for the mass function. Assigning to the running coupling the constant value at the IFP, $\alpha(Q^2) \equiv \alpha_c$, the SDE turns out to be an equation for eigenvalues and has only trivial solution with sufficiently small $\alpha_c$. In contrast, if $\alpha_c$ is larger than a critical value $\alpha_c = \pi/(3C_2(F))$, then the SDE has nontrivial solutions. The critical value $N^c_f$ is then determined from the relation $\alpha_c = \alpha_c(N_f, N_c)$, at constant $N_c$. However the situation changes when we use the running coupling. The kernel of the integral SDE, like the running coupling, is a function of the number of fermions and colors. And it is unclear what should be considered as a critical value of $\alpha_c$. The most obvious solution would be the introduction of a certain integral characteristic for the SDE (as has been done for the symmetric kernel in the theory of integral equations). We could then consider number of fermions as the free parameter, which would determine the value of $\alpha_c$.

If a chiral symmetry is broken then there exist boson degrees of freedom, which of course arise as Goldstone bosons. We examine the chiral phase transition by studying order parameters like the quark condensate, $(\bar{q}q)$, and the decay constant, $f$, of bound states. Examining these quantities near the point of phase transition may shed light on the nature of the transition. In this paper we study numerical solution of the SDE for a mass function using an exact expression for the running coupling. The existence of both trivial and nontrivial solutions of the SDE confirms that the phase transition takes place in an $SU(3)$ gauge theory.

In section II we discuss the equation for the running coupling in two loop approximation. In section III we briefly go over the conservation of Ward-Takahashi identities and set up the SDE in the local gauge. Section IV is devoted to the numerical calculations and discussion.

II. SOME PROPERTIES OF THE GAUGE THEORIES WITH THE INFRARED FIXED POINT

Let’s start from the Lagrangian of the $SU(N_c)$ gauge field theories. It appears as follows:

$$\mathcal{L} = \sum_{k=1}^{N_f} \bar{\psi}_k (i\hat{D}) \psi_k - \frac{1}{4} F_{\mu\nu}^{\alpha \beta} F^{\alpha \beta}_{\mu\nu},$$

where $\psi_k$ is a four component spinor of flavor $k$, $D^{\mu} = \partial^{\mu} - ig A_{\mu}^{\alpha} T_\alpha$, $T_\alpha$ are the generators of the gauge group and $g$ is the coupling constant. This Lagrangian is clearly

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invariant under the global symmetry group $SU(N_f)_L \times SU(N_f)_R \times U(1)_{L,R}$ because all fermions are massless. Nonetheless, this symmetry may be broken to diagonal subgroup $SU(N_f)_{L+R} \times U(1)_{L,R}$.

The next step is an analysis of the equation for the running coupling in two loop approximation (the first two coefficients are independent of the renormalization scheme, the higher-order coefficients are scheme dependent). It takes the form:

$$\frac{d\alpha}{d\ln(Q^2/\nu)} = -b\alpha^2 - c\alpha^3 - \ldots, \quad (II.2)$$

where $\alpha = g^2/4\pi$ and according to Ref.[4] coefficients $b$ and $c$ look as follows:

$$b = \frac{1}{12\pi}(11C_2(A) - 4T_fN_f), \quad (II.3)$$
$$c = \frac{1}{16\pi^2}\left(\frac{34}{3}C_2(A)^2 - \frac{20}{3}C_2(A)T_fN_f - 4C_2(F)T_fN_f\right), \quad (II.4)$$

The theory is asymptotically free if $b > 0$. The IFP exists if $c < 0$, which for $SU(3)$ takes place when $N_f > 8$. The running coupling at the IFP takes the value $\alpha_s = -b/c$. The fixed point coupling $\alpha_s$ can be made sufficiently small to perform a calculation by perturbation theory.

Certainly we assume that $0 < g \lesssim 1$ in the initial Lagrangian (II.1), which restricts the variation interval for $N_f$. For an $SU(3)$ gauge theory this variation interval is $10 \lesssim N_f < 33/2$ assuming an asymptotic freedom of the theory.

The equation (II.2) can be integrated as follows:

$$\frac{1}{\alpha(Q^2)} + \frac{1}{\alpha_s}\ln\left(b + \frac{c}{\alpha(Q^2)}\right) = b\ln\frac{Q^2}{\Lambda^2}, \quad (II.5)$$

where we have introduced the scale

$$\Lambda^2 = \nu^2\left(\frac{b}{\alpha_s(\nu^2)} + c\right)e^{b/c}\exp\left(-\frac{1}{b\alpha_s(\nu^2)}\right), \quad (II.6)$$

which has the same physical sense as the dimensional $\Lambda_{QCD}$ parameter in ordinary QCD. For further calculations we consider $\Lambda$ to be independent of $N_f$, for a fixed value of $N_f$. This is a good approximation when we are taking into consideration the small variation interval of $N_f$.

Equation (II.5) is a transcendental equation for $\alpha(Q^2)$. It can be solved analytically using complex Lambert $W_k(z)$ function [3]. Lambert’s function satisfies the transcendental equation $W_k(z) \exp(W_k(z)) = z$, where $k = 0, \pm 1, \pm 2, \ldots$. We note that there are only two possible real solutions $k = 0, -1$ which take place if the argument $z \geq -1/e$. The Lambert $W$ function has simple asymptotics: $W_0(z \to 0) \sim z - z^2$, $W_{-1}$ diverges when $z \to 0$, and $W_k(z \to \infty) \sim \ln z + 2\pi ik - \ln(\ln z + 2\pi ik)$.

Then we have:

$$\alpha(Q^2) = \alpha_s\left(1 + W_i\left(-\frac{(Q^2/\Lambda^2)^{b\alpha_s}}{ce}\right)\right)^{-1}, \quad (II.7)$$

$$i = \begin{cases} 1, & c > 0 \\ 0, & c < 0 \end{cases}. \quad (II.8)$$

The case $i = -1$ corresponds to the running coupling in ordinary QCD in two loop approximation (in the perturbative regime). Further, only the case $c < 0$ will be used for calculations. Based on the properties of Lambert’s function, it is possible to obtain the asymptotic form of (II.7):

$$\alpha(Q^2 \to 0) = \alpha_s\left(1 + \frac{(Q^2/\Lambda^2)^{b\alpha_s}}{ce}\right), \quad (II.8)$$
$$\alpha(Q^2 \to \infty) = \frac{1}{b\ln(Q^2/\Lambda^2)} \left(1 + \frac{1}{b\alpha_s}\ln\ln(Q^2/\Lambda^2)\right). \quad (II.9)$$

It is important to note that as the number of fermions decreases, the value of asymptote (II.8) increases, and the value of asymptote (II.9) decreases.

III. SCHWINGER-DYSON EQUATION FOR
THE MASS FUNCTION

Since the initial Lagrangian has only massless fermions, we must require conservation of the vector and axial vector Ward-Takahashi (WT) identities. It is essential for the conservation of the axial WT identity that the running coupling must depend on the same momentum as the gluon propagator [8]. The use of simple Landau gauge and other local covariant gauges violates the vector WT identity when the ladder approximation with a bare vertex is studied, and they are not suitable for this purpose. This problem can be solved by using a nonlocal gauge which depends on the momentum:

$$D_{\mu\nu} = -i\left(g_{\mu\nu} - \eta(p)\gamma^\mu\gamma^\nu\right)\frac{1}{p^2}. \quad (III.1)$$

This nonlocal gauge was proposed by T.Kugo and M.Mitchard [9], where $\eta(p)$:

$$\eta(p) = \frac{2}{p^2\alpha}\int_0^p dy(y\gamma(p) - y^2\gamma'(y)). \quad (III.2)$$

It is clear that function (III.2) coincides with the Landau gauge at small and large momenta, i.e., $\eta(0) = 1$, $\eta(\infty) = 1$.

The nonlocal Kugo gauge allows us to write the SDE for the dressing fermion propagator $S(p) = i/(A(p^2)\hat{p} - B(p^2))$. In the ladder approximation it is given by:
\[ iS(p)^{-1} = \hat{p} - iC_2(F) \int \frac{d^4k}{(2\pi)^4} g^2((p-k)^2) D^{\mu\nu}(p-k)i\gamma\mu S(k)i\gamma\nu, \]  

(III.3)

where we use a bare vertex \( ig\gamma^\mu T^\alpha \). The another advantage is that in the Kugo gauge the fermion wave function renormalisation constant \( A(p^2) \) is equal to one. Using

\[ B(p^2) = C_2(F) \int \frac{d^4k_E}{4\pi^3} \frac{1}{(p-k)^2k^2 + B^2(k^2)}. \]  

(III.5)

However, the obtained equation is rather complicated and can not be solved analytically without certain assumptions and approximations. One of these approximations is to use the constant value at the IFP for the running coupling, i.e., \( \alpha(Q^2) = \alpha_s \). It may be used in the region of momentum where the running coupling is slowly changing. Or in other words for \( \alpha_s \rightarrow \alpha_c \) from above and \( N_f \rightarrow N_c \) from below. Expression (III.2) in this approximation is equivalent to the Landau gauge and therefore (III.3) becomes:

\[ B(p^2) = 3C_2(F)\alpha_s \int \frac{d^4k}{4\pi^3} \frac{1}{(p-k)^2k^2 + B^2(k^2)}. \]  

(III.6)

Equation (III.5) can be integrated over angular part:

\[ \int d^4k \frac{1}{(p-k)^2} = \pi^2 \int_0^{\infty} k^2 dk^2 \frac{1}{p^2} + \pi^2 \int_0^{\infty} k^2 dk^2 \frac{1}{k^2}. \]  

(III.7)

The origin of the critical constant \( \alpha_c \) now becomes fairly clear. For recent reviews of the SDE and their application see for example Refs. [5, 6].

IV. NUMERICAL CALCULATIONS AND RESULTS

In this section we discuss the method which was used to solve the SDE (III.3). First of all we note that it is impossible to perform the angular integration analytically in this case. To solve the nonlinear SDE, we use a simple quadrature method. All calculations were performed using Mathematica software and consisted of the following steps. We set up a square lattice \((p_i, k_j)\) where both \(p_i\) and \(k_j\) pass the number of discrete values from lower boundary \(q_0\) to upper boundary \(\Lambda^2\) and then carry out
The cut-off $\Lambda$ of fermions decreases. There is a similar tendency in $\alpha$. In this area the approximation $\alpha(Q^2) \sim \alpha_s$ is useful and it is interesting to compare $B(0)$ with (III.10). We will do this below.

For better understanding of the physical nature of the phase transition, it is also useful to calculate other physical quantities. One of them is the value of the vacuum condensate, which can be easily found:

$$\langle \bar{q}q \rangle = -\lim_{x \to +0} tr S(x, 0) = -\frac{N_c N_F}{8\pi^2} \int_0^\Lambda dp E \frac{p^2 B(p_E)}{p_E^2 + B^2(p_E)}, \tag{IV.2}$$

and the value of meson decay constant (it is described by well known Pagels-Stokar formula)

$$f_\pi^2 = \frac{N_c N_F}{8\pi^2} \int_0^\Lambda dp \frac{p^2 B(p)}{(p^2 + B^2(p))^2} \left( B(p) - \frac{p^2 B(p)}{2 \frac{dp^2}{dp^2}} \right). \tag{IV.3}$$

These values have also been calculated using quadrature formulas and are illustrated in Table [I].

Let us analyze the quantities $B(0)$, $\langle \bar{q}q \rangle$, $f_\pi$ in more detail. These quantities are continuous functions near the critical point. This fact confirms that the phase transition is a transition of second or higher order, possibly of infinite order. For this reason, the quark condensate, decay constant and $B(0)$ may be fitted by polynomial functions near the critical point.

$$B(0) \sim (N_f^{cr} - N_f)^\alpha, \quad \langle \bar{q}q \rangle \sim (N_f^{cr} - N_f)^\beta, \quad f \sim (N_f^{cr} - N_f)^\rho.$$

If we find that the critical exponents $\alpha$, $\beta$, $\rho$ are small real numbers, it will confirm that the phase transition is of finite order. Note, that Eq. (III.10) describes a phase transition of infinite order as opposite to finite. The least squares fitting of the curves gives:

$$B(0) : \quad \alpha = 2.45, \quad N_f^{cr} = 11.43; \tag{IV.7}$$
$$\langle \bar{q}q \rangle : \quad \beta = 2.93, \quad N_f^{cr} = 11.21; \tag{IV.8}$$
$$f : \quad \rho = 1.69, \quad N_f^{cr} = 11.33. \tag{IV.9}$$

We indeed find that the critical exponents are small positive numbers. The value of the critical number of fermions is described by the well known formula:

$$N_f^{cr} = N_c \left( \frac{100 N_c^2 - 66}{25 N_c^2 - 15} \right),$$

obtained from the condition $\alpha_c$ and $\alpha_c$. In case $N_c = 3$, the critical number of fermions is $N_f^{cr} = 11.9$.

V. CONCLUDING REMARKS

In this paper we have shown numerically that the SDE for a fermion propagator with an exact expression for the running coupling has nontrivial solution $B(p)$. Based on the obtained solution, the value of chiral condensate and

![FIG. 1: Numerical solution of the Schwinger-Dyson equation. Here $B/\Lambda$ is dimensionless dynamical mass function](image)

**TABLE I**: Numerical values of the physical quantities obtained from the numerical solution of the SDE

| $N_f$ | $B(0)/\Lambda$ | $-\langle \bar{q}q \rangle/\Lambda^3 \cdot 10^{-4}$ | $f_\pi/\Lambda$ |
|-------|----------------|-----------------------------------------------|----------------|
| 10    | 0.0651         | 5.3224                                         | 0.02791        |
| 10.3  | 0.0540         | 2.2542                                         | 0.01845        |
| 10.5  | 0.0336         | 1.0331                                         | 0.0126         |
| 10.7  | 0.0186         | 0.3686                                         | 0.0077         |
| 11    | 0.0055         | 0.0473                                         | 0.0028         |
the decay constant of pseudoscalar bosons were calculated. These physical quantities are continuous functions near the critical point. Detailed analysis of $B(0)$ near the critical point shows that the phase transition is a transition of finite order.

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