On $q$–fractional derivatives of Riemann–Liouville and Caputo type

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Abstract. Based on the fractional $q$–integral with the parametric lower limit of integration, we define fractional $q$–derivative of Riemann–Liouville and Caputo type. The properties are studied separately as well as relations between them. Also, we discuss properties of compositions of these operators.

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1 Introduction

The fractional differential equations (FDE), as generalizations of integer-order ones, are used in describing various phenomena in the science, especially in physics, chemistry and material science, because of their ability to describe memory effects [5]. Today there are a number of concepts with different definitions of fractional integrals and derivatives and their applications in various mathematical areas (see, for example [8].

At the first moment, it was considered that it exists unique definition of fractional derivative until some confusion appeared in the conclusions. Now, we know that there are two basic types: Riemann-Liouville and Caputo fractional derivative. Hence two types of FDE are in use with very important difference in initial conditions: the first one requires initial conditions for fractional derivatives; on the contrary, the second one for integer order derivatives.
Many of continuous scientific problems have their discrete versions. A way of the treatment is from the point of view of $q$–calculus (see, for example [4]). W.A. Al-Salam [2] and R.P Agarwal [1] introduced several types of fractional $q$–integral operators and fractional $q$–derivatives, always with the lower limit of integration equal 0.

However, in some considerations, such as solving of $q$–differential equation of fractional order with initial values in nonzero point, it is of interest to allow that the lower limit of integration is variable. In our paper [9], we succeed to generalize this theory in that direction.

In continuation, our purpose in this paper is to define two types of the fractional $q$–derivatives based on the fractional $q$–integrals with the parametric lower limit of integration.

2 Preliminaries

In the theory of $q$–calculus (see [6]), for a real parameter $q \in \mathbb{R}^+ \setminus \{1\}$, we introduce a $q$–real number $[a]_q$ by

\[
[a]_q := \frac{1 - q^a}{1 - q} \quad (a \in \mathbb{R}) .
\]

The $q$–analog of the Pochhammer symbol ($q$–shifted factorial) is defined by:

\[
(a; q)_0 = 1 , \quad (a; q)_k = \prod_{i=0}^{k-1} (1 - aq^i) \quad (k \in \mathbb{N} \cup \{\infty\}) .
\]

Its natural expansion to the reals is

\[
(a; q)_\alpha = \frac{(a; q)_{\infty}}{(aq^\alpha; q)_{\infty}} \quad (a \in \mathbb{R}) . \tag{1}
\]

Also, $q$–binomial coefficient is given by

\[
\left[\alpha\right]_q^k = \frac{(q^{-\alpha}; q)_k}{(q; q)_k} \frac{(-1)^k q^{\alpha k}}{q^{\binom{k}{2}}} \quad (k \in \mathbb{N}, \; \alpha \in \mathbb{R}) . \tag{2}
\]

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The following formulas (see, for example, [6], [3] and [9]) will be useful:

\[
(a; q)_n = (q^{1-n}/a; q) \frac{(-1)^n}{a^n} q^{n(\frac{q}{a})}; \tag{3}
\]

\[
(bq^{-n}; q)_n = \frac{(q/a; q)_n}{(q/b; q)_n}; \tag{4}
\]

\[
(b/a; q)_\alpha = \sum_{n=0}^{\infty} (-1)^n \frac{\alpha}{n} q^{\binom{n}{2}} \frac{b^n}{a^n}; \tag{5}
\]

\[
\left(\frac{a}{q}\right)_{\alpha+n} = \left(\frac{aq}{q}\right)_{\alpha} \quad (n \in \mathbb{N}; a, b, q, \alpha \in \mathbb{R}); \tag{6}
\]

\[
\frac{(\mu q^k; q)_\alpha}{(\mu q; q)_\alpha} = \frac{(\mu q^k; q)_k}{(\mu q; q)_k} \quad (\mu, \alpha \in \mathbb{R}^+); \tag{7}
\]

\[
(q^{k-n}; q)_\alpha = 0 \quad (k, n \in \mathbb{N}_0, k \leq n). \tag{8}
\]

The next result will have an important role in proving the semigroup property of the fractional $q$–integral.

Let us denote

\[
S(\alpha, \beta, \mu) = \sum_{n=0}^{\infty} \frac{(\mu q^{1-n}; q)_{\alpha-1} (q^{1+n}; q)_{\beta-1} q^n}{(q; q)_{\alpha-1} (q; q)_{\beta-1}} q^\alpha. \tag{9}
\]

In the paper [9], the next lemma is proven.

**Lemma 1** For $\mu, \alpha, \beta \in \mathbb{R}^+$, the following identity is valid

\[
S(\alpha, \beta, \mu) = \frac{(\mu q; q)_\alpha}{(q; q)_\alpha}. \tag{10}
\]

The $q$–gamma function is defined by

\[
\Gamma_q(x) = \frac{(q; q)_x}{(q^x; q)_x} (1-q)^{1-x} \quad (x \in \mathbb{R} \setminus \{0, -1, -2, \ldots\}). \tag{11}
\]

Obviously,

\[
\Gamma_q(x+1) = [x]q \Gamma_q(x), \quad \Gamma_q(x) = (q; q)_{x-1}(1-q)^{1-x}. \tag{12}
\]

The $q$–hypergeometric function [6] is defined as

\[
\phi_1 \left(\begin{array}{c}
\scriptstyle a, b \\
\scriptstyle c
\end{array} | q; x \right) = \sum_{n=0}^{\infty} \frac{(a; q)_n (b; q)_n}{(c; q)_n (q; q)_n} x^n.
\]

The $q$–derivative of a function $f(x)$ is defined by

\[
(D_q f)(x) = \frac{f(x) - f(qx)}{x - qx} \quad (x \neq 0), \quad (D_q f)(0) = \lim_{x \to 0} (D_q f)(x),
\]
and \( q \)-derivatives of higher order:

\[
D_0^0 f = f , \quad D_0^n f = D_q(D_q^{n-1} f) \quad (n = 1, 2, 3, \ldots) .
\] (13)

For an arbitrary pair of functions \( u(x) \) and \( v(x) \) and constants \( \alpha, \beta \in \mathbb{R} \), we have linearity and product rules

\[
D_q(\alpha u(x) + \beta v(x)) = \alpha(D_q u)(x) + \beta(D_q v)(x),
\]

\[
D_q(u(x) \cdot v(x)) = u(qx)(D_q v)(x) + v(x)(D_q u)(x) .
\]

In this paper, very useful examples are the \( q \)-derivatives of the next functions:

\[
D_q(x^\lambda(a/x; q)_\lambda) = [\lambda]_q x^{\lambda-1}(a/x; q)_{\lambda-1} ,
\] (14)

\[
D_q(a^\lambda(x/a; q)_\lambda) = -[\lambda]_q a^{\lambda-1}(qx/a; q)_{\lambda-1} ,
\] (15)

\[
D_q(x^\lambda) = [\lambda]_q x^{\lambda-1} .
\] (16)

The \( q \)-integral is defined by

\[
(I_{q,a} f)(x) = \int_a^x f(t) \, dq_t = x(1-q) \sum_{k=0}^{\infty} f(xq^k) q^k \quad (0 \leq |q| < 1),
\] (17)

and

\[
(I_{q,a}^n f)(x) = \int_a^x f(t) \, dq_t = \int_0^x f(t) \, dq_t - \int_0^a f(t) \, dq_t .
\] (18)

However, these definitions cause troubles in research as they include the points outside of the interval of integration (see [7]). In the case when the lower limit of integration is \( a = xq^n \), i.e., when it is determined for some choice of \( x, q \) and positive integer \( n \), the \( q \)-integral (18) becomes

\[
\int_{xq^n}^x f(t) \, dq_t = x(1-q) \sum_{k=0}^{n-1} f(xq^k) q^k .
\] (19)

As for \( q \)-derivative, we can define \( I_{q,a}^n \) operator by

\[
I_{q,a}^0 f = f , \quad I_{q,a}^n f = I_{q,a}(I_{q,a}^{n-1} f) \quad (n = 1, 2, 3, \ldots) .
\]

For \( q \)-integral and \( q \)-derivative operators the following is valid:

\[
(D_q I_{q,a} f)(x) = f(x) , \quad (I_{q,a} D_q f)(x) = f(x) - f(a) ,
\]

and, more generally,

\[
(D_q^n I_{q,a} f)(x) = f(x) \quad (n \in \mathbb{N}) ,
\] (20)

\[
(I_{q,a}^n D_q f)(x) = f(x) - \sum_{k=0}^{n-1} \frac{(D_q^k f)(a)}{[\lambda]_q^k} x^k(a/x; q)_k \quad (n \in \mathbb{N}) .
\] (21)

The formula for \( q \)-integration by parts is

\[
\int_a^b u(x)(D_q v)(x) \, dq_x = [u(x)v(x)]_a^b - \int_a^b v(qx)(D_q u)(x) \, dq_x .
\] (22)
3 The fractional $q$–integral

In all further considerations we assume that the functions are defined in an interval $(0, b)$ ($b > 0$), and $a \in (0, b)$ is an arbitrary fixed point. Also, the required $q$–derivatives and $q$–integrals exist and the convergence of the series mentioned in the proofs is assumed.

Definition 1 The fractional $q$–integral is

$$ (I_{q,a}^\alpha f)(x) = \frac{x^{-\alpha}}{\Gamma_q(\alpha)} \int_a^x (qt/x; q)_{\alpha-1} f(t) \, dq \, t \quad (a < x; \ \alpha \in \mathbb{R}^+) \quad (23) $$

Lemma 2 The fractional $q$–integral (23) can be written in the equivalent form

$$ (I_{q,a}^\alpha f)(x) = \int_a^x f(t) \, dq \, w_\alpha(x, t) \quad (\alpha \in \mathbb{R}^+) \quad (24) $$

where $w_\alpha(x, t)$ is the function defined by

$$ w_\alpha(x, t) = \frac{1}{\Gamma_q(\alpha + 1)} (x^\alpha - x^\alpha(t/x; q)_\alpha) \quad (\alpha \in \mathbb{R}^+) \quad (25) $$

Proof. It is enough to notice that the $q$–differential of $w_\alpha(x, t)$ over variable $t$ is

$$ dq \, w_\alpha(x, t) = D_q w_\alpha(x, t) \, dq \, t = \frac{x^{-\alpha}(qt/x; q)_{\alpha-1}}{\Gamma_q(\alpha)} \, dq \, t \quad (26) $$

Using formula (25), the integral (23) can be written as

$$ (I_{q,a}^\alpha f)(x) = \frac{x^{-\alpha}}{\Gamma_q(\alpha)} \sum_{k=0}^{\infty} (-1)^k \left[ \frac{\alpha - 1}{k} \right] q^{(k+1)/2} x^{-k} \int_a^x t^k f(t) \, dq \, t \quad (\alpha \in \mathbb{R}^+) \quad (27) $$

Putting $\alpha = 1$ in (27), we get $q$–integral (18).

The fractional integral (see, for example [8]) is the limiting case of (23) when $q$ arises to 1, since

$$ \lim_{q \to 1} x^{\alpha-1}(qt/x; q)_{\alpha-1} = (x-t)^{\alpha-1} \quad (28) $$

Obviously, the next equality holds:

$$ (I_{q,a}^\alpha f)(a) = \frac{a^{-\alpha}}{\Gamma_q(\alpha)} \int_a^a (qt/a; q)_{\alpha-1} f(t) \, dq \, t = 0 \quad (28) $$

Lemma 3 For $\alpha \in \mathbb{R}^+$, the following is valid:

$$ (I_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha+1} D_q f)(x) + \frac{f(a)}{\Gamma_q(\alpha + 1)} x^{\alpha}(a/x; q)_\alpha \quad (a < x) \quad (29) $$
**Proof.** According to the formula (15), the $q$–derivative over the variable $t$ is

$$D_q(x^\alpha(t/x;q)_\alpha) = -[\alpha]_q x^{\alpha - 1}(qt/x;q)_\alpha^{-1}.$$  

Using the $q$–integration by parts (22), we obtain

$$
(I_{q,a}^\alpha f)(x) = -\frac{1}{[\alpha]_q \Gamma_q(\alpha)} \int_a^x D_q(x^\alpha(t/x;q)_\alpha) f(t)d_q t
= \frac{1}{\Gamma_q(\alpha + 1)} \left( x^\alpha(a/x;q)_\alpha f(a) + \int_a^x x^\alpha(qt/x;q)_\alpha (D_q f)(t)d_q t \right)
= (I_{q,a}^{\alpha + 1} D_q f)(x) + \frac{f(a)}{\Gamma_q(\alpha + 1)} x^\alpha(a/x;q)_\alpha . \tag{\Box}
$$

**Lemma 4** For $\alpha, \beta \in \mathbb{R}^+$, the following is valid:

$$
\int_0^a (qt/x;q)_\beta^{-1} (I_{q,a}^\alpha f)(t)d_q t = 0 \quad (a < x) .
$$

**Proof.** Using formulas (8) and (19), for $n \in \mathbb{N}_0$, we have

$$
(I_{q,a}^\alpha f)(aq^n) = \frac{1}{\Gamma_q(\alpha)} \int_a^{aq^n} (aq^n)^{\alpha - 1} ((qu)/(aq^n);q)_{\alpha - 1} f(u)d_q u
= -a^\alpha(1-q) \frac{\Gamma_q(\alpha)}{\Gamma_q(\alpha)} \sum_{j=0}^{n-1} (q^n)^{\alpha - 1}(q^{j+1-n};q)_{\alpha - 1} f(aq^j)q^j = 0 .
$$

From the other side, according to the definition of $q$–integral, we have

$$
\int_0^a (qt/x;q)_\beta^{-1} (I_{q,a}^\alpha f)(t)d_q t = a(1-q) \sum_{n=0}^{\infty} (aq^{n+1}/x;q)_{\beta - 1} (I_{q,a}^\alpha f)(aq^n)q^n ,
$$

what is obviously equal to zero . \(\Box\)

**Theorem 5** Let $\alpha, \beta \in \mathbb{R}^+$. The $q$–fractional integration has the following semigroup property

$$
(I_{q,a}^\alpha I_{q,a}^\beta f)(x) = (I_{q,a}^{\alpha + \beta} f)(x) \quad (a < x) .
$$

**Proof.** By previous lemma, we have

$$
(I_{q,a}^\alpha I_{q,a}^\beta f)(x) = \frac{x^\beta - 1}{\Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta - 1} (I_{q,a}^\alpha f)(t)d_q t,
$$

i.e.,

$$
(I_{q,a}^\alpha I_{q,a}^\beta f)(x) = \frac{x^\beta - 1}{\Gamma_q(\alpha) \Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta - 1} t^{\alpha - 1} \int_0^t (qu/t;q)_{\alpha - 1} f(u)d_q u
- \frac{x^\beta - 1}{\Gamma_q(\alpha) \Gamma_q(\beta)} \int_0^x (qt/x;q)_{\beta - 1} t^{\alpha - 1} \int_0^a (qu/t;q)_{\alpha - 1} f(u)d_q u .
$$
Since, as it was proven in the paper [1], the equality
\[(I_{\beta, q}^\alpha \Gamma_{\beta}^0 f)(x) = (I_{\alpha, q}^{\alpha+\beta} f)(x)\]
is valid, we conclude that
\[(I_{\beta, q,a}^\alpha \Gamma_{\beta}^0 f)(x) = (I_{\alpha, q,a}^{\alpha+\beta} f)(x)\]

Furthermore, we can write
\[(I_{\beta, q,a}^\alpha \Gamma_{\beta}^0 f)(x) = (I_{\alpha, q,a}^{\alpha+\beta} f)(x) + x^{\alpha+\beta-1} \frac{\Gamma_q(\alpha) \Gamma_q(\beta)}{\Gamma_q(\alpha+\beta)} \int_0^\infty (qt/x; q)_{\beta-1} t^{\alpha-1} \int_0^a (qu/t; q)_{\alpha-1} f(u) du .\]

wherefrom
\[(I_{\beta, q,a}^\alpha \Gamma_{\beta}^0 f)(x) = (I_{\alpha, q,a}^{\alpha+\beta} f)(x) + a(1-q) \sum_{j=0}^\infty c_j f(aq^j) q^j ,\]

with
\[c_j = \frac{x^{\alpha+\beta-1} (aq^{j+1}/x; q)_{\alpha+\beta-1}}{\Gamma_q(\alpha+\beta)} - \frac{x^{\alpha+\beta-1} (1-q) \sum_{n=0}^\infty (q^{n+1}; q)_{\beta-1} q^{n(\alpha-1)} (aq^{j+1-n}/x; q)_{\alpha-1} q^n)}{\Gamma_q(\alpha) \Gamma_q(\beta)} .\]

By using formulas (7) and (11), we get
\[c_j = ((1-q)x)^{\alpha+\beta-1}\]
\[\times \left\{ \frac{(aq^{j+1}/x; q)_{\alpha+\beta-1}}{(q; q)_{\alpha+\beta-1}} - \sum_{n=0}^\infty \frac{(q^{n+1}; q)_{\beta-1}}{(q; q)_{\beta-1}} \frac{(aq^{j+1-n}/x; q)_{\alpha-1}}{(q; q)_{\alpha-1}} q^n \right\} .\]

Putting \(\mu = q^j a/x\) into (10), we see that \(c_j = 0\) for all \(j \in \mathbb{N}\), which completes the proof. □

**Corollary 6** For \(\alpha \geq n \ (n \in \mathbb{N})\) the following is valid:
\[(D_q^n I_{\alpha, q,a}^n f)(x) = (I_{\alpha, q,a}^{\alpha-n} f)(x) \quad (a < x).\]

*Proof.* The statement follows from Theorem 5 and property (20). □
4 The fractional $q$–derivative of Riemann–Liouville type

On the basis of fractional $q$–integral, we can define $q$–derivative of real order.

**Definition 2** The fractional $q$–derivative of Riemann–Liouville type is

$$
(D_{q,a}^\alpha f)(x) = \begin{cases} 
(I_{q,a}^{-\alpha} f)(x), & \alpha \leq 0 \\
(D_q^{[\alpha]} I_{q,a}^{[\alpha]-\alpha} f)(x), & \alpha > 0,
\end{cases}
$$

(29)

where $[\alpha]$ denotes the smallest integer greater or equal to $\alpha$.

Notice that $(D_{q,a}^\alpha f)(x)$ has subscript $a$ to emphasize that it depends on the lower limit of integration used in definition (29). Since $[\alpha]$ is a positive integer for $\alpha \in \mathbb{R}^+$, then for $(D_q^{[\alpha]} f)(x)$ we apply definition (19).

According to definition and (29), we can easily prove that

$$
(D_{q,a}^\alpha f)(a) = 0 \quad (\forall \alpha \in \mathbb{R} \setminus \mathbb{N}_0).
$$

(30)

**Theorem 7** For $\alpha \in \mathbb{R}$, the following is valid:

$$
(D_q D_{q,a}^\alpha f)(x) = (D_{q,a}^{\alpha+1} f)(x) \quad (a < x).
$$

**Proof.** According to the formula (19), the statement is true for $\alpha \in \mathbb{N}_0$. For others, we will consider three cases.

For $\alpha \leq -1$, according to Theorem 5 we have

$$
(D_q D_{q,a}^\alpha f)(x) = (D_q I_{q,a}^{-\alpha} f)(x) = (D_q I_{q,a}^{1-\alpha} f)(x)
$$

$$
= (D_q I_{q,a}^{-\alpha-1} f)(x) = (I_{q,a}^{-(\alpha+1)} f)(x) = (D_{q,a}^{\alpha+1} f)(x).
$$

In the case $-1 < \alpha < 0$, i.e., $0 < \alpha + 1 < 1$, we obtain

$$
(D_q D_{q,a}^\alpha f)(x) = (D_q I_{q,a}^{-\alpha} f)(x) = (D_q I_{q,a}^{1-(\alpha+1)} f)(x) = (D_{q,a}^{\alpha+1} f)(x).
$$

At last, if $\alpha = n + \varepsilon$, $n \in \mathbb{N}_0$, $0 < \varepsilon < 1$, then $\alpha + 1 \in (n + 1, n + 2)$, so we get

$$
(D_q D_{q,a}^\alpha f)(x) = (D_q D_{q,a}^{\alpha+1} f)(x) = (D_q I_{q,a}^{1-\varepsilon} f)(x) = (D_{q,a}^{\alpha+1} f)(x).
\square
$$

**Theorem 8** For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$, the following is valid:

$$
(D_{q,a}^\alpha D_q f)(x) = (D_{q,a}^{\alpha+1} f)(x) - \frac{f(a)}{\Gamma_q(-\alpha)} x^{-\alpha-1}(a/x; q)_{-\alpha-1} \quad (a < x).
$$

**Proof.** Let us consider two cases. If $\alpha < 0$, then, with respect to Lemma 3 and formulas (14) and (20), the following holds:

$$
(D_{q,a}^{\alpha+1} f)(x) = (D_q D_{q,a}^\alpha f)(x) = (D_q I_{q,a}^{\alpha} f)(x)
$$

$$
= D_q \left( (I_{q,a}^{-\alpha+1} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha + 1)} x^{-\alpha}(a/x; q)_{-\alpha} \right)
$$

$$
= (D_q I_{q,a}^{1-\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha + 1)} [-\alpha]_q x^{-\alpha-1}(a/x; q)_{-\alpha-1}
$$

$$
= (D_{q,a} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} x^{-\alpha-1}(a/x; q)_{-\alpha-1}.
$$
If \( \alpha > 0 \), there exist \( n \in \mathbb{N}_0 \) and \( \varepsilon \in (0, 1) \), such that \( \alpha = n + \varepsilon \). Then, applying

the similar procedure, we get

\[
(D_{q,a}^{\alpha+1} f)(x) = (D_q D_{q,a}^{\alpha} f)(x) = (D_q D_{q,a}^{n+1} I_{q,a}^{1-\varepsilon} f)(x)
\]

\[
= D_{q}^{n+2} \left( (I_{q,a}^{2-\varepsilon} D_q f)(x) + \frac{f(a)}{\Gamma_q(2-\varepsilon)} x^{1-\varepsilon}(a/x; q)_{1-\varepsilon} \right)
\]

\[
= (D_q^{n+1} D_q I_{q,a} I_{q,a}^{1-\varepsilon} D_q f)(x) + \frac{f(a)}{\Gamma_q(2-\varepsilon)} D_q^{n+2} (x^{1-\varepsilon}(a/x; q)_{1-\varepsilon})
\]

\[
= (D_q^{n+1} I_{q,a}^{1-\varepsilon} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\varepsilon-n)} (x^{-\varepsilon-n-1}(a/x; q)_{-\varepsilon-n-1})
\]

\[
= (D_{q,a}^{\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} x^{-\alpha-1}(a/x; q)_{-\alpha-1}.
\]

**Corollary 9** The semigroup property for fractional \( q \)-derivative of Riemann–Liouville type is not valid, i.e., in general

\[
(D_{q,a}^{\alpha} D_{q,a}^{\beta} f)(x) \neq (D_{q,a}^{\alpha+\beta} f)(x).
\]

**Example 1** Notice that from

\[
D_{q,a}^{n+\varepsilon} \left( x^{\varepsilon-1}(a/x; q)_{\varepsilon-1} \right) = 0 \quad (n \in \mathbb{N}_0; \ 0 < \varepsilon < 1)
\]

we have two different conclusions. From one side, it is true

\[
\lim_{\varepsilon \to 1} D_{q,a}^{n+\varepsilon} \left( x^{\varepsilon-1}(a/x; q)_{\varepsilon-1} \right) = 0 = (D_q^{n+1} 1)(x) = D_q^{n+1}(x^{1}(a/x; q)_0).
\]

But, from the other side, it is

\[
\lim_{\varepsilon \to 0} D_{q,a}^{n+\varepsilon} \left( x^{\varepsilon-1}(a/x; q)_{\varepsilon-1} \right) = 0 \neq D_q^{n}(x^{-1}(a/x; q)_{-1}).
\]

So, we conclude that the mapping \( \alpha \mapsto D_{q,a}^{\alpha} f \) is not continuous from the right side over variable \( \alpha \).

**5 The fractional \( q \)-derivative of Caputo type**

If we change the order of operators, we can introduce another type of fractional \( q \)-derivative.

**Definition 2** The the fractional \( q \)-derivative of Caputo type is

\[
(\alpha D_{q,a}^{\alpha} f)(x) = \begin{cases} 
(I_q^{-\alpha} f)(x), & \alpha \leq 0 \\
(I_q^{[\alpha]} - \alpha D_q^{[\alpha]} f)(x), & \alpha > 0.
\end{cases}
\]
Theorem 10 For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ and $a < x$, the following is valid:

$$(\ast D_{q,a}^{\alpha+1} f)(x) - (\ast D_{q,a}^\alpha D_q f)(x) = \begin{cases} \frac{f(a)}{\Gamma_q(-\alpha)} x^{-\alpha-1}(a/x; q)_{-\alpha-1}, & \alpha \leq -1, \\ 0, & \alpha > -1. \end{cases}$$

Proof. As in Theorem 1 we will consider three cases. For $\alpha < -1$, according to Lemma 3 we have

$$(\ast D_{q,a}^{\alpha+1} f)(x) = (I_{q,a}^{-\alpha-1} f)(x) = (I_{q,a}^{-\alpha} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} x^{-\alpha-1}(a/x; q)_{-\alpha-1}$$

In the case $-1 < \alpha < 0$, i.e., $0 < \alpha + 1 < 1$, we obtain

$$(\ast D_{q,a}^{\alpha+1} f)(x) = (I_{q,a}^{-\alpha} D_q f)(x) = (\ast D_{q,a}^\alpha D_q f)(x)$$

Finally, if $\alpha = n + \varepsilon$, $n \in \mathbb{N}_0$, $0 < \varepsilon < 1$, then $\alpha + 1 \in (n + 1, n + 2)$, so we get

$$(\ast D_{q,a}^{\alpha+1} f)(x) = (I_{q,a}^{1-\varepsilon} D_q^{n+2} f)(x) = (I_{q,a}^{1-\varepsilon} D_q^{n+1} D_q f)(x) = (\ast D_{q,a}^\alpha D_q f)(x). \square$$

Theorem 11 For $\alpha \in \mathbb{R} \setminus \mathbb{N}_0$ and $a < x$, the following is valid:

$$(D_q \ast D_q^\alpha f)(x) - (\ast D_{q,a}^{\alpha+1} f)(x) = \begin{cases} 0, & \alpha < -1, \\ \frac{(D_q^\alpha f)(a)}{\Gamma_q([\alpha] - \alpha)} x^{[\alpha] - \alpha-1}(a/x; q)_{[\alpha]-\alpha-1}, & \alpha > -1. \end{cases}$$

Proof. At first, let $\alpha < 0$. Using Lemma 3, Theorem 10, and formulas 14 and 20, we get

$$(D_q \ast D_q^\alpha f)(x) = (D_q I_{q,a}^{-\alpha} f)(x)$$

$$= (D_q I_{q,a}^{-\alpha+1} D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha + 1)} D_q \left(x^{-\alpha}(a/x; q)_{-\alpha}\right)$$

$$= (\ast D_{q,a}^\alpha D_q f)(x) + \frac{f(a)}{\Gamma_q(-\alpha)} x^{-\alpha-1}(a/x; q)_{-\alpha-1}$$

The required equalities are valid both for $\alpha < -1$ or $-1 < \alpha < 0$, according to Lemma 10.

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If $\alpha > 0$, there exist $n \in \mathbb{N}_0$ and $\varepsilon \in (0, 1)$, such that $\alpha = n + \varepsilon$. Then, applying the similar procedure, we get

$$(D_q \ast D_q^\alpha f)(x) = (D_q I_q^{1-\varepsilon} D_q^{n+1} f)(x)$$

$$= (D_q I_q^{1-\varepsilon} D_q^{n+2} f)(x) + \frac{(D_q^{n+1} f)(a)}{\Gamma_q(2-\varepsilon)} D_q \left( x^{1-\varepsilon} (a/x; q)_{1-\varepsilon} \right)$$

$$= (D_q^{n+1} f)(x) + \frac{D_q^{n+1} f(a)}{\Gamma_q(n+1-\alpha)} x^{n-\alpha} (a/x; q)_{n-\alpha}. \quad \square$$

6 The fractional $q$–integrals and $q$–derivatives of some elementary functions

We will use previous results to evaluate fractional $q$–integrals and $q$–derivatives of some well-known functions in explicit form. Here, it is very useful to remind on $q$–form of Taylor theorem

$$f(x) = \sum_{k=0}^{\infty} \frac{(D_q f)(a)}{[k]_q!} x^k (a/x; q)_k$$

(32)

given by Jackson (see [3]). The next lemma will have crucial role in reaching of our goal.

**Lemma 12** For $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0$, $\lambda \in (1, \infty)$, the following is valid

$$I_q^{\alpha}(x^\lambda (a/x; q)_\lambda) = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\lambda + 1 + \alpha)} x^{\lambda+\alpha} (a/x; q)_{\lambda+\alpha} \quad (a < x),$$

$$D_q^{\alpha}(x^\lambda (a/x; q)_\lambda) = \frac{\Gamma_q(\lambda + 1)}{\Gamma_q(\lambda + 1 - \alpha)} x^{\lambda-\alpha} (a/x; q)_{\lambda-\alpha},$$

$$\ast D_q^{\alpha}(x^\lambda (a/x; q)_\lambda) = \begin{cases} 
0, & \lambda \in \mathbb{N}_0: \alpha > \lambda, \\
D_q^{\alpha}(x^\lambda (a/x; q)_\lambda), & \text{otherwise.}
\end{cases}$$

**Proof.** For $\lambda \neq 0$, according to the definition ([23]), we have

$$I_q^{\alpha}(x^\lambda (a/x; q)_\lambda) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \left( \int_0^\lambda (q t/x; q)_{\lambda-1} t^{\lambda-1} (a/t; q)_\lambda d_q t - \int_0^a (q t/x; q)_{\lambda-1} t^{\lambda-1} (a/t; q)_\lambda d_q t \right).$$

Also, the following is valid:

$$\int_0^a (q t/x; q)_{\lambda-1} t^{\lambda} (a/t; q)_\lambda d_q t = a^{\lambda+1}(1-q) \sum_{k=0}^{\infty} (aq^{k+1}/x; q)_{\alpha-1} q^{k\lambda}(q^{-k}; q)_\lambda q^k,$$
what vanishes because of (31). Therefrom, according to definition (17), we get
\[
\int_0^x (qt/x; q)_{\lambda-1} t^\lambda (a/t; q)_\lambda \, dt
\]
\[
= x^{\lambda+1} (1 - q) \sum_{k=0}^{\infty} (q^{1+k}; q)_{\lambda-1} \frac{1}{(q; q)_\lambda} \frac{a}{(xq^k; q)_\lambda} q^{(\lambda+1)k}.
\]
We notice presence of (9) in the previous formula, i.e.
\[
\int_0^x (qt/x; q)_{\lambda-1} t^\lambda (a/t; q)_\lambda \, dt
\]
\[
= (1 - q) x^{\lambda+1} (q; q)_{\lambda-1} (q; q)_\lambda S_\lambda (\lambda + 1, \alpha, a/(qx)).
\]
By using (10), we get
\[
\int_0^x (qt/x; q)_{\lambda-1} t^\lambda (a/t; q)_\lambda \, dt = (1 - q) \frac{(q; q)_{\lambda-1}(q; q)_\lambda}{(q; q)_{\lambda+\lambda}} x^{\lambda+1} (a/x; q)_{\lambda+\lambda},
\]
and applying (12), we obtain the required formula for \( I_{q,a}^\alpha \frac{1}{(q; q)_\lambda} \) when \( \lambda \neq 0 \).

In the case when \( \lambda = 0 \), using \( q \)-integration by parts (22), we have
\[
(I_{q,a}^\alpha 1)(x) = \frac{x^{\alpha-1}}{\Gamma_q(\alpha)} \int_a^x (qt/x; q)_{\alpha-1} dq dt = \frac{1}{\Gamma_q(\alpha)} \int_a^x D_q((t/x; q)_\alpha) (-\log q) \, dt.
\]
\[
= -\frac{1}{\Gamma_q(\alpha + 1)} \int_a^x D_q(x^\alpha (t/x; q)_\alpha) dt = -\frac{1}{\Gamma_q(\alpha + 1)} x^\alpha (a/x; q)_\alpha.
\]
The terms for \( q \)-derivatives can be obtained by applying definitions (29) and (30).

**Corollary 13** For \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0 \), \( n \in \mathbb{N}_0 \), and \( a < x \), the following is valid:
\[
I_{q,a}^\alpha(x^n) = (1 - q)^\alpha \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(q^{n-k+1}; q)_k \frac{x^{k+a}(a/x; q)_{k+a}}{(q; q)_{k+a}}.
\]
\[
D_{q,a}^\alpha(x^n) = (1 - q)^{-\alpha} \sum_{k=0}^{n} \binom{n}{k} a^{n-k}(q^{n-k+1}; q)_k \frac{x^{-\alpha}(a/x; q)_{k-\alpha}}{(q; q)_{k-\alpha}}.
\]
\[
\star D_{q,a}^\alpha(x^n) = (q^{n+1} - [\alpha]; q)_[\alpha] \Gamma_q(\alpha) \sum_{k=0}^{n} a^{n-k}(q^{n-k+1}; q)_k \frac{x^{\alpha}(a/x; q)_{k-\alpha}}{(q; q)_{k-\alpha}}.
\]
(Notice that \( \star D_{q,a}^\alpha(x^n) = 0 \) when \( \alpha > n \).

The \( q \)-exponential functions (see [4]) can be written like power series or, applying \( q \)-form of Taylor theorem (52), by
\[
e_q(x) = \sum_{n=0}^{\infty} \frac{x^n}{(q; q)_n} = e_q(a) \sum_{n=0}^{\infty} \frac{x^n(a/x; q)_n}{(q; q)_n} \quad (|x| < 1), \quad \text{(33)}
\]
\[
E_q(x) = \sum_{n=0}^{\infty} \frac{q^n}{(q; q)_n} x^n = E_q(a) \sum_{n=0}^{\infty} \frac{q^n(-a; q)_n}{(q; q)_n} \quad \text{(34)}
\]

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Corollary 14 For \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0 \) and \( 0 < a < x < 1 \), the following is valid:

\[
I_{q,a}^\alpha (e_q(x)) = (1 - q)^\alpha e_q(a) \sum_{n=0}^{\infty} \frac{x^{n+\alpha}(a/x; q)_{n+\alpha}}{(q; q)_{n+\alpha}},
\]

\[
D_{q,a}^\alpha (e_q(x)) = (1 - q)^{-\alpha} e_q(a) \sum_{n=0}^{\infty} \frac{x^{n-\alpha}(a/x; q)_{n-\alpha}}{(q; q)_{n-\alpha}},
\]

\[
* D_{q,a}^\alpha (e_q(x)) = (1 - q)^{-\alpha} e_q(a) \sum_{n=\lceil \alpha \rceil}^{\infty} \frac{x^{n-\alpha}(a/x; q)_{n-\alpha}}{(q; q)_{n-\alpha}}.
\]

Corollary 15 For \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0 \) and \( 0 < a < x \), the following is valid:

\[
I_{q,a}^\alpha (E_q(x)) = (1 - q)^\alpha E_q(a) \sum_{n=0}^{\infty} \frac{q^{(2)}(x)}{(-a; q)_n} \frac{x^{n+\alpha}(a/x; q)_{n+\alpha}}{(q; q)_{n+\alpha}},
\]

\[
D_{q,a}^\alpha (E_q(x)) = \frac{E_q(a)}{(1 - q)^\alpha} \sum_{n=0}^{\infty} \frac{q^{(2)}(x)}{(-a; q)_n} \frac{x^{n-\alpha}(a/x; q)_{n-\alpha}}{(q; q)_{n-\alpha}},
\]

\[
* D_{q,a}^\alpha (E_q(x)) = \frac{E_q(a)}{(1 - q)^\alpha} \sum_{n=\lceil \alpha \rceil}^{\infty} \frac{q^{(2)}(x)}{(-a; q)_n} \frac{x^{n-\alpha}(a/x; q)_{n-\alpha}}{(q; q)_{n-\alpha}}.
\]

7 The relationship between fractional q–integrals and q–derivatives

It is very important to establish the connection between two types of the fractional q–derivatives.

Theorem 16 Let \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0 \) and \( a < x \). The connection between Caputo type and Riemann-Liouville type fractional integral is

\[
(D_{q,a}^\alpha f(x)) = (I_{q,a}^{\alpha-1} D_{q,a} f(x)) + \sum_{k=0}^{[\alpha]-1} \frac{(D_{q,a}^k f)(a)}{\Gamma(q(1 + k - \alpha))} x^{k-\alpha} (a/x; q)_{k-\alpha}
\]

Proof. Any \( \alpha \in \mathbb{R}^+ \setminus \mathbb{N}_0 \) we can write in the form \( \alpha = n + \varepsilon \), where \( \varepsilon \in (0, 1) \). We will prove the statement by mathematical induction over \( n \in \mathbb{N}_0 \).

At first, let \( n = 0 \), i.e., \( \alpha \in (0, 1) \). According to Lemma 3, we have

\[
(I_{q,a}^{1-\alpha} f)(x) = (I_{q,a}^{2-\alpha} D_{q,a} f)(x) + \frac{f(a)}{\Gamma(q(2 - \alpha))} x^{1-\alpha} (a/x; q)_{1-\alpha}
\]

\[
= (I_{q,a} (D_{q,a}^{\alpha} f))(x) + \frac{f(a)}{\Gamma(q(2 - \alpha))} x^{1-\alpha} (a/x; q)_{1-\alpha}.
\]

By q–deriving, we get

\[
(D_{q,a} I_{q,a}^{1-\alpha} f)(x) = (D_{q,a} I_{q,a} (D_{q,a}^{\alpha} f))(x) + \frac{f(a)}{\Gamma(q(2 - \alpha))} D_{q} (x^{1-\alpha} (a/x; q)_{1-\alpha}),
\]

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and, with respect to \([14]\),
\[
(D^n_{q,a} f)(x) = (*D^n_{q,a} f)(x) + \frac{f(a)}{\Gamma_q(1 - \alpha)} x^{\alpha(a/x; q) - \alpha} .
\]

Suppose that the statement is valid for a real \(\alpha = n + \varepsilon, \varepsilon \in (0, 1)\), for a positive integer \(n \in \mathbb{N}\) and let us prove that it is valid for \(\alpha = n + 1 + \varepsilon\). Indeed, according to Theorem 17 the next equality is valid:
\[
(D_{q,a}^\alpha f)(x) = \left(D_q D_{q,a}^{n+\varepsilon} f\right)(x).
\]

With respect to the inductional assumption
\[
(D_{q,a}^{n+\varepsilon} f)(x) = (*D_{q,a}^{n+\varepsilon} f)(x) + \sum_{k=0}^{n} \frac{(D^k_q f)(a)}{\Gamma_q(1 + k - n - \varepsilon)} x^{k-n-\varepsilon(a/x; q)k-n-\varepsilon},
\]
and the formula \([14]\), we can write
\[
(D_{q,a}^\alpha f)(x)
= (D_q * D_{q,a}^{n+\varepsilon} f)(x) + \sum_{k=0}^{n} \frac{(D^k_q f)(a)}{\Gamma_q(1 + k - n - \varepsilon)} x^{k-n-\varepsilon(a/x; q)k-n-\varepsilon}
= (D_q * D_{q,a}^{n+\varepsilon} f)(x) + \sum_{k=0}^{n} \frac{(D^k_q f)(a)}{\Gamma_q(k - n - \varepsilon)} x^{k-n-1-\varepsilon(a/x; q)k-n-1-\varepsilon} .
\]

Using the Theorem 11 we obtain
\[
(D_q * D_{q,a}^{n+\varepsilon} f)(x) = (*D_{q,a}^{n+1+\varepsilon} f)(x) + \frac{(D^1_q f)(a)}{\Gamma_q(1 - \varepsilon)} x^{-\varepsilon(a/x; q) - \varepsilon} .
\]

So,
\[
(D_{q,a}^\alpha f)(x) = (*D_{q,a}^{n+1+\varepsilon} f)(x) + \frac{(D^1_q f)(a)}{\Gamma_q(1 - \varepsilon)} x^{-\varepsilon(a/x; q) - \varepsilon}
+ \sum_{k=0}^{n} \frac{(D^k_q f)(a)}{\Gamma_q(k - n - \varepsilon)} x^{k-n-1-\varepsilon(a/x; q)k-n-1-\varepsilon}
= (*D_{q,a}^\alpha f)(x) + \sum_{k=0}^{n+1} \frac{(D^k_q f)(a)}{\Gamma_q(k - n - \varepsilon)} x^{k-n-1-\varepsilon(a/x; q)k-n-1-\varepsilon} ,
\]
what is finishing the proof. \(\square\)

Here, we will discuss behavior of compositions of previously defined operators.

**Theorem 17** Let \(\alpha \in \mathbb{R}^+\). Then, for \(a < x\), the following is valid:
\[
(D_{q,a}^\alpha I_{q,a}^\alpha f)(x) = f(x) .
\]
Proof. With respect to Theorem 5 and the formulas (20) and (21), we have
\[
(D_q^α I_{q,a}^α f)(x) = (D_q^{[α]} I_{q,a}^{[α]} - α I_{q,a}^α f)(x) = (D_q^{[α]} I_{q,a}^{[α]} - α f)(x) \\
= (D_q^{[α]} I_{q,a}^α f)(x) = f(x). \quad \Box
\]

**Theorem 18** Let \( α ∈ \mathbb{R}^+ \setminus \mathbb{N} \). Then
\[
(I_{q,a}^α D_q^α f)(x) = f(x) \quad (α < x).
\]

**Proof.** Let \( α ∈ (0, 1) \). Since, according to (21), we can write
\[
f(x) = (I_{q,a} D_q f)(x) + f(a),
\]
and, by using Theorem 5 and Lemma 12 we have
\[
(I_{q,a}^{1-α} f)(x) = (I_{q,a}^{1-α} I_{q,a} D_q f)(x) + f(a)(I_{q,a}^{1-α} 1)(x) \\
= (I_{q,a}^{2-α} D_q f)(x) + \frac{f(a)}{Γ_q(2-α)} x^{1-α(a/x; q)_{1-α}}.
\]

Applying \( D_q \) on both sides of equality, we obtain
\[
(D_q^α f)(x) = (D_q^{1-α} f)(x) \\
= (D_q^{2-α} D_q f)(x) + \frac{f(a)}{Γ_q(2-α)} D_q(x^{1-α(a/x; q)_{1-α}}) \\
= (I_{q,a}^{1-α} D_q f)(x) + \frac{f(a)}{Γ_q(1-α)} x^{-α(a/x; q)_{1-α}}.
\]

Now, again with respect to Theorem 5 and Lemma 12 the following is valid:
\[
(I_{q,a}^α D_q^α f)(x) = (I_{q,a}^α I_{q,a}^{1-α} D_q f)(x) + \frac{f(a)}{Γ_q(1-α)} I_{q,a}^α (x^{-α(a/x; q)_{1-α}}) \\
= (I_{q,a} D_q f)(x) + f(a) = f(x).
\]

Let \( α = n + ε, \) with \( n ∈ \mathbb{N}, 0 < ε < 1. \) Putting \( α → α - 1 \) and \( f → D_q^{α-1} f \) into Lemma 3 and applying Theorem 4 we get
\[
(I_{q,a}^{α-1} D_q^{α-1} f)(x) = (I_{q,a}α D_q^{α-1} D_q^{α-1} f)(x) + \frac{(D_q^{α-1} f)(a)}{Γ_q(α)} x^{α-1(a/x; q)_{α-1}} \\
= (I_{q,a}^α D_q^α f)(x) + \frac{(D_q^{α-1} f)(a)}{Γ_q(α)} x^{α-1(a/x; q)_{α-1}}.
\]

According to property (30), we conclude that
\[
(I_{q,a}^α D_q^α f)(x) = (I_{q,a}^{α-1} D_q^{α-1} f)(x).
\]

Repeating the last identity \( n \) times, we get
\[
(I_{q,a}^α D_q^α f)(x) = (I_{q,a}^{α-n} D_q^{α-n} f)(x) = (I_q^ε D_q^ε f)(x) = f(x),
\]
what is finishing the proof. \( \Box \)
Theorem 19 Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, for $a < x$, the following is valid:

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = f(x) - \sum_{k=0}^{[\alpha]-1} \frac{(D_k f)(a)}{[k]_q!} x^{k}(a/x; q)_k .$$

Proof. With respect to Theorem 5 and the formulas (20) and (21), we have

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = (I_{q,a}^{\alpha} I_{q,a}^{[\alpha]-\alpha} D_{q}^{[\alpha]} f)(x) = (I_{q,a}^{[\alpha]} D_{q}^{[\alpha]} f)(x)$$

$$= f(x) - \sum_{k=0}^{[\alpha]-1} \frac{(D_k^{[\alpha]} f)(a)}{[k]_q!} x^{k}(a/x; q)_k . \square$$

Theorem 20 Let $\alpha \in \mathbb{R}^+ \setminus \mathbb{N}$. Then, for $a < x$, the following is valid:

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = f(x) .$$

Proof. Putting $f \mapsto I_{q,a}^{\alpha} f$ into Theorem 10 and using Theorem 18 Corollary 6 and formula (28), we get

$$(I_{q,a}^{\alpha} D_{q,a}^{\alpha} f)(x) = (D_{q,a}^{\alpha} I_{q,a}^{\alpha} f)(x) - \sum_{k=0}^{[\alpha]-1} \frac{(D_k^{[\alpha]} I_{q,a}^{\alpha} f)(a)}{[k]_q!(1+k-\alpha)} x^{k-\alpha}(a/x; q)_{k\alpha}$$

$$= f(x) - \sum_{k=0}^{[\alpha]-1} \frac{(D_{q,a}^{\alpha-k} f)(a)}{[k]_q!(1+k-\alpha)} x^{k-\alpha}(a/x; q)_{k\alpha} = f(x) . \square$$

Theorem 21 Let $\alpha \in \mathbb{R}$ and $\beta \in \mathbb{R}^+$. Then, for $a < x$, the following is valid:

$$(D_{q,a}^{\alpha} I_{q,a}^{\beta} f)(x) = (D_{q,a}^{\alpha-\beta} f)(x) .$$

Proof. Let $\alpha = n + \varepsilon$ and $\beta = m + \delta$, where $n > m$ and $\varepsilon, \delta \in [0,1)$ such $\varepsilon < \delta$. Then

$$(D_{q,a}^{\alpha} I_{q,a}^{\beta} f)(x) = (D_{q,a}^{n+1} I_{q,a}^{1-\varepsilon} I_{q,a}^{m+\delta} f)(x)$$

$$= (D_{q,a}^{n+1} I_{q,a}^{m+1+\delta-\varepsilon} f)(x)$$

$$= (D_{q,a}^{n+1} I_{q,a}^{m+1} I_{q,a}^{\delta-\varepsilon} f)(x)$$

$$= (D_{q,a}^{n-m} I_{q,a}^{\delta-\varepsilon} f)(x) .$$

From the other side

$$(D_{q,a}^{\alpha-\beta} f)(x) = (D_{q}^{[\alpha-\beta]} I_{q,a}^{[\alpha-\beta]-(\alpha-\beta)} f)(x) = (D_{q}^{n-m} I_{q,a}^{\delta-\varepsilon} f)(x) . \square$$

Theorem 22 Let $\alpha \in \mathbb{R} \setminus \mathbb{N}$ and $\beta \in \mathbb{R}^+$. Then, for $a < x$, the following is valid:

$$(I_{q,a}^{\beta} D_{q,a}^{\alpha} f)(x) = (D_{q,a}^{\alpha-\beta} f)(x) .$$
Applying the following identity holds:

\[ (I_{q,a}^\beta D_{q,a}^\alpha f)(x) = (I_{q,a}^{\beta-\alpha} f)(x) = (D_{q,a}^{\alpha-\beta} f)(x) . \]

Let \(0 < \alpha \leq \beta\). Then, with respect to Theorem 5 and Theorem 18 we have

\[ (I_{q,a}^\beta D_{q,a}^\alpha f)(x) = (I_{q,a}^{\beta-\alpha} I_{q,a}^\alpha D_{q,a}^\alpha f)(x) = (I_{q,a}^{\beta-\alpha} f)(x) = (D_{q,a}^{\alpha-\beta} f)(x) . \]

Finally, let \(\alpha > \beta\). According to Theorem 18 we can write

\[ f(x) = (I_{q,a}^\alpha D_{q,a}^\alpha f)(x) = (I_{q,a}^{\alpha-\beta} D_{q,a}^\alpha f)(x) . \]

Applying \(D_{q,a}^{\alpha-\beta}\) on both sides of the last equality, we finish the proof. □

Notice that statement of Theorem 22 is not valid for \(\alpha \in \mathbb{N}\). In that case, the following identity holds:

\[ (I_{q,a}^\beta D_{q,a}^n f)(x) = (D_{q,a}^{n-\beta} f)(x) - \sum_{k=0}^{n-1} \frac{(D_{q,a}^k f)(a)}{\Gamma_q(\beta - n + k + 1)} x^{\beta-n+k}(a/x; q)_{\beta-n+k} . \]

Indeed, if \(\alpha = n \leq \beta\), by using Theorem 5 formula (21) and Corollary 12 we get

\[ (I_{q,a}^\beta D_{q,a}^n f)(x) = (I_{q,a}^{\beta-n} f)(x) - \sum_{k=0}^{n-1} \frac{(D_{q,a}^k f)(a)}{[k]_q!} I_{q,a}^{\beta-n}(x^k(a/x; q)_k) \]

\[ = (D_{q,a}^{n-\beta} f)(x) - \sum_{k=0}^{n-1} \frac{(D_{q,a}^k f)(a)}{\Gamma_q(\beta - n + k + 1)} x^{\beta-n+k}(a/x; q)_{\beta-n+k} . \]

In similar way, by using Theorem 13 Theorem 18 Theorem 21 and Theorem 22 the next properties can be proven.

**Theorem 23** Let \(\alpha \in \mathbb{R} \setminus \mathbb{N}\) and \(\beta \in \mathbb{R}^+\). Then, for \(a < x\), the following is valid:

\[ (\star D_{q,a}^{\alpha} I_{q,a}^\beta f)(x) \]

\[ = (\star D_{q,a}^{\alpha} f)(x) + \sum_{k=0}^{[\alpha-\beta]-1} \frac{(D_{q,a}^k f)(a)}{\Gamma_q(k - \alpha + \beta + 1)} x^{k-\alpha+\beta}(a/x; q)_{k-\alpha+\beta} . \]

\[ (I_{q,a}^\beta \star D_{q,a}^{\alpha} f)(x) \]

\[ = (I_{q,a}^\beta f)(x) - \sum_{k=[\alpha-\beta]}^{[\alpha]} \frac{(D_{q,a}^k f)(a)}{\Gamma_q(k - \alpha + \beta + 1)} x^{k-\alpha+\beta}(a/x; q)_{k-\alpha+\beta} . \]

**Theorem 24** Let \(\alpha \leq c < x\) and \(\alpha \in \mathbb{R}^+ \setminus \mathbb{N}\). Then the following is valid:

\[ (I_{q,c}^\alpha D_{q,a}^\alpha f)(x) = (I_{q,c}^{\alpha-[\alpha]+1} D_{q,a}^{\alpha-[\alpha]+1} f)(x) \]

\[ - \sum_{k=1}^{[\alpha]} \frac{(D_{q,a}^{\alpha-k} f)(c)}{\Gamma_q(\alpha - k + 1)} x^{\alpha-k}(c/x; q)_{\alpha-k} . \]

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