On a classification of finite dimensional algebras with respect to the orthogonal (unitary) changes of basis

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Abstract. In this paper, we consider a classification, with respect to the orthogonal (unitary) change of basis, of finite dimensional algebras. A finite system of invariants, which separates nonequivalent algebras, whose systems of structural constants are from an invariant, open, dense set, is given.

1. Introduction
The classification problem of finite dimensional algebras is one of the important problems in Algebra [1, 2]. One of the most used approaches to this problem is the structural (basis free, invariant) method. As an example of such method one can consider a proof of the famous Artin-Wedderburn theorem on classification (semi)simple associative algebras or Cartan’s result on classification of (semi)simple Lie algebras. A structural approach to some more general, for example related to Genetics, algebras is not known. If even it can be done one should not expect to have an easy proof and simple formulation of the result. A disadvantage of the structural method is that the classification is understood only with respect to the general linear group.

Another approach to the classification problem is the coordinate (basis based, structural constants) method. In small dimensional cases, for such an approach we refer the reader to [3, 4, 5]. In case of any finite dimensional algebras, classifications with respect to the general linear groups and some other problems related to them are considered in [6]. In this paper, we also consider the classification problem for any finite dimensional algebras but only with respect to the orthogonal (unitary) change of basis. More exactly, we show how to construct an invariant, open, dense (in the Zariski topology) subset of the space of structural constants, and how to construct a finite system of invariants which separates any two nonequivalent algebras, with respect to the orthogonal (unitary) changes of basis, whose systems of structural constants are in that dense subset. To do it we consider the first the general classification problem of orbits, under one assumption on representation, which may be useful in many other classification problems as well. The main result concerning the classification of finite dimensional algebras (Theorem 2.5) will be derived from the general case though a proof of it may be presented straightforwardly also.

2. Classification and invariants under an Assumption
Let \( n, m \) be any natural numbers, \( \tau : (G, V) \rightarrow V \) be a fixed linear algebraic representation of an algebraic subgroup \( G \) of \( GL(m, F) \) over \( V \), where \( F \) is any field and \( V \) is \( n \)-dimensional
vector space over $F$. Further we consider this representation under the following assumption:

**Assumption.** There exists a nonempty open (in Zariski topology) $G$-invariant subset $V_0$ of $V$ and an algebraic map $P : V_0 \to G$ such that $P(\tau(g, v)) = P(v)g^{-1}$ whenever $v \in V_0$ and $g \in G$.

**Theorem 2.1.** Elements $u, v \in V_0$ are $G$-equivalent, that is $u = \tau(g, v)$ for some $g \in G$, if and only if

$$\tau(P(u), u) = \tau(P(v), v).$$

**Proof.** If $u = \tau(g, v)$ then

$$\tau(P(u), u) = \tau(P(\tau(g, v)), \tau(g, v)) = \tau(P(v)g^{-1}, \tau(g, v)) = \tau(P(v), \tau(g^{-1}, \tau(g, v))) = \tau(P(v), v).$$

Visa versa, if $\tau(P(u), u) = \tau(P(v), v)$ then

$$\tau(P(u)^{-1}P(v), v) = \tau((P(u))^{-1}, \tau(P(v), v)) = \tau((P(u))^{-1}, \tau(P(u), u)) = u$$

that is $u = \tau(g, v)$, where $g = P(u)^{-1}P(v)$. □

This theorem shows that the system of components of $\tau(P(x), x)$ is a finite separating system of invariants for the $G$-orbits in $V_0$, where $x = (x_1, x_2, \ldots, x_n)$ is an algebraic independent system of variables over $F$.

In what follows, it is assumed that $F$ is an algebraically closed field.

**Theorem 2.2.** The field of $G = GL(m, F)$-invariant rational functions $F(x)^{GL(m, F)}$ is generated over $F$ by the system of components of $\tau(P(x), x)$.

**Proof.** It is evident that all components of $\tau(P(x), x)$ are in $F(x)^{GL(m, F)}$. If $f(x) = f(\tau(g, x))$ for all $g \in G = GL(m, F)$ then

$$f(x) = f(\tau(g, x)) = f(\tau(P(\tau(g, x))^{-1}P(x), x)) = f(\tau(P(\tau(g, x))^{-1}, \tau(P(x), x))).$$

In particular it is true for $g = P(x)$. But the equality

$$P(\tau(P(x), x)) = I_m$$

is valid as far as $P(\tau(g, v)) = P(v)g^{-1}$ whenever $v \in V_0$ and $g \in GL(m, F)$. Therefore due to it one has $f(x) = f(\tau(P(x), x))$. □

**Corollary 2.3.** The field $F(x)$ is generated over $F(x)^{GL(m, F)}$ by the system of components of $P(x)$.

**Proof.** Indeed $F(x)^{GL(m, F)}(P(x)) = F(\tau(P(x), x))(P(x)) = F(\tau(P(x), x), P(x))$ and $\tau(P(x))^{-1}, \tau(P(x), x)) = x$ and therefore $F(x)^{GL(m, F)}(P(x)) = F(x)$. □

**Theorem 2.4.** The transcendence degree of $F(x)^{GL(m, F)}$ over $F$ equals to $n - m^2$ and the field extension $F(x)^{GL(m, F)} \subset F(x)$ is a pure transcendental extension.
Proof. We show that the system of components of \( P(x) \) is algebraic independent over \( F(x)^{GL(m, F)} \). Let \( f(y_{ij})_{i,j=1,2,...,m} \) be any polynomial over \( F(x)^{GL(m, F)} \) and \( f(P(x)) = 0 \) which means that \( f_v(P(v)) = 0 \) for all \( v \in V_1 \), where \( V_1 \) is a \( GL(m, F) \)-invariant nonempty open subset of \( V_0 \), \( f_v \) stands for the polynomial obtained from \( f(y_{ij})_{i,j=1,2,...,m} \) by substitution \( v \) for \( x \). Therefore for any \( v \in V_1 \) and \( g \in GL(m, F) \) one has \( 0 = f_v(P(\tau(g, v))) = f_v(P(v)g^{-1}) \). In particular \( 0 = f_v(g) \) for any \( g \in GL(m, F) \) that is \( 0 = f_v((y_{ij})_{i,j=1,2,...,m}) \) whenever \( v \in V_1 \). It implies that \( f((y_{ij})_{i,j=1,2,...,m}) \) is the zero polynomial. Now due to \( F \subset F(x)^{GL(m, F)} \subset F(x) \), tr.deg.\( F(x)/F = n \) implies the required result.

Now consider any \( m \) dimensional algebra \( W \), over the field \( F \), the multiplication of which is given by a bilinear map \( \cdot : W \times W \to W \) that is \( (u, v) \mapsto u \cdot v \), where \( u, v \in W \). If \( e = (e^1, e^2, ..., e^m) \) is a basis for the vector space \( W \) then one can represent the product (bilinear map) by a matrix \( A \in Mat(m \times m^2; F) \) such that \( u \cdot v = eA(u \otimes v) \) for any \( u = eu, v = ev \), where \( u = (u_1, u_2, ..., u_m), v = (v_1, v_2, ..., v_m) \) are column vectors. So the entries of \( A \) are the structural constants of the algebra \( W \) with respect to the basis \( e \).

If \( e' = (e^1, e'^2, ..., e'^m) \) is also a basis for \( W \), \( g \in GL(m, F), e'g = e \) and \( u \cdot v = e'B(u' \otimes v') \), where \( u = e'u', v = e'v' \), then \( u \cdot v = eA(u \otimes v) = e'B(u' \otimes v') = eg^{-1}B(gu \otimes gv) = eg^{-1}B(g \otimes g)(u \otimes v) \) as far as \( u = eu = e'u' = eg^{-1}u', v = ev = e'v' = eg^{-1}v' \). Therefore the equality \( B = gA(g^{-1})^{\otimes 2} \) is valid.

Let now \( \tau \) stand for the representation of \( GL(m, F) \) on \( n = m^3 \) dimensional vector space \( V = Mat(m \times m^2; F) \) defined by \( \tau : (g, A) \mapsto B = gA(g^{-1})^{\otimes 2} \).

We consider a subgroup \( G = O(m, F) \) of \( GL(m, F) \) defined by \( G = \{ g \in GL(m, F) : g^t = I_m \} \), where \( g^t \) stands for the transpose of \( g \).

We will construct an algebraic map \( P : V_0 \to GL(m, F) \), for which the equality of the Assumption holds true for any \( v \in V_0 \) and \( g \in G \), in the following way. It is known that

\[
\text{tr}_1(B) = \text{tr}_1(A)g^{-1}, \quad \text{tr}_2(B) = \text{tr}_2(A)g^{-1},
\]

where \( \text{tr}_1(A) \) stands for the row vector with entries \( \sum_{j=1}^{n} A_{1j}^t \), the contraction on the first upper and lower indices, \( \text{tr}_2(A) \) stands for the row vector with entries \( \sum_{j=1}^{n} A_{ij}^t \), the contraction on the first upper and second lower indices.

By induction it is easy to show that \( \text{tr}_i((BB^t)^k)B = g((AA^t)^kA)(g^{-1})^{\otimes 2} \) for any nonnegative integer \( k \). Therefore one has the following row equalities

\[
\text{tr}_i((BB^t)^k)B = \text{tr}_i((AA^t)^kA)g^{-1}, \quad i = 1, 2, k \in N.
\]

It implies that if \( P(A) \) is a matrix, for example, consisting of the first \( m \) these rows then \( P(\tau(g, A)) = P(A)g^{-1} \) for any \( g \in O(m, F) \). So we can state the following result.

**Theorem 2.5.** Two algebras with structural constants \( A, B \in Mat(m \times m^2, F) \) for which det \( P(A) \neq 0, \text{det} P(B) \neq 0, \) are \( O(m, F) \)- equivalent if and only if

\[
P(B)B(P(B)^{-1})^{\otimes 2} = P(A)A(P(A)^{-1})^{\otimes 2} \quad \text{and} \quad P(B)B(P(B)^{t}) = P(A)A(P(A)^{t}).
\]

**Proof.** Due to Theorem 2.1 the first equality is equivalent to \( B = \tau(g, A) \), where \( g = P(B)^{-1}P(A) \). The second equality is equivalent to \( g = P(B)^{-1}P(A) \in O(m, F) \).

**Corollary 2.6.** Two algebras with structural constants \( A, B \in Mat(m \times m^2, F) \) for which det \( P(A) \neq 0, \text{det} P(B) \neq 0, \) are \( SO(m, F) \)- equivalent if and only if

\[
P(B)B(P(B)^{-1})^{\otimes 2} = P(A)A(P(A)^{-1})^{\otimes 2}, \quad P(B)B(P(B)^{t}) = P(A)A(P(A)^{t})
\]

and det \( P(B) = \text{det} P(A) \).
Remark 2.7. It is evident that when $F$ is the field of complex numbers Theorem 2.5 and Corollary 2.6 hold true if one changes in them $\iota$ to $*$-the transpose with conjugation.

Theorem 2.5 shows that any $O(m, F)$-invariant function defined on $V_0$ is a function of the system of components $P(x)x(P(x)^{-1})^\otimes$ and $P(x)P(x)^t$. But all these components are rational functions of variables $x = (x_{ik})_{i,j,k=1,2,...,m}$ therefore one can ask the following question.

**Problem.** Let $f(x), f_1(x), f_2(x), ..., f_k(x)$ be rational functions in $x_1, x_2, ..., x_k$ over $F$. If it is known that the function $f(x)$ is a function, in the broad sense of function, of the functions $f_1(x), f_2(x), ..., f_k(x)$ over a dense subset of $F^k$, that is there exists function $h$ for which $f(u) = h(f_1(u), f_2(u), ..., f_k(u))$ for any $u$ from that subset, then does it imply that $f(x)$ is algebraic over the field $F(f_1(x), f_2(x), ..., f_k(x))$?

Remark 2.8. The same classification problem can be considered for all finite dimensional commutative (anti-commutative) algebras as well. Similar to Theorem 2.5 result for them can be obtained by appropriate construction of the map $P$.

**Example 2.9.** As an application let us consider two dimensional algebra case for $G = O(2, F)$. Let

$$ A = \begin{pmatrix} A_{1,1}^1 & A_{1,2}^1 & A_{2,1}^1 & A_{2,2}^1 \\ A_{1,1}^2 & A_{1,2}^2 & A_{2,1}^2 & A_{2,2}^2 \end{pmatrix} $$

be a matrix of structural constants. In this case

$$ \text{Tr}_1(A) = (A_{1,1}^1 + A_{2,1}^2, A_{1,2}^1 + A_{2,2}^2), \quad \text{Tr}_2(A) = (A_{1,1}^1 + A_{2,1}^2, A_{1,2}^1 + A_{2,2}^2) $$

and for $P(A)$ one can take the matrix $P(A) = \begin{pmatrix} A_{1,1}^1 + A_{2,1}^2 & A_{1,2}^1 + A_{2,2}^2 \\ A_{1,1}^2 + A_{2,1}^2 & A_{1,2}^2 + A_{2,2}^2 \end{pmatrix}$ and for $V_0$ all $A$ for which $\det P(A) = (A_{1,1}^1 + A_{2,1}^2)(A_{2,1}^1 + A_{1,2}^2) - (A_{1,2}^1 + A_{2,2}^2)(A_{1,1}^1 + A_{2,1}^2) \neq 0$. In this particular case the equality $P(\tau(g, A)) = P(A)g^{-1}$ is true whenever $g \in GL(2, F)$. To see the corresponding separating system of invariants in orthogonal group case one should evaluate $P(x)P(x)^t, P(x)x(P(x)^{-1})^\otimes$ and for $G = GL(2, F)$ case only $P(x)x(P(x)^{-1})^\otimes$, where $x = \begin{pmatrix} x_{1,1}^1 & x_{1,2}^1 & x_{2,1}^1 & x_{2,2}^1 \\ x_{1,1}^2 & x_{1,2}^2 & x_{2,1}^2 & x_{2,2}^2 \end{pmatrix}$.

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**References**

[1] Albert A A 2003 *Structure of algebras*, Colloquium publications 24, (AMS), 1–113.
[2] Erdmann K and Wildon M 2006 *Introduction to Lie Algebras*, (1st ed.) (Springer-Berlin)
[3] Michel G and Elisabeth R 2011 *African J. Math. Phys.* 10 81–91.
[4] Durán Díaz R et al 2003 *Lin. Alg. Appl.* 364 1–12.
[5] Hernández Encinas L et al 2004 *Lin. Alg. Appl.*, 387 69–82.
[6] Popov V 2011 *arXiv: 1411.6570*