Enhancement of $T_c$ in the superconductor–insulator phase transition on scale-free networks

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Abstract. A road map for understanding the relation between the onset of the superconducting state with a particular optimum heterogeneity in granular superconductors is provided by studying a random transverse Ising model on complex networks with a scale-free degree distribution regularized by an exponential cutoff $p(k) \propto k^{-\gamma} \exp[-k/\xi]$. In this paper we characterize in detail the phase diagram of this model, both on annealed and on quenched networks. To uncover the phase diagram of the model we use the tools of heterogeneous mean-field calculations for the annealed networks and the most advanced techniques of quantum cavity methods for the quenched networks. The phase diagram of the dynamical process depends on the temperature $T$, on the coupling constant $J$ and on the value of the branching ratio $\langle k(k-1) \rangle / \langle k \rangle$ where $k$ is the degree of the nodes in the network. For a fixed value of the coupling, the critical temperature increases linearly with $\langle k(k-1) \rangle / \langle k \rangle$, which diverges with increasing cutoff value $\xi$ for values of the $\gamma$ exponent $\gamma \leq 3$. This result suggests that the fractal disorder of the superconducting material can be responsible for an enhancement of the superconducting critical temperature. For low temperature and low couplings $T \ll 1$ and $J \ll 1$, however, we observe different behaviors for annealed and quenched networks. In the annealed networks there is no phase transition at zero temperature, while in the quenched network a Griffith phase dominated by extremely rare events and a phase transition at zero temperature are observed. The Griffiths critical region, however, decreases in size with increasing value of the cutoff $\xi$ of the degree distribution for values of the $\gamma$ exponents $\gamma \leq 3$.

Keywords: disordered systems (theory), phase diagrams (theory), cavity and replica method, random graphs, networks
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1. Introduction

The interplay between disorder and superconductivity has attracted the interest of physicists in the past few decades. Disorder is expected to compete with superconductivity by enhancing the electrical resistivity of a system. In this situation, on increasing the random disorder, the system undergoes a superconductor–insulator phase transition [1]. The theoretical explanation of the interplay between disorder and superconductivity is still a problem of intensive debate for granular superconductors.

Several authors have proposed that high-$T_c$ superconductors are intrinsically inhomogeneous [2]–[5]; however these materials have a multiphase complexity that is difficult to tackle by using analytical theoretical models in general. There is growing interest in a possible optimum inhomogeneity of superconducting cuprates that could enhance the superconducting critical temperature [6]. The control of defects and interstitials in heterostructures using new material science technologies can be used to design new granular superconductors with new functionalities [5]. Recent experiments have provided multiscale imaging of the granular structure of doped cuprate perovskites with a scale invariance that is reflected in the scale-free distribution of oxygen interstitials [7, 8]. Therefore the synthesis of novel granular superconductors made of superconducting networks with power-law connectivity distributions is now of great experimental interest. A road map for understanding these new possible materials is provided by constructing
heterogeneous mean-field models and studying the superconductor–insulator phase transition in complex scale-free networks. Recently the superconductor–insulator phase transition has been characterized on Bethe lattices by solving the random transverse Ising model on quenched Bethe lattices [9]–[11] with the use of advanced quantum cavity methods, which allow one to go beyond the mean-field prediction. These methods have been recently developed for studying the exact phase diagram of the random transverse Ising model [12, 13]. In this paper we use the quantum cavity mapping approximation of these methods, proposed in [9]–[11], which gives a physical interpretation of the phase diagram of the random transverse Ising model by mapping the equation determining the phase transition to a random polymer problem [14]. Furthermore we extend the results to networks with arbitrary degree distribution. In particular we focus on scale-free degree distributions mimicking the fractal disorder reported in cuprates. Already a mean-field calculation for the random transverse Ising model on complex networks has shown that the superconducting critical temperature is strongly enhanced for a scale-free degree distribution with power-law exponent $\gamma \leq 3$ [15]. This result might explain the experimental findings of Fratini et al [8] and might provide a new road map for designing new heterogeneous superconductors with enhanced superconducting critical temperatures.

On the theoretical side, this result is in line with the study of other phase transitions on scale-free network topologies [16, 17]. In fact it has been shown that the classical Ising model [18]–[20], the percolation transition [21] and epidemic spreading [22] change significantly when the second moment $\langle k(k - 1) \rangle$ of the degree distribution diverges. Recently it has been becoming clear that these effects of the topology apply also to quantum critical phenomena [23] and are found in the mean-field solutions for both the random transverse Ising model [15] and the Bose–Hubbard model [24]. An open problem that we approach in this paper is that of to what extent mean-field results are indicative of the behavior of the critical phenomena in quenched networks. In particular, we focus here on the random transverse Ising model and we study the difference between the phase diagrams on annealed and quenched scale-free networks. While for annealed networks a simple mean-field result is guaranteed to be valid, for quenched networks we have to approach the problem by using the recently introduced quantum cavity method and by the mapping this solution to the random polymer problem. We find that the phase diagram for annealed networks is significantly different from the phase diagram for the quenched networks as long as the second moment of the degree distribution remains finite. In particular, we found a phase transition at zero temperature not predicted by the mean-field approach and a replica symmetry broken phase at low temperatures. In contrast, as the second moment of the degree distribution diverges, $\langle k(k - 1) \rangle \to \infty$, the mean-field predictions approach the quenched solution. Therefore we identify the second moment of the degree distribution as a key quantity characterizing the disorder of the topology of the network. As the second moment of the degree distribution diverges, the superconducting critical temperature of the network diverges, both for quenched and for annealed complex networks.

2. The random transverse Ising model

We consider a system of spin variables $\sigma_i^z$, for $i = 1, \ldots, N$, defined on the nodes of a given network with adjacency matrix $a$ such that $a_{ij} = 1$ if there is a link between node $i$ and
node $j$, and otherwise $a_{ij} = 0$. The random traverse Ising model is defined as in [9]–[11] and is given by the Hamiltonian

$$\hat{H} = -\frac{J}{2} \sum_{ij} a_{ij} \sigma_i^z \sigma_j^z - \sum_i \epsilon_i \sigma_i^x - h \sum_i \sigma_i^z.$$  

(1)

This Hamiltonian is a simplification with respect to the $XY$ model Hamiltonian proposed by Ma and Lee [25] for describing the superconducting–insulator phase transition, but to leading order the equation for the order parameter is the same as that widely discussed in [9, 10]. The Hamiltonian describes the superconducting–insulator phase transition in terms of a ferromagnetic spin 1/2 spin system in a transverse field. We propose to use this Hamiltonian to describe for a granular superconductor a network with heterogeneity of the degree distribution. The spins $\sigma_i$ in equation (1) indicate states occupied or unoccupied by a Cooper pair or a localized pair; the parameter $J$ indicates the couplings between neighboring spins, the $\epsilon_i$ are quenched values of the on-site energy and $h$ is an external auxiliary field. To mimic the randomness of the on-site energy we draw the variables $\epsilon_i$ from a $\rho(\epsilon) = 1/2$ distribution with a finite support $\epsilon \in (-1, 1)$. Finally in this model the superconducting phase corresponds to the existence of a spontaneous magnetization in the $z$ direction.

### 3. Annealed complex networks

We consider networks of $N$ nodes $i = 1, \ldots, N$. We assign to each node $i$ a hidden variable $\theta_i$ from a $p(\theta)$ distribution indicating the expected number of neighbors of a node. An annealed complex network is a network that is dynamically rewired and where the probability of having a link between nodes $i$ and $j$ is given by $p_{ij}$:

$$p_{ij} = P(a_{ij} = 1) = \frac{\theta_i \theta_j}{\langle \theta \rangle N}$$  

(2)

where $a_{ij}$ is the matrix element $(i,j)$ of the adjacency matrix.

In this ensemble the degree $k_i$ of a node $i$ is a Poisson random variable with expected degree $\overline{k_i} = \theta_i$. Therefore we will have

$$\langle \theta \rangle = \langle k \rangle \quad \langle \theta^2 \rangle = \langle k(k - 1) \rangle.$$  

(3)

where $\langle \ldots \rangle$ indicates the average over the $N$ nodes of the network and the overline in equation (3) indicates the average over time-dependent degrees of the nodes. In order to mimic the fractal background present in cuprates [8] we assume that the expected degree distribution of the network is given by

$$p(\theta) = N \theta^{-\gamma} e^{-\theta/\xi}$$  

(4)

where $N$ is a normalization constant and $\xi$ can be modulated by an external parameter such as the doping or strain in cuprates.
4. The solution for the random transverse Ising model on annealed complex networks

4.1. The critical temperature

In order to study the random transverse Ising model on annealed complex networks above the percolation threshold, we consider the fully connected Hamiltonian given by

\[ \hat{H} = -\frac{J}{2} \sum_{ij} p_{ij} \sigma^z_i \sigma^z_j - \sum_i \epsilon_i \sigma^x_i - h \sum_i \sigma^z_i \]  

where in order to account for the dynamical nature of the annealed graph we have substituted the adjacency matrix \( a_{ij} \) in \( H \) with the matrix \( p_{ij} \) given by equation (2).

In order to evaluate the partition function we apply the Suzuki–Trotter decomposition in a number \( N_s \) of Suzuki–Trotter slices. Therefore we have in the limit \( N_s \to \infty \)

\[ \text{Tr} e^{-\beta \hat{H}} = \text{Tr} \left( e^{-\beta E/N_s} e^{-\beta \sum_i \epsilon_i \sigma^x_i/N_s} \right) \]  

where \( E \) is given by

\[ E = -\frac{J}{2} \sum_{ij} p_{ij} \sigma^z_i \sigma^z_j - h \sum_i \sum_\alpha \sigma^z_i. \]  

In order to perform this calculation we consider for each spin the sequence \( \sigma^z_i = \{ \sigma^z_i,1, \ldots, \sigma^z_i,N_s \} \) where each spin \( \sigma^z_i,\alpha \) represents the spin \( i \) in the Suzuki–Trotter slice \( \alpha \). The partition function is then defined as

\[ Z = \sum_{\{\sigma_i\}} \prod_{i=1}^N \prod_{\alpha=1}^{N_s} w(\sigma_i) e^{(\beta \epsilon_i/N_s) \sum_{\alpha=1}^{N_s} \sigma^z_i,\alpha} e^{(\beta J/2N_s) \sum_{ij} \sigma^z_i,\alpha \sigma^z_j,\alpha} \]  

where we have used

\[ w(\sigma_i) = \prod_{\alpha} (\sigma^z_i,\alpha | e^{(\beta \epsilon_i/N_s)} \sigma^x | \sigma^z_i,\alpha+1). \]  

In order to disentangle the quadratic terms we use \( N_s \) Hubbard–Stratonovich transformations:

\[ Z = \left( \frac{\beta N_s}{2\pi N_S} \right)^{N_s/2} \int D\mathbf{S} \exp \left[ -\frac{N_s}{2} \sum_{\alpha=1}^{N_s} (S_{\alpha})^2 \right] \times \exp \left[ N \int d\theta \left( \frac{p(\theta)}{2} \right) \right] \int_{-1}^1 d\epsilon \ln \text{Tr} \prod_{\alpha} e^{(\beta/N_s)(h+J \theta S_{\alpha}) \sigma^x} e^{(\beta/N_s)\epsilon \sigma^x}, \]  

where \( D\mathbf{S} = \prod_{\alpha=1}^{N_s} dS^\alpha \). The free energy \( f = -(1/\beta) \lim_{N \to \infty} \lim_{N_s \to \infty} (1/N) \ln Z \) can be evaluated at the stationary saddle point which is cyclically invariant. Therefore we get

\[ f = \inf_{S} \frac{J(\theta)}{2} S^2 - \frac{1}{\beta} \int d\theta \left( \frac{p(\theta)}{2} \right) \int_{-1}^1 d\epsilon \ln \left( 2 \cosh \left( \beta \sqrt{(h+JS\theta)^2 + \epsilon^2} \right) \right) \]

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where the value of $S$ which minimizes the free energy is given by the saddle point equation

$$S = \sum_{\theta} \frac{\theta}{p(\theta)} \int_0^1 d\epsilon \frac{JS\theta + h}{\sqrt{(JS\theta + h)^2 + \epsilon^2}} \tanh \left( \beta \sqrt{(JS\theta + h)^2 + \epsilon^2} \right).$$ \hspace{1cm} (10)

Finally the magnetizations along the axes $x$ and $z$ can be calculated by evaluating

$$m_{\theta,\epsilon}^x = \left. \frac{\text{Tr} \sigma_x e^{-\beta H}}{Z} \right|_{\theta_i = \theta, \epsilon_i = \epsilon},$$

$$m_{\theta,\epsilon}^z = \left. \frac{\text{Tr} \sigma_z e^{-\beta H}}{Z} \right|_{\theta_i = \theta, \epsilon_i = \epsilon}. \hspace{1cm} (11)$$

Performing these calculations we get

$$m_{\theta,\epsilon}^x = \frac{JS\theta + h}{\sqrt{(JS\theta + h)^2 + \epsilon^2}} \tanh \left( \beta \sqrt{(JS\theta + h)^2 + \epsilon^2} \right),$$

$$m_{\theta,\epsilon}^z = \frac{\epsilon}{\sqrt{(JS\theta + h)^2 + \epsilon^2}} \tanh \left( \beta \sqrt{(JS\theta + h)^2 + \epsilon^2} \right).$$

Therefore the magnetizations $m_{\theta,\epsilon}^x$ and $m_{\theta,\epsilon}^z$ depend on the value $\theta$ of the expected degree of the node and on the on-site energy $\epsilon$.

The order parameter for the superconducting–insulator phase transition is $S$ given by equation (10). From the self-consistent equation determining the order parameter for the transition it is immediately seen that the superconducting–insulator phase transition occurs for $h = 0$ at

$$1 = J \langle \theta^2 \rangle / \langle \theta \rangle \int_0^1 d\epsilon \frac{\tanh(\beta \epsilon)}{\epsilon} = \frac{J}{J_c(\beta)},$$ \hspace{1cm} (12)

which implies that for $\langle \theta^2 \rangle \to \infty$, we then have $\beta \to 0$ for any fixed value of the coupling $J > 0$, and the critical temperature for the paramagnetic ferromagnetic phase transition $T_c$ diverges. This implies that on annealed scale-free networks with $p(\theta) \propto \theta^{-\gamma}$ and $\gamma \leq 3$, the random transverse Ising model is always in the superconducting phase.

If we consider the special case of an expected degree distribution given by equation (4) and $\gamma \leq 3$, we have that $\langle \theta^2 \rangle / \langle \theta \rangle$ diverges in the limit $\xi \to \infty$. If we take $J(\langle \theta^2 \rangle / \langle \theta \rangle) \gg 1$ which can be achieved by decreasing $J$ constant and we go to the limit $\xi \to \infty$ with $\gamma \leq 3$, we find that the critical temperature for the superconductor–insulator transition diverges as

$$T_c \propto J \langle \theta^2 \rangle / \langle \theta \rangle = J \frac{\langle k(k-1) \rangle}{\langle k \rangle} \propto \begin{cases} \ln \xi & \text{if } \gamma = 3 \\ \xi^{3-\gamma} & \text{if } \gamma < 3. \end{cases}$$

Therefore on complex topologies, when the expected degree distribution is given by equation (4) and if $\gamma \leq 3$ we observe an enhancement of the superconducting temperature with increasing values of the cutoff $\xi$. In contrast, for $J(\langle \theta^2 \rangle / \langle \theta \rangle) \ll 1$ we have

$$T_c = \frac{4e^C}{\pi} \exp \left[ - \frac{\langle \theta \rangle}{J \langle \theta^2 \rangle} \right] = \frac{4e^C}{\pi} \exp \left[ - \frac{\langle k \rangle}{J \langle k(k-1) \rangle} \right]$$ \hspace{1cm} (13)

where $C \sim 0.577$ is the Euler number. Therefore in the annealed network model there is no phase transition at zero temperature. In fact from equation (13), when $\langle \theta^2 \rangle$ is finite we have that $T_c = 0$ only for $J = 0$. This phenomenon strongly depends on the assumption that the network is annealed, as was found in [9, 10] where the critical behavior of the random transverse Ising model was studied on a Bethe lattice. The critical indices of

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the phase transition will be given by the heterogeneous mean-field [19] approach to the phase transition as long as the cutoff $\xi = \infty$, and will depend non-trivially on the value of the power-law exponent of the expected degree distribution $\gamma$. The critical index of the susceptibility will be the only exception and remain a mean-field one as long as we do not include the dependence of the network on the embedding space [20].

5. Quenched networks

In this section we introduce quenched networks that have a structure that does not change in time. In section 6 we will characterize the random transverse Ising model on quenched networks and we compare the results with the heterogeneous mean-field results obtained in the previous sections. Therefore we consider a quenched network of $N$ nodes $i = 1, 2 \ldots, N$ and degree distribution $p(k)$. We indicate by $N(i)$ the set of nodes that are neighbors of node $i$. We assume that the distribution of quenched local energies $\epsilon_i$ is given by $\rho(\epsilon) = 1/2 \pi \epsilon$ with $\epsilon \in (-1, 1)$.

We will consider the configuration model in the particular case in which the degree distribution $p(k)$ of the network is scale-free, with power-law exponent $\gamma$, and regularized by an exponential cutoff $\xi$, i.e. we assume that

$$p(k) = N k^{-\gamma} e^{-k/\xi}. \quad (14)$$

where $N$ is a normalization constant. We will assume that the quenched network is random, meaning that the probability of reaching a node of degree $k$ by following a random link is given by $kp(k)/\langle k \rangle$.

6. The solution for a quenched network

In quenched networks, critical phenomena might show a phase diagram different to that for annealed networks. In particular this is true for the random transverse Ising model that on quenched Bethe lattices has a zero-temperature phase transition which is absent on the corresponding annealed network, as discussed in [9]–[11]. In order to study critical phenomena on quenched networks with a locally tree-like structure (vanishing clustering coefficient), the theory of the cavity method has been developed and applied to a large variety of classical critical phenomena on networks. Recently this approach has been extended to quantum critical phenomena without disorder using the Suzuki–Trotter formalism, and finding exact results [12, 13]. In this paper we take a different approach by following a method recently [9]–[11] proposed for studying the random transverse Ising model on Bethe lattices. This approach has the advantage that it can be applied to quantum critical phenomena with disorder and that it is analytically tractable. Moreover, as we will see, the solution of the problem can be mapped to a directed polymer problem, shedding light on the nature of the different phases for the critical dynamics.

6.1. Cavity mapping

In the cavity model, the cavity graph is considered. The cavity graph is the graph centered on a node $i$ when one of its neighbors $j \in N(i)$ is removed. It is therefore assumed that all the other neighbors $\alpha \in N(i) \setminus j$ of node $i$ are uncorrelated. It is assumed that on each
of these nodes $\alpha$, an efficient cavity field $B_{\alpha,i}$ is acting, such that the local Hamiltonian acting on the cavity graph centered at node $i$ is given by

$$H_{i,j}^{\text{cav}} = -\epsilon_i \sigma_x^i - \sum_{\alpha \in N(i) \setminus j} [\epsilon_\alpha \sigma_z^\alpha + B_{\alpha,i} \sigma_z^\alpha + J \sigma_z^i \sigma_z^\alpha]$$

(15)

where we assume here and in the following that the external field $h = 0$. It was recently shown that the cavity mean-field method is a very useful tool for studying the random transverse Ising model [9]–[11]. In fact, at the same time, it provides an analytical approach to the solution of the Hamiltonian, and it provides a physical interpretation of the phenomena occurring at low temperatures. Finally this approximation can be used to reproduce with good accuracy the phase diagram of the random transverse Ising model.

In the cavity mean-field approximation we replace the dynamical spin variables $\sigma_z^\alpha$ in $H_{i,j}^{\text{cav}}$ by their averages $\langle \sigma_z^\alpha \rangle$ with values

$$\langle \sigma_z^\alpha \rangle = B_{\alpha,i} \sqrt{B_{\alpha,i}^2 + \epsilon_{\alpha}^2} \tanh \beta \sqrt{B_{\alpha,i}^2 + \epsilon_{\alpha}^2}.$$ 

(16)

Therefore we assume that the cavity Hamiltonian acting on spin $i$ in the absence of spin $j$ can be approximated by

$$H_{i,j}^{\text{cavMF}} = -\epsilon_i \sigma_x^i - J \sum_{\alpha \in N(i) \setminus j} \langle \sigma_z^\alpha \rangle$$

(17)

which implies that $B_{i,j} = J \sum_{\alpha \in N(i) \setminus j} \langle \sigma_z^\alpha \rangle$ and therefore

$$B_{i,j} = J \sum_{\alpha \in N(i) \setminus j} \frac{B_{\alpha,i}}{\sqrt{B_{\alpha,i}^2 + \epsilon_{\alpha}^2}} \tanh \beta \sqrt{B_{\alpha,i}^2 + \epsilon_{\alpha}^2}.$$ 

(18)

This recursion induces a self-consistent equation for the distribution $P_k(B)$ of the cavity fields $B$ for nodes of degree $k$:

$$P_k(B) = \prod_{\alpha} \frac{p(k_\alpha) k_\alpha}{\langle k \rangle} \int \prod_{\alpha} [dB_{\alpha} d\epsilon_{\alpha} P_{k_\alpha}(B_{\alpha}) \rho(\epsilon_{\alpha})] \times \delta \left( B - J \sum_{\alpha=1}^{k-1} \frac{B_{\alpha}}{\sqrt{B_{\alpha}^2 + \epsilon_{\alpha}^2}} \tanh \beta \sqrt{B_{\alpha}^2 + \epsilon_{\alpha}^2} \right).$$

(19)

6.2. Mapping of the problem to a direct polymer

In order to characterize the critical behavior of the random transverse Ising model we can imagine iterating the relation (18) on the complex network $L \gg 1$ times. For $L$ finite and $N \to \infty$ the corresponding graph is locally a tree with branching ratio

$$K = \frac{\langle k(k-1) \rangle}{\langle k \rangle}.$$ 

(20)
Therefore the average number of paths of size $L$ starting from node $i$ and ending after $L$ steps is given by

$$\langle \sum_{i_1, i_2, \ldots, i_L} a_{i, i_1} a_{i_1, i_2} \cdots a_{i_{L-1}, i_L} \rangle = k_i \left( \frac{\langle k(k-1) \rangle}{\langle k \rangle} \right)^{L-1}. \tag{21}$$

The cavity field at the root of this tree is a function of the cavity fields on the boundary. If we assume that infinitesimal cavity fields $B \ll 1$ are acting on the boundary nodes, the cavity field $B_0$ at the root of the tree is given by

$$B_0 = \sum_{\mathcal{P}} \prod_{\alpha \in \mathcal{P}} J \tanh \beta \epsilon_{\alpha} \epsilon_{\alpha} = \Xi \tag{22}$$

where $\mathcal{P}$ indicates the path on the tree with heterogeneous degree distribution and branching ratio $K$ from the root to the leaf of the tree, and the product $\prod_{\alpha \in \mathcal{P}}$ is over all nodes along the path $\mathcal{P}$. The critical point of the dynamics is given by the parameters for which $(1/L) \ln \Xi = 0$, indicating when the cavity field propagates over large distances on the network. The expression (22) allows for a mapping of our problem to the directed polymer (DP) problem. In particular the function $\Xi$ is the partition function of the directed polymer where the energy on each edge is given by $e^{-E_{\alpha}} = J(tanh \beta \epsilon_{\alpha}/\epsilon_{\alpha})$ and the temperature is set equal to 1. This problem has been studied by Derrida and Spohn [14].

In the directed polymer problem there are two regimes:

- The replica symmetric (RS) regime in which the measure defined in equation (22) is more or less evenly distributed among the paths.
- The replica symmetry broken (RSB) regime in which the measure defined in equation (22) condenses on a small number of paths.

In order to find the phase transition between the two phases we need to study the behavior of $\ln \Xi$ over a typical sample of the disorder. This is done by evaluating the average over the quenched variables of $\ln \Xi$. This computation can be done by making use of the replica method [9]–[11], [26] by computing

$$\ln \Xi = \lim_{n \to 0} \frac{\Xi^n - 1}{n} \tag{23}$$

where the replicated partition function is a sum over $n$ paths:

$$\Xi^n = \sum_{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n} \prod_{\alpha} \left( J \tanh \beta \epsilon_{\alpha} \right)^{r_{\alpha}} \tag{24}$$

where the weight of each node $\alpha$ depends on the number of paths which go through this node and the overline symbol indicates the average over the quenched distribution of the variables $\epsilon$ and the quenched topology of the network. In the RS solution we assume that all of the paths starting from node $i$ are non-overlapping and independent; therefore,

$$\Xi^n = \sum_{\mathcal{P}_1, \mathcal{P}_2, \ldots, \mathcal{P}_n} \prod_{\alpha} \left( J \tanh \beta \epsilon_{\alpha} \right)^{r_{\alpha}}$$

$$= \left[ K^{L-1} k_i \left( \int_0^1 d\epsilon J \tanh \beta \epsilon \right)^L \right]^n \tag{25}$$
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where we have performed the averaging over the topology and the random distribution of energies $\epsilon$ of the replicated partition function. Using the replica trick we found that in the large $L$ limit $(1/L)\ln \Xi$ is given by

$$\frac{1}{L} \ln \Xi = \ln J + \ln \left( \frac{\langle k(k-1) \rangle}{\langle k \rangle} \right) \int_0^1 \frac{d\epsilon}{\epsilon} \frac{\tanh \beta \epsilon}{\epsilon} = \ln J + f(1).$$

(26)

In the RSB solution, according to the result of [14] which has shown that the overlap between paths is either 0 or 1, we assume instead that the $n$ paths consist of groups of $n/x$ identical paths, and therefore $r_\alpha = x$. Using the replica trick with this assumption we get in the large $L$ limit, and performing the averaging over the random quenched topology and the distribution of the quenched energies $\epsilon$,

$$\frac{1}{L} \ln \Xi = \ln J + \frac{1}{x} \ln \left( \frac{\langle k(k-1) \rangle}{\langle k \rangle} \right) \int_0^1 \frac{d\epsilon}{\epsilon} \left( \frac{\tanh \beta \epsilon}{\epsilon} \right)^x = \ln J + f(x)$$

(27)

where

$$f(x) = \frac{1}{x} \ln \left( \frac{\langle k(k-1) \rangle}{\langle k \rangle} \right) \int_0^1 \frac{d\epsilon}{\epsilon} \left( \frac{\tanh \beta \epsilon}{\epsilon} \right)^x.$$  

(28)

The value of $x$ is found by extremizing the function $f(x)$ with respect to $x$ with $x \in [0, 1]$. In particular, by minimizing $f(x)$ as a function of $x$ one finds that for $\beta > \beta_{\text{RSB}}$ the solution is replica symmetry broken with $x = m < 1$, and for $\beta > \beta_{\text{RSB}}$ the function is minimized at the boundary $m = 1$ and the solution is replica symmetric. We note here that a similar result can be obtained for a quenched network constructed with the hidden variable models.

6.3. The phase diagram

The critical behavior of the random transverse Ising model is found when $(1/L)\ln \Xi = 0$ and when $B_0/B$, the cavity field at the boundary, propagates at long distances on the network under the iteration equation (18). Therefore we have to distinguish between two regimes:

- The replica symmetric (RS) regime for $\beta < \beta_{\text{RSB}}$.
  In this regime we have that the critical line is dictated by the relation $(1/L)\ln \Xi = f(1) + \ln(J) = 0$; therefore the critical points are solutions to the equation

$$J \frac{\langle k(k-1) \rangle}{\langle k \rangle} \int_0^1 \frac{d\epsilon}{\epsilon} \frac{\tanh \beta \epsilon}{\epsilon} = 1.$$  

(29)

In the limit $J(\langle k(k-1) \rangle/\langle k \rangle) \gg 1$ we get

$$T_c = J \frac{\langle k(k-1) \rangle}{\langle k \rangle} \propto \begin{cases} \ln \xi & \text{if } \gamma = 3 \\ \xi^{3-\gamma} & \text{if } \gamma < 3 \end{cases}$$

where the last relation is valid if $p(k) = \mathcal{N}k^{-\gamma}e^{-k/\xi}$.

- The replica symmetry broken (RSB) regime for $\beta > \beta_{\text{RSB}}$.
  In this regime we have that the critical line is dictated by the relation $(1/L)\ln \Xi = \ln(J) + f(m) = 0$ and $f'(m) = 0$; therefore the critical points are solutions to the

$$J \frac{\langle k(k-1) \rangle}{\langle k \rangle} \int_0^1 \frac{d\epsilon}{\epsilon} \frac{\tanh \beta \epsilon}{\epsilon} = 1.$$  

(29)
6.3.1. The phase diagram at $T = 0$. It is interesting to study the critical point at $T = 0$. In particular, we observe relevant deviations of the phase diagram on a quenched network with respect to the behavior on an annealed network when the critical point is predicted to be at $J = 0$. The function $f(m)$ defined in equation (28) takes the simple form

$$f(m) = \frac{1}{m} \ln \left[ \frac{\langle k(k-1) \rangle}{\langle k \rangle} \right].$$

(31)

At $T = 0$ we can study the critical point of the transition by imposing $f'(m) = 0$ and $(1/L) \ln \Xi = \ln(J) + f(m) = 0$. By studying these equations we get the critical point

$$\frac{1}{J_c} = e^{\langle k(k-1) \rangle / \langle k \rangle} \ln \left( e^{\langle k(k-1) \rangle / \langle k \rangle} \right).$$

(32)

Therefore on any given network with finite branching ratio there is a quantum phase transition at $T = 0$. In the case in which $p(k) = Nk^{-\gamma}e^{-k/\xi}$ and $\gamma \leq 3$, we observe for large values of the exponential cutoff $\xi$ that $J_c$ goes to zero as

$${J_c} \propto \begin{cases} \frac{1}{\xi^{3-\gamma \ln(\xi)}} & \text{if } \gamma < 3 \\ \frac{1}{\ln(\xi) \ln(\ln(\xi))} & \text{if } \gamma = 3. \end{cases}$$

Therefore in the $\xi \to \infty$ limit, as long as the power-law exponent $\gamma$ is $\gamma \leq 3$ we recover the mean-field result that there is no phase transition at zero temperature.

6.3.2. The phase diagram for $T > 0$: the RS–RSB phase transition. It is interesting to find the critical point $T_c = T_{RSB} \ll 1$ where we have the transition between the replica symmetric phase and the replica symmetry broken phase. This point is found by solving the equations (30) for $m = 1$ (i.e. $f(1) = 0$ and $f'(1) = 0$) which we rewrite here for convenience:

$$J \langle k(k-1) \rangle / \langle k \rangle \int_0^1 d\epsilon \left( \frac{\tanh \beta \epsilon}{\epsilon} \right)^m = 1$$

(33)

We can find the critical point by expanding the equation for $J(\langle k(k-1) \rangle / \langle k \rangle) \ll 1$ and $T \ll 1$, finding

$$\frac{1}{T_{RS-\text{RSB}}} = 2 \langle k(k-1) \rangle / \langle k \rangle \ln \left( 2 \langle k(k-1) \rangle / \langle k \rangle \right)$$

$$T_c = \frac{\pi}{\varphi} \left[ \frac{\langle k(k-1) \rangle}{\langle k \rangle} \ln \left( 2 \frac{\langle k(k-1) \rangle}{\langle k \rangle} \right) \right]^{-2}$$

(34)
where $C$ is the Euler constant $C = 0.577...$. Therefore the critical temperature for the onset of the RSB phase for degree distributions given by equation (14) and $\gamma \leq 3$ goes to zero as the exponential cutoff $\xi$ diverges, and we have that $T_{c}^{RS-RSB}$ goes to zero with leading behavior

$$T_{c}^{RS-RSB} \propto \begin{cases} 
\frac{1}{\xi^{2(3-\gamma)\ln^{2}(\xi)}} & \text{if } \gamma < 3 \\
\frac{1}{\ln \xi \ln(\ln \xi + D)^{2}} & \text{if } \gamma = 3.
\end{cases}$$

where $D$ is a constant. In figure 1 we plot the critical temperature for the phase transition between the superconducting and insulator phases for small values of the temperature $T$ and coupling $J$. For temperature $T < T_{c}^{RS-RSB}$ the critical line is dictated by the replica symmetry broken equations (33). By plotting $T_{c} = T_{c}(J)$ for networks with different values of the branching ratio $K = \langle k(k-1) \rangle / \langle k \rangle$, we show that as $K$ grows the replica symmetry broken phase shrinks. In order to show how severe this effect is, in figure 2 we show the critical coupling constant $J_{c}^{RS-RSB}$ and the critical coupling constant at $T = 0$ $J_{c} = J_{c}(T = 0)$ as a function of the cutoff and the power-law exponent $\gamma$. We show that for the power-law exponent $\gamma \leq 3$, the branching ratio $K$ diverges with diverging values of the cutoff $\xi$, and in the limit $\xi \to \infty$ we recover the mean-field results. Moreover, as we show in figure 3, as the coupling is fixed and $J(\langle k(k-1) \rangle / \langle k \rangle) \gg 1$, the critical temperature for the superconductor–insulator phase transition diverges with diverging value of the branching ratio $K = \langle k(k-1) \rangle / \langle k \rangle$. Finally the critical indices of the phase transition will depend on the value of the exponential cutoff $\xi$ and the power-law exponent $\gamma$ of the degree distribution. Moreover, following arguments similar to those used in [10], we can show that the replica symmetric phase is a Griffith phase.
Enhancement of $T_c$ in the superconductor–insulator phase transition on scale-free networks

Figure 2. Critical value $J_c(T = 0)$ for the insulator–superconductor phase transition and critical value $J_c^{RS-RSB}$ for the replica symmetric and replica symmetry broken phases. For $\gamma = 2.5$ the Griffith phase comes to occupy a reduced fraction of the phase space as the exponential cutoff $\xi$ diverges, while for $\gamma = 3.5$ the Griffith phase survives also in the $\xi \to \infty$ limit.

Figure 3. The critical temperature of the superconductor–insulator phase transition as a function of the cutoff $\xi$ when the coupling constant $J = 1$ is kept constant. For $\gamma \leq 3$ the critical temperature for superconductivity is strongly enhanced by the scale-free topology of the underlying network.

7. Conclusions

In conclusion, we have studied the random transverse Ising model on networks with arbitrary degree distribution as a paradigm for studying superconductor–insulator phase transitions. We have studied the model both on annealed and on quenched networks using mean-field and quantum cavity methods. In particular, we have chosen a recently proposed approximation of the quantum cavity method that allows for a full analytical treatment of the problem while characterizing the phase diagram going beyond the mean-field treatment. This method is based on a mapping between the cavity equation and the...
random polymer problem in quenched media. We have therefore characterized fully the differences between the phase diagrams for the random transverse Ising model defined on annealed and quenched networks. The model shows a significant dependence on the second moment of the degree distribution $\langle k(k-1) \rangle$. In particular, the superconducting critical temperature is enhanced in networks with greater second moment of the degree distribution. Moreover in the limit $\langle k(k-1) \rangle \to \infty$, the phase diagram of the quenched networks is well approximated by the phase diagram of the annealed networks.

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