ABSTRACT

This study proposes ‘amigami’ as a new method of creating a general curved surface. It conducts the shape optimization of weaving paper strips based on the theory of nonlinear elasticity on Riemannian manifolds. The target surface is split into small curved strips by cutting the medium along with its coordinates, and each strip is embedded into a flat paper sheet to minimize a strain energy functional due to the in-plane deformation. The weak form equilibrium equation is derived from a Lie derivative with the virtual displacement vector field, and the equation is solved numerically using the Galerkin method with a non-uniform B-spline manifold. As a demonstration, we made catenoid and helicoid surfaces which are made by waving 54 paper strips (Fig.1). The papercraft reminds us of the isometric transformation from the catenoid to the helicoid and vice versa. We also provide strain estimates for paper strips with rigorous mathematical proof. This estimating process is a generalization of the classical beam theory of Euler-Bernoulli to a modern geometrical elasticity.
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1 Introduction

A curved surface appears in various fields such as nature, science, architecture, arts, and engineering products. One of the ever-lasting questions, especially from an engineering viewpoint, is how to create a curved geometry using a planar material. Typical processing is a combination of two deformations, in-plane stretching and out-of-plane bending, depending on the material we use. Perhaps, a paper sheet is the most common planar material used in everyday life. For a given paper sheet with a sufficiently small thickness, the energetic contribution of bending deformation is negligible compared to the stretching. In other words, the flexibility of a paper sheet is mainly due to the thin shape geometry. The traditional Japanese art ‘origami’ uses the most geometrical flexibility. It approximates a curved surface by a plane of zero Gaussian curvature, i.e. developable surface, using the out-of-plane plastic deformation \(^{(1)}\). Another paper art ‘kirigami’ makes a curved surface by controlling the cuts introduced into the paper sheet \(^{(2)}\). These paper constructions maintain the geometrical flexibility and, therefore, have received a great deal of attention for application to soft robots including robot arm \(^{(3)}\), crawler \(^{(4)}\), gripper \(^{(5)}\), shape morphing \(^{(6)}\), and solar panels \(^{(7)}\). However, there are several problems with these crafting methods \(^{(1)}\). For instance, a curved surface made by origami inevitably has edges and corners: it fails to make a smooth surface. On the other hand, curved surfaces made of kirigami are not filled and have voids and gaps. Engineering applications of origami and kirigami are limited by these geometric features.

In this study, we introduce a new method called ‘amigami’ \(^{(1)}\) for creating a general curved surface from a thin planar material such as a paper sheet. Amigami is based on the concept of maximizing the geometrical flexibility of a planar material and creates a smooth surface without having edges, corners, voids or gaps. Generally, a curved surface in the real world is expressed by a 2-dimensional Riemannian manifold embedded in 3-dimensional Euclidean space \(\mathbb{E}^3\). Here the Riemannian metric, or the first fundamental form of a curved surface, represents the in-plane stretching of a flat parameter space \(\mathbb{R}^2\). Similarly, the second fundamental form is related to the out-of-plane bending deformation. The planar material is thin enough, so the strain energy of out-of-plane bending deformation is relatively smaller than the energy of in-plane deformation, so the out-of-plane deformation can be ignored. Given this property of planar materials, a strategy of minimizing the strain energy of in-plane deformation is considered the most rational way to create curved surfaces.

Large strain energy may be required to obtain a general curved surface, and the deformation may exceed the elastic limit, leading to failure. A possible way to reduce the in-plane elastic strain is to split the target surface into narrow strips and which are then weaved together. This is the basic strategy of amigami to create curved surfaces, and the strain tensor and strain energy can be estimated with our theory based on elasticity on Riemannian manifolds. Recently, Ren et. al. made a curved surface by weaving elastic strips \(^{(8)}\). This method is similar to our theory as they incorporate mechanical force balance in the design of the strips. The advantages of our theory over this previous work are (i) the resulting surfaces are smooth and filled without gaps, (ii) the modeling and numerical calculation are truly based on a 2-dimensional manifold, (iii) the strain approximation can be estimated, which facilitates its application to engineering design.

The construction of the paper is as follows. In the next Section\(^{(2)}\) we provide a brief overview of the theory of elasticity on Riemannian manifolds. In Section\(^{(3)}\) we develop our theory of weaving paper strips. This theory includes modeling paper strips as Riemannian manifolds, numerical computing of its embeddings, and some approximation theorems. In Section\(^{(4)}\) some numerical results and papercrafts will be provided. Section\(^{(5)}\) is a brief conclusion of this paper. Appendix\(^{(6)}\) provides proof for the theorems provided in Section\(^{(3)}\) Appendix\(^{(7)}\) includes some papercraft kits.

2 Overview of elasticity on Riemannian manifold

2.1 Geometric modeling of elastic materials

First of all, we explain the classification for the geometrical modeling of elastic materials using Fig\(^{(2)}\). The simplest one is (i) discrete mass point approximation where the points are connected to (1) linear or (2) nonlinear elastic springs. The standard theories of continuum elasticity are established in (ii) Euclidean space, and it can be (iii) classified into (1) materially linearized model and (2) materially nonlinear model. The materially linearized model assumes that the strain in the medium is small enough, but it doesn’t need to assume its deformation is not small, and the equilibrium equation is still nonlinear PDE \(^{(10)}\). (iv) The geometrically linearized model \(^{(5)}\) assumes that the deformation is also small enough, and the problem will be linear PDE. (v) If the shape of the target object is special, some assumptions such as Euler-Bernoulli’s assumption can be adapted \(^{(11)}\). The class \(^{(5)}\) is geometrically linearized and can be adapted to some approximation based on its shape. This class includes standard theories of the strength of materials such as a

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1. ‘Amigami’ is a Japanese word which means weaving (編み) papers (紙).
As we will see in the following sections, the main models addressed in this study are those modeling on Riemannian manifolds. These modeling can be generalized to modeling on Riemannian manifolds: \( M, g^{[0]} \) and \( M, g^{[t]} \) are the same as a manifold as they share the smooth boundary and let \( g^{[0]} \) be a Riemannian metric on \( M \). We denote \( M^{[0]} = (M, g^{[0]}) \) as a reference state and \( M^{[t]} = (M, g^{[t]}) \) as a current state, where \( g^{[t]} \) is also a Riemannian metric. Without the loss of generality, we assume that the manifolds \( M^{[0]} \) and \( M^{[t]} \) are diffeomorphic. That is, \( M^{[0]} \) and \( M^{[t]} \) are the same as a manifold as they share the same \( M \), but different as a Riemannian manifold because their metrics are distinct. Throughout this study, we use a chart \((U, \varphi)\) and its coordinates \( u^i \). The notations \( g^{[0]} \) and \( g^{[t]} \) represent symbols that relate reference and current states just like \( M^{[0]} \), \( M^{[t]} \), \( g^{[0]} \), and \( g^{[t]} \). The Riemannian metrics are written with the local coordinates.

\[
g^{[0]} = g^{[0] ij} du^i \otimes du^j, \quad g^{[t]} = g^{[t] ij} du^i \otimes du^j. \tag{1}
\]

The dual metrics are written as

\[
g^{**[0]} = g^{**[0] ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}, \quad g^{**[t]} = g^{**[t] ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}. \tag{2}
\]

Figure 2: Classification of geometric modeling of elasticity. Modeling on Euclidean geometry includes (i) linear spring model, (ii) nonlinear spring model, (iii) materially linearized model, (iv) materially nonlinear model, (v) geometrically linearized model, (vi) standard theory of strength of materials, and (vii) simplified model accepting large deformation. These modeling can be generalized to modeling on Riemannian manifolds: \( M^{[0]} \), \( M^{[t]} \), \( g^{[0]} \), and \( g^{[t]} \). Our formulations in this paper are based on \( g^{[0]} \), \( g^{[t]} \), and \( g^{[t]} \).

As we will see in the following sections, the main models addressed in this study are \( g^{[0]} \), \( g^{[t]} \), and \( g^{[t]} \). The planar material is thin enough, so the 3-dimensional Euclidean model \( g^{[0]} \) will be approximated by a 2-dimensional manifold \( g^{[t]} \). The numerical calculation of the deformation of the curved piece of surface is based on \( g^{[t]} \). If the breadth of the paper strip is small enough, another shape approximation \( g^{[t]} \) can be adapted. This results in strain estimation (Theorem 3.2) and initial value determination (Theorem 3.4) of the Newton-Raphson method.

### 2.2 Tensor fields on reference and current states

From this section, we overview the theory of elasticity on Riemannian manifolds. Although most of the mathematical expressions and formulae follow those in [15, 16], and the theory on elasticity is mainly based on [13, 17] and [18], there are also good references for the elasticity theory from the geometric aspect. Some expressions in this paper have been modified to fit our research contents. Let \( M \) be an orientable and compact \( d \)-dimensional manifold with a piecewise smooth boundary and let \( g^{[0]} \) be a Riemannian metric on \( M \). We denote \( M^{[0]} = (M, g^{[0]}) \) as a reference state and \( M^{[t]} = (M, g^{[t]}) \) as a current state, where \( g^{[t]} \) is also a Riemannian metric. Without the loss of generality, we assume that the manifolds \( M^{[0]} \) and \( M^{[t]} \) are diffeomorphic. That is, \( M^{[0]} \) and \( M^{[t]} \) are the same as a manifold as they share the same \( M \), but different as a Riemannian manifold because their metrics are distinct. Throughout this study, we use a chart \((U, \varphi)\) and its coordinates \( u^i \). The notations \( g^{[0]} \) and \( g^{[t]} \) represent symbols that relate reference and current states just like \( M^{[0]} \), \( M^{[t]} \), \( g^{[0]} \), and \( g^{[t]} \). The Riemannian metrics are written with the local coordinates.

\[
g^{[0]} = g^{[0] ij} du^i \otimes du^j, \quad g^{[t]} = g^{[t] ij} du^i \otimes du^j. \tag{1}
\]

The dual metrics are written as

\[
g^{**[0]} = g^{**[0] ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}, \quad g^{**[t]} = g^{**[t] ij} \frac{\partial}{\partial u^i} \otimes \frac{\partial}{\partial u^j}. \tag{2}
\]

We don’t use \( \{x^i\} \) for the symbol of coordinates on \( M \) to avoid confusion with \( \{x^i_{[0]}\} \) and \( \{x^i_{[t]}\} \). These symbols are used for Euclidean space (Fig. 7).

The characters in \( g^{[0]} \) and \( g^{[t]} \) are inspired from time 0 (reference state) and time \( t \) (current state), but our theory in this paper is not time-dependent. Traditionally, upper- and lower-case letters are used to represent reference and current states, but this notation is confusing on state-independent symbols such as Green’s strain tensor field \( E \).
where the following conditions hold; \( g_{[0]} g^{*jk} = \delta^k_i \) and \( g_{[t]} g^{*jk} = \delta^k_i \). Similarly, \((r, s)\) type tensor field \( T \) is written as

\[
T = T_{ij}^{[0]} \frac{\partial}{\partial u^r} \otimes \cdots \otimes \frac{\partial}{\partial u^r} \otimes du^j \otimes \cdots \otimes du^r.
\]

We introduce orthonormal frames on the open subset \( U \) of the manifolds \( M_{[0]} \) and \( M_{[t]} \) by \( \{ e_i^{(0)} \} \) and \( \{ e_i^{(t)} \} \), respectively. The dual frames are \( \{ \theta^{(0)}m \} \) and \( \{ \theta^{(t)}m \} \). Then, the Riemannian metrics Eq. (1) and Eq. (2) become

\[
\begin{align*}
\theta^0_{[0]} &= g^{(0)}_{[0]ij} \otimes \theta^{(0)}_{[0]j}, \\
\theta^0_{[t]} &= g^{(0)}_{[t]ij} \otimes \theta^{(0)}_{[t]j}, \\
\theta^*_{[0]} &= g^{*}_{[0]ij} \otimes \theta^{(0)}_{[0]j}, \\
\theta^*_{[t]} &= g^{*}_{[t]ij} \otimes \theta^{(0)}_{[t]j}.
\end{align*}
\]

(4)

Obviously, some of the coefficients of the metrics will also be Kronecker delta; \( \theta^0_{[0]} = \delta_{ij} \) and \( \theta^0_{[0]} = \delta_{ij} \). Similarly, the tensor field \( T \) given in Eq. (3) becomes

\[
T = T_{ij}^{[0]} \frac{\partial}{\partial u^r} \otimes \cdots \otimes \frac{\partial}{\partial u^r} \otimes du^j \otimes \cdots \otimes du^r.
\]

(6)

Note that these symbols with the character decorations \( \{\}^{(0)} \) and \( \{\}^{(t)} \) are related to the orthonormal frames of the reference and current metrics.\(^4\) Volume elements of the manifolds \( M_{[0]} \) and \( M_{[t]} \) are given by the differential d-form \( v_{[0]} \) and \( v_{[t]} \) such that

\[
\begin{align*}
v_{[0]} &= \theta^{(0)}_{[0]} \wedge \cdots \wedge \theta^{(0)}_{[0]} = \sqrt{\text{det} g^{(0)}_{[0]ij}} du^1 \wedge \cdots \wedge du^d, \\
v_{[t]} &= \theta^{(t)}_{[t]} \wedge \cdots \wedge \theta^{(t)}_{[t]} = \sqrt{\text{det} g^{(0)}_{[t]ij}} du^1 \wedge \cdots \wedge du^d.
\end{align*}
\]

(7)

(8)

Similarly, the volume elements on the boundary \( \partial M \) are written by \( v_{\partial[0]} \) and \( v_{\partial[0]} \), respectively.

### 2.3 Stress, strain, stiffness, and strain energy

By definition, the reference state \( M_{[0]} \) is free from any stress. Let \( E \) be Green’s strain tensor field between the reference state \( M_{[0]} \) and the current state \( M_{[t]} \). Then, the \((0, 2)\)-type tensor field \( E \) is defined by the difference between the Riemannian metrics of the reference and the current state

\[
E = \frac{1}{2} (g_{[t]} - g_{[0]}).
\]

(9)

The stiffness tensor field \( C \) is a \((4, 0)\)-type tensor field, which defines materially-linearized strain energy and constitutive equation of materials.

\[
C = C^{ijkl} \frac{\partial}{\partial u^s} \otimes \frac{\partial}{\partial u^s} \otimes \frac{\partial}{\partial u^k} \otimes \frac{\partial}{\partial u^l}.
\]

(10)

These coefficients \( C^{ijkl} \) satisfy the following major and minor symmetries

\[
C^{ijkl} = C^{klji}, \quad C^{ijkl} = C^{ijkl} = C^{ikjl} = C^{jilk}.
\]

(11)

The stiffness tensor field \( C \) induces an inner product on \( \text{Sym}^2(T^*_p M) \) for each point \( p \in M \).\(^5\) If the stiffness tensor field \( C \) is isotropic, then the tensor can be characterized by two real values \( (\lambda, \mu) \) at each point \( p \in M \).

\[
C^{ijkl} = \lambda g^{*}_{[0]} g^{*kl}_{[0]} + \mu \left( g^{*}_{[0]} g^{*kl}_{[0]} + g^{*}_{[0]} g^{*jk}_{[0]} \right).
\]

(12)

These values \( (\lambda, \mu) \) are called Lamé parameters and can be represented by Young’s modulus \( Y \) and Poisson’s ratio \( \nu \):

\[
\lambda = \frac{\nu Y}{(1 + \nu)(1 - (d - 1)\nu)}, \quad \mu = \frac{Y}{2(1 + \nu)}.
\]

(13)

\(^4\)We don’t use notations such as \( \{\}^{(0)} \) and \( \{\}^{(t)} \) for less confusion especially on handwriting.

\(^5\)The symmetric tensor space \( \text{Sym}^2(T_p^* M) \) is a linear subspace of \( T_p^{(0, 2)} M \). The Green’s strain tensor \( E_p \) lives in the symmetric tensor space.
where $d$ is the dimensions of $M$. The materially linearized model assumes that the 2nd Piola-Kirchhoff stress tensor field $S$ is proportional to Green’s strain tensor field $E$:

$$S = C(E, \cdot) = C^{ijkl} E_{ij} \frac{\partial}{\partial u^k} \otimes \frac{\partial}{\partial u^l}. \quad (14)$$

Strain energy density $\mathcal{W}$ and strain energy $W$ are defined as

$$\mathcal{W} = \frac{1}{2} C(E, E) \nu[0] = \frac{1}{2} C_{ijkl} E_{ij} E_{kl} \sqrt{\det g_{ij[0]}}, \quad W = \int_M \mathcal{W}. \quad (15)$$

2.4 Equilibrium Equation

Let $N[\cdot] = (N, h[\cdot])$ be a Riemannian manifold and $\Phi : M[\cdot] \rightarrow N[\cdot]$ be an embedding which represents the deformation of the elastic material. Then, the Riemannian metric $g[\cdot]$ of the current state is induced by

$$g[\cdot] = \Phi^* h[\cdot]. \quad (16)$$

One of the most typical problems in elasticity theory is finding the equilibrium embedding $\Phi$ under some external forces.

Let $\Gamma_D$ be a Dirichlet boundary of $M$ and $\Gamma_N$ be a Neumann boundary of $M$. These boundaries satisfies $\partial M = \Gamma_D \cup \Gamma_N$ and $\emptyset = \Gamma_D \cap \Gamma_N$. We also use $\Gamma_D[0], \Gamma_N[0]$ and $\Gamma_D[t], \Gamma_N[t]$ to represent these boundaries on the reference and current states. Let $f_B$ be a body force on $M[t]$, $f_S$ be a surface force on $\Gamma_N[t]$. Then, the weak form PDE of the equilibrium equation is written as

$$\int_M \langle S, L_X g[t] \rangle \nu[0] - \int_M \langle f_B, X \rangle \nu[t] - \int_{\Gamma_N} \langle f_S, X \rangle \nu[\partial[t]] = 0 \quad (17)$$

where $X \in \mathfrak{X}(M)$ is a test vector field, $L_X$ is a Lie derivative operator along with the vector field $X$, and $\langle \cdot, \cdot \rangle$ is a product operator between dual spaces. In the next section, we will adapt this equilibrium equation to our weaving theory. (Proposition 3.3)

3 Theory of weaving paper strips

3.1 Weaving methods for paper strips

Before digging into our theory, let’s summarize some weaving methods to create a surface. There are several ways to construct a surface from strip shapes. See Fig.3 for example.

![Figure 3: Several methods to construct a surface by combining strips.](image)

Each of the three methods has the following benefits and restrictions.

(a) This is the simplest method to construct a curved surface from strips [19, 20]. Strictly speaking, this method is not weaving, and we need additional glue margins to assemble the paper strips.

In this paper, we formulated a force as a covector field. However, some studies such as [13] treat them as a covector field with volume forms. As we will see in the later section, the external forces will be treated as zero, so this will not be problematic.
Weaving paper strips for designing of general curved surface with geometrical elasticity

(b) This method can achieve smoothness and higher strength as a surface compared to (a) and (c) because the strips are weaved and there are no gaps or voids. However, this method requires the existence of global coordinates on the target surface. An example of a torus is [21].

(c) The gaps allow us to see the back side of the curved surface, and make it easier to weave the strips. The hexagon can be replaced with a pentagon or a heptagon to adapt the Gaussian curvature of the surface [9, 22, 23, 24].

We mainly use method (b), but our proposed method can also be adapted to (a) and (c) by adding a chart to each strip.

3.2 Modeling paper strips as 2-dimensional Riemannian manifolds

In general, when constructing a curved surface $S$ (e.g. a hemisphere in Fig.4a) from a flat material such as paper, we need to divide the curved surface into smaller pieces such as $M_t \subseteq S$ (Fig.4b) and construct each piece from a flat material $M_0$ (Fig.4c). This is because it is assumed that planar materials do not allow for in-plane large deformation, and are deformed mainly in the out-of-plane direction. Some previous researches such as [19, 20] assume that the planar materials deform only in the out-of-plane direction. In the language of differential geometry, this approach assumes that the first fundamental form of the surface is invariant before and after the deformation, and each surface piece $M_t$ should be approximated by a developable surface.

However, planer material can be deformed in-plane, and the strain energy of out-of-plane deformation can be ignored if the planar material is thin enough. In this situation, the elastic medium can be formulated as a 2-dimensional Riemannian manifold, and the strain energy $W$ with the deformation $\Phi$ should be as small as possible. Our question in this subsection is, “How can we find the reference state $M_0$ and the deformation $\Phi$ that minimizes the strain energy $W$?”

3.3 Swapping reference state for current state

As described in the previous Section 3.2, we consider the construction of a curved surface piece from a planar material (Fig.4c), which essentially means that the planar material is the reference state and the curved surface piece is the current state. This can be attributed to the problem of finding a reference state that minimizes the strain energy for the predetermined current state. However, it is counterintuitive and unwieldy to consider an elastic material with a predetermined current state, and it is more convenient to write the formulation by interchanging the reference state and the current state. The validity of this replacement has been proved in the next Proposition 3.1.

Proposition 3.1 (Swapping Reference State for Current State). Let $E = \alpha E$ be a Green’s strain tensor field with a real coefficient $\alpha$, and assume that the strain energy density $W$ is given as

$$W = \frac{1}{2} C(E, E)\psi_0.$$  \hspace{1cm} (18)

Then, the regular strain energy $W$ and the state-swapped strain energy $\hat{W}$ are equivalent under ignoring the residual term $O(\alpha^3)$. \[\blacksquare\]

The next Fig.5 illustrates the swapping of the reference and current states with the deformation $\hat{\Phi} = \Phi^{-1}$. 

8
Proof. Let $M_{[0]} = (M, g_{[0]}), M_{[t]} = (M, g_{[t]})$ be the regular reference and current state. Let $M_{[0]} = (M, g_{[0]}), M_{[t]} = (M, g_{[t]})$ be the swapped reference state and current state. They are just swapped each other, so $g_{[0]} = g_{[t]}, g_{[t]} = g_{[0]}, M_{[0]} = M_{[t]}, M_{[0]} = M_{[t]}$ are satisfied. The original strain energy $W$ of the deformation $\Phi : M_{[0]} \rightarrow M_{[t]}$ is given by
\begin{equation}
W = \int_M \mathcal{W} = \int_M \frac{1}{2} C(E, E) v_{[0]} = \frac{1}{2} \alpha^2 \int_M C(E, E) v_{[0]}.
\end{equation}

The swapped strain energy $\hat{W}$ of the deformation $\hat{\Phi} : M_{[0]} \rightarrow M_{[t]}$ is calculated straightforward by their definition as
\begin{align}
g_{[t]} &= g_{[0]} + 2\alpha \hat{E} \in g_{[0]} + O(\alpha), \\
\hat{E} &= \frac{1}{2}(g_{[t]} - g_{[0]}) = -1/2(g_{[t]} - g_{[0]}) = -E, \\
v_{[0]} &= v_{[0]} + O(\alpha), \\
\hat{C} &= C + O(\alpha), \\
\hat{W} &= \frac{1}{2} \hat{C}(\hat{E}, \hat{E}) v_{[0]} \in \frac{1}{2} \alpha^2 C(E, E) v_{[0]} + O(\alpha^3) = \mathcal{W} + O(\alpha^3), \\
\hat{W} &= \int_M \mathcal{W} \in \int_M \mathcal{W} + O(\alpha^3) = W + O(\alpha^3).
\end{align}

Thus, the strain energy $W$ and $\hat{W}$ are equivalent if the residual term $O(\alpha^3)$ is ignored. \hfill \blackslug

The material linearization (Fig.2) is an approximation that ignores the residual term $O(\alpha^3)$. The next Fig.6 shows how Proposition 3.1 works for elastic surface embedding.

Figure 6: Swapping states; the current state $M_{[t]}$ is now a planar shape.

Note that the tension part and compression part are also swapped, but their energies are equivalent under material linearization.

As discussed in Section 3.1, we can assume that the domain $D$ of coordinates can be regarded as a rectangular shape
\begin{align}
D &= I \times [c - b, c + b] \\
M_{[0]} &= (M, g_{[0]} = \left\{ p_{[0]}(u^1, u^2) \mid (u^1, u^2) \in D \right\} \\
M_{[t]} &= (M, g_{[t]} = \left\{ p_{[t]}(u^1, u^2) \mid (u^1, u^2) \in D \right\}.
\end{align}

\footnote{We treat Landau’s notation $O(f)$ as a function space. This is sometimes useful because we can use $\in$ and $\subseteq$ for strict evaluations.}
where $p_{[0]}$ is the parametric mapping to the surface $S \subseteq \mathbb{R}^3$; $p_{[0]} : D \to S$. Similarly for the current state mapping $p_{[t]} : D \to \mathbb{R}^2$. The second equalities in Eq. (27) and Eq. (28) are not strictly true from a set-theoretic point of view, but we equate them for convenience. However, we distinguish tangent vectors $p_{[0]}i = \partial p_{[0]}/\partial u^i$, $p_{[t]}i = \partial p_{[t]}/\partial u^i$, and $\partial / \partial u^i$. Note that these tangent vectors have some relationships such as

$$
\|p_{[0]}i\| = \left\| \frac{\partial}{\partial u^i} \right\|_{[0]}, \quad g_{[t]ij} = g_{[t]} \left( \frac{\partial}{\partial u^i}, \frac{\partial}{\partial u^j} \right) = p_{[t]}i \cdot p_{[t]}j, \quad p_{[t]}i = \Phi \ast p_{[0]i}.
$$

(29)

The following Fig.7 is a schematic diagram of the swapped states and their chart.

Intuitively, the mapping $\Phi : M_{[0]} \to \mathbb{R}^2$ is the unknown mapping, but in the numerical computing aspect, we need to find the unknown mapping $p_{[t]} : D \to \mathbb{R}^2$.

### 3.4 Determination of the breadth of the strip shape

If we take a coarser division of a surface $S$ such as in Fig 8a, then the strain on the material is expected to be larger. Therefore it is considered that it is better to divide the curved surface into smaller pieces such as in Fig 8b.

However, it is not sufficient to divide the surface into as many parts as possible. This is because the number of parts increases with the number of divisions, and thus the assembly of strips into the surface becomes more complicated. Therefore, it is very important to know the proper division of the curved surface $S$ before assembling the pieces of the surface. The following strain approximation formula is useful here.

---

8 We use boldface characters for symbols that live in Euclidean spaces $\mathbb{R}^2$ and $\mathbb{R}^3$.

9 The true domain of $p_{[0]}$ is larger than $D$, but our main interest is each embedding of the piece of surface.
Theorem 3.2 (Approximation of Strain). In the range of sufficiently small breadth $B$ of the curved piece, the piece is in an approximately $u^1$-directional uniaxial stress state at each point, and the principal strain can be approximated as

$$E_{11}^{(0)} \approx \frac{1}{2} K_0 [B^2 \left( r^2 - \frac{1}{3} \right) ], \quad E_{22}^{(0)} \approx -\nu E_{11}^{(0)}$$

where $K_0$ is the Gaussian curvature along the center curve $C_0$ of the reference state $M_0$, $r$ is a normalized breadth-directional coordinate $(-1 \leq r \leq 1)$.

The proof of this theorem will be given in the later Section A.2. The next Fig.9 shows how Theorem 3.2 works.

![Image of the strain approximation with Theorem 3.2](image)

Figure 9: Strain approximation with Theorem 3.2

The center curve $C_0$ is a submanifold of the reference state $M_0$ and equips the Riemannian metric induced from $g_0$ to the curve $C$. Here $C$ is a 1-dimensional manifold defined by

$$C = \{ \varphi^{-1}(u^1, u^2) \mid u^1 \in I, u^2 = c \} .$$

(31)

The breadth $B$ can be estimated by

$$B \approx b \cdot p_{02} \cdot e_{02}$$

(32)

where $b$ is the breadth parameter of the domain $D$ defined in Eq.(26), and $e_{02}$ is a breadth-directional unit vector on $C_0$. Empirically, the strain $E_{11}^{(0)}$ should be less than 0.01.

3.5 Weak form of the problem

In this section, we will provide the equilibrium equation for paper strips.

Proposition 3.3 (Weak Form PDE on Local Coordinates). The weak form equilibrium equation for the embedding of a surface piece $M_0$ into the Euclidean space $\mathbb{E}^2$ is represented in the local coordinates as follows.

$$\delta_r \int_M C^{ijkl} \left( \delta_{pq} \frac{\partial x_i^{[t]} }{\partial u^p} \frac{\partial x_j^{[t]} }{\partial u^q} - g_{[t]ij} \right) \frac{\partial \xi^r }{\partial u^k} \frac{\partial x_i^{[t]} }{\partial u^l} v_0^{[t]} = 0$$

(33)

where $\{x_i^{[t]}\}$ is the standard coordinates of the space $\mathbb{E}^2$ to be embedded, and $\{\xi^i\}$ are test functions.

Proof. In the equilibrium equations of weak form Eq.(17), we can treat the external forces $f_B$ and $f_S$ as zero. And also, there is no Dirichlet boundary, and the whole boundary is the Neumann boundary

$$\Gamma_D = \emptyset, \quad \Gamma_N = \partial M.$$ 

(34)

Therefore, for any test function $X \in \mathcal{X}(M)$, we have

$$\int_M C \left( E_x g_{[t]} \right) v_0^{[t]} = 0.$$ 

(35)

The Riemannian metric $g_{[t]}$ and the vector field $X$ can be represented on the local coordinates $\{u^i\}$ and $\{x_i^{[t]}\}$.

$$g_{[t]} = g_{[t]ij} du^i \otimes du^j = \delta_{ij} dx_i^{[t]} \otimes dx_j^{[t]}, \quad X = X^i \frac{\partial}{\partial u^i} = \xi^i \frac{\partial}{\partial x_i^{[t]}} .$$

(36)

The letter $C$ may be confusing with the symbol for the stiffness tensor field, but these can be distinguished in context.
Then, the Lie derivative of the Riemannian metric is also represented on the local coordinates
\[
\mathcal{L}_X g_{[t]} = \left( \delta_{kj} \frac{\partial \xi^k}{\partial x^l_{[t]}} + \delta_{kl} \frac{\partial \xi^k}{\partial x^j_{[t]}} \right) dx^j_{[t]} \otimes dx^l_{[t]} = \left( \delta_{kj} \frac{\partial \xi^k}{\partial u^l} + \delta_{kl} \frac{\partial \xi^k}{\partial u^j} \right) du^k \otimes du^l. \tag{37}
\]
Finally, we obtain the weak form on the local coordinates
\[
\delta_{rs} \int_M \sum_{ijkl} \left( \delta_{pq} \frac{\partial x^p}{\partial u^i} \frac{\partial x^q}{\partial u^j} - g_{ij} \right) \frac{\partial \xi^r}{\partial u^k} \frac{\partial \xi^s}{\partial u^l} v_{[0]} = 0 \tag{38}
\]
where \( \{\xi^i\} \) are arbitrary functions.

In most cases, the solution of the weak form cannot be obtained analytically, and we need some discretization. In the next two subsections, we will discuss discretization and solving the discretized equations numerically.

### 3.6 Approximation of the current state with B-spline surface

B-spline is a mathematical tool for geometric shape representation in affine space. It can be regarded as a generalization of Bézier curve and surface. With B-spline, we can create a C\(p\)-class smooth mapping from \(d\)-dimensional rectangular region, with piecewise polynomial of degree \(p\). Let \(N^I\) be a 2-dimensional B-spline basis function\(^{11}\) with index \(I\)
\[
N^I : D \rightarrow \mathbb{R}, \quad \sum_I N^I(u^1, u^2) = 1 \tag{40}
\]
where \(D\) is a rectangular domain defined by Eq. (26). With these functions \(N^I\), the unknown mapping \(p_{[t]}\) can be approximated by the following \(p_{[t]}\):
\[
p_{[t]}(u^1, u^2) = a_I N^I(u^1, u^2), \quad \tilde{x}^i_{[t]} = a^i_I N^I, \quad a_I = \begin{pmatrix} a^1_I \\ a^2_I \end{pmatrix}. \tag{41}
\]
We use upper-case indices such as \(I\) for function sequence, and lower-case indices such as \(i\) for geometric dimension. And, we also use the Einstein summation convention for both of these indices. Note that the character decoration \(\tilde{\cdot}\) represents that the symbol is related to B-spline approximation. The next Fig. [10] shows the approximated current state \(\tilde{M}_{[t]}\) as B-spline surface.

![Approximation with B-spline manifold](image)

(a) Discretize and approximate current state \(M_{[t]}\) with B-spline manifold \(\tilde{M}_{[t]}\).
(b) Refinement operation; increase number of control points for more approximation accuracy.

Figure 10: Approximation with B-spline manifold.

\(^{11}\)The function \(N^I\) is defined as
\[
N^I(u^1, u^2) = (B_{(i^1,p^1,k^1)} \otimes B_{(i^2,p^2,k^2)})(u^1, u^2) = B_{(i^1,p^1,k^1)}(u^1) B_{(i^2,p^2,k^2)}(u^2) \tag{39}
\]
where \(B_{(i,p,k)}\) is a B-spline basis function with index \(i\), polynomial degree \(p\) and knot vector \(k\), and \(I = (i^1, i^2)\) is a cartesian index. See [23] for our definitions and notations for B-spline. For polynomial degrees \(p^1, p^2 \leq 3\) must be satisfied because we will export the computed embedding \(\tilde{M}_{[t]}\) as SVG format. In the context of the Galerkin method, the specific definition of \(N^I\) is not very important, so we mainly use \(N^I\) instead of \(B_{(i^1,p^1,k^1)} \otimes B_{(i^2,p^2,k^2)}\).
We denote coordinates of where \( N \) (Approximation of Embedding) Theorem 3.4. Then the approximation \( g \) is the derivative of basis function, defined by \( \partial N / \partial u^i \). Here, we put functions \( F = \{ F^I \} \) as

\[
F_i^I = a_j^i a_k^I a_L^j \delta_{ij} \delta_{kl} A^{JJKL} - a_j^i \delta_{ij} B^{IJ} = 0 \tag{42}
\]

\[
A^{JJKL} = \int_M C_{ijkl} N_i^J N_J^k N_K^l N_L^I u[I] \tag{43}
\]

\[
B^{IJ} = \int_M C_{ijkl} N_i^J N_J^k g_0 kl u[I] \tag{44}
\]

where \( N_i^J \) is the derivative of basis function, defined by \( N_i^J = \partial N^J / \partial u^i \). Here, we put functions \( F = \{ F_i^J \} \) as

\[
F_i^J = a_j^i a_k^J a_L^j \delta_{ij} \delta_{kl} A^{JJKL} - a_j^i \delta_{ij} B^{IJ} \tag{45}
\]

Now, the problem is finding \( a = \{ a_i^J \} \) that satisfies \( F(a) = 0 \).

### 3.8 Newton-Raphson method

The Newton-Raphson method is a recursion formula to obtain a local solution of nonlinear smooth simultaneous equations. By using this method, the nonlinear simultaneous equations Eq. (42) can be solved numerically by the following formula

\[
a_i^j = a_i^j \frac{\partial a_i^j}{\partial F^I} F^I_j \tag{46}
\]

We denote coordinates of \( \nu \)-th iterated control points \( a_i^j \) as \( a_i^j \), and also \( F_i^I = F_i^I(a) \) with character decoration \( \frac{\partial F_i^I}{\partial a_i^j} \).

However, the determination of the initial values \( \{ a_i^j \} \) of the Newton-Raphson method is not obvious. We will discuss this in the next section.

#### 3.8.1 Determination of initial values

The next embedding approximation theorem allows us to compute an approximate embedding, which can be used to determine the initial value of the Newton-Raphson method.

**Theorem 3.4** (Approximation of Embedding). Let \( C_0 \) be the center curve of \( M_0 \), \( \kappa_0 \) be its geodesic curvature, \( B \) be the breadth from center curve of \( M_0 \). Similarly, let \( C_t \) be the center curve of \( M_t \), \( \kappa_t \) be its planer curvature. If the breadth \( B \) is sufficiently small, then the following approximation is satisfied.

\[
g_{[t]} c \approx g_{[0]} c \tag{47}
\]

\[
\kappa_{[t]} \approx \kappa_{[0]} \tag{48}
\]

\[\hfill \blacksquare\]

Let \( c_{[0]}(u^1) = p_{[0]}(u^1, c), c_{[t]}(u^1) = p_{[t]}(u^1, c) \) be the center curve parameterizations of \( C_{[0]}, C_{[t]} \). And let \( \{ e_{[0]} \}, \{ e_{[t]} \} \) be orthonormal bases on \( C_{[0]}, C_{[t]} \) defined by Gram-Schmidt orthonormalization of \( \{ p_{[0]} \}, \{ p_{[t]} \} \). Then the approximation \( g_{[t]} c \approx g_{[0]} c \) leads \( e_{[t]} \approx \Phi_* e_{[0]} \). The next Fig.11 illustrates how Theorem 3.4 works.

---

\[\text{12} \] This notation is useful especially when the symbol has superscripts and subscripts.

\[\text{13} \] Note that this bases \( \{ e_{[0]} \}, \{ e_{[t]} \} \) are different from the orthonormal frames \( \{ e_i^{(0)} \}, \{ e_i^{(t)} \} \) because the former is defined on \( C_{[0]}, C_{[t]} \), but the latter is defined on some open subset of \( M_{[0]}, M_{[t]} \).
This theorem shows that if the breadth $B$ is small enough, the shape of the embedding $M_t$ can be approximated only by geometric information such as the geodesic curvature $\kappa_{0[0]}$ and the Riemannian metric $g_{0[0]}$. The proof of this theorem will be given in a later Section\[\text{A.2}\]. Based on the above, we characterize initial state $M_{[s]}$ in the following. This state $M_{[s]}$ can be used for the determination of the initial values in the Newton-Raphson method. We use $\mathbb{C}_{[s]}$ notation\[\footnote{The letter in $\mathbb{C}_{[s]}$ is coming from the initial of “starting point.”} to represent a symbol that relates initial state similarly to $\mathbb{C}_{[0]}$ and $\mathbb{C}_{[t]}$.

(a-1) The Riemannian metric in the reference state and the initial state coincide on the center curve $C$. That is, $g_{[0][c]} = g_{[c][c]}$.

(a-2) The geodesic curvature $\kappa_{[0]}$ of the center curve $C_{[0]}$ in the reference state and the planar curvature $\kappa_{[s]}$ of the center curve $C_{[s]}$ in the initial state are equal.

(a-3) The tangent vector does not change in the breadth direction. That is, $\frac{\partial}{\partial u^2}p_{[s][2]} = 0$.

The validity of conditions (a-1) and (a-2) follows from the Theorem\[\text{3.4}\]. Condition (a-3) is not essential, but it is required for the uniqueness of $M_{[s]}$. Following the condition from (a-1) to (a-3), the initial state $M_{[s]}$ can be constructed explicitly.

**Proposition 3.5 (Construction of Initial State $M_{[s]}$).** The initial manifold $M_{[s]}$ can be constructed explicitly by the following ODE.

$$M_{[s]} = \{ p_{[s]}(u^1, u^2) \mid (u^1, u^2) \in D \}$$

$$p_{[s]}(u^1, u^2) = c_{[s]}(u^1) + \left( \frac{g_{0[0]}(u^1, c)}{\sqrt{\det(g_{0[0]}(u^1, c))}} \right) \left( u^2 - c \right) \hat{c}_{[s]}(u^1) \quad (51)$$

$$\| \hat{c}_{[s]} \| = s_{[0]} \quad (52)$$

where $\kappa_{[0]}$ is the geodesic curvature of the center curve $c_{[0]}$, $s_{[0]}$ is the speed of the parametrization $c_{[0]}$, and $\hat{c}_{[s]}$ is a differential operator with respect to $u^1$.

**Proof.** The the speed $s_{[s]}$ and the planar curvature $\kappa_{[s]}$ of the center curve $C_{[s]}$, and orthonormal basis $\{ e_{[s][i]} \}$ on the curve have the following properties.

$$s_{[s]} = \| \hat{c}_{[s]} \|, \quad e_{[s][1]} = \hat{c}_{[s]} s_{[s]}^{-1}, \quad e_{[s][2]} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e_{[s][1]}, \quad \hat{e}_{[s][1]} = s_{[s]} \kappa_{[s]} e_{[s][2]}.$$  

Then, Eq. (52) is obvious from the condition (a-1). The ODE Eq. (51) can be obtained with the condition (a-2):

$$\hat{c}_{[s]} = \hat{s}_{[s]} e_{[s][1]} + s_{[s]} \hat{e}_{[s][1]} = \left( \frac{\hat{s}_{[s]}}{\hat{s}_{[s]} s_{[s]}}, \frac{\hat{e}_{[s][1]}}{\hat{s}_{[s]} s_{[s]}}, \frac{\hat{e}_{[s][2]}}{\hat{s}_{[s]} s_{[s]}} \right) \cdot \frac{\hat{s}_{[s]}}{\hat{s}_{[s]} s_{[s]}}, \quad \hat{s}_{[s]} = s_{[s]} \kappa_{[s]} s_{[s]} \hat{e}_{[s][2]}.$$  

Here, let $q_{[s][j]}(u^1, c)$ be tangent vectors on $M_{[s]}$ on the center curve $C$. These tangent vectors can be obtained as

$$q_{[s][1]} = \hat{c}_{[s]}, \quad q_{[s][2]} = c_{[s]}.$$
As described above, the whole boundary of \( M \) where the control points \( \{ q_{i} \} \) are straight line in \( E^2 \) are defined as 
\( q_{i2} = (q_{i2} \cdot e_{[s]1})e_{[s]1} + (q_{i2} \cdot e_{[s]2})e_{[s]2} \)
\[ \begin{pmatrix} \frac{g_{012}}{\sqrt{\det(g_{0ij})}} - \frac{\sqrt{\det(g_{0ij})}}{g_{012}} \frac{q_{[s]1}}{g_{012}} \\ \frac{\sqrt{\det(g_{0ij})}}{g_{012}} \frac{q_{[s]1}}{g_{012}} \end{pmatrix} \]  
(56)

Then, the manifold \( M_{[s]} \) can be constructed by the condition (a-3).

\[ M_{[s]} = \{ p_{[s]}(u^1, u^2) \mid (u^1, u^2) \in D \} \]
(57)
\[ p_{[s]}(u^1, u^2) = c_{[s]}(u^1) + (u^2 - c)q_{[s]2}(u^1) \]
(58)

The above gives us all the formulas in the proposition.

The initial manifold \( M_{[s]} \) can be obtained numerically by solving the ODE with Runge-Kutta method. We can set the initial condition of the ODE arbitrarily, and that will produce a rigid transformation in \( E^2 \). See the next Fig.12 for the illustration of the construction of the initial manifold \( M_{[s]} \).

![Figure 12: Construction of the initial manifold \( M_{[s]} \).](image)

The next step of our proposed method is approximating the initial manifold \( M_{[s]} \) with B-spline manifold \( \tilde{M}_{[s]} \). Since the unknown values of the equations Eq.(46) are the control points of the B-spline manifold, we need to give the initial value of that. In this paper, these control points are determined by the following least-squares method.

\[ \tilde{M}_{[s]} = \{ \tilde{p}_{[s]}(u^1, u^2) \mid (u^1, u^2) \in D \} \]
(59)
\[ \tilde{p}_{[s]}(u^1, u^2) = \alpha_f N^T(u^1, u^2) \]
(60)
\[ \text{minimize} \quad \alpha_f \int_D \| p_{[s]}(u^1, u^2) - \tilde{p}_{[s]}(u^1, u^2) \|^2 \, du^1 du^2 \]
(61)

where the control points \( \{ \alpha_f \} \) are used for the initial values for the Newton-Raphson method, i.e. \( \alpha_f = \frac{a_{[s]}}{\theta} \). We also denote \( \tilde{M}_{[s]} \) for the approximated initial manifold.

### 3.8.2 Iteration with Newton-Raphson method

As described above, the whole boundary of \( M \) is the Neumann boundary, and hence the solution is not unique (Fig.13a). This leads that the matrix \( dF/da = (\partial F^i_j / \partial a_j) \) will be degenerate, and cannot compute \( \partial F^i_j / \partial a_j \) in Eq.(46). To solve this problem, we need to fix some control points. Fig.13b shows constraints to avoid rigid transformations. We use this type of constraint as default.

In some cases, the fixing condition is not good enough for convergence speed. Fixing three points (left endpoint, right endpoint, and center point) makes faster convergence, especially in the early stage of the iterations (Fig.13c). This is because the initial state \( M_{[s]} \) was based on only geometric properties, and it does not include information about

---

15 The tangent vectors on the center curve \( C_{[s]} \) in the figure are defined as \( q_{[s]}(u^1) = p_{[s]}(u^1) \).
16 The polynomial degrees of the B-spline manifold are determined as \( (p^3, p^3) = (3, 1) \) because the curves \( (u^1 : \text{const}) \) on \( M_{[s]} \) are straight line in \( E^2 \).
17 See Section 4.1.2 for an example of fixing three points.
elasticiy. This leads to unnatural strain distribution on each point on \( M_s \), causing the first step of the Newton-Raphson method to move in a strange direction. On the other hand, the center curve embedding \( C_s \) is roughly correct globally (Theorem 3.4). Therefore, fixing these three points is suitable as an auxiliary constraint.

(a) No constraints; the matrix \( \frac{dE}{dt} \) will be degenerate.
(b) The constraint for avoiding rigid transformations.
(c) Fixing three points; this makes faster convergence in the early stage.

![Various types of constraints.](image)

### 3.9 Overview of our method

The next Fig[14] is a flowchart of our proposed method. First, the shape of the target surface \( S \subseteq \mathbb{E}^3 \) is given. Then, the surface will be split into pieces not to have much strain on them. Each piece of the surface will be embedded into \( \mathbb{E}^2 \) by the numerical computation steps. Checking the convergence of the Newton-Raphson method is easy, but checking whether the refinement operation is enough is not straightforward. We can determine this by checking the strain distribution \( \bar{E}^{(0)} \). If the refinement is poor, we can find unnatural patterns along with its knot vectors. This unnatural pattern can be solved by inserting more knots around it.

![Flowchart of the whole computation process.](image)

---

18 See Section 4.2.2 for an example of the unnatural strain distribution pattern.
The next diagram shows relations between all symbols for manifolds.

\[
\begin{array}{c}
\text{Solve the ODE} \\
(\text{Proposition 3.5}) \\
M_0 \quad M_s \quad \Phi \\
\text{B-spline} \\
\text{approximation} \\
\tilde{M}_s = \tilde{M}_t \frac{0}{0} \quad \cdots \\
\text{Newton-Raphson} \\
\text{method and} \\
\text{refinement} \\
\tilde{M}_t \approx M_t
\end{array}
\]

\[ (62) \]

### 3.10 Implementation with Julia Language

We have implemented our method using the Julia Language \[29\], and its packages such as ForwardDiff.jl \[30\] and BasicBSpline.jl \[28\]. Our code is available on our GitHub repository \[19\].

### 4 Results

In this section, we will provide some results using our theory. During the computation, we assumed that the stiffness tensor field \( C \) of the paper medium is isotropic and its Poisson’s ratio of paper strips is \( \nu = 0.25 \). This value of Poisson’s ratio is based on \[31\]. The value of Young’s modulus \( Y \) does not affect the embedding shape \( M_t \) if the modulus is constant on \( M \). This is because we do not have external forces (Proposition 3.3). We also put \( Y = 1 \) during the computation.

#### 4.1 Paraboloid

##### 4.1.1 Parametric representation

A paraboloid can be parametrized as

\[
p_{[0]}(u^1, u^2) = \left( \begin{array}{c} u^1 \\ u^2 \\ u^1 u^2 \\ (u^1)^2 + (u^2)^2 \end{array} \right), \quad (u^1, u^2) \in [-1, 1] \times [-1, 1].
\]

The next Fig. 15a shows this parametrization with a checker pattern of width \( \delta = 0.1 \). This shape is four-fold symmetry, so we need to calculate the embeddings for the ten strip shapes as shown in Fig. 15b i.e. the we need to calculate the embeddings \( \tilde{\Phi}^{(i)} : M_{[0]}^{(i)} \to \tilde{M}_{[0]}^{(i)} \) for each domain

\[
D^{(i)} = [-1, 1] \times [(i - 1)\delta, i\delta] \quad (i = 1, \ldots, 10).
\]

(a) Coloring original surface \( S \) with \( \delta = 0.1 \).

(b) Split \( S \) into pieces \( M_{[0]}^{(i)} \).

Figure 15: Graph of \( z = x^2 + y^2 \) as a paraboloid surface \( S \).

\[19\] https://github.com/hyrodium/ElasticSurfaceEmbedding.jl
4.1.2 Numerical result

The computing process as shown in Section 3.9 is not just a simple Newton-Raphson method, but also includes refinement (Fig 10b) and another type of Newton-Raphson method (Fig 13). Thus, the computation process may compose a tree structure. The next Fig 16a shows the history of strain energies $\Delta W = \tilde{W} - W$ during computation of $\tilde{M}^{(10)}_t$ as a tree structure. Where $\tilde{W}$ is a strain energy of each approximated embedding $\tilde{M}^{(i)}_t$, and $W$ is the strain energy of the exact embedding $M_t$. The value of $W$ in the figure was calculated approximately with many refinements and many steps of the Newton-Raphson method. As discussed in Section 3.8.2, Fixing three points makes faster convergence, especially in the early stage of the iterations. Fig 16b shows the numerically calculated embeddings of all of the pieces of the surface in Fig 15b.

![Energy history during computation of $\tilde{M}^{(10)}_t$.](image)

(a) Energy history during computation of $\tilde{M}^{(10)}_t$.

![The embedded pieces of the paraboloid surface.](image)

(b) The embedded pieces of the paraboloid surface.

Figure 16: Numerical result of the paraboloid surface.

4.1.3 Papercraft model

The next Fig 17 shows the weaved papercraft model. Each piece of the surface was cut by a laser-cutting machine, and these paper strips were assembled with wood glue.

![Papercraft model of the paraboloid surface.](image)

Figure 17: Papercraft model of the paraboloid surface.

4.2 Hyperbolic paraboloid

4.2.1 Parametric representation

A hyperbolic paraboloid can be parametrized as

$$p_{(0)}(u^1, u^2) = \left( \begin{array}{c} u^1 \\ u^2 \\ (u^1)^2 - (u^2)^2 \end{array} \right), \quad (u^1, u^2) \in [-1, 1] \times [-1, 1].$$

(65)
The next Fig.15a shows this parametrization with a checker pattern of width \( \delta = 0.1 \). This shape also has a symmetry like a paraboloid in the previous section, so we need to calculate the embeddings for the ten strip shapes as shown in Fig.15b. The domain for each strip shape is the same as the paraboloid Eq.(64).

![Graph of \( z = x^2 - y^2 \) as a hyperbolic paraboloid surface](image)

Note that the hyperbolic surface \( S \) is a doubly ruled surface as shown in Fig.18c. This property makes the papercraft model interesting in the later section.

### 4.2.2 Numerical result

As discussed in Section 3.9, we can detect whether the refinement operation is enough by visualizing the strain distribution \( E_{11}^{(0)} \). The next Fig.19a shows these visualized distributions on \( \tilde{M}_{[3]}^{(3)} \) during the refinement operations. The fewer the number of control points, the larger the pattern along the knot vector appears in the strain distribution. Fig.19b shows the numerically calculated embeddings of all of the pieces of the surface in Fig.18b.

![Strain distribution on \( \tilde{M}_{[3]}^{(3)} \) and refinement operation.](image)

**Figure 19:** Numerical result of the hyperbolic paraboloid surface.

### 4.2.3 Papercraft model

Fig.20 is a picture of the papercraft model of the surface. By pulling the surface with two hands, we can feel there are straight lines on the surface. This behavior reminds us that the hyperbolic paraboloid is a ruled surface.
4.3 Catenoid and helicoid

4.3.1 Parametric representation

A catenoid can be parametrized with

\[
P_{[0]}(u^1, u^2) = \begin{pmatrix} \cosh(u^2) \cos(u^1) \\ \cosh(u^2) \sin(u^1) \\ u^2 \end{pmatrix}, \quad (u^1, u^2) \in [-\pi, \pi] \times [-\pi/2, \pi/2].
\]  
(66)

Similarly, a helicoid can be parametrized with

\[
P_{[0]}(u^1, u^2) = \begin{pmatrix} \sinh(u^2) \cos(u^1) \\ \sinh(u^2) \sin(u^1) \\ u^1 \end{pmatrix}, \quad (u^1, u^2) \in [-\pi, \pi] \times [-\pi/2, \pi/2].
\]  
(67)

Both of these surfaces have the same Riemannian metric \( g_{[0]} \).

\[
g_{[0]} = g_{[0]ij} du^i \otimes du^j,
\]
(68)

\[
(g_{[0]ij}) = \begin{pmatrix} \cosh(u^2)^2 & 0 \\ 0 & \cosh(u^2)^2 \end{pmatrix}.
\]
(69)

Thus, there exists a local isometric transformation between the catenoid (Fig.21a) and the helicoid (Fig.21b) \([32, 33]\). In the context of our amigami theory, this property leads that both embedded pieces of the surfaces will be equivalent.

4.3.2 Numerical result

The next Fig.21a and 21b show pieces of the catenoid and the helicoid. Their computed embedding is shown in Fig.21c and their strain distribution is shown in Fig.21d.
Weaving paper strips for designing of general curved surface with geometrical elasticity

4.3.3 Papercraft model

Fig. 1 is the weaved surfaces of the catenoid and the helicoid, and the next Fig. 22 shows the deformation between these surfaces. Two flexible steel plates are attached to each end of the curved surface, and some magnets are embedded in the wooden frame. These curved surfaces are fixed in this way.

Figure 22: Isometric transformation from the catenoid to the helicoid.

This papercraft was exhibited at the 2019 Joint Mathematics Meetings.

5 Conclusion

In this study, we conducted shape optimization of a 2-dimensional curved surface made of paper strips based on the theory of elasticity on Riemannian manifolds. The results are summarized as follows.

- The elasticity theory on Riemannian manifolds is appropriate for deformations of a surface.

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20 The video of the deformation is uploaded on YouTube: [https://www.youtube.com/watch?v=Gp6XkPLCw7s](https://www.youtube.com/watch?v=Gp6XkPLCw7s)
• Amigami (weaving paper strips) is suitable for constructing a thin smooth surface.

• The shapes of the embedded pieces of a surface are based on a 2-dimensional Riemannian manifold, and it does not relate to how the pieces deform in 3-dimensional Euclidean space.

• The strain in the paper can be estimated with Gaussian curvature.

• The embedded strip shape can be approximated with geodesic curvature.

• The deformation of the strip shape has properties that are generalizations of Euler-Bernoulli’s assumption. See Appendix A for more discussion.

A Proof of the approximation theorems

A.1 Rigorous version of the main theorems, and preparations

Actually, Theorem 3.2 and Theorem 3.4 are ambiguous statements that cannot be precisely evaluated with respect to the breadth of a curved surface shape. In this section, we will provide a rigorous version of these theorems and some preparations for their proof.

Definition A.1 (Centered Global Chart of Strip Shape). Let $M_{[0]} = (M, g_{[0]})$ be a 2-dimensional Riemannian manifold. If a chart $(U, \varphi)$ of $M_{[0]}$ satisfies the following properties, then the chart is called centered global chart.

- $U = M$, i.e. the chart $(U, \varphi)$ is global.
- The image $D = \varphi(M)$ satisfies
  \[
  D = \{ (u^1, u^2) \mid u^1 \in I, -B(u^1) \leq u^2 \leq B(u^1) \} \tag{70}
  \]
  where $I$ is a closed interval, and the function $B : I \to \mathbb{R}_{>0}$ represents the breadth of the strip shape along with the center curve $C = \{ \varphi^{-1}(u^1, u^2) \mid u^1 \in I, u^2 = 0 \}$.
- Let $(u^1, u^2) = \varphi(p)$ be coordinates of the chart. Then, the coordinates basis $\{ \partial/\partial u^1 \}$ is orthonormal frame w.r.t. $g_{[0]}$ on the center curve $C$. (i.e. $g_{[0]} C = \delta_{ij}$)
- The curves along with $(u^1 : \text{const.})$ are geodesic with unit tangent vector $\partial/\partial u^2$.  

If the manifold $M_{[0]}$ has a centered global chart, then, we can normalize the global domain $D$ by

\[
I \times [-1, 1] = \left\{ (s, r) \mid s = u^1, r = \frac{u^2}{B(u^1)}, (u^1, u^2) \in D \right\}. \tag{71}
\]

In most cases of the strip shape, the given manifold can be approximated by a 2-dimensional Riemannian manifold with a centered global chart. We would like to take a limit $B \to 0$ “uniformly”, so we will replace $B$ with $\beta B$ and take a limit $\beta \to 0$. Where $\beta$ is a positive real number.

Definition A.2 (Breadth-parametrized Strip Shape). Let $M_{[0]}$ has a centered global chart $\varphi$ and its domain $D = \varphi(M)$. Let $D_\beta$ be a narrowed domain with parameter $\beta$ defined by

\[
D_\beta = \{ (u^1, u^2) \mid u^1 \in I, -\beta B(u^1) \leq u^2 \leq \beta B(u^1) \} \quad (0 < \beta \leq 1). \tag{72}
\]

And let $M_\beta$ and $M_{[0], \beta}$ be narrowed manifolds with the domain $D_\beta$.

\[
M_\beta = \{ \varphi^{-1}(u^1, u^2) \mid (u^1, u^2) \in D_\beta \} \subseteq M, \quad M_{[0], \beta} = (M_\beta, g_{[0], \beta}), \quad g_{[0], \beta} = g_{[0]} | M_\beta. \tag{73}
\]

Then, $M_{[0], \beta}$ also has a centered global chart $\varphi|_{M_\beta}$ and its domain $D_\beta$. This parametrized shape $M_{[0], \beta}$ is called breadth-parametrized strip shape.

Let $M_{[0], \beta}$ be a breadth-parametrized strip shape, then we can take a normalized global domain like Eq.(71):

\[
I \times [-1, 1] = \left\{ (s, r) \mid s = u^1, r = \frac{u^2}{\beta B(u^1)}, (u^1, u^2) \in D_\beta \right\}. \tag{74}
\]

\[\text{There might exist an exact centered global chart of the given manifold, but we do not give proof for its existence or approximation of it in this paper.}\]
Weaving paper strips for designing of general curved surface with geometrical elasticity

The next Fig[23] illustrates breadth-parametrized strip shape $M_{[0]\beta}$, its centered global coordinates $(u^1, u^2)$ and its normalized coordinates $(s, r)$.

Figure 23: Centered global chart $D_\beta$ of reference state $M_{[0]\beta}$ and normalized domain $I \times [-1, 1]$.

We also assume that the stiffness tensor field $C$ is isotropic and its Lamé parameters are constant on $M$ in the following theorems.

Theorem A.3 (Approximation of Embedding). Let $M_{[0]\beta}$ be a breadth-parametrized strip shape and the stiffness tensor field $C$ is isotropic and its Lamé parameters are constant on $M$. Then, the following properties are satisfied.

$$g_{[\iota]\beta}|C = g_{[0]}|C + O(\beta^2)$$

$$\kappa_{[\iota]\beta} \in K_{[0]}(\beta^2)$$

Where $\kappa_{[0]}$ is a geodesic curvature of $C_{[0]}$, and $\kappa_{[\iota]\beta}$ is a planer curvature of $C_{[\iota]\beta}$.

Theorem A.4 (Approximation of Strain). Let $M_{[0]\beta}$ be a breadth-parametrized strip shape and the stiffness tensor field $C$ is isotropic and its Lamé parameters are constant on $M$. Then, the following properties are satisfied.

• The stress state is approximately $u^1$-directional uniaxial

$$S_{\beta}^{(0)12}, S_{\beta}^{(0)21}, S_{\beta}^{(0)22} \in O(\beta^3).$$

• The principal strain can be evaluated as

$$E_{\beta 11}^{(0)} \in \frac{1}{2} K_{[0]}(\beta B)^2 \left( r^2 - \frac{1}{3} \right) + O(\beta^3),$$

$$E_{\beta 22}^{(0)} \in -\nu E_{\beta 11}^{(0)} + O(\beta^3).$$

Where $K_{[0]}$ is the Gaussian curvature of $M_{[0]\beta}$ along with the center curve $C$.

These two theorems are rigorous versions of Theorem 3.4 and Theorem 3.2 and these can be proved in one proof. The proof consists of the following five parts:

(a) Geometry of the reference state
Approximate Riemannian metric of reference state $g_{[0]}$, using geodesic curvature $\kappa_{[0]}$ of center curve $C_{[0]}$ and Gaussian curvature $K_{[0]}$ of the piece of surface $M_{[0]\beta}$.

(b) Geometry of the current state
Calculate the embedding map $\Phi_\beta : M_{[0]\beta} \rightarrow E^2$ with unknown functions $\xi_\beta, \eta_\beta, \iota_{[\iota]\beta},$ and $\kappa_{[\iota]\beta}$.

(c) Strain tensor and strain energy
Based on the geometries of the reference and current states, the strain tensor and strain energy are specifically obtained.

(d) Minimization of the strain energy
Under these circumstances, the problem becomes an energy minimization problem. The embedding map $\varphi$, which was an unknown map, will be reduced to a problem with a finite number of unknown real numbers.

22This assumption about the stiffness tensor may probably be weakened, but for simplicity of proof we assume the above since it is not a practical problem.
(e) Approximation theorems
Based on the results of the minimization of the strain energy in the previous section, approximate evaluations of the embedding, the stress tensor, and the strain tensor will be evaluated.

Before proving the theorems, we introduce the following lemma.

**Lemma A.5 (Two Types of Order on Function Space and Their Relation).** Let \( \mathcal{F} \) be a set of some real-valued functions on a domain \([0, \epsilon] \subseteq \mathbb{R}\), and assume the following.

- Each function \( f \in \mathcal{F} \) is smooth enough that there exists its Taylor expansion with polynomial degree \( n \).
  \[
  f(t) \in a_0 + a_1 t + \frac{1}{2} a_2 t^2 + \cdots + \frac{1}{n!} a_n t^n + \mathcal{O}(t^{n+1}).
  \]
- Let \( \leq \) be a partial order on \( \mathcal{F} \) defined by
  \[
  f \leq g \overset{\text{def}}{\iff} \forall t \in [0, \epsilon], f(t) \leq g(t)
  \]
  and there exists minimum element \( f^* \in \mathcal{F} \). (i.e. \( \forall f \in \mathcal{F}, f^* \leq f \))
- Let \( \preceq \) be a total preorder on \( \mathcal{F} \) defined by
  \[
  f \preceq g \overset{\text{def}}{\iff} (a_m)_m \leq (b_m)_m \quad (\text{in dictionary order})
  \]
  where \((a_m)_m\) and \((b_m)_m\) are the coefficients of Taylor expansions of \( f \) and \( g \).

In this situation, the minimum element \( f^* \) with the order \( \leq \) is the minimum element with the order \( \preceq \).

**Proof.** Let us assume that the minimum element \( f^* \) in order \( \leq \) is not a minimum element of \( \mathcal{F} \) in order \( \preceq \). Then, there exists a function \( f'' \in \mathcal{F} \) such that \( f'' \preceq f^* \) and \( f'' \neq f^* \), and there exists a natural number \( j \) such that
\[
a_0 = a_0^*, \quad \ldots, \quad a_j = a_j^*, \quad a_{j+1}^* < a_{j+1}
\]
where \((a_m^*)_m \) and \((a_m)_m \) are the coefficients of Taylor expansions of \( f'' \) and \( f^* \). Thus
\[
\frac{f^*(t) - f''(t)}{t^{j+1}} \in \frac{a^*_{j+1} - a^*_{j+1}}{(j + 1)!} + \mathcal{O}(t)
\]
holds. Therefore, there exists \( \tau \in (0, \epsilon) \) such that \( f^*(\tau) > f''(\tau) \). This leads \( f^* \not\preceq f'' \) and derives a contradiction. 

### A.2 Proof of the main theorem

(a) Geometry of the reference state

In this part, we will calculate the Riemannian metric of the reference state \( g_{[0],[\beta]} \). The domain of \( g_{[0],[\beta]} \) depends on \( \beta \), but the values of \( g_{[0],[\beta]}(u^1, u^2) \) are independent from \( \beta \), so we sometimes don’t distinguish \( g_{[0],[\beta]} \) and \( g_{[0]} \).

\[
\begin{align*}
g_{[0]} &= g_{[0],ij} du^i \otimes du^j \\
(g_{[0],ij}) &= \begin{pmatrix} g_{[0],11} & g_{[0],12} \\
g_{[0],21} & g_{[0],22} \end{pmatrix}
\end{align*}
\]

The coordinates \((u^1, u^2)\) are on the centered global chart on \( M_{[0]} \), so for all points \( p \in M_{[0]} \),
\[
g_{[0],22} = \left\| \frac{\partial}{\partial u^2} \right\|^2_{[0]} = 1.
\]

Let \((u^1, u^2) = (u^1, \tau)\) be the geodesic curve with parameter \( \tau \), along with \((u^1 : \text{const})\). Then, the equation of the geodesic curve can be written as
\[
0 = \frac{d^2 u^i}{d\tau^2} + \Gamma_{[0],jk}^i \frac{du^j}{d\tau} \frac{du^k}{d\tau} = \Gamma_{[0],22}^1 \frac{\partial u^1}{\partial \tau} = 0, \quad \frac{\partial u^2}{\partial \tau} = 1.
\]
Where $\Gamma^{i}_{0j,k}$ is a Christoffel symbol of the reference state $M_{0,\beta}$. This can be calculated from Riemannian metric $g_{0}[0]$

$$\Gamma^{i}_{0j,k} = \frac{1}{2} g^{i\ell}_{0} \left( \frac{\partial g_{0\ell,j}}{\partial u^{k}} + \frac{\partial g_{0\ell,k}}{\partial u^{j}} - \frac{\partial g_{0\ell,j}}{\partial u^{k}} \right) = g^{i\ell}_{0} \frac{\partial g_{0\ell,j}}{\partial u^{k}}. \quad (88)$$

Here, the matrix $\begin{pmatrix} g_{0ij} \end{pmatrix}$ is invertible and $g_{0[12]}|_{C} = g_{0[21]}|_{C} = 0$ is satisfied, so $g_{0[12]} = g_{0[21]}$ is obtained as

$$\frac{\partial g_{0[12]}}{\partial u^{2}} = \frac{\partial g_{0[12]}}{\partial u^{2}} = 0, \quad g_{0[21]} = g_{0[12]} = 0. \quad (89)$$

Here, the geodesic curvature $\kappa_{0}[0]$ of the center curve $C_{0}[0]$ can be calculated.

$$\kappa_{0}[0](u^{1}) = \frac{1}{2 \sqrt{g_{0[1]}^{11} \sqrt{\det_{i,j}(g_{0ij})}}} \left( g_{0[1]}^{11} \left( 2 \frac{\partial g_{0[1]}^{12}}{\partial u^{1}} - \frac{\partial g_{0[1]}^{11}}{\partial u^{1}} \right) - g_{0[12]}^{11} \left( \frac{\partial g_{0[1]}^{11}}{\partial u^{1}} \right) \right) \bigg|_{u^{2} = 0} = -\frac{1}{2} \frac{\partial g_{0[1]}^{11}}{\partial u^{2}} \bigg|_{u^{2} = 0}. \quad (90)$$

And also, the Gaussian curvature $K_{0}[0]$ on $C_{0}[0] \subseteq M_{0}$ can be calculated as

$$K_{0}[0](u^{1}) = -\frac{1}{\sqrt{g_{0[1]}^{11} g_{0[22]}}} \left( \frac{\partial}{\partial u^{1}} \left( \frac{1}{\sqrt{g_{0[1]}^{11}}} \frac{\partial \sqrt{g_{0[22]}}}{\partial u^{1}} \right) + \frac{\partial}{\partial u^{2}} \left( \frac{1}{\sqrt{g_{0[22]}}} \frac{\partial \sqrt{g_{0[1]}^{11}}}{\partial u^{2}} \right) \right) \bigg|_{u^{2} = 0} = \kappa_{0}[0]^{2} - \frac{1}{2} \frac{\partial^{2} g_{0[1]}^{11}}{\partial (u^{2})^{2}} \bigg|_{u^{2} = 0}. \quad (91)$$

Hence

$$g_{0[11]}(u^{1}, u^{2}) \in g_{0[11]}(u^{1}, 0) + \frac{\partial g_{0[11]}^{11}}{\partial u^{2}} (u^{1}, 0) u^{2} + \frac{1}{2} \frac{\partial^{2} g_{0[11]}^{11}}{\partial (u^{2})^{2}} (u^{1}, 0) (u^{2})^{2} + O \left( (u^{2})^{3} \right)$$

$$= 1 - 2 \kappa_{0}[0] u^{2} + \left( \kappa_{0}[0]^{2} - K_{0}[0] \right) (u^{2})^{2} + O \left( (u^{2})^{3} \right) \quad (92)$$

holds. Finally, we get an approximation of the Riemannian metric of the reference state $g_{0}[0]$.

$$g_{0[1]}^{11} \left( 1 - 2 \kappa_{0}[0] u^{2} + \left( \kappa_{0}[0]^{2} - K_{0}[0] \right) (u^{2})^{2} + O \left( (u^{2})^{3} \right) \right) \bigg|_{u^{2} = 0} \quad (93)$$

And also, the orthonormal frame can be obtained as

$$e_{1}^{(0)} = \frac{1}{\sqrt{g_{0[1]}^{11}}} \frac{\partial}{\partial u^{1}}, \quad e_{2}^{(0)} = \frac{\partial}{\partial u^{2}}, \quad g^{(0)}_{11} = \sqrt{g_{0[1]}^{11}} du^{1}, \quad g^{(0)}_{22} = du^{2}. \quad (94)$$

(b) Geometry of the current state

Let $p_{t[\beta]} : D_{\beta} \to \mathbb{E}^{2}$ be a solution to the energy minimization problem.23 Let $c_{t[\beta]}$ be a parameterization of the center curve $C_{t[\beta]} \subseteq \mathbb{E}^{2}$, i.e. $c_{t[\beta]}(u^{1}) = p_{t[\beta]}(u^{1}, 0)$. Let $\gamma_{t[\beta]}$ be a differential operator with respect to $u^{1}$, $\{e_{t[\beta]1}, e_{t[\beta]2}\}$ be an orthonormal frame on $C_{t[\beta]}$, and $s_{t[\beta]}$ be a speed24 of the curve $c_{t[\beta]}$ and $\kappa_{t[\beta]}$ be a planar curvature of the curve $c_{t[\beta]}$. These functions are defined as

$$s_{t[\beta]} = \| \dot{c}_{t[\beta]} \| , \quad e_{t[\beta]1} = \frac{\dot{c}_{t[\beta]}}{s_{t[\beta]}}, \quad e_{t[\beta]2} = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} e_{t[\beta]1}, \quad \dot{c}_{t[\beta]} = s_{t[\beta]} \kappa_{t[\beta]} e_{t[\beta]2}. \quad (95)$$

In this situation, there exist functions $\xi_{\beta} : D_{\beta} \to \mathbb{R}, \eta_{\beta} : D_{\beta} \to \mathbb{R}$ which satisfy

$$p_{t[\beta]}(u^{1}, u^{2}) = c_{t[\beta]}(u^{1}) + \xi_{\beta}(u^{1}, u^{2}) e_{t[\beta]1}(u^{1}) + \eta_{\beta}(u^{1}, u^{2}) e_{t[\beta]2}(u^{1}), \quad \dot{c}_{t[\beta]}(u^{1}, 0) = 0. \quad (96, 97)$$

23 We do not give proof for the existence of the solution and its smoothness in this paper.
24 Also $s_{t[\beta]}$ can be defined as $s_{t[\beta]} = \| \dot{c}_{t[\beta]} \|$, but this is equal to 1 on the centered global chart. Note that these symbols $s_{t[\beta]}$ and $s_{t[\beta]}$ just represent speeds along with the center curves $C_{t[\beta]}$ and $C_{t[\beta]}$, and they are completely different from the symbol of the normalized coordinate $s$ of $(s, r)$.
\[ W_{ij} = \frac{1}{2} \frac{\partial g_{ij}}{\partial u^k} + \frac{1}{2} \frac{\partial g_{ij}}{\partial u^l} + \frac{1}{2} \frac{\partial g_{kl}}{\partial u^i} \frac{\partial g_{ik}}{\partial u^j} \] (99)

where \( \xi_{\alpha i} \) and \( \eta_{\beta j} \) are defined by \( \xi_{\alpha i} = \frac{\partial \xi_{\alpha}}{\partial u^i}, \eta_{\beta j} = \frac{\partial \eta_{\beta}}{\partial u^j} \). The coefficients of Riemannian metric \( g_{[i]j} = p_{[i]i} \cdot p_{[j]j} \) of the current state is obtained as follows.

\[ (g_{[i]j}) = (\eta_{\beta 1} + \kappa_{[\alpha]j} \xi_{\alpha 1})^2 + (\xi_{\alpha 1} - \kappa_{[\alpha]j} \eta_{\beta 1} + s_{[\alpha]j})^2 \] (101)

(c) Strain tensor and strain energy

By the definition Eq. (9), the Green’s strain tensor field \( E_\beta \) can be calculated as

\[ E_\beta = E_{[i]j}^{(0)} \theta_{[i]ji} \otimes \theta_{[j]ij}, \quad E_{[i]j}^{(0)} = \frac{1}{2} (g_{[i]j0} - g_{ij}^{(0)}) = \frac{1}{2} (g_{ij}^{(0)} - \delta_{ij}) \] (102)

And also, the strain energy density \( W_\beta \) can be obtained as

\[ W_\beta = \frac{1}{2} C^{ijkl} E_{[i]j}^{(0)} E_{[k]l}^{(0)} g_{[0]11} \] (103)

Then, the following strain energy \( W \) is a function of \( \beta \)

\[ W(\beta) = \int_{M_0} W_\beta = \int_1^1 (\beta B \int_{-1}^1 \frac{1}{2} C^{ijkl} E_{[i]j}^{(0)} E_{[k]l}^{(0)} g_{[0]11} dr) ds \] (104)

where \((s, r)\) are normalized global coordinates defined in Eq. (74). Note that the strain energy \( W \) is not just a function of \( \beta \), but also can be considered as a functional of the embedding \( \Phi \). The strain energy does not change with rigid transformations, so \( W \) can also be regarded as a functional of \( (\xi_{\alpha i}, \eta_{\beta j}, s_{[\alpha]j}, \kappa_{[\alpha]j}) \). This point of view is helpful in the later definition Eq. (110). Here, the strain energy \( W(\beta) \) can be approximated by Taylor’s theorem with Landau’s notation

\[ W(\beta) = \int_{a_0} a_0 ds + \beta \int_{a_1} a_1 ds + \frac{\beta^2}{2} \int_{a_2} a_2 ds + \cdots + \frac{\beta^n}{n!} \int_{a_n} a_n ds + O(\beta^{n+1}), \]

(105)

\[ a_m = \left( \frac{d}{d\beta} \right)^m (\beta B \int_{-1}^1 \frac{1}{2} C^{ijkl} E_{[i]j}^{(0)} E_{[k]l}^{(0)} g_{[0]11} dr) \bigg|_{\beta=0} \] (106)

Note that each coefficient \( a_m \) is a function of \( s \). Let \( \xi_{[i],j}^{(i,j)}, \eta_{[i],j}^{(i,j)}, s_{[i],j}^{(i,j)}, \) and \( \kappa_{[i],j}^{(i,j)} \) be functions of \( s \), defined as

\[ \xi_{[i],j}^{(i,j)}(s) = \lim_{\beta \to 0} \frac{\partial^{|i|+|j|} \xi_{\beta u^i}}{\partial u^i} \bigg|_{u_0^1 = s}, \quad \eta_{[i],j}^{(i,j)}(s) = \lim_{\beta \to 0} \frac{\partial^{|i|+|j|} \eta_{\beta u^j}}{\partial u^j} \bigg|_{u_0^1 = s}, \] (107)

\[ s_{[i],j}^{(i,j)}(s) = \lim_{\beta \to 0} \frac{\partial^{|i|+|j|} s_{[\beta u^i]}}{\partial u^i} \bigg|_{u_0^1 = s}, \quad \kappa_{[i],j}^{(i,j)}(s) = \lim_{\beta \to 0} \frac{\partial^{|i|+|j|} \kappa_{[\beta u^j]}}{\partial u^j} \bigg|_{u_0^1 = s}. \] (108)

Then, the real sequence \( \{a_m\} \) can be calculated with these unknown functions \( \xi_{[i],j}^{(i,j)}, \eta_{[i],j}^{(i,j)}, s_{[i],j}^{(i,j)}, \) and \( \kappa_{[i],j}^{(i,j)} \). Note that these unknown functions are not independent. For example,

\[ \xi_{[i+1],j}^{(i+1,j)}(s) = \frac{d}{ds} \xi_{[i],j}^{(i,j)}(s), \quad s_{[i+1],j}^{(i+1,j)}(s) = \frac{d}{ds} s_{[i],j}^{(i,j)}(s) \] (109)

holds. By the condition Eq. (97), \( \xi_{[i],j}^{(i,0,k)}(s) = 0 \) holds for any natural numbers \((i, k)\). Similarly, \( \eta_{[i],j}^{(i,0,k)}(s) = 0 \) also holds.
(d) Minimization of the strain energy

To apply Lemma A.5 here, we need to consider the following.

- Let us extend the domain of strain energy Eq. (104) with \( W(0) = 0 \). Then the domain of \( \beta \) will be an interval \([0, 1]\).
- Let \( \mathcal{F} \) be a set of functions defined by

\[
\mathcal{F} = \left\{ \beta \mapsto W(\beta; \xi, \eta, s, \kappa) \mid \xi, \eta \in C^{\infty}(D), \xi|_{\partial D} = \eta|_{\partial D} = 0, s|_{\partial D} > 0, \kappa|_{\partial D} \in C^{\infty}(I) \right\}
\]

where the function \( W(\beta; \xi, \eta, s, \kappa) \) is a positive function with variables \( \xi, \eta, s, \kappa \).

- The minimum element of the function space \( \mathcal{F} \) with the partial order \( \leq \) defined by Eq. (80) is obviously \( W(\beta) = W(\beta; \xi, \eta, s, \kappa) \).

Therefore, the lemma can be applied here, so the minimizing \( W(\beta) \) delivers minimizing the sequence \( \{a_m\} \) with dictionary order from \( a_0 \).

0th derivative

\[
a_0 = 0
\]

holds. This is obvious from Eq. (106).

1st derivative

\[
a_1 = \frac{2\nu \left(1 - \eta^{(1,0,0)} - \eta^{(0,1,0)} \right) + 2\nu \left(\eta^{(1,0,0)} - 1\right) \cdot \eta^{(0,1,0)}}{4(1 - \nu^2)/(\nu B)}
\]

holds.\(^{25}\) This is a positive function with variables \( \eta^{(1,0,0)}, \eta^{(0,1,0)} \). To minimize this function \( a_1 \), we get the following values.

\[
\eta^{(1,0,0)} = 0, \quad \eta^{(0,1,0)} = 1, \quad \eta^{(0,0)} = 1.
\]

With these values

\[
a_1 = 0
\]

holds. By the condition Eq. (109), \( \xi^{(1,0,0)} = 0 \) holds. Similarly, \( \eta^{(1,0,1)} = 0 \) and \( \eta^{(0,0)} = 0 \) also hold.

2nd derivative

\[
a_2 = 0
\]

holds. This is because \( E^{(0)}_{\beta_{ij}}(s, \beta Br) \in O(\beta) \) holds by the minimization of \( a_1 \).

3rd derivative

\[
a_3 = \frac{B^2 \left(4\nu \eta^{(0,2,0)}(s - \eta^{(0,0)}) + 2(\kappa(s) - \eta^{(0,0)})^2 + (1 - \nu)\xi^{(2,0,0)} + 2\eta^{(0,2,0)}\right)
\]

\[
+ 3 \left(1 - \nu\right) \left(\eta^{(0,1,1)}^2 + 2\eta^{(0,0)} + 2\eta^{(0,1,0)}^2 + 4\nu\eta^{(0,1,0)}\right) \right)}{(1 - \nu^2)/(\nu B)}
\]

\(^{25}\)This complicated expression was calculated with the Wolfram Engine, not by hand. See [ElasticSurfaceEmbedding-wolfram](https://github.com/hyrodium/ElasticSurfaceEmbedding-wolfram) for our scripts for the calculation.
With these values holds. This is a positive function with variables $\xi^{(0,1,1)}$, $\xi^{(0,2,0)}$, $\eta^{(0,1,1)}$, $\eta^{(0,2,0)}$, $s^{(0,1)}_{[t]}$, $\kappa^{(0)}_{[t]}$. To minimize this function $a_3$, we get the following values.

$$
\begin{align*}
\xi^{(0,2,0)} &= 0, & \xi^{(0,1,1)} &= 0, & \eta^{(0,2,0)} &= 0, & \eta^{(0,1,1)} &= 0, & s^{(0,1)}_{[t]} &= 0, & \kappa^{(0)}_{[t]} &= \kappa_{[0]} \\
\end{align*}
$$

(117)

With these values

$$
a_3 = 0
$$

(118)

holds. In the same discussion with $a_1$, $\xi^{(i,2,0)} = 0$, $\xi^{(i,1,1)} = 0$, $\eta^{(i,2,0)} = 0$, $\eta^{(i,1,1)} = 0$, $s^{(i,1)}_{[t]} = 0$ hold.

4th derivative

holds. This is because $E^{(0)}_{\beta \nu \omega} (s, \beta B r) \in O(\beta^2)$ holds by the minimization of $a_3$.

5th derivative

holds. This is a positive function with variables $\xi^{(0,1,2)}$, $\xi^{(0,2,1)}$, $\xi^{(0,3,0)}$, $\eta^{(0,1,2)}$, $\eta^{(0,2,1)}$, $\eta^{(0,3,0)}$, $s^{(0,2)}_{[t]}$, $\kappa^{(0)}_{[t]}$. To minimize this function $a_5$, we get the following values.

$$
\begin{align*}
\xi^{(0,1,2)} &= 0, & \xi^{(0,2,1)} &= 0, & \xi^{(0,3,0)} &= 0, & \eta^{(0,1,2)} &= \frac{1}{3} \nu K_{[0]} B^2, \\
\eta^{(0,2,1)} &= 0, & \eta^{(0,3,0)} &= -\nu K_{[0]}, & s^{(0,2)}_{[t]} &= -\frac{1}{3} K_{[0]} B^2, & \kappa^{(0,1)}_{[t]} &= 0. \\
\end{align*}
$$

(121)

With these values

$$
a_5 = \frac{8}{3} Y K_{[0]} B^5
$$

(122)

holds.

6th derivative and more

holds, and this cannot be calculated more. This is because we have only obtained up to a second-order approximation of the Riemannian metric $g_{[0]}$ in Eq. (93). For the same reason, $a_7, a_8, \ldots$ cannot be evaluated, and the rest of the unknown functions such as $\xi^{(0,4,1)}$ and $\kappa^{(0,2)}_{[t]}$ cannot be obtained.

(e) Approximation theorems

We have obtained some of the functions $\xi^{(i,j,k)}$, $\eta^{(i,j,k)}$, $s^{(i,j)}_{[t]}$, and $\kappa^{(i,j)}_{[t]}$ explicitly. By using these results, the following approximations can be evaluated.
• **Strain energy**
The strain energy $W(\beta)$ can be evaluated as
\[
W(\beta) \in \frac{Y}{45} \left( \int K_{[0]} B^5 ds \right) \beta^5 + O(\beta^6). \tag{124}
\]

• **Approximation of embedding** (Theorem A.3 Theorem 3.4)
The Riemannian metric $g_{[t]}$ and the planar curvature $\kappa_{[t]}$ on the center curve $C_{[t]}$ can be evaluated as
\[
g_{[t]}|_{C} \in g_{[0]}|_{C} + O(\beta^2) \tag{125}
\]
\[
\kappa_{[t]} \in \kappa_{[0]} + O(\beta^2). \tag{126}
\]
This approximation Eq.(125) can be improved with the higher order of $\beta$, but we don’t need it for constructing $M_{[\beta]}$.

• **Approximation of stress tensor** (Theorem A.4 Theorem 3.2)
The 2nd Piola-Kirchhoff stress tensor field $S_{\beta}$ is evaluated as
\[
S_{\beta}^{(0)11} \in Y E_{\beta}^{(0)11} + O(\beta^3), \quad S_{\beta}^{(0)12}, S_{\beta}^{(0)21}, S_{\beta}^{(0)22} \in O(\beta^3). \tag{127}
\]
This means the stress state is approximately $u^1$-directional uniaxial.

• **Approximation of strain tensor** (Theorem A.4 Theorem 3.2)
The Green’s strain tensor field $E_{\beta}$ is evaluated as
\[
E_{\beta}^{(0)} \in \frac{1}{2} K_{[0]} (\beta B)^2 \left( r^2 - \frac{1}{3} \right) + O(\beta^3), \tag{128}
\]
\[
E_{\beta}^{(0)} = E_{\beta}^{(0)} \in O(\beta^3), \quad E_{\beta}^{(0)} \in -\nu E_{\beta}^{(0)} + O(\beta^3).
\]

• **Relation to Euler-Bernoulli assumption**
The $\xi_{\beta}$ function which represents longitudinal deformation can be evaluated as
\[
\xi_{\beta}(u^1, u^2) = \xi_{\beta}(s, \beta r B) \in O(\beta^4). \tag{129}
\]
This means, if the breadth is sufficiently small, then the geodesic curve ($u^1$: const) on $M_{[0]}$ is still geodesic on $M_{[\beta]}$, and the geodesic curve is perpendicular to the center curve in both the reference state and the current state. These properties are natural generalizations of Euler-Bernoulli’s assumption.
**B Papercraft kit**

Bonus round! In this appendix, we will provide some papercraft kits from Section 4. You can print & cut this paper, and create your own papercraft models!

**B.1 Paraboloid**

The following Fig. 24 is a set of the embeddings calculated in Section 4.1.2. Printing this paper on an A4 paper four times and cutting it out yields 40 pieces of paper. Weaving them together will produce the curved surface as shown in Fig. 17.

![Figure 24: Papercraft kit of the paraboloid surface.](image-url)
B.2 Hyperbolic paraboloid

The following Fig 25 is a set of the embeddings calculated in Section 4.2.2. Printing this paper on an A4 paper four times and cutting it out yields 40 pieces of paper. Weaving them together will produce the curved surface as shown in Fig 17.

Figure 25: Papercraft kit of the hyperbolic paraboloid surface.

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