Hyperbolicity of arborescent tangles and arborescent links

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Abstract

In this paper, we study the hyperbolicity of arborescent tangles and arborescent links. We will explicitly determine all essential surfaces in arborescent tangle complements with non-negative Euler characteristic, and show that given an arborescent tangle \( T \), the complement \( X(T) \) is non-hyperbolic if and only if \( T \) is a rational tangle, \( T = Q_m \ast T' \) for some \( m \geq 1 \), or \( T \) contains \( Q_n \) for some \( n \geq 2 \). We use these results to prove a theorem of Bonahon and Seibermann which says that a large arborescent link \( L \) is non-hyperbolic if and only if it contains \( Q_2 \).

Key words: Arborescent tangles, arborescent links, hyperbolic manifolds

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1. Introduction

Arborescent tangles were defined by Conway [4] as tangles which can be obtained from the trivial tangles by certain operations. He used these to study a class of links which he called algebraic links. His purpose was to generalize 2-bridge links, also called rational links. Rational tangles make up the most basic class of such tangles; every rational tangle is associated with a unique rational number, \( p/q \), or \( \infty \), and Conway was the first to note that two rational tangles are isotopic if and only if they correspond to the same rational number. Later Gabai named Conway’s algebraic links arborescent links because the name algebraic links had already been used before Conway for another class of links. Arborescent links have also been studied by Montesinos [13], Hatcher and Thurston [7], Oertel [14], and many others.

Since arborescent tangles (resp. links) are built up from rational tangle components, we often want to decompose a tangle (link) into two arborescent tangle pieces. This involves cutting along a decomposing disk (sphere) called...
a Conway disk (Conway sphere), which cuts the tangle or link into a set of rational tangles. The length of an arborescent tangle or a large arborescent link is defined to be the minimum number of rational tangles among all such decompositions.

Wu classified all arborescent tangles without closed components whose exteriors are hyperbolic in the sense that such a tangle admits a hyperbolic structure with totally geodesic boundary \[17\]. The main purpose of this paper is to study the same problem for the complement of arborescent tangles, allowing closed components. Given an arborescent tangle \((B, T)\), define the tangle complement to be \(X(T) = B - T\), and the tangle exterior \(E(T) = B - \text{Int}N(T)\). Let \(Q_m\) be the tangle with two vertical strings and \(m\) horizontal circles, as shown in Figure 12. Given two tangles \(T_1, T_2\), define \(T_1 \ast T_2\) to be the tangle obtained by gluing \(T_1\) on top of \(T_2\). See the paragraph before Definition 3.11 for more details. We can now state the main theorem from Section 3.

**Theorem 3.22** Suppose \(T\) is an arborescent tangle. Then \(X(T)\) is non-hyperbolic if and only if one of the following holds.

1. \(T\) is a rational tangle.
2. \(T = Q_m \ast T'\) for some \(m \geq 1\).
3. \(T\) contains \(Q_n\) for some \(n \geq 2\).

A standard annulus in \(Q_m\) is an annulus separating the circles from the two vertical arcs. Similarly for standard torus. See Definition 3.11 for more details. The tangle complement \(X(T)\) is non-hyperbolic if and only if it contains an essential surface \(F\) which is a sphere, disk, annulus, or torus. These can be determined explicitly as follows.

**Addendum 3.23** Suppose \(T\) is an arborescent tangle.

1. \(X(T)\) contains no essential \(S^2\).
2. \(X(T)\) contains an essential disk \(D\) if and only if \(T\) is a rational tangle and \(D\) is the disk separating the two strings of \(T\).
3. \(X(T)\) contains an essential annulus \(A\) if and only if \(T = Q_m \ast T'\) for some \(m \geq 1\) and \(A\) is a standard annulus in \(Q_m\).
4. \(X(T)\) contains an essential torus \(F\) if and only if \(T\) contains a \(Q_m\) for some \(m \geq 2\) and \(F\) is a standard torus in \(Q_m\).

Bonahon-Siebenmann classified all non-hyperbolic arborescent links in an unpublished manuscript [1]. Oertel studied Montesinos links and found exactly which ones are hyperbolic. See Theorem 4.1 for his statement. We will use Theorem 3.22 to give a proof of the following theorem. Together with Oertel’s theorem, this gives a complete proof of Bonahon-Siebenmann’s theorem for the classification of non-hyperbolic arborescent links.

**Theorem 4.2** (Bonahon-Siebenmann) Suppose \(L\) is a large arborescent link. Then \(L\) is non-hyperbolic if and only if it contains \(Q_2\).
An alternative proof of Bonahon-Seibenhann’s theorem has been given by Futer and Gueritaud [5], using a different method.

Gabai’s definition for arborescent links uses tree diagrams (hence the use of the Latin word arbor, meaning tree). In this paper we define an arborescent link to be a Montesinos link or a link obtained by gluing two non-trivial arborescent tangles to each other. See Definition 2.1. The two definitions are equivalent for prime links, as shown in [15]. We will also show that if \( L \) is a large arborescent link then it is also prime. See Theorem 4.6.

2. Definitions and Preliminaries

Unless otherwise stated, in this paper surfaces are compact and orientable, and surfaces in 3-manifolds are properly embedded. A surface \( F \) in a 3-manifold \( M \) is essential means it is incompressible, \( \partial \)-incompressible, and not \( \partial \)-parallel. The manifold \( M \) is \( \partial \)-irreducible means \( \partial M \) is incompressible in \( M \). Given a set \( X \) in a manifold \( M \), let \( N(X) \) denote a regular neighborhood of \( X \) in \( M \). We use \( A \parallel B \) to denote that \( A \) is parallel to \( B \). Other classical definitions can be found in Hempel [10], Jaco [11], or Hatcher’s notes [8].

A tangle is a pair, \( (B, T) \), where \( B \) is a 3-ball and \( T \) is a properly embedded 1-manifold. In this paper we always assume that \( T \) consists of 2 arcs and possibly some circles, so \( T \) intersects \( \partial B \) in exactly 4 points. A marked tangle is a triple \( (B, T, \Delta) \) where \( (B, T) \) is a tangle and \( \Delta \) is a disk on \( \partial B \) containing exactly two endpoints of \( T \), called the gluing disk. We use \( T \) to describe a tangle when \( B \) and \( \Delta \) are not ambiguous.

Figure 1: The tangle \( T[2/3] \), first inscribed on a pillowcase and then progressively simplified.

A rational tangle \( T[p/q] \) is a tangle drawn by inscribing lines with slope \( p/q \) on a “pillowcase” with four holes at the corners. Figure 1 gives an example of the simplification of the rational tangle \( T[2/3] \) starting with the tangle drawn on a pillowcase. The class of rational tangles includes the two trivial tangles, \( T[0] \) and \( T[\infty] \), as shown in Figure 2.

Figure 2: The tangle on the left is \( T[0] \); on the right is \( T[\infty] \).
Given a rational tangle \((B, T)\) in standard position (as drawn on the pillowcase \(\partial B\)), define a horizontal circle as a simple closed curve on \(\partial B\) running horizontally and a vertical circle as a simple closed curve on \(\partial B\) running vertically. For example, the equator is a horizontal circle.

Given a tangle \((B, T)\), define the tangle complement to be \(X(T) = B - T\), and the tangle exterior to be \(E(T) = B - \text{Int} N(T)\). While they are homotopic, it is important to note that there are major differences between a surface in \(X(T)\) and a surface in \(E(T)\). For example, the boundary of \(X(T)\) is a 4-punctured sphere \(\partial B - T\), while the boundary of \(E(T)\) is a genus 2 surface. Also, a surface properly embedded in \(X(T)\) may be \(\partial\)-compressible in \(E(T)\) but not in \(X(T)\).

Given a string \(t_i\) from a tangle \(T\), the exterior of the string \(t_i\) is denoted \(E(t_i)\), i.e. \(E(t_i) = B - \text{Int} N(t_i)\). While \(\partial M\) usually denotes the boundary of the 3-manifold \(M\), it is convenient to use the notation \(\partial N(T)\) to denote the frontier of the regular neighborhood \(N(T)\) of \(T\) instead of the whole boundary of \(N(T)\). For example if \(T\) is a pair of arcs then \(\partial N(T)\) is a pair of annuli. Similarly, we define \(\partial N(t_i)\) to be the frontier of \(N(t_i)\) when \(t_i\) is a string of \(T\).

Two marked tangles, \((B_1, T_1, \Delta_1)\) and \((B_2, T_2, \Delta_2)\), are equivalent means there is an orientation preserving homeomorphism of triples from \((B_1, T_1, \Delta_1)\) to \((B_2, T_2, \Delta_2)\). For example, we can see that two rational tangles \(T[p_1/q_1]\) and \(T[p_2/q_2]\) are equivalent if and only if \(p_1/q_1 \equiv p_2/q_2 \mod \mathbb{Z}\). Given two tangles, \((B_1, T_1, \Delta_1)\) and \((B_2, T_2, \Delta_2)\), the sum \((B_1, T_1, \Delta_1) + (B_2, T_2, \Delta_2)\) is defined by choosing a gluing map \(\phi : \Delta_1 \to \Delta_2\) with \(\phi(\Delta_1 \cap T_1) = \Delta_2 \cap T_2\); write \((B, T) = (B_1, T_1, \Delta_1) + (B_2, T_2, \Delta_2)\) and say that \((B, T)\) is the sum of the two tangles. More simply, we say \(T = T_1 + T_2\). Note that this sum depends on the gluing map \(\phi\), but in most cases the property of \(T\) in which we are interested is not affected by the choice of \(\phi\). In the case that we want to be specific about the gluing disk, \(\Delta \approx \Delta_1 \approx \Delta_2\), we denote the sum as \(T = T_1 +_\Delta T_2\). Furthermore, we define the twice-punctured disk \(P(\Delta) = \Delta \cap X(T_1) = \Delta \cap X(T_2)\). A sum of two marked tangles, \(T = T_1 +_\Delta T_2\), is called nontrivial exactly when neither \((B_1, T_1, \Delta_1)\) nor \((B_2, T_2, \Delta_2)\) is \(T[0]\) or \(T[\infty]\).

An arborescent tangle \(T\) can be defined in terms of rational tangles as follows. Rational tangles are arborescent tangles, and any nontrivial sum of arborescent tangles is an arborescent tangle. Arborescent links are built from these arborescent tangles and will be explained more below.

Montesinos tangles are a smaller class within the class of arborescent tangles. They are characterized by the fact that their gluing disks are mutually disjoint. Tangles written in the form \(T(r_1, r_2, ..., r_n)\) with \(r_i\) a rational number for \(i = 1, ..., n\) are Montesinos tangles drawn by connecting each tangle \(T[r_i]\) in order from left to right and connecting the top strings and bottom strings. See Figure 3 for a diagram.

The length of an arborescent tangle \(T\), given by \(\ell(T)\), is the minimum number of (non-trivial) rational tangles from which \(T\) can be written as a sum. An example is shown in Figure 4.

As defined by Gabai [6], an arborescent link is the boundary of a surface constructed by plumbing (Murasugi sum along a 4-gon) as specified by a tree. The reader is referred to Gabai’s paper [6] for details on how the trees relate to
The links. An example of such a tree and associated link is shown in Figure 5.

A Conway sphere for a link \( L \) in \( S^3 \) is a sphere \( S \) intersecting \( L \) at 4 points, such that \( S - L \) is incompressible in \( S^3 - L \). Similarly, a Conway disk for a tangle \((B, T)\) is a disk \( D \) in \( B \) intersecting \( T \) at two points, such that \( D - T \) is incompressible in \( B - T \), and there is no disk \( E \) in \( B - T \) with \( \partial E \) a union of two arcs \( \alpha \cup \beta \), where \( \alpha \subset \partial B \), and \( \beta = E \cap D \) is an essential arc on \( D - T \). A Conway disk will also be called a decomposing disk.

An arborescent link can also be defined in terms of arborescent tangles. This is more convenient for our purposes. If \( T = T(p_1/q_1, \ldots, p_n/q_n) \) is a Montesinos tangle and \( q_i > 1 \) for all \( i \), then the numerator closure of \( T \) is called a Montesinos link of length \( n \). (See Figure 6) Note that a Montesinos link of length 1 or 2 is a 2-bridge link. Montesinos links have been studied in detail by Oertel [14], who called them star links since the tree diagrams (as in Gabai [6]) are star-shaped.
To denote a Montesinos link as in Figure 6, we write \( L(r_1, r_2, ..., r_n) \), where \( r_i = p_i/q_i \). Burde and Zieschang’s book \cite{2} gives more detail on Montesinos Links.

**Definition 2.1.** (Wu \cite{18}) A link is a small arborescent link if it is a Montesinos link of length at most 3. A link is a large arborescent link if it has a Conway sphere cutting it into two non-rational arborescent tangles. A link is an arborescent link if it is either a small arborescent link or a large arborescent link.

There is a slight difference between the definition for arborescent links given above and Gabai’s tree definition. For example, the diagram shown in Figure 7 is not arborescent by the above definition although it can be obtained from a tree diagram as defined by Gabai if an end-vertex with zero weight is allowed. However, notice that this link is a composite link. In this paper, we use the tangle-definition of arborescent link. This will not cause loss of generality when studying the hyperbolicity of arborescent links because we already know that composite links are non-hyperbolic.

![Figure 6: A Montesinos link \( L(r_1, r_2, ..., r_n) \).](image)

![Figure 7: This is an example of a link which is NOT an arborescent link by the tangle definition. However, if end-vertices with weight zero are allowed in Gabai’s definition, then this link is arborescent by that definition.](image)
3. Hyperbolicity of Tangle Complements

Recall that the complement of a tangle \((B,T)\) is the non-compact manifold \(X(T) = B - T\).

**Definition 3.1.** The complement \(X(T)\) of a tangle \(T\) is hyperbolic if it is irreducible, \(\partial\)-irreducible, atoroidal, and annular.

If \(X(T)\) is hyperbolic by the above definition then the double of \(X(T)\) along \(\partial X(T)\) with toroidal cusps removed is a compact manifold with toroidal boundary, which is irreducible, atoroidal, and cannot be Seifert fibered because \(\partial X(T)\) is a separating incompressible surface with negative Euler characteristic. (See the proof of Lemma 4.5.) Therefore the double of \(X(T)\) is hyperbolic, and hence \(X(T)\) admits a complete hyperbolic structure with totally geodesic boundary (see Thurston [16]). The main theorem of this paper is Theorem 3.22, which determines all non-hyperbolic arborescent tangle complements.

Proposition 3.6 shows that if \(X(T)\) is \(\partial\)-reducible then \(T\) is a rational tangle.

Proposition 3.7 shows that \(X(T)\) is always irreducible. Proposition 3.19 determines all \(X(T)\) which contain essential annuli, and Proposition 3.21 determines those containing essential tori. Theorem 3.22 follows from these propositions.

3.1. Essential disks and essential spheres in \(X(T)\)

Given a rational tangle \((B,T) = T[p/q] = (t_1 \cup t_2)\), a compressing disk for \(\partial B - T\) separates the strings \(t_1\) and \(t_2\). If \(T\) is the trivial tangle \(T[0]\), one can see that the horizontal disk with a horizontal circle as its boundary is the only compressing disk for \(\partial B - T\) up to isotopy. Similarly, up to isotopy the only compressing disk for the trivial tangle \(T[\infty]\) is the compressing disk with a vertical circle as its boundary. Since any rational tangle \(T = T[p/q]\) is homeomorphic to a trivial tangle, we can see that up to isotopy there is only one compressing disk for \(\partial X(T)\). This fact will be used in the proof of Lemma 3.3.

**Definition 3.2.** Let \(\alpha_1, \alpha_2\) be simple closed curves on a surface \(F\) such that \(\alpha_1 \cap \alpha_2 \neq \emptyset\). A bigon between \(\alpha_1, \alpha_2\) is a disk \(D \subset F\) such that \(\partial D = \alpha_1' \cup \alpha_2'\), where \(\alpha_i'\) is an arc on \(\alpha_i\).

The following Lemma is from Casson and Bleiler [3, pp. 26-30].

**Lemma 3.3.** Suppose \(\alpha, \beta\) are simple closed curves on a compact hyperbolic surface \(F\) such that there is no bigon between \(\alpha\) and \(\beta\). Let \(\beta'\) be a simple closed curve on \(F\) which is isotopic to \(\beta\). Then \(|\alpha \cap \beta| \leq |\alpha \cap \beta'|\), and equality holds iff there is no bigon between \(\alpha\) and \(\beta'\).

**Remark.** Lemma 3.3 also holds for non-hyperbolic surfaces.

**Lemma 3.4.** Suppose \(T\) is a \(p/q\) rational tangle, \((p,q) = 1\), \(q \geq 1\), and let \(S = \partial B - T\). Let \(P, Q \subset \partial B\) be twice punctured disks such that \(\alpha = \partial P = \partial Q = P \cap Q\) is a vertical circle and \(P\) is the left disk. If \(D\) is a compressing disk for \(S\) and neither \(D \cap P\) nor \(D \cap Q\) contains an arc which is inessential on \(P\) or \(Q\), respectively, then \(|\alpha \cap \partial D| = 2q\). In particular, \(P\) is incompressible.
**Proof.** Let $\alpha$ be a vertical circle on $\partial B$. By definition, $T$ is isotopic rel $\partial T$ to a pair of arcs $c_1 \cup c_2$ of slope $p/q$ on the pillowcase $\partial B$. Note that $|c_i \cap \alpha| = q$. Let $\beta$ be the boundary of a regular neighborhood of $c_1$ on $\partial B$. Then $\beta$ bounds a compressing disk of $\partial B - T$ in $B - T$. Figure 8 demonstrates an example of such a compressing disk.

Since $|c_1 \cap \alpha| = q$, we have $|\beta \cap \alpha| = 2q$. Note that $\beta$ intersects $P$ and $Q$ in essential arcs, hence there is no bigon between $\alpha$ and $\beta$. By the discussion above, $\partial D$ is isotopic to $\beta$, so by Lemma 3.3, $|\partial D \cap \alpha| = |\beta \cap \alpha| = 2q$ if and only if there are no bigons between $\partial D$ and $\alpha$, i.e., each component of $\partial D \cap P$ and $\partial D \cap Q$ is essential.

![Figure 8](image)

**Figure 8:** The shaded region represents the compressing disk (up to isotopy) for the rational tangle $T(2/3)$.

**Lemma 3.5.** Suppose $F$ is an essential surface in a 3-manifold $M$. If $M$ is $\partial$-reducible then there exists a $\partial$-reducing disk $D$ such that $D \cap F = \emptyset$. If $M$ is reducible then there is a reducing sphere $S$ such that $S \cap F = \emptyset$.

**Proof.** This is a standard innermost circle/outermost arc argument. Choose a $\partial$-reducing disk $D$ so that the number of components $|D \cap F|$ is minimal. (The proof for the reducing sphere is similar.) If $D \cap F$ has inessential circle components on $F$, let $D'$ be a disk on $F$ bounded by an innermost such component and let $D_1$ be the disk on $D$ bounded by $\partial D'$. Then $D'_1 = (D - D_1) \cup D'$ can be perturbed so that $|D'_1 \cap F| < |D \cap F|$, contradicting the fact that $|D \cap F|$ is minimal. If $D \cap F$ has some circle components and if they are all essential on $F$, then a disk on $D$ bounded by an innermost circle component of $D \cap F$ would be a compressing disk of $F$, contradicting the incompressibility of $F$. Therefore $D \cap F$ has no circle components. Similarly if $D \cap F$ has a trivial arc on $F$ then an outermost such arc $\alpha$ on $F$ would cut off a disk $D'$ on $F$ and $\alpha$ splits $D$ into $D_1$ and $D_2$; at least one of the $D'_i = D_i \cup D'$ is then a $\partial$-reducing disk of $M$, which can be isotoped to reduce $|D \cap F|$. If $D \cap F$ consists of essential arcs on $F$ then a disk on $D$ cut off by an outermost component of $D \cap F$ on $D$ would be a boundary compressing disk of $F$, contradicting the $\partial$-incompressibility of $F$. 

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Proposition 3.6. Let \( T = T_1 + \Delta T_2 \) be a non-trivial sum of arborescent tangles. Then \( P(\Delta) \) is essential in \( X(T) \) and \( \partial X(T) \) is incompressible in \( X(T) = B - T \).

Proof. Let \( T \) be a minimal counterexample in the sense that \( T \) is an arborescent tangle such that \( \ell(T) = n \) and there does not exist an arborescent tangle \( T' \) such that \( \ell(T') < n \) and \( T' \) is a counterexample. Then \( T = T_1 + \Delta T_2 \) is a nontrivial sum and \( n \geq 2 \). Let \( P = P(\Delta) \). We need to prove that \( P \) is incompressible and \( \partial \)-compressible, and \( \partial X(T) \) is incompressible.

Suppose \( P \) is compressible. Let \( D \) be a compressing disk for \( P \), so \( D \subset X(T_j) \) for \( j = 1 \) or \( 2 \). If \( \ell(T_j) = 1 \) then by Lemma 3.4, \( T_j = T[\infty] \). This contradicts the fact that \( T = T_1 + \Delta T_2 \) is a non-trivial sum. If \( \ell(T_j) > 1 \), then \( D \) is a compressing disk for \( \partial B - T_j \). Since \( \ell(T_j) < n \), this contradicts the fact that \( T \) is a minimal counterexample. Therefore, \( P \) is incompressible.

Next, note that a \( \partial \)-compressing disk for \( \partial B - T_j \) is also a compressing disk for \( \partial B - T_j \) for \( j = 1 \) or \( 2 \). Thus if there exists a \( \partial \)-compressing disk for \( P \), we will have the same contradiction as above unless \( T_j \) is a rational tangle. In this case, \( T_j \) must be an integral tangle and the sum is trivial. Hence \( P \) is \( \partial \)-incompressible.

Finally, suppose \( D' \) is a compressing disk for \( \partial B - T \). Since \( P \) is essential, by Lemma 3.5 we can find a compressing disk \( D'' \) such that \( D'' \cap P = \emptyset \). Thus \( D'' \) is a compressing disk for \( \partial B_k - T_k \) for \( k = 1 \) or \( 2 \), and \( \ell(T_k) < n \). Since \( T \) is a minimal counterexample, \( T_k \) cannot be a nontrivial tangle sum, so it is a rational tangle \( T(p/q) \). Since \( D'' \) is disjoint from \( P \), by Lemma 3.4 we must have \( q = 0 \), so \( T_k \) is a trivial tangle. This contradicts the assumption that \( T = T_1 + \Delta T_2 \) is a nontrivial sum. \( \square \)

Proposition 3.7. Arborescent tangle complements are irreducible.

Proof. If \( T \) is rational then \( E(T) \) is a handlebody and hence \( X(T) \) is irreducible. Suppose the result holds for any arborescent tangle \( T \) such that \( \ell(T) \leq n \). Suppose \( T' \) is an arborescent tangle such that \( \ell(T') = n + 1 \) and suppose there exists an essential sphere, \( S \subset X(T') \). Write \( T' = T_1 + \Delta T_2 \), where the sum is non-trivial. By Lemma 3.5 and Proposition 3.6 we can find an essential sphere \( S' \) which does not intersect \( P(\Delta) \). However, by the inductive hypothesis, this cannot happen. \( \square \)

3.2. Standard torus and standard annulus in \( Q_n \)

Definition 3.8. A curve \( \alpha \) on a planar surface \( F \) is of type I (resp. type II) means it bounds a once-punctured (resp. twice-punctured) disk on \( F \). Suppose \( A \subset E(T) \) is an annulus with \( \partial A \subset \partial B \). We call \( A \) a type I annulus (resp type II annulus) exactly when \( \partial_i A \) is a type I (resp type II) curve on \( \partial B - T \) for \( i = 1, 2 \). Furthermore, we note that there are two kinds of type I annuli. We call \( A \) a type I-A annulus exactly when \( A \) is a type I annulus such that \( \partial_1 A \parallel \partial_2 A \) on \( \partial B - T \); We say that \( A \) is a type I-B annulus exactly when \( \partial_1 A \not\parallel \partial_2 A \) on \( \partial B - T \). (See Figure 7.)
Lemma 3.9. Suppose $T = T_1 + \Delta T_2$ is an arborescent tangle, $A$ is an annulus in $X(T)$ with $\partial A \subset (\partial B - T)$, and $\partial A \cap \partial \Delta = \emptyset$. Then $(A \cap P(\Delta)) \cup (A \cap (\partial B - T))$ cannot contain curve components of both types I and II.

Proof. Suppose we have an annulus $A$ such that $(A \cap P(\Delta)) \cup (A \cap (\partial B - T))$ contains curves $\alpha_1$ and $\alpha_2$ with $\alpha_1$ a type I curve and $\alpha_2$ a type II curve. By an innermost disk/outermost arc argument, we may assume that $\alpha_1$ and $\alpha_2$ are disjoint essential circles on $A$. Thus, there must be an annulus $A' \subset A$ such that $\alpha_1 = \partial_1 A'$ bounds a disk $D_1$ on $\Delta \cup \partial B$ that intersects the tangle in one point and $\alpha_2 = \partial_2 A'$ bounds a disk $D_2$ on $\Delta \cup \partial B$ that intersects the tangle in two points (see Figure 10). Thus we have a sphere $S = (D_1 \cup A' \cup D_2) \subset B$ that intersects the tangle in three points. This is impossible.

Figure 9: The boundary of the different types of annuli as viewed on $S = \partial B - T$.

Figure 10: The arrow points to $S = D_1 \cup A' \cup D_2$.

Note that in the proof, $T_1$ could be trivial, in which case Lemma 3.9 says that every annulus $A \subset E(T)$ with $\partial A \subset \partial B - T$ such that $\partial_i A$ does not bound a disk on $\partial B - T$ for $i = 1, 2$, is of type I or type II.

Proposition 3.10. Suppose $T$ is an arborescent tangle and $A$ is an incompressible annulus of type I in $X(T)$. If $A$ is of type I-A then $A$ is parallel to the annulus $A' \subset \partial B - T$ with $\partial A' = \partial A$. If $A$ is of type I-B then $A = \partial N(t_i)$ for some string $t_i \in T$. In particular, $X(T)$ contains no essential annulus of type I.

Proof. We proceed by induction on the length of the tangle. Suppose $T = (t_1 \cup t_2)$ is a rational tangle and $A$ is an incompressible annulus of type I in $X(T)$. Define $P_1$ and $P_2$ as the respective left and right disks of the boundary sphere $\partial B$.

Suppose $A$ is a type I-A annulus with $\partial_1 A$, $\partial_2 A \subset P_1$. Consider the disk $A \cup D$ such that $D \subset B$, $\partial D = \partial_1 A$ and $\text{Int} D \cap A = \emptyset$. Push $D$ to $\text{Int} B$ to get a
disk $D_1 \cong A \cup D$ which intersects $T$ at a single point. Since $T$ is trivial, $D_1$ cuts off a ball $B_1$ such that $B_1 \cap T$ is a single unknotted arc $\tau$. Thus after removing a regular neighborhood of $\tau$, we get a solid torus bounded by $A \cup A'$ where $A'$ is an annulus on $\partial B$. Hence $A \parallel A'$.

If $A$ is of type I-B, then by definition, $\partial_i A$ bounds a disk $D_i \subset \partial B$ that intersects the tangle in exactly one point for $i = 1, 2$. Thus $A \cup D_1 \cup D_2$ is a sphere intersecting $T$ in exactly two points. It must be the case that $D_1$ and $D_2$ intersect the tangle in a single point from the same string, $t$. Since $T$ is rational, the string $t$ is unknotted and therefore $A = \partial N(t)$.

Suppose $T = T_1 +_\Delta T_2$ is a nontrivial sum and the result holds for any incompressible annulus of type I in $E(T_i)$ for $i = 1, 2$. Suppose $A$ is an incompressible annulus of type I in $E(T)$. Let $P = P(\Delta)$ and consider the intersection $A \cap P$ with minimal number of components. If $A \cap P = \emptyset$ then the result follows by induction. Suppose $A \cap P \neq \emptyset$. By Lemma 3.9, $P$ cuts $A$ into type I annulus components $A_1, A_2, \ldots, A_n$. If $A_i$ is a type I-A component for some $i \in \{1, 2, \ldots, n\}$, then by induction, $A_i$ is parallel to an annulus on $P$, hence we may reduce the intersection by an isotopy, contradicting the minimality of $|A \cap P|$. Thus $A_i$ is a type I-B component for $i = 1, 2, \ldots, n$. By induction, $A_i = \partial N(t_i)$ for some string $t_i$ in $T_1$ or $T_2$. Gluing the components back together, the result follows.

In particular, Proposition 3.10 tells us that for an arborescent tangle $T$, any essential annulus $A$ in $E(T)$ with $\partial A \subset \partial B$ must be a type II annulus.

Although $T(1/2, 1/2) = T[1/2] +_\Delta T[1/2]$ is equivalent to $T(1/2, -1/2) = T[1/2] +_\Delta T[-1/2]$ up to isotopy rel $\Delta$, it is important to recognize some special properties of the latter tangle. In particular, we want to develop a new notation to describe the sum: $T_1 +_\Delta T_2$ where $T_1 = T_2 = T(1/2, -1/2)$ and $\Delta'$ is the bottom disk of $B_1$ and the top disk of $B_2$. We denote this sum by $T(1/2, -1/2) * T(1/2, -1/2)$, i.e., this is the tangle where the $T(1/2, -1/2)$ tangle is glued on top of another $T(1/2, -1/2)$ tangle. In general, we call such a new tangle $T_1 * T_2$ the product of tangles $T_1$ and $T_2$. See Figure $11$. Note that Kauffman and Lambropoulou [12] use this notation and terminology in combining rational tangles.

![Figure 11: $T(1/2, -1/2) * T(1/2, -1/2)$ versus the Montesinos tangle $T(1/2, -1/2) + T(1/2, -1/2) = T(1/2, -1/2, 1/2, -1/2)$](image)

This leads to some new terms:
Definition 3.11. (1) For \( n \geq 0 \), define \( Q_0 \) to be the trivial tangle \( T[\infty] \), \( Q_1 = T(1/2, -1/2) \), and \( Q_n = Q_1 \ast Q_{n-1} \). Thus \( Q_n \) is the tangle with two vertical strings and \( n \) parallel horizontal circle components \( c_i, i = 1, \ldots, n \). For each \( c_i \), there exists a horizontal annulus \( A_i \) such that \( A_i \cap T = \partial_0 A_i = c_i \), and \( \partial_1 A_i \subset (\partial B - Q_n) \), where \( (\partial B - Q_n) \) is the punctured boundary sphere for the tangle \( Q_n \). See Figure 12 for an example.

(2) For \( n \geq 1 \), define the standard annulus in \( Q_n \) as the annulus in \( Q_n \) which separates the circles from the arcs of \( Q_n \), as in Figure 12.

(3) For \( n \geq 1 \), let \( A \) be the standard annulus in \( Q_n \). Let \( A' \) be the annulus on \( \partial B \) with \( \partial A' = \partial A \). Define the standard torus in \( Q_n \) as the torus obtained by pushing \( A \cup A' \) into the interior of \( X(Q_n) \). (Note for \( n = 1 \) this torus is inessential since it cuts off a cups in \( X(T) \).) See Figure 12 for an example.

(4) Since \( T \ast Q_1 = Q_1 \ast T \) up to isotopy rel \( \partial B \), we define switching as changing the order of \( T \) and \( Q_1 \). See Figure 13 for an example.

![Figure 12: The diagram on the left demonstrates the \( Q_3 \) tangle. The bold annulus in the middle is the standard annulus for \( Q_3 \) while the diagram on the right demonstrates the standard torus for \( Q_3 \).](image)

![Figure 13: The tangle on the left and the tangle on the right are equivalent by switching.](image)

Lemma 3.12. If \( A \) is a type II essential annulus in \( X(T) \) where \( T = T_1 + \Delta T_2 \) is a nontrivial sum, and \( A \) intersects \( P(\Delta) \) in a minimal nonempty collection of essential arcs on \( P(\Delta) \), then \( T = T(1/2) + \Delta T(1/2) \) up to isotopy rel \( \partial \Delta \) and \( A \) is the standard annulus for \( Q_1 \).

Proof. By assumption, \( A \) intersects \( P = P(\Delta) \) in essential arcs on \( P \). Since \( P \) is essential by Proposition 3.6, these arcs are also essential on \( A \). Let \( S_i = \partial B_1 - T_1 \), and \( P_i = S_i - P \), \( i = 1, 2 \).

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Suppose $|A \cap P| > 0$. Since $P$ is separating, $|A \cap P|$ is even. Let $\alpha_1, \alpha_2, ..., \alpha_n$ be arcs in $A \cap P$ such that $\alpha_j$ is adjacent to $\alpha_{j+1}$ on $A$ for $j = 1, 2, ..., n$ (where we define $\alpha_{n+1} = \alpha_1$). Then $\alpha_j \cup \alpha_{j+1}$ cuts off a disk $D_j$ from $A$ with $\partial D_j = \alpha_j \cup \beta_j \cup \alpha_{j+1} \cup \gamma_j$ where $\bigcup \beta_j = \partial_1 A$, $\bigcup \gamma_j = \partial_2 A$. Recall $\alpha_j$ and $\alpha_{j+1}$ are essential arcs on $P_i$ for $i = 1$ or 2, and $\text{Int}D_j \cap P = \emptyset$. Thus, $\partial D_j$ is an essential curve on $S_i$ for $i = 1$ or 2 and $D_j$ is a compressing disk for $S_i$.

Furthermore, $D_j$ intersects $P$ in two arcs. Without loss of generality, suppose $D_1$ is such a disk in $B_1 - T_1$. By Proposition 3.6, $T_1$ is rational, and by Lemma 3.13, $T_1 = T(1/2)$. Similarly, the disk $D_2$ is a compressing disk for $S_2$ intersecting $P$ in two arcs. By the same argument, $T_2 = T(1/2)$.

Since the compressing disk in $B_i - T_i$ is unique up to isotopy, $D_1, D_3, ..., D_{n-1}$ are all parallel in $B_1 - T_1$ while $D_2, D_4, ..., D_n$ are parallel in $B_1 - T_2$. Thus the two arcs of $P \cap A$ which are outermost on $\Delta$ belong to the same disk $D_{2r-1}$ in $A \cap (B_1 - T_1)$, and they belong to the same disk $D_{2s}$ in $A \cap (B_2 - T_2)$ for some $r, s$ so $D_{2r-1} \cup D_{2s}$ is a component of $A$. Since $A$ is connected, $D_{2r-1} \cup D_{2s} = A$, hence $|A \cap P| = 2$. By construction, one can see that $A$ is the standard annulus for $Q_1$.

3.3. Essential tori

**Definition 3.13.** Given a curve $\alpha$ on $\partial B$ separating $\partial B$ into two twice-punctured disks, we say that $T$ is $\alpha$-annular exactly when there exists an essential annulus $A$ with $\partial A \parallel \alpha$. Otherwise, we say that $T$ is $\alpha$-anannular.

**Lemma 3.14.** Suppose $T$ is an arborescent tangle.

1. If $A$ is an inessential annulus of type II in $B - T$ and $\partial A$ does not bound a disk in $B - T$, then $A$ is parallel to the annulus on $\partial B - T$ bounded by $\partial A$.

2. Suppose $F$ is an essential annulus or torus in $E(T)$, $T = T_1 + \Delta T_2$, $\partial F \cap \partial \Delta = \emptyset$, and $|F \cap P(\Delta)|$ is minimal. Then each component of $F \cap E(T_i)$ is essential.

3. Let $T = T_1 + \Delta T_2$, where $T_2$ is $\partial \Delta$-anannular. Suppose $F$ is an essential torus in $E(T)$. Up to isotopy, $F \subset E(T_i)$ for $i = 1$ or 2.

**Proof.** (1): Let $T$ be an arborescent tangle such that $A$ is an inessential annulus of type II in $B - T$ and $\partial A$ does not bound a disk in $B - T$. Suppose $A$ is compressible with compressing disk $C$. Then $\partial C$ cuts $A$ into two components, $A_1$ and $A_2$. Thus $C \cup A_1$ is a compressing disk for $\partial B - T$, contradicting the fact that $\partial A$ does not bound a disk in $B - T$.

Suppose $A$ is $\partial$-compressible. Let $D \subset E(T)$ be the boundary compressing disk with $\partial D = \alpha \cup \beta$, $\alpha = D \cap A$, $\beta = D \cap (B - T)$, and $\alpha$ is essential in $A$. Note that $\beta$ must lie in the annulus $A' \subset \partial(B - T)$ with $\partial A' = \partial A$. In this case, boundary compress $A$ along $D$ to get a disk $D'$ which has inessential boundary, $\alpha'$, bounding a disk $D''$ on $\partial(B - T)$. Since the tangle complement is irreducible by Proposition 3.7, the sphere $D' \cup D''$ bounds a ball. Therefore, $D' \parallel D''$ and hence $A$ is parallel to the annulus on $\partial(B - T)$ bounded by $\partial A$.

(2): Let $F$ be an essential annulus or torus in $E(T)$, $T = T_1 + \Delta T_2$ an arborescent tangle, $\partial F \cap \partial \Delta = \emptyset$, and $|F \cap P(\Delta)|$ minimal. Since $\partial F \cap \partial \Delta = \emptyset$,
The only type II essential annulus in $B - Q_1$ is the standard annulus.

Proof. Let $A$ be an essential annulus in $B - Q_1$, where $Q_1 = T_1 + \Delta T_2$, $T_i = T(1/2)$ for $i = 1, 2$, and $P = P(\Delta)$. Suppose $A$ intersects $P$ transversely and the number of components $|A \cap P|$ is minimal. By a standard innermost circle/outermost arc argument, the components of $A \cap P$ are either all circles or all arcs, essential on both $A$ and $P$. If the latter is true then we can apply Lemma 3.14 to get the result.

Assume that $A \cap P$ is all circles, thus $\partial A$ is disjoint from $\partial \Delta$. Let $A'$ be an annulus in $B$ with $\partial_1 A' = A' \cap T$ the circle component of $Q_1$, and $\partial_2 A'$ the horizontal circle on $\partial B$. Since $A$ is of type II, the two components in $\partial A$ are parallel to the vertical circle $\partial \Delta$, so $\partial A \cap \partial_2 A' \neq \emptyset$. Suppose the number of components $|A \cap A'|$ is minimal, and denote by $C$ the arc components of $A \cap A'$.

If there exists an arc component $\xi \subset C$ such that $\xi$ is an inessential outermost arc on $A'$, then $\xi$ cuts off an outermost disk $X$ from $A'$. If $\xi$ is essential on $A$, then $X$ is a $\partial$-compressing disk for $A$, contradicting the fact that $A$ is essential. If $\xi$ is inessential then as above we may reduce the number of components in the intersection $|A \cap A'|$. Thus we may assume that $C$ consists of essential arcs on $A'$. On the other hand, since $A \cap \partial_1 A' = \emptyset$ and $\partial A \cap \partial_2 A' \neq \emptyset$, $C$ is nonempty, and each component of $C$ has both endpoints on $\partial_2 A'$ and hence is inessential on $A'$, which is a contradiction.
Lemma 3.17. Given the \( Q_n \) tangle, \( n \geq 1 \) and any type II essential annulus \( A \subset B - Q_n \), there is an isotopy of \( Q_n \) so that \( Q_n = Q_m \ast Q_{n-m} \) and \( A \) is the standard annulus in \( Q_m \).

Proof. Let \( n \geq 1 \) and suppose \( A \subset B - Q_n \) is an essential annulus of type II with \( \partial A \subset \partial B - Q_n \). Let the \( n \) core circle components of \( Q_n \) be represented by \( c_i, i = 1, 2, ..., n \). Recall that for each \( c_i \), there exists an annulus \( A_i \) such that \( \partial_0 A_i = c_i, \partial_1 A_i \subset (\partial B - Q_n) \), where \( (\partial B - Q_n) \) is the boundary sphere for the tangle \( Q_n \).

By an innermost circle/outermost arc argument we may assume that \( (\cup A_i) \cap A \) consists of circles essential on both \( A \) and \( \cup A_i \). (See Figure 14.) By cutting and pasting along the annulus cut off by an outermost circle on \( A \), we may further assume that \( (\cup A_i) \cap A = \emptyset \). Since \( A \) is of type II, the two arc components of \( Q_n \) must be on the same component \( W \) of \( B - \text{Int}(N(A)) \), which is homeomorphic to \( D^2 \times I \) with \( A \) identified to \( \partial D^2 \times I \). Since the arc components of \( Q_n \) are unknotted in \( B \), we see that the other component \( X \) of \( B - \text{Int}(N(A)) \) is a solid torus, with \( A \) a longitudinal annulus.

Recall that the set of annuli \( A_i \) from the above argument do not intersect \( A \) and furthermore, \( \partial_0 A_i = c_i \) and \( \partial_1 A_i \subset (\partial B - Q_n) \). One can view \( \cup_i \partial_1 A_i \) as a set of nested circles with \( \partial A \) separating them into the two groups (see Figure 15).

The two boundary components of \( A \) break \( \partial B - Q_n \) into an annulus and two (twice-punctured) disks. After renumbering, we may assume that \( \partial_1 A_i \) \( (i = 1, ..., m) \) are contained in the annulus part (as in Figure 15).

Consequently, the other \( (n - m) \) nested circles correspond to the circle components in the two (twice-punctured) disk portions of \( \partial B - Q_n \). By adding a copy of \( A \) and doing an isotopy, we may assume these \( (n - m) \) circles all lie on the bottom disk (as in Figure 15). Similarly, for the \( m \) core circles described above, we know that these circles are parallel to circles on the boundary sphere \( \partial B - Q_n \), with the parallelisms given by the \( A_i \)’s. Using these parallelisms, we can perform isotopy to reorder the \( c_i \)’s, \( i = 1, ..., m \) and see that \( A \) is the standard annulus for \( Q_m \subset Q_n \).

Lemma 3.18. Let \( T \) be an arborescent tangle. For any type II curve \( \alpha \subset \partial X(T) \), \( T \) can be written as \( T = Q_n *_{\Delta} T' \) such that \( \partial \Delta = \alpha \) and \( T' \) is \( \partial \Delta \)-anannular (\( n \geq 0 \)).

Proof. If \( T \) is rational then there is no essential annulus in \( E(T) \), thus \( T = Q_0 * T' \) where \( T' = T \) is rational. Suppose the result holds for any arborescent tangle \( T \) having length \( \ell(T) \leq n \) and proceed by induction. Let \( T \) be an arborescent tangle such that \( \ell(T) = n + 1 \), and let \( \alpha \subset \partial B - T \) be a type II curve. If \( T \) is \( \alpha \)-anannular, then the result holds as \( T \) can be written as \( T = Q_0 *_{\Delta} T' \) with \( \partial \Delta = \alpha \) and \( T' = T \). Suppose, then, that \( T \) is \( \alpha \)-annular with \( A \) an essential annulus, \( \partial A \parallel \alpha \). Note that \( A \) must be a type-II annulus by Proposition 3.10.

Write \( T = T_1 +_{\Delta} T_2 \) where \( \ell(T_i) \leq n \) for \( i = 1, 2 \). If \( \partial \Delta \parallel \alpha \), then \( A \) intersects \( \Delta \) in arcs. By Lemma 3.12 \( T = T(1/2, 1/2) = T(1/2) +_{\Delta} T(1/2) \). Furthermore, we can write \( T = Q_1 *_{\Gamma} T' \) where \( \partial \Gamma = \alpha \) and \( T' \) is the trivial tangle.
If \( \partial \Delta \parallel \alpha \), choose a curve \( \alpha_i \parallel \alpha \). By the inductive hypothesis, we may write \( T_i = Q_{m_i} * \Delta_i, \) \( T'_i, i = 1, 2 \), where \( \partial \Delta_i = \alpha_i \) and \( T'_i \) is \( \Delta_i \)-annular. Now:

\[
T_1 + \Delta T_2 = (Q_{m_1} * T'_1) + \Delta (Q_{m_2} * T'_2) \text{ by definition}
\]

However because of how we have chosen \( \Delta, \Delta_1, \Delta_2 \), this is the same as \( (T'_1 * Q_{m_1}) * \Delta (Q_{m_2} * T'_2) \)

\[
= T'_1 * (Q_{m_1} * Q_{m_2}) * T'_2 \\
= Q_{m_1+m_2} * \Delta (T'_1 * T'_2) \text{ by switching}
\]

\[
= Q_{m_1+m_2} * \Delta T'' \text{ where } T'' = T'_1 * T'_2,
\]

and by Lemma 3.15 \( T'' \) is \( \partial \Delta \)-annular.

**Proposition 3.19.** Let \( T \) be an arborescent tangle. Then an annulus \( A \) in \( X(T) \) is essential if and only if \( T = Q_m * T'' \) for some \( m \geq 1 \) and \( A \) is a standard annulus in \( Q_m \).
Proof. A standard annulus of $Q_m$ is clearly an essential annulus in $Q_m \ast T'$, so assume that $A$ is an essential annulus in $X(T)$. By Proposition 3.10 $A$ is of type II. By Lemma 3.18 we can write $T$ as $Q_n \ast T''$, where $T''$ is $\partial \Delta$-anunnular, and $\partial \Delta \mid \partial A$. By Lemma 3.19 $A \subset X(T)$ can be isotoped into $Q_n$. By Lemma 3.17 up to isotopy we have $Q_n = Q_m \ast Q_{n-m}$ and $A$ is standard in $Q_m$. The result now follows by rewriting $T = Q_n \ast T''$ as $Q_m \ast T'$ with $T' = Q_n - m \ast T''$. \qed

Lemma 3.20. If $F$ is an incompressible torus in $Q_n$, $n \geq 1$, then $Q_n$ is isotopic to $Q_m \ast Q_{n-m}$ such that $F$ is standard in $Q_m$, $1 \leq m \leq n$. In particular if $F$ is essential in $X(Q_n)$, then $F$ is standard in $Q_m$, $2 \leq m \leq n$.

Proof. Let $n \geq 1$ and suppose $F \subset X(Q_n)$ is an incompressible torus. Define each of the core circle components of $Q_n$ by $c_1, c_2, \ldots, c_n$. Recall that for each $c_i$, there exists and annulus $A_i$ such that $\partial_0 A_i = c_i$, $\partial_1 A_i \subset (\partial B - Q_n)$, where $(\partial B - Q_n)$ is the boundary sphere for the tangle $Q_n$. As in the proof of Lemma 3.17 choose the $A_i$ pairwise disjoint and transverse to $F$ with $\sum_i |A_i \cap F|$ minimal.

If $(\cup_i A_i) \cap F = \emptyset$, then $F \subset E(Q_n - A_i) = E(T[\infty])$, hence $F$ is compressible. This contradicts the hypothesis, therefore $(\cup_i A_i) \cap F \neq \emptyset$. Suppose $F \cap A_1 \neq \emptyset$.

Let $\alpha$ be a such a circle in the intersection which is “outermost” on the annulus with respect to the ball. (In other words, it cuts off an annulus $A_1' \subset A_1$ such that $F \cap A_1' = \emptyset$.) Cut the torus along this arc $\alpha$ to get an annulus $F'$ with two copies of $\alpha$ as its boundary components. Glue each copy of $\alpha$ to one of two copies of $A_1'$ to make $F'$ an annulus with two parallel copies of $\partial_1 A_1 \subset (\partial B - Q_n)$ as its boundary. Since $F$ is incompressible, the new annulus $F'$ must be essential and moreover, $F'$ is an essential type-II annulus. Lemma 3.17 tells us that up to isotopy, $F'$ is a standard annulus in $Q_m$, $m \leq n$. Gluing the two copies of $\alpha$ back together we recover $F$, a standard torus in $Q_m$, $1 \leq m \leq n$ (since $m = 0$ implies that the torus is compressible). Furthermore, if $F$ is essential in $X(Q_n)$, then $2 \leq m \leq n$. \qed

Proposition 3.21. Let $T$ be an arborescent tangle. Then a torus $F$ in $X(T)$ is essential if and only if $F$ contains $Q_m$ for some $m \geq 2$ and $F$ is a standard torus in $Q_m$.

Proof. First assume that $T$ contains $Q_m$ with $m \geq 2$ and $F$ is a standard torus in $Q_m$. Then the solid torus $V$ in $B$ cut off by $F$ contains $m \geq 2$ circle components of $T$, which are the cores of $V$; hence $V - T$ is not a cusp and $F$ is incompressible in $V - T$. If $F$ is compressible in $B - \text{Int}V$, then after compression $F$ would become a reducing sphere of $B - T$, contradicting Proposition 3.7. Therefore, $F$ is an essential torus in $X(T)$.

We now assume that $F$ is an essential torus in $X(T)$ and proceed by induction on the length of the tangle $\ell(T)$. Suppose $T$ is a tangle having length $\ell(T) = 1$. $T$ is rational and hence atoroidal so the result is vacuously true. Suppose the result holds for a tangle $T$ having length $\ell(T) \leq k$. 

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Let $T$ be a tangle of length $k + 1$. By Lemma 3.18 $T$ can be written as $T = Q_s \ast \Delta' T'$ with $T' \partial \Delta$-annular and $P = P(\Delta)$. If $s > 0$, then Lemmas 3.14 Part (3) and 3.20 imply that $F$ is standard in $Q_n$ or $F \subset X(T')$. Furthermore, $\ell(T') \leq k$ so in the latter case, the inductive hypothesis gives the result.

Suppose $s = 0$ and moreover that $T$ cannot be written in such a sum with $s > 0$. Write $T = T_1 + \Delta T_2$ with $\ell(T_i) \leq k$ for $i = 1, 2$. Let $\alpha = \partial \Delta'$ and apply Lemma 3.18. The tangle $T_2$ can be written as $Q_m \ast \Delta' T_2'$ where $T_2$ is $\partial \Delta'$-annular. However, if $m > 0$, then by switching, $T$ can be written in the form $T = Q_s \ast \Delta T'$ with $T' \partial \Delta$-annular and $s > 0$ (by letting $s = m$). Since we assumed that was not the case, $m = 0$. Thus, $T_2$ is $\alpha$-annular. Now by Lemma 3.14 Part (3), $F \subset X(T_i)$ for $i = 1$ or 2. The result follows by induction.

We have now determined all arborescent tangles whose complement contains an essential surface which is an $S^2$, $D^2$, annulus or torus. These are summarized in the following theorem to determine all arborescent tangles whose complements are non-hyperbolic.

**Theorem 3.22.** Suppose $T$ is an arborescent tangle. Then $X(T)$ is non-hyperbolic if and only if one of the following holds.

1. $T$ is a rational tangle.
2. $T = Q_m \ast T'$ for some $m \geq 1$.
3. $T$ contains $Q_n$ for some $n \geq 2$.

**Proof.** By definition, $X(T)$ is non-hyperbolic if and only if it contains an essential surface $F$ which is a disk, sphere, annulus or torus. These are determined by Propositions 3.6, 3.7, 3.19, and 3.21 respectively.

**Addendum 3.23.** Suppose $T$ is an arborescent tangle.

1. $X(T)$ contains no essential $S^2$.
2. $X(T)$ contains an essential disk $D$ if and only if $T$ is a rational tangle and $D$ is the disk separating the two strings of $T$.
3. $X(T)$ contains an essential annulus $A$ if and only if $T = Q_m \ast T'$ for some $m \geq 1$ and $A$ is a standard annulus in $Q_m$.
4. If $X(T)$ contains an essential torus $F$ then $T$ contains a $Q_m$ for some $m \geq 2$ and $F$ is a standard torus in $Q_m$.

**Proof.** As above, this follows from Propositions 3.6, 3.7, 3.19 and 3.21.

### 3.4. Spheres intersecting $T$ at two points

As an application of the results in the previous sections, we will show that any sphere intersecting an arborescent tangle transversely at two points must be trivial in the sense that it bounds a 3-ball intersecting the tangle at a single trivial arc. The following lemma holds for any $n$-string tangle in a 3-ball.

**Lemma 3.24.** Let $(B, T)$ be a tangle, let $S$ be a sphere in $B$, and let $B'$ be the 3-ball in $B$ bounded by $S$. Suppose $|S \cap T| = 2$ and the component $t$ of $T$ intersecting $S$ is a circle. If $B' \cap T$ is not a single unknotted string in $B'$, then
either \( X(T) \) is reducible, or it is toroidal and the boundary of \( B' \cup N(t) \) is an essential torus in \( X(T) \).

Proof. Let \( t' = t \cap B' \), and \( t'' = t - \text{Int}(t') \). The union of \( B' \) and a regular neighborhood of \( t'' \) is a solid torus \( V \) in \( B \). Since a meridian disk of \( V \) intersects \( t \) at a single point, \( t \) is homologically nontrivial in \( V \), hence the torus \( F = \partial V \) is incompressible in \( V - T \). If \( F \) is compressible in the outside of \( V \) then a compression will produce a reducing sphere of \( B - T \), hence \( X(T) \) is reducible.

Now assume \( F \) is incompressible in \( B - T \). Since \( \partial X(T) \) is a punctured sphere, \( F \) is not boundary parallel. Hence either \( F \) is an essential torus and we are done, or it bounds a cusp, which means that \( V \cap T \) is the core of \( V \), so \( B' \cap T \) is a single unknotted arc in \( B' \), contradicting the assumption.

**Corollary 3.25.** If \( T \) is an arborescent tangle and \( S \) is a sphere in \( B \) intersecting \( T \) at two points, then the 3-ball \( B' \) in \( B \) bounded by \( S \) intersects \( T \) at a single unknotted string.

Proof. Let \( t' \) be the arc component of \( T \cap B' \), let \( t \) be the component of \( T \) containing \( t' \), and let \( t'' = t - \text{Int}(t') \).

If \( t \) is a circle component of \( T \) then by Lemma 3.24 either \( X(T) \) is reducible, which contradicts Proposition 3.7 or the torus \( F = \partial (B' \cup N(t)) \) is an essential torus in \( X(T) \). By Proposition 3.21 \( F \) must be the standard torus in \( Q_m \) for some \( m \geq 2 \), which contradicts the fact that the solid torus bounded by \( F \) has a meridian intersecting \( T \) at a single point.

We now assume that \( t \) is an arc component of \( T \). Then the frontier of \( B' \cup N(t') \) is a type I-B annulus \( F \) in \( B \). By Proposition 3.10 \( F \) cuts off a cusp in \( X(T) \), which implies that \( B' \cap T = t' \) and \( t' \) is an unknotted string in \( B' \).

Consider a tangle \( T \) with a single closed component \( t' \) that bounds a disk \( D \subset B \) such that \( \text{Int}(D) \) intersects \( T \) transversely in a single point. We call \( t' \) an **earring** of the tangle \( T \).

**Corollary 3.26.** Arborescent tangles cannot have earrings.

Proof. If this is not the case then a regular neighborhood of the disk \( D \) described above is a ball whose intersection with \( T \) is not a trivial arc since it contains a closed component. This contradicts Corollary 3.25.

4. Hyperbolicity of arborescent link complements

As mentioned in the introduction one great accomplishment of the work of Bonahon and Seibenmann [1] is that they classified all non-hyperbolic arborescent links. However, their work has remained incomplete and unpublished. Oertel [14] classified non-hyperbolic Montesinos links. See Theorem 4.1 below. The main theorem of this section is Theorem 4.2 which classifies all non-hyperbolic arborescent links of length at least 4. Together with Oertel’s theorem, this
gives an alternative proof of Bonahon-Seibenmann’s classification theorem. Another proof of Bonahon-Seibenmann’s classification theorem has been obtained recently by Futer and Gueritaud [5], using a completely different approach.

An arborescent link $L$ was defined in Section 2 as constructed from an arborescent tangle $(B, T)$ by adding two arcs on $\partial B$ to connect the boundary points of $T$. We proceed by recalling the precise definition from Section 2.

**Definition 2.1 (Wu [18])** A small arborescent link is a rational link or a Montesinos link of length 2 or 3; these are simply rational tangles connected along a band. A large arborescent link is obtained by gluing two nonrational arborescent tangles, $T_1$ and $T_2$, by an identification map of their boundary spheres (Conway spheres). A link is an arborescent link if it is either a small arborescent link or a large arborescent link.

In other words, if an arborescent tangle of length 2 or 3 is a Montesinos tangle, we may turn the tangle into an arborescent link by simply connecting the top two strings and the bottom two strings. (See Figure 6.) To be precise, however, note that an arborescent tangle of length 3 is not necessarily a Montesinos tangle. For example, the tangle in Figure 4 is not a Montesinos tangle, but after closing it appropriately one gets a Montesinos link. Montesinos links have been studied in detail by Oertel [14], who called them star links since the tree diagrams (as in Gabai [6]) are star-shaped. To denote a Montesinos link as in Figure 6, we write $K(r_1, r_2, \ldots, r_n)$, where $r_i = p_i/q_i$.

The following theorem of Oertel [14, Corollary 5] determines all non-hyperbolic Montesinos links.

**Theorem 4.1. (Oertel)** Suppose $K$ is a Montesinos link. $S^3 - K$ has complete hyperbolic structure if $K$ is not a torus link, and it is not equivalent to $L(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $L(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}, \frac{1}{2})$, $L(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$, $L(\frac{1}{2}, \frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or the mirror image of these links.

It should be noted that the term “torus link” in the above theorem and in Bonahon-Seibenmann’s unpublished manuscript [1] is not a torus link in the usual sense that it lies on a trivial torus $F$ in $S^3$; instead, it may contain one or both of the cores of the solid tori bounded by $F$. Bonahon and Seibenmann have given a classification of Montesinos links which are torus links in the above sense. For the convenience of the reader, these include $K(1/2, -1/2, 1/q)$ for $q \neq 0$, $K(1/4, -1/2, 1/3)$, and torus knots $(3, 4) \equiv K(-1/3, -1/2, 1/3)$ and $(3, 5) \equiv K(-1/5, 1/2, -1/3)$. Note that $K(1/4, -1/2, 1/3)$ is equivalent to the torus knot $(2, 3)$ union the axis linking this torus knot three times. See [1, Theorem A.8, Appendix] and Figure 16.

The following theorem determines all non-hyperbolic arborescent links of length at least 4. Together with the above theorem of Oertel, it gives an alternative proof of the classification theorem of Bonahon and Seibenmann for non-hyperbolic arborescent links. Futer and Gueritaud [5], have recently given another proof of Bonahon-Seibenmann’s Theorem using angled structures.
Recall that $Q_2$ denotes the tangle consisting of two vertical arcs and two horizontal circles. See Subsection 3.2.

**Theorem 4.2.** (Bonahon-Seibenmann) Suppose $L$ is a large arborescent link. Then $L$ is non-hyperbolic if and only if it contains $Q_2$.

**Proof.** A useful version of Thurston’s Hyperbolization Conjecture is stated in a survey paper by Allen Hatcher [9] as follows: *The interior of every compact irreducible atoroidal non-Seifert-fibered 3-manifold whose boundary consists of tori is hyperbolic.* Thurston’s Hyperbolization Conjecture has been proved for Haken manifolds and since the exterior $E(L)$ of a link in $S^3$ has non-empty boundary, it is either reducible or Haken. If $L$ contains $Q_2$ then either the standard torus $F$ in $Q_2$ is essential in $E(L)$ and hence $E(L)$ is toroidal, or $F$ is compressible, in which case $E(L)$ is reducible; in either case $E(L)$ is non-hyperbolic. Note that the exterior is irreducible and atoroidal if and only if $X(L)$ is irreducible and atoroidal. Therefore we need only show that if $L$ is a large arborescent link and if it does not contain $Q_2$ then the complement of $L$ is irreducible and atoroidal, and the exterior is not a Seifert fibered space. These will be proved in Lemmas 4.3, 4.4 and 4.5 below.

**Lemma 4.3.** Large arborescent link complements are irreducible.

**Proof.** Suppose $F$ is an essential sphere in $X(L)$ for a large arborescent link. Let $L = T_1 \cup S \cup T_2$, where $T_1$ and $T_2$ are each arborescent tangles of length $\geq 2$, and $S$ is a Conway sphere. By Proposition 3.7, $S - T = \partial X(T_i)$ is incompressible in both $X(T_1)$ and $X(T_2)$. Therefore, $S - T$ is incompressible in $X(L) = S^3 - L$. Hence we can apply Lemma 3.5 i.e. we can find an essential sphere $F'$ which does not intersect $S$. Hence $F' \subset X(T_i)$ for $i = 1$ or 2, however this contradicts Proposition 3.7.

**Lemma 4.4.** If $L$ is a large arborescent link, then $X(L)$ contains an essential torus if and only if $L$ contains $Q_2$. Furthermore, the essential torus is standard in $Q_m$ for some $m \geq 2$.

**Proof.** If $X(L)$ contains $Q_2$ then let $F$ be the standard torus for $Q_2$, with $V$ the solid torus bounded by $F$ intersecting $L$ in 2 core circles of $F$. Suppose $F$ is compressible. It cannot be compressible on the side containing the two core
curves, so suppose it is compressible on the other side. Compressing along a compressing disk \( D \) gives a sphere. This sphere bounds a ball which intersects the tangle in two disjoint circles, and there are also some components of the link outside of the ball; therefore it is an essential sphere, contradicting Lemma 1.6.

If \( F \) is boundary parallel, then it cuts off a cusp. The cusp cannot be in \( \text{Int}(V) \) since there are two core circles from \( L \) in \( V \). Thus it must be that \( F \) cuts off a cusp on the other side of \( F \). Suppose \( W \) is the solid torus bounded by \( F \) on the other side. Note that \( W \) contains the two vertical string in the definition of \( Q_n \), so there is a meridian disk of \( W \) intersecting \( W \cap L \) in two points. Therefore any meridian disk of \( W \) must bound a disk intersecting \( W \cap L \) an even number of times in \( W \), hence \( F \) cannot bound a cusp on this side either. Therefore, \( F \) is essential.

Now suppose \( L \) is a large arborescent link such that \( X(L) \) contains an essential torus, \( F \). We may write \( L = T_1 + S T_2 \) where \( T_i \) is an arborescent tangle with length \( \ell(T_i) \geq 2 \) for \( i = 1, 2 \), and \( S \) a Conway sphere, chosen so that \( S \) intersects \( F \) transversely, and the number of components, \( |S \cap F| \), is minimal. Let \( S' \) be the 4-punctured sphere \( X(L) \cap S \); note that \( |S \cap F| = |S' \cap F| \). If \( F \cap S' \) is empty, then \( F \subset X(T_i) \) for \( i = 1 \) or \( 2 \). By Proposition 3.21, \( T_i \) contains \( Q_2 \) and the torus \( F \) is standard in \( Q_m \) for some \( m \geq 2 \), hence \( L \) contains \( Q_2 \) and \( F \) is standard in \( Q_m \) also. Therefore we assume that the intersection \( F \cap S' \) is nonempty.

By Proposition 3.6, \( S' \) is incompressible. Similar to the proof of Lemma 3.14, Part (2), each component of \( F \cap S' \) is essential in \( X(T_i) \) for \( i = 1, 2 \); these are annulus components \( A_1, \ldots, A_n \subset F \) with \( \partial A_i \) essential curves on both \( F \) and \( S' \) (parallel on \( F \)). Thus they must be type-II annulus components by Proposition 3.10. (Note that \( n \) is even.) Without loss of generality, assume that \( A_j \subset X(T_1) \) for \( j \) odd, and \( A_j \subset X(T_2) \) for \( j \) even. By Proposition 3.19 for odd \( j \), we may write \( T_1 = Q_{m_j} \ast T_j' \) for some \( m_j \geq 1 \), and \( A_j \) the standard annulus in \( Q_{m_j} \). Similarly for even \( j \), we may write \( T_2 = Q_{m_j} \ast T_j' \) for some \( m_j \geq 1 \), and \( A_j \) the standard annulus in \( Q_{m_j} \).

If \( n = 2 \), then \( A_1 \) is the standard annulus for \( Q_{m_1} \) in \( B_1 \), \( m_1 \geq 1 \), and \( A_2 \) is the standard annulus for \( Q_{m_2} \) in \( B_2 \), \( m_2 \geq 1 \). Gluing them together, \( F \) is the standard torus for \( Q_{m_1+m_2} \), and \( m_1 + m_2 \geq 2 \).

Suppose, however, that \( n > 2 \), i.e., \( |S' \cap F| > 2 \). If the annuli on both sides of \( S \) are nested as in Figure 17, then numbering the components from the “inside-out” we have \( 1, 2, \ldots, k - 1, k, k - 1, \ldots, 2, 1 \). Gluing the \( k \) annuli on the \( B_1 \) side to the \( k \) annuli on the \( B_2 \) side of \( S \), we see that \( F \) has more than one component, contradicting the fact that \( F \) is a torus. Thus, without loss of generality, the annuli \( A_j \) (\( j \) odd) in \( B_1 \) are not all nested (as in Figure 18). Hence some \( A_j \), say \( A_1 \), is an “innermost” annulus and we may isotope the annulus \( A_1 \) (thus pulling the closed components from \( Q_{m_1} \) into \( B_2 \) past \( S \), reducing the number of components in the intersection, \( |F \cap S'| \). Furthermore, since not all the \( A_j \) are nested, there is still another annulus, say \( A_3 \), in \( B_1 \) which is the standard annulus for \( Q_{m_3} \), \( m_3 \geq 1 \). This contradicts the fact that we chose \( S \) with \( |S \cap F| \) minimal.

\[\]
Figure 17: Nested annuli in the proof of Lemma 4.4.

Figure 18: Non-nested annuli for the proof of Lemma 4.4.
Lemma 4.5. If \( L \) is a large arborescent link, then \( E(L) \) is not Seifert fibered.

Proof. Suppose \( L = T_1 \cup S T_2 \), where \( S \) is a Conway sphere and \( \ell(T_i) \geq 2 \) for \( i = 1, 2 \), and suppose \( E(L) \) is Seifert fibered. Let \( F = S \cap E(L) \). By Proposition 3.6, \( F \) is incompressible in \( E(T_i) \). Suppose \( F \) is \( \partial \)-compressible. Then there exists a disk \( D \subset E(L) \) such that \( \partial D = \alpha \cup \beta \) where \( \alpha = D \cap F, \beta = D \cap \partial E(L) \), and \( \alpha \) is essential in \( F \). Thus, \( \beta \) must run along \( \partial N(t) \) for a string \( t \in T_i \) for \( i = 1 \) or \( 2 \), and \( \alpha \) is an essential arc on \( F \). Hence the string \( t \) is parallel to an arc on \( S \) and therefore \( T_i \) is a rational tangle. This contradicts the fact that \( \ell(T_i) \geq 2 \). Thus \( F \) is not \( \partial \)-compressible, and \( F \) is essential in \( M \). By [8, Proposition 1.11], \( F \) must be a vertical or horizontal surface in the Seifert-fibered manifold \( E(L) \). Since vertical surfaces can only be annuli, tori, or Klein bottles (see Hatcher [8]), \( F \) must be a horizontal surface.

Next, notice that \( F \) is a separating (horizontal) surface and hence cuts \( E(L) \) into I-bundles. Filling in the regular neighborhood of the strings in \( T_i \) for \( i = 1 \) or \( 2 \) (these are simply I-bundles) on one side of \( F \) gives an I-bundle with boundary a sphere. This can only be an I-bundle over \( \mathbb{R}P^2 \), which has homotopy type the same as \( \mathbb{R}P^2 \). On the other hand, filling the regular neighborhood of the strings back into \( E(T_i) \) gives a 3-ball. This is impossible since a 3-ball and \( \mathbb{R}P^2 \) have different homotopy types. Hence \( E(L) \) cannot be Seifert fibered.

\[ \square \]

Theorem 4.6. Let \( L \) be a large arborescent link. Then \( L \) is a non-split prime link.

Proof. Let \( L \) be a large arborescent link such that \( L = T_1 \cup S T_2 \), where \( T_i \) is an arborescent tangle with \( \ell(T_i) \geq 2 \), and \( S \) a Conway sphere. Suppose \( B_i \) is the ball in \( S^3 \) with \( \partial B_i = S \) and \( B_i \cap L = T_i \) for \( i = 1, 2 \). By Lemma 4.3, \( L \) is a non-split link. We must show that \( L \) is prime. Suppose \( L = L_1 \# L_2 \), where \( L_i \) is nontrivial and let \( F \) be a decomposing sphere for \( L \) with \( L \cap F = \{ p_1, p_2 \} \). We may assume that \( F \) is transverse to \( S \), and that \( F \cap S \) consists of simple closed curves. Furthermore, we assume that \( F \) has been chosen so that the number of components \( |F \cap S| \) is minimal. If \( |F \cap S| = 0 \), then \( F \subset B_i \) for \( i = 1 \) or \( 2 \). Then by Corollary 3.26, \( L_i \) is trivial, a contradiction.

Suppose \( |F \cap S| \neq 0 \). Let \( \alpha \in F \cap S \) such that \( \alpha \) is an innermost curve on \( F \). Then \( \alpha \) bounds an innermost disk \( D \) on \( F \). We may choose \( \alpha \) so that \( D \cap L = \emptyset \) or \( D \cap L = p_i \) for \( i = 1 \) or \( 2 \) since \( F \) is a sphere. If \( D \cap L = \emptyset \), then we may reduce \( |F \cap S| \), contradicting minimality. If \( D \) intersects \( L \) in a single point, then \( \alpha \) bounds a disk \( D' \) on \( S \) which also intersects \( L \) in a single point. By Corollary 3.26, \( D \cup D' \) is a sphere which bounds a ball intersecting \( T_i \) in an unknotted string. Thus we may reduce \( |F \cap S| \) by an isotopy, contradicting the minimality of \( |F \cap S| \). Thus it must be the case that \( |F \cap S| = 0 \).

The Hopf link is an arborescent link since it is simply the boundary of a band with 2 twists, or equivalently, the integral rational tangle \( T(2) \) with numerator closure. By the definition of earring, either closed component can be called an earring of the link. However a large arborescent link cannot have an earring.

Corollary 4.7. If \( L \) is a large arborescent link then \( L \) cannot have earrings.
Proof. Suppose $L$ is a large arborescent link with an earring. A regular neighborhood of the earring is a ball, $B'$, whose boundary intersects the tangle in two points, but $B' \cap L$ is nontrivial. This contradicts Theorem 4.6.

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