Non-existence of measurable solutions of certain functional equations via probabilistic approaches

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Abstract. This paper deals with functional equations in the form of $f(x) + g(y) = h(x, y)$ where $h$ is given and $f$ and $g$ are unknown. We will show that if $h$ is a Borel measurable function associated with characterizations of the uniform or Cauchy distributions, then there is no measurable solutions of the equation. Our proof uses a characterization of the Dirac measure and it is also applicable to the arctan equation.

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1. Introduction

In this paper, we consider Borel measurable solutions $f, g : B \to \mathbb{R}$ of

$$f(x) + g(y) = h(x, y), \quad x, y \in B,$$

where $B$ is a Borel subset of $\mathbb{R}$ and $h : B^2 \to \mathbb{R}$ is a Borel measurable function.

Although in (1.1) there are numerous choices for the Borel measurable function $h$, we assume that $h$ is a composition of a non-constant Borel measurable function and a Borel measurable function $H$ satisfying that if $X$ and $Y$ are independent random variables with a common distribution, then $H(X, Y)$ and $Y$ are also independent. There would be numerous choices also for such $H$, however it is not necessarily easy to find such $H$. The more complicated $H$ is, the harder to show the independence of $H(X, Y)$ and $Y$ would be.

In this paper we assume that $H$ appears in the context of characterizations of the uniform distribution or the Cauchy distribution. Here characterization means that if $H(X, Y)$ and $Y$ are independent then $X$ follows the uniform distribution or the Cauchy distribution. We do not directly use those characterizations, instead we use a characterization of the Dirac measure.
Our approach is similar to that of Smirnov [9]. Smirnov [9] shows that any measurable solution of the Cauchy functional equation is locally-integrable, by applying the Kac-Bernstein theorem, which gives a characterization of the normal distribution. Recently, Mania [6] and Mania and Tikanadze [7] consider martingale characterizations of measurable solutions of the Cauchy, Abel and Lobachevsky functional equations.

The purpose of this paper is showing that a probabilistic approach is also applicable to more complicated functional equations than the Cauchy, Abel and Lobachevsky equations. Our proof uses a characterization of the Dirac measure, not of the normal distribution. Our proof is also applicable to the arctan equation.

Now we state our main results.

**Theorem 1.1.** Let \( h : (0, 1) \to (0, 1) \) be a Borel measurable map preserving the Lebesgue measure \( \ell \). Let \( j : [0, 1] \to \mathbb{R} \) be a Borel measurable map such that \( \ell(\{x \in [0, 1] : j(x) = c\}) < 1 \) for every \( c \in \mathbb{R} \). Then, there is no measurable solutions \( f, g : (0, 1) \to (0, 1) \) of

\[
f(x) + g(y) = j \left( \min \left\{ \frac{h(x)}{y}, \frac{1 - h(x)}{1 - y} \right\} \right), \quad x, y \in (0, 1). \tag{1.2}
\]

The conditions for the functions \( h \) and \( j \) are not restrictive. There are many candidates for measure-preserving transformations of \((0, 1)\). For example,

\[
h(x) = \begin{cases} 
2x & 0 < x < \frac{1}{2} \\
\frac{1}{2} & x = \frac{1}{2} \\
2x - 1 & \frac{1}{2} < x < 1
\end{cases}
\]

and

\[
h(x) = \begin{cases} 
x + a & 0 < x < 1 - a \\
a & x = 1 - a \\
x - 1 + a & 1 - a < x < 1
\end{cases}
\]

are measure-preserving transformations of \((0, 1)\).

**Theorem 1.2.** Let \( h : (0, 1) \to (0, 1) \) be a Borel measurable map preserving the Lebesgue measure \( \ell \). Let \( j : [0, 1] \to \mathbb{R} \) be a Borel measurable map such that \( \ell(\{x \in [0, 1] : j(x) = c\}) < 1 \) for every \( c \in \mathbb{R} \). Then, there is no Borel measurable solutions \( f, g : (0, 1) \to (0, 1) \) of

\[
f(x) + g(y) = j \left( \pi(h(x) + y) \right), \quad x, y \in (0, 1), \tag{1.3}
\]

where \( \pi(z) \) denotes the fractional part of \( z \).
Although (1.3) might look largely different from (1.2), our proofs of these theorems are very similar to each other.

Let the standard Cauchy measure be the probability measure on $\mathbb{R}$ with density $\frac{1}{\pi(1 + x^2)}$.

Theorem 1.3. Let $h : \mathbb{R} \to \mathbb{R}$ be a Borel measurable map preserving the standard Cauchy measure. Let $j : \mathbb{R} \to \mathbb{R}$ be a Borel measurable map such that $\ell(\{x \in \mathbb{R} : j(x) = c\}) < 1$ for every $c \in \mathbb{R}$. Then, there is no Borel measurable solutions $f, g : \mathbb{R} \to \mathbb{R}$ of

$$f(x) + g(y) = j\left(\frac{h(x) + y}{1 - h(x)y}\right), \quad x, y \in \mathbb{R}, \quad (1.4)$$

We can construct measurable maps preserving the standard Cauchy measure via measurable maps preserving the Lebesgue measure on an interval. If $X$ follows the standard Cauchy measure and the composition $z \mapsto f(\tan(z))$ preserves the uniform distribution on $(-\pi/2, \pi/2)$, then, $f(X)$ also follows the standard Cauchy measure.

The rest of this paper is organized as follows. In Sect. 2, we give a lemma for a characterization of the Dirac measure. In Sect. 3, we give proofs of Theorems 1.1, 1.2 and 1.3. Finally in Sect. 4, we give a probabilistic approach to the arctan equation as a corollary of Theorem 1.3.

2. A lemma

The following plays a crucial role in this paper. Although it is probably already known, we give a proof for the sake of completeness.

Lemma 2.1. (Characterization of Dirac measures) If $X$ and $Y$ are independent, and, $X + Y$ and $Y$ are independent, then, $Y$ is a constant a.s.

As we will see, this assertion is shown by solving the complex-valued Cauchy equation.

Proof. Throughout this proof, the symbol $i$ denotes the imaginary unit. Let

$$\phi_X(t) := E[\exp(itX)], \phi_Y(t) := E[\exp(itY)], \quad t \in \mathbb{R}. $$

Since $X$ and $Y$ are independent, we have that for every $s, t \in \mathbb{R},$

$$E[\exp(i(sX + Y) + tY))] = E[\exp(is(X + Y))]E[\exp(itY)] = \phi_X(s)\phi_Y(s)\phi_Y(t).$$

Furthermore,

$$E[\exp(i(s(X + Y) + tY))] = E[\exp(i(sX + (s + t)Y))] = \phi_X(s)\phi_Y(s + t).$$
Since $\phi_X$ is continuous on $\mathbb{R}$ and $\phi_X(0) = 1$, we have that for some $\epsilon_0 > 0$, $\phi_X(s) \neq 0$ for $s \in [-\epsilon_0, \epsilon_0]$.

Therefore,
\[
\phi_Y(s + t) = \phi_Y(s)\phi_Y(t), \quad s \in [-\epsilon_0, \epsilon_0], \ t \in \mathbb{R}.
\]

(2.1)

Let $s \in \mathbb{R}$. Then, for some $n$, $s/n \in [-\epsilon_0, \epsilon_0]$. By using (2.1) repeatedly, we have that
\[
\phi_Y(s) = n\phi_Y\left(\frac{s}{n}\right),
\]
and for every $t \in \mathbb{R}$,
\[
\phi_Y(s + t) = n\phi_Y\left(\frac{s}{n}\right) + \phi_Y(t) = \phi_Y(s) + \phi_Y(t).
\]

By using the fact that $\phi_Y(0) = 1$ and $\phi$ is continuous again, we have that for some constant $c \in \mathbb{C}$,
\[
\phi_Y(t) = \phi_Y(1)^t = \exp(ct), \ t \in \mathbb{R}.
\]

Let $c = a + bi$. Since $|\phi_Y(t)| \leq 1$ for every $t \in \mathbb{R}$, we have that $a = 0$.

By the Levy inversion formula, we see that for some constant $C$,
\[
P(Y = C) = 1.
\]

\section{3. Proofs}

\textbf{Proof of Theorem 1.1.} We assume there exist measurable maps $f, g : (0, 1) \to (0, 1)$ satisfying (1.2). Let $U$ and $V$ be two independent random variables with the uniform distribution on $(0, 1)$. Then, $f(U)$ and $g(V)$ are independent. By [3, Theorem 3.2], $\min\left\{\frac{h(U)}{V}, \frac{1-h(U)}{1-V}\right\}$ and $V$ are independent. By this and (1.2), $f(U)+g(V)$ and $g(V)$ are independent. Hence by Lemma 2.1, there exists a constant $c_0$ such that $g(V) = c_0$ a.s. Since $V$ is the uniform distribution on $(0, 1)$, $g(y) = c_0$, a.e. $y \in (0, 1)$.

For $x \in (0, 1)$, let
\[
H_x(y) := \min\left\{\frac{h(x)}{y}, \frac{1-h(x)}{1-y}\right\}, \ y \in (0, 1).
\]

Then, for each $x \in (0, 1)$, $H_x$ is continuous with respect to $y$ and
\[
\{H_x(y) | y \in (0, 1)\} = (\min\{h(x), 1-h(x)\}, 1].
\]

By the assumption, there exists a constant $c_1$ such that $\ell(\{z \in (0, 1] : j(z) < c_1\}) > 0$ and $\ell(\{z \in (0, 1] : j(z) > c_1\}) > 0$. Since the distribution of $h(U)$ is uniform on $(0, 1)$ if the distribution of $U$ is uniform on $(0, 1)$, we
see that for every $\epsilon > 0$ there exists $x$ such that $h(x) < \epsilon$. Hence there exists $x_0 \in (0,1)$ such that
\[
\ell \left( \{ z \in (\min \{ h(x_0), 1 - h(x_0) \}, 1] : j(z) < c_1 \} \right) > 0
\]
and
\[
\ell \left( \{ z \in (\min \{ h(x_0), 1 - h(x_0) \}, 1] : j(z) > c_1 \} \right) > 0.
\]
By this and $\ell(\{ y : g(y) \neq c_0 \}) = 0$, we can pick $y_1$ and $y_2$ such that $g(y_1) = g(y_2) = c_0$ and $j(H_x(y_1)) > c_1 > j(H_x(y_2))$. However, by (1.2),
\[
f(x) + c_0 = j(H_x(y_1)) = j(H_x(y_2)).
\]
Thus we have a contradiction. \hfill $\square$

**Proof of Theorem 1.2.** We assume there exist measurable maps $f, g : (0, 1) \to (0, 1)$ satisfying (1.3). Due to [3, Theorem 3.2] and Lemma 2.1, we can show that for some constant $c_0$, $g(y) = c_0$, a.e. $y \in (0,1)$, in the same manner as in the proof of Theorem 1.1.

For $x \in (0,1)$, let
\[
H_x(y) := \pi(h(x) + y), \ y \in (0, 1).
\]
Then, for each $x \in (0,1)$,
\[
\{ H_x(y) \mid y \in (0, 1) \} = [0,1) \setminus \{ h(x) \}.
\]
By the assumption, there exists $c_1$ such that $\ell(\{ z \in [0,1) : j(z) < c_1 \}) > 0$ and $\ell(\{ z \in [0,1) : j(z) > c_1 \}) > 0$. By this and $\ell(\{ y : g(y) \neq c_0 \}) = 0$, we can pick $y_1$ and $y_2$ such that $g(y_1) = g(y_2) = c_0$ and $j(H_x(y_1)) > c_1 > j(H_x(y_2))$. However, by (1.3),
\[
f(x) + c_0 = j(H_x(y_1)) = j(H_x(y_2)).
\]
Thus we have a contradiction. \hfill $\square$

**Proof of Theorem 1.3.** We assume there exist measurable maps $f, g : \mathbb{R} \to \mathbb{R}$ satisfying (1.4).

Let $X, Y$ be two independent standard Cauchy distributions. Then, by the assumption, $h(X)$ and $Y$ are independent standard Cauchy distributions. By [1], $h(X) + Y \over 1 - XY$ and $Y$ are independent. Hence, $j \left( \frac{h(X) + Y}{1 - h(X)Y} \right)$ and $g(Y)$ are independent. By Lemma 2.1 and (1.4), we have that for some constant $c_0$, $g(y) = c_0$, a.e. $y \in \mathbb{R}$.

For $x \in \mathbb{R}$, let
\[
H_x(y) := \frac{h(x) + y}{1 - h(x)y}, \ y \in \mathbb{R} \setminus \{ h(x) \}.
\]
Then,
\[
\{H_x(y) \mid y \in \mathbb{R} \setminus \{h(x)\}\} = \begin{cases} 
\mathbb{R} \setminus \{-\frac{1}{h(x)}\} & h(x) \neq 0 \\
\mathbb{R} & h(x) = 0
\end{cases}.
\]

By the assumption, there exists \(c_1\) such that \(\ell(\{z \in \mathbb{R} : j(z) < c_1\}) > 0\) and \(\ell(\{z \in \mathbb{R} : j(z) > c_1\}) > 0\). By this and \(\ell(\{y : g(y) \neq c_0\}) = 0\), we can pick \(y_1\) and \(y_2\) such that \(g(y_1) = g(y_2) = c_0\) and \(j(H_x(y_1)) > c_1 > j(H_x(y_2))\). However, by \((1.4)\),
\[
f(x) + c_0 = j(H_x(y_1)) = j(H_x(y_2)).
\]
Thus we have a contradiction.

\[
\text{Remark 3.1. (i) In general, if } Z \text{ is a real-valued random variable and } f : \mathbb{R} \to \mathbb{R} \text{ is Borel measurable, then, } f(X) \text{ is also a real-valued random variable. However if } f : \mathbb{R} \to \mathbb{R} \text{ is Lebesgue measurable, } f(X) \text{ may not be a real-valued random variable. We give an example. Let } \varphi : [0, 1] \to [0, 1] \text{ be the Cantor function. Indeed, let } X(w) := \inf\{y : \varphi(y) = w\}, \text{ then we can regard this as a random variable on } ([0, 1], \mathcal{B}([0, 1]), \ell), \text{ where } \mathcal{B}([0, 1]) \text{ is the completion of the Borel } \sigma\text{-algebra with respect to the Lebesgue measure. Let } A \text{ be a non Lebesgue measurable subset of } [0, 1]. \text{ Let } f = 1_{X(A)} \text{ then, } X(A) \text{ is contained in the Cantor set and hence the measure of } X(A) \text{ is zero and in particular it is Lebesgue measurable. Hence } f \text{ is Lebesgue measurable. However } f(X) \text{ is not Lebesgue measurable. This is a minor thing and it does not invalidate } [9, \text{ Section 2}], \text{ because there exists a Borel measurable function } g : \mathbb{R} \to \mathbb{R} \text{ such that } f = g, \text{ a.e. If } g \text{ is locally-integrable, then } f \text{ is also locally-integrable.}
\]
\[(ii) \text{ We can establish Theorems 1.1, 1.2 and 1.3 for the case that } j \text{ is Lebesgue measurable and } h \text{ is a measure-preserving map on the completed measure space. Since the Cauchy measure is equivalent to the Lebesgue measure, in the statements of Theorems 1.1, 1.2 and 1.3, we can replace the Borel sigma-algebra with its completion with respect to the Lebesgue measure.}\]

4. A probabilistic approach to the arctan equation

We can even show that every measurable solution of the arctan equation is zero by using Theorem 1.3. More specifically,

\[\text{Corollary 4.1. Every Borel measurable function } f : \mathbb{R} \to \mathbb{R} \text{ satisfying}
\]
\[
f(x) + f(y) = f\left(\frac{x + y}{1 - xy}\right), \quad xy \neq 1.
\]
\[(4.1)\]
\[\text{is limited to the function } f(x) = 0 \text{ for every } x \in \mathbb{R}.\]
(4.1) is called the arctan equation. Kiesewetter [4] showed that every continuous solution of (4.1) is the constant function taking zero at every point. Crstici, Muntean and Vornicescu [2] consider (4.1) on \{ (x, y) | xy < 1 \} under some additional assumptions Losonczi [5] shows that (4.1) has a form of \( A(\arctan x) \), where \( A \) is an additive function on \( \mathbb{R} \) with period \( \pi \).

**Proof of Corollary 4.1.** Assume that \( f \) is a Borel measurable solution of (4.1). By applying Theorem 1.3 to the case that \( g(x) = f(x), h(x) = x, j(x) = f(x) \), we see that there is no other cases than the case that there exists a constant \( C \) such that \( f(y) = C \) almost everywhere with respect to the Lebesgue measure.

Let \( x_0 \in (0, 1) \) be a real number such that \( f(x_0) = C \). Let \( F_0(y) := \frac{x_0 + y}{1 - x_0y} \). Then it is Lipschitz continuous on \([-x_0/2, x_0/2]\). Hence, by Rudin [8, Lemma 7.25],

\[
\ell \left( \left\{ F_0(y) : y \in \left[ -\frac{x_0}{2}, \frac{x_0}{2} \right], f(y) \neq C \right\} \right) = 0,
\]

where \( \ell \) denotes the one-dimensional Lebesgue measure. Since \( x_0 \in (0, 1) \), we have that

\[
F_0'(y) \geq 4x_0(1 + x_0) \left( 1 - \frac{x_0}{2} \right) > 0, \quad y \in \left[ -\frac{x_0}{2}, \frac{x_0}{2} \right],
\]

and hence,

\[
\ell \left( \left\{ F_0(y) : y \in \left[ -\frac{x_0}{2}, \frac{x_0}{2} \right] \right\} \right) > 0.
\]

Hence there exists \( y_0 \) such that

\[
f(y_0) = f(F_0(y_0)) = C.
\]

Therefore we have that

\[
2C = f(x_0) + f(y_0) = f(F_0(y_0)) = C,
\]

which implies \( C = 0 \).

We finally show that \( f(x) = 0 \) for every \( x \in \mathbb{R} \). Let \( x \neq 0 \) and \( x \neq -1 \). Let \( F(y) := \frac{x + y}{1 - xy} \). Then this function is well-defined and

\[
F'(y) = \frac{x + x^2 - x^2y + xy}{(1 - xy)^2} \neq 0
\]
on a neighborhood \([-\epsilon, \epsilon]\) of 0. In particular \( F \) is strictly monotone on \([-\epsilon, \epsilon]\), and hence, the inverse \( F^{-1} \) is Lipschitz on \( F([-\epsilon, \epsilon]) \).

Then, by using Rudin [8, Lemma 7.25] again,

\[
\ell \left( \left\{ y \in [-\epsilon, \epsilon] : f(F(y)) \neq 0 \right\} \right) = \ell \left( \left\{ F^{-1}(z) : z \in F([-\epsilon, \epsilon]), f(z) \neq 0 \right\} \right) = 0.
\]

Therefore we have that

\[
f(x) = f(x) + \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(y)dy = \frac{1}{2\epsilon} \int_{-\epsilon}^{\epsilon} f(F(y))dy = 0.
\]
By (4.1), we have that $f(0) + f(y) = f(y)$ and hence $f(0) = 0$. By (4.1) again, we have that

$$f(-1) + f(y) = f\left(\frac{y-1}{y+1}\right), \quad y \neq -1.$$  

By substituting $y = 1$ in this equation, we see that $f(-1) = 0$.

Thus we have that $f(x) = 0$ for every $x \in \mathbb{R}$. This completes the proof of Corollary 4.1.

\[\square\]

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