Modes of a stellar system II: non-ergodic systems

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ABSTRACT

An equation is derived for the energy of a small disturbance in a system that is generated by a distribution function (DF) of the form $f(J)$ – most galaxies and star clusters can be closely approximated by such a DF. The theory of van Kampen modes is extended to such general systems. A bilinear form on the space of DFs is defined such that the energy of a disturbance is its norm under this form. It is shown that van Kampen modes that differ in frequency are then orthogonal, with the consequence that the energies of van Kampen modes are additive. Consequently, most of the insight into the dynamics of ergodic systems that was gained in a recent paper on the van Kampen modes of ergodic systems applies to real clusters and galaxies.

Key words: Galaxy: kinematics and dynamics – galaxies: kinematics and dynamics – methods: analytical

1 INTRODUCTION

Stellar systems have hitherto been modelled in mean-field limit of an infinite number of constituent particles, when fluctuations vanish. That fluctuations play an essential role in the evolution of clusters was recognised over half a century ago (Henon, Spitzer, Chandra), but even now observations are fitted to mean-field models such as Michie-King models (Michie 1963; King 1966; McLaughlin et al. 2006; Piotto et al. 2015; Claydon et al. 2019).

Some fluctuations are internally generated by the shot noise inherent in a system with a finite number of particles, while other fluctuations are externally stimulated by the gravitational fields of neighbouring systems. As observational data become more precise, there must come a point at which fluctuations of either type can be detected. Detection of fluctuations would open the exciting possibility of using galaxies and star clusters to detect the passage of dark-matter haloes because the tidal fields of such haloes will excite the large-scale modes of globular clusters and dwarf spheroidal galaxies.

Modelling the small-scale effects of shot noise is rather straightforward because the system’s self-gravity is only important on the largest scales (e.g Fouvry et al. 2021). But following the derivation of the Balescu-Lenard equation by Heyvaerts (2010) and Chavanis (2012), it has become evident that self-gravity has a big impact on system-scale fluctuations even when the latter are stimulated by Poisson noise (Fouvry et al. 2015; Hamilton et al. 2018; Lau & Binney 2019, 2021a; Heggie et al. 2020). One expects externally generated fluctuations to be predominantly large-scale, so a viable theory of fluctuations must encompass self-gravity.

The standard approach to a theory of fluctuations is via normal modes: By linearising the equations of motion one derives a set of harmonically oscillating disturbances that is complete in the sense that any initial condition can be expressed as a linear combination of modes. This decomposition simultaneously characterises the initial condition in an physically significant way, and provides a convenient way to compute the system’s evolution by taking advantage of the almost trivial rule for evolving a normal mode.

The stability of stellar systems has traditionally been investigated by determining the frequencies of ‘Landau modes’ because the system is unstable if any of these frequencies has a positive imaginary part. Landau modes, however, lack the essential completeness property of normal modes. In the case of a homogeneous electrostatic plasma, van Kampen (1955) presented the true normal modes, now known as van Kampen modes in his honour, and Case (1959) proved that van Kampen’s modes are complete. Specification of the distribution function (DF) $f(x, v)$ of a collisionless system requires much more information than is required to specify the state of a fluid, and this fact is reflected in key differences between van Kampen modes and the modes of a fluid system. Crucially, the spectrum of van Kampen modes contains a continuum of real frequencies, and the DFs of individual modes contains a Dirac $\delta$-function associated with its frequency. Expression of an initial condition as a sum of normal modes involves an integral over frequency that eliminates the $\delta$-function.

Recently in the first paper in this series (Lau & Binney...
2 MATHEMATICAL BACKGROUND

Here we introduce essential mathematical tools and establish our notation. We focus on systems with integrable mean-field potentials, so their DFs can be written in the form \( f(J) \).

2.1 Angle-action variables

The role that Cartesian variables play for homogeneous systems is played for spheroidal systems by angle-action variables \( (\theta, J) \). The actions \( J_i \) are constants of motion while their conjugate variables, the angles \( \theta_i \), increase linearly in time, so \( \dot{\theta}(t) = \hat{\theta}(0) + \Omega t \). The particles’ Hamiltonian \( H(x, v) \) is a function \( H(J) \) of the actions only and the frequencies \( \Omega \), that control the rates of increase of the angles are given by \( \Omega = \partial H/\partial J \). Angle-action variables are canonical, so the volume element of phase space \( d^3w = d^3x d^3v = d^3\theta d^3J \) and Poisson brackets can be computed as

\[
[f, g] = \sum_{i} \left( \frac{\partial f}{\partial \theta_i} \frac{\partial g}{\partial J_i} - \frac{\partial f}{\partial J_i} \frac{\partial g}{\partial \theta_i} \right).
\]

Functions on phase space can be expressed as Fourier series:

\[
h(w) = \sum_n h_n(J)e^{in\theta}; \quad h_n(J) = \int \frac{d^3\theta}{(2\pi)^3} e^{-in\theta} h(w).
\]

Note that for real \( h \), \( h_{-n} = h_n \).

2.2 Potential-density pairs

Following Kalnajs (1976) we solve Poisson’s equation by introducing a basis of biorthogonal potential-density pairs. That is, a set of pairs \( (\rho^\alpha(x), \Phi^\alpha(x)) \) such that

\[
4\pi G \rho^\alpha(x) = \nabla^2 \Phi^\alpha(x) \quad \text{and} \quad \int d^3x \Phi^\alpha(x)^* \rho^\alpha(x) = -E \delta_{\alpha\alpha'}, \quad (3)
\]

where \( E \) is an arbitrary constant with the dimensions of energy. Given a density distribution \( \rho(x) \), we expand it in the basis

\[
\rho(x) = \sum_\alpha A_\alpha \rho^\alpha(x) \quad \Rightarrow \quad \Phi(x) = \sum_\alpha A_\alpha \Phi^\alpha(x), \quad (4)
\]

where

\[
A_\alpha = -\frac{1}{E} \int d^6w \Phi^\alpha(x)f(w). \quad (5)
\]

If \( \rho \) and \( \Phi \) are time-dependent, the \( A_\alpha \) become time-dependent. From equations (3) and (4) one can obtain an expression for \( \Phi \) in terms of \( \rho \). Comparison of this relation with Poisson’s integral, yields

\[
\frac{G}{|x' - x|} = \frac{1}{E} \sum_\alpha \Phi^\alpha(x) \Phi^\alpha(x'). \quad (6)
\]

The system’s potential energy is

\[
P = \frac{1}{2} \int d^3x \rho \Phi = \frac{1}{2} \int d^3x \sum_\alpha A_\alpha \Phi^\alpha \rho
\]

\[
= -\frac{E}{2} \sum_\alpha |A_\alpha|^2. \quad (7)
\]

A related calculation is the potential energy of one system when placed in the potential of another. This is

\[
P' = -G \int d^3x \frac{\rho(x)\rho'(x')}{|x - x'|}. \quad (8)
\]

The symmetry of this expression establishes that the energy of system a in the potential of system b is the same as that of system b placed in the potential of system a. Trivial adaptation of the derivation of equation (7) shows that

\[
P' = -\frac{E}{2} \sum_\alpha A_\alpha^* A_\alpha'. \quad (9)
\]

Now

\[
P' = \int d^6w f \Phi' = \int d^3J d^3\theta \sum_{mn} f_m \Phi_n e^{i(m+n)\theta}
\]

\[
= (2\pi)^3 \sum_n \int d^3J f_n \Phi_n. \quad (10)
\]

2.3 Linearised CBE

On dynamical timescales the DF of a stellar system satisfies the collisionless Boltzmann equation

\[
\frac{\partial f}{\partial t} + [f, H] = 0. \quad (11)
\]

When we split \( f(\theta, J, t) = f_0(J) + f_1(\theta, J, t) \) into its mean-field and fluctuating components and neglect terms quadratic and higher in the fluctuations, the CBE can be written

\[
\frac{\partial f_1}{\partial t} + [f_1, H_0] + [f_0, \Phi_1] = 0. \quad (12)
\]
When we use angle-action coordinates to evaluate the Poisson brackets and write \( f_1 \) and \( \Phi_1 \) as Fourier series in angles, equation (12) yields
\[
\frac{\partial f_n}{\partial t} + \mathbf{n} \cdot \mathbf{\Omega} f_n - \mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \Phi_{1n} = 0,
\]
where we have dropped the subscript 1 from \( f_{1n} \) for brevity but retained it on \( \Phi_{1n} \) for reasons that will soon become apparent.

3 ENERGY OF A PERTURBATION

Following Nelson & Tremaine (1999) we imagine using an externally applied gravitational field to impose a real perturbation \( f_1 \) on a mean-field model \( f_0(J) \). The perturbing gravitational potential \( \Phi_1 \) now has two components, the potential \( \Phi_e \) of the externally applied field and the potential \( \Phi_o \) generated via Poisson’s equation by the perturbed density distribution
\[
\rho_1 = \int d^3v f_1.
\]

The rate at which work is done by the external field is
\[
\frac{dE}{dt} = -\int d^3v \frac{\partial \Phi_e}{\partial \mathbf{J}} \rho_1 = -\int d^3v f_1(w) v \cdot \frac{\partial \Phi_e}{\partial \mathbf{x}}.
\]

Now
\[
\mathbf{v} \cdot \frac{\partial \Phi_e}{\partial \mathbf{x}} = -\left[ H_0, \Phi_e \right] = \mathbf{\Omega} \cdot \frac{\partial \Phi_e}{\partial \mathbf{J}},
\]
so equation (15) can be written
\[
\frac{dE}{dt} = -\int d^3v f_1(w) \mathbf{\Omega} \cdot \mathbf{n} \Phi_{1n}.
\]

With \( \Phi_1 \) decomposed into its two components, the linearised CBE (13) can be written
\[
\frac{\partial f_n}{\partial t} + \mathbf{n} \cdot \mathbf{\Omega} f_n - \mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \Phi_{1n} = \mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \Phi_{1n}.
\]

Eliminating \( \Phi_{1n} \) between the last two equations
\[
\frac{dE}{dt} \equiv -(2\pi)^3 \int d^3J \sum_n f_n \mathbf{n} \cdot \mathbf{\Omega} \frac{\mathbf{n} \cdot \mathbf{\Omega}}{\mathbf{n} \cdot \nabla J f_0}
\]
\[
\times \left( \frac{\partial f_n}{\partial t} + \mathbf{n} \cdot \mathbf{\Omega} f_n - \mathbf{n} \cdot \nabla J f_0 \Phi_{1n} \right).
\]

The integrand contains three terms. The middle term is proportional to \( \mathbf{n} \cdot \mathbf{\Omega} f_n \) and vanishes when summed over \( \mathbf{n} \) because the sum includes both \( \mathbf{n} \) and \( -\mathbf{n} \). The integral over the first term yields
\[
\frac{dK}{dt} = -\frac{2\pi^3}{2} \int d^3J \sum_n \frac{\mathbf{n} \cdot \mathbf{\Omega}}{\mathbf{n} \cdot \nabla J f_0} \frac{\partial f_n}{\partial t}.
\]

In preparation for handling the third term, we note that
\[
\int d^3\mathbf{\theta} f_1[\mathbf{\theta}, H_0] = \int d^3\mathbf{\theta} f_1 \frac{\partial \Phi_e}{\partial \mathbf{J}} \cdot \mathbf{\Omega} = (2\pi)^3 \int d^3J f_0 n \mathbf{\Omega} \Phi_{1n}.
\]

Hence the third term in equation (19) is
\[
\frac{dP}{dt} \equiv -(2\pi)^3 \int d^3J \sum_n \mathbf{n} \cdot \mathbf{\Omega} f_n \Phi_{1n}
\]
\[
= \int d^6w f_1[\mathbf{\theta}, H_0] = \int d^6w f_1 v \cdot \frac{\partial \Phi_e}{\partial \mathbf{x}}
\]
\[
= \int d^3x \Phi_e \frac{\partial f_0}{\partial t} = -\frac{d}{dt} \frac{1}{G} \int d^3x d^3x' \frac{\rho_1(x) \rho_1(x')}{|x - x'|}.
\]

where the penultimate equality uses integration by parts and the continuity equation \( \partial \rho/\partial t = -\nabla_x \cdot (\rho v) \).

Now that the two surviving contributions to the integral in equation (19) have proved to be total time derivatives, we can immediately integrate from \( E = K = P = 0 \) at \( t = 0 \) to obtain an equation for the energy of an arbitrary fluctuation
\[
E = -(2\pi)^3 \int d^3J \sum_n \frac{\mathbf{n} \cdot \mathbf{\Omega}}{\mathbf{n} \cdot \nabla J f_0} |f_n|^2 - \frac{i}{2} \int d^3x d^3x' \frac{|G\rho|^2}{|x - x'|}.
\]

This equation generalises to three dimensions the quantity Kalnajs (1971) shows to be constant in an isolated razor-thin disc (Kalnajs’ eqn. 48), and which he says will ‘identify as energy’. Our derivation shows that \( E \) is the work that must be done to establish a perturbation rather than just showing that \( E \) is constant when \( \Phi_e \) vanishes. Equation (23) differs from equation (5.130) in Binney & Tremaine (2008) for the energy of a disturbed ergodic system only by the replacement of \( df_0/df \) by \( \mathbf{n} \cdot \nabla J f_0/\mathbf{n} \cdot \mathbf{\Omega} \). When equation (6) is used to eliminate \( |x - x'| \) from equation (23) we obtain
\[
E = -(2\pi)^3 \int d^3J \sum_n \frac{\mathbf{n} \cdot \mathbf{\Omega}}{\mathbf{n} \cdot \nabla J f_0} |f_n|^2 - \frac{i}{2} \int A_0 |A_0|^2.
\]

3.1 Restriction of the perturbed DF

For certain combinations of \( n \) and \( J \), \( \mathbf{n} \cdot \nabla J f_0 \) will vanish and one might worry that such combinations will make the integral in equation (23) for \( E \) ill defined. However, from equation (18) it follows that for these combinations \( f_n(J) \) remains zero as the disturbance is excited, so the integrand in equation (23) vanishes at the apparently problematic points. Thus the disturbances \( f \) that can be induced in a system with equilibrium DF \( f_0(J) \) by an external potential are restricted in form.

Antonov (1961) broadened this result by showing (e.g. Binney & Tremaine 2008, p. 429) that the perturbation \( f_1 \) that is generated by applying any disturbing Hamiltonian \( H_1 \) to \( f_0 \) will be of the form
\[
f_1 = [h, f_0] = \frac{\partial h}{\partial \mathbf{J}} \frac{\partial f_0}{\partial \mathbf{J}},
\]
where \( h(w) \) is a function that depends on \( H_1 \). On expanding \( f \) and \( h \) in Fourier series we obtain
\[
f_n = \mathbf{n} \cdot \frac{\partial f_0}{\partial \mathbf{J}} \phi_n
\]
so when \( \mathbf{n} \cdot \nabla J f_0 = 0 \), \( f_n \) vanishes no matter how the system is disturbed.

3.2 A bilinear form for DFs

Additivity of energies is a fundamentally property of normal modes. It emerges naturally if we can express the energy of
a disturbance as the norm of the disturbance defined by a bilinear form under which normal modes are mutually orthogonal. Equation (24) suggests that the required form is
\[ (f|\tilde{f}) = -\frac{(2\pi)^3}{2} \int d^3J \sum_n \frac{n \cdot \Omega}{n \cdot \nabla J f_0} f_n \tilde{f}_n - \frac{\pi}{2} \sum_\alpha A_\alpha^* A_\alpha, \] (27)
where \( A_\alpha^*[f] \) is a functional of \( f \) while \( A_\alpha \) denotes the corresponding functional of \( \tilde{f} \).

4 NORMAL MODES OF A NON-ERGODIC SYSTEM

We now look for disturbances with exponential time dependence, so
\[ f_n(J, t) = f_n(J, \omega) e^{-i\omega t}, \quad \Phi_n(J, t) = \Phi_n(J, \omega) e^{-i\omega t} \] (28)
With this ansatz, the linearised CBE (13) becomes
\[ (n \cdot \Omega - \omega) f_n = (n \cdot \nabla J f_0) \Phi_n(J, \omega), \] (29)
This equation yields a well defined value for \( f_n \) in terms of \( \Phi_n \) when \( n \cdot \Omega \neq 0 \), but on resonant tori (tori on which \( n \cdot \Omega = 0 \)) it does not constrain \( f_n \). Therefore, as van Kampen (1955) pointed out, the solution to this equation must be written
\[ f_n(J, \omega) = \mathcal{P} \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \Phi_n(J, \omega) + g_n(J) \delta(n \cdot \Omega - \omega), \] (30)
where \( \mathcal{P} \) indicates that when integrated wrt \( n \cdot \Omega \) or \( \omega \), the Cauchy principal value around the singularity at \( \omega = n \cdot \Omega \) should be taken, and \( g_n(J) \) is an arbitrary function that specifies non-trivial distributions of stars on the resonant tori. In Section 3.1 we saw that the DFs of disturbances are restricted such that \( f_n = 0 \) when \( n \cdot \nabla J f_0 = 0 \). Since a system’s normal modes comprise possible disturbances, \( g_n \) must be restricted in the same way:
\[ n \cdot \nabla J f_0 = 0 \quad \Rightarrow \quad g_n = 0. \] (31)
In Paper I \( \Phi_n(J, \omega) \) in equation (30) is interpreted as the potential generated by the resonant stars and by the non-resonant stars that they excite. It is the dressed potential of the resonant stars. Consequently, Poisson’s equation imposes a relation between \( g_n \) and \( \Phi_n \). We obtain this relation by multiplying equation (30) by \( d^3w \, e^{in \theta} \Phi(\omega)^* \) and integrating through phase space to obtain the self-consistency condition (cf Paper I eqn. 42)
\[ \sum_\alpha M_{\alpha \alpha} A_\alpha = -B_{\alpha \alpha}, \] (32)
where
\[ M_{\alpha \alpha}(\omega) = \delta_{\alpha \alpha} - \frac{(2\pi)^3}{E} \mathcal{P} \int d^3J \sum_n \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \Phi_n(\omega)^* \Phi_n(\omega) \]
\[ B_{\alpha \alpha} = -\frac{(2\pi)^3}{E} \int d^3J \sum_n g_n(\omega) \Phi_n(\omega)^* \Phi_n(\omega) \delta(n \cdot \Omega - \omega). \] (33)
In any truly stable system (so excluding marginally stable systems such as those discussed by Mathur 1990), \( M(\omega) \) has an inverse for all real \( \omega \), so given \( g(\omega) \) we can solve for the amplitudes \( A_\alpha \) of the corresponding density and potential. The coefficients \( B_{\alpha \alpha} \) give the density and potential generated by the resonant stars alone, that is, after removing the contributions of stars driven by the gravitational field of the resonant stars (Paper I eqn. 46).

In principle for given functions \( g_n \) equation (30) yields a mode for every frequency of the form \( n \cdot \Omega \), but since vectors \( n \) with any large component generate very small densities in real space, the important modes are confined to frequencies that range from near zero \( (\Omega \approx 0 \) at large \( |J| \)) to about twice the system’s maximum circular frequency.

\( B \) necessarily vanishes for complex \( \omega \) because \( n \cdot \Omega \) is inherently real. Thus all van Kampen modes have real frequencies.

Any additional modes must arise when \( |M(\omega)| = 0 \) at some possibly complex frequency \( \omega \). If the underlying equilibrium \( f_0 \) is unchanged by reversing all velocities (in practice meaning that it has no net rotation), then if \( |M(\omega)| = 0 \) then \( |M(-\omega)| = 0 \) also, and at one of these two frequencies the disturbance will grow exponentially and the system is unstable. Hence every mode of a truly stable, time-reversible system is a van Kampen mode. This result extends to flattened systems and systems with velocity anisotropy a key result of Paper I. We defer discussion of marginally stable systems to Section 5.

The \( \mathcal{P} \) and \( \delta \) symbols in equation (30) signal that van Kampen modes make sense only within an integral wrt \( \omega \). Hence a physical disturbance will always be of the form
\[ F(\theta, J, t) = \sum_n \int d\omega f_n(J, \omega) e^{i(n \theta - \omega t)}. \] (34)
This disturbance is specified by one’s choice of the functions \( g_n(J) \).

4.1 Energies of normal modes

In Appendix A we show that in the case of two normal modes \( f(\omega) \) and \( \tilde{f}(\omega) \) the bilinear form (27) can be brought to the form
\[ (f|\tilde{f}) = -\frac{(2\pi)^3}{2} \sum_n \int d^3J \omega \left\{ \pi^2(n \cdot \nabla J f_0) \Phi_n(\omega)^* \Phi_n(\omega) + \frac{g_n(\omega)}{n \cdot \nabla J f_0} \delta(n \cdot \Omega - \omega) \right\}. \] (35)
Remarkably, on account of the factor \( \delta(n \cdot \Omega - \omega) \) the bilinear form is computed by integrating only over resonant stars, despite much of the energy lying with driven, non-resonant stars. Combining this equation for the bilinear form evaluated on two van Kampen modes with the equation (34) expressing a general disturbance as a sum of van Kampen modes, we obtain an alternative expression for the value of the form on any two disturbances \( F \) and \( \tilde{F} \):
\[ (F|\tilde{F}) = \left( \int d\omega f(\omega) \right) \left( \int d\omega \tilde{f}(\omega) \right) = \int d\omega d\tilde{\omega} \langle f(\omega) | \tilde{f}(\tilde{\omega}) \rangle \]
\[ = -\frac{(2\pi)^3}{2} \sum_n \int d^3J \omega_n(J), \]
\[ \times \left\{ \pi^2(n \cdot \nabla J f_0) \Phi_n(\omega_n)^* \Phi_n(\omega_n) + \frac{g_n(\omega_n)^* g_n(\omega_n)}{n \cdot \nabla J f_0} \right\}. \] (36)
where \( \omega_n(J) \equiv n \cdot \Omega \) and the potentials \( \Phi_n \) and driving terms \( g_n \) have acquired \( \omega_n \) as an argument to indicate that they are the potentials and driving terms of the van Kampen...
modes with that frequency in the decomposition of \( F \) into modes.

When we set \( \bar{F} = F \) in equation (36) we obtain the energy of a general disturbance as the sum of the energies of its component van Kampen modes:

\[
E[F] = \langle F|F \rangle = (2\pi)^3 \sum_n \int d^3 J \omega_n(J) N_n(J),
\]

where

\[
N_n(J) = -\frac{1}{2} \{ \pi^2 (\mathbf{n} \cdot \nabla_J f_0)|\Phi_n(\omega_n)|^2 + \frac{g_n(\omega_n)}{|\mathbf{n} \cdot \nabla_J f_0|} \} \tag{38}
\]

is the phase-space density of ‘plasmons’ associated with wavevector \( \mathbf{n} \) in the terminology of Hamilton & Heinemann (2020). Remarkably, this expression for the plasmon density involves only the driving term \( g_n \) and the resulting spatial structure \( \Phi_n \) of the component van Kampen modes – there is no mention of the kinetic energy of the driven stars. At first sight this is odd because in a stable system disturbances fade while \( E \) is constant. The equation works because the potentials in question belong not to the disturbance but to its constituent van Kampen modes, which do not change but nevertheless cause the disturbance to fade as they become more and more evenly distributed in phase.

5 DISCUSSION

There are close parallels between formulae derived here for general systems and ones presented in Paper I for ergodic systems. The essential difference is the universal replacement of \( df_0/dH \) in Paper I by \( \mathbf{n} \cdot \nabla_J f_0 / \mathbf{n} \cdot \mathbf{\Omega} \) as here. The other differences are more superficial, being caused by formulae in Paper I being derived from an operator that gives \( \partial^2 f / \partial t^2 \) rather than one that gives \( \partial f / \partial t \). This change leads to \( \delta((\mathbf{n} \cdot \mathbf{\Omega})^2 - \omega^2) \) in Paper I being replaced by \( \delta(\mathbf{n} \cdot \mathbf{\Omega} - \omega) \). These symbols have different dimensions:

\[
\delta((\mathbf{n} \cdot \mathbf{\Omega})^2 - \omega^2) = \delta((\mathbf{n} \cdot \mathbf{\Omega} - \omega)(\mathbf{n} \cdot \mathbf{\Omega} + \omega)) \approx \frac{\delta(\mathbf{n} \cdot \mathbf{\Omega} - \omega)}{2\omega}. \tag{39}
\]

As a consequence, \( g_n/2\omega \) in Paper I is equivalent to \( g_n \) here.

In Paper I we explored the consequences of reducing the self-gravity of a system by reducing the mass of its particles by a factor \( \xi < 1 \) and introducing a fraction \((1 - \xi)\) of the mean-field potential. This operation modified the structure of the system’s van Kampen modes by suppressing the dressing of the potential of resonant stars. The ensuing discussion applies equally to the general equilibria treated here.

In Paper I we stressed that Landau ‘modes’ are not really modes but zeroes of the function \( |\mathbf{M}| \). When such a zero lies just below the real axis (a ‘weakly damped mode’) the van Kampen modes on the adjacent stretch of the real axis are heavily damped and will make a large contribution to the system’s evolution. The time required for these modes to drift out of phase, and the associated disturbance to fade in real space, is inversely proportional to the extent of the heavily damped section of the real line, and therefore inversely proportional to the distance of the zero from the real line. This interpretation of ‘Landau damping’ applies equally to the general systems discussed here.

Although Mathur (1990) did not display an explicit example, he demonstrated the logical possibility of marginally stable systems, that is ones that can oscillate for ever. This possibility arises when frequencies of the form \( \mathbf{n} \cdot \mathbf{\Omega}(J) \) do not extend from a maximum frequency down to zero. The easiest way to engineer a gap in the frequency coverage is to impose a limit on the spatial extent of the system, so there is a minimal orbital frequency \( \Omega_{\text{min}} \), and to make the system effectively one-dimensional so small frequencies cannot be constructed by differencing frequencies greater than \( \Omega_{\text{min}} \). When these conditions are satisfied, Mathur (1990) shows that \( f_0 \) can be devised such that the matrix \( \mathbf{M} \) of equation (32) has vanishing determinant in a gap. Consequently, non-zero \( \mathbf{A} \) can then be found even though \( \mathbf{B} \) vanishes because we are in a gap. In these exceptional circumstances, a complete set of modes comprises the van Kampen modes plus any stable/unstable pairs of modes and any marginally stable modes in gaps.

6 CONCLUSIONS

We have extended results presented in Paper I from ergodic systems to systems with DFs of the form \( f(J) \). This is a major extension because ergodic systems probably do not occur in Nature while many galaxies and star clusters will have DFs that can be closely approximated by \( f(J) \).

We derived equation (23) for the energy of a disturbance to a general system, and motivated by this result defined a bilinear form on the space of DFs such that the energy of a disturbance is the norm of its DF.

Next we extended the concept of van Kampen modes to general systems. They exist for essentially the same range of real frequencies, and have the same physical interpretation and energy-additivity as in the ergodic case. There is again a sharp distinction between van Kampen modes, which have frequencies that lie in a real continuum, and classical modes, which have isolated, and generally complex, frequencies. When a system is time-reversible (lacks rotation) and stable, it can only have van Kampen modes.

The mathematical apparatus used here is simpler and less powerful than that deployed in Paper I. On the plus side, the formulae presented here are somewhat simpler than the corresponding formulae in Paper I. The downsides are (i) that modes do not emerge as eigenfunctions of a Hermitian operator; (ii) we have not shown that all van Kampen modes have positive energy. The lack of a connection to a Hermitian operator deprives us of the ability both to argue for the completeness of the modes and to confine the normal-mode frequencies to the axes of the complex plane as we could in the case of ergodic systems. Nonetheless, at least in the case of systems with \( f_0(J) \) that are unchanged by reversing all velocities, it seems likely that the frequencies are confined in the same way. This topic will be a major theme of Paper IV in this series.

The existence of modes with negative energy would not have important consequences on the dynamical timescale, because modes evolve independently of one another so long as the linear approximation holds. On the longer timescales associated with terms quadratic in the disturbance (the ‘two-body’ timescale), a system is likely to be rendered secularly unstable by the existence of negative-energy modes because non-linear terms could transfer energy from negative-
energy modes to positive-energy terms and thus causing the amplitudes of both types of mode to increase.

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APPENDIX A: THE BILINEAR FORM EVALUATED ON VAN KAMPEN MODES

Here we prove that the bilinear form (27) evaluated on two van Kampen modes can be expressed in the form (35). The product is the sum $K + \frac{1}{2}P'$ of kinetic- and potential-energy terms, where\(^1\)

$$K[f, \tilde{f}] = \frac{(2\pi)^3}{2} \sum_n \int d^3J \frac{n \cdot \Omega}{n \cdot \nabla J f_0} f_n \tilde{f}_n$$

$$P'[f, \tilde{f}] = -\mathcal{E} \sum_a A^a_\alpha \tilde{A}_\alpha.$$  \hspace{1cm} (A1)

Using equation (30) for the DF of a mode, we have

$$K = \frac{(2\pi)^3}{2} \sum_n \int d^3J \frac{n \cdot \Omega}{n \cdot \nabla J f_0}$$

$$\times \left\{ \mathcal{P} \left( \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \right) \phi_n^* + g_n^* \delta(n \cdot \Omega - \omega) \right\}$$

$$\times \left\{ \mathcal{P} \left( \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \right) \tilde{\phi}_n + \tilde{g}_n \delta(n \cdot \Omega - \omega) \right\}. \hspace{1cm} (A2)$$

With identities (B1) and (B2) the product of the two principal values yields

$$\int d^3J \frac{n \cdot \Omega}{n \cdot \nabla J f_0} \mathcal{P}(\cdot) \mathcal{P}(\cdot) = \int d^3J \frac{n \cdot \nabla J f_0 \phi_n \tilde{\phi}_n}{n \cdot \Omega}$$

$$\times \left\{ \mathcal{P} \left( \frac{1}{\omega - \omega} \right) \left\{ \mathcal{P} \left( \frac{\omega}{n \cdot \Omega - \omega} \right) - \mathcal{P} \left( \frac{\omega}{n \cdot \Omega - \omega} \right) \right\} \right\}$$

$$+ \pi^2 \delta(n \cdot \Omega - \omega) \delta(\omega - \omega). \hspace{1cm} (A3)$$

The cross terms yield

$$\int d^3J \left\{ \mathcal{P} \left( \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \right) \phi_n \tilde{\phi}_n \delta(n \cdot \Omega - \omega) \right\}$$

$$+ \mathcal{P} \left( \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \right) \tilde{\phi}_n g_n^* \delta(n \cdot \Omega - \omega) \right\}. \hspace{1cm} (A4)$$

The identity (B3) enables us to rewrite this in the form

$$\int d^3J \mathcal{P} \left( \frac{1}{\omega - \omega} \right) \int d^3J \left\{ \omega \phi_n \tilde{\phi}_n \delta(n \cdot \Omega - \omega) \right\}$$

$$- \omega \phi_n \tilde{\phi}_n \phi_n^* \delta(n \cdot \Omega - \omega) \right\}. \hspace{1cm} (A5)$$

When the cross terms are added to the reduced product of principal values (A3), the coefficient of $\mathcal{P}(1/(\omega - \omega))$ is

$$\left\{ \mathcal{P} \left( \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \right) \phi_n^* + g_n^* \delta(n \cdot \Omega - \omega) \right\} \omega \tilde{\phi}_n$$

$$- \left\{ \mathcal{P} \left( \frac{n \cdot \nabla J f_0}{n \cdot \Omega - \omega} \right) \tilde{\phi}_n + \tilde{g}_n \delta(n \cdot \Omega - \omega) \right\} \omega \phi_n$$

$$= f_n \omega \tilde{\phi}_n - \tilde{f}_n \omega \phi_n. \hspace{1cm} (A6)$$

When this expression is integrated wrt $J$, summed over $n$, and multiplied by $(2\pi)^3$, we obtain $\omega - \omega'$ times the mutual potential energy $P'$ (eqn. 10). Hence the denominator of the principal value symbol in equation (A5) is cancelled. Gathering together the contributions to $K$, we have this

\(^1\) $P'$ is the potential energy of the system defined by $f$ in the potential generated by $\tilde{f}$ or vice versa [eqn. 10]. The self-energy $P = \frac{1}{2}P'[f, \tilde{f}]$. 
potential-energy term, the product of δ-functions in equation (A3) and a similar term arising from the product of $\tilde{\Phi}_n$ and $\tilde{\Phi}_n$ in equation (A2). Thus

$$K = -\frac{1}{2}P' - \frac{(2\pi)^3}{2} \sum_n \int d^3 \mathbf{J}_\omega \left( \pi^2 \mathbf{n} \cdot \nabla J_0 \Phi_n^* \tilde{\Phi}_n \right)$$

$$+ \left( \mathbf{n} \cdot \mathbf{\Omega} \right) \left( \mathbf{n} \cdot \mathbf{\Omega} - \mathbf{\omega} \right) \delta(\mathbf{n} \cdot \mathbf{\omega}) \delta(\mathbf{\omega} - \tilde{\mathbf{\omega}}).$$  \hspace{1cm} (A7)

Thus $\langle f \tilde{f} \rangle = K + \frac{1}{2}P'$ is indeed given equation (35).

APPENDIX B: IDENTITIES INVOLVING PRINCIPAL VALUES

Here we list three identities that are required in Appendix A. For proofs see (e.g. Ramos & White 2018).

$$P\left( \frac{1}{x - x_1} \right) P\left( \frac{1}{x - x_2} \right) =$$

$$P\left( \frac{1}{x - x_1} \right) \{ P\left( \frac{1}{x - x_1} \right) - P\left( \frac{1}{x - x_2} \right) \}$$

$$+ \pi^2 \delta(x - x_1) \delta(x_1 - x_2),$$ \hspace{1cm} (B1)

$$P\left( \frac{x}{x - x_1} \right) = 1 + P\left( \frac{x_1}{x - x_1} \right).$$ \hspace{1cm} (B2)

$$P\left( \frac{x}{x - x_1} \right) \delta(x - x_1) + P\left( \frac{B}{x - x_1} \right) \delta(x - x_2)$$

$$= P\left( \frac{1}{x_1 - x_2} \right) \{ A\delta(x - x_1) - B\delta(x - x_2) \}.$$ \hspace{1cm} (B3)