Motion of membranes in space-times with torsion

Özgür Açık¹, Aytolun Çatakaya¹, Ümit Ertem² and Özgün Sütemen¹

¹ Department of Physics, Ankara University, Faculty of Sciences, 06100, Tandoğan-Ankara, Turkey
² Astronomer, Presidency of Religious Affairs, Üniversiteler Mah. Dumlupınar Bul. No:147/A 06800 Çankaya, Ankara, Turkey

E-mail: ozacik@science.ankara.edu.tr, aytolun.catalkaya@hotmail.com, umitertemm@gmail.com and ozgun.sutemen@gmail.com

Received 8 April 2019, revised 1 August 2019
Accepted for publication 9 August 2019
Published 11 September 2019

Abstract
The motion of membranes interacting with external fields in space-times with curvature and torsion is considered. The intrinsic and extrinsic properties of the immersion are fused together to form a stress tensor for the corresponding material hypersurface. This geometro-elastic stress tensor is part of the total stress tensor which is no longer symmetric or divergence-free because of the presence of torsion. The equation of motion of the membrane is given by equating the total stress tensor to a non-zero value determined by the curvature and torsion of the ambient space-time. Dirac and Önder–Tucker bubbles are considered as special cases. An example of the membrane motion on a manifold admitting a generalized Killing spinor is given.

Keywords: membrane motion, classical immersions, torsion, generalized Killing spinors

1. Introduction

In 1962, Dirac considered an extended model (bubble model) for the electron, the first excited state of which was described as the muon corresponding to the changes of the shape and size of the bubble [1]. Preceding Dirac, the shape and size fixed, extended general relativistic electron was considered by Lees [2], which contextually rules out the excited states of the electron. Dirac had thought that the surface tension of the electron in its rest frame could balance the electrical repulsion; but later, it was found that the Dirac bubble was not stable against quadruple deformations [3]. In another paper, Önder and Tucker generalized Dirac’s...
work by a model that can be put into the perspective of a class of Willmore-type immersions [4]. They added a term to Dirac’s action including the extrinsic curvature of the immersion. Extrinsic curvature terms appear in most theories of relativistic extended objects described by effective Lagrangian actions [5]. In [6], the equation of a membrane interacting with external fields in space-time with curvature was established. Tucker [6] generalizes Dirac’s work in geometrical terms as follows: Dirac encoded the role of surface tension by defining a stress tensor proportional to the induced metric on the hypersurface, though in [6] the additional geometrical term included the shape tensor of the immersion and in this way the external geometrical properties of the membrane become significant. It is important to note that the generality of the form for the stress tensor of the immersion in this model reduces to the model given in [4] for a special choice of geometric invariants of the membrane history. A crucial analysis for both the classical and quantum motion of free relativistic membranes was done in [7]. For the quantum regime, they developed both a covariant formalism with constraints and an alternative theory in terms of independent degrees of freedom. Later, Önder and Tucker investigated the semiclassical dynamics of a charged relativistic membrane that results with a harmonic estimate of radial quantum modes [8]; but their work was refined and improved in [9] by using a classical canonical approach relying on the Ostrogradski–Hamiltonian formalism and then applying Dirac’s constraint quantization scheme.

In this work we aim to generalize [6] by including the space-time torsion and deduce its possible effects on the equation of motion of a relativistic membrane subject to external fields in background spacetimes. Two important aspects of membrane motion are the stability and self-gravity of the membrane. Although we intend to analyze these effects in another work, we propose here that the inclusion of non-zero torsion in the dynamics of the membrane should contribute to the stability of the motion. In support of this idea we would like to point out that spin-generated torsion tends to stabilize compact objects in astrophysics [10]. Apart from astrophysical considerations, our model may be applied as a first approximation (i.e. without backreaction) to the motion of quasiparticles in solids containing dislocations and disclinations or to the motion of extended objects in supergravity theories with torsion. Also, in the self-gravitating case, our membrane may be treated as an element of a (Weyssenoff) spin fluid. Here the main scheme is based on a distribution valued total stress energy momentum tensor which is no longer symmetric or divergence-free because of the presence of torsion. So the equation of motion for the membrane is determined by equating the divergence of the stress tensor to a non-zero value that is constructed out of curvature and torsion of the ambient space-time. Since [6] includes the results of [1] and [4] as special cases, we also investigate the motions of both membranes which we call Dirac bubble and Önder–Tucker bubble respectively. The extra terms coming from the existence of torsion are determined for both cases. We also exemplify our investigations by considering a specific kind of space-time which admits a generalized Killing spinor field and a related non-zero torsion. Then, it is seen that the corresponding equations of motion simplify in terms of the Killing function; so it is easy to see the limiting case for vanishing torsion by taking this function to zero.

The organization of the paper is as follows. In section 2 we introduce the distribution valued total stress tensor and give the necessary differential operations in the language of coordinate-free differential geometry. After defining the equation of motion of the membrane by equating the divergence of the total stress tensor to a non-zero value determined by the curvature and torsion of the background, we deduce the normal and tangential jump conditions for the local physical momentum 1-form current. These conditions are given in terms of the divergence of the hypersurface stress tensor. The (immersed) hypersurface is a time-like 3-manifold corresponding to the space-time history of the membrane. In section 3, a convenient characterisation of the hypersurface stress tensor is given out of the extrinsic and intrinsic
geometric properties of the immersion and accompanying invariant functions. Here the topology of the membrane is restricted to be spherical and this bubble is coupled to an external electromagnetic field. The reduction to the Dirac bubble case is analyzed in section 3.1 and to the Önder–Tucker bubble case in section 3.2. Section 4 is devoted to the application of the previous results in a space-time admitting a generalized Killing spinor. The conclusion of our results constitutes section 5. There are three appendices that we think necessary for the completeness of the paper; where appendix A contains some information about curved space-time distributions, appendix B gives the Gauss–Codazzi formalism for codimension one immersions in the presence of torsion and appendix C contains a brief summary for the theory of Killing spinor bilinears.

2. Stress-energy-momentum tensors with discontinuities

Let $C : D \rightarrow M$ be a time-like 3-dimensional immersion into a 4-dimensional non-Riemannian space-time $(M, g, \nabla)$, where $D$ is some 3-dimensional parameter domain, $g$ is the space-time metric with Lorentzian signature and $\nabla$ the metric compatible connection with torsion. The induced metric and the induced connection on the image $\Sigma$ of $C$ are denoted respectively as $\hat{g}$ and $\hat{\nabla}$. If the local equation of the hypersurface is given by $\Phi = 0$ then the other parts of space-time are labelled by $+$ and $-$ corresponding to $\Phi > 0$ and $\Phi < 0$; so $M = M^+ \cup M^- \cup M^0$ where $M^0 = \Sigma = \partial M^- = -\partial M^+$. The unit space-like normal $N$ of the regular hypersurface $\Sigma$ is metrically related to the 1-form $d\Phi/|d\Phi|$ where $\| \cdot \|$ denotes the norm with respect to $g$. Usually a tilda over a tensor will denote the tensor associated to it by the space-time metric $g$ so that $\hat{N} = d\Phi/|d\Phi|$. For rank 2 covariant tensors we also have other usages for tilda where in a chart $x = (x^\mu)$ if $G = G_{\mu\nu}dx^\mu \otimes dx^\nu$ then we define the associated contravariant rank 2 tensor as $\tilde{G} = G_{\mu\nu}dx^\mu \otimes dx^\nu$. The Heaviside function with support on $\Phi > 0$ is $\Theta^+$ and the scalar Dirac distribution $\delta(\Phi)$ is given by the following relation

$$d\Theta^+ = \delta(\Phi)d\Phi$$

and we suppose that the g-dual of the total stress-energy-momentum tensor takes the form

$$\tilde{T} = \tilde{T}_+\Theta^+ + \tilde{T}_-\Theta^- + \tilde{T}_0\delta(\Phi)$$

such that the coefficient tensors are all smooth [11]. Here total stress-energy-momentum tensor $\tilde{T}$ contains all matter and fields and $\tilde{T}_0$ describes the stress properties of the material hypersurface $\Sigma$.

The covariant divergence of a bidegree $(2,0)$ tensor (i.e. with contravariant rank 2 and covariant rank 0) $\tilde{T}$ is a $(1,0)$ tensor $\nabla \cdot \tilde{T}$ and can be defined as

$$\nabla \cdot \tilde{T} = (\nabla x^\nu)(e^\alpha, -)$$

where $\{X_a\}$ and $\{e^a\}$ are arbitrary dual bases. If $f$ is a scalar on $M$ then

$$\nabla \cdot (f\tilde{T}) = e^a(\nabla \cdot (f\tilde{T}))X_a = (\nabla x^\nu(f\tilde{T})(e^b, e^a))X_a$$

$$= \tilde{T}(df, e^a)X_a + f((\nabla x^\nu(\tilde{T})(e^b, e^a))X_a$$

so we get

$$\nabla \cdot (f\tilde{T}) = \tilde{T}(df, -) + f\nabla \cdot \tilde{T}.$$
\[
\n\nabla \cdot G = -s^{-1} (T^q \wedge *_{Xq} R_{ab} \wedge *e^a) e^p .
\]

Here * is the Hodge map associated with \( g \), \( T^q \)'s are torsion 2-forms and \( R^a_{\mu} \)'s are the curvature 2-forms of \( \nabla \) and \( e^a_{\mu} := e^a \wedge e^\mu \wedge e_\rho \) where \( e_\rho := g_{\rho\mu} e^\mu \) and \( i_X \) is the internal contraction with respect to the frame basis element \( X_\rho \). The equation of motion of a hypersurface with possible (non-gravitational) interactions through the regions + and − may be given by equating the divergence of the dual stress tensor with the metric dual of the righthand side of equation (6) as

\[
\nabla \cdot \tilde{T} = -s^{-1} (T^q \wedge i_{Xq} R_{ab} \wedge *e^a) X_p := \tilde{\beta}
\]

in a background Riemann–Cartan spacetime. It is convenient at this point to say that the torsion may have a spinorial origin such that the spinor field has a vanishing stress tensor; so that it does not contribute to the total stress tensor. Thus if \( \nabla \cdot \tilde{T}_+ = 0 \) in \( M^+ \) and \( \nabla \cdot \tilde{T}_- = 0 \) in \( M^- \) we get

\[
[\tilde{T}](d\Phi, -) - \tilde{\beta} = -(\nabla \cdot \tilde{T}_0)|_{\Sigma}
\]

where naturally

\[
\tilde{T}_0(d\Phi, -)|_{\Sigma} = 0
\]

and \([\tilde{T}] := \tilde{T}_+ - \tilde{T}_-\) is the discontinuity of \( \tilde{T} \) across the hypersurface. Equation (9) comes from the assumption that \( T_0 \) is only defined on the hypersurface. Since \( d\Phi \) is space-like, equation (8) can be interpreted as the jump in the local (physical) momentum current \([J_{0\mu}] = -[T(d\Phi, -)]\) which gives the normal force on the hypersurface. If \( \dot{c} \) is a time-like future pointing observer curve then \( i_c * J \) is the local force 2-form belonging to any local momentum current 1-form \( J \), where \( \dot{c} \) is the observer velocity field and \( i_c \) is the internal contraction with respect to this vector field.

For a general mixed tensor \( S \) we may define \( \Pi_N S \) as

\[
(\Pi_N S)(Y_1, Y_2, ..., \beta_1, \beta_2, ...) = S(\Pi_N Y_1, \Pi_N Y_2, ..., \Pi_N \beta_1, \Pi_N \beta_2, ...)
\]

(10)

where \( \Pi_N = 1 - \{\tilde{N}(N)\}^{-1} \tilde{N} \otimes \tilde{N} \) and \( \Pi_N = 1 - \{\tilde{N}(N)\}^{-1} \tilde{N} \otimes N \) are the (1,1) projection tensors and one should remember that \( \tilde{N}(N) = g(N, N) = 1 \). Here \( Y_i \)'s are vector fields and \( \beta_j \)'s are 1-forms in spacetime. \( \Pi_N S \) will be referred to as being \( g \)-orthogonal to \( N \) or as the hypersurface tensor corresponding to \( S \). So with this definition, equation (9) says that the hypersurface stress-energy-momentum tensor must be \( g \)-orthogonal to its normal vector field, i.e.

\[
\Pi_N \tilde{T}_0 = \tilde{T}_0.
\]

(11)

The \( N \)-decomposition of \( \nabla \cdot \tilde{T}_0 \) is

\[
\nabla \cdot \tilde{T}_0 = \Pi_N(\nabla \cdot \tilde{T}_0) + \tilde{N}(\nabla \cdot \tilde{T}_0) N
\]

(12)

where the first term at the right hand side is \( g \)-orthogonal to \( N \) and the second term is parallel to \( N \). We can choose \( \{X_\mu\} \) and its dual \( \{e^\mu\} \) so that \( X_\mu \) is time-like and \( X_1 = N \) so from (3) we can write \( \tilde{N}(\nabla \cdot \tilde{T}_0) = (\nabla X_\mu \tilde{T}_0)(e^\mu, \tilde{N}) \) and from (11) we deduce

\[
(\nabla X_\mu \tilde{T}_0)(e^\mu, \tilde{N}) = \nabla X_\mu(\tilde{T}_0(e^\mu, \tilde{N})) - \tilde{T}_0(\nabla X_\mu e^\mu, \tilde{N}) - \tilde{T}_0(e^\mu, \nabla X_\mu \tilde{N})
\]

\[
= -\tilde{T}_0(e^\mu, \nabla X_\mu \tilde{N}) = -\tilde{T}_0(e^\mu, \nabla X_\mu \tilde{N})
\]

(13)

\[
(\nabla X_\mu \tilde{T}_0)(e^\mu, \tilde{N}) = \nabla X_\mu(\tilde{T}_0(e^\mu, \tilde{N})) - \tilde{T}_0(\nabla X_\mu e^\mu, \tilde{N}) - \tilde{T}_0(e^\mu, \nabla X_\mu \tilde{N})
\]

\[
= -\tilde{T}_0(e^\mu, \nabla X_\mu \tilde{N}) = -\tilde{T}_0(e^\mu, \nabla X_\mu \tilde{N})
\]
here \( i = 0, 2, 3 \) since \( \Pi_y \{ X_i \} = \{ X_i \} \). From (B3) it is seen that \( \nabla_X N = -A_N X_i \), and also using the metric compatibility \( \nabla g \) we reach

\[
\tilde{N}(\nabla \cdot \tilde{T}_0) = \tilde{T}_0(e^i, A_N X_i),
\]

(14)

and by further manipulation we write \( \tilde{T}_0(e^i, A_N X_i) = T_0(X^i, A_N X_i) \). If we also \( N \)-decompose the local physical momentum 1-form \( J_{d\Phi} \) as

\[
\tilde{N} \left( \nabla \cdot \tilde{T}_0 \right) = \tilde{T}_0(e^i, \tilde{A} N X_i)
\]

(15)

and by further manipulation we write

\[
\tilde{T}_0(e^i, \tilde{A} N X_i) = T_0(X^i, \tilde{A} N X_i).
\]

(16)

and by further manipulation we write

\[
\tilde{N} \left( \nabla \cdot \tilde{T}_0 \right) = \tilde{T}_0(e^i, \tilde{A} N X_i)
\]

(17)

3. Geometric elasticity

A convenient way to assign a tensor \( T_0 \) that satisfies the criterion (11) can be constructed from the first and second fundamental forms of the hypersurface as

\[
T_0 = |d\Phi| \left\{ L_1 \tilde{\Pi} g + L_2 \tilde{\Pi} H \right\}
\]

(18)

where the scalars \( L_i \) are selected as functions of \( \kappa_1, \kappa_2, \kappa_3, \Delta \kappa_1, \Delta \kappa_2, \Delta \kappa_3 \) such that \( \Delta \kappa \) is the hypersurface Laplacian. The invariant quantities \( \kappa_j \) are being taken as the elementary symmetric functions of the Weingarten map eigenvalues. If \( \{ X_i \mid i = 0, 2, 3 \} \) is a local \( g \)-orthonormal basis of tangent vector fields on \( \Sigma \) satisfying \( A_N X_i = \lambda_i X_i \) then \( \kappa_4 = \lambda_0 + \lambda_2 + \lambda_3, \kappa_5 = \lambda_0 \lambda_2 + \lambda_2 \lambda_3 + \lambda_3 \lambda_0 \) and \( \kappa_3 = \lambda_0 \lambda_2 \lambda_3 \) [6]. Because of the presence of torsion, \( T_0 \) will be non-symmetric with respect to its arguments whenever \( L_2 \) is non-vanishing. From now on we take the topology \( S^2 \times \mathbb{R} \) for the parameter domain \( D \) and couple the membrane to the Maxwell field \( F \) by taking \( F_- = 0 \) and \( T_+ = T_{\text{Maxwell}} = T_{ab} e^a \otimes e^b \) where

\[
T_{ab} = -\frac{1}{2} g_{ab} F_+^{cd} F_+^{cd} - F_+^{ab} F_+^{bc},
\]

(19)

where \( F_+ \) is the external Coulomb field of the spherical charged membrane in its proper frame. The proper frame is assumed to be determined by the center of mass of the bubble which is not obviously defined in relativistic space-times either flat or curved [12–14]. With this construction \( M_\pm \) appears as the world tube [15] traced out by the interior region of a space-like bubble. Now we can choose specific forms for \( T_0 \) and investigate the corresponding equations of motion.

3.1. Dirac bubble

For some coupling constant \( \kappa \) we take the hypersurface stress tensor as

\[
T_0 = \kappa |d\Phi| \Pi_y g
\]

\[
= \kappa |d\Phi| (g - \tilde{N} \otimes \tilde{N})
\]

(20)
and calculate
\[
(\nabla \cdot T_0)(N) = \kappa \{ g(\dd d\Phi|N) - \dd N(\dd d\Phi)|\dd N(N) \} - \kappa |d\Phi| \nabla \cdot (\dd N \otimes \dd N)(N)
\]  
(21)
where the terms in the curly parenthesis annihilate each other. If we use the definition of the divergence for the remaining term we get
\[
(\nabla \cdot T_0)(N) = -\kappa |d\Phi| \{ d\Phi[N] + g(N, X^a)\dd N(N) \}
\]  
(22)
where \(X^a := g^{ab} X^b\) and further manipulation gives
\[
(\nabla \cdot T_0)(N) = -\kappa |d\Phi| \{ g(\dd X^a, N^a) + g(\dd N, N^a) \}.
\]  
(23)
Since \(g(\dd N, N) = 0\) and \(g(\dd X^a, N) = -g(\dd N, \dd X^a) = -g(\dd N, \dd \Pi N X^a)\) we get
\[
(\nabla \cdot T_0)(N) = \kappa |d\Phi| \{ h(\dd \Pi N X^a) + g(\dd N, N^a) \} = \kappa |d\Phi| H(\Pi N X^a, \Pi N X^a).
\]  
(24)
Remembering that the mean curvature normal of the hypersurface is \(\eta := \frac{1}{3} h(\Pi N X^a, \Pi N X^a) = \frac{4}{3} \mathcal{H} N\) and with a little algebra
\[
[J_{\Phi\Phi}](N) = \frac{1}{2} |d\Phi| \{ ( -i_N F_+ \wedge *i_N F_+ + F_+ \wedge \dd N \wedge *(F_+ \wedge \dd N) ) \}.
\]  
(25)
Now we can use the formula for any \(p\)-forms \(\alpha\) and \(\beta\) and vector fields \(X\) and \(Y\)
\[
i_X \alpha \wedge *i_Y \beta = g(X, Y) g_p(\alpha, \beta) * 1 - (\alpha \wedge \dd Y) \wedge *(\beta \wedge \dd X)
\]  
(26)
for further manipulation and use the fact that \(F_+ \wedge \dd N\) is vanishing for spherical topology in bubble’s (local) rest frame. Also using \(* 1 = -1\) for Lorentzian signature in four dimensions and \(g_2(F_+, F_+) := *^{-1}(F_+ \wedge *F_+)\), then the equation of motion for our membrane is given by
\[
\kappa \mathcal{H} + *^{-1}(T^a \wedge i_X R_{ij} \otimes *e^a) / |d\Phi| = *^{-1}(F_+ \wedge *F_+) / 2.
\]  
(27)
This equation is the generalization of Dirac’s result \cite{1} to the case for the existence of torsion.

3.2. Önder–Tucker bubble

This time we set the hypersurface stress tensor to be
\[
T_0 = \kappa |d\Phi| \{ \text{Tr}(A_N) \dd \Pi N \dd H \}
\]  
(28)
which also includes the second fundamental form of the immersion, and this is not a symmetric term in the presence of torsion. Evaluation of the stress tensor on two vector fields \(X\) and \(A_N Y\) gives
\[
T_0(\Pi N X, A_N \Pi N Y) = \kappa |d\Phi| \{ \text{Tr}(A_N) g(\Pi N X, A_N \Pi N Y) - H(\Pi N X, A_N \Pi N Y) \}
\]  
(29)
Let us contract the first and the last arguments of the Riemann tensor,
\[
Ric(\Pi N Y, \Pi N X) = R(X^a, \Pi N Y, \Pi N X, X^a)
\]  
(30)
\[
= R(\Pi N X^a, \Pi N Y, \Pi N X^a, \Pi N X^a) + R(N, Y, X, N)
\]
from Gauss equation (B9). The first term at the right hand side can be decomposed as
\[
\tilde{Ric}(\Pi_X Y, \Pi_X X) + g(h(\Pi_X X^\flat), h(\Pi_X Y, \Pi_X X))
\]
\[\quad - g(h(\Pi_X Y, \Pi_X X), h(\Pi_X X^\flat, \Pi_X X)). \tag{31}\]

Let us analyze some terms separately. First recall that,
\[
g(h(\Pi_X Y, \Pi_X X), h(\Pi_X X^\flat, \Pi_X X_a)) = H(\Pi_X X^\flat, \Pi_X X_a) g(h(\Pi_X Y, \Pi_X X), N). \tag{32}\]

Since Tr(\(A_N\)) = \(\mathcal{H}\), it can be rewritten as Tr(\(A_N\))g(\(A_N\Pi_X Y, \Pi_X X\)) and this is equal to Tr(\(A_N\))\(\tilde{\Pi}_N g(A_N Y, X)\). So
\[
g(h(\Pi_X Y, \Pi_X X), h(\Pi_X X^\flat, \Pi_X X_a)) = \text{Tr}(A_N)\tilde{\Pi}_N g(A_N Y, X). \tag{33}\]

We can also expand \(A_N \Pi_X X\) as \(g(A_N \Pi_X X, \Pi_X X^\flat)\Pi_X X_a = g(h(\Pi_X X, \Pi_X X^\flat), N)\Pi_X X_a\) and similarly \(A_N \Pi_X Y = g(h(\Pi_X Y, \Pi_X X^\flat), N)\Pi_X X_a\), so we get
\[
g(A_N \Pi_X X, A_N \Pi_X Y) = g(h(\Pi_X X, \Pi_X X^\flat), h(\Pi_X Y, \Pi_X X_a)). \tag{34}\]

From (B11) the last equality can be expanded further as
\[
g(A_N \Pi_X X, A_N \Pi_X Y) = g(h(\Pi_X X^\flat, \Pi_X X), h(\Pi_X Y, \Pi_X X_a))
\[
+ g(T(\Pi_X X, \Pi_X X^\flat), h(\Pi_X Y, \Pi_X X_a)). \tag{35}\]

As a result, we find that
\[
\text{Tr}(A_N)\tilde{\Pi}_N g(A_N Y, X) - \tilde{\Pi}_N g(A_N X, A_N Y) = \tilde{Ric}(\Pi_X Y, \Pi_X X) - \tilde{Ric}(\Pi_X Y, \Pi_X X)
\]
\[+ g(R(N, \Pi_X Y)\Pi_X X_a, N) + g(T(\Pi_X X_a, \Pi_X X), h(\Pi_X Y, \Pi_X X_a^\flat)). \tag{36}\]

where the left hand side is easily seen to be \(T_0(X, A_N Y)/(\kappa|d\Phi|)\) and hence we get
\[
(\nabla \cdot T_0)(N) = \kappa|d\Phi| \{\tilde{\mathcal{R}} + \mathcal{R} + \tilde{Ric}(N, N) + g(R(N, \Pi_X X^\flat)\Pi_X X_a, N)
\]
\[+ g(T(\Pi_X X_a, \Pi_X X), \tilde{\Pi}_N h(X^\flat, X^\flat))\}. \tag{37}\]

Since \(g(\Pi_X X_a, N) = 0\) it follows that \(R(N, \Pi_X X^\flat)\Pi_X X_a, N) = 0\) which implies that
\[
g(R(N, \Pi_X X^\flat)\Pi_X X_a, N) = -g(\Pi_X X_a, R(N, \Pi_X X^\flat)N) = Ric(N, N). \]

So, the equation of motion of the membrane coupled to the Maxwell field in the presence of torsion is
\[
\kappa\{\tilde{\mathcal{R}} - \mathcal{R} + 2\tilde{Ric}(N, N) + g(\Pi_X T(X_a, X_b), \Pi_X h(X^\flat, X^\flat))\}
\[+ \ast^{-1}(T^a \wedge i_{X_a} R_g \wedge \ast^e h)/|d\Phi| = \ast^{-1}(F_+ \wedge \ast F_+)/2. \tag{38}\]

This is the equation of motion found by Önder and Tucker [4], generalized by the addition of torsion.

4. The investigation of membrane motions in a specific non-Riemannian space-time

We assume that our ambient space-time admits a general kind of Killing spinor field \(\psi \in \Gamma S^C M\) satisfying
\[
\nabla_X \psi = \frac{\alpha}{2} \tilde{X} \circ \psi \quad \forall X \in \Gamma TM \tag{39}\]
where $\alpha$ is a holomorphic function on $M$ and $S^C M$ is the complex spinor bundle and $\circ$ is the Clifford product \cite{16}. The smooth sections of the spinor bundle admit a Hermitian symmetric inner product $(.,.)$ with adjoint involution $\xi$ that is the main involutary anti-automorphism of the complex Clifford bundle which reverses the order of multiplication, so its action on a $p$-form $\omega$ is $\omega^\xi = (-1)^{(p/2)}\omega$ \cite{17}. Keeping in mind the action of the co-derivative $\delta$ on a Clifford form $\Phi$ in the presence of torsion as

$$\delta \Phi = -i \nabla_X \Phi + \frac{1}{2} i \nabla_X i_X T^a \iota_X (\Phi \wedge e_a)$$  \hspace{1cm} (40)$$

then it is easy to find the differential equations satisfied by the bilinears of the Killing spinor in the presence of torsion which is discussed in appendix C \cite{18,19}. For even values of $p$ these are given by

$$\nabla_X (\psi \overline{\psi})_p = \alpha i_X (\psi \overline{\psi})_{p+1}$$  \hspace{1cm} (41a)$$

$$d(\psi \overline{\psi})_p = \alpha (p + 1)(\psi \overline{\psi})_{p+1} + T^a \wedge i_X (\psi \overline{\psi})_p$$  \hspace{1cm} (41b)$$

$$\delta(\psi \overline{\psi})_p = -\frac{1}{2} (i_X i_X T^a)(i_X i_X (\psi \overline{\psi})_p \wedge e_b + (i_X i_X T^a)i_X (\psi \overline{\psi})_p)$$  \hspace{1cm} (41c)$$

and for odd values of $p$ they are

$$\nabla_X (\psi \overline{\psi})_p = \alpha e_a \wedge (\psi \overline{\psi})_{p-1}$$  \hspace{1cm} (42a)$$

$$d(\psi \overline{\psi})_p = T^a \wedge i_X (\psi \overline{\psi})_p$$  \hspace{1cm} (42b)$$

$$\delta(\psi \overline{\psi})_p = -\alpha (n - p + 1)(\psi \overline{\psi})_{p-1} - \frac{1}{2} (i_X i_X T^a)(i_X i_X (\psi \overline{\psi})_p \wedge e_b - (i_X i_X T^a)i_X (\psi \overline{\psi})_p).$$  \hspace{1cm} (42c)$$

We focus on the scalar and 1-form part of the inhomogeneous bilinear and if we denote the vector field corresponding to $\psi$ by $V_\psi$ then the following equations are immediate

$$d(\psi, \psi) = \alpha V_\psi$$  \hspace{1cm} (43a)$$

$$\delta(\psi, \psi) = 0$$  \hspace{1cm} (43b)$$

and

$$dV_\psi = T^a \wedge i_X V_\psi$$  \hspace{1cm} (44a)$$

$$\delta V_\psi = -4\alpha (\psi, \psi) - i_{V_\psi} i_X T^a.$$  \hspace{1cm} (44b)$$

The scalar bilinear $(\psi, \psi)$ is real and from $V_\psi = (\psi, e_a \circ \psi) e^a = (e_a \circ \psi, \psi) e^a = (\psi, e_a \circ \psi)^* e^a = \overline{V_\psi}$, it is seen that the 1-form part is also real which means that $\alpha$ is real too. As an application of our model, we consider a very simple background space-time with curvature and torsion. So, to give a spinorial origin to the torsion, we select the torsion 2-forms so that $T^a = e^a \wedge \frac{\omega}{\alpha}$ where $\alpha$ is the (Killing) function appearing in the Killing spinor equation. It is clear from equations (43a) and (44a) that this function also couples the scalar and

---

3 In \cite{16} the Killing function is considered as a pure imaginary function due to the sign convention in the Clifford algebra identity $e^a e^a + e^a e^a = -2 g^{ab}$. However our sign convention is $e^a e^a + e^a e^a = +2 g^{ab}$ which gives a real function.
the co-vector parts of the spinor bilinear. This form of torsion is the one that appears in the
generalized Brans–Dicke theory of gravity [20], where the Brans–Dicke scalar field that
corresponds to the function \( \alpha \) induces the torsion field. While in the generalized Brans–Dicke theory the scalar field has its own dynamics as a self-interacting field coupled to gravity via
the Einstein–Cartan field equations, in our case it is just the Killing function. It is interesting
to see that by this choice of torsion, equations (43a) and (44a) are consistent. The consistency
of equations (43a) and (44a) can be seen from

\[
d \tilde{V}_\psi = T^a \wedge i_{X_a} \tilde{V}_\psi = e^a \wedge \frac{d \alpha}{\alpha} \wedge i_{X_a} \tilde{V}_\psi = -\frac{d \alpha}{\alpha} \wedge \tilde{V}_\psi
\]

so, \( \alpha d \tilde{V}_\psi + d \alpha \wedge \tilde{V}_\psi = 0 \) which means that \( d(\alpha \tilde{V}_\psi) = 0 \) and this is in harmony with what we have in (43a).

Equation (44b) can be written as \( \alpha \delta(\tilde{V}_\psi) = -4(\psi, \psi) - 3V_\psi(\alpha) \) and it

\[
d \tilde{V}_\psi = T^a \wedge i_{X_a} \tilde{V}_\psi = e^a \wedge \frac{d \alpha}{\alpha} \wedge i_{X_a} \tilde{V}_\psi = -\frac{d \alpha}{\alpha} \wedge \tilde{V}_\psi
\]

(45)

and we obtain

\[
\delta(\alpha \tilde{V}_\psi) = -\alpha i_{X_a} \nabla_{X_a} (\alpha \tilde{V}_\psi) - 4i_{\tilde{d} \alpha} \tilde{V}_\psi.
\]

(47)

From (42a), for \( p = 1 \) we reach \( \delta(\alpha \tilde{V}_\psi) = -4(\alpha^2 \psi, \psi) + i_{\tilde{d} \alpha} \tilde{V}_\psi \) and finally we have

\[
\delta(\alpha \tilde{V}_\psi) = \alpha \delta \tilde{V}_\psi - 4g(d \alpha, \tilde{V}_\psi).
\]

(48)

We will also assume isotropic curvature 2-forms \( R^a_{bc} = fe^a \wedge e_b \) for simplicity, where \( f \) is a function. From the known action of curvature operator on spinor fields [17] we get

\[
R(X_c, X_d) \psi = -\frac{1}{4} i_{X_a} R_{ab} e^a \circ \psi = -\frac{f}{4} (g_{ab} g_{cb} - g_{ab} g_{ca}) e^a \circ \psi = -\frac{f}{2} e_{cd \circ \psi}
\]

(49)

and also from equation (39) we have

\[
R(X_c, X_d) \psi = \frac{1}{2} [X_c(\alpha) e_d - X_d(\alpha) e_c] \circ \psi + \alpha^2 e_{cd \circ \psi}
\]

(50)

and as a result we deduce a constraint equation

\[
\frac{1}{2} [X_c(\alpha) e_d - X_d(\alpha) e_c] \circ \psi = 0
\]

(51)

and the equality

\[
f = -\alpha^2.
\]

(52)

Equation (51) also implies that \( d \alpha \circ \psi = 0 \). From the considerations above, we can prominently reduce the equations of motion of our relativistic membrane step by step and can comment on the results.

First it should be useful to see that with our choice of torsion, the second fundamental form is still symmetric: For (B11) we have

\[
g(T(X_c, X_d), N) = H(X_c, X_d) - H(X_d, X_c)
\]

(53)
and since $\widetilde{N} = e^1$ we can write
\[
H_{ij} - H_{ji} = e^1(T(X_i, X_j)) = 2T^1(X_i, X_j) = i_{X_i}i_{X_j}(e^1 \wedge \frac{\partial \alpha}{\partial x^k}) = 0
\]  \hfill (54)
and hence we get the claim. It is also important to note that the covariant divergence of the total stress-energy-momentum tensor is exact
\[
\nabla \cdot T = -s^{-1}(T^i \wedge i_{X_i}R_{ab} \wedge s(e^{ab\rho})e_{\rho}) = -\alpha s^{-1}(\partial X \wedge \partial \alpha \wedge \partial e_{ab} \wedge \partial (e^{ab\rho})e_{\rho}) = d(-6\alpha^2).
\]  \hfill (55)
Note that the last term at the left hand side of both equations (27) and (38) is $i_N(\nabla \cdot T) = -6N(\alpha^2)$. So, the equation of motion in the presence of torsion and curvature for the Dirac bubble coupled to an electromagnetic field is
\[
\kappa \mathcal{H} + 6N(\alpha^2)/|d\Phi| = |F_+|^2/2.
\]  \hfill (56)
To reach the equation of motion of the Önder–Tucker bubble in this non-Riemannian spacetime we have to manipulate the terms in equation (38) separately. From (B9) we can contract the Gauss equation to get
\[
\text{Ric}(Y, Z) = \widehat{\text{Ric}}(Y, Z) + R(N, Y, Z, N) + H(Y, X_i)H(X^i, Z) - H(Y, Z)\mathcal{H}
\]  \hfill (57)
and one more contraction yields the Codazzi relation
\[
\mathcal{R} = \widehat{\mathcal{R}} + 2\text{Ric}(N, N) + H_{ij}H^{ij} - \mathcal{H}^2.
\]  \hfill (58)
This equation accounts for the first three terms in (29). The next term $g(\hat{\Pi}_i T(X_i, X_j), \hat{\Pi}_j h(X^i, X^j))$ reads $g(T(X_i, X_j), h(X^i, X^j))$ where the indices $i$ and $j$ run through 0, 2, 3. Since $h(X^i, X^j) = H^0/N$ we should use the normal part of $T(X_i, X_j) = \nabla X_i X_j - \nabla X_j X_i - [X_i, X_j]$; that is $T(X_i, X_j)^\perp = (H_{ij} - H_{ji})/N$ which vanishes according to our choice of the torsion tensor; these follow from the two equations (53) and (54). So, the equation of motion in the presence of torsion and curvature for the Önder–Tucker bubble coupled to an electromagnetic field is
\[
\kappa(\mathcal{H}^2 - H^2) + 6N(\alpha^2)/|d\Phi| = |F_+|^2/2,
\]  \hfill (59)
where $H^2 = H_{ij}H^{ij}$. This could also be rewritten by using the Codazzi relation (58) as
\[
\kappa(\widehat{\mathcal{R}} + 6\alpha^2) + 6N(\alpha^2)/|d\Phi| = |F_+|^2/2.
\]  \hfill (60)

5. Conclusion

We have analyzed the equations of motion of membranes under the action of external fields and under the influence of curvature and torsion. Our results specifically generalize the pioneering work of Dirac and also the invaluable work of Önder and Tucker for the terms arising from the existence torsion. Although Dirac’s work and also Önder and Tucker’s work were based on an action functional, our treatment assumed a total stress tensor for determining the motion of membranes. We also restricted our attention to four dimensional space-times and codimension one immersions; so the possible contributions that could come from the normal bundle connection forms disappeared automatically. Selecting a spherical topology for the membrane and coupling it to an external electromagnetic field and then fixing the geometric stress tensor of the time-like hypersurface have given equations of motion with extra curvature and torsion terms in comparison to Dirac’s and Önder and Tucker’s work respectively.

In Dirac’s paper the Coulombic stresses were balanced by the extrinsic curvature of the immersion, though in Önder and Tucker’s work these stresses were balanced by the intrinsic...
(Gaussian) curvature of the immersion. In our work there are also torsion terms, where in the absence of electromagnetic coupling, could counteract the remaining terms and stabilize the membrane dynamics in a different manner. For Riemann–Cartan backgrounds admitting generalized Killing spinors we have used the Killing function as the source of torsion and reached the equations dependent on this function. It is clear from the equations of motion that if the gradient of Killing function has no normal component with respect to the hypersurface, our result reduces to that of Dirac’s; but on the contrary the reduction to Önder and Tucker’s case is impeded by the remaining quadratic term of this function.

The next task should be to answer some important open questions in this setup. A Lagrangian formulation of this construction is crucial, which in the classical domain may generate the form of the hypersurface stress tensor by the usual Noetherian procedure. The motion of membranes in higher-dimensional space-times and with higher codimensions would be of interest, so that the contribution coming from the existence of the external twist potential could be analyzed and interpreted separately. Other interesting topologies could be selected for the membrane, for example the ones supporting axially symmetric configurations and also the membrane could be coupled to a gravitational field so that the background becomes active. For such a gravitating membrane the distributional Einstein–Cartan equations come into play. Axially symmetric configurations could appear in a rotating framework, where we believe the hypersurface stress tensor \( T_0 \) should be generalized so as to support rotations. Given that the hypersurface stress tensor is related to the stress 2-forms (of the membrane history), it is plausible that a contribution corresponding to the spin 2-forms should be added to the hypersurface stress tensor in this case. Another problem may be to deduce the spin structure of the membrane’s history and investigate the properties of spinor fields on the hypersurface induced from the generalized Killing spinor. These spinor fields may generate energy minimizing calibrations for the hypersurface.

The motion of membranes in the presence of curvature and torsion could also have applications in the field of condensed matter physics. Effective curvature and torsion are generated by disclinations and dislocations on two-dimensional graphene sheets. Since the electronic band structure of graphene is described by the Dirac equation, the presence of effective curvature and torsion changes the band structure by coupling with the Dirac equation. Since the problem in our paper is the motion of two-dimensional membranes in the presence of extrinsic curvature and torsion, the motion of graphene sheets in the presence of extrinsic curvature and torsion in addition to intrinsic disclinations and dislocations could also be investigated by the methods described here.

Acknowledgments

This work is supported by the Scientific and Technological Research Council of Turkey (TÜBİTAK) Research Project No. 118F086.

Appendix A. Curved space-time distributions

All the details of this appendix can be found in the [21]. Let \( M \) be an n-dimensional paracompact manifold without a metric structure then the test functions on \( M \) are smooth functions with compact support. The real or complex valued continuous linear functionals over the space of test functions are called the scalar distributions on \( M \). The effect of the scalar distribution \( D \) on the test function \( p \) is denoted as \( D : p \rightarrow D[p] \). Similarly tensors on \( M \) with compact support are called test tensors and the real or complex valued linear functionals over the space
of these quantities are called tensor distributions with effect \( T : U \to T[U] \); where the tensorial type of \( T \) is dual to that of the tensor \( U \). If \( f \) is a function, \( X \) a vector field, \( S \) a tensor and \( W \) is another tensor that has dual type with respect to \( S \) and \( T \) is a tensor distribution then the following equalities are well defined:

\[
(fT)[U] = T[fU],
\]

\[
(S \otimes T)(W \otimes U) = T[\langle S, W \rangle U],
\]

\[
T_X[U] = T[X \otimes U].
\]

Generally the tensor products of tensor distributions are not well defined. If the local frame \( \{X_a\} \) and the dual co-frame \( \{e^a\} \) are defined in an open subset \( \mathcal{N} \) of \( \mathcal{M} \) then the components of \( T \) are the scalar distributions defined in an obvious manner. If \( \alpha \) is a co-vector distribution then its component distributions are given by

\[
\alpha_a[p] = \alpha[pX_a]
\]

where \( \text{supp}(p) \subset \mathcal{N} \). For example on a test vector \( V \) with \( \text{supp}(V) \subset \mathcal{N} \) the \( \alpha_a[V] = \alpha_a\epsilon^a[V] \) equality holds. The components transform in the usual way under frame changes so tensor distributions can equally be regarded as distribution-valued tensors. As an additional structure if there is a globally defined n-form (volume form) \( \omega \) on \( \mathcal{M} \) then a distribution \( \hat{S} \) can be defined by

\[
\hat{S}[U] = \int_{\mathcal{M}} \langle S, U \rangle \omega.
\]

Let \( \mathcal{M} \) be orientable and be divided into two disjoint open regions \( \mathcal{M}^+ \) and \( \mathcal{M}^- \) by an (n-1)-dimensional submanifold \( \Sigma \), defined locally by the equation \( \Phi = 0 \) with \( d\Phi \neq 0 \) in a neighborhood of \( \Sigma \). The orientation is fixed by \( \Sigma = \partial \mathcal{M}^- = -\partial \mathcal{M}^+ \). While the Heaviside scalar distribution is defined as

\[
\Theta^\pm[p] = \int_{\mathcal{M}^\pm} p \omega
\]

the Dirac 1-form distribution is defined by

\[
\delta[V] = \int_{\Sigma} i_V \omega.
\]

The Leray form \( \sigma \) is defined by the equality \( \omega = d\Phi \wedge \sigma \) in a neighborhood of \( \Sigma \) and this makes it possible to define the usual scalar Dirac distribution as

\[
\delta(\Phi)[p] = \int_{\Sigma} p \sigma.
\]

One can show that \( \delta = \delta(\Phi)d\Phi \) and \( d\Theta^\pm = \pm \delta \).

A function \( f \) on \( \mathcal{M} \) is regularly \( C^k \) discontinuous at \( \Sigma \) if \( f \) and its first \( k \) derivatives are continuous on \( \mathcal{M}^\pm \) and converge uniformly to limits \( f_{\Sigma}^\pm \) etc. at \( \Sigma \). A regularly discontinuous tensor \( S \) is one whose components in any given chart intersecting \( \Sigma \) are regularly discontinuous functions. The discontinuity \( [S] \) of \( S \) is an ordinary continuous tensor over \( \Sigma \subset \mathcal{M} \) defined by

\[
[S] = S_{\Sigma}^+ - S_{\Sigma}^-.
\]

Let \( \hat{f} \) be the distribution related to a regularly discontinuous function \( f \). If \( f^\pm \) are arbitrary smooth extensions of \( f|_{\mathcal{M}^\pm} \) to \( \mathcal{M} \) then
\[ \hat{f} = \Theta^+ f^+ + \Theta^- f^- . \] (A.10)

It follows that
\[ \hat{d}f = \Theta^+ df^+ + \Theta^- df^- + [f] \delta, \] (A.11)

where \([f]\) is continuous on \(\text{supp}(\delta) = \Sigma\). For a smooth \(\nabla\) and regularly discontinuous tensor \(S\) we have
\[ \nabla \hat{S} = \Theta^+ \nabla S^+ + \Theta^- \nabla S^- + \delta \otimes [S], \] (A.12)

and if both \(\nabla\) and \(S\) are regularly discontinuous then
\[ \nabla \hat{S} = \Theta^+ \nabla S^+ + \Theta^- \nabla S^- + \delta \otimes [S]. \] (A.13)

**Appendix B. Gauss–Codazzi formalism with torsion**

Here, we give the embedding equations for a hypersurface into a space-time with torsion. If \((M, g, \nabla)\) is the ambient space-time, then the structure of the regular hypersurface is given by the triple \((\Sigma, \hat{g}, \hat{\nabla})\) where \(\hat{g} = \Pi_N g\) is the induced metric (the first fundamental form) which is compatible with the induced connection \(\hat{\nabla}\). The Gauss formula yields for a pair of tangent vector fields \(X, Y \in \Gamma \Sigma\)
\[ \nabla_X Y = \hat{\nabla}_X Y + h(X, Y) \] (B.1)

where \(h = H N\) is the shape operator of the immersion, \(H\) its second fundamental form and \(N\) is the unit normal field. The Weingarten formula generally is given by
\[ \nabla_X N = -A_N X + \nabla^\perp_X N \] (B.2)

where \(A_N\) is the Weingarten map satisfying \(g(h(X, Y), N) = g(A_N X, Y) = H(X, Y)\) and \(\nabla^\perp\) is the induced connection on the normal bundle. We will see that in the presence of torsion the Weingarten map will lose its symmetry as an endomorphism on each tangent space for the hypersurface. For the special case where \(N\) has constant norm then the term \(\nabla^\perp_X N\) vanishes automatically. So the Weingarten formula reads
\[ \nabla_X N = -A_N X. \] (B.3)

If \(R\) is the curvature operator and \(T\) torsion operator of \(\nabla, \hat{R}\) the curvature operator and \(\hat{T}\) the torsion operator of \(\hat{\nabla}\) then one can easily find the following equations
\[ (R(X, Y)Z)\| = \hat{R}(X, Y)Z + H(X, Z)A_N Y - H(Y, Z)A_N X \] (B.4)

and
\[ (R(X, Y)Z)^\perp = \{(\nabla_X H)(Y, Z) - (\nabla_Y H)(X, Z) + H(\hat{T}(X, Y), Z)\} N + H(Y, Z)\nabla^\perp_X N - H(X, Z)\nabla^\perp_Y N \] (B.5)

for non-constant \(g(N, N)\) and also from
\[ T(X, Y) = \hat{T}(X, Y) + h(X, Y) - h(Y, X) \] (B.6)

we reach
\[ (T(X, Y))\| = \hat{T}(X, Y) \] (B.7)
and

\[(T(X, Y))^\perp = h(X, Y) - h(Y, X).\]  \hspace{1cm} (B.8)

So the Gauss equation is

\[R(X, Y, Z, W) = \tilde{R}(X, Y, Z, W) + g(h(Y, W), h(X, Z)) - g(h(X, W), h(Y, Z))\]  \hspace{1cm} (B.9)

which is left unchanged in the presence of torsion where \(R(\tilde{R})\) is the Riemann tensor of \(M(\Sigma)\).

The Codazzi equation becomes (with unit \(N\))

\[R(X, Y, Z, W) = (\nabla_X H)(Y, Z) - (\nabla_Y H)(X, Z) + H(\tilde{T}(X, Y), Z)\]  \hspace{1cm} (B.10)

and also we have

\[g(T(X, Y), N) N = h(X, Y) - h(Y, X).\]  \hspace{1cm} (B.11)

**Appendix C. Bilinears of Killing spinors**

An inner product \((, )\) can be defined on the space of spinors depending on the involutions of the Clifford algebra. Spinor bilinears correspond to different degree differential forms that are constructed out of the spinor inner product. For a spinor \(\psi\) and the dual spinor \(\bar{\psi}\), their tensor product \(\psi \otimes \bar{\psi}\) is an endomorphism on the spinor space and can be written as a sum of differential forms as follows

\[
\psi \otimes \bar{\psi} = (\psi, \psi) + (\psi, e_a \circ \psi)e^a + (\psi, e_b \circ e_a \circ \psi)e^b + ... + (\psi, e_a \circ ... \circ e_{a_1} \circ e_{a_2} \circ \psi)e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_p} + ... + (-1)^{[n/2]}(\psi, z \circ \psi)z
\]  \hspace{1cm} (C.1)

where \(z\) is the volume form. The \(p\)-form bilinear \((\psi \bar{\psi})_p\) of a spinor \(\psi\) is defined as the \(p\)-form projection of \(\psi \otimes \bar{\psi}\) and is equal to

\[(\psi \bar{\psi})_p = (\psi, e_{a_1} \circ ... \circ e_{a_p} \circ \psi) e^{a_1} \wedge e^{a_2} \wedge ... \wedge e^{a_p}.\]  \hspace{1cm} (C.2)

For a Killing spinor \(\psi\) satisfying (39), we can find the special equations satisfied by the bilinear forms. The covariant derivative \(\nabla\) is compatible with the projection and duality operations and we have

\[\nabla_X (\psi \bar{\psi})_p = (\nabla_X \psi \bar{\psi})_p + (\psi \nabla_X \bar{\psi})_p.\]  \hspace{1cm} (C.3)

By using (39) and the relations between the Clifford product \(\circ\) and the wedge and interior products for the co-frame basis \(e^a\) and any Clifford form \(\omega\)

\[e^a \circ \omega = e^a \wedge \omega + i_X \omega\]  \hspace{1cm} (C.4)

\[\omega \circ e^a = e^a \wedge \eta \omega - i_X \eta \omega\]  \hspace{1cm} (C.5)

one can find the equation satisfied by the bilinear \(p\)-forms for the symmetric inner product with involution \(\xi\) in the following form

\[\nabla_X (\psi \bar{\psi})_p = \alpha \xi_{\psi}(\psi \bar{\psi})_{p+1}\]  \hspace{1cm} (C.6)

for \(p\) even and

\[\nabla_X (\psi \bar{\psi})_p = \alpha e_a \wedge (\psi \bar{\psi})_{p-1}\]  \hspace{1cm} (C.7)
for $p$ odd; here $\eta$ is the automorphism of the Clifford algebra which acts on a homogeneous element $\sigma$ of degree $p$ as $\eta \sigma = (-1)^p \sigma$. From the definitions of the exterior and co-derivatives in the presence of torsion

$$d \omega = e^a \wedge \nabla_a \omega + T^a \wedge i_X \omega$$

$$\delta \omega = -i_X \nabla_X \omega + \frac{1}{2} (i_X i_X T^a) i_X i_X (\omega \wedge e_a)$$

the following equalities satisfied by the bilinear $p$-forms of Killing spinors can be found by using (C6) and (C7)

$$d(\psi \overline{\psi})_p = \alpha (p+1)(\psi \overline{\psi})_{p+1} + T^a \wedge i_X (\psi \overline{\psi})_p$$

$$\delta(\psi \overline{\psi})_p = -\frac{1}{2} (i_X i_X T^b) (i_X i_X (\psi \overline{\psi})_p \wedge e_b + (i_X i_X T^a) i_X (\psi \overline{\psi})_p$$

for $p$ even and

$$d(\psi \overline{\psi})_p = T^a \wedge i_X (\psi \overline{\psi})_p$$

$$\delta(\psi \overline{\psi})_p = -\alpha (n-p+1)(\psi \overline{\psi})_{p-1} - \frac{1}{2} (i_X i_X T^b) (i_X i_X (\psi \overline{\psi})_p \wedge e_b - (i_X i_X T^a) i_X (\psi \overline{\psi})_p$$

for $p$ odd.

**References**

[1] Dirac P A M 1962 An extensible model of the electron *Proc. R. Soc. A* 268 57–67
[2] Lees A 1938 The electron in classical general relativity theory *Phil. Mag.* 28 385–95
[3] Gnädig P, Kunz Z, Hasenfratz P and Kuti J 1978 Dirac’s extended electron model *Ann. Phys., NY* 116 380–407
[4] Önder M and Tucker R W 1988 Membrane interactions and total mean curvature *Phys. Lett.* B 202 501–4
[5] Gregory R 1991 Effective actions for bosonic topological defects *Phys. Rev. D* 51 5839
[6] Tucker R W 1989 Motion of membranes in spacetime *Conf. on Mathematical Relativity and its Applications, Mathematical Sciences Institute, The Australian National University, Canberra AUS* pp 238–43
[7] Collins P A and Tucker R W 1976 Classical and quantum mechanics of free relativistic membranes *Nucl. Phys.* B 112 150–76
[8] Önder M and Tucker R W 1991 Semiclassical investigation of a charged relativistic membrane model *J. Phys. A: Math. Gen.* 21 3423–9
[9] Cordero R, Molgado A and Rojas E 2011 Quantum charged rigid membrane *Class. Quantum Grav.* 28 065010
[10] Wolf C 1995 The effect of spin generated torsion on the stability of compact astrophysical objects *Acta Phys. Slovaca* 45 583–90
[11] Lichnerowicz A 1980 Relativity and mathematical physics *Relativity, Quanta, and Cosmology* vol 2, ed M Pantaleo and F De Finis (New York: Johnson Reprint Corp) pp 403–72
[12] Giulini D 2015 Energy-momentum tensors and motion in special relativity *Equations of Motion in Relativistic Gravity* (Fundamental Theories of Physics vol 179) ed D Puetzfeld et al (Cham: Springer) (https://doi.org/10.1007/978-3-319-18335-0_3)

[13] Dixon W G 1979 Extended bodies in general relativity: their description and motion *Proc. Int. School of Physics ‘Enrico Fermi’: Isolated Gravitating Systems in General Relativity* ed J Ehlers (Amsterdam: North-Holland) (Course LXVII)

[14] Ehlers J and Rudolph E 1977 Dynamics of extended bodies in general relativity: center of mass description and quasirigidity *Gen. Relativ. Gravit.* **8** 197–217

[15] Gray A 2004 *Tubes* (Basel: Birkhäuser)

[16] Rademacher H-B 1991 Generalized Killing spinors with imaginary Killing function and conformal Killing fields *Global Differential Geometry and Global Analysis* (Lecture Notes in Mathematics vol 1481) ed D Ferus et al (Berlin: Springer) pp 192–8

[17] Benn I M and Tucker R W 1987 *An Introduction to Spinors and Geometry with Applications in Physics* (Bristol: IOP Publishing)

[18] Açık Ö and Ertem Ü 2015 Higher degree Dirac currents of twistor and Killing spinors in supergravity theories *Class. Quantum Grav.* **32** 175007

Aşık Ö and Ertem Ü 2015 Generating dynamical bosons from kinematical fermions CQG + (19 August) (https://cqgplus.com/2015/08/19/generating-dynamical-bosons-from-kinematical-fermions/)

[19] Açık Ö and Ertem Ü 2018 Generalized symmetry superalgebras (arXiv:1806.01079)

[20] Dereli T and Tucker R W 1982 Weyl scalings and spinor matter interactions in scalar-tensor theories of gravitation *Phys. Lett.* **110** 206–10

[21] Hartley D, Tucker R W, Tuckey P A and Dray T 2000 Tensor distributions on signature changing space-times *Gen. Relativ. Gravit.* **32** 491–503

[22] de Juan F, Cortijo A and Vozmediano M 2010 Dislocations and torsion in graphene and related systems *Nucl. Phys.* B **828** 625–37