A unifying approach to constrained and unconstrained optimal reinsurance

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Abstract

In this paper, we study two classes of optimal reinsurance models from perspectives of both insurers and reinsurers by minimizing their convex combination where the risk is measured by a distortion risk measure and the premium is given by a distortion premium principle. Firstly, we show that how optimal reinsurance models for the unconstrained optimization problem and constrained optimization problems can be formulated in a unified way. Secondly, we propose a geometric approach to solve optimal reinsurance problems directly. This paper considers a class of increasing convex ceded loss functions and derives the explicit solutions of the optimal reinsurance which can be in forms of quota-share, stop-loss, change-loss, the combination of quota-share and change-loss or the combination of change-loss and change-loss with different retentions. Finally, we consider two specific cases: Value at Risk (VaR) and Tail Value at Risk (TVaR).

Keywords: Distortion risk measure; Distortion premium principle; Geometric approach; Lagrangian dual method; Increasing convex function; Unconstrained optimization problem; Constrained optimization problem

1 Introduction

Reinsurance is an effective risk management tool for an insurance company. By balancing paid loss and reinsurance premium, the insurance company can control its risk by sharing a portion of loss. Let $X$ be the initial loss by an insurer. Assuming $X$ is a non-negative random variable with cumulative distribution function $F_X(x) = P(X \leq x)$, survival function $S_X(x) = P(X > x)$ and $0 < E[X] < \infty$. In order to avoid serious claims, the insurance company purchases reinsurance from another company and pay a certain amount of expense to the reinsurance company as compensation. Suppose the insurance
company separates the loss function as \( f(X) \), \( 0 \leq f(X) \leq X \), then the insurer retains the loss \( X - f(X) \), denoted as \( I_f(X) \). Let \( \Pi_f(X) \) denotes the reinsurance premium which corresponds to a ceded loss function \( f(X) \). \( T_{I_f}(X) \) and \( T_{R_f}(X) \) represent the total loss of the insurance company and the reinsurance company, respectively. Then we obtain the following relationships:

\[
T_{I_f}(X) = I_f(X) + \Pi_f(X) \quad (1.1)
\]

and

\[
T_{R_f}(X) = f(X) - \Pi_f(X). \quad (1.2)
\]

Let \( T(X) \) represents the convex combination of the total loss of the insurer and the reinsurer, as follows:

\[
T(X) = \beta T_{I_f}(X) + (1 - \beta)T_{R_f}(X), \quad \beta \in [0, 1]. \quad (1.3)
\]

The development of optimal reinsurance has gone through a long period of time. Borch (1960) demonstrated that the stop-loss reinsurance is optimal when the risk of the insurer is measured by variance under the expected value principle. Arrow (1963) showed that the stop-loss reinsurance is optimal when the insurer is an expected utility maximizer under the expected value principle. These basic conclusions have been extended to a number of interesting and important directions. For example, Young (1999), Kaluszka (2001), Kaluszka and Okolewski (2008). Cai and Tan (2007) proposed two optimization criterion that minimize total loss of the insurer by the Value at Risk (VaR) and the Conditional Tail Expectation (CTE). Cai et al. (2008) showed that quota-share and stop-loss reinsurance are optimal when they studied a class of increasing convex ceded loss functions by VaR and CTE under the expected value principle. Cheung (2010), Tan et al. (2011), Chi and Tan (2011), Chi and Tan (2013), Li et al. (2015) extended the fundamental results. Cheung et al. (2014) extended the conclusion of Tan et al. (2011) to the general convex risk measure that satisfied regular invariance. There are many studies about the distortion risk measures and orders, for example, Yin and Zhu (2016), Yin (2018). Chi and Tan (2013), Chi and Weng (2013) studied a class of premium principles which preserve the convex order. Zheng and Cui (2014) discussed the general model of the distortion risk measure and assumed that the distortion function is piecewise convex or concave. Cui et al. (2013) studied the general model with distortion risk measures and distortion premium principles. Cheung and Lo (2015) extended the model of Cui et al. (2013) under the cost-benefit framework. Assa (2015) studied the optimal reinsurance model of Cui et al. (2013) without the premium constraint by a marginal indemnification function (MIF) formula. Zhuang et al. (2016) studied the optimal reinsurance with premium constraint by combining the MIF formula and the Lagrangian dual method. Jiang et al. (2017) studied the Pareto-optimal reinsurance with risk constraints under the distortion risk measure. Motivated by Cai et al. (2008), Zhuang et al. (2016) and Jiang et al. (2017), we want to seek a unified way to solve this class of constrained optimization problems.

The rest of the paper is organized as follows. In Section 2, we give two classes of optimal models: unconstrained and constrained optimal models. Moreover, we propose a unified framework for two classes of problems. In Section 3, we give a geometric approach to solve the object function and derive the optimal reinsurance. In Section 4, we give numerical examples about VaR and TVaR. Section 5 concludes this paper.
2 The model

In this section, we set up the optimal reinsurance model for the insurer and the reinsurer. Moreover, we propose a unified way to solve the unconstrained optimization problem and constrained optimization problems. We start this section from giving a brief description of the distortion risk measure and premium.

2.1 Distortion risk measure and premium

Throughout the paper, we define VaR and TVaR as
\[ \text{VaR}_\alpha(X) = \inf \{ x : P(X > x) \leq \alpha \} \text{ and } \text{TVaR}_\alpha(X) = \mathbb{E}[X \mid X \geq \text{VaR}_\alpha(X)] \].

**Definition 2.1.** A distortion risk measure of a non-negative random variable \( X \) is defined as
\[ \varrho_g(X) = \int_0^\infty g(S_X(x))dx, \quad (2.1) \]
where function \( g : [0,1] \to [0,1] \) is non-decreasing, \( g(0)=0 \) and \( g(1)=1 \).

From the Fubini theorem, we have
\[ \varrho_g(X) = \int_0^1 \text{VaR}_\alpha(X)dg(\alpha). \quad (2.2) \]

**Definition 2.2.** A distortion premium principle of a non-negative random variable \( X \) is defined as
\[ \Pi(X) = (1 + \rho)\varrho_g(X), \]
where \( \rho > 0 \) is the safety loading.

We achieve the expression of reinsurance premium \( \Pi_f(X) \) which corresponds to a ceded loss function \( f(X) \) as follows:
\[ \Pi_f(X) = (1 + \rho)\varrho_g(f(X)). \quad (2.3) \]

If \( g(x) = x \), then the distortion premium principle recovers the expected value principle. If the distortion function is concave with \( \rho = 0 \), then the distortion premium principle recovers Wang's premium principle.

**Remark 2.1.** In this paper, assuming the confidence level of a distortion risk measure is \( 1 - \alpha \) \( (0 < \alpha < 1) \) and the confidence level of a distortion premium principle is \( 1 - \gamma \) \( (0 < \gamma < 1) \). For the convenience of discussion, we give the following definition
\[ K(t) = \frac{g_\alpha(t)}{g_\gamma(t)}, \quad t \in (0,1). \quad (2.4) \]

Note that \( K(t) \) may be convex, concave, piecewise convex or concave, where we only discuss the case that \( K(t) \) is a concave function. Other cases can take the same method to discuss.

In the following subsection, we will start from two optimization problems to study two classes of optimization problems.
2.2 Model setup

Let $\mathcal{H}$ denote a class of ceded loss functions, which consist of all $h(x)$ defined on $[0, \infty)$ with the form

$$h(x) = \sum_{j=1}^{n} C_{n,j} (x - d_{n,j})^+, \quad x \geq 0, \quad n = 1, 2, \ldots,$$

where $C_{n,j} \geq 0$ and $d_{n,j} \geq 0$ are constants such that $0 \leq \sum_{j=1}^{n} C_{n,j} \leq 1, \quad d_{n,1} \leq d_{n,2} \leq \cdots \leq d_{n,n}$.

Let $\mathcal{F} = \{f(x): f(x)$ is increasing convex function with $0 \leq f(x) \leq x$ for $x \in [0, \infty)\}$. Note that $\mathcal{H} \subset \mathcal{F}$.

Assuming that $f^*$ is an optimal reinsurance strategy, from Lemma 3.2 of Cai et al. (2008) and the Fubini Theorem, we have

$$\beta g(\mathbf{T}_I f^*(X)) + (1 - \beta) g(\mathbf{T}_R f^*(X)) = \min_{h \in \mathcal{H}} \{\beta g(\mathbf{T}_I h(X)) + (1 - \beta) g(\mathbf{T}_R h(X))\}. \quad (2.5)$$

Now we consider an idealized case. The insurer and the reinsurer have enough capital, so they do not worry the loss they will bear and the insurer can pay reinsurance premium without the budget constraint when they design the reinsurance contract. In this case, we give the unconstrained optimization model as follows.

**Mode 1 (Unconstrained optimization model)**

$$\min \{g_{g_a}(T(X))\}. \quad (2.6)$$

In realistic insurance application, risk regulators of the insurer and the reinsurer will require that their loss be limited in a range, and the insurer will have a budget constraint for the reinsurance premium. In this case, we give the constrained optimization model as follows.

**Mode 2 (Constrained optimization model)**

$$\begin{align*}
\min \{g_{g_a}(T(X))\}, \\
s.t. \quad g_{g_a}(T_{I_h}(X)) &\leq L_1, \\
g_{g_a}(T_{R_h}(X)) &\leq L_2, \\
\Pi_h(X) &\leq L_3,
\end{align*} \quad (2.7)$$

where $L_1$, $L_2$ and $L_3$ are some monetary levels.

It is important to find a unified approach to address the unconstrained optimization problem and constrained optimization problems. In the next we will show how the optimal reinsurance design for the unconstrained optimization problem and constrained optimization problems can be formulated in the same way.

Denoting the unconstrained optimization model as follows:

$$\min L^1_h(X) \triangleq \min \{g_{g_a}(T(X))\}. \quad (2.8)$$

By the Lagrangian dual method (From Jiang et al. (2017) with the formula (16)), (2.7) can be expressed as

$$\min L^2_h(X) \triangleq \min \{g_{g_a}(T(X)) + \lambda_1 (g_{g_a}(T_{I_h}(X)) - L_1) + \lambda_2 (g_{g_a}(T_{R_h}(X)) - L_2) + \lambda_3 (\Pi_h(X) - L_3)\}. \quad (2.9)$$
where \( \lambda_i > 0 \) for \( i = 1, 2, 3 \). We derive that \((2.8)\) and \((2.9)\) can be represented in the unified form

\[
\min L_h^3(X) \triangleq \min \{ \varrho_{g_a}(T(X)) + \lambda_1(\varrho_{g_a}(T_{I_h}(X)) - L_1) + \lambda_2(\varrho_{g_a}(T_{R_h}(X)) - L_2) + \lambda_3(\Pi_h(X) - L_3) \}, \tag{2.10}
\]

where \( \lambda_i \geq 0, i = 1, 2, 3 \).

From (2.10), we achieve the followings.

Case 1: \( \lambda_1 > 0, \lambda_2 = \lambda_3 = 0 \), which mean that the insurer limits its loss in a range.

Case 2: \( \lambda_2 > 0, \lambda_1 = \lambda_3 = 0 \), which mean that the reinsurer limits its loss in a range.

Case 3: \( \lambda_3 > 0, \lambda_1 = \lambda_2 = 0 \), which mean that the insurer has a reinsurance premium budget constraint.

For example, Zheng et al. (2014) and Zhuang et al. (2016).

Case 4: \( \lambda_1 > 0, \lambda_2 > 0, \lambda_3 = 0 \), which mean that two insurance companies all control their loss in a range. For example, Jiang et al. (2017).

Case 5: \( \lambda_1 > 0, \lambda_3 > 0, \lambda_2 = 0 \), which mean that the insurer has a loss constraint and a reinsurance premium budget constraint.

Case 6: \( \lambda_2 > 0, \lambda_3 > 0, \lambda_1 = 0 \), which mean that the insurer has a reinsurance premium budget constraint and the reinsurer limits its loss in a range.

We know that solving these optimal problems is transformed into solving (2.10). In the next section, we will solve (2.10) by a geometric approach. Before that we conclude this section by introducing the following notations. For \( \beta \in [0, 1] \), \( \lambda_i \geq 0, i = 1, 2, 3 \), we denote

\[
\beta + \lambda_1 = m_1, \quad 2\beta - 1 + \lambda_1 - \lambda_2 = m_2, \quad (1 + \rho)(2\beta - 1 + \lambda_1 - \lambda_2 + \lambda_3) = m_3, \tag{2.11}
\]

\[
\lambda_1L_1 + \lambda_2L_2 + \lambda_3L_3 = D, \quad M = m_3/m_2, \tag{2.12}
\]

\[
\mathcal{X} = \sup \{ X : X \in [0, \infty) \}, \tag{2.13}
\]

\[
\mathcal{K} = \sup \{ K(t) : t \in (0, 1) \}, \quad \mathcal{K} = \inf \{ K(t) : t \in (0, 1) \}, \tag{2.14}
\]

\[
H(x) = m_2g_a(S_X(x)) - m_3g_r(S_X(x)). \tag{2.15}
\]

### 3 Optimal reinsurance contract

In this section, we will derive the solution of these optimal problems. Now, we give the specific expression of (2.10).

From formulas (1.1)-(1.3), we have

\[
\varrho_{g_a}(T(X)) + \lambda_1(\varrho_{g_a}(T_{I_h}(X)) - L_1) + \lambda_2(\varrho_{g_a}(T_{R_h}(X)) - L_2) + \lambda_3(\Pi_h(X) - L_3) = m_1\varrho_{g_a}(X) - m_2\varrho_{g_a}(h(X)) + m_3\varrho_{g_r}(h(X)) - D,
\]

so (2.10) is expressed as

\[
\min L_h^3(X) = \min \{ m_1\varrho_{g_a}(X) - m_2\varrho_{g_a}(h(X)) + m_3\varrho_{g_r}(h(X)) - D \}. \tag{3.1}
\]

With the expression (2.5), we have

\[
L_f^3(X) = \min L_h^3(X). \tag{3.2}
\]

From Cai et al. (2008) with formulas (2.5) and (2.6), we obtain the following lemma.
Lemma 3.1. For any \( h(x) = \sum_{j=1}^{n} C_{n,j}(x-d_{n,j})+ \in \mathcal{H} \) and given the confidence levels \( 1 - \alpha \) with \( 0 < \alpha < S_X(0) \) and \( 1 - \gamma \) with \( 0 < \gamma < S_X(0) \), we obtain

\[
L_h^3(X) = m_1 \int_{0}^{1} S_X^{-1}(t) dg_\alpha(t) - m_2 \left[ \sum_{i=1}^{n-1} \sum_{j=1}^{i} C_{n,j} \int_{S_X(d_{n,i})}^{S_X(d_{n,i+1})} (S_X^{-1}(t) - d_{n,j}) dg_\alpha(t) \right] + \sum_{j=1}^{n} C_{n,j} \int_{0}^{1} (S_X^{-1}(t) - d_{n,j}) dg_\alpha(t) + \sum_{j=1}^{n} C_{n,j} \int_{0}^{1} (S_X^{-1}(t) - d_{n,j}) dg_\gamma(t) - D.
\]

Based on the expression of \( L_h^3(X) \), we analyze its minimum by discussing the magnitude not only of \( m_2 \) and 0, but also of \( M, \overline{K} \) and \( \underline{K} \). The results are summarized in the following lemma.

Lemma 3.2. Given the confidence levels \( 1 - \alpha \) with \( 0 < \alpha < S_X(0) \) and \( 1 - \gamma \) with \( 0 < \gamma < S_X(0) \), for any function \( h(x) = \sum_{j=1}^{n} C_{n,j}(x-d_{n,j})+ \in \mathcal{H} \) with given coefficients \( C_{n,j}, j=1,2,\ldots,n \):

1. If \( m_2 = 0 \), then \( h(x) = 0 \).
2. When \( m_2 \neq 0 \), considering the following cases.
   (i) If \( M \leq \underline{K} \), then
   \[
h(x) = x_{I_{\{m_2>0\}}} \sum_{j=1}^{n} C_{n,j}.
   \]
   (ii) If \( M \geq \overline{K} \), then
   \[
h(x) = x_{I_{\{m_2<0\}}} \sum_{j=1}^{n} C_{n,j}.
   \]
   (iii) When \( \underline{K} < M < \overline{K} \), considering the following six cases.
   Case A. If \( 0 < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) \leq \hat{a} \), then
   \[
h(x) = (x - S_X^{-1}(\hat{a}))_{+} I_{\{m_2<0\}} \sum_{j=1}^{n} C_{n,j}.
   \]
   Case B. If \( \hat{a} \leq S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) \leq \hat{b} \), then
   \[
h(x) = (x - S_X^{-1}(\hat{b}))_{+} I_{\{m_2>0\}} \sum_{j=1}^{n} C_{n,j} + (x - S_X^{-1}(\hat{a}))_{+} I_{\{m_2<0\}} \sum_{j=1}^{n} C_{n,j}.
   \]
   Case C. If \( \hat{b} \leq S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < 1 \), then
   \[
h(x) = (x - S_X^{-1}(\hat{b}))_{+} I_{\{m_2>0\}} \sum_{j=1}^{n} C_{n,j} + x_{I_{\{m_2<0\}}} \sum_{j=1}^{n} C_{n,j}.
   \]
   Case D. If \( 0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) \leq \hat{a} \leq S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,1}) \leq \hat{b} \), where \( k = 2,3,\ldots,n \), then
   \[
h(x) = (x - S_X^{-1}(\hat{b}))_{+} I_{\{m_2>0\}} \sum_{j=1}^{k-1} C_{n,j} + (x - S_X^{-1}(\hat{a}))_{+} I_{\{m_2<0\}} \left[ \sum_{j=1}^{k-1} C_{n,j} + \sum_{j=k}^{n} C_{n,j} \right].
   \]
   Case E. If \( \hat{a} \leq S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,l}) \leq \hat{b} \leq S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1 \), where \( l = 2,3,\ldots,n \), then
   \[
h(x) = (x - S_X^{-1}(\hat{b}))_{+} I_{\{m_2>0\}} \left[ \sum_{j=1}^{l-1} C_{n,j} + \sum_{j=l}^{n} C_{n,j} \right] + x_{I_{\{m_2<0\}}} \left[ x_{I_{\{m_2>0\}}} \sum_{j=1}^{l-1} C_{n,j} + (x - S_X^{-1}(\hat{a}))_{+} I_{\{m_2<0\}} \sum_{j=l}^{n} C_{n,j} \right].
   \]
Case F. If \(0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) \leq \tilde{a} \leq S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,l}) \leq \tilde{b} \leq S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1\), where \(l = 2, 3, \ldots, k-1, k = 3, 4, \ldots, n\), then

\[
h(x) = (x - S_X^{-1}(\tilde{b})) + \sum_{j=1}^{l-1} C_{n,j} + \sum_{j=l}^{k-1} C_{n,j} + \sum_{j=k}^{n} C_{n,j}
\]

\[
= \sum_{j=1}^{l-1} C_{n,j} + (x - S_X^{-1}(\tilde{a}))(\sum_{j=l}^{k-1} C_{n,j} + \sum_{j=k}^{n} C_{n,j})
\]

Proof. By Lemma 3.1, we have

\[
\frac{\partial L^3_h(X)}{\partial d_{n,1}} = C_{n,1} [m_2 g_\alpha(S_X(d_{n,1})) - m_3 g_\gamma(S_X(d_{n,1}))],
\]

\[
\frac{\partial L^3_h(X)}{\partial d_{n,2}} = C_{n,2} [m_2 g_\alpha(S_X(d_{n,2})) - m_3 g_\gamma(S_X(d_{n,2}))],
\]

\[\vdots\]

\[
\frac{\partial L^3_h(X)}{\partial d_{n,n}} = C_{n,n} [m_2 g_\alpha(S_X(d_{n,n})) - m_3 g_\gamma(S_X(d_{n,n}))].
\]

With the expression (2.15), if

\[
\frac{\partial L^3_h(X)}{\partial d_{n,j}} = 0,
\]

then \(H(d_{n,j}) = 0\). Let \(t = S_X(x)\), from (2.14) we know \(K \leq K(t) \leq \overline{K}\) for any \(t \in (0, 1)\). By the definitions of \(K(t)\), \(M\) and \(H(x)\), we derive four cases: if \(m_2 > 0\) and \(K(t) \geq M\), then \(L^3_h(X)\) is increasing; if \(m_2 > 0\) and \(K(t) \leq M\), then \(L^3_h(X)\) is decreasing; if \(m_2 < 0\) and \(K(t) \leq M\), then \(L^3_h(X)\) is decreasing; if \(m_2 < 0\) and \(K(t) \geq M\), then \(L^3_h(X)\) is increasing. In the next we will consider the following possible situations depending on above four cases.

1. If \(m_2 = 0\), then \(m_3 \geq 0\), \(L^3_h(X)\) is decreasing. Thus, the minimum of \(L^3_h(X)\) is attained at \(d_{n,1} = d_{n,2} = \ldots = d_{n,n} = \overline{X}\), \(L^3_h(X) = m_1 g_\alpha(X) - D\), \(h(x) = 0\).

2. When \(m_2 \neq 0\), we consider the following three cases: \(M \leq K\), \(M \geq \overline{K}\) and \(K < M < \overline{K}\).

(1) When \(M \leq K\), for any \(t \in (0, 1)\), \(K(t) \geq M\),

a) if \(m_2 > 0\), then \(L^3_h(X)\) is increasing, and the minimum of \(L^3_h(X)\) is attained at \(d_{n,1} = d_{n,2} = \ldots = d_{n,n} = 0\), so

\[
h(x) = \sum_{j=1}^{n} C_{n,j} x,
\]

\[
L^3_h(X) = m_1 g_\alpha(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_{0}^{\infty} H(x)dx \right] - D.
\]

b) if \(m_2 < 0\), then \(L^3_h(X)\) is decreasing, and the minimum of \(L^3_h(X)\) is attained at \(d_{n,1} = d_{n,2} = \ldots = d_{n,n} = \overline{X}\), so \(L^3_h(X) = m_1 g_\alpha(X) - D\), \(h(x) = 0\).

(2) When \(M \geq \overline{K}\), for any \(t \in (0, 1)\), \(K(t) \leq M\),

a) if \(m_2 > 0\), then \(L^3_h(X)\) is decreasing and the minimum \(L^3_h(X)\) is attained at \(d_{n,1} = d_{n,2} = \ldots = d_{n,n} = \overline{X}\), so \(L^3_h(X) = m_1 g_\alpha(X) - D\), \(h(x) = 0\).
b) if \( m_2 < 0 \), then \( L^3_h(X) \) is increasing and the minimum \( L^3_h(X) \) is attained at \( d_{n,1} = d_{n,2} = \ldots = d_{n,n} = 0 \), so

\[
h(x) = \sum_{j=1}^{n} C_{n,j} x,
\]

\[
L^3_h(X) = m_1 g_{a_1}(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_0^\infty H(x)dx \right] - D.
\]

(3) When \( K < M < \overline{K} \), to denote \( \hat{a} = \min\{ t : K(t) \geq M \} \) and \( \hat{b} = \max\{ t : K(t) \geq M \} \) for \( t \in (0, 1) \).

We obtain that \( K(t) \leq M \) on \( (0, \hat{a}] \) and \( \hat{b}, 1) \), \( K(t) \geq M \) on \( [\hat{a}, \hat{b}] \). If \( m_2 > 0 \), then \( L^3_h(X) \) is decreasing on \( (0, \hat{a}] \) and \( [\hat{b}, 1) \), increasing on \( [\hat{a}, \hat{b}] \); if \( m_2 < 0 \), then \( L^3_h(X) \) is increasing on \( (0, \hat{a}] \) and \( [\hat{b}, 1) \), and decreasing on \( [\hat{a}, \hat{b}] \). In the next, we consider the following six cases depending on the relative locations of \( S_X(d_{n,j}), \hat{a} \) and \( \hat{b} \).

Case A: \( 0 < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) \leq \hat{a} \), which is equivalent to \( d_{n,n} \geq d_{n,n-1} \geq \ldots \geq d_{n,1} \geq S_X^{-1}(\hat{a}) \), in this case \( K(t) \leq M \).

a) if \( m_2 > 0 \), then \( L^3_h(X) \) is decreasing, and the minimum \( L^3_h(X) \) is attained at \( d_{n,1} = d_{n,2} = \ldots = d_{n,n} = \overline{X} \), so \( h(x) = 0 \), \( L^3_h(X) = m_1 g_{a_1}(X) - D \).

b) if \( m_2 < 0 \), then \( L^3_h(X) \) is increasing, and the minimum \( L^3_h(X) \) is attained at \( d_{n,1} = d_{n,2} = \ldots = d_{n,n} = S_X^{-1}(\hat{a}) \), so

\[
h(x) = \sum_{j=1}^{n} C_{n,j} (x - S_X^{-1}(\hat{a}))_+,
\]

\[
L^3_h(X) = m_1 g_{a_1}(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_{S_X^{-1}(\hat{a})}^\infty H(x)dx \right] - D.
\]

Case B: \( \hat{a} \leq S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) \leq \hat{b} \), which is equivalent to \( S_X^{-1}(\hat{a}) \geq d_{n,n} \geq d_{n,n-1} \geq \ldots \geq d_{n,1} \geq S_X^{-1}(\hat{b}) \), in this case \( K(t) \geq M \).

a) if \( m_2 > 0 \), then \( L^3_h(X) \) is increasing, and the minimum \( L^3_h(X) \) is attained at \( d_{n,1} = d_{n,2} = \ldots = d_{n,n} = S_X^{-1}(\hat{b}) \), so

\[
h(x) = \sum_{j=1}^{n} C_{n,j} (x - S_X^{-1}(\hat{b}))_+,
\]

\[
L^3_h(X) = m_1 g_{a_1}(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_{S_X^{-1}(\hat{b})}^\infty H(x)dx \right] - D.
\]

b) if \( m_2 < 0 \), then \( L^3_h(X) \) is decreasing, and the minimum \( L^3_h(X) \) is attained at \( d_{n,1} = d_{n,2} = \ldots = d_{n,n} = S_X^{-1}(\hat{a}) \), so

\[
h(x) = \sum_{j=1}^{n} C_{n,j} (x - S_X^{-1}(\hat{a}))_+,
\]

\[
L^3_h(X) = m_1 g_{a_1}(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_{S_X^{-1}(\hat{a})}^\infty H(x)dx \right] - D.
\]

Case C: \( \hat{b} \leq S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) \leq 1 \), which is equivalent to \( S_X^{-1}(\hat{b}) \geq d_{n,n} \geq d_{n,n-1} \geq \ldots \geq d_{n,1} \), in this case \( K(t) \leq M \).
a) if $m_2 > 0$, then $L_h^3(X)$ is decreasing, and the minimum $L_h^3(X)$ is attained at $d_{n,1} = d_{n,2} = \ldots = d_{n,n} = S_X^{-1}(\hat{b})$, so

$$h(x) = \sum_{j=1}^{n} C_{n,j}(x - S_X^{-1}(\hat{b}))_+,$$

$$L_h^3(X) = m_1 g_{\alpha_n}(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_{S_X^{-1}(\hat{b})}^{\infty} H(x) dx \right] - D.$$ 

b) if $m_2 < 0$, then $L_h^3(X)$ is increasing, and the minimum $L_h^3(X)$ is attained at $d_{n,1} = d_{n,2} = \ldots = d_{n,n} = 0$, so

$$h(x) = \sum_{j=1}^{n} C_{n,j} x,$$

$$L_h^3(X) = m_1 g_{\alpha_n}(X) - \sum_{j=1}^{n} C_{n,j} \left[ \int_{0}^{\infty} H(x) dx \right] - D.$$ 

Case D: $0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) \leq \hat{a} \leq S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,1}) \leq \hat{b}$, which is equivalent to $d_{n,n} \geq \ldots \geq d_{n,k} \geq S_X^{-1}(\hat{a}) \geq d_{n,k-1} \geq \ldots \geq d_{n,1} \geq S_X^{-1}(\hat{b})$, where $k = 2, 3, \ldots, n$,

a) if $m_2 > 0$, then $L_h^3(X)$ is decreasing on $(0, \hat{a}]$ and increasing on $[\hat{a}, \hat{b}]$, the minimum $L_h^3(X)$ is attained at $d_{n,1} = \ldots = d_{n,k-1} = S_X^{-1}(\hat{a})$ and $d_{n,k} = \ldots = d_{n,n} = \overline{X}$, so

$$h(x) = \sum_{j=1}^{k-1} C_{n,j}(x - S_X^{-1}(\hat{b}))_+,$$

$$L_h^3(X) = m_1 g_{\alpha_n}(X) - \sum_{j=1}^{k-1} C_{n,j} \left[ \int_{S_X^{-1}(\hat{b})}^{\infty} H(x) dx \right] - D.$$ 

b) if $m_2 < 0$, then $L_h^3(X)$ is increasing on $(0, \hat{a}]$ and decreasing on $[\hat{a}, \hat{b}]$, the minimum $L_h^3(X)$ is attained at $d_{n,1} = \ldots = d_{n,k-1} = S_X^{-1}(\hat{a})$ and $d_{n,k} = \ldots = d_{n,n} = S_X^{-1}(\hat{a})$, so

$$h(x) = \sum_{j=1}^{k-1} C_{n,j}(x - S_X^{-1}(\hat{a}))_+ + \sum_{j=k}^{n} C_{n,j}(x - S_X^{-1}(\hat{a}))_+,$$

$$L_h^3(X) = m_1 g_{\alpha_n}(X) - \sum_{j=1}^{k-1} C_{n,j} \left[ \int_{S_X^{-1}(\hat{a})}^{\infty} H(x) dx \right] - \sum_{j=k}^{n} C_{n,j} \left[ \int_{S_X^{-1}(\hat{a})}^{\infty} H(x) dx \right] - D.$$ 

Case E: $\hat{a} \leq S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,l}) \leq \hat{b} \leq S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1$, which is equivalent to $S_X^{-1}(\hat{a}) \geq d_{n,n} \geq \ldots \geq d_{n,l} \geq S_X^{-1}(\hat{b}) \geq d_{n,l-1} \geq \ldots \geq d_{n,1}$, where $l = 2, 3, \ldots, n$,

a) if $m_2 > 0$, then $L_h^3(X)$ is increasing on $[\hat{a}, \hat{b}]$ and decreasing on $[\hat{b}, 1)$, the minimum $L_h^3(X)$ is attained at $d_{n,1} = \ldots = d_{n,l-1} = S_X^{-1}(\hat{b})$ and $d_{n,l} = \ldots = d_{n,n} = S_X^{-1}(\hat{b})$, so

$$h(x) = \sum_{j=1}^{l-1} C_{n,j}(x - S_X^{-1}(\hat{b}))_+ + \sum_{j=l}^{n} C_{n,j}(x - S_X^{-1}(\hat{b}))_+,$$

$$L_h^3(X) = m_1 g_{\alpha_n}(X) - \sum_{j=1}^{l-1} C_{n,j} \left[ \int_{S_X^{-1}(\hat{b})}^{\infty} H(x) dx \right] - \sum_{j=l}^{n} C_{n,j} \left[ \int_{S_X^{-1}(\hat{b})}^{\infty} H(x) dx \right] - D.$$
b) if $m_2 < 0$, then $L_h^3(X)$ is decreasing on $[\hat{a}, \hat{b}]$ and increasing on $[\hat{b}, 1)$, the minimum $L_h^3(X)$ is attained at $d_{n,1} = \ldots = d_{n,t-1} = 0$ and $d_{n,t} = \ldots = d_{n,n} = S_{X}^{-1}(\hat{\alpha})$, so

$$h(x) = \sum_{j=1}^{I-1} C_{n,j} x + \sum_{j=t}^{n} C_{n,j} (x - S_{X}^{-1}(\hat{a}))_+,$$

$$L_h^3(X) = m_1 g_{a_n}(X) - \sum_{j=1}^{I-1} C_{n,j} \left[ \int_{0}^{\infty} H(x) dx \right] - \sum_{j=t}^{n} C_{n,j} \left[ \int_{S_{X}^{-1}(\hat{b})}^{\infty} H(x) dx \right] - D.$$

Case F: $0 \leq S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) \leq \hat{a} \leq S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,l-1}) \leq \hat{b} \leq S_X(d_{n,l}) \leq \ldots \leq S_X(d_{n,1}) < 1$, which is equivalent to $d_{n,n} \geq \ldots \geq d_{n,k} \geq S_{X}^{-1}(\hat{a}) \geq d_{n,k-1} \geq \ldots \geq d_{n,l} \geq S_{X}^{-1}(\hat{b}) \geq d_{n,l-1} \geq \ldots \geq d_{n,1} \geq 0$, where $l = 2, 3, \ldots, k-1, k = 3, 4, \ldots, n$.

a) if $m_2 > 0$, then $L_h^3(X)$ is decreasing on $(0, \hat{a}]$ and $[\hat{b}, 1)$, and increasing on $[\hat{a}, \hat{b}]$, the minimum $L_h^3(X)$ is attained at $d_{n,1} = \ldots = d_{n,t-1} = S_{X}^{-1}(\hat{b})$, $d_{n,t} = \ldots = d_{n,k-1} = S_{X}^{-1}(\hat{b})$ and $d_{n,k} = \ldots = d_{n,n} = \underline{X}$, so

$$h(x) = \sum_{j=1}^{I-1} C_{n,j} (x - S_{X}^{-1}(\hat{b}))_+ + \sum_{j=t}^{n} C_{n,j} (x - S_{X}^{-1}(\hat{b}))_+,$$

$$L_h^3(X) = m_1 g_{a_n}(X) - \sum_{j=1}^{I-1} C_{n,j} \left[ \int_{S_{X}^{-1}(\hat{b})}^{\infty} H(x) dx \right] - \sum_{j=t}^{n} C_{n,j} \left[ \int_{S_{X}^{-1}(\hat{b})}^{\infty} H(x) dx \right] - D.$$

b) if $m_2 < 0$, then $L_h^3(X)$ is increasing on $(0, \hat{a}]$ and $[\hat{b}, 1)$, and decreasing on $[\hat{a}, \hat{b}]$, the minimum $L_h^3(X)$ is attained at $d_{n,1} = \ldots = d_{n,t-1} = 0$ and $d_{n,t} = \ldots = d_{n,k-1} = S_{X}^{-1}(\hat{a})$, $d_{n,k} = \ldots = d_{n,n} = S_{X}^{-1}(\hat{a})$, so

$$h(x) = \sum_{j=1}^{I-1} C_{n,j} x + \sum_{j=t}^{n} C_{n,j} (x - S_{X}^{-1}(\hat{a}))_+ + \sum_{j=k}^{n} C_{n,j} (x - S_{X}^{-1}(\hat{a}))_+,$$

$$L_h^3(X) = m_1 g_{a_n}(X) - \sum_{j=1}^{I-1} C_{n,j} \left[ \int_{0}^{\infty} H(x) dx \right] - \left( \sum_{j=t}^{n} C_{n,j} + \sum_{j=k}^{n} C_{n,j} \right) \left[ \int_{S_{X}^{-1}(\hat{b})}^{\infty} H(x) dx \right] - D. \square$

We are ready to present the key results of this section which are stated in Theorem 3.1. Lemma 3.2 is used to obtain the solution of (3.1) by determining the value of $\sum_{i,j} C_{n,j}$. The specific results are summarized in the following theorem.

**Theorem 3.1.** Given a confidence levels $1 - \alpha$ with $0 < \alpha < S_X(0)$ and $1 - \gamma$ with $0 < \gamma < S_X(0)$, for any function $h(x) = \sum_{j=1}^{n} C_{n,j} (x - d_{n,j})_+ \in \mathcal{H}$ with given coefficients $C_{n,j}$, $j = 1, 2, \ldots, n$:

1. If $m_2 = 0$, then $f^*(x) = 0$.
2. When $m_2 \neq 0$, considering the following cases.
   i. If $M < K$, then
      $$f^*(x) = \mathbb{1}_{(m_2 > 0)}.$$
   ii. When $M = K$, if there exist a point $t_0 \in (0,1)$ such that $K(t_0) = M$, then $f^* \in \mathcal{H}$; for other cases,
      $$f^*(x) = \mathbb{1}_{(m_2 > 0)}.$$
   iii. If $M > K$, then
      $$f^*(x) = \mathbb{1}_{(m_2 < 0)}.$$
(iv) When $M = \overline{K}$, if there exist a point $t^* \in (0, 1)$ such that $K(t^*) = M$, then $f^* \in \mathcal{H}$; for other cases,
\[ f^*(x) = x\mathbb{I}_{\{m_2 < 0\}}. \]
(v) When $K < M < \overline{K}$, we consider the following six cases.
Case A. If $t = \widehat{a}$, then $f^* \in \mathcal{H}$. For $0 < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \widehat{a}$,
\[ f^*(x) = (x - S_X^{-1}(\widehat{a})) + \mathbb{I}_{\{m_2 < 0\}}. \]
Case B. If $t = \widehat{a}$ or $t = \widehat{b}$, then $f^* \in \mathcal{H}$. For $\widehat{a} < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \widehat{b}$,
\[ f^*(x) = (x - S_X^{-1}(\widehat{b})) + \mathbb{I}_{\{m_2 > 0\}}. \]
Case C. If $t = \widehat{b}$, then $f^* \in \mathcal{H}$. For $\widehat{b} < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < 1$,
\[ f^*(x) = x\mathbb{I}_{\{m_2 < 0\}}. \]
Case D. If $t = \widehat{a}$ or $t = \widehat{b}$, then $f^* \in \mathcal{H}$. For $0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) < \widehat{a} < S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,1}) < \widehat{b}$, where $k = 2, 3, \ldots, n$,
\[ f^*(x) = (x - S_X^{-1}(\widehat{b})) + \mathbb{I}_{\{m_2 > 0\}} + (x - S_X^{-1}(\widehat{a})) + \mathbb{I}_{\{m_2 < 0\}}. \]
Case E. If $t = \widehat{a}$ or $t = \widehat{b}$, then $f^* \in \mathcal{H}$. For $\widehat{a} < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,l}) < \widehat{b} < S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1$, where $l = 2, 3, \ldots, n$,
\[ f^*(x) = (x - S_X^{-1}(\widehat{b})) + \mathbb{I}_{\{m_2 > 0\}} + [c_1 x + c_2 (x - S_X^{-1}(\widehat{a}))] + \mathbb{I}_{\{m_2 < 0\}}. \]
where $c_1 > 0, c_2 > 0$, and $c_1 x + c_2 (x - S_X^{-1}(\widehat{a})) \leq x$.

**Proof.** To obtain the specific form of the optimal ceded loss function $f^*$, we only need to judge the magnitude of $\sum C_{n,j}$ according to the sign of $H(x)$.

1. Since $m_2 = 0$, $L_0^k(X) = m_1 g_{a_0}(X) - D, h(x) = 0$. From (3.2) we derive that $L_{f_j}^k(X) = m_1 g_{a_0}(X) - D, f^* = 0$.

2. When $m_2 \neq 0$, we consider the following cases.
   (1) When $M < K$, for any $S_X(x) \in (0, 1)$, $K(S_X(x)) > M$, which is equivalent to
   \[ \frac{g_{a_0}(S_X(x))}{g_{r_0}(S_X(x))} > \frac{m_3}{m_2}, \]
   a) if $m_2 > 0$, then
   \[ H(x) = m_2 g_{a_0}(S_X(x)) - m_3 g_{r_0}(S_X(x)) > 0. \]
   We derive that when $\sum_{j=1}^n C_{n,j} = 1$, $L_{f_j}^3(X) = m_1 g_{a_0}(X) - \int_0^\infty H(x)dx - D$, the minimum of $L_{f_0}^3(X)$ is attained at $f^* = x$.
   b) if $m_2 < 0$, then $L_{f_j}^3(X) = m_1 g_{a_0}(X) - D$, the minimum of $L_{f_0}^3(X)$ is attained at $f^* = 0$.  

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(2) When $M = K$, if there exist a point $t_0$ such that $K(t_0) = M$, then $H(x) = 0$, $f^* \in \mathcal{H}$. For other cases, the proof is similar to the case of $M < K$ and we omit it.

(3) When $M > K$, for any $S_X(x) \in (0, 1)$, $K(S_X(x)) < M$,
   a) if $m_2 > 0$, then $L^3_f(X) = m_1 g_{g_0}(X) - D$, the minimum of $L^3_f(X)$ is attained at $f^* = 0$.
   b) if $m_2 < 0$, then $H(x) > 0$. We derive that when $\sum_{j=1}^n C_{n,j} = 1$, $L^3_f(X) = m_1 g_{g_0}(X) - \int_0^\infty H(x)dx - D$, the minimum of $L^3_f(X)$ is attained at $f^* = x$.

(4) When $M = K$, if there exist a point $t^*$ such that $K(t^*) = M$, then $H(x) = 0$, $f^* \in \mathcal{H}$. For other cases, the proof is similar to the case of $M > K$ and we omit it.

(5) When $K < M < K$, we consider the following six cases:
   i) for Case A: if $t = \hat{a}$, then $K(t) = M$, $H(x) = 0$, therefore, $f^* \in \mathcal{H}$. For any $S_X(d_{n,j}) \in (0, \hat{a})$, $K(S_X(d_{n,j})) < M$,
      a) if $m_2 > 0$, then $L^3_f(X) = m_1 g_{g_0}(X) - D$, the minimum of $L^3_f(X)$ is attained at $f^* = 0$.
      b) if $m_2 < 0$, then $H(x) > 0$, therefore, when $\sum_{j=1}^n C_{n,j} = 1$,
         $$L^3_f(X) = m_1 g_{g_0}(X) - \int_{S_X^{-1}(\hat{a})}^\infty H(x)dx - D,$$
         the minimum of $L^3_f(X)$ is attained at $f^* = (x - S_X^{-1}(\hat{a}))_+$.
   ii) for Case B: if $t = \hat{a}$ or $\hat{b}$, then $K(t) = M$, $H(x) = 0$, therefore, $f^* \in \mathcal{H}$. For any $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, $K(S_X(d_{n,j})) > M$,
      a) if $m_2 > 0$, then $H(x) > 0$, when $\sum_{j=1}^n C_{n,j} = 0$, $L^3_f(X) = m_1 g_{g_0}(X) - D$, the minimum of $L^3_f(X)$ is attained at $f^* = 0$.
      b) if $m_2 < 0$, then $H(x) < 0$, when $\sum_{j=1}^n C_{n,j} = 0$, $L^3_f(X) = m_1 g_{g_0}(X) - \int_{S_X^{-1}(\hat{b})}^\infty H(x)dx - D$, the minimum of $L^3_f(X)$ is attained at $f^* = 0$.
   iii) for Case C: if $t = \hat{b}$, then $K(t) = M$, $H(x) = 0$, therefore, $f^* \in \mathcal{H}$. For any $S_X(d_{n,j}) \in (\hat{b}, 1)$, $K(S_X(d_{n,j})) < M$,
      a) if $m_2 > 0$, then $H(x) < 0$, when $\sum_{j=1}^n C_{n,j} = 0$, $L^3_f(X) = m_1 g_{g_0}(X) - D$, the minimum of $L^3_f(X)$ is attained at $f^* = 0$.
      b) if $m_2 < 0$, then $H(x) > 0$, when $\sum_{j=1}^n C_{n,j} = 1$,
         $$L^3_f(X) = m_1 g_{g_0}(X) - \int_0^\infty H(x)dx - D,$$
         the minimum of $L^3_f(X)$ is attained at $f^* = x$.
   iv) for Case D: if $t = \hat{a}$ or $t = \hat{b}$, then $K(t) = M$, $H(x) = 0$, therefore, $f^* \in \mathcal{H}$. For other cases, we obtain that $K(S_X(d_{n,j})) < M$ for $S_X(d_{n,j}) \in (0, \hat{a})$, where $j = k, k + 1, \ldots, n$, $K(S_X(j)) > M$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = 1, 2, \ldots, k - 1$,
      a) if $m_2 > 0$, then $H(x) < 0$ for $S_X(d_{n,j}) \in (0, \hat{a})$, where $j = k, k + 1, \ldots, n$; $H(x) > 0$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = 1, 2, \ldots, k - 1$. So, when $\sum_{j=1}^{k-1} C_{n,j} = 1$ and $\sum_{j=k}^n C_{n,j} = 0$,
         $$L^3_f(X) = m_1 g_{g_0}(X) - \int_{S_X^{-1}(\hat{b})}^\infty H(x)dx - D,$$
         the minimum of $L^3_f(X)$ is attained at $f^* = (x - S_X^{-1}(\hat{b}))_+$.
      b) if $m_2 < 0$, then $H(x) > 0$ for $S_X(d_{n,j}) \in (0, \hat{a})$, where $j = k, k + 1, \ldots, n$; $H(x) < 0$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = 1, 2, \ldots, k - 1$.
(\hat{a}, \hat{b})$, where $j = 1, 2, \ldots, k - 1$. So, when $\sum_{j=1}^{k-1} C_{n,j} = 0$ and $\sum_{j=k}^{\infty} C_{n,j} = 1$,

$$L^3_f(X) = m_1 g_{S_n}(X) - \int_{S_X^{-1}(\hat{a})}^\infty H(x) dx - D,$$

the minimum of $L^3(X)$ is attained at $f^* = (x - S_X^{-1}(\hat{a}))_+$. 

v) for Case E: if $t = \hat{a}$ or $t = \hat{b}$, then $K(t) = M$, $H(x) = 0$, therefore, $f^* \in \mathcal{H}$. For other cases, we obtain that $K(S_X(d_{n,j})) > M$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = l, l + 1, \ldots, n$, $K(S_X(d_{n,j})) < M$ for $S_X(d_{n,j}) \in (\hat{b}, 1)$, where $j = 1, 2, \ldots, l - 1$,

a) if $m_2 > 0$, then $H(x) > 0$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = l, l + 1, \ldots, n$; $H(x) < 0$ for $S_X(d_{n,j}) \in (\hat{b}, 1)$, where $j = 1, 2, \ldots, l - 1$. We derive that $\sum_{j=1}^{l-1} C_{n,j} = 0$ and $\sum_{j=l}^{n} C_{n,j} = 1$,

$$L^3_f(X) = m_1 g_{S_n}(X) - \int_{S_X^{-1}(\hat{b})}^\infty H(x) dx - D,$$

the minimum of $L^3(X)$ is attained at $f^* = (x - S_X^{-1}(\hat{b}))_+$. 

b) if $m_2 < 0$, then $H(x) < 0$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = l, l + 1, \ldots, n$; $H(x) > 0$ for $S_X(d_{n,j}) \in (\hat{b}, 1)$, where $j = 1, 2, \ldots, l - 1$. We derive that $\sum_{j=1}^{l-1} C_{n,j} = 1$ and $\sum_{j=l}^{n} C_{n,j} = 0$,

$$L^3_f(X) = m_1 g_{S_n}(X) - \int_{0}^{\infty} H(x) dx - D,$$

the minimum of $L^3(X)$ is attained at $f^* = x$. 

vi) for Case F: if $t = \hat{a}$ or $t = \hat{b}$, then $K(t) = M$, $H(x) = 0$, therefore, $f^* \in \mathcal{H}$. For other cases, we obtain that $K(S_X(d_{n,j})) < M$ for $S_X(d_{n,j}) \in (0, \hat{a})$, where $j = k, k + 1, \ldots, n$; $K(S_X(d_{n,j})) > M$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = l, l + 1, \ldots, k - 1$; $K(S_X(d_{n,j})) < M$ for $S_X(d_{n,j}) \in (\hat{b}, 1)$, where $j = 1, 2, \ldots, l - 1$,

a) if $m_2 > 0$, then $H(x) < 0$ for $S_X(d_{n,j}) \in (0, \hat{a})$, where $j = k, k + 1, \ldots, n$; $H(x) > 0$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = l, l + 1, \ldots, k - 1$; $H(x) < 0$ for $S_X(d_{n,j}) \in (\hat{b}, 1)$, where $j = 1, 2, \ldots, l - 1$. We derive that $\sum_{j=1}^{l-1} C_{n,j} = 0$ and $\sum_{j=l}^{k-1} C_{n,j} = 1$,

$$L^3_f(X) = m_1 g_{S_n}(X) - \int_{S_X^{-1}(\hat{b})}^\infty H(x) dx - D,$$

the minimum of $L^3(X)$ is attained at $f^* = (x - S_X^{-1}(\hat{b}))_+$. 

b) if $m_2 < 0$, then $H(x) > 0$ for $S_X(d_{n,j}) \in (0, \hat{a})$, where $j = k, k + 1, \ldots, n$; $H(x) < 0$ for $S_X(d_{n,j}) \in (\hat{a}, \hat{b})$, where $j = l, l + 1, \ldots, k - 1$; $H(x) > 0$ for $S_X(d_{n,j}) \in (\hat{b}, 1)$, where $j = 1, 2, \ldots, l - 1$. We derive that $\sum_{j=1}^{l-1} C_{n,j} = c_1$; $\sum_{j=k}^{k-1} C_{n,j} = 0$ and $\sum_{j=k}^{n} C_{n,j} = c_2$,

$$L^3_f(X) = m_1 g_{S_n}(X) - c_1 \int_{0}^{\infty} H(x) dx - c_2 \int_{S_X^{-1}(\hat{a})}^{\infty} H(x) dx - D,$$

the minimum of $L^3(X)$ is attained at $f^* = c_1 x + c_2 (x - S_X^{-1}(\hat{a}))_+$, where $c_1 > 0, c_2 > 0$, and $c_1 x + c_2 (x - S_X^{-1}(\hat{a}))_+ \leq x$. \hfill \Box

**Remark 3.1.** When $\lambda_1 = \lambda_2 = \lambda_3 = 0$ and $\beta = 1$, if we adopt VaR risk measure and Wang’s premium principle, then our results recover Theorem 3 of Cheung (2010) and superior to them since their results only consist of quota-share reinsurance, but our results consist of the quota-share, stop-loss, change-loss, the combination of quota-share and change-loss or the combination of change-loss and change-loss with different retentions, which means that our results provide more options for reinsurance strategies.
Remark 3.2. In this paper, we only give the proof of the case that $K(t)$ is a concave function. When $K(t)$ has an irregular shape in the interval $(0,1)$, it is easy to derive the optimal reinsurance with the same way according to Remark 4.3 of Zhuang et al. (2016).

From Theorem 3.1, we obtain the other cases of unconstrained optimal problem and constrained optimal problems. Next, we will only give the unconstrained optimal reinsurance and other cases achieve with the same way.

Corollary 3.1. (Unconstrained optimization problem) Given the confidence levels $1 - \alpha$ with $0 < \alpha < S_X(0)$ and $1 - \gamma$ with $0 < \gamma < S_X(0)$, when $\lambda_1 = \lambda_2 = \lambda_3 = 0$, from Theorem 3.1 we achieve the followings.

1. If $\beta = \frac{1}{2}$, then $f^*(x) \in \mathcal{H}$.
2. When $\beta \neq \frac{1}{2}$, considering the following cases.
   (i) When $1 + \rho \leq K$, if there exist a point $t_a$ such that $K(t_a) = 1 + \rho$, then $f^*(x) \in \mathcal{H}$; for other cases,
   $$f^*(x) = x^\mathbb{I}_{\beta < \frac{1}{2}}.$$
   (ii) When $1 + \rho \geq K$, if there exist a point $t^*$ such that $K(t^*) = 1 + \rho$, then $f^*(x) \in \mathcal{H}$; for other cases,
   $$f^*(x) = x^\mathbb{I}_{\beta < \frac{1}{2}}.$$
   (iii) When $K < 1 + \rho < K$, considering the following six cases.

Case A. If $t = \hat{a}$, then $f^* \in \mathcal{H}$. For $0 < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \hat{a}$,
   $$f^*(x) = (x - S_X^{-1}(\hat{a}))^\mathbb{I}_{\beta < \frac{1}{2}}.$$

Case B. If $t = \hat{a}$ or $t = \hat{b}$, then $f^* \in \mathcal{H}$. For $\hat{a} < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \hat{b}$,
   $$f^*(x) = (x - S_X^{-1}(\hat{b}))^\mathbb{I}_{\beta > \frac{1}{2}}.$$

Case C. If $t = \hat{b}$, then $f^* \in \mathcal{H}$. For $\hat{b} < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < 1$,
   $$f^*(x) = x^\mathbb{I}_{\beta < \frac{1}{2}}.$$

Case D. If $t = \hat{a}$ or $t = \hat{b}$, then $f^* \in \mathcal{H}$. For $0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) < \hat{a} < S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,1}) < \hat{b}$, where $k = 2, 3, \ldots, n$,
   $$f^*(x) = (x - S_X^{-1}(\hat{b}))^\mathbb{I}_{\beta > \frac{1}{2}} + (x - S_X^{-1}(\hat{a}))^\mathbb{I}_{\beta < \frac{1}{2}}.$$

Case E. If $t = \hat{a}$ or $t = \hat{b}$, then $f^* \in \mathcal{H}$. For $\hat{a} < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,l}) < \hat{b} < S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1$, where $l = 2, 3, \ldots, n$,
   $$f^*(x) = (x - S_X^{-1}(\hat{b}))^\mathbb{I}_{\beta > \frac{1}{2}} + x^\mathbb{I}_{\beta < \frac{1}{2}}.$$

Case F. If $t = \hat{a}$ or $t = \hat{b}$, then $f^* \in \mathcal{H}$. For $0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,k}) < \hat{a} < S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,l}) < \hat{b} < S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1$, where $l = 2, 3, \ldots, k - 1, k = 3, 4, \ldots, n$,
   $$f^*(x) = (x - S_X^{-1}(\hat{b}))^\mathbb{I}_{\beta > \frac{1}{2}} + (\lambda_1 x + c_2(x - S_X^{-1}(\hat{a})))^\mathbb{I}_{\beta < \frac{1}{2}}.$$

where $c_1 > 0, c_2 > 0$, and $c_1 x + c_2(x - S_X^{-1}(\hat{a})) \leq x$.

Remark 3.3. If there no constraints, then $\lambda_1 = \lambda_2 = \lambda_3 = 0$. We know that $M = 1 + \rho, H(d_{n,j}) = (2\beta - 1)[g_\alpha(S_X(d_{n,j})) - (1 + \rho)g_\gamma(S_X(d_{n,j}))]$. If $\beta = \frac{1}{2}, t = t_0, t = t^*$, $t = \hat{a}$ or $t = \hat{b}$, then $f^* \in \mathcal{H}$, which means that optimal reinsurance strategy could be any increasing convex function.
4 Two special cases

In this section, we consider two special cases: Value at Risk (VaR) and Tail Value at Risk (TVaR). In order to simplify calculation, we take the expectation premium principle, which means that the risk of the insurer and the reinsurer are measured by VaR and TVaR risk measures under the expectation premium principle. Next, we will give the optimal reinsurance under the VaR risk measure and a corresponding numerical example.

4.1 Value at Risk

As we all know, the Value at Risk is a special example of the distortion risk measure with the distortion function

\[ g_\alpha(t) = \begin{cases} 
0, & t < \alpha, \\
1, & t \geq \alpha.
\end{cases} \]

When adopting the expectation premium principle \( g_\gamma(t) = t \), we derive

\[ K(t) = \frac{g_\alpha(t)}{g_\gamma(t)} = \begin{cases} 
0, & t < \alpha, \\
\frac{1}{t}, & t \geq \alpha.
\end{cases} \quad (4.1) \]

From Theorem 3.1, we obtain the following proposition.

**Proposition 4.1.** Assuming the risk is measured by the VaR risk measure under the expectation premium principle, for \( 0 < \alpha < S_X(0) \) and \( 0 < \alpha < S_X(0) \), \( \beta \in [0,1] \), \( \lambda_i \geq 0 \), where \( i = 1, 2, 3 \), we can derive the following results.

1. If \( m_2 = 0 \), then \( f^*(x) = 0 \).
2. When \( m_2 \neq 0 \), considering the following cases.
   (i) When \( M \leq 0 \), if \( t \in (0, \alpha) \) and \( M = 0 \), then \( K(t) = M \), \( f^*(x) \in \mathcal{H} \); for other cases, \( f^*(x) = 0 \).
   (ii) When \( M \geq \frac{1}{\alpha} \), if \( t^* = \alpha \) and \( M = \frac{1}{\alpha} \), then \( K(t^*) = M \), \( f^*(x) \in \mathcal{H} \); for other cases,
       \[ f^*(x) = xI_{\{m_2 < 0\}}. \]
   (iii) When \( 0 < M < \frac{1}{\alpha} \), we consider the following six cases.
       Case A. If \( t = \alpha \), then \( f^* \in \mathcal{H} \). For \( 0 < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \alpha \),
             \[ f^*(x) = (x - \text{VaR}_n(X)) + I_{\{m_2 < 0\}}. \]
       Case B. If \( t = \alpha \) or \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( \alpha < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \hat{b} \),
             \[ f^*(x) = (x - S_X^{-1}(\hat{b})) + I_{\{m_2 > 0\}}. \]
       Case C. If \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( \hat{b} < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < 1 \),
             \[ f^*(x) = xI_{\{m_2 < 0\}}. \]
Case D. If \( t = \alpha \) or \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( 0 < S_X(d_{n,m}) \leq \ldots \leq S_X(d_{n,k}) < \alpha < S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,1}) < \hat{b} \), where \( k = 2, 3, \ldots, n \),

\[
f^*(x) = (x - S_X^{-1}(\hat{b}))_+ I_{\{m_2 > 0\}} + (x - \text{VaR}_\alpha(X))_+ I_{\{m_2 < 0\}}.
\]

Case E. If \( t = \alpha \) or \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( \alpha < S_X(d_{n,m}) \leq \ldots \leq S_X(d_{n,l-1}) < \hat{b} < S_X(d_{n,l}) \leq \ldots \leq S_X(d_{n,1}) < 1 \), where \( l = 2, 3, \ldots, n \),

\[
f^*(x) = (x - S_X^{-1}(\hat{b}))_+ I_{\{m_2 > 0\}} + x I_{\{m_2 < 0\}}.
\]

Case F. If \( t = \alpha \) or \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( 0 < S_X(d_{n,m}) \leq \ldots \leq S_X(d_{n,k}) < \alpha < S_X(d_{n,k-1}) \leq \ldots \leq S_X(d_{n,l}) < \hat{b} < S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1 \), where \( l = 2, 3, \ldots, k - 1, k = 3, 4, \ldots, n \),

\[
f^*(x) = (x - S_X^{-1}(\hat{b}))_+ I_{\{m_2 > 0\}} + [c_1 x + c_2 (x - \text{VaR}_\alpha(X))]_+ I_{\{m_2 < 0\}}.
\]

where \( c_1 > 0, c_2 > 0 \), and \( c_1 x + c_2 (x - \text{VaR}_\alpha(X)) \leq x \).

**Remark 4.1.** Under the VaR risk measure, from (4.1) we obtain that \( \underline{K} = 0 \), \( \overline{K} = \frac{1}{\alpha} \) and \( K(t) = 0 \) for \( t \in (0, \alpha) \). If \( t \in (0, \alpha) \) and \( M = 0 \), then \( f^* \in \mathcal{H} \). This means that reinsurance contracts can be any increasing convex function for all \( t \in (0, \alpha) \). Moreover, we derive that \( S_X^{-1}(\hat{a}) = \text{VaR}_\alpha(X) \) since \( \hat{a} = \alpha \).

**Remark 4.2.** When \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) and \( \beta = 1 \), our results recover Theorem 3.1 of Cai et al. (2008), Theorem 1 of Cheung (2010) and Theorem 3.1 of Cai and Tan (2010). Moreover, our results are superior to them since optimal reinsurance contracts in our research include the combination of quota-share and change-loss and the combination of change-loss and change-loss with different retentions which do not exist in above paper, which means that our finding provides more options for reinsurance strategies.

**Example 4.1.** Suppose that the ground-up loss \( X \) follows an exponential distribution \( F_X(x) = 1 - e^{-0.001x} \) for \( x \geq 0 \), then \( E(X) = 1000 \), \( \text{VaR}_\alpha(X) = -1000 \ln \alpha \). Let \( \alpha = 0.05 \), then \( \text{VaR}_\alpha(X) = 2995.73 \), \( \underline{\lambda} = 20 \), \( \overline{K} = 0 \), \( \overline{K} = 20 \). Let \( \lambda_1 = 0.3 \), \( \lambda_2 = 0.4 \), \( \lambda_3 = 0.3 \), \( \rho = 0.2 \), then

\[
M = \frac{(2 \beta - 0.8) \times 1.2}{2 \beta - 1.1},
\]

if \( m_2 > 0 \) and \( m_3 > 0 \), then \( \beta > 0.55 \); if \( m_2 < 0 \) and \( m_3 < 0 \), then \( \beta < 0.4 \). From (4.1) we derive that \( \hat{a} = \alpha = 0.05 \), so we obtain the following optimal reinsurance.

(i) If \( m_2 = 0 \), then \( f^* = 0 \). The case for \( m_2 = 0 \) is equivalent to \( \beta = 0.55 \), which means it is the optimal option that an insurer undertakes all loss for the case \( \beta = 0.55 \).

(ii) When \( M \leq 0 \), if \( K(t) = M = 0 \) for \( t \in (0, 0.05) \), then \( f^*(x) \in \mathcal{H} \), which means that when \( \beta = 0.4 \) and \( t \in (0, 0.05) \), the optimal reinsurance could be any increasing convex function; for other cases, \( f^* = 0 \), for the case of \( M < 0 \), we have \( m_2 < 0 \) and \( m_3 > 0 \), which means that when \( 0.4 < \beta < 0.55 \), it is the optimal option that an insurer undertakes all loss.

(iii) When \( M \geq 20 \), if \( K(t^*) = M = 20 \) for \( t^* = 0.05 \), then \( f^*(x) \in \mathcal{H} \), which means that the optimal reinsurance could be any increasing convex function at the point \( t = t^* \) and \( \beta = 0.56 \); for other cases, \( \beta \leq 0.56 \), moreover, we obtain \( f^*(x) = 0 \) for \( 0.55 < \beta < 0.56 \) and \( f^*(x) = x \) for \( \beta < 0.4 \).

(iv) When \( 0 < M < 20 \), which is equivalent to \( \beta > 0.56 \), we derive that \( f^*(x) = 0 \) for Cases A and C, and \( f^*(x) = (x - S_X^{-1}(\hat{b}))_+ \) for Cases B and D-F. Therefore, it is the optimal option that an insurer undertakes all loss for Cases A and C, and the stop-loss reinsurance is optimal for Cases B and D-F. When \( t = 0.05 \) or \( t = \hat{b} \), the optimal reinsurance could be any increasing convex function. \( \square \)
4.2 Tail Value at Risk

Similar to Value at Risk, we consider the case of Tail Value at Risk. Due to the distortion function of TVaR can be described as follows:

\[ g_\alpha(t) = \begin{cases} 
\frac{t}{\alpha}, & t < \alpha, \\
1, & t \geq \alpha,
\end{cases} \]

when adopting the expectation premium principle \( g_\gamma(t) = t \), we have

\[ K(t) = \frac{g_\alpha(t)}{g_\gamma(t)} = \begin{cases} 
\frac{1}{\alpha}, & t < \alpha, \\
\frac{1}{t}, & t \geq \alpha.
\end{cases} \]

(4.2)

From Theorem 3.1, we obtain the following proposition.

**Proposition 4.2.** Assuming the risk is measured by the TVaR risk measure under the expectation premium principle, for \( 0 < \alpha < S_X(0) \) and \( 0 < \gamma < S_X(0) \), \( \beta \in [0, 1] \), \( \lambda_i \geq 0 \), where \( i = 1, 2, 3 \), reinsurance contracts are given as follows:

1. If \( m_2 = 0 \), then \( f^*(x) = 0 \).
2. When \( m_2 \neq 0 \), considering the following cases.
   (i) If \( M \leq 1 \), then
   \[ f^*(x) = x \mathbb{I}_{\{m_2 > 0\}}. \]
   (ii) When \( M \geq \frac{1}{\alpha} \), if \( t \in (0, \alpha] \) and \( M = \frac{1}{\alpha} \), then \( K(t) = M \), \( f^*(x) \in \mathcal{H} \). For other cases,
   \[ f^*(x) = x \mathbb{I}_{\{m_2 < 0\}}. \]
   (iii) When \( 1 < M < \frac{1}{\alpha} \), we consider the following three cases.
   Case B. If \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( 0 < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < \hat{b} \),
   \[ f^*(x) = (x - S_X^{-1}(\hat{b}))(\mathbb{I}_{\{m_2 > 0\}} + \mathbb{I}_{\{m_2 < 0\}}). \]
   Case C. If \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( \hat{b} < S_X(d_{n,n}) \leq S_X(d_{n,n-1}) \leq \ldots \leq S_X(d_{n,1}) < 1 \),
   \[ f^*(x) = x \mathbb{I}_{\{m_2 < 0\}}. \]
   Case E. If \( t = \hat{b} \), then \( f^* \in \mathcal{H} \). For \( 0 < S_X(d_{n,n}) \leq \ldots \leq S_X(d_{n,l}) < \hat{b} < S_X(d_{n,l-1}) \leq \ldots \leq S_X(d_{n,1}) < 1 \), where \( l = 2, 3, \ldots, n \),
   \[ f^*(x) = (x - S_X^{-1}(\hat{b}))(\mathbb{I}_{\{m_2 > 0\}} + \mathbb{I}_{\{m_2 < 0\}}) + x \mathbb{I}_{\{m_2 < 0\}}. \]

**Remark 4.3.** Due to \( K(t) = \frac{1}{\alpha} \) for \( t \in (0, \alpha] \), we derive that \( a \) do not exist for the case \( K < M < \frac{1}{\alpha} \).

Therefore, Cases A, D and F do not exist and we only consider above three cases.

**Remark 4.4.** When \( \lambda_1 = \lambda_2 = \lambda_3 = 0 \) and \( \beta = 1 \), our results is consistent with Theorem 4.1 of Cai et al. (2008) and Theorem 2 of Cheung (2010).

**Example 4.2.** Similar to Example 4.1, we consider the case of the TVaR risk measure and obtain the following optimal reinsurance.
(i) If \( m_2 = 0 \), then \( f^* = 0 \). The case for \( m_2 = 0 \) is equivalent to \( \beta = 0.55 \), which means that undertaking all loss for an insurer is the optimal option.

(ii) When \( M \leq 1 \), if \( m_2 > 0 \), then \( \beta \leq -0.35 \), therefore, there no reinsurance for the case \( \beta \in [0,1] \); if \( m_2 < 0 \) and \( m_3 \leq 0 \), then \( \beta \in [0,0.4] \), \( f^* = 0 \); if \( m_2 < 0 \) and \( m_3 \geq 0 \), then \( \beta \in [0.4,0.55] \), \( f^* = 0 \).

(iii) When \( M \geq 20 \), if \( K(t) = M = 20 \) for \( t \in (0,0.05] \), then \( f^*(x) \in \mathcal{H} \), which means that the optimal reinsurance could be any increasing convex function for \( t \in (0,0.05] \) and \( \beta = 0.56 \); for other cases, \( \beta \leq 0.56 \), furthermore, we obtain \( f^*(x) = 0 \) for \( 0.55 \leq \beta < 0.56 \) and \( f^*(x) = x \) for \( \beta < 0.4 \).

(iv) When \( 1 < M < 20 \), which is equivalent to \( \beta > 0.56 \), we derive that \( f^*(x) = 0 \) for Case C, and \( f^*(x) = (x - S_X^{-1}(\hat{b}))_+ \) for Cases B and E. Therefore, undertaking all loss for an insurer is the optimal option for Case C, and the stop-loss reinsurance is optimal for Cases B and E. When \( t = \hat{b} \), the optimal reinsurance could be any increasing convex function. \( \square \)

5 Conclusion

As we all know reinsurance is an effective risk management tool for the insurer to transfer part of its risk to the reinsurer. However, what we should do is to determine how much risk an insurer should transfer to the reinsurer. This paper discusses two classes of optimal reinsurance models by minimizing their convex combination where the risk is measured by a distortion risk measure and the premium is calculated by a distortion risk premium. We present a unified framework about the unconstrained optimization problem and constrained optimization problems, moreover, not only did we derive the optimal reinsurance strategy but also we derive the minimum of optimization problems by a geometric argument. Under the unified framework, we can derive the solution of the cases from Cases 1-6.

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