The strongly interacting Quark Gluon Plasma, and the critical behaviour of QCD at imaginary $\mu$

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We explore the highly non-perturbative hot region of the QCD phase diagram close to $T_c$ by use of an imaginary chemical potential $\mu$ which avoids the sign problem. The number density and the quark number susceptibility are consistent with a critical behaviour associated with the transition line in the negative $\mu^2$ half-plane. We compare the analytic continuation of these results with various phenomenological models, none of which provides a satisfactory description of data, a failure on which we make some comments. These results complement and extend the information obtained via the analysis of the susceptibilities evaluated at zero $\mu$, yielding a simple description of the candidate strongly interacting QGP phase. As a byproduct of our analysis we investigate the Polyakov loop and its hermitian conjugate. Our data offer a vivid evidence of the importance of the complex nature of the functional integral measure, which results in $L(\mu) \neq \bar{L}(\mu)$ for a real chemical potential $\mu$.

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I. INTRODUCTION

Theoretical arguments and experimental evidence suggest that hadronic matter undergoes a transition to a plasma of quarks and gluons at high temperature [1]. At extremely high temperatures quarks and gluons are nearly free, and should be described by the Stefan-Boltzmann law with the appropriate degrees of freedom. When temperature is not much larger than the critical temperature – say, $T_c < T < 2T_c$ – strong interactions among the constituents give rise to non-perturbative effects. In short, at large $T$ the QGP is a gas of nearly free quarks, which becomes strongly interacting at lower temperatures $T = (1 - 3)T_c$ [2, 3].

Several proposals have been made to characterise the properties of the system in such non-perturbative phase. For instance the above mentioned strong interactions might be enough to preserve bound states above $T_c$, while coloured states might appear, deeply affecting the thermodynamics of the system [4]. Analytic techniques are being refined more and more, so to be able to capture the features of the system closer and closer to $T_c$ [5]. Model theories of quasi particle physics have been considered as well [6, 7]. In this work we study this interesting dynamical region by lattice QCD simulations at $T \simeq 1.1T_c$ (the reason for this choice will be clear in the following), and a nonzero baryon density.

In principle, lattice QCD simulations at non-zero baryon density are plagued by the sign problem [8]. However, it has been realised that this problem can be circumvented thanks to physical fluctuations, which grow relatively large in the Quark Gluon Plasma phase. In this work we adopted the imaginary chemical potential approach [9, 10, 11, 12, 13, 14, 15, 16], which avoids the sign problem and makes it possible conventional Lattice QCD simulations. The interested reader might want to consult refs. [17, 18] for recent reviews and [19, 20] for more pedagogical introductions into the subject.

Other studies in the quark gluon plasma phase have addressed the higher temperature regime [21, 22, 23, 24, 25]. Here we analyze in detail the non-perturbative behaviour close to $T_c$, in the candidate strongly coupled Quark Gluon Plasma (sQGP) region (some preliminary results have appeared in [26]). We note that in the sQGP region the chiral critical line lies in the imaginary chemical potential plane, and that such a chiral line ends in the proximity of the endpoint of the Roberge–Weiss line [27]. We focus our analysis on the particle number and its susceptibility, on the chiral condensate, and on the Polyakov loop, and we find that the results are consistent with

FIG. 1: Schematic phase diagram for four flavor QCD in the $T, \mu^2$ plane. The candidate sQGP phase is bound by the chiral (pseudo)critical line in the negative $\mu^2$ half-plane
those expected of a critical behaviour associated with
the critical line at imaginary chemical potential. Hence,
the numerical results are compatible with simple power
law behaviour of the equation of state as a function of
the imaginary chemical potential $\mu_I$, yielding a modified
form of the Stefan-Boltzmann law.

The rest of this paper is organised as follows. Section
II is devoted to the analysis of the particle number and
the chiral condensate, which are related by the Maxwell
equation. In Section III we discuss the behaviour of the
Polyakov loop. It turns out that our approach offers
a particularly simple description of an apparent puzzle,
and, at the same time, gives a direct evidence of the phase
of the determinant at nonzero, real chemical potential.
The implication on the equation of state are summarised
in Section III, while Section IV discusses our results in
the light of phenomenological proposals, and alternative
lattice approaches. Last Section is a short summing up.

II. THERMODYNAMICS OF THE HOT PHASE
CLOSE TO $T_c$

Let us remind ourselves of the critical lines in the phase
diagram in the $T, \mu^2$ plane (Figure I): at high temperature
there is the Roberge Weiss transition at $\mu = \pi T/3$,
associated with the phase of the Polyakov loop, ending
at $T \simeq T_{RW}$. At lower temperatures the QGP region
is limited by a chiral transition at negative $\mu^2$, which
continues into the physical chiral transition at positive $\mu^2$,
i.e. real chemical potential [11, 12].

While $\mu$ approaches $\pi T/3$ at a constant temperature
$T \simeq T_{RW}$ the chiral transition approaches the Roberge
Weiss transition. Within the current numerical accuracy
the endpoint of the two transitions cannot be res-
olved, and the nature of the critical behaviour around
$T = T_{RW}, \mu = \pi T/3$ is an interesting question in it-
self. If $T$ is slightly larger than $T_{RW}$ we are approaching
the Roberge Weiss transition, if slightly lower we hit the
chiral transition, and at $T = T_{RW}$ we might expect inter-
esting critical phenomena whose universality class is not
known a priori. Note that in the chiral limit the $\mu = 0$
transition should be of first order, and we do not expect
any tricritical point along the critical line at a real chem-
ical potential. A possible occurrence of an endpoint at
finite mass in the $T, \mu$ plane depends on dynamical de-
tails, which are not known, and are not relevant for the
present study.

We have then carried out simulations on a $16^3 \times 4$
lattice and four flavor of staggered fermions at $\beta = 5.1$,
which, according to our previous results, yields $T \simeq T_{RW}$,end point of the RW transition. Fermions are fully de-
gegenerate, with a bare dimensionless mass ($a$ being the
lattice spacing) $\hat{m} = ma = 0.05$. For our lattice the
value of the (dimensionless) $\hat{\mu} \equiv \mu a$ which is relevant for
the Roberge Weiss transition reads $\hat{\mu} = \pi/12$ (the tem-
perature being $T = 1/(aN_t)$ and in our case $N_t = 4$).
With a slight abuse we will omit in the following the
hat-notation, nevertheless measuring $\mu$ and $T$ in unit of
inverse lattice spacing.

First, we check our data for the particle number against
a simple free field behaviour. We have numerically com-
puted the free field results for real chemical potential on a $16^3 \times 4$ lattice, and $m_q = 0.05$, and we have fitted them
to an expression motivated by one dimensional QCD [20],
which turns out to be an excellent parametrisation:

$$ n(\mu)_{\text{free}} = \frac{3 \sinh(\mu/T)}{K + \cosh(\mu/T)} \quad \text{(1)} $$

yielding the free field results for the number density as a
function of imaginary chemical potential

$$ n(\mu_I)_{\text{free}} = \frac{3 \sin(\mu/T)}{K + \cos(\mu/T)} \quad \text{(2)} $$

We then considered the ratio between the numerical
results and such free field results $R_F(\mu_I) = n(\mu_I)/n(\mu_I)_{\text{free}}$ (Figure 2). We observe a clear depen-
dence of $R_F(\mu_I)$ on $\mu_I$: the results are qualitatively dif-
ferent from a free field and the discrepancy cannot be ac-
counted for by any simple renormalisation of the de-
grees of freedom. This behaviour should be contrasted
with that of Fig. 12 of Ref. [22] where the results at high
temperature did differ from a free field behaviour by a
constant factor very close to one.

As a second attempt at interpreting our data in terms of
the simplest parameterisations, we consider an analogy
with the Hadron Resonance Gas model [30, 31]. We will
discuss more fully these aspects in the last Section.

In a nutshell, the HRG describes the system as a gas of
weakly interacting resonances. The pressure of the HRG
model reads:

$$ \frac{P(T, \mu) - P(T, 0)}{T^4} \simeq F(T) (\cosh(\frac{N_t m_q}{T}) - 1) \quad \text{(3)} $$

$$ F(T) \simeq \int dm \rho(m) \frac{m}{T} \, K_2 \left( \frac{m}{T} \right) \quad \text{(4)} $$

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{graph.png}
\caption{$R_F(\mu_I) = n(\mu_I)/n(\mu_I)_{\text{free}}$ as a function of $\mu_I$, showing a very clear evidence of a deviation from a free field behaviour.}
\label{fig:graph}
\end{figure}
and the argument of the hyperbolic cosine, $N_c\mu_q$, tells us that in the hadronic phase one can only excite baryonic degrees of freedom.

Let us remind ourselves here that general arguments guarantee that the partition function is periodic at imaginary $\mu$, and that the strong coupling analysis shows that periodicity is smooth at low temperature. Hence, the number density reads \[ n(\mu) = \sum_k b_{\mu} \sin(k N_c N_\mu) \] \[ n(\mu) = \frac{\partial P(\mu)}{\partial \mu} = K(T) \sin(N_c N_\mu) \] (5)

When HRG holds true, one term in the Fourier series should suffice. \[ n(\mu) = \frac{\partial P(\mu)}{\partial \mu} = K(T) \sin(N_c N_\mu) \] (6)

A cross check with the Hadron Resonance which uses the Taylor approach requires the computation of an infinite number of derivatives \[ \frac{\partial}{\partial \mu} = (\partial_\mu)^\gamma \] while the Fourier analysis – possible with the imaginary chemical potential approach – needs only one parameter fit.

In Figure 3 we display the ratios \[ R_B(\mu) = \frac{n(\mu)}{3 \sin(\mu_1/T)} \] \[ R_q(\mu) = \frac{n(\mu)}{3 \sin(\mu_1/T)} \] (7)

$R_B(\mu_1)$ should be a constant for a simple hadron gas (cfr. again Ref. 22), while mutatis mutandis, $R_q(\mu_1)$ should be a constant for a “hadron gas” made of quarks. Both $R_B$ and $R_q$ stay constant only for a short interval of chemical potential, indicating the region where $n(\mu_1) \propto \mu_1$. The deviation from a linear behaviour (for $\mu_1 > 0.05$), as well as from the simplest trigonometric parameterisations (for $\mu_1 > 0.15$), are evident from the plot.

We now move on and propose to describe the particle number in the critical region as \[ n(\mu) = A \mu_1 (\mu_1^2 - \mu_1^2)^\alpha \] (9)

This ansatz for $n(\mu_1)$ takes into account that $n(\mu)$ should be an odd function of $\mu$. Moreover, it reproduces a singular behaviour of the quark number susceptibility at a genuine critical point. Namely, the most divergent part of the quark number susceptibility $\chi_q$ behaves as

\[ \chi_q(\mu_1) \propto \frac{1}{(\mu_1^2 - \mu_1^2)^\gamma} \] (10)

where $\gamma = 1 - \alpha$, while $\mu_1^2 = \pi/12$ if the critical point coincides with the Roberge-Weiss line.

We then fit our data to Eq. 9 with $\mu_1^2$ either open or constrained. A fit to our entire interval with unconstrained $\mu_1^2$ gives $A = -0.94(4), \mu_1^2 = 0.0804(2), \alpha = 0.28(2)$ with a reduced $\chi^2 = 2.4$. We checked the stability of these results by choosing different ranges in chemical potential, and we obtained the exponent $\alpha$ ranging from 0.34(8) and 0.26(3), $\mu_1^2$ between 0.078(4) and 0.091(12), with reduced $\chi^2$ ranging between 1.8 and 5. (see Figure 4 for two representative fits).

If we constraint $\mu_1^2 = (\pi/12)^2$ the quality of the fits decreases giving a reduced $\chi^2 \approx 12$. If we limit the fitting interval to $\mu_1 > 0.15$, we need to add a constant to the function to approximate the regular component in this interval to obtain a reduced $\chi^2 = 3.5$, with $\alpha = 0.12(1)$. All in all, constraining $\mu_1^2$ does not improve the results.

We now go back to the quark number susceptibility as entailed in Eq. 10, for which our results indicate $\gamma = 1 - \alpha \approx 0.7$. Figure 5 shows $\chi_q$ obtained by numerical differentiation of $n(\mu_1)$. The numerical quality is of course poor, but, anyway, a fit to the form of Eq. 10 with an open $\mu_1^2$ gives $\gamma = 0.66(16)$, while a fit with constrained $\mu_1^2$ gives $\gamma = 0.44(22)$, in agreement with the above estimates, within the large errors.

The chiral condensate $<\bar{\psi}\psi>$ can be inferred from the chiral equation of state in either phases, and also in the presence on an explicit chiral symmetry breaking term. To make a closer contact with thermodynamics, we consider the Maxwell relation (the temperature is constant and its dependence is omitted) 32:

\[ \frac{\partial n(\mu, m)}{\partial m} = \frac{\partial <\bar{\psi}\psi>(\mu, m)}{\partial \mu} \] (11)

Considering the $m$ dependence in the expression for $n(\mu, T)$

\[ n(\mu, m) = F(m)\mu(\mu_1^2 - \mu^2)^\alpha \] (12)

where $F(m)$ depends only on the quark mass, and using Eq. 11 we arrive at

\[ <\bar{\psi}\psi>(\mu) = H(m)(\mu_1^2 - \mu^2)^{\alpha+1} \]

\[ + K(m)(\mu_1^2 - \mu^2)^\alpha + <\bar{\psi}\psi>_0 \] (13)
where $H(m), K(m)$ depends only on the mass and $\langle \bar{\psi} \psi \rangle_0$ is an integration constant which can be fixed e.g. by the chiral condensate at zero chemical potential.

We have then fitted the chiral condensate to the leading term at $\mu \simeq \mu_c$

$$\langle \bar{\psi} \psi \rangle = K(b - \mu^2)^{\alpha_c} + A$$

obtaining a nice fit with a reduced $\chi^2 = 0.79, A = 0.552(6), b = 0.06628(8), K = -0.63(2)$ and $\alpha_c = 0.47(2)$ in reasonable agreement with $\alpha$ estimated from the number density. By constraining the fitting interval $\mu_I > 0.2$ the sub-leading contributions are less important, and $\alpha_c = 0.32(12)$ gets even closer to $\alpha$.

It might be interesting to compare this critical behaviour with that of the endpoint of QCD from model field theories [33].

III. THE POLYAKOV LOOP

In the same spirit we have fitted the traced Polyakov loop $L = \text{Tr} P$ to a power law form

$$L(\mu_I) \propto (\mu_I^2 - \mu^2)^{\beta}$$

obtaining a reduced $\chi^2 = 0.79, A = 0.552(6), b = 0.06628(8), K = -0.63(2)$ and $\alpha_c = 0.47(2)$ in reasonable agreement with $\alpha$ estimated from the number density. By constraining the fitting interval $\mu_I > 0.2$ the sub-leading contributions are less important, and $\alpha_c = 0.32(12)$ gets even closer to $\alpha$.

It is interesting to look in more detail into the behaviour of $L = \text{Tr} P$.

The Polyakov loop $P$ satisfies the same relation as the quark propagator at nonzero chemical potential [38].

$$P(\mu) = P(\mu)$
This relation implies that, while both $L = \text{Tr} P$ and $\bar{L} = \text{Tr} P^\dagger$ are real at real chemical potential, $L \neq \bar{L}$, as noted in \cite{22, 34, 33, 36, 37}. We will show that the results at imaginary chemical potential offer a particularly simple illustration of these ideas, as well as a direct evidence of the complex phase of the determinant.

The asymmetry at real chemical potential is easily understood by considering the distribution of the Polyakov loops in the complex plane: for $\mu = 0$, since the $Z_3$ centre symmetry is broken by the dynamical quarks, the root corresponding to the phase $\phi = 0$ is preferred, and the average is non zero. A non zero, positive chemical potential encourages forward propagation: the distribution of the phases is further peaked at $\phi = 0$, while the two other phases have the same probability. Hence, the Polyakov loop remains real, and the final average is again real, different from zero, and slightly larger than the one at zero density. $\bar{L}$ instead describes backward propagation: again the Polyakov loop remains real, however its length is reduced, hence $L(\mu) \neq \bar{L}(\mu)$.

Notice that at a first naïve look it may sounds strange that, while configuration by configuration the Polyakov loop and its hermitian conjugate are always the complex conjugate of each other, their expectation values, even being real, differ from each other. However it should be clear that the complex nature of the functional integral measure plays an essential role in this respect, since the real part of the expectation value is not just the expectation value of the real part. In that sense the fact that $L(\mu) \neq \bar{L}(\mu)$ is directly linked to the complex nature of the fermion determinant, thus also giving a qualitative feeling about the severeness of the sign problem.

An apparent puzzle then arises when one considers the behaviour at imaginary chemical potential: there the measure is real and one can show that the absolute value of $L$ and $\bar{L}$ are equal as well as their real parts. What is then the fate of the asymmetry which is present at real chemical potential?

Consider
\begin{equation}
L_{\alpha/e}(\mu) \equiv L(\mu) \pm L(-\mu) = L(\mu) \pm \bar{L}(\mu),
\end{equation}
where Eq. (16) has been used in last equality. $L_{\alpha/e}$ are respectively even and odd in $\mu$. Remember that the analytic continuation to imaginary chemical potential of an even function is real, while the analytic continuation of an odd function is purely imaginary. Hence, the analytic continuation of the even observable $L_e(\mu) = L(\mu) + \bar{L}(\mu)$ at imaginary chemical potential is the real part of $L(\mu_1)$; while the analytic continuation of $L_o(\mu) = L(\mu) - \bar{L}(\mu)$ is the imaginary part of $L(\mu_1)$ at imaginary $\mu$. $L$ itself has no definite $\mu$-parity and its analytic continuation develops an imaginary part.

We conclude that we must search for the analytic continuation of the asymmetry $L \neq \bar{L}$ which is present at real chemical potential in the imaginary part of the Polyakov loop, which is non-zero in presence of an imaginary chemical potential. Figure 8 shows the imaginary part of $L(\mu_1)$: it is different from zero, offering a clean, direct evidence of the asymmetry $L \neq \bar{L}$, hence of the complex phase of the determinant at real chemical potential. In the same Figure 8 we have also plotted $n(\mu_1)$ (both $L$ and $n$ with an appropriate normalisation), to show its correlation with $L(\mu_1)$. This correlation is in agreement with the lattice interpretation of the number operator $n$ as ‘counting’ the links winding forwards minus those winding backwards, and it should become exact in the heavy quark limit when the model reduces to a Polyakov loop model \cite{33, 40}.

Let us now come back to the critical behaviour of $|L|$; since $\text{Im}(L(\mu_1)) \propto n(\mu_1)$, we might expect that $|L(\mu_1)|$ approaches zero with a similar power law. The fit of Figure 7 gives an exponent of 0.45(2), in the same range as $\gamma$. It would be interesting, of course, the study of the string tension and other correlators in the same range of chemical potentials.

IV. ANALYTIC CONTINUATION TO REAL CHEMICAL POTENTIAL

Finally, we can analytically continue\(^1\) our results for $n(\mu_1)$, obtaining
\begin{equation}
n(\mu) = A \mu (\mu_1^2 + \mu^2)^\alpha
\end{equation}

\(^1\) The singularity in the complex $\mu$ plane might well limit the radius of convergence of the Taylor expansion. However, the proposed – Pade’ like – parametrisation does not suffer from this problem \cite{16, 41} and it should not come as a surprise that we are able to analytically continue beyond the radius of convergence of the Taylor series: the analytic continuation is then valid for all the real values of the chemical potential $\mu$. 

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{fig8.pdf}
\caption{The imaginary part of the Polyakov loop, divided by the coefficient of the linear term of a fit at small $\mu$, as a function of the imaginary chemical potential, demonstrating the relevance of the phase of the determinant for a real chemical potential; in the same plot we show the number density, again normalised by the coefficient of the liner term.}
\end{figure}
with \( \alpha \approx 0.3 \). The results are shown in Figure 9. In the same diagram we also plot the free field results, as an indicative comparison. Note that coefficient \( A \) has a non trivial dimension, again indicating that the system is not free.

It is then amusing to notice that by using simple arguments from the theory of critical phenomena we arrive at a modified (lattice) Stefan-Boltzmann law, which would correspond to \( \alpha = 1 \), and a large \( \mu_I^2 \approx 0.5 \). In this framework a large \( \mu_I^2 \) can be interpreted as a spinodal point at imaginary chemical far away from the Roberge Weiss line.

Obviously, Eq. (15) accounts for a slower increase of the particle density closer to \( T_c \) than in the free case. This is expected on physical grounds, as well as from the behaviour of the susceptibilities \([35, 48, 47, 48, 49, 50]\), and of course accounts for the behaviour observed in Figure 2.

From the results above, we conclude that the data in the candidate region for a strongly coupled QCD are accounted for by a conventional critical behaviour: clearly, a free field behaviour would have been incompatible with it. In other words, the nonperturbative features of the plasma are closely related with the occurrence of the critical line at negative \( \mu^2 \).

\[ F_K(\mu_I) = \sum_{j=1}^{K} F_j \sin(j \mu_I / T) \]  

The fits suggest that the slower increase observed in the interacting case with respect to the free case can be described by an overall exponent smaller than one

\[ n(\mu) \]  


\[ \frac{\Delta P_C}{T^4} = (F_q(T)) \left( \cosh(\mu_u / T) + \cosh(\mu_d / T) \right) + F_q(T) \left( \cos(2\mu_u / T) + \cos(2\mu_d / T) \right) + \cosh((\mu_u + \mu_d) / T) \]  

The susceptibilities at zero chemical potential can be easily computed from Eq. (19), and we recognise that their ratios allow the identifications of the relevant degrees of freedom. These prediction for the susceptibilities ratio was contrasted with the numerical results, finding a poor agreement.

The imaginary chemical potential approach gives the possibility to check directly the consistence of various phenomenological models by analytically continuing from real to imaginary \( \mu \). We can subject our data to the same analysis by analytically continuing Eq. (19) from real to imaginary chemical potential. Setting \( \mu_{\text{isospin}} = \mu_u - \mu_d = 0 \), and including the contribution from baryons and tetraquarks we get:

\[ \frac{\Delta P}{T^4} = F_q(T) \cos(\mu / T) + F_q(T) \cos(2\mu / T) \]  

\[ + F_{qq}(T) \cos(3\mu / T) + + F_{qqq}(T) \cos(4\mu / T) \]  

giving in turn:

\[ n(\mu, T) = F_q(T) \sin(\mu / T) + 2F_{qq}(T) \sin(2\mu / T) \]  

\[ + 3F_{qqq}(T) \sin(3\mu / T) + 4F_{qqq}(T) \sin(3\mu / T) \]

From the point of view of the imaginary chemical potential analysis, checking these forms correspond to perform a Fourier analysis of our results. We have then fitted our data to the form

\[ F_K(\mu_I) = \sum_{j=1}^{K} F_j \sin(j \mu_I / T) \]

The results of the fits are shown in Figure 10. The reduced \( \chi^2 \) ranges from 84 to 2.85 (\( F_1 \) to \( F_4 \)) but the errors on the parameters grow big and the parameters themselves are not stable. We summarise the results in Table 1, and we conclude that, even if the trigonometric fits might eventually converge, it is hard to attach any simple physical interpretation of the parameters \( F_1, F_2, F_3, F_4 \) as contribution from free quarks, diquarks, baryons and tetraquarks.

This result is not unexpected, as the Fourier parametrisation Eq. (22) is not compatible with the critical fits Eq. (18). Does this mean that the occurrence of coloured bound states is ruled out?
Not really: apart from the fact that extending an Hadron Resonance Gas description to coloured states is anyway a non-trivial assumption, given the non-trivial interactions that one could expect. Consider that the masses themselves should depend on the chemical potential - the so-called BKS effect. From the perspective of the present study, the BKS effect is indeed very natural, in view of the phase diagram in Figure 1 and the related analysis of the data in term of critical behaviour which we have presented above. Remember, in fact, that the coefficients of the Hadron Gas parametrisation in Eq. (22), see text for details, cannot afford any definite conclusion. This does not come as a surprise. The masses themselves, hence the coefficients, will depend on \( \mu \) in some complicated way, which should anyhow conjure to give the simple behaviour inferred from the one of the masses.

For instance, in Ref. 4 it was proposed that

\[
M_{\text{coloured}} \approx 11.5 T_c \left( \left( T/3 T_c \right)^{0.5} + 0.1 T_c / (T - T_c) \right)
\]  

(23)

yielding the decoupling of the coloured masses at the critical point. A similar decoupling should take place at the critical line at negative \( \mu^2 \), which is not compatible with the simple factorisation of the terms depending of temperature and fugacities implied by the HRG model Eq. (19).

In Ref. 45, it was underscored that if the derivatives of the masses with respect to the chemical potential

\[
M''(T) = \frac{\partial^2 M(T, \mu)}{\partial \mu^2} (T, \mu = 0)
\]  

(24)

are large enough, the simple interpretation of the zero chemical potential susceptibilities as probes of particle contents has to be revised, and, more generally, the decoupling of the prefactor and the simple trigonometric factors predicted by the HRG model is no longer true. Hence, we cannot use simple forms as those of Eq (19) to assess the particle content of the sQGP gas.

Our analysis supports this point of view. It is interesting to note that indeed a recent direct calculation of the coloured spectrum indicates the survival of heavy coloured states above \( T_c \).

VI. SUMMARY AND OUTLOOK

We have studied the critical behaviour of the system in proximity of the critical endpoint of the chiral and RW line in the negative \( \mu^2 \). We have given a simple description of the non-perturbative features of the sQGP phase, based on the analysis of the critical behaviour in the imaginary \( \mu \) plane. We have proposed an EOS of the form

\[
n(\mu) = A \mu (\mu^2 + \mu^2)^{\alpha}
\]

with \( \alpha \approx 0.3 \), which accounts nicely for all the features of the numerical data. The exponent would read \( \alpha = 1 \) for a Stefan–Boltzmann-like law.

From a more mathematical point of view, the proposed parametrisation is a Pade’ approximant of order [2,1], as appropriate in the standard application to critical phenomena. The results thus obtained can be, in principle, analytically continued within the entire analyticity domain. Practical limitations - discussed at length in previous work - do arise because of numerical accuracy.

As for the particle content of the system, our results suggest that a fit to

\[
\frac{\Delta P}{T^4} = F_q(T \cos(\mu/T) + F_{qq}(T \cos(2\mu/T) + F_{qqq}(T \cos(3\mu/T) + F_{qqqq}(T \cos(4\mu/T)
\]

cannot afford any definite conclusion. This does not come as a surprise. The masses themselves, hence the coefficients, will depend on \( \mu \) in some complicated way, which should anyhow conjure to give the simple behaviour observed in the data. The fact that the masses themselves should depend on the chemical potential while approaching the critical endpoint offers a simple realisation of the BKS mechanism. It should also pointed out that extending a Hadron Resonance Gas description to coloured states is anyway a non-trivial assumption, given the non-trivial interactions that one could expect.

It would be very interesting to confront the numerical results in a broader range of temperatures with these ideas, as well as with analytic calculations and phenomenological models. Future work should hopefully be able to give a coherent account of critical behaviour, high
temperature expansions and particle contents in the region of the strongly interactive Quark Gluon Plasma. We hope that the simple description offered here might be of help in building such a complete picture.

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