A role of topology and quantum information in physics

Jaroslav HRUBY
Institute of Physics AV CR, Czech Republic
e-mail: jaroslav.hruby@iol.cz

Abstract
The role of topology in QIS with physical connection to the non-commutativity, discretization, supersymmetry, entanglement, nonseparability and CP violation in physics is discussed.

1 Introduction

In recent time it is believed that quantum mechanics (QM) has the potential to bring about a spectacular revolution in quantum information science (QIS) [1].

What is more interesting that QIS can give new ideas to QM and field theories. There are natural relationships between quantum entanglements and topological entanglements [2]. The violation of Bell inequalities by photons more than 10 km is well established [3] and so from QM the collapse of physical state is realized by the superluminal velocity, what is hard to believe. More interesting explanation is that only quantum information is nonlocal and topological connected and via measurement we obtain the collapse to the classical information. The interesting glue between quantum mathematics and topology appears as a candidate to show new way how to explain this old problem.
Topology studies global relationships in spaces, and how one space can be placed within another, such as knotting and linking of curves in three-dimensional space. This mathematical area is popular in physics namely with application of quantum groups. One way to study topological entanglement and quantum entanglement is to try making direct correspondences between patterns of topological linking and entangled quantum states.

A deeper method is to consider braid group representation (BGR) and unitary gates $R$ that are both universal for quantum computation and are also solutions to the condition for topological braiding. Such $R$-matrices are unitary solutions to the Quantum Yang-Baxter equation (QYBE).

In this way, we can study the apparently complex relationship among topological entanglement, quantum entanglement, and quantum computational universality. We can also show how it is connected with the non-commutative geometry, discretization and supersymmetries.

The basic question is arising:

how the field theory, space, topology and information can be connected?

A speculative study was presented by D. Deutsch and myself [4] and here we discuss another point of view.

It is known non-commutative geometry and quantum groups are of relevance of space-time quantization and discretization.

The idea of quantization of space-time using noncommutative coordinates like

$$x_\mu x_\nu - x_\nu x_\mu = i\hbar g_{\mu\nu} , \quad x_\mu x_\nu - qx_\nu x_\mu = 0$$

(1)

was presented half century ago.

For example it is natural to attempt to relate the noncommutativity parameter $q < 1$ to the minimal uncertainty in length measurement

$$\delta x > l_{PL} = \sqrt{\frac{2\kappa\hbar}{c^2}} \sim 10^{-35} \text{ m} ,$$

(2)

or time measurement

$$\delta t > \tau_{PL} = \frac{l_{PL}}{c} \sim 10^{-43} \text{ s} .$$

(3)

where $\kappa$, $\hbar$ are the gravitational and Planck constants and $c$ is the light velocity.
But this deformation parameter can be obtained also in the BGR, where b matrices have to satisfy braid relation \([5],[6]\):

\[
    b_{i}b_{i+1}b_{i} = b_{i+1}b_{i}b_{i+1}, \quad 1 \leq i \leq n - 1, \quad b_{i}b_{j} = b_{j}b_{i}, \quad |i - j| > 1, \tag{4}
\]

while QYBE in R-matrices can be written as follows:

\[
    \mathcal{R}_{i}(x)\mathcal{R}_{i+1}(xy)R_{i}(y) = R_{i+1}(y)R_{i}(xy)b_{i+1}(x), \tag{5}
\]

with the asymptotic condition \(R(x = 0) = b\). The b-matrix and R-matrix are \(n^2 \times n^2\) matrices acting on \(V \otimes V\), where \(V\) is an \(n\)-dimensional space. As \(b\) and \(R\) acts on the tensor product \(V_{i} \otimes V_{i+1}\) we denote them \(b_{i}\) and \(R_{i}\), respectively.

The association of a unitary operator with a braid that respects the topological structure of the braid and allows exploration of the entanglement properties of the operator. The entanglement between two physical states or two-qubit states are known and play crucial role in quantum physics and QIS. Our aim is to study the geometrical and algebraical fundament of physics and QIS. We want to understand the coincidence and try to show the way to the obtaining some knowledge about algebraic structure of the physical ”space” and the connection with quantum information. The algebraic structure of the ”space” give the possibility of discretization and quantization of the space.

Modern theory of quantum and braided groups can be applied in fractional supersymmetries and n-anyonic vector spaces with the generalized grassmannian variables \(\theta^{\alpha} = 0\), which also can give discretization.

For example there are possible discretization on the following bases:

1. fractional supersymmetry and paragrassmannian q-deformed superspace connected with fractional or anyonic statistics

2. on a model, with q-deformed Heisenberg uncertainty relation for the null sector[6].

At this moment is no known basic principle requiring space or time to be continuous or forbidding limitations on their units. There is also no known basic principle where is forbidden to combine physical ”space” with information n-qubit Hilbert space \(H_{2}^{n}\). But it is forbidden to transfer physical object via superluminal velocity like collapse of physical wave function realize. The
known EPR paradox must be explained more natural way via collapse of nonlocal quantum information, which can has some topological fundament.

The article is organized as follows:

we start in Sect. 2 from the q-deformed quantum mechanics (QM) and quantum space time to obtain quantum space. In Sect. 3 we discuss the quantum superspace, which can be extended. This is done to show the possibility for obtaining the richer structure in the fractional superspace and that the base of the quantization can be done on the level of such superspace in the general case. Superspace is also good example for starting the study an algebraic structure of "space". It is well known that the space variable for example Bose space variable $x_\mu$, $\mu = 1,...,4$ is a condensate of two Fermi variable $\theta_\mu, \theta$ in ordinary supersymmetries. The algebraic structure of "space" is important for showing some connections to the QIS.

In Sect. 4 we present basic information about quantum mathematics and the connection with topological anyonic theory and QIS is discussed. We also discuss basic information about supersymmetry and QIS.

In Sect. 5 we discuss the QYBE and universal quantum gate for two-qubit systems. We show the explanation the entanglement via nonlocal quantum information. Deformation of entanglement can be applied in physics on every state/antistate physical system. For example CP discrete symmetry violation can be explained via noncommutation or equivalently like $\varphi$ deformation on the braid connecting entangled states.

In Sect. 6 we show the application of topological entanglement on CP violation and in Sect. 7 the separability criterion in kaon system.

2 Q-deformed quantum mechanics and quantum space-time

Limitations on the precision of localization in spacetime have appeared in the recent literature as consequence of different approaches to quantum gravity or q-deformed calculus. In studies of quantum group (see for example S.Majid [6]) the commutation relation

$$ab - qba = 0, q \in C$$

is among the most typical, together with inhomogeneous relation

$$\tilde{a}b - q'\tilde{b}a = q''$$
where $q', q'' \in C$. For $q \neq 1$ these Eqs. can be transformed into another through
\begin{equation}
   b = \tilde{b}, a = \tilde{a} + \tilde{b}q'(q - 1)^{-1}, q = q',
\end{equation}
Let us remained that the commutation relation of the linear braided space (see S.Majid [6]) has the form:
\begin{equation}
   x_i x_j = x_b x_a R^{ab}_{ij},
\end{equation}
We now show the coincidence between our q-deformed QM and a model of the discretization of spacetime.
Let us suppose that $\Delta x_0 \equiv \tilde{q}^2 - 1 \approx 0$ is a parameter of the discretization of spacetime and $\tilde{q}$ a parameter of q-deformed QM.
Let us consider the discretization of standard differential calculus in one space dimension
\begin{equation}
   [x, dx] = dx \Delta x_0,
\end{equation}
and the action of the discrete translation group
\begin{equation}
   x^n dx = dx(x + \Delta x_0)^n,
\end{equation}
\begin{equation}
   \psi(x) dx = dx \psi(x + \Delta x_0),
\end{equation}
for any wave function $\psi$ of the Hilbert space of QM with the discrete space variable.
The discrete space variable can be defined as $x = n \Delta x_0$, where $n$ is an integer and $\Delta x_0$ is the interval between two discrete space points in this space variable.
If we define the derivatives by
\begin{equation}
   d\psi(x) = dx(\partial_x \psi)(x)(\tilde{\partial}_x \psi)(x)dx,
\end{equation}
\begin{equation}
   (\partial_x \psi)(x) = \frac{1}{\Delta x_0} [\psi(x + \Delta x_0) - \psi(x)],
\end{equation}
\begin{equation}
   (\tilde{\partial}_x \psi)(x) = \frac{1}{\Delta x_0} [\psi(x) - \psi(x - \Delta x_0)],
\end{equation}
then the ordinary one-dimensional Schrödinger equation will be
\[
\frac{1}{2} \frac{d^2 \psi(x)}{dx^2} + [E - U(x)]\psi(x) = 0, \tag{17}
\]
with the potential \( U(x) \) and wavefunction \( \psi(x) \equiv \psi(E,x) \), corresponding to energy value \( E \), has on the discrete space the form
\[
\frac{1}{2(\Delta x_0)^2} \left[ \psi((n + 1)\Delta x_0) - 2\psi(n\Delta x_0) + \psi((n - 1)\Delta x_0) \right] + [E - U(n\Delta x_0)]\psi(n\Delta x_0) = 0. \tag{18}\]

We now show the coincidence between such discretization model, non-commutative differential calculus and q-deformed QM, assuming \( \tilde{q}^2 \approx 1 \).

Let us suppose that ordinary continuum space variable \( y \) in QM has the form:
\[
y = \lim_{\Delta x_0 \to 0} (1 + \Delta x_0) \frac{\Delta x_0}{\Delta x_0} = e^x. \tag{19}\]

Using Eqs. (38-41) and (44) we get:
\[
\partial_y = y^{-1}\partial_x = (q_E + 1) \frac{1}{\tilde{q}^2} \partial_x. \tag{20}\]

Thus, using \( \Delta x_0 \equiv \tilde{q}^2 - 1 \), we have
\[
(\partial_y \psi)(y) = \frac{\psi((\Delta x_0 + 1)y) - \psi(y)}{\Delta x_0 y} = \frac{\psi(q^2 y) - \psi(y)}{(q^2 - 1)y}, \tag{21}\]
\[
(\partial_y \psi)(y) = (\Delta x_0 + 1) \frac{\psi(y) - \psi((\Delta x_0 + 1)y)}{\Delta x_0 y} = \frac{\psi(y) - \psi(q^2 y)}{(1 - q^{-2})y}. \tag{22}\]

what represents derivatives in the differential on the quantum hyperplane.

We can see that for \( \Delta x_0 \to 0 \) or \( \tilde{q}^2 \to 1 \) we have the ordinary QM and continuous space time.

There is unclear in the continuous space-time what is the “quantum line” because classically a coordinate always commutes with itself.

Quite different situation is in the quantum case of the discrete space-time or of space-time based on grassmannian variables.
3 Quantum superspace

We introduce supersymmetry (SUSY) with superspace \( \{ t, \Theta \} \), where \( t \) is the time variable and \( \Theta \) a Grassmann variable i.e. \( \Theta^2 = 1 \).

We define the supercoordinate

\[
X(t, \Theta) = x_{(0)}(t) + i \Theta x_{(1)}(t),
\]

(23)

where \( x_{(0)}(t) \) is the ordinary commuting space coordinate (Bose or null sector variable) and \( x_{(1)}(t) \) is the real anticommuting variable (Grassmann, Fermi or one-sector).

The changes of \( x_{(0)}(t) \) and \( x_{(1)}(t) \) follows from:

\[
\delta X(t, \Theta) = X(t', \Theta') - X(t, \Theta) = i \varepsilon QX(t, \Theta),
\]

(24)

where SUSY generator

\[
Q = \frac{\partial}{\partial \Theta} + i \Theta \frac{\partial}{\partial t} = \partial_\Theta + i \Theta \partial_t
\]

(25)

and \( \varepsilon \) is the infinitesimal Grassmann parameter.

We can see:

\[
\partial X = \varepsilon \partial_\Theta (x_{(0)} + i \Theta x_{(1)}) + i \varepsilon \Theta \partial_t (x_{(0)} + i \Theta x_{(1)})
= i \varepsilon x_{(1)} + i \varepsilon \Theta \partial_t x_{(0)}
\]

(26)

and SUSY transformations for the coordinates \( x_{(0)} \) and \( x_{(1)} \):

\[
\delta x_{(0)} = i \varepsilon x_{(1)}, \quad \delta x_{(1)} = \varepsilon \partial_t x_{(0)}
\]

(27)

It follows immediately:

\[
Q^2 X = Q [i x_{(1)} + i \Theta \partial_t x_{(0)}] = i \partial_t (x_{(0)} + i \Theta x_{(1)}) = i \partial_t X,
\]

(28)

or

\[
\frac{1}{2} \{ Q, Q \} X \equiv HX = i \partial_t X,
\]

(29)

which suggests that the Hamiltonian of the system be defined as \( H = \frac{1}{2} \{ Q, Q \} \) and the time translation is simply the Hamiltonian \( H = i \partial_t \).

In this sense the \( N = 1 \) SUSY (it means one Grassmann \( \Theta \)) is the square root of the time translation.

Let us now to turn to the general case i.e. the \( F-th \) roots of the time translation \( F = 1, 2, \ldots \).
We need $F$ real Grassmann coordinates $x_{(j)}(t)$, $j = 0, 1, \ldots, F - 1$, which belong to the following $(F-j)$-sectors $x_{(j)}(t)$ and the null sector $x_{(0)}(t) \equiv x(t)$ i.e. ordinary coordinate.

These sectors can be viewed as the components of a quantum superspace with fractional SUSY.

We denote fractional quantum superspace

$$X^F(t, \Theta) = \sum_{j=0}^{F-1} x_{(j)}(t) \Theta^j,$$  \hspace{1cm} (30)

where $\Theta$ is a real paragrassmannian variable satisfying $\Theta^F = 0$.

Let us introduce the $q$-commutation relation

$$x_{(j)}(t)x_{(F-j)}(t) = q^j x_{(F-j)}(t)x_{(j)}.$$  \hspace{1cm} (31)

In this sense the parameter $q^j$ connects different sectors.

Then fractional SUSY has the form:

$$\delta x_{(j-1)} = i \varepsilon (1 - q^j) x_{(j)},$$  \hspace{1cm} (32)

$$\delta x_{(F-1)} = \varepsilon (F \alpha^{F-1})^{-1} \partial_t x_{(0)},$$  \hspace{1cm} (33)

where $\alpha$ is a free constant.

We have

$$\delta^F x_{(j)}(t) = i^{(F-1)} \varepsilon_1 \ldots \varepsilon_F \partial_F x_{(j)}(t),$$  \hspace{1cm} (34)

since $\prod_{j=1}^F (1 - q^j) = F$ and $\varepsilon x_{(j)}(t) = q^{-j} x_{(j)}(t) \varepsilon$.

An invariant action is

$$S = \int dt \left[ \frac{1}{2} (\partial_t x)^2 + i (F \alpha^F) \sum_{j=0}^{F-1} (1 - q^{-j}) (\partial_t x_{(j)}) x_{(F-j)} \right]$$  \hspace{1cm} (35)

and fractional SUSY quantum mechanics (SSQM) of order $F$ is defined through the algebra

$$Q^F = H, \quad [H, Q] = 0, \quad F = 2, 3, \ldots,$$

where $H$ is the Hamiltonian.

The fractional SUSY can be extended by the following way:
For the $N = 2$ SUSY, the superspace is $(t, \Theta_1, \Theta_2)$ and SUSY transformation:

\[
\Theta_l' = \Theta_l + \varepsilon_l, \quad l = 1, 2; \quad t' = t + i \varepsilon_1 \Theta_1 + i \varepsilon_2 \Theta_2
\]  

(36)

Such extended SUSY can has application for anyonic or qubit superfields [4].

4 Some ideas from quantum mathematics

A quantum bit (qubit) is a quantum system with a two-dimensional Hilbert space, capable of existing in a superposition of Boolean states and of being entangled with the states of other qubits [1].

More precisely a qubit is the amount of the information which is contained in a pure quantum state from the two-dimensional Hilbert space $\mathcal{H}_2$.

A general superposition state of the qubit is

\[
|\psi\rangle = \psi_0 |0\rangle + \psi_1 |1\rangle,
\]

(37)

where $\psi_0$ and $\psi_1$ are complex numbers, $|0\rangle$ and $|1\rangle$ are kets representing two Boolean states. The superposition state has the propensity to be a 0 or a 1 and $|\psi_0|^2 + |\psi_1|^2 = 1$.

The eq.(1) can be written as

\[
|\psi\rangle = \psi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix} + \psi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix},
\]

(38)

where we labelled $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ and $\begin{pmatrix} 0 \\ 1 \end{pmatrix}$ two basis states zero and one.

The Clifford algebra relations of the $2 \times 2$ Dirac matrices is

\[
\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu},
\]

(39)

where

\[
\eta^{\mu\nu} = \begin{pmatrix} -1 & 0 \\ 0 & 1 \end{pmatrix}.
\]

(40)

We choose the representation

\[
\gamma^0 = i\sigma^2 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}
\]

(41)
and
\[ \gamma^1 = \sigma^3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \] (42)

where \( \sigma \) are Pauli matrices and \( \gamma^5 = \gamma^0 \gamma^1 \).

The projectors have the form
\[ P_0 = \frac{1 + \gamma^1}{2} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \] (43)
and
\[ P_1 = \frac{1 - \gamma^1}{2} = \begin{pmatrix} 0 & 0 \\ 0 & 1 \end{pmatrix} \] (44)

These projectors project qubit on the basis states zero and one:
\[ P_0 |\psi\rangle = \psi_0 \begin{pmatrix} 1 \\ 0 \end{pmatrix}, P_1 |\psi\rangle = \psi_1 \begin{pmatrix} 0 \\ 1 \end{pmatrix} \] (45)
and represent the physical measurements - the transformations of qubits to the classical bits.

In classical information theory the Shannon entropy is well defined:
\[ S_{\text{CL}}(\Phi) = -\sum_{\phi} p(\phi) \log_2 p(\phi), \] (46)

where the variable \( \Phi \) takes on value \( \phi \) with probability \( p(\phi) \) and it is interpreted as the uncertainty about \( \Phi \).

The quantum analog is the von Neumann entropy \( S_Q(\rho_{\Psi}) \) of a quantum state \( \Psi \) described by the density operator \( \rho_{\Psi} \):
\[ S_Q(\Psi) = -Tr_{\Psi} [\rho_{\Psi} \log_2 \rho_{\Psi}], \] (47)

where \( Tr_{\Psi} \) denotes the trace over the degrees of freedom associated with \( \Psi \). The von Neumann entropy has the information meaning, characterizing (asymptotically) the minimum amount of quantum resources required to code an ensemble of quantum states.

The density operator \( \rho_{\Psi} \) for the qubit state \( |\psi\rangle \) in (1) is given:
\[ \rho = |\psi\rangle \langle \psi| = |\psi_0|^2 |0\rangle \langle 0| + \psi_0 \psi_1^* |0\rangle \langle 1| + \psi_0^* \psi_1 |1\rangle \langle 0| + |\psi_1|^2 |1\rangle \langle 1| \] (48)
and corresponding density matrix is
\[ ρ_{kl} = \begin{pmatrix} |ψ_0|^2 & ψ_0 ψ_1^* \\ ψ_0^* ψ_1 & |ψ_1|^2 \end{pmatrix} \tag{49} \]
and \( k, l = 0, 1 \).

The von Neumann entropy reduces to a Shannon entropy if \( ρ_ψ \) is a mixed state composed of orthogonal quantum states.

A quantum gate is the analog of a logic gate in a classical computer. The NOT gate is \( X |k⟩ = |k ⊕ 1⟩ \), where the addition is \( \text{mod}(2) \). The unitary quantum gate defined
\[ M_- = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \tag{50} \]
defines an action \( M_- |0⟩ = |1⟩ \), \( M_- |1⟩ = |0⟩ \) is called a quantum not-gate.

The matrix
\[ \sqrt{M_-} = \begin{pmatrix} \frac{1 + i}{2} & \frac{1 - i}{2} \\ \frac{1 - i}{2} & \frac{1 + i}{2} \end{pmatrix} \tag{51} \]
If we denote \( |0⟩ = (1, 0)^T \) and \( |1⟩ = (0, 1)^T \) the action of
\[ \sqrt{M_-} |0⟩ = \frac{1 + i}{2} |0⟩ + \frac{1 - i}{2} |1⟩ \tag{52} \]
and
\[ \sqrt{M_-} |1⟩ = \frac{1 - i}{2} |0⟩ + \frac{1 + i}{2} |1⟩ \tag{53} \]
In general, an \( N \)-qubit can be in an arbitrary superposition of all \( 2^N \) classical states:
\[ |ψ_N⟩ = \sum α_x |x⟩ \quad x ∈ \{ |0⟩, |1⟩ \}^N \quad \text{and} \quad \sum_x |α_x|^2 = 1. \]

It is known that two bit gates are universal for quantum computation, which is likely to greatly simplify the technology required to build quantum computers.
Unitary gates, which play crucial role in QIS, are connected with R-matrices from QYBE and via Yang-Baxterization Hamiltonians can be expressed as square root of b-matrices. This established the connection between topology and QM, especially it gives topological interpretation of entanglement. Here we show it on the two-qubit state.

To understand this we start with a simple idea of quantum algorithm square root of not via partner–superpartner case. Supersymmetry is special case of anyonic Lie algebra $\mathbb{C}[\theta]/\theta^n$ with one coordinate $\theta, \theta^n = 0$ (for our Grassmann variable $\Theta^2 = 0$) [6] with anyonic variables fulfill:

$$\theta'' = \theta + \theta', \theta'\theta = e^{\frac{2\pi i}{n}} \theta \theta'$$

and the category of n-anyonic vector spaces where objects are $\mathbb{Z}_n$-graded spaces with the braided transposition:

$$\Psi(x \otimes y) = e^{\frac{2\pi i}{n} x|y|y \otimes x}$$

on elements $x, y$ of homogenous degree $|.|$.

Thus an anyonic braided group means as $\mathbb{Z}/n$-graded algebra B and coalgebra defined as $\varepsilon$:

$$B \rightarrow C, \Delta (x) = x_A \otimes x^A,$$

where summation understood over terms labelled by $A$ of homogeneous degree.

Coassociative and counital is in the sense:

$$x_{AB} \otimes x^B_A \otimes x^A = x^A \otimes x^A_B \otimes x^{AB}, \varepsilon(x_A) x^A = x = x_A \varepsilon(x^A)$$

and obeying

$$(xy)_A \otimes (xy)^A = x_A x_B \otimes x^A_B e^{\frac{2\pi i}{n} x^A |x_B|},$$

for all $x, y \in B$. The axioms for the antipode are as for usual quantum groups [6]. There is shown that anyonic calculus and anyonic matrices are generalization of quantum matrices and supermatrices.

The $N^2$ generators $t_{ij} = f(i) - f(j)$, where $f$ is a degree $\mathbb{Z}/n$ associated with the row or column fulfill

$$e^{(\frac{2\pi i}{n}) (f(i)f(k) + f(j)f(l))} R_{a \ b}^{k \ l} a^k \ b^l = e^{(\frac{2\pi i}{n}) (f(j)f(l) + f(i)f(k))} t^j \ _a a^j \ b^k \ l.$$
\begin{align}
\triangle t^i_j &= t^i_a \otimes t^a_j, \\
\varepsilon t^i_j &= \delta^i_j
\end{align}

(60)  

(61)  

It is required that $\mathcal{R}$ obeys certain anyonic QYBE. One method to obtain $\mathcal{R}$ is start with certain unitary solution $R$ of the usual QYBE and "transmute: them. In the diagrammatic notation this braided mathematics is known [5]. In QIS a "wiring" notation (which is known in physics like Feynman diagrams) was used for information flows for example for teleportation [4]. In the diagrammatic notation we "wire" the outputs of maps into the inputs of other maps to construct the algebraic operation Information flows along these wires except that under and over crossing are nontrivial operators, say $U$ and $U^{-1}$. Such operators can be universal quantum gates in QIS. Generally in anyonic we have new richer kind of "braided quantum field information mathematics".

To show it we begin with coincidence of the supersymmetric square root in SSQM $n = 2$ and square root of the not gate in QIS as an illustrative example.

It is well known SSQM is generated by supercharge operators $Q^+$ and $Q^- = (Q^+)^+$ which together with the Hamiltonian $H = 2H_{SSQM}$ of the system, where

$$H_{SSQM} = \frac{1}{L} \left( \begin{array}{cc} -\frac{d^2}{dt^2} + v^2 + v' & 0 \\ 0 & -\frac{d^2}{dt^2} + v^2 - v' \end{array} \right)$$

\begin{align}
H &= \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} = \begin{pmatrix} A^+ A^- & 0 \\ 0 & A^- A^+ \end{pmatrix} = - \left( \frac{d^2}{dx^2} \right) + \sigma_3 v' 
\end{align}

fulfil the superalgebra

$$Q^{\pm 2} = 0, \quad [H, Q^+] = [H, Q^-], \quad H = \{Q^+, Q^-\} = Q^2$$

(62)  

where

$$Q^- = \begin{pmatrix} 0 & 0 \\ A^- & 0 \end{pmatrix}, \quad Q^+ = \begin{pmatrix} 0 & A^+ \\ 0 & 0 \end{pmatrix}, \quad Q = Q^+ + Q^-$$
and

\[ A^\pm = \pm \frac{d}{dx} + v(x), \quad v' = \frac{dv}{dx}. \quad (63) \]

Such Hamiltonians \( H_0, H_1 \) fulfil

\[ H_0 A^+ = A^+ H_1, \quad A^- H_1 = H_0 A^-. \quad (64) \]

The eigenfunctions of the Hamiltonian

\[ H = \begin{pmatrix} H_0 & 0 \\ 0 & H_1 \end{pmatrix} \quad (65) \]

are \( \phi = (|0\rangle, |1\rangle)^T \) and then

\[ Q^0 \phi = |1\rangle, \quad (66) \]
\[ Q^1 \phi = |0\rangle. \quad (67) \]

where we denote \( Q^- = Q^0 \) and \( Q^+ = Q^1 \), respectively.

The previous relations in lead to the double degeneracy of all positive energy levels, of belonging to the “0” or “1” sectors specified by the grading state operator \( S = \sigma_3 \), where

\[ [S, H] = 0 \quad \text{and} \quad \{S, Q\} = 0. \]

The \( Q \) operator transforms eigenstates with \( S = +1 \), i.e. the null-state \( |0\rangle \) into eigenstates with \( S = -1 \), i.e. the one-state \( |1\rangle \) and vice versa.

With this notation the square root of not gate \( M_- = \sqrt{M_- \sqrt{M_-}} \) is represented by the unitary matrix \( \sqrt{M_-} \):

\[ \sqrt{M_-} = \frac{1}{2} \begin{bmatrix} 1 + i & 1 - i \\ 1 - i & 1 + i \end{bmatrix}, \]

that solves:

\[ \sqrt{M_-} \sqrt{M_-} |0\rangle = \sqrt{M_-} \left( \frac{1 + i}{2} |0\rangle + \frac{1 - i}{2} |0\rangle \right) = |1\rangle, \quad (68) \]
\[ \sqrt{M_-} \sqrt{M_-} |1\rangle = |0\rangle \quad (69) \]
In such way this supersymmetric double degeneracy represents two level quantum system and we can see the following:

\[ Q = Q^+ + Q^- = \begin{pmatrix} 0 & A^+ \\ A^- & 0 \end{pmatrix}, \]
\[ \tau = \sigma_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \]
\[ \{Q, \tau\} = 0. \]

It implies that the operator supercharge \( Q \) really transforms the state \( |0\rangle, |1\rangle \) as the operator square root of not quantum algorithm operator \( M_- \). In such a way supersymmetric "square root" corresponds the "square root of not" in QIS. We can ask if generally the Hamiltonians of the unitary braiding operators which leads to the Schrödinger equations are square root of QIS unitary gates. Such QIS unitary gates are unitary solutions of QYBE. The answer is positive and it can be explicitly shown for two-qubit system.

A system of two quantum bits is four dimensional space \( H_4 = H_2 \otimes H_2 \) having orthonormal basis \( |00\rangle, |01\rangle, |10\rangle, |11\rangle \).

A state of two-qubit system is a unit-length vector

\[ a_0|00\rangle + a_1|10\rangle + a_2|01\rangle + a_3|11\rangle, \tag{70} \]

so it is required \( |a_0|^2 + |a_1|^2 + |a_2|^2 + |a_3|^2 = 1 \).

We can see a state \( z \in H_4 \)

\[ z = \frac{1}{2}(|00\rangle + |01\rangle + |10\rangle + |11\rangle) = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle) \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle), \tag{71} \]

of a two-qubit system is decomposable, because it can be written as a product of states in \( H_2 \). A state that is not decomposable is entangled.

Consider the unitary matrix

\[ \mathcal{R} = \begin{pmatrix} a_0 & 0 & 0 & 0 \\ 0 & 0 & a_3 & 0 \\ 0 & a_2 & 0 & 0 \\ 0 & 0 & 0 & a_1 \end{pmatrix}, \tag{72} \]
defines a unitary mapping, whose action on the two-qubit basis is

\[ \Psi = R(\psi \otimes \psi) = R \left( \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \right) = \left( \begin{array}{c} a_0|00\rangle \\ a_3|10\rangle \\ a_2|01\rangle \\ a_1|11\rangle \end{array} \right). \] (73)

For \( a_0a_1 \neq a_2a_3 \) the state is entangled. For example the state is entangled \( \frac{1}{\sqrt{2}}(|01\rangle + |01\rangle) \) is entangled.

5 The QYBE and universal quantum gate

Matrix

\[ M_{CNOT} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, \] (74)

defines a unitary mapping, whose action on the two-qubit basis is

\[ M_{CNOT} \left( \begin{array}{c} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{array} \right) = \left( \begin{array}{c} |00\rangle \\ |01\rangle \\ |11\rangle \\ |10\rangle \end{array} \right). \] (75)

Gate \( M_{CNOT} \) is called controlled not, since the target qubit is flipped if and only if the control bit is 1.

Let \( R \)

\[ R = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 & 0 & 1 \\ 0 & 1 & -1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}, \] (76)

be the unitary solution to the QYBE.

Let \( M = M_1 \otimes M_2 \), where

\[ M_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}, \] (77)
and

\[ M_2 = \frac{1}{\sqrt{2}} \begin{pmatrix} -1 & 1 \\ i & i \end{pmatrix}, \]  

(78)

Let \( N = N_1 \otimes N_2 \), where

\[ N_1 = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ 1 & -i \end{pmatrix}, \]  

(79)

and

\[ N_2 = -\frac{1}{\sqrt{2}} \begin{pmatrix} 1 & 0 \\ 0 & i \end{pmatrix}, \]  

(80)

Then \( M_{\text{CNOT}} = M \cdot R \cdot N \) can be expressed in terms of \( R \). In the QYBE solution \( R \)-matrices usually depends on the deformation parameter \( q \) and the spectral parameter \( x \). Taking the limit of \( x \to 0 \) leads to the braided relation from the QYBE and the BGR \( b \)-matrix from the \( R \)-matrix. Yang-Baxterization is construct the \( R(x) \) matrix from a given BGR \( b \)-matrix. The BGR \( b \) of the eight-vertex model assumes the form

\[ b_{\pm} = \begin{pmatrix} 1 & 0 & 0 & q \\ 0 & 1 & \pm 1 & 0 \\ 0 & \mp 1 & 1 & 1 \\ -q^{-1} & 0 & 0 & 1 \end{pmatrix}, \]  

(81)

It has two eigenvalues \( \Lambda_{1,2} = 1 \pm i \). The corresponding \( R(x) \)-matrix via Yang-Baxterization is obtained to be

\[ R_{\pm}(x) = b + x \Lambda_1 \Lambda_2 = \begin{pmatrix} 1 + x & 0 & 0 & q(1 - x) \\ 0 & 1 + x & \pm(1 - x) & 0 \\ 0 & \mp (1 - x) & 1 + x & 1 \\ -q^{-1} (1 - x) & 0 & 0 & 1 + x \end{pmatrix}. \]  

(82)

Introducing the new variables \( \theta, \varphi \) as follows

\[ \cos \theta = \frac{1}{\sqrt{1 + x^2}}, \sin \theta = \frac{x}{\sqrt{1 + x^2}}, q = e^{i\varphi} \]  

(83)

the \( R \)-matrix has the form

\[ R_{\pm}(\theta) = \theta \cos(\theta) b_{\pm}(\varphi) + \sin(\theta) b_{\mp}^{-1}(\varphi) \]  

(84)
The time-independent Hamiltonian $H_{\pm}$ has the form

$$H_{\pm} = -\frac{i}{2} b_{\pm}^2(\varphi) = \frac{i}{2} \begin{pmatrix} 0 & 0 & 0 & -e^{i\varphi} \\ 0 & 0 & \mp 1 & 0 \\ 0 & \pm 1 & 0 & 0 \\ e^{-i\phi} & 0 & 0 & 0 \end{pmatrix},$$  \tag{85}$$

6 Topology and entanglement in physical application

The braid group representation $b_{\pm}(\varphi)$ yield the Bell states with the phase factor $e^{i\varphi}$

$$b_{\pm}(\varphi) \begin{pmatrix} |00\rangle \\ |01\rangle \\ |10\rangle \\ |11\rangle \end{pmatrix} = \frac{1}{\sqrt{2}} \begin{pmatrix} |00\rangle + e^{i\varphi}|11\rangle \\ |10\rangle \pm |01\rangle \\ \mp |01\rangle + |10\rangle \\ -e^{-i\varphi}|00\rangle + |11\rangle \end{pmatrix}. \tag{86}$$

which shows that $\varphi = 0$ leads to the Bell states, the maximum of entangled states:

$$\frac{1}{\sqrt{2}}(|00\rangle \pm |11\rangle), \tag{87}$$

$$\frac{1}{\sqrt{2}}(|10\rangle \pm |01\rangle). \tag{88}$$

In this chapter we want to show possible explanation of experimental measurements of the CP violation, using results from the QIS presented here.

For this purpose we consider in the $K\bar{K}$ systems entangled states in $H_4$:

$$|\Phi_1\rangle = \frac{1}{\sqrt{2}}(|KK\rangle + |\bar{K}\bar{K}\rangle), \quad |\Phi_2\rangle = \frac{1}{\sqrt{2}}(|KK\rangle - |\bar{K}\bar{K}\rangle),$$  \tag{89}$$

and

$$|\Phi_3\rangle = \frac{1}{\sqrt{2}}(|K\bar{K}\rangle + |\bar{K}K\rangle), \quad |\Phi_4\rangle = \frac{1}{\sqrt{2}}(|K\bar{K}\rangle - |\bar{K}K\rangle),$$  \tag{90}$$
The necessary codefinition of strangeness $\hat{S}$ and CP can be rigorously justified by applying local realism to a kaon belonging to a correlated kaon pair.

From this point of view the experimental measurement of $K \to 2\pi$ through the quantity

$$\eta_f \equiv \frac{A(\text{entagled L and S states } \to f)}{A(S \to f)}, \quad (91)$$

where $f$ is a pair of either two charged or two neutral pions, can be the effect from the measurement of the quantum nonseparability of the entangled kaon states and the discrete symmetry CP is preserved.

The above classical information analysis $K\bar{K}$ system via probabilities as is usual is not sufficient for the description of the quantum nonseparability.

The quantum nonseparability in the $K\bar{K}$ system needs to study this quantum system via the conditional density matrix and correlations between kaon states in QIS.

In the case of $\Phi$ and B-factories, where the neutral meson states produced ($K\bar{K}$ and $B\bar{B}$, respectively) constitute correlated Einstein-Podolsky-Rosen states (EPR), the knowledge that one of the two mesons decays at a given time through a charge specific channel ("tagging $\pm$") allows one to unambiguously infer the charge of the accompanying meson state at the same time.

If the separability criterion in kaon system is in the sense that in every moment we have correlation in every kaon system as two-qubit system in every moment (it means that every $|K\rangle$ is correlated with $|K\rangle$ or $|\bar{K}\rangle$), then we have four states $\Phi_{1,2,3,4}$ with "tagging" charges CP and $S \pm$.

It gives the possibility to explain CP-violation via deformation of entanglement (it means Bell-states $\Phi_{1,2,3,4}$ are not maximally entangled)

The violation can be realized via the phase factor $e^{i\varphi}$

$$b_{\pm}(\varphi) = \frac{1}{\sqrt{2}} \begin{pmatrix} |K\bar{K}\rangle + e^{i\varphi}|\bar{K}\bar{K}\rangle \\ |\bar{K}\bar{K}\rangle + |K\bar{K}\rangle \\ |K\bar{K}\rangle \pm |\bar{K}\bar{K}\rangle \\ -e^{-i\varphi}|K\bar{K}\rangle + |\bar{K}\bar{K}\rangle \end{pmatrix} \quad (92)$$

We can see two possibilities:

- one of the possible explanation is a topological origin of CP violation and the second possibility is the separability criteria in kaon system.
7 Separability criterion in kaon system

We consider a pair of $CP = \pm 1$, $|S\rangle$ and $|L\rangle$ in an impure mixing of pure entangled states, consisting a fraction $\lambda$.

We shall introduce the following pure $CP = \pm 1$ entangled states:

$$
|SL\rangle = \frac{1}{\sqrt{1+|\varepsilon|^2}} |S\rangle - \frac{\varepsilon}{\sqrt{1+|\varepsilon|^2}} |L\rangle
$$

and pure $\hat{S} = \pm 1$ entangled states:

$$
|S\bar{S}\rangle, |L\bar{L}\rangle,
$$

where $\varepsilon$ is a little complex parameter, which define a influence of the second state, which was initially present. Let us now consider a mixed states with a fraction parameter $\lambda$, i.e. a real number between 0 and 1:

$$
\rho(\varepsilon, \lambda) = \lambda P_{|SL\rangle} + \frac{1}{2}(1 - \lambda)(P_{|SS\rangle} + P_{|LL\rangle}),
$$

Using the Horodecki notations cit 8:

$$
\rho(\varepsilon, \lambda) = \frac{1}{4}\left[ I + \lambda \frac{1 - |\varepsilon|^2}{1 + |\varepsilon|^2} (\sigma_z \otimes I - I \otimes \sigma_z) + (1 - 2\lambda)\sigma_z \otimes \sigma_z - 2\lambda \frac{|\varepsilon|}{1 + |\varepsilon|^2} (\sigma_x \otimes \sigma_x + \sigma_y \otimes \sigma_y), \right]
$$

where $\sigma$ are Pauli matrices. Following cite 8 $\rho(\varepsilon, \lambda)$ can violate Bell inequality if

$$
M(\rho(\lambda, \varepsilon)) = \max\{ (2\lambda - 1)^2 + 4\lambda^2 \left( \frac{|\varepsilon|}{1 + |\varepsilon|^2} \right)^2 , 8\lambda^2 \left( \frac{|\varepsilon|}{1 + |\varepsilon|^2} \right)^2 \} > 1.
$$

So for

$$
\lambda > \frac{1}{2} \left( 1 - \frac{|\varepsilon|}{1 + |\varepsilon|^2} \right)^{-1},
$$

the quantum density matrix has the negative determinant, and therefore a negative eigenvalue. It is the inseparability criterion.

We can see that for $\varepsilon = 0$ there is no CP violation and in the experiments is measured rather the quantum nonseparability in the $K\bar{K}$ system.
It is easily seen that the fraction $\frac{1}{2}(1 - \lambda)$ connected with the $| S(t)\bar{S}(0)\rangle$ can have the experimental value for $\eta_{2\pi} \sim 2.27 \times 10^{-3}$. From this the value for the fraction $\lambda$ follows:

$$\lambda = 0.99546 > \frac{1}{2},$$

and this experimental value for $\lambda$ is in agreement with the nonseparability of the kaon states.

Of course we can use more complicated models simulating the contamination of the purely fully entangled states by random sources. We use two parameters: $\alpha$ allowing admixture of single $L$ and $S$ kaons and $v$ for pairs of entangled kaon states.

The contaminated source is given by

$$\rho_{LS} = \alpha[v\rho_E + (1 - v)\rho_R] + (1 - \alpha)[\frac{\rho_{LS}(t)\rho_v - \rho_v\rho_{LS}(0)}{2}],$$

where the idealized source with maximally entangled state $E$ is

$$\rho_E = | E\rangle\langle E |, | E \rangle = \frac{1}{2}(| S(t)\bar{S}(0)\rangle - | L(t)\bar{L}(0)\rangle)$$

the random source of $L$ and $S$ states

$$\rho_R = \frac{1}{4}(| S\bar{S}\rangle\langle \bar{S}S | + | SL\rangle\langle LS | + | LS\rangle\langle SL | + | L\bar{L}\rangle\langle \bar{L}L |).$$

The random source for single $L$ and $S$ states is

$$\rho_{LS} = (| S\rangle\langle \bar{S} | + | L\rangle\langle \bar{L} |),$$

and the vacuum term is

$$\rho_v = | 0\rangle\langle 0 |,$$

the same for the proper time $t$ and time 0.

Direct evaluation of the matrix element of the matrix elements of $\rho$ and then diagonalizing it gives for the entropy of the composite kaon system:

$$-S_Q(LS) = \frac{3}{4}\alpha(1 - v)\ln\frac{\alpha(1 - v)}{4} + \frac{1}{4}\alpha(1 + 3v)\ln\frac{\alpha(1 + 3v)}{4} + \alpha(1 - \alpha)\ln\frac{1 - \alpha}{4}$$

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and the entropy for the single kaon system

\[-S_Q(L, S) = \frac{1 + \alpha}{2} \ln \left(\frac{1 + \alpha}{4}\right) + \frac{1 - \alpha}{2} \ln \left(\frac{1 - \alpha}{4}\right)\]  \tag{106}\]

is the same for the proper time \(t\) and time \(0\).

The boundary criterion for the entanglement is obtained by equating both entropies \(S_Q(\langle L \rangle, S) = S_Q(L, S)\) and inseparability criterion is given [4]:

\[\alpha v > \frac{1}{\sqrt{2}}.\]  \tag{107}\]

If we have no single states \(\alpha = 1\) the parameter \(v = \lambda\) gives us the information about the nonseparability of kaon entanglement pairs. For the parameter \(\alpha\) we get the value

\[\alpha > 0.71033\]  \tag{108}\]

and for this value there is the nonseparability of the single kaon states.

8 Conclusions

The new development in QIS and appearing the topological fundament of entanglement. The concept of entanglement for pure quantum states was established in the early days of quantum mechanics, but the crucial role in QIS gives quite new interpretation. We hope that the geometrical and topological point of view on QIS can give quite new theories for physics.

Quite new is topological understanding of the violation of famous effect of the violation the CP discrete symmetry in physics.

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