Ricci flow on open 4-manifolds with positive isotropic curvature

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Abstract

In this note we prove the following result: Let \( X \) be a complete, connected 4-manifold with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form. Then \( X \) is diffeomorphic to \( S^4 \), or \( \mathbb{RP}^4 \), or \( S^3 \times S^1 \), or \( S^3 \tilde{\times} S^1 \), or a possibly infinite connected sum of them. This extends work of Hamilton and Chen-Zhu to the noncompact case. The proof uses Ricci flow with surgery on complete 4-manifolds, and is inspired by recent work of Bessières, Besson and Maillot.

Key words: uniformly positive isotropic curvature, bounded geometry, Ricci flow with surgery on complete manifolds

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1 Introduction

In a recent paper [BBM] Bessières, Besson and Maillot classified complete 3-manifolds with uniformly positive scalar curvature and with bounded geometry using a variant of Hamilton-Perelman’s Ricci flow with surgery. Inspired by their work we try to classify complete 4-manifolds with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form. More precisely we will show

Theorem 1.1. Let \( X \) be a complete, connected 4-manifold with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form. Then \( X \) is diffeomorphic to \( S^4 \), or \( \mathbb{RP}^4 \), or \( S^3 \times S^1 \), or \( S^3 \tilde{\times} S^1 \), or a possibly infinite connected sum of them.

(Here, \( S^3 \tilde{\times} S^1 \) is the only unorientable \( S^3 \) bundle over \( S^1 \). The notion of a (possibly infinite) connected sum will be given later in this section; cf. [BBM]. By [MW] it
is easy to see that the converse is also true: Any 4-manifold as in the conclusion of the theorem has no essential incompressible space form, and admits a complete metric with uniformly positive isotropic curvature and with bounded geometry.)

This extends work of Hamilton [H5] and Chen-Zhu [CZ2] to the noncompact case.

Recall ([MM]) that a Riemannian manifold $M$ is said to have positive isotropic curvature (PIC) if for all points $p \in M$ and all orthonormal 4-frames $\{e_1, e_2, e_3, e_4\} \subset T_p M$ the curvature tensor satisfies

$$R_{1313} + R_{1414} + R_{2323} + R_{2424} > 2R_{1234}. $$

Now we consider in particular a 4-dimensional manifold $X$. If we decompose the bundle $\Lambda^2 TX$ into the direct sum of its self-dual and anti-self-dual parts

$$\Lambda^2 TX = \Lambda^2_+ TX \oplus \Lambda^2_- TX,$$

then the curvature operator can be decomposed as

$$\mathcal{R} = \begin{pmatrix} A & B \\ B^T & C \end{pmatrix},$$

where $A = W_+ + \frac{R}{12}$, $C = W_- + \frac{R}{12}$, (here $W_+$ and $W_-$ are the self-dual part and the anti-self-dual part of the Weyl curvature respectively,) and $B$ gives the trace free part of the Ricci tensor. Denote the eigenvalues of the matrices $A, C$ and $\sqrt{BB^T}$ by $a_1 \leq a_2 \leq a_3$, $c_1 \leq c_2 \leq c_3$ and $b_1 \leq b_2 \leq b_3$ respectively. It is easy to see (cf. Hamilton [H5]) that for a Riemannian 4-manifold the condition of positive isotropic curvature is equivalent to the condition $a_1 + a_2 > 0$ and $c_1 + c_2 > 0$. A Riemannian 4-manifold $X$ is said to have uniformly positive isotropic curvature if there is a positive constant $c$ such that $a_1 + a_2 \geq c$ and $c_1 + c_2 \geq c$ everywhere.

As in [H5], an incompressible space form in a 4-manifold $X$ is a 3-dimensional submanifold $Y$ diffeomorphic to $S^3/\Gamma$ (where $\Gamma$ is a finite, fixed point free subgroup of isometries of $S^3$) such that $\pi_1(Y)$ injects into $\pi_1(X)$. The space form is called essential unless $\Gamma = 1$, or $\Gamma = \mathbb{Z}_2$ and the normal bundle is non-orientable. Also recall that a complete Riemannian manifold is said to have bounded geometry if the sectional curvature is bounded (in both sides) and the injectivity radius is bounded away from zero.

Now we explain the notion of (possibly infinite) connected sum, following [BBM]. Let $\mathcal{X}$ be a class of closed 4-manifolds. A 4-manifold $X$ is said to be a connected sum of members of $\mathcal{X}$ if there exists a locally finite graph $G$ and a map $v \mapsto X_v$ which associates to each vertex of $G$ a copy of some manifold in $\mathcal{X}$, such that by removing from each $X_v$ as many open 4-balls as vertices incident to $v$ and gluing the thus punctured $X_v$’s to each other along the edges of $G$ using diffeomorphisms of the boundary 3-spheres, one obtains a 4-manifold diffeomorphic to $X$.

Hamilton [H5] first used the Ricci flow with surgery to study compact 4-manifolds with positive isotropic curvature and with no essential incompressible space-form. (As Perelman [P2] pointed out, [H5] contains some unjustified statements. See also [CZ2].) Later in a breakthrough [P1], [P2] Perelman introduced
some important new ideas for the analysis of the Ricci flow, and devised a somewhat different surgery procedure for it: one of the differences lies in that Hamilton does surgery before curvature blows up, while Perelman does surgery exactly when curvature blows up. (For more details, variants and/or alternatives of Perelman’s arguments, see for examples [BBB⁺], [CaZ], [KL], [MT] and [Z].) Using Perelman’s ideas Chen-Zhu [CZ2] gave a complete proof of Hamilton’s main theorem in [H5]. Recently Chen-Tang-Zhu [CTZ] completely classified all compact 4-manifolds (and 4-orbifolds with isolated singularities) with positive isotropic curvature using Ricci flow with surgery on orbifolds. (Note that in [CZ2] and [CTZ] the surgeries are done exactly when curvature blows up as in [P2].)

Our proof of Theorem 1.1 uses a 4-dimensional analogue of a version of surgery constructed by Bessières, Besson and Maillot ([BBM]) in 3-dimension; see also [BBB⁺]. Their surgery procedure is closer to that of Hamilton in the sense that they do surgery before the curvature blows up; on the other hand, they also use crucial ideas from Perelman [P1], [P2]. However, I adopt a somewhat different approach from that in [BBM] to prove the existence of \((r, \delta, \kappa)\)-surgical solution with initial data a complete 4-manifold with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form, see Theorem 3.4. Note that Perelman’s proof of [P2, Proposition 5.1] uses the openness (w.r.t. time) property of canonical neighborhood assumption. In noncompact case it is not clear whether it is still true. It turns out that a weak openness (w.r.t. time) property of canonical neighborhood assumption holds in our noncompact situation; see Claim 1 in the proof of Proposition 3.6. We also need a slightly more general form of the persistence of almost standard cap (in the phrase of [BBB⁺]), see Proposition 3.1, which corresponds to [P2, Lemma 4.5]. With these tools in hand, we can adapt the original proof in [P2] and [CZ2] to our noncompact case. Our approach can be adapted to treat more general cases than that is considered in this note. Actually, I have used the method in this note to deal with complete 4-orbifolds with uniformly positive isotropic curvature, see [Hu1] and [Hu2]. (Those two papers were written before this note was, and the main results of this note are special case of those two papers, but I think maybe it is worth to write down the details of this more simple case, since in this case one needs not to worry about the additional complexity in the orbifold case, and the main idea is clearer.) I benefit much from [BBB⁺], [BBM], [H5], [P1], [P2] and [CZ2]. In particular, many definitions and proofs in this note are adapted from [BBB⁺], [BBM], [P2] and [CZ2].

In Section 2 we give some definitions and preliminary results, and in Section 3, we construct \((r, \delta, \kappa)\)-surgical solution with initial data a complete 4-manifold with uniformly positive isotropic curvature, with bounded geometry and with no essential incompressible space form, then Theorem 1.1 follows quickly. In Appendix A we collect some technical results on gluing \(\varepsilon\)-necks, and finally in Appendix B we give a version of bounded curvature at bounded distance for our surgical solution, following [P1], [P2]. In most cases we will follow the notations and conventions in [BBB⁺] and [BBM].
2 Surgical solutions on open 4-manifolds with uniformly PIC

Let \((X, g_0)\) be a complete 4-manifold with \(|Rm| \leq K\). Consider the Ricci flow (\([H1]\))

\[
\frac{\partial g}{\partial t} = -2Ric, \quad g|_{t=0} = g_0.
\]  

(2.1)

By Shi \([S]\), (2.1) has a short time solution with complete time slice and with bounded curvature. By Chen-Zhu \([CZ1]\) this solution is unique (in the category of complete solutions with bounded curvature).

Now we assume that the 4-manifold \((X, g_0)\) has uniformly positive isotropic curvature. Then we can easily generalize Hamilton’s pinching estimates in \([H5]\) to our situation, which plays a similar role in the category of 4-manifolds with uniformly positive isotropic curvature as the Hamilton-Ivey pinching estimate does in the category of 3-manifolds.

**Lemma 2.1.** (cf. Hamilton \([H5]\)) Let \((X, g_0)\) be a complete 4-manifold with uniformly positive isotropic curvature \((a_1 + a_2 \geq c, c_1 + c_2 \geq c)\) and with bounded curvature \((|Rm| \leq K)\). Then there exist positive constants \(\varrho, \Psi, L, P, S < +\infty\) depending only on the initial metric (through \(c, K\)), such that the complete solution to the Ricci flow (2.1) with bounded curvature satisfies

\[
a_1 + \varrho > 0, \quad c_1 + \varrho > 0, \\
\max\{a_3, b_3, c_3\} \leq \Psi(a_1 + \varrho), \quad \max\{a_3, b_3, c_3\} \leq \Psi(c_1 + \varrho), \\
\frac{b_3}{\sqrt{(a_1 + \varrho)(c_1 + \varrho)}} \leq 1 + \frac{Le^{Pt}}{\max\{\ln \sqrt{(a_1 + \varrho)(c_1 + \varrho)}, S\}}
\]

at all points and times.

**Proof** Note that Hamilton’s maximum principle for Ricci flow \([H2]\) holds in the case of complete manifolds with bounded curvature (see e.g. \([CCG+08, \text{Chapter 12}]\)). Then by inspecting Hamilton’s original proof in \([H5, \text{Section B}]\) we see that the lemma is true.

Since the 4-manifolds we consider have uniformly positive isotropic curvature, and in particular, have uniformly positive scalar curvature, the Ricci flow (2.1) will blow up in finite time. Using Lemma 2.1, we see that any blow-up limit (if it exists) coming from a solution as in Lemma 2.1 satisfies the following restricted isotropic curvature pinching condition

\[
a_3 \leq \Psi a_1, \quad c_3 \leq \Psi c_1, \quad b_3^2 \leq a_1 c_1,
\]

(2.3)

and in particular, has nonnegative curvature operator.
Following [H5], [BBB+] and [BBM], we will do surgery before the curvature blows up. Roughly speaking, the surgery procedure is: start with (2.1), at certain time before and near the first time when the curvature will blow up, cutoff necks in the manifold where the curvature is large, glue back caps, and remove some components with known topology to reduce the large curvature; continue the flow until one comes near the next time when the curvature will blow up, then do surgery as before, and continue $\cdots$.

Now we will adapt some definitions from [BBM].

Definition ([BBM]) Given an interval $I \subset \mathbb{R}$, an evolving Riemannian manifold is a pair $(X(t), g(t))$ $(t \in I)$, where $X(t)$ is a (possibly empty or disconnected) manifold and $g(t)$ is a Riemannian metric on $X(t)$. We say that it is piecewise $C^1$-smooth if there exists a discrete subset $J$ of $I$, such that the following conditions are satisfied:

i. On each connected component of $I \setminus J$, $t \mapsto X(t)$ is constant (in topology), and $t \mapsto g(t)$ is $C^1$-smooth;

ii. For each $t_0 \in J$, $X(t) = X(t_0)$ for any $t < t_0$ sufficiently close to $t_0$, and $t \mapsto g(t)$ is left continuous at $t_0$;

iii. For each $t_0 \in J \setminus \{\sup I\}$, $t \mapsto (X(t), g(t))$ has a right limit at $t_0$, denoted by $(X_+(t_0), g_+(t_0))$.

As in [BBM], a time $t \in I$ is regular if $t$ has a neighborhood in $I$ where $X(\cdot)$ is constant and $g(\cdot)$ is $C^1$-smooth. Otherwise it is singular. We also denote by $f_{\max}$ and $f_{\min}$ the supremum and infimum of a function $f$, respectively, as in [BBM].

Definition (Compare [BBM]) A piecewise $C^1$-smooth evolving Riemannian 4-manifold $\{(X(t), g(t))\}_{t \in I}$ with uniformly positive isotropic curvature, with bounded curvature and with no essential incompressible space form is called a surgical solution to the Ricci flow if it has the following properties:

i. The equation $\frac{\partial g}{\partial t} = -2\text{Ric}$ is satisfied at all regular times;

ii. For each singular time $t$ one has $(a_1 + a_2)_{\min}(g_+(t)) \geq (a_1 + a_2)_{\min}(g(t))$, $(c_1 + c_2)_{\min}(g_+(t)) \geq (c_1 + c_2)_{\min}(g(t))$, and $R_{\min}(g_+(t)) \geq R_{\min}(g(t))$;

iii. For each singular time $t$ there is a locally finite collection $\mathcal{S}$ of disjoint, embedded $S^3$'s in $X(t)$, and a manifold $X'$ such that

(a) $X'$ is obtained from $X(t) \setminus \mathcal{S}$ by gluing back $B^4$'s (closed 4-balls),

(b) $X_+(t)$ is a union of some connected components of $X'$ and $g_+(t) = g(t)$ on $X_+(t) \cap X(t)$, and

(c) Each component of $X' \setminus X_+(t)$ is diffeomorphic to $S^4$, or $\mathbb{RP}^4$, or $\mathbb{RP}^4 \setminus \mathbb{RP}^4$, or $S^3 \times S^1$, or $S^3 \times S^1$, or $\mathbb{R}^4$, or $\mathbb{RP}^4 \setminus B^4$, or $S^3 \times \mathbb{R}$.

Lemma 2.2. Any complete surgical solution with $a_1 + a_2 \geq c$, $c_1 + c_2 \geq c$ and starting at $t = 0$ must become extinct at some time $T < \frac{1}{2c}$. 


Proof From the evolution equation
\[ \frac{\partial R}{\partial t} = \Delta R + 2|\text{Ric}|^2 \] (2.4)
for the scalar curvature under Ricci flow, the maximum principle and the definition above, any complete surgical solution with \( a_1 + a_2 \geq c, \ c_1 + c_2 \geq c \) must become extinct at some time \( T \leq \frac{2}{R_{\text{min}}(0)} < \frac{1}{2c} \).

Let \( \{(X(t), g(t))\}_{t \in I} \) be a surgical solution and \( t_0 \in I \). As in [BBM], if \( t_0 \) is singular, we set \( X_{\text{reg}}(t_0) := X(t_0) \cap X_+(t_0) \), and \( X_{\text{sing}}(t_0) := X(t_0) \setminus X_{\text{reg}}(t_0) \). If \( t_0 \) is regular, \( X_{\text{reg}}(t_0) = X(t_0) \) and \( X_{\text{sing}}(t_0) = \emptyset \). Let \( t_0 \in [a, b] \subset I \) be a time, and \( Y \) be a subset of \( X(t_0) \) such that for every \( t \in [a, b] \), we have \( Y \subset X_{\text{reg}}(t) \). Then as in [BBM], we say the set \( Y \times [a, b] \) is unscathed.

In [H5] Hamilton devised a quantitative metric surgery procedure; later Perelman [P2] gave a somewhat different version, and in particular, he had the crucial notion of “canonical neighborhood”. To describe it we need some more notions such as \( \varepsilon \)-neck, \( \varepsilon \)-cap and strong \( \varepsilon \)-neck as given in [P2], [BBM], [CZ2].

Let \( (X, g) \) be a Riemannian 4-manifold, and \( x_0 \in M \). An open neighborhood \( N \subset X \) of \( x_0 \) is an \( \varepsilon \)-neck centered at \( x_0 \) if there is a diffeomorphism \( \psi : S^3 \times \mathbb{I} \rightarrow N \) such that the pulled back metric \( \psi^*g \), scaling with some factor, is \( \varepsilon \)-close (in \( C^{[\varepsilon^{-1}] \text{-close}} \) topology) to the standard metric \( S^3 \times \mathbb{I} \) with scalar curvature 1 and \( \mathbb{I} = (-\varepsilon^{-1}, \varepsilon^{-1}) \), and such that \( x_0 \in \psi(S^3 \times \{0\}) \).

An open subset \( U \) is an \( \varepsilon \)-cap centered at \( x_0 \) if \( U \) is the union of two sets \( V, W \) such that \( x_0 \in \text{Int} \ V, V \) is diffeomorphic to \( B^4 \) or \( \mathbb{RP}^4 \setminus (\text{Int} \ B^4) \), \( \overline{\partial V} \cap V = \partial V \), and \( W \) is an \( \varepsilon \)-neck.

Let \( (X(t), g(t)) \) be an evolving Riemannian 4-manifold, and \( (x_0, t_0) \) be a spacetime point. An open subset \( N \subset X(t_0) \) is a strong \( \varepsilon \)-neck centered at \( (x_0, t_0) \) if there is a number \( Q > 0 \) such that the set \( \{(x, t) | x \in N, t \in [t_0 - Q^{-1}, t_0]\} \) is unscathed, and there is a diffeomorphism \( \psi : S^3 \times \mathbb{I} \rightarrow N \) such that, the pulled back solution \( \psi^*g(\cdot, \cdot) \) scaling with the factor \( Q \) and shifting the time \( t_0 \) to 0, is \( \varepsilon \)-close (in \( C^{[\varepsilon^{-1}] \text{-close}} \) topology) to the subset \( (S^3 \times \mathbb{I}) \times [-1, 0] \) of the evolving round cylinder \( S^3 \times \mathbb{R} \), with scalar curvature one and length \( 2\varepsilon^{-1} \) to \( \mathbb{I} \) at time zero, and \( x_0 \in \psi(S^3 \times \{0\}) \).

Motivated by the structure theorems of 4-dimensional ancient \( \kappa \)-solution with restricted isotropic curvature pinching ([CZ2, Theorem 3.8]) and the standard solution ([CZ2, Corollary A.2]), following [P2], [BBM], [CZ2], we introduce the notion of canonical neighborhood.

Definition Let \( \varepsilon \) and \( C \) be positive constants. A point \( (x, t) \) in a surgical solution to the Ricci flow is said to have an \( (\varepsilon, C) \)-canonical neighborhood if it has an open neighborhood \( U, B_t(x, \sigma) \subset U \subset B_t(x, 2\sigma) \) with \( C^{-1}R(x, t)^{-\frac{1}{2}} < \sigma < CR(x, t)^{-\frac{1}{2}} \), which falls into one of the following three types:

(a) \( U \) is a strong \( \varepsilon \)-neck with center \( (x, t) \),
(b) \( U \) is an \( \varepsilon \)-cap with center \( x \) for \( g(t) \),
(c) at time $t$, $U$ is a compact 4-manifold with positive curvature operator, and moreover, the scalar curvature in $U$ at time $t$ is between $C^{-1}R(x,t)$ and $CR(x,t)$, and satisfies the derivative estimates

$$|\nabla R| < CR^2 \quad \text{and} \quad \left|\frac{\partial R}{\partial t}\right| < CR^2,$$

and the volume estimate

$$(CR(x,t))^{-2} < vol_t(U).$$

**Remark** Note that by [CZ2, Proposition 3.4 and Theorem 3.8] and [CZ2, Corollary A.2], for every $\varepsilon > 0$, there exists a positive constant $C(\varepsilon)$ such that each point in any ancient $\kappa$-solution with restricted isotropic curvature pinching or in the standard solution has an $(\varepsilon,C(\varepsilon))$-canonical neighborhood, except that for the standard solution, an $\varepsilon$-neck may not be strong.

We choose $\varepsilon_0 > 0$ such that $\varepsilon_0 < 10^{-4}$ and such that when $\varepsilon \leq 2\varepsilon_0$, Lemma A.1 in Appendix A and the results in the paragraph following its proof hold true. Let $\beta := \beta(\varepsilon_0)$ be the constant given by Lemma A.2 in Appendix A. Define $C_0 := \max\{100\varepsilon_0^{-1}, 2C(\beta\varepsilon_0/2)\}$, where $C(\cdot)$ is given in the Remark above. Fix $c_0 > 0$. Let $\varrho_0, \Psi_0, L_0, P_0, S_0$ be the constants given in Lemma 2.1 by setting $c = c_0$ and $K = 1$.

Now we consider some a priori assumptions, which consist of the pinching assumption and the canonical neighborhood assumption.

**Pinching assumption:** Let $\varrho_0, \Psi_0, L_0, P_0, S_0$ be positive constants as given above. A surgical solution to the Ricci flow satisfies the pinching assumption (with pinching constants $\varrho_0, \Psi_0, L_0, P_0, S_0$) if there hold

$$a_1 + \varrho_0 > 0, \quad c_1 + \varrho_0 > 0,$$

$$\max\{a_3, b_3, c_3\} \leq \Psi_0(a_1 + \varrho_0), \quad \max\{a_3, b_3, c_3\} \leq \Psi_0(c_1 + \varrho_0),$$

and

$$\frac{b_3}{\sqrt{(a_1 + \varrho_0)(c_1 + \varrho_0)}} \leq 1 + \frac{L_0e^{P_0t}}{\max\{\ln\sqrt{(a_1 + \varrho_0)(c_1 + \varrho_0)}, S_0\}} \quad (2.5)$$

at all points and times.

**Canonical neighborhood assumption:** Let $\varepsilon_0$ and $C_0$ be as given above. Let $r : [0, +\infty) \to (0, +\infty)$ be a non-increasing function. An evolving Riemannian 4-manifold $\{(X(t), g(t))\}_{t\in I}$ satisfies the canonical neighborhood assumption (CN), if any space-time point $(x,t)$ with $R(x,t) \geq r^{-2}(t)$ has an $(\varepsilon_0, C_0)$-canonical neighborhood.
Let \( \{(X(t), g(t))\}_{t \in I} \) be an evolving Riemannian 4-manifold. Recall [P1] that given \( \kappa > 0, r > 0, g(\cdot) \) is \( \kappa \)-noncollapsed at \( (x,t) \) (where \( t \geq r^2 \), and \( P(x,t,r,-r^2) \) is unscathed) on the scale \( r \) if

\[
|Rm| \leq r^{-2} \text{ on } P(x,t,r,-r^2) \text{ implies } \text{vol}B(x,t,r) \geq \kappa r^4,
\]

where \( P(x,t,r,-\Delta t) := \{(x',t')| x' \in B(x,t,r), t' \in [t - \Delta t, t]\} \).

Let \( \kappa : I \rightarrow (0, +\infty) \) be a function. We say \( \{(X(t), g(t))\}_{t \in I} \) has property \((NC)_\kappa\) if it is \( \kappa(t) \)-noncollapsed at any space-time point \((x,t)\) on all scales \( \leq 1 \).

The following proposition is analogous to [BBM, Theorem 6.5] and [BBB\(^+\), Theorem 6.2.1].

**Proposition 2.3.** Fix \( c_0 > 0 \). For any \( r, \delta > 0 \), there exist \( h \in (0, \delta r) \) and \( D > 10 \), such that if \((X(\cdot), g(\cdot))\) is a complete surgical solution with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with bounded curvature and with no essential incompressible space form, defined on an time interval \([a,b] \) \( (0 \leq a < b < \frac{1}{2c_0}) \) and satisfying the pinching assumption and the canonical neighborhood assumption \((CN)_r\), then the following holds:

Let \( t \in [a,b] \) and \( x, y, z \in X(t) \) such that \( R(x,t) \leq 2/r^2 \), \( R(y,t) = h^{-2} \) and \( R(z,t) \geq D/h^2 \). Assume there is a curve \( \gamma \) in \( X(t) \) connecting \( x \) to \( z \) via \( y \), such that each point of \( \gamma \) with scalar curvature in \([2C_0 r^{-2}, C_0^{-1} D h^{-2}]\) is the center of an \( \varepsilon_0 \)-neck. Then \((y,t)\) is the center of a strong \( \delta \)-neck.

**Proof.** We follow closely the proof of [BBM, Theorem 6.5] and [BBB\(^+\), Theorem 6.2.1]. (Compare [P2, Lemma 4.3], [CZ2, Lemma 5.2].) We argue by contradiction. Otherwise, there exist \( r, \delta > 0 \), sequences \( h_k \rightarrow 0, D_k \rightarrow +\infty \), a sequence of complete surgical solutions \((X_k(\cdot), g_k(\cdot))\) with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, a_1 + a_2 \geq c_0)\), with bounded curvature and with no essential incompressible space form, satisfying the pinching assumption (with constants \( \varrho_0, \Psi_0, L_0, P_0, S_0 \)) and \((CN)_r\), and sequences \( 0 \leq t_k < \frac{1}{2c_0}, x_k, y_k, z_k \in X_k(t_k) \) with \( R(x_k, t_k) \leq 2r^{-2}, R(y_k, t_k) = h_k^{-2} \) and \( R(z_k, t_k) \geq D_k h_k^{-2} \), and finally a sequence of curves \( \gamma_k \) in \( X_k(t_k) \) connecting \( x_k \) to \( z_k \) via \( y_k \), whose points of scalar curvature in \([2C_0 r^{-2}, C_0^{-1} D_k h_k^{-2}]\) are centers of \( \varepsilon_0 \)-necks, but \( y_k \) is not the center of a strong \( \delta \)-neck.

Consider the rescaled solution \((\overline{X}_k(\cdot), \bar{g}_k(\cdot))\), where \( \bar{g}_k(\cdot) = h_k^{-2}g_k(t_k + h_k^2 t) \). By Theorem B.1 in Appendix B (and the Remark after Theorem B.1), for any \( \rho > 0 \), there exists \( \Lambda(\rho) > 0 \) and \( k_0(\rho) > 0 \) such that the ball \((B(\bar{y}_k, 0, \rho)) \) has scalar curvature bounded above by \( \Lambda(\rho) \) for \( k > k_0(\rho) \). (Here and below, we adopt the convention in [BBM] to put a bar on the points when the relevant geometric quantities are computed w.r.t. the metric \( \bar{g}_k \).) Combined with the canonical neighborhood assumption, it implies that the parabolic neighborhoods \( P(\bar{y}_k, 0, \rho, -\frac{1}{2\Lambda(\rho)}) \) are unscathed, with scalar curvature bounded above by \( 2\Lambda(\rho) \) for all \( k \geq k_1(\rho) > k_0(\rho) \). By the pinching assumption, we get a uniform control of the
curvature operator there. Using a local version of Hamilton’s compactness theorem (see [BBB⁺, Theorem C.3.3]), we see that (a subsequence of) \((X_k(0), \bar{g}_k(0), \bar{y}_k)\) converges to some complete noncompact Riemannian 4-manifold \((\bar{X}_\infty, \bar{g}_\infty, \bar{y}_\infty)\). Clearly \(X_k\) must be diffeomorphic to \(S^3 \times \mathbb{R}\), and define the standard solution. Consider the semi-infinite cylinder limit and properties of strong necks, for any \(\rho > 0\) sufficiently large –a contradiction.

Clearly \(k\) that for any \(\rho > 0\), there exists \(k_2(\rho) \geq k_1(\rho)\), such that for any \(k \geq k_2(\rho)\) the parabolic neighborhoods \(P(\bar{y}_k, 0, \rho, -1/2)\) are unscathed, and have scalar curvature satisfying \(\frac{1}{2} \leq R \leq 2\). By the local compactness theorem ([BBB⁺, Theorem C.3.3]) again, it follows that \((X_k, \bar{g}_k(\cdot), (\bar{y}_k, 0))\) subconverges to some complete Ricci flow \(\bar{g}_\infty(\cdot)\) on \(\bar{X}_\infty\). This flow is defined on \((-\frac{1}{2}, 0]\), has \(R \leq 2\), and still satisfies (2.3).

Now set

\[
\tau_0 := \sup\{\tau > 0| \forall \rho > 0, \exists C(\rho, \tau) > 0, \exists k(\rho), \forall k \geq k(\rho), P(\bar{y}_k, 0, \rho, -\tau) \text{ is unscathed and } C(\rho, \tau)^{-1} \leq R \leq C(\rho, \tau) \text{ there}\}.
\]

We have shown \(\tau_0 \geq \frac{1}{2}\). It turns out that, as in Step 2 of the proof of [BBB⁺, Theorem 6.2.1], using the canonical neighborhood assumption one can show \(\tau_0 = +\infty\). This way we get an ancient solution which satisfies (2.3) and splits at the final time slice. By [CZ2, Lemma 3.2] it must be the standard flow on the round cylinder. This implies the point \((y_k, t_k)\) is the center of a strong \(\delta\)-neck when \(k\) is sufficiently large –a contradiction.

Now we describe more precisely Hamilton’s surgery procedure [H5]. We will follow [CZ2] closely. First we describe the model surgery on the standard cylinder, and define the standard solution. Consider the semi-infinite cylinder \(N_0 = (S^3 \times (-\infty, 4)]\) with the standard metric \(\bar{g}_0\) of scalar curvature 1. Let \(f\) be a smooth nondecreasing convex function on \((-\infty, 4]\) defined by

\[
\begin{align*}
 f(z) &= 0, \quad z \leq 0; \\
 f(z) &= w_0 e^{-\frac{w_0}{z}}, \quad z \in (0, 3]; \\
 f(z) &\text{ is strictly convex}, \quad z \in [3, 3.9]; \\
 f(z) &= -\frac{1}{2}\ln(16 - z^2), \quad z \in [3.9, 4].
\end{align*}
\]

(where \(w_0\) and \(W_0\) are universal positive constants given in Lemma 2.4 below). Replace the standard metric \(\bar{g}_0\) on the subspace \(S^3 \times [0, 4]\) in \(N_0\) by \(e^{-2f}\bar{g}_0\). The resulting metric will induce a complete, smooth metric (denoted by) \(\hat{g}_0\) on \(\mathbb{R}^4\). We call the complete Ricci flow \((\mathbb{R}^4, \hat{g}(\cdot))\) with initial data \((\mathbb{R}^4, \hat{g}_0)\) and with bounded curvature in any compact subinterval of \([0, \frac{4}{3}]\) the standard solution, which exists on the time interval \([0, \frac{3}{2}]\). Denote by \(p_0\) the tip of the standard solution, which is the fixed point of the \(SO(4)\)-action on the initial metric \((\mathbb{R}^4, \hat{g}_0)\). Note that by [CZ2, Appendix], there exists a constant \(\kappa_{st} > 0\) such that the standard solution is
κ-ut-noncollapsed on scales ≤ 1. We refer the reader to [CZ2, Appendix] for other properties of 4-dimensional standard solution.

Then we describe a similar surgery procedure for the general case. Suppose we have a δ-neck centered at x₀ in a Riemannian 4-manifold (X, g). Sometimes we will call \( R^{-\frac{1}{2}}(x_0) \) the radius of this neck. Let \( \Phi : S^3 \times [-l, l] \to V \subset N \) be Hamilton’s parametrization; see Appendix A. Assume the center \( x_0 \) of the δ-neck has \( R \)-coordinate \( z = 0 \). The surgery is to cut off the δ-neck along the middle 3-sphere and glue back two balls (caps) separately. We construct a new smooth metric on the glued back cap (say on the left hand side) as follows.

\[
\bar{g} = \begin{cases} 
  g(t_0), & z = 0; \\
  e^{-2f}g(t_0), & z \in [0, 2]; \\
  \varphi e^{-2f}g(t_0) + (1 - \varphi)e^{-2f}h^2\bar{g}_0, & z \in [2, 3]; \\
  e^{-2f}h^2\bar{g}_0, & z \in [3, 4].
\end{cases}
\]

where \( \varphi \) is a smooth bump function with \( \varphi = 1 \) for \( z \leq 2 \), and \( \varphi = 0 \) for \( z \geq 3 \), \( h = R^{-\frac{1}{2}}(x_0) \), and \( \bar{g}_0 \) is as above. We also perform the same surgery procedure on the right hand side with parameter \( \bar{z} \in [0, 4] \) (\( \bar{z} = 8 - z \)).

The following lemma of Hamilton justifies the pinching assumption of surgical solution.

**Lemma 2.4** (Hamilton [H5, Theorem D3.1]; compare [CZ2, Lemma 5.3]) There exist universal positive constants \( \delta_0, w_0 \) and \( W_0 \), and a constant \( h_0 \) depends only on \( c_0 \), such that given any surgical solution with uniformly positive isotropic curvature \( (a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0) \), satisfying the pinching assumption, defined on \( [a, t_0] \) \( (0 \leq a < t_0 < \frac{1}{2c_0}) \), if we perform Hamilton’s surgery as described above at a δ-neck (if it exists) of radius \( h \) at time \( t_0 \) with \( \delta < \delta_0 \) and \( h \leq h_0 \), then after the surgery, the pinching assumption still holds at all points at time \( t_0 \). Moreover, after the surgery, any metric ball of radius \( \delta^{-\frac{1}{2}}h \) with center near the tip (i.e. the origin of the attached cap) is, after scaling with the factor \( h^{-2} \), \( \delta^{\frac{1}{2}} \)-close to the corresponding ball of \( (\mathbb{R}^4, \bar{g}_0) \).

Usually we will be given two non-increasing step functions \( r, \delta : [0, +\infty) \to (0, +\infty) \) as surgery parameters. Let \( h(r, \delta), D(r, \delta) \) be the associated parameter as determined in Proposition 2.3, \( (h \) is also called the surgery scale,) and let \( \Theta := 2Dh^{-2} \) be the curvature threshold for the surgery process (as in [BBM]), that is, we will do surgery only when \( R_{\text{max}} \) reaches \( \Theta \).

Now we adapt two more definitions from [BBM].

**Definition** (compare [BBM]) Fix surgery parameter functions \( r, \delta \) and let \( h, D, \Theta = 2Dh^{-2} \) be the associated cutoff parameters. Let \( (X(t), g(t)) \) \( (t \in I \subset [0, \frac{1}{2c_0}) \) be an evolving Riemannian 4-manifold with uniformly positive isotropic curvature \( (a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0) \), with bounded curvature and with no
result follows by integrating the curvature derivative estimate \(|\Gamma_{t,r,\delta}|\) into the surgery scale \(a,b\). Let \(\Theta = 2Dh^{-2}\) be the associated cutoff parameters. A surgical solution \((X(\cdot), g(\cdot))\) with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with bounded curvature and no essential incompressible space form, defined on some time interval \(I \subset [0, \frac{1}{2c_0})\) is an \((r, \delta)\)-surgical solution if it has the following properties:

i. It satisfies the pinching assumption, and \(R(x, t) \leq \Theta(t)\) for all \((x, t)\);

ii. At each singular time \(t_0 \in I\), \((X_+(t_0), g_+(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery at time \(t_0\); and

iii. Condition \((CN)_{\kappa}\) holds.

Let \(k\) be a positive function (here, usually a nonincreasing step function). An \((r, \delta)\)-surgical solution which also satisfies Condition \((NC)_{\kappa}\) is called an \((r, \delta, \kappa)\)-surgical solution.

The following lemma is analogous to [BBM, Lemma 5.9].

**Lemma 2.5** Suppose we have fixed two constants \(r, \delta > 0\) as surgery parameters on an interval \([a, b]\). Let \((X(t), g(t))\) be an \((r, \delta)\)-surgical solution on \([a, b]\). Let \(a \leq t_1 < t_2 < b\) be two singular times (if they exist). Then \(t_2 - t_1\) is bounded from below by a positive number depending only on \(r, \delta\).

**Proof** We may assume that there are no other singular times between \(t_1\) and \(t_2\). Since \(R_{\max}(g_+(t_1)) \leq \Theta/2, R_{\max}(g(t_2)) = \Theta,\) and \(\Theta\) depends only on \(r, \delta\), the result follows by integrating the curvature derivative estimate \(|\frac{\partial \Gamma}{\partial t}| < C_0 R^2\) in the canonical neighborhood assumption (see [BBM, Lemma 5.9]).

The following proposition is similar to [BBM, Theorem 7.4], and it extends a result in [CZ2] to the noncompact case.

**Proposition 2.6** Let \(\varepsilon \in (0, 2\varepsilon_0]\). Let \((X, g)\) be a complete, connected 4-manifold. If each point of \(X\) is the center of an \(\varepsilon\)-neck or an \(\varepsilon\)-cap, then \(X\) is diffeomorphic to \(S^4\), or \(\mathbb{R}P^4\), or \(\mathbb{R}P^4 \# \mathbb{R}P^4\), or \(S^3 \times S^1\), or \(S^3 \tilde{\times} S^1\), or \(\mathbb{R}^4\), or \(\mathbb{R}P^4 \setminus B^4\), or \(S^3 \times \mathbb{R}\).
**Proof.** The result in the compact case has been shown in [CZ2]. So below we will assume that $X$ is not compact.

**Claim** Let $\varepsilon \in (0, 2\varepsilon_0]$. Let $(X,g)$ be a complete, noncompact, connected 4-manifold. If each point of $X$ is the center of an $\varepsilon$-neck, then $X$ is diffeomorphic to $\mathbb{S}^3 \times \mathbb{R}$.

Proof of Claim. Let $x_1$ be a point of $X$, and let $N_1$ be a $\varepsilon$-neck centered at $x_1$, given by some diffeomorphism $\psi_1 : \mathbb{S}^3 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_1$. Consider Hamilton’s canonical parametrization $\Phi_1 : \mathbb{S}^3 \times [-l_1, l_1] \to V_1 \subset N_1$ such that $V_1$ contains the portion $\psi_1(\mathbb{S}^3 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1}))$ in $N_1$. (See Appendix A.) Now choose a point $x_2$ in $\Phi_1(\mathbb{S}^3 \times \{0.9l_1\})$, and let $N_2$ be a $\varepsilon$-neck centered at $x_2$, given by some diffeomorphism $\psi_2 : \mathbb{S}^3 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_2$. Again consider Hamilton’s canonical parametrization $\Phi_2 : \mathbb{S}^3 \times [-l_2, l_2] \to V_2 \subset N_2$ such that $V_2$ contains the portion $\psi_2(\mathbb{S}^3 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1}))$ in $N_2$. Then by [H5, Theorem C2.4] we have Hamilton’s canonical parametrization $\Phi : \mathbb{S}^3 \times [-l, l] \to V_1 \cup V_2$, and for all $\alpha \in [-l_1, l_1]$ and all $\beta \in [-l_2, l_2]$, $\Phi_1(S^3 \times \{\alpha\})$ is isotopic to $\Phi_2(S^3 \times \{\beta\})$. (See also Appendix A.) Then we go on, choose $x_3, N_3, \Phi_3, \cdots$. This way the desired result follows.

Now consider the case that $X$ contains at least one $\varepsilon$-cap. In this case, since we are assuming $X$ is noncompact, $X$ contains only one cap. Then arguing as above, one see that $X$ is diffeomorphic to a cap. So in this case $X$ is diffeomorphic to $\mathbb{R}^4$ or $\mathbb{R}P^4 \setminus B^4$.

The following proposition is analogous to [BBM, Proposition A].

**Proposition 2.7** Fix $c_0 > 0$. There exists a positive constant $\tilde{\delta}$ (depending only on $c_0 > 0$) with the following property: Let $r, \delta$ be surgery parameters, let \{$(X(t), g(t))$\}$_{t \in [a,b]}$ ( $0 < a < b < \frac{1}{2c_0}$) be an $(r, \delta)$-surgical solution with uniformly positive isotropic curvature $(a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)$, with bounded curvature, and with no essential incompressible space form. Suppose that $\delta \leq \tilde{\delta}$, and $R_{\max}(b) = \Theta(b)$. Then there exists a Riemannian manifold $(X_+, g_+)$ which is obtained from $(X(-), g(-))$ by $(r, \delta)$-surgery at time $b$, such that

i. $g_+$ satisfies the pinching assumption at time $b$;

ii. $(a_1 + a_2)_{\min}(g_+(b)) \geq (a_1 + a_2)_{\min}(g(b)), (c_1 + c_2)_{\min}(g_+(b)) \geq (c_1 + c_2)_{\min}(g(b))$, and $R_{\min}(g_+(b)) \geq R_{\min}(g(b))$;

iii. $X_+$ has no essential incompressible space form.

**Proof** Let $\delta_0$ and $h_0$ be as given in Lemma 2.4. Set $\tilde{\delta} = \frac{1}{4} \min \{c_0^2 h_0, \delta_0\}$.

For the proof of i. and ii. we will follow that of [BBM, Proposition A]. Let $\mathcal{G}$ (resp. $\mathcal{O}$, resp. $\mathcal{R}$) be the set of points of $X(b)$ of scalar curvature less than $2r^{-2}$ (resp. $\in [2r^{-2}, \Theta(b)/2)$, resp. $\geq \Theta(b)/2$). The idea is to consider a maximal
collection \( \{N_i\} \) of pairwise disjoint cutoff necks in \( X(b) \), whose existence is guaranteed by Zorn’s Lemma. (Here, following [BBM], a cutoff neck is a strong \( \delta \)-neck centered at some point \((x, b)\) with \( R(x, b) = h^{-2} \).) It is easy to see that such a collection is locally finite by a volume argument.

**Claim 1** Any connected component of \( X(b) \setminus \cup_i N_i \) is contained either in \( G \cup \mathcal{O} \) or in \( R \cup \mathcal{O} \).

Proof of Claim 1. We argue by contradiction. Otherwise there is some component \( W \) of \( X(b) \setminus \cup_i N_i \) containing at least one point \( x \in G \) and one point \( z \in R \). Choose a minimizing geodesic path \( \gamma \) in \( W \) connecting \( x \) with \( z \). In the following Claim 2, we will show each point of \( \gamma \) with scalar curvature in \( [2C_0r^{-2}, C_0^{-1}Dh^{-2}] \) is the center of an \( \epsilon_0 \)-neck. Then we can apply Proposition 2.3 to conclude that there exists some point \( y \in \gamma \) with \( R(y, b) = h^{-2} \) which is the center of a strong \( \delta \)-neck. This will contradict the maximality of \( \{N_i\} \).

**Claim 2** Each point of such \( \gamma \) with scalar curvature in \( [2C_0r^{-2}, C_0^{-1}Dh^{-2}] \) is the center of an \( \epsilon_0 \)-neck.

Proof of Claim 2. The proof is a minor modification of that of the second claim in Lemma 7.7 of [BBM]. Let \( y \in \gamma \) be such a point. Then \( y \) is the center of an \((\epsilon_0, C_0)\)-canonical neighborhood \( U \). Clearly \( U \) cannot be a closed manifold by the curvature assumptions. We will show \( U \) cannot be an \((\epsilon_0, C_0)\)-cap either. Otherwise \( U = N \cup C \), where \( N \) is an \( \epsilon_0 \)-neck, \( N \cap C = \emptyset \), \( \overline{N} \cap C = \partial C \) and \( y \in \text{Int} \, C \). Let \( \psi : \mathbb{S}^3 \times (-\epsilon_0^{-1}, \epsilon_0^{-1}) \to N \) be the diffeomorphism which defines the neck \( N \). We use Hamilton’s method to give a canonical parametrization \( \Phi : \mathbb{S}^3 \times [-l', l'] \to V \subset N \) such that \( V \) contains the portion \( \psi(\mathbb{S}^3 \times (-0.98\epsilon_0^{-1}, 0.98\epsilon_0^{-1})) \) (cf. Lemma A.1 in Appendix A). Let \( S = \Phi(\mathbb{S}^3 \times \{0\}) \). We rescale the metric such that the scalar curvature of \( N \) is close to 1. Clearly \( \gamma \) is not minimizing in \( U \), since if \( x' \) (resp. \( z' \)) is an intersection of \( \gamma \) with \( S \) between \( x \) and \( y \) (resp. \( y \) and \( z \)), then \( d(x', z') < d(x', y) + d(y, z') \). The geodesic segment (in \( U \)) \([x'z']\) is not contained in \( W \) by the minimality of \( \gamma \) in \( W \). So \([x'z']\) \( \cap \partial W \neq \emptyset \). By definition of \( W \), the corresponding component of \( \partial W \) is a boundary component, denoted by \( S_i^+ \), of some cutoff neck \( N_i \). Then \( d(S_i^+, S) < \text{diam}(S) \) since \([x'z']\) \( \cap S_i^+ \neq \emptyset \). We use Hamilton’s method to give a canonical parametrization \( \Phi' : \mathbb{S}^3 \times [-l', l'] \to V' \subset N_i \) such that one of the ends of \( V' \), denoted by \( \partial_+ V' \), is at the rescaled distance \(< 0.03\epsilon_0^{-1} \) from the end \( S_i^+ \) of \( N_i \). Pick a point \( p' \) in \( V' \) which is at rescaled distance \( 0.2\epsilon_0^{-1} \) from \( \partial_+ V' \). Then \( d(p', S) \leq d(p', \partial_+ V') + d(\partial_+ V', S_i^+) + d(S_i^+, S) < 0.03\epsilon_0^{-1} + 0.2\epsilon_0^{-1} + \text{diam}(S) < 0.3\epsilon_0^{-1} \). Then it follows from the discussion after Lemma A.1 that the embedded \( \mathbb{S}^3 \) in the neck structure of \( V' \) which contains \( p' \) is isotopic to \( S \) in \( N \). It follows that \( \gamma \cap N_i \neq \emptyset \), which is impossible by the definition of \( W \).

Then we do Hamilton’s surgery along these \( N_i \)’s, and obtain an manifold \((X', g_+)\).
The components of \( X' \) consist of two types: Either they have curvature \( \leq \Theta(b)/2 \), or they are covered by canonical neighborhoods, whose diffeomorphism types are identified with the help of Proposition 2.6, and will be thrown away. We denote the resulting manifold by \((X_+, g_+)\). By Lemma 2.4 and our choice of \( \delta \) it satisfies the pinching assumption. Clearly ii) is also satisfied.

Now we show it satisfies iii. also. We will adapt an argument in [CZ2] to the noncompact case. We argue by contradiction. Suppose \( X_+ \) has an essential incompressible space form \( Y \approx S^3/\Gamma \), where \( \Gamma \) is a finite, fixed point free subgroup of isometries of \( S^3 \). After an isotopy, we may assume the intersection of \( Y \) with the union of all surgery caps is empty. Then \( Y \) may be seen as a submanifold in \( X(b) \) also. Below we will show \( Y \) is also an essential incompressible space form in \( X(b) \), which contradicts to our assumption on \( X(\cdot) \) and completes the proof.

Claim 3  \( Y \) is also an essential incompressible space form in \( X(b) \).

Proof of Claim 3. We argue by contradiction. Suppose \( Y \) is not an essential incompressible space form in \( X(b) \).

Case 1. \( Y \) is compressible in \( X(b) \). Then we can pick a loop \( \gamma \subset Y \) representing a nontrivial element in the kernel of \( i_* : \pi_1(Y) \rightarrow \pi_1(X(b)) \), where \( i \) is the inclusion map. So there is a map \( f : D^2 \rightarrow X(b) \) with \( f(\partial D^2) = \gamma \). Since \( f(D^2) \) is compact and the collection of our cutoff necks is locally finite, \( f(D^2) \) will intersects only a finite number of 3-spheres which lie in the middle of cutoff necks. Denote these 3-spheres by \( S_1, S_2, \ldots, S_m \). We perturb them slightly so that they meet \( f(D^2) \) transversely in a finite number of simple closed curves. By using an innermost circle argument we may assume (after modifying \( f \) suitably) that the enclosed disks in \( D^2 \) of all the circles in the preimage (of these intersection curves) are disjoint; denote these circles by \( C_1, C_2, \ldots, C_l \), and the enclosed 2-disks by \( D_1, D_2, \ldots, D_l \). Each \( f(C_j) \) bounds a homotopical 2-disk in \( S_1 \cup S_2 \cup \cdots \cup S_m \), since each \( S_k \) is a topological 3-sphere. So after a further modification of \( f \) we may assume that \( f(D_1 \cup D_2 \cup \cdots \cup D_l) \) is contained in \( S_1 \cup S_2 \cup \cdots \cup S_m \). On the other hand, since \( D^2 \setminus (D_1 \cup D_2 \cup \cdots \cup D_l) \) is connected, \( f(\partial D^2) = \gamma \subset Y \), we see that \( f(D^2 \setminus (D_1 \cup D_2 \cup \cdots \cup D_l)) \subset X_+ \). So \( \gamma \) bounds a homotopical disk in \( X_+ \). This contradicts to the choice of \( Y \).

Case 2. \( Y \) is incompressible in \( X(b) \), but not essential. If \( \Gamma = \{1\} \) then \( Y \) cannot be essential in \( X_+ \). So we may assume \( \Gamma = \mathbb{Z}_2 \) and the normal bundle of \( Y \) in \( X(b) \) is non-orientable. But the normal bundle of \( Y \) in \( X(b) \) is the same as in \( X_+ \). So \( Y \) again cannot be essential in \( X_+ \). A contradiction.

3 Existence of \((r, \delta, \kappa)\)-surgical solutions

As in [BBM], if \((X(\cdot), g(\cdot))\) is a piecewise \( C^1 \) evolving manifold defined on some interval \( I \subset \mathbb{R} \) and \([a, b] \subset I \), the restriction of \( g \) to \([a, b] \), still denoted by \( g(\cdot) \), is
the evolving manifold

\[ t \mapsto \begin{cases} (X_+(a), g_+(a)), & t = a, \\
(X(t), g(t)), & t \in (a, b]. \end{cases} \]

The following proposition is analogous to [P2, Lemma 4.5] and [BBM, Theorem 8.1], which is one of the key technical results in the process of constructing \((r, \delta, \kappa)\)-surgical solutions; compare [BBB, Theorem 8.1.2], [CaZ, Lemma 7.3.6], [KL, Lemma 74.1], [MT, Proposition 16.5] and [Z, Lemma 9.1.1], see also the formulation in the proof of [CZ2, Lemma 5.5]. We state it in a slightly more general form, which is applicable to our situation.

**Proposition 3.1** Fix \(c_0 > 0\). For all \(A > 0, \theta \in (0, \frac{3}{2})\) and \(\hat{r} > 0\), there exists \(\hat{\delta} = \hat{\delta}(A, \theta, \hat{r}) > 0\) with the following property. Let \(r(\cdot) \geq \hat{r}, \delta(\cdot) \leq \hat{\delta}\) be two positive step functions on \([a, b)\) \((0 \leq a < b < \frac{1}{2c_0})\), and let \((X(\cdot), g(\cdot))\) be a surgical solution with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with bounded curvature and with no essential incompressible space form, defined on \([a, b]\), such that it satisfies the pinching assumption on \([a, b]\), that \(R(x, t) \leq \Theta(r(t), \delta(t))\) for all space-time points with \(t \in [a, b]\), that at any singular time \(t_0 \in [a, b]\), \((X(t_0), g(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery, and that any point \((x, t)\) \((t \in [a, b])\) with \(R(x, t) \geq \frac{r(t)}{2}\) has a \((2\varepsilon_0, 2C_0)\)-canonical neighborhood. Let \(t_0 \in [a, b]\) be a singular time. Consider the restriction of \((X(\cdot), g(\cdot))\) to \([t_0, b]\). Let \(p \in X_+(t_0)\) be the tip of some surgery cap of scale \(h(t_0)\), and let \(t_1 \leq \text{min} \{b, t_0 + \theta h^2(t_0)\}\) be maximal (subject to this inequality) such that \(P(p, t_0, Ah(t_0), t_1 - t_0)\) is unscathed. Then the following holds:

i. The parabolic neighborhood \(P(p, t_0, Ah(t_0), t_1 - t_0)\) is, after scaling with factor \(h^{-2}(t_0)\) and shifting time \(t_0\) to zero, \(A^{-1}\)-close to \(P(p_0, 0, A, (t_1 - t_0)h^{-2}(t_0))\) (where \(p_0\) is the tip of the standard solution);

ii. If \(t_1 < \text{min} \{b, t_0 + \theta h^2(t_0)\}\), then \(B(p, t_0, Ah(t_0)) \subset X_{\text{sing}}(t_1)\) disappears at time \(t_1\).

We will follow the proof of [BBB, Theorem 8.1.2] and [BBM, Theorem 8.1].

Let \(\mathcal{M}_0 = (\mathbb{R}^4, g(\cdot))\) be the standard solution, and \(0 < T_0 < \frac{3}{2}\).

The following result is from [BBB\textsuperscript{+}], where the proof uses Chen-Zhu’s uniqueness theorem ([CZ1]).

**Lemma 3.2** ([BBB\textsuperscript{+}, Theorem 8.1.3]) For all \(A, \Lambda > 0\), there exists \(\rho = \rho(\mathcal{M}_0, A, \Lambda) > A\) with the following property. Let \(U\) be an open subset of \(\mathbb{R}^4\) and \(T \in (0, T_0]\). Let \(g(\cdot)\) be a Ricci flow defined on \(U \times [0, T]\), such that the ball \(B(p_0, 0, \rho) \subset U\) is relatively compact. Assume that

i. \(|\text{Rm}(g(\cdot))|_{0, U \times [0, T], g(\cdot)} \leq \Lambda,\)

ii. \(g(0)\) is \(\rho^{-1}\)-close to \(\hat{g}(0)\) on \(B(p_0, 0, \rho)\).

Then \(g(\cdot)\) is \(A^{-1}\)-close to \(\hat{g}(\cdot)\) on \(B(p_0, 0, A) \times [0, T]\).
Here, \(||Rm(g(\cdot))||_{k, U \times [0, T], g(\cdot)} := \sup_{U \times [0, T]} \{|\nabla^i Rm_{g(t)}|_{g(t)}| 0 \leq i \leq k\}||.

**Corollary 3.3** (Compare [BBM, Corollary 8.3]) Let \( A > 0 \). There exists \( \rho = \rho(M_0, A) > A \) with the following property. Let \( \{ (X(t), g(t)) \}_{t \in [0, T]} (T \leq T_0) \) be a surgical solution with uniformly positive isotropic curvature, with bounded curvature and with no essential incompressible space form. Assume that

i. \((X(\cdot), g(\cdot))\) is a parabolic rescaling of some surgical solution which satisfies the pinching assumption,

ii. \( |\frac{\partial R}{\partial t}| \leq 2C_0R^2 \) at any space-time point \((x, t)\) with \( R(x, t) \geq 1 \).

Let \( p \in X(0) \) and \( t \in (0, T] \) be such that

iii. \( B(p, 0, \rho) \) is \( \rho^{-1}\)-close to \( B(p_0, 0, \rho) \),

iv. \( P(p, 0, \rho, t) \) is unscathed.

Then \( P(p, 0, A, t) \) is \( A^{-1}\)-close to \( P(p_0, 0, A, t) \).

**Proof** The proof is similar to that of Corollaries 8.2.2 and 8.2.4 in [BBB⁺].

Using Corollary 3.3, one can easily adapt the arguments in the proof of [BBB⁺, Theorem 8.1.2] and [BBM, Theorem 8.1] to prove Proposition 3.1.

The following theorem is analogous to [P2, Proposition 5.1] and [BBM, Theorems 5.5 and 5.6]. We state it in a form similar to [MT, Theorem 15.9].

**Theorem 3.4** Given \( c_0, v_0 > 0 \), there are surgery parameter sequences

\[ K = \{\kappa_i\}_{i=1}^{\infty}, \quad \Delta = \{\delta_i\}_{i=1}^{\infty}, \quad r = \{r_i\}_{i=1}^{\infty} \]

such that the following holds. Let \( r(t) = r_i \) and \( \delta(t) = \delta_i \) on \([(i-1)2^{-5}, i \cdot 2^{-5})\) for all space-time points, \( i = 1, 2, \ldots \). Suppose that \( \delta : [0, \infty) \to (0, \infty) \) is a non-increasing step function with \( \delta(t) \leq \delta_i(t) \). Then the following holds: Suppose that we have a surgical solution \((X(\cdot), g(\cdot))\) with uniformly positive isotropic curvature, with bounded curvature and with no essential incompressible space form, defined on \([0, T]\) (for some \( T < \infty \)), which satisfies the following conditions:

1. the initial data \((X(0), g(0))\) is a complete 4-manifold with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with \(|Rm| \leq 1\), with no essential incompressible space form, and with \( \text{vol} B(x, 1) \geq v_0 \) at any point \( x \),

2. the solution satisfies the pinching assumption, and \( R(x, t) \leq \Theta(r(t), \delta(t)) \) for all space-time points,

3. it has only a finite number of singular times such that at each singular time \( t_0 \in (0, T) \), \((X_+(t), g_+(t))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery at time \( t_0 \), and

4. on each time interval \([(i-1)2^{-5}, i \cdot 2^{-5}] \cap [0, T]\) the solution satisfies \((CN)_{r_i}\) and \((NC)_{\kappa_i}\).
Then there is an extension of \((X(\cdot), g(\cdot))\) to a surgical solution defined for \(0 \leq t \leq T'\) (where \(T' < \frac{1}{2c_0}\) is the extinction time) and satisfying the above four conditions with \(T\) replaced by \(T'\).

To prove the theorem above, I will adapt the arguments in Perelman [P2] and Chen-Zhu [CZ2] to the noncompact case; compare [BBM].

The following lemma guarantees the non-collapsing under a weak form of the canonical neighborhood assumption, and is analogous to [P2, Lemma 5.2], [BBM, Proposition C], and [CZ2, Lemma 5.5]. We state it in a form close to [KL, Lemma 79.12].

**Lemma 3.5** Fix \(c_0 > 0\). Suppose that \(0 < r_- \leq \varepsilon_0, \kappa_- > 0\), and \(0 < E_- < E < \frac{1}{2c_0}\). Then there exists \(\kappa_+ = \kappa_+(r_-, \kappa_-, E_-, E) > 0\), such that for any \(r_+ > 0 < r_- \leq r_-,\) one can find \(\delta' = \delta'(r_-, r_+, \kappa_-, E_-, E) > 0\), with the following property. Suppose that \(0 \leq a < b < d < \frac{1}{2c_0}, b - a \geq E_- , d - a \leq E\). Let \(r\) and \(\delta\) be two positive step functions on \([a, d]\) with \(\varepsilon_0 \geq r \geq r_- \) on \([a, b]\), \(\varepsilon_0 \geq r \geq r_+ \) on \([b, d]\) and \(\delta \leq \delta'\) on \([a, d]\). Let \((X(\cdot), g(\cdot))\) be a surgical solution with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with bounded curvature and with no essential incompressible space form, defined on the time interval \([a, d]\), such that it satisfies the pinching assumption on \([a, d]\), that \(R(x, t) \leq \Theta(r(t), \delta(t))\) for all space-time points with \(t \in [a, d]\), that at any singular time \(t_0 \in [a, d]\), \((X_+(t_0), g_+(t_0))\) is obtained from \((X(\cdot), g(\cdot))\) by \((r, \delta)\)-surgery, that the conditions \((CN)_r\) and \((NC)_{\kappa_-}\) hold on \([a, b]\), and that any point \((x, t) \in [b, d]\) with \(R(x, t) \geq \left(\frac{r(t)}{2}\right)^{-2}\) has a \((2\varepsilon_0, 2C_0)\)-canonical neighborhood. Then \((X(\cdot), g(\cdot))\) satisfies \((NC)_{\kappa_+}\) on \([b, d]\).

**Proof** Using Proposition 3.1 and Perelman’s reduced volume, the proof of [CZ2, Lemma 5.2] can be adapted to our case without essential changes.

The following proposition justifies the canonical neighborhood assumption needed. We state it in a form similar to [MT, Proposition 17.1]. Compare [P2, Section 5], [BBM, Proposition B] and [CZ2, Proposition 5.4].

**Proposition 3.6** Given \(c_0 > 0\). Suppose that for some \(i \geq 1\) we have surgery parameter sequences \(\delta \geq \delta_1 \geq \delta_2 \geq \cdots \geq \delta_i \geq 0\), \(\varepsilon_0 \geq r_1 \geq \cdots \geq r_i > 0\) and \(\kappa_1 \geq \kappa_2 \geq \cdots \geq \kappa_i > 0\), where \(\delta\) is the constant given in Proposition 2.7. Then there are positive constants \(r_{i+1} \leq r_i\) and \(\delta_{i+1} \leq \min\{\delta_i, \delta'\}\), where \(\delta' = \delta'(r_i, r_{i+1}, \kappa_i)\) is the constant given in Lemma 3.5 by setting \(r_0 = r_i, \kappa_0 = \kappa_i, r_+ = r_{i+1}, E_- = 2^{-5}\) and \(E = 2^{-4}\), such that the following holds. Let \(r(t) = r_j\) and \(\delta(t) = \delta_j\) on \([j - 1)2^{-5}, j \cdot 2^{-5} \), \(j = 1, 2, \cdots, i + 1\). Suppose that \(\delta : [0, (i+1)2^{-5}] \to (0, \infty)\) is a non-increasing step function with \(\delta(t) \leq \tilde{\delta}(t)\). Let \((X(\cdot), g(\cdot))\) be any surgical solution to Ricci flow with uniformly positive isotropic curvature \((a_1 + a_2 \geq c_0, c_1 + c_2 \geq c_0)\), with bounded curvature and with no essential incompressible space...
form, defined on \([0, T]\) for some \(T \in (i \cdot 2^{-5}, (i + 1)2^{-5}]\), such that \(R(x, t) \leq \Theta(r(t), \delta(t))\) for all space-time points with \(t \in [0, T]\), that there are only a finite number of singular times, and at each singular time \(t_0 \in (0, T)\), \((X_+, g_+(t_0))\) is obtained from \((X_0, g(\cdot))\) by \((r, \delta)\)-surgery at time \(t_0\). Suppose that the restriction of the surgical solution to \([0, i \cdot 2^{-5}]\) satisfies the four conditions given in Theorem 3.4. Suppose also that \(\delta(t) \leq \delta_{i+1}\) for all \(t \in [(i - 1)2^{-5}, T]\). Then \((X(\cdot), g(\cdot))\) satisfies the condition \((CN)_{r_{i+1}}\) on \([i \cdot 2^{-5}, T]\).

**Proof** We argue by contradiction. Otherwise there exist \(r^\alpha \to 0\) as \(\alpha \to \infty\), and for each \(\alpha\) a sequence \(\delta^{\alpha\beta} \to 0\) as \(\beta \to \infty\), such that the following holds. For each \(\alpha, \beta\) there is a surgical solution \((X^{\alpha\beta}(\cdot), g^{\alpha\beta}(\cdot))\) to the Ricci flow defined for \(0 \leq t \leq T_{\alpha\beta}\) with \(i \cdot 2^{-5} < T_{\alpha\beta} \leq (i+1)2^{-5}\), such that it satisfies the conditions of the proposition w.r.t. these constants but not the conclusion.

By our assumption, Lemma 2.1 and Lemma 2.4, \((X^{\alpha\beta}(\cdot), g^{\alpha\beta}(\cdot))\) satisfies the pinching assumption on \([0, T_{\alpha\beta}]\). Note that by our assumption the scalar curvature of \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) on \([0, i \cdot 2^{-5}]\) are uniformly bounded above by a constant independent of \(\alpha, \beta\) (but depending on \(i\)). So by choosing \(r^\alpha\) sufficiently small we may assume that the condition \((CN)_{\alpha}\) holds on \([i \cdot 2^{-5}, i \cdot 2^{-5} + \theta]\) for some \(\theta > 0\). Also note that if for \((X^{\alpha\beta}(\cdot), g^{\alpha\beta}(\cdot))\) the condition \((CN)_{\alpha}\) holds in \([i \cdot 2^{-5}, \lambda]\) for some \(\lambda > i \cdot 2^{-5}\), then arguing as in the proof of [MT, Lemma 11.23] with some minor modifications, (note that the curvature of \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) are bounded on \([i \cdot 2^{-5}, \lambda]\)) we see that any point \((x, t)\) \((t \in [i \cdot 2^{-5}, \lambda])\) with \(R(x, t) \geq (\frac{\lambda^\alpha}{2})^{-2}\) has a \((2\varepsilon_0, 2C_0)\)-canonical neighborhood; cf. also [BBB+, Chapter 9]. This means that the canonical neighborhood condition has some sort of weak closeness (w.r.t. the time).

The following Claim 1 may be seen as some sort of weak openness (w.r.t. the time) property of the canonical neighborhood condition.

**Claim 1** Suppose for \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) the condition \((CN)_{\tau^\alpha}\) holds on \([i \cdot 2^{-5}, t_0]\) for some \(i \cdot 2^{-5} < t_0 < T_{\alpha\beta}\). Then there exists \(\tau > 0\) (depending on \(\alpha, \beta\)) such that any point \((x, t)\) \((t \in [i \cdot 2^{-5}, t_0 + \tau])\) with \(R(x, t) \geq (\frac{\tau^\alpha}{2})^{-2}\) has a \((2\varepsilon_0, 2C_0)\)-canonical neighborhood.

**Proof of Claim 1** We consider the following two cases.

Case i: \(R_{\text{max}}(g^{\alpha\beta}(t_0)) = \Theta = \Theta(r^\alpha, \delta^{\alpha\beta})\) (the curvature threshold for the surgery process). So a surgery occurs at \(t_0\). Then \(R_{\text{max}}(g^{\alpha\beta}(t_0)) \leq \Theta/2\), and the condition \((CN)_{\tau^\alpha}\) still holds in \((X^{\alpha\beta}_+(t_0), g^{\alpha\beta}_+(t_0))\) (cf. for example the proof of [KL, Lemma 73.7]). Also note that the curvature derivatives of \((X^{\alpha\beta}_+(t_0), g^{\alpha\beta}_+(t_0))\) are bounded.

The reason is as follows. If a point \(x \in X^{\alpha\beta}_+(t_0)\) lies within a distance of \(10\varepsilon_0^{-1}h\) from the added part \(X^{\alpha\beta}_+(t_0) \setminus X^{\alpha\beta}_0(t_0)\), (where \(h = h(r^\alpha, \delta^{\alpha\beta})\) is the surgery scale,) then it lies in an \(\varepsilon_0\)-cap, and the curvature derivatives at \((x, t_0)\) are bounded by the surgery construction. (Compare [MT, Claim 16.6].) If a point \(x \in X^{\alpha\beta}_+(t_0)\) lies at distance greater than or roughly equal to \(10\varepsilon_0^{-1}h\) from \(X^{\alpha\beta}_+(t_0) \setminus X^{\alpha\beta}_0(t_0)\),
then it has existed for a previous time interval since the set of singular times is a discrete subset of \( \mathbb{R} \), and by Shi’s local estimates [S] the curvature derivatives at \((x, t_0)\) are bounded also. Then by Shi’s theorem with initial curvature derivative bounds ([LT, Theorem 11], see also [MT, Theorem 3.29]), the curvature derivatives of \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) are bounded when restricted to \([t_0, t'']\) for some \( t'' > t_0 \) with \( t'' - t_0 \) sufficiently small.

Case ii: \( R_{\max}(g^{\alpha\beta}(t_0)) < \Theta = \Theta(r^\alpha, \delta^{\alpha\beta}) \). By the smoothing property of Ricci flow there exists \( t'' > t_0 \) such that we have \( R_{\max}(g^{\alpha\beta}(t)) < \Theta \) on \([t_0, t'']\). If there exist singular times before \( t_0 \), let \( t' \) be the last one. Otherwise let \( t' = (i - 1)2^{-5} + 2^{-6} \). Then there are no surgeries in the time interval \((t', t'')\). By Shi’s theorem with or without initial curvature derivative bounds we see that the curvature derivatives of \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) are bounded on \([t_0, t'']\).

Arguing this way, we see the following holds: Fix \( 0 < \sigma < t_0 \) and \( t'' > t_0 \) with \( t'' - t_0 \) sufficiently small as above. For any \( k \in \mathbb{N} \), the \( k \)-th covariant derivative of the curvature of \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) are bounded when restricted to a subinterval \([t_1, t_2]\) of \([\sigma, t'']\) with \( t_2 - t_1 \) sufficiently small. The bound depends on \( k, \alpha, \beta, \sigma \) and \( t'' \), but is independent of \( t_1 \) and \( t_2 \). Then arguing as in the proof of [H4, Lemma 2.4], for any \( k \in \mathbb{N} \), we have \( \sup_{x \in X^{\alpha\beta}(t_2)} \sum_{j=0}^{k} |\nabla^j g^{\alpha\beta}(t_2 - g^{\alpha\beta}(t_1))(x)|_{g^{\alpha\beta}(t_1)} \leq c_k \cdot (t_2 - t_1) \to 0 \) as \( t_2 \to t_1 \), where \( c_k \) is a constant (depending also on \( \alpha, \beta, \sigma \) and \( t'' \), but not on \( t_1 \) or \( t_2 \)). Moreover, from the curvature derivative estimates and the evolution equation (2.4), we know that if \( t - t_0 > 0 \) is sufficiently small and \( R(x, t) \geq (r^\alpha/2)^{-2} \) for some \( x \in X^{\alpha\beta}(t) \), then \( R(x, t_0) \geq (r^\alpha)^{-2} \) and \((x, t_0)\) has an \((\varepsilon_0, C_0)\)-canonical neighborhood. Then Claim 1 follows by combining the above estimates on metrics with the definition of canonical neighborhood.

From Claim 1, the weak closeness (w.r.t. the time) property of the canonical neighborhood condition mentioned above and the choice of our \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\) we have the following

Claim 2 For each \((X^{\alpha\beta}(t), g^{\alpha\beta}(t))\), there exists \( t^{\alpha\beta} \in (i \cdot 2^{-5}, T_{\alpha\beta}) \) such that the condition \((CN)_{\tau_{\alpha}}\) is violated at some space-time point \((x^{\alpha\beta}, t^{\alpha\beta})\), but any point \((x, t) \ (t \in [i \cdot 2^{-5}, t^{\alpha\beta}])\) with \( R(x, t) \geq (\frac{b^2}{2})^{-2} \) has a \((2\varepsilon_0, 2C_0)\)-canonical neighborhood.

Note that we are not claiming that \( t^{\alpha\beta} \) is the first time that the condition \((CN)_{\tau_{\alpha}}\) is violated.

Given \( \alpha \), we may assume that for all \( \beta, \delta^{\alpha\beta} \leq \delta'(r_i, r^\alpha/2, \kappa_i) \), the constant given in Lemma 3.5 by setting \( r_- = r_i, r_+ = r^\alpha/2, \kappa_- = \kappa_i, E_- = 2^{-5} \) and \( E = 2^{-4} \). Then Claim 2 allows one to apply Lemma 3.5 to get uniform \( \kappa \)-noncollapsing on all scales \( \leq 1 \) in \([0, t^{\alpha\beta}]\) with \( \kappa = \kappa_+(r_i, \kappa_i) > 0 \) independent of \( \alpha, \beta \). (Note that the \( \kappa \)-noncollapsed condition is closed w.r.t. the time (cf. [BBB+, Lemma 4.1.4]).)

Now similarly as in [P2], [CZ2], the idea is roughly as follows: Let \( \bar{g}^{\alpha\beta} \) be the solutions obtained by rescaling \( g^{\alpha\beta} \) with factors \( R(x^{\alpha\beta}, t^{\alpha\beta}) \) and shifting the times \( t^{\alpha\beta} \) to 0. If for any \( A > 0, b > 0 \), the sets \( B(x^{\alpha\beta}, 0, A) \times [-b, 0] \) are unscathed
when \( \alpha, \beta \) are sufficiently large, then with the help of the uniform \( \kappa \)-non-collapsing just obtained, the compactness theorem for Ricci flow, and Hamilton’s Harnack estimate [H3], we can get an ancient \( \kappa \)-solution with restricted isotropic curvature pinching as a limit, which will lead to a contradiction by the remark (after the definition of canonical neighborhood) in Section 2. If there exist \( A > 0, b > 0 \) such that there are arbitrarily large \( \alpha, \beta \) with \( \bar{B}(x^{\alpha \beta}, 0, A) \times [-b, 0] \) scattered, then using Proposition 3.1 one can show that the solution will be close to the standard solution (after suitable rescaling and time-shifting), which again leads to a contradiction. (Compare also, for example, the descriptions in [CaZ], [KL] and [MT].)

Actually one can argue similarly as in the proof of [CZ2, Proposition 5.4] with some minor modifications. (One can also check that the presentation in, for example, [Z, Theorem 9.2.1] can be adapted to our situation.) I only indicate some of the modifications (to arguments of [CZ2, Proposition 5.4]) needed in our situation.

1. One can use Theorem B.1 in Appendix B to simplify or replace some arguments in [CZ2, Proposition 5.4] (for example, Step 3 of the proof there).

2. One may use the neck-strengthening lemma A.2 in Appendix A to replace the remark after Lemma 5.2 in [CZ2] used in the second paragraph on p. 245 there.

**Proof of Theorem 3.4** (Compare Chapter 17, Section 2 in [MT].) We will construct the desired surgery parameter sequences inductively. By Shi’s work and the doubling time estimate (cf. [CLN, Lemma 6.1]), the initial condition that \( (X, g_0) \) is complete and has \( |Rm| \leq 1 \) guarantees a (smooth) complete solution exists on the time interval \([0, 2^{-5}]\), such that \( |Rm| \leq 2 \) on this time interval. By Lemma 2.1, the pinching assumption is satisfied on this time interval. Since we have the initial time curvature bound \( |Rm| \leq 1 \) and the volume lower bound on the initial time unit balls, by Perelman’s no-local-collapsing theorem (cf. [KL, Theorem 26.2]), the solution satisfies \((NC)_{\kappa_1}\) for some \( \kappa_1 > 0 \) on this time interval. By choosing \( r_1 \leq \varepsilon_0 \), it satisfies the condition \((CN)_{r_1}\) vacuously on this time interval. Pick any positive constant \( \delta \leq \hat{\delta} \). Now suppose we have constructed the surgery parameter sequences

\[
\mathbf{r}_i = \{r_1, \ldots, r_i\}, \quad \Delta_i = \{\delta_1, \ldots, \delta_i\}, \quad \mathbf{K}_i = \{\kappa_1, \ldots, \kappa_i\},
\]

with the desired property. We let \( r_{i+1} \) and \( \delta_{i+1} \) be as in Proposition 3.6. Then let \( \kappa_{i+1} = \kappa_+ (r_i, \kappa_i) \) be as given in Lemma 3.5. Set

\[
\mathbf{r}_{i+1} = \{r_i, r_{i+1}\}, \quad \Delta_{i+1} = \{\delta_1, \ldots, \delta_{i-1}, \delta_{i+1}, \delta_{i+1}\}, \quad \mathbf{K}_{i+1} = \{\kappa_i, \kappa_{i+1}\}.
\]

Let \( \delta : [0, (i + 1)2^{-5}] \to (0, \infty) \) be any non-increasing positive step function with \( \delta \leq \Delta_{i+1} \). Let \( (X(\cdot), g(\cdot)) \) be any surgical solution to Ricci flow defined on \([0, T]\) with \( T \in [i \cdot 2^{-5}, (i + 1)2^{-5}] \) satisfying the four conditions w.r.t. \( \delta \) and these sequences. We want to extend this surgical solution to one defined on \([0, (i+1)2^{-5}]\) such that it still satisfies the four conditions w.r.t. \( \delta, \mathbf{r}_{i+1}, \Delta_{i+1}, \) and \( \mathbf{K}_{i+1} \).

If \( R_{\max}(g(T)) < \Theta = \Theta(r_{i+1}, \delta_{i+1}) \), then we can run the Ricci flow with initial data \((X(T), g(T))\) for a while \([T, T + \theta]\) \((T + \theta \leq (i+1)2^{-5})\) until \( R_{\max}(T + \theta) \) reaches
the threshold Θ. By Lemma 2.1 the pinching assumption is satisfied. Note that by Proposition 3.6 the extended surgical flow still satisfies the condition \((CN)_{r_{i+1}}\). (There are no surgeries occurring in \([T, T+\theta]\), and the conditions of Proposition 3.6 hold.) Then by Lemma 3.5 it is \(\kappa_{i+1}\)-noncollapsed. So the extended flow satisfies all the four conditions w.r.t. these constants.

So we may assume that \(R_{\text{max}}(g(T)) = \Theta = \Theta(r_{i+1}, \delta_{i+1})\). Using Proposition 2.7 we do \((r_{i+1}, \delta_{i+1})\)-surgery and get \((X_+(T), g_+(T))\). If \(X_+(T) = \emptyset\), then we are done. Assume \(X_+(T) \neq \emptyset\), then we have \(R_{\text{max}}(g_+(T)) \leq \frac{1}{2}\Theta\), \(g_+(T)\) satisfies the pinching assumption at time \(T\), and \(X_+(T)\) has no essential incompressible space form. Then we run the Ricci flow with initial data \((X_+(T), g_+(T))\) for a little while \([T, \lambda]\) for some \(\lambda > T\) keeping \(R_{\text{max}}(g(t)) < \Theta\) on \([T, \lambda]\). By Lemma 2.1 the pinching assumption is satisfied. By Proposition 3.6 the condition \((CN)_{r_{i+1}}\) still holds for the extended flow. Then by Lemma 3.5 it is \(\kappa_{i+1}\)-noncollapsed. So the extended flow satisfies all the four conditions w.r.t. these constants.

In this way we can continue the surgical solution to the time interval \([0, (i+1) \cdot 2^{-5}]\) such that it always satisfies the four conditions, since by Lemma 2.5 the singular times cannot accumulate.

Finally note that by Lemma 2.2 the surgical solutions we have constructed must be extinct before the time \(t = \frac{1}{2e_0}\).

The following lemma is essentially due to [BBM].

**Lemma 3.7** (see [BBM, Proposition 2.6]) Let \(\mathcal{X}\) be a class of closed 4-manifolds, and \(X\) be a 4-manifold. Suppose there exists a finite sequence of 4-manifolds \(X_0, X_1, \ldots, X_p\) such that \(X_0 = X, X_p = \emptyset\), and for each \(i\) \((1 \leq i \leq p)\), \(X_i\) is obtained from \(X_{i-1}\) by cutting off along a locally finite collection of pairwise disjoint, embedded 3-spheres, gluing back \(B^4\)'s, and removing some components that are connected sums of members of \(\mathcal{X}\). Then each component of \(X\) is an connected sum of members of \(\mathcal{X}\).

**Proof** The proof of [BBM, Proposition 2.6] applies to 4-dimensional case.

Note that each manifold appeared in the list of our Proposition 2.6 is a (possibly infinite) connected sum of members of \(\mathcal{X} = \{S^4, \mathbb{R}P^4, S^3 \times S^1, S^3 \tilde{\times} S^1\}\). Then Theorem 1.1 follows from Theorem 3.4 and Lemma 3.7.

**Appendix A**

The following lemma is essentially due to Hamilton [H5].

**Lemma A.1** Let \(\varepsilon > 0\) be sufficiently small. Suppose that \(N\) is an \(\varepsilon\)-neck centered at \(x\), with a diffeomorphism \(\psi : S^3 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N\), in a Riemannian 4-manifold \((X, g)\). Then we have Hamilton’s canonical parametrization \(\Phi : S^3 \times \)
\([-l, l] \to V \subset N\), such that \(V\) contains the portion \(\psi(S^3 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1}))\) in \(N\).

**Proof** If \(\varepsilon > 0\) is sufficiently small, by the inverse function theorem every point (in the \(\varepsilon\)-neck \(N\)) at distance (w.r.t the rescaled metric \(R(x)g\)) at least \(0.01\varepsilon^{-1}\) from the ends lies on a unique constant mean curvature hypersurface. Then we can choose harmonic parametrizations of the spheres, choose the height function, and straighten out the parametrization by rotations of the horizontal spheres to obtain Hamilton’s canonical parametrization \(\Phi : S^3 \times [-l, l] \to V \subset N\) such that \(V\) contains the portion \(\psi(S^3 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1}))\) in \(N\). (Cf. the proof of [H5, Theorem C2.2].)

Now suppose that \(N_i\) is an \(\varepsilon\)-neck centered at \(x_i\), with a diffeomorphism \(\psi_i : S^3 \times (-\varepsilon^{-1}, \varepsilon^{-1}) \to N_i, i = 1, 2\), in a Riemannian 4-manifold \((X, g)\). Let \(\pi_i : N_i \to (-\varepsilon^{-1}, \varepsilon^{-1})\) be the composition of \(\psi_i^{-1}\) with the projection of \(S^3 \times (-\varepsilon^{-1}, \varepsilon^{-1})\) onto its second factor. Assume that \(N_1 \cap N_2\) contains a point \(y\) with \(-0.9\varepsilon^{-1} \leq \pi_i(y) \leq 0.9\varepsilon^{-1}\) \((i = 1, 2)\). Then by the above lemma we have Hamilton’s canonical parametrization \(\Phi_i : S^3 \times [-l_i, l_i] \to V_i \subset N_i\), such that \(V_i\) contains the portion \(\psi_i(S^3 \times (-0.98\varepsilon^{-1}, 0.98\varepsilon^{-1}))\) in \(N_i\). If \(\varepsilon\) is sufficiently small, we can use [H5, Theorem C2.4] to get Hamilton’s canonical parametrization \(\Phi : S^3 \times [-l, l] \to V_1 \cup V_2\) and diffeomorphisms \(F_1\) and \(F_2\) of the cylinders, such that \(\Phi_1 = \Phi \circ F_1\) and \(\Phi_2 = \Phi \circ F_2\). \(F_1\) and \(F_2\) are in fact isometries in the standard metrics on the cylinders by [H5, Lemma C2.1]. Moreover we know that for all \(\alpha \in [-l_1, l_1]\) and all \(\beta \in [-l_2, l_2]\), \(\Phi_1(S^3 \times \{\alpha\})\) is isotopic to \(\Phi_2(S^3 \times \{\beta\})\).

The following lemma is essentially due to [BBB⁺] and [BBM].

Let \(K_{st}\) be the supremum of the sectional curvatures of the (4-dimensional) standard solution on \([0, 4/3]\).

**Lemma A.2** (see [BBB⁺,Lemma 4.3.5] and [BBM, Lemma 4.11]) For any \(\varepsilon \in (0, 10^{-4})\) there exists \(\beta = \beta(\varepsilon) \in (0, 1)\) with the following property.

Let \(a, b\) be real numbers satisfying \(a < b < 0\) and \(|b| \leq \frac{3}{4}\), let \((X(\cdot), g(\cdot))\) be a surgical solution (with no essential incompressible space form) defined on \((a, 0]\), and \(x \in X(b)\) be a point such that:

i. \(R(x, b) = 1\);

ii. \((x, b)\) is the center of a strong \(\beta\varepsilon\)-neck;

iii. \(P(x, b, (\beta\varepsilon)^{-1}, |b|)\) is unscathed and satisfies \(|Rm| \leq 2K_{st}\).

Then \((x, 0)\) is center of a strong \(\varepsilon\)-neck.

**Proof** The proof is almost identical to that of [BBB⁺, Lemma 4.3.5].
Appendix B

Bounded curvature at bounded distance is one of the key ideas in Perelman [P1], [P2]; a 4-dimensional version appeared in [CZ2]. The following 4-dimensional version is very close to [MT, Theorem 10.2], [BBB+, Theorem 6.1.1] and [BBM, Theorem 6.4].

**Theorem B.1** For each $c, g, \Psi, L, P, S, A, C > 0$ and each $\varepsilon \in (0, 2\varepsilon_0]$, there exists $Q = Q(c, g, \Psi, L, P, S, A, \varepsilon, C) > 0$ and $\Lambda = \Lambda(c, g, \Psi, L, P, S, A, \varepsilon, C) > 0$ with the following property. Let $I = [a, b]$ (with $0 \leq a < b < \frac{1}{2}c$) and $\{ (X(t), g(t)) \}_{t \in I}$ be a surgical solution with uniformly positive isotropic curvature $(a_1 + a_2 \geq c, c_1 + c_2 \geq c)$, with bounded curvature, with no essential incompressible space form and satisfying the pinching condition (2.2) (with constants $g, \Psi, L, P, S$). Let $(x_0, t_0)$ be a space-time point such that:

1. $R(x_0, t_0) \geq Q$;
2. For each point $y \in B(x_0, t_0, AR(x_0, t_0)^{-1/2})$, if $R(y, t_0) \geq 4R(x_0, t_0)$, then $(y, t_0)$ has an $(\varepsilon, C)$-canonical neighborhood.

Then for any $y \in B(x_0, t_0, AR(x_0, t_0)^{-1/2})$, we have

$$\frac{R(y, t_0)}{R(x_0, t_0)} \leq \Lambda.$$

**Sketch of Proof** One can easily check that the proof of [BBB+, Theorem 6.1.1] and [BBM, Theorem 6.4] can be adapted to our situation. For some of the details one can also consult Step 2 of proof of [CZ2, Theorem 4.1] (for the smooth (without surgery) case) and Step 3 of proof of [CZ2, Proposition 5.4] (for the surgical case).

**Remark** For the estimate above, under a parabolic rescaling of the metrics, $c, g, P, R$, etc. will change in general, and $Q$ will change with the same scaling factor as $R$ does, but $\Lambda$ is scaling invariant.

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