A GLOBALLY CONVERGENT NUMERICAL METHOD FOR A 3D COEFFICIENT INVERSE PROBLEM FOR A WAVE-LIKE EQUATION

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Abstract. A version of the convexification globally convergent numerical method is constructed for a coefficient inverse problem for a wave-like partial differential equation. The presence of the Carleman Weight Function in the corresponding Tikhonov-like cost functional ensures the global strict convexity of this functional. Numerical results are presented to illustrate the effectiveness and efficiency of the proposed method.

Key words. Carleman Weight Function, convexification, global convergence, numerical results

AMS subject classifications. 35R30

1. Introduction. In [5] a version of the convexification globally convergent numerical method was analytically developed for a 3-D Coefficient Inverse Problem (CIP) for a wave-like Partial Differential Equation (PDE). In this paper we first provide some new analytical results, which significantly enhance the theory of [5]. The main goal of this paper is to test the numerical performance of the method of [5]. A numerical study was not conducted in [5]. An application of the CIP of this paper to the elasticity theory is discussed in the end of section 2.

Theorems 4.2 and 5.2 are new analytical results of this paper. The ideas of proofs of these results are also new. Even though a result, similar with the one of Theorem 5.2 was established in [5, Theorem 4], we lift here some restrictive conditions of [5]. In Theorem 4.2, we prove the Lipschitz stability estimate for a boundary value problem (BVP) for a nonlinear PDE with a non local term. The presence of this term causes both the nonlinearity and significant difficulties. Actually, we numerically solve this BVP by the convexification method. Therefore, the importance of the Lipschitz stability estimate is that it ensures that our BVP is sufficiently stable. The latter is confirmed in numerical Tests 2 and 4 in section 6, since these tests treat noisy data.

The key step of the convexification method is the construction of a weighted Tikhonov-like cost functional with the Carleman Weight Function (CWF) in it. The CWF is involved as the weight function in the Carleman estimate for a corresponding hyperbolic operator. This functional is strictly convex on a convex bounded set $P$ in a Hilbert space. A smallness condition is not imposed on the diameter of $P$, which,
in turn means that our numerical method converges globally. Unlike [5], we do not require here the existence of the minimizer on $P$ of our functional. In addition, unlike [5], we do not assume here that this minimizer belongs to the interior of $P$.

In addition, we do not impose anymore the assumption of Theorem 3 of [5] that all terms of the sequence generated by the gradient method of the minimization of that functional belong to the above mentioned bounded set. Also, we decrease the required smoothness of the solution of the forward problem from $H^8$ of [5] to $H^6$. In addition, the forward problem we consider here is an initial boundary value problem for our PDE rather than the Cauchy problem of [5]. Finally, we present our numerical results, which is our main goal here, and which was not done in [5].

Remarks 1.1:

1. It is well known that CIPs are both nonlinear and ill-posed. These two factors cause substantial challenges in their studies. Because of those challenges, as a rule, the minimal smoothness requirements are not of a primary concern of the authors of corresponding publications, see, e.g. [27] as well as Theorem 4.1 in [29].

2. In addition, our computational experience tells us that the smoothness conditions can be significantly relaxed. This is related to both the current paper and previous publications about the convexification cited below.

The convexification method addresses the question of the globally convergent numerical methods for CIPs with non overdetermined data. This is an important question since conventional numerical methods for CIPs rely on the minimizations of least squares cost functionals, see, e.g. [8, 10, 11]. However, these functionals are, as a rule, non convex. The latter leads to the well known phenomenon of local minima and ravines of those functionals. This phenomenon, in turn, means that in order to get a good approximation for the exact solution of a CIP, one needs to start the iterative optimization process of that functional in a sufficiently small neighborhood of that solution. However, such a neighborhood is rarely available in applications, also see [2, 3, 9] for similar comments. On the other hand, the global convergence of the convexification method is rigorously guaranteed, see, e.g. Theorem 5.3 as one of examples of such results as well as publications about the convexification cited below.

We call a numerical method for a CIP **globally convergent** if a theorem is proven, which guarantees that this method reaches a sufficiently small neighborhood of the true solution of that CIP without any advanced knowledge of this neighborhood. Recall that one of the main concepts of the regularization theory is the assumption of the existence of the true solution with the “ideal” noiseless data [4, 30]. We also refer to [9] for another globally convergent method for a similar CIP.

The convexification numerical method was first proposed in 1995 [14] and then in 1997 [15, 16]. Some computations were performed in [17] in the 1D case. Active computational studies of the convexification method have started after the publication of the paper [1] in 2017, since this work has analytically addressed a number of questions about the numerics. We refer to [12, 13, 19, 20, 21, 22] and references therein for some samples of publications, in which various analytical results about the convexification are combined with numerical studies, including the cases of experimentally collected backscattering data. We also refer to the recently published book of the first two authors [23]. All these works consider the cases when the wave field is generated by a point source. Even though these publications are generated by the ideas of the Bukhgeim-Klibanov method (BK), the case of the point source is outside of the framework of BK. We refer to [7] for the originating paper on BK as well as to, e.g. [4, 6, 18, 23] and references cited therein for some follow up publica-
tions. In [7], the method of Carleman estimates was introduced in the field of CIPs for the first time. The focus of [7] was only on the question of uniqueness theorems for multidimensional CIPs. First extensions of the idea of BK to the numerical side of the theory of CIPs were published later [14, 15, 16].

The framework of BK works only with the case when one of initial conditions in a wave-like PDE does not equal zero in the entire domain of interest. This assumption is used in [5]. Therefore, we also use it in the current paper, since we study here the numerical performance of the method of [5]. We also refer to works [2, 3], where a different version of the convexification method is used for CIPs for wave-like PDEs. Both publications [2, 3] work within the framework of the BK method.

The paper is structured as follows. In section 2 we formulate our Coefficient Inverse Problem. In section 3 we derive a BVP for a nonlinear PDE with a non-local term. If this problem is solved, then the target unknown coefficient can be easily found. In section 4 we formulate a Carleman estimate and prove the above mentioned Lipschitz stability estimate for that BVP. In section 5 we introduce the central functional of the convexification method and investigate its properties. Section 6 is devoted to a description of our numerical implementation as well as to the demonstration of numerical results.

2. Statement of the Coefficient Inverse Problem. In all Hilbert spaces considered below functions are real valued ones. Let $\Omega \subset \mathbb{R}^3$ be a convex bounded domain with a piecewise smooth boundary $\partial \Omega \in C^\infty$ and let the number $T > 0$. Denote $Q_T = \Omega \times (0,T)$, $S_T = \partial \Omega \times (0,T)$. Let the function $c(x)$ satisfies the following conditions:

\begin{align*}
(2.1) & \quad c(x) \in [1, b] \text{ for } x \in \Omega, \\
(2.2) & \quad c \in C^1(\Omega).
\end{align*}

We use “1” here in (2.1) for the normalization only. Here $b > 0$ is fixed number. In addition to (2.1), (2.2), we assume that there exists a point $x_0 \in \mathbb{R}^3 \setminus \Omega$ such that

\begin{align*}
(2.3) \quad \min_{\Omega} (\nabla c, x - x_0) \geq 0,
\end{align*}

where $(,) \text{ denotes the scalar product in } \mathbb{R}^3.$ Condition (2.3) means that the function $c(x)$ is increasing along the ray connecting any point $x \in \Omega$ with the point $x_0$ when $x$ moves away from $x_0$. This condition is the standard one in Carleman estimates for the hyperbolic operator $c(x) \partial^2_t - \Delta$, see [4, Theorem 1.10.2], [23, Theorem 2.5.1]. It is unclear whether or not that Carleman estimate is valid without condition (2.3).

We assume that

\begin{align*}
(2.4) \quad f \in C^3(\Omega).
\end{align*}

Let $a, k > 0$, $k > a$ be two numbers. We assume below that

\begin{align*}
(2.5) \quad k \geq \Delta f \geq a = \text{const.} > 0 \text{ for } x \in \Omega.
\end{align*}

Consider the following initial boundary value problem for a wave-like PDE:

\begin{align*}
(2.6) \quad & c(x) u_{tt} = \Delta u, (x, t) \in Q_T, \\
(2.7) \quad & u(x, 0) = f(x), u_t(x, 0) = 0,
\end{align*}
It is shown in [24, Chapter 4, Corollary 4.1] that one can impose some non restrictive conditions on $\partial \Omega$ and functions $f(x)$ and $s(x, t)$, which guarantee existence and uniqueness of the solution

$$u \in H^6(Q_T)$$

of problem (2.6)-(2.8). However, we do not discuss these conditions here for brevity. Rather, we assume smoothness (2.9) of the solution of that problem, also, see Remark 1.1 in section 1.

**Coefficient Inverse Problem (CIP).** Suppose that conditions (2.1)-(2.9) are satisfied. Assume that the coefficient $c(x)$ is unknown inside of the domain $\Omega$. Determine the function $c(x)$ for $x \in \Omega$, assuming that the Neumann boundary condition

$$\partial_n u |_{ST} = p(x, t),$$

where $n$ is the normal outward looking vector at $ST$.

Uniqueness of this CIP under conditions (2.5), (2.9) was established by the method of [7], see, e.g. [4, Theorem 1.10.5.1] and [23, Theorem 3.2.1, Theorem 3.3].

Equation (2.6) is the acoustic equation, and the function $1/\sqrt{c(x)}$ is the speed of propagation of sound waves, and $u(x, t)$ is the amplitude of sound waves. Equation (2.6) also appears in the elasticity theory for isotropic media [25]. Consider a simplified case when $u(x, t)$ is one of components of the displacement field, other components approximately equal zero and

$$c(x) = \frac{\rho(x)}{\lambda(x) + 2\mu(x)},$$

where $\rho(x)$ is the density of the medium, $\lambda(x)$ and $\mu(x)$ are Lamé coefficients [25]. Let $s(x, t)$ be the displacement at the boundary $\partial \Omega$ of the medium and $p(x, t)$ be the normal stress at that boundary. Suppose that we have a 3-D elastic slab occupying the domain $\Omega$. Suppose that, applying a certain force at the moment of time $\{t = 0\}$, we arrange the shape of this slab to become $u(x, 0) = f(x)$, and the slab does not fluctuate at $\{t = 0\}$, i.e. $u_t(x, 0) = 0$. For times $t \in (0, T)$ this slab is allowed to fluctuate without applying a force inside of $\Omega$, although the displacement at the boundary of the slab is controlled by the given function $s(x, t) = u |_{ST}$. Then we obtain forward problem (2.6)-(2.8). Assume that we measure the normal stress at the boundary of $\Omega$, i.e. we measure the function $p(x, t) = \partial_n u |_{ST}$. Then CIP (2.6)-(2.8), (2.10) is the problem of determining combination (2.11) of the above named parameters using boundary measurements (2.10).

### 3. Nonlinear Equation With a Non-Local Term.

In this section we derive from (2.6)-(2.8), (2.10) a nonlinear BVP with a non local term. This BVP does not contain the unknown coefficient $c(x)$. Having the solution of this BVP, one can straightforwardly compute $c(x)$. Therefore, we focus on this BVP below.

**3.1. Nonlinear boundary value problem.** Denote

$$q(x, t) = \partial_t^2 s(x, t), r(x, t) = \partial_t^2 p(x, t).$$

Let $w = u_{tt}$. By (2.9) $w \in H^4(Q_T)$. Since by (2.6)

$$w(x, 0) = \frac{\Delta f(x)}{c(x)},$$

we have

$$\begin{align*}
q(x, t) &= \partial_t^2 s(x, t), \\
r(x, t) &= \partial_t^2 p(x, t).
\end{align*}$$

Let

$$w = u_{tt}.$$
then (2.1) and (2.5) imply

\begin{align}
\frac{a}{b} \leq w(x, 0) \leq k, \\
\frac{a}{k} \leq \frac{\Delta f}{w(x, 0)} \leq \frac{kb}{a}.
\end{align}

We now establish an upper estimate on the constant \(k\). Fix an arbitrary number \(R > 0\). We will seek for the function \(w\) satisfying

\begin{equation}
w \in H^4(Q_T), \|w\|_{H^4(Q_T)} \leq R.
\end{equation}

By the embedding theorem

\begin{equation}
H^4(Q_T) \subset C^1(\overline{Q_T}) \text{ and } \|v\|_{C^1(\overline{Q_T})} \leq C \|v\|_{H^4(Q_T)}, \forall v \in H^4(Q_T).
\end{equation}

Hence, for all functions \(w\) satisfying (3.5)

\begin{equation}
\|w\|_{C^1(\overline{Q_T})} \leq CR,
\end{equation}

where the number \(C = C(Q_T) > 0\) in (3.6), (3.7) depends only on the domain \(Q_T\). Hence, (3.3) implies that we should have \(k \in (0, CR]\). Thus, we replace (3.3), (3.4) with:

\begin{equation}
\frac{a}{CR} \leq \frac{(\Delta f)(x)}{w(x, 0)} \leq CRb.
\end{equation}

Using (2.6)-(2.8), (2.10) and (3.1) we obtain

\begin{equation}
c(x) = \frac{(\Delta f)(x)}{w(x, 0)}, x \in \Omega,
\end{equation}

\begin{equation}
\frac{(\Delta f)(x)}{w(x, 0)}w_{tt} - \Delta w = 0, (x, t) \in Q_T,
\end{equation}

\begin{equation}
w_t(x, 0) = 0,
\end{equation}

\begin{equation}
w|_{S_T} = q(x, t), \partial_n w|_{S_T} = r(x, t).
\end{equation}

Thus, \(w(x, 0)\) is the non local term in the PDE (3.10). In addition, PDE (3.10) is nonlinear due to the presence of this term. We focus below on the numerical solution of BVP (3.10)-(3.12).

3.2. The set for \(w(x, t)\). Define the spaces \(H^2_0(Q_T), H^4_0(Q_T), H^{4,0}_0(Q_T)\) and the set \(\Omega_T\) as:

\begin{align}
H^2_0(Q_T) = \{ u \in H^2(Q_T) : u_t(x, 0) = 0 \}, \\
H^4_0(Q_T) = \{ u \in H^4(Q_T) : u_t(x, 0) = 0 \},
\end{align}
\( H^{1,0}_0(Q_T) = \{ u \in H^2(Q_T) : u_t(x,0) = 0, u|_{S_T} = 0, \partial_n u|_{S_T} = 0 \} \),
\[ \Omega_T = \{ (x,t) : x \in \Omega, t = T \} . \]

We now specify the set of functions on which we search for the solution \( w \) of BVP (3.10)-(3.12). Denote
\[
A(w)(x) = \Delta f(x) \frac{w(x,0)}{w(x,0)}.
\]

By (2.4), (3.5) and (3.6) the function \( A(w)(x) \in C^1(\Omega) \). We search for the function \( w \in P = P(a,b,R,f,q,r) \subset H^4_0(Q_T) \), where the set \( P \) depends only on listed parameters and is defined as:
\[
P = P(x_0,a,b,R,f,q,r) = \begin{cases} 
  \{ w(x,t) : w \in H^4_0(Q_T), \\
  \| w \|_{H^4(Q_T)} \leq R, \\
  a/CR \leq A(w) \leq CRb/a, \\
  \min_{T} \nabla A(w)(x), x - x_0 \geq 0, \\
  w|_{S_T} = q(x,t), \partial_n w|_{S_T} = r(x,t) \}. 
\end{cases}
\]

The inequality in the third line of (3.16) is due to (3.8) and (3.15). The inequality in the fourth line of (3.16) follows from (2.2), (2.3), (3.5), (3.6), (3.9) and (3.15).

Obviously
\[
P = \overline{P}.
\]

**Lemma 3.1** [5]. The set \( P = P(a,b,R,f,q,r) \) is convex.

Uniqueness of the solution \( w \in P \) of problem (3.10)-(3.12) follows from Theorem 4.2. Suppose that we have computed the solution \( w_{\text{comp}} \in P \) of this problem. Then, using (3.2) and (3.15), we set the computed coefficient \( c_{\text{comp}}(x) \) as:
\[
c_{\text{comp}}(x) = \frac{\Delta f(x)}{w_{\text{comp}}(x,0)}.
\]

**4. The Carleman Estimate and the Lipschitz Stability Estimate.** Since the convexification method is based on Carleman estimates, we formulate in this section a Carleman estimate, which was proven in [5]. Next, we formulate and prove the Lipschitz stability estimate for BVP (3.10)-(3.12).

**4.1. Carleman estimate.** Let the numbers \( \eta \in (0,1) \) and \( \lambda \geq 1 \). Consider functions \( \psi, \varphi_\lambda \),
\[
\psi(x,t) = |x - x_0|^2 - \eta t^2, \varphi_\lambda(x,t) = \exp(\lambda\psi(x,t)).
\]

Consider the numbers \( d, D \),
\[
d = \min_{x \in \Omega} |x - x_0|, \quad D = \max_{x \in \Omega} |x - x_0|.
\]

For any number \( \eta \in (0,1) \) we choose a sufficiently large \( T > D/\sqrt{\eta} \). Then
\[
M = M(\Omega, x_0, \eta, T) = \eta T^2 - d^2 > 0, \\
N = N(\Omega, x_0, \eta, T) = \eta T^2 - D^2 > 0,
\]

\[
M = M(\Omega, x_0, \eta, T) = \eta T^2 - d^2 > 0, \\
N = N(\Omega, x_0, \eta, T) = \eta T^2 - D^2 > 0,
\]
(4.5) \[ \varphi^2_N (x,t) \leq c^{2\lambda D^2}, \quad (x,t) \in Q_T. \]

Let the function \( g(x,t) \) be such that \( g, \partial_t g \in C \left( \overline{Q_T} \right) \).

**Theorem 4.1 (Carleman estimate [5])**. Assume that the function \( c(x) \) satisfies conditions (2.1)-(2.3). Let \( D^2 \) be the number defined in (4.2). Then there exist a number \( \eta_0 = \eta_0 \left( \Omega, x_0, \| c \|_{C^1(\Omega)} \right) \in (0,1) \), a number

\[ C_1 = C_1 \left( \eta, x_0, a, b, \| g \|_{C(\overline{Q_T})}, \| \partial_t g \|_{C(\overline{Q_T})}, Q_T \right) > 0 \]

and a sufficiently large number \( \lambda_0 = \lambda_0 \left( \eta, x_0, a, b, \| g \|_{C(\overline{Q_T})}, \| \partial_t g \|_{C(\overline{Q_T})}, Q_T \right) \geq 1 \), all three numbers depending only on listed parameters, such that if \( \eta \in (0, \eta_0) \) and the number \( T = T(\eta, x_0, \Omega) > 0 \) is so large that (4.4) holds, then for all \( \lambda \geq \lambda_0 \) the following Carleman estimate is valid

\[
\int_{Q_T} (c(x) u_{tt} - \Delta u)^2 \varphi_N^2 dxdt + \int_{Q_T} g(x,t) u(x,0) u_{tt}(x,t) \varphi_N^2 dxdt \\
+ C_1 \lambda^3 \exp \left( 2\lambda D^2 \right) \left( \| u \|_{H^1(S_T)}^2 + \| u \|_{L^2(S_T)}^2 \right) \\
+ C_1 \lambda^3 \exp \left( -2\lambda N \right) \left( \| u \|_{L^2(\Omega_T)}^2 + \| u \|_{H^1(\Omega_T)}^2 \right) + C_1 \exp \left( -2\lambda N \right) \| u \|_{H^1(Q_T)}^2
\]

(4.6) \[ \geq C_1 \int_{Q_T} \left( \lambda u_t^2 + \lambda (\nabla u)^2 + \lambda u^2 \right) \varphi_N^2 dxdt, \quad \forall u \in H^2_0(Q_T). \]

**Remarks 4.1:**

1. The nonlinear term \( g(x,t) u(x,0) u_{tt}(x,t) \) is introduced in (4.6) because its presence was used in [5] the proof of a direct analog of Theorem 1.6 about the strict convexity of the functional, which is constructed in section 5. If \( g(x,t) \equiv 0 \), then (4.6) is just the conventional Carleman estimate of [4, Theorem 1.10.2], [23, Theorem 2.5.1].

2. A non-conventional element here is the absence in (4.6) of the integral over \( \overline{Q_T} \cap \{ t = 0 \} \). Indeed, it follows from formulae (1.86), (1.87) of [4] as well as from Corollary 2.5.1 of [23] that this integral equals to zero as long as \( u_t(x,0) = 0 \), see (3.13).

3. Since the function \( \varphi_N^2 (x,t) \) is involved as the weight function in (4.6), then we call \( \varphi_N^2 \) the “Carleman Weight Function” (CWF) for the operator \( c(x) \partial^2_t - \Delta \).

**4.2. Lipschitz stability estimate.** In this section, we use Theorem 4.1 to establish the Lipschitz stability estimate for the nonlinear problem (3.10)-(3.12). The nonlinearity is due to the presence of the nonlocal term \( w(x,0) \) in (3.10). Suppose that we have two functions \( w_1, w_2 \in H^2(Q_T) \) satisfying the following conditions for \( i = 1, 2 \):

(4.7) \[ A(w_i)(x) \partial^2_t w_i - \Delta w_i = 0, \quad (x,t) \in Q_T, \]

(4.8) \[ \partial_t w_i(x,0) = 0, \quad i = 1, 2, \]

(4.9) \[ w_i |_{S_T} = q_i(x,t), \quad \partial_n w_i |_{S_T} = r_i(x,t), \]
Recall that the space $H^2_0(Q_T)$ is defined in (3.13).

**Theorem 4.2.** Assume that the function $f$ satisfies condition (2.4) and 
$\|f\|_{C^0(\overline{\Omega})} \leq f^0$. Assume that the function $\Delta f(x)$ satisfies conditions (2.5) with two positive constants $a,k$, where $0 < a < k$. For $i = 1,2$ let the functions $w_i \in H^2_0(Q_T)$ satisfy conditions (4.7)-(4.10), $w_i(x,0) \in C^1(\overline{\Omega})$ and $\|w_i(x,0)\|_{C^1(\overline{\Omega})} \leq w^0$. Also, assume that there exists a number $m > 0$ such that $w_i(x,0) \geq m$ in $\overline{\Omega}$. In addition, suppose that there exists a point $x_0 \in \mathbb{R}^3 \setminus \overline{\Omega}$ such that

\begin{equation}
(\nabla A(w_1)(x), x-x_0) \geq 0, \forall x \in \overline{\Omega}.
\end{equation}

Let $d$ and $D$ be two numbers defined in (4.2) and let the number $T > D/\sqrt{\eta}$, where the number $\eta \in (0,\eta_1)$, where the number $\eta_1 = \eta_1(\Omega,x_0,f^0,w^0,m) \in (0,1)$ depends only on listed parameters and is chosen the same way as the number $\eta_0 \in (0,1)$ in Theorem 4.1, in which $c(x)$ is replaced with $A(w_1)(x)$. Then there exists a number $Z = Z(\eta,f^0,w^0,m,a,k,x_0,Q_T) > 0$ depending only on listed parameters such that the following Lipschitz stability estimate holds:

\begin{equation}
\|w_1 - w_2\|_{H^1(Q_T)} \leq Z \left(\|q_1 - q_2\|_{H^1(S_T)} + \|r_1 - r_2\|_{L^2(S_T)}\right).
\end{equation}

**Proof.** Below in this proof $Z > 0$ denotes different numbers depending on parameters listed in the formulation of this theorem. It follows from (3.4) and (4.10) that $A(w_1) \in C^1(\overline{\Omega})$ and

\begin{equation}
\|A(w_1)(x)\|_{C^1(\overline{\Omega})} \leq Z_1, A(w_1)(x) \geq \frac{a}{w^0} > 0,
\end{equation}

where the constant $Z_1 = Z_1(f^0,w^0,m) > 0$ depends only on listed parameters. Denote

\begin{equation}
\bar{w} = w_1 - w_2, \bar{q} = q_1 - q_2, \bar{r} = r_1 - r_2.
\end{equation}

Obviously,

\begin{equation}
\bar{w}(x,0) = \bar{w}(x,t) - \int_0^t \bar{w}_t(x,\tau)\,d\tau.
\end{equation}

Using the formula $a_1b_1 - a_2b_2 = (a_1 - a_2)b_1 + (b_1 - b_2)a_2, \forall a_1,b_1,a_2,b_2 \in \mathbb{R}$, we obtain from (4.7)-(4.10), (4.14) and (4.15)

\begin{equation}
A(w_1)(x)\bar{w}_{tt} - \Delta \bar{w} + Y(x,t) \left(\bar{w}(x,t) - \int_0^t \bar{w}_t(x,\tau)\,d\tau\right) = 0,
\end{equation}

\begin{equation}
\bar{w}_t(x,0) = 0,
\end{equation}

\begin{equation}
\bar{w}|_{S_T} = \bar{q}(x,t), \quad \partial_\nu \bar{w}|_{S_T} = \bar{r}(x,t),
\end{equation}

where $\partial_\nu$ denotes the inward normal derivative at the boundary of $\overline{\Omega}$.
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\[ Y(x,t) = -\frac{\Delta f(x)}{w_1(x,t) w_2(x,t)}. \]

Hence,

(4.19) \[ |Y^2(x,t)| \leq Z \text{ in } Q_T. \]

Let \( \varphi_{\lambda}(x,t), \lambda \geq 1 \) be the function defined in (4.1), where the number \( \eta \in (0, \eta_1) \) and the number \( \eta_1 = \eta_1(\Omega, x_0, f^0, w^0, m) \in (0, 1) \) is chosen the same way as the number \( \eta_0 \) in Theorem 4.1, in which \( c(x) \) is replaced with \( A(w_1)(x) \). Using either Lemma 1.10.3 of [4] or Lemma 3.1.1 of [23] as well as (4.19), we obtain

\[
\int_{Q_T} Y^2 \left( \int_0^t \tilde{w}_t(x, \tau) \, d\tau \right) \varphi_{\lambda}^2 \, dx \, dt \leq \frac{Z}{\lambda} \int_{Q_T} \tilde{w}_t^2 \varphi_{\lambda}^2 \, dx \, dt.
\]

Hence, (4.16) and (4.19) imply:

(4.20) \[ \int_{Q_T} (A(w_1)(x) \tilde{w}_tt - \Delta \tilde{w})^2 \varphi_{\lambda}^2 \, dx \, dt \leq Z \int_{Q_T} (\tilde{w}_t^2 + \tilde{w}^2) \varphi_{\lambda}^2 \, dx \, dt. \]

In Theorem 4.1, set \( g(x,t) \equiv 0 \), see item 1 in Remarks 4.1. Then, keeping in mind (4.11), (4.13), (4.17) and (4.18), apply (4.6) to the left hand side of (4.20). We obtain

\[
Z \int_{Q_T} (\tilde{w}_t^2 + \tilde{w}^2) \varphi_{\lambda}^2 \, dx \, dt + Z \lambda^3 \exp \left( 2\lambda D^2 \left( \| \tilde{q} \|_{H^1(S_T)}^2 + \| \tilde{r} \|_{L^2(S_T)}^2 \right) \right)
+ Z \lambda^3 \exp (-2\lambda N) \left( \| \tilde{w} \|_{L^2(Q_T)}^2 + \| \tilde{w} \|_{H^1(Q_T)}^2 \right)
+ Z \exp (-2\lambda N) \| \tilde{w} \|_{H^1(Q_T)}^2
\geq \lambda \int_{Q_T} (\tilde{w}_t^2 + (\nabla \tilde{w})^2 + \lambda^2 \tilde{w}^2) \varphi_{\lambda}^2 \, dx \, dt, \forall \lambda \geq \lambda_0 \geq 1.
\]

Choose the number \( \lambda^{(1)} = \lambda^{(1)}(\eta, f^0, w^0, m, a, k, x_0, Q_T) \geq \lambda_0 \geq 1 \) depending only on listed parameters such that \( \lambda^{(1)} \geq 2Z \). Then the first term in the first line of (4.21) is absorbed by terms in the third line of (4.21). Hence,

\[
Z \exp (-3\lambda N/2) \left( \| \tilde{w}_t \|_{L^2(Q_T)}^2 + \| \tilde{w} \|_{H^1(Q_T)}^2 \right)
+ Z \exp (3\lambda D^2) \left( \| \tilde{q} \|_{H^1(S_T)}^2 + \| \tilde{r} \|_{L^2(S_T)}^2 \right)
\geq \lambda \int_{Q_T} (\tilde{w}_t^2 + (\nabla \tilde{w})^2 + \lambda^2 \tilde{w}^2) \varphi_{\lambda}^2 \, dx \, dt, \forall \lambda \geq \lambda^{(1)} \geq 1.
\]

Choose the number

(4.23) \[ t_1 = \frac{1}{\sqrt{2}} \sqrt{\frac{T^2 - D^2}{\eta}} < T. \]
Obviously \( Q_{t_1} \subset Q_T \) and \( \varphi_2^2(x,t) \geq -2\lambda \eta t_1^2 \) for \( (x,t) \in Q_{t_1} \). Also, using (4.4) and (4.23), we obtain

\[
(4.24) \quad -\frac{3}{2} \lambda N + 2\lambda \eta t_1^2 = -\frac{\lambda}{2} (\eta T^2 - D^2) = -\frac{\lambda N}{2}.
\]

Hence, (4.22) and (4.24) imply

\[
(4.25) \quad \|\bar{w}\|_{H^1(Q_{t_1})} \leq Z \exp \left( -\frac{\lambda N}{2} \right) \left[ \|\bar{w}_t\|_{L^2(\Omega_T)} + \|\bar{w}\|_{H^1(\Omega)} + \|\bar{w}\|_{H^1(Q_T)} \right]
+ Z \exp \left( 3\lambda (D^2 + T^2) \right) \left[ \|q\|_{H^1(S_T)} + \|\bar{r}\|_{L^2(S_T)} \right], \forall \lambda \geq \lambda^{(1)} \geq 1.
\]

Denote

\[
(4.26) \quad W(\lambda) = \text{the right hand side of (4.25)}.
\]

By the mean value theorem there exists a number \( t_0 \in (0, t_1) \) such that

\[
(4.27) \quad \|\bar{w}(x,t_0)\|_{H^1(\Omega)} + \|\bar{w}_t(x,t_0)\|_{L^2(\Omega)} = \frac{1}{t_1} \|\bar{w}\|_{H^1(Q_{t_1})} \leq Z \|\bar{w}\|_{H^1(Q_{t_1})}.
\]

Hence, by (4.25) and (4.26)

\[
(4.28) \quad \|\bar{w}(x,t_0)\|_{H^1(\Omega)} + \|\bar{w}_t(x,t_0)\|_{L^2(\Omega)} \leq W(\lambda).
\]

We now apply the method of energy estimates [24] to problem (4.16)-(4.18) in the domain \( Q_{t_0,T} = \Omega \times (t_0,T) \). Let \( x = (x_1, x_2, x_3) \in \mathbb{R}^3 \). Multiply both parts of equation (4.16) by \( 2\bar{w}_t \). Taking into account (4.19) and using Cauchy-Schwarz inequality, we obtain

\[
\partial_t \left( A(w_1) (x) \bar{w}_t^2 + (\nabla \bar{w})^2 \right) + \sum_{i=1}^3 \partial_{x_i} (-2\bar{w}_x \bar{w}_t) \leq Z \left( \bar{w}_t^2 + \bar{w}^2 + \int_{t_0}^{t} \bar{w}_t^2 (x,\tau) d\tau + \int_{0}^{t} \bar{w}_t^2 (x,\tau) d\tau \right).
\]

Integrating this inequality over the domain \( Q(t_0,t') \) for an arbitrary \( t' \in (t_0,T) \) and using the second inequality (4.13), we obtain

\[
(4.29) \quad \int_{Q(t_0,t')} \left( \bar{w}_t^2 + (\nabla \bar{w})^2 \right) (x,t') dx \leq Z \int_{t_0}^{t'} \int_{\Omega} \left( \bar{w}_t^2 + \bar{w}^2 \right) (x,t) dx dt + Z \|\bar{w}_t\|_{L^2(Q_{t_1})}^2
+ \|\bar{w}(x,t_0)\|_{H^1(\Omega)}^2 + \|\bar{w}_t(x,t_0)\|_{L^2(\Omega)}^2 + \|q\|_{H^1(S_T)}^2 + \|\bar{r}\|_{L^2(S_T)}^2 + \|\bar{w}\|_{H^1(Q_{t_1})}^2.
\]

Using (4.29), Gronwall inequality and substituting in (4.15) \( t = t_0 \) instead of \( t = 0 \), we obtain

\[
\int_{\Omega} \left( \bar{w}_t^2 + (\nabla \bar{w})^2 + \bar{w}^2 \right) (x,t') dx \leq Z \left( \|\bar{w}(x,t_0)\|_{H^1(\Omega)}^2 + \|\bar{w}_t(x,t_0)\|_{L^2(\Omega)}^2 \right)
+ Z \left( \|\bar{w}\|_{H^1(Q_{t_1})}^2 + \|q\|_{H^1(S_T)}^2 + \|\bar{r}\|_{L^2(S_T)}^2 \right), \forall t' \in (t_0,T).
\]
Setting $t' = T$ in (4.30), we obtain
\[
\|\tilde{w}_t\|_{L^2(\Omega_T)}^2 + \|\tilde{w}\|_{H^1(\Omega_T)}^2 \leq Z \left( \|\tilde{q}\|_{H^1(S_T)}^2 + \|\tilde{r}\|_{L^2(S_T)}^2 \right)
\]
(4.31) + $Z \left( \|\tilde{w}(x, t_0)\|_{H^1(\Omega)}^2 + \|\tilde{w}_t(x, t_0)\|_{L^2(\Omega)}^2 + \|\tilde{w}\|_{H^1(Q_{t_0})}^2 \right)$.

In addition, integrating (4.30) with respect to $t' \in (t_0, T]$, we obtain
\[
\|\tilde{w}\|_{H^1(Q_{t_0}, T)}^2 \leq Z \left( \|\tilde{w}(x, t_0)\|_{H^1(\Omega)}^2 + \|\tilde{w}_t(x, t_0)\|_{L^2(\Omega)}^2 + \|\tilde{w}\|_{H^1(Q_{t_0})}^2 \right)
\]
(4.32) + $Z \left( \|\tilde{q}\|_{H^1(S_T)}^2 + \|\tilde{r}\|_{L^2(S_T)}^2 \right)$.

Substituting (4.26) and (4.28) in (4.31) and (4.32), we obtain
\[
\|\tilde{w}_t\|_{L^2(\Omega_T)}^2 + \|\tilde{w}\|_{H^1(\Omega_T)}^2 \leq W(\lambda), \quad \forall \lambda \geq \lambda^{(1)} \geq 1,
\]
(4.33) and also
\[
\|\tilde{w}\|_{H^1(Q_{t_0}, T)}^2 \leq W(\lambda), \quad \forall \lambda \geq \lambda^{(1)} \geq 1.
\]
(4.34)

Let the number $\lambda^{(2)} = \lambda^{(2)}(\eta, J^0, w^0, m, a, k, x_0, Q_T) \geq \lambda^{(1)} \geq 1$ be so large that
\[
Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \leq \frac{1}{4}.
\]
(4.35)

Hence, (4.26) and (4.33) imply
\[
\frac{3}{4} \left( \|\tilde{w}_t\|_{L^2(\Omega_T)}^2 + \|\tilde{w}\|_{H^1(\Omega_T)}^2 \right) \leq \left( 1 - Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \right) \left( \|\tilde{w}_t\|_{L^2(\Omega_T)}^2 + \|\tilde{w}\|_{H^1(\Omega_T)}^2 \right)
\]
\[
\leq Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \|\tilde{w}\|_{H^1(Q_T)}^2
\]
\[
+ Z \exp \left( 3\lambda^{(2)} \left( D^2 + T^2 \right) \right) \left( \|\tilde{q}\|_{H^1(S_T)}^2 + \|\tilde{r}\|_{L^2(S_T)}^2 \right).
\]

Hence,
\[
\|\tilde{w}_t\|_{L^2(\Omega_T)}^2 + \|\tilde{w}\|_{H^1(\Omega_T)}^2 \leq Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \|\tilde{w}\|_{H^1(Q_T)}^2
\]
(4.36) + $Z \exp \left( 3\lambda^{(2)} \left( D^2 + T^2 \right) \right) \left( \|\tilde{q}\|_{H^1(S_T)}^2 + \|\tilde{r}\|_{L^2(S_T)}^2 \right)$.

Using (4.25), (4.26), (4.34) and (4.36), we obtain
\[
\|\tilde{w}\|_{H^1(Q_{t_0}, T)}^2 \leq Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \|\tilde{w}\|_{H^1(Q_T)}^2
\]
(4.37) + $Z \exp \left( 3\lambda^{(2)} \left( D^2 + T^2 \right) \right) \left( \|\tilde{q}\|_{H^1(S_T)}^2 + \|\tilde{r}\|_{L^2(S_T)}^2 \right)$.
Furthermore, combining (4.25) and (4.36) and recalling that since \( t_0 \in (0,t_1) \), then \( Q_{t_0} \subset Q_{t_1} \), we obtain

\[
\|\tilde{w}\|^2_{H^1(Q_{t_0})} \leq \|\tilde{w}\|^2_{H^1(Q_{t_1})} \leq Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \|\tilde{w}\|^2_{H^1(Q_T)} + Z \exp \left( 3\lambda^{(2)} \left( D^2 + T^2 \right) \right) \left( \|\tilde{q}\|^2_{H^1(S_T)} + \|\tilde{r}\|^2_{L_2(S_T)} \right).
\]

Obviously \( \|\tilde{w}\|^2_{H^1(Q_{t_0})} + \|\tilde{w}\|^2_{H^1(Q_{t_0})} = \|\tilde{w}\|^2_{H^1(Q_T)} \). Hence, summing up (4.37) and (4.38) and using (4.35), we obtain

\[
\frac{1}{2} \|\tilde{w}\|^2_{H^1(Q_T)} \leq \left( 1 - 2Z \exp \left( -\lambda^{(2)} \frac{N}{2} \right) \right) \|\tilde{w}\|^2_{H^1(Q_T)} + Z \exp \left( 3\lambda^{(2)} \left( D^2 + T^2 \right) \right) \left( \|\tilde{q}\|^2_{H^1(S_T)} + \|\tilde{r}\|^2_{L_2(S_T)} \right).
\]

The target estimate (4.12) of this theorem follows immediately from (4.14) and (4.39).

5. A Globally Strictly Convex Cost Functional. In this section we present analytical results of our paper. To solve BVP (3.10)-(3.12) numerically, we construct in this section a Tikhonov-like functional, which is strictly convex on the set \( P \) in (3.16). Since a smallness condition is not imposed on the number \( R \) in (3.16), then this functional is globally strictly convex.

5.1. Construction of a globally strictly convex cost functional. Let \( \alpha \in (0,1) \) be the regularization parameter. Recalling (3.17), we construct the following weighted Tikhonov-like regularization functional \( J_{\lambda,\alpha}(w) : P \to \mathbb{R} \):

\[
J_{\lambda,\alpha}(w) = \int_{Q_T} [A(w)(w)_t - \Delta(w)]^2 \varphi^2 \, dx \, dt + \alpha \|w\|^2_{H^4(Q_T)}, \ w \in P.
\]

Minimization Problem. Minimize the functional \( J_{\lambda,\alpha}(w) \) on the set \( P = P(a,b,R,f,q,r) \) defined in (3.16).

Theorem 5.1:

1. For any values of parameters \( \lambda, \alpha, \eta, T > 0 \) and for any function \( w \in P \) there exists the Fréchet derivative \( J'_{\lambda,\alpha}(w) \in H_0^1(Q_T) \) of the functional \( J_{\lambda,\alpha} \). Furthermore, this derivative is Lipschitz continuous, i.e. there exists a constant \( K > 0 \) such that

\[
\|J'_{\lambda,\alpha}(w_1) - J'_{\lambda,\alpha}(w_2)\|_{H^4(Q_T)} \leq K \|w_1 - w_2\|_{H^4(Q_T)}, \forall w_1, w_2 \in H_0^4(Q_T).
\]

2. Let \( \eta, T \) and \( \lambda_0 \) be the numbers of Theorem 4.1. Then there exists a constant \( C_2 = C_2(\eta, x_0, a, b, f, R, P, Q_T) > 0 \) and a sufficiently large number \( \lambda_1 = \lambda_1(\eta, x_0, a, b, f, R, P, Q_T) \geq \lambda_0 \), both depending only on listed parameters, such that for all \( \lambda \geq \lambda_1 \) and for all \( \alpha \in [2e^{-\lambda M}, 1) \) the functional \( J_{\lambda,\alpha}(w) \) is strictly convex on the set \( P \). More precisely,

\[
J_{\lambda,\alpha}(w_2) - J_{\lambda,\alpha}(w_1) - J'_{\lambda,\alpha}(w_1)(w_2 - w_1) \geq C_2e^{-2\lambda M} \|w_2 - w_1\|^2_{H^4(Q_T)} + \frac{\alpha}{2} \|w_2 - w_1\|^2_{H^4(Q_T)}.
\]
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\[
\geq \frac{\alpha}{2} \|w_2 - w_1\|_{H^4(Q_T)}, \forall w_1, w_2 \in P,
\]

where the constant \( M > 0 \) is defined in (4.3).

3. There exists unique minimizer \( w_{\min} \in P \) of the functional \( J_{\lambda, \alpha}(w) \) on the set \( P \) and the following inequality holds:

\[
J'_{\lambda, \alpha}(w_{\min})(w - w_{\min}) \geq 0, \forall w \in P.
\]

Item 1 of this theorem is proven in [5, Theorem 2], except of (5.2). However, (5.2) can be proven completely similarly with the proof of Theorem 3.1 of [1]. The most difficult part of the proof of Theorem 5.1 is the proof of item 2. This item is proven in [5, Theorem 2]. Item 3 follows from item 2 and an obvious combination of Lemma 2.1 and Theorem 2.1 of [1]. Therefore, we do not prove Theorem 5.1 here.

5.2. The accuracy of the minimizer. We now estimate the accuracy of the minimizer \( w_{\min} \in P \), which is found in Theorem 5.1. To do this, we recall first that one of main assumptions of the regularization theory is the a priori assumption of the existence of the exact solution of an ill-posed problem with the exact, i.e. noiseless data [4, 30].

Hence, let \( c^*(x) \) be the exact solution of our CIP with the exact boundary data \( s^*(x,t), p^*(x,t) \) in (2.8), (2.10). In this case the function \( u(x,t) \) should be replaced with the function \( u^*(x,t) \in H^6(Q_T) \) in (2.6)-(2.10), and the function \( c(x) \) in (2.6) should be replaced with \( c^*(x) \), where the function \( c^*(x) \) satisfies conditions (2.1)-(2.3). The initial condition \( f(x) \) in (2.6) remains the same. Let

\[
u_{tt}^* = w^* \in H^4_0(Q_T), w^*|_{S_T} = q^*, \partial_\nu w^*|_{S_T} = r^*.
\]

We assume that

\[
w^* \in P^* = P^*(x_0, a, b, R, f, q^*, r^*),
\]

where the set \( P^* \) is obtained from the set \( P \) in (3.16) via the replacement of functions \( q \) and \( r \) with functions \( q^* \) and \( r^* \) respectively.

Let a small number \( \delta \in (0, 1) \) be the level of the error in the boundary data. More precisely, we assume now that there exist functions \( F, F^* \in H^4_0(Q_T) \) such that

\[
F|_{S_T} = q(x,t), \partial_\nu F|_{S_T} = r(x,t),
\]

\[
F^*|_{S_T} = q^*(x,t), \partial_\nu F^*|_{S_T} = r^*(x,t),
\]

\[
\|F\|_{H^4(Q_T)}, \|F^*\|_{H^4(Q_T)} < R,
\]

\[
\|F - F^*\|_{H^4(Q_T)} < \delta < 1.
\]

It is clear from (2.4), (3.3) and (3.15) that

\[
|\nabla A(w), x - x_0| \leq B, \forall x \in \overline{\Omega}, \forall w \in P.
\]

Here and below \( B = B(x_0, a, b, R, f, Q_T) > 0 \) denotes different positive constants depending only on listed parameters. Choose a small number \( s_0 > 0 \). We impose
on the function \( w^* \) a slightly stronger assumption than the one in the fourth line of (3.16). More precisely, we assume that

\[
(5.12) \quad (\nabla A (w^*), x - x_0) \geq s_0. 
\]

**Remark 5.1.** Even though we assume in (5.10) that the noise is less than \( \delta \) in the \( H^4(Q_T) \) –norm, we need this assumption only for the theory. Indeed, in our computations we introduce a random noise in the boundary data and do not extend these data inside the time cylinder \( Q_T \). Computational results are successful, see section 6. In other words, computations are less pessimistic than the theory is. The latter has been always observed in all previous above cited publications about the convexification, including the most challenging cases of experimentally collected data [13, 21, 22].

For any function \( w \in P(a,b,x_0,R,f,q,r) \) as well as for the function \( w^* \in P^* (a,b,x_0,R,f,q^*,r^*) \) consider functions

\[
(5.13) \quad v_{\min} = w_{\min} - F, \quad v^* = w^* - F^*. 
\]

It follows from (2.4), (3.6), (3.7), (3.15) and (5.10)-(5.12) that

\[
(5.14) \quad (\nabla A (v^* + F), x - x_0) = (\nabla A (v^* + F^* + (F - F^*)), x - x_0) \geq s_0 - B\delta, 
\]

where numbers \( B \) and \( s_0 \) are defined in (5.11) and (5.12) respectively. Choose the number \( \delta_1 = \delta_1 (a,b,R,f,x_0,s_0,Q_T) \in (0,1) \) so small that

\[
(5.15) \quad s_0 - B\delta \geq 0, \forall \delta \in (0,\delta_1),
\]

Assume that in (5.10) \( \delta \in (0,\delta_1) \) as in (5.15). Then (3.6), (3.14), (3.16), (5.6)-(5.15) and the triangle inequality imply that

\[
(5.16) \quad v_{\min}, v^* \in P_0 = P_0 (a,b,x_0,2R,f,F) = \left\{ v(x,t) : v \in H^4_0 (Q_T), \right. \\
\left. \|v\|_{H^4(Q_T)} \leq 2R, \right. \\
\left. a/C R \leq A(v + F) \leq C R b/a, \right. \\
\left. \min_{\Omega T} (\nabla A (v + F) (x), x - x_0) \geq 0. \right. 
\]

Introduce the functional \( I_{\lambda,\alpha} : P_0 \rightarrow \mathbb{R} \) as (see (5.1)):

\[
I_{\lambda,\alpha} (v) = J_{\lambda,\alpha} (v + F) 
\]

\[
(5.17) \quad = \int_{Q_T} [A(v + F)(v + F)_{tt} - \Delta (v + F)]^2 \varphi^2 \Delta dxdt + \\
+ \alpha \|v + F\|^2_{H^4(Q_T)}, \quad v \in P_0.
\]

The following proposition is obvious:

**Proposition 5.1.** Let \( \lambda_1 = \lambda_1 (\eta,a,b,f,R,P,Q_T) \geq 1 \) be the number of Theorem 5.1. The obvious analog of Theorem 5.1 is valid for functional (5.17) for \( \lambda \geq \lambda_2 \), where

\[
(5.18) \quad \lambda_2 = \lambda_2 (\eta, x_0, a, b, f, R, P_0, F^*, Q_T) \geq \lambda_1 (\eta, x_0, a, b, f, 2R, P, Q_T) \geq 1.
\]
We have $F^*$ instead of $F$ in (5.18) due to (5.10).

**Theorem 5.2** (the accuracy of the minimizer). Let the numbers $\eta$ and $T$ be the same as in Theorem 4.1. Assume that the function $w^* \in P^*$ and conditions (5.7)-(5.10), (5.12) and (5.15) hold. Then:

1. In the functional $I_{\lambda,\alpha} (v)$, let $\lambda \geq \lambda_2$, where the number $\lambda_2$ is as in (5.18). Let $\alpha \in [2e^{-\lambda N}, 1)$, where the number $N$ is defined in (4.4). Let $v_{\text{min}} \in P_0$ be the minimizer of this functional on the set $P_0$. Denote $w_{\text{min}} = v_{\text{min}} + F$. Let the function $c_{\text{min}} (x)$ be the one reconstructed by the following analog of (3.9):

$$
(5.19) \quad c_{\text{min}} (x) = \frac{\Delta f}{w_{\text{min}} (x, 0)}.
$$

Then the following accuracy estimates hold:

$$
(5.20) \quad \|w_{\text{min}} - w^*\|_{H^1 (Q_T)} \leq C_3 (e^{\lambda (M + D^2) \delta} + \sqrt{\alpha}),
$$

$$
(5.21) \quad \|c_{\text{min}} - c^*\|_{L^2 (\Omega)} \leq C_3 (e^{\lambda (M + D^2) \delta} + \sqrt{\alpha}),
$$

where numbers $D^2$ and $M$ are defined in (4.2) and (4.3) respectively.

2. Assume that the number $\lambda_2$ is so large that $e^{-2\lambda_2 (M + D^2)} \leq \delta_1$, where $\delta_1$ is as in (5.15). Let the number $\delta_0 = \delta_0 (\eta, x_0, a, b, s_0, f, R, P_0, F^*, Q_T) \in (0, \delta_1) \subset (0, 1)$ be so small that

$$
(5.22) \quad \delta_0 = e^{-2\lambda_2 (M + D^2)}.
$$

For any $\delta \in (0, \delta_0)$, choose

$$
(5.23) \quad \lambda = \lambda (\delta) = \ln \delta^{-1/(2(M + D^2))}, \quad \alpha = 2e^{-\lambda N}.
$$

Define the number $\rho$ as

$$
(5.24) \quad \rho = \frac{1}{2} \min \left( \frac{N}{M + D^2}, 1 \right) \in \left( 0, \frac{1}{2} \right).
$$

Then accuracy estimates (5.20) and (5.21) become

$$
(5.25) \quad \|w_{\text{min}} - w^*\|_{H^1 (Q_T)} \leq C_3 \delta^\rho,
$$

$$
(5.26) \quad \|c_{\text{min}} - c^*\|_{L^2 (Q_T)} \leq C_3 \delta^\rho.
$$

Here and below $C_3 = C_3 (\eta, x_0, a, b, s_0, f, R, P_0, F^*, Q_T) > 0$ denotes different numbers depending only on listed parameters.

**Proof.** Recall that by (5.16) the function $v^* \in P_0$. Hence, we can use Theorem 5.1, (5.17) and Proposition 5.1 as:

$$
(5.27) \quad I_{\lambda,\alpha} (v^*) - I_{\lambda,\alpha} (v_{\text{min}}) - I_{\lambda,\alpha} (v_{\text{min}}) (v^* - v_{\text{min}}) \\
\geq C_3 e^{-2\lambda M} \|v^* - v_{\text{min}}\|^2_{H^1 (Q_T)}.
$$
By (5.4) \(-I_{\lambda,\alpha} (v_{\min}) (v^* - v_{\min}) \leq 0\). Since \(-I_{\lambda,\alpha} (v_{\min}) \leq 0\) as well, then (5.27) implies that
\[
C_3 e^{2\lambda M} I_{\lambda,\alpha} (v^*) \geq \|v^* - v_{\min}\|_{H^1(Q_T)}^2.
\]
Since the function \(w^*\) satisfies equation (3.10), then (5.1) implies that \(J_{\lambda,\alpha} (w^*) = \alpha \|w^*\|_{H^4(Q_T)}^2\). Next, using (4.5), (5.10) and (5.17), representing \(F = F^* + (F - F^*)\) and also using (3.6), we obtain
\[
I_{\lambda,\alpha} (v^*) = J_{\lambda,\alpha} (w^*) + \tilde{I}_{\lambda,\alpha} (v^*) = \alpha \|w^*\|_{H^4(Q_T)}^2 + \tilde{I}_{\lambda,\alpha} (v^*),
\]
where
\[
\left| \tilde{I}_{\lambda,\alpha} (v^*) \right| \leq C_3 \delta^2 \int_{Q_T} \varphi^2 \, dx \, dt \leq C_3 e^{2\lambda D^2} \delta^2.
\]
It follows from (5.29) and (5.30) that
\[
I_{\lambda,\alpha} (v^*) \leq C_3 \left( e^{2\lambda D^2} \delta^2 + \alpha \right).
\]
Combining this with (5.28), we obtain (5.20).

To obtain (5.21), we note that by (3.9) and (5.19)
\[
e_{\min} (x) - e^* (x) = \Delta f (x) \left( w^* (x, 0) - \overline{w}_{\min} (x, 0) \right),
\]
(5.32)
\[
\|u (x, 0)\|_{L_2(Q_T)} \leq C \|u\|_{H^1(Q_T)}, \forall u \in H^1 (Q_T).
\]
Thus, (3.8), (5.20), (5.31) and (5.32) imply (5.21). Estimates (5.25) and (5.26) follows from (5.20)-(5.24).

5.3. Gradient projection method. Lemma 3.1 implies that the set \(P_0 = P_0 (x_0, a, b, R, f, F)\) defined in (5.16) is convex in the space \(H^4_0 (Q_T)\). Therefore, there exists the projection operator \(Y : H^4_0 (Q_T) \rightarrow P_0\) of the space \(H^4_0 (Q_T)\) on the set \(P_0\) [26, Chapter 10, section 3.8]. We arrange the gradient projection method of the minimization of the functional \(I_{\lambda,\alpha} (v)\) on the set \(P_0\) as follows. Let \(v_0 \in P_0\) be an arbitrary point. Let the number \(\gamma \in (0, 1)\). Then we set
\[
v_n = Y \left( v_{n-1} - \gamma I_{\lambda,\alpha}' (v_{n-1}) \right), \quad n = 1, 2, ...
\]
Note that since by Theorem 5.1 \(I_{\lambda,\alpha}' (v_{n-1}) \in H^4_0 (Q_T)\), then all functions \(v_n \in H^4_0 (Q_T)\). Theorem 5.3 follows immediately from an obvious combination of Theorem 2.1 of [1] with Theorems 5.1 and 5.2. The convergence in this theorem is global because the starting point \(v_0\) of sequence (5.33) is an arbitrary point of the set \(P_0\) and a smallness condition is not imposed on the number \(R\) in (5.16), see Introduction.

**Theorem 5.3** (global convergence of the gradient projection method (5.33)). Assume that conditions and notations of Theorem 5.2 hold. Denote \(w_n = v_n + F, n = 0, 1, ...\) Also, let functions \(c_{n, \text{comp}} (x)\) be the ones computed by the following analog of formula (3.18):
\[
c_n (x) = \frac{\Delta f (x)}{w_n (x, 0)}.
\]
Then there exists a sufficiently small number $\gamma \in (0, 1)$ and a number $\theta = \theta(\gamma) \in (0, 1)$ such that the following convergence estimates hold:

$$
\|w_n - \bar{w}\|_{H^{s}(\Gamma_T)} \leq \theta^n \|w_0 - \bar{w}\|_{H^{s}(\Gamma_T)},
$$

$$
\|w_n - u^\ast\|_{H^{s}(\Gamma_T)} \leq C_3 \delta^p + \theta^n \|w_0 - \bar{w}\|_{H^{s}(\Gamma_T)},
$$

$$
\|c_n - c^\ast\|_{L_2(\Omega)} \leq C_3 \delta^p + \theta^n \|w_0 - \bar{w}\|_{H^{s}(\Gamma_T)}.
$$

6. Numerical Studies.

6.1. Numerical implementation.

6.1.1. The forward problem (2.6)-(2.8). In section 6 $x = (x, y, z)$. In our numerical tests, the computational domain is $\Omega = (0, 1) \times (0, 1) \times (0, 1)$ and $x_0 = (0.5, 0.5, -0.5)$ in (2.3). To generate the data (2.10) for the CIP, we have solved the forward problem (2.6)-(2.8) by the finite difference method. We set the total time $T = 1/4$. The mesh sizes with respect to spatial and time variables were the spatial mesh size $h = 1/32$ in all three directions and the mesh size in time is $\tau = 1/512$ respectively. Consider two partitions of the interval $[0, 1]$ and $[0, 1/4],$

$$
0 = x_0 < x_1 < ... < x_{32} = 1, \quad x_p - x_{p-1} = h,
$$

$$
0 = y_0 < y_1 < ... < y_{32} = 1, \quad y_p - y_{p-1} = h,
$$

$$
0 = z_0 < z_1 < ... < z_{32} = 1, \quad z_p - z_{p-1} = h,
$$

$$
0 = t_0 < t_1 < ... < t_{128} = 1/4, \quad t_p - t_{p-1} = \tau.
$$

Denote $u^k_{i,j,l} = u(x_i, y_j, z_l, t_k)$. We have used an implicit central difference scheme in time scale and central difference scheme in space scale. We formulate the implicit discrete scheme as:

$$
\begin{align*}
&c(x_i, y_j, z_l) \left( \frac{u^k_{i+1,j,l} + u^k_{i-1,j,l} - 2u^k_{i,j,l}}{\tau^2} + \frac{u^k_{i,j+1,l} + u^k_{i,j-1,l} - 2u^k_{i,j,l}}{h^2} \right) \\
&\quad - \frac{u^k_{i,j+1,l} + u^k_{i,j-1,l} - 2u^k_{i,j,l}}{h^2} = \frac{u^k_{i+1,j,l} + u^k_{i-1,j,l} - 2u^k_{i,j,l}}{h^2} - \frac{u^k_{i,j+1,l} + u^k_{i,j-1,l} - 2u^k_{i,j,l}}{h^2},
\end{align*}
$$

where $i, j, l$ are the indices with respect to $x, y, z$ and $k$ is the index respect to time, where $u^k = (u^{k-1} + u^{k+1})/2$. The coefficient $c(x)$ is assigned as a piecewise constant function to simulate the targets to be reconstructed, see the next section. As soon as problem (2.6)-(2.8) is solved, we obtain the data for the inverse problem, i.e. the function $p(x, t)$ in (2.10) in the discrete form. In all our numerical tests, we set in (2.7), (2.8) $f(x) = x^2 + y^2 + z^2$ for $x \in \Omega$ and $s(x, t) = f(x)(t+1)$ for $(x, t) \in S_T$ and use their discrete analogs in computations.

6.1.2. The inverse problem. The required second order $t-$ derivatives of the boundary data in (3.1) to solve the inverse problem, see (3.12), are computed by the finite differences with respect to $t$ on the boundary $S_T$. To solve the inverse problem, we numerically minimize functional (5.1). To do this, we write the differential operator in (5.1) in the finite differences and minimize it then with respect to the values of the function $w$ at grid points. Thus, we minimize the following functional for $w^\ast \in P^\ast$

$$
(6.2) \quad J_{x, \alpha}^\ast (w^\ast) = \int_{Q_T} \left[ A \left( w^\ast \right) \left( w^\ast \right)_t - \Delta \left( w^\ast \right) \right]^2 \left( \phi^\ast \right)_x^2 dx dt + \alpha \|w^h\|_{H^2(\Gamma_T)}^2,
$$
where the superscript $\tilde{H}$ means that derivatives in (6.2) are written via finite differences. The integration in (6.2) is the conventional summation over grid points. To avoid the “inverse crime” as well as to decrease the smoothness requirement from $H^4(Q_T)$ to $H^2(Q_T)$, we choose the mesh step sizes $\tilde{h}$ in space and $\tilde{\tau}$ in time larger than those in the forward problem, $\tilde{h} = 1/16 = 2h$ in all three directions, and $\tilde{\tau} = 1/64 = 8\tau$.

Note that even though the $H^4(Q_T)$—norm was used in the regularization term of (5.1), it was used for the theory only. In computations, however, we use the discrete analog of a simpler $H^2(Q_T)$—norm. This replacement works well numerically. We explain this by the fact that we use a rather coarse mesh in (6.2), and all norms in finite dimensional spaces are equivalent. Intuitively, as long as the mesh is rather coarse, one can still use the equivalence property of norms in finite dimensional spaces. We have used similar replacements in many of the above cited publications on the convexification, and they worked well.

To minimize functional (6.2), we apply the conjugate gradient method. Let $w_0^\tilde{H}$ be the resulting discrete function at the iteration number 0. Then $w_{s+1}^\tilde{H} = w_s^\tilde{H} + a_s d_s^\tilde{H}$, where $a_s$ is the iterative step size and $d_s^\tilde{H}$ is the conjugate gradient direction. Numbers $a_s$ are determined using the line search. The conjugate gradient directions $d_s^\tilde{H}$ are determined iteratively as follows: Let $g_s^\tilde{H}$ be the gradient of the discrete functional $J_{\lambda,\alpha}^\tilde{H}$ at the point $w_s$, i.e. $g_s^\tilde{H} = \nabla J_{\lambda,\alpha}^\tilde{H}(w_s^\tilde{H})$. At the iteration number 0 we have $g_0^\tilde{H} = -g_0^0$ and $\alpha_0$ is determined by the line search. Then $d_{s+1}^\tilde{H} = -g_{s+1}^\tilde{H} + \beta_s d_s^\tilde{H}$, where the step size $\beta_s$ is chosen by Polak-Ribiere-Polyak formula [28] $\beta_s = \left(\frac{\nabla g_{k+1}}{\nabla g_k^\tilde{H}}\right)^T (g_{k+1}^\tilde{H} - g_k^\tilde{H}) / \|g_k^\tilde{H}\|^2$.

The iterative process stops at the iteration number $s_0$, at which $\|g_{s_0}^\tilde{H}\|_{L_\infty} < \epsilon$, where the discrete $L_\infty$—norm is taken. In our numerical experiments, the tolerance number is $\epsilon = 10^{-2}$. The starting point $w_0^\tilde{H}$ for this iterative procedure is the background data, i.e. the function $w_0^\tilde{H}$ which corresponds to the solution of the forward problem (2.6)-(2.8) with $c(x) \equiv 1$. Finally, following (3.18), we set for computed functions: $w_{comp}^\tilde{H}(x, 0) = w_{s_0}^\tilde{H}(x, 0)$,

$$c_{comp}^\tilde{H}(x) = \frac{(\Delta f)(x)}{w_{comp}^\tilde{H}(x, 0)}.$$  

**Remark 6.1.** We point out that even though the global convergence of the gradient projection method is claimed in Theorem 5.3, we have established in our computations that the simpler to implement conjugate gradient method actually works well. Similar observations took place in all above cited numerical works of this research group on the convexification. An explanation of this can be found in [22, Theorem 4].

**6.2. Results.** In the tests of this subsection, we demonstrate the efficiency of our numerical method. To see how our method performs for complicated shapes of inclusions to be computed, we have chosen in Tests 1-4 complicated shapes of those inclusions. In all four tests, correct values of the function $c(x)$ are: $c(x) = 2$ inside of the inclusion and $c(x) = 1$ outside of it. Therefore, the correct inclusion/background contrast is 2 : 1 in all cases. Below values of all parameters were chosen by trial and error. In Tests 1-4, $T = 0.25$, $\lambda = 1$, $\eta = 0.1$. Note that even though our above theorems require large values of the parameter $\lambda$, the value $\lambda = 1$ works quite well computationally. This observation is similar to botherations in all above cited previous computational works on the convexification method. It was established in
these publications that \( \lambda \in [1,3] \) works well computationally. Indeed, it is well known that numerical results are often less pessimistic than analytical ones.

To test the stability of our method with respect to the random noise in the data, we introduce a random noise in the boundary data \( s(x,t), p(x,t) \) as:

\[
(6.3) \quad u^{\text{noise}}|_{S_T} = s(x,t) + \sigma \xi_x s(x,t), \quad (\partial_\nu u)^{\text{noise}}|_{S_T} = p(x,t) + \sigma \xi_x p(x,t),
\]

where \( \sigma \) is the noise level, and \( \xi_x \) is the random variable uniformly distributed in \([-1,1]\). Only points \( (x,t) \in S_T \) from the finite difference mesh are taken here. Here, the random variable \( \xi_x \) depends only on discrete points \( x \in \partial \Omega \). The noise was added to the boundary data \( s(x,t) \) only to solve the inverse problem. But it was not added when solving forward problem (2.6)-(2.8) to generate data (2.10) for the inverse problem.

It is well known that the regularization parameter for an ill-posed problem should depend on the noise level \([4, 30]\). In computational practice it also depends on the range of parameters of this problem. Thus, we take \( \alpha = 0 \) for noisy free cases of Tests 1,3. In the cases of noisy data of Tests 2 and 4 with the 3\% noise level we take \( \alpha = 0.05 \).

**Test 1.** Noisy free case. We test the reconstruction by our method of the case when the shape of our inclusion is the same as the shape of the letter ‘A’. The true function \( c(x) \) is depicted on Figures 6.1 (a), (b). We set \( c = 2 \) inside of this inclusion and \( c = 1 \) outside of it. Thus, the inclusion/background contrast is 2:1. See Figures 6.1 for the reconstruction results.

**Test 2.** We now use the noisy boundary data (6.3) with \( \sigma = 0.03 \), which means 3\% of random noise. We test the reconstruction by our method of the case when the shape of our inclusion is the same as the shape of the letter ‘Ω’. The function \( c(x) \) is depicted on Figures 6.2 (a) and (b). See Figures 6.2 for the reconstruction results.

**Test 3.** Noisy free case. We test the reconstruction by our method of the case when the shape of our inclusion is the same as the shape of the letter ‘C’. The function \( c(x) \) is depicted on Figures 6.3 (a) and (b). See Figures 6.3 for the reconstruction results.

**Test 4.** Just as in Test 2, we use the noisy data (6.3) with uniform distributed random variable on \([-1,1]\) of the 3\% level, i.e. \( \sigma = 0.03 \). The shape of our inclusion is the same as the shape of the letter ‘C’. The function \( c(x) \) is depicted on Figures 6.4 (a) and (b). See Figures 6.4 for the reconstruction results.

On Figures 6.1-6.4, each computed function \( c^{\text{comp}} \) is depicted by a 2-D slice and by the surface plot. We have not applied any postprocessing procedure to our computed images.

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Fig. 6.1. Results of Test 1. Noisy free case. The inclusion to be computed is ‘A’ shaped with $c = 2$ in this shape and $c = 1$ outside of it. (a) and (b) Correct images. (c) and (d) Computed images. One can see from (c) that $\max c_{\text{comp}}(x) \approx 2$, which is close to the true value $c_{\text{inclusion}} = 2$. Hence, the inclusion/background contrast of 2:1 is reconstructed accurately. Comparing (b) and (d), one can see that the shape of the inclusion is also accurately reconstructed.

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Fig. 6.2. Results of Test 2. Noisy data with $\sigma = 0.03$ in (6.3), i.e. 3% noise. The inclusion to be computed is 'Ω' shaped with $c = 2$ in this shape and $c = 1$ outside of it. (a) and (b) Correct images, (c) and (d) Computed images. In this example, we add 3% noise. One can see from (c) that $\max c_{\text{comp}}^2(x) \approx 2$, which is close to the true value $c(\text{inclusion}) = 2$. Hence, the inclusion/background contrast of 2:1 is reconstructed accurately. Comparing (b) and (d), one can see that the shape of the inclusion is also accurately reconstructed.

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Fig. 6.3. Results of Test 3. Noisy free case. The inclusion to be computed is ‘C’ shaped with $c = 2$ in this shape and $c = 1$ outside of it. (c) and (d) Computed images. One can see from (c) that $\max c_{\text{comp}}(x) \approx 2$, which is close to the true value $c(\text{inclusion}) = 2$. Hence, the inclusion/background contrast of $2:1$ is reconstructed accurately. Comparing (b) and (d), one can see that the shape of the inclusion is also accurately reconstructed.

Fig. 6.4. Results of Test 4. Noisy data with $\sigma = 0.03$ in (6.3), i.e. 3% noise. The inclusion to be computed is ‘C’ shaped with $c = 2$ in this shape and $c = 1$ outside of it. (c) and (d) Computed images. In this example, we add 3% noise. One can see from (c) that $\max c_{\text{comp}}(x) \approx 2$, which is close to the true value $c(\text{inclusion}) = 2$. Hence, the inclusion/background contrast of $2:1$ is reconstructed accurately. Comparing (b) and (d), one can see that the shape of the inclusion is also accurately reconstructed.