Geometry of Landau orbits in the absence of rotational symmetry

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The integer quantum Hall effect (IQHE) is usually modeled using Galilean-invariant or rotationally-invariant Landau levels. However, these are not generic symmetries of electrons moving in a crystalline background, even in the low-density continuum limit. We present a treatment of the IQHE which abandons Galilean dispersion and rotational symmetry, only keeping inversion symmetry, which exposes topological properties and more generic geometric structures. We define an emergent metric $g_{\alpha\beta}$ for each Landau level with a reformulation of the Hall viscosity. The metric is then used to define guiding-center coherent states where the corresponding single-particle wavefunctions are holomorphic functions of $z^+$ times a Gaussian. By numerically studying cases with quartic dispersion, we show that the number of the zeroes of the wavefunction encircled by the semiclassical orbit, denoted by $n$, defines a “topological spin” $s_n$ by $s_n = n + \frac{1}{2}$, while its original definition as an “intrinsic angular momentum” is no longer valid without rotational symmetry. We also present results for the density and current responses which differentiate between diagonal and Landau-level-mixing terms.

I. INTRODUCTION

Most treatments of the integer and fractional quantum Hall effects (IQHE and FQHE) in the low-density two-dimensional electron gas (2DEG) subjected to a uniform magnetic flux density begin by making the simplifying assumption that the electrons obey Newtonian dynamics with 2D Galilean invariance, and that the system has a continuous rotational symmetry around an axis normal to the 2D surface on which the electrons move.

Among the simplifications that follow from this is the well-known result that when rotational symmetry is present, the wavefunction of an electron in the lowest Landau level is a holomorphic function $f(z) = f(x + iy)$ times a Gaussian factor, which played a key role in Laughlin’s discovery$^1$ of a model wavefunction for the FQHE with Landau-level filling $\nu = 1/m$. The important role of rotational symmetry was further reinforced when it was observed$^2$ that the Laughlin state wavefunctions were mathematically related to “conformal blocks” of (Euclidean) 2D conformal field theory in which the global rotation symmetry SO(2) is promoted to a local Euclidean conformal symmetry $CO(2)$.

Much of the interest in the QHE relates to its robust topological properties, which by definition are not related to the presence or absence of symmetries such as rotational, Galilean or conformal symmetries, so the presence of such symmetries as a simplifying “toy model” feature is not problematic. However there has been recent interest in geometrical properties of the QHE$^3$–$^7$ in which symmetries play an important role, and the inclusion of non-generic symmetries may in this case be misleading.

The assumption of a local Galilean structure with an inertial mass tensor given by a scalar gravitational mass times the Euclidean metric of the flat plane has been extended by some authors to a treatment of the QHE that places the electrons on a Riemann surface that can be thought of as a curved 2D surface immersed in 3D Euclidean space, from which it receives an induced metric that defines a Laplacian operator. A related formulation of the problem has been given as “Newton-Cartan” theory$^11$. However the QHE is experimentally exhibited by electrons moving on a lattice plane inside crystalline condensed matter, in which there is generically no Galilean invariance or rotational symmetry, and the non-Newtonian electron kinetic energy derives from the band structure of an underlying atomic-scale crystal structure.

Models of the QHE that include non-generic symmetries are certainly useful models which will exhibit generic topological behavior, but cannot be relied upon to exhibit generic geometrical behavior. A useful analogy is the theory of metals, for which the simple model of a Galilean-invariant Fermi gas with a spherical Fermi surface is often used as an initial “toy model” that exhibits many generic features of metals (such as a $T$-linear specific heat), but fails to capture the intricate geometry of true metallic Fermi surfaces.

In an attempt to clarify the nature of geometric properties of the QHE, such as the “Hall viscosity”$^7$, we will here present a treatment of the integer QHE which abandons the Galilean dispersion relation of non-relativistic Newtonian dynamics, where

$$\varepsilon(p) = \frac{p_x^2 + p_y^2}{2m}$$  \hspace{1cm} (1)$$
keeping only the weaker condition

$$\varepsilon(-p) = \varepsilon(p)$$  \hspace{1cm} (2)$$
and the simplification that the classical Landau orbits $\varepsilon(p) = E$, constant, are compact closed curves in 2D momentum space, so that classical Landau orbits are uniquely fixed by their energy and guiding center. The Galilean-invariant model is a special case of these weaker restrictions, which simplify the semiclassical quantization. The weaker restrictions we impose could themselves be dropped at the expense of introducing some additional complications, but we believe that as they permit the complete removal of the non-generic Galilean and
rotational-symmetry structures from the theory of the QHE, they allow more generic geometrical structures to be revealed.

The “momentum space inversion symmetry” can be justified as a condition that derives from the assumption that the 2DEG has unbroken time-reversal symmetry in the absence of an applied magnetic field. It is also compatible with spatial inversion symmetry, which is the only metric-independent point symmetry compatible with generic crystalline condensed matter, as it is the only point symmetry of a generic Bravais lattice.

In this work, the geometrical description will be restricted to that of the integer QHE of non-interacting electrons in non-Galilean Landau levels, and the extension to the richer structure of the FQHE will be given elsewhere, building on the IQHE ingredients described here.

This paper is organized as follows. In Section II we briefly introduce a formulation of 2D spatial geometry using consistent covariant and contravariant index placement. With the algebra of Landau orbits and guiding centers reviewed in Section III we study the geometry of Landau orbits by emphasizing on a unimodular metric $g_{ab}$ in Section IV. Then in Section V we provide a natural choice of the metric along with a reformulation of the Hall viscosity in the absence of rotational symmetry. In Section VI we define a topological “Landau orbit spin” $s_n$ by investigating the root structure of the wavefunction, and the case of a quartic dispersion is numerically studied in Section VII. After a brief discussion of some other properties of the Landau orbits in Section VIII we describe in Section IX generic results for the density and current responses in the absence of rotational symmetry.

II. 2D GEOMETRY WITH CONSISTENT INDEX PLACEMENT

The Euclidean plane can be parametrized by a Cartesian coordinate system

$$\mathbf{x} = x^ae_a, \quad e_a \cdot e_b = \delta_{ab},$$

(3)

where $\{e_a, a = 1, 2\}$ is a basis of $\mathbf{x}$-independent orthonormal tangent vectors, and

$$e_a \times e_b = \epsilon_{ab} \mathbf{n},$$

(4)

where $\mathbf{n}$ is the unit normal to the plane, which is also $\mathbf{x}$-independent. Here $\delta_{ab}$ is the 2D Euclidean metric induced on the plane by its embedding in 3D Euclidean space, and $\epsilon_{ab}$ is the antisymmetric 2D Levi-Civita symbol, which has a sign ambiguity because the unit normal is only defined up to $\pm \mathbf{n}$ (the orientation choice).

It will prove very useful to use a formalism that distinguishes vectors, such as velocities $\mathbf{v} = v^ae_a = \dot{x}^ae_a$ which have components with upper indices (contravariant to those of $e_a$), from covectors, such as forces, which have lower (covariant) indices $F_a = e_a \cdot \mathbf{F}$. Then the scalar product $\mathbf{F} \cdot \mathbf{v} = F_a v^a \equiv F_1 v^1 + F_2 v^2$ is only defined as a bilinear relation between vectors and covectors, with Einstein summation convention on repeated upper/lower index pairs. In this formalism, the Euclidean metric $\delta_{ab}$, the inverse metric $\delta^{ab}$ and the Kronecker symbol $\delta^a_b$ are distinguished, although they are numerically equal when a Cartesian coordinate system is used.

The dual Levi-Civita symbol $\epsilon^{ab}$ is also numerically-equal to (but distinguished from) $\epsilon_{ab}$. Technically, $\epsilon_{ab}$ is a covariant pseudotensor density with weight $-1$, and $\epsilon^{ab}$ is a contravariant pseudotensor density with weight $+1$. If $g_{ab}$ is a metric tensor,

$$g = \det g = \epsilon^{ac} \epsilon^{bd} g_{ab} g_{cd}$$

(5)

is a scalar density with weight 2. A useful formal identity is

$$\epsilon^{ab} \epsilon_{cd} = \delta^a_c \delta^b_d - \delta^a_d \delta^b_c.$$

(6)

In any expression, covariant indices, contravariant indices, and tensor weights must balance.

The use of consistent index placement may seem pedantic in systems where the Euclidean metric plays a fundamental role, such as those governed by Newtonian dynamics, in which the inertial mass tensor of a particle $m_{ab}$ is given by $m \delta_{ab}$, where $m$ is its gravitational mass. However, in generic non-relativistic crystalline condensed matter (i.e., with a trigonal as opposed to a cubic lattice, and no point-group symmetry other than spatial inversion) the Euclidean metric plays no role whatsoever in the non-relativistic physics, except for the definition of the non-relativistic limit $\delta_{ab} v^a v^b \ll c^2$, where $\mathbf{v} = v^ae_a$ is the velocity of an electron relative to the preferred inertial frame of condensed matter, in which the underlying atomic lattice is at rest.

Consistent index placement exposes the presence of the Euclidean metric in physical equations: for example, the azimuthal angular momentum, which is conserved in Newtonian dynamics on the plane, is given by

$$L = \epsilon_{ab} \delta^{bc} \dot{x}^a p_c.$$

(7)

The presence of the inverse Euclidean metric (required to balance the index placement) immediately identifies the angular momentum as a quantity without validity in a generic crystalline environment.

III. THE ALGEBRA OF LANDAU ORBITS AND GUIDING CENTERS

The Schrödinger representation of the momentum of an electron is

$$e_a \cdot \mathbf{p} = p_a = -i \hbar \frac{\partial}{\partial x^a} - e A_a (\mathbf{x}),$$

(8)

where

$$\frac{\partial A_b}{\partial x^a} = \epsilon_{ab} B(x),$$

(9)
and \(x^a\) are the coordinates of the electron. With this definition, the Levi-Civita symbol is odd under time-reversal. We will concentrate on the case \(B(x) = B\), constant, leaving open the eventual extension to include a slow spatial variation \(B(x)\). In the case of constant \(B\), the commutation relations of the components of the momentum have the simple form of a Heisenberg algebra

\[
[p_a, p_b] = i\hbar B\epsilon_{ab},
\]

which has the exponentiated form

\[
U(a_1)U(a_2)\ldots U(a_n) = \prod_{i<j} e^{\frac{i}{\hbar}R(p_i,a_j)}U(\sum_i a_i),
\]

where \(U(a) = \exp(i a \cdot p)\) with real \(a^a\) are unitary operators, and

\[
\varphi(a, a') = -\hbar e B\epsilon_{a'b}a^a a^b.
\]

The remaining commutation relations are unaffected by \(B\):

\[
[x^a, x^b] = 0, \quad [x^a, p_b] = i\hbar \delta^a_b.
\]

In a system with uniform magnetic flux density, the electron coordinate may be decomposed as

\[
x = R + (eB)^{-1}e^a e_ap_b,
\]

where \(R = R^a e_a\) is the guiding center of the Landau orbit, with commutation relations

\[
[R^a, R^b] = -i\hbar (eB)^{-1} \epsilon^{ab} e_a p_b = 0,
\]

and which defines a second independent Heisenberg algebra. It is useful to write

\[
p_a = B\epsilon_{ab} d^b,
\]

where

\[
d = e(x - R)
\]

can be interpreted as the electric dipole moment defined by the displacement of the electron relative to its guiding center. The inversion symmetry \(\epsilon(p) = \epsilon(-p)\) means that in a Landau orbit eigenstate \(|\psi_{n\alpha}\rangle\) of \(\epsilon(p)\), the expectation value of this electric dipole vanishes, and

\[
\langle\psi_{n\alpha}|p|\psi_{n\alpha}\rangle = 0,
\]

which provides an unambiguous definition of the guiding center independent of the Landau level.

In a uniform magnetic field, a representation of the algebras of \(p_a\) and \(R^a\) can be given in terms of independent harmonic oscillator degrees of freedom

\[
p_a = \sqrt{\frac{\hbar}{2}} (eB)^{1/2} (e^a a^1 + e^b a^1)\),
\]

\[
R^a = \sqrt{\frac{\hbar}{2}} (eB)^{-1} (e^a b + e^b b^1)\),
\]

where

\[
\epsilon^{ab} e^a e^b = \epsilon_{ab} e^{a+b} = i,
\]

and \([a, a^1] = [b, b^1] = 1, [a, b] = [a^1, b^1] = 0\). A metric is said to be unimodular if it has determinant 1; The condition \(10\) means that two unimodular Euclidean-signature metrics and their inverses are defined by

\[
\bar{e}^a e^b = \frac{1}{\pi} (\bar{g}_{ab} + \epsilon_{ab}), \quad \bar{g}^{ab} = \epsilon^{ac} \epsilon^{bd} \bar{g}_{cd},
\]

\[
e^{a* a} = \frac{1}{\pi} (\bar{g}^{ab} + \epsilon^{ab}), \quad \bar{g}_{ab} = \epsilon_{ac} \epsilon_{bd} \bar{g}^{cd}.
\]

The standard treatments would immediately set both \(\bar{g}_{ab}\) and \(\bar{g}_{ab}\) equal to the Euclidean metric \(\delta_{ab}\), but we emphasize that they are independent free parameters of the representation \(19\).

It is now useful to introduce the generators of linear area-preserving \(SL(2, R)\) linear deformations of momentum space

\[
p_a \mapsto U(\beta) p_a U(\beta)^{-1} = \Lambda^b_a (\beta) p_b,
\]

where

\[
det \Lambda = \frac{1}{2} \epsilon^{ac} \epsilon_{bd} \Lambda^b_a \Lambda^d_c = 1.
\]

This unitary transformation leaves the guiding centers invariant, and preserves the commutation relation \(10\). The unitary operator \(U(\beta)\) is given by

\[
U(\beta) = \exp(i \pi \beta_{ab} \gamma_{ab}),
\]

where \(\beta_{ab} = \beta_{ba}\) is real, and the Hermitian generators \(\gamma_{ab} = \gamma_{ba}\) are given by

\[
\frac{1}{4} \{p_a, p_b\} = \hbar e B \gamma_{ab}.
\]

They obey the non-compact Lie algebra

\[
[\gamma_{ab}, \gamma_{cd}] = \frac{1}{2} i (\epsilon_{ac} \gamma_{bd} + \epsilon_{bc} \gamma_{ad} + \epsilon_{ad} \gamma_{bc} + \epsilon_{bd} \gamma_{ac})
\]

with a quadratic Casimir

\[
C_2 = \det \gamma = \frac{1}{2} \epsilon^{ac} \epsilon_{bd} \gamma_{ab} \gamma_{cd}.
\]

The representation \(25\) of this algebra has \(C_2 = -\frac{3}{16}\). Any unimodular metric \(\tilde{g}_{ab}\) with \(det \tilde{g} = 1\) defines a Cartan subalgebra generated by \(L_g = \tilde{g}^{ab} \gamma_{ab}\). The explicit form of this representation in terms of harmonic oscillator operators \(19\) is

\[
\gamma_{11} = \frac{1}{4} (a + a^1)^2,
\]

\[
\gamma_{22} = -\frac{1}{2} \left((a - a^1)^2\right),
\]

\[
\gamma_{12} = \frac{1}{4} i \left((a)^2 - (a^1)^2\right).
\]

Then one choice of a Cartan subalgebra is

\[
L = \gamma_{11} + \gamma_{22} = \frac{1}{4} (a^1 a + a a^1).
\]

This representation is in fact reducible into two irreducible representations of the Lie algebra \(26\) with the
same value of the quadratic Casimir, distinguished by parity under $a \mapsto -a$.

The transformation (22) is a hyperbolic “squeeze mapping” of momentum space if $\det \beta < 0$, a parabolic shear if $\det \beta = 0$, and an elliptic “generalized rotation” through an angle $\theta = \pi \sqrt{(\det \beta)}$ if $\det \beta > 0$. For all $\beta_{ab}$ with $\det \beta = 1$, $U(\beta)$ has the property $U(\beta)^2 = 1$, and reduces to the inversion mapping

$$U(\beta)p_aU(\beta) = -p_a, \quad \det \beta = 1. \quad (30)$$

This leaves $\gamma_{ab}$ unchanged, and thus, like the quadratic Casimir, $U(\beta)$ with $\det \beta = 1$ is in the center of the Lie algebra. The full $SL(2, R)$ group is thus spanned by $U(\beta)$ with

$$\det \beta \leq 1, \quad (31)$$

where all transformations with $\det \beta = 1$ are equivalent, and equal to the inversion. The analogous algebra that acts on the guiding centers has generators $\tilde{\gamma}_{ab}$ defined by

$$\frac{1}{4}\{P^a, R^b\} = -\hbar(eB)^{-1}e^{ac}e^{bd} \gamma_{cd}. \quad (32)$$

This obeys the algebra (26) with the same quadratic Casimir.

### IV. LANDAU ORBITS AND THEIR GEOMETRY

In keeping with the interpretation that electronic kinetic energy $\varepsilon(p)$ derives from an expansion of a band-structure around a band minimum, it will be assumed that

$$\varepsilon(p) = \varepsilon(-p) > \varepsilon(0), \quad p \neq 0. \quad (33)$$

A “regularity condition” on $\varepsilon(p)$ will also be assumed: this is that $\varepsilon(\lambda p)$ is analytic in $\lambda$ for all $p \neq 0$. In fact, for an operator $\varepsilon(p)$ to remain well-defined when it is a bivariate function of two non-commuting components $p_a$ that obey a Heisenberg algebra, it appears that $\varepsilon(p)$ should not just be analytic in each component of $p$, but should be an entire function, so that, for any c-number $p_0$, $\varepsilon(p_0 + (p - p_0))$ has an absolutely-convergent expansion in powers of the components of $\delta p(p_0) = p - p_0$, of the form

$$\varepsilon(p) = \sum_{n=0}^{\infty} e^{a_1 a_2 \ldots a_n}(p_0)\{\delta p_{a_1}(p_0), \ldots, \delta p_{a_n}(p_0)\} \quad (34)$$

(where $\{\ldots\}$ is a symmetrized product of the non-commuting variables) which is independent of the (arbitrary) choice of the expansion origin $p_0$. This “strong regularity” condition is satisfied if $\varepsilon(p)$ is a polynomial in the components $p_a$, which is the case investigated here.

A “strict monotonicity condition” will also be assumed so that, for all $p \neq 0$, $\varepsilon(\lambda p)$ is a strictly monotonically-increasing function in the range $\varepsilon(0) \leq \varepsilon(\lambda p) < \infty$ for $0 \leq \lambda < \infty$, that is unbounded as $\lambda \to \infty$. This implies that all curves $\varepsilon(p) = E > \varepsilon(0)$ are compact closed curves in the momentum plane $(p_1, p_2) \in \mathcal{R}_2$, with the topology of a circle $S_1$, and enclose the origin. These can be parametrized as $p(E, \theta) = p(E, \theta + 2\pi)$ and strict monotonicity implies that the sense of the orbit can be chosen so that, for all $\theta$,

$$w(E, \theta) = \varepsilon^{ab} p_a \partial_\theta p_b(E, \theta) > 0, \quad (35)$$

where $\partial_\theta p(E, \theta) = \partial_\theta p(E, \theta)/\partial \theta$.

The group velocity of the dispersion is

$$v(p) = e_\alpha \frac{\partial \varepsilon}{\partial p_\alpha}, \quad (36)$$

where $v(-p) = -v(p)$. The strict monotonicity and regularity conditions imply that the the only solution of the condition $v(p) = 0$ is $p = 0$, and that $p \cdot v(p) \equiv p_a v^a(p) > 0$ for $p \neq 0$. The classical Hamiltonian equation of motion is

$$\frac{dx}{dt} = v(p), \quad \frac{dp_a}{dt} = eB \varepsilon_{ab} v^b(p) \quad (37)$$

and a classical particle with kinetic energy $E$ moves on the closed momentum-space trajectory $p(E, \theta)$ with

$$\frac{d\theta}{dt} = \frac{eB \varepsilon_a(p) v^a(p)}{w(E, \theta)} > 0. \quad (38)$$

It is worth stressing that, in marked contrast to the case of Newtonian dynamics in free space, where $\varepsilon(p) = \frac{1}{2}m^{-1}\delta^{ab} p_a p_b$, the Euclidean metric $\delta_{ab}$ has no place in the Hamiltonian equations of motion of non-relativistic Landau orbits of electrons inside condensed matter.

While the classical Landau-orbit dynamics is fundamentally Hamiltonian, a classical Lagrangian description becomes possible with an additional restriction on the function $\varepsilon(p)$, that the odd function $v(p)$ defines an invertible map from the full momentum plane to a compact region $D_\nu$ of the velocity plane, so that the kinetic energy function has the form $\varepsilon(p) = \tilde{\varepsilon}(v(p))$, where $\tilde{\varepsilon}(\hat{x})$, $\hat{x} \in D_\nu$, is the Lagrangian kinetic energy.

The momentum-space area $\mathcal{A}(E)$ enclosed by the curve $p(E, \theta)$ is

$$\mathcal{A}(E) = \frac{1}{2} \int_0^{2\pi} d\theta e^{a b} p_a \partial_\theta p_b(E, \theta). \quad (39)$$

The strict monotonicity condition means that $\mathcal{A}(E)$ is a strictly monotonically increasing unbounded function of $E \geq \varepsilon(0)$, with $\mathcal{A}(E) \to 0$ as $E \to \varepsilon(0)^+$. The semiclassical quantization of Landau levels is

$$\mathcal{A}(E_n) = 2\pi \hbar eB s_n > 0, \quad (-1)^{2s_n} = -1. \quad (40)$$

The monotonicity condition ensures that for

$$s_n = (n + \frac{1}{2}) \text{sgn}(eB), \quad n = 0, 1, 2, \ldots, \quad (41)$$
Because the components of $p$ no longer commute with each other when $\hbar eB \neq 0$, the one-particle quantum Hamiltonian must be given as a series expansion in $p_a$ with an ordering convention. To reflect our choice to specialize to inversion-symmetric models, which singles out $p_0 = 0$ as a preferred origin for an expansion in $p - p_0$, the form used here will be

$$H = \sum_{n=0}^{\infty} \frac{1}{2n!} (p_{a_1} \ldots p_{a_n}) A_n^{(a_1), \ldots (a_n)} (p_{b_1} \ldots p_{b_n}), \quad (42)$$

where $A_n^{(a_1), \ldots (b_i)}$ is symmetric under any permutations of indices within the sets $\{a_1, \ldots, a_n\}$ or $\{b_1, \ldots, b_n\}$. In the limit $\hbar eB \to 0$, $H$ becomes the function $\varepsilon(p)$ in the momentum plane, which is then given in terms of the expansion coefficients $A_n^{(a_1), \ldots (b_i)}$. This form remains Hermitian if translational invariance is broken to make $A_n^{(a_1), \ldots (b_i)}(x)$ or $B(x)$ adiabatically-varying functions of position.

As a result of translational invariance, the Landau levels are macroscopically degenerate, with one independent state for each London quantum $\Phi_0 = \hbar/e$ of magnetic flux passing through the plane:

$$H(\Phi_0) = E_n(\hbar eB)/\Phi_0. \quad (43)$$

It will be useful to generalize this Hamiltonian eigenvalue problem to a Lagrangian eigenvalue problem

$$\dot{L}(v) = v \cdot p - H,$$

$$\dot{L}(v)|_{\Psi_{\alpha}}(\nu) = L_n(v, \hbar eB)|_{\Psi_{\alpha}}(\nu), \quad (44)$$

where the components of the velocity $v$ are $c$-number parameters of the Hermitian operator $\dot{L}(v, \hbar eB)$, and the eigenvalue $L_n(v)$ is a function of $v$ that characterizes each Landau level, and will be the Lagrangian for the motion of the guiding center of an electron in that level. For the Galilean-invariant Newtonian problem, this Lagrangian reduces to the simple form

$$L_n(v, \hbar eB) = \frac{1}{2} m\delta_{ab} v^a v^b - m^{-1} \hbar eB s_n. \quad (45)$$

The eigenstates of the Hamiltonian can be represented in a basis of harmonic oscillator states

$$|\Psi_{\alpha}| = \sum_{m=0}^{\infty} \frac{1}{\sqrt{m!}} a^{\dagger m} |\phi_{\alpha}| \equiv f_n(a^{\dagger})|\phi_{\alpha}|,$$

$$a|\phi_{\alpha}| = 0, \quad (46)$$

where both $f_n(a^{\dagger})$ and the reference state $|\phi_{\alpha}|$ depend on the choice of unimodular metric $\hat{g}_{ab}$. Note that while the diagonalization is carried out in a basis defined by a given choice of metric, it will turn out to be useful to represent each Landau-level eigenstate in this form using a Landau-level-specific choice $\hat{g}_{ab} \propto \hat{g}_{ab}^n$, where $\hat{g}_{ab}^n$ is a (non-unimodular) metric specific to the Landau level.

The question now arises, what is the natural basis to use to resolve the degeneracy of the Landau levels? Since $[H, R] = 0$, the guiding center is a classical constant of the motion. However, quantum mechanically, its components do not commute, and obey an uncertainty principle

$$\hat{g}_{ab} \left( \langle \Psi | [R^a R^b] | \Psi \rangle - \langle \Psi | R^a | \Psi \rangle \langle \Psi | R^b | \Psi \rangle \right) \geq \ell_B^2, \quad (48)$$

where $\hat{g}_{ab}$ is any Euclidean-signature unimodular metric, and

$$\ell_B = \sqrt{\hbar/eB}. \quad (49)$$

The closest quantum mechanical state to the classical state with a fixed guiding center is a guiding-center coherent state $|\Psi_n(x, \bar{\theta}) \rangle$ that obeys the inequality (48) as an equality, with

$$| \langle \Psi_n(x, \bar{\theta}) | R | \Psi_n(x, \bar{\theta}) \rangle \rangle = \| x \|, \quad (50)$$

The standard treatments of Landau levels based on a Newtonian kinetic energy $\frac{1}{2} m \delta_{ab} v^a v^b$ generally use a coherent state with $\hat{g}_{ab}$ given by the Euclidean metric $\delta_{ab}$, but since non-Newtonian Hamiltonian dynamics has no dependence on the Euclidean metric, this is no longer the “obvious” choice. Below, we will show that there is a “natural” choice that in general is different for each Landau level, but since any unimodular metric is a viable choice, we leave it unspecified for now, and use it to set both $\hat{g}_{ab}$ and $\bar{g}_{ab}$, so $\hat{g}_{ab} = \bar{g}_{ab}$.

A unimodular metric defines a complex structure through the factorization

$$\frac{1}{2} (\hat{g}_{ab} + e^{i\epsilon_{ab}}) = e^a e_b \quad (51)$$

where $e_a$ is a complex covariant vector with the property

$$e^a e_a = i. \quad (52)$$

Note that this factorization has a $U(1)$ ambiguity under $e_a \mapsto e_a \exp i\phi$. A dimensionless complex coordinate can be defined by

$$z = e_a x^a / \sqrt{2 \ell_B}. \quad (53)$$

The coherent state condition is then

$$b | \Psi_n(x, \bar{\theta}) \rangle = \bar{z} | \Psi_n(x, \bar{\theta}) \rangle, \quad (54)$$

with

$$\bar{z} = e_a \bar{x}^a / \sqrt{2 \ell_B}. \quad (55)$$

A common choice of electromagnetic gauge is the so-called “symmetric gauge”, which derives from the factorization [51] of the Euclidean metric $\delta_{ab}$ in terms of $(e_1, e_2) = 2^{-1} z(1, i)$. Then the symmetric gauge is given by

$$A_a(x) = 2^{-\frac{1}{2}} i (z^* e_a - ze_a). \quad (56)$$

For general $\hat{g}_{ab}$ with a factorization [51], [56] is still a valid gauge choice $A_a(x, \bar{\theta})$ that is now “symmetric” with
respect to the unimodular metric \( \tilde{g}_{ab} = e_a^* e_b + e_a e_b^* \). In this
gauge, the Schrödinger representation of the guiding-
center harmonic oscillator operators \( b \) and \( b^\dagger \) is
\[
 b = \frac{1}{2} z^* + \frac{\partial}{\partial z^*}, \quad b^\dagger = \frac{1}{2} z - \frac{\partial}{\partial z}.
\]
where \( \partial f(z^*)/\partial z = \partial f(z)/\partial z^* = 0 \). The corresponding
representation of the Landau-orbit harmonic oscillator operators is
\[
a = \frac{1}{2} z + \frac{\partial}{\partial z}, \quad a^\dagger = \frac{1}{2} z^* - \frac{\partial}{\partial z^*}.
\]
This immediately implies that the Schrödinger wave-
function \( \Psi_n(x; \bar{x}, \tilde{g}) = \langle x|\Psi_n(\bar{x}, \tilde{g}) \rangle \) of the coherent state has the functional form
\[
\Psi_n(x; \bar{x}, \tilde{g}) = f_n(z^* - \bar{z}^*) e^{\frac{i}{2}(\bar{z}^* z - 2 z^* \bar{z} + z^2 - \bar{z}^2)},
\]
where \( f_n(z^*) = (-1)^n f_n(-z^*) \) is a holomorphic function of \( z^* \). Here \((-1)^n\) is the parity of the Landau-orbit
coherent state with respect to spatial inversion around the point \( \bar{x} \) at which it is centered.

A continuous rotational invariance with respect to the metric \( \tilde{g}_{ab} \) exists if
\[
[H, L(\tilde{g})] = 0, \quad L(\tilde{g}) = -\frac{1}{2} \hbar c B \tilde{g}_{ab} \gamma_{ab}.
\]
In this case, the function \( f_n(z^*) \) is the homogeneous polynomial
\[
f_n(z^*) = \frac{(z^*)^n}{\sqrt{(n!)}}
\]
and an arbitrary lowest-Landau level \((n = 0)\) wavefunction has the well-known form
\[
\Psi_{o0a}(x) = F_o(z) e^{-\frac{i}{2} z^* \bar{z}},
\]
where \( F_o(z) \) is a holomorphic function. This is a familiar result in the rotationally-symmetric case \( \tilde{g}_{ab} = \delta_{ab} \) (when \( z^* \bar{z} = \frac{1}{2} \delta_{ab} x^a x^b / |\ell_B|^2 \)), but remains true if (60) holds for any \( \tilde{g}_{ab} \), when there is a “generalized rotational symmetry”.

However, such a continuous symmetry is not a generic property of electrons moving inside condensed matter, and the holomorphic lowest-Landau-level property (62) disappears in the absence of this continuous symmetry. Starting with the Laughlin wavefunction, many models for the FQHE have incorporated this holomorphic lowest-Landau-level structure, and it has often been regarded as a fundamental ingredient of a theory of the FQHE. However, (as evidenced by the fact that a Laughlin-like FQHE state also occurs in the second Landau level) the holomorphic structure (62) can only have “toy model” status as a convenient simplification, and thus can play no fundamental role in a generic theory of the FQHE.

In contrast, the holomorphic coherent-state property (59) is quite generic, with no requirement for a continuous rotational symmetry. In the generic absence of such a symmetry, the functions \( f_n(z) \) are holomorphic, but not polynomial. The coherent state has the structure
\[
|\Psi_n(x, \tilde{g})\rangle = f_n(a^\dagger)(\bar{x}, \tilde{g}),
\]
where
\[
a|\bar{x}, \tilde{g}\rangle = 0, \quad b|\bar{x}, \tilde{g}\rangle = \tilde{z}|\bar{x}, \tilde{g}\rangle.
\]
Holomorphic functions are characterized by their zeroes (plus their asymptotic behavior for large \( z \)). However, the location of the zeroes of the functions \( f_n(z^*) \) will vary continuously with the so-far arbitrary choice of the coherent state metric \( \tilde{g}_{ab} = \delta_{ab} \) (apart from the generic occurrence of a zero at \( z^* = 0 \) if \( n \) is odd, and its absence if \( n \) is even, as a consequence of momentum-space inversion symmetry (2)).

To use \( f_n(z^*) \) for a standardized description, a “natural choice” of metric must be uncovered. The key to this will be a property of the Landau levels that is now generally called their “Hall viscosity”, first discussed in the context of 2D Landau levels by Avron, Seiler and Zograf (ASZ) \(^7\). This will however need a significant reformulation of the Hall viscosity in the absence of a continuous rotational symmetry.

V. HALL VISCOSITY AND THE LANDAU-ORBIT METRIC

The viscosity of a continuous fluid is the usually-
dissipative relation between the stress (the current-
density of momentum) in a fluid and the gradient of its flow velocity. The use of a formalism with consistent index-placement that distinguishes between covariant and contravariant indices will be important in clarifying the structure of the stress tensor when rotational invariance is not present.

Momentum, like a force, is a covariant vector, and the momentum density \( T^n_a(x, t) \) is locally conserved in translationally-invariant continuum dynamics. If translational invariance is broken by coupling to background fields, the continuity relation is
\[
\partial_t T^n_a + \partial_b T^n_b = F_a,
\]
where \( T^n_a(x, t) \) is the mixed-index stress tensor, and \( F_a(x) \) is the body force exerted on the continuous medium by the background fields that break translational invariance.

As a mixed-index tensor, there is in general no symmetry relating the two indices. However, if the system has a rotational invariance under coordinate transformations that leave the Euclidean metric tensor \( \delta_{ab} \) invariant, the stress tensor has the Cauchy symmetry property
\[
\delta_{bc} T^n_b - \delta_{bc} T^n_c = 0.
\]
When there is such a rotational invariance, it is well-
known that it is possible to obtain the expression for the stress tensor from the variation of the action with respect to the metric that defines the invariance
\[
\delta S = -\frac{1}{2} \int d^4x \sqrt{(-g)} T_{\mu\nu} \delta g^\mu\nu.
\]
The model for this is the derivation of the stress-energy tensor \( T^\sigma_\nu \) in General Relativity (GR), where (local) rotational invariance generalizes to (local) Lorentz invariance, and the Cauchy relation becomes

\[
g_{\mu\nu} T^\sigma_\nu - T^\sigma_\mu g_{\sigma\nu} = 0, \quad (68)
\]

where \( g_{\mu\nu} \) is the gravitational metric.

The relation \( (67) \) (or its dual where \( g_{\mu\nu} \) is varied) is often taken as the fundamental definition of the stress-energy tensor, given in the form \( T_{\mu\nu} = g_{\mu\nu} T^\sigma_\sigma \) or \( T^{\mu\nu} = T^\sigma_\sigma g^{\sigma\nu} \), but this is a symptom of a common practice in GR to hide dependence on the metric by using it to raise or lower indices away from their “natural” positions (e.g. by forming metric-dependent constructions such as \( F^{\mu\nu} = g^{\mu\sigma} g^{\nu\tau} F_{\sigma\tau} \)).

However, on a fundamental level, stress (or stress-energy) is the response to strain of the matter fields in space (or space-time), and what is arguably its true defining relation is the variation of the action under the infinitesimal active diffeomorphism of the matter fields \( \varphi(x) \to \varphi(x') = \varphi(x + \delta u(x)) \), where \( \delta u(x) = u^\mu(x)\delta \lambda \). This definition of the stress-energy tensor is

\[
\delta S = \int d^4 x \sqrt{(-g)} T^\nu_\nu \nabla_\mu \delta u^\mu. \quad (69)
\]

The change in the inverse metric under the diffeomorphism (i.e., after the pull-back of the coordinate system to match the diffeomorphism) is

\[
g^{\mu\nu}(x) \to g^{\mu\nu}(x') + \delta g^{\mu\nu}(x') \quad (70)
\]

with

\[
\delta g^{\mu\nu} = L_a(g^{\mu\nu})\delta \lambda = -(g^{\sigma\sigma} \nabla_\sigma \delta u^\nu + g^{\nu\sigma} \nabla_\sigma \delta u^\mu), \quad (71)
\]

where \( L_a(\ldots) \) is the Lie derivative. Then the combination of the fundamental definition \( (69) \) with the local Lorentz-invariance property \( (68) \) leads to the familiar result \( (67) \).

The QHE is exhibited by incompressible 2D electron fluids. In the case of the integer QHE, this results from the Pauli principle, where (in the zero-temperature limit) all Landau levels are either filled or empty. In this case, there are no gapless collective degrees of freedom of the electron gas that can transmit forces through the bulk, or allow dissipation of energy.

This means that the gapped quantum incompressible liquid is quite different from the notional classical incompressible liquid, which is just the limiting case of a classical liquid with a large bulk modulus. The non-relativistic classical fluid supports sound waves (pressure waves), which travel with a group velocity that diverges in the limit in which the bulk modulus becomes infinite. (This of course violates causality as the speed of sound eventually exceeds the Lorentz speed \( c \), so the “thought experiment” of taking the incompressible limit of a classical non-relativistic fluid cannot in principle be realized.)

Formally, the classical incompressible fluid supports a hydrostatic pressure which is instantaneously spatially-uniform throughout the fluid, so the trace of the stress tensor is spatially homogeneous:

\[
T^a_\sigma(x, t) = p(t)\delta^a_\sigma. \quad (72)
\]

However the gapped quantum incompressible fluid does not support sound waves, and thus has a \textit{traceless} stress tensor:

\[
T^a_\sigma(x, t) = 0. \quad (73)
\]

The viscosity tensor is the linear relation between the stress in a fluid and a non-uniform flow velocity field \( v(x, t) = v^\sigma(x) \) that breaks translational invariance. It is also fundamentally a locally-defined mixed-index tensor

\[
T^a_\sigma(x, t) = \eta^a_\sigma(x, t) \partial_\nu v^\nu(x, t) + O(v^2). \quad (74)
\]

In the zero-temperature limit, the viscosity of the gapped incompressible fluid must be dissipationless, which is the antisymmetry condition

\[
\eta^a_\sigma = -\eta^c_b. \quad (75)
\]

The dissipationless viscosity must also be odd under time reversal.

In two dimensions, the tracelessness and antisymmetry of the viscosity tensor that is implied by quantum incompressibility means that it must have the structure

\[
\eta^{cf}_a = \epsilon^{ae} \epsilon^{cf} \eta^{H}_{ab}, \quad (76)
\]

\[
\eta^{H}_{ab} = \eta^{H}_{ac} = \eta^{H}_{ab} = -\eta^{H}_{cd} = \frac{1}{2} \epsilon (\epsilon_{ac} \eta^{H}_{bd} + \epsilon_{ad} \eta^{H}_{bc} + \epsilon_{bc} \eta^{H}_{ad} + \epsilon_{bd} \eta^{H}_{ac}). \quad (77)
\]

where \( \epsilon \equiv \epsilon_{12} \) is the Pfaffian of the 2D Levi-Civita symbol. Here the rank-4 tensor \( \eta^{H}_{ab} \) is odd under time-reversal, and \( \eta^{H}_{ab}(x, t) \) is a symmetric rank-2 covariant tensor that is also odd under time-reversal. This rank-2 tensor will here be called the 2D “Hall viscosity” tensor, and satisfies the stability condition

\[
\det \eta^{H} \geq 0. \quad (79)
\]

If there is rotational invariance, \( \eta^{H}_{ab} \) must be proportional to the Euclidean metric \( \delta_{ab} \). Previous work by other authors on the Hall viscosity has generally specialized to this case, with the correspondence

\[
\eta^{H}_{ab} = \bar{\eta}^{H} \delta_{ab}. \quad (80)
\]

In particular, Read\textsuperscript{73} has provided an interpretation of the time-reversal-odd scalar \( \bar{\eta}^{H} \) as

\[
\bar{\eta}^{H} = \frac{1}{2} \bar{n}_e \bar{s} = \frac{1}{2} \bar{I} \quad (81)
\]

where \( n_e \) is the 2D electron density, and \( \bar{s} \) is an “intrinsic orbital spin per electron”, or, equivalently, that \( \bar{I} \) is the (local) density of “intrinsic orbital angular momentum” of the electron fluid, a quantity that only has meaning if
there is a continuous rotational symmetry. Equivalently, Read’s formula can be stated as the result that, when there is such a rotational symmetry,
\[ \delta^a \eta^{H,ab} = I. \]  

To explicitly calculate the stress tensor, we must first obtain its generic expression in the absence of rotational symmetry using the fundamental definition \[109\). Similar derivation, though without consistent index placement, can be found in the reference.\[10\] We write down the action for 2D electron fluids in \(U(1)\) gauge fields \(A_0\) and \(A_a\),
\[ S = \int d^2xdt \mathcal{L}, \]  
\[ \mathcal{L} = \frac{i}{2} \hbar \psi^{\dagger} \partial_0 \psi - \frac{i}{2} \hbar (\partial_0 \psi^{\dagger}\psi + e A_0 \psi^{\dagger}\psi - \mathcal{H}) \]  
where we have assumed that the underlying manifold is the 2D Euclidean plane, i.e., \(g_{ab} = \delta_{ab}\) and \(\sqrt{g} = 1\). For a system with Galilean invariance the Hamiltonian density \(\mathcal{H}\) is just
\[ \mathcal{H} = \frac{\hbar^2}{2} m^{ab}(D_a \psi)^\dagger D_b \psi, \]  
\[ D_a = \partial_a - i \frac{e}{\hbar} A_a, \]  
where \(m^{ab}\) is the inverse mass tensor induced by the band structure, and therefore has nothing to do with the (inverse) manifold metric \(g^{ab}\). For the purpose of generality we leave the form of the Hamiltonian density unspecified. Under the infinitesimal spatial diffeomorphism \(x'^a = x^a + \delta u^a\) the \(U(1)\)-covariant derivatives transform as
\[ D'_a = (\delta^b_a - \partial_a \delta^b u^b) D_b, \]  
and the variation of the action is
\[ \delta S = \int d^2xdt T^a_b \delta \partial_b \delta u^b, \]  
\[ T^a_b = \frac{\delta \mathcal{H}}{\delta (D_a \psi)(D_b \psi)} + \frac{\delta \mathcal{H}}{\delta (M_{ab})} D_b \psi - L^a_b. \]

For the IQHE, which is essentially a single-particle problem, the stress tensor can be written in terms of single-particle operators,
\[ T^a_b = \frac{1}{2 \pi \hbar^2} \tau^a_b, \]  
\[ \tau^a_b = \frac{1}{2} \{\nu^a, p_b\}, \quad \nu^a = \frac{\partial \varepsilon(p)}{\partial p_a}, \]  
where \(\varepsilon(p)\) is the single-particle Hamiltonian. The traceless part of the stress tensor can also be obtained
\[ \tau^a_b = \delta^a_b - \frac{1}{2} \nu^a \delta^a_b = -i \epsilon^{abc} [\gamma_{bc}, \varepsilon(p)], \]
where the \(\gamma_{ab}\) operators are defined in Section (11).

Now we are ready to calculate the response of the stress tensor to a spatially-uniform but time-varying shear strain \(\partial_a \delta u^b(t) = \lambda^b_a(t)\). The perturbative Hamiltonian is
\[ \delta \varepsilon(t) = -\frac{1}{2} \nu^a \lambda^b_a(t) = -\frac{1}{2} \nu^a \lambda^b_a(t). \]  
According to linear response theory, the induced stress is determined by the retarded stress-stress correlation function,
\[ \tau^a_b(t) = \int dt' G^a_{b,c}(t - t') \lambda^c_d(t'), \]  
\[ G^a_{b,c}(t) = \sum_n \frac{\nu}{\hbar} \langle \Theta(t) \langle \hat{\tau}^n_d(t), \hat{\tau}^c_d(0) \rangle | n \rangle \]  
with \(\Theta(t)\) being the step function and \(\nu = 1\) if Landau level \(n\) is filled and \(\nu = 0\) if it is empty. We then Fourier-transform it into frequency space,
\[ G^a_{b,c}(\omega) = \int dt e^{i(\omega + \hbar \eta) t} G^a_{b,c}(t), \]  
where \(\eta\) is an infinitesimal positive number. To obtain the formula for the Hall viscosity, what we need is the coefficient of the first order term in the \(\omega\)-expansion,
\[ \frac{dG^a_{b,c}(\omega)}{d\omega} \bigg|_{\omega=0} = \sum_{n,n'} \frac{\nu \omega}{(E_n - E_{n'})^2} \left( \langle n | \hat{\tau}^a_b | n' \rangle \langle n' | \hat{\tau}^c_d | n \rangle - \langle n | \hat{\tau}^c_d | n' \rangle \langle n' | \hat{\tau}^a_b | n \rangle \right) \]  
\[ = \sum_n \hbar \nu \omega e^{i\epsilon} e^{i\xi} \langle n | [\gamma_{bc}, \gamma_{df}] | n \rangle. \]  
Comparing it to the definition of the viscosity \[107\), we arrive at the expression for the rank-4 tensor \(\eta^H_{abcd}\),
\[ \eta^H_{abcd} = -\frac{i}{2 \pi \hbar^2} \sum_n \nu_n \langle n | [\gamma_{bc}, \gamma_{df}] | n \rangle. \]  
Recall the commutator of the \(\gamma_{ab}\) operators \[26\), with which we obtain the expression for the 2D Hall viscosity tensor \(\eta^H_{ab}\),
\[ \eta^H_{ab} = \frac{1}{2 \pi \hbar^2} \sum_n \nu_n \langle n | \gamma_{ab} | n \rangle. \]  

Based on this calculation, in each Landau level we can define a metric \(g^H_{ab}\) by
\[ \langle n | \{p_a, p_b\} | n \rangle = -4 \pi \hbar^2 s_n \text{sgn}(eB) g^H_{ab} \]  
where
\[ s_n = -\text{sgn}(eB)(n + \frac{1}{2}) \]  
is a topological spin of the Landau level that appears in the semiclassical formula \[14\), and can also be obtained by inspection of the Landau-level coherent state calculated with \(g_{ab} \propto g^H_{ab}\), where \(g^H_{ab}\) is the (non-unimodular)
We observed that the polynomial roots

\[
\eta_{ab}^n = \frac{h}{2} \sum_n \nu_n s_n g_{ab}^n.
\]

(103)

For a rotationally-invariant system, all Landau levels have the same metric

\[
g_{ab} = \frac{|eB|}{2\pi\hbar} \delta_{ab}.
\]

(104)

We have now identified a “natural” unimodular metric \( \tilde{g}_{ab}^n \propto g_{ab}^n \) associated with each Landau level, that allows the definition of a standard localized Landau-orbit coherent state.

VI. ZEROES OF THE LANDAU-ORBIT WAVEFUNCTION AND TOPOLOGICAL SPIN

The identification of an intrinsic unimodular metric \( \tilde{g}_{ab}^n \), associated with the shape of the Landau orbit in a given Landau level now allows a systematic analysis of its wavefunction.

A basis of states in momentum space is defined by

\[
\frac{1}{2} \tilde{g}_{ab} p_a p_b |n, \tilde{g}\rangle = \hbar |eB|(n + \frac{1}{2})|n, \tilde{g}\rangle.
\]

(105)

where \( \tilde{g}_{ab} \) is some unimodular metric. Assuming that the expansion (102) is truncated at some finite maximum \( n = n_{\text{max}} \), there are only non-vanishing matrix elements \( \langle n_1 |H| n_2 \rangle \) for \( |n_1 - n_2| \leq 2n_{\text{max}} \). An arbitrary choice of \( \tilde{g}_{ab} \) can initially be made, and the problem can be diagonalized in a basis truncated to \( n \leq N \) for some suitably large \( N \). For each eigenstate of interest, the metric \( \tilde{g}_{ab}^n \) is calculated, and the problem is rediagonalized in a basis \( \tilde{g}_{ab}^n \) with \( \tilde{g}_{ab} = \tilde{g}_{ab}^n \propto g_{ab}^n \), and a suitably large truncation (this process can be iterated if necessary). A Taylor-series expansion that approximates \( |\psi_n\rangle \) by

\[
|\psi_n\rangle = f_n(a^\dagger)|0, \tilde{g}^n\rangle
\]

(106)

where \( f_n(z^*) \) is a finite-degree polynomial is then obtained, and the zeroes of the polynomial can be obtained with a standard root finder. We studied models with quartic dispersions, 2n_{\text{max}} = 4, and used MPACK\(^2\), a version of LAPACK adapted to use the GMP multiprecision library to get very accurate eigenstates, and studied how the patterns of zeroes varied with the truncation \( N \). We observed that the polynomial roots \( \{z_i\} \) converged to stable values inside some radius in the complex plane (which is also a representation of the momentum plane) which grew as \( N \) was increased. The use of MPACK allowed accurate determination of the zeroes for large \( N \) of order 1000, exposing them in a large “window” in momentum space.

The polynomial form

\[
|\Psi|^2 \propto \exp V, \quad V(z, z^*) = -|z|^2 + \sum_i \ln |z - z_i|^2
\]

(107)

gives the amplitude of the Landau-orbit wavefunction in momentum space in terms of a potential that is a solution of Poisson’s equation with a Laplacian defined by the metric \( \tilde{g}_{ab}^n \), with a uniform positive charge inside a circular disk of large radius (that fixes the asymptotic form of the holomorphic function \( f_n(z^*) \)), and unit negative point charges at the zeroes.

For a strictly-monotonic dispersion \( \varepsilon(p) \), the semiclassical quantization identifies the \( n \)th quantized Landau orbit with a region of momentum space with total area \( 2\pi|\hbar|eB| \) between the closed contour \( A(E^-) = 2\pi n|\hbar|eB| \) and \( A(E^+) = 2\pi (n+1)|\hbar|eB| \). This region has the topology of a disk (Euler characteristic \( \chi = 1 \)) for \( n = 0 \), or an annulus (Euler characteristic \( \chi = 0 \)) for \( n > 0 \), and can be thought of as a “fattened” semiclassical orbit. When \( f_n(z^+) \) is obtained using the metric \( \tilde{g}_{ab}^n \), we observed that it is possible to choose a threshold \( c \) so the region defined by \( |\Psi_n(z, z^*)|^2 = |\Psi_n(p)|^2 > c \) covers an area of order \( 2\pi|\hbar|eB| \), and defines a zero-free region with the same topology as the semiclassical prediction that is topologically a disk for \( n = 0 \), and for \( n > 0 \) an annulus that encircles \( n \) zeroes, hence defining a “topological spin” \( s_n = n + \frac{1}{2} \) of the Landau level. It is also possible to choose \( c \) so most of the weight of the wavefunction is within the region.

This picture remains true when \( f_n(z^*) \) is calculated using a different metric that is close to \( \tilde{g}_{ab}^n \), but eventually breaks down for a choice of metric sufficiently different from \( \tilde{g}_{ab}^n \); the pattern of zeroes changes with the metric, and for \( n > 0 \) the central zeroes eventually begin to “leak out” of the annulus defined by the “fattened” semiclassical orbit.

When a guiding-center coherent state of the Landau orbit is constructed using \( \tilde{g}_{ab} = \tilde{g}_{ab}^n \), there is also a mapping from the complex plane to real space, with the origin at the guiding center of the orbit, and the zeroes of \( f(z^*) \) become the zeroes of the coherent state wavefunction.

The analysis of the generic Landau orbits has now identified a topological spin \( s_n \) and a non-unimodular metric \( \tilde{g}_{ab}^n \) as well as its unimodular version \( \tilde{g}_{ab}^n \), which in the rotationally-invariant case reduces to \( \tilde{g}_{ab}^n = \delta_{ab} \).

VII. A CASE STUDY: THE QUARTIC DISPERSION

We will now present a detailed case study of the simplest model dispersion that exhibits the generic property
that there is no congruence between the shapes of different semiclassical orbits.

This is the model with quadratic and quartic terms in the dispersion:

$$H = A_{1}^{ab}p_{a}p_{b} + \frac{1}{2}A_{2}^{abcd}\{p_{a},p_{b}\}\{p_{c},p_{d}\}.$$  \hspace{1cm} (108)$$

Up to the addition of the quadratic Casimir, which is a constant, and does not affect the semiclassical orbits, there are four possible classes of quartic terms compatible with strict monotonicity. If $\tilde{g}_{i}^{ab}$ are the inverses of unimodular metrics:

\begin{align*}
\tilde{A}_{2,1}^{ab,cd} &= \tilde{g}_{i}^{ab}\tilde{g}_{j}^{cd} + \tilde{g}_{i}^{ab}\tilde{g}_{j}^{cd}, \\
\tilde{A}_{2,11}^{ab,cd} &= \tilde{g}_{i}^{ab}\tilde{u}_{d}^{i} + \tilde{u}_{d}^{i}\tilde{g}_{j}^{cd}, \\
\tilde{A}_{2,111}^{ab,cd} &= \tilde{u}_{1}^{a}\tilde{u}_{2}^{b}\tilde{u}_{d}^{i}\tilde{u}_{i}^{1} + \tilde{u}_{1}^{a}\tilde{u}_{2}^{b}\tilde{u}_{2}^{i} + \tilde{u}_{1}^{a}\tilde{u}_{1}^{b}\tilde{u}_{i}^{d} + \tilde{u}_{2}^{a}\tilde{u}_{2}^{b}\tilde{u}_{1}^{d}, \\
\tilde{A}_{2,111}^{ab,cd} &= \tilde{u}_{1}^{a}\tilde{u}_{1}^{b}\tilde{u}_{1}^{d}. \\
\end{align*}

(109)$$

where $\lambda > 0$, and $c \geq 1$. Strict monotonicity also requires that the quadratic term $A_{1}^{ab}$ has the form $\lambda^2\tilde{g}_{i}^{ab}$, $\lambda \geq 0$, but the weaker condition that the spectrum of $H$ is stable (i.e., bounded below) only requires that $\epsilon_{ac}\epsilon_{bd}A_{1}^{ab}u_{i}^{c}u_{i}^{d} \geq 0$ in classes II, III, IV.

The class I and III quartic terms have a $C_{4}$ symmetry with inverse metric $\tilde{g}_{i}^{ab}$. If $\tilde{A}_{1}^{ab} \propto \tilde{g}_{i}^{ab}$ the full model also has this symmetry, and all Landau orbits have the same metric given by the inverse of $\tilde{g}_{i}^{ab}$. If, in addition, $\tilde{g}_{i}^{ab} = \tilde{g}_{i}^{cd} = \tilde{g}_{i}^{ab}$, so $c = 1$, class I models have a continuous $SO(2)$ symmetry.

We first review the purely quadratic case, where the quartic term is absent, $A_{2}^{abcd} = 0$. In this case

$$H = \frac{1}{2}E_{0}\tilde{g}_{i}^{ab}\gamma_{ab}, \quad E_{0} > 0,$$

(110)$$

where $\tilde{g}_{i}^{ab}$ is the inverse of a unimodular metric $\tilde{g}_{i}^{ab}$ and there is a pseudo-Galilean invariance with $\delta_{ab}$ replaced by $\tilde{g}_{ab}$, and $m_{e}$ replaced by $(E_{0}\tilde{g}_{i}^{ab})^{-1}$. However, the resemblance to the Newtonian model is superficial, as $(E_{0}\tilde{g}_{i}^{ab})^{-1}$ and $\tilde{g}_{ab}$ can be spatially-varying (and their product can have Gaussian curvature) on a completely-geometrically-flat plane with a Cartesian coordinate system. In this non-generic case, the Landau-orbit metric is $\tilde{g}_{ab}$ independent of the Landau index, with all the zeroes of the Landau-orbit coherent state at its origin.

The next-simplest case is the purely-quartic class I model, which also has the non-generic feature of a common metric $\tilde{g}_{ab}$ for all Landau orbits, because of its hidden $C_{4}$ four-fold rotational symmetry. The generic model with the strict-monotonicity property is given by

$$H = \frac{1}{2}E_{0}\{\tilde{g}_{i}^{ab}\gamma_{ab}, \tilde{g}_{i}^{cd}\gamma_{cd}\},$$

(111)$$

where $\tilde{g}_{i}^{ab}$ and $\tilde{g}_{i}^{cd}$ are the inverses of two distinct unimodular metrics $\tilde{g}_{i}^{ab}$ and $\tilde{g}_{i}^{cd}$. In this case

$$\frac{1}{2}(\tilde{g}_{i}^{1} + \tilde{g}_{i}^{2}) = c\tilde{g}_{i}^{ab}, \quad c \geq 1,$$

(112)$$

where $\tilde{g}_{ab}$ is the common unimodular metric of all the Landau orbits. After a $SL(2, R)$ transformation the Hamiltonian is put into the form

$$H = E_{0}\{(\gamma_{11} + \gamma_{22})^{2} + (c - 1)(\gamma_{11}, \gamma_{22})\}$$

(113)
which has an explicit $C_4$ symmetry under $(p_1, p_2) \mapsto (-p_2, p_1)$, or $\gamma_{11} \leftrightarrow \gamma_{22}$. In the harmonic oscillator representation with $[a, a^\dagger] = 1$,

$$\gamma_{11} = \frac{1}{2}(a + a^\dagger)^2, \quad \gamma_{22} = -\frac{1}{2}(a - a^\dagger)^2,$$

and

$$|\psi_n\rangle = f_n(a^\dagger)|0\rangle, \quad a|0\rangle = 0. \quad (115)$$

The $C_4$ symmetry is then the symmetry of $H$ under $a^\dagger \mapsto ia^\dagger$.

Using the complex representation \[58\] the eigenproblem

$$H|\psi\rangle = E|\psi\rangle \quad (116)$$

can be written as a fourth order complex differential equation on the antiholomorphic part of the wavefunction:

$$(\partial_{z^*}^4 + a_2(z^*)\partial_{z^*}^2 + a_1(z^*)\partial_{z^*} + a_0(z^*))f(z^*) = 0,$$

$$a_2 = -(2 + \frac{8}{c-1})z^*^2,$$

$$a_1 = -(4 + \frac{16}{c-1})z^*,$$

$$a_0 = z^*^4 - 1 - \frac{2}{c-1}(1 - \frac{E}{E_0}), \quad (117)$$

whose solutions are entire functions \[13\] (antiholomorphic functions over the whole complex plane), and can be analyzed using Nevalinnna theory (see, e.g. the book Ref\[16\]). The solutions have a property that their orders $\rho(f)$, defined as

$$\rho(f) = \limsup_{r \to \infty} \frac{\ln \ln ||f||_{\infty, B_r}}{\ln r}, \quad (118)$$

$$||f||_{\infty, B_r} = \sup \{|f(z^*)| : z^* \in B_r\}, \quad (119)$$

where $B_r$ is the disk with radius $r$, are less or equal than 2. The functions $1/f(z^*)$ (the inverses of the solutions) are meromorphic functions and a similar notion of order incorporating the singularities can also be defined \[13\]. In that case the fact that the orders are less or equal than 2 basically means the number of roots of $f(z)$ (namely the singularities of $1/f(z^*)$) inside $B_r$ can go at most as $r^2(\ln r)^k$ for some $k$ when $r$ goes to infinity. We remark that this property is not unique to the quartic case, but valid for any finite-order expansion of the Hamiltonian, and presumably is related to the property that $f(z^*) \exp -\frac{1}{2}z^*z$ is a normalizable wavefunction.

We treated the problem numerically by projecting the Hamiltonian into the Hilbert space spanned by states where $f_n$ are polynomials of finite degree $N \gg n$, and solving a tridiagonal Hermitian eigenproblem to obtain the Taylor series for $f_n$, and then used a root finder to obtain approximate eigenstates in the form

$$|\psi_n\rangle \propto \prod_{i=1}^{N} (a^\dagger - z_i^* )|0\rangle. \quad (120)$$

To accurately explore the root structure $\{z_i^* \}$, we carried out the diagonalization using MPACK \[7\], an adaptation of the linear algebra routines from LAPACK to use the GMP library for arbitrary-precision floating-point calculations. These calculations reveal the zeroes of $f_n(z^*)$ in some region $|z| < R_N$, with truncation-dependent features tied to the boundary $|z| \approx R_N$, but with a structure for $|z| \ll R_N$ that becomes independent of $N$ as it, and consequently $R_N$, is increased, as illustrated in Fig.3.

We therefore believe that this calculational method reveals the true structure of roots of the holomorphic non-
FIG. 4. Contour plots of \(\ln|\Psi_{10}|^2\) for class II-IV quartic terms, with \(\{p_{x}^2, p_{y}^2 + p_{y}^4\}\), \(\{p_{x}^2, p_{y}^4\}\), \(p_{x}^4\) as their Hamiltonians respectively. Although classes II, III have closed contours that enclose part of the central zeroes (red pluses), none of the three has a clean separation between all the central zeroes and the rest ones. The contours inside the 2D-distribution regions also demonstrate qualitatively different features from class I: the puddle-like shapes indicate uniformly vanishing amplitudes.

FIG. 5. Restoration of the \(C_{4}\) symmetry as the Landau level index \(n\) increases. Here we adopted a different parametrization of the quartic terms. The Hamiltonian assumes the form
\[
20p_{x}^2 + 20p_{y}^4 + 2p_{x}^4 + 3p_{y}^4 + 4\{p_{x}^2, p_{y}^4\} + \{p_{x}, p_{y}^3\}.
\]

polynomial function \(f_n(z^*)\) in a range \(|z| < R\) that can be increased at will at the expense of increasing the floating-point precision of the numerical diagonalization.

Given that the exact eigenstate satisfies a three-term recurrence relation (or a five-term recurrence relation if quadratic terms are included) that depends on its eigen-
FIG. 6. Contour plots of $\ln |\Psi_{20}|^2$ for the Hamiltonian $p_x^2 + p_y^2 + 2p_x^2 + 3p_y^2 + 4(p_x^2, p_y^2) + \{p_x, p_y\}$. The lower plot is a zoom-in on the central region. The contours have the same piece-wise structures as in the case of pure quartic terms. Also note that the central cross still points to the maxima, and that the spikes to the saddle points.

value, it is possible that, once the eigenvalue has been accurately determined, the Taylor series could be extended beyond the truncation point using the recurrence relation, but we found that the eigenvector of the truncated tridiagonal matrix was accurate enough for finding the pattern of roots, when calculated with high precision.

A typical layout (Class I) of the roots is presented in Fig. 2. The central zeroes, the number of which determines the topological spin, are organized in a cross shape due to the $C_4$ symmetry. There are $n \mod 4$ degenerate zeroes at the origin, and the degeneracy will be lifted once we add quadratic terms to the Hamiltonian. Another noticeable feature is the four “spikes” that point to the origin and alternate with the central cross. The regions with quasi-uniform 2D distributions are bounded by line “charges” with constant density. If the patterns can be extrapolated to infinity, the total number of roots inside $B_r$ will go as $r^2$, in agreement with the mathematical bound mentioned above. The precise locations of the zeroes inside those 2D regions are sensitive to perturbation of the Hamiltonian, but we believe the very existence of such “dark” areas is of universal significance.

The structure of the roots strongly suggests that there is a semiclassical treatment in which $\ln f_n(z)$ is approximated by a piecewise-holomorphic function in regions separated by branch cuts along the apparent lines of zeroes, with a modification of the background 2D charge-density distribution of the Poisson equation in the regions where the roots seem to be a quasi-uniform 2D distribution. To visualize that, we show in Fig. 3 the contours of the 2D Coulomb potential $V(z, z^*)$, which is $\ln |f_n(z)|^2$ plus a quadratic potential derived from the Gaussian factor. Note that the actual topography has a “volcano crater” at the center enclosed by an annulus-like region containing the local maxima and saddle points. The fact that all the central zeroes are inside the crater and separated from the rest of the structure is critical for a semiclassical definition of the topological spin. We will see shortly that the same statement doesn’t apply to classes II, III, or IV. Additionally, for class I there is a free parameter $c - 1$ that controls the shapes of contours of constant $\varepsilon(p)$ in momentum space. A simple calculation reveals that $c - 1 > 2$ corresponds to contours with concave shapes, while $c - 1 < 2$ to convex ones.

Now we move on to the other three classes of quartic terms. As can be seen from the definitions, their qualitative difference from class I is that the classical dispersion
\( \varepsilon(p) \) is consistently zero along one (for classes II, IV) or two (for class III) radial directions on the momentum plane. As a result, the classical orbits for these three classes are not closed. Though the noncommutativity of \( p_a \) gives rise to effective quadratic terms for classes II and III, thus closing the quantum orbits, we expect that a clean separation between the central zeroes and the outer structure by a set of contours cannot be achieved for any of the three classes, which is verified in Fig. [3]

To complete the discussion, we add quadratic terms to the Hamiltonian. The parametrization now is

\[
H = Ap_x^2 + Bp_y^2 + Cp_z^4 + Dp_y^4 + E\{p_x^2, p_y^2\} + F\{p_x, p_y^3\} + G\{p_z^3, p_y\}. \tag{121}
\]

The \( C_4 \) symmetry is generically not present. However, as we go to high-energy states, the quartic terms will eventually dominate. Therefore we expect to see a quasi-\( C_4 \) symmetry gradually appear as the Landau index \( n \) increases, as shown in Fig. [4]. Due to this asymptotic \( C_4 \) symmetry, the generic features of the root structure and the contours still hold, as illustrated in Fig. [6].

We conclude this section by revisiting the semiclassical interpretation of the root structure and demonstrating that the region we pick to isolate the central zeroes can indeed encompass as much as 90 percent of the total probability while covering no zeroes.

**VIII. OTHER PROPERTIES OF THE LANDAU ORBIT**

There are some other important characteristic physical properties of the Landau orbit. The simplest is its diamagnetic magnetic moment normal to the plane, given by

\[
\mu_n = \left. \frac{\partial E_n}{\partial B} \right|_{\varepsilon(p)} \tag{122}
\]

which in the Newtonian model with Galilean invariance reduces to \( \mu_n = \hbar c B s_n / m_e \).

The second property is the effective mass that characterizes the kinetic energy of a guiding center that flows along a line of constant \( E_n(x) \), if there is slow adiabatic spatial variation of the Hamiltonian. This is given by the linear response to a perturbation that breaks the inversion symmetry of the Landau orbit around its guiding center. The perturbed Landau levels are given by

\[
(H - v^a p_a) |\Psi_{n,a}(v)\rangle = E_n(v) |\Psi_{n,a}(v)\rangle, \tag{123}
\]

where \( v \) is a parameter of the perturbed Hamiltonian. Then

\[
\langle \Psi_{n\alpha}(v) | H | \Psi_{n\alpha}(v) \rangle = E_n + \frac{1}{2} m_{ab} v^a v^b + O(v^4), \tag{124}
\]

and

\[
E_n(v) = E_n - \frac{1}{2} m_{ab} v^a v^b + O(v^4), \tag{125}
\]

For the Newtonian model, \( n_{ab}^\alpha = m_n \delta_{ab} \). Though both \( m_{ab}^a \) and \( g_{ab}^a \) are proportional to the Euclidean metric in the Newtonian model, there is in general no proportionality between them unless there is a discrete three-, four- or six-fold crystal rotational symmetry of the lattice plane on which the electrons move (and no tangential component of the magnetic flux), in which case they both remain proportional to the Euclidean metric which is invariant under the crystal symmetry.

The generic lack of any relation between the viscosity tensor and the effective mass tensor can be traced to the fact that the viscosity derives from perturbations that mix Landau levels with the same parity, while the effective mass derives from mixing of Landau levels with opposite parity.

Since \( p_a = \hbar c (x^a - R^a) / \ell_B^2 \), the perturbation can be written as

\[
H(v) = H - \epsilon_{ab} \hbar c v^a (x^b - R^b) / \ell_B^2. \tag{126}
\]

The effective mass tensor can therefore be interpreted as the part of the electric susceptibility associated with polarization of the Landau orbit (giving it an electric dipole moment defined as the displacement of its center of charge relative to the guiding center), with \( \epsilon^a = \epsilon_{ab} E_b / B \) (so \( \epsilon_a \) are the tangential components of the electric field in the comoving frame). The Landau-orbit polarizability contribution to the electric susceptibility per unit area is

\[
\chi_E^{ab}(x) = e^{ac} \epsilon_{bd} \frac{\epsilon_{ab}^2 \ell_B^2}{2 \pi \hbar^2} \sum_n \nu_n m_{cd}^n(x). \tag{127}
\]

**IX. DENSITY AND CURRENT RESPONSE**

With the properties of Landau levels introduced in previous sections, we now present a generic version of the electromagnetic responses of the particle density and current in the absence of rotational symmetry. These results agree with those found earlier\(^\text{[14]}\) to order \( q^2 \), but which have remained unpublished. However, we felt it was useful to provide (in Appendix \[\text{[A]}\]) a fuller and more explicit derivation than that available in Ref\[\text{[11]}\] using a somewhat different formalism to obtain the same results, and extending Ref\[\text{[17]}\] to give the formal expansion of the current-density operator of a non-Galilean system to all orders in \( q \).
Denote the particle density and current in the nth Landau level by $J_n^s$ and $J_n^b$. Our main results are

$$eJ_n^s = \frac{e^2}{2\pi\hbar} (1 + \sigma^{ab}\partial_a\partial_b)E_b - \chi_m\sigma^{ab}\partial_bB, \quad (128)$$

$$eJ_n^b = \frac{e^2B}{2\pi\hbar} (1 + \sigma^{ab}\partial_a\ln B) - \chi_E\sigma^{ab}\partial_aE_b, \quad (129)$$

where

$$\sigma^{ab} = \frac{1}{6}s_n g^{ab} - \left[ \frac{1}{6}(\tilde{R}_a, \{\tilde{R}^b, dh\})_n + a \leftrightarrow b \right]$$

$$+ \frac{1}{4}\langle(\tilde{R}_a, \tilde{R}^b), dh\rangle_n, \quad (130)$$

$$s_n g^{ab} = \frac{1}{2}\langle(\tilde{R}_a, \tilde{R}^b)\rangle_n, \quad (131)$$

$$\chi_m = \frac{e^2\tilde{v}_g^2}{2\pi\hbar^2} (\langle dh + d^2h\rangle_n + \langle dh, dh\rangle_n). \quad (132)$$

The notations used here are shown as follows:

$$(A, B)_n \equiv \sum_{n' \neq n} \langle n|A|n'\rangle\langle n'|B|n\rangle + \langle n|B|n\rangle\langle n'|A|n\rangle, \quad (133)$$

$$(A) \equiv \langle n|A|n\rangle, \quad (134)$$

where $|n\rangle$ and $E_n$ are the nth eigenstate and eigenvalue of the unperturbed single-particle Hamiltonian $h$. The scale derivative is defined as

$$dh = B_0\frac{\partial h}{\partial B_0}, \quad d^2h = B_0\frac{\partial(dh)}{\partial B_0}. \quad (135)$$

The first term in $\sigma^{ab}$ is proportional to the inverse Hall viscosity tensor and therefore universal. The second (third) term is due to Landau-level mixing between states with opposite (the same) parity, thus depending on the details of the Hamiltonian. When there is rotational symmetry, the third term vanishes because the eigenstates are invariant with respect to rescaling of $B_0$. When Galilean symmetry is present with a universal (inverse) metric $g^{ab}$, the second term can be explicitly calculated,

$$-\frac{1}{6}(\tilde{R}_a, \{\tilde{R}^b, dh\})_n + a \leftrightarrow b = \frac{4}{3}s_n g^{ab}, \quad (136)$$

which combined with the first term gives

$$\sigma^{ab} = \frac{3}{2}s_n g^{ab}. \quad (137)$$

In the Euclidean case where $g^{ab} = 1/\tilde{B}^2\delta^{ab}$, it agrees with existing results in the literature.\cite{5,11}

\section{Conclusion}

We have studied the geometry of Landau orbits in the absence of Galilean and rotational symmetries. With the help of consistent index placement, we reformulated the Hall viscosity and used it as a natural choice for the unimodular metric $g^{ab}$ that describes the shape of each Landau orbit. We thus demonstrated that the metric in the IQHE is derived from the intrinsic dynamics of the system rather than induced from the 3D space the system is embedded in. By investigating the root structure of the holomorphic part of the single-particle wavefunction, we defined a topological spin $s_n = n + \frac{1}{2}$ where $n$ is the number of central zeroes enclosed by a zero-free region that can be thought of as a “fattened” semiclassical orbit. Compared to the “intrinsic orbital angular momentum” interpretation, this definition is more generic and captures the topological nature of this quantity. We also introduced a mass tensor $m^{ab}_n$ associated with the kinetic energy of the guiding centers that only coincides with $g^{ab}_n$ when there is a continuous rotational symmetry.

With all the ingredients above, we presented the generic (without rotational symmetry) results for particle density and current responses. The results clearly showed that the Hall conductivity at finite wavevector consists of a universal term proportional to the inverse Hall viscosity tensor and terms related to Landau-level mixing. We believe that a generic version of the effective action should also possess these features which are usually hidden by the simplification of applying special symmetries.

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Appendix A: Density and current reponses in the absence of rotational symmetry

Let the two components of the dynamical momentum $p = p^\dagger$ have the commutation relation

$$[p_a, p_b] = i\hbar^2\epsilon_{ab}, \quad (A1)$$

where the antisymmetric Levi-Civita symbol $\epsilon_{ab} = -\epsilon_{ba}$ has det $\epsilon = 1$. With the Schrödinger representation

$$p_a = -i\hbar\partial_a - eA_a(x), \quad \partial_a \equiv \frac{\partial}{\partial x^a}, \quad (A2)$$

this represents the momentum of a charge-$e$ particle moving in a uniform background magnetic flux density

$$B_0 = \partial_1 A_2(x) - \partial_2 A_1(x) = \text{Pf}((\hbar/e)\ell_B^2), \quad (A3)$$

where the Pfaffian $\text{Pf}(\epsilon) = \epsilon_{12}$ of the Levi-Civita symbol is odd under time-reversal. A metric-independent formalism that consistently distinguishes between covariant (lower) and contravariant (upper) 2D spatial indices will be used here. The dual symbol $\epsilon^{ab}$ is defined by

$$\epsilon^{ab}\epsilon_{cd} = \delta^a_d \delta^b_c - \delta^a_c \delta^b_d, \quad (A4)$$

where $\delta^a_c$ is the Kronecker symbol. In a metric-independent formalism, there is no symbol “$\delta_{ab}$”, which if encountered, would represent the Euclidean metric in a Cartesian coordinate system.

The algebra [A11], in which the commutator is a c-number, is the Heisenberg algebra $\mathfrak{h}_2$. Its universal enveloping algebra $U(\mathfrak{h}_2)$ is spanned by the basis

$$\left\{ 1, p_a, \{p_a, p_b\}, \{p_a, p_b, p_c\}, \ldots \right\} \quad (A5)$$

where $\{A, B, \ldots, C\}$ is the symmetrized product of $n$ operators normalized so that $\{\lambda A, \lambda A, \ldots, \lambda A\} = n!/(\lambda A)^n$. The kinetic energy $h$ can then be given in the form

$$h = H(p) \equiv \frac{1}{\hbar^2} H_n^{a_1, a_2, \ldots, a_n} p_{a_1} p_{a_2} \cdots p_{a_n}, \quad (A6)$$

where the coefficients $H_n^{a_1, a_2, \ldots, a_n}$ are fully symmetric in all indices.

It is useful to define

$$\tilde{R}^a = (\tilde{R}^a)^\dagger = -\hbar^{-1}\ell_B^2\epsilon^{ab} p_b. \quad (A7)$$

Then

$$[\tilde{R}^a, p_b] = i\hbar\delta^a_b, \quad [\tilde{R}^a, \tilde{R}^b] = i\hbar^2\epsilon^{ab}. \quad (A8)$$

The unitary boost operator, parametrized by a real c-number vector $q$, is given by

$$U(q) = U(-q)^\dagger = e^{i q \cdot \tilde{R}}, \quad [q_a, q_b] = 0. \quad (A10)$$

Note that the scalar product $q \cdot \tilde{R} \equiv q_a \tilde{R}^a$ can only be formed between a covariant and a contravariant vector. Then

$$U(-q)pU(q) = p + \hbar q. \quad (A11)$$

and the derivatives $h^{a_1, a_2, \ldots, a_n}$ of $h$ are defined by

$$h(q) \equiv U(-q)hU(q) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} h^{a_1, a_2, \ldots, a_n} q_{a_1} q_{a_2} \cdots q_{a_n}, \quad (A12)$$

where the tensor-valued operators $h^{a_1, a_2, \ldots, a_n}$ are fully symmetric in their indices. In particular, the first derivative $h^a$ is the velocity operator $v^a$. Note also that

$$U(-q)U(-q')pU(q')U(q) = U(-(q + q'))pU(q + q'). \quad (A13)$$
From this
\[ U(-q)h^{a_1,\ldots,a_m}U(q) = \sum_{n=0}^{\infty} \frac{\hbar^n}{n!} h^{a_1,\ldots,a_mb_1,\ldots,b_n} q_{b_1} \cdots q_{b_n}. \] (A14)

as in the differentiation of c-number functions, where \((f')' = f''\).

Now let \(a \in U(\mathfrak{h})\) be a vector-valued operator and \(\lambda\) be a c-number. Then define the expansion
\[ H(p + \lambda a) = h + \lambda h_1(a) + O(\lambda^2). \] (A15)

The quantity \(h_1(a)\) can be expressed in two equivalent ways as the nested-commutator expansions
\[ h_1(a) = a_{a_1} h^{a_1} + \frac{1}{2!} [p_{a_1}, a_{a_2}] h^{a_1a_2} + \frac{1}{3!} [p_{a_1}, [p_{a_2}, a_{a_3}]] h^{a_1a_2a_3} + \ldots \] (A16)
\[ = h^{a_1} a_{a_1} + \frac{1}{2!} h^{a_1a_2} [a_{a_1}, p_{a_2}] + \frac{1}{3!} h^{a_1a_2a_3} [[a_{a_1}, p_{a_2}], p_{a_3}] + \ldots \] (A17)

The operator \(h\) is also a function of the scale factor \(B_0\), and more properly should be written as \(h(B_0)\). The magnetization operator \(m(p)\) is defined by
\[ m \equiv \left. \frac{\partial h}{\partial B_0} \right|_{(H_n)} = -\frac{1}{2} B_0^{-1} h_1(p) \]
\[ = -\frac{1}{2} B_0^{-1} v \cdot p = -\frac{1}{2} B_0^{-1} p \cdot v = \frac{1}{2} B_0^{-1} \left( h + i\ell_B^{-2} \epsilon_{ab} \tilde{R}^a h \tilde{R}^b \right). \] (A18)

If the kinetic energy is a quadratic operator, \(h = \frac{1}{2} H^a_{ab} p_a p_b\), then \(B_0 m = -h\). For any scalar operator \(h \in U(\mathfrak{h}_2)\), one can define the scale derivative
\[ dh \equiv B_0 \frac{\partial h}{\partial B_0} = \frac{1}{2} \left( -i\ell_B^{-2} \epsilon_{ab} \tilde{R}^a h \tilde{R}^b - h \right), \] (A19)
so
\[ dh = -B_0 m. \] (A20)

Higher orders of derivatives can be recursively defined as
\[ d^{n+1} h \equiv B_0 \frac{\partial (d^n h)}{\partial B_0}, \quad d^3 h \equiv dh. \] (A21)

Note that if \(Q = q \cdot \tilde{R} \in \mathfrak{h}\), then
\[ Q h \tilde{R}^a - \tilde{R}^a h Q = [Q, R^a] (h + 2dh) \] (A22)
where \([Q, R^a]\) is a c-number.

Now enlarge the algebra to add a second independent Heisenberg algebra, that of the guiding centers:
\[ [R^a, \tilde{R}^b] = -i\epsilon^{ab} \ell_B^2, \quad [R^a, \tilde{R}^b] = 0. \] (A23)

then
\[ r = R + \tilde{R} \] (A24)
is the classical coordinate of the charged particle:
\[ [r^a, r^b] = 0, \quad [r^a, p_b] = i\hbar \delta^a_b. \] (A25)

Note that
\[ \text{Tr}(e^{iqR}) = 2\pi \delta^2(q\ell_B) \] (A26)
where the trace is over the Hilbert space of the guiding-center, and \(\delta^2(x)\) is the 2D Dirac delta-function.
Now introduce an additional vector potential
\[ A_a(x) = \int \frac{d^2q}{(2\pi)^2} \hat{A}_a(q)e^{iq\cdot x}, \] (A27)
with
\[ \delta B(x) = \partial_1 A_2(x) - \partial_2 A_1(x). \] (A28)
Now define the operator
\[ A_a = A_a(r) = \int \frac{d^2q}{(2\pi)^2} \tilde{A}_a(q)e^{iq\cdot RU(q)}. \] (A29)
The problem of the charged particle moving in the 2D plane \( x \) in the presence of a non-uniform magnetic flux density \( B_0 + \delta B(x) \) is then described by the Hamiltonian
\[ H = H(p - eA) \] (A30)
The electric charge density operator is given by
\[ J^0(x) = e\int \frac{d^2q}{(2\pi)^2} e^{iq(R-x)}U(q), \] (A31)
and the electric current-density operator is given by
\[ J^a(x) = e\int \frac{d^2q}{(2\pi)^2} e^{iq(R-x)}J^a(q), \] (A32)
where, to leading order in \( \tilde{A} \),
\[ e_a J^a(q) = h_1(eU(q)) + O(\tilde{A}), \] (A33)
where \( e \) is a c-number unit vector.

The continuity equation (gauge invariance) requires that
\[ (i\hbar)^{-1}[J^0(x), H] + \partial_a J^a(x) = 0. \] (A34)
This must be satisfied at each order of an expansion in \( \tilde{A}(q) \). In the limit of uniform magnetic flux density \( B_0 \), the continuity relation is
\[ [h, U(q)] = hq_a J^a(q), \] (A35)
and the functional form of the expansion of \( J^a(q) \) must be
\[ hJ^a(q) = -i[\alpha^a(q), h] + i\epsilon_{ba} q_b \ell_B^2 \beta(q), \] (A36)
where
\[ iq_a \alpha^a(q) = U(q) - 1 \] (A37)
or
\[ \alpha^a(q) = -i \frac{\partial}{\partial q_a} \left( \sum_{n=1}^{\infty} \frac{1}{n!} \frac{1}{n!} \sum_{k=1}^{n} (iQ)^{k-1} \tilde{R}^a(iQ)^{n-k} \right). \] (A38)
Using (A16), the explicit form of \( J^a(q) \) can be written as
\[ i\hbar J^a = U(q) \left( [\tilde{R}^a, h] + \frac{1}{2!} [-iQ, [\tilde{R}^a, h]] + \frac{1}{3!} [-iQ, [-iQ, [\tilde{R}^a, h]]] + \cdots \right), \] (A39)
where \( Q = q \cdot \tilde{R} \) as mentioned above. Expanding \( U(q) \) and the nested commutators gives

\[
i\hbar J^a = \sum_{m=0}^{\infty} \frac{(iQ)^m}{m!} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)!} \sum_{k=0}^{n} (-1)^k \binom{n}{k} (iQ)^{n-k} [\tilde{R}^a, h](iQ)^k.
\] (A40)

The coefficient \( C_{l,k} \) for the term

\[
(iQ)^l [\tilde{R}^a, h](iQ)^k
\]

in the expansion is (using \( l + k = m + n \))

\[
C_{l,k} = \frac{1}{m!} \sum_{n=0}^{\infty} (-1)^n \frac{1}{(n+1)!} \binom{n}{k} (iQ)^l (iQ)^k
\]

\[
= \frac{1}{k!} \sum_{m=0}^{\infty} (-1)^m \frac{1}{k + m + 1} \binom{l}{m}
\]

\[
= \frac{1}{(k+l+1)!}.
\] (A41)

Now a significantly simplified form for \( J^a(q) \) can be written down

\[
i\hbar J^a = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \sum_{k=0}^{n} (iQ)^{n-k} [\tilde{R}^a, h](iQ)^k,
\] (A42)

which is further transformed into the desired form:

\[
i\hbar J^a = \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( (iQ)^n \tilde{R}^a h - h\tilde{R}^a (iQ)^n - i \sum_{k=0}^{n-1} (iQ)^{n-1-k} (Qh\tilde{R}^a - \tilde{R}^a h)(iQ)^k \right)
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \frac{1}{2} \{ (iQ)^n, \tilde{R}^a \} h + \frac{1}{2} [ (iQ)^n, \tilde{R}^a ] h - \frac{1}{2} h \{ (iQ)^n, \tilde{R}^a \} + \frac{1}{2} h [ (iQ)^n, \tilde{R}^a ] \right)
\]

\[
- iQ, \tilde{R}^a \right) \sum_{k=0}^{n-1} (iQ)^{n-k} (h + 2dh)(iQ)^k
\]

\[
= \sum_{n=0}^{\infty} \frac{1}{(n+1)!} \left( \left( \frac{1}{2} (iQ)^n, \tilde{R}^a \right) h - iQ, \tilde{R}^a \right) \sum_{k=0}^{n-1} (iQ)^{n-k} (2dh)(iQ)^k
\]

\[
+ iQ, \tilde{R}^a \right) \left( \frac{n}{2} (iQ)^n h + \frac{n}{2} h(iQ)^n - \sum_{k=0}^{n-1} (iQ)^{n-k} h(iQ)^k \right)
\]

\[
= [\alpha^a(q), h] + iQ, \tilde{R}^a \beta(q)
\] (A43)

where

\[
\alpha^a(q) = \sum_{n=0}^{\infty} \frac{1}{2(n+1)!} \{ (iQ)^n, \tilde{R}^a \},
\] (A44)

\[
\beta(q) = - \sum_{n=0}^{\infty} \frac{2}{(n+1)!} \sum_{k=0}^{n-1} (iQ)^{n-k} (dh)(iQ)^k + \sum_{n=0}^{\infty} \frac{1}{2(n+1)!} \sum_{k=0}^{n-1} [(iQ)^{n-k}, [(iQ)^k, h]]
\] (A45)

The expansion for \( \alpha^a(q) \) agrees with the universal form \( A38 \), and the leading term in \( \beta(q) \), \( -dh = B_0 m \), is just the expected magnetization term in the total current density

\[
J^0 = J^0_{\text{free}} - \partial_a P^a, \quad J^a = J^a_{\text{free}} + e^{ab} \partial_b M + \partial_a P^a.
\] (A46)

We are now in a position to give the perturbation result for a non-uniform magnetic flux density and a non-zero electric field. The perturbed Hamiltonian is

\[
H = h - e \int \frac{d^2q}{(2\pi)^2} \tilde{A}_0(q) U(q) e^{iq \cdot R} - e \int \frac{d^2q}{(2\pi)^2} \tilde{A}_0(q) J^a(q) e^{iq \cdot R}
\]

\[
- e^2 \int \frac{d^2q}{(2\pi)^2} \tilde{A}_a(-q) \Gamma^{ab}(q) \tilde{A}_b(q) e^{iq \cdot R} + O(A^3),
\] (A47)
\[
\Gamma^{ab}(q) = \frac{1}{\hbar^2} \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \left[ iQ, \cdots [iQ, [\tilde{R}^a, [\tilde{R}, [\tilde{R}^b]]]] \cdots \right] + \frac{1}{\hbar^2} \sum_{n=0}^{\infty} \frac{1}{(n+2)!} \left[ -iQ, \cdots [-iQ, [\tilde{R}^a, [\tilde{R}, [\tilde{R}^b]]]] \cdots \right] \tag{A48}
\]
\[
= \frac{1}{\hbar^2} \sum_{k=0}^{\infty} \frac{2}{(2k+2)!} \left[ iQ, \cdots [iQ, [\tilde{R}^a, [\tilde{R}, [\tilde{R}^b]]]] \cdots \right]. \tag{A49}
\]

The perturbed particle density and current density of a filled Landau level is
\[
J_0^a(x) = \frac{1}{2\pi \ell_B^2} \left( 1 - e^\frac{d^2 q}{(2\pi)^2} \chi^0_n(q) \hat{U}(q)e^{iq \cdot x} - e \int \frac{d^2 q}{(2\pi)^2} \chi^0_n(q) \hat{A}_q(q)e^{iq \cdot x} \right), \tag{A50}
\]
\[
J^a_n(x) = -\frac{1}{2\pi \ell_B^2} \left( e \int \frac{d^2 q}{(2\pi)^2} \chi^0_n(-q) \hat{A}_0(q)e^{iq \cdot x} + e \int \frac{d^2 q}{(2\pi)^2} \chi^{ab}_n(q) \hat{A}_b(q)e^{iq \cdot x} \right), \tag{A51}
\]
where
\[
\chi^0_n(q) = \sum_{n'(\neq n)} \frac{\langle n|U(-q)|n'\rangle \langle n'|U^a(q)|n \rangle + \langle n|U(q)|n'\rangle \langle n'|U(-q)|n \rangle}{E_n - E_{n'}} \tag{A52}
\]
\[
\chi^0_n(q) = \sum_{n'(\neq n)} \frac{\langle n|U(-q)|n'\rangle \langle n'|J^a(q)|n \rangle + \langle n|J^a(q)|n'\rangle \langle n'|U(-q)|n \rangle}{E_n - E_{n'}} \tag{A53}
\]
\[
\chi^{ab}_n(q) = \chi^{ab}_n(q) + \langle n|\Gamma^{ab}(q)|n \rangle, \tag{A54}
\]
\[
\chi^{ab}_n(q) = \sum_{n'(\neq n)} \frac{\langle n|J^a(-q)|n'\rangle \langle n'|J^b(q)|n \rangle + \langle n|J^b(q)|n'\rangle \langle n'|J^a(-q)|n \rangle}{E_n - E_{n'}}. \tag{A55}
\]

Below we will simplify our notation by defining
\[
(A, B)_n = \sum_{n' \neq n} \frac{\langle n|A|n'\rangle \langle n'|B|n \rangle + \langle n|B|n'\rangle \langle n'|A|n \rangle}{E_n - E_{n'}}, \tag{A56}
\]
so
\[
\chi^0_n(q) = (U(-q), U(q))_n, \tag{A57}
\]
\[
\chi^{0a}_n(q) = (U(-q), J^a(q))_n, \tag{A58}
\]
\[
\chi^{ab}_n(q) = (J^a(-q), J^b(q))_n. \tag{A59}
\]

We will also denote the expectation value of some operator \(\mathcal{O}\) in the \(n\)th Landau level by \(\langle \mathcal{O} \rangle_n\).

Using the expansions of \(U(q)\) and \(J^a(q)\), these correlation functions are readily obtained:
\[
\chi^0_n(q) = -(Q, Q)_n + O(|q|^4) \tag{A60}
\]
\[
\chi^{0a}_n(q) = -\frac{1}{\hbar} [Q, \tilde{R}^a] \left( 1 - \frac{1}{6} (Q^2)_n + \frac{1}{3} (Q, \{Q, dh\})_n - \frac{1}{2} (Q^2, dh)_n \right) + O(|q|^5), \tag{A61}
\]
\[
\chi^{ab}_n(q) = -\frac{1}{\hbar^2} [Q, \tilde{R}^a][Q, \tilde{R}^b] \left( (dh + d^2h)_n + (dh, dh)_n \right) + O(|q|^4) \tag{A62}
\]

If we define
\[
\chi^a_E = -\frac{e^2}{2\pi \ell_B^2} (\tilde{R}^a, \tilde{R}^b)_n, \tag{A63}
\]
\[
\sigma^{ab} = -\frac{1}{6} \delta_n \delta_n - \frac{1}{6} (\tilde{R}^a, \{\tilde{R}^b, dh\})_n + a \leftrightarrow b \tag{A64}
\]
\[
s_n \delta_n = \frac{1}{2} (\{\tilde{R}^a, \tilde{R}^b\})_n, \tag{A65}
\]
\[
\chi_m = \frac{e^2 \ell_B^2}{2\pi \hbar^2} \left( (dh + d^2h)_n + (dh, dh)_n \right), \tag{A66}
\]
then

\[ \chi_{n}^{00}(q) = -\frac{2\pi l_B^2}{e^2} q_a q_b \chi_{E}^{ab} + \mathcal{O}(|q|^4), \quad (A67) \]

\[ \chi_{n}^{0a}(q) = -\frac{1}{\hbar} [Q, \tilde{R}^a] (1 - \sigma^{ab} q_a q_b) + \mathcal{O}(|q|^5), \quad (A68) \]

\[ \tilde{\chi}_{n}^{ab}(q) = -\frac{2\pi}{e^2 l_B^2} [Q, \tilde{R}^a][Q, \tilde{R}^b] \chi_m + \mathcal{O}(|q|^4). \quad (A69) \]

Note that \( \chi_{E}^{ab} \) and \( g^{ab} \) are just the electric susceptibility tensor and the inverse of the Landau-orbit metric (up to normalization) defined in the main body of the paper. \( \sigma^{ab} \) is the tensor for the quadratic term in the \( q \) expansion of the Hall conductivity and \( \chi_m \) is the magnetic Hall conductivity.

Now using

\[ [Q, \tilde{R}^a] = il_B^2 e^{ba} q_b, \quad (A70) \]

\[ E(q) = iq \hat{A}_0(q), \quad (A71) \]

\[ \delta B(q) = i\epsilon^{ab} q_a \hat{A}_b(q), \quad (A72) \]

we are able to write down the generic responses (in the absence of rotational symmetry) of charge density and current density against a non-vanishing electric field and a non-uniform magnetic field:

\[ e J_n^a = \frac{e^2}{2\pi \hbar} (1 + \sigma^{cd} \partial_c \partial_d) e^{ab} E_b - \chi_m e^{ab} \partial_b B, \quad (A73) \]

\[ e J_n^0 = \frac{e^2 B}{2\pi \hbar} (1 + \sigma^{ab} \partial_a \partial_b \ln B) - \chi_{E}^{ab} \partial_a E_b. \quad (A74) \]

The Hall conductivity at finite wavevector \( q \) is thus

\[ \sigma_{H}^{ab} = \frac{e^2}{2\pi \hbar} (1 - \sigma^{cd} q_c q_d) e^{ab}, \quad (A75) \]

and we present here again the generic formula for \( \sigma^{ab} \):\n
\[ \sigma^{ab} = \frac{1}{6} s n g^{ab} - \left[ \frac{1}{6} (\tilde{R}^a, \{ \tilde{R}^b, dh \})_n + a \leftrightarrow b \right] + \frac{1}{4} (\{ \tilde{R}^a, \tilde{R}^b \}, dh)_n. \quad (A76) \]

The first term in \( \sigma^{ab} \) is proportional to the inverse of the Hall viscosity tensor \( \eta_{H}^{ab} \) and hence universal. The second and the third terms are due to inter-Landau-level mixing between states with opposite and the same parity respectively, so they depend on the details of the Hamiltonian. When there is rotational symmetry, the third term vanishes because the eigenstates are invariant with respect to rescaling of \( B_0 \). When Galilean symmetry is present, the second term can be explicitly calculated to be

\[ -\left[ \frac{1}{6} (\tilde{R}^a, \{ \tilde{R}^b, dh \})_n + a \leftrightarrow b \right] = \frac{4}{3} s n g^{ab}, \quad (A77) \]

which combined with the first term gives

\[ \sigma^{ab} = \frac{3}{2} s n g^{ab}. \quad (A78) \]

The case where \( g^{ab} = l_B^2 \delta^{ab} \) agrees with existing results in the literature.