Note on (conformally) semi-symmetric spacetimes

Ingemar Eriksson and José M M Senovilla

1 Google Switzerland GmbH, Brandschenkestrasse 110, CH-8002 Zürich, Switzerland
2 Física Teórica, Universidad del País Vasco, Apartado 644, 48080 Bilbao, Spain

E-mail: ineri3@gmail.com and josemm.senovilla@ehu.es

Received 22 August 2009, in final form 17 November 2009
Published 23 December 2009
Online at stacks.iop.org/CQG/27/027001

Abstract

We provide a simple proof that conformally semi-symmetric spacetimes are actually semi-symmetric. We also present a complete refined classification of the semi-symmetric spacetimes.

PACS numbers: 04.20.Cv, 02.40.Ky, 04.20.Jb

1. Introduction

Semi-symmetric spaces were introduced by Cartan [1]. They are characterized by the curvature condition

$$\nabla_{[a} \nabla_{b]} R_{cdef} = 0,$$

where $R_{cdef}$ denotes the Riemann tensor and round and square brackets enclosing indices indicate symmetrization and antisymmetrization, respectively. Geometrically, they satisfy the property that the sectional curvature relative to any tangent plane at any point remains invariant after parallel translation along an infinitesimal parallelogram [6, 8].

Semi-symmetric proper Riemannian manifolds were studied in [15, 16], and they are considered the natural generalization of locally symmetric spaces, i.e. those satisfying $\nabla_b R_{cdef} = 0$. This is not the case for general semi-Riemannian manifolds, where the intermediate condition

$$\nabla_a \nabla_b R_{cdef} = 0$$

is actually feasible [12, 13]. Semi-Riemannian manifolds satisfying (2) are called second-order symmetric, and are obviously semi-symmetric, but the converse is not true in general. Semi-symmetric spacetimes (four-dimensional Lorentzian manifolds) have been considered in [6] as a particular case of the more general pseudo-symmetric case [3].

A semi-Riemannian manifold is said to be conformally semi-symmetric if the Weyl tensor $C_{abcd}$ satisfies

$$\nabla_{[a} \nabla_{b]} C_{cdef} = 0,$$
and Ricci semi-symmetric if the Ricci tensor $R_{ab} \equiv R^c_{acb}$ satisfies
\[ \nabla_a \nabla_b R_{cd} = 0 \] (4)
see, e.g. [13] (section 2.4) and references therein. It is obvious that semi-symmetric spaces are automatically both conformally and Ricci semi-symmetric, but none of these two by itself should imply the former in general. Surprisingly, for dimensions greater than 4, conformal semi-symmetry implies semi-symmetry (for non-conformally flat spaces), see [2, 5].

However, in the four-dimensional proper Riemannian case there are examples of conformally semi-symmetric spaces which are not semi-symmetric (see lemma 1.1 in [4]). Thus, the question arises of what happens for other signatures in four dimensions. In this short note we want to consider the case of Lorentzian signature, providing a very simple proof of the non-trivial result that conformally semi-symmetric spacetimes (with non-zero Weyl tensor) are automatically Ricci semi-symmetric. Therefore, semi-symmetry and conformal semi-symmetry are essentially equivalent in this case. This equivalence is implicit in [7] (see theorem 95), but we would like to present herein a very simple and completely direct proof of the result.

2. Main results

Even though the calculations can be carried out in tensor formalism, it is much more efficient to resort to spinorial techniques. Thus, we will use the standard nomenclature and conventions in [10, 11], except for the scalar curvature $R \equiv R^c_{cc}$, as we will not use the usual $\Lambda \equiv R/24$ notation.

**Lemma 1.** Conformally semi-symmetric spacetimes are of Petrov type $N$, with
\[ \Psi_{ABCD} = \Psi_{A0B0C0D}, \quad \Phi_{ABA'B'} = \Phi_{2200A0A0B'}, \quad R = 0, \] (5)
or Petrov type $D$, with
\[ \Psi_{ABCD} = 6\Psi_{A0B0C0D}, \quad \Phi_{ABA'B'} = 4\Phi_{1101A0(AA'B'B'), \quad R = -12\Psi_2. \] (6)

**Proof.** In spinors (3) is equivalent to $\Box_{AB} \Psi_{CDEF} = 0$ and $\Box_{A'B'} \Psi_{CDEF} = 0$, or
\[ X_{ABC} G \Psi_{DEFG} = 0, \] (7)
\[ \Phi_{A'B'C} G \Psi_{DEFG} = 0. \] (8)
The first condition, (7), can be rewritten as
\[ 24 \Psi_{ABC} G \Psi_{DEFG} = -R(\epsilon_{ACC} \Psi_{DEFB} + \epsilon_{BCC} \Psi_{DEFB}). \] (9)
Contracting this over $BC$ yields
\[ 12 \Psi_{(AD\beta BG} \Psi_{EF)BG} = R \Psi_{ADEF}. \] (10)
This can only be satisfied for Petrov types $D$ and $N$.\(^3\) For type $N$, $\Psi_{ABCD} = \Psi_{A0B0C0D}$ and $R = 0$, while for type $D$ we have $\Psi_{ABCD} = 6\Psi_{A0B0C0D}$ and $R = -12\Psi_2$ (see [11], p 261). Condition (7) can then be verified to be satisfied in both cases.

For the second condition, (8), in type $N$ one gets $\Phi_{0} = 0$ and $\Phi_{1} = 0$, for all $i = 0, 1, 2$, after contracting with $o^C$ and $i^C i^D i^E i^F$, respectively. Hence,
\[ \Phi_{ABA'B'} = \Phi_{2200A0A0B'}. \] (11)
\(^3\) The reason is that, for a general $\Psi_{ABCD} = \alpha_{A} \beta_{B} \gamma_{C} \delta_{D}$, one gets non-vanishing terms like $\alpha_{A} \delta_{C} \beta_{B} (\gamma E \delta_{E})^2$ on the left-hand side which are not present in $\Psi_{ABCD}$.\(^2\)
Similarly, for type \( D \), contracting with \( \epsilon^A \epsilon^B \) and \( \epsilon^C \epsilon^D \), gives
\[
\Phi_{ABAB} = 4\Phi_1 o(AB)o(AB),
\]
which can then be checked to satisfy (8).

**Lemma 2.** Conformally semi-symmetric spacetimes are Ricci semi-symmetric.

**Proof.** In spinors (4) is equivalent to \( \epsilon_A \epsilon_B \Box_{AB} \Phi_{CDE} = 0 \). We show that \( \Box_{AB} \Phi_{CDE} = 0 \), or equivalently that
\[
X_{ABC}E/\Phi_1 ECD + X_{ABD}E/\Phi_1 ECD + \Phi_{ECD}E = 0.
\]
This is trivially satisfied for type \( N \), while substitution of (6) shows that it is also satisfied for type \( D \).

The above two lemmas imply our main result.

**Theorem 3.** In four-dimensional—non-conformally flat—spacetimes, conformal semi-symmetry is equivalent to semi-symmetry. Furthermore, the semi-symmetric spacetimes are of Petrov types \( D \), \( N \) or \( O \).

### 3. Classification

A classification of semi-symmetric spacetimes was given in [6] in the context of pseudo-symmetric spacetimes. However, this classification only considered the algebraic restrictions arising from condition (1), without analyzing the compatibility differential conditions derived thereof.

Here, we present a complete classification of the (conformally) semi-symmetric spacetimes. From the previous results, there are two main possibilities, types \( D \) and \( N \). Then we have the following.

#### 3.1. Type \( D \)

Taking into account the previous results the Ricci tensor can be written as
\[
R_{ab} = Ak_{(a} \ell_{b)} + Bm_{(a} \tilde{m}_{b)},
\]
where \( k_a = \phi A o A \) and \( \ell_a = \phi A A \) are the two multiple principal null directions and \( m_a = o A A \) is a complex null vector completing the usual null tetrad. The scalars \( A \) and \( B \) are proportional to \( 3\Psi_2 \pm 2\Phi_1 \), respectively.

Observe that \( \Psi_2 \) is real, so that these spacetimes are purely electric [14] with respect to the timelike direction defined by \( k_a + \ell_a \).

The Bianchi identities written in the Newman–Penrose (NP) formalism [9, 14] provide easily the following necessary conditions:
\[
A\sigma = A\lambda = A\mu = A\rho = 0, \quad (14)
\]
\[
B\kappa = B\nu = B\pi = B\tau = 0. \quad (15)
\]
The generic case is given by $A \neq 0 \neq B$, which leads to $\sigma = \lambda = \mu = \rho = \kappa = \nu = \pi = \tau = 0$. This immediately implies that $\nabla_v k_b = v_b k_b$ and $\nabla_a \ell_b = -v_b \ell_b$ for some vector field $v_a$. In other words, the two principal null directions are recurrent. It follows that the tensor $k_a \ell_b$ is covariantly constant and therefore that the spacetime is $2 \times 2$ decomposable, see, e.g. [13, 14]. On the other hand, one can easily check by direct computation that $2 \times 2$ decomposable spacetimes have Petrov type-D. Furthermore, their Riemann tensor decomposes as the sum of the corresponding ones for the two two-dimensional parts, so that each of them are clearly proportional to their Gaussian curvature and, therefore, they are automatically semi-symmetric. We conclude that the generic type-D semi-symmetric spacetimes are precisely the $2 \times 2$ decomposable ones.

The special cases are given by $3 \Psi_2 \pm 2 \Phi_{11} = 0$.

If $2 \Phi_{11} = -3 \Psi_2$ ($A = 0$), we get $\kappa = \nu = \tau = \pi = 0$. The principal null directions are geodesic but not necessarily shear free. One can then check that all remaining NP equations are compatible, their integrability conditions providing no new information.

If $2 \Phi_{11} = 3 \Psi_2$ ($B = 0$), we get $\rho = \mu = \sigma = \lambda = 0$, so that in general the principal null directions are not necessarily geodesic. Again all remaining NP equations are compatible. (In this case it is worth mentioning that, as one can easily see, the Einstein tensor $G_{ab} = -A m(a \bar{m}(b)$ can never satisfy the dominant energy condition.)

Note finally that $1 \times 3$ decomposable spacetimes which are the product of a real line times a semi-symmetric three-dimensional Riemannian manifold belong to these cases, because all static spacetimes are necessarily of Petrov types I, D or 0.

### 3.2. Type N

In this case, the Ricci tensor takes the null radiation form

$$R_{ab} = 2 \Phi_{22} k_a k_b.$$

The Bianchi identities in the NP form provide the following conditions:

$$\kappa = 0, \quad \sigma \Psi_4 = \rho \Phi_{22}$$

together with other differential relations. Thus, in general, the unique principal null direction $k^a$ is geodesic but can be shearing, twisting and expanding. The remaining Bianchi identities and the rest of the NP equations can be seen, once again, to be integrable, for the NP commutators do not lead to any extra conditions.

The second-order symmetric spacetimes—characterized by (2)—which are not locally symmetric must have a covariantly constant null vector field [13], from where one can infer that they belong to the particular class of type-N (or type-O) semi-symmetric spacetimes with $\sigma = \rho = 0$, so that $\Psi_4$ and $\Phi_{22}$ are in that case independent.

### Acknowledgments

We are grateful to S Haesen for bringing to our attention important references and relevant results, and to M Sánchez for some comments. JMMS is supported by grants FIS2004-01626 (MICINN) and GIU06/37 (UPV/EHU).

### References

[1] Cartan É 1946 *Leçons sur la Géométrie des Espaces de Riemann* 2nd edn (Paris: Gauthier-Villars)

[2] Deszcz R and Grycak W 1989 On manifolds satisfying some curvature conditions *Colloquium Math.* **57** 89–92
[3] Defever F, Deszcz R, Verstraelen L and Vrancken L 1994 On pseudo symmetric spacetimes J. Math. Phys. 35 5908–21

[4] Derdzinski A 1981 Examples de métriques de Kaehler et d’Einstein autoduales sur le plan complexe Géométrie riemannienne en dimension 4 (Séminaire Arthur Besse, 1978/79) (Cedic/Fernand Nathan, Paris) pp 334–46

[5] Grycak W 1987 Riemannian manifolds with a symmetry condition imposed on the 2nd derivative of the conformal curvature tensor Tensor N. S. 46 287–90

[6] Haesen S and Verstraelen L 2004 Classification of the pseudosymmetric space-times J. Math. Phys. 45 2343–6

[7] Haesen S and Verstraelen L 2005 Curvature and symmetries of parallel transport Differential Geometry and Topology, Discrete and Computational Geometry (Nato Science Series vol 197) ed M Boucetta and J-M Morvan (Amsterdam: IOS Press) chapter 8

[8] Haesen S and Verstraelen L 2007 Properties of a scalar curvature invariant depending on two planes Monuse. Math 122 59–72

[9] Newman E T and Penrose R 1962 An approach to gravitational radiation by a method of spin coefficients J. Math. Phys. 3 566–78

Newman E T and Penrose R 1962 J. Math. Phys. 4 998 (erratum)

[10] Penrose R and Rindler W 1984 Spinors and Spacetime vol 1 (Cambridge: Cambridge University Press)

[11] Penrose R and Rindler W 1986 Spinors and Spacetime vol 2 (Cambridge: Cambridge University Press)

[12] Senovilla J M M 2006 Second-order symmetric Lorentzian manifolds AIP Conf. Proc. 841 370–7

[13] Senovilla J M M 2008 Second-order symmetric Lorentzian manifolds: I. Characterization and general results Class. Quantum Grav. 25 245011

[14] Stephani H, Kramer D, MacCallum M A H, Hoenselaers C and Herlt E 2003 Exact Solutions to Einstein’s Field Equations 2nd edn (Cambridge: Cambridge University Press)

[15] Szabó Z I 1982 Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$: I. The local version J. Diff. Geom. 17 531–82

[16] Szabó Z I 1985 Structure theorems on Riemannian spaces satisfying $R(X, Y) \cdot R = 0$: II. Global versions Geom. Dedicata 19 65–108