Effective damping in the Raman cooling of trapped ions

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Abstract

We present a method of treating the interaction of a single three-level ion with two laser beams. The idea is to apply a unitary transformation such that the exact transformed Hamiltonian has one of the three levels decoupled for all values of the detunings. When one takes into account damping, the evolution of the system is governed by a master equation usually obtained via adiabatic approximation under the assumption of far-detuned lasers. To go around the drawbacks of this technique, we use the same unitary transformation to get an effective master equation.

1 Introduction

In recent years, much progress has been made in cooling and trapping techniques for neutral atoms and for ions \cite{1,2,3,4,5,6,7}. In fact, in a trapped ion the center-of-mass dynamics gets entangled with the internal degrees of freedom and some peculiar aspects of the vibrational response have been successfully exploited for experimentally generating Fock \cite{8}, coherent and squeezed \cite{9}, and Schrödinger catlike states \cite{10}, proposing theoretical schemes for engineering several nonclassical states \cite{11,12,13,14,15,16,17,18,19,20}, realizing tomographic reconstructions of the density matrix \cite{21,22,23,24}, and characterizing a variety of quantum effects \cite{25,26,27,28,29}. This is of fundamental interest, since it brings to the forefront issues involving the relationship between quantum and classical physics, but also offers potential applications for, e.g., precision spectroscopy \cite{30} or quantum computation \cite{31,32,33,34}.

In modeling typical experiments one considers a three-level atomic system interacting with two laser fields (Raman scheme \cite{35,36}) and reduces it to a two-level problem on the assumption of large detunings by using the adiabatic
elimination [37]: the effective Hamiltonian obtained in this way has the form of the usual Jaynes-Cummings model.

Adiabatic elimination has been criticized on several grounds [38,39,40,41], and other methods of deriving effective Hamiltonians exist [42,43]. In this spirit, we have recently proposed an alternative approach that involves using a unitary transformation (in fact, a nonlinear rotation) to obtain an equivalent Hamiltonian for which one level decouples [44,45]. The transformation can be exactly found and gives the same results as the adiabatic elimination (except for including intensity-dependent Stark shifts) when it is evaluated up to second-order terms in coupling constants.

To take into account the effects of damping in Raman cooling schemes, the standard way of proceeding is to start from the master equation for the three-level system and adiabatically eliminate the far-off resonant level. The details are described in many different text books [46,47]. Unfortunately, it is known that this treatment is not valid in many regimes of physical interest and other approximations are required [48,49,50,51,52,53].

The main purpose of this paper is to show how our approach of nonlinear rotations allows one to go around these drawbacks in a natural way. Our strategy can be stated in very simple terms: starting from the exact master equation for the three-level model, we apply to it the same unitary transformation leading to the effective Hamiltonian, obtaining in this way what we call an effective master equation [54]. Here we fully investigate this approach and present numerical evidences of its validity.

2 Physical system and model Hamiltonian

In the interest of retaining as much clarity as possible, we first recall some well-known facts [55,56] about the system we wish to treat here, which consists of a three-level trapped ion in the Λ configuration with energy levels $E_0 < E_1 < E_2$, as shown in figure 1. As usual, to describe this system we use the operators

$$\hat{S}_{ij} = \langle j | i \rangle ,$$

(1)

where $| i \rangle$ denotes the eigenstate of the $i$th atomic level. One can easily check that they satisfy

$$[\hat{S}_{ij}, \hat{S}_{kl}] = \delta_{jk} \hat{S}_{il} - \delta_{il} \hat{S}_{kj} ,$$

(2)

which correspond to the commutation relations of the algebra $u(3)$. Obviously, the three “diagonal” operators $\hat{S}_{ii}$ measure level populations, while the “off-
Fig. 1. Energy scheme of a three-level Λ-type ion interacting with two laser fields, coupling the two ground states to a common excited atomic state via a Raman transition. Here we illustrate the case of driving the first blue sideband.

Diagonal” \( \hat{S}_{ij} \) generate transitions from level \( i \) to level \( j \).

The levels \( |0\rangle \) and \( |1\rangle \) are metastable and coupled by stimulated Raman transitions via two classical optical fields (of frequencies \( \omega_a \) and \( \omega_b \)) of the form

\[
E_{\ell} = E_{\ell} \exp\left[i(k_{\ell} \cdot \hat{x} - \omega_{\ell} t - \varphi_{\ell})\right],
\]

where from now on the index \( \ell \) runs the values \( a \) and \( b \), \( \hat{x} \) is the position operator associated with the center-of-mass motion and \( \varphi_{\ell} \) is the phase of the laser field \( \ell \) at the mean position of the ion.

The ion is trapped in a harmonic potential. Therefore, the center-of-mass motion can be described in terms of annihilation and creation operators of vibrational quanta (phonons) in the usual way

\[
\hat{x}_q = \sqrt{\frac{\hbar}{2M\nu_q}} (\hat{a}_q + \hat{a}_q^\dagger) = \Delta x_q (\hat{a}_q + \hat{a}_q^\dagger),
\]

where \( \nu_q \) represents the oscillatory frequency along the \( q \)th direction, \( M \) is the ion mass, and \( \Delta x_q \) is the width of the ground-state wave function.

The Hamiltonian that describes the system is \( \hat{H} = \hat{H}_{cm} + \hat{H}_{ion} + \hat{V} \), where

\[
\begin{align*}
\hat{H}_{cm} &= \sum_q \hbar \nu_q \hat{a}_q^\dagger \hat{a}_q, \\
\hat{H}_{ion} &= \sum_i E_i \hat{S}_{ii}, \\
\hat{V} &= \hbar [g_a(\hat{x}) e^{-i\omega_a t} \hat{S}_{02} + g_a^*(\hat{x}) e^{i\omega_a t} \hat{S}_{20}] \\
&+ \hbar [g_b(\hat{x}) e^{-i\omega_b t} \hat{S}_{12} + g_b^*(\hat{x}) e^{i\omega_b t} \hat{S}_{21}].
\end{align*}
\]
The interaction term $\hat{V}$ is written in the rotating-wave approximation and the coupling constants are

$$g_\ell(\hat{x}) = \kappa_\ell \mathcal{E}_\ell \exp[i(k_\ell \cdot \hat{x} - \varphi_\ell)],$$  \hspace{1cm} (6)$$

where $\kappa_\ell$ is the corresponding dipole matrix element in the direction of the driving field.

Because one has the trivial constraint $\hat{S}_{00} + \hat{S}_{11} + \hat{S}_{22} = \hat{1}$, only two populations can vary independently. Eliminating the population of the level $|2\rangle$ we can recast the Hamiltonian as $\hat{H} = \hat{H}_0 + \hat{H}_{\text{int}}$, with

$$\hat{H}_0 = \sum_q \hbar \nu_q \hat{a}_q^\dagger \hat{a}_q - \hbar(\Delta_a + \omega_a) \hat{S}_{00} - \hbar(\Delta_b + \omega_b) \hat{S}_{11},$$  \hspace{1cm} (7)$$

$$\hat{H}_{\text{int}} = \hbar [g_a(\hat{x}) e^{-i\omega_a t} \hat{S}_{02} + g_a^*(\hat{x}) e^{i\omega_a t} \hat{S}_{20}]$$

$$+ \hbar [g_b(\hat{x}) e^{-i\omega_b t} \hat{S}_{12} + g_b^*(\hat{x}) e^{i\omega_b t} \hat{S}_{21}],$$

where we have defined the following detunings

$$\hbar \Delta_a = E_2 - E_0 - \hbar \omega_a, \hspace{1cm} \hbar \Delta_b = E_2 - E_1 - \hbar \omega_b.$$  \hspace{1cm} (8)$$

This Hamiltonian contains terms oscillating rapidly in time at frequencies $\omega_a$ and $\omega_b$, which can be eliminated by going to a rotating frame. The final result is

$$\hat{H}_0 = \sum_q \hbar \nu_q \hat{a}_q^\dagger \hat{a}_q - \hbar(\Delta_a \hat{S}_{00} + \Delta_b \hat{S}_{11}),$$  \hspace{1cm} (9)$$

$$\hat{H}_{\text{int}} = \hbar [g_a(\hat{x}) \hat{S}_{02} + g_a^*(\hat{x}) \hat{S}_{20}] + \hbar [g_b(\hat{x}) \hat{S}_{12} + g_b^*(\hat{x}) \hat{S}_{21}].$$

This is the basic Hamiltonian that will be used in our subsequent analysis.

3 Effective Raman Hamiltonian in the dispersive limit

The standard treatment assumes that level $|2\rangle$ is far off resonance and proceeds via adiabatic elimination to obtain an effective two-level Raman interaction Hamiltonian with an intensity-dependent coupling between levels $|0\rangle$ and $|1\rangle$. In Refs. [44] and [45] we have claimed that such a procedure has drawbacks
and proposed instead an alternative technique involving nonlinear rotations. For the problem at hand, we introduce the unitary transformation

\[ \hat{H} = \hat{T} \hat{H} \hat{T}^\dagger, \]  

where \( \hat{T} = \exp[\hat{J}(\hat{x})] \) and

\[ \hat{J}(\hat{x}) = [\varepsilon_a(\hat{x})\hat{S}_{02} - \varepsilon_a^*(\hat{x})\hat{S}_{20}] + [\varepsilon_b(\hat{x})\hat{S}_{12} - \varepsilon_b^*(\hat{x})\hat{S}_{21}]. \]  

Here the parameters \( \varepsilon_a(\hat{x}) \) and \( \varepsilon_b(\hat{x}) \) are defined by

\[ \varepsilon_\ell(\hat{x}) = \frac{g_\ell(\hat{x})}{\Delta_\ell}. \]  

To interpret this operator \( \hat{T} \), we note that \( \hat{S}_{02} \) and \( \hat{S}_{20} \) are raising and lowering su(2) operators that correspond to the allowed transitions \( 0 \leftrightarrow 2 \) and \( 1 \leftrightarrow 2 \). However, these two dipoles are not independent, since Eq. (2) imposes highly nontrivial couplings between them. In consequence, \( \hat{T} \) can be seen as a “rotation” acting on the subspace of these two dipoles.

By using the well-known expression

\[ e^{\hat{A}}\hat{B}e^{-\hat{A}} = \sum_{n=0}^{\infty} \frac{\hat{B}^{(n)}}{n!}, \]  

where \( \hat{B}^{(n)} = [\hat{A}, \hat{B}^{(n-1)}] \), and \( \hat{B}^{(0)} = \hat{B} \), the exact transformation law (10) has been found in Refs. [57] and [58] (see also [59]). The explicit expression is complicated although is valid for any values of the detunings. Since we are assuming that level \( |2\rangle \) is far off resonance and, consequently, the ratios \( \varepsilon_a(\hat{x}) \) and \( \varepsilon_b(\hat{x}) \) can be taken as small quantities, the series (13) can be evaluated keeping only up to second-order terms. By applying this to (9) we finally obtain an effective Hamiltonian \( \hat{H} = \hat{H}_0 + \hat{H}_{\text{int}} \), where

\[
\hat{H}_0 = \sum_q \hbar \nu_q \hat{a}_q^\dagger \hat{a}_q - \hbar \left( \Delta_a + \frac{|g_a(\hat{x})|^2}{\Delta_a} \right) \hat{S}_{00} - \hbar \left( \Delta_b + \frac{|g_b(\hat{x})|^2}{\Delta_b} \right) \hat{S}_{11},
\]

\[
\hat{H}_{\text{int}} = -\frac{\hbar}{2} \left( \frac{1}{\Delta_a} + \frac{1}{\Delta_b} \right) [g_a(\hat{x})g_b^*(\hat{x})\hat{S}_{01} + g_a^*(\hat{x})g_b(\hat{x})\hat{S}_{10}].
\]

In the usual experiments the wave vector difference is chosen to be parallel with the \( X \) direction of the trap, so \( (\mathbf{k}_a - \mathbf{k}_b) \cdot \hat{x} = \delta k \hat{x} \) and the interaction
couples only the motion in $X$ direction to the internal state of the trapped ion. Using the effective inversion between levels $|1\rangle$ and $|0\rangle$:

$$S_{01}^z = \frac{1}{2}(\hat{S}_{11} - \hat{S}_{00}),$$  \hspace{1cm} (15)

we can write

$$\hat{H}_0 = \hbar \nu \hat{a}^\dagger \hat{a} + \hbar \delta \hat{S}_{01}^z,$$  \hspace{1cm} (16a)

$$\hat{H}_{\text{int}} = -\frac{\hbar \Omega}{2}(e^{i\delta k \hat{x}} \hat{S}_{01} + e^{-i\delta k \hat{x}} \hat{S}_{10}),$$  \hspace{1cm} (16b)

with

$$\delta = \Delta_a - \Delta_b + \frac{|g_a|^2}{\Delta_a} - \frac{|g_b|^2}{\Delta_b}, \hspace{1cm} \Omega = |g_a g_b| \left(\frac{1}{\Delta_a} + \frac{1}{\Delta_b}\right).$$  \hspace{1cm} (17)

In terms of the phonon raising and lowering operators we rewrite (16b) as

$$\hat{H}_{\text{int}} = \frac{\hbar \Omega}{2}\{e^{i\eta(\hat{a} + \hat{a}^\dagger)} \hat{S}_{01} + e^{-i\eta(\hat{a} + \hat{a}^\dagger)} \hat{S}_{10}\},$$  \hspace{1cm} (18)

where the Lamb-Dicke parameter is defined as

$$\eta = \delta k \sqrt{\frac{\hbar}{2M\nu}},$$  \hspace{1cm} (19)

and represents the ratio between the recoil energy and the quantum vibrational energy, both taken in the $X$ direction. It is worth observing that the second-order corrections to this effective Hamiltonian vanish, so (18) accurately describes the system dynamics up to times $t \leq \hbar/(\Omega \varepsilon_0^3)$.

In the interaction picture relative to $\hat{H}_0$ we finally get

$$\hat{H}_{\text{int}} = -\frac{\hbar \Omega}{2}\{e^{i[\eta(\hat{a} + \hat{a}^\dagger) - \delta t]} \hat{S}_{01} + e^{-i[\eta(\hat{a} + \hat{a}^\dagger) - \delta t]} \hat{S}_{10}\}.$$  \hspace{1cm} (20)

By tuning the frequency difference $\delta$ to an integer multiple of the trap frequency $\nu$, $\delta = (n' - n)\nu$, we can resonantly drive transitions from $|0, n\rangle$ to $|1, n'\rangle$, where the ket $|j, n\rangle$ indicates the $n$th vibrational Fock state in the electronic state $j$ ($j = 0, 1$). In this case, $\hat{H}_{\text{int}}$ is dominated by a single stationary term. The exponent $\exp[i\eta(\hat{a} + \hat{a}^\dagger)]$ in Eq. (20) contains all powers of $\hat{a}$ and $\hat{a}^\dagger$. However, all contributions with $m \neq n' - n$ oscillate rapidly and average out when $\nu$ is much larger than $\Omega$. We assume the Lamb-Dicke limit, in which
In the relevant case of the first red sideband $\delta = \nu$, we get to lowest order in $\eta$

$$\dot{\mathcal{H}}_{\text{int}} = -i\eta\frac{\hbar\Omega}{2}(\hat{a}\hat{S}_{01} - \hat{a}^{\dagger}\hat{S}_{10}),$$

(21)

which is the familiar Jaynes-Cummings Hamiltonian. Similarly, there is a first blue sideband, corresponding to an anti-Jaynes-Cummings Hamiltonian and higher-order sidebands, but in the rest of this paper we shall be mainly concerned with the model Hamiltonian (21).

4 Damping in terms of an effective master equation

For many purposes, the coherent control of the vibrational dynamics plays a crucial role. For the model discussed in the previous section damping effects have been observed that even occurred under almost ideal conditions [60].

The presence of the two lasers causes the appearance of a coupling between internal (electronic) and external (center of mass) degrees of freedom of the trapped ion. However, when the Lamb-Dicke parameter is very small, the dynamics due to this coupling is slow compared with the internal dynamics that may be adiabatically eliminated. This reduction leads to a master equation for the motional degrees of freedom, where the involved transition rates depend on steady-state expectation values of internal operators [37].

To take full advantage of the method outlined in the previous section, we start from the density operator $\hat{\rho}$ for the external and internal degrees of freedom of the three-level ion and make the hypothesis that the dynamics is described by a general master equation of the Lindblad type [61]:

$$\frac{d}{dt}\hat{\rho} = \frac{1}{i\hbar}[\hat{H},\hat{\rho}] + \frac{\gamma_a}{2}\mathcal{L}[\hat{S}_{20}]\hat{\rho} + \frac{\gamma_b}{2}\mathcal{L}[\hat{S}_{21}]\hat{\rho},$$

(22)

where $\gamma_a$ and $\gamma_b$ represent the decoherence rates for the processes associated with the coupling of the dipoles with a zero-temperature bath and $\mathcal{L}[\hat{C}]$ is the Lindblad superoperator

$$\mathcal{L}[\hat{C}]\hat{\rho} = 2\hat{C}\rho\hat{C}^{\dagger} - \{\hat{C}^{\dagger}\hat{C},\hat{\rho}\}.$$  

(23)

Note that (22) describes an irreversible evolution of the system at different rates for each channel. It is implicitly assumed that the both dipole moments are orthogonal to each other; that is, $d_{02}\cdot d_{12}^{\dagger} = 0$, where $d_{ij}$ are the transition dipole matrix elements [62].
It seems natural to ask how this equation is transformed by the same unitary operator $\hat{T}$ leading to the effective Hamiltonian in (10). Let us denote the effective density matrix by

$$\hat{\rho} = \hat{T} \hat{\rho} \hat{T}^\dagger.$$  

(24)

Taking into account that $\hat{S}_{20} \hat{\rho} = \hat{S}_{21} \hat{\rho} = 0$, we get up to second-order terms

$$\hat{T} \hat{S}_{02} \hat{T}^\dagger = \hat{S}_{02} - \varepsilon_a^* (\hat{x}) \hat{S}_{00} - \varepsilon_b^* (\hat{x}) \hat{S}_{01},$$

(25)

$$\hat{T} \hat{S}_{12} \hat{T}^\dagger = \hat{S}_{12} - \varepsilon_b^* (\hat{x}) \hat{S}_{11} - \varepsilon_a^* (\hat{x}) \hat{S}_{10}.$$  

Then, if we apply $\hat{T}$ to (22), we obtain the effective master equation

$$\frac{d}{dt} \hat{\rho} = \frac{1}{i\hbar} [\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}] + \frac{\gamma_a}{2} \mathcal{L}[\varepsilon_a (\hat{x}) \hat{S}_{00} + \varepsilon_b (\hat{x}) \hat{S}_{10}] \hat{\rho} + \frac{\gamma_b}{2} \mathcal{L}[\varepsilon_b (\hat{x}) \hat{S}_{11} + \varepsilon_a (\hat{x}) \hat{S}_{01}] \hat{\rho},$$

(26)

where $\hat{\rho}$ is expressed in the interaction picture.

In the Lamb-Dicke limit we can make the approximation

$$\varepsilon(t) \approx \varepsilon(t) (1 + \eta_t \hat{x}),$$

(27)

with [compare with (19)]

$$\eta_t = k_t \sqrt{\frac{\hbar}{2M \nu}},$$

(28)

and the master equation takes the simpler form

$$\frac{d}{dt} \hat{\rho} = [\hat{\mathcal{H}}_{\text{int}}, \hat{\rho}] + \frac{\gamma_a |\varepsilon_a|^2}{2} \mathcal{L}[\hat{S}_{00}] \hat{\rho} + \frac{\gamma_b |\varepsilon_b|^2}{2} \mathcal{L}[\hat{S}_{11}] \hat{\rho} + \frac{\gamma_a |\varepsilon_a|^2}{2} \mathcal{L}[\hat{S}_{10}] \hat{\rho} + \frac{\gamma_b |\varepsilon_b|^2}{2} \mathcal{L}[\hat{S}_{01}] \hat{\rho} + \frac{\gamma_a}{2} \mathcal{K}[\varepsilon_a (\hat{x}), \hat{S}_{00}, \varepsilon_b (\hat{x}) \hat{S}_{10}] \hat{\rho} + \frac{\gamma_b}{2} \mathcal{K}[\varepsilon_b (\hat{x}), \hat{S}_{11}, \varepsilon_a (\hat{x}) \hat{S}_{01}] \hat{\rho}.$$  

(29)
This equation has a very suggestive and transparent physical meaning: the terms $L[\hat{S}_{00} \hat{S}]$ and $L[\hat{S}_{11} \hat{S}]$ describe pure phase dissipation, meanwhile the terms $L[\hat{S}_{10} \hat{S}]$ and $L[\hat{S}_{01} \hat{S}]$ describe dissipative-like transitions from the level $|0\rangle$ to $|1\rangle$ and back, which leads to a stationary distribution of population in these levels (incoherent mixing). These two terms seem to simulate the coupling to a thermal bath at finite temperature, but there is a strong difference: now the corresponding rates for the processes from level $|0\rangle$ to $|1\rangle$ and from level $|1\rangle$ to $|0\rangle$ are not related by a Maxwell-Boltzmann factor of the form $n/(n+1)$. All these terms have a purely atomic nature and appear in a similar master equation derived in a different context by Di Fidio and Vogel [63], who interpreted them in terms of quantum jumps.

Besides, we have also “crossed terms” described by superoperators $\mathcal{K}$, which contribute substantially (i.e., the corresponding terms are time independent in the rotating frame) only when some resonance conditions discussed after Eq. (20) hold. For the first red sideband we are considering here, they take the form

$$
\mathcal{K}[\varepsilon_a(\hat{x}) \hat{S}_{00}, \varepsilon_b(\hat{x}) \hat{S}_{10}] \hat{\rho} = i\varepsilon_a \varepsilon_b^* \left[ 2(\eta_a \hat{S}_{00} \hat{\rho} \hat{S}_{01} - \eta_b \hat{S}_{00} \hat{\rho} \hat{S}_{01}) \right. \\
- \eta(\hat{a} \hat{S}_{01} \hat{S}_{00} \hat{\rho} + \hat{\rho} \hat{a} \hat{S}_{01} \hat{S}_{00}) + \text{h. c.} \big],
$$

$$
\mathcal{K}[\varepsilon_b(\hat{x}) \hat{S}_{11}, \varepsilon_a(\hat{x}) \hat{S}_{01}] \hat{\rho} = i\varepsilon_b \varepsilon_a^* \left[ 2(\eta_b \hat{S}_{11} \hat{a}^\dagger \hat{\rho} \hat{S}_{10} - \eta_a \hat{S}_{11} \hat{a}^\dagger \hat{S}_{10}) \right. \\
+ \eta(\hat{a}^\dagger \hat{S}_{10} \hat{S}_{11} \hat{\rho} + \hat{\rho} \hat{a} \hat{S}_{11} \hat{S}_{10}) + \text{h. c.} \big],
$$

(30)

where we have retained only the dominant terms in the parameter $\eta = \eta_a - \eta_b$.

To test our theory, we have numerically integrated the master equation (29) using the Quantum Optics Toolbox produced by S. M. Tan [64]. In the typical experiments at NIST [22], a single $^9\text{Be}^+$ ion is stored in a RF Paul trap with a secular frequency along $X$ of $\nu/2\pi \simeq 11.2$ MHz, providing a spread of the ground state wave function of $\Delta x \simeq 7$ nm, with a Lamb-Dicke parameter of $\eta \simeq 0.202$. The two laser beams, with 0.5 W in each one, are approximately detuned $\Delta/2\pi \simeq 12$ GHz, so that $\Omega/2\pi \simeq 475$ kHz. With these data we find $\varepsilon_a \simeq \varepsilon_b \simeq 0.01$, so they can be considered as small parameters, as assumed in the previous Section. We take also $(\gamma_a + \gamma_b)/2\pi \simeq 19.4$ MHz (which is about one linewidth of the transition) and $\gamma_a = \gamma_b$. The observable measured in all these experiments is the fluorescence signal, which is the probability $P_{\downarrow}(t)$ of occupation of the electronic level $|0\rangle$. This probability may be written as

$$
P_{\downarrow}(t) = \sum_n \langle 0, n | \hat{\rho}(t) | 0, n \rangle.
$$

(31)

Let us consider first the case in which the ion starts in a Fock state with
An intriguing result found in the experiments of Ref. [8] is that the fluorescence signal, for initial Fock states, may be approximately modeled by

\[ P_\downarrow(t) \approx \frac{1}{2} \left[ 1 + \cos(2\Omega_{n_0} t)e^{-\gamma_{n_0} t} \right], \]

(32)

where \( \Omega_{n_0} \) is the associated Rabi frequency and the phenomenological decay constants \( \gamma_{n_0} \) were fitted as \( \gamma_{n_0} \approx \gamma_0(n_0 + 1)^{0.7} \). This exponential decay can be inferred with good accuracy from a numerical analysis of our simulated data [70].

In Fig. 3 we show our results for the evolution of the fluorescence signal for an initial coherent state with an average vibrational number \( \bar{n} = 3 \). The graphic reproduces all the salient features of the experiment [22], although for a perfect fitting a more precise value of our parameters \( \eta_a \) and \( \eta_b \) would be needed. The influence of the terms (30) is again very small.

In Fig. 4 we have plotted the dynamics of the same coherent state with \( \bar{n} = 3 \), but in a larger time scale and for two different values of \( \gamma_a/\gamma_b \). We see that
the stationary limit of the oscillations is not 0.5, due to the phenomenon of incoherent mixing mentioned above: while the details of the collapse are almost insensitive to the values of the ratio $\gamma_a/\gamma_b$, as time goes by the quasi-stationary values of the population tend to be different. This cannot be reproduced by using the standard master equation approach [20]. The “crossed terms” are important here in preventing that for larger values of $\gamma_a/\gamma_b$ the solutions would differ too much.

To confirm that this is, in fact, an incoherent effect that affects only to the population dynamics, in Fig. 5 we show the evolution of the coherence $\hat{\rho}_{01}$ for the same initial state, and we clearly see that the values of $\gamma_a/\gamma_b$ do not influence at all to the dynamics at the large of the dipole moment.

Fig. 4. Evolution of the $P_\downarrow(t)$ for the same initial coherent state with an average vibrational number $\bar{n} = 3$ as in Fig. 3 in a different time scale. The solid line represents the case $\gamma_a/\gamma_b = 1$, while the dashed line is for $\gamma_a/\gamma_b = 3$. 

Fig. 3. Evolution of the $P_\downarrow(t)$ for an initial coherent state with an average vibrational number $\bar{n} = 3$ driven by a Raman interaction. The rest of the parameters are the same as in Fig. 2.
5 Concluding remarks

What we expect to have accomplished in this paper is to present a comprehensive method of treating two-photon stimulated Raman transitions in a single trapped ion. Our approach is based on the application of a unitary transformation and leads in a natural way to an effective master equation of the Lindblad type with a clear physical interpretation.

In the framework of this description, we have shown that the evolution of populations and dipole moments predicted by our theory is in good agreement with the data of realistic experiments. The appearance of an incoherent mixing that leads to a stationary redistribution of populations has been also studied.

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