Linearization of two-state thruster models

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Abstract. A thruster is a device used for station keeping, attitude control, in the reaction control system, or long-duration, low-thrust acceleration. Thruster is one of the main components in autonomous surface vehicle. In this paper, we discuss the linearization of two-state thruster model by using the Jacobian method.

1. Introduction

A thruster is a device used in many equipments including Autonomous Surface Vehicle (ASV) for station keeping, attitude control, in the reaction control system, or long-duration, low-thrust acceleration. The thruster pushes the ASV so that the ASV can move to the desired position. In order to design a controller for thruster, we need the mathematical model of thruster. There are many papers in the literature that discuss the thruster model, such as [1, 2, 3, 4, 5].

In this paper, we focus on the two-state thruster model developed in [3]. The thruster model is a nonlinear ordinary differential equation. Usually, designing a controller for nonlinear systems is not easy task. As such, in this paper, we discuss the linearization of the thruster model by using Jacobian method. In Section 2, we describe the two-state thruster model in [3]. Then in Section 3, we discuss the linearization of the two-state thruster model. We use two conditions for linearization: tunnel thruster test and open-bladed thruster test. After obtaining the linearized two-state thruster model, we are planning to continue this work to the navigation and stability [6], state estimation [7, 8], control design [9, 10, 11, 12].

2. Two-state Thruster Models

The thruster model in [3] has two state variables, i.e. $\omega_m$ and $U_a$

$$\dot{\omega}_m = f_1(\omega_m, U_a, V_s, U_0) = -K_1\omega_m + K_2V_s - K_hQ,$$
$$\dot{U}_a = f_2(\omega_m, U_a, V_s, U_0) = -K_4K_3^{-1}|U_a|U_a + K_3^{-1}T,$$

where the input variables are $V_s$ and $U_0$. The expressions for $\omega_m$ and $U_a$ are denoted by $f_1(\omega_m, U_a, V_s, U_0)$ and $f_2(\omega_m, U_a, V_s, U_0)$, respectively. In this model, the output variable $T$ is given by

$$T = g(\omega_m, U_a, V_s, U_0) = \text{Lift}(\cos \theta) - \text{Drag}(\sin \theta).$$
The expressions for other variables and parameters in the model are as follows

\[ Q = 0.7R(\text{Lift}(\sin \theta) + \text{Drag}(\cos \theta)), \]
\[ U_a = U_a - U_0, \]  
\[ \text{Lift} = 0.5\rho V^2 A C_{L_{\text{max}}} \sin(2\alpha_e), \]  
\[ \text{Drag} = 0.5\rho V^2 A C_{D_{\text{max}}}(1 - \cos(2\alpha_e)), \]  
\[ \theta = p - \alpha_e, \]  
\[ \alpha_e = \left( \frac{\pi}{2} - p \right) - \arctan \left( \frac{U_a}{U_p} \right), \]  
\[ V^2 = U_p^2 + U_a^2, \]  
\[ U_p = \frac{0.7R\omega_m}{N}, \]  
\[ K_3 = \rho A L \gamma, \]  
\[ K_4 = \rho A \Delta \beta. \]

The interpretation of each variable and parameter in the model is as follows:

Table 1. Variables and parameters in the thruster model.

| Variable | Description |
|----------|-------------|
| Q | propeller torque (Nm) |
| N | reduction gear ratio |
| R | propeller radius (m) |
| p | blade pitch (rad) |
| \( U_p \) | propeller velocity (m/s) |
| A | tunnel cross sectional area (m²) |
| Drag | drag force (N) |
| \( \theta \) | angle of inlet to blades (rad) |
| \( \gamma \) | effective added mass ratio |
| \( U_0 \) | vehicle velocity (m/s) |
| \( \omega_m \) | motor rotational rate (rad/sec) |
| \( U_p \) | propeller velocity (m/s) |
| \( \alpha_e \) | effective angle of attack (rad) |
| \( U_a \) | section average flow velocity (m/s) |
| \( \rho \) | mass density of water (kg/m³) |
| \( \text{Lift} \) | lift force (N) |
| \( \Delta \beta \) | momentum coefficient |
| \( T \) | thrust force (N) |
| \( L \) | tunnel length (m) |

3. Linearization of the Thruster Model

In this section, we linearize the state and output equations in the thruster model (1)-(3). The thruster model is linearized around constant solution, i.e. \( \omega_m \) and \( U_a \) are constant functions. Because both \( \omega_m \) and \( U_a \) are constants, we obtain \( \dot{\omega}_m = 0 \) and \( \dot{U}_a = 0 \). By substituting \( \omega_m = 0 \) to (1), we have

\[ -K_1 \omega_m + K_2 V_s - K_h Q = 0, \]  
\[ \omega_m = \frac{K_2 V_s - K_h Q}{K_1}. \]

In order to obtain \( \omega_m \) as a constant function, the input \( V_s \) has to be a constant function that can be chosen arbitrarily. Next, \( \dot{U}_a = 0 \) is substituted to (2), as follows

\[ -K_4 K_3^{-1} U_a [U_a] + K_3^{-1} T = 0 \]
(U_a - U_0)|U_a - U_0| = \frac{T}{K_4} \tag{17}

U_a = \begin{cases} 
U_0 + \sqrt{\frac{T}{K_4}}, & \text{if } \frac{T}{K_4} \geq 0, \\
U_0 - \sqrt{\frac{T}{K_4}}, & \text{if } \frac{T}{K_4} < 0.
\end{cases} \tag{18}

Since \(U_a\) is a constant function, the input \(U_0\) must be a constant function which can be chosen freely. Then we determine the the Jacobian matrix of the system (1)-(3). As a first step, we determine the partial derivative of \(f\) freely. Then we compute the partial derivative of \(\omega\) w.r.t. state variable \(\omega_m\). More precisely, we compute \(\frac{\partial f_1}{\partial \omega_m}, \frac{\partial f_2}{\partial \omega_m}\) and \(\frac{\partial g}{\partial \omega_m}\):

\[
\frac{\partial f_1}{\partial \omega_m} = -K_1 - K_h \frac{\partial Q}{\partial \omega_m}, \tag{19}
\]

\[
\frac{\partial f_2}{\partial \omega_m} = 1 \frac{\partial g}{\partial \omega_m}, \tag{20}
\]

\[
\frac{\partial g}{\partial \omega_m} = \cos \theta \frac{\partial \text{Lift}}{\partial \omega_m} - \text{Lift}(\sin \theta) \frac{\partial \theta}{\partial \omega_m} - \sin \theta \frac{\partial \text{Drag}}{\partial \omega_m} - \text{Drag}(\cos \theta) \frac{\partial \theta}{\partial \omega_m}, \tag{21}
\]

\[
\frac{\partial Q}{\partial \omega_m} = 0.7R \left( \sin \theta \frac{\partial \text{Lift}}{\partial \omega_m} + \text{Lift}(\cos \theta) \frac{\partial \theta}{\partial \omega_m} + \cos \theta \frac{\partial \text{Drag}}{\partial \omega_m} - \text{Drag}(\sin \theta) \frac{\partial \theta}{\partial \omega_m} \right), \tag{22}
\]

\[
\frac{\partial \text{Lift}}{\partial \omega_m} = 0.5 \rho AC_{L_{\text{max}}} \left( \sin(2 \alpha_e) \frac{\partial}{\partial \omega_m} (V^2) + 2V^2 \cos(2 \alpha_e) \frac{\partial \alpha_e}{\partial \omega_m} \right), \tag{23}
\]

\[
\frac{\partial \text{Drag}}{\partial \omega_m} = 0.5 \rho AC_{D_{\text{max}}} \left( (1 - \cos(2 \alpha_e)) \frac{\partial}{\partial \omega_m} (V^2) + 2V^2 \sin(2 \alpha_e) \frac{\partial \alpha_e}{\partial \omega_m} \right), \tag{24}
\]

\[
\frac{\partial \theta}{\partial \omega_m} = -\frac{\partial \alpha_e}{\partial \omega_m}, \tag{25}
\]

\[
\frac{\partial \alpha_e}{\partial \omega_m} = \frac{U_a}{U_p^2 + U_a^2} \frac{\partial U_p}{\partial \omega_m}, \tag{26}
\]

\[
\frac{\partial (V^2)}{\partial \omega_m} = 2U_p \frac{\partial U_p}{\partial \omega_m}, \tag{27}
\]

\[
\frac{\partial U_p}{\partial \omega_m} = 0.7R \frac{\partial \omega_m}{\partial \omega_m} = N. \tag{28}
\]

Then, we compute the partial derivative of \(f_1, f_2\) and \(g\) w.r.t. state variable \(U_a\). More specifically, we determine \(\frac{\partial f_1}{\partial U_a}, \frac{\partial f_2}{\partial U_a}\) and \(\frac{\partial g}{\partial U_a}\):

\[
\frac{\partial f_1}{\partial U_a} = -K_h \frac{\partial Q}{\partial U_a}, \tag{29}
\]

\[
\frac{\partial f_2}{\partial U_a} = -2K_4K_3^{-1}|V_a| + K_3^{-1} \frac{\partial g}{\partial U_a}, \tag{30}
\]

\[
\frac{\partial g}{\partial U_a} = \cos \theta \frac{\partial \text{Lift}}{\partial U_a} - \text{Lift}(\sin \theta) \frac{\partial \theta}{\partial U_a} - \sin \theta \frac{\partial \text{Drag}}{\partial U_a} - \text{Drag}(\cos \theta) \frac{\partial \theta}{\partial U_a}, \tag{31}
\]

\[
\frac{\partial Q}{\partial U_a} = 0.7R \left( \sin \theta \frac{\partial \text{Lift}}{\partial U_a} + \text{Lift}(\cos \theta) \frac{\partial \theta}{\partial U_a} + \cos \theta \frac{\partial \text{Drag}}{\partial U_a} - \text{Drag}(\sin \theta) \frac{\partial \theta}{\partial U_a} \right), \tag{32}
\]

\[
\frac{\partial \text{Lift}}{\partial U_a} = 0.5 \rho AC_{L_{\text{max}}} \left( \sin(2 \alpha_e) \frac{\partial}{\partial U_a} (V^2) + 2V^2 \cos(2 \alpha_e) \frac{\partial \alpha_e}{\partial U_a} \right), \tag{33}
\]

\[
\frac{\partial \text{Drag}}{\partial U_a} = 0.5 \rho AC_{D_{\text{max}}} \left( (1 - \cos(2 \alpha_e)) \frac{\partial}{\partial U_a} (V^2) + 2V^2 \sin(2 \alpha_e) \frac{\partial \alpha_e}{\partial U_a} \right), \tag{34}
\]
\[
\frac{\partial \theta}{\partial U_a} = -\frac{\partial \alpha_e}{\partial U_a},
\]
\[
\frac{\partial \alpha_e}{\partial U_a} = -\frac{U_p}{U_p^2 + U_a^2},
\]
\[
\frac{\partial}{\partial U_a} (V^2) = 2U_a,
\]
\[
\frac{\partial U_p}{\partial U_a} = 0.
\]

Next, we calculate the partial derivative of \(f_1\), \(f_2\) and \(g\) w.r.t. input variable \(V_s\), i.e. we compute \(\frac{\partial f_1}{\partial V_s}\), \(\frac{\partial f_2}{\partial V_s}\) and \(\frac{\partial g}{\partial V_s}\):

\[
\frac{\partial f_1}{\partial V_s} = K_2, \quad \frac{\partial f_2}{\partial V_s} = 0, \quad \frac{\partial g}{\partial V_s} = 0.
\]

Finally, we determine the partial derivative of \(f_1\), \(f_2\) and \(g\) w.r.t. input variable \(U_0\). More precisely, we calculate \(\frac{\partial f_1}{\partial U_0}\), \(\frac{\partial f_2}{\partial U_0}\) and \(\frac{\partial g}{\partial U_0}\):

\[
\frac{\partial f_1}{\partial U_0} = 0, \quad \frac{\partial f_2}{\partial U_0} = 2K_4K_3^{-1}|U_a|, \quad \frac{\partial g}{\partial U_0} = 0.
\]

3.1. Tunnel Thruster Test

In this subsection, we determine the linear system obtained by using the parameters used in the tunnel thruster test [3]. The parameters are shown in Table 2.

**Table 2. Parameters for the tunnel thruster test.**

| Parameter   | Value             |
|-------------|-------------------|
| \(C_{L_{\text{max}}}\) | 1.75              |
| \(K_1\)     | 70.15             |
| \(A\)       | 0.00445 m²        |
| \(C_{D_{\text{max}}}\) | 1.2              |
| \(K_2\)     | 1133.2            |
| \(D\)       | 0.0762 m          |
| \(\Delta \beta\) | 0.2              |
| \(K_h\)     | 17.790            |
| \(R\)       | \(D/2\) m        |
| \(\gamma\)  | 0.5               |
| \(K_3\)     | 0.954             |
| \(L\)       | 0.4191 m          |
| \(K_4\)     | 0.910             |
| \(N\)       | 2                 |
| \(\rho\)    | 998 kg/m³        |
| \(p\)       | \(\pi/6\) rad     |

After substituting the parameters to the linearized system, we obtain the following linear system:

\[
\begin{bmatrix}
\dot{\omega}_m \\
\dot{U}_a
\end{bmatrix} = 
\begin{bmatrix}
-70.7 & 1.2 \\
-2.1 & -14.8
\end{bmatrix}
\begin{bmatrix}
\omega_m \\
U_a
\end{bmatrix} + 
\begin{bmatrix}
1133.2 & 0 \\
0 & 1.9
\end{bmatrix}
\begin{bmatrix}
V_s \\
U_0
\end{bmatrix},
\]
\[
T = 
\begin{bmatrix}
-2.03 & -12.2
\end{bmatrix}
\begin{bmatrix}
\omega_m \\
U_a
\end{bmatrix} + 
\begin{bmatrix}
0 & 0
\end{bmatrix}
\begin{bmatrix}
V_s \\
U_0
\end{bmatrix}
\]

3.2. Open-Bladed Thruster Test

In this subsection, we determine the linear system obtained by using the parameters used in the open-bladed thruster test [3]. The parameters are displayed in Table 3.
Table 3. Parameters for the open-bladed thruster test.

| Parameter | Value |
|-----------|-------|
| $C_{L_{\text{max}}}$ | 2     |
| $K_1$     | 10.8  |
| $C_{D_{\text{max}}}$ | 0.5   |
| $K_2$     | 0.65  |
| $\Delta \beta$ | 1.7   |
| $K_h$     | 8333  |
| $\gamma$  | 2.26  |
| $K_3$     | 4.0   |
| $L$       | 0.10 m|
| $K_4$     | 30.0  |
| $N$       | 2     |
| $\rho$    | 998 kg/m$^3$|
| $p$       | $\pi/4$ rad |

Then we substitute the parameters to the linearized system. We obtain the following linear system:

\[
\begin{bmatrix}
\dot{\omega}_m \\
\dot{U}_a
\end{bmatrix} =
\begin{bmatrix}
-523.7 & -1519.8 \\
-1.15 & -0.13
\end{bmatrix}
\begin{bmatrix}
\omega_m \\
U_a
\end{bmatrix} +
\begin{bmatrix}
0.65 & 0 \\
0 & 2.73
\end{bmatrix}
\begin{bmatrix}
V_s \\
U_0
\end{bmatrix},
\]

(43)

\[
T =
\begin{bmatrix}
-4.62 & 10.4
\end{bmatrix}
\begin{bmatrix}
\dot{\omega}_m \\
\dot{U}_a
\end{bmatrix} +
\begin{bmatrix}
0 & 0
\end{bmatrix}
\begin{bmatrix}
V_s \\
U_0
\end{bmatrix},
\]

(44)

4. Conclusions

In this paper, we have linearized the two-state thruster model by using the Jacobian method. We are planning to extend this work to navigation, stability, trajectory estimation and control design of the linearized model.

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