Abstract. Transverse-momentum dependent (TMD) hadronic wave functions develop light-cone divergences under QCD corrections, which are commonly regularized by the rapidity $\zeta$ of gauge vector defining the non-light-like Wilson lines. The yielding rapidity logarithms from infrared enhancement need to be resummed for both hadronic wave functions and short-distance functions, to achieve scheme-independent calculations of physical quantities. We briefly review the recent progress on the rapidity resummation for $B$-meson wave functions which are key ingredients of TMD factorization formulae for radiative-leptonic, semi-leptonic and non-leptonic $B$-meson decays. The crucial observation is that rapidity resummation induces a strong suppression of $B$-meson wave functions at small light-quark momentum, strengthening the applicability of TMD factorization in exclusive $B$-meson decays. The phenomenological consequence of rapidity-resummation improved $B$-meson wave functions is further discussed in the context of $B \to \pi$ transition form factors at large hadronic recoil.

1 Introduction

QCD factorization theorems are indispensable to the theoretical descriptions of high-energy processes probed at worldwide collider experiments, and the predictive power lies on the process-independence of non-perturbative hadronic functions, entering the factorization formulae. $B$-meson light-cone distribution amplitudes (LCDAs) and transverse-momentum dependent (TMD) wave functions are fundamental inputs for QCD calculations of exclusive $B$-meson decays applying collinear factorization and TMD factorization theorems. Great efforts have been devoted to the studies of renormalization properties and perturbative constraints of $B$-meson LCDAs in both momentum space [1-4] and dual space [5, 6], and to the explorations of QCD evolution equations of $B$-meson TMD wave functions [7, 8] with the standard renormalization-group (RG) method and the QCD resummation technique (for a recent review, see [9]).

Providing three-dimensional profile of a given hadron, TMD wave functions are more complicated due to the emergence of light-cone (or rapidity) singularities in the end-point region, which cancel in
the QCD corrections to LCDAs. Such rapidity divergences can be regularized by the rapidity ($\zeta$) of non-light-cone Wilson lines, at the price of generating infrared enhanced double logarithm $\ln^2 \zeta^2$ \cite{[10] [11]} for $B$-meson wave functions and single logarithm $\ln \zeta^2$ \cite{[12] [13]} for pion wave functions. Resummation of these rapidity logarithms in both hadronic wave functions and hard functions are essential to make scheme-independent predictions for physical observables. In the following, we will discuss the construction of rapidity evolution equation for $B$-meson TMD wave functions in the Mellin and impact parameter spaces in section 2 and present the resummation improved $B$-meson wave functions in momentum space by performing the inverse Mellin transformation in section 3 where the resummation effects on $B \to \pi$ transition form factors are also reported.

2 Rapidity Evolution Equation

The TMD wave functions of $B$-meson are defined by the non-local vacuum-to-hadron matrix element in coordinate space \cite{[8]}

$$\langle 0|\bar{q}(y)W_b(n)\gamma_{\mu}P_{\nu,0}|W_0(n)\Gamma h(0)|B(v)\rangle$$

$$= -\frac{i f_{BM_B}}{4} Tr \left\{ \frac{1 + y^2}{2} \left[ 2 \Phi_B^+(t, y^2) - \Phi_B^+(t, y^2) \right] \gamma_5 \right\},$$

which depends on longitudinal ($t = v \cdot y$) and transverse ($y^2$) variables as well as the non-light-like gauge vector $n$ defining the Wilson lines.

It is demonstrated in \cite{[10]} that both the double rapidity logarithm $\ln^2 \zeta^2$ with $\zeta^2 = 4(v \cdot n)^2/n^2$, due to the overlap of the collinear enhancement from a loop momentum $l$ collimated to the gauge vector $n$ and the soft enhancement, and the mixed logarithm $\ln \mu_f \ln \zeta^2$ ($\mu_f$ being the renormalization scale) which has the same origin of cusp divergence in $B$-meson LCDAs \cite{[1]} appear in the next-to-leading-order (NLO) QCD corrections to $B$-meson wave functions. Simultaneous resummation of two distinct types of rapidity logarithms for $B$-meson wavefunctions makes it subtler than that of the traditional Sudakov resummation for fast-moving light hadrons (see, however, \cite{[14]} for recent progress).

Recalling that the Sudakov resummation of $\ln^2 \zeta^2$ for pion wave function can be achieved by constructing the evolution equation from varying the gauge vector $n$, because the collinear divergence arises from the region with a loop momentum collimated to the pion momentum and varying the vector $n$ does not result in an additional collinear divergence. This trick however does not apply to the resummation of $B$-meson wave function due to the nature of rapidity logarithm explained in the above. To resum the rapidity logarithm for $B$-meson wave functions, we utilize the fact that the resulting collinear divergence is insensitive to the heavy-quark velocity $v$ and the effect from varying $v$ can then be factorized from the TMD wave functions. Trading the rapidity derivative for the differential of velocity yields

$$\zeta^2 \frac{d}{d \zeta^2} \Phi_B^{(b)}(x, k_T, \zeta^2, \mu_f) = \frac{v \cdot n}{2\epsilon_{ab}v^a\nu^b} v^+ \frac{d}{dv^+} \Phi_B^{(b)}(x, k_T, \zeta^2, \mu_f),$$

where $\epsilon_{ab}$ is an anti-symmetric tensor with $\epsilon_{+-} = -\epsilon_{-+} = 1$, $v^+$ is the plus component of the $b$-quark velocity and $x$ is the longitudinal momentum fraction of the light quark. Here, the crucial point is that the velocity derivative will be only applied to the Feynman rules involving an effective heavy $b$-quark

$$\frac{v \cdot n}{2\epsilon_{ab}v^a\nu^b} v^+ \frac{d}{dv^+} v^i \cdot v \cdot l = \partial^i \varepsilon_{ij} l^j,$$

inducing the special vertex

$$\partial^i \equiv \frac{v \cdot n}{2\epsilon_{ab}v^a\nu^b} \epsilon_{abcd} t^d \left( g^{a1} - \frac{i\nu^1}{v \cdot l} \right).$$
The rapidity evolution equation of $B$-meson wave function can then be written as
\[
\zeta^2 \frac{d}{d\zeta^2} \Phi_B(x, k_T, \zeta^2, \mu_f) = K^{(b,1)} \otimes \Phi_B(x, k_T, \zeta^2, \mu_f) - \frac{1}{Z_\Phi} \left( \frac{\alpha_s}{\lambda} \Phi_B(x, k_T, \zeta^2, \mu_f) \right),
\]

where $Z_\Phi$ is the counterterm for the ultraviolet renormalization of TMD wave function. The one-loop kernel $K^{(b,1)}$ collects the soft gluon dynamics as displayed in figure 1.

Computing the two effective diagrams yields
\[
K^{(b,1)}_1 = -\frac{\alpha_s C_F}{4\pi} \Gamma(\epsilon) \left( \frac{4\pi \mu_f^2}{\lambda^2} \right)^\epsilon \left( \frac{v \cdot n}{\epsilon^{\alpha_\beta} u^\alpha n^\beta} \right)^2,
\]
\[
K^{(b,1)}_2(N, b, \zeta^2) = \frac{\alpha_s C_F}{2\pi} \left( \frac{v \cdot n}{\epsilon^{\alpha_\beta} u^\alpha n^\beta} \right)^2 \left[ K_0(\lambda b) - K_0 \left( \sqrt{\frac{m_B b}{N}} \right) \right],
\]

where the gluon mass $\lambda$ is introduced to regularize the infrared divergence in each diagram, Mellin and Fourier transformations for the $B$-meson wave function are performed in the evaluation of irreducible contribution illustrated in the second diagram. It is then straightforward to derive
\[
\zeta^2 \frac{d}{d\zeta^2} \tilde{\Phi}_B(N, b, \zeta^2, \mu_f) = \tilde{K}^{(1)}(N, b, \zeta^2, \mu_f) \tilde{\Phi}_B(N, b, \zeta^2, \mu_f),
\]

with the renormalized evolution kernel
\[
\tilde{K}^{(1)}(N, b, \zeta^2, \mu_f) = -\frac{\alpha_s C_F}{2\pi} \left[ \ln \frac{\mu_f b}{2} + \gamma_E + K_0 \left( \sqrt{\frac{m_B b}{N}} \right) \right],
\]

where the infrared divergence cancels exactly between the two effective diagrams. The single logarithm of soft kernel $\tilde{K}^{(1)}$ can be organized by the RG resummation
\[
\mathcal{K}^{(1)}(N, b, \zeta^2, \mu_f) = \tilde{K}^{(1)}(N, b, \zeta^2, \mu_f) - \int_{\mu_c}^{\mu_f} \frac{d\mu}{\mu} A_K(\alpha_s(\mu)) \theta(\mu_f - \mu_c),
\]

where the scale $\mu_c = a \sqrt{\zeta^2 m_B / N}$, $a$ being an order-unity constant, is adjusted to diminish the logarithmic enhancement in the initial condition.

In addition, the factorization scale evolution of TMD wave function itself is computed as
\[
\tilde{\Phi}_B(N, b, \zeta^2, \mu_f) = \exp \left[ - \int_{\mu_0}^{\mu_f} \frac{d\mu}{\mu} \frac{\alpha_s(\mu)}{2\pi} C_F \left( \ln \zeta^2 - 2 \right) \right] \tilde{\Phi}_B(N, b, \zeta^2, \mu_0).
\]
The normalization conditions summarized as follows: to reduce the numerical analysis. The longitudinal parts are further taken from the so-called “free-parton” model [15] transparent, and then with running functions\(\hat{\Phi}_B(N, b, \zeta_0^2, \mu_0)\),

\[
\hat{\Phi}_B(N, b) = \exp \left[ \int_{\zeta_0}^{N\bar{z}_0^2} \frac{d\bar{z}^2}{\bar{z}^2} \mathcal{K}^{(1)}(N, b, \zeta_0^2, \mu_f) - \int_{\mu_0}^{\mu_f} \frac{d\mu}{\mu} \frac{\alpha_s(\mu)}{2\pi} C_F \left( \ln \zeta_0^2 - 2 \right) \right] \times \hat{\Phi}_B(N, b, \zeta_0^2, \mu_0),
\]

with the simplified rapidity kernel

\[
\mathcal{K}^{(1)}(N, b, \zeta_0^2, \mu_f) = -\frac{\alpha_s(\mu_c)}{2\pi} C_F \ln a - \int_{\mu_c}^{\mu_f} \frac{d\mu}{\mu} \frac{\alpha_s(\mu)}{2\pi} C_F \theta(\mu_f - \mu_c)
\]

which is valid in the large \(N\) limit.

### 3 Resummation Improved \(B\)-meson TMD Wave Functions

We are now in a position of discussing the resummation effect on \(B\)-meson wave functions in \(x\)-space by performing the inverse Mellin transformation

\[
\Phi_B^e(x, k_T) = \int_{c-\text{i}\infty}^{c+\text{i}\infty} \frac{dN}{2\pi i} (1 - x)^{-N} \hat{\Phi}_B^e(N, k_T),
\]

for the solution given by Eq. (12). To achieve this purpose, we need to know the initial conditions of TMD wave functions which are postulated to have the factorized form

\[
\Phi_B^e(x, k_T, \zeta_0^2) = \Phi_B^e(x, \zeta_0^2) \phi(k_T),
\]

to reduce the numerical analysis. The longitudinal parts are further taken from the so-called “free-parton” model [15]

\[
\phi_B^e(x, \zeta_0^2) = \frac{x}{2x_0^2} \theta(2x_0 - x) \quad \phi_B^e(x, \zeta_0^2) = \frac{2x_0 - x}{2x_0^2} \theta(2x_0 - x),
\]

which in Mellin space correspond to

\[
\tilde{\phi}_B^e(N, k_T, \zeta_0^2) = \frac{1 - (1 - 2x_0)^N}{2x_0^2 N(N + 1)} \quad \tilde{\phi}_B^e(N, k_T, \zeta_0^2) = \frac{(1 - 2x_0)^{N+1} + 2x_0 N + 2x_0 - 1}{2x_0^2 N(N + 1)}.
\]

The inverse Mellin transformation will be firstly performed for the resummation improved wave functions \(\Phi_B^e(N, k_T)\) with frozen strong coupling constant \(\alpha_s\), to make the analytical behavior more transparent, and then with running \(\alpha_s\). The resummation effects on the \(x\) dependence of \(B\)-meson wave functions \(\phi_B^e(x) = \Phi_B^e(x, k_T) / \phi(k_T)\) are illustrated in figure [2]. The primary observations are summarized as follows:

- Faster than linear attenuation of \(\phi^e(x)\) in the small \(x\) region is realized after the resummation improvement, albeit with the non-vanishing \(\phi^e(x)\) at \(x = 0\) initially.
- Resummation improved \(B\)-meson wave functions become smooth and develop radiative tails.
- The normalization conditions \(\int_0^1 dx \phi_B^e(x)(x) = 1\) are unaffected by the rapidity resummation.
It will be interesting to inspect the rapidity resummation effect for phenomenological observables which are relevant for the precision test of CKM mechanism (for recent reviews, see [16, 17]) and for the hunting of beyond standard model physics. In this respect, we consider the $B \to \pi$ transition form factors $f_{B \pi}^{\pm}(q^2)$ at large hadronic recoil as illuminative examples, which are essential to the golden-channel determination of matrix element $|V_{ub}|$ exclusively. Numerically the resummation effects are found to decrease the above two form factors by approximately 25% at $q^2 = 0$, due to the strong suppression of $B$-meson wave functions at the end-point. Notice that such improvement is sizeable and must be taken into account in the future calculations of $B \to \pi$ form factors applying TMD factorization theorem to catch up with the precision achieved in the calculations from light-cone QCD sum rules [18, 19].

4 Conclusion

Applying the QCD resummation technique with non-light-like Wilson lines, we construct rapidity evolution equation for TMD wave functions of $B$-meson collecting the double logarithm $\ln^2 \zeta^2$ and mixed logarithm $\ln \mu_f \ln \zeta^2$. Technically, the rapidity resummation of $B$-meson wave functions is novel due to the different nature of collinear divergence in QCD correction from that of energetic light mesons. The resummation improved $B$-meson wave functions induce a strong suppression in the small $x$ region and hence enhance the applicability of TMD factorization in exclusive $B$-meson decays. Sizeable corrections to the $B \to \pi$ transition form factors are also observed due to the resummation improvement. Many open questions concerning the TMD wave functions of $B$-meson remain to be answered, and these include (a) What are the relations between TMD wave functions and LCDAs at large transverse momentum? (b) What are the operator-product-expansion constraints of TMD wave functions? (c) Can we gain some insights of TMD wave functions at low energy scale from non-perturbative approaches? As a final remark, the rapidity resummation technique presented here can be generalized immediately to the TMD wave functions of $\Lambda_{b^-}$ baryon entering the factorization formulae of many exclusive decays [20, 22] which are of increasing interest at the LHC and Tevatron.
The Journal’s name

Acknowledgements

We are grateful to Hsiang-nan Li for a very fruitful collaboration. YMW would like to thank the organizers of QCD@work 2014 for inviting him to this stimulating workshop and for the generous financial support. This research is also partially supported by the National Science Foundation of China under Grant No. 11005100 (YLS), and by the DFG Sonderforschungsbereich/Transregio 9 “Computergestützte Theoretische Teilchenphysik” (YMW).

References

[1] B. O. Lange and M. Neubert, Nucl. Phys. B 690, 249 (2004) [Erratum-ibid. B 723, 201 (2005)] [hep-ph/0311345].
[2] V. M. Braun, D. Y. Ivanov and G. P. Korchemsky, Phys. Rev. D 69, 034014 (2004) [hep-ph/0309330].
[3] S. J. Lee and M. Neubert, Phys. Rev. D 72, 094028 (2005) [hep-ph/0509350].
[4] G. Bell and T. Feldmann, JHEP 0804, 061 (2008) [arXiv:0802.2221 [hep-ph]].
[5] G. Bell, T. Feldmann, Y. M. Wang and M. W. Y. Yip, JHEP 1311, 191 (2013) [arXiv:1308.6114 [hep-ph]].
[6] T. Feldmann, B. O. Lange and Y. M. Wang, Phys. Rev. D 89, 114001 (2014) [arXiv:1404.1343 [hep-ph]].
[7] H. n. Li and H. S. Liao, Phys. Rev. D 70, 074030 (2004) [hep-ph/0404050].
[8] H. N. Li, Y. L. Shen and Y. M. Wang, JHEP 1302, 008 (2013) [arXiv:1210.2978 [hep-ph]].
[9] H. n. Li, Phys. Part. Nucl. 45, no. 4, 756 (2014) [arXiv:1308.0413 [hep-ph]].
[10] H. n. Li, Y. L. Shen and Y. M. Wang, Phys. Rev. D 85, 074004 (2012) [arXiv:1201.5066 [hep-ph]].
[11] S. Cheng, Y. Y. Fan, X. Yu, C. D. Lü and Z. J. Xiao, Phys. Rev. D 89, 094004 (2014) [arXiv:1402.5501 [hep-ph]].
[12] H. n. Li, Y. L. Shen, Y. M. Wang and H. Zou, Phys. Rev. D 83, 054029 (2011) [arXiv:1012.4098 [hep-ph]].
[13] S. Cheng, Y. Y. Fan and Z. J. Xiao, Phys. Rev. D 89, 054015 (2014) [arXiv:1401.5118 [hep-ph]].
[14] H. N. Li, Y. L. Shen and Y. M. Wang, JHEP 1401, 004 (2014) [arXiv:1310.3672 [hep-ph]].
[15] H. Kawamura, J. Kodaira, C. F. Qiao and K. Tanaka, Phys. Lett. B 523, 111 (2001) [Erratum-ibid. B 536, 344 (2002)] [hep-ph/0109181].
[16] R. Aaij et al. [LHCb Collaboration], Eur. Phys. J. C 73, 2373 (2013) [arXiv:1208.3355 [hep-ex]].
[17] Wei Wang, Int. J. Mod. Phys. A 29, no. 20, 1430040 (2014) [arXiv:1407.6868 [hep-ph]].
[18] A. Khodjamirian, T. Mannel, N. Offen and Y.-M. Wang, Phys. Rev. D 83, 094031 (2011) [arXiv:1103.2655 [hep-ph]].
[19] A. Bharucha, JHEP 1205, 092 (2012) [arXiv:1203.1359 [hep-ph]].
[20] C. H. Chou, H. H. Shih, S. C. Lee and H. n. Li, Phys. Rev. D 65, 074030 (2002) [hep-ph/0112145].
[21] X. G. He, T. Li, X. Q. Li and Y. M. Wang, Phys. Rev. D 74, 034026 (2006) [hep-ph/0606025].
[22] C. D. Lü, Y. M. Wang, H. Zou, A. Ali and G. Kramer, Phys. Rev. D 80, 034011 (2009) [arXiv:0906.1479 [hep-ph]].