The equivalence Vitali cover integral with Lebesgue integral

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Abstract. Collection interval \( V \) is called Vitali Cover measurable set up \( A \subset \mathbb{R} \), if for every number \( \varepsilon > 0 \) and \( x \in A \) is an open interval \( I_x \in V \) so that \( x \in I_x \) and \( \mu(I_x) < \varepsilon \). By utilizing Vitali Cover can be constructed of a type of integral type, hereinafter referred Riemann Vitali Cover integral. A Function \( f : [a, b] \rightarrow \mathbb{R} \) is said to Lebesgue integrable if and only if there is a continuous function \( F : [a, b] \rightarrow \mathbb{R} \) so that \( F \) absolutely continuous on \([a, b]\) and

\[ F'(x) = f(x) \] almost everywhere in \([a, b]\). By using the properties of Vitali Cover can be shown that both types of the above integral are equivalent.

1. Introduction
Mathematics consist of a few study groups, one of them is analysis group. Integral theory is one of the analysis group that is study which is deductive and still grow well from theoritic side or application side [1]. It is shown that a Lebesgue integrable function comes equipped with a sequence of points which one can use in conjunction with a simple “first return-Riemann” integration procedure to compute the integral [2].

As the development of integral theory, until this time there are many kinds of integral which have been success to be organized by the expert. Each of the kind of integral is organized by using the different approachment and all of the kind of integral have a relationship to each other. In the jurnal of “Real Analysis Exchange”, has written a kind of integral, Riemann type in a strait line, which is continuously called Vitali integral or written as integral-

2. Methods
This research began with literature study. Because of that, material used is a few of research paper from the expert which publicate in seminar or written in jurnal, bulletin, or book. The way of research is done by studying and analyzing the research papers which have been collected. In other hand, the result of this research is done in the form of definition and theory which is supported by proof [1].
3. Vitali covers integral
In this part, Vitali Cover integral and its characteristic will be constructed. Definitions and theory used in this paper cited on “Absolute Integration Using Vitali Cover”, [1, 3] and before that, the definition of Vitali Cover and Vitali Lemma with its characteristic will be rememorized [3, 5].

I is free interval to the \( \mathbb{R} \), applies \( \mu (I) = |I| = \mu (I^\circ) = |I^\circ| \) with \( \mu \) Lebesgue value and \( I^\circ = \text{interior } I \) [4, 9]

3.1 Definition
Interval collection \( \nu \) is called Vitali Cover the set is measured by \( A \subset \mathbb{R} \), if for each number \( \varepsilon > 0 \) and \( x \in A \) there is opened interval \( I_x \in \nu \) so \( x \in I_x \) and \( \mu (I_x) < \varepsilon \) [1, 3, 9]

3.2 Lemma
If \( \nu \) Vitali Cover, a group is measured limited to \( A \) so for each number \( \varepsilon > 0 \) there is Vitali based interval which the amount is until: \( I_{x_1}, I_{x_2}, \ldots, I_{x_n} \in \nu \), so

\[
\mu \left( A - \bigcup_{j=1}^{n} I_{x_j} \right) < \varepsilon
\]  

(1)

Direct effect of lemma 1 is if \( \nu \) Vitali Cover of closed interval limited to \([a,b] \) for each number \( \varepsilon > 0 \) there is part of cover \( \Delta = \{ I_{x_1}, I_{x_2}, \ldots, I_{x_n} \} \) and \( \mu([a,b]) - \left( \bigcup_{j=1}^{n} I_{x_j} \right) < \varepsilon \). This condition could be considered as \( x_1 < x_2 < \ldots < x_n \) and after that

\[
P = \left\{ I_{x_1}, I_{x_2}, \ldots, I_{x_n} \right\}
\]

is called Vitali partition or partition-\( V \) on \([a,b]\). It is need to give attention for the next if \( \nu \) Vitali Cover of closed interval \([a, b]\) and \( P = \{[u, v], x\} \) partition- \( \nu \) which is related and measured by number \( \varepsilon > 0 \) same to the explanation above so:

\[
(b-a) - P \sum (v-u) = (b-a) - \sum_{i=1}^{n} (v_i - u_i) < \varepsilon \quad \text{and} \quad [u_i, u_{i+1}] \cap [u_{i+1}, u_{i+1}] \text{ for an } i \text{ maybe as an empty set.} \]  

(2)

[5, 7, 11]

3.3 Corollary
If \( \nu \) Vitali Cover interval to \([a,b]\) so for each number \( \varepsilon > 0 \) there is \( P = \{[u, v], x\} \) Vitali partition to \([a,b]\) with characteristic of (2). Vitali Lemma dan especially the corollary of (1) has guarantee the existence of partition- \( \nu \) to the interval \([a,b]\) which is related to Vitali Cover \( \nu \) interval \([a,b]\). Because of that, it can be organized as integral same like this:
3.4 Definition
Measured function $f: [a, b] \rightarrow \mathbb{R}$ is called Vitali-integrated ($V$-integrated) or written as $f \in V[a, b]$ if there is number of $A$ so for each number $\varepsilon > 0$ there is $V$ Vitali Cover interval $[a, b]$ and number $\eta > 0$ so for each partition-$V$ part to $[a, b]$ $P = \{[u, v], x\}$ with $b - a - P (v - u) < \eta$ causes $|A - P f(x)(v - u)| = |A - P f(x)(v - u)| < \varepsilon$

$A$ is called Vitali integral value or integral-$V$ function $f$ to $[a, b]$ [7, 11]

Theorem 3.5
If function $f$ is Vitali-integrated to $[a, b]$, so number $A$ which is mentioned in the definition is singular. For the next, if $f \in V[a, b]$ and the integral value $A$, is written as $A = (V) f$ [11].

Theorem 3.6
$\forall a, b \in [a, b]$ is linier area. Furthermore $(V) \int_a^b \left( a_1 f + a_2 g \right) = a_1 \int_a^b f + a_2 \int_a^b g$

Theorem 3.7
If $f \in V[a, b]$ if and if only for each number $\varepsilon > 0$ there is number $\eta > 0$ and Vitali Cover $V$ interval to $[a, b]$ so if $P_1 = \{[u, v] : x\}$ and $P_2 = \{[u, v] : x\}$ partition-$V$ is part to $[a, b]$ with $b - a - P \sum (v - u) < \eta$ ($i = 1, 2$) apply: $|P_1 \sum f(x)(v - u) - P_2 \sum f(x)(v - u)| < \varepsilon$

Theorem 3.8
If $f \in V[a, b]$ and $[a, b] \supset [c, d]$, so $f \in V[c, d]$ [11].

Corollary 3.9
If $f \in V[a, b]$ so $f \in V[a, x]$ for each $x \in [a, b]$ [1, 11].

If the value of function is real $f$ where $V$-integrated to $[a, b]$, so according to the Corollary of (4), $f$ is also $V$-integrated to $[a, x]$ for each $x \in [a, b]$. Following is given the definition:

Definition 3.10
If $f \in V[a, b]$ is mentioned as interval function $F : [a, b] \rightarrow \mathbb{R}$ with the formula: $F(x) = (V) \int_a^x f$ for each $x \in (a, b]$ and $F(a) = 0$

Function of $F$ in definition above is called $V$-primitive function $f$ to $[a, b]$. [1, 3]

Based on this definition of a primitive function, the result given is $F(b) = (V) \int_a^b f$ so $F(b) - F(a) = (V) \int_a^b f$

In addition, $F(b) - F(a)$ is noticed with $F(a, b)$ . If $P = \{[u, v] : x\}$ $V$-partition to $[a, b]$ with $b - a - P \sum (v - u) < \eta$ for each number $\eta > 0$ and $V$ Vitali Cover interval to $[a, b]$ so $F(a, b) = P \sum F(u, v)$, as a result:
The 2-nd International Seminar on Science and Technology 2020 (ISST-2) 2020
Journal of Physics: Conference Series 1763 (2021) 012084 doi:10.1088/1742-6596/1763/1/012084

So, if \( f \in V[a,b] \) with \( F \) primitive so for each number of \( \varepsilon > 0 \) there is number of \( \eta > 0 \) and \( V \) Vitali Cover interval to \([a,b]\) so if \( P = \{[u,v];x\} \) \( V \)-Vitali partition part to \([a,b]\) with \( b - a - \sum (v-u) < \eta \) apply: \( \sum \{F(u,v) - f(x)(v-u)\} < \varepsilon \)

**Theorem 3.11**

If \( a < c < b \) and \( f \in V[a,b] \), then \( (V)^{\frac{b}{a}} f = (V)^{\frac{c}{a}} f + (V)^{\frac{b}{c}} f \)

In the following will be given a few characteristic of Vitali integral which relates to its primitive function

[11, 13]

**Lemma 3.12**

If function \( f \) is \( V \)-integrated to \([a,b]\) with \( F \) as its \( V \)-primitive, so for each number \( \varepsilon > 0 \) there is complete full \( \text{Cover} \) to \([a,b]\) so for each \( \text{partition} \ \Sigma_i \) from \( \Sigma \)

**Theorem 3.13**

If \( f \in V[a,b] \) with \( F \) as its \( V \)-primitive, so \( F(x) = f(x) \) almost every where on \([a,b]\). [11]

4. Relation of the Vitali integral to the Lebesgue integral

This section will discuss the relationship between the Vitali integral and the Lebasque integral as outlined in the theorem

4.1 Theorem

If function \( f \) integrated to Lebesgue in \([a,b]\) so \( f \) is Vitali integrated to \([a,b]\) and

\[
(V)^{\frac{b}{a}} f = (L)^{\frac{b}{a}} f = F([a,b])
\]

Proof:

Because function \( f \) is Lebesgue integrated to \([a,b]\), so \( f \) is Henstock integrated to \([a,b]\) and

\[
(L)^{\frac{b}{a}} f = (H)^{\frac{b}{a}} f = F([a,b])
\]

\( F \) as the function \( f \) primitive

So, for each number \( \varepsilon > 0 \) there is complete full \( \text{Cover} \ \Delta \) to \([a,b]\) so for each \( \Delta \)- partition \( P = \{[u,v];x\} \) causes

\[
|H^{\frac{b}{a}} f - \sum f(x)(v-u)| < \varepsilon
\]
Need to be memorized that $\Delta$ is also Vitali Cover.

Have known that function $F \in AC[a, b]$, so for each number $\varepsilon > 0$ there is number $\eta > 0$. As a result, if $P' = \{I\}$ interval line that is not overlapping, its edge is member of $[a, b]$ and $P' \sum |I| < \eta$ causes $P' \sum |F(I)| < \varepsilon$

As a result, each $\Delta$-partition (Vitali partition) $P = \{[u, v] ; x\}$ to $[a, b]$ with $b - a - P\sum(v-u) < \eta$ causes

$$|P\sum F(u,v) - f(x)(v-u)| \leq P\sum|F(u,v) - f(x)(v-u)| + P' \sum[F(u,v)]$$

$$< \varepsilon + \varepsilon = 2\varepsilon$$

with $P'$ as the complement of $\Delta$-partition $P$ above that $P' \sum|(v-u)| < \eta$

4.2. Theorem

If $f \in V[a, b]$ with $F$ as its $V$-primitive, so $F'(x) = f(x)$ almost everywhere to $[a, b]$. [3, 11]

Proof:

$F$ $V$-primitive function $f \in V[a, b]$ if and if only for each number $\varepsilon > 0$ there is number $\delta > 0$ and Vitali Cover $\nu_0$ interval to $[a, b]$ so if $P = \{[u, v] ; x\}$ partition $\nu_0$ is part to $[a, b]$ with $b - a - P\sum(v-u) < \eta$ causes

$$\left|P\sum\left(F(\bar{I}) - f(x)\bar{I}\right)\right| < \frac{\varepsilon}{2}$$

Made a set of

$$X = \{x \in [a, b] : F'(x) \text{ nothing but } F'(x) \neq f(x) \text{ or } F'(x) \text{ no}\}$$

Proof finish if it is success to provide that $\mu(X) = 0$.

It does not decrease the meaning if it considered as $a, b \notin X$.

The author takes free $x \in X$ which means $\eta(x) > 0$ so for each Vitali Cover $V$ interval to $[a, b]$ there is $I \in \mathcal{V}$ so $x \in I \subset [a, b] \text{ apply}$

$$|F(u,v) - f(x)(v-u)| \geq (v-u)\eta(x)$$

or

$$|F(x) - f(x)| I \geq I\eta(x)$$

The set of $X$ is separated to part of sets as following. For each $n \in \mathbb{N}$ made

$$X_n = \left\{x : x \in X \text{ dan } \eta(x) \geq \frac{1}{n}\right\}$$

It is clear that $X_1 \subset X_2 \subset \ldots \text{ and } X = \bigcup_{n=1}^{\infty} X_n$

Taken $n$ (same) made:

$$V_n = \{I : I \in \mathcal{V} \cap V_0 \text{ dan } x \in X_n \cap I\}$$
Easy to understand that $V_n$ Vitali Cover is set of $X_n$. Based on Vitali Lemma for each $\varepsilon > 0$ there is a collection till of interval couple and a point that the till amount $(I_1,x_1),(I_2,x_2),\ldots,(I_n,x_n) \in V_n$ so,

$$\mu(X_n) \leq \sum_{i=1}^{n} |I_i| + \varepsilon$$

This condition causes

$$\mu(X_n) \leq \sum_{i=1}^{n} \left[ \frac{|F(I_i) - f(x_i)|}{\eta(x_i)} \right] + \varepsilon$$

$$< \varepsilon \frac{1}{\eta(x_i)} + \varepsilon$$

$$= ne + \varepsilon$$

Because of it is applied for each $\varepsilon > 0$ so $\mu(X_n) = 0$ and causes $\mu(X) = \sum_{n=1}^{\infty} \mu(X_n) = 0$

In other words, $F'(x)$ is almost everywhere to $[a,b]$ and $F'(x) = f(x)$ [4, 5, 11]

4.3 Theorem

If $f \in V[a,b]$, so $f$ is Lebesgue integrated to $[a,b]$ with $(L)^{\frac{b}{a}} f = (V)^{\frac{b}{a}} f$

Proof:

Because $f \in V[a,b]$ so function $F$ with

$$F(x) = \begin{cases} (V)^{\frac{b}{a}} f & \text{if } x \in (a,b) \\ 0 & \text{if } x = a \end{cases}$$

Have the characteristic of $F'(x) = f(x)$ which is almost everywhere to $[a,b]$ and $F \in AC [a,b]$.

This condition tells that $f$ is Lebesgue integrated to $[a,b]$ and $(L)^{\frac{b}{a}} f = (V)^{\frac{b}{a}} f$ [1, 3]

From a few last theories, which is Theorem 4.2 and Theorem 4.3, produced

4.4 Corollary

Function $f : [a,b] \to \mathbb{R}$ is Lebesgue integrated to $[a,b]$ if and if only $f \in V[a,b]$ and $(L)^{\frac{b}{a}} f = (V)^{\frac{b}{a}} f$

5. Conclusion

By utilizing the Vitali Cover, can be organized an integral of Riemann type on a straight line which is continuously called Vitali Integral. If Vitali cover used has specific characteristics, that is filtering properties and $\varepsilon$-fine properties, can be seen that Vitali integral is equivalence with Lebesgue integral and has the same integral value.

Acknowledgements

The author would like to thank for the funding support from the dean of the Teaching and Education Faculty of Tadulako University through DIPA funds.

References

[1] Soeparna D 1995 *Integral Dengan Liput Vitali* (Yogyakarta: Fakultas Matematika dan Ilmu Pengetahuan Alam UGM)
[2] Darji U B and Evans M J 2001 Real Analysis Exchange 27(2) 573-582
[3] Lee P Y and Seng C T 1993 18 (2) 409-419
[4] Wittaya N I 1976 Nonabsolute Integration On The Real Line (Philippines: University Of The Philippines)
[5] Yee L P 1989 Lanzhou Lectures On Henstock Integration (Singapore: World Scientific)
[6] Gordon R A 1994 The Integrals of Lebesgue, Denjoy, Perron, and Henstock (American: American Mathematical Society)
[7] Bruckner A M, Bruckner J B and Thomson B S 1997 Real Analysis (American: Prentice-Hall)
[8] Faure C A 29 (2) 947–951
[9] Federer H 1969 Geometric Measure Theory (Berlin: Springer-Verlag)
[10] Hagood J 2004 29 (05) 953–956
[11] Saks S 1937 Theory of the Integral (India: Osmania University)