Stability of Hill’s spherical vortex

Kyudong Choi

Department of Mathematical Sciences, Ulsan National Institute of Science and Technology, Ulsan, South Korea

Correspondence
Kyudong Choi, Ulsan National Institute of Science and Technology, Ulsan, South Korea.
Email: kchoi@unist.ac.kr

Funding information
NRF, Grant/Award Number: 2018R1D1A1B07043065; UNIST

Abstract
We study stability of a spherical vortex introduced by M. Hill in 1894, which is an explicit solution of the three-dimensional incompressible Euler equations. The flow is axi-symmetric with no swirl, the vortex core is simply a ball sliding on the axis of symmetry with a constant speed, and the vorticity in the core is proportional to the distance from the symmetry axis. We use the variational setting introduced by A. Friedman and B. Turkington (Trans. Amer. Math. Soc., 1981), which produced a maximizer of the kinetic energy under constraints on vortex strength, impulse, and circulation. We match the set of maximizers with the Hill’s vortex via the uniqueness result of C. Amick and L. Fraenkel (Arch. Rational Mech. Anal., 1986). The matching process is done by an approximation near exceptional points (so-called metrical boundary points) of the vortex core. As a consequence, the stability up to a translation is obtained by using a concentrated compactness method.

CONTENTS
1. INTRODUCTION .................................... 53
2. PRELIMINARIES .................................... 62
3. VARIATIONAL PROBLEM WITH COMPACTNESS AND UNIQUENESS ..... 74
4. EXISTENCE AND PROPERTIES OF MAXIMIZERS .......................... 79
5. STEADY VORTEX RINGS FROM MAXIMIZERS ........................... 90
6. COMPACTNESS ..................................... 105
7. UNIQUENESS OF HILL’S VORTEX .................................. 121
Acknowledgments ........................................ 128
References ............................................. 128
APPENDIX A: PROOF OF LEMMA 4.9. ........................... 132
1 | INTRODUCTION

1.1 | Hill’s spherical vortex: Hill (1894)

The three-dimensional incompressible Euler equations are written by

\[ \partial_t u + (u \cdot \nabla) u + \nabla P = 0, \]
\[ \text{div} u = 0, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (1.1) \]

where \( u(x,t) \in \mathbb{R}^3 \) is the fluid velocity and \( P(x,t) \in \mathbb{R} \) is the pressure. The Hill’s spherical vortex, which was discovered in 1894 [61], represents an axi-symmetric flow without swirl whose compactly supported vorticity is proportional to the distance from the symmetry axis. The vortex core, which means the support of the vorticity, is an unit ball which slides the axis in a constant speed forever without changing its shape or size. More precisely, we write the Euler equations in vorticity vector \( \omega \) form

\[ \partial_t \omega + (u \cdot \nabla) \omega = (\omega \cdot \nabla) u, \quad x \in \mathbb{R}^3, \quad t > 0. \quad (1.2) \]

Here, the fluid velocity \( u \) can be recovered from its vorticity \( \omega \) via the 3d Biot-Savart law \( u = \nabla \times (-\Delta)^{-1} \omega \), which makes the fluid at rest at infinity for compactly supported and bounded vorticities. When the velocity of a flow is axi-symmetric without swirl, the vorticity admits its angular component \( \omega^\theta \) only. By choosing the \( x_3 \)-axis as the axis of symmetry and by setting the relative vorticity \(^1\xi\) (in the cylindrical coordinates)

\[ \xi(r,z) = \frac{\omega^\theta(r,z)}{r}, \quad r = \sqrt{x_1^2 + x_2^2}, \quad z = x_3, \]

the symmetry transforms (1.2) into the active scalar equation

\[ \partial_t \xi + u \cdot \nabla \xi = 0, \quad x \in \mathbb{R}^3, \quad t > 0, \quad (1.3) \]

where the axi-symmetric velocity is determined by the axi-symmetric Biot-Savart law \( u = \mathcal{K}[\xi] \) introduced later in (2.5). In this setting, the Hill’s vortex \( \xi_H \) is simply defined by

\[ \xi_H(x) = 1_B(x), \quad (1.4) \]

where \( B \) is the unit ball in \( \mathbb{R}^3 \) centered at the origin. It has been well-known that

\[ \xi(t,x) = \xi_H(x + tu_\infty) = 1_B(x + tu_\infty) \]

\(^1\)In this paper, we use the terminology “relative vorticity”, which appeared in [87], even if it is not standard.
is a traveling wave solution of (1.3) where \( u_\infty \) is the constant velocity

\[
u_\infty = -W_H e_3, \quad W_H = (2/15).
\]

It produces the unique weak solution \( u(t) = \mathcal{K}[\xi(t)] \) of (1.1). The velocity \( u(t) \) lies on \( C^{\alpha}(\mathbb{R}^3) \), \( 0 < \alpha < 1 \) because the corresponding vorticity vector \( \omega = (r \xi) e_\theta \) lies on \( (L^1 \cap L^\infty)(\mathbb{R}^3) \). For more detail, we refer to Section 2 in this paper or the original paper [61], the classical textbooks [7, 71], the modern textbook [96]. In fluid mechanics, it is important to study such a localized vortex moving without changing shape or size because it might help to explain transport of mass, momentum and energy in large scale at a flow of high Reynolds number.

In this paper, we are interested in stability of the Hill’s vortex in axi-symmetric perturbations. Since such a traveling vortex can be easily observed experimentally, for example, when an ink is dropped in another fluid [57], smoke is ejected from a tube [109, p. 44], or a bubble rises in a liquid [70], it is natural to expect (or ask questions on) its stability in longer times. Expecting that the vortex (1.4) maximizes the kinetic energy among other axi-symmetric patch-type functions \( \xi \) having the same impulse condition

\[
\int_{\mathbb{R}^3} r^2 \xi dx = \int_{\mathbb{R}^3} r^2 \xi_H dx,
\]

Benjamin [12, Section I] in 1976 suggested variational principles for a broad class of steady vortex rings and inferred their stability up to a translation. Saffman also suggested in his textbook [96, footnote in p. 25] that one can employ conservation of mass and momentum to produce nonlinear stability in an \( L^1 \) and \( L^2 \) norm. However, to the best of our knowledge, there is still no rigorous proof for such stability. Wan’s paper [112] in 1988 contains an orbital stability statement2 in patch-type axi-symmetric perturbations as a corollary. The statement has most to do with our result in the sense that the variational principles used for both results are suggested by [12] and Friedman-Turkington [50].

Most of the other existing literature regarding on stability/instability issue focus on linearized (or approximated) response to a patch-type perturbation to its boundary and/or related numerical computations. Moffatt-Moore [85] in 1978 (also see [13]) analysed an approximate evolution equation for the patch boundary. Roughly speaking, the perturbation can produce a thin spike from the rear stagnation point when the initial vortex is either a prolate spheroid or an oblate spheroid. It might be understood that the irrotational flow outside the sphere tends to “sweep” the perturbation as mentioned in [85]. The situation is validated in a nonlinear setting by Pozrikidis [91] in 1986 numerically (also see [92] for an spectral approach). For non axi-symmetric perturbations, we refer to [51, 94]. Lastly, investigations by short-wavelength stability analysis can be found in [58, 76, 95].

Our main result (Theorem 1.2) says that the vortex is nonlinearly stable (up to a translation performed in the axis) in axi-symmetric perturbations which are allowed to be a non-patch type. Simply speaking, the amount swept by the irrotational flow outside the core can be controlled uniformly in all time by the initial difference. The key idea is to make a bridge between the

---

2 It was mentioned as a corollary [112, Corollary (H)] without a written proof. The metric used in the statement contains a non-invariance quantity, which makes the corollary incorrect. In particular, it fails when one simply compares two Hill’s vortices with different radii. The original statement of [112] will be reviewed in Remark 1.7 while a counterexample will be presented in Remark 2.1.
existence result of [50] based on variational method and the uniqueness result of Amick-Fraenkel [4] in order to apply the concentrated compactness method of Lions [77] into a maximizing sequence.

1.2 Main theorems 1.1, 1.2: stability of Hill’s vortex

By using the cylindrical coordinate system \((r, \theta, z)\), we say that a scalar function \(f : \mathbb{R}^3 \to \mathbb{R}\) is axi-symmetric if it has the form of \(f(x) = f(r, z)\), and a subset \(A \subset \mathbb{R}^3\) is axi-symmetric if the characteristic function \(1_A : \mathbb{R}^3 \to \mathbb{R}\) is axi-symmetric. Here is our main result for patch type data.

**Theorem 1.1.** The Hill’s vortex is stable up to a translation in the sense that for \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any axi-symmetric measurable subset \(A_0 \subset \mathbb{R}^3\) satisfying

\[
A_0 \subset \{0 \leq r < R\} \text{ for some } R < \infty
\]  

and

\[
\int_{A_0 \triangle B} (1 + r^2) \, dx \leq \delta,
\]

the corresponding solution \(\xi(t) = 1_{A_t}\) of (1.3) for the initial data \(\xi_0 = 1_{A_0}\) satisfies

\[
\inf_{\tau \in \mathbb{R}} \left\{ \int_{A_t \triangle B_{\tau}} (1 + r^2) \, dx \right\} \leq \varepsilon, \quad \forall t \geq 0,
\]

where

\[
B_{\tau} := \{x \in \mathbb{R}^3 \mid ||x - \tau e_z|| < 1\}
\]

is the unit ball centered at \((0, 0, \tau)\). Here, the symbol \(\triangle\) means the symmetric difference.

The above theorem is a particular case of the following stability theorem allowing non patch-type data:

**Theorem 1.2.** For \(\varepsilon > 0\), there exists \(\delta > 0\) such that for any non-negative axi-symmetric function \(\xi_0\) satisfying

\[
\xi_0, \ r\xi_0 \in L^\infty(\mathbb{R}^3) \tag{1.6}
\]

and

\[
\|\xi_0 - \xi_H\|_{L^1 \cap L^2(\mathbb{R}^3)} + \|r^2(\xi_0 - \xi_H)\|_{L^1(\mathbb{R}^3)} \leq \delta,
\]

where
the corresponding solution \(\xi(t)\) of (1.3) for the initial data \(\xi_0\) satisfies
\[
\inf_{\tau \in \mathbb{R}} \left\{ \|\xi(\cdot + \tau e_z, t) - \xi_H\|_{L^1 \cap L^2(\mathbb{R}^3)} + \|r^2(\xi(\cdot + \tau e_z, t) - \xi_H)\|_{L^1(\mathbb{R}^3)} \right\} \leq \varepsilon, \quad \forall t \geq 0.
\]
Here, \(\| \cdot \|_{L^1 \cap L^2}\) means \(\| \cdot \|_{L^1} + \| \cdot \|_{L^2}\).

Remark 1.3. In general, the Euler equations in velocity form (1.1) admits non-unique weak solutions for initial data \(u_0 \in L^2(\mathbb{R}^3)\) by [41, 115]. However, when the flow is axi-symmetric without swirl, we can consider the simpler equation (1.3) instead. Then for any axi-symmetric initial data
\[
0 \leq \xi_0 \in (L^1 \cap L^2)(\mathbb{R}^3) \quad \text{with} \quad r^2 \xi_0 \in L^1(\mathbb{R}^3),
\]
existence and uniqueness of a weak solution is guaranteed by imposing the extra condition (1.6) on the relative vorticity \(\xi_0\) by Ukhovskii-Yudovich [108] (also see [38, 97]). By the same reason, the assumption (1.5) is added in Theorem 1.1. We revisit the issue in detail in Subsection 3.3 (see Lemma 3.4 and Remark 3.5).

Remark 1.4. Theorem 1.2 deals with non-patch type solutions near the Hill’s vortex. It gives some advantage in the following sense: For any \(\delta > 0\), there is a \(C^\infty\)-smooth initial compactly supported axi-symmetric relative vorticity \(\xi_0\) satisfying the assumption of Theorem 1.2 whose compact support lies away from the axis \(\{r = 0\}\). Then, we have the initial vorticity \(\omega_0(x) = r \xi_0(r, z) e_\phi(\theta)\) lying on \(C^\infty_c(\mathbb{R}^3)\) which gives the global-in-time \(C^\infty\) solution \(u(t)\) of (1.1) by [97, 100, Theorem 2.4] since the axi-symmetric initial velocity \(u_0 := \nabla \times (-\Delta)^{-1} \omega\) has no swirl and lies on \(H^m(\mathbb{R}^3)\) for any integer \(m > 0\). As a result, the smooth solution \(u(t)\) is close in our sense (up to a translation) to the flow of the Hill’s vortex for all time.

Remark 1.5. By the scaling invariance of the Euler equations, we have a family of Hill’s vortices of two parameters in the following sense:
For any given \(0 < \lambda, a < \infty\), we define \(\xi_H(\lambda, a)\) by
\[
\xi_H(\lambda, a)(x) = \lambda 1_{B_a}(x) \quad (1.7)
\]
where \(B_a := \{x \in \mathbb{R}^3 \mid |x| < a\}\). Then
\[
\xi(t, x) = \lambda 1_{B_a}(x - t W_{H(\lambda, a)} e_3) \quad (1.8)
\]
is a traveling wave solution of (1.3) with its traveling speed
\[
W_{H(\lambda, a)} = W_H \cdot (\lambda a^2) = \frac{2}{15} \lambda a^2 \quad (1.9)
\]
(see Subsection 2.4 for more detail). The above theorems work for each fixed \(\lambda, a \in (0, \infty)\) by the scaling.

Remark 1.6. Dropping the non-negativity assumption on the initial relative vorticity \(\xi_0\) in Theorem 1.2 seems non-trivial in the sense that we do not exclude a possibility that small negative
part of $\xi_0$ might spoil the distribution of other positive part much at time infinity. Indeed, our variational method uses the fluid impulse $\int_{\mathbb{R}^3} r^2 \xi(t, x) \, dx$ conserved in time as the main reference quantity. Together with the non-negativity on $\xi$, the conservation of impulse plays a role of attraction or cohesion toward the symmetry axis $\{r = 0\}$. However, without assuming the sign condition, we do not expect any global-in-time bound on $\int_{\mathbb{R}^3} r^2 \xi^+(t, x) \, dx$ and $\int_{\mathbb{R}^3} r^2 \xi^-(t, x) \, dx$. There might be continued leakage of positive part and negative part of $\xi$ from the core of the Hill’s vortex. On the other hand, dropping the axis-symmetry assumption is more challenging. We do not even know global existence of solutions without the symmetry.

Remark 1.7. The paper [112] proved that the Hill’s vortex $\xi_H$ is a nondegenerate local maximum of the kinetic energy under certain constraints. It mentioned an orbital stability of the vortex as a corollary [112, Corollary (H)] without a written proof. The metric used in the corollary has the form

$$d(\xi_1, \xi_2) = \int r^2 |\xi_1 - \xi_2| \, dx + \left| \int zr^2 \xi_1 \, dx - \int zr^2 \xi_2 \, dx \right|.$$

We note that the term $\int zr^2 \xi(t, x) \, dx$ is not conserved in general when $\xi(t)$ is a solution of (1.3) while the impulse $\int r^2 \xi(t, x) \, dx$ is preserved. The statement [112, Corollary (H)] considers patch-type initial data and says that for $\varepsilon > 0$, there is $\delta > 0$ such that if $A_0$ is an axi-symmetric bounded subset of $\mathbb{R}^3$ satisfying

$$\int_{A_0 \triangle B} r^2 \, dx + \left| \int_{A_0} zr^2 \, dx - \int_{B} zr^2 \, dx \right| \leq \delta,$$

then the corresponding solution $\xi(t) = 1_{A_t}$ of (1.3) for the initial data $\xi_0 = 1_{A_0}$ satisfies

$$\inf_{\tau \in \mathbb{R}} \left( \int_{A_t \triangle B^\tau} r^2 \, dx + \left| \int_{A_t} zr^2 \, dx - \int_{B^\tau} zr^2 \, dx \right| \right) \leq \varepsilon \quad \text{for all} \quad t \geq 0. \tag{1.10}$$

However, when $A_0$ has a different impulse, that is, when

$$\int_{A_0} r^2 \, dx \neq \int_{B} r^2 \, dx,$$

the statement fails in general. The precise verification is postponed until Remark 2.1. Heuristically, the impulse part $\int_{A_t \triangle B^\tau} r^2 \, dx$ is minimized when two sets $A_t$ and $B^\tau$ share the same center. However, in the case, the non-invariance part $| \int_{A_t} zr^2 \, dx - \int_{B^\tau} zr^2 \, dx |$ can grow linearly in time due to the weight $z$ in the integrand (see (2.16)). It shows that the quantity in (1.10) cannot be small for large time.

Remark 1.8. It is natural to ask where the core of the perturbed solution should be at each time. Indeed, the current result merely says that the perturbed one is close to the Hill’s vortex up to some time-dependent $z-$translation $\tau(t)$. In this direction, one can estimate the core position $\tau(t)$ by a bootstrap argument (see [32]). Roughly speaking, the perturbed solution travels in a similar speed of the Hill’s vortex.
### 1.3 Ideas of proof

The stability up to a translation (or an orbital stability in general) is a proper notion of nonlinear stability of the Hill’s vortices. For instance, imagine the vortex (1.8) for $\lambda = 1$ and $0 < |a - 1| \ll 1$. The core travels with the speed (1.9) and eventually becomes disjoint from the other travelled core of the Hill’s vortex for $\lambda = 1, a = 1$. In that sense, a translation is necessary when comparing sliding vortices in longer times. We also refer to the explanation [36, p. 605] for the case of Kirchhoff ellipses asking stability up to a rotation.

To obtain such a stability, we follow the strategy via the variational method based on vorticity due to the idea of Kelvin [106] and Arnold [5] (also see the book [6]). The variational setting for vortex rings we use comes from [50]. More specifically, for given $\lambda, \mu, \nu \in (0, \infty)$, we consider an admissible function $\xi$ which is a characteristic function of strength $\lambda$

$$\xi = \lambda 1_A$$

for some axi-symmetric $A \subset \mathbb{R}^3$ (i.e., a patch-type data) and whose impulse satisfies the exact condition

$$\frac{1}{2} \int_{\mathbb{R}^3} r^2 \xi \, dx = \mu$$

while its total circulation (or its mass) is asked to hold the less strict condition

$$\int_{\mathbb{R}^3} \xi \, dx \leq \nu.$$  

(see (3.1)) (cf. for vortex pairs, see [2, 16, 107]). In this class of admissible functions, we pursue maximizing the kinetic energy

$$\int_{\mathbb{R}^3} |u|^2 \, dx,$$

where $u = \mathcal{K}[\xi]$ is the corresponding velocity field of the relative vorticity $\xi$. The approach prescribing impulse was first suggested by [12] (also see Burton [18]).

Under this setting, Friedman and Turkington in their paper [50, Theorem 2.1] showed that there exists a maximizer $\xi$ of the energy satisfying

$$\xi = \lambda f_H(\Psi), \quad \Psi := \mathcal{G}[\xi] - \frac{1}{2} W r^2 - \gamma,$$

(1.11)

for some $W > 0$ and $\gamma \geq 0$, where the vorticity function $f_H$ is defined by

$$f_H(s) = \begin{cases} 1, & s > 0, \\ 0, & s \leq 0, \end{cases}$$

(1.12)

and the stream function $\mathcal{G}[\xi]$ is defined later in (2.1). The constants $W, \gamma$ represent the propagation speed and the flux constant, respectively. The existence was obtained by a limiting argument ($\beta \to$...
0) via a penalized energy functional

\[
E_\beta[\xi] := (\text{kinetic energy of the fluid induced by } \xi) - \beta \lambda \int \left( \frac{\xi}{\lambda} \right)^{1+1/\beta} \, dx,
\]

(1.13)

where the definition of kinetic energy can be found in (2.7) (also see Remark 4.3).

On the other hand, Amick and Fraenkel in their paper [4, Theorem 1.1] showed that any \( \xi \) satisfying (1.11) with \( \gamma = 0 \) is the Hill’s vortex (1.7) with certain radius \( a > 0 \) (see (2.13)) up to a translation in \( z \)-variable. This uniqueness was proved by adapting the moving plane method (refer to [54, 99]). The main difficulty comes from discontinuity of the vorticity function \( f_H \) in (1.12) (also see Remark 7.3).

Before explaining the key ideas of our proof, we may assume \( \lambda = \nu = 1 \) without loss of generality. This is done by the scaling argument (e.g., see (4.4)). Now the impulse \( \mu \in (0, \infty) \) is the only free parameter. To obtain stability of the Hill’s vortex whose impulse is exactly the constant \( \mu \), we connect those two classical results [50] and [4] mentioned above by showing the following two statements:

1. For any impulse \( \mu > 0 \), every maximizer satisfies (1.11) for some speed \( W > 0 \) and some flux constant \( \gamma \geq 0 \) (see Theorem 5.1). It implies that every maximizer is a steady vortex ring related to the vorticity function \( f_H \) (1.12).
2. For any small impulse \( \mu > 0 \), every maximizer is the corresponding Hill’s vortex (1.7) up to a translation (Theorem 3.2).

When proving statement 1 above (Theorem 5.1), the discontinuity of \( f_H \) is the main obstacle. Indeed, for a maximizer \( \xi_{\text{max}} = 1_A \) with some axi-symmetric \( A \subset \mathbb{R}^3 \), by using variational principles, we have to find some constants \( W, \gamma \) satisfying

\[
A = \{ \Psi > 0 \},
\]

(1.14)

where the adjusted stream function \( \Psi \) is defined in (1.11). We note that for any choice of constants \( W, \gamma \), the set \( \{ \Psi > 0 \} \) is open so that its boundary \( \partial \{ \Psi > 0 \} \) is well-defined. Once the claim (1.14) is proved, such constants \( W, \gamma \) are uniquely and explicitly determined by any two points on the boundary \( \partial A \) because \( \Psi = 0 \) there (see (5.3)).

However, we merely know that \( A \) is measurable (i.e., \( A \) is defined up to almost everywhere) until proving the claim (1.14) for some \( W, \gamma \). It makes topological boundary \( \partial A \) unusable. Instead, we use the metrical\(^3\) boundary (see Definition 5.4) of the measurable set \( A \), whose element is called an exceptional point. Near an exceptional point of \( A \), there are non-trivial parts of \( A \) and \( A^c \) in the neighborhood (See Lemma 5.6) so that we can construct an approximation to the Dirac function at the point from the inside of \( A \) and from the outside of \( A \) (see Remark 5.8 for more detail). It is the key idea to solve the former statement (Theorem 5.1). As a minor difficulty, the restriction

\[
\xi(x) \in \{0, 1\}
\]

on admissible functions \( \xi \) prevents us from adding general \( L^\infty \)-perturbations into the maximizer \( \xi_{\text{max}} \). To overcome, we extend the class of admissible functions in advance so that non patch-type

---

\(^3\)These terminologies “metrical” and “exceptional” can be found in [37, p. 78] and [104, p. 765], respectively (also see [68]).
functions

\[ 0 \leq \xi(x) \leq 1 \]

are allowed (see (4.1)). Then we can freely add small negative perturbations supported on \( A \) and small positive perturbations supported on the complement \( A^c \) into \( \xi_{\text{max}} \).

The statement 2 above (Theorem 3.2) is decomposed into three steps:

\[ \mu \ll 1 \Rightarrow_1 \int \xi_{\text{max}} \, dx < 1 \Rightarrow_2 \gamma = 0 \Rightarrow_3 \xi_{\text{max}} = \xi_{H(1, a)} \text{ up to a translation}, \]

where the radius \( a > 0 \) satisfies \( \mu = (4/15)\pi a^5 \). To prove the first step (Proposition 7.4), we use the lower bound of the traveling speed \( W \) depending only on the impulse \( \mu \) (Proposition 5.13) (also see [50] or see [49, p. 42]). Together with an elementary estimate of stream functions (Lemma 7.6), it says that the total mass should be strictly smaller than the prescribed number \( \nu = 1 \). The second step is done at Lemma 5.14 by using variational principles. As noted in Remark 7.5, this strategy is essentially contained in [50, Remark 5.2] (cf. for vortex pairs, see [2, Remark 2.6]). For the third step, the uniqueness result due to [4, Theorem 1.1] is used. In Section 7, we carefully verify the setting and all the assumptions of [4, Theorem 1.1]. In sum, we make a bridge between the existence result [50] and the uniqueness result [4], which gives the characterization of the set of maximizers as a single orbit of the corresponding Hill’s vortex (Theorem 3.2).

The next part is to prove compactness (Theorem 3.1), which means that there is a limiting function (up to a subsequence) for any maximizing sequence, and the limit is a maximizer. As already used in the case of 2d vortex pairs by Burton-Nussenzveig Lopes-Lopes Filho [20] (and more recently by [2]), we exploit the concentrated compactness lemma of Lions [77, Lemma I.1]. Roughly speaking, the lemma says that for any sequence \( \{f_n\} \) of non-negative functions on the whole space, if each function of the sequence has the unit mass, then (up to a subsequence) one of the three following cases happens:\(^4\):

1. (compactness) The most mass is confined in a large ball up to a translation.
2. (vanishing) For any large ball, \( f_n \) has negligible mass on the ball as \( n \) goes to infinity.
3. (dichotomy) There is a real number \( \mu \) between 0 and 1 such that each \( f_n \) is decomposed into two functions having disjoint supports, and the supports are getting farther and farther away from each other. In particular, both the supports has non-trivial mass close to \( \mu \) and \( (1 - \mu) \), respectively.

In this context, our goal is to exclude both the case of vanishing and the case of dichotomy. There is a difficulty when avoiding the dichotomy case since sub-additivity of maximum energy in \( \mu > 0 \) is not known. In particular, since we are not allowed to use \( L^{\infty} \)-bound of the initial data \( \xi_0 \), we carefully use the particular form of the two functions \( \xi_1 = \xi_{1\Omega_1} \) from dichotomy (e.g., see (6.12)). It is important that the function class in the assumptions need to be large enough to cover our stability statement. As a consequence, the proof (Section 6.1) gets a bit technical.

Lastly, the orbital stability is obtained once we combine compactness (Theorem 3.1) together with uniqueness (Theorem 3.2). Indeed, we can use the classical contradiction argument. Loosely speaking, let’s suppose that there is a sequence of solutions obtained from perturbed initial data,

\(^4\) For the precise statements, see Lemma 6.1.
but at a certain positive time, it does not converge to the Hill’s vortex up to any translation. Then by the energy conservation, we can use the compactness result to obtain a limit function which is a maximizer. Then by the uniqueness result, the limit should be the Hill’s vortex up to a translation, which is a contradiction.

**Remark 1.9.** If one can extend the uniqueness result (Theorem 3.2) beyond the class of Hill’s vortices (i.e., \( \gamma > 0 \) case in (1.11)), then we can follow the same line of our proof in Subsection 3.4 of Theorem 1.2 to establish a similar stability for the vortex rings in certain (probably smaller) function class. However, such a uniqueness statement seems out of reach under the current technique (e.g., moving plane method [54]). For instance, while one may prove that the core set \( A \) is connected and symmetric (about the plane \( \{ z = c \} \) for some \( c \in \mathbb{R} \)) by following the approach from [12] (also see [22]) and from [20] respectively, it is not strong enough to guarantee the uniqueness.

### 1.4 Other vortices and waves

The Hill’s vortex can be considered as the limiting fattest case of the one parameter family from Fraenkel [48] and Norbury [88]. Except the limiting case, Fraenkel-Norbury’s solutions are genuine rings in the sense that their cores are away from the symmetry axis forming torus-type vortices. The Helmholtz’s rings of small cross-section [59] are related to the opposite limiting thinnest case. We refer to [49] for the existence of general vortex rings via a stream function method (also see [86] and references therein).

It is interesting that there are explicit spherical vortices with swirl traveling in a constant speed (e.g., see [60, 84]). We refer to the recent work [1] and references therein for diverse vortices with swirl. Their stability are generally open.

As a 2d analogue to Hill’s spherical vortex, there is an explicit traveling vortex pair (or “dipole”) introduced by H. Lamb [71, p. 231] in 1906 and, independently, by S. A. Chaplygin in 1903 [27, 28] (cf. [82]). For a simple presentation, let us consider the case when the vortex core is the unit disk in \( \mathbb{R}^2 \) and the traveling speed is exactly equal to 1. Then the vortex \( \omega_L \) has the form in the polar coordinates \((r, \theta)\):

\[
\omega_L(x) = \begin{cases} 
-\frac{2c_0}{J_0(c_0)} J_1(c_0 r) \sin \theta, & r \leq 1, \\
0, & r > 1,
\end{cases}
\]

where \( J_m(r) \) is the \( m \)-th order Bessel function of the first kind. The constant \( c_0 > 0 \) is the first zero point of \( J_1 \) and \( J_0(c_0) < 0 \). In particular, it satisfies

\[
\omega_L = c_0^2 f_L(\psi_L - W_L x_2), \quad \psi_L = (-\Delta \mathbb{R}^2)^{-1} \omega_L, \quad W_L = 1,
\]

where the vorticity function \( f_L \) is defined by

\[
f_L(s) = \begin{cases} 
s, & s > 0 \\
0, & s \leq 0.
\end{cases}
\]
Then
\[ \omega(t, x) = \omega_L(x - W_L t e_{x_2}) \]
is a traveling wave solution of the 2d Euler equations
\[ \partial_t \omega + u \cdot \nabla \omega = 0, \quad u = K * \omega \quad \text{in} \quad \mathbb{R}^2 \times (0, \infty), \]
with the 2d Biot-Savart kernel \( K(x) = (2\pi)^{-1} x^\perp |x|^{-2}, \ x^\perp = (-x_2, x_1) \). We note that the vorticity function \( f_L \) in (1.16) is smoother than \( f_H \) in (1.12) of the Hill’s vortex. The fact that \( f_L \) is Lipschitz helps to study certain properties of the dipole including the stability question. The orbital stability of the dipole (1.4) was recently obtained by [2]. We mention Burton’s work [17, 19] for other properties of the dipole. A similar stability for broader class of vortex pairs (not including the particular case (1.15)) was proved by [20]. In this paper, we follow a similar structure of [2, 20] to obtain the stability for the Hill’s vortex. For more properties of general vortex pairs, we refer to [16, 89, 107]. For general dimension \( N \geq 2 \), existence (and uniqueness) of a vortex generalizing (1.15) and (1.4) was proved by Burton-Preciso [21].

There are stability results for other exact solutions of the 2d Euler equations. For instance, we see [23, 102, 113] for a circular patch, [10] for a rectangular patch in the infinite strip. As simple applications of stability for a circular patch, we mention [29, 30] for winding number estimates and [31] for growth in perimeter. Surprisingly, there is an asymptotic stability result [9] for a 2d Couette flow (cf. [8] even for 3d Couette flow).

An orbital stability of solitary waves appeared first in [11, 14] for the generalized KdV equation. We also refer to [80, 81, 83] for instability and blow-ups (also see the survey paper [105]). For general dispersive equations, we refer to [15, 24, 55].

Such a stability up to a translation even occurs for inviscid/viscous shocks in conservation laws. In \( L^2 \)-setting in one dimensional space, we mention [35, 72] for the scalar case, [64, 65, 73] for systems such as compressible Euler/Navier-Stokes systems and [33, 34] for certain Keller-Segel type systems. We also refer to the classical paper [62] for asymptotic stability up to a translation in \( L^1 \)-setting (also see [67] and references therein).

In the rest of the paper, in Section 2, we introduce preliminary materials. Then, Theorems 3.1, 3.2 (compactness, uniqueness) are presented in Subsection 3.2. By assuming them for a moment, the proof of our main stability result (Theorems 1.1, 1.2) is given in Subsection 3.4. The rest of the paper (Sections 4–7) is devoted to prove Theorems 3.1, 3.2.

2  |  PRELIMINARIES

We introduce relevant mathematical background of the Hill’s vortex (more generally steady vortex rings). For more detail, we also refer to [49, Section 2], [4, Section 1], [50, Section 1].

2.1  |  Axi-symmetric Biot-Savart law

A vector field \( u \) is called \textit{axi-symmetric} if it has the form of
\[ u(x) = u^r(r, z)e_r(\theta) + u^\theta(r, z)e_\theta(\theta) + u^z(r, z)e_z, \]
for
\[ e_r(\theta) = (\cos \theta, \sin \theta, 0), \quad e_\theta(\theta) = (-\sin \theta, \cos \theta, 0), \quad e_z = (0, 0, 1), \]

where \((r, \theta, z)\) is the cylindrical coordinate to the Cartesian coordinate \(x = (x_1, x_2, x_3)\), that is, \(x_1 = r \cos \theta, x_2 = r \sin \theta, x_3 = z\). If \(u^\theta \equiv 0\), then \(u\) is called axi-symmetric without swirl. The divergence-free condition for \(u\) can be written as
\[ \partial_r(ru^r) + \partial_z(ru^z) = 0. \]

Then, there exists an axi-symmetric stream function \(\psi = \psi(r, z)\) such that
\[ u = \nabla \times \phi, \quad \phi = \begin{pmatrix} \psi \\ r e_\theta \end{pmatrix}. \]

By denoting the vorticity vector field
\[ \omega := \nabla \times u = (\partial_z u^r - \partial_r u^z)e_\theta(\theta) = \omega^\theta e_\theta(\theta), \]
the stream \(\psi\) satisfies
\[ -\frac{1}{r^2} L \psi = \xi \]
for the relative vorticity \(\xi := r^{-1} \omega^\theta\) with the operator
\[ L = r \frac{\partial}{\partial r} \left( \frac{1}{r} \frac{\partial}{\partial r} \right) + \frac{\partial^2}{\partial z^2} = \frac{\partial^2}{\partial r^2} - \frac{1}{r} \frac{\partial}{\partial r} + \frac{\partial^2}{\partial z^2}. \]

From \(-\Delta \phi = \omega^\theta e_\theta\), we may assume
\[ \phi = \frac{1}{4\pi |x|} \ast_{\mathbb{R}^3} (\omega^\theta e_\theta). \]

Then, by the axial symmetry (e.g., see [50, Section 1]), we have
\[ \psi(r, z) = \int_\Pi G(r, z, r', z') \xi(r', z') r' dr' dz' =: G[\xi](r, z), \quad (r, z) \in \Pi, \]
for the half space
\[ \Pi = \{(r, z) \in \mathbb{R}^2 \mid r > 0\} \]
and for the Green function
\[ G(x, y) = G(r, z, r', z') = \frac{rr'}{2\pi} \int_0^\pi \frac{\cos \theta}{\sqrt{r^2 + r'^2 - 2rr' \cos \theta + (z - z')^2}} d\theta, \]
where \( y_1^2 + y_2^2 = r'^2 \), \( y_3 = z' \) for \( y = (y_1, y_2, y_3) \in \mathbb{R}^3 \). We note that \( G(x, y) = G(y, x) \) and \( G \) is axi-symmetric for each variable. Following \cite{[111]}, we write

\[
G(r, z, r', z') = \frac{\sqrt{rr'}}{2\pi} F(s)
\]  

(2.2)

by setting

\[
s = \frac{(r - r')^2 + (z - z')^2}{rr'}, \quad F(s) = \int_0^\pi \frac{\cos \vartheta}{\sqrt{2(1 - \cos \vartheta) + s}} d\vartheta.
\]

We observe that the function \( F > 0 \) is strictly decreasing in \( s > 0 \). The function \( F \) (and \( G \)) can be estimated by using complete elliptic integrals of the first and second kind (e.g., see \cite[Lemma 3.3, [111, Section 19]). Indeed, by the asymptotic behavior which can be found in \cite[Lemma 2.1, Remark 2.2, [47, Lemmas 2.7, 2.8, [43, Section 2.3] we have

\[
F(s) = \frac{1}{2} \log \frac{1}{s} + \log 8 - 2 + O\left(s \log \frac{1}{s}\right) \quad \text{as} \quad s \to 0
\]

and

\[
F(s) = \frac{\pi}{2} \frac{1}{s^{3/2}} + O(s^{-5/2}) \quad \text{as} \quad s \to \infty.
\]

Thus we have

\[
F(s) \lesssim \frac{1}{s^\tau}, \quad 0 < \tau \leq 3/2, \quad s > 0
\]

and

\[
G(r, z, r', z') \lesssim \frac{(rr')^{\tau + \frac{1}{2}}}{\left(\sqrt{|r - r'|^2 + |z - z'|^2}\right)^{2\tau}}, \quad 0 < \tau \leq 3/2, \quad (r, z), (r', z') \in \Pi.
\]

(2.4)

We set the Biot-Savart law

\[
\mathcal{K}[^\xi] := \nabla \times \left(\frac{1}{r} G[^\xi] e_\vartheta\right) = \nabla \times \left(\frac{1}{r} \psi e_\vartheta\right).
\]

Then we recover the axi-symmetric velocity

\[
u = \mathcal{K}[^\xi] = \frac{-\partial_z \psi}{r} e_r + \frac{\partial_r \psi}{r} e_z.
\]

(2.5)

In Subsection 2.6, we present a natural class for \( \xi \) where the above formal computations can be valid.
2.2 | Vorticity equation

Once we assume that the flow \( u \) in (1.2) is axi-symmetric without swirl, we can derive the following active scalar equation for the relative vorticity \( \xi = \omega / r \):

\[
\begin{align*}
\partial_t \xi + u \cdot \nabla \xi &= 0, \quad u = \mathcal{K} [\xi], \quad x \in \mathbb{R}^3, \quad t > 0, \\
\xi |_{t=0} &= \xi_0, \quad x \in \mathbb{R}^3.
\end{align*}
\]  

(2.6)

In this paper, we consider only non-negative axi-symmetric weak solutions \( \xi(t) \) to (2.6) preserving the quantities (1)–(4) below (by noting \( dx = 2\pi rrdz \)):

(1) Impulse

\[
P[\xi] = \frac{1}{2} \int_{\mathbb{R}^3} r^2 \xi dx = \pi \int_{\Pi} r^3 \xi drdz.
\]

(2) Kinetic energy

\[
E[\xi] = \frac{1}{2} \int_{\mathbb{R}^3} \xi G(\xi) dx = \pi \iint_{\Pi \times \Pi} G(r, z, r', z') \xi(r', z') \xi(r, z) r' r dr' dz' dr dz.
\]  

(2.7)

(3) Circulation (or total mass)

\[
\Gamma[\xi] = \int_{\mathbb{R}^3} \xi dx = 2\pi \int_{\Pi} r \xi drdz.
\]

(4) Vortex strength

\[
\Lambda[\xi] = \text{ess sup}_{x \in \mathbb{R}^3} \xi(x).
\]

For the existence of such weak solutions, we see Lemma 3.4 in Subsection 3.3. In addition, we have conservation of any \( L^p \)-norm

\[
\| \xi \|_{L^p(\mathbb{R}^3)}, \quad p \in [1, \infty]
\]

and conservation of mass contained in any level sets

\[
\int_{\{ x \in \mathbb{R}^3 \mid a < \xi < b \}} \xi dx, \quad 0 < a < b < \infty.
\]

2.3 | Steady vortex rings

We call an axi-symmetric motion without swirl with vanishing velocity at infinity a steady vortex ring if the support of the vorticity is a bounded set (so-called the vortex core), and if the vortex moves at a constant speed along the axis without any change in its size and shape. In other words,
we are interested in axi-symmetric solutions $\xi$ to (2.6) of the form

$$\xi(x,t) = \xi_{ur}(x + tu_\infty)$$

for some compactly supported $\xi_{ur}$ and for some constant velocity $u_\infty = -We_z$. We set

$$U = \mathcal{C}[\xi_{ur}] + u_\infty$$

so that the pair $(\xi_{ur}, U)$ satisfies the stationary equation

$$U \cdot \nabla \xi_{ur} = 0, \quad x \in \mathbb{R}^3,$$

$$U \to u_\infty \quad \text{as} \quad |x| \to \infty.$$

By denoting $\psi_{ur} = \mathcal{C}[\xi_{ur}]$ and by using (2.5), we get

$$U = -\frac{\partial_z \psi_{ur}}{r} e_r + \left( \frac{\partial_r \psi_{ur}}{r} - W \right) e_z.$$

Thus the above stationary equation can be written as

$$\frac{\partial(\xi_{ur}, \psi_{ur})}{\partial(r,z)} = \frac{\partial \xi_{ur}}{\partial z} \frac{\partial \psi_{ur}}{\partial r} - \frac{\partial \xi_{ur}}{\partial r} \frac{\partial \psi_{ur}}{\partial z} = 0$$

for the adjusted stream function

$$\psi_{ur} = \psi_{ur} - \frac{1}{2} Wr^2 - \gamma \quad (2.8)$$

for any flux constant $\gamma$. In this sense, we shall seek a (time-independent) solution $\xi$ to (by dropping the subscript $\text{)}_{ur''}$

$$\xi = f(\Psi)$$

for some non-decreasing function $f : \mathbb{R} \to \mathbb{R}$ (so-called a vorticity function) satisfying $f(s) = 0$ for $s \leq 0$, $f(s) > 0$ for $s > 0$. Due to

$$-\frac{1}{r^2} \mathcal{L} \Psi = -\frac{1}{r^2} \mathcal{L} \psi = \xi,$$

the above system is reduced to a semilinear elliptic equation in the half-space $\Pi$:

$$-\frac{1}{r^2} \mathcal{L} \Psi = f(\Psi), \quad r > 0, \quad z \in \mathbb{R}, \Psi(0,z)$$

$$= -\gamma, \quad z \in \mathbb{R}, \frac{1}{r} \partial_r \Psi \to$$

$$-W, \quad \frac{1}{r} \partial_z \Psi \to 0 \quad \text{as} \quad \sqrt{r^2 + z^2} \to \infty. \quad (2.9)$$
2.4 Hill’s spherical vortex revisited

The stream function $\psi_H = \mathcal{G}[\xi_H]$ of the Hill’s vortex $\xi_H = 1_{[|x| \leq 1]}$ (1.4) is explicitly written by

$$\psi_H(x) = \begin{cases} 
\frac{1}{2} W r^2 \left(\frac{5}{2} - \frac{3}{2} |x|^2\right), & |x| \leq 1, \\
\frac{1}{2} W r^2 \frac{1}{|x|^3}, & |x| > 1,
\end{cases}$$

where the traveling speed $W$ is set by $2/15$. The corresponding axi-symmetric velocity $u = (u^r_H e_r + u^z_H e_z)$ is obtained via

$$u^r_H = -\frac{\partial_z \psi_H}{r}, \quad u^z_H = \frac{\partial_r \psi_H}{r}$$

(see (2.5)). As noted in [4], $\psi_H$ is simply obtained by solving the following O.D.E. problem for $\eta(|x|) = \psi(x)/r^2$:

$$\eta \in C^1[0, \infty) : \text{strictly decreasing,}$$

$$-\frac{1}{t^4} (t^4 \eta')' = 1, \quad 0 < t < 1,$n

$$-\frac{1}{t^4} (t^4 \eta')' = 0, \quad t > 1,$n

$$\eta(t)|_{t=1} = \frac{1}{2} W, \quad \eta(t)|_{t=\infty} = 0.$$n

Its relevant physical quantities are

$$\Lambda[\xi_H] = 1, \quad \Gamma[\xi_H] = \frac{4}{3} \pi, \quad P[\xi_H] = \frac{4}{15} \pi, \quad E[\xi_H] = \frac{8}{15 \cdot 21}. \quad (2.10)$$n

The adjusted stream function $\Psi_H := \psi_H - (1/2) W r^2$ (i.e., (2.8) with $\gamma = 0$) solves (2.9) for $\gamma = 0$ and for the vorticity function $f = f_H = 1_{(0, \infty)}$ (see (1.12)).

As mentioned in Remark 1.5, for any given $\lambda > 0$ and $\alpha > 0$, we set

$$\xi_{H(\lambda, \alpha)}(x) = \lambda 1_{B_\alpha}(x), \quad (2.11)$$

where $B_\alpha$ is the ball centered at the origin with radius $\alpha > 0$. In other words, we use the scaling

$$\xi_{H(\lambda, \alpha)}(x) = \lambda \xi_H \left(\frac{x}{\alpha}\right).$$

The corresponding stream function $\psi_{H(\lambda, \alpha)}$ is obtained by

$$\psi_{H(\lambda, \alpha)} = \lambda \alpha^4 \psi_H \left(\frac{x}{\alpha}\right), \quad (2.12)$$
and the corresponding traveling speed is set by

\[ W_{H(\lambda,a)} = \frac{2}{15} \lambda a^2. \]  

(2.13)

By the scaling, we see

\[ \Lambda[\xi_{H(\lambda,a)}] = \lambda, \Gamma[\xi_{H(\lambda,a)}] = \frac{4}{3} \pi \lambda a^3, P[\xi_{H(\lambda,a)}] = \frac{4\pi}{15} \lambda a^5, E[\xi_{H(\lambda,a)}] = \frac{8\pi}{15} \cdot \frac{21}{21} \lambda^2 a^7. \]  

(2.14)

**Remark 2.1.** As promised in Remark 1.7 of Subsection 1.2, here we verify that the orbital stability statement [112, Corollary (H)] fails by an example. Indeed, we can simply set the example

\[ A_0 = B_a \]

for \(0 < |a - 1| < 1/2\). In other words, we set the initial data \(\xi_0 = 1_{A_0} = 1_{B_a} = \xi_{H(1,a)}\). Then the solution has the form

\[ \xi(t, x) = 1_{B_a}(x - t \tilde{W}e_3) \]

with the traveling speed \(\tilde{W} := W_{H(1,a)} = (2/15)a^2\). We note that the speed \(\tilde{W}\) is different from the speed \(W_{H(1,1)} = (2/15)\) of the Hill’s vortex \(\xi_H = \xi_{H(1,1)}\) on the unit ball. Then we can check the quantity in the left-hand side of (1.10) admits a positive lower bound of order 1 for large time. More precisely, we first observe

\[ \xi(t) = 1_{A_t} \quad \text{with} \quad A_t = B_a^{(t\tilde{W})} := \{x \in \mathbb{R}^3 | ||x - (t\tilde{W})e_3|| < a\}, \quad t > 0. \]

For the impulse part in (1.10), for any \(t > 0\) and for any \(\tau \in \mathbb{R}\), if

\[ |t\tilde{W} - \tau| \geq (1 + a), \]  

(2.15)

then the two sets \(A_t\) and \(B^\tau\) become disjoint so that we get

\[ \int_{A_t \triangle B^\tau} r^2 dx = \left( \int_{A_t} r^2 dx + \int_{B^\tau} r^2 dx \right) - \left( \int_{A_0} r^2 dx + \int_{B} r^2 dx \right) = \frac{8\pi}{15} (a^5 + 1) \sim 1. \]

On the other hand, for the non-invariance part in (1.10), if

\[ |t\tilde{W} - \tau| < (1 + a), \]

then we can compute
\[ \left| \int_{A_t} zr^2dx - \int_{B^c} zr^2dx \right| = \left| \left( \int_{A_0} zr^2dx + t\tilde{W} \int r^2dx \right) - \left( \int_B zr^2dx + \tau \int r^2dx \right) \right| \\
= \left| t\tilde{W} \int_{A_0} r^2dx - \tau \int_B r^2dx \right| = \frac{8\pi}{15} |a^5 t\tilde{W} - \tau| \\
\geq \frac{8\pi}{15} \left( |a^5 t\tilde{W} - t\tilde{W}| - |t\tilde{W} - \tau| \right) \\
\geq \frac{8\pi}{15} \left( \frac{2}{15} a^2 |a^5 - 1| t - (1 + a) \right) \\
\geq \frac{8\pi}{15} (a + 1) \left( \frac{2}{15} a^2 |a - 1| t - 1 \right), \\
\]
which implies for \( t \geq 2((2/15)a^2|a - 1|)^{-1} \),
\[ \left| \int_{A_t} zr^2dx - \int_{B^c} zr^2dx \right| \geq \frac{8\pi}{15} (a + 1) \left( \frac{1}{15} a^2 |a - 1| t \right) \geq |a - 1| t \geq 1. \tag{2.16} \]
In sum, we have verified that for any \( t \geq 2((2/15)a^2|a - 1|)^{-1} \) (whether (2.15) holds or not),
\[ \inf_{\tau \in \mathbb{R}} \left( \int_{A_t} r^2dx + \left| \int_{A_t} zr^2dx - \int_{B^c} zr^2dx \right| \right) \geq 1. \]
However, the metric at the initial time has the estimate
\[ \int_{A_0} r^2dx + \left| \int_{A_0} zr^2dx - \int_B zr^2dx \right| = \int_{A_0} r^2dx - \int_B r^2dx \sim |a^5 - 1| \sim |a - 1|, \]
which we can make arbitrarily small by taking the limit \( a \to 1 \). Hence, the statement \([112, \text{ Corollary (H)}]\) (or see Remark 1.7) cannot be true for this example.

### 2.5 | Notations

We collect notations used in this paper.

\[ \|f\|_p := \|f\|_{L^p} = \|f\|_{L^p(\mathbb{R}^3)}, \quad p \in [1, \infty], \]
(weighted \( L^1 \)-space) \( L^1_w := \{f : \text{measurable} \mid \|r^2 f\|_1 < \infty\} \),

where \( \|r^2 f\|_1 := \int_{\mathbb{R}^3} (x_1^2 + x_2^2)|f(x)| \, dx \),
\[ \int dx := \int_{\mathbb{R}^3} dx, \quad \iint dydx := \int_{\mathbb{R}^3} \int_{\mathbb{R}^3} dydx, \]
\( \Pi := \{(r, z) \in \mathbb{R}^2 | r > 0\}, \quad \overline{\Pi} := \{(r, z) \in \mathbb{R}^2 | r \geq 0\}, \)

\[ \int dr dz := \int_{\Pi} dr dz, \quad \iint dr' dz' dr dz := \int_{\Pi} \int_{\Pi} dr' dz' dr dz. \]

We note

\[ dx = 2\pi r dr dz \]

when restricted to axi-symmetric integrands. For instance, for axi-symmetric \( f \),

\[ \| f \|_1 = 2\pi \int |f(r, z)| r dr dz. \]

For \( R > 0 \), we define

- (disks in \( \Pi \)) \( B_R(r, z) := \{(r', z') \in \Pi | |(r', z') - (r, z)| < R\}, \)
- (balls in \( \mathbb{R}^3 \)) \( B_R(x) := \{x' \in \mathbb{R}^3 | |x' - x| < R\}, \)
- (tori in \( \mathbb{R}^3 \)) \( T_R(r, z) := \{x' \in \mathbb{R}^3 | |(r', z') - (r, z)| < R\} \) \( (2.17) \)

when

\[ x_1'^2 + x_2'^2 = r'^2, \quad x_3' = z', \quad \text{for} \quad x' = (x_1', x_2', x_3'). \]

We note that \( T_R(r, z) \) is not a ball in \( \mathbb{R}^3 \) in general but a torus obtained by revolving \( B_R(r, z) \) with respect to the axis of symmetry.

We denote by \( BUC(\mathbb{R}^d) \) the space of all bounded uniformly continuous functions in \( \mathbb{R}^d \) and by \( C^\alpha(\mathbb{R}^d) \) for \( \alpha \in (0, 1) \) the space of all uniformly Hölder continuous functions of the exponent \( \alpha \) in \( \mathbb{R}^d \). For an integer \( k \geq 0 \), \( BUC^{k+\alpha}(\mathbb{R}^d) \) means the space of all \( \phi \in BUC(\mathbb{R}^d) \) such that \( \partial_l^l \phi \in BUC(\mathbb{R}^d) \cap C^\alpha(\mathbb{R}^d) \) for \( |l| \leq k \). For the half-space \( \overline{\Pi} \), we define \( BUC(\overline{\Pi}), C^\alpha(\overline{\Pi}), BUC^{k+\alpha}(\overline{\Pi}) \) in the same way as above.

### 2.6 Elementarystimates of stream functions

Before finishing the preliminary section, we collect some elementary estimates for axi-symmetric \( \xi \). First we study the decay rate of the stream function near \( r = 0 \) and \( r = \infty \) thanks to the estimate \( (2.4) \) of the kernel \( G \). These are essentially contained in \([50, \text{Lemma 3.4}].\)

**Lemma 2.2.** For axi-symmetric \( \xi \in (L^1_0 \cap L^2 \cap L^1)(\mathbb{R}^3) \), the stream function \( \psi = G[\xi] \) satisfies

\[ |\psi(r, z)| \lesssim r (\|r^2 \xi\|_1 + \|\xi\|_{L^1 \cap L^2}), \quad (r, z) \in \Pi, \]

\[ |\psi(r, z)| \lesssim r^{-1+\delta} (\|r^2 \xi\|_1 + \|\xi\|_{L^1 \cap L^2}), \quad (r, z) \in \Pi, \quad 0 < \delta \leq 1. \]

\( \square \)
Proof. We estimate
\[ |\psi(r, z)| \leq \int_{\Omega} G(r, z, r', z') |\xi(r', z')| r' dr' dz' = \int_{t<r/2} + \int_{t\geq r/2} =: I + II, \]
for \( t = \sqrt{(r-r')^2 + (z-z')^2} \). For the term \( I \), we take \( p \geq 2 \) and \( \tau = 1/(2p) \) in (2.4). Applying Hölder’s inequality with \((1/p) + (1/p') = 1\) implies that
\[
I \lesssim \int_{t<r/2} (r/r')^{1+1/2p} |\xi(r', z')| r' dr' dz' \lesssim \left( \int_{t<r/2} (r/r')^{(1+p)/2} t r' dr' dz' \right)^{1/p} \|\xi_1_{[t<r/2]}\|_{p'} \lesssim t^{3/p} \|\xi_1_{[t<r/2]}\|_{p'},
\]
where the last inequality follows from \( r' \sim r \) when \( t < r/2 \). Since \( 1 \leq 2r'/r \) for \( t < r/2 \),
\[
\|\xi_1_{[t<r/2]}\|_{p'} \leq \|\xi_1_{[t<r/2]}\|_{1}^{1-2/p} \|\xi_1_{[t<r/2]}\|_{2}^{2/p} \lesssim r^{-2+(4/p)} (\|r^2 \xi_1\|_1 + \|\xi_1\|_2).
\]
Thus we get
\[
I \lesssim r^{-1+(7/p)} (\|r^2 \xi_1\|_1 + \|\xi_1\|_2).
\]
For the term \( II \), we take \( \tau = 3/2 \) in (2.4). Since \( r' \leq 3t \) for \( t \geq r/2 \), we obtain
\[
II \lesssim \int_{t\geq r/2} (r/r')^2 |\xi(r', z')| r' dr' dz' \lesssim r \|\xi\|_1.
\]
Similarly, we get \( II \lesssim r^{-1} \|r^2 \xi\|_1 \). Thus, for any \( \delta \in [0, 1] \), we have
\[
II \lesssim r^{2\delta-1} (\|\xi\|_1 + \|r^2 \xi\|_1).
\]
By combining the estimates of \( I \) and \( II \), we get the estimate for \( 2 \leq p < \infty \) and \( 0 \leq \delta \leq 1 \),
\[
|\psi(r, z)| \leq C_p r^{-1+(7/p)} (\|r^2 \xi\|_1 + \|\xi\|_2) + Cr^{2\delta-1} (\|\xi\|_1 + \|r^2 \xi\|_1).
\]
Thus, (2.18) and (2.19) follow. \( \square \)

As a consequence, the energy defined in (2.7) is well-defined with the following estimates.

**Lemma 2.3.** For axi-symmetric \( \xi, \xi_1, \xi_2 \in (L^1_w \cap L^2 \cap L^1)(\mathbb{R}^3) \), we have
\[
|E[\xi]| \leq E[|\xi|] \lesssim (\|r^2 \xi\|_1 + \|\xi\|_{L^1 \cap L^2}) \|r^2 \xi\|_{1/2} \|\xi\|_{1/2},
\]
\[
\left| \int_{\Omega} \int_{\Omega} G(r, z, r', z') \xi_1 (r, z) \xi_2 (r', z') r r' dr' dz' dr dz \right| \lesssim (\|r^2 \xi_1\|_1 + \|\xi_1\|_{L^1 \cap L^2}) \|r^2 \xi_2\|_{1/2} \|\xi_2\|_{1/2},
\]
(2.20)
(2.21)
\[ |E[\xi_1] - E[\xi_2]| \lesssim (\|r^2(\xi_1 + \xi_2)\|_1 + \|\xi_1 + \xi_2\|_{L^1(L^2)}) \|r^2(\xi_1 - \xi_2)\|_1^{1/2} \|\xi_1 - \xi_2\|_1^{1/2}. \quad (2.22) \]

**Proof.** We use (2.18) to estimate
\[
\int_\Pi \left( \int_\Pi G(r, z, r', z') |\xi_1(r, z)| r \, dr \, dz \right) |\xi_2(r', z')| r' \, dr' \, dz' \leq \left( \|r^2 \xi_1\|_1 + \|\xi_1\|_{L^1(L^2)} \right) \int_\Pi r^2 |\xi_2(r, z)| dr \, dz.
\]
Since we have
\[
\|r^2 \xi\|_{L^1(\Pi)} \sim \|r \xi\|_1 \leq \|r^2 \xi\|_1^{1/2} \|\xi\|_1^{1/2},
\]
the estimate (2.21) holds. The estimate (2.20) follows from (2.21). By the symmetry \(G(r, z, r', z') = G(r', z', r, z)\) and by setting \(\tilde{\xi} = \xi_1 - \xi_2\), we obtain (by suppressing the measure \(rr' \, dr \, dz \, dr' \, dz'\))
\[
\frac{1}{\pi} (E[\xi_1] - E[\xi_2]) = \int G(r, z, r', z') \xi_1(r, z) \xi_1(r', z') - \int G(r, z, r', z') \xi_2(r, z) \xi_2(r', z') = \int G(r, z, r', z') \tilde{\xi}(r, z) \xi_1(r', z') + \int G(r, z, r', z') \xi_2(r, z) \tilde{\xi}(r', z') = \int G(r, z, r', z') \tilde{\xi}(r, z) (\xi_1(r', z') + \xi_2(r', z')).
\]
Applying (2.21) implies (2.22). \(\square\)

Next we show the energy defined in (2.7) is equal to the kinetic energy \(\frac{1}{2} \int |u|^2 dx\).

**Lemma 2.4.** For axi-symmetric \(\xi \in (L^1_0 \cap L^\infty \cap L^1)(\mathbb{R}^3)\), the stream function \(\psi = \mathcal{G}[\xi]\) is continuous on \(\Pi\),
\[
\psi(r, z) \to 0 \quad \text{as} \quad |(r, z)| \to \infty, \quad (2.23)
\]
\[
\psi \in H^2_{loc}(\Pi), \quad \text{and} \quad -\frac{1}{r^2} \mathcal{L}\psi = \xi \quad \text{a.e.}
\]
Moreover, the axi-symmetric velocity \(\mathcal{K}[\xi]\) is continuous on \(\mathbb{R}^3\) and lies on \(L^2(\mathbb{R}^3)\) with
\[
E[\xi] = \frac{1}{2} \|K[\xi]\|_2^2 = \frac{1}{2} \int_{\mathbb{R}^3} \frac{1}{r^2} (|\partial_r \psi|^2 + |\partial_z \psi|^2) \, dx. \quad (2.24)
\]
**Proof.** By setting \(\omega(x) = r \xi(r, z) e_\theta(\theta)\), we observe
\[
\int |\omega| \, dx = \int r |\xi|^{1/2} |\xi|^{1/2} \, dx \leq \|r^2 \xi\|_1^{1/2} \|\xi\|_1^{1/2} \quad (2.25)
\]
and
\[
\int |\omega|^2 \, dx = \int r^2 |\xi|^2 \, dx \leq \|\xi\|_\infty \int r^2 |\xi| \, dx < \infty.
\]
It implies $\omega \in L^p(\mathbb{R}^3)$ for any $p \in [1, 2]$. By setting

$$\phi = \frac{\psi}{r},$$

we have $\phi = (4\pi|x|)^{-1} * \omega$ (see Subsection 2.1). This representation implies $\nabla \phi \in L^q(\mathbb{R}^3)$ for any $q \in (3/2, 6]$ and $\phi \in L^q(\mathbb{R}^3)$ for any $q' \in (3, \infty)$ by the Hardy-Littlewood-Sobolev inequality (e.g., see [103, p. 354]). Hence

$$\phi \in W^{1,q''}(\mathbb{R}^3) \quad \text{for any} \quad q'' \in (3, 6]. \quad (2.27)$$

We also observe $\omega \in L^\infty_{\text{loc}}(\mathbb{R}^3)$ since $\sup_{x \in U} |\omega(x)| = \sup_{r \in U} r |\xi(r, z)| \leq C_U \|\xi\|_\infty$ for any bounded set $U \subset \mathbb{R}^3$. Together with (2.27), we get $\phi \in W^{2,6}_{\text{loc}}(\mathbb{R}^3)$ which implies $\phi$ and $\nabla \phi$ are continuous on $\mathbb{R}^3$ by the Sobolev embedding (e.g., see [46, p. 284]). Hence (2.27) implies $\phi \in BUC^{1/2}(\mathbb{R}^3)$ by the Morrey’s inequality (e.g., see [46, p. 280]). From (2.26), we have

$$\psi(r, z) = r \phi_2(\frac{r}{\sqrt{2}}e_{x_1} + ze_{x_3}), \quad (r, z) \in \Pi, \quad (2.28)$$

where $\phi(x) = \phi_1(x)e_{x_1} + \phi_2(x)e_{x_2} + \phi_3(x)e_{x_3}$. Since $\phi_2$ is continuous on $\mathbb{R}^3$, the stream function $\psi$ is continuous on $\Pi$ by defining $\psi|_{r=0} = 0$.

In particular, the velocity $\mathcal{K}[\xi] = (\nabla \times \phi)$ is continuous on $\mathbb{R}^3$ with $\mathcal{K}[\xi] \in L^2(\mathbb{R}^3)$. By integration by parts, we have

$$2E[\xi] = \int r \psi \frac{dx}{r} = \int \omega \cdot \phi \, dx = -\int \Delta \phi \cdot \phi \, dx = \int \nabla \times (\nabla \times \phi) \cdot \phi \, dx$$

$$= \int (\nabla \times \phi) \cdot (\nabla \times \phi) \, dx = \int \frac{1}{r^2} \left( |\partial_r \psi|^2 + |\partial_z \psi|^2 \right) \, dx = \|\mathcal{K}[\xi]\|_2^2.$$

The integration by parts we did in the above is justified in the following way:

We take a radial function $\varphi \in C^\infty(\mathbb{R}^3)$ satisfying $\varphi(x) = 1$ for $|x| \leq 1$ and $\varphi(x) = 0$ for $|x| \geq 2$, and set the cut-off function on $\mathbb{R}^3$ by $\varphi_M(x) = \varphi(x/M)$ for any $M > 0$. Then we have

$$\int \nabla \times (\nabla \times \phi) \cdot (\varphi_M \phi) \, dx = \int (\nabla \times \phi) \cdot (\nabla \times (\varphi_M \phi)) \, dx$$

$$= \int \varphi_M (\nabla \times \phi) \cdot (\nabla \times \phi) \, dx + \int (\nabla \times \phi) \cdot ((\nabla \varphi_M) \times \phi) \, dx.$$

Since we know $\omega = \nabla \times (\nabla \times \phi) \in L^1(\mathbb{R}^3), \phi \in L^\infty(\mathbb{R}^3)$, and $\nabla \times \phi \in L^2(\mathbb{R}^3)$, it is enough to show that the last integral above vanishes as $M \to \infty$. We simply compute

$$\left| \int (\nabla \times \phi) \cdot ((\nabla \varphi_M) \times \phi) \, dx \right| \lesssim \|\nabla \phi\|_2 \|\nabla \varphi_M\|_4 \|\phi\|_4 \lesssim M^{-1/4} \|\nabla \phi\|_2 \|\phi\|_4 \to 0$$

as $M$ goes to infinity. Thus (2.24) follows.

To show (2.23), let $\varepsilon > 0$. By (2.19) in Lemma 2.2, there exists $R > 0$ such that

$$\sup_{r \geq R} |\psi(r, z)| < \varepsilon.$$
On the other hand, from $\phi \in L^{q'}(\mathbb{R}^3)$ for any $q' \in (3, \infty)$ and $\phi \in BUC^{(1/2)}(\mathbb{R}^3)$, we get

$$\lim_{|x| \to \infty} |\phi(x)| = 0.$$ 

It implies, from $|\psi| = r|\phi|$, there exists $Z > 0$ such that

$$\sup_{r \leq R, |z| \geq Z} |\psi(r, z)| \leq R \cdot \sup_{x_1^2 + x_2^2 \leq R^2, |x_3| \geq Z} |\phi(x)| < \varepsilon.$$ 

Thus we get

$$\sup_{r^2 + z^2 \geq R^2 + Z^2} |\psi(r, z)| \leq \varepsilon,$$

which implies (2.23).

Since $\phi$ lies on $H^2_{loc}(\mathbb{R}^3)$, we have $-\Delta \phi = \omega$ a.e. in $\mathbb{R}^3$ by the elliptic regularity theory, which implies

$$\psi \in H^2_{loc}(\Pi) \quad \text{and} \quad -\frac{L\psi}{r^2} = \xi \quad \text{a.e. in } \Pi.$$ 

□

3 | VARIATIONAL PROBLEM WITH COMPACTNESS AND UNIQUENESS

3.1 | Variational setting: Friedman-Turkington (1981)

For $0 < \mu, \nu, \lambda < \infty$, we set the space of admissible functions

$$P_{\mu, \nu, \lambda} = \left\{ \xi = \lambda 1_A : A \text{ some axi-sym. set } \subset \mathbb{R}^3, \frac{1}{2} \| r \xi \|_1 = \mu, \| \xi \|_1 \leq \nu \right\},$$

(3.1)

and we study the variational problem of maximizing the energy $E$ on $P_{\mu, \nu, \lambda}$. In the rest of the paper, we set

$$I_{\mu, \nu, \lambda} = \sup_{\xi \in P_{\mu, \nu, \lambda}} E[\xi]$$

(3.2)

and denote $S_{\mu, \nu, \lambda}$ by the set of maximizers of (3.2), that is,

$$S_{\mu, \nu, \lambda} = \{ \xi \in P_{\mu, \nu, \lambda} | E[\xi] = I_{\mu, \nu, \lambda} \}.$$ 

(3.3)

We note that any $z$-directional translation of $\xi \in S_{\mu, \nu, \lambda}$ lie on the same set $S_{\mu, \nu, \lambda}$. 
3.2 Theorems 3.1, 3.2: compactness and uniqueness of the set of maximizers

We introduce the following compactness theorem and uniqueness theorem, whose proofs will be given in later sections. By assuming these theorems for a moment, we will produce our main result (Theorems 1.2 and 1.1) in Subsection 3.4.

Theorem 3.1 (Compactness of maximizing sequence). Let \(0 < \mu, \nu, \lambda < \infty\). Let \(\{\xi_n\}_{n=1}^{\infty}\) be a sequence of non-negative axi-symmetric functions in \(\mathbb{R}^3\) and let \(\{a_n\}_{n=1}^{\infty}\) be a sequence of positive numbers such that

\[
a_n \to 0 \quad \text{as} \quad n \to \infty,
\]

\[
\limsup_{n \to \infty} \|\xi_n\|_1 \leq \nu, \quad \lim_{n \to \infty} \int_{\{x \in \mathbb{R}^3 \mid \|\xi_n(x) - \lambda\| \geq a_n\}} \xi_n \, dx = 0, \quad \lim_{n \to \infty} \frac{1}{2} \|r^2 \xi_n\|_1 = \mu,
\]

\[
\sup_n \|\xi_n\|_2 < \infty, \quad \text{and} \quad \lim_{n \to \infty} E[\xi_n] = I_{\mu, \nu, \lambda}.
\]

Then there exist a subsequence \(\{\xi_{n_k}\}_{k=1}^{\infty}\), a sequence \(\{c_k\}_{k=1}^{\infty} \subset \mathbb{R}\), and a function \(\xi \in S_{\mu, \nu, \lambda}\) such that

\[
\|r^2 (\xi_{n_k}(\cdot + c_k e_z) - \xi)\|_1 \to 0 \quad \text{as} \quad k \to \infty.
\]

In particular, the set \(S_{\mu, \nu, \lambda}\) is non-empty.

Theorem 3.2. [Uniqueness] There exists a constant \(M_1 > 0\) such that for any constants \(0 < \mu, \nu, \lambda < \infty\) satisfying \(\mu \nu^{-5/3} \lambda^{2/3} \leq M_1\),

\[
S_{\mu, \nu, \lambda} = \{\xi_{H(\lambda, a)}(\cdot + ce_z) \mid c \in \mathbb{R}\},
\]

where \(\xi_{H(\lambda, a)}\) is the Hill’s vortex for the vortex strength constant \(\lambda\) with the radius \(a = a(\lambda, \mu) > 0\) solving the equation \(\mu = (4/15)\pi \lambda a^5\).

Remark 3.3. The optimal constant of \(M_1\) satisfying the above theorem can be explicitly computed even if we do not need the exact value in the sequel. Indeed, we consider the Hill’s vortex \(\xi_H = \xi_{H(1, 1)}\) of unit strength on the unit ball. By a direct computation (or see (2.10) in Subsection 2.4), we know

\[
\xi_H \in P_{(4/15)\pi, (4/3)\pi, 1}.
\]

On the other hand, by Theorem 3.2, for any \(\nu > 0\) satisfying \((4/15)\pi \nu^{-5/3} \leq M_1\), we obtain

\[
\xi_H \in S_{(4/15)\pi, \nu, 1}.
\]

Since the admissible class \(P_{(4/15)\pi, \nu, 1}\) is increasing in \(\nu > 0\), we conclude that (3.5) holds if and only if \(\nu \geq (4/3)\pi\), which was inferred by [12, p. 21] in 1976. Hence, we can set

\[
M_1 = \frac{4}{15} \pi \left(\frac{4}{3} \pi\right)^{-5/3},
\]

and it is sharp.
We consider the case when the Euler equations admit the active scalar transport equation form (2.6). Therefore existence and uniqueness of solutions \( \xi(t) \) can be studied analogously as the two-dimensional case. We refer to [3, 25, 26, 38, 63, 79, 97, 100, 108] in various settings.

For our stability result, we just need a weak solution preserving the quantities listed in Subsection 2.2. Since our main interest lies not on existence of such weak solutions but on stability of them, we only briefly explain the existence (and the uniqueness) here. We simply take initial data \( \xi_0 \) regular enough in order to have existence and uniqueness of the corresponding weak solution \( \xi(t) \) with desirable conservations. More precisely we consider a non-negative axi-symmetric initial data \( \xi_0 \in (L^1_\text{w} \cap L^\infty \cap L^1)(\mathbb{R}^3) \) satisfying \( (r\xi_0) \in L^\infty(\mathbb{R}^3) \). Such a regularity guarantees existence and uniqueness of the corresponding weak solution of the Euler equations (1.1) (see Remark 3.5 after Lemma 3.4). This axi-symmetric solution satisfies (2.6) both in its weak form and in the renormalized sense of DiPerna-Lions [42] (refer to [87, Definitions 2, 3] for precise notions of such solutions). As a result, we obtain all conservations listed in Subsection 2.2 by using [87, Theorems 1, 2, 3] (also see [2, Section 5.1] for 2d case). We state the result in the form of a lemma below.

**Lemma 3.4.** For any non-negative axi-symmetric \( \xi_0 \in (L^1_\text{w} \cap L^\infty \cap L^1)(\mathbb{R}^3) \) satisfying \( (r\xi_0) \in L^\infty(\mathbb{R}^3) \), there exists a unique weak solution \( \xi \in L^\infty(0, \infty; (L^1_\text{w} \cap L^\infty \cap L^1)(\mathbb{R}^3)) \) of (2.6) for the initial data \( \xi_0 \) such that

\[
\xi(t) \geq 0 : \text{ axi-symmetric},
\]

\[
||\xi(t)||_q = ||\xi_0||_q, \quad 1 \leq q \leq \infty,
\]

\[
||r^2\xi(t)||_1 = ||r^2\xi_0||_1,
\]

\[
E[\xi(t)] = E[\xi_0], \quad \text{for all } t > 0,
\]

and, for any \( 0 < a < b < \infty \) and for each \( t > 0 \),

\[
\int_{\{x \in \mathbb{R}^3 \mid a < \xi(x,t) < b\}} \xi(x,t) \, dx = \int_{\{x \in \mathbb{R}^3 \mid a < \xi_0(x) < b\}} \xi_0(x) \, dx.
\]

**Remark 3.5.** For axi-symmetric data \( \xi_0 \in (L^1_\text{w} \cap L^\infty \cap L^1)(\mathbb{R}^3) \) with \( (r\xi_0) \in L^\infty(\mathbb{R}^3) \), the initial velocity \( u_0 := \mathcal{K}[\xi_0] \) lies on \( L^2(\mathbb{R}^3) \) by Lemma 2.4, and the initial vorticity \( \omega_0 := (r\xi_0)e_\theta \) lies on \( (L^1 \cap L^\infty)(\mathbb{R}^3) \) (see the estimate (2.25)). In this setting, existence with uniqueness of a weak solution can be obtained similarly as in the two-dimensional case. For instance, if one use the earlier paper [108], then the existence of a weak solution for such data can be found in [108, Theorem 4.1]. Thanks to the transport structure for \( \xi \) in (2.6), we can show the norm \( ||\omega||_{L^\infty(0,T;L^\infty(\mathbb{R}^3))} \) is finite for any finite \( T > 0 \) (e.g., see the \textit{a priori} estimate (1.26) on p. 56 in [108]). Then, the uniqueness is obtained by [108, Theorem 2.2]. Or equivalently, one may simply use [97, Theorem 3.3] (also see [38, Theorem 1]) for both existence and uniqueness.

**Remark 3.6.** The assumption for initial data in Lemma 3.4 might be weakened if one does not ask uniqueness of solutions. To have a clear presentation toward nonlinear stability which is our main goal, here we do not seek such a generalization. For readers interested in the existence
issue (without asking uniqueness) with desirable conservations (3.6), (3.7), we refer to [87] and references therein.

Remark 3.7. Regarding on smooth solutions $u(t)$ of the velocity form (1.1) of the three-dimensional Euler equations, it is still an open problem whether it can develop a finite-time blow-up from a $C^\infty$ initial data. In particular, it looks very difficult to convince a blow-up via direct numerical experiments as explained in [110]. For classical solutions $u(t)$ of (1.1), local-in-time existence and uniqueness in Sobolev space $H^s$ for $s > 5/2$ [44, 66] and in Hölder space $C^{1,\alpha}$ for $\alpha > 0$ [56, 74] have been known. Very recently, [45] showed that the latter case for small $\alpha > 0$ admits a finite-time blow-up. It is interesting that the initial velocity $u_0 \in C^{1,\alpha}$ developing a blow-up in [45] is axi-symmetric without swirl while the corresponding initial relative vorticity $\xi_0(\cdot = \omega_0^\theta \alpha \xi_0 / r)$ does not lie on $L^\infty(\mathbb{R}^3)$. More precisely, it is not bounded on the axis $\{r = 0\}$. As noted in [45, p. 10], in order to have a $C^\infty$ vorticity $\omega = \omega_0^\theta e_\theta$, it is a necessary condition that $\omega_0^\theta$ vanishes linearly (so $\xi$ is at least bounded) on the axis $\{r = 0\}$ (also see [78]). Lastly, regarding on weak solutions $u(t)$ of (1.1), even uniqueness fails for any solenoidal vector field $u_0 \in L^2(\mathbb{R}^3)$ by [41, 115] (also see [40, 98, 101]).

3.4 Proof of nonlinear stability (Theorems 1.2, 1.1)

Now we are ready to prove Theorem 1.2 by assuming Theorem 3.1 (compactness) and Theorem 3.2 (uniqueness).

Proof of Theorem 1.2. We recall that $\xi_0 = \xi_0(1,1) = 1_B$ is the Hill’s vortex (2.11), where $B$ is the unit ball centered at the origin. Let us suppose that the conclusion of Theorem 1.2 were false. Then there exist a constant $\varepsilon_0 > 0$ and a sequence $\{\xi_{0,n}\}_{n=1}^\infty$ of non-negative axi-symmetric functions, and a sequence $\{t_n\}_{n=1}^\infty$ of non-negative numbers such that, for each $n \geq 1$, we have $\xi_{0,n} \in L^\infty(\mathbb{R}^3)$,

$$\|\xi_{0,n} - \xi_0\|_{L^1 \cap L^2} + \|r^2(\xi_{0,n} - \xi_0)\|_1 \leq \frac{1}{n^2},$$

and

$$\inf_{\tau \in \mathbb{R}} \{\|\xi_n(t_n, \cdot + \tau e_z) - \xi_0\|_{L^1 \cap L^2} + \|r^2(\xi_n(t_n, \cdot + \tau e_z) - \xi_0)\|_1\} \geq \varepsilon_0.$$

where $\xi_n(t)$ is the global-in-time weak solution of (2.6) for the initial data $\xi_{0,n}$ obtained by Lemma 3.4.

We set $\mu_0 = (4/15)\pi$, $\lambda_0 = 1$ and fix any $\nu_0 > 0$ satisfying $\mu_0 \nu_0^{-5/3} \lambda_0^{2/3} \leq M_1$, where $M_1$ is the constant in Theorem 3.2. Then the theorem says

$$S_{\mu_0, \nu_0, \lambda_0} = \{\xi_0(\cdot + ce_z) \mid c \in \mathbb{R}\}.$$  

By (3.8) and the estimate (2.22),

$$\lim_{n \to \infty} E[\xi_{0,n}] = E[\xi_0] = I_{\mu_0, \nu_0, \lambda_0}.$$
We write \( \xi_n = \xi_n(t_n) \) by suppressing \( t_n \). Thus, by the conservations \( (3.6) \), we get

\[
\begin{align*}
\lim_{n \to \infty} \frac{1}{2} \| r^2 \xi_n \|_1 &= \frac{1}{2} \| r^2 \xi_H \|_1 = \mu_0, \\
\lim_{n \to \infty} \| \xi_n \|_1 &= \| \xi_H \|_1 \leq \nu_0,
\end{align*}
\]

(3.11)

We claim

\[
\lim_{n \to \infty} \int_{\{ x \in \mathbb{R}^3 : |\xi_n(x) - \lambda_0| \geq 1/n \}} \xi_n \, dx = 0.
\]

(3.12)

To prove, we observe that the conservation \( (3.7) \) (together with \( (3.6) \) for \( q = 1 \)) implies

\[
\int_{\{ x \in \mathbb{R}^3 : |\xi_0,n(x) - \lambda_0| \geq 1/n \}} \xi_0,n \, dx =: I_n.
\]

By setting \( D(n) = \{ x \in \mathbb{R}^3 : |\xi_0,n(x) - \lambda_0| \geq 1/n \} \) and by recalling \( \xi_H = 1_B = \lambda_0 1_B \), we observe

\[
\| \xi_0,n - \xi_H \|_1 \geq \| \xi_0,n - \xi_H \|_{L^1(D(n) \cap B)}
\]

\[
= \int_{D(n) \cap B} |\xi_0,n(x) - \lambda_0| \, dx \geq \int_{D(n) \cap B} \frac{1}{n} \, dx = \frac{1}{n} |D(n) \cap B|.
\]

Thus we estimate

\[
I_n \leq \| \xi_0,n \|_{L^1(D(n))} = \| \xi_0,n \|_{L^1(D(n) \cap B)} + \| \xi_0,n \|_{L^1(D(n) \cap B^C)}
\]

\[
\leq \| \xi_0,n - \xi_H \|_{L^1(D(n) \cap B)} + \| \xi_H \|_{L^1(D(n) \cap B)} + \| \xi_0,n - \xi_H \|_{L^1(D(n) \cap B^C)}
\]

\[
\leq \| \xi_0,n - \xi_H \|_{L^1(B)} + \| \xi_H \|_{L^1(D(n) \cap B)} + \| \xi_0,n - \xi_H \|_{L^1(B^C)}
\]

\[
\leq \| \xi_0,n - \xi_H \|_1 + |D(n) \cap B| \leq (n+1) \| \xi_0,n - \xi_H \|_1 \leq \frac{n+1}{n^2} \to 0 \quad \text{as} \quad n \to \infty,
\]

which shows \( (3.12) \).

Now we apply Theorem 3.1 to the sequence \( \{ \xi_n \}_{n=1}^{\infty} \) with the choice \( a(n) = 1/n \) to obtain a subsequence (still denoted by \( \{ \xi_n \} \) after reindexing), \( \{ c_n \}_{n=1}^{\infty} \subset \mathbb{R} \), and \( \xi \in S_{\mu_0,\nu_0,\lambda_0} \) such that

\[
\| r^2(\xi(n)\cdot + c_ne_z) - \xi) \|_1 \to 0 \quad \text{as} \quad n \to \infty.
\]

(3.13)

By \( (3.10) \), we know \( \xi = \xi_H(\cdot + ce_z) \) for some \( c \in \mathbb{R} \). We may assume \( c = 0 \) by shifting \( c_n \) by the constant \( c \).

By uniform boundedness in \( L^2 \) from \( (3.11) \), the sequence \( \{ \xi_n(\cdot + c_ne_z) \} \) subsequently converges weakly in \( L^2(\mathbb{R}^3) \), and the weak limit agrees with \( \xi_H \) by \( (3.13) \). Thus, convergence of the norm

\[
\lim_{n \to \infty} \| \xi_n(\cdot + c_ne_z) \|_2 = \lim_{n \to \infty} \| \xi_n \|_2 = \| \xi_H \|_2
\]

from \( (3.11) \) gives the strong convergence in \( L^2 \) (still denoted by \( \{ \xi_n \} \))

\[
\xi_n(\cdot + c_ne_z) \to \xi_H \quad \text{in} \quad L^2(\mathbb{R}^3) \quad \text{as} \quad n \to \infty.
\]

(3.14)
In particular, $\xi_n(\cdot + c_ne_z) \to \xi_H$ in $L^1(U)$ for any bounded $U \subset \mathbb{R}^3$ by using Hölder's inequality. Since $\text{spt} \xi_H = B$ and

$$
\|\xi_n(\cdot + c_ne_z) - \xi_H\|_{L^1(\mathbb{R}^3)} = \|\xi_n(\cdot + c_ne_z) - \xi_H\|_{L^1(B)} + \|\xi_n(\cdot + c_ne_z)\|_{L^1(B^c)} \\
= \|\xi_n(\cdot + c_ne_z) - \xi_H\|_{L^1(B)} + \|\xi_n(\cdot + c_ne_z)\|_{L^1(\mathbb{R}^3)} - \|\xi_n(\cdot + c_ne_z)\|_{L^1(B)},
$$

the convergences (3.11) and (3.14) imply

$$
\limsup_{n \to \infty} \|\xi_n(\cdot + c_ne_z) - \xi_H\|_{L^1(\mathbb{R}^3)} \\
\leq \lim_{n \to \infty} \|\xi_n(\cdot + c_ne_z)\|_{L^1(\mathbb{R}^3)} - \liminf_{n \to \infty} \|\xi_n(\cdot + c_ne_z)\|_{L^1(B)} \\
\leq \|\xi_H\|_{L^1(\mathbb{R}^3)} - \|\xi_H\|_{L^1(B)} = 0.
$$

In sum, we have $\xi_n(\cdot + c_ne_z) \to \xi_H$ in $(L^1_w \cap L^2 \cap L^1)(\mathbb{R}^3)$, which contradicts to (3.9) because

$$
0 = \lim_{n \to \infty} \left\{ \|\xi_n(\cdot + c_ne_z) - \xi_H\|_{L^1(\mathbb{R}^3)} + \|r^2(\xi_n(\cdot + c_ne_z) - \xi_H)\|_1 \right\} \\
\geq \liminf_{n \to \infty} \left( \inf_{\tau \in \mathbb{R}} \left\{ \|\xi_n(\cdot + \tau e_z) - \xi_H\|_{L^1(\mathbb{R}^3)} + \|r^2(\xi_n(\cdot + \tau e_z) - \xi_H)\|_1 \right\} \right) \geq \varepsilon_0.
$$

\[ \square \]

**Proof of Theorem 1.2.** For axi-symmetric set $A_0$ satisfying (1.5), if we set the initial data $\xi_0 = 1_{A_0}$, then the data satisfies (1.6). Since

$$
\int_{A_0 \Delta B} 1 \, dx = \|\xi_0 - \xi_H\|_1 = \|\xi_0 - \xi_H\|_2^2, \quad \int_{A_0 \Delta B} r^2 \, dx = \|r^2(\xi_0 - \xi_H)\|_{L^1(\mathbb{R}^3)},
$$

we obtain Theorem 1.1 by applying Theorem 1.2 into the unique weak solution $\xi(t) = 1_{A_t}$ obtained from Lemma 3.4.

\[ \square \]

It remains to show Theorems 3.1 and 3.2. In Section 4, as a warm-up section, we revisit the existence result of a maximizer for (3.2) due to [50]. In fact, we show that such a maximizer maximizes the energy in a slightly larger class. Then, in Section 5, we prove that every maximizer gives a steady vortex ring by constructing a sequence of admissible perturbations. In Section 6.1, we obtain Theorem 3.1 via concentrated compactness due to [77]. Lastly, in Section 7, we apply the uniqueness result of [4] to prove Theorem 3.2.

## 4 EXISTENCE AND PROPERTIES OF MAXIMIZERS

In this section, our goal is to show the existence of a maximizer (Theorem 4.2) below, which will be used in Section 6.1 when proving Theorem 3.1.
Variational problem in larger spaces

Before stating the existence theorem, we introduce some other spaces of admissible functions. For $0 < \mu, \nu, \lambda < \infty$, we set the following spaces of admissible functions:

$$
P'_{\mu, \nu, \lambda} = \left\{ \xi \in L^\infty(\mathbb{R}^3) \mid \xi : \text{axi-symmetric}, 0 \leq \xi \leq \lambda, \frac{1}{2} \| r^2 \xi \|_1 = \mu, \| \xi \|_1 \leq \nu \right\},
$$

$$
P''_{\mu, \nu, \lambda} = \left\{ \xi \in L^\infty(\mathbb{R}^3) \mid \xi : \text{axi-symmetric}, 0 \leq \xi \leq \lambda, \frac{1}{2} \| r^2 \xi \|_1 \leq \mu, \| \xi \|_1 \leq \nu \right\}.
$$

(4.1)

**Remark 4.1.** We observe the set relations:

$$
P''_{\mu, \nu, \lambda} \supset P'_{\mu, \nu, \lambda} \supset P_{\mu, \nu, \lambda},
$$

(4.2)

and note that $P''_{\mu, \nu, \lambda}$ is closed under the weak-$L^2$ topology. That is, if $\{\xi_n\}$ is a sequence in $P''_{\mu, \nu, \lambda}$ and if $\xi_n \to \xi$ in $L^2(\mathbb{R}^3)$ for some $\xi \in L^2(\mathbb{R}^3)$ as $n \to \infty$, then the weak-limit $\xi$ lies on $P''_{\mu, \nu, \lambda}$.

As in (3.2), (3.3), we set the variational problems

$$
I'_{\mu, \nu, \lambda} = \sup_{\xi \in P'_{\mu, \nu, \lambda}} E[\xi], \quad I''_{\mu, \nu, \lambda} = \sup_{\xi \in P''_{\mu, \nu, \lambda}} E[\xi]
$$

(4.3)

and denote $S'_{\mu, \nu, \lambda}, S''_{\mu, \nu, \lambda}$ the sets of maximizers of (4.3), respectively.

**Theorem 4.2.** For $0 < \mu, \nu, \lambda < \infty$, we have

$$
I_{\mu, \nu, \lambda} = I'_{\mu, \nu, \lambda} = I''_{\mu, \nu, \lambda} \in (0, \infty) \quad \text{and} \quad S_{\mu, \nu, \lambda} = S'_{\mu, \nu, \lambda} = S''_{\mu, \nu, \lambda} \neq \emptyset.
$$

We prove the above theorem by building a series of lemmas in this section. In fact, we closely follow the approach of the proof of [50, Theorem 2.1]. Since some ingredient of the proof is needed later again (e.g., in Section 6.1 when proving Theorem 3.1), we reproduce the proof here. More specifically, the notion of Steiner symmetrization (see Proposition 4.8) and energy convergence lemma (see Lemma 4.10) will be used again in Section 6.1.

**Remark 4.3.** [50, Theorem 2.1] says, in our terminology, that there exists a compactly supported function

$$
\xi \in \left( S_{\mu, \nu, \lambda} \cap S'_{\mu, \nu, \lambda} \right)
$$

satisfying the symmetry $\xi(r, z) = \xi(r, -z)$ together with the property

$$
\xi = \lambda 1_{\{\Psi > 0\}}, \quad \Psi := G[\xi] - \frac{1}{2} Wr^2 - \gamma
$$

for some $W > 0$ and $\gamma \geq 0$. Such a function $\xi$ can be obtained from a weak-limit of a sequence $\{\xi_\beta\}_{\beta > 0}$ as $\beta \to 0$ where $\xi_\beta$ is a maximizer for the penalized energy functional (1.13) (see [50, Lemma 5.1]).
In the sequel, we frequently reduce the variational problems (3.2), (4.3) to the case \( \nu = \lambda = 1 \) by the scaling

\[
\xi_{\nu, \lambda}(x) = \frac{1}{\lambda} \xi \left( \left( \frac{\nu}{\lambda} \right)^{1/3} x \right).
\]

It is easy to check that if \( \xi \in P_{\mu, \nu, \lambda} \), then \( \xi_{\nu, \lambda} \in P_{M, 1, 1} \) for \( M := \mu \nu^{-5/3} \lambda^{2/3} \) and \( E[\xi_{\nu, \lambda}] = \lambda^{1/3} \nu^{-7/3} E[\xi] \) due to

\[
G[\xi_{\nu, \lambda}](x) = \frac{1}{\lambda} \left( \frac{\nu}{\lambda} \right)^{-4/3} G[\xi] \left( \left( \frac{\nu}{\lambda} \right)^{1/3} x \right).
\]

Thus we get

\[
\xi \in S_{\mu, \nu, \lambda} \text{ if and only if } \xi_{\nu, \lambda} \in S_{M, 1, 1}.
\] (4.4)

From now on, we abbreviate the notations as \( P_{\mu} = P_{\mu, 1, 1} \), \( I_{\mu} = I_{\mu, 1, 1} \), and \( S_{\mu} = S_{\mu, 1, 1} \). Similarly, we abbreviate the notations as \( P'_{\mu} = P'_{\mu, 1, 1} \), \( I'_{\mu} = I'_{\mu, 1, 1} \), \( S'_{\mu} = S'_{\mu, 1, 1} \), \( P''_{\mu} = P''_{\mu, 1, 1} \), \( I''_{\mu} = I''_{\mu, 1, 1} \), and \( S''_{\mu} = S''_{\mu, 1, 1} \).

As a warm-up, we first check that the maximum values \( I_{\mu}, I'_{\mu}, I''_{\mu} \) are non-trivial for each \( \mu > 0 \).

**Lemma 4.4.** Let \( \mu \in (0, \infty) \). Then

\[
0 < I_{\mu} \leq I'_{\mu} \leq I''_{\mu} < \infty.
\]

**Proof.** First we get

\[
0 < \sup_{\xi \in P_{\mu}} E[\xi] = I_{\mu}
\]

since any \( \xi \) in \( P_{\mu} \neq \emptyset \) is non-negative, and the kernel \( G \) is positive a.e. The set relations (4.2) give

\[
0 < I_{\mu} \leq I'_{\mu} \leq I''_{\mu}.
\]

Lastly, from the estimate (2.20), we have

\[
I''_{\mu} = \sup_{\xi \in P_{\mu}''} E[\xi] \lesssim (1 + \mu) \mu^{1/2} < \infty.
\]

\[
\square
\]

The following lemma is useful when we need convergence of the energy for a weak-convergent sequence \( \{\xi_n\} \) when the energy of each member \( \xi_n \) is uniformly concentrated in a fixed bounded set.
Lemma 4.5. For non-negative axi-symmetric functions $\xi_1, \xi_2 \in (L^1 \cap L^2 \cap L^1)(\mathbb{R}^3)$ and for axi-symmetric set $U \subset \mathbb{R}^3$, we have

$$|E[\xi_1] - E[\xi_2]| \leq \frac{1}{4\pi} \left| \int_U \int_U G(x, y)(\xi_1(x)\xi_1(y) - \xi_2(x)\xi_2(y)) \, dx \, dy \right|$$

\begin{align*}
&\quad + \int_{\mathbb{R}^3 \setminus U} \xi_1 G[\xi_1] \, dx + \int_{\mathbb{R}^3 \setminus U} \xi_2 G[\xi_2] \, dx. \\
\end{align*}

(4.5)

Proof. Let $\xi \in (L^1 \cap L^2 \cap L^1)(\mathbb{R}^3)$ be non-negative and axi-symmetric. By setting $\psi = G[\xi]$, we decompose

$$4\pi E[\xi] = \int 2\pi \psi \xi \, dx = \int_U + \int_{\mathbb{R}^3 \setminus U} =: I + II,$$

and

$$I = \int_U \xi(x) \int_{\mathbb{R}^3} G(x, y) \xi(y) \, dy \, dx = \int_U \int_U + \int_U \int_{\mathbb{R}^3 \setminus U} =: I_1 + I_2.$$ 

From the symmetry of $G(x, y) = G(y, x) > 0$, we estimate the last term $I_2$ by

$$I_2 = \int \xi(x) \int_{\mathbb{R}^3 \setminus U} G(x, y) \xi(y) \, dy \, dx = \int_{\mathbb{R}^3 \setminus U} \xi(x) \int G(x, y) \xi(y) \, dy \, dx$$

$$\leq \int_{\mathbb{R}^3 \setminus U} \xi(x) \int_{\mathbb{R}^3} G(x, y) \xi(y) \, dy \, dx = \int_{\mathbb{R}^3 \setminus U} 2\pi \psi(x) \xi(x) \, dx = II.$$ 

Thus we get

$$4\pi E[\xi] \leq I_1 + 2II,$$

which gives

$$0 \leq E[\xi] - \frac{1}{4\pi} \int_U \int_U G(x, y)\xi(x)\xi(y) \, dy \, dx \leq \int_{\mathbb{R}^3 \setminus U} \psi(x)\xi(x) \, dx.$$ 

By applying the above estimate into any pair $(\xi_1, \xi_2)$, we obtain (4.5). \qed

We check that the kernel $G(x, y)$ is locally square integrable due to its logarithm behavior (2.3) near $x = y$. This lemma will be used not only in this section but also in Section 6.1.

Lemma 4.6. The kernel $G(x, y)$ satisfies

$$G \in L^2_{loc}(\mathbb{R}^3 \times \mathbb{R}^3).$$

In particular, we have

$$\int_{B_M(r,z)} r'|G(r,z,r',z')|^2 \, dr' \, dz' \lesssim (Mr^4 + M^{7/2}r^{3/2}), \quad M > 0, \quad (r, z) \in \Pi,$$

where $B_M(r, z) = \{(r', z') \in \Pi \mid |(r, z) - (r', z')| < M\}$ as defined in (2.17).
Proof. We use the estimate (2.4) with $\tau = 1/4$:

$$G(r, z, r', z') \lesssim \frac{(rr')^{3/4}}{|(r, z) - (r', z')|^{1/2}}.$$  

Thus we estimate, by change of variables $r'/r = \tilde{r}$, $z'/r = \tilde{z}$,

$$\int_{B_M(r,z)} r'|G(r, z, r', z')|^2 dr'dz' \lesssim \int_{B_M(r,z)} r' \frac{(rr')^{3/2}}{|(r, z) - (r', z')|} dr'dz' = r^5 \int_{B_{Mr^{-1}}(1,0)} \frac{(\tilde{r})^{5/2}}{|(1, 0) - (\tilde{r}, \tilde{z})|} d\tilde{r}d\tilde{z} \leq r^5 (1 + Mr^{-1})^{5/2} \int_{B_{Mr^{-1}}(1,0)} \frac{1}{|{(1, 0)} - (\tilde{r}, \tilde{z})|} d\tilde{r}d\tilde{z} \lesssim r^5 (1 + M^{5/2}r^{-5/2}) \int_0^{Mr^{-1}} 1d\rho = (Mr^4 + M^{7/2}r^{3/2}).$$  

Thanks to the above estimate, we have $G \in L^2_{\text{loc}}(\mathbb{R}^3 \times \mathbb{R}^3)$. Indeed, for any $M > 0$, we estimate

$$\int_{B_M(0)} \int_{B_M(0)} |G(x, y)|^2 dydx = 4\pi^2 \int_{B_M(0)} r \int_{B_M(0)} r'|G(r, z, r', z')|^2 dr'dz' drdz \lesssim \int_{B_M(0)} r \int_{B_{2M}(r,z)} r'|G(r, z, r', z')|^2 dr'dz' drdz \lesssim \int_{B_M(0)} r(Mr^4 + M^{7/2}r^{3/2}) drdz \lesssim M^6 \int_{B_M(0)} 1dz \lesssim M^8.$$

\[\square\]

### 4.2 Existence of a maximizer

In this subsection, our goal is to prove the following existence lemma:

**Lemma 4.7.** Let $0 < \mu < \infty$. Then $S''_\mu \neq \emptyset$.

In order to prove Lemma 4.7, we introduce Steiner symmetrization (symmetrical rearrangement about the plane $\{z = 0\}$) with its property as in [50]. Here we say that a non-negative function $f$ on $\Pi$ satisfies the *monotonicity* condition (about the plane $\{z = 0\}$) if

$$f(r, z) = f(r, -z), \quad (r, z) \in \Pi \quad \text{and}$$

for each fixed $r > 0$, $f(r, z)$ is a non-increasing function of $z$ for $z > 0$.

**Proposition 4.8** (Steiner symmetrization). Let $p \in [2, \infty]$. For axi-symmetric $\zeta \geq 0$ satisfying $\zeta \in (L^1 \cap L^p \cap L^1)(\mathbb{R}^3)$, there exists an axi-symmetric $\zeta^* \geq 0$ satisfying the monotonicity
condition (4.7).

\[ \|\xi^*\|_q = \|\xi\|_q, \quad 1 \leq q \leq p, \]
\[ \|r^2 \xi^*\|_1 = \|r^2 \xi\|_1, \quad \text{and} \]
\[ \int_{\{x \in \mathbb{R}^3 \mid \xi^*(x) \in I\}} \xi^* \, dx = \int_{\{x \in \mathbb{R}^3 \mid \xi(x) \in I\}} \xi \, dx \quad \text{for any interval} \quad I \subset \mathbb{R}. \quad (4.8) \]

In particular, it satisfies

\[ E[\xi^*] \geq E[\xi]. \quad (4.9) \]

**Proof.** The symmetrical rearrangement \( \xi^* \) of \( \xi \) about the plane \( z = 0 \) satisfies the properties in (4.8) (e.g., see [75, Section 3.3]). The inequality (4.9) is a consequence of Riesz rearrangement inequality [93] (or see [75, p. 84]). We also refer to [49, Appendix I] which is an adaptation of Pólya-Szegö inequality [90]. \( \square \)

The following lemma says that the kinetic energy is concentrated in a bounded domain when \( \xi \) satisfies the monotonicity condition (4.7). We present its proof in Appendix A while a similar estimate can be found in [50, Lemma 3.5].

**Lemma 4.9.** Let \( \xi \in (L^1_w \cap L^2 \cap L^1)(\mathbb{R}^3) \) be an axi-symmetric nonnegative function satisfying the monotonicity condition (4.7). Then we have

\[ \int_{\mathbb{R}^3 \setminus Q} \xi \mathcal{G}[\xi] \, dx \lesssim \left( \frac{1}{\sqrt[4]{A}} + \frac{1}{R^2} \right) \left( \|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1 \right)^2, \quad (4.10) \]

where

\[ Q = Q_{A,R} = \{x \in \mathbb{R}^3 \mid |z| < AR, \quad r < R\} \]

provided \( R \geq 1 \) and \( A \geq 1 \).

The following lemma ensures convergence of the energy for any bounded sequence in \((L^1_w \cap L^2 \cap L^1)\) satisfying the monotonicity condition (4.7). This lemma was implicitly appeared in the proof of [50, Theorem 2.1] while it was not explicitly written in the form of a lemma. We will use the lemma both in this section and in Section 6.1.

**Lemma 4.10.** Let \( \{\xi_n\}_{n=1}^{\infty} \) be a sequence of axi-symmetric non-negative functions on \( \mathbb{R}^3 \) such that

\[ \xi_n \quad \text{satisfies the monotonicity condition} \quad (4.7) \quad \text{for each} \quad n, \]
\[ \sup_n \{ \|\xi_n\|_{L^1 \cap L^2} + \|r^2 \xi_n\|_1 \} < \infty, \quad \text{and} \]
\[ \xi_n \rightharpoonup \xi \quad \text{in} \quad L^2(\mathbb{R}^3) \quad \text{as} \quad n \to \infty \quad \text{for some non-negative axi-sym.} \quad \xi \in L^2(\mathbb{R}^3). \]

Then we have convergence of the energy:

\[ E[\xi_n] \to E[\xi] \quad \text{as} \quad n \to \infty. \]
Proof. First, we observe, by the weak convergence \( \xi_n \rightharpoonup \xi \) in \( L^2(\mathbb{R}^3) \),
\[
\|\xi\|_{L^1(\mathbb{R}^3)} + \|r^2 \xi\|_1 \leq C \quad \text{for some } C > 0.
\]
We set a bounded domain
\[
Q = Q_{A,R} = \{ x \in \mathbb{R}^3 \mid |z| < AR, \quad r < R \}.
\]
for \( R \geq 1 \) and \( A \geq 1 \). Then, by (4.5) of Lemma 4.5, we have
\[
|E[\xi_n] - E[\xi]| \leq \frac{1}{4\pi} \left| \int_Q \int_Q G(x,y)(\xi_n(x)\xi_n(y) - \xi(x)\xi(y)) \, dx \, dy \right|
+ \int_{\mathbb{R}^3 \setminus Q} \xi_n G[\xi_n] \, dx + \int_{\mathbb{R}^3 \setminus Q} \xi G[\xi] \, dx.
\]
Since \( \xi_n \) satisfies the monotonicity condition (4.7) for each \( n \geq 1 \), so does \( \xi \). Thus we can estimate, by (4.10) of Lemma 4.9,
\[
\int_{\mathbb{R}^3 \setminus Q} \xi G[\xi] \, dx \lesssim \left( \frac{1}{\sqrt{A}} + \frac{1}{R^2} \right) \quad \text{and} \quad \sup_n \int_{\mathbb{R}^3 \setminus Q} \xi_n G[\xi_n] \, dx \lesssim \left( \frac{1}{\sqrt{A}} + \frac{1}{R^2} \right).
\]
Since \( G(x,y) \in L^2(Q \times Q) \) by Lemma 4.6 and \( \xi_n(x)\xi_n(y) \rightharpoonup \xi(x)\xi(y) \) in \( L^2(Q \times Q) \), sending \( n \to \infty \) and \( A, R \to \infty \) imply convergence of the energy. \( \square \)

Now we are ready to prove Lemma 4.7.

Proof of Lemma 4.7. Let \( \{ \xi_n \} \subset P''_\mu \) be a sequence satisfying \( E[\xi_n] \nearrow I''_\mu \). By the Steiner symmetrization (Proposition 4.8 with \( p = \infty \)), we obtain the corresponding sequence \( \{ \xi^*_n \} \) in \( P''_\mu \) satisfying the monotonicity condition (4.7). Since \( \{ \xi^*_n \} \) is uniformly bounded in \( L^2(\mathbb{R}^3) \) by interpolation between \( L^1 \) and \( L^\infty \), by choosing a subsequence (still denoted by \( \{ \xi^*_n \} \) for simplicity), there exists a non-negative axi-symmetric function \( \xi \in L^2(\mathbb{R}^3) \) satisfying \( \xi^*_n \rightharpoonup \xi \) in \( L^2 \). Hence we can apply Lemma 4.10 for \( \{ \xi^*_n \} \) to get
\[
\lim_{n \to \infty} E[\xi^*_n] = E[\xi].
\]
Since the weak-limit \( \xi \) lies on \( P''_\mu \) and
\[
I''_\mu \geq \lim_{n \to \infty} E[\xi^*_n] \geq \lim_{n \to \infty} E[\xi_n] = I''_\mu,
\]
we conclude \( \xi \in S''_\mu \). \( \square \)

### 4.3 Properties of the set of maximizers

Next we show \( S'_\mu = S''_\mu \).
**Lemma 4.11.** Let $\mu \in (0, \infty)$. Then

\[ S'_\mu = S''_\mu \neq \emptyset \quad \text{and} \quad I'_\mu = I''_\mu. \]

**Proof.** Let us take any $\xi \in S''_\mu$ by recalling $S''_\mu \neq \emptyset$ from Lemma 4.7. We claim

\[ \xi \in P'_\mu. \quad (4.11) \]

For a contradiction, we suppose

\[ \mu > \frac{1}{2} \int r^2 \xi dx =: \mu_0, \]

that is, we assume $\xi \in P''_\mu \setminus P'_\mu$. From $I''_\mu > 0$ by Lemma 4.4, we have $\xi \not\equiv 0$ so $\mu_0 > 0$. If we define the (relative) translation $\xi_{\tau}$ of $\xi$ away from $z-$axis for $\tau > 0$ by

\[ \xi_{\tau}(r, z) = \begin{cases} \frac{r-\tau}{r} \xi(r-\tau, z) & \text{for } r \geq \tau, \\ 0 & \text{for } 0 < r < \tau, \end{cases} \quad (4.12) \]

then we have

\[ 0 \leq \xi_{\tau} \leq \sup_{r \geq \tau} \left( \frac{r-\tau}{r} \right) \cdot \|\xi\|_{\infty} \leq 1, \]

\[ \|\xi_{\tau}\|_1 = 2\pi \int r \xi_{\tau} dr dz = 2\pi \int_{r \geq \tau} (r - \tau) \xi(r - \tau, z) dr dz = 2\pi \int_{P} r \xi dr dz = \|\xi\|_1 \leq 1, \]

\[ \frac{1}{2} \int r^2 \xi_{\tau} dx = \pi \int_{r \geq \tau} (r - \tau)r^2 \xi(r - \tau, z) dr dz \]

\[ = \pi \int_{P} (r + \tau)^2 r \xi dr dz = \mu_0 + \frac{1}{2} \int (2\tau r + \tau^2) \xi dx. \quad (4.13) \]

Thus we can take some constant $\tau > 0$ such that $\frac{1}{2} \int r^2 \xi_{\tau} dx = \mu$, that is, we have $\xi_{\tau} \in P'_\mu$. On the other hand, we observe

\[ E[\xi_{\tau}] > E[\xi] \]

by exploiting the form (2.2) of the kernel $G$. Indeed, we observe, for $(r, z), (r', z') \in P$,

\[ G(r + \tau, z, r' + \tau, z') = \frac{\sqrt{(r + \tau)(r' + \tau)}}{2\pi} F\left( \frac{(r - r')^2 + (z - z')^2}{(r + \tau)(r' + \tau)} \right) \]

\[ > \frac{\sqrt{rr'}}{2\pi} F\left( \frac{(r - r')^2 + (z - z')^2}{rr'} \right) = G(r, z, r', z') \quad (4.14) \]

because $F(\cdot)$ is strictly decreasing (see Subsection 2.1). Thus, we have
\[
\frac{1}{\pi} E[\xi] = \int \int \int \int r r' G(r, z, r', z') \xi(r', z') \xi(r, z) dr' dz' dr dz \\
= \int \int \int_{r > \tau, r' > \tau} (r - \tau)(r' - \tau) G(r, z, r', z') \xi(r', z') \xi(r - \tau, z) dr' dz' dr dz \\
= \int \int \int r r' G(r + \tau, z, r' + \tau, z') \xi(r', z') \xi(r, z) dr' dz' dr dz \\
> \int \int \int r r' G(r, z, r', z') \xi(r', z') \xi(r, z) dr' dz' dr dz = \frac{1}{\pi} E[\xi],
\]

where the last inequality comes from (4.14) and non-triviality of \( \xi \geq 0 \). Hence we get

\[ I'_\mu \geq E[\xi] > E[\xi] = I''_\mu, \]

which contradicts to \( I'_\mu \leq I''_\mu \) obtained from Lemma 4.4. Hence \( \mu_0 = \mu \) so we get the claim (4.11).

Since \( \xi \in S''_\mu \cap P'_\mu \) implies

\[ I'_\mu \leq I''_\mu = E[\xi] \leq I'_\mu, \]

we get \( I'_\mu = I''_\mu \) and \( \xi \in S'_\mu \). In sum, we have shown \( S''_\mu \subset S'_\mu \). Due to \( P'_\mu \subset P''_\mu \) and \( I'_\mu = I''_\mu \), we get \( S'_\mu = S''_\mu \).

Now we show \( S'_\mu = S''_\mu \).

**Lemma 4.12.** Let \( \mu \in (0, \infty) \). Then

\[ S'_\mu \neq \emptyset \quad \text{and} \quad I'_\mu = I''_\mu. \]

**Proof.** Let us take any \( \xi \in S'_\mu \) by recalling \( S'_\mu \neq \emptyset \) by Lemma 4.11. In order to show \( \xi \in S_\mu \), we first show \( \xi \in P_\mu \). In other words, we claim

\[ |\{x \in \mathbb{R}^3 \mid 0 < \xi < 1\}| = 0. \quad (4.16) \]

For a contradiction, let us suppose \( |\{0 < \xi < 1\}| > 0 \). Then there exists \( \delta_0 > 0 \) such that

\[ |\{\delta_0 \leq \xi \leq 1 - \delta_0\}| > 0. \quad (4.17) \]

First, we take axi-symmetric compactly supported functions \( h_1, h_2 \in L^\infty(\mathbb{R}^3) \) such that for \( i = 1, 2 \),

\[
\text{spt } h_i \subset \{\delta_0 \leq \xi \leq 1 - \delta_0\},
\]

\[
\int h_1(x) dx = 1, \quad \frac{1}{2} \int r^2 h_1(x) dx = 0, \]

\[
\int h_2(x) dx = 0, \quad \frac{1}{2} \int r^2 h_2(x) dx = 1.
\]
We may consider them as a basis for our two constraints (mass, impulse) problem. Let \( h \in L^\infty(\mathbb{R}^3) \) be any axi-symmetric compactly supported function such that
\[
\text{spt } h \subset \{ \xi \leq 1 - \delta_0 \} \quad \text{and} \quad h \geq 0 \quad \text{on} \quad \{ 0 \leq \xi < \delta_0 \}.
\]
We set
\[
\eta := h - \left( \int h \, dx \right) h_1 - \left( \frac{1}{2} \int r^2 h \, dx \right) h_2
\]
so that \( \eta \in (L^1_w \cap L^\infty \cap L^1)(\mathbb{R}^3) \) is axi-symmetric, \( \int \eta \, dx = 0 \), and \( \frac{1}{2} \int r^2 \eta \, dx = 0 \). We consider \((\xi + \epsilon \eta)\) for \( \epsilon > 0 \). Thanks to the above construction of \( \eta \), we know
\[
\begin{align*}
(\xi + \epsilon \eta) &\in L^\infty(\mathbb{R}^3), \\
\frac{1}{2} \int r^2(\xi + \epsilon \eta) \, dx &= \frac{1}{2} \int r^2 \xi \, dx = \mu, \\
\text{and} \quad \int (\xi + \epsilon \eta) \, dx &= \int \xi \, dx \leq 1.
\end{align*}
\]
We claim
\[
0 \leq (\xi + \epsilon \eta) \leq 1 \quad \text{for small} \quad \epsilon > 0.
\]
We observe \((\xi + \epsilon \eta) = \xi \) on the set \( \{ \xi > 1 - \delta_0 \} \) while we have, for sufficiently small \( \epsilon > 0 \), on the set \( \{ \delta_0 \leq \xi \leq 1 - \delta_0 \} \),
\[
1 \geq 1 - \delta_0 + \epsilon \| \eta \|_\infty \geq (\xi + \epsilon \eta) \geq \delta_0 - \epsilon \| \eta \|_\infty \geq 0.
\]
On the remainder set \( \{ 0 \leq \xi < \delta_0 \} \), due to \( \eta = h \geq 0 \), we get \((\xi + \epsilon \eta) \geq 0 \) and, for small \( \epsilon > 0 \), we know \((\xi + \epsilon \eta) \leq 1 \). Hence we have shown
\[
(\xi + \epsilon \eta) \in P'_\mu \quad \text{for small} \quad \epsilon > 0.
\]
By the assumption \( \xi \in S'_\mu \), we have, for small \( \epsilon > 0 \),
\[
0 \geq \frac{E[\xi + \epsilon \eta] - E[\xi]}{\epsilon}.
\]
By taking the limit \( \epsilon \searrow 0 \), we have, for \( \psi = G[\xi] \),
\[
0 \geq \frac{1}{2} \int G[\xi] \eta \, dx + \frac{1}{2} \int \xi G[\eta] \, dx = \frac{1}{2} \int \psi \eta \, dx + \frac{1}{2} \int \xi G[\xi] \eta \, dx = \int \psi \eta \, dx,
\]
where we used the symmetry of the kernel \( G \). On the other hand, by the definition of \( \eta \), we have
\[
\int \psi \eta \, dx = \int \psi h \, dx - \left( \int \psi h_1 \, dx \right) \left( \int h \, dx \right) - \left( \int \psi h_2 \, dx \right) \left( \frac{1}{2} \int r^2 h \, dx \right).
\]
By denoting \( \beta := (\int \psi h_1 \, dx), \alpha := (\int \psi h_2 \, dx) \) and \( \tilde{\psi} := \psi - \frac{1}{2} \alpha r^2 - \beta \), we obtain
\[ 0 \geq \int \psi h dx - \beta \left( \int h dx \right) - \alpha \left( \frac{1}{2} \int r^2 h dx \right) = \int \tilde{\psi} h dx \]

\[ = \int_{0 \leq \xi < \delta_0} \tilde{\psi} h dx + \int_{1 - \delta_0 \geq \xi \geq \delta_0} \tilde{\psi} h dx, \]

where the decomposition is due to the assumption \( \text{spt } h \subset \{ \xi \leq 1 - \delta_0 \} \). Since \( h \) is an arbitrary function satisfying \( h \geq 0 \) on the set \( \{0 \leq \xi < \delta_0\} \), we have

\[ \tilde{\psi} = 0 \quad \text{a.e. on } \{\delta_0 \leq \xi \leq 1 - \delta_0\}, \]

\[ \tilde{\psi} \leq 0 \quad \text{a.e. on } \{0 \leq \xi < \delta_0\}. \tag{4.19} \]

On the other hand, since \( \psi \in H^2_{\text{loc}}(\Pi) \) by Lemma 2.4, we have \( \tilde{\psi} \in H^2_{\text{loc}} \), which implies, on the set \( \{\tilde{\psi} = 0\} \),

\[ \nabla \tilde{\psi} = 0 \quad \text{a.e. and } -\frac{1}{r^2} \mathcal{L} \tilde{\psi} = 0 \quad \text{a.e.} \]

Since \( -r^{-2} \mathcal{L} \tilde{\psi} = -r^{-2} \mathcal{L} \psi = \xi \), we get

\[ \xi = 0 \quad \text{a.e. on } \{\tilde{\psi} = 0\}. \]

Thus, by (4.19), we get

\[ \xi = 0 \quad \text{a.e. on } \{\delta_0 \leq \xi \leq 1 - \delta_0\}. \]

It contradicts the assumption (4.17). Hence we get the claim (4.16), which implies \( \xi \in P_\mu \). Due to \( \xi \in S'_\mu \), we get \( \xi \in S_\mu \). In sum, we have shown \( S'_\mu \subset S_\mu \). As in the last part of the proof of Lemma 4.11, we conclude \( I_\mu = I'_\mu \) and \( S_\mu = S'_\mu \). \( \square \)

Now we are ready to prove Theorem 4.2.

**Proof of Theorem 4.2.** By Lemmas 4.4, 4.11 and 4.12, and the scaling argument (4.4), we get the theorem. \( \square \)

Before closing this section, for a later use, we show that \( I_\mu \) is strictly increasing in the variable \( \mu > 0 \).

**Lemma 4.13.** Let \( 0 < \mu_0 < \mu < \infty \). Then

\[ I_{\mu_0} < I_\mu. \]

**Proof.** We take any function \( \xi \in S_{\mu_0} \) by recalling \( S_{\mu_0} \neq \emptyset \) from Theorem 4.2. We can find some \( \tau > 0 \) such that the relative translation \( \xi_\tau \) of \( \xi \) defined by (4.12) lies on \( P'_\mu \) as in the computation (4.13). We note \( E[\xi_\tau] > E[\xi] \) by (4.15). Hence we get

\[ I_{\mu_0} = E[\xi] < E[\xi_\tau] \leq I'_\mu = I_\mu. \]

\( \square \)
5 | STEADY VORTEX RINGS FROM MAXIMIZERS

In this section, our goal is to show Theorem 5.1 below, which is needed when proving Theorem 3.1 in Section 6.1 and Theorem 3.2 in Section 7.

5.1 | Every maximizer produces a vortex ring

Theorem 5.1. For $0 < \mu, \nu, \lambda < \infty$, each element $\xi$ of the set $S_{\mu,\nu,\lambda}$ satisfies

$$
\xi = \lambda 1_{\{\Psi > 0\}} \text{ a.e. for } \Psi = \mathcal{G}[\xi] - \frac{1}{2} W r^2 - \gamma
$$

(5.1)

for some constants $W > 0, \gamma \geq 0$, which are uniquely determined by $\xi$. Moreover, $\xi$ is compactly supported in $\mathbb{R}^3$.

Remark 5.2. As a consequence of (5.1), we obtain a steady vortex ring $\xi$ since we have (2.9) for the choice of $f(s) = \lambda f_H(s) = \lambda 1_{\{s > 0\}}$. In other words, the function $\xi(x - t W e_z)$ is an exact solution of (2.6).

We split the proof into 3 steps. First, Proposition 5.3 shows the existence (and uniqueness) of such a pair of constants $W \geq 0$ and $\gamma \geq 0$ for each maximizer $\xi$. Second, Proposition 5.11 proves that the constant $W$ is positive. Lastly, Proposition 5.13 gives compactness of the (essential) support of $\xi$.

5.2 | Exceptional points of a measurable set

As the first step, we prove that (5.1) holds for some unique non-negative $W, \gamma$:

Proposition 5.3. For $0 < \mu, \nu, \lambda < \infty$, each element $\xi$ of the set $S_{\mu,\nu,\lambda}$ satisfies

$$
\xi = \lambda 1_{\{\Psi > 0\}} \text{ a.e. for } \Psi = \mathcal{G}[\xi] - \frac{1}{2} W r^2 - \gamma
$$

(5.2)

for some constants $W, \gamma \geq 0$,

which are uniquely determined by $\xi$.

Proof. Let $\mu \in (0, \infty)$. By the scaling argument (4.4), it is enough to consider the case $\nu = \lambda = 1$.

We prove the uniqueness of $W, \gamma$ first by assuming the existence for a moment. Let $\xi \in S_{\mu}$ and set $\psi = \mathcal{G}[\xi]$. Suppose that there exist constants $W, \gamma \in \mathbb{R}$ satisfying

$$
\xi = 1_{\{(\psi - (1/2) W r^2 - \gamma) > 0\}} \text{ a.e.}
$$


We set \( A = \{ x \in \mathbb{R}^3 \mid (\psi(r,z) - (1/2)Wr^2 - \gamma) > 0 \} \). Since \( \Psi := (\psi - (1/2)Wr^2 - \gamma) \) is continuous on \( \Pi \) by Lemma 2.4, the axi-symmetric set \( A \) is open in \( \mathbb{R}^3 \). Clearly, we know

\[
0 < |A| < \infty
\]

because \( E[\xi] = I_\mu > 0 \) by Lemma 4.4 and \( |A| = \|\xi\|_1 \leq \nu \). We take any two points \( y', y'' \in \mathbb{R}^3 \) from the boundary \( \partial A \) satisfying \( r' > r'' > 0 \). Here we match \( y', y'' \in \mathbb{R}^3 \) into \( (r', z'), (r'', z'') \in \Pi \), respectively. Since \( y', y'' \in \partial A \) implies \( \Psi(y') = \Psi(y'') = 0 \), we get

\[
\psi(y') - \frac{1}{2}Wr'^2 - \gamma = 0 \quad \text{and} \quad \psi(y'') - \frac{1}{2}Wr''^2 - \gamma = 0.
\]

By solving these equations about \( W \) and \( \gamma \), we have

\[
W = 2 \frac{\psi(y') - \psi(y'')}{r'^2 - r''^2} \quad \text{and} \quad \gamma = \frac{r'^2}{r'^2 - r''^2} \psi(y'') - \frac{r''^2}{r'^2 - r''^2} \psi(y'),
\]

which produces the uniqueness of such \( W, \gamma \in \mathbb{R} \).

It remains to show the existence of such constants \( W, \gamma \geq 0 \) satisfying (5.2). Let \( \xi \in S_\mu \). Due to \( S_\mu \subset P_\mu \), we have, by the definition (3.1) of the class \( P_\mu \),

\[
\xi = 1_A \quad \text{for some axi-symmetric measurable subset} \quad A \subset \mathbb{R}^3.
\]

As before, we know \( |A| \in (0, \infty) \). Our goal is to find some \( W, \gamma \geq 0 \) satisfying

\[
A = \{ x \in \mathbb{R}^3 \mid \mathcal{G}[\xi](x) - (1/2)Wr^2 - \gamma > 0 \} \quad \text{a.e.} \quad (5.4)
\]

As observed when proving uniqueness, we might expect that any two points \( y', y'' \) on the boundary \( \partial A \) with different distances toward the axis (i.e., \( r' \neq r'' \)) play an important role when verifying (5.4). However, we know only that \( A \) is measurable (until showing (5.4)) so the set is defined up to measure zero. Thus, the notion of topological boundary is not useful as before. To overcome the difficulty, we use so-called metrical boundary, which is the set of exceptional points. These terminologies were used in [37, p. 78] and [104, p. 765] (also see [68]).

**Definition 5.4.** Let \( \Omega \) be a given fixed open subset of \( \mathbb{R}^N, N \geq 1 \). For any (Lebesgue) measurable subset \( U \subset \Omega \), we define the density \( D_e(U) \) of the set \( U \) by the collection of points \( x \in \Omega \) such that

\[
\lim \inf_{r \to 0} \frac{|B_r(x) \cap U|}{|B_r(x)|} = 1,
\]

where \( B_r(x) = \{ y \in \Omega \mid |x - y| < r \} \). Since \( 0 \leq |B_r(x) \cap U|/|B_r(x)| \leq 1, r > 0 \) holds,

\[
x \in D_e(U) \text{ if and only if } \lim_{r \to 0} \frac{|B_r(x) \cap U|}{|B_r(x)|} \text{ exists and the limit value is equal to } 1.
\]

Similarly, we define the dispersion \( D_i(U) \) by the set of points \( x \in \Omega \) such that

\[
\lim \sup_{r \to 0} \frac{|B_r(x) \cap U|}{|B_r(x)|} = 0.
\]
We call the set
\[
\Omega \setminus (D_e(U) \cup D_i(U)) =: \mathcal{E}(U)
\]
the \textit{metrical} boundary of $U$. Any point of the metrical boundary $\mathcal{E}(U)$ is called an \textit{exceptional point} of $U$. We note that $\mathcal{E}(U) = \mathcal{E}(V)$ when $V$ is equal to $U$ a.e.

The Lebesgue density theorem says

\textbf{Lemma 5.5.} For any measurable subset $U$ of $\mathbb{R}^N$, $N \geq 1$, almost every point of $U$ is a point of the density $D_e(U)$.

It is an immediate corollary of the well-known Lebesgue differentiation theorem. For example, a proof can be found in [114, Theorem 7.13]. The above lemma applied to the complement $U^c$ says that almost every point of $U^c$ is a point of the dispersion $D_i(U)$. As a result, the metrical boundary $\mathcal{E}(U)$ has always zero measure. However, the set is non-empty for typical cases:

\textbf{Lemma 5.6.} Let $\Omega \subset \mathbb{R}^N, N \geq 1$ be a non-empty connected open set. For any measurable set $U \subset \Omega$ satisfying $0 < |U| \leq \infty$ and $0 < |\Omega \setminus U| \leq \infty$, there exists an exceptional point of $U$.

A proof of the above lemma can be found in [37, Lemma 4] for the $1d$ case even with a sharper quantitative estimate. One can easily extend their existence proofs up to general connected open sets in $\mathbb{R}^N$. We present a similar proof in Appendix B for completeness.

On the other hand, an exceptional point has an interesting feature in the following sense:

If $U, U^c$ have positive measures and if $x$ is an exceptional point of $U$, then there exists a positive sequence $\{r_n\}$ satisfying $r_n \to 0$ while both $(B_{r_n}(x) \cap U)$ and $(B_{r_n}(x) \cap U^c)$ have positive measures for each $n$. An application, we can construct a sequence $\{f_n\}$ of non-negative bounded functions which are supported in $U$ converging to the Dirac mass at the exceptional point $x$. For instance, we can take

\[
f_n = \frac{1}{|B_{r_n}(x) \cap U|} 1_{B_{r_n}(x) \cap U}, \quad n \geq 1.
\]

Similarly, there is a sequence of functions supported in $U^c$ with the same property.

For our purpose, we need two exceptional points with different distances toward the axis of symmetry.

\textbf{Corollary 5.7.} For any measurable set $U \subset \mathbb{R}^3$ satisfying $0 < |U| \leq \infty$, there exist at least two exceptional points $y', y'' \in \mathbb{R}^3$ of the set $U$ satisfying $r' > r'' > 0$.

\textbf{Proof.} We take any $\bar{r} > 0$ such that both $U_A := U \cap \{0 < r < \bar{r}\}$ and $U_B := U \cap \{r > \bar{r}\}$ have finite positive measure. By Lemma 5.6 for $\Omega := \{0 < r < \bar{r}\}$, we have an exceptional point $y'$ of $U_A$. Definitely, the point $y'$ lies on $\{0 < r < \bar{r}\}$. Similarly, we get an exceptional point $y'' \in \{r > \bar{r}\}$ of $U_B$ by setting $\Omega = \{r > \bar{r}\}$. \hfill \Box

\footnote{As noted in [37, p. 78], this result may have been previously known.}
Coming back to the proof of existence of $W, \gamma \geq 0$, since $|A| \in (0, \infty)$, we can apply Corollary 5.7 into the set $A$ so that there are (at least) two exceptional points $y' = (y'_1, y'_2, y'_3), y'' = (y''_1, y''_2, y''_3) \in \mathbb{R}^3$ of the set $A$ whose cylindrical coordinates $(r', z'), (r'', z'') \in \Pi$ satisfy $r' > r'' > 0$. Set

$$r_0 = \min\{r' - r'', r''\} > 0. \quad (5.5)$$

• Step 1—construction of $W, \gamma$ from stream function via exceptional points:

We define

$$a = \frac{r'^2}{r'^2 - r''^2}, \quad b = \frac{r''^2}{r'^2 - r''^2}, \quad \text{and} \quad c = \frac{2}{r'^2 - r''^2}. \quad (5.6)$$

Then, $a > b > 0$ and $c > 0$. By using the stream function $\psi = C[\xi]$, we define $\gamma$ and $W$ by

$$\gamma = a\psi(y'') - b\psi(y') \quad \text{and} \quad W = c(\psi(y') - \psi(y'')). \quad (5.7)$$

Using such constants $W, \gamma$, we will show

$$0 \geq \int \left( \psi - W \frac{1}{2} r^2 - \gamma \right) h \, dx$$

for any axi-symmetric compactly supported $h \in L^\infty(\mathbb{R}^3)$ satisfying

$$h \geq 0 \quad \text{on} \quad A^c, \quad h \leq 0 \quad \text{on} \quad A. \quad (5.8)$$

Once we obtain it, it implies

$$\left( \psi - W \frac{1}{2} r^2 - \gamma \right) \leq 0 \quad \text{a.e. on} \quad A^c, \quad \left( \psi - W \frac{1}{2} r^2 - \gamma \right) \geq 0 \quad \text{a.e. on} \quad A.$$

Then the goal (5.4) of Proposition 5.3 for the existence part will follow once we show

$$\left( \psi - W \frac{1}{2} r^2 - \gamma \right) \neq 0 \quad \text{a.e. on} \quad A.$$

Remark 5.8. Here is the motivation for the choice of $W, \gamma$ in (5.7) (and $a, b, c$ in (5.6)):

If we define axi-symmetric functions $h_1$ in $\mathbb{R}^3$ by their cylindrical form:

$$h_1 = \frac{a}{2\pi r} \delta_{(r'', z'')} - \frac{b}{2\pi r} \delta_{(r', z')} \quad \text{and} \quad h_2 = \frac{c}{2\pi r} (\delta_{(r', z')} - \delta_{(r'', z'')}),$$

where $\delta_{(r,z)}$ is the Dirac-delta function in $\Pi$ at $(r, z)$, then we formally obtain

$$\int \psi h_1 \, dx = \int \psi (a \delta_{(r'', z'')} - b \delta_{(r', z')}) \, dr \, dz = \gamma. \quad (5.9)$$
Similarly, we get
\[ \int \psi h_2 \, dx = W. \] (5.10)

Due to the relations
\[ a - b = 1, \quad \frac{1}{2} (ar'' - br') = 0, \quad \frac{c}{2} (r'^2 - r'') = 1, \] (5.11)
we obtain
\[ \int h_1 \, dx = 1, \quad \int \frac{1}{2} r^2 h_1 \, dx = 0, \quad \int h_2 \, dx = 0, \quad \text{and} \quad \int \frac{1}{2} r^2 h_2 \, dx = 1. \] (5.12)

In order to search \( W, \gamma \) as Lagrange multipliers, we recall the proof of Lemma 4.12. As in (4.18), we set \( \eta \) by
\[ \eta = h - \left( \int h \, dx \right) h_1 - \left( \int \frac{1}{2} r^2 h \, dx \right) h_2 \] (5.13)
for arbitrary bounded \( h \) with (5.8). Then we want to have \((\xi + \epsilon \eta) \in \mathcal{P}_\mu\) (see the definition (3.1)) for small \( \epsilon > 0 \). From \( \int \eta \, dx = 0 \) and \( \int \frac{1}{2} r^2 \eta \, dx = 0 \), we have
\[ \int (\xi + \epsilon \eta) \, dx = \int \xi \, dx \leq 1 \quad \text{and} \quad \frac{1}{2} \int r^2 (\xi + \epsilon \eta) \, dx = \frac{1}{2} \int r^2 \xi \, dx = \mu. \] (5.14)

However, it is too optimistic to ask that \((\xi + \epsilon \eta)\) is of patch-type. Instead, thanks to Theorem 4.2, we are allowed to have \((\xi + \epsilon \eta)\) in the larger class \(\mathcal{P}'_\mu\) (or \(\mathcal{P}''_\mu\)) (see the definition (4.1)). Nevertheless, we still have a problem since \((\xi + \epsilon \eta)\) fails in general to satisfy the pointwise bound
\[ 0 \leq (\xi + \epsilon \eta) \leq 1 \quad \text{a.e.} \] (5.15)

In fact, \( \eta \) is not a measurable function but merely a measure. Hence, we are asked to approximate the formal perturbation \( \eta \) of (5.13) by a sequence of measurable functions \( \{\eta_n\} \) satisfying (5.14) and (5.15) at the same time. It raises technical difficulties. For instance, it asks certain sign condition for \( \eta_n \) due to the form \( \xi = 1_A \). Indeed, from \( \epsilon > 0 \), we need
\[ \eta_n \leq 0 \quad \text{on} \quad A, \quad \eta_n \geq 0 \quad \text{on} \quad A^c \] (5.16)
for arbitrary \( h \). Hence, we need sequences \( \{h_{i,n}\} \) converging to \( h_i \) for each \( i = 1, 2 \) such that the functions \( h_{i,n} \) satisfy not only (5.12) but also certain sign condition depending the choice of \( h \) (see (5.22)). As a result, \( \eta_n \) satisfies (5.16) for each \( n \). In short, for different \( h \), we have to construct different \( h_{i,n} \) (so different \( \eta_n \)) (see (5.20), (5.26)). The construction will be given below in detail since the process searching such a perturbation sequence seems not standard.

- **Step 2**—approximation (toward Dirac mass) supported on each side:

Since \( y', y'' \) are exceptional points of \( A \), we have a decreasing sequence \( \{r_n\}_{n=1}^\infty \) of positive numbers satisfying \( r_n \to 0 \) as \( n \to \infty \) and
\[ |B_{r_n}(y') \cap A|, |B_{r_n}(y') \cap A^c|, |B_{r_n}(y'') \cap A|, |B_{r_n}(y'') \cap A^c| > 0 \quad \text{for} \quad n \geq 1. \]
Here $B_{n}(y'), B_{n}(y'')$ are usual balls in $\mathbb{R}^{3}$ as defined in (2.17), and $| \cdot |$ is the Lebesgue measure for $\mathbb{R}^{3}$. Since $A \subset \mathbb{R}^{3}$ is axi-symmetric, we have

$$|T_{r_{n}}(r', z') \cap A|, |T_{r_{n}}(r', z') \cap A^{c}|, |T_{r_{n}}(r'', z'') \cap A|, |T_{r_{n}}(r'', z'') \cap A^{c}| > 0 \quad \text{for} \quad n \geq 1.$$

Here $T_{r_{n}}(r', z'), T_{r_{n}}(r'', z'')$ are tori in $\mathbb{R}^{3}$ defined in (2.17). We may assume

$$0 < r_{1} < \frac{r_{0}}{2}.$$

Then, due to (5.5) and the monotonicity of $\{r_{n}\}$, we know

$$T_{r_{n}}(r', z') \cap T_{r_{n}}(r'', z'') = \emptyset \quad \text{for} \quad n \geq 1.$$

We denote the above axi-symmetric sets of positive measures by

$$Y_{n}^{+} = T_{r_{n}}(r', z') \cap A, \quad Y_{n}^{-} = T_{r_{n}}(r', z') \cap A^{c}, \quad Z_{n}^{+} = T_{r_{n}}(r'', z'') \cap A, \quad Z_{n}^{-} = T_{r_{n}}(r'', z'') \cap A^{c}.$$

For each $n \geq 1$, we define the axi-symmetric, compactly supported functions $f_{n}^{\pm}, g_{n}^{\pm} \in L^{\infty}(\mathbb{R}^{3})$ by

$$f_{n}^{\pm}(x) = \frac{1}{|Y_{n}^{\pm}|} 1_{Y_{n}^{\pm}}(x), \quad g_{n}^{\pm}(x) = \frac{1}{|Z_{n}^{\pm}|} 1_{Z_{n}^{\pm}}(x).$$

The above functions are designed to satisfy

$$f_{n}^{\pm}, g_{n}^{\pm} \geq 0, \quad \int f_{n}^{\pm} \, dx = 1, \quad \int g_{n}^{\pm} \, dx = 1,$$

$$\text{spt}(f_{n}^{\pm}) \subset T_{r_{n}}(r', z'), \quad \text{spt}(g_{n}^{\pm}) \subset T_{r_{n}}(r'', z''),$$

$$\text{spt}(f_{n}^{\pm}), \text{spt}(g_{n}^{\pm}) \subset A, \quad \text{spt}(f_{n}^{-}), \text{spt}(g_{n}^{-}) \subset A^{c}, \quad n \geq 1,$$

where the last line means that each sequence is supported either on $A$ or $A^{c}$. Thanks to the convergence $r_{n} \to 0$, the sequences $\{f_{n}^{\pm}\}, \{g_{n}^{\pm}\}$ approximate the Dirac-masses on the circles

$$(x_1, x_2, x_3) \in \mathbb{R}^{3} \mid x_1^2 + x_2^2 = r'^2, x_3 = z''), \quad (x_1, x_2, x_3) \in \mathbb{R}^{3} \mid x_1^2 + x_2^2 = r''^2, x_3 = z''),$$

respectively in the following sense:

For any continuous axi-symmetric function $\phi$ in $\mathbb{R}^{3}$, we have

$$\int f_{n}^{\pm} \phi \, dx \to \phi(y'), \quad \int g_{n}^{\pm} \phi \, dx \to \phi(y'') \quad \text{as} \quad n \to \infty. \quad (5.17)$$

• Step 3—construction of a basis for two constraints (mass, impulse) problem with sign condition:

The impulse of the functions are estimated by

$$\int \frac{1}{2} r'^{2} f_{n}^{\pm} \, dx \in \left(\frac{1}{2}(r' - r_{n})^2, \frac{1}{2}(r' + r_{n})^2\right), \quad \int \frac{1}{2} r''^{2} g_{n}^{\pm} \, dx \in \left(\frac{1}{2}(r'' - r_{n})^2, \frac{1}{2}(r'' + r_{n})^2\right).$$
Thus, for each \( n \), we can set \( \tau_n^\pm \in (r' - r_n, r' + r_n) \) and \( \tau''_n^\pm \in (r'' - r_n, r'' + r_n) \) by solving
\[
\frac{1}{2}(\tau_n^\pm)^2 = \int \frac{1}{2} r^2 f_n^\pm \, dx, \quad \frac{1}{2}(\tau''_n^\pm)^2 = \int \frac{1}{2} r^2 g_n^\pm \, dx.
\]

We note \( \tau''_n^\pm > \tau_n^\pm \). Then we define
\[
a_n^\pm = \frac{(\tau_n^\pm)^2}{(\tau_n^\pm)^2 - (\tau''_n^\pm)^2}, \quad b_n^\pm = \frac{(\tau''_n^\pm)^2}{(\tau_n^\pm)^2 - (\tau''_n^\pm)^2}, \quad c_n^\pm = \frac{2}{(\tau_n^\pm)^2 - (\tau''_n^\pm)^2}.
\]

By recalling the definition of \( a, b, c \) in (5.6), we have
\[
a^\pm \to a, \quad b_n^\pm \to b, \quad c_n^\pm \to c \quad \text{as} \quad n \to \infty, \quad (5.18)
\]
due to
\[|\tau_n^\pm - r'|, |\tau''_n^\pm - r''| \leq r_n \to 0.\]

We simply observe (cf. (5.11))
\[
a_1^\pm > b_n^\pm > 0, \quad c_n^\pm > 0, \quad a_1^\pm - b_n^\pm = 1, \quad (5.19)
\]
\[
\frac{1}{2} (a_n^\pm (\tau''_n^\pm)^2 - b_n^\pm (\tau_n^\pm)^2) = 0, \quad \frac{c_n^\pm}{2} ((\tau_n^\pm)^2 - (\tau''_n^\pm)^2) = 1.
\]

Now, we define axi-symmetric, compactly supported functions \( h_{n,1}^\pm, h_{n,2}^\pm \in L^\infty(\mathbb{R}^3) \) by
\[
h_{n,1}^\pm = a_n^\pm g_n^\pm - b_n^\pm f_n^\pm, \quad h_{n,2}^\pm = c_n^\pm (f_n^\pm - g_n^\pm). \quad (5.20)
\]

By (5.19), they form a basis for two constraints (mass, impulse) problem:
\[
\int h_{n,1}^\pm \, dx = 1, \quad \int \frac{1}{2} r^2 h_{n,1}^\pm \, dx = 0, \quad \int h_{n,2}^\pm \, dx = 0, \quad \int \frac{1}{2} r^2 h_{n,2}^\pm \, dx = 1. \quad (5.21)
\]

Moreover, they satisfy the sign condition:
\[
h_{n,1}^+, h_{n,1}^+ \geq 0, \quad h_{n,1}^-, h_{n,1}^- \leq 0 \quad \text{on} \quad A,
\]
\[
h_{n,2}^+, h_{n,2}^+ \leq 0, \quad h_{n,2}^-, h_{n,2}^- \geq 0 \quad \text{on} \quad A^c. \quad (5.22)
\]

By setting
\[
W_n^\pm = \int \psi h_{n,2}^\pm \, dx \quad \text{and} \quad \gamma_n^\pm = \int \psi h_{n,1}^\pm \, dx \quad (5.23)
\]
(cf. (5.10) and (5.9)), we can show
\[
W_n^\pm \to W, \quad \gamma_n^\pm \to \gamma \quad \text{as} \quad n \to \infty, \quad (5.24)
\]

where \( W, \gamma \) are defined in (5.7). Indeed, since the stream function \( \psi \) is continuous by Lemma 2.4, we have, by (5.17), (5.18), as \( n \to \infty \),
\[
W_n^\pm = \int \psi h_{n,2}^\pm \, dx = c_n^\pm \left( \int \psi f_n^\pm \, dx - \int \psi g_n^\pm \, dx \right) \to c(\psi(y') - \psi(y'')) = W
\]
and

\[ y_n^\pm = \int \psi h_{n,1}^\pm \,dx = a_n^\pm \int \psi g_{n,1}^\pm \,dx - b_n^\pm \int \psi f_n^\pm \,dx \to a\psi(y'') - b\psi(y') = \gamma. \]

• Step 4—construction of a sequence of perturbations around patch-type data:

Now we are ready to define an approximation in \( \mathcal{P}_\mu' \) toward the formal perturbation (5.13) around our patch-type function \( \xi = 1_A \). Let us take and fix any axi-symmetric function \( h \in L^\infty(\mathbb{R}^3) \) satisfying

\[ h : \text{compactly supported, } h \geq 0 \text{ on } A^c, \text{ and } h \leq 0 \text{ on } A. \] (5.25)

Then there are four possible cases depending on the sign of the integrals \( \int h \,dx, \int \frac{1}{2}r^2h \,dx \):

- Case (I): \( \int h \,dx \geq 0, \int \frac{1}{2}r^2h \,dx \geq 0 \),
- Case (II): \( \int h \,dx \geq 0, \int \frac{1}{2}r^2h \,dx < 0 \),
- Case (III): \( \int h \,dx < 0, \int \frac{1}{2}r^2h \,dx \geq 0 \),
- Case (IV): \( \int h \,dx < 0, \int \frac{1}{2}r^2h \,dx < 0 \).

For each \( n \geq 1 \), we define

\[ \eta_n := \begin{cases} 
    h - (\int h \,dx)h_{n,1}^+ - (\int \frac{1}{2}r^2h \,dx)h_{n,2}^+ & \text{when Case (I)}, \\
    h - (\int h \,dx)h_{n,1}^- - (\int \frac{1}{2}r^2h \,dx)h_{n,2}^- & \text{when Case (II)}, \\
    h - (\int h \,dx)h_{n,1}^- - (\int \frac{1}{2}r^2h \,dx)h_{n,2}^+ & \text{when Case (III)}, \\
    h - (\int h \,dx)h_{n,1}^+ - (\int \frac{1}{2}r^2h \,dx)h_{n,2}^- & \text{when Case (IV)}. 
\end{cases} \] (5.26)

They are designed to satisfy

\[ \eta_n \leq 0 \text{ on } A, \ \eta_n \geq 0 \text{ on } A^c, \ \int \eta_n \,dx = 0, \text{ and } \int \frac{1}{2}r^2\eta_n \,dx = 0, \] (5.27)

for any possible cases.

For each \( n \geq 1 \), we claim

\[ (\xi + \varepsilon\eta_n) \in \mathcal{P}_\mu' \text{ for sufficiently small } \varepsilon > 0. \]

Indeed, by fixing the function \( h \) and \( n \geq 1 \), we have

\[ 0 \leq \xi + \varepsilon\eta_n \leq 1 \text{ for sufficiently small } \varepsilon > 0 \quad \text{(depending on } \|h_{n,i}^\pm\|_\infty, \ i = 1, 2) \]

thanks to the sign property (5.27). In addition, the property (5.27) on the integrals implies

\[ \int (\xi + \varepsilon\eta_n) \,dx = \int \xi \,dx \leq 1 \quad \text{and} \quad \int \frac{1}{2}r^2(\xi + \varepsilon\eta_n) \,dx = \int \frac{1}{2}r^2\xi \,dx = \mu. \]

As a result, we obtain the above claim for small \( \varepsilon > 0 \).
Step 5—verification of the boundary of the patch $1_A$ via Lagrange multipliers $W, \gamma$:

Since $\xi \in S_\mu = S'_\mu$ by Theorem 4.2, we obtain, for small $\varepsilon > 0$,

$$0 \geq \frac{E[\xi + \varepsilon \eta_n] - E[\xi]}{\varepsilon}.$$ 

Hence, by taking the limit $\varepsilon \downarrow 0$, we have

$$0 \geq \frac{1}{2} \int G[\xi] \eta_n \, dx + \frac{1}{2} \int \xi G[\eta_n] \, dx = \frac{1}{2} \int \psi \eta_n \, dx + \frac{1}{2} \int G[\xi] \eta_n \, dx = \int \psi \eta_n \, dx,$$  

(5.28)

where we used the symmetry of the kernel $G$. By the definitions (5.26) and (5.23), for the Case (I), we have

$$0 \geq \int \psi \eta_n \, dx = \int \psi h \, dx - \gamma + n \left( \int h \, dx \right) - W + n \left( \int \frac{1}{2} r^2 h \, dx \right) = \int \left( \psi - W + n \frac{1}{2} r^2 - \gamma + n \right) h \, dx.$$  

Similarly, we have

$$0 \geq \int \left( \psi - W_n - n \frac{1}{2} r^2 - \gamma_n^+ \right) h \, dx,$$  

for the Case (II),

$$0 \geq \int \left( \psi - W_n + n \frac{1}{2} r^2 - \gamma_n^- \right) h \, dx,$$  

for the Case (III),

$$0 \geq \int \left( \psi - W_n - n \frac{1}{2} r^2 - \gamma_n^- \right) h \, dx,$$  

for the Case (IV).

By (5.24), we can take limit $n \to \infty$ into the above inequalities to get

$$0 \geq \int \left( \psi - W + \frac{1}{2} r^2 - \gamma \right) h \, dx = \int \nabla \psi = \int \nabla \psi = \int A_c + \int A,$$  

(5.29)

for any possible cases. We set the adjusted stream function $\Psi = \psi - (1/2) W r^2 - \gamma$. Since $h$ is an arbitrary function satisfying the sign condition (5.25), the inequality (5.29) implies

$$\Psi \leq 0 \ a.e. \ on \ A_c \ and \ \Psi \geq 0 \ a.e. \ on \ A.$$

Thus we get

$$\{\Psi > 0\} \subset A \subset \{\Psi \geq 0\} \ \text{up to measure zero.}$$  

(5.30)

On the other hand, since $\psi \in H^2_{loc}(\Pi)$ by Lemma 2.4, we have $\Psi \in H^2_{loc}$, which implies

$$\nabla \Psi = 0 \ a.e. \ on \ \{\Psi = 0\}.$$  

Thus we obtain

$$-\frac{1}{r^2} L \Psi = 0 \ a.e. \ on \ \{\Psi = 0\}.$$
Due to \(-r^{-2} \mathcal{L} \Psi = -r^{-2} \mathcal{L} \psi = \xi = 1_A\), we have
\[
\{ \Psi = 0 \} \subset A^c \quad \text{up to measure zero.}
\]

Hence we conclude, by (5.30),
\[
A = \{ \Psi > 0 \} \quad \text{up to measure zero,}
\]
which gives the goal (5.4).

Lastly, let us show \(W \geq 0\) and \(\gamma \geq 0\). By the convergence \(\psi \to 0\) as \(|(r, z)| \to \infty\) due to Lemma 2.4 and by \(|\{\Psi > 0\}| < \infty\), we can take a sequence \(\{(r_n, z_n)\}\) in \(\Pi\) such that
\[
(r_n, z_n) \in \{\Psi \leq 0\}, \quad n \geq 1,
\]
\[
\psi(r_n, z_n) \to 0, \quad z_n \to \infty, \quad r_n \to 0 \quad \text{as} \quad n \to \infty.
\]

Thus we have
\[
\limsup_{n \to \infty} \left( \psi(r_n, z_n) - \frac{1}{2} W r_n^2 - \gamma \right) = \limsup_{n \to \infty} \Psi(r_n, z_n) \leq 0.
\]

Hence we obtain \(\gamma \geq 0\). Similarly, by taking another sequence \(\{(r_n, z_n)\}\) such that \((r_n, z_n) \in \{\Psi \leq 0\}, \psi(r_n, z_n) \to 0, z_n \to 0,\) and \(r_n \to \infty\) as \(n \to \infty\), we get (5.31) again, which cannot be true when \(W < 0\). Thus we obtain \(W \geq 0\). It finishes the proof of Proposition 5.3. \(\Box\)

### 5.3 Traveling speed is non-trivial

In this subsection, we prove that the traveling speed \(W\) of vortex rings from our variational problem (3.2) is positive. To prove, we use the energy identity (5.32) below due to [50, Lemma 3.1].

**Lemma 5.9.** For axi-symmetric \(\xi \in (L^1_{w} \cap L^\infty \cap L^1)(\mathbb{R}^3)\), if \(\xi\) is compactly supported, then we have
\[
E[\xi] = \int_{\mathbb{R}^3} (x \cdot \nabla \psi) \xi \, dx = \int_{\mathbb{R}^3} (r \partial_r \psi + z \partial_z \psi) \xi \, dx,
\]
where \(\psi = \mathcal{G}[\xi]\).

The above lemma can be formally obtained by using the integration by parts formula
\[
\int \frac{(\partial_r f)(\partial_r g) + (\partial_z f)(\partial_z g)}{r^2} \, dx = - \int f \frac{L g}{r^2} \, dx, \quad f, g : \text{axi-symmetric}
\]
twice. A proof in detail can be found in [50, p. 9–10].

**Remark 5.10.** The assumption on compactness (so boundedness) of \(\text{spt}(\xi)\) in Lemma 5.9 guarantees that the integral in (5.32) converges absolutely. Indeed, if \(\text{spt}(\xi) \subset B_K(0)\), then we
have
\[
\int |(r \partial_r \psi + z \partial_z \psi)\xi| \, dx \lesssim K^2 \int \left| \frac{\partial_r \psi}{r} + \frac{\partial_z \psi}{r} \right| \cdot |\xi| \, dx \lesssim K^2 \|K[\xi]\|_2 \|\xi\|_2 < \infty
\] (5.34)
by (2.24) of Lemma 2.4. If one wants to drop the assumption, the integral should be understood in a limit sense.

**Proposition 5.11.** For any \( \xi \in S_\mu \), the constant \( W \) in (5.2) of Proposition 5.3 is positive.

**Proof.** Let \( \xi \in S_\mu \). By Proposition 5.3, we have
\[
\xi = 1_{\{\psi - \frac{1}{2} W r^2 - \gamma > 0\}} \text{ a.e.}
\]
for some \( W \geq 0 \) and \( \gamma \geq 0 \). For a contradiction, let us assume \( W = 0 \). By setting
\[
A = \{ x \in \mathbb{R}^3 \mid \psi(x) - \gamma > 0 \},
\]
we know \( |A| = \|\xi\|_1 \leq 1 \). Thus we obtain \( \gamma > 0 \). Indeed, if \( \gamma = 0 \), then
\[
A = \{ \psi > 0 \} = \mathbb{R}^3 \setminus \{ r = 0 \}
\]
since the kernel \( G \) is positive a.e. and \( \xi \geq 0 \) is non-trivial. It implies \( |A| = \infty \), which is a contradiction. Thus \( \gamma = 0 \) is impossible.

By (2.23) in Lemma 2.4, we know the convergence \( \psi(x) \to 0 \) as \( |x| \to \infty \). It implies that the set \( A \) in (5.35) is bounded due to \( \gamma > 0 \). Thus \( \xi \) is (essentially) compactly supported. Now we can apply the identity (5.32) of Lemma 5.9 to have
\[
E[\xi] = \int (r \partial_r \psi + z \partial_z \psi)\xi \, dx = \int (r \partial_r \Psi + z \partial_z \Psi)\xi \, dx
\]
\[
= \int (r \partial_r F(\Psi) + z \partial_z F(\Psi)) \, dx,
\]
where we set \( \Psi = (\psi - \gamma) \) and \( F(s) = s^+ = \begin{cases} s, & s > 0 \\ 0, & s \leq 0 \end{cases} \), which is an antiderivative of the vorticity function \( f_H(s) = 1_{\{s > 0\}} \). Then, we can compute, by integration by parts which will be verified below,
\[
E[\xi] = \int x \cdot \nabla [F(\Psi)] \, dx = -\int (\nabla \cdot x) F(\Psi) \, dx = -3 \int F(\Psi) \, dx \leq 0,
\] (5.36)
which gives a contradiction due to \( E[\xi] = I_\mu > 0 \) from Theorem 4.2. Therefore we get \( W > 0 \) once we justify the integration by parts done in the above.

To justify, we take a radial function \( \varphi \in C_c^\infty(\mathbb{R}^3) \) satisfying \( \varphi(x) = 1 \) for \( |x| \leq 1 \) and \( \varphi(x) = 0 \) for \( |x| \geq 2 \), and set the cut-off function on \( \mathbb{R}^3 \) by \( \varphi_M(x) = \varphi(x/M) \) for any \( M > 0 \). Then we have
\[
\int (x \varphi_M) \cdot \nabla [F(\Psi)] \, dx = -\int \varphi_M (\nabla \cdot x) F(\Psi) \, dx - \int (x \cdot \nabla \varphi_M) F(\Psi) \, dx.
\] (5.37)
The first integral on the left-hand side of (5.37) converges absolutely as $M \to \infty$ since
\[ \int |(x \varphi_M) \cdot \nabla [F(\Psi)]| \, dx = \int |\varphi_M (r \partial_r \psi + z \partial_z \psi) \xi| \, dx \leq \int |(r \partial_r \psi + z \partial_z \psi) \xi| \, dx < \infty \]
by the computation (5.34) in Remark 5.10. We observe
\[ 0 \leq F(\Psi) = \Psi_1 \leq \psi_1 \]
and
\[ \|\psi\|_{\infty} \lesssim (\|r^2 \xi\|_1 + \|\xi\|_{L^1 \cap L^2}) \lesssim (\|r^2 \xi\|_1 + \|\xi\|_{L^1 \cap L^{\infty}}) \lesssim (\mu + 1) \]
by (2.19) for $\delta = 1$. Thus, the first integral on the right-hand side of (5.37) converges absolutely since
\[ \int |\varphi_M (\nabla \cdot x) F(\Psi)| \, dx \leq 3 \int |F(\Psi)| \, dx \lesssim (1 + \mu) |A| < \infty. \]
For the last integral of (5.37), we compute
\[ \left| \int (x \cdot \nabla \varphi_M) F(\Psi) \, dx \right| \lesssim (1 + \mu) \int_A |x| |\nabla \varphi_M| \, dx \lesssim (1 + \mu) |A \cap \{M \leq |x| \leq 2M\}|. \]
Since $A$ is bounded in $\mathbb{R}^3$, the intersection $A \cap \{M \leq |x| \leq 2M\}$ has zero measure for sufficiently large $M > 0$. We have justified the integration by parts done in (5.36).

5.4 \quad Vortex core is bounded

In order to show compactness of the vortex core, we prove first that $r^{-2} \mathcal{G}[\xi]$ for each maximizer $\xi$ vanishes at infinity.

**Lemma 5.12.** Let $\xi \in S_{\mu}$. Then, the stream function $\psi = \mathcal{G}[\xi]$ satisfies, for any $\alpha \in (0, 1)$,
\[ \frac{\psi}{r} \in BUC^{1+\alpha}(\Pi) \quad \text{and} \quad \frac{\psi}{r^2} \in BUC^{\alpha}(\Pi). \]
In particular, it satisfies
\[ \frac{\psi(r, z)}{r^2} \to 0 \quad \text{as} \quad |(r, z)| \to \infty. \quad \tag{5.38} \]

**Proof.** Let $\xi \in S_{\mu}$. By Propositions 5.3 and 5.11, it satisfies
\[ \xi = 1_{[\psi - \frac{1}{2} W r^2 - \gamma > 0]} \quad \text{a.e.} \]
for some $W > 0$ and $\gamma \geq 0$. We observe that if $x \in \text{spt}(\xi)$, then
\[ \psi(r, z) \geq \frac{1}{2} W r^2. \]
Since we have $\|\psi\|_\infty \lesssim (\mu + 1)$ by (2.19) for $\delta = 1$, there exists $R > 0$ such that

$$\text{spt}(\xi) \subseteq \{x \in \mathbb{R}^3 | r \leq R\}.$$ 

Thus, by setting $\omega(x) = r\xi(r, z)e_\theta(\theta)$, we get $|\omega(x)| \leq R\xi(r, z)$, which implies

$$\omega \in (L^1 \cap L^\infty)(\mathbb{R}^3).$$

By setting $\phi(x) = (\psi(r, z)/r)e_\theta(\theta)$, we have $\phi = (1/4\pi|x|) * \omega$ (recall Subsection 2.1). Since $\omega \in (L^1 \cap L^\infty)$, this representation implies

$$\nabla \phi \in L^p(\mathbb{R}^3), \quad p \in (3/2, \infty) \quad \text{and} \quad \phi \in W^{2,q}(\mathbb{R}^3), \quad q \in (3, \infty) \quad (5.39)$$

(e.g., see [103, p. 354]). It implies $\phi$ and $\nabla \phi$ are continuous on $\mathbb{R}^3$ (e.g., see [46, p. 284]). In particular, $\phi \in BUC^{1+\alpha}(\mathbb{R}^3)$ for any $\alpha \in (0, 1)$ (e.g., see [46, p. 280]). Since

$$\frac{\psi(r, z)}{r} = \frac{\phi_2(r e_{x_1} + z e_{x_3})}{r}, \quad (r, z) \in \Pi,$$

as in (2.28) where $\phi(x) = \phi_1(x)e_{x_1} + \phi_2(x)e_{x_2} + \phi_3(x)e_{x_3}$, we get $\psi/r \in BUC^{1+\alpha}(\overline{\Pi})$ and $\partial_r(\psi/r) \in BUC^\alpha(\overline{\Pi})$.

We observe that the form $\phi(x) = (\psi(r, z)/r)e_\theta(\theta)$ implies $\phi(x)|_{r=0} = 0$ thanks to continuity of $\phi$ on $\mathbb{R}^3$ and $\psi/r$ on $\overline{\Pi}$. Thus, we obtain

$$(\psi(r, z)/r)|_{r=0} = |\phi(x)|_{r=0} = 0.$$

By applying the identity

$$f(r) - f(0) = f'(rs)|_{s=0} = r \int_0^1 f'(rs)ds$$

into $f(r) = r^{-1}\psi(r, z)$ (for fixed $z$), we obtain, for each $(r, z) \in \Pi$,

$$\frac{\psi(r, z)}{r^2} = \int_0^1 \partial_r \left( \frac{\psi(r, z)}{r} \right) \bigg|_{(r, z)=(rs, z)} ds.$$

Due to $\partial_r(\psi/r) \in BUC^\alpha(\overline{\Pi})$, we get $\psi/r^2 \in BUC^\alpha(\overline{\Pi})$. As a consequence, we establish $|\phi|/r \in BUC^\alpha(\overline{\mathbb{R}^3})$.

On the other hand, by the weighted Hardy’s inequality for $\mathbb{R}^2$ (see e.g., [69] or [39, Corollary 14 (ii)]) and by (5.39), we have, for any $p \in (3/2, 2)$,

$$\left\| \frac{|\phi|}{r} \right\|_p^p \lesssim \sum_{i=1}^3 \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} \frac{|\phi_i|^p}{\sqrt{x_1^2 + x_2^2}} dx_1 dx_2 \right) dx_3 \lesssim \int_{\mathbb{R}} \left( \int_{\mathbb{R}^2} |\nabla x_1 x_2 \phi_i|^p dx_1 dx_2 \right) dx_3 \lesssim \left\| \nabla \phi \right\|_p^p < \infty.$$
Thus we have the convergence
\[
\frac{|\phi(x)|}{r} \to 0 \quad \text{as} \quad |x| \to \infty
\]
thanks to $|\phi|/r \in BUC^\alpha(\mathbb{R}^3)$. In other words, we obtain (5.38).

Thanks to Lemma 5.12, we can prove below that every maximizer is compactly supported. We also obtain a lower bound of the traveling speed $W$ depending only on its impulse $\mu$ together with an interesting identity. They are essentially contained in [50, Lemma 3.2] (for $\mu = 1$). We will need them when proving Proposition 7.4 and Theorem 3.2 in Section 7.

**Proposition 5.13.** For any $\xi \in S_\mu$, the support of $\xi$ in $\mathbb{R}^3$ is compact. Moreover, the constant $W$ in (5.2) of Proposition 5.3 satisfies
\[
0 < \frac{I_\mu}{2\mu} \leq W \quad \text{and} \quad 7I_\mu = 5W\mu + 3\gamma \int_{\mathbb{R}^3} \xi \, dx. \quad (5.40)
\]

**Proof.** Let $\xi \in S_\mu$. By Proposition 5.3 and 5.11, we get $\xi = 1_A$ a.e. where $A = \{\psi - \frac{1}{2}W r^2 - \gamma > 0\}$ for some $W > 0$ and $\gamma \geq 0$. If $x \in A$, then we have
\[
\frac{\psi(r,z)}{r^2} \geq \frac{1}{2}W > 0.
\]
Thus the convergence (5.38) from Lemma 5.12 implies that $A$ is bounded so that the (essential) support of $\xi$ is bounded in $\mathbb{R}^3$.

Now we use the identity (5.32) from Lemma 5.9. Here we follow a similar line of the proof of Proposition 5.11. By setting $\Psi = \psi - (1/2)W r^2 - \gamma$, we observe $\partial_t \Psi = \partial_r \psi - Wr$ and $\partial_z \Psi = \partial_z \psi$. Thus we have, by setting $F(s) = s^+$,
\[
E[\xi] = \int (r \partial_r \Psi + z \partial_z \Psi) \xi \, dx + W \int r^2 \xi \, dx = \int x \cdot \nabla[F(\Psi)] \, dx + 2W\mu.
\]
By integration by parts, we compute
\[
\int x \cdot \nabla[F(\Psi)] \, dx = -3 \int F(\Psi) \, dx,
\]
which can be justified as done in the proof of the integration by parts (5.36) in Proposition 5.11. In sum, we get
\[
I_\mu = E[\xi] = -3 \int F(\Psi) \, dx + 2W\mu \leq 2W\mu. \quad (5.41)
\]
Due to $I_\mu > 0$ from Theorem 4.2, the estimate in (5.40) is obtained.
To prove the identity in (5.40), we claim
\[
I_\mu = \frac{1}{2} \left( \int F(\Psi) \, dx + W\mu + \gamma \int \xi \, dx \right). \quad (5.42)
\]
Indeed, we compute
\[
\int F(\Psi)dx = \int \Psi^+ dx = \int \Psi \xi dx = \int \left( \psi - \frac{1}{2} W r^2 - \gamma \right) \xi dx
\]
\[
= 2E[\xi] - \frac{1}{2} W \int r^2 \xi dx - \gamma \int \xi dx = 2I_{\mu} - W \mu - \gamma \int \xi dx,
\]
which gives the above claim. By combining the identities (5.41) and (5.42), we get the identity
\[
7I_{\mu} = 5W \mu + 3\gamma \int \xi dx.
\]
□

Now we are ready to finish proving Theorem 5.1.

Proof of Theorem 5.1. By using Proposition 5.3, 5.11, 5.13, the proof is done. □

5.5 | Positive flux constant gives the full mass

Before finishing this section, we present the following lemma saying that positivity of \( \gamma \) in (5.1) for \( \xi \in S_{\mu} \) guarantees that \( \xi \) has the full mass. We put this lemma here since its proof follows a similar manner of the proof of Proposition 5.3 even though the result will be used only in Section 7.

Lemma 5.14. For each \( \xi \in S_{\mu} \) having \( \gamma > 0 \) in (5.1), we have
\[
\int_{\mathbb{R}^3} \xi dx = 1.
\]

Remark 5.15. The above lemma was first shown by [50, Remark 1 in Section 5] for a certain maximizer \( \xi \in S_{\mu} \) constructed in the paper. We simply adapt the proof here so that it works for every maximizer (cf. [2, Remark 2.6 (ii)] for 2d case).

Proof. Let \( \xi \in S_{\mu} \). Then, \( \xi \in S_{\mu}' \) by Theorem 4.2, and there exist unique \( W > 0, \gamma \geq 0 \) such that \( \xi = 1_A \) with \( A = \{ \psi - (1/2)W r^2 - \gamma > 0 \} \) by Theorem 5.1 which we just have proved. We recall that the unique constants \( W, \gamma \) are simply obtained by (5.3) or (5.7) (in the proof of Proposition 5.3).

Let us suppose
\[
\int \xi dx < 1.
\]

Then our goal is to show \( \gamma = 0 \). For each \( n \geq 1 \), we define (cf. see (5.26))
\[
\eta_n = \begin{cases} 
    h - \frac{1}{2} \int r^2 h dx & \text{when } \frac{1}{2} \int r^2 h dx \geq 0 \text{ that is, Case (I),(III)}, \\
    h - \frac{1}{2} \int r^2 h dx & \text{when } \frac{1}{2} \int r^2 h dx < 0 \text{ that is, Case (II),(IV)}, 
\end{cases}
\]
where \( h, h_{n,2}^\pm \in L^\infty(\mathbb{R}^3) \) are axi-symmetric, compactly supported functions appeared in (5.20) and
(5.25) during the proof of Proposition 5.3. Thanks to the properties (5.21), (5.22) of \( h^{\pm}_{n,2} \), the function \( \eta_n \) for each \( n \) satisfies (cf. (5.27))

\[
\eta_n \leq 0 \quad \text{on} \quad A, \quad \eta_n \geq 0 \quad \text{on} \quad A^c, \quad \frac{1}{2} \int r^2 \eta_n \, dx = 0.
\]

We fix \( n \geq 1 \) and consider \( (\xi + \varepsilon \eta_n) \) for \( \varepsilon > 0 \). Due to the assumption \( \int \xi \, dx < 1 \), we get

\[
(\xi + \varepsilon \eta_n) \in P'_\mu \quad \text{for any sufficiently small} \quad \varepsilon > 0.
\]

Thus we have \( E[\xi + \varepsilon \eta_n] - E[\xi] \leq 0 \) for such \( \varepsilon > 0 \). Hence, by taking limit \( \varepsilon \downarrow 0 \), we have

\[
0 \geq \int \psi \eta_n \, dx
\]

as in (5.28). From the definition of \( W^+_n \) in (5.23), we have

\[
0 \geq \int \left( \psi - W^+_n \frac{1}{2} r^2 \right) h \, dx \quad \text{when} \quad \frac{1}{2} \int r^2 h \, dx \geq 0,
\]

\[
0 \geq \int \left( \psi - W^-_n \frac{1}{2} r^2 \right) h \, dx \quad \text{when} \quad \frac{1}{2} \int r^2 h \, dx < 0.
\]

By the convergence \( W^{\pm}_n \to W \) in (5.24), we can take the limit to the above inequalities to obtain

\[
0 \geq \int \left( \psi - \frac{1}{2} W r^2 \right) h \, dx = \int_{A^c} + \int_A
\]

for any case. Since \( h \) is an arbitrary function satisfying the sign condition (5.25), we follow the same approach as in (the last part of) the proof of Proposition 5.3 to arrive at

\[
A = \{ \psi - \frac{1}{2} Wr^2 > 0 \} \quad \text{a.e.}
\]

By uniqueness of the pair of constants \( (W, \gamma) \) from Proposition 5.3, we get \( \gamma = 0 \).

\[\square\]

6 | COMPACTNESS

In this section, we prove compactness (Theorem 3.1), which is needed when proving the stability (Theorem 1.2) in Subsection 3.4.

6.1 | Concentrated compactness lemma: Lions (1984)

We start the proof by stating a slight variation of the concentrated compactness lemma [77, Lemma I.1]. For instance, such a variation can be found in [20, Lemma 1]. Here, \( B_R(r', z') = \{(r, z) \in \Pi \mid |(r, z) - (r', z')| < R \} \) is the disk in the half-space \( \Pi \) centered at \((r', z')\) with radius \( R \) as defined in (2.17).
Lemma 6.1. Let $0 < \mu < \infty$. Let $\{\rho_n\}_{n=1}^{\infty} \subset L^1(\Pi)$ satisfy

$$\rho_n \geq 0 \quad \text{for} \quad n \geq 1 \quad \text{and} \quad \int_{\Pi} \rho_n \, dr \, dz \to \mu \quad \text{as} \quad n \to \infty.$$  

Then, there exists a subsequence $\{\rho_{n_k}\}_{k=1}^{\infty}$ satisfying one of the three following possibilities:

(i) (Compactness) There exists a sequence $\{(r_k, z_k)\}_{k=1}^{\infty} \subset \Pi$ such that for arbitrary $\varepsilon > 0$, there exist $R > 0$ and an integer $k_0 \geq 1$ such that

$$\int_{B_R(r_k, z_k)} \rho_{n_k} \, dr \, dz \geq \mu - \varepsilon \quad \text{for} \quad k \geq k_0.$$  

(ii) (Vanishing) For each $R > 0$,

$$\lim_{k \to \infty} \sup_{(r', z') \in \Pi} \int_{B_R(r', z')} \rho_{n_k} \, dr \, dz = 0.$$  

(iii) (Dichotomy) There exists a constant $\alpha \in (0, \mu)$ such that for arbitrary $\varepsilon > 0$, there exist an integer $k_0 \geq 1$ and sequences $\{\rho_k^{(1)}\}_{k=1}^{\infty}, \{\rho_k^{(2)}\}_{k=1}^{\infty} \subset L^1(\Pi)$ such that for each $k \geq k_0$,

$$\rho_k^{(1)} = 1_{\Omega_k^{(1)}} \rho_{n_k}, \quad \rho_k^{(2)} = 1_{\Omega_k^{(2)}} \rho_{n_k} \quad \text{for some disjoint measurable sets} \quad \Omega_k^{(1)}, \Omega_k^{(2)} \subset \Pi,$$

$$||\rho_{n_k} - (\rho_k^{(1)} + \rho_k^{(2)})||_{L^1(\Pi)} + \left| \int_{\Pi} \rho_k^{(1)} \, dr \, dz - \alpha \right| + \left| \int_{\Pi} \rho_k^{(2)} \, dr \, dz - (\mu - \alpha) \right| \leq \varepsilon,$$

and

$$\text{dist} (\Omega_k^{(1)}, \Omega_k^{(2)}) \to \infty \quad \text{as} \quad k \to \infty.$$  

Proof of Lemma 6.1. We first observe that the concentrated compactness lemma [77, Lemma I.1] holds even for the half space $\Pi = \{(r, z) \in \mathbb{R}^2 \mid r, z, \in \mathbb{R}, \ r > 0\}$. Then, we apply the result into

$$\bar{\rho}_n := \frac{\mu}{\int_{\Pi} \rho_n \, dr \, dz} \rho_n,$$

due to $\int_{\Pi} \bar{\rho}_n \, dr \, dz = \mu$. \hfill \Box

6.2 Proof of compactness theorem (Theorem 3.1)

Proof of Theorem 3.1. Let $\mu \in (0, \infty)$. As in the previous sections, it is enough to show Theorem 3.1 for the case $\nu = \lambda = 1$ since the general case will follow the scaling argument (4.4).
Let \( \{\xi_n\} \) be a sequence of non-negative axi-symmetric functions and let \( \{a_n\} \) be a sequence of positive numbers such that

\[
a_n \to 0 \quad \text{as} \quad n \to \infty,
\]

\[
\limsup_{n \to \infty} \|\xi_n\|_1 \leq 1, \quad \lim_{n \to \infty} \int_{\{x \in \mathbb{R}^3 \mid ||\xi_n(x) - 1|| \geq a_n\}} \xi_n \, dx = 0, \quad \lim_{n \to \infty} \frac{1}{2} \|r^2 \xi_n\|_1 = \mu,
\]

\[
\sup_n \|\xi_n\|_2 < \infty, \quad \text{and} \quad \lim_{n \to \infty} E[\xi_n] = I_\mu.
\]

We set

\[
K_0 = \sup_n \|\xi_n\|_2 < \infty.
\]

By taking a subsequence if necessary (still denoted by \( \{\xi_n\} \) for simplicity), we may assume

\[
\|\xi_n\|_1 \leq 2, \quad \frac{1}{2} \mu < \mu_n < 2\mu \quad \text{for each} \quad n \geq 1,
\]

where we set \( \mu_n = \frac{1}{2} \|r^2 \xi_n\|_1 \). By setting

\[
\rho_n(r, z) = \pi r^3 \xi_n(r, z) \geq 0, \quad (r, z) \in \Pi,
\]

we have

\[
\int_{\Pi} \rho_n \, dr \, dz = \pi \int_{\Pi} r^3 \xi_n \, dr \, dz = \frac{1}{2} \int_{\mathbb{R}^3} r^2 \xi_n \, dx = \mu_n \to \mu \quad \text{as} \quad n \to \infty.
\]

Now we can apply Lemma 6.1 into the sequence \( \{\rho_n\} \). Then, for a certain subsequence (still using the same parameter \( n \)), one of the three cases, (ii) Vanishing, (iii) Dichotomy, (i) Compactness, should occur. First, we shall exclude the cases (ii) Vanishing, (iii) Dichotomy in order to get the Case (i) Compactness.

- **Elimination of Case (ii) Vanishing:**

  Let us suppose that the vanishing Case (ii) happens. that is, we assume

  \[
  \lim_{n \to \infty} \sup_{(r', z') \in \Pi} \int_{B_R(r', z')} r^3 \xi_n \, dr \, dz = 0 \quad \text{for each} \quad R > 0.
  \]

  To contradict, it is enough to show

  \[
  \lim_{n \to \infty} E[\xi_n] = 0
  \]

  since this implies \( I_\mu = 0 \) by (6.1), which gives a contradiction to \( I_\mu > 0 \) by Theorem 4.2 To show (6.5), we recall the estimate (2.4) with \( \tau = 3/2 \):

  \[
  G(r, z, r', z') \leq C_1 \frac{(rr')^2}{|r(z) - (r', z')|^3},
  \]
where $C_1 > 0$ is a universal constant. Then, for any $R \geq 1$, we decompose

\[
E[\xi_n] = \iint \pi G(r, z, r', z') \xi_n(r', z') \xi_n(r, z) r r' dr' dz' dr dz
= \iint_{|r, z| - (r', z') \geq R} + \iint_{|r, z| - (r', z') < R, G(r, z, r', z') < C_1 R r^2} + \iint_{|r, z| - (r', z') < R, G(r, z, r', z') \geq C_1 R r^2} =: I_{R, n} + II_{R, n} + III_{R, n}.
\]

For the integral $I_{R, n}$, we have, by (6.3) and (6.6),

\[
I_{R, n} \lesssim \frac{\mu_n^2}{R^3} \lesssim \frac{\mu^2}{R^3}.
\]

For the integral $II_{R, n}$, we estimate

\[
II_{R, n} \lesssim R \int_{\Pi} \int_{B_R(r, z)} r'^3 r^3 \xi_n(r', z') \xi_n(r, z) dr' dz' dr dz \lesssim R \left( \int_{\Pi} r^3 \xi_n(r, z) dr dz \right) \left( \sup_{(r, z) \in \Pi} \int_{B_R(r, z)} r'^3 \xi_n(r', z') dr' dz' \right) \lesssim R \mu \left( \sup_{(r, z) \in \Pi} \int_{B_R(r, z)} r'^3 \xi_n(r', z') dr' dz' \right).
\]

For the integral $III_{R, n}$, we observe that the condition $G \geq C_1 R r^2 r^2$ together with (6.6) implies $|(r, z) - (r', z')| \leq R^{-1/3} \leq 1 \leq R$. Thus we get

\[
III_{R, n} \lesssim \iint_{|(r, z) - (r', z')| \leq R^{-1/3}} G(r, z, r', z') r^r r' \xi_n(r', z') r \xi_n(r, z) dr' dz' dr dz
= \int_{\Pi} r \xi_n(r, z) \left( \int_{B_{R^{-1/3}}(r, z)} G(r, z, r', z') r^r r' \xi_n(r', z') dr' dz' \right) dr dz \lesssim \int_{\Pi} r \xi_n(r, z) \left( \int_{B_{R^{-1/3}}(r, z)} r'^3 |G(r, z, r', z')|^2 dr' dz' \cdot \int_{B_{R^{-1/3}}(r, z)} r' |\xi_n|^2 dr' dz' \right)^{1/2} dr dz \lesssim \|\xi_n\|_2 \int_{\Pi} r \xi_n(r, z) \left( \int_{B_{R^{-1/3}}(r, z)} r'^3 |G(r, z, r', z')|^2 dr' dz' \right)^{1/2} dr dz.
\]

On the other hand, we recall the estimate (4.6) of Lemma 4.6:

\[
\int_{B_M(r, z)} r'^r |G(r, z, r', z')|^2 dr' dz' \lesssim (M r^4 + M^{7/2} r^{3/2}), \quad M > 0, \quad (r, z) \in \Pi.
\]
By plugging $M = R^{-1/3}$ into the above, we have

$$
\left( \int_{B_{R^{-1/3}}(r,z)} |r' G(r, z, r', z')|^2 dr' dz' \right)^{1/2} \lesssim (1 + r^2) R^{-1/6}, \quad R \geq 1, \quad (r, z) \in \Pi.
$$

Hence, we get

$$
III_{R,n} \lesssim \|\xi_n\|_2 \int_{\Pi} r \xi_n(r, z) (R^{-1/6} (1 + r^2)) dr dz.
$$

Collecting the above estimates, we get, for $R \geq 1$ and for $n \geq 1$,

$$
E[\xi_n] \lesssim \frac{\mu^2}{R^3} + R\mu \left( \sup_{(r,z) \in \Pi} \int_{B_R(r,z)} r^3 \xi_n(r', z') dr' dz' \right) + R^{-1/6} K_0(1 + \mu).
$$

We take $\limsup_{n \to \infty}$ to get, by (6.4),

$$
\limsup_{n \to \infty} E[\xi_n] \lesssim \frac{\mu^2}{R^3} + R^{-1/6} K_0(1 + \mu), \quad R \geq 1.
$$

Then sending $R \to \infty$ implies (6.5). Thus the Case (ii) Vanishing cannot occur.

- Elimination of Case (iii) Dichotomy:

Let us suppose that the dichotomy Case (iii) happens with some $\alpha \in (0, \mu)$.

* Step 1—Applying Steiner symmetrization into each half:

We fix $\varepsilon > 0$. Then there exist an integer $k_0 \geq 1$ and sequences $\{\xi_{1,n}\}, \{\xi_{2,n}\} \subset L^1_{\omega}(\mathbb{R}^3)$ such that

$$
\xi_{1,n} = \Omega_{n}^{(1)} \xi_n, \quad \xi_{2,n} = \Omega_{n}^{(2)} \xi_n \quad \text{for some disjoint axi-sym. sets} \quad \Omega_{n}^{(1)}, \Omega_{n}^{(2)} \subset \mathbb{R}^3, \quad (6.7)
$$

$$
||r^2 \xi_{3,n}||_1 + |\alpha_n - \alpha| + |\beta_n - (\mu - \alpha)| \leq \varepsilon \quad \text{for} \quad n \geq k_0, \quad \text{and} \quad (6.8)
$$

$$
d_n \to \infty \quad \text{as} \quad n \to \infty, \quad (6.9)
$$

where we set

$$
\xi_{3,n} = \xi_n - \xi_{1,n} - \xi_{2,n}, \quad d_n = \text{dist} (\Omega_{n}^{(1)}, \Omega_{n}^{(2)}),
$$

$$
\alpha_n = \frac{1}{2} \int r^2 \xi_{1,n} dx, \quad \text{and} \quad \beta_n = \frac{1}{2} \int r^2 \xi_{2,n} dx.
$$

By choosing a subsequence (still using the same index $n$), we may assume that

$$
\alpha_n \to \bar{\alpha} \quad \text{and} \quad \beta_n \to \bar{\beta} \quad \text{as} \quad n \to \infty \quad (6.10)
$$
for some \( \bar{\alpha} \in [\alpha - \varepsilon, \alpha + \varepsilon] \) and \( \bar{\beta} \in [(\mu - \alpha) - \varepsilon, (\mu - \alpha) + \varepsilon] \). Recalling \( \xi_n = \xi_{1,n} + \xi_{2,n} + \xi_{3,n} \), we split the energy of \( \xi_n \) into

\[
E[\xi_n] = \pi \iint rr'G(r, z, r', z')\xi_n(r, z)\xi_n(r', z')dr'dz'dr dz
= E[\xi_{1,n}] + E[\xi_{2,n}] + 2\pi \iint rr'G(r, z, r', z')\xi_{1,n}(r, z)\xi_{2,n}(r', z')dr'dz'dr dz
+ \pi \iint rr'G(r, z, r', z')(2\xi_n(r, z) - \xi_{3,n}(r, z))\xi_{3,n}(r', z')dr'dz'dr dz
= : E[\xi_{1,n}] + E[\xi_{2,n}] + I_n + II_n.
\]

The estimate (2.4) for \( \tau = 3/2 \) implies

\[
I_n \lesssim (\mu/n)^2 \lesssim \mu^2/(d/n)^3.
\]

For the integral \( II_n \), we apply (2.21) to get

\[
|II_n| \lesssim (r^2(2\xi_n - \xi_{3,n})_1 + 2\xi_n - \xi_{3,n}\|L^1 \cap L^2\|\xi_{1,2,n})_1/2\|\xi_{3,n}\|_1/2
\lesssim (r^2\xi_n_1 + \xi_n\|L^1 \cap L^2\|\xi_{1,2,n})_1/2\xi_1/2 \lesssim (1 + \mu + K_0)\xi^{1/2}, \quad n \geq k_0,
\]

where we used (6.8), (6.2) and (6.3). Hence, we get

\[
E[\xi_n] \leq E[\xi_{1,n}] + E[\xi_{2,n}] + C\mu^2/(d/n)^3 + C(1 + \mu + K_0)\xi^{1/2},
\]

where \( C > 0 \) is a universal constant. Now we take Steiner symmetrization \( \xi_{i,n}^* \) (Proposition 4.8 with \( p = 2 \)) of \( \xi_{i,n} \) for \( i = 1, 2 \) to see that for \( n \geq k_0 \),

\[
E[\xi_n] \leq E[\xi_{1,n}^*] + E[\xi_{2,n}^*] + C\mu^2/(d/n)^3 + C(1 + \mu + K_0)\xi^{1/2},
\]

\[
\alpha_n = 1/2\|r^2\xi_{1,n}^*\|_1, \quad \beta_n = 1/2\|r^2\xi_{2,n}^*\|_1.
\]

\[\text{Step 2—Taking limit on } n \to \infty:\]

We observe that \( \xi_{i,n}^* \) for each \( i \) and for each \( n \) is axi-symmetric and non-negative. We also have

\[
\int_{\{x \in \mathbb{R}^3 \mid |\xi_{i,n}^*(x)| \geq \alpha_n \}} \xi_{i,n}^* \ dx = \int_{\{x \in \mathbb{R}^3 \mid |\xi_{i,n}(x)| \geq \alpha_n \}} \xi_{i,n} \ dx \leq \int_{\{x \in \mathbb{R}^3 \mid |\xi_n(x)| \geq \alpha_n \}} \xi_n \ dx,
\]

where the equality follows from the property (4.8) of Steiner symmetrization (Proposition 4.8) while the inequality comes from the form

\[
\xi_{i,n} = 1_{\xi_{i,n}^*}^* \xi_n.
\]
in (6.7). Thus, by (6.1), we have, for each \( i = 1, 2 \),
\[
\lim_{n \to \infty} \int_{\{x \in \mathbb{R}^3 \mid ||\xi_{i,n}^*||_2 \geq a_n\}} \xi_{i,n}^* \, dx = 0. \tag{6.13}
\]
We also have, by (4.8),
\[
||\xi_{1,n}^*||_2^2 + ||\xi_{2,n}^*||_2^2 = ||\xi_1||_2^2 + ||\xi_{2,n}||_2^2 = ||\xi_{1,n} + \xi_{2,n}||_2^2 \leq ||\xi_n||_2^2.
\]
Thus, by (6.2), we have the uniform \( L^2 \)-bound
\[
\sup_{n} \left( ||\xi_{1,n}^*||_2 + ||\xi_{2,n}^*||_2 \right) \lesssim K_0.
\]
By choosing a subsequence (still denoted by \( \{\xi_{i,n}^*\} \)), we obtain
\[
\xi_{i,n}^* \rightharpoonup \bar{\xi}_i \quad \text{in} \quad L^2(\mathbb{R}^3) \quad \text{as} \quad n \to \infty \tag{6.14}
\]
for some non-negative axi-symmetric \( \bar{\xi}_i \in L^2(\mathbb{R}^3) \) for \( i = 1, 2 \). We note
\[
||\bar{\xi}_i||_2 \leq \liminf_{n \to \infty} ||\xi_{i,n}^*||_2 \lesssim K_0 \quad \text{for} \quad i = 1, 2. \tag{6.15}
\]
We observe, for any bounded set \( U \subset \mathbb{R}^3 \), we have
\[
\xi_{i,n}^* \rightharpoonup \bar{\xi}_i \quad \text{in} \quad L^1(U) \quad \text{as} \quad n \to \infty \tag{6.16}
\]
due to \( L^\infty(U) \subset L^2(U) \). Thus we get, for bounded \( U \subset \mathbb{R}^3 \),
\[
||\bar{\xi}_i||_{L^1(U)} \leq \liminf_{n \to \infty} ||\xi_{i,n}^*||_{L^1(U)} \leq \liminf_{n \to \infty} ||\xi_{i,n}^*||_1,
\]
which implies
\[
||\bar{\xi}_i||_1 \leq \liminf_{n \to \infty} ||\xi_{i,n}^*||_1,
\]
for \( i = 1, 2 \). Thus we have, by (6.1) and (6.11),
\[
||\bar{\xi}_1||_1 + ||\bar{\xi}_2||_1 \leq \liminf_{n \to \infty} ||\xi_{1,n}^*||_1 + \liminf_{n \to \infty} ||\xi_{2,n}^*||_1 \leq \limsup_{n \to \infty} \left( ||\xi_{1,n}^*||_1 + ||\xi_{2,n}^*||_1 \right) \leq \limsup_{n \to \infty} ||\xi_{i,n}||_1 \leq 1. \tag{6.17}
\]
By a similar argument, we have
\[
r^2 \xi_{i,n}^* \rightharpoonup r^2 \bar{\xi}_i \quad \text{in} \quad L^1(U) \quad \text{as} \quad n \to \infty
\]
for bounded \( U \subset \mathbb{R}^3 \) so that
\[
||r^2 \bar{\xi}_i||_1 \leq \liminf_{n \to \infty} ||r^2 \xi_{i,n}^*||_1, \quad i = 1, 2.
\]
Thus, by recalling (6.10) and (6.11), we have
\[ \alpha \geq \frac{1}{2} \| r^2 \xi_1 \|_1, \quad \beta \geq \frac{1}{2} \| r^2 \xi_2 \|_1. \] (6.18)

We claim
\[ \| \xi_i \|_\infty \leq 1 \quad \text{for} \quad i = 1, 2. \] (6.19)

To prove, let us suppose that the case \( \| \xi_i \|_\infty > 1 \) happens either \( i = 1 \) or \( 2 \). We may assume that the case for \( i = 1 \) occurs since the other case can be solved in the same way. Then there exist a measurable subset \( U \subset \mathbb{R}^3 \) and a small constant \( \eta > 0 \) such that
\[ |U| > 0 \quad \text{and} \quad \xi_1 \geq 1 + \eta \quad \text{in} \quad U. \]

We may assume that \( U \) is bounded in \( \mathbb{R}^3 \). From the weak convergence (6.16), we have
\[ (1 + \eta)|U| \leq \| \bar{\xi}_1 \|_{L^1(U)} \leq \liminf_{n \to \infty} \| \xi^*_{1,n} \|_{L^1(U)}. \] (6.20)

On the other hand, by setting \( A^*_{1,n} = \{ x \in \mathbb{R}^3 \mid \| \xi^*_{1,n} (x) - 1 \| \geq a_n \} \), we estimate
\[ \| \xi^*_{1,n} \|_{L^1(U)} = \int_{U \cap A^*_{1,n}} \xi^*_{1,n} \, dx + \int_{U \setminus A^*_{1,n}} \xi^*_{1,n} \, dx \leq \int_{A^*_{1,n}} \xi^*_{1,n} \, dx + (1 + a_n)|U|. \]

By taking \( \limsup_{n \to \infty} \), we get, by (6.13) and (6.1),
\[ \limsup_{n \to \infty} \| \xi^*_{1,n} \|_{L^1(U)} \leq \limsup_{n \to \infty} \int_{A^*_{1,n}} \xi^*_{1,n} \, dx + \limsup_{n \to \infty} (1 + a_n)|U| = |U|. \]

It contradicts to (6.20). Thus we have (6.19).

Next, we claim
\[ \lim_{n \to \infty} E[\xi^*_{1,n}] = E[\bar{\xi}_1] \quad \text{for} \quad i = 1, 2. \] (6.21)

Indeed, since \( \xi^*_{1,n} \) is from Steiner symmetrization (Proposition 4.8) by definition, it satisfies the monotonicity condition (4.7) and
\[ \| \xi^*_{1,n} \|_{L^1 \cap L^2} + \| r^2 \xi^*_{1,n} \|_1 = \| \xi_{1,n} \|_{L^1 \cap L^2} + \| r^2 \xi_{1,n} \|_1 \leq \| \xi_n \|_{L^1 \cap L^2} + \| r^2 \xi_n \|_1 \leq 1 + K_0 + \mu \]
by (6.2), (6.3), (6.7). Since we have the weak-convergence (6.14), we can apply Lemma 4.10 into the sequence \{\xi_{1,n}\} for \( i = 1, 2 \) so that we obtain the convergence (6.21) of the kinetic energy.

By sending \( n \to \infty \) in (6.11) and by using (6.1), (6.21), (6.9), (6.17), (6.15), (6.18), and (6.19), we obtain
\[ 0 \leq \bar{\xi}_i \leq 1 \quad i = 1, 2, \]
\[ I_\mu \leq E[\bar{\xi}_1] + E[\bar{\xi}_2] + C(1 + \mu + K_0)\varepsilon^{1/2}, \]
\[ \| \bar{\xi}_1 \|_1 + \| \bar{\xi}_2 \|_1 \leq 1, \quad \| \bar{\xi}_1 \|_2 + \| \bar{\xi}_2 \|_2 \leq CK_0, \]
\[ \alpha \geq \frac{1}{2} \| r^2 \bar{\xi}_1 \|_1, \quad \beta \geq \frac{1}{2} \| r^2 \bar{\xi}_2 \|_1. \] (6.22)
Step 3—Dropping the parameter $\varepsilon$:

By summarizing what we have done in Step 1 and Step 2, for each $\varepsilon > 0$, there exist functions $\bar{\xi}_1, \bar{\xi}_2, \tilde{\bar{\xi}}_1, \tilde{\bar{\xi}}_2$ and the constants $\bar{\alpha} = \bar{\alpha}, \bar{\beta} = \bar{\beta}$ satisfying (6.22) while the constants $C > 0$ in (6.22) are independent of the choice of $\varepsilon > 0$. Then we can apply a similar argument in Step 2 for the sequences

$$\{\xi_i\}_{i=1}^\infty, \quad \varepsilon_m = \frac{1}{m}$$

for $i = 1, 2$ in order to obtain the following claim:

There exist axi-symmetric non-negative functions $\hat{\xi}_i \in L^2(\mathbb{R}^3)$ for $i = 1, 2$ such that

$$0 \leq \hat{\xi}_i \leq 1, \quad i = 1, 2,$$

$$I_\mu \leq E[\hat{\xi}_1] + E[\hat{\xi}_2],$$

$$||\hat{\xi}_1||_1 + ||\hat{\xi}_2||_1 \leq 1,$$

$$\alpha \geq \frac{1}{2} ||r^2\hat{\xi}_1||_1, \quad \mu - \alpha \geq \frac{1}{2} ||r^2\hat{\xi}_2||_1.$$

Indeed, by the uniform $L^2$-bound in (6.22), we can first extract subsequential weak-limits $\hat{\xi}_i \in L^2(\mathbb{R}^3)$ in $L^2$ for $i = 1, 2$, that is,

$$\hat{\xi}_i^m \rightharpoonup \hat{\xi}_i \quad \text{in} \quad L^2 \quad \text{as} \quad m \to \infty \quad \text{(by reindexing)}.$$

Thanks to (6.22), such limits satisfies the same pointwise estimate:

$$0 \leq \hat{\xi}_i \leq 1.$$

We also have $\hat{\xi}_i^m \rightharpoonup \hat{\xi}_i$ in $L^1(U)$ for any bounded set $U \subset \mathbb{R}^3$ as we proved (6.16). Thus, as in (6.17), we get the same $L^1$-bound:

$$||\hat{\xi}_1||_1 + ||\hat{\xi}_2||_1 \leq 1.$$}

Similarly, since $\bar{\alpha}^m \in [\alpha - \varepsilon, \alpha + \varepsilon]$ and $\bar{\beta}^m \in [(\mu - \alpha) - \varepsilon, (\mu - \alpha) + \varepsilon]$, we get

$$\alpha \geq \frac{1}{2} ||r^2\hat{\xi}_1||_1, \quad \mu - \alpha \geq \frac{1}{2} ||r^2\hat{\xi}_2||_1.$$

Since $\hat{\xi}_i$ satisfies the monotonicity condition (4.7) for each $m$ and for $i = 1, 2$, we can apply Lemma 4.10 to the sequences to obtain $E[\hat{\xi}_i^m] \to E[\hat{\xi}_i]$ for $i = 1, 2$. Applying the convergence into (6.22), we get

$$I_\mu \leq E[\hat{\xi}_1] + E[\hat{\xi}_2].$$

Hence, we get the claim (6.23).

Step 4: Contradiction to the dichotomy case.
We first observe that if both $\hat{\xi}_1$ and $\hat{\xi}_2$ are identically zero, then we have
\[
I_\mu \leq E[\hat{\xi}_1] + E[\hat{\xi}_2] = 0
\]
which is a contradiction to $I_\mu > 0$ in Theorem 4.2. Therefore, either $\hat{\xi}_1$ or $\hat{\xi}_2$ should be nontrivial. Without loss of generality, we may assume $\hat{\xi}_1 \neq 0$. Then we have
\[
0 < ||\hat{\xi}_1||_1 \leq 1 - ||\hat{\xi}_2||_1 =: \nu_1.
\tag{6.24}
\]
By using Theorems 4.2 and 5.1, we take a compactly supported function
\[
\zeta_1 \in S_{\alpha, \nu_1, 1}.
\]
We recall $S_{\alpha, \nu_1, 1} = S''_{\alpha, \nu_1, 1}$ (by Theorem 4.2) and $\hat{\xi}_1 \in P''_{\alpha, \nu_1, 1}$ (by (6.23) and (6.24)). Thus we have $E[\zeta_1] \geq E[\hat{\xi}_1]$. Now we have
\[
0 \leq \zeta_1 \leq 1, \quad 0 \leq \hat{\xi}_2 \leq 1,
\]
\[
I_\mu \leq E[\zeta_1] + E[\hat{\xi}_2],
\]
\[
||\zeta_1||_1 + ||\hat{\xi}_2||_1 \leq 1,
\]
\[
\alpha = \frac{1}{2} ||r^2 \zeta_1||_1, \quad \mu - \alpha \geq \frac{1}{2} ||r^2 \hat{\xi}_2||_1.
\]
Next, we observe that if $\hat{\xi}_2 \equiv 0$, we have
\[
I_\mu \leq E[\hat{\zeta}_1] = I_{\alpha, \nu_1, 1} \leq I_{\alpha, 1, 1} = I_\alpha,
\]
where the last inequality comes from $\nu_1 \leq 1$. Thus it is a contradiction to $I_\alpha < I_\mu$ by Lemma 4.13. Thus we may assume $\hat{\xi}_2 \neq 0$, which implies
\[
0 < ||\hat{\xi}_2||_1 \leq 1 - ||\hat{\zeta}_1||_1 =: \nu_2.
\]
By using Theorems 4.2 and 5.1 again, we take a compactly supported function
\[
\zeta_2 \in S_{\mu - \alpha, \nu_2, 1}.
\]
As before, we have $E[\zeta_2] \geq E[\hat{\zeta}_2]$ due to $S_{\mu - \alpha, \nu_2, 1} = S''_{\mu - \alpha, \nu_2, 1}$ and $\hat{\xi}_2 \in P''_{\mu - \alpha, \nu_2, 1}$. In sum, we have
\[
0 \leq \zeta_i \leq 1 \quad \text{for} \quad i = 1, 2,
\]
\[
I_\mu \leq E[\zeta_1] + E[\zeta_2],
\]
\[
||\zeta_1||_1 + ||\zeta_2||_1 \leq 1,
\]
\[
\alpha = \frac{1}{2} ||r^2 \zeta_1||_1, \quad \mu - \alpha = \frac{1}{2} ||r^2 \zeta_2||_1.
\tag{6.25}
\]
By a translation in $z$-variable (if necessary), we may assume that
\[
\text{spt} \zeta_1 \cap \text{spt} \zeta_2 = \emptyset.
\]
Then we have
\[ 0 \leq (\xi_1 + \xi_2) \leq 1, \quad \int (\xi_1 + \xi_2) \, dx \leq 1, \quad \text{and} \quad \frac{1}{2} \int r^2 (\xi_1 + \xi_2) \, dx = \mu, \]
which implies \((\xi_1 + \xi_2) \in \mathcal{D}'_\mu\) so that
\[ E[(\xi_1 + \xi_2)]] \leq I'_\mu = I_\mu \]
by Theorem 4.2. Together with (6.25), we get
\[ I_\mu \leq E[\xi_1] + E[\xi_2] = E[\xi_1 + \xi_2] - 2\pi \int r^2 G(r, z, r', z') \xi_1(r, z) \xi_2(r', z') \, dr' \, dz' \, dr \, dz \]
\[ \leq I_\mu - 2\pi \int r^2 G(r, z, r', z') \xi_1(r, z) \xi_2(r', z') \, dr' \, dz' \, dr \, dz. \]
Hence, either \(\xi_1 \equiv 0\), or \(\xi_2 \equiv 0\) holds due to \(G(r, z, r', z') > 0\) a.e. This contradicts to the last line of (6.25) due to \(\alpha \in (0, \mu)\). Thus the Case (iii) Dichotomy cannot occur.

* Case (i) Compactness:

Up to now, we have shown that the Case (i) Compactness should occur. That means: There exists a sequence \(\{r_n, z_n\} \subset \bar{\Pi}\) such that for arbitrary \(\epsilon > 0\), there exist \(R = R(\epsilon) > 0\) and \(k_0 = k_0(\epsilon) \geq 1\) such that
\[ \frac{1}{2} \int_{T_n^\epsilon} r^2 \xi_n \, dx \geq \mu - \epsilon, \quad \text{for all } n \geq k_0(\epsilon), \quad (6.26) \]
where axi-symmetric \(T_n^\epsilon \subset \mathbb{R}^3\) is defined by
\[ T_n^\epsilon = T_{R(\epsilon)}(r_n, z_n) = \{ x \in \mathbb{R}^3 \mid |(r, z) - (r_n, z_n)| < R(\epsilon) \} \]
when \(x_1^2 + x_2^2 = r^2, \quad x_3 = z\) as defined in (2.17).

** Step 1—Boundedness of \(\{r_n\}\):

We note that there are only two cases whether (a) \(\limsup_{n \to \infty} \bar{r}_n = \infty\) or (b) \(\sup_n \bar{r}_n < \infty\). We shall first show that the Case (a) cannot occur.

Let us suppose the Case (a) \(\limsup_{n \to \infty} \bar{r}_n = \infty\) happens. We may assume that \(\lim_{n \to \infty} \bar{r}_n = \infty\) by choosing a subsequence (and by reindexing). We claim
\[ \lim_{n \to \infty} E[\xi_n] = 0. \]
This claim implies \(I_\mu = 0\) by (6.1), which is a contradiction to \(I_\mu > 0\) in Theorem 4.2. To prove the claim, we set \(\psi_0(x) = G(\xi_n)\). Then, for \(\epsilon > 0\), we decompose
\[ E[\xi_n] = \frac{1}{2} \int \psi_0 \xi_n \, dx = \frac{1}{2} \int_{T_n^\epsilon} + \frac{1}{2} \int_{\mathbb{R}^3 \setminus T_n^\epsilon} =: I_{n, \epsilon} + II_{n, \epsilon}, \]
For the first term $I_{n,\varepsilon}$, by using (2.18), we estimate, for $n \geq 1$,

$$I_{n,\varepsilon} \leq \frac{1}{2} \left\| \frac{\psi_n}{r} \right\|_{\infty} \int_{T^*_n} r^{\xi_n} dx \lesssim (1 + K_0 + \mu) \left( \sup_{|r-R_n^*|<R(\varepsilon)} \frac{1}{r} \right) \int_{T^*_n} r^{2\xi_n} dx \lesssim \frac{(1 + K_0 + \mu) \mu}{(\tilde{r}_n - R(\varepsilon))}$$

by using (6.2) and (6.3), which gives $I_{n,\varepsilon} \to 0$ as $n \to \infty$. For the second term $II_{n,\varepsilon}$, by using Hölder’s inequality, for $n \geq k_0(\varepsilon)$,

$$II_{n,\varepsilon} \leq \frac{1}{2} \left\| \frac{\psi_n}{r} \right\|_{\infty} \int_{\mathbb{R}^3 \setminus T^*_n} r^{\xi_n} dx \leq \frac{1}{2} \left\| \frac{\psi_n}{r} \right\|_{\infty} \left( \int_{\mathbb{R}^3 \setminus T^*_n} r^{2\xi_n} dx \right)^{1/2} \left( \int \xi_n dx \right)^{1/2} \lesssim (1 + K_0 + \mu)(\mu_n - (\mu - \varepsilon))^{1/2},$$

(6.27)

which gives $\limsup_{n \to \infty} II_{n,\varepsilon} \lesssim (1 + K_0 + \mu)\varepsilon^{1/2}$. Collecting the above estimates, we get, for any $\varepsilon > 0$,

$$\limsup_{n \to \infty} E[\xi_n] \lesssim (1 + K_0 + \mu)\varepsilon^{1/2},$$

which implies

$$E[\xi_n] \to 0 \quad \text{as} \quad n \to \infty.$$  

Thus the Case (a) cannot occur, and the Case (b)

$$\sup_n \tilde{r}_n < \infty$$

should occur.

* Step 2—Reformulated goal via translations:

We may assume that $\tilde{r}_n = 0$ for $n \geq 1$ by replacing $R(\varepsilon)$ with

$$\left( R(\varepsilon) + \sup_n \tilde{r}_n \right) > 0$$

due to the set inclusion $T^*_n \subset \{ x \in \mathbb{R}^3 \mid |(r, z) - (0, \tilde{z}_n)| < (R(\varepsilon) + \sup_n \tilde{r}_n) \}$. Now we have $(\tilde{r}_n, \tilde{z}_n) = (0, \tilde{z}_n) \in \partial \Pi$ for $n \geq 1$. By redefining $\xi_n$ by translation of $\xi_n$ in $x_3$-variable, our goal (3.4) is transformed into (by setting $c_n = \tilde{z}_n$) the following new goal:

$$\text{to extract a subsequence of } \{\xi_n\} \text{ converging to } \xi \text{ in } L^1_\omega \text{ for some } \xi \in S_{\mu}. \quad (6.28)$$

We note that this translation in $x_3$—variable does not break (6.1), (6.2), (6.3). Moreover, (6.26) can be re-written by the following form:

For arbitrary $\varepsilon > 0$, there exist $R = R(\varepsilon) > 0$ and $k_0 = k_0(\varepsilon) \geq 1$ such that

$$\mu_n \geq \frac{1}{2} \int_{B^\varepsilon} r^{2\xi_n} dx \geq (\mu - \varepsilon), \quad \text{for all } n \geq k_0(\varepsilon), \quad (6.29)$$

by setting $B^\varepsilon = B_R(0) = T_{R(\varepsilon)}(0, 0) = \{ x \in \mathbb{R}^3 \mid |x| < R(\varepsilon) \}$ (see the definition (2.17)).

* Step 3—Extracting a weak-limit:
Since the sequence \( \{\xi_n\} \) is uniformly bounded in \( L^2 \) from (6.2), by choosing a subsequence (still denoted by \( \{\xi_n\} \)), we get
\[
\xi_n \rightharpoonup \xi \quad \text{in} \quad L^2(\mathbb{R}^3) \quad \text{as} \quad n \to \infty
\]
for some non-negative axi-symmetric function \( \xi \in L^2(\mathbb{R}^3) \). The weak convergence in \( L^2 \) implies, for any bounded subset \( U \subset \mathbb{R}^3 \),
\[
\xi_n \rightharpoonup \xi \quad \text{and} \quad r^2 \xi_n \rightharpoonup r^2 \xi \quad \text{in} \quad L^1(U) \quad \text{as} \quad n \to \infty.
\]
Hence, we get, by (6.1), (6.2), (6.29),
\[
\|\xi\|_2 \leq K_0, \quad \int \xi \, dx \leq 1, \quad (6.30)
\]
\[
(\mu - \varepsilon) \leq \frac{1}{2} \int_{B_{\varepsilon}} r^2 \xi \, dx \leq \mu \quad \text{for} \quad \varepsilon > 0, \quad \text{and} \quad \frac{1}{2} \int r^2 \xi \, dx = \mu.
\]

* Step 4—Verifying the pointwise bound: We claim
\[
\|\xi\|_\infty \leq 1. \quad (6.31)
\]
Indeed, we can follow the same approach in the proof of (6.19) in the dichotomy case. For a contradiction, we assume that there is a bounded set \( U \subset \mathbb{R}^3 \) and a constant \( \eta > 0 \) such that
\[
|U| > 0 \quad \text{and} \quad \xi \geq 1 + \eta \quad \text{in} \quad U.
\]
Then we get
\[
(1 + \eta)|U| \leq \|\xi\|_{L^1(U)} \leq \liminf_{n \to \infty} \|\xi_n\|_{L^1(U)}. \quad (6.32)
\]
On the other hand, by setting
\[
A_n = \{x \in \mathbb{R}^3 \mid |\xi_n(x) - 1| \geq a_n\}, \quad (6.33)
\]
we get
\[
\|\xi_n\|_{L^1(U)} = \int_{U \cap A_n} \xi_n \, dx + \int_{U \cap A_n^c} \xi_n \, dx \leq \int_{A_n} \xi_n \, dx + (1 + a_n)|U|. \quad (6.34)
\]
By taking \( \limsup_n \), and by using (6.1), we get
\[
\limsup_{n \to \infty} \|\xi_n\|_{L^1(U)} \leq |U|.
\]
It contradicts to (6.32). We have proved (6.31). Thanks to (6.30), we know
\[
\xi \in \mathcal{P}_\mu' \quad (6.34)
\]
(recall the definition (4.1) of \( \mathcal{P}_\mu' \) in Subsection 4.1).

* Step 5—Establishing convergence in energy:
Next, we claim

$$\lim_{n \to \infty} E[\xi_n] = E[\xi].$$

(6.35)

Indeed, we estimate, as in (6.27),

$$\int_{\mathbb{R}^3 \setminus B_\varepsilon} \xi_n G[\xi_n] \, dx \leq \left\| \frac{G[\xi_n]}{r} \right\|_{\infty} \int_{\mathbb{R}^3 \setminus B_\varepsilon} r \xi_n \, dx$$

$$\leq \left\| \frac{G[\xi_n]}{r} \right\|_{\infty} \left( \frac{\int_{\mathbb{R}^3 \setminus B_\varepsilon} r^2 \xi_n \, dx}{\int \xi_n \, dx} \right)^{1/2}$$

$$\leq C(1 + K_0 + \mu)(\mu_n - (\mu - \varepsilon))^{1/2}, \quad n \geq k_0(\varepsilon),$$

by (2.18), (6.2), (6.3), and (6.29). In the same way, we obtain

$$\int_{\mathbb{R}^3 \setminus B_\varepsilon} \xi G[\xi] \, dx \leq C(1 + K_0 + \mu)\varepsilon^{1/2}$$

thanks to the estimate (6.30). Then, by using (4.5) of Lemma 4.5, we can estimate difference in energy for $n \geq k_0(\varepsilon)$ by

$$|E[\xi_n] - E[\xi]| \leq \frac{1}{4\pi} \left| \int_{B_\varepsilon} \int_{B_\varepsilon} G(x, y)(\xi_n(x)\xi_n(y) - \xi(x)\xi(y)) \, dx \, dy \right|$$

$$+ C(1 + K_0 + \mu)|\mu_n - \mu|^{1/2} + C(1 + K_0 + \mu)\varepsilon^{1/2}.$$

Since $G \in L^2(B_\varepsilon \times B_\varepsilon)$ by Lemma 4.6 and $\xi_n(x)\xi_n(y) \rightharpoonup \xi(x)\xi(y)$ in $L^2(B_\varepsilon \times B_\varepsilon)$, we get the claim (6.35) by sending $n \to \infty$ first and then $\varepsilon \to 0$.

Step 6—It is of a patch-type: Since the convergence (6.35) implies $E[\xi] = I_\mu = I'_\mu$ by (6.1), we obtain $\xi \in S'_\mu$ due to (6.34). By using $S_\mu = S'_\mu$ from Theorem 4.2, we get

$$\xi \in S_\mu,$$

which implies $\xi \in P_\mu$. That means

$$\xi = 1_A$$

for some axi-symmetric measurable set $A \subset \mathbb{R}^3$.

Step 7—Weak convergence $\sqrt{r^2 \xi_n} \rightharpoonup \sqrt{r^2 \xi}$, in $L^2(\mathbb{R}^3)$:

We observe, as $n \to \infty$,

$$\frac{1}{\sqrt{2}} \| \sqrt{r^2 \xi_n} \|_2 = \sqrt{\mu_n} \to \sqrt{\mu} = \frac{1}{\sqrt{2}} \| \sqrt{r^2 \xi} \|_2,$$

(6.36)

which implies $\sup_n \| \sqrt{r^2 \xi_n} \|_2 < \infty$. Thus there exists a non-negative axi-symmetric function $g \in L^2(\mathbb{R}^3)$ such that

$$\sqrt{r^2 \xi_n} \rightharpoonup g$$
in $L^2(\mathbb{R}^3)$ (by reindexing). We claim

$$g = \sqrt{r^2\xi} \text{ a.e. in } \mathbb{R}^3. \quad (6.37)$$

Indeed, for any bounded set $U \subset \mathbb{R}^3$, we have

$$\sqrt{r^2\xi_n} \rightharpoonup g \text{ in } L^1(U).$$

On the other hand, for any $\phi \in L^\infty(U)$, we can prove

$$\int_U \sqrt{r^2\xi_n} \phi \, dx \to \int_U \sqrt{r^2\xi} \phi \, dx \quad \text{as } n \to \infty \quad (6.38)$$

in the following way:

First we estimate, by using $\sqrt{\xi} = \xi$,

$$\left| \int_U \sqrt{r^2\xi_n} \phi \, dx - \int_U \sqrt{r^2\xi} \phi \, dx \right| = \left| \int_U r\phi \left( \sqrt{\xi_n} - \xi \right) \, dx \right| \leq \int_U r|\phi| \left( \sqrt{\xi_n} - \xi \right) \, dx + \int_U r\phi(\xi_n - \xi) \, dx =: I_n + II_n.$$

We observe $II_n \to 0$ as $n \to \infty$ due to $(r\phi) \in L^2(U)$ and $\xi_n \to \xi$ in $L^2(U)$. For $I_n$, by recalling the definition (6.33) of $A_n$, we estimate

$$I_n \leq \int_{U \cap A_n} r|\phi| \left( \sqrt{\xi_n} + \xi_n \right) \, dx + \int_{U \cap A^c_n} r|\phi| \left( |\sqrt{\xi_n} - 1| + |\xi_n - 1| \right) \, dx$$

$$\leq \left( \sup_{r \in U} \|\phi\|_{L^\infty(U)} \right) \left[ |U|^\frac{1}{2} \left( \int_{A_n} \xi_n \, dx \right)^\frac{1}{2} + \int_{A_n} \xi_n \, dx \right] + 2 \int_{U \cap A^c_n} r|\phi| |\xi_n - 1| \, dx$$

$$\leq \left( \sup_{r \in U} \|\phi\|_{L^\infty(U)} \right) \left[ |U|^\frac{1}{2} \left( \int_{A_n} \xi_n \, dx \right)^\frac{1}{2} + \int_{A_n} \xi_n \, dx \right] + 2 \left( \sup_{r \in U} \|\phi\|_{L^\infty(U)} \right) a_n |U|.$$

Thus, by (6.1), we have $I_n \to 0$ as $n \to \infty$. Now we have (6.38), which implies

$$g = \sqrt{r^2\xi} \text{ a.e. in } U.$$

for any bounded set $U \subset \mathbb{R}^3$. Hence we get the claim (6.37).

* Step 8—Strong convergence $r^2\xi_n \rightharpoonup r^2\xi$, in $L^1(\mathbb{R}^3)$:

Since we have $\sqrt{r^2\xi_n} \rightharpoonup \sqrt{r^2\xi}$, in $L^2(\mathbb{R}^3)$ by (6.37) and $\|\sqrt{r^2\xi_n}\|_2 \to \|\sqrt{r^2\xi}\|_2$, by (6.36), we have the strong convergence

$$\sqrt{r^2\xi_n} \to \sqrt{r^2\xi} \text{ in } L^2(\mathbb{R}^3) \quad \text{as } n \to \infty. \quad (6.39)$$
We note that for the ball $B^\varepsilon$ in (6.29) and for all $n \geq k_0(\varepsilon)$,

\[
\int r^2 |\xi_n - \xi| \, dx \leq \int_{B^\varepsilon} r^2 |\xi_n - \xi| \, dx + \int_{\mathbb{R}^3 \setminus B^\varepsilon} r^2 \xi_n \, dx + \int_{\mathbb{R}^3 \setminus B^\varepsilon} r^2 \xi \, dx \leq \int_{B^\varepsilon} r^2 |\xi_n - \xi| \, dx + 2[\mu_n - (\mu - \varepsilon)] + 2\varepsilon.
\]

We claim, for each $\varepsilon > 0$,

\[
\lim_{n \to \infty} \int_{B^\varepsilon} r^2 |\xi_n - \xi| \, dx = 0
\]

for each fixed $\varepsilon > 0$. Indeed, we estimate

\[
\int_{B^\varepsilon} r^2 |\xi_n - \xi| \, dx \leq \int_{B^\varepsilon} r^2 |\xi_n - 1_{A^n} \xi| \, dx + \int_{B^\varepsilon} r^2 |1_{A^n} - \xi| \, dx =: I_n^\varepsilon + II_n^\varepsilon.
\]

For the integral $I_n^\varepsilon$, we have, by recalling the definition (6.33) of $A_n$,

\[
I_n^\varepsilon = \int_{B^\varepsilon \cap A_n} r^2 |\xi_n - 1_{A^n} \xi| \, dx + \int_{B^\varepsilon \cap A_n^c} r^2 |\xi_n - 1_{A^n} | \, dx
\]

\[
= \int_{B^\varepsilon \cap A_n} r^2 \xi_n \, dx + \int_{B^\varepsilon \cap A_n^c} r^2 |\xi_n - 1| \, dx \leq \left( R(\varepsilon)^2 \int_{A_n} |\xi_n| \, dx + R(\varepsilon)^2 a_n |B^\varepsilon| \right) \to 0 \text{ as } n \to \infty
\]

by (6.1). For the integral $II_n^\varepsilon$, we estimate, by recalling $\xi = 1_A$,

\[
II_n^\varepsilon = \int_{B^\varepsilon} r^2 |1_{A^n} - 1_A| \, dx = \int_{B^\varepsilon} r^2 |1_{A^n} - 1_A|^2 \, dx = \int_{B^\varepsilon} r^2 |\sqrt{1_{A^n}} - \sqrt{1_A}|^2 \, dx
\]

\[
\leq 2 \int_{B^\varepsilon} r^2 |\sqrt{1_{A^n} - \sqrt{\xi_n}}|^2 \, dx + 2 \int_{B^\varepsilon} r^2 |\sqrt{\xi_n} - \sqrt{1_A}|^2 \, dx =: II_{n,1}^\varepsilon + II_{n,2}^\varepsilon.
\]

As in (6.42), we estimate

\[
II_{n,1}^\varepsilon = 2 \int_{B^\varepsilon \cap A_n} r^2 |\sqrt{\xi_n}|^2 \, dx + 2 \int_{B^\varepsilon \cap A_n^c} r^2 |1 - \sqrt{\xi_n}|^2 \, dx
\]

\[
\leq 2 \int_{B^\varepsilon \cap A_n} r^2 \xi_n \, dx + 2 \int_{B^\varepsilon \cap A_n^c} r^2 |1 - \xi_n|^2 \, dx \leq \left( 2R(\varepsilon)^2 \int_{A_n} |\xi_n| \, dx + 2(a_n R(\varepsilon)^2 |B^\varepsilon|) \right) \to 0 \text{ as } n \to \infty
\]

by (6.1). Lastly, we observe, by recalling $\xi = 1_A$,

\[
II_{n,2}^\varepsilon = 2 \int_{B^\varepsilon} r^2 |\sqrt{\xi_n} - \sqrt{\xi}|^2 \, dx = 2 \int_{B^\varepsilon} \left| \sqrt{r^2 \xi_n} - \sqrt{r^2 \xi} \right|^2 \, dx \to 0
\]
as \( n \to \infty \) by (6.39). Hence we get the claim (6.41).

Combining the claim (6.41) with (6.40), we get, for each \( \varepsilon > 0 \),

\[
\limsup_{n \to \infty} \int r^2 |\xi_n - \xi| \, dx \leq \limsup_{n \to \infty} \int r^2 |\xi_n - \xi| \, dx + 4\varepsilon \leq 4\varepsilon,
\]

Sending \( \varepsilon \to 0 \), we obtain the convergence \( r^2 \xi_n \to r^2 \xi \) in \( L^1(\mathbb{R}^3) \) as \( n \to \infty \), which is our goal (6.28). Lastly, the set \( S_{\mu, \nu, \lambda} \) is non-empty thanks to Theorem 4.2. It finishes the proof of Theorem 3.1.

\[\square\]

7 | **UNIQUENESS OF HILL’S VORTEX**

7.1 | **Hill’s problem and uniqueness result: Amick-Fraenkel (1986)**

The final goal is to prove Theorem 3.2. First, we introduce the setting of Amick-Fraenkel [4, Theorem 1.1].

The paper [4] denotes \( H(\Pi) \) the completion of \( C^\infty_c(\Pi) \) (the class of infinitely smooth and compactly supported functions in \( \Pi \)) in the norm \( \| \cdot \|_H \) from the inner product defined by

\[
\langle \phi, \psi \rangle_H = \int_\Pi \frac{1}{r^2} ((\partial_r \phi)(\partial_r \psi) + (\partial_z \phi)(\partial_z \psi)) \, r \, dr \, dz.
\]  

Note if \( \phi, \psi \in C^\infty_c(\Pi) \), then we can integrate by parts (e.g., as in (5.33)) to get another representation

\[
\langle \phi, \psi \rangle_H = \int_\Pi \left( -\frac{1}{r^2} L \psi \right) \phi \, r \, dr \, dz.
\]  

For instance, we have \( E[-r^{-2} L \psi] = \pi \| \psi \|_H^2 \) for any \( \psi \in C^\infty_c(\Pi) \) (see (2.7) and (2.24)). This setting can be embedded in \( \mathbb{R}^5 \) in the following sense:

We denote \( y = (y_1, y_2, y_3, y_4, y_5) = (y', y_5) \in \mathbb{R}^5 \). For a function \( \phi : \Pi \to \mathbb{R} \), we define the cylindrical symmetric function \( T[\phi] : \mathbb{R}^5 \to \mathbb{R} \) by

\[
T[\phi](y) = \frac{\phi(r, z)}{r^2},
\]

where \( r^2 = |y'|^2 = y_1^2 + y_2^2 + y_3^2 + y_4^2 \) and \( z = y_5 \). When \( \phi \in C^\infty_c(\Pi) \), we get \( T \phi \in C^\infty_c(\mathbb{R}^5 \setminus \{ r = 0 \}) \), and it satisfies

\[
\Delta_{\mathbb{R}^5} (T \phi) = \frac{1}{r^2} L \phi = T[L \phi].
\]  

Moreover, for \( \phi, \psi \in C^\infty_c(\Pi) \), we can compute

\[
\langle \phi, \psi \rangle_H = \int_\Pi \left[ r^3 (\partial_r (\phi/r^2) \partial_r (\psi/r^2) + \partial_z (\phi/r^2) \partial_z (\psi/r^2)) + 2\partial_r (\phi \psi/r^2) \right] \, dr \, dz
\]

\[
= \int_\Pi \left[ r^3 \left( \frac{\partial_r \phi}{r^2} \frac{\partial_r \psi}{r^2} + \frac{\partial_z \phi}{r^2} \frac{\partial_z \psi}{r^2} \right) \right] \, dr \, dz = \frac{1}{2\pi^2} \int_{\mathbb{R}^5} \nabla T \phi \cdot \nabla T \psi \, dy,
\]  

(7.4)
which produces the identity
\[
\|\phi\|_H = \frac{1}{\sqrt{2\pi^2}} \|\nabla T \phi\|_{L^2(\mathbb{R}^5)}.
\] (7.5)

For given constants \(\lambda, W > 0\), the authors of [4] defined the Hill’s problem for \((\lambda, W)\) in the following way (also see [49]):

To find \(\psi\) such that
\[
-\frac{1}{r^2} \mathcal{L} \psi = \lambda f_H(\Psi), \quad \Psi := \psi - \frac{1}{2} W r^2
\]
\[
\psi|_{r=0} = 0, \quad \psi(r, z) \to 0 \quad \text{as} \quad r^2 + z^2 \to \infty \quad \text{in} \quad \bar{\Pi},
\]
for the vorticity function \(f_H = 1_{(0, \infty)}\) as defined in (1.12).

In [4], any function \(\psi \in H(\Pi) \setminus \{0\}\) is said to be a weak solution of the Hill’s problem for \((\lambda, W)\) if it satisfies
\[
\langle \phi, \psi \rangle_H = \lambda \int_{A(\psi)} \phi r \, dr \, dz \quad \text{for any} \quad \phi \in H(\Pi),
\] (7.6)

where
\[
A(\psi) = \{(r, z) \in \Pi \mid \psi(r, z) - \frac{1}{2} W r^2 > 0\}.
\]

Thanks to (7.2), we expect that a weak solution \(\psi\) satisfies
\[
-\frac{1}{r^2} \mathcal{L} \psi = \lambda 1_{\{\psi(r, z) - (1/2) W r^2 > 0\}}
\] (7.7)
in a certain weak sense. In our variational setting, for any maximizer \(\xi \in S_{\mu, \gamma, \lambda}\), the stream function \(\psi = \mathcal{G}[\xi]\) satisfies (7.7) once we assume \(\gamma = 0\) in (5.1). Thus it becomes a weak solution of the Hill’s problem. We write the statement in the form of a lemma, whose proof in detail is given in Appendix C (even if it looks very natural and trivial):

**Lemma 7.1.** If \(\xi \in S_{\mu}\) satisfies
\[
\xi = 1_{\{(\psi - (1/2) W r^2 > 0)\}} \quad \text{a.e. for some} \quad W > 0,
\] (7.8)
then the stream function \(\psi = \mathcal{G}[\xi]\) is a weak solution of the Hill’s problem for \((1, W)\).

Now we borrow the uniqueness result of Amick-Fraenkel [4]:

**Theorem 7.2.** [Theorem 1.1 in [4]] If \(\psi \in H(\Pi) \setminus \{0\}\) is a weak solution of the Hill’s problem for \((\lambda, W)\), then we have
\[
\psi(r, z) = \psi_{H(\lambda, a)}(r, z - c)
\]
for some \(c \in \mathbb{R}\) and for the constant \(a = a(\lambda, W) > 0\) solving the equation \(W = (2/15) \lambda a^2\) where \(\psi_{H(\lambda, a)}\) is the stream function (2.12) of the Hill’s vortex
\[
\xi_{H(\lambda, a)} = \lambda 1_{B_a}(x), \quad B_a \subset \mathbb{R}^3 : \text{the ball centered at the origin with radius} \ a.
Remark 7.3. One of the key ideas of [4] is to use the observation (7.3). Indeed, for a weak solution \( \psi \) of the Hill’s problem for \((\lambda, W)\), we expect (7.7) (i.e., (5.1) for \( \gamma = 0 \)). It implies, by (7.3), in \( \mathbb{R}^5 \),

\[-\Delta_{\mathbb{R}^5}(\mathcal{T}\psi) = -\mathcal{T}(\mathcal{L}\psi) = \lambda f_H(\psi - (1/2)Wr^2) = \lambda f_H(\mathcal{T}\psi - (1/2)W) \geq 0.\]

One may expect spherical symmetry of \( \mathcal{T}\psi \) in \( \mathbb{R}^5 \) (up to a translation in \( y_5 \)-direction) via the moving plane method due to [54, 99]. The main difficulty lies on the fact that the vorticity function \( f_H \) is not regular enough to satisfy the original setting of [54] directly. By overcoming the obstacle (see Section 3 of [4]), spherical symmetry of \( \mathcal{T}\psi \) is obtained. We may assume that \( \mathcal{T}\psi \) is radially symmetric by shifting in \( y_5 \)-variable if necessary. Moreover, it is strictly decreasing in the radial direction. Then, it only remains to solve for \( \eta(|y|) = (\mathcal{T}\psi)(y) \) and some unknown \( a > 0 \) to the following O.D.E. problem:

\[\eta \in C^1[0, \infty) : \text{strictly decreasing},\]

\[-\frac{1}{t^4}(t^4\eta')(t) = \lambda, \quad 0 < t < a,\]

\[-\frac{1}{t^4}(t^4\eta')(t) = 0, \quad t > a,\]

\[\eta(a) = \frac{1}{2}W, \quad \eta(\infty) = 0.\]

It has the unique solution \( \eta \) so that \( \psi(x) = r^2\eta(|x|) \) is equal to \( \psi_{H(\lambda, a)}(x) \) in (2.12). The radius \( a \) is determined by (2.13) (cf. for the circular vortex pair (1.15), refer to [17] or Section 6.2 of [2]).

7.2 Every maximizer with small impulse loses certain mass

In order to prove Theorem 3.2, we first show the following proposition saying that every maximizer with small impulse has zero flux constant \( \gamma \).

**Proposition 7.4.** There exists a constant \( M_1 > 0 \) such that for any \( 0 < \mu \leq M_1 \) and for each \( \xi \in S_\mu \), we have

\[\int_{\mathbb{R}^3} \xi \, dx < 1.\]

In that case, the flux constant \( \gamma \) in (5.1) of Theorem 5.1 is equal to 0.

Remark 7.5. The same result for a certain maximizer \( \xi \in S_\mu \) can be found in [50, Remark 5.2], which was obtained from some uniform estimates for a sequence of maximizers for the penalized energy functional (1.13). Here we adapt the proof so that it works for every maximizer (cf.[2, Remark 2.6 (iii)] for the circular vortex pair (1.15)).

To prove Proposition 7.4, we need the following estimate of the kernel \( G \) (cf. Lemma 4.8 in [50]).
Lemma 7.6. For $\alpha > 0$, we have
\[
\int_{r' < \alpha} G(r, z, r', z') r' d r' d z' \lesssim \alpha^4, \quad (r, z) \in \Pi.
\]

Proof. Let $\alpha > 0$ and $(r, z) \in \Pi$. Since we have
\[
\int_{r' < \alpha} G(r, z, r', z') r' d r' d z' = \int_{r' < \alpha} G(r, 0, z', z - z) r' d r' d z' = \int_{r' < \alpha} G(r, 0, r', z') r' d r' d z',
\]
we may assume $z = 0$. We denote
\[
t = \sqrt{(r - r')^2 + (z - z')^2} = r \sqrt{\left(1 - \frac{r'}{r}\right)^2 + \left(\frac{z - z'}{r}\right)^2}.
\]

When $r > 2\alpha$, we can estimate, by setting $\tau = 3/2$ in (2.4),
\[
\int_{r' < \alpha} G(r, 0, r', z') r' d r' d z' \lesssim \int_{r' < \alpha} \frac{r^2 (r')^3}{t^3} d r' d z'
\]
\[
= r^2 \int_{r' < \alpha/r} \left(\frac{r'}{r}\right)^3 \left(\sqrt{\left(1 - \frac{r'}{r}\right)^2 + \left(\frac{z}{r}\right)^2}\right)^{-3} d r' d z'
\]
\[
= r^4 \int_{r' < \alpha/r} (r')^3 \left(\sqrt{(1 - r')^2 + z'^2}\right)^{-3} d r' d z'
\]
\[
= r^4 \left(\frac{\alpha}{r}\right)^3 \int_0^{\alpha/r} \int_{\mathbb{R}} \left(\sqrt{(1 - r')^2 + z'^2}\right)^{-3} d z' d r' \lesssim \alpha^3 r \int_0^{\alpha/r} \frac{1}{(1 - r')^2} d r' \lesssim \alpha^4, \quad (7.9)
\]
where we used change of variables $\frac{r'}{r} \mapsto r'$, $\frac{z'}{r} \mapsto z'$.
Now we consider the remained case $r \leq 2\alpha$. We split the integral
\[
\int_{r' < \alpha} G(r, 0, r', z') r' d r' d z' = \int_{r' < \alpha, \quad t < r/2} + \int_{r' < \alpha, \quad t \geq r/2} = I + II.
\]
For $I$, we have, by (2.4) with $\tau = 1/2$,

$$
\int_{t < r/2} G(r, 0, r', z') r' dr' dz' \lesssim \int_{t < r/2} \frac{r(r')^2}{t} dr' dz'
$$

$$
= r^2 \int_{t < r/2} \left( \frac{r'}{r} \right)^2 \left\{ \sqrt{\left(1 - \frac{r'}{r}\right)^2 + \left(\frac{z'}{r}\right)^2} \right\}^{-1} dr' dz'
$$

$$
\leq r^4 \int_{\sqrt{(1-r')^2+z'^2} < 1/2} \left( \frac{r'}{r} \right)^2 \left( \sqrt{\left(1 - \frac{r'}{r}\right)^2 + z'^2} \right)^{-1} dr' dz'
$$

$$
\leq r^4 \int_{\sqrt{(1-r')^2+z'^2} < 1/2} \left( \sqrt{(1 - \frac{r'}{r})^2 + z'^2} \right)^{-1} dr' dz' \lesssim r^4 \lesssim x^4.
$$

For $II$, we estimate, by (2.4) with $\tau = 3/2$ as in (7.9),

$$
\int_{t \geq r/2} G(r, 0, r', z') r' dr' dz' \lesssim \int_{t \geq r/2} \frac{r^2(r')^3}{t^3} dr' dz'
$$

$$
= r^2 \int_{t \geq r/2} \left( \frac{r'}{r} \right)^3 \left\{ \sqrt{\left(1 - \frac{r'}{r}\right)^2 + \left(\frac{z'}{r}\right)^2} \right\}^{-3} dr' dz'
$$

$$
= r^4 \int_{\sqrt{(1-r')^2+z'^2} \geq 1/2} \left( \frac{r'}{r} \right)^3 \left( \sqrt{(1 - \frac{r'}{r})^2 + z'^2} \right)^{-3} dr' dz'
$$

$$
\lesssim r^4 \left( \frac{x}{r} \right)^3 \int_{\sqrt{(1-r')^2+z'^2} \geq 1/2} \left( \sqrt{(1 - \frac{r'}{r})^2 + z'^2} \right)^{-3} dr' dz' \lesssim r^4 \left( \frac{x}{r} \right)^3 \lesssim x^4.
$$

□

Now we prove Proposition 7.4.

**Proof of Proposition 7.4.** Let $\xi \in S_\mu$ for some $\mu \in (0, \infty)$. Then, by Theorem 5.1, there exist unique $W = W_\xi > 0, Y = Y_\xi \geq 0$ such that $\xi = 1_{\{\psi - (1/2)W r^2 - Y > 0}\}}$. Moreover, $\xi$ is compactly supported. By

$$
\mu = \frac{1}{2} \int r^2 \xi dx \geq \frac{1}{2} \int_{r \geq 2\sqrt{\mu}} r^2 \xi dx \geq 2\mu \int_{r \geq 2\sqrt{\mu}} \xi dx,
$$

we get $\int_{r \geq 2\sqrt{\mu}} \xi dx \leq \frac{1}{2}$, which implies

$$
\int \xi dx \leq \int_{0 < r < 2\sqrt{\mu}} \xi dx + \frac{1}{2}.
$$

(7.10)
Since we have

$$0 \leq \xi = 1_{\{0 \leq \frac{1}{2} W r^2 - \gamma > 0\}} \leq \frac{2\psi}{W r^2},$$

we estimate, for any $\alpha > 0$,

$$\int_{0 \leq r < 2\alpha} \xi \, dx = 2\pi \int_{0 \leq r < 2\alpha} \xi \, r \, dr \, dz \leq 2\pi \int_{0 \leq r < 2\alpha} \frac{2\psi}{W r^2} \, r \, dr \, dz$$

$$= \frac{4\pi}{\alpha^2 W} \int_{0 \leq r < 2\alpha} \left( \int_{\Pi} G(r, z, r', z') \xi(r', z') \, dr' \, dz' \right) \, r \, dr \, dz$$

$$\leq \frac{1}{\alpha^2 W} \sup_{(r, z) \in \Pi} \left( \int_{0 \leq r' < 2\alpha} G(r, z, r', z') \, r' \, dr' \, dz' \right) \int \xi \, dx,$$

where we used the symmetry of $G$ in the last equality. Using Lemma 7.6, we get

$$\int_{0 \leq r < 2\alpha} \xi \, dx \lesssim \frac{\alpha^2}{W} \int \xi \, dx \leq \frac{\alpha^2}{W}.$$ 

Thus we get

$$\int_{0 \leq r < 2\sqrt{\mu}} \xi \, dx = \sum_{i=0}^{\infty} \int_{(\sqrt{\mu})2^{-i-1} \leq r < (\sqrt{\mu})2^{-i}} \xi \, dx \lesssim \frac{1}{W} \sum_{i=0}^{\infty} ((\sqrt{\mu})2^{-i-1})^2$$

$$\lesssim \frac{\mu}{W} \sum_{i=0}^{\infty} \left( \frac{1}{4} \right)^i \lesssim \frac{\mu}{W}. \quad (7.11)$$

Now we recall the estimate (5.40) of Proposition 5.13:

$$I_\mu \leq 2W\mu.$$

On the other hand, we claim

$$I_\mu \geq I_1\mu^{7/5}$$

for any $\mu \leq 1$. Indeed, it is a simple consequence from scaling. Let us take and fix any $\xi_1 \in S_1$ whose existence is guaranteed by Theorem 4.2. By setting

$$\xi_\mu(x) = \xi_1(\mu^{-1/5} x),$$

we get $\xi_\mu \in P_\mu$ for any $\mu \leq 1$. Thus, we get the above claim due to $I_\mu \geq E(\xi_\mu) = \mu^{7/5} E(\xi_1) = \mu^{7/5} I_1$.

Hence, we have $2W\mu \geq I_\mu \geq I_1\mu^{7/5}$ for any $\mu \leq 1$, which gives

$$W \geq \frac{I_1}{2} \mu^{2/5}, \quad \mu \leq 1. \quad (7.12)$$
In sum, for any $\mu \leq 1$ and for any $\xi \in S_\mu$, we have, by (7.10), (7.11), (7.12),
\[
\int \xi \, dx \leq \int_{0<r<2\sqrt{\mu}} \xi \, dx + \frac{1}{2} \leq C \mu^{3/5} + \frac{1}{2}
\]
for some universal constant $C > 0$. Thus there exists a sufficiently small constant $M_1 > 0$ such that for any $0 < \mu \leq M_1$ and for any $\xi \in S_\mu$, we have
\[
\int \xi \, dx < 1,
\]
which implies
\[
\gamma = \gamma_\xi = 0
\]
by Lemma 5.14.

Now we are ready to prove Theorem 3.2.

7.3 Proof of uniqueness theorem (Theorem 3.2)

Proof of Theorem 3.2. Thanks to the scaling argument (4.4), it is enough to show the theorem for general $\mu > 0$ with fixed $\lambda = \nu = 1$.

Let $\mu \in (0, M_1]$ where $M_1 > 0$ is the constant from Proposition 7.4. Due to $S_\mu \neq \emptyset$ from Theorem 4.2, we can take any $\xi \in S_\mu$. Then, by Theorem 5.1, we get
\[
\xi = 1_{\{\psi-(1/2)W_\xi r^2-\gamma_\xi > 0\}}.
\]
for some constants $W_\xi > 0$ and $\gamma_\xi \geq 0$. By Proposition 7.4, we get $\gamma_\xi = 0$, which implies
\[
7I_\mu = 5W_\xi \cdot \mu
\]
by the identity (5.40) of Proposition 5.13. In other words, $W_\xi$ is determined by knowing only the value of $\mu$. Let us denote $W_\xi$ by $W_\mu$ from now on. Then, by Lemma 7.1, the stream function $\psi = \mathcal{G}[\xi]$ is a weak solution of the Hill’s problem for $(\lambda, W) = (1, W_\mu)$. We set the radius $a_\mu > 0$ solving $W_\mu = (2/15)(a_\mu)^2$. Then, by Theorem 7.2, there exists a constant $c' \in \mathbb{R}$ such that
\[
\psi(x) = \psi_{H(1,a_\mu)}(x + c'e_z),
\]
where $\psi_{H(1,a_\mu)}$ is the stream function (2.12) of the Hill’s vortex $\xi_{H(1,a_\mu)} = 1_{B_{a_\mu}}$. Thus we get $\xi(x) = \xi_{H(1,a_\mu)}(x + c'e_z)$. In sum, we have shown, for any $0 < \mu \leq M_1$,
\[
\emptyset \neq S_\mu \subset \{\xi_{H(1,a_\mu)}(\cdot + ce_z) \mid c \in \mathbb{R}\}.
\]
In particular, the radius $a_\mu$ is explicitly computed by
\[
\mu = \frac{1}{2} \int r^2 \xi \, dx = \frac{1}{2} \int r^2 \xi_{H(1,a_\mu)} \, dx = \frac{4\pi}{15}(a_\mu)^5
\]
(e.g., see (2.14)). To show the reverse inclusion, we recall that any translation in $z-$variable does not change the quantities involved in the variational problem (3.2). Thus, from $\xi_{H(1,a,\mu)}(\cdot + e_z) = \xi \in S_\mu$ for some $c' \in \mathbb{R}$, we obtain

$$S_\mu \supset \{ \xi_{H(1,a,\mu)}(\cdot + ce_z) \mid c \in \mathbb{R} \}.$$ 

\[ \square \]

**ACKNOWLEDGMENTS**

The work is partially supported by the NRF-2018R1D1A1B07043065 (National Research Foundation of Korea) and by the Research Fund 1.200085.01 of UNIST (Ulsan National Institute of Science & Technology). We thank Ken Abe for many helpful discussions, including discussions on the identity (5.3). Without his encouragement, this research work would not have been possible. We also thank the anonymous referee for valuable comments.

**REFERENCES**

1. K. Abe, *Existence of vortex rings in Beltrami flows*, Comm. Math. Phys. 391 (2022), no. 2, 873–899.
2. K. Abe and K. Choi, *Stability of Lamb dipoles*, Arch. Ration. Mech. Anal. 244 (2022), no. 3, 877–917.
3. H. Abidi, T. Hmidi, and S. Keraani, *On the global well-posedness for the axisymmetric Euler equations*, Math. Ann. 347 (2010), no. 1, 15–41.
4. C. J. Amick and L. E. Fraenkel, *The uniqueness of Hill’s spherical vortex*, Arch. Rational Mech. Anal. 92 (1986), 91–119.
5. V. I. Arnold, *Sur la géométrie différentielle des groupes de Lie de dimension infinie et ses applications à l’hydrodynamique des fluides parfaits*, Ann. Inst. Fourier 16 (1966), 319–361.
6. V. I. Arnold and B. A. Khesin, *Topological methods in hydrodynamics*, Applied Mathematical Sciences, vol. 125, Springer-Verlag, New York, 1998.
7. G. K. Batchelor, *An introduction to fluid dynamics*, Cambridge Mathematical Library, Cambridge University Press, Cambridge, 1999.
8. J. Bedrossian, P. Germain, and N. Masmoudi, *On the stability threshold for the 3D Couette flow in Sobolev regularity*, Ann. of Math. (2) 185 (2017), no. 2, 541–608.
9. J. Bedrossian and N. Masmoudi, *Inviscid damping and the asymptotic stability of planar shear flows in the 2D Euler equations*, Publ. Math. Inst. Hautes Études Sci. 122 (2015), 195–300.
10. J. Beichman and S. Denisov, *2D Euler equation on the strip: stability of a rectangular patch*, Comm. Partial Differ. Equ. 42 (2017), 100–120.
11. T. B. Benjamin, *The stability of solitary waves*, Proc. Roy. Soc. Ser. A 328 (1972), 153–183.
12. T. B. Benjamin, *The alliance of practical and analytical insights into the nonlinear problems of fluid mechanics*, Lecture Notes in Math. 503 (1976), 8–29.
13. D. Bliss, *The dynamics of flows with high concentrations of vorticity*, PhD thesis, Department of Aeronautics and Astronautics, Massachusetts Institute of Technology, 1973.
14. J. Bona, *On the stability theory of solitary waves*, Proc. Roy. Soc. London Ser. A 344 (1975), 363–374.
15. J. L. Bona, P. E. Souganidis, and W. A. Strauss, *Stability and instability of solitary waves of Korteweg-de Vries type*, Proc. Roy. Soc. London Ser. A 411 (1987), 395–412.
16. G. R. Burton, *Steady symmetric vortex pairs and rearrangements*, Proc. Roy. Soc. Edinburgh Sect. A 108 (1988), 269–290.
17. G. R. Burton, *Uniqueness for the circular vortex-pair in a uniform flow*, Proc. Roy. Soc. London Ser. A 452 (1996), 2343–2350.
18. G. R. Burton, *Vortex-rings of prescribed impulse*, Math. Proc. Cambridge Philos. Soc. 134 (2003), no. 3, 515–528.
19. G. R. Burton, *Isoperimetric properties of Lamb’s circular vortex-pair*, J. Math. Fluid Mech. 7 (2005), S68–S80.
20. G. R. Burton, H. J. Nussenzweig Lopes, and M. C. Lopes Filho, *Nonlinear stability for steady vortex pairs*, Comm. Math. Phys. 324 (2013), 445–463.
21. G. R. Burton and L. Preciso, *Existence and isoperimetric characterization of steady spherical vortex rings in a uniform flow in $\mathbb{R}^N$*, Proc. Roy. Soc. Edinburgh Sect. A 134 (2004), no. 3, 449–476.
22. L. A. Caffarelli and A. Friedman, *The shape of axisymmetric rotating fluid*, J. Funct. Anal. 35 (1980), 109–142.
23. D. Cao, J. Wan, and G. Wang, *Nonlinear orbital stability for planar vortex patches*, Proc. Amer. Math. Soc. 147 (2019), 775–784.
24. T. Cazenave and P.-L. Lions, *Orbital stability of standing waves for some nonlinear Schrödinger equations*, Comm. Math. Phys. 85 (1982), 549–561.
25. D. Chae and O. Y. Imanuvilov, *Generic solvability of the axisymmetric 3-D Euler equations and the 2-D Boussinesq equations*, J. Differ. Equ. 156 (1999), 1–17.
26. D. Chae and N. Kim, *Axisymmetric weak solutions of the 3-D Euler equations for incompressible fluid flows*, Nonlinear Anal. 29 (1997), 1393–1404.
27. S. A. Chaplygin, *One case of vortex motion in fluid*, Trudy Otd. Fiz. Nauk Imper. Mosk. Obsch. Lyub. Estest. 11 (1903), no. 11–14.
28. S. A. Chaplygin, *One case of vortex motion in fluid*, Regul. Chaotic Dyn. 12 (2007), 219–232.
29. K. Choi, *On the estimate of distance traveled by a particle in a disk-like vortex patch*, Appl. Math. Lett. 97 (2019), 67–72.
30. K. Choi and I.-J. Jeong, *On the winding number for particle trajectories in a disk-like vortex patch of the Euler equations*, preprint, arXiv:2008.05085.
31. K. Choi and I.-J. Jeong, *Growth of perimeter for vortex patches in a bulk*, Appl. Math. Lett. 113 (2021), 106 857, 9.
32. K. Choi and I.-J. Jeong, *Filamentation near Hill’s vortex*, Comm. Partial Differ. Equ. 48 (2023), no. 1, 54–85.
33. K. Choi, M. Kang, Y. Kwon, and A. Vasseur, *Contraction for large perturbations of traveling waves in a hyperbolic-parabolic system arising from a chemotaxis model*, Math. Models Methods Appl. Sci. 30 (2020), no. 2, 387–437.
34. K. Choi, M. Kang, and A. Vasseur, *Global well-posedness of large perturbations of traveling waves in a hyperbolic-parabolic system arising from a chemotaxis model*, J. Math. Pures Appl. 142 (2020), 266–297.
35. K. Choi and A. Vasseur, *Short-time stability of scalar viscous shocks in the inviscid limit by the relative entropy method*, SIAM J. Math. Anal. 47 (2015), 1405–1418.
36. P. Constantin, *On the Euler equations of incompressible fluids*, Bull. Amer. Math. Soc. 44 (2007), no. 4, 603–621.
37. H. T. Croft, *Three lattice-point problems of Steinhaus*, Quart. J. Math. Oxford Ser. (2) 33 (1982), no. 129, 71–83.
38. R. Danchin, *Axisymmetric incompressible flow with bounded vorticity*, Uspekhi Mat. Nauk 62 (2007), 73–94.
39. E. B. Davies and A. M. Hinz, *Explicit constants for Rellich inequalities in $L_p(\Omega)$*, Math. Z. 227 (1998), no. 3, 511–523.
40. C. De Lellis and L. Székelyhidi, Jr., *The Euler equations as a differential inclusion*, Ann. of Math. (2) 170 (2009), no. 3, 1417–1436.
41. C. De Lellis and L. Székelyhidi, Jr, *Admissibility criteria for weak solutions of the Euler equations*, Arch. Ration. Mech. Anal. 195 (2010), no. 1, 225–260.
42. R. J. DiPerna and P. L. Lions, *Ordinary differential equations, transport theory and Sobolev spaces*, Invent. Math. 98 (1989), 511–547.
43. T. Do, *On vorticity gradient growth for the axisymmetric 3D Euler equations without swirl*, Arch. Ration. Mech. Anal. 234 (2019), no. 1, 181–209.
44. D. G. Ebin and J. Marsden, *Groups of diffeomorphisms and the motion of an incompressible fluid*, Ann. of Math. (2) 92 (1970), no. 1, 102–163.
45. T. Elgindi, *Finite-time singularity formation for $C^{1,\alpha}$ solutions to the incompressible Euler equations on $\mathbb{R}^3$*, Ann. of Math. (2) 194 (2021), no. 3, 647–727.
46. L. C. Evans, *Partial differential equations*, Graduate Studies in Mathematics, vol. 19, 2nd ed., American Mathematical Society, Providence, RI, 2010.
47. H. Feng and V. Šverák, *On the Cauchy problem for axi-symmetric vortex rings*, Arch. Ration. Mech. Anal. 215 (2015), 89–123.
48. L. E. Fraenkel, *On steady vortex rings of small cross-section in an ideal fluid*, Proceedings of the Royal Society of London. Series A, Mathematical and Physical Sciences 316 (1970), 29–62.
49. L. E. Fraenkel and M. S. Berger, *A global theory of steady vortex rings in an ideal fluid*, Acta Math. 132 (1974), 13–51.
50. A. Friedman and B. Turkington, *Vortex rings: existence and asymptotic estimates*, Trans. Amer. Math. Soc. 268 (1981), 1–37.
51. A. Fukuyu, T. Ruzi, and A. Kanai, The response of Hill’s vortex to a small three dimensional disturbance, J. Phys. Soc. Jpn. 63 (1994), 510–527.
52. G. P. Galdi, An introduction to the mathematical theory of the Navier-Stokes equations: Steady-State Problems, Springer Monographs in Mathematics, 2nd ed., Springer, New York, 2011.
53. T. Gallay and V. Šverák, Remarks on the Cauchy problem for the axisymmetric Navier-Stokes equations, Confluentes Math. 7 (2015), no. 2, 67–92.
54. B. Gidas, W. M. Ni, and L. Nirenberg, Symmetry and related properties via the maximum principle, Comm. Math. Phys. 68 (1979), 209–243.
55. M. Grillakis, J. Shatah, and W. Strauss, Stability theory of solitary waves in the presence of symmetry. I, J. Funct. Anal. 74 (1987), 160–197.
56. N. Gunther, On the motion of fluid in a moving container, Isvestia Akad. Nauk SSSR. Ser. Fiz.-Mat. 20 (1927), 1323–1348.
57. J. F. Harper and D. W. Moore, The motion of a spherical liquid drop at high reynolds number, J. Fluid Mech. 32 (1968), no. 2, 367–391.
58. Y. Hattori and K. Hijiya, Short-wavelength stability analysis of Hill’s vortex with/without swirl, Phys. Fluids 22 (2010), 074104.
59. H. Helmholtz, Über Integrale der hydrodynamischen Gleichungen, welche den Wirbelbewegungen entsprechen, J. Reine Angew. Math. 55 (1858), 25–55.
60. W. M. Hicks, Researches in vortex motion. part iii: On spiral or gyrostatic vortex aggregates, Phil. Trans. R. Soc. A. 192 (1899), 33–99.
61. M. J. M. Hill, On a spherical vortex, Philos. Trans. Roy. Soc. London Ser. A 185 (1894), 213–245.
62. A. M. Il’in and O. A. Oleinik, Behavior of solutions of the Cauchy problem for certain quasilinear equations for unbounded increase of the time, Dokl. Akad. Nauk SSSR 120 (1958), 25–28.
63. Q. Jiu, J. Wu, and W. Yang, Viscous approximation and weak solutions of the 3D axisymmetric Euler equations, Math. Methods Appl. Sci. 38 (2015), 548–558.
64. M.-J. Kang and A. Vasseur, Contraction property for large perturbations of shocks of the barotropic Navier-Stokes system, J. Eur. Math. Soc. (JEMS) 23 (2021), no. 2, 585–638.
65. M.-J. Kang and A. F. Vasseur, Uniqueness and stability of entropy shocks to the isentropic Euler system in a class of inviscid limits from a large family of Navier-Stokes systems, Invent. Math. 224 (2021), no. 1, 55–146.
66. T. Kato, Nonstationary flows of viscous and ideal fluids in R³, J. Functional Analysis 9 (1972), 296–305.
67. C. E. Kenig and F. Merle, Asymptotic stability and Liouville theorem for scalar viscous conservation laws in cylinders, Comm. Pure Appl. Math. 59 (2006), no. 6, 769–796.
68. V. I. Kolyada, On the metric Darboux property, Anal. Math. 9 (1983), no. 4, 291–312.
69. A. Kufner, Weighted Sobolev spaces, A Wiley-Interscience Publication, John Wiley & Sons, Inc., New York, 1985. Translated from the Czech.
70. T. Laker and S. Ghaasiaan, Monte-carlo simulation of aerosol transport in rising spherical bubbles with internal circulation, J. Aerosol Sci. 35 (2004), no. 4, 473–488.
71. H. Lamb, Hydrodynamics, 3rd ed., Cambridge University Press, Cambridge, 1906.
72. N. Leger, L² stability estimates for shock solutions of scalar conservation laws using the relative entropy method, Arch. Ration. Mech. Anal. 199 (2011), no. 3, 761–778.
73. N. Leger and A. Vasseur, Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non-BV perturbations, Arch. Ration. Mech. Anal. 201 (2011), no. 1, 271–302.
74. L. Lichtenstein, Über einige Existenzprobleme der Hydrodynamik, Math. Zeit. Phys. 23 (1925), 89–154.
75. E. H. Lieb and M. Loss, Analysis, graduate studies in mathematics, vol. 14, American Mathematical Society, Providence, RI, 1997.
76. A. Lifschitz, Instabilities of ideal fluids and related topics, Z. Angew. Math. Mech. 75 (1995), no. 6, 411–422.
77. P. L. Lions, The concentration-compactness principle in the calculus of variations. The locally compact case. I, Ann. Inst. H. Poincaré Anal. Non Linéaire 1 (1984), 109–145.
78. J.-G. Liu and W.-C. Wang, Characterization and regularity for axisymmetric solenoidal vector fields with application to Navier-Stokes equation, SIAM J. Math. Anal. 41 (2009), no. 5, 1825–1850.
79. A. Majda, Vorticity and the mathematical theory of incompressible fluid flow, Comm. Pure Appl. Math. 39 (1986), no. S, suppl., S187–S220.
80. Y. Martel and F. Merle, Instability of solitons for the critical generalized Korteweg-de Vries equation, Geom. Funct. Anal. 11 (2001), 74–123.
81. Y. Martel and F. Merle, Blow up in finite time and dynamics of blow up solutions for the $L^2$-critical generalized KdV equation, J. Amer. Math. Soc. 15 (2002), 617–664.
82. V. V. Meleshko and G. J. F. van Heijst, On Chaplygin's investigations of two-dimensional vortex structures in an inviscid fluid, J. Fluid Mech. 272 (1994), 157–182.
83. F. Merle, Existence of blow-up solutions in the energy space for the critical generalized KdV equation, J. Amer. Math. Soc. 14 (2001), 555–578.
84. H. K. Moffatt, Degree of knottedness of tangled vortex lines, J. Fluid Mech. 35 (1969), 117–129.
85. H. K. Moffatt and D. W. Moore, The response of Hill's spherical vortex to a small axisymmetric disturbance, J. Fluid Mech. 87 (1978), 749–760.
86. W. M. Ni, On the existence of global vortex rings, J. Analyse Math. 37 (1980), 208–247.
87. C. Nobili and C. Seis, Renormalization and energy conservation for axisymmetric fluid flows, Math. Ann. 382 (2022), 1–36.
88. J. Norbury, A steady vortex ring close to Hill's spherical vortex, Proc. Cambridge Philos. Soc. 72 (1972), 253–284.
89. J. Norbury, Steady planar vortex pairs in an ideal fluid, Comm. Pure Appl. Math. 28 (1975), 679–700.
90. G. Pólya and G. Szegő, Isoperimetric Inequalities in Mathematical Physics, Annals of Mathematics Studies, No. 27, Princeton University Press, Princeton, N J, 1951.
91. C. Pozrikidis, The nonlinear instability of hill's vortex, J. Fluid Mech. 168 (1986), 337–367.
92. B. Protas and A. Elcrat, Linear stability of Hill's vortex to axisymmetric perturbations, J. Fluid Mech. 799 (2016), 579–602.
93. F. Riesz, Sur Une Inegalite Integrale, J. London Math. Soc. 5 (1930), no. 3, 162–168.
94. T. Rozi, Evolution of the surface of hill's vortex subjected to a small three-dimensional disturbance for the cases of $m = 0, 2, 3$ and $4$, Phys. Fluids 68 (1999), 2940–2955.
95. T. Rozi and Y. Fukumoto, The most unstable perturbation of wave-packet form inside hill's vortex, J. Phys. Soc. Japan 69 (2000), 2700–2701.
96. P. G. Saffman, Vortex dynamics, Cambridge Monographs on Mechanics and Applied Mathematics, Cambridge University Press, New York, 1992.
97. X. Saint Raymond, Remarks on axisymmetric solutions of the incompressible Euler system, Comm. Partial Differ. Equ. 19 (1994), no. 1-2, 321–334.
98. V. Scheffer, An inviscid flow with compact support in space-time, J. Geom. Anal. 3 (1993), no. 4, 343–401.
99. J. Serrin, A symmetry problem in potential theory, Arch. Rational Mech.Anal. 43 (1971), 304–318.
100. T. Shirota and T. Yanagisawa, Note on global existence for axially symmetric solutions of the Euler system, Proc. Japan Acad. Ser. A Math. Sci. 70 (1994), 299–304.
101. A. Shnirelman, On the nonuniqueness of weak solution of the Euler equation, Comm. Pure Appl. Math. 50 (1997), no. 12, 1261–1286.
102. T. C. Sideris and L. Vega, Stability in $L^1$ of circular vortex patches, Proc. Amer. Math. Soc. 137 (2009), 4199–4202.
103. E. M. Stein, Harmonic analysis: real-variable methods, orthogonality, and oscillatory integrals, Princeton Mathematical Series, vol. 43, Princeton University Press, Princeton, NJ, 1993.
104. A. Szenes, Exceptional points for Lebesgue's density theorem on the real line, Adv. Math. 226 (2011), no. 1, 764–778.
105. T. Tao, Why are solitons stable? Bull. Amer. Math. Soc. 46 (2009), 1–33.
106. W. Thomson (Lord Kelvin), Maximum and minimum energy in vortex motion, Nature 22 (1880), no. 574, 618–620, Mathematical and Physical Papers 4, pp. 172–183, Cambridge University Press, Cambridge, 1910.
107. B. Turkington, On steady vortex flow in two dimensions. I, II, Comm. Partial Differ. Equ. 8 (1983), 999–1030, 1031–1071.
108. M. R. Ukhovskii and V. I. Yudovich, Axially symmetric flows of ideal and viscous fluids filling the whole space, J. Appl. Math. Mech. 32 (1968), 52–61.
109. M. Van Dyke, An album of fluid motion, The Parabolic Press, Stanford, California, 1982.
110. A. F. Vasseur and M. Vishik, Blow-up solutions to 3D Euler are hydrodynamically unstable, Comm. Math. Phys. 378 (2020), no. 1, 557–568.
111. V. Šverák, Selected topics in fluid mechanics (an introductory graduate course taught in 2011/2012), http://www-users.math.umn.edu/%7Esverak/course-notes2011.pdf.
112. Y. H. Wan, *Variational principles for Hill's spherical vortex and nearly spherical vortices*, Trans. Amer. Math. Soc. *308* (1988), 299–312.

113. Y. H. Wan and M. Pulvirenti, *Nonlinear stability of circular vortex patches*, Comm. Math. Phys. *99* (1985), 435–450.

114. R. L. Wheeden and A. Zygmund, *Measure and integral*, 2nd ed., CRC Press, Boca Raton, FL, 2015.

115. E. Wiedemann, *Existence of weak solutions for the incompressible Euler equations*, Ann. Inst. H. Poincaré Anal. Non Linéaire *28* (2011), no. 5, 727–730.

**APPENDIX A: PROOF OF LEMMA 4.9**

We begin with proving the following estimate of the stream function \( \psi = G[\xi] \) when \( \xi \) satisfies the monotonicity condition (4.7). The result is essentially due to [50, Lemma 3.5]. Here we follow the approach of [2, Proposition 3.3].

**Lemma A.1.** Let \( \xi \in (L^1_w \cap L^2 \cap L^1)(\mathbb{R}^3) \) be an axi-symmetric nonnegative function satisfying the monotonicity condition (4.7). Then, \( \psi = G[\xi] \) satisfies

\[
\psi(r, z) \lesssim \left( \|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1 \right) \cdot \left( \frac{r^2}{\sqrt{A}} + \frac{1}{A} + r^2 \left( \frac{A}{|z|} \right)^3 \right), \quad (r, z) \in \Pi
\]

(A.1)

provided \( r \leq \frac{|z|}{A}, \) \( |z| > 0, \) and \( A \geq 1. \)

**Proof.** By replacing \( A \) to \( A/2 \), it is equivalent to show (A.1) for \( r \leq 2|z|/A \) and \( A \geq 2 \). Let us take \( (r, z) \in \Pi \) satisfying \( r \leq 2|z|/A \). We may assume that \( z > 0 \). By setting

\[
t = |(r, z) - (r', z')|,
\]

we split the integral

\[
\psi(r, z) = \int_{\Pi} G(r, z, r', z') \xi(r', z') r' dr' dz' = \int_{t < r/2} \cdots + \int_{t \geq r/2} \cdots = : I + II.
\]

For the term \( I \), we estimate, by (2.4) with \( \tau = 1/6 \) and by Hölder’s inequality,

\[
I \lesssim \int_{t < r/2} \left( \sqrt{rr'} \left( \frac{rr'}{t^2} \right)^{1/6} (r')^{1/3} \right) (r')^{2/3} \xi(r', z') dr' dz'
\]

\[
\lesssim \left( \int_{t < r/2} \frac{(rr')^2}{t} r' dr' dz' \right)^{1/3} \|\xi 1_{\{t < r/2\}}\|_{3/2} = : I_1 \cdot I_2.
\]

For \( t < r/2 \), we have \( r \sim r' \) so we estimate

\[
I_1 \lesssim r^{5/3} \left( \int_{t < r/2} \frac{1}{t} dr' dz' \right)^{1/3} \lesssim r^2.
\]
We observe that the conditions \( t < \frac{r}{2} \) and \( r \leq 2z/A \) imply \( |z - z'| < z/A \). We also observe that, for any function \( g : \mathbb{R}_{>0} \to \mathbb{R}_{\geq 0} \) which is non-increasing,

\[
\int_{s-(s/A)}^{s+(s/A)} g(\sigma) \, d\sigma \leq \frac{4}{A} \|g\|_{L^1(0,\infty)}, \quad s > 0, \ A \geq 2, \tag{A.2}
\]

thanks to \( sg(s) \leq \|g\|_{L^1(0,\infty)} \) for any \( s > 0 \). By the assumption (4.7), we can apply (A.2) into the one-dimensional function \( \xi'(r', \cdot ; z') \) of \( z' \)-variable (by fixing \( r' \)), which produces

\[
I_2 \leq \left( \| \xi \mathbb{1}_{\{t<\frac{r}{2}\}} \|_1 + \| \xi \mathbb{1}_{\{t<\frac{r}{2}\}} \|_2 \right) \leq \left( \| \xi \mathbb{1}_{\{|z-z'|<\frac{z}{A}\}} \|_1 + \| \xi \mathbb{1}_{\{|z-z'|<\frac{z}{A}\}} \|_2 \right)
\]

\[
\lesssim \left( \frac{1}{A} \| \xi \|_1 + \frac{1}{\sqrt{A}} \| \xi \|_2 \right) \lesssim \frac{1}{\sqrt{A}} \| \xi \|_{L^1 \cap L^2}.
\]

Thus we get

\[
I \lesssim \frac{r^2}{\sqrt{A}} \| \xi \|_{L^1 \cap L^2}.
\]

For the term \( II \), by (2.4) with \( \tau = 3/2 \), we estimate

\[
II \lesssim \int_{t \geq \frac{r}{2}} \left( \sqrt{rr'} \left( \frac{r'r'}{t^2} \right)^{3/2} \right) \xi'(r', z') \, dr' \, dz' = \int_{t \geq \frac{r}{2}} \left( \frac{r^2(r')^3}{t^3} \right) \xi'(r', z') \, dr' \, dz' \\
= \int_{t \geq \frac{r}{2}, \ |z-z'|<\frac{z}{A}} \int_{t \geq \frac{r}{2}, \ |z-z'|<\frac{z}{A}} =: II_1 + II_2.
\]

Since \( t \geq \frac{r}{2} \) implies \( r' \leq |r' - r| + r \leq 3t \), we have, by (A.2),

\[
II_1 \lesssim \int_{|z-z'|<\frac{z}{A}} (r')^2 \xi'(r', z') \, dr' \, dz' \leq \int_{|z-z'|<\frac{z}{A}} ((r')^1 + (r')^3) \xi'(r', z') \, dr' \, dz' \\
\lesssim \frac{1}{A} (\| \xi \|_1 + \| r^2 \xi \|_1).
\]

For \( II_2 \), since \( |z - z'| \geq \frac{z}{A} \) implies \( t \geq \frac{z}{A} \), we have

\[
II_2 \leq r^2 \int_{t \geq \frac{z}{A}} \left( \frac{(r')^3}{t^3} \right) \xi'(r', z') \, dr' \, dz' \leq r^2 \left( \frac{A}{z} \right)^3 \| r^2 \xi \|_1.
\]

In sum, we obtained

\[
\phi(r, z) \lesssim \frac{r^2}{\sqrt{A}} \| \xi \|_{L^1 \cap L^2} + \frac{1}{A} (\| \xi \|_1 + \| r^2 \xi \|_1) + r^2 \left( \frac{A}{z} \right)^3 \| r^2 \xi \|_1,
\]

which implies (A.1).
Now we are ready to prove Lemma 4.9.

**Proof of Lemma 4.9.** We decompose

\[
\int_{\mathbb{R}^3 \setminus Q} \psi \xi \, dx = \int_{r \geq R} + \int_{r < R, \ |z| \geq AR} =: I + II,
\]

and estimate, by (2.19) with \(\delta = 1\),

\[
I \lesssim (\|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1) \int_{r \geq R} \frac{r^2}{r^2} \xi \, dx \lesssim \frac{1}{R^2} (\|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1)^2.
\]

For \(II\), since \(r < R\) and \(|z| \geq AR\) imply \(r \leq |z|/A\), applying (A.1) yields

\[
II \lesssim (\|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1) \int_{r < R, \ |z| \geq AR} \left( \frac{r^2}{\sqrt{A}} + \frac{1}{A} + \frac{r^2}{A^2} \right)^3 \xi \, dx
\]

\[
\lesssim (\|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1) \int \left( \frac{r^2}{\sqrt{A}} + \frac{1}{A} + \frac{r^2}{A^2} \right)^3 \xi \, dx
\]

\[
\lesssim \left( \frac{1}{\sqrt{A}} + \frac{1}{R^3} \right) \cdot (\|\xi\|_{L^1 \cap L^2} + \|r^2 \xi\|_1)^2.
\]

Combining the above estimates, we obtain the conclusion (4.10).

**APPENDIX B: PROOF OF LEMMA 5.6**

**Proof of Lemma 5.6.** Let \(\Omega \subset \mathbb{R}^N\) be a non-empty connected open set and let \(U \subset \Omega\) satisfy \(|U| > 0\) and \(|\Omega \setminus U| > 0\). Since \(|D_e(U)| = |U| > 0\) by Lemma 5.5, we can take some point \(y \in D_e(U)\). Similarly, from \(|D_1(U)| = |\Omega \setminus U| > 0\), we take another point \(z \in D_1(U)\). We connect \(y\) and \(z\) by a polygonal line \(L\) consisting of a finite number of line segments joined end to end lying in \(\Omega\) and set

\[
r_0 = \text{dist}(L, \Omega^c) > 0.
\]

(For the case \(\Omega = \mathbb{R}^N\), we simply take \(r_0 = 1\).) Let us show the existence of an exceptional point. For \(r \in (0, r_0)\), we define \(f_r : L \to \mathbb{R}\) by

\[
f_r(x) = \frac{|B_r(x) \cap U|}{|B_r(x)|}, \quad x \in L.
\]

Since \(y \in D_e(U)\) and \(z \in D_1(U)\), there exists \(r_1 \in (0, r_0/2)\) such that

\[
f_{r_1}(y) \geq 3/4, \quad f_{r_1}(z) \leq 1/4.
\]
Since \( f_{r_1} \) is continuous on \( L \), there exists \( x_1 \in L \) satisfying

\[
\frac{|B_{r_1}(x_1) \cap U|}{|B_{r_1}(x_1)|} = \frac{1}{2}.
\]

We note \( B_{r_1}(x_1) \subset \Omega \).

By an induction, we can construct a sequence of positive numbers \( \{r_n\}_{n=1}^\infty \) and a sequence \( \{x_n\}_{n=1}^\infty \) of points in \( \Omega \) such that (by setting \( x_0 = x_1 \))

\[
0 < r_n < \frac{r_{n-1}}{2}, \quad B_{r_n}(x_n) \subset B_{r_{n-1}}(x_{n-1}), \quad \frac{|B_{r_n}(x_n) \cap U|}{|B_{r_n}(x_n)|} = \frac{1}{2} \quad \text{for any } n \geq 1.
\]

Indeed, we assume that there exist \( r_k, x_k \) satisfying the above conditions up to \( k = 1, 2, \ldots, n \). Then, by applying Lemma 5.5 to \( U \cap B_{r_n}(x_n) \), we have \( |D_e[U] \cap B_{r_n}(x_n)| > 0 \). Similarly, \( |D_l[U] \cap B_{r_n}(x_n)| = |D_e[\Omega \setminus U] \cap B_{r_n}(x_n)| > 0 \). Therefore we can take \( y', z' \in B_{r_n}(x_n) \) such that

\[
y' \in D_e(U) \quad \text{and} \quad z' \in D_l(U).
\]

We take sufficiently small \( r_{n+1} \in (0, r_n / 2) \) such that \( B_{r_{n+1}}(y') \), \( B_{r_{n+1}}(z') \subset B_{r_n}(x_n) \), and

\[
\frac{|B_{r_{n+1}}(y') \cap U|}{|B_{r_{n+1}}(y')|} \geq \frac{3}{4} \quad \text{and} \quad \frac{|B_{r_{n+1}}(z') \cap U|}{|B_{r_{n+1}}(z')|} \leq \frac{1}{4}.
\]

By the same argument in the above, we have a point \( x_{n+1} \) on the line segment connecting \( y' \) and \( z' \) with

\[
\frac{|B_{r_{n+1}}(x_{n+1}) \cap U|}{|B_{r_{n+1}}(x_{n+1})|} = \frac{1}{2}.
\]

Clearly, we have \( B_{r_{n+1}}(x_{n+1}) \subset B_{r_n}(x_n) \).

This construction guarantees that \( \lim_n x_n = x \) exists and \( \{x\} = \cap_{n \geq 1} B_{r_n}(x_n) \subset \Omega \). In particular, we can verify that the limit \( x \) is an exceptional point of \( U \). Indeed, from \( B_{r_n}(x_n) \subset B_{2r_n}(x) \), we observe

\[
|B_{2r_n}(x) \cap U| \leq |B_{r_n}(x_n) \cap U| + |B_{2r_n}(x) \setminus B_{r_n}(x_n)| \leq \frac{1}{2}|B_{r_n}(x_n)| + (2^N - 1)|B_{r_n}(x_n)|
\]

\[
= \left( 2^N - \frac{1}{2} \right)|B_{r_n}(x_n)| = \frac{2^N - (1/2)}{2^N}|B_{2r_n}(x)| \quad \text{for any } n \geq 1.
\]

Similarly, we get

\[
|B_{2r_n}(x) \cap (\Omega \setminus U)| \leq \frac{2^N - (1/2)}{2^N}|B_{2r_n}(x)| \quad \text{for any } n \geq 1.
\]

It implies

\[
0 < \frac{1}{2} \frac{2^N}{2^N} \leq \frac{|B_{2r_n}(x) \cap U|}{|B_{2r_n}(x)|} \leq \frac{2^N - (1/2)}{2^N} < 1 \quad \text{for any } n \geq 1.
\]

Thus \( x \not\in (D_e(U) \cup D_l(U)) \), which means \( x \in E(U) \). \( \Box \)
APPENDIX C: PROOF OF LEMMA 7.1

We begin with the following observations:

For any $\phi \in \mathcal{C}^\infty_c(\Pi)$, we have that $T\phi \in L^{10/3}(\mathbb{R}^5)$ is compactly supported with

$$\|T\phi\|_{L^{10/3}(\mathbb{R}^5)} \lesssim \|\nabla T\phi\|_{L^2(\mathbb{R}^5)} \tag{C.1}$$

by Gagliardo-Nirenberg-Sobolev inequality (e.g., see [46, p. 277]). In addition, we have

$$\int_\Pi |\phi|^{10/3}r^{-11/3}drdz \lesssim \|\phi\|_{H}^{10/3} \tag{C.2}$$

by (7.5) and by the computation

$$\int_{\mathbb{R}^5} |T\phi|^{10/3}dy = 2\pi^2 \int_\Pi |\phi/r^2|^{10/3}r^3drdz = 2\pi^2 \int_\Pi |\phi|^{10/3}r^{-11/3}drdz.$$

Let $E$ be the Hilbert space which is the completion of $C^\infty_c(\mathbb{R}^5)$ in the norm $\| \cdot \|_E$ from the inner product

$$\langle f, g \rangle_E = \frac{1}{2\pi^2} \int_{\mathbb{R}^5} \nabla f \cdot \nabla g dy.$$

We denote $E_s$ the closed linear subspace of $E$ which is formed by completing $C^\infty_{c,s}(\mathbb{R}^5)$ (the class of infinitely smooth and compactly supported functions $f$ in $\mathbb{R}^5$ with the cylindrical symmetry $f(y) = f(r,z)$) in the norm $\| \cdot \|_E$. We observe, for $f, g \in E_s$,

$$\langle f, g \rangle_E = \int_\Pi ((\partial_r f)(\partial_r g) + (\partial_z f)(\partial_z g))r^3 drdz.$$ 

Then, [4, Lemma 2.2] says that the space $\mathcal{H}(\Pi)$ defined from (7.1) can be identified with the space $E_s$ via the transform $T$.

**Lemma C.1.** [Lemma 2.2 in [4]] The spaces $\mathcal{H}(\Pi)$ and $E_s$ are isometrically isomorphic under the transformation $f = T[\phi]$ of any $\phi \in \mathcal{H}(\Pi)$ or $f \in E_s$.

The proof follows the computation (7.4) once we note that the space $C^\infty_{c,s}(\mathbb{R}^5 \setminus \{r = 0\})$ is dense in $C^\infty_{c,s}(\mathbb{R}^5)$ under the norm $\| \cdot \|_E$, hence in $E_s$. For the detail, we refer to the proof of [4, Lemma 2.2].

By Lemma C.1 and by the definition of $\mathcal{H}(\Pi) = \overline{C^\infty_c(\Pi)}$, the identity (7.5) holds for any $\phi \in \mathcal{H}(\Pi)$. Similarly, the estimates (C.1) and (C.2) hold for $\phi \in \mathcal{H}(\Pi)$. For deep discussions about such homogeneous spaces $E, E_s$, we recommend [52, Sections II.6, II.7].

Now we are ready to prove Lemma 7.1.

**Proof of Lemma 7.1.** Let $\xi \in S_{\mu}$ satisfy (7.8). We recall the definition (7.1) of the norm $\| \cdot \|_H$. By (2.24) of Lemma 2.4 and by Theorem 4.2, we have

$$\pi \|\psi\|_H^2 = E[\xi] = I_\mu \in (0, \infty),$$

Here we use the subscript ‘s’ for cylindrical symmetry and ‘c’ for compact support while the original paper [4] used subscript ‘c’ for the symmetry and ‘0’ for compact support.
where \( \psi := \mathcal{G}[\xi] \) is the stream function \( \xi \). Now we show \( \psi \in \mathcal{H}(\Pi) \) which is equivalent to prove \( \mathcal{T}\psi \in E_s \) due to Lemma C.1. In the proof below, we denote \( x \in \mathbb{R}^3 \) and \( y \in \mathbb{R}^5 \). Since the axisymmetric function \( \xi(\cdot, x) \) lies on \( (L^1_{w} \cap L^\infty)(\mathbb{R}^3) \), we get
\[
\xi(\cdot, y) \in (L^1 \cap L^\infty)(\mathbb{R}^5)
\]
by
\[
\frac{1}{\pi} \int_{\mathbb{R}^5} |\xi| dy = 2\pi \int_{\Pi} r^3 |\xi| dr dz = \int_{\mathbb{R}^3} r^2 |\xi| dx.
\]

Thanks to Theorem 5.1 and Lemma 5.12, we know that
\[
\xi(\cdot, y) \text{ is compactly supported in } \mathbb{R}^5,
\]
\[
\mathcal{T}\psi \in BUC^\alpha(\bar{\mathbb{R}}^5), \quad 0 < \alpha < 1, \quad \text{and} \quad (\mathcal{T}\psi)(y) \to 0 \quad \text{as} \quad |y| \to \infty, \quad y \in \mathbb{R}^5. \tag{C.3}
\]
We also observe, by Lemma 2.4 and by the identity (7.3), in \( \mathbb{R}^5 \setminus \{r = 0\} \),
\[
\mathcal{T}\psi \in H^2_{loc}(\mathbb{R}^5 \setminus \{r = 0\}) \quad \text{and} \quad -\Delta_{\mathbb{R}^5}(\mathcal{T}\psi) = -\mathcal{T}[\mathcal{L}\psi] = \xi \quad \text{a.e.}
\]
Moreover, for any cylindrical symmetric bounded subset \( U \subset \mathbb{R}^5 \) with the corresponding axisymmetric bounded set \( \tilde{U} \subset \mathbb{R}^3 \), we have
\[
\int_{U} |\mathcal{T}\psi|^2 dy \leq \int_{\tilde{U}} \left( \frac{|\psi|}{r} \right)^2 dx \leq |U||\mathbb{R}^3||\psi/r||^2_{L^\infty(\mathbb{R}^3)} < \infty
\]
by (2.18) and
\[
\int_{U} |\nabla_{\mathbb{R}^5} \mathcal{T}\psi|^2 dy \leq \int_{\tilde{U}} \left( \frac{\psi}{r^2} \right)^2 + \frac{\partial_y \psi}{r^2} + \frac{|\psi|}{r^3} dy
\]
\[
\lesssim \int_{\tilde{U}} \left( \frac{\psi}{r^2} \right)^2 + \frac{\partial_y \psi}{r^2} + \frac{|\psi|}{r^3} dx + \int_{\tilde{U}} \left( \frac{\psi}{r^2} \right)^2 dx \leq E[\xi] + |\tilde{U}||\mathbb{R}^3||\mathcal{T}\psi||^2_{L^\infty(\mathbb{R}^5)} < \infty
\]
by (2.24) and (C.3). It gives \( \mathcal{T}\psi \in H^1_{loc}(\mathbb{R}^5) \). Hence the Poisson equation
\[
-\Delta_{\mathbb{R}^5}(\mathcal{T}\psi) = \xi
\]
is satisfied in a weak sense in any ball in \( \mathbb{R}^5 \), which gives the following representation of \( \mathcal{T}\psi \) via the fundamental solution in \( \mathbb{R}^5 \):
\[
\mathcal{T}\psi = \frac{1}{8\pi^2 |\cdot, y|^3} \ast_{\mathbb{R}^5} \xi(\cdot, y).
\]
This representation implies (as in the proof of Lemma 2.4)
\[
\mathcal{T}\psi \in W^{2,p}(\mathbb{R}^5) \cap BUC^{1+\alpha}(\mathbb{R}^5), \quad p > 5/3, \quad 0 < \alpha < 1.
\]
In particular, we have
\[
\mathcal{T}\psi \in H^1(\mathbb{R}^5).
\]
Since $\mathcal{T} \psi$ is cylindrical symmetric, we conclude $T \psi \in E_*$, which implies $\psi \in H(\Pi)$ by Lemma C.1.

It remains to show that the weak formulation (7.6) holds. Due to our assumption (7.8), it is equivalent to show

$$\langle \phi, \psi \rangle_H = \int_\Pi \phi \xi r \, drdz \quad \text{for any } \phi \in H(\Pi).$$

(C.4)

First, we observe that the right-hand side of (C.4) makes sense for any $\phi \in H(\Pi)$ thanks to (C.2) and the fact $\xi \in (L^\infty \cap L^1_w)(\mathbb{R}^3)$. Indeed, we can estimate, by (C.2),

$$\int_\Pi |\phi| \xi r \, drdz = \int_\Pi |\phi| r^{-11/10} \xi r^{21/10} \, drdz$$

$$\leq \left( \int_\Pi |\phi|^{10/3} r^{-11/3} \, drdz \right)^{3/10} \left( \int_\Pi \xi^{10/7} r^3 \, drdz \right)^{7/10}$$

$$\leq \|\phi\|_H \left( \|\xi\|_\infty^{3/7} \int_{\mathbb{R}^3} r^2 \xi \, dx \right)^{7/10} = \|\phi\|_H \|\xi\|_\infty^{3/10} \|r^2 \xi\|_1^{7/10} \leq \mu^{7/10} \|\phi\|_H.$$  

(C.5)

Second, (C.4) is clear for any $\phi \in C^\infty_c(\Pi)$ by integration by parts thanks to

$$\xi = -\frac{1}{r^2} \mathcal{L} \psi \quad \text{a.e.}$$

from Lemma 2.4. Lastly, for a general $\phi \in H(\Pi)$, we take a sequence $\{\phi_n\} \subset C^\infty_c(\Pi)$ such that $\phi_n \to \phi$ in $H(\Pi)$. For the left-hand side of (C.4), we know $\langle \phi_n, \psi \rangle_H \to \langle \phi, \psi \rangle_H$ as $n \to \infty$. For the right-hand side of (C.4), as in the computation (C.5), we have the convergence

$$\int_\Pi |\phi_n - \phi| \xi r \, drdz \lesssim \mu^{7/10} \|\phi_n - \phi\|_H \to 0 \quad \text{as } n \to \infty.$$  

Hence we obtain (C.4) for any $\phi \in H(\Pi)$.  

\[\Box\]