Time-Bridge Estimators of Integrated Variance

A. Saichev$^{1,3}$, D. Sornette$^{1,2}$

$^1$ETH Zurich – Department of Management, Technology and Economics, Switzerland
$^2$Swiss Finance Institute, 40, Boulevard du Pont-d’Arve, Case Postale 3, 1211 Geneva 4, Switzerland
$^3$Nizhni Novgorod State University – Department of Mathematics, Russia.

E-mail addresses: saichev@hotmail.com & dsornette@ethz.ch
Time-Bridge Variance Estimators

Abstract

We present a set of log-price integrated variance estimators, equal to the sum of open-high-low-close bridge estimators of spot variances within $n$ subsequent time-step intervals. The main characteristics of some of the introduced estimators is to take into account the information on the occurrence times of the high and low values. The use of the high’s and low’s of the bridge associated with the original process makes the estimators significantly more efficient that the standard realized variance estimators and its generalizations. Adding the information on the occurrence times of the high and low values improves further the efficiency of the estimators, much above those of the well-known realized variance estimator and those derived from the sum of Garman and Klass spot variance estimators. The exact analytical results are derived for the case where the underlying log-price process is an Itô stochastic process. Our results suggests more efficient ways to record financial prices at intermediate frequencies.

Didier Sornette
Department of Management, Technology and Economics
(D-MTEC, KPL F38.2) ETH Zurich
Kreuzplatz 5
CH-8032 Zurich
Switzerland
1 Introduction

The integrated variance is a crucial risk indicator of the stochastic log-price process within specific time intervals. Most of the existing high-frequency integrated variance estimators are modifications of the well-known realized volatility (see, for instance, Andersen et al. (2003), Aït-Sahalia (2005), Zhang et al. (2005)), and are based on the knowledge of the open and close prices of \( n \) time-step intervals dividing the whole time interval of interest. Another common practice to estimate the variance of a log-price process is to use not two (open-close) log-prices within a given time-step, but four values, the so-called the open-high-low-close (OHLC) of the log-prices. Well-known examples are the Garman and Klass (G&K) (1980) and Parkinson (Park) (1980) spot variance estimators.

The main goal of this paper is to demonstrate the efficiency of bridge OHLC integrated variance estimators, that use the knowledge of the high and low values of the bridge process derived from the original log-price process, as well as possibly the random occurrence times of these extrema within each time-step interval. We compare the efficiencies of these time-OHLC bridge estimators with the efficiency of the standard realized variance and with the efficiency of the integrated variance estimators based on the G&K estimators of the variance within each elementary time-step interval. We show that some time-OHLC integrated variance estimators achieve a very significant improvement in efficiency compared with the realized variance and the G&K integrated variance estimators. Another remarkable property of the proposed time-OHLC bridge estimators is that they depend much less on the drift of the log-price process than the realized variance and G&K integrated variance estimators. This has the great advantage of essentially removing the biases that affect the standard estimators, given that the drift (expected return) is in general the most poorly constrained statistical variable. We compare the efficiencies of the introduced integrated variance estimators using the Itô process as our workhorse to model the stochastic behavior of log-prices.

Present databases record either all prices associated with transactions or prune the data to keep the OHLC at given time steps, for instance, seconds, minutes or days. The later records giving the OHLC of the realized log-prices do not allow the reconstruction of the OHLC (and even less the occurrence times of the high’s and low’s) for the associated bridge process in each elementary interval. Of course, one could construct the OHLC and any other useful information from the full time series of all transaction prices. But then, one could question the value of deriving new estimators based on a reduced information set. Therefore, the present paper can be considered as a normative exercise to learn about the fundamental limits of integrated
variance estimators. Our results are also useful in suggesting more efficient ways to record financial prices at intermediate frequencies: instead of recording the OHLC at the daily scale for instance, we propose that data centers and vendors should store to open and close of the real log-price and the high and low of the corresponding bridge in each day (or in any other chosen frequency). Our calculations below show that this information, which has the same cost and is as easy to obtain at the end of the day from the high frequency data, provides much more efficient estimators of the variance that can be stored for future use. The same conclusion holds true for other risk measures beyond variance such as higher order moments, but this is not explored in the present paper.

The paper is organized as follows. Section 2 describes the properties of the well-known realized variance estimator, which we need in order to compare its efficiency with the efficiencies of the suggested time-OHLC bridge integrated variance estimators. Section 3 is devoted to the discussion of the efficiencies of the simple bridge integrated variance estimators, illustrating the comparative efficiency and unbiasedness of the bridge integrated variance estimators. This section written in a pedagogical style gradually introduces the readers in the area of homogeneous most efficient variance estimators. Section 4 provides a detailed analysis of the efficiency of the OHL and time-OHLC bridge integrated variance estimators, which turn out to be significantly more efficient than the realized variance and the G&K integrated variance estimators. Section 5 describes the results of numerical simulations demonstrating the comparative efficiency of the proposed estimators. Section 6 concludes. The paper is completed by three appendix. Appendix A presents the essential properties of the canonical bridge. Appendix B derives the joint probability density function (pdf) of the high value and of its occurrence time. Appendix C derives and gives the statistical properties of the joint distribution of the high and low values and of the occurrence time of the last extremum for the canonical bridge.

2 Realized variance and beyond

Henceforth, we assume that the log-price $X(t)$ of a given security follows an Itô process

$$dX(t) = \mu(t)dt + \sigma(t)dW(t), \quad X(0) = X_0,$$ (1)

where $W(t)$ is a realization of the standard Wiener process, while $\mu(t)$ is the drift process, and $\sigma^2(t)$ is the instantaneous variance of the log-price process $X(t)$. 
2.1 Definitions and basic properties of realized variance

Let us provide first some basic definitions and properties.

**Definition 1** The integrated variance of the process \( X(t) \) within the time interval \( t \in (0, T) \) is

\[
D(T) := \int_{0}^{T} \sigma^2(t) dt .
\]  

**Definition 2** The spot variance is defined within the time-step interval

\[
S_i : (t_{i-1}, t_i]
\]

by

\[
\hat{D}_{\text{real}} \{X(t) : t \in S_i\} := (X_i - X_{i-1})^2, \quad X_i := X(t_i), \quad t_i := i\Delta, \quad \Delta = \frac{T}{n} .
\]  

**Definition 3** The well-known statistical estimator of the integrated variance is the so-called realized variance defined as

\[
[X, X]_T := \sum_{i=1}^{n} \hat{D}_{\text{real}} \{X(t) : t \in S_i\} .
\]  

**Remark 1** For Itô processes and for \( n \to \infty \), it is well-known that the realized variance converges in probability to the integrated one. However, for real data, the number \( n \) of available data points is always limited, ultimately by the discreteness of the transaction flow and the associated microstructure noise. Such structures, which are not taken into account in the Itô log-price model, can be neglected in the use of the realized variance estimator if the discrete time step \( \Delta \) is much larger than the inverse of the mean frequency \( \nu \) of the tick-by-tick transactions, so that \( n \ll \nu T \).

**Assumption 1** While \( \Delta \gg 1/\nu \), we assume that \( \Delta \) is sufficiently small in comparison with the time scales over which the drift process \( \mu(t) \) and the instantaneous variance \( \sigma^2(t) \) vary, so that one may replace the original Itô process by Wiener processes with drift

\[
dX^i(t) \simeq \mu_i dt + \sigma_i dW(t), \quad X^i(t_{i-1}) = X_{i-1}, \quad t \in S_i,
\]

\[
\mu_i = \text{const}, \quad \sigma_i = \text{const} .
\]
Consider the special case of the Wiener process with drift

\[ X(t, \mu, \sigma) = \mu t + \sigma W(t) . \] (7)

Using the scale-invariance property of the Wiener process, the following identity holds in law (represented by the symbol \( \sim \))

\[ \hat{D}_{\text{real}}\{X(t, \mu, \sigma) : t \in \mathbb{S}_i\} \sim \sigma^2 \Delta [\gamma + W(1)]^2 = \sigma^2 \Delta \cdot X^2(1; \gamma) , \] (8)

where

\[ X(t; \gamma) = \gamma t + W(t), \quad \gamma = \frac{\mu}{\sigma} \sqrt{\Delta}, \quad t \in (0,1) , \] (9)

is the canonical Wiener process with drift. Applying the identity in law (8) to the realized variance expression (5), (4), we obtain

\[ [X, X]_T \sim \Delta \sum_{i=1}^{n} \sigma_i^2 (\gamma_i + W_i)^2 , \] (10)

where \( \{W_i\} \) are iid Gaussian variables \( \mathcal{N}(0,1) \). Accordingly, the expected value of the realized variance is

\[ \mathbb{E}[[X, X]_T] = \Delta \sum_{i=1}^{n} \sigma_i^2 (1 + \gamma_i^2), \quad \gamma_i = \frac{\mu_i}{\sigma_i} \sqrt{\Delta} . \] (11)

This recovers the well-known fact that the realized variance is in general biased for non-zero drift, and is non-biased only for zero-drift \( (\mu(t) \equiv 0) \).

### 2.2 Beyond realized variance with new estimated variance estimators \( \hat{D}_{\text{est}}(T) \)

The essential idea of the present work is that it is possible to improve on the realized variance estimator of the integrated variance estimator, for a fixed \( n \ll \nu T \) of time-steps with durations \( \Delta \), by replacing it by

\[ \hat{D}_{\text{est}}(T) = \sum_{i=1}^{n} \hat{D}_{\text{est}}\{X(t) : t \in \mathbb{S}_i\} , \] (12)

where the functional \( \hat{D}_{\text{est}}\{X(t) : t \in \mathbb{S}_i\} \) is an improved estimator of the spot variance given by definition \( 2 \). The subscript \( \text{est} \) is used to refer to some particular estimator and the subscript \( \text{real} \) means that this estimator reduces to the realized variance estimator.
Definition 4: The estimator $\hat{D}_{\text{est}}(T)$ defined by (12) is said to be unbiased if, for all intervals $i = 1, \ldots, n$,

$$
\mathbb{E}\left[\hat{D}_{\text{est}}\{X(t) : t \in S_i\}\right] = \Delta \cdot \sigma_i^2 ,
$$

which implies

$$
\mathbb{E}\left[\hat{D}_{\text{est}}(T)\right] = \Delta \sum_{i=1}^{n} \sigma_i^2 .
$$

When there exists at least one interval $j$, such that condition (13) does not hold, the estimator is considered biased.

2.3 Estimator efficiency

Let $\hat{D}_{\text{est}}(T)$ be some unbiased variance estimator. We propose to quantify its efficiency in terms of the coefficient of variation

$$
\rho[\hat{D}_{\text{est}}(T)] = \sqrt{\frac{\text{Var}[\hat{D}_{\text{est}}(T)]}{\mathbb{E}[\hat{D}_{\text{est}}(T)]}} .
$$

As an illustration, the coefficient of variation of the realized variance for a Wiener process with zero drift ($\mu(t) \equiv 0$) is equal to

$$
\rho[[X, X]_T] = \sqrt{\frac{2 \sum_{i=1}^{n} \sigma_i^4}{\sum_{i=1}^{n} \sigma_i^2}}.
$$

We will need the following theorem:

**Theorem 2.1** The lower bound of the function

$$
f(s) := \sqrt{\frac{\sum_{i=1}^{n} s_i^2}{\sum_{i=1}^{n} s_i}}, \quad s = \{s_1, s_2, \ldots, s_n\}, \quad \forall s_i > 0
$$

is equal to

$$
\rho(n) := \inf_{\forall s_i > 0} f(s) = \frac{1}{\sqrt{n}} .
$$

And this lower bound is attained iff all $s_i$ are identical: $s_i \equiv s > 0$. 

7
Proof. Let \( \{s_i\} \) be a realization of some random variable \( S \) with probabilities \( \Pr\{S = s_i\} = \frac{1}{n} \), \( i = 1, \ldots, n \). Expected and mean square values of the random variable \( S \) are equal to

\[
E[S] = \frac{1}{n} \sum_{i=1}^{n} s_i, \quad E[S^2] = \frac{1}{n} \sum_{i=1}^{n} s_i^2.
\] (19)

Since, for any random variable \( S \), the inequality \( \sqrt{E[S^2]} \geq E[S] \) holds, this implies \( f(s) \geq \frac{1}{\sqrt{n}} \). The inequality becomes an equality iff all \( s_i \equiv s \) for \( \forall s > 0 \). ■

Applying this theorem to the right-hand-side of expression (16) shows that \( \rho[[X, X]_T] \) satisfies the inequality

\[
\rho[[X, X]_T] \geq \rho_{\text{real}}(n), \quad \rho_{\text{real}}(n) = \sqrt{\frac{2}{n}},
\] (20)

where the lower bound \( \rho_{\text{real}}(n) \) of the efficiency is attained only if all \( \{\sigma_i\} \) are identical.

Below, we will compare the efficiencies of different estimators via the comparison of their lower bounds

\[
\rho_{\text{est}}(n) = \inf_{\forall \sigma_i} \rho_{\text{est}}[\hat{D}(T)].
\] (21)

3 Realized bridge variance estimators

3.1 Basic definitions

An important motivation for the introduction of a new class of so-called “realized bridge variance estimators” is to obtain much reduced biases compared that of the realized variance (5) observed for nonzero drifts \( \mu(t) \neq 0 \).

Definition 5 The bridge \( Y(t, S_i) \) in discrete time steps of the original process \( X(t) \) is defined by

\[
Y(t, S_i) := X(t) - X_{i-1} - \frac{t - t_{i-1}}{\Delta} (X_i - X_{i-1}), \quad t \in S_i,
\] (22)

where \( X_i := X(t_i), \ t_i := i\Delta \) and \( \Delta = \frac{T}{n} \).

As an example, let \( X(t) \) be the Wiener process with drift \( X(t, \mu, \sigma) \) defined by (7). Using the transition and scale invariant properties of the Wiener process leads to

\[
Y(t, S_i) \sim \sigma \sqrt{\Delta} (W(\zeta) - \zeta W(1)), \quad \zeta = \frac{t - t_{i-1}}{\Delta} \in (0, 1).
\] (23)
This means that the bridge $Y(t, S_i)$ \((22)\) is identical in law to

$$Y(t, S_i) \sim \sigma \sqrt{\Delta} Y(\zeta), \quad \text{(24)}$$

where

$$Y(t) := W(t) - t \cdot W(1), \quad t \in (0, 1], \quad \text{(25)}$$

is the \textit{canonical bridge} whose basic properties are given in Appendix A.

**Remark 2** The canonical bridge $Y(t)$ is completely independent of the drift $\mu$. This property is the fundamental reason for the better performance of the variance bridge estimators compared with the realized variance: the biases and efficiencies of bridge variance estimators do not depend on the drift $\mu$.

In the following, we explore the statistical properties of the bridge variance estimators

$$\hat{D}_{\text{est}}(T) = \sum_{i=1}^{n} \hat{D}_{\text{est}}\{Y(t, S_i) : t \in S_i\}, \quad \text{(26)}$$

obtained from the general expression \((12)\) by replacing the initial process \{$X(t_i)$\} by its corresponding bridge \{$Y(t_i, S_i)$\}.

**Definition 6** The estimator \((26)\) is called \textit{homogeneous} if, when applied to the Wiener processes with drift \((6)\), the following identity in law holds

$$\hat{D}_{\text{est}}\{Y(t, S_i) : t \in S_i\} \sim \sigma_i^2 \Delta \cdot \hat{d}_{\text{est}}, \quad \text{(27)}$$

where

$$\hat{d}_{\text{est}} := \hat{D}_{\text{est}}\{Y(t) : t \in (0, 1]\} \quad \text{(28)}$$

is the \textit{canonical estimator} of the spot variance depending on the canonical bridge $Y(t)$ \((25)\). Obviously, the estimator \((26)\) is unbiased if and only if $E[\hat{d}_{\text{est}}] = 1$.

**Theorem 3.2** Under Assumption \(1\), the lower bound of the efficiency of the unbiased homogeneous integrated bridge variance estimator \((27)\) is

$$\rho_{\text{est}}(n) = \sqrt{\frac{\text{Var}[\hat{d}_{\text{est}}]}{n}}, \quad \text{(29)}$$

where $\text{Var}[\hat{d}_{\text{est}}]$ is the variance of the canonical spot variance estimator $\hat{d}_{\text{est}}$ \((28)\).
Proof. Under Assumption 1, the unbiased homogeneous bridge variance estimator (26) is identical in law to
\[
\hat{D}_{\text{est}}(T) \sim \Delta \sum_{i=1}^{n} \sigma_{i}^2 \cdot \hat{d}_{i}^{\text{est}},
\] (30)
where \{\hat{d}_{i}^{\text{est}}\} are iid random variables with mean value \(E[\hat{d}_{\text{est}}] = 1\) and variance \(\text{Var}[\hat{d}_{\text{est}}]\). Accordingly, the expected value and variance of the unbiased bridge variance estimator (26) are equal to
\[
E[\hat{D}_{\text{est}}(T)] = \Delta \sum_{i=1}^{n} \sigma_{i}^2, \quad \text{Var}[\hat{D}_{\text{est}}(T)] = \Delta^2 \text{Var}[\hat{d}_{\text{est}}] \sum_{i=1}^{n} \sigma_{i}^4. \] (31)
Substitute these relations into (15), we obtain
\[
\rho[\hat{D}_{\text{est}}(T)] = \sqrt{\text{Var}[\hat{d}_{\text{est}}] \sum_{i=1}^{n} \sigma_{i}^4 / \sum_{i=1}^{n} \sigma_{i}^2}. \] (32)
Using theorem 2.1, this yields the result (29).

3.2 Simplest bridge variance estimator

Our first example of an homogeneous bridge variance estimator is
\[
\hat{D}_{\text{simple}}(T) = \sum_{i=1}^{n} \hat{D}_{\text{simple}}\{Y(t, S_i) : t \in S_i\},
\] (33)
where the estimator of the spot variance is given by
\[
\hat{D}_{\text{simple}}\{Y(t, S_i) : t \in S_i\} = AY^2(t_i(\eta)), \quad t_i(\eta) = t_{i-1} + \eta \cdot \Delta, \quad \eta \in (0, 1), \] (34)
and \(A\) is a normalizing factor. The estimator \(\hat{D}_{\text{simple}}(T)\) is homogeneous and, if relations (3) are valid, then
\[
Y^2(t_i(\eta), S_i) \sim \sigma_i^2 \Delta \cdot Y_i^2(\eta), \] (35)
where \{\(Y_i(\eta)\)\} are iid random variables that are identical in law to the canonical bridge (25). Substituting relation (35) into (33) leads to the identity in law
\[
\hat{D}_{\text{simple}}(T) \sim A \Delta \sum_{i=1}^{n} \sigma_{i}^2 Y_i^2(\eta). \] (36)
The fact that the canonical bridge \( Y(\eta) \) is Gaussian with mean value \( \mathbb{E}[Y^2(\eta)] = \eta(1 - \eta) \) implies that the estimator (33) is unbiased in the sense of definition 4 if \( A = 1/\eta(1 - \eta) \). Accordingly, the variance of the estimator (33) is equal to the variance of the realized variance obtained for zero drift \( (\mu(t) \equiv 0) \):

\[
\text{Var}[\hat{D}_{\text{simple}}(T)] = 2\Delta^2 \sum_{i=1}^{n} \sigma_i^4.
\]

This result means that the lower bound of the efficiency of the simplest bridge estimator (33) is equal to the lower bound of the efficiency of the realized variance estimator at zero drift:

\[
\rho_{\text{simple}}(n) = \rho_{\text{real}}(n) = \sqrt{\frac{2}{n}}.
\]

The shortcoming of the estimator (33) is that it is actually less efficient than the realized variance at zero drift in a sense discussed below.

### 3.3 Comparative efficiencies of realized variance estimators

**Definition 7** Let the estimator of the spot variance

\[
\hat{D}_{\text{est}}\{X(t) : t \in S_i\} \quad \text{or} \quad \hat{D}_{\text{est}}\{Y(t, S_i) : t \in S_i\}
\]

depends on \( \kappa_{\text{est}} \) values of the process \( X(t) \) or \( Y(t, S_i) \) at \( \kappa_{\text{est}} \) time-step within the time interval \( t \in S_i \). The corresponding estimators of the realized volatility \( \hat{D}_{\text{est}}(T) \) (12) or (33) are then using a total number \( n_{\text{eff}} = \kappa_{\text{est}} \cdot n \) of time-steps.

**Example 1** The realized variance corresponds to \( \kappa_{\text{real}} = 1 \). Indeed, the two values \( \{X_{i-1}, X_i\} \) are used to estimate the spot realized variance (4), and the first value is excluded from the semi-closed interval \( S_i \) (3).

**Example 2** For the simplest bridge estimator (33) with (34), \( \kappa_{\text{simple}} = 2 \). Indeed, the estimator (34) depends on the bridge \( Y(t_i(\eta), S_i) \) for \( t_i(\eta) \in S_i \) and \( Y(t_i(\eta), S_i) \) (22) is defined by the open and close values \( \{X_{i-1}, X_i\} \) of the original stochastic process \( X(t) \). Excluding the open value, this yields \( \kappa_{\text{simple}} = 2 \).

**Example 3** Consider the Garman & Klass (G&K) variance estimator based on open, high, low and close prices, used as the spot variance estimator in
expression (12):
\[
\hat{D}_{\text{GK}} \{X(t) : t \in \mathbb{S}_i\} = k_1(H_i - L_i)^2 - k_2(C_i(H_i - L_i) - 2H_iL_i) - k_3C_i^2,
\]
where \( \{O_i, C_i, H_i, L_i\} \) are the open, close, high and low values
\[
O_i = X_{i-1}, \quad C_i = X_i, \quad H_i = \sup_{t \in \mathbb{S}_i}[X(t) - O_i], \quad L_i = \inf_{t \in \mathbb{S}_i}[X(t) - O_i].
\]
Excluding the open value leads to \( \kappa_{\text{GK}} = 3 \).

**Definition 8** We characterize the efficiencies of the novel variance estimators by comparing them with that of the standard realized variance estimator. The corresponding comparative efficiency \( R_{\text{est}} \) is constructed as the ratio of the lower bounds of the efficiencies of the realized variance and novel variance estimator:
\[
R_{\text{est}} = \frac{\rho_{\text{real}}(\kappa_{\text{est}} \cdot n)}{\rho_{\text{est}}(n)}.
\]
Putting in this expression \( \rho_{\text{real}}(n) \) given by equation (20) and \( \rho_{\text{est}}(n) \) given by expression (29) yields
\[
R_{\text{est}} = \sqrt{\frac{2}{\kappa_{\text{est}} \cdot \text{Var}[\hat{d}_{\text{est}}]}}.
\]

**Remark 3** For a given duration \( T \) used to define the integrated variance (2), relation (41) takes into account that the typical waiting time between successive data samples is given by \( \Delta_{\text{eff}} \approx T / n_{\text{eff}} \). Such waiting time should be approximately the same for the different generalized variance estimators proposed below, leading to similar distortions to the adequacy of the Itô process (1) in its ability to describe the real price process in the presence of discrete tick-by-tick and other microstructure noise.

**Example 4** Let us come back to the simple variance estimator based on expression (34) for \( \hat{D}_{\text{simple}} \{Y(t, \mathbb{S}_i) : t \in \mathbb{S}_i\} \). The result (38) is equivalent to \( \text{Var}[\hat{d}_{\text{simple}}] = 2 \). Substituting this value in (41) yields
\[
R_{\text{simple}} = \frac{1}{\sqrt{\kappa_{\text{simple}}}} = \frac{1}{\sqrt{2}} \approx 0.707.
\]
The efficiency of the simplest bridge estimator is smaller than that of the realized variance.
Example 5  Let us evaluate the comparative efficiency of the generalized realized variance estimator based on the spot G&K variance estimator in the case of zero drift $\mu(t) \equiv 0$. It is known that the variance of the spot G&K variance estimator given by (39) is equal to

$$\text{Var}\left[\tilde{D}_{GK}\{X(t) : t \in S_i\}\right] = \sigma_i^2 \Delta \cdot 0.2693 \quad \Rightarrow \quad \text{Var}\left[\tilde{d}_{GK}\right] = 0.2693.$$ (43)

This gives

$$R_{GK} = \sqrt{\frac{2}{\kappa_{GK} \cdot 0.2693}} = \sqrt{\frac{2}{3 \cdot 0.2693}} \approx 1.573.$$ (44)

Therefore, for zero drift, the G&K realized variance estimator is approximately 1.6 times more efficient than the realized variance estimator.

3.4 High bridge variance estimator

The fact that the G&K realized variance estimator based on open-high-low-close prices is significantly more efficient than the standard realized variance, at least for Itô process $X(t)$ (11) with zero drift $\mu(t) \equiv 0$, suggests to study other estimators using different combinations of the open-high-low-close prices. Let us start by analyzing the simplest case of what we will refer to as the “high bridge variance estimator”, defined through its spot variance given by

$$\tilde{D}_{\text{high}}\{Y(t, S_i) : t \in S_i\} = A \cdot H_i^2,$$ (45)

where $A$ is normalizing factor and

$$H_i = \sup_{t \in S_i} Y(t, S_i),$$ (46)

is the high value of the bridge $Y(t, S_i)$. Note that we use here the same notation for the high value of the bridge $Y(t, S_i)$ as for that of the original process $X(t)$, hoping that this will not give rise to any confusion.

It follows from (24) that

$$\tilde{D}_{\text{high}}\{Y(t, S_t) : t \in S_i\} \sim \sigma_i^2 \Delta \cdot \tilde{d}_{\text{high}}, \quad \tilde{d}_{\text{high}} = AH^2,$$ (47)

where the high value $H$ of the canonical bridge $Y(t)$ (25) has the following probability density function (pdf)

$$\varphi_{\text{high}}(h) = 4he^{-2h^2}, \quad h > 0.$$ (48)
The derivation of the pdf (48) is given in Jeanblanc et al. (2009) (see also the derivations presented in Appendix B). Accordingly, the expected value and the variance of the square of $H$ are given by

$$E[H^2] = \frac{1}{2}, \quad \text{Var}[H^2] = \frac{1}{4}.$$  \hspace{1cm} (49)

In order for the high spot bridge variance estimator to be unbiased, we have to choose in (51) the value $A = 2$ for the normalizing factor. This gives $\text{Var}[\hat{d}_{\text{high}}] = 1$. With $\kappa_{\text{high}} = 2$, we find that the comparative efficiency (11) of the high bridge realized variance estimator is $R_{\text{high}} = 1$. Thus, the high bridge realized variance estimator has the same efficiency as the standard realized variance. But the advantage of the former is that, under Assumption (11), it is unbiased for any drift $\mu(t) \neq 0$.

**Remark 4** Let us give the intuition for the above result, obtained despite the larger value of $\kappa_{\text{high}} = 2$ compared to $\kappa_{\text{real}} = 1$. The reason is that the pdf of the random variable $2H^2$ is narrower than that of the random variable $W^2$ defining the spot realized variance at zero drift. The same reason underlies the comparative efficiency of the G&K as well the other high and low bridge realized variance estimators discussed below. The narrowing of the pdf’s of high’s and low’s compared with the pdf’s of the increments of the original stochastic process $X(t)$ results from a weak version of the Law of Large Numbers, in the sense that the high’s and low’s incorporate significant additional information about the underlying process within a given time-step, thus leading to narrower pdfs’.

### 3.5 Time-high bridge variance estimator

We now introduce a novel ingredient to improve further the estimation of the variance. In addition to using only the high $H_i$ of the bridge $Y(t, S_i)$, we also assume that the time $t_{\text{high}}^i$ of the occurrence of this high is recorded:

$$t_{\text{high}}^i : H_i = Y(t_{\text{high}}^i, S_i).$$  \hspace{1cm} (50)

The corresponding time-high bridge spot variance estimator is given by

$$\hat{D}_{\text{est}} \{ Y(t, S_i) : t \in S_i \} = A \cdot s \left( \frac{t_{\text{high}}^i - t_{i-1}}{\Delta} \right) \cdot H_i^2,$$  \hspace{1cm} (51)

where $A$ is a normalizing factor, while $s(t), t \in (0,1)$ is some function that remains to be determined so as to make the above spot variance estimator
as efficient as possible. Before providing the solution of this problem, let us note that the following identify in law follows from (24)

\[
\hat{D}_{est} \{ Y(t, S_i) : t \in S_i \} \sim \sigma_i^2 \Delta \cdot \hat{d}_{est},
\]

where

\[
\hat{d}_{est} = A \cdot s(t_{high}) \cdot H^2
\]

is the canonical time-high bridge estimator of the spot variance, \( H \) is the high value of the canonical bridge \( Y(t) \) (25), and \( t_{high} \) is the corresponding time-point (50).

The expected value of the canonical estimator (53) is equal to

\[
E \left[ \hat{d}_{est} \right] = A \int_0^1 s(t)\alpha(t; 2)dt, \quad \alpha(t; \lambda) := \int_0^\infty h^\lambda \varphi_{high}(h, t)dh
\]

where \( \varphi_{high}(h, t) \) is the joint pdf of \( H \) and \( t_{high} \). Taking

\[
A = \frac{1}{\int_0^1 s(t)\alpha(t; 2)dt},
\]

we obtain an unbiased time-high canonical bridge estimator:

\[
\hat{d}_{est} = \frac{s(t_{high})H^2}{\int_0^1 s(t)\alpha(t; 2)dt}.
\]

Its variance is

\[
\text{Var} \left[ \hat{d}_{est} \right] = \frac{\int_0^1 s^2(t)\alpha(t; 4)dt}{\left( \int_0^1 s(t)\alpha(t; 2)dt \right)^2} - 1.
\]

**Theorem 3.3** The function \( s(t) \) that minimizes the variance (57) of the unbiased time-high canonical bridge estimator (56) is

\[
s_{t, high}(t) = \frac{\alpha(t; 2)}{\alpha(t; 4)}.
\]

The corresponding minimal variance is equal to

\[
\text{Var} \left[ \frac{s_{t, high}(t_{high})H^2}{\int_0^1 s(t)\alpha(t; 2)dt} \right] = \inf_{\forall s(t)} \text{Var} \left[ \hat{d}_{est} \right] = \frac{1}{\mathcal{E}_{t, high}} - 1,
\]

\[
\mathcal{E}_{t, high} = \int_0^1 \frac{\alpha^2(t; 2)}{\alpha(t; 4)}dt.
\]
Proof. We use the Schwarz inequality

\[
\left( \int_0^1 A(t)B(t)dt \right)^2 \leq \int_0^1 A^2(t)dt \int_0^1 B^2(t)dt
\]  

(60)

with

\[
A(t) = s(t)\sqrt{\alpha(t; 4)}, \quad B(t) = \frac{\alpha(t; 2)}{\sqrt{\alpha(t; 4)}},
\]

(61)

to obtain

\[
\left( \int_0^1 s(t)\alpha(t; 2)dt \right)^2 \leq \int_0^1 s^2(t)\alpha(t; 4)dt \int_0^1 \frac{\alpha^2(t; 2)}{\alpha(t; 4)}dt.
\]  

(62)

After simple transformations, we rewrite the last inequality in the form

\[
\text{Var}\left[\hat{d}_{t\text{-high}}\right] = \int_0^1 \frac{s^2(t)\alpha(t; 4)dt}{\left( \int_0^1 s(t)\alpha(t; 2)dt \right)^2} - 1 \geq \frac{1}{\int_0^1 \frac{\alpha^2(t; 2)}{\alpha(t; 4)}dt} - 1. 
\]  

(63)

The equality in (63) is reached by substituting in it \(s(t) = s_{t\text{-high}}(t)\) given by expression (58).

The joint pdf of \(H\) and \(t_{\text{high}}\) is derived in Appendix B and reads

\[
\varphi_{\text{high}}(h, t) = \sqrt{\frac{2}{\pi}} \frac{h^2}{\sqrt{t^3(1-t)^3}} \exp\left(-\frac{h^2}{2t(1-t)}\right), \quad h > 0, \quad t \in (0, 1).
\]

(64)

Substituting this expression for \(\varphi_{\text{high}}(h, t)\) into (54) yields

\[
\alpha(t; \lambda) = \frac{2}{\sqrt{\pi}} \left[2t(1-t)\right]^\frac{3}{2} \Gamma\left(\frac{3 + \lambda}{2}\right).
\]

(65)

Therefore,

\[
s_{t\text{-high}}(t) = \frac{1}{5t(1-t)}, \quad \mathcal{E}_{t\text{-high}} = \frac{3}{5} \Rightarrow \text{Var}\left[\hat{d}_{t\text{-high}}\right] = \frac{2}{3},
\]

(66)

and

\[
\mathcal{R}_{t\text{-high}} = \sqrt{\frac{3}{2}} \simeq 1.225.
\]

(67)

Thus, the time-high bridge realized variance estimator is less efficient than the corresponding G&K estimator at zero drift, but is more efficient than the realized variance.

Remark 5 The numerical result (67) takes into account that the use of \(t_{\text{high}}^i\) does not increase the number of sample values used in the spot estimator (51). Thus, \(\kappa_{t\text{-high}} = \kappa_{\text{high}} = 2\).
4 Bridge time-high-low estimators

4.1 Bridge Parkinson estimator

Definition 9 The bridge realized variance estimator \(\hat{D}_{bPark}\) that uses as spot variance estimator

\[
\hat{D}_{bPark}\{Y(t, S_i) : t \in S_i\} = A \cdot (H_i - L_i)^2
\]

(68)
is called the bridge Parkinson estimator. In expression (68), \(H_i\) and \(L_i\) are the high and low values of the bridges \(Y(t, S_i)\) (22).

The bridge Parkinson estimator is identical in law to

\[
\hat{D}_{bPark}\{Y(t, S_i) : t \in S_i\} \sim \sigma_i^2 \Delta \cdot \hat{d}_{bPark}, \quad \hat{d}_{bPark} = A \cdot (H - L)^2,
\]

(69)

where \(H, L\) are the high and low values of the canonical bridge \(Y(t)\) (25). The joint pdf of \(H\) and \(L\) have been derived by Saichev et al. (2009) and reads

\[
\varphi(h, \ell) = \sum_{m=-\infty}^{\infty} m [m\mathcal{I}(m(h - \ell)) + (1 - m)\mathcal{I}(m(h - \ell) + \ell)], \quad \mathcal{I}(h) = 4(4h^2 - 1) e^{-4h^2}.
\]

(70)

It will be clear below that it is convenient to describe the joint statistical properties of the high \(H\) and low \(L\) by using polar coordinates

\[
H = R \cos \Theta, \quad L = R \sin \Theta, \quad R \in (0, \infty), \quad \theta \in \left(-\frac{\pi}{2}, 0\right). \quad (71)
\]

Accordingly, we rewrite the canonical estimator (69) in the form

\[
\hat{d}_{bPark} = AR^2 (1 - \sin 2\Theta).
\]

(72)

Choosing the constant \(A\) that makes the estimator (69) unbiased, we obtain

\[
\alpha(\theta; \lambda) = \int_0^\infty r^{\lambda+1} \varphi(r \cos \theta, r \sin \theta) dr.
\]

(73)

Substituting expression (70) yields

\[
\alpha(\theta; \lambda) = \sum_{m=-\infty}^{\infty} m \left[ m \beta(m(\cos \theta - \sin \theta); \lambda) + (1 - m) \beta(m(\cos \theta - \sin \theta) + \sin \theta; \lambda) \right], \quad \beta(y; \lambda) = \frac{C(\lambda)}{|y|^{2+\lambda}}, \quad C(\lambda) = \frac{1 + \lambda}{\sqrt{2\lambda}} \Gamma \left(\frac{2 + \lambda}{2}\right).
\]

(74)
The variance of the canonical bridge Parkinson estimator is equal to

$$\text{Var} \left[ \hat{d}_{bPark} \right] = \frac{\int_{-\pi/2}^{0} (1 - \sin 2\theta)^2 \alpha(\theta; 4) d\theta}{\left( \int_{-\pi/2}^{0} (1 - \sin 2\theta) \alpha(\theta; 2) d\theta \right)^2} - 1 \simeq 0.2000 \ . \quad (75)$$

Substituting this value into (71) and taking into account that \( \kappa_{bPark} = 3 \) for the bridge Parkinson estimator, we obtain the comparative efficiency

$$\text{Var} \left[ \hat{d}_{bPark} \right] = 0.2000, \quad \kappa_{bPark} = 3, \quad \Rightarrow \quad \mathcal{R}_{bPark} \simeq 1.823 \ , \quad (76)$$

which means that the bridge Parkinson estimator is significantly more efficient than the G&K estimator at zero drift.

**Remark 6** We stress that the canonical estimator \( \hat{d}_{bPark} \) is significantly different from the well-known canonical Parkinson estimator (see Parkinson (1980))

$$\hat{d}_{Park} = \frac{(H - L)^2}{4 \ln 2} \ , \quad (77)$$

where \( H \) and \( L \) are the high and low values of the canonical Wiener process with drift \( X(t, \gamma) \) (9). In contrast with the bridge Parkinson estimator (73) which is unbiased for any \( \gamma \), the standard Parkinson estimator is biased at nonzero drift. Moreover, the variance of the standard Parkinson estimator at zero drift is

$$\text{Var} \left[ \hat{d}_{Park} \right] \simeq 0.4073 \ , \quad (78)$$

which is approximately twice the variance of the bridge Parkinson estimator (76).

### 4.2 Non-quadratic homogeneous estimators

Until now, we have considered homogeneous (in the sense of definition 6) high-low estimators that are quadratic functions of the high and low values. We now consider the more general class of homogeneous estimators, whose spot variance estimators have the form

$$\hat{D}_{\text{est}} \{ Y(t, S_i) : t \in S_i \} = D_{\text{est}}(H_i, L_i) \ , \quad (79)$$

where \( D_{\text{est}}(h, \ell) \) is an arbitrary homogeneous function of second order.

**Example 6** To illustrate the notion of non-quadratic homogeneous functions of second order, consider the typical example

$$D_{\text{est}}(H_i, L_i) = \frac{(H_i - L_i)^3}{\sqrt{H_i^2 + L_i^2}} \ , \quad (80)$$
which satisfies the scaling property
\[ D_{\text{est}}(\delta \cdot H_i, \delta \cdot L_i) \equiv \delta^2 \cdot D_{\text{est}}(H_i, L_i) \quad \forall \delta > 0. \quad (81) \]

The following theorem states that the spot variance estimator (79) satisfies the relations (27), (28) of definition 6 for homogeneous estimators.

**Theorem 4.4** The spot variance estimator (79) is homogeneous in the sense of definition 6.

**Proof.** Let \( H_i \) and \( L_i \) be the high and low values of the bridge \( Y(t, S_i) \). Due to relation (24) and Assumption 1, the following identity in law holds
\[ \{H_i, L_i\} \sim \sigma_i \sqrt{\Delta} \cdot \{H, L\}, \quad (82) \]
where \( \{H, L\} \) are the high and low values of the canonical bridge \( Y(t) \) (25).
Substituting this last relation into (79) yields
\[ D_{\text{est}}(H_i, L_i) \sim D_{\text{est}}(\sigma_i \sqrt{\Delta} H, \sigma_i \sqrt{\Delta} L). \quad (83) \]
Using the homogeneity of the function \( D(h, \ell) \), we rewrite the previous relation in the form
\[ D_{\text{est}}(H_i, L_i) \sim \sigma^2 \Delta_{\text{est}} \cdot D(H, L) , \quad (84) \]
which is analogous to expression (27), where the canonical estimator of the spot variance is equal to
\[ \hat{d}_{\text{est}} = D_{\text{est}}(H, L). \quad (85) \]

Using the polar coordinates (71), the canonical estimator \( \hat{d}_{\text{est}} \) reads
\[ \hat{d}_{\text{est}} = D_{\text{est}}(R \cos \Theta, R \sin \Theta). \quad (86) \]
Using the homogeneity of the function \( D_{\text{est}} \), we obtain
\[ \hat{d}_{\text{est}} = R^2 \cdot s(\theta), \quad s(\theta) = D_{\text{est}}(\cos \theta, \sin \theta). \quad (87) \]
Its expected value is equal to
\[ \mathbb{E}[\hat{d}_{\text{est}}] = \int_{-\pi/2}^{\pi/2} s(\theta) \alpha(\theta; 2) d\theta , \quad (88) \]
where the function \( \alpha(\theta, \lambda) \) is given by the equality (73). Thus, the homogeneous non-quadratic canonical estimator reads
\[ \hat{d}_{\text{est}} = \frac{R^2 s(\Theta)}{\int_{-\pi/2}^{\pi/2} s(\theta) \alpha(\theta; 2) d\theta} \quad \Rightarrow \quad \mathbb{E}[\hat{d}_{\text{est}}] = 1. \quad (89) \]
Accordingly, the variance of the unbiased estimator is equal to

\[
\text{Var}\left[\hat{d}_{\text{est}}\right] = \frac{\int_{-\pi/2}^{0} s^2(\theta)\alpha(\theta; 4)d\theta}{\left(\int_{-\pi/2}^{0} s(\theta)\alpha(\theta; 2)d\theta\right)^2} - 1. \tag{90}
\]

One can easily prove the result analogous to theorem 3.3 that the minimum value of the variance (90) of the canonical estimator (89) with respect to all possible functions \(s(\theta)\) is given by

\[
\text{Var}\left[\hat{d}_{\text{me}}\right] = \inf_{\forall s(\theta)} \text{Var}\left[\hat{d}_{\text{est}}\right] = \frac{1}{\mathcal{E}_{\text{me}}} - 1, \quad \mathcal{E}_{\text{me}} = \int_{-\pi/2}^{0} \frac{\alpha^2(\theta; 2)}{\alpha(\theta; 4)}d\theta, \tag{91}
\]

where \(\hat{d}_{\text{est}}\) is an arbitrary homogeneous canonical estimator of the form (89), while \(\hat{d}_{\text{me}}\) is the corresponding most efficient estimator given by

\[
\hat{d}_{\text{me}} = \frac{1}{\mathcal{E}_{\text{me}}} R^2 s_{\text{me}}(\Theta), \quad s_{\text{me}}(\theta) = \frac{\alpha(\theta; 2)}{\alpha(\theta; 4)}. \tag{92}
\]

Calculating the numerical value of the integral in expression (91) yields

\[
\hat{d}_{\text{me}} = 0.1974, \quad \kappa_{\text{me}} = 3, \quad \Rightarrow \quad \mathcal{R}_{\text{me}} \simeq 1.838, \tag{93}
\]

which shows a high efficiency compared with the standard realized variance.

### 4.3 Time-high-low homogeneous estimator

Let us consider the unbiased homogeneous time-high-low canonical estimator

\[
\hat{d}_{\text{est}} = \frac{R^2 s(\Theta, t_{\text{last}})}{\int_{0}^{1} dt \int_{-\pi/2}^{0} d\theta \ s(\theta, t)\alpha_{\text{last}}(\theta, t; 2)}, \tag{94}
\]

where \(s(\theta, t)\) is an arbitrary function, \(t_{\text{last}} = \sup\{t_L, t_H\}\) is the larger of the two times at which occur the high and low values of the canonical bridge and \(\alpha_{\text{last}}(\theta, t; \lambda)\) is given by (C.17) in Appendix C.3.

It is easy to prove the result analogous to theorem 3.3 that the most efficient estimator of the form (94) is

\[
\hat{d}_{t\text{-me}} = \frac{R^2}{\mathcal{E}_{t\text{-me}}} \frac{\alpha_{\text{last}}(\Theta, t_{\text{last}}; 2)}{\alpha_{\text{last}}(\Theta, t_{\text{last}}; 4)}, \quad \mathcal{E}_{t\text{-me}} = \int_{0}^{1} dt \int_{-\pi/2}^{0} d\theta \frac{\alpha^2_{\text{last}}(\theta, t; 2)}{\alpha_{\text{last}}(\theta, t; 4)}, \tag{95}
\]

and the variance of this estimator is equal to

\[
\text{Var}\left[\hat{d}_{t\text{-me}}\right] = \frac{1}{\mathcal{E}_{t\text{-me}}} - 1. \tag{96}
\]
The numerical calculation of $E_{t}\text{me}$ gives

$$\text{Var}\left[\hat{d}_{t}\text{me}\right] \simeq 0.1873, \quad \kappa_{t}\text{me} = 3, \quad \Rightarrow \quad R_{t}\text{me} \simeq 1.887. \quad (97)$$

The estimator of the realized variance based on the canonical estimator is significantly more efficient than that based on the G&K estimator at zero drift.

4.4 High-low-close bridge estimator

Until now, we have not used explicitly the information contained in the close values $X_{i}$ of the time-step intervals $S_{i}$. The close values $X_{i}$ have been used only for the construction of the bridge $Y(t, S_{i})$. It seems plausible that taking into account explicitly the close values $X_{i}$ in the construction of spot variance estimators may produce bridge realized variance estimators $\hat{D}_{\text{est}}(T) = \sum_{i=1}^{n} \hat{D}_{\text{est}}\{Y(t, S_{i}) : t \in S_{i}; X_{i}\}$ that are even more efficient than those considered until now. We show that this is indeed the case by studying the example associated with the spot variance estimator given by

$$\hat{D}_{\text{est}}\{Y(t, S_{i}) : t \in S_{i}; X_{i}\} = D_{\text{est}}(H_{i}, L_{i}, X_{i}), \quad (98)$$

where $D_{\text{est}}(h, \ell, x)$ is an arbitrary homogeneous function satisfying relation (81). Due to its homogeneity, the following identity in law holds true

$$D_{\text{est}}(H_{i}, L_{i}, X_{i}) \sim \sigma_{i}^{2}\Delta \cdot \hat{d}_{\text{est}}, \quad \hat{d}_{\text{est}} = D_{\text{est}}(H, L, X), \quad (99)$$

where $H$ and $L$ are the high and low values of the canonical bridge, while $X = \gamma + W$ is the close value of the underlying canonical Wiener process with drift (9). It is known (see, for instance, Jeanblanc et al. (2009)) that the canonical bridge $Y(t)$ and $W$ are statistically independent. Thus, the joint pdf $\varphi(h, \ell, x)$ of the three random variables $\{H, L, X\}$ is equal to

$$\varphi(h, \ell, x; \gamma) = \frac{1}{\sqrt{2\pi}} \exp\left(-\frac{(x-\gamma)^2}{2}\right) \varphi(h, \ell), \quad (100)$$

where the joint pdf $\varphi(h, \ell)$ of high and low values is given by expression (70).

Analogously to (86), it is convenient to represent the canonical estimator $\hat{d}_{\text{est}}(99)$ in the spherical coordinate system

$$H = R \cos \Upsilon \cos \Theta, \quad L = R \cos \Upsilon \sin \Theta, \quad X = R \sin \Upsilon,$$

$$\Upsilon \in (-\pi/2, \pi/2), \quad \Theta \in (-\pi/2, 0). \quad (101)$$
The canonical estimator $\hat{d}_{\text{est}}$ then takes the form

$$\hat{d}_{\text{est}} = R^2 s(\Theta, \Upsilon),$$

(102)

where

$$s(\Theta, \Upsilon) = D_{\text{est}}(\cos \Upsilon \cos \Theta, \cos \Upsilon \sin \Theta, \sin \Upsilon).$$

(103)

Analogously to (94) and (95), the unbiased most efficient high-low-close canonical estimator is given by

$$\hat{d}_{\text{me-x}} = \frac{1}{E_{\text{me-x}}} R^2 s_{\text{me-x}}(\Theta, \Upsilon; \gamma), \quad s_{\text{me-x}}(\theta, \upsilon; \gamma) = \frac{\alpha(\theta, \upsilon; 2; \gamma)}{\alpha(\theta, \upsilon; 4; \gamma)}.$$  

(104)

The function $\alpha(\theta, \upsilon; \lambda; \gamma)$ is defined by the equality

$$\alpha(\theta, \upsilon; \lambda; \gamma) = \int_0^\infty r^{\lambda+2} \varphi(r \cos \upsilon \cos \theta, r \cos \upsilon \sin \theta, r \sin \upsilon; \gamma) dr.$$  

(105)

The variance of the most efficient canonical estimator $\hat{d}_{\text{me-x}}$ is equal to

$$\text{Var}\left[\hat{d}_{\text{me-x}}\right] = \frac{1}{E_{\text{me-x}}} - 1,$$

(106)

with

$$E_{\text{me-x}} = \int_{-\pi/2}^{\pi/2} d\theta \int_{-\pi/2}^{\pi/2} d\upsilon \cos \upsilon \frac{\alpha^2(\theta, \upsilon; 2; \gamma)}{\alpha(\theta, \upsilon; 4; \gamma)}.$$  

(107)

The calculation of the integral (107) for $\gamma = 0$ gives

$$\text{Var}\left[\hat{d}_{\text{me-x}}\right] \simeq 0.1794, \quad \kappa_{\text{me-x}} = 3, \quad \Rightarrow \quad R_{\text{me-x}} \simeq 1.928.$$  

(108)

This estimator is definitely better than the most efficient time-high-low-canonical estimator, as can be seen by comparing (108) with (97).

4.5 Time-high-low-close bridge estimator

The last example we present here is the realized variance estimator that uses in each interval $S_i$ the high and low values $H_i, L_i$ of the bridge $Y(t, S_i)$ (22), the close value $X_i$ of the original stochastic process $X(t)$ and the time instant $t_{last} = \sup\{t_{L}, t_{H}\}$ defined as the larger of the two times at which occur the high and low values of the canonical bridge.

One can rigorously prove that, analogously to (104), the homogeneous time-OHLC bridge canonical estimator that is most efficient for some given value of $\gamma$ value is equal to

$$\hat{d}_{t,\text{me-x}}(\Theta, \Upsilon, t_{last}; \gamma) = R^2 s_{t,\text{me-x}}(\Theta, \Upsilon, t_{last}; \gamma),$$

$$s_{t,\text{me-x}}(\theta, \upsilon, t; \gamma) = \frac{1}{E_{t,\text{me-x}}(\gamma)} \frac{\alpha(\theta, \upsilon, t; 2; \gamma)}{\alpha(\theta, \upsilon, t; 4; \gamma)},$$

(109)
where
\[
E_{t\text{-me-x}}(\gamma) = \int_0^1 dt \int_{-\pi/2}^{\pi/2} d\theta \int_{-\pi/2}^{\pi/2} dv \cos v \frac{\alpha^2(\theta, v, t; 2; \gamma)}{\alpha(\theta, v, t; 4; \gamma)}
\] (110)
and
\[
\alpha(\theta, v, t; \lambda; \gamma) = \int_0^\infty r^{\lambda+2} \varphi_{\text{last}}(r \cos v \cos \theta, r \cos v \sin \theta, r \sin v, t; \gamma) dr.
\] (111)
The joint pdf \(\varphi(h, \ell, x, t; \gamma)\) is
\[
\varphi_{\text{last}}(h, \ell, x, t; \gamma) = \frac{1}{\sqrt{2\pi}} \exp \left( -\frac{(x - \gamma)^2}{2} \right) \varphi_{\text{last}}(h, \ell, t),
\] (112)
where \(\varphi_{\text{last}}(h, \ell, t)\) is given by expression (C.15) in Appendix C.2.

**Remark 7** Recall that the parameter factor \(\gamma\) is unknown, because both the drift \(\mu_i\) and the instantaneous variances \(\sigma^2_i\) in equations (6) are generally unknown. Therefore, our strategy below is to choose, for definiteness, \(\gamma = 0\) and then explore the dependence on \(\gamma\) of the bias and efficiency of the different “zero drift” estimators. Accordingly, we will use below the following shorthand notations, omitting the argument \(\gamma\), such as
\[
\hat{d}_{t\text{-me-x}}(\Theta, \Upsilon, t) := \hat{d}_{t\text{-me-x}}(\Theta, \Upsilon, t; \gamma = 0).
\]
The calculation of the integral (110), where \(\alpha(\theta, v, t; \lambda)\) is given by expression (C.18) in Appendix C.4 yields for \(\gamma = 0\)
\[
\text{Var} \left( \hat{d}_{t\text{-me-x}} \right) = \frac{1}{E_{t\text{-me-x}}} - 1 \simeq 0.1710, \quad \kappa_{t\text{-me-x}} = 3, \quad \Rightarrow \quad R_{t\text{-me-x}} \simeq 1.975.
\] (113)
This estimator is more efficient than all the previous one discussed until now.

## 5 Numerical simulations and comments

### 5.1 Description of numerical simulations

The goal of this section is to check by numerical simulations some analytical results obtained above. Realizations of the canonical Wiener process \(X(t; \gamma)\) with drift for time \(t \in [0, 1]\) are obtained numerically as cumulative sums of a number \(I(t) = 10^5\) of Gaussian summands, corresponding to a discrete time step \(\Delta = 10^{-5}\). For each numerical realization, we calculate
the values of the open-close spot variance canonical estimator, equal in this case to
\[
\hat{d}_{\text{real}} = (\gamma + W)^2,
\]
and the values of the G&K canonical estimator
\[
\hat{d}_{\text{GK}} = k_1(H - L)^2 - k_2(W(H - L) - 2HL) - k_3(\gamma + W)^2,
\]
where $H$ and $L$ are the high and low values of the simulated process $X(t; \gamma)$.

We also constructed numerical realizations of the bridge process $Y(t)$ (25) and calculated the corresponding values of the canonical estimator $\hat{d}_{t\text{-me-x}}$ (109). This estimator depends on the function $\alpha(\theta, \nu, t; \lambda)$ defined by expression (C.18) in Appendix C.4, which is explicitly obtained by summing a double-infinite series (C.19). In practice, we estimate this double-sum by keeping only the 101 first terms in each dimension, corresponding to estimating $101 \times 101 \simeq 10^4$ summands in (C.18).

**Remark 8** At first glance, it would seem that the calculation of the G&K estimator (115), which needs only a few simple arithmetic operations, is much easier than the evaluation of the large number of summands in the series (C.18) that define the estimator $\hat{d}_{t\text{-me-x}}$ (109). In our computerized world, it turns out that there is actually no significant difference from the computational point of view.

### 5.2 Statistics of the estimators in the case of zero drift ($\gamma = 0$)

Figure 1 shows 5000 realizations of the open-close estimator $\hat{d}_{\text{real}}$ (114), of the G&K estimator (115) and of the estimator $\hat{d}_{t\text{-me-x}}$ in the case the Wiener process with zero drift ($\gamma = 0$). It is clear that the last estimator is the most efficient in comparison with the open-close and the G&K estimators. The expected values and variances of these three estimators obtained by statistical averaging over $10^4$ samples are

\[
\begin{align*}
\mathbb{E}[\hat{d}_{\text{real}}] &\simeq 1.0110, & \mathbb{E}[\hat{d}_{\text{GK}}] &\simeq 1.0058, & \mathbb{E}[\hat{d}_{t\text{-me-x}}] &\simeq 1.0001, \\
\text{Var}[\hat{d}_{\text{real}}] &\simeq 1.9947, & \text{Var}[\hat{d}_{\text{GK}}] &\simeq 0.2669, & \text{Var}[\hat{d}_{t\text{-me-x}}] &\simeq 0.1696.
\end{align*}
\]

These values are consistent with the theoretical analytical predictions obtained in previous sections:

\[
\begin{align*}
\mathbb{E}[\hat{d}_{\text{real}}] = \mathbb{E}[\hat{d}_{\text{GK}}] = \mathbb{E}[\hat{d}_{t\text{-me-x}}] &= 1, \\
\text{Var}[\hat{d}_{\text{real}}] = 2, & \text{Var}[\hat{d}_{\text{GK}}] &\simeq 0.2693, & \text{Var}[\hat{d}_{t\text{-me-x}}] &\simeq 0.1710.
\end{align*}
\]
In order to have truly comparable efficiencies of these realized variance estimators, bearing in mind that their effective sample sizes are different ($\kappa_{\text{real}} = 1$, $\kappa_{\text{GK}} = \kappa_{t\text{-me-x}} = 3$), we performed moving averages with $r = 30$ subsequent samples for the open-close estimator (114) and with $r = 10$ subsequent samples for the G&K estimator (115) and estimator $\hat{d}_{t\text{-me-x}}$ (109). Figure 2 presents there moving averages, which mimick the normalized estimators of the integrated variance in the case where all instantaneous variances are the same ($\sigma_i^2 = \sigma^2 = \text{const}$). It is clear that the open-close estimator of the realized variance remains significantly less efficient than the G&K estimator, and much less efficient than the most efficient estimator $\hat{d}_{t\text{-me-x}}$.

5.3 $\gamma$-dependence of biases and efficiencies of canonical estimators

In the previous subsection, we presented detailed calculations of the comparative efficiency of unbiased variance estimators for the particular case of Wiener processes with zero drift. In real financial markets, the drift process $\mu(t)$ is unknown and there is not reason for it to vanish. Thus, it is important to explore quantitatively the dependence on the parameter $\gamma$ (9) of the biases and efficiencies of the spot variance canonical estimators described above.

We begin with the open-close spot variance canonical estimator $\hat{d}_{\text{real}}$ (114). It is easy to show that its expected value and variance are quadratic functions of $\gamma$:

$$E\left[\hat{d}_{\text{real}}\right] = 1 + \gamma^2, \quad \text{Var}\left[\hat{d}_{\text{real}}\right] = 2 + 4\gamma^2. \quad (116)$$

The spot variance homogeneous time-open-high-low canonical bridge estimators, such as the Park estimator $\hat{d}_{\text{bPark}}$ (73) and the time-high-low estimator $\hat{d}_{t\text{-me}}$ (95), are unbiased for all $\gamma$:

$$E\left[\hat{d}_{\text{bPark}}\right] = E\left[\hat{d}_{t\text{-me}}\right] = 1. \quad (117)$$

Their variances do not depend on $\gamma$ at all:

$$\text{Var}\left[\hat{d}_{\text{bPark}}\right] \simeq 0.2000, \quad \text{Var}\left[\hat{d}_{t\text{-me}}\right] \simeq 0.1873 \quad \forall \gamma. \quad (117)$$

To obtain the $\gamma$-dependence of the biases and variances of the G&K canonical estimator $\hat{d}_{\text{GK}}$ (113) and of the canonical estimator $\hat{d}_{t\text{-me-x}}$ (109), we generate $10^4$ numerical realizations of the canonical Wiener process $X(t, \gamma)$ (9) with drift, for $\gamma = 0; 0.1; \ldots 1.5; 1.6$. Then, we calculated the statistical averages and variances of the corresponding $10^4$ realizations of the canonical estimators $\hat{d}_{\text{GK}}$ and $\hat{d}_{t\text{-me-x}}$, which are shown in figure 3. The continuous lines
are respectively the expected value (116) of the open-close estimator $\hat{d}_{\text{real}}$ given by expression (114) and the fitted curves

$$E[\hat{d}_{\text{est}}] = a_{\text{est}} \gamma^2 + b_{\text{est}}$$

for the averaged values of the canonical estimators $\hat{d}_{\text{GK}}$ and $\hat{d}_{\text{t-me-x}}$. Their fitted parameters are

$$a_{\text{GK}} \simeq 0.126, \quad a_{\text{t-me-x}} \simeq 0.082, \quad b_{\text{GK}} \simeq b_{\text{t-me-x}} \simeq 1.$$

Figure 4 shows the statistical average of the variances of the canonical estimators $\hat{d}_{\text{GK}}$ and $\hat{d}_{\text{t-me-x}}$. The two horizontal lines indicate the variance values (117). The continuous lines show the fitted curves

$$\text{Var}[\hat{d}_{\text{est}}] = c_{\text{est}} \gamma^2 + d_{\text{est}}$$

of the variances of the canonical estimators $\hat{d}_{\text{GK}}$ and $\hat{d}_{\text{t-me-x}}$. Their parameters are

$$c_{\text{GK}} \simeq 0.089, \quad c_{\text{t-me-x}} \simeq 0.0272, \quad d_{\text{GK}} \simeq 0.271, \quad b_{\text{t-me-x}} \simeq 0.170.$$

### 5.4 Construction of general variance estimators

We have introduced the canonical estimator $\hat{d}_{\text{t-me-x}}$ given by expression (109) that includes the information on the value of the time $t_{\text{last}} = \sup \{t_L, t_H \}$ defined as the larger of the two times at which occur the high and low values of the canonical bridge. It seems that the canonical estimator

$$\hat{d}_{\text{tt-me-x}}(\Theta, \Upsilon, t_{\text{high}}, t_{\text{low}}; \gamma) = R^2 s_{\text{tt-me-x}}(\Theta, \Upsilon, t_{\text{high}}, t_{\text{low}}; \gamma),$$

$$s_{\text{tt-me-x}}(\theta, v, t_1, t_2; \gamma) = \frac{1}{\mathcal{E}_{\text{tt-me-x}}(\gamma)} \frac{\alpha(\theta, v, t_1, t_2; 2; \gamma)}{\alpha(\theta, v, t_1, t_2; 4; \gamma)},$$

(118)

taking into account both high’s and low’s and their corresponding occurrence times ($t_{\text{high}} : H = Y(t_{\text{high}}), t_{\text{low}} : L = Y(t_{\text{low}})$) is even more efficient than the estimator (109). In expression (118), we have used the notation

$$\alpha(\theta, v, t_1, t_2; \lambda, \gamma) = \int_0^\infty r^{\lambda+2} \varphi(r \cos v \cos \theta, r \cos v \sin \theta, r \sin v, t_1, t_2; \gamma)dr,$$

(119)

where $\varphi(h, \ell, x, t_1, t_2; \gamma)$ is the joint pdf of the high-low-close-$t_{\text{high}}$-$t_{\text{low}}$ random variables.

We have not explored the statistical properties of the estimator (118) because we have made not yet the effort of deriving the exact analytical
expression of \( \phi(h, \ell, x, t_1, t_2; \gamma) \). We can however construct the function \( \alpha_{\text{\text{119}}} \) using statistical averaging:

\[
\alpha(\theta, \nu, t_1, t_2; \lambda, \gamma) \cos \nu d\theta dt_1 dt_2 \simeq \frac{1}{K} \sum_{k=1}^{K} R^k I(\Upsilon_k, \Theta_k, t_{\text{high}}, t_{\text{low}}) .
\]

(120)

In this expression, the values \( \{\Upsilon_k, \Theta_k, t_{\text{high}}, t_{\text{low}}\} \) are parameters of numerically simulated \( k \)-th sample of the canonical Wiener process with drift \( X(t, \gamma) \) (9), and \( I \) is the indicator of the set

\[
(\nu, \nu + d\nu) \times (\theta, \theta + d\theta) \times (t_1, t_1 + dt_1) \times (t_2, t_2 + dt_2) .
\]

We would like to point out that it is possible to construct the function \( \alpha \) by an analogous statistical treatment for more general log-price process that extend the Wiener process with drift to include more adequately the microstructure noise, the presence of heavy tails of returns and other stylized facts that can be found for various financial assets. In others words, relations such as (120) offer the possibility of constructing novel most efficient variance estimators of the form (118), extending the standard approach of econometricians looking for new constructions of efficient volatility estimators. The requisite is to be able to simulate numerically the underlying stochastic process that is representing a given financial asset dynamics. Then, the use of statistical averaging, similar to (120), will enable the construction of high-frequency realized estimators that use the most efficient estimators described above as elementary “bricks”.

6 Conclusion

We have introduced a variety of integrated variance estimators, based on the open-high-low values of the bridges \( Y(t, S_i) \) (22), and close values \( X_i \) (1) of the underlying log-price process \( X(t) \). The main peculiarity of some of the introduced estimators is to take into account not only the high and low values but additionally their occurrence time. This last piece of information lead to estimators that are even more efficient. We discussed quantitatively the statistical properties of the estimators for the class off Itô model for the log-price stochastic process.

Our work opens the road to the construction of novel types of integrated variance estimators of log-price processes of real financial markets that take into account the microstructure noise, heavy power tails of returns, and chaotic jumps.
Acknowledgements: We are grateful to Fulvio Corsi for valuable discussions of some aspects of this paper.

A Basic properties of the canonical bridge

A.1 Symmetry properties

The canonical bridge $Y(t)$ exhibits the following time reversibility and reflection properties

$$Y(t) \sim Y(1-t), \quad Y(t) \sim -Y(t). \tag{A.1}$$

Some statistical consequences of these symmetry properties are as follows. Let

$$H = \sup_{t \in (0,1)} Y(t), \quad L = \inf_{t \in (0,1)} Y(t), \tag{A.2}$$

be the high and low values of the canonical bridge, while $t_{\text{high}}$ and $t_{\text{low}}$ are their corresponding occurrence times:

$$t_{\text{high}} : H = Y(t_{\text{high}}), \quad t_{\text{low}} : L = Y(t_{\text{low}}). \tag{A.3}$$

Consider the cumulative distribution (cdf)

$$\Phi_{\text{high}}(t) = \Pr\{t_{\text{high}} < t\}$$

of the occurrence time $t_{\text{high}}$ of the high value of the canonical bridge. Due to the reversibility property (A.1), one has

$$\Pr\{t_{\text{high}} < t\} = \Pr\{t_{\text{high}} > 1-t\} \quad \Rightarrow \quad \Phi_{\text{high}}(t) + \Phi_{\text{high}}(1-t) = 1. \tag{A.4}$$

Accordingly, the pdf of $t_{\text{high}}$

$$\varphi_{\text{high}}(t) := \frac{d\Phi_{\text{high}}(t)}{dt}$$

presents the symmetry

$$\varphi_{\text{high}}(t) = \varphi_{\text{high}}(1-t). \tag{A.5}$$

Due to the reversibility property of the canonical bridge, the cdf $\Phi_{\text{low}}(t)$ of $t_{\text{low}}$ [A.3] coincides with the cdf of $t_{\text{high}}$:

$$\Phi_{\text{low}}(t) = \Phi_{\text{high}}(t) \quad \Rightarrow \quad \varphi_{\text{low}}(t) = \varphi_{\text{high}}(t) = \varphi_{\text{high}}(1-t).$$
A.2 Interplay between bridge and Wiener processes

We will need below the well-known identity in law for the canonical bridge

\[ Y(t) \sim Y(t) := (1-t)W\left(\frac{t}{1-t}\right). \]

Using the change of time variable

\[ \tau = \frac{t}{1-t} \quad \iff \quad t = \frac{\tau}{1+\tau} \]

and the scaling properties of the Wiener process, we can replace the compounded process

\[ Y(t(\tau)) = Y\left(\frac{\tau}{1+\tau}\right) \]

by the more convenient process, which is identical in law and reads

\[ Y(t(\tau)) \sim Z(\tau) = \frac{1}{1+\tau}W(\tau). \quad (A.6) \]

In turn, the following identity in law holds

\[ Y(t) \sim Z(\tau(t)) = Z\left(\frac{t}{1-t}\right). \quad (A.7) \]

B Joint pdf of the high value and its occurrence time

B.1 Reflection method

Let us consider the function \( f(\omega; \tau, h) \) such that

\[ \Pr\{W(\tau) \in (\omega, \omega + d\omega) \cap W(\tau') < h(1+\tau') : \tau' \in (0, \tau)\} = f(\omega; \tau, h)d\omega. \quad (B.1) \]

This function \( f(\omega; \tau, h) \) satisfies to the following diffusion equation

\[ \frac{\partial f}{\partial \tau} = \frac{1}{2}\frac{\partial^2 f}{\partial \omega^2} \quad (B.2) \]

with initial and absorbing boundary conditions

\[ f(\omega; \tau = 0, h) = \delta(\omega), \quad f(\omega = h + h\tau; \tau, h) = 0. \quad (B.3) \]
We solve the initial-boundary problem (B.2) with (B.3) using the reflection method, which amounts to searching for a solution of the form

\[ f(\omega; \tau, h) = \frac{1}{\sqrt{2\pi \tau}} \left[ \exp\left(-\frac{\omega^2}{2\tau}\right) - A \exp\left(-\frac{(\omega - 2h)^2}{2\tau}\right) \right], \]

where the factor \( A \) is defined from the absorbing boundary condition (B.3), i.e.

\[ \exp\left(-\frac{(h + h\tau)^2}{2\tau}\right) = A \exp\left(-\frac{(h - h\tau)^2}{2\tau}\right) \Rightarrow A = e^{-2h^2}. \]

We thus obtain

\[ f(\omega; \tau, h) = \frac{1}{\sqrt{2\pi \tau}} \left[ \exp\left(-\frac{\omega^2}{2\tau}\right) - \exp\left(-2h^2 - \frac{(\omega - 2h)^2}{2\tau}\right) \right]. \]

(B.4)

### B.2 Pdf of the maximal value of the canonical bridge

In view of (B.1) and (A.6), the joint pdf of \( W(\tau) \) and of the high value \( H(\tau) = \sup_{\tau' \in (0, \tau)} Z(\tau') \) (B.5) of the stochastic process \( Z(\tau') \) within the interval \( \tau' \in (0, \tau) \) is equal to

\[ Q(\omega, h; \tau) = \frac{\partial f(\omega; \tau, h)}{\partial h}. \]

Substituting in the above equation the expression (B.4) yields

\[ Q(\omega, h; \tau) = \frac{1}{\tau} \sqrt{\frac{2}{\pi \tau}} (2h(1 + \tau) - \omega) \exp\left[-2h^2 - \frac{(\omega - 2h)^2}{2\tau}\right], \quad \omega < h(1 + \tau), \quad h > 0. \]

(B.6)

In particular, the pdf of the high value \( H(\tau) \) (B.5)

\[ Q(h; \tau) = \int_{-\infty}^{h(1+\tau)} Q(\omega, h, \tau) d\omega \]

is equal to

\[ Q(h; \tau) = \sqrt{\frac{2}{\pi \tau}} \exp\left(-\frac{h^2(1 + \tau)}{2\tau}\right) + 2he^{-2h^2} \text{erfc}\left(\frac{h(1 - \tau)}{\sqrt{2\tau}}\right). \]
Using the identity in law (A.7), the pdf $Q_{\text{high}}(h; t)$ of the high value

$$H(t) = \sup_{t^\prime \in (0, t)} Y(t^\prime)$$

is equal to

$$Q_{\text{high}}(h; t) = Q \left( h; \frac{t}{1-t} \right). \quad \text{(B.7)}$$

In particular, the pdf’s of the high values $H$ (B.5) and $H$ (A.2) are the same and equal to

$$\varphi_{\text{high}}(h) = \lim_{\tau \to \infty} Q(h; \tau) = 4he^{-2h^2}. \quad \text{(B.8)}$$

### B.3 Pdf of the high value of the bridge and of its occurrence value

In order to derive the joint pdf of the maximal value $H$ (A.2) and of the occurrence time $t_{\text{high}}$ (A.3), we first consider the related joint pdf of the high value $H$ (B.5) of the auxiliary process $Z(\tau)$ (A.6) and of its occurrence time $\tau_{\text{high}} : H = Z(\tau_{\text{high}})$.

The function $F(h, \tau)$ that defines the probability

$$F(h, \tau)dh = \Pr \{ H \in (h, h + dh), \tau_{\text{high}} < \tau \}.$$

is given by

$$F(h, \tau) = \int_{-\infty}^{h(1+\tau)} Q(\omega, h; \tau)P(\omega, h, \tau) d\omega, \quad \text{(B.9)}$$

where $Q(\omega, h; \tau)$ is the joint pdf of $W(\tau)$ and $H(\tau)$, given by equality (B.6), while

$$P(\omega, h, \tau, \theta) = \lim_{\theta \to \infty} P(\omega, h, \tau, \theta),$$

$$P(\omega, h, \tau, \theta) = \Pr \{ W(\tau^\prime|\tau, \omega) < h(1 + \tau^\prime) : \tau^\prime \in (\tau, \tau + \theta) \}. \quad \text{(B.10)}$$

Here, $W(\tau^\prime|\tau, \omega)$ is the conditioned Wiener process that takes the value $\omega$ at $\tau^\prime = \tau$. Due to the identity in law (A.6), $P(\omega, h, \tau)$ is equal to the probability that the following inequality holds

$$Z(\tau^\prime|\tau, \omega) < h, \quad \tau^\prime \in (\tau, \infty),$$

where $Z(\tau^\prime|\tau, \omega)$ is the conditioned stochastic process $Z(\tau^\prime)$, which is equal to $\omega/(1 + \tau)$ at $\tau^\prime = \tau$. 

31
The probability $P(\omega, h, \tau, \theta)$ (B.10) is given by

$$P(\omega, h, \tau, \theta) = \int_{-\infty}^{h(1+\tau+\theta)} f(x; \omega, h, \tau, \theta) dx, \quad (B.11)$$

where the pdf $f(x; \omega, h, \tau, \theta)$ satisfies the initial-boundary value problem

$$\frac{\partial f}{\partial \theta} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2},$$

$$f(x; \omega, h, \tau, \theta = 0) = \delta(x - \omega), \quad f(x = h(1 + \tau + \theta); \omega, h, \tau, \theta) = 0.$$ 

Its solution, obtained by the reflection method, is

$$f(x; \omega, h, \tau, \theta) = \frac{1}{\sqrt{2\pi\theta}} \left[ \exp \left( \frac{-(x - \omega)^2}{2\theta} \right) - \exp \left( -2h(h(1 + \tau) - \omega) - \frac{(x + \omega - 2h(1 + \tau))^2}{2\theta} \right) \right].$$

Substituting this last expression into (B.11) yields

$$P(\omega, h, \tau, \theta) =$$

\[
\frac{1}{2} \left[ \text{erfc} \left( \frac{\omega - h(1 + \tau + \theta)}{\sqrt{2\theta}} \right) - e^{-2h(h(1+\tau) - \omega)} \text{erfc} \left( \frac{h(1 + \tau - \theta) - \omega}{\sqrt{2\theta}} \right) \right].
\]

In particular, in the limiting case $\theta \to \infty$, one has

$$P(\omega, h, \tau) = 1 - e^{-2h(h(1+\tau) - \omega)}. \quad (B.12)$$

Substituting $Q(\omega, h; \tau)$ (B.6) and $P(\omega, h, \tau)$ (B.12) into (B.9), after integration, we obtain

$$F(h, \tau) = 2he^{-2h^2} \text{erfc} \left( \frac{h(1 - \tau)}{\sqrt{2\tau}} \right). \quad (B.13)$$

Consider now the probability

$$\Phi_{\text{high}}(h, t) dh = \Pr\{H \in (h, h + dh), t_{\text{high}} < t\}.$$ 

Due to the identity in law (A.7), $\Phi_{\text{high}}(h, t)$ is equal to

$$\Phi_{\text{high}}(h, t) = F \left( h, \frac{t}{1-t} \right) = 2he^{-2h^2} \text{erfc} \left( \frac{h(1 - 2t)}{\sqrt{2t(1-t)}} \right). \quad (B.14)$$
The integration over $h \in (0, \infty)$ gives the cumulative distribution function (cdf) of the random occurrence times $t_{\text{high}}$ (A.3):

$$
\Phi_{\text{high}}(t) = \Pr\{t_{\text{high}} < t\} = \int_0^\infty \Phi(h, t) dh = t, \quad t \in (0, 1).
$$

This means that the occurrence time $t_{\text{high}}$ of the high value of the canonical bridge is uniformly distributed. The above cdf satisfies the symmetry property (A.4). The corresponding pdf $\varphi_{\text{high}}(t) = 1$ satisfies obviously to symmetry property (A.5).

The sought joint pdf of the high value $H$ of canonical bridge $Y(t)$ and of its corresponding occurrence time $t_{\text{high}}$ is

$$
\varphi_{\text{high}}(h, t) = \frac{\partial \Phi_{\text{high}}(h, t)}{\partial t}. \quad \text{(B.15)}
$$

Substituting here $\Phi_{\text{high}}(h, t)$ (B.14) yields

$$
\varphi_{\text{high}}(h, t) = \sqrt{\frac{2}{\pi}} \frac{h^2}{\sqrt{t^3(1-t)^3}} \exp\left(-\frac{h^2}{2t(1-t)}\right). \quad \text{(B.16)}
$$

### C Statistics of the high, low and occurrence time of the last extremum of the canonical bridge

#### C.1 Statistical description of the joint pdf of the high, low and occurrence time of the last extremum

The occurrence times of the first and last absolute extremes (A.3) of canonical bridge $Y(t)$ are formally defined as

$$
t_{\text{first}} = \inf\{t_L, t_H\}, \quad t_{\text{last}} = \sup\{t_L, t_H\}. \quad \text{(C.1)}
$$

The joint pdf of the high $H$ and low $L$ (A.2) together with the cdf of the occurrence time $t_{\text{last}}$ is given by

$$
\Phi_{\text{last}}(h, \ell, t) dh d\ell = \Pr\{H \in (h, h + dh) \cap L \in (\ell, \ell + d\ell) \cap t_{\text{last}} < t\}. \quad \text{(C.2)}
$$

We derive the function $\Phi_{\text{last}}(h, \ell, t)$ by using a natural generalization of the reasoning presented in Appendix B that led to the joint pdf $\Phi_{\text{high}}(h, t)$ (B.14) of the high value $H$ and of the cdf of the occurrence time $t_{\text{high}}$. Namely, we calculate first the probability

$$
F(h, \ell, \tau) dh d\ell = \Pr\{H \in (h, h + dh), L \in (\ell, \ell + d\ell), \tau_{\text{last}} < \tau\}. \quad \text{(C.3)}
$$
where
\[ H = \sup_{\tau \in (0, \infty)} Z(\tau), \quad L = \inf_{\tau \in (0, \infty)} Z(\tau), \]
\[ \tau_{\text{last}} = \sup \{ \tau_{\text{low}}, \tau_{\text{high}} \}, \quad \tau_{\text{low}} : L = Z(\tau_{\text{low}}), \quad \tau_{\text{high}} : H = Z(\tau_{\text{high}}). \]

Analogously to (B.9), \( F(h, \ell, \tau) \) is equal to
\[ F(h, \ell, \tau) = \int_{\ell(1+\tau)}^{h(1+\tau)} Q(\omega, h, \ell, \tau) P(\omega, h, \ell, \tau) d\omega, \quad (C.4) \]
where
\[ Q(\omega, h, \ell, \tau) = -\frac{\partial^2 f(\omega; h, \ell, \tau)}{\partial h \partial \ell}, \quad (C.5) \]
and the pdf \( f(\omega; h, \ell, \tau) \) satisfies the initial-boundary problem
\[ \frac{\partial f}{\partial \tau} = \frac{1}{2} \frac{\partial^2 f}{\partial \omega^2}, \quad f(\omega; h, \ell, \tau = 0) = \delta(\omega), \quad (C.6) \]
\[ f(\omega = h(1+\tau); h, \ell, \tau) = 0, \quad f(\omega = \ell(1+\tau); h, \ell, \tau) = 0, \quad \tau > 0. \]

Similarly to \( P(\omega, h, \tau) \) (B.10), the probability \( P(\omega, h, \ell, \tau) \) is given by
\[ P(\omega, h, \ell, \tau) = \lim_{\theta \to \infty} P(\omega, h, \ell, \tau, \theta), \quad P(\omega, h, \ell, \tau, \theta) = \Pr \{ \ell(1+\tau') < B\tau'|\tau, \omega) < h(1+\tau') : \tau' \in (\tau, \tau+\theta) \}. \]

Analogously to (B.11), the last probability \( P(\omega, h, \ell, \tau, \theta) \) is equal to
\[ P(\omega, h, \ell, \tau, \theta) = \int_{\ell(1+\tau+\theta)}^{h(1+\tau+\theta)} f(x; \omega, h, \ell, \tau, \theta) dx, \quad (C.7) \]
where \( f(x; \omega, h, \ell, \tau, \theta) \) is the solution of the initial-boundary problem
\[ \frac{\partial f}{\partial \theta} = \frac{1}{2} \frac{\partial^2 f}{\partial x^2}, \quad f(x; \omega, h, \ell, \tau, \theta = 0) = \delta(x - \omega), \quad (C.8) \]
\[ f(x = h(1+\tau + \theta); \omega, h, \ell, \tau, \theta) = 0, \quad f(x = \ell(1+\tau + \theta); \omega, h, \ell, \tau, \theta) = 0. \]

Knowing the function \( F(h, \ell, \tau) \) defined by equality (C.3), one can find the sought function \( \Phi_{\text{last}}(h, \ell, t) \) (C.2) using the following relation
\[ \Phi_{\text{last}}(h, \ell, t) = F \left( h, \ell, \frac{t}{1-t} \right), \quad (C.9) \]
which is analogous to (B.14). In turn, one can find the joint pdf of the high \( H \), low \( L \) values (A.2) and occurrence time of the last absolute extremum \( t_{\text{last}} \) (C.1) of the canonical bridge \( Y(t) \) using, analogously to (B.15), the relation
\[ \varphi_{\text{last}}(h, \ell, t) = \frac{\partial \Phi_{\text{last}}(h, \ell, t)}{\partial t}. \quad (C.10) \]
C.2 Solutions of boundary-value problems

Using the initial-boundary problem (C.6) with the reflection method, we obtain

\[
f(\omega; h, \ell, \tau) = \sum_{m=-\infty}^{\infty} \left[ e^{-2(h-\ell)^2m^2 g(\omega + 2(h-\ell)m; \tau)} - e^{-2((h-\ell)m+h)^2 g(\omega - 2((h-\ell)m); \tau)} \right],
\]

where

\[
g(\omega; \tau) = \frac{1}{\sqrt{2\pi \tau}} \exp \left( -\frac{\omega^2}{2\tau} \right).
\]

In turn, the solution of the initial-boundary problem (C.8) is given by

\[
f(x; \omega, h, \ell, \tau, \theta) = \sum_{m=-\infty}^{\infty} \left[ e^{-2(h-\ell)^2m^2(1+\tau)} - e^{-2((h-\ell)m+h)^2} g(y - \omega + 2m(h-\ell)(1+\tau); \theta) - e^{-2((h-\ell)m+h)^2(1+\tau)+2\omega((h-\ell)m+h)} g(y + \omega - 2((h-\ell)m+h)(1+\tau); \theta) \right].
\]

After substituting \( f(\omega; h, \ell, \tau) \) (C.11) into (C.5), we obtain

\[
Q(\omega, h, \ell, \tau) = 4 \tau^2 \sum_{m=-\infty}^{\infty} m \left[ e^{-2(h-\ell)^2m^2(1+\tau)} - e^{-2((h-\ell)m+h)^2} \right].
\]

After substituting \( f(x; \omega, h, \ell, \tau, \theta) \) (C.12) into (C.7), and taking the limit \( \theta \to \infty \), we obtain

\[
P(\omega, h, \ell, \tau) = \sum_{m=-\infty}^{\infty} \left[ e^{-2(h-\ell)^2(1+\tau)m^2+2(h-\ell)m\omega} - e^{-2(h+(h-\ell)m)^2(1+\tau)+2(h+(h-\ell)m)\omega} \right].
\]

After substituting \( Q(\omega, h, \ell, \tau) \) (C.13) and \( P(\omega, h, \ell, \tau) \) (C.14) into (C.4), we obtain the explicit expression for \( F(h, \ell, \tau) \). Substituting it into (C.9)
and using relation (C.9), we obtain the pdf of the high $H$, low $L$ values and occurrence time $t_{\text{last}}$ of the last extremum under the form

$$
\varphi_{\text{last}}(h, \ell, t) = \sum_{m=-\infty}^{\infty} \sum_{n=-\infty}^{\infty} \left( m^2 \left[ g(h, t, 2(h - \ell)m, 2(h - \ell)n) - g(\ell, t, 2(h - \ell)m, 2(h - \ell)n) \right] - m(m + 1) \left[ g(h, t, -2(h - \ell)m, 2(h - \ell)n) - g(\ell, t, -2(h - \ell)m, 2(h - \ell)n) \right] -
\right)
\times
\left[ g(h, t, 2(h + (h - \ell)n) + g(\ell, t, 2(h + (h - \ell)n)) \right] -
\times
\left[ g(h, t, -2(h + (h - \ell)n), 2(h + (h - \ell)n)) + g(\ell, t, -2(h + (h - \ell)n), 2(h + (h - \ell)n)) \right]
\right),
$$

$$
g(y, t, a, c) = \sqrt{\frac{2}{\pi(1-t)^3}} \exp \left( \frac{-(a+y)^2 - (a+c)(a-c+2y)t}{2t(1-t)} \right) \times
\left[ (a+y)^3 - (a+y)(3 + (a+y)(a-c+2y))t + (3a-c+4y)t^2 \right].
$$

(C.15)

### C.3 Function $\alpha_{\text{last}}(\theta, t; \lambda)$

Some of the most efficient estimators introduced in this paper are defined through the function

$$
\alpha_{\text{last}}(\theta, t; \lambda) = \int_0^\infty r^{\lambda+1} \varphi_{\text{last}}(r\cos \theta, r\sin \theta, t) dr,
$$

(C.16)
which is analogous to (73). The calculation of the integral (C.16) yields

\[
\alpha_{\text{last}}(\theta, t; \lambda) = -\frac{1}{\sqrt{8\pi(1-t)^3t^3}} \Gamma\left(\frac{3+\lambda}{2}\right) \sum_{m,n=-\infty}^{\infty} (m^2 \times \\
[\beta(co, t, 2sc \cdot m, 2sc \cdot n; \lambda) - \beta(si, t, 2sc \cdot m, 2sc \cdot n; \lambda) - \\
\beta(co, t, 2sc \cdot m, 2(co + sc \cdot n); \lambda) + \\
\beta(si, t, 2sc \cdot m, 2(co + sc \cdot n); \lambda)] - \\
m(m+1) \left[ \beta(co, t, -2(co + sc \cdot m), 2sc \cdot n; \lambda) - \\
\beta(si, t, -2(si + sc \cdot m), 2sc \cdot n; \lambda) - \\
\beta(co, t, -2(co + sc \cdot m), 2(co + sc \cdot n); \lambda) + \\
\beta(si, t, -2(co + sc \cdot m), 2(co + sc \cdot n); \lambda) \right],
\]

where

\[
co = \cos \theta, \quad si = \sin \theta, \quad sc = co - si,
\]

\[
\beta(y, t, a, c; \lambda) = \left[ (a+y)^2[a+y-(a-c+2y)t](3+\lambda) + \\
\delta t[(6a-2c+8y)t-6(a+y)] \right],
\]

\[
\delta = \delta(y, t, a, c) = \frac{(a+y)^2 - (a+c)(a-c+2y)t}{2t(1-t)}.
\]

C.4 Function \( \alpha(\theta, \upsilon, t; \lambda) \)

Consider the function

\[
\alpha(\theta, \upsilon, t; \lambda) = \int_0^{\infty} r^{\lambda+2} \varphi_{\text{last}}(r \cos \upsilon \cos \theta, r \cos \upsilon \sin \theta, r \sin \upsilon, t; \gamma = 0) dr,
\]

(C.18)

that enters into the definition of the canonical estimator (109) in the case of zero drift \( \gamma = 0 \). Using expression (112) for the pdf \( \varphi_{\text{last}}(h, \ell, x, t; \gamma) \), we
obtain after calculations the following expression

\[
\alpha(\theta, \nu, t; \lambda) = \frac{\Gamma\left(\frac{4+\lambda}{2}\right)}{4\pi\sqrt{t'(1-t)^3}} \sum_{m,n=-\infty}^{\infty} \left( m^2 \times \right.
\]

\[
\left[ \beta'(x, co, t, sc \cdot m, sc \cdot n; \lambda) - \beta'(x, si, t, sc \cdot m, sc \cdot n; \lambda) - \\
\beta'(x, co, t, cc + sc \cdot n; \lambda) + \beta'(x, si, t, cc + sc \cdot n; \lambda) \right] - \\
m(m+1) \left[ \beta'(x, co, t, -cc - sc \cdot m, sc \cdot n; \lambda) - \\
\beta'(x, si, t, -cc - sc \cdot m, sc \cdot n; \lambda) - \\
\beta'(x, co, t, -cc - sc \cdot m, cc + sc \cdot n; \lambda) + \\
\beta'(x, si, t, -cc - sc \cdot m, cc + sc \cdot n; \lambda) \right].
\]

(C.19)

which is analogous to (C.17). Here, we have set

\[
x = \sin \nu, \quad co = \cos \theta \cos \nu, \quad si = \sin \theta \cos \nu, \\
cc = 2 \cos \theta \cos \nu, \quad sc = 2(\cos \theta - \sin \theta) \cos \nu, \\
\beta'(x, y, t, a, c, \lambda) = \left[ r(4+\lambda)(a+y-(a-c+2y)t)+ \\
\delta t((6a-2c+8y)t-6(a+y)) \right] \delta^{-\lambda/2}, \\
\delta = \frac{r-(a+c)(a-c+2y)t}{2t(1-t)} + \frac{x^2}{2}, \quad r = (a+y)^2.
\]
References

Aït-Sahalia, Y., P.A. Mykland, and L. Zhang (2005). How often to sample a continuous-time process in the presence of market microstructure noise. *Review of Financial Studies* 18, 351-416.

Andersen, T. G., T. Bollerslev, F. X. Diebolt and P. Labys (2003). Modeling and Forecasting Realized Volatility. *Econometrica* 71, 529-626.

Garman, M. and M. J. Klass (1980). On the Estimation of Security Price Volatilities From Historical Data. *Journal of Business* 53, 67-78.

Jeanblanc J. & M. Yor, and M. Chesney (2009). *Mathematical Methods for Financial Markets*. Springer.

Parkinson, M. (1980). The Extreme Value Method for Estimating the Variance of the Rate of Return. *Journal of Business* 53, 61-65.

Saichev, A., D. Sornette, V. Filimonov F. Corsi (2009). Homogeneous Volatility Bridge Estimators. *ETH Zurich working paper*, http://ssrn.com/abstract=1523225

Saichev A., Y. Malevergne, D. Sornette (2010) *Theory of Zipf’s Law and Beyond* (Lecture Notes in Economics and Mathematical Systems), Springer.

Zhang, L., Mykland, P.A. and At-Sahalia, Y. (2005). A tale of two time scales: determining integrated volatility with noisy high-frequency data. *Journal of the American Statistical Association* 100, 1394-1411.
Fig. 1: 5000 realizations of the open-close estimator \( \hat{d}_{\text{real}} \), of the G&K estimator \( \hat{d}_{\text{GK}} \), and of the most efficient estimator \( \hat{d}_{\text{me-x}} \).
Fig. 2: Moving averages of the open-close (top panel), G&K (middle panel) and time-OHLC \cite{109} (lower panel) estimators over respective windows sizes of 30 samples for the top panel and 10 samples for the two other panels. As explained in the text, this moving average mimicks the normalized estimators of the integrated variance in the case where all instantaneous variances are the same ($\sigma_i^2 = \sigma^2 = \text{const}$).
Fig. 3: Top to bottom, $\gamma$-dependence of the expected values of the open-close $\hat{d}_{\text{real}}$ (114), G&K $\hat{d}_{\text{GK}}$ (115) and most efficient $\hat{d}_{\text{t-me}}$ (109) canonical estimators. The horizontal line is the expected value of the canonical estimators $\hat{d}_{\text{bPark}}$ and $\hat{d}_{\text{t-me}}$.
Fig. 4: $\gamma$-dependence of the statistical average of the variances of the canonical estimators $d_{GK}$ (upper open circles) and $d_{t\text{-}me\text{-}x}$ (lower open circles). The two horizontal lines are the variances (117) of the canonical estimators $d_{bPark}$ (top) and $d_{t\text{-}me}$ (bottom), respectively.