The geometry of constant mean curvature surfaces
in \( \mathbb{R}^3 \)

William H. Meeks III* Giuseppe Tinaglia†

Abstract
We derive intrinsic curvature and radius estimates for compact disks embedded in \( \mathbb{R}^3 \) with nonzero constant mean curvature and apply these estimates to study the global geometry of complete surfaces embedded in \( \mathbb{R}^3 \) with nonzero constant mean curvature.

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1 Introduction.
A longstanding problem in classical surface theory is to classify the complete, simply-connected surfaces embedded in \( \mathbb{R}^3 \) with constant mean curvature. In the case the surface is simply-connected and compact, this classification follows by work of either Hopf [15] in 1951 or of Alexandrov [1] in 1956, who gave different proofs that a round sphere is the only possibility. In this paper we prove that if a complete, embedded simply-connected surface has nonzero constant mean curvature, then it is compact.

**Theorem 1.1.** Complete, simply-connected surfaces embedded in \( \mathbb{R}^3 \) with nonzero constant mean curvature are compact, and thus are round spheres.

Theorem 1.1, together with results of Colding and Minicozzi [11] and Meeks and Rosenberg [22] that show that the complete, simply-connected minimal surfaces embedded in \( \mathbb{R}^3 \) are planes and helicoids, finishes the classification of complete simply-connected surfaces embedded in \( \mathbb{R}^3 \) with constant mean curvature.

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In the first section of this paper we observe how combining results in our previous papers, namely [25, 26, 30], leads to the following intrinsic radius and curvature estimates for embedded disks in $\mathbb{R}^3$ with nonzero constant mean curvature, where the radius of a compact Riemannian surface with boundary is the maximum intrinsic distance of points in the surface to its boundary.

**Theorem 1.2** (Radius Estimates). There exists an $R \geq \pi$ such that any compact disk embedded in $\mathbb{R}^3$ of constant mean curvature $H > 0$ has radius less than $\frac{R}{2}$.

**Theorem 1.3** (Curvature Estimates). Given $\delta, \mathcal{H} > 0$, there exists a $K(\delta, \mathcal{H}) \geq \sqrt{2H}$ such that any compact disk $M$ embedded in $\mathbb{R}^3$ with constant mean curvature $H \geq \mathcal{H}$ satisfies

$$\sup_{\{p \in M \mid d_M(p, \partial M) \geq \delta\}} |A_M| \leq K(\delta, \mathcal{H}),$$

where $|A_M|$ is the norm of the second fundamental form and $d_M$ is the intrinsic distance function of $M$.

The radius estimate in Theorem 1.2 implies that a complete, simply-connected surface $M$ embedded in $\mathbb{R}^3$ with nonzero constant mean curvature is compact. In this way Theorem 1.1 follows from Theorem 1.2.

We wish to emphasize to the reader that the curvature estimates for embedded constant mean curvature disks given in Theorem 1.3 depend only on the lower positive bound $\mathcal{H}$ for their mean curvature. Previous important examples of curvature estimates for constant mean curvature surfaces, assuming certain geometric conditions, can be found in the literature; see for instance [3, 4, 6, 10, 11, 35, 36, 37, 38, 40, 41].

Our investigation is inspired by the pioneering work of Colding and Minicozzi in the minimal case [7, 8, 9, 10]; however in the constant positive mean curvature setting this description leads to the existence of radius and curvature estimates. Since the plane and the helicoid are complete simply-connected minimal surfaces properly embedded in $\mathbb{R}^3$, a radius estimate does not hold in the minimal case. Moreover rescalings of a helicoid give rise to a sequence of embedded minimal disks with arbitrarily large norms of their second fundamental forms at points that can be arbitrarily far from their boundary curves; therefore in the minimal setting, curvature estimates also do not hold.

For clarity of exposition, we will call an oriented surface $M$ immersed in $\mathbb{R}^3$ an $H$-surface if it is embedded, connected and it has positive constant mean curvature $H$. We will call an $H$-surface an $H$-disk if the $H$-surface is homeomorphic to a closed disk in the Euclidean plane.

The next corollary is an immediate consequence of Theorem 1.3.
Corollary 1.4. If $M$ is a complete $H$-surface with positive injectivity radius $r_0$, then

$$\sup_M |A_M| \leq K(r_0, H).$$

As complete $H$-surfaces of bounded norm of the second fundamental form are properly embedded in $\mathbb{R}^3$ by Theorem 6.1 in [23], Corollary 1.4 implies the next result.

Corollary 1.5. A complete $H$-surface with positive injectivity radius is properly embedded in $\mathbb{R}^3$.

Since there exists an $\varepsilon > 0$ such that for any $C > 0$, every complete immersed surface $\Sigma$ in $\mathbb{R}^3$ with $\sup_{\Sigma} |A_\Sigma| < C$ has injectivity radius greater than $\varepsilon/C$, Corollary 1.4 also demonstrates that a necessary and sufficient condition for an $H$-surface to have bounded norm of the second fundamental form is that it has positive injectivity radius.

Corollary 1.6. A complete $H$-surface has positive injectivity radius if and only if it has bounded norm of the second fundamental form.

In Section 3 we obtain curvature estimates for $H$-surfaces that are annuli; these estimates are analogous to the curvature estimates in Theorem 1.3 for $H$-disks but necessarily must also depend on the flux\(^1\) of a given annulus. We then apply these new curvature estimates to prove the next Theorem 1.7 on the properness of complete $H$-surfaces of finite topology. Earlier as the main result in [11], Colding and Minicozzi proved the similar theorem that complete minimal surfaces of finite topology embedded in $\mathbb{R}^3$ are proper, thereby solving the classical Calabi-Yau problem in the minimal setting.

Theorem 1.7. A complete $H$-surface with smooth compact boundary (possibly empty) and finite topology has bounded norm of the second fundamental form and is properly embedded in $\mathbb{R}^3$.

Theorem 1.7 shows that certain classical results for $H$-surfaces hold when the hypothesis of “properly embedded” is replaced by the weaker hypothesis of “complete and embedded.” For instance, in the seminal paper [17], Korevaar, Kusner and Solomon proved that the ends of a properly embedded $H$-surface of finite topology in $\mathbb{R}^3$ are asymptotic to the ends of surfaces of revolution defined by Delaunay [14] in 1841, and that if such a surface has two ends, then it must be a Delaunay surface. Earlier Meeks [19] proved that a properly embedded $H$-surface of finite topology

\(^1\)See Definitions 3.2 and 3.3 for the definition of this flux.
in $\mathbb{R}^3$ cannot have one end. In particular, this last result together with Theorem 1.7 gives a generalization of Theorem 1.1.

The theory developed in this manuscript also provides key tools for understanding the geometry of $H$-disks in a Riemannian three-manifold, especially in the case that the manifold is locally homogeneous. These generalizations and applications of the work presented here will appear in our forthcoming paper [27]. See [24] for applications of the present paper to obtain area estimates for closed $H$-surfaces of fixed genus embedded in a flat 3-torus; see [12, 13, 34] for examples that demonstrate that Theorem 1.7 does not hold in the hyperbolic 3-space $\mathbb{H}^3$ when $H \in [0, 1)$ and in the Riemannian product $\mathbb{H}^2 \times \mathbb{R}$ when $H \in [0, 1/2)$.

2 The Intrinsic Curvature and Radius Estimates.

In [26] we proved the following extrinsic curvature and radius estimates for compact disks embedded in $\mathbb{R}^3$ with constant mean curvature.

**Theorem 2.1 (Extrinsic Curvature Estimates).** Given $\delta, H > 0$, there exists a constant $K_0(\delta, H)$ such that for any $H$-disk $D$ with $H \geq H$,

$$\sup_{\{p \in D \mid d_{\mathbb{R}^3}(p, \partial D) \geq \delta\}} |A_D| \leq K_0(\delta, H).$$

**Theorem 2.2 (Extrinsic Radius Estimates).** There exists a constant $R_0 \geq \pi$ such that any $H$-disk $D$ has extrinsic radius less than $\frac{R_0}{H}$. In other words, for any point $p \in D$,

$$d_{\mathbb{R}^3}(p, \partial D) < \frac{R_0}{H}.$$

Thus, Theorems 1.2 and 1.3 are immediate consequences of a chord-arc type result from [25], namely Theorem 2.4 below, and Theorems 2.2 and 2.1. Key ingredients in the proof of Theorem 2.4 include results in [30] and the main theorem in [28]. The results in [28] and [30] that are needed to prove Theorem 2.4 below make use of theorems and tools discussed in [26]. The results in [26, 25, 28, 30] are inspired by and generalize the main results of Colding and Minicozzi in [10, 11] for minimal disks to the case of $H$-disks.

**Definition 2.3.** Given a point $p$ on a compact surface $\Sigma \subset \mathbb{R}^3$, $\Sigma(p, R)$ denotes the closure of the component of $\Sigma \cap B(p, R)$ passing through $p$.

**Theorem 2.4 (Weak chord arc property).** There exists a $\delta_1 \in (0, \frac{1}{2})$ such that the following holds.

Let $\Sigma$ be an $H$-disk in $\mathbb{R}^3$. Then for all closed intrinsic balls $\overline{B}_\Sigma(x, R)$ in $\Sigma - \partial \Sigma$. 

\begin{align*}
\text{Proof:} & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & & \end{align*}
1. $\Sigma(x, \delta_1R)$ is a disk with piecewise smooth boundary $\partial \Sigma(x, \delta_1R) \subset \partial B(\delta_1R)$.

2. $\Sigma(x, \delta_1R) \subset B(\Sigma(x, \delta_1R))$.

We begin by applying Theorem 2.4 to prove the intrinsic radius estimate.

**Proof of Theorem 1.2.** Without loss of generality, fix $H = 1$. Arguing by contradiction, if the radius estimates were false then there would exist a sequence of 1-disks containing arbitrarily large geodesic balls with centers at the origin $\vec{0} \in \mathbb{R}^3$. Let $\Sigma_n$ denote the sequence of such 1-disks and let $B_\Sigma(\vec{0}, n) \subset \Sigma_n$ be the sequence of geodesic balls. Theorem 2.4 implies $B_\Sigma(\vec{0}, n) \subset \Sigma_n$ contains a 1-disk centered at $\vec{0}$ of extrinsic radius $\delta_1n$. For $n$ large enough, this contradicts the Extrinsic Radius Estimate and completes the proof of Theorem 1.2.

We next prove the intrinsic curvature estimate.

**Proof of Theorem 1.3.** Let $\varepsilon = \delta_1\delta$, where $\delta_1 \in (0, \frac{1}{2})$ is given in Theorem 2.4 and let $K(\delta, \mathcal{H}) := K_0(\varepsilon, \mathcal{H})$, where $K_0(\varepsilon, \mathcal{H})$ is given in Theorem 2.1. Let $D$ be an $H$-disk with $H \geq \mathcal{H}$ and let $p \in D$ be a point with $d_D(p, \partial D) \geq \delta$. By Theorem 2.4, the closure of the component $E$ of $D \cap B(p, \varepsilon) \subset D$ containing $p$ is an $H$-disk in the interior of $D$ with $\partial E \subset B(p, \varepsilon)$. By Theorem 2.1,

$$|A_E|(p) \leq K_0(\varepsilon, \mathcal{H}) = K(\delta, \mathcal{H}).$$

This completes the proof of Theorem 1.3.

### 3 Curvature estimates for $H$-annuli and properness of $H$-surfaces with finite topology.

A classical conjecture in the global theory of minimal surfaces, first stated by Calabi in 1965 [5] and later revisited by Yau [42, 43], is the following:

**Conjecture 3.1** (Calabi-Yau Conjecture). There do not exist complete immersed minimal surfaces in a bounded domain in $\mathbb{R}^3$.

Based on earlier work of Jorge and Xavier [16], Nadirashvili [33] proved the existence of a complete, bounded, immersed minimal surface in $\mathbb{R}^3$, thereby disproving the above conjecture. In contrast to these results, Colding and Minicozzi proved in [11] that complete, finite topology minimal surfaces *embedded* in $\mathbb{R}^3$ are proper. Thus, the Calabi-Yau conjecture holds in the classical setting of complete, embedded, finite topology minimal surfaces.
In this section we will apply Proposition 3.4 below to obtain Theorem 1.7, a result that generalizes the properness result of Colding and Minicozzi for embedded minimal surfaces of finite topology to the setting of $H$-surfaces. In the proof of this proposition we will need to apply the main theorems in [29], whose proofs depend on the results in the first three sections of the present paper, as well as results in [25, 28, 30].

Recall first the definition of flux of a 1-cycle in an $H$-surface; see for instance [17, 18, 39] for further discussion of this invariant.

**Definition 3.2.** Let $\gamma$ be a piecewise-smooth 1-cycle in an $H$-surface $M$. The flux of $\gamma$ is $\int_{\gamma} (H\gamma + \xi) \times \dot{\gamma}$, where $\xi$ is the unit normal to $M$ along $\gamma$ and $\gamma$ is parameterized by arc length.

The flux is a homological invariant and we say that $M$ has zero flux if the flux of any 1-cycle in $M$ is zero; in particular, since the first homology group of a disk is zero, the flux of an $H$-disk is zero.

**Definition 3.3.** Let $E$ be a compact $H$-annulus. Then the flux $F(E)$ of $E$ is the length of the flux vector of either generator of the first homology group of $E$.

The next proposition implies that given a compact 1-annulus with a fixed positive (or zero) flux, then given $\delta > 0$, the injectivity radius function on this annulus is bounded away from zero at points of distance greater than $\delta$ from its boundary.

**Proposition 3.4.** Given $\rho > 0$ and $\delta \in (0, 1)$ there exists a positive constant $I_0(\rho, \delta)$ such that if $E$ is a compact 1-annulus with $F(E) \geq \rho$ or with $F(E) = 0$, then
\[
\inf_{\{p \in E \mid d_E(p, \partial E) \geq \delta\}} I_E \geq I_0(\rho, \delta),
\]
where $I_E: E \to [0, \infty)$ is the injectivity radius function of $E$.

**Proof.** Arguing by contradiction, suppose there exist a $\rho > 0$ and a sequence $E(n)$ of compact 1-annuli satisfying $F(E(n)) \geq \rho > 0$ or $F(E(n)) = 0$, with injectivity radius functions $I_n: E(n) \to [0, \infty)$ and points $p(n)$ in $\{q \in E(n) \mid d_{E(n)}(q, \partial E(n)) \geq \delta\}$ with
\[
I_n(p(n)) \leq \frac{1}{n}.
\]
We next use the fact that the injectivity radius function on a complete Riemannian manifold with boundary is continuous.

For each $p(n)$ consider a point $q(n) \in \overline{B}_{E(n)}(p(n), \delta/2)$ where the following positive continuous function obtains its maximum value:
\[
f: \overline{B}_{E(n)}(p(n), \delta/2) \to (0, \infty),
\]
Let \( r(n) = \frac{1}{2} d_{E(n)}(q(n), \partial B_{E(n)}(p(n), \delta/2)) \) and note that

\[
\frac{\delta/2}{I_n(q(n))} \geq \frac{2r(n)}{I_n(q(n))} = f(q(n)) \geq f(p(n)) \geq n\delta/2.
\]

Moreover, if \( x \in \overline{B}_{E(n)}(q(n), r(n)) \), then by the triangle inequality,

\[
\frac{r(n)}{I_n(x)} \leq \frac{d_{E(n)}(x, \partial B_{E(n)}(p(n), \delta/2))}{I_n(x)} = f(x) \leq f(q(n)) = \frac{2r(n)}{I_n(q(n))}.
\]

Therefore, for \( n \) large the \( H_n \)-surfaces \( M(n) = \overline{1 - n\delta} [\overline{B}_{E(n)}(q(n), r(n)) - q(n)] \) satisfy the following conditions:

- \( I_{M(n)}(\tilde{0}) = 1 \);
- \( d_{M(n)}(\tilde{0}, \partial M(n)) \geq \frac{n\delta}{4} \);
- \( I_{M(n)}(x) \geq \frac{1}{2} \) for any \( x \in \overline{B}_{M(n)}(\tilde{0}, \frac{n\delta}{4}) \).

By Theorem 3.2 in [25], for any \( k \in \mathbb{N} \), there exists an \( n(k) \in \mathbb{N} \) such that the closure of the component \( \Delta(n(k)) \) of \( M(n(k)) \cap B(k) \) containing the origin is a compact \( H_n(k) \)-surface with boundary in \( \partial B(k) \) and the injectivity radius function of \( \Delta(n(k)) \) restricted to points in \( \Delta(n(k)) \cap B(k - \frac{1}{2}) \) is at least \( \frac{1}{2} \). By Theorem 1.3 of [29], for \( k \) sufficiently large, \( \Delta(n(k)) \) contains a simple closed curve \( \Gamma(n(k)) \) with the length of its nonzero flux vector bounded from above by some constant \( C > 0 \). Since the curves \( \Gamma(n(k)) \) are rescalings of simple closed curves \( \tilde{\Gamma}(n(k)) \subset E(n(k)) \), then the \( \tilde{\Gamma}(n(k)) \) are simple closed curves with nonzero flux. Hence these simple closed curves are generators of the first homology group of the annuli \( E(n(k)) \). This immediately gives a contradiction in the case that \( F(E(n(k))) = 0 \).

If \( F(E(n(k))) \geq \rho > 0 \), we have that

\[
C \geq |F(\Gamma(n(k)))| = |F\left(\frac{1}{I_{n(k)}(q(n(k)))}\tilde{\Gamma}(n(k))\right)| = |F(\tilde{\Gamma}(n(k)))| \geq \frac{F(E(n(k)))}{I_{n(k)}(q(n(k)))} \geq \frac{\rho}{I_{n(k)}(q(n(k)))} \geq \rho n(k).
\]

These inequalities lead to a contradiction for \( n(k) > \frac{C}{\rho} \), which completes the proof of the proposition.
An immediate consequence of Proposition 3.4 and the intrinsic curvature estimates for $H$-disks is the following result.

**Corollary 3.5.** Given $\rho > 0$ and $\delta \in (0, 1)$ there exists a positive constant $A_0(\rho, \delta)$ such that if $E$ is a compact 1-annulus with $F(E) \geq \rho$ or with $F(E) = 0$, then

$$\sup_{\{p \in E \mid d_E(p, \partial E) \geq \delta\}} |A_E| \leq A_0(\rho, \delta).$$

When $M$ has finite topology, the flux of each of its finitely many annular ends is either zero or bounded away from zero by a fixed positive number. Thus, Proposition 3.4 implies that the injectivity radius function of $M$ is positive, and so the norm of its second fundamental is bounded by Theorem 1.3. The next corollary is a consequence of this last property and the fact that a complete embedded $H$-surface of bounded norm of the second fundamental form is properly embedded in $\mathbb{R}^3$; see Theorem 6.1 in [23] or item 1 of Corollary 2.5 in [32] for this properness result.

**Corollary 3.6.** A complete surface $M$ with finite topology embedded in $\mathbb{R}^3$ with nonzero constant mean curvature has bounded norm of the second fundamental form and is properly embedded in $\mathbb{R}^3$.

**Remark 3.7.** With slight modifications, the proof of the above corollary generalizes to the case where the $H$-surface $M$ above is allowed to have smooth compact boundary; for example, see [31] for these types of adaptations. Thus, Theorem 1.7 holds as well.

Corollary 3.6 motivates our conjecture below concerning the properness and at most cubical area growth estimates for complete surfaces $M$ embedded in $\mathbb{R}^3$ with finite genus and constant mean curvature.

**Conjecture 3.8.** A complete surface $M$ of finite genus embedded in $\mathbb{R}^3$ with constant mean curvature has at most cubical area growth in the sense that such an $M$ has area less than $CR^3$ in ambient balls of radius $R$ for some $C$ depending on $M$. In particular every such surface is properly embedded in $\mathbb{R}^3$.

Conjecture 3.8 holds for complete minimal surfaces embedded in $\mathbb{R}^3$ with a countable number of ends and finite genus. This cubical volume growth result follows from the properness of such minimal surfaces (by Meeks, Perez and Ros in [20]), because properly embedded minimal surfaces in $\mathbb{R}^3$ of finite genus have bounded norm of the second fundamental form (by Meeks, Perez and Ros in [21]) and because properly embedded minimal surfaces in $\mathbb{R}^3$ with bounded norm of the second fundamental form have at most cubical volume growth (by Meeks and Rosenberg in [23]).
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