Wave equations on q-Minkowski space

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Abstract

We give a systematic account of the exterior algebra of forms on q-Minkowski space, introducing the q-exterior derivative, q-Hodge star operator, q-coderivative, q-Laplace-Beltrami operator and the q-Lie-derivative. With these tools at hand, we then give a detailed exposition of the q-d’Alembert and q-Maxwell equation. For both equations we present a q-momentum-indexed family of plane wave solutions. We also discuss the gauge freedom of the q-Maxwell field and give a q-spinor analysis of the q-field strength tensor.

1 Introduction

In a previous paper [11], we gave a detailed account of the q-deformation of spacetime and its symmetry group (see [10] for a detailed comparison with the approach of [2, 1, 12]). The physical motivation behind this was to develop a q-regularisation scheme and/or give a toy model for Planck scale corrections to the geometry of spacetime. As a next step we now investigate wave equations on this non-commutative spacetime.

The key idea in [8, 11] was that q-Minkowski space should be given by 2 × 2 braided Hermitean matrices, which were introduced by S. Majid in [8] as a non-commutative deformation of the algebra of complex-valued polynomial functions on the space of ordinary Hermitean matrices, i.e. on Minkowski space. Braided matrices have a central and grouplike element, the so-called braided determinant, which plays the rôle of a q-norm and which determines a q-deformed Minkowski metric. One considers braided matrices and not the algebra of 2 × 2 quantum matrices because of the insufficient covariance properties of the latter [8].
However, the braided matrices as given in [3] did not generalise the additive group structure of Minkowski space. The addition of matrices should be reflected in our dual and q-deformed setting by a braided coaddition as introduced in [6], by which one means a braided coproduct of the form \( \Delta x = x \otimes 1 + 1 \otimes x \) which extends as an algebra map with respect to a braided tensor product \( \otimes \) and not the ordinary tensor product \( \otimes \). A braided tensor product \( \otimes \) is like the super tensor products encountered in the theory of superspaces, but with the \( \pm 1 \) factors replaced by braid statistics. Explicitly, there is a braiding \( \Psi \) such that 

\[
(a \otimes b)(c \otimes d) = a \Psi(b \otimes c)d,
\]
i.e. \( \Psi \) measures how two independent copies of a system fail to commute. In the case of the commutative algebra of polynomial functions on a space, \( \Psi \) is simply given by the twist map \( \Psi(a \otimes b) = b \otimes a \). This braiding is determined by a background quantum group, which acts as the symmetry group of the system. See [5] and the references therein for an introduction to the theory of braided matrices and braided groups.

In [11] we found such a braiding and background quantum group, which allowed for quantum Minkowski space to have a braided coaddition. This gave rise to a natural quantum Lorentz group which preserves the entire structure of quantum Minkowski space, i.e. both its braided coaddition and its non-commutative algebra structure. The final result is given in terms of two solutions of the 4-dimensional QYBE:

\[
\begin{align*}
R_{M}^{ab\,cd} &= R^{-1_{LB}}B_{J}^{A}R^{B_{J}^{A}D}R^{A_{J}^{I}C}C_{L}^{I}D
R_{L}^{ab\,cd} &= R_{C_{J}^{I}B}^{C_{J}^{I}A}R^{A_{K}^{L}D}R^{L_{K}^{I}C}C_{I}^{I}D
\end{align*}
\]

Here \( P \) denotes the permutation map and 

\[
R = \begin{pmatrix} q & 0 & 0 & 0 \\ 0 & 1 & q^{-1} & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & q \end{pmatrix}, \quad q \in \mathbb{R}
\]

is the standard \( SU_q(2) \) R-matrix. The matrix \( \tilde{R} \) is defined as \((R^{t_2})^{-1}t_2\), where \( t_2 \) denotes transposition in the second tensor component. We also used multi-indices \( a = (AA') = (11), (12), (21), (22) \). These two matrices obey the relation

\[
0 = (PR_L + 1)(PR_M - 1),
\]
which ensures the existence of a braided coaddition \([\mathfrak{B}]\). In terms of these data, q-Minkowski space \(M_q\) is given as the algebra of quantum covectors \(M_q = V'(\mathbb{R}_M)\) in the notation of \([\mathfrak{B}]\). It has generators \(x_a\) and a star structure \(\bar{x}_a = x_{\bar{a}}\), where \(\bar{a} = (A'A)\) denotes the twisted multi-index. We denote the star structure by a bar in order to avoid confusion with the Hodge star operator.

The quantum Lorentz group \(L_q\) is defined as a quotient of the FRT algebra \(A(\mathbb{R}_L)\) with generators \(\lambda^a_b\) by the metric relation \(\lambda^a_c\lambda^b_d g^{cd} = g^{ab}\), where the q-metric \(g^{ab}\) is given by

\[
g^{ab} = \frac{1}{q + q^{-1}} \varepsilon_{AC} R^{A'}_{DB} \varepsilon^{DB'}
\]

in terms of the \(SL_q(2, \mathbb{C})\)-spinor metric

\[
\varepsilon_{AB} = \begin{pmatrix} 0 & 1/\sqrt{q} \\ -\sqrt{q} & 0 \end{pmatrix}.
\]

We are working in a ‘spinorial basis’, where the metric has two negative and two positive eigenvalues. In this basis, the injective \(*\)-Hopf algebra morphism \([\mathfrak{A}]\) (which induces a push forward of \(L_q\)-comodules)

\[
L_q \hookrightarrow SL_q(2, \mathbb{C})
\]

has the simple form \(\lambda^a_b \mapsto t^B_A \lambda^a_b \varsigma\), where \(t\) are the generators of \(SL_q(2, \mathbb{C})\). One can also choose a ‘\(x, y, z, t\)-basis’ by a simple change of generators \([\mathfrak{A}]\), but for our purposes it is more convenient to stay in the spinorial basis.

If we now consider the coaction of \(L_q\) on \(M_q\), one problem arises: In order to obtain full covariance under the coaction by the q-Lorentz group, we have to adjoin \(L_q\) slightly by a single invertible central and grouplike element \(\varsigma\) \([\mathfrak{B}]\). The extended q-Lorentz group is denoted by \(\tilde{L}_q\), and its covariant right coaction on q-Minkowski space is given by

\[
\beta_{M_q} : x_a \mapsto x_b \otimes \lambda^b_a \varsigma.
\]

Equation \([\mathfrak{B}]\) implies that we also have a covariant coaction by the extended algebra \(\tilde{SL}_q(2, \mathbb{C})\) given by \(x^A_A \mapsto x^B_B \otimes t^A_B t^A_{B'} \varsigma\). Since the element \(\varsigma\) measures the degree of elements on \(M_q\), it is often called dilaton element \([\mathfrak{L}]\).

Thus q-Lorentz group and q-Minkowski space are given as non-commutative deformations of polynomial function algebras. This means that if we are interested in wave equations on
quantum Minkowski space, we also have to dualise and q-deform the notion of fields. Classically, a Lorentz field $\varphi$ on Minkowski space is a left Lorentz group module morphism

$$
\varphi : M \rightarrow V \\
x \mapsto \varphi(x),
$$

where $V$ is some finite dimensional vector space and left Lorentz module. If we denote the linear coordinate functionals on $V$ by $\xi_i$, then $\varphi$ induces a right $\mathcal{P}(L)$-comodule morphism

$$
\Phi : \mathcal{P}(V) \rightarrow \mathcal{P}(M) \\
\xi_i \mapsto \Phi(\xi_i) = \xi_i \circ \varphi,
$$

which is also an algebra map. However, this algebra homomorphism property is somewhat accidental since all algebras are commutative, and we do not have any reason to expect this to hold in the non-commutative case. Thus if $V_q = \bigoplus_{i=0}^{\infty} V_i$ is a non-commutative deformation of $\mathcal{P}(V)$ as a graded $\ast$-algebra, then we call a linear $\ast$-map

$$
\Phi : V_1 \rightarrow M_q
$$

from the linear component of $V_q$ to $M_q$ a q-Lorentz field if it is a right $\tilde{\mathcal{L}}_q$-comodule morphism, irrespectively whether it extends as an algebra map or not.

A solution to a Lorentz covariant wave equation $O\varphi = 0$, where $O$ is some linear differential operator then induces on the dual level a solution to the $\mathcal{P}(L)$-covariant equation $O\Phi = 0$. We take this as the general form of a covariant wave equation also in the q-deformed case. But the necessity to extend the q-Lorentz group by the dilaton element suggests to consider only massless wave equations, since a mass term would destroy the scaling invariance. However, this is not too serious a restriction if we are interested in a toy model for Planck scale corrections to the geometry.

An outline of the paper is as follows. In section 2 we analyse differential forms on quantum Minkowski space and introduce the q-exterior derivative and the q-Hodge star operator. This leads to a natural definition of a q-coderivative, a q-Laplace-Beltrami operator and a q-Lie derivative. The crucial ingredient in these constructions is an $\tilde{\mathcal{L}}_q$-covariant antisymmetrisation operation, which is developed in 2.1. Section 3 then applies these results to the q-d’Alembert equation as the simplest example of a wave equation on quantum Minkowski space. Solutions of this equation define a conserved current, but unlike in the classical case this current does
not vanish for real solutions. We also give a family of q-deformed plane wave solutions to this equation. The fact that these plane waves exist only on the q-light cone gives further support to our claim that wave equations on q-Minkowski space should be massless. In section 4 we analyse the q-Maxwell equations, and again give a family of q-deformed plane wave solutions. We also discuss the q-gauge freedom of the q-Maxwell equation. Finally we then give an $SL_q(2,\mathbb{C})$-spinor decomposition of the self-dual and anti-self-dual parts of the q-field strength tensor. The spinor formulation of the q-Maxwell equation in the last section also allows for a generalisation to arbitrary spin.

2 Differential forms on quantum Minkowski space

Since quantum Minkowski space has a braided coaddition \cite{[1]}, we can immediately apply the results of \cite{[2]}, where a braided differential calculus was developed. The key idea was to obtain braided differential operators $\partial^a$ by ‘differentiating’ the braided coaddition. The resulting algebra of braided differential operators $\mathcal{D}$ acts on quantum Minkowski space with an action $\alpha : \mathcal{D} \otimes M_q \rightarrow M_q$ such that the braided Leibnitz rule holds \cite{[2]}

$$\partial^a f g = (\partial^a f) g + \cdot \circ \Psi^{-1}_L(\partial^a \otimes f) g.$$  

In \cite{[2]} it was also shown that the braided differential operators $\partial^a$ obey the relations of $V(R_M)$

$$\partial^a \partial^b = R_{M \, a \, d}^\epsilon \partial^d \partial^\epsilon.$$  

This means that we can define a covariant coaction $\beta_D : \mathcal{D} \rightarrow \mathcal{D} \otimes \tilde{L}_q$ by $\partial^a \mapsto \partial^a \otimes \lambda_b^a \zeta^{-1}$ and thus obtain a coaction

$$\beta_{\mathcal{D} \otimes M_q} : \mathcal{D} \otimes M_q \rightarrow (\mathcal{D} \otimes M_q) \otimes \tilde{L}_q,$$

which is an algebra map because of the covariance properties of the braided tensor product. This covariant coaction then makes the action $\alpha$ into an $\tilde{L}_q$-comodule morphism. Note that the $\partial^a$ transform with a scaling factor $\zeta^{-1}$ inverse to the one for the coordinates, as appropriate for a derivative.

Thus there is well-developed theory of $\tilde{L}_q$-covariant ‘q-partial derivatives’ on quantum Minkowski space, but q-exterior derivative, q-Hodge star operator and the q-coderivative are far less well understood. In this section, we introduce these operations.
2.1 Antisymmetrisers and the q-exterior algebra

In the classical examples we would like to q-deform, fields are often forms on Minkowski space. Thus we need a dual and q-deformed generalisation of the exterior algebra. We shall construct this algebra explicitly, and not merely define it abstractly as in [12]. The essential ingredient in our approach is an $\tilde{\mathcal{L}}_q$-covariant q-antisymmetrisation operation. For this, we have to use a q-deformed notion of antisymmetry, where we call an $\tilde{\mathcal{L}}_q$-tensor $T_{...ab...}$ q-antisymmetric in the adjacent indices $a$ and $b$ if

$$T_{...ab...} = -T_{...cd...} R^{dc}_{ab}. \quad (7)$$

Here we use an index notation for tensors, where $T_{ab}$ and $T^{ab}$, etc. denote any elements of right $\tilde{\mathcal{L}}_q$-comodules, which transform as $T_{ab} \mapsto T_{cd} \otimes \lambda^c_a \lambda^d_b \varsigma^n$ and $T^{ab} \mapsto T^{cd} \otimes S \lambda^a_c S \lambda^b_d \varsigma^m$, where $S$ denotes the antipode in $\mathcal{L}_q$. We do not require a tensor to have a specific $\varsigma$-scaling property, and therefore $n$ and $m$ can be any integers.

As shown in [11], we can use the q-metric $g^{ab}$ and its inverse to raise and lower indices.

Due to the relation between the R-matrix and the q-metric [11]

$$R^k_{\ c}^{\ d} = g_{pf} g_{qc} R^q_{\ ab} g^{ak} \ g^{bl}, \quad (8)$$

this raising and lowering of indices preserves the q-antisymmetry of tensors. If for example a tensor $T_{...ab...}$ is q-antisymmetric in $a$ and $b$ then the tensor with upper indices $T^{...ab...} = T^{...ij...} g^{ia} g^{jb} \ldots$ obeys

$$T^{...ab...} = -R^{ab}_{\ dc} T^{...cd...},$$

A tensor is called q-symmetric in $a$ and $b$ if

$$T_{...ab...} = T_{...cd...} R^{dc}_{\ ab}. \quad (7)$$

Again, this translates into the corresponding formula for upper indices by virtue of a relation of the type (3) for the matrix $R_M$. Note that we are using two different R-matrices for the definition of q-symmetry and q-antisymmetry.

Since $R_L$ and $R_M$ obey the relation (2), one might suspect that $(P R_M - 1)$ would be a good candidate for a q-antisymmetriser. However, this operator is not a projector, and it is also not quite clear how to obtain higher antisymmetrisers. We shall therefore take a
different approach. Recall that in the classical case, the space of totally antisymmetric tensors of valence four is one-dimensional, and one can choose a basis vector \( \varepsilon_{abcd} \) with \( \varepsilon_{1234} = 1 \), which defines a projector (antisymmetriser) \( \frac{1}{4!} \delta_{[abc} \varepsilon_{dfgh]} \) onto this one-dimensional space. By successively contracting indices, one obtains the lower antisymmetrisers. The q-deformed case is quite similar:

**Lemma 2.1** Up to a factor, there is exactly one complex valued tensor \( \varepsilon_{abcd} \), which is totally q-antisymmetric in any two adjacent indices.

**Proof.** One can show explicitly that the system of linear equations \( \varepsilon_{abcd} = -\varepsilon_{ijcd} R_{L}^{ji} = -\varepsilon_{aijcd} R_{L}^{ji} \) has a one-dimensional solution space. The non-zero entries of \( \varepsilon_{abcd} \) in the normalisation \( \varepsilon_{1234} = 1 \) are:

\[
\begin{align*}
\varepsilon_{1234} &= 1 & \varepsilon_{1243} &= -q^2 & \varepsilon_{1324} &= -1 & \varepsilon_{1342} &= q^2 \\
\varepsilon_{1423} &= 1 - q^2 & \varepsilon_{1432} &= -1 & \varepsilon_{1441} &= 1 - q^{-2} \\
\varepsilon_{2134} &= -1 & \varepsilon_{2143} &= q^{-2} & \varepsilon_{2314} &= 1 & \varepsilon_{2341} &= -1 \\
\varepsilon_{2413} &= -q^{-2} & \varepsilon_{2431} &= q^{-2} & \varepsilon_{2434} &= q^{-2} - 1 & \varepsilon_{3124} &= 1 \\
\varepsilon_{3142} &= -q^2 & \varepsilon_{3214} &= -1 & \varepsilon_{3241} &= 1 & \varepsilon_{3412} &= q^2 \\
\varepsilon_{3421} &= -q^2 & \varepsilon_{3424} &= 1 - q^2 & \varepsilon_{4123} &= -1 & \varepsilon_{4132} &= 1 \\
\varepsilon_{4141} &= q^2 - 1 & \varepsilon_{4144} &= q^{-2} - 1 & \varepsilon_{4213} &= 1 & \varepsilon_{4231} &= -1 \\
\varepsilon_{4243} &= 1 - q^{-2} & \varepsilon_{4312} &= -1 & \varepsilon_{4321} &= 1 & \varepsilon_{4342} &= q^2 - 1 \\
\varepsilon_{4414} &= 1 - q^{-2} & \varepsilon_{4411} &= q^{-2} - 1 & & & \\
\end{align*}
\]

Unlike the \( \varepsilon \)'s for \( SU_q(n) \) where only the \( \pm 1 \) entries of \( \varepsilon \) are changed to powers of \( q \) and all zero entries remain unchanged, we here obtain non-zero entries like \( \varepsilon_{4414} \), etc. In terms of this q-antisymmetric tensor, we define q-antisymmetrisers by successively contracting indices:

**Definition 2.2** Let \( T_{a_1...a_n} \) be an \( \tilde{L}_q \)-tensor. We define its q-antisymmetrisation in the adjacent indices \( a_1...a_n \) by

\[
T_{[a_1...a_n]} := T_{c_1...c_n} A_{\{k\}_{a_1...a_n}}
\]

for \( n < 5 \) and zero otherwise. The \textbf{q-antisymmetrisers} \( A_{\{k\}} \) are defined as

\[
\begin{align*}
A_{\{4\} fgh} &:= -\frac{1}{n_4} \varepsilon^{[abcd} \varepsilon_{efgh]} & A_{\{3\} fgh} &:= -\frac{1}{n_3} \varepsilon^{[abcd} \varepsilon_{d|efgh]} \\
A_{\{2\} gh} &:= -\frac{1}{n_2} \varepsilon^{[ab} \varepsilon_{cd]gh} & A_{\{1\} k} &:= -\frac{1}{n_1} \varepsilon^{[ab} \varepsilon_{bcd]k},
\end{align*}
\]

where the normalisation factors \( n_k \)

\[
\begin{align*}
n_1 &= 2(1 + q^2 + q^4), & n_2 &= (1 + q^2)(1 + q^4) \\
n_3 &= n_1, & n_4 &= q^{-2} 2(1 + q^2 + q^4)(1 + q^2)(1 + q^4)
\end{align*}
\]
are a q-deformation of \((4-k)!k!\).

The q-antisymmetrisation of a tensor is clearly q-antisymmetric in the sense of (7), but also has the other properties one might expect:

**Proposition 2.3** By explicit calculation, one can show:

1. The antisymmetrisers \(\mathcal{A}_{\{k\}}\) are projectors:
   \[
   \mathcal{A}_{\{k\}}^2 = \mathcal{A}_{\{k\}}, \quad \text{i.e.} \quad T_{\cdots[a_1\ldots a_n]\cdots} = T_{\cdots[a_1\ldots a_n]\cdots} \quad \text{(10)}
   \]
   for \(k = 1 \ldots 4\).

2. Lower dimensional q-antisymmetrisers cancel on higher dimensional ones:
   \[
   T_{\cdots[a_1\ldots a_k\ldots a_l\ldots a_n]\cdots} = T_{\cdots[a_1\ldots a_n]\cdots} \quad \text{(11)}
   \]

3. The one-dimensional projector is trivial:
   \[
   \mathcal{A}_{\{1\}} = 1, \quad \text{i.e.} \quad T_{\cdots[a]\cdots} = T_{\cdots a\cdots} \quad \text{(12)}
   \]

4. There are two invertible matrices \(B\) and \(B'\) such that
   \[
   \mathcal{A}_{\{2\}} = (PR_{M} - 1)B = B'(PR_{M} - 1).
   \]
   I.e., the two-dimensional antisymmetriser \(\mathcal{A}_{\{2\}}\) factors through \((PR_{M} - 1)\).

In addition to (10), one usually requires a linear operator to be Hermitean with respect to a given inner product before calling it ‘projector’. We can show something similar in our case, e.g. \(\mathcal{A}_{\{2\}}\) is ‘Hermitean’ with respect to the q-deformed metric in the sense that

\[
\mathcal{A}_{\{2\}}^{ab} = g_{cj}g_{di} \mathcal{A}^{ij}_{\{2\}} g^{al} g^{bk}
\]

and similar relations for the other q-antisymmetrisers, but these properties are not important for our purposes and we therefore do not discuss them in detail. One might regard (10) as the minimal requirement for a q-antisymmetrisation operation to make sense, but for the following all the other properties are needed as well, in particular (12). This relation ensures that q-symmetric tensors are in the kernel of the q-antisymmetrisers. To make this point explicit, note that (11) and (12) imply:
Corollary 2.4 If an $\tilde{\mathcal{L}}_q$-tensor $T_{a_1 \ldots a_k}$ is $q$-symmetric in two adjacent indices $a_i$ and $a_{i+1}$ then

$$T_{(a_1 \ldots a_i a_{i+1} \ldots a_k)} = 0.$$  \hspace{1cm} (13)

Thus although $q$-symmetry and $q$-antisymmetry are defined in terms of two different R-matrices, the two notions are compatible in this sense. The relation (13) shall be of importance in section 2.2, where we introduce the external derivative $d$ and show that $d^2 = 0$. Note also that one can easily define a $q$-symmetriser on two adjacent indices as $S_{\{2\}} = \frac{1}{2}(1 - A_{\{2\}})$, but in our non-Hecke case it is not quite clear how to obtain higher $q$-symmetrisers.

So far, we have used the index notation for $q$-antisymmetrised tensors without addressing the problem of $\tilde{\mathcal{L}}_q$-covariance of the $q$-antisymmetrisation. This question is answered by the following proposition:

Proposition 2.5 The coaction $\beta$ by the $q$-Lorentz group commutes with the operation of $q$-antisymmetrisation. Symbolically,

$$\beta \circ [ ] = [ ] \circ \beta.$$

This means in particular that the property of $q$-antisymmetry is covariant under the coaction by $\tilde{\mathcal{L}}_q$.

Proof. We need to show that monomials of generators of $\mathcal{L}_q$ commute with the $q$-antisymmetriser $A_{\{n\}}$. The generators of the $q$-Lorentz group obey

$$\lambda^a_e \lambda^b_f \lambda^c_g \lambda^d_h e^{hgf} e_{cdekl} = n^{-1}_4 \varepsilon_{mnpq} \lambda^m_e \lambda^n_f \lambda^p_g \lambda^q_h e^{hgf} e_{cdekl}.$$

This implies for the $q$-antisymmetriser $A_{\{2\}}$:

$$\lambda^a_e \lambda^b_f e^{cdefe} e_{cdkl} = \lambda^a_e \lambda^b_f \lambda^c_g \lambda^d_h e^{hgf} e_{cdkl} = n^{-1}_4 \varepsilon_{mnpq} \lambda^m_e \lambda^n_f \lambda^p_g \lambda^q_h e^{hgf} e_{cdkl} = n^{-1}_4 \varepsilon_{mnpq} \lambda^m_e \lambda^n_f \lambda^p_g \lambda^q_h e^{hgf} e_{cdkl} = \varepsilon^{mnpq} \lambda^m_e \lambda^n_f \lambda^p_g \lambda^q_h e^{hgf} e_{cdkl} = \varepsilon^{mnpq} \lambda^m_e \lambda^n_f \lambda^p_g \lambda^q_h e^{hgf} e_{cdkl}.$$

and similar for $A_{\{3\}}$ and $A_{\{4\}}$. The case of $A_{\{1\}}$ is trivial. \hfill $\Box$

In terms of these $q$-antisymmetrisers, one can now define a $q$-deformation of the wedge product of copies of $M_q$, realised in the braided tensor product. Explicitly, let $M_q \wedge \ldots \wedge M_q$ be
the subalgebra of $M_q \otimes \ldots \otimes M_q$ generated by

$$dx_{a_1} \wedge \ldots \wedge dx_{a_n} := dx_{[a_1} \otimes \ldots \otimes dx_{a_n]}$$

$$= dx_{b_1} \otimes \ldots \otimes dx_{b_n} A_{\{a \}_{a_1 \ldots a_n}},$$

where $dx$ denotes a copy of the generators of $M_q$. This means that the linear component

$$\Lambda_n := (M_q \wedge \ldots \wedge M_q)_{1/n}$$

is spanned by the totally $q$-antisymmetric

$$e_{a_1 \ldots a_n} := dx_{a_1} \wedge \ldots \wedge dx_{a_n}$$

with relations of the form (8). We note that $\Lambda_n$ is very similar to the general form of the exterior algebras discussed in [15] for the special case where all R-matrices are of Hecke type. In the framework set out in section [9] we now define a $p$-form on quantum Minkowski space as an $\bar{L}_q$-comodule morphism

$$w : \Lambda_p \rightarrow M_q.$$ 

For $p = 0$, we define $\Lambda_0 := \mathbb{C}$ as the linear component of $P(\mathbb{R})$, the algebra of complex valued polynomial functions on $\mathbb{R}$, and choose a basis element $e$.

Over the ring of all $\bar{L}_q$-scalars in $M_q$, $p$-forms form a linear space, which is denoted by $\Omega_p$. A simple analysis of the rank of the antisymmetrisers $A_{\{p\}}$ shows that these spaces have the dimensions 1, 4, 6, 4, 1 for $p = 0, 1, 2, 3, 4$, and dimension 0 for $p > 4$. Thus the dimensions are the same as in the classical case. The one-dimensional space $\Omega_4$ is spanned by the top form $\varepsilon$ defined as

$$\varepsilon : e_{a_1 \ldots a_4} \mapsto \varepsilon_{a_1 \ldots a_4}.$$ 

(14)

As linear maps, $p$-forms $w$ are determined by their value on the elements $e_{a_1 \ldots a_p}$ of $\Lambda_p$, but we also have:

**Proposition 2.6** All $p$-forms on quantum Minkowski space are of the form

$$w(e_{a_1 \ldots a_p}) = w_{\{a_1 \ldots a_p\}},$$

for some $w_{a_1 \ldots a_p} \in M_q$. 

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Proof. The proposition is tantamount to claiming that all totally q-antisymmetric tensors are in the image of the q-antisymmetrisers. Hence the proposition can be proved by verifying that the dimension of totally q-antisymmetric tensors over the ring of \( \tilde{L}_q \)-scalars in \( M_q \) coincides with the ranks of the q-antisymmetrisers \( A_{\{k\}} \). \( \square \)

As in the classical case, the space of forms on quantum Minkowski space
\[
\Omega := \bigoplus_{p=0}^{4} \Omega_p,
\]
can be equipped with an algebra structure. Given a \( p \)-form \( w \) and an \( r \)-form \( v \), their \( q \)-wedge product \( w \wedge v \in \Omega_{p+r} \) is defined as:
\[
w \wedge v : e_{a_1 \ldots a_{p+r}} \mapsto w[a_1 \ldots a_p]v_{a_{p+1} \ldots a_{p+r}}.
\]
By virtue of (11), this bilinear operation ‘\( \wedge \)’ defines an associative algebra structure on \( \Omega \), the \( q \)-exterior algebra, with identity \( 1 \in \Omega_0 \) given by \( 1 : e \mapsto 1 \).

2.2 The q-exterior derivative

In this section we define an \( q \)-exterior derivative on forms on \( q \)-Minkowski space. Essential building blocks are the braided differential operators \( \partial^a \) and the q-antisymmetrisers.

**Definition 2.7** The \( q \)-exterior derivative \( d : \Omega_p \mapsto \Omega_{p+1} \) is defined as
\[
dw(e_{a_1 \ldots a_{p+1}}) := \partial_{[a_1} w_{a_2 \ldots a_{p+1}]}.
\]
Forms whose \( q \)-exterior derivative vanishes are called closed and forms which are themselves \( q \)-exterior derivatives are said to be exact. The crucial test for a definition of a ‘\( q \)-deformed exterior derivative’ is whether exact forms are closed. In [15], for example, this could be shown only in the (trivial) Hecke case. Due to the careful construction of the q-antisymmetrisers, we obtain in our non-Hecke case:

**Proposition 2.8** Exact forms on \( q \)-Minkowski space are closed:
\[
d^2 = 0.
\]

**Proof.** Relation (11) implies for exact forms \( dw \):
\[
d^2 w(e_{a_1 \ldots a_{p+2}}) = \partial_{[a_1} \partial_{a_2} w_{a_3 \ldots a_{p+1}]} = \partial_{[[a_1} \partial_{a_2]} w_{a_3 \ldots a_{p+1}]} = 0.
\]
Here we used that braided differential operators $\partial_a$ obey the relations of $V'(\mathbf{R}_M)$ \cite{9}, i.e. $\partial_{a_1}\partial_{a_2}$ is q-symmetric and thus $\partial_{[a_1}\partial_{a_2]} = 0$ by virtue of corollary 2.4.

Using the recent results on q-integration by A. Kempf and S. Majid \cite{3} one might also be able to prove a q-Poincaré lemma for q-Minkowski space, which would then imply that all closed forms are exact.

As a consequence of the braided Leibnitz rule (5) we find for the act ion of the q-exterior derivative $d$ on wedge products of forms:

**Corollary 2.9** The q-exterior derivative $d$ acts as

$$dw \wedge v = (d w) \wedge v + (-1)^p w \wedge dv$$

on wedge products $w \wedge v$, where $w$ is a p-form.

**Proof.** The crucial point is that the inverse braiding brings up R-matrices, which cancel on the q-antisymmetriser because of the symmetry property (7). We prove the corollary only for 1-forms $w$, the general case follows immediately by using the hexagon identity for the braiding $\Psi$. Thus let $w$ be a 1-form and $v$ a p-form. On $\Lambda_{p+2}$ we have by virtue of (5) and the q-antisymmetry of $\varepsilon^{abcd}$:

$$\partial_{[a_1}w_{a_2}v_{a_3...a_{p+2}]} = (\partial_{[a_1}w_{a_2]}v_{a_3...a_{p+2}]} + \cdot \circ \Psi^{-1}(\partial_{[a_1} \otimes w_{a_2]}v_{a_3...a_{p+2}]}$$

$$= (\partial_{[a_1}w_{a_2]}v_{a_3...a_{p+2}]} + w_c \otimes \partial_{d} R^{-1cd}_{L[a_2a_1}v_{a_3...a_{p+2}]}$$

Here we used the inverse braiding $\Psi^{-1}(\partial_{a_1} \otimes w_{a_2}) = w_c \otimes \partial_{d} R^{-1cd}_{L[a_2a_1}$ (See \cite{3} Prop. 3.2) for a useful list of braidings between various standard algebras). \hfill \Box

### 2.3 q-Hodge star operator

The other operation on forms on q-Minkowski space one can define with the tools at hand is the q-Hodge star operator. It is defined in terms of the metric $g^{ab}$ and the tensor $\varepsilon_{abcd}$.

**Definition 2.10** The q-Hodge star operator $\ast : \Omega_p \rightarrow \Omega_{4-p}$ is defined by

$$\ast w(e_{a_1...a_{4-p}}) = n_0^{-1/2} \varepsilon_{a_1...a_{4-p}b_1...b_p} g^{b_1c_1} \cdots g^{b_pc_p} w_{[c_1...c_p]}$$

$$= w_{[c_1...c_p]} H_{\{p\} a_1...a_{4-p}},$$

with the normalisation factor $n_0 := n_4$. 12
Again, this definition will only be justified if we can show some non-trivial properties. We shall use the q-Hodge star to define a q-coderivative $\delta$ and a q-Laplace-Beltrami operator $\Delta$. For these operators to have reasonable properties, it is necessary for the square of the q-Hodge star to have a sufficiently well-behaved form. It turns out that we recover exactly the classical result:

**Proposition 2.11** The square of the q-Hodge star operator on $p$-forms is given by:

$$\ast \circ \ast = (-1)^{p(4-p)}.$$  \hfill (16)

Furthermore, one finds

$$\ast 1 = \epsilon, \quad \ast \epsilon = 1$$

for the special cases of the top form $\epsilon$ and the identity form $1$.

**Proof.** By explicit calculation, one can verify the following relations between the q-antisymmetrisers $A_{\{k\}}$ and the matrices $H_{\{k\}}$, which implement the q-Hodge star operation:

$$H_{\{0\}}H_{\{4\}} = 1$$

$$-H_{\{1\}}H_{\{3\}} = A_{\{1\}}$$

$$H_{\{2\}}H_{\{2\}} = A_{\{2\}}$$

$$-H_{\{3\}}H_{\{1\}} = A_{\{3\}}$$

$$H_{\{4\}}H_{\{0\}} = A_{\{4\}}$$ \hfill (17)

Together with (11), these relations imply the proposition. \hfill $\square$

Since we are working in a ‘spinorial basis’ we do not obtain an additional $(-1)$-factor on the right hand side of relation (16), as in the case of a ‘$x,y,z,t$-basis’. Another property of the q-Hodge star operator is that one can ‘shift’ the q-Hodge star operator in the q-wedge product of two $p$-forms:

**Lemma 2.12** If $w$ and $v$ are both $p$-forms, then

$$\ast w \wedge v = (-1)^p w \wedge^* v.$$  

**Proof.** It is sufficient to verify the relations

$$-H_{\{1\}b}^a A_{\{4\}klmn}^{bcde} = H_{\{1\}b}^a A_{\{4\}klmn}^{abcd}$$

$$H_{\{2\}c}^{ab} A_{\{4\}klmn}^{cdef} = H_{\{2\}c}^{ab} A_{\{4\}klmn}^{abcd}$$

$$-H_{\{3\}d}^{abc} A_{\{4\}klmn}^{defg} = H_{\{3\}d}^{abc} A_{\{4\}klmn}^{abcd}$$

which establishes the lemma. \hfill $\square$
Now that we are given both a well-behaved exterior derivative and a q-Hodge star operator, it is straightforward to define a suitable notion of q-coderivative, q-Laplace-Beltrami operator and q-Lie derivative.

**Definition 2.13** The **q-coderivative** $\delta : \Omega_p \rightarrow \Omega_{p-1}$ and the **q-Laplace-Beltrami operator** $\Delta : \Omega_p \rightarrow \Omega_p$ on $p$-forms on quantum Minkowski space are defined as

$$
\delta := *d^*, \quad \Delta := \delta d + d\delta.
$$

Forms $w$ on quantum Minkowski space which satisfy $\delta w = 0$ are called **co-closed**, and forms which are themselves q-coderivatives are called **co-exact**. As a corollary of proposition 2.11 and proposition 2.8, one finds:

**Corollary 2.14** Co-exact forms on quantum Minkowski space are co-closed:

$$
\delta^2 = 0.
$$

Thus, although $d$, $\delta$ and $\Delta$ are defined in terms of deformed antisymmetrisers and differential operators on a non-commutative space, their abstract properties resemble very much the classical case. It is straightforward to verify:

$$
\begin{align*}
    d\Delta &= \Delta d, & \delta \Delta &= \Delta \delta, \\
    \Delta^* &= *\Delta, & \delta^* &= (-1)^p *d \\
    *\delta &= (-1)^{p+1}d^*, & d\delta^* &= *\delta d \\
    *d\delta &= \delta d^*, & \Delta^* &= *\Delta
\end{align*}
$$

Further, and less trivial, properties will be given in the following sections. In particular we will analyse the explicit action of these operators on zero and 1-forms, which is of interest to physical applications.

The q-Hodge star operator also enables us to generalise the idea of a Lie derivative. For this purpose, we introduce a **q-inner product** on the q-exterior algebra $\Omega$ as a bilinear map $(\cdot, \cdot) : \Omega_p \times \Omega_r \rightarrow \Omega_{p-r}$ defined by

$$
(w, v) := i_v w := *(v \wedge i_w^{-1}(w)),
$$

Further, and less trivial, properties will be given in the following sections. In particular we will analyse the explicit action of these operators on zero and 1-forms, which is of interest to physical applications.
where \( w \) is a \( p \)-form. The q-inner product is ‘transposed’ to the q-wedge product in the sense that
\[
(v \wedge w, u) = i_{v \wedge w} u = i_v (i_w u) = (w, i_v u),
\]
and one can also show the formulae
\[
^*w = i_w \varepsilon
\]
\[
\delta i_v w = i_v \delta w + (-1)^p i_i d_v w,
\]
where \( v \) is a \( p \)-form. Furthermore, lemma 2.12 implies
\[
(v, w) = i_v w = i_v *w = (*v, *w),
\]
for any two \( p \)-forms \( v \) and \( w \). In terms of this q-inner product, we now introduce:

**Definition 2.15** Let \( v \) be a 1-form. The **q-Lie derivative** with respect to \( v \) is defined as:
\[
L_v := i_v \circ d + d \circ i_v,
\]
and is obviously a linear map \( L_v : \Omega_p \rightarrow \Omega_p \).

The q-Lie derivative commutes with the q-exterior derivative
\[
L_v d = d L_v,
\]
and we also have
\[
L_{f \wedge v} w = f \wedge L_v w + df \wedge i_v w
\]
for zero forms \( f \) and 1-forms \( v \). For the action of the q-Lie derivative on zero forms on q-Minkowski space, we find:

**Proposition 2.16** Let \( f \in \Omega_0 \) and let \( v \in \Omega_1 \). Then the action of \( L_v \) on \( f \) is given by
\[
L_v f : e \mapsto v^a \partial_a f.
\]

**Proof.** First note that \( L_v f = i_v df \), since \( f \) is a zero form. Then show by explicit calculation, that
\[
H^a_{(1)cde} H^{bde}_{(4)} = -g^{ba}.
\]
Therefore
\[
L_v f(e) = -v_b \partial_a f H^a_{(1)cde} A^{bde}_{(4)kln} H^{kln}_{(4)} = -v_b \partial_a f H^a_{(1)cde} H^{bde}_{(4)} = v_b \partial_a g^{ba} f,
\]
\[
\square
\]
3 q-Scalar field

3.1 The q-d’Alembert equation

The simplest case of a wave equation on q-Minkowski space is the q-d’Alembert equation, where fields are 0-forms \( \phi \) and the wave equation is given by the q-Laplace-Beltrami operator:

**Definition 3.1** A solution of the q-d’Alembert equation is a 0-form \( \phi \) such that

\[
\Delta \phi = 0. \tag{20}
\]

This equation can be written less abstractly in terms of the braided differential operators and the value \( \phi \) on \( e \) of \( \phi \).

**Proposition 3.2** Equation (20) is equivalent to

\[
\Box \phi = 0, \tag{21}
\]

where \( \Box := \partial_a \partial_b g^{ab} \) is the q-d’Alembert operator.

**Proof.** Since \( ^* \phi \) is a 4-form, \( d \delta \phi \) vanishes and thus \( \Delta \phi = \delta d \phi \). With relation (19), we find:

\[
0 = \delta d \phi(e) = \partial_j \partial_a \phi H_a^{(1)_{cde}} A_{(4)_{klmn}} H_{(4)}^{klmn} = \partial_j \partial_a \phi H_a^{(1)_{cde}} H_{(4)}^{cde} = \partial_j \partial_a g^{fa} \phi
\]

which proves the equivalence of (20) and (21). \( \square \)

In the form (21), it is easy to see that the q-d’Alembert equation is \( \widetilde{L}_q \)-covariant. Keeping in mind the various transformation properties, one can show that the action \( \alpha \) of the operator \( \Box \) commutes with the coaction by \( \widetilde{L}_q \):

\[
\beta_{M_q} \circ \alpha \circ (\Box \otimes \phi) = \alpha \circ \beta_{\mathcal{D}_q} \circ (\Box \otimes \phi).
\]

One could also write down a q-Klein Gordon equation of the form

\[
(\Box + m^2) \phi = 0,
\]

but this equation would only be \( \widetilde{L}_q \)-covariant if the ‘mass’ \( m \) transformed as \( m \mapsto m \otimes \varsigma^{-1} \), i.e. not as an \( \widetilde{L}_q \)-scalar. One might argue that this transformation property in itself is not
necessarily harmful, but the results of the next section on plane wave solutions seem to suggest
to us that \( \mathcal{L}_q \)-covariant wave equations on \( M_q \) are inherently massless.

The requirement for \( \varphi \) to be a 0-form corresponds classically to a restriction to real-valued
solutions. A complex valued solution of the \( q \)-d’Alembert equation is a linear *-map

\[
\varphi : \text{span}_\mathbb{C}\{e, \bar{e}\} \to M_q
\]
such that (20) holds. The space \( \text{span}_\mathbb{C}\{e, \bar{e}\} \) is simply the linear component of \( \mathcal{P}(\mathbb{C}) \). For both
0-forms and complex valued solutions of the \( q \)-d’Alembert equation, there exists a conserved
current:

**Proposition 3.3** Let \( \varphi \) be a solution of the \( q \)-d’Alembert equation. Then the current 1-form \( j \)

\[
j := \varphi \wedge d\varphi - q^{-2} d\varphi \wedge \varphi
\]
is conserved:

\[
\delta j = 0.
\]

**Proof.** Equation (17), corollary 2.9, and the relation \( q^{-2} R_L^{-1cd} g_{ab} g^{ba} = g^{cd} \) imply:

\[
\delta j(e_a) = \partial_b(\varphi \partial_a \varphi - q^{-2}(\partial_b \varphi \partial_a \varphi) H^q_{a}^{cde} A^{bde} A^{cd} H^{klmn}_{(1)} H^{klmn}_{(4)} + H^q_{a}^{cd} H^{klmn}_{(1)} H^{klmn}_{(1)} H^{klmn}_{(4)} = H^q_{a}^{cd} H^{klmn}_{(1)} H^{klmn}_{(1)} H^{klmn}_{(4)} = ((\partial_b \varphi (\partial_a \varphi) - (\partial_a \varphi) (\partial_b \varphi) q^{-2} R_L^{-1cd} g^{ba} g^{ba}) = 0
\]
Here we used repeatedly the relations (17).

The interesting feature of this current is that unlike in the classical case, it does not
vanish if \( \varphi \) is real. If we assume that a non-trivial \( \varphi \) is given by a central and real \( \varphi \in M_q \), then

\[
j : e_a \mapsto (1 - q^{-2}) \varphi \partial_a \varphi,
\]
vanishes only in the commutative case.

### 3.2 Plane wave solutions

We shall now construct a family of plane wave solutions to the \( q \)-d’Alembert equation. For this
purpose, we regard a copy of \( V(R_M) \) as momentum space and denote its generators by \( p^a \). It
is a \( \mathcal{L}_q \)-comodule algebra with coaction \( p^a \mapsto p^b \otimes S_{b}^{a} \), i.e. has the \( \varsigma \)-scaling property as appropriate for momenta. The relations between the \( p \)'s are described in terms of \( R_{M} \), but on the \( q \)-deformed light cone \( P_{0} \) defined as the quotient of \( V(R_{M}) \) by the relation \( g_{ab}p^ap^b = 0 \), one also has:

**Lemma 3.4** There is an isomorphism

\[
P_{0} \cong V(q^{-2}R_{L})/(g_{ab}p^ap^b = 0).
\]

Thus on the quotient \( P_{0} \), the generators also obey the relations

\[
p^ap^b = q^{-2}R_{L}^{ab}p^dp^c.
\]

**Proof.** Let \( p = (a,b,c,d) \) be the vector of generators. The algebra \( V(q^{-2}R_{L}) \) has the same relations as \( V(R_{M}) \) except for \( cb = q^2bc + (1 - q^2)dd \), which differs from the corresponding relation \( cb = bc - (1 - q^2)ad - (1 - q^{-2})dd \). However, in the quotients \( V(q^{-2}R_{L})/(g_{ab}p^ap^b = 0) \) and \( P_{0} \), the generators obey \( ad - q^{-2}cb = 0 \) and we can rewrite both relations as \( ad = bc - (1 - q^{-2})dd \). \( \square \)

The \( q \)-light cone is invariant under the coaction by \( \mathcal{L}_q \), i.e. the coaction \( \beta \) by the \( q \)-Lorentz group on \( V(R_{M}) \) descends to a covariant coaction \( \beta : P_{0} \to P_{0} \otimes \tilde{\mathcal{L}}_q \). Using \( P_{0} \) as an 'index set' we define a family of \( q \)-deformed plane waves:

\[
\exp(ix.p) := \sum_{n=0}^{\infty} \frac{i^n}{[n]!} x_1 \ldots x_n p_n \ldots p_1
\]

as a formal power series in \( M_{q} \otimes P_{0} \), where \([n] = 1 + q + \ldots + q^{n-1} \) and \([n]! = [1] \ldots [n] \). This exponential is different from the one proposed in [3], but is based on the same idea.

**Proposition 3.5** The family of \( P_{0} \)-indexed complex valued plane waves

\[
\varphi(p) : \text{span}_C \{e, \bar{e}\} \to M_{q}
\]

\[
e \mapsto \exp(ix.p)
\]

\[
\bar{e} \mapsto \exp(-ix.p)
\]

are solutions of the \( q \)-d’Alembert equation.

**Proof.** First note that \( \exp(ix.p) \) transforms as a scalar under the coaction by \( \tilde{\mathcal{L}}_q \), since the dilaton terms always cancel. Thus \( \varphi(p) \) are \( \tilde{\mathcal{L}}_q \)-comodule morphisms, and it is also obvious
that they are *-maps. It remains to show $\Box \exp(ix.p) = 0$:

$$
\partial^2 \exp(ix.p) = \sum_{n} \frac{1}{n!} \partial^2 x_1 \ldots x_n p_n \ldots p_1 \\
= \sum_{n} \frac{1}{n!} \partial_{x_1} x_2 \ldots x_n (1 + q^{-1} PR_{L_{12}}^{-1} + \ldots + q^{-(n-1)} PR_{L_{1(n-1)}}^{-1}) p_n \ldots p_1 \\
= \sum_{n} \frac{1}{n!} \partial_{x_1} x_2 \ldots x_n p_n \ldots p_2 p^3 \\
= \exp(x.p) \ i p^3
$$

and thus $\Box \exp(ix.p) = 0$. □

These plane wave type solutions exist only on the q-light cone, giving further support to our claim that wave equations on quantum Minkowski space should be massless. In this special case of plane wave solutions, we can also show a stronger statement than proposition 3.5. In general, we require $\varphi$ only to be a comodule morphism on the linear component of $\mathcal{P}(\mathbb{C})$, but the plane wave solutions can be extended to the whole algebra:

**Proposition 3.6** The linear maps $\varphi(p)$ extend to *-algebra maps

$$
\varphi(p) : \mathcal{P}(\mathbb{C}) \rightarrow M_q,
$$

which are also $\mathcal{L}_q$ comodule morphisms.

**Proof.** It is sufficient to show that $\exp(-ix.p)$ and $\exp(ix.p)$ commute. Using the statistics relations $p_1 x_2 = x_2 R_{L_{12}}^{-1} p_1$ and lemma 3.4, which implies

$$
q R_{L_{12}}^{-1} p_1 p_2 = q^{-1} p_2 p_1, \quad R_{M_{12}} p_2 p_1 = q^2 R_{L_{12}}^{-1} p_2 p_1,
$$

we can show that the monomials commute:

$$
(x_1 \ldots x_n p_n \ldots p_1) (x_{1'} \ldots x_{m'} p_{m'} \ldots p_{1'}) = x_1 \ldots x_n x_{1'} \ldots x_{m'} (q R_{L_{nm}}^{-1} \ldots q R_{L_{1m'}}^{-1}) (q R_{L_{n1'}}^{-1} \ldots q R_{L_{11'}}^{-1}) p_n \ldots p_1 p_{m'} \ldots p_{1'}
$$

$$
=q^{-nm} x_1 \ldots x_n x_{1'} \ldots x_{m'} p_{m'} \ldots p_{1'} p_n \ldots p_1
$$

$$
=q^{-nm} x_{1'} \ldots x_{m'} x_1 \ldots x_n (R_{M_{1m'}} R_{M_{1n'}} \ldots R_{M_{nn'}}) (R_{M_{11'}} R_{M_{1n'}}) p_{m'} \ldots p_{1'} p_n \ldots p_1
$$

$$
=x_1' \ldots x_{m'} x_1 \ldots x_n (q R_{L_{m'n}}^{-1} \ldots q R_{L_{1n'}}^{-1}) (q R_{L_{1'm'}}^{-1} \ldots q R_{L_{11'}}^{-1}) p_{m'} \ldots p_{1'} p_n \ldots p_1
$$

$$
=(x_{1'} \ldots x_{m'} p_{m'} \ldots p_{1'}) (x_1 \ldots x_n p_n \ldots p_1)
$$

We used primed and unprimed indices to distinguish the two monomials. □

4 q-Vector field
4.1 The q-Maxwell equation

For q-Maxwell equations, we apply a similar strategy as for the q-d’Alembert equation: we first give a more abstract definition in terms of $\delta$ and $d$ and then show how this equation looks in terms of the maybe more familiar braided differential operators $\partial$.

**Definition 4.1** A solution of the q-Maxwell equation is a 1-form $A$ such that

$$\delta dA = 0.$$ \hfill (23)

Using the results from the preceding sections, we can rewrite this rather abstract relation to resemble the classical equation $\partial^\mu \partial_\mu A_\nu - \partial^\mu \partial_\nu A_\mu = \partial^\mu \partial_{[\mu} A_{\nu]} = 0$:

**Proposition 4.2** The equation (23) is equivalent to the set of four equations

$$\partial^c \partial_{[c} A_{z]} = 0,$$ \hfill (24)

or alternatively

$$\Box A_z - \partial_z \partial^c A_c = 0.$$ \hfill (25)

Here $\partial$ denotes the braided differential operators on $M_q$, $\Box$ the q-d’Alembert operator and ‘[ ]’ the q-antisymmetriser.

**Proof.** First verify by explicit calculation that

$$H_{[2]}^{ab} H_{[3]}^{cd} H_{[3]}^{ef} \frac{(1 + q^2)^2}{q(2(1 + q^2 + q^4))^{1/2}} g^{xc} A_{[2]}^{ab} A_{[3]}^{cd} A_{[3]}^{ef}$$ \hfill (26)

By virtue of this relation, we obtain

$$0 = \delta dA(e_z)$$

$$= \partial_x \partial_0 A_0 A_{[2]}^{ab} H_{[2]}^{cd} H_{[3]}^{ef} A_{[3]}^{cd} h_{klm} H_{[3]}^{klm}$$

$$= \partial_x \partial_0 A_0 H_{[2]}^{ab} H_{[3]}^{ef} H_{[3]}^{klm} H_{[3]}^{klm}$$

$$= \partial_x \partial_0 A_0 H_{[2]}^{ab} H_{[3]}^{ef} H_{[3]}^{klm} H_{[3]}^{klm}$$

$$= \partial_x \partial_0 A_0 g^{xc} A_{[2]}^{ab}$$

$$= \partial^c \partial_{[c} A_{z]},$$

where we used the definitions of $\delta$ and $d$, the relations (17) and (11), and finally (26). This establishes the equivalence of (24) and (23). In order to prove (25) note that the generators of
\[ V(R_M) \text{ obey} \]
\[ \alpha^a \alpha^b = \alpha^c \alpha^d R_{M \, ca}^{ bd}. \quad (27) \]

Hence with (12), we find:
\[
\begin{align*}
0 & = \partial^c \partial_{[c} A_{z]} \\
& = \partial^c (\partial_{[c} A_{z]} - \partial_{i} A_{j} R_{M \, i z}^{ j}) \\
& = \partial^c \partial_{c} A_{z} - \partial^c \partial_{i} A_{j} R_{M \, i z}^{ j} \\
& = \Box A_{z} - \partial_{z} \partial^c A_{c}.
\end{align*}
\]

In the form (24), the \( \tilde{L}_q \)-covariance of the q-Minkowski equations can be easily established. Again, a massive field equation, i.e. a q-Proca equation

\[ \partial^c \partial_{[c} A_{z]} = m^2 A_{z} \]

would be \( \tilde{L}_q \)-covariant only if \( m \) transformed as \( m \rightarrow m \otimes \zeta^{-1} \), and again we shall find q-deformed plane wave solutions only on the q-light cone.

As in the undeformed case, solutions to the q-Maxwell equation have a gauge freedom. If \( A \) is a solution of the q-Maxwell equation and \( \phi \) a 0-form then by virtue of theorem 2.8 the 1-form \( A + d \phi \) is also a solution. Provided it is possible to solve the inhomogeneous equation

\[ \Delta \phi = -\delta A, \]

we can use this gauge freedom to arrange for \( A \) to satisfy the q-Lorentz gauge condition

\[ \delta A = 0. \]

Using an argument similar to the proof of proposition 3.2, one can show that the q-Lorentz gauge condition is satisfied iff

\[ \partial^c A_{c} = 0. \quad (28) \]

Proposition 4.2 implies that in this case \( A \) obeys \( \Delta A = 0 \) or equivalently

\[ \Box A_{z} = 0. \]

As in the classical case, a field \( A \) satisfying the q-Lorentz gauge has a residual gauge freedom

\[ A \mapsto A + d \phi, \]
where \( \varphi \) is a solution of the q-d’Alembert equation.

The q-Maxwell equation also has a family of plane wave solutions. However, in this case the solutions are indexed by the q-momentum \( p^a \) (the generators of the q-light cone \( P_0 \)) and the ‘q-amplitude’ \( A_z \), which are the generators of a copy of \( M_q \). We define the algebra \( Y \) as the quotient of \( P_0 \otimes M_q \) by the relation
\[
p^c A_c = 0.
\]
This algebra \( Y \) labels the plane wave solutions to the q-Maxwell equation:

**Proposition 4.3** Then the family of \( Y \)-indexed 1-forms
\[
A : e_z \mapsto \exp(ix.p) \otimes A_z
\]
viewed as a formal power series in \( M_q \otimes Y \) are solutions of the q-Maxwell equation and satisfy the q-Lorentz gauge condition \( \delta A = 0 \).

**Proof.** Using the q-Maxwell equations in the form (24), one finds:
\[
\partial^c \partial_c \exp(ix.p) \otimes A_z = \partial^c (\partial_c \exp(ix.p) \otimes A_z - \partial_m \exp(ix.p) \otimes A_n R^m_{cz} R^c_{mn}) = \exp(ix.p) p^p p^m \otimes A_n R^m_{cz}
\]
\[
= 0.
\]
Here we used (12), proposition 3.5 and relation (27). These solutions obviously satisfy the q-Lorentz gauge condition. \( \square \)

A solution \( A \) of the q-Maxwell equation (24) defines a 2-form \( F = dA \), the q-field strength tensor which obeys the two equations
\[
dF = 0, \quad \delta F = 0 \tag{29}
\]
Proposition 4.2 implies that the second relation is equivalent to
\[
\partial^c F_{cd} = 0. \tag{30}
\]
Furthermore, we find
\[
\Box F_{ab} = \partial^c F_{ab} = \partial^c \partial_{[a} A_{b]} = \partial^c \partial_{[a} \partial A_{b]} = \partial^c \partial_{[a} \partial^e A_c = 0,
\]
\[22\]
using the fact that $\Box$ is central in $\mathcal{D}$ and the q-Maxwell equation for $A$. Since at present we do not have a q-Poincaré lemma, we only know that q-Maxwell equations in the form (29) are implied by (23), but we cannot prove that they are equivalent. But nevertheless, we shall proceed by investigating the q-Maxwell equation in terms of this q-field strength tensor.

4.2 Elements of the $SL_q(2,\mathbb{C})$-spinor calculus

In the next section we shall give a $SL_q(2,\mathbb{C})$-spinor description of the q-field strength tensor $F$ similar to the the classical case. For this purpose, we need some elements of the $SL_q(2,\mathbb{C})$-spinor calculus, some aspects of which were already discussed in [13]. This case is very simple since the R-matrix [1] is of Hecke-type, i.e. obeys

$$0 = (PR + q^{-1})(PR - q).$$

Relation (31) ensures that after a suitable normalisation these operators are projectors. Furthermore, one does not have any problems with higher q-antisymmetrisers, since they do not exist. One could also define a q-antisymmetriser by first identifying a q-antisymmetric $\varepsilon_{AB}$, similar to the procedure in section 2.1, but this approach gives the same result. The q-antisymmetric spinor $\varepsilon_{AB}$ is simply the $SL_q(2,\mathbb{C})$-spinor metric (3), which obeys

$$q\varepsilon_{AB} = -\varepsilon_{CD}R^{-1DC}_{AB}.$$  

One can easily verify that then

$$A^{AB}_{CD} = \frac{1}{q + q^{-1}}\varepsilon^{BA}_{CD} = \frac{1}{q + q^{-1}}(PR^{-1} - q^{-1})^{AB}_{CD}. $$

obeys $A^2 = A$ by virtue of (31). We also define define a q-symmetriser

$$S := \frac{1}{2}(1 - A) = \frac{1}{q^{-1} + q}(PR^{-1} + q),$$

and accordingly the q-symmetrisation `(` `)` and q-antisymmetrisation `[ ]` of a multivalent q-spinor $T_{...AB...}$ with two adjacent lower indices $A$ and $B$ as

$$T_{...(AB)...} = T_{...CD...}S_{AB}^{CD}, \quad T_{...[AB]...} = T_{...CD...}A_{AB}^{CD}$$
and similarly for upper indices. Due to (31), the $q$-(anti)-symmetrisation of a $q$-spinor is $q$-(anti)-symmetric:

$$T_{\ldots (AB) \ldots} = 0, \quad T_{\ldots [AB] \ldots} = 0,$$

and we also have a decomposition

$$T_{\ldots AB \ldots} = T_{\ldots (AB) \ldots} + T_{\ldots [AB] \ldots}. \quad (34)$$

The $q$-(anti)-symmetrisation is also $SL_q(2, \mathbb{C})$-covariant, i.e. both operations ‘( )’ and ‘[ ]’ commute with the coaction by $SL_q(2, \mathbb{C})$. Furthermore, if $T_{\ldots CD \ldots}$ is a multivalent $q$-spinor, then

$$T_{\ldots [CD] \ldots} = \frac{1}{q^{-1} + q} \varepsilon_{CD} T_{\ldots B \ldots}. \quad (35)$$

In this formula we do not violate the index notation by writing $\varepsilon_{CD}$ on the left since $C$ and $D$ are adjacent indices and the generators of $SL_q(2, \mathbb{C})$ preserve the spinor metric.

4.3 $q$-Spinor analysis of the $q$-field strength tensor

We now apply the results from the last section to the field strength tensor $F$, or more generally, to any $q$-antisymmetric tensor $F_{ab} \in M_q$. Similar to the classical case, such a tensor decomposes into $SL_q(2, \mathbb{C})$-spinors:

**Proposition 4.4** Let $F_{ab} \in M_q$ be an $q$-antisymmetric $\widetilde{L}_q$-tensor in the sense of [7]. Then

$$f^{AB}_{A' B'} := F_{AI' B'} R^I_{A' I}$$

admits a decomposition

$$f^{AB}_{A' B'} = \phi^{AB} \varepsilon_{A' B'} + \varepsilon^{AB} \psi_{A' B'},$$

where $\phi^{AB}$ and $\psi_{A' B'}$ are $q$-symmetric $SL_q(2, \mathbb{C})$-spinors.

**Proof.** Since the tensor $F_{ab}$ is $q$-antisymmetric, $f^{AB}_{A' B'}$ obeys:

$$f^{AB}_{A' B'} = F_{AI' B'} R^I_{A' I}$$

$$= -F_{CC' DD'} R^D_{LA'I'B'} R^L_{A' I}$$

$$= -F_{CC' DD'} R^D_{LA'I'B'} R^L_{A' I}$$

$$= -f^{CD}_{C' D'} R^D_{A'B'} R^{AB}_{DC} \quad (36)$$

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Due to relation (34), we also have
\[ f^{AB} A'B' = f^{(AB)} (A'B') + f^{[AB]} (A'B') + f^{(AB)} [A'B'] + f^{[AB]} [A'B']. \]
This implies with (36), (33) and (31):
\[ f^{AB} A'B' = -f^{CD} C'D' R^{D'}_{A'B'} R^{A'B}_{DC} \]
\[ = -q^2 f^{(AB)} (A'B') + f^{[AB]} (A'B') + f^{(AB)} [A'B'] - q^{-2} f^{[AB]} [A'B'], \]
and also
\[ f^{AB} A'B' = -f^{CD} C'D' R^{-1D'}_{B'A'} R^{-1BA} \]
\[ = -q^{-2} f^{(AB)} (A'B') + f^{[AB]} (A'B') + f^{(AB)} [A'B'] - q^2 f^{[AB]} [A'B'], \]
and therefore 0 = f^{(AB)} (A'B') + f^{[AB]} [A'B']. With relation (35), it follows
\[ f^{AB} A'B' = \phi^{AB} \varepsilon_{A'B'} + \varepsilon^{AB} \psi_{A'B'}, \]
where
\[ \phi^{AB} = f^{(AB)} C^C, \quad \psi_{A'B'} = f^C_{A'B'} \]
are q-symmetric $SL_q(2,C)$-spinors.

In the case of a real tensor, the two components $\phi$ and $\psi$ are not independent, but are related by the star structure on $M_q$. Recall that the conjugate of a tensor $T_{a...d} \in M_q$ is given by $\overline{T}_{a...d} = T_{\bar{d}...\bar{a}}$. Such a tensor is called real if $T = \overline{T}$.

**Proposition 4.5** A q-antisymmetric tensor $F_{ab}$ is real iff
\[ \psi_{DC} = -\overline{\phi^{CD}}. \]

Thus a real tensor can be written as
\[ f^{AB} A'B' = \phi^{AB} \varepsilon_{A'B'} + \varepsilon_{BA} \phi_{B'A'} \]
in terms of the q-symmetric $SL_q(2,C)$-spinor $\phi^{AB}$.

**Proof.** For the proof, we need the symmetry property
\[ R^{AB}_{CD} = R^{DC}_{BA} \] (37)
of the $SU_q(2)$ R-matrix, which can be verified by inspection of (1). A similar relation holds for $R^{-1}$ and $\tilde{R}$. Hence if $F_{ab}$ is real then

$$
\bar{f}_{AB}^A = F_{A'B'} R_{A'I}^B R_{A'I}^B
= F_{A'B'} R_{A'I}^B
= F_{A'B'} R_{A'I}^B
= \tilde{f}_{AB}^A.
$$

In components, this means that

$$
\bar{\phi}^{AB}_{A'B'} + \varepsilon^{AB} \bar{\psi}_{A'B'} = \phi^{A'A'}_{B} \varepsilon_{BA} + \varepsilon^{B'A'}_{A} \psi_{BA}.
$$

Due to the q-symmetry of $\phi$ and $\psi$ and the q-antisymmetry of $\varepsilon_{AB}$, multiplication of this equation by $q R_{BA}^{CD}$ yields by virtue of (33):

$$
q^2 \bar{\phi}^{CD}_{A'B'} - \varepsilon^{CD} \bar{\psi}_{A'B'} = -\phi^{B'A'}_{CD} \varepsilon_{DC} + q^2 \varepsilon^{B'A'}_{DC} \psi_{DC},
$$

again using the relation (37). Thus

$$
\bar{\phi}^{CD}_{A'B'} = \varepsilon^{B'A'} \psi_{DC},
$$

which implies the proposition, since $\varepsilon^{B'A'} = -\varepsilon_{A'B'}$.

Classically, this decomposition of the field strength tensor into spinors coincides with the decomposition into its self-dual and anti-self-dual part. The same result holds in the non-commutative case. By virtue of proposition 2.11, any two-form $F$ on quantum Minkowski space can be decomposed uniquely as

$$
F = F^+ + F^-,
$$

where $F^+ = \frac{1}{2} (F + *F)$ and $F^- = \frac{1}{2} (F - *F)$ are self-dual and anti-self-dual, i.e. obey $*F^\pm = \pm F^\pm$. The q-Maxwell equation (29) are then equivalent to either

$$
dF^+ = 0, \quad dF^- = 0,
$$

or the two equations

$$
\delta F^+ = 0, \quad \delta F^- = 0.
$$

(38)
Proposition 4.6 Let $F_{AB}$ be a $q$-antisymmetric tensor. Then

$$f^+_{AB} A'B' = \phi_{AB} \varepsilon_{A'B'}, \quad f^-_{AB} A'B' = \varepsilon^{AB} \psi_{A'B'}$$

are the self-dual and anti-self-dual parts of $f$.

**Proof.** It suffices to show that $f^\pm$ are self-dual and anti-self-dual, respectively. On the tensor $f_{AB} A'B'$, the $q$-Hodge star operation is implemented by the matrix

$$U_{(2)CDC'D'} := \tilde{R}_{A'C'}^{I'B'} H_{(2)C'J'D'} R_{J'D'}^{I'B'}.$$

By explicit calculation, one verifies that this operator satisfies the relations

$$S^{A'B'}_{EF} \varepsilon_{A'B'} U_{(2)CDC'D'} = S^{CD}_{EF} \varepsilon_{C'D'},$$

$$\varepsilon_{A'B'} S^{E'F'} U_{(2)CDC'D'} = \varepsilon^{CD} S^{E'F'}_{C'D'}.$$

Since $\phi_{AB}$ and $\psi_{A'B'}$ are $q$-symmetric and thus eigenvectors of the $q$-symmetriser $S$, this implies that $\phi_{A'B'}$ and $\varepsilon^{AB} \psi_{A'B'}$ are self-dual and anti-self-dual, respectively. $\square$

If we are looking for real solutions of the $q$-Maxwell equations (29), it is thus sufficient to solve one of the two equations in (38). In terms of the $SL_q(2, \mathbb{C})$-spinor $\psi$ this means:

**Corollary 4.7** For real $F$, the $q$-Maxwell equation $\delta F^-$ is equivalent to either

$$\nabla^{B'I'} \psi_{I'B'} = 0 \quad (39)$$

or the conjugate equation

$$\phi^{B'I'} \nabla'_{I'B'} = 0$$

where $\nabla^{C'} := \tilde{R}_{A'A}^{C'} \partial^{A}$, and its conjugate $\nabla'_{C'} := \partial'_{A'} \tilde{R}_{CC'}^{AA'}$ acts from the right.

**Proof.** Proposition 4.6 implies with (34)

$$0 = \partial^a F^-_{ab} = \tilde{R}_{A'B'}^{I'B'} \partial'_{A'} \varepsilon^{AI} \psi_{I'B'} = \tilde{R}^{I'B'}_{A'A'} \partial'_{A'} \psi_{I'B'} = \nabla^{B'I'} \psi_{I'B'}.$$
Since the tensor $F_{ab}$ is real, proposition 4.5 implies for the conjugate equation:

$$0 = \overline{\psi}_{I'B'}\partial_{A'}^{A}\varepsilon_{AI}\overline{R}_{A'I}^{I'B'} = \phi_{B'I'}\partial_{A'}^{A}\varepsilon_{AI}\overline{R}_{A'I}^{I'B'} = \phi_{B'I'}\partial_{A'A}\overline{R}_{B'I}^{A'A} = \phi_{B'I'}\nabla_{I'B'}.$$ 

We used relation (37) and $\varepsilon_{AB} = -\varepsilon_{BA}$.

Equation (39) is also quite easy to generalise to an arbitrary q-spinor field. We call a totally q-symmetric spinor $\psi_{A_1...A_n}$ satisfying

$$\nabla^{BA_1}\psi_{A_1...A_n} = 0$$

a massless q-spinor field of spin $\frac{1}{2}n$. Thus in particular, we find the q-Weyl equation

$$\nabla^{BA}\psi_A = 0.$$

Details of the general case will be discussed elsewhere.

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