N-dimensional gravitational collapse of Type I matter fields with non-zero radial pressure

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Abstract

We study the gravitational collapse of Type I matter field in $N$ dimensional spacetime with radial pressure $p_r$ as a function of $r$. We find that for a given smooth initial data set satisfying physical requirements, naked singularities exist for spacetime dimensions $N = 4$ and 5 while for $N \geq 6$ Cosmic Censorship Conjecture withholds its ground. We, also, study the collapse with linear equation of state and find that, similar to dust collapse with appropriate choice of initial data naked singularities occur in all dimensions.

Keywords: Type I matter field, gravitational collapse, central-singularity

I. Introduction

In the recent paper [1] R. Goswami and P. S. Joshi pointed out that the occurrence of naked singularity (NS) as end state of dust collapse beginning from smooth initial data with physically relevant restrictions, can be removed when one goes to higher dimensional space-time. This has restored Cosmic Censorship Conjecture (CCC) (articulated by Roger Penrose [2]) in dust collapse for sufficiently high dimension of the space-time. Many other cases of the dust collapse have been studied by various authors [3-4].

A lot of importance is being given to the physically relevant models such as Type I matter fields that include most of the known physical forms of matter like dust, perfect fluid, etc. The authors in [5-7] have discussed the role of initial data in spherically symmetric gravitational collapse for Type I matter fields. Recently, in the paper [8] A. Mahajan et al have shown the restoration of CCC in Einstein cluster spacetime for spacetime dimensions $N \geq 6$ with non-zero tangential pressure $p_\theta$ and radial pressure $p_r = 0$.

F.I. Cooperstock and et al. [9] have shown that if the radial pressure $p_r$ is positive, $r > 0$ then non-central singularity is covered by the horizon irrespective of the sign of tangential pressure $p_\theta$.

In recent years, special attention is given to higher dimensional gravitational collapse since the development in string theory indicate that gravity may be a higher dimensional interaction. Hence, it would be interesting to know the status of CCC in higher dimension.
spacetime. These studies may help in possible appropriate mathematical formulation of the conjecture or assist in providing any possible proof of CCC.

We take up the problem of Type I matter field collapse with radial pressure $p_r$ as a function of $r$ and arbitrary tangential pressure $p_\theta$. The apparent horizon for the evolving collapse has been studied and it is found that the final state of collapsing cloud results in the black hole (BH) for the dimension of the spacetime $N \geq 6$.

Introduction of equation of state has importance as it makes the collapsing models more physically relevant. Does the linear equation of state $p = k \rho$ stimulates the outcome of NS/BH phases as end state of gravitational collapse? We study, this case in section V.

In section II, we briefly summarize the field equations and state the conditions on the initial data under which the collapse evolves. The evolving field equations with $p_r = p_r(r)$ have been illustrated in section III. In section IV, the behaviour of the apparent horizon is studied. The case with linear equation of state has been analyzed in section V. Finally, we make some concluding remarks.

II. Field equations and initial data

The general spherically symmetric metric describing $N$ dimensional space-time geometry within the collapsing cloud can be described in comoving coordinates $(t, r, \theta, \phi)$ by

$$ds^2 = -e^{2\nu(t,r)}dt^2 + e^{2\psi(t,r)}dr^2 + R^2(t,r)d\Omega^2_{N-2}$$

where

$$d\Omega^2_{N-2} = \sum_{i=1}^{N-2} \left[ \prod_{j=1}^{i-1} \sin^2(\theta^j) \right] (d\theta^i)^2$$

is the metric on an $(N-2)$ sphere. The stress-energy tensor for Type I field in a diagonal form is given by

$$T^t_t = -\rho, \quad T^r_r = p_r, \quad T^\theta_\theta = T^\phi_\phi = p_\theta$$

The quantities $\rho$, $p_r$, and $p_\theta$ are the energy density, radial and tangential pressures respectively. We take the matter field to satisfy weak energy condition i.e. the energy density measured by any local observer be non-negative, so for any vector $V^i$, we must have, $T_{ik}V^iV^k \geq 0$ which means $\rho \geq 0$; $\rho + p_r \geq 0$ and $\rho + p_\theta \geq 0$.

Einstein field equations for the metric (1) are described by

$$\rho = \frac{(N-2)\dot{F}'}{2R^{N-2}R'}$$

$$p_r = -\frac{(N-2)\ddot{F}}{2R^{N-2}R}$$

$$\nu' = \frac{(N-2)(p_\theta - p_r)}{(\rho + p_r)R} - \frac{\dot{p}_r}{\rho + p_r}$$

$$-2\dot{R} + R\frac{\dot{G}}{G} + \frac{\dot{R}H'}{H} = 0$$

$$G - H = 1 - \frac{F}{R^{N-3}}$$

where we have defined $G(t, r) = e^{-2\psi(R')^2}$ and $H(t, r) = e^{-2\nu(R)^2}$. 

The arbitrary function $F(t, r)$ represents the total mass in a shell of collapsing cloud of comoving radius $r$. The weak energy conditions imply $F \geq 0$. The regularity at the initial epoch $t = t_i$ is preserved by $F(t_i, 0) = 0$ i.e. the mass function should vanish at the centre of the cloud. The space-time density singularity is due to $R = 0$ and $R' = 0$, the later one is due to shell-crossings and can possibly be removed from the space-time [11], hence we consider here only a physically relevant singularity $R = 0$ known as shell-focusing singularity where all matter shells collapse to a zero physical radius. The scaling freedom available for the radial co-ordinate $r$ is being used to introduce the function $v(t, r)$ by the relation

$$R(t, r) = vv(t, r),$$

this relation is due to the definition $v(t, r) = R/r$ [12]. We have $v(t_i, r) = 1$; $v(t_s(r), r) = 0$ and for collapse $\dot{v} < 0$. The time $t = t_s(r)$ corresponds to the shell-focusing singularity $R = 0$. The six arbitrary functions of the shell radius $r$ as given by $\nu(t_i, r) = \nu_o(r)$, $\psi(t_i, r) = \psi_o(r)$, $R(t_i, r) = r$, $\rho(t_i, r) = \rho_o(r)$, $p_r(t_i, r) = p_r(r)$, $p_\theta(t_i, r) = p_\theta(r)$ evolve the dynamics of the initial data prescribed at the initial epoch $t = t_i$. We have a total of five equations with seven unknowns, namely $\rho, p_r, \rho_\theta, \nu, \psi, R$ and $F$ giving us the freedom of choice of two functions. Selection of these two free functions, subject to the given initial data and the weak energy condition above, determines the matter distribution and the metric of the spacetime and thus, leads to a particular time evolution of the initial data. The existence and uniqueness of solution of the system of field equations with above mentioned initial data has been discussed by Joshi and Dwivedi [13]. The solution continues to exist in the neighbourhood of the singularity given by $R = 0$.

### III. Radial pressure $p_r = p_r(r)$

We consider radial pressure $P_r$ as a function of $r$. The two allowed free functions $p_r$ and $\nu(t, r)$ are chosen as follows:

$$p_r = p_r(r), \quad \nu(t, r) = c(t) + \eta(R)$$

Now, let us see, how the Einstein’s field equations react to this choice which will decide the end state of collapse. As $p_r$ is a function of $r$ alone, integrating equation (9), we obtain

$$F(t, r) = -\frac{2}{(N-1)(N-2)} p_r R^{N-1} + z(r)$$

where $z(r)$ is another arbitrary function of $r$. We need to choose $z(r)$,

$$z(r) = \frac{4}{(N-1)(N-2)} p_r (N-1) p_r(r)$$

so that the mass function takes the form

$$F = \frac{2}{(N-1)(N-2)} p_r (2r^{N-1} - R^{N-1})$$

that satisfies the regularity conditions at the initial epoch $F(t_i, r) = 2 p_r r^{N-1}/(N-1)(N-2)$ and $F(t_i, 0) = 0$. From equation (10), it is evident that non-negativity of radial pressure maintains non-negativity of mass function throughout the gravitational collapse i.e. $F \geq 0$ provided $p_r \geq 0$. Next, equation (8) becomes

$$\rho = \frac{(N-1) p_r (2r^{N-2} - R^{N-2}R') + p_r'(2r^{N-1} - R^{N-1})}{(N-1) R^{N-2}R'}$$

At the initial epoch $t = t_i$,

$$\rho_o = p_r + \frac{1}{N-1} t p_r' \geq 0$$

\[12\]
an all important relation between energy density, radial pressure and pressure gradient. In the limit of approach to the singularity i.e. as $r \to 0$ and $t \to t_s$

$$\rho = \frac{p_r(0)(2 - v^{N-1})}{v^{N-1}},$$

(13)

the density blows up at the singularity. Equation (13) can be written in the form

$$G = e^2 \eta(R) d(r)$$

(14)

with $d(r) = 1 + r^2 b(r)$ where $b(r)$ is the energy distribution function for the collapsing shells. Using equation (10) in equation (5), we obtain

$$\rho + p_r \eta, R R' = (N - 2)(p_\theta - p_r) R' - R p'_r$$

(15)

The tangential pressure blows up in the limit of approach to the singularity as the density diverges at the singularity and we observe that all of initial data is not independent at the initial epoch $t = t_i$

$$\rho + p_r \eta, R = (N - 2)(p_\theta - p_r) - r p'_r$$

(16)

Using the initial data prescribed at the initial epoch $t = t_i$ to evolve the collapse, we can integrate equation (16) and obtain,

$$\eta(R) = \int_0^R \frac{(N - 2)(p_\theta - p_r) - R p'_r}{R(\rho_0 + p_r)} dR$$

(17)

so, velocity distribution function $\eta(R)$ is determined by smooth functions $p_r$ and $p_\theta$. Using equations (10) and (14), equation (6) can be put in the form

$$R^{(N-2)/2} \dot{R} = - a(t) e^{\eta} \sqrt{[1 + r^2 b(r)] R^{N-3} c^{2\eta} - R^{N-3} + \frac{2 p_r (2 - v^{N-1})}{(N - 1)(N - 2)}}$$

(18)

where $a(t)$ is a function of time and we choose $a(t) = 1$ by suitable scaling of time co-ordinate.

We define

$$h(R) = \frac{e^{2\eta(R)} - 1}{R^2} = 2 g(R) + O(R^2)$$

(19)

Using this definition, equation (18) becomes

$$v^{(N-3)/2} \dot{v} = -e^{\eta} \sqrt{v^{N-3} b(r) c^{2\eta} + v^{N-1} h(rv) + \frac{2 p_r (2 - v^{N-1})}{(N - 1)(N - 2)}}$$

(20)

Integrating above equation, we obtain

$$t(v, r) = \int_v^1 \frac{v^{(N-3)/2} dv}{e^{\eta} \sqrt{v^{N-3} c^{2\eta} b(r) + v^{N-1} h(rv) + \frac{2 p_r (2 - v^{N-1})}{(N - 1)(N - 2)}}}$$

(21)

where the variable $r$ is treated as a constant. Further, the time taken for the central shell to reach the singularity is given by

$$t_{x_0} = \int_0^1 \frac{v^{(N-3)/2} dv}{\sqrt{v^{N-3} b(0) + v^{N-1} h(0) + \frac{2 p_r(0) (2 - v^{N-1})}{(N - 1)(N - 2)}}}$$

(22)
where \( dv \) turn, depend on the given set of density, tangential pressure and velocity function \( r \) and \( p \) and initial data set for the collapse. Now, we assume that the initial data functions \( \chi \) is well defined provided \([v^{N-3}b(0) + v^{N-1}h(0)] + [2 p_r(0) (2 - v^{N-1})]/[(N-1)(N-2)] > 0\). The Taylor expansion of \( t(v, r) \) around the center \( r = 0 \) is given by

\[
t(v, r) = t(v, 0) + r \chi_p(v) + O(r^2)
\]

The time taken for other shells close to center \( r = 0 \) to reach the singularity as \( t \rightarrow t_s \) can be determined from

\[
t_s(r) = t_{s, r} + r \chi_p(0) + O(r^2)
\]

and we have

\[
\chi_p(v) = -\frac{N - 3}{2} \int_v^1 \frac{v^{(N-3)/2}}{[v^{N-3}b(0) + v^{N-1}h(0)] + [2 p_r(0) (2 - v^{N-1})]/[(N-1)(N-2)]} \left[ v^{N-3}b(0) + v^{N-1}h(0) + \frac{2 p_r(0) (2 - v^{N-1})}{(N-1)(N-2)} \right]^{3/2} dv.
\]

It is clear that \( \chi_p(v) \) depends on the functions \( p_r(0), h(0) \) and \( b(0) \) and these functions, in turn, depend on the given set of density, tangential pressure and velocity function \( \nu \). For any constant \( v \) surface, we have \( dv = 0 \) which implies \( \dot{v} \chi_p(v) = -v' \).

Now, equation (20) takes the form

\[
v^{(N-3)/2} v' = [\chi_p(v)] D(0, v)
\]

where

\[
D(0, v) = \sqrt{v^{N-3}b(0) + v^{N-1}h(0) + \frac{2 p_r(0) (2 - v^{N-1})}{(N-1)(N-2)}}
\]

From equations (10) and (12), it is evident that all of initial data is not independent. Hence, we conclude that \( b(r) \), radial and tangential pressures \( p_r \) and \( p_\theta \) will be sufficient to form initial data set for the collapse. Now, we assume that the initial data functions \( b(r), p_\theta(r, v) \) and \( p_r(r) \) are at least \( C^2 \) functions and that satisfy the regularity conditions at the center \( r = 0 \) i.e. \( p'_r(0) = p'_\theta(0) = 0 \) and \( p_{r, r}(0) - p_{\theta, \theta}(0) = 0 \).

Using equation (20), we can write

\[
\chi_p(0) = \lim_{v \to 0} \chi_p(v) = \lim_{v \to 0} \frac{v^{N-3} v'}{D(0, v)}
\]

Since the function \( D(0, v) > 0 \) and \( v \in [0, 1] \) and if the above limit exists, then the sign of \( \chi_p(0) \) as positive or negative shall depend on the behaviour of \( v' \). Hence, \( v' > 0 \) may give rise to \( \chi_p(0) > 0 \) as limit of sequence of positive terms can not be negative. If \( \chi_p(0) = 0 \) then we need to consider the higher derivative of \( t(v, r) \) at \( r = 0 \) for similar analysis.

**IV. Study of apparent horizon**

The end state of gravitational collapse, in terms of either a naked singularity or a black hole, is determined by the causal structure of non-spacelike curves in the neighbourhood of singularity. The singularity is visible if there exist future directed families of non-spacelike curves emanating from the singularity that reach the far away observers in the future, and which have past end point at the singularity. Otherwise event horizon forms early enough to cover the singularity and thus, the end state of collapse is a black hole. To determine this characteristic of singularity, we study the behaviour of apparent horizon which is the boundary of trapped surfaces as the collapse evolves. This boundary of the trapped region of the space-time within the collapsing cloud is given by the equation,

\[
\frac{F}{R^{N-3}} = 1
\]
and which is the equation for the apparent horizon, for details, we refer the reader to [8].
Firstly to know about the NS/BH phases for any \( r > 0 \) along the singularity curve \( t = t_s(r) \).
For example, from above equation along the singularity curve (which corresponds to \( R = 0 \)),
for any \( r > 0 \), the mass function \( F(t, r) \) takes the form
\[
F(t, r) = c_1(2 - v^{N - 1}),
\]
for a specific choice of \( r \), whereas the physical area radius \( R(= c_2v) \to 0 \) as \( v \to 0 \) where \( c_1 \) and \( c_2 \) are constants. Therefore, trapped surfaces form that cover the neighbourhood of the center before the development of singularity.

Using equation (29), the apparent horizon \( v_{Nah} \) is obtained through the equation
\[
\left( N - 1 \right)(N - 2) + 2r^2 p_r v^{N - 3} - 2 = 0.
\]
(30)

We know from above that all the points \( r > 0 \) on the singularity curve are already covered and the behaviour of apparent horizon is being studied close to the central singularity at \( r = 0, R = 0 \), therefore, the values of \( r \) in this analysis must be sufficiently small. With this background, the equation (30) is a polynomial equation in \( v \). For dimensions \( N = 4 \) and \( N = 5 \) the \( v_{Nah} \) are determined as positive roots of above polynomial equation and we obtain
\[
v_{4ah} = \frac{A_{1}^{1/3}}{r^{1/3}} - \frac{A_{1}^{-1/3}}{r^{2/3}}, \quad A_{1} = r^3 p_r + \sqrt{1 + r^6 p_r^2}.
\]
(31)

and
\[
v_{5ah} = \frac{\sqrt{p_r \left[ -3 + \sqrt{9 + 2r^4 p_r^2} \right]}}{r^{1/3}}.
\]
(32)

For \( N \geq 6 \), the process of determining the solution of equation (30) goes tedious. Here, we argue that by choosing \( r \) sufficiently small, \( v^{N - 1} \) in equation (30) can be neglected. For example, for \( N = 6, r = 0.01 \) and \( p_r(r) = 1 + r^2 + r^3 + r^4 + r^5 + r^6 \), equation (30) gives \( v = 0.02714509001 \). With this, we observe that for \( N = 6 \) the term \( v^{N-3} [(N - 1)(N - 2)]/[2r^2 p_r] = 1.999999985 \) which is very close to 2, while the contribution from the term \( v^{N-1} = 0.1473860672 \times 10^{-7} \) is negligible, hence, we can do way with the later term. Therefore, for \( N \geq 6 \), we have
\[
v_{Nah} = \left[ \frac{4r^2 P_r}{(N - 1)(N - 2)} \right]^{1/3}.
\]
(33)

Now, we discuss the nature of central singularity \( r = 0, t = t_s(0) \) through the behaviour of causal paths emerging from the singularity and reaching out to a far away observer in future or otherwise. Using equation (21), the equation of apparent horizon in \( (t, r) \) is written as follows
\[
t_{ah}(r) = t_s(r) - B(r)
\]
(34)
where
\[
B(r) = \int_{0}^{v_{ah}} \frac{v^{(N-3)/2} dv}{e^{\eta \sqrt{v^{N-3}e^{2\eta n_h(r) + v^{N-1}h(rv) + \frac{2p_r(2-v^{N-1})}{(N-1)(N-2)}}}}}
\]
(35)
Since the behaviour of apparent horizon is being studied close to the central singularity at \( r = 0, R = 0 \), the upper limit of integration in the above equation is small. Hence, the
The integrand in above equation can be expanded in a power series in \( v \) and keeping only leading term, we obtain

\[
t_{ab}(r) = t_{ss} + r \chi_p(0) - \sqrt{\frac{N-2}{N-1}} (p_r)^{\frac{1}{2}} (v_{Nab})^{\frac{1}{N-1}}
\]  

(36)

For \( N = 4, 5 \) and if \( \chi_p(0) \) is non-zero positive, then in above equation the second term dominates over the last negative term so that the apparent horizon curve is increasing as we move away from the singularity, therefore the singularity is naked. On the other hand for \( N \geq 6 \), the negative term in equation (36) starts dominating and thus, advancing the time of trapped surface formation. Hence for \( N \geq 6 \), we have a black hole solution. These results coincide with the results obtained in [8] with radial pressure \( p_r = 0 \).

These results can be verified by plotting curves for \( \chi_p(0) = 0 \).

V. Collapse with linear equation of state

In this section, we aim at the question, whether the choice of equation of state \( p = k \rho \), \( k \in [0, 1] \) contributes through \( k \) in the development of BH/NS phases. R. Goswami and P. S. Joshi have studied the case of an isentropic perfect fluid with linear equation of state in four dimensional spacetime, wherein, they pointed that the occurrence of NS/BH evolving from regular initial data depend on the choice of rest of the free function available [15]. The stress-energy tensor for Type I field in a diagonal form is given by [10]

\[
T^t_t = -\rho, \quad T^r_r = T^\theta_\theta = T^\phi_\phi = p
\]  

(37)

The quantities \( \rho \) and \( p = p_r = p_\theta \) are the energy density and pressure respectively. We take the matter field to satisfy weak energy condition which implies \( \rho \geq 0; \rho + p \geq 0 \). Linear equation of state for perfect fluid is

\[
p(t, r) = k \rho(t, r) \text{ where } k \in [0, 1]
\]  

(38)

Einstein field equations for the metric (1) are derived as

\[
\rho = \frac{(N-2)F'}{2R^{N-2}R'} = -\frac{1}{k} \frac{(N-2)F'}{2R^{N-2}R}
\]  

(39)

\[
\nu' = -\frac{k}{k+1} [\ln(\rho)]'
\]  

(40)

\[
R'\tilde{G} - 2\tilde{G} \nu' = 0
\]  

(41)

\[
G - H = 1 - \frac{F}{R^{N-3}}.
\]  

(42)

As before, we have \( R(t, r) = r \nu(t, r) \) and the choice of physically reasonable initial data is maintained herein as in section II. For further details, we refer the reader to [5]. A general mass function for the cloud can be considered as

\[
F(t, r) = r^{N-1}M(r, v)
\]  

(43)

where \( M(r, v) \) is regular and continuously twice differentiable. Using equation [8] in equations [5], we obtain

\[
\rho = \frac{N-2}{2} \times \frac{(N-1)M + r[M_v + M_v v']}{v^{N-2}(v + rv')} = -\frac{(N-2)M_v}{2k v^{N-2}}
\]  

(44)
Then as $v \to 0$, $\rho \to \infty$ and $p \to \infty$ i.e. both the density and pressure blow up at the singularity. At the initial epoch, the regular density distribution takes the form

$$
\rho_o(r) = \frac{N-2}{2} [rM(r,1), r + (N-1)M(r,1)].
$$

(45)

From equation (44), it is clear that $\rho = \rho(r, v)$ and therefore, $v' = f(r, v)$. We rearrange equation (44) as follows

$$(N-1)kM + krM_r + Z(r, v)M_v = 0$$

(46)

where

$$Z(r, v) = (k + 1)rv' + v$$

(47)

Equation (46) has general solution of the form,

$$S(X, Y) = 0$$

(48)

where $X(r, v, M)$ and $Y(r, v, M)$ are the solutions of the system of equations (44).

Equation (48) has many classes of solutions but only those classes of solutions should be considered which satisfy energy conditions, which are regular and gives $\rho \to \infty$ as $v \to 0$. This means the energy conditions and equation of state $p = k\rho$ isolate the class of functions $M(r, v)$ so that the mass function $\mathcal{F}(t, r)$, the metric function $\nu(t, r)$ and the function $b(r)$ (to follow) evolve as the collapse begins according to the Type I field equations.

On integrating equation (40), we obtain the general metric function,

$$v(r, v) = -\frac{k}{k+1} [\ln(\rho)]$$

(50)

Define a suitably differentiable function $A(r, v)$, $A(r, v)_v = v'/R'$ so that equation (50) takes the form

$$A(r, v) = -\frac{k\rho'}{(k+1)\rho R'}$$

(51)

At the initial epoch, we have

$$[A(r, v)_v]_{v=1} = -\frac{k\rho_o'(r)}{(k+1)\rho_o(r)}$$

(52)

whereas using equation (44), the relation between the function $M$ and $A$, at all the epochs is given by

$$A(r, v)_v R' = -\frac{k}{(k+1)} \left[ \ln \left( \frac{(N-2)M_v}{2k v^{N-2}} \right) \right]'$$

(53)

From above, it is clear that $M(r, v)$ is a decreasing function with respect to $v$ and if we consider a smooth initial profile for the density in such way that density gradient vanishes at the center then we have $A(r, v) = r g_o(r, v)$ where $g_o(r, v)$ is another suitably differentiable function.

Also, the use of definition of $A(r, v)$ in equation (41) yields

$$G(t, r) = d(r)e^{2rA}$$

(54)

where $d(r)$ is another arbitrary continuously differentiable function of $r$. We can write

$$d(r) = 1 + r^2b(r).$$

(55)
where $b$ is the energy distribution function for the collapsing shells. Define a function $h(r, v)$ as
\[ h(r, v) = \frac{e^{2rA} - 1}{r^2} \] (56)
and substituting this equation, together with (55) and (54) in equation (42), we get
\[ v^{(N-3)/2} \dot{v} = -\rho^{-k/(k+1)} \sqrt{v^{N-3}h(r, v) + bv^{N-3}e^{2rA} + M}. \] (57)
where negative sign is chosen since, for the collapse, $\dot{v} < 0$. Integrating the above equation, we have
\[ t(v, r) = \int_v^1 \frac{v^{(N-3)/2} dv}{\rho^{-k/(k+1)} \sqrt{v^{N-3}h(r, v) + bv^{N-3}e^{2rA} + M}} \] (58)
In above equation, the variable $r$ is treated as a constant. Expanding $t(v, r)$ around the center, we get
\[ t(v, r) = t(v, 0) + r\chi(v) + O(r^2) \] (59)
where the function
\[ \chi(v) = \frac{dt}{dr} \bigg|_{r=0} = -\frac{1}{2} \int_v^1 \frac{v^{(N-3)/2}B_r(0, v)}{B(0, v)^{3/2}} dv. \] (60)
and
\[ B(r, v) = \rho^{-k/(k+1)} \sqrt{v^{N-3}h(r, v) + bv^{N-3}e^{2rA} + M(r, v)} \] (61)
\[ t_{sa} = \int_0^1 \frac{v^{(N-3)/2} dv}{B(0, v)} \] (62)
\[ t_s(r) \equiv t(0, r) = t_{sa} + r\chi(0) + O(r^2) \] (63)
Now, it is clearly seen that the value of $\chi(0)$ depends on the free functions $b(0), M(0, v)$ and $h(0, v)$, which in turn, depend on the initial data at the initial surface $t = t_s$. Thus, a tangent to the singularity curve $t = t_{sa}$ is completely determined by the given set of density, pressure, velocity function $\nu$ and function $b$. Further, from equation (57), we can write
\[ v^{(N-3)/2}v' = \chi(v)B(0, v) + O(r). \] (64)
Hence,
\[ \chi(0) = \lim_{v \to 0} \chi(v) = \lim_{v \to 0} \frac{v^{(N-3)/2} v'}{B(0, v)}. \] (65)
The existence of the limit depends on the cumulative effect of all the terms present in above equation but since $B(0, v) > 0$ as $v \to 0$ provided $M(0, 0) > 0$ and $v \in [0, 1]$, therefore positive or negative sign of $\chi(0)$ absolutely depends on $v'$. So, if $v' > 0$ then $\chi(0) > 0$ and otherwise. If $\chi(0) = 0$ then we will have to take into account next higher order non-zero term in the singularity curve equation and do a similar analysis. Thus, we reckon that value of $\chi(0)$ depends on the initial data profiles of density and pressure (through the free function $M(r, 1)$), the energy function $b(r)$ and the other free function $h(r, 1)$ (for determination of the metric function $\psi$). Secondly, its sign depends on the nature of gradient of $v$.
Now, we assume that these free functions to be at least $C^2$ and satisfy other physical requirements. Let us choose the function $M(r, v)$ as follows
\[ M(r, v) = 1 + (rv) + (rv)^2 + (rv)^3 + ....... \]
Now, the equation of apparent horizon $\rho = 1$ give rise to a polynomial equation in $v$
\[ v^{N-3} - rv^{N-2} = r^2. \] (66)
The apparent horizon $V_{N\text{ah}}$ for space dimensions $N = 4, 5$ and 6 are derived as follows

$$V_{4\text{ah}} = \frac{1 \pm \sqrt{1 - 4r^3}}{2r},$$

next for $N = 5$, we have

$$A = -108r^4 + 8 + 12\sqrt{3} \sqrt{27r^4 - 4r^2}$$

which is an imaginary term for sufficiently small values of $r$, $0 < r < \sqrt{4/27}$. Hence, apparent horizon may occur at some value of $r \geq \sqrt{4/27}$ for a chosen initial data of $M(r,v)$. So, for such value of $r$, we can write

$$V_{5\text{ah}} = \frac{V_{4\text{ah}}^{1/3}}{r^{1/3}} + \frac{2/3}{r} + \frac{1/3}{r},$$

and

$$V_{6\text{ah}} = \frac{1}{4r} + \frac{\sqrt{3S}}{12r} + \frac{1}{12r} \sqrt{-18ST + 6(12)^{1/3}rST^{2/3} + 24(12)^{2/3}r^4S - 18\sqrt{3T^{1/3}}}{ST^{1/3}}$$

where $T = r^2(9 + \sqrt{-768r^5 + 81})$ and $S = \sqrt{\frac{3T^{1/3} + (212)^{1/3}rT^{2/3} + 8(12)^{2/3}r^4}{T^{1/3}}}$.

The nature of central singularity $r = 0, t = t_s(0)$ as NS/BH is studied through the behaviour of geodesics emanating from the singularity to reach an asymptotic observer in future or otherwise. Using equation (58), the equation of apparent horizon in $(t, r)$ is written as

$$t_{\text{ah}}(r) = t_s(r) - B_1(r)$$

where

$$B_1(r) = \int_0^{V_{\text{ah}}} \frac{v^{(N-3)/2} dv}{\rho^{-k/(k+1)}\sqrt{\nu^{N-3}h(r,v) + b\nu^{N-3}e^{2rA} + M}}$$

The upper limit of integration in the above equation is small as discussed earlier. Using the value of $\rho$ from equation (44) and expanding the terms in the integrand in a power series in $v$ and keeping only leading term, we obtain

$$t_{\text{ah}}(r) = t_s(r) - \frac{[(N-1)(N-2)]^m}{2m} \int_0^{V_{\text{ah}}} \frac{v^{(2m-N-3)/2} dv}{(1 - rv)^{-(1-2m)/2}}$$

where we have put $m = k/(k+1)$ and $0 \leq m \leq 1/2$. Further, since $r$ is sufficiently small, therefore, we can do away with the terms $O(rv)^2$, hence for $N \geq 4$, we obtain

$$t_{\text{ah}}(r) = t_{s_0} + r\chi 0 + \frac{[(N-1)(N-2)]^m}{2m} \left[ \frac{2}{N+1-2m} - \frac{r(1-2m)V_{\text{Nah}}}{N-2m-1} \right] (V_{\text{Nah}})^{-(N+1-2m)/2}$$

The second term in above equation dominates over the last for sufficiently small values of $r$ and for any value of $k$ (the parameter of equation of state) and independent of any choice of dimension of the spacetime. Hence, causal paths emerging from the singularity reach out to an asymptotic observer at infinity, thus naked singularities appear in all dimensions for sufficiently small values of $r$. The results for dust case i.e. with $m = 0$ agree with the results obtained in [3].
VI. Conclusion

We have considered the radial pressure $p_r$ as a function of $r$, due to which the mass function $F(t, r)$ becomes the function of both $r$ and $v$, while the tangential pressure remains arbitrary throughout the analysis. Through the study of apparent horizon, it is found that NS forms at the center of the collapsing cloud for dimensions $N = 4$ and $N = 5$, whereas for $N \geq 6$ there exists a critical value of $\chi(0)$ below which a BH solution persists. This statement supports the understanding that gravity may be a higher dimensional interaction. This generalizes the results obtained in [3].

Next, we have analyzed the effects of the equation of state on the formation of apparent horizon in continual gravitational collapse, the parameter $k$ involves in determination of the curve $t = t_{ab}(r)$ and assist in strengthening the outgoing causal path to reach out to an asymptotic observer at infinity. It is analyzed that naked singularities exist in all the dimensions. $k = 0$ is the usual case of dust collapse.

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