P-Resolutions of Cyclic Quotients from the Toric Viewpoint

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1 Introduction

(1.1) The break through in deformation theory of (two-dimensional) quotient singularities $Y$ was Kollár/Shepherd-Barron’s discovery of the one-to-one correspondence between so-called P-resolutions, on the one hand, and components of the versal base space, on the other hand (cf. [KS], Theorem (3.9)). It generalizes the fact that all deformations admitting a simultaneous (RDP-) resolution form one single component, the Artin component.

According to definition (3.8) in [KS], P-resolutions are partial resolutions $\pi : \tilde{Y} \to Y$ such that

- the canonical divisor $K_{\tilde{Y}|Y}$ is ample relative to $\pi$ (a minimality condition) and
- $\tilde{Y}$ contains only mild singularities of a certain type (so-called T-singularities).

Despite their definition as those quotient singularities admitting a $\mathbb{Q}$-Gorenstein one-parameter smoothing ([KS], (3.7)), there are at least three further descriptions of the class of T-singularities: An explicit list of their defining group actions on $\mathbb{A}^2$ ([KS], (3.10)), an inductive procedure to construct their resolution graphs ([KS], (3.11)), and a characterization using toric language ([Al], (7.3)).

The latter one starts with the observation that affine, two-dimensional toric varieties (given by some rational, polyhedral cone $\sigma \subseteq \mathbb{R}^2$) provide exactly the two-dimensional cyclic quotient singularities. Then, T-singularities come from cones over rational intervals of integer length placed in height one (i.e. contained in the affine line $(\bullet, 1) \subseteq \mathbb{R}^2$).
If the affine interval is of length \( \mu + 1 \), then the corresponding T-singularity will have Milnor number \( \mu \) (on the \( Q \)-Gorenstein one-parameter smoothing).

(1.2) In [Ch] and [St] Christophersen and Stevens gave a combinatorial description of all P-resolutions for two-dimensional, cyclic quotient singularities. Using an inductive construction method (going through different cyclic quotients with step-by-step increasing multiplicity) they have shown that there is a one-to-one correspondence between P-resolutions, on the one hand, and certain integer tuples \((k_2, \ldots, k_{e-1})\) yielding zero if expanded as a (negative) continued fraction (cf. (1.2)), on the other hand.

The aim of the present paper is to provide an elementary, direct method for constructing the P-resolutions of a cyclic quotient singularity (i.e. a two-dimensional toric variety) \( Y_\sigma \). Given a chain \((k_2, \ldots, k_{e-1})\) representing zero, we will give a straight description of the corresponding polyhedral subdivision of \( \sigma \). (In particular, the bijection between those 0-chains and P-resolutions will be proved again by a different method.)

2 Cyclic Quotient Singularities

In the following we want to remind the reader of basic notions concerning continued fractions and cyclic quotients. It should be considered a good chance to fix notations. References are [Od] (§1.6) or the first sections in [Ch] and [St], respectively.

(2.1) **Definition:** To integers \( c_1, \ldots, c_r \in \mathbb{Z} \) we will assign the continued fraction \([c_1, \ldots, c_r] \in Q\), if the following inductive procedure makes sense (i.e. if no division by 0 occurs):

- \([c_r] := c_r\)
- \([c_1, \ldots, c_r] := c_1 - 1/[c_i+1, \ldots, c_r]\).

If \( c_i \geq 2 \) for \( i = 1, \ldots, r \), then \([c_1, \ldots, c_r]\) is always defined and yields a rational number greater than 1. Moreover, all these numbers may be represented by those continued fractions in a unique way.

(2.2) Let \( n \geq 2 \) be an integer and \( q \in (\mathbb{Z}/n\mathbb{Z})^* \) be represented by an integer between 0 and \( n \). These data provide a group action of \( \mathbb{Z}/n\mathbb{Z} \) on \( \mathbb{C}^2 \) via the matrix \[
\begin{pmatrix}
\xi & 0 \\
0 & \xi^q
\end{pmatrix}
\] (with \( \xi \) a primitive \( n \)-th root of unity). The quotient is denoted by \( Y(n, q) \).
In toric language, $Y(n, q)$ equals the variety $Y_\sigma$ assigned to the polyhedral cone $\sigma := \langle (1, 0); (-q, n) \rangle \subseteq \mathbb{R}^2$. $(Y_\sigma$ is defined as $\text{Spec} \mathcal{O}[\sigma^\vee \cap \mathbb{Z}^2]$ with $\sigma^\vee := \{ r \in (\mathbb{R}^2)^* | r \geq 0 \text{ on } \sigma \} = \langle [0, 1]; [n, q] \rangle \subseteq (\mathbb{R}^2)^* \cong \mathbb{R}^2$.)

**Notation:** Just to distinguish between $\mathbb{R}^2$ and its dual $(\mathbb{R}^2)^*$, we will denote these vector spaces by $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$, respectively. (Hence, $\sigma \subseteq N_{\mathbb{R}}$ and $\sigma^\vee \subseteq M_{\mathbb{R}}$.) Elements of $N_{\mathbb{R}} \cong \mathbb{R}^2$ are written in paranthesis; elements of $M_{\mathbb{R}} \cong \mathbb{R}^2$ are written in brackets. The natural pairing between $N_{\mathbb{R}}$ and $M_{\mathbb{R}}$ is denoted by $\langle , \rangle$ which should not be mixed up with the symbol indicating the generators of a cone. Finally, all these remarks apply for the lattices $N \cong \mathbb{Z}^2$ and $M \cong \mathbb{Z}^2$, too.

Let $n, q$ as before. We may write $n/(n-q)$ and $n/q$ (both are greater than 1) as continued fractions

$$n/(n-q) = [a_2, \ldots, a_{e-1}] \quad \text{and} \quad n/q = [b_1, \ldots, b_r] \quad (a_i, b_j \geq 2).$$

The $a_i$’s and the $b_j$’s are mutually related by Riemenschneider’s point diagram (cf. [R]).

Take the convex hull of $(\sigma^\vee \cap M) \setminus \{0\}$ and denote by $w^1, w^2, \ldots, w^e$ the lattice points on its compact edges. If ordered the right way, we obtain $w^1 = [0, 1]$ and $w^e = [n, q]$ for the first and the last point, respectively.

Then, $E := \{ w^1, \ldots, w^e \}$ is the minimal generating set (the so-called Hilbert basis) of the semigroup $\sigma^\vee \cap M$. These point are related to our first continued fraction by

$$w^{i-1} + w^{i+1} = a_i w^i \quad (i = 2, \ldots, e-1).$$

**Remark:** The surjection $N^E \rightarrow \sigma^\vee \cap M$ provides a minimal embedding of $Y_\sigma$. In particular, $e$ equals its embedding dimension.

In a similar manner we can define $v^0, \ldots, v^{r+1} \in \sigma \cap N$ in the original cone; now we have $v^0 = (1, 0)$, $v^{r+1} = (-q, n)$, and the relation to the continued fractions is $v^{j-1} + v^{j+1} = b_j v^j$ (for $j = 1, \ldots, r$).
Drawing rays through the origin and each point $v^j$, respectively, provides a polyhedral subdivision $\Sigma$ of $\sigma$. The corresponding toric variety $Y_\Sigma$ is a resolution of our singularity $Y_\sigma$. The numbers $-b_j$ equal the self intersection numbers of the exceptional divisors; since $b_j \geq 2$, the resolution is the minimal one.

3 The Maximal Resolution

(3.1) **Definition:** ([KS], (3.12)) For a resolution $\pi: \tilde{Y} \to Y$ we may write $K_{\tilde{Y}|Y} := K_{\tilde{Y}} - \pi^*K_Y = \sum_j (\alpha_j - 1)E_j$ ($E_j$ denote the exceptional divisors, $\alpha_j \in \mathbb{Q}$). Then, $\pi$ will be called maximal, if it is maximal with respect to the property $0 < \alpha_j < 1$.

The maximal resolution is uniquely determined and dominates all the P-resolutions. Hence, for our purpose, it is more important than the minimal one. It can be constructed from the minimal resolution by successive blowing up of points $E_i \cap E_j$ with $\alpha_i + \alpha_j \geq 0$ (cf. Lemma (3.13) and Lemma (3.14) in [KS]).

(3.2) **Proposition:** The maximal resolution of $Y_\sigma$ is toric. It can be obtained by drawing rays through 0 and all interior lattice points (i.e. $\in \mathbb{N}$) of the triangle $\Delta := \text{conv}(0, v^0, v^{r+1})$, respectively.

**Proof:** We have to keep track of the rational numbers $\alpha_j$. Hence, we will show how they can be “seen” in an arbitrary toric resolution of $Y_\sigma$. Let $\Sigma < \sigma$ be a subdivision generated by one-dimensional rays through the points $u^0, \ldots, u^{s+1} \in \sigma \cap \mathbb{N}$. (In particular, $u^0 = v^0 = (1, 0)$ and $u^{s+1} = v^{r+1} = (-q, n)$; moreover, for the minimal resolution we would have $s = r$ and $u^j = v^j$ ($j = 0, \ldots, r+1$).) Denote by $c_1, \ldots, c_s$ the integers given by the relations

$$u^{j-1} + u^{j+1} = c_j u^j \quad (j = 1, \ldots, s).$$
(In particular, \(c_j = b_j\) for the minimal resolution again.)

As usual, the numbers \(-c_j\) equal the self intersection numbers of the exceptional divisors \(E_j\) in \(Y_\Sigma\): Indeed, \(D := \sum_i u^i E_i\) is a principal divisor (if you do not like coefficients \(u^i\) from \(N\), evaluate them by arbitrary elements of \(M\)); hence,

\[
0 = E_j \cdot D = E_j \cdot (w^{j-1}E_{j-1} + w^jE_j + w^{j+1}E_{j+1}) = w^{j-1} + (E_j)^2 w^j + w^{j+1} = (c_j + (E_j)^2) \cdot w^j \quad (j = 1, \ldots, s).
\]

On the other hand, we can use the projection formula to obtain

\[
-2 = 2g(E_j) - 2 = K_{\tilde{Y}_Y} \cdot E_j + (E_j)^2 = \sum_i (\alpha_i - 1) (E_i \cdot E_j) + (E_j)^2 = (\alpha_{j-1} - 1) + (\alpha_j - 1) (E_j)^2 + (\alpha_{j+1} - 1) + (E_j)^2,
\]

hence

\[
\alpha_{j-1} + \alpha_{j+1} = c_j \alpha_j \quad (j = 1, \ldots, s; \ \alpha_0, \alpha_{s+1} := 1).
\]

Looking at the definition of the \(c_j\)'s (via relations among the lattice points \(u^i\)), there has to be some \(R \in M_R\) such that

\[
\alpha_j = \langle u^j, R \rangle \quad (j = 0, \ldots, s + 1).
\]

The conditions \(\langle u^0, R \rangle = \alpha_0 = 1\) and \(\langle u^{s+1}, R \rangle = \alpha_{s+1} = 1\) determine \(R\) uniquely. Now, we can see that \(\alpha_j\) measures exactly the quotient between the length of the line segment \(0u^j\), on the one hand, and the length of the \(\Delta\)-part of the line through \(0\) and \(u^j\), on the other hand. In particular, \(\alpha_j < 1\) if and only if \(u^j\) sits below the line connecting \(u^0\) and \(u^{s+1}\).

![Diagram](image)

This explains how to construct the maximal resolution: Start with the minimal one and continue subdividing each small cone \(\langle u^j, u^{j+1} \rangle\) into \(\langle u^j, u^j + u^{j+1} \rangle \cup \langle u^j + u^{j+1}, u^{j+1} \rangle\) as long as it contains interior lattice points below the line \([R = 1]\), i.e.
Corollary: Every P-resolution is toric.

Proof: P-resolutions are obtained by blowing down curves in the maximal resolution.

(3.3) Example: We take the example $Y(19, 7)$ from [KS], (3.15). Since

$$\sigma = \langle (1, 0), (-7, 19) \rangle,$$

the interior of $\Delta$ is given by the three inequalities

$$y > 0, \quad 19x + 7y > 0, \quad \text{and} \quad 19x + 8y < 19 \quad \text{corresponding to} \quad R = [1, 8/19].$$

The only primitive (i.e. generating rays) lattice points contained in int $\Delta$ are

$$u^1 = (0, 1), \quad u^2 = (-1, 4), \quad u^3 = (-2, 7), \quad u^4 = (-1, 3), \quad u^5 = (-5, 14), \quad u^6 = (-4, 11).$$

They provide the maximal resolution. The corresponding $\alpha$’s can be obtained by taking the scalar product with $R = [1, 8/19]$, i.e. they are $8/19, 13/19, 18/19, 5/19, 17/19, \text{and} \ 12/19$.

The minimal resolution uses only the rays through $u^1 = (0, 1), \ u^4 = (-1, 3), \ \text{and} \ u^6 = (-4, 11)$, respectively.

4 P-Resolutions

(4.1) In this section we will speak about partial toric resolutions $\pi : Y_\Sigma \to Y_\sigma$.

Nevertheless, we use the same notation as we did for the maximal resolution: The fan $\Sigma$ subdividing $\sigma$ is generated by rays through $u^0, \ldots, u^s \in \sigma \cap N$; each ray $u^j$ corresponds to an exceptional divisor $E_j \subseteq Y_\Sigma$. However, since $u^{j-1} + u^{j+1}$ need not to be a multiple of $u^j$, the numbers $c_j$ do not make sense anymore.

Lemma: ([Re], (4.3)) For $K := K_{Y_\Sigma}$ or $K := K_{Y_\Sigma | Y_\sigma}$ the intersection number $(E_j \cdot K)$ is positive, zero, or negative, if the line segments $\overline{u^{j-1}u^j}$ and $\overline{u^ju^{j+1}}$ form a strict concave, flat, or strict convex “roof” over the two cones, respectively.

\[ \begin{align*}
\text{Diagram:} & \quad (E_j \cdot K) > 0 \\
\text{Diagram:} & \quad (E_j \cdot K) = 0 \\
\text{Diagram:} & \quad (E_j \cdot K) < 0
\end{align*} \]
**Proof:** Using $K := K_{Y_2} = - \sum_{i=0}^{s+1} E_i$ (cf. (2.1)) we have

$$(E_j \cdot K) = -(E_j \cdot E_{j-1}) - (E_j)^2 - (E_j \cdot E_{j+1}).$$

On the other hand, as in the proof of Proposition (3.2), we know that

$$0 = (E_j \cdot E_{j-1}) u^{i-1} + (E_j)^2 u^i + (E_j \cdot E_{j+1}) u^{i+1}.$$ 

Combining both formulas yields the final result

$$(E_j \cdot K) u^j = (E_j \cdot E_{j-1}) (u^{i-1} - u^i) + (E_j \cdot E_{j+1}) (u^{i+1} - u^i).$$

$$\blacksquare$$

**Remark:** The previous lemma together with Proposition (3.2) illustrate again the fact that all P-resolutions (and we just need the fact that the canonical divisor is relatively ample) are dominated by the maximal resolution.

(4.2) In [Ch] Christophersen has defined the set

$$K_{e-2} := \{(k_2, \ldots, k_{e-1}) \in \mathbb{N}^{e-2} \mid [k_2, \ldots, k_{e-1}] \text{ is well defined and yields } 0 \}$$

of chains representing zero. To every such chain there are assigned non-negative integers $q_1, \ldots, q_e$ characterized by the following mutually equivalent properties:

- $q_1 = 0$, $q_2 = 1$, and $q_{i-1} + q_{i+1} = k_i q_i$ ($i = 2, \ldots, e-1$);
- $q_{e-1} = 1$, $q_e = 0$, and $q_{i-1} + q_{i+1} = k_i q_i$ ($i = 2, \ldots, e-1$);
- $q_e = 0$ and $[k_i, \ldots, k_{e-1}] = q_{i-1}/q_i$ with $\gcd(q_{i-1}, q_i) = 1$ ($i = 2, \ldots, e-1$).

(The two latter properties do not even use the fact that the continued fraction $[k_2, \ldots, k_{e-1}]$ yields zero.)

**Remark:** The elements of $K_{e-2}$ correspond one-to-one to triangulations of a (regular) $(e-1)$-gon with vertices $P_2, \ldots, P_{e-1}, P_*$. Then, the numbers $k_i$ tell how many triangles are attached to $P_i$. The numbers $q_i$ have an easy meaning in this language, too.

Finally, for a given $Y_\sigma$ with embedding dimension $e$, Christophersen defines

$$K(Y_\sigma) := \{(k_2, \ldots, k_{e-1}) \in K_{e-2} \mid k_i \leq a_i \} .$$

**Theorem:** Each P-resolution of $Y_\sigma$ (i.e. the corresponding subdivision $\Sigma$ of $\sigma$) is given by some $k \in K(Y_\sigma)$ in the following way:

(1) $\Sigma$ is built from the rays that are orthogonal to $w^i/q_i - w^{i-1}/q_{i-1} \in M_\mathbb{R}$ (for $i = 3, \ldots, e-1$). In some sense, if the occurring divisions by zero are interpreted well,
\( \Sigma \) may be seen as dual to the Newton boundary generated by \( w^i/q_i \in \sigma^i \) \((i = 1, \ldots, e)\).

(2) The affine lines \([\langle \cdot, w^i \rangle = q_i] \) form the “roofs” of the \( \Sigma \)-cones. In particular, the (possibly degenerate) cones \( \tau^i \in \Sigma \) correspond to the elements \( w^1, \ldots, w^e \in E \) The “roof” over the cone \( \tau^i \) has length \( \ell_i := (a_i - k_i) q_i \) (the lattice structure \( M \subseteq M_{\mathbb{R}} \) induces a metric on rational lines). In particular, \( \tau^i \) is degenerated if and only if \( k_i = a_i \). The Milnor number of the T-singularity \( Y_{t^i} \) equals \((a_i - k_i - 1)\).

\[(4.3) \text{ Proof:} \] According to the notation introduced in (4.1), the fan \( \Sigma \) consists of (non-degenerate) cones \( \tau^j := \langle w^{j-1}, w^j \rangle \) with \( j = 1, \ldots, s + 1 \). (Except \( w^0 = (1, 0) \) and \( w^e = (-q, n) \), their generators \( w^j \) are primitive lattice points (i.e. \( \in N \)) contained in \( \text{int} \Delta \subseteq \sigma \).)

Step 1: For each \( \tau^j \) there are \( w \in E, d \in \mathbb{N} \) such that \( \langle w^{j-1}, w \rangle = \langle w^j, w \rangle = d \).

First, it is very clear that there are a primitive lattice point \( w \in M \) and a non-negative number \( d \in \mathbb{R}_{\geq 0} \) admitting the desired properties. Moreover, since \( w \in N \), \( d \) has to be an integer, and Reid’s Lemma (4.1) tells us that \( w \in \sigma' \). It remains to show that \( w \) belongs even to the Hilbert basis \( E \subseteq \sigma' \cap M \).

Denote by \( \ell \) the length of the line segment \( \overline{w^{j-1}w^j} \) on the “roof” line \([\langle \cdot, w \rangle = d]\).

Since \( \tau^j \) represents a T-singularity, we know from (7.3) of [A] (cf. [B]) of the present paper) that \( d|\ell \). In particular, \( \overline{w^{-1}w^j} \) contains the \( d \)-th multiple \( d \cdot u \) of some lattice point \( u \in \tau^j \cap M \) (w.l.o.g. not belonging to the boundary of \( \sigma \)). Hence, \( \langle u, w \rangle = 1 \) and \( u \in \text{int} \sigma \cap M \), and this implies \( w \in E \).

Step 2: Knowing that each of the cones \( \tau^1, \ldots, \tau^{s+1} \in \Sigma \) is assigned to some element \( w \in E \), a slight adaption of the notation (a renumbering) seems to be very useful: Let \( \tau^i := \langle w^{i-1}, w^i \rangle \) be the cone assigned to \( w^i \in E \), and denote by \( d_i, \ell_i \) the height and the length of its “roof” \( \overline{w^{i-1}w^i} \), respectively. Some of these cones might be degenerated, i.e. \( \ell_i = 0 \). This it at least true for the extremal \( \tau^1 \) and \( \tau^e \) coinciding with the two rays spanning \( \sigma \). Here we have even \( d_1 = d_e = 0 \); in particular \( w^0 = w^1 = (1, 0) \) and \( w^{e-1} = w^e = (-q, n) \).

Since \( d_i|\ell_i \), we may introduce integers \( k_i \leq a_i \) yielding \( \ell_i = (a_i - k_i) d_i \). For \( i = 2, \ldots, e - 1 \) they are even uniquely determined.

Step 3: Using the following three ingredients

(i) \( \langle w^{i-1}, w^i \rangle = \langle w^i, w^i \rangle = d_i \) \((i = 1, \ldots, e)\),

(ii) \( w^{i-1} + w^{i+1} = a_i w^i \) \((i = 2, \ldots, e - 1)\); cf. (2.3)), and

(iii) \( \langle w^i - w^{i-1}, w^{i-1} \rangle = \ell_i = (a_i - k_i) d_i \) (since \( \{w^{i-1}, w^i\} \) forms a \( \mathbb{Z} \)-basis of \( M \)),
we obtain

\[ d_{i+1} + d_{i+1} = (a_i d_i + d_{i-1}) - (a_i d_i - d_{i+1}) \]
\[ = (a_i d_i + (u^{i-1}, w^{i-1})) - (u^i, a_i w^i - w^{i+1}) \]
\[ = a_i d_i + (u^{i-1}, w^{i-1}) - (u^i, w^{i-1}) \]
\[ = a_i d_i + (u^{i-1} - u^i, w^{i-1}) \]
\[ = a_i d_i - (a_i - k_i) d_i = k_i d_i \quad \text{(for } i = 2, \ldots, e-1). \]

In particular, \( k_i \geq 0 \) (and even \( \geq 1 \) for \( e > 3 \)). Moreover, since \( \{w^{i-1}, w^i\} \)
forms a basis of \( M \) and \( u^{i-1} \in N \) is primitive, we have \( \gcd(d_{i-1}, d_i) = 1 \). Hence, \( d_i = q_i \) (both series of integers satisfy the second of the three properties mentioned in the beginning of \((4.2)\)). Finally, the third of these properties yields

\[ [k_2, \ldots, k_{e-1}] = q_1/q_2 = d_1/d_2 = 0, \text{ i.e. } k \in K_{e-2}. \]

The reversed direction (i.e. the fact that each \( K(Y_\sigma) \)-element indeed yields a P-resolution) follows from the above calculations in a similar manner. \( \square \)

**Remark:** Subdividing each \( \tau_i \) further into \( (a_i - k_i) \) equal cones (with “roof” length \( q_i \) each) yields the so-called M-resolution (cf. \([BC]\)) assigned to a P-resolution. It is defined to contain only \( T_0 \)-singularities (i.e. \( T \)-singularities with Milnor number 0); in exchange, \( K_{Y|Y'} \) does not need to be relatively ample anymore. This property is replaced by “relatively nef”.

**Examples:**

(1) The continued fraction \([1, 2, 2, \ldots, 2, 1]\) = 0 yields \( q_1 = q_e = 0 \) and \( q_i = 1 \) otherwise. In particular, the “roof” lines equal \([\langle \bullet, w^i \rangle = 1\) (for \( i = 2, \ldots, e-1) \) describing the RDP-resolution of \( Y_\sigma \). The assigned M-resolution equals the minimal resolution mentioned at the end of \((2.3)\).

(2) Let us return to Example \((3.3)\): The embedding dimension \( e \) of \( Y_\sigma \) is 6, the vector \((a_2, \ldots, a_{e-1})\) equals \((2, 3, 2, 3)\), and, except the trivial RDP element mentioned in \((1)\), \( K(Y_\sigma) \) contains only \((1, 3, 1, 2)\) and \((2, 2, 1, 3)\).

In both cases we already know that \( q_1 = q_6 = 0 \) and \( q_2 = q_5 = 1 \). The remaining values are given by the equation \( q_3/q_4 = [k_4, k_5]\), i.e. we obtain \( q_3 = 1, q_4 = 2 \) or \( q_3 = 2, q_4 = 3 \), respectively.

Hence, in case of \((1, 3, 1, 2)\) the fan \( \Sigma \) is given by the additional rays through \((0, 1)\) and \((-4, 11)\). For \( k = (2, 2, 1, 3) \) we need the only one through \((-1, 4)\).

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