REALIZING THE LOCAL WEIL REPRESENTATION OVER A NUMBER FIELD

GERALD CLIFF AND DAVID McNEILY

ABSTRACT. We show that the Weil representation of the symplectic group $Sp(2n, F)$, where $F$ is a non-archimedian local field, can be realized over the field $K = \mathbb{Q}(\sqrt{p}, \sqrt{-p})$, where $p$ is the residue characteristic of $F$.

1. Introduction

Our main result is that the Weil representation of the symplectic group $Sp(2n, F)$, where $F$ is a non-archimedian local field of residue characteristic $\neq 2$, can be realized over a number field $K$. We take an infinite-dimensional complex vector space $V$ such that the Weil representation is given by $\rho: Sp(2n, F) \to PGL(V)$ and we find a $K$-subspace $V_0$ of $V$ such that $\rho(g)(V_0) = V_0$ for all $g \in Sp(2n, F)$.

This answers a question raised by D. Prasad [P]. Indeed, we show that we can take $K = \mathbb{Q}(\sqrt{p}, \sqrt{-p})$ where $p$ is the residue characteristic of $F$. We assume that $p$ is odd. A consequence of this, also pointed out by Prasad, is that the local theta correspondence can be defined for representations which are realized over $K$.

Let $W$ be the Weil representation of $Sp(2n, F)$. The Weil representation can be defined using the Schrödinger representation of the Heisenberg group $H$. Let $\lambda$ be a fixed complex character on the additive group of the field $F$. Suppose that $F^{2n}$ is the direct sum $X \oplus Y$ of totally isotropic $F$-subspaces. The Schrödinger model is realized in the Bruhat-Schwartz space $S(X)$ of locally constant functions $f: X \to \mathbb{C}$ of compact support. For $h \in H$, there are operators $S_\lambda(h)$ on $S(X)$ such that $S_\lambda: H \to GL(S(X))$ is the unique smooth irreducible representation of $H$ with central character $\lambda$. The natural action of the symplectic groups extends to an action on $H$, and the Weil representation is given by operators $W_\lambda(g)$ on $S(X)$, $g \in Sp(2n, F)$, such that

$$W_\lambda(g)^{-1}S_\lambda(h)W_\lambda(g) = S_\lambda(hg) \quad h \in H, \ g \in Sp(2n, F).$$

Let $Q(\lambda)$ be the field obtained by adjoining all the character values of $\lambda$ to $\mathbb{Q}$, and let $E = Q(\lambda)(\sqrt{-1})$. In the case that $F$ has characteristic 0, $E$ is the field obtained from $Q$ by adjoining $\sqrt{-1}$ and all $p$-power roots of unity. For a subfield $L$ of $C$, define $S(X, L)$ to be the space of locally constant functions on $X$ of compact support having values in $L$. We show that there is an explicit choice of Weil operators $W_\lambda(g)$ on $S(X)$ which leave $S(X, E)$ invariant.

The Galois group of $E$ over $Q$ acts on $S(X, E)$ and on $\text{End}(S(X, E))$. In Section 7 we define a 1-cocycle $\delta$ on $\text{Gal}(E/Q(\sqrt{p}, \sqrt{-p}))$ with values in $GL(S(X, E))$ such that

$$(I) \quad \sigma W_\lambda(g) = \delta(\sigma)^{-1}W_\lambda(g)\delta(\sigma), \quad g \in \text{Sp}(V).$$

Using Galois descent, we show that there exists $\alpha \in GL(S(X, E))$ such that $\delta(\sigma) = \alpha^{-1}\sigma\alpha$ for $\sigma \in \text{Gal}(E/Q(\sqrt{p}, \sqrt{-p}))$. 

1
Main Theorem. The operators $\alpha W_\lambda(g)\alpha^{-1}$ leave $S(X, Q(\sqrt{p}, \sqrt{-p}))$ invariant, and provide a form of the Weil representation realized over $Q(\sqrt{p}, \sqrt{-p})$.

To indicate how we find the 1-cocycle satisfying (I), for the rest of the introduction we assume that $F$ has characteristic 0. The Galois group of $Q(\lambda)/Q$ is isomorphic to the units $Z_p^*$ of the $p$-adic integers. For an element $s$ of $Z_p^*$, we let $\sigma_s$ denote the corresponding element of $\text{Gal}(Q(\lambda)/Q)$. For an element $t \in F^*$, we define the character $\lambda[t]$ of $F$ by $\lambda[t](r) = \lambda(tr)$, $r \in F$.

For $t \in F^*$, let $g_t \in \text{Sp}(2n, F)$ be defined by $(x + y)g_t = t^{-1}x + ty$, and $f_t \in \text{GL}(2n, F)$ by $(x + y)f_t = x + ty$, where $x \in X, y \in Y$. Then $f_t$ is not in general in $\text{Sp}(2n, F)$, but conjugation by $f_t$ leaves $\text{Sp}(2n, F)$ invariant. We have

$$W_\lambda(g^t) = W_{\lambda[t]}(g), \quad g \in \text{Sp}(V).$$

Furthermore, observing $f_t^s$ is the composite $tI \circ g_t$, we show

$$W_\lambda(g^{t\epsilon^s}) = W_\lambda(g_t)^{-1}W_\lambda(g)W_\lambda(g_t).$$

For $\sigma \in \text{Gal}(E/Q(\sqrt{p}, \sqrt{-p}))$, $\sigma|_{Q(\lambda)}$ is a square, so $\sigma|_{Q(\lambda)} = \sigma_{\epsilon^s}$ where $\epsilon$ is a primitive $p - 1$ root of unity, $s$ is a principal unit of $O$, and $i$ is an integer, $1 \leq i \leq (p - 1)/2$. We note

$$\sigma W_\lambda(g) = W_\lambda(\epsilon^s g)(g).$$

In light of (II) and (III), we deduce

$$\sigma W_\lambda(g) = W_\lambda(\epsilon^s g)^{-1}W_\lambda(g)W_\lambda(\epsilon^s g).$$

The last equation is used to show that $\delta(\sigma) = W_\lambda(\epsilon^s g)$ satisfies (I) and almost satisfies the one-cocycle condition.

Equations (II), (III), and (IV) are proved using an integral formula for Weil operators due to Ranga Rao [RR]; see equation (3) in Section 3. This formula is also used to show that conjugation by the Weil operators $W_\lambda(g)$ leaves $S(X, E)$ invariant.

2. Preliminary remarks on local fields, characters and measures

We fix some notation and recall some elementary facts about the characters of the additive group of a local field. Further details can be found in the first two chapters of [W].

Let $F$ be a non-Archimedean local field, $O$ its ring of integers, and $m$ the maximal ideal of $O$. The order of the residue class field $\kappa = O/m$ shall be denoted $q$; we note that $q$ is power of $p = \text{char} \kappa$. We assume throughout that $p$ is different from 2; in particular, 2 is a unit of $O$.

Given a fractional $O$-ideal $a$, there exists an unique integer $v(a)$, the valuation of $a$, such that

$$a = m^{v(a)}.$$

If $s \in F$ is non-zero, the valuation of the ideal $sO$ is refered to as the valuation of $s$, denoted $v(s)$. The absolute value on $F$ is related to the valuation $v$ on $F$ by

$$|s| = q^{-v(s)}, \quad s \in F, s \neq 0.$$

Let $\lambda$ be a non-trivial, continuous, complex linear (unitary) character of $F^+$. The continuity of $\lambda$ ensures that its kernel contains a fractional $O$-ideal. The fact that $\lambda$ is non-trivial allows one to deduce that the set of all such fractional $O$-ideals has a unique maximal element $i = i_\lambda$, the conductor of $\lambda$. The level of $\lambda$ is defined to be the valuation of $i_\lambda$. 


Given \( n \geq 1 \), let
\[
\nu_{pn} = \{ z \in \mathbb{C} : z^{p^n} = 1 \}, \quad \nu_{p\infty} = \bigcup_{n=1}^{\infty} \nu_{pn}.
\]
(The more customary symbol \( \mu \) will be used to denote a measure.)

**Lemma 1.** We have
\[
\text{im} \lambda = \begin{cases} \nu_p, & \text{if char } F = p; \\ \nu_{p\infty}, & \text{if char } F = 0. \end{cases}
\]

**Proof.** Take \( x \in F \). If char \( F = p \) then
\[
1 = \lambda(0) = \lambda(px) = \lambda(x)^p.
\]
This shows \( \text{im} \lambda \subseteq \nu_p \). Equality follows from the fact \( \text{im} \lambda \) is a non-trivial subgroup of the simple abelian group \( \nu_p \).

If char \( F = 0 \) then, since \( p \in \mathfrak{m} \), there exists an \( n \geq 0 \) such that \( p^n x \in \mathfrak{i}_\lambda \). For such \( n \),
\[
1 = \lambda(p^n x) = \lambda(x)^{p^n}.
\]
Then \( \text{im} \lambda \subseteq \nu_{p\infty} \). If the inclusion were proper then there would exist \( m \geq 0 \) such that \( \text{im} \lambda = \nu_{pm} \). In this case, if \( x \in F \) then
\[
\lambda(x) = \lambda \left( p^m \cdot \frac{x}{p^m} \right) = \lambda \left( \frac{x}{p^m} \right)^{p^m} = 1
\]
since \( \lambda(x/p^m) \) is a \( p^m \)-th root of unity. As this would contradict the non-triviality of \( \lambda \), \( \text{im} \lambda = \nu_{p\infty} \). \( \square \)

Define \( Q(\lambda) \) to be the field obtained by adjoining to \( Q \) all the character values \( \lambda(x), x \in F \).

Define
\[
\mathcal{P} \simeq \begin{cases} \mathbb{Z}/p\mathbb{Z}, & \text{if char } F = p; \\ \mathbb{Z}_p, & \text{if char } F = 0. \end{cases}
\]

Note that \( \mathcal{P} \) is the topological closure of the prime ring of \( F \).

**Lemma 2.** There is a canonical topological isomorphism
\[
\text{Gal}(Q(\lambda)/Q) \simeq \mathcal{P}^*.
\]

**Proof.** The preceding lemma ensures that \( \text{im} \lambda \) is invariant under the action of Galois, hence restriction yields a homomorphism
\[
\text{Gal}(Q(\lambda)/Q) \to \text{Aut}(\text{im} \lambda) \simeq \begin{cases} (\mathbb{Z}/p\mathbb{Z})^*, & \text{if char } F = p; \\ \mathbb{Z}_p^*, & \text{if char } F = 0. \end{cases}
\]
It is readily checked that this map is an isomorphism of topological groups. The proof is completed by appealing to the description of \( \mathcal{P} \) given above. \( \square \)

The pairing
\[
(s, t) \to \lambda(st), \quad s, t \in F,
\]
is non-degenerate and leads to an identification of \( F^+ \) with its Pontryagin dual [W, II.5]. The image of \( s \in F \) in the dual shall be denoted \( \lambda[s] : \)
\[
\lambda[s](t) = \lambda(st), \quad t \in F.
\]
Let \( \mu = dt \) be a Haar measure on \( F^+ \). If \( \phi \) is a locally constant, complex valued function on \( F \) of compact support, the Fourier transform \( \mathcal{F}_\lambda \phi \) is the complex valued function on \( F \) defined by
\[
\mathcal{F}_\lambda \phi(s) = \int_F \lambda[s](t)\phi(t) \, dt, \quad s \in F.
\]
It can be shown that \( \mathcal{F}_\lambda \phi \) is locally constant and has compact support. Furthermore, the general theory of Fourier transforms asserts the existence of a positive constant \( c \), depending only on the Haar measure \( dt \), such that
\[
(\mathcal{F}_\lambda \mathcal{F}_\lambda \phi)(t) = c\phi(-t), \quad t \in F.
\]
There is a unique Haar measure on \( F^+ \) for which \( c = 1 \); it shall be denoted \( d_\lambda \) and will be referred to as the self-dual Haar measure associated with \( \lambda \). [W, VII.2]

**Lemma 3.** If \( \lambda \) has level \( l \) then the associated self-dual Haar measure is characterized by the condition
\[
\int_{O} d_\lambda t = q^{l/2}.
\]

**Proof.** This follows from [W, Corollary 3, VII.2]. \( \square \)

**Corollary.** If \( s \in F^* \) then
\[
d_{\lambda[s]} t = |s|^{1/2} d_\lambda t.
\]

**Proof.** Since \( i_\lambda = si_{\lambda[s]} \), the levels \( l_1 \) of \( \lambda \) and \( l_2 \) of \( \lambda[s] \) satisfy the relation \( l_1 = v(s) + l_2 \). Therefore, Lemma 3 yields
\[
\int_{O} d_{\lambda[s]} t = q^{l_2/2} = q^{-v(s)/2} q^{l_1/2} = |s|^{1/2} \int_{O} d_\lambda t.
\]
This completes the proof of the corollary. \( \square \)

### 3. The Schrödinger and Weil Representations

Let \((\cdot,\cdot)\) be a non-degenerate, alternating, \( F \)-bilinear form on a finite dimensional \( F \)-vector space \( V \). The Heisenberg group \( H \) is the group on \( V \times F \) having multiplication
\[
(v, t)(v', t') = (v + v', t + t' + \langle v, v' \rangle / 2), \quad t, t' \in F, \, v, v' \in V.
\]

Let \( \lambda \) be a non-trivial, continuous, complex linear character of \( F^+ \). Since \( Z(H) = 0 \times F \simeq F^+ \), it may be viewed as a character of the center of the Heisenberg group \( H \).

**Stone-von Neumann Theorem.** There exists a smooth, irreducible representation of \( H \) having central character \( \lambda \). Such a representation is necessarily admissible, and is unique up to isomorphism.

A proof of the Stone-von Neumann Theorem can be found in [MVW, 2.1]. The representation provided by the Stone-von Neumann Theorem is referred to as the Schrödinger representation of type \( \lambda \).

The symplectic group
\[
\text{Sp}(V) = \{ g \in GL(V) : \langle vg, wg \rangle = \langle v, w \rangle, \, v, w \in V \}
\]
acts on the Heisenberg group \( H \) as a group of automorphisms as follows: if \( g \in \text{Sp}(V) \) and \((t, v) \in H \) then
\[
(t, v)g = (t, vg).
\]
Given a Schrödinger representation $S_\lambda$ of type $\lambda$ and $g \in \text{Sp}(V)$, consider the representation $S_\lambda^g$ of $H$ defined by

$$S_\lambda^g(h) = S_\lambda(hg), \quad h \in H.$$ 

It is readily verified that $S_\lambda^g$ is a smooth, irreducible representation of $H$. Furthermore, observing that $g$ acts trivially on $Z(H)$, $S_\lambda^g$ has central character $\lambda$. The Stone-von Neumann Theorem allows us to conclude that the representation $S_\lambda$ and $S_\lambda^g$ are equivalent, hence the ambient space affording $S_\lambda$ admits an operator $W_\lambda(g)$ for which

$$S_\lambda^g(h) = W_\lambda(g)^{-1} S_\lambda(h) W_\lambda(g), \quad h \in H,$$

In light of Schur’s Lemma, the operator $W_\lambda(g)$ is uniquely defined up to multiplication by a non-zero constant. As a result, the map

$$g \mapsto W_\lambda(g), \quad g \in \text{Sp}(V),$$

is a projective representation of $\text{Sp}(V)$, called a Weil representation of type $\lambda$.

In this paper we will consider the Schrödinger models of $S_\lambda$ and $W_\lambda$ ([K, Lemma 2.2, Proposition 2.3], [MVW, 2.I.4(a), 2.II.6], [RR, §3]). Let

$$V = X + Y$$

where $X$ and $Y$ are maximal, totally isotropic subspaces. The Schrödinger model is realized in the Bruhat-Schwartz space $S(X)$ of locally constant functions $f : X \to \mathbb{C}$ of compact support: if $x \in X$, $y \in Y$ and $t \in F$ then $S_\lambda((x + y, t))$ is the operator defined by

$$[S_\lambda((x + y, t))\phi](x') = \lambda \left( t + \frac{(x,y)}{2} + \langle x', y \rangle \right) \phi(x + x'), \quad \phi \in S(X), x' \in X.$$ 

The description of the Weil representation requires some additional notation. Viewing $x + y \in V$ as a row vector $(x,y)$, each $g \in \text{Sp}(V)$ can be expressed in the matrix form

$$(2) \quad g = \begin{pmatrix} a & b \\ c & d \end{pmatrix},$$

where $a : X \to X$, $b : X \to Y$, $c : Y \to X$, and $d : Y \to Y$. With this notation, set

$$Y_g = Y / \ker c.$$

If $\mu_g$ is a Haar measure on $Y_g$ then the action of $W_\lambda(g)$ on $S(X)$ is given by

$$(3) \quad [W_\lambda(g)\phi](x) = \int_{Y_g} \lambda \left( \frac{\langle xa, x b \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \, d\mu_g y, \quad \phi \in S(X), \quad x \in X.$$ 

Note that the integral appearing in $(3)$ is well-defined, for the integrand is constant on the cosets of $\ker c$, hence can be viewed as a function on $Y_g$. The fact $\phi \in S(X)$ can be used to show that the integrand belongs to $S(Y_g)$, hence the integral converges, and that the resulting function $W_\lambda(g)\phi$ belongs to $S(X)$.

We now recall a particular choice of Haar measures $\mu_{\lambda,g}$ on $Y_g$, $g \in \text{Sp}(V)$ [RR, §3.3]. Fix a basis $x_1, \ldots, x_n$ of $X$ and let $y_1, \ldots, y_n$ be the dual basis of $Y$ defined by the conditions

$$\langle x_i, y_j \rangle = \delta_{ij}, \quad 1 \leq i, j \leq n.$$
Let \( \tau_i, 0 \leq i \leq n, \) be the element of \( \text{Sp}(V) \) defined by

\[
x_j \tau_i = \begin{cases} 
-y_j, & \text{if } j \leq i; \\
x_j, & \text{if } i < j.
\end{cases} \quad \text{and} \quad y_j \tau_i = \begin{cases} 
x_j, & \text{if } j \leq i; \\
y_j, & \text{if } i < j.
\end{cases}
\]

We note that \( Y_{\tau_i} \) can be identified with the subspace of \( Y \) spanned by the elements \( y_1, \ldots, y_i \).

We define

\[
d\mu_{\lambda, \tau_i} y = \prod_{k=1}^{i} d\lambda y_k.
\]

where \( d\lambda y_k \) is the self-dual Haar measure associated with \( \lambda \).

Let

\[
P = \{ g \in \text{Sp}(V) : Yg = g \},
\]

the parabolic subgroup that leaves \( Y \) invariant. If \( \dim Y_g = i \) then [RR, Theorem 2.14] ensures the existence of elements \( p_1 \) and \( p_2 \) of \( P \) such that

\[
g = p_1 \tau_i p_2.
\]

Observing that the operator \( p_1 \) induces an isomorphism \( \overline{p_1} : Y_g \to Y_{\tau_i} \), we set

\[
\mu_{\lambda, g} = |\det(p_1 p_2|Y)|^{-1/2} \overline{p_1} : \mu_{\lambda, \tau_i}.
\]

Here, \( \overline{p_1} : \mu_{\lambda, \tau_i} \) denotes the pullback of the Haar measure \( \mu_{\lambda, \tau_i} \) to \( Y_g \) via \( \overline{p_1} : \) if \( E \) is a measurable subset of \( Y_g \) then

\[
\overline{p_1} : \mu_{\lambda, \tau_i}(O) = \mu_{\lambda, \tau_i}(O \overline{p_1}).
\]

**Theorem 4.** The measures \( \mu_{\lambda, g} \), \( g \in \text{Sp}(V) \), are well-defined. Furthermore, the projective representation \( W_\lambda \) of \( \text{Sp}(V) \) defined by (3) with the Haar measures \( \mu_g = \mu_{\lambda, g} \) has the following properties.

(i) If \( g \in \text{Sp}(V) \) and \( p_1, p_2 \in P \) then \( W_\lambda(p_1 g p_2) = W_\lambda(p_1) W_\lambda(g) W_\lambda(p_2) \); in particular \( W_\lambda \) restricts to an ordinary representation of \( P \).

(ii) If \( \phi \in \mathcal{S}(X) \) and \( p = \begin{pmatrix} a & b \\ 0 & d \end{pmatrix} \in P \) then

\[
[W_\lambda(p)\phi](x) = |\det a|^{1/2} \lambda \left( \frac{\langle xa, xb \rangle}{2} \right) \phi(xa), \quad x \in X.
\]

**Proof.** This follows from [RR, Theorem 3.5] \( \square \)

**Lemma 5.** If \( s \in F^* \) and \( g \in \text{Sp}(V) \) then \( \mu_{\lambda[s], g} = [s]^{1/2} \mu_{\lambda, g} \).

**Proof.** In light of the Corollary to Lemma 3, (4) yields

\[
d\mu_{\lambda[s], \tau_i} y = \prod_{k=1}^{i} d\lambda[s] y_k = \prod_{k=1}^{i} [s]^{1/2} d\lambda y_k = [s]^{i/2} \prod_{k=1}^{i} d\lambda y_k = [s]^{i/2} d\mu_{\lambda, \tau_i} y.
\]

Therefore, (6) gives

\[
\mu_{\lambda[s], g} = |\det(p_1 p_2|Y)|^{-1/2} \overline{p_1} : \mu_{\lambda[s], \tau_i} = |s|^{i/2} |\det(p_1 p_2|Y)|^{-1/2} \overline{p_1} : \mu_{\lambda, \tau_i} = |s|^{i/2} \mu_{\lambda, g} = [s]^{1/2} \mu_{\lambda, g},
\]

since \( Y_g \) has dimension \( i \) over \( F \). \( \square \)
4. Action of Symplectic Similitudes

Given \( s \in F^* \), let \( f_s \) be the element of \( GL(V) \) defined by
\[
(x + y)f_s = x + sy, \quad x \in X, y \in Y.
\]
Conjugation by \( f_s \) leaves the symplectic group \( Sp(V) \) invariant. In fact, if \( g \in Sp(V) \) is expressed in the matrix form (2) then
\[
g^{f_s} = \begin{pmatrix} a & sb \\ s^{-1}c & d \end{pmatrix}.
\]
In particular, we note that the spaces \( Y_g \) and \( Y_{g^{f_s}} \) are equal, since \( \ker c = \ker s^{-1}c \).

**Lemma 6.** If \( s \in F^* \) then \( \mu_{\lambda,g^{f_s}} = \mu_{\lambda,g} \).

**Proof.** Let \( p_{i,s}, 0 \leq i \leq n \), be the elements of \( Sp(V) \) defined by
\[
x_jp_{i,s} = \begin{cases} s^{-1}x_j, & \text{if } j \leq i; \\ x_j, & \text{if } i < j. \end{cases}
\]
y_jp_{i,s} = \begin{cases} sy_j, & \text{if } j \leq i; \\ y_j, & \text{if } i < j. \end{cases}
\]
Note that \( p_{i,s} \in P \) and
\[\det(p_{i,s}|_Y) = s^i.\]
Moreover, one readily verifies that
\[\tau_i^{f_s} = \tau_ip_{i,s}.\]
Let \( g \in G \). If \( g = p_1\tau_ip_2, p_1, p_2 \in P \), then
\[g^{f_s} = (p_1\tau_ip_2)^{f_s} = p_1^{f_s}\tau_i^{f_s}p_2 = p_1^{f_s}\tau_i(p_{i,s}p_2^{f_s}).\]
Observing that both \( p_1^{f_s} \) and \( p_{i,s}p_2^{f_s} \) belong to \( P \), (6) yields
\[\mu_{\lambda,g^{f_s}} = |\det(p_1^{f_s}p_{i,s}p_2^{f_s}|_Y)|^{-1/2}p_1^{f_s} \cdot \mu_{\lambda,\tau_i}.\]
Using (7), if \( p \in P \) then \( p^{f_s}|_Y = p|_Y \). As a consequence,
\[\overline{p_1^{f_s}} = \overline{p}: Y_g \to Y_{\tau_i}.\]
In light of these observations,
\[\det(p_1^{f_s}p_{i,s}p_2^{f_s}|_Y) = \det(p_1p_ip_2|_Y) \cdot \det(p_{i,s}|_Y) \cdot \det(p_1p_2|_Y) = s^i \det(p_1p_2|_Y),\]
hence
\[\mu_{\lambda,g^{f_s}} = |s^i \det(p_1p_2|_Y)|^{-1/2}p_1^{f_s} \cdot \mu_{\lambda,\tau_i} = |s|^{-i/2}\mu_{\lambda,g} = |s|_{Y_g}^{-1/2}\mu_{\lambda,g},\]
since \( Y_g \) has dimension \( i \) over \( F \).

Let \( W^{f_s}_\lambda \) be the projective representation of \( Sp(V) \) defined by
\[W^{f_s}_\lambda(g) = W_\lambda(g^{f_s}).\]
For the proof of the next result, let \( |\alpha|_V \) denote the module of an automorphism \( \alpha \) of an \( F \)-vector space \( V \) [W, I.2]. We have
\[|\alpha|_V = |\det \alpha|.
\]
In particular, the module of left multiplication by \( s \in F^* \) on \( V \) satisfies
\[|s|_V = |s|^{\dim V}.
\]

**Proposition 7.** If \( s \in F^* \) then \( W^{f_s}_\lambda = W_{\lambda[s]} \).
Lemma 8. If \( \phi \in \mathcal{S}(X) \) and \( x \in X \) then the integral formula (3) and Lemma 6 yield

\[
\left[ W_\lambda(g^f) \phi \right](x) = \int_{Y_g} \lambda \left( \frac{\langle xa, sxb \rangle - 2 \langle sxb, s^{-1}yc \rangle + \langle s^{-1}yc, yd \rangle}{2} \right) \phi(xa + s^{-1}yc) \, d\mu_{\lambda,g}, y
\]

Replacing \( y \) by \( sy \), the definition of \( |s|_{Y_y} \) and Lemma 4 yield

\[
\left[ W_\lambda(g^f) \phi \right](x) = |s|_{Y_y}^{-1/2} \int_{Y_y} \lambda \left( \frac{\langle xa, sxb \rangle - 2 \langle sxb, yc \rangle + \langle yc, syd \rangle}{2} \right) \phi(xa + yc) \, d\mu_{\lambda,g}
\]

This completes the proof of the lemma. \( \square \)

5. Action of Galois

Let \( \mu \) be a Haar measure on a totally disconnected topological group \( A \). If \( O_1 \) and \( O_2 \) are non-empty compact open sets in \( A \) then the ratio

\[
(O_1 : O_2) = \frac{\mu(O_1)}{\mu(O_2)}
\]

is a rational number [C, I.1.1.]. Hence, if \( \mu(O) \) lies in a subfield \( L \) of \( C \) for some non-empty compact open set \( O \) then the same is true for all non-empty compact open sets. The measure \( \mu \) is said to \( L \)-rational if this is the case.

Lemma 8. The measures \( \mu_{\lambda,g}, g \in \text{Sp}(V) \), are \( Q(\sqrt{q}) \)-rational.

Proof. If \( t \in F^* \) then \( |t| \) is a power of \( q \). Therefore, (6) shows that it is sufficient to verify that the measures \( \mu_{\lambda,\tau_i} \) are \( Q(\sqrt{q}) \)-rational. Formulas (1) and (4) ensure that this is indeed the case: if \( \mathcal{Y}_i = \sum_{k=1}^{\infty} O_{y_k} \) then

\[
\int_{\mathcal{Y}_i} d\mu_{\lambda,\tau_i} y = q^{il/2}.
\]

This completes the proof of the lemma. \( \square \)

Let \( A \) be a totally disconnected topological group. If \( L \) is a subfield of \( C \), let \( \mathcal{S}(A, L) \) denote the space of locally constant, \( L \)-valued functions on \( A \) of compact support.

Lemma 9. Let \( A \) be a totally disconnected topological group, \( L \subseteq K \) an extension of fields, and \( \mu \) a \( L \)-rational Haar measure on \( A \). If \( \phi \in \mathcal{S}(A, K) \) then \( \int_A \phi \, d\mu \) belongs to \( K \).
**Proposition 12.** Let $\phi \in \mathcal{S}(A,K)$, there exists compact open subsets $A_1, \ldots, A_k$ of $A$ and scalars $c_1, \ldots, c_k$ in $K$ such that

$$\phi = \sum_{i=1}^{k} c_i \chi_{A_i}.$$ 

Here, $\chi_{A_i}$ denotes the characteristic function of $A_i$. Since $\mu(A_i) \in L \subset K$, it follows that

$$\int_A \phi \, d\mu = \sum_{i=1}^{k} c_i \mu(A_i)$$

lies in $K$. \hfill $\square$

Let $Q(\lambda)$ be the character field of $\lambda$ and set

$$E = Q(\lambda)(\sqrt{-1}).$$

Observe that Lemma 1 ensures that $Q(\sqrt{q})$ is a subfield of $E$.

**Proposition 10.** The operators $W_\lambda(g)$, $g \in \text{Sp}(V)$, leave the subspace $S(X,E)$ invariant.

**Proof.** If $\phi \in S(X,E)$ then the integrand in (3) lies in $S(Y_q,E)$, since $Q(\lambda) \subseteq E$. In light of Lemma 8, Lemma 9 applied in the case $A = Y_q$, $K = E$, $L = Q(\sqrt{q})$, and $\mu = \mu_{\lambda,g}$ allows us to deduce that the integral (3) lies in $E$. It follows immediately that $W_\lambda(g) \phi \in S(X,E).$ \hfill $\square$

By Lemma 1, $E$ is a Galois extension of $Q$. Its Galois group acts on $S(X,E)$: if $\sigma \in \text{Gal}(E/Q)$ and $\phi \in S(X,E)$ then

$$(\sigma(\phi))(x) = \sigma(\phi(x)), \quad x \in X.$$ 

There is an associated Galois action on $\text{End}S(X,E)$: if $\sigma \in G$ and $T \in \text{End}S(X,E)$ then

$$(^{\sigma}T) = \sigma[T(\sigma^{-1}(\phi))], \quad \phi \in S(X,E).$$

The Galois group also permutes the unitary characters of $F^+$: if $\sigma \in \text{Gal}(E/Q)$ and $\lambda$ is a unitary character of $F^+$ then $^{\sigma}\lambda$ is the character defined by

$$^{\sigma}\lambda(t) = \sigma(\lambda(t)), \quad t \in F^+.$$ 

Let $\mathcal{P}$ be the topological closure of the prime ring of $F$. The image of $s \in \mathcal{P}^*$ in $\text{Gal}(Q(\lambda)/\lambda)$ under the canonical isomorphism of Lemma 2 will be denoted $\sigma_s$.

**Lemma 11.** Let $\sigma \in \text{Gal}(E/Q)$. If $\sigma|_{Q(\lambda)} = \sigma_s$ then $^{\sigma}\lambda = \lambda[s]$.

**Proof.** (char $F = 0$) Let $i$ be the conductor of $\lambda$. Given $t \in F$, fix $n \geq 1$ such that $t \in p^{-n}i$. Since $p^n t \in i$,

$$1 = \lambda(p^n t) = \lambda(t)^{p^n},$$

thus $\lambda(t) \in \nu_{p^n}$. Fixing $r \in \mathbb{Z}$ such that $s \equiv r \mod p^n\mathcal{P}$,

$$^{(\sigma)}(\lambda)(t) = \sigma(\lambda(t)) = \lambda(t)^{\sigma} = \lambda(rt) = \lambda(st),$$

the last equality following from the fact $rt \equiv st \mod i$. \hfill $\square$

Given $\sigma \in \text{Gal}(E/Q)$, let $^{\sigma}W_\lambda$ be the projective representation defined by

$$(^{\sigma}W_\lambda)(g) = ^{\sigma}(W_\lambda(g)), \quad g \in \text{Sp}(V).$$

**Proposition 12.** Let $\sigma \in \text{Gal}(E/Q(\sqrt{q}))$. If $\sigma|_{Q(\lambda)} = \sigma_s$ then $^{\sigma}W_\lambda(g) = W_{\lambda[s]}(g)$.

The proof of Proposition 12 is based on the integral formula (3) and the following
Lemma 13. Let $A$ be a totally disconnected topological group, $L \subseteq K$ an extension of fields, and $\mu$ a $L$-rational Haar measure on $A$. If $\sigma$ is an $L$-automorphism of $K$ then, for all $\phi \in \mathcal{S}(A, K)$,

\[
\int_A \sigma(\phi) \, d\mu = \sigma \left( \int_A \phi \, d\mu \right).
\]

Proof. Using the notation introduced in the proof of Lemma 5, if $\phi = \sum_{i=1}^{k} c_i \chi_{A_i}$ then

\[
\sigma(\phi) = \sum_{i=1}^{k} \sigma(c_i) \chi_{A_i}.
\]

Therefore, since $\mu(A_i) \in L$ is fixed by $\sigma$,

\[
\int \sigma(\phi) \, d\mu = \sum_{i=1}^{k} \sigma(c_i) \mu(A_i) = \sigma \left( \sum_{i=1}^{k} c_i \mu(A_i) \right) = \sigma \left( \int A \phi \, d\mu \right).
\]

This completes the proof of the lemma.

Proof of Proposition 12. Let $g \in \text{Sp}(V)$, $\phi \in \mathcal{S}(X, E)$, and $x \in X$. We assume $g$ has the matrix representation (2). Lemma 8 asserts that the measure $\mu_{\lambda,g}$ is $Q(\sqrt{q})$-rational. Applying Lemma 13 to the case $A = Yg$, $L = Q(\sqrt{q})$, $K = E$, and $\mu = \mu_{\lambda,g}$, the definition of $W_\lambda$, the formula (3), and Lemma 11 yield

\[
[W_\lambda(g)\phi](x) = \sigma \left[ W_\lambda(g)(\sigma^{-1}\phi)(x) \right]
\]

\[
= \sigma \left[ \int_{Y_g} \left( \lambda \left( \frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) (\sigma^{-1}\phi)(xa + yc) \right) \, d\mu_{\lambda,g}y \right]
\]

\[
= \int_{Y_g} \left[ \lambda \left( \frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] \, d\mu_{\lambda,g}y
\]

\[
= \int_{Y_g} \left[ \lambda[s] \left( \frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] \, d\mu_{\lambda,g}y.
\]

Observing $s \in P^* \subseteq O^*$, Lemma 5 implies that $\mu_{\lambda[s],g} = \mu_{\lambda,g}$. The preceding calculation thus gives

\[
[W_\lambda(g)\phi](x) = \int_{Y_g} \left[ \lambda[s] \left( \frac{\langle xa, xb \rangle - 2 \langle xb, yc \rangle + \langle yc, yd \rangle}{2} \right) \phi(xa + yc) \right] \, d\mu_{\lambda[s],g}y
\]

\[
= [W_{\lambda[s]}(g)\phi](x).
\]

This completes the proof.

6. The Fundamental Identity

Let

\[
\mathcal{G} = \{ \sigma \in \text{Gal}(E/Q(\sqrt{q})) : \exists s \in O^* \text{ such that } \sigma|_{Q(\lambda)} = \sigma s^2 \}.
\]
Note that $\mathcal{G}$ is a subgroup of $\text{Gal}(E/\mathbb{Q}(\sqrt{q}))$. Given $s \in F^*$, let $g_s \in \text{Sp}(V)$ be the map defined by $$(x + y)g_s = s^{-1}x + sy, \quad x \in X, y \in Y.$$ We observe that $g_s$ lies in the parabolic subgroup $P$ that leaves $Y$ invariant and is related to the operator $f_{s^2}$ defined earlier by the identity $f_{s^2} = sI \circ g_s$.

**Proposition 14.** Let $\sigma \in \mathcal{G}$ and $g \in \text{Sp}(V)$. If $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{s^2}$, $s \in \mathcal{O}^*$, then $\sigma W_\lambda(g) = W_\lambda(g_s)^{-1}W_\lambda(g)W_\lambda(g_s)$.\[\]

**Proof.** In light of Propositions 7 and 12, $\sigma W_\lambda(g) = W_\lambda[\sigma^2]g = W_\lambda f_{s^2}(g) = W_\lambda(g_f^s)$. Applying Theorem 4(i) with $p^{-1}_1 = p_2 = g_s$, $W_\lambda(g_f^s) = W_\lambda(g_s)^{-1}W_\lambda(g)W_\lambda(g_s) = W_\lambda(g_s)^{-1}W_\lambda(g)W_\lambda(g_s)$. This completes the proof of the proposition.\[\]

**Corollary.** If $t \in F^*$ and $\sigma \in \mathcal{G}$ then $\sigma W_\lambda(g_t) = W_\lambda(g_t)$.

**Proof.** Fix $s \in \mathcal{O}^*$ such that $\sigma|_{\mathbb{Q}(\lambda)} = \sigma_{s^2}$. Observing that $g_s$ and $g_t$ are commuting elements of $P$, the preceding proposition combines with Theorem 4(i) to yield $\sigma W_\lambda(g_t) = W_\lambda(g_s)^{-1}W_\lambda(g_t)W_\lambda(g_s) = W_\lambda(g_s)^{-1}g_tg_s = W_\lambda(g_t)$, as required.\[\]

### 7. The Cocycle

Define $\mathfrak{H} = \text{Gal} \left( \mathbb{E}/\mathbb{Q}(\sqrt{p}, \sqrt{-p}) \right)$. Our aim in this section is the construction of a 1-cocycle $\delta : \mathfrak{H} \to \text{GL}(\mathcal{S}(X,E))$ such that $\sigma W_\lambda(g) = \delta(\sigma)^{-1}W_\lambda(g)\delta(\sigma), \quad g \in \text{Sp}(V), \sigma \in \mathfrak{H}$. Let $p^* = (-1)^{(p-1)/2}p$. When combined with restriction to $\mathbb{Q}(\lambda)$, the canonical isomorphism of Lemma 2 yields $\mathfrak{H} \simeq \text{Gal} \left( \mathbb{Q}(\lambda)/\mathbb{Q}(\sqrt{p}) \right) \simeq (\mathbb{P}^*)^2$. If $\epsilon \in \mathbb{P}^*$ is a primitive $p - 1$ root of unity then $\mathbb{P}^* = \langle \epsilon \rangle \times U_1$ with $U_1 = \{1\}$ if char $F = p$; $\{r \in \mathbb{P} : r \equiv 1 \mod p\}$ if char $F = 0$, a pro-$p$ group. As $p$ is assumed to be odd, the map $r \mapsto r^2$ is an automorphism of $U_1$, hence $\mathbb{P}^{*^2} = \langle \epsilon^2 \rangle \times U_1$. The isomorphism (11) identifies $U_1$ with $\text{Gal} \left( \mathbb{E}/\mathbb{Q}(\nu_p, \sqrt{-1}) \right)$, where $\nu_p$ is the group of complex $p$-th roots of unity. This in turn leads to an identification of $\langle \epsilon^2 \rangle$ with $\mathfrak{H}/\text{Gal} \left( \mathbb{E}/\mathbb{Q}(\nu_p, \sqrt{-1}) \right) \simeq \text{Gal} \left( \mathbb{Q}(\nu_p, \sqrt{-1})/\mathbb{Q}(\sqrt{p}, \sqrt{-p}) \right)$.\[\]
In particular, the element $\eta$ of $\mathfrak{M}$ characterized by
\[
\eta|_{\mathbb{Q}(\lambda)} = \sigma e^2
\]
has order $(p - 1)/2$ and restricts to a generator of $\text{Gal} \left( \mathbb{Q}(\sqrt{-1})/\mathbb{Q}(\sqrt{-p}) \right)$.

Given $\sigma \in \mathfrak{M}$, there is a unique integer $i$, $1 \leq i \leq (p - 1)/2$, and a unique element $s \in U_1$, such that
\[
\sigma|_{\mathbb{Q}(\lambda)} = \sigma e^{2is^2}.
\]
If $\tau$ is a second element of $\mathfrak{M}$, say
\[
\tau|_{\mathbb{Q}(\lambda)} = \sigma e^{2it^2}, \quad 1 \leq j \leq (p - 1)/2, \quad t \in U_1,
\]
then
\[
\sigma \tau|_{\mathbb{Q}(\lambda)} = \sigma e^{2k(s^2t^2)},
\]
where $st \in U_1$ and
\[
k = \begin{cases} 
i + j, & \text{if } i + j \leq (p - 1)/2; \\
i + j - \frac{p - 1}{2}, & \text{if } i + j > (p - 1)/2.
\end{cases}
\]

Our initial attempt at the construction of the cocycle is to define
\[
D(\sigma) = W_\lambda(g_{e^2s}), \quad \sigma|_{\mathbb{Q}(\lambda)} = \sigma e^{2is^2}, \quad 1 \leq i \leq (p - 1)/2, \quad s \in U_1.
\]
Proposition 14 ensures that
\[
^\sigma W_\lambda(g) = D(\sigma)^{-1}W_\lambda(g)D(\sigma), \quad g \in \text{Sp}(V), \sigma \in \mathfrak{M}.
\]
Assuming $\sigma$ and $\tau$ are as above, the definition of $D$ yields
\[
D(\sigma \tau) = W_\lambda(g_{e^2st}).
\]
On the other hand, the Corollary to Proposition 14 gives
\[
^\sigma D(\tau) = ^\sigma W_\lambda(g_{e^2t}) = W_\lambda(g_{e^2t}),
\]
hence Theorem 4(i) yields
\[
D(\sigma)^{\sigma} D(\tau) = W_\lambda(g_{e^2s})W_\lambda(g_{e^2t}) = W_\lambda(g_{e^2+is^2+it^2}).
\]
If $i + j \leq (p - 1)/2$ then
\[
W_\lambda(g_{e^2+is^2+it^2}) = W_\lambda(g_{e^k}).
\]
If $i + j > (p - 1)/2$ then, since $e^{(p-1)/2} = -1$, Theorem 4(i) yields
\[
W_\lambda(g_{e^2+is^2+it^2}) = W_\lambda(g_{e^k})W_\lambda(g_{e^k}),
\]
where $\iota = g_{-1}$ is the central involution of $Sp(V)$ that maps $v \in V$ to $-v$. In summary,
\[
D(\sigma)^{\sigma} D(\tau) = \begin{cases} 
D(\sigma \tau), & \text{if } i + j \leq (p - 1)/2; \\
W_\lambda(\iota)D(\sigma \tau), & \text{if } i + j > (p - 1)/2.
\end{cases}
\]
In particular, $D$ is not a 1-cocycle; to get one we must account for the factor $W_\lambda(\iota)$.

Since $\iota \in P$, Theorem 4(i) implies that if $\phi$ belongs to $S(X, E)$ then
\[
[W_\lambda(\iota)\phi](x) = \phi(-x), \quad x \in X.
\]
In particular, $W_\lambda(\iota)$ is an involution, hence the operators
\[
\rho_e = \frac{1}{2} (I + W_\lambda(\iota)) \quad \text{and} \quad \rho_o = \frac{1}{2} (I - W_\lambda(\iota))
\]
are orthogonal idempotents. Furthermore, recalling \( \iota = g_1 \), the Corollary to Proposition 14 shows that both \( \rho_e \) and \( \rho_o \) are fixed by the action of Galois. Finally, since \( I = \rho_e + \rho_o \), it is easily verified that the operators
\[
\rho_e + c\rho_o, \quad c \in E, c \neq 0,
\]
are invertible.

**Lemma 15.** The norm equation
\[
N(u) = -1, \quad N : \mathbb{Q}(\nu_p, \sqrt{-1}) \to \mathbb{Q}(\sqrt{p}, \sqrt{-p})
\]
has a solution.

**Proof.** The case \( p \equiv 1 \mod 4 \) is covered by [CMS, Lemma 24(2)]. In the case \( p \equiv 3 \mod 4 \) one has
\[
N(-1) = (-1)^{[\mathbb{Q}(\nu_p, \sqrt{-1}) : \mathbb{Q}(\sqrt{p}, \sqrt{-p})]} = (-1)^{(p-1)/2} = -1,
\]
since \( (p - 1)/2 \) is odd.

Let \( u \) be a solution of the norm equation of the preceding lemma. Given \( \sigma \in \mathfrak{H} \), set
\[
A(\sigma) = \rho_e + \left( \prod_{l=0}^{k-1} \eta^l(u) \right) \rho_0,
\]
where \( \sigma|_{\mathbb{Q}(\lambda)} = \sigma_{e^{2i\pi s}}, \quad 1 \leq i \leq (p - 1)/2, \quad s \in U_1 \)
where \( \eta \) satisfies (12). The remarks preceding Lemma 15 ensure that \( A(\sigma) \in \text{GL}(S(X, E)) \).

With the notation introduced earlier, if \( \sigma \) and \( \tau \) belong to \( \mathfrak{H} \) then
\[
A(\sigma \tau) = \rho_e + \left( \prod_{l=0}^{k-1} \eta^l(u) \right) \rho_0.
\]

On the other hand, observing
\[
\sigma \eta^{-i}|_{\mathbb{Q}(\lambda)} = \sigma_{e^{2i\pi s}} \sigma^{-i} = \sigma_{e^{2i\pi s}} \sigma_{e^{-2i}} = \sigma_{e^2},
\]
the fact (11) identifies \( U_1 \) with \( \text{Gal}(E/\mathbb{Q}(\nu_p, \sqrt{-1})) \) allows us to deduce that the restrictions of \( \sigma \) and \( \eta^i \) to \( \mathbb{Q}(\nu_p, \sqrt{-1}) \) coincide. Therefore,
\[
\sigma A(\tau) = \left[ \rho_e + \left( \prod_{l=0}^{j-1} \eta^l(u) \right) \rho_0 \right] = \rho_e + \sigma \left( \prod_{l=0}^{j-1} \eta^l(u) \right) \rho_0
\]
\[
= \rho_e + \eta^{i} \left( \prod_{l=0}^{j-1} \eta^l(u) \right) \rho_0 = \rho_e + \left( \prod_{l=i}^{i+j-1} \eta^l(u) \right) \rho_0,
\]
hence
\[
A(\sigma) \sigma A(\tau) = \left[ \rho_e + \left( \prod_{l=0}^{i+j-1} \eta^l(u) \right) \rho_0 \right] = \rho_e + \left( \prod_{l=0}^{i+j-1} \eta^l(u) \right) \rho_0
\]
\[
= \rho_e + \left( \prod_{l=0}^{i+j-1} \eta^l(u) \right) \rho_0.
\]
If \( i + j \leq (p - 1)/2 \) then
\[
\prod_{l=0}^{i+j-1} \eta^l(u) = \prod_{l=i}^{k-1} \eta^l(u),
\]
hence

\[ A(\sigma)^\circ A(\tau) = A(\sigma \tau). \]

If \( i + j > (p - 1)/2 \) then the choice of \( \eta \) and \( u \) yield

\[
\prod_{l=0}^{i+j-1} \eta^l(u) = \left( \prod_{l=0}^{(p-3)/2} \eta^l(u) \right) \left( \prod_{l=(p-1)/2}^{i+j-1} \eta^l(u) \right) = N(u) \prod_{l=0}^{k-1} \eta^l(u) = -\prod_{l=0}^{k-1} \eta^l(u).
\]

Observing that \( \rho_e = \rho_e W_\lambda(i) \) and \( -\rho_o = \rho_o W_\lambda(i) \),

\[ A(\sigma)^\circ A(\tau) = \rho_e - \left( \prod_{l=0}^{k-1} \eta^l(u) \right) \rho_0 = \left[ \rho_e + \left( \prod_{l=0}^{k-1} \eta^l(u) \right) \rho_0 \right] W_\lambda(i) = A(\sigma \tau) W_\lambda(i). \]

In summary,

\[
A(\sigma)^\circ A(\tau) = \begin{cases} A(\sigma \tau), & \text{if } i + j \leq (p - 1)/2; \\ A(\sigma \tau) W_\lambda(i), & \text{if } i + j > (p - 1)/2. \end{cases}
\]

Consider the map \( \delta : \mathfrak{H} \to \text{GL}(S(X, E)) \) given by

\[ \delta(\sigma) = A(\sigma) D(\sigma). \]

If \( \sigma, \tau \in \mathfrak{H} \) are as above

\[ \delta(\sigma)^\circ \delta(\tau) = (A(\sigma) D(\sigma))^\circ (A(\tau) D(\tau)) = A(\sigma) D(\sigma)^\circ A(\tau) D(\tau). \]

By Theorem 4(1), \( ^{\circ}A(\tau) \in E[W_\lambda(i)] \) commutes with \( D(\sigma) = W_\lambda(g_{v,s}) \), hence

\[ A(\sigma) D(\sigma)^\circ A(\tau) D(\tau) = A(\sigma)^\circ A(\tau) D(\sigma)^\circ D(\tau). \]

If \( i + j > (p - 1)/2 \) then (14) and (15) yield

\[ A(\sigma)^\circ A(\tau) D(\sigma)^\circ D(\tau) = A(\sigma \tau) W_\lambda(i) W_\lambda(i) D(\sigma \tau) = A(\sigma \tau) D(\sigma \tau). \]

Since this is trivially true if \( i + j \leq (p - 1)/2 \), we conclude

\[ \delta(\sigma)^\circ \delta(\tau) = A(\sigma \tau) D(\sigma \tau) = \delta(\sigma \tau). \]

This shows that \( \delta \) is a 1-cocycle. Furthermore, if \( g \in \text{Sp}(V) \) then Theorem 4(i) shows that \( A(\sigma) \in E[W_\lambda(i)] \) commutes with \( W_\lambda(g) \), hence (12) yields

\[
\delta(\sigma)^{-1} W_\lambda(g) \delta(\sigma) = (A(\sigma) D(\sigma))^{-1} W_\lambda(g) A(\sigma) D(\sigma)
\]
\[ = D(\sigma)^{-1} A(\sigma)^{-1} W_\lambda(g) A(\sigma) D(\sigma)
\]
\[ = D(\sigma)^{-1} W_\lambda(g) D(\sigma)
\]
\[ = \sigma W_\lambda(g) \]

which verifies that (10) is satisfied.
8. The Triviality of the Cocycle

Let $\delta : \mathfrak{H} \to \text{GL}(\mathcal{S}(X, E))$ be the 1-cocycle satisfying (10) constructed above.

**Lemma 16.** If $\phi \in \mathcal{S}(X, E)$ then there exists an open subgroup $\mathfrak{K}$ of $\mathfrak{H}$ such that

$$\delta(\sigma)\phi = \phi, \quad \sigma \in \mathfrak{K}.\)

*Proof.* If char $F = p$ then $\mathfrak{H}$ is a finite discrete group, so one may take $\mathfrak{K}$ to be the trivial subgroup.

Assume char $F = 0$. If $\mathcal{X}$ is a lattice in $X$ then the subgroups

$$p^k\mathcal{X}, \quad k \in \mathbb{Z},$$

form a local base at the origin. Therefore, given $x \in X$, there exist $i_x \in \mathbb{Z}$ such that $\phi$ is constant on the coset $x + p^{i_x}\mathcal{X}$. As the family $\{x + p^{i_x}\mathcal{X} : x \in X\}$ is an open cover of $X$, there exists $x_1, \ldots, x_m$ in $X$ such that

$$\text{supp} \phi \subseteq \bigcup_{j=1}^m x_j + p^{i_j}\mathcal{X}.$$

Set

$$i = \max \{i_{x_1}, \ldots, i_{x_m}\}$$

and consider $x + p^i\mathcal{X} \cap \text{supp} \phi$, $x \in X$. If it is empty then the restriction of $\phi$ to the coset $x + p^i\mathcal{X}$ is identically 0. If not, there exists $j$ such that $x + p^i\mathcal{X} \cap x_j + p^{i_j}\mathcal{X}$ is non-empty, hence

$$x + p^i\mathcal{X} \subseteq x_j + p^{i_j}\mathcal{X}$$

by choice of $i$. The choice of $i_{x_j}$ thus ensures that the restriction of $\phi$ to $x + p^i\mathcal{X}$ is the constant function with value $\phi(x_j)$. We conclude that $\phi$ is constant on the $p^i\mathcal{X}$-cosets of $X$.

Let $\sigma \in \mathfrak{H}$. If $\sigma |_{\mathcal{Q}(\lambda)} = \sigma_r^2$, $r \in U_1$, then by construction $\delta(\sigma) = W_\lambda(g_r)$. Observing

$$g_r = \begin{pmatrix} r^{-1} \cdot 1_X & 0 \\ 0 & r \cdot 1_Y \end{pmatrix} \in P,$$

if $x \in X$ then Theorem 4(i) yields

$$(\delta(\sigma)\phi)(x) = (W_\lambda(g_r)\phi)(x) = |r|^{-\dim X/2} \lambda \left(\frac{\langle r^{-1}x, rx \rangle}{2}\right) \phi(r^{-1}x) = \phi(r^{-1}x),$$

since $r$ is a unit and $\langle , \rangle$ is $F$-bilinear and alternating. Fix $j \in \mathbb{Z}$ such that $i > j$ and $\text{supp} \phi \subseteq p^i\mathcal{X}$.

If $x \not\in p^i\mathcal{X}$ then neither is $r^{-1}x$, so the choice of $j$ ensures that

$$(\delta(\sigma)\phi)(x) = \phi(r^{-1}x) = 0 = \phi(x).$$

On the other hand, suppose $x \in p^i\mathcal{X}$. In this case, if $r \equiv 1 \mod p^{-j}$ then

$$r^{-1}x + p^i\mathcal{X} = x + p^{-j}p^i\mathcal{X} + p^i\mathcal{X} = x + p^i\mathcal{X},$$

hence the choice of $i$ ensures that

$$(\delta(\sigma)\phi)(x) = \phi(r^{-1}x) = \phi(x).$$

In light of the preceding discussion,

$$\mathfrak{K} = \{\sigma \in \mathfrak{H} : \sigma |_{\mathcal{Q}(\lambda)} = \sigma_r^2, r \equiv 1 \mod p^{-j}\} = \text{Gal}(E/\mathbb{Q}(\nu_{p^{-j}}, \sqrt{-1}))$$

where $\nu_{p^{-j}}$ is a primitive $p^j$th root of unity.
Let $K/k$ be a Galois extension and $M$ a $K$-vector space equipped with an semi-linear action of the Galois group $\text{Gal}(K/k)$: if $\sigma \in \text{Gal}(K/k)$, $m \in M$ and $e \in K$ then
\[ \sigma(em) = \sigma(e)\sigma(m). \]

For such an action, the fixed-point set
\[ M^{\text{Gal}(K/k)} = \{ m \in M : m = \sigma(m) \forall \sigma \in \text{Gal}(K/k) \} \]
is a $k$-vector space. The canonical action of $\text{Gal}(K/k)$ on $K$ yields a semi-linear action on the tensor product $K \otimes_k M^{\text{Gal}(K/k)}$:
\[ \sigma(e \otimes m) = \sigma(e) \otimes m, \quad \sigma \in \text{Gal}(K/k), e \in E, m \in M^{\text{Gal}(K/k)}. \]
The action of Galois on $M$ is said to be smooth if the stabilizer of each $m \in M$ is open in $\text{Gal}(K/k)$.

**Proposition 17** (Galois Descent). If $M$ is a $K$-vector space equipped with a semi-linear, smooth action of $\text{Gal}(K/k)$ then the canonical map
\[ \psi : K \otimes_k M_k \rightarrow M \]
is a $K$-linear isomorphism of $\text{Gal}(K/k)$-modules.

**Proof.** The case $K = k_s$, the separable closure of $k$, is proved in [B, AG.14.2]. The general case is proved using the same argument, mutatis mutandis. \qed

**Proposition 18.** There exists $\alpha \in \text{GL}(S(X,E))$ such that
\[ (16) \quad \delta(\sigma) = \alpha^{-1}\sigma\alpha, \quad \sigma \in \mathfrak{H}. \]

**Proof.** The canonical action (8) of $\mathfrak{H}$ on $S(X,E)$ is clearly semi-linear. It is furthermore smooth, since each element of $S(X,E)$ takes only finitely many values in $E$.

On the other hand, since $\delta$ is a 1-cocyle, then
\[ (\sigma, \phi) \mapsto \delta(\sigma)\sigma(\phi), \quad \sigma \in \mathfrak{H}, \phi \in S(X,E), \]
is also an action of $\mathfrak{H}$ on $S(X,E)$, referred to as the twisted action by $\delta$. It is semi-linear, since $\delta$ takes values in $\text{GL}(S(X,E))$. Since the original action is smooth, if $\phi \in S(X,E)$ then there exists an open subgroup $\mathfrak{H}_1$ such that
\[ \sigma(\phi) = \phi, \quad \sigma \in \mathfrak{H}_1. \]
Furthermore, Lemma 16 asserts that there is an open subgroup $\mathfrak{K}$ of $\mathfrak{H}$ such that
\[ \delta(\sigma)\phi = \phi, \quad \sigma \in \mathfrak{K}. \]

Therefore, if $\sigma \in \mathfrak{H}_1 \cap \mathfrak{K}$ then
\[ \delta(\sigma)\sigma(\phi) = \delta(\sigma)\phi = \phi. \]
This shows that the stabilizer of $\phi$ under the twisted action contains the open subgroup $\mathfrak{H}_1 \cap \mathfrak{K}$. Since it is the union of its $\mathfrak{H}_1 \cap \mathfrak{K}$-cosets, it follows that the stabilizer of $\phi$ under the twisted action is open. We conclude that the twisted action is smooth.

Using $S(X,E)$ and $\delta S(X,E)$ to denote the $\mathfrak{H}$-modules defined by the natural and twisted actions, respectively, Galois Descent asserts the existence of $E$-linear, $\mathfrak{H}$-equivariant isomorphisms
\[ \delta S(X,E) \simeq E \otimes_{\mathbb{Q}(\sqrt{p}, \sqrt{-p})} S(X,E)_{\mathfrak{H}} \quad \text{and} \quad E \otimes_{\mathbb{Q}(\sqrt{p}, \sqrt{-p})} S(X,E)^{\delta} \simeq S(X,E). \]
In particular,
\[ \dim_{\mathbb{Q}(\sqrt{p}, \sqrt{-p})} \delta S(X, E)^{\delta} = \dim_{E} S(X, E) = \dim_{\mathbb{Q}(\sqrt{p}, \sqrt{-p})} S(X, E)^{\delta}, \]
so \( \delta S(X, E)^{\delta} \) and \( S(X, E)^{\delta} \) are \( \mathbb{Q}(\sqrt{p}, \sqrt{-p}) \)-isomorphic. As any such isomorphism extends by scalars to a \( E \)-linear, \( \mathfrak{H} \)-equivariant isomorphism
\[ E \otimes_{\mathbb{Q}(\sqrt{p}, \sqrt{-p})} \delta S(X, E)^{\delta} \cong E \otimes_{\mathbb{Q}(\sqrt{p}, \sqrt{-p})} S(X, E)^{\delta}, \]
we conclude
\[ \delta S(X, E)^{\delta} \cong S(X, E). \]

Let \( \alpha \in \text{GL}(S(X, E)) \) be a \( \mathfrak{H} \)-equivariant isomorphism \( \delta S(X, E) \to S(X, E) \). If \( \sigma \in \mathfrak{H} \) and \( \phi \in \mathfrak{H} \) then the definition of the twisted action ensures that
\[ \alpha \delta(\sigma) \sigma(\phi) = \sigma(\alpha \phi), \]
hence
\[ \delta(\sigma) \phi = \alpha^{-1} \alpha \delta(\sigma) \sigma(\sigma^{-1}(\phi)) = \alpha^{-1} \sigma(\sigma^{-1}(\phi(\alpha))) = \alpha^{-1} \sigma(\phi). \]
This completes the proof of the proposition. \( \square \)

9. Proof of the Main Theorem

Fix \( \alpha \in \text{GL}(S(X, E)) \) satisfying the conclusion of Proposition 18. In light of (11) and (16), if \( \sigma \in \mathfrak{H} \) and \( g \in \text{Sp}(V) \) then
\[ \sigma(\alpha W_{\lambda}(g)\alpha^{-1}) = \sigma \alpha \sigma W_{\lambda}(g)\sigma(\alpha)^{-1} = \sigma \delta(\sigma)^{-1} W_{\lambda}(g)\delta(\sigma)(\sigma)^{-1} = \alpha W_{\lambda}(g)\alpha^{-1}. \]
The compatatability of the Galois actions (8) and (9) allows us to deduce that the operators
\[ \alpha W_{\lambda}(g)\alpha^{-1}, \quad g \in \text{Sp}(V), \]
leave
\[ S(X, E)^{\delta} = S(X, E^{\delta}) = S(X, \mathbb{Q}(\sqrt{p}, \sqrt{-p})) \]
invariant, hence provide a projective Weil representation realized over \( \mathbb{Q}(\sqrt{p}, \sqrt{-p}) \).

References

[B] A. Borel, Linear Algebraic Groups, 2nd ed, Springer-Verlag, New York, 1991.
[C] P. Cartier, Representations of \( \mathfrak{p} \)-adic groups: A survey, in “Automorphic Forms, Representations, and \( \mathcal{L} \)-functions”, Proc. Sympos. Pure. Math. 33 part 1, Amer. Math. Soc. (1979), 111–155.
[CMS] G. Cliff, D. McNeilly, and F. Szechtman, Character fields and Schur indices of irreducible Weil characters, J. Group Theory 7 (2004), 39–64.
[J] N Jacobson, Basic Algebra I, 2nd ed., W. H. Freeman and Company, San Francisco, 1985.
[K] S. S. Kudla, Notes on the local theta correspondence, \texttt{http://www.math.toronto.edu/~skudla/castle.pdf}
[MVW] C. Moeglin, M.-F. Vignéras, and J.-L. Waldspurger, Correspondance de Howe sur un corps \( \mathfrak{p} \)-adique, Lecture Notes in Math., vol. 1291, Springer-Verlag, Berlin, 1970.
[P] D. Prasad, A brief survey on the theta correspondence, Number theory (Tiruchirapalli, 1996), 171–193, Contemp. Math., 210, Amer. Math. Soc., Providence, RI, 1998.
[RR] R. Ranga Rao, On some explicit formulas in the theory of the Weil representation. Pacific J. Math. 157 (1993), 335–371.
[S] J. P. Serre, Linear Representations of Finite Groups, Springer-Verlag, New York, 1977.
[W] A. Weil, Basic Number Theory, Springer-Verlag, Berlin, 1973.
University of Alberta, Department of Mathematical and Statistical Sciences, Edmonton, Alberta, Canada T6G 2G1

E-mail address: gcliff@math.ualberta.ca, dam@math.ualberta.ca