Minimality of invariant laminations for partially hyperbolic attractors

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Abstract

Let \( f : M \to M \) be a \( C^1 \)-diffeomorphism over a compact boundaryless Riemannian manifold \( M \), and \( \Lambda \) a compact \( f \)-invariant subset of \( M \) admitting a partially hyperbolic splitting \( T\Lambda = E^s \oplus E^{cu} \oplus E^u \) over the tangent bundle \( T\Lambda \).

It's known from the Hirsch–Pugh–Shub theory that \( \Lambda \) admits two invariant laminations associated to the extremal bundles \( E^s \) and \( E^u \). These laminations are families of dynamically defined immersed submanifolds of the \( M \) tangent, respectively, to the bundles \( E^s \) and \( E^u \) at every point in \( \Lambda \). In this work, we prove that at least one of the invariant laminations of a transitive partially hyperbolic attractor with a one-dimensional center bundle is minimal: the orbit of every leaf intersects \( \Lambda \) densely. This result extends those in Bonatti et al (2002 J. Inst. Math. Jussieu 1 513–41) and Hertz et al (2007 Fields Institute Communications vol 51 (Providence, RI: American Mathematical Society) pp 103–9) about minimal foliations for robustly transitive diffeomorphisms.

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1. Introduction

We say that a property \( P \) of a system is robust (relative to the \( C^1 \)-topology) if \( P \) holds for any small \( C^1 \)-perturbation of the initial system. It is well known that hyperbolic sets exhibit many robust topological properties, as they are structurally stable. Given that non-hyperbolic sets appear in relevant subsets of dynamical systems (open sets of \( \text{Diff}^1(M) \)), it is natural to investigate whether these properties still appear, robustly, for non-hyperbolic systems.

An example of robust property that hold for transitive hyperbolic sets is the fact that the orbit of its invariant manifolds has a dense intersection with the set.
In this work, we obtain a non-hyperbolic counterpart of the above property for transitive attractors presenting a weaker form of hyperbolicity known as partial hyperbolicity. In this setting, we also have invariant submanifolds associated to the expanding and contracting bundles. As in the hyperbolic case, we can wonder if the orbit of these submanifolds intersect the attractor in a dense subset of it.

A simpler case to start with is when the center bundle is one-dimensional, which is the case that most resembles the hyperbolic one. This gives two properties of the attractor that play an important role in our study. First, this guarantees that the attractor is far from homoclinic tangencies (a technical requirement that enables us to use some results about heterodimensional cycles with blender-like properties as in [7]). Secondly, it gives rise to invariant central curves for the hyperbolic periodic points of the attractor, allowing us to split our study into different cases, according to the Morse–Smale dynamics on these curves (see section 8).

In [7], the case of 3-dimensional robustly transitive diffeomorphisms was investigated (when the attractor is the whole manifold). It was proved that the leaves of at least one of the invariant foliations is dense in the ambient manifold. Later in [13], this result was extended to diffeomorphisms on manifolds of higher dimensions, still with one-dimensional center bundle.

Our main result translates [7, 13] to the context of robustly transitive attractors. Let us state our results in a more precise way.

Let $\text{Diff}^1(M)$ denote the space of $C^1$ diffeomorphisms from a compact boundaryless Riemannian manifold $M$ to itself, endowed with the usual uniform $C^1$-topology. Consider a diffeomorphism $f \in \text{Diff}^1(M)$ and an $f$-invariant set $\Lambda$ over whom the tangent bundle has a partially hyperbolic splitting $T\Lambda = E^s \oplus E^c \oplus E^u$. This means that the extremal subbundles $E^s$ and $E^u$ are, respectively, uniformly contracting and uniformly expanding by the action of the derivative $Df$ of $f$, and that $E^c$ has an intermediate behaviour\footnote{We refer to appendix B of [8] for a precise definition and a list of elementary properties of partially hyperbolic systems.}. If both $E^s$ and $E^u$ are nontrivial, we say that $\Lambda$ is a strongly partially hyperbolic set. When the whole manifold $M$ is strongly partially hyperbolic, we call $f$ a strongly partially hyperbolic diffeomorphism.

In what follows we assume that the dimensions of $E^s(x)$, $E^c(x)$, and $E^u(x)$ do not depend on the point $x \in \Lambda$. Due to the $Df$-invariance and the continuity of the splitting, this assumption is automatically satisfied when $\Lambda$ is transitive.

According to [14], strong partial hyperbolicity leads to the existence of two laminations $F^s$ and $F^u$ on the set $\Lambda$ (see section 3), named strong stable and strong unstable laminations, respectively.

When $\Lambda = M$, the laminations are commonly referred to as foliations. We say that the foliation is minimal if each leaf is a dense subset of $M$. Assuming that $M$ is connected, it is enough to require that the orbit of each leaf is dense in $M$ (see lemma 4.5 of [7]). When the strong stable (resp. unstable) foliation of a partially hyperbolic diffeomorphism is minimal we speak of $s$-minimality (resp. $u$-minimality).

Let us denote by $\text{RTPH}_1^s(M)$ the open subset of $\text{Diff}^1(M)$ consisting of robustly transitive, robustly non-hyperbolic, partially hyperbolic diffeomorphisms with one-dimensional center bundle. Our results is motivated by the following theorem.

**Theorem 1.1** ([7], [13]). There is an open and dense subset of $\text{RTPH}_1^s(M)$ consisting of diffeomorphisms which are either robustly $s$-minimal or robustly $u$-minimal.
principle, this accumulation could be done ‘outside’ $\Lambda$ (that is, with a small or none intersection with $\Lambda$). A more strong property would be that the orbit of each leaf intersect $\Lambda$ in a dense subset. Unlike the case of transitive diffeomorphisms, this requirement do not implies that each leaf itself has a dense intersection with the set. Nevertheless, there is a fixed number $d \in \mathbb{N}$ such that the union of the $d$ firsts iterates of each leaf intersect $\Lambda$ densely. Altogether, these considerations lead to the definition of $u$- and $s$-minimality we introduce in this paper (see section 7).

Fixed an open set $U \subset M$, denote by $\text{RTPHA}_1(U)$ (resp. $\text{GTPHA}_1(U)$) the subset of $\text{Diff}^1(M)$ of diffeomorphisms $f$ for which the maximal $f$-invariant subset $\Lambda_f(U) = \bigcap_{n \in \mathbb{Z}} f^n(U)$ of $U$ is a robustly (resp. generically) transitive attractor that is robustly non-hyperbolic and partially hyperbolic with one-dimensional center bundle. Observe that $\text{RTPHA}_1(U)$ is an open subset of $\text{Diff}^1(M)$, and that $\text{GTPHA}_1(U)$ is locally residual in $\text{Diff}^1(M)$. The next two theorems summarize the main results in this paper.

**Theorem A (generically transitive case).** For every open subset $U \subset M$, there is a residual subset of $\text{GTPHA}_1(U)$ consisting of diffeomorphisms $g$ for which $\Lambda_g(U)$ is either generically $s$-minimal or generically $u$-minimal.

**Theorem B (robustly transitive case).** For every open subset $U \subset M$, there is an open and dense subset of $\text{RTPHA}_1(U)$ consisting of diffeomorphisms $g$ for which $\Lambda_g(U)$ is either robustly $s$-minimal or robustly $u$-minimal.

Note that theorem 1.1 is a particular case of theorem B when $U = M$.

The main difficulty in adapting the global theorem 1.1 to our local case is that the partial hyperbolicity is defined only in the attractor. This requires an initial preparation on verifying that the leaves behave ‘nicely’ inside an extended splitting defined in a neighborhood of the attractor. Another difference is that we can not saturate a strong stable leaf with unstable ones (which is a key step in [7]), since not all the points in the stable leaf belongs to the attractor. Also, our proof do not rely on orientability assumptions.

Concerning possible generalizations of theorems A and B, one may ask if these results could be obtained to the broader setting of robustly or generically transitive sets (instead of attractors). In the next lines we see that, by an example in [3], this generalization is not possible (at least for the generic case).

In [3] it is proved that every manifold $M$ of dimension $\geq 3$ supports a generically transitive set $\Lambda_f(U)$ that is not robustly transitive. In addition, $\Lambda_f(U)$ is strongly partially hyperbolic with one-dimensional center bundle, which is the case we treat in this paper. In their construction, there are two hyperbolic periodic points $p$ and $q$ in $\Lambda_f(U)$, with $\text{index}(p) = \text{index}(q) + 1$ such that $\mathcal{F}^+(p) \cap \Lambda_f(U) = \{ p \}$ and $\mathcal{F}^-(q) \cap \Lambda_f(U) = \{ q \}$, preventing both laminations to be minimal (see definition of cuspidal points in [3]). Moreover, the existence of these cuspidal points is a robust property, so the continuations $\Lambda_g(U)$ do not have minimal laminations either.

Although theorems A and B guarantee the existence of a minimal lamination, the verification of which of them (stable or unstable) are minimal is a difficult problem. Let us discuss the minimality in some examples.

An easy way to produce examples of proper robustly transitive attractors is to consider the product map $F = f \times g$ of a robustly transitive diffeomorphism $f : M \to M$ with a north–south dynamics $g : S^1 \to S^1$ in the circle. By denoting $\Lambda_F$ the attractor $M \times \{ p \}$ of $F$, where $p$ is the hyperbolic attracting fixed point of $g$, one easily deduces that if $\mathcal{F}^+(\text{resp. } \mathcal{F}^-)$ is minimal for $f$, then $\Lambda_F$ is $s$-minimal (resp. $u$-minimal).

A different class of attractors that are not the product map of transitive diffeomorphisms is obtained by skew-products in [5]. It is derived from the product map $f \times Id$ where
$f : M \to M$ has a hyperbolic transitive attractor $\Lambda \subset M$ and $Id$ is the identity map on $S^1$. In [5] they prove that we can perturb this map to obtain robustly non-hyperbolic transitive attractors homeomorphic to $\Lambda \times S^1$. These attractors have one-dimensional center bundle and all its central curves are compact. These attractors provide examples which are both $u$- and $s$-minimal (see proposition 9.1).

As far as we know, the following question is still open when the center bundle is one-dimensional.

**Question 1.** Is there a robustly transitive diffeomorphism for which one of the minimality fails?

If such $f$ does exist, then the construction above for $F = f \times g$ and $\tilde{F} = f^{-1} \times g$ gives examples of strictly $s$-minimal and strictly $u$-minimal attractors. By ‘strictly’ we mean that the attractor has only that kind of minimality. On the other hand, the existence of strictly $u$-minimal or $s$-minimal attractors do not guarantees the same for diffeomorphisms. This lead us to pose the following question.

**Question 2.** Is there a strictly $u$-minimal (resp. $s$-minimal) partially hyperbolic attractor with one-dimensional center bundle?

Observe that if question 2 has a negative answer, so has question 1.

2. Preliminaries

We introduce in this section the basic definitions and terminology that we use throughout this paper.

Fix $f \in \text{Diff}^1(M)$. Given a subset $X \subset M$, we denote the orbit, the forward orbit, and the backward orbit of $X$ by $O_f(X)$, $O_f^+(X)$, and $O_f^-(X)$, respectively. For every open subset $U$ of $M$, we define the maximal $f$- invariant set of $f$ in $U$ by

$$\Lambda_f(U) := \bigcap_{n \in \mathbb{Z}} f^n(U).$$

With respect to $f$, a compact invariant set $\Lambda \subset M$ is said:

- **Isolated or locally maximal:** If there is an open neighborhood $U$ of $\Lambda$ such that $\Lambda = \Lambda_f(U)$. Equivalently, $\Lambda$ is the maximal invariant subset of $f$ in $U$. Any open neighborhood $U$ of $\Lambda$ satisfying $\Lambda = \Lambda_f(U)$ is called an isolating block of $\Lambda$.
- **An attractor:** If there is an open neighborhood $U$ of $\Lambda$ such that $f(U) \subset U$ and $\Lambda = \bigcap_{n \in \mathbb{N}} f^n(U)$. We call $\Lambda$ a proper attractor if $U \neq M$, and thus $\Lambda \neq M$.
- **Transitive:** If there is $x \in \Lambda$ such that its forward orbit $O^+_f(x)$ is dense in $\Lambda$. In our setting, this is equivalent to the following property: Given any pair $V_1$, $V_2$ of (relative) nonempty open sets of $\Lambda$, there is $n \in \mathbb{Z}$ such that $f^n(V_1) \cap V_2 \neq \emptyset$.
- **Robustly transitive set (resp. attractor):** If there are an isolating block $U$ of $\Lambda$ and a neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}^1(M)$ such that, for every $g \in \mathcal{U}$, the set $\Lambda_g(U)$ is a compact transitive set (resp. attractor) with respect to $g$.
- **Generically transitive set (resp. attractor):** If there are an isolating block $U$ of $\Lambda$, a neighborhood $\mathcal{U}$ of $f$ in $\text{Diff}^1(M)$, and a residual subset $\mathcal{R} \subset \mathcal{U}$ such that, for every $g \in \mathcal{R}$, the set $\Lambda_g(U)$ is a compact transitive set (resp. attractor) with respect to $g$.

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2 For partially hyperbolic diffeomorphisms with center bundle of dimension 2, section 6.2 of [10] give examples of robustly transitive diffeomorphism without stable (resp. unstable) bundle, which are strictly $u$-minimal (resp. $s$-minimal).
Remark 2.1. Isolated sets vary, \emph{a priori}, just upper semicontinuously with respect to the Hausdorff distance. Following the standard terminology, we call the set $\Lambda_g(U)$ the \textit{continuation} of the set $\Lambda_f(U)$ when $g$ varies in a small $C^1$ neighborhood of $f$.

Remark 2.2. An attractor $\Lambda$ of a diffeomorphism $f$ is an isolated set, so we also denote it by $\Lambda_f(U)$ for some isolating block $U$ of $\Lambda$. Observe that if $g$ is close enough to $f$, then the continuation $\Lambda_g(U)$ of $\Lambda_f(U)$ is also an attractor for $g$. Clearly, if $\Lambda_f(U)$ is a proper set, so is its continuation $\Lambda_g(U)$.

Remark 2.3. Some authors require the addition property of transitivity in the definition of attractors. We do not follow this requirement. The reason is that we want to talk about the continuation of attractors as in remark 2.2, and transitivity is not a robust property in general.

Remark 2.4. By the forward invariance $f(U) \subset U$ in the definition of attractors, one gets that $\Lambda_f(U)$ contain every unstable manifold and unstable leaf of its points.

Let $\Lambda$ be a partially hyperbolic set of a $C^1$ diffeomorphism $f$. Given a hyperbolic periodic point $p \in \Lambda$, we denote its period by $\pi(p)$. The local and global stable manifolds of $p$ are denoted by $W^s_r(p, f)$ and $W^s(p, f)$, respectively. Also, $W^s_r(\mathcal{O}_f(p), f)$ and $W^s(\mathcal{O}_f(p), f)$ stand for the global and local stable manifolds of the orbit of $p$. The dimension of $W^s(p, f)$ as a submanifold of $M$ is called the \textit{index} of $p$ and is denoted by $\text{index}(p)$.

If $g \in \text{Diff}^1(M)$ is sufficiently close to $f$, then there is a hyperbolic continuation of $p$ for $g$ that we denote by $p_g$, and satisfies $\pi(p_g) = \pi(p)$ and $\text{index}(p_g) = \text{index}(p)$.

Remark 2.5. Given a partially hyperbolic splitting $E^s \oplus E^c \oplus E^u$ of a compact invariant set $\Lambda$, we denote the dimensions of the bundles $E^s$, $E^c$ and $E^u$ by $d^s$, $d^c$, and $d^u$, respectively. Clearly, $d^c \leq \text{index}(p) \leq d^s + d^u$, for every hyperbolic periodic point $p$. In particular, when $d^c = 1$, there are only two possibilities for $\text{index}(p)$, which are $d^s$ or $d^s + 1$. The set of all periodic points in $\Lambda$ with index $\sigma$ are denoted by $\text{Per}_\sigma(f|_{\Lambda})$.

When dealing with perturbations of a diffeomorphism $f$, we usually want that the new diffeomorphism is sufficiently close to $f$ to inherit the robust properties of $f$. Hence, we introduce the following definition.

Definition 2.6. Let $\Lambda$ be an isolated set of a diffeomorphism $f$ and $U \subset M$ be an isolated block of $\Lambda$. We say that a neighborhood $\mathcal{U}$ of $\Lambda$ is \textit{compatible} (with respect to $U$) if it is sufficiently small so that, for all $g \in \mathcal{U}$, we have:

\begin{itemize}
  \item the set $\Lambda_g(U)$ is an isolated set;
  \item if $\Lambda_f(U)$ is an attractor of $f$, then $\Lambda_g(U)$ is an attractor of $g$;
  \item if $\Lambda_f(U)$ is a partially hyperbolic set, then $\Lambda_g(U)$ is a partially hyperbolic set of $g$ with the same bundle dimensions;
  \item if $\Lambda_f(U)$ is a generically (resp. robustly) transitive set of $f$, then $\Lambda_g(U)$ is a generically (resp. robustly) transitive set of $g$.
\end{itemize}

3. Invariant laminations

In this section we introduce the main object of our study, the \textit{invariant laminations} of a partially hyperbolic set.
Definition 3.1. Let $\Lambda \subset M$ be a compact subset of the manifold $M$. A Lamination $\mathcal{F}$ of $\Lambda$ is a family of immersed submanifolds of $M$ of the same dimension (called leaves), satisfying the following:

(a) each leaf of the lamination intersects $\Lambda$, and for every point $x \in \Lambda$ there is a leaf of $\mathcal{F}$ containing $x$ that we denote by $\mathcal{F}(x)$.
(b) if $\mathcal{F}(x) \cap \mathcal{F}(y) \neq \emptyset$, then $\mathcal{F}(x) = \mathcal{F}(y)$.
(c) $\mathcal{F}(x)$ depends continuously on $x \in \Lambda$.

The continuous dependence in item (c) means the following. For every $x \in \Lambda$, there exist a neighborhood $V_x \subset \Lambda$ of $x$, a disk $W_x \subset \mathcal{F}(x)$ centered at $x$ and with same dimension as $\mathcal{F}(x)$, and a continuous map $\phi_x : V_x \to \text{Emb}(W_x, M)$ such that $\phi_x(x)$ is the inclusion of $W_x$ in $M$ and $\phi_x(z)(W_z)$ is a neighborhood of $z$ inside $\mathcal{F}(z)$. Here $\text{Emb}(W_x, M)$ denotes the space of all embeddings from $W_x$ to $M$.

Given $f \in \text{Diff}^1(M)$ and an $f$-invariant set $\Lambda$, the foliation $\mathcal{F}$ is said to be $f$-invariant if $f(\mathcal{F}(x)) = \mathcal{F}(f(x))$ for every $x \in \Lambda$.

The following result summarizes classical properties about strongly partially hyperbolic sets (see [14], appendix IV of [16], and appendix B of [8]).

Proposition 3.2. Let $\Lambda$ be a partially hyperbolic set of a diffeomorphism $f \in \text{Diff}^1(M)$. Then there are $f$-invariant foliations $\mathcal{F}^s$ and $\mathcal{F}^u$ of $\Lambda$ satisfying the following:

(a) The leaves of $\mathcal{F}^s$ and $\mathcal{F}^u$ are $C^1$ immersed submanifolds of $M$ of dimensions $d^s$ and $d^u$, respectively, called strong stable and strong unstable leaves.
(b) The strong stable and strong unstable leaves are tangent, respectively, to the strong stable and strong unstable bundles. That is, for every $x \in \Lambda$, it holds that $T_x \mathcal{F}^s(x) = E^s(x)$ and $T_x \mathcal{F}^u(x) = E^u(x)$.
(c) The diffeomorphism $f$ exponentially contracts the leaves of $\mathcal{F}^s$, and its inverse $f^{-1}$ exponentially contracts the leaves of $\mathcal{F}^u$.
(d) These two laminations are unique: there is no other $f$-invariant lamination satisfying the properties in items (a)–(c).
(e) Let $\mathcal{F}^s_r(w)$ denote the local stable disk inside $\mathcal{F}^s(w)$ of radius $r > 0$ centered at $w$. Then, for every $r > 0$, $\mathcal{F}^s_r(x) = \bigcup_{n \in \mathbb{N}} f^{-n}(\mathcal{F}^s_r(f^n(x)))$ and $\mathcal{F}^u_r(x) = \bigcup_{n \in \mathbb{N}} f^n(\mathcal{F}^u_r(f^{-n}(x)))$.

When we deal with the continuations $\Lambda_g(U)$ of an isolating set $\Lambda = \Lambda_f(U)$, we denote by $\mathcal{F}^s(g)$ and $\mathcal{F}^u(g)$ the strong stable and unstable laminations of $\Lambda_g(U)$ for $g$. In this case, $\mathcal{F}^s(x, g)$ stands for the leaf of $\mathcal{F}^s(g)$ that contains $x \in \Lambda_g(U)$, and $\mathcal{F}^u_r(x, g)$ stands for the local stable disk of radius $r > 0$ centered at $x$. Similar notations are considered to the unstable lamination.

Remark 3.3. The leaves of $\mathcal{F}^s(g)$ and $\mathcal{F}^u(g)$ depend continuously on the diffeomorphism $g$. This means that, fixed $r > 0$ and $\varepsilon > 0$, if $g$ is sufficiently close to $f$ and $x \in \Lambda_f(U)$ is sufficiently close to $y \in \Lambda_g(U)$, then the disk $\mathcal{F}^s_r(y, g)$ is $\varepsilon$-close to the disk $\mathcal{F}^s_r(x, f)$ with respect to the $C^1$-topology.

4. $C^1$-Generic dynamics

In this section we gather some useful $C^1$-generic properties of homoclinic classes, transitive sets and attractors. We say that a property $P$ is $C^1$-generic if $P$ holds for every diffeomorphism in a residual ($G_\delta$ and dense) subset $R$ of $\text{Diff}^1(M)$. 
Definition 4.1 (Homoclinic class). Let \( p \) be a hyperbolic periodic point of a diffeomorphism \( f \). A homoclinic point \( x \) of \( p \) is a point whose forward and backward iterates converge to the orbit \( O_f(p) \) of \( p \) (i.e. \( x \in W^s(O_f(p)) \cap W^u(O_f(p)) \)). If the stable and unstable manifolds of the orbit of \( p \) meet transversely at \( x \), we say that \( x \) is a transverse homoclinic point. Otherwise, we say that \( x \) is a homoclinic tangency.

The homoclinic class of \( p \), denoted by \( H(p, f) \), is the closure of the set of all transverse homoclinic points of \( p \). That is,
\[
H(p, f) = \overline{W^s(O_f(p)) \cap W^u(O_f(p))}.
\]

Remark 4.2. Homoclinic classes are transitive sets and contain a dense subset of periodic points (of the same index of \( p \)). By the persistence of the transverse intersections between the invariant manifolds of \( p \), we find that homoclinic classes vary lower semicontinuously (that is, the map \( g \mapsto H(p_g, g) \) is a lower semicontinuous map). See chapter 10.4 of [8] for a detailed discussion about homoclinic classes.

Remark 4.3. If \( \Lambda_f(U) \) is an attractor, then remark 2.4 gives that \( H(p, f) \subset \Lambda_f(U) \) for every hyperbolic periodic point \( p \in \Lambda_f(U) \).

Theorem 4.4 ([11, 12]). There is a residual subset \( \mathcal{R}_0 \) of \( \text{Diff}^1(M) \) such that, for every \( f \in \mathcal{R}_0 \), the following holds:

(a) The diffeomorphism \( f \) is Kupka–Smale: the set \( \text{Per}(f) \) of periodic points of \( f \) consist of hyperbolic periodic points only, and their invariant manifolds met transversely.

(b) \( \text{Per}(f) \) is dense in the nonwandering set \( \Omega(f) \) of \( f \). In particular, any isolated transitive compact set has a dense subset of periodic points.

(c) Any transitive set intersecting a homoclinic class is contained in it. Consequently, any pair of homoclinic classes are either disjoint or coincide.

Item (a) is the well known Kupka–Smale Theorem, item (b) is the main theorem in [12], and item (c) is item (c) of theorem A in [11].

Remark 4.5. By item (c), \( C^1 \)-generic transitive attractors are homoclinic classes.

Proposition 4.6. There is a residual subset \( \mathcal{R}_1 \) of \( \text{Diff}^1(M) \) such that, if \( f \in \mathcal{R}_1 \) and \( \Lambda_f(U) \) is an isolated subset of \( M \), then the following hold:

(a) If \( \Lambda_f(U) \) is a transitive attractor, then there is a neighborhood \( U \) of \( f \) such that, for every \( g \in \mathcal{R}_1 \cap U \), the set \( \Lambda_g(U) \) is a transitive attractor. In other words, the set \( \Lambda_f(U) \) is a generically transitive attractor.

(b) If \( \Lambda_f(U) \) is non-hyperbolic, then it contains a pair of (hyperbolic) saddles of different indices.

Item (a) is theorem B of [1]. Item (b) is due to Mané in the proof of the Ergodic Closing Lemma [15].

Remark 4.7. Every generically (or robustly) transitive set that is robustly non-hyperbolic and partially hyperbolic with one-dimensional center bundle must be strongly partially hyperbolic. Indeed, if one of the bundles \( E^s \) or \( E^u \) is trivial, then item (b) of proposition 4.6 shows that the set must contain a sink or a source, preventing it to be transitive.

Next proposition is a translation of some results in section 2 of [7] about generic transitive diffeomorphisms to the context of generic transitive attractors. The arguments in [7] can be applied to our context without substantial amendments, so here we only sketch the proof.
Proposition 4.8. There is a residual subset \( \mathcal{R}_2 \) of \( \text{Diff}^1(M) \), with \( \mathcal{R}_2 \subset \mathcal{R}_1 \), satisfying the following. Let \( f \in \mathcal{R}_2 \) and \( \Lambda_f(U) \) be a transitive isolated set of \( f \) that is partially hyperbolic with one-dimensional center bundle. For every pair of hyperbolic periodic points \( p, q \in \Lambda_f(U) \) with indices \( d' \) and \( d' + 1 \), respectively, there is an open set \( \mathcal{V}_{p,q} \subset \text{Diff}^1(M) \), with \( f \in \mathcal{V}_{p,q} \), such that, for every \( g \in \mathcal{V}_{p,q} \), it holds:

(a) \( W^s(O_g(q_g)) \subset W^s(O_g(p_g)) \) and \( W^u(O_g(p_g)) \subset W^u(O_g(q_g)) \).

(b) if \( \Lambda_f(U) \) is robustly transitive, then \( \Lambda_g(U) \subset H(p_g,g) \).

Sketch of the proof. The corresponding statements in [7] are proposition 2.6 and corollary 2.5, and the proofs go similarly. Let us verify that we are in the same hypotheses as in [7].

By proposition 4.6, the set \( \Lambda_f(U) \) is generically transitive and contains saddles \( p \) and \( q \) of indices, respectively, \( d' \) and \( d' + 1 \) (see remark 2.5). Observe that for every \( g \) close to \( f \) one has \( p_g, q_g \in \Lambda_g(U) \). We can assume that there is an heterodimensional cycle\(^3\) associated to \( p_g \) and \( q_g \) for every \( g \) in a dense subset of a neighbourhood of \( f \) (see proposition 1.1 of [4]). Since \( E^c \) is one-dimensional, there is no homoclinic tangencies associated to the periodic points of \( \Lambda_g(U) \).

Altogether, these observations put us in the same context as in section 2 of [7]. Note that in [7] the attractor is the whole manifold, so the conclusion that \( H(p_g,g) = M \) in [7] corresponds to the inclusion \( \Lambda_g(U) \subset H(p_g,g) \) in the context of proper attractors. \( \blacksquare \)

Corollary 4.9. There is an open and dense subset \( \mathcal{B} \) of \( \text{RTPHA}_1(U) \) such that, for every \( f \in \mathcal{B} \), the attractor \( \Lambda_f(U) \) is a homoclinic class and depends continuously on \( f \in \mathcal{B} \).

Proof. Applying proposition 4.8 to a dense subset of diffeomorphisms in \( \text{RTPHA}_1(U) \), we get an open and dense subset \( \mathcal{B} \) of \( \text{RTPHA}_1(U) \) such that, for every \( f \in \mathcal{B} \), there are a periodic point \( p \in \Lambda_f(U) \) and a neighborhood \( U \) of \( f \) such that \( \Lambda_g(U) \subset H(p_g,g) \) for every \( g \in U \). On the other hand, remark 4.3 gives that \( H(p_g,g) \subset \Lambda_g(U) \) for every \( g \in U \). Therefore, we conclude that \( \Lambda_g(U) = H(p_g,g) \) for every \( g \in U \).

To get the continuous dependence of \( \Lambda_f(U) \) on \( f \in \mathcal{B} \), recall that homoclinic classes vary lower semicontinuously and attractors vary upper semicontinuously (see remarks 2.1 and 4.2). \( \blacksquare \)

5. Invariant extensions of the partially hyperbolic splitting

In this section we study the behavior of the leaves of a partially hyperbolic set \( \Lambda \). Our goal is to show that these leaves are tangent to an extended splitting in a neighborhood of \( \Lambda \) and satisfies the accumulation property in proposition 5.4.

A partially hyperbolic splitting \( E^s \oplus E^c \oplus E^u \) defined over a compact invariant set \( \Lambda \) can always be extended to a continuous splitting in a neighborhood of \( \Lambda \). Although its not always possible to make this extension \( Df \)-invariant, in some particular cases it is indeed feasible, as we see in the next theorem.

Theorem 5.1 ([2, 9]). There is a residual subset \( \mathcal{R}_3 \subset \text{Diff}^1(M) \) with the following property. Let \( f \in \mathcal{R}_3 \) and \( \Lambda = H(p, f) \) be a partially hyperbolic homoclinic class. Then there is an extension of the partially hyperbolic splitting on \( \Lambda \) to a continuous splitting on a compact neighborhood \( U \) of \( \Lambda \) that is invariant in the following sense: for every \( x \in U \) such that \( f(x) \in U \), we have that \( Df_i(E^i(x)) = E^i(f(x)) \), for \( i \in \{s, c, u\} \).

\(^3\) See [6] for the definition and a general overview on this topic.
Theorem 5.1 is a combination of two important results. First, this extension holds for a class of sets known as chain recurrent classes (see theorem 7 of [9]). Second, $C^1$-generic homoclinic classes are chain recurrent classes (see remark 1.10 of [2]).

Remark 5.2. For robustly transitive attractors, the existence of such extended splitting holds not only generically, but openly in $\text{Diff}^1(M)$. The reason is that transitive attractors are chain recurrent classes.

Next lemma is a consequence of lemma 3.6 in [9].

Lemma 5.3. Let $f \in \text{Diff}^1(M)$ and $\Lambda = H(p, f)$ be a partially hyperbolic homoclinic class with index $(p) = d^s$. Assume that there is an extension of the splitting $E^s \oplus E^c \oplus E^u$ to a compact neighborhood $U$ of $\Lambda$. Then, for every $x \in \Lambda$, the leaf $\mathcal{F}^s(x)$ is tangent to $E^s$ at every point in $U$.

Proof. By lemma 3.6 of [9], $\mathcal{F}^s(p)$ is tangent to $E^s$ at every point in $U$. Take any $x \in H(p, f)$. There exists a sequence $\{x_n\}_{n \in \mathbb{N}}$ of transverse homoclinic points of $p$ that accumulates at $x$. Given $r > 0$, the disks $\mathcal{F}^s_r(x_n)$ are also tangent to $E^s$ in $U$. By the continuity of the strong stable lamination in $\Lambda$, the sequence $\{\mathcal{F}^s_r(x_n)\}_{n \in \mathbb{N}}$ accumulates (with respect to the $C^1$-topology) to a disk $\mathcal{F}^s_r(x)$, so $\mathcal{F}^s_r(x)$ is tangent to $E^s$ in $U$. By the arbitrary choice of $r$, we conclude the proof of this proposition.

Proposition 5.4. Under the hypotheses of lemma 5.3, if $y \in \mathcal{F}^s(x) \cap \Lambda$, then $\mathcal{F}^s(y) \subset \mathcal{F}^s(x)$.

Proof. Consider $y \in \mathcal{F}^s(x) \cap \Lambda$, $n \in \mathbb{N}$ and $z = f^n(y)$. Clearly, $\mathcal{F}^s(f^n(x))$ accumulates to $z$. Consider a sequence $\{x_k\}_{k \in \mathbb{N}}$ of points in $\mathcal{F}^s(f^n(x))$ converging to the point $z$, and fix $r > 0$ sufficiently small so the sequence of disks $\{\mathcal{F}^s_r(x_k)\}_{k \in \mathbb{N}}$ is contained in $U$. By lemma 5.3, each disk $\mathcal{F}^s_r(x_k)$ is tangent to the extended bundle $E^s$ in $U$. Hence, there is a subsequence of $\{\mathcal{F}^s_r(x_k)\}_{k \in \mathbb{N}}$ that converges (in the $C^1$-topology) to a $C^1$-topological disk $D$ of dimension $d^s$, tangent to the extended bundle $E^s$ in $U$. Clearly, we must have $D = \mathcal{F}^s_r(z)$ (see item (d) of proposition 3.2), thus $\mathcal{F}^s_r(z) \subset \mathcal{F}^s(f^n(x))$. Taking the $n$th pre-image of this inequality, and using the $f$-invariance of the lamination, we obtain that

$$f^{-n}(\mathcal{F}^s_r(f^n(y))) \subset \mathcal{F}^s(x).$$

As this inequality holds for every $n \in \mathbb{N}$, item (e) of proposition 3.2 gives that $\mathcal{F}^s(y) \subset \mathcal{F}^s(x)$, ending the proof of the proposition.

Remark 5.5. Note that lemma 5.3 implies that if $\mathcal{F}^s(x)$ contains some point that is sufficiently close to a hyperbolic periodic point $q \in \Lambda$ of index $d^s$, then $\mathcal{F}^s(x)$ intersects transversely the local unstable manifold of $q$.

In what follows, we fix the set $\mathcal{R} = \mathcal{R}_0 \cap \mathcal{R}_1 \cap \mathcal{R}_2 \cap \mathcal{R}_3$ and assume that the isolating block $U$ of an attractor $\Lambda_f(U)$ is always endowed with an extension of the partially hyperbolic splitting of $\Lambda_f(U)$ as in proposition 5.1.

4 We do not know a priori if the sequence $\{x_k\}_{k \in \mathbb{N}}$ could be taken inside $\Lambda$. That is the reason why this proposition is not an immediate consequence of the continuity of $\mathcal{F}^s$.
6. Generic absence of local strong homoclinic intersections

In this section we prove that, \( C^1 \)-generically, the local strong stable and the local strong unstable leaves of a hyperbolic periodic point are disjoint. This is a technical condition that will be useful later in the proof of theorems A and B.

Given \( n \in \mathbb{N} \), let \( \text{Per}(n, f_{|U}) \) be the set of periodic points \( p \in \Lambda_f(U) \) with period \( \pi(p) \leq n \). By item (a) of theorem 4.4, for every \( f \in \mathcal{R} \) and \( n \in \mathbb{N} \), the set \( \text{Per}(n, f_{|U}) \) is a finite hyperbolic set.

Remark 6.1. Given \( f \in \mathcal{R} \) and \( n \in \mathbb{N} \), there is a neighborhood \( U_n \) of \( f \) such that, for every \( g \in U_n \), the set \( \text{Per}(n, g_{|U}) \) consists of the continuation of \( \text{Per}(n, f_{|U}) \) as a hyperbolic set.

In the next lemma we follow some standard Kupka–Smale-like arguments.

Lemma 6.2. Let \( \mathcal{U} \) be a compatible neighborhood of \( f \in \mathcal{R} \) with respect to a partially hyperbolic attractor \( \Lambda_f(U) \) with center dimension \( d^c \geq 1 \). Fixed \( \varepsilon > 0 \), there is a residual subset \( \mathcal{G} \) of \( \mathcal{U} \) such that, for every \( g \in \mathcal{G} \) and every pair of distinct periodic points \( a, b \) of \( \Lambda_g(U) \), it holds that

\[
\mathcal{F}_\varepsilon^s(a, g) \cap \mathcal{F}_\varepsilon^u(b, g) = \emptyset.
\]

Proof. Fix \( f \in \mathcal{R} \) and a compatible neighborhood \( \mathcal{U} \) of \( f \). We need the following claim.

Claim 6.3. Let \( n \in \mathbb{N} \), \( f_* \in \mathcal{U} \cap \mathcal{R} \) and \( U_n \) be a compatible neighborhood of \( f_* \) such that, for every \( g \in U_n \), the set \( \text{Per}(n, g_{|U}) \) is the continuation of the hyperbolic set \( \text{Per}(n, f_*{\mid}_U) \). Fixed \( \varepsilon > 0 \), there is an open and dense subset \( \mathcal{V}_n \) of \( U_n \) such that, for every \( g \in \mathcal{V}_n \) and every pair of distinct points \( p_g, q_g \in \text{Per}(n, g_{|U}) \) it holds that

\[
\mathcal{F}_\varepsilon^s(p_g, g) \cap \mathcal{F}_\varepsilon^u(q_g, g) = \emptyset.
\]

Proof of the claim. Given \( p, q \in \text{Per}(n, f_*{\mid}_U) \) with \( p \neq q \), if there is \( g \in U_n \) such that \( \mathcal{F}_\varepsilon^s(p_g, g) \cap \mathcal{F}_\varepsilon^u(q_g, g) = \emptyset \), then, by the continuous dependence of the leaves on \( g \), the same holds for any diffeomorphism in an open neighborhood of \( g \).

On the other hand, since \( d^n + d^1 \) is less than the ambient dimension, if \( \mathcal{F}_\varepsilon^s(p_g, g) \cap \mathcal{F}_\varepsilon^u(q_g, g) \neq \emptyset \), then this intersection is not transverse. Hence, after an arbitrarily small perturbation, we can assume that the disks \( \mathcal{F}_\varepsilon^s(p_g, g) \) and \( \mathcal{F}_\varepsilon^u(q_g, g) \) are disjoint. As a conclusion, there is an open and dense subset \( \mathcal{V}_{p,q} \) of \( U_n \) such that

\[
\mathcal{F}_\varepsilon^s(p_g, g) \cap \mathcal{F}_\varepsilon^u(q_g, g) = \emptyset, \text{ for every } g \in \mathcal{V}_{p,q}. \tag{6.1}
\]

By remark 6.1, the set

\[
\mathcal{V}_n = \bigcap_{(p, q) \in \mathcal{B}} \mathcal{V}_{p,q}, \text{ where } \mathcal{B} = \{(p, q) \in \text{Per}(n, f_{|U})^2 \mid p \neq q\},
\]

is a finite intersection of open and dense subsets of \( U_n \), which means that \( \mathcal{V}_n \) is also open and dense in \( U_n \). By construction, the set \( \mathcal{V}_n \) satisfies the required property in claim 6.3.

To conclude the proof of the lemma, we use a genericity argument as follows. Let \( \{f_i\}_{i \in \mathbb{N}} \) be a dense subset of \( \mathcal{U} \cap \mathcal{R} \). Fixed \( n \in \mathbb{N} \), we apply claim 6.3 to each \( f_i, i \in \mathbb{N} \). In this way, we obtain an open and dense subset \( \mathcal{V}_n^i \) of some neighborhood \( U_n^i \) of \( f_i \), satisfying the non intersection condition in claim 6.3. Note that \( \mathcal{G}_n = \bigcup_{i \in \mathbb{N}} \mathcal{V}_n^i \) is an open and dense subset of \( \mathcal{U} \). Finally, we set \( \mathcal{G} = \bigcap_{n \in \mathbb{N}} \mathcal{G}_n \), which is the desired residual subset of \( \mathcal{U} \) satisfying the conclusion of the lemma.

\[\square\]
7. Minimality

In this section we define \( u - \) and \( s - \)minimal sets and give a sufficient condition to obtain this property.

Given a partially hyperbolic set \( \Lambda \), denote
\[
F^s_\Lambda(x) = F^s(x) \cap \Lambda \quad \text{and} \quad F^u_\Lambda(x) = F^u(x) \cap \Lambda.
\]

**Definition 7.1 (dynamically minimal lamination).** Let \( \Lambda \) be a partially hyperbolic set of a diffeomorphism \( f \) with nontrivial stable bundle \( E^s \). We say that the lamination \( F^s \) is **dynamically minimal** if there is \( d \in \mathbb{N} \) such that, for all \( x \in \Lambda \), it holds that
\[
\bigcup_{i=1}^d F^s_\Lambda(f^i(x)) = \Lambda.
\]
In this case we say that \( \Lambda \) is an \( s - \)minimal set.

We say that \( \Lambda \) is a **robustly \( s - \)minimal set** if \( \Lambda = \Lambda_f(U) \) is an isolated set, and \( \Lambda_g(U) \) is \( s - \)minimal for all \( g \) in a neighborhood \( U \) of \( f \).

If \( s - \)minimality is verified only in a residual subset of \( U \), then we say that \( \Lambda_f(U) \) is a **generically \( s - \)minimal set**.

The definition of \( u - \)minimality is analogous, considering the strong unstable lamination \( F^u \).

**Remark 7.2.** The term **dynamically** in definition 7.1 stress that it depends on the diffeomorphism \( f \), as we consider not only one leaf, but also some of its iterates. When \( M \) is a connected manifold and \( \Lambda = M \), this definition coincides with the notion of \( s - \)minimality we gave in the introduction (when \( F^s \) is a minimal foliation). Recall that, in this case, the density of the orbits of all leaves implies the density of each leaf itself.

7.1. A sufficient condition for minimality

Next theorem and its corollary establish a sufficient condition for \( u - \) or \( s - \)minimality of partially hyperbolic homoclinic classes. They are key ingredients in the proofs of theorems A and B.

For the next statements, recall the notation in remark 2.5.

**Theorem 7.3.** Let \( f \in \text{Diff}^1(M) \) and \( \Lambda = H(p, f) \) be a partially hyperbolic homoclinic class admitting an extension of the invariant splitting \( E^s \oplus E^c \oplus E^u \) to a compact neighborhood \( U \) of \( \Lambda \). Then,

- (a) If \( \text{index}(p) = d^s \) and \( p \in \overline{\mathcal{O}_f(F^s(x))} \) for every \( x \in \Lambda \), then \( \Lambda \) is \( s - \)minimal.
- (b) If \( \text{index}(p) = d^s + d^c \) and \( p \in \overline{\mathcal{O}_f(F^u(x))} \) for every \( x \in \Lambda \), then \( \Lambda \) is \( u - \)minimal.

**Corollary 7.4 (generic version).** Let \( f \in \mathcal{R} \) and \( \Lambda = H(p, f) \) be a partially hyperbolic homoclinic class. Then,

- (a) If \( \text{index}(p) = d^s \) and \( \text{index}(p) = d^s \) and \( p \in \overline{\mathcal{O}_f(F^c(x))} \) for every \( x \in \Lambda \), then \( \Lambda \) is \( s - \)minimal.
- (b) If \( \text{index}(p) = d^s + d^c \) and \( \text{index}(p) = d^s + d^c \) and \( p \in \overline{\mathcal{O}_f(F^u(x))} \) for every \( x \in \Lambda \), then \( \Lambda \) is \( u - \)minimal.

Here we only treat the \( s - \)minimal case in item (a) of these statements. To get item (b), we just apply item (a) to the inverse map \( f^{-1} \).

**Lemma 7.5.** Let \( \Lambda = H(p, f) \) be as in theorem 7.3 with \( \text{index}(p) = d^s \) and \( \pi(p) = d \). If there is \( x \in \Lambda \) such that \( p \in \overline{\mathcal{O}_f(F^s(x))} \), then \( \Lambda = \bigcup_{i=1}^d F^s_\Lambda(f^i(x)) \).

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Proof. Note that the inclusion \( \bigcup_{i=1}^{d} F^\Lambda_i(f^i(x)) \subseteq \Lambda \) is immediate (recall the notation \( F^\Lambda_i(x) = F^i(x) \cap \Lambda \)). To prove the inclusion \( \Lambda \subseteq \bigcup_{i=1}^{d} F^\Lambda_i(f^i(p)) \), we first note that
\[
\Lambda = H(p, f) = W^s(\partial_j(p)) \cap \Lambda = \bigcup_{i=1}^{d} F^\Lambda_i(f^i(p)).
\] (7.1)

To conclude this lemma, it suffices to prove that \( F^\Lambda_i(p) \subset F^\Lambda_i(x) \).

As \( p \in F^\Lambda_i(x) \), proposition 5.4 gives that \( F^\Lambda_i(p) \subset F^\Lambda_i(x) \). Given \( z \in F^\Lambda_i(p) \), consider a transverse homoclinic point \( \tilde{z} \) of \( p \) close to \( z \). Since \( F^\Lambda_i(p) \subset F^\Lambda_i(x) \), the leaf \( F^\Lambda_i(x) \) accumulates at \( \tilde{z} \) and intersect \( W^s(\partial_j(p)) \) at a point \( w \) that can be chosen arbitrarily close to \( \tilde{z} \). By equation (7.1), there is \( j \in \mathbb{N} \) such that \( F^\Lambda_i(f^j(p)) \) accumulates at \( x \) and, by proposition 5.4, it also accumulates at \( w \). Then, \( F^\Lambda_i(f^j(p)) \) meets transversely \( W^s(\partial_j(f^i(p))) \) in a sequence of homoclinic points of \( p \) converging to \( w \), which means that \( w \in H(p, f) \). By construction, \( w \in F^\Lambda_i(x) \) can be taken arbitrarily close to \( z \), so \( z \in F^\Lambda_i(x) \). From the arbitrary choice of \( z \in F^\Lambda_i(p) \), we conclude that \( F^\Lambda_i(p) \subset F^\Lambda_i(x) \).

This last inclusion and equation (7.1) imply that \( \Lambda \subseteq \bigcup_{i=1}^{d} F^\Lambda_i(f^i(x)) \), finishing the proof of the lemma. \( \square \)

In what follows, given a compact set \( X \subset M \), we denote by \( B_r(X) \) the set of points in \( M \) whose distance to \( X \), with respect to a Riemannian metric on \( M \), is less than \( r \).

Proof of theorem 7.3. Suppose that \( \text{index}(p) = d^i \) and \( p \in \partial_j(f^i(x)) \). We prove item (a) of the theorem.

Fix \( x \in \Lambda \). Since \( p \in \partial_j(f^i(x)) \), given \( \varepsilon > 0 \) there is \( j_1 \in \mathbb{Z} \) such that \( p \in B_r(f^{j_1}(x)) \). By remark 5.5, the leaf \( F^\Lambda_i(f^{j_1}(x)) \) intersects transversely the local unstable manifold of \( p \), provided \( \varepsilon \) is small enough. By the \( \lambda \)-lemma, the forward iterates of \( F^\Lambda_i(f^{j_1}(x)) \) start to accumulate at \( F^\Lambda_i(\partial_j(p)) \), so there are \( r > 0 \) and \( j_2(x) \in \mathbb{N} \) such that, for every \( j \geq j_2(x) \), it holds that
\[
\Lambda \subset B_r \left( \bigcup_{i=1}^{d} f^{-j+i}(F^\Lambda_i(x)) \right), \quad \text{where } \pi(p) = d.
\] (7.2)

By the continuity of the lamination \( F^\Lambda \) and the \( \lambda \)-lemma, there is a neighborhood \( U_x \) of \( x \) such that, for every \( y \in U_x \cap \Lambda \) and \( j \geq j_2(x) \), it holds
\[
\Lambda \subset B_r \left( \bigcup_{i=1}^{d} f^{-j+i}(F^\Lambda_i(y)) \right) \subset B_r \left( \bigcup_{i=1}^{d} F^\Lambda_i(f^{j+i}(y)) \right).
\] (7.3)

In this way, for each \( x \in \Lambda \) we get a number \( j_2(x) \) and a neighborhood \( U_x \) of \( x \) satisfying inclusion (7.2). Using these open sets as a covering for the compact set \( \Lambda \), we extract a finite subcovering \( \bigcup_{i=1}^{n} U_{x_i} \) of \( \Lambda \).

Set \( J = \max_{i=1}^{n} (j_2(x_i)) \). By construction,
\[
\Lambda \subset B_r \left( \bigcup_{i=1}^{d} F^\Lambda_i(f^{J+i}(y)) \right), \quad \text{for all } y \in \Lambda.
\] (7.3)

Applying inclusion (7.3) to \( y = f^J(x) \), and observing that \( \varepsilon \) can be taken arbitrarily small, we conclude that \( \Lambda \subseteq \bigcup_{i=1}^{d} F^\Lambda_i(f^J(x)) \). Thus, there is \( j_3(x) \in \{1, \ldots, d\} \) such that \( p \in F^\Lambda_i(f^{j_3(x)}(x)) \), which means that \( f^{-j_3(x)}(p) \in F^\Lambda_i(x) \). Applying lemma 7.5 to the periodic point \( f^{-j_3(x)}(p) \), and observing that \( H(p, f) = H(f^{-j_3(x)}(p), f) = \Lambda \), we obtain that \( \bigcup_{i=1}^{d} F^\Lambda_i(f^J(x)) = \Lambda \). As it holds for all \( x \in \Lambda \), the set \( \Lambda \) is \( s \)-minimal. \( \square \)
Proof of corollary 7.4. Suppose that index(p) = d^s and, for every x ∈ Λ, \( \partial_f^i(F^s(x)) \cap \text{Per}_{d^u}(f^n_i) \neq \emptyset \). We prove item (a) of the corollary.

Since we are in the generic context, we can assume that Λ admits an invariant extension of the splitting to a compact neighborhood U (see theorem 5.1). By hypothesis, for every x ∈ Λ there is a periodic point \( p_\ast \in \partial_f^i(F^s(x)) \cap \text{Per}_{d^u}(f^n_i) \). By remark 5.5, \( \partial_f^i(F^s(x)) \) intersect \( W^s(\mathcal{O}(p_\ast)) \) transversely. From the invariance of the set \( \partial_f^i(F^s(x)) \) and the λ-Lemma, we get that

\[
\partial_f^i(F^s(p_\ast)) \subset \partial_f^i(F^s(x)).
\]

(7.4)

Note that \( F^s(p_\ast) = W^s(p_\ast) \), so

\[
H(p_\ast, f) \subset \partial_f^i(F^s(p_\ast)).
\]

(7.5)

Clearly, \( H(p, f) \cap H(p_\ast, f) \neq \emptyset \), since \( p_\ast \in \Lambda = H(p, f) \). By item (c) of theorem 4.4, every two non-disjoint homoclinic classes of \( f \in \mathcal{R} \) coincide, so \( \Lambda = H(p_\ast, f) \). Putting together this fact, equations (7.4) and (7.5), we obtain that \( \Lambda \subset \partial_f^i(F^s(x)) \).

As this holds for every \( x \in \Lambda \), theorem 7.3 gives that \( \Lambda \) is s-minimal.

\[\square\]

8. Central curves: classification of periodic points

A first step in proving theorems A and B is to classify the dynamics on certain central invariant curves. According to this classification, we investigate \( u \)- and \( s \)-minimality case by case.

Unlike the strong stable and unstable bundles, we can not guarantee the existence of an invariant center lamination tangent to the center bundle. Nevertheless, if \( \Lambda \) is a partially hyperbolic attractor with one-dimensional center bundle, we guarantee the existence of invariant central curves for the hyperbolic periodic points of \( \Lambda \) (proposition 8.1). A central curve is a \( C^1 \) map \( \gamma : \mathbb{R} \to M \) that is tangent to the (extended) center bundle \( E^c \) at every point of \( \gamma \subset U \) (see section 5).

Next result is an adaptation\(^5\) of Theorem 2 in [13] for the context of partially hyperbolic attractors.

**Proposition 8.1.** Let \( f \in \mathcal{R} \) and \( \Lambda_f(U) \) be a partially hyperbolic attractor of \( f \) with one-dimensional center bundle. Then there exists \( K > 0 \) such that, for every hyperbolic periodic point with period \( N \geq K \), there exists an \( f^N \)-invariant central curve \( L(p) \) (i.e. \( f^N(L(p)) = L(p) \)) containing \( p \) in its interior.

**Sketch of the proof.** The proof of this result is almost identical to the one of Theorem 2 of [13]. It involves only local arguments, which are still valid inside the isolating block \( U \) of the attractor \( \Lambda_f(U) \). Following their arguments, for each periodic point \( p \in \Lambda_f(U) \) with period \( N \) sufficiently big, we obtain a local central curve \( \gamma(p) \) inside \( U \) with the following property: \( \gamma(p) \backslash p \) is the union of two connected components \( \gamma^+(p) \) and \( \gamma^-(p) \), each one invariant either by \( f^N \) or by \( f^{-N} \). By taking forward and backward iterates of these components, we can extend \( \gamma(p) \) to a curve \( L(p) \) such that \( f^N(L(p)) = L(p) \). In the process, we may assume that \( L(p) \) do not end in a periodic point inside \( U \) by extending \( L(p) \) if necessary.

Observe that \( L(p) \) may escape from the region \( U \), so let us prove that \( L(p) \) is tangent to \( E^c \) at every point intersecting \( U \). By definition, if \( z \in L(p) \cap U \), then there is \( n \in \mathbb{Z} \) such that \( f^n(z) \in \gamma^+(p) \cup \gamma^-(p) \). Since \( \Lambda \) is an attractor, the forward orbit of every point in \( U \) lies in \( U \). Hence, if \( n < 0 \), then \( f^n(f^n(z)) \in U \) for every \( i \in \{ 1, \ldots, |n| \} \), and \( z = f^n(f^n(z)) \). If \( n > 0 \), then \( f^{-i}(f^n(z)) = f^{n-i}(z) \in U \) for every \( i \in \{ 1, \ldots, n \} \), and \( z = f^{-n}(f^n(z)) \).

\(^5\) In [13], the partial hyperbolicity is defined over the whole manifold.
In any case, the invariance of $E^c$ given in proposition 5.1 and the fact that $\gamma^+(p) \cup \gamma^-(p)$ is tangent to $E^c$ gives the announced property. □

**Remark 8.2.** Recall that $\mathcal{R}$ consists of Kupka–Smale diffeomorphisms, so the set of periodic points $p$ with $\pi(p) \leq K$ is a finite set. Moreover, if $p$ is a periodic point of period $d$, then the period of any periodic point in the curve $L(p)$ is a divisor of $2d$. Hence, there are only finitely many periodic points in $L(p)$, and the dynamics on this curve is Morse–Smale.

In general, there is not a unique invariant central curve passing through $p$. We consider a choice of these invariant curves that is coherent with the dynamics on $\Lambda_f(U)$, that is, satisfying $L(f(p)) = f(L(p))$.

We denote by $L_U(p)$ the connected component of $L(p) \cap U$ containing $p$ and by $\Gamma_p \subset L_U(p)$ the smallest compact and connected subset of $L_U(p)$ that contains all periodic points and all periodic closed curves of $L_U(p)$ (it may happens that $\Gamma_p = \{p\}$). There are three possibilities for the boundary $\partial \Gamma_p$ of $\Gamma_p$, relative to the set $L_U(p)$: either it is empty, a unitary set, or a two points set. If $\partial \Gamma_p = \emptyset$, then $\Gamma_p$ is a closed curve. When $\partial \Gamma_p \neq \emptyset$, we say that $\partial \Gamma_p$ are the extremal points of $\Gamma_p$. A periodic point $q$ is called extremal if there is some $p \in \Lambda_f(U)$ such that $q \in \partial \Gamma_p$.

**Remark 8.3.** Since $U$ is a neighborhood of the compact set $\Lambda_f(U)$, the length of $L_U(p)$ is uniformly bounded from below, and the point $p$ is uniformly far from the edges of $L_U(p)$, if any. Hence, there is $\delta > 0$ such that, for every periodic point $p \in \Lambda$, the set $L_U(p)$ contains a disk centered at $p$ of length bigger than $\delta$.

Now we classify the periodic points of $f$ in $U$ as follows:

$$P_1 \cup P_2 \cup P_3 \cup P_4 = \{ p \in \text{Per}(f) \cap U, \quad \pi(p) \geq K \}$$

where

- $p \in P_1$ if the extremal points of $\Gamma_p$ are attracting in the central direction,
- $p \in P_2$ if the extremal points of $\Gamma_p$ are repelling in the central direction,
- $p \in P_3$ if there are one attracting and one repelling extremal points of $\Gamma_p$, and
- $p \in P_4$ if $\Gamma_p$ is a closed curve. (see figure 1)

**Remark 8.4.** Since the dynamics in $L(p)$ is Morse–Smale, there are finitely many periodic points $a_1, \ldots, a_m \in \Gamma_p$ such that

$$\Gamma_p \subset \bigcup_{i=1}^{m} W^s(a_i, f) \quad \text{and} \quad \Gamma_p \subset \bigcup_{i=1}^{m} W^u(a_i, f),$$

where the stable and unstable manifolds $W^s$ and $W^u$ refer to the dynamics restricted to $L(p)$. Observe that $W^s(a_i, f) = \{a_i\}$ if $\text{index}(a_i) = d^s$, and $W^u(a_i, f) = \{a_i\}$ if $\text{index}(a_i) = d^u + 1$.

**Remark 8.5.** If $p \in P_1 \cup P_2$, then $\partial \Gamma_p$ is either the empty set or consists of (at most two) points of index $d^u + 1$. Hence

$$L_U(p) \subset \Gamma_p \cup W^s(\partial \Gamma_p).$$
Figure 1. Classification of periodic points according to its central curve.

Similarly, if \( p \in P_2 \cup P_4 \), then \( \partial \Gamma_p \) is either the empty set or consists of (at most two) points of index \( d^s \). Hence
\[
L_U(p) \subset \Gamma_p \cup W^u(\partial \Gamma_p) \subset \Lambda.
\]

**Lemma 8.6.** For every \( p \in \text{Per}(f) \cap U \) with \( \pi(p) > K \), the following holds:

1. \( \Gamma_p \subset \Lambda_f(U) \),
2. \( f(\Gamma_p) = \Gamma_{f(p)} \),
3. \( f(P_i) = P_i \).

**Proof.** By definition, the periodic points of \( \Gamma_p \) belong to \( U \). Since \( \Lambda_f(U) \) is an attractor, remarks 8.4 and 2.4 gives that
\[
\Gamma_p \subset \bigcup_{q \in \Gamma_p} W^u(q) \subset \Lambda_f(U),
\]
proving item (a).

Items (a) and (c) follow from the coherent choice of the central curves. Observe that \( f \) sends closed curves to closed curves and extremal points of \( \Gamma_p \) to extremal points of \( \Gamma_{f(p)} \).

**Lemma 8.7.** \( P_i = \Lambda_f(U) \) for some \( i \in \{1, \ldots, 4\} \).

**Proof.** By item (b) of theorem 4.4, the periodic points of \( f \) are dense in \( \Lambda_f(U) \). By remark 8.2, the periodic points of \( \Lambda_f(U) \) with period less than \( K \) is a finite set. Since \( \Lambda_f(U) \) is infinite and transitive, it has no isolated periodic orbits. Hence the set of periodic points of \( \Lambda_f(U) \) with period bigger than \( K \) is also dense in \( \Lambda_f(U) \). In other words, \( \Lambda_f(U) = P_1 \cup P_2 \cup P_3 \cup P_4 \).

Let \( x \in \Lambda_f(U) \) be a point with a dense orbit. Clearly, \( x \in P_i \) for some \( i \in \{1, \ldots, 4\} \). From item (c) in lemma 8.6, the whole orbit of \( x \) lies in \( P_i \), which implies that \( P_i = \Lambda_f(U) \).

9. **Proof of theorems A and B**

In this section we prove theorems A and B. We start investigating \( u \)- and \( s \)-minimality for generic attractors in section 9.1, and then we investigate their continuations in section 9.2.
9.1. Generic $u$- and $s$-minimal attractors

Recall the notation $\text{GTPHA}_1(U)$ as the subset of $\text{Diff}^1(M)$ of diffeomorphisms $f$ such that $\Lambda_f(U)$ is a robustly non-hyperbolic, partially hyperbolic set with one-dimensional center bundle that is generically transitive. Given $f_0 \in \mathcal{R} \cap \text{GTPHA}_1(U)$, let $U_0$ be a compatible neighborhood of $f_0$, and $\mathcal{G}_0$ be the residual subset of $U_0$ given by lemma 6.2.

Fixed $f \in \mathcal{G}_0 \cap \mathcal{R}$ and $\Lambda = \Lambda_f(U)$, we verify that $\Lambda$ has one minimal lamination. We split this verification into three cases, according to which sets $P_i$’s are dense in $\Lambda$ (see lemma 8.7).

**Proposition 9.1.**

(a) If the set $P_1 \cup P_4$ is dense in $\Lambda$, then $\Lambda$ is $u$-minimal.

(b) If the set $P_2 \cup P_4$ is dense in $\Lambda$, then $\Lambda$ is $s$-minimal.

(c) If the set $P_3 \cup P_4$ is dense in $\Lambda$, then $\Lambda$ is $u$-minimal.

Fix $x \in \Lambda_f(U)$. Since $\Lambda_f(U)$ is an attractor, for every $\epsilon > 0$, the disk $F^u_\epsilon(x)$ is a subset of $\Lambda_f(U)$. Hence every point $z \in F^u_\epsilon(x)$ has a local stable disk $F^s_\epsilon(z)$, and we define the topological disk of codimension one

$$\Delta(x, \epsilon) = \bigcup_{z \in F^u_\epsilon(x)} F^s_\epsilon(z).$$

(9.1)

**Remark 9.2.** By remark 8.3, for any periodic point $p$, the curve $L_U(p)$ contains a disk of length $\delta$ centered at $p$ ($\delta$ does not depend on $p$). Thus, if $p$ is sufficiently close to $x$, then $L_U(p)$ meets topologically transversely the disk $\Delta(x, \epsilon)$ at some point $z_p = z_p(x)$ (see figure 2).

**Lemma 9.3.** If $z_p$ lies in the stable manifold of some periodic point $\tilde{p} \in \Gamma_p$, then $\tilde{p} \in \overline{\mathcal{O}(F^u(x))}$.

**Proof.** By the definitions of $\Delta(x, \epsilon)$ and $z_p$, there is a point $w_p \in F^u_\epsilon(x)$ such that $z_p \in F^s_\epsilon(w_p)$ (see figure 2). Then, considering a Riemannian metric $d(\cdot, \cdot)$ in $U$, the contraction on $F^s_\epsilon(w_p)$ by positive iterates of $f$ gives that

$$\lim_{n \to \infty} d(f^n(z_p), f^n(w_p)) = 0.$$  

(9.2)

By hypothesis, $z_p \in W^s(\tilde{p}, f)$, so we also have that

$$\lim_{n \to \infty} d(f^n(z_p), f^n(\tilde{p})) = 0.$$  

(9.3)

Combining (9.2) and (9.3), we obtain

$$\lim_{n \to \infty} d(f^n(\tilde{p}), f^n(w_p)) \to 0.$$  

As $w_p \in F^u(x)$ and $\tilde{p}$ is periodic, this implies that the orbit of $F^u(x)$ accumulates at $\tilde{p}$. □

Now we start to prove proposition 9.1.

**Proof of (a).** As $f \in \mathcal{R}$, $\Lambda$ is a homoclinic class (remark 4.5). By corollary 7.4, to prove the $u$-minimality of $\Lambda$, it suffices to see that, for every $x \in \Lambda$, it holds that

$$\overline{\mathcal{O}(F^u(x))} \cap \text{Per}_{d^u+1}(f_{\mid x}) \neq \emptyset.$$  

(9.4)

Fix $x \in \Lambda$. Since $P_1 \cup P_4$ is dense in $\Lambda$, we can take $p \in P_1 \cup P_4$ sufficiently close to $x$ so that, as in remark 9.2, there is a transverse intersection point $z_p$ between $L_U(p)$ and $\Delta(x, \epsilon)$.

By remarks 8.4 and 8.5, the point $z_p$ either lies in a stable manifold of some periodic point $\tilde{p} \in \Gamma_p$ of index $d^u+1$, or $z_p$ is a hyperbolic periodic point of index $d^u$. If the first possibility holds, then lemma 9.3 implies equation (9.4) and we are done.
Let us suppose then that $z_p$ is a hyperbolic periodic point of index $d'$. From equation (9.2) and proposition 5.4, the orbit of $F^u(x)$ accumulates at the orbit of $F^u(z_p)$. Thus, to get equation (9.4) it is enough to show that

$$\overline{O(F^u(z_p))} \cap \text{Per}_{d'+1}(f_{\Lambda_1}) \neq \emptyset.$$  

(9.5)

Consider the topological manifold $\Delta(z_p, \varepsilon)$. For every $q \in P_1 \cup P_4$ sufficiently close to $z_p$, the curve $L_U(q)$ meets topologically transversely $\Delta(z_p, \varepsilon)$ at some point $z_q$. Note that $F^u(z_p) \cap F^s(z_q) \neq \emptyset$. Since $f \in G_0$ and $z_p$ is periodic, lemma 6.2 gives that $z_q$ is not periodic. By remark 8.4, $z_q$ belongs to the stable manifold of some periodic point in $L_U(q)$ with index $d' + 1$. Now lemma 9.3 implies equation (9.5) and, consequently, equation (9.4). This ends the proof of Case (a).

**Proof of (b).** Note that this case is not as symmetrical to the Case (a) as it may seem. We cannot saturate a strong stable disk with strong unstable leaves, since not all of the points in the strong stable disk belong to $\Lambda$. Instead, for each point in $p \in P_2 \cup P_4$, we introduce the topological disk $\nabla(p, \varepsilon)$ as follows.

Let $L_U(p)$ be a one-dimensional disk of length $\varepsilon$ centered at $p$. We define

$$\nabla(p, \varepsilon) = \bigcup_{y \in L_U(p)} F^u_\varepsilon(y) \quad \text{(see figure 3).}$$

(9.6)

Recall that in this case the curve $L_U(p)$ is contained in $\Lambda$ (see remark 8.5). Thus, for every $y \in L_U(p)$ there exists $F^u(y)$, so $\nabla(p, \varepsilon)$ is well defined.

By corollary 7.4, to prove the $s$-minimality of $\Lambda$ it is enough to see that, for every $x \in \Lambda$, it holds that

$$\overline{O(F^u(x))} \cap \text{Per}_{d'}(f_{\Lambda_1}) \neq \emptyset.$$  

(9.7)

Recall that, by remark 8.3, the curve $L_U(p)$ contains a disk centered at $p$ with length $\delta$, so the disk $\nabla(p, \varepsilon)$ has a minimum size (inner radius at $p$) that do not depend on $p$. Hence, given any $x \in \Lambda$, if $p \in P_2 \cap P_4$ is sufficiently close to $x$, then $F^u_\varepsilon(x)$ intersects topologically...
transversely $\nabla(p, \varepsilon)$ at some point $w_p$ (see figure 3). By the definition of $\nabla(p, \varepsilon)$, there is $z_p \in L_U(p)$ such that $w_p \in F^s_u(z_p)$, which means that

$$\lim_{n \to \infty} d(f^{-n}(z_p), f^{-n}(w_p)) = 0. \tag{9.8}$$

By remarks 8.4 and 8.5, either $z_p \in W^u(\tilde{p})$ for some periodic point $\tilde{p} \in \Gamma_p \cap \text{Per}_{d_l}(f_{l_1})$, or $z_p \in \text{Per}_{d_{l+1}}(f_{l_1})$. In the first situation, we get that

$$\lim_{n \to \infty} d(f^{-n}(z_p), f^{-n}(\tilde{p})) = 0. \tag{9.9}$$

Combining equations (9.8) and (9.9) it gives that

$$\lim_{n \to \infty} d(f^{-n}(z_p), f^{-n}(w_p)) \to 0,$$

which implies equation (9.7), as $w_p \in F^s(x)$.

If $z_p \in \text{Per}_{d_{l+1}}(f_{l_1})$, then the orbit of $F^s(x)$ accumulates at the orbit of $F^s(z_p)$. Thus, to get equation (9.7), it is enough to prove that

$$\overline{O}(F^s(z_p)) \cap \text{Per}_{d_l}(f_{l_1}) \neq \emptyset. \tag{9.10}$$

Consider a periodic point $q \in P_3 \cup P_4$ close to $z_p$ such that $F^s(z_p)$ intersects topologically transversely $\nabla(q, \varepsilon)$ at a point $w_q$. Then, there is a point $z_q \in L_U(q)$ with $w_q \in F^s(z_q)$. By lemma 6.2, $z_q$ is not periodic, so $z_q$ lies in the unstable manifold of some periodic point of index $d'$ in $\Gamma_q$. In this situation, equations (9.8) and (9.9) hold replacing $z_p$, $w_p$, and $\tilde{p}$ by $z_q, w_q$, and $q$, respectively. Arguing as before, these equations lead to equation (9.10) and, consequently, equation (9.7). This ends the proof of Case (b).

Proof of (c). By corollary 7.4, to prove the $u$-minimality of $\Lambda$ it is enough to see that, for every $x \in \Lambda$, equation (9.4) holds.

Consider the codimension one topological disk $\Delta(x, \varepsilon)$ as in equation (9.1). Fix $\tilde{\varepsilon} > 0$ and $p \in P_3$ sufficiently close to $x$ so that, as in remark 9.2, $L_U(p)$ intersects topologically transversely $\Delta(x, \varepsilon)$ at a point $z_p$. 

![Figure 3. Case (b).](image-url)
Let \( l \) be the curve joining \( z_p \) and \( p \) inside \( L_U(p) \) (we consider \( l = \{ p \} \) if \( z_p = p \)). We can assume that there is no periodic points in the interior of \( l \) (otherwise, we replace \( p \) by a periodic point in \( L_U(p) \) with this property).

There are three possible situations:

(a) \( \Delta(x, \epsilon) \cap \text{Per}_{\pi(x)}(f_{\lambda}) = \emptyset \) and \( z_p \in W^s(p) \).

(b) \( \Delta(x, \epsilon) \cap \text{Per}_{\pi(x)}(f_{\lambda}) = \emptyset \) and \( z_p \in W^u(p) \).

(c) \( \Delta(x, \epsilon) \cap \text{Per}_{\pi(x)}(f_{\lambda}) \neq \emptyset \).

In (a), observe that if \( z_p \neq p \), then \( W^s(p) \) is non-trivial and index \( (p) = d^s + 1 \). If \( z_p = p \), then \( p \in \Delta(x, \epsilon) \) and, since \( \Delta(x, \epsilon) \cap \text{Per}_{\pi(x)}(f_{\lambda}) = \emptyset \), it also leads to index \( (p) = d^s + 1 \). Hence, we can apply lemma 9.3 to obtain equation (9.4), and we are done.

Now, let us assume that (b) holds. We also assume that \( z_p \neq p \), since otherwise it also fits in situation (a), that we already treat.

As \( z_p \in W^u(p) \), the segment \( l \subset L_U(p) \) joining \( p \) and \( z_p \) is a subset of \( W^u(p) \), so it is contained in the attractor \( \Lambda \). Let \( \tilde{z}_p = f^{-2d}(z_p) \in l \) and \( \tilde{\Delta}(x, \epsilon) = f^{-2d}(\Delta(x, \epsilon)) \), where \( \pi(p) = d \) (see figure 4). Denote by \( \tilde{l} \) the curve joining \( z_q \) and \( \tilde{z}_p \) inside \( l \). Since the curve \( \tilde{l} \) is a subset of \( \Lambda \), it is accumulated by periodic points of \( \Lambda \).

Observe that \( \Delta(x, \epsilon) \) is also a codimension one topological disk that intersects \( L_U(p) \) topologically transversely at \( z_p \). Thus, arguing as in remark 9.2, if a periodic point \( q \) is sufficiently close to \( z_p \), then \( L_U(q) \) intersects \( \tilde{\Delta}(x, \epsilon) \). Moreover, we can assume that \( q \) is close enough to \( x \) so that \( L_U(q) \) also intersects \( \Delta(x, \epsilon) \) (this assumption is guaranteed if \( p \) is chosen sufficiently close to \( x \)). We take such point \( q \in P_3 \) lying between two points \( z_q, \tilde{z}_q \) in \( L_U(q) \) satisfying

\[
z_q \in L_U(q) \cap \Delta(x, \epsilon) \quad \text{and} \quad \tilde{z}_q \in L_U(q) \cap \tilde{\Delta}(x, \epsilon) \quad \text{(see figure 4).} \tag{9.11}
\]

If the point \( z_q \) belongs to the stable manifold of some periodic point of index \( d^s + 1 \), then, lemma 9.3 implies equation (9.4), and we are done. Thus, we can assume that \( z_q \) does not belong to the stable manifold of any periodic point of \( L_U(q) \). As \( q \in P_3 \), this assumption implies that \( z_q \) lies in the unstable manifold of the extremal point of \( \Gamma_q \) of index \( d^s \). Recall that \( \Delta(x, \epsilon) \cap \text{Per}_{\pi(x)}(f_{\lambda}) = \emptyset \), so \( \tilde{z}_q \) is not a periodic point of index \( d^s \). Hence, the only possibility for \( \tilde{z}_q \) is to be in the stable manifold of some periodic point of \( \Gamma_q \) of index \( d^s + 1 \).

By the coherent choice of the central curves and equation (9.11), the curve \( L_U(f^{2d}(q)) \subset f^{2d}(L_U(q)) \) meets \( \Delta(x, \epsilon) \) at the point \( f^{2d}(\tilde{z}_q) \). Clearly, this intersection lies in the stable manifold of some periodic point of \( \Gamma_q \) of index \( d^s + 1 \). Hence, we are in the same situation as in lemma 9.3, replacing \( z_p \) and \( p \) by \( f^{2d}(\tilde{z}_q) \) and \( f^{2d}(q) \), respectively. As in the lemma, this leads to equation (9.4), ending the proof of item (b).

We are left with the last possibility of item (c). Let \( a \) be a periodic point in \( \Delta(x, \epsilon) \) of index \( d^s \). We can assume that \( \epsilon \) is sufficiently small so that, by lemma 6.2, \( a \) is the only periodic point in \( \Delta(a, \epsilon) \). Then, we can choose \( \hat{x} \in F^u_a(a) \) and \( \hat{\epsilon} > 0 \) such that \( \Delta(\hat{x}, \hat{\epsilon}) \) has no periodic points. It means that, replacing \( x \) by \( \hat{x} \) and \( \epsilon \) by \( \hat{\epsilon} \) in the beginning of the proof, and following the same steps, situation (c) do not occur. Arguing as before, we get the equivalent of equation (9.4) to \( \hat{x} \), that is, \( \overline{O(F^u(\hat{x}))} \cap \text{Per}_{\pi+1}(f_{\lambda}) \neq \emptyset \). As \( \hat{x} \in F^u_a(a) \), this implies that

\[
\overline{O(F^u(a))} \cap \text{Per}_{\pi+1}(f_{\lambda}) \neq \emptyset. \tag{9.12}
\]

Finally, as \( a \in \Delta(x, \epsilon) \), there is \( w \in F^u_a(x) \) such that \( a \in F^u_w(w) \), so the orbit of \( F^u_a(x) \) accumulates at the orbit of \( F^u_a(a) \). This fact together with equation (9.12) implies equation (9.4).

This completes the proof of the three cases in proposition 9.1. \( \square \)
9.2. Proof of theorems A and B

In this subsection we complete the proof of theorems A and B, investigating the continuations of a generic $u$- or $s$-minimal attractor. We start with the following auxiliary lemma.

**Lemma 9.4.** Let $f \in \mathcal{R}$ and $\Lambda_1 f(U)$ be an $s$-minimal isolated partially hyperbolic set with one-dimensional center bundle. Let $U$ be a compatible neighborhood of $f$. For every hyperbolic periodic point $p \in \Lambda_1 f(U)$, there is an open set $W_p \subset U$, such that, for every $g \in W_p$ and every strong stable disk $D$ centered at some point $x \in \Lambda_1 g(U)$, we have

$$H(pg, g) \subset O^{-g}(D).$$

Moreover, if $\text{index}(p) = d^s$, then $W_p$ is a neighborhood of $f$.

**Remark 9.5.** Lemma 9.4 has a dual version for $u$-minimal sets and the forward orbit $O^+ g(D)$ of an unstable disk $D$. The proof is analogous.

**Proof of lemma 9.4.** Note that remark 2.1 implies the upper semicontinuity of the sets $\Lambda_1 f(U)$, so we can assume that every diffeomorphism in $\mathcal{R}$ is a continuity point of the map $g \mapsto \Lambda_1 g(U)$.

First we treat the case where $p \in \Lambda_1 f(U)$ has index $d^s$.

Consider the local unstable manifold $W^u_f(O_f(p))$. By $s$-minimality, for each $x \in \Lambda_1 f(U)$ the leaf $\mathcal{F}^s(x)$ intersects $W^u_f(O_f(p))$ transversely (see remark 5.5). Hence, by the continuity of the strong stable lamination (see remark 3.3), there are open neighborhoods $U_x$ of $x$ and $V_x$ of $f$ such that, for every $y \in U_x \cap \Lambda_1 g(U)$ and every $g \in V_x$, the leaf $\mathcal{F}^s(y, g)$ intersects $W^u_{\sigma}(O_{\sigma}(p_\sigma), g)$ transversely.

Covering the compact set $\Lambda_1 f(U)$ with the open sets $U_x$, we can extract a finite covering $(U_{x_1}, \ldots, U_{x_n})$ of $\Lambda_1 f(U)$. Let $B = \bigcup_{i=1}^n U_{x_i}$ and $V$ denote the open neighborhood $\bigcap_{i=1}^n V_{x_i}$ of $f$. Since $f$ is a continuity point of the map $g \mapsto \Lambda_1 g(U)$, after shrinking $V$ if necessary, the inclusion $\Lambda_1 g(U) \subset B$ holds for every $g \in V$. By construction, for every point $y \in \Lambda_1 g(U)$ and every $g \in V$, the leaf $\mathcal{F}^s(y, g)$ intersects $W^u_{\sigma}(O_{\sigma}(p_\sigma), g)$ transversely. Applying the $\lambda$-Lemma to a disk in $\mathcal{F}^s(y, g)$ transverse to $W^u_{\sigma}(O_{\sigma}(p_\sigma), g)$, we get

$$O_{\sigma}(\mathcal{F}^s(p_\sigma, g)) \subset \overline{O_{\sigma}(\mathcal{F}^s(y, g))}.$$
Since \( \text{index}(p) = d^* \), we have \( H(p_q, g) \subset \overline{\mathcal{O}_{g}(F^i(p_q, g))} \). Therefore,
\[
H(p_q, g) \subset \overline{\mathcal{O}_{g}(F^i(y, g))}. 
\tag{9.13}
\]
Note that if \( D \) is any strong stable disk centered at some point \( x \in \Lambda_{\varepsilon}(U) \) and \( y \) is an accumulation point of the pre-orbit of \( x \), then
\[
\overline{\mathcal{O}_{g}(F^i(y, g))} \subset \overline{\mathcal{O}_{g}(D)}. 
\tag{9.14}
\]
Combining equations (9.13) and (9.14), we conclude the lemma for the case of \( \text{index}(p) = d^* \), where \( \mathcal{W}_p = \mathcal{V} \). Observe that in this case \( \mathcal{W}_p \) is a neighborhood of \( f \).

When \( \text{index}(p) = d^* + 1 \), consider another periodic point \( q \in \Lambda_{\varepsilon}(U) \) with index \( d^* \) (the existence of such point is assured by (b) of proposition 4.6). By proposition 4.8, there is an open subset \( \mathcal{W}_{p,q} \) of \( \mathcal{U} \) such that, if \( g \in \mathcal{W}_{p,q} \), then
\[
H(p_q, g) \subset \mathcal{W}^s(\overline{\mathcal{O}_{g}(p_q, g)}) \subset \overline{\mathcal{O}_{g}(F^i(q_q, g))}. 
\]
Applying the first part of the proof for \( q \) (which has index \( d^* \)), we obtain a neighborhood \( \mathcal{W}_q \) of \( f \) such that, for every \( g \in \mathcal{W}_q \), it holds that \( \overline{\mathcal{O}_{g}(F^i(q_q, g))} \subset \overline{\mathcal{O}_{g}(D)} \). By setting \( \mathcal{W}_p = \mathcal{W}_q \cap \mathcal{W}_{p,q} \), we conclude that the inclusion \( H(p_q, g) \subset \overline{\mathcal{O}_{g}(D)} \) holds for every \( g \in \mathcal{W}_p \).

Recall that \( f \in \mathcal{W}_{p,q,q} \) and \( \mathcal{W}_q \) is a neighborhood of \( f \), so \( f \in \mathcal{W}_p \). □

Now we are ready to finish the proof of theorems A and B.

**Proof of theorem A.** By proposition 9.1, there is a residual subset \( \mathcal{G} \subset \mathcal{R} \) of \( \text{GTPHA}_1(U) \) such that, if \( f \in \mathcal{G} \), then \( \Lambda_f(U) \) is either \( s \)-minimal or \( u \)-minimal.

Suppose that \( f \in \mathcal{G} \) is such that \( \Lambda_f(U) \) is \( s \)-minimal (the case where \( \Lambda_f(U) \) is \( u \)-minimal goes similarly). Let \( p \in \Lambda_f(U) \) be a periodic point of index \( d^* \) (given by item (b) of proposition 4.6) and \( \mathcal{W}_p \) be the neighborhood of \( f \) given in lemma 9.4. For every \( g \in \mathcal{W}_p \) and every disk \( D = F^i_s(x, g) \), with \( x \in \Lambda_{\varepsilon}(U) \), it holds that \( H(p_g, g) \subset \overline{\mathcal{O}_{g}(D)} \).

By remark 4.5, a transitive attractor is, generically, a homoclinic class. Hence, there is a residual subset \( \mathcal{Z} \) of \( \mathcal{W}_p \) such that, for every \( g \in \mathcal{Z} \) and every disk \( D = F^i_s(x, g) \), with \( x \in \Lambda_{\varepsilon}(U) \), it holds that \( \Lambda_{\varepsilon}(U) \subset \overline{\mathcal{O}_{g}(D)} \). In particular, it holds that \( \Lambda_{\varepsilon}(U) \subset \overline{\mathcal{O}(F^i_s(x, g))} \) for every \( x \in \Lambda_{\varepsilon}(U) \). By corollary 7.4, \( \Lambda_{\varepsilon}(U) \) is \( s \)-minimal. Since it holds for every \( g \in \mathcal{G} \cap \mathcal{W}_p \), the set \( \Lambda_f(U) \) is generically \( s \)-minimal. □

**Proof of theorem B.** Let \( B \) be the subset of \( \text{GTPHA}_1(U) \) given by corollary 4.9. We can assume that, for every \( f \in B \), the attractor \( \Lambda_f(U) \) has periodic points with index \( d^* \) and \( d^* + 1 \), as it holds generically (see proposition 4.6) and the existence of such points persist by small perturbations. By theorem A, there are two locally generic sets \( \mathcal{G}_0 \) and \( \mathcal{G}_1 \) such that, if \( g \in \mathcal{G}_0 \), then \( \Lambda_{\varepsilon}(U) \) is a generically \( s \)-minimal set, and if \( g \in \mathcal{G}_1 \) then \( \Lambda_{\varepsilon}(U) \) is a generically \( u \)-minimal set.

Given \( f \in B \cap \mathcal{G}_0 \), and \( p \in \Lambda_f(U) \) with index \( d^* \), we apply lemma 9.4 to obtain a neighborhood \( \mathcal{W}_p \) of \( f \) satisfying the following. For every \( g \in \mathcal{W}_p \) and every disk \( D = F^i_s(x, g) \), with \( x \in \Lambda_{\varepsilon}(U) \), it holds that \( H(p_g, g) \subset \overline{\mathcal{O}_{g}(D)} \).

Consider the open neighborhood \( B_f = \mathcal{W}_p \cap B \) of \( f \). By corollary 4.9, \( \Lambda_{\varepsilon}(U) = H(p_g, g) \) for every \( g \in B_f \). Hence, \( \Lambda_{\varepsilon}(U) \subset \overline{\mathcal{O}_{g}(D)} \) for every \( x \in \Lambda_{\varepsilon}(U) \). In particular, it holds that \( \Lambda_{\varepsilon}(U) \subset \overline{\mathcal{O}(F^i_s(x, g))} \) for every \( x \in \Lambda_{\varepsilon}(U) \). By remark 5.2, theorem 7.3 can be applied in an open subset \( \mathcal{C}_f \) of \( B_f \), so that \( \Lambda_{\varepsilon}(U) \) is an \( s \)-minimal set for every \( g \in \mathcal{C}_f \). Setting \( B_s = \bigcup_{f \in B \cap \mathcal{G}_0} \mathcal{C}_f \), it gives an open subset of \( B \) such that, for every \( g \in B_s \), the attractor \( \Lambda_{\varepsilon}(U) \) is robustly \( s \)-minimal. In the same way, we can obtain an open subset \( B_u \) of \( B \) such that, for every \( g \in B_u \), the attractor \( \Lambda_{\varepsilon}(U) \) is robustly \( u \)-minimal. By construction, the union \( B_s \cup B_u \) is an open and dense subset of \( \text{GTPHA}_1(U) \). □
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