Lorentz Surfaces and Lorentzian CFT
—— with an appendix on quantization of string phase space

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(Dedicated to Ling-Miao Chou, whom I can never thank enough.)

Abstract

The interest in string Hamiltonian system has recently been rekindled due to its application to target-space duality. In this article, we explore another direction it motivates. In Sec. 1, conformal symmetry and some algebraic structures of the system that are related to interacting strings are discussed. These lead one naturally to the study of Lorentz surfaces in Sec. 2. In contrast to the case of Riemann surfaces, we show in Sec. 3 that there are Lorentz surfaces that cannot be conformally deformed into Mandelstam diagrams. Lastly in Sec. 4, we discuss speculatively the prospect of Lorentzian conformal field theory.

Additionally, to have a view of what quantum picture a string Hamiltonian system may lead to, we discuss independently in the Appendix a formal geometric quantization of the string phase space.

MSC number 1991: 05C90, 53C50, 53Z05, 57M50, 81S10, 81T40.

Acknowledgements. This work follows from numerous discussions with Orlando Alvarez, who helped me clarify all sorts of premature ideas and generously proofread and commented the draft. It is also under the shadow of Bill Thurston, whose insight on geometry greatly influences me. To both of them I am deeply indebted. I would also like to express my gratitude to Jørgen E. Andersen, Martin Halpern, Duong Phong, and Alan Weinstein for their courses, from which I had my first contact with CFT and quantization. Besides, I want to thank Hung-Wen Chang, Thom Curtright, Marco Monti, Rafael Nepomechie, and Radu Tătar for discussions and assistance.

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0. Introduction and Outline.

Introduction.

The interest in string Hamiltonian system has recently been rekindled due to its application to target-space duality (e.g. [C-Z], [A-AG-B-L]). In this article, we explore another direction it provides.

Interacting strings can be realized as collections of partially ordered integral filaments in the string Hamiltonian system $LT^*M$. They can be regarded as maps from Lorentzian world-sheets $\Sigma$ into the target-space $M$. Together with the conformal symmetry in the system, one is motivated to the study of Lorentz surfaces. Depending on the role singularities on Lorentz surfaces play, there are coarse and fine Lorentz surfaces. We discuss only coarse ones due to technical reasons. Like pants decompositions for Riemann surfaces, one has rompers decompositions for Lorentz surfaces. Such decompositions provide a way to study their moduli spaces. In contrast to Riemannian case, we show that there are Lorentz surfaces that cannot be rectified into Mandelstam diagrams. As theory of Riemann surfaces to conformal field theory (CFT), theory of Lorentz surfaces should lead to a Lorentzian counterpart of CFT. We explore this prospect speculatively at the end.

Additionally, to have a view of what quantum picture the string Hamiltonian system may lead to, we discuss independently in the Appendix a formal geometric quantization of the string phase space.

Readers are referred to [AG-G-M-V], [At1-2], [F-S], [G-S-W], [Ka1-2], [Mo-S1-2], [L-T], [Se1-3], [Thor], and [Zw] for strings, CFT, and string fields; [B-E], [H-E] and [Pe] for Lorentzian manifolds; [C-B] and [St] for surface theory; [Bo], [Kö] for graph theory; [Br], [G-S], [Mi], [Sn] and [Wo] for geometric quantization.

Outline.

1. Beginning with string Hamiltonian systems.
2. Lorentz surfaces and their moduli.
   2.1 Lorentz surfaces and Mandelstam diagrams.
   2.2 Basic structures and coarse conformal groups.
   2.3 Rompers decompositions and coarse moduli spaces.
3. Rectifiability into Mandelstam diagrams.
   3.1 Branched coverings, positive cones, and rectifiability.
   3.2 Electrical circuits and examples of unrectifiability.
4. Toward Lorentzian conformal field theories.
   Appendix. Quantization of string phase space.
1 Beginning with string Hamiltonian systems.

For introduction of notations, let us recall first the following basic objects:

| physical object         | mathematical presentation                                                                 |
|-------------------------|-------------------------------------------------------------------------------------------|
| target-space            | $M = (M, ds^2, B)$ with the metric $ds^2$, also denoted by $\langle \ , \ \rangle$, either Riemannian or Lorentzian, and $B$ a 2-form, (a $B$-field in physicists’ terminology) |
| particle                | a smooth map $\phi : S^1 \to M$                                                            |
| configuration space     | $LM =$ the loop space of $M$, which contains all $\phi$                                     |
| (momentum) phase space  | $LT^*M$; it has a canonical symplectic structure $\omega$ given by $\omega_\gamma(\eta, \xi) = \int_{S^1} d\sigma \omega(\eta_{\gamma(\sigma)}, \xi_{\gamma(\sigma)})$, where $\omega$: the canonical symplectic structure on $T^*M$, $\gamma \in LT^*M$, and $\eta, \xi$: tangent vectors at $\gamma$ |
| Lagrangian $\mathcal{L}$ on $T_sLM$ ($LT_sM$) | $\mathcal{L}(\phi, X) = \int_{S^1} d\sigma \mathcal{L}(\phi, X; \sigma)$ with $\mathcal{L}(\phi, X; \sigma)$ being $\frac{1}{2} (\langle X(\sigma), X(\sigma) \rangle - \langle \phi_* \partial_\sigma, \phi_* \partial_\sigma \rangle) + B(X(\sigma), \phi_* \partial_\sigma)$ |
| Legendre transformation | the map $\quad LT_sM \quad \longrightarrow \quad LT^*M$ $\quad (\phi, X) \quad \longmapsto \quad (\phi, \pi)$ with $\quad \pi(\sigma) = \frac{\delta \mathcal{L}}{\delta X}(\sigma) = \langle \cdot , X(\sigma) \rangle + B(\cdot, \phi_* \partial_\sigma)$ |
| Hamiltonian $\mathcal{H}$ on $LT^*M$ | the push-forward of $\mathcal{L}$ under the Legendre transformation; explicitly $\mathcal{H}(\phi, \pi; \sigma)$ equals $\frac{1}{2} \langle \pi - B_\phi(\sigma), \pi - B_\phi(\sigma) \rangle + \frac{1}{2} \langle \phi_* \partial_\sigma, \phi_* \partial_\sigma \rangle$, where $B_\phi(\sigma) = B(\cdot, \phi_* \partial_\sigma)$ and $\langle \ , \ \rangle^\sim$ is the induced metric on fibers of $T^*M$ from $\langle \ , \ \rangle$ |
In principle, the Hamiltonian system \((LT^*M, \omega, \mathcal{H})\) contains all the classical information of a free closed bosonic string. The Liouville 1-form \(\theta\) on \(LT^*M\) is given similarly by
\[
\theta_{\gamma}(\eta) = \int_{S^1} d\sigma \theta_{\gamma(\sigma)}(\eta_{\gamma(\sigma)}),
\]
where \(\eta\) is a tangent vector at \(\gamma\); and it satisfies \(d\theta = \omega\).

Completeness and symmetry of string system.

For a general target-space \(M\), the string Hamiltonian flow \(\rho_t\) may not be complete. The integral trajectories of the Hamiltonian vector field \(X_\mathcal{H}\) may not be all definable to the whole \(\mathbb{R}\). Such incompleteness arises for two reasons: (1) \(M\) itself may not be complete. (2) Singularities (i.e. nonsmoothness) of string could arise when evolving toward either future or past; and hence \(X_\mathcal{H}\) is no longer defined there. The collection of singular loops that appear in Case (2) forms by definition the \(s\)-boundary \(\partial_sLT^*M\) of \((LT^*M, X_\mathcal{H})\). The union \(LT^*M \cup \partial_sLT^*M\) shall be denoted by \(\overline{LT^*M}\). One can define the life-span map \((\tau^-, \tau^+)\) from \(LT^*M\) to \([-\infty, 0] \times [0, +\infty]\) by assigning to a \(\gamma \in LT^*M\) the pair \((\inf \{t\}, \sup \{t\})\) where \(t\) is such that \(\rho_t(\gamma)\) is well-defined.

Let \(S_I\) be the space of inextendable extrema of \(I\). As maps from intervals into \(LM\), these extrema are naturally lifted to \(LT_\mathcal{H}M\); and the map
\[
\kappa_0 : S_I \rightarrow LT_\mathcal{H}M, \quad f \mapsto f_*\left(\partial_t|_{S^1 \times \{0\}}\right)
\]
gives an identification of \(S_I\) to \(LT_\mathcal{H}M\). The Legendre transformation now takes them further to \(LT^*M\). The image \(J\) consists of parametrized maximal integral trajectories of \(X_\mathcal{H}\). The map
\[
\iota_0 : J \rightarrow LT^*M, \quad j \mapsto j(0)
\]
identifies $\mathcal{J}$ with $LT^*M$; and the interval $(\tau^-(j(0)), \tau^+(j(0)))$ parametrizes $j$.

Let $Conf(S^1 \times \mathbb{R})$ be the group of conformal diffeomorphisms of $(S^1 \times \mathbb{R}, dt^2 - d\sigma^2)$. Recall that it has four components: $Conf^{(+,\uparrow)}(S^1 \times \mathbb{R})$, $Conf^{-,\uparrow}(S^1 \times \mathbb{R})$, $Conf^{(\downarrow,\uparrow)}(S^1 \times \mathbb{R})$, and $Conf^{(+,\downarrow)}(S^1 \times \mathbb{R})$, where $\pm$ (resp. $\uparrow \downarrow$) indicates whether it preserves the space- (resp. time-)orientation of $S^1 \times \mathbb{R}$. The identity component $Conf^{(+,\uparrow)}(S^1 \times \mathbb{R})$ is a $\mathbb{Z}$-covering of $Diff^+(S^1) \times Diff^+(S^1)$. One can define an action of $Conf(S^1 \times \mathbb{R})$ on $S_I$ by precomposition: Let $h \in Conf(S^1 \times \mathbb{R})$ and

$$S^h_I = \{ f \in S_I | h(S^1 \times \{0\}) \subseteq \text{Dom}(f) \}.$$  

For $f \in S_I$, one defines $h \cdot f$ by $f \circ h$. This right action is in general pseudo since the defined domain $S^h_I$ of $h$ may not be the whole $S_I$. Conjugated to $LT^*M$ by $\kappa_0$ and Legendre transformation, it then becomes a (pseudo-)action on $LT^*M$. Notice that this action leaves $X_H$ invariant; thus it commutes with the flow $\rho_t$ whenever defined. In fact, $\rho_t$ is the restriction of this action to the subgroup $\mathbb{R}$ of pure translations on $S^1 \times \mathbb{R}$ along the $t$-direction.

With these prerequisites, we now demonstrate the following well-known folklore (e.g. [Se3]).

**Proposition 1.1 [symplecticity].** The symplecticity of the $Conf(S^1 \times \mathbb{R})$-action on $LT^*M$ depends on its components as follows.

- $Conf^{(+,\uparrow)}(S^1 \times \mathbb{R}) :$ symplectic,
- $Conf^{-,\uparrow}(S^1 \times \mathbb{R}) :$ symplectic when $B = 0$,
- $Conf^{(\downarrow,\uparrow)}(S^1 \times \mathbb{R}) :$ anti-symplectic,
- $Conf^{(+,\downarrow)}(S^1 \times \mathbb{R}) :$ anti-symplectic when $B = 0$.

**Proof.** Assume first that $h \in Conf^{(+,\uparrow)}(S^1 \times \mathbb{R})$ and that $S^1 \times \{0\}$ and $h(S^1 \times \{0\})$ are disjoint, say, $h(S^1 \times \{0\})$ lies in the chronological future domain of $S^1 \times \{0\}$. Together they bound a compact annulus $\Sigma$ in $S^1 \times \mathbb{R}$. Let $I_\Sigma$ be the functional on $S_I$ defined by

$$I_\Sigma(f) = \int_\Sigma \left\{ \frac{1}{2} \text{tr}(f^*ds^2) * 1 + f^*B \right\},$$

where $\Sigma$ is endowed with the standard submanifold metric from $S^1 \times \mathbb{R}$. This functional depends only on the conformal class of the metric on $\Sigma$. Consider a new metric on $\Sigma$ that is conformal to the standard metric and satisfies the following two properties: (1) It is smoothly extendable to a metric on $S^1 \times \mathbb{R}$ that is conformal to the standard one; (2) it is the standard metric in a neighborhood of $S^1 \times \{0\}$ and coincides with the push-forward metric $h_*(dt^2 - d\sigma^2)$ in a neighborhood of
(\Omega^1 \times \{0\})$. Such a metric can be obtained by a partition of unity argument. Denote by $I^\ast_{\Sigma}$ the functional on $S_I$ associated to one such metric.

Let $U$ be a Jacobi field along $f \in S_I$ and $f_\varepsilon$ be a one-parameter family of elements in $S_I$ that gives rise to $U$ with $f_0 = f$. One has the following first variation formula

$$dI^\ast_{\Sigma}(U) = dI^\ast_{\Sigma}(U) = \frac{d}{d\varepsilon} I^\ast_{\Sigma}(f_\varepsilon)$$

$$= \int_{\Sigma} \left[ -\langle \nabla' (\nabla df) , U \rangle + f^* (i_U dB) \right] + \int_{\partial\Sigma} \left[ \langle f_* \nu', U \rangle - f^* (i_U B) \right],$$

where all quantities with "" are with respect to the new metric and $\nu'$ is the outward unit normal along $\partial\Sigma$ (with respect to the new metric). The contribution to the formula from the metric part has been well-known in the literature of harmonic maps ([E-L1]). That from the $B$-field results from the following manipulation. Define $F : \Sigma \times (-\varepsilon_0, \varepsilon_0) \to M$ by $F(\sigma, t, \varepsilon) = f_\varepsilon(\sigma, t)$. Then

$$\frac{d}{d\varepsilon} \int_{\Sigma} f^\ast_B = \int_{\Sigma \times \{\varepsilon\}} \mathcal{L}^B f^\ast_B = \int_{\Sigma \times \{\varepsilon\}} \left[ i_\nu^\ast dF^\ast_B + di^\ast_B F^\ast_B \right]$$

$$= \int_{\Sigma} f^\ast (i_U dB) + \int_{\partial\Sigma} f^\ast (i_U B) \quad \text{at} \quad \varepsilon = 0.$$  

Back to the first variation formula. Since $f \in S_I$, the integrand of the integral over $\Sigma$ vanishes. From the fact that, with respect to the new metric, the outward unit normal $\nu'$ for $\Sigma$ is $-\partial t$ at $S^1 \times \{0\}$ and $h_* \partial t$ at $h(S^1 \times \{0\})$, the boundary term can be expressed more explicitly as

$$\int_{S^1} \left[ (U|_{h(S^1 \times \{0\})}, (f \circ h)_* (\partial t|_{t=0})) + B \left( U|_{h(S^1 \times \{0\})}, (f \circ h)_* \partial \sigma \right) \right]$$

$$- \int_{S^1} \left[ (U|_{S^1 \times \{0\}}, f_* (\partial t|_{t=0})) + B \left( U|_{S^1 \times \{0\}}, f_* \partial \sigma \right) \right].$$

This indicates that, if we define a 1-form $\Xi_{\mathcal{L}}$ on $S_I$ by

$$\Xi_{\mathcal{L}}|_f (\cdot) = \int_{S^1} \left[ \langle (\cdot)|_{S^1 \times \{0\}}, f_* (\partial t|_{t=0}) \rangle + B \left( (\cdot)|_{S^1 \times \{0\}}, f_* (\partial \sigma|_{t=0}) \right) \right],$$

then $h$ as a map on $S_I$ by pre-composition satisfies

$$h^\ast \Xi_{\mathcal{L}} - \Xi_{\mathcal{L}} = dI^\ast_{\Sigma}$$

since $h_* U = U$. Thus the 2-form $d\Xi_{\mathcal{L}}$ on $S_I$ is invariant under $h$.

If $S^1 \times \{0\}$ and $h(S^1 \times \{0\})$ intersect, they together bound a $\Sigma = \cup_i \Omega_i$ with each $\Omega_i$ a lens domain in $S^1 \times \mathbb{R}$ bounded by two simple spacelike arcs. The first variation formula still holds since the non-smoothness of corners of $\Omega_i$ is insignificant.
under \( f \). And the rest hence follows as well. This shows that \( d\Xi_L \) is invariant under \( \text{Conf}^{(+,1)}(S^1 \times \mathbb{R}) \).

For the other three components of \( \text{Conf}(S^1 \times \mathbb{R}) \), they are the \( \text{Conf}^{(+,1)}(S^1 \times \mathbb{R}) \)-cosets of the following simple reflections:

- \( \text{Rx}_{(-,\uparrow)} : (\sigma, t) \mapsto (-\sigma, t) \),
- \( \text{Rx}_{(-,\downarrow)} : (\sigma, t) \mapsto (-\sigma, -t) \),
- \( \text{Rx}_{(+,\downarrow)} : (\sigma, t) \mapsto (\sigma, -t) \).

Denote \( \Xi_L = \Xi^m_L + \Xi^B_L \), where \( \Xi^m_L \) involves only the metric and \( \Xi^B_L \) only the \( B \)-field. It is clear that

- \( \text{Rx}_{(-,\uparrow)}^* \Xi_L = \Xi^m_L - \Xi^B_L \),
- \( \text{Rx}_{(-,\downarrow)}^* \Xi_L = -\Xi^m_L - \Xi^B_L \),
- \( \text{Rx}_{(+,\downarrow)}^* \Xi_L = -\Xi^m_L + \Xi^B_L \).

Thus they satisfy the symplecticity properties described in the proposition for the components of \( \text{Conf}(S^1 \times \mathbb{R}) \) in which they lie. Therefore so do these components.

On the other hand, the pullback 1-form \( \theta_L \) on \( LT^* M \) of \( \theta \) by the Legendre transformation is exactly \( (\kappa_0^{-1})^* \Xi_L \). Consequently, after conjugating all back to \( LT^* M \) via the Legendre transformation, one concludes that \( \text{Conf}(S^1 \times \mathbb{R}) \) acts on \( LT^* M \) as indicated. This proves the proposition.

\( \square \)

**Single-phase-space-description of interacting strings.**

An element of \( LT^* M \) can be regarded as a parametrized string in \( M \) with an infinitesimal intent of motion. Such objects can join or split in various ways (Figure 1-1) (cf. [St1] and [Wi1]). Formally, the joining operation "\( \cdot \)" induces a product \( \ast \) of string fields by convolution:

\[
\psi_1 \ast \psi_2(\gamma) = \int_{\gamma_1 = \gamma} [D\gamma_1] [D\gamma_2] \psi_1(\gamma_1) \psi_2(\gamma_2);
\]

and the splitting operation induces a coproduct by an inverse of convolution:

\[
\psi \mapsto \{\psi_1, \psi_2\} \quad \text{with either} \quad \psi_1 \ast \psi_2 = \psi \quad \text{or} \quad \psi_2 \ast \psi_1 = \psi.
\]

So far the coproduct as defined could be multi-valued. These operations describe the basic interaction of strings or string fields. It is possible to define an \( A_{\infty} \)-structure or a co-\( A_{\infty} \)-structure associated to many-to-one joinings or splittings ([St1-2], [Ko]).

The Hamiltonian system \( (L^* M, \omega, \mathcal{H}) \) together with these operations are in principle enough to give a description of interacting strings. At the classical level, a
Loops in $\mathcal{T}^*M$ can join or split in various ways. The upper ones involve loops in the s-boundary of $L\mathcal{T}^*M$; while the lower ones loops in $LT^*M$.

The process of interactions of strings corresponds to a partially ordered collection of integral filaments to $X_{\mathcal{H}}$ (Figure 1-2). At quantum level, it corresponds to a partially ordered collection of one-parameter families of string fields governed by the Schrödinger equation (cf. Appendix).

For string fields ([W-Z], [Zw]), one also likes to know what action governs their dynamics. The form of such actions for string fields are constrained among other things by conditions due to symmetry of the theory. In the above phase-space-picture of interacting strings, there doesn’t seem to have any limitation to the pattern of interaction of strings at a single instant, i.e. the type of string vertices. However, an action for string field should set a limitation to possible string vertices. For example, if interacting part of the action involves only $\psi \ast \psi \ast \psi$, then only simple trivalent string vertices could appear. There are details to be worked out to make this picture solid, which we are not ready. However, when turning to the Lagrangian counterpart of this picture, one is naturally led to the study of Lorentz surfaces.

### 2 Lorentz surfaces and their moduli.

A partially ordered collection of integral filaments can be realized as a map from a Lorentzian 2-manifold $\Sigma$ to the target-space $M$. Such $\Sigma$ has to admit metric singularities. Conformal symmetry in the theory indicates that only the conformal structure of $\Sigma$ matters. The whole setting thus leads to a study of to-be-defined Lorentz surfaces. This is a fundamental ingredient toward an un-Wick-rotated string theory. We like to mention the papers [De1-2] by T. Deck, which are likely a pi-
Figure 1-2. A partially ordered collection of integral filaments in \((LT^*M, X_H)\) represents a process of string interactions. The partially ordered labelling set indicates the events when strings join or split, i.e. the occurrence of loop operations. The dotted line \(\cdots\) in \(LT^*M\) indicates the Hamiltonian flow.

2.1 Lorentz surfaces and Mandelstam diagrams.

Analogous to a Riemann surface being a conformal class of Riemannian 2-manifolds, a Lorentz surface is meant to be a conformal class of Lorentzian 2-manifolds. However, while a Riemannian structure resides on every paracompact smooth manifold, a Lorentzian structure on a smooth manifold in general has to accommodate singularities where the metric is degenerate due to topological reasons.

Definition 2.1.1 [s-d-l Lorentzian 2-manifold]. A string-diagram-like (s-d-l) Lorentzian 2-manifold is a smooth 2-manifold of finite type with a smooth Lorentzian structure that satisfies the following conditions: (1) All of its singularities are isolated points and they lie in the interior of the manifold. (2) There is no trapped set in the interior of the manifold. (3) The complement of singularities is time-orientable and contains no causal loops. (4) Every boundary component is spacelike and every end is conformal to either \(S^1 \times (-\infty, 0]\) or \(S^1 \times [0, \infty)\).
The conditions listed in the definition reflect the assumptions for interacting strings:
(1) Points on a string are causally independent. (2) Interacting strings are both past and future asymptotically non-interacting. (3) Each interaction takes place only at an instant.

For simplicity of phrasing, in this article we shall mainly consider s-d-l Lorentzian 2-manifolds with ends unless otherwise noted. We shall call a such time-oriented $\Sigma$ of type $(+, \hat{\chi}, m, n)$ (resp. $(-, \hat{\chi}, m, n)$) if $\Sigma$ is orientable (resp. non-orientable) with $m$ past- and $n$ future-ends such that the Euler characteristic of the surface after all the ends are capped by a disk is $\hat{\chi}$.

**Singularities.**

Let $\Sigma$ be a time-oriented s-d-l Lorentzian 2-manifold. One can assign an index $\text{ind}_p(C)$ to the light-cone field $C$ at any $p$ in $\Sigma$ to be the index of any causal line field in a neighborhood of $p$. The Poincaré-Bendixson theorem implies that a generic singularity $p$ could only have index $-1$ in order not to violate either the time-orientability or no-trapped-set condition. A non-generic one arises from fusion of some $s$ generic ones and is of index $-s$. The light-cone structure around such is illustrated in Figure 2 - 1. In a neighborhood of such $p$, each of the causal future

\[ J^+(p) \] and causal past \[ J^-(p) \] of $p$ has $s+1$ components. Together they form a causal flower at $p$ with the petals alternating between future and past. Such singularity can be perturbed and disintegrated back into a collection of generic ones.
The coarse and fine conformal equivalences.

Unlike the case of Riemannian manifolds, singularities distinguish themselves in a Lorentzian manifold. In defining a conformal equivalence \( f \) between two generic Lorentzian manifolds, one has the following two choices: (1) The coarse sense: up to some mild extra demand, \( f \) is required to be a homeomorphism but is smooth only in the complement of singularities. (2) The fine sense: \( f \) is required to be a diffeomorphism.

For the coarse conformal Lorentzian geometry, one may as well enlarge the scope of the metric tensors under consideration to those that are smooth in the complement of their singularities and satisfy only some weaker conditions around the singularities - basically that an appropriate metric completion of the complement around a singularity should give back exactly that original singularity. Due to this looseness, the coarse conformal Lorentzian geometry is conceivably much less rigid than the fine conformal Lorentzian geometry. In the 2-dimensional case, associated to each singularity of the metric, there is a fine conformal invariant, i.e. the modulus of the smooth orbital equivalence class of the transverse pair of null line fields around that singularity. This may not be preserved under a coarse conformal deformation. Though both are interesting, details concerning the above moduli of singularities does not seem available at the moment. Fortunately, both our string action and its conformal invariance can be readily extended to Lorentzian 2-manifolds in the coarse category.

**Definition 2.1.2 [Lorentz surface]**. A coarse (resp. fine) Lorentz surface is a coarse (resp. fine) conformal equivalence class of coarse (resp. fine) Lorentzian 2-manifolds.

In this article we shall discuss only coarse s-d-l Lorentz surfaces unless otherwise noted.

**Mandelstam diagrams.**

Let \( \mathbb{R}^{1+1} \) be the standard 2-dimensional Minkowski space-time with metric \( d\sigma^+ \cdot d\sigma^- \), where \((\sigma^+, \sigma^-)\) are the light-cone coordinates of \( \mathbb{R}^{1+1} \). The global time function \( t \) is then given by \( \frac{1}{2}(\sigma^+ + \sigma^-) \). Notice that all infinite Minkowskian cylinders \( \mathbb{R}^{1+1}/\mathbb{Z}v \) with \( v \) spacelike are homothetic to each other. We shall fix \( v_0 = (2\pi, -2\pi) \) throughout the paper. The time function \( t \) on \( \mathbb{R}^{1+1} \) then descends to the standard cylinder \( S^1 \times \mathbb{R} = \mathbb{R}^{1+1}/\mathbb{Z}v_0 \). So do the two 1-forms \( d\sigma^+ \) and \( d\sigma^- \). A Mandelstam diagram is a coarse s-d-l Lorentzian 2-manifold \( \Xi \) that satisfies the following conditions: (1) \( \Xi \) admits an annuli decomposition \( \{ A_\alpha \} \) with each annulus \( A_\alpha \) homothetic via an \( f_\alpha \)
to one of the standard annuli: $S^1 \times \mathbb{R}$, $S^1 \times (\mathbb{R}, 0]$, $S^1 \times [0, \infty)$, and $S^1 \times [0, T]$ for some $T$. (2) When boundaries of these annuli are pasted in $\Xi$, the pulled-back local 1-forms $\{ f_\alpha^* d\sigma^+ \}$ and $\{ f_\alpha^* d\sigma^- \}$ on $\Xi$ can also be pasted. The result is a bi-valued 1-form $\mu$ on $\Xi$ singular only at the singularities of $\Xi$ (or better a bi-valued transverse measure to the null foliations of $\Xi$, cf. Sec. 2.2). For $\Xi$ orientable, $\mu$ splits into two single-valued 1-forms $\mu_L$, $\mu_R$ on $\Xi$. For $\Xi$ non-orientable, $\mu$ can be lifted to the orientation covering space $\Xi^{\text{orient}}$ and becomes single-valued. We shall call these 1-forms characteristic 1-forms on $\Xi$. (3) Up to constant shifts, one for each $A_\alpha$, the collection of local time functions $\{ f_\alpha^* t \}$ on $\Xi$ form a globally well-defined time function (still denoted by $t$) on $\Xi$.

2.2 Basic structures and coarse conformal groups.

The light-cone structure determines the conformal structure of a regular Lorentzian manifold. In two-dimensions it determines the coarse conformal structure of an s-d-l Lorentzian manifold and hence a coarse s-d-l Lorentz surface $\Sigma$. There are some basic structures on $\Sigma$ that follow from its light-cone structure. They play important roles in our study.

Basic structures for $\Sigma$ orientable.

Let $\Sigma$ be both oriented and time-oriented. One can then define a left (resp. right) null line element on $\Sigma$ as one whose future direction to the time-orientation is opposite to (resp. accordant with) the given orientation of the surface. They form the left null line field on $\Sigma$. Its integral trajectories give the left null foliations $F_L$ of $\Sigma$. It is a foliation with singularities the same as $\Sigma$. The time-orientation of $\Sigma$ gives a direction for leaves of $F_L$. Generically, they emit from past-ends of $\Sigma$ and go into future-ends. There are finitely many directed leaves that either emit from or land on some singularity. We shall call them characteristic leaves. The complement of these leaves and singularities is a collection of strips, which we shall call the left characteristic strips.

Let $\Sigma_0$ be the complement of singularities. Then the left leaf-space $\Sigma_0/ F_L$ with the quotient topology is a smooth non-Hausdorff 1-manifold whose non-Hausdorff points correspond exactly to the characteristic leaves of $F_L$. A smooth 1-form on $\Sigma_0/ F_L$ can be pulled back to a directed transverse measures on $\Sigma_0$ with respect to $F_L$. It has the property that the total measure along any small loop encircling a singularity is 0. Consequently, homologous 1-cycles on $\Sigma$ have the same transverse measures.
Analogously, one has the right null foliation $\mathcal{F}_R$, right characteristic strips, and the right leaf-space $\Sigma_0/\mathcal{F}_R$ for $\Sigma$. They share similar properties as their corresponding left partners. The two foliations $\mathcal{F}_L$ and $\mathcal{F}_R$ are transverse to each other.

For generic $\Sigma$, all its singularities are of index $-1$ and there are no characteristic leaves that connect any two of them. One can then identify to a point each collection of non-Hausdorff points in $\Sigma_0/\mathcal{F}_L$ associated to a singularity. The result is a regular 4-valence graph. It can be embedded as a graph $\Gamma_L$ in $\Sigma$ such that its vertex set coincides the set of singularities of $\Sigma$ and that its edges are transverse to $\mathcal{F}_L$. Any two different such embeddings are isotopic in $\Sigma$ relative to the vertex set and $\Gamma_L$ is a deformation retract of $\Sigma$ along the leaves of $\mathcal{F}_L$. For non-generic $\Sigma$, $\Sigma$ can be obtained after squashing a finite collection of characteristic strips of some generic $\Sigma^\parallel$ along their partner foliation. The $\Gamma_L^\parallel$ in $\Sigma^\parallel$ then descends to a $\Gamma_L$ in $\Sigma$ following a corresponding sequence of either null-operation (which preserves topology of graphs) if squashing a left strip or pinching of some part of an edge if squashing a right strip. Different $\Sigma^\parallel$’s that lead to same $\Sigma$ could however give different $\Gamma_L$. In both cases, the complement of $\Gamma_L$ consists of topological cylinders, one for each end of $\Sigma$. Similarly, one has also $\Gamma_R$ with the same properties. (FIGURE 2 - 2.)

FIGURE 2 - 2. Non-generic $\Sigma$ can be obtained from a generic one $\Sigma^\parallel$ by a finite sequence of squashing. For clarity, the strip to be squashed at each step is shadowed.

The time-orientation on $\Sigma$ as a timelike vector field on $\Sigma_0$ is transverse to both $\mathcal{F}_L$ and $\mathcal{F}_R$. Hence it induces an orientation for $\Sigma_0/\mathcal{F}_L$ and $\Sigma_0/\mathcal{F}_R$. This in turn
leads to an orientation for $\Gamma_L$ and $\Gamma_R$. One can thus define positive 1-forms on $\Sigma_0/F_L$ and $\Sigma_0/F_R$ to be those whose evaluation along the orientation is positive.

The union of the left and right set of characteristic leaves together bind $\Sigma$. Its complement is a disjoint union of light-cone-diamonds, each of the form $I^+(q_1) \cap I^-(q_2)$ for some chronological pair of points $(q_1, q_2)$. This gives the (characteristic) light-cone-diamond (l-c-d) tessellation of $\Sigma$ (Figures 2 - 2, 2 - 3, 3 - 4, and 4 - 3).

The union of all left characteristic leaves and strips that come from the same past-end will be called a past-left crown of $\Sigma$ (Figure 2 - 3). One can border it by the left characteristic leaves that lie in the closure of the strips involved. Similarly, one has past-right, future-left, and future-right crowns.

![Figure 2 - 3. The left characteristic leaves and strips of $\Sigma$ and the past-left crown associated to one of its past-ends.](image)

Basic structures for $\Sigma$ non-orientable.

When $\Sigma$ is non-orientable, the above pairs of structures can be associated to its orientation covering space $\Sigma^{\text{ornt}}$ with the lifted Lorentzian conformal structure. The latter is also s-d-l. The non-trivial deck transformation on $\Sigma^{\text{ornt}}$ is an involution on $\Sigma$ that preserves time-orientation while exchanging left and right. Any left structure for $\Sigma^{\text{ornt}}$ and its right partner are isomorphic to each other under this involution. A characteristic leaf to the light-cone field on $\Sigma$ is defined to be the projection of one on $\Sigma^{\text{ornt}}$. Similarly for a characteristic strip. They are embedded in $\Sigma$. Either foliation, $F_L$ or $F_R$, on $\Sigma^{\text{ornt}}$ projects to the locally transverse bi-foliation on $\Sigma$ associated to the light-cone field. We shall take the leaf space $\Sigma_0^{\text{ornt}}/\mathcal{F}$, where $\mathcal{F}$ is either $F_L$ or $F_R$, as the leaf space associated to $\Sigma$. Recall that it is oriented. As in the orientable case, non-generic $\Sigma$ can be obtained from generic one by squashing. The
graph $\Gamma(\Sigma)$ is taken to be either $\Gamma_L(\Sigma)_{\text{ornt}}$ or $\Gamma_R(\Sigma)_{\text{ornt}}$. When $\Sigma$ is non-generic, it is understood that one considers squashings invariant under the deck transformation. The collection of characteristic leaves on $\Sigma$ give likewise the l-c-d tessellation of $\Sigma$.

Remark 2.2.1. The l-c-d tessellation resembles a circle packing on a Riemann surface ([R-S], [Thu]). While the latter is only an approximate conformal structure to a Riemann surface, the former is intrinsic to and completely determines the coarse conformal structure of a Lorentz surface.

Coarse conformal groups.

The following discussion refines and recasts Theorems 2.2 and 2.3 in [De] to the current setting.

Let $\Sigma$ be a time-oriented s-d-l Lorentz surface and $Conf^{(c)}(\Sigma)$ be the group of coarse conformal automorphisms of $\Sigma$. Assume first that $\Sigma$ is oriented. Let $\text{Diff}^\pm \left( \Sigma_0/\mathcal{F}_L \amalg \Sigma_0/\mathcal{F}_R \right)$ be the group of diffeomorphisms of the disjoint union $\Sigma_0/\mathcal{F}_L \amalg \Sigma_0/\mathcal{F}_R$ that are either orientation-preserving or orientation-reversing and

$$\text{Diff}_0 \left( \Sigma_0/\mathcal{F}_L \amalg \Sigma_0/\mathcal{F}_R \right) = \text{Diff}_0(\Sigma_0/\mathcal{F}_L) \times \text{Diff}_0(\Sigma_0/\mathcal{F}_R)$$

be its identity component. Let $\text{Aut}(\text{Tessln}_\Sigma)$ be the group of isotopy classes of (topological) automorphisms of $\Sigma$ that preserve the l-c-d tessellation by sending tiles only to tiles. Since an automorphism of surface $\Sigma$ is coarse conformal if and only if it preserves $\{\mathcal{F}_L, \mathcal{F}_R\}$, hence their characteristic leaves, and is smooth outside singularities, there are two group homomorphisms

$$\varphi_1 : Conf^{(c)}(\Sigma) \longrightarrow \text{Diff}^\pm \left( \Sigma_0/\mathcal{F}_L \amalg \Sigma_0/\mathcal{F}_R \right)$$

and

$$\varphi_2 : Conf^{(c)}(\Sigma) \longrightarrow \text{Aut}(\text{Tessln}_\Sigma).$$

When $\Sigma$ is topologically an infinite cylinder, it is represented by the standard $S^1 \times \mathbb{R}$ and hence $Conf^{(c)}(\Sigma)$ is $Conf(S^1 \times \mathbb{R})$. In this case, $\text{Diff}^\pm \left( \Sigma_0/\mathcal{F}_L \amalg \Sigma_0/\mathcal{F}_R \right) = \text{Diff}^\pm (S^1 \amalg S^1)$ has four components: $\text{Diff}^+(S^1) \times \text{Diff}^+(S^1)$ (resp. $\text{Diff}^-(S^1) \times \text{Diff}^-(S^1)$) for orientation-preserving (resp. -reversing) diffeomorphisms with each $S^1$ mapped to itself; and $\text{Diff}^{++}(S^1 \amalg S^1)$ (resp. $\text{Diff}^{--}(S^1 \amalg S^1)$) for orientation-preserving (resp. -reversing) diffeomorphisms with the two $S^1$ exchanged. And

$$\varphi_1 :$

- $Conf^{(++,+)}(S^1 \times \mathbb{R}) \longrightarrow \text{Diff}^+(S^1) \times \text{Diff}^+(S^1)$
- $Conf^{(--)}(S^1 \times \mathbb{R}) \longrightarrow \text{Diff}^-(S^1) \times \text{Diff}^-(S^1)$
- $Conf^{(+-)}(S^1 \times \mathbb{R}) \longrightarrow \text{Diff}^{++}(S^1 \amalg S^1)$
- $Conf^{(-+)}(S^1 \times \mathbb{R}) \longrightarrow \text{Diff}^{--}(S^1 \amalg S^1)$
is a non-trivial \( \mathbb{Z} \)-covering. On the other hand, no canonical l-c-d tessellations exist for \( S^1 \times \mathbb{R} \) and \( \varphi_2 \) is vacuous.

For all other topologies, while \( \varphi_1 \) may not be surjective (this can be seen particularly using the concept of ”grafting” (cf. Sec. 2.3)), \( \varphi_2 \) is always surjective since any surface automorphism of \( \Sigma \) that preserves the l-c-d tessellation can be isotoped into a coarse conformal one. Its kernel is isomorphic to \( \text{Diff}_0(\Sigma_0/\mathcal{F}_L) \times \text{Diff}_0(\Sigma_0/\mathcal{F}_R) \) via \( \varphi_1 \). It is contractable and hence is the identity component of \( \text{Conf}^{(c)}(\Sigma) \).

For \( \Sigma \) non-orientable, one has likewise the surjective group homomorphism \( \varphi_2 \)

\[
\varphi_2 : \text{Conf}^{(c)}(\Sigma) \longrightarrow \text{Aut}(\text{Tessln}_\Sigma).
\]

Its kernel is isomorphic to \( \text{Diff}_0(\Sigma^\text{ornt}_0/\mathcal{F}) \) and is the identity component of \( \text{Conf}^{(c)}(\Sigma) \).

Remark 2.2.2. For \( \Sigma \) not a cylinder, \( \text{Aut}(\text{Tessln}_\Sigma) \) is finite. (In fact, since for \([f]\) in \( \text{Aut}(\text{Tessln}_\Sigma) \), \( f \) takes a singularity to a singularity and its restriction to a tile determines the whole isotopy class \([f]\) by tilewise continuation, the order of \( \text{Aut}(\text{Tessln}_\Sigma) \) is bounded by \( 2 \cdot (-8\chi(\Sigma)) \), where \(-8\chi(\Sigma)\) is the bound counted with multiplicity for the number of tiles that are adjacent to some singularity and factor 2 is due to a possible flip under \( f \)). Comparing the theory of Riemann surfaces, it is instructive to regard \( \text{Aut}(\text{Tessln}_\Sigma) \) as the true symmetry group of a Lorentz surface. The infinite dimensional identity component \( \text{Diff}_0(\Sigma_0/\mathcal{F}_L) \times \text{Diff}_0(\Sigma_0/\mathcal{F}_R) \) or \( \text{Diff}(\Sigma^\text{ornt}_0/\mathcal{F}) \) of \( \text{Conf}^{(c)}(\Sigma) \) reflects the local non-rigidity of a Lorentzian conformal map in two dimensions. This is contrasted by the local rigidity of a holomorphic map in the Riemannian case. Since \( \text{Aut}(\text{Tessln}_\Sigma) \) is finite, elements in \( \text{Conf}^{(c)}(\Sigma) \) are of periodic type in the Nielsen-Thurston’s classification of surface automorphisms. This is a parallel to the fact that automorphisms of Riemann surfaces of negative Euler characteristic are also of periodic type and they form a finite group.

Remark 2.2.3. The algebra \( \text{Vect}_\mathbb{C}(S^1) \) of complex-valued smooth vector fields on the circle and its central extensions, Virasoro algebras, have been of importance to string theory. Their generalizations to a Riemann surface have been studied and led to some algebras of Virasoro type ([K-N1 - 3], [J-K-L], [M-N-Z]). In the current Lorentzian setting, one may regard \( \text{Conf}^{(c)}(\Sigma) \) as a generalization of \( \text{Diff}(S^1) \) and central extensions of its complexified Lie algebra as other algebras of Virasoro type. Assume \( \Sigma \) has \( m \) past- and \( n \) future-ends. By restricting automorphisms to the ends of \( \Sigma \), one has the following double inclusion:

\[
\text{Conf}^{(c)}(\Sigma) \supseteq \Pi_m \text{Conf}(S^1 \times \mathbb{R}) \quad \Pi_n \text{Conf}(S^1 \times \mathbb{R}) \supseteq \text{Conf}^{(c)}(\Sigma).
\]

A representation of such diagrams or their central extensions into the category of Hilbert spaces should give a picture of how a \( \Sigma \) selects incoming past states, how it
transmutes them, and then produces outgoing future states. Unfortunately, we are still far from realizing this goal.

2.3 Rompers decompositions and coarse moduli spaces.

Pants decompositions have played important roles in understanding Riemann surfaces. We shall now discuss their Lorentzian analogue and use it to understand the coarse moduli spaces of Lorentz surfaces.

Rompers decompositions of an s-d-l Lorentz surface.

A loop $C$ in an s-d-l Lorentz surface $\Sigma$ is called achronal if $I^+(C) \cap C$ is empty, in other words if there are no two points of $C$ with timelike separation ([H-E]) (Figure 2-4). A loop in $\Sigma$ is called peripheral if it can be homotoped into either a boundary component or an end of $\Sigma$; otherwise it is called non-peripheral.

(a) achronal

(b) non-achronal

Figure 2-4. Achronal and non-achronal spacelike simple loops in an s-d-l Lorentz surface $\Sigma$. Notice that a non-achronal one could travel in a complicated way in $\Sigma$.

Lemma 2.3.1 [simplest loop]. Let $\Sigma$ be an s-d-l Lorentz surface that has more than one singularity. Then there exists an achronal spacelike simple loop $C$ in $\Sigma$ that is non-peripheral. Furthermore, there are only finitely many free homotopy classes of such loops.

We shall call a non-peripheral achronal spacelike simple loop in $\Sigma$ a simplest loop.
Proof. Since there are only finitely many singularities in $\Sigma$, the no-causal-loop condition implies that there exists a singularity $p_0$ whose causal past $J^-(p_0)$ contains no singularities. Let $N$ be a closed neighborhood of $J^-(p_0)$ in the form of a submanifold-with-boundary in $\Sigma$ that contains only the singularity $p_0$ and is deformation retractable to $J^-(p_0)$.

Since both $N$ and $\Sigma - N$ contain a singularity, the future boundary $\partial^+N$ of $N$ contains at least a simple loop $C_0$ that is non-peripheral. As a boundary of a submanifold in $\Sigma$, $C_0$ has an orientable neighborhood. Since $N$ is deformation retractable to $J^-(p_0)$, the complement $N - J^-(p_0)$ is a collection of annuli $A_i$. The completion $\overline{A}_i$ of $A_i$ (with respect to the topology of $\Sigma$) contains as the past boundary component a broken null loop alternating between future and past directions. This is the boundary shared with $J^-(p_0)$. Such a boundary loop can be isotoped into a spacelike loop in $A_i$. The latter in turn is isotopic to the future boundary component of $A_i$ that serves also as a future boundary component of $N$. Applying this to the $A_i$ that has $C_0$ as the future boundary, one then obtains a simple spacelike loop $C$. Since $C$ comes from perturbing a broken null loop in $\partial J^-(p_0)$, which has to be achronal, $C$ itself has empty $I^-(C) \cap C$. This shows that $C$ is achronal.

To see that there are only finitely many free homotopy classes of such $C$, first notice that any achronal spacelike loop that crosses a tile in the l-c-d tessellation of $\Sigma$ that lies far enough in an end has to be peripheral. Consequently, there are only finitely many tiles that can admit some non-peripheral achronal spacelike simple loops to cross them. Up to free homotopy, we may assume that all the loops considered do not hit the corners of any tiles. If $C$ is achronal, then it can pass through a tile at most once. Any two such loops with the same pattern of crossing the edges of tiles are free homotopic; and there are only finitely many such patterns since there are only finitely admissible tiles.

This completes the proof. \[\square\]

Notice that a spacelike simple loop $C$ in an s-d-l $\Sigma$ cannot be null-homotopic. Its tubular neighborhood is always a cylinder; and the complement $\Sigma - C$ remains s-d-l (after appropriately bordered). By applying the above lemma finitely many times with slight modification in the proof to take $\Sigma$ with spacelike boundary also into account, one then has

**Corollary 2.3.2 [simple cut system].** Let $\Sigma$ be an s-d-l Lorentz surface that has more than one singularity. Then there exists a system of achronal spacelike simple loops $C_\alpha$ such that the complement $\Sigma - \{C_\alpha\}_\alpha$ is a collection of s-d-l Lorentz surfaces that have exactly one singularity.
We shall call such \( \{C_\alpha\}_\alpha \) a \textit{simple cut system} of \( \Sigma \). Two such systems are \textit{equivalent} if they differ by a homotopy of loops. The following lemma characterizes the component of its complement.

\textbf{Lemma 2.3.3 [s-d-l with one singularity].} \textit{Let} \( \Sigma \) \textit{be an s-d-l Lorentz surface (without boundary) that has only one singularity. Then} \( \Sigma \) \textit{is topologically a sphere with some} \( m + n \geq 3 \) \textit{punctures, where} \( m \) \textit{and} \( n \) \textit{are the number of past- and future-ends respectively. All positive integer pairs} \((m, n)\) \textit{with} \( m + n \geq 3 \) \textit{are admissible; and, up to coarse conformal equivalences, there are only finitely many such} \( \Sigma \) \textit{for each admissible} \((m, n)\).

\textit{Proof.} Let \( p \) \textit{be the singularity. Let us first show that} \( \Sigma \) \textit{must be orientable. Since there is only one singularity, no-causal-loop condition implies that every characteristic strip} \( \Omega \) \textit{of} \( \Sigma \) \textit{must be of the either form in} \textbf{Figure 2 - 5} (a). The singularity \( p \) \textit{as appears exactly once on each side of} \( \Omega \) \textit{must be of spacelike separation in} \( \Omega \). \textit{On the other hand, every loop at} \( p \) \textit{can be deformed into a product of loops at} \( p \) \textit{that lie completely in some characteristic strips. These two observations imply that one can choose a set of generators for} \( \pi_1(\Sigma, p) \) \textit{that consists only of spacelike loops. Such loops must have orientable tubular neighborhood. Consequently,} \( \Sigma \) \textit{is orientable.}

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure2-5.png}
\caption{Admissible characteristic strips on a Lorentz surface with only one singularity.}
\end{figure}

\textit{Let} \( \Sigma \) \textit{now be oriented. Then one may reconstruct it from, say, its past-left crowns. Since} \( \Sigma \) \textit{has only one singularity, each of the crowns with their own characteristic leaves of the restricted Lorentz structure is as indicated in} \textbf{Figure 2 - 6}. \textit{To get back} \( \Sigma \), \textit{one has to identify some future bordering characteristic leaf labelled by} \( A \) \textit{to another labelled by} \( B \). \textit{There are only finitely many such pairings. Each pairing determines the topology of the resulting 2-complex with a tessellation by extending the characteristic leaves following the causal structure of the crowns. For}
a pairing that gives a connected manifold structure, the resulting topology has to be a punctured sphere. The existence of singularity $p$ implies that the total number of ends has to be at least three. The coarse conformal structure is also determined by the pairing since different ways of pasting along the same paired characteristic leaves will give the same l-c-d tessellation pattern. Figure 2-7 indicates how one can construct such Lorentz surface with arbitrary $m$ past- and $n$ future-ended as long as $m + n \geq 3$.

This completes the proof.

\[
\begin{array}{c}
\text{Figure 2-6. A bordered past-left crown that appears in an s-d-l Lorentz surface with one singularity. For clarity, it is drawn as a planar domain.}
\end{array}
\]

Due to its topology, we shall call an s-d-l Lorentz surface with one singularity a set of rompers and $(m, n)$ its type. Each component of the complement of a simple cut system of $\Sigma$ is a set of rompers with ends truncated. And we shall call such decomposition a rompers decomposition. Notice that since there are only finitely many achronal spacelike simple loops up to homotopy, there are only finitely many non-equivalent simple cut systems for $\Sigma$.

Rompers, and hence all Lorentz surfaces, admit foliations whose generic leaves are simplest loops. By pinching these leaves, one then obtains a directed network $\text{Net}(\Sigma, \{C_\alpha\}_\alpha)$ associated to a rompers decomposition of $\Sigma$. We shall also called it a sewing-diagram (Figure 2-9).

Graftings and coarse moduli spaces.
By cut-and-paste with $S^1 \times \mathbb{R}$ along a timelike ray at the singularity, one can obtain an s-d-l Lorentz surface with one singularity that has any $m$ past- and $n$ future-ends for $m + n \geq 3$. These surfaces are called rompers.

**Definition 2.3.4 [grafting].** Let $\Sigma$ be an s-d-l Lorentz surface and $C$ be a simplest loop in $\Sigma$. Recall that its tubular neighborhood is a cylinder. Define a *grafting* of $\Sigma$ along $C$ of step $k$, $k \in \mathbb{N}$, by cutting $\Sigma$ along $C$ and then pasting to it conformally the standard Minkowskian cylinder $S^1 \times [0, 2\pi k]$, as illustrated in **Figure 2-8**. This leads to a new s-d-l Lorentz surface, denoted by $(\Sigma, C; k)$. The reverse of this shall be called an *excision* of step $k$.

Notice that a grafting on $\Sigma$ determines uniquely a conformal structure on the new surface $\Sigma'$. The effect of grafting along $C$ on $\Gamma_L$ and $\Gamma_R$ (or, with suitable mod-
ification, $\Gamma(\Sigma)$ if $\Sigma$ non-orientable) is to make their edges that cross $C$ wind more; but it leaves their topologies unchanged. The identity component of $Conf^{(c)}(\Sigma')$ is canonically isomorphic to that of $Conf^{(c)}(\Sigma)$ by the restriction map from $\Sigma'$ to $\Sigma$.

**Remark 2.3.5.** We define graftings only for $k \in \mathbb{N}$ so that the l-c-d tessellation remains the same on the original part of the surface. It should be noted that a grafting for $k \in \mathbb{R}_+$ is also well-defined in exactly the same way.

Let $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$ be the moduli space of s-d-l (coarse) Lorentz surfaces of type $(\pm, \hat{\chi}, m, n)$. It is a discrete set. Using grafting, one can define a relation $\prec$ on $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$ by setting $\Sigma \prec \Sigma'$ if $\Sigma'$ can be obtained by a finite sequence of graftings along simplest loops beginning with $\Sigma$. (We shall say that "$\Sigma$ precedes $\Sigma'$" or that "$\Sigma'$ follows $\Sigma$".) This defines a partial ordering on $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$. We shall call a minimal element relative to $\prec$ a primitive Lorentz surface.

**Proposition 2.3.6 [primitive finite].** There are only finitely many minimal elements in $\left(\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n), \prec\right)$.

**Proof.** Let $\Sigma$ be a Lorentz surface in $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$. Then any of the components of its rompers decompositions must have Euler characteristic negative but greater than $\hat{\chi} - (m + n)$. There are only finitely many of them. Up to grafting and excision, their different ways of pasting will lead only to finitely many different l-c-d tessellation patterns on Lorentz surfaces of type $(\pm, \hat{\chi}, m, n)$. This proves the proposition. \qed

To capture its geometry, it is instructive to define a directed graph structure on $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$. The vertex set is the moduli space itself. $\Sigma_1$ and $\Sigma_2$ is connected by a directed edge $(\Sigma_1, \Sigma_2)$ if $\Sigma_2$ is obtained from $\Sigma_1$ by a grafting of step 1 along a simplest loop. This is an infinite graph with finitely many source but no sink vertices. Its valence at a vertex $\Sigma$ is bounded by twice the number of homotopy classes of simplest loops in $\Sigma$. In general, points in $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$ are not related just by graftings and excisions; and hence this graph can have several components. Roughly, the set of rays in the directed graph gives a compactification of the moduli space. The geometry associated to an added point is an s-d-l Lorentz surface obtained from a $\Sigma$ in $\mathcal{M}^{(c)}_{Lorz}(\pm, \hat{\chi}, m, n)$ by cutting along some collection of simplest loops and then attach an $S^1 \times [0, \infty)$ or $S^1 \times (-\infty, 0]$ to each pair of the newly created boundary components (FIGURE 2-10).

**Remark 2.3.7.** Technically, a ray can have more than one limit geometry; hence the above prescription of compactification needs to be refined. The fine moduli space
$\mathcal{M}^{(f)}_{\text{Lorz}}(\pm, \check{\chi}, m, n)$ is stratified by the l-c-d tessellation patterns of Lorentz surfaces of type $(\pm, \check{\chi}, m, n)$; since in addition graftings can be defined for all $\mathbb{R}_+$ (cf. Remark 2.3.5), one conceives that the directed graph for $\mathcal{M}^{(c)}_{\text{Lorz}}(\pm, \check{\chi}, m, n)$ can be embedded in $\mathcal{M}^{(f)}_{\text{Lorz}}(\pm, \check{\chi}, m, n)$ uniformly, picking one point for the image of a vertex in each stratum.

Remark 2.3.8. When compared with the Fenchel-Nielsen coordinates for the Teichmüller and moduli spaces of Riemann surfaces, the set of primitive elements in $\mathcal{M}^{(c)}_{\text{Lorz}}(\pm, \check{\chi}, m, n)$ is a parallel to a coordinate-plane for the parameters of twisting angles, while the step of graftings is a parallel to the inverse of the length parameters.

Sketch of examples.

To illustrate the ideas, let us give a brief sketch of how to build the coarse moduli spaces $\mathcal{M}^{(c)}_{\text{Lorz}}(\pm, -2, 1, 1)$ and $\mathcal{M}^{(c)}_{\text{Lorz}}(-, -2, 1, 1)$. For $\Sigma$ of type $(\pm, -2, 1, 1)$, by considering first pants decompositions and then their degenerates, one finds that the rompers that build $\Sigma$ can only be of types $(1, 3), (1, 2), (2, 2), (2, 1),$ and $(3, 1)$. They together create seven admissible sewing-diagrams (FIGURE 2-9).

![Admissible rompers and sewing-diagrams](image)

**FIGURE 2-9** Admissible rompers and sewing-diagrams for an s-d-l Lorentz surface of type $(\pm, -2, 1, 1)$. Characteristic leaves in each rompers and degeneracy relations between diagrams are also indicated.

For these simpler types, each $(m, n)$ gives only one set of rompers. We will think of them as Mandelstam diagrams (cf. Sec. 3.1 and Remark 3.1.3). To build
a primitive element, we truncate these Mandelstam diagrams by equal-time curves so that, after truncation, any collar of the boundary that is singularity-free and has spacelike boundary contains no larger cylinders than $S^1 \times [0, 2\pi)$. Twists (and also flips for $M^{(c)}_{\text{Lorz}}(-, -2, 1, 1)$) when sewing truncated rompers following an admissible sewing-diagram will create a collection of Lorentz surfaces of type $(\pm, -2, 1, 1)$. After sewn, the characteristic leaves on each set of rompers extend to those on the whole surface by following null leaves in other rompers. After excisions of step at most 1 along sewing loops, one obtains all the primitive elements. Together with the homotopy classes of simplest loops thereon, one can then build the whole moduli space. Each simple cut system (or, equivalently, sewing-pattern) gives a Fenchel-Nielsen type coordinate chart on the moduli space by the step number of graftings. These charts look like a collection of lattice cones and they overlap at most marginally at where some grafting step remains small. (Figure 2-10.)

We omit the details here since it is tedious. We would like to know if there are more efficient ways to understand these moduli spaces.

Though the discussion here on the moduli spaces of Lorentz surfaces is far from enough nor complete, we shall be contented to stop here. These moduli spaces and their asymptotic behaviors when $(\hat{\chi}, m, n)$ get large should be important for string theory.

3 Rectifiability into Mandelstam diagrams.

Mandelstam diagrams are related to both Hamiltonian and light-cone string theory. In the Riemannian case, it is known that a Riemannian 2-manifold with at least two punctures is coarse conformal to a Euclidean Mandelstam diagram [G-W]. In contrast to this, we shall show in this section that the analogue does not hold for the Lorentzian case.

3.1 Branched coverings, positive cones, and rectifiability.

An s-d-l Lorentz surface $\Sigma$ is called rectifiable if it has a Mandelstam diagram $\Xi$ as a representative. The following proposition characterizes rectifiability.

**Proposition 3.1.1 [rectifiability].** Let $\Sigma$ be an s-d-l Lorentz surface. Then (1), for $\Sigma$ orientable, $\Sigma$ is rectifiable if and only if $\Sigma$ is a coarse-conformal branched covering over a Minkowskian cylinder; (2), for $\Sigma$ non-orientable, $\Sigma$ is rectifiable if and only if its orientation covering $\Sigma^{\text{ornt}}$ with the lifted coarse conformal structure is rectifiable.
Figure 2-10. Directed graph gives a way to describe the geometry of $\mathcal{M}_{\text{Lorz}}^{(c)}(\pm, \hat{\chi}, m, n)$ and its compactification. Only some components of $\mathcal{M}_{\text{Lorz}}^{(c)}(+, -2, 1, 1)$ are shown sketchily with the Fenchel-Nielsen type of coordinate charts. Due to symmetry, there can be more redundancy of the coordinates, e.g. $01010 = 10001$, $01110 = 10101$, etc. Two examples of the changes of the geometry along a ray and their limit are also indicated.
Remark 3.1.2. When $\Sigma$ is orientable and rectifiable, coarse conformalness implies that the branched points over the cylinder are exactly the singularities of $\Sigma$.

Proof of Proposition 3.1.1. We shall assume that $\chi(\Sigma) < 0$ since otherwise $\Sigma$ is an s-d-l cylinder and the proposition is readily true. Recall that the standard cylinder $S^1 \times \mathbb{R}$ is given by $\mathbb{R}^{1+1}/\mathbb{Z}v_0$ with $v_0 = (2\pi, -2\pi)$ in the light-cone coordinates.

Assume first that $\Sigma$ is generic and oriented. Recall that a positive 1-form $\mu_L$ on $\Sigma$ and a positive 1-form $\mu_R$ on $\Sigma$ together give a pair $(\mu_L, \mu_R)$ of measures on $\Sigma$ that are transverse to each other. It defines a map $\text{Hol}_0$ from the based path space $\text{Path}(\Sigma, q_0)$ to $\mathbb{R}^{1+1}$ by

$$\text{Hol}_0(\gamma) = (\int_\gamma \mu_L, \int_\gamma \mu_R).$$

Due to homotopy invariance, $\text{Hol}_0$ descends to a map $\text{Hol}_1$ from the universal covering $\tilde{\Sigma}$ of $\Sigma$ to $\mathbb{R}^{1+1}$. It descends further to a branched covering map from $\Sigma$ to the cylinder $\mathbb{R}^{1+1}/\mathbb{Z}v_0$ if and only if

$$(*) \text{ [covering]} \quad \text{Hol}_0(\gamma) \in \mathbb{Z} \cdot v_0, \text{ for all } \gamma \in \pi_1(\Sigma, q_0).$$

This condition contains two parts:

$$(*)_1 \text{ [slope]} \quad \frac{\int_\gamma \mu_R}{\int_\gamma \mu_L} = -1, \text{ for all } \gamma \in \pi_1(\Sigma, q_0);$$

and

$$(*)_2 \text{ [integral]} \quad \int_\gamma \mu_L \in 2\pi\mathbb{Z}, \text{ for all } \gamma \in \pi_1(\Sigma, q_0).$$

Observe that $(\Sigma, \mu_L \cdot \mu_R)$, where $\cdot$ is the symmetric product, is a representative of $\Sigma$. The slope condition $(*)_1$ means that the 1-form $\mu_L + \mu_R$ is exact. Its integral $t$ is then a global time function on $\Sigma$ whose level curves are transverse to both leaves of $\mathcal{F}_L$ and $\mathcal{F}_R$. The equal time trajectories through singularities thus give an annuli decomposition of $\Sigma$ and the $\text{Hol}_0$-type maps on these annuli send them to standard cylinders in a way that meets the requirements for a Mandelstam diagram. This shows that $\Sigma$ is rectifiable if and only if there exists a positive $(\mu_L, \mu_R)$ that satisfies Cond. $(*)_1$.

Let $\{E^L_i\}_{i=1}^k$, $\{E^R_i\}_{i=1}^k$ be the set of directed edges of $\Gamma_L$, $\Gamma_R$ respectively. Let $\{\gamma\}_{i=1}^{k_0}$ be a set of generators of $\pi_1(\Sigma, q_0)$. One has $k = -2\chi(\Sigma)$ and $k_0 = -\chi(\Sigma) + 1$. Cond. $(*)_1$ for a positive pair $(\mu_L, \mu_R)$ is equivalent to the existence of a positive solution to the following system of linear equations

$$\sum_{j=1}^k a^R_{ij} x^R_j = -\sum_{j=1}^k a^L_{ij} x^L_j \quad \text{for} \quad i = 1, \ldots, k_0,$$
where $a^R_{ij}$ (resp. $a^L_{ij}$) is the multiplicity of $\gamma_i$ with respect to $E^R_j$ (resp. $E^L_j$) defined as the signed number of times that $\gamma_i$ would go over $E^R_j$ (resp. $E^L_j$) under the deformation retract that takes $\Sigma$ to $\Gamma^R$ (resp. $\Gamma^L$) and the unknowns $x^R_j$ (resp. $x^L_j$) are the prospect values for $\int_{E^R_j} \mu_R$ (resp. $\int_{E^L_j} \mu_L$). Due to homogeneity of the system and integralness of its coefficients, the set $Sol^+$ of positive solutions to the system, if not empty, is a cone in the $2k-k_0$ dimensional solution space of the linear system in $\mathbb{R}^k \oplus \mathbb{R}^k$ with non-empty $Sol^+ \cap ((2\pi \mathbb{Z})^k \oplus (2\pi \mathbb{Z})^k)$. From this intersection, one obtains $(\mu_L, \mu_R)$ that satisfies both Cond. $(\ast_1)$ and Cond. $(\ast_2)$; and hence Cond. $(\ast)$. This shows that the existence of a positive $(\mu_L, \mu_R)$ that satisfies the covering condition $(\ast)$ is equivalent to the existence of a positive $(\mu_L, \mu_R)$ that satisfies the slope condition $(\ast_1)$. This concludes the proof of Part (1) for $\Sigma$ generic.

When $\Sigma$ is non-generic, recall that it can be obtained from a generic $\Sigma^\parallel$ by squashing strips along transverse foliation. Applying the above argument to $\Sigma^\parallel$ with the modification that allows non-negative solutions to the linear system, whose zero components correspond to the squashed strips, one obtains the same result for $\Sigma$. This completes the proof of Part (1).

For $\Sigma$ non-orientable, since the orientation covering $\Xi^{\text{ornt}}$ of a Mandelstam diagram $\Xi$ with the lifted structure is also a Mandelstam diagram, that $\Sigma$ is rectifiable implies that $\Sigma^{\text{ornt}}$ is also rectifiable. Assume now the converse that $\Sigma^{\text{ornt}}$ is rectifiable to a Mandelstam diagram $\Xi'$. Let $f$ from $\Xi'$ to itself be the involution that comes from the non-trivial deck transformation on $\Sigma^{\text{ornt}}$. Since the Lorentz structure on $\Sigma$ is simply the average of that in $\Sigma^{\text{ornt}}$ with respect to the covering projection, we only need to show that the quotient $\Xi = \Xi'/f$ with the averaged Lorentzian structure from that of $\Xi'$ under the locally conformal quotient map is also a Mandelstam diagram.

Let $\Xi'$ be oriented and $\mu'_L$, $\mu'_R$ be the left and right characteristic 1-forms. Recall that the Lorentzian metric on $\Xi'$ is then $\mu'_L \cdot \mu'_R$. For a simply-connected region $\Delta$ in $\Xi$, let $\Delta_1$, $\Delta_2$ be the two components of its corresponding region in $\Xi'$. The transverse pair of local 1-forms on $\Sigma$

$$
\mu_1|\Delta = \frac{1}{2} (\mu'_L|_{\Delta_1} + \mu'_R|_{\Delta_2}) \quad \text{and} \quad \mu_2|\Delta = \frac{1}{2} (\mu'_R|_{\Delta_1} + \mu'_L|_{\Delta_2})
$$

cannot be globally well-defined since extending the local $\mu_1$, $\mu_2$ along a loop with non-orientable neighborhood will turn $\mu_1$, $\mu_2$ into each other. Nevertheless, the two mixed locally defined objects,

$$
\mu|\Delta = \frac{1}{2} (\mu_1|\Delta + \mu_2|\Delta) \quad \text{and} \quad dh^2|\Delta = \mu_1|\Delta \cdot \mu_2|\Delta,
$$

can always be globally extended to a $\mu$ and $dh^2$. Exactness of $\mu'_L + \mu'_R$ implies exactness of $\mu$; and its integral gives a time function $t$ on $\Xi$ that can be realized as
the average of one on $\Xi'$. Its level curves have to be transverse to the bi-foliation on $\Xi$ coming from the projection of, say, $F_L$ on $\Xi'$. Hence, as in showing the relation of Cond. ($\ast_1$) to rectifiability, time level trajectories through singularities give an annuli decomposition of $\Xi$ and the $\text{Hol}_0$-type maps using the local $(\mu_1, \mu_2)$, now well-defined on each annulus, provide the required homotheties to standard cylinders. This shows that $(\Xi, dh^2)$ is indeed a Mandelstam diagram.

This concludes the proof of Part (2); and hence the proposition.

$\square$

Remark 3.1.3. Since an s-d-l Lorentz surface with only one singularity has spacelike generators for its fundamental group, the slope condition can be satisfied by some positive $(\mu_L, \mu_R)$; and hence all such Lorentz surfaces are rectifiable.

For $\Sigma$ generic and oriented, there is some geometry related to the linear system that appears in the above proof. We shall now take a look at this. It will be used in the next subsection for checking rectifiability of $\Sigma$.

Let $\Omega^L_i$ (resp. $\Omega^R_j$) be the left (resp. right) characteristic strip associated to $E^L_i$ (resp. $E^R_j$). One can define a pairing between $\{E^L_i\}$ and $\{E^R_j\}$ as illustrated in Figure 3-1. Each characteristic strip $\Omega$ contains a unique light-cone-diamond $D$ that have the two singularities at the border of $\Omega$ as two of its four corners. The pairing $(E_i, E_j)$ of two edges $E_i$ and $E_j$, one left and one right, takes an integer value that counts how many times $\Omega_j$ crosses $D_i$ with the orientation of $E_i$ taken into consideration. An equivalent definition is given by assigning to each $E^L_i, E^R_i$ a directed positive measure $\mu^L_i, \mu^R_i$ of total mass 1 compatible with the orientation of

![Figure 3-1](image-url)
edges. Then
\[(E^R_i, E^L_j) = \int_{E^R_i} \mu^R_j \quad \text{and} \quad (E^L_j, E^R_i) = \int_{E^L_j} \mu^L_i,\]
where \(\mu^R_j, \mu^L_i\) in the integrand are now regarded as the induced directed transverse measures with respect to \(\mathcal{F}_L, \mathcal{F}_R\).

Let \(V_L\) be the real left-edge space \(\text{Span}_\mathbb{R} \{E^L_1, \ldots, E^L_k\}\) with the positive definite inner product that takes \(\{E^L_1, \ldots, E^L_k\}\) as an orthonormal basis. Let \(Z_L\) and \(U_L\) be respectively the cycle space and the cut space of \(\Gamma_L\). (Recall that the cut space of a graph \(\Gamma\) can be regarded as the space of functions on the vertex set of \(\Gamma\) modulo \(\mathbb{R}\); hence as the space of exact 1-cocycles of \(\Gamma\) as a simplicial 1-complex.) Let \(V_R, Z_R,\) and \(U_R\) be defined similarly. Recall the orthogonal decomposition [Bo]:
\[V_L = Z_L \oplus U_L, \quad V_R = Z_R \oplus U_R.\]

Define \(T_{R \to L}\) from \(V_R\) to \(V_L\) by linearly extending
\[T_{R \to L}(E^R_i) = \sum_j (E^R_i, E^L_j) E^L_j;\]
and, similarly, \(T_{L \to R}\) from \(V_L\) to \(V_R\) by linearly extending
\[T_{L \to R}(E^L_i) = \sum_j (E^L_i, E^R_j) E^R_j.\]

Notice that \(T_{R \to L}(E^R_i)\) is the edge-path in \(\Gamma_L\) obtained by homotoping \(E^R_i\) into \(\Gamma_L\) relative to its end-points; and similarly for \(T_{L \to R}(E^L_i)\). Hence, \(T_{R \to L} \circ T_{L \to R}(E^R_i)\) and \(E^R_i\) are homotopic relative to the end-points; and so are \(T_{L \to R} \circ T_{R \to L}(E^L_i)\) and \(E^L_i\). This implies that
\[T_{R \to L} \circ T_{L \to R} = \text{Id}_{V_L} \quad \text{and} \quad T_{L \to R} \circ T_{R \to L} = \text{Id}_{V_R}.\]

Thus we can define the transition matrix \(T\) to be \(T_{R \to L}\) and its inverse \(T_{L \to R}\). In terms of the bases \(\{E^L_i\}\) and \(\{E^R_i\}\), we may let
\[A^L = \left( a^L_{ij} \right)_{k_0 \times k}, \quad A^R = \left( a^R_{ij} \right)_{k_0 \times k}, \quad \text{and} \quad T = \left( (E^R_i, R^L_j) \right)_{k \times k},\]
then \(Z_L\) (resp. \(Z_R\)) is the subspace in \(V_L\) (resp. \(V_R\)) generated by the row vectors of \(A^L\) (resp. \(A^R\)) and the two matrices \(A^L, A^R\) are related by
\[A^R T = A^L.\]

The slope condition now reads
\[A^R (v_1 + T v_2) = 0 \quad \text{for some column vectors } v_1, v_2 \in \mathbb{R}^k_>,\]
where $\mathbb{R}_{>0}^k$ is the strictly positive orthant of $\mathbb{R}^k$. Equivalently,
\[
\left( \mathbb{R}_{>0}^k + T\mathbb{R}_{>0}^k \right) \cap \mathcal{U}_R \neq \emptyset.
\]

**Remark 3.1.4.** Let $\{e_j\}$ be the standard basis for $\mathbb{R}^k$; then $Te_j$ lies in the hyperplane
\[
\{(x_1, \ldots, x_k)^t \mid \sum x_i = -1\}
\]
for $j = 1, \ldots, k$. To see this, since $Te_j = \sum_i (E_i^R, E_j^L) e_i$, it suffices to show that
\[
\sum_i (E_i^R, E_j^L) = -1 \quad \text{for } j = 1, \ldots, k.
\]

Let $\Omega_j^L$ be the left strip associated to $E_j^L$ and
\[
\cdots, \Omega_{j-1}^R, \Omega_j^R, \Omega_{j+1}^R, \cdots
\]
be the sequence of right-strips that $\Omega_j^L$ crosses following the future-direction. (Note that same strip appears in general more than once in this sequence.) For $s$ negatively (resp. positively) large enough, $\Omega_j^L$ must cross both the boundary components of $\Omega_j^R$ from the halves causally before (resp. after) the singularities. Thus, if let $c_{sj}$ be the contribution to $\sum_i (E_i^R, E_j^L)$ at each occurrence of crossing, then the sequence $\{c_{sj}\}_s$ must be of the form
\[
\cdots, 0, \ldots, -1, 0, 1, \ldots, -1, 1, 0, \ldots, -1, \ldots 0, \ldots,
\]
namely, a sequence obtained by inserting 0’s to a finite 1, $-1$ alternating sequence beginning and ending with $-1$ (Figure 3-2). Consequently,
\[
\sum_i (E_i^R, E_j^L) = \sum_s c_{sj} = -1
\]
as required. Incidentally, it follows from this that the intersection of $\left( \mathbb{R}_{>0}^k + T\mathbb{R}_{>0}^k \right)$ with $\{(x_1, \ldots, x_k)^t \mid \sum x_i = 0\}$ is $\text{Span}_{\mathbb{R}_{>0}^k}\{e_i + Te_j\}_{ij}$.

### 3.2 Electrical circuits and examples of unrectifiability.

Given an s-d-l Lorentz surface with a simple cut system $(\Sigma, \{C_{\alpha}\}_\alpha)$. Recall from Sec. 2.3 the network $\text{Net}(\Sigma, \{C_{\alpha}\}_\alpha)$ associated to it, whose edge $E_\alpha$ has a **favored direction** induced from the time-orientation of $\Sigma$. Let $A_\alpha = S^1 \times [0, 2\pi k_\alpha]$ be grafted to $\Sigma$ along $C_\alpha$. If the new surface $\Sigma'$ is rectifiable to a Mandelstam diagram $\Xi$ with time function $t$ and characteristic 1-forms $\mu_L, \mu_R$ (for $\Sigma$ orientable; otherwise
Figure 3-2. The pattern in which a left-strip crosses right-strips and the contribution at each occurrence of crossing to the sum of pairings.

$(\mu_L, \mu_R)$ is the local splitting of the characteristic bi-valued 1-form $\mu$ for $\Sigma$ non-orientable. Then on $A_\alpha$

$$\left( \int_{\{0\} \times [0, 2\pi \kappa_\alpha]} dt \right) \Big/ \left( \int_{S^1 \times \{0\}} \mu_L \right) = k_\alpha .$$

And similarly for $\mu_R$. These relations resemble the Ohm's law:

$$V / I = R ,$$

where $V$ is the (electrical) potential difference at the end points of a conducting rod, $I$ the current through it, and $R$ the total resistance thereon. The behavior of the transverse measures to null foliations on $\Sigma$ reminds one the Kirchhoff’s first law which states that the algebraic sum of the currents at each vertex must be zero. These considerations suggest one the following approximate dictionary:
A grafted s-d-l Lorentz surface \((\Sigma; \{(C_\alpha, k_\alpha)\}_\alpha)\):

- total left (or right) transverse measure \(\mu_\alpha\) along \(C_\alpha\)
- time-orientation on \(\Sigma\)
- step \(k_\alpha\) of grafting along \(C_\alpha\) (\(k_\alpha\) large)
- time function on \(\Sigma\)

An electrical circuit supported on the network \(\text{Net}(\Sigma, \{C_\alpha\}_\alpha)\):

- current through edge \(E_\alpha\)
- favored direction of current; direction of the edges of \(\text{Net}(\Sigma, \{C_\alpha\}_\alpha)\)
- resistance \(R_\alpha\) at edge \(E_\alpha\)
- potential on \(\text{Net}(\Sigma, \{C_\alpha\}_\alpha)\)

This realization explains why there are unrectifiable s-d-l Lorentz surfaces. Suppose we begin with any \((\Sigma, \{C_\alpha\}_\alpha)\) whose associated electrical circuit contains a bridge. Then by varying the step \(k_\alpha\) of grafting along \(C_\alpha\), one varies the resistance \(R_\alpha\) of the circuit and can manage to force the current through the bridge go in the unfavored direction. If, in addition, these \(k_\alpha\) are kept large enough, then trying to rectify the new s-d-l Lorentz surface \(\Sigma' = (\Sigma; \{(C_\alpha, k_\alpha)\}_\alpha)\) will render some characteristic strip in \(\Sigma'\) reverse its original restricted time-orientation. Hence it cannot be rectified. The following example illustrates this.

**Example 3.2.1.** Let \(\Sigma\) be an s-d-l Lorentz surface of genus 2 with one past- and one future-end. Let \(\{C_\alpha\}\) be a simple cut system as indicated in (Figure 3-3). Assume that each \(C_\alpha\) is oriented from right to left so that the transverse measure \(\int_{C_\alpha} \mu_R\) is positive for any positive 1-form on \(\Sigma_0^{\text{grat}}/\mathcal{F}_R\). The associated electrical circuit is then a simple bridge. Recall that

\[
I_3 > (\text{resp. } =, <) 0 \quad \text{if and only if} \quad R_2 R_4 > (\text{resp. } =, <) R_1 R_5.
\]

For clarity of observation, let’s recast the surface into an immersed planar domain (using a Morse function and its associated handlebody decomposition of \(\Sigma\)). Assume that the basic structures on \(\Sigma\) are as indicated in Figure 3-4. Up to (topological) automorphism of the surface, the pair of graphs \((\Gamma_L, \Gamma_R)\) in \(\Sigma\) with their edges labelled are shown in Figure 3-5. It follows from this, with careful examination
Figure 3-3. An s-d-l Lorentz surface with a simple cut system \((\Sigma, \{C_\alpha\}_\alpha)\) and its associated electrical circuit.

of this pair of graphs, that the transition matrix \(T\) for \(\Sigma\) is

\[
T = \begin{bmatrix}
-1 & 0 & 0 & 0 & 0 & -1 & -1 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\
0 & -1 & -2 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 1 & 2 & 0 \\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 0 & 0 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & -2 & 0 \\
-1 & 0 & 0 & -1 & -1 & 0 & 0 & 0
\end{bmatrix}
\]

and the right cut space \(\mathcal{U}_R = \text{Span}_\mathbb{R}\{u_a, u_b, u_c, u_d\}\), where

\[
\begin{align*}
  u_a &= ( -1, 1, -1, 1, 0, 0, 0, 0)^t \\
  u_b &= ( 1, 0, 0, 0, 1, -1, 0, 0)^t \\
  u_c &= ( 0, -1, 1, 0, 1, -1, 0, 1)^t \\
  u_d &= ( 0, 0, 0, -1, 0, 1, 1, -1)^t
\end{align*}
\]

with \((\cdot)^t\) meaning transpose. One can check that, say,

\[
\left(\frac{3}{8}e_1 + e_2 + 2e_3 + \frac{1}{2}e_4 + e_5 + e_6 + \frac{1}{2}e_7 + \frac{5}{4}e_8\right) + \left(\frac{1}{4}Te_1 + Te_2 + 2Te_3 + Te_4 + Te_5 + Te_6 + \frac{1}{8}Te_7 + Te_8\right) = 3u_a + 2u_b + u_d.
\]

Thus \((\mathbb{R}^8_{>0} + T\mathbb{R}^8_{>0}) \cap \mathcal{U}_R \neq \emptyset\) and \(\Sigma\) is rectifiable.
Figure 3-4. Basic structures of $\Sigma$. 
To see how $T$ varies after grafting, one needs also to know both the right edges $E_{i}^R$ and the left strips $\Omega_{j}^L$ that go across a given $C_{\alpha}$. This can be obtained from Figure 3-4. The result is listed in the following table:

|         | $E_{i}^R$ through $C_{\alpha}$ | $\Omega_{j}^L$ through $C_{\alpha}$ |
|---------|---------------------------------|-------------------------------------|
| $C_1$   | $-E_{1}^R, \ E_{4}^R$          | $\Omega_{1}^L, \ \Omega_{4}^L, \ \Omega_{5}^L, \ \Omega_{6}^L, \ \Omega_{7}^L, \ \Omega_{8}^L$ |
| $C_2$   | $E_{2}^R, \ -E_{3}^R$          | $\Omega_{2}^L, \ \Omega_{3}^L$    |
| $C_3$   | $E_{5}^R, \ -E_{6}^R$          | $\Omega_{1}^L, \ \Omega_{4}^L, \ \Omega_{5}^L, \ \Omega_{8}^L$ |
| $C_4$   | $E_{4}^R, \ -E_{7}^R$          | $\Omega_{6}^L, \ \Omega_{7}^L$    |
| $C_5$   | $-E_{6}^R, \ E_{8}^R$          | $\Omega_{1}^L, \ \Omega_{2}^L, \ \Omega_{3}^L, \ \Omega_{4}^L, \ \Omega_{5}^L, \ \Omega_{8}^L$ |

The "-"-sign before an $E_{i}^R$ indicates that $E_{i}^R$ crosses a $C_{\alpha}$ from the future domain of $C_{\alpha}$ to its past domain. When no indication (i.e. a hidden "+"-sign), it crosses a $C_{\alpha}$ from the past domain of $C_{\alpha}$ to its future domain. Consequently, after grafted along $C_{\alpha}$ by step $k_{\alpha}$, the transition matrix $T'$ for the new s-d-l Lorentz surface $\Sigma'$ becomes
As suggested by its associated electrical circuit, assume that

\[(k_2 + 2)(k_4 + 2) \ll (k_1 - 2)(k_5 - 2).\]

We want to show that \(\Sigma'\) is then not rectifiable.

Suppose otherwise, then there exist \(x_i > 0, i = 1, \ldots, 8\), and \(a, b, c, d\) real such that

\[
\sum_i x_i T' e_i < au_a + bu_b + cu_c + du_d
\]

(meaning that each component of the former vector is less than the corresponding one of the latter). Explicitly,

(1) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq -a + b \)
(2) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq -a + c \)
(3) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq a - d \)
(4) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq b - c \)
(5) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq b - c \)
(6) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq c - d \)
(7) \(-x_1 - x_6 + x_7 - 2k_1(x_1 + x_4 + x_5 + x_6 + x_7 + x_8) \leq c - d \)

That (2), (4), (5), (8) are positive implies that either (i) \(d < c < a \leq b\), or (ii) \(d < c < b < a\). Case (i) is ruled out by taking (4) + (7). For case (ii), let \(u = x_1 + x_4 + x_5 + x_8\) (resp. \(v = x_2 + x_3, w = x_6 + x_7\)) be the total right transverse measure along \(C_3\) (resp. \(C_2, C_4\)). By considering (3), (4)+(7), (7), and(8), one has

\[
0 < (u + v)(2k_5 - 1) < c - d < b - d < 2w(k_4 + 1),
0 < (u + w)(2k_1 + 1) < a - b < a - c < 2v(k_2 + 1).
\]

This implies that

\[
[(u + v)(2k_5 - 1)] [(u + w)(2k_1 + 1)] < [2w(k_4 + 1)] [2v(k_2 + 1)].
\]

On the other hand, since \(u, v, w > 0\), the assumption that \((k_2 + 2)(k_4 + 2) \ll (k_1 - 2)(k_5 - 2)\) implies the opposite. This leads to a contradiction; and hence \(x_i\) cannot be all positive. Consequently, \(\Sigma'\) is not rectifiable.

\[\square\]
4 Toward Lorentzian conformal field theories.

""This is a very deep business," · · · "There are a thousand details which I should desire to know before I decide upon our course of action. · · ·.""

——— from The adventure of the speckled band, in Adventures of Sherlock Holmes by Sir A.C. Doyle.

This last section contains discussions on some ingredients of prospect definitions of Lorentzian conformal field theory (LCFT). It serves to provoke some thoughts for further investigations and is by no means complete. We shall give first a proto-definition of LCFT after Atiyah and Segal ([At1], [At2], [Ge], [Se2], [Se3]; see also [AG-G-M-V], [F-S], [Mo-S]); and then discuss its refinements. All the Lorentz surfaces in the discussion are in the coarse category.

Proto-definition 4.1 [LCFT]. A prototype for an abstract (coarse) Lorentzian conformal field theory after Atiyah and Segal consists of the following data:

• $\mathcal{C}_{\text{Lorz}}$, the category of related geometries:
  
  **Objects**: An object in $\mathcal{C}_{\text{Lorz}}$ is a finite disjoint union of unparametrized circles, $\sqcup S^1$. We shall denote the set of objects by $\text{Obj}(\mathcal{C}_{\text{Lorz}})$. It is isomorphic to $\mathbb{N} \cup \{0\}$, the set of nonnegative integers.

  **Morphisms**: A morphism from $\sqcup^m S^1$ to $\sqcup^n S^1$ is a time-oriented s-d-l Lorentz surface $\Sigma$ (not necessarily connected or orientable) with $m$ past- and $n$ future-ends decorated with an ordered collection of $m + n$ parametrized simple spacelike loops $C_\alpha$ such that each of the first $m$ (resp. last $n$) loops can be homotoped into a different past- (resp. future-) end. When there is no risk of confusion, we shall denote a morphism $\{\Sigma; (C_1, \ldots, C_{m+n})\}$ simply by $\Sigma$. Also we shall denote the set of such by $\text{Mor}(m, n)$ and their union by $\text{Mor}(\mathcal{C}_{\text{Lorz}})$.

**Composition of morphisms**: One can compose a morphism $\Sigma_2$ from $\sqcup^m S^1$ to $\sqcup^n S^1$ to a morphism $\Sigma_1$ from $\Sigma_2$ from $\sqcup^l S^1$ to $\sqcup^p S^1$ by sewing orderly the last $m$ loops of $\Sigma_1$ to the first $m$ loops of $\Sigma_2$ by identifying points of the same parametrization. Notice that this determines uniquely a new s-d-l Lorentz surface $\Sigma_1 \diamond \Sigma_2$.

• A functor $Z$ from $\mathcal{C}_{\text{Lorz}}$ to $\mathcal{C}_{\text{mod,}R}$, the tensor category of modules over a ring $R$ that satisfies essentially the following two properties, in addition to some naturality requirements of the autofunctors on $\mathcal{C}_{\text{mod,}R}$ induced by $\text{Diff}(S^1)$:

  **Multiplicativity**: $Z(S^1 \sqcup S^1) = Z(S^1) \otimes Z(S^1)$.

  **Associativity under sewing**: $Z(\Sigma_1 \diamond \Sigma_2) = Z(\Sigma_2) \diamond Z(\Sigma_1)$, up to a multiple factor by an element in $R$.
For string theory, $Z(S^1)$ is in principle the state space for a string moving in a given target-space.

The above proto-definition for LCFT is a plain parroting from the Riemannian case. One likes to know if there are refinements that make it more akin to the nature of s-d-l Lorentz surfaces. To simplify the argument and make the essential points prominent, we shall restrict ourselves to oriented Lorentz surfaces for the rest of the discussions.

Let’s reflect first on the following question:

Q. What could distinguish a would-be Lorentzian string theory from Riemannian ones?

Let $\Sigma$ be an oriented s-d-l Lorentz surface. Recall from Sec. 2 the basic structures associated to $\Sigma$. They either have or suggest some natural physical interpretations:

(1°) The singular set $\text{Sing}(\Sigma)$: It corresponds to the interacting points of closed strings. By tracing along the two sets of characteristic leaves from singularities, $\text{Sing}(\Sigma)$ provides two collections of labelled marked points on the incoming and outgoing strings. These marked points indicate either the prospective or historical interacting points on the strings; and their label indicates the type of interactions as designated by the index of corresponding singularities. The directed graph $\text{Net}(\Sigma, \{C_\alpha\}_\alpha)$ associated to any simple cut system has $\text{Sing}(\Sigma)$ as the vertex set. It should be thought of as a Feynman diagram in the space-time depicting the interacting process of particles associated to $\Sigma$.

(2°) The l-c-d tessellation $\text{Tess}_{\Sigma}$: It gives a grid pattern $\text{Grid}(\Sigma)$ (the 1-complex in $\Sigma$ made of characteristic leaves), which suggests a statistical mechanical treatment of interacting strings. It also reminds one of adelic string theories, in which worldsheets for simple case could be trees, instead of 2-dimensional manifolds (e.g. [B-F]).

(3°) Foliations $\mathcal{F}_L$ and $\mathcal{F}_R$: They suggest some natural fields and operators in the theory and hint at a connection with Connes’ non-commutative geometry (e.g. [Co]). Furthermore, they provide two ways of identifying incoming strings to outgoing strings in a piecewise manner. This relates $\Sigma$ to a sequence of bonded directed links, with adjacent ones differing by a simple twist of bond. Thus a connection to knot theory is hinted at. (Example 4.2.)
Example 4.2 [Lorentz surfaces and bonded directed links]. The sequence of bonded directed links associated to the Lorentz surface depicted in Figure 3 - 4 is illustrated below (Figure 4 - 1).

\[\begin{array}{cccccc}
\includegraphics[width=0.2\textwidth]{fig1}\quad & \includegraphics[width=0.2\textwidth]{fig2}\quad & \includegraphics[width=0.2\textwidth]{fig3}\quad & \includegraphics[width=0.2\textwidth]{fig4}\quad & \includegraphics[width=0.2\textwidth]{fig5}\quad\\
\end{array}\]

Figure 4 - 1. The sequence of bonded directed links associated to $\Sigma$ in Figure 3 - 4. Notice that twists of bond resemble splicings of links [Kau].

In view of these features, the following categories should play roles in the final picture of LCFT. We list their $Obj$ and $Mor$. Composition of morphisms are essentially obtained by pasting appropriately along the geometric objects involved.

(a) $C_{t\text{-graph}}$ (cf. Item (1$^o$)): A time-directed graph ($t\text{-graph}$) is a directed graph that contains no cycles. Any graph, each of whose edges is attached to distinct vertices, can be made time-directed (Figure 4 - 2).

$Obj = \mathbb{N}$.

$Mor$: A morphism from $m$ to $n$ is a $t$-graph (not necessarily connected) with $m$ incoming and $n$ outgoing external edges. These external edges are ordered, first incoming ones and then outgoing ones.

\[\begin{array}{cc}
\includegraphics[width=0.2\textwidth]{fig6}\quad & \includegraphics[width=0.2\textwidth]{fig7}\quad\\
\end{array}\]

Figure 4 - 2. A $t$-graph interpolates two collections of points. It represents a particle interacting process.
(b) $C_{\text{tes-Lorz}}$ (cf. Item (2°)): A characteristic broken null (CBN-) loop in a Lorentz surface $\Sigma$ is a simple oriented broken null loop that lies in Grid ($\Sigma$) and whose arbitrary small tubular neighborhood contains a simple spacelike loop. One can assign to it a label $[i_1, \ldots, i_{2k}] \in (\mathbb{Z} - \{0\})^{2k}/\mathbb{Z}_{2k}$ for some $k$ by counting the signed number of steps it goes following the orientation with $+$ for future headed steps and $-$ for past headed steps. For the present case, we require that they be oriented from left to right relative to the future direction. (Figure 4 - 3).

Obj $= \mathbb{N}$.

Mor: A morphism is an oriented s-d-l Lorentz surface decorated with an ordered collection of peripheral CNB-loops $C_\alpha$, one for each end (cf. $C_{\text{Lorz}}$). Only $C_\alpha$’s that are labelled the same are sewable.

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure4-3.png}
\caption{A tessellated oriented s-d-l Lorentz surface decorated with CBN-loops. The latter can be cyclically labelled.}
\end{figure}

(c) Category $B\text{-}\text{Link}^\dagger$ of bonded d-links (cf. Item (3°)) (Figures 4 - 1 and 4 - 4): Here a d-link means a finite disjoint union of directed circles. A $k$-bond is an ordered $k$-prong. A bonded d-link is a d-link with a finite collection of bonds whose ends are attached to the link. There are two kinds of simple twists of bonds: forward, $\text{twist}^+$; and backward, $\text{twist}^-$, as indicated in Figure 4 - 4. Simple twists associated to a same bond are inverse of each other; and they transform one bonded d-link to another.

Obj: The set of bonded d-links.

Mor: A morphism is a finite sequence of bonded d-links obtained by a sequence of simple twists.
It is instructive to think of $B$-Link$^\dagger$ as a dual category of $C_{t-graph}$. In some sense, $B$-Link$^\dagger$ and $C_{t-graph}$ are transverse to each other. In the orientable case, the bonded directed link associated to $\Sigma$ appears as a pair - (left, right) - (cf. Example 4.2).

![Figure 4-4. Bonded d-links and simple twists of bond.](image)

(d) \textit{Extended structures} (cf. Items (1$^\circ$), (2$^\circ$), and (3$^\circ$)).

Let us explain Item (d) in brief. Recall that in \textit{extended topological quantum field theories} (ETQFT) at dimension $d$, one considers generalized path-integrals by associating \textit{higher algebraic structures} to higher co-dimensional manifolds in a way that satisfies complicated consistency relations due to different ways of decomposing a manifold (e.g. [Fr1], [Fr2], [La] for more details and terminology). For LCFT that involve only orientable Lorentz surfaces, the geometric objects involved at various dimensions are:

| dimension | geometric object |
|-----------|------------------|
| 2         | oriented s-d-l Lorentz surface $\Sigma$ as in $\text{Mor}(\mathcal{C}_{Lorz})$ with the decorating loops $C_\alpha$ parametrized from left to right relative to the future direction |
| 1         | $S^1$ with either the left- or the right-set of marked points labelled with natural numbers (cf. Item (1$^\circ$)) |
| 0         | coarse conformal class $[p]$ of germs of Lorentzian disks around singularity $p$ |
For the coarse category, \([p]\) is determined by the index of \(p\); hence, the set of geometric objects at dimension zero is parametrized by \(\mathbb{N}\) by assigning \(s\) to \([p]\) for \(p\) of index \(-s\). For higher dimensions, consider the left sector first. At dimension one, one concerns about \(S^1\) with a left set of labelled marked points only up to \(\text{Diff}^+(S^1)\). The set of such classes is parametrized by \(O_L = \prod_k \mathbb{N}^k/\mathbb{Z}_k\), where \(\mathbb{Z}_k\) acts on \(\mathbb{N}^k\) by cyclically shifting the coordinates generated by \((i_1, i_2, \ldots, i_k) \mapsto (i_k, i_1, i_2, \ldots, i_{k-1})\). Denote an element in \(O_L\) by \([i_1, \ldots, i_k]_L\). There are forgetful maps defined on \(O_L\) by deletions of entries. This makes \(O_L\) a directed set by defining \([j_1, \ldots, j_l]_L \rightarrow [i_1, \ldots, i_k]_L\) if \([i_1, \ldots, i_k]_L\) is obtainable from \([j_1, \ldots, j_l]_L\) by a forgetful map. At dimension two, there is a unique \([i_1, \ldots, i_k]_L\) associated to each \(C_\alpha\) in \((\Sigma; (C_\alpha)_\alpha))\) obtained by considering the left characteristic leaves through \(C_\alpha\) following its orientation (cf. Item (1°)). And similarly for the right sector.

Thus, as in ETQFT, a prototype for LCFT that takes all dimensions into account consists of the following data. For simplicity, we assume all the categories of algebraic structures appearing in the setting are subcategories in \(\mathcal{C}_{\text{mod}, R}\):

- An assignment to each \(s \in \mathbb{N}\) a category \(Z(s)\), (e.g. \(\text{Rep}(\text{Sym}_s)\) or \(\text{Rep}(\mathbb{Z}_s)\)).
  The choice of \(Z(s)\) should be related to the interacting of \(s\) strings.

- A rule
  \[ V : \prod_k \left(2^{\mathcal{C}_{\text{mod}, R}}\right)^k/\mathbb{Z}_k \rightarrow 2^{\mathcal{C}_{\text{mod}, R}}, \]

  where \(\mathbb{Z}_k\) acts on \(\left(2^{\mathcal{C}_{\text{mod}, R}}\right)^k\) by cyclic shift; and an assignment to each
  \([i_1, \ldots, i_k]_L\) in \(O_L\) an element \(Z[i_1, \ldots, i_k]_L\) in \(V[Z(i_1), \ldots, Z(i_k)]\). Similarly
  for the right sector.

- An assignment to each \((\Sigma; (C_\alpha)_\alpha))\) a pair of homomorphisms - left and right - , one from a tensor product of \(Z[i_1, \ldots, i_k]_L\)'s related to the incoming \(C_\alpha\) to that related to the outgoing ones and the other similarly for the right sector.

- Consistency conditions:

  (i) Naturality from forgetful maps: Let \(I\) indicate the type of forgetfulness. The following diagrams commute:

  \[
  \begin{array}{ccc}
  \left(2^{\mathcal{C}_{\text{mod}, R}}\right)^l/\mathbb{Z}_l & \Rightarrow & \left(2^{\mathcal{C}_{\text{mod}, R}}\right)^k/\mathbb{Z}_k \\
  \searrow & V & \nearrow \\
  & 2^{\mathcal{C}_{\text{mod}, R}} &
  \end{array}
  \]

  where \(l > k\) and \(\Rightarrow\) are induced by the forgetful maps of deletion of components. And, with abuse of terminology, there exists a family
of forgetful functors $F = \{ f_I \}_I$ defined on $\prod_k \mathcal{V} \left( \frac{(2^{c_{\text{mod},R}})^k}{\mathbb{Z}_k} \right)$, that conjugates $\implies$ with

given a function $f_I : \mathcal{V} \left( \frac{(2^{c_{\text{mod},R}})^l}{\mathbb{Z}_4} \right) \to \mathcal{V} \left( \frac{(2^{c_{\text{mod},R}})^k}{\mathbb{Z}_4} \right)$

for some appropriate $k, l$ such that if $[j_1, \ldots, j_l] \to [i_1, \ldots, i_k]$ is of type $I$ then

$$f_I (\mathcal{V} [Z(j_1), \ldots, Z(j_l)]) = \mathcal{V} [Z(i_1), \ldots, Z(i_k)]$$

and

$$f_I (Z[j_1, \ldots, j_l]) = Z[i_1, \ldots, i_k].$$

Similarly for the right sector. Different decompositions of a forgetful map are required to lead to the same result.

(ii) From sewings: Due to the fact that sewing increases marked points associated to $C_\alpha$, sewing of two composable $\Sigma_1, \Sigma_2$ now leads to a $\Sigma_1 \diamond \Sigma_2$ whose $Z(\Sigma_1 \diamond \Sigma_2)$ has a domain and image modules different from the domain of $\Sigma_1$ and the image of $\Sigma_2$ respectively (Figure 4 - 5). Consistency conditions arise from the fact that a $\Sigma$ could have more than one (though, recall that, at most finitely many) non-equivalent simple cut system (cf. Sec. 2.3) and, hence, could admit different sewing patterns. $Z(\Sigma)$ should be indifferent of ways of such decompositions.

The whole format extends that in Proto-definition 4.1.

**Remark 4.3.** Notice that there are "pinching functors" from $\mathcal{C}_{\text{tes-Lorz}}, B\text{-Link}^\uparrow$, and $\mathcal{C}_{\text{Lorz}}$ respectively to $\mathcal{C}_{\text{graph}}$. Physically, $\mathcal{C}_{\text{graph}}$ is the most fundamental category in the theory; all the rest should be its extensions.

**Remark 4.4.** It should be noted that the space of morphisms in these categories and natural bundles thereover are among the major things for study, following the spirit of [F-S] for CFT.

We shall leave more thorough and detailed studies to the future and conclude the paper here with the wish of rich Lorentzian CFT and un-Wick-rotated string theory as their Riemannian siblings.
Appendix. Quantization of string phase space.

A most natural quantization for string phase space is through geometric quantization. Though a complete setting is still beyond grasp at the moment, some manipulations for the case of finite dimensional phase spaces go through formally. We shall explore them in this appendix. We discuss only closed strings and assume that $M$ is Riemannian.

Prequantum line bundles.

Let $ev : LM \times S^1 \to M$ be the evaluation map $ev(\phi, \sigma) = \phi(\sigma)$. A $k$-form $\nu$ on $M$ induces a $(k - 1)$-form, still denoted by $\nu$, on $LM$ by setting [Br]

$$\nu|_{LM} = \int_{S^1} ev^*\nu|_{M}.$$ 

Explicitly, for $Z_i, i = 1, \cdots, k - 1$, in $T_\phi LM$,

$$\nu|_{LM}(Z_1, \cdots, Z_{k-1}) = \int_{S^1} d\sigma \nu|_{M}(Z_1(\sigma), \ldots, Z_{k-1}(\sigma), \phi_\sigma \partial_\sigma).$$

In this way $B$ is regarded as a 1-form on $LM$ and $dB$ a 2-form. Their pull-back to $LT^*M$ via projection map shall be denoted the same. There is also a section-evaluation map $sev : S^1 \times LT^*M \to S^1 \times T^*M$ with $sev(\sigma, \gamma) = (\sigma, \gamma(\sigma))$. Recall
θ the Liouville 1-form on $T^*M$ and $\theta$ be the Liouville 1-form on $LT^*M$. One has $\theta = \int_{S^1} \text{ev}^* d\sigma \wedge \theta$ and $\omega = \int_{S^1} \text{ev}^* d\sigma \wedge \omega$. Let $\theta_B = \theta + B$, $\omega_B = \omega + dB$, and $H_0(\phi, \pi; \sigma) = \frac{1}{2} (\pi(\sigma) - \partial_\sigma, \pi(\sigma)) + \frac{1}{2} (\phi, \partial_\sigma, \phi, \partial_\sigma)$. Then the map $(\phi, \pi) \mapsto (\phi, \pi - B)$ is an equivalence from $(LT^*M, \omega, H)$ to $(LT^*M, \omega_B, H_0)$ since it pulls back $\theta_B$ to $\theta$ and $H_0$ to $H$. The string system as given resembles that of a particle moving in a Riemannian manifold with an external electromagnetic field.

**Remark A.1.** Implicit in the validity of the same notation for a $k$-form on $M$ and its induced $(k-1)$-form on $LM$ is the commutativity relation:

$$d \int_{S^1} \text{ev}^* = \int_{S^1} \text{ev}^* d,$$

which follows from the fact that the difference of the two sides is $\int_{S^1} \mathcal{L}_{\partial_\sigma}$, where $\mathcal{L}$ here means the Lie derivative, and this integral vanishes. (cf. Gysin sequence of sphere-fibration.)

**Remark A.2.** Observant readers may notice that the $\text{Diff}(S^1)$-action on $(LT^*M, \omega)$ by reparametrization is not symplectic. This is an obvious defect of the setting.

Recall that a *prequantum line bundle* over a symplectic manifold is a Hermitian line bundle with a connection over that manifold whose curvature equals the symplectic 2-form up to a conventional factor ($\hbar^{-1}$ in [Wo]). Such a line bundle with connection does not always exist. When it does, the symplectic manifold is said to be *quantizable*.

**Assertion A.3 [quantizability].** The infinite dimensional symplectic manifold $(LT^*M, \omega_B)$ is quantizable.

**Reason.** Consider the trivial Hermitian line bundle $\mathbb{L} = LT^*M \times \mathbb{C}$ over $LT^*M$. Let $\gamma_\tau$ be a path in $LT^*M$ and $\gamma_\tau(\sigma)$ be its realization in $T^*M$. Given $z_0 \in \mathbb{C}$, the unique solution to the first order differential equation

$$\frac{dz}{d\tau}(\tau) = \frac{i}{\hbar} \int_{S^1} d\sigma \left\{ \theta_{\gamma_\tau}(\sigma) (\partial_\tau) + B(\partial_\tau, \gamma_\tau, \partial_\sigma) \right\} \quad \text{with} \quad z(0) = z_0$$

defines a unique lifting $\tilde{\gamma_\tau} = (\gamma_\tau, z(\tau))$ of $\gamma_\tau$ in $\mathbb{L}$ and hence a horizontal distribution therein. The parallel transports it generates are unitary due to the factor $i$. This defines a compatible connection $\nabla^B$ in $\mathbb{L}$ with $\nabla^B = d - \frac{i}{\hbar} \theta_B$. We now check that the curvature of $\nabla^B$ is indeed $\hbar^{-1} \omega_B$.

Let $\alpha$ be the $\mathbb{C}$-valued connection 1-form in $\mathbb{L}$ associated to $\nabla^B$ and $\text{Pr}_\mathbb{C} : \mathbb{L} \to \mathbb{C}$ be the projection to the $\mathbb{C}$-component. Then, explicitly,

$$\alpha = \text{Pr}_\mathbb{C^*} - \frac{i}{\hbar} \theta_B,$$
where we use the same notation to denote the pullback 1-form in $\mathbb{L}$ of $\theta_B$ and identify the tangent space of any point in $\mathbb{C}$ with $\mathbb{C}$ itself canonically. Let $\text{Pr}_H$ be the horizontal projection of tangent vectors in $\mathbb{L}$ to the horizontal distribution. Then the 2-form $i \, d\alpha \circ \text{Pr}_H$ on $\mathbb{L}$ descends to the curvature 2-form on $L^*M$. On the other hand, observe that $d \, \text{Pr}_C = 0$ due to the fact that $\text{Pr}_C$ is a coordinate function and hence its differential as a 1-form has to be closed. Thus,

$$i \, d\alpha = i \, d \, \text{Pr}_C \ast + \hbar^{-1} \int_{S^1} \{sev^* d\sigma \wedge \theta + ev^* B\}$$

$$= \hbar^{-1} \int_{S^1} \{sev^* d\sigma \wedge \omega + ev^* dB\} = \hbar^{-1} \omega_B.$$

Consequently, $i \, d\alpha \circ \text{Pr}_H$ descends to $\hbar^{-1} \omega_B$ on $L^*M$. This concludes the reason.

Remark A.4. Notice that $\nabla^B$ is flat along every fiber $T^*_0LM$ of $L^*M$. Let $\nabla$ be $\nabla^B$ with $B = 0$; then the map $(\phi, \pi; z) \mapsto (\phi, \pi - B\phi; z)$ gives a bundle-with-connection isomorphism from $(\mathbb{L}, \nabla)$ to $(\mathbb{L}, \nabla^B)$. In general, $\pi_1(M)$ and $\pi_2(M)$ are non-trivial; and hence there can be non-equivalent prequantum line bundles over $(L^*M, \omega_B)$.

Geometric quantization and string field theory.

Geometric quantization of the string phase space $(L^*M, \omega, H)$ (or its equivalent) can be regarded as a geometrization of string field theory. Sections in $\mathbb{L}$ are candidates for string fields (or string wave functions). An observable corresponding to a measurable physical quantity given by a real-valued function $F$ on $L^*M$ is the operator $\hat{F}$ acting on sections $s$ of $\mathbb{L}$ by

$$\hat{F} s = -i\hbar \nabla_{X_F} s + F s,$$

where $X_F$ is the Hamiltonian vector field generated by $F$. It is the infinitesimal generator for the one-parameter group action $\hat{\rho}_t$ on sections in $\mathbb{L}$ by

$$\hat{\rho}_t s(\gamma) = s(\rho_t \gamma) e^{-\frac{i}{\hbar} \int_0^t dt' L_{X_F}},$$

where $\rho_t$ is the flow generated by $X_F$ and $L_F = \theta(X_F) - F$ is the Lagrangian of $F$ and the integration $\int_0^t dt'$ is taken along the flow $\rho_t$ from $\gamma$ to $\rho_t \gamma$. However, there are subtleties in this naive picture.

From the standard geometric quantization ([Wo] for details), one learns that the polarization $\mathcal{P}$ of $L^*M$ by vertical fibers $T^*_0LM$ has to be introduced. Only those sections in $(\mathbb{L}, \nabla)$ that are flat along $\mathcal{P}$ could be physical. They are called $\mathcal{P}$-polarized sections and are string fields that come from those over $LM$. The canonical line
bundle \( K_P \) associated to \( \mathcal{P} \) and its square root \( \delta_P = \sqrt{K_P} \) also have to be introduced. Sections \( \nu \) in \( \delta_P \) are half-forms on \( LM \) and one replaces \( \mathbb{L} \) by \( \mathbb{L}_P = \mathbb{L} \otimes \delta_P \). Since most observables do not preserve \( \mathcal{P} \), one needs to introduce a pairing between \( \mathcal{P} \)-polarized sections \( \tilde{s} = \psi \nu \) in \( \mathbb{L}_P \) and \( \mathcal{P}' \)-polarized sections \( \tilde{s}' = \psi' \nu' \) in \( \mathbb{L}_{P'} \) for another polarization \( \mathcal{P}' \) transverse to \( \mathcal{P} \). It is defined by

\[
(\tilde{s}, \tilde{s}') = \int_{LT^*M} \overline{\psi'} \psi' (\nu, \nu') \mathrm{vol}_\omega,
\]

where \((\nu, \nu') = \sqrt{\nu'^2 \wedge \nu^2 / \mathrm{vol}_\omega} \) and \( \mathrm{vol}_\omega \) is the symplectic volume form on \( LT^*M \). Such pairing allows one to project \( \mathcal{P}' \)-polarized sections in \( \mathbb{L}_{P'} \) to \( \mathcal{P} \)-polarized sections in \( \mathbb{L}_P \). Finally, there is the metaplectic correction to give a more coherent treatment of the half-forms with respect to various polarizations.

Another subtlety arises from symmetries: (1) the missing but required symmetry of \( \text{Diff}(S^1) \) due to reparametrizations; and (2) the manifest conformal symmetry of the theory (cf. Sec. 1). A complete program should contain a prescription of how to restore the first symmetry and the final quantities extracted from the setting should be parametrization-independent. The second symmetry suggests an extension \( \mathbb{L}' \) of \( \mathbb{L} \) to include anti-commuting fields (ghosts) and a BRST operator that acts on sections of \( \mathbb{L}' \). Only the BRST (co)homology classes are significant. The Hilbert space \( \mathcal{H} \) of physical states of the theory has now a trinity nature: first, it appears usually as a representation of a graded algebra depending on the target-space \( M \); second, its elements as \( \mathcal{P} \)-polarized sections in \( \mathbb{L}_P \) should be a generalization of square-integrable functions in the case of finite dimensional configuration spaces; and third, these elements are BRST-(co)homology classes. Unfortunately, not all subtleties are resolvable at the moment. Nevertheless, the most fundamental object - the Hamiltonian operator on string fields - can be constructed at the formal level.

**BKS-construction and the Schrödinger equation.**

The string Hamiltonian \( \mathcal{H} \) is quadratic with respect to the momentum variable \( \pi \); and hence the flow it generates does not preserve the vertical polarization \( \mathcal{P} \) in \( LT^*M \). The Blattner-Kostant-Sternberg- (BKS-) construction is developed to remedy this ([Sn], [Wo]).

The metric \( ds^2 \) on \( \hat{M} \) induces a metric on \( LM \) via the map \( \text{sev} \). Let \( \mathrm{vol}_{LM} \) be the metric volume form on \( LM \) and \( \sqrt{\mathrm{vol}_{LM}} \) be a fixed half-form associated to \( \mathrm{vol}_{LM} \). A physical section \( \tilde{s} \) in \( \mathbb{L}_P \) can now be written as the pull-back of

\[
\tilde{s} = \psi \sqrt{\mathrm{vol}_{LM}}
\]
by the projection from $LT^*M$ to $LM$. Denote the pull-back section by the same notation. Let $\rho_t$ be the string Hamiltonian flow on $LT^*M$ and $\tilde{\rho}_t$ be its induced action on physical sections in $\mathbb{L}_P$ defined by

$$\tilde{\rho}_t \tilde{s} = \psi_t \rho_t^* \sqrt{\text{vol}_{LM}},$$

where

$$\psi_t (\gamma) = \psi(\rho_t \gamma) e^{-\frac{\pi}{2} \int_0^t dt' L_H}$$

with $L_H$ the phase-space string Lagrangian. Due to the fact that the flow $\rho_t$ does not preserve the polarization, the driven section $\tilde{\rho}_t \tilde{s}$ is in general no longer physical. The pairing between driven and not-driven physical sections, $\tilde{\rho}_t \tilde{s}$ and $\tilde{s}'$ now becomes

$$(\tilde{\rho}_t \tilde{s}, \tilde{s}') = \int_{LT^*M} e^{\frac{\pi}{2} \int_0^t dt' L_H} \psi \circ \rho_t \psi' \sqrt{(\rho_t^* \text{vol}_{LM}, \text{vol}_{LM}) \text{vol}_{\omega}}.$$ 

One would like to rewrite this integral as an integral over the configuration space $LM$ of the form:

$$\int_{LM} \left\{ \text{Id} - \frac{it}{\hbar} \mathcal{O}_H + O(t^2) \right\} \psi \psi' \text{vol}_{LM},$$

where $\mathcal{O}_H$ depends only on $H$. The string Hamiltonian operator is then $\mathcal{O}_H$; and the Schrödinger equation reads

$$i\hbar \frac{\partial \psi}{\partial t} = \mathcal{O}_H \psi.$$

Given $t \in \mathbb{R} - \{0\}$. To carry out the above construction, it turns out more natural to work on the equivalent system $(LT^*M, \omega_B^{(t)}, \mathcal{H}_0^{(t)})$, where

$$\omega_B^{(t)} = \frac{1}{t} \omega + dB$$

and

$$\mathcal{H}_0^{(t)} (\phi, \pi) = \frac{1}{2t^2} \int_{S^1} d\sigma \langle \pi, \pi \rangle^\sim + \frac{1}{2} \int_{S^1} d\sigma \langle \phi_\ast \partial_\sigma, \phi_\ast \partial_\sigma \rangle.$$

The potential 1-form associated to $\omega_B^{(t)}$ is now $\theta_B^{(t)} = \frac{1}{t} \theta + B$. Assume that $\dim M \geq 3$ and that $\phi$ is generic; hence, an embedding. Fix a Fermi coordinate system $x$ [Hi] in a tubular neighborhood $U$ of $\phi$ in $M$ by choosing an orthonormal frame $\{e_i\}$ along $\phi$ with $e_1$ the unit tangent vector of $\phi$. This then induces a trivialization $\{(\phi(\sigma), \pi_1(\sigma)) | \sigma \in [0, 2\pi)\}$ of $LT^*_U M$. With respect to this, for $(Y, Z) \in T_{(\phi, \pi)}LT^*M$ with $Y$ the horizontal component and $Z$ the vertical component, one has

$$d\mathcal{H}_0^{(t)}|_{(\phi, \pi)}(Y, Z) = \frac{1}{t^2} \int_{S^1} d\sigma \langle Z, \pi(\sigma) \rangle^\sim - \int_{S^1} d\sigma \langle Y, \nabla_{\partial_\sigma} \partial_\sigma \rangle.$$
The correspondence between $T^*LT^*M$ and $T_LLT^*M$ induced from $\omega_B^{(t)}$ is given by
\[
\begin{align*}
&d\phi^i \rightarrow -t \frac{\partial}{\partial \pi_i} \\
&d\pi_i \rightarrow t \frac{\partial}{\partial \phi^i} - t^2 \left( i \frac{\partial}{\partial \phi^i} i_{\phi^i} dB \right)_j \frac{\partial}{\partial \pi_j};
\end{align*}
\]
and hence
\[
X_{H_0^{(t)}}^{(t)} \bigg|_{(\phi, \pi)} = \frac{1}{t} \pi^\sim - i_{\pi^\sim} i_{\phi^i} dB + t \left( \nabla_{\phi^i} \phi^i \nabla_{\phi^i} \phi^i \right)^\sim
\]
\[
= \frac{1}{t} \pi^i \frac{\partial}{\partial \phi^i} - \pi^i \left( i \frac{\partial}{\partial \phi^i} i_{\phi^i} dB \right)_j \frac{\partial}{\partial \pi_j} + t \left( \nabla_{\phi^i} \phi^i \nabla_{\phi^i} \phi^i \right)^\sim \frac{\partial}{\partial \pi_j}.
\]
We shall denote its horizontal part by $Y_\mathcal{H}$ and its vertical part by $Z_\mathcal{H}$.

The phase $L_\mathcal{H}$ now becomes $\theta_B^{(t)}(X_{H_0^{(t)}}) - H_0^{(t)}$. Straightforward computation gives
\[
L_{H_0^{(t)}} (\phi, \pi) = H_0^{(t)} (\phi, \pi) + \int_{S^1} d\sigma \left\{ \frac{1}{t} \langle \pi, B^\sim \rangle - \langle \phi^i \nabla_{\phi^i} \phi^i \rangle \right\}.
\]
Since $H_0^{(t)}$ is invariant along the Hamiltonian flow, one may simply evaluate it at $t' = 0$; and the phase factor becomes
\[
e^{\frac{i}{\hbar} \int_{S^1} d\sigma \langle \pi, \pi^\sim \rangle} e^{\frac{i}{\hbar} \int_{S^1} d\sigma \langle \psi^i \nabla_{\phi^i} \phi^i \rangle} e^{\frac{i}{\hbar} \int_0^t dt' \int_{S^1} d\sigma \langle \psi^i (\sigma, t'), \phi^i (\sigma, t') \rangle} e^{- \frac{i}{\hbar} \int_0^t dt' \int_{S^1} d\sigma \langle \phi^i (\sigma, t'), \phi^i (\sigma, t') \rangle}.
\]
The first factor $e^{\frac{i}{\hbar} \int_{S^1} d\sigma \langle \pi, \pi^\sim \rangle}$ makes the integral along $T^*LM$ Gaussian. Since what matters is the result after taking $\frac{d}{dt} \bigg|_{t = 0}$, one only needs to expand everything else in the integrand of $\int_{LT^*M} \cdots$ vertically up to orders $t$ and $\pi^2$ around the vertical critical set $\Lambda_c = \{ \pi = 0 \}$ of $H_0^{(t)}$ and then integrate out $\pi$ by applying the stationary phase approximation formula
\[
\left( \frac{1}{2 \pi \hbar} \right)^{\frac{1}{2} \text{dim} L \mathbb{R}^n} \int_{L \mathbb{R}^n} [D\pi] e^{\frac{i}{\hbar} \int_{S^1 \times S^1} d\sigma d\sigma' g^{ab} (\sigma) \delta (\sigma - \sigma') \pi_a (\sigma) \pi_b (\sigma) \mathcal{F}(\pi)}
\]
\[
\approx \frac{e^{\frac{i t}{\hbar}} \text{sign} (g^{ab} (\sigma) \delta (\sigma - \sigma'))}{\sqrt{\det (g^{ab} (\sigma) \delta (\sigma - \sigma'))}} \left[ \sum_{k=0}^{\infty} \frac{(\hbar t)^k}{k!} \left( \sum_{a,b} \int_{S^1 \times S^1} d\sigma d\sigma' g^{ab} (\sigma) \delta (\sigma - \sigma') \frac{\delta^2}{\delta \pi_a (\sigma) \delta \pi_b (\sigma')} \right)^k \mathcal{F}(\pi) \right]_{\pi = 0}.
\]
The outcome will be an integral over $LM$ and $O(H)$ can thus be obtained.

The details.
In the following computation, \(\text{pr} : LT^*M \to LM\) is the cotangent bundle projection map and \(T^*_\phi LM\) the fibre at \(\phi\). And we shall denote \(\frac{\partial}{\partial x^i}\) by \(\partial_i\).

(a) The phase factor. First one has

\[
\frac{d}{dt}\bigg|_{t' = 0} \int_{S^1} d\sigma \langle \pi(\sigma, t'), B_\phi(\sigma, t') \rangle \sim = \int_{S^1} d\sigma \langle Y_H g^{ij}(\sigma) \pi_i B_{\phi j} \rangle + \int_{S^1} d\sigma \langle Z_H, B_\phi \rangle \sim + \int_{S^1} d\sigma \langle \pi, i Y_H di_\phi \partial_\sigma B \rangle \sim .
\]

The first term vanishes since the only possible non-zero \(Y_H g^{ij}\) at \(t' = 0\) under Fermi coordinates is \(Y_H g^{11}\), in which case \(B_{\phi 1} = B(e_1, \phi_\sigma \partial_\sigma) = 0\). The third term also vanishes since \(\langle \pi, i Y_H di_\phi \partial_\sigma B \rangle \sim = \frac{1}{i} i \pi \sim di_\phi \partial_\sigma B = 0\). Hence only the second term remains and

\[
\frac{i}{\hbar} \int_0^t dt' \int_{S^1} d\sigma \langle \pi(\sigma, t'), B_\phi(\sigma, t') \rangle \sim = \frac{i}{\hbar} \int_{S^1} d\sigma \langle \pi, B_\phi \rangle \sim + \frac{it}{2\hbar} \int_{S^1} d\sigma B \left(-i\pi \sim i\phi_\sigma \partial_\sigma B\right) + t \nabla_{\phi_\sigma \partial_\sigma} \phi_\sigma \partial_\sigma + O(t^2)
\]

\[
= \frac{i}{\hbar} \int_{S^1} d\sigma \langle \pi, B_\phi \rangle \sim + O(t\pi, t^2) .
\]

Next, by first variation,

\[
-\frac{i}{\hbar} \int_0^t dt' \int_{S^1} d\sigma \langle \phi_{t'}, \partial_\sigma, \phi_{t'} \partial_\sigma \rangle
\]

\[
= -\frac{i}{\hbar} \int_{S^1} d\sigma \langle \phi_\sigma \partial_\sigma, \phi_\sigma \partial_\sigma \rangle + \frac{it}{\hbar} \int_{S^1} d\sigma \langle \nabla_{\phi_\sigma \partial_\sigma} \phi_\sigma \partial_\sigma, Y_H(\sigma, t) \rangle + O(t^2)
\]

\[
= -\frac{it}{\hbar} \int_{S^1} d\sigma \langle \phi_\sigma \partial_\sigma, \phi_\sigma \partial_\sigma \rangle + O(t\pi, t^2) .
\]

Altogether and explicitly in local trivialization,

\[
e^{\frac{i\pi}{\hbar}} \int_{S^1} d\sigma \langle \phi_\sigma \partial_\sigma, \phi_\sigma \partial_\sigma \rangle . \ e^{\frac{i\pi}{\hbar}} \int_0^t dt' \int_{S^1} d\sigma \langle \pi(\sigma, t'), B_\phi(\sigma, t') \rangle \sim . \ e^{-\frac{i\pi}{\hbar}} \int_0^t dt' \int_{S^1} d\sigma \langle \phi_{t'}, \partial_\sigma, \phi_{t'} \partial_\sigma \rangle
\]

\[
= \left[ 1 + \frac{i}{\hbar} \int_{S^1} d\sigma B_\phi^i(\sigma) \pi_i(\sigma) - \frac{1}{2\hbar^2} \int_{S^1} d\sigma d\sigma' B_\phi^i(\sigma) B_\phi^j(\sigma') \pi_i(\sigma) \pi_j(\sigma') - \frac{it}{2\hbar} \int_{S^1} d\sigma \langle \phi_\sigma \partial_\sigma, \phi_\sigma \partial_\sigma \rangle + O(t\pi, t^2) \right] .
\]

(b) The \(\psi \circ \rho_t\) part. Regard \(d\psi|_\phi\) as a complex-valued 1-form along \(\phi\) in \(M\). Let \(\text{grad} \psi\) be the vector field on \(LM\) with \(\text{grad} \psi|_\phi\) the metric equivalent of \(d\psi|_\phi\). Then

\[
(\psi \circ \rho_t)|_{T^*_\phi LM(\pi)} = \psi \circ \rho_t(\phi, \pi) = \psi (\text{pr} \circ \rho_t(\phi, \pi)) = e^{i Y_H} \psi(\phi)
\]

49
\[
\psi(\phi) + tY_H\psi(\phi) + \frac{t^2}{2}Y_H^2\psi(\phi) + O(\pi^3)
\]

\[
= \psi(\phi) + t \int_{S^1} d\sigma \left( Y_H, \text{grad} \psi \right)_{\phi(\sigma)} + \frac{t^2}{2} \int_{S^1} d\sigma \left( \nabla_{Y_H} Y_H, \text{grad} \psi \right)_{\phi(\sigma)}
\]

\[
+ \frac{t^2}{2} \int_{S^1} d\sigma \left( Y_H, \nabla_{Y_H} \text{grad} \psi \right)_{\phi(\sigma)} + O(\pi^3).
\]

The term
\[
\frac{t^2}{2} \int_{S^1} d\sigma \left( \nabla_{Y_H} Y_H, \text{grad} \psi \right)_{\phi(\sigma)}
\]

\[
= \frac{t^2}{2} \int_{S^1} d\sigma \left( (Y_H Y_H^r) \partial_r + Y_H^r Y_s^s \nabla_{\partial_r} \partial_s, \text{grad} \psi \right)_{\phi(\sigma)}.
\]

Observe that
\[
Y_H Y_H^r = \left. \frac{d}{dt'} \right|_{t'=0} \pi'(\sigma, t')
\]

is the vertical component \(Z_H\) of \(X_H\) up to metrical dual; thus \(\frac{t^2}{2} \int_{S^1} d\sigma \left( (Y_H Y_H^r) \partial_r \right. \text{grad} \psi \left. \right)_{\phi(\sigma)}\) is of order \(O(t^2 \pi, t^3)\). Consequently,

\[
(\psi \circ \rho_t)|_{\mathbb{T}^*_{vol}(\pi)}
\]

\[
= \psi(\phi) + \int_{S^1} d\sigma \left( \pi(\text{grad} \psi) \right)_{\phi(\sigma)} + \frac{1}{2} \int_{S^1} d\sigma \left( \nabla_{\partial_1} \partial_1, \text{grad} \psi \right)_{\phi(\sigma)} \pi^1 \pi^1
\]

\[
+ \frac{1}{2} \int_{S^1} d\sigma \left( \pi^\sim, \nabla_{\pi^\sim} \text{grad} \psi \right)_{\phi(\sigma)} + O(t^2 \pi, t^3).
\]

(c) The volume factor \(\sqrt{\rho^*_{vol_{LM}}} \cdot \rho_{vol_{LM}}\). In terms of the coordinates \(x\), one may write locally and formally that

\[
vol_{LM} = \sqrt{\det O_g} \int_{\{1, \ldots, n\} \times S^1} dx^i(\sigma)
\]

\[
vol_\omega = \left( \frac{1}{2\pi h} \right)^{\text{dim} L \mathbb{R}^n} \int_{\{1, \ldots, n\} \times S^1} \left( dp_i(\sigma) \wedge dx^i(\sigma) \right)
\]

where \(O_g\) is the linear operator on \(T_uLM\) defined by

\[
\xi^i(\sigma) \left. \partial_i \right|_{x(\sigma)} \mapsto \delta^{ij} g_{rs}(x(\sigma)) \xi^s(\sigma) \left. \partial_j \right|_{x(\sigma)}
\]

and the curly wedge \(\wedge\) represents a formal continuous wedge product.

Denote \(pr \circ \rho_t(\phi, \pi)\) in coordinates by \(x^i(\sigma, t)\). Then

\[
x^i(\sigma, t) = x^i(\sigma) + t \frac{\delta \mathcal{H}}{\delta p_i(\sigma)} + O(t^2).
\]
Hence
\[ dx^i(\sigma, t) = dx^i(\sigma) + t \int_{S^1} d\sigma^1 \frac{\delta^2 \mathcal{H}}{\delta p_j(\sigma^1) \delta p_i(\sigma)} \bigg|_{(\phi, \pi)} dp_j(\sigma^1) \]
\[ + t \int_{S^1} d\sigma^1 \frac{\delta^2 \mathcal{H}}{\delta x^j(\sigma^1) \delta p_i(\sigma)} \bigg|_{(\phi, \pi)} dx^j(\sigma^1) + O(t^2) \]
and
\[(\rho^* \text{vol}_{LM}) \wedge \text{vol}_{LM} \]
= \sqrt{\det \mathcal{O}_g(pr \circ \rho_t(\phi, \pi))} \sqrt{\det \mathcal{O}_g(\phi)} \left( \lambda_{i, \sigma} \in \{1, \ldots, n\} \times S^1 dx^i(\sigma, t) \right) \wedge \left( \lambda_{j, \sigma} \in \{1, \ldots, n\} \times S^1 dx^j(\sigma) \right)
= \sqrt{\det \mathcal{O}_g(pr \circ \rho_t(\phi, \pi))} \sqrt{\det \mathcal{O}_g(\phi)} \det \left( t \frac{\delta^2 \mathcal{H}}{\delta p_j(\sigma^1) \delta p_i(\sigma)} \bigg|_{(\phi, \pi)} \right) \lambda_{j, \sigma} \wedge (dp_j(\sigma^1) \wedge dx^j(\sigma^1))
= (2\pi h)^{\dim \mathbb{R}^n} \sqrt{\det \mathcal{O}_g(pr \circ \rho_t(\phi, \pi))} \sqrt{\det \mathcal{O}_g(\phi)} \det \left( t \mathcal{O}_g^{-1} \right) \text{vol}_{\omega}
= (2\pi h)^{\dim \mathbb{R}^n} \sqrt{\det \mathcal{O}_g(pr \circ \rho_t(\phi, \pi))} \cdot \det \mathcal{O}_g^{-1}(\phi) \text{vol}_{\omega}.

To get the expansion at \((\phi, \pi)\), we need the following digression. Recall that, with respect to the normal coordinate system \(y\) at a point \(q\) in \(M\), one has
\[ g_{ab}(y) = g_{ab}(q) - \frac{1}{3} R_{abcd} y^c y^d + o(|y|^2), \]
where \(R\) is the curvature tensor evaluated at \(q\). Observe that for \(q\) at loop \(\phi\), the Fermi coordinates \(x\) and the normal coordinates \(y\) around \(q\) satisfies
\[ (y^1, y^2, \ldots, y^n) = (x^1 - x^1(q), x^2, \ldots, x^n) + o(|y|) \]
since the induced map of the coordinate transformation, say from \(x\) to \(y\), on the tangent space is the identity map at \(q\). Consequently, in coordinates \(x\),
\[ g_{ij}(x(\sigma, t)) = g_{ij}(x(\sigma, 0)) \]
\[- \frac{t^2}{3} R \left( \partial_i|_{x(\sigma, 0)} , Y_{\mathcal{H}}(\sigma, 0) , \partial_j|_{x(\sigma, 0)} , Y_{\mathcal{H}}(\sigma, 0) \right) + O(t^3) \]
= \[ g_{ij}(x(\sigma, 0)) - \frac{1}{3} R \left( \partial_i|_{x(\sigma, 0)} , \pi^\sim , \partial_j|_{x(\sigma, 0)} , \pi^\sim \right) + O(t^3). \]

With the above curvature term denoted by \(R(\partial_i, \pi^\sim, \partial_j, \pi^\sim)\), one has in terms of local distributions
\[ \det \mathcal{O}_g(pr \circ \rho_t(\phi, \pi)) \cdot \det \mathcal{O}_g^{-1}(\phi) \]
\[
\begin{align*}
&= \det \left( g_{ij}(x(\sigma, t)) \delta(\sigma^1 - \sigma^2) \right) \cdot \det \left( g^{kl}(x(\sigma)) \delta(\sigma^3 - \sigma^4) \right) \\
&= \det \left( \left( g_{ij}(x(\sigma^1, 0)) - \frac{1}{3} R(\partial_i, \pi^\sim, \partial_j, \pi^\sim) + O(t^3) \right) \delta(\sigma^1 - \sigma^2) \right) \\
&\quad \cdot \det \left( g^{kl}(x(\sigma)) \delta(\sigma^3 - \sigma^4) \right) \\
&= \det \left( \Id - \frac{1}{3} \mathcal{O}_{\text{Riemann}}(\pi^\sim, \pi^\sim) + O(t^3) \right) \\
&= 1 - \frac{1}{3} \text{tr} \mathcal{O}_{\text{Riemann}}(\pi^\sim, \pi^\sim) + O(t^3),
\end{align*}
\]

where \( \Id \) is the identity map at \( T_\phi LM \) and

\[
\mathcal{O}_{\text{Riemann}}(\pi^\sim, \pi^\sim) : T_\phi LM \to T_\phi LM \\
\xi \mapsto R(\cdot, \pi^\sim, \xi, \pi^\sim)
\]

with "\( \sim \)" representing the metrically equivalent vector field to a 1-form along \( \phi \). Formally,

\[
\text{tr} \mathcal{O}_{\text{Riemann}}(\pi^\sim, \pi^\sim) = \int_{T_\phi LM} [\mathcal{D} \xi] \langle \xi, \mathcal{O}_{\text{Riemann}} \xi \rangle = \int_{S^1} d\sigma \mathcal{O}_{\text{Ric}}(\phi(\sigma))(\pi^\sim, \pi^\sim),
\]

where \( \mathcal{O}_{\text{Ric}}(\phi(\sigma)) \) is the local density functional of \( \text{tr} \mathcal{O}_{\text{Riemann}} \) along \( S^1 \). Thus

\[
\sqrt{\det \mathcal{O}_g(p \circ \rho_t(\phi, \pi)) \cdot \det \mathcal{O}_g^{-1}(\phi)} = 1 - \frac{1}{6} \int_{S^1} d\sigma (\mathcal{O}_{\text{Ric}}(\phi(\sigma)))^s \pi_r(\sigma) \pi_s(\sigma) + O(t^2)
\]

and

\[
\sqrt{(\rho_t^* \text{vol}_{LM}, \text{vol}_{LM})} = (2\pi\hbar t)^{\frac{1}{2} \dim \mathbb{R}^n} \cdot \left( 1 - \frac{1}{12} \int_{S^1} d\sigma (\mathcal{O}_{\text{Ric}}(\phi(\sigma)))^s \pi_r(\sigma) \pi_s(\sigma) + O(t^2) \right).
\]

(d) **All together.** Putting all these expansions together and extracting terms of type 1, \( t \), and \( \pi^2 \) from the product, one obtains

\[
\int_{LT^*M} e^{\frac{i}{\hbar} \int_0^1 d\tau \mathcal{L}_H} \psi \circ \rho_t \psi' \sqrt{\left( \rho_t^* \text{vol}_{LM}, \text{vol}_{LM} \right)} \text{vol} \omega
\]

\[
= \int_{LM} \left( \frac{1}{2\pi\hbar} \right)_{\dim \mathbb{R}^n} \left[ \mathcal{D}_{\pi_\sigma} \right] \int_{T_{2\tau} LM} \left( \frac{1}{t} \right)^{\dim \mathbb{R}^n} \left[ \mathcal{D}_{\pi_\sigma} \right] e^{\frac{2i\pi}{\hbar} \int_{S^1} d\sigma \langle \pi(\sigma), \pi(\sigma) \rangle} \cdot
\]

\[
\left[ 1 + \frac{i}{\hbar} \int_{S^1} d\sigma B_i^i(\sigma) \pi_i(\sigma) \pi^i(\sigma) - \frac{1}{2\hbar^2} \int_{S^1 \times S^1} d\sigma_1 d\sigma_2 B_i^j(\sigma^1) B_j^i(\sigma^2) \pi_i(\sigma^1) \pi_j(\sigma^2)
\right.
\]

\[
- \frac{it}{2\hbar} \int_{S^1} d\sigma \langle \phi^* \partial_\sigma, \phi^* \partial_\sigma \rangle + O(t^2, t\pi) \bigg].
\]

\[
\tilde{\psi}(\phi) + \int_{S^1} d\sigma \pi(\sigma)(\text{grad} \tilde{\psi})_{|\phi(\sigma)} + \frac{1}{2} \int_{S^1} d\sigma \langle \nabla_{\partial_\tau} \partial_\tau, \text{grad} \tilde{\psi} \rangle_{|\phi(\sigma)} \pi^1(\sigma) \pi^i(\sigma)
\]

52
\[
\left(2\pi \hbar t\right)^{\frac{1}{2}} \int_{\dim L \mathcal{R}^n} e^{\frac{i}{\hbar} t} \int_{T \times L \mathcal{R}^n} \left[\int_{\dim L \mathcal{R}^n} \psi \right] \cdot \left[\int_{\dim L \mathcal{R}^n} \psi \right] 
\]

Let \( \text{Sec}_p(\mathbb{L}) \) be the space of \( P \)-polarized sections in \( \mathbb{L} \). Define the following operators from \( (T_\ast L \mathcal{R} \otimes T_\ast L \mathcal{R}) \otimes \text{Sec}_p(\mathbb{L}) \) to \( \text{Sec}_p(\mathbb{L}) \):

\[
\begin{align*}
\mathcal{O}_{d^2}(\eta, \xi) \psi &= \int_{S^1 \times S^1} d\sigma_1 d\sigma_2 \left( \eta \phi(\sigma_1), \nabla_{\xi(\sigma_2)} \text{grad} \psi \right)_{\phi(\sigma_2)}, \\
\mathcal{O}_{B, d}(\eta, \xi) \psi &= \int_{S^1} d\sigma B_\phi(\eta) \int_{S^1} d\sigma d\psi(\xi), \\
\mathcal{O}_{B^2}(\eta, \xi) \psi &= \psi \int_{S^1} d\sigma B_\phi(\eta) \int_{S^1} d\sigma B_\phi(\xi), \quad \text{and} \\
\mathcal{O}_{h, d}(\eta, \xi) \psi &= \int_{S^1 \times S^1} d\sigma_1 d\sigma_2 \left( \nabla_{\partial_i} \partial_i, \text{grad} \psi \right)_{\phi(\sigma_1)} \eta^1_{\phi(\sigma_1)} \xi^2_{\phi(\sigma_2)};
\end{align*}
\]

and their trace \( \text{Tr} \)

\[
\text{Tr} \mathcal{O}_{\phi} = \int_{S^1 \times S^1} d\sigma_1 d\sigma_2 \mathcal{O}_{ij}(\phi(\sigma_1), \phi(\sigma_2)) g^{ij}(\phi(\sigma_1)) \delta(\sigma_1 - \sigma_2),
\]

where \( \mathcal{O}_{ij}(\phi(\sigma_1), \phi(\sigma_2)) \) are components of the density of \( \mathcal{O} \) at \( \phi \) along \( S^1 \times S^1 \) with respect to the Fermi coordinates. Then, after applying the Gaussian integration, one has

\[
\int_{LT^\ast M} e^{\frac{it}{\hbar}} \int_0^t dt' \left( \psi \right) \left( \psi' \right) \sqrt{\left( \rho^* \text{vol}_{LM}, \text{vol}_{LM} \right) \text{vol}}
\]

\[
\sim \int_{LM} [Dx_\sigma] e^{\frac{it}{\hbar}} \text{sign} \left( g^{ab}(\sigma) \delta(\sigma - \sigma') \right) \frac{1}{\sqrt{\det(g^{ab}(\sigma) \delta(\sigma - \sigma'))}} \left\{ \psi - \left( \frac{it}{\hbar} \right) \left[ \frac{1}{2} \int_{S^1} d\sigma \left( \phi_\ast \partial_\phi \phi_\ast \partial_\phi \right) \right. \\
\left. - \frac{\hbar}{2} \text{Tr} \mathcal{O}_{d^2} - ih \text{Tr} \mathcal{O}_{B, d} + \frac{1}{2} \text{Tr} \mathcal{O}_{B^2} \\
\left. - \frac{\hbar}{2} \text{Tr} \mathcal{O}_{h, d} + \frac{\hbar^2}{12} \text{Tr} \mathcal{O}_{\text{Ric}} \right) \right\} \left( \psi' \right).
\]
The factor $\frac{e^{\frac{ir}{4} \text{sign}(g^{ab}(\sigma)\delta(\sigma-\sigma'))}}{\sqrt{|\det (g^{ab}(\sigma)\delta(\sigma-\sigma'))|}}$ should be absorbed into the pairing by the further metaplectic correction, which we won’t discuss. Hence

$$\mathcal{O}_H = -\frac{\hbar^2}{2} \text{Tr} \left( \mathcal{O}_{d^2} + \frac{2i}{\hbar} \mathcal{O}_{B,d} - \frac{1}{\hbar^2} \mathcal{O}_{B^2} \right)$$

$$+ \frac{1}{2} \int_{S^1} d\sigma \langle \phi_\ast \partial_\sigma, \phi_\ast \partial_\sigma \rangle - \frac{\hbar^2}{2} \text{Tr} \mathcal{O}_{h,d} + \frac{\hbar^2}{12} \text{Tr} \mathcal{O}_{\text{Ric}},$$

and the Schrödinger equation follows. Notice that the part $\text{Tr} (\mathcal{O}_{d^2} + \frac{2i}{\hbar} \mathcal{O}_{B,d} - \frac{1}{\hbar^2} \mathcal{O}_{B^2})$ resembles $(\nabla - \frac{i}{\hbar} B)(\nabla - \frac{i}{\hbar} B)$ and should be regarded as the Laplacian operator on sections of $\mathbb{L}$. The only term in $\mathcal{O}_H$ that does not have a parallel to the case for a point-like charged particle is $-\frac{\hbar^2}{2} \text{Tr} \mathcal{O}_{h,d}$, which one may call the holonomy term. The restriction of this term to a geodesic loop $\phi$ in $M$ with trivial holonomy is zero.

**Remark A.5.** Presumably, the Feynman propagator for string field theory can also be worked out in the BKS method [Wo]. What is presented in this Appendix is the easy part. The hard part is to introduce further some regularizations to make physical sense of these traces. That we haven’t yet succeeded.
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