Interface in Kerr-AdS black hole spacetime

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Abstract

A defect solution in the AdS$_5 \times S^5$ black hole spacetime is given. This is a generalization of the previous work [1] to another spacetime. The equation of motion for a sort of non-local operator, “an interface,” is given and its numerical solution is shown. This result gives a new example of holographic relation of complexity and will be a clue for solving problems about black hole complexity.

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1 Introduction

Complexity is important concept for studying the inside of black holes [2]. Complexity-Action relation [3, 4] is a very interesting conjecture. This conjecture is recently studied in various spacetimes [5]. According to this conjecture the quantity, holographic complexity, which describes the development of the inside of the black hole is equal to the action calculated in a region called the Wheeler-DeWitt patch. This is a certain spacetime region defined by a given time at each boundary of the black hole spacetime. This is a holographic relation. Then there should be a quantity which describes the counterpart in the boundary gauge theories. By this motivation, the definition of complexity is tried to find recent works [6, 7, 8, 9, 10, 11, 12, 13].

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The study of complexity by using nonlocal operators is very useful method to reveal unknown properties of complexity. Effects of non local operators are also studied in recent researches, for example [14]. We studied the effect of the fundamental string for the complexity growth in the previous works [15, 16]. So our next interest is more various dimensional non-local operators.

In the previous work [17] we found that an interface can enter the horizon only for no gauge flux. This is a sort of non-local operator which has one codimension. This operator is introduced in gauge theories as a defect to separate the whole space into two gauge theories and very studied recently, for example, in [18, 19]. In our past work [1] we studied the interface solution in AdS spacetime \( \text{AdS}_5 \times S^5 \) formed by D3-branes with a probe D5-brane. This work showed that nonlocal operators introduce a new parameter which enables us to compare the results from gauge theory and gravity theory, and then actually proved that the results from both theories agree in the first order. This parameter corresponds to the flux on the probe brane. Introducing the flux is important also in the context of the compactification [20]. In this paper we treat an interface in black hole spacetime. This analysis reveals the behavior of the interface in black hole spacetimes. Then the result of this paper will give a new tool to reveal complexity by using nonlocal operators.

In our assumption the probe brane occupies the \( S^3 \) subspace of \( \text{AdS}_5 \). There are two ways for expressing this space. One is hyperspherical coordinates where the \( S^3 \) metric is written as

\[
    ds^2 = d\vartheta^2 + \sin^2 \vartheta (d\psi_1^2 + \sin^2 \psi_1 d\psi_2^2).
\]

On the other hand we can choose other polar coordinates (Hopf coordinates) as

\[
    x^1 = \sin \theta \sin \phi_1, \quad x^2 = \sin \theta \cos \phi_1, \quad x^3 = \cos \theta \sin \phi_2, \quad x^4 = \cos \theta \cos \phi_2,
\]

where the metric is expressed as

\[
    ds^2 = d\theta^2 + \sin^2 \theta d\phi_1^2 + \cos^2 \theta d\phi_2^2.
\]

We used former coordinates system [1] in the previous analysis [17]. If we take the former case (hyperspherical coordinates), the equation of motion is

\[
    \frac{d}{dz} \left( \frac{\varphi' z^2 f(z)}{\sqrt{1 + \varphi'^2 z^2 f(z)}} \right) = \frac{4}{z} \frac{\varphi' z^2 f(z)}{\sqrt{1 + \varphi'^2 z^2 f(z)}} - \frac{4K}{z} - \frac{2 \tan \varphi}{\sqrt{1 + \varphi'^2 z^2 f(z)}}.
\]

We use the latter coordinates system [3] in this paper. This coordinate system has \( T^2 \) structure. The radii of these circles changes depending on the latitude angle \( \vartheta \).

This paper is organized as follows. In section 2 we consider the interface in \( \text{AdS}_5 \times S^5 \) black hole spacetime. We use the Hopf coordinate where the \( S^3 \) metric is written as [3]. This black hole spacetime corresponds to the case where the angular momentum is put to zero in the Kerr-AdS black hole solution. This is warm up for the next section. In section 3 we introduce the angular momentum. This is the Kerr-AdS black hole spacetime. We investigate the gauge flux dependence of the interface solution. The D5-brane do not inter the horizon for non zero gauge flux in the same way as before. We also find the angular momentum dependence of the interface solution.

## 2 Interfaces in the AdS-Schwarzschild black hole spacetime

In this section we find the interface solution in \( \text{AdS}_5 \times S^5 \) black hole spacetime. This is the solution in the same background to [17] with different coordinates and assumptions.


2.1 Action and Equation of motion

We set the coordinates as

\[
\begin{align*}
\text{AdS}_5 : & \quad t, \phi_1, \phi_2, r, \vartheta; \quad \vartheta \in [0, \pi/2], \quad \phi_1, \phi_2 \in [0, 2\pi), \\
S^5 : & \quad \theta, \phi, \varphi_1, \varphi_2, \varphi_3; \quad \theta, \phi, \varphi_1, \varphi_2 \in [0, \pi], \quad \varphi_3 \in [0, 2\pi).
\end{align*}
\]

The worldvolume of the D5-brane extends the subspace spanned by \((t, \phi_1, \phi_2, r, \vartheta, \theta, \phi_3)\), where it forms one-dimensional subspace in \((r, \vartheta)\) plane. The compact part \(S^5\) is induced to

\[
ds^2_{S^5} = d\theta^2 + \sin^2 \theta d\phi^2.
\]

The non-compact part \(\text{AdS}_5\) is

\[
ds^2_{\text{AdS}_5} = -f(r) dt^2 + r^2 (\sin^2 \vartheta d\phi_1^2 + \cos^2 \vartheta d\phi_2^2) + \frac{dr^2}{f(r)} + r^2 d\vartheta^2,
\]

where

\[
f(r) = 1 + r^2 - \frac{r_m^2}{r^2}.
\]

In the above the parameter \(r_m\) is related to the black hole mass as

\[
r_m^2 = \frac{16\pi GM}{5\text{vol}[S^3]} = \frac{8GM}{5\pi}.
\]

We assume that the \(r\) dependence of \(\vartheta\) is

\[
\vartheta = \vartheta(r).
\]

The assumption for the gauge flux is

\[
F = -\kappa d\theta \wedge \sin \theta d\phi.
\]

There is the Ramond-Ramond 4-form defined as

\[
C_4 = -r^2 dt \wedge d\theta \wedge (\sin \vartheta d\phi_1) \wedge (\cos \vartheta d\phi_2)
\]

to satisfy \(d\text{vol}[\text{AdS}_5] = dC_4\). From these the induced metric is

\[
ds^2_{\text{ind}} = -f(r) dt^2 + \left(\frac{1}{f(r)} + r^2 \vartheta'(r)^2\right) dr^2 + r^2 (\sin^2 \vartheta d\phi_1^2 + \cos^2 \vartheta d\phi_2^2) + d\theta^2 + \sin^2 \theta d\phi^2.
\]

Taking the summation with the gauge flux, its determinant is

\[
G_{\text{ind}} + F = \begin{bmatrix}
-f(r) & 1/f(r) + r^2 \vartheta' \\
1/f(r) + r^2 \vartheta' & r^2 \sin^2 \theta \\
r^2 \sin^2 \theta & r^2 \cos^2 \theta
\end{bmatrix},
\]

\[
\Rightarrow \sqrt{-\det[G + F]} = \sqrt{1 + \kappa^2 \sin \theta \ r^2 \sin \vartheta \ \cos \vartheta \ \sqrt{1 + \vartheta'^2 r^2 f(r)}} = -L_{\text{DBI}}/T_5.
\]

The Wess-Zumino term is, from the gauge flux and the Ramond-Ramond 4-form,

\[
S_{WZ}/T_5 = \int F \wedge C_4 = (2\pi)^2 \text{vol}[S^2] T \kappa \int dr \sin \vartheta \ \cos \vartheta r^2 \vartheta'(r),
\]

where \(\text{vol}[S^2]\) is the volume spanned by \(\theta\) and \(\phi\) and \((2\pi)^2\) comes from the integral on the torus spanned by \(\phi_1\) and \(\phi_2\). The D5 brane action is

\[
S_{D5}/T_5 = -(2\pi)^2 \text{vol}[S^2] T \sqrt{1 + \kappa^2} \int dr \sin \vartheta \ \cos \vartheta r^2 (\sqrt{1 + r^2 f(r) \vartheta'(r)^2} - K r^2 \vartheta'(r)),
\]

where \(K\) is a constant.
where we use $K := \kappa/\sqrt{1 + \kappa^2}$ for simplicity. This action is rewritten in coordinates $z = 1/r$ and $\varphi = \pi/4 - \vartheta$ as

$$
\frac{S_{D5}}{T_5(2\pi)^2 \text{vol}[S^2]} = -\sqrt{1 + \kappa^2} \int dr \sin \vartheta \cos \vartheta r^2 \left(\sqrt{1 + r^2 f(r) \vartheta'(r)^2} - K r \vartheta'(r)\right) \\
= \sqrt{1 + \kappa^2} \int \frac{dz}{2\pi^4} \cos(2\varphi)(\sqrt{1 + z^2 f(z) \varphi'(z)^2} - K \varphi'(z)) \\
= \frac{\sqrt{1 + \kappa^2}}{2} \int dz \cos(2\varphi) \left(\sqrt{1 + g(z) \varphi'(z)^2} - K \varphi'(z)\right),
$$

where we defined a function, $g(z) := z^2 f(z) = 1 + z^2 - r_m^2 z^4$. From this the equation of motion is

$$
\frac{d}{dz} \left(\frac{\cos(2\varphi)}{z^4} \left(\frac{g(z) \varphi'(z)}{\sqrt{1 + g(z) \varphi'(z)^2}} - K\right)\right) + \frac{2 \sin(2\varphi)}{z^4} \left(\sqrt{1 + g(z) \varphi'(z)^2} - K \varphi'(z)\right) = 0. \tag{15}
$$

This equation is changed as follows.

$$
\frac{d}{dz} \left(\frac{\varphi' z^2 f(z)}{\sqrt{1 + \varphi' z^2 f(z)}}\right) = 4 \frac{\varphi' z^2 f(z)}{z} \frac{4K - 2 \tan(2\varphi)}{\sqrt{1 + \varphi' z^2 f(z)}}. \tag{16}
$$

Eq.\(16\) differs only in the factor two in the argument of the tangent function from the hyperspherical coordinates case (eq.\(4\)).

Introducing a parameter expression $(r(\tau), \varphi(\tau))$, the Lagrangian is

$$
\frac{S_{D5}}{T_5(2\pi)^2 \text{vol}[S^2]} = -\int d\tau L, \quad L = \frac{1 + \kappa^2}{2} \frac{\cos(2\varphi)}{z^4} \left(\sqrt{\dot{z}^2 + g \dot{\varphi}^2} - K \dot{\varphi}\right). \tag{17}
$$

Since there is a gauge invariance we can choose the gauge where $\dot{z}^2 + g \dot{\varphi}^2 = 1$. The equations of motion are

$$
\begin{align*}
z : & \quad \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} \frac{\dot{z}}{\sqrt{1 + \varphi' z^2 f(z)}}\right) + \frac{4 \cos(2\varphi)}{z^5} \left(1 - K \dot{\varphi}\right) - \frac{\cos(2\varphi)}{2z^4} g \varphi'^2 = 0, \quad \text{(18a)} \\
\varphi : & \quad \frac{d}{d\tau} \left(\frac{\cos(2\varphi)}{z^4} (g \varphi - K)\right) + \frac{2 \sin(2\varphi)}{z^4} (1 - K \varphi') = 0. \quad \text{(18b)}
\end{align*}
$$

Define $u_z := dz/d\tau, u_\varphi := d\varphi/d\tau$. Then we obtained the following equations.

$$
\begin{align*}
\frac{dz}{d\tau} & = u_z, \quad \text{(19a)} \\
\frac{d\varphi}{d\tau} & = u_\varphi, \quad \text{(19b)} \\
\frac{du_z}{d\tau} & = 2u_z u_\varphi \tan(2\varphi) + \frac{4}{z} (K u_\varphi - gu_\varphi^2) + \frac{g' u_z^2}{2}, \quad \text{(19c)} \\
\frac{du_\varphi}{d\tau} & = -\frac{g'}{g} u_z u_\varphi + 2 \tan(2\varphi) (u_\varphi^2 - \frac{1}{g}) + \frac{4u_z}{z} (u_\varphi - \frac{K}{g}). \quad \text{(19d)}
\end{align*}
$$

Initial condition  Since near the boundary $z \approx 0$ the solution looks the same as [1], then $\varphi = \arcsin(\kappa z)$. Taking account of the gauge condition, $u_z^2 + u_\varphi^2 g = 1$, the initial condition is for small $z_0$

$$
\begin{align*}
\varphi(z_0) & = \arcsin(\kappa z_0), \quad \text{(20a)} \\
u_z(z_0) & = \sqrt{\frac{1 - \kappa^2 z_0^2}{1 + \kappa^2 (1 - r_m^2 z_0^4)}}, \quad \text{(20b)} \\
u_\varphi(z_0) & = \frac{\kappa}{\sqrt{1 + \kappa^2 (1 - r_m^2 z_0^4)}}. \quad \text{(20c)}
\end{align*}
$$
Here we have to confirm the validity of the constraint. Let us examine the evolution of the constraint. By substituting the equations (19),

\[
\frac{d}{d\tau}(u_z^2 + gu_z^2) = 2u_z\ddot{u}_z + 2gu_\varphi\dot{u}_\varphi + u_\varphi^2g'u_z \\
= 2u_z(2u_\varphi u_\varphi \tan(2\varphi) + \frac{4}{z}(Ku_\varphi + u_z^2 - 1) + \frac{g'u_z^2}{2}) \\
+ 2gu_\varphi(-\frac{g'}{g}u_z u_\varphi + 2\tan(2\varphi)(u_\varphi^2 - \frac{1}{g}) + \frac{4u_z}{z}(u_\varphi - \frac{K}{g})) + u_\varphi^2g'u_z \\
= 4u_\varphi^2 u_\varphi \tan(2\varphi) + \frac{8u_u u_\varphi}{z} K + \frac{8u_z^3}{z} - \frac{8u_z}{z} + g'u_z u_\varphi^2 \\
- 2g'u_z u_\varphi^2 + 4\tan(2\varphi)(gu_\varphi^3 - u_\varphi) + \frac{8gu_z u_\varphi^2}{z} - \frac{8u_z u_\varphi K}{z} + u_\varphi^2 g'u_z \\
= 4u_\varphi(u_z^2 + gu_\varphi^2 - 1) \tan(2\varphi) + \frac{8u_z}{z}(u_\varphi^2 + gu_\varphi^2 - 1) \\
= (4u_\varphi \tan(2\varphi) + \frac{8u_z}{z})(u_z^2 + gu_\varphi^2 - 1) = 0.
\]

where in the last equality we use the constraint $u_z^2 + gu_\varphi^2 = 1$. Then this constraint is valid.

**2.2 Interface solution**

By numerical calculation, we solve the equations of motion (19) under the initial condition (20). The result is shown in the following three figures. The interface touches the boundary at the angle $\varphi = 0$ and extends toward the inside of the AdS spacetime.

Figure 1 shows the flux dependence of the D5-brane solution. The black hole mass is fixed to $r_m = 10$. The degree of the slope at the boundary changes depending on the value of gauge flux. We can see that the D5-brane can penetrate the horizon only for no gauge flux, $K = 0$.

Figure 2 shows the mass dependence for fixed gauge flux $K = 0.2$. The D5-brane curves tightly for large masses to avoid the horizon. For examples, $z_h = 0.32$ for $r_m = 10$ and $z_h = 0.10$ for $r_m = 90$.

The whole sketch of the D5-brane is drawn in Figure 3. As we can see from the metric (6), the D5-brane has a torus structure. The radii of two circles change depending on $\varphi = \pi/4 - \vartheta$. At the edges $\varphi = \pm \pi/4$ one of these circles shrinks to zero. We assumed that one side of the brane touches to the boundary at $\varphi = 0$ since $\varphi = 0$ is the solution that can penetrate the horizon and we want to follow the changes according to the value of the gauge flux. Then we found if the brane has the non zero gauge flux the brane can not approach the horizon. The smaller the value of the gauge flux, the closer the D5-brane can approach the horizon.

**3 Interface in the Kerr-AdS black hole spacetime**

Here we introduce the black hole angular momentum to the black hole spacetime considered in the previous section. This is Kerr-AdS black hole.

**3.1 Action and Equation of motion**

We set the coordinates as

\[
\text{AdS}_5: \quad t, \phi_1, \phi_2, r, \vartheta; \quad \vartheta \in [0, \pi/2], \quad \phi_1, \phi_2 \in [0, 2\pi), \\
S^5: \quad \theta, \phi, \varphi_1, \varphi_2, \varphi_3; \quad \theta, \phi, \varphi_1, \varphi_2 \in [0, \pi], \quad \varphi_3 \in [0, 2\pi).
\]

(22)
The worldvolume of the D5-brane is extend the subspace spanned by \((t,r,\vartheta,\phi_1,\phi_2,\theta,\phi)\), where it forms one-dimensional subspace in \((r, \vartheta)\) plane. As before the embedding of the D5-brane is expressed as a function of \(r\)

\[
\vartheta = \vartheta(r).
\]  

(23)

We take the assumption for the gauge flux

\[
\mathcal{F} = -\kappa d\theta \wedge \sin \theta d\phi.
\]  

(24)

The compact part is induced to \(ds_{\text{K5}}^2 = d\theta^2 + \sin^2 \theta d\phi^2\) on the D5-brane. The non-compact part is the 5-dimensional Kerr-AdS black hole \([21]\). Its metric is

\[
\begin{align*}
    ds_{\text{KAS}}^2 &= -\frac{\Delta r}{\rho^2} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi_1 - \frac{b \cos^2 \theta}{\Xi_b} d\phi_2 \right)^2 \\
    &\quad + \frac{\Delta_\theta \sin^2 \theta}{\rho^2} \left( adt - \frac{r^2 + a^2}{\Xi_a} d\phi_1 \right)^2 \\
    &\quad + \frac{\Delta_\theta \cos^2 \theta}{\rho^2} \left( bdt - \frac{r^2 + b^2}{\Xi_b} d\phi_2 \right)^2 \\
    &\quad + \frac{\rho^2}{\Delta_r} dr^2 + \frac{\rho^2}{\Delta_\theta} d\theta^2 + \frac{1 + \rho^2}{r^2 \rho^2} \left( abd\theta - \frac{b(r^2 + a^2) \sin^2 \theta}{\Xi_a} d\phi_1 - \frac{a(r^2 + b^2) \cos^2 \theta}{\Xi_b} d\phi_2 \right)^2.
\end{align*}
\]
In the above
\[\rho^2(r) = r^2 + a^2 \cos^2 \theta + b^2 \sin^2 \theta,\]
\[\Delta_r(r) = \frac{1}{r^2} (r^2 + a^2)(r^2 + b^2)(r^2 + 1) - 2m, \quad \Delta_\theta(\theta) = 1 - a^2 \cos^2 \theta - b^2 \sin^2 \theta,\]
\[\Xi_a = 1 - a^2, \quad \Xi_b = 1 - b^2.\]

Since the axis of the rotation, \(\phi_1\) and \(\phi_2\), are equivalent, we focus on the case where the black hole has the angular momentum only for one axis \((a \neq 0, b = 0)\). Then the metric is
\[
ds_{\text{KAdS}}^2 = -\frac{\Delta_{\text{ra}}}{\rho_a^2} \left( dt - \frac{a \sin^2 \theta}{\Xi_a} d\phi_1 \right)^2 + \frac{\Delta_{\text{\phi}_1} \sin^2 \theta}{\rho_a^2} \left( a dt - r \frac{r^2 + a^2}{\Xi_a} d\phi_1 \right)^2 + \frac{\Delta_{\text{\phi}_2} \cos^2 \theta}{\rho_a^2} r^4 d\phi_2^2
\]
\[+ \frac{\rho_a^2}{\Delta_{\text{ra}}} dr^2 + \frac{\rho_a^2}{\Delta_{\text{\phi}_1}} d\theta^2 + \frac{1}{\rho_a^2} r^2 a^2 \cos^4 \theta d\phi_2^2
\]
\[= -\frac{\Delta_{\text{ra}} - \Delta_{\text{\phi}_1} a^2 \sin^2 \theta}{\rho_a^2 \Xi_a} dt^2 + \frac{-\Delta_{\text{ra}} a \sin^2 \theta + \Delta_{\text{\phi}_1} (r^2 + a^2)^2}{\rho_a^2 \Xi_a} \sin^2 \theta d\phi_1^2
\]
\[+ \frac{2 \Delta_{\text{ra}} - a^2}{\rho_a^2 \Xi_a} \Delta_{\text{\phi}_1} a \sin^2 \theta dt d\phi_1 + r^2 \cos^2 \theta d\phi_2^2 + \frac{\rho_a^2}{\Delta_{\text{ra}}} dr^2 + \frac{\rho_a^2}{\Delta_{\text{\phi}_1}} d\theta^2, \quad (26)
\]
where
\[\rho_a^2 = r^2 + a^2 \cos^2 \theta, \quad \Delta_{\text{ra}} = (r^2 + a^2)(r^2 + 1) - 2m, \quad \Delta_{\text{\phi}_1} = 1 - a^2 \cos^2 \theta, \quad \Xi_a = 1 - a^2. \quad (27)
\]

The horizon is located at
\[\Delta_{\text{ra}} = 0 \Rightarrow (2m + a^2)z^4 - (1 + a^2)z^2 - 1 = 0,
\]
\[z_h^2 = \frac{1 + a^2 + \sqrt{(1 + a^2)^2 + 8m + 4a^2}}{4m + 2a^2}. \quad (28)
\]

The Ramond-Ramond 4-form is determined by
\[d\text{vol}[\text{AdS}_5] = dC_4. \quad (29)
\]

Let us find the volume form of the AdS part. The above metric \((26)\) can be written as
\[
ds_{\text{KAdS}}^2 = -\frac{\Delta_{\text{ra}}}{\rho_a^2} dt^2 + \frac{\Delta_{\text{\phi}_1} \sin^2 \theta}{\rho_a^2} d\phi_1^2 + \text{diag}(r, \theta, \phi_2), \quad \left[ \frac{dt}{d\phi_1} \right] = M \left[ \frac{dt}{d\phi_1} \right], \quad (30)
\]
where
\[M := \begin{bmatrix} 1 & -a \sin^2 \theta / \Xi_a \\ a & -(r^2 + a^2) / \Xi_a \end{bmatrix},
\]
\[\det M = -\frac{r^2 + a^2(1 - \sin^2 \theta)}{\Xi_a} = -\frac{r^2 + a^2 \cos^2 \theta}{\Xi_a} = -\rho_a^2. \quad (30)
\]

Then the determinant of the metric for \(t\) and \(\phi_1\) part is
\[\det G = \det(M^T G' M) = (\det M)^2 \det G' = -\frac{\Delta_{\text{ra}} \Delta_{\text{\phi}_1} \sin^2 \theta \rho_a^4}{\rho_a^2 \Xi_a^2} = -\frac{\Delta_{\text{ra}} \Delta_{\text{\phi}_1} \sin^2 \theta \rho_a^4}{\Xi_a^2}, \quad (31)
\]
and the total factor in AdS is
\[-\frac{\Delta_{\text{ra}} \Delta_{\text{\phi}_1}}{\Xi_a^2} r^2 \sin^2 \theta \cos^2 \theta \rho_a^2 \rho_a^2 = -\frac{\rho_a^4}{\Xi_a^2} r^2 \sin^2 \theta \cos^2 \theta. \quad (32)
\]
The volume form of the AdS part is
\[ d\text{vol}_{\text{AdS}} = \frac{\rho_a^2 r \sin \vartheta \cos \vartheta}{\Xi_a} dt dr d\vartheta d\varphi_1 d\varphi_2. \] (33)

Then the Ramond-Ramond 4-form is
\[ C_4 = -\frac{\rho_a^4}{\Xi_a^2} \sin \vartheta \cos \vartheta dt d\vartheta d\varphi_1 d\varphi_2. \] (34)

From the assumption for the embedding (23) and the gauge flux (24), the sum of the induced metric and the gauge flux is
\[ G_{\text{ind}} + F = \begin{bmatrix}
\rho_a^2 \frac{\Delta_{ra} - \Delta_{\vartheta a} a^2 \sin^2 \vartheta}{\rho_a^2 \Xi_a} & \Delta_{ra} - \Delta_{\vartheta a} (r^2 + a^2) \frac{a \sin^2 \vartheta}{\rho_a^2 \Xi_a} \\
\rho_a^2 \frac{\Delta_{\vartheta a} a \sin^2 \vartheta}{\rho_a^2 \Xi_a} & \rho_a^2 \frac{\varphi'^2}{\Xi_a} - \Delta_{\vartheta a} a^2 \sin^2 \vartheta \frac{\Delta_{\vartheta a} (r^2 + a^2) \sin^2 \vartheta}{\rho_a^2 \Xi_a} - \frac{r^2 \cos^2 \vartheta}{\rho_a^2 \Xi_a}
\end{bmatrix}. \]

Its determinant is
\[ \sqrt{-\det(G_{\text{ind}} + F)} = \sqrt{1 + \kappa^2} \cos \vartheta \sin \vartheta \frac{\left(\frac{\rho_a}{\Xi_a}\right)^2 \sqrt{\Delta_{ra} \varphi'^2 + \Delta_{\vartheta a} \varphi'^2}}{\frac{\rho_a}{\Xi_a}}. \]

In the above, since \( \vartheta \in [0, \pi/2] \), then \( \cos \vartheta \geq 0 \). The D5-brane action consists of the following.
\[ S_{\text{D5}} = S_{\text{DBI}} + S_{\text{WZ}}, \quad (35) \]
\[ S_{\text{DBI}} = -T_5 \int \sqrt{-\det(G_{\text{ind}} + F)}, \]
\[ S_{\text{WZ}} = T_5 \int F \wedge C_4. \]

We change the variables \( z = 1/r \), \( \varphi = \pi/4 - \vartheta \). Introducing a parameter \( \tau, (z(\tau), \varphi(\tau)) \),
\[ \sqrt{-\det(G_{\text{ind}} + F)} = \sqrt{1 + \kappa^2} d\tau \cos(2\varphi) \frac{\rho_a}{2\Xi_a} \sqrt{\Delta_{ra} \varphi'^2 + \Delta_{\vartheta a} \varphi'^2}, \] (36a)
\[ F \wedge C_4 = \frac{\rho_a^4}{2\Xi_a} d\varphi \cos(2\varphi). \] (36b)

The action is
\[ S_{\text{D5}} = -T_5(2\pi)^2 \text{vol}[S^2] T \int d\tau \mathcal{L}, \] (37)

where the Lagrangian is
\[ \mathcal{L} = \frac{\sqrt{1 + \kappa^2}}{2\Xi_a} \cos(2\varphi) \left( \frac{\rho_a}{z^3} \sqrt{\Delta_{ra} z^2 + \Delta_{\vartheta a} \varphi'^2} - K \rho_a^4 \varphi \right) \]
\[ = \frac{\sqrt{1 + \kappa^2}}{2\Xi_a} \cos(2\varphi) \left( \frac{p_a}{z^4} \sqrt{\Delta_{ra} z^2 + \Delta_{\vartheta a} \varphi'^2} - K p_a^4 \varphi \right). \] (38)

In the second line of the above we defined
\[ \rho_a^2 = \frac{1}{z^2} + \frac{a^2}{2} (1 + \sin(2\varphi)) =: \frac{p_a^2}{z^2}, \quad \Delta_{ra} = \left( \frac{1}{z^2} + a^2 \right) \left( \frac{1}{z^2} + 1 \right) - 2m =: \frac{\Delta_{ra}}{z^4}, \] (39a)
\[ \Delta_{\vartheta a} := 1 - \frac{a^2}{2} (1 + \sin(2\varphi)). \] (39b)
We choose the gauge where $\Delta_{ca} \dot{z}^2 + \Delta_{za} \dot{\varphi}^2 = 1$ to fix the gauge degrees of freedom.

The equations of motion are

$$z : \frac{d}{d\tau} \left( \frac{\cos(2\varphi)}{z^4} p_a \Delta_{ca} \dot{z} \right) + 4 \frac{\cos(2\varphi)}{z^5} p_a (p_a - Kp_a^4 \dot{\varphi})$$

$$\varphi : \frac{d}{d\tau} \left( \frac{\cos(2\varphi)}{z^4} p_a \Delta_{za} \dot{\varphi} \right) - \frac{d}{d\tau} \left( \frac{\cos(2\varphi)}{z^4} Kp_a^4 \right) + 2 \frac{\sin(2\varphi)}{z^4} (p_a - Kp_a^4 \dot{\varphi})$$

Let us define variables $u_z := dz/d\tau$ and $u_\varphi := d\varphi/d\tau$. Then the equations are

$$\frac{dz}{d\tau} = u_z,$$

$$\frac{d\varphi}{d\tau} = u_\varphi,$$

$$\frac{du_z}{d\tau} = u_z \left( \frac{4u_z}{z} - \frac{\dot{p}_a}{p_a} - \frac{\Delta_{ca}}{\Delta_{za}} + 2u_\varphi \tan(2\varphi) \right)$$

$$+ \Delta_{ca}^{-1} \left( \frac{\partial_\varphi p_a}{p_a} (1 - 4Kp_a^3 u_\varphi) + \frac{1}{2} \partial_z \Delta_{za} u_\varphi^2 \right) - \frac{4}{z} \Delta_{ca}^{-1} (1 - Kp_a^3 u_\varphi),$$

$$\frac{du_\varphi}{d\tau} = u_\varphi \left( \frac{4u_z}{z} - \frac{\dot{p}_a}{p_a} - \frac{\Delta_{za}}{\Delta_{ca}} + 2u_\varphi \tan(2\varphi) \right) + \left( \frac{\cos(2\varphi)}{z^4} - \frac{p_a \Delta_{ca}}{\Delta_{za}} \right) - \frac{d}{d\tau} \left( \frac{\cos(2\varphi)}{z^4} Kp_a^4 \right)$$

$$+ \Delta_{za}^{-1} \left( \frac{\partial_\varphi p_a}{p_a} (1 - 4Kp_a^3 u_\varphi) + \frac{1}{2} \partial_\varphi \Delta_{za} u_\varphi^2 \right) - 2 \tan(2\varphi) \Delta_{za}^{-1} (1 - Kp_a^3 u_\varphi).$$

Let us confirm the validity of the constraint. By substituting these equations of motion,

$$\frac{d}{d\tau} (\Delta_{ca} u_z^2 + \Delta_{za} u_\varphi^2) = 2\Delta_{ca} u_z \dot{u}_z + 2\Delta_{za} u_\varphi \dot{u}_\varphi + \Delta'_{ca}(\varphi) u_z^2 u_\varphi + \Delta'_{za}(z) u_z u_\varphi^2$$

$$= -2\Delta_{ca} u_z \frac{d}{d\tau} \log \left( \frac{\cos(2\varphi)}{z^4} p_a \Delta_{ca} \right) - \frac{8u_z}{z} (1 - Kp_a^3 u_\varphi) + 2u_z \left( \frac{\partial_\varphi p_a}{p_a} + \frac{1}{2} \partial_z \Delta_{za} u_\varphi^2 - 4Kp_a^2 \partial_\varphi p_a u_\varphi \right)$$

$$- 2\Delta_{za} u_\varphi \frac{d}{d\tau} \log \left( \frac{\cos(2\varphi)}{z^4} p_a \Delta_{za} \right) - 4u_\varphi \tan(2\varphi) (1 - Kp_a^3 u_\varphi)$$

$$+ 2u_\varphi \left( \frac{\partial_\varphi p_a}{p_a} + \frac{1}{2} \partial_\varphi \Delta_{za} u_\varphi^2 - 4Kp_a^2 \partial_\varphi p_a u_\varphi \right) + 2Kp_a^3 u_\varphi \frac{d}{d\tau} \log \left( \frac{\cos(2\varphi)}{z^4} Kp_a^4 \right) + \Delta'_{ca} u_z^2 u_\varphi + \Delta'_{za} u_z u_\varphi^2$$

$$= -2(1 - Kp_a^3 u_\varphi)(-2 \tan(2\varphi) - \frac{4u_z}{z}) - \frac{8u_z}{z} (1 - Kp_a^3 u_\varphi) - 4u_\varphi \tan(2\varphi) (1 - Kp_a^3 u_\varphi) = 0.$$

Then this constraint is valid.

**Initial condition** As in the previous section, we use the approximation $\varphi = \arcsin(\kappa z)$ near the boundary. Taking account of the constraint, $\Delta_{ca} z^2 + \Delta_{za} \varphi^2 = 1$, for small $z_0$

$$\varphi_0 = \arcsin(\kappa z_0),$$

$$u_z(\tau_0) = \sqrt{\frac{1 - \kappa^2 z_0^2}{\Delta_{ca}(1 - \kappa^2 z_0^2) + \Delta_{za} \kappa^2}},$$

$$u_\varphi(\tau_0) = \frac{\kappa}{\sqrt{\Delta_{ca}(1 - \kappa^2 z_0^2) + \Delta_{za} \kappa^2}}.$$
3.2 Interface solution

We perform numerical calculation to solve the equations of motion (41) under the initial condition (42). The whole structure of bulk spacetime is the same as figure 3. The result is shown in the following two figures.

Figure 4 is the angular momentum dependence. This shows that the D5-brane can not enter the horizon in the same to the AdS Schwarzschild black holes. The location of the horizon is $z_h \approx 0.047$ for $a = 0$.

Figure 5 shows the angular momentum dependence for negative $a$ values. From this plot we can see that the behaviors are the same between the positive direction and its opposite rotation.

Figure 6 shows the $a$ dependence for $K = 0$. We see that the D5-brane can enter the horizon only for $a = 0$. Even if the gauge flux is zero $K = 0$, the D5-brane avoids the horizon for the black holes which have the non zero angular momentum. For example the horizon is located at $z_h \approx 0.047$ for $a = 0.05$.

Figure 4: Angular momentum $a$ dependence (1): $m = 10^5$, $K = 0.01$

Figure 5: Angular momentum $a$ dependence (2): $m = 10^5$, $K = 0.01$

Figure 6: Angular momentum $a$ dependence (3): $m = 10^5$, $K = 0$
4 Summary and discussion

In this paper we treat the interface which is a co-dimension one object realized by the D5-brane in IIB string theory. We obtained the interface solution in the AdS black hole spacetime by numerical calculation. The boundary condition is given in the same way for the flat AdS$S_5 \times S^5$ case imposed in \cite{1}, that is, the degree of the slope at the boundary changes according to the gauge flux. The D5-brane is bent inside of spacetime by the effect of the black hole. The D5-brane penetrates the horizon only for the case $K = 0$ and $a = 0$. We found in \cite{17} that this probe brane can enter the horizon only for the case $K = 0$ and in this paper that it can not enter the horizon if the black holes have the angular momentum, either.

For boundary theories, we can also have an interesting conclusion. By the gauge/gravity correspondence, our bulk has super Yang-Mills theories with gauge groups SU$(N)$ and SU$(Nk)$ on its boundary. Here the difference $k$ is related to the parameter $\kappa$ by integrating the flux on the D5-brane as \cite{1}

$$k = -\frac{T_5}{T_3} \int F = \frac{\kappa}{\pi \alpha'}.$$ (43)

The edge of the probe D5-brane touching the boundary corresponds the interface which separates the above two gauge theories. As we saw in this paper, the probe D5-brane without the gauge flux penetrates the horizon directly. Then in this case, there is the interface located at $\varphi = 0$ and there are two gauge theories at $-\pi/4 \leq \varphi \leq 0$ and $0 \leq \varphi \leq \pi/4$ (see figure 3). If the gauge flux is non zero, the probe brane is bent tightly and touches the boundary at two positions. Then in this case there are two interfaces and there are three gauge theories with gauge groups SU$(N)$, SU$(N - k)$ and SU$(N)$ at the boundary. The analysis of these boundary theories and verification of the holography is left as our future work.

In this paper we considered the case the black holes has the angular momentum but the probe brane is static. Considering a moving defect \cite{22} is important for studying CA relation. As stated in \cite{23} the extremal surfaces do not penetrate the horizon in a static case, whereas it may penetrate the horizon in the time developing case. Then there is a possibility that the probe brane can enter the horizon if we consider the time developing interface or other non-local operators. Since the growth of the action in the WDW patch is calculated by integrating inside of the horizon, these problem is important to use these non-local operators for studying black hole complexity.

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