BOUNDS FOR THE REGULARITY OF LOCAL COHOMOLOGY OF BIGRADED MODULES

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Abstract. Let $M$ be a finitely generated bigraded module over the standard bigraded polynomial ring $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$, and let $Q = (y_1, \ldots, y_n)$. The local cohomology modules $H^k_Q(M)$ are naturally bigraded, and the components $H^k_Q(M)_j = \bigoplus_i H^k_Q(M)_{(i,j)}$ are finitely generated graded $K[x_1, \ldots, x_m]$-modules. In this paper we study the regularity of $H^k_Q(M)_j$, and show in several cases that $\text{reg} H^k_Q(M)_j$ is linearly bounded as a function of $j$.

Introduction

In this paper we study the regularity of local cohomology of bigraded modules. Let $K$ be a field and $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the polynomial in the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$. We consider $S$ to be a standard bigraded $K$-algebra with $\deg x_i = (1,0)$ and $\deg y_j = (0,1)$ for all $i$ and $j$. Let $I \subset S$ be a bigraded ideal. Then $R = S/I$ is again a standard bigraded $K$-algebra. Let $M$ be a finitely generated bigraded $R$-module. We consider the local cohomology modules $H^k_Q(M)$ with respect to $Q = (y_1, \ldots, y_n)$. This module has a natural bigraded $S$-module structure. For all integers $j$ we set

$$H^k_Q(M)_j = \bigoplus_i H^k_Q(M)_{(i,j)}.$$ 

Notice that $H^k_Q(M)_j$ is a finitely generated graded $S_0$-module, where $S_0$ is the polynomial ring $K[x_1, \ldots, x_m]$.

The main purpose of this paper is to study the regularity of the $S_0$-modules $H^k_Q(M)_j$ as a function of $j$. In all known cases, $\text{reg} H^k_Q(M)_j$ is bounded above (or equal to) a linear function $aj + b$ for suitable integers $a$ and $b$ with $a \leq 0$, see [5]. Various cohomological conditions on $M$ are known that guarantee that for suitable $k$ the regularity of $H^k_Q(M)_j$ as a function of $j$ is actually bounded, see the papers [3], [7] and [8]. For example, in [8] Corollary 2.8 it is shown that if $M$ is sequentially Cohen–Macaulay with respect to $Q$, then there exists a number $c$ such that $| \text{reg} H^k_Q(M)_j | \leq c$ for all $k$ and $j$. In general however one can expect only linear bounds. As shown in [3] Theorem 5.3 and Corollary 5.4 the regularity of the local cohomology modules $H^k_Q(R)_j$ is linearly bounded, if $R = S/(f)$ is a hypersurface ring for which the content ideal $c(f) \subset S_0$ is $m_0$-primary, where $m_0 = (x_1, \ldots, x_m)$ is the graded maximal ideal of $S_0$. Here, for a bihomogeneous polynomial $f = \sum_{|\beta|=b} f_\beta y^\beta$, 

2000 Mathematics Subject Classification. 13D45, 13D02, 16W50. This paper was written during the visit of the second author at Universität Duisburg-Essen, Campus Essen. He is grateful for its hospitality. The second author was in part supported by a grant from IPM (No. 91130029).
the ideal $c(f)$ is the ideal in $S_0$ defined by the polynomials $f_\beta$. If, more generally, $R = S/I$, where $I$ is a bihomogeneous ideal, one defines the content ideal $c(I)$ of $I$ to be the ideal generated in $S_0$ by the polynomials $c(f)$ with $f$ in $I$. As the main result of Section $1$ we show in Theorem $1.1$ that there exists a linear function $\ell : \mathbb{Z}_- \to \mathbb{Z}_+$ such that $c(I)^{\ell(j)}H^n_Q(R)_j = 0$ for all $j$. This result is then used to generalize the above quoted theorem about hypersurface rings and to obtain that $\text{reg} H^n_Q(R)_j$ is linearly bounded, provided $c(I)$ is $m_0$-primary, see Corollary $1.2$. Our Theorem $1.5$ generalizes $[5, \text{Corollary 5.4}]$ in a different direction. It is shown in this theorem that the regularity of $H^n_Q(R)_j$ is linearly bounded provided that $\dim S_0/c(I) \leq 1$. It is an open question whether the condition on the dimension of $S_0/c(I)$ can be dropped in this statement.

Some more explicit results concerning the regularity of $H^n_Q(R)_j$, in the case that $R = S/(f)$ is a hypersurface ring, are obtained in Section $1$ In Proposition $1.6$ we show that $\text{reg} H^n_Q(R)_j$ is a linear function if $f = \sum_{i=1}^n f_i y_i$ where $f_1, \ldots, f_n$ is a regular sequence, and in Corollary $1.8$ it is shown that $\text{reg} H^n_Q(R)_j$ is a linear function if $f = f_1 y_1 + f_2 y_2$ and $\deg \gcd(f_1, f_2) < \deg f_i$.

Unfortunately, the methods used in Section $1$ to bound the regularity can be used only for the top local cohomology $H^n_Q(R)$ and only when the content ideal of the defining ideal of $R$ is $m_0$-primary, or in the hypersurface case an ideal of height $m - 1$. The situation is much better when we consider local cohomology of multigraded $S$-modules. Indeed, in Section $2$ of the paper it is shown in Theorem $2.3$ that if $M$ is a finitely generated $\mathbb{Z}^m \times \mathbb{Z}^n$-graded $S$-module. Then there exists an integer $c$, which only depends on the $x$-shifts of the bigraded resolution of $M$, such that $|\text{reg} H^k_Q(M)_j| \leq c$ for all $k$ and all $j$. As mentioned above such a bound also exists when $M$ is only bigraded, but sequentially Cohen-Macaulay with respect to $Q$. The more it is surprising that in the multigraded case, no other cohomological condition on $M$ is required to obtain such a global bound. The proof of Theorem $2.3$ uses essentially a result of Bruns and the first author $[2, \text{Theorem 3.1}]$ which says that the multigraded shifts in the resolution of a multigraded $S$-module $M$ can be bounded in terms of the multigraded degrees of the generators of the first relation module of $M$.

1. On the Annihilation of the Graded Components of Top Local Cohomology with Applications to Regularity Bounds

Let $I = (f_1, \ldots, f_r)$ where the $f_i$ are bihomogeneous polynomials. Then the content ideal $c(I)$ of $I$ is defined to be the ideal $c(f_1) + \cdots + c(f_r) \subset S_0$ where for bihomogeneous polynomial $f = \sum_{|\beta|=b} f_\beta y^\beta$ the ideal $c(f)$ is the ideal in $S_0$ defined by the polynomials $f_\beta$. Obviously, the definition of $c(I)$ does not depend on the chosen set of generators of $I$. In this section we show that a certain power of the content ideal of $I$ annihilates $H^n_Q(R)_j$, and use this fact to bound the regularity of $H^n_Q(R)_j$ in some cases.

**Theorem 1.1.** Let $R = S/I$ where $I$ is a bigraded ideal in $S$. Then there exists a linear function $\ell : \mathbb{Z}_- \to \mathbb{Z}_+$ such that $c(I)^{\ell(j)}H^n_Q(R)_j = 0$ for all $j \leq 0$. 
Proof. Let \( f_1, \ldots, f_r \) be a minimal set of bihomogeneous polynomials generating the ideal \( I \), and assume that \( f_i \) is homogeneous of bidegree \((-a_i, -b_i)\). Then \( R \) has the following free presentation
\[
\cdots \longrightarrow \bigoplus_{i=1}^{r} S(-a_i, -b_i) \longrightarrow S \longrightarrow R \longrightarrow 0.
\]
By [5, Theorem 1.1], the \( S_0 \)-module \( H^n_Q(R)_j \) has then the following free \( S_0 \)-presentation
\[
\cdots \longrightarrow \bigoplus_{i=1}^{r} H^n_Q(S)(-a_i, -b_i)_j \xrightarrow{\varphi} H^n_Q(S) \longrightarrow H^n_Q(R) \longrightarrow 0.
\]
The map \( \varphi \) can be described as follows: We may write \( f_i = \sum_{|\beta|=b_i} f_i, \beta y^\beta \) with \( \deg f_i, \beta = a_i \) for \( i = 1, \ldots, r \) and set \( R = K[y_1, \ldots, y_n] \). By Formula (1) in [5] we have
\[
G_0 := H^n_Q(S)_j = \bigoplus_{|c|=-n-j} S_0 z^c,
\]
where \( z \in \text{Hom}_K(R_{-n-j}, K) \) is the \( K \)-linear map with
\[
z^a(y^b) = \begin{cases} 
z^{a-b}, & \text{if } b \leq a, \\
0, & \text{if } b < a.\end{cases}
\]
We set \( G_1 = \bigoplus_{i=1}^{r} F_i \) where
\[
F_i = H^n_Q(S)(-a_i, -b_i)_j = \bigoplus_{|c|=-n-j+b_i} S_0(-a_i) z^c.
\]
For \( G_0 \) the basis consists the elements \( z^c \) with \( |c| = -n - j \) and \( G_1 \) has a basis consisting of the elements \( e_i z^c \) with \( \deg e_i = a_i \) and \( |c| = -n - j + b_i \). We have \( \varphi(e_i z^c) = f_i, \beta z^{c-\beta} \) with \( |\beta| = b_i \) if \( \beta \leq c \), and otherwise 0. We set \( \varphi(F_i) = U_i \) for \( i = 1, \ldots, r \). Then \( \text{Im} \varphi = \sum_{i=1}^{r} U_i \). We set \( T_i = K[x_{i, \beta}]_{\beta \in N^n, |\beta|=b_i} \), \( P_i = T_i[y_1, \ldots, y_n] \) and \( g_i = \sum_{|\beta|=b_i} x_{i, \beta} y^\beta \) for \( i = 1, \ldots, r \). By [5, Proposition 5.2] there exists a linear function \( l_i(j) \) such that
\[
[H^n_Q(P_i/(g_i))_j]_{l_i(j)} = 0 \quad \text{for} \quad i = 1, \ldots, r.
\]
Since the graded \( T_i \)-module \( H^n_Q(P_i/(g_i))_j \) is generated in degree 0, it follows that \( m_i^{l_i(j)} H^n_Q(P_i/(g_i))_j = 0 \) where \( m_i = \{ x_{i, \beta} \}_{\beta \in N^n, |\beta|=b_i} \) for \( i = 1, \ldots, r \). Replacing \( x_{i, \beta} \) by \( f_i, \beta \) we obtain
\[
(1) \quad c(f_i)^{l_i(j)} H^n_Q(S/(f_i))_j = 0 \quad \text{for} \quad i = 1, \ldots, r.
\]
Indeed, in order to prove (1), we consider the map \( \pi : T_i \rightarrow S_0 \) where \( x_{i, \beta} \mapsto f_i, \beta \). It follows from the free presentations of \( H^n_Q(P_i/(g_i))_j \) and \( H^n_Q(S/(f_i))_j \) that
\[
(2) \quad H^n_Q(S/(f_i))_j \cong H^n_Q(P_i/(g_i))_j \otimes_{T_i} S_0.
\]
Notice that if \( m \in H^n_Q(P_i/(g_i))_j \) and \( h \in T_i \), then we have
\[
(hm) \otimes 1 = m \otimes \pi(h) = \pi(h)(m \otimes 1).
\]
In particular, if \( h \in m_i^{l(i)} \), then \( \pi(h)(m \otimes 1) = (hm) \otimes 1 = 0 \), and hence by (2) we have \( \pi(h)H^n_Q(S/(f_i))_j = 0 \) for all \( h \in m_i^{l(i)} \). Therefore \( c(f_i)^{l(i)}H^n_Q(S/(f_i))_j = 0 \), because \( \pi(m_i^{l(i)}) = c(f_i)^{l(i)} \).

Since the functor \( H^n_Q(\cdot) \) is right exact, the canonical epimorphism \( S/(f_i) \to S/I = R \) induces and epimorphism \( H^n_Q(S/(f_i))_j \to H^n_Q(R)_j \) for all \( j \). It follows that

\[
c(f_i)^{l(i)}H^n_Q(R)_j = 0 \quad \text{for} \quad i = 1, \ldots, r.
\]

We set \( \ell(j) = \sum_{i=1}^r l_i(j) \). Then \( \ell(j) \) is again a linear function of \( j \), and

\[
c(I)^{\ell(j)}H^n_Q(R)_j = \left( \sum_{i=1}^r c(f_i) \right)^{\ell(j)}H^n_Q(R)_j = 0,
\]

as desired. \( \square \)

Let \( N \) be a \( \mathbb{Z} \)-graded \( S_0 \)-module, and

\[
0 \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0,
\]

be the minimal graded free \( S_0 \)-resolution of \( N \) with \( F_i = \bigoplus_{j=1}^{t_i} S_0(-a_{ij}) \) for \( i = 1, \ldots, k \). Then the \textit{Castelnuovo-Mumford regularity} \( \text{reg} N \) of \( N \) is defined to be the integer

\[
\text{reg} N = \max_{i,j} \{ a_{ij} - i \}.
\]

**Corollary 1.2.** Assume in addition to Theorem [1.1] that \( c(I) \) is an \( m_0 \)-primary ideal where \( m_0 \) is the graded maximal ideal of \( S_0 \). Then there exists a linear function \( \ell : \mathbb{Z}_- \to \mathbb{Z}_+ \) such that

\[
0 \leq \text{reg} H^n_Q(R)_j \leq \ell(j)
\]

for all \( j \).

**Proof.** The lower bound for the regularity follows from the fact that \( H^n_Q(R)_j \) is generated in degree 0. Since \( c(I) \) is an \( m_0 \)-primary, it follows that \( m_0^k \subseteq c(I) \) for some \( k \) and hence by Theorem [1.1] there exists a linear function \( \ell' \) such that

\[
m_0^{k\ell'(j)}H^n_Q(R)_j = c(I)^{\ell'(j)}H^n_Q(R)_j = 0.
\]

Therefore \( H^n_Q(R)_j \) is of finite length for all \( j \) and \( \text{reg} H^n_Q(R)_j \leq k\ell'(j) - 1 = \ell(j) \).

**Corollary 1.3.** With the assumptions and the notation of Theorem [1.1] we have

\[
\dim S_0 H^n_Q(R)_j = \dim S_0/c(I) \quad \text{for all} \quad j.
\]

**Proof.** As we have already seen, \( H^n_Q(R)_j \) has the following \( S_0 \)-presentation

\[
\cdots \to \bigoplus_{i=1}^r S_0^{n_i}(-a_i) \xrightarrow{\varphi_j} S_0^{n_0} \to H^n_Q(R)_j \to 0,
\]

where \( n_0 = (\binom{j-1}{n-1}) \) and \( n_i = (\binom{j+b_i}{n-1}) \) for \( i = 1, \ldots, r \). Let \( U_j \) be the matrix describing \( \varphi_j \) with respect to the canonical bases. Notice that \( I_{n_0}(U_j) \subseteq c(I) \) where \( I_{n_0}(U_j) \) is the ideal generated by the \( n_0 \)-minors of matrix \( U_j \). Thus

\[
\dim S_0/c(I) \leq \dim S_0/I_{n_0}(U_j) = \dim H^n_Q(R)_j.
\]
Here the second equality follows from Formula (5) in [5]. On the other hand, Theorem 1.1 implies that \( \dim H^0_Q(R)_j \leq \dim S_0/c(I) \). Therefore the desired equality follows. \( \Box \)

The following known fact is needed for the proof of the next corollary. For the convenience of the reader we include its proof.

Lemma 1.4. Let \( M \) be a graded \( S_0 \)-module with \( \dim M > 0 \) and \( |K| = \infty \). Suppose \( f \) be a linear form such that \( 0 :_M f \) has finite length. Then

\[
\dim M/fM = \dim M - 1.
\]

Proof. We denote by \( H_M(t) = \sum_{i \in \mathbb{Z}} \dim_K M_it^i \) the Hilbert-series of \( M \). Consider the exact sequence \( 0 \to 0 :_M f \to M(-1) \to M \to M/fM \to 0 \). Since \( (0 :_M f) \) has finite length, it follows that \( H_{(0:_M f)}(t) = Q_0(t) \), where \( Q_0(t) \) is a polynomial in \( t \) with \( Q_0(1) \neq 0 \). We may also write \( H_M(t) = Q(t)/(1 - t)^d \) where \( d = \dim M \) with \( Q(1) \neq 0 \). Hence

\[
H_{M/fM}(t) = H_M(t) - tH_M(t) + H_{0:Mf}(t)
= P(t)/(1 - t)^{d - 1}
\]
where \( P(t) = Q(t) + (1 - t)^{d - 1}Q_0(t) \) with \( P(1) \neq 0 \). Thus the desired equality follows from [1 Corollary 4.1.8]. \( \Box \)

As another application of Theorem 1.1 we have

Theorem 1.5. Consider the hypersurface ring \( R = S/fS \) where \( f \) is a bihomogeneous polynomial in \( S \). Suppose that \( \dim S_0/c(f) \leq 1 \). Then there exists a linear function \( \ell : \mathbb{Z}_- \to \mathbb{Z}_+ \) such that

\[
0 \leq \text{reg} \ H^0_Q(R)_j \leq \ell(j)
\]
for all \( k \) and \( j \).

Proof. Note that \( R \) has only two non-vanishing local cohomology modules \( H^0_Q(R)_j \), namely for \( k = n - 1 \) and \( k = n \). A similar argument as that used in the proof of [5 Proposition 5.1](b) shows that \( H^{n-1}_Q(R)_j \) is linearly bounded, provided \( H^n_Q(R)_j \) is linearly bounded. Thus it suffices to consider the case \( k = n \). The desired result follows from Corollary 1.2 in the case that \( \dim S_0/c(f) = 0 \). Now let us assume that \( \dim S_0/c(f) = 1 \). We may assume that \( K \) is infinite. Otherwise, we apply a suitable base field extension. Thus we can choose a linear form \( g \) such that \( 0 :_H^0_Q(R)_j g \) and \( 0 :_{S_0/c(f)} g \) are finite length modules. After a change of coordinate we may assume that \( g = x_1 \). For any \( S_0 \)-module \( M \) we set \( \overline{M} = M/x_1M \). We first observe that \( \overline{H^n_Q(R)}_j = H^n_Q(\overline{R})_j \). Indeed, since the functor \( H^n_Q(-) \) is right exact, the exact sequence \( R \to R \to \overline{R} \to 0 \) induces the exact sequence \( H^n_Q(R)_j \to H^n_Q(\overline{R})_j \to 0 \), which yields the desired isomorphism.

Next observe that \( \overline{S_0/c(f)} = \overline{S_0/c(\overline{f})} \) where \( \overline{f} \) is the image of \( f \) under the canonical epimorphism \( S_0 \to \overline{S_0} \). Identifying \( S_0 \) with \( K[x_2, \ldots, x_n] \) the map \( f \mapsto \overline{f} \) is obtained by substituting \( x_1 \) by 0. Since \( \dim S_0/c(f) = 1 \) and since \( 0 :_{S_0/c(f)} \) \( x_1 \) has
Lemma 1.4. By [4, Proposition 20.20] we have
\[ \text{reg } H^n_Q(R_j) = \max \{ \text{reg} (0 : H^n_Q(R_j), x_1), \text{reg} H^n_Q(R_j) \} \]

Moreover, by Corollary 1.3, we have \( \dim H^n_Q(R_j) = \dim \overline{S_0}/c(\overline{f}) = 0 \). Hence, it follows from [5, Theorem 5.3] that \( \text{reg} H^n_Q(R_j) \) is linearly bounded as a function of \( j \). Now we show that \( \text{reg} (0 : H^n_Q(R_j), x_1) \) is linearly bounded, as well. Then by (3) the desired conclusion follows. The exact sequence \( 0 \to R \xrightarrow{x_1} R \to R/c(\overline{f}) \to 0 \) induces the exact sequence

\[ H^{n-1}_Q(R_j) \to H^{n-1}_Q(R_j) \to 0 : H^n_Q(R_j), x_1 \to 0. \]

In particular, we conclude that the highest degree of a generator of \( 0 : H^n_Q(R_j), x_1 \) is less than or equal to the highest degree of a generator of \( H^{n-1}_Q(R_j) \). As \( \text{reg} H^n_Q(R_j) \) is linearly bounded, it follows that there exists a linear function \( \ell' \) such that \( \text{reg} H^{n-1}_Q(R_j) \leq \ell'(j) \), see the proof of [5] Proposition 5.1(b). Thus this linear function \( \ell' \) also bounds the highest degree of a generator of \( H^{n-1}_Q(R_j) \), and hence the highest degree of a generator of \( 0 : H^n_Q(R_j), x_1 \).

By Theorem 1.4 there exists a linear function \( \ell'' \) such that
\[ c(f)^{\ell''(j)}(0 : H^n_Q(R_j), x_1) \subseteq c(f)^{\ell''(j)} H^n_Q(R_j) = 0. \]

Since \( \dim \overline{S_0}/c(\overline{f}) = 0 \), it follows that there exists an integer \( k \) such that \( m_0^{k(\ell''(j) + 1)}(0 : H^n_Q(R_j), x_1) \subseteq c(\overline{f})^{\ell''(j)}(0 : H^n_Q(R_j), x_1) = 0 \).

Hence we obtain
\[ m_0^{k(\ell''(j) + 1)}(0 : H^n_Q(R_j), x_1) = m_0^{k(\ell''(j))} = k(\ell''(j)) \subseteq c(\overline{f})^{\ell''(j)}(0 : H^n_Q(R_j), x_1) = 0. \]

We conclude that \( (0 : H^n_Q(R_j), x_1)_i = 0 \) for \( i \geq \ell'(j) + k \ell''(j) \) and therefore,
\[ \text{reg} (0 : H^n_Q(R_j), x_1) \leq \ell'(j) + k \ell''(j). \]

\( \square \)

For a hypersurface ring of bidegree \((d, 1)\) we have the following more precise result.

**Proposition 1.6.** Let \( R = S/f S \) where \( f = \sum_{i=1}^n f_i y_i \) with \( \deg f_i = d \) for all \( i \). If \( f_1, \ldots, f_n \) is a regular sequence, then
\[ \text{reg} H^n_Q(R_j) = \text{reg} S_0/c(I)^{-n-j+1} = -dj - n. \]

**Proof.** We first assume that \( f_i = x_i \) for \( i = 1, \ldots, n \) and set \( R' = S/g S \) where \( g = \sum_{i=1}^n x_i y_i \). By the statement after [5] Proposition 4.5, \( H^n_Q(R')_j \) has the following free \( S_0 \)-resolution
\[ 0 \to S_0^{\beta_1}(j) \to \cdots \to S_0^{\beta_j}(n+j-3) \to S_0^{\beta_2}(n+j-2) \to S_0^{\beta_1}(1) \to S_0^{\beta_0} \to H^n_Q(R')_j \to 0. \]
Hence reg $S$ we first assume that $f$ graded maximal ideal of $S$.

Let Lemma 1.7. It follows that reg $H^i_Q(R)_j = -dj - n$. Now we prove the second equality. As before, we first assume that $f_i = x_i$ for $i = 1, \ldots, n$ and hence $c(I) = m_0$ where $m_0$ is the graded maximal ideal of $S_0$. We set $k = -n - j + 1$. Due to the well-known fact that $m^k$ has a linear resolution, the minimal graded free $S_0$-resolution of $S_0/m_0$ is of the form

$$0 \to S^\beta_0(n + k - 1) \to \cdots \to S^\beta_0(k + 1) \to S^\beta_0(k) \to S_0 \to S_0/m_0^k \to 0.$$

Hence reg $S_0/m_0^k = k - 1$. As before, by using flatness of $\pi$, we obtain the following free resolution for $S_0/c(I)^k$

$$0 \to S^\beta_0(d(n + k - 1)) \to \cdots \to S^\beta_0(d(k + 1)) \to S^\beta_0(dk) \to S_0 \to S_0/c(I)^k \to 0.$$

Hence reg $S_0/c(I)^k = d(n + k - 1) - n = -dj - 1$. \hfill $\Box$

For the proof of the next corollary we need the following

**Lemma 1.7.** Let $R = S/ghS$ and $R' = S/hs$ where $g \in S_0$ is a homogeneous polynomial and $h \in S$ is a bihomogeneous polynomial. Then

$$\text{reg } H^n_Q(R)_j = \text{reg } H^n_Q(R')_j + \deg g.$$

**Proof.** Let $F/U_j$ be the standard presentation of $H^n_Q(R)_j$, as described in [5, Section 3, page 322]. Then it follows that $F/gU_j$ is the standard presentation of $H^n_Q(R)_j$. This yields the desired conclusion. \hfill $\Box$

**Corollary 1.8.** Let $R = S/fS$ where $f = f_1y_1 + f_2y_2$ is a bihomogeneous polynomial in $S$ of bidegree $(d, 1)$, and set $g = \gcd(f_1, f_2)$. Then

$$\text{reg } H^2_Q(R)_j = \begin{cases} -(d - \deg g)j + \deg g - 2 & \text{if } \deg g < d, \\ \deg g & \text{if } \deg g = d. \end{cases}$$

**Proof.** If $\deg g < d$, then we may write $f = gh$ where $h = h_1y_1 + h_2y_2$ with $\gcd(h_1, h_2) = 1$ and $\deg h_i > 0$ for $i = 1, 2$. Note that $h_1, h_2$ is a regular sequence. Hence by Proposition 1.6 and Lemma 1.7 we have the first equality. If $\deg g = d$, then $\deg h_1 = \deg h_2 = 0$. Hence $H^2_Q(R') = 0$ where $R' = S/hs$. Therefore, the second equality follows from Lemma 1.7. \hfill $\Box$

2. **An upper bound for the regularity of local cohomology of multigraded modules**

Let $K$ be a field and $S = K[x_1, \ldots, x_m, y_1, \ldots, y_n]$ be the polynomial ring over $K$ in the variables $x_1, \ldots, x_m, y_1, \ldots, y_n$. We consider $S$ as standard $\mathbb{Z}^m \times \mathbb{Z}^n$-graded $K$-algebra. Let $M$ be a finitely generated $\mathbb{Z}^m \times \mathbb{Z}^n$-graded $S$-module. Computing local
cohomology by using the Čech complex shows that \( H^s(M) \) is naturally \( \mathbb{Z}^m \times \mathbb{Z}^n \)-graded. Therefore, in view of the fact that \( H^s(M) = \bigoplus_k H^s(M)_{(k,j)} \) we see that the \( \mathbb{Z} \)-graded components \( H^s_j(M) \) of local cohomology modules \( H^s(M) \) are naturally \( \mathbb{Z}^m \)-graded \( S_0 \)-modules where \( S_0 \) is the standard \( \mathbb{Z}^m \)-graded \( K \)-algebra \( K[x_1, \ldots, x_m] \), see \cite{6} Section 1] for more details. In particular, \( H^s(M) \) may also be viewed as a \( \mathbb{Z} \)-graded module over \( S_0 \). We recall the following theorem from \cite{2}.

**Theorem 2.1.** Let \( N \) be a finitely generated \( \mathbb{Z}^m \)-graded \( S_0 \)-module. Let

\[
0 \to F_k \to \cdots \to F_1 \to F_0 \to N \to 0,
\]

be the minimal graded free \( S_0 \)-resolution of \( N \) with \( F_i = \bigoplus_{j=1}^{t_i} S_0(-a_{ij}) \) for \( i = 1, \ldots, k \). Assume that the first multigraded shifts \( a_{0j} \) of \( N \) belong to \( \mathbb{N}^m \). Then for all \( i \) and all \( j = 1, \ldots, t_i \) we have

\[
x^{a_{ij}} \mid \text{lcm}(x^{a_{i1}}, \ldots, x^{a_{it_i}}).
\]

As an immediate consequence we obtain

**Corollary 2.2.** The regularity of \( N \) is bounded by a constant \( c \) which only depends on the shifts \( a_{ij} \) with \( i \leq 1 \).

As a main result of this section we have

**Theorem 2.3.** Let \( M \) be a finitely generated \( \mathbb{Z}^m \times \mathbb{Z}^n \)-graded \( S \)-module. Then there exists an integer \( c \), which only depends on the \( x \)-shifts of the bigraded resolution of \( M \), such that

\[
| \text{reg} H^s(M)_j | \leq c \quad \text{for all } s \text{ and all } j.
\]

**Proof.** By applying a suitable multigraded shift to \( M \) we may assume that all generators of \( M \) have multidegrees belonging to \( \mathbb{N}^m \times \mathbb{N}^n \). Then all shifts in the multigraded resolution of \( M \) belong to \( \mathbb{N}^m \times \mathbb{N}^n \). Let

\[
\mathbb{F} : 0 \to F_l \xrightarrow{\varphi_l} \cdots \to F_1 \xrightarrow{\varphi_1} F_0 \xrightarrow{\varphi_0} M \to 0,
\]

be a \( \mathbb{Z}^m \times \mathbb{Z}^n \)-graded free resolution of \( M \) where \( F_i = \bigoplus_{k=1}^{t_i} S(-a_{ik}, -b_{ik}) \) for \( i = 1, \ldots, l \). Applying the functor \( H^n(M)_j \) to this resolution yields a graded complex of free \( \mathbb{Z}^m \)-graded modules

\[
H^n(M)_j : 0 \to H^n(F_l)_j \xrightarrow{\varphi^*_l} \cdots \to H^n(F_1)_j \xrightarrow{\varphi^*_1} H^n(F_0)_j \xrightarrow{\varphi^*_0} H^n(M)_j \to 0.
\]

Notice that

\[
H^n(F_i)_j = \bigoplus_{k=1}^{t_i} \bigoplus_{|a|=n-j+|b_{ik}|} S_0(-a_{ik})z^a,
\]

is a finitely generated free \( S_0 \)-module. It follows that the \( \mathbb{Z} \)-graded modules \( H^s_j(M) \) are all generated in non-negative degrees. In particular, \( \text{reg} H^s(M)_j \geq 0 \). Thus it suffices to show that there exists an integer \( c \) such that \( \text{reg} H^s(M)_j \leq c \).

For each \( i = 1, \ldots, l \), consider the exact sequence

\[
0 \to \text{Ker} \varphi^*_i \to H^n(F_i)_j \xrightarrow{\varphi^*_i} H^n(F_{i-1})_j \to N_{i-1,j} \to 0,
\]

where
where $N_{i-1,j} = \text{Coker} \varphi_i^*$. It follows from (11) and Corollary 2.2 that there exists an integer $c_{i-1}$ such that $\text{reg} N_{i-1,j} \leq c_{i-1}$. Note that the constant $c_{i-1}$ does not depend on $j$. Hence from (3) one obtains $\text{reg} \varphi_i^* = \text{reg} N_{i-1,j} + 1$ and $\text{reg} \ker \varphi_i^* = \text{reg} N_{i-1,j} + 2$. By [5, Theorem 1.1] we have

$$H_Q^i(M)_j \cong H_{Q^*}^i(H_Q^j(F)_j) \cong \ker \varphi_{n-s}/\text{Im} \varphi_{n-s+1},$$

which is an isomorphism of $\mathbb{Z}^m$-graded $S_0$-modules. Therefore, the exact sequence $0 \to \text{Im} \varphi_{n-s+1} \to \ker \varphi_{n-s} \to H_Q^i(M)_j \to 0$ yields

$$\text{reg} H_Q^i(M)_j \leq \max\{\text{reg} \ker \varphi_{n-s}, \text{reg} \text{Im} \varphi_{n-s+1} - 1\}$$

$$= \max\{\text{reg} N_{n-s-1,j} + 2, \text{reg} N_{n-s,j}\}$$

$$\leq \max\{c_{n-s-1} + 2, c_{n-s}\} \leq c,$$

where $c = \max_i\{c_i\} + 2$. □

**Corollary 2.4.** Let $I \subseteq S$ be a monomial ideal. Then there exists an integer $c$ such that

$$|\text{reg} H_Q^i(S/I)_j| \leq c \quad \text{for all } i \text{ and all } j.$$

For the top local cohomology of $K$-algebras with monomial relations we have the following more precise statement.

**Proposition 2.5.** Let $I = (u_1v_1, \ldots, u_rv_r)$ be a monomial ideal where the $u_i$ are monomials in $K[x_1, \ldots, x_n]$ and the $v_j$ are monomials in $K[y_1, \ldots, y_n]$. We let $J$ be the monomial ideal $K[x_1, \ldots, x_n]$ generated by $u_1, \ldots, u_r$. Then

$$H_Q^n(S/I)_j \cong (S_0/J)^{(-1)}_{(-1)}.$$

In particular, the regularity of $H_Q^n(S/I)_j$ is constant, namely equal to $\text{reg} S_0/J$, for $j \leq -n$.

**Proof.** We set $\deg(u_i v_i) = (a_i, b_i)$ for $i = 1, \ldots, r$. Let

$$\cdots \to \bigoplus_{i=1}^r S(-a_i, -b_i) \to S \to S/I \to 0,$$

be the free presentation of $S/I$. Then $H_Q^n(S/I)_j$ has the following $S_0$-presentation

$$\cdots \to G_1 \xrightarrow{\varphi} G_0 \to H_Q^n(S/I)_j \to 0$$

by free $S_0$-modules, where

$$G_0 = \bigoplus_{|c| = -n-j} S_0 z^c \quad \text{and} \quad G_1 = \bigoplus_{|c| = -n-j + b_i} \bigoplus_{i=1}^r S_0(-a_i) z^c,$$

with $\varphi$ as described in the proof Theorem [11]. For $G_0$ the basis consists the monomials $z^c$ with $|c| = -n - j$ and $G_1$ has a basis consisting of the elements $e_i z^c$ with $\deg e_i = a_i$ and $|c| = -n - j + b_i$. We have $\varphi(e_i z^c) = u_i z^{c - \beta_i}$ if $\beta_i \leq c$ where $v_i = y^{\beta_i}$, and otherwise 0. It follows from this description that each column of the matrix describing $\varphi$ with respect to this basis has only one non-zero entry and the entries of each of this matrix generates $J$. This implies that $H_Q^n(S/I)_j \cong G_0/JG_0$, as desired. □
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