STRING EQUATIONS OF THE Q-KP HIERARCHY

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ABSTRACT. Based on the Lax operator $L$ and Orlov-Shulman’s $M$ operator, the string equations of the $q$-KP hierarchy are established from special additional symmetry flows, and the negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of the 2-reduced $q$-KP hierarchy are also obtained.

Keywords: $q$-KP hierarchy, additional symmetry, string equations, Virasoro constraint
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1. Introduction

The $q$-deformed integrable system (also called $q$-analogue or $q$-deformation of classical integrable system) is defined by means of $q$-derivative $\partial_q$ [1,2] instead of usual derivative $\partial$ with respect to $x$ in a classical system. It reduces to a classical integrable system as $q \to 1$. Recently, the $q$-deformed Kadomtsev-Petviashvili ($q$-KP) hierarchy is a subject of intensive study in the literature from [3] to [14]. Its infinite conservation laws, bi-Hamiltonian structure, $\tau$ function, additional symmetries and its constrained sub-hierarchy have already been reported in [4,5,11,12,14].

The additional symmetries, string equations and Virasoro constraints of the KP hierarchy are important as they are involved in the matrix models of the string theory [15]. For example, there are several new works [16–20] on this topic. The additional symmetries were discovered independently at least twice by Sato School [21] and Orlov-Shulman [22], in quite different environments and philosophy although they are equivalent essentially. It is well-known that L.A.Dickey [23] presented a very elegant and compact proof of Adler-Shiota-van Moerbeke (ASvM) formula [24,25] based on the Lax operator $L$ and Orlov and Shulman’s $M$ operator [22], and gave the string equation and the action of the additional symmetries on the $\tau$ function of the classical KP hierarchy. S.Panda and S.Roy gave the Virasoro and $W$-constraints on the $\tau$ function of the $p$-reduced KP hierarchy by expanding the additional symmetry operator in terms of the Lax operator [26,27]. It is quite interesting to study the analogous properties of $q$-deformed KP hierarchy by this expanding method . The main purpose of this article is to give the string equations of the $q$-KP hierarchy, and then study the negative Virasoro constraint generators $\{L_{-n}, n \geq 1\}$ of 2-reduced $q$-KP hierarchy.

The organization of this paper is as follows. We recall some basic results and additional symmetries of $q$-KP hierarchy in Section 2. The string equations are given in Sections 3. The Virasoro constraints on the $\tau$ function of the 2-reduced ($q$-KdV) hierarchy are studied in Section 4. Section 5 is devoted to conclusions and discussions.

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At the end of this section, we shall collect some useful facts of $q$-calculus [2] to make this paper be self-contained. The $q$-derivative $\partial_q$ is defined by

$$\partial_q(f(x)) = \frac{f(qx) - f(x)}{q-1}$$

and the $q$-shift operator is

$$\theta(f(x)) = f(qx).$$

$\partial_q(f(x))$ recovers the ordinary differentiation $\partial_x(f(x))$ as $q$ goes to 1. Let $\partial_q^{-1}$ denote the formal inverse of $\partial_q$. In general the following $q$-deformed Leibnitz rule holds

$$\partial_q^n \circ f = \sum_{k \geq 0} \left( \begin{array}{c} n \\ k \end{array} \right)_q \theta^{n-k}(\partial_q^n f) \partial_q^{n-k}, \quad n \in \mathbb{Z}$$

where the $q$-number and the $q$-binomial are defined by

$$(n)_q = \frac{q^n - 1}{q - 1},$$

$$\left( \begin{array}{c} n \\ k \end{array} \right)_q = \frac{(n)_q(n-1)_q \cdots (n-k+1)_q}{(1)_q(2)_q \cdots (k)_q}, \quad \left( \begin{array}{c} n \\ 0 \end{array} \right)_q = 1.$$  

For a $q$-pseudo-differential operator ($q$-PDO) of the form $P = \sum_{i=-\infty}^{i=\infty} p_i \partial_q^i$, we separate $P$ into the differential part $P_+ = \sum_{i \geq 0} p_i \partial_q^i$ and the integral part $P_- = \sum_{i \leq -1} p_i \partial_q^i$. The conjugate operation “$*$” for $P$ is defined by $P^* = \sum_{i} (\partial_q^i)^* p_i$ with $\partial_q^* = -\partial_q \theta^{-1} = -\frac{1}{q} \partial_q$, $(\partial_q^{-1})^* = (\partial_q^*)^{-1} = -\theta \partial^{-1}$. The $q$-exponent $e^x_q$ is defined as follows

$$e^x_q = \sum_{n=0}^{\infty} \frac{x^n}{(n)_q!}, \quad (n)_q! = (n)_q(n-1)_q(n-2)_q \cdots (1)_q.$$  

Its equivalent expression is of the form

$$e^x_q = \exp \left( \sum_{k=1}^{\infty} \frac{(1-q)^k}{k(1-q^k)} x^k \right),$$

which is crucial to develop the $\tau$ function of the $q$-KP hierarchy [11].

2. $q$-KP Hierarchy and its Additional Symmetries

Similar to the general way of describing the classical KP hierarchy [21, 28], we first give a brief introduction of $q$-KP hierarchy and its additional symmetries based on [11, 12].

Let $L$ be one $q$-PDO given by

$$L = \partial_q + u_0 + u_{-1} \partial_q^{-1} + u_{-2} \partial_q^{-2} + \cdots,$$  

which are called Lax operator of $q$-KP hierarchy. There exist infinite number of $q$-partial differential equations related to dynamical variables $\{u_i(x, t_1, t_2, t_3, \cdots), i = 0, -1, -2, -3, \cdots \}$ and can be deduced from the generalized Lax equation,

$$\frac{\partial L}{\partial t_n} = [B_n, L], \quad n = 1, 2, 3, \cdots,$$
which are called $q$-KP hierarchy. Here $B_n = (L^n)_+ = \sum_{i=0}^{n} b_i \partial^i_q$ and $L^n = L^n - L^n_1$. $L$ in eq. (2.1) can be generated by dressing operator $S = 1 + \sum_{k=1}^{\infty} s_k \partial^{-k}_q$ in the following way
\[
L = S \circ \partial_q \circ S^{-1}.
\]

Dressing operator $S$ satisfies Sato equation
\[
\frac{\partial S}{\partial t_n} = -(L^n)_- S, \quad n = 1, 2, 3, \ldots.
\]

The $q$-wave function $w_q(x, t; z)$ and the $q$-adjoint function $w^*_q(x, t; z)$ are given by
\[
w_q = S e^{xz} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right),
\]
\[
w^*_q(x, t; z) = (S^*)^{-1}|_{x/q} e^{-xz} \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right),
\]

which satisfies following linear $q$-differential equations
\[
L w_q = z w_q, \quad L^*|_{x/q} w^*_q = z w^*_q.
\]

Here the notation $P|_{x/t} = \sum_i P_i(x/t) t^i \partial^i_q$ is used for $P = \sum_i p_i(x) \partial^i_q$.

Furthermore, $w_q(x, t; z)$ and $w^*_q(x, t; z)$ can be expressed by sole function $\tau_q(x; t)$ [11] as
\[
w_q = \frac{\tau_q(x; t - [z^{-1}])}{\tau_q(x; t)} e^{xz} \exp\left(\sum_{i=1}^{\infty} t_i z^i\right) = \frac{e^{xz} e^{\xi(t, z)} e^{-\sum_{i=1}^{\infty} \frac{z^i}{i} \partial_i \tau_q}}{\tau_q},
\]
\[
w^*_q = \frac{\tau_q(x; t + [z^{-1}])}{\tau_q(x; t)} e^{-xz} \exp\left(-\sum_{i=1}^{\infty} t_i z^i\right) = \frac{e^{-xz} e^{-\xi(t, z)} e^{\sum_{i=1}^{\infty} \frac{z^i}{i} \partial_i \tau_q}}{\tau_q},
\]

where
\[
[z] = \left(z, \frac{z^2}{2}, \frac{z^3}{3}, \ldots\right).
\]

The following Lemma shows there exist an essential correspondence between $q$-KP hierarchy and KP hierarchy.

**Lemma 1.** [11] Let $L_1 = \partial + u_{-1} \partial^{-1} + u_{-2} \partial^{-2} + \cdots$, where $\partial = \partial/\partial x$, be a solution of the classical KP hierarchy and $\tau$ be its tau function. Then
\[
\tau_q(x, t) = \tau(t + [x]_q)
\]
is a tau function of the $q$-KP hierarchy associated with Lax operator $L$ in eq. (2.1), where
\[
[x]_q = (x, \frac{(1 - q)^2}{2(1 - q^2)} x^2, \frac{(1 - q)^3}{3(1 - q^3)} x^3, \ldots, \frac{(1 - q)^i}{i(1 - q^i)} x^i, \ldots).
\]

Define $\Gamma_q$ and Orlov-Shulman’s $M$ operator
\[
\Gamma_q = \sum_{i=1}^{\infty} \left( i t_i + \frac{(1 - q)^i}{(1 - q^i)} x^i \right) \partial_q^{-1},
\]
$M = S\Gamma_q S^{-1}.$ \hfill (2.7)

Dressing $[\partial_k - \partial^k_q, \Gamma_q] = 0$ gives

$\partial_k M = [B_k, M].$ \hfill (2.8)

Eq. (2.2) together with eq. (2.8) implies that

$\partial_k (M^m L^n) = [B_k, M^m L^n].$ \hfill (2.9)

Define the additional flows for each pair $m, n$ as follows

$\frac{\partial S}{\partial t^*_{m,n}} = -(M^m L^n)_- S,$ \hfill (2.10)

or equivalently

$f\frac{\partial L}{\partial t^*_{m,n}} = -(M^m L^n)_-, L,$ \hfill (2.11)

$f\frac{\partial M}{\partial t^*_{m,n}} = -(M^m L^n)_-, M.$ \hfill (2.12)

The additional flows $\partial^*_m = \frac{\partial}{\partial t^*_m}$ commute with the hierarchy, i.e. $[\partial^*_{m,n}, \partial_k] = 0$ but do not commute with each other, so they are additional symmetries \cite{12}. $(M^m L^n)_-$ serves as the generator of the additional symmetries along the trajectory parametrized by $t^*_{m,n}$.

3. String equations of the $q$-KP hierarchy

In this section we shall get string equations for the $q$-KP hierarchy from special additional symmetry flows. For this, we need a lemma.

**Lemma 2.** The following equation

$[M, L] = -1$ \hfill (3.1)

holds.

**Proof.** Direct calculations show that

$[\Gamma_q, \partial_q] = \left\{ \sum_{i=1}^{\infty} \left[ it_i + \frac{(1-q)^i}{1-q^i} x^i \right] \partial_q^{i-1}, \partial_q \right\}$

$= \sum_{i=1}^{\infty} \left[ \frac{(1-q)^i}{1-q^i} x^i \partial_q^{i-1}, \partial_q \right]$ \hfill

$= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left( x^i \partial_q^{i-1} - (\partial_q \circ x^i) \partial_q^{i-1} \right)$ \hfill

$= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left( (\partial_q x^i) + q^i x^i \partial_q \partial_q^{i-1} \right)$ \hfill

$= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left( (1-q^i) x^i \partial_q^{i-1} - \frac{1-q^i}{1-q} x^{i-1} \partial_q^{i-1} \right)$ \hfill

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$= \sum_{i=1}^{\infty} \frac{(1-q)^i}{1-q^i} \left( (1-q^i) x^i \partial_q^{i-1} - \frac{1-q^i}{1-q} x^{i-1} \partial_q^{i-1} \right)$ \hfill
\[
\sum_{i=1}^{\infty} ((1 - q)^i x^i \partial_q^i - (1 - q)^{i-1} x^{i-1} \partial_q^{i-1})
\]

\[
= -1,
\]

where we have used \([t_i, \partial_q] = 0\) in the second step and \(\partial_q \circ x^i = (\partial_q x^i) + q^i x^i \partial_q\) in the fourth step. Then

\[
[M, L] = [S \Gamma_q S^{-1}, S \partial_q S^{-1}] = S[\Gamma_q, \partial_q]S^{-1} = -1.
\]

By virtue of Lemma 2, we have

Corollary 1. \([M, L] = -1\) implies \([M, L^n] = -n L^{n-1}\). Therefore,

\[
[ML^{-n+1}, L^n] = -n.
\]

The action of additional flows \(\partial_{1, -n+1}^*\) on \(L^n\) are \(\partial_{1, -n+1}^* L^n = -[(ML^{-n+1})_+, L^n]\), which can be written as

\[
\partial_{1, -n+1}^* L^n = [(ML^{-n+1})_+, L^n] + n.
\]

The following theorem holds by virtue of eq.(3.3).

Theorem 1. If an operator \(L\) does not depend on the parameters \(t_n\) and the additional variables \(t_{1, -n+1}\), then \(L^n\) is a purely differential operator, and the string equations of the \(q\)-KP hierarchy are given by

\[
[L^n, \frac{1}{n} (ML^{-n+1})_+] = 1, \ n = 2, 3, 4, \ldots
\]

In view of the additional symmetries and string equations, we can get the following corollary, which plays a crucial role in the study of the constraints on the \(\tau\) function of the \(p\)-reduced \(q\)-KP hierarchy.

Corollary 2. If \(L^n\) is a differential operator, and \(\partial_{1, -n+1}^* S = 0\), then

\[
(ML^{-n+1})_+ = \frac{n-1}{2} L^{-n}, \ n = 2, 3, 4, \ldots
\]

Proof. Since \([M, L] = -1\), it is not difficult to obtain

\[
[M, L^{-n+1}] = (n-1)L^{-n},
\]

and hence

\[
(ML^{-n+1})_- = (L^{-n+1}M)_- = (n-1)L^{-n}.
\]

Noticing \([L^{-n}, L^n] = 0\), then

\[
[(ML^{-n+1})_-, (L^{-n+1}M)_-, L^n] = 0, \ \text{i.e.,}
\]

\[
[(ML^{-n+1})_-, L^n] = [(L^{-n+1}M)_-, L^n].
\]

Thus

\[
\partial_{1, -n+1}^* L^n = -[(L^{-n+1}M)_-, L^n]
\]

\[
= -\frac{1}{2} [(ML^{-n+1})_+ + (L^{-n+1}M)_-, L^n],
\]
or equivalently

$$\partial_{n+1} L = \frac{1}{2}(ML^{-n+1} + L^{-n+1}M)_- S.$$  

Therefore, it follows from the equation $$\partial_{n+1} L = 0$$ that

$$(ML^{-n+1} + L^{-n+1}M)_- = 0.$$  

Combining this with (3.6) finishes the proof. \qed

4. Constraints on the $\tau$ function of the $q$-KdV hierarchy

In this section, we mainly study the associated constraints on $\tau$-function of the 2-reduced $q$-KP ($q$-KdV) hierarchy from string equations eq. (3.4). To this end, we first define residue $res L = u_{-1}$ of $L$ given by eq. (2.1) and state two very useful lemmas.

**Lemma 3.** For $n = 1, 2, 3, \cdots$,

$$res L^n = \frac{\partial^2 \log \tau_q}{\partial t_1 \partial t_n}.$$  

(4.1)

where $\tau_q$ is the tau function of the $q$-KP hierarchy.

**Proof.** Taking the residue of $\frac{\partial s}{\partial t_n} = -(L^n)_- S$, we get

$$\frac{\partial s_1}{\partial t_n} = -res((L^n)_- (1 + s_1 \partial^{-1} + s_2 \partial^{-2} + \cdots )) = -res(L^n)_- = -res L^n.$$  

Noting that $u_0 = s_1 - \theta(s_1) = -x(q-1)\partial_q s_1 = x(q-1)\partial_q \log \tau_q$, $s_1 = -\frac{\partial \log \tau_q}{\partial t_1}$ (see [14]), then

$$res L^n = -\frac{\partial s_1}{\partial t_n} = \frac{\partial^2 \log \tau_q}{\partial t_1 \partial t_n}.$$  

\qed

**Lemma 4.** Orlov-Shulman’s $M$ operator has the expansion of the form

$$M = \sum_{i=1}^{\infty} \left(it_i + \frac{(1-q)^i}{1-q} x^i\right) L^{i-1} + \sum_{i=1}^{\infty} V_{i+1} L^{-i-1},$$  

(4.2)

where

$$V_{i+1} = -i \sum_{a_1 + 2a_2 + 3a_3 + \cdots = i} (-1)^{a_1 + a_2 + \cdots} \frac{(\partial t_1)^{a_1} (\frac{1}{2} \partial t_2)^{a_2} (\frac{1}{3} \partial t_3)^{a_3}}{a_1! a_2! a_3!} \cdots \log \tau_q.$$  

**Proof.** First, we assert $Mw_q = \frac{\partial w_q}{\partial z}$. Indeed, from the identity $\partial_{n}^{-1} e_{q}^{xz} = z^{-1} e_{q}^{xz}$ we have that

$$Mw_q = ST_q S^{-1} S e_q^{xz} e^{\xi(t,z)} = S \left(\sum_{i=1}^{\infty} \left(it_i + \frac{(1-q)^i}{1-q} x^i\right) z^{-1}\right) e_q^{xz} e^{\xi(t,z)},$$  

where $\xi(t,z) = \sum_{i=1}^{\infty} t_i z^i$. On the other hand,

$$\frac{\partial w_q}{\partial z} = \frac{\partial (S e_q^{xz} e^{\xi(t,z)})}{\partial z} = S \left(\frac{\partial e_q^{xz}}{\partial z} e^{\xi(t,z)} + e_q^{xz} \frac{\partial e^{\xi(t,z)}}{\partial z}\right)$$

$$= S \left(\sum_{i=1}^{\infty} \left(it_i + \frac{(1-q)^i}{1-q} x^i\right) z^{-1}\right) e_q^{xz} e^{\xi(t,z)}.$$  


Thus the assertion is verified. Next, by a direct calculation from eq. (4.4) and eq. (2.5), we have

$$\log w_q = \sum_{k=1}^{\infty} \frac{(1 - q)^k}{k(1 - q^k)} (xz)^k + \sum_{n=1}^{\infty} t_n z^n + \sum_{N=0}^{\infty} \frac{1}{N!} \left( - \sum_{i=1}^{\infty} \frac{z^{-i}}{i} \partial_i \right)^N \log \tau_q - \log \tau_q. \quad (4.3)$$

Let $M = \sum_{n=1}^{\infty} a_n L^{-n} + \sum_{n=1}^{\infty} b_n L^{-n}$. Then in light of $L_w = zw_q$ and the assertion mentioned in above, we obtain

$$\frac{\partial w_q}{\partial z} = M w_q = \left( \sum_{n=1}^{\infty} a_n L^{-n} + \sum_{n=1}^{\infty} b_n L^{-n} \right) w_q,$n

and hence

$$\frac{\partial \log w_q}{\partial z} = \frac{1}{w_q} \frac{\partial w_q}{\partial z} = \sum_{n=1}^{\infty} a_n z^{-n} + \sum_{n=1}^{\infty} b_n z^{-n}. \quad (4.4)$$

Thus by comparing the coefficients of $z$ in $\frac{\partial \log w_q}{\partial z}$ given by eq. (4.3) and eq. (4.4), $a_i$ and $b_i$ are determined such that $M$ is obtained as eq. (4.2).

To be an intuitive glance, the first few $V_{i+1}$ are given as follows.

$$V_2 = -\frac{\partial \log \tau_q}{\partial t_1},$$
$$V_3 = -\frac{\partial \log \tau_q}{\partial t_2} - \frac{\partial^2 \log \tau_q}{\partial t_1^2},$$
$$V_4 = \left( \frac{1}{2} \frac{\partial^3}{\partial t_1^3} - \frac{3}{2} \frac{\partial^2}{\partial t_1 \partial t_2} + \frac{\partial}{\partial t_3} \right) \log \tau_q,$n
$$V_5 = \left( -\frac{1}{3!} \frac{\partial^4}{\partial t_1^4} - \frac{1}{2} \frac{\partial^3}{\partial t_1^2 \partial t_2} - \frac{4}{3} \frac{\partial^2}{\partial t_1 \partial t_3} + \frac{\partial}{\partial t_4} \right) \log \tau_q,$n
$$V_6 = \left( \frac{1}{4!} \frac{\partial^5}{\partial t_1^5} - \frac{5}{12} \frac{\partial^4}{\partial t_1^3 \partial t_3} + \frac{5}{6} \frac{\partial^3}{\partial t_1^2 \partial t_3} - \frac{5}{4} \frac{\partial^2}{\partial t_1 \partial t_4} - \frac{5}{6} \frac{\partial^2}{\partial t_2 \partial t_3} + \frac{\partial}{\partial t_5} \right) \log \tau_q.$n

Now we consider the 2-reduced $q$-KP hierarchy ($q$-KdV hierarchy), by setting $L^2_\tau = 0$ or setting

$$L^2 = \partial_q^2 + (q - 1)xu\partial_q + u. \quad (4.5)$$

To make the following theorem be a compact form, introduce

$$L_{-n} = \frac{1}{2} \sum_{i=2n+1 \atop i \neq 0 \mod 2}^{\infty} i\tilde{t}_i \frac{\partial}{\partial \tilde{t}_{i-2n}} + \frac{1}{4} \sum_{k+l=n+1} (2k - 1)(2l - 1)\tilde{t}_{2k-1}\tilde{t}_{2l-1} \quad (4.6)$$

and

$$\tilde{t}_i = t_i + \frac{(1 - q)^i}{i(1 - q^i)} x^i, \quad i = 1, 2, 3, \cdots. \quad (4.7)$$

**Theorem 2.** If $L^2$ satisfies eq. (3.4), the Virasoro constraints imposed on the $\tau$-function of the $q$-KdV hierarchy are

$$L_{-n} \tau_q = 0, \quad n = 1, 2, 3, \cdots, \quad (4.8)$$

and the Virasoro commutation relations

$$[L_{-n}, L_{-m}] = (-n + m)L_{-(n+m)}, \quad m, n = 1, 2, 3, \cdots \quad (4.9)$$

hold.
Proof. For \( n = 1, 2, 3, \cdots \), we have

\[
\text{res}(ML^{-2n+1}) = \text{res}(ML^{-2n+1})_\tau = \text{res}\left(-\frac{2n+1}{2}L^{-2n}\right)_\tau = 0 \quad (4.10)
\]

with the help of eq. (3.5). Substituting the expansion of \( M \) in eq. (4.2) into eq. (4.10), then

\[
\sum_{i=1}^{\infty} \left( it_i + \frac{(1-q)^i}{1-q^i} x^i \right) \text{res} L^{-2n} + \sum_{i=1}^{\infty} \text{res}(V+iL^{-i-2n}) = 0,
\]

which implies

\[
\sum_{i=2n+1 \atop i \neq 0 \text{(mod 2)}}^{\infty} \left( it_i + \frac{(1-q)^i}{1-q^i} x^i \right) \text{res} L^{-2n} + (2n-1)t_{2n-1} + \frac{(1-q)^{2n-1}}{1-q^{2n-1}} x^{2n-1} = 0. \quad (4.11)
\]

Substituting \( \text{res} L^{-2n} = \frac{\partial^2 \log \tau_n}{\partial t_{i+1} \partial t_{-2n}} \) into eq. (4.11), then performing an integration with respect to \( t_1 \) and multiplying by \( \frac{d}{dt_1} \), it becomes

\[
\tilde{L}_{-n} \tau_q = 0, \quad n = 1, 2, 3, \cdots,
\]

where

\[
\tilde{L}_{-n} = \frac{1}{2} \sum_{i=2n+1 \atop i \neq 0 \text{(mod 2)}}^{\infty} \left( it_i + \frac{(1-q)^i}{1-q^i} x^i \right) \frac{\partial}{\partial t_{i-2n}} + \frac{(1-q)^{2n-1}}{1-q^{2n-1}} \cdot \frac{1}{2} t_1 x^{2n-1}
\]

\[
+ \frac{1}{2} (2n-1)t_1 t_{2n-1} + C(t_2, t_3, \cdots ; x). \quad (4.12)
\]

The integration constant \( C(t_2, t_3, \cdots ; x) \) with respect to \( t_1 \) could be the arbitrary function with the parameters \( (t_2, t_3, \cdots ; x) \). What we will do is to determine \( C(t_2, t_3, \cdots ; x) \) such that \( \tilde{L}_{-n} \) satisfy Virasoro commutation relations.

Let

\[
\tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q^i)} x^i, \quad i = 1, 2, 3, \cdots,
\]

and choose \( C(t_2, t_3, \cdots ; x) \) as

\[
C(t_2, t_3, \cdots ; x) = -\frac{1}{4} \sum_{k=3}^{2n-3} (2k-1)(2n-2k+1) \left( t_{2k-1} + \frac{(1-q)^{2k-1}}{(2k-1)(1-q^{2k-1})} x^{2k-1} \right)
\]

\[
\cdot \left( t_{2n-2k+1} + \frac{(1-q)^{2n-2k+1}}{(2n-2k+1)(1-q^{2n-2k+1})} x^{2n-2k+1} \right)
\]

\[
- \frac{1}{2} (2n-1)x \left( t_{2n-1} + \frac{(1-q)^{2n-1}}{(2n-1)(1-q^{2n-1})} x^{2n-1} \right),
\]

Then

\[
\tilde{L}_{-n} = \frac{1}{2} \sum_{i=2n+1 \atop i \neq 0 \text{(mod 2)}}^{\infty} \frac{\partial}{\partial \tilde{t}_{i-2n}} + \frac{1}{4} \sum_{k+l=n+1} (2k-1)(2l-1) \tilde{t}_{2k-1} \tilde{t}_{2l-1} \equiv L_{-n}
\]

and

\[
L_{-n} \tau_q = 0, \quad n = 1, 2, 3, \cdots
\]
as we expected. By a straightforward and tedious calculation, the Virasoro commutation relations

\[ [L_{-n}, L_{-m}] = (-n + m)L_{-(n+m)}, m, n = 1, 2, 3, \ldots \]

can be verified.

\[ \square \]

**Remark 1.** As we know, the \( q \)-deformed KP hierarchy reduces to the classical KP hierarchy when \( q \to 1 \) and \( u_0 = 0 \). The parameters \((\tilde{t}_1, \tilde{t}_2, \ldots, \tilde{t}_i, \ldots)\) in eq. (4.6) tend to \((t_1 + x, t_2, \ldots, t_i, \ldots)\) as \( q \to 1 \). One can further identify \( t_1 + x \) with \( x \) in the classical KP hierarchy, i.e. \( t_1 + x \to x \), therefore the Virasoro generators \( L_{-n} \) in eq. (4.6) of the 2-reduced \( q \)-KP hierarchy tend to \[ 
\hat{L}_{-n} = \frac{1}{2} \sum_{i=2n+1}^{\infty} \sum_{i \neq 0 \pmod{2}} \tilde{t}_i \frac{\partial}{\partial \tilde{t}_{i-2n}} + \frac{1}{4} \sum_{k+l=n+1} (2k-1)(2l-1)t_{2k-1}t_{2k-1}, n = 2, 3, \ldots 
\]
and

\[ \hat{L}_{-1} = \frac{1}{2} \sum_{i=3}^{\infty} \sum_{i \neq 0 \pmod{2}} \tilde{t}_i \frac{\partial}{\partial \tilde{t}_{i-2}} + \frac{1}{4} \tilde{t}^2, \]

which are identical with the results of the classical KP hierarchy given by L.A. Dickey [29] and S. Panda, S. Roy [26].

5. Conclusions and discussions

To summarize, we have derived the string equations in eq. (3.4) and the negative Virasoro constraint generators on the \( \tau \) function of 2-reduced \( q \)-KP hierarchy in eq. (4.8) in Theorem 2. The results of this paper show obviously that the Virasoro generators \( \{L_{-n}, n \geq 1\} \) of the \( q \)-KP hierarchy are different with the \( \{\hat{L}_{-n}, n \geq 1\} \) of the KP hierarchy, although they satisfy the common Virasoro commutation relations. Furthermore, one can find the following interesting relation between the \( q \)-KP hierarchy and the KP hierarchy

\[ L_{-n} = \hat{L}_{-n}\big|_{t_i \to \tilde{t}_i = t_i + \frac{(1-q)^i}{i(1-q)^j} t_i}, \]

and it seems to demonstrate that \( q \)-deformation is a non-uniform transformation for coordinates \( t_i \to \tilde{t}_i \), which is consistent with results on \( \tau \) function [11] and the \( q \)-soliton [14] of the \( q \)-KP hierarchy.

For the p-reduced \((p \geq 3)\) \( q \)-KP hierarchy, which is the \( q \)-KP hierarchy satisfying the reduction condition \( (L^p)_- = 0 \), we can obtain \( (ML^{m+1})_- = 0 \). Using the similar technique in \( q \)-KdV hierarchy, we can deduce the Virasoro constraints on the \( \tau \) function of the p-reduced \( q \)-KP hierarchy for \( p \geq 3 \). Moreover, for \( \{L_n, n \geq 0\} \) we find a subtle point at the calculation of \( res(V_{i+1}L^{-i+2n}) \), and will try to study it in the future.

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