Toral rank conjecture for moment-angle complexes

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Abstract

We consider an operation $K \mapsto L(K)$ on the set of simplicial complexes, which we call the “doubling operation”. This combinatorial operation has been recently brought into toric topology by the work of Bahri, Bendersky, Cohen and Gitler on generalised moment-angle complexes (also known as $K$-powers). The crucial property of the doubling operation is that the moment-angle complex $Z_K$ can be identified with the real moment-angle complex $RZ_{L(K)}$ for the double $L(K)$. As an application we prove the toral rank conjecture for $Z_K$ by estimating the lower bound of the cohomology rank (with rational coefficients) of real moment-angle complexes $RZ_K$. This paper extends the results of our previous work, where the doubling operation for polytopes was used to prove the toral rank conjecture for moment-angle manifolds.\(^1\)

1 Doubling operations

Here we give the definition of the “doubling operation” and discuss its main properties.

The author learned about the definition below from a communication with the authors of the recent work [1], in which the importance of the “doubling operation” in toric topology has been first demonstrated.

Definition 1.1. Let $K$ be an arbitrary simplicial complex on the vertex set $[m] = \{v_1, \ldots, v_m\}$. The double of $K$ is the simplicial complex $L(K)$ on the vertex set $[2m] = \{v_1, v'_1, \ldots, v_m, v'_m\}$ determined by the following condition: $\omega \subset [2m]$ is the minimal (by inclusion) missing simplex of $L(K)$ iff $\omega$ is of the form $\{v_i, v'_1, \ldots, v'_k\}$, where $\{v_i, \ldots, v_k\}$ is a missing simplex of $K$.

If $K = \partial P^*$ is a boundary of the dual of the simple polytope $P$, then $L(K)$ coincides with $L(P)^*$, see Definition 1 in [7].

The doubling operation for simple polytopes has also featured in the recent work [5] of Gitler and Lopes de Medrano.

Examples.

- If $K = \Delta^m$ is the $(m - 1)$-dimensional simplex, then $L(K) = \Delta^{2m}$.
- If $K = \partial \Delta^m$ is the boundary of the $(m - 1)$-dimensional simplex, then $L(K) = \partial \Delta^{2m}$.

It is easy to see that “doubling operation” respects join of the simplicial complexes i.e. $L(K_1 \ast K_2) = L(K_1) \ast L(K_2)$.

Given a simplicial complex $K$ we denote by $\text{mdim} K$ the minimal dimension of the maximal by inclusion simplices. Thus, for any $K$ $\text{mdim} K \leq \dim K$, and $K$ is pure iff $\text{mdim} K = \dim K$.

The following lemma is the direct corollary from the definitions.

\(^1\)While preparing this paper, the author learned that the toral rank conjecture for moment-angle complexes has been also recently proved in a work of Cao and Lu [3], albeit by a different method.
Lemma 1.2. Let $K$ be a simplicial complex on $[m]$, then $\dim L(K) = m + \dim K$ and $\mtext{mdim} L(K) = m + \mtext{mdim} K$.

2 $K$-powers

Definition 2.1. Let $(X, A)$ be a pair of CW—complexes. For a subset $\omega \subset [m]$ we define
\[(X, A)^\omega := \{(x_1, \ldots, x_m) \in X^m | x_i \in A \text{ for i } \notin \omega \}.
\]Now let $K$ be a simplicial complex on $[m]$. The $K$-power of the pair $(X, A)$ is
\[(X, A)^K := \bigcup_{\omega \in K} (X, A)^\omega.
\]

In this paper we shall consider two examples of $K$-powers (see [2]):

- Moment-angle complexes $Z_K = (D^2, S^1)^K$.
- Real moment-angle complexes $\mathbb{R}Z_K = (I^1, S^0)^K$.

The next lemma explains the usefulness of the notion of "doubling operation" in the studying of the relationship between moment-angle complexes and real moment-angle complexes.

Lemma 2.2. Let $(X, A)$ be a pair of CW—complexes and $K$ be a simplicial complex on the vertex set $[m]$. Consider a pair $(Y, B) = (X \times X, (X \times A) \cup (A \times X))$. For this pair we have:
\[(Y, B)^K = (X, A)^{L(K)}.
\]In particular $Z_K = \mathbb{R}Z_{L(K)}$.

Proof. For a point $y = (y_1, \ldots, y_m) \in Y^m$ we set
\[\omega_Y(y) = \{v_i \in [m] | y_i \in Y \setminus B \} \subset [m].\]For a point $x = (x_1, x_1', \ldots, x_m, x_m') \in X^{2m}$ the subset $\omega_X(x) \subset [2m]$ is defined in a similar way. Let $y = (y_1, \ldots, y_m) = ((x_1, x_1'), \ldots, (x_m, x_m')) \in Y^m \subset X^{2m}$. It follows from the definition of the $K$-powers that $y \notin (Y, B)^K$ iff $\omega_Y(y) \notin K$. The latter is equivalent to the condition $\omega_X(x) \notin L(K)$, where $x = (x_1, x_1', \ldots, x_m, x_m')$, since if $\omega_Y(y) = \{v_{i_1}, \ldots, v_{i_k}\}$ then $\omega_X(x) \supset \{v_{i_1}, v_{i_1}', \ldots, v_{i_k}, v_{i_k}'\}$. Therefore
\[y \notin (Y, B)^K \iff x \notin (X, A)^{L(K)}\]and the statement of the lemma is proved.

Example. Let $K = \partial \Delta^2$ be the boundary of 1-simplex. Then we get decomposition of 3-dimensional sphere:
\[Z_K = D^2 \times S^1 \cup S^1 \times D^2 = S^3.
\]On the other hand $L(K) = \partial \Delta^4$ and $\mathbb{R}Z_{L(K)} = \partial I^4 = S^3$ is the boundary of the standard 4-dimensional cube. So, in accordance with the lemma, $Z_K = \mathbb{R}Z_{L(K)}$. 2
3 Toral rank conjecture

Let $X$ be a finite-dimensional topological space. Denote by $\text{trk}(X)$ the largest integer for which $X$ admits an almost free $T^{\text{trk}(X)}$ action.

**Conjecture** (Halperin’s toral rank conjecture, [6]).

$$\text{hrk}(X, \mathbb{Q}) := \sum \dim H^i(X, \mathbb{Q}) \geq 2^{\text{trk}(X)}$$

Moment-angle complexes provide a big class of spaces with torus action, since there is natural coordinatewise $T^m$ action on the space $\mathcal{Z}_K$. In fact for some $r$ one can choose subtorus $T^r \subset T^m$ such that the action $T^r : \mathcal{Z}_K$ is almost free. Our aim is to estimate the maximal rank of such subtorus and the lower bound of $\text{hrk}(\mathcal{Z}_K, \mathbb{Q})$.

**Lemma 3.1.** Let $K$ be $(n-1)$-dimensional simplicial complex on the vertex set $[m]$. Then the rank of subtorus $T^r \subset T^m$ that acts almost freely on $\mathcal{Z}_K$ is less or equal to $m - n$.

**Proof.** For a subset $\omega \subset [m]$ we set $T^\omega = (T, e)^\omega$ (see definition of $K$-powers), where $e \in T$ is identity. It is easy to see that isotropy subgroups of the action $T^m : \mathcal{Z}_K$ are of the form $T^\omega$, $\omega \in K$. Therefore $T^r \subset T^m$ acts almost freely iff the set $T^r \cap T^\omega$ is finite for any $\omega \in K$.

Let $\sigma$ be the simplex of the dimension $(n-1)$. Since the intersection $T^r \cap T^\sigma$ of two subtori in $T^m$ is finite,

$$\text{rk} T^r + \text{rk} T^\sigma \leq \text{rk} T^m,$$

thus $r \leq m - n$. \hfill \Box

**Remark.** In fact for any $(n-1)$-dimensional complex $K$ there is subtorus $T^r \subset T^m$ of the rank $r = m - n$ that acts on $\mathcal{Z}_K$ almost freely, [4, §7.1].

Now we prove our main result about the cohomology rank of the real moment-angle complexes.

**Theorem 3.2.** Let $K$ be a simplicial complex on the vertex set $[m]$ with $\text{mdim} K = n - 1$. Then

$$\text{hrk}(\mathbb{R} \mathcal{Z}_K, \mathbb{Q}) \geq 2^{m-n}.$$
After summing these inequalities over \( k \) we obtain:

\[
hrk(Y, \mathbb{Q}) = \sum \dim H^k(Y) \geq \sum (\dim H^{k-1}(A) - \dim H^{k-1}(X) + \dim H^k(X)) = \sum \dim H^{k-1}(A) = hrk(A, \mathbb{Q}).
\]

\[
\square
\]

**Proof of the theorem 3.2.** We shall prove this fact by induction on \( m \). The base of induction is trivial.

Assume this statement is true for the complexes with less than \( m \) vertices and \( K \) is the complex with \( m \) vertices.

The real moment-angle complex is a subspace of the \( m \)-dimensional cube \( \mathbb{R}Z_K \subset [-1;1]^m \). Denote by \((x_1, \ldots, x_m)\) coordinates in \([-1;1]^m\). Assume that the vertex \( v_1 \) belong to the maximal (by inclusion) simplex of \( K \) of the dimension \( \dim K = n - 1 \).

Consider the decomposition of \( \mathbb{R}Z_K = M_+ \cup X \cup M_- \), where

\[
M_+ = \{ \vec{x} \in \mathbb{R}Z_K \subset \mathbb{R}^m \mid x_1 \geq 0 \},
\]

\[
M_- = \{ \vec{x} \in \mathbb{R}Z_K \subset \mathbb{R}^m \mid x_1 \leq 0 \},
\]

\[
X = \{ \vec{x} \in \mathbb{R}Z_K \subset \mathbb{R}^m \mid x_1 = 0 \}.
\]

It is easy to see that the pair \((M_+, X)\) satisfies the hypothesis of the lemma 3.3, so

\[
hrk(\mathbb{R}Z_K, \mathbb{Q}) \geq hrk(X, \mathbb{Q}).
\]

Now lets describe the space \( X \) more explicitly. Let \( k \) be the number of vertices in the complex \( \text{lk} v_1 \). Then \( X \) is just the disjoint union of the \( 2^{m-k-1} \) copies of the space \( \mathbb{R}Z_{\text{lk} v_1} \).

Moreover, since \( v_1 \) is vertex of the maximal (by inclusion) simplex of the minimal dimension \( n - 1 \), so \( \dim \text{lk} v_1 = n - 2 \). Thus, by the hypothesis of induction

\[
hrk(X, \mathbb{Q}) = 2^{m-k-1} hrk(\mathbb{R}Z_{\text{lk} v_1}, \mathbb{Q}) \geq 2^{m-k-1} \cdot 2^{k-(n-1)} = 2^{m-n}
\]

The step of induction is proved.

Now let’s turn our attention to the moment-angle complexes. Combining the results of lemma 1.2, lemma 2.2 and theorem 3.2 we have:

\[
hrk(\mathbb{Z}_K, \mathbb{Q}) = hrk(\mathbb{R}Z_{\text{lk}(K)}) \geq 2^{m-\dim \text{lk}(K)-1} = 2^{m-\dim K-1} \geq 2^{m-\dim K-1}.
\]

Thus the *toral rank conjecture* holds for the action of subtori of \( T^m \) on the moment-angle complexes \( \mathbb{Z}_K \).

The cohomology ring of \( \mathbb{Z}_K \) was calculated in [2]. One of the corollaries of this computation and Hochster’s theorem states (see [2], theorem 8.7):

**Theorem 3.4.**

\[
H^*(\mathbb{Z}_K, \mathbb{Q}) \cong \bigoplus_{\omega \subseteq [m], \ p \geq -1} \tilde{H}^p(K_\omega, \mathbb{Q}),
\]

where \( K_\omega \) is the restriction of \( K \) on the subset \( \omega \subset [m] \).

In view of this theorem we can reformulate our main result as follows:

\[
\dim \bigoplus_{\omega \subseteq [m]} \tilde{H}^*(K_\omega, \mathbb{Q}) \geq 2^{m-n},
\]

for any simplicial complex \( K \) on \([m]\) with \( \dim K = n - 1 \).

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