The discrete analogue of the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \)

A.R. Hayotov,

Institute of Mathematics, National University of Uzbekistan, Do’rmon yo’li str., 29, Tashkent 100125, Uzbekistan

Abstract

In the present paper we construct the discrete analogue \( D_m(h\beta) \) of the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \). The discrete analogue \( D_m(h\beta) \) plays the main role in construction of optimal quadrature formulas and interpolation splines minimizing the semi-norm in the \( K_2(P_m) \) Hilbert space.

Key words: Differential operator, discrete analogue, Hilbert space, discrete argument functions

1991 MSC: 41A05, 41A15

1 Introduction and Preliminaries

The optimization problem of approximate integration formulas in the modern sense appears as the problem of finding the minimum of the norm of an error functional \( \ell \) given on some set of functions.

The minimization problem of the norm of the error functional by coefficients was reduced in [13,15] to the system of difference equations of Wiener-Hopf type in the space \( L_2^{(m)} \), where \( L_2^{(m)} \) is the space of functions with square integrable \( m \)-th generalized derivative. Existence and uniqueness of the solution of this system was proved by S.L. Sobolev. In the works [13,15] the description of some analytic algorithm for finding the coefficients of optimal formulas is given. For this S.L. Sobolev defined and investigated the discrete analogue \( D_{h_H}^{(m)}(h\beta) \) of the polyharmonic operator \( \Delta^m \). The problem of construction of the discrete operator \( D_{h_H}^{(m)}(h\beta) \) for \( n \)-dimensional case was very hard. In one

---

Email address: abdullo_hayotov@mail.ru (A.R. Hayotov).
dimensional case the discrete analogue $D^{(m)}_h(h\beta)$ of the differential operator $\frac{d^{2m}}{dx^{2m}}$ was constructed by Z.Zh. Zhamalov [17] and Kh.M. Shadimetov [8].

Further, in the work [9] the discrete analogue of the differential operator $\frac{d^{2m-2}}{dx^{2m-2}}$ was constructed. The constructed discrete analogue of the operator $\frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}}$ was applied for finding the coefficients of the optimal quadrature formulas (see [10,12]) and for construction of interpolation splines minimizing the semi-norm (see [11]) in the space $W_{2}^{(m,m-1)}(0,1)$, where $W_{2}^{(m,m-1)}(0,1)$ is the Hilbert space of functions $\varphi$ which $\varphi^{(m-1)}$ is absolutely continuous, $\varphi^{(m)}$ belongs to $L_{2}(0,1)$ and $\int_{0}^{1}(\varphi^{(m)}(x) + \varphi^{(m-1)}(x))^{2}dx < \infty$.

Here, we mainly use a concept of functions of a discrete argument and the corresponding operations [13, Chapter VII]. For completeness we give some of definitions.

Let $\beta \in \mathbb{Z}$, $h = \frac{1}{N}$, $N = 1, 2, \ldots$. Assume that $\varphi$ and $\psi$ are real-valued functions defined on the real line $\mathbb{R}$.

**Definition 1.** The function $\varphi(h\beta)$ is a *function of discrete argument* if it is defined on some set of integer values of $\beta$.

**Definition 2.** The *inner product* of two discrete argument functions $\varphi(h\beta)$ and $\psi(h\beta)$ is defined as

$$\langle \varphi, \psi \rangle = \sum_{\beta = -\infty}^{\infty} \varphi(h\beta) \cdot \psi(h\beta).$$

Here we assume that the series on the right hand side converges absolutely.

**Definition 3.** The *convolution* of two functions $\varphi(h\beta)$ and $\psi(h\beta)$ is defined as the inner product

$$\varphi(h\beta) \ast \psi(h\beta) = \langle \varphi(h\gamma), \psi(h\beta - h\gamma) \rangle = \sum_{\gamma = -\infty}^{\infty} \varphi(h\gamma) \cdot \psi(h\beta - h\gamma).$$

**Definition 4.** The function $\varphi(x) = \sum_{\beta = -\infty}^{\infty} \varphi(h\beta)\delta(x-h\beta)$ is called the *harrow-shaped function* corresponding to the function of discrete argument $\varphi(h\beta)$, where $\delta$ is Dirac’s delta function.

We are interested in to find a discrete function $D_{m}(h\beta)$ that satisfies the following equation

$$D_{m}(h\beta) \ast G_{m}(h\beta) = \delta_{d}(h\beta), \quad (1.1)$$

where
\[ G_m(h\beta) = \frac{(-1)^m \text{sign}(h\beta)}{4\omega^{2m-1}} \left[ (2m-3) \sin(h\omega\beta) - h\omega\beta \cos(h\omega\beta) \\
+ 2 \sum_{k=1}^{m-2} \frac{(-1)^k (m - k - 1)(h\omega\beta)^{2k-1}}{(2k - 1)!} \right], \]  
(1.2)

\( m \geq 2, \omega > 0, \delta_d(h\beta) \) is equal to 1 when \( \beta = 0 \) and is equal to 0 when \( \beta \neq 0 \), i.e. \( \delta_d(h\beta) \) is the discrete delta function.

The discrete function \( D_m(h\beta) \) plays an important role in the calculation of the coefficients of the optimal quadrature formulas and interpolation splines minimizing a semi-norm in the Hilbert space

\[ K_2(P_m) = \left\{ \varphi : [0, 1] \to \mathbb{R} \bigg| \varphi^{(m-1)} \text{ is abs. cont. and } \varphi^{(m)} \in L_2(0, 1) \right\}, \]
equipped with the norm

\[ \| \varphi \| = \left\{ \int_0^1 P_m \left( \frac{d}{dx} \varphi(x) \right)^2 \, dx \right\}^{\frac{1}{2}}, \]  
(1.3)

where \( P_m \left( \frac{d}{dx} \right) = \frac{d^m}{dx^m} + \omega^2 \frac{d^{m-2}}{dx^{m-2}} \), \( \omega > 0 \) and \( \int_0^1 \left( P_m \left( \frac{d}{dx} \varphi(x) \right) \right)^2 \, dx < \infty \).

Note that the equality (1.3) is semi-norm, moreover \( \| \varphi \| = 0 \) if and only if \( \varphi(x) = c_1 \sin \omega x + c_2 \cos \omega x + Q_{m-3}(x), \) with \( Q_{m-3}(x) \) a polynomial of degree \( m-3 \). In particular, the optimal quadrature formulas and interpolation splines in the \( K_2(P_m) \) space are exact for the trigonometric functions \( \sin \omega x, \cos \omega x \) and polynomials of degree \( m-3 \).

It should be noted that for a linear differential operator of order \( m \),

\[ L := P_m \left( \frac{d}{dx} \right) = \frac{d^m}{dx^m} + a_{m-1}(x) \frac{d^{m-1}}{dx^{m-1}} + \ldots + a_1(x) \frac{d}{dx} + a_0(x), \]

Ahlberg, Nilson, and Walsh in the book [1, Chapter 6] investigated the Hilbert spaces in the context of generalized splines. Namely, with the inner product

\[ \langle \varphi, \psi \rangle = \int_0^1 L \varphi(x) \cdot L \psi(x) \, dx, \]

\( K_2(P_m) \) is a Hilbert space if we identify functions that differ by a solution of \( L \varphi = 0 \).

Note that the equation (1.1) is the discrete analogue to the following equation

\[ \left( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \right) G_m(x) = \delta(x), \]  
(1.4)
where

\[ G_m(x) = \frac{(-1)^m \text{sign}(x)}{4 \omega^{2m-1}} \left[ (2m - 3) \sin(\omega x) - \omega x \cdot \cos(\omega x) \right. \]

\[ + 2 \sum_{k=1}^{m-2} \frac{(-1)^k (m - k - 1)(\omega x)^{2k-1}}{(2k-1)!} \],

and \( \delta \) is the Dirac’s delta function.

Moreover the discrete function \( D_m(h\beta) \) has analogous properties the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \): the zeros of the discrete operator \( D_m(h\beta) \) coincides with the discrete functions corresponding to the zeros of the operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \). The discrete function \( D_m(h\beta) \) is called the discrete analogue of the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \).

The rest of the paper is organized as follows: in Section 2 we give some known formulas and auxiliary results which will be used in the construction of the discrete function \( D_m(h\beta) \). Section 3 is devoted to the construction of the discrete analogue \( D_m(h\beta) \) of the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \).

2 Auxiliary results

In this section we explain some known formulas (see, for instance, \([13,16]\)) and auxiliary results which we use in the construction of the discrete analogue \( D_m(h\beta) \) of the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \).

For the Fourier transformations and their properties:

\[ F[\varphi(x)] = \int_{-\infty}^{\infty} \varphi(x)e^{2\pi ipx} \, dx, \quad F^{-1}[\varphi(p)] = \int_{-\infty}^{\infty} \varphi(p)e^{-2\pi ipx} \, dp, \quad (2.1) \]

\[ F[\varphi \ast \psi] = F[\varphi] \cdot F[\psi], \quad (2.2) \]

\[ F[\varphi \cdot \psi] = F[\varphi] \ast F[\psi], \quad (2.3) \]

\[ F[\delta^{(\alpha)}(x)] = (-2\pi i)^\alpha, \quad F[\delta(x)] = 1, \quad (2.4) \]

where * is the convolution and for two functions \( \varphi \) and \( \psi \) is defined as follows

\[ (\varphi \ast \psi)(x) = \int_{-\infty}^{\infty} \varphi(x-y)\psi(y) \, dy = \int_{-\infty}^{\infty} \varphi(y)\psi(x-y) \, dy. \]
For the Dirac’s delta function:

\[
\begin{align*}
\delta(hx) &= h^{-1}\delta(x), \\
\delta(x-a) \cdot f(x) &= \delta(x-a) \cdot f(a), \\
\delta^{(\alpha)}(x) \cdot f(x) &= f^{(\alpha)}(x), \\
\phi_0(x) &= \sum_{\beta=-\infty}^{\infty} \delta(x-\beta) = \sum_{\beta=-\infty}^{\infty} e^{2\pi i x \beta}.
\end{align*}
\] (2.5)

It is known [2,14,15] that the Euler-Frobenius polynomials \(E_k(x)\) are defined by the following formula

\[
E_k(x) = \frac{(1-x)^{k+2}}{x} \left( x \frac{d}{dx} \right)^k \frac{x}{(1-x)^2}, \quad k = 1, 2, ...
\] (2.9)

\(E_0(x) = 1\), moreover all roots \(x_j^{(k)}\) of the Euler-Frobenius polynomial \(E_k(x)\) are real, negative and simple, i.e.

\[
x_1^{(k)} < x_2^{(k)} < ... < x_k^{(k)} < 0.
\] (2.10)

Furthermore, the roots equally spaced from the ends of the chain (2.10) are reciprocal, i.e.

\[
x_j^{(k)} \cdot x_{k+1-j}^{(k)} = 1.
\]

Euler obtained the following formula for the coefficients \(a_s^{(k)}\), \(s = 0, 1, ..., k\) of the polynomial \(E_k(x) = \sum_{s=0}^{k} a_s^{(k)} x^s\):

\[
a_s^{(k)} = \sum_{j=0}^{s} (-1)^j \binom{k+2}{j} (s+1-j)^{k+1}.
\] (2.11)

The polynomial \(E_k(x)\) satisfies the following identity

\[
E_k(x) = x^k E_k \left( \frac{1}{x} \right), \quad x \neq 0,
\] (2.12)

i.e. \(a_s^{(k)} = a_k^{(k-s)}\), \(s = 0, 1, 2, ..., k\).

Now we consider the following polynomial of degree \(2m - 2\)

\[
\mathcal{P}_{2m-2}(x) = \sum_{s=0}^{2m-2} p_s^{(2m-2)} x^s = (1-x)^{2m-4} \left[ (2m-3) \sin h\omega - h\omega \cos h\omega \right] x^2
\]

\[+ \left[ 2h\omega - (2m-3) \sin(2h\omega) \right] x + \left[ (2m-3) \sin h\omega - h\omega \cos h\omega \right]\]
\[ +2(x^2 - 2x \cos \omega + 1)^2 \sum_{k=1}^{m-2} \frac{(-1)^k (m - k - 1)(h\omega)^{2k-1}(1 - x)^{2m-2k-4} E_{2k-2}(x)}{(2k - 1)!} \]  

where \( E_{2k-2}(x) \) is the Euler-Frobenius polynomial of degree \( 2k - 2 \), \( \omega > 0 \), \( h\omega \leq 1 \), \( h = 1/N \), \( N \geq m - 1 \), \( m \geq 2 \).

The polynomial \( \mathcal{P}_{2m-2}(x) \) has the following properties.

**Lemma 2.1** The Polynomial \( \mathcal{P}_{2m-2}(x) \) satisfies the identity

\[ \mathcal{P}_{2m-2}(x) = x^{2m-2} \mathcal{P}_{2m-2} \left( \frac{1}{x} \right). \]

**Lemma 2.2** For the derivative of the polynomial \( \mathcal{P}_{2m-2}(x) \) the following equality holds

\[ \mathcal{P}'_{2m-2} \left( \frac{1}{x_k} \right) = -\frac{1}{x_k^{2m-4}} \mathcal{P}'_{2m-2}(x_k), \]

where \( x_k \) is a root of the polynomial \( \mathcal{P}_{2m-2}(x) \).

Further we prove Lemmas 2.1-2.2.

**Proof of Lemma 2.1.** If we replace \( x \) by \( \frac{1}{x} \) in (2.13), we obtain

\[
\mathcal{P}_{2m-2} \left( \frac{1}{x} \right) = \left( 1 - \frac{1}{x} \right)^{2m-4} \left[ (2m - 3) \sin h\omega - h\omega \cos h\omega \right] \frac{1}{x^2} \]

\[+ 2 \left( \frac{1}{x^2} - 2 \frac{x}{x} \cos h\omega + 1 \right) \sum_{k=1}^{m-2} \frac{(-1)^k (m - k - 1)(h\omega)^{2k-1}(1 - \frac{1}{x})^{2m-2k-4} E_{2k-2}(\frac{1}{x})}{(2k - 1)!}.
\]

Hence, taking into account (2.12) and the fact that \( (1 - \frac{1}{x})^{2k} = \frac{x^k(1-x)^{2k}}{x^k} \), we get the statement of the lemma. Lemma 2.1 is proved.

**Proof of Lemma 2.2.** Let \( x_1, x_2, \ldots, x_{2m-2} \) be the roots of the polynomial \( \mathcal{P}_{2m-2}(x) \). Then from Lemma 2.1 we immediately get that if \( x_k \) is the root then \( \frac{1}{x_k} \) is also the root of the polynomial \( \mathcal{P}_{2m-2}(x) \) and if \( |x_k| < 1 \) then \( |\frac{1}{x_k}| > 1 \).

Suppose the roots \( x_1, x_2, \ldots, x_{2m-2} \) are situated such that

\[ x_k \cdot x_{2m-1-k} = 1. \] (2.14)

Therefore we have

\[ \mathcal{P}_{2m-2}(x) = \mathcal{P}_{2m-2}^{(2m-2)}(x-x_1)(x-x_2)\ldots(x-x_{2m-2}). \]
where \( p_{2m-2}^{(2m-2)} \) is the leading coefficient of the polynomial \( P_{2m-2}(x) \).

Hence
\[
P'_{2m-2}(x) = p_{2m-2}^{(2m-2)} \sum_{j=1}^{2m-2} \frac{\prod_{i=1}^{2m-2} (x - x_i)}{x - x_j}.
\]

Then for \( x = x_k \) we get
\[
P'_{2m-2}(x_k) = p_{2m-2}^{(2m-2)} \prod_{i=1, i \neq k}^{2m-2} (x_k - x_i) \tag{2.15}
\]

and for \( x = \frac{1}{x_k} \), taking into account (2.14), using the equality
\[
\prod_{i=1, i \neq 2m-1 - k}^{2m-2} \frac{1}{x_{2m-1 - i}} = \prod_{i=1, i \neq 2m-1 - k}^{2m-2} \frac{1}{x_{2m-1 - i}} = x_k
\]

we have
\[
P'_{2m-2} \left( \frac{1}{x_k} \right) = p_{2m-2}^{(2m-2)} \sum_{j=1}^{2m-2} \frac{\prod_{i=1}^{2m-2} \left( \frac{1}{x_k} - x_i \right)}{x_k - x_j}
\]
\[
= p_{2m-2}^{(2m-2)} \frac{1}{x_k^{2m-3}} \prod_{i=1, i \neq 2m-1 - k}^{2m-2} (1 - x_k x_i)
\]
\[
= -p_{2m-2}^{(2m-2)} \frac{1}{x_k^{2m-3}} \prod_{i=1, i \neq 2m-1 - k}^{2m-2} \frac{x_k - x_{2m-1 - i}}{x_{2m-1 - i}}
\]
\[
= -\frac{1}{x_k^{2m-4}} p_{2m-2}^{(2m-2)} \prod_{i=1, i \neq k}^{2m-2} (x_k - x_i). \tag{2.16}
\]

From (2.15) and (2.16) we get the statement of the lemma. Lemma 2.2 is proved. \( \square \)

In the proof of the main result we have to calculate the following series

\[
S_1 = \sum_{\beta = -\infty}^{\infty} \frac{1}{[\beta - h(p + \frac{\omega}{2\pi})][\beta - h(p - \frac{\omega}{2\pi})]}, \tag{2.17}
\]
\[
S_2 = \sum_{\beta = -\infty}^{\infty} \frac{1}{[\beta - h(p + \frac{\omega}{2\pi})]^2}, \tag{2.18}
\]
\[
S_3 = \sum_{\beta = -\infty}^{\infty} \frac{1}{[\beta - h(p - \frac{\omega}{2\pi})]^2}, \tag{2.19}
\]
\[
F[h \bar{G}_{k,1}] = \frac{(-1)^k}{(2\pi)^{2k}} \sum_{\beta = -\infty}^{\infty} \frac{1}{(p - h^{-1}\beta)^{2k}}. \tag{2.20}
\]
We denote $\lambda = e^{2\pi iph}$ then the following holds

**Lemma 2.3** For the series (2.17)-(2.20) the following are taken place

$$ S_1 = \frac{-(2\pi)^2 \lambda \sin h\omega}{h\omega(\lambda^2 - 2\lambda \cos h\omega)}, \quad S_2 = \frac{-(2\pi)^2 \lambda}{(\lambda^2 + 1) \cos h\omega - 2\lambda + i(\lambda^2 - 1) \sin h\omega}, \quad S_3 = \frac{-(2\pi)^2 \lambda}{(\lambda^2 + 1) \cos h\omega - 2\lambda + i(\lambda^2 - 1) \sin h\omega}, $$

where $E_{2k-2}(\lambda)$ is the Euler-Frobenius polynomial of degree $2m - 2$.

**Proof.** To calculate the series (2.17)-(2.20) we use the following well-known formula from the residual theory (see [6], p.296)

$$ \sum_{\beta = -\infty}^{\infty} f(\beta) = -\sum_{z_1, z_2, \ldots, z_n} \text{res}(\pi \cot(\pi z) \cdot f(z)), \quad (2.21) $$

where $z_1, z_2, \ldots, z_n$ are poles of the function $f(z)$.

At first we consider $S_1$. We denote $f_1(z) = \frac{1}{(z - h(p + \frac{\omega}{2\pi}))(z - h(p - \frac{\omega}{2\pi}))}$. It is clear that $z_1 = h(p + \frac{\omega}{2\pi})$ and $z_2 = h(p - \frac{\omega}{2\pi})$ are the poles of order 1 of the function $f_1(z)$. Then taking into account the formula (2.21) we have

$$ S_1 = \sum_{\beta = -\infty}^{\infty} f_1(\beta) = -\sum_{z_1, z_2} \text{res}(\pi \cot(\pi z) \cdot f_1(z)). \quad (2.22) $$

Since

$$ \text{res}_{z = z_1}(\pi \cot(\pi z) \cdot f_1(z)) = \lim_{z \to z_1} \frac{\pi \cot(\pi z)}{z - h(p - \frac{\omega}{2\pi})} = \frac{\pi^2}{h\omega} \cot(\pi hp + \frac{h\omega}{2}), $$

and

$$ \text{res}_{z = z_2}(\pi \cot(\pi z) \cdot f_1(z)) = \lim_{z \to z_2} \frac{\pi \cot(\pi z)}{z - h(p + \frac{\omega}{2\pi})} = -\frac{\pi^2}{h\omega} \cot(\pi hp - \frac{h\omega}{2}). $$

Using the last two equalities, from (2.22) we get

$$ S_1 = \frac{\pi^2}{h\omega} \left( \cot(\pi hp - \frac{h\omega}{2}) - \cot(\pi hp + \frac{h\omega}{2}) \right) = \frac{\pi^2 \sin(h\omega)}{h\omega \sin(\pi hp - \frac{h\omega}{2}) \sin(\pi hp + \frac{h\omega}{2})}. $$

Taking into account that $\lambda = e^{2\pi iph}$ and using the well known formulas

$$ \cos z = \frac{e^{zi} + e^{-zi}}{2}, \quad \sin z = \frac{e^{zi} - e^{-zi}}{2i}, $$

after some simplifications for $S_1$ we have

$$ S_1 = \frac{-(2\pi)^2 \lambda \sin h\omega}{h\omega(\lambda^2 - 2\lambda \cos h\omega + 1)}. $$
Similarly for $S_2$ and $S_3$ we arrive the following equalities:

$$S_2 = \frac{-(2\pi)^2 \lambda}{(\lambda^2 + 1) \cos h\omega - 2\lambda + i(\lambda^2 - 1) \sin h\omega}$$

and

$$S_3 = \frac{-(2\pi)^2 \lambda}{(\lambda^2 + 1) \cos h\omega - 2\lambda - i(\lambda^2 - 1) \sin h\omega}.$$

The series (2.20) was calculated in [8], where the following expression has been obtained

$$F[h \overrightarrow{G}_{k,1}] = \frac{h^{2k} \lambda E_{2k-2}(\lambda)}{(2k-1)! (1-\lambda)^{2k}}.$$

So, Lemma 2.3 is proved.  

\[\square\]

3 Construction of the discrete analogue of the operator

In this section we construct the discrete analogue $D_m(h\beta)$ of the operator

$$\frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$$

and obtain some properties of the constructed discrete function $D_m(h\beta)$.

The result is the following one:

**Theorem 3.1** The discrete analogue to the differential operator

$$\frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}}$$

satisfying equation (1.1) has the form

$$D_m(h\beta) = \frac{2\omega^{2m-1}}{(-1)^m p_{2m-2}^{(2m-2)}} \begin{cases} 
\sum_{k=1}^{m-1} A_k \lambda_k^{[\beta]-1}, & |\beta| \geq 2, \\
1 + \sum_{k=1}^{m-1} A_k, & |\beta| = 1, \\
C + \sum_{k=1}^{m-1} \frac{A_k}{\lambda_k}, & \beta = 0, 
\end{cases} \quad (3.1)$$

where

$$A_k = \frac{(1 - \lambda_k)^{2m-4}(\lambda_k^2 - 2\lambda_k \cos h\omega + 1)^2 p_{2m-2}^{(2m-2)}}{\lambda_k p_{2m-2}^{(2m-2)}}, \quad (3.2)$$

$$C = 4 - 4 \cos h\omega - 2m - \frac{p_{2m-3}^{(2m-2)}}{p_{2m-2}^{(2m-2)}}, \quad (3.3)$$

$$p_{2m-2}^{(2m-2)} = (2m - 3) \sin h\omega - h\omega \cos h\omega + 2 \sum_{k=1}^{m-2} \frac{(-1)^k (m - k - 1)(h\omega)^{2k-1}}{(2k-1)!}, \quad (3.4)$$
$P_{2m-2}(\lambda)$ is the polynomial of degree $2m-2$ defined by (2.13), $p_{2m-2}^{(2m-2)}, p_{2m-3}^{(2m-2)}$ are the coefficients and $\lambda_k$ are the roots of the polynomial $P_{2m-2}(\lambda)$, $|\lambda_k| < 1$.

Before proving this result we present the following properties that are not difficult to verify.

**Lemma 3.2** The discrete analogue $D_m(h\beta)$ of the differential operator \( \frac{d^{2m}}{dx^{2m}} + 2\omega^2 \frac{d^{2m-2}}{dx^{2m-2}} + \omega^4 \frac{d^{2m-4}}{dx^{2m-4}} \) satisfies the following equalities

1) \( D_m(h\beta) * \sin(h\omega\beta) = 0, \)
2) \( D_m(h\beta) * \cos(h\omega\beta) = 0, \)
3) \( D_m(h\beta) * (h\omega\beta) \sin(h\omega\beta) = 0, \)
4) \( D_m(h\beta) * (h\omega\beta) \cos(h\omega\beta) = 0, \)
5) \( D_m(h\beta) * (h\beta)^\alpha = 0, \alpha = 0, 1, ..., 2m - 5. \)

**Proof of Theorem 3.1.** According to the theory of periodic distribution (generalized) functions and Fourier transformations, instead of the function $D_m(h\beta)$ it is convenient to search the harrow-shaped function (see [13,15])

\[
\overline{D}_m(x) = \sum_{\beta=-\infty}^{\infty} D_m(h\beta)\delta(x - h\beta).
\]

In the class of harrow-shaped functions, equation (1.1) takes the following form

\[
\overline{D}_m(x) * \overline{G}_m(x) = \delta(x),
\]

where $\overline{G}_m(x) = \sum_{\beta=-\infty}^{\infty} G_m(h\beta)\delta(x - h\beta)$ is the harrow-shaped function corresponding to the discrete function $G_m(h\beta)$.

Applying the Fourier transformation to both sides of equation (3.5) and taking into account (2.2) and (2.4) we have

\[
F[\overline{D}_m(x)] = \frac{1}{F[\overline{G}_m(x)]}. \tag{3.6}
\]

First, we calculate the Fourier transformation $F[\overline{G}_m(x)]$ of the harrow-shaped function $\overline{G}_m(x)$.

Using equalities (2.5), (2.6) and (2.8), we get
\[
\tilde{G}_m(x) = \sum_{\beta = -\infty}^{\infty} G_m(h\beta)\delta(x - h\beta)
= h^{-1}G_m(x) \sum_{\beta = -\infty}^{\infty} \delta(h^{-1}x - \beta).
\]

Hence
\[
\tilde{G}_m(x) = h^{-1}G_m(x) \cdot \phi_0(h^{-1}x). \quad (3.7)
\]

Taking into account (2.1) and (2.8), for \(F[\phi_0(h^{-1}x)]\) we obtain
\[
F[\phi_0(h^{-1}x)] = \sum_{\beta = -\infty}^{\infty} \int_{-\infty}^{\infty} \delta(h^{-1}x - \beta)e^{2\pi ipx} \, dx
= h \sum_{\beta = -\infty}^{\infty} \int_{-\infty}^{\infty} \delta(x - h\beta)e^{2\pi ipx} \, dx
= h \sum_{\beta = -\infty}^{\infty} e^{2\pi iph\beta} = h \sum_{\beta = -\infty}^{\infty} \delta(h\beta - \beta) = h\phi_0(hp). \quad (3.8)
\]

Applying the Fourier transformation to both sides of (3.7) and using (2.3), (3.8), we get
\[
F[\tilde{G}_m(x)] = F[G_m(x)] \ast \phi_0(hp). \quad (3.9)
\]

To calculate the Fourier transformation \(F[G_m(x)]\), we use equation (1.4). Taking into account (2.7), we rewrite equation (1.4) in the following form
\[
(\delta(x) + 2\omega^2 \delta(2m-2)(x) + \omega^4 \delta(2m-4)(x)) \ast G_m(x) = \delta(x).
\]

Hence, keeping in mind (2.2) and (2.4), we have
\[
F[G_m(x)] = \frac{1}{(2\pi ip)^{2m} + 2\omega^2(2\pi ip)^{2m-2} + \omega^4(2\pi ip)^{2m-4}}.
\]

Using the last equality, from (3.9) we arrive to
\[
F[\tilde{G}_m(x)] = \frac{1}{(2\pi ip)^{2m} + 2\omega^2(2\pi ip)^{2m-2} + \omega^4(2\pi ip)^{2m-4}} \ast \phi_0(hp)
= \frac{1}{h} \sum_{\beta = -\infty}^{\infty} \int_{-\infty}^{\infty} \delta(y - h^{-1}\beta) \, dy
= \frac{1}{h} \sum_{\beta = -\infty}^{\infty} \int_{-\infty}^{\infty} \frac{\delta(y - h^{-1}\beta)}{(2\pi i(p - y))^{2m} + 2\omega^2(2\pi i(p - y))^{2m-2} + \omega^4(2\pi i(p - y))^{2m-4}}.
\]

Hence, after some calculations we get
\[
F[\tilde{G}_m(x)] = \sum_{\beta = -\infty}^{\infty} \frac{h^{-1}}{(2\pi i(p - \beta h))^{2m} + 2\omega^2(2\pi i(p - \beta h))^{2m-2} + \omega^4(2\pi i(p - \beta h))^{2m-4}}. \quad (3.10)
\]
Now, expanding to partial fractions of the right hand side of (3.10) and taking into account (3.6), we conclude

\[
F[\overset{\leftrightarrow}{D}_m](p) = h \left[ \frac{(-1)^{m-1} (2m - 3) h^2}{(2\pi)^2 \cdot 2\omega^{2m-2}} S_1 + \frac{(-1)^{m-2} h^2}{(2\pi)^2 \cdot 4\omega^{2m-2}} (S_2 + S_3) \right. \\
\left. + \frac{1}{\omega^{2m}} \sum_{k=1}^{m-2} (-1)^{m-k} (m-k-1) \omega^{2k} F[h \overset{\rightarrow}{G}_{k,1}] \right]^{-1}, \quad (3.11)
\]

where \( S_1, S_2, S_3 \) and \( F[h \overset{\rightarrow}{G}_{k,1}] \) are defined by (2.17)-(2.20).

It is clear that the function \( F[\overset{\leftrightarrow}{D}_m](p) \) is a periodic function with respect to the variable \( p \), and the period is \( h^{-1} \). Moreover it is real and analytic for all real \( p \). The zeros of the function \( F[\overset{\leftrightarrow}{D}_m](p) \) are \( p = h^{-1} \beta + \frac{\omega}{2\pi}, \ p = h^{-1} \beta - \frac{\omega}{2\pi} \) and \( p = h^{-1} \beta \).

The function \( F[\overset{\leftrightarrow}{D}_m](p) \) can be expanded to the Fourier series as follows

\[
F[\overset{\leftrightarrow}{D}_m](p) = \sum_{\beta=-\infty}^{\infty} \hat{D}_m(h\beta) e^{2\pi i p h \beta}, \quad (3.12)
\]

where \( \hat{D}_m(h\beta) \) are the Fourier coefficients of the function \( F[\overset{\leftrightarrow}{D}_m](p) \), i.e.

\[
\hat{D}_m(h\beta) = \int_0^{h^{-1}} F[\overset{\leftrightarrow}{D}_m](p) e^{-2\pi i p h \beta} dp. \quad (3.13)
\]

Applying the inverse Fourier transformation to both sides of (3.12), we obtain the following harrow-shaped function

\[
\overset{\leftrightarrow}{D}_m(x) = \sum_{\beta=-\infty}^{\infty} \hat{D}_m(h\beta) \delta(x - h\beta).
\]

Then, according to the definition of harrow-shaped functions, the discrete function \( \hat{D}_m(h\beta) \) is the discrete argument function \( D_m(h\beta) \), which we are searching. So, we have to find the discrete argument function \( D_m(h\beta) \).

But here, to find the function \( \hat{D}_m(h\beta) \) we will not use the formula (3.13). In this case to obtain \( \hat{D}_m(h\beta) \) we have to calculate the series \( S_1, S_2, S_3 \) and \( F[h \overset{\rightarrow}{G}_{k,1}] \) defined by (2.17)-(2.20). We have calculated these series in Lemma 2.3. Therefore, using Lemma 2.3, from (3.13) we get

\[
F[\overset{\leftrightarrow}{D}_m](p) = \frac{2\omega^{2m-1}}{(-1)^m} \frac{(1 - \lambda)^{2m-4} (\lambda^2 + 1 - 2\lambda \cos h\omega)^2}{\lambda \mathcal{P}_{2m-2}(\lambda)}, \quad (3.14)
\]
where $\lambda = e^{2\pi i \phi}$ and $P_{2m-2}(\lambda)$ is the polynomial of degree $2m - 2$ defined by (2.13).

Now we will get the Fourier series of the function $F[\bar{D}_m](\rho)$. Dividing the polynomial $(1 - \lambda)^{2m-4}(\lambda^2 + 1 - 2\lambda \cos h\omega)^2$ by the polynomial $\lambda P_{2m-2}(\lambda)$ we obtain

$$F[\bar{D}_m](\rho) = \frac{2\omega^{2m-1}}{(1-m)(-1)^m \lambda P_{2m-2}(\lambda)} \left(1 - \lambda\right)^{2m-4}(\lambda^2 + 1 - 2\lambda \cos h\omega)^2$$

$$= \frac{2\omega^{2m-1}}{(1-m)(-1)^m p_{2m-2}^{(2m-2)}} \left[\lambda + 4 - 4 \cos h\omega - 2m - \frac{p_{2m-3}^{(2m-2)}}{p_{2m-2}^{(2m-2)}}\right] + \frac{S_{2m-2}(\lambda)}{\lambda P_{2m-2}(\lambda)}, \quad (3.15)$$

where $S_{2m-2}(\lambda)$ is a polynomial of degree $2m - 2$.

It is clear that $\frac{S_{2m-2}(\lambda)}{\lambda P_{2m-2}(\lambda)}$ is the proper fraction. Since the roots of the polynomial $\lambda P_{2m-2}(\lambda)$ are real and simple (see Appendix), then the rational fraction $\frac{S_{2m-2}(\lambda)}{\lambda P_{2m-2}(\lambda)}$ is expanded to the sum of partial fractions, i.e.

$$\frac{S_{2m-2}(\lambda)}{\lambda P_{2m-2}(\lambda)} = \frac{1}{p_{2m-2}^{(2m-2)}} \left[\frac{A_0}{\lambda} + \sum_{k=1}^{m-1} \left(\frac{A_{1,k}}{\lambda - \lambda_{1,k}} + \frac{A_{2,k}}{\lambda - \lambda_{2,k}}\right)\right], \quad (3.16)$$

where $A_0$, $A_{1,k}$ and $A_{2,k}$ are unknown coefficients, $\lambda_{1,k}$, $\lambda_{2,k}$ are the roots of the polynomial $P_{2m-2}(\lambda)$ and $|\lambda_{1,k}| < 1$, $|\lambda_{2,k}| > 1$ and

$$\lambda_{1,k} \lambda_{2,k} = 1. \quad (3.17)$$

Keeping in mind equality (3.16) from (3.15), we obtain

$$F[\bar{D}_m](\rho) = \frac{2\omega^{2m-1}}{(1-m)(-1)^m \lambda P_{2m-2}(\lambda)} \left(1 - \lambda\right)^{2m-4}(\lambda^2 + 1 - 2\lambda \cos h\omega)^2$$

$$= \frac{2\omega^{2m-1}}{(1-m)(-1)^m p_{2m-2}^{(2m-2)}} \left[\lambda + 4 - 4 \cos h\omega - 2m - \frac{p_{2m-3}^{(2m-2)}}{p_{2m-2}^{(2m-2)}}\right]$$

$$+ \frac{1}{p_{2m-2}^{(2m-2)}} \left[\frac{A_0}{\lambda} + \sum_{k=1}^{m-1} \left(\frac{A_{1,k}}{\lambda - \lambda_{1,k}} + \frac{A_{2,k}}{\lambda - \lambda_{2,k}}\right)\right]. \quad (3.18)$$

Multiplying both sides of the last equality by $\lambda P_{2m-2}(\lambda)$ we have
\[
\frac{2\omega^{2m-1}}{(-1)^m} \left(1 - \lambda\right)^{2m-4}(\lambda^2 + 1 - 2\lambda \cos h\omega)^2 \\
= \lambda \mathcal{P}_{2m-2}(\lambda) \left[ \frac{2\omega^{2m-1}}{(-1)^m \mathcal{P}_{2m-2}^{(2m-2)}} \left[ \lambda + 4 - 4 \cos h\omega - 2m - \frac{p_{2m-3}^{(2m-2)}}{p_{2m-2}} \right] + \frac{1}{p_{2m-2}} \left[ \frac{A_0}{\lambda} + \sum_{k=1}^{m-1} \left( \frac{A_{1,k}}{\lambda - \lambda_{1,k}} + \frac{A_{2,k}}{\lambda - \lambda_{2,k}} \right) \right] \right].
\] (3.19)

To find unknown coefficients \(A_0, A_{1,k}\) and \(A_{2,k}\) in (3.19) we put \(\lambda = 0, \lambda = \lambda_{1,k}\) and \(\lambda = \lambda_{2,k}\). Then we get

\[
A_0 = \frac{2\omega^{2m-1}}{(-1)^m},
\]
\[
A_{1,k} = \frac{2\omega^{2m-1}(1 - \lambda_{1,k})^{2m-4}(\lambda_{1,k}^2 + 1 - 2\lambda_{1,k} \cos h\omega)^2 p_{2m-2}^{(2m-2)}}{(-1)^m \lambda_{1,k} \mathcal{P}_{2m-2}^{(2m-2)}(\lambda_{1,k})},
\]
\[
A_{2,k} = \frac{2\omega^{2m-1}(1 - \lambda_{2,k})^{2m-4}(\lambda_{2,k}^2 + 1 - 2\lambda_{2,k} \cos h\omega)^2 p_{2m-2}^{(2m-2)}}{(-1)^m \lambda_{2,k} \mathcal{P}_{2m-2}^{(2m-2)}(\lambda_{2,k})}.
\] (3.20)

Hence, taking into account (3.17) and Lemma 2.2, we get

\[
A_{2,k} = -\frac{A_{1,k}}{\lambda_{1,k}^2}.
\] (3.21)

Since \(|\lambda_{1,k}| < 1\) and \(|\lambda_{2,k}| > 1\) the expressions

\[
\sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} \text{ and } \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}}
\]
can be expanded to the Laurent series on the circle \(|\lambda| = 1\), i.e.

\[
\sum_{k=1}^{m-1} \frac{A_{1,k}}{\lambda - \lambda_{1,k}} = \frac{1}{\lambda} \sum_{k=1}^{m-1} \frac{A_{1,k}}{1 - \frac{\lambda_{1,k}}{\lambda}}, \quad \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}} = \frac{1}{\lambda} \sum_{k=1}^{m-1} A_{1,k} \sum_{\beta=0}^{\infty} \left( \frac{\lambda_{1,k}}{\lambda} \right)^\beta
\]

and taking into account (3.17), we deduce

\[
\sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \lambda_{2,k}} = \sum_{k=1}^{m-1} \frac{A_{2,k}}{\lambda - \frac{1}{\lambda_{2,k}}} = -\sum_{k=1}^{m-1} \frac{A_{2,k} \lambda_{1,k}}{\lambda - \lambda_{1,k}} = -\sum_{k=1}^{m-1} A_{2,k} \lambda_{1,k} \sum_{\beta=0}^{\infty} (\lambda \lambda_{1,k})^\beta.
\]

Using the last two equalities and taking into account (3.3) and \(\lambda = e^{2\pi i ph}\) from (3.18) we get

14
\[ F[\overline{D}_m](p) = \frac{1}{p^{2m-2}} \left\{ \frac{2\omega^{2m-1}}{(-1)^m} e^{2\pi iph} + C \right\} + A_0 e^{-2\pi iph} \]
\[ + \sum_{k=1}^{m-1} A_{1,k} \sum_{\beta=0}^{\infty} \lambda_{1,k}^\beta e^{-2\pi iph(\gamma+1)} - \sum_{k=1}^{m-1} A_{2,k} \sum_{\beta=0}^{\infty} \lambda_{1,k}^{\beta+1} e^{2\pi iph \gamma} \} \]

Thus the Fourier series for \( F[\overline{D}_m](p) \) has the following form

\[ F[\overline{D}_m](p) = \sum_{\beta=-\infty}^{\infty} D_m(h\beta) e^{2\pi ihp\beta}, \]

where

\[ D_m(h\beta) = \frac{1}{p^{2m-2}} \left\{ \begin{array}{ll}
\sum_{k=1}^{m-1} A_{1,k} \lambda_{1,k}^{-\beta-1}, & \beta \leq -2, \\
A_0 + \sum_{k=1}^{m-1} A_{1,k}, & \beta = -1, \\
\frac{2\omega^{2m-1}}{(-1)^m} C - \sum_{k=1}^{m-1} A_{2,k} \lambda_{1,k}, & \beta = 0, \\
\frac{2\omega^{2m-1}}{(-1)^m} - \sum_{k=1}^{m-1} A_{2,k} \lambda_{1,k}^2, & \beta = 1, \\
- \sum_{k=1}^{m-1} A_{2,k} \lambda_{1,k}^{\beta+1}, & \beta \geq 2. \end{array} \right. \]

Thus, using (3.21), we rewrite the discrete function \( D_m(h\beta) \) in the following form

\[ D_m(h\beta) = \frac{1}{p^{2m-2}} \left\{ \begin{array}{ll}
\sum_{k=1}^{m-1} A_{1,k} \lambda_{1,k}^{|\beta|-1}, & |\beta| \geq 2, \\
\frac{2\omega^{2m-1}}{(-1)^m} + \sum_{k=1}^{m-1} A_{1,k}, & |\beta| = 1, \\
\frac{2\omega^{2m-1}}{(-1)^m} C + \sum_{k=1}^{m-1} A_{1,k}, & \beta = 0. \end{array} \right. \] (3.22)

Combining (3.2) and (3.20) we get

\[ A_{1,k} = \frac{2\omega^{2m-1}}{(-1)^m} A_k. \] (3.23)

Finally, denoting \( \lambda_k = \lambda_{1,k} \) and taking into account (3.23) from (3.22) we get the statement of the Theorem. Theorem 3.1 is proved. \( \square \)

**Remark 1.** From (3.1) we note that \( D_m(h\beta) \) is an even function, i.e.

\[ D_m(h\beta) = D_m(-h\beta) \]

and since \( |\lambda_k| < 1 \), then the function \( D_m(h\beta) \) decreases exponentially as \( |\beta| \to \infty \).

**Remark 2.** It should be noted that as consequence Theorem 3.1 generalizes previous results given in [3,4] for the particular cases \( m = 2, 3 \).
Here we obtain explicate formulas for the coefficients of the polynomials \( P_{2m-2}(x) \) defined by (2.13), for the cases \( m = 2, 3, ..., 6 \). Furthermore, for these cases, we show connection of the polynomials \( P_{2m-2}(x) \) with the Euler-Frobenius polynomials \( E_{2m-2}(x) \) defined by (2.9).

Using equality (2.11), when \( m = 2, 3, ..., 6 \), for the Euler-Frobenius polynomials \( E_{2m-2}(x) \) we have

\[
E_2(x) = x^2 + 4x + 1, \\
E_4(x) = x^4 + 26x^3 + 66x^2 + 26x + 1, \\
E_6(x) = x^6 + 120x^5 + 1191x^4 + 2416x^3 + 1191x^2 + 120x + 1, \\
E_8(x) = x^8 + 502x^7 + 14608x^6 + 88234x^5 + 156190x^4 + 88234x^3 + 14608x^2 + 502x + 1, \\
E_{10}(x) = x^{10} + 2036x^9 + 152637x^8 + 2203488x^7 + 9738114x^6 + 15724248x^5 + 9738114x^4 + 2203488x^3 + 152637x^2 + 2036x + 1.
\]

Now, by direct computation from (2.13) when \( m = 2, 3, ..., 6 \), for the coefficients \( p_s^{(2m-2)} \), \( s = 0, 1, ..., 2m-2 \), of the polynomials \( P_{2m-2}(x) \) we consequently get the following results.

1) For \( m = 2 \): \( P_2(x) = \sum_{s=0}^{2} p_s^{(2)} \) and

\[
p_2^{(2)} = p_0^{(2)} = \sin(h\omega) - h\omega \cos(h\omega), \\
p_1^{(2)} = 2h\omega - \sin(2h\omega), \\
\frac{p_2^{(2)}}{p_2^{(2)}} = \frac{p_0^{(2)}}{p_2^{(2)}} = 1, \\
\frac{p_1^{(2)}}{p_2^{(2)}} = 4 - \frac{2}{5}(h\omega)^2 + O(h^4).
\]

2) For \( m = 3 \): \( P_4(x) = \sum_{s=0}^{4} p_s^{(4)} \) and
\[ p_4^{(4)} = p_0^{(4)} = 3\sin(h\omega) - h\omega \cos(h\omega) - 2h\omega, \]
\[ p_3^{(4)} = p_1^{(4)} = 10h\omega \cos(h\omega) + 2h\omega - 3\sin(2h\omega) - 6\sin(h\omega), \]
\[ p_2^{(4)} = -8h\omega(1 + \cos^2(h\omega)) + 6\sin(h\omega) - 2h\omega \cos(h\omega) + 6\sin(2h\omega), \]
\[ \frac{p_4^{(4)}}{p_4^{(4)}} = \frac{p_0^{(4)}}{p_4^{(4)}} = 1, \] (4.3)
\[ \frac{p_3^{(4)}}{p_4^{(4)}} = \frac{p_1^{(4)}}{p_4^{(4)}} = 26 - \frac{18}{7}(h\omega)^2 + O(h^4), \]
\[ \frac{p_2^{(4)}}{p_4^{(4)}} = 66 - \frac{64}{7}(h\omega)^2 + O(h^4). \]

3) For \( m = 4 \): \( \mathcal{P}_6(x) = \sum_{s=0}^{6} p_s^{(6)} \) and

\[ p_6^{(6)} = p_0^{(6)} = 5\sin(h\omega) - h\omega \cos(h\omega) - 4h\omega + \frac{1}{3}(h\omega)^3, \]
\[ p_5^{(6)} = p_1^{(6)} = 10h\omega - 5\sin(2h\omega) - 20\sin(h\omega) + 4h\omega \cos(h\omega) + \frac{4}{3}(h\omega)^3 \]
\[ -8\cos(h\omega) \left(-2h\omega + \frac{1}{6}(h\omega)^3\right), \]
\[ p_4^{(6)} = p_2^{(6)} = 35\sin(h\omega) - 7h\omega \cos(h\omega) - 12h\omega + 20\sin(2h\omega) + \frac{1}{3}(h\omega)^3 \]
\[ -8\cos(h\omega) \left(4h\omega + \frac{2}{3}(h\omega)^3\right) + 4(1 + 2\cos^2(h\omega)) \left(-2h\omega + \frac{1}{6}(h\omega)^3\right), \]
\[ p_3^{(6)} = -16\cos(h\omega) \left(-2h\omega + \frac{1}{6}(h\omega)^3\right) + 4(1 + 2\cos^2(h\omega)) \left(4h\omega + \frac{2}{3}(h\omega)^3\right) \]
\[ -40\sin(h\omega) + 8h\omega \cos(h\omega) + 12h\omega - 30\sin(2h\omega), \]
\[ \frac{p_6^{(6)}}{p_6^{(6)}} = \frac{p_0^{(6)}}{p_6^{(6)}} = 1, \] (4.4)
\[ \frac{p_5^{(6)}}{p_6^{(6)}} = \frac{p_1^{(6)}}{p_6^{(6)}} = 120 - \frac{26}{3}(h\omega)^2 + O(h^4), \]
\[ \frac{p_4^{(6)}}{p_6^{(6)}} = \frac{p_2^{(6)}}{p_6^{(6)}} = 1191 - \frac{470}{3}(h\omega)^2 + O(h^4), \]
\[ \frac{p_3^{(6)}}{p_6^{(6)}} = 2416 - \frac{1108}{3}(h\omega)^2 + O(h^4). \]

4) For \( m = 5 \): \( \mathcal{P}_8(x) = \sum_{s=0}^{8} p_s^{(8)} \) and
\[ \begin{align*}
p_8^{(8)} &= p_0^{(8)} = 7 \sin(h\omega) - h\omega \cos(h\omega) - 6h\omega + \frac{2}{3}(h\omega)^3 - \frac{1}{60}(h\omega)^5, \\
p_7^{(8)} &= p_1^{(8)} = -\frac{13}{30}(h\omega)^5 + 26h\omega + \frac{4}{3}(h\omega)^3 - 8\cos(h\omega) \left(-3h\omega + \frac{1}{3}(h\omega)^3 - \frac{1}{120}(h\omega)^5\right) \\
&- 7\sin(2h\omega) - 42\sin(h\omega) + 6h\omega \cos(h\omega), \\
p_6^{(8)} &= p_2^{(8)} = -\frac{11}{10}(h\omega)^5 - 4(h\omega)^3 - 48h\omega - 8\cos(h\omega) \left(-\frac{13}{60}(h\omega)^5 + 12h\omega + \frac{2}{3}(h\omega)^3\right) \\
&+ 4(1 + 2\cos^2(h\omega)) \left(-3h\omega + \frac{1}{3}(h\omega)^3 - \frac{1}{120}(h\omega)^5\right) + 112\sin(h\omega) \\
&- 16h\omega \cos(h\omega) - 105\sin(2h\omega), \\
p_5^{(8)} &= p_3^{(8)} = -\frac{13}{30}(h\omega)^5 + 54h\omega + \frac{4}{3}(h\omega)^3 + 8\cos(h\omega) \left(\frac{67}{120}(h\omega)^5 + \frac{5}{3}(h\omega)^3 + 21h\omega\right) \\
&+ 4(1 + 2\cos^2(h\omega)) \left(-\frac{13}{60}(h\omega)^5 + 12h\omega + \frac{2}{3}(h\omega)^3\right) - 182\sin(h\omega) \\
&+ 26h\omega \cos(h\omega) - 105\sin(2h\omega), \\
p_4^{(8)} &= 210 \sin(h\omega) - 30h\omega \cos(h\omega) - 52h\omega + 140\sin(2h\omega) + \frac{4}{3}(h\omega)^3 - \frac{1}{30}(h\omega)^5 \\
&- 16\cos(h\omega) \left(-\frac{13}{60}(h\omega)^5 + 12h\omega + \frac{2}{3}(h\omega)^3\right) \\
&+ 4(1 + 2\cos^2(h\omega)) \left(-\frac{11}{20}(h\omega)^5 - 2(h\omega)^3 - 18h\omega\right), \\
p_8^{(8)} &= p_0^{(8)} = 1, \\
p_7^{(8)} &= p_1^{(8)} = 502 - \frac{1426}{55}(h\omega)^2 + O(h^4), \\
p_6^{(8)} &= p_2^{(8)} = 14608 - \frac{86664}{55}(h\omega)^2 + O(h^4), \\
p_5^{(8)} &= p_3^{(8)} = 88234 - \frac{141798}{11}(h\omega)^2 + O(h^4), \\
p_4^{(8)} &= 156190 - \frac{273872}{11}(h\omega)^2 + O(h^4). \\
\end{align*} \]

5) For \( m = 6 \): \( \mathcal{P}_{10}(x) = \sum_{s=0}^{10} p_s^{(10)} \) and
\[ p_{10}^{(10)} = p_{0}^{(10)} = 9 \sin(h\omega) - h\omega \cos(h\omega) - 8h\omega + \frac{1}{2520}(h\omega)^7 + (h\omega)^3 - \frac{1}{30}(h\omega)^5, \]

\[ p_{9}^{(10)} = p_{1}^{(10)} = 50h\omega - 9 \sin(2h\omega) - 72 \sin(h\omega) + 8h\omega \cos(h\omega) - \frac{4}{5}(h\omega)^5 + \frac{1}{21}(h\omega)^7 \]

\[ -8 \cos(h\omega) \left( -4h\omega + \frac{1}{5040}(h\omega)^7 + \frac{1}{2}(h\omega)^3 - \frac{1}{60}(h\omega)^5 \right), \]

\[ p_{8}^{(10)} = p_{2}^{(10)} = 261 \sin(h\omega) - 29h\omega \cos(h\omega) - 136h\omega + 72 \sin(2h\omega) + \frac{397}{840}(h\omega)^7 - \frac{1}{2}(h\omega)^5 \]

\[ -9(h\omega)^3 - 8 \cos(h\omega) \left( 24h\omega - \frac{2}{5}(h\omega)^5 + \frac{1}{42}(h\omega)^7 \right) \]

\[ +4(1 + 2 \cos^2(h\omega)) \left( -4h\omega + \frac{1}{5040}(h\omega)^7 + \frac{1}{2}(h\omega)^3 - \frac{1}{60}(h\omega)^5 \right), \]

\[ p_{7}^{(10)} = p_{3}^{(10)} = 8 \cos(h\omega) \left( 64h\omega - \frac{149}{630}(h\omega)^7 - \frac{4}{15}(h\omega)^5 - 4(h\omega)^3 \right) + 216h\omega + \frac{8}{3}(h\omega)^5 \]

\[ +16(h\omega)^3 + \frac{302}{315}(h\omega)^7 + 4(1 + 2 \cos^2(h\omega)) \left( 24h\omega - \frac{2}{5}(h\omega)^5 + \frac{1}{42}(h\omega)^7 \right) \]

\[ -576 \sin(h\omega) + 64h\omega \cos(h\omega) - 252 \sin(2h\omega), \]

\[ p_{6}^{(10)} = p_{4}^{(10)} = 882 \sin(h\omega) - 98h\omega \cos(h\omega) - 240h\omega + 504 \sin(2h\omega) + \frac{149}{315}(h\omega)^7 - \frac{8}{15}(h\omega)^5 \]

\[ -8(h\omega)^3 - 8 \cos(h\omega) \left( 104h\omega + \frac{14}{15}(h\omega)^5 + 8(h\omega)^3 + \frac{151}{315}(h\omega)^7 \right) \]

\[ +4(1 + 2 \cos^2(h\omega)) \left( -60h\omega + \frac{397}{1680}(h\omega)^7 - \frac{1}{4}(h\omega)^5 - \frac{9}{2}(h\omega)^3 \right), \]

\[ p_{5}^{(10)} = -1008 \sin(h\omega) + 112h\omega \cos(h\omega) + 236h\omega - 630 \sin(2h\omega) - \frac{8}{5}(h\omega)^5 + \frac{2}{21}(h\omega)^7 \]

\[ -16 \cos(h\omega) \left( -60h\omega + \frac{397}{1680}(h\omega)^7 - \frac{1}{4}(h\omega)^5 - \frac{9}{2}(h\omega)^3 \right) \]

\[ +4(1 + 2 \cos^2(h\omega)) \left( 80h\omega + \frac{4}{3}(h\omega)^5 + 8(h\omega)^3 + \frac{151}{315}(h\omega)^7 \right), \]

\[ \frac{p_{10}^{(10)}}{p_{10}^{(10)}} = \frac{p_{0}^{(10)}}{p_{10}^{(10)}} = 1, \]

\[ \frac{p_{9}^{(10)}}{p_{10}^{(10)}} = \frac{p_{1}^{(10)}}{p_{10}^{(10)}} = 2036 - \frac{998}{13}(h\omega)^2 + O(h^4), \]

\[ \frac{p_{8}^{(10)}}{p_{10}^{(10)}} = \frac{p_{2}^{(10)}}{p_{10}^{(10)}} = 152637 - \frac{170920}{13}(h\omega)^2 + O(h^4), \]

\[ \frac{p_{7}^{(10)}}{p_{10}^{(10)}} = \frac{p_{3}^{(10)}}{p_{10}^{(10)}} = 2203488 - \frac{3644480}{13}(h\omega)^2 + O(h^4), \]

\[ \frac{p_{6}^{(10)}}{p_{10}^{(10)}} = \frac{p_{4}^{(10)}}{p_{10}^{(10)}} = 9738114 - \frac{19460312}{13}(h\omega)^2 + O(h^4), \]

\[ \frac{p_{5}^{(10)}}{p_{10}^{(10)}} = 15724248 - \frac{33280180}{13}(h\omega)^2 + O(h^4). \]
From (4.1)-(4.6) we state the following.

**Conjecture 4.1** For the coefficients $p_s^{2m-2}$, $s = 0, 1, ..., 2m - 2$ of the polynomial $P_{2m-2}(x)$ the following holds

$$
\frac{p_s^{(2m-2)}}{p_{2m-2}^{(2m-2)}} = a_s^{(2m-2)} + O(h^2), \quad s = 0, 1, ..., 2m - 2,
$$

where $a_s^{(2m-2)}$ are the coefficients of the Euler-Frobenius polynomial $E_{2m-2}(x)$ of degree $2m - 2$.

**Acknowledgements**

The author thanks professor A.Cabada for discussion of the results. The present work was done in the University of Santiago de Compostela, Spain. A.R. Hayotov thanks the program Erasmus Mundus Action 2, Lot 10, Marco XXI for financial support (project number: Lot 10 - 20112572).

**References**

[1] Ahlberg, J.H., Nilson, E.N., Walsh, J.L.: The Theory of Splines and Their Applications, Academic Press, New York – London (1967).

[2] Frobenius, F.G.: On Bernoulli numbers and Euler polynomials, Sitzungsberichte der Königlich Preußischen Akademie der Wissenschaften zu Berlin, pp. 809-847, (1910).

[3] Hayotov, A.R.: Construction of discrete analogue of the differential operator $\frac{d^4}{dx^4} + 2\cdot\frac{d^2}{dx^2} + 1$ and its properties (Russian). Uzbek. Math. Zh. 2009, no. 3, 81–88. arxiv:1212.3672v1[math.NA] (2009).

[4] Hayotov, A.R.: Discrete analogues of some differential operators (Russian). Uzbek. Math. Zh. 2012, no. 1, 151-155. (2012).

[5] Hayotov, A.R., Milovanović, G.V., Shadimetov, Kh.M.: On an optimal quadrature formula in the sense of Sard, Numerical Algorithms, 57, 487–510 (2011).

[6] Hayotov, A.R., Milovanović, G.V., Shadimetov, Kh.M.: Interpolation splines minimizing a semi-norm, Calcolo, DOI: 10.1007/s10092-013-0080-x (2013)

[7] Maqsudov, Sh., Salokhitdinov, M.S., Sirojiddinov, S.H.: The theory of complex variable functions. -Tashkent, (1976).
[8] Shadimetov, Kh.M.: The discrete analogue of the differential operator \( \frac{d^{2m}}{dx^{2m}} \) and its construction. Questions of Computations and Applied Mathematics. Tashkent, (1985) 22-35. ArXiv:1001.0556.v1 [math.NA] Jan. 2010.

[9] Shadimetov, Kh.M., Hayotov, A.R.: Construction of the discrete analogue of the differential operator \( \frac{d^{2m}}{dx^{2m}} - \frac{d^{2m-2}}{dx^{2m-2}} \). Uzbek Math. Zh., 2004, no 2, pp. 85-95.

[10] Shadimetov, Kh.M., Hayotov, A.R.: Computation of coefficients of optimal quadrature formulas in the space \( W_{2}^{(m,m-1)}(0,1) \). Uzbek Math. Zh. 2004, no 3, pp.67-82.

[11] Shadimetov, Kh.M., Hayotov, A.R.: Construction of interpolation splines minimizing semi-norm in \( W_{2}^{(m,m-1)}(0,1) \) space, BIT Numer Math. DOI 10.1007/s10543-012-0407-z (2012)

[12] Shadimetov, Kh.M., Hayotov, A.R.: Optimal quadrature formulas in the sense of Sard in \( W_{2}^{(m,m-1)} \) space. Calcolo, DOI: 10.1007/s10092-013-0076-6 (2013)

[13] Sobolev S.L. Introduction to the theory of cubature formulas. Nauka, Moscow, 1974.

[14] Sobolev, S.L.: On the roots of Euler polynomials, in: Selected Works of S.L.Sobolev. Springer, pp.567-572, (2006).

[15] Sobolev, S.L., Vaskevich, V.L.: The Theory of Cubature Formulas. Kluwer Academic Publishers Group, Dordrecht (1997).

[16] Vladimirov, V.S.: Generalized Functions in Mathematical Physics (Russian). Nauka, Moscow (1979).

[17] Zhamalov Z.Zh. A difference analogue of the operator \( \frac{d^{2m}}{dx^{2m}} \). Direct and inverse problems for partial differential equations and their applications, pp. 97-108, 186, "Fan", Tashkent, (1978).