ANTI-ASSOCIATIVE ALGEBRAS

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Abstract An anti-associative algebra is a nonassociative algebra whose multiplication satisfies the identity \( a(bc) + (ab)c = 0 \). Such algebras are nilpotent. We describe the free anti-associative algebras with a finite number of generators. Other types of nonassociative algebras, obtained either by the polarization process, such as Jacobi-Jordan algebras, or obtained by deformation quantization, are associated with this class of algebras. Following Markl-Remm's work [7], we describe the operads associated with these algebra classes and in particular the cohomology complexes related to deformations.

TABLE DES MATIÈRES

1. Definition and basic properties 2
  1.1. Definition 2
  1.2. Nilpotency of anti-associative algebras 3
  1.3. The associative and Lie multiplication algebras 3
  1.4. Derivations and anti-derivations 4
2. Free anti-associative algebras 6
  2.1. The free algebra \( F_{\text{AA}}(1) \) 7
  2.2. The free algebra \( F_{\text{AA}}(2) \) 7
  2.3. The free algebra \( F_{\text{AA}}(k) \) 8
3. Homology of anti-associative algebras 10
  3.1. The standard complex of homology 10
  3.2. Homology of the free anti-associative algebra on one generator 11
4. Commutative and anti-commutative anti-associative algebras 12
  4.1. Commutative case 12
  4.2. Anti-commutative case 13
5. Polarization and depolarization of anti-associative algebras 14
  5.1. Polarization of an anti-associative algebra : the commutative part 14
  5.2. Polarization of an anti-associative algebra : the anti-commutative part 17
  5.3. Deformation quantization of anti-Poisson algebras 18
6. The operad \( \mathcal{AA} \) 19
  6.1. The operad \( \mathcal{AA} \) 19
  6.2. The minimal model 20
Références 23
1. Definition and basic properties

Let $\mathbb{K}$ be a field of characteristic 0. Recall that an algebra $(A, \mu)$ over $\mathbb{K}$ (often simply called an algebra) is a $\mathbb{K}$-vector space $A$ equipped with a bilinear multiplication $\mu$.

1.1. Definition.

**Definition 1.** We say that the algebra $(A, \mu)$ is anti-associative if the product $\mu$ satisfies the quadratic identity

$$\mu(\mu(x, y), z) + \mu(x, \mu(y, z)) = 0$$

for any $x, y, z \in A$.

Such an algebra cannot be unitary.

**Examples**

1. The only non trivial 2-dimensional anti-associative algebra is isomorphic to the algebra defined by

$$e_1^2 = e_2, \ e_1e_2 = e_2e_1 = e_2^2 = 0.$$  

2. In dimension 3 we have the following non isomorphic nontrivial anti-associative algebras $(A, \cdot)$:

   (a) $e_1 \cdot e_1 = 0, \ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_3,$
   (b) $e_1 \cdot e_1 = e_2, \ e_1 \cdot e_2 = -e_2 \cdot e_1 = e_3,$
   (c) $e_1 \cdot e_1 = e_2, \ e_1 \cdot e_3 = ae_2, \ e_3 \cdot e_1 = be_2, \ e_3 \cdot e_3 = e_2,$
   (d) $e_1 \cdot e_1 = e_2, \ e_1 \cdot e_3 = ae_2, \ e_3 \cdot e_1 = be_2,$

with $a, b \in \mathbb{K}$ and where $\{e_1, e_2, e_3\}$ is a basis of $A$.

3. Let us denote by $(\mathcal{F}_{AA}(X), \cdot)$ the free anti-associative algebra on one generator. It is a graded algebra

$$\mathcal{F}_{AA}(X) = \bigoplus_{k \geq 0} \mathcal{F}_{AA}(X)(k),$$

where $\mathcal{F}_{AA}(X)(k)$ consists of the vector space of elements of degree $k$. The anti-associativity implies that the degree 0 component is trivial. Let us determine each of these components.

   (a) Degree 2 : We put $X \cdot X = X^2$. Thus $\mathcal{F}_{AA}(X)(2) = \mathbb{K}\{X^2\}$.
   (b) Degree 3 : We have $X \cdot (X^2) = -X^2 \cdot X$. We put $X^3 = X \cdot X^2$. Then $X^2 \cdot X = -X^3$ and $\mathcal{F}_{AA}(X)(3) = \mathbb{K}\{X^3\}$.
   (c) Degree 4 :

$$\begin{cases} X \cdot X^3 = -X^2 \cdot X^2 \\ X \cdot X^3 = -X \cdot (X^2 \cdot X) = (X \cdot X^2) \cdot X = -(X^2 \cdot X) \cdot X = X^2 \cdot X^2. \end{cases}$$

Then $X^2 \cdot X^2 = 0$ and any product of degree 4 is zero. Then $\mathcal{F}_{AA}(X)(4) = \{0\}$

We deduce that $\dim \mathcal{F}_{AA}(X) = 3$. It corresponds to the previous example (b).
1.2. Nilpotency of anti-associative algebras. Recall that an algebra $A$ is called nilpotent if there exists an integer $k$ such that all products of $k$ elements of $A$ are 0. The smallest integer $k$ such that the subspace $A^k$ generated by all the products of degree $k$ is zero is called the nilindex of $A$. To simplify the notations, we write $xy$ the multiplication of two elements of $A$ in place of $x \cdot y$.

**Proposition 2.** An anti-associative algebra is nilpotent of nilindex 4.

**Proof.** Let $a, b, c, d$ be elements of the anti-associative algebra $A$. We have

$$ (ab)(cd) = -a(b(cd)) = a((bc)d) = -(a(bc))d = ((ab)c)d = -(ab)(cd). $$

Then $(ab)(cd) = 0$ and all the other products of four elements are also 0.

1.3. The associative and Lie multiplication algebras. Let $a$ be an element of the anti-associative algebra $A$. We denote by $L_a$ and $R_a$ the left and right translation in $A$:

$$ L_a(x) = ax, \quad R_a(x) = xa. $$

Since $A$ is a nilpotent algebra of nilindex 4, these linear maps are nilpotent of degree 3:

$$ L_a^3 = R_a^3 = 0. $$

Let $\mathcal{M}(A)$ be the subalgebra of $\text{End}(A)$ generated by the linear maps $L_a$ and $R_a$ for all $a \in A$. It is an associative algebra called the multiplication algebra of $A$ which is nilpotent of nilindex 3. We have the identities

$$ \begin{cases} 
L_aL_b = -L_{ab}, \\
R_aR_b = -R_{ba}, \\
R_aL_b = -L_bR_a, 
\end{cases} $$

resulting from the anti-associativity in $A$ then every element of $\mathcal{M}(A)$ can be written as linear combination of $L_x, R_y, L_xR_y$.

**Example.** Let us consider the free anti-associative algebra on one generator. It is of dimension 3 and its multiplication satisfies

$$ e_1e_1 = e_2, \quad e_1e_2 = -e_2e_1 = e_3 $$

(with $e_1 = X, e_2 = X^2, e_3 = XX^2$). The associative algebra $\mathcal{M}(A)$ is 3-dimensional generated by $L_{e_1}, R_{e_1}, L_{e_2}$.

We can consider also the Lie subalgebra $\mathcal{L}(A)$ of $gl(A)$ generated by the left and right multiplication of $A$. It corresponds to the Lie algebra associated with the associative algebra $\mathcal{M}(A)$. Since it is generated by the Lie brackets between these nilpotent linear maps, we have

**Proposition 3.** The Lie multiplication algebra associated with the anti-associative algebra is a 2-step nilpotent Lie algebra.

**Example.** In the previous example, the Lie algebra $\mathcal{L}(A)$ is isomorphic to the 3-dimensional Heisenberg algebra.
1.4. Derivations and anti-derivations. Recall that a derivation of the algebra $A$ is a linear operator which satisfies

$$f(xy) = xf(y) + f(x)y.$$ 

The set of derivations of $A$, $D(A)$, is a Lie algebra. A derivation $f$ is called inner if $f \in D(A) \cap L(A)$. For example, we consider for any $x \in A$ the linear map $L_x - R_x$ which belongs to $L(A)$. In [3], it is proved that this linear map is a derivation of the nonassociative algebra $A$ if and only if $A$ is weakly associative. Then if $A$ is anti-associative algebra, $L_x - R_x$ is a derivation if and only if

$$x(yz) + y(zx) - y(xz) = 0$$

for any $x, y, z \in A$.

**Definition 4.** Let $A$ be an algebra (not necessarily anti-associative). A linear map $f \in \text{End}(A)$ is an anti-derivation if this map satisfies

$$f(xy) + xf(y) + f(x)y = 0$$

for any $x, y \in A$.

We denote by $\tilde{D}(A)$ the set of anti-derivations of $A$.

**Proposition 5.** If $A$ is a nonassociative algebra (anti-associative or not) then $[\tilde{D}(A), \tilde{D}(A)] \subset D(A)$.

**Proof.** Let $f, g$ be two anti-derivations of $A$.

$$f \circ g(xy) = f(g(xy)) = -f(xyg) + g(xy) = f(x)(g(y) + xf(g(y))) + f(g(xy))y + g(x)f(y).$$

Then

$$[f, g](xy) = f(x)g(y) + xf(g(y)) + f(g(x)y + g(x)f(y) - g(x)f(y) - xg(f(y)))$$

and $[f, g]$ is a derivation of $A$.

**Proposition 6.** Let $A$ be an anti-associative algebra. Then for all $x \in A$, the linear map $L_x + R_x$ is an anti-derivation of $A$.

**Proof.** We have

$$(L_x + R_x)(yz) + (L_x + R_x)(yz) + y(L_x + R_x)(z) = x(yz) + (yz)x + (xy)z + (yx)z + y(xz) + y(zx) = 0$$

since $A$ is anti-associative.

**Remark.** Let $A$ be an algebra (not necessarily anti-associative). Then $L_x + R_x$ is an anti-derivation if and only if the multiplication of $A$ satisfies:

$$\tilde{A}(x, y, z) + \tilde{A}(y, x, z) + \tilde{A}(y, z, x) = 0$$

where $\tilde{A}$ is the trilinear map

$$\tilde{A}(x, y, z) = x(yz) + (xy)z.$$

An algebra whose operator $\tilde{A}$ satisfies Identity (2) will be called $(Id + c + \tau)$-anti-associative. (the terminology comes from the analogy with $\nu$-associative algebra where $\nu$ is a vector of $\mathbb{K}[\Sigma_3]$, the group algebra of the symmetric group of order 3, see [5]).
Definition 7. Let \( A \) be an anti-associative algebra. For \( x \in A \), the anti-derivation \( \widetilde{ad}_x = L_x + R_x \) will be called inner and \( \widetilde{I}(A) \) denotes the set of inner anti-derivations of \( A \).

For any derivation \( g \) of \( A \) we have
\[
[\widetilde{ad}_x, g](y) = \widetilde{ad}_x(g(y)) - g(xy + yx) \\
= xg(y) + g(y)x - g(xy) - yg(x) = \widetilde{g}(x)y - yg(x) \\
= \widetilde{ad}_{-g(x)}(y).
\]

Then \([\widetilde{ad}_x, g] = \widetilde{ad}_{-g(x)}\) and

Proposition 8. Let \( A \) be an anti-associative algebra. The set of inner anti-derivations \( \widetilde{I}(A) \) and the set of derivations \( D(A) \) satisfy
\[
[\widetilde{I}(A), D(A)] \subset \widetilde{I}(A).
\]

Remark. The inner anti-derivations are defined in the same way for \( Id + c + \tau_{12} \)-anti-associative algebra and Proposition \( \exists \) is also true for this type of algebras.

Assume now that the linear map \( L_x + R_x \) belongs to \( D(A) \) in the anti-associative algebra \( A \). This is equivalent to write that, for all \( y, z \) in \( A \),
\[
x(yz) + (yz)x - y(xz + zx) - (xy + yx)z = 0,
\]
that is,
\[
x(yz) + (yz)x = 0 \iff \widetilde{ad}_x(yz) = 0
\]

Then \( \widetilde{ad}_x \in D(A) \) if and only if \( \widetilde{ad}_x(u) = 0 \) forall \( u \in A^2 \). We deduce:

Proposition 9. Let \( A \) be an anti-associative algebra. The set of anti-derivations \( \widetilde{I}(A) \) and the set of anti-derivations \( D(A) \) satisfy
\[
\widetilde{I}(A) \cap D(A) = \{ \widetilde{ad}_x \text{ such that forall } u \in A^2, \widetilde{ad}_x(u) = 0 \}.
\]

In particular for any \( x \in A^2 \), the inner anti-derivation \( \widetilde{ad}_x \) is also a derivation. More generally, if \( f \) belongs to \( \widetilde{D}(A) \cap D(A) \) then
\[
f(xy) = xf(y) + f(x)y = -xf(y) - f(x)y
\]
or equivalently
\[
f(x^2) = 0, \quad xf(y) + f(x)y = 0
\]
for any \( x, y \in A \).

For two endomorphisms \( f \) and \( g \) of \( A \), we define the symmetric product
\[
f \bullet g = f \circ g + g \circ f.
\]

Then
\[
\widetilde{ad}_x \bullet \widetilde{ad}_y = L_x L_y + L_x R_y + R_x L_y + R_x R_y + L_y R_x + R_y L_x + R_y R_x \\
= -L_{xy} + L_x R_y - L_y R_x - R_y x - L_y x + L_y R_x - L_x R_y - R_{xy} \\
= -L_{xy+yx} - R_{xy+yx} = -\widetilde{ad}_{xy+yx} = \widetilde{ad}_{-(xy+yx)}.
\]

Likewise, if \( f \in \widetilde{D}(A) \),
\[
f \bullet \widetilde{ad}_x = -\widetilde{ad}_{f(x)} = \widetilde{ad}_{-f(x)}.
\]
Proposition 10. Let $A$ be an anti-associative algebra and $f \in \widetilde{D}(A)$. Then

1. $\widetilde{\text{ad}}(x) \cdot \widetilde{\text{ad}} y = -\widetilde{\text{ad}}(xy + yx)$ for any $x, y \in A$,
2. $f \cdot \widetilde{\text{ad}}(x) = -\widetilde{\text{ad}}(f(x))$ for any $x \in A$.

In general, the symmetric product of two anti-associative derivations is not an anti-associative derivation. In fact, if $f, g \in \widetilde{D}(A)$,

$$
(f \circ g)(xy) = f(g(xy)) + g(f(xy)) = -f(g(x)y + xf(y)) - g(f(x)y + xf(y))
$$

$$
= f(g(x))y + g(x)f(y) + f(x)g(y) + g(f(x)y + xg(y))
$$

$$
= 2(f(x)g(y) + g(x)f(y)) + f(g(x)) + g(f(x)y)
$$

and

$$
x(f \circ g)(y) + (f \circ g)(x)y = x(f(g(y)) + g(f(y))) + (f(g(x)) + g(f(x)y).
$$

Thus $f \circ g$ is an anti-associative derivation if and only if these linear maps satisfy

$$
f(x)g(y) + g(x)f(y) = -(f(g(x)) + g(f(x)y) - x(f(g(y)) + g(f(y))
$$

Remark: on the inner derivations of nonassociative algebras Usually we cannot define directly the notion of inner derivation on a nonassociative algebra. If $U$ is such algebra, we consider as above the Lie multiplication algebra $L(U)$ generated by the left and right multiplication of $U$. We say that a derivation $f$ of $U$ is inner if $f \in L(U)$. The set $L(U) \cap D(U)$ is an ideal of $D(U)$. If $U$ is a Lie algebra, then $L(U) \cap D(U)$ corresponds to the set of adjoint operators, that is $L_x - R_x$. For an associative algebra, an element of $L(U) \cap D(U)$ is of type $L_x - R_x$ as soon as $U$ is an unitary associative algebra. It is not always the case when $U$ is not unitary. For example, if we consider the 2-dimensional associative algebra given in a basis $\{e_1, e_2\}$ by $e_1e_1 = e_2$ (other non defined products are 0), then all derivations of $L_x - R_x$ are trivial, $D(U)$ is the set of endomorphisms $f$ satisfying $f(e_1) = ae_1 + be_2$, $f(e_2) = 2ae_2$, $L(U)$ is the set of endomorphism $f$ satisfying $f(e_1) = be_2$, $f(e_2) = 0$ and $L(U) \cap D(U)$ is the set of endomorphisms $f$ satisfying $f(e_1) = be_2$, $f(e_2) = 0$. Recall also that $L_x - R_x$ is a derivation of $U$ if and only if $U$ is a weakly associative algebra.

When $A$ is an anti-associative algebra, the brackets $[\widetilde{\text{ad}}_x, \widetilde{\text{ad}}_y], [L_x - R_x, \widetilde{\text{ad}}_y], [L_x - R_x, L_y - R_y]$ are always derivations of $A$ and also anti-associative derivations because any product of 4 elements of $A$ is zero. For the same reason, $L_x R_y$ and $R_x L_y$ are derivations and anti-derivations of $A$. This implies that $L_x + R_y$ is an anti-derivation as soon as $y = x + v$ with $z(vt) = 0$ for any $z, t \in A$. In particular $L_x + R_x + v$ is an anti-derivation for any $v \in A^2$ and $L(A) = \widetilde{L}(A)$ as soon as $R_y = 0$ for any $v \in A^2$.

2. **Free anti-associative algebras**

We denote by $\mathcal{F}_{AA}(k)$ the free anti-associative algebra with $k$ generators. This algebra is graded by the degree of its elements:

$$
\mathcal{F}_{AA}(k) = \bigoplus_{1 \leq n \leq 3} \mathcal{F}_{AA}(k)^{(n)}
$$

because all products of degree 4 are null. Recall that the anti-associativity prevents having elements of degree 0.
2.1. The free algebra $\mathcal{F}_{A\mathcal{A}}(1)$. We have already described this algebra in the first paragraph. Let us recall this construction briefly. Let $X$ be the generator. Then $X^2 \in \mathcal{F}_{A\mathcal{A}}(1)$. For the elements of degree 3, since we have $X \cdot X^2 = -X^2 \cdot X$, we deduce that $\mathcal{F}_{A\mathcal{A}}(1)(3) = \mathbb{K}\{X \cdot X^2\}$. Then $\dim \mathcal{F}_{A\mathcal{A}}(1)$ and a basis is given by
\[ X, X^2, X \cdot X^2 \]
and the multiplication
\[ X \cdot X = X^2, X^2 \cdot X = -X \cdot X^2. \]
To use conventional conventions we denote $e_1^{(1)} = X, e_1^{(2)} = X^2, e_1^{(3)} = X \cdot X^2$. Then we have
\[ e_1^{(1)} \cdot e_1^{(1)} = e_1^{(2)}, e_1^{(1)} \cdot e_1^{(2)} = -e_1^{(2)} \cdot e_1^{(1)} = e_1^{(3)}. \]
The Lie algebra $D(\mathcal{F}_{A\mathcal{A}}(1))$ is 3-dimensional constituted of matrices
\[
\begin{pmatrix}
\alpha & 0 & 0 \\
0 & 2\alpha & 0 \\
0 & 0 & 3\alpha
\end{pmatrix}
\]
while
\[
\tilde{D}(\mathcal{F}_{A\mathcal{A}}(1)) \text{ is the space of matrices }
\begin{pmatrix}
\alpha & 0 & 0 \\
\beta & -2\alpha & 0 \\
\gamma & 0 & \alpha
\end{pmatrix}
\]
and an anti-associative derivation is a derivation when it is trivial on $(\mathcal{F}_{A\mathcal{A}}(1))^2 = \mathbb{K}\{e_1^{(2)}, e_1^{(3)}\}$. Any inner anti-associative derivation is collinear to $ad_{e_1^{(1)}}$.

2.2. The free algebra $\mathcal{F}_{A\mathcal{A}}(2)$. Let $\mathcal{F}_{A\mathcal{A}}(2) = \oplus_{1 \leq n \leq 3} \mathcal{F}_{A\mathcal{A}}(2)^{(n)}$ the graded decomposition of $\mathcal{F}_{A\mathcal{A}}(2)$. If we denote by $X, Y$ the generators of this free algebra, we have
\[
(1) \quad \mathcal{F}_{A\mathcal{A}}(2)^{(1)} = \mathbb{K}\{X, Y\}
\]
\[
(2) \quad \mathcal{F}_{A\mathcal{A}}(2)^{(2)} = \mathbb{K}\{X^2, XY, YX, Y^2\}
\]
\[
(3) \quad \mathcal{F}_{A\mathcal{A}}(2)^{(3)} = \mathbb{K}\{X(X^2), X(XY), X(YX), X(Y^2), YX^2, YY^2\}
\]
and $\dim \mathcal{F}_{A\mathcal{A}}(2) = 14$. Moreover, we have
\[
\left\{ \begin{array}{c}
\mathcal{F}_{A\mathcal{A}}(2)^{(1)} \cdot \mathcal{F}_{A\mathcal{A}}(2)^{(1)} = \mathcal{F}_{A\mathcal{A}}(2)^{(2)} \\
\mathcal{F}_{A\mathcal{A}}(2)^{(1)} \cdot \mathcal{F}_{A\mathcal{A}}(2)^{(2)} = \mathcal{F}_{A\mathcal{A}}(2)^{(2)} \cdot \mathcal{F}_{A\mathcal{A}}(2)^{(1)} = \mathcal{F}_{A\mathcal{A}}(2)^{(3)}
\end{array} \right.
\]
all other non defined products being zero. Now let’s look at the anti-derivations. Any anti-derivation of $A = \mathcal{F}_{A\mathcal{A}}(2)$ is entirely determined by its values $f(X)$ and $f(Y)$. In fact for any generators $uv$ of $\mathcal{F}_{A\mathcal{A}}(2)^{(2)}$ with $u, v \in \{X, Y\}$ we have $f(uv) = -uf(v) - f(u)v$. Any generators of $\mathcal{F}_{A\mathcal{A}}(2)^{(3)}$ is written $u(vw)$ with $u, v, w \in \{X, Y\}$. Since $(uv)w = -u(vw)$ we have
\[
f(u(vw)) = u(f(v)w), \quad u(vf(w)) - f(u)(vw) = 0.
\]
If we take $f(X) = a_1 X + a_2 Y + Z_1$ and $f(Y) = b_1 X + b_2 Y + Z_2$ with $Z_i \in \mathcal{F}_{A\mathcal{A}}(2)^{(2)} \oplus \mathcal{F}_{A\mathcal{A}}(2)^{(3)}$, then the previous equations give $a_2 = b_1 = 0, a_1 = b_2$. As a consequence for all $u \in \mathcal{F}_{A\mathcal{A}}(2)^{(3)}$
\[
f(u) = a_1 u,
\]
and for any $v \in \mathcal{F}_{A\mathcal{A}}(2)^{(2)}$,
\[
f(v) = -2a_1 v + Z, \quad Z \in \mathcal{F}_{A\mathcal{A}}(2)^{(3)}.
\]
In particular
\[
\dim \tilde{D}(\mathcal{F}_{A\mathcal{A}}(2)) = 25 = 2 (\dim \mathcal{F}_{A\mathcal{A}}(2)^{(2)} \oplus \mathcal{F}_{A\mathcal{A}}(2)^{(3)}) + 1.
\]
Now, concerning the inner anti-derivations, we can easily see that the linear maps \( \tilde{ad}_x \) for any \( x \in \{X, Y, X^2, XY, YX, Y^2\} \) are linearly independent. We deduce that

\[
\dim \tilde{I}(F_{AA}(2)) = 6.
\]

2.3. The free algebra \( F_{AA}(k) \). The free anti-associative algebra with \( k \) generators \( X_1, \cdots, X_k \) admits the following grading associated with the degree of the products of generators:

\[
F_{AA}(k) = \bigoplus_{1 \leq n \leq 3} F_{AA}(k)^{(n)}
\]

and we have

\[
\begin{align*}
(1) & \quad F_{AA}(k)^{(1)} = \mathbb{K}\{X_1, \cdots, X_k\}, \\
(2) & \quad F_{AA}(k)^{(2)} = \mathbb{K}\{X_iX_j, 1 \leq i, j \leq k\}, \\
(3) & \quad F_{AA}(k)^{(3)} = \mathbb{K}\{X_i(X_jX_l), 1 \leq i, j, l \leq k\}.
\end{align*}
\]

Then

\[
\begin{align*}
(1) & \quad \dim F_{AA}(k)^{(1)} = k, \\
(2) & \quad \dim F_{AA}(k)^{(2)} = k^2, \\
(3) & \quad \dim F_{AA}(k)^{(3)} = k^3.
\end{align*}
\]

and

\[
\dim F_{AA}(k) = k + k^2 + k^3.
\]

By induction, we prove that any anti-derivation \( f \) is completely defined by the vectors \( f(X_i), i = 1, \cdots, k \) and these vectors satisfy

\[
X_i(X_jf(X_k)) - f(X_i)(X_jX_k) = 0
\]

which implies that the restriction of \( f \) to \( F_{AA}(k)^{(1)} \) is equal to \( a \cdot \text{Id} \). In particular

\[
\dim \tilde{D}(F_{AA}(k)) = 1 + k \left( \dim F_{AA}(k)^{(2)} \oplus F_{AA}(k)^{(3)} \right) = 1 + k^3 + k^4.
\]

Concerning the inner anti-derivations, we can easily see that the linear maps \( \tilde{ad}_x \) for any generator \( x \in F_{AA}(k)^{(1)} \oplus F_{AA}(k)^{(2)} \) are linearly independent. We deduce that

\[
\dim \tilde{I}(F_{AA}(2)) = k + k^2.
\]

The free algebra \( F_{AA}(k) \) admits a natural grading associated with the nilpotency property:

\[
F_{AA}(k) = F_{AA}(k)^{(1)} \oplus F_{AA}(k)^{(2)} \oplus F_{AA}(k)^{(3)}
\]

with

\[
F_{AA}(k)^{(1)} \cdot F_{AA}(k)^{(1)} = F_{AA}(k)^{(2)}, \quad F_{AA}(k)^{(1)} \cdot F_{AA}(k)^{(2)} = F_{AA}(k)^{(3)}, \quad F_{AA}(k)^{(1)} \cdot F_{AA}(k)^{(3)} = 0.
\]

If \( \{e_1, \cdots, e_k\} \) is a basis of \( F_{AA}(k)^{(1)} \), that is a basis of generators of the free algebra, \( \{f_{i,j} = e_i e_j, 1 \leq i, j \leq k\} \) a basis of \( F_{AA}(k)^{(2)} \), \( \{g_{i,j,l} = e_i f_{j,l} \} \) a basis of \( F_{AA}(k)^{(3)} \), then we have

\[
\begin{align*}
e_i e_j &= f_{i,j} \\
e_i f_{j,l} &= g_{i,j,l} \\
e_i g_{j,l,s} &= g_{j,l,s} e_i = f_{i,j} f_{l,s} = f_{i,j} g_{l,s,t} = g_{l,s,t} f_{i,j} = 0 \\
f_{i,j} e_l &= -e_i f_{j,l}
\end{align*}
\]

for all \( i, j, l, s, t \in [i, k] \).
Application: Construction of anti-associative algebras. From this free anti-associative algebra, we deduce the anti-associative algebras with $k$ generators:

$$ A = A^{(1)} \oplus A^{(2)} \oplus A^{(3)} $$

where

$$ A^{(1)} = F_{AA}(k)^{(1)}, \quad A^{(2)} = A^{(1)} \cdot A^{(1)}, \quad A^{(3)} = A^{(1)} \cdot A^{(2)} \oplus A^{(2)} \cdot A^{(1)}. $$

This allows the construction of the anti-associative algebras from the data of the generators. For example, if $\dim A^{(1)} = 2$, then $\dim A^{(2)} \leq 4$. If this dimension is equal to 4, we have $F_{AA}(2)$. If this dimension is equal to 1, then $\dim A^{(3)} \leq 2$. We complete the table of multiplication putting $f_{ij}e_k = -e_if_{j,k}$.

Example: dim $A^{(1)} = 2$. In this case dim $A \leq 14$. If dim $A = 14$, then $A$ is isomorphic to $F_{AA}(2)$. We can describe the other anti-associative algebras considering the dimensions of $A^{(2)}$ and $A^{(3)}$. We will not develop here a complete classification but show how to obtain it. We put $A^{(1)} = \mathbb{K}\{e_1, e_2\}$.

1. If $\dim A^{(2)} = 1$, then $A^{(3)} = 0$. In fact
   1. If $A^{(2)} = \mathbb{K}\{f_{1,1}\}$, then
      $$ e_1f_{1,1} = f_{1,1}e_1 = 0, \quad e_2f_{1,1} = \alpha f_{1,1}e_1 = 0, \quad f_{1,1}e_2 = -\beta e_1f_{1,1} = 0 $$
      and $A^{(3)} = 0$.
   2. If $A^{(2)} = \mathbb{K}\{f_{1,2}\}$, then we can assume that $e_1e_1 = e_2e_2 = 0$ (if not we are in the previous case). Then
      $$ e_1f_{1,2} = 0, \quad f_{1,2}e_1 = 0, \quad f_{1,2}e_2 = 0, \quad e_2f_{1,2} = -b f_{1,2}e_2, = 0 $$
      This implies that $A^{(3)} = 0$. We obtain the following 3-dimensional anti-associative algebras:
      \begin{itemize}
      \item[(a)] $e_i e_j = \alpha_{ij} f_{1,1}, \alpha_{1,1} = 1, e_1f_{1,1} = f_{1,1}e_1 = 0.$
      \item[(b)] $e_i e_i = 0, e_1e_2 = -e_2e_1 = f_{1,2}, e_1f_{1,2} = f_{1,2}e_1 = 0.$
      \end{itemize}

2. If $A^{(2)} = \mathbb{K}\{f_{1,1}, f_{1,2}\}$ then $\dim A^{(3)} \leq 1$ and if this dimension is not 0, then $A^{(3)} = \mathbb{K}\{g_{1,1,2}\}$. We obtain the following algebras:
   \begin{itemize}
   \item[(a)] $e_1e_1 = f_{1,1}, \quad e_1e_2 = f_{1,2}, \quad e_2e_1 = af_{1,1} + bf_{1,2}, \quad e_2e_2 = cf_{1,1} + df_{1,2}$.
   \item[(b)]
   $$ \begin{cases}
   e_1e_1 = f_{1,1}, \quad e_1e_2 = f_{1,2}, \quad e_2e_1 = af_{1,1} + bf_{1,2}, \quad e_2e_2 = cf_{1,1} + df_{1,2} \\
   e_1f_{1,1} = 0, \quad e_1f_{1,2} = g_{1,1,2}, \quad e_2f_{1,1} = b^2 g_{1,1,2}, \quad e_2f_{1,2} = (a + bd) g_{1,1,2} \\
   f_{1,1}e_1 = 0, \quad f_{1,2}e_1 = -bg_{1,1,2}, \quad f_{1,1}e_2 = -g_{1,1,2}, \quad f_{1,2}e_2 = -dg_{1,1,2}
   \end{cases} $$
   This algebra is 5-dimensional and can be considered as the quotient of the free algebra $F_{AA}(2)$ modulo the relations $XY - aX^2 - bXY, \quad Y^2 - cX^2 - dXY$.
   \item[(c)] If we assume that $\dim A^{(2)} = 2$ and that $A$ is not isomorphic to the previous one, then such an algebra is isomorphic to the quotient of $F_{AA}(2)$ modulo the relations $XY = aX^2, YX = cX^2$. In this case $A^{(3)} = 0$ and we have the algebra
   $$ \begin{cases}
   e_1e_1 = f_{1,1}, \quad e_1e_2 = af_{1,1}, \quad e_2e_1 = cf_{1,1}, \quad e_2e_2 = f_{2,2} \\
   e_if_{j,k} = 0, \quad f_{j,k}e_i = 0.
   \end{cases} $$
   This algebra is 4-dimensional.

All the other cases can be treated in a similar way. It will be a next work.
3. Homology of Anti-Associative Algebras

3.1. The standard complex of homology. Let $A$ be a finite-dimensional anti-associative algebra, $T(A) = \oplus T^n(A)$ its tensorial algebra. For any $n \geq 4$, we consider the subspace $R^n(A)$ of $T^n(A)$ generated by the vectors

$$x_{i_1} \otimes \cdots \otimes x_{i_{k-1}} \otimes x_{i_k} x_{i_{k+1}} \otimes x_{i_{k+2}} \otimes \cdots \otimes x_{i_{l-1}} \otimes x_{i_l} x_{i_{l+1}} \otimes x_{i_{l+2}} \otimes \cdots \otimes x_{i_{n+2}}$$

and we denote by $\tilde{T}^n(A)$ the factor space

$$\tilde{T}^n(A) = \frac{T^n(A)}{R^n(A)}$$

for $n \geq 4$ and $\tilde{T}^i(A) = T^i(A)$ for $1 \leq i \leq 3$. We denote by $a \otimes b$ the multiplication on $\tilde{T}(A) = \oplus \tilde{T}^n(A)$ deduced from the tensor product.

Let $H_*(A, A)$ be the homology group of the chain complex $(\tilde{T}^n(A), b_n)$ where $b_n : \tilde{T}^{n+1}(A) \rightarrow \tilde{T}^n(A)$ is defined by

$$b_n(x_1 \otimes x_2 \otimes \cdots \otimes x_{n+1}) = x_1 x_2 \otimes x_3 \otimes \cdots \otimes x_{n+1} + x_1 \otimes x_2 x_3 \otimes x_4 \otimes \cdots \otimes x_{n+1} + \cdots$$

$$+ x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{n+1} + \cdots$$

$$+ x_{n+1} x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

$$= \sum_{i=1}^n x_1 \otimes \cdots \otimes x_{i-1} \otimes x_i x_{i+1} \otimes x_{i+2} \otimes \cdots \otimes x_{n+1}$$

$$+ x_{n+1} x_1 \otimes x_2 \otimes \cdots \otimes x_n$$

For example

$$b_1(x_1 \otimes x_2) = x_1 x_2 + x_2 x_1$$

$$b_2(x_1 \otimes x_2 \otimes x_3) = x_1 x_2 \otimes x_3 + x_1 \otimes x_2 x_3 + x_3 x_1 \otimes x_2$$

$$b_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = x_1 x_2 \otimes x_3 \otimes x_4 + x_1 \otimes x_2 x_3 \otimes x_4 + x_1 \otimes x_2 \otimes x_3 x_4 + x_1 \otimes x_2 \otimes x_3 \otimes x_4 + x_1 \otimes x_2 \otimes x_3 \otimes x_4 + x_1 \otimes x_2 \otimes x_3 \otimes x_4$$

**Lemma 11.** For every $n \geq 2$, $b_{n-1} \circ b_n = 0$.

**Proof.**

$$b_1(x_1 \otimes x_2) = x_1 x_2 + x_2 x_1$$

$$b_2(x_1 \otimes x_2 \otimes x_3) = (x_1 x_2) x_3 + x_1 x_2 x_3 + x_3 x_1 x_2$$

$$b_3(x_1 \otimes x_2 \otimes x_3 \otimes x_4) = (x_1 x_2) x_3 x_4 + (x_1 x_2) x_4 x_3 + x_4 (x_1 x_2) x_3 + x_3 (x_1 x_2) x_4 + x_4 (x_1 x_2) x_3 + x_1 x_2 x_3 x_4$$

$$+(x_1 x_2) x_3 x_4 + x_4 x_1 x_2 x_3 + x_3 x_4 x_1 x_2$$

$$= 2 [(x_1 x_2) \otimes (x_3 x_4) + (x_4 x_1) \otimes (x_2 x_3)] = 0$$

and for the general case
\[(b_{n-1} \circ b_n)(x_1 \bar{\otimes} \cdots \bar{\otimes} x_{n+1})\]
\[= \sum_{i=1}^{n} b_{n-1}(x_1 \bar{\otimes} \cdots \bar{\otimes} x_{i-1} \bar{\otimes} x_i x_{i+1} \bar{\otimes} \cdots \bar{\otimes} x_{n+1}) + b_{n-1}(x_{n+1} x_1 \bar{\otimes} \bar{\otimes} \cdots \bar{\otimes} x_n)\]
\[= \sum_{i=2}^{n} x_1 \bar{\otimes} \cdots \bar{\otimes} x_{i-1} (x_i x_{i+1}) \bar{\otimes} \cdots \bar{\otimes} x_{n+1} + \sum_{i=1}^{n-1} x_1 \bar{\otimes} \cdots \bar{\otimes} (x_i x_{i+1}) x_{i+2} \bar{\otimes} \cdots \bar{\otimes} x_{n+1}\]
\[+ x_{n+1} (x_1 x_2 \bar{\otimes} x_3 \bar{\otimes} \cdots \bar{\otimes} x_n) + (x_{n+1} x_1) x_2 \bar{\otimes} x_3 \bar{\otimes} \cdots \bar{\otimes} x_n + (x_n x_{n+1}) x_1 \bar{\otimes} x_2 \bar{\otimes} \cdots \bar{\otimes} x_{n-1}\]
\[+ x_n (x_{n+1} x_1) \bar{\otimes} x_2 \bar{\otimes} \cdots \bar{\otimes} x_{n-1}\]
\[= 0\]

3.2. Homology of the free anti-associative algebra on one generator. Let \(F_{AA}(1)\) be the free anti-associative algebra on one generator. This algebra admits a basis \(\{e_1, e_2, e_3\}\) with
\[e_1 e_1 = e_2, \quad e_1 e_2 = -e_2 e_1 = e_3\]
all the other non defined products are zero. We have
\begin{enumerate}
  \item \(\widetilde{T}^1(F_{AA}(1)) = F_{AA}(1)\)
  \item \(\widetilde{T}^2(F_{AA}(1)) = \mathbb{K}\{e_1 \bar{\otimes} e_2, e_2 \bar{\otimes} e_1, e_1 \bar{\otimes} e_3, e_3 \bar{\otimes} e_1, e_1 \bar{\otimes} e_1\}\)
    In fact \(e_2 \bar{\otimes} e_2 = e_1 e_1 \bar{\otimes} e_1 = 0, \quad e_2 \bar{\otimes} e_3 = e_1 e_3 \bar{\otimes} e_1 = 0\) and also \(e_3 \bar{\otimes} e_2 = e_3 \bar{\otimes} e_3 = 0\).
  \item \(\widetilde{T}^3(F_{AA}(1)) = \mathbb{K}\{e_1 \bar{\otimes} e_i, i = 1, 2, 3, e_1 \bar{\otimes} e_2 \bar{\otimes} e_1, e_1 \bar{\otimes} e_3 \bar{\otimes} e_1, e_2 \bar{\otimes} e_1 \bar{\otimes} e_1, e_3 \bar{\otimes} e_1 \bar{\otimes} e_1\}\)
  \item for \(n \geq 3\),
    \[\widetilde{T}^n(F_{AA}(1)) = \mathbb{K}\{e_{i_1} \bar{\otimes} \cdots \bar{\otimes} e_{i_n}, j \in [1, n], (i_1, \cdots, i_j, \cdots, i_n) = (1, 1, \cdots, 1), i_j \in [1, 3]\}\]
\end{enumerate}
To simplify the notations, we denote
\[
\begin{cases}
  e_1^n = e_1 \bar{\otimes} \cdots \bar{\otimes} e_1 (n \text{ factors}) \\
  e_i^{n,k} = e_1 \bar{\otimes} \cdots \bar{\otimes} e_i \bar{\otimes} e_i \bar{\otimes} \cdots \bar{\otimes} e_1, \; i = 2, 3, \; k = 1, \cdots, n, \; k \text{ is the place of } e_i
\end{cases}
\]
With this notation \(\widetilde{T}^n(F_{AA}(1))\) is generated by the vectors \(e_1^n, e_i^{n,k}\).

We deduce in particular
\begin{enumerate}
  \item \(\text{Im}(b_1) = \mathbb{K}\{e_2\}\),
  \item \(\text{Im}(b_2) = \mathbb{K}\{e_1 \bar{\otimes} e_2 + 2e_2 e_1 \bar{\otimes} e_1, e_1 \bar{\otimes} e_3 - e_3 \bar{\otimes} e_1\} = \mathbb{K}\{e_2^{2,2} + 2e_2^{2,1}, e_3^{2,2} - e_3^{2,1}\}\),
  \item \(\text{Ker}(b_1) = \mathbb{K}\{e_1 \bar{\otimes} e_2, e_2 \bar{\otimes} e_1, e_1 \bar{\otimes} e_3, e_3 \bar{\otimes} e_1\} = \mathbb{K}\{e_2^{2,2}, e_2^{2,1}, e_3^{2,2}, e_3^{2,1}\}\)
\end{enumerate}
and we have
\[\dim H_1(F_{AA}(1), F_{AA}(1)) = 2\]

More generally
\begin{enumerate}
  \item \(b_n(e_1^{n+1}) = 2e_2^{n,1} + \sum_{k=2}^{n} e_2^{n,k}\)
  \item \(b_n(e_3^{n+1,k}) = e_3^{n,k-1} - e_3^{n,k}, \; k \in [2, \cdots, n]\),
  \item \(b_n(e_2^{n+1,n+1}) = e_3^{n,n} - e_3^{n,1}\),
  \item \(b_n(e_2^{n+1,1}) = 0\),
  \item \(b_n(e_3^{n+1,k}) = 0, \; k \geq 1\).
\end{enumerate}
We deduce
\[ \dim \text{Im}(b_n) = n. \]

The vector space \( \text{Ker}(b_{n-1}) \) is generated by the vectors \( e_3^{n,k} \) for \( k = 1, \ldots, n \), \( e_2^{n,1} \) and \( \sum_{k=2}^{n} e_2^{n,k} \).

Then
\[ \dim \text{Ker}(b_{n-1}) = n + 2. \]

We deduce

**Lemma 12.**
\[ \dim H_n(\mathcal{F}_{AA}(1), \mathcal{F}_{AA}(1)) = 2. \]

A basis of this space is given by the classes of the vectors \( e_2^{n+1,1} \) and \( e_3^{n+1,1} \).

4. Commutative and anti-commutative anti-associative algebras

4.1. **Commutative case.** Let \( A \) be an anti-associative algebra. Assume that the product is commutative. Then
\[ a(bc) = -(ab)c = -c(ab) = (ca)b = b(ca) = -(bc)a = -a(bc) = -(cb)a = c(ba) \]

and
\[ a(bc) = 0. \]

We deduce that any commutative anti-associative algebra is 3-nilpotent. It is also a commutative associative algebra. If for any \( x \in A \) we have \( x^2 = 0 \), then \( xy = 0 \) for any \( x, y \in A \) and \( A \) is abelian. The classification of nilpotent associative algebras is established up to the dimension 5 \([6]\). We deduce from these lists the following examples:

1. Dimension 2:
   \[ e_1e_1 = e_2. \]

2. Dimension 3:
   \[ e_1e_2 = e_2e_1 = e_3. \]

3. Dimension 4
   - (a) \( e_1e_1 = e_2, \ e_1e_3 = e_3e_1 = e_4 \)
   - (b) \( e_1e_2 = e_2, \ e_3e_3 = e_4 \)
   - (c) \( e_1e_1 = e_4, \ e_2e_3 = e_3e_2 = e_4. \)

The free algebra with \( p \) generators \( X_1, \ldots, X_p \) admits a basis \( \{X_i, 1 \leq i \leq p, X_{ij}, 1 \leq i \leq j \leq p\} \) with the multiplication
\[ X_iX_j = X_jX_i = X_{ij}, \ X_iX_{jk} = 0, \ X_{jk}X_{rs} = 0. \]

Its dimension is equal to \( p(p+3)/2 \). For example, if \( p = 2 \), this algebra is of 5-dimensional and its structure is given by
\[ e_1e_1 = e_3, \ e_1e_2 = e_2e_1 = e_4, \ e_2e_2 = e_5 \]

and we have in particular \( \dim H_1 = 6. \)
4.2. Anti-commutative case. An anti-associative algebra is anti-commutative if for any \( a,b \in A \):
\[
ab = -ba.
\]
In particular \( a^2 = 0 \) for any \( a \in A \). The first non abelian example is in dimension 3 and given by
\[
e_1e_2 = -e_2e_1 = e_3.
\]
In this case, this algebra is also a Lie algebra isomorphic to the Heisenberg algebra.

**Proposition 13.** An anti-commutative anti-associative algebra is a Lie algebra if and only if this algebra is 2-step nilpotent.

In fact if \( A \) is an anti-commutative anti-associative algebra, then for all \( a,b,c \in A \) we have
\[
\begin{align*}
    a(bc) &= -(cb)a = (ca)b = -(ab)c = -(ca)b = (ac)b = -(ba)c.
\end{align*}
\]
We deduce
\[
    a(bc) + b(ca) + c(ab) = 3a(bc).
\]
Then \( a(bc) + b(ca) + c(ab) = 0 \) if and only if \( a(bc) = 0 \) for any \( a,b,c \in A \).

We deduce that any 2-step nilpotent Lie algebra is an anti-commutative anti-associative algebra. This gives a lot of examples of anti-commutative anti-associative algebras.

Concerning a general classification of the finite dimensional anti-commutative anti-associative algebras, the previous result show that this problem is not easy to solve because a general classification of the 2-step nilpotent Lie algebras is still not known. In very small dimensions, the computations are not very difficult (the problems arise as soon as the dimension is greater than 7). We obtain that \( A \) is abelian or isomorphic to
\[
\begin{align*}
    1. & \text{ The free anti-commutative algebra on one generator } X \text{ is one-dimensional } (X^2 = 0) \text{ and corresponds to the one-dimensional Lie algebra.} \\
    2. & \text{ The free algebra on two generators } X,Y \text{ is the vector space } \mathbb{K}\{X,Y,XY\} \text{ (all products of degree 3 are zero). It is isomorphic to the 3-dimensional Heisenberg algebra.} \\
    3. & \text{ The free algebra on } p \text{ generators } X_1, \ldots, X_p \text{ admits the basis } \{X_i, 1 \leq i \leq p, X_iX_j, 1 \leq i < j \leq p, X_iX_jX_k, 1 \leq i < j < k \leq p\}. \text{ If we denote by } e_i = X_i, e_{ij} = X_iX_j \text{ and } e_{i,j,k} = X_i(X_jX_k), \text{ the anti-associative multiplication is given by} \\
    & \quad \left\{ 
        \begin{align*}
            e_ie_j &= e_{ij}, & 1 \leq i < j \leq p, \\
            e_ie_j &= e_{ij}, & 1 \leq i < j < k \leq p, \\
            e_ie_j &= e_{ij} + e_{kl} = e_{ij}e_{kl} = e_{ijkl} = e_{ijkl}e_{lsr} = 0.
        \end{align*}
    \right.
\end{align*}
\]
If \( p \geq 3 \), this algebra is not a Lie algebra. For example, if \( p = 3 \), the dimension is 7, a basis is given by the family \( \{e_1, e_2, e_3, e_{12}, e_{13}, e_{23}, e_{123}\} \) and the multiplication:
\[
\begin{align*}
    e_1e_2 &= e_{12}, e_1e_3 = e_{13}, e_2e_3 = e_{23}, \\
    e_1e_{23} &= -e_2e_{13} = e_3e_{12} = e_{123}.
\end{align*}
\]
It is not 2-step nilpotent then it is not a Lie algebra.
5. Polarization and depolarization of anti-associative algebras

Let \((A,\mu)\) be a \(\mathbb{K}\)-algebra with multiplication \(\mu\). The polarization process determines two other multiplications \(\rho_\mu\) and \(\psi_\mu\) on \(A\) define by

\[
\rho_\mu(x,y) = \mu(x,y) + \mu(y,x), \quad \psi_\mu(x,y) = \mu(x,y) - \mu(y,x)
\]

for any \(x,y \in A\). The multiplication \(\rho_\mu\) is commutative and the second one, \(\psi_\mu\), is anti-commutative (i.e. skew-symmetric).

The depolarization process permits to find again \(\mu\) starting of a commutative multiplication \(\rho\) and of an anti-commutative multiplication by

\[
\mu(x,y) = \frac{1}{2}(\rho(x,y) + \psi(x,y)).
\]

For example, if \(\mu\) is associative, then \(\rho_\mu\) is a Jordan multiplication and \(\psi_\mu\) a Lie bracket. An example of depolarization process concerns the Poisson algebras which can be considered as nonassociative algebras by this process.

To simplify we will write

\[
x \cdot y = \rho_\mu(x,y) = xy + yx, \quad [x,y] = \psi_\mu(x,y) = xy - yx.
\]

Then the axiom of anti-associativity is equivalent to

\[(3) \quad (x \cdot y) \cdot z + x \cdot (y \cdot z) + x \cdot [y,z] + z \cdot [x,y] + [x,y \cdot z] - [z,x \cdot y] + [x,[y,z]] - [z,[x,y]] = 0.
\]

5.1. Polarization of an anti-associative algebra : the commutative part. Let \(A\) be an anti-associative algebra with multiplication \(xy\). If we denote by \(\mathcal{AA}_*\) the trilinear map

\[
\mathcal{AA}_*(x_1, x_2, x_3) = x_1 \cdot (x_2 \cdot x_3) + (x_1 \cdot x_2) \cdot x_3,
\]

then

\[(4) \quad \mathcal{AA}_*(x_1, x_2, x_3) = x_1(x_3x_2) - x_2(x_3x_1) - x_2(x_1x_3) + x_3(x_1x_2)
\]

and we easily obtain from Equation (4) the following equation :

\[
\mathcal{AA}_*(x_1, x_2, x_3) + \mathcal{AA}_*(x_2, x_3, x_1) + \mathcal{AA}_*(x_3, x_1, x_2)) = 0.
\]

But \(\cdot\) is commutative implying that

\[(5) \quad \mathcal{AA}_*(x_1, x_2, x_3) - \mathcal{AA}_*(x_3, x_2, x_1) = 0.
\]

Then considering \(\alpha_1, \alpha_2, \alpha_3, \alpha_4, \alpha_5, \alpha_6 \in \mathbb{K}\), the identity \(\sum_{\sigma_i \in \Sigma_3} \alpha_i \mathcal{AA}_*(x_{\sigma_i(1)}, x_{\sigma_i(2)}, x_{\sigma_i(3)}) = 0\), as \(\alpha_5 = \alpha_1 + \alpha_3 - \alpha_4\) and \(\alpha_6 = \alpha_1 - \alpha_2 + \alpha_3\), is equivalent to the system

\[
\begin{align*}
\alpha_1(\mathcal{AA}_*(x_1, x_2, x_3) + \mathcal{AA}_*(x_2, x_3, x_1) + \mathcal{AA}_*(x_3, x_1, x_2)) \\
+ \alpha_2(\mathcal{AA}_*(x_2, x_1, x_3) - \mathcal{AA}_*(x_3, x_1, x_2)) \\
+ \alpha_3(\mathcal{AA}_*(x_3, x_2, x_1) + \mathcal{AA}_*(x_2, x_3, x_1) + \mathcal{AA}_*(x_3, x_1, x_2)) \\
+ \alpha_4(\mathcal{AA}_*(x_1, x_3, x_2) - \mathcal{AA}_*(x_2, x_3, x_1)) = 0
\end{align*}
\]

which is equivalent to the system

\[
\begin{align*}
\mathcal{AA}_*(x_1, x_2, x_3) + \mathcal{AA}_*(x_2, x_3, x_1) + \mathcal{AA}_*(x_3, x_1, x_2) = 0, \\
\mathcal{AA}_*(x_1, x_2, x_3) - \mathcal{AA}_*(x_3, x_2, x_1) = 0.
\end{align*}
\]

which is clearly satisfied. Since \(\cdot\) is commutative, this system is equivalent to

\[(6) \quad x_1 \cdot (x_2 \cdot x_3) + x_2 \cdot (x_3 \cdot x_1) + x_3 \cdot (x_1 \cdot x_2) = 0.
\]
Proposition 14. Let \((A, \mu)\) be an anti-associative algebra. Then the commutative multiplication \(\bullet\) associated with \(\mu\) satisfies the Jacobi identity
\[
x_1 \bullet (x_2 \bullet x_3) + x_2 \bullet (x_3 \bullet x_1) + x_3 \bullet (x_1 \bullet x_2) = 0.
\]
Such algebras are called Jacobi-Jordan algebras \([2, 3]\).

Examples on Jacobi-Jordan algebras \([2]\):

1. Dimension 2: \(e_1 e_1 = e_2\)
2. Dimension 3: \(e_1 e_1 = e_2, e_3 e_3 = e_2\)
3. Dimension 4:
   - (a) \(e_1 e_1 = e_2, e_1 e_3 = e_4\)
   - (b) \(e_1 e_1 = e_2, e_3 e_4 = e_2\)

An algebraic study of these algebras can be found in \([2, 3]\). A Jacobi-Jordan algebra is a nilalgebra of nilindex 3. In fact for any \(x \in A\) we have \(x^3 = 0\). More precisely, in \([2]\), one proves that the class of Jacobi-Jordan algebras and the class of commutative nilalgebras of nilindex at most three coincides. This implies, from an Albert result, that

Proposition 15. Any finite dimensional Jordan-Jacobi algebra is nilpotent.

Definition 16. We will said that an algebra \((A, \mu)\) is Jacobi-Jordan admissible is the commutative algebra \((A, \bullet)\) define by \(x \bullet y = \mu(x, y) + \mu(y, x)\) is a Jordan-Jacobi algebra.

A necessary and sufficient condition for a multiplication \(\mu\) to be Jacobi-Jordan algebra is
\[
\sum_{\sigma \in \Sigma_3} AA_{\mu}(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0
\]
where \(\Sigma_3\) is the symmetric group of degree 3.

In particular, anti-associative algebras are Jacobi-Jordan admissible. A Jacobi-Jordan algebra which comes from an anti-associative algebras will be called a special Jacobi-Jordan algebra. In the following, we will give some examples of non special Jacobi-Jordan algebras.

Proposition 17. If a Jacobi-Jordan algebra is special, then its nilindex is smaller or equal to 4.

Example: the free Jacobi-Jordan algebra \(F_{JJ}(p)\). Let us consider the free Jacobi-Jordan algebra \(F_{JJ}(p)\) on \(p\) generators. For \(p = 1\), we have
\[
F_{JJ}(1) = \mathbb{K}\{X\} \oplus \mathbb{K}\{X^2\}
\]
and the multiplication corresponds to
\[
e_1 e_1 = e_2.
\]
This algebra is special because it is a commutative anti-associative algebra and any commutative anti-associative algebra is a special Jordan-Jacobi algebra.

For \(p = 2\), we have
\[
F_{JJ}(2) = \bigoplus_{k \geq 1} F_{JJ}(2)^{(k)}
\]
where \(F_{JJ}(2)^{(k)}\) is the vector space of terms of degree \(k\) and
\[
F_{JJ}(2)^{(k)} \cdot F_{JJ}(2)^{(k')} = F_{JJ}(2)^{(k+k')}.
\]
We have
We deduce that the free Jordan-Jacobi algebra with two generators is not special.

**Proposition 18.** The free Jordan-Jacobi algebra with two generators is not special.

Using the previous remark which precise that any term of partial degree greater or equal to 3 is zero, we deduce that \( \mathcal{F}_{JJ}(2)^{(5)} = \{0\} \) and more generally \( \mathcal{F}_{JJ}(p)^{(k)} = \{0\} \) as soon as \( k \geq 2p + 1 \).
5.2. Polarization of an anti-associative algebra: the anti-commutative part. Let $A$ be an anti-associative algebra with multiplication $xy$ and $\bullet$ and $[,]$ the associated commutative and anti-commutative multiplications. These two multiplications satisfy the relation (3) which implies

\begin{equation}
-y \bullet (z \bullet x) + x \bullet [y, z] + z \bullet [x, y] + [x, y \bullet z] - [z, x \bullet y] + [x, [y, z]] - [z, [x, y]] = 0.
\end{equation}

This equation is solved as soon as $(A, \bullet)$ is nilpotent with an nilindex equal to 3 and $[x, y] = 0$ for any $x, y \in A$. Then

**Proposition 19.** Any Jacobi-Jordan algebra with a nilindex equal to 3 is special.

We can also examine some consequences of Equation (3). Using the natural action of the symmetric group $\Sigma_3$ on the triple $(x, y, z)$, we obtain

**Proposition 20.** The commutative and anti-commutative multiplication associated with the anti-associative multiplication satisfy the identities

\begin{enumerate}
  \item $[x, y \bullet z] + [y, z \bullet x] + [z, x \bullet y] = 0,$
  \item $[x, [y, z]] - 2[y, [z, x]] + [z, [x, y]] = x \bullet (y \bullet z) - z \bullet (x \bullet y),$
  \item $x \bullet [y, z] + [y, z \bullet x] + [x, y \bullet z] = 0$
\end{enumerate}

for any $x, y, z \in A$.

**Remarks.**

1. The identity $x \bullet [y, z] = -[y, z \bullet x] - [x \bullet y, z]$ implies

\[ x \bullet [y, z] + y \bullet [z, x] + z \bullet [x, y] = 0. \]

Since $[x, y \bullet z] + [y, z \bullet x] + [z, x \bullet y] = 0$ and $x(yz) = x(y \bullet z) + [x, y \bullet z] + x(z, y) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]]$, we have

\[ x(yz) + y(zx) + z(xy) = [x, [y, z]] + [y, [z, x]] + [z, [x, y]] \]

and the algebra $(A, [,])$ is a Lie algebra if and only if the anti-associative algebra $A$ is a non-commutative Jacobi-Jordan algebra.

2. If we denote by $f_x$ the linear map $f_x(y) = x \bullet y$, then the second relation is written

\[ f_x[y, z] + [y, f_x(z)] + [f_x(y), z] = 0 \]

that is $f_x$ is an anti-derivation of the algebra $(A, [,])$. If $\overline{Der}(A, [,])$ denotes the subvector space of $\text{End}(A)$ whose elements are the anti-derivation of $(A, [,])$, then

\[ [\overline{Der}(A, [,]), \overline{Der}(A, [,])] \subset \overline{Der}(A, [,]). \]

If we call inner anti-derivation the anti-derivations of type $f_x$ for $x \in A$, then

\[ f_{x \bullet y} = -f_x \circ f_y - f_y \circ f_x \]

and the subspace of inner derivation is a subalgebra of the Jordan algebra $\text{End}(A)$.

**Proposition 21.** Let $A$ be a Jacobi-Jordan algebra and $f$ an anti-derivation of $A$. We can provide on the one dimensional extension $A' = A \oplus \mathbb{K}\{X\}$ of $A$ a Jordan-Jacobi structure $\mu$ given by

\[ \begin{cases} 
\mu(x, y) = xy, \ \forall x, y \in A \\
\mu(x, X) = f(x) \ \forall x \in A, \\
\mu(X, X) = 0.
\end{cases} \]

This proposition is a consequence of Corollary 4.6 of [1].
5.3. Deformation quantization of anti-Poisson algebras.

**Definition 22.** An anti-Poisson algebra is a triple \((A, \psi, \rho)\) where \(A\) is a \(K\)-vector space, \(\psi\) an anti-commutative (or skew-symmetric) multiplication on \(A\), \(\rho\) a Jacobi-Jordan multiplication on \(A\) such that these two bilinear maps satisfy the anti-Leibniz identity:

\[
L_{\rho}(\psi, \rho)(x, y, z) = \rho(\psi(x, y), z) + \psi(x, \rho(y, z)) + \psi(\rho(x, z), y) = 0.
\]

for any \(x, y, z \in A\).

Recall that an antiderivation of an algebra \((A, \mu)\) is a linear map \(f\) such that

\[
f(\mu(x, y)) + \mu(x, f(y)) + \mu(f(x), y) = 0
\]

for any \(x, y\) in \(A\). The anti-Leibniz identity can be interpreted saying that for any \(z \in A\), the linear map \(x \mapsto \rho(x, z)\) is an antiderivation of the algebra \((A, \psi)\).

A formal deformation of an anti-associative algebra \((A, \mu_0)\) is given by a family

\[
\varphi_i : A \otimes A \to A, \quad i \in \mathbb{N}
\]

satisfying \(\varphi_0 = \mu_0\) and

\[
(D_k) : \quad \sum_{i+j=k, i,j \geq 0} \varphi_i(\varphi_j(a, b), c) + \sum_{i+j=k, i,j \geq 0} \varphi_i(a, \varphi_j(b, c)) = 0
\]

for any \(a, b, c \in A\) and for each \(k \geq 1\). Now expand \(\mu(a, b)\), for \(a, b \in A\), into the power series

\[
\mu(a, b) = \mu_0(a, b) + t\varphi_1(a, b) + t^2\varphi_2(a, b) + \cdots
\]

for some \(k\)-bilinear functions \(\varphi_i : A \otimes A \to A\). Then \(\mu\) is anti-associative if and only if \((D_k)\) are satisfied for each \(k \geq 1\).

In we denote as before \(\mu_0(a, b) = ab\) the multiplication of the anti-associative algebra \(A\), then

\[
(D_1) : \quad \varphi_1(a, bc) + a(\varphi_1(b, c)) + \varphi_1(ab, c) + (\varphi_1(a, b))c = 0,
\]

\[
(D_2) : \quad \varphi_1(a, \varphi_1(b, c)) + \varphi_1(\varphi_1(a, b), c) + \varphi_2(a, bc) + a(\varphi_2(b, c)) + \varphi_2(ab, c) + (\varphi_2(a, b))c = 0.
\]

To make a link with the classical cohomological approach of deformation theories, we put

\[
Z^2_{\mathcal{A}A}(A, A) = \{ \varphi : A \otimes A \to A, \delta^2(\varphi)(a, b, c) = \varphi(a, bc) + a(\varphi(b, c)) + \varphi(ab, c) + (\varphi(a, b))c = 0 \}.
\]

Then \((D_1)\) is satisfied as soon as \(\varphi_1 \in Z^2_{\mathcal{A}A}(A, A)\). For example, if \(f\) is a linear endomorphism of \(A\), the bilinear map \(\varphi_f(a, b) = f(ab) - f(a)b - af(b)\) is in \(Z^2_{\mathcal{A}A}(A, A)\).

Assume now that \(\mu_0\) is anti-commutative : \(ba = -ab\) and consider the condition \(D_2\) that we write

\[
\varphi_1(x_1, \varphi_1(x_2, x_3)) + \varphi_1(\varphi_1(x_1, x_2), x_3)) + \delta^2(\varphi_2)(x_1, x_2, x_3) = 0.
\]

Let us determine the constants \(a_\sigma\) such that

\[
\sum_{\sigma \in \Sigma_3} a_\sigma \delta^2(\varphi_2)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0.
\]

If we develop this equation we obtain the non-trivial solution \(a_\sigma = 1\) for every \(\sigma \in \Sigma_3\). We deduce from \(D_2\) that

\[
\sum_{\sigma \in \Sigma_3} \varphi_1(x_{\sigma(1)}, \varphi_1(x_{\sigma(2)}, x_{\sigma(3)})) + \varphi_1(\varphi_1(x_{\sigma(1)}, x_{\sigma(2)}), x_{\sigma(3)}) = 0.
\]
and $\varphi_1$ is an anti-associative admissible multiplication. We deduce that the commutative multiplication $\rho_1$ associated with $\varphi_1$ is an anti-associative multiplication. Now, from $(D_1)$ we have
\[
\sum_{\sigma \in \Sigma_3} a_\sigma \delta^2(\varphi_1)(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)}) = 0
\]
for any $a_\sigma$. In particular, if
\[
a_{Id} = a_{\tau(12)} = a_c, \quad a_{\tau(13)} = a_{\tau(23)} = a_c^2
\]
where $\tau(ij)$ is the transposition between $i$ and $j$ and $c$ the cycle $(123)$, then we have
\[
\varphi_1(x_1, x_2 x_3) + x_2 \varphi_1(x_3, x_1) - x_3 \varphi_1(x_1, x_2) + x_2 \varphi_1(x_1, x_3) - x_3 \varphi_1(x_2, x_1) + \varphi_1(x_2 x_3, x_1) = 0
\]
that is
\[
\rho_1(x_1, x_2 x_3) + x_2 \rho_1(x_1, x_3) - x_3 \rho_1(x_1, x_2) = 0
\]
or since $\mu_0$ is anti-commutative,
\[
\rho_1(x_1, x_2 x_3) + x_2 \rho_1(x_1, x_3) + \rho_1(x_1, x_2) x_3 = 0.
\]
We shall say that the multiplication $\rho_1$ and $\mu_0$ are connected by the anti-Leibniz relation.

**Theorem 23.** Let $(A, \mu_0)$ be an anti-symmetric anti-associative algebra and $\mu_t = \mu_0 + \sum t^k \varphi_k$ an anti-associative formal deformation of $\mu_0$, that is $\mu_t$ is an anti-associative multiplication. Then $(A, \mu_0, \rho_1)$ is an anti-Poisson algebra, that is

1. $(A, \rho_1)$ is a Jordan-Jacobi algebra,
2. The products $\mu_t$ and $\rho_1$ are connected by the anti-Leibniz identity :
\[
\mathcal{L}_y(\mu_0, \rho_1)(x, y, z) = \rho_1(xy, z) + x \rho_1(y, z) + \rho_1(x, y)z = 0.
\]

We will say that $\mu_t$ is a deformation quantization of the anti-Poisson algebra $(A, \mu_0, \rho_1)$.

### 6. The operad $\mathcal{AAss}$

**6.1. The operad $\mathcal{AAss}$.** Recall that deformations of an associative algebra are controlled by the Hochschild cohomology $H^*(A, A)$. This means that $H^1(A, A)$ classifies infinitesimal deformations and $H^2(A, A)$ contains the obstructions for their extensions. In this case this cohomology coincides with the operadic cohomology because the corresponding quadratic operad $\mathcal{AAss}$ is a Koszul operad. In this section we recall some results given in [7] and we refer to this paper for all the proofs. Let $\mathcal{AAss} = \oplus_{n \geq 1} \mathcal{AAss}(n)$ the quadratic operad associated with anti-associative algebras. We have

1. $\mathcal{AAss}(1) = \mathbb{K}$
2. $\mathcal{AAss}(2) = \mathbb{K}\{x_1 x_2, x_2 x_1\}$
3. $\mathcal{AAss}(3) = \mathbb{K}\{x_1(x_2 x_3), x_2(x_1 x_3), x_2(x_2 x_1), x_1(x_3 x_2), x_2(x_3 x_1), x_3(x_1 x_2)\}$
4. $\mathcal{AAss}(n) = \{0\}$ for $n \geq 4$.

The Poincaré or generating series of $\mathcal{AAss}$ is the series
\[
g_{\mathcal{AAss}}(t) = \sum (-1)^n \frac{\dim(\mathcal{AAss}(n)) t^n}{n!} = -t + t^2 - t^3.
\]
Since $\mathcal{AAss}(1) = \mathbb{K}$, $\mathcal{AAss}$ admits a minimal model, unique up to isomorphism. The generating series $g_M$ associated with this minimal model is the formal inverse of $g_{\mathcal{AAss}}(t)$ taken with the opposite sign
\[
g_{\mathcal{AAss}}(-g_M(t)) = t.
We deduce
\[ g_M(t) = -t + t^2 - t^3 + 4t^5 - 14t^6 + 30t^7 - 33t^8 - 55t^9 + \cdots \]
and this series cannot be the Poincaré Series of the dual operad. In fact, if \( \mathcal{A} \) is the ideal of relations which defines \( \mathcal{A} \mathcal{A} \text{ss} \) and which is generated as a \( \Sigma_3 \)-module by the relation \( (x_1(x_2x_3) + (x_1x_2)x_3) \), the ideal \( \mathcal{R} \) which generates the dual operad \( \mathcal{A} \mathcal{A} \text{ss}^! \) is the orthogonal of \( \mathcal{R} \) for the classical inner product

\[
\begin{aligned}
&< (x_i \cdot x_j) \cdot x_k, (x_{i'} \cdot x_{j'}) \cdot x_{k'} > = 0, \text{ if } (i, j, k) \neq (i', j', k'), \\
&< (x_i \cdot x_j) \cdot x_k, (x_i \cdot x_j) \cdot x_k > = \varepsilon(\sigma), \quad \text{with } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}, \\
&< x_i \cdot (x_j \cdot x_k), x_{i'} \cdot (x_{j'} \cdot x_{k'}) > = 0, \text{ if } (i, j, k) \neq (i', j', k'), \\
&< x_i \cdot (x_j \cdot x_k), x_i \cdot (x_j \cdot x_k) > = -\varepsilon(\sigma) \quad \text{with } \sigma = \begin{pmatrix} 1 & 2 & 3 \\ i & j & k \end{pmatrix}, \\
&< (x_i \cdot x_j) \cdot x_k, x_{i'} \cdot (x_{j'} \cdot x_{k'}) > = 0,
\end{aligned}
\]

where \( \varepsilon(\sigma) \) is the signature of \( \sigma \). Then \( \mathcal{R}^! \) is also generated as a \( \Sigma_3 \)-module by the relation \( (x_1(x_2x_3) + (x_1x_2)x_3) \) and \( \mathcal{A} \mathcal{A} \text{ss} \) is selfdual. We deduce that the generating series of \( \mathcal{A} \mathcal{A} \text{ss}^! \) is \( -t + t^2 - t^3 \). We deduce

**Theorem 24.** [7]. *The quadratic operad \( \mathcal{A} \text{Ass} \) is not a Koszul operad.*

Concerning the problem of deformation of anti-associative algebras, the ‘standard’ cohomology of an anti-associative algebra \( A \) with coefficients in itself is described in [7] and compared to the relevant part of the deformation cohomology based on the minimal model of the anti-associative operad \( \mathcal{A} \text{Ass} \). Since \( \mathcal{A} \text{Ass} \) is not Koszul, these two cohomologies differ. The standard cohomology \( H^*_\text{st}(A, A) \) is the cohomology of the complex

\[
C^1(A, A) \xrightarrow{\delta^1_{\text{AA}}} C^2(A, A) \xrightarrow{\delta^2_{\text{AA}}} C^3(A, A) \xrightarrow{\delta^3_{\text{AA}}} 0 \xrightarrow{0} 0 \xrightarrow{0} \cdots
\]

in which \( C^p(A, A) := Hom(A \otimes^p, A) \) for \( p = 1, 2, 3 \), and all higher \( C^p \)'s are trivial. The two nontrivial pieces of the differential are basically the Hochschild differentials with “wrong” signs of some terms:

\[
\begin{aligned}
\delta^1_{\text{AA}}(f)(x, y) &:= xf(y) - f(xy) + f(x)y, \quad \text{and} \\
\delta^2_{\text{AA}}(\varphi)(x, y, z) &:= x\varphi(y, z) + \varphi(xy, z) + \varphi(x, yf) + \varphi(x, yz),
\end{aligned}
\]

for \( f \in Hom(V, V), \varphi \in Hom(A \otimes^2, V) \) and \( x, y, z \in V \). One sees, in particular, that

\[ H^*_\text{st}(A, A)^p = 0 \]

for \( p \geq 4 \).

### 6.2. The minimal model.
In case of anti-associative algebras, the deformation cohomology is described by the cohomology defined by the minimal model. This minimal model is an homology isomorphism \( (\Gamma(E), \partial) \to \mathcal{A} \mathcal{A} \text{ss} \) from the free operad \( \Gamma(E) \) equipped to a differential and \( \mathcal{A} \mathcal{A} \text{ss} \). The Poincaré series, studied above as the inverse of the Poincaré series of \( \mathcal{A} \mathcal{A} \text{ss} \), that is

\[ g_M(t) = -t + t^2 - t^3 + 4t^5 - 14t^6 + 30t^7 - 33t^8 - 55t^9 + \cdots \]
corresponds to the series associated with the minimal model. The deformation cohomology of anti-associative algebras, based on the study of a minimal model is also studied in [7]. We summarize the results: we consider the complex

\[ C^1_{AAss}(A, A) \xrightarrow{\delta_1} C^2_{AAss}(A, A) \xrightarrow{\delta_2} C^3_{AAss}(A, A) \xrightarrow{\delta_3} C^4_{AAss}(A, A) \xrightarrow{\delta_4} \cdots \]

- \( C^1_{AAss}(A, A) = \text{Hom}(A, A) \)
- \( C^2_{AAss}(A, A) = \text{Hom}(A^{\otimes 2}, \text{AAss}) \)
- \( C^3_{AAss}(A, A) = \text{Hom}(A^{\otimes 3}, \text{AAss}) \), and
- \( C^4_{AAss}(A, A) = \text{Hom}(A^{\otimes 4}, \text{AAss}) \) \( \oplus \) \( \text{Hom}(A^{\otimes 5}, \text{AAss}) \) \( \oplus \) \( \text{Hom}(A^{\otimes 5}, \text{AAss}) \) \( \oplus \) \( \text{Hom}(A^{\otimes 5}, \text{AAss}) \).

Observe that \( C^p_{AAss}(A, A) = C^p_{AAss} \) for \( p = 1, 2, 3 \), while \( C^4_{AAss}(A, A) \) consists of 5-linear maps. The differential \( \delta^p \) agrees with \( \delta^p_{AAss} \) for \( p = 1, 2 \) while, for \( g \in C^3_{AAss}(A, A) \), one has

\[ \delta^3(g) = (\delta^3_1(g), \delta^3_2(g), \delta^3_3(g), \delta^3_4(g)), \]

where

\[ \delta^3_1(g)(x, y, z, t, u) := xg(y, z, tu) - g(x, y, z(tu)) + (xy)g(z, t, u) - g(xy, zt, u) \]
\[ + g(xy, z, tu) - g((xy)z, t, u) + g(x, y, z(tu)) - g(x, yz, tu), \]
\[ \delta^3_2(g)(x, y, z, t, u) := g((xy)z, t, u) - g(xy, z, tu) + g(x, yz, tu) - g(x, y(zt), u) \]
\[ + xyg(z, t, u) - g(xy, zt, u) + g(x, yz, tu) - g(xy, z, tu), \]
\[ \delta^3_3(g)(x, y, z, t, u) := g(x, yz, tu) - xg(yz, t, u) + g(x, y(zt), u) - xg(y, z, tu) \]
\[ + g(x, y, zt)u - g(xy, z, tu) + (g(x, y, zt)u - g(xy, z, tu) - g(xy, z, tu), \text{ and} \]
\[ \delta^3_4(g)(x, y, z, t, u) := g(xy, zt, u) - g(x, y, z(tu)) + xg(y, zt, u) - g(xy, z, tu) \]
\[ + (xg(y, z, tu) - g(xy, zt, u) + g(x, y, zt)u - g(xy, z, tu) - g(xy, z, tu), \]

for \( x, y, z, t, u \in V \).

**Case of Jacobi-Jordan algebra.** Let \( J_{\text{aJo}} = \oplus_{n \geq 2} J_{\text{aJo}}(n) \) be the quadratic operad associated with the Jacobi-Jordan multiplication. It is clear that

(1) \( \dim J_{\text{aJo}}(2) = 1 \) from the commutativity of the multiplication,

(2) \( \dim J_{\text{aJo}}(3) = 2 \) and more precisely \( J_{\text{aJo}}(3) = \mathbb{K}\{x_1(x_2x_3), x_2(x_3x_1)\} \).

The vector space \( J_{\text{aJo}}(4) \) is generated by the products \( (x_i(x_j(x_kx_l))) \) and \( (x_i(x_j)(x_kx_l)) \). From the Jacobi-Jordan axiom, we have

\[ (x_i(x_j)(x_kx_l)) = -(x_i(x_j)(x_kx_l)) \]
\[ = -(x_k(x_i(x_j))) - x_i((x_i(x_j)x_k)) \]

then \( (x_i(x_j(x_kx_l))) \) generates \( J_{\text{aJo}}(4) \). Moreover, the previous identity implies

\[ x_1(x_2(x_3x_4)) + x_2(x_1(x_3x_4)) - x_3(x_4(x_1x_2)) - x_4(x_3(x_1x_2)) = 0, \]
\[ x_3(x_2(x_1x_4)) + x_2(x_3(x_1x_4)) - x_1(x_4(x_3x_2)) - x_4(x_1(x_3x_2)) = 0, \]
\[ x_4(x_2(x_3x_1)) + x_2(x_4(x_3x_1)) - x_3(x_1(x_4x_2)) - x_1(x_3(x_4x_2)) = 0. \]

This shows that \( (x_i(x_j(x_kx_l))) \) generates \( J_{\text{aJo}}(4) \) for \( i = 1, 2, 3 \). Since

\[ (x_i(x_k(x_ix_j))) = -(x_i(x_j(x_kx_i))) - (x_i(x_j(x_kx_l))) \]
we deduce, by adding the 3 identities, that
\[
x_1(x_2(x_3x_4)) + x_1(x_3(x_2x_4)) + x_2(x_1(x_3x_4)) + x_2(x_3(x_1x_4)) \\
+ x_3(x_1(x_2x_4)) + x_3(x_2(x_1x_4)) = 0,
\]
and it is the only relation on this 6 terms then \(\mathcal{J}_a\mathcal{J}o(4)\) is the space
\[
\mathbb{K}\{(x_1(x_2(x_3x_4)), (x_1(x_3(x_2x_4)), (x_2(x_1(x_3x_4)), (x_2(x_3(x_1x_4)), (x_3(x_1(x_2x_4))\}
\]
and
\[
\dim \mathcal{J}_a\mathcal{J}o(4) = 5.
\]
Since the product is commutative, \(\mathcal{J}_a\mathcal{J}o(5)\) is generated by the products represented by
\[
\bullet(\bullet(\bullet(\bullet))) , \bullet(\bullet(\bullet)), \bullet(\bullet), \bullet) .
\]
The Jordan-Jacobi condition gives (we use now the letters \(a, b, c, \cdots\) in place of \(x_1, x_2, x_3, \cdots\) to shorten the formulae) :
\[
(ab)((cd)e) + a(b((cd)e) + b(a((cd)e)) = 0 \\
&a((bc)(de)) + (bc)((de)a) + (de)(a(bc)) = 0
\]
this shows that \(\mathcal{J}_a\mathcal{J}o(5)\) is generated by the products \(\bullet(\bullet(\bullet)))\). Since \(\dim \mathcal{J}_a\mathcal{J}o(4) = 5\),
we have 25 generators of this type. We have
\[
(ab)((cd)e) + a(b((cd)e) + b(a((cd)e)) = 0
\]
and also
\[
(ab)((cd)e) = -(cd)(e(ab)) - e((ab)(cd)) \\
= c(d(e(ab))) + d(c(e(ab))) + e(a(b(cd))) + e(b(a(cd)))
\]
that implies
\[
a(b((cd)e) + b(a((cd)e)) + c(d(e(ab))) + d(c(e(ab))) + e(a(b(cd))) + e(b(a(cd))) = 0
\]
and by symmetry
\[
a(c((bd)e) + c(a((bd)e)) + d(e(ac))) + d(b(e(ac))) + e(a(c(bd))) + e(c(a(bd))) = 0.
\]
Since the other relations are consequences of the previous relations, we deduce that
\[
\dim \mathcal{J}_a\mathcal{J}o(5) = 23.
\]
At this step, the generating series is
\[
g(t) = -t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{4!} + \frac{23t^5}{5!} + o(t^6)
\]
Computing the ideal of relations of the dual operad \(\mathcal{J}_a\mathcal{J}o^!\), we see that the associated multiplication is anti-commutative and satisfies
\[
x_1(x_2x_3) - x_3(x_1x_2) = 0
\]
The anti-commutativity implies that
\[
x_1(x_2x_3) + (x_1x_2)x_3 = 0
\]
and a \(\mathcal{J}_a\mathcal{J}o\)-algebra is an anti-associative anti-commutative algebra. Then
\[
\mathcal{J}_a\mathcal{J}o^!(4) = \{0\}.
\]
Its generating series is
\[
g_{\mathcal{J}_a\mathcal{J}o^!} = -t + \frac{t^2}{2} - \frac{t^3}{6}
\]
and the inverse series
\[-t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{24} - \frac{t^5}{12} - \frac{7t^6}{144} + \frac{13t^7}{72} + O(t^8).\]
Such series cannot be the generating series of a quadratic operad because the sign of $t^6$ is negative. Consequently, $JaJo$ is not a Koszul operad and

**Theorem 25.** The quadratic operad $JaJo$ corresponding to the Jacobi-Jordan algebra is not a Koszul operad.

We deduce, as in the previous case, that the deformation cohomology is described by the cohomology defined by the minimal model. Since the Poincaré series of $JaJo$ is $g(t) = -t + \frac{t^2}{2} - \frac{t^3}{3} + \frac{5t^4}{4!} + o(t^5)$, the inverse is $\tilde{g}(t) = -t + \frac{t^2}{2} - \frac{t^3}{6} + o(t^5)$. We consider the complex

\[
C^1_{JaJo}(A, A) \xrightarrow{\delta^1} C^2_{JaJo}(A, A) \xrightarrow{\delta^2} C^3_{JaJo}(A, A) \xrightarrow{\delta^3} \cdots
\]

with
- $C^i_{JaJo}(A, A) = Hom(A, A)$
- $C^2_{JaJo}(A, A) = Sym(A^{\otimes 2}, A)$
- $C^3_{JaJo}(A, A) = Sym(A^{\otimes 3}, A)$, and
- $C^4_{JaJo}(A, A) = Sym(A^{\otimes 5}, A) \oplus Sym(A^{\otimes 5}, A) \oplus Sym(A^{\otimes 5}, A)$

where $Sym(A^{\otimes n}, A)$ denotes the space of symmetric $n$-linear maps, that is invariant by the action of the symmetric group $\Sigma_n$. The differentials $\delta^i$ of this cohomological complex coincide with the standard cohomology in degree 1 and 2:

$\delta^1(f)(x_1, x_2) = f(x_1x_2) - x_1f(x_2) - f(x_1)x_2$

and

$\delta^2\varphi(x_1, x_2, x_3) = x_1\varphi(x_2, x_3) + \varphi(x_1, x_2x_3) + x_2\varphi(x_3, x_1) + \varphi(x_2, x_3x_1) + x_3\varphi(x_1, x_2) + \varphi(x_3, x_1x_2)$

where $\varphi$ is a symmetric bilinear form, that is invariant by $\Sigma_2$. We have

$\delta^2\varphi(x_1, x_2, x_3) = \sum_{\sigma \in \Sigma_3} \delta^2_{AA}\varphi(x_{\sigma(1)}, x_{\sigma(2)}, x_{\sigma(3)})$

where $\delta^2_{AA}$ is the differential associated with the cohomological complex of the anti-associative algebras. Then, in degree 4, we will have

$\delta^3(\psi) = \sum_{\sigma \in \Sigma_4} \delta^3_{1,AA}(\psi), \delta^3_{2,AA}(\psi), \delta^3_{3,AA}(\psi), \delta^3_{4,AA}(\psi)$,

where $\psi$ is invariant by $\Sigma_4$ and $\delta^3_{1,AA}$ corresponding to the anti-associative case.

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