UNIFICATION OF EXTREMAL LENGTH GEOMETRY ON TEICHMÜLLER SPACE VIA INTERSECTION NUMBER

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Abstract. In this paper, we give a framework for the study of the extremal length geometry of Teichmüller space after S. Kerckhoff, F. Gardiner and H. Masur. There is a natural compactification using extremal length geometry introduced by Gardiner and Masur. The compactification is realized in a certain projective space. We develop the extremal length geometry in the cone which is defined as the inverse image of the compactification via the quotient mapping. The compactification is identified with a subset of the cone by taking an appropriate lift. The cone contains canonically the space of measured foliations in the boundary.

We first extend the geometric intersection number on the space of measured foliations to the cone, and observe that the restriction of the intersection number to Teichmüller space is represented explicitly by the formula in terms of the Gromov product with respect to the Teichmüller distance. From this observation, we deduce that the Gromov product extends continuously to the compactification.

As an application, we obtain an alternative approach to Earle-Ivanov-Kra-Markovic-Royden’s characterization of isometries. Namely, with some few exceptions, the isometry group of Teichmüller space with respect to the Teichmüller distance is canonically isomorphic to the extended mapping class group. We also obtain a new realization of Teichmüller space, a hyperboloid model of Teichmüller space with respect to the Teichmüller distance.

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1. Introduction

The Teichmüller distance is a canonical and important distance on Teichmüller space. The geometry of the Teichmüller distance is deeply related to the extremal length geometry on that space (cf. [23]). To the author’s knowledge, in [15], S. Kerckhoff first studied the boundary of Teichmüller space at infinity via extremal length. The extremal length geometry on Teichmüller space was formulated precisely by F. Gardiner and H. Masur in [9]. (cf. §1.2).

1.1. Motivation. Unification of Teichmüller geometry in terms of intersection number. To define the Thurston compactification of Teichmüller space, we first recognize each point of Teichmüller space as a function on the set of simple closed curves by assigning the hyperbolic lengths of geodesic representatives, and then, we take the closure of the set of projective classes of such functions in the projective space (cf. [4]). In a broad sense, completions due to Thurston carry out with recognizing each point of Teichmüller space as a function on the set of simple closed curves (see also [3]). Hence, the Gardiner-Masur compactification is considered as an object in the category “Thurston’s completion” (cf. §3.1). Thus, it is expected that every boundary point of the Gardiner-Masur compactification is recognized as the projective class of a function defined by (a kind of) intersection number.

In [2], F. Bonahon realized the Thurston compactification in the space of geodesic currents. Indeed, in his method, any point of Teichmüller space is associated to an equivariant Radon measure on the space of hyperbolic geodesics on the universal cover of the base surface of Teichmüller space. He extended the intersection
number function to the space of geodesic currents, and gave a unified treatment for the Thurston compactification in terms of the intersection number. His theory is broadly applied in many fields in mathematics and yields enormous rich results (cf. e.g. [1] and [5]). Thus, it is natural to ask:

**Question 1.** Can we develop extremal length geometry in terms of intersection number?

**Relation to the geometry of the Teichmüller distance.** As discussed in the previous section, the space \( \mathbb{R}^+_S \) of non-negative functions on the set \( S \) of simple closed curves is the ambient space of Thurston’s completions. The interior \((0, \infty)^S\) of \( \mathbb{R}^+_S \) admits a pseudo-distance

\[
d_\infty(f, g) = \log \sup_{\alpha \in S} \left\{ \frac{f(\alpha)}{g(\alpha)}, \frac{g(\alpha)}{f(\alpha)} \right\}
\]

which is perceived as the product distance of countably many 1-dimensional hyperbolic spaces. Possibly \( d_\infty(f, g) = \infty \) for some \( f, g \in \mathbb{R}^+_S \) and the topology from (1.1) is different from the product topology on \( \mathbb{R}^+_S \). From Kerckhoff’s formula (2.8), a natural lift given in (1.2) of the Gardiner-Masur embedding gives an isometric embedding from Teichmüller space to the ambient space \((0, \infty)^S, d_\infty\). One may ask:

**Question 2.** How is the geometry of Teichmüller distance related to the geometry of the Gardiner-Masur compactification (embedding)?

### 1.2. Results

In this paper, we attempt to unify the extremal length geometry via intersection number, aiming for a counterpart for Bonahon’s theory on geodesic currents.

We fix the notation to give our results precisely. Henceforth, we fix a Riemann surface \( X = X_{g,m} \) of genus \( g \) with \( m \) punctures such that \( 2g - 2 + m > 0 \). Denote by \( T_{g,m} \) the Teichmüller space of \( X \). When the argument depends on the basepoint, we consider the Teichmüller space \( T_{g,m} \) as a pointed space \((T_{g,m}, x_0)\), where \( x_0 = (X, id) \).

Let \( S \) be the set of non-peripheral and non-trivial simple closed curves on \( X \), and \( \mathcal{MF} \) the space of measured foliations. The space \( \mathcal{MF} \) is contained in \( \mathbb{R}^+_S \) (cf. §2.2).

We refer readers to §3 for details on the Gardiner-Masur closure. We consider the cone \( C_{GM} \) which is defined as the inverse image of the Gardiner-Masur closure \( c_{GM}(T_{g,m}) \) via the projection \( \mathbb{R}^+_S - \{0\} \to \mathbb{P}\mathbb{R}^+_S \) (cf. §4.1). It is known that the space \( \mathcal{PMF} \) of projective measured foliations is contained in the Gardiner-Masur boundary \( \partial_{GM} T_{g,m} \) and hence \( \mathcal{MF} \subset C_{GM} \) (cf. §9). One of our aims in this paper is to define the intersection number function on \( C_{GM} \). In order to avoid any confusion, we denote by \( I(\cdot, \cdot) \) the original geometric intersection number function on \( \mathcal{MF} \).

### 1.2.1. Unification by intersection number

The Gardiner-Masur embedding (3.1) admits a natural lift

\[
\tilde{\Phi}_{GM} : T_{g,m} \ni y \mapsto [S \ni \alpha \mapsto \text{Ext}_y(\alpha)^{1/2}] \in C_{GM} \subset \mathbb{R}^+_S.
\]

Our unification is stated as follows.
Theorem 1 (Unification). There is a unique continuous function
\[ i(\cdot, \cdot) : C_{GM} \times C_{GM} \to \mathbb{R} \]
with the following properties.

(i) For any \( y \in T_{g,m} \), the projective class of the function \( S \ni \alpha \mapsto i(\tilde{\Phi}_{GM}(y), \alpha) \)
is exactly the image of \( y \) under the Gardiner-Masur embedding. Actually, it holds
\[ i(\tilde{\Phi}_{GM}(y), \alpha) = \text{Ext}_g(\alpha)^{1/2} \]
for all \( \alpha \in S \).
(ii) For \( a, b \in C_{GM} \), \( i(a, b) = i(b, a) \).
(iii) For \( a, b \in C_{GM} \) and \( t, s \geq 0 \), \( i(ta, sb) = ts i(a, b) \).
(iv) For any \( y, z \in T_{g,m} \),
\[ i(\tilde{\Phi}_{GM}(y), \Phi_{GM}(z)) = \exp(d_T(y, z)). \]
In particular, we have \( i(\tilde{\Phi}_{GM}(y), \Phi_{GM}(y)) = 1 \) for \( y \in T_{g,m} \).
(v) For \( F, G \in MF \subset C_{GM} \), the value \( i(F, G) \) is equal to the geometric intersection number \( I(F, G) \).

For a technical reason, instead of proving Theorem 1, we will show Theorem 4, which is the basepoint dependent version of Theorem 1 (cf. §8). Actually, we will consider another lift
(1.3) \( \Psi_{x_0} : T_{g,m} \ni y \mapsto \left[ S \ni \alpha \mapsto \exp(-d_T(x_0, y)) \cdot \text{Ext}_y(\alpha)^{1/2} \right] \in C_{GM} \)
of the Gardiner-Masur embedding in Theorem 4 in place of \( \tilde{\Phi}_{GM} \). Namely,
(1.4) \( \Psi_{x_0}(y) = \exp(-d_T(x_0, y)) \cdot \tilde{\Phi}_{GM}(y) \)
for all \( y \in T_{g,m} \). One of advantages to use the embedding \( \Psi_{x_0} \) is that \( \Psi_{x_0} \) admits a continuous extension to \( \text{cl}_{GM}(T_{g,m}) \), whereas \( \tilde{\Phi}_{GM} \) diverges at infinity (cf. Proposition 3.1 and (1.3)).

1.2.2. Hyperboloid model of Teichmüller space. We represent the situations of our theorems schematically in Figure 1. For any \( y \in T_{g,m} \), \( \tilde{\Phi}_{GM}(y) \) and \( \Psi_{x_0}(y) \) are projectively equivalent in \( \mathbb{R}^S_+ \). From (iv) in Theorem 1 the image under \( \tilde{\Phi}_{GM} \) coincides with the “hyperboloid”
(1.5) \( \{ a \in C_{GM} \mid i(a, a) = 1 \} \),
and the boundary of the cone \( C_{GM} \) is represented as the “light cone”
(1.6) \( \{ a \in C_{GM} \mid i(a, a) = 0 \} \)
from (iii) and (iv) in Theorem 1 and the continuity of the intersection number on \( C_{GM} \). The image of \( \Psi_{x_0} \) looks like a section in the cone. These images contact only at the images of the basepoint. In the hyperboloid model, the Teichmüller distance is represented as
\[ d_T(y, z) = \log i(\Phi_{GM}(y), \Phi_{GM}(z)). \]

This hyperboloid model might be a comparable object with Bonahon’s realization of the Thurston compactification of Teichmüller space in the space of geodesic currents (cf. §4 in [2]).
1.2.3. **Extension of the Gromov product.** The following corollary confirms that the Gardiner-Masur boundary is a kind of a canonical boundary for the geometry of the Teichmüller distance.

**Corollary 1** (Extension of the Gromov product for $d_T$). For any $x_0 \in \mathcal{T}_{g,m}$, there is a unique continuous function

$$\langle \cdot | \cdot \rangle_{x_0} : \text{cl}_{GM}(\mathcal{T}_{g,m}) \times \text{cl}_{GM}(\mathcal{T}_{g,m}) \to [0, +\infty]$$

such that

1. for $y, z \in \mathcal{T}_{g,m}$,

$$\langle y | z \rangle_{x_0} = \frac{1}{2}(d_T(x_0, y) + d_T(x_0, z) - d_T(y, z)),$$

2. for $[F], [G] \in \mathcal{PMF} \subset \partial_{GM} \mathcal{T}_{g,m}$,

$$\exp(-2\langle [F] | [G] \rangle_{x_0}) = \frac{I(F, G)}{\text{Ext}_{x_0}(F)^{1/2} \cdot \text{Ext}_{x_0}(G)^{1/2}}.$$  

The conclusion in Corollary 1 is somewhat surprising because Teichmüller space with the Teichmüller distance is believed to be a metric space with less “good natures” for geodesic triangles. For instance, Teichmüller space is neither a metric space with Busemann negative curvature nor a Gromov hyperbolic space (cf. [21], [24] and [26]). Recently, C. Walsh informed that there is a geodesic metric space with the property that the Gromov product does not extend to the horofunction boundary (cf. [11]).

1.2.4. **Rigidity theorem for mappings of bounded distortion for triangles.** Our unified treatment of extremal length geometry in terms of intersection number links the geometry of the Teichmüller distance (an analytical aspect in Teichmüller theory) with the geometry on $\mathcal{MF}$ via intersection number (a topological aspect in Teichmüller theory).

We will deal with a mapping of bounded distortion for triangles which is defined as a mapping $\omega : \mathcal{T}_{g,m} \to \mathcal{T}_{g,m}$ satisfying

$$\frac{1}{D_1}(x | y)_{x_0} - D_2 \leq \langle \omega(x) | \omega(y) \rangle_{\omega(z)} \leq D_1(x | y)_{x_0} + D_2.$$
for all \( x, y, z \in T_{g,m} \) and some constants \( D_1, D_2 > 0 \) independent of the choice of points of \( T_{g,m} \). A mapping \( \omega': T_{g,m} \to T_{g,m} \) is said to be a quasi-inverse of a mapping \( \omega: T_{g,m} \to T_{g,m} \) if there is a constant \( D_3 > 0 \) such that
\[
\sup_{x \in T_{g,m}} \{ d_T(x, \omega \circ \omega'(x)), d_T(x, \omega' \circ \omega(x)) \} \leq D_3.
\]

One can easily check that any quasi-inverse \( \omega' \) of \( \omega \) is also a mapping of bounded distortion for triangles. In \([9.3]\) we prove the following.

**Theorem 2** (Asymptotic Rigidity). Suppose that the complex dimension of \( T_{g,m} \) is at least two. Let \( \omega: T_{g,m} \to T_{g,m} \) be a mapping of bounded distortion for triangles. Assume the following two conditions:

(a) The map \( \omega \) admits a continuous extension to \( \partial GM T_{g,m} \).

(b) The map \( \omega \) has a quasi-inverse \( \omega' \) which admits a continuous extension to \( \partial GM T_{g,m} \).

Then, the following hold:

(1) The map \( \omega \) acts homeomorphically on \( \mathcal{PMF} \subset \partial GM T_{g,m} \) and \( \omega' = \omega^{-1} \) on \( \mathcal{PMF} \).

(2) The restriction of \( \omega \) to \( \mathcal{PMF} \) preserves \( S \) and induces a simplicial automorphism of the complex of curves.

By definition, a quasi-invertible mapping of bounded distortion for triangles is a quasi-isometry. However, the author does not know whether Theorem 2 holds for quasi-isometries on \( T_{g,m} \). We remark that (1) in Theorem 2 holds when the complex dimension of \( T_{g,m} \) is equal to one. In this case, \( (T_{g,m}, d_T) \) is isometric to the hyperbolic plane, and both the Gardiner-Masur boundary and \( \mathcal{PMF} \) coincide with the boundary at infinity of the hyperbolic plane (cf. e.g. [27]). Hence any quasi-isometry on \( (T_{g,m}, d_T) \) induces a homeomorphism of \( \mathcal{PMF} \). However, the assertion (2) does not hold because the isometry group of \( (T_{g,m}, d_T) \) acts transitively in this case.

**1.2.5. Isometries on \( T_{g,m} \).** Theorem 2 allows us to give an alternative approach to Earle-Ivanov-Kra-Markovic-Royden’s characterization of the isometry group of \( (T_{g,m}, d_T) \) via the Gardiner-Masur compactification. Namely, we show the following in \([9.4]\).

**Corollary 2** (Royden [34], Earle-Kra [7], Ivanov [14], and Earle-Markovic [8]). Suppose that \( 3g - 3 + m \geq 2 \) and \((g,m)\) is neither \((1,2)\) nor \((2,0)\). Then, the isometry group of \( (T_{g,m}, d_T) \) is canonically isomorphic to the extended mapping class group.

Actually, our proof of Corollary 2 is somewhat modelled on Ivanov’s proof. We outline the idea of his proof. The essential part is to show that an isometric action on \( (T_{g,m}, d_T) \) induces an automorphism of the complex of curves. After then, from a theorem by Ivanov, Korkmaz and Luo, we see that such an automorphism of the complex of curves is induced by an element of the extended mapping class group (cf. [13], [16] and [19]). Finally, it is checked that the action of the given isometry coincides with the action of the element of the extended mapping class group.

As noted before, our proof of Corollary 2 also follows the same line. However, our proof of the essential part above follows from Theorem 2 which holds for mappings of bounded distortion for triangles. To show the essential part above, Ivanov induces
a self-homeomorphism of $\mathcal{P}\mathcal{M}\mathcal{F}$. To do this, he identifies $\mathcal{P}\mathcal{M}\mathcal{F}$ with the unit sphere in the tangent space, and defines the self-homeomorphism by passing the “exponential maps” (cf. the discussion after the proof of Lemma 5.2 in [14]).

1.3. Plan of this paper. This paper is organized as as follows. In §§2 and 3 we recall basic notions in Teichmüller theory and known results for the Gardiner-Masur compactification.

In §4 we define the cones which are essential objects in this paper. We also define the (topological) models of cones, and canonical identifications between cones and their models. We use such models when we develop an argument which depends on the choice of the basepoint of $T_{g,m}$.

From §5 to §8 we devote to constructing the intersection number on the cone $C_{G_M}$. In §5 we define the extremal length $\text{Ext}^x_0(\cdot)$ and the intersection number $i^x_0(\cdot, \cdot)$ associated to the basepoint $x_0$ on a part of each model. The definition of this “new” extremal length is motivated by the following formula

\[(1.7) \quad \text{Ext}^x_0(G) = \sup_{F \in \mathcal{MF} \setminus \{0\}} \frac{I(G, F)^2}{\text{Ext}_y(F)}\]

for $G \in \mathcal{MF}$ (cf. (2.6)). We first define the intersection number between elements of $C_{GM}$ and measured foliations (§5.1), and the extremal length for elements of $C_{GM}$ (§5.2). In §6 we discuss the topology of models of cones. In §§7 and 8 the intersection number on the cone is defined by extending the functions defined in earlier sections. We prove Corollary 1 in §8.2. In §9 we show Theorem 2 and give an alternative approach to Earle-Kra-Ivanov-Markovic-Royden’s characterization in Corollary 2.

2. Teichmüller theory

2.1. Teichmüller space. The Teichmüller space $T_{g,m}$ of Riemann surfaces of analytically finite type $(g,m)$ is the set of equivalence classes of marked Riemann surfaces $(Y, f)$ where $Y$ is a Riemann surface and $f : X \to Y$ a quasiconformal mapping. Two marked Riemann surfaces $(Y_1, f_1)$ and $(Y_2, f_2)$ are said to be Teichmüller equivalent if there is a conformal mapping $h : Y_1 \to Y_2$ which is homotopic to $f_2 \circ f_1^{-1}$.

Teichmüller space $T_{g,m}$ has a canonical complete distance, called the Teichmüller distance $d_T$, which is defined by

\[(2.1) \quad d_T(y_1, y_2) = \frac{1}{2} \log \inf \{K(h) \mid h \text{ is q.c. homotopic to } f_2 \circ f_1^{-1}\}\]

for $y_i = (Y_i, f_i) \in T_{g,m} (i = 1, 2)$, where $K(h)$ is the maximal dilatation of $h$ (e.g. [12, §4.1.1]).

2.2. Measured foliations. Denote by $\mathbb{R}_+ \otimes S$ the set of formal products $t\alpha$ where $t \geq 0$ and $\alpha \in S$. The set $\mathbb{R}_+ \otimes S$ is embedded into $\mathbb{R}_+^S$ by

\[(2.2) \quad \mathbb{R}_+ \otimes S \ni t\alpha \mapsto [S \ni \beta \mapsto t I(\alpha, \beta)] \in \mathbb{R}_+^S.\]

We topologize $\mathbb{R}_+^S$ with the pointwise convergence (i.e. the product topology). The space $\mathcal{MF}$ of measured foliations on $X$ is the closure of the image of the mapping (2.2). The intersection number of any two weighted curves in $\mathbb{R}_+ \otimes S$ is defined by $I(t\alpha, s\beta) = ts I(\alpha, \beta)$. It is known that the intersection number function extends continuously on $\mathcal{MF} \times \mathcal{MF}$ (cf. [32]).
The positive numbers $\mathbb{R}_{>0}$ acts on $\mathbb{R}_+^S$ by multiplication. Let

\begin{equation}
\text{proj}: \mathbb{R}_+^S - \{0\} \to \mathbb{P}\mathbb{R}_+^S = (\mathbb{R}_+^S - \{0\})/\mathbb{R}_{>0}
\end{equation}

be the quotient mapping. The space $\mathcal{PMF}$ of projective measured foliations is defined to be the quotient

$$\mathcal{PMF} = \text{proj}(\mathcal{MF} - \{0\}) = (\mathcal{MF} - \{0\})/\mathbb{R}_{>0}.$$ 

It is known that $\mathcal{MF}$ and $\mathcal{PMF}$ are homeomorphic to $\mathbb{R}^{6g-6+2n}$ and $S^{6g-7+2n}$ respectively (cf. [4]).

\section{Extremal length.}

For $y = (Y, f) \in T_{g,m}$ and $\alpha \in S$, the extremal length of $\alpha$ on $y$ is defined by

\begin{equation}
\text{Ext}_{y}(\alpha) = 1/\sup_A \{\text{Mod}(A) \mid A \subset Y \text{ and the core is homotopic to } f(\alpha)\},
\end{equation}

where $\text{Mod}(A)$ is the modulus of an annulus $A$, which is equal to $(\log r)/2\pi$ if $A$ is conformally equivalent to a round annulus $\{1 < |z| < r\}$. For $t\alpha \in \mathbb{R}_+ \otimes S$, we set

$$\text{Ext}_{y}(t\alpha) = t^2 \text{Ext}_{y}(\alpha).$$

In [15], Kerckhoff showed that the extremal length function extends continuously on $\mathcal{MF}$. Let

\begin{equation}
\mathcal{MF}_1 = \{F \in \mathcal{MF} \mid \text{Ext}_y(F) = 1\}.
\end{equation}

The extremal length of measured foliations satisfies the following inequality, which is called Minsky’s inequality:

\begin{equation}
I(F, G)^2 \leq \text{Ext}_y(F) \cdot \text{Ext}_y(G)
\end{equation}

for all $y \in T_{g,m}$ and $F, G \in \mathcal{MF}$ (cf. [31]). Minsky’s inequality is sharp in the sense that for any $y \in T_{g,m}$ and $F \in \mathcal{MF} - \{0\}$, there is a unique $G \in \mathcal{MF} - \{0\}$ up to positive multiple such that

\begin{equation}
I(F, G)^2 = \text{Ext}_y(F) \cdot \text{Ext}_y(G).
\end{equation}

Furthermore, such a pair $F$ and $G$ of measured foliations are realized by the horizontal and vertical foliations of a holomorphic quadratic differential on a marked Riemann surface $y$, and vice versa (cf. [9]).

\section{Kerckhoff’s formula.}

In [15], Kerckhoff gave the following formula:

\begin{equation}
d_T(y, z) = \frac{1}{2} \log \sup_{F \in \mathcal{MF}-\{0\}} \frac{\text{Ext}_y(F)}{\text{Ext}_z(F)} = \frac{1}{2} \log \max_{F \in \mathcal{MF}_1} \frac{\text{Ext}_y(F)}{\text{Ext}_z(F)}.
\end{equation}

In fact, for any $y_1, y_2 \in T_{g,m}$, there is a unique pair $(F, G)$ of measured foliations in $\mathcal{MF}_1$ such that

\begin{equation}
\frac{\text{Ext}_{y_1}(F)}{\text{Ext}_{y_2}(F)} = \frac{\text{Ext}_{y_2}(G)}{\text{Ext}_{y_1}(G)} = e^{2d_T(y_1, y_2)}.
\end{equation}
3. The Gardiner-Masur closure

In [9], Gardiner and Masur proved that a mapping

\[ \Phi_{GM}: T_{g,m} \ni y \mapsto [S \ni \alpha \mapsto \operatorname{Ext}_y (\alpha)^{1/2}] \in \mathbb{R}_+^S \]

is an embedding and the image is relatively compact, where \( \operatorname{Ext}_y (\alpha) \) is the extremal length of \( \alpha \in S \) on \( y \in T_{g,m} \). The closure \( \operatorname{cl}_{GM}(T_{g,m}) \) of the image is called the Gardiner-Masur closure or compactification, and the complement of the image in the closure is said to be the Gardiner-Masur boundary which we denote by \( \partial_{GM} T_{g,m} \).

For \( y \in T_{g,m} \), we define a continuous function \( \mathcal{E}_y \) on \( MF \) by

\[ \mathcal{E}_y (F) = \left\{ \frac{\operatorname{Ext}_y (F)}{K_y} \right\}^{1/2} \]

where \( K_y = \exp (2d_T(x_0, y)) \). In [28], the author showed that for any \( p \in \partial_{GM} T_{g,m} \), there is a continuous function \( \mathcal{E}_p \) on \( MF \) such that

(E1) the projective class of the assignment \( S \ni \alpha \mapsto \mathcal{E}_p (\alpha) \) is equal to \( p \);

(E2) if a sequence \( \{y_n\}_{n=1}^\infty \) converges to \( p \in \operatorname{cl}_{GM}(T_{g,m}) \), there are \( t_0 > 0 \) and a subsequence \( \{y_{n_j}\}_j \) such that \( \mathcal{E}_{y_{n_j}} \) converges to \( t_0 \mathcal{E}_p \) uniformly on any compact set of \( MF \).

Notice that \( t \mathcal{E}_p \) also satisfies (E1) and (E2) above for all \( t > 0 \) and \( p \in \partial_{GM} T_{g,m} \), and the function \( \mathcal{E}_p \) depends on the choice of basepoint \( x_0 \). When we emphasize the dependence, we write \( \mathcal{E}_{x_0} \) instead of \( \mathcal{E}_p \).

We first sharpen the condition (E2) above as follows (cf. [30]).

**Proposition 3.1.** For any \( p \in \partial_{GM} T_{g,m} \), one can choose \( \mathcal{E}_p \) appropriately such that the function

\[ \operatorname{cl}_{GM}(T_{g,m}) \times MF \ni (p, F) \mapsto \mathcal{E}_p (F) \]

is continuous.

**Proof.** We normalize \( \mathcal{E}_p \) such that

\[ \max_{F \in MF_1} \mathcal{E}_p (F) = 1. \tag{3.3} \]

Notice from (2.9) that \( \max_{F \in MF_1} \mathcal{E}_y (F) = 1 \) for all \( y \in T_{g,m} \). Let \( \{y_n\}_{n=1}^\infty \) be a sequence that converges to \( p \in \partial_{GM} T_{g,m} \). From the condition (E2) above, there are a subsequence \( \{y_{n_j}\}_j \) and \( t_0 > 0 \) such that \( \mathcal{E}_{y_{n_j}} \) converges to \( t_0 \mathcal{E}_p \) uniformly on any compact set of \( MF \), and hence

\[ 1 = \max_{F \in MF_1} \mathcal{E}_{y_{n_j}} (F) \rightarrow t_0 \max_{F \in MF_1} \mathcal{E}_p (F) = t_0. \]

This implies that \( \mathcal{E}_{y_n} \) converges to \( \mathcal{E}_p \) on any compact set of \( MF \). \( \square \)

**Convention 1.** In what follows, we normalize \( \mathcal{E}_p \) as in (3.3) for all \( p \in \partial_{GM} T_{g,m} \).

For instance, for \( G \in MF \) it holds

\[ \mathcal{E}_{[G]} (F) = \mathcal{E}_{x_0} (F) = \frac{I(F, G)}{\operatorname{Ext}_{x_0} (G)^{1/2}} \quad (F \in MF). \tag{3.4} \]

Indeed, by definition, there is a positive number \( t_0 \) such that \( \mathcal{E}_{[G]} (F) = t_0 I(F, G) \) for all \( F \in MF \). By (2.7) and Convention 1 we obtain

\[ 1 = \max_{F \in MF_1} \mathcal{E}_{[G]} (F) = t_0 \max_{F \in MF_1} I(F, G) = t_0 \operatorname{Ext}_{x_0} (G)^{1/2}. \]
Figure 2. Models and the model map of the cone $C_{GM}$.

The following is proven in [30] by the similar argument as the case of the Thurston compactification (cf. [4]).

**Proposition 3.2.** For $p \in \text{cl}_{GM}(T_{g,m})$, the following are equivalent.

1. $p \in \partial_{GM} T_{g,m};$
2. there is an $F \in \mathcal{MF} - \{0\}$ with $E_p(F) = 0$.

4. Cones $C_{GM}$, $T_{GM}$ and $\tilde{\partial}_{GM}$

4.1. Cones. Define

\[
\begin{align*}
C_{GM} &= \text{proj}^{-1}(\text{cl}_{GM}(T_{g,m})) \cup \{0\} \subset \mathbb{R}_+^S \\
T_{GM} &= \text{proj}^{-1}(T_{g,m}) \cup \{0\} \subset \mathbb{R}_+^S \\
\tilde{\partial}_{GM} &= \text{proj}^{-1}(\partial_{GM} T_{g,m}) \cup \{0\} \subset C_{GM} \subset \mathbb{R}_+^S.
\end{align*}
\]

We topologize $C_{GM}$, $T_{GM}$ and $\tilde{\partial}_{GM}$ with the topology induced from $\mathbb{R}_+^S$. Notice that $\mathcal{MF}$ is contained in $\tilde{\partial}_{GM}$ as a closed subset since $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$. In particular, any $G \in \mathcal{MF}$ is nothing other than an assignment

\[
S \ni \alpha \mapsto I(\alpha, G).
\]

4.2. Models of $C_{GM}$, $T_{GM}$ and $\tilde{\partial}_{GM}$. We define models of cones by

\[
\begin{align*}
\mathcal{MC}_{GM} &= \text{cl}_{GM}(T_{g,m}) \times \mathbb{R}_+ / (\text{cl}_{GM}(T_{g,m}) \times \{0\}) \\
\mathcal{MT}_{GM} &= T_{g,m} \times \mathbb{R}_+ / (T_{g,m} \times \{0\}) \\
\mathcal{M\tilde{\partial}}_{GM} &= \partial_{GM} T_{g,m} \times \mathbb{R}_+ / (\partial_{GM} T_{g,m} \times \{0\}) \\
\mathcal{MF} &= \mathcal{PMF} \times \mathbb{R}_+ / (\mathcal{PMF} \times \{0\}) \\
\mathcal{MF}_1 &= \mathcal{PMF} \times \{1\}
\end{align*}
\]

(see Figure 2). Since $\mathcal{PMF} \subset \partial_{GM} T_{g,m}$, $\mathcal{MF}_1 \subset \mathcal{MF} \subset \mathcal{M\tilde{\partial}}_{GM}$. In this setting, we often identify $\text{cl}_{GM}(T_{g,m})$ with the slice $\text{cl}_{GM}(T_{g,m}) \times \{1\}$ of $\mathcal{MC}_{GM}$.

We abbreviate the point $(p, t) \in \mathcal{MC}_{GM}$ to $tp$. We denote $1p$ by $p$ for the simplicity. For $s \geq 0$ and $\zeta = tp \in \mathcal{MC}_{GM}$ with $t \geq 0$ and $p \in \text{cl}_{GM}(T_{g,m})$, we define the multiplication $s\zeta$ by $s\zeta = (st)p$. 
From Proposition 3.1 and 3.2, the embedding (1.3) is continuous. Therefore, we have a continuous bijection (the *model map*)

\[ \tilde{\Psi}_{x_0} : \mathcal{M}_{GM} \to \mathcal{C}_{GM} \]

defined by

\[ (4.5) \quad \tilde{\Psi}_{x_0}(tp) = \tilde{\Psi}_{x_0}(p, t) = t \cdot \Psi_{x_0}(p) = [S \ni \alpha \mapsto tE_p(\alpha)]. \]

By definition, \( \tilde{\Psi}_{x_0} \) is homogeneous in the sense that

\[ \tilde{\Psi}_{x_0}(t\zeta) = t\tilde{\Psi}_{x_0}(\zeta) \]

for \( t \geq 0 \) and \( \zeta \in \mathcal{M}_{GM} \) and satisfies \( \tilde{\Psi}_{x_0}(p) = \Psi_{x_0}(p) \) for \( p \in \text{cl}_{GM}(T_{g,m}) \). Since \( \text{cl}_{GM}(T_{g,m}) \) is compact, the bijection \( \tilde{\Psi}_{x_0} \) is a homeomorphism. It follows from (3.3) that

\[ (4.6) \quad \tilde{\Psi}_{x_0}(s[F]) = s \operatorname{Ext}_{x_0}(F)^{-1/2} \cdot F \in \mathcal{M}F \]

for \( s[F] \in \mathcal{M}F \) and hence \( \tilde{\Psi}_{x_0}(\mathcal{M}F) = \mathcal{M}F \). In particular, we deduce the following.

**Lemma 4.1** (Image of \( \mathcal{M}F_1 \)). For \([G] \in \mathcal{M}F_1 \), we have \( \tilde{\Psi}_{x_0}([G]) \in \mathcal{M}F_1 \subset \partial_{GM} \).

**Remark 4.1.** From the identification (4.5), we recognize \( \mathcal{C}_{GM} \), \( \mathcal{T}_{GM} \) and \( \partial_{GM} \) as cones with slices \( \text{cl}_{GM}(T_{g,m}) \), \( T_{g,m} \) and \( \partial_{GM}T_{g,m} \), respectively. Notice that this identification depends on the choice of the basepoint \( x_0 \) (cf. (1.3)).

5. **Intersection number and Extremal length associated to a basepoint**

In this section, we define the intersection number on \( \mathcal{M}_{GM} \times \mathcal{M}F \) and the extremal length for elements in \( \mathcal{M}_{GM} \) associated to the basepoint \( x_0 \). We will extend the intersection number given here to the whole \( \mathcal{M}_{GM} \times \mathcal{M}_{GM} \) in §8.1.

5.1. **Intersection number associated to the basepoint**. For \( \zeta = tp \in \mathcal{M}_{GM} \) \((t \geq 0 \) and \( p \in \text{cl}_{GM}(T_{g,m}) \)) and \( \eta \in \mathcal{M}F \), we define the *intersection number associated to the basepoint \( x_0 \)* by

\[ (5.1) \quad i_{x_0}(\zeta, \eta) = i_{x_0}(tp, \eta) = t \cdot \mathcal{E}_p \left( \tilde{\Psi}_{x_0}(\eta) \right) = t \cdot \mathcal{E}_p \left( \tilde{\Psi}_{x_0}(\eta) \right). \]

The intersection number (5.1) depends on the basepoint \( x_0 \). Indeed, By (4.5), we have

\[ i_{x_0}(ty, s[F]) = t \cdot \mathcal{E}_y \left( \tilde{\Psi}_{x_0}(s[F]) \right) = t \cdot \frac{\operatorname{Ext}_y(s \cdot \operatorname{Ext}_{x_0}(F)^{-1/2} \cdot F)}{\mathcal{E}_y} \]

\[ = ts \cdot e^{-d_{\partial GM}(x_0,y)} \left( \frac{\operatorname{Ext}_y(F)}{\operatorname{Ext}_{x_0}(F)} \right)^{1/2} \]

for \( ty \in \mathcal{M}T_{GM} \) and \( s[F] \in \mathcal{M}F \). By (4.5), \( \zeta \in \mathcal{M}_{GM} \) corresponds to the function

\[ (5.2) \quad S \ni \alpha \mapsto i_{x_0}(\zeta, \tilde{\Psi}_{x_0}^{-1}(\alpha)) \]

in \( \mathcal{C}_{GM} \) via \( \tilde{\Psi}_{x_0} \). From Proposition 3.1, the assignment

\[ \mathcal{M}_{GM} \times \mathcal{M}F \ni (\zeta, \eta) \mapsto i_{x_0}(\zeta, \eta) \]
is continuous. Furthermore, the intersection number \( 5.1 \) is homogeneous since
\[
i_{x_0}(s_1 \zeta, s_2 \eta) = i_{x_0}((s_1 t)p, s_2 \eta) = (s_1 t)\mathcal{E}_p(\Psi_{x_0}(s_2 \eta))
\]
\[
= s_1 s_2 \cdot t \mathcal{E}_p(\Psi_{x_0}(\eta)) = s_1 s_2 i_{x_0}(\zeta, \eta)
\]
where \( s_1, s_2 \geq 0, \zeta = tp \) with \( t \geq 0 \) and \( p \in \text{cl}_{GM}(T_{g,m}) \), and \( \eta \in \text{MF} \).

Proposition 5.1 (Intersection number on \( \mathcal{M}F \)). The intersection number function \( 5.1 \) coincides with the original intersection number function on \( \mathcal{M}F \times \mathcal{M}F \) via \( \Psi_{x_0} \). Namely, when \( G = \Psi_{x_0}(\zeta) \) and \( F = \Psi_{x_0}(\eta) \) with \( \zeta, \eta \in \text{MF} \),
\[
i_{x_0}(\zeta, \eta) = I(G, F).
\]

Proof. Notice from (4.6) that
\[
i_{x_0}(\zeta, \eta) = \mathcal{E}_{x_0}(\mathcal{E}_{\eta}) = \mathcal{E}_{x_0}(\mathcal{E}_{\eta}) = I(G, F).
\]

□ □

(2) This follows from Proposition 5.1 and (1.7).

5.2. Extremal length on \( \mathcal{M}C_{GM} \) associated to the basepoint. For \( \zeta \in \mathcal{M}C_{GM} \), we define the extremal length of \( \zeta \) on \( ty \in \mathcal{M}T_{GM} \) associated to the basepoint \( x_0 \) by
\[
\mathcal{E}_{xt_{y_0}}(\zeta) = t^2 \cdot \max_{\eta \in \text{MF}_1} \frac{i_{x_0}(\zeta, \eta)^2}{\mathcal{E}_{\eta}(\Psi_{x_0}(\eta))} = t^2 \cdot \sup_{F \in \mathcal{M}F - \{0\}} \frac{i_{x_0}(\zeta, \eta)^2}{\mathcal{E}_{\eta}(\Psi_{x_0}(\eta))}.
\]

Then, \( \mathcal{E}_{xt_{y_0}}(\cdot) \) is homogeneous and satisfies
\[
i_{x_0}(\zeta, \eta)^2 \leq \mathcal{E}_{xt_{y_0}}(\zeta) \cdot \mathcal{E}_{\eta}(\Psi_{x_0}(\eta))
\]
for all \( y \in T_{g,m}, \zeta \in \mathcal{M}C_{GM} \) and \( \eta \in \text{MF} \). Since \( \text{MF}_1 \) is compact, for every \( \zeta \in \mathcal{M}C_{GM} \), there is an \( \eta \in \text{MF} - \{0\} \) such that
\[
\mathcal{E}_{xt_{y_0}}(\zeta) = t^2 \frac{i_{x_0}(\zeta, \eta)^2}{\mathcal{E}_{\eta}(\Psi_{x_0}(\eta))}
\]
or
\[
i_{x_0}(\zeta, \eta)^2 = \mathcal{E}_{xt_{y_0}}(\zeta) \cdot \mathcal{E}_{\eta}(\Psi_{x_0}(\eta))
\]
5.2.1. Basic properties. We can easily see the following.

Lemma 5.1. The following two properties hold.

(1) For \( ty, sz \in \mathcal{M}T_{GM} \) with \( t, s \geq 0 \) and \( y, z \in T_{g,m} \),
\[
\mathcal{E}_{xt_{y_0}}(sz) = t^2 s^2 \exp(-2d_T(x_0, z) + 2d_T(y, z)).
\]

(2) For \( \zeta \in \text{MF} \) and \( y \in T_{g,m} \),
\[
\mathcal{E}_{xt_{y_0}}(\zeta) = \mathcal{E}_{\eta}(\Psi_{x_0}(\zeta)).
\]

Proof. (1) Since \( K_z = \exp(2d_T(x_0, z)) \), from Kerckhoff’s formula, we have
\[
\mathcal{E}_{xt_{y_0}}(sz) = t^2 \cdot \sup_{\eta \in \text{MF} - \{0\}} \frac{i_{x_0}(sz, \eta)^2}{\mathcal{E}_{\eta}(\Psi_{x_0}(\eta))} = t^2 s^2 \sup_{F \in \mathcal{M}F - \{0\}} \frac{\mathcal{E}_{\eta}(F)}{K_z \mathcal{E}_{\eta}(F)}
\]
\[
= t^2 s^2 \exp(-2d_T(x_0, z) + 2d_T(y, z)).
\]
(2) This follows from Proposition 5.1 and (1.7). □ □
We notice the following non-triviality of the extremal length \((5.3)\).

**Lemma 5.2 (Non-triviality).** Let \(\zeta \in \mathcal{MC}_{GM}\). If \(\mathcal{E}xt_{y}^{\zeta_{0}}(\zeta) = 0\) for some \(y \in \mathcal{T}_{g,m}\), then \(\zeta = 0\).

**Proof.** Take \(t \geq 0\) and \(p \in \mathcal{cl}_{GM}(\mathcal{T}_{g,m})\) with \(\zeta = tp\). Suppose \(\mathcal{E}xt_{y}^{\zeta_{0}}(\zeta) = 0\). From \((5.1)\), we have

\[
0 = \mathcal{E}xt_{y}^{\zeta_{0}}(\zeta) = \sup_{\eta \in \mathcal{MF} \setminus \{0\}} \frac{i_{x_{0}}(\zeta, \eta)}{\mathcal{E}xt_{y}(\Psi_{x_{0}}(\eta))} = \sup_{\eta \in \mathcal{MF} \setminus \{0\}} \frac{e_{\zeta}(\Psi_{x_{0}}(\eta))}{\mathcal{E}xt_{y}(\Psi_{x_{0}}(\eta))}
\]

\[
= \sup_{F \in \mathcal{MF} \setminus \{0\}} \frac{e_{\zeta}(F)}{\mathcal{E}xt_{y}(F)} = \sup_{F \in \mathcal{MF} \setminus \{0\}} \frac{t^{2} e_{p}(F)}{\mathcal{E}xt_{y}(F)}.
\]

Therefore, we obtain

\[
t e_{p}(F) = 0
\]

for all \(F \in \mathcal{MF} \setminus \{0\}\). On the other hand, since \(p \in \mathcal{cl}_{GM}(\mathcal{T}_{g,m})\), \(e_{p}(\alpha) \neq 0\) for some \(\alpha \in \mathcal{S}\), and we get \(t = 0\). Therefore, \(\zeta = tp = 0\). □ □

5.2.2. **Continuity.** Notice that the extremal length given in \((5.3)\) satisfies the distortion property:

\[
e^{-2d_{y}(y_{1}, y_{2})} \mathcal{E}xt_{y_{1}}^{\zeta_{0}}(\zeta) \leq \mathcal{E}xt_{y_{2}}^{\zeta_{0}}(\zeta) \leq e^{-2d_{y}(y_{1}, y_{2})} \mathcal{E}xt_{y_{1}}^{\zeta_{0}}(\zeta)
\]

for \(y_{1}, y_{2} \in \mathcal{T}_{g,m}\) and \(\zeta \in \mathcal{MC}_{GM}\). Indeed, since \(\tilde{\Psi}_{x_{0}}(\eta) \in \mathcal{MF}\) for \(\eta \in \mathcal{MF}\), we have

\[
\mathcal{E}xt_{y_{1}}(\tilde{\Psi}_{x_{0}}(\eta)) \geq e^{-2d_{y}(y_{1}, y_{2})} \mathcal{E}xt_{y_{2}}(\tilde{\Psi}_{x_{0}}(\eta))
\]

for all \(\eta \in \mathcal{MF}\). Therefore, we obtain

\[
\mathcal{E}xt_{y_{2}}^{\zeta_{0}}(\zeta) = \sup_{\eta \in \mathcal{MF} \setminus \{0\}} \frac{i_{x_{0}}(\zeta, \eta)}{\mathcal{E}xt_{y_{2}}(\Psi_{x_{0}}(\eta))} \leq e^{-2d_{y}(y_{1}, y_{2})} \mathcal{E}xt_{y_{1}}^{\zeta_{0}}(\zeta).
\]

The following lemma immediately follows from Proposition \(8.1\) and the above observation, and we omit the proof.

**Lemma 5.3 (Continuity).** The function

\[
\mathcal{M}_{GM} \times \mathcal{MC}_{GM} \ni (t y, \zeta) \mapsto \mathcal{E}xt_{t y}^{\zeta_{0}}(\zeta)
\]

is continuous.

5.3. **Extremal length is intrinsic.** The extremal length \((5.3)\) is intrinsic in the following sense.

**Theorem 3 (Extremal length is intrinsic).** For \(y \in \mathcal{T}_{g,m}\), there is a continuous function

\[
\mathcal{E}xt_{y} : C_{GM} \to \mathbb{R}_{+}
\]

such that

1. \(\mathcal{E}xt_{y}^{\zeta_{0}}(\zeta) = \mathcal{E}xt_{y} \circ \tilde{\Psi}_{x}(\zeta)\) for \(\zeta \in \mathcal{MC}_{GM}\) and \(x \in \mathcal{T}_{g,m}\), and
2. For \(F \in \mathcal{MF} \subset C_{GM}\), the value \(\mathcal{E}xt_{y}(F)\) is equal to the original extremal length of \(F\).

**Remark 5.1.** From the property (2) in Theorem \(8\), the extremal length obtained in Theorem \(8\) is a continuous extension of the original extremal length on \(\mathcal{MF}\). Thus, the author believes that no confusion occurs when we use the same symbol to denote the extension of the extremal length in Theorem \(8\).
Proof of Theorem 3. We only check the existence and the property (1) because the property (2) follows from Lemma 5.1.

Let \( t, s > 0 \) and \( x_1, x_2, z, w \in \mathcal{T}_{g, m} \). Suppose that \( \hat{\Psi}_{x_1}(tz) = \hat{\Psi}_{x_2}(sw) \). Then,
\[
te^{-d_T(x_1, z) \text{Ext}_z(\alpha)^{1/2}} = se^{-d_T(x_2, w) \text{Ext}_w(\alpha)^{1/2}}
\]
for all \( \alpha \in \mathcal{S} \). From the injectivity of the Gardiner-Masur embedding \( \hat{\Psi} \) we have \( z = w \) (cf. Lemma 6.1 in [9]). Hence
\[(5.8) \quad \quad t = s \exp(d_T(x_1, z) - d_T(x_2, z)).\]

By Lemma 5.1, we obtain
\[
\text{Ext}_g^0(\alpha) = \text{Ext}_g^0(\alpha) \quad \text{and} \quad (5.9) \text{ holds for all } \alpha \in \mathcal{C}_{GM}.
\]

6. Topology of the Model

Notice that \( \text{cl}_{GM}(\mathcal{T}_{g, m}) \) is separable and metrizable (cf. [28]). Hence, \( \mathcal{M}_{GM} \) and \( \mathcal{C}_{GM} \) are locally compact, separable and metrizable.

6.1. Bounded sets are precompact. We shall begin with the following proposition.

Proposition 6.1 (Boundedness implies compactness). For any \( R > 0 \),
\[
\mathcal{M}_{GM}(R) = \{ \zeta \in \mathcal{M}_{GM} \mid \text{Ext}_{x_0}^\zeta(\zeta) \leq R \}
\]
is a compact set in \( \mathcal{M}_{GM} \). Furthermore, the level set
\[
\{ \zeta \in \mathcal{M}_{GM} \mid \text{Ext}_{x_0}^\zeta(\zeta) = 1 \}
\]
coincides with \( \text{cl}_{GM}(\mathcal{T}_{g, m}) \times \{ 1 \} \). In particular \( \text{Ext}_{x_0}^\zeta(\zeta) = 1 \) for \( \zeta \in \mathcal{M}_{F_1} \).

Proof. From the definition (5.3), the condition \( \text{Ext}_{x_0}^\zeta(\zeta) \leq R \) implies that
\[
i_{x_0}(\zeta, \hat{\Psi}_{x_0}^{-1}(\alpha)) \leq R^{1/2} \text{Ext}_{x_0}(\alpha)^{1/2}
\]
for all \( \alpha \in \mathcal{S} \). By Tikhonov’s theorem, the product of closed intervals
\[
\prod_{\alpha \in \mathcal{S}} [0, R^{1/2} \text{Ext}_{x_0}(\alpha)^{1/2}]
\]
is a compact set in \( \mathbb{R}^\mathcal{S} \). From (5.4), the image of \( \mathcal{M}_{GM}(R) \) by \( \hat{\Psi}_{x_0} \) is contained in the above product. Thus, by Lemma 5.3, \( \mathcal{M}_{GM}(R) \) is closed and hence compact. The second claim immediately follows from the first and Lemma 5.1. \qed \qed
6.2. A system of neighborhoods. Let \((\zeta, \xi) \in \text{MC}_{GM} \times \text{MF}\) with \(\zeta, \xi \neq 0\) and \(\delta > 0\). We define
\[
U_\delta(\zeta : \xi) = \{ \eta \in \text{MC}_{GM} \mid |i_{x_0}(\eta, \xi) - i_{x_0}(\zeta, \xi)| < \text{Ext}^0_{x_0}(\eta)^{1/2}\text{Ext}^0_{x_0}(\zeta)^{1/2}\delta \}
\]
\[
U_\delta(0 : \xi) = \{ \eta \in \text{MC}_{GM} \mid i_{x_0}(\eta, \xi) < \text{Ext}^0_{x_0}(\xi)^{1/2}\delta \}.
\]
Notice that
\[
\text{Ext}^0_{x_0}(\zeta)^{1/2} = \text{Ext}^0_{x_0}(\xi)^{1/2}.
\]

(6.1)
\[
U_\delta(\zeta : t\xi) = U_\delta(\zeta : \xi)
\]
for \(t > 0\) and \((\zeta, \xi) \in \text{MC}_{GM} \times \text{MF}\) with \(\xi \neq 0\). We set
\[
U_\delta(\zeta) = \cap_{\xi \in \text{MF} - \{0\}} U_\delta(\zeta : \xi).
\]
We start with the following lemma.

**Proposition 6.2.** Let \(\delta > 0\) and \(\zeta \in \text{MC}_{GM}\). Then
\[
(1 - \delta)\text{Ext}^0_{x_0}(\zeta)^{1/2} < \text{Ext}^0_{x_0}(\eta)^{1/2} < (1 + \delta)\text{Ext}^0_{x_0}(\zeta)^{1/2}
\]
for \(\eta \in U_\delta(\zeta)\).

**Proof.** From (5.5) and Proposition 6.1 we can find \(\xi \in \text{MF}_1\) such that
\[
i_{x_0}(\zeta, \xi)^2 = \text{Ext}^0_{x_0}(\zeta) \cdot \text{Ext}^0_{x_0}(\xi) = \text{Ext}^0_{x_0}(\zeta).
\]
By (3.21), for \(\eta \in U_\delta(\zeta)\), we have
\[
\text{Ext}^0_{x_0}(\zeta)^{1/2} = i_{x_0}(\zeta, \xi) < i_{x_0}(\eta, \xi) + \text{Ext}^0_{x_0}(\zeta)^{1/2}\delta \leq \text{Ext}^0_{x_0}(\eta)^{1/2} + \text{Ext}^0_{x_0}(\zeta)^{1/2}\delta
\]
and hence
\[
(1 - \delta)\text{Ext}^0_{x_0}(\zeta)^{1/2} \leq \text{Ext}^0_{x_0}(\eta)^{1/2}.
\]
Similarly, we take \(\xi \in \text{MF}_1\) with \(i(\eta, \xi)^2 = \text{Ext}^0_{x_0}(\eta)\text{Ext}^0_{x_0}(\xi) = \text{Ext}^0_{x_0}(\eta)\). This means that
\[
\text{Ext}^0_{x_0}(\eta)^{1/2} = i_{x_0}(\eta, \xi) < i_{x_0}(\zeta, \xi) + \text{Ext}^0_{x_0}(\xi)^{1/2}\delta \leq \text{Ext}^0_{x_0}(\xi)^{1/2} + \text{Ext}^0_{x_0}(\zeta)^{1/2}\delta,
\]
and we are done. \(\square\)

We claim the following (compare Lemma 4.1 of [28]. See also [15]).

**Lemma 6.1.** Let \(\zeta \in \text{MC}_{GM}\). For any \(\delta > 0\), \(U_\delta(\zeta)\) is an open neighborhood of \(\zeta\) with compact closure. Furthermore, we have that \(\cap_{\delta > 0} U_\delta(\zeta) = \{\zeta\}\).

**Proof.** It is clear that \(\zeta \in U_\delta(\zeta)\) for all \(\delta > 0\). Let \(\zeta' \in U_\delta(\zeta)\). We suppose on the contrary that there is a sequence \(\{\zeta_n\}_{n=1}^\infty\) in the complement \(\text{MC}_{GM} \setminus U_\delta(\zeta)\) which converges to \(\zeta'\). For any \(n\), there is \(\xi_n \in \text{MF}_1\) such that
\[
|i_{x_0}(\zeta_n, \xi_n) - i_{x_0}(\zeta, \xi)| \geq \text{Ext}^0_{x_0}(\zeta)^{1/2}\text{Ext}^0_{x_0}(\xi)^{1/2}\delta = \text{Ext}^0_{x_0}(\zeta)^{1/2}\delta
\]
Since \(\text{MF}_1\) is compact, we may assume that \(\xi_n\) converges to \(\xi_\infty \in \text{MF}_1\). Since \(\zeta_n \to \zeta'\) as \(n \to \infty\), by Proposition 6.1 and (6.2), we have
\[
|i_{x_0}(\zeta', \xi_\infty) - i_{x_0}(\zeta, \xi)| \geq \text{Ext}^0_{x_0}(\zeta)^{1/2}\delta,
\]
and we get a contradiction by Lemma 6.2. Hence \(U_\delta(\zeta)\) is open. By Lemma 6.2, \(U_\delta(\zeta)\) is contained in \(\text{MC}_{GM}(1 + \delta)\text{Ext}^0_{x_0}(\zeta)\). Therefore, by Proposition 6.1, the closure of \(U_\delta(\zeta)\) is compact.

To show the remaining claim, we only treat the case \(\zeta \neq 0\). The other case is dealt with the same manner. Suppose that \(\eta \in U_\delta(\zeta)\) for all \(\delta > 0\). By definition, we have
\[
|i_{x_0}(\eta, \xi) - i_{x_0}(\zeta, \xi)| < \text{Ext}^0_{x_0}(\zeta)^{1/2}\delta
\]
for all $\xi \in \mathcal{M}F_1$ and $\delta > 0$. This means that $i_{x_0}(\eta, \xi) = i_{x_0}(\zeta, \xi)$ for all $\xi \in \mathcal{M}F_1$ and $\eta = \zeta$.  

7. The Gromov Product and Extension of $E_\zeta$

For $\eta = ty \in \mathcal{M}F_{GM}$ and $\zeta \in \mathcal{M}C_{GM}$, we define

$$E_\eta(\zeta) = \left\{ \frac{E_{xt_0^y}^{x_0}(\zeta)}{K_y} \right\}^{1/2} = t \cdot \exp(-dT(x_0, y)) \cdot E_{xt_0^y}^{x_0}(\zeta)^{1/2}. \tag{7.1}$$

After identifying $\mathcal{M}F$ and $\mathcal{M}F$ via $\tilde{\Psi}_{x_0}$, by Lemma 5.1 the function $E_\eta$ in (7.1) is recognized as an extension of the function (3.2) to $\mathcal{M}C_{GM}$. By definition, the function $E_\eta$ satisfies the homogeneous property

$$E_{sy}(t\zeta) = st \cdot \left\{ \frac{E_{xt_0^y}^{x_0}(\zeta)}{K_y} \right\}^{1/2} = st \cdot E_y(\zeta), \tag{7.2}$$

for $sy \in \mathcal{M}F_{GM}$, $t \geq 0$ and $\zeta \in \mathcal{M}C_{GM}$.

Notice from Lemma 5.1 that

$$E_{sy}(tz) = st \cdot \exp(-dT(x_0, z) + d_T(y, z) - d_T(x_0, y)) \tag{7.3}$$

$$= st \cdot \exp(-2|y|z_{x_0})$$

for $sy, tz \in \mathcal{M}F_{GM}$ where $(y|z)_{x_0}$ is the Gromov product

$$|y|z_{x_0} = \frac{1}{2}(d_T(x_0, z) + d_T(x_0, y) - d_T(y, z))$$

with basepoint $x_0$. In particular, we have the following symmetry

$$E_{sy}(tz) = E_{tz}(sy) \tag{7.4}$$

for $sy, tz \in \mathcal{M}F_{GM}$.

The following was observed for the extremal length function on $\mathcal{M}F$ in [28].

**Proposition 7.1 (Equicontinuity).** The family $\{E_y\}_{y \in \mathcal{T}_{y,m}}$ is an equicontinuous family of continuous functions on $\mathcal{M}C_{GM}$. In fact, for $\delta > 0$ and $\zeta \in \mathcal{M}C_{GM}$, we have

$$|E_y(\zeta) - E_y(\eta)| \leq \max\{1, E_{xt_0^x}^{x_0}(\zeta)^{1/2}\} \delta \tag{7.5}$$

for all $\eta \in U_\delta(\zeta)$ and $y \in \mathcal{T}_{y,m}$.

**Proof.** We first assume that $\zeta \neq 0$. Take $\xi \in \mathcal{M}F_1$ with $i_{x_0}(\zeta, \xi) = E_{xt_0^y}^{x_0}(\zeta)^{1/2}E_{xt_0^y}^{x_0}(\xi)^{1/2}$ (cf. (5.6)). If $\eta \in U_\delta(\zeta)$,

$$E_{xt_0^y}^{x_0}(\zeta)^{1/2}E_{xt_0^y}^{x_0}(\xi)^{1/2} = i_{x_0}(\zeta, \xi) \leq i_{x_0}(\eta, \xi) + E_{xt_0^y}^{x_0}(\xi)^{1/2}\delta$$

$$\leq E_{xt_0^y}^{x_0}(\eta)^{1/2}E_{xt_0^y}^{x_0}(\xi)^{1/2} + E_{xt_0^y}^{x_0}(\xi)^{1/2}\delta.$$

Hence we get

$$E_{xt_0^y}^{x_0}(\zeta)^{1/2} \leq E_{xt_0^y}^{x_0}(\eta)^{1/2} + \frac{E_{xt_0^y}^{x_0}(\xi)^{1/2}}{E_{xt_0^y}^{x_0}(\xi)^{1/2}} \delta \tag{7.6}$$

$$\leq E_{xt_0^y}^{x_0}(\eta)^{1/2} + K_y^{1/2}E_{xt_0^y}^{x_0}(\xi)\delta,$$

since $E_{xt_0^y}^{x_0}(\xi) \geq K_y^{-1}E_{xt_0^y}^{x_0}(\xi) = K_y^{-1}$ (cf. (5.6)).
We also take $\xi' \in \mathcal{MF}_1$ with $i_{x_0}(\eta, \xi') = \mathcal{Ex}t^{x_0}_y(\eta)^{1/2} \mathcal{Ex}t^{x_0}_y(\xi')^{1/2}$. Then,

$$
\mathcal{Ex}t^{x_0}_y(\eta)^{1/2} \mathcal{Ex}t^{x_0}_y(\xi')^{1/2} = i_{x_0}(\eta, \xi') \leq i_{x_0}(\zeta, \xi') + \mathcal{Ex}t^{x_0}_y(\zeta)\delta
\leq \mathcal{Ex}t^{x_0}_y(\zeta)^{1/2} \mathcal{Ex}t^{x_0}_y(\xi')^{1/2} + \mathcal{Ex}t^{x_0}_y(\zeta)^{1/2}\delta.
$$

Hence, by the same argument as above,

$$
\mathcal{Ex}t^{x_0}_y(\eta)^{1/2} \leq \mathcal{Ex}t^{x_0}_y(\zeta)^{1/2} + K_y^{1/2} \mathcal{Ex}t^{x_0}_y(\zeta)^{1/2}\delta.
$$

Thus, (7.6) and (7.7) implies (7.5).

Thus, (7.6) and (7.7)

$$
|\mathcal{Ex}t^{x_0}_y(\eta)^{1/2} - \mathcal{Ex}t^{x_0}_y(\zeta)^{1/2}| \leq K_y^{1/2} \mathcal{Ex}t^{x_0}_y(\zeta)^{1/2}\delta.
$$

Suppose next that $\zeta = 0$. If we take $\xi' \in \mathcal{MF}_1$ with $i_{x_0}(\eta, \xi') = \mathcal{Ex}t^{x_0}_y(\xi')^{1/2} \mathcal{Ex}t^{x_0}_y(\eta)^{1/2}$,

$$
\mathcal{Ex}t^{x_0}_y(\eta)^{1/2} \cdot \mathcal{Ex}t^{x_0}_y(\xi')^{1/2} = i_{x_0}(\eta, \xi') < \delta.
$$

Therefore, we conclude

$$
|\mathcal{Ex}t^{x_0}_y(\eta)^{1/2} - \mathcal{Ex}t^{x_0}_y(\xi')^{1/2}| = \mathcal{Ex}t^{x_0}_y(\eta) \leq \frac{\delta}{\mathcal{Ex}t^{x_0}_y(\xi')^{1/2}} \leq K_y^{1/2}\delta.
$$

Thus, (7.8) and (7.9) implies (7.5).

\[\square\]  

8. Extension of the intersection number

One of the purpose of this section is to show the following theorem.

**Theorem 4** (Intersection number on $C_{GM}$). There exists a unique continuous function

$$
i(\cdot, \cdot) : C_{GM} \times C_{GM} \to \mathbb{R}_+
$$

independent of the choice of basepoint $x_0$ satisfying the following properties.

1. For any $\zeta, \eta \in \mathcal{MC}_{GM}$,

$$
i(\Psi_{x_0}(\zeta), \Psi_{x_0}(\eta)) = i_{x_0}(\zeta, \eta).
$$

In particular, we have

$$
i(\tilde{\Psi}_{x_0}(ty), \tilde{\Psi}_{x_0}(sp)) = ts e^{-d_T(x_0, y)} \text{Ext}_y (\Psi_{x_0}(p))^{1/2}
$$

$$
i(\tilde{\Psi}_{x_0}(p), F) = i (\Psi_{x_0}(p), F) = \mathcal{E}_p(F)
$$

for $x_0, y \in T_{g,m}$, $p \in \partial_{GM} T_{g,m}$, $F \in \mathcal{MF}$ and $t, s \geq 0$.

2. $i(a, b) = i(b, a)$ for $a, b \in \mathcal{CGM}$.

3. $i(sa, tb) = st \cdot i(a, b)$ for $s, t \geq 0$ and $a, b \in \mathcal{CGM}$.

4. For $x_0, y, z \in T_{g,m}$,

$$
i(\Psi_{x_0}(y), \Psi_{x_0}(z)) = \exp(-2(y \mid z)_{x_0}).
$$

5. The self-intersection number satisfies

$$
i(a, a) = \begin{cases} 
    t^2 \exp(-2d_T(x_0, y)) & \text{if } a = \tilde{\Psi}_{x_0}(ty) \in T_{GM} \\
    0 & \text{if } a \in \partial_{GM}
\end{cases}
$$

for $x_0 \in T_{g,m}$. 

For $F, G \in \mathcal{MF} \subset \mathcal{CGM}$,
\[ i(F, G) = I(F, G), \]
where we recall that the intersection number in the right-hand side is the original intersection number function on $\mathcal{MF} \times \mathcal{MF}$.

Theorem 3 follows from Theorem 3. Indeed, the only difference is the item (iv) in Theorem 3. From (3) in Theorem 3, we have
\[ i(F_{GM}(y), F_{GM}(z)) = \exp(d_T(x_0, y)) \cdot \exp(d_T(x_0, z)) \cdot i(\Psi_{x_0}(y), \Psi_{x_0}(z)) = \exp(d_T(y, z)) \]
for $y, z \in T_{g,m}$.

Corollaries. We give two corollaries of Theorem 4 before proving the theorem.

**Corollary 3** (Minsky’s inequality). For $x \in T_{g,m}$ and $a, b \in \mathcal{CGM}$, we have
\[ i(a, b)^2 \leq \text{Ext}_x(a) \cdot \text{Ext}_x(b). \]
The equality holds if the projective classes of $a$ and $b$ are on a common Teichmüller geodesic in this order.

**Proof.** Suppose that $a, b \in T_{g,m}$, and $a, b \in \mathcal{CGM}$ with $a = \Psi_{x_0}(ty)$ and $b = \Psi_{x_0}(sz)$. Then, by Lemma 3.1, we have
\[ i(a, b)^2 = i_{x_0}(\Psi_{x_0}(ty), \Psi_{x_0}(sz))^2 = t^2 s^2 \exp(-4(y | z)_{x_0}) = t^2 s^2 \exp(2d_T(y, z) - 2d_T(x_0, y) - 2d_T(x_0, z)) \leq \text{Ext}_{x_0}^y(ty) \cdot \text{Ext}_{x_0}^z(sz) = \text{Ext}_x(a) \cdot \text{Ext}_x(b). \]
Since $T_{g,m}$ is dense in $\mathcal{CGM}$, we have the desired inequality.

Suppose the projective classes of $a, x$ and $b$ are on a common Teichmüller geodesic $\gamma : \mathbb{R} \to T_{g,m}$ in this order. We may assume that $a, b \in \partial_{GM} T_{g,m}$ since intersection number and extremal length are homogeneous. From the assumption, we may choose $\gamma$ such that $\gamma(t) \to a$ and $\gamma(-t) \to b$ when $t \to \infty$. Therefore, from (8.4) we have
\[ i(\gamma(t), \gamma(-t))^2 = \text{Ext}_x(\gamma(t)) \cdot \text{Ext}_x(\gamma(-t)) \]
for sufficiently large $t > 0$. By letting $t \to \infty$, we get the equality in (8.3). □ □

**Corollary 4** (Intrinsic representation of extremal length). For $y \in T_{g,m}$ and $a \in \mathcal{CGM}$, we have
\[ \text{Ext}_y(a) = \sup_{F \in \mathcal{MF} - \{0\}} \frac{i(a, F)^2}{\text{Ext}_y(F)} = \sup_{b \in \mathcal{CGM} - \{0\}} \frac{i(a, b)^2}{\text{Ext}_y(b)}. \]

**Proof.** Notice that in the definition 5.3 of the extremal length, the measured foliation $F$ in the denominator in 5.3 is taken in $\mathcal{MF} - \{0\} \subset \mathcal{CGM}$. Therefore, by (2) of Theorem 3 and Theorem 4 for $a \in \mathcal{CGM}$, we have
\[ \text{Ext}_y(a) = \mathcal{E}_{y} x_0^{-1} \circ \Psi_{x_0}^{-1} (a) = \sup_{F \in \mathcal{MF} - \{0\}} \frac{i_{x_0}(\tilde{\Psi}_{x_0}^{-1}(a), \tilde{\Psi}_{x_0}^{-1}(F))^2}{\text{Ext}_y(F)} = \sup_{F \in \mathcal{MF} - \{0\}} \frac{i(a, F)^2}{\text{Ext}_y(F)}. \]
The second equality follows from Corollary 3. □ □
8.1. Extension of the intersection number \( i_{x_0} \). To show Theorem 8.1 we first extend the intersection number \( \langle \cdot, \cdot \rangle \) to the whole \( \mathcal{MC}_{GM} \times \mathcal{MC}_{GM} \).

**Proposition 8.1** (Extension of \( i_{x_0} \)). For any \( x_0 \in T_{g,m} \), there exists a unique continuous function

\[
(8.6) \quad i_{x_0}(\cdot, \cdot) : \mathcal{MC}_{GM} \times \mathcal{MC}_{GM} \to \mathbb{R}_+
\]

such that

1. For \( ty \in MT_{GM} \) and \( sp \in \tilde{\mathcal{M}}_{GM} \) with \( y \in T_{g,m}, p \in \partial GM T_{g,m} \) and \( t, s \geq 0 \),

\[
i_{x_0}(ty, sp) = ts \mathcal{E}_y(p) = ts e^{-d{r(x_0, y)}E_{xt_y}(p)\frac{1}{2}};
\]

2. \( i_{x_0}(\zeta, \eta) = i_{x_0}(\eta, \zeta) \) for \( \zeta, \eta \in \mathcal{MC}_{GM} \);

3. \( i_{x_0}(s\zeta, t\eta) = st \cdot i_{x_0}(\zeta, \eta) \) for \( s, t \geq 0 \) and \( \zeta, \eta \in \mathcal{MC}_{GM} \);

4. For \( y, z \in T_{g,m} \), \( i_{x_0}(y, z) = \exp(-2|y - z|_{x_0}) \);

5. For \( \zeta = tp \in \mathcal{MC}_{GM} \) with \( p \in c_{GM}(T_{g,m}) \),

\[
i_{x_0}(\zeta, \zeta) = \begin{cases} i^2 \exp(-2d_{r(x_0, p)}) & \text{if } \zeta \in MT_{GM} \\ 0 & \text{if } \zeta \in \tilde{\mathcal{M}}_{GM} \end{cases}
\]

6. \( i_{x_0}(\tilde{\Psi}_{x_0}^{-1}(F), \tilde{\Psi}_{x_0}^{-1}(G)) = I(F, G) \) for all \( F, G \in \mathcal{M}_{F} \).

**Proof.** Consider the equicontinuous family \( \{\mathcal{E}_y\}_{y \in T_{g,m}} \) given in Proposition 7.1. For any \( \zeta \in \mathcal{MC}_{GM} \),

\[
\mathcal{E}_y(\zeta) = \left( \frac{\mathcal{E}_{xt_y}^z(\zeta)}{K_y} \right)^{1/2} \leq \mathcal{E}_{xt_y}^z(\zeta)^{1/2}.
\]

By Proposition 6.1 the family \( \{\mathcal{E}_y\}_{y \in T_{g,m}} \) is uniformly bounded on any compact set. Therefore, the family is a normal family.

Let \( \zeta \in \mathcal{MC}_{GM} \). Let \( p \in c_{GM}(T_{g,m}) \) and \( t \geq 0 \) such that \( \zeta = tp \). Let \( \{y_n\}_{n=1}^\infty \) be a sequence converging to \( p \). Take a sequence \( \{t_n\}_{n=1}^\infty \) of positive numbers with \( t_n \to t \). By Ascoli-Arzelà theorem, there is a subsequence \( \{y_{n_j}\} \) such that a sequence \( \{\mathcal{E}_{t_n y_{n_j}}\}_{j} \) converges to the continuous function \( \mathcal{E}' \) on \( \mathcal{MC}_{GM} \) uniformly on any compact set. Notice from Lemma 5.3 and 7.1 that for \( sz \in MT_{GM} \),

\[
(8.7) \quad \mathcal{E}'(sz) = \lim_{j \to \infty} \mathcal{E}_{t_n y_{n_j}}(sz) = \lim_{j \to \infty} \mathcal{E}_{sz}(t_n y_{n_j}) = \mathcal{E}_{sz}(\zeta).
\]

Take another sequence \( \{t'_k y'_k\}_{k} \) in \( MT_{GM} \) which tends to \( \zeta \) such that \( \mathcal{E}_{t'_k y'_k} \) converges to a continuous function \( \mathcal{E}'' \) on \( \mathcal{MC}_{GM} \) uniformly on any compact set of \( \mathcal{MC}_{GM} \). Since the right-hand side of \( (8.7) \) is independent of converging sequences, the same conclusion holds for \( \mathcal{E}'' \). Namely, we have

\[
\mathcal{E}''(sz) = \mathcal{E}_{sz}(\zeta) = \mathcal{E}'(sz)
\]

for all \( sz \in MT_{GM} \). Since \( MT_{GM} \) is dense in \( \mathcal{MC}_{GM} \) and both \( \mathcal{E}'' \) and \( \mathcal{E}' \) are continuous on \( \mathcal{MC}_{GM} \), \( \mathcal{E}'' = \mathcal{E}' \) on \( \mathcal{MC}_{GM} \). This means that the limit \( \mathcal{E}' \) above is dependent only on \( \zeta \), independent of the choice of the sequence \( \{y_n\}_{n=1}^\infty \) converging to \( \zeta \). We denote by \( i_{x_0}(\zeta, \cdot) \) the limit.

For any \( R > 0 \), notice again that \( \{\mathcal{E}_{y}\}_{y \in \mathcal{MC}_{GM}(R)} \) is a normal family of continuous functions on \( \mathcal{MC}_{GM} \). From the argument above,

\[
(8.8) \quad \mathcal{MC}_{GM} \times \mathcal{MC}_{GM} \ni (\zeta, \eta) \mapsto i_{x_0}(\zeta, \eta)
\]

is continuous in two variables. The condition (1) in the statement follows from the construction and \( (7.1) \). From the density of \( MT_{GM} \times \mathcal{MC}_{GM} \) in \( \mathcal{MC}_{GM} \times \mathcal{MC}_{GM} \) we deduce the uniqueness of our function \( i_{x_0}(\cdot, \cdot) \).
Let us check that our function \( i_{x_0}(\cdot, \cdot) \) satisfies the remaining conditions (2) to (5) in the statement. Indeed, (2) and (3) follows from the density of \( MT_{G,M} \) in \( MC_{G,M} \) and equations (7.2) and (7.4). We get (4) from (7.3). The condition (5) is verified from

\[
i_{x_0}(\zeta, \zeta) = t^2 \exp(-2(y \mid y)_{x_0}) = t^2 \exp(-d_T(x_0, y))
\]

when \( \zeta = ty \in MT_{G,M} \) and the continuity of the function \( i_{x_0}(\cdot, \cdot) \). The last condition (6) follows from Proposition 5.1.

\[\Box \Box\]

**Proof of Theorem 4.** Theorem 4 immediately follows from Proposition 8.1. Indeed, we define

\[
i(a, b) = i_{x_0}(\hat{\Psi}^{-1}(a), \hat{\Psi}^{-1}(b))
\]

for \( a, b \in GM \). By applying the similar argument as that in the proof of Theorem 3, one can see that the intersection number (8.9) is intrinsic in the sense that the value is independent of the choice of basepoint \( x_0 \).

\[\Box\]

8.2. **Extension of the Gromov product.** In this section, we give a proof of Corollary 1. The uniqueness of the extension follows from the density of \( T_{g,m} \) in \( cl_{GM}(T_{g,m}) \) and the condition (1) in Corollary 1. Hence it suffices to show the existence.

Define

\[
\langle p \mid q \rangle_{x_0} = -\frac{1}{2} \log i_{x_0}(p, q)
\]

for \( p, q \in cl_{GM}(T_{g,m}) \), where \( cl_{GM}(T_{g,m}) \) is identified with a subset via the embedding (1.3). Notice from Proposition 6.1 and Corollary 3 that \( i_{x_0}(p, q) \leq 1 \) for \( p, q \in cl_{GM}(T_{g,m}) \). Therefore, the pairing \( \langle \cdot \mid \cdot \rangle_{x_0} \) defined above is continuous with value in \([0, \infty)\). From (4) of Proposition 8.1, the pairing (8.10) coincides with the Gromov product with basepoint \( x_0 \). Since

\[
I(F, G) = i_{x_0}(\hat{\Psi}^{-1}(F), \hat{\Psi}^{-1}(G)) = i_{x_0}(\text{Ext}_{x_0}(F)^{1/2}, \text{Ext}_{x_0}(G)^{1/2}) = \text{Ext}_{x_0}(F)^{1/2} \cdot \text{Ext}_{x_0}(G)^{1/2} i_{x_0}([F], [G]),
\]

we conclude (2) of Corollary 1.

\[\Box\]

9. **Isometric action on Teichmüller space**

An orientation preserving homeomorphism \( h: X \to X \) induces a homeomorphic action \( h_* \) on \( \partial GM T_{g,m} \) by the equation

\[
E_{h_*}(p)(F) = t E_p(h^{-1}(F))
\]

for all \( F \in MF \), where \( t > 0 \) is independent of \( F \). Indeed, the action \( h_* \) is the homeomorphic extension of the Teichmüller modular group action on \( T_{g,m} \) induced by \( h \) (cf. §5.4 of [28]). In this section, we give a necessary condition for a mapping of \( \partial GM T_{g,m} \) to be induced from a homeomorphism on \( X \).
9.1. Null space. For \( a \in C_{GM} \), we define the null space of \( a \) by
\[
\mathcal{N}(a) = \{ b \in C_{GM} \mid i(a, b) = 0 \}.
\]
By definition, \( 0 \in \mathcal{N}(a) \) for all \( a \in C_{GM} \). We remark the following simple claim.

**Proposition 9.1.** The following hold.
1. For \( a \in C_{GM} - \{ 0 \} \), \( \mathcal{N}(a) = \{ 0 \} \) if and only if \( a \in \partial GM \).
2. \( \mathcal{N}(a) \subset \partial GM \) for all \( a \in C_{GM} \).
3. \( \mathcal{N}(a) \cap \mathcal{MF} = \{ 0 \} \) for \( a \in \partial GM \).

**Proof.** (1) If \( a \in \mathcal{T}_{GM} \), \( \mathcal{N}(a) = \{ 0 \} \) from Lemma 5.2 and (1) of Theorem 1. If \( a \in \partial GM \), from (5) of Theorem 1, \( a \in \mathcal{N}(a) \) and \( \mathcal{N}(a) = \{ 0 \} \).

(2) From (1) above, \( \mathcal{N}(a) = \{ 0 \} \subset \partial GM \) for \( a \in \mathcal{T}_{GM} \). Let \( a \in \partial GM \). For any \( b \in \mathcal{N}(a) \), \( a \in \mathcal{N}(b) = \{ 0 \} \). This means that \( \mathcal{N}(a) \cap \mathcal{T}_{GM} = \{ 0 \} \) for all \( a \in C_{GM} \).

(3) Let \( a \in \partial GM \). Suppose \( a = \tilde{\Psi}_{GM}(tp) \) for some \( t \geq 0 \) and \( p \in \partial GM \mathcal{T}_{g,m} \). If \( \mathcal{N}(a) \cap \mathcal{MF} = \{ 0 \} \),
\[
t_{\mathcal{E}_{p}}(F) = i(\tilde{\Psi}_{x_0}(tp), \tilde{\Psi}_{x_0}(F)) = i(a, F) = 0
\]
for all \( F \in \mathcal{MF} - \{ 0 \} \) by Theorem 6.2. By Proposition 5.2, this implies \( p \in \mathcal{T}_{g,m} \), which is a contradiction. □

Let \( \omega \) be a mapping \( \omega : cl_{GM}(\mathcal{T}_{g,m}) \to cl_{GM}(\mathcal{T}_{g,m}) \). We extend the action of \( \omega \) to \( MC_{GM} \) by
\[
H_{\omega} : MC_{GM} \ni tp \mapsto t \omega(p) \in MC_{GM}
\]
where \( t \geq 0 \) and \( p \in cl_{GM}(\mathcal{T}_{g,m}) \). Let \( x_0 \in \mathcal{T}_{g,m} \) be the basepoint as before. We define a homeomorphism \( h_{\omega} \) on \( C_{GM} \) by
\[
h_{\omega} = \tilde{\Psi}_{x_0} \circ H_{\omega} \circ \tilde{\Psi}_{x_0}^{-1}.
\]

**Proposition 9.2.** Let \( \omega : \mathcal{T}_{g,m} \to \mathcal{T}_{g,m} \) be a mapping of bounded distortion for triangles. Suppose that \( \omega \) admits a continuous extension to \( cl_{GM}(\mathcal{T}_{g,m}) \). Then, for \( a, b \in \partial GM \), \( i(h_{\omega}(a), h_{\omega}(b)) = 0 \) if and only if \( i(a, b) = 0 \). Furthermore, if \( \omega \) has a quasi-inverse \( \omega' \) which also admits a continuous extension to \( cl_{GM}(\mathcal{T}_{g,m}) \), then
\[
h_{\omega} \circ h_{\omega}(\mathcal{N}(a)) \subset \mathcal{N}(a)
\]
when \( a \in \partial GM \).

**Proof.** Let \( D_1 \) and \( D_2 \) be the distortion constants of \( \omega \). A formal calculation yields
\[
2 \langle \omega(y) | \omega(z) \rangle_{\omega(x_0)} = 2 \langle \omega(y) | \omega(z) \rangle_{x_0} - 2 \langle \omega(x_0) | \omega(y) \rangle_{x_0}
\]
\[
- 2 \langle \omega(x_0) | \omega(z) \rangle_{x_0} - 2 d_{\mathcal{T}}(x_0, \omega(x_0))
\]
for every \( x, y, \omega \in \mathcal{T}_{g,m} \). Since \( \omega \) is a mapping of bounded distortion for triangles with constant \( D_1, D_2 > 0 \), we conclude that
\[
e^{-2D_1}J_{x_0}(y, z) i_{x_0}(y, z)^{D_1} \leq i_{x_0}(\omega(y), \omega(z)) \leq e^{-2D_2}J_{x_0}(y, z) i_{x_0}(y, z)^{D_2},
\]
where
\[
J_{x_0}(y, z) = e^{2d_{\mathcal{T}}(x_0, \omega(x_0))} i_{x_0}(\omega(x_0), \omega(y)) i_{x_0}(\omega(x_0), \omega(z)).
\]

Let \( \zeta, \eta \in \partial GM \mathcal{T}_{g,m} \). Since \( \omega \) has a continuous extension to \( cl_{GM}(\mathcal{T}_{g,m}) \), by letting \( y \to \zeta \) and \( z \to \eta \) in (9.3), we get
\[
e^{-2D_1}J_{x_0}(\zeta, \eta) i_{x_0}(\zeta, \eta)^{D_1} \leq i_{x_0}(\omega(\zeta), \omega(\eta)) \leq e^{-2D_2}J_{x_0}(\zeta, \eta) i_{x_0}(\zeta, \eta)^{D_2}
\]
from Proposition [8.1] where

\[
J_{x_0}(\zeta, \eta) = \lim_{y \to \zeta} \lim_{z \to \eta} e^{2D_3(x_0, \omega(x_0))} i_{x_0}(\omega(x_0), \omega(y)) i_{x_0}(\omega(x_0), \omega(z)) = e^{2D_3(x_0, \omega(x_0))} \xi_{x_0}(\omega(\zeta))^{1/2} \xi_{x_0}(\omega(\eta))^{1/2} \neq 0
\]

since \(\omega(x_0) \in T_{g,m}\) (cf. Lemma [5.2]). Therefore, (9.4) implies that \(i_{x_0}(\omega(\zeta), \omega(\eta)) = 0\) if and only if \(i_{x_0}(\zeta, \eta) = 0\) for \(\zeta, \eta \in \partial_{GM} T_{g,m}\).

Let \(a, b \in \partial_{GM}\). Take \(\zeta, \eta \in \partial_{GM} T_{g,m}\) and \(t, s \geq 0\) with \(a = \tilde{\Psi}_x(t\zeta)\) and \(b = \tilde{\Psi}_x(s\eta)\). Then, by (8.3),

\[
i(a, b) = i_{x_0}(t\zeta, s\eta) = ts i_{x_0}(\zeta, \eta).
\]

Therefore, \(i(a, b) = 0\) if and only if \(i(h_\omega(a), h_\omega(b)) = 0\).

Suppose \(\omega\) has a quasi-inverse \(\omega'\) of quasi-inverse constant \(D_3\) which extends continuously to \(c_{GM}(T_{g,m})\). Then,

\[
2(y \mid z)_{x_0} - 2D_3 \leq 2(y \mid \omega' \circ \omega(z))_{x_0} \leq 2(y \mid z)_{x_0} + 2D_3
\]

and

\[
e^{-2D_3} i_{x_0}(y, z) \leq i_{x_0}(y, \omega' \circ \omega(z)) \leq e^{2D_3} i_{x_0}(y, z).
\]

Therefore, by letting \(y \to \zeta\) and \(z \to \eta\), we have

\[
e^{-2D_3} i_{x_0}(\zeta, \eta) \leq i_{x_0}(\zeta, \omega' \circ \omega(\eta)) \leq e^{2D_3} i_{x_0}(\zeta, \eta)
\]

for all \(\zeta, \eta \in \partial_{GM} T_{g,m}\), which implies

\[
e^{-2D_3} i(a, b) \leq i(a, h_\omega \circ h_\omega(b)) \leq e^{2D_3} i(a, b)
\]

for \(a, b \in \partial_{GM}\). Let \(b \in h_\omega \circ h_\omega(N(a))\). Take \(c \in N(a)\) with \(b = h_\omega \circ h_\omega(c)\). Since \(i(a, b) = i(a, h_\omega \circ h_\omega(c)) \leq e^{2D_3} i(a, c) = 0\),

we have \(b \in N(a)\). \(\square\)

9.2. \(\omega\) preserves \(\mathcal{PMF}\). This section is devoted to showing (1) in Theorem [2] Namely, we prove the following.

Proposition 9.3 (\(\omega\) preserves \(\mathcal{PMF}\)). Let \(\omega : T_{g,m} \to T_{g,m}\) be a mapping of bounded distortion for triangles with continuous extension to \(c_{GM}(T_{g,m})\). Suppose that \(\omega\) has a quasi-inverse \(\omega'\) which also extends continuously to \(c_{GM}(T_{g,m})\). Then, the restriction of \(\omega\) to \(\mathcal{PMF}\) is a self-homeomorphism of \(\mathcal{PMF}\). Furthermore, \(\omega' = \omega^{-1}\) on \(\mathcal{PMF}\).

The proof of Proposition [9.3] is given in [9.2.2]. Before showing Proposition [9.3] we deal with uniquely ergodic measured foliations as elements in \(C_{GM}\) in the next section.

9.2.1. Uniquely ergodic measured foliations. In this paper, \(G \in \mathcal{MF} - \{0\}\) is said to be uniquely ergodic if every \(F \in (\mathcal{N}(G) - \{0\}) \cap \mathcal{MF}\) is projectively equivalent to \(G\). It is known that the set of uniquely ergodic measured foliations is dense in \(\mathcal{MF}\) (cf. [4]. See also [22] and [36]).

In the Gardiner-Masur boundary, simple closed curves and uniquely ergodic measured foliations are rigid in the following sense.
Lemma 9.1 (Theorem 3 of [29]). Let \( p \in \text{cl}_{GM}(T_{g,m}) \). Let \( G \in \mathcal{MF} \) be a simple closed curve or a uniquely ergodic measured foliation. Suppose that \( E_p(F) = 0 \) for all \( F \in N(G) \cap \mathcal{MF} \). Then there is \( t > 0 \) such that
\[
E_p(F) = t i(F,G)
\]
for all \( F \in \mathcal{MF} \). Namely, \( p = [G] \) as points in \( \text{cl}_{GM}(T_{g,m}) \).

We give a characterization of uniquely ergodic measured foliations as follows.

Lemma 9.2 (Uniquely ergodic points). The following four conditions are equivalent for \( a \in C_{GM} - \{0\} \):

(i) There exists \( b \in C_{GM} - \{0\} \) such that \( N(a) = \{tb \mid t \geq 0\} \).
(ii) \( N(a) = \{ta \mid t \geq 0\} \).
(iii) \( a \in \mathcal{MF} \) and \( a \) is uniquely ergodic.
(iv) \( N(a) \) contains a uniquely ergodic measured foliation.

Proof. (i) is equivalent to (ii). Clearly (ii) implies (i). Since \( N(a) \neq \{0\} \), \( a \in \partial_{GM} \).
Thus, (ii) follows from (i) since \( i(a,a) = 0 \) (cf. Theorem 4).

(ii) implies (iii). By (1) and (3) of Proposition 9.1, \( a \in \partial_{GM} \) and \( N(a) \cap \mathcal{MF} \neq \{0\} \).
Therefore, we have \( a \in \mathcal{MF} \).
Thus, if \( F \in \mathcal{MF} \) satisfies \( I(F,a) = 0 \), \( F \) is projectively equivalent to \( a \).
This means that \( a \) is a uniquely ergodic measured foliation.

(iii) implies (ii). Let \( G \in \mathcal{MF} \subset C_{GM} \) be a uniquely ergodic measured foliation.
Let \( b \in N(G) - \{0\} \). From Proposition 9.1, \( b \in \partial_{GM} \).
Let \( p \in \partial_{GM} T_{g,m} \) and \( t > 0 \) with \( b = \tilde{\Psi}_{x_0}(tp) \).
Then, by Theorem 1
\[
t E_p(G) = i(\tilde{\Psi}_{x_0}(tp),G) = i(b,G) = 0.
\]
By Lemma 9.1, \( b \) is projectively equivalent to \( G \).
This means that \( N(G) = \{tG \mid t \geq 0\} \).

(iii) is equivalent to (iv). Clearly (iii) implies (iv) since \( a \in N(a) \).
Suppose \( N(a) \) contains a uniquely ergodic measured foliation \( G \).
Since \( i(a,G) = 0 \), by applying the same argument in “(iii) implies (ii)” above, we deduce \( a \) is projectively equivalent to \( G \) and \( a \) is a uniquely ergodic measured foliation.

9.2.2. Proof of Proposition 9.3. Let \( G \in \mathcal{MF} \subset C_{GM} \) be a uniquely ergodic measured foliation.
Since \( N(G) = \{tG \mid t \geq 0\} \), we have from Proposition 9.2 that
\[
h_{\omega'} \circ h_{\omega}(N(G)) \subset N(G) = \{tG \mid t \geq 0\}.
\]
Since \( h_{\omega'} \circ h_{\omega}(G) \in h_{\omega'} \circ h_{\omega}(N(G)), h_{\omega'} \circ h_{\omega}(N(G)) \neq \{0\} \).
Therefore,
\[
h_{\omega'} \circ h_{\omega}(N(G)) = N(G) = \{tG \mid t \geq 0\}.
\]
This implies that \( \omega' \circ \omega([G]) = [G] \).
Since the set \( \mathcal{PMF}^{h_{\omega}} \) of uniquely ergodic measured foliations is dense in \( \mathcal{PMF} \)
and \( \omega \) and \( \omega' \) are continuous, we conclude that \( \omega' \circ \omega \) is the identity mapping on \( \mathcal{PMF} \).
By applying the same argument, we deduce that \( \omega \circ \omega' \) is also the identity on \( \mathcal{PMF} \).
In particular, since
\[
\mathcal{PMF} = \omega \circ \omega'(\mathcal{PMF}) \subset \omega(\partial_{GM} T_{g,m}),
\]
\( \mathcal{MF} \) is contained in both \( h_{\omega}(\partial_{GM}) \) and \( h_{\omega'}(\partial_{GM}) \).
Let \([G] \in \mathcal{PMF}^{UE}\) again. By Proposition 9.4, we can take \(F \in \mathcal{N}(h_\omega(G)) \cap \mathcal{MF}\) with \(F \neq 0\). Since \(\mathcal{MF} \subset h_\omega(\tilde{\mathcal{GM}})\), there is an \(a \in \tilde{\mathcal{GM}}\) such that \(F = h_\omega(a)\). Since \(i(h_\omega(a), h_\omega(G)) = i(F, h_\omega(G)) = 0\), we have from Proposition 9.2 that \(i(a, G) = 0\). Hence, it follows from Lemma 9.2 that \(a = tG\) for some \(t > 0\). Therefore \(h_\omega(G) = t^{-1}F \in \mathcal{MF}\), and \(\omega([G]) \in \mathcal{PMF}\) for all \([G] \in \mathcal{PMF}^{UE}\). By applying the same argument to \(h_\omega\), we conclude that \(\omega(\mathcal{PMF}) \subset \mathcal{PMF}\) and \(\omega'(\mathcal{PMF}) \subset \mathcal{PMF}\) from the density of uniquely ergodic measured foliations in \(\mathcal{MF}\).

On the other hand, since \(\omega \circ \omega' = \omega' \circ \omega\) are the identity on \(\mathcal{PMF}\), we deduce
\[
\mathcal{PMF} = \omega \circ \omega'(\mathcal{PMF}) \subset \omega(\mathcal{PMF}) \subset \mathcal{PMF}
\]
and we are done. \(\square\)

9.2.3. Null space in \(\mathcal{MF}\). From Proposition 9.3, we have the following observation.

**Proposition 9.4.** Let \(\omega\) be as Proposition 9.3. For \(G \in \mathcal{MF}\),
\[
h_\omega(\mathcal{N}(G) \cap \mathcal{MF}) = \mathcal{N}(h_\omega(G)) \cap \mathcal{MF}.
\]

**Proof.** Take a quasi-inverse \(\omega'\) of \(\omega\). Notice as in Proposition 9.3 that \(\omega' = \omega^{-1}\) on \(\mathcal{PMF}\). Therefore, the restrictions of \(h_\omega\) and \(h_{\omega'}\) to \(\mathcal{MF}\) are self-homeomorphisms of \(\mathcal{MF}\) and \(h_\omega = h_{\omega'}^{-1}\) on \(\mathcal{MF}\).

Take \(F \in \mathcal{N}(h_\omega(G)) \cap \mathcal{MF}\). Since \(i(h_\omega \circ h_{\omega'}(F), h_\omega(G)) = i(F, h_\omega(G)) = 0\), we have \(i(h_{\omega'}(F), G) = 0\) and \(i(h_\omega(F), G) = 0\). Therefore, from Proposition 9.2, we obtain \(F \in \mathcal{N}(h_\omega(G)) \cap \mathcal{MF}\) and
\[
\mathcal{N}(h_\omega(G)) \cap \mathcal{MF} \subset \mathcal{N}(h_\omega(G)) \cap \mathcal{MF},
\]
and we are done. \(\square\)

9.3. Proof of Theorem 2. From Proposition 9.3, it suffices to check the assertion (2) in the theorem.

We identify \(\alpha \in \mathcal{S}\) as an element of \(\tilde{\mathcal{GM}}\) by (2.2). Then, by Proposition 9.3, \(h_\omega(\alpha) \in \mathcal{MF}\). Notice that \(\mathcal{N}(\alpha) \cap \mathcal{MF}\) is a subset of codimension one in \(\mathcal{MF}\). By Proposition 9.4, so is \(\mathcal{N}(h_\omega(\alpha)) \cap \mathcal{MF}\) since \(h_\omega\) is a self-homeomorphism of \(\mathcal{MF}\). Since the complex dimension of \(\mathcal{T}_{g,m}\) is at least 2, by virtue of Theorem 4.1 in [14], we deduce that \(h_\omega(\alpha) \in \mathbb{R}_+ \otimes \mathcal{S}\). By applying the same argument to the quasi-inverse \(\omega'\), we conclude that the action of \(\omega\) on \(\mathcal{PMF}\) preserves \(\mathcal{S}\). Namely, \(\omega\) is a bijection from \(\mathcal{S}\) onto \(\mathcal{S}\).

Let \(\alpha, \beta \in \mathcal{S}\) with \(i(\alpha, \beta) = 0\). Then, \(\beta \in \mathcal{N}(\alpha) \cap \mathcal{MF}\). By the argument above, \(h_\omega(\beta) \in \mathcal{N}(h_\omega(\alpha)) \cap \mathcal{MF}\) and hence \(i(h_\omega(\alpha), h_\omega(\beta)) = 0\). This means that \(\omega: \mathcal{S} \to \mathcal{S}\) induces an automorphism of the complex of curves of \(X\). \(\square\)

9.4. Proof of Corollary 2. The purpose of this section is to prove Corollary 2 by applying Theorem 2. It is known that any isometry of \((\mathcal{T}_{g,m}, d_T)\) extends to \(\partial_\text{GM} \mathcal{T}_{g,m}\) as a homeomorphism (cf. [18]. See also [3], [10] and [33]).
9.4.1. **Action of the extended mapping class group.** Before proving Corollary 2, we shall recall the action of the extended mapping class group on Teichmüller space (cf. [12] and [25]).

The **extended mapping class group** $\text{Mod}^*(X)$ is defined by

$$
\text{Mod}^*(X) = \text{Diff}(X) / \text{Diff}_0(X)
$$

where $\text{Diff}(X)$ is the group of diffeomorphisms of $X$ and $\text{Diff}_0(X)$ is the normal subgroup of $\text{Diff}(X)$ consisting of diffeomorphisms which are isotopic to the identity. Here, we may choose $X$ so that it admits an antiholomorphic reflection $j_X: X \to X$.

Let $\psi \in \text{Diff}(X)$. If $\psi$ is an orientation preserving diffeomorphism, the action of the mapping class of $\psi$ is defined by

$$
\psi_\ast (Y, f) = (Y, f \circ \psi^{-1}).
$$

If $\psi$ is represented by an orientation reversing diffeomorphism, there is an orientation preserving diffeomorphism $\vartheta_\psi$ such that $\psi$ is isotopic to $\vartheta_\psi \circ j_X$. Then, the action of $\psi$ is defined by

$$
\psi_\ast (Y, f) = (Y^*, \overline{r}_Y \circ f \circ j_X \circ \overline{\vartheta}_\psi^{-1}),
$$

where $Y^*$ is the conjugate Riemann surface to $Y$, that is, the coordinate charts of $Y^*$ are those of $Y$ followed by complex conjugations, and $\overline{r}_Y : Y \to Y^*$ is the anticonformal mapping induced by the identity mapping on the underlying surface of $Y$.

The following is well-known. However, we give a proof here because the author cannot find a suitable reference in the case of the action of orientation reversing diffeomorphisms.

**Lemma 9.3 (Isometry).** Any element in the extended mapping class group acts isometrically on $(T_{g,m}, d_T)$.

**Proof.** Let $\psi \in \text{Mod}^*(X)$. If $\psi$ is represented by an orientation preserving diffeomorphism, the assertion is well-known (cf. e.g [12]).

Suppose that $\psi$ is represented by an orientation reversing diffeomorphism. Let $\vartheta_\psi$ as above. From the original definition of the Teichmüller distance (2.1), we have

$$
d_T(\psi_\ast (Y_1, f_1), \psi_\ast (Y_2, f_2)) = \frac{1}{2} \log \inf_{h'} \text{K}(h')
$$

where $h'$ which runs over all quasiconformal mapping from $Y_1^*$ to $Y_2^*$ homotopic to

$$(f_2 \circ j_X \circ \vartheta_\psi) \circ (f_1 \circ j_X \circ \vartheta_\psi)^{-1} = \overline{r}_{Y_2} \circ f_2 \circ f_1^{-1} \circ \overline{r}_{Y_1}^{-1}.
$$

Since each $\overline{r}_{Y_i}$ are anticonformal, the action of $\psi_\ast$ is an isometry. \hfill $\square$

In the proof of the following lemma, we use the following simple formula: For any simple closed curve $\alpha$ on a Riemann surface $Y$,

$$
\text{Ext}_{Y^*}(\overline{r}_Y (\alpha)) = \text{Ext}_Y (\alpha).
$$

Indeed, the modulus of an annulus does not change under taking the complex conjugation (cf. (2.4)).

**Lemma 9.4 (Action at the boundary).** For $\psi \in \text{Mod}^*(X)$, the restriction of the action of $\psi$ to $PMF \subset \partial_{GM} T_{g,m}$ coincides with the canonical action of $\psi$ on $PMF$, that is, the continuous extension of the action $S \ni \alpha \mapsto \psi(\alpha) \in S$.
Proof. Let \( \psi \in \text{Mod}^*(X) \). We only check the case where \( \psi \) corresponds to an orientation reversing diffeomorphism. The other case can be treated in a similar way (cf. e.g. Theorem 1.3 of \([28]\)).

For \( \alpha \in \mathcal{S} \), we denote by \( \mathcal{R}_{\alpha,y} : [0, \infty) \to \mathcal{T}_{g,m} \), the Teichmüller geodesic ray which emanates from \( y \) and is defined by the Jenkins-Strebel differential on \( y \) whose vertical foliation is \( \alpha \). Let \( (X_t, f_t) = \mathcal{R}_{\alpha,x_0}(t) \) for \( t \geq 0 \). Let \( p_{\infty} \in \partial_{GM} \mathcal{T}_{g,m} \) be the limit of the Teichmüller geodesic ray \( t \mapsto \psi_\ast(\mathcal{R}_{\alpha,x_0}(t)) \).

Take \( \beta \in \mathcal{S} \) with \( i(\alpha, \beta) = 0 \). From the proof of Theorem 5.1 of \([9]\),

\[
\text{Ext}_{X_t}(f_t(\beta)) = \text{Ext}_{\mathcal{R}_{\alpha,x_0}(t)}(\beta) = O(1)
\]
as \( t \to \infty \) (see also \([15]\)). Take \( \vartheta_\psi \) as above. Since \( \vartheta_\psi \circ j_X \) is isotopic to \( \psi \),

\[
\text{Ext}_{\vartheta_\psi(\mathcal{R}_{\alpha,x_0}(t))}(\psi(\beta)) = \text{Ext}_{X_t}(\vartheta_\psi \circ f_t \circ j_X \circ \vartheta_\psi^{-1}(\psi(\beta)))
\]
\[
= \text{Ext}_{X_t}(\vartheta_\psi \circ f_t(\beta)) = \text{Ext}_{X_t}(f_t(\beta)) = O(1)
\]
as \( t \to \infty \) (cf. \([6,25]\)). This means that the corresponding function \( \mathcal{E}_{p_\infty} \) at the limit \( p_\infty \) satisfies

\[
\mathcal{E}_{p_\infty}(\beta') = \lim_{t \to \infty} \mathcal{E}_{\vartheta_\psi(\mathcal{R}_{\alpha,x_0}(t))}(\beta') = \lim_{t \to \infty} e^{-\vartheta_\psi \circ f_t \circ j_X \circ \vartheta_\psi^{-1}(\psi(\beta))) = 0}
\]

for all \( \beta' \in \mathcal{S} \) with \( i(\alpha, \beta') = 0 \). Since the set \( \{ t\beta' \in \mathbb{R}_+ \otimes \mathcal{S} \mid i(\psi(\alpha), \beta') = 0 \} \) is dense in \( \mathcal{N}(\psi(\alpha)) \cap \mathcal{MF} \), by Lemma \([9]\) the limit \( p_\infty \) is equal to the projective class of \( \psi(\alpha) \). \( \square \)

9.4.2. Proof of Corollary \([4]\). Let \( \omega \) be an isometry of \( \mathcal{T}_{g,m} \). Then, \( \omega \) extends homeomorphically to \( \text{cl}_{GM}(\mathcal{T}_{g,m}) \) (cf. \([18]\)). We denote by the same symbol \( \omega \) the extension. By Theorem \([6]\) and Theorems by Ivanov, Korkmaz and Luo in \([13,16]\), there is a diffeomorphism \( h \) on \( X \) which induces the action of the complex of curves above. By Lemma \([9]\) \( h \) acts on \( \mathcal{T}_{g,m} \) isometrically and the action extends on \( \text{cl}_{GM}(\mathcal{T}_{g,m}) \). We denote by \( \varpi \) the action of \( h \) to \( \text{cl}_{GM}(\mathcal{T}_{g,m}) \). Let \( \varpi = \omega \circ h^{-1} \).

By Lemma \([9]\) \( \varpi \) acts on \( \mathcal{T}_{g,m} \) isometrically and coincides with the identity on \( \mathcal{PMF} \subset \partial_{GM} \mathcal{T}_{g,m} \).

The following argument is impressed with the proof of Theorem A in \([14]\). However, our situation is different from that in Ivanov’s proof as we mentioned in \([12,25]\). For completeness, we proceed to prove the theorem.

Claim 9.1. \( \varpi \) has a fixed point in \( \mathcal{T}_{g,m} \).

Proof. Take \( \alpha, \beta \in \mathcal{S} \) which fill up \( X \). Consider a holomorphic quadratic differential \( q \) whose horizontal and vertical foliations are \( \alpha \) and \( \beta \) respectively (cf. \([11]\)). Consider the Teichmüller disk \( \varphi : \mathbb{D} \to \mathcal{T}_{g,m} \) corresponding to the quadratic differential \( q \). It is well-known that the Teichmüller disk \( \varphi \) is invariant under the action of a pseudo-Anosov mapping \( \tau_\alpha \circ \tau_\beta^{-1} \) where \( \tau_\alpha \) and \( \tau_\beta \) are Dehn-twists along \( \alpha \) and \( \beta \), respectively (cf. \([35]\)). Let \( \mu_1 \) and \( \mu_2 \) be the stable and unstable foliations of the pseudo-Anosov mapping. For simplifying of the notation, we set \( \{ \lambda_i \}_{i=1}^4 = \{ \alpha, \beta, \mu_1, \mu_2 \} \), where the equality holds as unordered sets. Let \( \theta_i \in \partial \mathbb{D} \) be the corresponding point to \( \lambda_i \) via \( \varphi \). This means that the radial ray of direction \( \theta_i \) terminates at the projective class of \( \lambda_i \in \partial_{GM} \mathcal{T}_{g,m} \) (cf. \([29]\)). See also Theorem 5.1 of \([9]\) and Lemma \([3,1] \). We may assume that \( \theta_i \) lies on \( \partial \mathbb{D} \) counterclockwise. For \( i = 1, 2 \), let \( g_i \) be the hyperbolic geodesic connecting \( \theta_i \) and \( \theta_{i+2} \) in \( \mathbb{D} \). Then,
Proof. As in the previous section, for \( \lambda \) the Teichmüller geodesic which terminates at the projective classes of \( g \) by Proposition 1.3 in [7], \( T \) of the complex of curves. Hence, by Ivanov-Korkmaz-Luo’s theorem, Comments on the exceptional cases.

9.5. It is known that the canonical homomorphism from the extended mapping class group \( \text{Mod}^*(X) \) is isometric to the extended mapping class group \( \text{Mod}^*(X_{g,m}) \) since \( \text{Mod}^*(X_{g,m}) \) is isometric to the isometry group \( \text{Isom}(T_{g,m}, d_T) \) of \( (T_{g,m}, d_T) \). From Lemma 9.3 there is a natural homomorphism

\[
\text{Mod}^*(X) \ni h \mapsto h_* \in \text{Isom}(T_{g,m}, d_T). \tag{9.7}
\]

From Claim 9.2 the homomorphism (9.7) is surjective. Let \( h \in \text{Mod}^*(X) \) and assume that \( h_* = \text{id} \) on \( T_{g,m} \). Then, from Lemma 9.3 the extension of \( h_* \) to \( \partial \text{GM} T_{g,m} \) fixes \( S \) pointwise. From Theorem 2 \( h_* \) induces the identity automorphism of the complex of curves. Hence, by Ivanov-Korkmaz-Luo’s theorem, \( h \) should be the identity from the topological assumption of \( X \). □

9.5. Comments on the exceptional cases. Suppose first that \( (g, m) = (1, 2) \). It is known that the canonical homomorphism from the extended mapping class group on \( X_{1,2} \) to the isometry group is neither injective nor surjective. Indeed, by Proposition 1.3 in [7], \( T_{1,2} \) admits a biholomorphic mapping to the Teichmüller space \( \mathcal{T}_{0,5} \) of a sphere \( X_{0,5} \) with five punctures which is induced by the quotient mapping \( X_{1,2} \to X_{0,5} \) of the action of the hyperelliptic involution (double branched points are considered as punctures). Hence, from Corollary 2 the isometry group of \( T_{1,2} \) is isometric to the extended mapping class group \( \text{Mod}^*(X_{0,5}) \) of \( X_{0,5} \) since the Teichmüller distance coincides with the Kobayashi distance. Therefore, the canonical homomorphism from the extended mapping class group \( \text{Mod}^*(X_{1,2}) \) to the isometry group of \( T_{1,2} \) is not surjective (cf. Corollary 3 in §4.3 of [7]). By a theorem due (independently) to Birman and Viro, the hyperelliptic involution of \( X_{1,2} \) fixes every non-trivial and non-peripheral simple closed curves on \( X_{1,2} \) (cf.
Figure 3. The metric space $X$.

Hence, the hyperelliptic involution acts trivially on $T_{1,2}$ and the canonical homomorphism is not injective (cf. [19]).

When $(g,m) = (2,0)$, any automorphism of the complex of curves induces a homeomorphism on $X_{2,0}$. However, the hyperelliptic involution fixes every non-trivial simple closed curves on $X_{2,0}$ and hence the action of the extended mapping class group is not faithful (cf. e.g. [19]). In fact, it is known that the hyperelliptic involution generates the kernel of the canonical homomorphism (e.g. [19]).

9.6. Comments on the characterization of biholomorphisms. The problem of characterizing isometries and biholomorphisms makes sense for Teichmüller spaces of arbitrary Riemann surfaces. In the case where the Teichmüller space is of infinite dimension, Earle and Gardiner [6] obtained the characterization for Riemann surfaces of topologically finite type. In [17], N. Lakic obtained the characterization for Riemann surfaces of finite genus. Finally, in [20], Markovic settled the characterization for biholomorphisms of Teichmüller space of arbitrary Riemann surfaces.

10. Appendix: A proper geodesic metric space without extendable Gromov product

This section is devoted to giving a geodesic metric space on which the Gromov product does not extend to the horofunction boundary. The following example is given by Cormac Walsh (cf. [37]). Notice that the Gardiner-Masur compactification coincides with the horofunction compactification with respect to the Teichmüller distance (cf. [18]).

Let $C_n$ be the frame $\partial([-n,n] \times [0,n])$ with the standard Euclidean metric. We construct a space $X$ by gluing each frame $C_n$ to $\mathbb{R}$ along the bottom edge $[-n,n] \times \{0\}$ of $C_n$ and the interval $[-n,n]$ of $\mathbb{R}$ isometrically. The space $X$ is a proper geodesic space (cf. Figure 3). Let $b_0$, $x^1_n, y^1_n, x^2_n$ and $y^2_n$ be points in $X$ corresponding to $0 \in \mathbb{R}$, $(-n,0)$, $(-n,n)$, $(n,0)$ and $(n,n)$ in $C_n$ respectively. We consider $b_0$ as the basepoint of $X$. Then, one can see that for $i = 1, 2$, $\{x^i_n\}_n$ and $\{y^i_n\}_n$ converges to the same Busemann point in the horofunction boundary of $X$ though $\{y^i_n\}_n$ is not an almost geodesic (cf. [33]). On the other hand, we see

$$\lim_{n \to \infty} \langle y^1_n | y^2_n \rangle_{b_0} = \lim_{n \to \infty} \frac{1}{2}(2n + 2n - 2n) = \infty$$

while $\langle x^1_n | x^2_n \rangle_{b_0} = (n + n - 2n)/2 = 0$ for all $n$. 
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