An Adaptive Observer for Uncertain Linear Time-Varying Systems with Unknown Additive Perturbations

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Abstract

In this paper we are interested in the problem of adaptive state observation of linear time-varying (LTV) systems where the system and the input matrices depend on unknown time-varying parameters. It is assumed that these parameters satisfy some known LTV dynamics, but with unknown initial conditions. Moreover, the state equation is perturbed by an additive signal generated from an exosystem with uncertain constant parameters. Our main contribution is to propose a globally convergent state observer that requires only a weak excitation assumption on the system.

1 Introduction

In view of its wide practical application the problem of disturbance cancellation, has been extensively studied in the literature. The interested reader is referred to [32] for a recent, comprehensive review of the literature. A scenario that is now universally adopted to tackle this problem is the output regulation paradigm—formally articulated in [9, Chapter 4]—which treats in a unified framework the problems of output tracking and disturbance rejection. In this scenario it is assumed that the disturbance to be rejected (or the output to be tracked) is generated by an exosystem that is an autonomous dynamical system. It is assumed that only part of the state is available for measurement, which constitutes the output signal. Consequently, a first task to be solved in the output regulation problem is the one of state observation.

In this paper we adopt the mathematical formulation of output regulation addressing the problem of adaptive state observation of LTV systems where the system and the input matrices depend on unknown time-varying parameters and the state equation is perturbed by an additive signal generated from an exosystem with uncertain constant parameters. More precisely, in the paper we assume the following conditions.

C1 The system is LTV with unknown time-varying parameters.

C2 These parameters satisfy an autonomous LTV system with known dynamics but unknown initial conditions.

1See [7] for an extensive survey on the problem of state observation of LTV systems.
The measurable output is only one of the components of the state vector.

The disturbance to be rejected is generated by an exosystem whose system matrix depends on a vector of unknown constant parameters.

The objective is to design an adaptive observer that will reconstruct the system state and all the unknown parameters.

The problem formulation adopted in the paper is very similar to the estimation part of the work [32]. However, it is important to underscore that some critical assumptions [32, Assumptions 1-4] are avoided in the present paper. Specifically, in [32] it is assumed that

- the plant is minimum-phase;
- the sign of the high-frequency gain is known;
- the relative degree of the plant and upperbound of the plant order are known;
- the disturbance is the sum of a known amount of harmonics.

In the present work we are able to work with nonminimum-phase system. The high-frequency gain may be unknown. We assume only that the plant order is given, relative degree may be uncertain. Disturbance in the present work is arbitrary for a class of LTI exogenous systems. The present work constitutes a non-trivial extension of our previous research, e.g. [24, 25, 26], where we addressed the problems of disturbance estimation and cancellation for a plant with input delay and partial measurement of the state. But in mentioned works we considered linear time invariant systems with known constant parameters and all uncertainties were localized in the disturbance. In this paper we consider the system with unknown time-varying parameters and uncertain disturbance.

To tackle the problem at hand we propose in this paper to use a new procedure to design state observers for state-affine systems, called parameter estimation-based observers (PEBO), first reported in [16]. The main novelty of PEBO is that the state observation problem is reformulated as a problem of parameter estimation of a regressor equation. A drawback of the PEBO proposed in [16] is that it involves a non-robust open-loop integration. This shortcoming was later removed in [19] with the definition of the generalized (G)PEBO, where the properties of the principal matrix solution of an unforced LTV system \( \dot{x}(t) = A(t)x(t) \) are exploited. This novel technique has been successfully applied to reaction systems [20], power systems [14], systems with delayed measurements [2] and distributed state estimation [21]—see also [23] for a related adaptive observer design for a class of nonlinear systems.

In this paper we use the GPEBO technique to derive the regressor equation needed for the estimation of the state and the system parameters—including the parameters of the aforementioned exosystem. To estimate these parameters we use the dynamic regressor extension and mixing (DREM) parameter adaptation algorithm introduced in [1] and later further elaborated in [17].

One central feature of DREM is that it generates, out of a \( q \)-dimensional regression equation, \( q \) scalar regression equations. This property is essential in our problem since, as we will show below, the regressor equation associated to this problem is nonlinear and it consists of a linear and a nonlinear part, and we need to invoke DREM to concentrate our attention on the linear part of the regression equation.

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2It should be clarified that in [32], besides the estimation of the disturbance, a step of control via adaptive backstepping is added, while in this paper we restrict ourselves to the observation part.

3The interested readers are referred to the aforementioned papers for further details on GPEBO and DREM.
Combining GPEBO and DREM has already been explored by the authors using gradient estimators in [4, 11]. An important contribution of the present work is that we propose to use—instead of the gradient scheme—a new least-squares (LS)-based algorithm, referred in sequel as [LS+DREM] estimator. The use of LS advocated in the paper removes the need to calculate a, computationally demanding state-transition matrix in the estimator. Moreover, the superior convergence properties of LS estimators, as opposed to gradient-based, are widely recognized [13, 28]. In the paper we show that the parameters of the regressor equation can be—globally and exponentially—estimated under the weak assumption of interval excitation (IE) [12, 31] of the regressor. This should be contrasted with the highly demanding persistency of excitation required in classical (non-DREM-based) gradient and LS estimators [8, 30].

The remainder of the paper is organized as follows. In Section 2 we present the problem formulation. In Section 3 we derive the regression equation used in the estimator, which is presented in Appendix B. The main result is given in Section 4. In Section 5 we summarize all the equations needed for the implementation of the adaptive observer. We wrap up the paper with concluding remarks in Section 6. In Appendix A we give some preliminary Lemmata instrumental for the proof of our main results.

Notation. Given $n \in \mathbb{Z}_+, q \in \mathbb{Z}_+, I_n$ is the $n \times n$ identity matrix and $0_{n \times q}$ is an $n \times q$ matrix of zeros. For $x \in \mathbb{R}^n$, we denote the square of the Euclidean norm as $|x|^2 := x^\top x$. $e_q \in \mathbb{R}^n$ denotes the $q$-th vector of the $n$-dimensional Euclidean basis. Given a smooth signal $z : \mathbb{R}_+ \to \mathbb{R}$ we denote $[z(t)]^{(i)} = \frac{d^i z(t)}{dt^i}$, for $i \in \mathbb{Z}_+$.

## 2 Problem Formulation

In this paper we are interested in the problem of adaptive state observation of the following uncertain, additively perturbed, LTV system

$$
\begin{align*}
\dot{x}(t) &= [A + \theta(t)e_1^\top]x(t) + B(t)u(t) + e_n \delta(t), \\
y(t) &= e_1^\top x(t),
\end{align*}
$$

with state $x(t) \in \mathbb{R}^n$, input $u(t) \in \mathbb{R}$ and measurable output $y(t) \in \mathbb{R}$, where

$$
A := \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \\ 0_{1 \times n} & 1 \\ \end{bmatrix}.
$$

It is assumed that the unknown time-varying vectors $\theta(t) \in \mathbb{R}^n$ and $B(t) \in \mathbb{R}^n$ satisfy the LTV dynamics

$$
\begin{align*}
\dot{x}_\theta(t) &= A_\theta(t)x_\theta(t), \\
\dot{x}_B(t) &= A_B(t)x_B(t), \\
\theta(t) &= h_\theta x_\theta(t), \\
B(t) &= h_B x_B(t),
\end{align*}
$$

with $x_\theta(t) \in \mathbb{R}^{n_\theta}$, $x_B(t) \in \mathbb{R}^{n_B}$, and the matrices $h_\theta \in \mathbb{R}^{n \times n_\theta}$, $h_B \in \mathbb{R}^{n \times n_B}$, $A_\theta(t) \in \mathbb{R}^{n_\theta \times n_\theta}$, $A_B(t) \in \mathbb{R}^{n_B \times n_B}$ are known. On the other hand, the disturbance $\delta(t) \in \mathbb{R}$ is the output of the exogenous system

$$
\begin{align*}
\dot{w}(t) &= S(\rho)w(t) \\
\delta(t) &= h_B^\top w(t),
\end{align*}
$$

where $S(\rho)$ is a symmetrical matrix, $\rho > 0$ and $h_B$ is a vector.
with \( w(t) \in \mathbb{R}^n_w \). It is assumed that the mapping \( S : \mathbb{R}^{n_\rho} \rightarrow \mathbb{R}^{n_w \times n_w} \) is known, but the vectors of constant parameters \( \rho \in \mathbb{R}^{n_\rho} \) and \( h_\delta \in \mathbb{R}^{n_w} \) are unknown. Furthermore, without loss of generality [9], it is assumed that all the eigenvalues of \( S(\rho) \) have nonnegative real part.

The objective is to reconstruct asymptotically (or in finite time) the vectors \( \theta(t) \), \( B(t) \), \( \rho \), and the state \( x(t) \).

**Remark 1.** It should be underscored that the initial conditions of all the differential equations that describe the systems dynamics are unknown.

### 3 Reducing the Observation Problem to Estimation of Parameters

In this section we apply the GPEBO procedure [19] to translate the problem of state observation to one of parameter estimation that will be combined with the task of estimation of the system parameters, to develop an adaptive observer. As will be shown below, the regression equation that should be used for parameter estimation is nonlinear in the parameters. Indeed, it consists of the sum of a “classical” linear regression equation (LRE) with all the unknown constant parameters and a term which depends nonlinearly on these parameters. As it has been shown in [18], thanks to the use of the DREM technique— that generates scalar LREs for separable nonlinearly parameterized regressions—it is possible to use for the parameter estimation only the first, linear part of the regression equation and disregard the nonlinear part of it.

**Proposition 1.** There exists measurable signals \( z(t) \in \mathbb{R}^n \), \( \Omega_L(t) \in \mathbb{R}^s \), \( \Omega_N(t) \in \mathbb{R}^{n_s} \), \( Y(t) \in \mathbb{R} \), two known mappings \( \mathcal{W} : \mathbb{R}_+ \times \mathbb{R}^s \rightarrow \mathbb{R}^n \), \( \mathcal{K} : \mathbb{R}^s \rightarrow \mathbb{R}^{n_s} \) and a vector of constant parameters \( \theta \in \mathbb{R}^s \) such that the state of the system (1) can be written as

\[
x(t) = z(t) + \mathcal{W}(t, \theta) + \epsilon_x(t)
\]

where \( \theta \) satisfies the separable, nonlinearly parameterized regression equation

\[
Y(t) = \Omega_L^T(t)\theta + \Omega_N^T(t)\mathcal{K}(\theta) + \epsilon(t)
\]

with \( \epsilon_x(t) \in \mathbb{R}^n \) and \( \epsilon(t) \in \mathbb{R} \) exponentially decaying to zero and \( s := n + n_\theta + n_B \).

**Proof.** Following the GPEBO construction we invoke [29, Property 4.4] to rewrite the vectors \( x_\theta(t) \) and \( x_B(t) \) of (2) as

\[
x_\theta(t) = \Phi_\theta(t)x_\theta(0),
\]

\[
x_B(t) = \Phi_B(t)x_B(0),
\]

\( x_\theta(0) = x_\theta(0) \) and \( x_B(0) = x_B(0) \), and \( \Phi_\theta(t) \), \( \Phi_B(t) \) are the principal matrix solutions of the LTV systems (2), which are the solutions of the auxiliary filters

\[
\dot{\Phi}_\theta(t) = A_\theta(t)\Phi_\theta(t), \quad \Phi_\theta(0) = I_{n_\theta},
\]

\[
\dot{\Phi}_B(t) = A_B(t)\Phi_B(t), \quad \Phi_B(0) = I_{n_B}.
\]

Thus, the vectors \( \theta(t) \) and \( B(t) \) may be rewritten as

\[
\theta(t) = h_\theta\Phi_\theta(t)x_\theta(0)
\]

\[
B(t) = h_B\Phi_B(t)x_B(0).
\]
Choose a vector $K \in \mathbb{R}^n$ such that the matrix
\[
A_K := \begin{bmatrix}
-K & I_{n-1} \\
0_{1 \times n-1} & 0
\end{bmatrix}.
\] (7)
is Hurwitz, and define the set of filters
\[
\dot{z}(t) = A_K z(t) + Ky(t),
\]
\[
\dot{\Omega}(t) = A_K \Omega(t) + h_\theta \Phi_\theta(t)y(t),
\]
\[
\dot{P}(t) = A_K P(t) + h_B \Phi_B(t)u(t),
\]
\[
\] (8)
with states $z(t) \in \mathbb{R}^n$, $\Omega(t) \in \mathbb{R}^{n \times n_\theta}$ and $P(t) \in \mathbb{R}^{n \times n_B}$.

Define the auxiliary variable
\[
e(t) := x(t) - z(t) - \Omega(t) x_{\theta_0} - P(t) x_{B_0}
\] (9)
whose derivative is given by\(^4\)
\[
\dot{e}(t) = Ax + \theta y + Bu + e_n \delta - [A_K z + K y] - [A_K \Omega + h_\theta \Phi_\theta y] x_{\theta_0} - [A_K P + h_B \Phi_B u] x_{B_0},
\]
\[
= Ax + h_\theta \Phi_\theta(t) x_{\theta_0} y + h_B \Phi_B(t) x_{B_0} u + e_n \delta - [A_K z + K y] - [A_K \Omega + h_\theta \Phi_\theta y] x_{\theta_0}
\]
\[
- [A_K P + h_B \Phi_B u] x_{B_0},
\]
\[
= Ax + e_n \delta - [A_K z - K e_1^\top x] - A_K \Omega x_{\theta_0} - A_K P x_{B_0},
\]
\[
= A_K (x - z - \Omega x_{\theta_0} - P x_{B_0}) + e_n \delta,
\]
consequently
\[
\dot{e}(t) = A_K e(t) + e_n h_\delta^\top w(t).
\] (10)
Combining the equation above with (3) we get
\[
\dot{w}(t) = S(\rho) w(t)
\]
\[
\dot{e}(t) = A_K e(t) + e_n h_\delta^\top w(t).
\]
We apply now to this cascaded system the transformation discussed in Lemma 1 of Appendix A. Towards this end, consider the Sylvester equation
\[
\Pi(\rho) S(\rho) = A_K \Pi(\rho) + e_n h_\delta^\top,
\]
that, given the fact that the spectra of $S(\rho)$ and $A_K$ are disjoint, has a unique solution. We underscore the fact that that $\Pi(\rho)$ depends on the unknown vector $\rho$, hence it is unknown. Defining the signal
\[
e(t) := e(t) - \Pi(\rho) w(t),
\]
we get the decoupled dynamics
\[
\dot{w}(t) = S(\rho) w(t)
\]
\[
\dot{e}(t) = A_K e(t).
\] (11)
\(^4\)To simplify the notation, whenever clear from the context, the time time argument will be omitted.
Hence
\[ e(t) = \Pi(\rho)w(t) + \epsilon(t) \]
in which the second term is exponentially decaying. Multiplying the equation above by \( e_1^\top \) we get
\[
e_1^\top e(t) = e_1^\top [\Pi(\rho)w(t) + \epsilon(t)] = e_1^\top [x(t) - z(t) - \Omega(t)x_{\theta_0} - P(t)x_{B_0}],
\]
where we have used (9) to get the second identity. Defining the measurable signals
\[
\zeta(t) := y(t) - e_1^\top z(t), \\
\varphi(t) := \begin{bmatrix} \Omega_\top(t)e_1 \\ P_\top(t)e_1 \end{bmatrix},
\]
we obtain the first key relationship
\[
\zeta(t) = \varphi_\top(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} + e_1^\top \Pi(\rho)w(t) + e_1^\top \epsilon(t). \tag{13}
\]

In the next step we will reparameterize (13) in a different way to represent the unknown term \( \Pi(\rho)w(t) \) in a bona fide regression equation form. Towards this end, consider the auxiliary (not-realizable) filter with the state \( \Psi(t) \in \mathbb{R}^{n_w} \):
\[
\dot{\Psi}(t) = A_f \Psi(t) + e_n \begin{bmatrix} e_1^\top \Pi(\rho)w(t) + e_1^\top \epsilon(t) \end{bmatrix}, \tag{14}
\]
where \( A_f \in \mathbb{R}^{n_w \times n_w} \) is given as
\[
A_f := \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \end{bmatrix}, \tag{15}
\]
for some constant vector \( f \in \mathbb{R}^n \) such that \( A_f \) is a Hurwitz matrix. Using (13) the system (14) may be rewritten as
\[
\dot{\Psi}(t) = A_f \Psi(t) + e_n \begin{bmatrix} \zeta(t) - \varphi_\top(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} \end{bmatrix}. \tag{16}
\]
On the other hand, invoking Lemma 2 of Appendix A, the system (14) may be rewritten as
\[
\dot{\Psi}(t) = A_\Gamma \Psi(t) + e_n \epsilon(t). \tag{17}
\]
where \( \epsilon(t) \) is exponentially decaying and \( A_\Gamma \) is given as
\[
A_\Gamma := \begin{bmatrix} 0_{(n-1) \times 1} & I_{n-1} \end{bmatrix}, \tag{18}
\]
for some constant vector \( \Gamma \in \mathbb{R}^n \) such that the spectra of \( A_\Gamma \) and \( S \) coincide. Comparing (16) and (17), and taking into account the special form of matrices the \( A_f \) and \( A_\Gamma \), one can deduce that the last rows in the right hand sides of (16) and (17) are equal, that is,
\[
f_\top \Psi(t) + \zeta(t) - \varphi_\top(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} = \Gamma_\top \Psi(t) + \epsilon(t). \tag{19}
\]
To remove the unknown vector $\Gamma$ from the identity above we introduce the realizable filters with states $L \in \mathbb{R}^{nw}$ and $Q \in \mathbb{R}^{n \times (n_\theta + n_B)}$,

$$\dot{L}(t) = A_f L(t) + e_n \zeta(t),$$
$$\dot{Q}(t) = A_f Q(t) + e_n \varphi^\top(t)$$

(20)

and define the signal

$$E(t) := \Psi(t) - L(t) + Q(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix},$$

(21)

which, in fact, is exponentially decaying since it satisfies

$$\dot{E}(t) = A_f \Psi + e_n \zeta - e_n \varphi^\top \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} - A_f L - e_n \zeta + A_f Q \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} + e_n \varphi^\top \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix}$$

$$= A_f \left[ \Psi - L + Q \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} \right]$$

$$= A_f E(t).$$

From (21) we have

$$\Psi(t) = L(t) - Q(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} + E(t),$$

(22)

Substitution of (22) into (19) yields the second key relationship

$$(f - \Gamma)^\top \left[ L(t) - Q(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} + E(t) \right] + \zeta(t) - \varphi^\top(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} = \varepsilon(t),$$

(23)

which one can rewrite in the nonlinear regression form (5) with the definitions

$$Y(t) := \zeta(t) + L^\top(t)f,$$
$$\Omega_L(t) := \begin{bmatrix} Q^\top(t)f + \varphi(t) \\ L(t) \end{bmatrix},$$

(24)

with the unknown parameter vector

$$\theta := \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \\ \Gamma \end{bmatrix},$$

(25)

for the linear term and

$$\Omega_N^\top(t)K(\theta) := -\Gamma^\top(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix}.$$  

for the quadratic one, which can be clearly written as a separable NLPR via

$$\Gamma^\top(t)Q(t) \begin{bmatrix} x_{\theta_0} \\ x_{B_0} \end{bmatrix} = \sum_{i=1}^{n} \sum_{j=1}^{n_\theta + n_B} Q_{ij}(t) \theta_i \theta_j.$$  

(26)
To complete the proof we need prove the relation (4). Towards this end, we recall the dynamical model of $e(t)$ (10) and the key relation (13), that we repeat here for ease of reference as

$$
\dot{e}(t) = A_K e(t) + e_n h_\delta w(t),
$$
$$
e_1^\top e(t) = \zeta(t) - \varphi^\top(t) \begin{bmatrix} x_{g0} \\ x_{B0} \end{bmatrix},
$$

where $A_K$ is given in (7). Then, using the fact that $e_1^\top A^i_K e_n \equiv 0$ for all $i = 0, \ldots, n - 2$, we can compute

$$
e_1(t) = e_1^\top e(t),
$$
$$
\dot{e}_1(t) = e_1^\top \dot{e}(t) = e_1^\top A_K e(t) + e_1^\top e_n h_\delta^\top w(t) = e_1^\top A_K e(t),
$$
$$
\dot{e}_2(t) = e_1^\top \dot{e}(t) = e_1^\top A_K^2 e + e_1^\top e_n h_\delta^\top w(t) = e_1^\top A_K^2 e(t),
$$
$$
\vdots
$$
$$
e_1^{(n-1)}(t) = e_1^\top e^{(n-1)}(t) = e_1^\top A_K^{n-1} e(t) + e_1^\top A_K^{n-2} e_n h_\delta^\top w(t) = e_1^\top A_K^{n-1} e(t),
$$

that we write compactly as

$$
\begin{bmatrix}
e_1^\top e(t) \\
e_1^\top \dot{e}(t) \\
\vdots \\
e_1^\top e^{(n-1)}(t)
\end{bmatrix} = \begin{bmatrix}
e_1^\top \\
e_1^\top A_K \\
\vdots \\
e_1^\top A_K^{n-1}
\end{bmatrix} e(t)
$$

(27)

In the sequel we will express the left hand side vector of (27) in an alternative form. For, we rearrange (19) as

$$
\zeta(t) - \varphi^\top(t) \begin{bmatrix} x_{g0} \\ x_{B0} \end{bmatrix} = (\Gamma - f)^\top \Psi(t) + \varepsilon(t),
$$

which replaced in (13) yields

$$
e_1^\top e(t) = (\Gamma - f)^\top \Psi(t) + \varepsilon(t).
$$

(28)

Now, the dynamics of $\Psi$ and $\varepsilon$ are defined in (17) and (38), (39), respectively. For ease of reference we repeat them here as

$$
\dot{\Psi}(t) = A_\Gamma \Psi(t) + e_n \varepsilon(t),
$$
$$
\dot{\xi}(t) = F_c \xi(t).
$$
$$
\varepsilon(t) = h_c \xi(t).
$$

Using these equations and (28) we can compute the left hand side vector of (27) as

$$
\begin{bmatrix}
e_1^\top e(t) \\
e_1^\top \dot{e}(t) \\
\vdots \\
e_1^\top e^{(n-1)}(t)
\end{bmatrix} = \begin{bmatrix}
(\Gamma - f)^\top \Psi(t) + \varepsilon(t) \\
(\Gamma - f)^\top \dot{\Psi}(t) + \dot{\varepsilon}(t) \\
\vdots \\
(\Gamma - f)^\top \Psi^{(n-1)}(t) + \varepsilon^{(n-1)}(t)
\end{bmatrix} = \begin{bmatrix}
(\Gamma - f)^\top \\
(\Gamma - f)^\top A_\Gamma \\
\vdots \\
(\Gamma - f)^\top A_\Gamma^{n-1}
\end{bmatrix} \Psi(t) + \dot{\varepsilon}(t)
$$

(29)
where $\bar{e}(t)$ is a suitably defined exponentially decaying signal. Equating the right hand sides of (27) and (29) yields the identity

$$
\begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix} e(t) =
\begin{bmatrix}
    (\Gamma - f)^T \\
    (\Gamma - f)^T A_{\Gamma} \\
    \vdots \\
    (\Gamma - f)^T A_{\Gamma}^{n-1}
\end{bmatrix} \Psi(t) + \bar{e}(t),
$$

(30)

where we notice that the matrix premultiplying $e(t)$ is full rank. Using the latter fact, recalling (9) and replacing it in (30) finally yields

$$
x(t) =
\begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix}^{-1}
\begin{bmatrix}
    (\Gamma - f)^T \\
    (\Gamma - f)^T A_{\Gamma} \\
    \vdots \\
    (\Gamma - f)^T A_{\Gamma}^{n-1}
\end{bmatrix} \Psi(t) +
\begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix}^{-1}
\begin{bmatrix}
    L(t) - Q(t) \begin{bmatrix} x_0 \\ x_0 \end{bmatrix} + E(t) + \begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix} \bar{e}(t).
\end{bmatrix}

$$

where have used (22) to replace $\Psi(t)$ in the second identity. The proof of the claim (4) is completed from the equation above defining

$$
\mathcal{W}(t, \theta) := \begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix}^{-1}
\begin{bmatrix}
    (\Gamma - f)^T \\
    (\Gamma - f)^T A_{\Gamma} \\
    \vdots \\
    (\Gamma - f)^T A_{\Gamma}^{n-1}
\end{bmatrix} \left\{ L(t) - [Q(t) - [\Omega(t) \ P(t)] \begin{bmatrix} x_0 \\ x_0 \end{bmatrix}, \right\},
$$

$$
\mathcal{E}_x(t) := \begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix}^{-1}
\begin{bmatrix}
    (\Gamma - f)^T \\
    (\Gamma - f)^T A_{\Gamma} \\
    \vdots \\
    (\Gamma - f)^T A_{\Gamma}^{n-1}
\end{bmatrix} E(t) + \begin{bmatrix}
    e_1^T \\
    e_1^TA_K \\
    \vdots \\
    e_1^T A_{K}^{n-1}
\end{bmatrix}^{-1} \bar{e}(t).
$$



4 Adaptive State Observer with LS+DREM Estimator

In this section we complete the design of the proposed adaptive observer combining—in the classical certainty-equivalent way—the results of Proposition 1 with the new LS+DREM estimator proposed in Appendix B. More precisely, we apply the LS+DREM estimator of Proposition 3 to the NLPRE (5) and replace the estimated parameters in the expression of the state (4) to generate the observed state.
The first step is to identify from the NLPRE (5) the function $\Omega(t)$ and the mapping $G(\theta)$ of (40). This follows trivially defining
\[
\Omega(t) := \left[ \Omega_L^T(t) \quad \Omega_N^T(t) \right] \in \mathbb{R}^{n(s+1)}
\]
and
\[
G(\theta) := \begin{bmatrix} \theta \\ K(\theta) \end{bmatrix}.
\] (31)

In this case $\Omega(t) \in \mathbb{R}^{n(s+1)}$ and $G: \mathbb{R}^s \to \mathbb{R}^{n(s+1)}$.

It is easy to see that the monotonicity Assumption A2 is trivially satisfied with the matrix
\[
Q = \begin{bmatrix} I_s & 0_{s \times ns} \end{bmatrix},
\] (32)
which selects only the linear part of the NLPRE (5).

We are in position to present the main result of the paper.

**Proposition 2.** Consider the LTV system (1) and the associated NLPRE (5). Define the certainty equivalent adaptive observer
\[
\hat{x}(t) = z(t) + \mathcal{W}(t, \hat{\theta}(t))
\]
\[
= z(t) + \begin{bmatrix} \mathbf{e}_1^T \\ \mathbf{e}_1^T A_K \\ \vdots \\ \mathbf{e}_1^T A_{K}^{n-1} \end{bmatrix}^{-1} \begin{bmatrix} (\hat{\Gamma}(t) - f)^T \\ (\hat{\Gamma}(t) - f)^T A_F(t) \\ \vdots \\ (\hat{\Gamma}(t) - f)^T A_{F}^{n-1}(t) \end{bmatrix} \left\{ L(t) - [Q(t) - [\Omega(t) \quad P(t)]] [\hat{x}_{B_0}(t)] \right\},
\] (33)
where
\[
\hat{A}_{\Gamma}(t) := \begin{bmatrix} 0_{(n-1) \times 1} \\ \hat{\Gamma}^T(t) \\ I_{n-1} \end{bmatrix},
\]
and the estimated parameters
\[
\hat{\theta}(t) = \begin{bmatrix} \hat{x}_{B_0}(t) \\ \hat{x}_{B_0}(t) \\ \hat{\Gamma}(t) \end{bmatrix}
\]
are generated with the LS+DREM interlaced algorithm (43) with the definitions (31) and (32).

Assume the vector $\Omega$ satisfies Assumption A3 of Appendix B and the system does not exhibit finite escape time. Then, for all initial conditions we have the following.

(i) All signals remain bounded.

(ii) The parameter errors satisfy (45).

(iii) The state observation error $\tilde{x}(t) := \hat{x}(t) - x(t)$ satisfies
\[
\lim_{t \to \infty} |\tilde{x}(t)| = 0.
\]

**Proof.** As shown in Proposition 1, the state $x(t)$ satisfies (up to additive exponentially decaying signals) the equation (4), hence (33) is a certainty equivalent version of it which is obtained replacing the unknown parameters by their current estimate. The proof then is a direct consequence of the exponential convergence of the parameter estimation errors (45), smoothness of all maps and the assumption that there is no finite escape time. □□□

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As usual in all adaptive designs [8, 30], without loss of generality, we disregard the presence of additive terms that exponentially converge to zero. See [8, Subsection 4.3.7] for the analysis of its effect.
5 Summary of the Adaptive Observer Design

To facilitate its practical implementation in this section we give a summary of the proposed design.

Data: \( A_\theta(t), A_B(t), h_\theta, h_B, u(t) \) and \( y(t) \).

Tuning parameters for the LRE generator: \( K \in \mathbb{R}^n, f \in \mathbb{R}^{n_\omega} \) such that \( A_K \) (7) and \( A_f \) (15) are Hurwitz.

Tuning parameters for the parameter estimator: Scalars \( \alpha > 0, f_0 > 0 \) and \( \gamma > 0 \).

State equations for the LRE generator: (6), (8), (20).

State equations for the parameter estimator: (43).

Algebraic equations for the LRE generator: (12).

Algebraic equations for the parameter estimator: (24), (26), (31).

6 Numerical example

In this section we will show an example, illustrating the procedure of parameterization of the plant model with time-varying parameters and disturbance generated by an exogeneous system.

Plant model. Consider the plant (1) with parameters (2), where

\[
A = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A_\theta = \begin{bmatrix} -0.001 & 0 \\ 0 & -0.002 \end{bmatrix}, \quad h_\theta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_\theta(0) = \begin{bmatrix} -2 \\ -1 \end{bmatrix},
\]

\[
A_B(t) = \begin{bmatrix} 0 \\ -1 + 0.1 \sin t \end{bmatrix}, \quad h_B = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad x_B(0) = \begin{bmatrix} 0.7 \\ 0.2 \end{bmatrix},
\]

the control signal \( u(t) = 10 + \sin(0.5t) \), and the disturbance (3) with parameters

\[
S(\rho) = \begin{bmatrix} 0 & 1 \\ \rho & 0 \end{bmatrix}, \quad h_\delta = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad w(0) = \begin{bmatrix} -10 \\ 1 \end{bmatrix}, \quad \rho = -1.
\]

Parameterization. We recall that the selection of the vector \( \Gamma \) is such that the spectra of \( A_\Gamma \) and \( S(\rho) \) should coincide. In this example the characteristic polynomial of \( S(\rho) \) is \( s^2 - \rho \). Therefore, \( \Gamma \) can be selected as \( \Gamma = \rho e_1 \) to comply with this requirement. This implies that only one element of \( \Gamma \) is unknown and in \( \theta \) (25) only one unknown parameter \( \rho \) should be included and it takes the form

\[
\theta = \begin{bmatrix} x_\theta(0) \\ x_B(0) \\ \rho \end{bmatrix} = \begin{bmatrix} -2 \\ -1 \\ 0.7 \\ 0.2 \\ -1 \end{bmatrix}.
\]
Regarding the nonlinearly parameterized term (26) it takes the form

\[ \Gamma^\top Q(t) \begin{bmatrix} x_{\theta_0} \\ x_B \end{bmatrix} = -e_1^\top Q(t) \begin{bmatrix} \rho x_{\theta}(0) \\ \rho x_B(0) \end{bmatrix}, \]

hence

\[ \Omega_N(t) = -e_1^\top Q(t), \quad K(\theta) = \begin{bmatrix} \rho x_{\theta}(0) \\ \rho x_B(0) \end{bmatrix} = \begin{bmatrix} 2 \\ 1 \\ -0.7 \\ -0.2 \end{bmatrix}. \]

The description of the regressor equation (5) is completed with the definitions given in (5).

**Parameter estimator and observed state** The estimated parameters \( \hat{\theta} \) are generated with the LS+DREM algorithm of Appendix B with the selection matrix \( Q = [I_5 \mid 0_{5 \times 4}] \), which is chosen to retain only the linear part of the regression equation (5). The observation of the states is, finally, given by (33).

**Simulation results** The tuning gains used in the simulation are given in the caption of the various figures. In Fig. 1 we show plots of the norm of the estimation error \( |\theta - \hat{\theta}(t)| \) and the observation error \( x(t) - \hat{x}(t) \). To evaluate the effect of the tuning parameters on the transient performance we repeated the previous simulation significantly changing the gains \( f_0 \) and \( \alpha \). The results are given in Fig. 2, which clearly reveal the transient performance degradation—notice the difference in time scales.

Finally, to test the robustness of the adaptive observer to measurement noise we added to the plant output the signal depicted in Fig. 3. The resulting simulations are given in Fig. 4, showing very little degradation of the performance. Observing, however, a small steady state error.

**Figure 1:** The estimation errors for \( K = \text{col}(7.5, 25) \), \( f = \text{col}(-1, -2) \), \( f_0 = 0.001 \), \( \alpha = 100 \), \( \gamma = 100 \).
Figure 2: The estimation errors for $K = \text{col}(7.5, 25), f = \text{col}(-1, -2), f_0 = 0.1, \alpha = 1, \gamma = 100.$

Figure 3: Plot of the measurement noise

7 Concluding Remarks

We have presented in the article a new general technique of state observation of LTV systems disturbed by an additive signal generated by an LTI exosystem. The system matrices and the matrix of the exosystem are assume to be unknown. The main analytical tool employed for the solution of the problem is the GPEBO technique that translates the state observation problem in a parameter estimation task, for which a regression equation—in the present case nonlinearly parameterized—is developed. To overcome the difficulty of estimating the parameters in this case we propose to use the DREM technique. In particular, we propose a new least-squares-based DREM-like estimator that allows us to solve this task.

Thanks to this [LS+DREM] scheme the estimation of the unknown constant parameters is achieved very rapidly and robustly with respect to measuring noise. It is also shown that the speed of convergence may be easily regulated suitably selecting the tuning coefficients of the estimator—which consists only of four scalar parameters, whose role is clearly identified. Some simulation results reveal the efficiency of the proposed approach.

In future works we are planning to investigate the extension to the nonlinear setting of our work that is mentioned in Remark 2.
Figure 4: The estimation errors with noisy measurement for $K = \text{col}(7.5, 25)$, $f = \text{col}(-1, -2)$, $f_0 = 0.001$, $\alpha = 100$, $\gamma = 100$.

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A Auxiliary Lemmata

In this Appendix we give two preliminary lemmata that are used in the construction of the LRE of Section 3. Although the first one of these may be found in the literature [9], for the sake of completeness we also give its proof.

Lemma 1. Consider a stable LTI system

$$\dot{x}(t) = Ax(t) + Bu(t)$$

with $x(t) \in \mathbb{R}^n$, driven by the exosystem (34)

$$\dot{w}(t) = Sw(t)$$
$$u(t) = Qw(t),$$

(34)

with $w(t) \in \mathbb{R}^n$, where the spectra of $A$ and $S$ are disjoint. Then, $x(t)$ can be split as

$$x(t) = \Pi w(t) + \tilde{x}(t)$$

where $\Pi$ is the unique solution of the Sylvester equation

$$\Pi S = A\Pi + BQ$$

and $\tilde{x}(t)$ is exponentially decaying.
Proof. Form the composition
\[
\dot{w}(t) = Sw(t) \\
\dot{x}(t) = Ax(t) + BQw(t)
\]
The change of variables
\[
\dot{\tilde{x}}(t) = x(t) - \Pi w(t)
\]
yields the diagonal system
\[
\dot{w}(t) = Sw(t) \\
\dot{\tilde{x}}(t) = A\tilde{x}(t)
\]
The proof is completed recalling that \(A\) is Hurwitz. \(\square\)

The construction introduced in the proof of Lemma 1 can also be used in the proof of the following Lemma, from which the expression (17) can be deduced.

Lemma 2. Consider a stable LTI system with the state \(x(t) \in \mathbb{R}^n\)
\[
\dot{x}(t) = A_f x(t) + e_n u(t),
\]
driven by the exosystem
\[
\begin{align*}
\dot{u}(t) &= Qw(t) + Me(t), \\
\dot{w}(t) &= Sw(t), \\
\dot{e}(t) &= A_K e(t)
\end{align*}
\]
where \(w(t) \in \mathbb{R}^n\), the spectra of \(A_f\) and \(S\) are disjoint, \(A_f\) is a Hurwitz matrix having the form (15) and the matrix \(A_K\) is an arbitrary Hurwitz matrix. Then,

(i) \(x(t)\) can be split as
\[
x(t) = \Pi_f w(t) + \tilde{x}(t)
\]
where \(\Pi_f\) is the unique solution of the Sylvester equation
\[
\Pi_f S = A_f \Pi_f + e_n Q
\]
and \((\tilde{x}, e)\) satisfies
\[
\begin{bmatrix}
\dot{\tilde{x}}(t) \\
\dot{e}(t)
\end{bmatrix} =
\begin{bmatrix}
A_f & e_n M \\
0 & A_K
\end{bmatrix}
\begin{bmatrix}
\tilde{x}(t) \\
e(t)
\end{bmatrix}.
\]

(ii) \(x(t)\) satisfies
\[
\dot{x}(t) = A_G x(t) + e_n \varepsilon(t),
\]
where \(A_G\) is given in (18), the spectra of \(A_G\) and \(S\) coincide, and \(\varepsilon(t)\) is exponentially decaying.

Proof. With Lemma 1 in mind, consider the composition
\[
\begin{align*}
\dot{w}(t) &= Sw(t) \\
\dot{e}(t) &= A_K e(t) \\
\dot{\tilde{x}}(t) &= A_f x(t) + e_n Qw(t) + e_n Me(t).
\end{align*}
\]
The change of variables
\[
\dot{\tilde{x}}(t) = x(t) - \Pi w(t)
\]
proves (35).
In order to prove claim (ii), observe that

\[ x_1(t) = e_1^T x(t) = h^T w(t) + e_1^T \tilde{x}(t) \]

where \( h^T = e_1^T \Pi f \). Moreover, in view of the special form of \( A_f \), we have \( x_{i+1}(t) = \dot{x}_i(t) \) for \( i = 1, \ldots, n - 1 \).

As a consequence

\[ x_1 = h^T w(t) + e_1^T \tilde{x}(t), \]
\[ x_2 = h^T S w(t) + e_1^T [\tilde{x}(t)]^{(1)} \]
\[ x_3 = h^T S^2 w(t) + e_1^T [\tilde{x}(t)]^{(2)} \]
\[ \vdots \]
\[ x_n = h^T S^{n-1} w(t) + e_1^T [\tilde{x}(t)]^{(n-1)} \]

and, finally,

\[ \dot{x}_n = h^T S^n w(t) + e_1^T [\tilde{x}(t)]^{(n)}. \] (36)

Consider now the characteristic polynomial of matrix \( S \):

\[ \gamma(\lambda) = \det(\lambda I - S) = \lambda^n + \gamma_1 \lambda^{n-1} + \gamma_2 \lambda^{n-2} + \cdots + \gamma_n \lambda^0. \]

and recall that, thanks to the Cayley–Hamilton theorem,

\[ S^n = -\gamma_1 S^{n-1} - \gamma_2 S^{n-2} - \cdots - \gamma_n S^0. \]

Substituting the latter into (36) yields

\[ \dot{x}_n = h^T (-\gamma_1 S^{n-1} - \gamma_2 S^{n-2} - \cdots - \gamma_n S^0) w(t) + e_1^T [\tilde{x}(t)]^{(n)} \]
\[ = -\gamma_n x_1 - \gamma_{n-1} x_2 - \cdots - \gamma_1 x_n + e_1^T [\tilde{x}(t)]^{(n)} + \sum_{i=1}^{n} \gamma_i [\tilde{x}(t)]^{(n-i)} \]
\[ = -\gamma_n x_1 - \gamma_{n-1} x_2 - \cdots - \gamma_1 x_n + \varepsilon(t), \]

where

\[ \varepsilon(t) = e_1^T [\tilde{x}(t)]^{(n)} + \sum_{i=1}^{n} \gamma_i [\tilde{x}(t)]^{(n-i)}. \] (37)

In this way, we have shown that

\[ \dot{x}(t) = A_\Gamma x(t) + e_n \varepsilon(t), \]

in which \( \Gamma = [ -\gamma_n \quad -\gamma_{n-1} \quad \ldots \quad -\gamma_1 ] \).

To complete the proof of (ii) it remains to check that \( \varepsilon(t) \) is exponentially decaying. But this is a straightforward consequence of (35), which we rewrite as

\[ \dot{\xi}(t) = F_c \xi(t) \] (38)
with
\[ \xi(t) := \begin{bmatrix} \tilde{x}(t) \\ e(t) \end{bmatrix}, \quad F_c := \begin{bmatrix} A_f & e_nM \\ 0 & A_K \end{bmatrix}, \]
and \( F_c \) a Hurwitz matrix. Now, \( \tilde{x}(t) \) can expressed as \( \tilde{x}(t) = H_c \xi(t) \) with \( H_c := \begin{bmatrix} I_n & 0_{n \times n} \end{bmatrix} \). Thus
\[ [\tilde{x}(t)]^{(i)} = H_c F_c^i \xi(t). \]
Replacing the identity above in (37) yields
\[ \varepsilon(t) = h_\varepsilon^\top \xi(t), \]
where we defined the vector
\[ h_\varepsilon := e_1^\top H_c \left[ F_c^n + \sum_{i=1}^n \gamma_i F_c^{n-i} \right]. \]
Since \( \xi(t) \) is exponentially decaying, claim (ii) is proven. \( \Box \)

**Remark 2.** It is worth stressing that, taking advantage of an important result of [15], the property indicated in Lemma 2 can be extended to the case in which a stable \( d \)-dimensional system
\[ \dot{x}(t) = A_f x(t) + e_d u(t) \]
is driven by a nonlinear exosystem of the form
\[
\begin{align*}
    u(t) &= q(w(t)) + Me(t), \\
    \dot{w}(t) &= s(w(t)), \\
    \dot{e}(t) &= A_K e(t)
\end{align*}
\]
Also in this case, in fact, it is possible to show that - if \( d \) and \( A_f \) are appropriately chosen - \( x(t) \) can be seen as the solution of a system \( \dot{x} = \tilde{s}(x) + e_d \varepsilon \) in which \( \tilde{s}(\cdot) \) is a suitable “copy” of \( s(\cdot) \) and \( \varepsilon \) is asymptotically decaying to 0.

### B A Least-Squares-Based DREM Parameter Estimator

In this appendix we propose a new parameter estimator for separable, NLPRE which is a combination of a least-squares search and the mixing procedure of DREM. The least-squares search is used to construct the extended regressor—this in contrast with previous versions of DREM where this regressor is obtained applying a bank of linear operators to the original regressor equation, see [17] for a recent overview of the DREM procedure. There are two significant advantages of the new estimator with respect to the previous DREM algorithms. First, we avoid the step of selecting the bank of linear operators, which significantly affect the performance of the estimator and whose choice is far from clear. Second, it is widely recognized that the transient performance of the least-squares search significantly outperforms the gradient-based one.

The following standing assumptions are imposed throughout the appendix.

**Assumption A1 [A NLPRE]** The existence of a separable, NLPRE
\[ Y(t) = \Omega(t)G(\theta), \]
where the signals $Y(t) \in \mathbb{R}^m$ and $\Omega(t) \in \mathbb{R}^{m \times p}$ are measurable, the mapping $G : \mathbb{R}^q \rightarrow \mathbb{R}^p$, $q \leq p$ is known but the constant parameters $\theta \in \mathbb{R}^q$ are unknown.

**Assumption A2** [Monotonicity] There exists a matrix $Q \in \mathbb{R}^{q \times p}$ such that mapping $G(\theta)$ verifies the linear matrix inequality

$$Q \nabla G(\theta) + \nabla^T G(\theta) Q^T \geq \rho I_q, \quad \forall \theta \in \mathbb{R}^q,$$

for some $\rho \in \mathbb{R}_{>0}$. Consequently [6, 22], The mapping $QG(\theta)$ is strongly monotone, that is,

$$(a - b)^T [QG(a) - QG(b)] \geq \rho |a - b|^2 > 0, \quad \forall a, b \in \mathbb{R}^q, \ a \neq b. \quad (42)$$

**Assumption A3** [Interval Excitation] [31, Definition 3.1] The regressor matrix $\Omega$ of the NLPRE (5) is interval exciting (IE). That is, there exists constants $C_c > 0$ and $t_c \geq t_0 > 0$ such that

$$\int_{t_0}^{t_0 + t_c} \Omega^\top(s) \Omega(s) ds \geq C_c I_p.$$

**Proposition 3.** Consider the NLPRE (40) with $G(\theta)$ satisfying Assumption A2 and $\Omega$ verifying Assumption A3. Define the LS+DREM interlaced estimator

$$\dot{\hat{\theta}}_g(t) = \alpha F(t) \Omega^\top(t)(Y(t) - \Omega(t) \hat{\theta}_g(t)), \quad \hat{\theta}_g(0) =: \theta_{g0} \in \mathbb{R}^p \quad (43a)$$

$$\dot{F}(t) = -\alpha F(t) \Omega^\top(t) \Omega(t) F(t), \quad F(0) = \frac{1}{f_0} I_p \quad (43b)$$

$$\dot{\hat{\theta}}(t) = \gamma Q \Delta(t) [Y(t) - \Delta(t) G(\hat{\theta}(t))], \quad \hat{\theta}(0) =: \theta_0 \in \mathbb{R}^q, \quad (43c)$$

with tuning gains $f_0 > 0$, $\alpha > 0$ and $\gamma > 0$, and we defined

$$\Delta(t) := \det\{I_p - f_0 F(t)\} \quad (44a)$$

$$Y(t) := \text{adj}\{I_p - f_0 F(t)\}(\hat{\theta}_g(t) - f_0 F(t) \theta_{g0}), \quad (44b)$$

where adj{·} denotes the adjugate matrix. Define the parameter error $\tilde{\theta}(t) := \hat{\theta}(t) - \theta$. Then, for all initial conditions $\theta_{g0} \in \mathbb{R}^p$ and $\theta_0 \in \mathbb{R}^q$, we have that

$$\lim_{t \to \infty} \tilde{\theta}(t) = 0, \ (exp), \quad (45)$$

with all signals bounded.

**Proof.** With some abuse of notation, define the signal

$$\tilde{\theta}(t) := \hat{\theta}_g(t) - G(\theta),$$

whose derivative is given by

$$\dot{\tilde{\theta}}(t) = -\alpha F(t) \Omega^\top(t) \Omega(t) \tilde{\theta}(t),$$

where we used (40) and (43a). Now, from the fact that [8, Theorem 4.3.4],

$$\frac{d}{dt}(F^{-1}(t) \tilde{\theta}(t)) = 0$$

with all signals bounded.
we have
\[ \tilde{G}(t) = f_0 F(t) \tilde{G}(0), \]
which may be rewritten as the extended NLPRE
\[ (I_p - f_0 F(t)) \tilde{G}(\theta) = \dot{\theta}_g(t) - f_0 F(t) \theta_g, \]
(46)
Following the DREM procedure we multiply (46) by \( \text{adj}\{I_p - f_0 F(t)\} \) to get the following NLPRE
\[ \mathcal{Y}(t) = \Delta(t) \mathcal{G}(\theta), \]
(47)
where we used (44a) and (44b). Replacing (47) in (43c) we get
\[ \dot{\theta}(t) = -\gamma \Delta^2(t) Q [\mathcal{G}(\dot{\theta}(t)) - \mathcal{G}(\theta)]. \]
To analyze the stability of the latter system define the Lyapunov function candidate
\[ U(\tilde{\theta}) := \frac{1}{2\gamma} |\tilde{\theta}|^2. \]
Computing its time derivative yields
\[ \dot{U}(t) = -\Delta^2(\theta - \tilde{\theta})^T Q [\mathcal{G}(\dot{\theta}(t)) - \mathcal{G}(\theta)] \]
\[ \leq -\Delta^2(t) \rho |\dot{\theta}(t)|^2 \]
\[ = -2\rho \gamma \Delta^2(t) U(t), \]
where we invoked the **Assumption A.2** of strong monotonicity property (42) of \( Q \mathcal{G}(\theta) \) to get the first bound. To complete the proof, we invoke the Comparison Lemma [10, Lemma 3.4] that yields the bound
\[ U(t + t_c) \leq e^{-2\rho \gamma \int_{t}^{t + t_c} \Delta^2(s) ds} U(t), \]
which ensures \( \lim_{t \to \infty} \tilde{\theta}(t) = 0 \) (exp) if \( \Delta(t) \) is PE. The latter condition follows from the assumption that \( \Omega(t) \) is IE and [31, Lemma 3.5], which ensures the following
\[ \Omega(t) \in IE \Rightarrow \Delta(t) > 0, \forall t \geq t_0 + t_c. \]
(48)
Consequently, \( \Delta(t) \) is PE.
\[ \square \square \square \]
**Remark 3.** Proposed estimator (43b) may be compared with the adaptive estimation algorithm presented in [27], where the problem of frequency estimation of a periodical signals corrupted by a noise was considered. Indeed, for the case \( m = p = 1 \) with \( F \in \mathbb{R} \), \( \Omega \in \mathbb{R} \) the algorithm (43b) becomes
\[ \dot{F} = -\alpha \Omega^2 F^2. \]
In [27] a similar tuning rule was deduced (see equation (45)). But in [27] was derived with other (heuristic) arguments, from an idea of iterative decreasing an adaptation gain at special time instants [5]. Continuous extention of the iterative approach [5] yielded [27] which is a very particular scenario of the general approach (43b) proposed in this paper.