Detecting Intermediate-Mass Ratio Inspirals From The Ground And Space

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The detection of a gravitational capture of a stellar-mass compact object by a massive black hole (MBH) will allow us to test gravity in the strong regime. The repeated, accumulated bursts of gravitational radiation from these sources can be envisaged as a geodesic mapping of spacetime around the MBH. These sources form via two-body relaxation, by exchanging energy and angular momentum, and inspiral in a slow, progressive way down to the final merger. The range of frequencies is localised in the range of millihertz in the case of MBH of masses $\sim 10^6 M_\odot$, i.e. that of space-borne gravitational-wave observatories such as LISA. In this article I show that, depending on their orbital parameters, intermediate-mass ratios (IMRIs) of MBH of masses between a hundred and a few thousand have frequencies that make them detectable (i) with ground-based observatories, or (ii) with both LISA and ground-based ones such as advanced LIGO/Virgo and third generation ones, with ET as an example. The binaries have a signal-to-noise ratio large enough to ensure detection. More extreme values in their orbital parameters correspond to systems detectable only with ground-based detectors and enter the LIGO/Virgo band in particular in many different harmonics for masses up to some 2000, $M_\odot$. I show that environmental effects are negligible, so that the source should not have this kind of complication. The accumulated phase-shift is measurable with LISA and ET, and for some cases also with LIGO, so that it is possible to recover information about the eccentricity and formation scenario. For IMRIs with a total mass $\lesssim 2000 M_\odot$ and initial eccentricities up to 0.999, LISA can give a warning to ground-based detectors with enough time in advance and seconds of precision. The possibility of detecting IMRIs from the ground alone or combined with space-borne observatories opens new possibilities for gravitational wave astronomy.

I. INTRODUCTION

The typical size of a massive black hole (MBH), i.e. its Schwarzschild radius, is from the point of view of the host galaxy extremely tiny. For a $10^6 M_\odot$ MBH, this difference spans over ten orders of magnitude. However, we have discovered a deep link between the properties of the galaxy and those of the MBH, in particular between the mass of the MBH and the velocity dispersion $\sigma$ of the spheroidal component of the galaxy $[1]$. Because the region of interest is difficult to resolve, the lower end of this correlation is uncertain. However, if we extend these correlations to smaller systems, globular clusters, or ultra-compact dwarf galaxies should harbour black holes with masses ranging between $10^2$ and $10^4, M_\odot$, i.e. intermediate-mass black holes, IMBHs (for a review, see the work of $[2,3]$), although such black holes have never been robustly detected.

The best way to probe the nature of the MBH is with gravitational waves, which allow us to extract information that is unavailable electromagnetically. The gravitational capture and plunge of a compact object through the event horizon is one of the main goals of the Laser Interferometer Space Antenna (LISA) mission $[4]$. A compact object of stellar mass, so dense that it defeats the tidal forces of the MBH, is able to approach very closely the central MBH, emitting a large amount of gravitational radiation as orbital energy is radiated away. This causes the semi-major axis to shrink. This “doomed” object spends many orbits around the MBH before it is swallowed. The radiated energy which can be thought of as a snapshot containing detailed information about the system will allow us to probe strong-field gravitational physics. Depending on the mass ratio $q$, we talk about either extreme-mass ratio inspirals, $q \gtrsim 10^4 : 1$ (EMRI, see $[5,6]$) or intermediate-mass-ratio inspirals, $q \sim 10^2 - 10^4 : 1$ (IMRI, see e.g. $[7,9]$).

In galactic nuclei the predominant way of producing EMRIs is via two-body relaxation $[5]$. At formation, these sources have extremely large eccentricities, particularly if the MBH is Kerr $[10]$, which is what we should expect from nature. However, in globular clusters, which harbour MBH in the range of IMBHs, the loss-cone theory, which is our tool to understand how EMRIs form, [see e.g. $[11,13]$] becomes very complex, mostly due to the fact that the IMBH is not fixed at the centre of the system. It becomes even more difficult when we add the emission of GWs—another layer of complication to the Newtonian problem. As of now, we must rely on computer simulations to address this problem.

The joint detection of a GW source with different observatories has been already discussed in the literature but not in the mass ratio range that is addressed in this work. The series of works $[14,17]$ investigated the formation, evolution, inspiraling and merger of IMBH binaries with a mass ratio not larger than 10 and the prospects of
II. FORMATION OF INTERMEDIATE-MASS RATIO INSPIRALS IN GLOBULAR CLUSTERS

In this work the sources of interest are inspirals of compact objects on to an IMBH with a mass ratio of about $10^5 - 10^4:1$. The most accurate simulations of a globular cluster are the so-called direct-summation $N$-body algorithms. In this scheme, one directly integrates Newton’s equations of motion between all stars in a cluster at every timestep, with a regularisation algorithm for binaries, so that any phenomenon associated with gravity naturally arises [see e.g. 32, 33] and the latter for the concept of regularisation. Following the first implementation of 31, many modern direct-summation codes can mimic the effects of general relativity via a post-Newtonian expansion of the forces to be integrated [see section 9 of 33] for a review of stellar-dynamical relativistic integrators.

The first dynamical simulation that presented the formation and evolution of an IMRI down to a few Schwarzschild radii from coalescence using this scheme is the work of 39. In one of the simulations we presented, we observed and tracked the spontaneous production of an IMRI between an IMBH of mass $M_{BH} = 500 M_{\odot}$ and a stellar-mass black hole of mass $m_{CO} = 26 M_{\odot}$. After a few Myrs, the IMRI merges and the IMBH receives a relativistic recoil [40, 42] and escapes the whole cluster. It must be noted that the IMBH was in a binary for almost all of the simulation time with another compact object, a stellar-mass black hole. The IMBH exchanged companions a few times and was ionised for a last time very abruptly to form the last binary. This binary started at a very small semi-major axis, of about $a \sim 10^{-5}$ pc, and a very large eccentricity, of $e = 0.999$, which fits in the parabolic capture mechanism of 23. A few years later, 43 find similar results for a close range of masses but with a different approach. The work of 44 follows very closely the initial setup of 39 and reproduces our results with a different integrator, which corroborates our findings. Last, the numerical experiments of 45 explore IMBHs in a lighter range, of masses around $M_{BH} = 150 M_{\odot}$. They however also report that the IMBH forms a binary for about 90% of the time. The probability distribution of semi-major axis peaks at about $\lesssim 10^{-5}$ pc.

III. LIGHT AND MEDIUM-SIZE IMRIS

The characteristic amplitude and the GW harmonics in the quadrupolar radiation approximation can be calculated following the scheme of 46, in which the orbital parameters change slowly due to the emission of radiation. This is emitted at every integer multiple of the orbital frequency, $\omega_{n} = n \sqrt{GM_{BH}/a^3}$, with $a$ the semi-major axis. The strain amplitude in the $n$-th harmonic at a given distance $D$, normalized to the typical values of this work is

$$h_n = g(n, e) \frac{G^2 M_{BH} m_{CO}}{D a c^4} \left( \frac{D}{500 \text{ Mpc}} \right)^{-1} \left( \frac{a}{10^{-5} \text{ pc}} \right)^{-1} \left( \frac{M_{BH}}{10^3 M_{\odot}} \right) \left( \frac{m_{CO}}{10 M_{\odot}} \right).$$

In this expression $M_{BH}$ is the mass of the IMBH, $m_{CO}$ is the mass of the compact object (CO), and $g(n, e)$ is a function of the harmonic number $n$ and the eccentricity $e$ [see 46]. We consider the RMS amplitude averaged over the two GW polarizations and all directions. Other alternatives to this approach, such as the works of 47, 50 give a more accurate description of the very few last orbits, but remain substantially equivalent to 46 at previous stages of the evolution. This approach gives a correct...
estimation of the frequency cutoff at the innermost stable circular orbit (ISCO) frequency and is enough for the main goal of this work [and see the work of [51] for a discussion about the detection of binaries with mass ratios of 0.1 with advanced ground-based detectors using aligned-spin effective-one-body waveforms].

With this approximation, I show in Fig. [1] $h_c$ as function of the frequency of two different IMRIs, and a few moments in the evolution before the final merger, which happens at a time $T_{\text{merg}}$. For the kind of eccentricities that I am considering in this work, this time can be estimated following [54] for typical values as

$$T_{\text{merg}} \approx \frac{24\sqrt{2}}{85} \frac{(1-e_0)^{7/2}e^5}{\mathcal{G}^3 \mathcal{M}_\text{BH}^2 \mathcal{m}_\text{CO}^4} a_0^4 \approx 6.4 \times 10^5 \text{yrs} \quad (3)$$

$$\times \left( \frac{\mathcal{M}_\text{BH}}{10^4 \mathcal{M}_\odot} \right)^2 \left( \frac{\mathcal{m}_\text{CO}}{10 \mathcal{M}_\odot} \right)^{-1} \left( \frac{R^D}{200 R_\odot} \right)^4 \left( \frac{1-e_0}{10^{-5}} \right)^{-1/2}, \quad (4)$$

where $R^D$ and $e_0$ are the initial pericenter distance and eccentricity, respectively. In this Fig. [1] the IMBH has a mass of $M_{\text{BH}} = 100 \mathcal{M}_\odot$ and the mass of the compact object (CO) is set to $10 \mathcal{M}_\odot$. I depict the LISA sensitivity curve and those of Advanced LIGO (LIGO, henceforth) and the ET in its D configuration [22], although I have shortened the characteristic amplitude to start at lower values for clarity, since none of the sources I have considered achieves higher values. For reference, I include as well the full waveform in the LIGO sensitivity curve as estimated by the IMRPhenomD approach of [52, 53], which has been developed to study systems with mass ratios of up to $q = 18$. This curve is close to the peak of harmonics in amplitude for this specific case but in general this is not true, and depends on the specifics of the binary such as periastron argument, inclination angle, precession of the orbital plane, to mention a few.

We can see that eccentricities corresponding to those that we can expect for a dynamical capture as described in the introduction produce IMRIs which are observable with LISA and both the ET and LIGO. In particular, the left panel corresponds to an IMRI which spends half a minute in LIGO. For lighter masses of the CO, this time becomes larger. For higher eccentricities, which can be achieved via two-body relaxation or in the parabolic braking scenario, at these masses the IMRIs can be seen only by ground-based detectors, with a significant amount of time and the vast majority of the harmonics in band. It is interesting to note that the ET has been estimated to be able to detect up to several hundred events per year, see [55, 56].

In Fig. [2] I show a more massive system, with a total mass of $310 \mathcal{M}_\odot$. The source recedes in frequency due to the larger mass. For the systems considered in the upper panels, this allows IMRIs to spend more time in LISA and accumulate more SNR, with the resulting shortened time in the ground-based detectors which, however, is still significant. For the lower panels, however, LISA is again deaf to these sources.

Finally, in Fig. [3] I show a system similar to what is found in the numerical simulations of [23]. The mass of the IMBH is set to $500 \mathcal{M}_\odot$ Higher frequencies lead the source to be observable by only ground-based detectors.

### IV. LARGE-MASS IMRIS

In Figs. [4], [5] and [6] we can see IMBHs with masses $M_{\text{BH}} = 1000 \mathcal{M}_\odot$, $2000 \mathcal{M}_\odot$ and $3000 \mathcal{M}_\odot$, respectively. For more moderate eccentricities, the IMRIs in the examples can be detected with LISA and the ET, but they do not enter the LIGO detection band. More extreme eccentricities lead to a large amount of harmonics entering the ET band for significant amounts of time. In the case of a $2000 \mathcal{M}_\odot$ IMBH, it can spend as much as 10 minutes in band in different harmonics. Larger masses, i.e. $3000 \mathcal{M}_\odot$ produce short-lived sources that however spend up to one minute in band of the ET.

### V. ENVIRONMENTAL EFFECTS

In the previous sections I have shown the evolution of an IMRI under the assumption that the binary is perfectly isolated from the rest of the stellar system. I.e. the binary evolves only due to the emission of GWs. The reason for this is that the problem is cleaner and easier to understand. However, the binary is located in a dense stellar system, typically a globular cluster. While the role of gas is negligible, since the gas density in these systems is very low. Hence, so as to assess whether surrounding stars could vary or modify the evolution after the IMRI has formed, in this section I investigate the impact of the stellar system in a semi-analytical approach. The basic idea is to split the evolution of both the semi-major axis and the eccentricity in two contributions, one driven by the dynamical interactions with stars (subscript D) and one due to emission of GWs (subscript GW), $\dot{a} = \dot{a}_{\text{GR}} + \dot{a}_D$, and $\dot{e} = \dot{e}_{\text{GR}} + \dot{e}_D$ with dots representing the time derivative.

From [54],

$$\dot{a}_{\text{GW}} = - \frac{64}{5} \frac{\mathcal{G}^3 \mathcal{M}_\text{BH} \mathcal{m}_\text{CO}(M_{\text{BH}} + m_{\text{CO}})}{\mathcal{e}^5 a^3 (1-e^2)^{7/2}} \left( 1 + \frac{73}{24} e^2 + \frac{37}{96} e^4 \right)$$

$$\dot{e}_{\text{GW}} = - \frac{304}{15} \frac{\mathcal{G}^3 \mathcal{M}_\text{BH} \mathcal{m}_\text{CO}(M_{\text{BH}} + m_{\text{CO}})}{\mathcal{e}^5 a^4 (1-e^2)^{5/2}} \left( 1 + \frac{121}{304} e^2 \right)$$

The GW terms are as given in [54]. Using the relationships of [57], we have that
FIG. 1. Characteristic amplitude of the first ten harmonics emitted during the evolution of an IMRI. The left, solid V-shaped curve corresponds to LISA’s intrinsic noise, and the two right U-shaped curves to the ET (lower $h_c$ values) and to Advanced LIGO. The mass of the IMBH is fixed to $M_{\text{MBH}} = 100 M_\odot$ and the mass of the compact object is $m_{\text{CO}} = 10 M_\odot$. The source is assumed to be located at a distance of $D = 500 \text{ Mpc}$. Each panel corresponds to a binary with different initial values for the semi-major axis or eccentricity. I localise and show on the second harmonic a few instants of time in the evolution of the binary before the final merger. The total amount of time for the binary to merge from the initial values of the semi-major axis and eccentricity is given in each panel, $T_{\text{mrg}}$. The square symbol corresponds to one year before it. The rest of the harmonics also display the same instants of time using the same symbol but without a text label. I show the value of the eccentricity in that particular moment and the pericentre value $R_p$ in function of the Schwarzschild radius $R_S$. Additionally, I depict in the right panel with a dashed, orange curve the full waveform of the system in the LIGO sensitivity curve as approximated by the IMRPhenomD algorithm presented in [52, 53].

Following the usual notation, $G$ is the gravitational constant, $\rho$ is the stellar density around the binary, $\sigma$ the corresponding velocity dispersion of the cluster and $H$ the so-called hardening constant, as introduced in the work of [57]. For the kind of binaries I am considering in this work, i.e. hard ones, we have that $(de/d\ln(1/a))_D = K(e)$. Since the density drops significantly during the evolution, we can regard $\sigma$ as approximately constant and hence $de = K(e) d\ln(1/a) = -K(e) da / da$, so that $H \simeq 16$, as in the original work of [57] and see also [58]. Therefore,

$$\dot{a}_D = -H \frac{G\rho}{\sigma} a^2.$$  \hspace{1cm} (7)

VI. LOUDNESS OF THE SOURCES

A. Low-eccentricity sources: LIGO

As it progresses in the inspiral, a compact binary becomes observable and more circular. The characteristic amplitude $h_c$ of an IMRI emitting at a given frequency $f$ is given by

$$h_c = \frac{2\pi f B (1 - 12e^2)}{1 - e^2}.$$  \hspace{1cm} (8)
FIG. 2. Same as in Fig. (1) but for $M_{\text{MBH}} = 300 \, M_\odot$ and $m_{\text{CO}} = 30 \, M_\odot$ and different labels in the evolution. Notice the displacement of the peak of frequencies, which wanders from the LISA band to the LIGO/Virgo one (top, left panel to the bottom, right one).

$$h_c = \sqrt{\frac{2 \dot{E}/\dot{f}}{\pi D}},$$  \hspace{1cm} (9)

with $\dot{E}$ the power emitted, $\dot{f}$ the time derivative of the frequency and $D$ the distance to the source [60]. The sky and orientation-averaged SNR of a monochromatic source with the ansatz of ideal signal processing is given by the equation

$$\left(\frac{S}{N}\right)^2 = \frac{4}{\pi D^2} \int \frac{\dot{E}}{f S^2_h(f)} \, df$$  \hspace{1cm} (10)

as derived in [60], where $D$ is the distance to the source, $\dot{E}$ is the rate of energy lost by the source, $\dot{f}$ is the rate of change of frequency and $S^2_h(f) \approx 5 \, S_h(f)$ is the sky and orientation average noise spectral density of the detector.
For a source with multiple frequency components, the total SNR$^2$ is obtained by summing the above expression over each mode.

In Fig. 3 I show the Fourier-transformed waveform of both panels of Fig 1, as approximated by the algorithm of [61]. Theirs is a time-domain waveform that describes binaries of black holes evolving on mildly eccentric orbits, not exceeding $e \lesssim 0.2$. When the binaries enter the LIGO/Virgo band, even if they start with initially high eccentricities, they reach values below the threshold of the algorithm, which therefore is a good approximant to estimate the waveform and compute the SNR.

FIG. 3. Same as in Fig. 1 but for $M_{\text{MBH}} = 500, M_\odot$ and $m_{\text{CO}} = 26$, which is based in the relativistic stellar-dynamical simulation of [39]. The left panel corresponds to the kind of eccentricity in that work and the right one to a more extreme one, and I show different labels in the evolution.

FIG. 4. Same as in Fig. 1 but for $M_{\text{MBH}} = 1000, M_\odot$ and $m_{\text{CO}} = 30$ and different labels in the evolution.
FIG. 5. Same as in Fig. (1) but for $M_{\text{MBH}} = 2000, M_\odot$ and $m_{\text{CO}} = 5$ and different labels in the evolution.

FIG. 6. Same as in Fig. (1) but for $M_{\text{MBH}} = 3000, M_\odot$ and $m_{\text{CO}} = 30$ and different labels in the evolution.

B. High-eccentricity sources

When moving to lower frequencies, the eccentricity exceeds by far the limit of the approximation of [61] that I have used to derive the SNR. To calculate it when the IMRIs sweep the LISA bandwidth, I use the expression (derived from Eq. 20 of [46], Eq. 2.1 of [60] and Eq. 56 of [49])

\[
\left( \frac{S}{N} \right)_n^2 = \int f_n(t_{\text{ini}}) \left( \frac{h_{c,n}(f_n)}{h_{\text{det}}(f_n)} \right)^2 \frac{1}{f_n} d(\ln(f_n)). \tag{11}
\]

In this Eq. $f_n(t)$ is the (redshifted) frequency of the $n$th harmonic at time $t$ ($f_n = n \times f_{\text{orbital}}$), $h_{c,n}(f_n)$ is the characteristic amplitude of the $n$th harmonic when the frequency associated to that component is $f_n$, and $h_{\text{det}}$ is the square root of the sensitivity curve of the detectors.

A few examples of the SNRs for the IMRI systems in the LISA band of the previous sections (and ET in parentheses for the same source), assuming a distance of 500 Mpc and taking the contribution of the first 100 harmonics are: Fig. (1) 15 (1036), left panel, and virtually 0, 0.01 (1087) for the right one. For Fig. (2), the upper, left panel 50 (1994) and the upper, right panel 24 (1995), while the lower, left panel has 2 (1991), and the lower, right one approximately 0, 0.01 (2231). In Fig. (3), the left panel yields an SNR of 36 (1449), and the right one of about zero, 0.05 (1461). In Fig. (4), the left panel has 79 (328), and the right one approximately zero, 0.4 (305). Fig. (5) has 7 (15) in the left panel and approximately 0 in the right one, 0.1 (37). Finally, Fig. (6) has 5 (1).

For the IMRI examples given in Figs. (1), assuming a distance of $D = 500$ Mpc, I find a SNR in the LIGO bandwidth of 42.87 and 42.55, for the left and right panels, respectively. In Figs. (2), at the same distance, I find 17.12, 17.13 for the top-left, and top-right panels, respectively and 17.15, 16.40 for the lower-left and lower-right ones.
FIG. 7.  **Left panel:** As in Fig. (1) but for $M_{MBH} = 100$, $M_\odot$ and $m_{CO} = 30$ and different labels in the evolution. **Right panel:** Same as the left one but taking into account stellar dynamics (see text). I adopt an ambient stellar density of $2 \times 10^5 M_\odot pc^{-3}$, $K_0 = 0.1$ and a one-dimensional velocity dispersion of $\sigma = 15$ km/s.

In Figs. (9) and (10) I give three examples of the accumulated SNR as calculated in this section. In the first figure I display in the left panel the SNR in ET of the system of Fig. (2), bottom, right panel and, on the right panel, of Fig. (3), right panel, also for ET. In the second one I show the accumulated SNR of the system depicted in Fig. (3), left panel, for LISA.

However, and for the case of LISA, this is the total accumulated SNR for the total time that the source spends on band. The observational time, the time during which we retrieve data from the source, is in all cases shorter and, hence, the accumulated, observed SNR is lower. As an example, for Fig. (3), left panel, if we integrate all of the time the source spends on band, we obtain the aforementioned SNR of 36. However, if we integrate the last 10 yrs before merger, the SNR goes down to 23, and to 19 for the last 5 yrs. If we observed the source earlier in the evolution, say, e.g. 10 yrs before merger to 5 yrs before it, the SNR would be 14 and 100 yrs before merger to 95 yrs, 3. I show an example for the accumulated SNR for this system in Fig. (11), 10 and 5 yrs before the final plunge. This only applies to LISA, because the time spent on the ground-based detector ET is much shorter.

So as to assess whether this approach is robust, I give now the SNR for the systems of Sec. (VI A) in the LIGO band, which have been calculated with the waveform model introduced in that section. In Fig. (1) as estimated with this approach, the SNR is 41 and 40, for the left and right panels, respectively. In Fig. (2) I find, from left to right, top to bottom, 12, 12, 11 and 14. These results are very close to those of Sec. (VI A). The small differences arise from the fact that eccentricity tends to enhance the amount of energy emitted during the inspiral as the system radiates in band for longer. It is reasonable to take these estimates for circular orbits as a guideline.
for eccentric systems of similar mass to these. If the source is eccentric, since $a = R_{\text{per}}/(1 - e)$, $a$ is larger at the time the source reaches a frequency of 10 Hz. The inspiral time depends on the value of $a$, and is larger for larger $a$. Another way to see this is that $dE/dt$ is smaller when $e$ is larger at fixed periapsis (or frequency in our approximation). This is because at fixed periapsis, increasing the eccentricity puts more of the orbit further from the MBH and hence the energy flux is on average reduced. As $dE/dt$ is smaller, it takes longer to inspiral. This also explains why the SNR is slightly lower – $dE/dt$ is lower at fixed periapsis and thus at fixed frequency in this approximate model (physically, energy is being radiated out of band so we do not detect it all).

VII. ACCUMULATED PHASE SHIFT

Understanding how IMRIs form and what are their orbital parameters can help us to reverse-engineer the environmental properties of the host cluster. Although the IMRIs considered in this work have very large initial eccentricities, when they reach the LIGO/Virgo band the eccentricity is virtually zero. It is however important to measure a non-zero eccentricity, because it can be a constraint on the formation mechanism as well as the stellar environment of the IMRI. If a residual eccentricity is present, it will induce a difference in the phase evolution of the signal as compared to a circular inspiral. Thanks to the derivation of [62] of the phase correction due to non-zero eccentricities, we can estimate the accumulated phase shift to lowest post-Newtonian order and to first order in $e^2$ with

$$\Delta \Psi_e(f) = \Psi_{\text{last}} - \Psi_1 \cong -\Psi_1 = \frac{7065}{187136} e_i^2 (\pi f M_z)^{-5/3}. \tag{12}$$

In the last equation $e_i$ is the eccentricity at the frequency of the dominant harmonic at which it enters the detector bandwidth, $f$ is the frequency for the $n = 2$ harmonic, and I have introduced the quantity $M_z := (1 + z)G (M_{\text{BH}} \times m_{\text{CO}})^{3/5} (M_{\text{BH}} + m_{\text{CO}})^{-1/5} / c^3$. Also, I make the approximation that $\Delta \Psi_e(f) = \Psi_{\text{last}} - \Psi_1 \cong -\Psi_1$, with $\Psi_{\text{last}}$ and $\Psi_1$ the final and initial phase. This is so because of the pronounced fall-off of $\Psi_e(f)$ with increasing frequency, see discussion in section B.2 of [63].

So as to derive the accumulated phase shift in terms of $f$ and the remaining time to merger, we now recall from [64] that the semi-major axis of the binary is

$$a^3 = \frac{G(M_{\text{BH}} + m_{\text{CO}})}{(\pi f)^2}. \tag{13}$$

The time for merger for $e \ll 1$ can be derived from [64] as follows,

$$T_{\text{merg}} \approx \frac{5}{256} \frac{c^5}{G^3 M_{\text{BH}} \times m_{\text{CO}} (M_{\text{BH}} + m_{\text{CO}})} \left[ \frac{G(M_{\text{BH}} + m_{\text{CO}})}{(\pi f)^2} \right]^{4/3}. \tag{14}$$
Last, let us recall that

\[ e^2 f^{19/9} \approx \text{constant}, \tag{15} \]

which can be derived from relation 5.12 of [54] with \( 1/(1-e^2) \approx 1 \) combined with Eq. (13), i.e. \( a \propto f^{-2/3} \).

Therefore, if we use Eq. (13) in Eq. (14), we obtain

\[ \pi f \approx \left( \frac{5}{256} \right)^{3/8} M_z^{-5/8} \tau_{\text{merg}}^{-3/8}. \tag{16} \]

Hence, using Eqs. (12, 13, 16), we have that the accumulated phase shift in terms of \( f, e_i(f), M_z \) and \( \tau_{\text{merg}} \) is

\[ \Delta \Psi_e(f) = \left( \frac{5}{256} \right)^{-17/12} \frac{7065}{187136} (\pi f_i)^{19/9} e_i^2 M_z^{25/36} \tau_{\text{merg}}^{17/12} \approx 10 (\pi f_i)^{19/9} e_i^2 M_z^{25/36} \tau_{\text{merg}}^{17/12} \tag{17} \]

The accumulated phase shift is detectable if \( \gtrsim \pi \). With this approximation, I find the following phase shifts in radians, for the IMRI systems presented in the previous sections, imposing a minimum threshold SNR of 5 (the numbers correspond to the panels of the figures from the top to the bottom, left to right):

(i) For LISA, and taking into account only the last five years before merger, Fig. (1) has a negligible phase shift. Fig. (2) \( 180, 3.4 \times 10^5 \), while the other two panels have a negligible phase shift. Fig. (3) \( 1.5 \times 10^6 \) and the right panel is negligible. Fig. (4) \( 5200 \) and the right panel is negligible. Least, Fig. (5) has also a negligible phase shift.

(ii) For the ET, Fig. (1) \( \sim 5.1 \times 10^{-3}, 19000 \) for the left and right panels, Fig. (2) \( \sim 2.6 \times 10^{-7}, \sim 3.4 \times 10^{-3}, 0.66 \) and 4600. Fig. (3) \( 1.3 \times 10^{-3} \) and 3900. Fig. (4) \( 3.5 \times 10^{-6} \) and 450. Fig. (5) \( 1.3 \times 10^{-2} \) and 2600. Fig. (6) has a negligible phase shift.

(iii) For LIGO, Fig. (1) \( 4 \times 10^{-6} \) and 1.2. Fig. (2) \( 1.1 \times 10^{-10}, 1.4 \times 10^{-6}, 2.3 \times 10^{-4} \) and 10. The rest of the cases have negligible phase shifts.

VIII. CONCLUSIONS

Intermediate-mass ratio inspirals are typically formed in dense stellar systems such as galactic nuclei and globular clusters, with typically very large eccentricities (from \( e = 0.999 \)) and small semi-major axis (below \( a \sim 10^{-5}\text{pc} \)), as found in a number of stellar-dynamics simulations of globular clusters [25, 39, 43–45]. Besides classical two-body relaxation, an interesting way of explaining the formation of these sources is the parabolic capture mechanism described by [23, 25].

In this work I show that IMRIs in clusters are detectable not only by space-borne observatories such as LISA. Depending on the properties of the IMRI, it can be detected in conjunction with LIGO/Virgo or the ET, so that ground-based and space-borne observatories should be envisaged as one instrument if they are simulaneously operative.

I have considered IMBHs with masses ranging between \( M_{\text{BH}} = 100 M_\odot \) up to 3000 \( M_\odot \) and COs with different masses. I have separated them in light and medium-size IMRIs, for IMBHs with masses up to 500 \( M_\odot \) (which is a particular case based on the findings of [39]) and large-mass IMRIs, for masses between 1000 \( M_\odot \) and 3000 \( M_\odot \).

I find that light and medium-size IMRIs can be observed by LISA and ground-based detectors for eccentricities starting at 0.99 and up to 0.9995. In the range of frequencies of LIGO/Virgo they spend a maximum of about one minute on band. Higher eccentricity sources, however, can only be detected by ground-based detectors (see [39] for a discussion on the role of eccentricity for low mass ratio binaries). This is due to the fact that, as the eccentricity increases, the pericenter distance decreases, so that the characteristic frequency of the GWs emitted at the pericenter increases (see [65], Eq. 37 for a derivation of the peak frequency in the same approximation.)
FIG. 11. Same as in Fig. (10) but taking into account only the SNR accumulated 10 (left panel) and 5 (right panel) years before the merger. See discussion in text.

used in this work). In some cases, the full cascade of harmonics falls entirely in the bandwidth of the ground-based detectors.

The peak of large-mass IMRIs recedes in frequency as compared to light and medium-size ones, so that the cascade of harmonics is shifted towards the LISA domain. However, for eccentricities below 0.9995, IMRIs with IMBHs covering the full range of masses considered in this work ($100 M_\odot$ up to $3000 M_\odot$) should be detectable with LISA with modest to large SNRs, from a few to tens, depending on the eccentricity and duration of the observation. For ground-based detectors, I compute the SNR for LIGO using the waveforms from a Fourier-transformation of the time-domain Taylor T4 algorithm of [61] (limited to eccentricities $\lesssim 0.2$) and derive large enough SNRs, always of a few tens.

Lower-frequency sources require larger eccentricities, and we cannot use these waveforms. For these detectors I use an approximate scheme to calculate the SNR, and I have compared it with the previous results for LIGO and I find that the approach is robust. The values for ET can reach as much as $\sim 2000$, and are of typically a few hundred and of tens for masses up to $2000 M_\odot$. LISA has SNRs of a few tens to then significantly drop when the IMRI system has the peak of harmonics closer to the ground-based regime.

By combining ground-based and space-borne observations we can impose better constraints on the system’s parameters. On the one hand, LISA can observe the inspiral and hence provide us with measurements of parameters such as the chirp mass. On the other hand, ground-base detectors detect the merger and ringdown, and therefore measure other parameters such as the final mass and spin. Thanks to this joint detection, one can split various degeneracies and get better measurements of the parameters, as compared to individual detections.

I have estimated with a semi-analytical approach the possible influence of the environment after their formation and I find no impact, which will make it easier to detect and interpret these sources.

By looking at the accumulated phase shift, one could investigate the origin of light IMRIs thanks to a residual eccentricity. I find that LISA binaries accumulate typically hundred of thousands and up to millions of radians, while ET binaries can accumulate up to 19000 radians, and typically of a few thousands. While IMRI binaries in LIGO live much shorter time, there is a case which does accumulate enough radians.

LISA can warn ground-based detectors with at least one year in advance and seconds of precision, so that this observatory and LIGO/Virgo and the ET should be thought of as a single detector, if they are operating at the same time. Until LISA is launched, the perspective of detecting IMRIs from the ground opens new possibilities.

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