Model checking for Process Rewrite Systems and a class of action-based regular properties

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Abstract

We consider the model checking problem for Process Rewrite Systems (PRSs), an infinite-state formalism (non Turing-powerful) which subsumes many common models such as Pushdown Processes and Petri Nets. PRSs can be adopted as formal models for programs with dynamic creation and synchronization of concurrent processes, and with recursive procedures. The model-checking problem for PRSs and action-based linear temporal logic (ALTL) is undecidable. However, decidability for some interesting fragment of ALTL remains an open question. In this paper we state decidability results concerning generalized acceptance properties about infinite derivations (infinite term rewriting) in PRSs. As a consequence, we obtain decidability of the model-checking (restricted to infinite runs) for PRSs and a meaningful fragment of ALTL.

Keywords: Infinite-state systems, process rewrite systems, petri nets, pushdown processes, model checking, action–based linear temporal logic.

1 Introduction

Automatic verification of systems is nowadays one of the most investigated topics. A major difficulty to face when considering this problem is that reasoning about systems in general may require dealing with infinite state models. Software systems may introduce infinite states both manipulating data ranging over infinite domains, and having unbounded control structures such as recursive procedure calls and/or dynamic creation of concurrent processes (e.g. multi–treading). Many different formalisms have been proposed for the description of infinite state systems. Among the most popular are the well known formalisms of Context Free Processes, Pushdown Processes, Petri Nets, and Process Algebras. The first two are models of sequential computation, whereas Petri Nets and Process Algebra explicitly take into account concurrency. The model checking problem for these infinite
state formalisms have been studied in the literature. As far as Context Free Processes and Pushdown Processes are concerned, decidability of the modal $\mu$-calculus, the most powerful of the modal and temporal logics used for verification, has been established (see [2, 7, 10, 11, 13]). In [6, 8, 9], model checking for Petri nets has been studied. The branching temporal logic as well as the state-based linear temporal logic are undecidable even for restricted logics. Fortunately, the model checking for action-based linear temporal logic (ALTL) [8, 9, 12] is decidable.

Verification of formalisms which accommodate both parallelism and recursion is a challenging problem. In order to formally study this kind of systems, recently the formal framework of Process Rewrite Systems (PRSs) has been introduced [12]. This framework (non Turing-powerful), which is based on term rewriting, subsumes many common infinite states models such as Pushdown Processes and Petri Nets. PRSs can be adopted as formal models for programs with dynamic creation and (a restricted form of) synchronization of concurrent processes, and with recursive procedures. The decidability results already known in the literature for the general framework of PRSs concern reachability analysis [12] and symbolic reachability analysis [3, 4]. Unfortunately, the model checking of action-based linear temporal logic becomes undecidable [1, 12]. It remains undecidable even for restricted models such as PA processes [1]. However, decidability for some interesting fragment of ALTL and the general framework of PRSs remains an open question.

Our contribution: In this paper we state a decidability result concerning generalized acceptance properties about infinite derivations (infinite term rewriting) in PRSs. In order to formalize these properties we introduce the notion of Multi Büchi Rewrite Systems (MBRS) that is, informally speaking, a PRS with a finite number of accepting components, where each component is a subset of the PRS. Moreover, as a consequence of our decidability result, we obtain decidability of the model checking (restricted to infinite runs) for PRSs and a meaningful fragment of ALTL. Within this fragment we can express important classes of properties like invariant, as well as strong and weak fairness constraints.

Plan of the paper: In Section 2, we recall the framework of Process Rewrite Systems and ALTL logic. In Section 3, we introduce the notion of Multi Büchi Rewrite System, and show how our decidability result about generalized acceptance properties of infinite derivations in PRSs can be used in model-checking for a meaningful ALTL fragment. In Section 4, we prove our decidability result. Several proofs are omitted for lack of space. They can be found in the extended version of this paper.

Related Work: Our decidability result extends one stated in [5], regarding classical acceptance properties (a la Büchi) of derivations in PRSs. In particular, our ALTL fragment is strictly more expressive (and surely more interesting in the applications) than one considered in [5].
2 Preliminaries

2.1 Process Rewrite Systems

Definition 2.1 (Process Term). Let $\text{Var} = \{X, Y, \ldots\}$ be a finite set of process variables. The set of process terms $t$ over $\text{Var}$, denoted by $T$, is defined by the following syntax:

$$t ::= \varepsilon \mid X \mid t \cdot t \mid t\parallel t$$

where $X \in \text{Var}$, $\varepsilon$ denotes the empty term, “$\cdot$” denotes parallel composition, and “$\parallel$” denotes sequential composition.

We always work with equivalence classes of process terms modulo commutativity and associativity of “$\parallel$”, and modulo associativity of “$\cdot$”. Moreover $\varepsilon$ will act as the identity for both parallel and sequential composition.

Definition 2.2 (Process Rewrite System). A Process Rewrite System (or PRS, or Rewrite System) over a finite alphabet of atomic actions $\Sigma$ and the set of process variables $\text{Var}$ is a finite set of rewrite rules $\mathcal{R} \subseteq T \times \Sigma \times T$ of the form $t \xrightarrow{a} t'$, where $t$ ($\neq \varepsilon$) and $t'$ are terms in $T$, and $a \in \Sigma$.

A PRS $\mathcal{R}$ over $\text{Var}$ and the alphabet $\Sigma$ induces a labelled transition system (LTS) over $T$ with a transition relation $\rightarrow \subseteq T \times \Sigma \times T$ that is the smallest relation satisfying the following inference rules:

$$
\frac{(t \xrightarrow{a} t') \in \mathcal{R}}{t \xrightarrow{a} t'} \quad \frac{t \xrightarrow{a} t'} {t \parallel t \xrightarrow{a} t'} \quad \frac{t \xrightarrow{a} t'} {t \parallel t \parallel t' \xrightarrow{a} t'} \quad \frac{t \xrightarrow{a} t'} {t \cdot t \xrightarrow{a} t \cdot t'}
$$

where $t, t', t_1, t_1'$ are process terms and $a \in \Sigma$.

In similar way we define for every rule $r \in \mathcal{R}$ the notion of one-step derivation by $r$ relation, denoted by $\xrightarrow{r}$.

A path in $\mathcal{R}$ from $t_0 \in T$ is a (finite or infinite) sequence of LTS edges of the form $t_0 \xrightarrow{a_0} t_1, t_1 \xrightarrow{a_1} t_2, \ldots$, denoted by $t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \ldots$. A run in $\mathcal{R}$ from $t_0$ is a maximal path from $t_0$, i.e., a path from $t_0$ which is either infinite or has the form $t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} \ldots \xrightarrow{a_{n-1}} t_n$ and there is no edge $t_n \xrightarrow{a} t \in \rightarrow$, for any $a \in \Sigma$ and $t \in T$. We write $\text{runs}_\mathcal{R}(t_0)$ (resp., $\text{runs}_\mathcal{R}(\infty)(t_0)$) to refer to the set of runs (resp., infinite runs) in $\mathcal{R}$ from $t_0$, and $\text{runs}(\mathcal{R})$ to refer to the set of all the runs in $\mathcal{R}$.

A finite derivation in $\mathcal{R}$ from a term $t$ to a term $t'$ (through a finite sequence $\sigma = r_1 r_2 \ldots r_n$ of rules in $\mathcal{R}$), is a sequence $d$ of one-step derivations of the form $t_0 \xrightarrow{r_1} t_1, t_1 \xrightarrow{r_2} t_2, \ldots, t_{n-1} \xrightarrow{r_n} t_n$, with $t_0 = t$ and $t_n = t'$, and it is denoted by $t_0 \xrightarrow{\sigma} t_1 \xrightarrow{r_2} t_2 \ldots \xrightarrow{r_n} t_n$. The derivation $d$ is a $n$-step derivation (or a derivation of length $n$), and for succinctness is also denoted by $t \xrightarrow{\mathcal{R}^*} t'$. Moreover, we say that $t'$ is reachable in $\mathcal{R}$ from the term $t$ (through derivation $d$). If $\sigma$ is empty, we say that $d$ is a null derivation.

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1When we look at terms we think of it as right-associative. So, when we say that a term has the form $t_1 \cdot t_2$, then we mean that $t_1$ is either a single variable or a parallel composition of process terms.
An infinite derivation in $\mathcal{R}$ from a term $t_0$ (through an infinite sequence $\sigma = r_1 r_2 \ldots$ of rules in $\mathcal{R}$), is an infinite sequence of one step derivations of the form $t_0 \xrightarrow{a_1} t_1 \xrightarrow{a_2} t_2 \ldots$, denoted by $t_0 \xrightarrow{\sigma}\ast$. For succinctness such a derivation is also denoted by $t_0 \xrightarrow{\sigma^{\ast}}$.

For technical reasons, we shall also consider PRSs in a syntactical restricted form called normal form $[12]$. A PRS $\mathcal{R}$ is said to be in normal form if every rule $r \in \mathcal{R}$ has one of the following forms:

**PAR rules:** $X_1 \parallel X_2 \ldots \parallel X_p \xrightarrow{a} Y_1 \parallel Y_2 \ldots \parallel Y_q$ where $p \in N \setminus \{0\}$ and $q \in N$.

**SEQ rules:** $X \xrightarrow{a} Y.Z$ or $X.Y \xrightarrow{a} Z$ or $X \xrightarrow{a} Y$ or $X \xrightarrow{a} \varepsilon$.

with $X,Y,Z,X_i,Y_j \in \text{Var}$. A PRS where all the rules are SEQ (resp., PAR) rules is called sequential (resp., parallel) PRS.

### 2.2 ALTL (Action–based LTL)

Given a finite set $\Sigma$ of atomic actions, the set of formulae $\varphi$ of ALTL over $\Sigma$ is defined as follows:

$$\varphi ::= \text{true} \mid \neg \varphi \mid \varphi \land \varphi \mid \langle a \rangle \varphi \mid \varphi U \varphi$$

where $a \in \Sigma$, $\langle a \rangle \varphi$ denotes the one–step next operator, and $U$ denotes the strong until operator. We also consider the derived operators $F \varphi ::= \text{true} U \varphi$ ("eventually $\varphi$") and its dual $G \varphi ::= \neg F \neg \varphi$ ("always $\varphi$").

In order to give semantics to ALTL formulae on a PRS $\mathcal{R}$, we need some additional notation. Given a path $\pi = t_0 \xrightarrow{a_0} t_1 \xrightarrow{a_1} t_2 \xrightarrow{a_2} \ldots$ in $\mathcal{R}$, $\pi^i$ denotes the suffix of $\pi$ starting from the $i$–th term in the sequence, i.e. the path $t_i \xrightarrow{a_i} t_{i+1} \xrightarrow{a_{i+1}} \ldots$. If the path $\pi$ is non–trivial (i.e., the sequence contains at least two terms) we denote the first action $a_0$ by firstact($\pi$).

ALTL formulae over a PRS $\mathcal{R}$ are interpreted in terms of the set of the runs in $\mathcal{R}$ satisfying the given ALTL formula. The denotation of a formula $\varphi$ relative to $\mathcal{R}$, in symbols $[\varphi]_\mathcal{R}$, is defined inductively as follows:

- $[[\text{true}]_\mathcal{R}] = \text{runs}(\mathcal{R})$,
- $[[\neg \varphi]]_\mathcal{R} = \text{runs}(\mathcal{R}) \setminus [[\varphi]]_\mathcal{R}$,
- $[[\varphi_1 \land \varphi_2]]_\mathcal{R} = [[\varphi_1]]_\mathcal{R} \cap [[\varphi_2]]_\mathcal{R}$,
- $[[\langle a \rangle \varphi]]_\mathcal{R} = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{firstact}(\pi) = a \text{ and } \pi^i \in [[\varphi]]_\mathcal{R} \}$,
- $[[\varphi_1 U \varphi_2]]_\mathcal{R} = \{ \pi \in \text{runs}(\mathcal{R}) \mid \text{for some } i \geq 0 \pi^i \in [[\varphi_2]]_\mathcal{R} \text{ and for all } j < i \pi^j \in [[\varphi_1]]_\mathcal{R} \}$.

For any term $t \in T$ and ALTL formula $\varphi$, we say that $t$ satisfies $\varphi$ (resp., satisfies $\varphi$ restricted to infinite runs) (w.r.t $\mathcal{R}$), in symbols $t \models_\mathcal{R} \varphi$ (resp., $t \models_{\mathcal{R},\infty} \varphi$), if $\text{runs}_\mathcal{R}(t) \subseteq [[\varphi]]_\mathcal{R}$ (resp., $\text{runs}_{\mathcal{R},\infty}(t) \subseteq [[\varphi]]_\mathcal{R}$).
The model-checking problem (resp., model–checking problem restricted to infinite runs) for ALTL and PRSs is the problem of deciding if, given a PRS $\mathcal{R}$, an ALTL formula $\varphi$ and a term $t$ of $\mathcal{R}$, $t \models_\mathcal{R} \varphi$ (resp., $t \models_{\mathcal{R},\infty} \varphi$). The following is a well–known result:

**Proposition 2.1** (see [2, 8, 12]). The model–checking problem for ALTL and parallel (resp., sequential) PRSs, possibly restricted to infinite runs, is decidable.

### 3 Multi Büchi Rewrite Systems

**Definition 3.1 (Multi Büchi Rewrite System).** A Multi Büchi Rewrite System (MBRS) (with $n$ accepting components) over a finite set of process variables $\text{Var}$ and an alphabet $\Sigma$ is a tuple $M = \langle \mathcal{R}, \langle \mathcal{R}_A^1, \ldots, \mathcal{R}_A^n \rangle \rangle$, where $\mathcal{R}$ is a PRS over $\text{Var}$ and $\Sigma$, and for all $i = 1, \ldots, n$ $\mathcal{R}_i^A \subseteq \mathcal{R}$. $\mathcal{R}$ is called the support of $M$.

In the definition above, if $n = 1$, then $M$ is also called Büchi Rewrite System (BRS) [5], and every rule $r \in \mathcal{R}_1^A$ is called accepting rule of $M$.

We say that $M$ is a MBRS in normal form (resp., sequential MBRS, parallel MBRS) if the underlying PRS $\mathcal{R}$ is in normal form (resp., is sequential, is parallel). For a rule sequence $\sigma$ in $\mathcal{R}$ the finite maximal of $\sigma$ as to $M$, denoted by $\Upsilon_f^M(\sigma)$, is the set $\{i \in \{1, \ldots, n\} | \sigma$ contains some occurrence of rule in $\mathcal{R}_i^A\}$. The infinite maximal of $\sigma$ as to $M$, denoted by $\Upsilon_\infty^M(\sigma)$, is the set $\{i \in \{1, \ldots, n\} | \sigma$ contains infinite occurrences of some rule in $\mathcal{R}_i^A\}$. Given $K, K^\omega \subseteq \{1, \ldots, n\}$ and a derivation $t \overset{\sigma}{\Rightarrow}_{\mathcal{R}}$, we say that $t \overset{\sigma}{\Rightarrow}_{\mathcal{R}}$ is a $(K, K^\omega)$-accepting derivation in $M$ if $\Upsilon_f^M(\sigma) = K$ and $\Upsilon_\infty^M(\sigma) = K^\omega$.

For all $n \in \mathbb{N} \setminus \{0\}$ let us denote by $P_n$ the set $2^\{1,\ldots,n\}$ (i.e., the set of the subsets of $\{1, \ldots, n\}$).

### 3.1 Model-checking of PRSs

The main result of the paper concerns the decidability of the following problem:

**Problem 1:** Given a MBRS $M = \langle \mathcal{R}, \langle \mathcal{R}_1^A, \ldots, \mathcal{R}_n^A \rangle \rangle$ over $\text{Var}$ and the alphabet $\Sigma$, given a process term $t$ and two sets $K, K^\omega \in P_n$, to decide if there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $t$.

Without loss of generality we can assume that the input term $t$ in Problem 1 is a process variable in $\text{Var}$. In fact, if $t \notin \text{Var}$, then, starting from $M$, we construct a new MBRS $M'$ by adding a new variable $X$ and a rule of the form $X \rightarrow t$ whose finite maximal as to $M'$ is the empty set.

Before proving the decidability of Problem 1 in Section 4, we show how a solution to this problem can be effectively exploited for automatic verification of some meaningful
(action-based) linear time properties of infinite runs in PRSs. In particular, we consider the following ALTL fragment

\[ \varphi ::= F \psi \mid GF \psi \mid \neg \varphi \mid \varphi \land \varphi \]  

(1)

where \( \psi \) denotes an ALTL propositional formula. For succinctness, we denote an ALTL propositional formula of the form \(<a>\ true\) (with \(a \in \Sigma\)) simply by \(a\).

Within this fragment, property patterns frequent in system verification can be expressed. In particular, we can express safety properties (e.g., \(G \psi_1\)), guarantee properties (e.g., \(F \psi_1 \rightarrow F \psi_2\), or \(G \psi_1 \rightarrow G \psi_2\)), obligation properties (e.g., \(F \psi_1 \rightarrow F \psi_2\), \(G \psi_1 \rightarrow G \psi_2\)), response properties (e.g., \(GF \psi_1\)), persistence properties (e.g., \(FG \psi_1\)), and finally reactivity properties (e.g., \(GF \psi_1 \rightarrow GF \psi_2\)). Notice that important classes of properties like invariants, as well as strong and weak fairness constraints, can be expressed.

In order to prove decidability of the model-checking problem restricted to infinite runs for this fragment of ALTL we need some definitions. Given a propositional formula \(\psi\) over \(\Sigma\), we denote by \([\psi]_\Sigma\) the subset of \(\Sigma\) inductively defined as follows

- for all \(a \in \Sigma\) \([a]_\Sigma = \{a\}\),
- \([\neg \psi]_\Sigma = \Sigma \setminus [\psi]_\Sigma\),
- \([\psi_1 \land \psi_2]_\Sigma = [\psi_1]_\Sigma \cap [\psi_2]_\Sigma\).

Evidently, given a PRS \(\mathcal{R}\) over \(\Sigma\), an ALTL propositional formula \(\psi\) and an infinite run \(\pi\) of \(\mathcal{R}\), we have that \(\pi \in [\psi]_\mathcal{R}\) iff \(\text{firstact}(\pi) \in [\psi]_\Sigma\). Given a rule \(r = t \xrightarrow{a} t' \in \mathcal{R}\), we say that \(r\) satisfies \(\psi\) if \(a \in [\psi]_\Sigma\). We denote by \(AC(\psi)\) the set of rules in \(\mathcal{R}\) that satisfy \(\psi\).

Now, we can prove the following result

**Theorem 3.1.** The model-checking problem for PRSs and the fragment ALTL (1), restricted to infinite runs, is decidable.

**Proof.** Given a PRS \(\mathcal{R}\), a process term \(t\) and a formula \(\varphi\) belonging to ALTL fragment (1), we have to decide if \(t \models_{\mathcal{R}, \infty} \varphi\) or, equivalently, if there exists an infinite run \(\pi \in \text{runs}_{\mathcal{R}, \infty}(t)\) satisfying the formula \(\neg \varphi\).

Let us consider the derived operator \(F^+ \varphi := F \varphi \land \neg GF \varphi\). Pushing negation inward, and using the following logic equivalences

- \(G \varphi_1 \land G \varphi_2 \equiv G(\varphi_1 \land \varphi_2)\)
- \(\neg F \varphi_1 \equiv G \neg \varphi_1\)
- \(\neg G \varphi_1 \equiv F \neg \varphi_1\)
- \(F \varphi_1 \equiv F^+ \varphi_1 \lor GF \varphi_1\)

The set of ALTL propositional formulae \(\psi\) over the set \(\Sigma\) of atomic actions is defined as follows:

\[ \psi ::= <a>\ true \mid \psi \land \psi \mid \neg \psi \]  

where \(a \in \Sigma\).
\[ \neg \varphi \equiv \bigvee_i \left( \bigwedge_j F^+ \psi_j \land \bigwedge_k G \eta_k \land G \zeta \right) \]  

where \( \psi_j, \eta_k, \) and \( \zeta \) are ALTL propositional formulae. Evidently, we can restrict ourselves to consider a single disjunct in (2). In other words, our starting problem is reducible to the problem of deciding, given a formula having the following form

\[ F^+ \psi_1 \land \ldots \land F^+ \psi_{m_1} \land GF \eta_1 \land \ldots \land GF \eta_{m_2} \land G \zeta^3 \]  

if there exists an infinite run \( \pi \in \text{runs}_{R,\infty}(t) \) satisfying formula (3).

Let us consider the MBRS in normal form \( M = (R, (R_{A1}, \ldots, R_{An})) \) where \( n = m_1 + m_2 + 1 \) and

\[
\begin{align*}
\text{for all } i = 1, \ldots, m_1 & \quad R_{Ai} = AC_R(\psi_i) \\
\text{for all } j = 1, \ldots, m_2 & \quad R_{j+m_1} = AC_R(\eta_j) \\
R_{m_1+m_2+1} & = AC_R(\neg \zeta)
\end{align*}
\]

Let \( K = \{1, \ldots, m_1 + m_2\} \) and \( K^\omega = \{m_1 + 1, \ldots, m_1 + m_2\} \). It is easy to show that there exists a run \( \pi \in \text{runs}_{R,\infty}(t) \) satisfying formula (3) if and only if there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( t \). By the decidability of Problem 1, we obtain the assertion. \( \square \)

## 4 Decidability results on MBRSs

In this section we prove the main result of the paper, i.e. the decidability of Problem 1 defined in Subsection 3.1. We proceed in two steps. First, in Subsection 4.1 we decide the problem for the class of MBRSs in normal form. Then, in Subsection 4.2 we extend the result to the whole class of MBRSs. For the proof we need some preliminary results, represented by the following Propositions 4.1–4.3, that easily follow from the decidability of ALTL model-checking problem for parallel (resp., sequential) PRSs (see Proposition 2.1).

**Proposition 4.1.** Given a parallel MBRS \( M_P = (R_P, (R^A_{P_1}, \ldots, R^A_{P_n})) \) over \( Var \), given two variables \( X, Y \in Var \) and \( K \in P_n \), it is decidable whether there exists a finite derivation in \( R_P \) starting from \( X \) (resp., of the form \( X \xrightarrow[\sigma]{s_P} Y \), of the form \( X \xrightarrow[\sigma]{s_P} \varepsilon \), of the form \( X \xrightarrow[\sigma]{s_P} t \| Y \) with \(|\sigma| > 0\)) such that \( \Upsilon^f_{M_P}(\sigma) = K \).

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\( ^3\psi_j, \eta_k \) and \( \zeta \) are ALTL propositional formulae
Proposition 4.2. Let us consider two parallel MBRSs \( M_{P_1} = \langle \mathcal{R}_P, \langle \mathcal{R}_{P_1}^A, \ldots, \mathcal{R}_{P_1}^n \rangle \rangle \) and \( M_{P_2} = \langle \mathcal{R}_P, \langle \mathcal{R}_{P_2}^A, \ldots, \mathcal{R}_{P_2}^n \rangle \rangle \) over \( \mathcal{V} \mathcal{A} \mathcal{R} \), and with the same support \( \mathcal{R}_P \). Given a variable \( X \in \mathcal{V} \mathcal{A} \mathcal{R} \), two sets \( K, K^\omega \in P_n \), and a subset \( \mathcal{R}_P^* \) of \( \mathcal{R}_P \) it is decidable whether there exists a derivation in \( \mathcal{R}_P \) of the form \( X \xrightarrow{\omega} \mathcal{R}_P^* \) such that \( \mathcal{Y}^f_{M_{P_1}}(\sigma) = K \), \( \mathcal{Y}^f_{M_{P_2}}(\sigma) = K^\omega \), and \( \sigma \) is either infinite or contains some occurrence of rule in \( \mathcal{R}_P \setminus \mathcal{R}_P^* \).

Now, let us give an additional notion of reachability (for variables) in sequential PRSs.

Definition 4.1. Given a sequential PRS \( \mathcal{R}_S \) over \( \mathcal{V} \mathcal{A} \mathcal{R} \), and \( X, Y \in \mathcal{V} \mathcal{A} \mathcal{R} \), \( Y \) is reachable from \( X \) in \( \mathcal{R}_S \) if there exists a term \( t \) of the form \( X_1.X_2.\ldots.X_n.Y \) such that \( X \xrightarrow{\omega} \mathcal{R}_S t \).

Proposition 4.3. Let us consider a sequential MBRS \( M_S = \langle \mathcal{R}_S, \langle \mathcal{R}_{S_1}^A, \ldots, \mathcal{R}_{S_n}^A \rangle \rangle \) over \( \mathcal{V} \mathcal{A} \mathcal{R} \). Given two variables \( X, Y \in \mathcal{V} \mathcal{A} \mathcal{R} \) and two sets \( K, K^\omega \in P_n \), it is decidable whether

1. \( Y \) is reachable from \( X \) in \( \mathcal{R}_S \) through a derivation having finite maximal \( K \) as to \( M_S \).

2. There exists a \( (K, K^\omega) \)-accepting infinite derivation in \( M_S \) from \( X \).

4.1 Decidability of Problem 1 for MBRSs in normal form

In this subsection we prove the decidability of Problem 1 restricted to the class of MBRSs in normal form. We shall use the following result stated in [5].

Theorem 4.1 (see [5]). Given a BRS \( M = \langle \mathcal{R}, \mathcal{R}_F \rangle \) in normal form and a process variable \( X \) it is decidable whether there exists an infinite derivation in \( \mathcal{R} \) from \( X \) of the form \( X \xrightarrow{\omega} \mathcal{R}^* \) such that \( \sigma \) does not contain occurrences of accepting rules.

Let \( M = \langle \mathcal{R}, \langle \mathcal{R}_1^A, \ldots, \mathcal{R}_n^A \rangle \rangle \) be a MBRS in normal form over \( \mathcal{V} \mathcal{A} \mathcal{R} \) and the alphabet \( \Sigma \), and \( K \) and \( K^\omega \) be elements in \( P_n \). Given \( X \in \mathcal{V} \mathcal{A} \mathcal{R} \), we have to decide if there exists a \( (K, K^\omega) \)-accepting infinite derivation in \( M \) from \( X \). The proof of decidability is by induction on \( |K| + |K^\omega| \).

Base Step: \( |K| = 0 \) and \( |K^\omega| = 0 \). Let \( M_F = \langle \mathcal{R}, \mathcal{R}_F \rangle \) be the BRS with \( \mathcal{R}_F = \bigcup_{i=1}^n \mathcal{R}_i^A \). Given an infinite derivation \( X \xrightarrow{\omega} \mathcal{R} \) in \( \mathcal{R} \) from a variable \( X \), then this derivation is \( (\emptyset, \emptyset) \)-accepting in \( M \) if, and only if, it does not contain occurrences of accepting rules in \( M_F \). So, the decidability result follows from Theorem 4.1.

Inductive Step: \( |K| + |K^\omega| > 0 \). By the inductive hypothesis, for each \( K' \subseteq K \) and \( K^\omega \subseteq K^\omega \) with \( |K'| + |K^\omega'| < |K| + |K^\omega| \) the result holds. Starting from this assumption we shall show that Problem 1, with input the sets \( K \) and \( K^\omega \), can be reduced to (a combination of) two similar, but simpler, problems (that are decidable): the first (resp., the second) is a decidability problem on infinite derivations of parallel (resp., sequential) MBRSs. Before illustrating our approach, we need few additional definitions and notation.

Remark 4.1. Since \( M \) is in normal form we can limit ourselves to consider only terms \( t \), called terms in normal form, defined as \( t ::= X \mid t\|$ | X, t \) (where \( X \in \mathcal{V} \mathcal{A} \mathcal{R} \)). In fact, given a term in normal form \( t \), each term \( t' \) reachable from \( t \) in \( M \) is still in normal form.
In the following, $M_{P} = \langle \mathcal{R}_{P}, (\mathcal{R}_{P,1}^{A}, \ldots, \mathcal{R}_{P,n}^{A}) \rangle$ denotes the restriction of $M$ to the PAR rules, i.e. $\mathcal{R}_{P}$ (resp., $\mathcal{R}_{P,i}^{A}$ for $i = 1, \ldots, n$) is the set $\mathcal{R}$ (resp., $\mathcal{R}_{i}^{A}$ for $i = 1, \ldots, n$) restricted to the PAR rules. Moreover, we shall use two new variables $\hat{Z}_{F}$ and $\hat{Z}_{\infty}$, and denote by $T$ (resp., $T_{PAR}$, $T_{SEQ}$) the set of process terms in normal form (resp., in which no sequential composition occurs, in which no parallel composition occurs) over $\text{Var} \cup \{ \hat{Z}_{F}, \hat{Z}_{\infty} \}$.

**Definition 4.2 (Subderivation).** Let $t \xrightarrow{\lambda_{s}} t \parallel (X.s) \xrightarrow{\sigma_{s}}$ be a derivation in $\mathcal{R}$ from $t \in T$. The set of the subderivations $d'$ of $d = (t \parallel (X.s) \xrightarrow{\sigma_{s}})$ from $s$ is inductively defined as follows:

1. if $d$ is a null derivation or $s = \varepsilon$ or $d$ is of the form $t \parallel (X.Z) \xrightarrow{\sigma_{s}} Y \xrightarrow{\sigma'_{s}}$ (with $r = X.Z \xrightarrow{a_{s}} Y$ and $s = Z \in \text{Var}$), then $d'$ is the null derivation from $s$;
2. if $d$ is of the form $t \parallel (X.s) \xrightarrow{\sigma_{s}} t \parallel (X.s') \xrightarrow{\sigma'_{s}}$ (with $s \xrightarrow{\sigma_{s}} s'$) and $s' \xrightarrow{\sigma'_{s}}$ is a subderivation of $t \parallel (X.s') \xrightarrow{\sigma'_{s}}$ from $s'$, then $s \xrightarrow{\sigma_{s}} s' \xrightarrow{\sigma'_{s}}$ is a subderivation of $d$ from $s$;
3. if $d$ is of the form $t \parallel (X.s) \xrightarrow{\sigma_{s}} t' \parallel (X.s) \xrightarrow{\sigma'_{s}}$ (with $t \xrightarrow{\sigma_{s}} t'$), then every subderivation of $t' \parallel (X.s) \xrightarrow{\sigma'_{s}}$ from $s$ is also a subderivation of $d$ from $s$.

Moreover, we say that $d'$ is a subderivation of $\overline{t} \xrightarrow{\lambda_{s}} t \parallel (X.s) \xrightarrow{\sigma_{s}}$

Given a rule sequence $\sigma$ in $\mathcal{R}$, and a subsequence $\sigma'$ of $\sigma$, $\sigma \setminus \sigma'$ denotes the rule sequence obtained by removing from $\sigma$ all and only the occurrences of rules in $\sigma'$. Let us denote by $\Pi_{PAR, \infty}^{K,K_{\omega}}$ the set of derivations in $\mathcal{R}$ such that there does not exist a subderivation of $d$ that is a $(K, K_{\omega})$-accepting infinite derivation in $M$.

Let us sketch the main idea of our technique. At first, let us focus on the class of derivations $\Pi_{PAR, \infty}^{K,K_{\omega}}$. Let $p \xrightarrow{\rho_{s}}$ be a $(\overline{K}, \overline{K}_{\omega})$-accepting derivation in $M$ belonging to $\Pi_{PAR, \infty}^{K,K_{\omega}}$ with $p \in T_{PAR}$, $K \subseteq K$ and $\overline{K}_{\omega} \subseteq K_{\omega}$. The idea is to mimic this derivation by using only PAR rules belonging to extensions of the parallel MBRS $M_{P}$. If $\sigma$ contains only PAR rule occurrences, then $p \xrightarrow{\rho_{s}}$ is a $(\overline{K}, \overline{K}_{\omega})$-accepting derivation in the parallel MBRS $M_{P}$. Otherwise, $p \xrightarrow{\rho_{s}}$ can be written in the form:

$$p \xrightarrow{\lambda_{s}} \overline{p} \parallel X \xrightarrow{\sigma_{s}} p \parallel (Y.Z) \xrightarrow{\omega_{s}}$$ \hspace{1cm} (1)

where $r = X \xrightarrow{a_{s}} Y.Z$, $\lambda$ contains only occurrences of PAR rules in $\mathcal{R}$, $\overline{p} \in T_{PAR}$ and $X, Y, Z \in \text{Var}$. Let $Z \xrightarrow{\rho_{s}}$ be a subderivation of $p \parallel (Y.Z) \xrightarrow{\omega_{s}}$ from $Z$. By the definition of subderivation only one of the following four cases may occur:

**A** $Z \xrightarrow{\rho_{s}}$ is finite and $\overline{p} \xrightarrow{\omega_{s}}$.

**B** $Z \xrightarrow{\rho_{s}}$ leads to the term $\varepsilon$, and $p \xrightarrow{\sigma_{s}}$ is of the form $p \xrightarrow{\lambda_{s}} \overline{p} \parallel X \xrightarrow{\sigma_{s}} p \parallel (Y.Z) \xrightarrow{\omega_{s}} t \parallel Y \xrightarrow{\omega_{s}}$ where $\rho$ is a subsequence of $\omega_{1}$ and $\overline{p} \xrightarrow{\omega_{1}} \rho_{s} \parallel t$. 

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C $Z \xrightarrow{\rho^*} \text{ leads to a variable } W \in Var$, and $p \xrightarrow{\sigma^*} \text{ can be written as}$

$$p \xrightarrow{\lambda^*} \text{ implies } X \xrightarrow{r^*} (Y.Z) \xrightarrow{\omega^*} t \parallel (Y.W) \xrightarrow{r^*} t \parallel W' \xrightarrow{\omega^*}$$ (2)

where $r' = Y.W \xrightarrow{b} W'$ (with $W' \in Var$), $\rho$ is a subsequence of $\omega_1$ and $p \xrightarrow{\omega^*} t$.

D $Z \xrightarrow{\rho^*}$ is infinite, and $p \xrightarrow{\omega^*} \text{.}$

Cases A, B and C are similar, so for brevity we examine only cases C and D. At first, let us consider case C. The derivation in equation (2) is $(\overline{K}, \overline{K}^\omega)$-accepting if, and only if, the following derivation, obtained by anticipating the application of the rules in $\rho$ before the application of the rules in $\xi = \omega_1 \setminus \rho$, is $(\overline{K}, \overline{K}^\omega)$-accepting

$$p \xrightarrow{\lambda^*} \overline{p} \parallel X \xrightarrow{r^*} \overline{p} \parallel (Y.Z) \xrightarrow{\rho^*} \overline{p} \parallel (Y.W) \xrightarrow{\rho^*} \overline{p} \parallel W' \xrightarrow{\omega^*}$$ (3)

The idea is to collapse the finite derivation $X \xrightarrow{r^*} Y.Z \xrightarrow{\rho^*} Y.W \xrightarrow{\rho^*} W'$ into a single PAR rule of the form $r'' = X \xrightarrow{K_1'} W'$ where $K' = \Upsilon_M(r^* \rho) \subseteq K$. So, the label of $r''$ keeps track of the finite maximal of $rr'\rho$ in $M$. Now, we can apply recursively the same reasoning to the derivation in $\mathcal{K}$ from $p \parallel W' \in T_{PAR}$ given by $p \parallel W' \xrightarrow{\lambda^*} \overline{p} \parallel W' \xrightarrow{\omega^*}$, which belongs to $\Pi_{PAR,\infty}^{K,K^\omega}$ and whose finite (resp., infinite) maximal as to $M$ is contained in $K$ (resp., $K^\omega$).

Now, let us consider case D. Since $p \xrightarrow{\omega^*}$ belongs to $\Pi_{PAR,\infty}^{K,K^\omega}$, we have that $\Upsilon_M(r) = K_2 \subseteq K$, $\Upsilon_M(\rho) = K_1 \subseteq K$, $Y_M(r) = \emptyset$ and $|K_1^r| < |K| + |K^\omega|$. From our assumptions (inductive hypothesis) it is decidable whether there exists a $(K_1^r, K^\omega)$-accepting infinite derivation in $M$ from variable $Z$. Then, we keep track of the infinite rule sequence $r\rho$ by adding a PAR rule of the form $r' = X \xrightarrow{K_1'} Z_\infty$ with $K' = K_1 \cup K_2$. So, the label of $r'$ keeps track of the finite and infinite maximal of $r\rho$ in $M$. Now, we can apply recursively the same reasoning to the derivation $p \parallel Z_\infty \xrightarrow{\omega^*}$ in $\mathcal{K}$ from $p \parallel Z_\infty \in T_{PAR}$, which belongs to $\Pi_{PAR,\infty}^{K,K^\omega}$ and whose finite (resp., infinite) maximal as to $M$ is contained in $K$ (resp., $K^\omega$).

In other words, all subderivations in $p \xrightarrow{\omega^*}$ are abstracted away by PAR rules not belonging to $\mathcal{K}$, according to the intuitions given above.

For keeping track of the finite subderivations of the forms A, B and C, we define a first extension of the parallel MBRS $M_P$ in the following way.

**Definition 4.3.** The MBRS $M_{PAR}^K = (\mathcal{R}_{PAR}, (\mathcal{R}_{PAR,1}^{K,A}, \ldots, \mathcal{R}_{PAR,n}^{K,A}))$ is the least parallel MBRS with $n$ accepting components, over $Var \cup \{\hat{Z}_F\}$ and the alphabet $\hat{\Sigma} = \Sigma \cup P_n^4$, satisfying the following properties:

1. $\mathcal{R}_{PAR} \supseteq \mathcal{R}_P$ and $\mathcal{R}_{PAR,i} \supseteq \mathcal{R}_{Por,i}^4$ for all $i = 1, \ldots, n.$

\(^4\) let us assume that $\Sigma \cap P_n = \emptyset$
2. Let \( r = X \overset{a}{\rightarrow} Y.Z \in \mathcal{R} \), \( Z \overset{\sigma}{\rightarrow}_{\mathcal{R}_{\text{PAR}}}^* \varepsilon \), and \( K' = \Upsilon_f^M(r) \cup \Upsilon_{\mathcal{M}_{\text{PAR}}}^f(\sigma) \). If \( K' \subseteq K \), then \( r' = X \overset{K'}{\rightarrow} \hat{Z}_F \in \mathcal{R}_{\text{PAR}}^K \) (resp., \( r' = X \overset{K'}{\rightarrow} Y \in \mathcal{R}_{\text{PAR}}^K \)) and \( \Upsilon_{\mathcal{M}_{\text{PAR}}}^f(r') = K' \).

3. Let \( r = X \overset{a}{\rightarrow} Y.Z \in \mathcal{R} \), \( r' = Y.W \overset{b}{\rightarrow} W' \in \mathcal{R} \), \( Z \overset{\sigma}{\rightarrow}_{\mathcal{R}_{\text{PAR}}}^* W \), and \( K' = \Upsilon_f^M(r) \cup \Upsilon_{\mathcal{M}_{\text{PAR}}}^f(\sigma) \). If \( K' \subseteq K \), then \( r'' = X \overset{K'}{\rightarrow} W' \in \mathcal{R}_{\text{PAR}}^K \) and \( \Upsilon_{\mathcal{M}_{\text{PAR}}}^f(r'') = K' \).

**Lemma 4.1.** The parallel MBRS \( M_{\text{PAR}}^K = \langle \mathcal{R}_{\text{PAR}}^K, \langle \mathcal{R}_{\text{PAR},1}^{K,A}, \ldots, \mathcal{R}_{\text{PAR},n}^{K,A} \rangle \rangle \) can be effectively constructed.

**Proof.** Figure 1 reports the procedure BUILD-PARALLEL-MBRS\((M,K)\), which, starting from the MBRS \( M \) (in normal form) and the set \( K \in P_n \), builds the parallel MBRS \( M_{\text{PAR}}^K = \langle \mathcal{R}_{\text{PAR}}^K, \langle \mathcal{R}_{\text{PAR},1}^{K,A}, \ldots, \mathcal{R}_{\text{PAR},n}^{K,A} \rangle \rangle \). The algorithm uses the routine UPDATE\((r',K')\) that is defined as follows:

\[
\mathcal{R}_{\text{PAR}}^K := \mathcal{R}_{\text{PAR}}^K \cup \{r'\};
\]

for each \( i \in K' \) do \( \mathcal{R}_{\text{PAR},i}^{K,A} := \mathcal{R}_{\text{PAR},i}^{K,A} \cup \{r'\} \);

Notice that by Proposition 4.1, the conditions in each of the if statements in lines 7, 9 and 13 are decidable, therefore, the procedure is effective. Moreover, since the set of rules of the form \( X \overset{K'}{\rightarrow} Y \) with \( X \in \text{Var} \), \( Y \in \text{Var} \cup \{\hat{Z}_F\} \) and \( K' \in P_n \) is finite, termination immediately follows.

In order to simulate infinite subderivations of the form \( D \), we need to add additional PAR rules in \( M_{\text{PAR}}^K \). The following definition provides an extension of \( M_{\text{PAR}}^K \) suitable for our purposes.

**Definition 4.4.** By \( M_{\text{PAR}}^{K,K'} = \langle \mathcal{R}_{\text{PAR}}^{K,K'}, \langle \mathcal{R}_{\text{PAR},1}^{K,K',A}, \ldots, \mathcal{R}_{\text{PAR},n}^{K,K',A} \rangle \rangle \) and \( M_{\text{PAR},\infty}^{K,K'} = \langle \mathcal{R}_{\text{PAR}}^{K,K'}, \langle \mathcal{R}_{\text{PAR},1}^{K,K',A}, \ldots, \mathcal{R}_{\text{PAR},\infty,n}^{K,K',A} \rangle \rangle \) we denote the parallel MBRSs over \( \text{Var} \cup \{\hat{Z}_F, \hat{Z}_\infty\} \) and the alphabet \( \Sigma \cup P_n \cup P_n \times P_n \) (with the same support), defined by \( M \) and \( M_{\text{PAR}}^K \) in the following way:

- \( \mathcal{R}_{\text{PAR}}^{K,K'} := \mathcal{R}_{\text{PAR}}^K \cup \{X \overset{\mathcal{R}_{\text{PAR}}^{K,K'}}{\rightarrow} \hat{Z}_\infty \mid \mathcal{R} \subseteq K, \mathcal{R}^\omega \subseteq K^\omega, \text{ there exists a rule } r = X \overset{a}{\rightarrow} Y.Z \in \mathcal{R} \) and an infinite derivation \( Z \overset{\sigma}{\rightarrow}_{\mathcal{R}}^* \varepsilon \) such that \( |\Upsilon_f^M(\sigma)| + |\Upsilon_\infty^M(\sigma)| < |K| + |K^\omega| \) and \( \Upsilon_f^M(\sigma) \cup \Upsilon_f^M(r) = \overline{K} \) and \( \Upsilon_\infty^M(\sigma) = \overline{K}^\omega \})

- \( \mathcal{R}_{\text{PAR},i}^{K,K',A} := \mathcal{R}_{\text{PAR},i}^{K,A} \cup \{X \overset{\mathcal{R}_{\text{PAR}}^{K,K',A}}{\rightarrow} \hat{Z}_\infty \in \mathcal{R}_{\text{PAR}}^{K,K'} \mid i \in K \} \) for all \( i = 1, \ldots, n \)
Algorithm BUILD-PARALLEL-MBRS($M,K$)

1 \( \mathcal{R}^{K}_{PAR} := \mathcal{R}_{P} \);
2 for \( i = 1, \ldots, n \) do \( \mathcal{R}^{A,K}_{PAR,i} := \mathcal{R}^{A}_{P,i} \);
3 repeat
4     \( \text{flag} := \text{false}; \)
5     for each \( r = X \xrightarrow{a} Y.Z \in \mathcal{R} \) and \( K_1 \subseteq K \) such that \( \Upsilon^{f}_{M}(r) \subseteq K \) do
6         Set \( K' = K_1 \cup \Upsilon^{f}_{M}(r) \);
7         if \( Z \xrightarrow{\delta}^{*}_{K'_{PAR}} \) \( p \) for some \( p \) such that \( \Upsilon^{f}_{M_{PAR}}(\sigma) = K_1 \) then
8             if \( r' = X \xrightarrow{K'}_{\mathcal{R}_{F}} Y \notin \mathcal{R}^{K}_{PAR} \) then UPDATE\((r', K')\); \( \text{flag} := \text{true}; \)
9             if \( Z \xrightarrow{\epsilon}^{*}_{K'_{PAR}} \) \( \in \) such that \( \Upsilon^{f}_{M_{PAR}}(\sigma) = K_1 \) then
10                if \( r' = X \xrightarrow{K'}_{\mathcal{R}_{F}} Y \notin \mathcal{R}^{K}_{PAR} \) then UPDATE\((r', K')\); \( \text{flag} := \text{true}; \)
11                for each \( r' = Y.W \xrightarrow{b} W' \in \mathcal{R} \) such that \( \Upsilon^{f}_{M}(r') \subseteq K \) do
12                    Set \( K' = K_1 \cup \Upsilon^{f}_{M}(r'); \)
13                    if \( Z \xrightarrow{\delta}^{*}_{K'_{PAR}} \) \( W \) such that \( \Upsilon^{f}_{M_{PAR}}(\sigma) = K_1 \) then
14                        if \( r'' = X \xrightarrow{K'}_{\mathcal{R}_{F}} W' \notin \mathcal{R}^{K}_{PAR} \) then UPDATE\((r'', K')\); \( \text{flag} := \text{true}; \)
15 until \( \text{flag} = \text{false} \)

Figure 1: Algorithm to build the parallel MBRS \( M^{K}_{PAR} \).

- \( \mathcal{R}^{K,K_{\omega}}_{PAR,i,\infty} = \{ X \xrightarrow{\underline{K}_{\omega}} Z_{\infty} \in \mathcal{R}^{K,K_{\omega}}_{PAR} | i \in K_{\omega} \} \) for all \( i = 1, \ldots, n \)

By the inductive hypothesis on decidability of Problem 1 for sets \( K' \), \( K'_{\omega} \in P_{n} \) such that \( K' \subseteq K \), \( K'_{\omega} \subseteq K_{\omega} \) and \( |K'| + |K'_{\omega}| < |K| + |K_{\omega}| \), it follows that

**Lemma 4.2.** \( M^{K,K_{\omega}}_{PAR} \) and \( M^{K,K_{\omega}}_{PAR,\infty} \) can be built effectively.

The following two lemmata establish the validity of our construction.

**Lemma 4.3.** Let \( p \xrightarrow{\sigma}^{*}_{K} \) be a \((K, K'_{\omega})\)-accepting derivation in \( M \) belonging to \( \Pi^{K,K'_{\omega}}_{PAR,\infty} \), with \( p \in T_{PAR} \), \( K \subseteq K \) and \( K_{\omega} \subseteq K_{\omega} \). Then, there exists in \( \mathcal{R}^{K,K_{\omega}}_{PAR} \) a derivation of the form \( p \xrightarrow{\delta}^{*}_{K,K_{\omega}^{P}_{PAR}} \) such that \( \Upsilon^{f}_{M,K_{\omega}^{P}_{PAR}}(\rho) = K_{\omega} \) and \( \Upsilon^{\infty}_{M_{PAR}}(\rho) \cup \Upsilon^{f}_{M_{PAR}}(\rho) = K_{\omega} \). Moreover, if \( \sigma \) is infinite, then \( p \) is either infinite or contains some occurrence of rule in \( \mathcal{R}^{K,K_{\omega}}_{PAR} \setminus \mathcal{R}^{K}_{PAR} \).

**Lemma 4.4.** Let \( p \xrightarrow{\sigma}^{*}_{K,K_{\omega}^{P}_{PAR}} \) with \( p \in T_{PAR} \). Then, there exists in \( \mathcal{R} \) a derivation of the form \( p \xrightarrow{\delta}^{*}_{K_{\omega}^{P}_{PAR}} \) such that \( \Upsilon^{f}_{M}(\delta) = \Upsilon^{f}_{M,K_{\omega}^{P}_{PAR}}(\sigma) \) and \( \Upsilon^{\infty}_{M}(\delta) = \Upsilon^{\infty}_{M_{PAR}}(\rho) \cup \Upsilon^{f}_{M_{PAR}}(\sigma) \). Moreover, if \( \sigma \) is either infinite or contains some occurrence of rule in \( \mathcal{R}^{K,K_{\omega}}_{PAR} \setminus \mathcal{R}^{K}_{PAR} \), then \( \delta \) is infinite.

Now, let us go back to Problem 1 and consider a \((K, K'_{\omega})\)-accepting infinite derivation in \( M \) from a variable \( X \) of the form \( X \xrightarrow{\delta}^{*}_{K} \) and non belonging to \( \Pi^{K,K'_{\omega}}_{PAR,\infty} \). In this case, the
derivation \( \xrightarrow{s_r} \) can be written in the form \( X \xrightarrow{s_r} \tau \| (Y,Z) \xrightarrow{s_r} \), with \( Z \in \text{Var} \), and such that there exists a subderivation of \( \tau \| (Y,Z) \xrightarrow{s_r} \) from \( Z \) that is a \((K,K^{\omega})\)-accepting infinite derivation in \( M \). In order to manage this kind of derivation, we build, starting from the MBRSs \( M \) and \( M_{\text{PAR}}^K \), a sequential MBRS \( M_{\text{SEQ}}^K \) according to the following definition:

**Definition 4.5.** By \( M_{\text{SEQ}}^K = \langle \mathcal{R}_{\text{SEQ}}^K, \langle \mathcal{R}_{\text{SEQ},1}^K, \ldots, \mathcal{R}_{\text{SEQ},n}^K \rangle \rangle \) we denote the sequential MBRS over \( \text{Var} \) and the alphabet \( \Sigma = \Sigma \cup P_n \) defined as follows:

- \( \mathcal{R}_{\text{SEQ}}^K = \{ X \xrightarrow{a} Y,Z \in \mathcal{R} \} \cup \{ X \xrightarrow{K'} Y \mid X,Y \in \text{Var}, K' \subseteq K \text{ and there exists a derivation } X \xrightarrow{s_r} p \| Y \text{ in } \mathcal{R}_{\text{PAR}}^K \text{ for some } p \in T_{\text{PAR}}, \text{ with } |\sigma| > 0 \text{ and } \gamma^f_{M_{\text{PAR}}^K}(\sigma) = K' \} \)

- \( \mathcal{R}_{\text{SEQ},i}^K = \{ X \xrightarrow{a} Y,Z \in \mathcal{R}_i^A \} \cup \{ X \xrightarrow{K'} Y \in \mathcal{R}_{\text{SEQ}}^K \mid i \in K' \} \) for all \( i = 1, \ldots, n \)

By Proposition 4.1 we obtain the following result

**Lemma 4.5.** \( M_{\text{SEQ}}^K \) can be built effectively.

Soundness and completeness of the procedure described above is stated by the following two theorems.

**Theorem 4.2.** Let \( K \neq K^{\omega} \). Given \( X \in \text{Var} \), there exists a \((K,K^{\omega})\)-accepting infinite derivation in \( M \) from \( X \) if, and only if, the following property is satisfied:

- There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K',\emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{s_r} p \| Y \) such that \( \gamma^f_{M_{\text{PAR}}^K}(\rho) = K \) and \( \gamma^\infty_{M_{\text{PAR}}^K}(\rho) \cup \gamma^f_{M_{\text{PAR}}^K}(\rho) = K^{\omega} \). Moreover, \( \rho \) is either infinite or contains some occurrence of rule in \( \mathcal{R}_{\text{PAR}}^K \setminus \mathcal{R}_{\text{PAR}}^K \).

**Theorem 4.3.** Let \( K = K^{\omega} \). Given \( X \in \text{Var} \), there exists a \((K,K^{\omega})\)-accepting infinite derivation in \( M \) from \( X \) if, and only if, one of the following conditions is satisfied:

1. There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K',\emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{s_r} p \| Y \) such that \( \gamma^f_{M_{\text{PAR}}^K}(\rho) = K \) and \( \gamma^\infty_{M_{\text{PAR}}^K}(\rho) \cup \gamma^f_{M_{\text{PAR}}^K}(\rho) = K^{\omega} \). Moreover, \( \rho \) is either infinite or contains some occurrence of rule in \( \mathcal{R}_{\text{PAR}}^K \setminus \mathcal{R}_{\text{PAR}}^K \).

2. There exists a \((K,K^{\omega})\)-accepting infinite derivation in \( M_{\text{SEQ}}^K \) from \( X \).

These two results, together with Propositions 4.2 and 4.3, allow us to conclude that Problem 1 restricted to the class of MBRSs in normal form is decidable.
4.2 Decidability of Problem 1 for unrestricted MBRSs

In this section we extend the decidability result stated in the previous Subsection to the whole class of MBRSs, showing that Problem 1 for unrestricted MBRSs is reducible to the Problem 1 for MBRSs in normal form. We use a construction very close to one used in [12] to solve the reachability problem for PRSs. Remember that we can assume that the input term in Problem 1 is a process variable.

Let $M$ be a MBRS over $Var$ and the alphabet $\Sigma$, and with $n$ accepting components. Now, we describe a procedure that transforms $M$ into a new MBRS $M'$ with the same number of accepting components. Moreover, this procedure has in input also a finite set of rules $R_{AUX}$, and transforms it in $R'_{AUX}$. If $M$ is not in normal form, then there exists a rule in $M$ that is neither a PAR rule nor a SEQ rule. We call such rules bad rules [12]. There are five types of bad rules:

1. The bad rule is $r = u \rightarrow u_1||u_2$. Let $Z_1, Z_2, W$ be new variables (non belonging to $Var$). We get $M'$ replacing the bad rule $r$ with the rules $r' = u \rightarrow W$, $r_1 = Z_1 \rightarrow u_1$, $r_2 = Z_2 \rightarrow u_2$ such that $\gamma^f_{M'}(r') = \gamma^f_M(r)$, $\gamma^f_{M'}(r_1) = \gamma^f_M(r_2) = 0$. If $r \in R_{AUX}$, then $R'_{AUX} = (R_{AUX} \setminus \{r\}) \cup \{r', r_1, r_2\}$. Otherwise, $R'_{AUX} = R_{AUX}$.

2. The bad rule is $r = u_1||(u_2,u_3) \rightarrow u$. Let $Z_1, Z_2$ be new variables. We get $M'$ replacing the bad rule $r$ with the rules $r_1 = u_1 \rightarrow Z_1$, $r_2 = u_2,u_3 \rightarrow Z_2$, $r' = Z_2 \rightarrow u$ such that $\gamma^f_{M'}(r') = \gamma^f_M(r)$, $\gamma^f_{M'}(r_1) = \gamma^f_M(r_2) = 0$. If $r \in R_{AUX}$, then $R'_{AUX} = (R_{AUX} \setminus \{r\}) \cup \{r', r_1, r_2\}$. Otherwise, $R'_{AUX} = R_{AUX}$.

3. The bad rule is $r = u \rightarrow u_1,u_2$ (resp., $r = u_1,u_2 \rightarrow u$) where $u_1$ is not a single variable. Let $Z$ be a new variable. We get $M'$ and $R'_{AUX}$ in two steps. First, we substitute $Z$ for $u_1$ in (left-hand and right-hand sides of) all the rules of $M$ and $R_{AUX}$. Then, we add the rules $r_1 = Z \rightarrow u_1$ and $r_2 = u_1 \rightarrow Z$ such that $\gamma^f_{M'}(r_1) = \gamma^f_{M'}(r_2) = 0$.

4. The bad rule is $r = u_1 \rightarrow X.u_2$ where $u_2$ is not a single variable. Let $Z,W$ be new variables. We get $M'$ replacing the bad rule $r$ with the rules $r' = u_1 \rightarrow W$, $r_1 = W \rightarrow X.Z$, $r_2 = Z \rightarrow u_2$ such that $\gamma^f_{M'}(r') = \gamma^f_M(r)$ and $\gamma^f_{M'}(r_1) = \gamma^f_{M'}(r_2) = 0$. If $r \in R_{AUX}$, then $R'_{AUX} = (R_{AUX} \setminus \{r\}) \cup \{r', r_1, r_2\}$. Otherwise, $R'_{AUX} = R_{AUX}$.

5. The bad rule is $r = X.u_1 \rightarrow u_2$ where $u_1$ is not a single variable. Let $Z$ be a new variable. We get $M'$ replacing the bad rule $r$ with the rules $r_1 = u_1 \rightarrow Z$, $r' = X.Z \rightarrow u_2$, such that $\gamma^f_{M'}(r') = \gamma^f_M(r)$ and $\gamma^f_{M'}(r_1) = 0$. If $r \in R_{AUX}$, then $R'_{AUX} = (R_{AUX} \setminus \{r\}) \cup \{r', r_1\}$. Otherwise, $R'_{AUX} = R_{AUX}$.

After a finite number of applications of this procedure, starting from $R_{AUX} = \emptyset$, we obtain a MBRS $M'$ in normal form and a finite set of rules $R'_{AUX}$. Let $M' = (R', \langle R^A_1, \ldots, R^A_n \rangle)$. Now, let us consider the MBRS $M'$ in normal form with $n + 1$ accepting

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5. Remember that we assume that sequential composition is right-associative. So, when we write $t_1.t_2$, then $t_1$ is either a single variable or a parallel composition of process terms.

6. Note that we have not specified the label of the new rules, since it is not relevant.
components given by $M_F = \langle \mathcal{R}', \langle \mathcal{R}'^A_1, \ldots, \mathcal{R}'^A_n, \mathcal{R}' \setminus \mathcal{R}'_{AUX} \rangle \rangle$. We can prove that, given a variable $X \in Var$ and two sets $K, K^w \in P_n$, there exists a $(K, K^w)$-accepting infinite derivation in $M$ from $X$ if, and only if, there exists a $(K \cup \{n+1\}, K^w \cup \{n+1\})$-accepting infinite derivation in $M_F$ from $X$.

### Conclusion

In this paper we have stated decidability about generalized acceptance properties of infinite derivations in PRSs. Our result has an immediate application to the model-checking within a meaningful fragment of ALTL logic. In order to obtain this result we have used an approach different from classical automata-theoretic one. The reason is that PRSs are not closed under intersection with state finite ($\omega$-star-free) automaton \cite{1} (and in fact model-checking for full ALTL is undecidable). Future work should aim to extend our result to a larger fragment of ALTL. In particular, we are working on the ALTL fragment (closed under boolean operations) which uses the temporal operators $G$ ("always") and $F$ ("eventually") without restrictions (i.e. nested arbitrarily).

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APPENDIX

A Definitions and simple properties

In this section we give some definitions and deduce simple properties that will be used in sections B–C for the proof of Lemmata 4.3–4.4 and Theorems 4.2–4.3.

In the following \( \text{Var} \) denotes the set of variables \( \text{Var} \cup \{ Z_F, Z_{\infty} \} \), \( T \) denotes the set of terms in normal form over \( \text{Var} \), and \( T_{\text{PAR}} \) (resp., \( T_{\text{SEQ}} \)) the set of terms in \( T \) not containing sequential (resp., parallel) composition.

Definition A.1. The set of subterms of a term \( t \in T \), denoted by \( \text{SubTerms}(t) \), is defined inductively as follows:

- \( \text{SubTerms}(\varepsilon) = \{ \varepsilon \} \).
- \( \text{SubTerms}(X) = \{ X \} \), for all \( X \in \text{Var} \).
- \( \text{SubTerms}(X.t) = \text{SubTerms}(t) \cup \{ X.t \} \), for all \( X \in \text{Var} \) and \( t \in T \setminus \{ \varepsilon \} \).
- \( \text{SubTerms}(t_1 || t_2) = \bigcup \{(t_1', t_2') \in S : \text{SubTerms}(t_1') \cup \text{SubTerms}(t_2') \} \cup \{ t_1 || t_2 \} \), with \( S = \{(t_1', t_2') \in T \times T \mid t_1', t_2' \neq \varepsilon \text{ and } t_1 || t_2 = t_1' || t_2' \} \) for all \( t_1, t_2 \in T \setminus \{ \varepsilon \} \).

Definition A.2. The set of terms obtained from a term \( t \in T \) substituting an occurrence of a subterm \( st \) of \( t \) with a term \( t' \in T \), denoted by \( t[st \rightarrow t'] \), is defined inductively as follows:

- \( t[t \rightarrow t'] = \{ t' \} \).
- \( X.t[st \rightarrow t'] = \{ X.s \mid s \in t[st \rightarrow t'] \} \), for all \( X \in \text{Var} \), \( t \in T \setminus \{ \varepsilon \} \) and \( st \in \text{SubTerms}(X.t) \setminus \{ X.t \} \).
- \( t_1 || t_2[st \rightarrow t'] = \{ t'' || t' \mid (t_1', t_2') \in T \times T, t_1', t_2' \neq \varepsilon, t_1' || t_2' = t_1 || t_2, st \in \text{SubTerms}(t_1'), t'' \in t_1'[st \rightarrow t'] \} \), for all \( t_1, t_2 \in T \setminus \{ \varepsilon \} \) and \( st \in \text{SubTerms}(t_1 || t_2) \setminus \{ t_1 || t_2 \} \).

Definition A.3. For a term \( t \in T \), the set of terms \( \text{SEQ}(t) \) is the subset of \( T_{\text{SEQ}} \setminus \{ \varepsilon \} \) defined inductively as follows:

- \( \text{SEQ}(\varepsilon) = \emptyset \).
- \( \text{SEQ}(X) = \{ X \} \), for all \( X \in \text{Var} \).
- \( \text{SEQ}(X.t) = \{ X.t' \mid t' \in \text{SEQ}(t) \} \), for all \( X \in \text{Var} \) and \( t \in T \setminus \{ \varepsilon \} \).
- \( \text{SEQ}(t_1 || t_2) = \text{SEQ}(t_1) \cup \text{SEQ}(t_2) \).

\(^7\)Remember that we identify terms with their equivalence classes. In particular, \( t_1 = t_2 \) (resp., \( t_1 \neq t_2 \)) is used to mean that \( t_1 \) is equivalent (resp., not equivalent) to \( t_2 \).
For a term \( t \in T_{SEQ} \setminus \{\varepsilon\} \) having the form \( t = X_1.X_2.\ldots.X_n.Y \), we denote by \( last(t) \) the variable \( Y \). Given two terms \( t, t' \in T_{SEQ} \setminus \{\varepsilon\} \), with \( t = X_1.X_2.\ldots.X_n.Y \) and \( t' = X'_1.X'_2.\ldots.X'_{k}.Y' \), we denote by \( t \circ t' \) the term \( X_1.X_2.\ldots.X_n.X'_1.X'_2.\ldots.X'_{k}.Y' \). Notice that \( t \circ t' \) is the only term in \( t[Y \rightarrow t'] \), and that the operation \( \circ \) on terms in \( T_{SEQ} \setminus \{\varepsilon\} \) is associative.

The proof of the following two Propositions is simple

**Proposition A.1.** The following properties hold:

1. If \( t \xrightarrow{\sigma_s^*} t' \) and \( t \in SubTerms(s) \), for some \( s \in T \), then it holds \( s \xrightarrow{\sigma_s^*} s' \) for all \( s' \in s[t \rightarrow t'] \);

2. If \( t \xrightarrow{\sigma_s^*} \) is an infinite derivation in \( \mathcal{R} \) and \( t \in SubTerms(s) \), for some \( s \in T \), then it holds \( s \xrightarrow{\sigma_s^*} \).

**Proposition A.2.** Let \( \mathcal{R}_S \) be a sequential PRS over \( \text{Var} \). If \( t, t' \in T_{SEQ} \setminus \{\varepsilon\} \) such that \( last(t) \xrightarrow{\mathcal{R}_S^*} t' \), then it holds that

1. \( t \xrightarrow{\mathcal{R}_S^*} t \circ t' \);

2. \( t'' \circ t \xrightarrow{\mathcal{R}_S^*} t'' \circ t \circ t' \) for all \( t'' \in T_{SEQ} \setminus \{\varepsilon\} \).

Now, we give the notion of Interleaving of a (finite or infinite) sequence of rule sequences in a PRS \( \mathcal{R}' \). In order to formalize this concept and facilitate the proof of some connected results, we redefine the notion of sequence rule. Precisely, a sequence rule in \( \mathcal{R}' \) can be seen as a mapping \( \sigma : N' \rightarrow \mathcal{R}' \) where \( N' \) can be a generic subset of \( N \). A rule sequence \( \sigma' : N'' \rightarrow \mathcal{R}' \) is a subsequence of \( \sigma : N' \rightarrow \mathcal{R}' \) iff \( N'' \subseteq N' \) and \( \sigma' = \sigma|_{N''} \), that is \( \sigma' \) is the restriction of \( \sigma \) to the set \( N'' \). For a rule sequence \( \sigma : N' \rightarrow \mathcal{R}' \), we denote by \( pr(\sigma) \) the set \( N' \). For a set \( N' \subseteq N \) we denote by \( \text{min}(N') \) the smallest element of \( N' \). Given two rule sequences \( \sigma \) and \( \sigma' \), we say that they are disjoint if \( pr(\sigma) \cap pr(\sigma') = \emptyset \).

Let \( n \in N \setminus \{0\} \) and \( (K_h)_{h=0}^m \) be a sequence of elements in \( P_n \) (where \( m \in N \cup \{\infty\} \)). Let us denote by \( \bigoplus_{h=0}^m K_h \) the element of \( P_n \) given by \( \{i\} \) for all \( j \in N \) there exists a \( h > j \) such that \( i \in K_h \). Evidently, if \( m \) is finite, then \( \bigoplus_{h=0}^m K_h \) is empty.

**Definition A.4.** Let \( (\rho_h)_{h=0}^m \) be a sequence of rule sequences in a PRS \( \mathcal{R}' \) (where \( m \in N \cup \{\infty\} \)). The Interleaving of \( (\rho_h)_{h=0}^m \), denoted by \( \text{Interleaving}((\rho_h)_{h=0}^m) \), is the set of rule sequences \( \sigma \) in \( \mathcal{R}' \) such that there exists an injective mapping \( M_\sigma : \bigcup_{h=0}^m \{h\} \times pr(\rho_h) \rightarrow N \) (depending on \( \sigma \)) satisfying the following properties (where \( \Delta \) is the set \( \bigcup_{h=0}^m \{h\} \times pr(\rho_h) \))

- For all \( h = 1, \ldots, m \) and for all \( n, n' \in pr(\rho_h) \) with \( n < n' \), then \( M_\sigma(h,n) < M_\sigma(h,n') \);

- \( pr(\sigma) = M_\sigma(\Delta) \);

- for all \( (h,n) \in \Delta \), \( \sigma(M_\sigma(h,n)) = \rho_h(n) \).
The proof of the following two Propositions is simple.

**Proposition A.3.** Let $M'$ be a MBRS with support $\mathcal{R}'$, and $(\sigma_h)_{h=0}^m$ be a sequence of rule sequences in $\mathcal{R}'$ (where $m \in N \cup \{\infty\}$). Then, for all $\pi \in \text{Interleaving}((\sigma_h)_{h=0}^m)$ we have

1. $\Upsilon^f_{M'}(\pi) = \bigcup_{h=0}^m \Upsilon^f_{M'}(\sigma_h)$.
2. $\Upsilon^\infty_{M'}(\pi) = \bigcup_{h=0}^m \Upsilon^\infty_{M'}(\sigma_h)$ \bigcup \bigoplus_{h=0}^m \Upsilon^f_{M'}(\sigma_h).

**Proposition A.4.** Let $\sigma$ be a rule sequence in a PRS $\mathcal{R}$ and $(\rho_h)_{h=0}^m$ (where $m \in N \cup \{\infty\}$) be a sequence of subsequences of $\sigma$ two by two disjoints and such that $\bigcup_{h=0}^m \text{pr}(\rho_h) = \text{pr}(\sigma)$. Then, $\sigma \in \text{Interleaving}((\rho_h)_{h=0}^m)$.

### B Proof of Lemmata 4.3 and 4.4

**Remark B.1.** By construction, the following properties hold:

- for all $r \in \mathcal{R}^K_{\text{PAR}}$ \quad $\Upsilon^f_{M_{\text{PAR}}} (r) = \Upsilon^f_{M_{\text{PAR}}} (r)$ and $\Upsilon^f_{M_{\text{PAR}}} (r) = \emptyset$.
- for all $r \in \mathcal{R}^K_{\text{PAR}} \cap \mathcal{R}$ \quad $\Upsilon^f_{M_{\text{PAR}}} (r) = \Upsilon^f_{M_{\text{PAR}}} (r)$ and $\Upsilon^f_{M_{\text{PAR}}} (r) = \emptyset$.
- for all $r = X \Rightarrow Z_{\infty} \in \mathcal{R}^K_{\text{PAR}}$ \quad $\Upsilon^f_{M_{\text{PAR}}} (r) = K$ and $\Upsilon^f_{M_{\text{PAR}}} (r) = K'$.

The following lemma easily follows by the definition of subderivation.

**Lemma B.1.** Let $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ be a derivation in $\mathcal{R}$, and let $s \Rightarrow^*_\mathcal{R}$ be a subderivation of $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ from $s$. Then, one of the following conditions is satisfied:

1. $s \Rightarrow^*_\mathcal{R}$ is infinite and $t \Rightarrow^*_\mathcal{R}$. Moreover, if $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ is in $\Pi^K_{\text{PAR},\infty}$, then also $t \Rightarrow^*_\mathcal{R}$ is in $\Pi^K_{\text{PAR},\infty}$.

2. $s \Rightarrow^*_\mathcal{R}$ leads to $\varepsilon$ and the derivation $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ can be written in the form

$$t \parallel (X.s) \Rightarrow^*_\mathcal{R} t' \parallel X \Rightarrow^*_\mathcal{R}$$

where $t \Rightarrow^*_\mathcal{R} t'$ and $\sigma_1 \in \text{Interleaving}(\lambda, \sigma')$. Moreover, if $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ is in $\Pi^K_{\text{PAR},\infty}$, there is a derivation of the form $t \parallel X \Rightarrow^*_\mathcal{R} t' \parallel X \Rightarrow^*_\mathcal{R}$ belonging to $\Pi^K_{\text{PAR},\infty}$.

3. $s \Rightarrow^*_\mathcal{R}$ leads to a term $s' \neq \varepsilon$ and $t \Rightarrow^*_\mathcal{R}$. If $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ is in $\Pi^K_{\text{PAR},\infty}$, then also $t \Rightarrow^*_\mathcal{R}$ is in $\Pi^K_{\text{PAR},\infty}$. Moreover, if $t \parallel (X.s) \Rightarrow^*_\mathcal{R}$ is finite and leads to $T$, then $T = (X.s') \parallel t'$ where $t \Rightarrow^*_\mathcal{R} t'$.
4. \( s \xrightarrow{\sigma', \text{r}} \) leads to a variable \( W \in \text{Var} \) and the derivation \( t\|(X.s) \xrightarrow{\sigma} \) can be written in the form
   \[ t\|(X.s) \xrightarrow{\sigma} t'\|(X.W) \xrightarrow{\text{r}} t'' \xrightarrow{\sigma', \text{r}} \]
   where \( r = X.W \xrightarrow{\alpha} W' \in \text{R}, t \xrightarrow{\lambda, \sigma} \) and \( \sigma_1 \in \text{Interleaving}(\lambda, \sigma') \). Moreover, if \( t\|(X.s) \xrightarrow{\sigma, \text{r}} \) is in \( \Pi^{K,K,\omega}_{\text{PAR,\infty}} \), there is a derivation of the form \( t\|W' \xrightarrow{\lambda} t'' \xrightarrow{\sigma', \text{r}} \) belonging to \( \Pi^{K,K,\omega}_{\text{PAR,\infty}} \).

### B.1 Proof of Lemma 4.3

In order to prove Lemma 4.3, we need the following Lemma.

**Lemma B.2.** Let \( p \xrightarrow{\sigma, \text{r}} t\|p' \) with \( p, p' \in T_{\text{PAR}} \) and \( \Upsilon^f_M(\sigma) \subseteq K \). Then, there exists a \( s \in T_{\text{PAR}} \) such that \( p \xrightarrow{\sigma, K, K, \omega}_{\text{PAR}} s\|p' \) with \( \Upsilon^f_M(\sigma) = \Upsilon^f_{M, \text{PAR}}(\rho) \), and \( s = \varepsilon \) if \( t = \varepsilon \).

**Proof.** The proof is by induction on the length of finite derivations \( p \xrightarrow{\sigma, \text{r}} \) in \( \text{R} \) from terms in \( T_{\text{PAR}} \) with \( \Upsilon^f_M(\sigma) \subseteq K \), and uses Lemma B.1 Properties 2–3 in the Definition of \( M^K_{\text{PAR}} \) and Remark B.1. For brevity, we omit it. □

Now, we can prove Lemma 4.3. Let \( p \xrightarrow{\sigma, \text{r}} \) be a \((\overline{K}, \overline{K})\)-accepting non–null derivation in \( M \) belonging to \( \Pi^{K,K,\omega}_{\text{PAR,\infty}} \), with \( p \in T_{\text{PAR}} \). \( K \subseteq \overline{K} \) and \( \overline{K} \subseteq K^\omega \). We have to prove that there exists in \( \text{R}^{K,K,\omega}_{\text{PAR}} \) a derivation of the form \( p \xrightarrow{\rho, \omega}_{\text{PAR}} \) such that \( \Upsilon^f_M(\rho) = \overline{K} \) and \( \Upsilon^f_{M, \text{PAR}}(\rho) = \overline{K}^\omega \). Moreover, if \( \sigma \) is infinite, then \( \rho \) is either infinite or contains some occurrence of rule in \( \Pi^{K,K,\omega}_{\text{PAR}} \). At first, let us prove the following property

**A** There exists a \( p' \in T_{\text{PAR}} \), a non empty finite rule sequence \( \lambda \in \text{R}^{K,K,\omega}_{\text{PAR}} \), and a non empty subsequence \( \eta \) (possibly infinite) of \( \sigma \) such that \( \text{min}(pr(\eta)) = \text{min}(pr(\sigma)) \) (i.e. the first rule occurrence in \( \eta \) is the first rule occurrence in \( \sigma \)), \( p \xrightarrow{\lambda, \sigma}_{\text{PAR}} p', \Upsilon^f_{M, \text{PAR}}(\lambda) = \Upsilon^f_M(\eta), \Upsilon^f_{M, \text{PAR}}(\lambda) \). Moreover, if \( \sigma \) is infinite, then either \( \sigma \setminus \eta \) is infinite or \( \lambda \) is a rule in \( \text{R}^{K,K,\omega}_{\text{PAR}} \setminus \text{R}^{K}_\text{PAR} \).

The derivation \( p \xrightarrow{\sigma, \text{r}} \) can be rewritten as

\[ p \xrightarrow{\text{r}} t \xrightarrow{\sigma', \text{r}} \]  

(1)

At first, let us assume that \( r \) is a PAR rule. In this case \( t \in T_{\text{PAR}} \) and \( r \in \text{R}^{K,K,\omega}_{\text{PAR}} \). By Remark B.1 \( \Upsilon^f_{M, \text{PAR}_\infty}(r) = \Upsilon_M(r) \), and \( \Upsilon^f_{M, \text{PAR}_\infty}(r) = \emptyset = \Upsilon_M^\infty(r) \). Moreover, \( t \xrightarrow{\sigma', \text{r}} \) is in \( \Pi^{K,K,\omega}_{\text{PAR,\infty}} \), with \( \sigma' = \sigma \setminus r \). Thus, since \( \sigma' \) is infinite if \( \sigma \) is infinite, property A follows, setting \( p' = t, \lambda = r \) and \( \eta = r \). If \( r \) is not a PAR rule, then \( r = Z \xrightarrow{\alpha} Y.Z' \) (since \( p \in T_{\text{PAR}} \))
for some $Z,Y,Z' \in \text{Var}$ and $a \in \Sigma$. So, $p = p''\|Z$ and $t = p''\|(Y.Z')$ with $p'' \in T_{\text{PAR}}$. From (1), let $Z' \xrightarrow{\nu} \ast_{g}$ be a subderivation of $t = p''\|(Y.Z') \xrightarrow{\nu} \ast_{g}$ from $Z'$. By Lemma [B.1] we can distinguish four subcases. Since cases 2–4 (of Lemma [B.1]) are similar, for brevity, we consider only cases 1 and 4.

**Case 1:** $Z' \xrightarrow{\nu} \ast_{g}$ is infinite, and $p'' \xrightarrow{\nu} \ast_{g}$. Moreover, $p'' \xrightarrow{\nu} \ast_{g}$ is in $\Pi_{\text{PAR},\infty}^{K,K'}$. By the hypothesis, $(\mathcal{Y}_{M}(\nu), \mathcal{Y}_{M}(\nu)) \neq (K,K')$, $\mathcal{Y}_{M}(\nu) \subseteq K$ and $\mathcal{Y}_{M}(\nu) \subseteq K'$. Hence, $|\mathcal{Y}_{M}(\nu)| + |\mathcal{Y}_{M}(\nu)| < |K| + |K'|$. Moreover, $r = Z^{a_{1}}Y.Z'$ with $\mathcal{Y}_{M}(r) \subseteq K$. By the definition of $\mathcal{R}_{\text{PAR}},$ it follows that $r' = Z^{K_{1},K_{1}'} \omega_{1} \in \mathcal{R}_{\text{PAR}}^{K,K'} \mathcal{R}_{\text{PAR}}$ where $K_{1} = \mathcal{Y}_{M}(\nu) \cup \mathcal{Y}_{M}(r)$ and $K' = \mathcal{Y}_{M}(\nu)$. By Remark [B.1] we have that $\mathcal{Y}_{M}(r') = K_{1}$ and $\mathcal{Y}_{M}(r') = K'$. So, we have that $p = p''\|Z^{a_{1}}Y.Z' \xrightarrow{\nu} \ast_{g}$ and this derivation is in $\Pi_{\text{PAR},\infty}^{K,K'}$.

Since $\nu \in \text{Interleaving}(\nu, \sigma_{1})$, $p''\|Z^{a_{1}}Y.Z' \xrightarrow{\nu} \ast_{g}$ and this derivation is in $\Pi_{\text{PAR},\infty}^{K,K'}$. Since $Z' \xrightarrow{\nu} \ast_{g} W$ and $\mathcal{Y}_{M}(\nu) \subseteq K$, by Lemma [B.2] it follows that $Z' \xrightarrow{\nu} \ast _{g} W$ with $\mathcal{Y}_{M}(\nu) = \mathcal{Y}_{M}(\nu)$. Since $r = Z^{a_{1}}Y.Z' \in \mathcal{R}$ and $r' = Y.W^{a_{1}}W' \in \mathcal{R}$, where $\mathcal{Y}_{M}(r') \subseteq K$ and $\mathcal{Y}_{M}(r') \subseteq K$, by the definition of $\mathcal{R}_{\text{PAR}}$ it follows that $r'' = Z^{K_{1},K_{1}'} \omega_{1} \in \mathcal{R}_{\text{PAR}}^{K,K'} \mathcal{R}_{\text{PAR}}$ where $K' = \mathcal{Y}_{M}(r') \cup \mathcal{Y}_{M}(\nu r')$ and $\mathcal{Y}_{M}(r'') = K'$. By construction, $r'' = A \mathcal{R}_{\text{PAR}}$, and by Remark [B.3] $\mathcal{Y}_{M}(r''') = K'$ and $\mathcal{Y}_{M}(K)(r''') = \emptyset$. Since $\sigma \setminus \nu r' = \sigma_{1} \sigma_{2}$, $\mathcal{Y}_{M}(K)(r''') = \emptyset = \mathcal{Y}_{M}(\nu r')$, and $\sigma_{1} \sigma_{2}$ is infinite if $\sigma$ is infinite, property A follows setting $p'' = p''\|W'$, $\lambda = r''$ and $\eta = r\nu r'$.

Therefore, Property A is satisfied. Since $\sigma \setminus \eta$ is a subsequence of $\sigma$, we have $\mathcal{Y}_{M}(\sigma \setminus \eta) \subseteq K$ and $\mathcal{Y}_{M}(\sigma \setminus \eta) \subseteq K'$. Thus, if $\sigma \neq \eta$ we can apply property A to the derivation $p' \xrightarrow{\eta} \ast_{g}$. Repeating this reasoning it follows that there exists a $m \in N \cup \{\infty\}$, a sequence $(p_{h})_{h=0}^{m+1}$ of terms in $T_{\text{PAR}}$, a sequence $(\lambda_{h})_{h=0}^{m}$ of non empty finite rule sequences in $\mathcal{R}_{\text{PAR}}^{K,K'}$, two sequences $(\sigma_{h})_{h=0}^{m}$ and $(\eta_{h})_{h=0}^{m}$ of non empty rule sequences in $\mathcal{R}$ such that for all $h = 0, \ldots, m$

1. $p = p_{0}$ and $\sigma = \sigma_{0}$.

2. $\eta_{h}$ is a subsequence of $\sigma_{h}$, $\min(pr(\eta_{h})) = \min(pr(\sigma_{h}))$, and if $h \neq m$ then $\sigma_{h+1} = \sigma_{h} \setminus \eta_{h}$.

3. $p_{h} \xrightarrow{\lambda_{h}} \ast_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g} \ x_{g}$
4. If \( m \) is finite, then \( \sigma_m = \eta_m \). If \( \sigma \) is infinite, then either \( m \) is infinite or there exists an \( h \) such that \( \lambda_h \) is a rule in \( \mathcal{R}_{\text{PAR}}^{K,K} \setminus \mathcal{R}_{\text{PAR}}^K \).

By setting \( \rho = \lambda_0 \lambda_1 \ldots \) we have that \( p \xrightarrow{\rho^*}_{\mathcal{R}_{\text{PAR}}^{K,K}} \). By Property 4 it follows that if \( \sigma \) is infinite, then either \( \rho \) is infinite or \( \rho \) contains some occurrence of rule in \( \mathcal{R}_{\text{PAR}}^{K,K} \setminus \mathcal{R}_{\text{PAR}}^K \).

Let us assume that \( m = \infty \). The proof for \( m \) finite is simpler. By Properties 1–2 \( \eta_0, \eta_1, \ldots \) are non empty subsequences of \( \sigma \) two by two disjoints. Since \( \sigma \) is infinite, we can assume that \( \text{pr}(\sigma) = N \). Now, let us show that

\[
5. \quad \sigma \in \text{Interleaving}((\eta_h)_{h \in \mathbb{N}})
\]

By Proposition \( \text{A.4} \) it suffices to prove that for all \( h \in \mathbb{N} \) there exists an \( i \in \mathbb{N} \) such that \( h \in \text{pr}(\eta_i) \). By Property 2 it follows that for all \( h \in \mathbb{N} \) \( \min(\text{pr}(\sigma_h)) < \min(\text{pr}(\sigma_{h+1})) \).

Let \( h \in \mathbb{N} \), then there exists the smallest \( i \in \mathbb{N} \) such that \( h \notin \text{pr}(\sigma_i) \). Since \( \sigma_0 = \sigma \), \( i > 0 \) and \( h \in \text{pr}(\sigma_{i-1}) \). Since \( \sigma_i = \sigma_{i-1} \setminus \eta_{h-1} \), \( h \notin \text{pr}(\sigma_i) \) and \( h \in \text{pr}(\sigma_{i-1}) \), it follow that \( h \in \text{pr}(\eta_{h-1}) \). Thus, Property 5 holds. By Properties 3, 5, and Proposition \( \text{A.3} \) it follows that \( \Upsilon^f_{M_{\text{PAR}}} \rho \cup \Upsilon^f_{M_{\text{PAR}}} \rho = \bigoplus_{h \in \mathbb{N}} \Upsilon^f_{M_{\text{PAR}}} \rho \cup \bigcup_{h \in \mathbb{N}} \Upsilon^f_{M_{\text{PAR}}} \rho = \Upsilon^f_{M_{\text{PAR}}} \rho = \mathcal{K} \).

This concludes the proof.

**B.2 Proof of Lemma 4.4**

In order to prove Lemma 4.4, we need the following Lemma.

**Lemma 4.3.** Let \( p \xrightarrow{\rho^*}_{\mathcal{R}_{\text{PAR}}^{K,i}} p' \| p'' \) with \( p, p', p'' \in T_{\text{PAR}} \), \( p' \) not containing occurrences of \( \hat{Z}_F \) and \( \hat{Z}_\infty \), and \( p'' \) not containing occurrences of variables in \( \text{Var} \). Then, there exists a \( t \in T \) such that \( p \xrightarrow{\rho^*_{\mathcal{R}}} p' \| t \) with \( \Upsilon^f_{M_{\text{PAR}}} (\rho) = \Upsilon^f_{M_{\text{PAR}}} (\sigma) \), and \( |\rho| > 0 \) if \( |\sigma| > 0 \).

**Proof.** Let \( \mathcal{R}_{\text{PAR}}^{K,i} \setminus \mathcal{R} = \{ r_1, \ldots, r_m \} \), where for all \( i = 1, \ldots, m \) \( r_i \) is the \( i \)-th rule added into \( \mathcal{R}_{\text{PAR}}^{K,i} \) during the computation of algorithm of Lemma 4.1. For all \( i = 1, \ldots, m \) let us denote by \( M_{\text{PAR}}^{K,i} \) (with support \( \mathcal{R}_{\text{PAR}}^{K,i} \)) the parallel MBRS \( M_{\text{PAR}}^{K} \) soon before the rule \( r_i \) is added during the computation. Then, it suffices to prove that the following two properties are satisfied:

1. Let \( p \xrightarrow{\rho^*}_{\mathcal{R}_{\text{PAR}}^{K,i}} p' \| p'' \) with \( p, p', p'' \in T_{\text{PAR}} \), \( p' \) not containing occurrences of \( \hat{Z}_F \) and \( \hat{Z}_\infty \), and \( p'' \) not containing occurrences of variables in \( \text{Var} \). Then, there exists a \( t \in T \) such that \( p \xrightarrow{\rho^*_{\mathcal{R}}} p' \| t \) with \( \Upsilon^f_{M_{\text{PAR}}} (\rho) = \Upsilon^f_{M_{\text{PAR}}} (\sigma) \), and \( |\rho| > 0 \) if \( |\sigma| > 0 \).
2. If \( r_i = X^{K'} \rightarrow Y \) with \( Y \in \text{Var} \) (resp., \( Y = \bar{Z_F} \)), then there exists a derivation of the form \( X \xrightarrow{\delta^*} Y \) (resp., \( X \xrightarrow{\delta^*} t \) for some term \( t \)) such that \( \Upsilon_M^i(\eta) = K' \) and \( |\eta| > 0 \).

The proof is by induction on \( i \) (for the base step it is suffices observe that \( M_{\text{PAR}}^{K,1} = M_{\text{P}} \) where \( M_{\text{P}} \) is the restriction of \( M \) to the PAR rules). For the inductive step Property 1 can be easily proved by induction on \( \sigma \), while Property 2 follows immediately by Property 1 and algorithm of Lemma 4.1.

In order to prove lemma 4.4 we use a mapping for coding pairs of integers by single integers. In particular, we consider the following bijective mapping from \( N \times N \) to \( N \):

\[
< >: (x, y) \in N \times N \rightarrow 2^x(2y + 1) - 1
\]

Let \( \ell \) (resp. \( \wp \)) be the first (resp., second) component of \( < >^{-1} \). Then,

1. for all \( x, y \in N \) \( \ell(<x, y>) = x \) and \( \wp(<x, y>) = y \),
2. for all \( z \in N \) \( \ell(z), \wp(z) > z \),
3. for all \( z \in N \) \( \ell(z), \wp(z) \leq z \),
4. for all \( z, z' \in N \) if \( z > z' \) and \( \ell(z) = \ell(z') \) then \( \wp(z) > \wp(z') \).

Now, we introduce a new function \( \text{next} : N \times N \rightarrow N \times N \) defined as

\[
\text{next}(x, 0) = (x, 0)
\]

\[
\text{next}(x, y + 1) = \begin{cases} 
(\ell(y), \wp(y) + 1) & \text{if next}(x, y) = (\ell(y), \wp(y)) \\
\text{next}(x, y) & \text{otherwise}
\end{cases}
\]

For all \( x, y \in N \) let us denote by \( \text{next}_x(y) \) the second component of \( \text{next}(x, y) \). The following lemma establishes some properties of \( \text{next} \). The proof is simple.

**Lemma B.4.** The function \( \text{next} \) satisfies the following properties:

1. For all \( x, y \in N \) if \( y \leq x \) then \( \text{next}(x, y) = (x, 0) \).
2. For all \( x, y \in N \) \( \text{next}(x, y) = (x, z_{x,y}) \) for some \( z_{x,y} \in N \).
3. For all \( x, y \in N \) \( \text{next}_x(y) \leq \text{next}_x(y + 1) \).
4. Let \( x, y_1, y_2 \in N \) with \( \text{next}_x(y_1) < \text{next}_x(y_2) \). Then, there exists a \( k \in N \) such that \( \text{next}(x, k) = (\ell(k), \wp(k)), \wp(k) = \text{next}_x(y_2) - 1 \) and \( y_1 \leq k < y_2 \).
5. For all \( x, n \in N \) there exists a \( y \in N \) such that \( \text{next}(x, y) = (x, n) \).
6. For all \( x \in N \) \( \text{next}(\ell(x), x) = (\ell(x), \wp(x)) \).
7. For all \( x, i \in N \) if \( i \neq \ell(x) \) then \( \text{next}(i, x + 1) = \text{next}(i, x) \).
Now, we can prove Lemma 4.4. Let \( p \xrightarrow{\delta} p^* \) with \( p \in \text{T}_{PAR} \). We have to prove that there exists in \( \mathcal{R} \) a derivation of the form \( p \xrightarrow{\delta^*} \) such that \( \Upsilon^f_M(\delta) = \Upsilon^f_{M_{PAR}}(\sigma) \) and \( \Upsilon^\infty_M(\delta) = \Upsilon^\infty_{M_{PAR}}(\sigma) \). Moreover, if \( \sigma \) is either infinite or contains some occurrence of rule in \( \mathcal{R}_{PAR} \), then \( \delta \) is infinite.

Let \( \lambda \) be the subsequence of \( \sigma \) containing all, and only, the occurrences of rules in \( \mathcal{R}_{PAR} \). Let us assume that \( \lambda \) is infinite. The proof for \( \lambda \) finite (and possibly empty) is similar. Now, \( \lambda = r_0r_1r_2 \ldots \), where for all \( h \in N \) \( r_h \in \mathcal{R}_{PAR} \). Moreover, \( \sigma \) can be written in the form \( \rho_0\rho_1\rho_2 \ldots \), where \( \sigma \setminus \lambda = \rho_0\rho_1\rho_2 \ldots \) and for all \( h \in N \) \( \rho_h \) is a finite rule sequence (possibly empty) in \( \mathcal{R}_{PAR} \). For all \( h \in N \) we denote by \( \sigma^h \) the suffix of \( \sigma \) given by \( \rho_0r_hr_{h+1}r_{h+1} \ldots \). Now, we prove that there exists a sequence of terms in \( \text{T}_{PAR} \), \( (p_h)_{h \in N} \), a sequence of variables \( (X_h)_{h \in N} \) and a sequence of terms \( (t_h)_{h \in N} \) such that for all \( h \in N \):

i. \( p_0 = p \).

ii. \( p_h \xrightarrow{\sigma^h} \).

iii. \( p_h \xrightarrow{\eta^h} p_{h+1} || t_h || X_h \) with \( \Upsilon^f_M(\eta_h) = \Upsilon^f_{M_{PAR}}(\rho_h) \).

iv. \( X_h \xrightarrow{\pi^h} \) with \( \pi_h \) infinite, \( \Upsilon^f_M(\pi_h) = \Upsilon^f_{M_{PAR}}(\rho_h) \) and \( \Upsilon^\infty_M(\pi_h) = \Upsilon^\infty_{M_{PAR}}(\rho_h) \).

Setting \( p_0 = p \), property ii is satisfied for \( h = 0 \). So, let us assume that the statement is true for all \( h = 0, \ldots, k \). Then, it suffices to prove that

A. there exists a \( p_{k+1} \in \text{T}_{PAR} \), a term \( t_k \) and a variable \( X_k \) such that \( p_k \xrightarrow{\eta^h} p_{k+1} || t_k || X_k \), \( p_{k+1} \xrightarrow{\sigma^{k+1} \eta^h} \), and \( X_k \xrightarrow{\pi^h} \) with \( \pi_k \) infinite. Moreover, \( \Upsilon^f_M(\eta_k) = \Upsilon^f_{M_{PAR}}(\rho_k) \), \( \Upsilon^f_M(\pi_k) = \Upsilon^\infty_{M_{PAR}}(\rho_k) \).

By the inductive hypothesis we have \( p_k \xrightarrow{\sigma^k} \), that can be written as

\[
p_k \xrightarrow{\eta^k} p' || p'' || X \xrightarrow{\eta^k} p' || p'' || \xrightarrow{\rho_k} \xrightarrow{\sigma^{k+1}} \xrightarrow{\rho_k}
\]

where \( r_k = X^{K',K''} \xrightarrow{\rho_k} \xrightarrow{\rho_k} \xrightarrow{\rho_k} \xrightarrow{\eta^k} \eta_k \) with \( X \in \text{Var} \) and \( K',K'' \in \text{P}^\infty \). Moreover, \( p'' \) does not contain occurrences of \( \hat{Z} \) and \( \hat{Z}_\infty \), and \( \rho_k \) doesn’t contain occurrences of variables in \( \text{Var} \). By the definition of \( \mathcal{R}_{PAR} \) we have \( X \xrightarrow{\pi_k} \) with \( \pi_k \) infinite, \( \Upsilon^{f}_{M}(\pi_k) = K' \) and \( \Upsilon^\infty_{M}(\pi_k) = K'' \). By Remark 4.4 we have \( \Upsilon^{f}_{M}(r_k) = K' \) and \( \Upsilon^{f}_{M}(r_k) = K'' \). Since the left-hand side of each rule in \( \mathcal{R}_{PAR} \) does not contain occurrences of \( \hat{Z} \) and \( \hat{Z}_\infty \), it follows that \( p' \xrightarrow{\sigma^{k+1} \eta^h} \). Since \( \rho_k \) is a rule sequence in \( \mathcal{R}_{PAR} \), by Lemma 4.3 it follows that \( p_k \xrightarrow{\eta^k} p' || t || X \) for some
term \( t \) and \( \Upsilon^f_M(\eta_k) = \Upsilon^f_{M_{PAR}}(\rho_k) \). By Remark B.1 we deduce that \( \Upsilon^f_M(\eta_k) = \Upsilon^f_{M_{PAR}}(\rho_k) \).

So, property A follows, setting \( p_{k+1} = p', \ell_k = t \) and \( X_k = X \). Thus, Properties iii-iv are satisfied.

For all \( h \in N \) the infinite derivation \( X_h \xrightarrow{r_{h,0}} \sigma \) can be written as

\[
s(h,0) \xrightarrow{r_{h,0}} s(h,1) \xrightarrow{r_{h,1}} s(h,2) \ldots
\]

where \( s(h,0) = X_h \) and for all \( k \in N \) \( r(h,k) \in R \). For all \( k, h \in N \) we denote by \( r_k \) the rule \( r(\ell(k), \varphi(k)) \), and by \( s_h(k) \) the term \( s_{next(h,k)} \). Now, we show that for all \( k \in N \)

\[
p_{k+1} \xrightarrow{t_0} \ldots \xrightarrow{t_{k-1}} s_0(k) \xrightarrow{r_k} s_1(k) \xrightarrow{t_k} \ldots \xrightarrow{t_{k+1}} s_{k+1}(1)\]

By Lemma B.4 it follows that \( s_k(k) = s_{next(k,k)} = s_{(k,0)} = X_k \). So, by Property iii we deduce that

\[
p_{k+1} \xrightarrow{t_0} \ldots \xrightarrow{t_{k-1}} s_0(k) \xrightarrow{r_k} s_1(k) \xrightarrow{t_k} \ldots \xrightarrow{t_{k+1}} s_{k+1}(1)\]

So, in order to obtain (2) it suffices to prove that

\[
s_0(k) \xrightarrow{r_k} s_1(k) \xrightarrow{t_0} \ldots \xrightarrow{t_{k-1}} s_{k+1}(1)\]

By Property 6 of Lemma B.4 for all \( k \in N \) \( next(\ell(k), k) = (\ell(k), \varphi(k)) \). Moreover, \( next(\ell(k), k + 1) = (\ell(k), \varphi(k) + 1) \). Therefore, \( s_{\ell(k)(k)} = s_{\ell(k), \varphi(k)} \) and \( s_{\ell(k), \varphi(k) + 1} = s_{\ell(k)}(k) \). By Property 7 of Lemma B.4 for all \( i \neq \ell(k) \) \( next(i, k + 1) = next(i, k) \). So, for all \( i \neq \ell(k) \) \( s_i(k + 1) = s_i(k) \). Since \( \ell(k) \leq k \), we obtain evidently (4). So, (2) is satisfied for all \( k \in N \). Moreover, since \( s_0(0) = X_0 \), we have

\[
p = p_0 \xrightarrow{r_{0,0}} p_1 \xrightarrow{t_0} \ldots
\]

Setting \( \delta = \eta_0 \eta_1 \tau_0 \eta_2 \tau_1 \eta_3 \tau_2 \ldots \), from (2) and (5) we obtain that \( p \xrightarrow{r_{\delta}} \sigma \) with \( \delta \) infinite. Therefore, it remains to prove that \( \Upsilon^f_M(\delta) = \Upsilon^f_{M_{PAR}}(\sigma) \) and \( \Upsilon^\infty_M(\delta) = \Upsilon^\infty_{M_{PAR}}(\sigma) \).

Let \( \mu = \tau_0 \tau_1 \tau_2 \ldots \) Evidently, \( \mu \in \text{Interleaving}(\pi_h \delta) \). By Properties iii-iv, Proposition A.3 and remembering that \( \sigma = \rho_0 \tau_0 \tau_1 \ldots \), we obtain

\[
\Upsilon^f_M(\delta) = \bigcup_{h \in N} \Upsilon^f_M(\eta_h) \cup \Upsilon^f_M(\mu) = \bigcup_{h \in N} \Upsilon^f_{M_{PAR}}(\rho_h) \cup \bigcup_{h \in N} \Upsilon^f_M(\pi_h) = \bigcup_{h \in N} \Upsilon^f_{M_{PAR}}(\rho_h) \cup \bigcup_{h \in N} \Upsilon^\infty_M(\pi_h) \cup \bigcup_{h \in N} \Upsilon^f_M(\pi_h) = \bigcup_{h \in N} \Upsilon^f_{M_{PAR}}(\rho_h) \cup \bigcup_{h \in N} \Upsilon^\infty_M(\pi_h) \cup \bigcup_{h \in N} \Upsilon^f_M(\pi_h)
\]

By Remark B.1 for all \( r \in R_{M_{PAR}} \) \( \Upsilon^f_{M_{PAR}}(r) = \emptyset \). Remembering that \( \lambda = r_0 r_1 r_2 \ldots \), by Properties iii-iv and Proposition A.3 we obtain

\[
\Upsilon^\infty_M(\delta) = \bigcup_{h \in N} \Upsilon^f_M(\eta_h) \cup \Upsilon^\infty_M(\mu) = \bigcup_{h \in N} \Upsilon^f_{M_{PAR}}(\rho_h) \cup \bigcup_{h \in N} \Upsilon^\infty_M(\pi_h) \cup \bigcup_{h \in N} \Upsilon^f_M(\pi_h)
\]

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C Proof of Theorems 4.2 and 4.3

In order to prove Theorems 4.2 and 4.3 we need the following lemmata C.1 C.3

Remark C.1. By construction the following properties hold

- for all $r \in \mathbb{R} \cap \mathbb{R}^{K}_{SEQ}$ \hspace{1pt} $\Upsilon^{f}_{M}(r) = \Upsilon^{f}_{M_{SEQ}}(r)$.
- for all $r = X^{K'}Y \in \mathbb{R}^{K}_{SEQ} \setminus \mathbb{R}$ \hspace{1pt} $\Upsilon^{f}_{M_{SEQ}}(r) = K'$.

Lemma C.1. Let $t, t' \in T_{SEQ}$ and $s$ be any term in $T$ such that $t \in SEQ(s)$. The following results hold

1. If \hspace{1pt} $\frac{r}{s}_{SEQ}$ \hspace{1pt} $t'$ with $r \in \mathbb{R}^{K}_{SEQ}$, then there exists a $s' \in T$ with $t' \in SEQ(s')$ such that $s \xrightarrow{\sigma^{*}}_{r} s'$, with \hspace{1pt} $\Upsilon^{f}_{M}(\sigma) = \Upsilon^{f}_{M_{SEQ}}(r)$ and \hspace{1pt} $|\sigma| > 0$.

2. If \hspace{1pt} $\frac{r}{s}_{SEQ}$ \hspace{1pt} $t'$ with $t \neq \varepsilon$, then there exists a $s' \in T$ with $t' \in SEQ(s')$ such that $s \xrightarrow{\rho^{*}}_{r} s'$, with \hspace{1pt} $\Upsilon^{f}_{M}(\rho) = \Upsilon^{f}_{M_{SEQ}}(\sigma)$, and \hspace{1pt} $|\rho| > 0$ if \hspace{1pt} $|\sigma| > 0$.

3. If \hspace{1pt} $\frac{r}{s}_{SEQ}$ \hspace{1pt} is a $(K, K^{\omega})$-accepting infinite derivation in $M^{K}_{SEQ}$ from $t \in T_{SEQ}$, then there exists a $(K, K^{\omega})$-accepting infinite derivation in $M$ from $s$.

Proof. At first, we prove Property 1. There are two cases:

- $r = Y \xrightarrow{a} Z_1.Z_2 \in \mathbb{R}$. By Remark C.1 \hspace{1pt} $\Upsilon^{f}_{M}(r) = \Upsilon^{f}_{M_{SEQ}}(r)$. Since $t \in SEQ(s)$ and $t \xrightarrow{r}_{SEQ} \hspace{1pt} t'$, we deduce that there exists a $s' \in s[Y \rightarrow Z_1.Z_2]$ such that $t' \in SEQ(s')$. Since $Y \xrightarrow{r} Z_1.Z_2$, by Proposition A.1 it follows that $s \xrightarrow{r} s'$. Therefore, Property 1 is satisfied.

- $r = Y^{K'}Z$ with $Y, Z \in Var$, \hspace{1pt} $\Upsilon^{f}_{M_{SEQ}}(r) = K'$, last($t$) = $Y$ and last($t'$) = $Z$. By the definition of $\mathbb{R}^{K}_{SEQ}$ there exists a derivation in $\mathbb{R}^{K}_{PAR}$ of the form $Y \xrightarrow{\sigma^{*}}_{SEQ} p\|Z$ for some $p \in T_{PAR}$, with \hspace{1pt} $\Upsilon^{f}_{M_{SEQ}}(\sigma) = \Upsilon^{f}_{M_{SEQ}}(r)$ and \hspace{1pt} $|\sigma| > 0$. By Lemma B.3 there exists a term $st$ such that $Y \xrightarrow{\sigma^{*}}_{r} st\|Z$ with \hspace{1pt} $\Upsilon^{f}_{M}(\rho) = \Upsilon^{f}_{M_{SEQ}}(\sigma)$ and \hspace{1pt} $|\rho| > 0$. So,
\( \Upsilon_M^f(\rho) = \Upsilon_M^{f_{SEQ}}(r) \). Since \( t \in SEQ(s) \) and \( t \xrightarrow{\rho_{SEQ}^s} t' \), we deduce that there exists a \( s' \in s[Y \rightarrow st][Z] \) such that \( t' \in SEQ(s') \). Since \( Y \xrightarrow{\rho^s} st[Z] \), by Proposition \[A.1\] we conclude that \( s \xrightarrow{\rho^s} s' \) with \( |\rho| > 0 \). Thus, Property 1 is satisfied.

Property 2 can be easily proved by induction on the length of \( \sigma \), and using Property 1. Finally, Property 3 easily follows from Property 1 and Proposition \[A.3\] □

The following definition introduces the notion of level of application of a rule in a derivation:

**Definition C.1.** Let \( t \xrightarrow{r} t' \) be a single-step derivation in \( \mathbb{R} \) with \( t \in T \). We say that \( r \) is applicable at level 0 in \( t \xrightarrow{r} t' \), if \( t = t||s \), \( t' = t||s' \) (for some \( t, s, s' \in T \), and \( r = s^n \rightarrow s' \), for some \( n \in \Sigma \).

We say that \( r \) is applicable at level \( k > 0 \) in \( t \xrightarrow{r} t' \), if \( t = t||s \), \( t' = t||s' \) (for some \( t, s, s' \in T \), \( s \xrightarrow{r} s' \), and \( r \) is applicable at level \( k - 1 \) in \( s \xrightarrow{r} s' \).

The level of application of \( r \) in \( t \xrightarrow{r} t' \) is the greatest level of applicability of \( r \) in \( t \xrightarrow{r} t' \).

The definition above extends in the obvious way to \( n \)-step derivations and to infinite derivations.

**Lemma C.2.** Let \( i \in K \), \( X \in Var \) and \( X \xrightarrow{\sigma^*_{PAR}} \) be a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( X \). Then, one of the following conditions is satisfied:

1. There exists a variable \( Y \in Var \) reachable from \( X \) in \( \mathbb{R}^K_{SEQ} \) through a \((K', \emptyset)\)-accepting derivation in \( M^K_{SEQ} \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{\sigma^*_{PAR}} \) such that \( \Upsilon_M^{K,K'}(\rho) = K \) and \( \Upsilon_M^{\infty,K,K'}(\rho) \cup \Upsilon_M^{\infty,K,K^\omega}(\rho) = K^\omega \). Moreover, \( \rho \) is either infinite or contains some occurrence of rule in \( \mathbb{R}^{K,K^\omega}_{PAR} \setminus \mathbb{R}^K_{PAR} \).

2. There exists a variable \( Y \in Var \) reachable from \( X \) in \( \mathbb{R}^K_{SEQ} \) through a \((K, \emptyset)\)-accepting derivation in \( M^K_{SEQ} \) with \( \{i\} \subseteq K_i \subseteq K \), and there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( Y \).

**Proof.** The proof is by induction on the level \( k \) of application of the first occurrence of a rule \( r \) of \( \Pi^{K,K^\omega}_{PAR} \) in a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from a variable. If \( X \xrightarrow{\sigma^*_{PAR}} \in \Pi^{K,K^\omega}_{PAR} \), by Lemma \[4.3\] Property 1 follows, setting \( Y = X \). Otherwise, it is easy to deduce that the derivation \( X \xrightarrow{\sigma^*_{PAR}} \) can be written in the form

\[
X \xrightarrow{\sigma^*_{PAR}} t||Z \xrightarrow{r'} t||W.Z' \xrightarrow{\sigma^*_{PAR}}
\]

where \( r' = Z^n \rightarrow W.Z' \) (with \( W, Z, Z' \in Var \)), and there exists a subderivation of \( t||W.Z' \xrightarrow{\sigma^*_{PAR}} \) from \( Z' \), namely \( Z' \xrightarrow{\sigma^*_{PAR}} \), that is a \((K, K^\omega)\)-accepting infinite derivation in \( M \).

**Base Step:** \( k = 0 \). In this case \( r \) must occur in the rule sequence \( \sigma_1 r' (\sigma_2 \setminus \sigma_2') \). By Lemma \[B.1\] we have \( t \xrightarrow{\sigma_2 \setminus \sigma_2'} \). Therefore, there exists a derivation of the form \( X \xrightarrow{\sigma_{PAR}^*} t'||Z \xrightarrow{r'} t'||W.Z' \).
with \( \{i\} \subseteq \Upsilon^f_M(\lambda r') \subseteq K \). By Lemma 3.2 applied to the derivation \( X \xrightarrow{r*} t\|Z \), there exists a \( p \in T_{\text{PAR}} \) such that \( X \xrightarrow{r*}_{K,K} p\|Z \), with \( \Upsilon^f_{M_{\text{PAR}}} (\rho) = \Upsilon^f_M (\lambda) \). By the definition of \( \mathcal{R}_{\text{SEQ}}^K \) we have that \( X \xrightarrow{r*}_{K,K} Z \xrightarrow{r*}_{K,K} W.Z' \), with \( \Upsilon^f_{M_{\text{SEQ}}} (\mu) = \Upsilon^f_{M_{\text{PAR}}} (\rho) \) and \( \Upsilon^f_{M_{\text{SEQ}}} (r') = \Upsilon^f_M (\lambda r') \). Therefore, \( \Upsilon^f_{M_{\text{SEQ}}} (\mu r') = \Upsilon^f_M (\lambda r') \). Thus, variable \( Z' \) is reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K_i, \emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( \{i\} \subseteq K_i \subseteq K \), and there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( Z' \). This is exactly what Property 2 states.

**Induction Step:** \( k > 0 \). If the rule sequence \( \sigma_1 r' (\sigma_2 \setminus \sigma'_2) \) contains some occurrence of \( r \), then the thesis follows by reasoning as in the base step. Otherwise, \( \sigma'_2 \) contains the first occurrence of \( r \) in \( \sigma \). Clearly, this occurrence is the first occurrence of a rule of \( \mathcal{R}_A \) in the \((K, K^\omega)\)-accepting infinite derivation \( Z' \xrightarrow{r*}_K \), and it is applied at level \( k' \) in \( Z' \xrightarrow{r*}_K \) with \( k' < k \). By inductive hypothesis, the thesis holds for the derivation \( Z' \xrightarrow{r*}_K \). Therefore, it suffices to prove that \( Z' \) is reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K', \emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K' \subseteq K \). By Lemma 3.2 applied to the derivation \( X \xrightarrow{r*}_K \), there exists a \( p \in T_{\text{PAR}} \) such that \( X \xrightarrow{r*}_{K,K} p\|Z \), with \( \Upsilon^f_{M_{\text{PAR}}} (\rho) = \Upsilon^f_M (\sigma_1) \subseteq K \). By the definition of \( \mathcal{R}_{\text{SEQ}}^K \) we obtain that \( X \xrightarrow{r*}_{K,K} Z \xrightarrow{r*}_{K,K} W.Z' \), with \( \Upsilon^f_{M_{\text{SEQ}}} (\mu) = \Upsilon^f_{M_{\text{PAR}}} (\rho) \) and \( \Upsilon^f_{M_{\text{SEQ}}} (r') = \Upsilon^f_M (r') \subseteq K \). So, \( \Upsilon^f_{M_{\text{SEQ}}} (\mu r') \subseteq K \). This concludes the proof. \( \square \)

**Lemma C.3.** Let \( X \in \text{Var} \) and \( X \xrightarrow{r*}_K \) be a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( X \). Then, one of the following conditions is satisfied:

1. There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K', \emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K' \subseteq K \), and there exists a derivation \( Y \xrightarrow{r*}_{K,K^\omega} \) such that \( \Upsilon^f_{M_{\text{PAR}}} K^\omega (\rho) = K \) and \( \Upsilon^f_{M_{\text{PAR}}} (\rho) \cup \Upsilon^f_{M_{\text{PAR}}} (\rho) = K^\omega \). Moreover, \( \rho \) is either infinite or contains some occurrence of rule in \( \mathcal{R}_{\text{PAR}} \). \( \mathcal{R}_{\text{PAR}} \) \( \mathcal{R}_{\text{PAR}} \).

2. There exists a variable \( Y \in \text{Var} \) reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K, \emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \), and there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( Y \).

**Proof.** It suffices to prove that, assuming that Property 1 is not satisfied, Property 2 must hold. If \(|K| = 0\), property 2 is obviously satisfied. So, let us assume that \(|K| > 0\). Let \( K = \{j_1, \ldots, j_{|K|}\} \), and for all \( p = 1, \ldots ,|K| \) let \( K_p = \{j_1, \ldots, j_p\} \). Let us prove by induction on \( p \) that the following property is satisfied:

**A** There exists a variable \( Y \) reachable from \( X \) in \( \mathcal{R}_{\text{SEQ}}^K \) through a \((K', \emptyset)\)-accepting derivation in \( M_{\text{SEQ}}^K \) with \( K_p \subseteq K' \subseteq K \), and there exists a \((K, K^\omega)\)-accepting infinite derivation in \( M \) from \( Y \).
Base Step: $p = 1$. Considering that Property 1 isn’t satisfied, the result follows from Lemma \ref{C.2} setting $i = j_1$.

**Induction Step:** $1 < p \leq |K|$. By the inductive hypothesis there exists a $t \in T_{\text{SEQ}} \setminus \{\varepsilon\}$ such that $X \xrightarrow{\sigma^*_K} t$ with $K_{p-1} \subseteq \Upsilon_{M_{\text{SEQ}}}^f (\rho) \subseteq K$, and there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ of the form $\text{last}(t) \xrightarrow{\rho^*_K}$. By Lemma \ref{C.2} applied to the derivation $\text{last}(t) \xrightarrow{\rho^*_K}$ and considering that Property 1 is not satisfied, it follows that there exists a $\overline{t} \in T_{\text{SEQ}} \setminus \{\varepsilon\}$ such that $\text{last}(t) \xrightarrow{\rho^*_K} \overline{t}$ with $\{j_p\} \subseteq \Upsilon_{M_{\text{SEQ}}}^f (\overline{\rho}) \subseteq K$, and there exists a $(K, K^\omega)$-accepting infinite derivation in $M$ from $\text{last}(\overline{t})$. So, we have $X \xrightarrow{\sigma^*_K} t \circ \overline{t}$ with $K_p \subseteq \Upsilon_{M_{\text{SEQ}}}^f (\overline{\rho}) \subseteq K$. Therefore, Property A follows, setting $Y = \text{last}(\overline{t})$.

By property A, the thesis follows.

\[\Box\]

**C.1 Proof of Theorem 4.2**

$(\Rightarrow)$ Since $K \neq K^\omega$ and $K \supseteq K^\omega$, it follows that $K \supseteq K^\omega$. Let $d = X \xrightarrow{\sigma^*_K}$ be a $(K, K^\omega)$-accepting infinite derivation in $M$ from $X$. Evidently, $K \setminus K^\omega = \{i \in \{1, \ldots, n\}| \sigma \text{ contains a finite non-null number of occurrences of rules in } R^A_i\}$. Then, for all $i \in K \setminus K^\omega$ it is defined the greatest application level, denoted by $h_i(d)$, of occurrences of rules of $R^A_i$ in the derivation $d$. The proof is by induction on $\max_{i \in K \setminus K^\omega} \{h_i(d)\}$.

**Base Step:** $\max_{i \in K \setminus K^\omega} \{h_i(d)\} = 0$. In this case each subderivation of $d = X \xrightarrow{\sigma^*_K}$ does not contain occurrences of rules in $\bigcup_{i \in K \setminus K^\omega} R^A_i$. So, $d$ belongs to $\Pi_{\text{PAR}, \infty}^{K, K^\omega}$. Then, by Lemma \ref{4.3} we obtain the assertion setting $Y = X$.

**Induction Step:** $\max_{i \in K \setminus K^\omega} \{h_i(d)\} > 0$. If $d = X \xrightarrow{\sigma^*_K}$ is in $\Pi_{\text{PAR}, \infty}^{K, K^\omega}$, by Lemma \ref{4.3} we obtain the assertion setting $Y = X$. Otherwise, it is easy to deduce that the derivation $X \xrightarrow{\sigma^*_K}$ can be written in the form

$$X \xrightarrow{\sigma^*_K} t || Z \xrightarrow{\tau_K} t || W. Z' \xrightarrow{\sigma^*_K}$$

where $r = Z \xrightarrow{\sigma_i} W. Z'$ (with $W, Z, Z' \in \text{Var}$), and there exists a subderivation of $t || W. Z' \xrightarrow{\sigma^*_K}$ from $Z'$, namely $d' = Z' \xrightarrow{\sigma^*_K}$, that is a $(K, K^\omega)$-accepting infinite derivation in $M$. Evidently, $\max_{i \in K \setminus K^\omega} \{h_i(d')\} < \max_{i \in K \setminus K^\omega} \{h_i(d)\}$. By inductive hypothesis, the thesis holds for the derivation $d'$. Therefore, it suffices to prove that $Z'$ is reachable from $X$ in $R^K_{\text{SEQ}}$ through a $(K', \emptyset)$-accepting derivation in $M^K_{\text{SEQ}}$ with $K' \subseteq K$. By Lemma \ref{B.2} applied to the derivation $X \xrightarrow{\sigma^*_K} t || Z$ where $\Upsilon_{M}(\sigma_1) \subseteq K$, there exists a $p \in T_{\text{PAR}}$ such that $X \xrightarrow{\rho_1^*_K} Z$ with $\Upsilon_{M_{\text{PAR}}}^f (\rho_1) = \Upsilon_{M}^f (\sigma_1)$. By the definition of $R^K_{\text{SEQ}}$ we obtain that $X \xrightarrow{\rho_1^*_K} Z$ with $\Upsilon_{M_{\text{SEQ}}}^f (\gamma) = \Upsilon_{M_{\text{SEQ}}}^f (\rho_1)$ and $\Upsilon_{M_{\text{SEQ}}}^f (\gamma r) = \Upsilon_{M}^f (r) \subseteq K$. So, $\Upsilon_{M_{\text{SEQ}}}^f (\gamma r) \subseteq K$. Therefore, the thesis holds.

$(\Leftarrow)$ By the hypothesis we have
1. \( X \overset{\rightarrow}{\mapsto}_{K^*_{SEQ}} t \) with \( t \in T_{SEQ} \setminus \{\varepsilon\} \), \( \text{last}(t) = Y \) and \( \gamma^f_{M} K_{SEQ} (\lambda) \subseteq K \).

2. \( Y \overset{\rightarrow}{\mapsto}_{K^*_{SEQ}} \) with \( \gamma^f_{K,K^*_{PAR}} \) \( \rho = K \) and \( \gamma^\infty_{K,K^*_{PAR}} \rho \) \( \cup \gamma^f_{K,K^*_{PAR}} \) \( \rho = K^* \). Moreover, \( \rho \) is either infinite or contains some occurrence of a rule in \( R_{PAR} \setminus R_{PAR}^K \).

Since \( X \in SEQ(X) \), by condition 1 and Lemma C.1, it follows that there exists a \( s \in T \) such that \( t \in SEQ(s) \) and \( X \overset{s}{\mapsto}_s \) \( \gamma^f_\eta \) \( \subseteq K \). By condition 2 and Lemma 4.4, it follows that there exists a \( (K,K^*) \)-accepting infinite derivation in \( M \) of the form \( Y \overset{\rightarrow}{\mapsto}_{\rho} \).

Since \( Y \in SubTerms(s) \), by Proposition A.1 we have that \( s \overset{\rightarrow}{\mapsto}_{\rho} \). After all, we obtain \( X \overset{\rightarrow}{\mapsto}_s \) \( \overset{\rightarrow}{\mapsto}_{\rho} \), that is a \( (K,K^*) \)-accepting infinite derivation in \( M \) from \( X \). This concludes the proof.

C.2 Proof of Theorem 4.3

(\( \Rightarrow \)) It suffices to prove that, assuming that condition 1 (in the enunciation) does not hold, condition 2 must hold. Under this hypothesis, we show that there exists a sequence of terms \( (t_h)_{h \in N} \) in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying the following properties for all \( h \in N \):

i. \( t_0 = X \),

ii. \( \text{last}(t_h) \overset{\rightarrow}{\mapsto}_{K^*_{SEQ}} t_{h+1} \) with \( \gamma^f_{M} (\rho_h) = K \),

iii. there exists a \( (K,K^*) \)-accepting infinite derivation in \( M \) from \( \text{last}(t_h) \),

iv. \( \text{last}(t_h) \) is reachable from \( X \) in \( R^K_{SEQ} \) through a \( (K',\emptyset) \)-accepting derivation in \( M^K_{SEQ} \) with \( K' \subseteq K \).

For \( h = 0 \) properties iii and iv are satisfied, by setting \( t_0 = X \). So, assume the existence of a finite sequence of terms \( t_0,t_1,\ldots,t_h \) in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying properties i-iv. It suffices to prove that there exists a term \( t_{h+1} \) in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying iii and iv, and such that \( \text{last}(t_h) \overset{\rightarrow}{\mapsto}_{K^*_{SEQ}} t_{h+1} \) with \( \gamma^f_{M} (\rho_h) = K \). By the inductive hypothesis, \( \text{last}(t_h) \) is reachable from \( X \) in \( R^K_{SEQ} \) through a \( (K',\emptyset) \)-accepting derivation in \( M^K_{SEQ} \) with \( K' \subseteq K \), and there exists a \( (K,K^*) \)-accepting infinite derivation in \( M \) from \( \text{last}(t_h) \). By Lemma C.3 applied to variable \( \text{last}(t_h) \), and the fact that condition 1 does not hold, it follows that there exists a term \( t \rightarrow T_{SEQ} \setminus \{\varepsilon\} \) such that \( \text{last}(t_h) \overset{\rightarrow}{\mapsto}_{K^*_{SEQ}} t \) with \( \gamma^f_{M} (\rho_h) = K \), and there exists a \( (K,K^*) \)-accepting infinite derivation in \( M \) from \( \text{last}(t) \). Since \( \text{last}(t_h) \) is reachable from \( X \) in \( R^K_{SEQ} \) through a \( (K',\emptyset) \)-accepting derivation in \( M^K_{SEQ} \) with \( K' \subseteq K \), it follows that \( \text{last}(t) \) is reachable from \( X \) in \( R^K_{SEQ} \) through a \( (K,\emptyset) \)-accepting derivation in \( M^K_{SEQ} \). Thus, setting \( t_{h+1} = t \), we obtain the result.

Let \( (t_h)_{h \in N} \) be the sequence of terms in \( T_{SEQ} \setminus \{\varepsilon\} \) satisfying properties i-iv. Since in this case \( |K| > 0 \) (remember that \( |K| + |K^*| > 0 \)), we have \( |\rho_h| > 0 \) for all \( h \in N \). Then, by Proposition A.2 we obtain that for all \( h \in N \)

\[
t_0 \circ t_1 \circ \ldots \circ t_h \overset{\rightarrow}{\mapsto}_{K^*_{SEQ}} t_0 \circ t_1 \circ \ldots \circ t_h \circ t_{h+1}
\]
Therefore,

\[ X = t_0 \xrightarrow{\rho_0^*} \underbrace{t_0 \circ t_1 \xrightarrow{\rho_1^*} t_0 \circ t_1 \circ t_2 \xrightarrow{\rho_2^*} \ldots}_{\rho_0 \circ \ldots \circ \rho_h} t_0 \circ t_1 \circ \ldots \circ t_h \]

is an infinite derivation in \( \mathfrak{R}^K \) from \( X \). Setting \( \delta = \rho_0 \rho_1 \ldots \), from ii and Proposition A.3 we obtain that \( \Upsilon_{M_{\underline{\mathcal{R}}}^{\underline{S}}}^f (\delta) = \bigcup_{h \in N} \Upsilon_{M_{\underline{\mathcal{R}}}^{\underline{S}}}^f (\rho_h) = K \) and \( \Upsilon_{M_{\underline{\mathcal{R}}}^{\underline{S}}}^{\infty} (\delta) = \bigoplus_{h \in N} \Upsilon_{M_{\underline{\mathcal{R}}}^{\underline{S}}}^f (\rho_h) = K = K^\omega \). Hence, condition 2 (in the enunciation) holds.

(\( \Leftarrow \)) At first, let us assume that condition 2 holds. Then, since \( X \in \mathcal{S}EQ(X) \), the result follows directly by Lemma C.1. Assume that condition 1 holds instead. Then, we reason as in the proof of Theorem 4.2.