Z IS UNIVERSAL

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Abstract. We use order zero maps to express the Jiang-Su algebra \( Z \) as a universal C\(^*\)-algebra on countably many generators and relations, and we show that a natural deformation of these relations yields the stably projectionless algebra \( W \) studied by Kishimoto, Kumjian and others. Our presentation is entirely explicit and involves only \(^*\)-polynomial and order relations.

1. Introduction

In Elliott’s programme to classify simple, nuclear C\(^*\)-algebras using \( K \)-theoretic invariants, the Jiang-Su algebra \( Z \) plays a particularly prominent role (see [18]). While there are various ways of characterizing \( Z \) (see for example [4] and [14]), its most concise description (due to the second named author, in [20]) is as the unique initial object in the category of strongly self-absorbing C\(^*\)-algebras. Here, a separable, unital C\(^*\)-algebra \( D \neq \mathbb{C} \) is strongly self-absorbing if there is an isomorphism \( \varphi : D \to D \otimes D \) that is approximately unitarily equivalent to the first factor embedding, cf. [16]. The statement that \( Z \) is an initial object in this category is equivalent to saying that every strongly self-absorbing C\(^*\)-algebra absorbs \( Z \) tensorially (i.e. is ‘\( Z \)-stable’).

Apart from \( Z \), the known strongly self-absorbing algebras are: the Cuntz algebras \( O_2 \) and \( O_\infty \), UHF algebras of infinite type, and such UHF algebras tensored with \( O_\infty \). These all admit presentations as universal C\(^*\)-algebras (see Section 5 for a discussion), and Theorem 3.1 of this article provides such a description for \( Z \) which, although complicated, is explicit and algebraic in the sense that it involves only \(^*\)-polynomial and order relations. The proof relies on the ‘order zero’ presentations of prime dimension drop algebras described in [14] (see Section 2), and gives a construction of \( Z \) as an inductive limit of such algebras with connecting maps defined in terms of generators and relations.

The Jiang-Su algebra may be thought of as a stably finite analogue of \( O_\infty \), and the C\(^*\)-algebra \( W \) constructed in [3] (and studied in another form in [5]) has been similarly proposed as a stably finite analogue of \( O_2 \). The conjecture that \( W \otimes W \cong W \), while still open, is known to have interesting consequences. For example, it is shown in [3] that among the C\(^*\)-algebras classified in [11], those that are simple and have trivial \( K \)-theory would absorb \( W \) tensorially. On the other hand, L. Robert proves in [12] that the Cuntz semigroup of a \( W \)-stable C\(^*\)-algebra is determined by the cone of its lower semicontinuous 2-quasitraces. These results indicate that \( W \) may play an important role in the classification of nuclear, stably projectionless C\(^*\)-algebras. In this article, we examine the structure of \( W \) rather than its role in classification, by showing in Theorem 4.3 how to obtain \( W \) as a nonunital deformation of \( Z \).

The paper is organized as follows. In Section 2 we establish notation and recall various basic facts about order zero maps and dimension drop algebras. Section 3 contains the presentation of \( Z \) as a universal C\(^*\)-algebra (Theorems 3.1 and 3.3), and Section 4 contains the corresponding description of \( W \) (Theorem 4.3). We conclude with some open questions in Section 5.

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In this section, we collect some well-known facts about order zero maps and dimension drop algebras that are used throughout the article. (Detailed exposition of order zero maps can be found in [21] and [22].) We denote by $\epsilon_{ij}$ (or $\epsilon_{ij}^{(n)}$) the canonical $(i,j)$-th matrix unit in $M_n = M_n(\mathbb{C})$.

Recall that a completely positive (c.p.) map $\varphi : B \to A$ has order zero if it preserves orthogonality. Every completely positive and contractive (c.p.c.) order zero map $\varphi : B \to A$ (for $B$ unital) is of the form $\varphi(\cdot) = \pi_\varphi(\cdot)\varphi(1_B) = \varphi(1_B)\pi_\varphi(\cdot)$ for a *-homomorphism $\pi_\varphi : B \to A^{**}$ called the supporting *-homomorphism of $\varphi$. We frequently use the notion of positive functional calculus provided by this decomposition: if $f \in C_0(0,1]$ is positive with $\|f\| \leq 1$ then the map $f(\varphi) : B \to A$ given by $f(\varphi)(\cdot) := \pi_\varphi(\cdot)f(\varphi(1_B))$ is a well-defined c.p.c. order zero map. It is easy to see that if $p \in B$ is a projection, then $f(\varphi)(p) = f(\varphi(p))$. On the other hand, if $\varphi(1_B)$ is a projection, then $\varphi$ is in fact a *-homomorphism.

Finally, c.p.c. order zero maps $B \to A$ correspond bijectively to *-homomorphisms $C_0(0,1], B \to A$. For $B = M_n$, one way of interpreting this fact is to say that the cone $C_0(0,1], M_n)$ is the universal C*-algebra generated by a c.p.c. order zero map on $M_n$. Equivalently, it is easy to check that $C_0(0,1], M_n)$ is the universal C*-algebra on generators $x_1, \ldots, x_n$ subject to the relations $R_n$ given by

$$\|x_i\| \leq 1, \quad x_i \geq 0, \quad x_i x_i^* = x_i^2, \quad x_i^* x_j \perp x_i^2 x_i \quad \text{for} \quad 1 \leq i \neq j \leq n \quad (2.1)$$

(for example by mapping $x_j$ to $t^{1/2} \otimes e_{ij}$, so that $t \otimes e_{ij}$ corresponds to $x_i^2 x_j$). One can therefore view the statement

$$C_0(0,1], M_n) = C^*(\varphi \mid \varphi \text{ c.p.c. order zero on } M_n) \quad (2.2)$$

as an abbreviation for these relations.

**Remark 2.1.** In the case $n = 2$, $C_0(0,1], M_2)$ is the universal C*-algebra $C^*(x \mid \|x\| \leq 1, x^2 = 0)$. Therefore, if $A$ is a C*-algebra and $v \in A$ is a contraction with $v^2 = 0$, then there is a unique c.p.c. order zero map $\psi : M_2 \to A$ with $\psi(1/2)(e_{12}) = v$ (so that $\psi(e_{11}) = vv^*$ and $\psi(e_{22}) = v^*v$).

By a prime dimension drop algebra, we mean a C*-algebra of the form

$$Z_{p,q} := \{f \in C([0,1], M_p \otimes M_q) \mid f(0) \in M_p \otimes 1_q, f(1) \in 1_p \otimes M_q\}, \quad (2.3)$$

where $p$ and $q$ are coprime natural numbers. The Jiang-Su algebra $\mathcal{Z}$ is the unique inductive limit of prime dimension drop algebras which is simple and has a unique tracial state (see [1]).

The order zero notation (2.2) essentially appears in [13] Proposition 2.5, where the presentation of prime dimension drop algebras described in [3] Proposition 7.3 is reinterpreted in terms of order zero maps. Specifically, the prime dimension drop algebra $Z_{p,q}$ is the universal unital C*-algebra

$$C^*(\alpha, \beta \mid \alpha \text{ c.p.c. order zero on } M_p, \beta \text{ c.p.c. order zero on } M_q, \alpha(1_p) + \beta(1_q) = 1, [\alpha(M_p), \beta(M_q)] = 0),$$

with generators corresponding to the obvious embeddings of $C_0([0,1], M_p)$ and $C_0([0,1], M_q)$ into $Z_{p,q}$.

When $q = p + 1$, there is another presentation of $Z_{p,p+1}$ in terms of order zero maps that does not involve a commutation relation. The following is essentially contained in [14] Proposition 5.1, and we note that these relations have already proved highly useful, for example in [18], [21], [15] and [8].

**Proposition 2.2.** Let $Z^{(n)}$ be the universal unital C*-algebra $C^*(\varphi, \psi \mid \mathcal{R}_n)$, where $\mathcal{R}_n$ denotes the set of relations:

1. $\varphi$ and $\psi$ are c.p.c. order zero maps on $M_n$ and $M_q$ respectively;
2. $\psi(e_{11}) = 1 - \varphi(1_n)$;
3. $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

Then $Z^{(n)} \cong Z_{n,n+1}$.

In Section 2 we use Proposition 2.2 to write $\mathcal{Z}$ as a limit of dimension drop algebras in a universal way. We make analogous use of Proposition 1.1 in a nonunital version of Proposition 2.2 to present $\mathcal{W}$.
3. Generators and Relations for the Jiang-Su Algebra

In this section, we will construct an inductive system \((Z^{(q(k))}, \alpha_k)\), where \(q(k) = p^k\) for some fixed \(p \geq 2\) \((p = 2\) will do\) and \(Z^{(q(k))} = C^*(\varphi_k, \psi_k) | \mathcal{R}_{q(k)} \cong Z_{q(k), q(k)+1}\) (as in Proposition 2.2), and we will check that the inductive limit is simple with a unique tracial state. It will then follow from the classification theorem of \([4]\) that \(Z \cong \lim Z^{(q(k))}, (\alpha_k)\).

If this procedure is to provide an explicit presentation of \(Z\) as a universal \(C^*\)-algebra, we need to be able to describe the connecting maps \(\alpha_k\) in terms of generators and relations. (This is perhaps the key difference between our construction and the original construction of \(Z\) as an inductive limit in \([4]\).) In other words, for every \(k \in \mathbb{N}\) we will find c.p.c. order zero maps \(\hat{\varphi}_k : M_{q(k)} \to Z^{(q(k)+1)}\) and \(\hat{\psi}_k : M_2 \to Z^{(q(k+1))}\) that satisfy the relations \(\mathcal{R}_{q(k)}\) of Proposition 2.2. By universality, we will then have unital connecting maps \(\alpha_k : Z^{(q(k))} \to Z^{(q(k)+1)}\) with \(\alpha_k \circ \varphi_k = \hat{\varphi}_k\) and \(\alpha_k \circ \psi_k = \hat{\psi}_k\).

Before giving the connecting maps, it is instructive to note that there are obvious choices for \(\hat{\varphi}_k\) and \(\hat{\psi}_k\). Since \(q(k+1) = q(k)^3\), we can identify \(M_{q(k)+1}\) with \(M_{q(k)} \otimes M_{q(k)} \otimes M_{q(k)}\) (and \(e_{q(k)}^{(q(k)+1)}\) with \(e_{q(k)}^{(q(k))} \otimes e_{q(k)}^{(q(k))} \otimes e_{q(k)}^{(q(k))}\)). We could then set \(\hat{\varphi}_k = \varphi_{k+1} \circ (\id_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)})\) and \(\hat{\psi}_k = \psi_{k+1}\); it is easy to see that these maps satisfy the relations \(\mathcal{R}_{q(k)}\), but the corresponding inductive limit certainly would not be simple. The idea is therefore to define \(\hat{\varphi}_k\) in such a way as to ensure that \([0, 1]\) is chopped up into suitably small pieces under the induced \(*\)-homomorphism \(\alpha_k\); \(\psi_k^{(1/2)}\) \((e_2)\) will then be some partial-isometry-like element that facilitates the relations \(\mathcal{R}_{q(k)}\).

One way of doing this is as follows. Define \(\rho_k : M_{q(k)} \to M_{q(k)+1}\) by

\[
\rho_k = (\id_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)}) \oplus ((1_k) \otimes \bigoplus_{i=1}^{q(k)} (\id_{M_{q(k)}} \otimes e_{q(k), q(k)} \otimes e_{ii})).
\]

Note that \(\rho_k\) is c.p.c. order zero, with supporting \(*\)-homomorphism \(\pi_{\rho_k} = \id_{M_{q(k)}} \otimes 1_{q(k)} \otimes 1_{q(k)}\). We may then define \(\hat{\varphi}_k := \varphi_{k+1} \circ \rho_k\). For this to work, we need to be able to transport the defect \(1 - \varphi_{k+1}(\rho_k(1_{q(k)})) = 1 - \varphi_{k+1}(1_{q(k)+1}) + \varphi_{k+1}(1_{q(k)+1}) - \rho_k(1_{q(k)})\) underneath \(\varphi_{k+1}(\rho_k(e_{11}^{(q(k))}))\), and the basic idea is to do this in two steps.

Step 1. Use \(\psi_{k+1}(e_2)\) to transport the corner \(\pi_{\varphi_{k+1}(e_{12})}(1_{11}) - \varphi_{k+1}(\rho_k(1_{q(k)}))\) \(\pi_{\psi_{k+1}(e_{22})}\varphi_{k+1}(e_{11}^{(q(k)+1)})\) \(\pi_{\psi_{k+1}(e_{22})}\varphi_{k+1}(e_{11}^{(q(k)+1)})\) underneath \(\varphi_{k+1}(e_{11}^{(q(k)+1)})\) \(\pi_{\varphi_{k+1}(e_{11})}\varphi_{k+1}(e_{11}^{(q(k)+1)})\) \(\pi_{\varphi_{k+1}(e_{11})}\varphi_{k+1}(e_{11}^{(q(k)+1)})\).

Step 2. Use a partial isometry \(v_{k+1} \in M_{q(k)+1}\) to transport (a projection bigger than) \(1_{q(k)+1} - \rho_k(1_{q(k)})\) underneath (a projection smaller than) \(\rho_k(e_{11}^{(q(k)+1)}) - e_{11}^{(q(k)+1)}\), so that \(\varphi_{k+1}(v_{k+1})\) transports the rest of \(1 - \varphi_{k+1}(\rho_k(1_{q(k)}))\) underneath \(\varphi_{k+1}(\rho_k(e_{11}^{(q(k)+1)})) - \varphi_{k+1}(e_{11}^{(q(k)+1)})\).

Although this is essentially the right idea, it needs fine-tuning in the guise of functional calculus. We achieve this in Theorem 3.3 by adjusting the relations for \(Z^{(q(k))}\), while for Theorem 3.1 we modify \(\hat{\varphi}_k\) and \(\hat{\psi}_k\) using the following piecewise linear functions:

\[
\begin{align*}
d(t) &= \begin{cases} 
1 & t < 0.5 \\
1 - t & 0.5 \leq t \leq 1
\end{cases} \\
f(t) &= \begin{cases} 
1 & t < 0.5 \\
0 & 0.5 \leq t \leq 1
\end{cases} \\
g(t) &= \begin{cases} 
1 & t < 0.5 \\
0 & 0.5 \leq t \leq 1
\end{cases} \\
h(t) &= \begin{cases} 
1 & t < 0.5 \\
1 - t & 0.5 \leq t \leq 1
\end{cases}
\end{align*}
\]

These are chosen so that, writing \(\tilde{d}(t) = d(1 - t)\), we have

\[
\begin{align*}
g &= f - h, & hf &= h, & (1 - f)\tilde{d} &= 1 - f & \text{and} & \quad g\tilde{d} &= g.
\end{align*}
\]
For use in Section 3 we also note that if \( \hat{d} \) is the function \( \hat{d}(t) = d(t(1 - t)) \) then we have

\[
(f - f^2) \hat{d} = f - f^2 \quad \text{and} \quad g \hat{d} = g.
\] (3.3)

Finally, to accomplish Step 2, we choose a partial isometry

\[ v_{k+1} \in M_{q(k+1)} \]

such that

\[
v_{k+1} v_{k+1}^* = 1_{q(k)} \otimes e_{q(k),q(k)} \otimes 1_{q(k)-1}
\]

and

\[
v_{k+1}^* v_{k+1} = (e_{11} \otimes 1_{q(k)-1} \otimes 1_{q(k)}) + (e_{11} \otimes e_{q(k),q(k)} \otimes e_{q(k),q(k)}) - (e_{11} \otimes e_{11} \otimes e_{11}).
\]

This is possible since both of these projections have rank \( q(k)^2 - q(k) \); since they are orthogonal, we moreover have \( v_{k+1}^2 = 0 \). This \( v_{k+1} \) then satisfies:

(i) \( v_{k+1}^* v_{k+1} \perp e_{11} \otimes e_{11} \otimes e_{11} = e_{11}^{(q(k+1))} \) (in fact, \( v_{k+1} v_{k+1}^* \) is orthogonal to \( e_{11}^{(q(k+1))} \) too);

(ii) \( v_{k+1}^* v_{k+1} \) is dominated by \( \rho_k(e_{11}^{(q(k))}) \) (and therefore by \( \rho_k(e_{11}^{(q(k-1))} - e_{11}^{(q(k+1))}) \)); and

(iii) \( v_{k+1} v_{k+1}^* \) acts like a unit on

\[
1_{q(k+1)} - \rho_k(1_{q(k)}) = \bigoplus_{i=1}^{q(k)} \left( 1 - \frac{i}{q(k)} \right) (1_{q(k)} \otimes e_{q(k),q(k)} \otimes e_{ii}).
\] (3.5)

**Theorem 3.1.** Let the functions \( d, f, g, h \in C_0([0,1], \) the partial isometries \( v_{k+1} \in M_{q(k+1)} \), and the c.p.c. order zero maps \( \rho_k : M_{q(k)} \to M_{q(k+1)} \) be as above for each \( k \in \mathbb{N} \). Define \( Z_U \) to be the universal unital C*-algebra generated by c.p.c. order zero maps \( \varphi_k \) on \( M_{q(k)} \) (\( k \in \mathbb{N} \)) and \( \psi_k \) on \( M_2 \) (\( k \in \mathbb{N} \)) such that for each \( k \), these maps satisfy the relations \( \mathcal{R}_{q(k)} \), i.e.

\[
\psi_k(e_{11}) = 1 - \varphi_k(1_{q(k)})
\] (3.6)

and

\[
\psi_k(e_{22}) \varphi_k(e_{11}) = \psi_k(e_{22}),
\] (3.7)

together with the additional relations \( \mathcal{S}_{q(k)} \) given by

\[
\varphi_k = f(\varphi_{k+1}) \circ \rho_k,
\] (3.8)

\[
\psi_k^{1/2}(e_{12}) = \left( 1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) \right)^{1/2} d(\psi_{k+1})(e_{12}) + h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2} f(\varphi_{k+1})(e_{k+1}).
\] (3.9)

Then \( Z_U \cong Z \).

**Proof.** For each \( k \), define \( \hat{\varphi}_k : M_{q(k)} \to Z(q^{(k+1)}) = C^*(\varphi_{k+1}, \psi_{k+1} | \mathcal{R}_{q(k+1)}) \) and \( \hat{\psi}_k : M_2 \to Z(q^{(k+1)}) \) by

\[
\hat{\varphi}_k = f(\varphi_{k+1}) \circ \rho_k
\] (3.10)

and

\[
\hat{\psi}_k^{1/2}(e_{12}) = \gamma_k + \delta_k,
\] (3.11)

where

\[
\gamma_k := \left( 1 - f(\varphi_{k+1})(1_{q(k+1)}) + g(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})) \right)^{1/2} d(\psi_{k+1})(e_{12})
\] (3.12)

and

\[
\delta_k := h(\varphi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2} f(\varphi_{k+1})(e_{k+1}).
\] (3.13)

We need to check that \( \hat{\varphi}_k \) and \( \hat{\psi}_k \) satisfy the relations \( \mathcal{R}_{q(k)} \). First, it is obvious that \( \hat{\varphi}_k \) is c.p.c. order zero since \( \varphi_{k+1} \) and \( \rho_k \) are, and \( f \) is contractive. Next, to show that \( \hat{\psi}_k \) genuinely defines a c.p.c. order zero map \( \hat{\psi}_k \), it suffices to check that \( \gamma_k + \delta_k \) is a contraction that squares to zero (see Remark 2.1). In fact, this would follow automatically from the relations (3.3) and (3.7) for \( \hat{\varphi}_k \) and \( \hat{\psi}_k \) (where, for the moment, we interpret \( \hat{\psi}_k(e_{11}) \) and \( \hat{\psi}_k(e_{22}) \) as notation for \( \hat{\psi}_k^{1/2}(e_{12}) \hat{\psi}_k^{1/2}(e_{12})^* \) and
where \( \delta \) is a partial isometry. Claim 2: If \( \psi \) and \( \hat{\psi} \) satisfy the relations (3.7) and (3.6), respectively. Indeed, 1 - \( \hat{\varphi}_k(1_q(k)) \) is certainly a contraction, and (3.9) and (3.7) would imply that

\[
\hat{\psi}_k(e_{22})\hat{\varphi}_k(e_{11}) = \hat{\psi}_k(e_{22})(1 - \hat{\varphi}_k(1_q(k))) = \psi_k(e_{22}) - \sum_{i=1}^{n} \hat{\psi}_k(e_{22})\hat{\varphi}_k(e_{11})\hat{\varphi}_k(e_{ii}) = 0, \tag{3.14}
\]

and hence that \( \left( \hat{\psi}_k^{1/2}(e_{12}) \right)^2 = 0 \). Let us now check that \( \hat{\varphi}_k \) and \( \hat{\psi}_k \) really do satisfy these relations.

Claim 1: \( \hat{\psi}_k(e_{11}) = 1 - \hat{\varphi}_k(1_q(k)) \).

Proof of Claim 1. First note that, using (3.7) and property (ii) of the partial isometry \( v_{k+1} \), we have

\[
d(\psi_{k+1})(e_{12})f(\varphi_{k+1})(v_{k+1}) = d^{1/2}(\psi_{k+1})(e_{12})d^{1/2}(\psi_{k+1})(e_{22})\varphi_{k+1}(e_{11})f(\varphi_{k+1})(v_{k+1})v_{k+1}v_{k+1} = 0.
\]

Therefore, the cross terms \( \gamma_k\delta_k^e \) and \( \delta_k\gamma_k^e \) in the expansion of \( \hat{\psi}_k(e_{11}) = \hat{\psi}_k^{1/2}(e_{12})\hat{\psi}_k^{1/2}(e_{12}) \) vanish.

Using the fact that \( fh = h \), and property (iii) of \( v_{k+1} \), we have

\[
h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k)))f(\varphi_{k+1})(v_{k+1})f(\varphi_{k+1})(v_{k+1}^*) = h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k))v_{k+1}v_{k+1}^*) = h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k))).
\]

Thus, \( \delta_k\delta_k^e = h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k))) \). From (3.6) we have

\[
d(\psi_{k+1})(e_{12}) = d(\psi_{k+1})(e_{11}) = d(1 - \varphi_{k+1}(1_q(k+1))) = d(\varphi_{k+1}(1_q(k+1))),
\]

where \( d(t) = d(1 - t) \) as in (3.2), whence we also obtain

\[
(1 - f(\varphi_{k+1})(1_q(k+1)))d(\psi_{k+1})(e_{11}) = (1 - f(\varphi_{k+1}(1_q(k+1))))d(\psi_{k+1}(1_q(k+1)))
\]

\[
= (1 - f(\varphi_{k+1}(1_q(k+1))))
\]

\[
= 1 - f(\varphi_{k+1}(1_q(k+1))).
\]

Similarly, we have \( g(\varphi_{k+1})(1_q(k+1))d(\psi_{k+1})(e_{11}) = g(\varphi_{k+1}(1_q(k+1))), \) hence

\[
g(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k)))d(\psi_{k+1})(e_{11}) = g(\varphi_{k+1}(1_q(k+1) - \rho_k(1_q(k))).
\]

We therefore have \( \gamma_k\gamma_k^e = 1 - f(\varphi_{k+1}(1_q(k+1)) + g(\varphi_{k+1}(1_q(k+1) - \rho_k(1_q(k))), \) since \( g + h = f \), it follows that

\[
\hat{\psi}_k(e_{11}) = \gamma_k\gamma_k^e + \delta_k\delta_k^e
\]

\[
= 1 - f(\varphi_{k+1}(1_q(k+1))) + g(\varphi_{k+1}(1_q(k+1) - \rho_k(1_q(k))) + h(\varphi_{k+1}(1_q(k+1) - \rho_k(1_q(k))))
\]

\[
= 1 - f(\varphi_{k+1}(1_q(k+1))\rho_k(1_q(k)))
\]

\[
= 1 - \varphi_{k+1}(1_q(k))).
\]

Claim 2: \( \hat{\psi}_k(e_{22})\hat{\varphi}_k(e_{11}) = \hat{\psi}_k(e_{22}). \)

Proof of Claim 2. Since \( fh = h \) and \( v_{k+1} \) is a partial isometry with property (ii), we have

\[
h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k)))f(\varphi_{k+1})(v_{k+1})f(\varphi_{k+1})(\rho_k(e_{11}))
\]

\[
= h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k)))v_{k+1}
\]

\[
= h(\varphi_{k+1})(1_q(k+1) - \rho_k(1_q(k)))f(\varphi_{k+1})(v_{k+1}).
\]

Thus, \( \delta_k\hat{\varphi}_k(e_{11}) = \delta_k \). Next, it follows from (3.7), upon approximating \( d^{1/2} \) and \( f \) uniformly by polynomials, that

\[
d^{1/2}(\psi_{k+1})(e_{22})f(\varphi_{k+1})(e_{11}) = f(1)d^{1/2}(\psi_{k+1})(e_{22}) = d^{1/2}(\psi_{k+1})(e_{22}).
\]

Since \( e_{11}^{q(k+1)} \perp (\rho_k(e_{11}^{q(k+1)}) - e_{11}^{q(k+1)}) \) and \( f(\varphi_{k+1}) \) is order zero, we therefore have \( d^{1/2}(\psi_{k+1})(e_{22})f(\varphi_{k+1})(\rho_k(e_{11})) = d^{1/2}(\psi_{k+1})(e_{22}), \) hence \( d(\psi_{k+1})(e_{22})f(\varphi_{k+1})(\rho_k(e_{11})) = d(\psi_{k+1})(e_{22}). \) Therefore, \( \gamma_k\hat{\varphi}_k(e_{11}) = \gamma_k \), and so \( \psi_k(e_{22})\hat{\varphi}_k(e_{11}) = (\gamma_k + \delta_k^e)(\gamma_k + \delta_k)\hat{\varphi}_k(e_{11}) = \psi_k(e_{22}). \)

We have now shown that \( \hat{\varphi}_k \) and \( \hat{\psi}_k \) satisfy the relations \( R_q(k) \). This means that, for any \( k \in \mathbb{N}, (3.8) \) and (3.9) do not introduce any new relations on \( \varphi_{k+1} \) and \( \psi_{k+1} \); thus, the sub-C*-algebra generated...
by $\varphi_{k+1}$ and $\psi_{k+1}$ within $\mathcal{Z}U$ is isomorphic to the universal $C^*$-algebra on relations $\mathcal{R}(q(k+1))$ (that is, to $Z(q(k+1))$), and moreover contains the sub-$C^*$-algebra generated by $\varphi_k$ and $\psi_k$. Therefore, by Proposition 2.2, $\mathcal{Z}U$ is isomorphic to an inductive limit of prime dimension drop algebras.

The strategy for the remainder of the proof is to pass from the abstract picture of $\mathcal{Z}U$ as a universal $C^*$-algebra, to a concrete description as an inductive limit $\varinjlim Z(q(k)), \alpha_k$, where the (unital) connecting maps $\alpha_k : Z(q(k)) \to Z(q(k+1))$ are determined by (3.8) and (3.9) (i.e. $\alpha_k \circ \varphi_k = \hat{\varphi}$ and $\alpha_k \circ \psi_k = \hat{\psi}$). We will obtain explicit descriptions of the maps $\alpha_k$, and use these to show that $\mathcal{Z}U$ is simple and has a unique tracial state.

For each $k \in \mathbb{N}$, let us fix an identification of $Z(q(k))$ with $Z_q(k), q(k)+1$ via the order zero map $M_{q(k)} \to Z_q(k), q(k)+1$ (which, abusing notation, we also call $\varphi_k$) defined by:

$$\varphi_k(a)(t) = u_k(t)(a \otimes 1_{q(k)})u_k(t)^* \oplus (1-t)(a \otimes e_{q(k)+1, q(k)+1})$$

for $a \in M_{q(k)}$ and $t \in [0,1]$. (Here, $u_k$ is a unital or the algebra $C([0,1], M_{q(k)} \otimes M_{q(k)+1})$, included nonunitarily in the top left corner of $C([0,1], M_{q(k)} \otimes M_{q(k)+1})$, with $u_k(0) = 1$ and $u_k(1)$ implementing the flip in $M_{q(k)} \otimes M_{q(k)+1}$.) It is easy to write down a suitable $\psi_k$, but for the purpose of computing the connecting map $Z_q(k), q(k)+1 \to Z_q(k+1), q(k)+1$ (also called $\alpha_k$), this is not necessary.

For each $t \in [0,1]$, let us write $\alpha_k^t$ for the map $ev_t \circ \alpha_k : Z_q(k), q(k)+1 \to M_{q(k)} \otimes M_{q(k)+1}$, where $ev_t$ denotes evaluation at $t$. Then $\alpha_k^t$ is a finite-dimensional representation of $Z_q(k), q(k)+1$, so is a direct sum of finitely many irreducible representations $\pi_{1, t}, \ldots, \pi_{m(t)}$ of $Z_q(k), q(k)+1$ (corresponding up to unitary equivalence and, at the endpoints, up to multiplicity, to point evaluations). Since $C^*(\varphi_k(1_{q(k)})) \subset Z_q(k), q(k)+1$ separates the points of $[0,1]$, it is easy to see that the unitary equivalence classes of $\pi_{1, t}, \ldots, \pi_{m(t)}$ can be determined by computing $\alpha_k^t_\pi(\varphi_k(1_{q(k)}))$. To do this, note that

$$f(\varphi_{k+1})(b)(t) = u_{k+1}(t)(b \otimes 1_{q(k)+1})u_{k+1}(t)^* \oplus f(1-t)(b \otimes e_{q(k)+1, q(k)+1})$$

for $b \in M_{q(k)+1}$, and recall the definition (5.1) of $\rho_k$. We then have, for $a \in M_{q(k)}$ and $t \in [0,1]$

$$\alpha_k^t(\varphi_k(a)) = f(\varphi_{k+1})(\rho_k(a))(t) = u_{k+1}(t)(a \otimes 1_{q(k)+1} \otimes 1_{q(k)} \otimes 1_{q(k)+1})u_{k+1}(t)^*$$

$$\oplus u_{k+1}(t)\left(\bigoplus_{i=1}^{q(k)} iq(k) (a \otimes e_{q(k), q(k)} \otimes e_{ii} \otimes 1_{q(k)+1})\right)u_{k+1}(t)^*$$

$$\oplus f(1-t)(a \otimes 1_{q(k)+1} \otimes 1_{q(k)} \otimes e_{q(k)+1, q(k)+1})$$

$$\oplus f(1-t)\left(\bigoplus_{i=1}^{q(k)} iq(k) (a \otimes e_{q(k), q(k)} \otimes e_{ii} \otimes e_{q(k)+1, q(k)+1})\right)$$

$$\sim_u \left(\bigoplus_{m=1}^{q(k)} \bigoplus_{i=1}^{q(k)} \varphi_k(a) \left(1 - iq(k)\right)\right) \oplus \left(\bigoplus_{m=1}^{q(k)} \varphi_k(a) (1-f(1-t))\right)$$

$$\oplus \left(\bigoplus_{i=1}^{q(k)} \varphi_k(a) \left(1 - if(1-t)\right)\right),$$

where $\sim_u$ denotes unitary equivalence. Write $h_t = 1 - i\frac{f(1-t)}{q(k)}$ (so that, in fact, $h_q(k) = 1 - f(1-t) = h(t))$. By our earlier reasoning it then follows that, for every $t \in [0,1]$, there is a unitary $w_k(t) \in M_{q(k)+1} \otimes M_{q(k)+1}$ such that

$$\alpha_k^t = w_k(t) \left(\bigoplus_{m=1}^{q(k)} \bigoplus_{i=1}^{q(k)-1} ev_{q(k), t}^{m, i} \right) \oplus \left(\bigoplus_{m=1}^{q(k)} \varphi_k(a) (1-f(1-t))\right) \oplus \left(\bigoplus_{i=1}^{q(k)} ev_{h(t)}(i)\right) \otimes w_k(t)^*.$$

(3.17)

It could be that $t \mapsto w_k(t)$ is not continuous, but this does not matter. (Moreover, it is not difficult to show that, up to approximate unitary equivalence, continuity may be assumed anyway.)
We can also give a description of the connecting map \( \alpha_{k,k+n} = \alpha_{k+n-1} \circ \cdots \circ \alpha_k \). For each \( j \in \mathbb{N} \), let \( \Lambda_j \) be the sequence of continuous functions given by listing each constant function \( i/q(j) \) (for \( 1 \leq i \leq q(j) - 1 \)) with multiplicity \( q(j) \), then \( h \) with multiplicity \( q(j)(q(j) - 1) \) and then each \( h \), for \( 1 \leq i \leq q(j) \). Then \( \alpha_{k,k+n} \) is fibrewise unitarily equivalent to the direct sum of all maps of the form \( ev_{F_0} \circ \cdots \circ ev_n \) with \( F_j \in \Lambda_{k+j-1} \) for \( 1 \leq j \leq n \).

Let us write \( T(A) \) for the space of tracial states on a C*-algebra \( A \). Recall that every tracial state on \( Z_{q(j),q(j)+1} \) is of the form \( \int \tau \circ ev_i(\cdot) d\mu(t) \) for some Borel probability measure \( \mu \) on \([0,1]\), where \( \tau \) is the unique tracial state on \( M_{q(j)} \otimes M_{q(j)+1} \). In particular, every such trace extends to a trace on \( C([0,1], M_{q(j)} \otimes M_{q(j)+1}) \), and is invariant under fibrewise unitary equivalence.

Since \( Z_U = \varprojlim Z_{q(k),q(k)+1} \) with unital connecting maps \( \alpha_k \), we have \( T(Z_U) \cong \varprojlim T(Z_{q(k),q(k)+1}) \). Thus \( T(Z_U) \) is an inverse limit of nonempty compact Hausdorff spaces, so is nonempty. That is, \( Z_U \) has at least one tracial state. For uniqueness, we need to show that for every \( k \in \mathbb{N} \), every \( \epsilon > 0 \), and every \( b \in Z_{q(k),q(k)+1} \) we have

\[
|\tau_1(\alpha_{k,k+n}(b)) - \tau_2(\alpha_{k,k+n}(b))| < \epsilon
\]

for all sufficiently large \( n \) and every \( \tau_1, \tau_2 \in T(Z_{q(k),q(k)+1}) \). The key observation for this is that for each \( j \), most of the elements in the sequence \( \Lambda_j \) defined above are constant functions. Thus (3.18) holds, and so \( Z_U \) has a unique tracial state.

It is well known that, to establish simplicity, it suffices to show the following (see for example [13, Theorem 3.4]): if \( b \) is a nonzero element of \( Z_{q(k),q(k)+1} \), then \( \alpha_{k,r}(b) \) generates \( Z_{q(r),q(r)+1} \) as a (closed, two-sided) ideal for every sufficiently large \( r \) which is the case if and only if \( \alpha_{k,r}(b) \) is nonzero for every \( t \in [0,1] \). Suppose that \( b \) is such an element, so that there is an interval in \((0,1)\) of width \( \epsilon > 0 \) on which \( b \) is nonzero. For each \( n \in \mathbb{N} \) and \( t \in [0,1] \), \( \alpha_{k,k+n+1}(b) \) contains summands unitarily equivalent to \( b \left( h^{(n)} \left( \frac{t}{q(k+n)} \right) \right) \) for \( 1 \leq i \leq q(k+n) - 1 \), where \( h^{(n)} := h \circ \cdots \circ h \). Moreover, \( h^{(n)}(t) = \begin{cases} 0, & 0 \leq t \leq l_n/4^n \\ 4^n t - l_n, & l_n/4^n \leq t \leq (1 + l_n)/4^n \\ 1, & (1 + l_n)/4^n \leq t \leq 1 \end{cases} \) for some \( l_n \), and so it suffices to show that for large \( n \) we have \( \frac{1}{q(k+n)} < \frac{1}{4^n} \). But this is true for all large \( n \) since \( \frac{4^n}{q(k+n)} = \frac{4^n}{p^n} \to 0 \) as \( n \to \infty \). Thus \( Z_U \) is simple.

It now follows from the classification theorem [1, Theorem 6.2] that \( Z_U \cong Z \).

Remark 3.2. One point that should be emphasized is that, despite the use of functional calculus, the relations of Theorem (3.5) are algebraic, or at least C*-algebraic in the sense that they involve only *-polynomial and order relations. This can be made explicit by encoding the relations (3.2) satisfied by the functions \( d, f, g \) and \( h \) into the relations for the building blocks \( Z^{(q(k))} \).

More specifically, it is not difficult to derive from Proposition (2.2) that the dimension drop algebra \( Z_{n,n+1} \) is isomorphic to the universal C*-algebra on generators \( \varphi, \psi \) and \( h \) with relations:

(i) \( \varphi, \psi \) and \( h \) are c.p.c. order zero maps on \( M_n, M_2 \) and \( \mathbb{C} \) respectively (in particular, \( h \) is just a positive contraction);
(ii) \( [\psi(e_{11}), \varphi(M_n)] = [h, \varphi(M_n)] = 0; \)
(iii) \( \psi(e_{11}) h = h; \)
(iv) \( h(1 - \varphi(1_n)) = 1 - \varphi(1_n); \)
(v) $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

(It is a straightforward exercise in functional calculus to write down inverse isomorphisms between the universal $C^*$-algebra determined by these relations and $Z^{(n)} \cong Z_{n,n+1}$.) The following is then proved in exactly the same way as Theorem 5.1

**Theorem 3.3.** The Jiang-Su algebra $Z$ is isomorphic to the universal unital $C^*$-algebra generated by c.p.c. order zero maps $\varphi_k$ on $M_q(k)$ $(k \in \mathbb{N})$ and $\psi$ on $M_2$ $(k \in \mathbb{N})$, and positive contractions $h_k$ $(k \in \mathbb{N})$, together with (for each $k \in \mathbb{N}$) the relations:

$$
[\psi_k(e_{11}), \varphi_k(M_q(k))] = [h_k, \varphi_k(M_q(k))] = 0,
$$

$$
\psi_k(e_{11})h_k = h_k,
$$

$$
h_k(1 - \varphi_k(1_q(k))) = 1 - \varphi_k(1_q(k)),
$$

$$
\psi_k(e_{22})\varphi_k(e_{11}) = \psi_k(e_{22}),
$$

$$
\varphi_k = \varphi_{k+1} \circ \rho_k,
$$

$$
\frac{1}{\sqrt{2}}(1 + h_k)^{1/2} \psi_k^{1/2}(e_{12}) = (h_{k+1} + (1 - h_{k+1})\varphi_{k+1}(v_k v_k^*)^{1/2} \psi_{k+1}^{1/2}(e_{12})
$$

$$
+ (1 - \psi_{k+1}(e_{11}))^{1/2} \varphi_{k+1}^{1/2}(v_k),
$$

where the c.p.c. order zero maps $\rho_k : M_q(k) \to M_q(k+1)$ and the partial isometries $v_k \in M_q(k)$ are as in (3.1) and (3.4) respectively. \qed

4. **$W$ as a universal $C^*$-algebra**

The article [10] (or, in a much more general setting, [11]) contains a classification by tracial data of simple inductive limits of building blocks

$$
W_{n,m} := \{f \in C([0,1], M_n \otimes M_m) \mid f(0) = a \otimes 1_m, f(1) = a \otimes 1_{m-1}, a \in M_n\}, \quad n, m \in \mathbb{N}, m > 1. \quad (4.1)
$$

Such building blocks are easily seen to be stably projectionless, and it can moreover be shown that they have trivial $K$-theory (this is why the classifying invariant is purely tracial). The classification is also complete in the sense that every permissible value of the invariant is attained—see [17] or [8] Proposition 5.3. Then, $W$ may be defined as the unique $C^*$-algebra in this class which has a unique tracial state (and no unbounded traces).

An explicit construction of $W$ is given in [3], and in this section we obtain another one by adapting the previous section’s universal characterization of $Z$. To begin with, notice that $W_{n,n+1}$ is isomorphic to a subalgebra of the dimension drop algebra $Z_{n,n+1}$; the following indicates that it in fact may be thought of as its nonunital analogue (compare with Proposition 2.2).

**Proposition 4.1.** Let $W^{(n)}$ be the universal $C^*$-algebra $C^*(\varphi, \psi \mid \hat{\mathcal{R}}_n)$, where $\hat{\mathcal{R}}_n$ denotes the set of relations:

(i) $\varphi$ and $\psi$ are c.p.c. order zero maps on $M_n$ and $M_2$ respectively;

(ii) $\psi(e_{11}) = \varphi(e_{11})(1 - \varphi(e_{11}))$;

(iii) $\psi(e_{22})\varphi(e_{11}) = \psi(e_{22})$.

Then $W^{(n)} \cong W_{n,n+1}$.

**Proof.** The proof is almost identical to that of Proposition 2.2 but we include it here for completeness. Define $\varphi : M_n \to W_{n,n+1}$ by

$$
\varphi(a)(t) = (a \otimes 1_n) \oplus (1 - t)(a \otimes e_{n+1,n+1})
$$

for $a \in M_n$ and $t \in [0,1]$. Then $\varphi$ is clearly a c.p.c. order zero map. Equivalently, if we write

$$
x_i(t) = (e_{11} \otimes 1_n) \oplus (1 - t)^{1/2}(e_{11} \otimes e_{n+1,n+1}) = \varphi(1_{e_{11}})(t)
$$

for $1 \leq i \leq n$, then the $x_i$ satisfy the order zero relations $\mathcal{R}_n^{(0)}$ and $\varphi(e_{ij}) = x_i^* x_j$. Next, define

$$
v(t) = t^{1/2}(1 - t)^{1/2} \sum_{j=1}^n e_{j1} \otimes e_{n+1,j}.
$$
We have unique c.p.c. order zero map $\psi : M_2 \to W_{n,n+1}$ with $\psi^{1/2}(e_{12}) = v$, i.e.

$$\psi(e_{12})(t) = t(1-t) \sum_{j=1}^n e_{j1} \otimes e_{n+1,j},$$

so that $\psi(e_{11}) = v^*v$, $\psi(e_{22}) = v^*v$ and $\varphi$ and $\psi$ satisfy all of the relations $\mathcal{R}_n$.

Next, we check that $v$ and the $x_i$, $1 \leq i \leq n$, generate $W_{n,n+1}$ as a $C^*$-algebra. Write $A := C^*(\{v, x_1, \ldots, x_n\})$. We have

$$v^*x_i(t) = t^{1/2}(1-t)(e_{ii} \otimes e_{1,n+1})$$

and

$$v^*x_i(x_j)(t) = t(1-t)^{3/2}(e_{ij} \otimes e_{1i})$$

for $1 \leq i, j \leq n$. Thus, for $t \in (0,1)$, the elements $v^*x_i(t)$ and $v^*x_ix_j(t)$ give all matrix units $\{e_{ii} \otimes e_{jj}\}_{1 \leq i,j \leq n}$, so generate all of $M_n \otimes M_{n+1}$, and so the irreducible representation $ev_1 : W_{n,n+1} \to M_n \otimes M_{n+1}$ restricts to an irreducible representation of $A$. For $t \in [0,1]$, the $x_i$ generate all the matrix units of $M_n$ in the endpoint irreducible representation $ev_\infty : W_{n,n+1} \to M_n$. Thus every irreducible representation of $W_{n,n+1}$ restricts to an irreducible representation of $A$. Also, since $x_i(s)$ is not unitarily equivalent to $x_i(t)$ for distinct $s,t \in (0,1)$, it follows that inequivalent irreducible representations of $W_{n,n+1}$ restrict to inequivalent irreducible representations of $A$. Therefore, by Stone-Weierstrass (i.e. [2] Proposition 11.1.6), we do indeed have $C^*(\{v, x_1, \ldots, x_n\}) = W_{n,n+1}$.

It remains to show that these generators of $W_{n,n+1}$ enjoy the appropriate universal property: for every representation $\{(\hat{\varphi}, \hat{\psi})\}$ of the given relations, we need to show that there is a $^*$-homomorphism $W_{n,n+1} \to C^*(\hat{\varphi}, \hat{\psi})$ sending $\varphi$ to $\hat{\varphi}$ and $\psi$ to $\hat{\psi}$. By [2] Lemma 3.2.2, it suffices to consider the case where $\{(\hat{\varphi}, \hat{\psi})\}$ is an irreducible representation on some Hilbert space $H$ (i.e. has trivial commutant in $\mathcal{B}(H)$). Note that the irreducible representations of $W_{n,n+1}$ are (up to unitary equivalence), the evaluation maps $ev_t : W_{n,n+1} \to M_n(t+1)$ for $t \in (0,1)$ together with the endpoint representation $ev_\infty : W_{n,n+1} \to M_n$. We will therefore show that (again, up to unitary equivalence) $\hat{\varphi} = ev_t \circ \varphi$ and $\hat{\psi} = ev_t \circ \psi$ for some $t \in (0,1) \cup \{\infty\}$.

For each $i \in \{1, \ldots, n\}$, let $\hat{\psi}_i : M_2 \to C^*(\hat{\varphi}, \hat{\psi})$ be the c.p.c. order zero map defined by $\hat{\psi}_i(e_{12}) = \hat{\psi}_{i,12}(e_{12}) \hat{\varphi}_{i,12}(e_{11})$, so that $\hat{\psi}_i(e_{11}) = \hat{\psi}_i(1_n) = (1 - \hat{\varphi}(1_n))$ and $\hat{\psi}_i(e_{22}) = \hat{\psi}_i(1_n) = (1 - \hat{\varphi}(1_n))$. Define

$$z := \hat{\psi}(e_{11}) + \sum_{i=1}^n \hat{\psi}_i(e_{22}) \in C^*(\hat{\varphi}, \hat{\psi}).$$

Then

$$[z, \hat{\varphi}(e_{11})] = \hat{\psi}_1(e_{22})\hat{\varphi}(e_{11}) - \hat{\varphi}(e_{11})\hat{\psi}_1(e_{22}) = 0,$$

and

$$[z, \hat{\varphi}(e_{12})] = \hat{\psi}^2(e_{12}) + \sum_{i=1}^n \hat{\psi}_i(e_{22})\hat{\psi}_{i,12}(e_{12})\hat{\varphi}_{i,12}(e_{11}) - \sum_{i=1}^n \hat{\psi}_i(e_{11})\hat{\varphi}_{i,12}(e_{12})\hat{\psi}_i(e_{22})$$

$$= \hat{\psi}^2(e_{12}) + 0 - \hat{\psi}^2(e_{12}) = 0,$$

so $z$ is central in $C^*(\hat{\varphi}, \hat{\psi})$, and is therefore $\zeta 1$ for some scalar $\zeta$. Moreover, $z$ is positive with $\|z\| = \|\hat{\psi}(e_{11})\| = \|\hat{\psi}(1_n)(1 - \hat{\varphi}(1_n))\| \leq 1/4$, so $0 \leq \zeta \leq 1/4$.

If $\zeta = 0$ then $\hat{\psi} = 0$ and $\hat{\varphi}(1_n)$ is a projection. It follows that $\hat{\varphi}$ is a $^*$-homomorphism giving an irreducible representation of $M_n$ on $H$. Thus (up to unitary equivalence) $H = \mathbb{C}^n$ and $\hat{\varphi} = ev_\infty \circ \varphi$.

Suppose that $\zeta > 0$. Then $\zeta\hat{\psi}(e_{11}) = z\hat{\psi}(e_{11}) = (\hat{\psi}(e_{11}))^2$, so $p := \zeta^{-1}\hat{\psi}(e_{11})$ and $q_i := \zeta^{-1}\hat{\psi}_i(e_{22})$ are equivalent orthogonal projections with $p + q_1 + \cdots + q_n = 1$. Since $p$ commutes with $\hat{\varphi}(M_n)$, the maps $p\hat{\varphi}(\cdot)p$ and $(1-p)\hat{\psi}(\cdot)(1-p)$ are c.p.c. order zero. In fact,

$$\zeta \hat{\varphi}(1_n)(1-p) = \hat{\varphi}(1_n)(z - \hat{\psi}(e_{11})) = z - \hat{\psi}(e_{11}) = \zeta(1-p),$$

and

$$\zeta \hat{\varphi}(1_n)(1-p) = \hat{\varphi}(1_n)(z - \hat{\psi}(e_{11})) = z - \hat{\psi}(e_{11}) = \zeta(1-p),$$

therefore $\zeta 1$ in $C^*(\hat{\varphi}, \hat{\psi})$. Hence, $\hat{\varphi}$ is a $^*$-homomorphism giving an irreducible representation of $M_n$ on $H$. Thus (up to unitary equivalence) $H = \mathbb{C}^n$ and $\hat{\varphi} = ev_\infty \circ \varphi$.
i.e. \((1 - p)\hat{\phi}(1_n)(1 - p) = 1 - p\). Thus, \((1 - p)\hat{\phi}(\cdot)(1 - p)\) is a unital c.p.c. order zero map into the corner \((1 - p)\mathcal{B}(H)(1 - p) \cong \mathcal{B}((1 - p)H)\), so is a *-homomorphism into this corner. Also, \(p_\phi(\hat{1}_n)p\) commutes with (the WOT-closure of) the corner \(pC^*(\hat{\phi}, \hat{\psi})p = pC^*(\hat{\phi})p\) (which, by irreducibility, is all of \(p\mathcal{B}(H)p\) so \(p_\phi(\hat{1}_n)p = tp\) for some \(t \in [0, 1]\). Yet \(t^{-1}p_\phi(\cdot)p\) is also a *-homomorphism, and is in fact an irreducible representation of \(M_n\) on \(pH\). In particular, up to unitary equivalence, \(pH = \mathbb{C}^n\) and \(p_\phi(\cdot)p = t \cdot \text{id}_{M_n}\).

Moreover, since every \(q_1\) is equivalent to \(p\), they all have trace \(n = \text{tr}(p)\). Thus (again up to unitary equivalence) \((1 - p)H = \mathbb{C}^{n^2}\) (so \(H = \mathbb{C}^{n(n + 1)}\)) and \((1 - p)\hat{\phi}(\cdot)(1 - p) : M_n \rightarrow M_{n^2}\) is just \(a \mapsto \text{diag}(a, \ldots, a)\). Finally, since

\[
 t(1 - t)p = tp(p - tp) = \hat{\phi}(1_n)p(p - \hat{\phi}(1_n)p) = p\phi(1_n)(1 - \hat{\phi}(1_n)) = p\phi(e_{11}) = \zeta p,
\]

we have \(t(1 - t) = \zeta\). Therefore, \(\hat{\phi} = p_\phi(\cdot)p + (1 - p)\hat{\phi}(\cdot)(1 - p) = \text{ev}_{1-1} \circ \phi\) and, since \(\zeta^{-1/2}\psi^{1/2}(e_{12})\) is a partial isometry implementing an equivalence between \(q_1\) and \(p\), \(\hat{\psi} = \text{ev}_{1-1} \circ \psi\) (up to conjugation by a unitary). Thus \(W_{n,n+1}\) has the required universal property. \(\square\)

**Remark 4.2.** It should also be possible to detect *-homomorphisms from \(W_{n,n+1}\) to a stable rank one \(\mathcal{C}^*\)-algebra \(A\) at the level of the Cuntz semigroup \(W(A)\) (just as for \(Z_{n,n+1}\) in [13] Proposition 5.1]). The existence of \(\langle x \rangle \in W(A)\) and a positive contraction \(y \in A\) with \(n \langle x \rangle = \langle y \rangle\) and \((y - y^2) \ll \langle x \rangle\) (where \(\ll\) denotes the relation of compact containment) is probably necessary and sufficient, but perhaps this is not the most useful characterization.

Finally, we present \(W\) as a nonunital deformation of \(Z\).

**Theorem 4.3.** Choose positive functions \(d, f, g, h \in \mathcal{C}_0(0, 1]\), partial isometries \(v_{k+1} \in M_{q(k+1)}\), and c.p.c. order zero maps \(\rho_k : M_{q(k)} \rightarrow M_{q(k+1)}\) as in Theorem 3.1. Define \(W_U\) to be the universal \(\mathcal{C}^*\)-algebra generated by c.p.c. order zero maps \(\phi_k\) on \(M_{q(k)}\) \((k \in \mathbb{N})\) and \(\psi_k\) on \(M_2\) \((k \in \mathbb{N})\) such that for each \(k\), these maps satisfy the relations \(\mathcal{R}_{q(k)}\), i.e.

\[
\psi_k(e_{11}) = \phi_k(1_{q(k)})(1 - \phi_k(1_{q(k)}))
\] (4.2)

and

\[
\psi_k(e_{22})\phi_k(e_{11}) = \psi_k(e_{22}),
\] (4.3)

together with the additional relations \(\mathcal{S}_{q(k)}\) given by

\[
\phi_k = f(\phi_{k+1}) \circ \rho_k,
\] (4.4)

\[
\psi_k^{1/2}(e_{12}) = f(\phi_{k+1})(\rho_k(1_{q(k)}))^{1/2} \left( h(\phi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2} f(\phi_{k+1})(v_{k+1}) + (1 - f(\phi_{k+1})(1_{q(k+1)}) + g(\phi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})))^{1/2} d(\psi_{k+1})(e_{12}) \right).
\] (4.5)

Then \(W_U \cong W\).

**Proof.** The proof is essentially the same as that of Theorem 3.1, so we omit most of the details. As before, let us write \(\hat{\phi}_k = f(\phi_{k+1}) \circ \rho_k\) and \(\psi_k^{1/2}(e_{12}) = \gamma_k + \delta_k\), where this time

\[
\gamma_k := f(\phi_{k+1})(\rho_k(1_{q(k)}))^{1/2} \lambda_k d(\psi_{k+1})(e_{12})
\]

and

\[
\delta_k := f(\phi_{k+1})(\rho_k(1_{q(k)}))^{1/2} \mu_k f(\phi_{k+1})(v_{k+1}),
\]

with

\[
\lambda_k := (1 - f(\phi_{k+1})(1_{q(k+1)}) + g(\phi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)})))^{1/2}
\]

and

\[
\mu_k := h(\phi_{k+1})(1_{q(k+1)} - \rho_k(1_{q(k)}))^{1/2}.
\]

To show that \(\hat{\psi}_k(e_{11}) = \hat{\phi}_k(1_{q(k)})(1 - \hat{\phi}_k(1_{q(k)}))\), we proceed exactly as in the proof of Claim 1. The only difference is that we now have

\[
d(\psi_{k+1})(e_{11}) = d(\psi_{k+1}(e_{11})) = d(\phi_{k+1}(1_{q(k+1)})(1 - \phi_{k+1}(1_{q(k+1)}))) = d(\phi_{k+1}(1_{q(k+1)})�)
\]
where \( \hat{d}(t) = d(t(1-t)) \) as in (3.3). We also have

\[
f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1-f(\varphi_{k+1})(1_{q(k+1)})) = \pi_{q_{k+1}}(\rho_k(1_{q(k)}))(f - f^2)(\varphi_{k+1}(1_{q(k+1)})).
\]

Since \( \hat{d}(f-f^2) = f - f^2 \), this therefore gives

\[
f(\varphi_{k+1})(\rho_k(1_{q(k)}))(1-f(\varphi_{k+1})(1_{q(k+1)})) = f(\rho_k(1_{q(k)}))(1-f(\varphi_{k+1})(1_{q(k+1)})),
\]

and the rest of the argument carries over mutatis mutandis. (Note in particular that \( \lambda_k \) and \( \mu_k \) both commute with \( f(\varphi_{k+1})(\rho_k(1_{q(k)}))^{1/2} \).) The proof that \( \hat{\psi}(v_{22})\varphi_k(e_{11}) = \hat{\psi}(v_{22}) \) is literally the same as the proof of Claim 2.

We now know that \( W_U \) is isomorphic to an inductive limit \( \lim_{\to}(W_{q(k)}q(k+1), \beta_k) \). Moreover, arguing exactly as before, we see that the connecting maps \( \beta_k \) are (fibrewise) unitarily equivalent to the connecting maps \( \alpha_k \) obtained earlier. That is, there are unitaries \( z_k(t) \in M_{q(k+1)} \otimes M_{q(k+1)+1} \) such that

\[
\beta_k = z_k(t) \begin{pmatrix} q(k+1) & q(k) \\ m=1 & i=1 \end{pmatrix} \begin{pmatrix} \psi & \psi \\ e & e \end{pmatrix} \begin{pmatrix} q(k+1) & q(k) \\ m=1 & i=1 \end{pmatrix} \begin{pmatrix} \psi & \psi \\ e & e \end{pmatrix} z_k(t)^* \tag{4.6}
\]

for every \( t \in [0,1] \).

The same arguments as with \( Z_U \) show that \( W_U \) is simple and has a unique tracial state. (One has to perhaps be slightly careful about the existence of a trace, since the space of tracial states of a nonunital C*-algebra need not be compact. But this is not an issue.) The only minor technicality is that, since the building blocks \( W_{q(k)}q(k+1) \) are nonunital and the connecting maps \( \beta_k \) are degenerate, \( W_U \) may have unbounded traces. However, one can easily show, using (3.9), that this is not the case. It therefore follows from the classification theorem of [11] (or indeed from the more general result proved in [11]) that \( W_U \cong W \).

\[\square\]

**Corollary 4.4.** There exists a trace-preserving embedding of \( W \) into \( Z \). Such an embedding is canonical at the level of the Cuntz semigroup, and is unique up to approximate unitary equivalence.

**Proof.** This follows immediately from Theorem [13] and Theorem [22]. The result can already be deduced from the main theorem of [11], which also gives the uniqueness statement. \[\square\]

### 5. Outlook

#### 5.1. It might be interesting to characterize other C*-algebras as we have done for \( Z \) and \( W \). It should in particular be possible, for any \( n \geq 2 \), to obtain a universal construction of a simple, monotracial, stably projectionless C*-algebra \( W_n \) with \( (K_0(W_n), K_1(W_n)) = (0, \mathbb{Z}/(n-1)\mathbb{Z}) \). Candidate building blocks could be of the form

\[
\{ f \in C([0,1], M_m \otimes M_{(n-1)(m+1)}) : f(0) = a \otimes 1_{(n-1)(m+1)}, f(1) = a \otimes 1_{(n-1)m}, a \in M_1 \},
\]

which at least have the right \( K \)-theory. Of course, \( W_2 \) is just \( W \), obtained as in Theorem [4.3].

It was proved in [11] that \( W \otimes \mathcal{K} \cong \mathcal{O}_2 \rtimes \mathbb{R} \) for certain ‘quasi-free’ actions of \( \mathbb{R} \) on the Cuntz algebra \( \mathcal{O}_2 \) (see for example [3] and [11]). More generally, one would expect (i.e. the Elliott conjecture predicts) that \( W_n \otimes \mathcal{K} \cong \mathcal{O}_n \rtimes \mathbb{R} \), and in this sense \( W_n \) might be thought of as a stably projectionless analogue of \( \mathcal{O}_n \). (Similar speculation is made in the article [1] on the Jiang-Su algebra.)

It is unclear what interpretation the corresponding universal unital algebras might have. Note for example that the Jiang-Su algebra is not stably isomorphic to a crossed product of a Kirchberg algebra by \( \mathbb{R} \) (when simple, such a crossed product is either traceless or stably projectionless—see [6, Proposition 4]).

#### 5.2. One of our motivations for presenting \( Z \) as a universal C*-algebra was to find a direct proof of its strong self-absorption (i.e. one that does not rely on classification). To put this problem into context, consider the other strongly self-absorbing C*-algebras. On the one hand, UHF algebras of infinite type can also be described in terms of order zero generators and relations, for example:

\[
M_{2^\infty} \cong C^*(\{\varphi_k\}_{k=1}^\infty | \varphi_k \text{ order zero on } M_{q(k)}, \varphi_k(1_{q(k)}) = \varphi_k(1_{q(k)})^2, \varphi_k = \varphi_{k+1} \circ \text{id}_{q(k)} \otimes 1_{q(k)} \otimes 1_{q(k)})
\]

(where \(q(k)\) is still \(2^k\)), and the proof of strong self-absorption in this case amounts to linear algebra. On the other hand, while \(O_2\) and \(O_\infty\) are presented simply as \(C^*(s_1, s_2 | s^*_1 s_1 = 1 = s_1 s^*_1 + s_2 s^*_2)\) and \(C^*((s_i)_{i=1}^\infty | s^*_i s_j = \delta_{ij})\) respectively, the proofs that \(O_2 \otimes O_2 \cong O_2\) and \(O_\infty \otimes O_\infty \cong O_\infty\) require some difficult analysis (see for example [13]). It is conceivable that our presentation of \(Z\) lies somewhere in the middle of this spectrum.

That being said, it is at least possible to show from our relations, in connection with [11], that the \(C^*\)-algebra \(Z_{U}^{\infty}\) is strongly self-absorbing. (One can show that \(Z_{U}^{\infty}\) has stable rank one and strict comparison, and then use the main theorem of [11] to deduce strong self-absorption.)

Meanwhile, it remains an open problem to prove that \(W \otimes W \cong W\).

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