An Overview of Rationalization Theories of Non-simply Connected Spaces and Non-nilpotent Groups

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Dedicated to Professor Banghe Li on His 80th Birthday

Abstract We give an overview of five rationalization theories for spaces (Bousfield-Kan’s $\mathbb{Q}$-completion; Sullivan’s rationalization; Bousfield’s homology rationalization; Casacuberta-Peschke’s $\Omega$-rationalization; Gómez-Tato-Halperin-Tanré’s $\pi_1$-fiberwise rationalization) that extend the classical rationalization of simply connected spaces. We also give an overview of the corresponding rationalization theories for groups ($\mathbb{Q}$-completion; $\mathbb{H}\mathbb{Q}$-localization; Baumslag rationalization) that extend the classical Malcev completion.

Keywords Rationalisation, localisation, rational homotopy theory, group theory

MR(2010) Subject Classification 55-02, 55P62, 55P60

1 Introduction

A simply connected space $X$ is called rational if one of two equivalent properties is satisfied: (1) homotopy groups $\pi_\ast(X)$ are $\mathbb{Q}$-vector spaces; (2) reduced integral homology groups $\tilde{H}_\ast(X, \mathbb{Z})$ are $\mathbb{Q}$-vector spaces. It is important for the rational homotopy theory that for a simply connected space $X$ there exists a universal map to a rational space $X_\mathbb{Q} \to X_{\mathbb{Q}}$ that induces isomorphisms $\pi_\ast(X_\mathbb{Q}) \cong \pi_\ast(X) \otimes \mathbb{Q}$ and $\tilde{H}_\ast(X_\mathbb{Q}, \mathbb{Z}) \cong \tilde{H}_\ast(X, \mathbb{Q})$. The space $X_\mathbb{Q}$ is called rationalization of $X$.

This note is devoted to an overview of five rationalization theories of all spaces that extend the classical rationalization of simply connected spaces:

1. Bousfield-Kan’s $\mathbb{Q}$-completion $Q_\infty[11]$;  
2. Sullivan’s rationalization $\text{Sul} [7, 18, 33]$;  
3. Bousfield’s homology rationalization $L_{\mathbb{H}\mathbb{Q}} [10]$;  
4. Casacuberta-Peschke’s $\Omega$-rationalization $L_{\Omega\mathbb{Q}} [13]$;  
5. Gómez-Tato-Halperin-Tanré’s $\pi_1$-fiberwise rationalization $L^{\pi_1}_\mathbb{Q}[17]$.

We do not aim to provide an overview of the history of the topic. We just want to point out that there are five different approaches to rationalization of a space list their properties and show some relations between them.

Received February 3, 2022, accepted July 8, 2022
Supported by the Ministry of Science and Higher Education of the Russian Federation, agreement 075-15-2019-1619
For spaces of finite rational type the Sullivan rationalization coincides with the $\mathbb{Q}$-completion
\[ \text{Sul}(X) \cong \mathbb{Q}_\infty X. \]
(1.1)
However, all other pairs of functors are non-isomorphic even for spaces of finite rational type
but there are natural transformations:
\[ L^\pi_1 \mathbb{Q} \longrightarrow L^\Omega \mathbb{Q} \longrightarrow L^H \mathbb{Q} \longrightarrow \mathbb{Q}_\infty. \]
(1.2)
Some properties of these constructions can be summarised in the following table.

|                          | $\mathbb{Q}_\infty$ | Sul | $L^H \mathbb{Q}$ | $L^\Omega \mathbb{Q}$ | $L^\pi_1 \mathbb{Q}$ |
|--------------------------|----------------------|-----|------------------|----------------------|----------------------|
| Idempotent               | $-$                  | $-$ | $+$              | $+$                  | $+$                  |
| Rational homology equivalence | $-$              | $-$ | $+$              | $+$                  | $+$                  |
| “Right” for nilpotent spaces | $+$              | $-$ | $+$              | $-$                  | $+$                  |
| “Right” for nilpotent spaces of finite type | $+$              | $+$ | $+$              | $+$                  | $-$                  |
| Has a simplicial group description | $+$              | $-$ | $?$              | $+$                  | $+$                  |
| Aspherical for the wedge of two circles | $+$              | $+$ | $?$              | $+$                  | $+$                  |

The three functors $L^\pi_1 \mathbb{Q}$, $L^\Omega \mathbb{Q}$, $L^H \mathbb{Q}$ are localizations, and they are defined by three classes of maps:
\[ \{\pi_1\text{-fiberwise rational eq.}\} \subseteq \{\Omega\text{-rational eq.}\} \subseteq \{\text{homology rational eq.}\}, \]
(1.3)
and three classes of spaces:
\[ \{\pi_1\text{-fiberwise rational sp.}\} \supseteq \{\Omega\text{-rational sp.}\} \supseteq \{\text{homology rational sp.}\}, \]
(1.4)
that we also describe.
Since we deal with non-simply connected spaces, we focus attention on the fundamental
groups of all these constructions. These rationalization theories for spaces correspond to the
following rationalization theories for groups: $\mathbb{Q}$-completion; $H\mathbb{Q}$-localization; Baumslag rationalization.

| Spaces                          | Groups                      |
|---------------------------------|-----------------------------|
| Bousfield-Kan’s $\mathbb{Q}$-completion | $\mathbb{Q}$-completion     |
| Sullivan’s rationalization      |                             |
| Homology rationalization        | $H\mathbb{Q}$-localization  |
| $\Omega$-rationalization        | Baumslag rationalization    |
| $\pi_1$-fiberwise rationalization | $-$                         |

These three rationalization theories for groups generalize the classical Malcev completion [25]. The $\pi_1$-fiberwise rationalization of a space does not correspond to a rationalization theory on groups because it does not affect the fundamental group. All these constructions can be generalized to localizations in any set of primes $P$ but for simplicity we focus our attention on rationalizations.
2 General Localizations in Categories

Let $C$ be a category. For a morphism $f : c_1 \to c_2$ an object $l \in C$ is called $f$-local, if the map $f^* : C(c_2, l) \to C(c_1, l)$ is a bijection. An object is called $W$-local for some class of morphisms $W$, if it is $w$-local for any $w \in W$. A $W$-localization of an object $c \in C$ is a morphism $w : c \to l$ from the class $W$ to a local object. It is easy to check that a $W$-localization satisfies two universal properties: (1) it is universal (initial) morphism to a $W$-local object with domain $c$; (2) it is universal (terminal) morphism from $W$ with domain $c$. If the $W$-localization exists for any object in $C$, it defines a coaugmented functor $L : C \to C$, i.e., a functor with a natural transformation $\eta : \text{Id} \to L$. Moreover, this coaugmented functor is a localization functor in the following sense: the maps $\eta L, L\eta : L \to L^2$ are equal isomorphisms. In particular, $L \cong L^2$. In this case the full subcategory of $W$-local objects is a reflective subcategory and the restriction of the functor $L$ is the reflector. Moreover, this reflector induces an equivalence of categories

$$C[W^{-1}] \simeq \text{Loc}(W),$$

(2.1)

where $\text{Loc}(W)$ is the subcategory of local objects.

Some of these definitions are also useful in the context of categories enriched over spaces (or simplicially enriched categories). We denote by $\text{Map}_C(c_1, c_2)$ the hom-space between objects $c_1, c_2 \in C$. In the enriched setting we say that an object $l$ is $f$-local if $f^* : \text{Map}_C(c_2, l) \to \text{Map}_C(c_1, l)$ is a weak equivalence. The theory of $f$-localizations of spaces (in this enriched sense) can be found in [15, 21].

3 The Malcev Completion of Nilpotent Groups

3.1 Definition

Recall that the lower central series of a group $G$ is a series of normal subgroups given by

$$G_{n+1} = [G_n, G], \quad G = G_1 \supseteq G_2 \supseteq \ldots.$$  

(3.1)

We say that a group is $n$-nilpotent if $G_n = 1$. A group is called nilpotent of class $n$ if it is $(n+1)$-nilpotent but not $n$-nilpotent. A group is called nilpotent if is is $n$-nilpotent for some $n$.

We say that a group $G$ is rational (or uniquely divisible, or complete) if for any $n \geq 1$ the map $G \to G, g \mapsto g^n$ is a bijection. For example, an abelian group is rational iff it a vector space over $\mathbb{Q}$. In particular, there is a natural way of rationalization of an abelian group $A$: the map

$$A \to A \otimes \mathbb{Q}, \quad a \mapsto a \otimes 1$$

(3.2)

is the universal map from an abelian group to a rational group.

Malcev [24, 25] (see also [19, 34]) extended the functor of rationalization from the category of abelian groups to the category of nilpotent groups

$$- \otimes \mathbb{Q} : (\text{nilpotent groups}) \to (\text{nilpotent groups}).$$

(3.3)

This functor is called the Malcev completion. The Malcev completion can be defined as the universal map to a rational nilpotent group

$$G \to G \otimes \mathbb{Q}.$$ 

(3.4)
The Malcev completion shares a lot of nice properties with the rationalization on abelian groups. For example, it takes short exact sequences to short exact sequences and central extensions to central extensions \[20, \text{Prop. 1.10}\]. It induces an isomorphism on homology \[19, \text{Th. 4.8}\]

\[H_*(G, \mathbb{Q}) \cong H_*(G \otimes \mathbb{Q}, \mathbb{Z}), \quad (3.5)\]

and others. There is no such a nice generalization of this functor to all groups.

3.2 The Malcev Correspondence

The Malcev completion sends nilpotent groups to rational nilpotent groups. Rational nilpotent groups is a convenient class of groups because the category of rational nilpotent groups is equivalent to the category of nilpotent Lie algebras over \( \mathbb{Q} \) \[24, \text{Th. 9}\], \[34, \text{Th. 12.11}\], \[3, \text{Th. 4.6}\], \[29, \text{App. A}\].

\[(\text{rational nilpotent groups}) \cong (\text{nilpotent Lie algebras over } \mathbb{Q}). \quad (3.6)\]

Moreover, if \( G \) is a rational nilpotent group and \( \mathfrak{g} \) is the corresponding Lie algebra, then the associated graded Lie algebras \( \text{gr}(G) = \bigoplus_{n \geq 1} G_n/G_{n+1} \) and \( \text{gr}(\mathfrak{g}) = \bigoplus_{n \geq 1} \mathfrak{g}_n/\mathfrak{g}_{n+1} \) are isomorphic. It follows that the category of rational \( n \)-nilpotent groups is equivalent to the category of \( n \)-nilpotent Lie algebras

\[(\text{rational } n\text{-nilpotent groups}) \cong (\text{\( n \)-nilpotent Lie algebras over } \mathbb{Q}). \quad (3.7)\]

The functor from nilpotent Lie algebras to rational nilpotent groups is very explicit. For any nilpotent Lie algebra \( \mathfrak{g} \) over \( \mathbb{Q} \) we define a new binary operation by the Baker-Campbell-Hausdorff formula

\[a * b = a + b + \frac{1}{2} [a, b] + \frac{1}{12} ([a, [a, b]] - [b, [a, b]]) + \cdots \quad (3.8)\]

This sum is finite because the Lie algebra is nilpotent. The Lie algebra together with this new binary operation is the corresponding rational nilpotent group.

4 \( \mathbb{Q} \)-completion of Groups

4.1 Definitions

\( \mathbb{Q} \)-completion of a group \( G \) can be defined as follows. For each \( n \) we consider the nilpotent group \( G/G_n \) and take its Malcev’s completion \( G/G_n \otimes \mathbb{Q} \). These groups form an inverse tower

\[G/G_1 \otimes \mathbb{Q} \leftarrow G/G_2 \otimes \mathbb{Q} \leftarrow G/G_3 \otimes \mathbb{Q} \leftarrow \cdots. \quad (4.1)\]

The inverse limit of this tower is the \( \mathbb{Q} \)-completion of \( G \)

\[\widehat{G}_\mathbb{Q} = \varprojlim (G/G_n \otimes \mathbb{Q}). \quad (4.2)\]

There are several equivalent definitions of the \( \mathbb{Q} \)-completion. Instead of using the ordinary lower central series one can use the rational lower central series. It is defined as follows,

\[G_n^\mathbb{Q} = \ker(G_n^\mathbb{Q} \rightarrow G_n^\mathbb{Q}/[G_n^\mathbb{Q}, G] \otimes \mathbb{Q}), \quad (4.3)\]

where \( G_1^\mathbb{Q} = G \). The map \( G/G_n \otimes \mathbb{Q} \rightarrow G/G_n^\mathbb{Q} \otimes \mathbb{Q} \) is an isomorphism (it follows from the fact that \( G_n^\mathbb{Q}/G_n \) is torsion). It follows that

\[\widehat{G}_\mathbb{Q} = \varprojlim (G/G_n^\mathbb{Q} \otimes \mathbb{Q}). \quad (4.4)\]
Another equivalent definition is more categorical, it can be found in [11, Ch. IV, §2.2]. Consider the category of all nilpotent rational groups \( \text{Nil}_\mathbb{Q} \) and the category \( G \downarrow \text{Nil}_\mathbb{Q} \) of homomorphisms \( G \to N \), where \( N \in \text{Nil}_\mathbb{Q} \). Then the \( \mathbb{Q} \)-completion is the limit of the functor \( G \downarrow \text{Nil}_\mathbb{Q} \to \text{Gr} \) given by \( (G \to N) \mapsto N \).

\[
\hat{G}_\mathbb{Q} = \lim_{G \to N \in \text{Nil}_\mathbb{Q}} N.
\] (4.5)

4.2 (Non)-idempotency

Generally \( \mathbb{Q} \)-completion is not idempotent. For example, if \( F_\infty \) is the countably generated free group, then its double \( \mathbb{Q} \)-completion is not isomorphic to its \( \mathbb{Q} \)-completion. However the functor of completion is idempotent for the class of groups with finite dimensional \( H_1(G, \mathbb{Q}) \) [9, §13]. Note that all finitely generated groups are in this class. Moreover, the completion preserves this class of groups and for groups in this class there is an isomorphism \( H_1(G, \mathbb{Q}) \cong H_1(\hat{G}_\mathbb{Q}, \mathbb{Q}) \).

4.3 \( \mathbb{Q} \)-completion of the Free Group

Let \( F = F(X) \) be a free group generated by some set \( X \). Then its \( \mathbb{Q} \)-completion can be described on the language of Lie algebras using Baker-Campbell-Hausdorff formula.

Denote by \( L = L(X) \) the free Lie algebra over \( \mathbb{Q} \) generated by \( X \). Then it has a natural grading \( L = \bigoplus_{i \geq 1} L^i \), where \( L^i \) is the vector space generated by commutators of weight \( i \). We consider the complete free Lie algebra, which is given by

\[
\hat{L} = \prod_{i \geq 1} L^i.
\] (4.6)

Note that the lower central series of \( L \) can be described as follows \( L_n = \bigoplus_{i \geq n} L^i \). So, the complete free Lie algebra can be described as

\[
\hat{L} = \lim_{\leftarrow} L/L_n.
\] (4.7)

There is a natural structure of a group on this Lie algebra given by the Baker-Campbell-Hausdorff formula (3.8). Then there is an isomorphism

\[
\hat{F}_\mathbb{Q} \cong (\hat{L}, \ast).
\] (4.8)

In order to prove this it is enough to prove that \( F/F_n \otimes \mathbb{Q} \cong (L/L_n, \ast) \). It follows from the Malcev correspondence because \( F/F_n \otimes \mathbb{Q} \) is the free object in the category of \( n \)-nilpotent rational groups and \( L/L_n \) is the free object in the category of \( n \)-nilpotent Lie algebras over \( \mathbb{Q} \).

For a free group of rank \( \geq 2 \) the second homology group of the completion is non-trivial (and even uncountable) [22, Th.1]

\[
H_2(\hat{F}_\mathbb{Q}, \mathbb{Q}) \neq 0.
\] (4.9)

However nothing is known about the homology groups for \( n \geq 3 \).

**Question 4.1** Is it true that \( H_n(\hat{F}_\mathbb{Q}, \mathbb{Q}) = 0 \) for a free group \( F \) and \( n \geq 3 \)?

5 \( \mathbb{H}_\mathbb{Q} \)-localization of Groups

5.1 Definition

A homomorphism \( f : G \to G' \) is called rationally 2-acyclic (or \( \mathbb{H}_\mathbb{Q} \)-homomorphism) if the map \( H_1(G, \mathbb{Q}) \to H_1(G', \mathbb{Q}) \) is an isomorphism and the map \( H_2(G, \mathbb{Q}) \to H_2(G', \mathbb{Q}) \) is an...
epimorphism. This definition is natural because a homomorphism $f : G \to G'$ is rationally 2-acyclic if and only if there exists a map of spaces $F : X \to X'$ that induces an isomorphism of all homology groups $H_*(X, \mathbb{Q}) \cong H_*(X', \mathbb{Q})$ such that $f : G \to G'$ is isomorphic to $\pi_1(F) : \pi_1(X) \to \pi_1(X')$ [10, Lemma 6.1]. Roughly speaking, the class of rationally 2-acyclic morphisms is the image of the class of rational homology equivalences under $\pi_1$.

We denote by $\mathcal{W}_{H\mathbb{Q}}$ the class of rationally 2-acyclic homomorphisms. Then a group is called $H\mathbb{Q}$-local if it is $\mathcal{W}_{H\mathbb{Q}}$-local and $H\mathbb{Q}$-localization is the $\mathcal{W}_{H\mathbb{Q}}$-localization. The $H\mathbb{Q}$-localization exists for all groups [10, Th.5.2] and defines a functor

$$\ell_{H\mathbb{Q}} : \mathcal{Gr} \to \mathcal{Gr}.$$  \hspace{1cm} (5.1)

The class of $H\mathbb{Q}$-local groups can be described more explicitly. We say that a central extension $E \twoheadrightarrow G$ is rational if the kernel is a $\mathbb{Q}$-vector space. Then the class of $H\mathbb{Q}$-local groups is the least class containing the trivial group, closed under small limits and rational central extensions [9, Th. 3.10]. This description implies that an $H\mathbb{Q}$-local group is rational.

The functor $\ell_{H\mathbb{Q}}$ is right exact [1]. There is a transfinite limit construction of $H\mathbb{Q}$-localization [9].

5.2 $H\mathbb{Z}$-localization of $G$-modules

There is a theory of $H\mathbb{Z}$-localizations of $G$-modules, which is similar to the theory of $H\mathbb{Q}$-localizations of groups. Let $G$ be a fixed group and consider the category of modules $\mathcal{Mod}(G)$. Then a homomorphism $f : M \to M'$ is called 1-acyclic if $H_0(G, M) \to H_0(G, M')$ is iso and $H_1(G, M) \to H_1(G, M')$ is epi. A module $M$ is called $H\mathbb{Z}$-local if it is local with respect to the class of 1-acyclic homomorphisms. The class of $H\mathbb{Z}$-local modules can be described as the least class containing the trivial module, closed small limits and central extensions of modules (an extension of a $G$-module $E \to M$ is called central if the action of $G$ on the kernel is trivial) [9].

6 Baumslag Rationalization of Groups

6.1 Definition

Recall that a group $G$ is called rational if the power map $G \to G$, $g \mapsto g^n$ is bijective for any $n \geq 1$. It is easy to see that for a group $G$ the following statements are equivalent: (1) $G$ is rational; (2) $G$ is local with respect to the maps $n : \mathbb{Z} \to \mathbb{Z}$ for $n \geq 1$; (3) $G$ is local with respect to the embedding $\mathbb{Z} \hookrightarrow \mathbb{Q}$.

For any group $G$ there exists a universal map to a rational group that we call the Baumslag rationalization

$$G \to \text{Bau}(G)$$  \hspace{1cm} (6.1)

(see [27, 30]). This defines a coaugmented functor

$$\text{Bau} : \mathcal{Gr} \to \mathcal{Gr}.$$  \hspace{1cm} (6.2)

The functor of Baumslag rationalization satisfies the following properties:

$\bullet$ $\text{Bau}$ sends epimorphisms to epimorphisms;
$\bullet$ Moreover, $\text{Bau}$ is right exact [1];
$\bullet$ $|\text{Bau}(G)| \leq |G|$ for any $G$;
$\bullet$ $\text{Bau}$ commutes with filtered colimits.
The first two statements follow from [1, Th. 4.1]. For a finite group $\text{Bau}(G) = 1$. For infinite groups the fact $|\text{Bau}(G)| \leq |G|$ follows from the equation $|\text{Bau}(F(S))| = |F(S)| = |S|$ for an infinite set $S$ [6, Th. 36.2] and the fact that the epimorphism $F(G) \rightarrow G$ induces an epimorphism $\text{Bau}(F(G)) \rightarrow \text{Bau}(G)$. The commuting with filtered colimits follows from the obvious fact that a filtered colimit of rational groups is rational and the formula $\text{Hom}(\text{colim} G_i, H) = \text{lim} \text{Hom}(G_i, H)$.

6.2 Baumslag Rationalization of a Free Group

Baumslag did not consider the functor but he studied rational groups, and he called them $D$-groups [4–6]. In particular, he studied the free rational group $\text{Bau}(F)$. He proved that for a free group $F$ there is an increasing family of subgroups $(G_{\alpha})_{\alpha \leq \alpha_0}$ of $\text{Bau}(F)$ indexed by ordinals $\leq \alpha_0$ for some ordinal $\alpha_0$

$$F = G_0 \leq G_1 \leq \cdots \leq G_{\alpha} \leq \cdots \leq G_{\alpha_0} = \text{Bau}(F)$$

together with a family of elements $(g_{\alpha} \in G_{\alpha})_{\alpha < \alpha_0}$ such that

- $G_{\alpha} = \bigcup_{\beta < \alpha} G_{\beta}$ for any limit ordinal $\alpha \leq \alpha_0$;
- for $\alpha < \alpha_0$ the centraliser $A_{\alpha} = C(g_{\alpha}, G_{\alpha})$ is isomorphic to a subgroup of $\mathbb{Q}$ and $G_{\alpha+1}$ is isomorphic to the free product with amalgamation $G_{\alpha+1} \cong G_{\alpha} \ast_{A_{\alpha}} \mathbb{Q}$.

(see [6, Th. 36.1]).

Baumslag also computed the abelianization of this group. He proved that if $F = F(S)$ is a free group freely generated by a set $S$, then

$$\text{Bau}(F)_{ab} \cong \mathbb{Q}^{\oplus S} \oplus (\mathbb{Q}/\mathbb{Z})^{\oplus T}$$

where the torsion free summand $\mathbb{Q}^{\oplus S}$ is generated by the images of $S$ and $T$ is a set, which is countable, if $1 < |S| < \infty$, and $|T| = |S|$, if $S$ is infinite. (see [6, Th. 37.3]). In particular, this implies that for a free group $F$ there is an isomorphism

$$H_*(\text{Bau}(F), \mathbb{Q}) \cong H_*(F, \mathbb{Q}). \quad (6.3)$$

6.3 Equivariantly Rational Modules

Let $G$ be a group and $M$ be a $G$-module (i.e. $\mathbb{Z}[G]$-module). We say that a $G$-module $M$ is equivariantly rational if the multiplication map

$$(1 + g + g^2 + \cdots + g^{n-1}) : M \rightarrow M$$

is a bijection for any $g \in G$ and $n \geq 1$. It is easy to see that an equivariantly rational $G$-module is a $\mathbb{Q}$-vector space. If $G$ is rational, a $G$-module $M$ is equivariantly rational if and only if the semidirect product $G \rtimes M$ is a rational group.

There is an obvious functor of rationalization on the category of modules $\text{Mod}(G)$ which is just the ring theoretical localization with respect to the set of elements in $\mathbb{Z}[G]$:

$$\Sigma = \{1 + g + g^2 + \cdots + g^{n-1} \mid g \in G, n \geq 1\}. \quad (6.4)$$
Casacuberta and Peschke generalized the equation (6.3). They proved [13, Th. 8.7, Cor. 7.3] that for a free group $F$ and an equivariantly rational $\text{Bau}(F)$-module $M$ the map $F \to \text{Bau}(F)$ induces isomorphisms

$$H_*(F, M) \cong H_*(\text{Bau}(F), M), \quad H^*(F, M) \cong H^*(\text{Bau}(F), M).$$  \hspace{1cm} (6.5)

7 Comparison of Rationalizations of Groups

Any rational nilpotent group is $H\mathbb{Q}$-local because it can be obtained from the trivial group as a multiple rational central extension. Hence, the inverse limit of rational nilpotent groups is also $H\mathbb{Q}$-local. In particular, the $\mathbb{Q}$-completion of any group $G$ is $H\mathbb{Q}$-local. From the universal property of $H\mathbb{Q}$-localization we obtain a homomorphism $\ell_{H\mathbb{Q}}(G) \to \widehat{G}_{\mathbb{Q}}$ (see (5.1)). On the other hand, it is easy to see that any $H\mathbb{Q}$-local group is rational, so we have a map $\text{Bau}(G) \to \ell_{H\mathbb{Q}}(G)$. Therefore, we obtain two natural transformations

$$\text{Bau}(G) \to \ell_{H\mathbb{Q}}(G) \to \widehat{G}_{\mathbb{Q}}. \hspace{1cm} (7.1)$$

If $H_1(G, \mathbb{Q})$ is finite dimensional, then the right hand map is an epimorphism and its kernel is the intersection of the rational lower central series:

$$\frac{\ell_{H\mathbb{Q}}(G)}{\bigcap_n (\ell_{H\mathbb{Q}}(G))^n} \cong \widehat{G}_{\mathbb{Q}}. \hspace{1cm} (7.2)$$

By the definition we have that the map $H_2(G, \mathbb{Q}) \to H_2(\ell_{H\mathbb{Q}}(G), \mathbb{Q})$ is an epimorphism. In particular, for a free group $F$ we have $H_2(\ell_{H\mathbb{Q}}(F), \mathbb{Q}) = 0$. On the other hand $H_2(\widehat{F}_{\mathbb{Q}}, \mathbb{Q}) \neq 0$ for a free group of rank $\geq 2$. Therefore $\ell_{H\mathbb{Q}}(F)$ and $\widehat{F}_{\mathbb{Q}}$ are not isomorphic $\ell_{H\mathbb{Q}}(F) \neq \widehat{F}_{\mathbb{Q}}$.

On the other hand we know that for a countable group $G$ the group $\text{Bau}(G)$ is at most countable. But for a finitely generated free group $F$ of rank $\leq 2$ the groups $\widehat{F}_{\mathbb{Q}}$ and $\ell_{H\mathbb{Q}}(F)$ are uncountable. Therefore, for a finitely generated free group $F$ the groups

$$\text{Bau}(F), \ell_{H\mathbb{Q}}(F), \widehat{F}_{\mathbb{Q}}, \text{ are not isomorphic.} \hspace{1cm} (7.3)$$

8 Rationalization of Nilpotent Spaces

Let $G$ be a group and $M$ be a $G$-module. The module $M$ is called nilpotent if $MI^n = 0$ for some $n$, where $I$ is the augmentation ideal of $\mathbb{Z}[G]$. Note that if $G$ is a nilpotent group, then $M$ is a nilpotent $G$-module if and only if $G \ltimes M$ is a nilpotent group.

A connected space $X$ is called nilpotent if $\pi_1(X)$ is a nilpotent group and $\pi_n(X)$ is a nilpotent $\pi_1(X)$-module for any $n$. In other words a space $X$ is nilpotent if and only if $\pi_1(X) \ltimes \pi_n(X)$ is a nilpotent group for any $n$.

Denote by $\mathcal{W}_{H\mathbb{Q}}$ the class of all rational homology equivalences, i.e., the class of all maps $f : X \to Y$ that induce an isomorphism $H_*(X, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$. A nilpotent space $X$ is called rational if one of the following equivalent conditions hold:

- $\tilde{H}_*(X, \mathbb{Z})$ is rational;
- $\pi_*(X)$ is rational;
- $X$ is $\mathcal{W}_{H\mathbb{Q}}$-local

(see [11, Ch.V, Prop.3.3]). A map between nilpotent spaces $f : X \to Y$ is called rational homotopy equivalence if one of the following equivalent conditions hold:

- $f$ is a rational homology equivalence;
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\[ \pi_* (X) \otimes Q \to \pi_* (Y) \otimes Q \text{ is iso.} \]

Then a rationalization of a space \( X \) is a rational homotopy equivalence to a rational space

\[ X \to X_Q. \] (8.1)

In particular

\[ H_* (X_Q, \mathbb{Z}) \cong H_* (X, Q), \quad \pi_* (X_Q) \cong \pi_* (X) \otimes Q. \] (8.2)

Rationalization exists for any nilpotent space \([20, \text{Th. 3A}], [11, \text{Ch. V, Prop. 4.2}], [7]\). It is easy to see that it is the \( W_{nil}^H Q \)-localization in the homotopy category of nilpotent spaces, where \( W_{nil}^H Q \) is the class of rational homotopy equivalences. Therefore the map \( X \to X_Q \) satisfies two universal properties: (1) it is the universal map (initial) to a rational space; (2) it is the universal (terminal) rational homotopy equivalence.

9 Bousfield-Kan \( \mathbb{Q} \)-completion of Spaces

Here by a space we mean a simplicial set.

For a set \( S \) we denote by \( Q(S) \) the \( Q \)-vector space freely generated by elements of \( S \). For a pointed set \( S = (S, s_0) \) we denote by \( Q(S) \) the quotient \( Q(S)/\{Q \cdot s_0\} \). This defines a functor on the category of pointed sets

\[ Q : \text{Sets}_* \to \text{Sets}_*. \] (9.1)

This functor has a natural structure of a monad with the obvious unit \( \eta_S : S \to Q(S) \) and multiplication \( \mu : Q(Q(S)) \to Q(S) \), whose restriction on the basis \( Q(S) \setminus \{0\} \) of \( Q(Q(S)) \) is identical. Then any pointed space \( S \) has a canonical cosimplicial resolution corresponding to this monad that we denote by \( Q(S) \).

For a pointed simplicial set \( X \) we can apply this construction component-wise and obtain a cosimplicial space \( Q(X) \). Then the Bousfield-Kan \( \mathbb{Q} \)-completion \( Q_\infty X \) is the total space of the cosimplicial space \( Q(X) \).

\[ Q_\infty X = \text{Tot}(Q(X)) \] (9.2)

(see [11] for details).

Equivalently Bousfield-Kan \( \mathbb{Q} \)-completion of connected spaces can be defined via simplicial groups. The homotopy category of connected pointed spaces is equivalent to the homotopy category of simplicial groups

\[ (\text{connected pointed spaces}) \simeq (\text{simplicial groups}). \] (9.3)

The equivalence is given by the functors of Kan loop group \( X \mapsto \mathcal{G}(X) \) and the functor of simplicial classifying space of a simplicial group \( G \mapsto \mathcal{W}(G) \). Then the Bousfield-Kan \( \mathbb{Q} \)-completion can be defined as the composition of these equivalences with the functor of component-wise \( \mathbb{Q} \)-completion on simplicial groups \([11, \text{Ch. IV, \$4}]\).

\[ (\text{connected pointed spaces}) \xrightarrow{Q_\infty} (\text{connected pointed spaces}) \]

\[ \xrightarrow{\mathcal{G}} (\text{simplicial groups}) \xrightarrow{(\cdot)_Q} (\text{simplicial groups}) \] (9.4)
The \( \mathbb{Q} \)-completion can be also defined “axiomatically” via towers of fibrations \([11, \S III.6.3]\), \([7, \S 12.1]\). Let

\[
\begin{array}{ccccccc}
X & \leftarrow & \text{id} & X & \leftarrow & \text{id} & X & \leftarrow & \text{id} & \ldots \\
\downarrow & & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \downarrow & \ldots \\
N_1 & \leftarrow & N_2 & \leftarrow & N_3 & \leftarrow & N_4 & \leftarrow & \ldots \\
\end{array}
\]

be a commutative diagram such that

1. \( N_i \) are \( \mathbb{Q} \)-nilpotent spaces;
2. the map \( \lim_{\rightarrow} H^*(N_i, \mathbb{Q}) \rightarrow H^*(X, \mathbb{Q}) \) is an isomorphism;
3. \( N_i \rightarrow N_{i-1} \) is a fibration.

Then the map \( X \rightarrow \mathbb{Q}_\infty X \) is isomorphic to the map \( X \rightarrow \lim_{\rightarrow} N_i \).

The Bousfield-Kan \( \mathbb{Q} \)-completion can be used to detect rational homology equivalences. For a map \( f : X \rightarrow Y \) the following statements are equivalent:

- \( f \) is a rational homology equivalence;
- \( \mathbb{Q}_\infty f : \mathbb{Q}_\infty X \rightarrow \mathbb{Q}_\infty Y \) is a weak homotopy equivalence.

The \( \mathbb{Q} \)-completion of a nilpotent space is the ordinary rationalization but it is not idempotent for general spaces. A space is called \( \mathbb{Q} \)-good if the map \( X \rightarrow \mathbb{Q}_\infty X \) is a rational homology equivalence and \( \mathbb{Q} \)-bad otherwise. For \( \mathbb{Q} \)-bad spaces \( \mathbb{Q}_\infty(\mathbb{Q}_\infty X) \) is not homotopy equivalent to \( \mathbb{Q}_\infty(X) \) \([11, \text{Ch.I, 5.1}]\). It is known that the wedge of two circles \( S^1 \vee S^1 \) is \( \mathbb{Q} \)-bad \([22]\). Hence

\[
\mathbb{Q}_\infty(\mathbb{Q}_\infty(S^1 \vee S^1)) \not\simeq \mathbb{Q}_\infty(S^1 \vee S^1).
\]  \tag{9.6}

For any space \( X \) there is an epimorphism

\[
\pi_1(\mathbb{Q}_\infty X) \twoheadrightarrow \pi_1(X)_\mathbb{Q}
\]  \tag{9.7}

which is generally not an isomorphism (compare with \([23]\)).

### 10 Sullivan’s Rationalization of Spaces

The strength of the classical rational homotopy theory is due to linear algebraic models of spaces, in particular, minimal Sullivan models \([16, 18]\). A rationalization of a space can be defined on this language. Namely, Sullivan’s rationalization of a space \( X \) is the geometric realization of the minimal Sullivan model of \( X \). For spaces of finite rational type Sullivan’s rationalization coincides with Bousfield-Kan’s \( \mathbb{Q} \)-completion.

Let us explain it in more details following Bousfield and Gugenheim \([7]\) (see also \([18]\)). By a graded algebra we mean a non-nenegatively graded algebra \( A = \bigoplus_{n \geq 0} A^n \). A differential graded algebra is a graded algebra with a differential \( \partial : A \rightarrow A \) of degree 1 (i.e. a linear map such that \( \partial(A^n) \subseteq A^{n+1} \); \( \partial^2 = 0 \) and \( \partial(ab) = \partial(a)b + (-1)^{|a|a}\partial(b) \) for all homogeneous \( a, b \in A \)). A graded algebra \( A \) is called commutative if it satisfies \( ab = (-1)^{|a||b|}ba \) for any homogeneous \( a, b \in A \). Cdg-algebra is an abbreviation for commutative differential graded algebra. A morphism of cdg-algebras \( f : A \rightarrow B \) is called quasi-isomorphism, if it induces an isomorphism on cohomology \( H^*(A) \cong H^*(B) \).
10.1 Minimal Cdg-algebras

For any cdg-algebra $A$ and $n \geq 0$ we denote by $A(n) \subseteq A$ the subalgebra generated by $A^0, A^1, \ldots, A^n, \partial(A^n)$; and let $A(-1) = \mathbb{Q} \cdot 1_A \subseteq A_0$. So we obtain a filtration by cdg-subalgebras

$$\mathbb{Q} \cdot 1_A = A(-1) \subseteq A(0) \subseteq A(1) \subseteq \cdots \subseteq A.$$  \hfill (10.1)

Further, for any $m \geq 0$ we consider a cdg-subalgebra $A(n, m) \subseteq A(n)$ defined inductively:

- $A(n, 0) = A(n-1)$
- $A(n, m+1)$ is the subalgebra generated by $A(n, m)$ and the set $\{a \in A^n \mid \partial(a) \in A(n, m)\}$

$$A(n-1) = A(n, 0) \subseteq A(n, 1) \subseteq A(n, 2) \subseteq \cdots \subseteq A(n).$$  \hfill (10.2)

A cdg-algebra $M$ is called minimal if

- it is connected $\mathbb{Q} \cong M^0$;
- it is a free commutative graded algebra (generated by some positively graded vector space);
- $M(n) = \bigcup_m M(n, m)$ for any $n \geq 1$.

If a cdg-algebra $M$ is simply connected (i.e., $M^0 = \mathbb{Q}$ and $M^1 = 0$), then $M$ is minimal if and only if it is free commutative graded algebra and $\partial(M) \subseteq \mathcal{M} \cdot \mathcal{M}$, where $\mathcal{M} = \bigoplus_{n \geq 1} M^n$.

For any homologically connected (i.e., $H^0(A) = \mathbb{Q}$) cdg-algebra $A$ there exists a quasi-isomorphism from a minimal cdg-algebra $M \to A$. Moreover, the minimal cdg-algebra $M$ is defined uniquely up to isomorphism. However, $M$ is not natural in $A$ but it is natural “up to homotopy”.

**Remark 10.1** There is a model structure on the category of cdg-algebras, where weak equivalences are quasi-isomorphisms and fibrations are surjective morphisms. Minimal algebras are cofibrant with respect to this model structure. Moreover, any cofibrant cdg-algebra $C$ can be decomposed as a tensor product

$$C \cong M \otimes \bigotimes_{\alpha} T(n_{\alpha}),$$  \hfill (10.3)

where $M$ is a minimal cdg-algebra and for $n \geq 0$ by $T(n)$ we denote the cdg-algebra freely generated by two elements $a, \partial a$, where $|a| = n$, $|\partial a| = n + 1$ and $\partial$ is the unique differential such that $\partial(a) = \partial a$ (see [7, Proposition 7.11]).

10.2 PL de Rham Complex

For any $n \geq 0$ we denote by $\nabla^*_n$ the cdg-algebra generated by elements $t_0, \ldots, t_n$ of degree 1 and elements $\partial t_0, \ldots, \partial t_n$ of degree 2 modulo relations

$$t_0 + t_1 + \cdots + t_n = 1, \quad \partial t_0 + \partial t_1 + \cdots + \partial t_n = 0$$  \hfill (10.4)

with the unique differential $\partial : \nabla^m_n \to \nabla^{m+1}_n$ such that $\partial(t_i) = \partial t_i$. We also consider a simplicial object $\nabla^*_n$ in the category of cdg-algebras, whose components are $\nabla^*_n$ and the face and degeneracy maps are defined by

$$d_j(t_i) = \begin{cases} t_i, & i < j, \\ 0, & i = j, \\ t_{i-1}, & i > j, \end{cases} \quad s_j(t_i) = \begin{cases} t_i, & i < j, \\ t_i + t_{i+1}, & i = j, \\ t_{i+1}, & i > j. \end{cases}$$  \hfill (10.5)
Note that for each fixed $n$ we have a cdg-algebra $\nabla^*_n$ and for each fixed $m$ we have a simplicial vector space $\nabla^m$. Then we can define two adjoint functors
\[ A_{PL} : \text{sSets} \rightleftarrows \text{cdga}^{\text{op}} : F \] (10.6)
such that
\[ A_{PL}(X)^m = \text{Hom}_{\text{sSets}}(X, \nabla^m), \quad F(A)_n = \text{Hom}_{\text{cdga}}(A, \nabla^*_n), \] (10.7)
where $A_{PL}(X) = \bigoplus_{m \geq 0} A_{PL}(X)^m$ is considered as a cdg-algebra with the pointwise product and $F(A)$ is in fact the composition of two functors $\nabla : \Delta^{\text{op}} \rightarrow \text{cdga}$ and $\text{Hom}(A, -) : \text{cdga} \rightarrow \text{Sets}$. The cdg-algebra $A_{PL}(X)$ is called the PL de Rham complex of $X$ or Sullivan de Rham complex of $X$. The cohomology of $A_{PL}(X)$ is isomorphic to $H^*(X, \mathbb{Q})$.

10.3 Sullivan’s Rationalization

For a connected simplicial set $X$ we consider a quasi-isomorphism $M(X) \rightarrow A_{PL}(X)$ from a minimal cdg-algebra $M(X)$. The algebra $M(X)$ is called the Sullivan minimal model of $X$. $M(X)$ is not natural by $X$ in the category of cdg-algebras but it is natural “up to homotopy”. Sullivan’s rationalization of $X$ is the geometric rationalization of Sullivan’s minimal model
\[ \text{Sul}(X) = F(M(X)). \] (10.8)
The composition of the unit of the adjunction (10.6) $\eta : X \rightarrow F(A_{PL}(X))$ and the map $F(A_{PL}(X)) \rightarrow F(M(X))$ defines the map
\[ X \rightarrow \text{Sul}(X). \] (10.9)

If $X$ is a space of rational finite type (i.e., $H^n(X, \mathbb{Q})$ is finite dimensional for any $n$), then Sullivan’s rationalization coincides with Bousfield-Kan’s $\mathbb{Q}$-completion ([7, Th.12.2])
\[ \text{Sul}(X) \cong \mathbb{Q}_\infty X. \] (10.10)

A similar construction of rationalization can be done via Lie algebra models and it also coincides with Bousfield-Kan’s $\mathbb{Q}$-completion [12].

11 Homological Rationalization of Spaces

Bousfield developed a theory of $h_*$-localization of spaces with respect to any homology theory $h_*$. The homology rationalization of a space is the $H_*(-, \mathbb{Q})$-localization.

Recall that we denote by $\mathcal{W}_{H\mathbb{Q}}$ the class of rational homology equivalences in the homotopy category of spaces, i.e., the class of maps $f : X \rightarrow Y$ that induce an isomorphism $H_*(X, \mathbb{Q}) \cong H_*(Y, \mathbb{Q})$. Then the homology rationalization of a space is the $\mathcal{W}_{H\mathbb{Q}}$-localization. It is denoted by
\[ X \rightarrow L_{H\mathbb{Q}}(X). \] (11.1)
It exists for any space [10] and defines a functor on the homotopy category.

We say that a space is homologically rational if it is $\mathcal{W}_{H\mathbb{Q}}$-local. Hence the homologically rationalization satisfies two universal properties: (1) it is the universal rational homology equivalence; (2) it is the universal map to a homologically rational space.
The homology rationalization corresponds to the $H\mathbb{Q}$-localization of groups in the following sense. The fundamental group of a homological rationalization of a space $X$ is the $H\mathbb{Q}$-localization of the fundamental group of $X$.

$$\pi_1(L_{H\mathbb{Q}}(X)) = \ell_{H\mathbb{Q}}(\pi_1(X)).$$

(11.2)

There is a description of connected homologically rational spaces on the language of their homotopy groups. Namely, a connected space $X$ is homologically rational if and only if

- $\pi_n(X)$ is $H\mathbb{Q}$-local for $n \geq 1$;
- $\pi_n(X)$ is $H\mathbb{Z}$-local as a $\pi_1(X)$-module for $n \geq 2$ (see [10, Th. 5.5]).

There is a transfinite limit construction for homology rationalization of a space [14].

Note that for the moment there is no a description of $L_{H\mathbb{Q}}$ in terms of simplicial groups, and the following question is open.

**Question 11.1** Is the space $L_{H\mathbb{Q}}(S^1 \vee S^1)$ aspherical?

### 12 $\Omega$-rationalization of Spaces

#### 12.1 Definition

Casacuberta and Peschke developed their own theory of localization of spaces [13] (see also [28]) further developed by Bastardas and Casacuberta [2], which was initially hinted by Farjoun and Bousfield [8, Example 7.3].

A connected space $X$ is called $\Omega$-rational if one of the following equivalent conditions hold:

- the power map $(-)^m : \Omega X \to \Omega X$ is a homotopy equivalence for any $m \geq 1$;
- $\pi_1(X)$ is a rational group and $\pi_n(X)$ is an equivariantly rational $\pi_1(X)$-module for $n \geq 2$;
- $\pi_1(X)$ is rational and $\pi_1(X) \rtimes \pi_n(X)$ is rational for any $n \geq 2$.

Then the $\Omega$-rationalization of a connected space $X$ is the universal map to an $\Omega$-rational space (in the homotopy category)

$$X \longrightarrow L_{\Omega\mathbb{Q}}(X).$$

(12.1)

It exists for any connected space.

#### 12.2 $\Omega$-rationalization as an Enriched Localization

The $\Omega$-rationalization is a particular case of the general theory of $f$-localization of spaces [15], [21]. In this theory one should consider the category of spaces as a category enriched over itself, and it is not enough to think about the homotopy category. If $f : A \to B$ is a map of pointed spaces, a space $X$ is called $f$-local, if the map

$$\text{Map}_*(B, X) \longrightarrow \text{Map}_*(A, X)$$

(12.2)

is a weak equivalence. The $f$-localization of spaces is the universal map to an $f$-local space. A space is $\Omega$-rational if and only if it is $f$-local, where $f$ is the wedge of the maps $(-)^m : S^1 \to S^1$ for $m \geq 1$; and $\Omega$-rationalization is just the $f$-localization. So the $\Omega$-rationalization shares all properties of $f$-localizations that can be found in [15].

#### 12.3 $\Omega$-rationalization as a Localization in the Homotopy Category

We described $\Omega$-rational spaces as local spaces with respect to the maps $(-)^m : S^1 \to S^1$ in the enriched sense. It is also possible to describe them as local spaces in the strict sense with
respect to a wider set of maps in the homotopy category, without any enrichment. Consider
the sphere with one added point \( S^n_+ = S^n \cup \text{pt} \). This additional point is considered as the base
point of this space. For \( n \geq 2 \) we also consider the suspension
\[
S^n_\tau = \Sigma S^{n-1}_+. \tag{12.3}
\]
For convenience we set \( S^1_\tau := S^1 \). The space \( S^n_\tau \) has the obvious co-\( H \)-structure because it is a
suspension. There is a homotopy equivalence \( S^n_\tau \simeq S^n \vee S^1 \) but this is not an equivalence of
co-\( H \)-spaces. The space \( S^n_\tau \) is interesting because there is a group isomorphism
\[
\pi_1(X) \times \pi_n(X) \cong [S^n_\tau, X], \tag{12.4}
\]
where the structure of a group on \([S^n_\tau, X]\) comes from the co-\( H \)-structure on \( S^n_\tau \). Then a space
\( X \) is \( \Omega \)-rational if and only if it is local with respect to the maps \( (-)^n : S^n_\tau \to S^n_\tau, n \geq 1 \) in the
homotopy category (here the map \( (-)^n : S^n_\tau \to S^n_\tau \) is induced by the map \( (-)^n : S^1 \to S^1 \)).

12.4 \( \Omega \)-rational Equivalences

We say that a map of connected spaces \( f : X \to Y \) is an \( \Omega \)-equivalence if \( L_{\Omega Q}(f) \) is an
isomorphism in the homotopy category. There is a description of the class \( \Omega \)-equivalences. For
a group \( G \) we set
\[
\mathbf{R}[G] = \mathbb{Z}[[\text{Bau}(G)][\Sigma^{-1}], \tag{12.5}
\]
where
\[
\Sigma = \{ 1 + g + g^2 + \cdots + g^{n-1} \mid g \in \text{Bau}(G), n \geq 1 \}. \tag{12.6}
\]
Then a map \( f : X \to Y \) is an \( \Omega \)-equivalence if and only if it induces isomorphisms

(1) \( \text{Bau}(\pi_1(X)) \cong \text{Bau}(\pi_1(Y)) \);
(2) \( H^*(Y, \mathcal{A}) \cong H^*(X, \mathcal{A}) \), for any local system \( \mathcal{A} \) over \( \mathbf{R}[\pi_1(Y)] \).

The condition (2) is equivalent to the condition

(2') \( H_*(X, \mathbf{R}[\pi_1(X)]) \cong H_*(Y, \mathbf{R}[\pi_1(Y)]) \) (see [13, Th. 3.2]).

12.5 \( \Omega \)-rationalization and Simplicial Groups

Casacuberta and Bastardas proved that the \( \Omega \)-rationalization can be defined via simplicial
groups as the component-wise Baumslag rationalization [2].

\[
\begin{array}{ccc}
\text{(connected pointed spaces)} & \xrightarrow{L_{\Omega Q}} & \text{(connected pointed spaces)} \\
\text{\downarrow g} & & \text{\downarrow \text{Bau}} \\
\text{(simplicial groups)} & \xrightarrow{\text{\text{Bau}}} & \text{(simplicial groups)}
\end{array} \tag{12.7}
\]

13 \( \pi_1 \)-fiberwise Rationalization of Spaces

Gómez-Tato, Halperin and Tanrée developed a version of rationalizations of spaces that I call it
\( \pi_1 \)-fiberwise rationalization [17]. They also developed a theory of algebraic models that extends
the theory of minimal Sullivan modules.

A space \( X \) is called \( \pi_1 \)-fiberwise rational if its universal cover \( \tilde{X} \) is rational in the classical
sense. In other words, a space \( X \) is \( \pi_1 \)-fiberwise rational if \( \pi_n(X) \) is a \( \mathbb{Q} \)-vector space for \( n \geq 2 \).
fibration $X$ exists for any space and can be computed as the fiberwise $Q$-description of the classes of equivalences, it is easy to see that there are inclusions:

\[ \text{space (see [10, Prop. 12.10, Lemma 3.5] and [14]). On the other hand, looking on homology completion is a homologically rational}

Then the $\pi_1$-fiberwise rationalization is the $\pi_1$-fiberwise rational homotopy equivalence to a $\pi_1$-fiberwise rational space

\[ X \longrightarrow L^\pi_1_Q(X). \quad (13.1) \]

It exists for any space and can be computed as the fiberwise $Q$-completion applied to the fibration $X \to B\pi_1(X)$. The class of $\pi_1$-fiberwise rational homotopy equivalences can be described in terms of rational chain coalgebra $C_*(X, Q)$ and in terms of homology. Namely for a map of connected spaces $f : X \to Y$ the following statements are equivalent [31, Th. 16] (see also [32]):

- $f$ is a $\pi_1$-fiberwise rational homotopy equivalence;
- $\pi_1(X) \to \pi_1(Y)$ is an isomorphism and $H_*(X, f^*A) \to H_*(Y, A)$ is an isomorphism for any local system $A$ over $Q[\pi_1(Y)]$;
- $\text{Cobar}(C_*(X, Q)) \to \text{Cobar}(C_*(Y, Q))$ is a quasi-isomorphism of dg-algebras.

### 14 Comparison of Rationalizations of Spaces

It is not difficult to check that the Bousfield-Kan $Q$-completion is a homologically rational space (see [10, Prop. 12.10, Lemma 3.5] and [14]). On the other hand, looking on homology description of the classes of equivalences, it is easy to see that there are inclusions:

\[ \{ \text{fiberwise rational eq.} \} \subseteq \{ \Omega\text{-rational eq.} \} \subseteq \{ \text{homology rational eq.} \}, \quad (14.1) \]

and for the classes of spaces:

\[ \{ \text{fiberwise rational sp.} \} \supseteq \{ \Omega\text{-rational sp.} \} \supseteq \{ \text{homology rational sp.} \}. \quad (14.2) \]

This implies that there are natural transformations

\[ L^\pi_1_Q \longrightarrow L_{\Omega Q} \longrightarrow L_{HQ} \longrightarrow Q_\infty. \quad (14.3) \]

For the wedge of two circles we have

\[ L^\pi_1_Q(S^1 \vee S^1) = S^1 \vee S^1 \quad (14.4) \]
\[ L_{\Omega Q}(S^1 \vee S^1) = K(\text{Bau}(F), 1) \quad (14.5) \]
\[ L_{HQ}(S^1 \vee S^1) = ? \quad (14.6) \]
\[ Q_\infty(S^1 \vee S^1) = K(\tilde{F}_Q, 1), \quad (14.7) \]

where $F$ is the free group of rank two. Indeed, for $L^\pi_1_Q$ it is obvious. For $L_{\Omega Q}, Q_\infty$ this follows from the simplicial group construction and the fact that $S^1 \vee S^1 = \mathbb{W}(F)$, where $F$ is the constant free simplicial group. For $L_{HQ}$ it still is an open question, if $L_{HQ}(S^1 \vee S^1)$ is aspherical or not. However, we know that $\pi_1(L_{HQ}(S^1 \vee S^1)) = \ell_{HQ}(F)$. Thus these four rationalizations of $S^1 \vee S^1$ are non-homotopy equivalent.

### 15 An Example: the Classifying Space of the Burnside Group

Let $G = B(2, n)$ be the free Burnside group of rank two and exponent $n$. Since the group is generated by torsion elements, $\text{Bau}(G) = \ell_{HQ}(G) = 1$. Therefore, the spaces $L_{\Omega Q}(BG)$ and
$L_{HQ}(BG)$ are simply connected. However they are not contractible for large enough $n$. Indeed, by the Theorem of Ol’shanskii [26, Cor.31.2] for large enough $n$ the second homology $H_2(G,\mathbb{Z})$ is a free abelian group of countable rank, and hence, 
\[ H_2(L_{HQ}(BG),\mathbb{Q}) = H_2(L_{HQ}(BG),\mathbb{Q}) = H_2(G,\mathbb{Q}) \neq 0. \] (15.1)

By the Hurewicz theorem we obtain 
\[ \pi_2(L_{HQ}(BG)) \neq 0, \quad \pi_2(L_{HQ}(BG)) \neq 0. \] (15.2)

**Acknowledgements** I am grateful to Emmanuel Farjoun and Stephen Halperin for useful discussions. We also thank the referees for their time and comments.

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