H-Distributions via Sobolev Spaces

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Abstract. H-distributions associated with weakly convergent sequences in Sobolev spaces are determined. It is shown that a weakly convergent sequence \((u_n)\) in \(W^{-k,p}(\mathbb{R}^d)\) has the property that \(\theta u_n\) converges strongly in \(W^{-k,p}(\mathbb{R}^d)\) for every \(\theta \in \mathcal{S}(\mathbb{R}^d)\) if and only if all H-distributions related to this sequence are equal to zero. Results are applied on a weakly convergent sequence of solutions to a family of linear first-order PDEs.

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1. Introduction

H-measures, or Microlocal defect measures, of Tartar [21] and Gérard [8] obtained for weakly convergent sequences in \(L^2(\mathbb{R}^d)\), and their generalization to \(L^p(\mathbb{R}^d), p \in (1, \infty)\), called H-distributions [5], are widely used to determine whether a weakly convergent sequence of solutions to certain classes of equations converges strongly. For example, by using H-measures the authors of [3] obtained \(L^1_{\text{loc}}\)-precompactness of solutions to diffusion–dispersion approximation for a scalar conservation law. In homogenization theory, applications of these objects can be found, e.g., in [4,12]. In [14], H-measures are applied to family of entropy solutions of a first order quasilinear equation and in [18] to ultraparabolic equation. The list of applications of these objects is far from being complete.

Our aim in this paper is to extend the concept of H-distributions to the Sobolev spaces. For the purposes of this paper, we introduce in Sect. 2.1 new tensor product—spaces of test functions and distributions. For the reader’s convenience, we give full description of such spaces in the Appendix (Propositions 4.1 and 4.2).
To use the duality $W^{-k,p} - W^{k,q}$, $q = \frac{p}{p-1}$, $k \in \mathbb{N}_0$, we prove the existence result for H-distributions associated with a weakly convergent sequence in $L^p(\mathbb{R}^d)$; in Theorem 2.1 we extend the result of [5, Theorem 2.1] since we did not use the localization coming from the compactly supported test functions. H-distributions of Theorem 2.1 are defined on the space of rapidly decreasing functions. This leads to the improvements of results of [5] in the case of $L^p$-spaces. In Theorem 3.1, we prove the existence of H-distributions for weakly convergent sequences in Sobolev spaces. Our main theorem, Theorem 3.2, shows that if for a given weakly convergent sequence $u_n \rightharpoonup 0$ in $W^{-k,p}(\mathbb{R}^d)$ and every weakly convergent sequence $v_n \rightharpoonup 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding H-distributions are equal to zero, then for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $(\varphi u_n)$ converges strongly to zero in $W^{-k,p}(\mathbb{R}^d)$. Clearly, the converse assertion also holds. As an application, we analyze in Theorem 3.3 a weakly convergent sequence $(u_n)$ of solutions to $\sum_{i=1}^{d} \partial_i(A_i(x)u_n) = f_n$ in $W^{-k,p}(\mathbb{R}^d)$, $d > \frac{p}{p-1}$, and show that the supports of the corresponding H-distributions are concentrated on the characteristic set $\{(x, \xi) : \sum_{i=1}^{d} A_i(x)\xi_i = 0\}$, under the new condition that for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$, $(\varphi f_n)$ strongly converges to zero in $W^{-k-1,p}(\mathbb{R}^d)$. Moreover, if all H-distributions assigned to this equation are equal to zero, then $(\varphi u_n)$ converges strongly to zero in $W^{-k,p}(\mathbb{R}^d)$. The corresponding results for $L^2(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$ are obtained in [5, 21], respectively. Even in mentioned cases, our results with $k = 0$ extend previous results since the non-locality is the essential part of our approach. Moreover, the results from the recent contributions in which the H-distributions were used cf. [11, 13, 17] can be extended to a more general situations (in the Sobolev spaces with negative coefficients) by using results from this paper.

2. Basic Definitions and Assertions

2.1. Some Spaces of Distributions

We refer to [2] for the Sobolev spaces $W^{k,q}(\mathbb{R}^d)$. If $k > \frac{d}{q}$, then $W^{k,q}(\mathbb{R}^d) \subset C_0(\mathbb{R}^d)$, where $C_0(\mathbb{R}^d)$ is the space of continuous functions vanishing at infinity. The dual $(W^{k,q}(\mathbb{R}^d))' =: W^{-k,p}(\mathbb{R}^d)$ is isometrically isomorphic to the Banach space consisting of distributions $u \in \mathcal{S}'(\mathbb{R}^d)$ of the form $u = \sum_{|\alpha| \leq k} \partial^\alpha u_\alpha$, where all $u_\alpha \in L^p(\mathbb{R}^d)$, normed by $\|u\| := \inf\{(\sum_{|\alpha| \leq k} \|u_\alpha\|_{L^p})^{1/p} : u = \sum_{|\alpha| \leq k} \partial^\alpha u_\alpha\}$, cf. [2, Theorem 3.10, p. 50].

To give clear explanations concerning a new space, which will be denoted by $\mathcal{S}\mathcal{E}(\mathbb{R}^d \times S^{d-1})$, and its dual $\mathcal{S}\mathcal{E}'(\mathbb{R}^d \times S^{d-1})$, we will use some classical results, [7, 16], of $L^2$ and Sobolev theory for the unit sphere $S^{d-1}$ as well as of [6] for some results for $C^k$ and $L^2$ functions on $S^{d-1}$. Concerning Sobolev spaces and distributions on a manifold, we refer to [19] and for tensor product of test spaces, to [22].

We define the space of smooth functions $\mathcal{S}\mathcal{E}(\mathbb{R}^d \times S^{d-1})$ by the sequence of norms
\[ p_{\mathbb{R}^d \times S^{d-1}}^{\infty,1}(\theta) = \sup_{(x,\xi) \in \mathbb{R}^d \times S^{d-1}, |\alpha + \beta| \leq k} \langle x \rangle^k |(\Delta^*_\xi)\partial_x^\beta \theta(x, \xi)|, \quad (2.1) \]

where \( \langle x \rangle^k = (1 + |x|^2)^{k/2} \) and \( \Delta^* \) is the Laplace–Beltrami operator. The space \( \mathcal{S}(\mathbb{R}^d \times S^{d-1}) \) is a Fréchet space and can be identified with the completion of tensor product \( \mathcal{S}(\mathbb{R}^d)^* \otimes \mathcal{E}(S^{d-1}) \), as was shown in Proposition 4.2 in the Appendix. Complete description of this space can be found in the Appendix.

### 2.2. H-Distributions on \( L^p \) Spaces

A bounded function \( \psi \), on \( \mathbb{R}^d \), is called \( L^p \)-Fourier multiplier if \( f \mapsto \mathcal{A}_\psi(f) := (\psi \hat{f}) \) is a bounded mapping from \( \mathcal{S}(\mathbb{R}^d) \) to \( L^p(\mathbb{R}^d) \) and can be continuously extended to a mapping from \( L^p(\mathbb{R}^d) \) to \( L^p(\mathbb{R}^d) \). Here \( \hat{f}(\xi) = \mathcal{F}[f](\xi) = \int_{\mathbb{R}^d} e^{-2\pi i x \cdot \xi} f(x) \, dx \) denotes the Fourier transform on \( \mathbb{R}^d \), while \( \hat{g}(x) = \mathcal{F}^{-1} [g](x) = \int_{\mathbb{R}^d} e^{2\pi i x \cdot \xi} g(\xi) \, d\xi \) denotes the inverse Fourier transform. The space of \( L^p \)-Fourier multipliers, denoted by \( \mathcal{M}_p(\mathbb{R}^d) \), is supplied by the norm \( \| \psi \|_{\mathcal{M}_p} := \| \mathcal{A}_\psi \|_{L^p \to L^p} \), where \( \| \cdot \|_{L^p \to L^p} \) is the standard operator norm.

If \( \psi \in C^\kappa(\mathbb{R}^d, \{0\}) \), \( \kappa = [\tfrac{d}{2}] + 1 \), is homogeneous of zero degree (i.e., \( \psi(\lambda \xi) = \psi(\xi), \lambda > 0 \)), then \( \psi \in L^\infty(\mathbb{R}^d) \) and

\[
|\partial^\alpha \psi(\xi)| \leq A|\xi|^{-|\alpha|}, \quad \xi \in \mathbb{R}^d \setminus \{0\},
\]

for every \( |\alpha| \leq \kappa \) (with \( A = \max_{|\beta| \leq \kappa} \sup_{\xi \neq 0} |\xi|^{\alpha} |\partial^\beta \psi| \), cf. [1, p. 120]). Thus, \( \psi \) fulfills conditions from the Mihlin theorem (cf. [9]): Let \( \psi \) be a complex-valued bounded function on \( \mathbb{R}^d \setminus \{0\} \) that satisfies (2.2) for all multi-indices \( |\alpha| \leq [\tfrac{d}{2}] + 1 \). Then \( \psi \in \mathcal{M}_p(\mathbb{R}^d) \) for any \( 1 < p < \infty \) and

\[
\| \psi \|_{\mathcal{M}_p} \leq C_d \max \left\{ p, \frac{1}{p-1} \right\} (A + \| \psi \|_\infty). \tag{2.3}
\]

Moreover, if \( \psi \in C^\kappa(S^{d-1}) \), then constant \( A \) in (2.3) can be replaced by \( \| \psi \|_{C^\kappa(S^{d-1})} \).

Fourier multiplier operators \( \mathcal{A}_\psi \) with symbol \( \psi \in C^\kappa(S^{d-1}) \) can be defined on \( W^{-k,p}(\mathbb{R}^d) \), via duality

\[
W^{-k,p}(\mathcal{A}_\psi u, v)_{W^{k,q}} := W^{-k,p}(u, \mathcal{A}_\psi v)_{W^{k,q}}.
\]

Since \( \partial^\alpha \mathcal{A}_\psi v = \mathcal{A}_\psi(\partial^\alpha v) \), we know that \( \mathcal{A}_\psi v \in W^{k,q}(\mathbb{R}^d) \). If \( u \in W^{-k,p}(\mathbb{R}^d) \) is of the form \( u = \sum_{|\alpha| \leq k} \partial^\alpha u_\alpha \), then for all \( v \in W^{k,q}(\mathbb{R}^d) \),

\[
W^{-k,p}(\mathcal{A}_\psi u, v)_{W^{k,q}} = \sum_{|\alpha| \leq k} W^{-k,p}(\partial^\alpha u_\alpha, \mathcal{A}_\psi v)_{W^{k,q}}
= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \|\mathcal{A}_\psi(u_\alpha), \partial^\alpha v\|_{L^q} = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \|\mathcal{A}_\psi(u_\alpha), \partial^\alpha v\|_{L^q}.
\]

One can see that every \( L^p \)-multiplier operator \( \mathcal{A}_\psi \) with symbol \( \psi \in \mathcal{M}_p(\mathbb{R}^d) \) is a bounded operator from \( W^{-k,p}(\mathbb{R}^d) \) to \( W^{-k,p}(\mathbb{R}^d) \).

In order to prove the existence of an H-distributions of Theorem 3.1 given below, we need Tartar’s First commutation lemma [21] and the modification of this lemma given in [5].
Let $\psi \in C(S^{d-1})$ and $b \in C_0(\mathbb{R}^d)$ define the Fourier multiplier operator $A_{\psi}$ and the operator of multiplication $B$, acting on $u \in L^2(\mathbb{R}^d)$, as follows: $F(A_{\psi} u)(\xi) = \psi(\frac{\xi}{|\xi|}) F(u)(\xi)$, $\xi \in \mathbb{R}^d \setminus \{0\}$, and $Bu(x) = b(x)u(x)$, $x \in \mathbb{R}^d$. Then the operators $A_{\psi}$ and $B$ are bounded on $L^2(\mathbb{R}^d)$, and their commutator $C := A_{\psi}B - BA_{\psi}$ is a compact operator from $L^2$ into itself.

Moreover, [5]: If a sequence $(v_n)$ is bounded in both $L^2(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, for some $r \in (2, \infty)$ and $v_n \to 0$ in the sense of distributions, then the sequence $(Cv_n)$ strongly converges to zero in $L^q(\mathbb{R}^d)$, for any $q \in [2, r] \setminus \{\infty\}$.

**Theorem 2.1.** If $u_n \to 0$ in $L^p(\mathbb{R}^d)$, and $v_n \to 0$ in $L^q(\mathbb{R}^d)$, then there exist subsequences $(u_{n'})$, $(v_{n'})$ and a distribution $\mu(x, \xi) \in SE'(\mathbb{R}^d \times S^{d-1})$ of order not more than $\kappa = \lfloor d/2 \rfloor + 1$ in $\xi$, such that for every $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$ and $\psi \in C^\kappa(S^{d-1})$,

$$
\lim_{n' \to \infty} \int_{\mathbb{R}^d} A_{\psi}(\varphi_1 u_{n'})(x) \overline{(\varphi_2 v_{n'})}(x) dx = \lim_{n' \to \infty} \int_{\mathbb{R}^d} (\varphi_1 u_{n'})(x) \overline{A_{\psi}(\varphi_2 v_{n'})}(x) dx
=: \langle \mu, \varphi_1 \overline{\varphi_2} \psi \rangle,
$$

where $A_{\psi} : L^p(\mathbb{R}^d) \to L^p(\mathbb{R}^d)$ is a Fourier multiplier operator with the symbol $\psi \in C^\kappa(S^{d-1})$.

By the order of $\mu \in SE(\mathbb{R}^d \times S^{d-1})$ we mean that for any $\varphi \in S(\mathbb{R}^d)$, $\langle \mu(x, \xi), \varphi(x) \psi(\xi) \rangle$ can be extended on $C^\kappa(S^{d-1})$ [see (2.9) and (2.10) below].

**Proof.** First, notice that the Fourier multiplier operator $A_{\psi}$ with $\psi \in C^\kappa(S^{d-1})$ is well defined on both $\varphi_1 u_n \in L^p(\mathbb{R}^d)$ and $\varphi_2 v_n \in L^q(\mathbb{R}^d)$, and that the adjoint operator of $A_{\psi}$ is $A_{\overline{\psi}}$. Thus, the first equality in (2.4) holds.

Let $1 < p \leq 2$. Consider a sequence of sesquilinear (linear in $\varphi \in S(\mathbb{R}^d)$ and anti-linear in $\varphi \in S(\mathbb{R}^d)$) functionals

$$
\mu_n(\varphi, \psi) = \int_{\mathbb{R}^d} u_n A_{\overline{\psi}}(\varphi v_n) dx.
$$

By the continuity of $A_{\psi}$ and the boundedness of $(u_n)$ and $(v_n)$ in $L^p(\mathbb{R}^d)$ and $L^q(\mathbb{R}^d)$, it follows that there exists $c > 0$, such that for every $n \in \mathbb{N}$,

$$
|\mu_n(\varphi, \psi)| \leq \|u_n\|_{L^p} \|A_{\overline{\psi}}(\varphi v_n)\|_{L^q} \leq c \|\psi\|_{C^\kappa(S^{d-1})} \|\varphi\|_{L^\infty}.
$$

Fix $\varphi \in S(\mathbb{R}^d)$ and denote by $(B_n \varphi)$ the sequence of functions defined on $C^\kappa(S^{d-1})$ by

$$
\langle B_n \varphi, \cdot \rangle = \mu_n(\varphi, \cdot).
$$

For every $n \in \mathbb{N}$, the linearity of $B_n \varphi$ is clear and the continuity follows from (2.6):

$$
|\langle B_n \varphi, \psi \rangle| \leq c_\varphi \|\psi\|_{C^\kappa(S^{d-1})}, \quad \text{where } c_\varphi = c \|\varphi\|_{L^\infty}.
$$

If we fix $\psi \in C^\kappa(S^{d-1})$, then (2.5) implies that the mapping $S(\mathbb{R}^d) \to \mathbb{C}$, $\varphi \mapsto \langle B_n \varphi, \psi \rangle$ is anti-linear and, again by (2.6), continuous.

We continue with fixed $\varphi$ and apply the Sequential Banach Alaoglu theorem to obtain weakly star convergent subsequence $(B_k \varphi)$ in $(C^\kappa(S^{d-1}))'$. We denote the weak star limit of $B_k \varphi$ by $B \varphi$, i.e., for every $\psi \in C^\kappa(S^{d-1})$,
\[ \langle B \varphi, \psi \rangle = \lim_{k \to \infty} \langle B_k \varphi, \psi \rangle. \]

We are going to show that \( B \) can be defined on the whole \( S(\mathbb{R}^d) \), so that \( S(\mathbb{R}^d) \ni \varphi \mapsto B \varphi \in (C^\infty(\mathbb{R}^{d-1}))' \) is linear and continuous.

By the diagonalization argument, we define \( B \) on a countable dense set \( M = \{ \varphi_m : m \in \mathbb{N} \} \subset S(\mathbb{R}^d) \). For that purpose extract a subsequence \((B_{1,k})_k \subset (B_n)_n\) such that \((B_{1,k} \varphi_1)\) is weakly star convergent in \((C^\infty(\mathbb{R}^{d-1}))'\) and denote the limit as \( B \varphi_1 \). Then extract a subsequence \((B_{2,k})_k \subset (B_1)_k\) such that \((B_{2,k} \varphi_2)\) is weakly star convergent in \((C^\infty(\mathbb{R}^{d-1}))'\) and denote the limit as \( B \varphi_2 \). Notice also that \( B_{2,k} \varphi_1 \) converges weakly star to \( B \varphi_1 \). Repeating this procedure (extracting subsequences for all \( \varphi_n \in M \)), we obtain diagonal (sub)sequence \( B_{k,m} \in \mathcal{L}(S(\mathbb{R}^d), (C^\infty(\mathbb{R}^{d-1}))') \), such that for all \( \varphi_m \in M \)

\[ \langle B \varphi_m, \psi \rangle = \lim_{k \to \infty} \langle B_{k,m} \varphi_m, \psi \rangle, \quad \psi \in C^\infty(\mathbb{R}^{d-1}). \]

Denote \( B_{k,m} =: b_k \) and fix \( \psi \in C^\infty(\mathbb{R}^{d-1}) \). By (2.7), \( \varphi \mapsto \langle b_k \varphi, \psi \rangle \), is a pointwise bounded sequence in \( S'(\mathbb{R}^d) \) which converges on a dense set \( M \subset S(\mathbb{R}^d) \). By the Banach–Steinhaus theorem, see, e.g., [10, p. 169], \( (b_k(\cdot), \psi) \) converges to \( \langle B(\cdot), \psi \rangle \) on \( S(\mathbb{R}^d) \). In this way, we show that for every \( \varphi \in S(\mathbb{R}^d) \) and every \( \psi \in C^\infty(\mathbb{R}^{d-1}) \)

\[ \lim_{k \to \infty} \langle b_k \varphi, \psi \rangle = \langle B \varphi, \psi \rangle. \]

Moreover, by (2.7),

\[ |\langle B \varphi, \psi \rangle| \leq c ||\varphi||_{L^\infty} ||\psi||_{C^\infty(\mathbb{R}^{d-1})}. \quad (2.9) \]

By [22, Part III, Chap. 50, Proposition 50.7, p. 524] (it is a version of the Schwartz kernel theorem) we have that there exists \( \mu \in \mathcal{S}'(\mathbb{R}^d \times \mathbb{R}^{d-1}) \) defined as

\[ \langle \mu(x, \xi), \varphi(x)\psi(\xi) \rangle = \lim_{k \to \infty} \langle b_k \varphi, \psi \rangle = \lim_{k \to \infty} \int u_k \overline{\mathcal{A}_\psi(\varphi v_k)}dx, \quad (2.10) \]

for all \( \varphi \in S(\mathbb{R}^d), \psi \in C^\infty(\mathbb{R}^{d-1}) \), where \( (u_k) \) is a subsequence of \( (u_n) \) and \( (v_k) \) is a subsequence of \( (v_n) \) corresponding to \( (b_k) \). Now, we will use the factorization property of \( S(\mathbb{R}^d) \), [15]: Every \( \varphi \in S(\mathbb{R}^d) \) can be written as \( \varphi = \overline{\varphi_1} \varphi_2 \), for some \( \varphi_1, \varphi_2 \in S(\mathbb{R}^d) \). Then

\[ \langle \mu, \varphi \psi \rangle = \lim_{k \to \infty} \int u_k \overline{\mathcal{A}_\psi(\overline{\varphi_1} \varphi_2 v_k)}dx. \]

Since \( \|\varphi_2 v_k\|_{L^2} \leq \|v_k\|_{L^2} \|\varphi_2\|_{L^{\frac{2d}{d-2}}} \), we can apply the commutation lemma to \( \varphi_2 v_k \in L^2 \cap L^q \) and \( \overline{\varphi_1} \in S(\mathbb{R}^d) \subset C_0(\mathbb{R}^d) \) to obtain that for every \( \varphi_1, \varphi_2 \in S(\mathbb{R}^d) \) and \( \psi \in C^\infty(\mathbb{R}^{d-1}) \),

\[ \langle \mu, \overline{\varphi_1} \varphi_2 \psi \rangle = \lim_{k \to \infty} \int_{\mathbb{R}^d} \varphi_1 u_k \overline{\mathcal{A}_\psi(\varphi_2 v_k)}dx. \quad (2.11) \]

This completes the proof of Theorem 2.4 for \( 1 < p \leq 2 \).

In the case when \( p > 2 \), we define \( \mu_n(\varphi, \psi) := \int_{\mathbb{R}^d} \mathcal{A}_\psi(\varphi u_n)\overline{v_n}dx \).
Then, in the same way as above, but now with the change of the roles of \((u_n)\) and \((v_n)\), we use factorization \(\varphi = \varphi_1 \varphi_2\), then the commutation lemma on \(\varphi_1 u_n \in L^2(\mathbb{R}^d)\) and apply the preceding proof. \(\square\)

**Remark 1.** The formulation of the previous theorem can be slightly changed in the case when \(p \in (1, 2)\). Then, instead of \(v_n \to 0\) in \(L^q\) we can assume that \(v_n \to 0\) in \(L^r\) for some \(r \geq q\) and obtain the same result as in Theorem 2.1. In that case for every \(\varphi \in \mathcal{S}(\mathbb{R}^d)\), \(\varphi v_n \in L^q \cap L^2\) and the same proof can be applied. The same idea but with compactly supported \(\varphi\) was used in [5].

### 3. H-Distribution and Sobolev Spaces

The next theorem determines H-distributions associated with sequences in Sobolev space.

**Theorem 3.1.** If a sequence \(u_n \to 0\) weakly in \(W^{-k,p}(\mathbb{R}^d)\) and \(v_n \to 0\) weakly in \(W^{k,\tilde{q}}(\mathbb{R}^d)\), then there exist subsequences \((u_{n'})\), \((v_{n'})\) and a distribution \(\mu \in \mathcal{S}'(\mathbb{R}^d)\) such that for every \(\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)\) and every \(\psi \in C^\infty(\mathbb{S}^{d-1})\),

\[
\lim_{n' \to \infty} \langle A_\psi(\varphi_1 u_{n'}), \varphi_2 v_{n'} \rangle = \lim_{n' \to \infty} \langle \varphi_1 u_{n'}, A_\psi(\varphi_2 v_{n'}) \rangle = \langle \mu, \varphi_1 \varphi_2 \psi \rangle.
\]

(3.1)

**Proof.** Since \(u_n \to 0\) in \(W^{-k,p}(\mathbb{R}^d)\), there exist a subsequence \(u_{n'} \to 0\) such that \(u_{n'} = \sum_{|\alpha| \leq k} \partial^\alpha g_{\alpha,n'}\), where for every \(|\alpha| \leq k\), \((g_{\alpha,n'})\) is a sequence of \(L^p\)-functions such that \(g_{\alpha,n'} \to 0\) in \(L^p(\mathbb{R}^d)\). Indeed, since a weakly convergent sequence forms a bounded set in \(W^{-k,p}(\mathbb{R}^d)\), using the same proof of the representation theorem for elements of \(W^{-k,p}(\mathbb{R}^d)\), one can obtain the existence of bounded sets \(\{F_{\alpha,n}, n \in \mathbb{N}\}, |\alpha| \leq k\), such that \(u_n = \sum_{|\alpha| \leq k} \partial^\alpha F_{\alpha,n}\). Now, since \(\{F_{\alpha,n}, n \in \mathbb{N}\}\) are bounded in \(L^p(\mathbb{R}^d)\), these sets are weakly precompact and every \(\{F_{\alpha,n}, n \in \mathbb{N}\}\) has a weakly convergent subsequence. By the diagonalization method one can find a subsequence such that \(F_{\alpha,n'} \to f_\alpha \in L^p(\mathbb{R}^d), n' \to \infty, |\alpha| \leq k\), in \(L^p(\mathbb{R}^d)\). Since \(\sum_{|\alpha| \leq k} \partial^\alpha F_{\alpha,n'} \to 0\), it follows that \(\sum_{|\alpha| \leq k} \partial^\alpha f_\alpha = 0\). Thus we obtain required subsequence \(u_{n'} = \sum_{|\alpha| \leq k} \partial^\alpha (F_{\alpha,n'} - f_\alpha)\). In the sequel we will not relabel subsequences, so we will use \(u_n\) instead of \(u_{n'}\).

Since

\[
\partial^\alpha [A_\psi(u)] = A_{\psi_\alpha}(u) = A_\psi(\partial^\alpha u), \quad \text{for } \psi_\alpha(\xi) = (2\pi i)^{|\alpha|} \xi^\alpha \psi(\xi),
\]

we have that

\[
A_\psi(\varphi_1 \partial^\alpha F_{\alpha,n}) = (-1)^{|\alpha|} \sum_{0 \leq |\beta| \leq |\alpha|} (-1)^{|\beta|} \binom{\alpha}{\beta} \partial^\beta [A_\psi(\partial^\alpha F_{\alpha,n} \partial^\alpha - \varphi_1)],
\]

and so

\[
\langle A_\psi(\varphi_1 u_n), \varphi_2 v_n \rangle = \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \sum_{0 \leq |\beta| \leq |\alpha|} \binom{\alpha}{\beta} \langle A_\psi(F_{\alpha,n} \partial^\alpha - \varphi_1), \partial^\beta \varphi_2 v_n \rangle
\]

\[
= \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \sum_{0 \leq |\beta| \leq |\alpha|} \sum_{0 \leq |\gamma| \leq |\beta|} \binom{\alpha}{\beta} \partial^\gamma \langle A_\psi(F_{\alpha,n} \partial^\alpha - \varphi_1), \partial^\beta \varphi_2 \partial^\gamma v_n \rangle.
\]

(3.2)
For the moment, we fix $\alpha$ and apply Theorem 2.1 to $F_{\alpha,n} \to 0$ in $L^p(\mathbb{R}^d)$ and $v_n \to 0$ in $L^q(\mathbb{R}^d)$, thus obtaining subsequences $(F_{\alpha,n_0})_{n_0}$, $(v_{\alpha,n_0})_{n_0}$ and an H-distribution $\mu_{\alpha,0} \in SE'(\mathbb{R}^d \times \mathbb{S}^{d-1})$, such that

$$\langle \mu_{\alpha,0}, \varphi_2 \rangle := \lim_{n_0 \to \infty} \langle A_\psi (\varphi_1 F_{\alpha,n_0}), \varphi_2 v_{\alpha,n_0} \rangle.$$ 

Then, applying Theorem 2.1 to $F_{\alpha,n_0} \to 0$ in $L^p(\mathbb{R}^d)$, and $\partial^{(1,0,\ldots,0)} v_{\alpha,n_0} \to 0$ in $L^q(\mathbb{R}^d)$, we obtain subsequences $(F_{\alpha,n_{(1,0,\ldots,0)}})_{n_{(1,0,\ldots,0)}}$, $(v_{\alpha,n_{(1,0,\ldots,0)}})_{n_{(1,0,\ldots,0)}}$ and an H-distribution $\mu_{\alpha,(1,0,\ldots,0)} \in SE'(\mathbb{R}^d \times \mathbb{S}^{d-1})$. Thus, we obtain finitely many H-distributions $\mu_{\alpha,\gamma}$, $0 \leq \gamma \leq \alpha$, such that

$$\langle \mu_{\alpha,\gamma}, \varphi_2 \rangle := \lim_{n_\gamma \to \infty} \langle A_\psi (\varphi_1 F_{\alpha,n_\gamma}), \varphi_2 \partial^{\gamma} v_{\alpha,n_\gamma} \rangle.$$ 

The last one $\mu_{\alpha,\alpha}$ is obtained together with subsequences $(F_{\alpha,n_\alpha})_{n_\alpha}$, $(v_{\alpha,n_\alpha})_{n_\alpha}$ which we are going to use to define H-distribution $\mu^\alpha$ in the following way: For $\varphi_1, \varphi_2 \in S(\mathbb{R}^d)$, $\psi \in C^\infty(\mathbb{S}^{d-1})$,

$$\langle \mu^\alpha, \varphi_1 \varphi_2 \rangle := (-1)^{|\alpha|} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \langle \mu_{\alpha,\gamma}, \partial^{\alpha-\beta} \varphi_1 \partial^{\beta-\gamma} \varphi_2 \psi \rangle.$$ 

The sum on the right-hand side is finite and all H-distributions $\mu_{\alpha,\gamma}$ can be defined via $(F_{\alpha,n_\alpha})_{n_\alpha}$ which is subsequence of $F_{\alpha,n_\gamma}$ and $(v_{\alpha,n_\alpha})_{n_\alpha}$ which is subsequence of $(v_{\alpha,n_\gamma})_{n_\gamma}$ so the H-distribution $\mu^\alpha$ is well defined.

Let us emphasize that we have obtained $\mu^\alpha$ for a fixed $\alpha$. Now if we take first $\alpha = 0$ with previous procedure, we can obtain H-distribution $\mu^0$ defined via $(F_{0,n_0})_{n_0}$ and $(v_{n_0})_{n_0}$. Then, starting with $(F_{e_1,n_0})_{n_0}$ and $(v_{e_1,n_0})_{n_0}$ we obtain (by the same procedure) H-distribution $\mu^{e_1}$ defined via $(F_{e_1,n_{e_1}})_{n_{e_1}}$ and $(v_{e_1,n_{e_1}})_{n_{e_1}}$. Here $e_1 = (1,0,\ldots,0)$. Then we proceed with $e_2 = (0,1,0,\ldots,0)$ to obtain H-distribution $\mu^{e_2}$ and so on with all $|\alpha| \leq k$.

At the end we obtain H-distribution $\mu$ defined by

$$\langle \mu, \varphi_1 \varphi_2 \rangle := \sum_{|\alpha| \leq k} (-1)^{|\alpha|} \sum_{0 \leq \beta \leq \alpha} \binom{\alpha}{\beta} \sum_{0 \leq \gamma \leq \beta} \binom{\beta}{\gamma} \langle \mu_{\alpha,\gamma}, \partial^{\alpha-\beta} \varphi_1 \partial^{\beta-\gamma} \varphi_2 \psi \rangle.$$ 

and subsequences $(\sum_{|\alpha| \leq k} \partial^\alpha F_{\alpha,n_{(0,\ldots,0,k)}})_{n_{(0,\ldots,0,k)}}$ and $(v_{n'}) \equiv v_{(0,\ldots,0,k),n_{(0,\ldots,0,k)}}$.

Distribution $\mu$ obtained in Theorem 3.1 is called $H$-distribution corresponding to the (sub)sequence $(u_{n'}, v_{n'})$.

Assume that the distributions $\mu$ determined by Theorem 3.1 are equal to zero. Then the local strong convergence in $W^{-k,p}(\mathbb{R}^d)$ easily follows. We will prove a more delicate assertion in the next theorem.

**Theorem 3.2.** Let $u_n \to 0$ in $W^{-k,p}(\mathbb{R}^d)$. If for every sequence $v_n \to 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding H-distribution is zero, then for every $\theta \in S(\mathbb{R}^d)$, $\theta u_n \to 0$ strongly in $W^{-k,p}(\mathbb{R}^d)$, $n \to \infty$.

**Proof.** For the strong convergence we need to prove that for every $\theta \in S(\mathbb{R}^d)$ sup $\{\langle \theta u_n, \phi \rangle : \phi \in B \} \to 0$, $n \to \infty$, for every bounded $B \subseteq W^{k,q}(\mathbb{R}^d)$.
If it would not be true, then there would exist $\theta \in \mathcal{S}(\mathbb{R}^d)$, a bounded set $B_0$ in $W^{k,q}(\mathbb{R}^d)$, an $\varepsilon_0 > 0$ and a subsequence $(\theta u_k) \subset (\theta u_n)$, such that 

$$\sup\{ |\langle \theta u_k, \phi \rangle | : \phi \in B_0 \} \geq \varepsilon_0, \quad \text{for every} \; k \in \mathbb{N}.$$ 

Choose $\phi_k \in B_0$ such that $|\langle \theta u_k, \phi_k \rangle | > \varepsilon_0/2$. Since $\phi_k \in B_0$ and $B_0$ is bounded in $W^{k,q}(\mathbb{R}^d)$, $(\phi_k)$ is weakly precompact in $W^{k,q}(\mathbb{R}^d)$, i.e., up to a subsequence, $\phi_k \rightharpoonup \phi_0$ in $W^{k,q}(\mathbb{R}^d)$. Moreover, since $\phi_0$ is fixed, $(u_k, \phi_0) \to 0$ and

$$|\langle \theta u_k, \phi_k - \phi_0 \rangle | > \frac{\varepsilon_0}{4}, \quad k > k_0. \quad (3.3)$$

Applying Theorem 3.1 on $u_k \to 0$ and $\phi_k - \phi_0 \to 0$, we obtain that for every $\varphi_1, \varphi_2 \in \mathcal{S}(\mathbb{R}^d)$

$$\lim_{k \to \infty} W^{-k,p} A_\psi(\varphi_1 u_k), \varphi_2(\phi_k - \phi_0))_{W^{k,q}} = 0. \quad (3.4)$$

With $\psi \equiv 1$ on $\mathbb{S}^{d-1}$, (3.4) implies

$$\lim_{k \to \infty} \langle \varphi_1 u_k, \varphi_2(\phi_k - \phi_0) \rangle = 0.$$ 

Again, we use the factorization property of $\mathcal{S}(\mathbb{R}^d)$. So if $\theta \in \mathcal{S}(\mathbb{R}^d)$, then $\theta = \phi_1 \phi_2$, for some $\phi_1, \phi_2 \in \mathcal{S}(\mathbb{R}^d)$, and we have that $\lim_{k \to \infty} \langle \phi_1 u_k, \phi_2(\phi_k - \phi_0) \rangle = 0$, i.e., $\lim_{k \to \infty} \langle \theta u_k, \phi_0 \rangle = 0$. This contradicts (3.3) and completes the proof.

3.1. Localization Property

Recall [20, p. 117], the Riesz potential of order $s$, $\text{Re}(s) > 0$ is the operator $I_s = (-\Delta)^{-\frac{s}{2}}$, see also [9]. Consideration of the Fourier transform and convolution theorem reveals that $I_s$, for $0 < \alpha < d$, is a Fourier multiplier, i.e., $\mathcal{F}[I_s f](\xi) = (2\pi |\xi|)^{-\alpha} \mathcal{F}[f](\xi)$. We will use the potential $I_1$ with the following properties:

$$\| I_1(f) \|_{L^q} \leq C \| f \|_{L^q}, \quad \text{for} \; f \in L^q(\mathbb{R}^d), \quad 1 < q < d; \quad (3.5)$$

$$\partial_j I_1(f) = - R_j(f), \quad f \in L^q(\mathbb{R}^d), \quad \text{where} \; R_j := A_{\xi_j/|\xi|}. \quad (3.6)$$

Moreover, $R_j : L^q \to L^q$ is continuous.

Consider now a sequence $u_n \to 0$ in $W^{-k,p}(\mathbb{R}^d)$ satisfying the following sequence of equations:

$$\sum_{i=1}^d \partial_i \left( A_i(x) u_n(x) \right) = f_n(x), \quad (3.7)$$

where $A_i \in \mathcal{S}(\mathbb{R}^d)$ and $f_n$ is a sequence of temperate distributions such that $\varphi f_n \to 0$ in $W^{-k-1,p}(\mathbb{R}^d)$, for every $\varphi \in \mathcal{S}(\mathbb{R}^d)$. \quad (3.8)

**Theorem 3.3.** Let $1 < q < d$. If $u_n \to 0$ in $W^{-k,p}(\mathbb{R}^d)$ satisfies (3.7), (3.8), then for any sequence $v_n \to 0$ in $W^{k,q}(\mathbb{R}^d)$ the corresponding $H$-distribution $\mu$ satisfies

$$\sum_{j=1}^d A_j(x) \xi_j \mu(x, \xi) = 0 \quad \text{in} \; \mathcal{SE}'(\mathbb{R}^d \times \mathbb{S}^{d-1}). \quad (3.9)$$
Moreover, if \(3.9\) implies \(\mu(x, \xi) = 0\), we have the strong convergence \(\theta u_n \to 0\), in \(W^{-k,p}(\mathbb{R}^d)\), for every \(\theta \in \mathcal{S}(\mathbb{R}^d)\).

**Proof.** Let \(v_n \to 0\) in \(W^{k,q}(\mathbb{R}^d)\), \(\varphi_1 \in \mathcal{S}(\mathbb{R}^d)\), \(\varphi_2 \in \mathcal{S}(\mathbb{R}^d)\) and let \(\psi \in C^\infty(\mathbb{S}^{d-1})\). We have to prove \(3.9\), i.e., [multiplying \(3.9\) by \((i\xi)^{-1}\), \(|\xi| \neq 0\)] that, up to a subsequence,

\[
0 = \sum_{j=1}^{d} \left( \mu \cdot A_j \varphi_1 \varphi_2 \frac{\xi_j}{i|\xi|} \psi \right) = \lim_{n \to \infty} \sum_{j=1}^{d} \left( u_n A_j \varphi_1, A_{\Psi_j} (\varphi_2 v_n) \right), \tag{3.10}
\]

where \(\Psi_j = \frac{\xi_j}{i|\xi|} \psi(\frac{\xi}{|\xi|})\). Moreover, \(A_{\Psi_j} = -R_j \circ A_{\bar{\psi}} = \partial_j I_1 \circ A_{\bar{\psi}}\), see \(3.6\). Thus, \(3.10\) is equivalent to

\[
\lim_{n \to \infty} \left( \sum_{j=1}^{d} \partial_j (u_n A_j), \varphi_1 I_1(A_{\bar{\psi}}(\varphi_2 v_n)) \right) + \sum_{i=1}^{d} \lim_{n \to \infty} \left( u_n A_j, \partial_j (\varphi_1) I_1(A_{\bar{\psi}}(\varphi_2 v_n)) \right) = 0. \tag{3.11}
\]

Since \(A_{\bar{\psi}}(\varphi_2 v_n) \in W^{k,q}(\mathbb{R}^d)\) it follows from \(3.5\) that

\[
\partial^\alpha I_1(A_{\bar{\psi}}(\varphi_2 v_n)) = I_1(A_{\bar{\psi}}(\partial^\alpha(\varphi_2 v_n))) \in L^{\frac{qd}{d-q}}(\mathbb{R}^d), \quad \text{for all } 0 \leq |\alpha| \leq k. \tag{3.12}
\]

Now, since \(q < \frac{qd}{d-q}\), we have that for all \(\varphi \in \mathcal{S}(\mathbb{R}^d)\),

\[
\|\varphi I_1(A_{\bar{\psi}}(\varphi_2 v_n))\|_{L^q} \leq \|I_1(A_{\bar{\psi}}(\varphi_2 v_n))\|_{L^{\frac{qd}{d-q}}} \|\varphi\|_{L^d}. \tag{3.13}
\]

From \(3.12\) and \(3.13\) we see that

\[
\partial^\alpha [\varphi I_1(A_{\bar{\psi}}(\varphi_2 v_n))] \in L^q(\mathbb{R}^d), \quad \text{for all } 0 \leq |\alpha| \leq k.
\]

Now,

\[
\partial^{\alpha+\epsilon_j} [I_1(A_{\bar{\psi}}(\varphi_2 v_n))] = -R_j(A_{\bar{\psi}}(\partial^\alpha(\varphi_2 v_n))) \in L^q(\mathbb{R}^d),
\]

which gives us that for all \(\varphi \in \mathcal{S}(\mathbb{R}^d)\),

\[
\varphi I_1(A_{\bar{\psi}}(\varphi_2 v_n)) \in W^{k+1,q}(\mathbb{R}^d),
\]

and moreover

\[
\varphi I_1(A_{\bar{\psi}}(\varphi_2 v_n)) \to 0 \quad \text{in } W^{k+1,q}(\mathbb{R}^d). \tag{3.14}
\]

Take \(\bar{\varphi}_1 = \bar{\varphi}_{11} \bar{\varphi}_{12}\) all in \(\mathcal{S}(\mathbb{R}^d)\). From \(3.8\) and \(3.14\) we conclude that

\[
\left( \varphi_{11} f_n, \bar{\varphi}_{12} I_1(A_{\bar{\psi}}(\varphi_2 v_n)) \right) \to 0.
\]

From here and \(3.7\) we conclude that the first term in \(3.11\) converges to zero.

Now we analyze the second term in \(3.11\). We will prove that \(\partial_j(\bar{\varphi}_1) I_1(A_{\bar{\psi}}(\varphi_2 v_n))\) converges strongly to zero in \(W^{k,q}(\mathbb{R}^d)\). For that purpose we write \(\partial_j \bar{\varphi}_1 = \bar{\varphi}_{13} \bar{\varphi}_{14}\), all in \(\mathcal{S}(\mathbb{R}^d)\), and denote by \(L_m\) the open ball centered at the origin with radius \(m \in \mathbb{N}\). By Rellich lemma \(W^{k+1,q}(L_m)\) is compactly embedded in \(W^{k,q}(L_m)\). Since \(\bar{\varphi}_{14} I_1(A_{\bar{\psi}}(\varphi_2 v_n))\) weakly converges
to zero in $W^{k+1,q}(\mathbb{R}^d)$, by the diagonalization procedure we can extract a subsequence (not relabeled) such that for all $m \in \mathbb{N}$

$$\varphi_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n)) \longrightarrow 0 \text{ in } W^{k,q}(L_m).$$ (3.15)

Take smooth cutoff functions $\chi_m$ such that $\chi_m(x) = 1$ for $x \in L_m$ and $\chi_m(x) = 0$ for $x \in \mathbb{R}^d \setminus L_{m+1}$ and write $\varphi_{13} = \chi_m \varphi_{13} + (1 - \chi_m)\varphi_{13}$. We have that

$$\|\varphi_{13}\varphi_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n))\|_{W^{k,q}} \leq \sup_{|\alpha| \leq k, |x| > m} |\partial^\alpha \varphi_{13}| \|\varphi_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n))\|_{W^{k,q}}$$

(3.16)

$$+ \|\chi_m \varphi_{13}\varphi_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n))\|_{W^{k,q}}.$$ (3.17)

Let $\varepsilon > 0$. The sequence $\varphi_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n))$ is bounded in $W^{k,q}(\mathbb{R}^d)$, i.e. there is $M > 0$ such that $\|\varphi_{14}I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n))\|_{W^{k,q}} \leq M$. Since $\varphi_{13} \in \mathcal{S}(\mathbb{R}^d)$, there exists $m_0 \in \mathbb{N}$ such that for all $m \geq m_0$

$$\sup_{|\alpha| \leq k, |x| > m} |\partial^\alpha \varphi_{13}| < \frac{\varepsilon}{2M}.$$ 

Next, from (3.15) we have that (3.17) goes to zero as $n \to \infty$. So, for given $\varepsilon$, there exists $n_0 \in \mathbb{N}$ such that (3.17) is less than $\varepsilon/2$ for all $n \geq n_0$. Thus the left hand side in (3.16) is less than $\varepsilon$ for $n > n_0$, i.e., $\partial_j(\varphi_1)I_1(\mathcal{A}_{\bar{\psi}}(\varphi_2 v_n))$ converges strongly in $W^{k,q}(\mathbb{R}^d)$ and (3.11) holds, which completes the proof of (3.9).

If coefficients $A_j$ are such that $\sum_{j=1}^d A_j(x)\xi_j \neq 0, \xi \in \mathbb{S}^{d-1}$, then Theorem 3.2 implies the strong convergence $\theta u_n \longrightarrow 0$, for every $\theta \in \mathcal{S}(\mathbb{R}^d).$  

4. Appendix

We give here the full description of spaces $\mathcal{S}\mathcal{E}$ and $\mathcal{S}\mathcal{E}'$ introduced in Sect. 2.1. We recall from [6, Section 3.8.] the basic properties of Sobolev spaces on the unit sphere with respect to the surface measure $dS^{d-1}$. In the sequel we assume that $d > 2$. Let $\Omega_l = \{x \in \mathbb{R}^d; |x| \in [1 - l, 1 + l]\}, 0 < l < 1$, and $k \in \mathbb{N}_0$. Then $\phi \in C^k(\mathbb{S}^{d-1})$ if for some and hence for all $0 < l < 1$, $\phi^* \in C^k(\Omega_l)$, where $\phi^*(x) = \phi(x/|x|)$. Moreover, [6, p. 9], $C^k(\mathbb{S}^{d-1})$ is equipped with the norm

$$p_{\mathbb{S}^{d-1},k}(\phi) = |\phi|_{C^k(\mathbb{S}^{d-1})} = \sup_{|\alpha| \leq k, x \in \Omega_l} |\partial^\alpha \phi^*(x)|,$$ (4.1)

and this norm does not depend on $l \in (0, 1)$. Then $C^\infty(\mathbb{S}^{d-1}) = \bigcap_{k \in \mathbb{N}_0} C^k(\mathbb{S}^{d-1})$. The completion of $C^\infty(\mathbb{S}^{d-1})$ with respect to the norm

$$\|v\|_{H^s(\mathbb{S}^{d-1})} = \left\|\left(-\Delta^* + \left(\frac{d - 2}{2}\right)^2\right)^{s/2} v\right\|_{L^2(\mathbb{S}^{d-1})},$$

where $\Delta^*$ is the Laplace–Beltrami operator, is the Sobolev space $H^s(\mathbb{S}^{d-1})$, $s \in \mathbb{N}_0$. The case $d = 2$, when $\mathbb{S}^1$ is given by $x = \cos \theta, y = \sin \theta, \theta \in [0, 2\pi),$
is the simple one which we do not consider. Note, in this case one can take $-\Delta^* + 1$ instead of $-\Delta^* + ((d-2)/2)^2$.

Denote by $\{Y_{n,j}, 1 \leq j \leq N_{n,d}, n \in \mathbb{N}_0\}$ the orthonormal basis of $L^2(S^{d-1})$ (cf. [6, p. 121] or [19, Proposition 10.2, p. 92]), where $N_{n,d} \sim O(n^{d-2})$, [6, p. 16], is the dimension of the set of independent spherical harmonics $Y_{n,j}$ of order $n$. Then we have

$$v \in H^s(S^{d-1}) = \left\{ v \in L^2(S^{d-1}) : \left( \sum_{n=0}^{\infty} \sum_{j=1}^{N_{n,d}} \left( n + \frac{d-2}{2} \right)^{2s} \|v_{n,j}\|^2 \right)^{1/2} < \infty \right\},$$

where $v_{n,j} = \int_{S^{d-1}} v Y_{n,j} \, dS^{d-1}$.

The space $C^\infty(S^{d-1})$, supplied by the sequence of norms (4.1), $k \in \mathbb{N}_0$, is denoted by $\mathcal{E}(S^{d-1})$. By the Sobolev lemma for compact manifolds [7, Theorems 2.20, 2.21 (see also Theorem 2.10)], explicitly written in [19, Theorem 7.6, p. 61], we have that

$$\mathcal{E}(S^{d-1}) = \bigcap_{s \in \mathbb{N}_0} H^s(S^{d-1}).$$

This is a Fréchet space. Since all elements of $C^\infty(S^{d-1})$ are compactly supported, we also have that $\mathcal{E}(S^{d-1}) = D(S^{d-1})$.

By [16, Theorem II.10, p. 52], if we have orthonormal bases $(\psi_n)_{n \in \mathbb{N}}$ and $(\tilde{\psi}_m)_{m \in \mathbb{N}}$ for $L^2(\mathbb{R}^d, dx)$ with Lebesgue measure $dx$ and $L^2(S^{d-1}, dS^{d-1})$ with surface measure $dS^{d-1}$ respectively, then $\psi_n(t_1)\tilde{\psi}_m(t_2)$, $(t_1, t_2) \in \mathbb{R}^d \times S^{d-1}$, is an orthonormal basis for $L^2(\mathbb{R}^d \times S^{d-1})$.

By [16, Appendix to V.3, p. 141] (where the case $d = 1$ is treated), one has that the product of one-dimensional harmonic oscillators $N_x = N_1 \ldots N_d$, $N_i = x_i^2 - (d/dx_i)^2, i = 1, \ldots, d$, and the Hermite basis $h_n(x) = h_{n_1}(x_1) \ldots h_{n_d}(x_d), n \in \mathbb{N}_0^d$, of $L^2(\mathbb{R}^d)$ satisfy

$$N_x^k h_n = (2n_1 + 1)^k \ldots (2n_d + 1)^k h_n, \quad n \in \mathbb{N}_0^d, \quad k \in \mathbb{N}_0.$$  

Moreover, $S(\mathbb{R}^d)$ is determined by the sequence of norms

$$|||\phi|||_k = |||N^k \phi|||_2 = \sum_{n \in \mathbb{N}_0^d} (2n_1 + 1)^{2k} \ldots (2n_d + 1)^{2k}|a_n|^2, \quad k \in \mathbb{N},$$

where $\phi = \sum_{n \in \mathbb{N}_0^d} a_n h_n \in S(\mathbb{R}^d)$. This sequence of norms is equivalent to the usual one for $S(\mathbb{R}^d)$.

Now, we define the space of smooth functions $\mathcal{SE}(\mathbb{R}^d \times S^{d-1})$ by the sequence of norms (2.1). By the quoted Sobolev lemma for compact manifolds [7] and (4.3), we have the next proposition.

**Proposition 4.1.** The family of norms (2.1) is equivalent to any of the following two families of norms:

$$p^2_{R^d \times S^{d-1}, k}(\theta) = \left( \int_{R^d \times S^{d-1}} |N_x^k(\Delta^*_x)^\alpha \partial_x^\beta \theta(x, \xi)|^2 \, dx d\xi \right)^{1/2},$$

$$p_{R^d \times S^{d-1}, k}(\theta) = \sup_{(x, \xi) \in R^d \times \Omega, |\alpha + \beta| \leq k} \langle x \rangle^k |\partial_x^\alpha \partial_\xi^\beta \theta^*(x, \xi)|,$$
where \( \theta^*(x, \xi) = \theta(x, \xi/|\xi|) \), \( \langle x \rangle^k = (1 + |x|^2)^{k/2} \) and the derivatives with respect to \( \xi \) are defined as above, with fixed \( x \).

In particular, \( SE(\mathbb{R}^d \times S^{d-1}) \) is a Fréchet space.

Note that \( SE(\mathbb{R}^d \times S^{d-1}) \) induces the \( \pi \)-topology on \( S(\mathbb{R}^d) \otimes E(S^{d-1}) \), see [22, Chap. 43] for the \( \pi \)-topology. Since \( S(\mathbb{R}^d) \) is nuclear, the completion \( S(\mathbb{R}^d) \otimes E(S^{d-1}) \) is the same for the \( \pi \) and the \( \varepsilon \) topologies, cf. [22, Part III, Chap. 50, Theorem 50.1, p. 511].

Proposition 4.2.

\[
S(\mathbb{R}^d) \otimes E(S^{d-1}) = SE(\mathbb{R}^d \times S^{d-1}). \tag{4.7}
\]

Proof. Clearly the embedding \( S(\mathbb{R}^d) \otimes E(S^{d-1}) \rightarrow SE(\mathbb{R}^d \times S^{d-1}) \) is continuous. Thus for the proof of (4.7), it is enough to prove that the left space is dense in the right one. As for general manifolds, we have that \( h_m(x) \times Y_{n,j}(\xi), \) \( 1 \leq j \leq N_{n,d}, n, m \in \mathbb{N}, \) is an orthonormal basis for \( L^2(\mathbb{R}^d \times S^{d-1}) \). Now, by (4.3)–(4.5), it follows that

\[
\theta(x, \xi) = \sum_{n=0}^{\infty} \sum_{j=1}^{N_{n,d}} \sum_{m \in \mathbb{N}_0^d} a_{n,j,m} h_m(x) Y_{n,j}(\xi) \in SE(\mathbb{R}^d \times S^{d-1}) \tag{4.8}
\]

if and only if for every \( r > 0 \),

\[
\sum_{n=0}^{\infty} \sum_{j=1}^{N_{n,d}} \sum_{m \in \mathbb{N}_0^d} |a_{n,j,m}|^2 (1 + n^2 + |m|^2)^r < \infty \tag{4.9}
\]

(cf. [23, Chapter 9]). Now taking finite sums of the right-hand side of (4.8), we obtain that \( S(\mathbb{R}^d) \otimes E(S^{d-1}) \) is dense in \( SE(\mathbb{R}^d \times S^{d-1}) \). This completes the proof. \( \square \)

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