The Generalized Bloch Conjecture for the quotient of certain Calabi-Yau varieties

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Abstract

In this paper, the generalized Bloch Conjecture on zero cycles for the quotient of certain complete intersections with trivial canonical bundle is proved to hold.

As an application of Bloch-Srinivas method on the decomposition of the diagonal, we compute the rational coefficient Lawson homology for 1-cycles and codimension two cycles for these quotient varieties. The (Generalized) Hodge Conjecture is proved to hold for codimension two cycles (and hence also for 2-cycles) on these quotient varieties.

Contents

1 Introduction 1
2 Main results 2
3 The proof of main theorems 3
4 Application to 1-cycles and codimension two cycles 8
5 Low dimensional examples 10

1 Introduction

In this paper, all varieties are defined over \(\mathbb{C}\). For a projective variety \(X\), denote by \(Z_p(X)\) the spaces of algebraic \(p\)-cycles and \(\text{Ch}_p(X)\) the Chow group of \(p\)-cycles on \(X\), i.e., \(\text{Ch}_p(X) = Z_p(X)/\{\text{rational equivalence}\}\). Let \(cl_p : \text{Ch}_p(X) \to H_{2p}(X, \mathbb{Z})\) be the cycle class map. Tensoring with \(\mathbb{Q}\), we have \(cl_p \otimes \mathbb{Q} : \text{Ch}_p(X) \otimes \mathbb{Q} \to H_{2p}(X, \mathbb{Q})\). Let \(\text{Ch}_p(X)_{\text{hom}} \subset \text{Ch}_p(X)\) be the subgroup of \(p\)-cycles homologous to zero. Set \(\text{Ch}^q(X) := \text{Ch}_{n-q}(X)\).
In 1968, D. Mumford showed that $\text{Ch}_0(X)_{\text{hom}}$ is not finite dimensional for a smooth projective surface $X$ with non-vanishing geometric genus $p_g(X)$ (cf. [M]). This result was generalized by Roitman to arbitrary dimension (cf. [R]). In this situation, a nontrivial conjecture of Bloch asserts that if a smooth projective surface $X$ with $p_g(X) = 0$, then $\text{Ch}_0(X)_{\text{hom}}$ is finite dimensional ([B1]). Equivalently, if $p_g(X) = 0$, then there is a curve $C \subset X$ such that the natural map $\text{Ch}_0(C) \to \text{Ch}_0(X)$ is surjective.

This conjecture can be generalized as follows (cf. [PV]):

**Conjecture 1.1 (Generalized Bloch Conjecture)** Let $X$ be a smooth projective variety satisfying $H^i,0(X) = 0$ for all $i > r$. Then there is a subvariety $i : Z \hookrightarrow X$, where $\dim Z = r$, such that $i_* : \text{Ch}_0(Z) \to \text{Ch}_0(X)$ is surjective.

Some examples are known in support of these conjectures. For example, Bloch’s conjecture is true for surfaces which are not of general type [BKL]. This conjecture also holds for some surfaces of general type which are quotients of some special surfaces by a free finite group action (cf. [IM], [Vs]). In higher dimensional case, it was proved by Roitman [R] that $\text{Ch}_0(X) \cong \mathbb{Z}$ for smooth projective complete intersection with $H^i,0(X) = 0$ for all $i > 0$. In [BS], it was proved by Bloch and Srinivas that the Generalized Bloch Conjecture holds for Kummer varieties of odd dimensions.

In this paper, the Generalized Bloch Conjecture is proved to hold for quotients of certain even dimensional complete intersection with trivial canonical bundle by a free involution and for the resolution of singularities to the quotients of certain odd dimensional complete intersection with trivial canonical bundle by an involution with isolated fixed points. Based on these results, we compute the rational Lawson homology and verify the Generalized Hodge Conjecture on 1-cycles and codimension-2 cycles for these varieties.

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### 2 Main results

Now let $X^n \subset \mathbb{P}^{2n+1}$ be the complete intersection of quadrics

$$Q_i(z_0, z_1, \ldots, z_n) + Q'_i(z_{n+1}, z_{n+2}, \ldots, z_{2n+1}) = 0, \quad i = 0, 1, \ldots, n,$$

where $Q_i$ and $Q'_i$ are quadratic forms in $n + 1$ variables. For our propose, we assume

$$Q_i(z_0, z_1, \ldots, z_n) = \sum_{j=0}^{n} a_{ij}z_j^2$$

for $i = 0, 1, \ldots, n$. We also assume that $X^n$ is smooth, which holds for the generic choice of $Q'_i$ and the choice of $a_{ij}, i, j = 0, 1, \ldots, n$ such that $\det(a_{ij}) \neq 0$. From the
direct calculation we know \( X^n \) is a Calabi-Yau \( n \)-fold, i.e., \( K_{X^n} \) is trivial and hence \( h^{n,0}(X^n) = \dim H^{n,0}(X^n) = 1 \).

For \( n = 2m \) a positive even integer, we define an involution of \( \mathbb{P}^{2n+1} \) by

\[
\sigma : (z_0 : z_1 : \cdots : z_{2n+1}) \mapsto (-z_0 : -z_1 : \cdots : -z_n, z_{n+1}, \cdots, z_{2n+1})
\]

which takes \( X^n \) to itself. The quotient \( Y^n = X^n/\langle \sigma \rangle \) is a smooth projective variety with \( H^{i,0}(Y^n) = 0 \) for all \( i \geq 1 \) for \( n = 2m \) even (cf. Lemma \[3,2\]). Denote by \( \pi : X^n \to Y^n \) the projection.

Our first main result is following theorem.

**Theorem 2.1** Let \( n \) be a positive even integer. The Generalized Bloch Conjecture holds for \( Y^n \), i.e., for the projective variety \( Y^n = X^n/\langle \sigma \rangle \) above, we have \( \text{Ch}_0(Y^n) = \mathbb{Z} \).

If \( n = 2m - 1 \) is a positive odd integer, then we define another involution of \( \mathbb{P}^{2n+1} \) by

\[
\rho : (z_0 : z_1 : \cdots : z_{2n+1}) \mapsto (z_0 : -z_1 : \cdots : -z_n, z_{n+1}, \cdots, z_{2n+1})
\]

which takes \( X^n \) to itself. The involution \( \rho \) has \( 2^{2m} \) isolated fixed points. Hence the quotient \( Y^{2m-1} = X^{2m-1}/\langle \rho \rangle \) is a projective variety with \( 2^{2m} \) isolated singular points, denote by \( q_i, i = 1, 2, \cdots, 2^{2m} \). Denote also by \( \pi : X^{2m-1} \to Y^{2m-1} \) the projection. Each singular point is a cyclic quotient singular point. Let \( Y^{2m-1} \to Y^{2m-1} \) be a resolution of singularity, then the exceptional divisor \( E_i \) at each singular point \( q_i \) has only normal crossings in \( Y^{2m-1} \) and every irreducible component of \( E_i \) is nonsingular and rational \[Fu\].

Our second main result is the following theorem.

**Theorem 2.2** Let \( n = 2m - 1 \) be a positive odd integer. The Generalized Bloch Conjecture holds for \( Y^n \), i.e., for \( Y^n \) above, we have \( \text{Ch}_0(Y^{2m-1}) \cong \mathbb{Z} \). Moreover, \( \text{Ch}_0(Y^{2m-1}) \cong \mathbb{Z} \).

The application of the main results on algebraic cycles and Lawson homology is given in section \[4\].

### 3 The proof of main theorems

**Lemma 3.1** For the generic choice of \( Q_i' \) and the choice of \( a_{ij}, i, j = 0, 1, \cdots, n \) such that the determinant \( \det(a_{ij}) \) of the matrix \( (a_{ij}) \) is nonzero, then \( X^n \) is a smooth projective variety of dimension \( n \).

**Proof.** It follows from the definition of smoothness of projective variety. Note that \( \det(a_{ij}) \neq 0 \) implies that there is no common solution for the system of equations by \( Q_i = 0 \) and those of partial derivatives. Similarly for the generic choice of \( Q_i' \), there is no common solution for the system of equations \( Q_i', i = 1, 2, \cdots, n + 1 \). \( \square \)
Lemma 3.2 For $n = 2m$ a positive even integer, the quotient $Y^n = X^n/\langle \sigma \rangle$ by the involution $\sigma : X^n \to X^n$ is a smooth projective variety. Moreover, $H^{i,0}(Y^n) = 0$ for all $i > 0$.

Proof. The involution $\sigma : X^n \to X^n$ is induced by the involution of $\mathbb{P}^{2n+1}$ defined by

$$\sigma : (z_0 : z_1 : \cdots : z_{2n+1}) \mapsto (-z_0 : -z_1 : \cdots : -z_n, z_{n+1}, \cdots, z_{2n+1}).$$

By the assumption, the fixed point set of $\sigma : \mathbb{P}^{2n+1} \to \mathbb{P}^{2n+1}$ and $X^n$ have no intersection since the system of equations $Q_i = 0$, $i = 0, 1, \cdots, n$ has no common solution in $\mathbb{P}^n$ by the assumption that $\det(a_{ij}) \neq 0$. Similarly for a generic choice of $Q'$, $i = 0, 1, \cdots, n$. Therefore, $\sigma : \mathbb{P}^{2n+1} \to \mathbb{P}^{2n+1}$ induces a fixed point free involution on $X$ and hence the quotient $Y^n = X^n/\langle \sigma \rangle$ is a smooth projective variety. Note that $H^{i,0}(Y^n) = 0$ for $i > n$ for the reason of dimension. For $i < n$, $\dim H^{i,0}(Y^n) \leq \dim H^{i,0}(X^n)$ and the latter is zero by Lefschetz hyperplane Theorem. For $i = n$, we have $2 = \chi(O_{X^n}) = 2\chi(O_{Y^n}) = 2(1 - \dim H^{1,0}(X^n) + \cdots + (-1)^n \dim H^{n,0}(Y^n)) = 2(1 + \dim H^{n,0}(Y^n))$, where the second equality holds since $X^n \to Y^n$ is a étale morphism (cf. Example 18.3.9 in [Ful]) and the last equality holds since $n$ is an even integer and so $\dim H^{n,0}(Y^n) = 0$. \hfill \Box

Remark 3.3 Since $H^{i,0}(Y^n) = 0$ for all $i > 0$, the Generalized Bloch Conjecture implies $\text{Ch}_0(Y^n) \otimes \mathbb{Q} = \mathbb{Q}$. The statement in Theorem 2.1 is slightly stronger than this.

A well-known result is needed in our computation.

Lemma 3.4 Suppose a finite group $G$ acts on a variety $X$ with nonsingular quotient variety $Y = X/G$. Let $\pi : X \to Y$ be the quotient map. Then there exist two homomorphisms $\pi_* : \text{Ch}_0(X)_{\text{hom}} \to \text{Ch}_0(Y)_{\text{hom}}$ and $\pi^* : \text{Ch}_0(Y)_{\text{hom}} \to \text{Ch}_0(X)_{\text{hom}}$ such that

$$\begin{cases} 
\pi^* \pi_* & = \sum_{g \in G} g_* \\
\pi_* \pi^* & = N
\end{cases} \tag{4}$$

where $N = |G|$ means the multiplication by $N$ in $\text{Ch}_0(X)_{\text{hom}}$. In particular, $\text{Ch}_0(Y)_{\text{hom}} = 0$ if and only if $\sum_{g \in G} g_* = 0$ in $\text{End}(\text{Ch}_0(Y)_{\text{hom}})$.

Proof. See, e.g. [IM], Lemma 1 and Lemma 2, where $X$ is a surface. The point is that both $\pi_*$ and $\pi^*$ are well-defined. The proof works in higher dimensional case and the case that $Y$ is singular (cf. [Ful]). \hfill \Box

Now we can apply Lemma 3.4 to the quotient map $\pi : X^n \to Y^n$. Let $\pi_* : \text{Ch}_0(X^n) \to \text{Ch}_0(Y^n)$ be the push forward map and let $\pi^* : \text{Ch}_0(Y^n) \to \text{Ch}_0(X^n)$ be the pull back. Then we have $\pi_* \pi^* = 2 : \text{Ch}_0(Y)_{\text{hom}} \to \text{Ch}_0(Y)_{\text{hom}}$ and $\pi^* \pi_* = \sigma_* + 1 : \text{Ch}_0(X)_{\text{hom}} \to \text{Ch}_0(X)_{\text{hom}}$. \hfill \Box
Since $\text{Ch}_0(Y^n)_{\text{hom}}$ is divisible (cf. [B2], [R]), it suffices to show

$$2\text{Ch}_0(Y)_{\text{hom}} = \pi_*\pi^*\text{Ch}_0(Y)_{\text{hom}} = 0.$$ 

Since $\pi_*$ is surjective, it suffices to show $\pi^*\pi_* : \text{Ch}_0(X^n)_{\text{hom}} \to \text{Ch}_0(X^n)_{\text{hom}}$ is the zero map. That is, we need to show that $\sigma_* = -1 : \text{Ch}_0(X^n)_{\text{hom}} \to \text{Ch}_0(X^n)_{\text{hom}}$.

Therefore, Theorem 2.1 follows from the following proposition.

**Proposition 3.5** Let $\sigma : X^n \to X^n$ be induced by the involution in Equation (2). Then $\sigma_* = -1 : \text{Ch}_0(X^n)_{\text{hom}} \to \text{Ch}_0(X^n)_{\text{hom}}$.

To prove this proposition, we need some auxiliary results. Let $\tau_i : X^n \to X^n$ be the automorphism of $X^n$ induced by $\sigma_i : \mathbb{P}^{2n+1} \to \mathbb{P}^{2n+1}$, where

$$\sigma_i : (z_0 : \cdots : z_{i-1} : z_i : z_{i+1} : \cdots : z_{2n+1}) \mapsto (z_0 : \cdots : z_{i-1} : -z_i : z_{i+1} : \cdots : z_{2n+1})$$

for $i = 0, 1, \cdots, n$. Note that $\tau_i$ maps $X^n$ to itself by the assumption of $Q_i$. Then $\tau_i : X^n \to X^n$ induces a homomorphism on Chow group of 0-cycles $\tau_{i*} : \text{Ch}_0(X^n) \to \text{Ch}_0(X^n)$ and so $\tau_{i*} : \text{Ch}_0(X^n)_{\text{hom}} \to \text{Ch}_0(X^n)_{\text{hom}}$. Then we have the following result.

**Lemma 3.6** The homomorphism $\tau_{i*} = -1 : \text{Ch}_0(X^n)_{\text{hom}} \to \text{Ch}_0(X^n)_{\text{hom}}$ for all $i = 0, 1, \cdots, n$.

**Proof.** By the symmetry of $t_0, t_1, \cdots, t_n$, we only need to show the case for $i = 0$. From the definition of $X^n$ and the assumption that the matrix $(a_{ij})$ is non-degenerated, we can make a linear transformation such that there is only one quadratic, say $Q_1 + Q_1'$, depending on the variable $t_0$. Then one can see that the function field of $X^n/\langle \tau_1 \rangle$ is

$$\mathbb{C}(t_1, t_2, \cdots, t_{2n}, t_{2n+1})/\langle Q_i + Q_i' = 0, i = 2, 3, \cdots, n + 1 \rangle.$$ 

Note that the variety $Y_0 \subset \mathbb{P}^{2n}$ defined by equations $Q_i + Q_i' = 0, i = 2, 3, \cdots, n+1$ is a smooth complete intersection. Since the sum of the degrees of the defining equations of $Y_0$ is $\sum_{i=1}^{n} 2 = 2n$, $Y_0$ is a smooth Fano variety. This implies $Y_0$ is rationally connected (cf. [C], [KMM]). Therefore, for any two generic points $p_1, p_2$ on $Y_0$, there is a rational curve $C$ passing through $p, q$.

From the definition, the rational function field of $Y_0$ is also

$$\mathbb{C}(t_1, t_2, \cdots, t_{2n}, t_{2n+1})/\langle Q_i + Q_i' = 0, i = 2, 3, \cdots, n + 1 \rangle.$$ 

So $Y_0$ is birational equivalent to $X^n/\langle \tau_1 \rangle$. Hence, for any generic two points on $X^n/\langle \tau_1 \rangle$, there also exists a curve passing through the two points, i.e., $X^n/\langle \tau_1 \rangle$ is a rationally connected variety. So $\text{Ch}_0(X^n/\langle \tau_1 \rangle) \cong \mathbb{Z}$ and $\text{Ch}_0(X^n/\langle \tau_1 \rangle)_{\text{hom}} = 0$.

This together with Lemma 3.4 implies that $\tau_{i*} + 1 = 0 \in \text{End}(\text{Ch}_0(X^n))_{\text{hom}}$. This completes the proof of the Lemma. \qed
Note that from the definition we have \( \sigma = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n \) and so \( \sigma_* = \tau_0 \circ \tau_1 \circ \cdots \circ \tau_n = (-1)^{n+1} = -1 \) since \( n = 2m \) is an even integer. This completes the proof of Proposition 3.5 and hence Theorem 2.1.

In the following, we focus on the proof of Theorem 2.2. Note that \( n = 2m - 1 \) in the below of this section.

Lemma 3.7 For the generic choice of \( Q_i' \) and \( (a_{ij}) \), the variety \( Y^{2m-1} = X^{2m-1}/(\rho) \) has exact \( 2^m \) isolated singular points.

Proof. Note that the set of singular points on \( Y^{2m-1} \) is exact the set of fixed points of the involution \( \rho : X^{2m-1} \to X^{2m-1} \). This fixed points set is defined by equations

\[
t_1 = t_2 = \cdots = t_n = 0 \quad \text{and} \quad Q_i + Q_i' = 0, \quad i = 0, 1, \ldots, n,
\]

i.e., the intersection of \( \mathbb{P}^{n+1} \) and \( X^n \) in \( \mathbb{P}^{2n+1} \). Note that the degree of \( X^n \) is \( 2^{n+1} = 2^{2m} \) and for a generic choice of \( Q_i' \), equation (5) has no solution of multiplicity bigger than 1. 

By Lemma 3.6, the map \( \rho : X^{2m-1} \to X^{2m-1} \) induces the push forward map \( \rho_* = (-1)^n = (-1)^{2m-1} = -1 : \operatorname{Ch}_0(X^{2m-1}) \to \operatorname{Ch}_0(X^{2m-1}) \). By Lemma 3.3, \( \pi^* \pi_* = \rho_* + 1 = 0 : \operatorname{Ch}_0(Y^{2m-1}) \to \operatorname{Ch}_0(Y^{2m-1}) \) and \( \operatorname{Ch}_0(Y^{2m-1}) = 0 \).

Lemma 3.8 Let \( \phi : \overline{\mathbb{Y}^{2m-1}} \to Y^{2m-1} \) be a resolution of singularity. Then \( \operatorname{Ch}_0(\overline{\mathbb{Y}^{2m-1}}) \cong \mathbb{Z} \).

Proof. For each singular point \( q_i \in \operatorname{Sing}(\mathbb{Y}^{2m-1}) \), \( i = 1, 2, \ldots, 2^{2m} \), the exceptional divisor \( E_i = \phi^{-1}(q_i) \) has normal crossings in \( \overline{\mathbb{Y}^{2m-1}} \) and every irreducible component of \( E_i \) is nonsingular and rational (cf. Corollary after Theorem 1 in [Fuj]). Since each singular point in our case is a quotient singularity of type \( \mathbb{C}^{2m-1}/\mathbb{Z}_2 \), the exceptional divisor \( E_i \) contains exactly one irreducible component, which is isomorphic to \( \mathbb{P}^{2m-2} \) (cf. Remark 3.9).

Set \( E = \bigcup_{i=0}^{2^{2m}} E_i \). Since \( \operatorname{Ch}_0(E_i) = 0 \) and that \( E_i \) are mutually disjoint to each other, we get \( \operatorname{Ch}_0(E) = 0 \). Set \( U = \overline{\mathbb{Y}^{2m-1}} - E \cong \mathbb{Y}^{2m-1} - \bigcup_{i=0}^{2^{2m}} q_i \). Then the isomorphism \( \operatorname{Ch}_0(\overline{\mathbb{Y}^{2m-1}}) \cong \mathbb{Z} \) follows from the fact that \( \operatorname{Ch}_0(\mathbb{Y}^{2m-1}) = 0 \). This fact can be seen from the commutative diagram of Chow groups

\[
\begin{array}{cccccc}
\operatorname{Ch}_0(E) & \xrightarrow{\cong} & \operatorname{Ch}_0(\overline{\mathbb{Y}^{2m-1}}) & \xrightarrow{\phi_*} & \operatorname{Ch}_0(U) & \xrightarrow{\cong} 0 \\
\downarrow{\cong} & & \downarrow{\phi_*} & & \downarrow{\cong} & \\
\operatorname{Ch}_0(\bigcup_{i=0}^{2^{2m}} q_i) & \xrightarrow{\cong} & \operatorname{Ch}_0(\overline{\mathbb{Y}^{2m-1}}) & \xrightarrow{\cong} & \operatorname{Ch}_0(U) & \xrightarrow{\cong} 0.
\end{array}
\]
Remark 3.9 Each singular point of $Y^{2m-1}$ is the quotient singularity of the same type as that of $\mathbb{C}^{2m-1}/\mathbb{Z}_2$, where $\mathbb{Z}_2$ acts on $\mathbb{C}^{2m-1}$ as

$$(x_1, x_2, \cdots, x_{2m-1}) \mapsto (-x_1, -x_2, \cdots, -x_{2m-1}).$$

The singular point of $\mathbb{C}^n/\mathbb{Z}_2$ can be resolved by one blow up with the exceptional divisor $E \cong \mathbb{P}^{n-1}$.

To see this, we first note that all the $\mathbb{Z}_2$-invariant monomials of $x_1, x_2, \cdots, x_n$ are $x_i x_j, 1 \leq i \leq j \leq n$. This gives an embedding $X := \mathbb{C}^n/\mathbb{Z}_2 \hookrightarrow \mathbb{C}^N$, where $N = \binom{n+1}{2} = \frac{1}{2}n(n+1)$. Let $u_{ij}, 1 \leq i \leq j \leq n$ be the coordinates of $\mathbb{C}^N$. Then $\mathbb{C}^n/\mathbb{Z}_2$ is the locus of the ideal generated by all $2 \times 2$ minors of the symmetric matrix $(u_{ij})_{1 \leq i,j \leq n}$, where $u_{ij} := u_{ji}$ if $i > j$.

Let $\widetilde{\mathbb{C}}^N$ be the blow up of $\mathbb{C}^N$ at the origin and let $\tilde{X}$ be the proper transform of $X = \mathbb{C}^n/\mathbb{Z}_2$. A direct calculation shows that $\tilde{X}$ is smooth. The explicit equations for $n = 3$ will be given below while the general case is similar. The exceptional divisor $E$ of $\tilde{X} \rightarrow X$ is just the quadric equation given by those $2 \times 2$ minors in $\mathbb{P}^{N-1}$, i.e., the intersection of $\mathbb{P}^{N-1}$ and $\tilde{X}$. Note that $E \subset \mathbb{P}^{N-1}$ with the above defining equations is exactly the image of the Plücker embedding $\mathbb{P}^{n-1} \hookrightarrow \mathbb{P}^{N-1}$. Therefore, $E \cong \mathbb{P}^{n-1}$.

Now we write down the details for the case that $n = 3$. In this case $N = \binom{4}{2} = 6$. Let $\widetilde{\mathbb{C}}^6 \subset \mathbb{C}^6(u_1, \cdots, u_6) \times \mathbb{P}^5[v_1: \cdots: v_6]$ be defined by $u_i v_j = u_j v_i, 1 \leq i \neq j \leq 6$. Note that $X = \mathbb{C}^3/\mathbb{Z}_2 \subset \mathbb{C}^6$ is defined by

$$\begin{cases} u_1 u_2 = u_6^2, \\ u_1 u_3 = u_5^2, \\ u_2 u_3 = u_4^2. \end{cases}$$

Note that $\widetilde{\mathbb{C}}^6$ is covered by affine open sets $(v_i \neq 0), i = 1, 2, \cdots, 6$. On the affine open piece $v_1 \neq 0$, $\tilde{X}$ is defined by

$$\begin{cases} u_i = u_1 v_i, & i = 2, 3, \cdots, 6, \\ v_2 = v_6^2, \\ v_3 = v_5^2, \\ v_2 v_3 = v_4^2. \end{cases}$$

It is easy to check by the definition of smoothness that this piece of $\tilde{X}$ is smooth. Similarly for all other pieces of $X$. Therefore $\tilde{X}$ is smooth.

The exceptional divisor $E$ is defined by the following equations:

$$\begin{cases} u_i = 0, & i = 1, 2, \cdots, 6, \\ v_1 v_2 = v_6^2, \\ v_2 v_3 = v_5^2, \\ v_2 v_3 = v_4^2. \end{cases}$$

Hence $E$ is isomorphic to the image of Plücker embedding $\mathbb{P}^2 \hookrightarrow \mathbb{P}^5$ and therefore $E \cong \mathbb{P}^2$. 
4 Application to 1-cycles and codimension two cycles

In this section, we deduce a sequence of results on algebraic cycles and cohomology theories for $Y^n$ as the application of the decomposition of the diagonal given by Bloch [B1], Bloch and Srinivas [BS] and the generalization by many others.

First we consider the case that $n = 2m$ is an even positive integer.

**Corollary 4.1** $\text{Ch}^p(Y^{2m})$ is weakly representable for $p \leq 2$.

**Proof.** It follows from Theorem 2.1 and Theorem 1 in [BS]. 

The Hodge Conjecture for codimension $p$ cycles (denote by $\text{Hodge}^{q,q}(X)$): The rational cycle class map

$$cl^q \otimes \mathbb{Q} : \text{Ch}^q(X) \otimes \mathbb{Q} \to H^{q,q}(X) \cap H^{2q}(X, \mathbb{Q})$$

is surjective.

More generally, let $N^p H^k(X, \mathbb{Q}) \subset H^k(X, \mathbb{Q})$ be the arithmetic filtration defined by Grothendieck [G] and let $F^p H^k(X, \mathbb{C}) \subset H^k(X, \mathbb{C})$ be the Hodge filtration. Set $F^p H^k(X, \mathbb{Q}) := F^p H^k(X, \mathbb{C}) \cap H^k(X, \mathbb{Q})$ and denote by $\tilde{F}^p H^k(X, \mathbb{Q})$ the maximal sub-Hodge structure in $F^p H^k(X, \mathbb{Q})$. It was shown in [G] that $N^p H^k(X, \mathbb{Q}) \subset \tilde{F}^p H^k(X, \mathbb{Q})$.

The generalized Hodge Conjecture can be stated as follows (denote by $\text{GHC}(p, k, X)$):

$$N^p H^k(X, \mathbb{Q}) = \tilde{F}^p H^k(X, \mathbb{Q}).$$

**Corollary 4.2** The Hodge Conjecture for $Y^4$ holds. The generalized Hodge Conjecture for $\text{GHC}(1, 4, Y^4)$ holds. More generally, the generalized Hodge Conjecture $\text{GHC}(1, 2m, Y^{2m})$ for $Y^{2m}$ holds.

**Proof.** The first statement follows from Theorem 2.1 and Theorem 1 in [BS]. Similar method can be used to prove $\text{GHC}(1, 4, Y^4)$ and more general statement $\text{GHC}(1, 2m, Y^{2m})$. By Theorem 2.1, $\text{Ch}_0(Y^{2m}) \cong \mathbb{Z}$, we have $\text{GHC}(1, 2m, Y^{2m})$ by Corollary 15.23 in [Le] or Proposition 5.5 in [Vo].

**Remark 4.3** $\text{GHC}(1, 4, Y^4)$ is the only non-trivial part of the generalized Hodge Conjecture for $Y^4$. The Hodge Conjecture for 2-cycles and codimension 2 cycles on $Y^{2m}$ holds, i.e., both $\text{Hodge}^{2m-2, 2m-2}(Y^{2m})$ and $\text{Hodge}^{2, 2}(Y^{2m})$ hold for all positive integer $m$. However, both $\text{Hodge}^{2m-2, 2m-2}(Y^{2m})$ and $\text{Hodge}^{2, 2}(Y^{2m})$ are trivial if $m > 2$ since both $H^4(Y^{2m})$ and $H^{4m-4}(Y^{2m})$ are isomorphic to $\mathbb{Z}$.

Recall that the Lawson homology $L_p H_k(X)$ of $p$-cycles is defined by

$$L_p H_k(X) := \pi_{k-2p}(\mathbb{Z}_p(X)) \quad \text{for} \quad k \geq 2p \geq 0,$$
where \( Z_p(X) \) is provided with a natural topology (cf. [LT], [L1] and [L2]). For general background on Lawson homology, the reader is referred to [L2]. There are natural maps, called \textbf{cycle class maps} \( \Phi_{p,k} : L_pH_k(X) \rightarrow H_k(X) \).

Define

\[
L_pH_k(X)_{\text{hom}} := \ker\{\Phi_{p,k} : L_pH_k(X) \rightarrow H_k(X)\};
\]

\[
L_pH_k(X, \mathbb{Q})_{\text{hom}} := L_pH_k(X)_{\text{hom}} \otimes \mathbb{Q};
\]

\[
T_pH_k(X) := \text{Image}\{\Phi_{p,k} : L_pH_k(X) \rightarrow H_k(X)\};
\]

\[
T_pH_k(X, \mathbb{Q}) := T_pH_k(X) \otimes \mathbb{Q}.
\]

The \textbf{Griffiths group} of \( p \)-cycles is defined to

\[
\text{Griff}_p(X) := \text{Ch}_p(X)_{\text{hom}}/\text{Ch}_p(X)_{\text{alg}},
\]

where \( \text{Ch}_p(X)_{\text{alg}} \) denotes the space of cycles in \( \text{Ch}_p(X) \) which are algebraically equivalent to zero. Set \( \text{Griff}^p(X) := \text{Griff}_{\dim X-p}(X) \). It was shown in [Fr] that \( L_pH_{2p}(X)_{\text{hom}} \simeq \text{Griff}_p(X) \) for any projective variety.

**Corollary 4.4** For every positive integer \( m \), we have \( L_pH_k(Y^{2m}, \mathbb{Q})_{\text{hom}} = 0 \) for \( p \leq 1 \), \( p \geq 2m-2 \) and \( k \geq 2p \). In particular, \( \text{Griff}_1(Y^{2m}) \otimes \mathbb{Q} = 0 \) and \( \text{Griff}^2(Y^{2m}) = 0 \).

**Proof.** Recall that a theorem of Peters [Pe] says that if \( \text{Ch}_0(Y) \otimes \mathbb{Q} \cong \mathbb{Q} \) for a smooth projective variety \( Y \), then \( L_pH_*(Y)_{\text{hom}} = 0 \) for \( p \leq 1 \) and all \(*\). Hence the \( p \leq 1 \) part follows from Theorem 2.1 and Peters’ result. It was observed, independently by M. Voineagu [Vo] and the author [H], that Peters’ method could be used to show \( L_pH_*(Y)_{\text{hom}} = 0 \) for \( p \geq \dim(Y) - 2 \) and all \(*\) under the same assumption. So the \( p \geq 2m-2 \) part follows from Theorem 2.1 and the observation. In particular, \( \text{Griff}_1(Y^{2m}) \otimes \mathbb{Q} = 0 \) and \( \text{Griff}^2(Y^{2m}) \otimes \mathbb{Q} = 0 \). Since \( \text{Griff}^2(Y^n) \) has no torsion [BS], therefore \( \text{Griff}^2(Y^{2m}) = 0 \), i.e., homological equivalence and algebraic equivalence coincide for codimension-2 cycles on \( Y^{2m} \). The completes the proof of Corollary 4.4.

As applications of Theorem 2.2 we have similar results for \( Y^{2m-1} \) as those in Corollary 4.4.

**Corollary 4.5** \( \text{Ch}^p(Y^{2m-1}) \) is weakly representable for \( p \leq 2 \).

**Corollary 4.6** The Generalized Hodge Conjecture for \( \text{GHC}(1, 3, Y^3) \) holds. More generally, the Generalized Hodge Conjecture \( \text{GHC}(1, 2m-1, Y^{2m-1}) \) for \( Y^{2m} \) holds.

**Corollary 4.7** For every integer \( m \geq 2 \), we have \( L_pH_k(Y^{2m-1}, \mathbb{Q})_{\text{hom}} = 0 \) for \( p \leq 1 \), \( p \geq 2m-3 \) and \( k \geq 2p \). In particular, \( \text{Griff}_1(Y^{2m-1}) \otimes \mathbb{Q} = 0 \) and \( \text{Griff}^2(Y^{2m-1}) = 0 \).
Recall that for $V \subset U$ a Zariski open subset of a quasi-projective variety $U$, we have the long exact sequence for Lawson homology, i.e.,
\[ \cdots \to L_pH_k(Z) \to L_pH_k(U) \to L_pH_k(V) \to L_pH_{k-1}(Z) \to \cdots \]
where $Z = U - V$ (cf. [LE]).

By Corollary 4.7 and Equation (6), we get

**Corollary 4.8** For every integer $m \geq 2$, we have $L_pH_k(Y^{2m-1}, \mathbb{Q})_{\text{hom}} = 0$ for $p \leq 1$, $p \geq 2m - 3$ and $k \geq 2p$.

**Proof.** Since $Y^{2m-1}$ is a singular variety, the Bloch-Srinivas method on decompositions of the diagonal does not work for $Y^{2m-1}$. So we try to compute $L_pH_k(Y^{2m-1}, \mathbb{Q})_{\text{hom}}$ by the localization sequences for Lawson homology.

Set $V := Y^{2m-1} - E$ and $E = \cup_{i=1}^{2m} E_i$. Then $V \cong Y^{2m-1} - \cup_{i=1}^{2m} P_i$, where $P_i$, $i = 1, \ldots, 2^m$ are singular points of $Y^{2m-1}$ and $E_i$, $i = 1, \ldots, 2^m$ are the corresponding exceptional divisors. By using Equation (6) to the $U \subset Y^{2m-1}$, we get $L_1H_k(U) \cong L_1H_k(Y^{2m-1})$ and so $L_1H_k(U)_{\text{hom}} \cong L_1H_k(Y^{2m-1})_{\text{hom}}$.

From the following commutative diagram (cf. [LE2], Prop. 4.9)

\[
\begin{array}{cccccc}
L_1H_k(E) & \xrightarrow{\cong} & L_1H_k(Y^{2m-1}) & \xrightarrow{\phi_{1,k}} & L_1H_k(U) & \xrightarrow{\cong} & L_1H_k(Y^{2m-1}) \\
H_k(E) & \xrightarrow{\cong} & H_k(Y^{2m-1}) & \xrightarrow{\Psi_{1,k}} & H_k(\mathbb{B}M) & \xrightarrow{\cong} & H_k(Y^{2m-1}) \\
\end{array}
\]

where $H_k(\mathbb{B}M)$ is the Borel-Moore homology of $U$, and the injectivity of $\Phi_{1,k} \otimes \mathbb{Q}$ (i.e., $L_1H_k(Y^{2m-1}, \mathbb{Q})_{\text{hom}} = 0$ by Corollary 4.7), we get the injectivity of $\Psi_{1,k} \otimes \mathbb{Q}$ by the Five Lemma.

\[
\square
\]

5 **Low dimensional examples**

For a smooth complex projective variety, we set $h^{i,j}(X) := \dim_{\mathbb{C}} H^{i,j}(X)$.

The case $n = 1$ is trivial. In this case $Y^1 \cong \mathbb{P}^1$. In the case $n = 2$, all $Y^2$ are Enrique surfaces. It was proved in [BKL] that all Enrique surfaces $S$ satisfy $\text{Ch}_0(S) \cong \mathbb{Z}$.

The next case is $n = 3$. In this case, $X^3$ (for simplicity denote by $X$ in this paragraph) is the complete intersection of 4 quadric hypersurfaces in $\mathbb{P}^7$. By the adjunction formula, the canonical bundle $K_X$ of $X$ is trivial and so $H^{3,0}(X) \cong \mathbb{C}$ and $h^{3,0}(X) = 1$. The Euler class $\chi(X)$ of $X$ is the top Chern class of $X$ (Gauss-Bonnet Theorem). Let $h$ be the hyperplane class of $X$. The total Chern class $c(X) := 1 + c_1(X) + c_2(X) + c_3(X) = (1 + h)^8(1 + 2h)^{-4} |X|$ and so $c_3(X) = -8h^3 = -128$ since $h^3|_X = \deg(X) = 16$. Hence $\chi(X) = -128$. This together with Lefschetz hyperplane theorem implies $b_3(X) = 132$. By
the Hodge decomposition of $H^3(X, \mathbb{C})$, we get $h^{3,0}(X) + h^{2,1}(X) + h^{1,2}(X) + h^{0,3}(X) = 132$. So $h^{2,1}(X) = h^{1,2}(X) = 65$.

Since $\pi : X^3 \to Y^3$ is an involution with 16 isolated fixed point, we have $\chi(X^3) - 16 = 2(\chi(Y^3) - 16)$ and so $\chi(Y^3) = -56$. Note that $\widetilde{Y}^3 \to Y^3$ is the resolution of singularity with exceptional divisor $E = \bigcup_{i=1}^{16} E_i$, where $E_i \cong \mathbb{P}^2$ since it is the exceptional divisor of the resolution of singularity on $\mathbb{C}^3 / \mathbb{Z}_2$ (cf. Remark 3.9). So $\chi(\widetilde{Y}^3) - 16\chi(\mathbb{P}^2) = \chi(Y^3) - 16$, i.e., $\chi(\widetilde{Y}^3) = -24$. Since $b_2(Y^3) = 1$, we get $b_2(\widetilde{Y}^3) = 17$. Hence $b_1(\widetilde{Y}^3) = 0$, we get $b_3(\widetilde{Y}^3) = 60$. We get $h^{1,2}(\widetilde{Y}^3) = h^{2,1}(\widetilde{Y}^3) = 30$ since $h^{3,0}(\widetilde{Y}^3) = 0$.

Recall that Suslin’s Conjecture on Lawson homology states that: For any abelian group $A$ and smooth quasi-projective variety $X$ of dimension $n$, the map $L_pH_k(X, A) \to H_k^{BM}(X, A)$ is an isomorphism for $k \geq n + p$ and a monomorphism for $k = n + p - 1$. Here $H_k^{BM}(X, A)$ means Borel-Moore homology with coefficient in $A$.

By the above computation, Theorem 2.1 and the Proposition 5.3 in [V o], we have the following result.

**Corollary 5.1** For $\widetilde{Y}^3$ above, we have the following statements:

1. $L_1H_2(\widetilde{Y}^3) \cong H_2(\widetilde{Y}^3) \cong \mathbb{Z}^{17}$.
2. $L_1H_3(\widetilde{Y}^3) \cong H_3(\widetilde{Y}^3) \cong \mathbb{Z}^{60}$.
3. $L_2H_4(\widetilde{Y}^3) \cong L_1H_4(\widetilde{Y}^3) \cong H_4(\widetilde{Y}^3) \cong \mathbb{Z}^{17}$.
4. $L_3H_6(\widetilde{Y}^3) \cong L_2H_6(\widetilde{Y}^3) \cong L_1H_6(\widetilde{Y}^3) \cong H_6(\widetilde{Y}^3) \cong \mathbb{Z}$.
5. All other $L_pH_*(\widetilde{Y}^3)$ ($p \geq 1$) are trivial.

In particular, Suslin’s Conjecture for $\widetilde{Y}^3$ holds.

Next case is $n = 4$. In this case, it can be calculated that $h^{4,0}(X^4) = 1$, $h^{3,1}(X^4) = h^{1,3}(X^4) = 151$ and $h^{2,2}(X^4) = 652$ since $X^4$ is a complete intersection. Since $h^{4,0}(Y^4) = 0$, we get $h^{3,1}(Y^4) = h^{1,3}(Y^4) = 75$ and $h^{2,2}(Y^4) = 326$ by Riemann-Roch-Hirzebruch theorem for orbit spaces (cf. [AS], 4.7), Since $h^{3,1}(Y^4) \neq 0$, $\text{Ch}_1(Y^4)$ is not weakly representable. In particular, $\text{Ch}_1(Y^4)_\text{hom} \otimes \mathbb{Q}$ is nontrivial. From this and the proof of Theorem 2.1 we obtain the Chow group of 1-cycles for $X^4/\langle \tau_i \rangle$ is not weakly representable, although $X^4/\langle \tau_i \rangle$ is a rationally connected variety for each $i = 0, 1, \ldots, n$.

In this case we can say a little more about Lawson homology of $Y^4$. By the above computation, Theorem 2.1 and the Proposition 5.3 in [V o], we have the following result.

**Corollary 5.2** For $Y^4$ above, we have the following statements

1. $L_1H_2(Y^4)_\mathbb{Q} \cong H_2(Y^4)_\mathbb{Q} \cong \mathbb{Q}$.
2. $L_2H_4(Y^4) \hookrightarrow L_1H_4(Y^4) \cong H_4(Y^4) \cong \mathbb{Z}^{476}$.
3. \( L_3H_6(Y^4) \cong L_2H_6(Y^4) \cong L_1H_6(Y^4) \cong H_6(Y^4) \cong \mathbb{Z} \).

4. \( L_4H_8(Y^4) \cong L_3H_8(Y^4) \cong L_2H_8(Y^4) \cong L_1H_8(Y^4) \cong H_8(Y^4) \cong \mathbb{Z} \).

5. All other \( L_pH_s(Y^4) \) (\( p \geq 1 \)) are trivial.

In particular, Suslin’s Conjecture for \( Y^4 \) holds.

Remark 5.3 One can show that \( H^{1,n-1}(Y^{2m}, \mathbb{Q}) \) is nontrivial for all \( n = 2m \geq 4 \). Therefore \( Ch_1(Y^{2m})_{hom} \otimes \mathbb{Q} \neq 0 \) and Theorem 2.7 is the best result we could obtain. Similarly, one can show that \( H^{1,n-1}(Y^{2m-1}, \mathbb{Q}) \) is nontrivial for all \( n = 2m - 1 \geq 3 \) and Theorem 2.2 is the best result.

References

[AS] M. F. Atiyah and I. M. Singer, The index of elliptic operators. III. Ann. of Math. (2) 87 1968 546–604.

[B1] S. Bloch, Lectures on algebraic cycles. Duke University Mathematics Series, IV. Duke University, Mathematics Department, Durham, N.C., 1980. 182 pp. (not consecutively paged).

[B2] S. Bloch, Some elementary theorems about algebraic cycles on Abelian varieties. Invent. Math. 37 (1976), no. 3, 215–228.

[BKL] S. Bloch, A. Kas, and D. Lieberman, Zero cycles on surfaces with \( p_g = 0 \). Compositio Math. 33 (1976), no. 2, 135–145.

[BS] S. Bloch and V. Srinivas, Remarks on correspondences and algebraic cycles. Amer. J. Math. 105 (1983), no. 5, 1235–1253.

[C] F. Campana, Connexité rationnelle des variétés de Fano. Ann. Sci. École Norm. Sup. (4) 25 (1992), no. 5, 539–545.

[Fr] E. Friedlander, Algebraic cycles, Chow varieties, and Lawson homology. Compositio Math. 77 (1991), no. 1, 55–93.

[Fuj] A. Fujiki, On resolutions of cyclic quotient singularities. Publ. Res. Inst. Math. Sci. 10 (1974/75), no. 1, 293–328.

[Ful] W. Fulton, Intersection theory. Second edition, Springer-Verlag, Berlin, 1998.

[G] A. Grothendieck, Hodge’s general conjecture is false for trivial reasons. Topology 8 1969 299–303.
[H] W. Hu, A note on Lawson homology for smooth varieties with small Chow groups. arxiv:math/0602516

[IM] H. Inose and M. Mizukami, Rational equivalence of 0-cycles on some surfaces of general type with $p_g = 0$. Math. Ann. 244 (1979), no. 3, 205–217.

[KMM] J. Kollár, Y. Miyaoka and S. Mori, Rational connectedness and boundedness of Fano manifolds. J. Differential Geom. 36 (1992), no. 3, 765–779.

[L1] B. Lawson, Algebraic cycles and homotopy theory., Ann. of Math. 129 (1989), 253-291.

[L2] B. Lawson, Spaces of algebraic cycles. pp. 137-213 in Surveys in Differential Geometry, 1995 vol.2, International Press, 1995.

[Le] J. D. Lewis, A survey of the Hodge conjecture. (English summary) Second edition. Appendix B by B. Brent Gordon. CRM Monograph Series, 10. American Mathematical Society, Providence, RI, 1999. xvi+368 pp. ISBN: 0-8218-0568-1

[LF] P. Lima-Filho, Lawson homology for quasiprojective varieties. Compositio Math. 84 (1992), no. 1, 1–23.

[LF2] P. Lima-Filho, On the generalized cycle map. (English summary) J. Differential Geom. 38 (1993), no. 1, 105–129.

[M] D. Mumford, Rational equivalence of 0-cycles on surfaces. J. Math. Kyoto Univ. 9 1968 195–204.

[Pe] C. Peters, Lawson homology for varieties with small Chow groups and the induced filtration on the Griffiths groups. Math. Z. 234 (2000), no. 2, 209–223.

[PV] K. H. Paranjape and V. Srinivas, Algebraic cycles. Current trends in mathematics and physics, 71–86, Narosa, New Delhi, 1995.

[R] A.A. Roitman, Rational equivalence of zero-dimensional cycles. (Russian) Mat. Sb. (N.S.) 89(131) (1972), 569–585, 671.

[Vo] Mircea Voineagu, Semi-topological K-theory for certain projective varieties. Preprint. arxiv.org/abs/math/0601008

[Vs] C. Voisin, Sur les zéro-cycles de certaines hypersurfaces munies d’un automorphisme. Ann. Scuola Norm. Sup. Pisa Cl. Sci. (4) 19 (1992), no. 4, 473–492.