Stagnation Point Flow in Copper Nanofluid over Slippery Cylinder with Viscous Dissipation

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Abstract. This study investigates the heat transfer dissipation on stagnation point flow over a slippery stretching/shrinking cylinder in a copper nanofluid by considering the effect of viscous dissipation. A system of nonlinear partial differential equations is modelled and transformed into ordinary differential equations using similarity transformations. The governing equations with the corresponding boundary conditions are analysed numerically using a bvp4c solver in MATLAB. The solutions are found to be dependent on the Eckert number and slip parameters. The results are represented by the velocity and temperature profiles as well as the skin friction coefficient and the Nusselt number. Dual solutions are observed for the shrinking cylinder in the presence of Eckert number. Velocity profile and skin friction coefficient consistently increase while temperature profile increases initially and then decreases with the increase of slip parameter for both first and second solutions. Moreover, the presence of copper nanoparticles reduces the thermal boundary layer thickness. This research can be enhanced by using hybrid nanofluids to further improve the heat transfer.

1. Introduction

A nanofluid becomes a major contemporary interest issue for the last few years in convection transfer of heat. Choi and Eastmen (1995) first introduced the nanofluid term and it is defined as suspending nanoscale particles of fluid capable in the base fluid. These fluids are scientifically believed to have a unique feature, higher thermal conductivity compared to the base fluid due to its nanoparticle characteristics (Soid et al. 2017). This fact has sparked many researchers to investigate further on heat transfer in nanofluids such as Ahmad and Pop (2010), Bachok et al. (2011) and Ali et al. (2020).

Recently, a shrinking/stretching surface on the boundary layer flows has attracted many researchers from engineering fields. This study focuses on a shrinking cylinder with viscous dissipation that allows liquids and gases disperse in the fluid which enhances the heat transfer. Miklavčič and Wang (2006) were the first who initiated the concept of flow over a shrinking surface where the fluid was stretched towards a slot. Since the viscous dissipation would act an energy source, thus the changed in temperature distribution will affect the rate of heat transfer consequently. Viscous dissipation effect had influences on the cooling/heating process was described by Alinejad and Samarbakhsh (2012).

The most important aspect of boundary layer flow over a shrinking/stretching cylinder is the heat transfer characteristics. It is essential to comprehend the fluid flow and heat transfer characteristic...
of the shrinking cylinder to achieve highly standard end products. When fluids flow past immersed bodies, such as plates, cylinders, and spheres, heat is often transferred between the boundary and the fluid. Najib et al. (2017) studied the steady stagnation point flow and heat transfer in nanofluids over a shrinking cylinder with slip effect without viscous dissipation. The importance of the imposition of viscous dissipation will affect the rate of heat transfer and articulate stronger gravitational fields. Cooling and heating processes significantly influence the effect of viscous dissipation (Alinejad & Samarbakhsh, 2012) as done by Ahmed et al. (2018) and Arthur and Seini (2014). Hence, this study imposes viscous dissipation in the energy equation.

This research extends the problem of Najib et al. (2017) by considering the effects of viscous dissipation on the steady stagnation point flow and heat transfer in nanofluids over a slippery shrinking cylinder which focuses on dual solutions and also stability analysis. The governing partial differential equation (PDE) is transformed into a non-linear ordinary differential equation (ODE). The numerical results are obtained by using bvp4c in MATLAB for various governing parameters such as Eckert number, Prandtl number and slip parameter. The determination of the second (lower branch) solution is more difficult compared to the first (upper branch) solution because the bvp4c method will converge to the first solution even for poor guesses. Then, the stability analysis is done to determine the stability of the dual solutions.

2. Mathematical formulation
Consider a steady stagnation-point flow and heat transfer over a slippery cylinder with radius R placed in copper nanofluids. This study extends the work done by Najib et al. (2017) with the inclusion of the viscous dissipation and slip effects. The governing equations that govern the mathematical problem can be written as below (Najib et al. 2017; Kalteh, Ghorbani and Khademinejad 2016).

\[
\frac{\partial}{\partial x} (ru) + \frac{\partial}{\partial r} (rv) = 0 \tag{1}
\]

\[
u \frac{\partial u}{\partial x} + v \frac{\partial v}{\partial r} = u_x + \frac{\mu_{nf}}{\rho_{nf}} \left( \frac{\partial^2 u}{\partial r^2} + \frac{1}{r} \frac{\partial u}{\partial r} \right) \tag{2}
\]

\[
\left( \rho C_p \right)_{nf} \left( u \frac{\partial T}{\partial x} + v \frac{\partial T}{\partial r} \right) = \frac{1}{r} \frac{\partial}{\partial r} \left( r k_{nf} \frac{\partial T}{\partial r} \right) + \Phi \tag{3}
\]

subjected to the boundary conditions

\[
\begin{align*}
  u &= u_w + A \left( \frac{\partial u}{\partial r} \right)_{r=R} , v = 0, T = T_w \quad \text{at } r = R , \\
  u &\to u_x , T \to T_\infty \quad \text{as } r \to \infty 
\end{align*} \tag{4}
\]

where \( u \) is the velocity component along the \( x \) – direction, \( v \) is the velocity component along the \( y \) – direction, \( r \) is the radius of the cylinder, \( u_x = x U_\infty / L \) is the free stream velocity, \( T \) is the temperature of the fluid, \( \rho \) is the density, \( C_p \) is the specific heat at constant pressure, \( \Phi \) is the viscous dissipation contribution, \( T_\infty \) is the ambient fluid temperature, \( u_w = x U_w / L \) is the velocity at the wall, \( A \) is the unsteadiness parameter, \( T_w \) is the constant temperature distribution near the surface and \( R \) is the radius of the cylinder. Further, \( \mu_{nf} \) is the dynamic thermal viscosity of the nanofluid, \( \rho_{nf} \) is the density of the nanofluid, \( \left( \rho C_p \right)_{nf} \) is the heat capacity of nanofluid and \( k_{nf} \) is an effective thermal conductivity of
the nanofluid which are given in table, Oztop and Abu-Nada (2008). The physical characteristics of the nanofluids are given by

\[
\rho_{nf} = (1 - \phi) \rho_f + \phi \rho_s, \quad \mu_{nf} = \frac{\mu_f}{(1 - \phi)\tau_s}, \quad \alpha_{nf} = \frac{k_{nf}}{(\rho C_p)_{nf}},
\]

\[
(\rho C^s_p)_{nf} = (1 - \phi)(\rho C^s_p)_f + \phi(\rho C^s_p)_s, \quad \frac{k_{nf}}{k_f} = \frac{k_s + 2k_f - 2\phi(k_f - k_s)}{(k_s + 2k_f) + \phi(k_f - k_s)} \tag{5}
\]

where \( \phi, \rho_f, \rho_s, \mu_f, \mu_s, \alpha_{nf}, (\rho C^s_p)_f \) and \((\rho C^s_p)_s\) are the nanoparticle volume fraction, density of fluid and of solid fraction, dynamic viscosity of fluid and of solid fraction, the effective thermal diffusivity of the nanofluid and other constant physical characteristics, as well as the heat capacity of fluid and of solid fraction, respectively.

The similarity solutions of equations (1) – (3) along with the boundary conditions (4) can be written as

\[
\eta = \frac{r^2 - R^2}{2R} \left( \frac{u}{v_{fmax}} \right)^{1/2}, \quad \psi = \left( \frac{v_f u_{xnf}}{u_{fmax}} \right)^{1/2} Rf(\eta), \quad \theta = \frac{T - T_{\infty}}{T_w - T_{\infty}} \tag{6}
\]

where \( \eta \) is the similarity variable, \( \theta \) is the dimensionless temperature \( \psi \) is the stream function defined as \( u = \frac{1}{r} \frac{\partial \psi}{\partial r} \) and \( v = -\frac{1}{r} \frac{\partial \psi}{\partial x} \), which satisfies equation (1). By defining \( \eta \) in this form, the boundary conditions at \( r = R \) reduce to the boundary conditions at \( \eta = 0 \), which is more convenient for numerical computations.

Substituting (6) into equations (2) and (3), we get the nonlinear ordinary differential equations:

\[
\frac{1}{(1 - \phi)^2} \left[ \frac{1}{(1 - \phi) + \phi \frac{\rho_f}{\rho_s}} \right] \left[ (1 + 2\kappa \eta) \theta^*(\eta) + 2\kappa \theta^*(\eta) \right] + 1 - (f(\eta))^2 + f(\eta)f^*(\eta) = 0. \tag{7}
\]

\[
\frac{k_{nf}}{k_f} \left[ \frac{1}{(1 - \phi) + \phi \frac{\rho C^s_p}{\rho C^s_p}_f} \right] \left[ (1 + 2\kappa \eta) \theta^*(\eta) + 2\kappa \theta^*(\eta) \right] + \frac{\text{Pr} \cdot \kappa \cdot \rho_s (1 + 2\kappa \eta) f^*(\eta)}{\left( 1 - \phi \right) \rho_f + \phi \rho_s} \tag{8}
\]

\[
+ \text{Pr} \cdot f(\eta) \theta^*(\eta) = 0.
\]

While the boundary conditions (4) are converted to

\[
f(0) = 0, \quad f'(0) = \varepsilon + \sigma f^*(0), \quad \theta(0) = 1, \quad f'(\infty) \rightarrow 1, \quad \theta(\infty) \rightarrow 0. \tag{9}
\]

The prime denotes differentiation with respect to \( \eta \). Moreover, \( \kappa \) is the curvature parameter, \( \text{Pr} \) is the Prandtl number, \( \text{Ec} \) is the Eckert number, \( \varepsilon \) is the stretching/shrinking parameter and \( \sigma \) is the slip parameter defined as

\[
\kappa = \left( \frac{L v_f}{U_{fmax} R^2} \right)^{1/2}, \quad \text{Pr} = \frac{v_f}{\alpha_f}, \quad \text{Ec} = \frac{(u_{fmax})^2}{C_p (T_w - T_{\infty})}, \quad \varepsilon = \frac{U_{fmax}}{U_{\infty}}, \quad \sigma = A \left( \frac{U_{fmax}}{v_f L} \right)^{1/2} \tag{10}
\]

where \( \varepsilon > 0 \) corresponding to stretching velocity and \( \varepsilon < 0 \) corresponding to shrinking velocity.
The physical quantities of interest are the local skin friction coefficients $C_f$ and the local Nusselt number $Nu_x$, which are defined as

$$C_f = \frac{\tau_w}{\rho_f \left( \frac{u}{r} \right)^2}, \quad Nu_x = \frac{xq_w}{k_f \left( T_w - T_0 \right)}, \quad (11)$$

where $\tau_w$ is the skin friction or the shear stresses on the stretching or shrinking sheet, and $q_w$ is the heat flux from the surface of the plate, which are given by

$$\tau_w = \mu_{nf} \left( \frac{\partial u}{\partial r} \right)_{r=R}, \quad q_w = -k_{nf} \left( \frac{\partial T}{\partial r} \right)_{r=R}. \quad (12)$$

Using (6) in (11) and (12), we obtain the skin friction and local Nusselt number:

$$C_f \text{Re}^{1/2} = \frac{1}{(1-\phi)^{1.5}} f''(0) \quad (13)$$

$$Nu_x \text{Re}^{-1/2} = \frac{k_{nf}}{k_f} \theta'(0)$$

3. Numerical results and discussion

This study explores the problem of the stagnation point flow over a slippery stretching/shrinking cylinder in a copper Cu nanofluids by considering the viscous dissipation effect. The nonlinear ordinary differential equations (7) and (8) along with the boundary conditions (9) were solved numerically by using the boundary value problem solver, bvp4c in MATLAB software with the thermophysical properties of fluid and nanoparticles are tabulated in Table 1. In this method, the bvp4c needs more than one initial guess that satisfy the boundary conditions (9) since the present problem may have more than one solution. In the numerical computations, when $\eta \to \infty$ this study choose $\eta = 7$ for the first solution indicated by solid line and $\eta = 20$ for the second solution represented by dash line to fulfil the boundary conditions in equation (9). The tolerance was set at $10^{-10}$.

The effects of Eckert number $Ec$ and slip parameter $\sigma$ where curvature parameter $\kappa$, volume fraction parameter $\phi$ and Prandtl number $Pr$ are set to be 0.2, 0.1 and 6.2 respectively throughout this study. The results are described in terms of the skin friction and the local Nusselt number as well as the velocity and temperature profiles. The existence of dual solutions for each parameter is also presented graphically on the velocity and temperature profiles.

Table 1. Thermophysical properties of fluid and nanoparticles from Oztop and Abu-Nada (2008)

| Physical properties | Fluid phase (water) | $C_{Cu}$ |
|--------------------|---------------------|---------|
| $C_f (J/kgK)$      | 4179                | 385     |
| $\rho (kg/m^3)$    | 997.1               | 8933    |
| $K(W/mK)$          | 0.613               | 400     |
| $\alpha \times 10^7 (m^2/s)$ | 1.47               | 1163.1  |
| $\beta \times 10^{-8} (1/K)$ | 21                 | 1.67    |

The numerical computations are conducted for several values of the dimensionless parameters which are Eckert $Ec$ and slip parameter $\sigma$ with constant curvature parameter $\kappa$, volume fraction parameter...
$\varphi$, stretching/shrinking parameter $\varepsilon$ and Prandtl number $Pr$. Comparative study for the skin friction coefficient $f''(0)$ with numerical results of Omar et al. (2015) and Najib et al. (2017) were performed to prove the numerical results obtained for the case of non slip $\sigma = 0$ and viscous dissipation effects $Ec = 0$. The comparison presents a complimentary agreement, as can be seen in Table 2. Therefore, the numerical results obtained are considerably reliable and valid.

**Table 2.** Comparison of the values of $f''(0)$ for copper nanofluid for several values of $\varepsilon$ at $\varphi = 0.1$, $\kappa = 0.2$, $Ec = 0$, $\sigma = 0$ and $Pr = 6.2$.

| $\varepsilon$ | Omar et al. (2015) | Najib et al. (2017) | Present Results |
|---------------|---------------------|---------------------|-----------------|
| 2             | -2.307873           | -2.307873           | -2.30787251     |
| 1             | 0                   | 0                   | 0               |
| 0.5           | 0.887754            | 0.887754            | 0.88775406      |
| 0             | 1.553805            | 1.553805            | 1.55380537      |
| -0.5          | 1.933203            | 1.933203            | 1.93320347      |
| -1            | 1.866443            | 1.866429            | 1.86642950      |
| -1.15         | 1.684248            | 1.684248            | 1.68424843      |
|               | [-0.026540]         | [-0.02654]          | [-0.02654004]   |
| -1.2          | 1.590181            | 1.590181            | 1.59018207      |
|               | [0.028451]          | [0.028451]          | [0.02845129]    |
| -1.2465       | 1.478323            | 1.478322            | 1.47832423      |
|               | [0.109422]          | [0.109422]          | [0.10942231]    |

**Figure 1.** Variation of the Skin Friction Coefficient $f''(0)$ with $\varepsilon$ for various values of $Ec$ when $\varphi = 0.1$, $\kappa = 0.2$, $\sigma = 0.2$, $Pr = 6.2$. 

![Figure 1](image-url)
Figure 2. Variation of the Nusselt Number $-\theta'(0)$ with $\varepsilon$ for various value of $Ec$ when $\varphi = 0.1, \kappa = 0.2, \sigma = 0.2, Pr = 6.2$.

Figures 1 and 2 show the variation of the skin friction coefficient $f''(0)$ and the rate of heat transfer $-\theta'(0)$ for the copper $Cu$ nanoparticle with $\varepsilon < 0$ (shrinking cylinder) for several values of $Ec$. From both figures, there exist dual solutions for equations (7) and (8) subject to boundary conditions (9). Both figures also show the critical value $\varepsilon_c$ of shrinking parameter for which the solution exists for the corresponding values of $Ec$. The solutions for $f''(0)$ are unique when $\varepsilon > -1$, dual solutions (first and second solutions) exist when $-1 \leq \varepsilon < -1$ and no solution when $\varepsilon < \varepsilon_c$. The critical shrinking value $\varepsilon_c$ remains unchanged as $Ec$ parameter increases. Apparently, as the Eckert number $Ec$ increases, the value of skin friction coefficient remains unchanged for the first solutions as well as the second solution as expected from the nonexistence of $Ec$ in the momentum equation. Although there is no change in the values of skin friction coefficient for various $Ec$, but there is an existence of first and second (dual) solutions. The positive values of the first solution of skin friction coefficient indicate that the fluid exerts a drag force on the surface of the cylinder. The increasing in the Eckert number $Ec$ causes the increases in the rate of heat transfer for the first solution. However, the second solutions reduce rapidly from the critical point. The values of Nusselt number are positive for the first and second solution. This practically means that the heat is transferred from the surface to the fluid.

Figure 3. Velocity profiles $f'(\eta)$ for various value of $Ec$ when $\varphi = 0.1, \varepsilon = -1.2, \kappa = 0.2, \sigma = 0.2, Pr = 6.2$. 
There is no change in the value of the skin friction coefficient as shown in Figure 3 due to the absence of viscous dissipation effects in momentum equation (7). Even though the velocity profile remains unchanged for different $Ec$, but there is an existence of first and second (dual) solutions.

The temperature profiles $\theta(\eta)$ in Figure 4 shows a change for the different number of $Ec$ where $Ec > 0$. The larger Eckert number $Ec > 0$ will affect the dimensionless temperature distribution within the boundary layer region as stated by Gangadhar et al. (2018) that the fluid temperature is enhanced due to the viscous heating, hence the temperature is increasing as the Eckert number increases for both the first and second solutions. The increasing $Ec$ leads to more heat energy is kept in the fluid due to frictional heating. The thickness of the boundary layer for the second solution in Figures 3 and 4 is larger compared to the first solution. The velocity and temperature profiles satisfy the boundary conditions (9) asymptotically and thus verify the validity of the numerical results acquired.

Figure 5. Velocity profiles $f'(\eta)$ for various value of $\sigma$ when $\varphi = 0.1$, $\varepsilon = -1.2$, $\kappa = 0.2$, $Ec = 1$, $Pr = 6.2$. 
Figure 6. Temperature profile $\theta(\eta)$ for various of $\sigma$ when $\varphi = 0.1$, $\varepsilon = -1.2$, $\kappa = 0.2$, $Ec = 1$, $Pr = 6.2$.

Figure 5 illustrates the velocity profiles for the first solution increases as the slip parameter $\sigma$ increases. According to Najib et al. (2017), the presence of slip parameter $\sigma$ causes the skin friction coefficient to decrease. The temperature profile in Figure 6 increases initially and leads to decrease smoothly at approximately $\eta \approx 0.5$ for the first solution. A similar pattern for the second solution where the temperature decreases at a certain point and then increases as the slip parameter $\sigma$ increases for the second solution. The slower movement of fluid leads to the decrease of the rate of heat transfer and thus causes the slight increases of the thermal boundary layer thickness with the increase of the slip parameter of the second solution.

4. Conclusion

The problem of two-dimensional steady stagnation point flow and heat transfer in nanofluids over a shrinking cylinder with slip effect and viscous dissipation is studied in this paper. With the aid of the similarity transformation, the governing equations are reduced to the self-similar nonlinear ordinary differential equations which are then solved numerically by using the function bvp4c in MATLAB for different values of governing parameters. The existence and duality (first and second) of the solutions are clearly presented and described by considering the combination slip parameter $\sigma$, curvature parameter $\kappa$, volume fraction parameter $\varphi$ and Eckert number $Ec$ and Prandtl number $Pr$ with Copper $Cu$ as a nanoparticle. The existence of duality solutions was found in the shrinking case $\varepsilon < 0$.

Some of the following conclusions are drawn from figures and tables:

- There exist dual solutions of Copper with different values of Eckert number and slip parameter.
- There is no change in the velocity profiles and skin friction coefficient for the increase or decrease of the Eckert number since Eckert number is independent to the momentum equation.
- Temperature profiles and Local Nusselt number increase as Eckert number increases for the first solution and second solution.
- Velocity profile and skin friction coefficient consistently increase while temperature profile increases initially and then decreases with the increase of slip parameter for the first solution. Similar behavior for the second solution.
- The boundary layer thickness of the second solution is always thicker than the first solution for all parameters.
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References

1. Ahmad, S., & Pop, I. (2010). Mixed convection boundary layer flow from a vertical flat plate embedded in a porous medium filled with nanofluids. *International Communications in Heat and Mass Transfer, 37*(8), 987-991.
2. Ahmed, A., Siddique, J. I., & Sagheer, M. (2018). Dual solutions in a boundary layer flow of a power law fluid over a moving permeable flat plate with thermal radiation, viscous dissipation and heat generation/absorption. *Fluids, 3*(1), 1-16.
3. Ali, M., Shahzad, M., Sultan, F., Khan, W.A., & Rashid, S. (2020). Exploring the feature of stratification phenomena 3D flow of Cross nanofluid considering activation energy. International Communications in Heat and Mass Transfer, 116, 104674(1-7).
4. Alinejad, J., & Samarbakhsh, S. (2012). Viscous flow over nonlinearly stretching sheet with effects of viscous dissipation. *Journal of Applied Mathematics, 1*-10.
5. Arthur, E. M., & Seini, I. Y. (2014). Hydromagnetic stagnation point flow over a porous stretching surface in the presence of radiation and viscous dissipation. *Applied and Computational Mathematics, 3*(5), 191-196.
6. Bachok, N., Ishak, A., & Pop, I. (2011). Stagnation point flow over a stretching/shrinking sheet in a nanofluid. *Nanoscale Research Letters, 6*(1), 1-10.
7. Choi, S. U. S., & Eastmen, J. A. (1995). Enhancing thermal conductivity of fluids with nanoparticles. *Proceedings of ASME International Mechanical Engineering Congress & Exposition*(pp. 12-17). San Francisco, C. A.
8. Gangadhar, K., Kannan, T., Dasaradharamiah, K., & Sakhivel, G. (2018). Boundary layer flow of nanofluids to analyse the heat absorption/generation over a stretching sheet with variable suction/injection in the presence of viscous dissipation. *International Journal of Ambient Energy, 1*-12.
9. Kalteh, M., Ghorbani, S., & Khademinejad, T. (2016). Viscous dissipation and thermal radiation effects on the magnetohydrodynamic (MHD) flow and heat transfer over a stretching slender cylinder. *Прикладная механика и техническая физика, 57*(3), 463-472.
10. Miklavčič, M., & Wang, C. (2006). Viscous flow due to a shrinking sheet. *Quarterly of Applied Mathematics, 64*(2), 283-290.
11. Najib, N., Bachok, N., & Arifin, N. M. (2017a). Stagnation point flow and heat transfer in nanofluids over a stretching/shrinking with slip effect and stability analysis. *Proceedings of Researchfora (6-11). Perth, Australia.
12. Omar, N. S., Bachok, N., & Arifin, N. M. (2015). Stagnation point flow over a stretching or shrinking cylinder in a copper-water nanofluid. *Indian Journal of Science and Technology, 8*(31), 1-7.
13. Oztop, H. F., & Abu-Nada, E. (2008). Numerical study of natural convection in partially heated rectangular enclosures filled with nanofluids. *International Journal of Heat and Fluid Flow, 29*(5), 1326-1336.
14. Soid, S. K., Ishak, A., & Pop, I. (2017). Boundary layer flow past a continuously moving thin needle in a nanofluid. *Applied Thermal Engineering, 114*, 58-64.