ALGEBRAIC K-THEORY AND PARTITION FUNCTIONS IN CONFORMAL FIELD THEORY

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Notation

\[ g \] — Lie algebra of rank \( r \)
\[ \alpha_1, \ldots, \alpha_r \] — Simple roots of \( g \)
\[ \omega_1, \ldots, \omega_r \] — Fundamental weights of \( g \)
\[ \Delta^+ \] — Set of positive roots of \( g \)
\[ \rho \] — Weyl vector
\[ C(g) \] — Cartan matrix of \( g \)
\[ W(g) \] — Weyl group of \( g \)
\[ g \] — Coxeter number of \( g \)
\[ h \] — Dual Coxeter number of \( g \)
\[ V(\lambda) \] — Irreducible representation of \( g \), of highest weight \( \lambda \)
\[ \chi_\lambda = ch(\lambda) \] — Character of the representation \( V(\lambda) \)
\[ Y(g) \] — Yangian associated with \( g \)
\[ W^i_j \] — Irreducible representation of \( Y(g) \)
\[ Q^i_j \] — Character of the representation \( W^i_j |_g \)
\[ Q_0 \] — Weyl denominator
\[ n(\chi_\lambda) \] — Weyl numerator of the character \( \chi_\lambda \)
Chapter 1

Introduction

Quantum field theory (QFT) plays a major role in modern theoretical physics. From its beginnings in the late 1920s, it has grown to be the most successful physical theory in existence today. It provides the best working description of the fundamental laws of physics and is an extremely useful tool for investigating the behaviour of complex systems. Its unrivalled ability to accurately calculate physical quantities only adds to its reputation. Therefore it is hardly surprising that quantum field theory plays such a central role in our description of nature.

In addition to being a significant theory in its own right, QFT provides essential tools to many other branches of physics, for example to condensed matter physics. The influence of QFT is far-reaching, with its very mathematical nature helping to build new bridges between physics and mathematics. Despite its many great successes, very little is known about the deep mathematical structure underlying this theory. Progress in this area would be beneficial to both mathematics and physics.
With any quantum field theory, the main aim is to find an exact solution, by no means an easy task. In fact the solution of any non-trivial QFT, whether or not it is physically relevant, would be a major step forward in the search for a better understanding of the subject. By an exact solution we mean the explicit calculation of the n-point functions of the theory. This is sufficient since Wightman’s reconstruction theorem \[1\] states that with this knowledge the entire field content and physical state space of the theory can be computed.

The attempt to solve any QFT exactly is a very ambitious undertaking. The most likely candidates for success are those QFTs whose symmetries give rise to a large number of conservation laws. These laws might impose enough restrictions on the theory to allow it to be solved exactly.

Unfortunately this approach is useless in 3+1 dimensions because of the Coleman-Mandula theorem \[2\]. This states that besides Poincaré invariance and an internal gauge group describing the degeneracy of the particle spectrum, any additional symmetries cause the scattering matrix of each massive QFT to be trivial. There is a similar result due to Lochlainn O’Raifeartaigh \[3\], which states that it is impossible to combine internal and relativistic symmetries other than in a trivial way.

**1+1 dimensional integrable models**

This motivates us to consider the 1+1 dimensional case, in which the Coleman-Mandula theorem no longer holds. Here there is nothing to rule out the existence of a theory with an infinite number of conservation laws, meaning that the hope of finding exact solutions is much more realistic. Such theories are called integrable. Integrable models are hugely relevant in physics, having found applications in off-
critical descriptions of statistical mechanics and condensed matter systems reduced to two dimensions.

Restrictions to this low-dimensional case may seem unrealistic. However, by facilitating the construction of exact solutions, 1+1 dimensional theories can be used to shed new light on the structure of more general QFTs. This easily justifies the study of QFT in 1+1 dimensions.

**Links to conformal field theory**

The recent wave of interest in integrable models lies in their interpretation as deformed conformal field theories (CFTs) [4]. CFTs form a particular class of integrable field theories. They are characterised by scale invariance and describe massless relativistic particles or statistical mechanical systems at a critical point. As with other theories, they become extremely powerful in two dimensions, being characterised by an infinite number of conserved currents that ensure their solvability. Here the infinite set of conservation laws corresponds to conformal space-time symmetry. Interest in this topic was rekindled when Belavin, Polyakov and Zamolodchikov [5] showed that the so-called minimal models are particular examples of solvable massless QFTs.

Given a CFT, what happens to the infinite set of conservation laws arising from conformal invariance when the system moves away from the critical point and scale invariance is lost? When the action of the critical point theory is perturbed by particular fields, the conformally invariant structure is in general destroyed. However, Zamolodchikov showed that for particular deformations of the CFT, an infinite set of the conserved charges may survive the breaking of conformal symmetry. This
results in the corresponding massive field theory being integrable.

Scattering matrices

The scattering matrix (or S-matrix) is central to the study of quantum field theory. It determines the on-shell structure of the model and describes the collision of quantum particles. It must obey certain constraints inspired by the physics of the system. These include crossing symmetry, Lorentz invariance, analyticity in the energy variables, and the conservation of probability. In 1+1 dimensions these constraints are often restrictive enough that they enable one to conjecture the S-matrix and hence determine the theory completely. An essential feature of any 1+1 dimensional integrable field theory is an infinite set of conserved charges. This strongly restricts the dynamics of the system, imposing the following conditions on the scattering process: conservation of the total number of particles, conservation of individual particle momenta, and factorisation of the S-matrix into 2-particle scattering amplitudes.

Consistency of different ways of decomposing an amplitude into two-particle amplitudes leads to cubic relations between the two-particle amplitudes. These cubic relations are essentially the Yang-Baxter (or star-triangle) relations [6, 7].

In principle, exact construction of the scattering matrix serves as a first step in the calculation of the n-point or correlation functions of a system, for example via the form factor programme [8, 9, 10]. Although the relevant calculations are highly non-trivial and have been carried out in only the simplest cases, this nevertheless demonstrates the importance of the S-matrix in the search for a complete solution of any quantum field theory.
The thermodynamic Bethe ansatz

On the other hand it is possible to start with a massive integrable field theory and proceed in the opposite direction to recover conformal invariance. In the high-energy limit the masses of particles become negligible, with the result that scale and therefore conformal invariance are approximately restored. The high energy limit of an integrable QFT is studied by means of the thermodynamic Bethe ansatz (TBA).

The TBA was developed over twenty years ago by Yang and Yang \[11, 12\], as a technique to calculate thermodynamic quantities for a system of bosons interacting dynamically through factorisable scattering. The method was later generalized \[13, 14, 15\] to a system of relativistic particles interacting dynamically through the scattering matrix of an integrable QFT. It has become one of the most effective techniques for exploring the close relationship between conformal and integrable field theories.

Using the TBA approach, information can be extracted from a massive integrable quantum field theory once its scattering matrix is known. In particular, the effective central charge of the corresponding UV conformal field theory can be calculated. The information gained in this way is often sufficient to determine the field content at the critical point. Once the perturbing operator has been determined, the integrable field theory can then be formally described in terms of a classical Lagrangian. The TBA also has applications in scattering theory, where it can be used to test conjectured S-matrices for consistency.

This brief account of quantum field theory sets the scene for the work in this thesis.
Outline of PhD thesis

By a mixture of analytical and numerical techniques, physicists have formulated many beautiful and well-supported hypotheses for the integrable massive perturbations of conformally invariant theories, though mathematical proofs are not yet available. In particular it has been found that conformal dimensions are given in terms of the dilogarithm formulas [16], by finite order elements of the Bloch group, a basic tool in algebraic K-theory. This shows a deep connection between physics and a very active domain of number theory.

Among the integrable models for which this relationship holds are those described by pairs $(X, Y)$ of ADET Dynkin diagrams [17]. Such models have equations of the form $AU = V$, where $A = C(X)^{-1} \otimes C(Y)$, $C$ denotes a Cartan matrix, and $X$ and $Y$ are the Dynkin diagrams of simple Lie algebras of ranks $m$ and $n$ respectively. Moreover $U = \log(x)$ and $V = \log(1 - x)$ satisfy $e^U + e^V = 1$, the vector $x$ is given by $x = (x_{11}, \ldots, x_{mn})$, and $f(x)$ is to be interpreted as $(f(x_{11}), \ldots, f(x_{mn}))$. The relationship between the matrix $A$ and the scattering matrix of the integrable quantum field theory is described in chapter 3.

It seems that the resulting algebraic equations for $x$ are solvable in terms of roots of unity. The case $(X, Y) = (A_m, A_n)$ has already been studied [17]. In this thesis we consider mainly the case $(X, Y) = (D_m, A_n)$. We study the algebraic equations of the model and find all solutions in explicit form using the representation theory of Lie algebras and related quantum groups. The solutions seem to be torsion elements of the Bloch group, allowing the effective central charge of the corresponding CFT to be calculated using the dilog formula mentioned above.
On a more mathematical note, we are interested in a closely related problem involving certain q-hypergeometric series. We investigate the overlap between series of this type and modular functions. We study a particular class of r-fold q-hypergeometric series, denoted $f_{A,B,C}$, depending on some $2 \times 2$ matrix $A$. It turns out that for certain choices of the matrix $A$, the function $f$ can be made modular. We calculate the corresponding values of $B$ and $C$. It is expected that functions $f$ arising in this way are characters of some rational conformal field theory. We show that this is true in at least one case. The content of the thesis is arranged as follows:

**Chapter 2** introduces the mathematical and physical concepts important for subsequent chapters. The first section presents the theory of simple Lie algebras, as well as a number of other useful mathematical definitions. The second section is a short introduction to conformal field theory, concentrating on the aspects most relevant to this thesis.

**Chapter 3** is the core of the thesis. We study the integrable models described by pairs of Dynkin diagrams $(D_m, A_n)$. The equations of these models are solved in the general case, and their relation to Yangian representation theory is discussed. Results of the effective central charge calculations for many different models are summarised. The realisation of such models as coset models is also discussed.

**Chapter 4** places our results in a more general context. We examine the relationship between certain q-hypergeometric series and modular functions. This is very closely related to the conformal field theory of the previous chapter.

**Chapter 5** contains initial steps for the solutions of the remaining cases. A number of integrable models described by pairs of ‘non-classical’ Dynkin diagrams is
studied. In particular the pairs $(E_6, T_1)$, $(E_7, T_1)$, and $(E_8, T_1)$ are considered. In each case the equations of the models are solved, and corresponding values of the effective central charge are calculated. All calculations in this chapter are carried out using only elementary algebra.
Chapter 2

Overview

2.1 Mathematical Overview

2.1.1 Lie algebras

This section gives a brief introduction to simple Lie algebras, focusing mainly on concepts that arise in the study of conformal field theory. As far as possible the discussion is self-contained. There are numerous good books on the subject, and for a more detailed account of the material presented here see e.g. [18] [19].
Simple Lie algebras, generators, and roots

A vector space \( g \) is called a **Lie algebra** if it is equipped with an anti-symmetric bilinear map \([\cdot, \cdot] : g \times g \to g\), called the **Lie bracket**, satisfying the Jacobi identity

\[
[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0 ,
\]

for all \( x, y, z \in g \).

We consider only finite-dimensional Lie algebras, meaning that \( g \) is finite-dimensional when viewed as a vector space. Moreover we restrict ourselves to **simple Lie algebras**. These are non-abelian, i.e. \([g, g] \neq 0\), and contain no proper ideal, i.e. there is no subalgebra \( \mathfrak{k} \) of \( g \) such that \([\mathfrak{k}, g] \subset \mathfrak{k}\). This particular class of Lie algebras is special in that each of its members is classified uniquely by its Dynkin diagrams.

Below are some examples of simple Lie algebras that can be realised in terms of matrix algebras.

\[
A_n = \mathfrak{su}(n + 1) ,
\]

\[
B_n = \mathfrak{so}(2n + 1) ,
\]

\[
C_n = \mathfrak{sp}(2n) ,
\]

\[
D_n = \mathfrak{so}(2n) .
\]

Here \( A_n \) denotes the set of anti-hermitian complex \((n + 1) \times (n + 1)\) matrices of trace zero. Elements of \( B_n \) and \( D_n \) are anti-symmetric complex matrices of trace zero.
zero. Elements of $C_n$ are $2n \times 2n$ matrices of form

$$\begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

where $B$ and $C$ are symmetric and $D = -A^T$.

A Lie algebra can be specified by a set of generators \{I^a\} and their commutation relations $[I^a, I^b] = \sum_c f^{abc} I^c$, where $f^{abc}$ denote the structure constants of $\mathfrak{g}$. There are $\text{dim}(\mathfrak{g})$ generators in total.

In the **Cartan-Weyl basis** generators are constructed as follows:

- Find a maximal set of commuting generators $\{H^1, \ldots, H^r\}$. These generators form a Cartan subalgebra $\mathfrak{h}$ of $\mathfrak{g}$. The dimension, $r$, of $\mathfrak{h}$ is called the **rank** of $\mathfrak{g}$.

- The remaining generators are chosen to be the particular combinations $X^\beta$ of the $I^a$ that satisfy $[H^i, X^\beta] = \beta^i X^\beta$.

The vector $\beta = (\beta^1, \ldots, \beta^r)$ is called a **root** of $\mathfrak{g}$. Roots are elements of $\mathfrak{h}^*$, the dual space of the Cartan subalgebra $\mathfrak{h}$ (since the root $\beta$ naturally maps an element $H^i \in \mathfrak{h}$ to the number $\beta^i$, by $\beta(H^i) = \beta^i$). The set of all roots is denoted by $\Delta$.

A set of **simple roots**, $\Pi = \{\alpha_1, \ldots, \alpha_r\}$, is a convenient basis for the space of all roots. It is defined up to automorphisms of the maximal torus. Any root $\beta$ can be
written as an integral linear combination of simple roots

\[ \beta = \sum_{i=1}^{r} n_i \alpha_i , \]

where \( n_i \in \mathbb{Z} \), and either all \( n_i \geq 0 \) or all \( n_i \leq 0 \). For such a root, we define its
length by

\[ l(\beta) = \sum_{i=1}^{r} n_i . \]

The length functional \( l : \Delta \rightarrow \mathbb{Z} \) naturally induces a \( \mathbb{Z} \)-grading on the space of roots. This allows us to define a partial ordering, \(<\), on the set of roots as follows. For \( \alpha, \beta \in \Delta \) we write

\[ \alpha > \beta \iff l(\alpha) > l(\beta) . \] (2.1)

Then we say that \( \alpha \) is a positive root if \( l(\alpha) > 0 \), and \( \alpha \) is a negative root if \( l(\alpha) < 0 \). The sets of positive and negative roots are denoted by \( \Delta^+ \) and \( \Delta^- \) respectively. Clearly \( \Delta^- = -\Delta^+ \) and \( \Delta = \Delta^+ \cup \Delta^- \).

The adjoint representation and the Killing form

The most natural representation of a Lie algebra is the adjoint representation, in which \( \mathfrak{g} \) is represented as an operator algebra acting on itself. Every element \( x \in \mathfrak{g} \) can be viewed as an operator by means of the adjoint action, \( x \mapsto ad(x) \), which acts on \( \mathfrak{g} \) as

\[ ad(x)(y) = [x, y] . \]
This leads in a natural way to the introduction of a symmetric bilinear form on \( g \). This is the so-called Killing form \( K : g \times g \rightarrow \mathbb{C} \), defined by

\[
(x, y) \mapsto K(x, y) = \frac{1}{2h} Tr (ad(x) \circ ad(y)) ,
\]

(2.2)

for all \( x, y \in g \). The factor \( h \) in the normalisation constant is the dual Coxeter number of \( g \), defined later in (2.3).

The Killing form establishes an isomorphism between \( h \) and \( h^\ast \), and can be used to induce a positive definite scalar product on \( h^\ast \) through \( (\alpha, \beta) = K(H^\alpha, H^\beta) \). Here the element \( H^\alpha \in h \) is uniquely defined by the relation \( \alpha(H) = K(H^\alpha, H) \). Since all roots of \( g \) sit in \( h^\ast \), this defines a scalar product on the root space. From now on this is the scalar product we use between roots.

The partial ordering introduced in (2.1) identifies a unique highest root \( \theta \) for which

\[
l(\theta) > l(\alpha) ,
\]

for all other roots \( \alpha \). The particular normalisation (2.2) of the Killing form was chosen to ensure that

\[
(\theta, \theta) = 2 .
\]

This follows the standard convention in which the square length of a long root is 2.

The highest root can be written as

\[
\theta = \sum_{i=1}^{r} a_i \alpha_i = \sum_{i=1}^{r} a_i^\vee \alpha_i^\vee ,
\]
where the \( a_i \) and \( \alpha_i^\vee \) are natural numbers, called the **Kac labels** and **dual Kac labels** respectively. Here \( \alpha_i^\vee \) is the **simple coroot** corresponding to the simple root \( \alpha_i \). It is defined as

\[
\alpha_i^\vee := \frac{2\alpha_i}{\alpha_i^2} .
\]

The sums of the Kac labels and dual Kac labels define two important constants in the theory of simple Lie algebras. These are the **Coxeter number**

\[
g = \sum_{i=1}^{r} a_i + 1 ,
\]

and the **dual Coxeter number**

\[
h = \sum_{i=1}^{r} \alpha_i^\vee + 1 .
\] (2.3)

**Cartan matrices and Dynkin diagrams**

To each simple Lie algebra \( \mathfrak{g} \) we can associate a unique **Cartan matrix** \( C(\mathfrak{g}) \). This is an positive definite \( r \times r \) matrix, whose elements \( C_{ij} \) are defined in terms of the simple roots by

\[
C_{ij} = (\alpha_i, \alpha_j^\vee) .
\] (2.4)

The Cartan matrix has the following properties:

- \( C_{ij} \in \mathbb{Z} \),
- The diagonal entries satisfy \( C_{ii} = 2 \),
• The off-diagonal entries $C_{ij}$ can take values 0, -1, -2, or -3 ,

• $C_{ij} = 0 \iff C_{ji} = 0$ ,

• $\det(C) > 0$ .

The **Dynkin diagram** of a simple Lie algebra is a connected planar diagram encoding the Cartan matrix in the following way. To every simple root $\alpha_i$ we associate a vertex, and vertices $i$ and $j$ are joined by $C_{ij}C_{ji}$ lines. Let $\theta_{ij}$ be the angle between the simple roots $\alpha_i$ and $\alpha_j$. It can easily be shown that $C_{ij}C_{ji} = 4 \cos^2 \theta_{ij}$, and hence that any two vertices can be joined by either 0, 1, 2, or 3 links. Where there is only one link joining two vertices, the corresponding simple roots are both of the same length. Where there is more than one link joining two vertices, an arrow is drawn pointing from the longer to the shorter root. The Dynkin diagrams of simple Lie algebras are separated into two classes, those in which only single links occur (the ADE series), and those where multiple links are allowed (the BCFG series). For obvious reasons these are called **simply-laced** and **non simply-laced** respectively. Clearly in the former case all roots must have the same length, while in the latter they can be different. In the set of all roots of a given Lie algebra, at most two different lengths are possible.

A simple Lie algebra uniquely determines its Dynkin diagram, and hence the classification of simple Lie algebras corresponds exactly to the classification of Dynkin diagrams. The Dynkin diagram contains all of the information necessary to reconstruct the entire root system of the algebra. For simple Lie algebras the set of all allowed Dynkin diagrams is as follows.
Simply-laced:

\[ A_n \]

\[ D_n \]

\[ E_6 \]

\[ E_7 \]

\[ E_8 \]
Non simply-laced:

\[
B_n \quad \bullet \quad \bullet \quad \bullet \quad \ldots \ldots \quad \bullet \quad \bullet
\]

\[
C_n \quad \bullet \quad \bullet \quad \bullet \quad \ldots \ldots \quad \bullet \quad \bullet
\]

\[
F_4 \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet \quad \bullet
\]

\[
G_2 \quad \bullet \quad \bullet
\]

The algebras of types \( A, B, C, \) and \( D \) are known as \textbf{classical Lie algebras}, while \( E_6, E_7, E_8, F_4, \) and \( G_2 \) are the \textbf{exceptional Lie algebras}.

Weights and the Weyl vector

Let \( \{ \omega_1, \ldots, \omega_r \} \) be the basis dual to the basis of simple coroots, i.e.

\[
(\omega_i, \alpha_j^\vee) = \delta_{ij}.
\]

Then \( \omega_1, \ldots, \omega_r \) are called the \textbf{fundamental weights} of \( g \).

For an arbitrary representation of \( g \), a basis \( \{ |\lambda\rangle \} \) can always be found such that

\[
H^i|\lambda\rangle = \lambda^i|\lambda\rangle.
\]
The eigenvalues $\lambda_i$ form a vector $\lambda = (\lambda^1, \ldots, \lambda^r)$ called a weight. As in the case of roots, weights live in the space $\mathfrak{h}^*$; $\lambda(H^i) = \lambda^i$. Therefore, the scalar product between weights is also fixed by the Killing form. That roots and weights occupy the same space makes perfect sense, since roots are just a special name given to the weights of the adjoint representation.

Every weight can be written as an integral linear combination of the fundamental weights
\[ \lambda = \sum_{i=1}^{r} \lambda_i \omega_i , \]
where $\lambda_i \in \mathbb{Z}$. The expansion coefficients $\lambda_i$ of a weight $\lambda$ in the fundamental weight basis are called Dynkin labels. From now on when a weight is written in component form
\[ \lambda = (\lambda_1, \ldots, \lambda_r) , \]
it is assumed that the components are the Dynkin labels.

One weight of particular importance is the Weyl vector, $\rho$. This is written in terms of fundamental weights as
\[ \rho = \sum_{i=1}^{r} \omega_i = (1,1,\ldots,1) . \]

Alternatively it can be expressed in terms of positive roots as
\[ \rho = \frac{1}{2} \sum_{\alpha \in \Delta^+} \alpha . \]

The Weyl vector is uniquely determined by the property that $(\rho, \alpha^\vee_i) = 1$. This
follows immediately from the definition of \( \rho \) in terms of fundamental weights.

### Weyl reflections and the Weyl group

Given roots \( \alpha \) and \( \beta \) of \( g \), it can be shown that the quantity \( \beta - (\alpha^\vee, \beta) \alpha \) is also a root. This leads to the introduction of a new operation \( s_\alpha : \Delta \to \Delta \) defined by

\[
s_\alpha \beta := \beta - (\alpha^\vee, \beta) \alpha.
\]

Such a mapping is called a Weyl reflection. It is a reflection with respect to the hyperplane perpendicular to \( \alpha \). The set of all such reflections with respect to roots forms the Weyl group of \( g \), denoted \( W(g) \). The \( r \) simple Weyl reflections

\[
s_i \equiv s_{\alpha_i},
\]

generate the whole of the Weyl group via composition. Since only a limited number of composite simple Weyl reflections can differ from the identity, \( W(g) \) is always a finite group. The action of the Weyl group on the simple roots yields the set of all roots of the algebra

\[
\Delta = W(g)\{\alpha_1, \ldots, \alpha_r\}.
\]

### The Weyl character formula

A character is a useful way of encoding all of the information about a representation. Suppose \( \pi : g \to End(V) \) is a representation of \( g \). Then the character of \( \pi \) is the
function

\[ \chi : \mathfrak{g} \to \mathbb{C} \]

\[ x \mapsto Tr(\pi(x)) . \]

The character is independent of the choice of basis for \( V \).

The **Weyl character formula** allows us to calculate the character of an irreducible representation given its highest weight. The character of the irreducible representation of highest weight \( \lambda \) is given by

\[
\chi_{\lambda} = \frac{\sum_{w \in W(\mathfrak{g})} \det(w) e^{w(\lambda + \rho)}}{\sum_{w \in W(\mathfrak{g})} \det(w) e^{w\rho}} .
\]

The Weyl denominator can be written as

\[
\sum_{w \in W(\mathfrak{g})} \det(w) e^{w\rho} = \prod_{\alpha \in \Delta^+} (e^{\alpha/2} - e^{-\alpha/2}) .
\]

In many calculations it is more convenient to use the multiplicative form of this expression.

### 2.1.2 Affine Kac-Moody algebras

To every finite-dimensional Lie algebra \( \mathfrak{g} \), we can associate an affine extension \( \hat{\mathfrak{g}} \) simply by adding an extra node to the Dynkin diagram of \( \mathfrak{g} \). The resulting algebra is called an **affine Kac-Moody algebra**. Affine Kac-Moody algebras have important
applications in conformal field theory, in particular in the study of WZW models.

The fundamental concepts of roots, weights, Cartan matrices and the Weyl group extend easily from the finite to the affine case. However the addition of the extra simple root in the affine case results in both the root system and the Weyl group of \( \hat{\mathfrak{g}} \) becoming infinite. Consequently all highest-weight representations are infinite-dimensional. For simplicity these representations are arranged in terms of a new parameter, \( k \), called the \textbf{level}. The level of a weight (now described by \( r + 1 \) Dynkin labels) is just the sum of all of its Dynkin labels, each multiplied by the corresponding comark.

Affine Kac-Moody algebras form a large subclass of the more general Kac-Moody algebras. In particular they consist of those Kac-Moody algebras whose Cartan matrix is positive semi-definite. For a good overview of affine Kac-Moody algebras see e.g. [18, 20].

2.1.3 Yangian algebras

Throughout this discussion \( \mathfrak{g} \) will denote a finite-dimensional complex simple Lie algebra of rank \( r \) and dimension \( d \), whose generators \( I^1, \ldots , I^d \) satisfy the commutation relations \([I^a, I^b] = f^{abc} I^c\). To every such \( \mathfrak{g} \) we can associate an infinite dimensional quantum group, \( Y(\mathfrak{g}) \), called a Yangian. One has \( U \mathfrak{g} \subset Y(\mathfrak{g}) \), where \( U \mathfrak{g} \) is the universal enveloping algebra of \( \mathfrak{g} \). \( Y(\mathfrak{g}) \) is constructed as follows:
First notice that $\mathfrak{g}$ has a coproduct $\Delta : U\mathfrak{g} \to U\mathfrak{g} \otimes U\mathfrak{g}$ given by

$$\Delta(I^a) = I^a \otimes 1 + 1 \otimes I^a .$$

$\Delta$ is coassociative

$$(\Delta \otimes 1)\Delta(x) = (1 \otimes \Delta)\Delta(x) , \quad \forall x \in \mathfrak{g} ,$$

and a homomorphism

$$\Delta([x, y]) = [\Delta(x), \Delta(y)] , \quad \forall x, y \in \mathfrak{g} .$$

Let $\{ J^a \}$ be a second set of generators in the adjoint representation of $\mathfrak{g}$, such that

$$[I^a, J^b] = f^{abc} J^c .$$

There is a non-trivial coproduct $\Delta : Y_\hbar(\mathfrak{g}) \to Y_\hbar(\mathfrak{g}) \otimes Y_\hbar(\mathfrak{g})$ given by

$$\Delta(J^a) = J^a \otimes 1 + 1 \otimes J^a + \frac{\hbar}{2} f^{abc} I^c \otimes I^b .$$

Let $Y_\hbar(\mathfrak{g})$ be the algebra generated by $\{ I^a, J^a \}$, with the additional constraints that

$$[J^a, [J^b, I^c]] - [I^a, [J^b, J^c]] = \hbar^2 a_{abcde} \{ I^d, I^e, I^g \} ,$$

and

$$[[J^a, J^b], [I^l, J^m]] + [[J^l, J^m], [I^a, J^b]] = \hbar^2 \left( a_{abcde} f^{lme} + a_{lmcde} f^{abc} \right) \{ I^d, I^e, I^g \} ,$$

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where
\[ a_{abcdefg} = \frac{1}{24} f^{adi} f^{bej} f^{cgk} f^{ijkl}, \]
and
\[ \{x_1, x_2, x_3\} = \sum_{i \neq j \neq k} x_i x_j x_k. \]

These constraints are chosen such that \( \Delta \) becomes a homomorphism. The Yangian of \( g \), denoted \( Y(g) \), is defined to be the Hopf algebra over \( \mathbb{C} \), got by setting \( h = 1 \) above.

Yangians were introduced in this way by Drinfeld [21] as part of his work on solutions of the Yang-Baxter equation. He later gave a second description [22] of Yangians, in terms of generators and relations, which we now show.

Let \( C = (C_{ij}) \) denote the Cartan matrix of \( g \), and let \( B_{ij} = \frac{(\alpha_i, \alpha_j)}{2} \) be the symmetrised version. Then \( Y_h(g) \) is isomorphic to the associative algebra with generators \( \xi_{ik}^\pm \) and \( \phi_{ik}, i = 1, \ldots r \) and \( k \in \mathbb{Z} \), and defining relations

- \([\phi_{ik}, \phi_{jl}] = 0,\]
- \([\phi_{i0}, \xi_{jl}^\pm] = \pm 2B_{ij} \xi_{jl}^\pm,\]
- \([\xi_{ik}^\pm, \xi_{jl}^\mp] = \delta_{ij} \phi_{i,k+l},\]
- \([\phi_{i,k+1}, \xi_{jl}^\pm] - [\phi_{i,k}, \xi_{jl+1}^\pm] = \pm hB_{ij} (\phi_{ik} \xi_{jl}^\pm + \xi_{jl}^\pm \phi_{ik}),\]
- \([\xi_{i,k+1}^\pm, \xi_{jl}^\pm] - [\xi_{i,k}^\pm, \xi_{jl+1}^\mp] = \pm hB_{ij} (\xi_{ik}^\pm \xi_{jl}^\mp + \xi_{jl}^\pm \xi_{ik}^\mp),\]
- \( i \neq j \) and \( n = 1 - A_{ij} \Rightarrow \text{Sym} [\xi_{i,k_1}^\pm [\xi_{i,k_2}^\pm \ldots [\xi_{i,k_n}^\pm, \xi_{jl}^\mp] \ldots]] = 0.\]
where \( Sym \) is the sum over all permutation of \( k_1, \ldots, k_n \). As before, our definition depends on a parameter \( h \). This means that we have defining relations for a whole family of Hopf algebras, \( Y_h(\mathfrak{g}) \). For all \( h \neq 0 \), the different \( Y_h(\mathfrak{g}) \) are isomorphic. For this reason we choose, as in the first definition, to specialise to \( h = 1 \), and we call this the Yangian of \( \mathfrak{g} \).

For a more complete introduction to Yangians and other quantum group see [23, 24].

### 2.1.4 The dilogarithm and related functions

The **dilogarithm** is the function defined by the power series

\[
Li_2(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^2}, \quad z \in \mathbb{C}, \ |z| < 1.
\]

It has an analytic continuation to \( z \in \mathbb{C} - (1, \infty) \), given by

\[
Li_2(z) = -\int_{0}^{z} \frac{\log(1-u)}{u} \, du.
\]

There are only eight special values for which this function can be computed exactly. These are

\[
0, \pm 1, \frac{1}{2}, \frac{3 - \sqrt{5}}{2}, -\frac{1 + \sqrt{5}}{2}, \frac{1 - \sqrt{5}}{2}, -\frac{1 - \sqrt{5}}{2}.
\]

Nevertheless it satisfies many functional equations, for example

\[
Li_2\left(\frac{1}{z}\right) = -Li_2(z) - \frac{\pi^2}{6} - \frac{1}{2} \log^2(-z),
\]

\[
Li_2(1 - z) = -Li_2(z) + \frac{\pi^2}{6} - \log(z) \log(1 - z).
\]
A more detailed discussion of the dilogarithm is found in [25, 26].

The Rogers dilogarithm is defined by

\[
L(x) = -\frac{1}{2} \int_0^x \left[ \frac{\log(1-u)}{u} + \frac{\log(u)}{1-u} \right] du.
\]

On the interval \((0, 1)\) it is related to the ordinary dilogarithm \(Li_2\) by

\[
L(x) = Li_2(x) + \frac{1}{2} \log(x) \log(1-x).
\]

This can be extended to the whole real axis by setting \(L(0) = 0, L(1) = \pi^2/6,\) and

\[
L(x) = \begin{cases} 
\frac{\pi^2}{3} - L(x^{-1}) & \text{if } x > 1, \\
L \left( \frac{1}{1-x} \right) - \frac{\pi^2}{6} & \text{if } x < 0.
\end{cases}
\] (2.5)

This function has many intriguing properties. It appears in various branches of mathematics, including number theory, algebraic K-theory, and the geometry of hyperbolic 3-manifolds.

Define a Riemann surface \(\hat{C}\) by

\[
\hat{C} = \{ (u, v) \in \mathbb{C}^2 \mid e^u + e^v = 1 \} \cup (\infty, 0) \cup (0, \infty).
\]

The function \(L\) has a certain analytic continuation that provides a natural group homomorphism

\[
L : \hat{C} \to \mathbb{C}/(2\pi i)^2 \mathbb{Z}.
\] (2.6)
This function, sometimes called the **enhanced dilogarithm**, is discussed in more detail by Zagier [27]. It is characterised by

\[ L(\infty, 0) = 0 \quad \text{and} \quad dL = \frac{1}{2} (udv - vdu) , \]

and has the following properties:

\[
\begin{align*}
L(0, \infty) & = \frac{\pi^2}{6} , \\
L(u + 2\pi i, v) & = L(u, v) + \pi i v , \\
L(u, v + 2\pi i) & = L(u, v) - \pi i u , \\
\end{align*}
\]

where \( e^u + e^v = 1 \). There is an embedding \((0, 1) \to \hat{\mathbb{C}}\) given by

\[
\begin{align*}
u & = \log(x) , \\
v & = \log(1 - x) .
\end{align*}
\]

Clearly \( L(u, v) = L(x) \) for \( x \in (0, 1) \).

### 2.1.5 The Bloch group and related structures

Consider the free abelian group with basis \([z], \ z \in \mathbb{C}^* - \{1\} \). Let \( A \) be the subgroup of elements

\[
\sum_{i=1}^{n} n_i [z_i], \quad n_i \in \mathbb{Z} ,
\]
such that $z_1, \ldots, z_n \in \mathbb{C}^* - \{1\}$ satisfy

$$\sum_{i=1}^{n} z_i \wedge (1 - z_i) = 0.$$ 

Here the sum is taken in the abelian group $\Lambda^2 \mathbb{C}^*$ (the set of all formal linear combinations of symbols $x \wedge y$, for $x, y \in \mathbb{C}^*$, subject to relations of linearity in $x, y$, of $x \wedge x = 0$ and $(x_1 x_2) \wedge y = x_1 \wedge y + x_2 \wedge y$).

For all $x, y \in \mathbb{C}^* - \{1\}$ with $xy \neq 1$, $\mathcal{A}$ contains the elements

$$[x] + \left[ \frac{1}{x} \right],$$

$$[x] + [1 - x],$$

$$[x] + [y] + \left[ \frac{1 - x}{1 - xy} \right] + [1 - xy] + \left[ \frac{1 - y}{1 - xy} \right].$$

Let $\mathcal{C}$ be the subgroup of $\mathcal{A}$ generated by all such elements. Then the **Bloch group** is defined as

$$B(\mathbb{C}) = \mathcal{A}/\mathcal{C}.$$ 

One weakness of the Bloch group is that it does not take into account the multi-valued nature of the log function. To deal with this problem we introduce an extension $B(\hat{\mathbb{C}})$ of $B(\mathbb{C})$, called the **extended Bloch group**. For a more detailed explanation see [17, 27].

The **torsion subgroup** of the Bloch group is the subgroup containing all elements of finite order.
In quantum field theory we deal with the torsion subgroup of $B(\mathbb{C})$, for which $(2\pi i)^2 L$ takes values in $\mathbb{Q}/\mathbb{Z}$. These values yield the conformal dimensions (more precisely the exponents $h_i - c/24$) of the fields of the theory.

### 2.2 Introduction to Conformal Field Theory

Here we give a short account of some important aspects of conformal field theory. The emphasis is on the ideas relevant to subsequent discussions. For a more detailed introduction to the subject see e.g. [18, 28, 29].

#### 2.2.1 Conformal invariance

Let $g_{\mu\nu}$ denote the metric tensor in $d$-dimensional space-time. Then a conformal coordinate transformation is an invertible mapping $x \rightarrow x'$, which leaves the metric tensor invariant up to scale

$$g_{\mu\nu}(x) \rightarrow g'_{\mu\nu}(x') = \Lambda(x)g_{\mu\nu}(x).$$  \hspace{1cm} (2.7)

The set of all conformal transformations forms the conformal group.

Abstractly a conformal field theory is a Euclidean quantum field theory whose symmetry group contains local conformal transformations (as well as the usual Euclidean symmetries). Local conformal symmetry is especially important in two dimensions since the corresponding symmetry algebra is infinite-dimensional. This
property means that two-dimensional conformal field theories have an infinite num-
ber of conserved quantities and are therefore completely solvable by symmetry con-
siderations alone. We now take a closer look at this important special case.

Conformal group in two dimensions

In two dimensions there is an infinite number of coordinate transformations that, al-
though not everywhere well-defined, are locally conformal. These are the analytic
maps from the complex plane to itself. This set is known as the local conformal
group, although strictly speaking it is not a group since the mappings are not nec-
essarily one-to-one and do not map the Riemann sphere to itself. Hence the need
to distinguish these local transformations from global conformal transformations,
which are well-defined everywhere.

The set of all analytic maps contains the six-parameter global conformal group,
which is the subset of mappings that are invertible and defined everywhere. The
group formed by these global transformations is often called the special conformal
group. It can be shown that the functions $f(z) = (az + b)/(cz + d)$, satisfying
$ad - bc = 1$, are the only globally defined invertible analytic maps. Hence we can
write the special conformal group as the set

$$\left\{ f(z) = \frac{az + b}{cz + d}; \quad ad - bc = 1, \quad a, b, c, d \in \mathbb{C} \right\}.$$ 

It is easy to see that this group can be parametrised by the set of complex $2 \times 2$
matrices with unit determinant, modulo the negative unit matrix, i.e. $SL(2, \mathbb{C})/\mathbb{Z}_2$. 
The distinction between local and global conformal groups is unique to the two-dimensional case. In higher dimensions all local conformal transformations are global. As already stated, local properties of conformal invariance are of more immediate interest than global ones, since it is the infinite-dimensionality of the local conformal group that allows so much to be known about conformally invariant field theories in two dimensions. With this in mind we now proceed to find the algebra of generators of the local conformal group. This is the Witt algebra, also known as the conformal algebra.

**Conformal generators and the Witt algebra**

To calculate the commutation relations of generators of the conformal algebra, we first note that any holomorphic infinitesimal transformation, \( z \to f(z) \), \( \bar{z} \to \bar{f}(\bar{z}) \), can be written as

\[
z' = z + \epsilon(z) \quad \text{where} \quad \epsilon(z) = \sum_{n=\infty}^{-\infty} c_n z^{n+1}, \quad n \in \mathbb{Z}.
\]

Then the complexified infinitesimal conformal symmetry transformations are locally generated by

\[
l_n = -z^{n+1} \partial_z \quad \text{and} \quad \bar{l}_n = -\bar{z}^{n+1} \partial_{\bar{z}}.
\]

The holomorphic generators \( \{l_n\} \) form a Lie algebra with commutation relations

\[
[l_n, l_m] = (n - m) l_{n+m}.
\]
This is the so-called \textbf{Witt algebra}. The corresponding anti-holomorphic generators form an isomorphic Lie algebra with commutation relations

\[ [\bar{l}_n, \bar{l}_m] = (n - m) \bar{l}_{n+m}. \]

These generators also satisfy the relation

\[ [l_n, \bar{l}_m] = 0. \]

Each of these two infinite-dimensional algebras contains a finite subalgebra generated by \{\(l_{-1}, l_0, l_1\}\}. This is the subalgebra associated with the global conformal group. In the quantum case the Witt algebra will be corrected to include an extra term proportional to the central charge (the so-called \textbf{central extension}). The unique central extension will be given by the \textbf{Virasoro algebra}.

\textbf{Primary and quasi-primary fields}

Given a field with scaling dimension \(\Delta\) and planar spin \(s\), define the holomorphic and antiholomorphic \textbf{conformal dimensions} \(h\) and \(\bar{h}\) by

\[ h = \frac{1}{2}(\Delta + s), \quad \bar{h} = \frac{1}{2}(\Delta - s). \]
Any field $\phi$ that satisfies the transformation property

$$\phi(z, \bar{z}) \rightarrow \phi'(f, \bar{f}) = \left(\frac{df}{dz}\right)^{-h} \left(\frac{d\bar{f}}{d\bar{z}}\right)^{-\tilde{h}} \phi(z, \bar{z}),$$

under any local conformal transformation $z \rightarrow f(z)$, $\bar{z} \rightarrow \bar{f}(\bar{z})$, is a primary field. Not all fields in a conformal field theory are primary fields. The rest are known as secondary fields. As we will see later, the importance of primary fields lies in their ability to generate the whole field content of a theory.

**Operator product expansion and the central charge**

The operator product expansion (OPE) expresses a product of two operator-valued fields at different points $z$ and $w$ as an infinite sum of single fields. In two-dimensional CFTs it is a convergent expansion. Although written without brackets, it is understood that the OPE is meaningful only within correlation functions.

For a single holomorphic field $\phi$ of conformal dimensions $(h, \tilde{h})$, the OPE of the energy-momentum tensor with $\phi$ is given by

$$T(z)\phi(w, \bar{w}) \sim \frac{h}{(z-w)^2}\phi(w, \bar{w}) + \frac{1}{z-w}\partial_w\phi(w, \bar{w}),$$

$$\bar{T}(\bar{z})\phi(w, \bar{w}) \sim \frac{\tilde{h}}{(\bar{z}-\bar{w})^2}\phi(w, \bar{w}) + \frac{1}{\bar{z}-\bar{w}}\partial_{\bar{w}}\phi(w, \bar{w}),$$

where the symbol $\sim$ means equality modulo expressions regular as $w \rightarrow z$.

In general the OPE of a holomorphic field $A(z)$ with an arbitrary field $B(w)$ can be
written as

\[ A(z) B(w) = \sum_{n=-\infty}^{N} \frac{A_{n-h(A)} B(w)}{(z-w)^n} . \]

Usually one writes

\[ A(z) = \sum z^{-n} A_{n-h(A)} , \]

for a holomorphic field \( A \), where \( A_n \) are linear maps and \( h \) denotes the holomorphic conformal dimension of the field \( A \).

The general OPE of the energy-momentum tensor with itself is

\[ T(z) T(w) \sim \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{holomorphic terms} . \]

\( c \) is a constant that depends on the specific model under consideration. This constant is called the central charge. Physically the central charge describes how a specific system reacts to the introduction of a macroscopic length scale.

**Virasoro algebra**

We have already seen that the classical generators of local conformal transformations obey the Witt algebra. We now show that the corresponding quantum generators obey a similar algebra with an added central extension term. This is the well-known **Virasoro algebra**.

The energy-momentum tensor has the following Laurent expansion in terms of
modes \( L_n, \bar{L}_n \):

\[
T(z) = \sum_{n \in \mathbb{Z}} z^{-n-2} L_n, \quad T(\bar{z}) = \sum_{n \in \mathbb{Z}} \bar{z}^{-n-2} \bar{L}_n .
\]

These modes are themselves operators and can be written as

\[
L_n = \frac{1}{2\pi i} \oint dz z^{n+1} T(z) , \quad \bar{L}_n = \frac{1}{2\pi i} \oint d\bar{z} \bar{z}^{n+1} \bar{T}(\bar{z}) .
\]

The mode operators \( L_n \) and \( \bar{L}_n \) are the quantum generators of the local conformal transformations on the Hilbert space. They obey the algebra

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} ,
\]

\[
[L_n, \bar{L}_m] = 0 ,
\]

\[
[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12} n(n^2 - 1) \delta_{n+m,0} .
\]

Each set of generators \( \{L_n\} \) and \( \{\bar{L}_n\} \) constitutes a copy of the so-called Virasoro algebra. It is worth noting that the central term is absent for the subalgebra \( \{L_{-1}, L_0, L_1\} \) belonging to the global conformal group.
Proof of commutation relations:

\[
[L_n, L_m] = \frac{1}{(2\pi i)^2} \oint_0 dw \int_w dz \frac{1}{(z-w)^{n+1}} \left\{ \frac{c/2}{(z-w)^4} + \frac{2T(w)}{(z-w)^2} + \frac{\partial T(w)}{z-w} + \text{reg.} \right\}
\]

\[
= \frac{1}{2\pi i} \oint_0 dw \int_w dz \frac{1}{(z-w)^{n+1}} \left\{ \frac{c}{12} (n+1)n(n-1)w^{n-2} + 2(n+1)w^n T(w) + w^{n+1} \partial T(w) \right\}
\]

\[
= \frac{c}{12} n(n^2-1) \delta_{n+m,0} + 2(n+1)L_{m+n} - \frac{1}{2\pi i} \oint_0 dw (n+m+2)w^{n+m+1}T(w)
\]

\[
= \frac{c}{12} n(n^2-1) \delta_{n+m,0} + (n-m)L_{m+n}.
\]

Every conformal field theory determines a representation of the Virasoro algebra for some value of \(c\). For \(c = 0\) there is no central extension term, and the Virasoro algebra reduces to the classical Witt algebra.

Conformal families

Primary fields are central to the study of conformal field theory as they allow such theories to be classified via the highest weight representations of the Virasoro algebra. By introducing operators

\[
L_{-n}(z) = \frac{1}{2\pi i} \oint_w dz \frac{1}{(z-w)^{n-1}} T(z),
\]

we can assign to each primary field \(\phi\) a conformal family \([\phi]\), obtained by successive actions of \(L_{-n}\) on \(\phi\)

\[
[\phi] := \{L_{-k_1} \ldots L_{-k_n} \phi : k_1 \geq k_2 \ldots \geq k_n > 0\}.
\]
Members of $[\phi]$ are called **descendant fields** or **secondary fields**. It can be shown that every conformal family defines a highest weight representation of the Virasoro algebra.

### Verma modules

We mentioned briefly that conformal families form representations of the Virasoro algebra

$$[L_n, L_m] = (n - m)L_{n+m} + \frac{c}{12}n(n^2 - 1)\delta_{n+m,0}.$$  

We now construct these representations. Representations of the antiholomorphic counterpart can be constructed similarly, and since the holomorphic and antiholomorphic components of the overall algebra decouple, its representations are got by taking tensor products.

To introduce the Hilbert space of states, we begin by defining the **vacuum state**, $|0\rangle$, of the theory by the condition

$$L_n|0\rangle = 0, \quad \text{for all } n \geq 0.$$  

To each primary field $\phi$ of conformal dimension $h$, we associate a **highest-weight state** $|h\rangle = \phi(0)|0\rangle$ satisfying

$$L_0|h\rangle = h|h\rangle \quad \text{and} \quad L_n|h\rangle = 0, \forall n > 0.$$
The remaining Virasoro generators produce an infinite space of *descendant states* by acting on the highest weight state. The space generated in this way is called a **Verma module**

\[ V_{c,h} := \{ L_{-k_1} \ldots L_{-k_n} |h\rangle : 0 \leq k_1 \leq \ldots \leq k_n, n \in \mathbb{N} \} . \]

Let \( V_{c,h} \) and \( \tilde{V}_{c,\tilde{h}} \) be the Verma modules generated by the sets \( \{ L_n \} \) and \( \{ \tilde{L}_n \} \) respectively, for central charge \( c \) and highest weights \( h \) and \( \tilde{h} \). Then the total state space is the direct sum

\[ \sum_{h, \tilde{h}} V_{c,h} \otimes \tilde{V}_{c,\tilde{h}} , \]

over all conformal dimensions of the theory.

Each Verma module corresponds to some representation of the Virasoro algebra. There is no guarantee that this representation is irreducible. This depends on the structure of the Virasoro algebra for given values of \( c \) and \( h \).

**Singular vectors**

A Verma module \( V_{c,h} \) is **reducible** if it contains a submodule that is itself a representation of the Virasoro algebra. Such a submodule is generated by a highest-weight state \( |v\rangle \) such that \( L_n |v\rangle = 0 \) for \( n > 0 \). Generally any state \( |v\rangle \), other than the highest-weight state, that is annihilated by all \( L_n \) (\( n > 0 \)) is called a **singular vector**.

\[ \text{Notice that the descendant state } L_{-k_1} \ldots L_{-k_n} |h\rangle \text{ is itself an eigenvector of } L_0 \text{ through } L_0 L_{-k_1} \ldots L_{-k_n} |h\rangle = (h + l) L_{-k_1} \ldots L_{-k_n} |h\rangle \text{ Here the integer } l = \sum_{i=1}^n k_i (k_i > 0) \text{ is called the level of the state.} \]
tor or null vector. It generates its own Verma module $V_v$ contained inside $V_{c,h}$.

From a Verma module $V_{c,h}$ containing one or more singular vectors, we can construct an irreducible representation of the Virasoro algebra by quotienting out of $V_{c,h}$ the null submodule. We denote by $M_{c,h}$ the representation obtained in this way.

Then the physical space has the structure

$$
\mathcal{H} = \bigoplus_{h,\bar{h}} n_{h,\bar{h}} M_{c,h} \otimes M_{c,\bar{h}},
$$

(2.8)

where $n_{h,\bar{h}}$ is the multiplicity with which the conformal weights $(h, \bar{h})$ occur.

Characters

To every highest weight module $V_{c,h}$ we associate a character $\chi_{c,h}(\tau)$ defined by

$$
\chi_{c,h}(\tau) = \text{Tr} q^{L_0 - c/24} = \sum_{n=0}^{\infty} \dim(h + n) q^{n + h - c/24},
$$

where $q \equiv \exp(2\pi i \tau)$ and $c, h \in \mathbb{Q}$. As before $h$ denotes the holomorphic conformal dimension, and $c$ is the central charge. Here $\dim(n + h)$ denotes the number of linearly independent states at level $n$ in the module, and it follows that characters can be viewed as generating functions for the number of states at any given level.

The formula for a generic Verma module character is

$$
\chi_{c,h}(\tau) = \frac{q^{h-c/24}}{\phi(q)},
$$

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where \( \phi(q) \) denotes the Euler \( \phi \) -function given by

\[
\frac{1}{\phi(q)} = \prod_{n=1}^{\infty} \frac{1}{1-q^n} = \sum_{n=0}^{\infty} p(n)q^n.
\]

Here \( p(n) \) denotes the number of partitions of the integer \( n \).

**Unitarity and the Kac determinant**

A Virasoro algebra representation is **unitary** if it contains no negative-norm states. A representation being unitary imposes certain constraints on \( c \) and \( h \). To analyse the unitarity of a given Virasoro algebra representation we consider the matrix of inner products between all basic states, the so-called **Gram matrix**, denoted \( (M_{ij}) \). This is a block diagonal matrix, with blocks \( M^{(l)} \) corresponding to states of level \( l \). The **Kac determinant** is the general formula for the determinant of the Gram matrix. It is given by

\[
det M^{(l)} = \alpha_l \prod_{r,s \geq 1, rs \leq l} \left[ h - h_{r,s}(c) \right]^{p(l-rs)} ,
\]

where \( p(l - rs) \) is the number of partitions of the integer \( l - rs \) and

\[
\alpha_l = \prod_{r,s \geq 1, rs \leq l} [(2r)^s s!]^{m(r,s)} ,
\]

\[
m(r, s) = p(l - rs) - p(l - r(s + 1)) .
\]
The roots of the Kac determinant can be expressed as

\[ c = 1 - \frac{6}{m(m+1)}, \]

\[ h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}. \]

The Kac determinant can be used to show that representations satisfying \( c \geq 1 \) and \( h \geq 0 \) are always unitary. For \( 0 < c < 1 \) and \( h > 0 \), all representations are non-unitary apart from the discrete set

\[ c = 1 - \frac{6}{m(m+1)}, \]

\[ h_{r,s}(m) = \frac{[(m+1)r - ms]^2 - 1}{4m(m+1)}, \]

where \( 1 \leq r < m \) and \( 1 \leq s \leq r \).
2.2.2 Models in conformal field theory

Minimal models

The family of conformal field theories defined by the conditions

\[ c = 1 - 6 \frac{(p - p')^2}{pp'}, \]

\[ h_{r,s} = \frac{(pr - p's)^2 - (p - p')^2}{4pp'}, \]

with \( 1 \leq r < p' \) and \( 1 \leq s < p \), are called \textbf{minimal models}\(^2\). They are particularly simple conformal field theories, this simplicity in principle allowing for a complete solution. Such theories are characterised by a Hilbert space made of a finite number of Verma modules. They describe discrete statistical models at their critical points.

Here we have only dealt with the holomorphic sector. As seen in (2.8) a physical theory is built from a tensor product of holomorphic and antiholomorphic modules, and a typical Hilbert space has the form

\[ \mathcal{H} = \bigoplus_{h, \bar{h}} M(c, h) \otimes \bar{M}(c, \bar{h}) . \]

How to combine the components of a minimal model is a more difficult question, but one particularly simple solution is to associate to each holomorphic module \( M_{c,h_{r,s}} \) the corresponding antiholomorphic module \( \bar{M}_{c,h_{r,s}} \). The Hilbert space of such

\(^2\)Every minimal model has central charge \( c < 1 \).
a theory is

\[ \mathcal{H} = \bigoplus_{1 \leq r < p', 1 \leq s < p} M(c, h_{r,s}) \otimes \bar{M}(c, h_{r,s}). \]

The resulting theory is called \textit{diagonal} since the two factors in each tensor product are identical.

In general we denote by \( \mathcal{M}(p, p') \) the minimal model associated with the pair \((p, p')\), and take as our convention \( p > p' \). A minimal model that satisfies \(|p - p'| = 1\) is called \textit{unitary}.

\textbf{WZW models}

A \textbf{Wess-Zumino-Witten model} is a simple model of conformal field theory whose solutions are realised by affine Kac-Moody algebras. Such models can be formulated directly in terms of an action.

The non-linear sigma model action is

\[ S_0 = \frac{k}{16\pi} \int d^2 x Tr'((\partial^\mu g^{-1} \partial_\mu g), \]

where \( g(x) \) is a matrix bosonic field living on the group manifold \( G \) associated to the Lie algebra \( g \), and \( Tr' \) indicates a representation independent normalisation. Quantisation of the model with this action leads to a theory without conformal invariance. To restore conformal invariance to the quantised theory we add the \textbf{Wess-Zumino} term

\[ S_{WZ} = -\frac{i}{24\pi} \int_B d^3 y \epsilon_{\alpha\beta\gamma} Tr'(\tilde{g} \partial^\alpha \tilde{g}^{-1} \partial^\beta \tilde{g}^{-1} \partial^\gamma \tilde{g}). \]
This is defined on a three-dimensional manifold $B$, whose boundary is the compactification of our original two-dimensional space. We have denoted by $\tilde{g}$ the extension of the field $g$ to this three-dimensional manifold. This extension is not unique, leading to the possibility of ambiguities in the Wess-Zumino term. Such ambiguities take values in $k\mathbb{Z}$. Hence the action $S_{WZW}$, defined below, is well-defined modulo discrete numbers $k\mathbb{Z}$. These discrete ambiguities vanish in the derivation of the classical equations of motion, hence the classical theory is well-defined.

The model with action

$$S_{WZW} = S_0 + kS_{WZ}, \quad k \in \mathbb{Z},$$

is called the **Wess-Zumino-Witten** model. It is often referred to as the $\hat{g}_k$-WZW model, where $k$ is the level of the affine Kac-Moody algebra $\hat{g}$.

The **Sugawara construction** gives an energy-momentum tensor of the form

$$T(z) = \frac{1}{2(k + h)} \sum_a (J^a J^a)(z),$$

(2.9)

where the $J^a$ are the components of the conserved currents, $h$ is the dual Coxeter number of the Lie algebra $g$, and $k$ is the level of $\hat{g}$. The central charge\(^3\) is given by

$$c(\hat{g}_k) = \frac{k \dim(g)}{k + h(g)}.$$  

(2.10)

An alternative definition of the $\hat{g}_k$-WZW model is the conformal field theory whose

\(^3\)Every WZW-model has central charge $c \geq 1$.  

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energy momentum tensor is given by (2.9).

**Coset models**

We now introduce a third class of models by means of the **coset construction**. This greatly increases the number of known solvable models. A **coset** is a quotient of two WZW models, with the central charge of the coset being the difference of the central charges of the two WZW components. It is expected that the coset construction will provide a framework for the complete classification of all rational conformal field theories. Coset models incorporate the two classes of models that we have already looked at

- WZW models are represented by trivial cosets,
- Clearly models with $c < 1$ can be represented by the coset construction, since the central charge of a coset is the difference of the central charges of the two WZW components. However all RCFTs with $c < 1$ are known to be minimal models. Hence any coset with $c < 1$ must provide a new representation of a minimal model.

An outline of the construction is as follows:

Let $\mathfrak{g}$ be a simple Lie algebra and $\hat{\mathfrak{g}}_k$ the corresponding affine Lie algebra. Let $\mathfrak{h}$ be a subalgebra of $\mathfrak{g}$. Denote by $L^0_m$ the Virasoro modes got from the $\hat{\mathfrak{g}}_k$ Sugawara construction. Then $L^0_m$ satisfies the Virasoro algebra with

$$c(\hat{\mathfrak{g}}_k) = \frac{k \dim(\mathfrak{g})}{k + h(\mathfrak{g})}.$$
The Virasoro modes $L_m^h$ for $h$ are defined analogously and satisfy the Virasoro algebra with
\[ c(\hat{h}) = \frac{l \dim(h)}{l + h(h)}. \]
Here $l$ denotes the level of the affine Lie algebra $\hat{h}$. The coset field theory contains the fields that are annihilated by $L_m^h$.

It can be shown that modes $L_m^{g/h} := L_m^g - L_m^h$ satisfy the Virasoro algebra with central charge given by
\[ c(\hat{g}/\hat{h}) = \frac{k \dim(g)}{k + h(g)} - \frac{l \dim(h)}{l + h(h)}. \tag{2.11} \]

This construction is often called the Goddard-Kent-Olive construction, for details see [30]. From now on, the quotient theory characterised by the energy-momentum tensor $T^g - T^h$ will be referred to as the coset $\hat{g}/\hat{h}$.

The central charge $c$ given in (2.11) must be non-negative. If $h$ is not simple but has two simple factors $h_1$ and $h_2$ (such that $h = h_1 \oplus h_2$) then (2.11) is modified to
\[ c = c(\hat{g}) - c(\hat{h}_1) - c(\hat{h}_2). \]

### 2.2.3 Partition functions and modular invariance

We have so far assumed conformal field theories to be defined on the whole complex plane. Physically this is not a very realistic situation as the holomorphic and antiholomorphic parts of the theory decouple completely. To impose more realistic
physical constraints on a conformal field theory, we look to couple the holomorphic and antiholomorphic sectors through the geometry of the space on which the theory is defined. For this purpose we consider conformal field theory on a torus. The interaction of the holomorphic and antiholomorphic sectors in this case is given by modular transformations.

Conformal field theory on the torus

Define a torus on the complex plane by specifying two linearly independent lattice vectors. These vectors are represented by complex numbers $\omega_1$ and $\omega_2$ respectively, called the periods of the lattice. Identify points that differ by an integer combination of these vectors. The properties of the conformal field theory defined on the torus do not depend on the overall scale of the lattice or on the absolute orientation of the lattice vectors. The relevant scale parameter is $\tau = \omega_2/\omega_1$, called the modular parameter.

The partition function

The partition function $Z$ of a CFT can be expressed in terms of the Virasoro characters (of the Verma modules forming the Hilbert space of the theory) as

$$Z = \sum_{h,\bar{h}} n_{h,\bar{h}} \chi_h(\tau) \chi_{\bar{h}}(\bar{\tau}),$$
where \( h \) and \( \bar{h} \) label a certain highest-weight state \(|h, \bar{h}\rangle\), and \( n_{h, \bar{h}} \) is the multiplicity of such a state. For rational conformal field theories this sum\(^4\) is always finite.

**Modular invariance**

The partition function must be independent of the particular periods \( \omega_{1,2} \) chosen for a given torus. This has the advantage of imposing certain constraints on the conformal field theory defined on the torus.

Suppose \( \omega'_{1,2} \) are two periods describing the same lattice as \( \omega_{1,2} \). Since they belong to the same lattice, the points \( \omega'_1 \) and \( \omega'_2 \) must be integer combinations of \( \omega_1 \) and \( \omega_2 \), and vice versa. Moreover, the unit cell of the lattice should have the same area no matter what periods are chosen. Writing

\[
\begin{pmatrix}
\omega'_1 \\
\omega'_2
\end{pmatrix}
= 
\begin{pmatrix}
a & b \\
c & d
\end{pmatrix}
\begin{pmatrix}
\omega_1 \\
\omega_2
\end{pmatrix},
\]

the conditions above restrict the choice of entries to \( a, b, c, d \in \mathbb{Z} \) with \( ad - bc = 1 \).

This directs us to consider the group of matrices \( SL(2, \mathbb{Z}) \).

Under the change of period above, the modular parameter transforms as

\[
\tau \rightarrow \frac{a \tau + b}{c \tau + d}, \quad ad - bc = 1.
\]

Clearly the signs of all parameters \( a, b, c, d \) can be simultaneously changed without affecting the overall transformation. Hence the symmetry of interest here is the

\[^4\text{For simplicity we will usually write this sum as } Z = \sum_{i,j} n_{ij} \chi_i(\tau)\chi_j(\bar{\tau}).\]
modular group $SL(2, \mathbb{Z})/\mathbb{Z}_2$. It can be shown that the modular transformations
$T: \tau \to \tau + 1$ and $S: \tau \to -\frac{1}{\tau}$ generate the whole of the modular group.

Modular invariance and characters

Conformal invariance requires the partition function $Z$ to be invariant under the
modular group. This places some restrictions on the characters $\chi_i$. One essential
property is the invariance of characters under the particular modular transformation
$\tau \to -1/\tau$, i.e. the fact that the $\chi_i$ can be written as

$$\chi_i(\tau) = \sum_k \tilde{a}_{ik} \tilde{q}^k,$$

where $\tilde{q} = \exp\left(-\frac{2\pi i}{\tau}\right)$, and the sum is over all $k$ of the form $h_i - c/24 + n$, where
$n \in \mathbb{N}$.

The connection to algebraic K-theory

Apart from the factor $q^{h_i-c/24}$, the characters $\chi_i$ can also be described combinatorially. We can write

$$\chi_i(\tau) = \sum_m q^{Q_i(m)} (q)_m,$$

where $Q_i(m) = \frac{mA_m}{2} + b_i m + h_i - \frac{c}{24}$, for characters of certain conformal field theories with integrable deformations. Here the matrix $A$ is the same for all characters $\chi_i$ of a given CFT.
It can be shown \cite{17} that a general sum of the form

\[ \sum_q \frac{q^{Q(m)}}{(q)_m}, \]

with \( Q(m) = \frac{mA_m}{2} + bm + h \), and rational coefficients, can only be modular when all solutions of the equations

\[ \sum_j A_{ij} \log(x_j) = \log(1 - x_i), \]

give finite order elements \( \sum_i [x_i] \) of the Bloch group of the algebraic numbers (or more precisely in the extension of the Bloch group that takes the multivaluedness of the logarithm into account).

There are many matrices \( A \) for which \( \sum_j A_{ij} \log(x_j) = \log(1 - x_i) \) yield torsion elements of the extended Bloch group. The best known examples are related to Dynkin diagrams. Given a pair of Dynkin diagrams \((X, Y)\), the matrix \( A \) is given by \( A(X, Y) = C(X) \otimes C(Y)^{-1} \). In chapter 3 we will consider the case \((X, Y) = (D_m, A_n)\).

It is expected that whenever \( \sum_j A_{ij} \log(x_j) = \log(1 - x_i) \) yields finite order elements of the Bloch group, the resulting \( x_i \) should be rational linear combinations of roots of unity, possibly apart from some \( \mathbb{Z}_2 \) extension. This has turned out to be true for all examples considered so far. Moreover there is a map from finite order elements of the Bloch group to the central charges and scaling dimensions of conformal field theories. This mapping is given by the dilogarithm function and will be described in chapter 3.
This suggests a clear, and very interesting, relationship between certain integrable quantum field theories in two dimensions and the algebraic K-theory of the complex numbers.
Chapter 3

Yangians and the \((D_m, A_n)\) Models

It is clear from the previous chapter that a study of integrable models described by pairs of Dynkin diagrams should yield interesting results. In this chapter we consider the particular models described by the pairs \((D_m, A_n)\). Using the representation theory of Yangians we solve the equations of these models in the general case. We demonstrate how to calculate the effective central charge using the dilogarithm formula, and finally we relate these models to the coset models described in chapter 2.
3.1 Overview

Equations of the model \((X, Y)\)

Consider the integrable model described by the pair of Dynkin diagrams \((X, Y)\). The equations of this model are of the form \(AU = V\), where the matrix \(A\) is given by

\[
A(X, Y) = C(X)^{-1} \otimes C(Y),
\]

and \(X\) and \(Y\) are Dynkin diagrams of ranks \(m\) and \(n\) respectively. Here \(U = \log(x)\) and \(V = \log(1 - x)\), such that the equation \(e^U + e^V = 1\) is satisfied. \(x\) denotes the vector \((x_{11}, \ldots, x_{mn})\).

Exponentiation of \(AU = V\) leads to a set of purely algebraic equations.

\[
AU = V \equiv \sum_j A_{ij} \log(x_j) = \log(1 - x_i) \quad (3.2)
\]

\[
\Rightarrow \prod_j x_j^{A_{ij}} = 1 - x_i. \quad (3.3)
\]

Notice that this exponentiation transforms a set of equations with infinitely many solutions \((3.2)\) into a set with a finite number of solutions \((3.3)\). For simplicity we choose to solve the equations in the form \((3.3)\), however care must be taken to choose logarithms in such a way that the original equations \((3.2)\) of the model are satisfied.

In some cases the algebraic equations \((3.3)\) can be solved using nothing more than elementary algebra. However as the matrix \(A\) grows in size this becomes more
difficult. By a suitable change of variables, the equations (3.3) can be written in a form that allows them to be solved relatively easily using the representation theory of Lie algebras and related quantum groups.

For this purpose we introduce the new variable $z = (z_{11}, \ldots, z_{mn})$, where

$$x = z^{-C(X) \otimes I_Y}.$$  \hfill (3.4)

The algebraic equations (3.3) can be rewritten in terms of $z$.

$$x^A = 1 - x \Rightarrow x^{-C(X)^{-1} \otimes C(Y)} = 1 - x$$

$$\Rightarrow z^{-I_X \otimes C(Y)} = 1 - z^{-C(X) \otimes I_Y}$$

$$\Rightarrow z^{2-C(Y)} + z^{2-C(X)} = z^2.$$  \hfill (3.5)

It turns out that the solutions of (3.5) arise naturally in representation theory, see [17] and references therein. In particular, for the model $(D_m, A_n)$, the components $z_{ij}$ of $z$ are the characters $Q^i_j$ of the Yangian $Y(D_m)$, that satisfy $Q^i_{n+1} = 1$ for $i = 1, 2, \ldots, m$. Hence a solution of (3.5) amounts to finding a matrix $g \in SO(2m)$, whose Yangian characters satisfy $Q^i_j(g) = 1$ for $i = 1, 2, \ldots, m$. Having found such a matrix, the equations (3.5) can be solved using the relation

$$z_{ij} = Q^i_j(g).$$

These $z_{ij}$ can easily be transformed into solutions $x_{ij}$ of (3.3) using equation (3.4).
Effective Central Charge Calculations

Denote solutions of the equations (3.3) by $x^i = (x_{i1}^1, \ldots, x_{imn}^i)$, where the superscript $i$ denotes the number of distinct solutions. In particular there is a unique solution, denoted $x^0$, whose components satisfy $0 < x_{jk}^0 \in \mathbb{R} < 1$ for all $j$ and $k$. For future reference we refer to $x^0$ as the ‘minimal solution’.

We are interested in the values taken by the solutions $x^i$ under the mapping

$$
\frac{6}{\pi^2} \sum_{jk=1, \ldots, mn} L(u_{jk}^i, v_{jk}^i) = c - 24h_i \mod 24\mathbb{Z} .
$$

(3.6)

Here $u_{jk}^i = \log(x_{jk}^i)$ and $v_{jk}^i = \log(1 - x_{jk}^i)$ form solutions of the equations $AU = V$, and torsion elements of the Bloch group. $L$ denotes the analytic continuation (2.6) of the Rogers dilogarithm satisfying

$$
L(u_{jk}^0, v_{jk}^0) = L_2(x_{jk}^0) + \frac{1}{2}u_{jk}^0v_{jk}^0 .
$$

(3.7)

The effective central charge is defined as the particular value of $c - 24h_i$ that arises from the minimal solution $x^0$. In this case we have

$$
c_{\text{eff}} = \frac{6}{\pi^2} \sum_{jk=1, \ldots, mn} L(u_{jk}^0, v_{jk}^0) .
$$

(3.8)

The factor of $24\mathbb{Z}$ in (3.6) essentially arises because the dilogarithm function is multi-valued outside its region of convergence $|z| < 1$. The only value of $c - 24h_i$ that can be calculated exactly is $c_{\text{eff}}$, since it corresponds to the minimal solution
(all of whose components are real and between 0 and 1). Although it will not always be mentioned, this mod \(24\mathbb{Z}\) term of course applies throughout the thesis.

**The matrix \(A\)**

The matrix \(A\) defined in (3.1) is in fact related to scattering matrices as follows. Suppose we take a system containing \(r\) different species of particles. Consider the scattering of two particles of types \(i\) and \(j\), and rapidities \(\theta_i\) and \(\theta_j\) respectively. For the type of system of interest to us, the particles merely pass through each other with some time delay (i.e. there is no exchange of particle quantum number). This time delay is described by the scattering matrix, with the scattering being described by an energy-dependent phase

\[
S_{ij} = e^{i f_{ij}(\theta_j - \theta_i)}. 
\]

The scattering matrix takes particular values at \(\pm \infty\). In terms of these values \(A\) is defined as

\[
A_{ij} = \frac{f_{ij}(-\infty) - f_{ij}(+\infty)}{2\pi}. 
\]

### 3.2 Quantum Groups and Yangians

Quantum groups were first introduced by Fadeev and collaborators. They arose from the quantum inverse scattering method [31], developed to construct and solve quantum integrable systems. In their original form quantum groups are associative
algebras whose defining relations are expressed in terms of a matrix of constants called a quantum R-matrix. This matrix depends on the particular integrable system under consideration. Quantum groups facilitate the understanding of solutions (R-matrices) of the quantum Yang-Baxter equation associated with such integrable systems. Furthermore they provide a general framework for finding new solutions. Of special importance are those solutions that depend on a spectral parameter. In particular those which are rational functions of this parameter arise from the family of quantum groups called Yangians. More recently quantum groups have arisen in connection with 1+1 dimensional integrable quantum field theories, as the algebras satisfied by certain non-local conserved currents. For example, Yangians appear as ‘quantum symmetry algebras’ in G-invariant Wess-Zumino-Witten models [32].

The term Yangian was introduced by V.G. Drinfeld [21] to specify those quantum groups related to rational solutions of the quantum Yang-Baxter equation. In fact Yangians are named after C.N. Yang who found the simplest such solution [6]. It is worth noting that Lüscher [33] effectively found much of the Yangian $Y(so_n)$ well in advance of the general construction.

Although quantum groups first appeared in the physics literature and many of the fundamental papers are written in the language of integrable systems, their properties are still accessible through more mainstream mathematical techniques. There are many unexpected connections between quantum groups and other seemingly unrelated areas of mathematics (for example knot theory and the representation theory of algebraic groups in characteristic p). In recent years these connections have sparked considerable interest in quantum groups.
3.2.1 Representation theory of Yangians

As mentioned above, the importance of Yangians stems from the fact that their finite-dimensional representations can be used to construct rational solutions of the quantum Yang-Baxter equation. The problem of describing all finite-dimensional irreducible representations of $Y(\mathfrak{g})$ was solved by Drinfeld himself. He gave a classification of such representations, similar to that for the Lie algebra $\mathfrak{g}$ in terms of highest weights (but without giving an ‘explicit’ realisation of these representations). Such a realisation was given for $\mathfrak{g} = \mathfrak{sl}_2$ by V. Chari and A. Pressley in [34].

An alternative approach to obtaining a better understanding of such representations would be to find a Yangian character formula, analogous to the Weyl character formula for Lie algebras. In [35, 36] Chari and Pressley give examples of such a formula for $Y(\mathfrak{sl}_2)$. Unfortunately their proof does not extend to other cases, and at present no general formula is known.

Irreducible Yangian representations

Suppose $\mathfrak{g}$ is a Lie algebra of rank $r$. Then $\mathfrak{g}$ has $r$ fundamental weights, denoted $\omega_1, \ldots, \omega_r$, one corresponding to each node on its Dynkin diagram. The fundamental representations of $\mathfrak{g}$ are the $r$ irreducible representations of highest weights $\omega_i$ ($i = 1, 2, \ldots, r$). Similarly $Y(\mathfrak{g})$ has $r$ fundamental (finite-dimensional) irreducible representations.

Since $\mathfrak{g} \subset Y(\mathfrak{g})$, any representation of $Y(\mathfrak{g})$ is automatically a representation of $\mathfrak{g}$. However, a representation which is $Y(\mathfrak{g})$-irreducible may become reducible when...
restricted to \( \mathfrak{g} \). In fact this is typically the case for the fundamental representations of \( Y(\mathfrak{g}) \).

Nevertheless, in some cases an irreducible \( Y(\mathfrak{g}) \)-representation remains \( \mathfrak{g} \)-irreducible. In the simplest situation, given an irreducible representation \( \rho \) of \( \mathfrak{g} \), in certain cases we can construct a representation \( \tilde{\rho} \) of \( Y(\mathfrak{g}) \) by

\[
\tilde{\rho}(I_a) = \rho(I_a), \quad \tilde{\rho}(J_a) = 0.
\]

These cases are described as follows.

Let \( a_i \) be the coefficient of the simple root \( \alpha_i \) in the expansion of the highest root \( \theta \) of \( \mathfrak{g} \). Put \( k_i = (\theta, \theta)/(\alpha_i, \alpha_i) \), and let \( \omega_i \) be the corresponding fundamental weight of \( \mathfrak{g} \). Then the irreducible representation of \( \mathfrak{g} \), of highest weight \( \lambda \), can be extended to an irreducible representation of \( Y(\mathfrak{g}) \) using (3.9) in the following cases:

1. \( \lambda = \omega_i \), when \( a_i = k_i \).
2. \( \lambda = t \omega_i \) (\( t \in \mathbb{N} \)), when \( a_i = 1 \).

Included in these cases are all fundamental representations of \( A_n \) and \( C_n \), and the vector and (half)-spinor representations of \( B_n \) and \( D_n \).

The more general case, in which an irreducible \( Y(\mathfrak{g}) \)-representation is \( \mathfrak{g} \)-reducible is significantly more complicated. For more details see [24].
3.3 The Lie Algebra $D_r$

3.3.1 Some properties of $D_r$

The Lie algebra $D_r$ has $r$ simple roots $\alpha_1, \ldots, \alpha_r$ given by

\[
\begin{align*}
\alpha_1 &= e_1 - e_2 , \\
\alpha_2 &= e_2 - e_3 , \\
&\vdots \\
\alpha_{r-1} &= e_{r-1} - e_r , \\
\alpha_r &= e_{r-1} + e_r ,
\end{align*}
\]

and $r$ fundamental weights $\omega_1, \ldots, \omega_r$ given by

\[
\begin{align*}
\omega_1 &= e_1 , \\
\omega_2 &= e_1 + e_2 , \\
&\vdots \\
\omega_{r-2} &= e_1 + \ldots + e_{r-2}, \\
\omega_{r-1} &= \frac{1}{2}(e_1 + e_2 + \ldots + e_{r-1} - e_r) , \\
\omega_r &= \frac{1}{2}(e_1 + e_2 + \ldots + e_{r-1} + e_r) .
\end{align*}
\]

The Weyl vector $\rho$ is given by

\[
\rho = (r - 1)e_1 + (r - 2)e_2 + \ldots + 2e_{r-2} + e_{r-1} .
\]
The quantity \( e_i \) is defined by its action on a diagonal \( n \times n \) matrix as

\[
e_i (\text{diag}(a_1, \ldots, a_n)) = a_i.
\]

The Weyl group of \( D_r \) is denoted \( W(D_r) \). It is an extension of \( S_r \) by \( (\mathbb{Z}_2)^{r-1} \). Its elements act on \( (e_1, \ldots, e_r) \) as

\[
(e_1, \ldots, e_r) \rightarrow (\epsilon_1 e_{s(1)}, \ldots, \epsilon_r e_{s(r)}),
\]

where \( s \in S_r, \epsilon_i \in \{\pm 1\}, \) and \( \prod_{i=1}^r \epsilon_i = 1 \). The order of \( W(D_r) \) is \( r! 2^{r-1} \).

**Representations of \( Y(D_r) \)**

In this thesis we follow the notation of Kirillov and Reshetikhin [37]. Again \( \mathfrak{g} \) is a simple Lie algebra of rank \( r \), with fundamental weights \( \omega_1, \ldots, \omega_r \). \( Y(\mathfrak{g}) \) is the corresponding Yangian.

Let \( W^j_i \) denote the irreducible representation of \( Y(\mathfrak{g}) \), of highest weight \( j \omega_i \). Here \( j \in \mathbb{N} \) and \( i = 1, 2, \ldots, r \). As mentioned in the previous section, \( W^j_i \) may become reducible when restricted to the Lie algebra \( \mathfrak{g} \).

For \( \mathfrak{g} = D_r \), the representations \( W^j_i \big|_{\mathfrak{g}} \) are irreducible in the cases

\[
(i, j) = (1, j), \quad (i, j) = (r-1, j), \quad (i, j) = (r, j).
\]
These correspond to the vector and half-spinor representations.

Define polynomials $Q^i_j$ by

$$Q^i_j = \text{ch} \left( W^i_j \right),$$

where $\text{ch}$ denotes the character of a representation.

It is claimed in [37] that for $g = D_r$, the functions $Q^i_j$ form the unique solution of the system of recurrence relations

$$(Q^i_j)^2 - Q^i_{j-1} Q^i_{j+1} = Q^{i-1}_j Q^{i+1}_j, \quad 1 \leq i \leq r - 3,$$  

$$(Q^{r-2}_j)^2 - Q^{r-2}_{j-1} Q^{r-2}_{j+1} = Q^{r-3}_j Q^{r-1}_j,$$  

$$(Q^{r-1}_j)^2 - Q^{r-1}_{j-1} Q^{r-1}_{j+1} = Q^{r-2}_j,$$  

$$(Q^r_j)^2 - Q^r_{j-1} Q^r_{j+1} = Q^{r-2}_j,$$  

(3.11)

with initial data given by

$$Q^0_j = 1,$$

$$Q^i_1 = \text{ch} \left( V(\omega_i) + V(\omega_{i-2}) + \ldots \right), \quad i = 1, \ldots, r - 2,$$

$$Q^{r-1}_1 = \text{ch} \left( V(\omega_{r-1}) \right),$$

$$Q^r_1 = \text{ch} \left( V(\omega_r) \right).$$  

(3.12)

Here $V(\omega_i)$ denotes the $i^{th}$ fundamental representation of $D_r$.

In future we refer to the equations (3.11) as the Kirillov-Reshetikhin (KR) equations. In the following section we prove a formula for the quantities $Q^1_1$ that agrees with the Yangian interpretation.
3.3.2 Solutions of the Kirillov-Reshetikhin equations

The paper [38] studies a class of 2-dimensional Toda equations on discrete space-time. These arise as functional relations for commuting families of transfer matrices in solvable lattice models associated with any classical Lie algebra $X_r$. For $D_r$ ($r \geq 4$) the relevant system of Toda equations is

$$
T^a_k(u-1)T^a_k(u+1) = T^a_{k+1}(u)T^a_{k-1}(u) + T^{a-1}_k(u)T^{a+1}_k(u)
$$

$$
1 \leq a \leq r - 3,
$$

$$
T^{r-2}_k(u-1)T^{r-2}_k(u+1) = T^{r-2}_{k+1}(u)T^{r-2}_{k-1}(u)
$$

$$
+ T^{r-3}_k(u)T^{r-1}_k(u)T^r_k(u),
$$

$$
T^a_k(u-1)T^a_k(u+1) = T^a_{k+1}(u)T^a_{k-1}(u) + T^{r-2}_k(u) a = r - 1, r.
$$

Here $k \in \mathbb{Z}_{\geq 0}$, $u \in \mathbb{C}$, and $a \in \{1, 2, \ldots, r\}$. The system is considered with the initial conditions $T^a_0(u) = 1$ for any $1 \leq a \leq r$. (Note that we also take $T^0_k = 1$).

The system of equations (3.13) can be solved iteratively to express an arbitrary $T^a_k(u)$ ($k \geq 1$) as a determinant or a Pfaffian of a matrix with entries 0 and $\pm T^b_k(u + \text{const.})$ for $0 \leq b \leq r$. Such solutions are given explicitly for the cases $B_r$, $C_r$, and $D_r$ in [38]. The proof is shown only for the case $C_r$, with the claim that it extends to the other cases. We carry out the proof for the $D_r$ case and it indeed works nicely. Notation and method follow exactly that of [38].
Notation

For any \( l \in \mathbb{C} \), put

\[
x_a^l = \begin{cases} 
  T_1^a(u + l) & \text{if } 1 \leq a \leq r , \\
  1 & \text{if } a = 0 .
\end{cases}
\] (3.14)

Introduce the infinite-dimensional matrices \( \mathcal{T} = (T_{ij})_{i,j \in \mathbb{Z}} \) and \( \mathcal{E} = (E_{ij})_{i,j \in \mathbb{Z}} \) by

\[
T_{ij} = \begin{cases} 
  x_{i,j+1}^{i,j+1} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i+j}{2} \in \{1,0,\ldots,3-r\} , \\
  -x_{i,j-1}^{i,j-1} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i+j}{2} = \frac{5}{2} - r , \\
  -x_{i,j-3}^{i,j-3} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i+j}{2} = \frac{3}{2} - r , \\
  -x_{i,j+3}^{i,j+3} & \text{if } i \in 2\mathbb{Z} + 1 \text{ and } \frac{i+j}{2} \in \{1-r,-r,\ldots,3-2r\} , \\
  0 & \text{otherwise .}
\end{cases}
\]

\[
E_{ij} = \begin{cases} 
  \pm 1 & \text{if } i = j - 2 \pm 2 \text{ and } i \in 2\mathbb{Z} , \\
  x_i^{r-1} & \text{if } i = j - 3 \text{ and } i \in 2\mathbb{Z} , \\
  x_i^{r-2} & \text{if } i = j - 1 \text{ and } i \in 2\mathbb{Z} , \\
  0 & \text{otherwise .}
\end{cases}
\]

We use the notation \( \mathcal{T}_k(i,j,\pm x_i^a) \) to mean the \( k \times k \) sub-matrix of \( \mathcal{T} \) whose \((i,j)\) element is exactly \( \pm x_i^a \). This definition is unambiguous since for any \( 1 \leq a \leq r \) and \( l \), the quantity \( \pm x_i^a \) is contained in \( \mathcal{T} \) exactly once as a matrix element. For a more detailed explanation refer to the original paper.
Theorem 1 For \( k \in \mathbb{Z}_{\geq 1} \),

\[
T_k^a(u) = \det \left( T_{2k-1}(1,1,x_{-k+1}^a) + \mathcal{E}_{2k-1}(2,3,x_{-k+r+a+4}) \right) \quad (3.15)
\]

\[1 \leq a \leq r - 2 , \]

\[
T_k^{r-1}(u) = \operatorname{pf} \left( T_{2k}(2,1,-x_{-k+1}^{r-1}) + \mathcal{E}_{2k}(1,2,x_{-k+1}^{r-1}) \right) , \quad (3.16)
\]

\[
T_k^r(u) = (-1)^k \operatorname{pf} \left( T_{2k}(1,2,-x_{-k+1}^r) + \mathcal{E}_{2k}(2,1,x_{-k+1}^r) \right) . \quad (3.17)
\]

solve the \( D_r \) system of equations

\[
T_k^a(u - 1)T_k^a(u + 1) = T_{k+1}^a(u)T_{k-1}^a(u) + T_k^{a-1}(u)T_k^{a+1}(u) \quad (3.18)
\]

\[1 \leq a \leq r - 3 , \]

\[
T_k^{r-2}(u - 1)T_k^{r-2}(u + 1) = T_{k+1}^{r-2}(u)T_{k-1}^{r-2}(u) + T_k^{r-3}(u)T_k^{r-1}(u) \quad (3.19)
\]

\[
T_k^{r-1}(u - 1)T_k^{r-1}(u + 1) = T_{k+1}^{r-1}(u)T_{k-1}^{r-1}(u) + T_k^{r-2}(u) \quad (3.20)
\]

Proof

Using equations (3.15), (3.16) and (3.17) we can show that

\[
T_k^{r-1}(u)T_k^r(u) = (-1)^k \det \left( T_{2k}(1,1,-x_{-k+1}^{r-1}) + \mathcal{E}_{2k}(2,2,x_{-k+1}^r) \right) , \quad (3.22)
\]

\[
T_k^{r-1}(u + 1)T_k^r(u - 1) = (-1)^k \det \left( T_{2k}(2,1,x_{1-k}^{r-2}) + \mathcal{E}_{2k}(1,1,x_{-k}^r) \right) , \quad (3.23)
\]

\[
T_{k+1}^{r-1}(u)T_k^r(u - 1) = (-1)^{k+1} \det \left( T_{2k+1}(1,1,-x_{-k+1}^{r-1}) + \mathcal{E}_{2k+1}(2,2,x_{-k}^r) \right) \quad (3.24)
\]

\[
T_k^{r-1}(u + 1)T_{k+1}^r(u) = (-1)^k \det \left( T_{2k+1}(2,1,x_{-k+1}^{r-2}) + \mathcal{E}_{2k+1}(1,1,x_{-k}^r) \right) . \quad (3.25)
\]
This is done by taking matrices

\[ M = T_{2k+1}(2, 1, -x_{r_{k+1}}^{-1}) + E_{2k+1}(1, 2, x_{r_{k+1}}^{-1}), \]
\[ M = T_{2k+1}(1, 2, -x_{r_{k+1}}^{-1}) + E_{2k+1}(2, 1, x_{r_{k+1}}^{-1}), \]
\[ M = T_{2k+2}(2, 1, -x_{r_{k+1}}^{-1}) + E_{2k+2}(1, 2, x_{r_{k+1}}^{-1}), \]
\[ M = T_{2k+2}(1, 2, -x_{r_{k+1}}^{-1}) + E_{2k+2}(2, 1, x_{r_{k+1}}^{-1}), \]

respectively, and using Jacobi’s identity as in [38].

**Jacobi’s Identity:**

\[
D_M \begin{bmatrix} 1 \\ 1 \end{bmatrix} D_M \begin{bmatrix} n \\ n \end{bmatrix} = D_M D_M \begin{bmatrix} 1, & n \\ 1, & n \end{bmatrix} + D_M \begin{bmatrix} 1 \\ n \end{bmatrix} D_M \begin{bmatrix} n \\ 1 \end{bmatrix}. 
\]

Here \( D_M \) is the determinant of any \( n \times n \) matrix \( M \), and \( D_M \begin{bmatrix} i_1, & i_2, & \ldots \\ j_1, & j_2, & \ldots \end{bmatrix} \) denotes its minor removing the \( i_k \)'s rows and the \( j_k \)'s columns.

The following facts are used in the proof:

1. If \( M \) is an odd anti-symmetric matrix then \( \det(M) = 0 \).
2. If \( M \) is an odd anti-symmetric matrix then \( D_M \begin{bmatrix} 1 \\ n \end{bmatrix} = D_M \begin{bmatrix} n \\ 1 \end{bmatrix} \).
3. If \( M \) is an even anti-symmetric matrix then \( D_M \begin{bmatrix} 1 \\ n \end{bmatrix} = -D_M \begin{bmatrix} n \\ 1 \end{bmatrix} \).
Having done this, (3.18), (3.19), (3.20) and (3.21) can be proved by again using the Jacobi identity, this time setting

\[
D_M = T_{k+1}^r(u),
\]
\[
D_M = T_{k+1}^{r-2}(u),
\]
\[
D_M = T_{k+1}^{r-1}(u)T_k^r(u - 1),
\]
\[
D_M = T_k^{r-1}(u + 1)T_{k+1}^r(u),
\]

respectively, with \( M \) taken as in the right hand sides of (3.22-3.25). This proves the theorem.

**A special case**

Of particular interest to us is the special case in which \( T \) is a constant, i.e. there is no \( u \)-dependence. In this case the structure of the \( T \)-system is exactly that of the Kirillov-Reshetikhin equations (3.11). In the case where each \( T \) is \( u \)-independent, the subindex \( l \) in the quantity \( x_l^a \) can be omitted, as this refers to the shift in the \( u \) argument, which is now irrelevant. In fact from now on we will abandon the \( x_l^a \) notation altogether as it is no longer necessary in this simpler special case. We revert to using \( T_1^a \) in place of \( x^a \). We get the following corollary of theorem 1.
Corollary 1 For $k \in \mathbb{Z}_{\geq 1}$,

$$T^a_k = \det \left( T_{2k-1}(1, 1, T^a_1) + \mathcal{E}_{2k-1}(2, 3, T^r_1) \right) ; 1 \leq a \leq r - 2 ,$$

$$T^{r-1}_k = \text{pf} \left( T_{2k}(2, 1, -T^{r-1}_1) + \mathcal{E}_{2k}(1, 2, T^r_1) \right) , \quad (3.26)$$

$$T^r_k = (-1)^k \text{pf} \left( T_{2k}(1, 2, -T^r_1) + \mathcal{E}_{2k}(2, 1, T^r_1) \right) ,$$

solve the $D_r$ Kirillov-Reshetikhin equations

$$ (T^a_k)^2 = T^a_{k+1} T^a_{k-1} + T^{a-1}_k T^{a+1}_k ; 1 \leq a \leq r - 3 ,$$

$$ (T^{r-2}_k)^2 = T^{r-2}_k T^{r-2}_{k-1} + T^{r-3}_k T^{r-1}_k T^r_k ,$$

$$ (T^{r-1}_k)^2 = T^{r-1}_k T^{r-1}_{k-1} + T^{r-2}_k ,$$

$$ (T^r_k)^2 = T^r_{k+1} T^r_{k-1} + T^{r-2}_k .$$

We now proceed to show that, under the initial conditions (3.12) used by Kirillov and Reshetikhin, the quantities $T^1_1, \ldots, T^r_1$ are exactly the Yangian characters $Q^1_1, \ldots, Q^r_1$.  

Recursion relation for $T^1_n$

(3.26) is a solution of the Kirillov-Reshetikhin equations in terms of determinants and Pfaffians of certain matrices with entries 0 and $\pm T^1_1, \ldots, \pm T^r_1$. In particular, for $n \geq 1$, $T^1_n$ is given by the determinant

$$T^1_n = \det \left( T_{2n-1}(1, 1, T^1_1) + \mathcal{E}_{2n-1}(2, 3, T^r_1) \right) .$$
Expansion of this determinant leads to the following recursion relation for the quantity $T_n^1$ ($n \geq 1$) in terms of $T_{n-1}^1$, $T_{n-2}^1$, $\ldots$, $T_1^1$ and $T_1^r, \ldots, T_r^r$:

$$T_n^1 = T_{n-1}^1 - T_{n-2}^1 - T_{n-3}^1 - \cdots - T_{n-r}^1 + T_r^r - \cdots$$

For simplicity we define the quantity $R_{n-k}$ by

$$R_{n-k} = T_{n-k}^1 + T_{n-k-2}^1 + T_{n-k-3}^1 + \ldots$$

Then $R_n = T_n^1 + R_{n-2}$.

**Theorem 2** The recursion equation (3.27) can be written in matrix form as

$$\begin{bmatrix}
T_n^1 \\
T_{n-1}^1 \\
T_{n-2}^1 \\
\vdots \\
T_{n-2r+3}^1 \\
R_{n-r+1} \\
R_{n-r}
\end{bmatrix} = \mathcal{M}
\begin{bmatrix}
T_{n-1}^1 \\
T_{n-2}^1 \\
T_{n-3}^1 \\
\vdots \\
T_{n-2r+2}^1 \\
R_{n-r} \\
R_{n-r-1}
\end{bmatrix}$$

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where $\mathcal{M}$ is the $2r \times 2r$ matrix

$$
\begin{pmatrix}
T^1_1 & -T^2_1 & \ldots & -T^{r-2}_1 & T^{r-1}_1 & T^{r-2}_1 & \ldots & T^2_1 & -T^1_1 & 1 & -(T^{r-1}_1 - (T^r_1)^2) & 2T^{r-1}_1 - T^r_1 \\
1 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 1 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 1 & 0 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 0 \\
\vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \ldots & \vdots & \vdots & \vdots & \vdots & \vdots \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & \ldots & 0 & 1 & 0 & \ldots & 0 & 0 & 0 & 0 & 1 \\
0 & 0 & \ldots & 0 & 0 & 0 & \ldots & 0 & 0 & 0 & 1 & 0
\end{pmatrix}
$$

with initial values $T^1_0 = T^{1-2r+2}_1 = 1$ and $T^1_{-j} = 0$ for $j = 1, 2, \ldots, 2r - 3$.

Note: This is the matrix $\mathcal{M}$ for $r$ even. For $r$ odd the matrix has the same structure with slight changes of sign. This does not affect subsequent calculations, and results are identical.

The eigenvalues of the matrix $\mathcal{M}$ are the solutions of the equation

$$
0 = 1 + \lambda^{2r} - T^1_1(\lambda^{2r-1} + \lambda) + (T^2_1 - 1)(\lambda^{2r-2} + \lambda^2) \\
- (T^3_1 - T^1_1)(\lambda^{2r-3} + \lambda^3) + \ldots + (T^{r-2}_1 - T^r_1)(\lambda^{r+2} + \lambda^{r-2}) \\
- (T^{r-1}_1 - T^r_1)(\lambda^{r+1} + \lambda^{r-1}) + ((T^{r-1}_1)^2 + (T^r_1)^2 - 2T^{r-2}_1)\lambda^r. \quad (3.28)
$$
Suppose the equation (3.28) has roots $a_1, \ldots, a_{2r}$. Notice that $0, \pm 1$ are not roots.

By the symmetry of equation (3.28), and the fact that $\prod_{i=1}^{2r} a_i = 1$, the roots can be written as $a_1, \ldots a_{r}^{-1}, a_r, \ldots, a_{r}^{-1}$, where for the purposes of ordering we identify $a_{r+1} = a_1^{-1}, \ldots, a_{2r} = a_{r}^{-1}$. Then the following equations must be satisfied:

\[
T_1^1 = \sum_{i=1}^{2r} a_i ,
\]

\[
T_1^2 - 1 = \sum_{i<j} a_i a_j ,
\]

\[
T_1^k - T_1^{k-2} = \sum_{i_1<\ldots<i_k} a_{i_1} \ldots a_{i_k} \text{ for } k = 3, \ldots, r - 2,
\]

\[
T_1^{r-1}T_1^r - T_1^{r-3} = \sum_{i_1<\ldots<i_{r-1}} a_{i_1} \ldots a_{i_{r-1}},
\]

\[
(T_1^{r-1})^2 + (T_1^r)^2 - 2T_1^{r-2} = \sum_{i_1<\ldots<i_r} a_{i_1} \ldots a_{i_r}.
\]

We consider the orthogonal group, $SO(2r)$, of linear transformations of $\mathbb{C}^{2r}$ which preserve the form $z_1 z_2 + z_3 z_4 + \ldots + z_{2r-1} z_{2r}$. A maximal torus consists of elements $g = \text{diag}(a_1, a_1^{-1}, \ldots, a_r, a_r^{-1}), a_i \in \mathbb{C}$.

**Theorem 3** For the choice (3.12) of initial conditions, and $g = \text{diag}(a_1, a_1^{-1}, \ldots, a_r, a_r^{-1})$, the quantities $T_1^i$ are exactly the Yangian characters $Q_i^1(g)$ for $i = 1, 2, \ldots, r$.

**Proof**

It is a well known fact that for $1 \leq k \leq r - 2$, the characters of the Lie algebra $D_r$ have the structure

\[
\chi(\omega_k) = \sum_{i_1<\ldots<i_k} a_{i_1} \ldots a_{i_k}.
\]
where $a \in \{a_1, a_1^{-1}, \ldots, a_r, a_r^{-1}\}$, as before. The final two characters have a slightly different structure: $\chi(\omega_{r-1}) = \sum a_1^{\pm \frac{1}{2}} \ldots a_r^{\pm \frac{1}{2}}$, where the sum is taken over all possible combinations with an odd number of negative exponents, and $\chi(\omega_r)$ is defined similarly but with an even number of negative exponents. This follows immediately from the structure of the fundamental weights of $D_r$, and the action of the Weyl group, see (3.10).
Now suppose we take initial conditions as in equations (3.12). We can then conclude the following:

\[ T^1_1 = \sum_{i=1}^{2r} a_i \Rightarrow T^1_1 = \chi(\omega_1) = Q^1_1 , \]

\[ T^2_1 - 1 = \sum_{i<j} a_ia_j \Rightarrow T^2_1 - 1 = \chi(\omega_2) \]
\[ \Rightarrow T^2_1 = \chi(\omega_2) + 1 \]
\[ \Rightarrow T^2_1 = Q^2_1 , \]

\[ T^3_1 - T^1_1 = \sum_{i<j<k} a_ia_ja_k \Rightarrow T^3_1 - T^1_1 = \chi(\omega_3) \]
\[ \Rightarrow T^3_1 = \chi(\omega_3) + T^1_1 \]
\[ \Rightarrow T^3_1 = \chi(\omega_3) + \chi(\omega_1) \]
\[ \Rightarrow T^3_1 = Q^3_1 , \]

\[ T^k_1 - T^{k-2}_1 = \sum_{i_1<\ldots<i_k} a_{i_1} \ldots a_{i_k} \Rightarrow T^k_1 - T^{k-2}_1 = \chi(\omega_k) \]
\[ \Rightarrow T^k_1 = T^{k-2}_1 + \chi(\omega_k) \]
\[ \Rightarrow T^k_1 = Q^{k-2}_1 + \chi(\omega_k) \quad \text{by induction on } k \]
\[ \Rightarrow T^k_1 = Q^k_1 \quad \text{for } 4 \leq k \leq r - 2 . \]
The final two equations

\[ T_r^{-1}T_r - T_r^{-3} = \sum_{i_1 < \ldots < i_{r-1}} a_{i_1} \ldots a_{i_{r-1}} \]

and

\[ T_r^{-1}T_r = \sum_{{i_1 < \ldots < i_{r-1}}} a_{i_1} \ldots a_{i_{r-1}} + Q_r^{-3} , \]

are satisfied by

\[ T_1^{r-1} = \chi(\omega_{r-1}) = Q_1^{r-1} , \]

and

\[ T_1^r = \chi(\omega_r) = Q_1^r . \]

Hence we can conclude that \( T_1^1, \ldots, T_1^r \) are in fact the Yangian characters \( Q_1^1, \ldots, Q_1^r \).

**Remark:** Using the Weyl character formula and the recursion relation (3.27), we can conclude that

\[ T_i^1(g) = Q_i^1(g) , \]

for \( g = \text{diag} \left( a_1, a_1^{-1}, \ldots, a_r, a_r^{-1} \right) \). This is done in the following theorem.

**Theorem 4** \( T_i^1(g) = Q_i^1(g) \) for all \( i \), where \( g = \text{diag} \left( a_1, a_1^{-1}, \ldots, a_r, a_r^{-1} \right) \).
Proof

Since $Q^1_i$ is irreducible for all $i$, we can write

$$Q^1_i = ch\left(V(i\omega_1)\right) = \sum_{w \in W(D_r)} sgn(w) w(a_1^{r-1+i}a_2^{r-2}\ldots a_{r-1})$$

by the Weyl character formula. Then clearly $Q^1_i = 1$ for $i = 0$ and $-2r + 2$, and $Q^1_i = 0$ for $i = -1, -2, \ldots, -2r + 3$.

We already know that $T^1_{i+1} = M T^1_i$, where $M$ is a $2r \times 2r$ matrix. Hence, if $T^1_i = ch\left(V(i\omega_1)\right)$ is true for any $2r$ values of $i$, then $T^1_i = ch\left(V(i\omega_1)\right)$ must be true for all $i$.

Notice that the recursion relation (3.27) is true for $n = 1, 0, -1, \ldots, -2r + 2$ if one puts $T^1_0 = T^1_{-2r+2} = 1, R_0 = R_{-1} = 0, and T^1_{-j} = 0$ for $j = 1, \ldots, 2r - 3$.

Then clearly for the $2r - 1$ values $i = 0, \ldots, -2r + 2$ we have $Q^1_i(g) = T^1_i(g)$. We also know from the previous theorem that $T^1_i = Q^1_i$. Hence $Q^1_i(g) = T^1_i(g)$ for all $i$. This proves the theorem.

3.4 Solving the Equations of the $(D_m, A_n)$ Model

In this section we study the integrable model described by the pair of Dynkin diagrams $(D_m, A_n)$. Using the representation theory of Yangians we solve the equations of the model.
3.4.1 Equations of the model

The equations of the model \((D_m, A_n)\) are \(AU = V\), where \(A = C(D_m)^{-1} \otimes C(A_n)\), \(U = \log(x)\), \(V = \log(1 - x)\), and \(x = (x_{11}, \ldots, x_{mn})\). By exponentiation and a change of variables they can be rewritten as

\[
z^{2-C(D_m)} + z^{2-C(A_n)} = z^2.
\] (3.29)

We seek a matrix \(g \in SO(2m)\) whose Yangian characters satisfy \(Q_{n+1}^i(g) = 1\) for \(i = 1, 2, \ldots, m\). This leads to the relation

\[
z_{ij} = Q_j^i(g),
\]

when \(z_{ij} = Q_j^i(g)\) as above. Here \(z_{ij}\) are the components of the solutions of (3.29). Write

\[
g = \text{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m) .
\]

**Theorem 5** Suppose \(g \in SO(2m)\) is a matrix that satisfies

\[
Q_{n+1}^i(g) = 1 ,
\] (3.30)

for \(i = 1, 2, \ldots, m\), and moreover \(Q_j^i(g) \neq 0\) for \(j = 1, \ldots, n\). Then \(g\) also satisfies
the set of equations

\[ Q_{n+2}^i(g) = 0, \]
\[ Q_{n+3}^i(g) = 0, \]
\[ \vdots \]
\[ Q_{n+2m-2}^i(g) = 0, \]
\[ Q_{n+2m-1}^i(g) = \pm 1, \]

for \( i = 1, 2, \ldots, m \). In particular

\[ Q_{n+2m-1} = \begin{cases} 
+1 & \text{for } i = 1, 2, \ldots, m - 2, \\
+1 & \text{for } i = m - 1, m, \text{ if } m \equiv 0 \text{ or } 1 \text{mod} 4, \\
-1 & \text{for } i = m - 1, m, \text{ if } m \equiv 2 \text{ or } 3 \text{mod} 4.
\end{cases} \]  

(3.32)

**Proof**

This has been proved by Nahm. The idea of the proof is the following. First one considers the variety of generic solutions of equations (3.11) with initial data given by (3.12) and its closure. This excludes solutions with an unwanted pattern of vanishing \( Q_{ij} \). Interchanging \( i \) and \( j \) in equations (3.11) allows to write down explicit algebraic relations between \( Q_{ij}^1 \) and \( Q_{n+2}^i \). These yield \( Q_{n+j}^1 = 0 \) for \( j = 2, \ldots, m - 1 \). Then one considers an analogous algebraic variety of solutions of equations (3.11) without imposing initial data at \( j = 0 \). The character formula for \( Q_j^1 \) is easily generalised to this larger variety. The algebraic equations remain valid and yield \( dQ_{n+m}^1 = 0 \). When one uses the character formula for \( Q_{n+m}^1 \) this
implies $Q_{n+m+j}^1 = Q_{n+m-j}^1$.

**Lemma 1** Any matrix $g \in SO(2m)$ that satisfies the equations

\[
Q_{n+1}^1(g) = 1,
Q_{n+2}^1(g) = 0,
Q_{n+3}^1(g) = 0,
\vdots
Q_{n+2m-2}^1(g) = 0,
Q_{n+2m-1}^1(g) = 1,
\]

and the two equations

\[
Q_{n+1}^m = 1, \tag{3.34}
Q_{n+2m-1}^m = \pm 1, \tag{3.35}
\]

also satisfies the set of equations (3.31).

**Proof**

This follows immediately from substitution of (3.33, 3.34, 3.35) in the Kirillov-Reshetikhin equations (3.11) for $D_m$.

One can conclude that the problem of finding a matrix $g$ whose characters satisfy (3.30) is equivalent to the problem of finding a matrix $g$ whose characters satisfy (3.33, 3.34, 3.35). We choose to work with the second set of conditions as
these involve only those irreducible Yangian representations that remain irreducible as representations of the corresponding Lie algebra. This simplifies matters greatly. To find a matrix \( g \in SO(2m) \) whose Yangian characters satisfy the equations (3.33, 3.34, 3.35), we first find a solution of the set of equations (3.33), and show that it is unique. We then show that the same solution satisfies (3.34) and (3.35) when square roots are chosen appropriately.

### 3.4.2 Some useful facts

**Notation**

Let \( W(D_{m-1})[k] \) denote the Weyl group of \( D_{m-1} \) acting on the elements

\[
\pm 1, \pm 2, \ldots, \pm (k - 1), \pm (k + 1), \ldots, \pm m .
\]

Any element of \( W(D_m) \) can be factorised uniquely as \( \sigma_k \omega \), \( k = 1, \ldots, m \), where \( \sigma_1 = 1, \sigma_k = (1k) \) for \( k \neq 1 \), and \( \omega \in W(D_{m-1})[k] \). Define the quantity \( A_k \) by

\[
A_k = \sum_{w \in W(D_{m-1})[k]} sgn(1k).sgn(w).w ((1k)(a_2^{-2}a_3^{-3} \cdots a_{m-2}^{-2}a_{m-1})) .
\]

**Lemma 2**

\[
A_k = (-1)^{k-1} \prod_{t=1}^{m} a_t^{2-m} \prod_{i<j} (a_ia_j - 1)(a_i - a_j) ,
\]

where \( t \neq k \) and \( i, j \in \{1, \ldots, m\} \) with \( i, j \neq k \).
Proof

For $k \neq 1$ we have

$$A_k = - \sum_{w \in W(D_{m-1})[k]} \text{sgn}(w) \cdot w \left((1k)(a_{m-2}m-3 \cdots a_{m-2}^2a_{m-1})\right)$$

$$= - \sum_{w \in W(D_{m-1})[k]} \text{sgn}(w) \cdot w \left((a_{m-2}^2 \cdots a_k^{m-k+1}a_{k+1}^{m-k-1} \cdots a_{m-2}^2a_{m-1})\right)$$

$$= -(-1)^{k-2} \sum_{w \in W(D_{m-1})[k]} \text{sgn}(w) \cdot w \left((a_{m-2}^2 \cdots a_k^{m-k+1}a_{k+1}^{m-k-1} \cdots a_{m-2}^2a_{m-1})\right).$$

This final expression in fact holds for all $k$ (including $k = 1$). Up to a factor of $(-1)^{k-1}$ this is the Weyl denominator for $D_{m-1}[k]$. Therefore, by the multiplicative formula for the Weyl denominator we can write

$$A_k = (-1)^{k-1}(a_1^2 \cdots a_k^{m-k+1}a_{k+1}^{m-k-1} \cdots a_{m-1}) \prod_{i<j} (a_i a_j - 1)(a_i a_j^{-1} - 1)$$

$$= (-1)^{k-1} \prod_{t=1}^{m} a_t^{2-m} \prod_{t<j} (a_i a_j - 1)(a_i - a_j),$$

where again $t \neq k$ and $i, j \in \{1, 2, \ldots, m\}$ with $i, j \neq k$.

Lemma 3

$$A_1A_2 \cdots A_m = \frac{\prod_{i>j}(a_i - a_j)^{m-2}(a_i a_j - 1)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}}.$$
Proof

\[ A_1 A_2 \ldots A_m = \left( \prod_{k=1}^{m} (-1)^{k-1} \right) \left( \prod_{i=1}^{m} a_i^{m-1} \right) \prod_{i<j}^{m} (a_i a_j - 1)^{m-2} (a_i - a_j)^{m-2} \]

\[ = \left( \prod_{k=1}^{m} (-1)^{k-1} \right) \left( \frac{\prod_{i<j}^{m} (a_i a_j - 1)^{m-2} (a_i - a_j)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}} \right) \]

\[ = (-1)^{(m-1)m} \left( \frac{\prod_{i<j}^{m} (a_i a_j - 1)^{m-2} (a_i - a_j)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}} \right) \]

\[ = (-1)^{(m-1)m} \left( (-1)^{m-2} \right)^{\frac{m(m-1)}{2}} \left( \frac{\prod_{i<j}^{m} (a_i a_j - 1)^{m-2} (a_i - a_j)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}} \right) \]

\[ = (-1)^{\frac{m(m-1)^2}{2}} \left( \frac{\prod_{i<j}^{m} (a_i a_j - 1)^{m-2} (a_i - a_j)^{m-2}}{\prod_{i=1}^{m} a_i^{(m-1)(m-2)}} \right) \]

\[ = \pm \prod_{i<j}^{m} (a_i a_j - 1)^{m-2} (a_i - a_j)^{m-2} \]

\[ \prod_{i=1}^{m} a_i^{(m-1)(m-2)} . \]

Lemma 4

\[
\begin{vmatrix}
1 & \ldots & \ldots & 1 \\
(a_1 + a_1^{-1}) & \ldots & \ldots & (a_m + a_m^{-1}) \\
\vdots & \vdots & \vdots & \vdots \\
(a_1^{m-1} + a_1^{1-m}) & \ldots & \ldots & (a_m^{m-1} + a_m^{1-m})
\end{vmatrix}
= \prod_{i>j}^{m} (a_i - a_j) (a_i a_j - 1).
\]
Proof

Clearly both sides of this equation are polynomials. To prove that they are equivalent, we must show that their zeros coincide. This implies equality up to a constant, which we show to be 1. The proof is in three steps.

1. Show that every zero of the RHS is also a zero of the LHS,

2. Show that both sides of the equation have the same order,

3. Show that the constant is equal to 1.

The RHS has a zero when \( a_i = a_j \) or when \( a_i = a_j^{-1} \) for \( i \neq j \). Both of these cases lead to rows \( i \) and \( j \) being equal in the determinant on the LHS, which implies the whole LHS is zero. Hence every zero of the RHS is also a zero of the LHS.

The left hand side can be rewritten as

\[
LHS = \begin{vmatrix}
 a_1^{m-1} & \ldots & \ldots & a_m^{m-1} \\
 a_1^m + a_1^{m-2} & \ldots & \ldots & a_m^m + a_m^{m-2} \\
 \vdots & \vdots & \vdots & \vdots \\
 a_1^{2m-2} + 1 & \ldots & \ldots & a_m^{2m-2} + 1
\end{vmatrix}.
\]

The term of highest order on the LHS is \( a_1^{m-1}a_2^m \ldots a_m^{2m-2} \). Hence the LHS has order

\[
\sum_{k=m-1}^{2m-2} k = \sum_{k=1}^{2m-2} k - \sum_{k=1}^{m-2} k = \frac{3m(m - 1)}{2} = 3\binom{m}{2}.
\]

Now for the order of the RHS. There are \( \binom{m}{2} \) terms of the form \( (a_i - a_j)(a_ia_j - 1) \), and each has order three. Hence the RHS has order \( 3\binom{m}{2} \). Thus the LHS and
the RHS have the same order. This proves that the two sides of the equation are equivalent up to a constant. To calculate this constant compare the coefficient of the term \( a_1^{m-1} a_2^m a_3^{m+1} \cdots a_{m-1}^{2m-3} a_m^{2m-2} \) on both sides of the equation. On the LHS this term occurs exactly once, coming from a product along the main diagonal of the matrix. On the RHS it arises from the product

\[
\prod_{i>j} (a_i a_j)(a_i) = \left( \prod_{i=1}^m a_i^{m-1} \right) (a_2 a_3^2 \cdots a_{m-1}^{m-2} a_m^{m-1}) = a_1^{m-1} a_2^m \cdots a_m^{2m-2}.
\]

Hence it occurs exactly once on both sides. If the coefficients on both sides agree for one term they must agree for every term, hence the constant is 1. This proves the theorem.

**Lemma 5** Let \( \chi \) denote some character of a Lie algebra \( \mathfrak{g} \), and let \( g \) be an element of some maximal torus. Suppose \( w_1 \in W(\mathfrak{g}) \) satisfies \( sgn(w_1) = -1 \) such that \( \chi(w_1(g)) = \chi(g) \). Then \( \sum_{w \in W} sgn(w) \chi(w(g)) = 0 \).

**Proof**

\[
\sum_{w \in W} sgn(w) \chi(w(g)) = \sum_{w \in W} sgn(ww_1) \chi(ww_1(g)) = - \sum_{w \in W} sgn(w) \chi(w(g)) = 0.
\]

**Lemma 6**

\[
\sum_{k=1}^m \left( a_k^{m-j} + a_k^{j-m} \right) A_k = 0
\]

for \( j = 2, \ldots, m \).
Proof

For notational simplicity set $\xi_j^{m-i} = (a_j^{m-i} + a_j^{-m+i})$. Then $A_k$ can be written as

$$A_k = sgn(1k) \sum_{S_{m-1}} sgn(w) w ((1k)\xi_2^{m-2} \xi_3^{m-3} \ldots \xi_{m-1}^1) .$$

Then for $j = 2, \ldots, m$ we get the following:

$$\sum_{k=1}^{m} (a_k^{m-j} + a_k^{-m+j}) A_k$$
$$= \sum_{k=1}^{m} \xi_k^{m-j} A_k$$
$$= \xi_1^{m-j} A_1 + \sum_{k=2}^{m} \xi_k^{m-j} A_k$$
$$= \sum_{S_{m-1}} sgn(w) \xi_1^{m-j} w (\xi_2^{m-2} \xi_3^{m-3} \ldots \xi_{m-1}^1)$$
$$- \sum_{k=2}^{m} \left( \sum_{S_{m-1}} sgn(w) \xi_k^{m-j} w (\xi_2^{m-2} \ldots \xi_{k-1}^{m-k+1} \xi_{k+1}^{m-k-1} \ldots \xi_{m-1}^1) \right)$$
$$= \sum_{S_m} sgn(w) w (\xi_1^{m-j} \xi_2^{m-2} \ldots \xi_j^{m-j} \ldots \xi_k^{m-k} \ldots)$$
$$= 0 .$$
3.4.3 Solution of the equations

By theorem 5 we need a matrix \( g = \text{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m) \) whose characters satisfy

\[
\begin{align*}
Q_{n+1}^{1}(g) & = 1, \\
Q_{n+j}^{1}(g) & = 0, \text{ for } j = 2, \ldots, m - 1, \\
Q_{n+2m}^{1}(g) & = 0, \text{ for } j = 2, \ldots, m - 1, \\
Q_{n+2m-1}^{1}(g) & = 1.
\end{align*}
\]
Using the Weyl character formula these can be rewritten as follows

\[ Q_{n+1}^1(g) = 1 \]
\[ \iff \sum_{i=1}^{m} (a_{i}^{n+m} + a_{i}^{-n-m}) A_{i} = \sum_{i=1}^{m} (a_{i}^{m-1} + a_{i}^{-1-m}) A_{i} \]
\[ \iff \sum_{i=1}^{m} ((a_{i}^{n+2m-1} - 1)a_{i}^{-1-m} + (a_{i}^{-n-2m+1} - 1)a_{i}^{m-1}) A_{i} = 0 , \]

\[ Q_{n+j}^1(g) = 0 \quad j = 2, \ldots, m-1 \]
\[ \iff \sum_{i=1}^{m} (a_{i}^{n+m+j-1} + a_{i}^{-n-m-j+1}) A_{i} = 0 \]
\[ \iff \sum_{i=1}^{m} ((a_{i}^{n+2m-1} - 1)a_{i}^{-j-m} + (a_{i}^{-n-2m+1} - 1)a_{i}^{m-j}) A_{i} = 0 , \]

\[ Q_{n+m}^1(g) = 0 \]
\[ \iff \sum_{i=1}^{m} (a_{i}^{n+2m-1} + a_{i}^{-n-2m+1}) A_{i} = 0 \]
\[ \iff \sum_{i=1}^{m} (a_{i}^{n+2m-1} - 2 + a_{i}^{-n-2m+1}) A_{i} = 0 , \]

\[ Q_{n+2m-j}^1(g) = 0 \quad j = 2, \ldots, m-1 \]
\[ \iff \sum_{i=1}^{m} (a_{i}^{n+3m-j-1} + a_{i}^{-n-3m+j+1}) A_{i} = 0 \]
\[ \iff \sum_{i=1}^{m} ((a_{i}^{n+3m-1} - 1)a_{i}^{-j-m} + (a_{i}^{-n-3m+1} - 1)a_{i}^{m-j}) A_{i} = 0 , \]

\[ Q_{n+2m-1}^1(g) = 1 \]
\[ \iff \sum_{i=1}^{m} (a_{i}^{n+3m-2} + a_{i}^{-n-3m+2}) A_{i} = \sum_{i=1}^{m} (a_{i}^{m-1} + a_{i}^{-1-m}) A_{i} \]
\[ \iff \sum_{i=1}^{m} ((a_{i}^{n+2m-1} - 1)a_{i}^{m-1} + (a_{i}^{-n-2m+1} - 1)a_{i}^{-1-m}) A_{i} = 0 . \]
Note: In obtaining the above equations we used Lemma 6.

We now have a system of $2m - 1$ equations in $2m$ variables. The variables are $a_i$ and $a_i^{-1}$ with $i = 1, \ldots, m$. By summing the equations for $Q^1_{n+j}$ and $Q^1_{n+2m-j}$ for $j = 1, \ldots, m - 1$, we get a system of $m$ equations in $m$ variables. This time the variables are $a_i^{n+2m-1} + a_i^{-n-2m+1}$ for $i = 1, \ldots, m$. These equations can be solved exactly. They can be written in matrix form as

$$
\begin{pmatrix}
A_1 & \ldots & \ldots & A_m \\
(a_1 + a_1^{-1})A_1 & \ldots & \ldots & (a_m + a_m^{-1})A_m \\
\vdots & \vdots & \vdots & \vdots \\
(a_1^{m-1} + a_1^{1-m})A_1 & \ldots & \ldots & (a_m^{m-1} + a_m^{1-m})A_m
\end{pmatrix}
\begin{pmatrix}
a_1^{n+2m-1} - 2 + a_1^{-n-2m+1} \\
a_2^{n+2m-1} - 2 + a_2^{-n-2m+1} \\
\vdots \\
a_m^{n+2m-1} - 2 + a_m^{-n-2m+1}
\end{pmatrix}
= \begin{pmatrix}
0 \\
\vdots \\
0 \\
0
\end{pmatrix}
$$

(3.36)

For $i = 1, \ldots, m$ this implies

$$
a_i^{n+2m-1} - 2 + a_i^{-n-2m+1} = 0$$

$$
a_i^{-n-2m+1} \left( a_i^{n+2m-1} - 1 \right)^2 = 0$$

$$
a_i^{n+2m-1} = 1$$

since $a_i \neq 0$. 

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Theorem 6 Any matrix $g = \text{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m)$, whose entries satisfy
\[ a_i^{n+2m-1} = 1, \quad (3.37) \]
for $i = 1, \ldots, m$, is a solution of the equations (3.33).

Proof

\[ a_i^{n+2m-1} = 1 \quad \Rightarrow \quad a_i^{n+m} = a_i^{1-m} \]
\[ \Rightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w(a_i^{n+m}a_2^{m-2} \ldots a_{m-1}) \]
\[ = \sum_{w \in W(D_m)} \text{sgn}(w).w(a_1^{1-m}a_2^{m-2} \ldots a_{m-1}) \]
\[ = \sum_{w \in W(D_m)} \text{sgn}(w).w(w_1(a_1^{1-m}a_2^{m-2} \ldots a_{m-1})) \]
where $w_1$ is the identity with signs $(- + \ldots + -)$
\[ = \sum_{w \in W(D_m)} \text{sgn}(w).w(a_1^{m-1}a_2^{m-2} \ldots a_{m-1}) \]
\[ \Rightarrow Q^1_{n+1}(g) = 1. \]

\[ a_i^{n+2m-1} = 1 \quad \Rightarrow \quad a_i^{n+3m-2} = a_i^{m-1} \]
\[ \Rightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w(a_1^{n+3m-2}a_2^{m-2} \ldots a_{m-1}) \]
\[ = \sum_{w \in W(D_m)} \text{sgn}(w).w(a_1^{m-1}a_2^{m-2} \ldots a_{m-1}) \]
\[ \Rightarrow Q^1_{n+2m-1}(g) = 1. \]
\[ a_i^{n+2m-1} = 1 \Rightarrow a_i^{n+m+j-1} = a_i^{j-m} \]
\[ \Rightarrow \sum_{w \in W(D_m)} sgn(w).w(a_i^{n+m+j-1}a_2^{m-2} \cdots a_{m-1}) \]
\[ = \sum_{w \in W(D_m)} sgn(w).w(a_i^{-m+j}a_2^{m-2} \cdots a_{m-1}) \]
\[ = 0 \text{, by Lemma 5 using } w_1 = (1j) \]
\[ \Rightarrow Q_{n+j}^1(g) = 0 \text{ for } j = 2, \ldots, m - 1. \]

\[ a_i^{n+2m-1} = 1 \Rightarrow a_i^{n+3m-j-1} = a_i^{m-j} \]
\[ \Rightarrow \sum_{w \in W(D_m)} sgn(w).w(a_i^{n+3m-j-1}a_2^{m-2} \cdots a_{m-1}) \]
\[ = \sum_{w \in W(D_m)} sgn(w).w(a_i^{m-j}a_2^{m-2} \cdots a_{m-1}) \]
\[ = 0 \text{, by Lemma 5 using } w_1 = (1j) \]
\[ \Rightarrow Q_{n+2m-j}^1(g) = 0 \text{ for } j = 2, \ldots, m - 1. \]

\[ a_i^{n+2m-1} = 1 \Rightarrow \sum_{w \in W(D_m)} sgn(w).w(a_i^{n+2m-1}a_2^{m-2} \cdots a_{m-1}) \]
\[ = \sum_{w \in W(D_m)} sgn(w).w(a_2^{m-2} \cdots a_{m-1}) \]
\[ = 0 \text{, by Lemma 5 using } w_1 = (1m) \]
\[ \Rightarrow Q_{n+m}^1(g) = 0. \]

**Theorem 7** Condition (3.37) is necessary for solutions of the \((D_m, A_n)\) equations, provided \(a_i \neq a_j^{\pm 1}\) for \(i, j = 1, 2, \ldots, m\) and \(i \neq j\).

**Proof**

The solution is unique if and only if the determinant of the matrix in equation (3.36) is non-zero.
\[
\det = \begin{vmatrix}
A_1 & \ldots & \ldots & A_m \\
(a_1 + a_1^{-1})A_1 & \ldots & \ldots & (a_m + a_m^{-1})A_m \\
\vdots & \vdots & \ddots & \vdots \\
(a_1^{m-1} + a_1^{-m})A_1 & \ldots & \ldots & (a_m^{m-1} + a_m^{-m})A_m \\
\end{vmatrix}
\]

\[
= \prod_{i=1}^{m} A_i \left( \prod_{i>j} (a_i - a_j)(a_i a_j - 1) \right) \left( \prod_{i=1}^{m} a_i^{m-1} \right)
\]

This is non-zero provided \( a_i \neq a_j \) and \( a_i \neq a_j^{-1} \) for \( i \neq j \).

**Remark:** Nahm’s proof of theorem 5 also implies that \( a_i \neq a_j^{\pm 1} \) is necessary for \( i \neq j \).

We now show that this particular choice of the matrix \( g \) also satisfies the equations (3.34, 3.35).

**Theorem 8** A matrix \( g = \text{diag}(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1}) \in SO(2m) \), whose entries satisfy \( a_i^{n+2m-1} = 1 \) for \( i = 1, 2, \ldots, m \), is a solution of the equation \( Q_{n+1}^m(g) = 1 \),
provided square roots of the \( a_i \) are chosen to satisfy

\[
\prod_{i=1}^{m} a_i^{n+2m-1} = \begin{cases} 
+1 & \text{if } m \equiv 0 \text{ mod } 4 \text{ or } m \equiv 1 \text{ mod } 4 , \\
-1 & \text{if } m \equiv 2 \text{ mod } 4 \text{ or } m \equiv 3 \text{ mod } 4 .
\end{cases}
\]  

(3.38)

**Proof**

\[
Q_{n+1}^m(g) = 1
\]

\[
\Leftrightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{n+2m-1} a_2^{n+2m-1} \cdots a_{m-1}^{n+2m-1} a_m^{n+2m-1-(m-2)} \right) = 1
\]

\[
\Leftrightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right) = \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right)
\]

\[
\Leftrightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right) = \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right)
\]

\[
\Leftrightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right) = \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right)
\]

\[
\Leftrightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right) = \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right)
\]

\[
\Leftrightarrow \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right) = \sum_{w \in W(D_m)} \text{sgn}(w).w \left( a_1^{m-1} a_2^{m-2} \cdots a_{m-2}^{2} a_{m-1} \right)
\]
where
\[
\hat{w} = \begin{cases} 
(1m)(2, m - 1) \ldots \left( \frac{m}{2}, \frac{m}{2} + 1 \right) & \text{for } m \text{ even}, \\
(1m)(2, m - 1) \ldots \left( \frac{m-1}{2}, \frac{m+1}{2} \right) & \text{for } m \text{ odd}.
\end{cases}
\] (3.39)

Clearly for the equation \(Q_{n+1}^m(g) = 1\) to be satisfied, we must choose the signs \(\pm \sqrt{a_i}\) such that the equation
\[
\prod_{i=1}^{m} a_i^{n+2m-1} = sgn(\hat{w})
\]
is satisfied. This amounts to choosing square root signs that satisfy
\[
\prod_{i=1}^{m} a_i^{n+2m-1} = \begin{cases} 
+1 & \text{if } m \equiv 0 \mod 4 \text{ or } m \equiv 1 \mod 4, \\
-1 & \text{if } m \equiv 2 \mod 4 \text{ or } m \equiv 3 \mod 4.
\end{cases}
\] (3.40)

**Theorem 9** Let \(g = diag(a_1, a_1^{-1}, \ldots, a_m, a_m^{-1})\) be a matrix whose entries satisfy the conditions (3.37) and (3.40). Then \(g\) satisfies the equation \(Q_{n+2m-1}^m = \pm 1\).

**Proof**

The character \(Q_{n+2m-1}^m(g)\) has numerator
\[
\sum_{w \in W(D_m)} sgn(w) \cdot w \left( \prod_{i=1}^{m} a_i^{n+2m-1} \right) \cdot w \left( a_1^{m-1} a_2^{m-2} \ldots a_{m-2}^2 a_{m-1} \right)
= \pm \sum_{w \in W(D_m)} sgn(w) \cdot w \left( a_1^{m-1} a_2^{m-2} \ldots a_{m-2}^2 a_{m-1} \right)
= \pm Q_0,
\]

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where $Q_0$ denotes the Weyl denominator. Hence by (3.40)

$$Q_{n+2m-1}(g) = \begin{cases} 
+1 & \text{if } m \equiv 0 \mod 4 \text{ or } m \equiv 1 \mod 4 \\
-1 & \text{if } m \equiv 2 \mod 4 \text{ or } m \equiv 3 \mod 4.
\end{cases}$$

### 3.5 Effective Central Charge Calculations

Given a model described by a pair of Dynkin diagrams, we can calculate the effective central charge of the corresponding conformal field theory using the dilogarithm formulae. In this section we carry out a detailed example of this calculation for the $(D_3, A_2)$ case, and we summarise the results of such calculations for numerous other cases.

#### 3.5.1 Detailed example

To show how these calculations work we look at one particular example in more detail. Consider the model described by the pair of Dynkin diagrams $(D_3, A_2)$. We solve the equations $AU = V$ and use the solutions to calculate the effective central charge (and other values of $c - 24h$) of the corresponding conformal field theory.

The equations of this model are $AU = V$ where $A = C(D_3)^{-1} \otimes C(A_2)$, $U = \log(x)$, $V = \log(1 - x)$, and $x = (x_{11}, x_{12}, x_{21}, x_{22}, x_{31}, x_{32})$. Exponentiation leads to a set of algebraic equations

$$x^A = 1 - x .$$  
(3.41)
By the change of variable
\[ x = z^{-C(D_3)\otimes I_2}, \]
the equations (3.41) can be written as
\[ z(2-C(D_3))\otimes I_2 + z I_3 \otimes (2-C(A_2)) = z^2 \Rightarrow \]
\[ \begin{align*}
(z_{11})^2 &= z_{21} z_{31} + z_{12}, \\
(z_{12})^2 &= z_{22} z_{32} + z_{11}, \\
(z_{21})^2 &= z_{11} + z_{22}, \\
(z_{22})^2 &= z_{12} + z_{21}, \\
(z_{31})^2 &= z_{11} + z_{32}, \\
(z_{32})^2 &= z_{12} + z_{31},
\end{align*} \] (3.43)
where \( z = (z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}) \). The first step is to solve these equations. The \( z_{ij} \) are the characters \( Q_j^i(g) \) of representations of the Yangian \( Y(D_3) \), for some specially chosen matrix \( g \in SO(6) \).

**Solution**

Choose the matrix \( g = diag(a_1, a_1^{-1}, a_2, a_2^{-1}, a_3, a_3^{-1}) \) \( \in SO(6) \) so that the entries satisfy
\[ a_i^7 = 1, \text{ with } a_i \neq a_j^{\pm 1} \text{ for } i \neq j. \]
As discussed earlier the square root signs must be carefully chosen in order to satisfy the equations \( a_1^7 a_2^7 a_3^7 = -1 \). This results in four choices for the matrix \( g \).

\[
(a_1, a_2, a_3) = \begin{cases} 
(1, e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}), & \\
(1, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}), & \\
(1, e^{\frac{8\pi i}{7}}, e^{\frac{10\pi i}{7}}), & \\
(e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}). & 
\end{cases}
\]

Characters

The three fundamental Yangian characters are written in terms of entries of \( g \) as

\[
\begin{align*}
Q_1^1(g) &= a_1 + a_2 + a_3 + a_1^{-1} + a_2^{-1} + a_3^{-1}, \\
Q_1^2(g) &= a_1^{-1/2} a_2^{-1/2} a_3^{-1/2} + a_1^{-1/2} a_2^{-1/2} a_3^{-1/2} + a_1^{-1/2} a_2^{-1/2} a_3^{-1/2} + a_1^{-1/2} a_2^{-1/2} a_3^{-1/2}, \\
Q_1^3(g) &= a_1^{1/2} a_2^{1/2} a_3^{1/2} + a_1^{1/2} a_2^{1/2} a_3^{1/2} + a_1^{1/2} a_2^{1/2} a_3^{1/2} + a_1^{1/2} a_2^{1/2} a_3^{1/2}.
\end{align*}
\]

Having computed these characters (using the Weyl character formula), the Kirillov-Reshetikhin equations are used to calculate the characters \( Q_2^1(g), Q_2^2(g) \) and \( Q_2^3(g) \).

The KR equations for \( D_3 \) are

\[
\begin{align*}
(Q_j^1)^2 - Q_{j-1}^1 Q_{j+1}^1 &= Q_j^2 Q_j^3, \\
(Q_j^2)^2 - Q_{j-1}^2 Q_{j+1}^2 &= Q_j^1, \\
(Q_j^3)^2 - Q_{j-1}^3 Q_{j+1}^3 &= Q_j^1.
\end{align*}
\]

The identification \( z_{ij} = Q_j^i(g) \) means that we have now found the solutions \( z = (z_{11}, z_{12}, z_{21}, z_{22}, z_{31}, z_{32}) \) of the equations \( z^{(2-C(D_3)) \otimes I_2} + z^{I_3 \otimes (2-C(A_2))} = z^2 \).
We compute the values of $x_{ij}$ using the relation

$$x = z^{-C(D_3) \otimes I_2} \Rightarrow \begin{cases} 
  x_{11} &= z_{11}^{-2} z_{21} z_{31} , \\
  x_{12} &= z_{12}^{-2} z_{22} z_{32} , \\
  x_{21} &= z_{11} z_{21}^{-2} , \\
  x_{22} &= z_{12} z_{22}^{-2} , \\
  x_{31} &= z_{11} z_{31}^{-2} , \\
  x_{32} &= z_{12} z_{32}^{-2} .
\end{cases}$$

The logarithms of these solutions must be chosen so as to satisfy

$$\log(x) = ( -C(D_3) \otimes I_2 ) \log(z) ,$$

which are equivalent to the equations

$$\begin{cases} 
  u_{11} = \log(x_{11}) &= -2 \log(z_{11}) + \log(z_{21}) + \log(z_{31}) , \\
  u_{12} = \log(x_{12}) &= -2 \log(z_{12}) + \log(z_{22}) + \log(z_{32}) , \\
  u_{21} = \log(x_{21}) &= \log(z_{11}) - 2 \log(z_{21}) , \\
  u_{22} = \log(x_{22}) &= \log(z_{12}) - 2 \log(z_{22}) , \\
  u_{31} = \log(x_{31}) &= \log(z_{11}) - 2 \log(z_{31}) , \\
  u_{32} = \log(x_{32}) &= \log(z_{12}) - 2 \log(z_{32}) ,
\end{cases}$$

and

$$\log(1 - x) = ( -I_3 \otimes C(A_2) ) \log(z) ,$$
which is equivalent to

\[
\begin{align*}
  v_{11} &= \log(1 - x_{11}) = -2 \log(z_{11}) + \log(z_{12}), \\
  v_{12} &= \log(1 - x_{12}) = \log(z_{11}) - 2 \log(z_{12}), \\
  v_{21} &= \log(1 - x_{21}) = -2 \log(z_{21}) + \log(z_{22}), \\
  v_{22} &= \log(1 - x_{22}) = \log(z_{21}) - 2 \log(z_{22}), \\
  v_{31} &= \log(1 - x_{31}) = -2 \log(z_{31}) + \log(z_{32}), \\
  v_{32} &= \log(1 - x_{32}) = \log(z_{31}) - 2 \log(z_{32}).
\end{align*}
\]

Then the pairs \((u_{jk}, v_{jk}) \equiv (\log(x_{jk}), \log(1 - x_{jk}))\) are the solutions of the equations \(AU = V\).

In this particular example there are four solutions of the equations \(x^A = 1 - x\). We label these \(x^i = (x^i_{11}, \ldots, x^i_{32})\) for \(i = 0, 1, 2, 3\). We calculate the values of \(c - 24h_i\) using the formula

\[
c - 24h_i = \frac{6}{\pi^2} \sum_{jk=11,\ldots,32} L(u^i_{jk}, v^i_{jk}).
\]

For any given model, the effective central charge, \(c_{eff}\), is the value of \(c - 24h_i\) arising from the unique solution \(x^0\), whose components \(x^0_{jk} \in \mathbb{R}\) all satisfy \(0 < x^0_{jk} < 1\).

For the case \((D_3, A_2)\) the four choices of the matrix \(g\) give rise to four different values of \(c - 24h_i\). These are

\[
\begin{align*}
g &= \left(1, e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}\right) \quad \Rightarrow \quad c_{eff} = \frac{24}{7}, \\
g &= \left(1, e^{\frac{2\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) \quad \Rightarrow \quad c - 24h_1 = -\frac{72}{7} \mod 24\mathbb{Z}, \\
g &= \left(1, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) \quad \Rightarrow \quad c - 24h_2 = -\frac{120}{7} \mod 24\mathbb{Z}, \\
g &= \left(e^{\frac{2\pi i}{7}}, e^{\frac{4\pi i}{7}}, e^{\frac{6\pi i}{7}}\right) \quad \Rightarrow \quad c - 24h_3 = 0 \mod 24\mathbb{Z}.
\end{align*}
\]
### 3.5.2 Effective central charge for other models

These calculations have been carried out for many different models. The results are summarised in the following table:

| Pair        | $c_{eff}$ | $c - 24h_1$ | $c - 24h_2$ | $c - 24h_3$ |
|-------------|-----------|-------------|-------------|-------------|
| $(A_1, A_1)$| $1/2$     | -           | -           | -           |
| $(A_1, A_2)$| $4/5$     | $-44/5$     | -           | -           |
| $(A_1, A_3)$| $1$       | -           | -           | -           |
| $(A_1, A_4)$| $8/7$     | $32/7$      | $-40/7$     | -           |
| $(A_2, A_1)$| $6/5$     | $-54/5$     | -           | -           |
| $(A_2, A_2)$| $2$       | -           | -           | -           |
| $(A_2, A_3)$| $18/7$    | $6$         | $162/7$     | -           |
| $(A_2, A_4)$| $3$       | $9$         | $27$        | $-9$        |
| $(A_3, A_1)$| $2$       | -           | -           | -           |
| $(A_3, A_2)$| $24/7$    | $0$         | $-72/7$     | $-120/7$    |
| $(A_3, A_3)$| $9/2$     | -           | -           | -           |
| $(D_3, A_1)$| $2$       | -           | -           | -           |
| $(D_3, A_3)$| $9/2$     | -           | -           | -           |
| $(D_4, A_1)$| $3$       | -           | -           | -           |
| $(D_4, A_2)$| $16/3$    | $-32/3$     | $-16$       | $-307/3$    |
| $(D_4, A_3)$| $36/5$    | $24/5$      | -           | -           |
\section{3.6 \( (D_m, A_n) \) as coset models}

Consider the model described by the pair of Dynkin diagrams \((X, Y)\). Its effective central charge is known or conjectured to be

\[ c_{\text{eff}}(X, Y) = \frac{r(X)r(Y)h(X)}{h(X) + h(Y)}, \]

where \(r\) denotes the rank, and \(h\) the dual Coxeter number of a Lie algebra. Then for the case \((D_m, A_n)\) we expect the effective central charge to be

\[ c_{\text{eff}}(D_m, A_n) = \frac{(2m - 2)mn}{2m + n - 1}. \]

Now consider the coset model

\[ \frac{(D_m)_{n+1}}{u(1)^m}. \tag{3.44} \]

From (2.11) we know that such a coset model has central charge

\[ \frac{(n + 1)(2m^2 - m)}{n + 1 + 2m - 2} - n = \frac{(2m - 2)mn}{2m + n - 1}. \]

Clearly this coincides with the value of \(c_{\text{eff}}(D_m, A_n)\).

This is extremely good evidence to suggest that the model described by the Dynkin diagrams \((D_m, A_n)\) is a unitary model described by the coset

\[ \frac{(D_m)_{n+1}}{u(1)^m}. \]
In all cases checked so far, the h-values calculated from the dilog formulae (3.6, 3.8) for the model \((D_m, A_n)\) also arise as h-values of the coset model (3.44).
Chapter 4

Nahm’s Conjecture

In the previous chapter we studied a family of matrices related to pairs of Dynkin diagrams. Each matrix $A$ had the special property that solutions of $A \log x = \log(1 - x)$ gave torsion elements of the Bloch group. This lead to a nice relationship with conformal field theory. In this chapter we consider a number of $2 \times 2$ matrices that don’t fall into this Cartan matrix pattern but nevertheless yield torsion elements of the Bloch group, at least for the special ‘minimal’ solution. We show that these matrices are again related to conformal field theory.

4.1 Overview

A q-hypergeometric series is a series of the form $\sum_{n=0}^{\infty} A_n(q)$, where $A_0(q)$ is a rational function, and $A_n(q) = R(q, q^n)A_{n-1}(q)$ for all $n \geq 1$ for some rational function $R(x, y)$ with $\lim_{x \to 0} \lim_{y \to 0} R(x, y) = 0$. 

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The question of when a q-hypergeometric series is also modular is an interesting open question in mathematics. There are a handful of known examples of such series, the most famous ones being given by the Rogers-Ramanujan identities

\[
\sum_{n=0}^{\infty} \frac{q^n}{(q)_n} = \prod_{n \equiv \pm 1 \mod 5} \frac{1}{1 - q^n} \quad (|q| < 1),
\]

\[
\sum_{n=0}^{\infty} \frac{q^{n^2+n}}{(q)_n} = \prod_{n \equiv \pm 2 \mod 5} \frac{1}{1 - q^n} \quad (|q| < 1).
\]

These are modular functions up to factors \(q^{-1/60}\) and \(q^{11/60}\) respectively.

In general, the problem of understanding the overlap between q-hypergeometric series and modular functions is still completely unsolved. Nahm’s conjecture takes a first step towards tackling this problem by considering a special case involving r-fold hypergeometric series. (These are defined as above but with \(n\) running over \((\mathbb{Z}_\geq 0)^r\) rather than \(\mathbb{Z}_\geq 0\).

Let \(A\) be a positive definite symmetric \(r \times r\) matrix, \(B\) a vector of length \(r\), and \(C\) a scalar, all three with rational coefficients. Define a function \(f_{A,B,C}\) by the r-fold q-hypergeometric series

\[
f_{A,B,C}(z) = \sum_{n=(n_1,\ldots,n_r) \in (\mathbb{Z}_\geq 0)^r} \frac{q^{\frac{1}{2}nA^n+Bn+C}}{(q)_{n_1} \cdots (q)_{n_r}},
\]

where \((q)_n = (1 - q)(1 - q^2) \cdots (1 - q^n)\). We can ask the question, when is \(f_{A,B,C}\) a modular function? Nahm’s conjecture doesn’t fully answer this question, but predicts which matrices \(A\) can occur.
Given any positive definite symmetric $r \times r$ matrix $A = (A_{ij})$ with real coefficients, we can consider the system of equations

$$x_i = \prod_{j=1}^{r} (1 - x_j)^{A_{ij}}. \quad (4.1)$$

This system has a finite number of solutions $x = (x_1, \ldots, x_r)$. As it will be relevant later, it is worth noting that there is always a unique solution that satisfies $x_i \in \mathbb{R}$ and $0 \leq x_j \leq 1$ for all $x_i$, see [27]. Denote this solution by $x^0 = (x^0_1, \ldots, x^0_r)$.

**Note:** The related set of equations $1 - x_i = \prod_{j=1}^{r} x_j^{A_{ij}}$ corresponds to the set of equations (4.1) with $A$ replaced by $A^{-1}$. There is a duality between these two cases. In particular the effective central charges are related by

$$c_{eff}(A) + c_{eff}(A^{-1}) = r.$$

Let $F$ denote the number field $\mathbb{Q}(x_1, \ldots, x_r)$. Given any solution $x = (x_1, \ldots, x_r)$ of (4.1), define the element $\xi_x \in \mathbb{Z}(F)$ by

$$\xi_x = [x_1] + \ldots + [x_r].$$
\( \xi_x \) in an element of the Bloch group \( B(F) \):

\[
\partial(\xi_x) = \sum_{i=1}^{r} x_i \wedge (1 - x_i) \\
= \sum_{i=1}^{r} \left( \prod_{j=1}^{r} (1 - x_j)^{A_{ij}} \right) \wedge (1 - x_i) \\
= \sum_{i=1}^{r} \sum_{j=1}^{r} A_{ij} (1 - x_j) \wedge (1 - x_i) \\
= \sum_{i=1}^{r} \sum_{j=1}^{r} A_{ij} x_j \wedge x_i \\
= 0 ,
\]

since \( A_{ij} \) is symmetric and \( x_i \wedge x_j \) is anti-symmetric in \( i \) and \( j \). Here \( \partial \) denotes the boundary function

\[
\partial : [x] \to x \wedge (1 - x) ,
\]

used to define the Bloch group.

**Conjecture 1 (Nahm’s Conjecture)** Let \( A \) be a positive definite symmetric \( r \times r \) matrix with rational coefficients. Then the following are equivalent:

1. The element \( \xi_x \) is a torsion element of \( B(F) \) for all solutions \( x = (x_1, \ldots, x_r) \).

2. There exist \( B \in \mathbb{Q}^r \) and \( C \in \mathbb{Q} \) such that \( f_{A,B,C}(z) \) is a modular function.

For the case \( r = 1 \) this conjecture is proved in [27]. It is expected that modular functions \( f_{A,B,C} \) that arise in this way are characters of certain rational conformal field theories. We show later that this is certainly true in at least one case.
4.2 Terhoeven’s Matrices

In his PhD thesis [39], Michael Terhoeven looked for rational $2 \times 2$ matrices for which $\xi_{x^0}$ was a torsion element of the Bloch group. To do this he carried out a systematic search of all $2 \times 2$ matrices of the form

$$A = \frac{1}{m} \begin{pmatrix} a & b \\ b & c \end{pmatrix} \in M_2(\mathbb{Q}),$$

with $a, b, c, m \leq 11$. With twelve exceptions, the resulting matrices fall into three main groups according to the pattern of their entries. Of these twelve matrices, three are products of Cartan matrices, with the remaining nine as follows:

$$A = \begin{pmatrix} 11 & 9 \\ 9 & 8 \end{pmatrix}, \quad A = \begin{pmatrix} 8 & 5 \\ 5 & 4 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix},$$

$$A = \begin{pmatrix} 8 & 3 \\ 3 & 2 \end{pmatrix}, \quad A = \frac{1}{2} \begin{pmatrix} 5 & 4 \\ 4 & 4 \end{pmatrix}, \quad A = \frac{1}{3} \begin{pmatrix} 8 & 1 \\ 1 & 2 \end{pmatrix},$$

$$A = \frac{1}{9} \begin{pmatrix} 8 & 3 \\ 3 & 0 \end{pmatrix}, \quad A = \begin{pmatrix} 4 & 1 \\ 1 & 1 \end{pmatrix}, \quad A = \frac{1}{3} \begin{pmatrix} 1 & 1 \\ 1 & 0 \end{pmatrix}.$$  

For each of these matrices we solve the equations $x = (1 - x)^A$, and in cases where $\xi_x$ is a torsion element of the Bloch group for all solutions $x$, we calculate the values of $B$ and $C$ for which $f_{A,B,C}$ is modular.
Note: For simplicity we solve the equations \( x^A = 1 - x \) instead of \( x = (1 - x)^A \) in some cases. To recover the desired solutions we need only replace \( x \) by \( 1 - x \).

### 4.2.1 Equations

\( A = (11, 9, 9, 8) \)

We get the following set of equations in two variables

\[
\begin{align*}
1 - x_1 &= x_1^{11} x_2^9, \\
1 - x_2 &= x_1^9 x_2^8.
\end{align*}
\]

These reduce to the one-variable equation

\[
(x_1^8 - x_1^7 + 8x_1^6 - 9x_1^5 - x_1^4 + 2x_1^3 + 3x_1^2 - 3x_1 + 1)(x_1^4 - 8x_1^3 + 3x_1^2 + 2x_1 + 1) = 0.
\]

\( \xi \) is not a torsion element of \( B(F) \) for all solutions \( x \). In fact it turns out that \( \xi \) is torsion for four out of the twelve solutions, namely the solutions that annihilate the second factor. This second factor has Galois group \( D_4 \).

\( A = (8, 5, 5, 4) \)

We get the following set of equations in two variables

\[
\begin{align*}
1 - x_1 &= x_1^8 x_2^5, \\
1 - x_2 &= x_1^5 x_2^4.
\end{align*}
\]
These reduce to the one-variable equation

\[(x_1^4 - x_1^3 + 3x_1^2 - 3x_1 + 1)(x_1^4 + x_1^3 + 3x_1^2 - 3x_1 - 1) = 0.\]

\(\xi_x\) is only a torsion element of \(B(F)\) for half of the solutions, namely those that annihilate the second factor. This second factor has Galois group \(D_4\).

\[ \textbf{A} = (4, 3, 3, 3) \]

We get the following set of equations in two variables

\[
\begin{align*}
1 - x_1 &= x_1^4x_2^3, \\
1 - x_2 &= x_1^3x_2^3.
\end{align*}
\]

These reduce to the one-variable equation

\[(4x_1^2 - 2x_1 - 1)(2x_1^2 - 2x_1 + 1) = 0.\]

\(\xi_x\) is a torsion element of \(B(F)\) for the two solutions that annihilate the first factor, but for no other solutions. The first factor has Galois group \(S_2\).

\[ \textbf{A} = (8, 3, 3, 2) \]

We get the following set of equations in two variables

\[
\begin{align*}
1 - x_1 &= x_1^8x_2^3, \\
1 - x_2 &= x_1^3x_2^2.
\end{align*}
\]
These reduce to the one-variable equation

\[(x_1^2 - x_1 + 1)(x_1^4 + 2x_1^3 + x_1^2 - 2x_1 - 1) = 0 .\]

\(\xi_x\) is a torsion element of \(B(F)\) for the four solutions that annihilate the second factor, but for no other solutions. The second factor has Galois group \(D_4\).

\[A = 1/2(5, 4, 4, 4)\]

We get the following set of equations in two variables

\[1 - x_1 = x_1^{5/2} x_2^2 , \]
\[1 - x_2 = x_1^2 x_2^2 .\]

These reduce to the one-variable equation

\[(1 - x_1 + 2x_1^2 - 2x_1^3 - x_1^{3/2} + x_1^{5/2} - x_1^{7/2}) = 0 .\]

Since solving the equations in this case requires a careful choice of square roots, it is perhaps more appropriate to consider this equation as a polynomial of degree 7 in the variable \(y = x_1^{1/2}\). The resulting equation is

\[(y^3 - y + 1)(y^4 + 2y^3 - y - 1) = 0 .\]

Again \(\xi_x\) is not torsion for all solutions, but only for solutions that annihilate the second factor. The Galois group of the second factor is \(D_4\).
\[ A = \frac{1}{3}(8, 1, 1, 2) \]

We get the following set of equations in two variables

\[
1 - x_1 = x_1^{8/3} x_2^{1/3}, \\
1 - x_2 = x_1^{1/3} x_2^{2/3}.
\]

These reduce to the one-variable equation

\[
(x_1^2 - x_1 + 1)(x_1^6 + x_1^5 - 2x_1^4 + 2x_1 - 1) = 0.
\]

\( \xi_x \) is a torsion element for six of the eight solutions, namely the six that annihilate the second factor. The Galois group of the second factor is \( A_4 \times C_2 \).

\[ A = \frac{1}{9}(8, 3, 3, 0) \]

We get the following set of equations in two variables

\[
1 - x_1 = x_1^{8/9} x_2^{1/3}, \\
1 - x_2 = x_1^{1/3}.
\]

These reduce to the one-variable equation

\[
x_2(x_2^3 - 2x_2^2 + x_2 + 1)(x_2^4 - 6x_2^3 + 12x_2^2 - 9x_2 + 1) = 0.
\]

\( \xi_x \) is a torsion element for four of the seven non-zero solutions, the four annihilating
the second factor. This factor has Galois group $D_4$.

\[ A = (4, 1, 1, 1) \]

We get the following set of equations in two variables

\[
\begin{align*}
1 - x_1 &= x_1^4 x_2, \\
1 - x_2 &= x_1 x_2.
\end{align*}
\]

These reduce to the one-variable equation

\[
x_1^4 + x_2 - 1 = 0,
\]

whose Galois group is again $D_4$. $\xi_x$ is a torsion element of $B(F)$ for all solutions.

\[ A = 1/2(1, 1, 1, 0) \]

We get the following set of equations in two variables

\[
\begin{align*}
1 - x_1 &= x_1^{1/2} x_2^{1/2}, \\
1 - x_2 &= x_1^{1/2}.
\end{align*}
\]

These reduce to the one-variable equation

\[
x_2 (x_2^3 - 5 x_2^2 + 6 x_2 - 1) = 0.
\]

$\xi_x$ is a torsion element of $B(F)$ for all solutions. Ignoring the trivial factor (whose
Galois group is $S_1$, this equation has Galois group $A_3$.

*Note: All Galois group calculations were carried out using the programme Pari. This can be downloaded free from http://pari.math.u-bordeaux.fr/*.  

### 4.2.2 Torsion elements of the Bloch group

In light of Nahm’s conjecture, the two cases of greatest interest are $A = (4, 1, 1, 1)$ and $A = 1/2(1, 1, 1, 0)$. In both of these cases all solutions of $x = (1 - x)^A$ yield torsion elements of the Bloch group, and hence the conjecture suggests that there exist values $B \in \mathbb{Q}^2$ and $C \in \mathbb{Q}$ such that the function $f_{A,B,C}(z)$ is modular. We now proceed to calculate these $B$ and $C$ values.

### 4.2.3 $B$ and $C$ values

Given a matrix $A$, we calculate the associated $B$-values by some experimentation! This is done by considering the functions $f_{A,B,0}$ for different values of $B$, and examining the coefficients of the resulting expansion (with small coefficients suggesting a modular function). As a useful check, we expect that any realistic candidate for a $B$-value should give rise to a rational $C$-value.
Given the vector \( B = (B_1, B_2) \), \( C \) can then be calculated using the equation:

\[
C = \frac{\phi_2(B_i)}{2} \left( \frac{1 - x_i}{x_i} \right) - \frac{1}{2} \phi_1(B_i) \left( \frac{1 - x_i}{x_i} \right) T_{ij}^{-1} \phi_1(B_j) \left( \frac{1 - x_j}{x_j} \right) \\
+ \frac{1}{2} \phi_1(B_i) \left( \frac{1 - x_i}{(x_i)^2} \right) T_{ij}^{-1} - \frac{1}{2} \left( \frac{1 - x_i}{(x_i)^2} \right) T_{ii}^{-1} T_{ij}^{-1} \phi_1(B_j) \left( \frac{1 - x_j}{x_j} \right) \\
+ \frac{1}{8} \left( \frac{(1 - x_i)(2 - x_i)}{(x_i)^3} \right) (T_{ii})^{-1} \\
- \frac{1}{12} \left( \frac{1 - x_i}{(x_i)^2} \right) (T_{ij}^{-1})^3 \left( \frac{1 - x_j}{(x_j)^2} \right) \\
- \frac{1}{8} \left( \frac{1 - x_i}{(x_i)^2} \right) T_{ii}^{-1} T_{ij}^{-1} T_{jj}^{-1} \left( \frac{1 - x_j}{(x_j)^2} \right).
\]

Here \( \phi_i \) denotes the \( i \)th Bernoulli polynomial, and the matrix \( T = (T_{ij}) \) is given by

\[
T_{ij} = (A)^{-1}_{ij} + \delta_{ij} \left( \frac{1 - x_i}{x_i} \right).
\]

For the matrix \( A = (4, 1, 1, 1) \), we find that the values of \( B \) and \( C \) for which \( f_{A,B,C}(z) \) is modular are

\[
B = (0, 1/2), \quad C = 1/120, \\
B = (2, 1/2), \quad C = 49/120.
\]

These values agree with the calculations of Don Zagier [27].

For the matrix \( A = 1/2(1, 1, 1, 0) \), the corresponding \( B \) and \( C \) values are

\[
B = (0, 1/2), \quad C = -1/84, \\
B = (1/2, 1/2), \quad C = 5/84, \\
B = (0, 1), \quad C = -1/21.
\]
4.2.4 \( f_{A,B,C} \) as characters of rational CFTs

For the case \( A = (4, 1, 1, 1) \) we calculate the functions \( f_{A,B,C} \) explicitly. Up to order 6 these are

\[
\begin{align*}
\quad & f_{A,(2,\frac{1}{2}),\frac{49}{120}} = q^{\frac{49}{120}} \left( 1 + q + q^2 + 2q^3 + 3q^4 + 4q^5 + 6q^6 + \ldots \right), \\
\quad & f_{A,(0,\frac{1}{2}),\frac{1}{120}} = q^{\frac{1}{120}} \left( 1 + q + 2q^2 + 3q^3 + 4q^4 + 6q^5 + 8q^6 + \ldots \right).
\end{align*}
\]

These are indeed characters of a rational CFT, being two of the characters of the \((5, 4)\)-minimal model (see [18], pg.243).

Further to this, it is expected that the matrix \( A \) is related to some integrable perturbation of this conformal field theory. It would be interesting to explore this connection in more detail at a later stage.

For the case \( A = 1/2(1, 1, 1, 0) \) the functions \( f_{A,B,C} \) are more complicated. They are
We have yet to find a conformal field theory with these characters, but are optimistic that one exists!

**Product expansions of the functions** $f_{A,B,C}$

In many cases the functions $f_{A,B,C}$ have nice product expansions. Consider for example $A = (2, 2, 2, 4) = C(A_4) \otimes C(T_2)^{-1}$, and $B = (0, 0)$. In this case $f$ can be written in the form

$$f_{A,B,0} = \prod_{n = \pm 1 \text{ or } \pm 2 \mod 7} \frac{1}{1 - q^n}.$$
Unfortunately this does not work in general. For example consider the case $A = 1/2(1, 1, 1, 0), B = (1/2, 1/2)$. While $f$ can be expanded as a product quite easily, it does not appear to follow any particular pattern. In this case $f$ is given by

$$f_{A,B,0} = \left(1 - (q^{1/4})^2\right)^{-1} \left(1 - (q^{1/4})^3\right)^{-1} \left(1 - (q^{1/4})^5\right) \left(1 - (q^{1/4})^7\right)^{-2} \left(1 - (q^{1/4})^8\right)^{-2} \left(1 - (q^{1/4})^{12}\right)^{-6} \left(1 - (q^{1/4})^{13}\right)^2 \left(1 - (q^{1/4})^{14}\right)^9 \left(1 - (q^{1/4})^{16}\right)^{-12} \left(1 - (q^{1/4})^{17}\right)^{-7} \left(1 - (q^{1/4})^{18}\right)^{13} \left(1 - (q^{1/4})^{19}\right)^{19} \left(1 - (q^{1/4})^{20}\right)^{-15} \ldots .$$

4.2.5 Effective central charge calculations

We have already seen that, given a solution $x = (x_1, \ldots, x_r)$ of (4.1) for which $\xi_x$ is a torsion element of the Bloch group, we can apply the mapping

$$\frac{6}{\pi^2} \sum_{i=1}^r L(x_i) = c - 24h \mod 24 .$$

This gives some information about the corresponding conformal field theory.

Since $\xi_{x,0}$ is a torsion element of the Bloch group for each of Terhoeven’s matrices (and their inverses), we can calculate the corresponding value of the effective central charge in each case. Furthermore we can calculate $c - 24h \mod 24$ for any solution $x$ of (4.1) for which $\xi_x$ is a torsion element. The results of some such calculations are summarised below.
4.2.6 Significance of the Galois group

The Kronecker-Weber theorem states that every finite abelian extension $F$ of $\mathbb{Q}$ (i.e. every algebraic number field $F$ whose Galois group over $\mathbb{Q}$ is abelian) is a subfield of a cyclotomic field.

For a given abelian extension $F$ of $\mathbb{Q}$ there is in fact a minimal cyclotomic field that contains $F$. The conductor of $F$ is defined to be the smallest integer $n$ such that $F$ lies inside the field generated by the $n^{th}$ roots of unity. Hence we can say that $Gal(F)$ is abelian if and only if $F$ is generated by $n^{th}$ roots of unity. We can conclude that if an equation has abelian Galois group, its solutions can be expressed in terms of roots of unity.
Now let $G$ be a group and $com(G)$ be its commutator subgroup. $G/com(G)$ is abelian, so $G$ is an extension of an abelian group by $com(G)$. We expect that in all cases of interest to us, $com(G)$ is either the identity or some $\mathbb{Z}_2$ stuff. Of the cases considered so far, the pairs of Cartan matrices give rise to equations with abelian Galois groups, while the equations arising from Terhoeven’s matrices have Galois groups that are abelian or have $com(G) = \mathbb{Z}_2$ or $\mathbb{Z}_2 \times \mathbb{Z}_2$. The pattern is not yet understood completely.
Chapter 5

Integrable Models Described by Exceptional Lie Algebras

In this chapter we consider the integrable models described by pairs of Dynkin diagrams \((E_m, T_1)\). There are three distinct models of this type since \(m\) can take values 6, 7, or 8. For each model the equations are of the form \(U = AV\), where the matrix \(A\) is given by \(A = C(E_m) \otimes C(T_1)^{-1}\), \(U = \log(x)\), \(V = \log(1 - x)\), and \(x = (x_1, \ldots, x_m)\). In each case we solve the equations and compute the effective central charge of the corresponding conformal field theory.
5.1 Exceptional Lie Algebras

The $E$ family of exceptional Lie algebras has three members, namely $E_6$, $E_7$, and $E_8$. Their Dynkin diagrams are

\[ E_6 \]

\[ E_7 \]

\[ E_8 \]

The corresponding Cartan matrices are

\[
C(E_6) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]
Coxeter matrices:

\[
C(E_7) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

\[
C(E_8) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]

Their Coxeter numbers are

\[h(E_6) = 12,\]
\[h(E_7) = 18,\]
\[h(E_8) = 30.\]
5.2 Dynkin Diagrams $T_r$

The ‘Tadpole’ diagram, $T_r$, is got by folding the diagram $A_{2r}$ in the middle, to get a pairwise identification of the vertices

$$T_r = A_{2r}/\mathbb{Z}_2.$$ 

This gives the Dynkin diagram

$$\begin{array}{c}
\bullet \quad \cdots \quad \cdots \quad \bullet \\
1 & 2 & \cdots & r
\end{array}$$

The Cartan matrix of $T_r$ is identical to that of $A_r$ except for the entry $C(T_r)_{rr} = 1$. In particular for $T_1$ we get the $1 \times 1$ Cartan matrix

$$C(T_1) = 1.$$ 

The Coxeter number of $T_r$ is $h(T_r) = 2r + 1$. 

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5.3 Pairs of Dynkin Diagrams \((E_m, T_1)\)

5.3.1 Solving the equations of the model

Case 1: \(A = A(E_6, T_1)\)

Here we consider the pair of Dynkin diagrams \(X = E_6\) and \(Y = T_1\). The matrix \(A = A(X, Y)\) is given by \(A(E_6, T_1) = C(E_6) \otimes C(T_1)^{-1}\). Its entries are

\[
A(E_6, T_1) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]
Exponentiating the equations $U = AV$ leads to the following set of algebraic equations:

\[
\begin{align*}
  x_1 &= \frac{(1 - x_1)^2}{1 - x_3}, \\
  x_2 &= \frac{(1 - x_2)^2}{1 - x_4}, \\
  x_3 &= \frac{(1 - x_3)^2}{(1 - x_1)(1 - x_4)}, \\
  x_4 &= \frac{(1 - x_4)^2}{(1 - x_2)(1 - x_3)(1 - x_5)}, \\
  x_5 &= \frac{(1 - x_5)^2}{(1 - x_4)(1 - x_6)}, \\
  x_6 &= \frac{(1 - x_6)^2}{1 - x_5}.
\end{align*}
\]

To solve these equations we write all variables in terms of $x_6$. There are two possible solutions. Either $x_6 = x_1$, and $x_6$ satisfies the equation

\[
(5x_6^2 - 5x_6 + 1)(x_6 - 1)^3 = 0,
\]

or $x_6 \neq x_1$, and $x_6$ satisfies the following polynomial of degree 7

\[
x_6(x_6^2 - 3x_6 + 1)(x_6 - 1)^4 = 0.
\]

Both of these equations can be solved using MAPLE. Excluding ‘solutions’ that imply $x_i = \infty$ for any $i = 1, \ldots, 6$, there are two distinct solutions given by

\[
x_1 = \frac{1}{2} \pm \frac{\sqrt{5}}{10}.
\]
The remaining $x_i$ are given by

\[ x_2 = \frac{5}{2} x_1 - 1, \]
\[ x_3 = 4x_1 - 2, \]
\[ x_4 = 10 - \frac{25}{2} x_1, \]
\[ x_5 = 4x_1 - 2, \]
\[ x_6 = x_1. \]

Clearly these solutions reflect the symmetry of the $E_6$ Dynkin diagram.

More explicitly we have

\[ x_1 = x_6 = \frac{1}{2} + \frac{\sqrt{5}}{10} \Rightarrow x_2 = \frac{1 + \sqrt{5}}{4} \]
\[ \Rightarrow x_3 = \frac{2}{\sqrt{5}} = x_5 \]
\[ \Rightarrow x_4 = \frac{15 - 5\sqrt{5}}{4}. \]

\[ x_1 = x_6 = \frac{1}{2} - \frac{\sqrt{5}}{10} \Rightarrow x_2 = \frac{1 - \sqrt{5}}{4} \]
\[ \Rightarrow x_3 = \frac{2}{\sqrt{5}} = x_5 \]
\[ \Rightarrow x_4 = \frac{15 + 5\sqrt{5}}{4}. \]
Case 2: $A = A(E_7, T_1)$

In this case $(X, Y) = (E_7, T_1)$. The matrix $A(X, Y)$, given by $A(E_7, T_1) = C(E_7) \otimes C(T_1)^{-1}$, has the following entries

$$A(E_7, T_1) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}$$

We want to solve the following set of algebraic equations arising from the model

$$x_1 = \frac{(1 - x_1)^2}{1-x_3},$$
$$x_2 = \frac{(1 - x_2)^2}{1-x_4},$$
$$x_3 = \frac{(1 - x_3)^2}{(1 - x_1)(1 - x_4)},$$
$$x_4 = \frac{(1 - x_4)^2}{(1 - x_2)(1 - x_3)(1 - x_5)},$$
$$x_5 = \frac{(1 - x_5)^2}{(1 - x_4)(1 - x_6)},$$
$$x_6 = \frac{(1 - x_6)^2}{(1 - x_5)(1 - x_7)},$$
$$x_7 = \frac{(1 - x_7)^2}{(1 - x_6)}.$$
As before, these equations can be solved analytically. We reduce the equations to a polynomial of degree 14 in \( x_7 \). This can be factorised using MAPLE. We then solve the equation

\[(2x_7 - 1)(3x_7^2 - 9x_7 + 5)(x_7^3 - x_7^2 - 2x_7 + 1)(x_7^2 - 3x_7 + 1)(x_7 - 1)^6 = 0 .\]

Excluding irrelevant solutions we find that there are five distinct solutions given by

\[
x_1 = -\frac{3}{2} + \frac{\sqrt{21}}{2},
\]

\[
x_1 = -\frac{3}{2} - \frac{\sqrt{21}}{2},
\]

\[
x_1 = -\frac{1}{12} (756 + 84\sqrt{3}i)^{1/3} - \frac{7}{(756 + 84\sqrt{3}i)^{1/3}} + 2
\]

\[
+ \frac{\sqrt{3}i}{2} \left\{ \frac{1}{6} (756 + 84\sqrt{3}i)^{1/3} - \frac{14}{(756 + 84\sqrt{3}i)^{1/3}} \right\},
\]

\[
x_1 = -\frac{1}{12} (756 + 84\sqrt{3}i)^{1/3} - \frac{7}{(756 + 84\sqrt{3}i)^{1/3}} + 2
\]

\[
- \frac{\sqrt{3}i}{2} \left\{ \frac{1}{6} (756 + 84\sqrt{3}i)^{1/3} - \frac{14}{(756 + 84\sqrt{3}i)^{1/3}} \right\},
\]

\[
x_1 = \frac{1}{6} (756 + 84\sqrt{3}i)^{1/3} + \frac{14}{(756 + 84\sqrt{3}i)^{1/3}} + 2 .
\]

Using the method described in Appendix A, these solutions can be written in the
somewhat nicer form

\[ x_1 = \frac{-3}{2} + \frac{\sqrt{21}}{2}, \]

\[ x_1 = \frac{-3}{2} - \frac{\sqrt{21}}{2}, \]

\[ x_1 = -6 \cos \left( \frac{2\pi}{7} \right) + 2 \cos \left( \frac{3\pi}{7} \right) + 4 \cos \left( \frac{\pi}{7} \right), \]

\[ x_1 = -4 \cos \left( \frac{2\pi}{7} \right) + 6 \cos \left( \frac{3\pi}{7} \right) + 2 \cos \left( \frac{\pi}{7} \right), \]

\[ x_1 = -2 \cos \left( \frac{2\pi}{7} \right) + 4 \cos \left( \frac{3\pi}{7} \right) + 6 \cos \left( \frac{\pi}{7} \right). \]

Case 3: \( A = A(E_8, T_1) \)

In this case \((X, Y) = (E_8, T_1)\). The matrix \( A(X, Y) = C(E_8) \otimes C(T_1)^{-1} \) is given by

\[
A(E_8, T_1) = \begin{pmatrix}
2 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\
0 & 2 & 0 & -1 & 0 & 0 & 0 & 0 \\
-1 & 0 & 2 & -1 & 0 & 0 & 0 & 0 \\
0 & -1 & -1 & 2 & -1 & 0 & 0 & 0 \\
0 & 0 & 0 & -1 & 2 & -1 & 0 & 0 \\
0 & 0 & 0 & 0 & -1 & 2 & -1 & 0 \\
0 & 0 & 0 & 0 & 0 & -1 & 2 & -1 \\
0 & 0 & 0 & 0 & 0 & 0 & -1 & 2
\end{pmatrix}
\]
The algebraic equations obtained from the model are

\[ x_1 = \frac{(1 - x_1)^2}{1 - x_3}, \]
\[ x_2 = \frac{(1 - x_2)^2}{1 - x_4}, \]
\[ x_3 = \frac{(1 - x_3)^2}{(1 - x_1)(1 - x_4)}, \]
\[ x_4 = \frac{(1 - x_4)^2}{(1 - x_2)(1 - x_3)(1 - x_5)}, \]
\[ x_5 = \frac{(1 - x_5)^2}{(1 - x_4)(1 - x_6)}, \]
\[ x_6 = \frac{(1 - x_6)^2}{(1 - x_5)(1 - x_7)}, \]
\[ x_7 = \frac{(1 - x_7)^2}{(1 - x_6)(1 - x_8)}, \]
\[ x_8 = \frac{(1 - x_8)^2}{(1 - x_7)}. \]

As in the previous case we get a polynomial of degree 17 in \( x_8 \). After factorisation this is

\[ (2x_8 - 1)(x_8^3 - 2x_8^2 - x_8 + 1)(x_8^5 - 3x_8^4 - 3x_8^3 + 4x_8^2 + x_8 - 1)(x_8 - 1)^8 = 0. \]
There are five relevant solutions

\[
x_1 = 4 \cos \left( \frac{2\pi}{11} \right) - 2 \cos \left( \frac{4\pi}{11} \right) - 2 \cos \left( \frac{5\pi}{11} \right) - 2 \cos \left( \frac{\pi}{11} \right),
\]
\[
x_1 = 2 \cos \left( \frac{2\pi}{11} \right) + 4 \cos \left( \frac{4\pi}{11} \right) + 2 \cos \left( \frac{3\pi}{11} \right) - 2 \cos \left( \frac{\pi}{11} \right),
\]
\[
x_1 = -4 \cos \left( \frac{5\pi}{11} \right) + 2 \cos \left( \frac{4\pi}{11} \right) - 2 \cos \left( \frac{3\pi}{11} \right) + 2 \cos \left( \frac{\pi}{11} \right),
\]
\[
x_1 = -4 \cos \left( \frac{3\pi}{11} \right) + 2 \cos \left( \frac{4\pi}{11} \right) + 2 \cos \left( \frac{2\pi}{11} \right) + 2 \cos \left( \frac{5\pi}{11} \right),
\]
\[
x_1 = -4 \cos \left( \frac{\pi}{11} \right) - 2 \cos \left( \frac{2\pi}{11} \right) - 2 \cos \left( \frac{5\pi}{11} \right) - 2 \cos \left( \frac{3\pi}{11} \right).
\]

### 5.3.2 Effective central charge calculations

Each \( x = (x_1, x_2, \ldots, x_m) \) satisfies the set of algebraic equations \( x = (1 - x)^A \). Logarithms of these solutions must be chosen so that they satisfy the original equations of the model, namely \( \log(x) = A \log(1 - x) \). To do this we define \( \log(x) \) in terms of \( \log(z) \), where \( z = (z_1, \ldots, z_m) \) is the variable introduced earlier

\[
1 - x = z^{-C(Y)} \Rightarrow v_i = \log(1 - x) = -C(Y) \log(z),
\]
\[
x = z^{-C(X)} \Rightarrow u_i = \log(x) = -C(X) \log(z).
\]

The effective central charge is calculated using the formula

\[
c_{eff} = \frac{6}{\pi^2} \sum_{i=1}^{m} L(u_i^0, v_i^0).
\]  

(5.2)
Here \((u^0_i, v^0_i)\) corresponds to the special solution whose components satisfy \(x_i \in \mathbb{R}\) and \(0 < x_i < 1\) for all \(i\).

Other values of \(c - 24h\) can be calculated (modulo \(24\mathbb{Z}\)) from the remaining solutions by

\[
c - 24h = \frac{6}{\pi^2} \sum_{i=1}^m L(u_i, v_i) .
\]  

(5.3)

**Case 1:** \(A(E_6, T_1)\)

The relations

\[
1 - x = z^{-C(T_1) \otimes I_6} \quad \text{and} \quad x = z^{-C(E_6)} ,
\]

give rise to the following set of equations for \(\log(x_i)\) and \(\log(1 - x_i)\).

\[
v_i = \log(1 - x_i) = -\log(z_i) , \quad \text{for } i = 1, \ldots, 6 ,
\]

and

\[
u_1 = \log(x_1) = -2 \log(z_1) + \log(z_3) ,
\]
\[
u_2 = \log(x_2) = -2 \log(z_2) + \log(z_4) ,
\]
\[
u_3 = \log(x_3) = \log(z_1) - 2 \log(z_3) + \log(z_4) ,
\]
\[
u_4 = \log(x_4) = \log(z_2) + \log(z_3) - 2 \log(z_4) + \log(z_5) ,
\]
\[
u_5 = \log(x_5) = \log(z_4) - 2 \log(z_5) + \log(z_6) ,
\]
\[
u_6 = \log(x_6) = \log(z_5) - 2 \log(z_6) .
\]
This choice of logs must (and does) satisfy the following equations \((U = AV)\).

\[
\log(x_1) = 2 \log(1 - x_1) - \log(1 - x_3), \\
\log(x_2) = 2 \log(1 - x_2) - \log(1 - x_4), \\
\log(x_3) = - \log(1 - x_1) + 2 \log(1 - x_3) - \log(1 - x_4), \\
\log(x_4) = - \log(1 - x_2) - \log(1 - x_3) + 2 \log(1 - x_4) - \log(1 - x_5), \\
\log(x_5) = - \log(1 - x_4) + 2 \log(1 - x_5) - \log(1 - x_6), \\
\log(x_6) = - \log(1 - x_5) + 2 \log(1 - x_6).
\]

Substituting the two solutions (5.1) into the equations (5.2) and (5.3) gives the following results:

\[
x_1 = \frac{1}{2} - \frac{\sqrt{5}}{10} \Rightarrow c - 24h = -\frac{24}{5} \mod 24\mathbb{Z}, \\
x_1 = \frac{1}{2} + \frac{\sqrt{5}}{10} \Rightarrow c_{eff} = \frac{24}{5}.
\]
Case 2: $A(E_7, T_1)$

Using the same method in this case gives the following results:

\[
x_1 = -\frac{3}{2} + \frac{\sqrt{21}}{2} \Rightarrow c_{\text{eff}} = 6 ,
\]
\[
x_1 = -\frac{3}{2} - \frac{\sqrt{21}}{2} \Rightarrow c - 24h = -18 \quad \text{mod} \ 24\mathbb{Z} ,
\]
\[
x_1 = -6 \cos\left(\frac{2\pi}{7}\right) + 2 \cos\left(\frac{3\pi}{7}\right) + 4 \cos\left(\frac{\pi}{7}\right) \Rightarrow c - 24h = -\frac{30}{7} \quad \text{mod} \ 24\mathbb{Z} ,
\]
\[
x_1 = -4 \cos\left(\frac{2\pi}{7}\right) + 6 \cos\left(\frac{3\pi}{7}\right) + 2 \cos\left(\frac{\pi}{7}\right) \Rightarrow c - 24h = -\frac{78}{7} \quad \text{mod} \ 24\mathbb{Z} ,
\]
\[
x_1 = -2 \cos\left(\frac{2\pi}{7}\right) + 4 \cos\left(\frac{3\pi}{7}\right) + 6 \cos\left(\frac{\pi}{7}\right) \Rightarrow c - 24h = -\frac{102}{7} \quad \text{mod} \ 24\mathbb{Z} .
\]
Case 3: $A(E_8, T_1)$

Again the same calculation gives

$$x_1 = 4 \cos \left( \frac{2\pi}{11} \right) - 2 \cos \left( \frac{4\pi}{11} \right) - 2 \cos \left( \frac{5\pi}{11} \right) - 2 \cos \left( \frac{\pi}{11} \right)$$

$$\Rightarrow c - 24h = -\frac{40}{11} \mod 24\mathbb{Z},$$

$$x_1 = 2 \cos \left( \frac{2\pi}{11} \right) + 4 \cos \left( \frac{4\pi}{11} \right) + 2 \cos \left( \frac{3\pi}{11} \right) - 2 \cos \left( \frac{\pi}{11} \right)$$

$$\Rightarrow c - 24h = -\frac{160}{11} \mod 24\mathbb{Z},$$

$$x_1 = -4 \cos \left( \frac{5\pi}{11} \right) + 2 \cos \left( \frac{4\pi}{11} \right) - 2 \cos \left( \frac{3\pi}{11} \right) + 2 \cos \left( \frac{\pi}{11} \right)$$

$$\Rightarrow c_{eff} = \frac{80}{11},$$

$$x_1 = -4 \cos \left( \frac{3\pi}{11} \right) + 2 \cos \left( \frac{4\pi}{11} \right) + 2 \cos \left( \frac{2\pi}{11} \right) + 2 \cos \left( \frac{5\pi}{11} \right)$$

$$\Rightarrow c - 24h = -\frac{112}{11} \mod 24\mathbb{Z},$$

$$x_1 = -4 \cos \left( \frac{\pi}{11} \right) - 2 \cos \left( \frac{2\pi}{11} \right) - 2 \cos \left( \frac{5\pi}{11} \right) - 2 \cos \left( \frac{3\pi}{11} \right)$$

$$\Rightarrow c - 24h = -\frac{208}{11} \mod 24\mathbb{Z}.$$
Summary

Certain integrable models are described by pairs $(X, Y)$ of ADET Dynkin diagrams. At high energy these models are expected to have a conformally invariant limit. The S-matrix of the model determines algebraic equations, whose solutions are mapped to the central charge and scaling dimensions of the corresponding conformal field theory. We study the equations of the $(D_m, A_n)$ model and find all solutions explicitly using the representation theory of Lie algebras and related Yangians. These mathematically rigorous results are in agreement with the expectations arising from physics.

We also investigate the overlap between certain q-hypergeometric series and modular functions. We study a particular class of r-fold q-hypergeometric series, denoted $f_{A,B,C}$. Here $A$ is a positive definite, symmetric, $2 \times 2$ matrix, $B$ is a vector of length 2, and $C$ is a scalar, all three with rational entries. It turns out that for certain choices of the matrix $A$, the function $f_{A,B,C}$ can be made modular. We calculate the corresponding values of $B$ and $C$. It is expected that functions $f_{A,B,C}$ arising in this way are characters of some rational conformal field theory. We show that this is true in at least one case.
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Appendix A

Useful Method of Solving Equations

This appendix describes a useful method of solving equations. In chapter 5 it is used to simplify certain solutions arising from equations of the \((E_m, T_1)\) model.

A.1 General Method

Suppose we wish to find the roots of an irreducible polynomial

\[x^3 + a_1 x^2 + a_2 x + a_3 = 0, \quad (A.1)\]

with rational \(a_i\). This equation has three roots, say \(x_1, x_2, \) and \(x_3\). Suppose that

\[x_1 = \sum_{i=1}^{3} b_i (\omega^i + \omega^{-i}) = b_1 (\omega + \omega^{-1}) + b_2 (\omega^2 + \omega^{-2}) + b_3 (\omega^3 + \omega^{-3}),\]
with rational $b_i$, and $\omega = exp\left(\frac{2\pi i}{7}\right)$.

**Galois group**

The Galois group of $\omega$ is generated by $\gamma : \omega \mapsto \omega^3$. This action extends to other elements as

$$
\gamma : \omega \mapsto \omega^3 \mapsto \omega^2 \mapsto \omega^6 \mapsto \omega^4 \mapsto \omega^5 \mapsto \omega.
$$

Hence the Galois group is $\mathbb{Z}_6$

$$
\gamma : 1 \mapsto 3 \mapsto 2 \mapsto 6 \mapsto 4 \mapsto 5 \mapsto 1.
$$

**Solutions**

Using the action of the Galois group we can find the other two roots by

$$
x = \sum_{k=1}^{3} b_k \left( \omega^{\gamma(k)} + \omega^{-\gamma(k)} \right),
$$

It follows that the three roots of the equation are

$$
x_1 = b_1(\omega + \omega^{-1}) + b_2(\omega^2 + \omega^{-2}) + b_3(\omega^3 + \omega^{-3}) ,
x_2 = b_1(\omega^3 + \omega^{-3}) + b_2(\omega + \omega^{-1}) + b_3(\omega^2 + \omega^{-2}) ,
x_3 = b_1(\omega^2 + \omega^{-2}) + b_2(\omega^3 + \omega^{-3}) + b_3(\omega + \omega^{-1}) .
$$
**Coefficients**

In terms of the three roots (A.2) the equation (A.1) becomes

\[ x^3 + a_1 x^2 + a_2 x + a_3 = x^3 + (b_1 + b_2 + b_3) x^2 + \left(-2 \sum b_i^2 + (-2 + 5) \sum_{i<j} b_i b_j\right) x + \ldots \]

Solving for \(\{a_1, a_2, a_3\}\) in terms of \(\{b_1, b_2, b_3\}\) gives

\[
\begin{align*}
    a_1 &= b_1 + b_2 + b_3, \\
    a_2 &= -2 \sum b_i^2 + 3 \sum b_i b_j = -\frac{7}{2} \sum b_i^2 + \frac{3}{2} \left(\sum b_i\right)^2.
\end{align*}
\]

This can be rearranged to get

\[
\begin{align*}
    \sum_{i=1}^{3} b_i &= a_1, \\
    \sum_{i=1}^{3} b_i^2 &= -\frac{2}{7} \left(a_2 - \frac{3}{2} a_1^2\right).
\end{align*}
\]

**A.2 Example**

We use the method described above to find the roots of the equation

\[ x^3 - 6x^2 + 5x - 1 = 0. \]
Solution

We can solve for \( \{b_1, b_2, b_3\} \) to get

\[
(a_1, a_2) = (-6, 5) \implies b_1 + b_2 + b_3 = -6 \quad \text{and} \quad b_1^2 + b_2^2 + b_3^2 = 14
\]

\[
\implies (b_1, b_2, b_3) = (-3, -2, -1) .
\]

The 3 roots of the equation turn out to be

\[
x = -4 \cos \left( \frac{2\pi}{7} \right) + 6 \cos \left( \frac{3\pi}{7} \right) + 2 \cos \left( \frac{\pi}{7} \right),
\]

\[
x = -2 \cos \left( \frac{2\pi}{7} \right) + 4 \cos \left( \frac{3\pi}{7} \right) + 6 \cos \left( \frac{\pi}{7} \right),
\]

\[
x = -6 \cos \left( \frac{2\pi}{7} \right) + 2 \cos \left( \frac{3\pi}{7} \right) + 4 \cos \left( \frac{\pi}{7} \right) .
\]