Likelihood-based tests on linear hypotheses of large dimensional mean vectors with unequal covariance matrices.

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Abstract

This paper considers testing linear hypotheses of a set of mean vectors with unequal covariance matrices in large dimensional setting. The problem of testing the hypothesis \( H_0 : \sum_{i=1}^q \beta_i \mu_i = \mu_0 \) for a given vector \( \mu_0 \) is studied from the view of likelihood, which makes the proposed tests more powerful. We use the CLT for linear spectral statistics of a large dimensional \( F \)-matrix in Zheng [21] to establish the new test statistics in large dimensional framework, so that the proposed tests can be applicable for large dimensional non-Gaussian variables in a wider range. Furthermore, our new tests provide more optimal empirical powers due to the likelihood-based statistics, meanwhile their empirical sizes are closer to the significant level. Finally, the simulation study is provided to compare the proposed tests with other high dimensional mean vectors tests for evaluation of their performances.

Keywords: Large dimensional data, Linear hypothesis, Mean vectors tests, Random matrix theory

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1. Introduction

Testing on mean vectors endures as an old, yet active research field with the applications of multiple comparisons, MANOVA and classification. Continuing improvements on data acquisition techniques and the ease of access to high computation power pose constant challenges in applying the traditional statistical methods to these emerging data sets, because they are established on the basis of fixed dimension $p$ as the sample size $n$ tends to infinity. Within this context, more and more attention is paid to find the efficient testing methods for high dimension data and much progress has been made in this respect. A special attention has been given to the linear hypothesis test of mean vectors, which is an important part of multivariate statistical analysis and widely used in the biology, finance and etc. Suppose $X_i = (x_{i1}, \ldots, x_{in})'$, $i = 1, \ldots, q$ to be the independent sample from $q$ population with mean $\mu_i$ and covariance matrix $\Sigma_i$, $i = 1, \ldots, q$, respectively, where $\Sigma_i$'s are unequal $p \times p$ covariance matrix. Consider the test hypothesis

\[ H_0 : \sum_{i=1}^{q} \beta_i \mu_i = \mu_0 \quad \text{v.s.} \quad H_1 : \text{not } H_0, \quad (1.1) \]

where $\beta_1, \ldots, \beta_q$ are the given scalars and $\mu_0$ is a known vector. Of course, the multivariate Behrens-Fisher problem and MANOVA are covered as the special cases. A classical solution was first proposed by Bennett [7], see also in Anderson [1], which was an extension of the methodology for two-sample case in Scheffe [14] to the multiple case. Then many efforts have been devoted to develop the solutions in the large dimensional data setting. To be specific, Bai and Saranadasa [4] investigated the two-sample case under the normal assumption and equal covariance matrices, and extended Hotelling’s $T^2$ test to the $p > N$ setting. Motivated by this work, Chen and Qin [8] proposed a two-sample test for the equality of the means of high dimensional data with unequal covariance matrices. Aoshima and Yata [2] derived a nonparametric test for which the significant levels are not effected by the population distribution assumption. Also, Fujikoshi et al. [10], Srivastava and Fujikoshi [19], Srivastava [16], Schott [15], Srivastava and Du [18], Srivastava [17], Srivastava et al. [20] and Hu et al. [11] were proposed for the test on the equality of high dimensional mean vectors. Nishiyamaa et al. [13] focused on the testing linear hypothesis on the mean vectors of normal populations with unequal covariance matrices when the dimensionality $p$ exceeds the sample
size $n_i$. They proposed a new test procedure based on the Dempster trace criterion and showed its consistency in high dimension setting.

Different from the previous works, we proposed new tests based on likelihood for the hypothesis (1.1) by the CLT for LSS of a large dimensional $F$-matrix in Zheng [21]. The tests in this work were suitable for non-Gaussian variables in a wider range. More important, our proposed tests provided the more accurate sizes and achieved much better performance on the empirical powers than other high dimensional test methods, which had been sustained by the simulation. Finally, the restricted condition was relaxed to the finite 4-th moment compared with Chen and Qin [8] and Nishiyamaa et al. [13], which made our tests more applicable.

The remainder of the article is organized as follows. Section 2 gives a quick review of the linear hypothesis test of mean vectors, then the CLT for LSS of a large dimensional $F$-matrix in Zheng [21] is also provided in this part. In Section 3, we propose the new testing statistics in large dimensional setting based on the classical likelihood test. Simulation results are presented to evaluate the performance of our test compared with other high dimensional mean vectors tests in Section 4. Finally, a conclusion is drawn in the Section 5, and the proofs and derivations are listed in the Appendix A.

2. Problem Description and Preliminary

In this section, the problem of testing the linear hypothesis of mean vectors is described in details. As mentioned above, a classical test statistic is proposed by Bennett (1951) under Gaussian assumption, i.e. $X_i = (x_{i1}, \cdots, x_{in_i})'$, $i = 1, \cdots, q$ be $n_i$ independent samples from $N_p(\mu_i, \Sigma_i)$. Without loss of generality, assume $n_1$ is the least one if all the sample sizes $n_1, \cdots, n_q$ are different. Set

$$y_k = \beta_1 x_{1k} + \sum_{i=2}^{q} \beta_i \sqrt{\frac{n_1}{n_i}} \left( x_{ik} - \frac{1}{n_1} \sum_{l=1}^{n_1} x_{il} + \frac{1}{\sqrt{n_1 n_i}} \sum_{m=1}^{n_i} x_{im} \right). \quad (2.1)$$

where $k = 1, \cdots, n_1$. It is simplified as $y_k = \sum_{i=1}^{q} \beta_i x_{ik}$, if $n_1 = n_2 = \cdots = n_q$. Then it is obviously that

$$E y_k = \sum_{i=1}^{q} \beta_i \mu_i$$
and
\[ E(y_k - E y_k)(y_l - E y_l)' = \delta_{kl} \left( \sum_{i=1}^{q} \frac{\beta_i^2 n_1}{n_i} \Sigma_i \right), \]
where \( \delta_{kl} \) is a Kronecker's delta function. Meanwhile, make the denotations as below
\[ \bar{y} = \frac{1}{n_1} \sum_{k=1}^{n_1} y_k = \sum_{i=1}^{q} \beta_i \bar{x}_i, \quad \bar{x}_i = \frac{1}{n_i} \sum_{i=1}^{n_i} x_{il}, \]
and
\[ S = \frac{1}{n_1 - 1} \sum_{k=1}^{n_1} (y_k - \bar{y})(y_k - \bar{y})'. \tag{2.2} \]
So the classical test statistic according to Bennett (1951) is
\[ T^2 = n_1(\bar{y} - \mu_0)'S^{-1}(\bar{y} - \mu_0), \tag{2.3} \]
which follows a \( p \)-dimensional \( T^2 \) with \( n_1 - 1 \) freedom degree for any fixed \( p \).
However, it is not the case when the dimension \( p \) grows larger. Denote
\[ \sum_{i=1}^{q} \beta_i \mu_i \equiv \mu, \quad \sum_{i=1}^{q} \frac{\beta_i^2 n_1}{n_i} \Sigma_i \equiv \Sigma, \]
then we note that \( y_1, \cdots, y_{n_1} \) follows the distribution \( N_p(\mu, \Sigma) \) independently under the Gaussian assumption. Because \( \sqrt{n_1}(\bar{y} - \mu) \sim N_p(0, \Sigma) \),
then it is obtained that \( n_1(\bar{y} - \mu_0)'S^{-1}(\bar{y} - \mu_0)' \sim W_p(1, \Sigma) \) and \( (n_1 - 1)S \sim W_p(n_1 - 1, \Sigma) \) under the null hypothesis, and they are independent with each other, where \( W_p(f, \Sigma) \) means a \( p \) dimensional Wishart distribution with freedom degree \( f \) and parameter \( \Sigma \). Define the matrix
\[ F = \frac{n_1(\bar{y} - \mu_0)(\bar{y} - \mu_0)'}{S}, \tag{2.4} \]
according to the definition of \( F \)-matrix, \( F \) is an \( F \)-matrix with freedom degree \( (1, n_1 - 1) \). Thus the classical \( T^2 \)-test statistic in (2.3) can be represented as
\[ T^2 = \text{tr} \left( n_1(\bar{y} - \mu_0)(\bar{y} - \mu_0)'S^{-1} \right) = \text{tr}(F) \]
Under the suitable 4-th moments constrains, using the results on the limiting spectral distribution of \( F \)-matrix in eq. (4.4.1) in Bai and Silverstein [6], it can be obtained with probability 1
\[ \frac{1}{p} T^2 = \frac{1}{p} \sum_{i=1}^{p} \lambda_i^F \rightarrow \int_{a}^{b} \frac{x(1-\gamma_2)\sqrt{(b-x)(x-a)}dx}{2\pi x(\gamma_1 + \gamma_2 x)} = \frac{1}{1-\gamma_2} \equiv d(\gamma_2) > 1 \tag{2.5} \]
where \( \{ \lambda_i^F, i = 1, \ldots, p \} \) are the eigenvalues of the matrix \( F \), \( p/1 = \gamma_p \to \gamma_1 \in (0, +\infty), p/(n_1 - 1) = \gamma_{n_1} \to \gamma_2 \in (0, 1) \), \( h = \sqrt{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2} \) and

\[
a = \left( \frac{1 - h}{1 - \gamma_2} \right)^2, \quad b = \left( \frac{1 + h}{1 - \gamma_2} \right)^2.
\]

This result is derived in the Appendix A.1. As seen from above, it shows that almost surely

\[ T^2 = p \cdot d(\gamma_2) \]

Thus, any test that assumes asymptotic \( T \)-square distribution of \( T^2 \) will result in a serious error when \( p \) grows higher and higher. Therefore, we intend to make some amendments to the classical test by the CLT of LSS (linear spectral statistic) of large dimensional \( F \)-matrices, which is Theorem 3.2 in Zheng [21]. In order to introduce it, we first make clear some preliminary preparations.

Let \( \{ \xi_{ki} \in \mathbb{C}, i, k = 1, 2, \ldots \} \) and \( \{ \eta_{kj} \in \mathbb{C}, j, k = 1, 2, \ldots \} \) be either both real or both complex random arrays. Write \( \xi_i = (\xi_{1i}, \xi_{2i}, \ldots, \xi_{pi})' \) and \( \eta_j = (\eta_{1j}, \eta_{2j}, \ldots, \eta_{pj})' \). Also, for any positive integers \( n_1, n_2, \xi = (\xi_1, \ldots, \xi_{n_1}) \) and \( \eta = (\eta_1, \ldots, \eta_{n_2}) \) can be thought as two independent samples of a \( p \)-dimensional observations of size \( n_1 \) and \( n_2 \), respectively. Let \( S_1 \) and \( S_2 \) be the associated sample covariance matrices, i.e.

\[
S_1 = \frac{1}{n_1} \sum_{i=1}^{n_1} \xi_i \xi_i^* \quad \text{and} \quad S_2 = \frac{1}{n_2} \sum_{j=1}^{n_2} \eta_j \eta_j^*.
\]

where \( * \) stands for complex conjugate and transpose. Then, the following so-called \( F \)-matrix generalizes the classical Fisher-statistic to the present \( p \)-dimensional case,

\[ V_n = S_1 S_2^{-1} \quad (2.6) \]

where \( n = (n_1, n_2) \) and \( n_2 > p \) is required to ensure that almost surely the matrix \( S_2 \) is invertible.

Let us also make some assumptions as below:

**Assumption [A]** For any fixed \( \epsilon_0 > 0 \)

\[
\frac{1}{n_1 p} \sum_{i=1}^{n_1} \sum_{j=1}^{p} \mathbb{E}|\xi_{ij}|^4 I(|\xi_{ij}| \geq \epsilon_0 \sqrt{n_1}) \to 0;
\]

\[
\frac{1}{n_2 p} \sum_{i=1}^{n_2} \sum_{j=1}^{p} \mathbb{E}|\eta_{ij}|^4 I(|\eta_{ij}| \geq \epsilon_0 \sqrt{n_2}) \to 0.
\]
**Assumption [B]** The sample size $n_1, n_2$ and the dimension $p$ increase to infinity in such a large dimensional limiting scheme that

\[
y_{n_1} = \frac{p}{n_1} \to y_1 \in (0, +\infty), \quad y_{n_2} = \frac{p}{n_2} \to y_2 \in (0, 1).
\] (2.7)

Let $F^V_n$ denote the empirical spectral distribution (ESD) of the matrix $V_n$. Under the assumptions above, the ESD $F^V_n$ almost surely converges to the LSD (limiting spectral distribution) $F_{y_1,y_2}$ with the density function represented as

\[
\ell(x) = \begin{cases} 
\frac{(1 - y_2)\sqrt{(b' - x)(x - a')}}{2\pi x(y_1 + y_2 x)}, & a' \leq x \leq b', \\
0, & \text{otherwise},
\end{cases}
\] (2.8)

and has a point mass $1 - \frac{1}{y_1}$ at the origin if $y_1 > 1$, where $h' = \sqrt{y_1 + y_2 - y_1y_2}$

\[
a' = \left(\frac{1 - h'}{1 - y_1}\right)^2, \quad b' = \left(\frac{1 + h'}{1 - y_2}\right)^2.
\]

See p.72 of Bai and Silverstein [6]. We use $F_{y_{n_1}, y_{n_2}}$ to mark an analog representation of $F_{y_1,y_2}$ by substituting the index $y_{n_1}, y_{n_2}$ for $y_1, y_2$. Let $\mathcal{A}$ be a set of functions $f_1, f_2, \cdots$, which are analytic in an open region in the complex plane containing the support of the continuous part of the LSD $F_{y_1,y_2}$ defined in (2.8). A linear spectral statistic (LSS) of the random matrix $V_n$ is expressed as

\[
\int f(x) dF^V_n(x) = \frac{1}{p} \sum_{i=1}^{p} f(\lambda^V_i), \quad f \in \mathcal{A},
\]

where $(\lambda^V_i)$ are the real eigenvalues of the $p \times p$ square matrix $V_n$. Then based on the empirical process $G_n := \{G_n(f)\}$ indexed by $\mathcal{A}$,

\[
G_n(f) = p \cdot \int_{-\infty}^{+\infty} f(x) \left[F^V_n - F_{y_{n_1}, y_{n_2}}\right] (dx), \quad f \in \mathcal{A},
\] (2.9)

the CLT for LSS of large dimensional $F$-matrices (Theorem 3.2 in Zheng [21]) is provided as following, which will play a fundamental role in next derivations.
Let
\[
\kappa = \begin{cases} 
2, & \text{if the } \xi, \eta - \text{variables are real}, \\
1, & \text{if the } \xi, \eta - \text{variables are complex}.
\end{cases}
\]

Lemma 2.1 (Theorem 3.2 in Zheng [21]). Assume that

1. Assumptions [A]–[B] are satisfied;
2. For any positive integers \(n_1, n_2\), \(\xi = (\xi_1, \cdots, \xi_{n_1})\) and \(\eta = (\eta_1, \cdots, \eta_{n_2})\) can be thought as two independent samples of a \(p\)-dimensional observations, where \(\xi_i = (\xi_{i1}, \xi_{i2}, \cdots, \xi_{ip})'\) and \(\eta_j = (\eta_{j1}, \eta_{j2}, \cdots, \eta_{jp})'\). For all \(i, j, k\), \(E\xi_{ki} = E\eta_{kj} = 0\), \(E|\xi_{ki}|^2 = E|\eta_{kj}|^2 = \kappa - 1\), \(E|\xi_{ki}|^4 = \beta_x + \kappa - 1 < \infty\) and \(E|\eta_{kj}|^4 = \beta_y + \kappa - 1 < \infty\), where \(\beta_x\) and \(\beta_y\) are contains concerned with the 4-th moments.

Let \(f_1, \cdots, f_s \in A\), then the random vector \((G_n(f_1), \cdots, G_n(f_s))\) weakly converges to a \(s\)-dimensional Gaussian vector with the mean vector
\[
\mu(f_j) = \frac{\kappa - 1}{4\pi i} \int f_j(z) d\log \left( \frac{(1 - y_2)m_0^2(z) + 2m_0(z) + 1 - y_1}{(1 - y_2)m_0^2(z) + 2m_0(z) + 1} \right) 
\]
\[
+ \frac{\kappa - 1}{4\pi i} \int f_j(z) d\log (1 - y_2m_0^2(z)(1 + m_0(z))^{-2}) 
\]
\[
+ \frac{\beta_x y_1}{2\pi i} \int f_j(z) (1 + m_0(z))^{-3} dm_0(z) 
\]
\[
+ \frac{\beta_y}{4\pi i} \int f_j(z) \left( 1 - \frac{y_2m_0^2(z)}{(1 + m_0(z))^2} \right) d\log \left( 1 - \frac{y_2m_0^2(z)}{(1 + m_0(z))^2} \right) 
\]
and covariance function
\[
v(f_j, f_\ell) = -\frac{\kappa}{4\pi^2} \iint \frac{f_j(z_1)f_\ell(z_2)}{(m_0(z_1) - m_0(z_2))^2} dm_0(z_1)dm_0(z_2) 
\]
\[
- \frac{\beta_x y_1 + \beta_y y_2}{4\pi^2} \iint \frac{f_j(z_1)f_\ell(z_2)}{(1 + m_0(z_1))^2(1 + m_0(z_2))^2} dm_0(z_1)dm_0(z_2)
\]
where \(j, \ell \in \{1, \cdots, s\}\), \(m_0(z) = m_{y_2}(-m(z))\). Here \(m(z)\) is the Stieltjes Transform of \(E_{y_1, y_2} = (1 - y_1)I_{[0, \infty)} + y_1F_{y_1, y_2}\) and \(m_{y_2}(z)\) is the Stieltjes Transform of \(E_{y_2} = (1 - y_2)I_{[0, \infty)} + y_2F_{y_2}\), where \(F_{y_2}\) is the LSD of the matrix \(S_2\). The contours all contain the support of \(F_{y_1, y_2}\) and non overlapping in both \(2.14\) and \(2.15\).
The expression of the asymptotic mean and covariance in Lemma 2.1 is complicated to figure it out. So further steps were given to help the evaluation of the asymptotic mean and covariance in the Corollary 3.2 in Zheng [21]. However, the result provided in the Corollary 3.2 in Zheng [21] is not correct, I think it is a typo mistake. In order to obtain an accurate and simplified form for computing the asymptotic mean and covariance, we reviewed the Corollary 3.2 in Zheng [21], and give the corrected result in the following Lemma 2.2, which is proved in the Appendix A.2.

Lemma 2.2. Under the assumptions of Lemma 2.1, the asymptotic means and covariances of the limiting random vector can be computed as follows

\[
\mu(f_j) = \lim_{\tau \uparrow 1} \frac{\kappa - 1}{4\pi i} \oint_{|\xi|=1} f_j \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi)}{(1 - y_2^2)} \right) \left[ \frac{1}{\xi - \frac{1}{\tau}} + \frac{1}{\xi + \frac{1}{\tau}} - \frac{2}{\xi + \frac{y_2}{h'}} \right] d\xi
\]  

(2.16)

\[
+ \frac{\beta_x \cdot y_1 (1 - y_2^2)}{2\pi i \cdot h'^2} \oint_{|\xi|=1} f_j \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi)}{(1 - y_2^2)} \right) \frac{1}{(\xi + \frac{y_2}{h'})^3} d\xi
\]  

(2.17)

\[
+ \frac{\beta_y \cdot y_2 (1 - y_2^2)}{2\pi i \cdot h'} \oint_{|\xi|=1} f_j \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi)}{(1 - y_2^2)} \right) \frac{\xi + \frac{y_2}{h'}}{(\xi + \frac{y_2}{h'})^3} d\xi,
\]  

(2.18)

where \( j, \ell \in \{1, \ldots, s\} \), "Re" represents the real part of \( \xi \) and \( \tau \downarrow 1 \) means that \( \tau \) approaches 1 from above.

\[
\nu(f_j, f_\ell) = -\lim_{\tau \uparrow 1} \frac{\kappa}{4\pi^2} \oint_{|\xi|=1} \oint_{|\xi|=1} f_j \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi_1)}{(1 - y_2^2)} \right) f_\ell \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi_2)}{(1 - y_2^2)} \right) d\xi_1 d\xi_2
\]

(2.19)

\[
- \frac{(\beta_x y_1 + \beta_y y_2) (1 - y_2^2)^2}{4\pi^2 h'^2} \oint_{|\xi|=1} \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi_1)}{(1 - y_2^2)} \right) \frac{1}{(\xi_1 + \frac{y_2}{h'})^2} d\xi_1 \oint_{|\xi|=1} \left( \frac{1 + h'^2 + 2h' \text{Re}(\xi_2)}{(1 - y_2^2)} \right) \frac{1}{(\xi_2 + \frac{y_2}{h'})^2} d\xi_2
\]

(2.20)

where \( j, \ell \in \{1, \ldots, s\} \), "Re" represents the real part of \( \xi \) and \( \tau \downarrow 1 \) means that \( \tau \) approaches 1 from above.

3. The Proposed Testing Statistics

Based on the CLT for the \( F \)-matrices in Lemma 2.1, a corrected scaling for the classical test statistic is established. Recall that

\[
T^2 = \text{tr}(F), \quad F = \frac{n_1(\mathbf{y} - \mu_0)(\mathbf{y} - \mu_0)'}{S}
\]
Under the null hypothesis $H_0$, we have

$$n_1(\mathbf{\bar{y}} - \mu_0)(\mathbf{\bar{y}} - \mu_0)' \sim W_p(1, \Sigma), \quad (n_1 - 1)S \sim W_p(n_1 - 1, \Sigma)$$

and they are independent with each other. According to the definition of $F$-matrix, standardization of the entries cannot effect on the values of $F$, because both the numerator and denominator are already centralized and have the same covariance parameter $\Sigma$. Consequently, $F$ is exactly distributed as the $F$-matrix $V_n$ with freedom degree $(1, n_1 - 1)$, where in addition they have the same limiting spectral distributions. Thus, our proposed test statistic is given by Lemma 2.1 under the large dimensional setting $p/1 = \gamma_p \to \gamma_1 \in (0, +\infty)$ and $p/(n_1 - 1) = \gamma_{n_1} \to \gamma_2 \in (0, 1)$, which means that our method is valid for moderate high dimensionality. However, it still works for ultra high dimensional data if there is a more larger sample size.

**Theorem 3.1.** Assuming that the conditions of Lemma 2.1 hold under $H_0$ in (1.1), $T^2$ is defined as in (2.3) and $f(x) = x$. Let $p/1 = \gamma_p \to \gamma_1 \in (0, +\infty)$ and $p/(n_1 - 1) = \gamma_{n_1} \to \gamma_2 \in (0, 1)$. Then, under $H_0$

$$T_{\text{ours}} = v(f)^{-1/2} [T^2 - p \cdot d(\gamma_{n_1}) - \mu(f)] \Rightarrow N(0, 1).$$

(3.1)

where $d(\gamma_{n_1})$ is derived in (3.3), and $\mu(f), v(f)$ are depicted as (3.4) and (3.5), respectively.

**Proof 3.1.** According to the definition in (2.3), we have

$$T^2 = \text{tr}(F) = \sum_{i=1}^{p} \lambda_i^F = p \cdot \int xdF_n^F(x)$$

where $F_n^F(x)$ is the ESD of the matrix $F$ in (2.4).

Since $F$ is exactly distributed as the $F$-matrix $V_n$ with freedom degree $n = (1, n_1 - 1)$, $F$ has the same limiting spectral distribution with the $F$-matrix $V_n$. Furthermore, the unbiased estimator of the covariance matrix of $y_k, k = 1, \cdots, n_1$ is adopted for the denominator $S$ in $F$, which is the only item subtracting sample mean. So it is equivalent to apply the CLT for LSS of large dimensional $F$-matrix to either $F$ or $V_n$ with freedom degree $n = (1, n_1 - 1)$. Then define $f(x) = x$ and

$$G_n(f) = p \cdot \int f(x)d (F_n^F(x) - F_{\gamma_p, \gamma_{n_1}}(x)), \quad (3.2)$$
where $F_{\gamma_p, \gamma_n}$ is analogous to LSD of the matrix $F$, which has a density in (2.8) but with $\gamma_p, \gamma_n$ instead of $y_k, k = 1, 2.,$ respectively. Consequently, $F_{\gamma_p, \gamma_n} (f) = \int f(x)dF_{\gamma_p, \gamma_n}(x)$ is exactly analogous to the $d(\gamma_2)$ calculated in (2.5) by substituting $\gamma_n$ for $\gamma_2$, i.e.

$$F_{\gamma_p, \gamma_n} (f) = \int f(x)dF_{\gamma_p, \gamma_n}(x) = \frac{1}{1 - \gamma_n} \equiv d(\gamma_n) \quad (3.3)$$

By Lemma 2.1, $G_n(f)$ weakly converges to a Gaussian vector with mean

$$\mu(f) = \frac{\gamma_2}{(1 - \gamma_2)^2} + \frac{\beta_y \gamma_2}{1 - \gamma_2} \quad (3.4)$$

and variance

$$v(f) = \frac{2h^2}{(1 - \gamma_2)^4} + \frac{\beta_x \gamma_1 + \beta_y \gamma_2}{(1 - \gamma_2)^2} \quad (3.5)$$

where $h = \sqrt{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2}$, $\beta_x$ and $\beta_y$ here are the Kurtosis of the standardized $\bar{y}$ and $y_i$, respectively, which can be calculated from (2.1). But it is complicated and we simulated them in the simulation study. (3.4) and (3.5) are calculated by Lemma 2.1 in the Appendix A.3 and Appendix A.4. From

$$T^2 = p \cdot \int f(x)dF_n^F(x)$$

$$= p \cdot \int f(x)d(F_n^F(x) - F_{\gamma_p, \gamma_n}(x)) + p \cdot F_{\gamma_p, \gamma_n}(f), \quad (3.6)$$

we get

$$G_n(f) = T^2 - p \cdot d(\gamma_n) \Rightarrow N(\mu(f), v(f)). \quad (3.7)$$

Therefore,

$$T_{ours} = v(f)^{-\frac{1}{2}} [T^2 - p \cdot d(\gamma_n) - \mu(f)] \Rightarrow N(0, 1).$$

The test statistic we proposed for testing (1.1) is based on the likelihood ratio test statistic $T^2$ the and its asymptotic distribution is derived in the theorem above. However, it is worth noticing that in the above proof, we used the Gaussian assumption for entry variables to fit $F$-matrix definition, but Lemma 2.1 does not need this Gaussian assumption. Therefore, the
asymptotic distribution for in Theorem 3.1 could be applied more generally to non-Gaussian variables. The simulations could certainly make out a case for this point of view.

Next, we consider some special cases of the test hypothesis (1.1), and derive their test statistics and asymptotic distributions in some corollaries. First, we focus on the testing the equality of two population mean vectors with unequal covariance matrices. That is

\[ H_0 : \mu_1 = \mu_2 \quad \text{v.s.} \quad H_1 : \mu_1 \neq \mu_2, \quad (3.8) \]

which is a special case of the hypothesis (1.1) with \( q = 2, \beta_1 = 1, \beta_2 = -1 \) and \( \mu_0 = 0 \). Then define

\[ y_k = x_{1k} - \sqrt{\frac{n_1}{n_2}} x_{2k} + \frac{1}{\sqrt{n_1 n_2}} \sum_{l=1}^{n_1} x_{2l} - \frac{1}{n_2} \sum_{m=1}^{n_2} x_{2m}, \]

and \( n_1 < n_2 \) without loss of generality. Thus, we also have \( \{y_k, k = 1, \cdots, n_1\} \) are independent and

\[ \mu \equiv E y_k = \mu_1 - \mu_2 \]

and

\[ \Sigma \equiv E(y_k - E y_k)(y_l - E y_l)' = \delta_{kl} \left( \Sigma_1 + \frac{n_1}{n_2} \Sigma_2 \right), \]

where \( \delta_{kl} \) is a Kronecker’s delta function. So it is equivalent to test

\[ H_0 : \mu = 0 \quad \text{v.s.} \quad H_1 : \mu \neq 0 \]

and the classical test statistic is

\[ T_{BF} = n_1 y' S^{-1} y, \quad (3.9) \]

where \( S \) is defined in (2.2) with \( q = 2 \). Applying the Theorem 3.1, we have the following corollary:

**Corollary 3.1.** For testing \( H_0 : \mu_1 = \mu_2 \) with unequal covariance matrix \( \Sigma_i, i = 1, 2 \), under the assumption of Theorem 3.1, we have the conclusion of Theorem 3.7 still holds, only with the test statistic \( T^2 \) in (3.7) is revised by \( T_{BF} \).
For more simplicity, we assume all of the variables have the common covariance matrix, that is $\Sigma_1 = \cdots = \Sigma_q = \Sigma$. Then for testing the hypothesis (1.1) with the common covariance matrix assumption, we set

$$\bar{y} = \sum_{i=1}^{q} \beta_i \bar{x}_i \quad S = \frac{1}{q} \sum_{i=1}^{q} \sum_{k=1}^{n_i} (x_{ik} - \bar{x}_i)(x_{ik} - \bar{x}_i)'$$

where $\bar{x}_i = \frac{1}{n_i} \sum_{k=1}^{n_i} x_{ik}$. So the classical likelihood test statistic is

$$T_M = \sum_{i=1}^{q} \frac{\beta_i^2}{n_i} (\bar{y} - \mu_0)' S^{-1} (\bar{y} - \mu_0),$$

(3.10)

Define the matrix

$$F_1 = \sum_{i=1}^{q} \frac{\beta_i^2}{n_i} (\bar{y} - \mu_0)(\bar{y} - \mu_0)' S^{-1}$$

(3.11)

according to the definition of $F$-matrix, $F_1$ is an $F$-matrix with freedom degree $(1, \sum_{i=1}^{q} n_i - q)$. Thus the test statistic $T_M$ can be written as

$$T_M = \text{tr} \left( \sum_{i=1}^{q} \frac{\beta_i^2}{n_i} (\bar{y} - \mu_0)(\bar{y} - \mu_0)' S^{-1} \right) = \text{tr}(F_1)$$

Applying the Theorem 3.1, the corresponding corollary are given as below.

**Corollary 3.2.** For testing $H_0 : \sum_{i=1}^{q} \beta_i \mu_i = \mu_0$ with common covariance matrix $\Sigma$, assume that $p/1 = \gamma_p \rightarrow \gamma_1 \in (0, +\infty)$ and $p/(\sum_{i=1}^{q} n_i - q) = \gamma_n \rightarrow \gamma_2 \in (0, 1)$ and other conditions of Theorem 3.1 still hold, then we have the conclusion of Theorem 3.1 only with the test statistic $T^2$ in (3.7) is revised by $T_M$.

For the test on the equality of two mean vectors with common covariance matrix, we have

$$T_D = \frac{n_1 n_2}{n_1 + n_2} \bar{y}' S^{-1} \bar{y} = \text{tr}(F_2)$$
where 
\[
\bar{y} = \bar{x}_1 - \bar{x}_2; \quad S = \frac{1}{n_1 + n_2 - 2} \sum_{i=1}^{n_1} \sum_{k=1}^{n_2} (x_{ik} - \bar{x}_i)(x_{ik} - \bar{x}_i)'
\]

and
\[
F_2 = \frac{n_1 n_2}{n_1 + n_2} \bar{y} \bar{y}' S^{-1}
\]
is satisfied for the definition of $F$-matrix with freedom degree $(1, n_1 + n_2 - 2)$. Applying the Theorem 3.1, the corresponding corollary is given as below.

**Corollary 3.3.** For testing $H_0 : \mu_1 = \mu_2$ with equal covariance matrix $\Sigma$, assume that $p/1 = \gamma_p \to \gamma_1 \in (0, +\infty)$ and $p/(n_2 + n_2 - 2) = \gamma_n \to \gamma_2 \in (0, 1)$ and other conditions of Theorem 3.1 still hold, then we have the conclusion of Theorem 3.1 only with the test statistic $T^2$ in (3.1) is revised by $T_D$.

### 4. Simulation Study

In this section, simulations are conducted to evaluate the test statistics that we proposed based on likelihood $T^2$ test statistic. Two hypotheses $H_{0a} : \sum_{i=1}^{3} \beta_i \mu_i = 0$ and $H_{0b} : \mu_1 = \mu_2$ with unequal covariance matrices are investigated without loss of generality. We also present the corresponding simulation results of other tests as a comparison, for example tests in Nishiyama et al. [13](TNT) for $H_{0a}$ and the tests in Nishiyama et al. [13](TNT) and Chen and Qin [8] (CQT) for $H_{0b}$. The samples are generated from the model

\[
x_{ij} = \Gamma_i z_{ij} + \mu_i, \quad i = 1, \cdots, q, j = 1, \cdots, n_i
\]

where $z_{ij} = (z_{ij1}, \cdots, z_{ijp})'$ and \{z_{ijk}, k = 1, \cdots, p\} are independently distributed as one of the following distribution assumptions:

(i) $N(0, 1)$; (ii) Gamma$(4, 0.5) - 2$.

For the covariance matrix $\Sigma_i, i = 1, 2, 3$, the following cases concerned with the dimension $p$ are taken into account by

\[
\Sigma_i = \Gamma_i^2 = W_i \Phi_i W_i \quad (4.1)
\]

\[
W_i = \text{diag}(w_{i1}, \cdots, w_{ip}), \quad w_{ij} = 2 \times i + (p - j + 1)/p \quad (4.2)
\]

\[
\Phi_i = (\phi_{jk}^{(i)}), \quad \phi_{jj}^{(i)} = 1, \phi_{jk}^{(i)} = (-1)^{(j+k)}(0.2 \times i)^{|j-k|^{0.1}}, j \neq k, \quad (4.3)
\]
which is cited from Hu et al. [11]. The suitable mean vectors \( \mu_i, i = 1, 2, 3 \) are chosen for different hypotheses.

First, for two-sample problem \( H_{0a} : \mu_1 = \mu_2 \), the null hypothesis is assumed to be \( \mu_1 = \mu_2 = 0 \) without loss of generality. Denote \( \Delta \mu = \left( \epsilon \sqrt{2 \log(p)} \cdot 1_{[p^0]}, 0'_{p-\lfloor p^0 \rfloor} \right)' \), where \( 1_p \) represents a vector with all elements are 1, \( \lfloor \rfloor \) denotes the integer truncation function and \( \epsilon, v_0 \) are varying constants. For the alternative hypothesis, the sparse model similar to the one in Chen and Qin [8] is applied, which describes \( x_{ij} = \Gamma_i z_{ij} + \mu_i, i = 1, 2, j = 1, \cdots, n_i \) and \( \mu_1 = 0, \mu_2 = \Delta \mu \).

Secondly, we consider the three groups testing problem \( H_{0b} : \sum_{i=1}^{3} \beta_i \mu_i = 0 \). Under the null hypothesis, we choose two cases of \( \beta_i, i = 1, 2, 3 \). One is \( \beta_1 = \beta_2 = \beta_3 = 1 \), and the corresponding mean vectors are generally selected as \( \mu_1 = 1, \mu_2 = 1, \mu_3 = -2 \). The other one is \( \beta_1 = \beta_2 = -\frac{1}{2}, \beta_3 = 1 \), and the corresponding mean vectors are given as \( \mu_1 = 1, \mu_2 = 3, \mu_3 = 2 \) without loss of generality. The alternative hypotheses are designed that \( \mu_3 \) is the value under the null hypothesis added \( \Delta \mu \) described as above, while \( \mu_1 \) and \( \mu_2 \) remain unchanged.

For each set of the scenarios, we report both empirical Type I errors and powers with 10,000 replications at \( \alpha = 0.05 \) significance level. Different pair values of \( p, n_1, n_2, n_3 \) are selected, and \( \epsilon \) is varying from 0 to 0.9 or 1 to show the empirical sizes and powers. The mean parameter is supposed to be unknown and substituted by the sample mean during the calculations. Simulation results of empirical Type I errors and powers for the three group tests are listed in the Table 1 and Table 2. Simulation results of empirical Type I errors and powers for the two-sample test are represented in Table 3.

For three groups tests, as seen from the Table 1 and Table 2, the advantages of our proposed tests compared with the tests in Nishiyamaa et al. [13] (TNT) for \( H_{0a} \) are listed in two aspects. First, almost all of the empirical Type I errors of our proposed test are around the nominal size 5%, which are better than that of TNT. Although, the empirical size of the proposed test is slightly higher for the case of \( p = 40, n_1 = 90, n_2 = n_3 = 100 \) under the Gamma assumption, it can be accepted and understood due to both asymptotic and nonparametric. Further more, it decreases with the increasing dimension \( p \) and sample size \( n_1 \).

Secondly, it is obvious that our proposed tests give a much better performance on the empirical powers, which uniformly dominates that of the TNT over the entire range. For examples, under the Normal assumption
Table 1: Empirical sizes and powers of the comparative tests for $H_0 : \sum_{i=1}^{3} \beta_i \mu_i = 0$ with $\beta_1 = \beta_2 = \beta_3 = 1$ at $\alpha = 0.05$ significance level for normal and gamma random vectors with 10,000 replications. The alternative hypothesis is $\mu_3 = -(\mu_1 + \mu_2) + \mu_\Delta$, $\mu_\Delta = (\epsilon \sqrt{2 \log(p)} \cdot 1_{[p-v_0]}^{\prime} (p \cdot n_1, n_2, n_3)$ (40, 90, 100, 100) (40, 180, 200, 200) (80, 180, 200, 200) (120, 180, 200, 200)

\[
\begin{array}{cccccc}
(p, n_1, n_2, n_3) & \text{Normal} & \epsilon = 0 \text{ (size)} & 0.0647 & 0.0697 & 0.0614 & 0.0707 \\
& & 0.2 & 0.1056 & 0.0772 & 0.1059 & 0.0721 \\
& & 0.4 & 0.2568 & 0.0872 & 0.3042 & 0.0855 \\
& & 0.6 & 0.5795 & 0.1235 & 0.6888 & 0.1181 \\
& & 0.8 & 0.8739 & 0.2055 & 0.9496 & 0.2038 \\
& & 1 & 0.9854 & 0.4134 & 0.9986 & 0.4014 \\
\hline
& \text{Gamma} & \epsilon = 0 \text{ (size)} & 0.0758 & 0.0726 & 0.0641 & 0.0690 \\
& & 0.2 & 0.1022 & 0.0748 & 0.0960 & 0.0738 \\
& & 0.4 & 0.2071 & 0.0838 & 0.2400 & 0.0865 \\
& & 0.6 & 0.4491 & 0.1194 & 0.5571 & 0.1106 \\
& & 0.8 & 0.7804 & 0.1609 & 0.8848 & 0.1660 \\
& & 1 & 0.9588 & 0.2896 & 0.9912 & 0.2780 \\
\hline
(p, n_1, n_2, n_3) & \text{Normal} & \epsilon = 0 \text{ (size)} & 0.0643 & 0.0714 & 0.0678 & 0.0705 \\
& & 0.2 & 0.0906 & 0.0750 & 0.1085 & 0.0757 \\
& & 0.4 & 0.2093 & 0.0781 & 0.2869 & 0.0944 \\
& & 0.6 & 0.4895 & 0.0985 & 0.6701 & 0.1143 \\
& & 0.8 & 0.8224 & 0.1329 & 0.9490 & 0.1798 \\
& & 1 & 0.9773 & 0.1890 & 0.9980 & 0.3558 \\
\hline
& \text{Gamma} & \epsilon = 0 \text{ (size)} & 0.0662 & 0.0683 & 0.0743 & 0.0732 \\
& & 0.2 & 0.0986 & 0.0738 & 0.0980 & 0.0758 \\
& & 0.4 & 0.2387 & 0.0847 & 0.2392 & 0.0848 \\
& & 0.6 & 0.6743 & 0.1127 & 0.5462 & 0.1096 \\
& & 0.8 & 0.9572 & 0.1776 & 0.8808 & 0.1615 \\
& & 1 & 0.9993 & 0.3008 & 0.9928 & 0.2566 \\
\end{array}
\]
Table 2: Empirical sizes and powers of the comparative tests for $H_0: \sum_{i=1}^{3} \beta_i \mu_i = 0$ with $\beta_1 = \beta_2 = -\frac{1}{2}, \beta_3 = 1$ at $\alpha = 0.05$ significance level for normal and gamma random vectors with 10,000 replications. The alternative hypothesis is $\mu_3 = \frac{1}{2}(\mu_1 + \mu_2) + \mu_{\Delta}$, $\mu_{\Delta} = (\epsilon \sqrt{2 \log(p)} \cdot 1_{[p^{\cdot}v_0], 0_{p^{\cdot}[-p^{\cdot}v_0]}})$.

| $(p, n_1, n_2, n_3)$ | $(40, 90, 100, 100)$ | $(40, 180, 200, 200)$ |
|---------------------|----------------------|-----------------------|
|                     | Ours | TNT | Ours | TNT |
| Normal $\epsilon = 0$ (size) |   |   |   |   |
| $v_0=0.3$  | 0.0691 | 0.0705 | 0.0670 | 0.0692 |
| 0.2 | 0.1107 | 0.0742 | 0.0965 | 0.0713 |
| 0.4 | 0.3162 | 0.0895 | 0.2204 | 0.0845 |
| 0.6 | 0.6827 | 0.1194 | 0.4947 | 0.0961 |
| 0.8 | 0.9433 | 0.1935 | 0.8139 | 0.1299 |
| 1 | 0.9972 | 0.3499 | 0.9686 | 0.1958 |
| $v_0=0.1$  |   |   |   |   |
| Normal $\epsilon = 0$ (size) |   |   |   |   |
| $v_0=0.1$  | 0.0730 | 0.0715 | 0.0672 | 0.0729 |
| 0.2 | 0.1093 | 0.0723 | 0.1093 | 0.0723 |
| 0.4 | 0.2748 | 0.0894 | 0.6591 | 0.0967 |
| 0.6 | 0.6045 | 0.1067 | 0.9886 | 0.1465 |
| 0.8 | 0.9083 | 0.1448 | 1 | 0.2561 |
| 1 | 0.9933 | 0.1947 | 1 | 0.5925 |
| $(p, n_1, n_2, n_3)$ | $(80, 180, 200, 200)$ | $(120, 180, 200, 200)$ |
|---------------------|----------------------|-----------------------|
|                     | Ours | TNT | Ours | TNT |
| Normal $\epsilon = 0$ (size) |   |   |   |   |
| $v_0=0.2$  | 0.0626 | 0.0676 | 0.0682 | 0.0678 |
| 0.2 | 0.1173 | 0.0772 | 0.1081 | 0.0706 |
| 0.4 | 0.3596 | 0.0799 | 0.2471 | 0.0814 |
| 0.6 | 0.7925 | 0.1062 | 0.5629 | 0.0942 |
| 0.8 | 0.9848 | 0.1526 | 0.8832 | 0.1167 |
| 1 | 0.9999 | 0.2524 | 0.9900 | 0.1717 |
| $v_0=0.1$  |   |   |   |   |
| Normal $\epsilon = 0$ (size) |   |   |   |   |
| $v_0=0.1$  | 0.0658 | 0.0684 | 0.0726 | 0.0712 |
| 0.2 | 0.1324 | 0.0746 | 0.1107 | 0.0709 |
| 0.4 | 0.4737 | 0.0812 | 0.3199 | 0.0789 |
| 0.6 | 0.9214 | 0.1070 | 0.7183 | 0.0978 |
| 0.8 | 0.9994 | 0.1457 | 0.9655 | 0.1225 |
| 1 | 1 | 0.2375 | 0.9993 | 0.1725 |
Table 3: Empirical sizes and powers of the comparative tests for $H_0 : \mu_1 = \mu_2$ at $\alpha = 0.05$ significance level for normal and gamma random vectors with 10,000 replications. The alternative hypothesis is $\mu_1 = 0, \mu_2 = (\epsilon \sqrt{2 \log(p)} \cdot 1'_{[p, v_0]}, 0'_{p-1'})'$

| $(p, n_1, n_2)$ | $(v_0=0.2)$ | $(v_0=0.1)$ | $(v_0=0.1)$ | $(v_0=0.2)$ |
|-----------------|--------------|--------------|--------------|--------------|
|                 | Ours | TNT | CQT | Ours | TNT | CQT |
| $(40, 90, 100)$ | Normal | Normal | Normal | Normal | Normal | Normal |
| $\epsilon = 0$ (size) | 0.0678 | 0.0689 | 0.0679 | 0.0645 | 0.0663 | 0.0677 |
| 0.2 | 0.1068 | 0.0805 | 0.0696 | 0.2 | 0.0953 | 0.0782 | 0.0701 |
| 0.4 | 0.2403 | 0.1179 | 0.0705 | 0.4 | 0.2444 | 0.1195 | 0.0750 |
| 0.6 | 0.5222 | 0.2344 | 0.0725 | 0.6 | 0.5588 | 0.2344 | 0.0770 |
| 0.8 | 0.8323 | 0.5470 | 0.0776 | 0.8 | 0.8715 | 0.5360 | 0.0790 |
| 1 | 0.9748 | 0.8863 | 0.0926 | 0.9 | 0.9549 | 0.7267 | 0.0850 |
| $\epsilon = 0$ (size) | 0.0681 | 0.0711 | 0.0710 | 0.0618 | 0.0670 | 0.0687 |
| 0.2 | 0.1030 | 0.0786 | 0.0717 | 0.2 | 0.0938 | 0.0788 | 0.0702 |
| 0.4 | 0.2313 | 0.1161 | 0.0709 | 0.4 | 0.2312 | 0.1217 | 0.0752 |
| 0.6 | 0.5200 | 0.2382 | 0.0732 | 0.6 | 0.5532 | 0.2396 | 0.0801 |
| 0.8 | 0.8309 | 0.5633 | 0.0821 | 0.8 | 0.8704 | 0.5371 | 0.0897 |
| 1 | 0.9697 | 0.8915 | 0.0943 | 0.9 | 0.9531 | 0.7279 | 0.0982 |
| $\epsilon = 0$ (size) | 0.0661 | 0.0706 | 0.0680 | 0.0638 | 0.0707 | 0.0641 |
| 0.2 | 0.0872 | 0.0713 | 0.0743 | 0.2 | 0.1041 | 0.0746 | 0.0698 |
| 0.4 | 0.1769 | 0.0978 | 0.0790 | 0.4 | 0.2687 | 0.1125 | 0.0704 |
| 0.6 | 0.4007 | 0.1510 | 0.0802 | 0.6 | 0.6170 | 0.2288 | 0.0773 |
| 0.8 | 0.6993 | 0.2966 | 0.0803 | 0.8 | 0.9166 | 0.6044 | 0.0814 |
| 1 | 0.9280 | 0.5999 | 0.0820 | 0.9 | 0.9781 | 0.8514 | 0.0823 |
| $\epsilon = 0$ (size) | 0.0606 | 0.0722 | 0.0694 | 0.0647 | 0.0674 | 0.0666 |
| 0.2 | 0.0828 | 0.0724 | 0.0752 | 0.2 | 0.1007 | 0.0798 | 0.0705 |
| 0.4 | 0.1720 | 0.0959 | 0.0786 | 0.4 | 0.2609 | 0.1131 | 0.0699 |
| 0.6 | 0.3856 | 0.1486 | 0.0795 | 0.6 | 0.6093 | 0.2282 | 0.0781 |
| 0.8 | 0.6970 | 0.2865 | 0.0798 | 0.8 | 0.9114 | 0.6149 | 0.0815 |
| 1 | 0.9279 | 0.6010 | 0.0818 | 0.9 | 0.9741 | 0.8499 | 0.0842 |
with $\beta_1 = \beta_2 = -\frac{1}{2}, \beta_3 = 1$ in Table 2, our empirical powers is 96.86% closing to 1, while the one of TNT is only around 20% for the case of $p = 40, n_1 = 180, n_2 = n_3 = 200$ and $\epsilon = 1, \nu_0 = 0.1$. When the dimension increases to $p = 120$, the empirical power of our proposed test rises up to 99%, but the one of TNT remains under 20%.

For two-samples test, we compared our test with the ones in Nishiyama et al. [13] (TNT) and Chen and Qin [8] (CQT) for $H_0$ together. As seen from the Table 3, it was the same thing for the comparison to TNT. First, all the empirical sizes of our proposed test are one upon that of TNT, and our empirical powers grows up to 1 rapidly, which are superior to that of TNT. Then let us make a comparative analysis between CQT and our test. It can be easily found that the empirical sizes of the CQT are slightly higher than that of our proposed test for almost all cases. Further, CQT behaves even worse on the empirical powers, like $p = 120, n_1 = 180, n_2 = n_3 = 200$ and $\epsilon = 0.9, \nu_0 = 0.2$ under the Normal assumption, the empirical power of the CQT remains under 10% when our empirical power increases to 1.

Finally, It must be pointed that the proposed test cannot be use for the ultra high dimension $p > n_1$. On one hand, the condition of $p < n_1$ is requested to guarantee the inverse of sample covariance matrix of $y_i$. On the other hand, the limiting variance $\nu(f)$ is related with $\gamma_1$. If the dimension $p$ is large enough, it will make the proposed test unstable.

5. Conclusion

In this paper, the new testing statistics based on likelihood were proposed for the linear hypotheses test of the large dimensional mean vectors with unequal covariance matrices. By using the CLT for LSS of a large dimensional $F$-matrix in Zheng [21], we guaranteed that the tests proposed were feasible for the non-Gaussian variables in a wider range. Furthermore, our test methods provided the more optimal powers due to the likelihood based statistics, meanwhile the empirical sizes were closer to the significant level. However, it is limited by the constrain $p < n_1$, which is requested for the existence of the inverse of sample covariance matrix. For future works, maybe we can extend this work to other forms of test statistics by large dimensional spectral analysis in random matrix theory, and make it more powerful and applicable for different situations.
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Appendix A. Proofs

Appendix A.1. Derivation of \( d(\gamma_2) \) in (2.5).

Let \( F_{\gamma_1,\gamma_2}(x) \) be the LSD of the matrix \( \mathbf{F} \), and denote the integral
\[
F_{\gamma_1,\gamma_2}(f) = \int_a^b \frac{x \cdot (1 - \gamma_2) \sqrt{(b - x)(x - a)}}{2\pi x(\gamma_1 + \gamma_2 x)} dx,
\]
where \( f(x) = x \) and
\[
a = \left( \frac{1 - \sqrt{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2}}{1 - \gamma_2} \right)^2, \quad b = \left( \frac{1 + \sqrt{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2}}{1 - \gamma_2} \right)^2.
\]

Set \( x = \frac{(1 + h^2 + 2h \cos \theta)}{(1 - \gamma_2)^2}, 0 < \theta < \pi \), where \( h = \sqrt{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2} \). Then
\[
\sqrt{(b - x)(x - a)} = \frac{2h \sin \theta}{(1 - \gamma_2)^2}, \quad dx = -\frac{2h \sin \theta}{(1 - \gamma_2)^2} d\theta;
\]
\[
x = \frac{|1 + h e^{i\theta}|^2}{(1 - \gamma_2)^2}, \quad \gamma_1 + \gamma_2 x = \frac{|h + \gamma_2 e^{i\theta}|^2}{(1 - \gamma_2)^2}.
\]

So we have
\[
\int_a^b \frac{x \cdot (1 - \gamma_2) \sqrt{(b - x)(x - a)}}{2\pi x(\gamma_1 + \gamma_2 x)} dx
\]
\[
= -\frac{2}{\pi(1 - \gamma_2)} \int_0^\pi \frac{h^2 \sin^2 \theta}{|h + \gamma_2 e^{i\theta}|^2} d\theta
\]
\[
= \frac{1}{\pi(1 - \gamma_2)} \int_0^{2\pi} \frac{h^2 \sin^2 \theta}{|h + \gamma_2 e^{i\theta}|^2} d\theta
\]
\[
= -\frac{1}{4\pi i(1 - \gamma_2)} \int_{|\xi| = 1} \frac{h^2(\xi - \xi^{-1})^2}{|h + \gamma_2 \xi|^2 \xi} d\xi
\]
\[
= -\frac{h}{4\pi i(1 - \gamma_2)\gamma_2} \int_{|\xi| = 1} \frac{(\xi^2 - 1)^2}{(\xi + \frac{h}{\gamma_2})(\xi + \frac{h}{\gamma_2})\xi^2} d\xi
\]
There are two poles inside the unit circle: 0, $-\frac{\gamma_2}{h}$. Their corresponding residues are
\[
\operatorname{Res}(0) = \frac{-h^2 - \gamma_2^2}{\gamma_2 h},
\]
\[
\operatorname{Res}\left(-\frac{\gamma_2}{h}\right) = \frac{h^2 - \gamma_2^2}{\gamma_2 h}.
\]
Therefore
\[
F_{\gamma_1,\gamma_2}(f) = -\frac{h}{4\pi i(1 - \gamma_2)\gamma_2} \cdot 2\pi i \left( \operatorname{Res}(0) + \operatorname{Res}\left(-\frac{\gamma_2}{h}\right) \right) = \frac{1}{1 - \gamma_2} \equiv d(\gamma_2).
\]
Similarly, $d(\gamma_n)$ in Theorem 3.1 is exactly analogous to the $d(\gamma_2)$ by substituting $\gamma_n$ for $\gamma_2$, i.e.
\[
F_{\gamma_p,\gamma_n}(f) = \int f(x) dF_{\gamma_p,\gamma_n}(x) = \frac{1}{1 - \gamma_n} \equiv d(\gamma_n)
\]
where $F_{\gamma_p,\gamma_n}$ is analogous to LSD of the matrix $F$, which has a density in (2.8) but with $\gamma_p, \gamma_n$ instead of $y_k, k = 1, 2$, respectively.

Appendix A.2. Derivations of the corrected Corollary 3.2 in Zheng [21].

Because it is difficult to apply Lemma 2.1 directly, which has the complex form of the asymptotic mean and covariance. So the Corollary 3.2 in Zheng [21] was proposed to help the evaluation of the asymptotic mean and covariance. However, the result of the Corollary 3.2 in Zheng [21] is not correct, I think it is a typo mistake. In order to obtain the accurate and simplified form for computing the asymptotic mean and covariance, we reviewed it and gave some derivations and calculations as below.

First, make clear some notations:

- $m(z)$ is the Stieltjes Transform of the LSD $F_{y_1,y_2}$, where $F_{y_1,y_2}$ is the LSD of the $F$-matrix $V_n$. Define
  \[
  m(z) = -\frac{1}{z} + y_1 m(z), \quad (A.1)
  \]
then \( m(z) \) is the Stieltjes Transform of \( F_{y_1, y_2} \equiv (1 - y_1)I_{[0, \infty)} + y_1 F_{y_1, y_2} \), which has an inverse equation as

\[
  z = -\frac{1}{m(z)} + y_1 \int \frac{dF_{y_2}(x)}{x + m(z)},
\]

(A.2)

where \( F_{y_2} \) is the LSD of the matrix \( S_2 \).

- Denote \( m_{y_2}(z) \) is the Stieltjes Transform of \( F_{y_2} \), consequently

\[
  m_{y_2}(z) = -\frac{1 - y_2}{z} + y_2 m_{y_2}(z)
\]

is the Stieltjes Transform of \( F_{y_2} \equiv (1 - y_2)I_{[0, \infty)} + y_2 F_{y_2} \), which has an inverse

\[
  z = -\frac{1}{m_{y_2}(z)} + \frac{y_2}{1 + m_{y_2}(z)}.
\]

(A.4)

Therefore, equation (A.2) can be written as

\[
  z = -\frac{1}{m(z)} + y_1 \int \frac{dF_{y_2}(x)}{x + m(z)};
\]

\[
  = -\frac{1}{m(z)} + y_1 \cdot m_{y_2}(-m(z))
\]

\[
  = -\frac{1}{m(z)} - \frac{y_1(1 - y_2)}{y_2 m(z)} + \frac{y_1}{y_2} m_{y_2}(-m(z))
\]

\[
  = -\frac{y_1 + y_2 - y_1 y_2}{y_2 m(z)} + \frac{y_1}{y_2} m_{y_2}(-m(z))
\]

(A.5)

- Let \( m_0(z) = m_{y_2}(-m(z)) \), for simplicity denote it as \( m_0 \) if no confusion. By the inverse equation (A.4), we have

\[
  m(z) = \frac{(1 - y_2) (m_0 + 1/(1 - y_2))}{m_0 (1 + m_0)}.
\]

(A.6)

Combine equation (A.5) and (A.6), the relationship between \( z \) and \( m_0 \) is obtained

\[
  z = -\frac{m_0 (m_0 + 1 - y_1)}{(1 - y_2) (m_0 + 1/(1 - y_2))}.
\]

(A.7)
Since the contour enclosed the supporting set of the LSD $F_{y_1,y_2}(x)$ of the $F$-matrix $V_n$, which contains the interval
\[
a' = \frac{(1 - h')^2}{1 - y_2^2}, \quad b' = \frac{(1 + h')^2}{1 - y_2^2}
\]
if $y_1 \leq 1$, where $h' = \sqrt{y_1 + y_2 - y_1 y_2}$. When $y_1 > 1$, the contour should enclose the whole support $\{0\} \cup [a', b']$, because the $F_{y_1,y_2}$ has a positive mass at the origin at this time. However, due to the exact separation theorem in Bai and Silverstein [5], for large enough $p$ and $n$, the discrete mass at the origin will coincide with that of $F_{y_1,y_2}$. So we can restrict the integral on the contours only enclosed the continuous part of the LSD $F_{y_1,y_2}$. Therefore, solve the real roots of the equation (A.7) at two points $a', b'$, we obtain
\[
m_0(a') = -\frac{1 - h'}{1 - y_2}, \quad m_0(b') = -\frac{1 + h'}{1 - y_2}.
\]
It is obviously that when $z$ runs in the positive direction around the interval $[a', b']$, $m_0(z)$ runs in the same direction around the interval $[-\frac{1 - h'}{1 - y_2}, -\frac{1 + h'}{1 - y_2}]$. So define $m_0(z) = -\frac{1 + h'\tau \xi}{1 - y_2}$, where $\tau > 1$ but very close to 1, and $|\xi| = 1$. By (A.7),
\[
z = \frac{1 + h'^2 + h'\tau^{-1}\xi + h'\tau \xi}{(1 - y_2)^2}.
\]
(A.8)

Further,
\[
m'(z) = -\frac{(1 - y_2)m_0^2 + 2m_0 + 1}{m_0(1 + m_0)^2} \cdot dm_0 = \frac{(1 - y_2)^2}{h'\tau} \cdot \frac{\xi + \frac{y_2}{h'\tau}}{(\xi + \frac{y_2}{h'\tau})^2 (\xi + \frac{1}{h'\tau})^2} \cdot d\xi
\]
and
\[
m(z) = -\frac{(1 - y_2)^2}{h'\tau} \cdot \frac{\xi}{(\xi + \frac{y_2}{h'\tau})(\xi + \frac{1}{h'\tau})}.
\]

Put these results into the expressions of the asymptotic mean and covariance in Lemma 2.1. According to the definition (A.8), when $z$ anticlockwise runs along the unit circle, $z$ anticlockwise runs around a contour closely enclosed the interval $[a', b']$ when $\tau$ is closed to 1. Thus, letting $\tau \downarrow 1$, we have

22
\[
\mu(f_j) = \frac{\kappa - 1}{4 \pi i} \int f_j(z) d \log \left( \frac{(1-y_2)m_0^2(z) + 2m_0(z) + 1 - y_1}{(1-y_2)m_0^2(z) + 2m_0(z) + 1} \right)
\]
\[+ \frac{\kappa - 1}{4 \pi i} \int f_j(z) d \log \left( 1 - y_2m_0^2(z)(1 + m_0(z))^{-2} \right)
\]
\[+ \frac{\beta_x y_1}{2 \pi i} \int f_j(z) (1 + m_0(z))^{-3} dm_0(z)
\]
\[+ \frac{\beta_y}{4 \pi i} \int f_j(z) \left( 1 - \frac{y_2m_0^2(z)}{(1 + m_0(z))^2} \right) d \log \left( 1 - \frac{y_2m_0^2(z)}{(1 + m_0(z))^2} \right)
\]
\[= \lim_{\tau \downarrow 1} \frac{\kappa - 1}{4 \pi i} \int \int |\xi| = 1 \left[ f_j(z_1) f_i(z_2) \right] \frac{1}{(z_1 - z_2)^2} dm_0(z_1) dm_0(z_2)
\]
\[+ \frac{\beta_x y_1}{2 \pi i} \int \int |\xi| = 1 \left[ f_j(z_1) f_i(z_2) \right] \frac{1}{(1 + m_0(z_1))^2 (1 + m_0(z_2))^2} dm_0(z_1) dm_0(z_2)
\]
\[v(f_j, f_k) = - \frac{\kappa}{4 \pi^2} \int \int f_j(z_1) f_k(z_2) \frac{(m_0(z_1) - m_0(z_2))^2}{dm_0(z_1) dm_0(z_2)}
\]
\[v(f_j, f_k) = - \frac{\beta_x y_1 + \beta_y y_2}{4 \pi^2} \int \int f_j(z_1) f_k(z_2) \frac{1}{(1 + m_0(z_1))^2 (1 + m_0(z_2))^2} dm_0(z_1) dm_0(z_2)
\]
where \( j, k \in \{1, \ldots, s\} \), "Re" represents the real part of \( \xi \) and \( \tau \downarrow 1 \) means that "\( \tau \) approaches 1 from above'.

Appendix A.3. Calculation of \( \mu(f) \) in (3.4).

For the function \( f(x) = x \), the computation of \( \mu(f) \) is divided into three parts. Still use the denotation \( h = \sqrt{\gamma_1 + \gamma_2 - \gamma_1 \gamma_2} \), then the first part is
\[ I_1 = \lim_{\tau \downarrow 1} \frac{\kappa - 1}{4\pi i} \oint_{|\xi| = 1} f \left( \frac{1 + h^2 + 2h\text{Re}(\xi)}{(1 - \gamma_2)^2} \right) \left[ \frac{1}{\xi - \frac{1}{\tau}} + \frac{1}{\xi + \frac{1}{\tau}} - \frac{2}{\xi + \frac{2\gamma_2}{h\tau}} \right] d\xi \]

\[ = \lim_{\tau \downarrow 1} \frac{\kappa - 1}{4\pi i} \oint_{|\xi| = 1} \frac{|1 + h\xi|^2}{(1 - \gamma_2)^2} \left( \frac{1}{\xi - \frac{1}{\tau}} + \frac{1}{\xi + \frac{1}{\tau}} - \frac{2}{\xi + \frac{2\gamma_2}{h\tau}} \right) d\xi \]

\[ = \lim_{\tau \downarrow 1} \frac{(\kappa - 1)h}{4\pi i(1 - \gamma_2)^2} \cdot 2\pi i \left[ \text{Res}(0) + \text{Res} \left( -\frac{1}{\tau} \right) + \text{Res} \left( \frac{1}{\tau} \right) - 2\text{Res} \left( -\frac{\gamma_2}{h\tau} \right) \right] \]

\[ = \frac{(\kappa - 1)h}{4\pi i(1 - \gamma_2)^2} \cdot 2\pi i \left[ -\frac{2h}{\gamma_2} + 2 + \frac{1 + h^2}{h} - 2 + \frac{1 + h^2}{h} + \frac{2(1 - \gamma_2)^2\gamma_1}{h\gamma_2} \right] \]

\[ = \frac{\gamma_2}{(1 - \gamma_2)^2} \]

The second part is

\[ I_2 = \frac{\beta_1 \cdot \gamma_1 (1 - \gamma_2)^2}{2\pi \cdot h^2} \oint_{|\xi| = 1} f \left( \frac{1 + h^2 + 2h\text{Re}(\xi)}{(1 - \gamma_2)^2} \right) \frac{1}{(\xi + \frac{2\gamma_2}{h\tau})^3} d\xi \]

\[ = \frac{\beta_1 \cdot \gamma_1 (1 - \gamma_2)^2}{2\pi i \cdot h^2} \oint_{|\xi| = 1} \frac{|1 + h\xi|^2}{(1 - \gamma_2)^2} \frac{1}{(\xi + \frac{2\gamma_2}{h\tau})^3} d\xi \]

\[ = \frac{\beta_1 \cdot \gamma_1}{2\pi ih} \oint_{|\xi| = 1} \frac{(\xi + \frac{1}{h}) (\xi + h)}{\xi} \frac{1}{(\xi + \frac{2\gamma_2}{h\tau})^3} d\xi \]

\[ = \frac{\beta_1 \cdot \gamma_1}{2\pi ih} \cdot 2\pi i \left[ \text{Res}(0) + \text{Res} \left( -\frac{\gamma_2}{h} \right) \right] \]

\[ = \frac{\beta_1 \cdot \gamma_1}{2\pi ih} \cdot 2\pi i \left[ \frac{h^3}{y_2^3} - \frac{h^3}{y_2^3} \right] \]

\[ = 0 \]

The third part is

\[ I_3 = \frac{\beta_2 \cdot \gamma_2 (1 - \gamma_2)}{2\pi i \cdot h} \oint_{|\xi| = 1} f \left( \frac{1 + h^2 + 2h\text{Re}(\xi)}{(1 - \gamma_2)^2} \right) \frac{\xi + \frac{1}{h}}{(\xi + \frac{2\gamma_2}{h\tau})^3} d\xi \]

\[ = \frac{\beta_2 \cdot \gamma_2 (1 - \gamma_2)}{2\pi i \cdot h} \oint_{|\xi| = 1} \frac{|1 + h\xi|^2}{(1 - \gamma_2)^2} \frac{\xi + \frac{1}{h}}{(\xi + \frac{2\gamma_2}{h\tau})^3} d\xi \]
\[
\frac{\beta_2 \gamma_2}{2\pi i(1 - \gamma_2)} \int_{|\xi| = 1} \frac{(\xi + \frac{1}{h}) (\xi + h)}{\xi} \frac{\xi + \frac{1}{h}}{(\xi + \frac{\gamma_2}{h})^2} d\xi
\]

\[
= \frac{\beta_2 \gamma_2}{2\pi i(1 - \gamma_2)} \cdot 2\pi i \left[ \text{Res}(0) + \text{Res}(\frac{-\gamma_2}{h}) \right]
\]

\[
= \frac{\beta_2 \gamma_2}{2\pi i(1 - \gamma_2)} \cdot 2\pi i \left[ \frac{h^2}{y_2^3} - \frac{h^2}{y_2^3} \right]
\]

\[
= \frac{\beta_2 \gamma_2}{1 - \gamma_2}
\]

Finally,

\[
\mu(f) = \frac{\gamma_2}{(1 - \gamma_2)^2} + \frac{\beta_2 \gamma_2}{1 - \gamma_2}
\]

**Appendix A.4. Calculation of \( v(f) \) in (3.5).**

The computation of \( v(f) \) in (3.5) is divided into two parts. For the first part

\[
- \lim_{\tau \downarrow 1} \frac{\kappa}{4\pi^2} \int_{|\xi_2| = 1} \int_{|\xi_1| = 1} \frac{f \left( \frac{1 + h^2 + 2h \Re(\xi_1)}{(1 - \gamma_2)^2} \right) f_\ell \left( \frac{1 + h^2 + 2h \Re(\xi_2)}{(1 - \gamma_2)^2} \right)}{(\xi_1 - \tau \xi_2)^2} d\xi_1 d\xi_2
\]

the following integral is computed firstly

\[
\int_{|\xi_1| = 1} f \left( \frac{1 + h^2 + 2h \Re(\xi_1)}{(1 - \gamma_2)^2} \right) \frac{1}{(\xi_1 - \tau \xi_2)^2} d\xi_1
\]

\[
= \int_{|\xi_1| = 1} \frac{|1 + h \xi_1|^2}{(1 - \gamma_2)^2 (\xi_1 - \tau \xi_2)^2} d\xi_1
\]

\[
= \frac{h}{(1 - \gamma_2)^2} \int_{|\xi_1| = 1} \frac{(\xi + \frac{1}{h}) (\xi + h)}{\xi(\xi_1 - \tau \xi_2)^2} d\xi_1
\]

\[
= \frac{2\pi i h}{(1 - \gamma_2)^2} \cdot \frac{1}{\tau^2 \xi_2^2}
\]

Then we obtained

\[
- \lim_{\tau \downarrow 1} \frac{\kappa}{4\pi^2} \int_{|\xi_2| = 1} \int_{|\xi_1| = 1} \frac{f \left( \frac{1 + h^2 + 2h \Re(\xi_1)}{(1 - \gamma_2)^2} \right) f_\ell \left( \frac{1 + h^2 + 2h \Re(\xi_2)}{(1 - \gamma_2)^2} \right)}{(\xi_1 - \tau \xi_2)^2} d\xi_1 d\xi_2
\]

\[
= - \lim_{\tau \downarrow 1} \frac{\kappa}{4\pi^2} \cdot \frac{2\pi i h}{(1 - \gamma_2)^2} \int_{|\xi_2| = 1} \frac{|1 + h \xi_2|^2}{(1 - \gamma_2)^2} \frac{1}{\tau^2 \xi_2^2} d\xi_2
\]
\[
\lim_{\tau \downarrow 1} \frac{\kappa h^2}{2\pi i (1 - \gamma_2)^4 \tau^2} \oint_{|\xi_2| = 1} \frac{(\xi_2 + \frac{1}{h}) (\xi_2 + h)}{\xi_2^3} d\xi_2 = \frac{\kappa h^2}{(1 - \gamma_2)^4}
\]

For the second part, we calculate

\[
\oint_{|\xi_1| = 1} f \left( \frac{1 + h^2 + 2h \text{Re}(\xi_1)}{(1 - \gamma_2)^2} \right) \left( \xi_1 + \frac{1}{h} \right) (\xi_1 + h) d\xi_1 = \frac{2\pi ih}{(1 - \gamma_2)^2} \left( \frac{\gamma_2^2 - h^2}{\gamma_2^2} \right)
\]

Then the second part is

\[
- \frac{(\beta_1 \gamma_1 + \beta_2 \gamma_2) (1 - \gamma_2)^2}{4\pi^2 h^2} \oint_{|\xi_1| = 1} f_j \left( \frac{1 + h^2 - 2h \text{Re}(\xi_1)}{(1 - \gamma_2)^2} \right) \xi_1 d\xi_1 \oint_{|\xi_2| = 1} f_j \left( \frac{1 + h^2 + 2h \text{Re}(\xi_2)}{(1 - \gamma_2)^2} \right) \frac{2\pi ih}{(1 - \gamma_2)^2} \cdot \frac{2\pi ih}{(1 - \gamma_2)^2}
\]

\[
= \frac{\beta_1 \gamma_1 + \beta_2 \gamma_2}{(1 - \gamma_2)^2}
\]

Finally, the covariance is

\[
v(f) = \frac{\kappa h^2}{(1 - \gamma_2)^4} + \frac{\beta_1 \gamma_1 + \beta_2 \gamma_2}{(1 - \gamma_2)^2}.
\]

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