THE EVALUATION OF THE SUMS OF MORE GENERAL SERIES BY BERNSTEIN POLYNOMIALS

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Abstract

Let $n, k$ be the positive integers, and let $S_k(n)$ be the sums of the $k$-th power of positive integers up to $n$: $S_k(n) = \sum_{l=1}^{n} l^k$. By means of that we consider the evaluation of the sum of more general series by Bernstein polynomials. Additionally we show the reality of our idea with some examples.

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1. Introduction

The history of Bernstein polynomials depends on Bernstein in 1904. It is well known that Bernstein polynomials play a crucial important role in the area of approximation theory and the other areas of mathematics, on which they have been studied by many researchers for a long time [1, 3, 5-7, 10, 11, 16, 17]. These polynomials also take an important role in physics.

Recently the works including applications of umbral calculus to Genocchi numbers and polynomials [2], the Legendre polynomials associated with Bernoulli, Euler, Hermite and Bernstein polynomials [3], the applications of umbral calculus to extended Kim’s $p$-adic $q$-deformed fermionic integrals in the $p$-adic integer ring [4], the integral of the product of several Bernstein polynomials [5], the generating function of Bernstein polynomials [6], a theorem concerning Bernstein polynomials [10], new generating function of the $(q)$-Bernstein type polynomials and their interpolation function [11], $q$-analogues of the sums of powers of consecutive integers, squares, cubes, quarts and quints [12-15, 18-20], have been investigated extensively.
In the complex plane, the Bernoulli polynomials $B_n(x)$ are known by the following generating series:

$$\sum_{n=0}^{\infty} B_n(x) \frac{t^n}{n!} = \frac{t}{e^t - 1} e^{xt}, \quad |t| < 2\pi. \quad (1.1)$$

In the case $x = 0$ in (1.1), we have $B_n(0) := B_n$ that stands for Bernoulli numbers. By (1.1), we have

$$B_n(x) = \sum_{k=0}^{n} \binom{n}{k} B_k x^{n-k}. \quad (1.2)$$

The Bernoulli numbers satisfy the following identity

$$B_0 = 1 \quad \text{and} \quad (B + 1)^n - B_n = \delta_{1,n}$$

where $\delta_{1,n}$ stands for Kronecker’s delta and we have used $B^n := B_n$ (for details, see [3], [7], [9], [17]).

Recently, Acikgoz and Araci has constructed the generating function for the Bernstein polynomials $B_{k,n}(x)$ by the following rule:

$$\sum_{n=k}^{\infty} B_{k,n}(x) \frac{t^n}{n!} = \frac{(tx)^k}{k!} e^{(1-x)t} \quad (t \in \mathbb{C} \text{ and } k = 0, 1, 2, \ldots, n). \quad (1.3)$$

By (1.3), we see that

$$\sum_{n=k}^{\infty} B_{k,n}(x) \frac{t^n}{n!} = \sum_{n=k}^{\infty} \binom{n}{k} x^k (1-x)^{n-k} \frac{t^n}{n!}$$

by comparing the coefficients of $\frac{t^n}{n!}$ in the above, we derive well known expression of Bernstein polynomials, as follows: For $k, n \in \mathbb{Z}_+$

$$B_{k,n}(x) = \binom{n}{k} x^k (1-x)^{n-k} \quad (1.4)$$

where $x \in [0, 1]$ and $\binom{n}{k}$ is known as

$$\binom{n}{k} = \left\{ \begin{array}{ll} \frac{n!}{k!(n-k)!}, & \text{if } n \geq k \\ 0, & \text{if } n < k. \end{array} \right.$$

It follows from (1.4) that a few Bernstein polynomials are as follows:

$$B_{0,0}(x) = 1, B_{0,1}(x) = 1 - x, B_{1,1}(x) = x, B_{0,2}(x) = (1-x)^2, B_{1,2}(x) = 2x(1-x)$$

$$B_{2,2}(x) = x^2, B_{0,3}(x) = (1-x)^3, B_{1,3}(x) = 3x(1-x)^2, B_{2,3}(x) = 3x^2(1-x), B_{3,3}(x) = x^3.$$ 

In the same time, the Bernstein polynomials $B_{k,n}(x)$ have several properties of interest:

- $B_{k,n}(x) \geq 0$, for $0 \leq x \leq 1$ and $k = 0, 1, \ldots, n$
- Bernstein polynomials have the symmetry property $B_{k,n}(x) = B_{n-k,n}(1-x)$
- $\sum_{k=0}^{n} B_{k,n}(x) = 1$, which is know a part of unity.
- $B_{k,n}(x) = (1-x)B_{k,n-1}(x) + xB_{k-1,n-1}(x)$ with $B_{k,n}(x) = 0$ for $k < 0, k > n$ and $B_{0,0}(x) = 1$ cf. [1], [3], [5], [6], [7], [10], [16], [17].
From (1.1), a few Bernoulli polynomials can be generated as

\[ B_0(x) = 1, B_1(x) = x - \frac{1}{2}, B_2(x) = x^2 - x + \frac{1}{6}, B_3(x) = x^3 - \frac{3}{2}x^2 + \frac{1}{2}x. \]

For any positive integer \( n \), followings are the most known first three sums of powers of integers:

\[ 1 + 2 + 3 + \cdots + n = \frac{n(n + 1)}{2}, \]
\[ 1^2 + 2^2 + 3^2 + \cdots + n^2 = \frac{n(n + 1)(2n + 1)}{6} \]
and
\[ 1^3 + 2^3 + 3^3 + \cdots + n^3 = (1 + 2 + 3 + \cdots + n)^2 = \left( \frac{n(n + 1)}{2} \right)^2. \]

Formulas for sums of integer powers were first given in generalizable form by mathematician Thomas Harriot (c. 1560-1621) of England. At about the same time, Johann Faulhaber (1580-1635) of Germany gave formulas for these sums, but he did not make clear how to generalize them. Also Pierre de Fermat (1601-1665) and Blaise Pascal (1623-1662) gave the formulas for sums of powers of integers.

The Swiss mathematician Jacob Bernoulli (1654-1705) is perhaps best and most deservedly known for presenting formulas for sums of integer powers. Because he gave the most explicit sufficient instructions for finding the coefficients of the formulas [12-15, 18-20].

So, we are interested in finding a method to derive a formula for the sums of powers of integers. Following an idea due to J. Bernoulli, we aim to obtain a Theorem which gives the method for the evaluation of the sums of more general series by Bernstein polynomials.

2. The Evaluation of the Sums of More General Series by Bernstein Polynomials

In the 17th century a topic of mathematical interest was finite sums of power of integers such as the series \( 1 + 2 + 3 + \cdots + (n - 1) \) or the series \( 1^2 + 2^2 + 3^2 + \cdots + (n - 1)^2 \). The closed form for these finite sums were known, but the sums of the more general series \( 1^k + 2^k + 3^k + \cdots + (n - 1)^k \) was not. It was the mathematician Jacob Bernoulli who would solve this problem with the following equality [12-15, 18-20]. The sum of the \( k \)-th powers of the first \( (n - 1) \) integers is given by the formula

\[ 1^k + 2^k + 3^k + \cdots + (n - 1)^k = \int_1^n B_k(x) \, dx \quad (2.1) \]

using the integral of the Bernoulli polynomials, \( B_n(x) \), under integral from 1 to \( n \).

We are now in a position to express our aim as Theorem 1 for the evaluation of the sum of more general series by Bernstein polynomials.

**Theorem 1.** Let \( n, k \) and \( m \) be positive integers and let \( S_m(n) = \sum_{l=1}^n l^m \), then we have

\[ S_m(n) = \frac{(-n^{-1})^k}{(m + k + 1)!} \sum_{l=k}^{m+k+1} \binom{m + k + 1}{l} B_{m+k-l+1} B_{k,l}(-n) - \frac{1}{(m + 1)!} \sum_{l=0}^{m+1} \binom{m + 1}{l} 2^{m+1-l} B_{l+1}. \]
Proof. To prove this Theorem, we take \( \sum_{k=0}^{\infty} \frac{k^t}{k!} \) in the both sides of the Eq. (2.1), so it yields to

\[
e^t + e^{2t} + \ldots + e^{(n-1)t} = \int_{1}^{n} \left( \sum_{k=0}^{\infty} B_k(x) \frac{t^k}{k!} \right) dx
\]

\[
= \int_{1}^{n} \left[ \frac{t}{e^t - 1} e^{xt} \right] dx
\]

\[
= \left[ \sum_{m=0}^{\infty} B_m \frac{t^{m-1}}{m!} \right] \left[ e^{nt} - e^t \right]
\]

from the last identity, we see that

\[
e^{2t} + e^{3t} + \ldots + e^{nt} = \frac{1}{t} \left[ \sum_{m=0}^{\infty} B_m \frac{t^m}{m!} \right] \left[ \frac{n-k}{t^k} \sum_{m=0}^{\infty} B_{k,m} (-n) \frac{t^m}{m!} - \sum_{m=0}^{\infty} 2^m \frac{t^m}{m!} \right]
\]

by using Cauchy product rule in the right hand side of Eq. (2.2), we have

\[
I_1 = \sum_{m=0}^{\infty} \left( n-k \right) \frac{k!}{t^k} \sum_{l=k}^{m} \binom{m}{l} B_{m-l, l} \left( -n \right) \frac{t^{m-l}}{m!} - \sum_{m=0}^{\infty} \left( \sum_{l=0}^{m} \binom{m}{l} 2^{m-l} \frac{B_l}{m!} \right) \frac{t^{m-1}}{m!}.
\]

By (2.2), we derive the following

\[
I_2 = \sum_{m=0}^{\infty} \left( 2^m + 3^m + \ldots + n^m \right) \frac{t^m}{m!}.
\]

When we equate \( I_1 \) and \( I_2 \), we have

\[
1^m + 2^m + 3^m + \ldots + n^m = \frac{\left( -n \right)^k}{(m+k+1)!} \sum_{l=k}^{m+k+1} \binom{m+k+1}{l} B_{m+k-l+1, l} \left( -n \right) - \frac{1}{(m+1)!} \sum_{l=0}^{m+1} \binom{m+1}{l} 2^{m+1-l} B_l + 1.
\]

Thus, we complete the proof of the Theorem. \( \square \)

Let \( m = k \) in Theorem \( \square \) we readily get the following Corollary 1.

**Corollary 1.** Let \( n \) and \( k \) be positive integers and let \( S_k \left( n \right) \) be \( \sum_{i=1}^{n} i^k \), then we have

\[
S_k \left( n \right) = \frac{\left( -n \right)^k}{(2k+1)!} \sum_{l=k}^{2k+1} \binom{2k+1}{l} B_{2k-l+1, l} \left( -n \right) - \frac{1}{(k+1)!} \sum_{l=0}^{k+1} \binom{k+1}{l} 2^{k+1-l} B_l + 1.
\]
**Example 1.** Taking $k = 1$ in Corollary 1, we see that

$$1 + 2 + 3 + ... + n = \frac{1}{6} \sum_{l=1}^{n-1} \binom{3}{l} B_{3-l} B_{1,l} (-n) - \frac{1}{2} \sum_{l=0}^{2} \binom{2}{l} 2^{2-l} B_{l} + 1$$

$$= \frac{n (n + 1)}{2}.$$  

For $k = 2$ in Corollary 1, we have

$$1^2 + 2^2 + 3^2 + ... + n^2 = \frac{n^2}{120} \sum_{l=2}^{5} \binom{5}{l} B_{5-l} B_{2,l} (-n) - \frac{1}{6} \sum_{l=0}^{3} \binom{3}{l} 2^{3-l} B_{l} + 1$$

$$= \frac{n (n + 1) (2n + 1)}{6}.$$  

By similar way, it can be easily shown for $k = 3, 4, \ldots$.

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