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\title{p-Completely bounded weighted homomorphisms on the p-analog of the Fourier-Stieltjes algebra}
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\begin{abstract}
In this paper we study p-completely boundedness of weighted maps on the p-analog of the Fourier-Stieltjes algebras, based on the p-operator structure defined by the authors. We study p-completely boundedness of homomorphisms, which are induced by continuous and proper piecewise affine maps, that is a generalization of Ilie’s results on Figà-Talamanca-Herz algebras.
\end{abstract}

\textbf{Keywords:} Completely bounded homomorphisms, p-analog of the Fourier-Stieltjes algebras, QSL\(_p\)-spaces, Piecewise affine maps

\textbf{MSC2010:} Primary 46L07; Secondary 43A30, 47L10.

\section{Introduction}
Let \(G\) be a locally compact group. The Fourier algebra, \(A(G)\), and the Fourier-Stieltjes algebra, \(B(G)\), on the locally compact group \(G\), have been found by Eymard in 1964 \cite{13}. The general form of special type of maps on the Fourier and Fourier-Stieltjes algebras has been studied extensively. For example, when \(G\) is an Abelian topological group, \(A(G)\) is nothing except \(L_1(\hat{G})\), where \(\hat{G}\) is the Pontrjagin dual group of \(G\), and \(B(G)\) is isometrically isomorphic to \(M(\hat{G})\), the measure algebra. In this case, Cohen in \cite{5} and \cite{6} studied homomorphisms from \(L_1(G)\) to \(M(H)\), for Abelian groups \(G\) and \(H\), and gave the general form of these maps, as the weighted maps by a piecewise affine map on the underlying groups.

By \cite{11,12}, we know that \(A(G)\) and \(B(G)\) are operator spaces as the predual of a von Neumann algebra, and the dual of a \(C^*\)-algebra, respectively. Ilie in \cite{21} and \cite{22} studied the completely bounded homomorphisms from the Fourier to the Fourier-Stieltjes algebras. It is shown that for a continuous piecewise affine map \(\alpha : Y \subseteq H \to G\), the homomorphism \(\Phi_\alpha : A(G) \to B(H)\), defined through

\[\Phi_\alpha u = \begin{dcases} u \circ \alpha & \text{on } Y \\ 0 & \text{o.w.}\end{dcases}, \quad u \in A(G),\]

is a \(p\)-completely bounded homomorphism of \(A(G)\) to \(B(H)\).
is completely bounded. Moreover, in the cases that $\alpha$ is an affine map and a homomorphism, the homomorphism $\Phi_\alpha$ is completely contractive and completely positive, respectively.

The Figà-Talamanca-Herz algebras were introduced by Figà-Talamanca for Abelian locally compact groups [14], and it was generalized for general locally compact group by Herz [19]. For $p \in (1, \infty)$, coefficient functions of the left regular representation of a locally compact group $G$ on $L_p(G)$ give us the Figà-Talamanca-Herz algebra $A_p(G)$, and we have $A_2(G) = A(G)$. Therefore, Figà-Talamanca-Herz algebras can be seen as the $p$-analog of the Fourier algebras.

Daws in [9] introduced the $p$-operator space structure, with an extensive application to $A_p(G)$, that generalizes the operator space structure of $A(G)$. Oztop and Spronk in [24], and Ilie in [20] studied the $p$-completely bounded homomorphisms on the Figà-Talamanca-Herz algebras, using the $p$-operator space structure. In [20] it is shown that the map $\Phi_\alpha : A_p(G) \to A_p(H)$, defined via

$$\Phi_\alpha u = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{o.w} \end{cases}, \quad u \in A_p(G),$$

is a $p$-completely (bounded) contractive homomorphism for a continuous proper (piecewise) affine map $\alpha : Y \subseteq H \to G$ in the case that the locally compact group $H$ is amenable.

Runde in [26] found $p$-analog of the Fourier-Stieltjes algebras, $B_p(G)$. He used extensively the theory of $QSL_p$-spaces and representations on these spaces. Additionally, the $p$-operator space structure of $B_p(G)$ is fully described by authors in [1] and it is shown that $B_p(G)$ is a $p$-operator space, as the dual space of the algebra of universal $p$-pseudofunctions $UPF_p(G)$. The second author of this paper studied the $p$-analog of the Fourier-Stieltjes algebras on the inverse semigroups in [28].

In this paper, for a continuous proper piecewise affine map $\alpha : Y \subseteq H \to G$, we study the weighted maps $\Phi_\alpha : B_p(G) \to B_p(H)$ which is defined by

$$\Phi_\alpha u = \begin{cases} u \circ \alpha & \text{on } Y \\ 0 & \text{o.w} \end{cases}, \quad u \in B_p(G). \quad (1)$$

We will show that when $\alpha$ is an affine map, $\Phi_\alpha$ is a $p$-completely contraction, and in the case that $\alpha$ is a piecewise affine map, it is $p$-completely bounded homomorphism. For this aim, we put amenability assumption on open subgroups of $H$. Our approach to the concept of $p$-operator space structure on the $p$-analog of the Fourier-Stieltjes algebra, is the $p$-operator structure that can be implemented on this space from its predual.

The paper is constructed as follows. First, we give required definitions and theorems about the $p$-analog of the Fourier-Stieltjes algebras and representations on $QSL_p$-spaces in Section 2. In Section 3 some crucial and previously obtained properties of the algebra $B_p(G)$ will be listed, and as a consequence in Section 4 we will investigate $p$-completely boundedness of some well-known operators on the $p$-analog of the Fourier-Stieltjes algebras (Theorem 5). The obtained results will be applied in the final section, Section 5 to generalize Ilie’s results on homomorphisms of the Figà-Talamanca-Herz algebras in [20].
weighted homomorphisms of $B_p(G)$

2 Preliminaries

Throughout this paper, $G$ and $H$ are locally compact groups, and for $p \in (1, \infty)$, the number $p'$ is its complex conjugate, i.e. $1/p + 1/p' = 1$. We commence by the essential notions and definitions on $QSL_p$-spaces, and representations of groups on such spaces. For more information one can see [26].

**Definition 1.** A representation of a locally compact group $G$ is a pair $(\pi, E)$, where $E$ is a Banach space and $\pi$ is a group homomorphism that maps each element $x \in G$ to an invertible isometric operator $\pi(x)$ on $E$. This homomorphism is continuous with respect to the given topology on $G$ and the strong operator topology on $B(E)$.

**Remark 1.** Each representation $(\pi, E)$ can be lifted to a representation of the group algebra $L_1(G)$ on $E$. Denoting this homomorphism with the same symbol $\pi$, it is defined through

$$\pi(f) = \int f(x)\pi(x)dx, \ f \in L_1(G),$$

$$\langle \pi(f)\xi, \eta \rangle = \int f(x)\langle \pi(x)\xi, \eta \rangle dx, \ \xi \in E, \ \eta \in E^*,$$

where the integral (2) converges with respect to the strong operator topology.

**Definition 2.** For two representations $(\pi, E)$ and $(\rho, F)$ of the locally compact group $G$. We have the following terminologies.

1. $(\pi, E)$ and $(\rho, F)$ are called equivalent, if there exists an invertible isometric map $T : E \to F$ for which the following diagram commutes for each $x \in G$,

```
  E  \pi(x)\ E
    |  \downarrow T
    F  \rho(x)\ F
```

2. the $(\pi, E)$ has a subrepresentation $(\rho, F)$, if $F$ is a closed subspace of $E$, and for each $x \in G$, the operator $\rho(x)$ is the restriction of $\pi(x)$ to the subspace $F$.

3. we say $(\pi, E)$ contains $(\rho, F)$ and write $(\rho, F) \subseteq (\pi, E)$, if it is equivalent to a subrepresentation of $(\pi, E)$.

**Definition 3.** 1. A Banach space is called an $L_p$-space if it is of the form $L_p(X)$ for some measure space $X$.

2. A Banach space is called a $QSL_p$-space if it is isometrically isomorphic to a quotient of a subspace of an $L_p$-space.

We denote by $\text{Rep}_p(G)$ the collection of all (equivalence classes) of representations of $G$ on a $QSL_p$-space.
Definition 4. A representation of a Banach algebra $\mathcal{A}$ is a pair $(\pi, E)$, where $E$ is a Banach space, and $\pi$ is a contractive algebra homomorphism from $\mathcal{A}$ to $\mathcal{B}(E)$. We call $(\pi, E)$ isometric if $\pi$ is an isometry, and essential if the linear span of $\{\pi(a)\xi : a \in \mathcal{A}, \xi \in E\}$ is dense in $E$.

Remark 2. For a locally compact group $G$ there exists a one-to-one correspondence between representations of $G$ and essential representations of $L^1(G)$.

Definition 5. 1. A representation $(\pi, E) \in \text{Rep}_p(G)$ is called cyclic, if there exists $\xi_0 \in E$ such that $\pi(L^1(G))\xi_0$ is dense in $E$. The set of cyclic representations of group $G$ on $QSL_p$-spaces is denoted by $\text{Cyc}_p(G)$.

2. A representation $(\pi, E) \in \text{Rep}_p(G)$ is called $p$-universal, if it contains every cyclic representation.

Remark 3. By [17, Remark 2.9-(3)], and [17, Proposition 2.4], it is easy to see that every $p$-universal representation of $G$, contains every cyclic representation of $G$ on a $QSL_p$-space, in the sense of equivalency. In Addition, every representation in $\text{Rep}_p(G)$ is contained in a $p$-universal representation. Actually, one could make a new $p$-universal representation by constructing $l_p$-direct sum of an arbitrary representation with a $p$-universal representation.

Definition 6. We say that the function $u : G \to \mathbb{C}$ is a coefficient function of a representation $(\pi, E)$, if there exist $\xi \in E$ and $\eta \in E^*$ such that $u(x) = \langle \pi(x)\xi, \eta \rangle$, $x \in G$.

Now, we state the definition of the $p$-analog of the Fourier-Stieltjes algebras.

Definition 7. The space of all coefficient functions of representations in $\text{Rep}_p(G)$ endowed with the infimum norm below is called the $p$-analog of the Fourier-Stieltjes algebra and is denoted by $B_p(G)$,

$$\|u\| = \inf \left\{ \sum_{n=1}^{\infty} ||\xi_n|| \|\eta_n\| < \infty : u(\cdot) = \sum_{n=1}^{\infty} \langle \pi_n(\cdot)\xi_n, \eta_n \rangle, (\pi_n, E_n) \in \text{Cyc}_p(G) \right\}$$

for every $u \in B_p(G)$.

The left regular representation of $G$ on $L_p(G)$, is denoted by $\lambda_p$ and defined as following,

$$\lambda_p : G \to \mathcal{B}(L_p(G)), \quad \lambda_p(x)\xi(y) = \xi(x^{-1}y), \quad \xi \in L_p(G), x, y \in G.$$  

and it obtains a description of the Figà-Talamanca-Herz algebras.

Definition 8. Figà-Talamanca-Herz algebra on the locally compact group $G$, which is denoted by $A_p(G)$, is the collection of functions $u : G \to \mathbb{C}$ which are infinite sum of coefficient functions of the representation $(\lambda_p, L_p(G))$ like

$$u(\cdot) = \sum_{n=1}^{\infty} \langle \lambda_p(\cdot)\xi_n, \eta_n \rangle,$$  \text{(3)}
weighted homomorphisms of $B_p(G)$

with

$$(\xi_n)_{n \in \mathbb{N}} \subseteq L_p(G), \quad (\eta_n)_{n \in \mathbb{N}} \subseteq L_{p'}(G), \quad \text{and} \quad \sum_{n=1}^{\infty} ||\xi_n|| ||\eta_n|| < \infty. \quad (4)$$

The norm of $A_p(G)$ is defined as

$$||u|| = \inf \left\{ \sum_{n=1}^{\infty} ||\xi_n|| ||\eta_n|| : u(\cdot) = \sum_{n=1}^{\infty} \langle \lambda_p(\cdot)\xi_n, \eta_n \rangle \right\},$$

where the infimum is taken over all expressions of $u$ in (3) with (4).

Remark 4. 1. The $p$-analog of the Fourier-Stieltjes algebra has been studied, for example in [7, 16, 23] and [25], as the multiplier algebra of the Figà-Talamanca-Herz algebra. In this paper, we follow the construction of Runde in definition and notation (See [26]) which we have swapped indexes $p$ and $p'$.

2. By [26, Lemma 4.6], the space $B_p(G)$ can be defined to be the set of all coefficient functions of a $p$-universal representation $(\pi, E)$, and the norm of an element $u \in B_p(G)$ is the infimum of all values $\sum_{n=1}^{\infty} ||\xi_n|| ||\eta_n|| < \infty$, which such vectors exist in the representation of $u$ as a coefficient function of $(\pi, E)$, i.e. $u(\cdot) = \sum_{n=1}^{\infty} \langle \pi(\cdot)\xi_n, \eta_n \rangle$.

3. By [26, Theorem 4.7], the space $B_p(G)$ equipped with the norm defined as above, and pointwise operations is a commutative unital Banach algebra, and by [26, Corollary 5.3], by denoting the multiplier algebra of $A_p(G)$ by $\mathcal{M}(A_p(G))$, we have the following contractive embeddings

$$A_p(G) \subseteq B_p(G) \subseteq \mathcal{M}(A_p(G)).$$

4. In [27, Lemma 2.4], the following identification is shown for an open subgroup $G_0$ of a locally compact group $G$

$$A_p(G_0) \cong \{ f \in A_p(G) : \text{supp}(f) \subset G_0 \},$$

and through this fact, one can assume that functions in $A_p(G_0)$ are restriction of functions in $A_p(G)$ to the open subgroup $G_0$.

Definition 9. Let $(\pi, E) \in \text{Rep}_p(G)$.

1. For each $f \in L_1(G)$, let $\|f\|_\pi := \|\pi(f)\|_{\mathcal{B}(E)}$, then $\| \cdot \|_\pi$ defines an algebra seminorm on $L_1(G)$.

2. By $\text{PF}_{p,\pi}(G)$, we mean the $p$-pseudofunctions associated with $(\pi, E)$, which is the closure of $\pi(L_1(G))$ in $\mathcal{B}(E)$.

3. If $(\pi, E) = (\lambda_p, L_p(G))$, we denote $\text{PF}_{p,\lambda_p}(G)$ by $\text{PF}_{\pi}(G)$.

4. If $(\pi, E)$ is $p$-universal, we denote $\text{PF}_{p,\pi}(G)$ by $\text{UPF}_{\pi}(G)$, and call it the algebra of universal $p$-pseudofunctions.

Remark 5. 1. For $p = 2$, the algebra $\text{PF}_{p}(G)$ is the reduced group $C^*$-algebra, and $\text{UPF}_{p}(G)$ is the full group $C^*$-algebra of $G$. 

2. If $(\rho, F) \in \text{Rep}_p(G)$ is such that $(\pi, E)$ contains every cyclic subrepresentation of $(\rho, F)$, then $\| \cdot \|_\rho \leq \| \cdot \|_\pi$ holds. In particular, the definition of $\text{UPF}_p(G)$ is independent of a particular $p$-universal representation.

3. With $\langle \cdot, \cdot \rangle$ denoting $L_1(G) - L_\infty(G)$ duality, and with $(\pi, E)$ a $p$-universal representation of $G$, we have
\[
\|f\|_\pi = \sup\{\langle f, g \rangle : g \in B_p(G), \|g\|_{B_p(G)} \leq 1\}, \quad f \in L_1(G).
\]

Next lemma states that $B_p(G)$ is a dual space.

**Lemma 1.** [26 Lemma 6.5] Let $(\pi, E) \in \text{Rep}_p(G)$. Then, for each $\phi \in PF_{p, \pi}(G)^*$, there is a unique $g \in B_p(G)$, with $\|g\|_{B_p(G)} \leq \|\phi\|$ such that
\[
\langle \pi(f), \phi \rangle = \int_G f(x)g(x)dx, \quad f \in L_1(G). \tag{5}
\]

Moreover, if $(\pi, E)$ is $p$-universal, we have $\|g\|_{B_p(G)} = \|\phi\|$.

The $p$-operator space structure which is used in this paper is Daws’ approach for $A_p(G)$ [9]. A concrete $p$-operator space is a closed subspace of $B(E)$, for some $QSL_{p}$-space $E$. In this case for each $n \in \mathbb{N}$ one can define a norm $\| \cdot \|_n$ on $M_n(X) = M_n \otimes X$ by identifying $M_n(X)$ with a subspace of $B(l^p_n \otimes p E)$. So, we have the family of norms $(\| \cdot \|_n)_{n \in \mathbb{N}}$ satisfying:

$D_\infty$: For $u \in M_n(X)$ and $v \in M_m(X)$, we have that $\|u \oplus v\|_{n+m} = \max\{\|u\|_n, \|v\|_m\}$.

Here $u \oplus v \in M_{n+m}(X)$ has block representation $\begin{pmatrix} u & 0 \\ 0 & v \end{pmatrix}$.

$M_p$: For every $u \in M_m(X)$ and $\alpha \in M_{n,m}, \beta \in M_{m,n}$ we have that
\[
\|\alpha \beta \| \leq \|\alpha\|_{B(l^p_n, l^p_m)} \|u\|_m \|\beta\|_{B(l^p_m, l^p_n)}.
\]

**Definition 10.** A linear operator $\Psi : X \to Y$ between two $p$-operator spaces is called $p$-completely bounded if $\|\Psi\|_{p, cb} = \sup_{n \in \mathbb{N}} \|\Psi^{(n)}\| < \infty$, and $p$-completely contractive if $\|\Psi\|_{p, cb} = \sup_{n \in \mathbb{N}} \|\Psi^{(n)}\| \leq 1$, where $\Psi^{(n)} : M_n(X) \to M_n(Y)$ is defined in the natural way.

**Theorem 1.** [4 Theorem 4.3] Let $X$ be a $p$-operator space. There exists a $p$-complete isometry $\varphi : X^* \to B(l_p(I))$ for some index set $I$.

**Lemma 2.** [3 Lemma 4.5] If $\Psi : X \to Y$ is $p$-completely bounded map between two operator spaces $X$ and $Y$, then $\Psi^* : Y^* \to X^*$ is $p$-completely bounded, with $\|\Psi^*\|_{p, cb} \leq \|\Psi\|_{p, cb}$.

**Remark 6.**

1. It should be noticed that, converse of Lemma 2 is not necessarily true, unless $X$ be a closed subspace of $B(E)$, for some $L_p$-space $E$.

2. In comparison to [20], because of above explanations, a major difference in our work is that we need to study predual of some crucial $p$-completely bounded maps (See Theorem 5), instead of their duals.
In Section 5, we will study the homomorphisms on the \( p \)-analog of the Fourier-Stieltjes algebras induced by the continuous map \( \alpha : Y \subseteq H \to G \), on locally compact groups \( G \) and \( H \), in the cases that \( \alpha \) is homomorphism, affine and piecewise affine map, and \( Y \) in the coset ring of \( H \). So, we give some preliminaries here.

For a locally compact topological group \( H \), let \( \Omega_0(H) \) denote the ring of subsets which generated by open cosets of \( H \). By \([20]\) we have

\[
\Omega_0(H) = \left\{ Y \cup \bigcup_{i=1}^n Y_i : Y \text{ is an open coset of } H, \quad Y_1, \ldots, Y_n \text{ open subcosets of infinite index in } Y \right\}.
\]

Moreover, for a set \( Y \subseteq H \), by \( \text{Aff}(Y) \) we mean the smallest coset containing \( Y \), and if \( Y = Y_0 \cup \bigcup_{i=1}^n Y_i \in \Omega_0(H) \), then \( \text{Aff}(Y) = Y_0 \). Similarly, let us denote by \( \Omega_{\text{am}-0}(H) \) the ring of open cosets of open amenable subgroups of \( H \), that is the ring of subsets of the form \( Y \setminus \bigcup_{i=1}^n Y_i \) where \( Y \) is an open coset of an open amenable subgroup of \( H \) and \( Y_i \) is an open subcosets of infinite index in \( Y \) (which has to be an open coset of an open amenable subgroup), for \( i = 1 \ldots, n \).

**Definition 11.** Let \( G \) and \( H \) be locally compact groups, and let \( \alpha : Y \subseteq H \to G \) be a map.

1. The map \( \alpha \) is called an affine map on an open coset \( Y \) of an open subgroup \( H_0 \), if
   \[
   \alpha(xy^{-1}z) = \alpha(x)\alpha(y)^{-1}\alpha(z), \quad x, y, z \in Y,
   \]
   where \( y \in Y \setminus \bigcup_{i=1}^n Y_i \).
2. The map \( \alpha \) is called a piecewise affine map if
   \( a \) there are pairwise disjoint \( Y_i \in \Omega_0(H) \), for \( i = 1, \ldots, n \), such that \( Y \setminus \bigcup_{i=1}^n Y_i \),
   \( b \) there are affine maps \( \alpha_i : \text{Aff}(Y_i) \subseteq H \to G \), for \( i = 1, \ldots, n \), such that
   \[
   \alpha|_{Y_i} = \alpha_i|_{Y_i}.
   \]

**Definition 12.** If \( X \) and \( Y \) are locally compact spaces, then a map \( \alpha : Y \to X \) is called proper, if \( \alpha^{-1}(K) \) is compact subset of \( Y \), for every compact subset \( K \) of \( X \).

**Proposition 1.** \([11, Proposition 4]\) Let \( \alpha : H \to G \) be a continuous group homomorphism. Then \( \alpha \) is proper if and only if the bijective homomorphism \( \tilde{\alpha} : H/\ker \alpha \to \alpha(H) = G_0 \), is a topological group isomorphism, and \( \ker \alpha \) is compact.

**Remark 7.** 1. Proposition \([11]\) implies that every continuous proper homomorphism is automatically a closed map. Therefore, \( \alpha(H) \) is a closed subgroup of \( G \). Also, \( \ker \alpha \) is a compact normal subgroup of \( H \).
2. It is well-known that \( \tilde{\alpha} \) is a group isomorphism, if and only if \( \alpha \) is an open homomorphism into \( \alpha(H) \), with the relative topology.
3. [21] Remark 2.2] If \( Y = h_0H_0 \) is an open coset of an open subgroup \( H_0 \subseteq H \), and \( \alpha : Y \subseteq H \to G \) is an affine map, then there exists a group homomorphism \( \beta \) associated to \( \alpha \) such that

\[
\beta : H_0 \subseteq H \to G, \quad \beta(h) = \alpha(h_0)^{-1}\alpha(hh_0), \quad h \in H_0.
\]

4. It is clear that, \( \alpha \) is a proper affine map, if and only if \( \beta \) is a proper homomorphism.

5. [20] Lemma 8] Let \( Y \in \Omega_0(H) \), and \( \alpha : \text{Aff}(Y) \to G \) be an affine map such that \( \alpha|Y \) is proper, then \( \alpha \) is proper.

3 Properties of the \( p \)-analog of the Fourier-Stieltjes algebras

As the main purpose of this paper is to study \( p \)-completely boundedness of operators on the \( p \)-analog of the Fourier-Stieltjes algebras, priorly, we need to determine \( p \)-operator structure governed on these algebras. In fact, by [1] it has been shown that the predual of \( B_p(G) \), that is the universal algebra of \( p \)-pseudo functions \( UPF_p(G) \), is a \( p \)-operator space and by applying results from [9], it is obtained that the space \( B_p(G) \) inherits \( p \)-operator structure through being dual of a \( p \)-operator space. We briefly explain about this structure in the following.

- **\( p \)-operator space structure of \( UPF_p(G) \) and \( B_p(G) \).**

For a representation \((\Pi, \mathcal{E}) \in \text{Rep}_p(G)\), a matrix representation is an \( n \)-folded representation \((\Pi^{(n)}, \mathcal{E}^{(n)})\) in the following way

\[
\pi^{(n)} : M_n(L_1(G)) \to \mathcal{B}(\mathcal{E}^{(n)})
\]

\[
\Pi^{(n)}([f_{ij}]) := [\Pi(f_{ij})], \quad [f_{ij}] \in M_n(L_1(G)).
\]

For \((\pi, \mathcal{E}) \in \text{Rep}_p(G)\), we can determine cyclic subrepresentations \( (\pi_{r,g}, E_{r,g})\) \( r \in \mathbb{N}, g \in L_1(G) \) associated with cyclic vectors \( (\xi_{r,g})_{r \in \mathbb{N}, g \in L_1(G)} \) such that

\[
\|\pi_{g,r}(g)\xi_{g,r}\| > \|\pi(g)\| - \frac{1}{r}, \quad \|\xi_{g,r}\| \leq 1.
\]

Consider the following \( l_p \)-direct sum of aforementioned cyclic subrepresentations associated with the \( l_p \)-direct sum of their paired spaces.

\[
\mathcal{E} = l_p^\infty \oplus_{g \in L_1(G)} \oplus_{r=1}^\infty E_{g,r} \quad \Pi = l_p^\infty \oplus_{g \in L_1(G)} \oplus_{r=1}^\infty \pi_{g,r}.
\]

In [1] Proposition 3.11], it is indicated that the algebra \( PF_{p,\pi}(G) \) is \( p \)-completely isomorphic to a closed subspace of \( \mathcal{B}(\mathcal{E}) \). Indeed, it is shown that the cyclic matrix representation \((\Pi^{(n)}, \mathcal{E}^{(n)})\) is an isometric map from \( M_n(PF_{p,\pi}(G)) \) onto a closed subspace of \( M_n(\mathcal{B}(\mathcal{E})) = \mathcal{B}(\mathcal{E}^{(n)}) \). As a consequence, it has been obtained that for a representation \((\pi, \mathcal{E}) \in \text{Rep}_p(G)\), the algebra of \( p \)-pseudo functions \( PF_{p,\pi}(G) \) is a \( p \)-operator space [1 Theorem 3.12]. The next proposition is applicable and illustrate the path in which we have a well-defined \( p \)-operator space structure on the algebra \( UPF_p(G) \).
Proposition 2. If \((ρ, F)\) is a subrepresentation of \((π, E)\), then the natural map
\[
Ψ : PF_{p, π}(G) \to PF_{p, ρ}(G), \quad π(f) \to ρ(f), \quad f ∈ L_1(G),
\]
is \(p\)-completely contractive.

Proof. This is Proposition 4.4 in [1].

Through Proposition 2, it is concluded that the algebra of universal \(p\)-pseudofunctions \(UPF_p(G)\) is an abstract \(p\)-operator space and is independent of choosing specific universal representation [1, Theorem 4.5]. As a consequence of this theorem, we give an immensely important theorem below and proposition afterwards (see [1, Theorem 4.7 and Proposition 4.8]).

Theorem 2. For \(p ∈ (1, \infty)\), the Banach algebra \(B_p(G)\) is a \(p\)-operator space.

Next proposition is the output of all materials mentioned before.

Proposition 3. For a locally compact group \(G\), and a complex number \(p ∈ (1, \infty)\), the identification \(B_p(G) = UPF_p(G)^*\) is \(p\)-completely isometric isomorphism.

• Spaces \(A_{p, π}\) and \(B_{p, π} = PF_{p, π}(G)^*\).

Now, it is the turn for some specific spaces to be introduced as they will be used later. A \(p\)-generalization of \(π\)-Fourier space introduced by Arsac [3] can be presented as below.

Definition 13. For a representation \((π, E) ∈ \text{Rep}_p(G)\), we define the \(p\)-analog of the \(π\)-Fourier space, \(A_{p, π}\), to be closed linear span of the collection of the coefficient functions of representation \((π, E)\), i.e. functions of the form
\[
u(x) = \sum_{n}⟨π(x)ξ_n, η_n⟩, \quad x ∈ G, (ξ_n)_{n ∈ N} ⊆ E, (η_n)_{n ∈ N} ⊆ E^*
\]
equipped with the norm
\[
∥u∥_{A_{p, π}} = \inf \left\{ \sum_{n=1}^{∞} ∥t_n∥∥s_n∥ : u(x) = \sum_{n}⟨π(x)t_n, s_n⟩, \quad x ∈ G \right\},
\]
and evidently, infimum is taken over all possible equivalent representative of \(u\) so that the value is convergent.

Next proposition unravels significance of the latest space. To clarify the notation, we introduce the representations \((π_U, (E)_U)\), \((π_∞, l_p(E))\) and \((π_∞^U, l_p(E)_U)\) of the locally compact group \(G\), associated to a representation \((π, E) ∈ \text{Rep}_p(G)\). To do this, for a Banach space \(E\), it has been denoted by \((E)_U\) the ultrapower space of \(E\), with elements of the form \((ξ_i)\) with the norm \(∥(ξ_i)∥ = \lim_U ∥ξ_i∥\) (see [18] as a classical reference). Furthermore, for a complex number \(p ∈ (1, ∞)\), the Banach space \(L_p(E) = L_p(\mathbb{N}, E)\), is as following
\[
L_p(E) = \left\{ (ξ_n)_n : ∥(ξ_n)_n∥ = \left( ∑_n ∥ξ_n\parallel^p \right)^{\frac{1}{p}} < ∞ \right\},
\]
which by Proposition 4 it is super-reflexive whenever $E$ is so. Additionally, in the case that $E$ is a $QSL_p$-space, then $L_p(E)$ is again a $QSL_p$-space as well as $(E)_{\mathcal{U}}$.

Now, for a representation $(\pi, E) \in \text{Rep}_p(G)$, we introduce two representations $(\pi^\infty, L_p(E))$ and $(\pi^\infty_{\mathcal{U}}, (E)_{\mathcal{U}})$, as following:

\[
\pi^\infty : G \to \mathcal{B}(L_p(E)), \quad \pi^\infty(x)(\xi_n) = (\pi(x)\xi)_n, \quad x \in G, \; (\xi_n)_n \in L_p(E),
\]

\[
\pi^\infty_{\mathcal{U}} : G \to \mathcal{B}((E)_{\mathcal{U}}), \quad \pi^\infty_{\mathcal{U}}(x)(\xi)_{\mathcal{U}} = (\pi(x)\xi)_{\mathcal{U}}, \quad x \in G, \; (\xi)_{\mathcal{U}} \in (E)_{\mathcal{U}}.
\]

Similarly, $(\pi^\infty_{\mathcal{U}}, (L_p(E))_{\mathcal{U}})$ can be defined.

**Proposition 4.** Let $(\pi, E) \in \text{Rep}_p(G)$. Then

1. there exists a free ultrafilter $\mathcal{U}$, such that the canonical representation of $\text{PF}_{p,\pi}(G)$ on $F = (L_p(E))_{\mathcal{U}}$ is weak-weak$^*$ continuous, essential and isometric,

2. the identification $\text{PF}_{p,\pi}(G)^* = \text{Rep}_{p,\pi}(G)^* = A_{\pi^\infty}^\infty$ holds. Moreover, it can be concluded that $\text{PF}_{p,\pi}(G)^* = A_{\pi^\infty}^\infty = A_{\pi^\infty_{\mathcal{U}}}$

**Proof.** These results can be derived by careful examination of the proof of Lemma 6.5 in [26] together with adapting our notation of the space $A_{\pi^\infty}^\infty$. However, for more details, one can see Proposition 2 and results afterwards in [26].

The dual space $\text{PF}_{p,\pi}(G)^*$ will be denoted by $B_{p,\pi}$ to follow the tradition initiated in [3], and it is called $p$-analog of the $\pi$-Fourier-Stieltjes algebra. The duality between $B_{p,\pi}$ and $\text{PF}_{p,\pi}(G)$ can be stated via relation below,

\[
\langle \pi(f), u \rangle = \int_G u(x)f(x)dx, \quad f \in L_1(G), \; u \in B_{p,\pi}.
\]

On top of that,

\[
\|u\| = \sup_{\|f\|_1 \leq 1} |\langle \pi(f), u \rangle| = \sup_{\|f\|_1 \leq 1} \left| \int_G u(x)f(x)dx \right|, \quad u \in B_{p,\pi},
\]

\[
\|f\|_p = \sup_{\|u\| \leq 1} |\langle \pi(f), u \rangle| = \sup_{\|u\| \leq 1} \left| \int_G u(x)f(x)dx \right|, \quad f \in L_1(G).
\]

Moreover, by weak-weak$^*$ continuity of the essential and isometric representation $(\pi^\infty_{\mathcal{U}}, F)$ of $\text{PF}_{p,\pi}(G)$, we have

\[
\text{PF}_{p,\pi}(G) = \text{PF}_{p,\pi^\infty_{\mathcal{U}}}(G),
\]

and in the case that the representation $(\pi, E)$ is a $p$-universal representation, then since $(\pi^\infty_{\mathcal{U}}, (L_p(E))_{\mathcal{U}})$ is also a $p$-universal representation, our notation coincides with Runde’s one in [26]. Additionally, the fashionable notation for $\text{PF}_{p,\lambda_p}(G)^*$ is $\text{PF}_{p}(G)$, instead of $B_{p,\lambda_p}$.

Now, we state a result from [26] adapted to our notations.
Theorem 3. 1. For a representation $(\pi, E) \in \text{Rep}_p(G)$, the inclusion $B_{p,\pi} \subseteq B_p(G)$ is contractive, and is an isometric isomorphism whenever $(\pi, E)$ is a $p$-universal representation.

2. We have the following contractive inclusions

$$PF_p(G) = B_{p,\lambda_p} \subseteq B_p(G) \subseteq M(A_p(G)),$$

and all inclusions will become equalities in the case that $G$ is amenable.

Proof. See [26, Theorem 6.6 and Theorem 6.7]).

• Extension Theorem.

In the next stage of preparation for our results, we bring some facts about extension of a function $u \in B_p(G)$ to a function $B_p(G)$, where $G_0 \subseteq G$ is an open subgroup of the locally compact group $G$. We refer the interested reader to [2, Subsection 3.3] for detailed description. Let $G_0 \subseteq G$, be any subset, and $u : G_0 \to \mathbb{C}$ be a function. Let $u^\circ$ denote the extension of $u$ to $G$ by setting value zero outside of $G_0$, i.e.

$$u^\circ = \begin{cases} u \text{ on } G_0 \\ 0 \text{ o.w.} \end{cases}.$$ 

Following lemma has intricate importance in the sequel.

Lemma 3. Let $(\pi, E) \in \text{Rep}_p(G)$. Then the restriction of $\pi$ to the open subgroup $G_0$, which is denoted by $(\pi_{G_0}, E)$ belongs to $\text{Rep}_p(G_0)$. Moreover, for each $f \in L_1(G_0)$ and each $g \in L_1(G)$, we have the following relations

$$\pi_{G_0}(f) = \pi(f^\circ), \quad \text{and} \quad \pi_{G_0}(g|_{G_0}) = \pi(g \chi_{G_0}). \quad (9)$$

Proof. This is Lemma 3 in [2].

Next proposition is the main building block of the extension problem.

Proposition 5. Let $G$ be a locally compact group and $G_0$ be its open subgroup, and let $(\pi, E) \in \text{Rep}_p(G)$. Then the following statements hold.

1. The map $S_{\pi_{G_0}} : PF_{p,\pi_{G_0}}(G_0) \to PF_{p,\pi}(G)$ defined via $S_{\pi_{G_0}}(\pi_{G_0}(f)) = \pi(f^\circ)$, for $f \in L_1(G_0)$ and each $g \in L_1(G)$, is an isometric homomorphism. In fact, we have the following isometric identification

$$PF_{p,\pi_{G_0}}(G_0) = \{\pi(f) : f \in L_1(G), \text{ supp}(f) \subseteq G_0\} \subseteq PF_{p,\pi}(G).$$

2. The linear restriction map $R_\pi : B_{p,\pi} \to B_{p,\pi_{G_0}}$ which is defined for $u \in B_{p,\pi}$, as $R_\pi(u) = u|_{G_0}$ is the dual map of $S_{\pi_{G_0}}$, and is a quotient map.

3. The extension map $E_\pi : B_{p,\pi_{G_0}} \to B_{p,\pi}$, defined via $E_\pi(u) = u^\circ$ is an isometric map.

4. The restriction map $R : B_p(G) \to B_p(G_0)$ is a contraction.
5. When \((\pi, E)\) is also a \(p\)-universal representation, we have the following contractive inclusions

\[ PF_p(G_0)^* \subseteq B_{p, \pi G_0} \subseteq B_p(G_0) \subseteq M(A_p(G_0)). \]

Under the assumption that \(G_0\) is amenable, we have isometric identification below

\[ PF_p(G_0)^* = B_{p, \pi G_0} = B_p(G_0) = M(A_p(G_0)). \]

**Proof.** This is Proposition 4 in [2].

Next theorem has a key role in studying of weighted homomorphisms [1].

**Theorem 4 (Extension Theorem).** Let \(G\) be a locally compact group and \(G_0\) be its open subgroup. Then

1. the extension map \(E_{MM} : M(A_p(G_0)) \to M(A_p(G))\), defined for \(u \in M(A_p(G_0))\) via \(E_{MM}(u) = u^\circ\) is an isometry.
2. for every \(u \in B_p(G_0)\), we have \(u^\circ \in M(A_p(G))\), and the map \(E_{BM} : B_p(G_0) \to M(A_p(G))\), with \(u \mapsto u^\circ\), is a contraction.
3. if \(G_0\) is also an amenable subgroup, then for every \(u \in B_p(G_0)\), we have \(u^\circ \in B_p(G)\), and the associated extending map \(E_{BB} : B_p(G_0) \to B_p(G)\) is an isometric one.

**Proof.** This is Proposition 5 in [2].

**Remark 8.** It is worthwhile to take notice of the fact that when the subgroup \(G_0 \subseteq G\) is an amenable open one, a \(p\)-universal representation of \(G_0\) can be induced by restriction of a \(p\)-universal representation of \(G\) to \(G_0\).

## 4 Special \(p\)-completely bounded operators on \(B_p(G)\)

The essential features of the \(p\)-analog of the Fourier-Stieltjes algebras and the algebra of \(p\)-pseudofunctions have been provided so far, and this section is allocated to maintain critical tools in dealing with operators on \(B_p(G)\). We take our initial step towards our purposes. Upcoming proposition is going to reveal the relation between \(p\)-universal representation of a locally compact group \(G\) and the quotient group \(G/N\), where \(N\) is a closed normal subgroup.

Recall the canonical quotient map below

\[ q : G \to G/N, \quad q(x) = xN, \quad x \in G, \]

which is a continuous and onto homomorphism, whenever \(N\) is a closed normal subgroup of \(G\).

**Proposition 6.** Let \(N \subseteq G\) be a closed normal subgroup. Then
1. \((\rho, F) \in \text{Rep}_p(G/N)\) implies that \((\rho \circ q, F) \in \text{Rep}_p(G)\) and the identification
\[
PF_{p,\rho}(G/N) = PF_{p,\rho \circ q}(G),
\]
is an isometric isomorphism.

2. each representation \((\pi, E) \in \text{Rep}_p(G)\) induces a representation \((\tilde{\pi}_K, K) \in \text{Rep}_p(G/N)\) so that
\[
(\tilde{\pi}_K \circ q, K) \subseteq (\pi, E).
\]

3. with aforementioned notations, each \(p\)-universal representation \((\rho, F)\) of \(G/N\) is contained in the representation \((\tilde{\pi}_K, K)\) induced by a \(p\)-universal representation \((\pi, E)\) of \(G\).

4. the map \(\Phi_q : B_p(G/N) \to B_p(G)\) defined through \(\Phi_q(u) = u \circ q\), for \(u \in B_p(G/N)\) is an isometric map.

Proof. 1. Let \((\rho, F) \in \text{Rep}_p(G/N)\). Evidently, we have \((\rho \circ q, F) \in \text{Rep}_p(G)\). Recall the natural map \(\hat{P} : L_1(G) \to L_1(G/N)\) from [13],
\[
\hat{P} f(xN) = \int_N f(xn)dn, \quad xN \in G/N.
\]
For each \(x \in G\), \(z \in F\), and \(\psi \in F^*\) we have \(\langle \rho \circ q(x)z, \psi \rangle = \langle \rho(x)z, \psi \rangle\), and since for each continuous function \(u : G/N \to \mathbb{C}\), and \(f \in L_1(G)\), we have \(\hat{P}(u \circ q) = u \cdot \hat{P} f\), it follows that
\[
\langle \rho \circ q(f)z, \psi \rangle = \langle \rho(P f)z, \psi \rangle, \quad f \in L_1(G), \quad z \in F, \quad \psi \in F^*.
\]
Consequently, we have
\[
\rho \circ q(x) = \rho(xN), \quad \text{and} \quad \rho \circ q(f) = \rho(P f), \quad x \in G, \quad f \in L_1(G). \tag{10}
\]
The relation \[(10)\] means that \(PF_{p,\rho \circ q}(G) = PF_{p,\rho}(G/N)\) isometrically, by the fact that the map \(\hat{P}\) is onto.

2. Let \((\pi, E) \in \text{Rep}_p(G)\). Set the closed subspace \(K\) of \(E\), that is itself a QSL\(p\)-space, as following
\[
K = \{ x \in E : \pi(x) = \xi, \quad \text{for all} \ x \in N \}\.
\]
The space \(K\) is invariant under \(\pi\), i.e.
\[
\pi(x)K \subseteq K, \quad \text{and} \quad \pi(f)K \subseteq K, \quad x \in G, \ f \in L_1(G).
\]
So, if we let \((\pi_K, K)\) denote the representation of \(G\) that takes every element \(x \in G\) to \(\pi_K(x) = \pi(x)|_K\), then
\[
(\pi_K, K) \in \text{Rep}_p(G), \quad \text{and} \quad (\pi_K, K) \subseteq (\pi, E).
\]
Now, put
\[
\tilde{\pi}_K : G/N \to B(K), \quad \tilde{\pi}_K(xN) = \pi(x), \quad xN \in G/N.
\]
Then by the definition of \(K\), the pair \((\tilde{\pi}_K, K)\) is well-defined and belongs to \(\text{Rep}_p(G/N)\). Additionally,
\[
(\tilde{\pi}_K \circ q, K) = (\pi_K, K) \subseteq (\pi, E).
\]
3. Let \((\rho, F) \in \text{Rep}_p(G/N)\) be a \(p\)-universal representation, then by Part 411 \((\rho \circ q, F) \in \text{Rep}_p(G)\), and consequently it is contained in a \(p\)-universal representation \((\pi, E)\) of \(G\), and we have (up to an isometry)

\[
F \subseteq E, \quad \rho \circ q(f) = \pi(f)|_E.
\]

Moreover, by the definition of \(K\), we have \(F \subseteq K\), and it is obtained that

\[
(\rho \circ q, F) \subseteq (\pi_K, K), \quad \text{and consequently} \quad (\rho, F) \subseteq (\tilde{\pi}_K, K),
\]

and since \((\tilde{\pi}_K, K)\) contains the \(p\)-universal representation \((\rho, F)\), then it is a \(p\)-universal representation.

4. It is straightforward through previous parts.

Next theorem is our first main result of this paper, and it will be applied to give the results on weighted homomorphisms on the \(p\)-analog of the Fourier-Stieltjes algebras. For more clarification, we need to introduce the notion of the \(p\)-tensor product \(E \otimes_p F\) of two QSL\(_p\)-spaces \(E\) and \(F\), that is defined in \([26]\). In fact, Runde introduced the norm \(\| \cdot \|_p\) on the algebraic tensor product \(E \otimes F\) which benefits from pivotal properties. As an important property of the norm \(\| \cdot \|_p\), is the fact that the completion \(E \otimes_p F\) of \(E \otimes F\) with respect to \(\| \cdot \|_p\) is a QSL\(_p\)-space. Furthermore, for two representations \((\pi, E)\) and \((\rho, F)\) of the locally compact group \(G\) in \(\text{Rep}_p(G)\), the representation \((\pi \otimes \rho, E \otimes_p F)\) is well-defined and belongs to \(\text{Rep}_p(G)\). As a result, for two functions \(u(\cdot) = \langle \pi(\cdot)\xi, \eta \rangle\) and \(v(\cdot) = \langle \rho(\cdot)\xi', \eta' \rangle\), the pointwise product of them is a coefficient function of the representation \((\pi \otimes \rho, E \otimes_p F)\), i.e. \(u \cdot v(\cdot) = \langle (\pi(\cdot) \otimes \rho(\cdot))(\xi \otimes \xi'), \eta \otimes \eta' \rangle\). For more details on \(p\)-tensor product \(\otimes_p\) see \([26]\) Theorem 3.1 and Corollary 3.2.

**Theorem 5.** Let \(p \in (1, \infty)\) and \(G\) be a locally compact group. Then we have the following statements:

1. For any \((\rho, F) \in \text{Rep}_p(G)\), the identity map \(I : B_{p,p} \rightarrow B_p(G)\) is a \(p\)-completely contractive map.
2. For an open subgroup \(G_0\) of \(G\), the restriction map \(R : B_p(G) \rightarrow B_p(G_0)\), is a \(p\)-completely contractive homomorphism.
3. For an element \(a \in G\), the translation map \(L_a : B_p(G) \rightarrow B_p(G)\), defined through \(L_a(u) = au\), where \(au(x) = u(ax)\), for \(x \in G\), is a \(p\)-completely contractive map.
4. The homomorphism \(\Phi_q : B_p(G/N) \rightarrow B_p(G)\), with \(\Phi_q(u) = u \circ q\), is a \(p\)-completely contractive homomorphism.
5. For an open amenable subgroup \(G_0\) of \(G\), the extension map \(E_{BB} : B_p(G_0) \rightarrow B_p(G)\) is a \(p\)-completely contractive homomorphism.
6. For an open coset \(Y\) of an open subgroup \(G_0\) of \(G\), the map \(M_Y : B_p(G) \rightarrow B_p(G)\), with \(M_Y(u) = u \cdot \chi_Y\), is \(p\)-completely contractive homomorphism. More generally, for a set \(Y \in \Omega_0(G)\), the map \(M_Y\) is a \(p\)-completely bounded homomorphism.
Proof. 1. We want to prove that for each \((\rho,F) \in \text{Rep}_p(G)\), the following map is a \(p\)-complete contraction,
\[
I : B_{p,\rho} \to B_p(G), \quad I(u) = u. \quad (11)
\]

Let \((\pi,E)\) be a \(p\)-universal representation of \(G\) that contains the representation \((\rho,F)\). Following relations hold between \((\rho,F)\), and \((\pi,E)\),
\[
F \subseteq E, \quad \rho(x) = \pi(x)|_F, \quad \text{and} \quad \rho(f) = \pi(f)|_F, \quad x \in G, \ f \in L_1(G).
\]

Since \(\rho(f) = \pi(f)|_F\), then \(\|\rho(f)\| \leq \|\pi(f)\|\). Additionally, the map \(I\) is weak*-weak* continuous, and it is a contraction by Theorem 5 Define
\[
\ast I : UP\text{PF}_p(G) \to PF_{p,\rho}(G), \quad \ast I(\pi(f)) = \pi(f)|_F = \rho(f),
\]
then \(\ast I\) is the predual of the map \((11)\). Because, we have \((\pi(f), I(u)) = \langle \rho(f), u \rangle\), for every \(f \in L_1(G)\) and \(u \in B_{p,\rho}\). Following calculations indicate that \(\ast I\) is a \(p\)-complete contraction: for each \(n \in \mathbb{N}\), and \(\|\rho(f)\|_n \in \mathbb{M}_n(\mathbb{F}_p,\rho(G))\) we have
\[
\|\rho(f)\|_n = \sup \left\{ \|\rho(f)\| \in \mathbb{M}_n(F), \sum_{j=1}^{n} \|\xi_j\|^p \leq 1 \right\}
\]
\[
\leq \sup \left\{ \|\pi(f)\| \in \mathbb{M}_n(E), \sum_{j=1}^{n} \|\xi_j\|^p \leq 1 \right\}
\]
\[
= \|\pi(f)\|_n,
\]
so, we have \(\|\rho(f)\|_n \leq \|\pi(f)\|_n\), and by this, it is concluded that
\[
\|I\|_{p,\text{cb}} \leq \|\ast I\|_{p,\text{cb}} \leq 1.
\]

2. Let \(G_0 \subseteq G\) be an open subgroup and recall the maps \(R\) and \(S = S_\pi\) (for a \(p\)-universal representation \((\pi,E)\) of \(G\) in Proposition 5 together with the relation \((9)\) from Lemma 5. Since we have \(S^* = R\), and the map \(S\) is indeed an identity, therefore, \(S\) is a \(p\)-completely isometric map, and consequently,
\[
\|R\|_{p,\text{cb}} = \|S^*\|_{p,\text{cb}} \leq \|S\|_{p,\text{cb}} = 1.
\]

3. For \(a \in G\), consider the following map
\[
L_a : B_p(G) \to B_p(G), \quad L_a(u) = _a u, \quad _a u(x) = u(ax), \ x \in G.
\]

Preadual of the map \(L_a\) is as following
\[
\ast L_a : UP\text{PF}_p(G) \to UP\text{PF}_p(G), \quad \ast L_a(\pi(f)) = \pi(\lambda_p(a)f),
\]
and it is clearly \(p\)-completely contractive, and consequently, this is true for \(L_a\). On the other hand, the map \(L_a\) has the inverse \(L_{a^{-1}}\), and it is \(p\)-completely contractive as well as \(L_a\), which makes it a \(p\)-completely isometric map.
4. Let \( N \subseteq G \) be a closed normal subgroup. Recalling all notations from Proposition \( \mathfrak{A} \) we let \((\pi, E) \in \text{Rep}_p(G)\) be a \( p \)-universal representation and \((\tilde{\pi}_K, K)\) be the induced \( p \)-universal representation of \( G/N \). For functions \( f \in L_1(G) \), and \( u \in B_p(G/N) \), we have

\[
\langle \pi(f), \Phi_q(u) \rangle = \langle \pi(f), u \circ q \rangle = \langle \tilde{\pi}(f), u \rangle .
\] (12)

This implies that the map \( \Phi_q \) is weak\(^*\)-weak\(^*\) continuous, and by this we define the predual map \( \Phi_q^* \), as following,

\[
* \Phi_q : P F_p, \pi \circ q(G) \to UPF_p(G/N), \quad \Phi_q(f \circ q(f)) = \tilde{\pi}(f), \quad f \in L_1(G),
\]

which by (12) we have \((* \Phi_q)^* = \Phi_q^*\). By using similar relation to (10), we have \( \tilde{\pi} \circ q(f) = \tilde{\pi}(f) \), which means that the predual map \( \Phi_q^* \) is an identity map that is \( p \)-completely isometric map via the following computation

\[
\|(* \Phi_q^* (f \circ q(f)))\|_n = \|\tilde{\pi}(P f)\|_n = \|\tilde{\pi} \circ q(f)\|_n .
\]

Therefore, we have \( \|\Phi_q\|_{p-cb} \leq 1 \).

5. Let \( G_0 \subseteq G \), be an open amenable subgroup, and \( u \in B_p(G_0) \). By Theorem \( \mathfrak{A} \) the extension map \( E_{BB} \) is well-defined,

\[
E_{BB} : B_p(G_0) \to B_p(G), \quad E_{BB}(u) = u^0 .
\]

Let \((\pi, E)\) be a \( p \)-universal representation of \( G \). We denote the restriction of \((\pi, E)\) to \( G_0 \) by \((\pi|_{G_0}, E)\), which is a \( p \)-universal representation of \( G_0 \) via Remark \( \mathfrak{A} \). We note that by the relation

\[
\langle \pi(f), u^0 \rangle = \langle \pi|_{G_0}(f|_{G_0}), u \rangle, \quad f \in L_1(G), \ u \in B_p(G_0) ,
\] (13)

the map \( E_{BB} \) is weak\(^*\)-weak\(^*\) continuous. So, we define the predual map \( E_{BB}^* \), as following:

\[
* E_{BB} : UPF_p(G) \to UPF_p(G_0), \quad * E_{BB}(\pi(f)) := \pi|_{G_0}(f|_{G_0}) ,
\]

which by (13) we have \((* E_{BB})^* = E_{BB}^*\). We need to take notice of the fact that since \( \chi_{G_0} \in B_p(G_0) \), via \( \mathfrak{A} \) Theorem 2, \( \chi_{G_0} \) is a normalized coefficient function of \((\pi, E)\), i.e. there are \( \xi, \eta \in E^* \) so that

\[
\|\xi\| = \|\eta\| = 1, \text{ and } \chi_{G_0}(x) = \langle \pi(x) \xi, \eta \rangle, \quad x \in G,
\] (14)

(or it can be directly concluded from \( \mathfrak{A} \) Theorem 1.51.)

On the other hand, for \( f \in L_1(G_0) \), \( \xi \in E \), and \( \eta \in E^* \), we have

\[
\langle \pi(f \chi_{G_0}), \xi, \eta \rangle = \int_G f(x) \chi_{G_0}(x) \langle \pi(x) \xi, \eta \rangle dx
\]

\[
= \int_G f(x) \langle \pi(x) \xi, \eta \rangle \langle \pi(x) \xi, \eta \rangle dx
\]

\[
= \int_G f(x) \langle \pi(x) \chi, \eta \rangle \langle \pi(x) \chi, \eta \rangle dx
\]

\[
= \langle \pi \circ \pi(f) \rangle \langle \xi, \eta \rangle,
\]

\[
= \langle \pi \circ \pi(f) \rangle \langle \xi, \eta \rangle,
\]
which implies that
\[ \langle \pi(f \chi_{G_0}) \xi, \eta \rangle = \langle (\pi \otimes \pi(f)) (\xi \otimes \xi), \eta \otimes \eta \rangle, \quad f \in L_1(G), \ \xi, \eta \in E^*. \tag{15} \]

Therefore, by combining equality \(15\) with \(9\), we have
\[ \langle \pi_{G_0}(f|_{G_0}) \xi, \eta \rangle = \langle (\pi \otimes \pi(f)) (\xi \otimes \xi), \eta \otimes \eta \rangle, \quad f \in L_1(G), \ \xi, \eta \in E^*. \tag{16} \]

Additionally, since \((\pi, E)\) is a p-universal representation, and we have
\[ (\pi, E) \subseteq (\pi \otimes \pi, E \tilde{\otimes}_p E), \]
thus, \((\pi \otimes \pi, E \tilde{\otimes}_p E)\) can be assumed as a p-universal of \(G\). Let
\[ \ast E_{BB}^{(n)} : \mathbb{M}_n(UPF_p(G)) \to \mathbb{M}_n(UPF_p(G_0)), \quad \ast E_{BB}^{(n)}(\pi(f_{ij})) := (\pi_{G_0}(f_{ij}|_{G_0})). \]
Then via \(16\) we have
\[
\| \ast E_{BB}^{(n)}(\pi(f_{ij})) \|_n^p = \| \| \pi_{G_0}(f_{ij}|_{G_0}) \|_n^p \\
= \sup \left\{ \left| \sum_{i,j=1}^{n} (\pi_{G_0}(f_{ij}|_{G_0}) \xi_j, \eta_i) \right| : \sum_{j=1}^{n} \| \xi_j \|_p \leq 1, \sum_{i=1}^{n} \| \eta_i \|_p' \leq 1 \right\} \\
\leq \sup \left\{ \left| \sum_{i,j=1}^{n} (\pi \otimes \pi(f_{ij})) \phi_j, \psi_i) \right| : \sum_{j=1}^{n} \| \phi_j \|_{E \tilde{\otimes}_p E} \leq 1, \sum_{i=1}^{n} \| \psi_i \|_{E^* \tilde{\otimes}_p E^*} \leq 1 \right\} \\
= \| \| (\pi \otimes \pi(f_{ij})) \|_n^p, 
\]
and since norm of \(UPF_p(G)\) is independent of choosing p-universal representation (see \(\mathbb{A}\) Theorem 4.5) then we have \(\| E_{BB} \|_{p-cb} \leq 1\), which implies that \(\| E_{BB} \|_{p-cb} \leq 1\).

6. Let \((\pi, E)\) be a p-universal representation. By \(\mathbb{A}\) Corollary 2, for \(Y \in \Omega_0(G)\) the map \(M_Y : B_p(G) \to B_p(G)\) with \(M_Y(u) = u \cdot \chi_Y\) is well-defined, and
\[ \| M_Y \| \leq 2^{mY}, \]
(or directly can be obtained from \(\mathbb{B}\) Theorem 1.5 via fussy calculations). On the other hand, by the following relation this map is weak*-weak* continuous
\[ \langle \pi(f), u \cdot \chi_Y \rangle = \langle \pi(f \cdot \chi_Y), u \rangle, \quad f \in L_1(G), \ u \in B_p(G). \tag{17} \]
So, one may define its predual map as following
\[ \ast M_Y : UPF_p(G) \to UPF_p(G), \quad \ast M_Y(\pi(f)) = \pi(f \cdot \chi_Y), \]
and by \(17\) we have \((\ast M_Y)^* = M_Y\).
Step 1: To prove the claim, first we let $Y$ be an open coset itself. Similar to (14), the function $\chi_Y$ is a normalized coefficient function of the representation $(\pi, E)$ which means that there are elements $\xi_Y \in E$, and $\eta_Y \in E^*$ such that

$$\|\xi_Y\| = \|\eta_Y\| = 1, \quad \text{and} \quad \chi_Y(x) = \langle \pi(x) \xi_Y, \eta_Y \rangle, \quad x \in G.$$ 

So, for a matrix $[\pi(f_{ij})] \in M_n(\text{UPF}_p(G))$, through the relation (15), we have

$$\|\pi(f_{ij} \cdot \chi_Y)\|_n = \sup \left\{ \left| \sum_{i,j=1}^n \langle \pi(f_{ij} \cdot \chi_Y) \xi_j, \eta_i \rangle : \sum_{j=1}^n \|\xi_j\|_p^p \leq 1, \sum_{i=1}^n \|\eta_i\|_p^p \leq 1 \right| \right\} \leq \sup \left\{ \left| \sum_{i,j=1}^n \langle \pi \otimes \pi(f_{ij}) \phi_j, \psi_i \rangle : \sum_{j=1}^n \|\phi_j\|_{E_p \otimes E_p} \leq 1, \sum_{i=1}^n \|\psi_i\|_{E^* \otimes E^*} \leq 1 \right| \right\}$$

$$= \|\pi(f_{ij})\|_n.$$ 

By these computations, we obtain that the map $M_Y$ is a $p$-complete contraction. Therefore, we have $\|M_Y\|_{p-cb} \leq 1$, through the fact that $p$-operator norm of UPF$_p(G)$ is independent of choosing $p$-universal representation.

Step 2: Now let $Y = Y_0 \setminus \bigcup_{i=1}^m Y_i \in \Omega_0(G)$, and we have,

$$M_Y = M_{Y_0} - \sum_{i=1}^m M_{Y_i} - \sum_{i,j} M_{Y_i \cap Y_j} + \sum_{i,j,k} M_{Y_i \cap Y_j \cap Y_k} + \ldots + (-1)^{m+1} M_{Y_1 \cap \ldots \cap Y_m}.$$ 

Therefore, we have $\|M_Y\|_{p-cb} \leq 2^m Y$.

Remark 9. The importance of Theorem 5-(1) is that while we are working with maps with ranges as subspaces of the $p$-analog of the Fourier-Stieltjes algebras, we just need to restrict ourselves to their ranges, as what we have done in the rest of Theorem 5. Another proof of this part can also be derived via Proposition 2.

5 $p$-Completely homomorphisms on $B_p(G)$ induced by proper piecewise affine maps

As an application of previous sections, we are ready to study on homomorphisms $\Phi_\alpha : B_p(G) \to B_p(H)$ of the form

$$\Phi_\alpha u = \begin{cases} u \circ \alpha \text{ on } Y & u \in B_p(G), \\ 0 & \text{otherwise} \end{cases}$$

for the proper and continuous piecewise affine map $\alpha : Y \subseteq H \to G$ with $Y = \bigcup_{i=1}^n Y_i$ and $Y_i \in \Omega_{am-0}(H)$, which are pairwise disjoint, for $i = 1, \ldots, n$. We will give some results in the sequel, and in this regard, we need the following lemma. For general form of this lemma, see [20, Lemma 1], and related references there, e.g. [10].
Lemma 4. Let $G$ and $H$ be locally compact groups and $\alpha : H \to G$ be a proper homomorphism that is onto, then there is a constant $c_\alpha > 0$, such that
\[
\int_H f \circ \alpha(h)dh = c_\alpha \int_G f(x)dx, \quad f \in L_1(G).
\]

Proposition 7. Let $G$ and $H$ be locally compact groups and $\alpha : H \to G$ be a proper continuous group homomorphism. Then the homomorphism $\Phi_\alpha : B_p(G) \to B_p(H)$, of the form $\Phi_\alpha(u) = u \circ \alpha$, is well-defined and $p$-completely contractive homomorphism.

Proof. Let $(\pi, E)$ be a $p$-universal representation of $G$. Obviously, $(\pi \circ \alpha, E) \in \text{Rep}_p(H)$, and $\Phi_\alpha$ is a contractive homomorphism so that its range is the subspace of $B_p(H)$ of functions which are coefficient functions of the representation $(\pi \circ \alpha, E)$. We divide our proof into two steps.

Step 1: First, we suppose that $\alpha : H \to G$ is a continuous isomorphism. In this case, $(\pi \circ \alpha, E)$ is a $p$-universal representation of $H$, and by Lemma 4, for every $f \in L_1(H)$ and $u \in B_p(G)$, we have
\[
\langle \pi \circ \alpha(f), u \circ \alpha \rangle = \int_H f(h)u \circ \alpha(h)dh
= \int_H (f \circ \alpha^{-1}) \circ \alpha(h)u \circ \alpha(h)dh
= c_\alpha \int_G f \circ \alpha^{-1}(x)u(x)dx
= c_\alpha \langle \pi(f \circ \alpha^{-1}), u \rangle.
\]
Consequently, the map $\Phi_\alpha$ is weak*-weak* continuous, and we define
\[
\Phi_\alpha : UPF_p(H) \to UPF_p(G), \quad \Phi_\alpha(\pi \circ \alpha(f)) := c_\alpha \pi(f \circ \alpha^{-1}).
\]
According to the above relation, we have $(\Phi_\alpha)^* = \Phi_\alpha$. On the other hand, for every $\xi \in E$ and $\eta \in E^*$, we have
\[
\langle \pi \circ \alpha(f)\xi, \eta \rangle = \int_H f(h)\langle \pi \circ \alpha(h)\xi, \eta \rangle dh
= \int_H f \circ \alpha^{-1} \circ \alpha(h)\langle \pi \circ \alpha(h)\xi, \eta \rangle dh
= c_\alpha \int_G f \circ \alpha^{-1}(x)\langle \pi(x)\xi, \eta \rangle dx
= \langle c_\alpha \pi(f \circ \alpha^{-1})\xi, \eta \rangle,
\]
which means $\pi \circ \alpha(f) = c_\alpha \pi(f \circ \alpha^{-1})$. Consequently, $\Phi_\alpha$ is an identity map, so is a $p$-complete isometric map,
\[
\|\Phi_\alpha^n(\pi \circ \alpha(f_{i,j}))\|_n = \|(c_\alpha \pi(f_{i,j} \circ \alpha^{-1}))\|_n = \|(\pi \circ \alpha(f_{i,j}))\|_n.
\]
Therefore, $\|\Phi_\alpha\|_{cb} \leq \|\Phi_\alpha\|_{cb} = 1$. 

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Step 2: Now let $\alpha : H \to G$ be any proper continuous homomorphism. Let $G_0 = \alpha(H)$, and $N = \ker \alpha$. Let us define

$$\tilde{\alpha} : H/N \to G_0, \quad \tilde{\alpha}(xN) = \alpha(x), \quad x \in G,$$

then by Proposition 7, the map $\tilde{\alpha}$ is a continuous isomorphism, $N$ is a compact normal subgroup of $H$, and $G_0$ is an open subgroup of $G$. Therefore, $\alpha = \tilde{\alpha} \circ q$. By Step 1, the map $\Phi_{\tilde{\alpha}}$ is $p$-completely contractive, and because of the following composition, $\Phi_{\alpha}$ is $p$-completely contractive, via Theorem 5-(2)-(4).

$$\Phi_{\alpha} = \Phi_q \circ \Phi_{\tilde{\alpha}} \circ R.$$

For the next proposition, we have to put the amenability assumption on the subgroups of $H$, because of Theorem 4.

**Proposition 8.** Let $G$ and $H$ be two locally compact groups, $Y$ be an open coset of an open amenable subgroup of $H$, and $\alpha : Y \subseteq H \to G$ be a continuous proper affine map. Then the map $\Phi_{\alpha} : B_p(G) \to B_p(H)$, defined as

$$\Phi_{\alpha}(u) = \begin{cases} u \circ \alpha, & \text{on } Y, \\ 0, & \text{o.w.} \end{cases}, \quad u \in B_p(G), \quad (18)$$

is $p$-completely contractive. More generally, if $\alpha$ is a continuous proper piecewise affine map, and $Y = \bigcup_{i=1}^n Y_i$, where disjoint sets $Y_i$ belong to $\Omega_{am-0}(H)$, then the map $\Phi_{\alpha}$ is $p$-completely bounded.

**Proof.** In [2, Theorem 3], which is a consequence of Theorem 4, it has been proven that the map $(18)$ is well-defined and it is contractive when the map $\alpha$ is affine on an open coset $Y$ of an open amenable subgroup $H_0$ of $H$, and is bounded in the case that the map $\alpha$ is piecewise affine on the set $Y = \bigcup_{i=1}^n Y_i$, with disjoint $Y_i \in \Omega_{am}(H)$, for $i = 1, \ldots, n$, of the form $Y_i = Y_{i,0} \cup \bigcup_{j=1}^{m_i} Y_{ij}^m$.

Let $\alpha : Y = y_0 H_0 \to G$ be a continuous proper affine map on the open coset $Y = y_0 H_0$, and $H_0$ be an open amenable subgroup of $H$, for which by Remark 7-[4], there exists a continuous group homomorphism $\beta : H_0 \subseteq H \to G$ associated to $\alpha$ such that

$$\beta(h) = \alpha(y_0)^{-1} \alpha(y_0 h), \quad h \in H_0,$$

which is proper via Remark 7-[4]. Now, consider the following composition

$$\Phi_{\alpha} = L_{\alpha(y_0)^{-1}} \circ E_{BB} \circ \Phi_\beta \circ L_\alpha(y_0),$$

then by Proposition 7 and Theorem 5-[3]-[4], in place, the map $\Phi_{\alpha}$ is $p$-completely contractive homomorphism.

Next, we consider the piecewise affine case. Let the map $\alpha : Y \subseteq H \to G$ be a proper and continuous piecewise affine map. Then for some $n \in \mathbb{N}$, and $i = 1, \ldots, n$, there are disjoint sets $Y_i \in \Omega_{am-0}(H)$, such that $Y = \bigcup_{i=1}^n Y_i$, and
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for $i = 1, \ldots, n$, the map $\alpha_i : \text{Aff}(Y_i) \to G$ is an affine map, together with the fact that $\alpha_i|_{Y_i} = \alpha|_{Y_i}$. On top of that, by Remark 7-5, each affine map $\alpha_i$ is proper. Therefore, by considering

$$\Phi_\alpha = \sum_{i=1}^n M_{Y_i} \circ \Phi_{\alpha_i},$$

and through the above computations for the maps $\Phi_{\alpha_i}$, we have

$$\|\Phi_\alpha\|_{p-cb} \leq \sum_{i=1}^n 2^{m_{Y_i}}.$$

Here the value $m_{Y_i}$ is the corresponding number to each $Y_i$, as it is in Theorem 3.6.

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