TVERBERG’S THEOREM AND GRAPH COLORING
ALEXANDER ENGSTRÖM AND PATRIK NORÉN

Abstract. The topological Tverberg theorem have been generalized in several directions by setting extra restrictions on the Tverberg partitions. This was initiated by Vrećica and Zivaljević who used the chessboard complexes studied by them together with Björner and Lovász. They were motivated both by combinatorial applications of a colored Tverberg theorem and the enumeration of Tverberg partitions.

Restricted Tverberg partitions defined from that certain points cannot be in the same part, are encoded with graphs. When two points are adjacent in the graph, they are not in the same part. If the restrictions are too harsh, then the topological Tverberg theorem fails. The colored Tverberg theorem corresponds to graphs constructed as disjoint unions of small complete graphs. Hell studied the case of paths and cycles.

In graph theory these partitions are usually viewed as graph colorings. As explored by Aharoni, Haxell, Meshulam and others there are fundamental connections between several notions of graph colorings and topological combinatorics.

For ordinary graph colorings it is enough to require that the number of colors $q$ satisfy $q > \Delta$, where $\Delta$ is the maximal degree of the graph. It was proven by the first author using equivariant topology, that if $q > \Delta^2$ then the topological Tverberg theorem still works. It is conjectured that $q > K\Delta$ is also enough for some constant $K$, and in this paper we prove a fixed-parameter version of that conjecture.

The required topological connectivity results are proven with shellability, which also strengthens some previous partial results where the topological connectivity was proven with the nerve lemma.

1. Introduction

Tverberg’s theorem [16] asserts that for any affine map $f$ from a simplex on $(d + 1)(q - 1) + 1$ vertices to $\mathbb{R}^d$ there is a partition of the vertices into $q$ parts such that

$$\bigcap_{i=1}^{q} f\text{(simplex spanned by part } i) \neq \emptyset.$$ 

It was generalized by Bárány, Schlosman and Szűcs [1] to continuous $f$, but then the equivariant topology used in the proof requires $q$ to be a prime. Later this was modified to $q$ a prime power by Özaydin [14] (unpublished) and Volovikov [20]. The topological version was later extended to several versions where more conditions on the partition are required [11, 12, 15, 18, 19], and there are several exciting new directions [2, 3, 4, 9, 10].

The extra conditions can be stated in forms of colorings in different ways, and in this article we focus on the most general one: The partition of the points in Tverberg’s theorem is considered a coloring where every color correspond to a part. The known results at that time, all followed from this local condition:
Theorem (5). Let $G$ be a graph with $(d + 1)(q - 1) + 1$ vertices and $q$ a prime-power satisfying

$$q > \max_{v \in V(G)}(|N^2(v)| + 2|N(v)|)$$

where $N^2(v)$ is the set of vertices on distance two from $v$ and $N(v)$ is the set of vertices adjacent to $v$. Then for any continuous map $f$ from a simplex with the same vertex set as $G$ to $\mathbb{R}^d$ there is a $q$-coloring of $G$ such that

$$\bigcap_{i=1}^q f(\text{simplex spanned by color } i) \neq \emptyset.$$ 

The equivariant topology used to prove that theorem builds on that certain spaces are enough topologically connected. That was proven by topological methods, as the nerve lemma, in [8]. But the question was raised, if, as was done for the chessboard complexes by Zieger [21], this could be proven by vertex decomposability and shellability. We prove that this is possible in Corollary 2.11.

With the previous known versions of Tverberg’s theorem the following natural conjecture was made in [8].

Conjecture. There is a constant $K$ such that the following holds: Let $G$ be a graph on $(d + 1)(q - 1) + 1$ vertices and maximal degree $\Delta$, and let $f$ be a continuous map from a simplex $\Sigma$ with the same vertex set as $G$ to $\mathbb{R}^d$. If

$$q > K\Delta$$

then there is a $q$-coloring of $G$ satisfying

$$\bigcap_{i=1}^q f(\text{simplex spanned by color } i) \neq \emptyset.$$ 

The emeritus of the field, Helge Tverberg, believes in the conjecture [17]. In Corollary 3.4 we prove the following fixed-parameter version of it.

Theorem. For every $\varepsilon > 0$ there exists a constant $K_\varepsilon$ such that the following holds: Let $G$ be a graph on $((d + 1)(q - 1) + 1)(1 + \varepsilon)$ vertices and maximal degree $\Delta$ (with $d$ and $\Delta$ large enough depending on $\varepsilon$), and let $f$ be a continuous map from a simplex $\Sigma$ with the same vertex set as $G$ to $\mathbb{R}^d$. If

$$q > K_\varepsilon\Delta$$

then there is a $q$-coloring of $G$ satisfying

$$\bigcap_{i=1}^q f(\text{simplex spanned by color } i) \neq \emptyset.$$ 

The crucial statements in equivariant topology of Section 3 builds on graphs being vertex decomposable. In Section 2 we introduce this concept and prove some fairly technical statements about it. We’ve made an effort to make Section 2 completely independent and only about graph theory, allowing experts in this field to improve on our results without a deep understanding of the equivariant topology used in Section 3.
1.1. Some notation. The neighborhood $N^2_G(v)$ in a graph $G$ of a vertex $v$ is the set of vertices of $G$ adjacent to $v$; and $N^*_G(v) = N^2_G(v) \cup \{v\}$. The vertices on distance two from $v$ in $G$, $N^2_G(v)$, are all vertices $u$ with a path on two edges to $v$. Usually we drop the $G$ subscript if the graph containment is clear.

2. Decomposing skeletons

In topological combinatorics a central notion is shellability. A simplicial complex is shellable if its facets can be peeled off in a controlled manner, providing a certificate that the space topologically is a collection of equidimensional spheres wedged together at a point. One method to prove a complex shellable is by the stronger notion of vertex decomposable. For simplicial complexes determined by graphs, we introduce a filtrated version of vertex decomposable right off on the level of graphs, and then return to its topological interpretation and consequences in Section 3.

Definition 2.1. For every non-negative integer $k$ we define the graph property $\mathcal{V}D_k$. Any graph $G$ is $\mathcal{V}D_0$, and a graph $G$ on $k$ vertices and no edges is $\mathcal{V}D_k$. If $G$ is a graph with a vertex $v$ such that $G \setminus v$ is $\mathcal{V}D_k$ and $G \setminus N^*(v)$ is $\mathcal{V}D_{k-1}$, then $G$ is $\mathcal{V}D_k$.

Note that if $G$ is $\mathcal{V}D_k$ and $k \geq l \geq 0$ then $G$ is also $\mathcal{V}D_l$.

Remark. In Proposition 3.2 in Section 3 it will be proven that the $(k - 1)$-skeleton of the independence complex of $G$ is pure $(k - 1)$-dimensional and vertex decomposable if $G$ is $\mathcal{V}D_k$.

The cartesian product of two graphs $G$ and $H$, denoted $G \square H$, is the graph with vertex set $V(G) \times V(H)$ and edge set

$\{(u, v)(u', v') \mid uu' \in E(G), v \in V(H)\} \cup \{(u, v)(u', v') \mid u \in V(G), vv' \in E(H)\}$.

Our goal in preparation of Section 3 and the equivariant topology, is to prove that cartesian products $G \square K_q$ are $\mathcal{V}D_k$ for as high $k$ as possible. There is a procedure for general $G$ that is not strong enough for the products of interest, but since our approach builds on it, we explain it. First we need a lemma that in the simplicial complex setting is due to Ziegler [21].

Lemma 2.2. If $G$ has an isolated vertex $v$ and $G \setminus v$ is $\mathcal{V}D_{k-1}$, then $G$ is $\mathcal{V}D_k$.

Proof. We prove this by induction on the number of vertices of $G$. If $G$ has only isolated vertices then we are done.

Otherwise there is a vertex $u$ of $G \setminus v$ verifying that it is $\mathcal{V}D_{k-1}$. By definition $(G \setminus v) \setminus u = (G \setminus u) \setminus v$ is $\mathcal{V}D_{k-1}$, and $(G \setminus v) \setminus N^*_G(u) = (G \setminus N^*_G(u)) \setminus v$ is $\mathcal{V}D_{k-2}$. By induction $G \setminus u$ is $\mathcal{V}D_k$ and $G \setminus N^*_G(u)$ is $\mathcal{V}D_{k-1}$ since $v$ is an isolated vertex of them, and this shows that $G$ is $\mathcal{V}D_k$. □

Lemma 2.2 indicates that one way to recursively prove that a graph is $\mathcal{V}D_k$ for a non-trivial $k$, is to turn vertices isolated by removing their adjacent vertices, and then increase $k$ by applying Lemma 2.2. Here is one way to formalize that.
Lemma 2.3. Let $G$ be a graph with a vertex $v$ whose neighborhood is $N(v) = \{u_1, u_2, \ldots, u_n\}$. If $G \setminus N^\circ(v)$ and

$$G \setminus (N^\circ(u_i) \cup \{u_1, u_2, \ldots, u_{i-1}\})$$

for $1 \leq i \leq n$ are $\mathcal{VD}_{k-1}$, then $G$ is $\mathcal{VD}_{k}$.

Proof. From Lemma 2.2 and that $G \setminus N^\circ(v)$ is $\mathcal{VD}_{k-1}$ we get that $G \setminus N^\circ(v) = G \setminus \{u_1, u_2, \ldots, u_n\}$ is $\mathcal{VD}_{k}$.

For $i = n, n-1, \ldots, 2, 1$, use Definition 2.1 on $G \setminus \{u_1, u_2, \ldots, u_{i-1}\}$ with the vertex $u_i$. It follows that $G \setminus \{u_1, u_2, \ldots, u_{i-1}\}$ is $\mathcal{VD}_{k}$ from that $(G \setminus \{u_1, u_2, \ldots, u_{i-1}\}) \setminus u_i = G \setminus \{u_1, u_2, \ldots, u_i\}$ is $\mathcal{VD}_{k}$ and $(G \setminus \{u_1, u_2, \ldots, u_{i-1}\}) \setminus N^\circ (u_i) = G \setminus (N^\circ(u_i) \cup \{u_1, u_2, \ldots, u_{i-1}\})$ is $\mathcal{VD}_{k-1}$.

With the last step of $i = 1$ we got that $G \setminus \{u_1, u_2, \ldots, u_{i-1}\} = G$ is $\mathcal{VD}_{k}$. □

For generic graphs, avoiding global structures as in cartesian products, the following proposition is efficient.

Proposition 2.4 (Dochtermann & Engström [6], Theorem 5.9). Let $G$ be a graph on $n$ vertices and maximal degree $\Delta > 0$. Then $G$ is $\mathcal{VD}_{\lfloor n/2\Delta \rfloor}$.

Proof. We do induction on the number of vertices. If $n < 2\Delta$ then the statement is true since all graphs are $\mathcal{VD}_0$.

If $n \geq 2\Delta$ then fix some vertex $v$ of $G$ with neighborhood $N^\circ(v) = \{u_1, u_2, \ldots, u_m\}$. Now consider the following subgraphs: $G \setminus N^\circ(v)$ and $G \setminus (N^\circ(u_i) \cup \{u_1, u_2, \ldots, u_{i-1}\})$ for $1 \leq i \leq m$. All of them have less vertices than $G$, but the difference is at most $2\Delta$ vertices. Thus by induction, and by that the maximal degree never increases by taking subgraphs, all of them are $\mathcal{VD}_{\lfloor n/2\Delta \rfloor - 1}$. By Lemma 2.3, the graph $G$ is $\mathcal{VD}_{\lfloor n/2\Delta \rfloor}$. □

In Engström [8] a much weaker version of our main theorems was proved by removing squids. Our approach follows this idea, but is much more technically involved. To begin with we define a class of algorithms to remove squids, called $DF$-algorithms. Then we prove that any $DF$-algorithm provides certificates that graphs are of the right $\mathcal{VD}_{k}$ class.

But first we define squids.

Definition 2.5. A squid with body $w$ in $G \Box K_q$ is a subset of $V(G \Box K_q)$ that is either

(i) a subset of

$$(N^\circ_G(v) \cup N^\circ_G(w)) \times \{i\} \cup \{w\} \times \{1, 2, \ldots, q\}$$

for two adjacent vertices $v$ and $w$, and $1 \leq i \leq q$, or

(ii) a subset of

$$N^\circ_G(w) \times \{i, j\} \cup \{w\} \times \{1, 2, \ldots, q\}$$

where $1 \leq i < j \leq q$.

The vertices not of the form $(w, k)$ are arms. The heart of a squid of type (i) is $(w, i)$ and the hearts of a squid of type (ii) are $(w, i)$ and $(w, j)$.

The heart, body and head is part of the squid data, and two squids could be on the same subset of $V(G \Box K_q)$ but differ in that regard. If $S$ is a squid, then we also use the symbol $S$ for the subset of $V(G \Box K_q)$ in set theoretic statements if no confusion occurs.

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1 Since this is a math paper we allow squids with less than three hearts.
An instance of squids removed from a cartesian product \( G \square K_q \) is modeled as a DF-tuple.

**Definition 2.6.** A DF-tuple is a five tuple \( (G, q, j, \{S_1, S_2, \ldots, S_j\}, m) \) consisting of

1. a finite graph \( G \) with vertices in \( \mathbb{N} \);
2. integers \( |G| \geq m \geq j \geq 0 \), and \( q > 0 \); and
3. squids \( S_1, S_2, \ldots, S_j \) in \( G \square K_q \).

In a DF-tuple \( (G, q, j, \{S_1, S_2, \ldots, S_j\}, m) \) at most \( j \) squids have been removed from \( G \square K_q \). A DF-algorithm is a collection of DF-tuples with an instruction for how to remove one more squid if \( j < m \). The squid to be removed is defined by a map from the collection of tuples into itself.

**Definition 2.7.** A DF-algorithm \((A, G)\) is a set \( G \) of DF-tuples, and a map

\[
A : \{(G, q, j, \{S_1, S_2, \ldots, S_j\}, m) \in G | j < m \} \rightarrow \mathbb{N} \times \mathbb{N}
\]

that for any \( T = (G, q, j, \{S_1, S_2, \ldots, S_j\}, m) \in G \) with \( j < m \), satisfies

1. \((v, i) := A(T) \in V(H)\) where \( H = G \square K_q \setminus \bigcup_{i=1}^j S_i\), and
2. if
   a. \( S \subseteq N^o_H(v, i) \cup N^o_H(v, j) \) for some \( (v, j) \in H \), or
   b. \( S \subseteq (N^o_H(v, i) \cap G \{i\}) \cup N^o_H(u, i) \) for some \( u \in N^o_G(v) \) with \((u, i) \in H\),
then \((G, q, j + 1, \{S_1, S_2, \ldots, S_j, S\}, m) \in G\).

After setting up the definitions and notations for removing squids with DF-algorithms, we now prove that they certify that the relevant cartesian products are \( \text{VD}_k \).

**Theorem 2.8.** Let \((A, G)\) be a DF-algorithm. If \((G, q, j, \{S_1, S_2, \ldots, S_j\}, m) \) is in \( G \) then \( G \square K_q \setminus \bigcup_{i=1}^j S_i \) is \( \text{VD}_{(m-j)} \).

**Proof.** Set \( H = G \square K_q \setminus \bigcup_{i=1}^j S_i \). The proof is by induction on \( m - j \). The base case \( m = j \), that \( H \) is \( \text{VD}_0 \), follows from Definition 2.1.

Now assume that \( m > j \) and set \((v, i) = A((G, q, j, \{S_1, S_2, \ldots, S_j\}, m))\). The neighbors of \((v, i)\) in \( H \) are either in \( G \times \{i\} \) or in \( \{v\} \times K_q \). Order them

\[
N^o_H(v, i) = \{(u_1, j_1), (u_2, j_2), \ldots, (u_n, j_n)\}
\]
such that \( u_1 = u_2 = \cdots u_k = v \) and \( j_{k+1} = j_{k+2} = \cdots j_n = i \) for some \( k \).

For \( l = 1, 2, \ldots, n \) define squids

\[
S'_l = N^o_H(u_l, j_l) \cup \{(u_1, j_1), (u_2, j_2), \ldots, (u_l, j_l)\}
\]
and note that \((G, q, j + 1, \{S_1, S_2, \ldots, S_j, S'_l\}, m) \) is in \( G \)

by (2.b) in Definition 2.7 for \( 1 \leq l \leq k \), and
by (2.a) in Definition 2.7 for \( k < l \leq n \).

Define one more squid \( S''_l = N^o_H(v, i) \cap G \times \{i\} \) and note that \((G, q, j + 1, \{S_1, S_2, \ldots, S_j, S''_l\}, m) \) is in \( G \) by (2.a) in Definition 2.7.

By induction, \( H \setminus S''_l \) for \( 1 \leq l \leq n \), are \( \text{VD}_{(m-j-1)} \). We can now conclude by Lemma 2.3 that \( H = G \square K_q \setminus \bigcup_{i=1}^j S_i \) is \( \text{VD}_{(m-j)} \).

**Corollary 2.9.** Let \((A, G)\) be a DF-algorithm. If \((G, q, \emptyset, m) \in G \) then \( G \square K_q \) is \( \text{VD}_m \).

**Proof.** This is a special case of Theorem 2.8. □
2.1. Two DF-algorithms. We now introduce two DF-algorithms. Using the first one, we later show the same Tverberg type results as in Engström [8], but employ only the combinatorial topology of shellability instead of stronger abstract tools from algebraic topology. Proving this conjecture from [8] gives a result in the same spirit as Ziegler’s [21], where he proved that the optimal connectivity bounds of chessboard complexes can be proved by shelling skeletons of chessboard complexes.

Theorem 2.10. Let \( G \) be the set of DF-tuples \((G, q, j, \{S_1, S_2, \ldots, S_j\}, m)\) such that

1. the squids \( S_1, S_2, \ldots, S_j \) have bodies \( s_1, s_2, \ldots, s_j \) and satisfy
   \[
   (G \square K_q \setminus \bigcup_{i=1}^k S_i) \cap (\{s_k\} \times K_q) = \emptyset
   \]
   for all \( 1 \leq k \leq j \),
2. and \( q > \left| N^2(v) \right| + 2\left| N^0(v) \right| \) for all vertices \( v \) of \( G \).

Then \( G \square K_q \setminus \bigcup_{i=1}^k S_j \) is non-empty if \( j < m \) and any map

\[
\mathbf{A} : \{(G, q, j, \{S_1, S_2, \ldots, S_j\}, m) \in G| j < m \} \rightarrow \mathbb{N} \times \mathbb{N}
\]

sending \((G, q, j, \{S_1, S_2, \ldots, S_j\}, m)\) to any vertex of \( G \square K_q \setminus \bigcup_{i=1}^k S_j \) defines a DF-algorithm \((\mathbf{A}, G)\).

Proof. We only have to prove that \( H = G \square K_q \setminus \bigcup_{i=1}^k S_j \) is non-empty if \( j < m \), and then it follows right off by the definition that this gives DF-algorithms.

By assumption \( j < m \), and there is a \( s \in V(G) \setminus \{s_1, s_2, \ldots, s_j\} \). We claim that \( s \square K_q \cap H \) is non-empty. If it was empty, it should have been deleted by arms of squids. There are two types of squids: Those with to sets of arms reaching only neighbors (those in \( N(v) \)), and those with one set of arms reaching neighbors of neighbors (those in \( N^2(v) \)). Since \( q > \left| N^2(v) \right| + 2\left| N(v) \right| \) there are not enough arms to delete all of \( s \square K_q \), and we are done. \( \square \)

Corollary 2.11. Let \( q \) be an integer and \( G \) a graph on \( m \) vertices with \( q > \left| N^2(v) \right| + 2\left| N(v) \right| \) for all vertices \( v \). Then \( G \square K_q \) is \( \mathcal{VD}_m \).

Proof. This follows directly from Corollary 2.9 and Theorem 2.10 \( \square \)

To prove the second main theorem of this paper, we need a more dynamic way to remove squids. We will use the following strategy to remove squids from \( G \square K_q \): We first remove \( s_1 \) squids with hearts on the top row \( G \times r_1 \) where \( r_1 = 1 \). The removal of these squids will have different effect on the rows \( G \times j \) with \( j > 1 \). If a large number of squids have arms also on row \( G \times j \), then this row is a bad choice for continuing the removal of squids from. So the next step is to let \( r_2 \) be the top-most row with the most number of preserved vertices. We remove \( s_2 \) squids with hearts on the row \( G \times r_2 \) and proceed in the same manner, until \( s_1 + s_2 + \cdots + s_k \) is large enough. To ensure that we simply don’t run out of vertices, the sizes \( s_i \) are specified with a dynamic DF-size scheme.

Definition 2.12. A sequence \((s_1, s_2, \ldots, s_k)\) is a dynamic DF-size scheme given the positive integers \( n, q, \Delta \) if

1. \( q \geq k > 0 \) and all \( s_i > 0 \); and
Theorem 2.13. For every \( \varepsilon > 0 \) there exists a constant \( K_\varepsilon \) such that for every graph \( G \) with \( N(1 + \varepsilon) \) vertices (with \( N \) and \( \Delta \) large enough depending on \( \varepsilon \)) and

\[
q > K_\varepsilon \Delta,
\]

there is a dynamic DF-size scheme \((s_1, s_2, \ldots, s_k)\) with \( n = N(1 + \varepsilon) \) and \( N \leq \sum_{i=1}^{k} s_i \).

Proof. We only need asymptotic estimates and disregard that several of the variables should be integers. To satisfy (2) of Definition 2.12 we prove that

\[
a \left( \sum_{i=1}^{j-1} s_i \right) + 2\Delta s_j \leq n = N(1 + \varepsilon)
\]

for some \( a \) when the \( s_j \) are defined properly. To satisfy this inequality, with equality for all \( j \), we set

\[
s_j = \frac{N(1 + \varepsilon)}{2\Delta} \left( \frac{2\Delta - a}{2\Delta} \right)^{j-1}.
\]

Now set \( k = 2\Delta \gamma \) and \( a = \sqrt{1 + \varepsilon} \) in

\[
\sum_{j=1}^{k} s_j = \frac{N(1 + \varepsilon)}{2\Delta} \frac{1 - \left( \frac{2\Delta - a}{2\Delta} \right)^k}{1 - \left( \frac{2\Delta - a}{2\Delta} \right)}
\]

\[
= \frac{N(1 + \varepsilon)}{2\Delta} \frac{1 - \left( \frac{2\Delta - a}{2\Delta} \right)^k}{a}
\]

\[
= N \sqrt{1 + \varepsilon} \left( 1 - \left( 1 - \frac{\gamma \sqrt{1 + \varepsilon}}{2\Delta \gamma} \right)^{2\Delta \gamma} \right)
\]

\[
\geq N \sqrt{1 + \varepsilon} \left( 1 - e^{-\gamma \sqrt{1 + \varepsilon}} \right)
\]

\[
= N
\]

with \( \gamma = -\frac{1}{\sqrt{1 + \varepsilon}} \ln \left( 1 - \frac{1}{\sqrt{1 + \varepsilon}} \right) \). Finally, the variable \( a \) should satisfy

\[
\sqrt{1 + \varepsilon} = a = 1 + \frac{\Delta}{q - k} = 1 + \frac{1}{K_\varepsilon - 2\gamma}
\]

and we set

\[
K_\varepsilon = \sqrt{1 + \varepsilon} - 1 + 2\gamma = \sqrt{1 + \varepsilon} - 1 - \frac{2}{\sqrt{1 + \varepsilon}} \ln \left( 1 - \frac{1}{\sqrt{1 + \varepsilon}} \right).
\]

Now we describe how to get a DF-algorithm from a dynamic DF-size scheme.

**Definition 2.14.** Given a graph \( G \) with vertices in \( N \) of maximal degree \( \Delta \), and a dynamic DF-size scheme \((s_1, s_2, \ldots, s_k)\) with \( n, q \); the dynamic DF-scheme is the set \( \mathcal{G} \) of DF-tuples \((G, q, j, \{S_1, S_2, \ldots, S_j\}, n)\) such that:
for each \( l \) with \( s_1 + s_2 + \ldots + s_{l-1} \leq j \) all the squids
\[
S_{s_1+s_2+\ldots+s_{j-1}+1}, \ldots, S_{\max s_1+s_2+\ldots+s_{j},j}
\]
have hearts on the same row \( G \times r_1 \),
• all the \( r_j \) are different,
• when the squids with hearts on rows \( G \times r_1, G \times r_2, \ldots, G \times r_k \) are deleted, then \( G \times r_1 \) is the top most row with maximal number of preserved vertices.

together with a map
\[
A : \{(G, q, j, \{S_1, S_2, \ldots, S_j, \}, n) \in \mathcal{G}| q < n \} \to \mathbb{N} \times \mathbb{N}
\]
defined as the vertex \( (v, r_i) \in G \square K_q \setminus (S_1 \cup S_2 \cup \cdots \cup S_j) \) for which \( s_1 + s_2 + \cdots + s_{i-1} < j \leq s_1 + s_2 + \cdots + s_i \) and
\[
v = \min(u \ | \ (u, r_i) \in G \square K_q \setminus (S_1 \cup S_2 \cup \cdots \cup S_j)).
\]

A dynamic DF-scheme is a DF-algorithm, since the dynamic DF-scheme guarantees
\[
\{u \ | \ (u, r_i) \in G \square K_q \setminus (S_1 \cup S_2 \cup \cdots \cup S_j)\}
\]
to be non-empty.

**Corollary 2.15.** For every \( \varepsilon > 0 \) there exists a constant \( K_\varepsilon \) such that for every graph \( G \) with \( N(1 + \varepsilon) \) vertices (with \( N \) and \( \Delta \) large enough depending on \( \varepsilon \)) \( G \square K_q \) is \( \mathcal{VD}_N^q \) if \( q > K_\varepsilon \Delta \).

**Proof.** This follows directly from Corollary 2.9, Theorem 2.13 and Definition 2.14. \( \square \)

### 3. Equivariant Topology

In this section we will use the facts about vertex decomposable graphs derived in Section 2 to derive new theorems of Tverberg type. Recall that a set of vertices of a graph \( G \) is independent if none of them are adjacent. The *independence complex* of a graph \( G \), denoted \( \text{Ind}(G) \), is the simplicial complex on the same vertex set as \( G \) whose faces are the independent sets of \( G \). For basic combinatorial topology we refer to Björner’s excellent survey [5], but we collect a few useful facts. The link of a vertex \( v \) of \( \Sigma \) is \( \text{lk}_\Sigma(v) = \{\sigma \in \Sigma \ | \ v \notin \sigma, \ \sigma \cup \{v\} \in \Sigma\} \), and the deletion of \( v \) is \( \text{dl}_\Sigma(v) = \Sigma \setminus v = \{\sigma \in \Sigma \ | \ v \notin \sigma\} \).

For independence complexes \( \text{lk}_{\text{Ind}(G)}(v) = \text{Ind}(G \setminus N^*(v)) \) and \( \text{dl}_{\text{Ind}(G)}(v) = \text{Ind}(G \setminus v) \).

A more comprehensive introduction to basic operations on independence complexes is given in [7]. The \( k \)-skeleton of \( \Sigma \) is \( \Sigma^{\leq k} = \{\sigma \in \Sigma \ | \ \dim \sigma \leq k\} \), and an easy exercise is \( \text{lk}_{\Sigma}^{\leq k}(v) = \text{lk}_{\Sigma^{\leq k}}(v) \) and \( \text{dl}_{\Sigma}^{\leq k}(v) = \text{dl}_{\Sigma^{\leq k}}(v) \).

**Definition 3.1.** A simplicial complex \( \Sigma \) is *vertex decomposable* if it is pure, and either \( \Sigma = \{\emptyset\} \) or it has a vertex \( v \) with \( \text{lk}_\Sigma(v) \) and \( \text{dl}_\Sigma(v) \) vertex decomposable.

The most important consequences of a pure \( d \)-dimensional complex being vertex decomposable, is that it is shellable, homotopically a wedge of \( d \)-dimensional spheres, and in particular, \((d-1)\)-connected.

**Proposition 3.2.** If \( G \) is a \( \mathcal{VD}_k \) graph then \( \text{Ind}(G)^{\leq k-1} \) is pure \((k-1)\)-dimensional and vertex decomposable.
Proof. We first prove that if \( G \) is \( \mathbb{V}d_k \) then \( \text{Ind}(G) \leq k \) is pure \((k - 1)\)-dimensional.

The first case is that \( \text{Ind}(G) \leq 1 \) is pure \((-1)\)-dimensional for all \( G \).

The second case is when \( G \) is a \( k \)-vertex graph without edges. Then \( \text{Ind}(G) \leq k \) is a \((k - 1)\)-simplex and pure \((k - 1)\)-dimensional.

The third case is when \( G \) is \( \mathbb{V}d_k \) since \( G \setminus v \) is \( \mathbb{V}d_k \) and \( G \setminus N^*(v) \) is \( \mathbb{V}d_{k-1} \). Say that \( \sigma \in \text{Ind}(G) \leq k \) would be a facet of dimension less than \( k - 1 \) to reach a contradiction. If \( v \notin \sigma \) then we get a contradiction right off since \( \sigma \) is in the pure \((k - 1)\)-dimensional complex \( \text{Ind}(G \setminus v) \leq k - 1 \). If \( v \in \sigma \), then \( \sigma \setminus v \) is not a facet of \( \text{Ind}(G \setminus N^*(v)) \leq k - 2 \) since it is pure and \((k - 2)\)-dimensional. If we extend \( \sigma \setminus v \) to a facet \( \tau \) in \( \text{Ind}(G \setminus N^*(v)) \leq k - 2 \), then \( \sigma \) is strictly included in the facet \( \tau \cup \{v\} \) of \( \text{Ind}(G) \leq k - 1 \) and we have a contradiction.

Now we prove that \( \text{Ind}(G) \leq k - 1 \) is vertex decomposable if \( G \) is \( \mathbb{V}d_k \).

The complex \( \{\emptyset\} \) is vertex decomposable by definition, and simplices are by an easy argument left to the reader.

Now to the case that \( G \) is \( \mathbb{V}d_k \) since \( G \setminus v \) is \( \mathbb{V}d_k \) and \( G \setminus N^*(v) \) is \( \mathbb{V}d_{k-1} \). The deletion \( dl_{\text{Ind}(G) \leq k - 1}(v) = dl_{\text{Ind}(G \setminus v) \leq k - 1} = \text{Ind}(G \setminus v) \leq k - 1 \) is vertex decomposable since \( G \setminus v \) is \( \mathbb{V}d_k \).

The link \( lk_{\text{Ind}(G) \leq k - 1}(v) = lk_{\text{Ind}(G \setminus v) \leq k - 1} = \text{Ind}(G \setminus N^*(v)) \leq k - 2 \) is vertex decomposable since \( G \setminus N^*(v) \) is \( \mathbb{V}d_{k-1} \). We conclude that \( \text{Ind}(G) \leq k - 1 \) is vertex decomposable. \( \square \)

**Theorem 3.3.** Let \( q \geq 2 \) be a prime power, \( d \geq 1 \), and set \( N = (d + 1)(q - 1) + 1 \). Let \( \Sigma \) be a simplex on the same vertex set as \( G \) and \( f \) a continuous function from \( \Sigma \) to \( \mathbb{R}^d \).

If \( G \Box K_q \) is \( \mathbb{V}d_N \), then there is a \( q \)-coloring of \( G \)

\[
C_1 \cup C_2 \cup \cdots \cup C_q = V(G)
\]

such that

\[
\bigcap_{i=1}^{q} f(\text{simplex spanned by } C_i)
\]

is non-empty.

**Proof.** The complex \( \text{Ind}(G \Box K_q) \leq N - 1 \) is vertex decomposable by Proposition 3.2 since \( G \Box K_q \) is \( \mathbb{V}d_N \). The complex \( \text{Ind}(G \Box K_q) \) is \((N - 2)\)-connected since \( \text{Ind}(G \Box K_q) \leq N - 1 \) is that.

Now the remaining part of the proof is standard equivariant topology, a minor modification of Theorem 2.2 in \( \mathbb{S}^3 \), and we only sketch the proof.

The map \( f \) from \( \Sigma \) to \( \mathbb{R}^d \) induces a map \( f^\Sigma \) from the \( q \)-fold join \( \Sigma^q \) to the \( q \)-fold join \( (\mathbb{R}^d)^q \). If we restrict \( \Sigma^q \) to the \( \sigma_1 \ast \sigma_2 \ast \cdots \ast \sigma_q \) where all pairs \( \sigma_i, \sigma_j \) are disjoint, then we get the 2-wise \( q \)-fold deleted join

\[
\text{Ind}(G' \Box K_q)
\]

where \( G' \) is the graph on the same vertex set as \( G \) but with no edges. If we further restrict the deleted join to require that all \( \sigma_i \) are independent sets, then we get

\[
\text{Ind}(G \Box K_q).
\]

To prove the theorem by contradiction, suppose that there is no \( q \)-coloring whose images of the faces given by the colors intersect in a non-empty set. Then the image of the map can be restricted, and we have a map

\[
f^\Sigma : \text{Ind}(G \Box K_q) \to (\mathbb{R}^d)^q \setminus \{\gamma_1 x + \cdots + \gamma_q x \mid x \in \mathbb{R}^d\}.
\]
By assumption \( q \) is a prime power \( p^k \), and there is a free \( \mathbb{Z}_p^k \) action on \( \text{Ind}(G \square K_q) \) and \((\mathbb{R}^d)^q\) by permuting the \( q \) coordinates. This action extends to the map \( f^\ast q \). By a Borsuk-Ulam type argument of Volovikov \[20\], such an equivariant map into \((\mathbb{R}^d)^q \setminus \{\gamma_1 x + \cdots + \gamma_q x \mid x \in \mathbb{R}^d\}\) forces the connectivity of \( \text{Ind}(G \square K_q) \) to be at most \( N - 3 = (d+1)(q-1) - 2 \).

But since it is \((N-2)\)-connected we have a contradiction. \( \square \)

**Corollary 3.4.** For every \( \varepsilon > 0 \) there exists a constant \( K_\varepsilon \) such that the following holds: Let \( G \) be a graph on \((d+1)(q-1)+1\) vertices and maximal degree \( \Delta \) (with \( d \) and \( \Delta \) are large enough depending on \( \varepsilon \)), and let \( f \) be a continuous map from a simplex \( \Sigma \) with the same vertex set as \( G \) to \( \mathbb{R}^d \). If

\[
q > K_\varepsilon \Delta
\]

then there is a \( q \)-coloring of \( G \) satisfying

\[
\bigcap_{i=1}^{q} f(\text{simplex spanned by color } i) \neq \emptyset.
\]

**Proof.** According to Corollary 2.15 there is a constant \( K'_\varepsilon \) such that \( G' \square K_{q'} \) is \( \text{VD} \) \((d+1)(q'-1)+1\) for every induced subgraph of \( G' \) of \( G \) on \((d+1)(q'-1)+1\) vertices, whenever \( q' > K'_\varepsilon \Delta \).

Set \( K_\varepsilon = 2K'_\varepsilon \). Then, by Bertrand’s postulate, there is a prime \( q_p \) such that \( q \geq q_p > K_\varepsilon \Delta \). Let \( G_p \) be an induced subgraph of \( G \) on \((d+1)(q_p-1)+1\) vertices. By Corollary 2.15, \( G_p \square K_{q_p} \) is \( \text{VD} \) \((d+1)(q_p-1)+1\). By Theorem 3.3 we have that there is a \( q_p \)-coloring with the intersection of the images of monochromatic simplices non-empty. Any \( q_p \)-coloring is also a \( q \)-coloring of \( G_p \) since \( q > \Delta \).

We may assume that \( K_\varepsilon \geq 1 \). The \( q \)-coloring of \( G_p \) extends to a \( q \)-coloring of \( G \) since \( q > \Delta \), and the intersection

\[
\bigcap_{i=1}^{q} f(\text{simplex spanned by color } i) \neq \emptyset.
\]

expands if it changes when extending from \( G_p \) to \( G \). \( \square \)

**Remark.** By the prime number theorem we could have done much better than a factor 2 with Bertrand’s postulate in the proof of Corollary 3.4. We refer to the excellent textbook by Ingham \[13\] for classical results.

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Department of Mathematics, University of California, Berkeley, CA 94720
E-mail address: alex@math.berkeley.edu

KTH – The Royal Institute of Technology, Stockholm, Sweden
E-mail address: pnore@math.kth.se