EMBEDDABLE ALGEBRAS INTO ZINBIEL ALGEBRAS VIA THE COMMUTATOR

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Abstract. A criterion for elements of free Zinbiel algebras to be Lie or Jordan is established. This criterion is used in studying speciality problems of Tortkara algebras. We construct a base of free special Tortkara algebras. Further, we prove analogue of classical Cohn’s and Shirshov’s theorems for Tortkara algebras.

1. Introduction

An algebra with identity

\[ a \circ (b \circ c) = (a \circ b + b \circ a) \circ c \]

is called (right)-Zinbiel algebra. These algebras were introduced by J-L. Loday in [9] as dual of Leibniz algebras. In some papers Zinbiel algebras are called dual Leibniz, chronological or pre-commutative algebras [1],[2],[6],[7].

Anti-commutative algebra with so-called Tortkara identity

\[ (ab)(cb) = J(a,b,c)b \]

is called Tortkara algebra. Over a field of characteristic different from two \([2]\) has the following multi-linear form

\[ (ab)(cd) + (ad)(cb) = J(a,b,c)d + J(a,d,c)b. \]

Tortkara algebra was defined in [5], as minus-algebra \(A^{(-)} = (A, [], +)\) of Zinbiel algebra \((A, \circ, +)\), in other words, any Zinbiel algebra under the commutator product \([x,y] = x \circ y - y \circ x\) satisfies the Tortkara identity. Any Zinbiel algebra under the anti-commutator product \(\{x,y\} = x \circ y + y \circ x\) is commutative and associative [9].

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2000 Mathematics Subject Classification. 17A30, 17A50.

Key words and phrases. Zinbiel algebra, Lie elements, Cohn’s and Shirshov’s theorems.

The authors were supported by grant AP05131123 and the second author was supported by the FAPESP Proc.2017/21429-6.
Metabelian Lie and dual mock-Lie algebras are examples of Tortkara algebras, see [12]. Let \( A \) be an algebra with identities
\[
(4) \quad a(bc) = b(ac), \quad ([a, c], b, a) + ([b, a], c, a) + ([c, b], a, a) = 0.
\]
It is not a Zinbiel algebra, but its minus-algebra \( A^{(-)} \) is a Tortkara algebra [5].

As far as connections of Tortkara algebras with other classes of algebras we have the following facts. The variety of Tortkara algebras \( T \) is not subvariety of the variety of binary-Lie algebras. Consequently, it is not a subvariety of variety of Malcev algebras. In [10] it is studied a variety of algebras defined by the following identity
\[
[a, b, cd] - [a, b, c]d - c[a, b, d] = 0.
\]
It contains Jordan, Lie, right-symmetric and LT-algebras. Tortkara algebras are out of this variety.

Tortkara algebra \( B \) is called special if there exists a Zinbiel algebra \( A \) such that \( B \) is a subalgebra of \( A^{(-)} \), otherwise it is called exceptional.

Let \( A = C[x] \) be an algebra with multiplication
\[
a \star b = b \int_0^x (\int_0^x a \, dx) \, dx.
\]
Then \((A, \star)\) is not a Zinbiel algebra, but \( A^{(-)} \) is a Tortkara algebra [5]. It is an example of special Tortkara algebra (see Remark 1 at the end of our paper).

In our paper we give a base of free special Tortkara algebras in terms of polynomials of free Zinbiel algebras.

Let \( ST \) be the class of special Tortkara algebras. By \( ST \) denote the homomorphic closure of \( ST \). In general,
\[
ST \subseteq ST \subseteq T
\]
and \( ST \) is the smallest variety that contains \( ST \).

In [7] P. Kolesnikov proved that
\[
ST \subsetneq ST.
\]
He constructed a Tortkara algebra on four generators which is in \( ST \) but not in \( ST \). Further, he asked the question in [7] about the maximal number of free generators for which all homomorphic images of free special Tortkara algebra are special. In this paper we show that there exists an algebra in \( ST \) on three generators which is not in \( ST \), and any algebra in \( ST \) on two generators is also in \( ST \). This is the analogue of classical Cohn’s theorem [4] in Jordan theory for free special Tortkara algebras. The analogue of this theorem for a free special Jordan dialgebra is studied in [11].
Further, we prove that a free Tortkara algebra on two generators is special. As a consequence, taking account above result, we have that any Tortkara algebra on two generators is special. This is the analogue of Shirshov's theorem in theory of Jordan algebras. Shirshov's theorem for Malcev algebras was considered in [8].

An identity is called special if it holds in any Tortkara algebra in $ST$ but does not hold in all Tortkara algebras. In general, we still do not know whether there exists a special identity. In [3], M. Bremner by computer algebraic methods has studied special identities in terms of Tortkara triple product $[a, b, c] = [[a, b], c]$ in a free Zinbiel algebra and discovered one identity in degree 5 and one identity in degree 7 which do not follow from the identities of lower degree. We prove that there is no special identity in two variables. It is known that there is no special identity up to degree seven in case a field has zero characteristic.

All our algebras are considered over a field $K$ of characteristic 0.

2. Main results

2.1. Definitions and notations. Let $Zin(X)$ be a free Zinbiel algebra on a set $X$. For $a_1, \ldots, a_n \in Zin(X)$ denote by $a_1 a_2 \cdots a_n$ a left-bracketed element $(\cdots (a_1 \circ a_2) \cdots) \circ a_n$. In [9] it was proved that the following set of elements

$$\mathcal{V}(X) = \bigcup_n \{x_{i_1} x_{i_2} \cdots x_{i_n} | x_1, \ldots, x_n \in X\}$$

forms a base of the free Zinbiel algebra $Zin(X)$.

Define a linear map $p : Zin(X) \to Zin(X)$ on base elements as follows

$$p(x_i) = -x_i,$$

$$p(x_{i_1} x_{i_2} \cdots x_{i_{m-2}} y z) = x_{i_1} x_{i_2} \cdots x_{i_{m-2}} y z,$$  

$m > 1$.

where $y, z \in X$.

For $a \in Zin(X)$ set

$$\overline{a} \overset{\text{def}}{=} a - p(a)$$

Since $p^2 = id$, it is clear that

$$p(\overline{a}) = -\overline{a}.$$

Let $\Gamma$ be set of sequences $\alpha = i_1 i_2 \cdots i_{n-2} i_{n-1} i_n$ such that $i_{n-1} < i_n$. For $\alpha = i_1 \cdots i_{n-2} i_{n-1} i_n \in \Gamma$ set

$$x_\alpha = x_{i_1} \cdots x_{i_{n-2}} x_{i_{n-1}} x_{i_n}.$$  

We call the element $x_\alpha$, where $\alpha \in \Gamma$, skew-right-commutative or shortly skew-rcom element of $Zin(X)$.

For example, if $X = \{a, b, u, v\}$, then
\[
(ab)(uv) = (ab)(uv) - p((ab)(uv)) = 
((ab)u)v + (u(ab))v - p(((ab)u)v + (u(ab))v) = 
\begin{align*}
abuv + uabv + aubv &- p(abuv + uabv + aubv) = 
\frac{\abuv + uabv + aubv}{x_{3124}} = uabv - uavb.
\end{align*}
\]

An element \( u \in \text{Zin}(X) \) is called Lie if it can be presented as a word on \( X \) under Lie product \([a, b] = ab - ba\). Similarly, an element \( u \in \text{Zin}(X) \) is called Jordan if it can be presented as a word on \( X \) in terms of Jordan product \( \{a, b\} = ab + ba \).

Let \( ST(X) \) be free special Tortkara algebra on \( X \) under Lie commutator, i.e., subalgebra of \( \text{Zin}(X)^{-} = (\text{Zin}(X), [ , ] ) \) generated by \( X \). Let \( J(X) \) be subalgebra of \( \text{Zin}(X)^{+} = (\text{Zin}(X), \{ , \} ) \) generated by \( X \).

Define Dynkin map \( D : \text{Zin}(X) \to \text{Zin}(X) \) on base elements as follows
\[
D : x_{i_{1}}x_{i_{2}}\cdots x_{i_{n}} \mapsto \{ \cdots \{ x_{i_{1}}, x_{i_{2}} \} \cdots , x_{i_{n}} \}.
\]

2.2. Formulations of main results.

2.1. Theorem. Let \( f \) be a Zinbiel polynomial of \( \text{Zin}(X) \). Then \( f \) is a Lie polynomial if and only if \( p(f) = -f \).

2.2. Theorem. The set of skew-rcom elements \( x_{\alpha} \), where \( \alpha \in \Gamma \), forms base of \( ST(X) \).

Let \( ST(X)_{m_{1},\ldots,m_{q}} \) be the homogenous part of \( ST(X) \) generated by \( m_{i} \) generators \( x_{i} \) where \( i = 1, \ldots, q \). Then
\[
\dim ST(X)_{m_{1},\ldots,m_{q}} = \sum_{i<j} \frac{(n-2)!}{m_{i}! \cdots m_{q}!} m_{i}m_{j}
\]
where \( n = m_{1} + \cdots + m_{q} \). In particular, the multilinear part of \( ST(X) \) has dimension \( \frac{n!}{2} \).

2.3. Theorem. Let \( a, b, c \in \text{Zin}(X) \). Then a non-zero element \( abc - acb \) and \( bc - cb \) are Lie if and only if \( b \) and \( c \) are Lie.

2.4. Theorem. Let \( f \) be a homogenous Zinbiel polynomial of degree \( n \) in \( \text{Zin}(X) \). Then \( f \) is a Jordan element if and only if \( D(f) = n!f \). The algebra \( J(X) \) is isomorphic to polynomial algebra \( K[X] \).

Let \( T(X) \) be a free Tortkara algebra generated by a set \( X \).

2.5. Theorem. The free Tortkara algebra \( T(\{x, y\}) \) is special.

The next theorem is an analogue of Cohn’s theorem on speciality of homomorphic images of special Jordan algebras in two generators [4].
2.6. Theorem. Any homomorphic image of a free special Tortkara algebra on two generators is special. For three generators case this statement is not true: a homomorphic image of special Tortkara algebra with three generators might be non special.

2.7. Corollary. Any Tortkara algebra on two generators is special.

Proof. It follows from Theorems 2.5 and 2.6.

□

This result is analogue of Shirshov theorem for Jordan algebras [13].

3. Skew-right-commutative Zinbiel elements and Lie elements in $\text{Zin}(X)$

3.1. Shuffle permutations. Let $Sh_{m,n}$ be set of shuffle permutations, i.e.,

$$Sh_{m,n} = \{ \sigma \in S_{m+n} | \sigma(1) < \cdots < \sigma(m), \sigma(m+1) < \cdots < \sigma(m+n) \}.$$ 

For any positive integers $i_1, \ldots, i_m, j_1, \ldots, j_n$ denote by $Sh(i_1 \ldots i_m; j_1 \ldots j_n)$ set of sequences that constructed by shuffle permutations $\sigma \in Sh_{m,n}$ by changing $\sigma(l)$ to $i_l$ if $l \leq k$ and to $j_{l-k}$ if $k < l \leq k+n$.

For example,

$$Sh(1, 2; 3, 4) = \{1234, 1324, 1423, 2314, 2413, 3412\},$$

$$Sh(2, 3; 4, 1) = \{2341, 2431, 2134, 3421, 3124, 4123\}.$$ 

The following proposition is proved in [9].

3.1. Proposition.

$$(x_{i_1} \cdots x_{i_p}) \circ (x_{j_1} \cdots x_{j_q}) = \sum_{\sigma \in Sh(i_1 \ldots i_p; j_1 \ldots j_{q-1})} x_{\sigma(1)} \cdots x_{\sigma(p+q-1)} x_{j_q}.$$ 

For any two left-bracketed elements $u = a_1 \cdots a_p, v = a_{p+1} \cdots a_{p+q} \in Zin(X)$ define their shuffle product by

$$u \ll v = \sum_{\sigma \in S_{p,q}} a_{\sigma(1)} \cdots a_{\sigma(p+q)}.$$ 

3.2. Proposition. The shuffle product on $Zin(X)$ has the following properties

a. the shuffle product is commutative and associative

$$a \ll b = b \ll a, \quad (a \ll b) \ll c = a \ll (b \ll c)$$

b. $(a_1 \cdots a_p) \circ (b_1 \cdots b_q) = (a_1 \cdots a_{p-1} \ll b_1 \cdots b_{q-1}) \circ b_q$
c. 
\[(a_1 \cdots a_p) \sqcup (b_1 \cdots b_q) = (a_1 \cdots a_{p-1} \sqcup b_1 \cdots b_q) \circ a_p + (a_1 \cdots a_p \sqcup b_1 \cdots b_{q-1}) \circ b_q.\]

For example,
\[(ab) \circ (cd) = (abc + acbd + cab) d = (ab \sqcup c) \circ d,\]
\[\begin{align*}
(ab) \sqcup (cd) &= abcd + acbd + cabd + cadb + cdab = \\
&= (abc + acb + cab) d + (acd + cad + cda) b = (ab \sqcup c) d + (a \sqcup cd) b.
\end{align*}\]

Proof. All three properties directly follow from the definition of the shuffle product and proposition 3.2. □

3.2. Products of skew-right-commutative elements.

3.3. Lemma. Zinbiel product of skew-right-commutative elements can be presented as follows
\[x_1 \cdots x_k \circ x_{k+1} \cdots x_n =\]
\[\begin{align*}
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{n-1} x_n - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{n-2} x_n + \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-1}\right) x_k x_n - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-1}\right) x_{n-1} x_n = \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} + \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{k-1} x_n.
\end{align*}\]

Proof. By part b of Proposition 3.2,
\[\begin{align*}
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-1}\right) x_k x_n - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-1}\right) x_{n-1} x_n = \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} + \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{k-1} x_n.
\end{align*}\]

(by part c of Proposition 3.2)
\[\begin{align*}
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-1}\right) x_k x_n - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{n-1} x_n = \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} + \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{k-1} x_n = \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{n-1} x_n + \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} + \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{k-1} x_n = \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{n-1} x_n + \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} + \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{k-1} x_n = \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} - \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{n-1} x_n + \\
&\left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_k x_{n-1} + \left(x_1 \cdots x_k \sqcup x_{k+1} \cdots x_{n-2}\right) x_{k-1} x_n.
\end{align*}\] □
3.4. Lemma. 
\[
[x_{i_1} \cdots x_{i_n}, x_{i_j}] =
\]
\[
x_{i_1} \cdots x_{i_{n-1}} x_{i_n} - x_{i_1} \cdots x_{i_{n-1}} x_{i_n} - (x_{i_n} \lrcorner x_{i_1} \cdots x_{i_{n-1}}) x_{i_n - 1}.
\]

Proof. 
\[
[x_{i_1} \cdots x_{i_{n-1}}, x_{i_n}] =
\]
\[
x_{i_1} \cdots x_{i_{n-1}} x_{i_n} - x_{i_n} x_{i_1} \cdots x_{i_{n-1}} =
\]
\[
(x_{i_1} \cdots x_{i_{n-1}}) x_{i_n} - (x_{i_1} \cdots x_{i_{n-1}} x_{i_n - 1}) x_{i_n} - x_{i_n} (x_{i_1} \cdots x_{i_{n-1}} x_{i_n - 1}) =
\]
\[
(x_{i_1} \cdots x_{i_{n-1}} x_{i_n} - x_{i_1} \cdots x_{i_{n-1}} x_{i_n - 1}) x_{i_n} - (x_{i_1} \cdots x_{i_{n-1}} x_{i_n - 1}) x_{i_n - 1}.
\]
(by part b of Proposition 3.2)

3.5. Lemma. Commutator of skew-right-commutative elements is a linear combination of skew-right-commutative elements.

Proof. 
\[
[x_1 \cdots x_k, x_{k+1} \cdots x_n] =
\]
\[
x_1 \cdots x_k \circ x_{k+1} \cdots x_n - x_{k+1} \cdots x_n \circ x_1 \cdots x_k =
\]
(by Lemma 3.3)
\[
(x_1 \cdots x_k \lrcorner x_{k+1} \cdots x_n - (x_1 \cdots x_k \lrcorner x_{k+1} \cdots x_k x_{k-1} \cdots x_n - x_{k+1} \cdots x_n) x_{n-1}) +
\]

(by part a of Proposition 3.2)
\[
\begin{align*}
&\left( x_1 \cdots x_{k-1} x_{k+1} \cdots x_n - (x_1 \cdots x_{k-2} x_k x_{k+1} \cdots x_{n-1}) x_{k-1} x_n - (x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n-2} x_n) x_{k-1} x_{n-1} \right) x_{k-2} x_k x_{k+1} x_{k+1} \cdots x_{n-1} x_n \\
&= (x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n-1}) x_{k-1} x_n - (x_1 \cdots x_{k-2} x_k x_{k+1} \cdots x_{n-1}) x_{k-1} x_n - (x_1 \cdots x_{k-1} x_{k+1} \cdots x_{n-2} x_n) x_{k-1} x_{n-1}.
\end{align*}
\]

3.6. **Lemma.** If \( f \in ST(X) \), then \( p(f) = -f \).

**Proof.** Since \( ST(X) \) is generated by the commutator products on \( X \) and \([x, y] = \bar{xy} \) for any \( x, y \in X \), Lemmas 3.4 and 3.5 complete the proof. □

Below we prove that any skew-right-commutative element of \( Zin(X) \) is Lie.

3.7. **Lemma.** Let \( f \in Zin(X) \) with \( p(f) = -f \). Then \( f \in ST(X) \).

**Proof.** Write \( a \equiv b \) if \( a - b \in ST(X) \). If \( p(f) = -f \), then \( f \) can be written as a linear combination of skew-right-commutative elements. To get a proof of our statement, it is sufficient to show that

\[
(5) \quad \bar{x}_1 \cdots \bar{x}_n \equiv 0.
\]

We prove it by induction on \( n \).

If \( n = 2 \) then

\[
\bar{x}_1 \bar{x}_2 = [x_1, x_2].
\]

Assume that (5) is true for elements whose degree less than \( n \). So,

\[
\bar{z} \bar{x}_{i_k+1} \cdots \bar{x}_{i_n} \equiv 0
\]

for any Lie element \( z \) such that degree of \( z \) is no more than \( k \). Take \( z := \bar{x}_1 \cdots \bar{x}_{i_k} \). We have

\[
\bar{x}_1 \cdots \bar{x}_{i_k} \bar{x}_{i_{k+1}} \cdots \bar{x}_{i_n} \equiv 0
\]

for \( 1 < k < n - 1 \). Hence,

\[
\bar{x}_1 \cdots \bar{x}_{i_k-1} \bar{x}_i \bar{x}_{i_{k+1}} \cdots \bar{x}_{i_n} \equiv \bar{x}_1 \cdots \bar{x}_{i_k} \bar{x}_{i_{k-1}} \bar{x}_{i_{k+1}} \cdots \bar{x}_{i_n}.
\]

Since the symmetric group \( S_{n-2} \) is generated by transpositions \( (12), (23), \ldots, (n-3 n-2) \), for any \( \sigma \in S_{n-2} \) we have

\[
(6) \quad \bar{x}_1 \cdots \bar{x}_i \equiv \bar{x}_{\sigma(i_1)} \cdots \bar{x}_{\sigma(i_{n-2})} \bar{x}_{i_{n-1}} \bar{x}_{i_n}.
\]

By (6) and Lemma (3.4) we have

\[
[x_1 \cdots x_{i_{n-2}} \bar{x}_i \bar{x}_{i_{n-1}}, \bar{x}_{i_n}] = x_1 \cdots x_{i_{n-3}} \bar{x}_i \bar{x}_{i_{n-2}} \bar{x}_{i_{n-1}} \bar{x}_{i_n} - x_1 \cdots x_{i_{n-3}} \bar{x}_i \bar{x}_{i_{n-2}} \bar{x}_{i_{n-1}} \bar{x}_{i_n} - (n-2) x_1 \cdots x_{i_{n-3}} \bar{x}_i \bar{x}_{i_{n-2}} \bar{x}_{i_{n-1}} \bar{x}_{i_n} \equiv 0
\]

and

\[
[x_1 \cdots x_{i_{n-2}} \bar{x}_i \bar{x}_{i_{n-1}}, \bar{x}_{i_{n-1}}] =
\]

\[
= x_1 \cdots x_{i_{n-3}} x_{i_{n-2}} \bar{x}_i \bar{x}_{i_{n-1}} - x_1 \cdots x_{i_{n-3}} x_{i_{n-2}} \bar{x}_{i_{n-1}} x_{i_{n-2}} \bar{x}_{i_{n-1}} - (n-2) x_1 \cdots x_{i_{n-3}} x_{i_{n-2}} x_{i_{n-1}} x_{i_{n-2}} \bar{x}_{i_{n-1}} =
\]

Take sum of these two elements. We have

\[ [x_1 \cdots x_{i-3} x_{i-2} x_{i-1}, x_{i}] + [x_1 \cdots x_{i-3} x_{i-2} x_{i-1}, x_{i-1}] = -(n-1)x_1 \cdots x_{i-3} x_{i-2} x_{i} x_{i-1} = 0. \]

In other words,

\[ x_1 \cdots x_{i-3} x_{i-2} x_{i} x_{i-1} \equiv -x_1 \cdots x_{i-3} x_{i-2} x_{i-1} x_{i}. \]

By (6) and (7)

\[ x_1 \cdots x_{i-3} x_{i-2} x_{i-1} x_{i} \equiv -x_1 \cdots x_{i-3} x_{i-2} x_{i-1} x_{i-2}. \]

Hence,

\[ x_1 \cdots x_{i-3} x_{i-2} x_{i} x_{i-1} \equiv 0 \]

and this completes the proof. \(\square\)

3.3. **Proof of Theorem 2.1.** It follows from Lemmas 3.6 and 3.7. \(\square\)

3.4. **Proof of Theorem 2.2.** Since a skew-rcom element is defined as a difference of two base elements, a linear combination of skew-rcom elements is trivial, hence, they are linear independent in \( \text{Zin}(X) \). By Lemma 3.6 any element of \( ST(X) \) is a linear combination of skew-rcom elements. So we have proved that the set of skew-rcom elements, generated by a set \( X \), form a basis of \( ST(X) \). The dimension of \( ST(X) \) is calculated easily using the first part of this theorem. \(\square\)

4. **Proof of Theorem 2.3**

We give proof of our Theorem for elements of a form \( abc - acb \). The case \( bc - cb \) can be established in a similar way.

Let \( a \in \text{Zin}(X) \) and suppose that \( b, c \in ST(X) \). Then by Theorem 2.2

\[ b = \sum_{\alpha} \lambda_{\alpha} x_{\alpha}, \quad c = \sum_{\beta} \mu_{\beta} y_{\beta}. \]

We have

\[ abc - acb = \sum_{\alpha} \lambda_{\alpha} x_{\alpha} \sum_{\beta} \mu_{\beta} y_{\beta} - \sum_{\beta} \mu_{\beta} y_{\beta} \sum_{\alpha} \lambda_{\alpha} x_{\alpha} = \]
\[ \sum_{\alpha, \beta} \lambda_{\alpha} \mu_{\beta}(ax_{\alpha}y_{\beta} - ay_{\beta}x_{\alpha}). \]

It can be easily checked that \( ax_{\alpha}y_{\beta} - ay_{\beta}x_{\alpha} \in ST(X) \) and hence \( abc - acb \in ST(X) \).

Conversely, suppose that \( abc - acb \) is a non-zero element of \( ST(X) \) and
\[ b = \sum_{\alpha} \lambda_{\alpha} x_{\alpha}, \quad c = \sum_{\beta} \mu_{\beta} y_{\beta}. \]

We have
\[ abc - acb = a \sum_{\alpha} \lambda_{\alpha} x_{\alpha} \sum_{\beta} \mu_{\beta} y_{\beta} - a \sum_{\beta} \mu_{\beta} y_{\beta} \sum_{\alpha} \lambda_{\alpha} x_{\alpha} = \]
\[ \sum_{\alpha, \beta} \lambda_{\alpha} \mu_{\beta}(ax_{\alpha}y_{\beta} - ay_{\beta}x_{\alpha}). \]

Since the given element is non-zero, there exists \( \alpha = i_1 \ldots i_k, \beta = j_1 \ldots j_l \) so that
\[ \lambda_{\alpha} \mu_{\beta}(ax_{\alpha}y_{\beta} - ay_{\beta}x_{\alpha}) \neq 0. \]

Then
\[ a(x_{i_1} \ldots x_{i_k})(y_{j_1} \ldots y_{j_l}) - a(y_{j_1} \ldots y_{j_l})(x_{i_1} \ldots x_{i_k}) = \]
(by part b of Proposition 3.2)
\[ ((a_{\sqcup} x_{i_1} \cdots x_{i_{k-1}}) x_{i_k} \sqcup y_{j_1} \cdots y_{j_{l-1}}) y_{j_l} - ((a_{\sqcup} y_{j_1} \cdots y_{j_{l-1}}) y_{j_l} \sqcup x_{i_1} \cdots x_{i_{k-1}}) x_{i_k} \]
(by part c of Proposition 3.2)
\[ (a_{\sqcup} y_{j_1} \cdots y_{j_{l-1}} \sqcup x_{i_1} \cdots x_{i_{k-1}}) y_{j_l} x_{i_k} + ((a_{\sqcup} x_{i_1} \cdots x_{i_{k-1}}) x_{i_k} \sqcup y_{j_1} \cdots y_{j_{l-1}}) y_{j_l} x_{i_k} - \]
(by part a of Proposition 3.2)
\[ (a_{\sqcup} y_{j_1} \cdots y_{j_{l-1}} \sqcup x_{i_1} \cdots x_{i_{k-1}}) y_{j_l} x_{i_k} - ((a_{\sqcup} y_{j_1} \cdots y_{j_{l-1}}) y_{j_l} \sqcup x_{i_1} \cdots x_{i_{k-1}}) x_{i_k-1} x_{i_k} = \]

We note that no two summands above form a skew-rcm element with each other. So, if \( abc - acb \) is Lie, then \( b \) and \( c \) must be in \( ST(X) \).
5. Proof of Theorem 2.4

In this section we prove a Jordan criterion for \( Zin(X) \).

5.1. Lemma. \( \{ \{ \cdots \{ x_{i_1}, x_{i_2} \} \cdots \}, x_{i_n} \} = \sum_{\sigma \in S_n} x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_n)}. \)

Proof. It is proved by induction on \( n \). If \( n = 2 \), then \( \{ x_{i_1}, x_{i_2} \} = x_{i_1} x_{i_2} + x_{i_2} x_{i_1} \). Suppose that it is true for \( n - 1 \). Then

\[
\{ \{ \cdots \{ x_{i_1}, x_{i_2} \} \cdots \}, x_{i_n} \} =
\{ \{ \cdots \{ x_{i_1}, x_{i_2} \} \cdots \}, x_{i_{n-1}} \} x_{i_n} + x_{i_n} \{ \{ \cdots \{ x_{i_1}, x_{i_2} \} \cdots \}, x_{i_{n-1}} \} =
\sum_{\sigma \in S_{n-1}} x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_{n-1})} + x_{i_n} \sum_{\sigma \in S_{n-1}} x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_{n-1})}
\]

(by Proposition 3.1)

\[
\sum_{\sigma \in S_n} x_{\sigma(i_1)} x_{\sigma(i_2)} \cdots x_{\sigma(i_n)}.
\]

\( \square \)

Proof of Theorem 2.4. Recall that \( A^{(+)} \) is associative and commutative algebra if \( A \) is Zinbiel. Any Jordan polynomial in \( Zin(X) \) can be written as linear combination of left-normed Jordan monomials in \( X \) by anti-commutators. Then the proof follows from Lemma 5.1 and definition of the map \( D \).

Let \( \varphi : K[X] \to J(X) \) be a canonical homomorphism from polynomial algebra generated by \( X \) to \( J(X) \) defined as \( x_{i_1} x_{i_2} \cdots x_{i_n} \mapsto \{ \{ \cdots \{ x_{i_1}, x_{i_2} \} \cdots \}, x_{i_n} \} \). Then it is clear that \( \text{Ker} \varphi \) is empty and therefore \( K[X] \) and \( J(X) \) are isomorphic.

Denote by \( J(X)_{m_1, \ldots, m_q} \) the homogenous part of \( J(X) \) generated by \( m_i \) generators \( x_i \) where \( i = 1, \ldots, q \).

5.2. Corollary. The dimension of the homogenous part \( J(X)_{m_1, \ldots, m_q} \) of \( J(X) \) is equal to

\[
dim J(X)_{m_1, \ldots, m_q} = 1.
\]

Proof. It is an immediate consequence of Theorem 2.4. \( \square \)

6. Speciality of \( T(\{ x, y \}) \)

In this section we prove that the free Tortkara algebra on two generators \( T(\{ x, y \}) \) is special. As a corollary, we have a base of \( T(\{ x, y \}) \).

6.1. Lemma. Let \( T_n \) be the \( n \)-th homogenous part of \( T(\{ x, y \}) \). Then \( T_{n+1} = T_n T_1 \) for any \( n \).
Proof. Clearly, $T_{n+1} \supseteq T_n T_1$.

We write $a \equiv b$ if $a - b \in T_n T_1$. We prove the statement by induction on degree $n$. We have

$$(ab)(cd) = \frac{1}{2}J(b, c, d)a - \frac{1}{2}J(a, c, d)b - \frac{1}{2}J(a, b, d)c + \frac{1}{2}J(a, b, c)d \equiv 0$$

This is the basis of induction for $n$. Suppose that our statement is true for fewer than $n > 4$. Let $C \in T_n$ and $C = A_k B_l$ where $A_k$ and $B_l$ be elements of $T(\{x, y\})$ whose degrees are $k$ and $l$, respectively, and $k + l = n$. Now we consider induction on $l$. By induction on $n$ we may assume that they are left-normed and write

$$C = A_k B_l = (A_{k-1} a_k)(B_{l-1} b_l)$$

where $a_k, b_l \in \{x, y\}$. Suppose $l = 2$ and $b_1 = x, b_2 = y$. Assume $a_k = x$. Then by identity (2) and induction on $n$ we have

$$C = (A_{k-1} x)(y x) = J(A_{k-1}, x, y)x \equiv 0.$$

Suppose that our statement is true for fewer than $l > 2$. We have

$$(A_{k-1} a_k)(B_{l-1} b_l) =$$

(by anti-commutativity identity)

$$-(A_{k-1} a_k)(b_l B_{l-1}) =$$

(by identity (3))

$$(A_{k-1} B_{l-1})(b_l a_k) - J(A_{k-1}, a_k, b_l) B_{l-1} - J(A_{k-1}, B_{l-1}, b_l)a_k.$$

We note that by base of induction on $l$ $(A_{k-1} B_{l-1})(b_l a_k) \equiv 0$, and $J(A_{k-1}, a_k, b_l) B_{l-1} \equiv 0$. By induction on $n$ we have $J(A_{k-1}, B_{l-1}, b_l)a_k \equiv 0$. Hence,

$$(A_{k-1} a_k)(B_{l-1} b_l) \equiv 0.$$

Proof of Theorem 2.5. In order to prove the statement it is sufficient to show that algebras $T(\{x, y\})$ and $ST(\{x, y\})$ are isomorphic. Let $\varphi$ be a natural homomorphism from $T(\{x, y\})$ to $ST(\{x, y\})$. By Lemma 6.1 the vector space $T(\{x, y\})$ is spanned by the set of left-normed elements. We note that number of left-normed elements in two generators is equal to the number of skew-rcom elements in two generators. Suppose that the kernel of $\varphi$ is not zero. Then we have a linear combination of skew-rcom elements which is zero in $ST(\{x, y\})$. It contradicts to the first part of Theorem 2.2. Therefore, $\text{Ker } \varphi = (0)$.

6.2. Corollary. The set of left-normed elements is a basis of $T(\{x, y\})$. 

7. Speciality of homomorphic images of $ST(\{x, y\})$

Let $\alpha$ be an ideal of $ST(X)$. By Cohn's criterion (Theorem 2.2 of [3]) $ST(X)/\alpha$ is special if and only if $\{\alpha\} \cap ST(X) \subseteq \alpha$ where $\{\alpha\}$ is the ideal of $Zin(X)$ generated by the set $\alpha$.

Proof of Theorem 2.6. Assume that $g_1, g_2, \ldots$ are generators of the ideal $\alpha$. It is clear that if $xy \in \alpha$ then $ST(\{x, y\})/\alpha$ is special.

Therefore, by Theorem 2.3 we can assume that each element $g_i$ has a form $f_i xy$ for some $f_i \in Zin(\{x, y\})$.

Let $w$ be a non-zero element of $\{\alpha\} \cap ST(\{x, y\})$. Then $p(w) = -w$ and $w$ is a linear combination of left-normed monomials in $x, y, g's$ that each monomial is linear at least one generator of $\alpha$. Let $a_1 \cdots a_n$ be a term of $w$ in the linear combination. To prove the statement we consider two cases, depending on what position a generator appear in $a_1 \cdots a_n$.

Case 1. Suppose that generators of $\alpha$ appear only in the first $n - 2$ positions in $a_1 \cdots a_n$. Then write all $a'_i's$ in terms of elements of $X$. Since $w \in ST(\{x, y\})$, $w$ must have the term $p(a_1 \cdots a_n)$ with opposite sign. Hence, $a_1 \cdots a_n \in \alpha$.

Case 2. Suppose that generators of $\alpha$ appear in either $n - 1$-th or $n$-th positions in $a_1 \cdots a_n$, (a generator of $\alpha$ may appear in the first $n - 2$-positions), namely,

$$a_1 \cdots a_{n-i} f_i xy$$

or

$$a_1 \cdots a_{n-2} f_i xxy$$

for some $i$. If the generators of $\alpha$ appear both $n-1$-th and $n$-th positions in $a_1 \cdots a_n$, then write one of them in terms of $x's$ and $y's$. We also express $a_1, \ldots, a_{n-2}$ in terms of $x's$ and $y's$, therefore we can assume that $a_1, \ldots, a_{n-2} \in X$. By $u$ denote $a_1 \cdots a_{n-2}$.

Now we show that if $w f_i xy$ is a term of $w$ then $w$ has the term $u f_i xyy$ with opposite sign.

We have

$$w f_i xy = f_i xxy + u f_i xyy + u f_i xy + f_i xxy + f_i xxy + u f_i xyy - f_i uyxx - f_i yuxx - u f_i yxx.$$

As we see that $w f_i xy$ has the part that is not in $ST(\{x, y\})$. Since $w \in ST(\{x, y\})$, $w$ must have terms $f_i uyxx, f_i yuxx, u f_i yxx$ with opposite signs, and for each $s \in \{f_i xxy, f_i xyy, u f_i xxy\}$, $w$ must have terms $s$ or $p(s)$ with opposite sign. We note that $w$ must have some terms in which $g's$ appear in either $n - 1$-th or $n$-th positions that delete the terms $f_i uyxx, f_i yuxx, u f_i yxx$. These kind of terms are generated by $u, f, x, y$. Then by Theorem 2.3 all possibilities are $u f_i xxy, f_i xxy$
and $f_iuxyx$. We have

\[ uf_iuxyx = f_iuxyx + f_iuxy + uf_ixy - f_iuuxx - f_iuyxx - uf_iyyx, \]
\[ f_iuxyx = f_iuxxy + f_iuxxy + uf_ixxy + f_iuxxy + uf_ixxy - \]
\[ f_iuuxx - uf_iyxx - uf_iyxx, \]
\[ f_iuxyx = f_iuxyx + uf_iuxxy + uf_iuxxy - f_iuyxx - uf_iyyx - uf_iyyx. \]

We see that the element $f_iuyxx$ appears only in $uf_iuxyx$ and moreover,

\[ xf_iuxy - uf_iuxyx = f_iuuxy + uf_iuxxy + uf_iuxxy + uf_iuxxy + \]
\[ uf_iuxxy + uf_iuxxy + uf_iuxxy \in ST(\{x, y\}). \]

Therefore, $w$ has the term $uf_iuxyx$ with opposite sign. Since $f_iuxy$ is a generator of $\alpha$, and by Theorem 2.3

\[ uf_iuxy - uf_iuxyx \in \alpha. \]

Hence, $a_1 \cdots a_n - a_1 \cdots a_{n-2} a_n a_{n-1} \in \alpha$. If $xf_iuxy$ is a nonzero term of $w$, then by similar way one can show that $w$ must have term $f_iuxy$.

So, we obtain $w \in \alpha$. It follows $\{\alpha\} \cap ST(\{x, y\}) \subseteq \alpha$. Hence by Cohn’s criterion $ST(\{x, y\})/\alpha$ is special.

Now we show that a homomorphic image of $ST(\{x, y, z\})$ may be not special.

Let $\alpha$ be an ideal of $ST(\{x, y, z\})$ generated by elements

\[ g_1 = yyz, g_2 = yxz, g_3 = yxy. \]

Consider element

\[ w = 2xyyz - 2yyxz + 2zyxy. \]

Then

\[ w = 2xg_1 - 2yg_2 + 2zg_3. \]

It follows

\[ w \in \{\alpha\} \cap ST(\{x, y, z\}). \]

One can easily check that there is no $\lambda_1, \lambda_2, \lambda_3 \in K$ so that

\[ w = \lambda_1 [x, g_1] + \lambda_2 [y, g_2] + \lambda_3 [z, g_3]. \]

Then $w \notin \alpha$. Hence, by P. Cohn’s criterion (Theorem 2.2 of [4]),

\[ ST(\{x, y, z\})/\alpha \text{ is not special.} \]
8. Remarks and open questions

1. Let $A = C[x]$ be an algebra with multiplication

$$a \star b = b \int_0^x (\int_0^x a \, dx) \, dx.$$  \hspace{1cm} (8)

$(A, \star)$ is not a Zinbiel algebra. This algebra was considered in [5]. It was proved that it satisfies the following identities

$$a \star (b \star c) - b \star (a \star c) = 0,$$

$$(a, b, c, d) + (b, c, a, d) + (c, a, b, d) = 0$$

where $(a, b, c) = a \star (b \star c) - (a \star b) \star c$. Moreover, it was proved that algebra $A$ with respect to commutator $[a, b]_\star = a \star b - b \star a$ is a Tortkara algebra. A question on speciality of $(A, [\cdot, \cdot]_\star)$ was posed.

We show that answer is positive. Let $B = C[x]$ be an algebra with multiplication

$$a \triangledown b = b \int_0^x (\int_0^x a \, dx) \, dx + \int_0^x a \, dx \int_0^x b \, dx.$$  

Then $(B, \triangledown)$ is a Zinbiel algebra. For $\triangledown$ multiplication we define commutator $[a, b]_\triangledown = a \triangledown b - b \triangledown a$. Note that $[a, b]_\star = [a, b]_\triangledown$. So, $A^{(-)}$ is isomorphic to $B^{(-)}$. Hence, $(A, [\cdot, \cdot]_\star)$ is special.

2. It is shown in [5] that an algebra with identities (4) is not Zinbiel but under the commutator product is Tortkara. What about speciality of these algebras?

3. Let $k_m$ be kernel of the natural homomorphism from free Tortkara algebra to free special Tortkara algebra on $m$ generators. An element of the ideal $k_m$ is called a $s$-identity. We showed that $k_2 = (0)$. Are there $s$-identities for $m > 2$?

4. Is it true the analogue of Lemma 6.1 for $m > 2$ generators? Whenever it is valid for $m$ generators, it immediately follows speciality of $T(\{x_1, \ldots, x_m\})$, in particular, $k_m = (0)$.

5. What is the analogue of classical Artin’s theorem for Tortkara algebras? In other words, if $A$ is a Tortkara algebra and $alg < a, b >$ is the subalgebra generated by $a, b \in A$, then what is the characterization of $Var(alg < a, b >)$?

Acknowledgments. The authors are grateful to Professor I.P. Shestakov for his discussion and for essential comments.

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