PERTURBATION THEORY AND TECHNIQUES¹

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1. Introduction. There are several equivalent formulations of the problem of quantizing an interacting field theory. The list includes canonical quantization, path integral (or functional) techniques, stochastic quantization, “unified” methods such as the Batalin-Vilkovisky formalism, and techniques based on the realizations of field theories as low energy limits of string theory. The problem of obtaining an exact nonperturbative description of a given quantum field theory is most often a very difficult one. Perturbative techniques, on the other hand, are abundant, and common to all of the quantization methods mentioned above is that they admit particle interpretations in this formalism.

The basic physical quantities that one wishes to calculate in a relativistic \((d+1)\)-dimensional quantum field theory are the S-matrix elements

\[
S_{ba} = \langle \psi_a(t) | S | \psi_b(t) \rangle_{\text{in}} \quad (1)
\]

between in and out states at large positive time \(t\). The scattering operator \(S\) is then defined by writing \(1\) in terms of initial free particle (descriptor) states as

\[
S_{ba} =: \langle \psi_b(0) | S | \psi_a(0) \rangle . \quad (2)
\]

Suppose that the hamiltonian of the given field theory can be written as \(H = H_0 + H'\), where \(H_0\) is the free part and \(H'\) the interaction hamiltonian. The time evolution of the in and out states are governed by the total hamiltonian \(H\). They can be expressed in terms of descriptor states which evolve in time with \(H_0\) in the interaction picture and correspond to free particle states. This leads to the Dyson formula

\[
S = T \exp \left( -i \int_{-\infty}^{\infty} dt \ H_I(t) \right) , \quad (3)
\]

where \(T\) denotes time-ordering and \(H_I(t) = \int d^d x \ H_{\text{int}}(x,t)\) is the interaction hamiltonian in the interaction picture, with \(H_{\text{int}}(x,t)\) the interaction hamiltonian density, which deals with essentially free fields. This formula expresses \(S\) in terms of interaction picture operators acting on free particle states in \(2\) and is the first step towards Feynman perturbation theory.

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For many analytic investigations, such as those which arise in renormalization theory, one is interested instead in the Green’s functions of the quantum field theory, which measure the response of the system to an external perturbation. For definiteness, let us consider a free real scalar field theory in \((d + 1)\)-dimensions with lagrangian density

\[
\mathcal{L} = \frac{1}{2} \partial_{\mu} \phi \partial^{\mu} \phi - \frac{1}{2} m^2 \phi^2 + \mathcal{L}_{\text{int}}
\]

(4)

where \(\mathcal{L}_{\text{int}}\) is the interaction lagrangian density which we assume has no derivative terms. The interaction hamiltonian density is then given by \(\mathcal{H}_{\text{int}} = -\mathcal{L}_{\text{int}}\). Introducing a real scalar source \(J(x)\), we define the normalized “partition function” through the vacuum expectation values

\[
Z[J] = \frac{\langle 0 | S[J] | 0 \rangle}{\langle 0 | S[0] | 0 \rangle}
\]

(5)

where \(|0\rangle\) is the normalized perturbative vacuum state of the quantum field theory given by (4) (defined to be destroyed by all field annihilation operators), and

\[
S[J] = T \exp \left( i \int d^{d+1} x \left( \mathcal{L}_{\text{int}} + J(x) \phi(x) \right) \right)
\]

(6)

from the Dyson formula. This partition function is the generating functional for all Green’s functions of the quantum field theory, which are obtained from (5) by taking functional derivatives with respect to the source and then setting \(J(x) = 0\). Explicitly, in a formal Taylor series expansion in \(J\) one has

\[
Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \prod_{i=1}^{n} \int d^{d+1} x_i \ J(x_i) \ G^{(n)}(x_1, \ldots, x_n),
\]

(7)

whose coefficients are the Green’s functions

\[
G^{(n)}(x_1, \ldots, x_n) := \frac{\langle 0 | T \left[ \exp \left( i \int d^{d+1} x \ \mathcal{L}_{\text{int}} \right) \phi(x_1) \cdots \phi(x_n) \right] | 0 \rangle}{\langle 0 | T \exp \left( i \int d^{d+1} x \ \mathcal{L}_{\text{int}} \right) | 0 \rangle}.
\]

(8)

It is customary to work in momentum space by introducing the Fourier transforms

\[
\tilde{J}(k) = \int d^{d+1} x \ e^{i k \cdot x} \ J(x), \quad \tilde{G}^{(n)}(k_1, \ldots, k_n) = \prod_{i=1}^{n} \int d^{d+1} x_i \ e^{i k_i \cdot x_i} \ G^{(n)}(x_1, \ldots, x_n),
\]

(9)

in terms of which the expansion (7) reads

\[
Z[J] = \sum_{n=0}^{\infty} \frac{i^n}{n!} \prod_{i=1}^{n} \int \frac{d^{d+1} k_i}{(2\pi)^{d+1}} \ \tilde{J}(-k_i) \ \tilde{G}^{(n)}(k_1, \ldots, k_n).
\]

(10)

The generating functional (10) can be written as a sum of Feynman diagrams with source insertions. Diagrammatically, the Green’s function is an infinite series of graphs which can be represented symbolically as

\[
\tilde{G}^{(n)}(k_1, \ldots, k_n) = \text{Diagrammatic Representation}
\]

(11)
where the \( n \) external lines denote the source insertions of momenta \( k_i \) and the bubble denotes the sum over all Feynman diagrams constructed from the interaction vertices of \( L_{\text{int}} \).

The Green’s functions can also be used to describe scattering amplitudes, but there are two important differences between the graphs \([11]\) and those which appear in scattering theory. In the present case, external lines carry propagators, i.e. the free field Green’s functions

\[
\Delta(x - y) = \langle 0 \| T \left[ \phi(x) \phi(y) \right] \| 0 \rangle = \langle x | (\Box + m^2)^{-1} | y \rangle = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{i}{p^2 - m^2 + i \epsilon} e^{-ip(x-y)}
\]

where \( \epsilon \to 0^+ \) regulates the mass shell contributions, and their momenta \( k_i \) are off-shell in general \((k_i^2 \neq m^2)\). By the LSZ theorem, the S-matrix element is then given by the multiple on-shell residue of the Green’s function in momentum space as

\[
\langle k_1', \ldots, k_n' | S - \Pi | k_1, \ldots, k_l \rangle = \lim_{k_1, \ldots, k_n \to m^2} \prod_{i=1}^{n} \frac{1}{\sqrt{c_i}} (k_i^2 - m^2) \prod_{j=1}^{l} \frac{1}{\sqrt{c_j}} (k_j^2 - m^2)
\]

\[
\times G^{(n+m)}(-k_1', \ldots, -k_n', k_1, \ldots, k_l),
\]

where \( i c_i, i c_j \) are the residues of the corresponding particle poles in the exact two-point Green’s function.

This article deals with the formal development and computation of perturbative scattering amplitudes in relativistic quantum field theory, along the lines outlined above. Initially we deal only with real scalar field theories of the sort \([1]\) in order to illustrate the concepts and technical tools in as simple and concise a fashion as possible. These techniques are common to most quantum field theories. Fermions and gauge theories are then separately treated afterwards, focusing on the methods which are particular to them.

2. Diagrammatics. The pinnacle of perturbation theory is the technique of Feynman diagrams. Here we develop the basic machinery in a quite general setting and use it to analyse some generic features of the terms comprising the perturbation series.

Wick’s theorem. The Green’s functions \([8]\) are defined in terms of vacuum expectation values of time-ordered products of the scalar field \( \phi(x) \) at different spacetime points. Wick’s theorem expresses such products in terms of normal-ordered products, defined by placing each field creation operator to the right of each field annihilation operator, and in terms of two-point Green’s functions \([12]\) of the free field theory (propagators). The consequence of this theorem is the haffnian formula

\[
\langle 0 \| T \left[ \phi(x_1) \cdots \phi(x_n) \right] \| 0 \rangle = \left\{ \begin{array}{ll} 0 & , \quad n = 2k - 1, \\
\sum_{\pi \in S_{2k}} \prod_{i=1}^{k} \langle 0 \| T \left[ \phi(x_{\pi(2i-1)}) \phi(x_{\pi(2i)}) \right] \| 0 \rangle & , \quad n = 2k. \end{array} \right.
\]

The formal Taylor series expansion of the scattering operator \( S \) may now be succinctly summarized into a diagrammatic notation by using Wick’s theorem. For each spacetime integration \( \int d^{d+1}x_i \) we introduce a vertex with label \( i \), and from each vertex there emanates some lines corresponding to field insertions at the point \( x_i \). If the operators represented by two lines appear in a two-point function according to \((14)\), i.e. they are contracted, then these two lines are connected together. The \( S \) operator is then represented as a sum over all such Wick diagrams,
bearing in mind that topologically equivalent diagrams correspond to the same term in $S$. Two diagrams are said to have the same pattern if they differ only by a permutation of their vertices. For any diagram $\mathcal{D}$ with $n(\mathcal{D})$ vertices, the number of ways of interchanging vertices in $n(\mathcal{D})!$. The number of diagrams per pattern is always less than this number. The symmetry number $S(\mathcal{D})$ of $\mathcal{D}$ is the number of permutations of vertices that give the same diagram. The number of diagrams with the pattern of $\mathcal{D}$ is then $n(\mathcal{D})!/S(\mathcal{D})$.

In a given pattern, we write the contribution to $S$ of a single diagram $\mathcal{D}$ as $\frac{1}{n(\mathcal{D})!} \theta(\mathcal{D})^\circ$, where the combinatorial factor comes from the Taylor expansion of $S$, the colons denote normal ordering of quantum operators, and $\theta(\mathcal{D})^\circ$ contains spacetime integrals over normal-ordered products of the fields. Then all diagrams with the pattern of $\mathcal{D}$ contribute $\theta(\mathcal{D})^\circ / S(\mathcal{D})$ to $S$. Only the connected diagrams $\mathcal{D}_r, r \in \mathbb{N}$ (those in which every vertex is connected to every other vertex) contribute and we can write the scattering operator in a simple form which eliminates contributions from all disconnected diagrams as

$$S = \exp \left( \sum_{r=1}^{\infty} \frac{\theta(\mathcal{D}_r)}{S(\mathcal{D}_r)} \right)^\circ.$$  \hfill (15)

**Feynman rules.** Feynman diagrams in momentum space are defined from the Wick diagrams above by dropping the labels on vertices (and also the symmetry factors $S(\mathcal{D})^{-1}$), and by labelling the external lines by the momenta of the initial and final particles that the corresponding field operators annihilate. In a spacetime interpretation, external lines represent on-shell physical particles while internal lines of the graph represent off-shell virtual particles ($k^2 \neq m^2$). Physical particles interact via the exchange of virtual particles. An arbitrary diagram is then calculated via the Feynman rules

$$p_1 \cdots p_n = \int \frac{d^{d+1}p}{(2\pi)^{d+1}} \frac{i}{p^2 - m^2 + i\varepsilon},$$

for a monomial interaction $L_{\text{int}} = \frac{g}{m} \phi^n$.

**Irreducible Green's functions.** A one-particle irreducible (1PI) or proper Green’s function is given by a sum of diagrams in which each diagram cannot be separated by cutting one internal line. In momentum space, it is defined without the overall momentum conservation delta-function factors and without propagators on external lines. For example, the two particle 1PI Green’s function

$$k \quad \text{1PI} \quad k =: \Sigma(k)$$ \hfill (17)

is called the self-energy. If $G(k)$ is the complete two-point function in momentum space, then
one has
\[ G(k) := \frac{k}{k^2 - m^2 - \Sigma(k)} \]  
and thus it suffices to calculate only 1PI diagrams.

The 1PI effective action, defined by the Legendre transformation \( \Gamma[\phi] := -i \ln Z[J] - \int d^{d+1}x \ J(x) \phi(x) \) of (14), is the generating functional for proper vertex functions and it can be represented as a functional of only the vacuum expectation value of the field \( \phi \), i.e. its classical value. In the semi-classical (WKB) approximation, the one-loop effective action is given by
\[ \Gamma[\phi] = S[\phi] + \frac{i}{\hbar} \Tr \ln (1 + \Delta V''[\phi]) + O(\hbar^2) \]
where we have denoted \( S[\phi] = \int d^{d+1}x \ L \) and \( V[\phi] = -L_{\text{int}} \), and for each term in the infinite series we define \( x_{n+1} := x_1 \). The first term in (19) is the classical contribution and it can be represented in terms of connected tree diagrams. The second term is the sum of contributions of one-loop diagrams constructed from \( n \) propagators \( -i \Delta(x - y) \) and \( n \) vertices \( -i V''[\phi] \). The expansion may be carried out to all orders in terms of connected Feynman diagrams, and the result of the above Legendre transformation is to select only the one-particle irreducible diagrams and to replace the classical value of \( \phi \) by an arbitrary argument. All information about the quantum field theory is encoded in this effective action.

**Parametric representation.** Consider an arbitrary proper Feynman diagram \( D \) with \( n \) internal lines and \( v \) vertices. The number \( \ell \) of independent loops in the diagram is the number of independent internal momenta in \( D \) when conservation laws at each vertex have been taken into account, and it is given by \( \ell = n + 1 - v \). There is an independent momentum integration variable \( k_i \) for each loop, and a propagator for each internal line as in (16). The contribution of \( D \) to a proper Green’s function with \( r \) incoming external momenta \( p_i \), with \( \sum_{i=1}^r p_i = 0 \), is given by
\[ \hat{I}_D(p) = \frac{V(D)}{S(D)} \prod_{i=1}^n \int \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \frac{i}{k_i^2 - m^2 + i\epsilon} \prod_{j=1}^v (2\pi)^{d+1} \delta^{(d+1)}(P_j - K_j) \]  
where \( V(D) \) contains all contributions from the interaction vertices of \( L_{\text{int}} \), and \( P_j \) (resp. \( K_j \)) is the sum of incoming external momenta \( p_{lj} \) (resp. internal momenta \( k_{lj} \)) at vertex \( j \) with respect to a fixed chosen orientation of the lines of the graph. After resolving the delta-functions in terms of independent internal loop momenta \( k_1, \ldots, k_\ell \) and dropping the overall momentum conservation delta-function along with the symmetry and vertex factors in (20), one is left with
a set of momentum space integrals

$$I_\mathcal{D}(p) = \prod_{i=1}^\ell \int \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \prod_{j=1}^n \frac{i}{a_j(k, p) + i\epsilon}$$

(21)

where $a_j(k, p)$ are functions of both the internal and external momenta.

It is convenient to exponentiate propagators using the Schwinger parametrization

$$\frac{i}{a_j + i\epsilon} = \int_0^{\infty} d\alpha_j \ e^{i\alpha_j(a_j + i\epsilon)},$$

(22)

and after some straightforward manipulations one may write the Feynman parametric formula

$$\prod_{j=1}^n \frac{i}{a_j(k, p) + i\epsilon} = (n-1)! \prod_{j=1}^n \int_0^1 d\alpha_j \ \delta\left(1 - \sum_{j} \alpha_j\right) \frac{\Delta \mathcal{D}(k; \alpha, p)^n}{D_\mathcal{D}(k; \alpha, p)}$$

(23)

where $D_\mathcal{D}(k; \alpha, p) := \sum_{j} \alpha_j [a_j(k, p) + i\epsilon]$ is generically a quadratic form

$$D_\mathcal{D}(k; \alpha, p) = \frac{1}{2} \sum_{i,j=1}^\ell k_i \cdot Q_{ij}(\alpha)k_j + \sum_{i=1}^\ell L_i(p) \cdot k_i + \lambda (p^2).$$

(24)

The positive symmetric matrix $Q_{ij}$ is independent of the external momenta $p_i$, invertible, and has non-zero eigenvalues $Q_1, \ldots, Q_\ell$. The vectors $L_i^\mu$ are linear combinations of the $p_i^\mu$, while $\lambda(p^2)$ is a function of only the Lorentz invariants $p_i^2$. After some further elementary manipulations, the loop diagram contribution $I_\mathcal{D}(p)$ may be written as

$$I_\mathcal{D}(p) = \prod_{j=1}^n \int_0^1 d\alpha_j \ \frac{1}{Q_i(\alpha)^2} \int \frac{d^{d+1}k_i}{(2\pi)^{d+1}} \delta\left(1 - \sum_{j} \alpha_j\right) \times \left(\frac{1}{2} \sum_i k_i^2 + \lambda (p^2) - \frac{1}{2} \sum_{i,j} L_i(p) \cdot Q^{-1}(\alpha)_{ij} L_j(p)\right)^{-n}.$$ 

(25)

Finally, the integrals over the loop momenta $k_i$ may be performed by Wick rotating them to euclidean space and using the fact that the combination of $\ell$ integrations in $\mathbb{R}^{d+1}$ has $O((d+1)\ell)$ rotational invariance. The contribution from the entire Feynman diagram $\mathcal{D}$ thereby reduces to the calculation of the parametric integrals

$$I_\mathcal{D}(p) = \frac{\Gamma\left(n - \frac{(d+1)\ell}{2}\right)}{(2\pi)^{\frac{(d+1)\ell}{2}}} \prod_{j=1}^\ell \frac{1}{Q_i(\alpha)^2} \int_0^1 d\alpha_j \ \delta\left(1 - \sum_{j} \alpha_j\right) \left(\lambda (p^2) - \frac{1}{2} \sum_{i,j} L_i(p) \cdot Q^{-1}(\alpha)_{ij} L_j(p)\right)^{-n} \frac{1}{\left(\frac{(d+1)\ell}{2}\right)}$$

(26)

where $\Gamma(s)$ is the Euler gamma-function.

### 3. Regularization

The parametric representation (20) is generically convergent when $2n - (d + 1)\ell > 0$. When divergent, the infinities arise from the lower limits of integration $\alpha_j \to 0$. This is just the parametric representation of the large $k$ divergence of the original Feynman amplitude (21). Such ultraviolet divergences plague the very meaning of a quantum field theory and must be dealt with in some way. We will now quickly tour the standard methods
of ultraviolet regularization for such loop integrals, which is prelude to the renormalization program that removes the divergences (in a renormalizable field theory). Here we consider regularization simply as a means of justification for the various formal manipulations that are used in arriving at expressions such as \([26]\).

**Momentum cutoff.** Cutoff regularization introduces a mass scale \(\Lambda\) into the quantum field theory and throws away the Fourier modes of the fields for spatial momenta \(k\) with \(|k| > \Lambda\). This regularization spoils Lorentz invariance. It is also non-local. For example, if we restrict to a hypercube in momentum space, so that \(|k_i| < \Lambda\) for \(i = 1, \ldots, d\), then
\[
\int_{|k| < \Lambda} \frac{d^d k}{(2\pi)^d} e^{i k \cdot x} = \prod_{i=1}^d \sin(\Lambda x^i)/\pi x^i
\]
which is a delta-function in the limit \(\Lambda \to \infty\) but is non-local for \(\Lambda < \infty\). The regularized field theory is finite order by order in perturbation theory and depends on the cutoff \(\Lambda\).

**Lattice regularization.** We can replace the spatial continuum by a lattice \(\mathcal{L}\) of rank \(d\) and define a lagrangian on \(\mathcal{L}\) by
\[
L_{\mathcal{L}} = \frac{1}{2} \sum_{i \in S(\mathcal{L})} \dot{\phi}_i^2 + J \sum_{\langle i,j \rangle \in L(\mathcal{L})} \phi_i \phi_j + \sum_{i \in S(\mathcal{L})} V(\phi_i)
\]
where \(S(\mathcal{L})\) is the set of sites \(i\) of the lattice on each of which is situated a time-dependent function \(\phi_i\), and \(L(\mathcal{L})\) is the collection of links connecting pairs \(\langle i, j \rangle\) of nearest-neighbour sites \(i, j\) on \(\mathcal{L}\). The regularized field theory is now local, but still has broken Lorentz invariance. In particular, it suffers from broken rotational symmetry. If \(\mathcal{L}\) is hypercubic with lattice spacing \(a\), i.e. \(\mathcal{L} = (\mathbb{Z}a)^d\), then the momentum cutoff is at \(\Lambda = a^{-1}\).

**Pauli-Villars regularization.** We can replace the propagator \(i(k^2 - m^2 + i \epsilon)^{-1}\) by \(i(k^2 - m^2 + i \epsilon)^{-1} + i \sum_j c_j (k^2 - M_j^2 + i \epsilon)^{-1}\), where the masses \(M_j \gg m\) are identified with the momentum cutoff as \(\min\{M_j\} = \Lambda \to \infty\). The mass-dependent coefficients \(c_j\) are chosen to make the modified propagator decay rapidly as \((k^2)^{-N-1}\) at \(k \to \infty\), which gives the \(N\) equations \((m^2)^i + \sum_j c_j (M_j^2)^i = 0, i = 0, 1, \ldots, N - 1\). This regularization preserves Lorentz invariance (and other symmetries that the field theory may possess) and is local in the following sense. The modified propagator can be thought of as arising through the alteration of the lagrangian density \([1]\) by \(N\) additional scalar fields \(\varphi_j\) of masses \(M_j\) with
\[
L_{PV} = \frac{1}{2} \partial_\mu \varphi \partial^\mu \varphi - \frac{1}{2} m^2 \varphi^2 + \sum_{j=1}^N \left( \frac{1}{2} \partial_\mu \varphi_j \partial^\mu \varphi_j - \frac{1}{2} M_j^2 \varphi_j^2 \right) + L_{\text{int}}[\Phi]
\]
where \(\Phi := \phi + \sum_j \sqrt{c_j} \varphi_j\). The contraction of the \(\Phi\) field thus produces the required propagator. However, the \(c_j\)'s as computed above are generically negative numbers and so the lagrangian density \([28]\) is not hermitian (as \(\Phi \neq \Phi^\dagger\)). It is possible to make \([28]\) formally hermitian by redefining the inner product on the Hilbert space of physical states, but this produces negative norm states. This is no problem at energy scales \(E \ll M_j\) on which the extra particles decouple and the negative probability states are invisible.

**Dimensional regularization.** Consider a euclidean space integral \(\int d^4 k (k^2 + a^2)^{-r}\) arising after Wick rotation from some loop diagram in \((3+1)\)-dimensional scalar field theory. We replace this integral by its \(D\)-dimensional version
\[
\int \frac{d^D k}{(k^2 + a^2)^r} = \frac{\pi^D}{(2\pi)^D} \frac{(a^2)^{\frac{D}{2} - r}}{(r-1)!} \Gamma \left( r - \frac{D}{2} \right) .
\]
This integral is absolutely convergent for $D < 2r$. We can analytically continue the result of this integration to the complex plane $D \in \mathbb{C}$. As an analytic function, the only singularities of the Euler function $\Gamma(z)$ are poles at $z = 0, -1, -2, \ldots$. In particular, $\Gamma(z)$ has a simple pole at $z = 0$ of residue 1. If we write $D = 4 + \epsilon$ with $|\epsilon| \to 0$, then the integral is proportional to $\Gamma(r - 2 - \frac{\epsilon}{2})$ and $\epsilon$ plays the role of the regulator here. This regularization is Lorentz invariant (in $D$ dimensions) and is distinguished as having a dimensionless regularization parameter $\epsilon$. This parameter is related to the momentum cutoff $\Lambda$ by $\Lambda^{-1} = \ln(\Lambda/m)$, so that the limit $\epsilon \to 0$ corresponds to $\Lambda \to \infty$.

Infrared divergences. Thus far we have only considered the ultraviolet behaviour of loop amplitudes in quantum field theory. When dealing with massless particles ($m = 0$ in (11)) one has to further worry about divergences arising from the $k \to 0$ regions of Feynman integrals. After Wick rotation to euclidean momenta, one can show that no singularities arise in a given Feynman diagram as some of its internal masses vanish provided that all vertices have superficial degree of divergence $d + 1$, the external momenta are not exceptional (i.e. no partial sum of the incoming momenta $p_i$ vanishes), and there is at most one soft external momentum. This result assumes that renormalization has been carried out at some fixed euclidean point. The extension of this property when the external momenta are continued to physical on-shell values is difficult. The Kinoshita-Lee-Nauenberg theorem states that, as a consequence of unitarity, transition probabilities in a theory involving massless particles are finite when the sum over all degenerate states (initial and final) is taken. This is true order by order in perturbation theory in bare quantities or if minimal subtraction renormalization is used (to avoid infrared or mass singularities in the renormalization constants).

4. Fermion Fields. We will now leave the generalities of our pure scalar field theory and start considering the extensions of our previous considerations to other types of particles. Henceforth we will primarily deal with the case of (3 + 1)-dimensional spacetime. We begin by indicating how the rudiments of perturbation theory above apply to the case of Dirac fermion fields. The lagrangian density is

$$\mathcal{L}_F = \overline{\psi}(i \partial - m)\psi + \mathcal{L}'$$

(30)

where $\psi$ are four-component Dirac fermion fields in (3+1)-dimensions, $\overline{\psi} := \psi^\dagger \gamma^0$ and $\partial = \gamma^\mu \partial_\mu$ with $\gamma^\mu$ the generators of the Clifford algebra $\{\gamma^\mu, \gamma^\nu\} = 2\eta^{\mu\nu}$. The lagrangian density $\mathcal{L}'$ contains couplings of the Dirac fields to other field theories, such as the scalar field theories considered previously.

Wick’s theorem for anticommuting Fermi fields leads to the pfaffian formula

$$\langle 0 \mid T[\psi(1) \cdots \psi(n)] \mid 0 \rangle = \begin{cases} 0, & n = 2k - 1, \\ \frac{1}{2^k k!} \sum_{\pi \in S_{2k}} \text{sgn}(\pi) \prod_{i=1}^{k} \langle 0 \mid T[\psi(\pi(2i-1)) \psi(\pi(2i))] \mid 0 \rangle, & n = 2k \end{cases}$$

(31)

where for compactness we have written in the argument of $\psi(i)$ the spacetime coordinate, the Dirac index, and a discrete index which distinguishes $\psi$ from $\overline{\psi}$. The non-vanishing contractions in (31) are determined by the free fermion propagator

$$\Delta_F(x - y) = \langle 0 \mid T[\psi(x) \overline{\psi}(y)] \mid 0 \rangle = \langle x | (i \partial - m)^{-1} | y \rangle = i \int \frac{\text{d}^4 p}{(2\pi)^4} \frac{\hat{p} + m}{p^2 - m^2 + i \epsilon} e^{-i p(x-y)}. \quad (32)$$

Perturbation theory now proceeds exactly as before. Suppose that the coupling lagrangian density in (30) is of the form $\mathcal{L}' = \overline{\psi}(x) V(x) \psi(x)$. Both the Dyson formula (8) and the
The diagrammatic formula (15) are formally the same in this instance. For example, in the formal expansion in powers of \( \int d^4x \mathcal{L}' \), the vacuum-to-vacuum amplitude (the denominator in (16)) will contain field products of the form \( \prod_{i=1}^{n} \int d^4x_i \bra{0} T [ \bar{\psi}(x_i) V(x_i) \psi(x_i)] \ket{0} \) which correspond to fermion loops. Before applying Wick’s theorem the fields must be rearranged as \( \text{tr} \prod_{i=1}^{n} V(x_i) \psi(x_i) \bar{\psi}(x_{i+1}) \) (with \( x_{n+1} := x_1 \)), where tr is the \( 4 \times 4 \) trace over spinor indices. This reordering introduces the familiar minus sign for a closed fermion loop, and one has

\[
(-1)^n \prod_{i=1}^{n} \int d^4x_i \text{tr} \prod_{j=1}^{n-1} \Delta_F(x_j - x_{j+1}) V(x_{j+1}) \Delta_F(x_{j+1} - x_{j+2}).
\]

(F33)

Feynman rules are now described as follows. Fermion lines are oriented to distinguish a particle from its corresponding antiparticle, and carry both a four-momentum label \( p \) as well as a spin polarization index \( r = 1, 2 \). Incoming fermions (resp. antifermions) are described by the wavefunctions \( u_p^{(r)} \) (resp. \( \bar{u}_p^{(r)} \)), while outgoing fermions (resp. antifermions) are described by the wavefunctions \( \bar{u}_p^{(r)} \) (resp. \( u_p^{(r)} \)). Here \( u_p^{(r)} \) and \( \bar{u}_p^{(r)} \) are the classical spinors, i.e. the positive and negative energy solutions of the Dirac equation \( (\not{p} - m)u_p^{(r)} = (\not{p} + m)\bar{u}_p^{(r)} = 0 \). Matrices are multiplied along a Fermi line, with the head of the arrow on the left. Closed fermion loops produce an overall minus sign as in (32), and the multiplication rule gives the trace of Dirac matrices along the lines of the loop. Unpolarized scattering amplitudes are summed over the spins of final particles and averaged over the spins of initial particles using the completeness relations for spinors

\[
\sum_{r=1,2} u_p^{(r)} \bar{u}_p^{(r)} = \not{p} + m, \quad \sum_{r=1,2} u_p^{(r)} \bar{u}_p^{(s)} = \not{p} - m, \tag{34}
\]

leading to basis independent results. Polarized amplitudes are computed using the spinor bilinears \( \bar{u}_p^{(r)} \gamma^\mu u_p^{(s)} = \bar{u}_p^{(r)} \gamma^\mu v_p^{(s)} = 2 p^\mu \delta^{rs} \), \( \bar{u}_p^{(r)} v_p^{(s)} = -\bar{u}_p^{(r)} v_p^{(s)} = 2 m \delta^{rs} \), and \( \bar{u}_p^{(r)} v_p^{(s)} = 0 \).

When calculating fermion loop integrals using dimensional regularization, one utilizes the Dirac algebra in \( D \) dimensions

\[
\gamma^\mu \gamma_\mu = \eta^\mu_\mu = D, \\
\gamma_\mu \not{p} \gamma^\mu = (2 - D) \not{p}, \\
\gamma^\mu \not{k} \gamma_\mu = 4 p \cdot k + (D - 4) \not{k} \not{k}, \\
\gamma^\mu \not{q} \gamma_\mu = -2 \not{q} \not{q} - (D - 4) \not{p} \not{q}, \\
\text{tr} \not{1} = 4, \quad \text{tr} \gamma^{\mu_1} \cdots \gamma^{\mu_{2k-1}} = 0, \quad \text{tr} \gamma^{\mu_1} \gamma^{\nu_1} = 4 \eta^{\mu_1 \nu_1}, \\
\text{tr} \gamma^{\mu_1} \gamma^{\nu} \gamma^\sigma = 4 (\eta^{\mu_1 \nu} \eta^{\rho \sigma} - \eta^{\mu_1 \rho} \eta^{\nu \sigma} + \eta^{\mu_1 \sigma} \eta^{\nu \rho}), \tag{35}
\]

Specific to \( D = 4 \) dimensions are the trace identities

\[
\text{tr} \gamma^5 = \text{tr} \gamma^\mu \gamma^\nu \gamma^5 = 0, \quad \text{tr} \gamma^\mu \gamma^\nu \gamma^\rho \gamma^\sigma \gamma^5 = -4 i \epsilon^{\mu \nu \rho \sigma} \tag{36}
\]

where \( \gamma^5 := i \gamma^0 \gamma^1 \gamma^2 \gamma^3 \). Finally, loop diagrams evaluated with the fermion propagator \( \frac{1}{(k^2 - m^2 + i\epsilon)^D} \) require a generalization of the momentum space integral (20) given by

\[
\int \frac{d^Dk}{(2\pi)^D} \frac{1}{(k^2 + 2k \cdot p + a^2 + i\epsilon)^r} = \frac{(-\pi)^{D/2} \Gamma(r - \frac{D}{2})}{(2\pi)^D (r - 1)!} \frac{1}{(a^2 - p^2 + i\epsilon)^{r - \frac{D}{2}}}. \tag{37}
\]
From this formula we can extract expressions for more complicated Feynman integrals which are tensorial, i.e. which contain products of momentum components $k^\mu$ in the numerators of their integrands, by differentiating (37) with respect to the external momentum $p^\mu$.

5. Gauge Fields. The issues we have dealt with thus far have interesting difficulties when dealing with gauge fields. We will now discuss some general aspects of the perturbation expansion of gauge theories using as prototypical examples quantum electrodynamics (QED) and quantum chromodynamics (QCD) in four spacetime dimensions.

Quantum electrodynamics. Consider the QED lagrangian density

$$\mathcal{L}_{\text{QED}} = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \mu^2 A_\mu A^\mu + \bar{\psi} (i \slashed{\partial} - e \slashed{A} - m) \psi$$

(38)

where $A_\mu$ is a $U(1)$ gauge field in (3 + 1)-dimensions and $F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu$ is its field strength tensor. We have added a small mass term $\mu \to 0$ for the gauge field which at the end of calculations should be taken to vanish in order to describe real photons (as opposed to the soft photons described by (38)). This is done in order to cure the infrared divergences generated in scattering amplitudes due to the masslessness of the photon, i.e. the long range nature of the electromagnetic interaction. The Bloch-Nordsieck theorem in QED states that infrared divergences cancel for physical processes, i.e. for processes with an arbitrary number of undetectable soft photons.

Perturbation theory proceeds in the usual way via the Dyson formula, Wick’s theorem, and Feynman diagrams. The gauge field propagator is given by

$$\langle 0 | T[A_\mu(x) A_\nu(y)] | 0 \rangle = \langle x | \left[ \eta_{\mu\nu} \left( \Box + \mu^2 \right) - \partial_\mu \partial_\nu \right]^{-1} | y \rangle = i \int \frac{d^4 p}{(2\pi)^4} \frac{-\eta_{\mu\nu} + p_\mu p_\nu}{p^2 - \mu^2 + i\epsilon} e^{-ip(x-y)}$$

(39)

and is represented by a wavy line. The fermion-fermion-photon vertex is

$$\begin{array}{c}
\gamma \\
\mu
\end{array} = -ie\gamma_\mu .$$

(40)

An incoming (resp. outgoing) soft photon of momentum $k$ and polarization $r$ is described by the wavefunction $\psi^{(r)}(k)$ (resp. $\psi^{(r)}(k)^*$), where the polarization vectors $\epsilon^{(r)}_\mu(k)$, $r = 1, 2, 3$ solve the vector field wave equation $(\Box + \mu^2)A_\mu = \partial_\mu A^\mu = 0$ and obey the orthonormality and completeness conditions

$$\epsilon^{(r)}_\mu(k)^* \cdot \epsilon^{(s)}_\nu(k) = -\delta^{rs} , \sum_{r=1}^{3} \epsilon^{(r)}_\mu(k) \epsilon^{(r)}_\nu(k)^* = -\eta_{\mu\nu} + \frac{k_\mu k_\nu}{\mu^2} ,$$

(41)

along with $k \cdot \epsilon^{(r)}(k) = 0$. All vector indices are contracted along the lines of the Feynman graph. All other Feynman rules are as previously.

Quantum chromodynamics. Consider nonabelian gauge theory in (3 + 1)-dimensions minimally coupled to a set of fermion fields $\psi^A$, $A = 1, \ldots, N_f$ each transforming in the fundamental representation of the gauge group $G$ whose generators $T^a$ satisfy the commutation relations $[T^a, T^b] = f^{abc} T^c$. The lagrangian density is given by

$$\mathcal{L}_{\text{QCD}} = -\frac{1}{4} F_{\mu\nu}^{a} F^{a \mu\nu} + \frac{1}{2\alpha} (\partial_\mu A^a_\mu)^2 + \partial_\mu \bar{\eta} D^\mu \eta + \sum_{A=1}^{N_f} \bar{\psi}^A (i \slashed{\partial} - m_A) \psi^A ,$$

(42)
where \( F^{\alpha}_{\mu\nu} = \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + f^{abc} A^b_\mu A^c_\nu \) and \( D_\mu = \partial_\mu + i e R(T^a) A^a_\mu \) with \( R \) the pertinent representation of \( G \) (\( R(T^a)_{bc} = f^a_{bc} \) for the adjoint representation and \( R(T^a) = T^a \) for the fundamental representation). The first term is the Yang-Mills lagrangian density, the second term is the covariant gauge-fixing term, and the third term contains the Faddeev-Popov ghost fields \( \eta \) which transform in the adjoint representation of the gauge group.

Feynman rules are straightforward to write down and are given by

\[
\frac{-i \delta^{ab} \eta_{\mu\nu} + (\alpha - 1) \frac{k_\mu k_\nu}{k^2}}{k^2 + i \epsilon}
\]

\[
e f^{abc} \left[ \eta_{\lambda\mu}(p_\nu - k_\nu) + \eta_{\mu\nu}(q_\lambda - p_\lambda) + \eta_{\nu\lambda}(k_\mu - q_\mu) \right]
\]

\[
- i e^2 \left[ f^{eab} f^{ecd} (\eta_{\mu\lambda} \eta_{\nu\rho} - \eta_{\mu\rho} \eta_{\nu\lambda}) + f^{eac} f^{ebd} (\eta_{\mu\nu} \eta_{\lambda\rho} - \eta_{\mu\rho} \eta_{\lambda\nu}) + f^{ead} f^{ebc} (\eta_{\mu\nu} \eta_{\lambda\rho} - \eta_{\nu\rho} \eta_{\lambda\mu}) \right]
\]

\[
\frac{i \delta^{ab}}{k^2 + i \epsilon}
\]

\[
e k_\mu f^{abc}
\]

(43)

where wavy lines represent gluons and dashed lines represent ghosts. Feynman rules for the fermions are exactly as before, except that now the vertex (40) is multiplied by the colour matrix \( T^a \). All colour indices are contracted along the lines of the Feynman graph. Colour factors may be simplified by using the identities

\[
\text{Tr} \ R^a R^b = \frac{\dim R}{\dim G} C_2(R) \delta^{ab}, \quad R^a R^a = C_2(R), \quad R^a R^b R^a = \left( C_2(R) - \frac{1}{2} C_2(G) \right) R^b
\]

(44)

where \( R^a := R(T^a) \) and \( C_2(R) \) is the quadratic Casimir invariant of the representation \( R \) (with value \( C_2(G) \) in the adjoint representation). For \( G = SU(N) \), one has \( C_2(G) = N \) and \( C_2(N) = (N^2 - 1)/2N \) for the fundamental representation.

The cancellation of infrared divergences in loop amplitudes of QCD is far more delicate than in QED, as there is no analog of the Bloch-Nordsieck theorem in this case. The Kinoshita-Lee-Nauenberg theorem guarantees that, at the end of any perturbative calculation, these divergences must cancel for any appropriately defined physical quantity. However, at a given order of perturbation theory, a physical quantity typically involves both virtual and real emission contributions that are separately infrared divergent. Already at two-loop level these divergences have a highly
intricate structure. Their precise form is specified by the Catani colour-space factorization formula which also provides an efficient way of organizing amplitudes into divergent parts, which ultimately drop out of physical quantities, and finite contributions.

The computation of multi-gluon amplitudes in nonabelian gauge theory is rather complicated when one uses polarization states of vector bosons. A much more efficient representation of amplitudes is provided by adopting a helicity (or circular polarization) basis for external gluons. In the spinor-helicity formalism, one expresses positive and negative helicity polarization vectors in terms of massless Weyl spinors $|k^\pm\rangle := \frac{1}{2} (\mathbb{1} \pm \gamma_5) \psi_k = \frac{1}{2} (\mathbb{1} \pm \gamma_5) \psi_k$ through

$$e_\mu'^\pm (k; q) = \pm \frac{\langle q^\mp | \gamma_\mu | k^\mp \rangle}{\sqrt{2} \langle q^\mp | k^\mp \rangle}$$

(45)

where $q$ is an arbitrary null reference momentum which drops out of the final gauge invariant amplitudes. The spinor products are crossing symmetric, antisymmetric in their arguments, and satisfy the identities

$$\langle k_i^- | k_j^+ \rangle \langle k_j^- | k_i^+ \rangle = 2 k_i \cdot k_j ,$$

$$\langle k_i^- | k_j^+ \rangle \langle k_i^- | k_j^+ \rangle = \langle k_i^- | k_i^- \rangle \langle k_j^+ | k_j^+ \rangle + \langle k_i^- | k_j^+ \rangle \langle k_j^- | k_i^+ \rangle .$$

(46)

Any amplitude with massless external fermions and vector bosons can be expressed in terms of spinor products. Conversely, the spinor products offer the most compact representation of helicity amplitudes which can be related to more conventional amplitudes described in terms of Lorentz invariants. For loop amplitudes, one uses a dimensional regularization scheme in which all helicity states are kept four-dimensional and only internal loop momenta are continued to $D = 4 + \epsilon$ dimensions.

6. Computing Loop Integrals. At the very heart of perturbative quantum field theory is the problem of computing Feynman integrals for multi-loop scattering amplitudes. The integrations typically involve serious technical challenges and for the most part are intractable by straightforward analytical means. We will now survey some of the computational techniques that have been developed for calculating quantum loop amplitudes which arise in the field theories considered previously.

Asymptotic expansion. In many physical instances one is interested in scattering amplitudes in certain kinematical limits. In this case one may perform an asymptotic expansion of multi-loop diagrams whose coefficients are typically non-analytic functions of the perturbative expansion parameter $\hbar$. The main simplification which arises comes from the fact that the expansions are done before any momentum integrals are evaluated. In the limits of interest, Taylor series expansions in different selected regions of each loop momentum can be interpreted in terms of subgraphs and co-subgraphs of the original Feynman diagram.

Consider a Feynman diagram $\mathcal{D}$ which depends on a collection $\{Q_i\}$ of large momenta (or masses), and a collection $\{m_i, q_i\}$ of small masses and momenta. The prescription for the large momentum asymptotic expansion of $\mathcal{D}$ may be summarized in the diagrammatic formula

$$\lim_{Q \to \infty} \mathcal{D}(Q; m, q) = \sum_{\mathcal{D} \subset \mathcal{D}} (\mathcal{D} / \mathcal{D})(m, q) \star (T_{\{m_i, q_i\}} \mathcal{D})(Q; m, q) ,$$

(47)

where the sum runs through all subgraphs $\mathcal{D}$ of $\mathcal{D}$ which contain all vertices where a large momentum enters or leaves the graph and is one-particle irreducible after identifying these
vertices. The operator $T_{\{m_2,q_0\}}$ performs a Taylor series expansion before any integration is carried out, and the notation $({\mathcal D}/d) \ast (T_{\{m_2,q_0\}})\,d$ indicates that the subgraph $d \subset {\mathcal D}$ is replaced by its Taylor expansion in all masses and external momenta of $d$ that do not belong to the set $\{Q_i\}$. The external momenta of $d$ which become loop momenta in $\mathcal{D}$ are also considered to be small. The loop integrations are then performed only after all these expansions have been carried out. The diagrams $\mathcal{D}/d$ are called co-subgraphs.

The subgraphs become massless integrals in which the scales are set by the large momenta. For instance, in the simplest case of a single large momentum $Q$ one is left with integrals over propagators. The co-subgraphs may contain small external momenta and masses, but the resulting integrals are typically much simpler than the original one. A similar formula is true for large mass expansions, with the vertex conditions on subdiagrams replace by propagator conditions. For example, consider the asymptotic expansion of the two-loop double bubble diagram

\[ \begin{align*} \quad = & \quad + 2 \quad + \quad \star \quad \star \quad \star \end{align*} \]

in the region $q^2 \ll m^2$, where $m$ is the mass of the inner loop. The subgraphs (to the right of the stars) are expanded in all external momenta including $q$ and reinserted into the fat vertices of the co-subgraphs (to the left of the stars). Once such asymptotic expansions are carried out, one may attempt to reconstruct as much information as possible about the given scattering amplitude by using the method of Padé approximation which requires knowledge of only part of the expansion of the diagram. By construction, the Padé approximation has the same analytic properties as the exact amplitude.

Brown-Feynman reduction. When considering loop diagrams which involve fermions or gauge bosons, one encounters tensorial Feynman integrals. When these involve more than three distinct denominator factors (propagators), they require more than two Feynman parameters for their evaluation and become increasingly complicated. The Brown-Feynman method simplifies such higher-rank integrals and effectively reduces them to scalar integrals which typically require fewer Feynman parameters for their evaluation.

To illustrate the idea behind this method, consider the one-loop rank 3 tensor Feynman integral

\[ J^{\mu\nu\lambda} = \int \frac{d^D k}{(2\pi)^D} \frac{k^\mu k^\nu k^\lambda}{k^2 (k^2 - \mu^2) (q - k)^2 ((k - q)^2 + \mu^2) (k^2 + 2k \cdot p)} \]  

(49)

where $p$ and $q$ are external momenta with the mass-shell conditions $p^2 = (p - q)^2 = m^2$. By Lorentz invariance, the general structure of the integral \[ \text{(49)} \] will be of the form

\[ J^{\mu\nu\lambda} = a^{\mu\nu} p^\lambda + b^{\mu\nu} q^\lambda + c^\mu s^{\nu\lambda} + c^\nu s^{\mu\lambda} \]  

(50)

where $a^{\mu\nu}$, $b^{\mu\nu}$ are tensor-valued functions and $c^\mu$ a vector-valued function of $p$ and $q$. The symmetric tensor $s^{\mu\nu}$ is chosen to project out components of vectors transverse to both $p$ and
\[ q_\mu s^{\mu
u} = q_\mu s^{\mu
u} = 0, \text{ with the normalization } s^{\mu
u} = D - 2. \]

Solving these constraints leads to the explicit form

\[ s^{\mu
u} = \eta^{\mu
u} - \frac{m^2 q^\mu q^\nu + q^2 p^\mu p^\nu - (p \cdot q) (q^\mu p^\nu + p^\mu q^\nu)}{m^2 q^2 - (p \cdot q)^2}. \]  

(51)

To determine the as yet unknown functions \( a^{\mu
u}, b^{\mu
u} \) and \( c^\mu \) above, we first contract both sides of the decomposition (50) with \( p^\mu \) and \( q^\mu \) to get

\[ 2p_\lambda J^{\mu\nu\lambda} = 2m^2 a^{\mu
u} + 2(p \cdot q) b^{\mu
u}, \quad 2q_\lambda J^{\mu\nu\lambda} = 2(p \cdot q) a^{\mu
u} + 2q^2 b^{\mu
u}. \]

(52)

Inside the integrand of (49), we then use the trivial identities

\[ 2k \cdot p = (k^2 + 2k \cdot p) - k^2, \quad 2q \cdot k = k^2 + q^2 - (k - q)^2 \]

(53)

to write the left-hand sides of (52) as the sum of rank 2 Feynman integrals which, with the exception of the one multiplied by \( q^2 \) from (53), have one less denominator factor. This formally determines the coefficients \( a^{\mu
u} \) and \( b^{\mu
u} \) in terms of a set of rank 2 integrations. The vector function \( c^\mu \) is then found from the contraction

\[ J^{\mu\nu}\nu = p_\nu a^{\mu\nu} + q_\nu b^{\mu\nu} + (D - 2) c^\mu. \]

(54)

This contraction eliminates the \( k^2 \) denominator factor in the integrand of (49) and produces a vector-valued integral. Solving the system of algebraic equations (52) and (54) then formally determines the rank 3 Feynman integral (49) in terms of rank 1 and rank 2 Feynman integrals. The rank 2 Feynman integrals thus generated can then be evaluated in the same way by writing a decomposition for them analogous to (50) and solving for them in terms of vector-valued and scalar-valued Feynman integrals. Finally, the rank 1 integrations can be solved for in terms of a set of scalar-valued integrals, most of which have fewer denominator factors in their integrands.

Generally, any one-loop amplitude can be reduced to a set of basic integrals by using the Passarino-Veltman reduction technique. For example, in supersymmetric amplitudes of gluons any tensor Feynman integral can be reduced to a set of scalar integrals, i.e. Feynman integrals in a scalar field theory with a massless particle circulating in the loop, with rational coefficients. In the case of \( \mathcal{N} = 4 \) supersymmetric Yang-Mills theory, only scalar box integrals appear.

**Reduction to master integrals.** While the Brown-Feynman and Passarino-Veltman reductions are well-suited for dealing with one-loop diagrams, they become rather cumbersome for higher loop computations. There are other more powerful methods for reducing general tensor integrals into a basis of known integrals called master integrals. Let us illustrate this technique on a scalar example. Any scalar massless two-loop Feynman integral can be brought into the form

\[ I(p) = \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D k'}{(2\pi)^D} \prod_{j=1}^t \Delta_j^{-l_j} \prod_{i=1}^q \Sigma_i^{n_i}, \]

(55)

where \( \Delta_j \) are massless scalar propagators depending on the loop momenta \( k, k' \) and the external momenta \( p_1, \ldots, p_n \), and \( \Sigma_i \) are scalar products of a loop momentum with an external momentum or of the two loop momenta. The topology of the corresponding Feynman diagram is uniquely determined by specifying the set \( \Delta_1, \ldots, \Delta_t \) of \( t \) distinct propagators in the graph, while the integral itself is specified by the powers \( l_j \geq 1 \) of all propagators, by the selection \( \Sigma_1, \ldots, \Sigma_q \) of \( q \) scalar products and by their powers \( n_i \geq 0 \).
The integrals in a class of diagrams of the same topology with the same denominator dimension \( r = \sum_j l_j \) and same total scalar product number \( s = \sum_i n_i \) are related by various identities. One class follows from the fact that the integral over a total derivative with respect to any loop momentum vanishes in dimensional regularization as
\[
\int \frac{d^D k}{(2\pi)^D} \frac{\partial J(k)}{\partial k^\mu} = 0,
\]
where \( J(k) \) is any tensorial combination of propagators, scalar products and loop momenta. The resulting relations are called integration by parts identities and for two-loop integrals can be cast into the form
\[
\int \frac{d^D k}{(2\pi)^D} \int \frac{d^D k'}{(2\pi)^D} \frac{\partial f(k, k', p)}{\partial k^\mu} = 0 = \int \frac{d^D k}{(2\pi)^D} \int \frac{d^D k'}{(2\pi)^D} \frac{\partial f(k, k', p)}{\partial k'^\mu},
\]
where \( f(k, k', p) \) is a scalar function containing propagators and scalar products, and \( v^\mu \) is any internal or external momentum. For a graph with \( \ell \) loops and \( n \) independent external momenta, this results in a total of \( \ell (n + \ell) \) relations.

In addition to these identities, one can also exploit the fact that all Feynman integrals (55) are Lorentz scalars. Under an infinitesimal Lorentz transformation \( p^\mu \to p^\mu + \delta p^\mu \), with \( \delta p^\mu = p^\nu \delta \epsilon^\mu_\nu \), \( \delta \epsilon^\mu_\nu = -\delta \epsilon^\nu_\mu \), one has the invariance condition \( I(p + \delta p) = I(p) \), which leads to the linear homogeneous differential equations
\[
\sum_{i=1}^n \left( p^\nu_i \frac{\partial}{\partial p^\mu_i} - p^\mu_i \frac{\partial}{\partial p^\nu_i} \right) I(p) = 0 .
\]
This equation can be contracted with all possible antisymmetric combinations of \( p_{i\mu} p_{j\nu} \) to yield linearly independent Lorentz invariance identities for (55).

Using these two sets of identities, one can either obtain a reduction of integrals of the type (55) to those corresponding to a small number of simpler diagrams of the same topology and diagrams of simpler topology (fewer denominator factors), or a complete reduction to diagrams with simpler topology. The remaining integrals of the topology under consideration are called irreducible master integrals. These momentum integrals cannot be further reduced and have to be computed by different techniques. For instance, one can apply a Mellin-Barnes transformation of all propagators given by
\[
\frac{1}{(k^2 + a)^l} = \frac{1}{(l - 1)!} \int_{-i\infty}^{i\infty} dz \frac{a^{-z}}{(k^2)^l z} \Gamma(l + z) \Gamma(-z) ,
\]
where the contour of integration is chosen to lie to the right of the poles of the Euler function \( \Gamma(l + z) \) and to the left of the poles of \( \Gamma(-z) \) in the complex \( z \)-plane. Alternatively, one may apply the negative dimension method in which \( D \) is regarded as a negative integer in intermediate calculations and the problem of loop integration is replaced with that of handling infinite series. When combined with the above methods, it may be used to derive powerful recursion relations among scattering amplitudes. Both of these techniques rely on an explicit integration over the loop momenta of the graph, their differences occurring mainly in the representations used for the propagators.

The procedure outlined above can also be used to reduce a tensor Feynman integral to scalar integrals, as in the Brown-Feynman and Passarino-Veltman reductions. The tensor integrals are expressed as linear combinations of scalar integrals of either higher dimension or with propagators raised to higher powers. The projection onto a tensor basis takes the form (55) and can thus be reduced to master integrals.
7. String Theory Methods. The realizations of field theories as the low-energy limits of string theory provides a number of powerful tools for the calculation of multi-loop amplitudes. They may be used to provide sets of diagrammatic computational rules, and they also work well for calculations in quantum gravity. In this final part we shall briefly sketch the insights into perturbative quantum field theory that are provided by techniques borrowed from string theory.

String theory representation. String theory provides an efficient compact representation of scattering amplitudes. At each loop order there is only a single closed string diagram, which includes within it all Feynman graphs along with the contributions of the infinite tower of massive string excitations. Schematically, at one-loop order one has

\[
\begin{array}{c}
\includegraphics[width=0.5\textwidth]{string_diagram}\end{array}
\]

\[= + + + \ldots . \tag{59}\]

The terms arising from the heavy string modes are removed by taking the low-energy limit in which all external momenta lie well below the energy scale set by the string tension. This limit picks out the regions of integration in the string diagram corresponding to particle-like graphs, but with different diagrammatic rules.

Given these rules, one may formulate a purely field theoretic framework which reproduces them. In the case of QCD, a key ingredient is the use of a special gauge originally derived from the low-energy limit of tree-level string amplitudes. This is known as the Gervais-Neveu gauge and it is defined by the gauge-fixing lagrangian density

\[
\mathcal{L}_{\text{GN}} = -\frac{1}{2} \text{Tr} \left( \partial_\mu A^\mu - \frac{i e}{\sqrt{2}} A_\mu A^\mu \right)^2.
\] (60)

This gauge choice simplifies the colour factors that arise in scattering amplitudes. The string theory origin of gauge theory amplitudes is then most closely mimicked by combining this gauge with the background field gauge, in which one decomposes the gauge field into a classical background field and a fluctuating quantum field as \(A_\mu = A^{\text{cl}}_\mu + A^{\text{qu}}_\mu\), and imposes the gauge-fixing condition \(D^{\text{cl}}_\mu A^{\text{qu}}_\mu = 0\) where \(D^{\text{cl}}_\mu\) is the background field covariant derivative evaluated in the adjoint representation of the gauge group. This hybrid gauge is well-suited for computing the effective action, with the quantum part describing gluons propagating around loops and the classical part describing gluons emerging from the loops. The leading loop momentum behaviour of one-particle irreducible graphs with gluons in the loops is very similar to that of graphs with scalar fields in the loops.

Supersymmetric decomposition. String theory also suggests an intimate relationship with supersymmetry. For example, at tree-level QCD is effectively supersymmetric because a multi-gluon tree amplitude contains no fermion loops, and so the fermions may be taken to lie in the adjoint representation of the gauge group. Thus pure gluon tree-amplitudes in QCD are identical to those in supersymmetric Yang-Mills theory. They are connected by supersymmetric Ward identities to amplitudes with fermions (gluinos) which drastically simplify computations. In supersymmetric gauge theory these identities hold to all orders of perturbation theory.

At one-loop order and beyond, QCD is not supersymmetric. However, one can still perform a supersymmetric decomposition of a QCD amplitude for which the supersymmetric components of the amplitude obey the supersymmetric Ward identities. Consider, for example, a one-loop multi-gluon scattering amplitude. The contribution from a fermion propagating in the loop can
be decomposed into the contribution of a complex scalar field in the loop plus a contribution from an $\mathcal{N} = 1$ chiral supermultiplet consisting of a complex scalar field and a Weyl fermion. The contribution from a gluon circulating in the loop can be decomposed into contributions of a complex scalar field, an $\mathcal{N} = 1$ chiral supermultiplet, and an $\mathcal{N} = 4$ vector supermultiplet comprising three complex scalar fields, four Weyl fermions and one gluon all in the adjoint representation of the gauge group. This decomposition assumes the use of a supersymmetry preserving regularization.

The supersymmetric components have important cancellations in their leading loop momentum behaviour. For instance, the leading large loop momentum power in an $n$-point 1PI graph is reduced from $|k|^n$ down to $|k|^{n-2}$ in the $\mathcal{N} = 1$ amplitude. Such a reduction can be extended to any amplitude in supersymmetric gauge theory and is related to the improved ultraviolet behaviour of supersymmetric amplitudes. For the $\mathcal{N} = 4$ amplitude, further cancellations reduce the leading power behaviour all the way down to $|k|^{n-4}$. In dimensional regularization, $\mathcal{N} = 4$ supersymmetric loop amplitudes have a very simple analytic structure owing to their origins as the low-energy limits of superstring scattering amplitudes. The supersymmetric Ward identities in this way can be used to provide identities among the non-supersymmetric contributions. For example, in $\mathcal{N} = 1$ supersymmetric Yang-Mills theory one can deduce that fermion and gluon loop contributions are equal and opposite for multi-gluon amplitudes with maximal helicity violation.

Scattering amplitudes in twistor space. The scattering amplitude in QCD with $n$ incoming gluons of the same helicity vanishes, as does the amplitude with $n-1$ incoming gluons of one helicity and one gluon of the opposite helicity for $n \geq 3$. The first non-vanishing amplitudes are the maximal helicity violating (MHV) amplitudes involving $n-2$ gluons of one helicity and two gluons of the opposite helicity. Stripped of the momentum conservation delta-function and the group theory factor, the tree-level amplitude for a pair of gluons of negative helicity is given by

$$A(k) = e^{n-2} \langle k_r^- | k_i^+ \rangle \prod_{i=1}^{n} \langle k_r^- | k_{i+1}^+ \rangle^{-1}.$$  

(61)

This amplitude depends only on the holomorphic (negative chirality) Weyl spinors. The full MHV amplitude (with the momentum conservation delta-function) is invariant under the conformal group $SO(4,2) \cong SU(2,2)$ of four dimensional Minkowski space. After a Fourier transformation of the positive chirality components, the complexification $SL(4,\mathbb{C})$ has an obvious four dimensional representation acting on the positive and negative chirality spinor products. This representation space is isomorphic to $\mathbb{C}^4$ and is called twistor space. Its elements are called twistors.

Wavefunctions and amplitudes have a known behaviour under the $\mathbb{C}^\times$-action which rescales twistors, giving the projective twistor space $\mathbb{CP}^3$ or $\mathbb{RP}^3$ according to whether the twistors are complex-valued or real-valued. The Fourier transformation to twistor space yields (due to momentum conservation) the localization of an MHV amplitude to a genus 0 holomorphic curve $\mathbb{CP}^1$ of degree 1 in $\mathbb{CP}^3$ (or to a real line $\mathbb{RP}^1 \subset \mathbb{RP}^3$). It is conjectured that, generally, an $\ell$-loop amplitude with $p$ gluons of positive helicity and $q$ gluons of negative helicity is supported on a holomorphic curve in twistor space of degree $q + \ell - 1$ and genus $\leq \ell$. The natural interpretation of this curve is as the worldsheet of a string. The perturbative gauge theory may then be described in terms of amplitudes arising from the couplings of gluons to a string. This twistor string theory is a topological string theory which gives the appropriate framework for understanding the twistor properties of scattering amplitudes. This framework has been used to analyse MHV tree diagrams and one-loop $\mathcal{N} = 4$ supersymmetric amplitudes of gluons.
See Also. Effective field theories. Perturbative renormalization theory and BRST. Analytic properties of the S-matrix. Dispersion relations. Scattering: Asymptotic completeness and bound states. Scattering: Fundamental concepts and tools. Stationary phase approximation. Supersymmetric particle models. Gauge theories from strings.

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