Decay and fragmentation in an open Bose-Hubbard chain

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We analyze the decay of ultracold atoms from an optical lattice with loss form a single lattice site. If the initial state is dynamically stable a suitable amount of dissipation can stabilize a Bose-Einstein condensate, such that it remains coherent even in the presence of strong interactions. A transition between different dynamical phases is observed if the initial state is dynamically unstable. This transition is analyzed here in detail. For strong interactions, the system relaxes to an entangled quantum state with remarkable statistical properties: The atoms bunch in a few “breathers” forming at random positions. Breathers at different positions are coherent, such that they can be used in precision quantum interferometry and other applications.

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I. INTRODUCTION

Decoherence and dissipation, caused by the irreversible coupling of a quantum system to its environment, represent a major obstacle for a long-time coherent control of quantum states. Sophisticated methods have been developed to maintain coherence also in the presence of dissipation with applications in quantum control and quantum information processing [1, 2]. Only recently a new paradigm has been put forward: Dissipation can be used as a powerful tool to steer the dynamics of complex quantum systems if it can be accurately controlled. It was shown theoretically that dissipative processes can be constructed which allows one to prepare pure states for quantum computation [3, 4], to implement universal quantum computation [5], or to deterministically generate entangled quantum states [6, 7].

Ultracold atoms in optical lattices provide a distinguished system to realize new methods of quantum control and quantum state engineering [8]. Both the coherent dynamics of the atoms as well as dissipative processes can be accurately controlled, including the localized manipulation with single-site resolution [9, 10]. In this article we analyze the dynamics induced by the interplay of localized particle dissipation and strong atom-atom interactions. If the interaction strength exceeds a threshold, two meta-stable equilibria emerge which can be used to prepare either an almost pure Bose-Einstein condensate or a macroscopically entangled “breather” state.

The meta-stable breather states show remarkable statistical properties: The atoms relax to a coherent superposition of bunches localized at different lattice positions. Driven by particle loss and interactions, almost all atoms localize in one of the non-dissipative wells. The meta-stable state corresponds to a coherent superposition of these localized modes and thus to a macroscopically entangled quantum state. Because of the tunable large number of atoms forming the breather state, they may serve as a distinguished probe of decoherence and the emergence of classicality. Furthermore, the breather states generalize the so-called NOON states enabling interferometry beyond the standard quantum limit [11, 12]. As particle loss is an elementary and omnipresent dissipation process, this method may be generalized to a variety of open quantum systems well beyond the dynamics of ultracold atoms, e.g., to optical fiber setups [13] or hybrid quantum systems [14, 15].

The paper is organized as follows. After introducing the model system in Sec.II we analyze breather states in small systems, which allow for a numerically exact simulation of the quantum many-body dynamics in Sec.III. In extended lattices discussed in Sec.IV the localized modes correspond to so-called discrete breathers. The emerging meta-stable quantum state is more complex, as the atoms can localize in a variety of lattice sites. Nevertheless, one can identify “breather-states” by the number fluctuations and the correlations between neighboring sites. The formation of breather states can be understood to a large extent within a semi-classical phase space picture introduced in Sec.V. We analyze the flow of phase space distribution functions such as the Wigner or the Husimi function. To leading order it is given by a classical Liouvillian flow which is equivalent to a dissipative Gross-Pitaevskii equation. The emergence of breather states can then be linked to a classical bifurcation of the associated mean-field dynamics. While this semiclassical approach obviously cannot describe the coherence of the quantum state or the formation of entanglement, it correctly predicts the critical interaction strength above which breather states are formed.

II. PARTICLE LOSS IN AN OPTICAL LATTICE

Optical lattices offer unique possibilities in controlling the quantum dynamics of ultracold atoms [16, 22]. In particular, experimental parameters such as the strength of the atom-atom interactions can be readily tuned by a
variation of the lattice depth. Recently, several experiments demonstrated a local control of the atomic dynamics. Single site access can be implemented optically either by increasing the lattice period [14, 23] or by pushing the resolution of the optical imaging system to the limit [12, 13]. Furthermore, the advanced imaging systems in these experiments enable a precise measurement of the atom number per site. An even higher resolution can be realized by a focused electron beam [10, 11]. However, the interaction of the electron beam with the atomic cloud is generally dissipative: Atoms are ionized and then removed from the lattice by a static electric field. At the same time, this method enables the detection of single atoms with outstanding spatial resolution.

The coherent dynamics of ultracold atoms in deep optical lattices is described by the celebrated Bose-Hubbard Hamiltonian [21]

\[ \hat{H} = -J \sum_j \left( \hat{a}_j \hat{a}^\dagger_{j+1} + \hat{a}_j^\dagger \hat{a}_{j+1} \right) + \frac{U}{2} \sum_j \hat{a}_j^\dagger \hat{a}_j^\dagger \hat{a}_j \hat{a}_j, \]

where \( \hat{a}_j \) and \( \hat{a}_j^\dagger \) are the bosonic annihilation and creation operators in mode \( j \), \( J \) denotes the tunneling matrix element between the wells and \( U \) the interaction strength. We set \( \hbar = 1 \), thus measuring energy in frequency units. This model assumes that the lattice is sufficiently deep, such that the dynamics takes place in the lowest Bloch band only. Throughout this paper we consider finite lattices with \( M \) sites with periodic boundary conditions, i.e., we identify the sites \( j = 0 \) and \( j = M \).

In this article we analyze the non-equilibrium dynamics triggered by localized dissipation implemented either by a resonant laser or a focused electron beam. The atoms are removed rapidly and irreversibly from the lattice, such that the dissipative dynamics can be described by a Markovian Master equation,

\[ \frac{d}{dt} \hat{\rho} = -i[\hat{H}, \hat{\rho}] + \mathcal{L}\hat{\rho}. \] (2)

Particle loss is described by the Liouvillian [22, 23, 24]

\[ \mathcal{L}_{\text{loss}}\hat{\rho} = -\frac{1}{2} \sum_j \gamma_j \left( \hat{a}_j^\dagger \hat{a}_j \hat{\rho} + \hat{\rho} \hat{a}_j^\dagger \hat{a}_j - 2 \hat{a}_j \hat{a}_j^\dagger \hat{\rho} \right), \] (3)

where \( \gamma_j \) denotes the loss rate at site \( j \). Furthermore, the atoms experience phase noise due to collisions with the background gas [30–32] or the absorption and spontaneous emission of photons from the lattice beams [33]. This dissipation process is described by the Liouvillian

\[ \mathcal{L}_{\text{noise}}\hat{\rho} = -\frac{\kappa}{2} \sum_j \left( \hat{n}_j^2 \hat{\rho} + \hat{\rho} \hat{n}_j^2 - 2 \hat{n}_j \hat{n}_j \hat{\rho} \right). \] (4)

Phase noise can be made very small, e.g., by detuning the optical lattice far from the atomic resonance such that we can assume \( \kappa = 0 \) in most simulations. A detailed analysis of decoherence due to phase noise is then provided in Sec. III D.

For numerical simulations, we will make use of the quantum jump method [34, 35], where the density matrix \( \hat{\rho} \) is decomposed into state vectors,

\[ \hat{\rho} = \frac{1}{L} \sum_{\ell=1}^L |\Psi\rangle \langle \Psi|, \] (5)

whose continuous evolution is interrupted by stochastic quantum jumps. The continuous evolution is determined by the Schrödinger equation with the effective non-hermitian Hamiltonian

\[ \hat{H}_{\text{eff}} = \hat{H} - i \left( \frac{\delta p}{\delta t} \right) = \hat{H} - \frac{i}{2} \sum_j \gamma_j \hat{a}_j \hat{a}_j^\dagger - \frac{i\kappa}{2} \sum_j \hat{n}_j^2. \] (6)

Since \( \hat{H}_{\text{eff}} \) is non-hermitian, the state vector \( |\Psi\rangle \) must be renormalized after every time-step. In the case of particle loss, the state vector jumps according to

\[ |\Psi\rangle \rightarrow \frac{\hat{a}_j |\Psi\rangle}{\| \hat{a}_j |\Psi\rangle \|} \] (7)

with a probability

\[ \delta p = \gamma_j \langle \Psi| \hat{a}_j^\dagger \hat{a}_j |\Psi\rangle \delta t \] (8)

during a short time interval \( \delta t \). The full density matrix is recovered by averaging over many of these random trajectories in state space.

III. DECAY IN AN OPEN TRIPLE-WELL TRAP

To begin with, we consider the open Bose-Hubbard trimer as an elementary model system, allowing for numerical exact solutions for rather large particle numbers. Still, this model already exhibits the different dynamical phases we aim to understand. A sketch of this system is provided in Fig. 1 (a). The bosons tunnel at a rate \( J \) between three lattice sites with periodic boundary conditions. Loss occurs only from the site \( j = 2 \) at a rate \( \gamma_2 \). The system is mirror-symmetric with respect to an exchange of sites 1 and 3.

![FIG. 1: (Color online) The model systems studied in the present paper: (a) An open Bose-Hubbard trimer with loss from site 2 and periodic boundary conditions, (b) An extended one-dimensional optical lattice with localized loss from a single lattice site.](image)
A. Atomic correlations

The most obvious effect of particle dissipation is the decrease of the total particle number $n_{\text{tot}}$ in the lattice, which is shown in the top row of Fig. 2. We simulate the dynamics for three different initial states for weak ($U = 0.01J$) and strong ($U = 0.1J$) interactions. A pure Bose-Einstein condensate with an (anti-) symmetric wavefunction

$$|\Psi_\pm\rangle = \frac{1}{\sqrt{N!}} (\hat{a}_1^\dagger \pm \hat{a}_3^\dagger)^N |0\rangle$$

and the Fock state

$$|\Psi_F\rangle = \frac{1}{2\sqrt{(N/2)!}} (\hat{a}_1^\dagger)^{(N/2)} (\hat{a}_3^\dagger)^{(N/2)} |0\rangle,$$

assuming that the initial particle number $N$ is even. The most interesting observation here is that the decay is very slow for the anti-symmetric initial state $|\Psi_-\rangle$. Indeed, this state is a stationary state of the master equation for $U = 0$, such that decay is absent in the non-interacting limit (cf. [28]). The physical reason for this is the destructive interference of atoms tunneling from sites 1 and 3 to the leaky site 2. In the case of strong interactions, tunneling is allowed but weak. Localized states, which will be referred to as breather states, form at the non-dissipative sites. The formation and properties of these states are analyzed in detail in the present paper.

The dissipative dynamics drives the atoms to a very different quantum state depending on the initial state and the interaction strength $U$. To characterize these states we analyze the first and second order correlation functions between different sites of the lattice. The coherence of the many-body quantum state is characterized by the first-order correlation function between the wells $j$ and $\ell$,

$$g_{j,\ell}^{(1)} = \frac{\langle \hat{a}_j^\dagger \hat{a}_\ell \rangle}{\langle \hat{n}_j \rangle \langle \hat{n}_\ell \rangle},$$

which is plotted in the second row of Fig. 2. The symmetric initial state $|\Psi_+\rangle$ is stable for all values of the interaction strength $U$ and the BEC remains approximately pure. In this case, particle dissipation can even increase the purity and coherence of the condensate. This counter-intuitive feature was discussed in detail in [26, 27, 37, 38]. The anti-symmetric state $|\Psi_-\rangle$ is stable only if interactions are weak. For $U = 0.1J$ one observes a sharp decrease of first-order correlation which indicates the destruction of the condensate. The initial state is dynamically unstable such that the atoms relax to a different meta-stable equilibrium state, the breather state.

Density fluctuations and correlations are characterized by the second order correlation function

$$g_{j,\ell}^{(2)} = \frac{\langle \hat{n}_j \hat{n}_\ell \rangle}{\langle \hat{n}_j \rangle \langle \hat{n}_\ell \rangle}.$$

For $j = \ell$, this expression reduces to the normalized second moment of the number operator $\langle \hat{n}_j^2 \rangle / \langle \hat{n}_j \rangle^2$, which quantifies the number fluctuations in the $j$th well. The evolution of the number fluctuations and correlations are
shown in Fig. 2 in the bottom panels. While these quantities are essentially constant for a BEC with a symmetric wave function |Ψ₊⟩, strong anti-correlations develop for the initial state |Ψ₋⟩ in the regime of strong interactions. The (anti-) correlations are also found for the Fock state |Ψ_F⟩, whose experimental preparation can be significantly easier. These results show that the atoms bunch at one of the non-dissipative lattice sites, while the other sites are essentially empty. Nevertheless, as we are going to discuss in detail in section IV, the two contributions localized either at site 1 or 3 remain coherent. The atoms thus relax deterministically to a macroscopically entangled state, also called a Schrödinger cat state (cf. [29]). We will refer to these states as “breather” states as they correspond to the so-called discrete breathers in extended lattices in the semiclassical limit [40–42]. This correspondence will be discussed in detail in Sec. IV.

B. Transition to the breather regime

The meta-stable breather states exists only for strong atomic interactions. The onset of breather formation is analyzed in Fig. 3 where we have plotted the total particle number as well as the first and second order correlations after a fixed propagation time t_{final} = 50 J⁻¹ as a function of the interaction strength U. As one can see for weak interactions, U ≲ 0.01 s⁻¹, the BEC remains almost pure and the density-density correlation function are approximately equal to unity. The characteristic properties of a breather state, strong number fluctuations and anti-correlations between neighboring sites, are observed only for U ≥ 0.01 s⁻¹. The transition to the breather regime can be understood within a semiclassical phase space picture which will be discussed in detail in Sec. IV. This approach predicts a bifurcation of meta-stable states at a critical interaction strength Un_{tot} = 0.4J. Before we come back to this issue, we first characterize the quantum properties of the breather states in more detail.

C. Characterization and Interferometry of the breather state

In a breather state a large number of atoms localize at a single lattice site, leaving the neighboring sites essentially empty. To make this statement more precise, we analyze the full counting statistics and the coherence of the many-body quantum state in detail. Figure 4 (a,b) shows the full counting statistics of the atom number in well 1 and 2, respectively, at time t = 10 J⁻¹ after a breather state has formed. The most important result is that the probability distribution P(n_1) becomes bimodal: Either a breather forms in the first well (n_1 large) or in the third well (n_1 almost zero). The second well is almost empty for large values of the interaction constant U. This stabilizes the breather state as only few atoms are subject to particle loss. For intermediate values of the interaction constant U, one also finds the characteristic bimodal number distribution in the first well. However, the atom number in the second well is larger, such that decay is much stronger.

The two breathers in site 1 and 3 are fully coherent, even for large interactions. To analyze the coherence of the many-body quantum state ρ(t) in more detail, we first note that ρ(t) can be written as the incoherent sum...
precision interferometry beyond the standard quantum limit. In the present setup, the coherence of wells 1 and 3 is guaranteed as in an ordinary NOON state but the total number of atoms forming the NOON state varies statistically. Nevertheless, this is sufficient for precision interferometry.

Starting from the breather state analyzed in the preceding section, we consider an interferometric measurement, where the modes (lattice sites) 1 and 3 are mixed. Assuming that interactions (by tuning a Feshbach resonance) and losses are switched off, the dynamics during the interferometer stage is given by the time evolution operator

$$\hat{U}_{\text{interferometer}} = \exp[-i\hat{H}_{\text{mix}}t],$$

where $\hat{H}_{\text{mix}} = iJ(\hat{a}_1^\dagger \hat{a}_3 - \hat{a}_1^\dagger \hat{a}_3^\dagger)$. In analogy to the parity observable in NOON state interferometry \cite{20}, we record the probability to detect either an even or an odd number of atoms in site 1. Such a measurement is automatically realized by the optical imaging apparatus in the experiments \cite{12,13}.

This probability $P_{\text{even},odd}$ to detect an even or an odd number of atoms is plotted in Fig. 5 (a) as a function of time. $P_{\text{even}}$ approaches unity periodically at times

$$t_{\text{even}} = \left( n + \frac{1}{4} \right) \pi J^{-1}, \quad n = 0, 1, 2, \ldots,$$

which unambiguously proves the coherence of the breather state. Figure 5 (b,c) shows the full counting statistics in site 1 at the beginning of the interferometer stage at $t = 10J^{-1}$ and during the interferometer stage at $t = 13.92J^{-1}$. Destructive interference forbids to detect an odd number of atoms at this time, such that $P_{\text{even}}(t)$ approaches unity.

The interference fringes observed for $P_{\text{even},odd}(t)$ are extremely sharp, which enables precision measurement beyond the standard quantum limit. In the present setup, the detection of a fringe reveals the value of the tunneling rate $J$ with ultra-high precision via equation \cite{14}. Different quantities can be measured by a modified interferometry scheme as described in \cite{20}. An important but very difficult goal is to increase the number of atoms forming a NOON state (see, e.g., \cite{14}), as the measurement uncertainty of this method scales inversely with the particle number $N$. This goal may be archived with the breather states discussed here which are readily generated also for large samples.

D. Entanglement and decoherence

The atoms in a breather or NOON state are strongly entangled: If some atoms are measured at one site, then the remaining atoms will be projected onto the same site
with overwhelming probability. To unambiguously detect this form of multi-partite entanglement, we analyze the variance of the population imbalance \( \Delta (\hat{n}_3 - \hat{n}_1)^2 \), which scales as \( \sim \hat{n}_{\text{tot}}^2 \) for a breather state, while it is bounded by \( \hat{n}_{\text{tot}} \) for a pure product state, \( \hat{n}_{\text{tot}} \) being the total atom number. The variance can thus serve as an entanglement criterion, if the quantum state is pure or, more importantly, if one can assure that a large value of the variance is not due to an incoherent mixture of states localized at site 1 or 3.

We assume that a quantum state is decomposed into pure states, \( \hat{\rho} = L^{-1} \sum_{a=1}^L |\psi_a\rangle \langle \psi_a| \), as it is automatically the case in a quantum jump simulation \[3,4\]. We then introduce the entanglement parameter

\[
E_{r,q} := \langle (\hat{n}_r - \hat{n}_q)^2 \rangle - \langle \hat{n}_r \rangle^2 \langle \hat{n}_q \rangle - \langle \hat{n}_r \rangle \langle \hat{n}_q \rangle \tag{18}
\]

for the wells \( (r,q) \), where \( \langle \cdot \rangle_{a,b} \) denotes the expectation value in the pure state \( |\psi_{a,b}\rangle \). The last term in the parameter \( E_{r,q} \) corrects for the possibility of an incoherent superposition of states localized at sites 1 and 3. For a separable quantum state one can now show that \( E_{1,k} < 0 \) such that a value \( E_{1,k} > 0 \) unambiguously proves entanglement of the atoms. The detailed derivation is given in appendix A.

Figure 6 shows the evolution of the entanglement parameter \( E_{1,3}(t) \) for three different initial states. The symmetric state \( |\Psi_+\rangle \) remains close to a pure BEC, such that \( E_{1,3}(t) \approx 0 \) for all times. In contrast, the anti-symmetric state \( |\Psi_-\rangle \) and the Fock state \( |\Psi_F\rangle \) relax to strongly entangled breather states if interactions are sufficiently strong. In this case we observe large positive values of the entanglement parameter \( E_{1,3}(t) \approx 1500 \) and \( E_{1,3}(t) \approx 500 \), respectively, which clearly reveals the presence of many-particle entanglement. Notably, entanglement is also generated for the Fock state \( |\Psi_F\rangle \) in the regime of weak interactions \( U = 0.01J \). However, this is only a transient phenomena caused by interference effects. The breather states formed in the case of strong interactions are metastable such that the generated entanglement persists for long times until all atoms decay from the trap. Thus, localized particle dissipation enables the robust, deterministic generation of entanglement only in the presence of strong interactions.

Furthermore, entangled breather states provide a sensitive probe for environmentally induced decoherence. Figure 7(a) shows the evolution of the entanglement parameter \( E_{1,3}(t) \) for three different values of the strength of phase noise \( \kappa \) starting from the anti-symmetric initial state \( |\Psi_-\rangle \). Entanglement is generated in all cases, but \( E_{1,3}(t) \) rapidly decreases again when \( \kappa \) is large due to the decoherence of the breathers. Notably, one finds strong number fluctuations \( \langle n_1^{(2)} \rangle > 1 \) and anti-correlations \( \langle n_1^{(2)} \rangle < 1 \) also in the presence of strong phase noise, but interferometry is no longer possible. Figure 7(b) shows the maximum value of \( E_{1,3}(t) \) realized in the presence of phase noise. Entanglement decreases with the noise rate \( \kappa \), in which breather states with large particle numbers are most sensitive. However, entanglement persists up to relatively large values of \( \kappa \approx 10^{-2}J \) in all cases.

### IV. SEMICLASSICAL INTERPRETATION

The formation of breather states can be understood to a large extent within a semi-classical phase space picture. Any quantum state can be represented by a quasi distribution function on the associated classical phase space without loss of information, such as the Wigner or the Husimi function [2]. In the following, we make use of
both distribution functions which are defined as

\[ Q(\alpha_1, \ldots, \alpha_M; t) := \langle \alpha_1, \ldots, \alpha_M | \hat{\rho}(t) | \alpha_1, \ldots, \alpha_M \rangle \tag{19} \]

and

\[
W := \frac{1}{\pi^M} \int \prod_j d^2 \beta_j \exp \left[ \sum_j \alpha_j \beta_j^* - \alpha_j^* \beta_j \right] \\
\times \langle \alpha_1 - \beta_1, \ldots, \alpha_M - \beta_M | \hat{\rho} | \alpha_1 + \beta_1, \ldots, \alpha_M + \beta_M \rangle
\]

respectively. Here, |\alpha_j\rangle is a Glauber coherent state in the jth well and M is the number of lattice sites. The evolution equations of these distribution functions can be calculated systematically using the operator correspondence discussed in [2]. The evolution equation for the Wigner function is discussed in detail in appendix B.

A general feature is that the dynamics of the phase space quasi distribution functions is, to leading order in \(1/N\), given by a classical Liouville equation,

\[
\frac{\partial Q}{\partial t} = -\sum_j \left( \frac{\partial}{\partial \alpha_j} \hat{\alpha}_j + \frac{\partial}{\partial \alpha_j^*} \hat{\alpha}_j^* \right) Q + \text{noise.} \tag{20} \]

Due to the structure of the evolution equation (20), the ‘classical’ Liouvillian flow provides the skeleton of the quantum dynamics (see M. Berry’s quote in [36]) of the Husimi or Wigner function, whereas the quantum corrections vanish with increasing particle number as \(1/N\) [42]. In particular, the Liouvillian approximation neglects phase-space interference effects as well as (anti-)diffusion terms which lead to an elongation of the Wigner- and the Husimi-function [40]. The associated classical flow is given by the dissipative discrete Gross-Pitaevskii equation (DGPE) [42, 47, 48, 53]

\[
i \dot{\alpha}_j = -J(\alpha_{j+1} - \alpha_{j-1}) + U|\alpha_j|^2 \alpha_j - i\gamma_j \alpha_j/2. \tag{21} \]

Figure 8 (a) shows three trajectories of the DGPE for different initial values of the \(\alpha_j = |\alpha_j|e^{i\phi_j}\). We have plotted the evolution of the population imbalance between the first and third site \(z = (|\alpha_3|^2 - |\alpha_1|^2)/n_{\text{tot}}\) vs. the relative phase \(\Delta \phi = \phi_3 - \phi_1\). One observes that the trajectory starting at \(\Delta \phi = 0\) (red) is dynamically stable, such that it remains in the vicinity of the point \((z, \Delta \phi) = (0, 0)\) for all times. In contrast, trajectories starting close to \((z, \Delta \phi) = (0, \pi)\) converge to regions with either \(z > 0\) or \(z < 0\). These regions correspond to self-trapped states, which are known from the non-dissipative case [23, 49, 50]. For \(\gamma_j > 0\), these states become attractively stable, which enables the dynamic formation of breather states.

The corresponding quantum dynamics of an initially pure BEC with a (anti-) symmetric |\Psi_\pm\rangle wave function is shown in Fig. 8 (b,e) and (c,f), respectively. The Husimi function of the initial states are localized around \((z, \Delta \phi) = (0, 0)\) and \((z, \Delta \phi) = (0, \pi)\) as shown in Fig. 8 (b,c). The DGPE then predicts the flow of the Husimi function on a coarse grained scale. Trajectories starting in the vicinity of \((z, \Delta \phi) = (0, 0)\) remain close to their initial states and so does the Husimi function of the symmetric state |\Psi_+\rangle. In contrast, the Husimi function splits up into two fragments localized in the self-trapping regions of phase space for the anti-symmetric initial state |\Psi_-\rangle – a breather state is formed. Finally, in Fig. 8 (d)
FIG. 9: (Color online) Properties of the meta-stable solutions of the non hermitian DGPE \([22]\) for \(J = 1\) and \(\gamma_2 = 0.2\) as a function of the interaction strength \(g = Un_{\text{tot}}\). (a) Decay rate per atom \(\Gamma\) and (b) relative occupation of the first well \(|\alpha_1|^2\). The icons on the right indicate the density distribution in the three wells and the dynamical stability for large \(g\).

the Husimi function of a Fock state is depicted. In this case also the dynamics leads to the split of the function in two parts, as Fig. 8(g) illustrates. However, the number of fluctuations and correlations is less pronounced.

The semi-classical picture predicts the fragmentation of the condensate but, of course, cannot assert the coherence and thus the entanglement of the fragments which is a genuine quantum feature. However, it correctly predicts the stability of an initial state and the emergence of breathers. Thus we can infer the critical interaction strength for the transition to the breather regime from the associated “classical” dynamics. To this end we analyze the meta-stable states of the DGPE which are defined as the solutions of the equation

\[
-J(\alpha_{\ell-1} + \alpha_{\ell+1}) + U|\alpha_\ell|^2\alpha_\ell - \frac{i}{2}\delta_{\ell,2}\alpha_\ell = (\mu - i\Gamma/2)\alpha_\ell.
\]

(22)

Here and in the following, we denote by \(\vec{\alpha} = (\alpha_1, \ldots, \alpha_M)^t\) the vector of all amplitudes \(\alpha_j\). The meta-stable states are not stationary states of the DGPE in a strict sense, as the norm and thus the effective nonlinearity \(g = U||\vec{\alpha}||^2\) decays with a rate \(\Gamma\). However, if decay is slow enough and if the solutions are dynamically stable, the time evolution will follow these quasi steady states adiabatically (cf., e.g., \([51]\)).

The properties of the meta-stable states, their decay rate and their density distribution are summarized in Fig. 9 as a function of the effective nonlinearity \(g\). In the linear case \(g = 0\), three solution exist which are obtained by a simple diagonalization of the single-particle Hamiltonian. Of particular interest is the anti-symmetric state

\[
\vec{\alpha}_\text{as} = \frac{1}{\sqrt{2}}(1, 0, -1),
\]

(23)

which exists for all \(g\) and has a vanishing decay rate \(\Gamma\). With increasing interaction strength \(g\), new solutions come into being. At a critical value \(g_{\text{cr}} = 0.4\), the anti-symmetric state \(\vec{\alpha}_\text{as}\) bifurcates and two breather solutions emerge. These breathers are strongly localized in one of the non-decaying wells \(j = 1, 3\). Due to the symmetry of the system, both have the same decay rate \(\Gamma\).

For weak interactions, the state \(\vec{\alpha}_\text{as}\) dominates the dynamics as its decay rate \(\Gamma\) vanishes. However, this is no longer possible for \(g > g_{\text{cr}}\) as these states become dynamically unstable as shown in Fig. 10 (a). Instead, the breathers dominate the dynamics. Their decay rate is rather small \([44, 46]\) and, most importantly, they are attractively stable as shown in Fig. 10 (b). Thus, a breather is formed dynamically during the time evolution for most initial conditions if \(g\) is large enough. The remaining meta-stable states are marginally unstable as shown in Fig. 10 (c,d).

V. DECAY IN EXTENDED LATTICES

Next, we are going to discuss how localized single particle loss affects the dynamics in a more realistic extended lattice. Also in this case a breather emerges when the interaction strength exceeds a critical value. In the following we will analyze the breather formation quantitatively and derive a formula for the critical interaction strength, which depends on the size of the optical lattice. The results presented in this section should be observable in ongoing experiments with ultracold bosons in quasi one dimensional optical lattices \([10, 11, 52]\). As exact numerical simulations of the many-body quantum dynamics are no longer possible for extended lattices with
many atoms, we use the truncated Wigner method (see appendix B for details). This approximate method is appropriate for a system with large filling factors, since in this case the error induced from the truncation vanishes as $1/N$. More importantly for our case, the truncated Wigner method can describe the deviation from a pure BEC state, in contrast to pure mean-field models.

**A. Breather state formation**

In the following we consider an extended optical lattice consisting of $M = 50$ sites with periodic boundary conditions unless states otherwise. Loss occurs from the lattice site $j = 1$ only. As an initial state we assume a pure BEC which is moved at constant speed or accelerated to the edge of the first Brillouin zone. Hence the quantum state of the atoms at $t = 0$ is given by

$$\ket{\Psi(0)} = \frac{1}{\sqrt{N!}} \left( \sum_j \psi_j \hat{a}_j \right)^N \ket{0}$$

with $\psi_j = (-1)^j / \sqrt{M}$, which generalizes the antisymmetric initial state $\ket{\Psi_-}$ discussed for the triple-well trap. We consider the case of large filling factors, with $N/M = 1000$ in all simulations.

For weak interactions the quantum state remains close to a pure BEC during the decay, such that all coherence functions are approximately one. The dynamics changes dramatically for strong interactions as shown in Fig. 11. The phase coherence $g_{j,k}^{(1)}$ between adjacent wells is lost after a short transient period, indicating the dynamical instability of the condensate. At the same time the number fluctuations $g_{j,j+1}^{(2)}$ rapidly increase as shown in Fig. 11 (b). This reveals a strong spatial bunching of the atoms as expected for a breather state. This feature of the dissipative equilibrium state is in strong contrast to the non-dissipative case, where repulsive interactions suppress number fluctuations in thermal equilibrium. Part (d) of the figure reveals the second characteristic trait of the breather state. Strong anti-correlations with $g_{j,j+2}^{(2)} \approx 0.5$ are observed between the site $j = 25$ and the next-to-nearest neighbor. No anti-correlations are observed for the direct neighbor, as breathers can extend over more than one site in an extended lattice. We thus conclude that the atoms tend to bunch at one site of the lattice, leaving the neighboring sites essentially empty. This is exactly the signature of the breather state in the extended lattice, which we have discussed above for the trimer case. The position of the individual breathers in this breather state is random due to the quantum fluctuations. We note that it can be experimentally easier to prepare breather states starting from a Mott insulator instead of a BEC at the band edge. Simulations for small lattices show the development of strong density anti-correlations and multi-particle entanglement also in this case.

The transition to the breather regime for strong interactions is further analyzed in Fig. 12 which shows the first and second order coherence functions as a function of the interaction strength for a fixed propagation time $t_{\text{final}} = 50J^{-1}$ at the reference site $j = 9$. For $UN(0) \lesssim 2.5J$ the phase coherence between neighboring sites is preserved, while the atoms decay from the lattice. For stronger interactions, however, phase coherence is lost and the BEC fragments into a breather state. The second order correlation function $g_{j,j+2}^{(2)}$ reveals the

**FIG. 11:** (Color online) Dynamics of a leaky Bose-Hubbard chain with 50 wells. We have plotted (a) the atomic density $\langle n_j(t) \rangle$ and (b) the density fluctuations $g_{j,j}^{(2)}(t)$ in each lattice site as well as (c) the phase coherence $g_{j,j}^{(1)}(t)$ and (d) the density-density correlations $g_{j,j+k}^{(2)}(t)$ between site $j = 25$ and the neighboring sites $k = 1$ (solid red line), $k = 2$ (dashed blue line). Parameters are $UN(0) = 25J$, $\gamma_1 = 2J$, $M = 50$ and $\rho(t = 0) = N/M = 1000$.

**FIG. 12:** (Color online) Transition to the breather regime in an open optical lattice. Shown is (a) the phase coherence $g_{j,j}^{(1)}$ and (b) the number correlation function $g_{j,j}^{(2)}$ between site $j = 9$ and the neighboring sites $k = 1$ (solid red line) and $k = 2$ (dashed blue line), respectively, after a fixed propagation time $t_{\text{final}} = 50J^{-1}$. One observes a sharp transition when the interaction strength $UN(0)$ exceeds a critical value of approx. $2.5J$. Parameters are $\gamma_1 = 2J$, $M = 50$ and the atomic density is $\rho(t = 0) = N/M = 1000$. 
existence of strong anti-correlations for large values of \( U \). However, we observe \( g_{j,j+2}^{(2)} > 1 \) in the vicinity of the transition point. This is a consequence of the localization of the breathers, which becomes tighter with increasing \( U \) \cite{41}. Directly above the transition breathers exist, but typically extend over several lattice sites, such that we observe positive correlations at this length scale. Moreover, the formation of breathers suppresses the decay from the lattice, that is, the total particle number decreases more slowly. This is due to the strong localization of the breathers preventing atoms from tunneling to the leaky lattice sites.

Note that the coherence functions show the same qualitative behaviour if another lattice site is chosen as a reference site instead of \( j = 25 \) or \( j = 9 \). The oscillations we observe in Fig. 12(b) and for intermediate values of \( U \) are just a manifestation of the temporal oscillations of the \( g^{(1)} \) and \( g^{(2)} \), as shown in Fig. 11(c) and (d).

### B. Critical interaction strength

Breather formation sets in abruptly when the interaction strength exceeds a critical value \( U_{\text{crit}} \). Extensive numerical simulations show that the transition point depends on the size of the lattice, i.e. the number of sites \( M \), as shown in Fig. 13. As the lattice becomes larger, breather formation is facilitated such that the critical value \( U_{\text{crit}} \) decreases rather rapidly. In these simulations, \( U_{\text{crit}} \) was determined as follows. After a fixed propagation time we find the values of the density fluctuations \( \rho_{U,j}^{(2)} \) for different interaction strengths \( U \) and for various lattice sites \( j \). We identified the critical interaction as the maximum interaction strength in which the density fluctuations at all sites \( j \) differ from the value in the non-interacting case, \( g_{j,j}^{(2)} = 1 \), by less than 5%. In all simulations we have used \( \gamma_{1} = 2J \) and the same initial density, \( \rho(t = 0) = N/M = 1000 \). In the following we derive a formula for the critical interaction strength, which will also clarify the microscopic origin of breather formation and its connection to the self-trapping effect. Our considerations follows the reasoning presented in \cite{58} for the analogous mean-field system.

As shown in \cite{58}, breathers formation is a local process, which occurs if the local effective nonlinearity exceeds a critical value \( L \)

\[
U n_{j} / J \leq L
\]

for at least one lattice site \( j \). Then the nonlinearity is strong enough to induce self-trapping at the respective lattice site (cf. also \cite{23,19,50}). Starting from this local ansatz, the critical interaction strength can be inferred as follows. Breathers are observed if the probability to satisfy condition (26) exceeds a certain threshold value

\[
\text{prob}(\exists j : n_{j} > JL/U) \geq P_{\text{th}}.
\]

\[
U \rho \geq U_{\text{crit}} \rho = \frac{-JL}{\ln[1 - (1 - P_{\text{th}})^{1/M}]},
\]

where \( \rho = N/M \) is the atomic density.

The analytic prediction (27) depends on two parameters \( J \) and \( P_{\text{th}} \), which are used as fit parameters to model the numeric results. This fit yields an excellent agreement with the numeric results as shown in Fig. 13. We stress that the decrease of \( U_{\text{crit}} \) with increasing lattice site cannot be modeled by a simple algebraic or exponential decay. Notably, we obtain significantly smaller values for \( U_{\text{crit}} \) than in \cite{58}. This is attributed to the fact that the unstable initial state considered in this paper, a BEC at the band edge, has a higher energy and thus fragments into breathers much more easily.

![Fig. 13: The critical nonlinearity \( \rho U_{\text{crit}} \), at which breathers start to form, as a function of the lattice size \( M \). Numerical results using the truncated Wigner method (blue circles) are compared to a fit using equation (27) (red line). The fitting parameters we found are \( L = 0.075 \) with bounds \( (0.054, 0.096) \) and \( P_{\text{th}} = 1 - 2.4 \times 10^{-7} \) with bounds \( (1 + 4.5 \times 10^{-7}, 1 - 9.3 \times 10^{-7}) \), while the summed square of residuals is \( SSE = 1.2 \times 10^{-3} \). The other parameters are \( \gamma_{1} = 2J \) and \( \rho(t = 0) = N/M = 1000 \).][10]
VI. CONCLUSIONS AND OUTLOOK

Recently, there has been a great interest in engineering quantum dynamics by using dissipation in various systems [3, 4, 10-11, 14, 17, 24, 26, 28-29, 30, 32-34, 44, 47, 48, 52, 58]. In this paper, we report the effects of an elementary dissipation mechanism, the localized single particle loss, in a BEC loaded in a deep optical lattice. Particle losses combined with strong interparticle interactions and discrete geometry can deterministically lead to the formation of quantum superpositions of discrete breathers. For a small trimer system we have discussed the properties of these “breather states” in detail including entanglement, decoherence and possible applications in precision quantum interferometry. A semiclassical interpretation of breather state formation has revealed the connection to a classical bifurcation of the associated mean-field dynamics. Furthermore, we have studied the dynamical formation of breather states in extended lattices and we have derived a formula which predicts the critical interaction strength, in which the breathers start to form, in lattices with different size. The formation and the properties of these structures could be readily observed in ongoing experiments with ultracold atoms in optical lattices [10, 13, 15, 23].

Nonlinear structures, like bright or dark solitons, are well known in the context of the Gross-Pitaevskii equation (see for example [59-61]). However, this equation cannot give us any information about the quantum nature of the problem, these structures are “classical” objects. With the present work we open a new direction: stable nonlinear structures that exhibit purely quantum properties, like entanglement. These properties cannot be studied anymore with a simple GP equation (or DGPE for discrete systems) and one should go beyond them to support state-of-the-art experiments.

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Appendix A: Entanglement criterion

In this section we provide a detailed derivation of the entanglement criterion based on [18] which is adapted to the NOON states discussed in the present paper. This result generalizes established entanglement criteria in terms of spin squeezing [62] and is derived in a similar way. In contrast to spin squeezing inequalities, it shows that a state is entangled if the variance defined below in (A2) is larger than a certain threshold value.

We assume that the many-body quantum state \( \hat{\rho} \) is decomposed into a mixture of pure states

\[
\hat{\rho} = \sum_a p_a \hat{\rho}_a
\]

where every pure state \( \hat{\rho}_a = |\psi_a\rangle \langle \psi_a| \) has a fixed particle number \( N_a \). Note that the quantum jump simulation of the dynamics directly provides such a decomposition. We define the entanglement parameter

\[
E_{r,q} := \langle (\hat{n}_r - \hat{n}_q)^2 \rangle - \langle \hat{n}_r \rangle^2 - \langle \hat{n}_q \rangle^2
\]

for the sites \( r \) and \( q \). In this expression \( \langle \cdot \rangle_{a,b} \) denotes the expectation value in the pure state \( |\psi_{a,b}\rangle \). Now we can proof that \( E_{r,q} < 0 \) for every separable state such that a value \( E_{r,q} > 0 \) unambiguously reveals the presence of many-particle entanglement. Note that \( E_{r,q} \) provides and entanglement criterion, it is not a quantitative entanglement measure in the strict sense.

To proof this statement we consider an arbitrary separable state and show that \( E_{r,q} < 0 \) for this class of states. If a pure state \( \hat{\rho}_a \) is separable, it can be written as a tensor product of single particle states

\[
\hat{\rho}_a = \hat{\rho}_a^{(1)} \otimes \hat{\rho}_a^{(2)} \otimes \cdots \otimes \hat{\rho}_a^{(N_a)}
\]

We furthermore introduce the abbreviation

\[
\hat{S}_\pm := \hat{n}_r \pm \hat{n}_q.
\]

This operator is also written as a symmetrized tensor product of single-particle operators

\[
\hat{S}_\pm = \sum_{k=1}^{N_a} \mathbb{I} \otimes \cdots \otimes \mathbb{I} \otimes \hat{s}_\pm^{(k)} \otimes \mathbb{I} \otimes \cdots \otimes \mathbb{I},
\]

where the superscript \( (k) \) denotes that the single-particle operator \( \hat{s}_\pm^{(k)} \) acts on the \( k \)th atom. The single-particle operators are given by

\[
\hat{s}_\pm = |r\rangle \langle r| \pm |q\rangle \langle q|,
\]

where \( |r\rangle \) is the quantum state where the particle is localized in site \( r \).

For a separable pure state \( \hat{\rho}_a \), the expectation values of the population imbalance \( \langle S_- \rangle_a = \text{tr}[\hat{\rho}_a \hat{S}_-] \) and its square can be expressed as (dropping the subscript \( a \) for
notational clarity)

\[ \langle \hat{S}_- \rangle = \sum_{k=1}^{N} \text{tr} \left[ \rho^{(k)} \hat{s}_-^{(k)} \right] \]

\[ \langle \hat{S}_-^2 \rangle = \sum_{j \neq k}^{N} \text{tr} \left[ \rho^{(j)} \otimes \rho^{(k)} (\hat{s}_-^{(j)} \otimes \hat{s}_-^{(k)}) \right] + \sum_{j=1}^{N} \text{tr} \left[ \rho^{(j)} \hat{s}_-^{(j)} \right] \text{tr} \left[ \rho^{(k)} \hat{s}_-^{(k)} \right] \]

\[ = \sum_{j,k=1}^{N} \text{tr} \left[ \rho^{(j)} \hat{s}_-^{(j)} \right] \text{tr} \left[ \rho^{(k)} \hat{s}_-^{(k)} \right] - \sum_{j=1}^{N} \text{tr} \left[ \rho^{(j)} \hat{s}_-^{(j)} \right] + \sum_{j=1}^{N} \text{tr} \left[ \rho^{(j)} \hat{s}_-^{(j)} \right]^2 \]

\[ = \langle \hat{S}_- \rangle^2 + \sum_{j=1}^{N} \text{tr} \left[ \rho^{(j)} \hat{s}_-^{(j)} \right]^2 - \sum_{j=1}^{N} \text{tr} \left[ \rho^{(j)} \hat{s}_-^{(j)} \right] \}

Using \( \text{tr}[\rho^{(j)} \hat{s}_-^{(j)}] = \text{tr}[\rho^{(j)} \hat{s}_-^{(j)}] \) we thus find that every pure products state \( \hat{\rho}_a \) satisfies the condition

\[ \langle \hat{S}_-^2 \rangle_a - \langle \hat{S}_- \rangle_a^2 \leq \langle \hat{S}_+ \rangle_a. \]  

(A7)

If the total quantum state \( \hat{\rho} \) is separable, such that it can be written as a mixture of separable pure states \( \sum_{a=1}^{N} p_a \hat{\rho}_a \), the expectation values are given by

\[ \langle \hat{S}_-^2 \rangle = \sum_{a} p_a \langle \hat{S}_-^2 \rangle_a \]  

(A8)

\[ \leq \langle \hat{S}_+ \rangle + \sum_{a} p_a \langle \hat{S}_- \rangle_a^2 \]

\[ \langle \hat{S}_- \rangle^2 = \sum_{a,b} p_a p_b \langle \hat{S}_- \rangle_a \langle \hat{S}_- \rangle_b \]  

(A9)

\[ = \sum_{a} p_a \langle \hat{S}_- \rangle_a^2 - \frac{1}{2} \sum_{a,b} p_a p_b \left[ \langle \hat{S}_- \rangle_a - \langle \hat{S}_- \rangle_b \right]^2. \]

We thus find that every separable quantum state satisfies the following inequality for the variance of the population imbalance \( \hat{S}_- \):

\[ \langle \hat{S}_-^2 \rangle - \langle \hat{S}_- \rangle^2 \leq \langle \hat{S}_+ \rangle + \frac{1}{2} \sum_{a,b} p_a p_b \left[ \langle \hat{S}_- \rangle_a - \langle \hat{S}_- \rangle_b \right]^2. \]  

(A10)

This inequality for separable quantum states can be rewritten as

\[ E_{r,q} < 0 \]  

(A11)

in terms of the entanglement parameter

Appendix B: Truncated Wigner function dynamics

In this appendix, we will explicitly derive the evolution equation for the Wigner function which corresponds to the master equation (A2). To this end we use the following operator correspondences (A2):

\[ \hat{a}_j \hat{\rho} \leftrightarrow \left( \alpha_j + \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) W, \]

(B1)

\[ \hat{\rho} \hat{a}_j \leftrightarrow \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) W, \]

(B2)

\[ \hat{a}_j^\dagger \hat{\rho} \leftrightarrow \left( \alpha_j^* - \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) W, \]

(B3)

\[ \hat{\rho} \hat{a}_j^\dagger \leftrightarrow \left( \alpha_j^* + \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) W, \]

(B4)

where \( \alpha_j \) are the eigenvalues of the destruction operator:

\[ \hat{a}_j |\alpha_j\rangle = \alpha_j |\alpha_j\rangle, \quad \langle \alpha_j|\hat{a}_j^\dagger = \alpha_j^* \langle \alpha_j|. \]  

(B5)

Substituting these correspondences in the master equation (A2), we obtain the following evolution equation for the Wigner function:

\[ \partial_t W = 2J \sum_{j=1}^{M-1} \text{Im} \left[ \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \left( \alpha_j^* + \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right) - \left( \alpha_{j+1} - \frac{1}{2} \frac{\partial}{\partial \alpha_{j+1}} \right) \left( \alpha_j + \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \right] W \]

\[ + U \sum_{j=1}^{M} \text{Im} \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right)^2 \left( \alpha_j^* + \frac{1}{2} \frac{\partial}{\partial \alpha_j^*} \right)^2 - \sum_{j=1}^{M} \frac{\gamma_j}{2} \left[ \left( \alpha_j - \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \left( \alpha_j + \frac{1}{2} \frac{\partial}{\partial \alpha_j} \right) \right] W. \]  

As one can easily see the above equation includes not only first and second order derivatives, but also third order ones arising from the interaction term (the \( U \)-dependent term in the second line of equation (B6)). These third order
derivatives makes the equation quickly unstable, so an approximate method is needed. One technique that is widely used in optical systems is the truncated Wigner method [55, 56], which is a good approximation as far as the mode occupation numbers are large. In this approximation one neglects all the terms that include third order derivatives, thus we have the equation

$$\partial_t W = \sum_j \frac{\partial}{\partial x_j} \left[ J(y_{j+1} + y_{j-1}) U(y_j - x_j^2 y_j - y_j^3) + \frac{\gamma_j}{2} x_j \right] W$$

$$+ \sum_j \frac{\partial}{\partial y_j} \left[ -J(x_{j+1} + x_{j-1}) - U(x_j y_j^2 - x_j^3) + \frac{\gamma_j}{2} y_j \right] W + \frac{1}{2} \sum_j \frac{\gamma_j}{2} \left( \frac{\partial^2}{\partial x_j^2} + \frac{\partial^2}{\partial y_j^2} \right) W,$$

where $x_j, y_j$ are the real and imaginary part of $\alpha_j$ respectively.

Equation (B7) is a Fokker-Planck equation, thus it can be rewritten in the language of stochastic differential or Langevin equations. To be more precise, consider the Fokker-Planck equation of the form [57]:

$$\partial_t W = -\sum_j \frac{\partial}{\partial z_j} A_j(z, t) W$$

$$+ \frac{1}{2} \sum_{j,k} \frac{\partial}{\partial z_j} \frac{\partial}{\partial z_k} \left[ B(z, t) B^T(z, t) \right]_{jk} W,$$

where the diffusion matrix $D = BB^T$ is positive definite. Now, we can write equation (B8) as a system of stochastic equations:

$$\frac{dz}{dt} = A(z, t) + B(z, t) E(t),$$

where the real noise sources $E_j(t)$ have zero mean and satisfy $\langle E_j(t) E_k(t') \rangle = \delta_{jk} \delta(t - t')$. In our case, equation (B7) can be rewritten:

$$\frac{dx_j}{dt} = -J(y_{j+1} + y_{j-1}) - U(y_j - x_j^2 y_j - y_j^3)$$

$$\quad - \frac{\gamma_j}{2} x_j + \sqrt{\gamma_j} \xi_j(t),$$

$$\frac{dy_j}{dt} = J(x_{j+1} + x_{j-1}) + U(x_j y_j^2 - x_j^3)$$

$$\quad - \frac{\gamma_j}{2} y_j + \sqrt{\gamma_j} \eta_j(t),$$

where $\xi_j(t), \eta_j(t)$ for $j = 1, ..., M$ are $\delta$-correlated in time with zero mean. Here it must be noted that $\xi_j(t), \eta_j(t)$ are not real noise sources, but are included only to recapture the commutation relations of the operators.

As an initial state one uses a product state of the form

$$|\Psi(t = 0)\rangle = |\psi_1\rangle |\psi_2\rangle ... |\psi_M\rangle,$$

where $|\psi_j\rangle$ is a Glauber coherent state in the $j$th well. This state represents a pure BEC in a grand-canonical framework. The Wigner function of a Glauber coherent state $|\psi_j\rangle$ is a Gaussian,

$$W(\alpha_j, \alpha^*_j) = \frac{2}{\pi} e^{-|\alpha_j - \psi_j|^2}.$$
ensemble;

\[
\langle O_1 \ldots O_k \rangle_{\text{sym}} = \int \prod_{i=1}^{M} d^2 \alpha_i O_{j_i} \ldots O_k W(\alpha_1, \alpha_2^*, \ldots) \]

\[= \frac{1}{N_T} \sum_{i=1}^{N_T} O_{j_1} \ldots O_k \]  

(B15)

where \(O_{j_i}\) stands for \(\alpha_i\) or \(\alpha_i^*\), \(N_T\) is the number of trajectories and the subscript sym reminds us that only expectations values of symmetrized observables can be calculated.

In Fig. 14 we compare the results of the truncated Wigner approximation with the results of the exact quantum jump method for the triple-well trap studied in Sec. III. The simulations show a very good agreement also in the regime of strong interactions. The only small discrepancy is that oscillations of the correlation functions are slightly less pronounced. As the truncated phase space approximations become more accurate with increasing filling factors \cite{4546}, we expect that the truncated Wigner simulations discussed in Sec. IV are reliable both qualitatively and quantitatively.

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