Photons in a nested Mach-Zehnder interferometer

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Abstract

In a paper titled “Asking photons where they have been?” [3] Danan, Farfurnik, Bar-Ad and Vaidman describe an experiment with pre and post selected photons going through nested Mach-Zehnder interferometers. They find that some of the mirrors leave no footprints on the signal and interpret this as evidence that the photon skipped these mirrors and by a discontinuous path. I review the experiment and analyze it within the orthodox framework of quantum mechanics. The picture of interfering multiple trajectories provide a satisfactory explanation of the experimental findings and the absence of footprints.

Alex Grossmann has been a beacon of light and warmth to me, a teacher, mentor, father figure and dear friend.

1 The experiment

Consider the nested Mach-Zehnder interferometers shown in Fig. 1. Each one of the five mirrors \{A, B, C, E, F\} oscillates with its characteristic frequency. The intensity of light falling on the top half of the detector surface is compared with the intensity falling on the bottom half. The power spectrum of the signal bears evidence to the oscillation frequencies of the mirrors.
Figure 1: A nested Mach-Zehnder interferometer: The four squares represent beam-splitters. The external Mach-Zehnder has $1:2$ beam splitters and the internal $1:1$ beam splitters. The black lines marked \{A,...,E\} represent mirrors that slightly oscillate, each with its own specific frequency. Light is fed in the upper blue port and measured by a “quad-detector” D which compares the signal on its top half with the bottom half.

The authors tune the interferometer so that frequencies corresponding to mirrors A, B, C show up in the power spectrum but those of E, F do not. The appearance of a characteristic frequency in the signal gives, of course, evidence that photons hit the corresponding mirror. The authors of [3] go one step further and interpret the absence of a characteristic frequency as evidence that no photon visited the mirror. Since the E, F mirrors are the gate keepers of the A, B mirrors, the authors conclude that photons may have discontinuous paths.

2 Discontinuous trajectories vs interference

Theoretical support of the (unorthodox) notion of discontinuous trajectories comes from a combination of the “two state” formulation of quantum mechanics [2] and the “weak values” formulation of weak quantum measurement [1] where trajectories are defined as the overlap of forward and backward propagating paths.

There is a small, but substantial community of adherents of “two state” and “weak values”, but also critics [5] and parts of the controversy are about semantics. The theoretical analysis of the experiment given in [3] and the
Figure 2: A circuit diagram corresponding to the experiment with the nested Mach-Zehnder interferometer in Fig 1. A, B, C, E, F represents the mirror gates and D is a quadrature detector that compares the signal on its top half with the signal in its bottom half. The beam splitters are represented by crossings. The deflection angle of the mirror C is $\gamma$.

supporting material, will be foreign to most readers.

Danan et. al. qualify the interpretation in terms of discontinuous paths. They show that the same results are obtained by considering interfering classical waves. They do not argue with orthodox quantum mechanics, but rather promote a picture of discontinuous trajectories as the simplest explanation of the experiment.

Here I shall preset an orthodox quantum mechanical analysis of the experiment [4]. The analysis is based on a quantum circuit model. I recover the main results in [3], and explain the absence of footprints of some of the mirrors without resorting to discontinuous paths.

3 A quantum circuit model

In the experiment, the orientations of the mirrors A, B, C, E, F are functions of time. However, as the interaction time of the photon with each mirror is very short, one may consider the mirrors to be effectively at rest. The angles of the mirrors may then be viewed as parameters. The measurement of the power spectrum in [3] is, for the purpose of the analysis, just a clever trick to gain information about the instantaneous angles $\alpha, \beta, \gamma, \eta, \phi$, of the mirrors.

A quantum mechanical model corresponding to experiment is the circuit of gates shown in Fig. 2. We model the photon (annihilation) operator by two coordinates representing the channel and wave-vector in the plane:

$$a_j(k), \quad j \in \{1, 2, 3\}, \quad k \in \mathbb{R}^2$$  (1)
$j$ labels the three channels, marked blue, red and green in Fig. 2. $k$ describes
the photon wave vector in a given channel. The modes satisfy the canonical
commutation relations
\[
[a_j(k), a_k(k')^\dagger] = \delta_{jk} \delta(k - k')
\] (2)
The beam splitter $S_{jk}$ acts trivially on the mode $k$ and non-trivially on the
channel mode indexes $jk$. Since a beam splitter is time-reversal invariant the
corresponding matrix may be chosen symmetric\footnote{A scattering matrix $S$ is time reversal invariant if $TS = S^{-1}T$ with $T$ the anti-unitary
time reversal.}. The two beam splitters in
the circuit are
\[
S_{12} = \begin{pmatrix} \sqrt{p} & i \sqrt{q} \\ i \sqrt{q} & \sqrt{p} \end{pmatrix}, \quad S_{23} = \frac{1}{\sqrt{2}} \begin{pmatrix} 1 & i \\ i & 1 \end{pmatrix}
\] (3)
with $p + q = 1$, representing the distribution of the incoming photon into the
outgoing channels.

The mirrors act trivially on the channel index $j$ and non-trivially on the
wave vector $k$. The mirror $A$, with angle $\alpha/2$ (and angle of deflection $\alpha$),
acts on the vector $k$ as a reflection, associated with the symmetric matrix $M_\alpha$:
\[
k \mapsto M_\alpha k, \quad M_\alpha = \begin{pmatrix} \cos \alpha & \sin \alpha \\ \sin \alpha & -\cos \alpha \end{pmatrix}
\] (4)
The mirror $A$ acts on $|k\rangle$ as a unitary map,
\[
U_A = \int dk \langle M_\alpha k | k \rangle
\] (5)
Similarly, the mirror $B$, with angle $\beta/2$, will be represented by the matrix $M_\beta$ and the unitary $U_B$, etc.

Two reflections make a rotation, and three reflections give again a reflection. The total deflection associated with the path $E - A - F$ is
\[
M_\phi M_\alpha M_\eta = M_\tilde{\alpha}, \quad \tilde{\alpha} = -\alpha + \eta + \phi
\] (6)
and similarly for the path $E - B - F$
\[
M_\phi M_\beta M_\eta = M_\tilde{\beta}, \quad \tilde{\beta} = -\beta + \eta + \phi
\] (7)
One can further decorate the circuit with unitaries that represent phase delays between channels. This is important in practice as it allows to tune the interferometers. (This will affect the phases in Eq. 25.) However, for the theoretical analysis and the sake of simplicity it is best not to.
4 No footprints of the gate keepers

The footprints of the gate keepers $E,F$ are erased when the nested beam splitters are $S_{23}$. This can be simply seen without calculations.

Photons have 3 routes from the input port to the detector: The route through $C$, the route through $E - A - F$ and the route through $E - B - F$. When the two gates $A,B$ are identical, $A = B$, only the route through $C$ is open. The other two routes are dumped into the green port.

To see this note first that the two inner beam splitters $S_{23}$ swap the two channels:

$$S_{23} \cdot S_{23} = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad (8)$$

This remains true when the two gates $A,B$ are identical, so $U_A = U_B$. The dashed box in the figure corresponds to a unitary that swaps the 2-3 channels

$$(S_{23} \otimes 1)(1 \otimes U_A)(S_{23} \otimes 1) = i \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \otimes U_A \quad (9)$$

The signal $I(\tilde{\alpha}, \tilde{\beta}, \gamma)$ is a function of the total angles of deflection along the three paths, $\alpha, \beta, \gamma$, see Eq. 6, 7. $I(\tilde{\alpha}, \tilde{\beta}, \gamma)|_{\alpha = \beta}$ is only a function of $\gamma$ since the paths through $A,B$ do not reach the detector. It follows that when $\alpha$ is close to $\beta$

$$\partial_\eta I = O(\beta - \alpha), \quad \partial_\phi I = O(\beta - \alpha), \quad (10)$$

The footprints of $E,F$ on the signal are weak when $A \approx B$. This does not depend on $\alpha$ and $\beta$ being small and how the mirrors $A$ and $B$ are manipulated. It only depends on their synchronization so they are approximately identical.

5 The amplitudes of interfering paths

The unitary $U$ associated with the circuit in Fig. 2 is a product of operators acting on $\mathbb{C}^4 \otimes L^2(\mathbb{R}^2)$. We are interested in the matrix element

$$\langle 1, k' | U | 1, k \rangle \quad (11)$$

Photons have 3 routes to the detector: The route through $C$, the route through $A$ and the route through $B$. We can then write

$$\langle 1, k' | U | 1, k \rangle = \langle 1, k' | U_C | 1, k \rangle + \langle 1, k' | U_A | 1, k \rangle + \langle 1, k' | U_B | 1, k \rangle \quad (12)$$
Since the beam splitters and mirrors act on different parts of the tensor product they can be computed separately. We have

\begin{align*}
\langle 1, k' | U_C | 1, k \rangle &= \left( \langle 1 | S_{12} | 1 \rangle \right)^2 \delta(k' - M, k) \\
&= p \delta(k' - M, k) \quad (13) \\
\langle 1, k' | U_A | 1, k \rangle &= \langle 1 | S_{12} | 2 \rangle \langle 2 | S_{23} | 2 \rangle^2 \langle 2 | S_{12} | 1 \rangle \delta(k' - M_\alpha k) \\
&= -\frac{q}{2} \delta(k' - M_\alpha k) \quad (14) \\
\langle 1, k' | U_B | 1, k \rangle &= \langle 1 | S_{12} | 2 \rangle \langle 2 | S_{23} | 3 \rangle \langle 3 | S_{23} | 2 \rangle \langle 2 | S_{12} | 1 \rangle \delta(k' - M_\beta k) \\
&= q^2 \delta(k' - M_\beta k) \quad (15)
\end{align*}

5.1 The incoming state

To compute the amplitudes for detection we need to describe the incoming state and detector.

The photon creation operator corresponding to the one-particle state $|\varphi\rangle$ is

\[ a^\dagger = a_1^\dagger(\varphi) = \int dk \langle k|\varphi\rangle a_1^\dagger(k) \quad (16) \]

We assume that $|\varphi\rangle$ is a (normalized) wave packet in k-space representing a narrow beam of light propagating in the x-direction, i.e.

\[ \langle k_x \rangle \gg \sqrt{\langle k_y^2 \rangle}, \quad \langle k_y \rangle \approx 0, \quad \langle k_j \rangle = \langle \varphi | k_j | \varphi \rangle \quad (17) \]

We choose for the incoming state a single photon state

\[ |\varphi\rangle = a^\dagger |0\rangle, \quad a(k) |\varphi\rangle = \varphi(k) |0\rangle \quad (18) \]

5.2 The quad-detector

For the sake of simplicity we assume that the quad-detector is a collection of single photon detectors associated with the mutually orthogonal state

\[ |f_j\rangle \langle f_j| \quad (19) \]

The state with $j > 0$ lie in the upper half of the detector and the state with $j < 0$ in the lower half. The two projections

\[ W_\pm = \sum_{\pm j > 0} |f_j\rangle \langle f_j| \quad (20) \]
project on the top and bottom half of the detector. The signal $I$ is the observable associated with the difference of projections

$$I = W_+ - W_-$$  \hspace{1cm} (21)$$

As no projection, other than the identity, is close to the identity, $W_{\pm}$ describe “strong measurement”.

6 The detection amplitude

The second quantized version of $U$ is

$$U = \sum \int dk dk' \langle k, k | U | j, k' \rangle a_k^+(k)a_j(k')$$ \hspace{1cm} (22)$$

The amplitude for the $j$-th detector to absorb the photon$^2$ is

$$\langle 0 | a_1(f_j)Ua_1^+(\varphi) | 0 \rangle = \int dk' dk \langle f_j | k \rangle \langle 1, k | U | 1, k' \rangle \langle k' | \varphi \rangle$$

$$= \langle f_j | \tilde{\varphi} \rangle$$ \hspace{1cm} (23)$$

where

$$\langle k | \tilde{\varphi} \rangle = p\varphi(M_{\tilde{\alpha}}k) - q^2 \varphi(M_{\tilde{\beta}}k) + q \varphi(M_{\gamma}k)$$ \hspace{1cm} (24)$$

In the case $p : q = 1 : 2$ this simplifies to

$$\langle k | \tilde{\varphi} \rangle = \frac{1}{3} \left( \varphi(M_{\tilde{\alpha}}k) - \varphi(M_{\tilde{\alpha}}k) + \varphi(M_{\gamma}k) \right)$$ \hspace{1cm} (25)$$

For the sake of writing uncluttered formulas we focus on this case.

6.1 The case of large overlap: $\tilde{\alpha} \approx \tilde{\beta} \approx \gamma$

When $\tilde{\alpha} \approx \tilde{\beta} \approx \gamma$ the three terms in Eq. 25 have substantial overlap leading to interference. With $\tilde{\alpha}$ close to $\gamma$, (without assuming that either one is small)

$$M_{\tilde{\alpha}} \approx M_{\gamma} + (\tilde{\alpha} - \gamma)M'_{\gamma}$$ \hspace{1cm} (26)$$

$^2$In principle, one needs to take into account the free propagation from the interferometer to the detector. This turns out to be equivalent to a unitary transformation of the states $|f_j\rangle$, and affect the final result, Eq. 33, by changing the overall constant in the big bracket.
and the amplitude is\(^3\)

\[
\varphi(M_\alpha k) \approx \varphi(M_\gamma k) + (\tilde{\alpha} - \gamma)(M_\gamma' k) \cdot \nabla \varphi(M_\gamma k)
\]  
(27)

Similarly for \(\varphi(M_\beta)\). It follows that the total amplitude is

\[
\varphi(M_\gamma k) - \varphi(M_\alpha k) + \varphi(M_\beta k) \approx \varphi(M_\gamma k) + (\tilde{\beta} - \tilde{\alpha})(M_\gamma' k) \cdot \nabla \varphi(M_\gamma k)
\]  
(28)

The projection of the amplitude on the j-th detector is

\[
\langle f_j | U_C | \varphi \rangle + (\alpha - \beta) \langle f_j | M_\gamma' k \cdot \nabla U_C | \varphi \rangle , \quad U_C | k \rangle = | M_\gamma k \rangle
\]  
(29)

The footprints of the \(E,F\) mirrors have been erased in this approximation.

### 6.2 The detection amplitude of small angles

In the case that \(\gamma\) is small Eq. 29 reduces to

\[
\langle f_j | U_Z | \varphi \rangle + (\gamma + \alpha - \beta) \langle f_j | X k \cdot \nabla U_Z | \varphi \rangle
\]  
(30)

where

\[
U_Z | k \rangle = | Z k \rangle = | k_x, -k_y \rangle , \quad X k = (k_y, k_x)
\]  
(31)

For the sake of simplicity, and comparison with [3], we focus on this special case. note that \(\gamma\) small forces \(\tilde{\alpha}\) and \(\tilde{\beta}\) to be small as well. But, \(\alpha\) and \(\beta\) need to not be small, but \(\beta - \alpha\) is small.

### 6.3 Amplification

The first term in Eq. 30 dominates the second, the latter being of (relative) order \(O(\gamma + \alpha - \beta)\). To make the quad-signal sensitive to the sub-dominant term one uses the freedom to shift the quad-detector up and down so that signal from the top face cancel the signal from the bottom face when \(\gamma - \alpha + \beta = 0\). Namely,

\[
\sum_{j>0} | \langle f_j | U_C | \varphi \rangle |^2 - \sum_{j<0} | \langle f_j | U_C | \varphi \rangle |^2 = 0
\]  
(32)

\(^3\)Use \(M = M^\dagger\) to get rid of the dagger.
It follows that the signal is
\[
I(\tilde{\alpha}, \tilde{\beta}, \gamma) \approx (\gamma + \alpha - \beta) \left( \frac{1}{9} \sum \langle \varphi | U_Z | f_j \rangle \text{sgn}(j) \langle f_j | (Xk) \cdot \nabla U_Z | \varphi \rangle \right) (33)
\]

The term in the big brackets is a constant, independent of the angle \( s \), but depending on the details of the detector\(^4\). This recovers the main result in [3].

7 Concluding remarks

- The authors of [3] interpreted the weak footprints of the mirrors \( E, F \) as support for the picture of “discontinuous quantum trajectories”. The picture of “multiple interfering trajectories” offers an alternative simple interpretation.

- “Weak values” and “weak measurements” did not enter the discussion.

- As even amateur detective know, the absence of footprints does not rule out the crime. A trivial example is when one mirror erases the footprints of the other as is the case when \( AF = 1 \). The absence of footprints of a mirror in the detector does not imply that the photon did not hit the mirror. Similarly, the absence of the \( E, F \) footprints, does not imply that the photons avoided the \( E, F \) mirrors.

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\(^4\)Including the distance of the detector from the interferometer.
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