Estimating the distribution of jumps in regular affine models: uniform rates of convergence

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Abstract

The problem of separating the jump part of a multidimensional regular affine process from its continuous part is considered. In particular, we present an algorithm for a nonparametric estimation of the jump distribution under the presence of a nonzero diffusion component. An estimation methodology is proposed which is based on the log-affine representation of the conditional characteristic function of a regular affine process and employs a smoothed in time version of the empirical characteristic function in order to estimate the derivatives of the conditional characteristic function. We derive almost sure uniform rates of convergence for the estimated Lévy density and prove that these rates are optimal in the minimax sense. Finally, the performance of the estimation algorithm is illustrated in the case of the Bates stochastic volatility model.

Keywords: Regular affine processes, estimation, empirical characteristic function

1 Introduction

The problem of nonparametric statistical inference for jump processes or more generally for semimartingales models has long history and goes back to the works of Rubin and Tucker (1959) and Basawa and Brockwell (1982). The recent revival of interest in this topic documented, for example, in Figueroa-López (2004) and Figueroa-López (2009), is mainly related to a wide availability of financial and economical time series data and new types of statistical issues that have not been addressed before. For instance, there is now considerable evidence (see, e.g. Cont and Mancini (2007)) that most financial time series contain a continuous martingale component. This is why a number of works appeared in recent years deal with the problem of estimating some characteristics of jumps in the general semimartingale models with a nonzero continuous part. Without further assumptions such kind of statistical inference is not possible because the behavior of the jump component becomes statistically indistinguishable from the behavior of the diffusion part as the activity of small jumps tends to infinity. In the case of Lévy processes the activity of small jumps can be measured by the so called Blumenthal-Getoor index. The nearer is the Blumenthal-Getoor index to 2, the more difficult becomes the problem of separating the jump component from the diffusion component and hence the problem of statistical inference on the characteristics of jumps (see, e.g. Neumann and Reiß (2007)). Suppose that values of a log-return process $X(t) = \log(S(t))$ on a time grid $\pi = \{t_0, t_1, \ldots, t_N\}$ are observed. If $|\pi|$
is small (high-frequency data) then a large increment \( X(t_i) - X(t_{i-1}) \) indicates that a jump occurred between time \( t_{i-1} \) and \( t_i \). Based on this insight and the continuous-time observation analogue, inference for various characteristics of jumps of the underlying semimartingale can be conducted. For example, in Aït-Sahalia and Jacod (2008) the problem of statistical inference on the degree of jump activity in the general semimartingale models based on high-frequency data is considered. They proposed an estimation procedure which is able to “see through” the continuous part of semimartingale and consistently estimate the degree of small jump activity under some restrictions on the structure of the underlying semimartingale. In fact, these restrictions keep the highest degree of activity of small jumps away from 2, thus allowing for a consistent estimation of the degree of jump activity.

In this work we consider the problem of estimating the characteristics of jumps in the class of regular affine models with a nonzero continuous part. Affine Itô-Lévy processes are nowadays rather popular in financial and econometric modeling. Due to their analytical tractability on the one hand and their rather rich dynamics and implied volatility patterns on the other hand, they are particularly useful in the context of option pricing. Many well known models such as Heston and Bates stochastic volatility models fall into the class of affine Itô-Lévy processes. Option pricing in these models can be conveniently done via the Fourier method. The literature on affine processes is rather extensive. Let us mention two seminal papers of Duffie, Pan and Singleton (2000) and Duffie, Filipović and Schachermayer (2003), where theoretical analysis of regular affine models has been conducted. More recent literature includes Glasserman and Kim (2007) and Keller-Ressel and Steiner (2008).

We propose an approach based on the log-affine representation of the conditional characteristic function of a regular affine process. This representation together with some transformation allows one to consistently estimate the characteristics of the jump component under some prior bound on the highest degree of activity of small jumps. We present uniform convergence rates for the so constructed estimate of a transformed Lévy density which turn out to be optimal in minimax sense. As a main technical result that may be of independent interest we provide uniform convergence rates for a smoothed in time version of the empirical characteristic function. The problem of parametric estimation of the characteristics of an affine jump-diffusion process (processes with finite intensity of jumps) \( X(t) \) from a high-frequency time series of the asset \( S(t) = \exp(X(t)) \) has been recently considered in the literature by Singleton (2000) and Bates (2005). In Singleton (2000) the general method of moments (GMM) based on the empirical characteristic function is employed and asymptotic properties of the estimator are investigated. Bates (2005) proposed a filtration-based maximum likelihood methodology for estimating the parameters and realizations of latent affine processes. Filtration is conducted in the transform space of characteristic functions, using a version of Bayes rule for recursively updating the joint characteristic function of latent variables and the data conditional upon past data. Since the characteristics of an affine process are a priori an infinite-dimensional object, a parametric approach is always exposed to the problem of misspecification, in particular when there is no inherent economic foundation of the parameters and they are only used to generate different shapes of possible jump distributions. The problem of a semi-parametric inference for the characteristics of some special type affine processes \( X(t) \) has been studied in the literature as well. In the case of high-frequency observations the problem of a nonparametric inference on the Lévy measure for time changed Lévy processes, belonging sometimes to the class of affine processes, has been recently studied in Figueroa-López (2009) where consistency was proved. In Jongbloed, van der Meulen and van der Vaart (2005) the case of a one-dimensional Lévy driven
Ornstein-Uhlenbeck process, affine process with zero diffusion part, is considered. The authors assume that the corresponding jump component is self-decomposable, i.e. the degree of the jump activity is less than 1. They propose a cumulant $M$-estimator to estimate the so called canonical function of the driving self-decomposable process from low-frequency data and prove consistency of the resulting estimate. As to the special case of Lévy processes, a semi-parametric estimation problems for Lévy models under low-frequency data has recently been studied in Neumann and Reiß (2007). Let us mention that in Neumann and Reiß (2007) a diffusion component is assumed to be known. So, the above works do not encounter the problem of separating diffusion and jump components as the activity of small jumps increases. Furthermore, the challenge of devising nonparametric estimation methods for the Lévy density in general regular affine models lies in the fact that the structure of the conditional characteristic function of a regular affine process has not such explicit form as in the case of Lévy processes and is related to the parameters of the underlying affine process not directly but via a Riccati equation. The last but not the least: the increments of an affine process are not any longer independent, hence advanced tools from time series analysis have to be used.

The paper is organized as follows. In Section 2 we recall the definition and basic properties of regular affine processes. In Section 3 we formulate a spectral estimation algorithm which applies as soon as an estimates of the corresponding conditional characteristic function and its time derivatives are available. Section 5 is devoted to a nonparametric estimation of the above characteristic function and its derivatives. We derive uniform rates of convergence and prove that these rates are optimal in minimax sense. Section 5 concludes the paper.

2 Affine processes

Let us fix a probability space $(\Omega, \mathcal{F}, P)$ and an information filtration $(\mathcal{F}_t)_{t \geq 0}$. The process $X(t)$ is an affine process if it is stochastically continuous, time-homogenous Markov process with state space $\mathcal{D} \subset \mathbb{R}^d$, such that the characteristic function of $X(t)$ given $X(0)$ is an affine function of the initial state $X(0)$:

\[ \phi(u|s,x) := \mathbb{E}\left(e^{iu^\top X(s)}|X(0) = x\right) = e^{\psi_0(u,s)+x^\top \psi_1(u,s)}, \quad u \in \mathbb{R}^d. \]

The affine process $X(t)$ is called regular, if the derivatives

\[ F_0(u) := \frac{\partial \psi_0(u,s)}{\partial s}\bigg|_{s=0}, \quad F_1(u) := \frac{\partial \psi_1(u,s)}{\partial s}\bigg|_{s=0} \]

exist and are continuous at $u = 0$. The following theorem provides the characterization of affine processes and is proved in Duffie, Filipović and Schachermayer (2003).

**Theorem 2.1.** If $(X(t))_{t \geq 0}$ is a regular affine process, then $\psi_0$ and $\psi_1$ satisfy the generalized Riccati equations

\begin{align}
\frac{\partial \psi_0(u,s)}{\partial s} &= F_0(\psi_1(u,s)), \quad \psi_0(u,0) = 0, \\
\frac{\partial \psi_1(u,s)}{\partial s} &= F_1(\psi_1(u,s)), \quad \psi_1(u,0) = iu,
\end{align}
where\[ F_0(z) = (\alpha^{(0)} z, z) + (z, \beta^{(0)}) - \gamma^{(0)} + \int_{D \setminus \{0\}} \left( e^{z^T u} - 1 - (\chi(u), z) \right) \nu^{(0)}(du) \]
\[ F_{1,j}(z) = (\alpha_j^{(1)} z, z) + (z, \beta_j^{(1)}) - \gamma_j^{(1)} + \int_{D \setminus \{0\}} \left( e^{z^T u} - 1 - (\chi(u), z) \right) \nu_j^{(1)}(du) \]
for \( j = 1, \ldots, d \) and \( \chi(u) = (\chi_1(u), \ldots, \chi_d(u)) \) with
\[ \chi_k(u) = \begin{cases} 0, & u_k = 0, \\ (1 \wedge |u_k|) \frac{u_k}{|u_k|}, & \text{otherwise} \end{cases} \]
for \( k = 1, \ldots, d \). Here \( \alpha = (\alpha^{(0)}, \alpha^{(1)}) \in \mathbb{R}^{d \times d} \times \mathbb{R}^{d \times d} \), \( \beta = (\beta^{(0)}, \beta^{(1)}) \in \mathbb{R}^d \times \mathbb{R}^d \), \( \gamma = (\gamma^{(0)}, \gamma^{(1)}) \in \mathbb{R}^d \) and \( \nu = (\nu^{(0)}, \nu_1^{(1)}, \ldots, \nu_d^{(1)}) \) is a vector of measures on \( \mathbb{R}^d \), satisfying
\[ \int_{D \setminus \{0\}} \|\chi(u)\|^2_2 \nu^{(0)}(du) < \infty, \quad \int_{D \setminus \{0\}} \|\chi(u)\|^2_2 \nu_j^{(1)}(du) < \infty, \quad j = 1, \ldots, d, \]
where here and in the sequel \( \|x\|_2 := \sqrt{x_1^2 + \ldots + x_d^2} \) for any \( x \in \mathbb{R}^d \).

Under some admissibility conditions a regular affine process \( X(t) \) is a Feller process in the domain \( D = \mathbb{R}_+^m \times \mathbb{R}^{d-m} \) (see Duffie, Filipović and Schachermayer, 2003, Section 2) with the infinitesimal generator
\[ Af(x) = \sum_{k,l=1}^d \left( \alpha_{kl}^{(0)} + \sum_{i=1}^m \alpha_{kl,i}^{(1)} x_i \right) \frac{\partial^2 f(x)}{\partial x_k \partial x_l} + \sum_{i=1}^m \alpha_{ii}^{(1)} x_i \frac{\partial^2 f(x)}{\partial x_i^2} \\
+ \sum_{k=1}^d \beta_k^{(0)} \frac{\partial f(x)}{\partial x_k} + \sum_{k=1}^d \sum_{i=1}^m \beta_{k,i}^{(1)} x_i \frac{\partial f(x)}{\partial x_k} + \sum_{k=m+1}^d \sum_{i=m+1}^d \beta_{k,i}^{(1)} x_i \frac{\partial f(x)}{\partial x_k} \\
+ \int_{D \setminus \{0\}} \left( f(x + \xi) - f(x) - \sum_{k=m+1}^d \chi_k(\xi) \nabla_k f(x) \right) \nu^{(0)}(d\xi) \\
+ \sum_{i=1}^m x_i \int_{D \setminus \{0\}} \left( f(x + \xi) - f(x) - \chi_i(\xi) \nabla_i f(x) - \sum_{k=m+1}^d \chi_k(\xi) \nabla_k f(x) \right) \nu_i^{(1)}(d\xi) \]
where \( 0 \leq m \leq d \) and \( f \in C^2(D) \). In the sequel we assume that the above admissibility conditions hold. Moreover, we restrict our analysis to the class of regular affine processes with
\[ \nu_1^{(1)} = \ldots = \nu_d^{(1)} = 0. \]

On the one hand this assumption reduces the dimensionality of the jump component of \( X \). On the other hand the class of affine models satisfying (4) remains rather large and includes such well known models as Heston, Bates and Barndorff-Nielsen and Shephard stochastic volatility models.

The goal of this paper is to investigate the problem of a nonparametric inference for the Lévy measure \( \nu^{(0)} \) based on a time series of the asset prices \( S(t) = (S_1(t), \ldots, S_k(t)) \) that follow exponential regular affine model:
\[ S_k(t) = S_k(0) e^{X_k(t)}, \quad t \in [0, T], \quad k = 1, \ldots, d, \]
where \( X(t) = (X_1(t), \ldots, X_d(t)) \) is a regular affine process with characteristics \( \chi = (\alpha, \beta, \gamma, \nu) \).
2.1 Regularity properties of affine processes

In the sequel we shall need to know how fast the derivatives of the conditional characteristic function of an affine process can grow. The following lemma provides the corresponding bounds.

**Lemma 2.2.** Let for some natural \( k > 0 \) the Lévy measure \( \nu^{(0)} \) satisfies

\[
\int_{\{\|x\| > 1\}} \|x\|^k \nu^{(0)}(dx) < \infty,
\]

then functions \( \psi_0(u,s) \) and \( \psi_1(u,s) \) from the representation \( (1) \) are in \( C^{k+1}(\mathbb{R}_+) \) as functions of \( s \). Moreover, for any fixed \( x \in \mathcal{D} \) and any \( u \in \mathbb{R}^d \) the following estimates hold

\[
\sup_{s \in [0,T]} \left| \frac{\partial^l \phi(u|s,x)}{\partial s^l} \right| \leq C \|u\|^{2l} \quad l = 0, \ldots, k + 1,
\]

where \( C \) is a positive constant depending on \( T \) and \( x \).

3 Spectral estimation

If values of the log-return process \( X(t) \) on a time grid \( \pi = \{ t_0, t_1, \ldots, t_N \} \) are observed and \( |\pi| \) is small (high-frequency data) then a large increment \( X(t_i) - X(t_{i-1}) \) indicates that a jump occurred between time \( t_{i-1} \) and \( t_i \). Considering different functions of increments and averaging over all increments, we can estimate various integrals w.r.t. the underlying Lévy measure and hence various characteristics of the jump component. Another possibility to estimate the characteristics of jumps is to make use of the spectral representation \( (1) \) by first estimating the c.f. \( \phi(u|s, x) \) and then its derivative \( \partial_s \log(\phi(u|s, x)) \) at the point \( s = 0 \). Finally, the representation of Theorem 2.1 and some transformation allow one to eliminate the diffusion part of \( X \) and to estimate the Lévy measure \( \nu^{(0)} \). Suppose that for any fixed \( x \in \mathcal{D} \) two sequences of estimates \( \{ \hat{\phi}_N(u|s, x) \} \) and \( \{ \hat{\phi}_{s,N}(u|s, x) \} \) of the conditional characteristic function \( \phi(u|s, x) \) and its derivative \( \partial_s \phi(u|s, x) \) respectively, where \( u \in \mathbb{R}^d \) and \( s, x \in [0,T] \) for some \( T > 0 \), are constructed. At this stage it is not important how they were obtained. We assume only that the estimates \( \hat{\phi}_N(u|s, x) \) and \( \hat{\phi}_{s,N}(u|s, x) \) are uniformly consistent in \( u \), i.e. for any fixed \( x \in \mathcal{D} \) and \( s \in S \)

\[
\sup_{u \in \mathbb{R}^d} \left| w(\|u\|) \left| \hat{\phi}_N(u|s, x) - \phi(u|s, x) \right| \right| = O_{a.s.}(\zeta_{0,N}),
\]

\[
\sup_{u \in \mathbb{R}^d} \left| w(\|u\|) \left| \hat{\phi}_{s,N}(u|s, x) - \partial_s \phi(u|s, x) \right| \right| = O_{a.s.}(\zeta_{1,N}),
\]

where \( w \) is a non-negative weighting function on \( \mathbb{R}_+ \), \( \|u\| = \max_{i=1,\ldots,d} |u_i| \) for any \( u \in \mathbb{R}^d \) and \( \zeta_N := \max \{ \zeta_{0,N}, \zeta_{1,N} \} \to 0 \) as \( N \to \infty \). In the next sections we discuss how to construct such consistent estimates from a time series data under random sampling using a smoothed it time version of empirical characteristic function.

3.1 Algorithm

Our aim is to consistently estimate the Lévy measure \( \nu^{(0)} \) using the estimates \( \hat{\phi}_N(u|s, x) \) and \( \hat{\phi}_{s,N}(u|s, x) \). Note that in this formulation other parameters of the affine process \( X(t) \) are
viewed as nuisance components which are to be filtered out during estimation. The main idea of estimation algorithm is as follows. Denote \( \psi(u,s,x) := \psi_0(u,s) + x^\top \psi_1(u,s) \) and introduce function

\[
\Psi(u) := \int_{[-1,1]^d} (\partial_s \psi(u|0, x) - \partial_s \psi(u + w|0, x)) \, dw
\]

that fulfills (due to (2) and (3))

\[
\Psi(u) = 2d^3 \text{tr}(\alpha(0)) + 2d^3 \sum_{j=1}^d x_j \text{tr}(\alpha_j^{(1)}) + \int_D e^{iz^\top u} \rho(0)(dz),
\]

where

\[
\rho(0)(dz) := 2d^d \prod_{k=1}^d \left(1 - \sin \frac{z_k}{z_k}\right) \nu(0)(dz)
\]

is a finite measure. Function \( \Psi(u) \) satisfies, due the Riemann-Lebesgue theorem, the following asymptotic relation

\[
\lim_{\|u\| \to \infty} \Psi(u) = 2d^3 \text{tr}(\alpha(0)) + 2d^3 \sum_{j=1}^d x_j \text{tr}(\alpha_j^{(1)}) =: L.
\]

So, in order to reconstruct a transformed Lévy measure \( \rho(0) \) and hence \( \nu(0) \) one can first estimate function \( \Psi(u) \) then estimate its limit as \( \|u\| \to \infty \) and finally the Fourier transform of \( \rho(0) \) using (10).

**Remark 3.1.** Transformation (10) which transforms the Lévy measure \( \nu(0) \) to a finite measure is not unique. Motivated by the results of Neumann and Reiß (2007) for the case of one dimensional Lévy processes, one can consider the derivative \( \frac{\partial \psi(u|0, x)}{\partial u_1...\partial u_d} \) instead. However, the latter type of transformation would require existence of the second order moments of the process \( X(t) \). Moreover, empirical results indicate that integration in (10) may reduce the variance of an estimate for \( \partial \psi(u|0, x) \).

**Remark 3.2.** If we are interested in reconstructing only one particular component of \( \nu \) then it is enough to construct the estimates for the corresponding marginal conditional c.f. and its derivative in time. These estimates in turn can be constructed using only a time series of the corresponding component of \( X \) (see Section 6 for numerical illustration).

In the sequel we shall assume that the measure \( \rho(0) \) is absolutely continuous w.r.t. Lebesgue measure on \( \mathbb{R}^d \) and possesses bounded density denoted (with some abuse of notations) by \( \rho(0)(x) \). In fact, this means that the index of jump activity of the process \( X(t) \) introduced in Aït-Sahalia and Jacod (2008) for general semimartingales is smaller than 1, i.e.

\[
\max_{k=1,...,d} \inf_{\{r \geq 0 : \int_{\{|x_1|>1,...,|x_{k-1}|>1,|x_k|\leq 1,|x_{k+1}|>1,...,|x_d|>1\}} |x_k|^r \nu(0)(dx) < \infty} \ < 1.
\]

Note that according to the admissibility conditions in Duffie, Filipović and Schachermayer (2003) it must hold

\[
\max_{k=1,...,m} \inf_{\{r \geq 0 : \int_{\{|x_1|>1,...,|x_{k-1}|>1,|x_k|\leq 1,|x_{k+1}|>1,...,|x_d|>1\}} |x_k|^r \nu(0)(dx) < \infty} \ < 1.
\]
So, our assumption puts an upper bound on the degree of activity of small jumps for components $X_{m+1}(t), \ldots, X_d(t)$. Namely, we consider the case of medium activity of small jumps. The case of highly active small jumps can be treated in our framework as well. In this case, however, we can not any longer use the Fourier inversion formulas for densities since the density $\rho^{(0)}$ becomes unbounded. But we can still employ the Fourier inversion formula for distributions instead. For the sake of the brevity we shall not pursue this possibility in this paper. Let $q_0(v)$ and $q_1(v)$ be any two functions satisfying

$$q_0(v) \leq \inf_{\|u\|=v} \inf_{s \in S} |\phi(u,s,x)|,$$

$$q_1(v) \geq \sup_{\|u\|=v} \sup_{s \in S} |\partial_s \phi(u,s,x)| = \sup_{\|u\|=v} \sup_{s \in S} |\partial_s \phi(u,s,x)/\phi(u,s,x)|$$

with $v \in \mathbb{R}_+$. Functions $q_0(v)$ and $q_1(v)$ can be found if some prior bounds on the eigenvalues of the matrix $\alpha^{(0)}$ are available. The following lemma can be used to construct $q_0(v)$ and $q_1(v)$.

**Lemma 3.3.** Let $X$ be a regular affine process with admissible characteristics $\chi = (\alpha, \beta, \gamma, \nu)$ such that (1) is fulfilled, then the following estimates hold

$$\|u\|_2^{-2} |\log(\phi(u,s,x))| \lesssim \lambda_{\max}(\mathfrak{A}) \int_0^s \lambda_{\max}(e^{t\mathfrak{B}}) \ dt, \quad \|u\|_2 \to \infty,$$

$$\|u\|_2^{-2} |\partial_s \phi(u,s,x)| \lesssim \lambda_{\max}(\mathfrak{A}) \lambda_{\max}(e^{s\mathfrak{B}}), \quad \|u\|_2 \to \infty,$$

where $\mathfrak{A} := (a_{ij}^{(0)})_{m+1 \leq i,j \leq d}$, $\mathfrak{B} := (\beta_{i,j}^{(1)})_{m+1 \leq i,j \leq d}$ and for any matrix $a$ $\lambda_{\max}(a)$ stands for the maximal eigenvalue of $a$.

Let us now formulate our estimation procedure. It will be, for reader convenience, separated into several steps.

**Step 1** Construct estimates for $\psi(u,s,x)$ and $\partial_s \psi(u,s,x)$ as follows

$$\hat{\psi}_N(u,s,x) = \log(T_{q_0}([\hat{\phi}_N])(u,s,x)),$$

$$\hat{\psi}_{s,N}(u,s,x) = T_{q_1}([\hat{\phi}_{s,N}]/[\hat{\phi}_N])(u,s,x),$$

where

$$T_{q_0}([\hat{\phi}_N]) = \begin{cases} 1, & |\hat{\phi}_N| > 1, \\ \hat{\phi}_N, & q_0 \leq |\hat{\phi}_N| \leq 1, \\ 0_0[\hat{\phi}_N]/|\hat{\phi}_N|, & |\hat{\phi}_N| < q_0 \end{cases}$$

and

$$T_{q_1}([\hat{\phi}_{s,N}]/[\hat{\phi}_N]) = \begin{cases} \hat{\phi}_{s,N}/[\hat{\phi}_N], & |\hat{\phi}_{s,N}| \leq q_1|\hat{\phi}_N|, \\ q_1[\hat{\phi}_{s,N}]/|\hat{\phi}_N|/(\phi_N[\hat{\phi}_{s,N}]), & |\hat{\phi}_{s,N}| > q_1|\hat{\phi}_N|. \end{cases}$$
Step 2 Define
\[ \hat{\Psi}_N(u) := \int_{[-1,1]^d} \left( \psi_{s,N}(u|0,x) - \psi_{s,N}(u+w|0,x) \right) dw. \]

Let \( K(u) \) be a non-negative function supported on \([-1,1]\) that satisfies
\[ \int_{-1}^{1} K(u) du = 1. \]

For any \( U > 0 \) put
\[ K_U(u) = \frac{U}{U^2 - 1} K \left( \frac{u}{U^2 - 1} \right) \]
and define an estimate for the limit \( L \) as
\[ L_{U,N} = \int_{\mathbb{R}^d} \left[ K_U(u_1) \times \ldots \times K_U(u_d) \right] \hat{\Psi}_N(u) du, \]

Step 3 Reconstruct density \( \rho^{(0)}(x) \) using the Fourier inversion formula
\[ \tilde{\rho}_N^{(0)}(x;U) = \frac{1}{(2\pi)^d} \int_{-U}^{U} \ldots \int_{-U}^{U} e^{-iu^\top x} \left[ \hat{\Psi}_N(u) - L_{U,N} \right] du. \]

In the next section we present uniform convergence rates for the estimate \( \tilde{\rho}_N^{(0)}(x;U) \).

4 Theoretical properties

First, the following lemma shows that both estimates \( \hat{\psi}_N(u,s,x) \) and \( \hat{\psi}_{s,N}(u|s,x) \) are uniformly consistent

**Lemma 4.1.** If estimates \( \hat{\phi}_N(u,s,x) \) and \( \hat{\phi}_{s,N}(u|s,x) \) fulfill \( \text{[\text{a}] - [\text{b}]} \), then it holds for any fixed \( s \in S \) and \( x \in D \)
\[ \sup_{u \in \mathbb{R}^d} \left[ w_0(||u||)(\hat{\psi}_N(u,s,x) - \psi(u,s,x)) \right] = O_{a.s.}(\zeta_N), \]
\[ \sup_{u \in \mathbb{R}^d} \left[ w_1(||u||)(\hat{\psi}_{s,N}(u|s,x) - \partial_s \psi(u|s,x)) \right] = O_{a.s.}(\zeta_N), \]
where \( w_0(||u||) = \varrho_0(||u||)w(||u||) \) and \( w_1(||u||) = \varrho_0(||u||)(2^{-1} \varrho_1^{-1}(||u||) \wedge 1)w(||u||) \).

Next we prove minimax upper and lower risk bounds for the estimate \( \tilde{\rho}_N^{(0)}(x;U) \).

4.1 Upper risk bounds

For any \( \Lambda > 0, 1 \leq \kappa < 2 \) and \( R > 0 \) let \( A(\Lambda, \kappa, R) \) stand for a class of regular affine models with admissible characteristics \( \chi = (\alpha, \beta, \gamma, \nu) \) satisfying \( \text{[\text{c}]} \) and

1. It holds for any \( s \in S \)
\[ \max \left\{ \lambda_{\max}(A) \int_{0}^{s} \lambda_{\max}(\mathcal{B}) \, dt, \lambda_{\max}(A)\lambda_{\max}(\mathcal{B}) \right\} \leq \Lambda, \]
where matrices \( A \) and \( \mathcal{B} \) are defined in Lemma \( \text{[3]} \).
2. For any \( u \in \mathbb{R}^d \)

\[
\prod_{k=1}^{d} u_k \leq R,
\]

where measure \( \rho^{(0)} \) is related to the Lévy measure \( \nu^{(0)} \) via (11).

Here and in the sequel \( \mathcal{F}[\rho](u) \) stands for the Fourier transform of a measure \( \rho \).

**Remark 4.2.** It can be shown that if the Lévy measure \( \nu^{(0)} \) satisfies

\[
\max_{k=1,\ldots,d} \inf_{r \geq 0} \left\{ \int_{|x|>1, \ldots, |x_{k-1}|>1, |x_k| \leq 1, |x_{k+1}|>1, \ldots, |x_d|>1} |x_k|^r \nu^{(0)}(dx) < \infty \right\} < 2 - \kappa,
\]

i.e. the degree of jump activity of each component of the process \( X(t) \) is less than \( 2 - \kappa \),

then inequality (14) holds with some \( R > 0 \) and

\[
\text{sup}_{u \in \mathbb{R}^d} \left| |u|^{-\kappa} |\mathcal{K}(u)| \right| < \infty.
\]

Based on the above assumption and Lemma 4.1 the following proposition on a uniform convergence of the estimate \( \hat{\rho}_N^{(0)}(x;U) \) can be proved.

**Proposition 4.3.** Let \( X(t) \) be a regular affine process with characteristics \( \chi = (\alpha, \beta, \lambda, \nu) \in \mathcal{A}(\Lambda, \kappa, R) \) and with a conditional characteristic function \( \phi(u | s, x) \). Suppose that

\[
\int_{-1}^{1} |u|^{-\kappa} |\mathcal{K}(u)| du < \infty.
\]

If the sequences of estimates \( \{ \hat{\phi}_N(u | s, x) \} \) and \( \{ \hat{\phi}_{s,N}(u | s, x) \} \) for \( \phi(u | s, x) \) and \( \partial_s \phi(u | s, x) \) respectively are available and fulfill (8) and (9), then for any large enough \( U \) the estimate \( \rho_N^{(0)}(x;U) \) satisfies

\[
\sup_{x \in \mathcal{D}} |\rho^{(0)}(x) - \hat{\rho}_N^{(0)}(x;U)| = O_{a.s.} \left( \zeta_N \log^{-1}(U)U^{2+d}e^{\Lambda U^2} + U^{-(\kappa-1)} \right).
\]

**Discussion**

As will be shown in Section 5 one can construct estimates \( \hat{\phi}_N(u | s, x) \) and \( \hat{\phi}_{s,N}(u | s, x) \) such that (8) and (9) hold with \( w(v) = \min\{1, v^{-4}\} \) and \( \zeta_N = \max\{\zeta_0, \zeta_1, \zeta_N\} = O(N^{-r} \log^q N) \) for some \( r > 0 \) and \( q > 0 \). As a result the rates in Theorem 4.3 are logarithmic if \( \Lambda > 0 \). More precisely, setting

\[
U_N := \Lambda^{-1/2} \sqrt{r \log N - \left( \frac{(\kappa - 1)}{2} + 3 + d/2 + q \right) \log \log N},
\]

we get as \( N \to \infty \)

\[
\sup_{x \in \mathcal{D}} |\rho^{(0)}(x) - \hat{\rho}_N^{(0)}(x,U_N)| = O_{a.s.} \left( (\log N)^{-(\kappa-1)/2} \right).
\]

Several observations can be made from the inspection of these rates. First, the convergence rates are logarithmic and correspond to a severely ill-posed problem. The reason for the severe ill-posedness is that we face a deconvolution like problem: the law of the continuous part of \( X(T) \) is convolved (in generalized sense) with that of the jump part to give the distribution of \( X(T) \). Second, the nearer is \( \kappa \) to 1 the more complex becomes the estimation problem. Since the degree of jump activity is equal to \( 2 - \kappa \) this means that estimating the characteristics of jumps becomes more difficult as the degree of jump activity increases to 1.
4.2 Lower risk bounds

The rates in (17) are in fact optimal in the minimax sense for the class $\mathcal{A}(\Lambda, \kappa, R)$.

**Proposition 4.4.** The following minimax lower risk bounds hold

$$\liminf_{N \to \infty} \inf_{\hat{\rho}_N(0)} \sup_{\chi \in \mathcal{A}(\Lambda, \kappa, R)} P_{\chi} \left( (\log N)^{(\kappa-1)/2} \sup_{x \in D} |\rho(0)(x) - \hat{\rho}_N(0)(x)| > \varepsilon \right) > 0,$$

where $\varepsilon$ is some positive number and $\hat{\rho}_N(0)$ is any estimator of $\rho(0)$ based on $N$ observations.

5 Nonparametric estimation of $\phi$ and $\partial_s \phi$

In this section we present a method for estimating the conditional characteristic function $\phi(u|s, x)$ and its derivatives in time from a time series of $X(t)$. For our theoretical study we adopt the so-called random design observational model, i.e. we assume that for some $x \in D$ a trajectory of the process $X$ containing pairs

$$(X(t_n), X(t_n + \delta_n)), \quad X(t_n) = x, \quad n = 1, \ldots, N,$$

is available, where $\delta_n, n = 1, \ldots, N$ are i.i.d. random variables on $[0, T]$ for some fixed $T > 0$ with a common bounded density $p_\delta(x)$ and $\min_n(t_{n+1} - t_n) \geq 2T$. This assumption implies that the time horizon $T + t_N$ of observations tends to $\infty$ as $N \to \infty$, a condition which is not unusual even in the literature on statistical inference from high-frequency data (see, e.g. Figueroa-López (2008)). In Figure 1 a typical trajectory of a one-dimensional Ornstein-Uhlenbeck process (without jump component) is shown together with the level line $x = 0$. We estimate $\phi_N(u|s, x)$ and

![Figure 1: A typical path of the one-dimensional Ornstein-Uhlenbeck process $dX(t) = -5X(t) \, dt + 3.5 \, dW(t)$ along with the line $x = 0$.](image-url)
\[ \partial_s \phi_N(u|s,x) \] by a local linear smoothing of empirical characteristic process. Define

\begin{align}
\tilde{\phi}_N(u|s,x) &:= \sum_{n=1}^{N} \tau_{0,n}(s) \exp(iu^\top X(t_n + \delta_n)), \\
\tilde{\phi}_{s,N}(u|s,x) &:= \sum_{n=1}^{N} \tau_{1,n}(s) \exp(iu^\top X(t_n + \delta_n)),
\end{align}

where

\[ \tau_{j,n}(s) = \frac{b_{j,n}(s)}{\sum_{k=1}^{N} b_{0,k}(s)}, \quad j = 0, 1, \]

\[ b_{0,n}(s) = K \left( \frac{\delta_n - s}{h} \right) (S_{N,2} - (\delta_n - s)S_{N,1}), \]

\[ b_{1,n}(s) = K \left( \frac{\delta_n - s}{h} \right) ((\delta_n - s)S_{N,0} - S_{N,1}) \]

and

\[ S_{N,j} = \sum_{n=1}^{N} K \left( \frac{\delta_n - s}{h} \right) (\delta_n - s)^j, \quad j = 0, 1, 2. \]

Here \( K \) is a kernel and \( h \) is a bandwidth. Denote

\[ \Gamma(s) = N^{-1} \begin{pmatrix} h^{-1}S_{N,0} & h^{-2}S_{N,1} \\ h^{-2}S_{N,1} & h^{-3}S_{N,2} \end{pmatrix}, \quad \bar{\Gamma}(s) = \begin{pmatrix} \mu_0(s) & \mu_1(s) \\ \mu_1(s) & \mu_2(s) \end{pmatrix} \]

with \( \mu_l(s) = \int_{\mathbb{R}} z^l K(z)p_\delta(s + hz) \, dz \). It easy to show that if matrix \( \Gamma \) is invertible then it holds \((\tilde{\phi}_N, h\tilde{\phi}_{s,N}) = \Gamma^{-1}Y\), where \( Y \) is a two-dimensional vector with components

\[ Y_k = \frac{1}{Nh} \sum_{n=1}^{N} \exp(iu^\top X(t_n + \delta_n)) \left( \frac{\delta_n - s}{h} \right)^k K \left( \frac{\delta_n - s}{h} \right), \quad k = 0, 1. \]

In order to be able to prove the convergence of estimates \( \tilde{\phi}_N(u|s,x) \) and \( \tilde{\phi}_{s,N}(u|s,x) \) we need the following assumptions

**Assumptions**

**(AX1)** The sequence \( X_n := X(t_n + \delta_n) - X(t_n) \) is exponentially strongly mixing, i.e. the mixing coefficients \( \alpha_X \) (see Appendix for definition) satisfy

\[ \alpha_X(n) \leq \tilde{\alpha} \exp(-cn), \quad n \geq 1, \]

for some \( \tilde{\alpha} > 0 \) and \( c > 0 \).

**(AX2)** The Lévy measure \( \nu^{(0)} \) satisfies for some \( p > 2 \)

\[ \int_{\{\|x\| > 1\}} \|x\|^p \nu^{(0)}(dx) < \infty. \]
The minimal eigenvalue of the matrix $\bar{\Gamma}(s)$ is bounded away from below by some positive constant $\gamma_0$ uniformly in $s$, i.e.

$$\min_{s \in S} \lambda_{\text{min}}(\bar{\Gamma}(s)) \geq \gamma_0 > 0.$$ 

**Remark 5.1.** Exponentially strongly mixing holds for a wide class of Itô-Lévy processes. In Masuda (2007) conditions are formulated that ensure that a multidimensional Itô-Lévy process is exponentially $\beta$-mixing (hence exponentially $\alpha$-mixing). Suppose that $X$ is a regular affine process with characteristics $\chi$ satisfying admissibility conditions, the condition (24) and $\int_{|x| > 1} |x|^q \nu(0)(dx) < \infty$ for some $q > 0$. Moreover, assume that

$$\beta_{k,i}^{(1)} = 0, \quad \alpha_{k,i}^{(1)} = 0, \quad i, k \in \mathcal{I} := \{1, \ldots, d\} \setminus \mathcal{I},$$

where $\mathcal{I}$ is a subset of $\{1, \ldots, d\}$. This situation is typical for stochastic volatility models. If the maximal eigenvalue of the matrix $(\beta_{i,j})_{i,j \in \mathcal{I}}$ is negative then both sequences $X_{\mathcal{I}}(t_n + \delta_n)$ and $\bar{X}_{\mathcal{I}}(t_n)$ are exponentially $\beta$-mixing and ergodic. Taking the initial distribution for $X_{\mathcal{I}}(0)$ to be the invariant one, both sequences $X_{\mathcal{I}}(t_n + \delta_n)$ and $\bar{X}_{\mathcal{I}}(t_n + \delta_n)$ become stationary.

**Remark 5.2.** Assumption (AS) imposes some regularity conditions on the design $\{\delta_n\}_{n=1}^\infty$. In particular, it ensures that points arbitrary close to 0 appear in the sample with positive probability as $N \to \infty$. This in turn allows us to consistently estimate the derivative $\partial_s \phi(u|s, x)|_{s=0}$.

### 5.1 Asymptotic properties of $\hat{\phi}$ and $\tilde{\phi}_s$. 

In this section we investigate the asymptotic properties of estimates $\hat{\phi}_N(u)$ and $\tilde{\phi}_s,N(u)$ of estimates $\bar{\phi}_N(u)$ and $\bar{\phi}_s,N(u)$ respectively which are defined as follows. If the smallest eigenvalue of the matrix $\Gamma$ is greater than $\gamma_0/2$ we set $\hat{\phi}_N(u) = \phi_N(u)$ and $\tilde{\phi}_s,N(u) = \phi_s,N(u)$. Otherwise, we put $\hat{\phi}_N(u) = \tilde{\phi}_s,N(u) = 0$. The following proposition shows that both estimates $\hat{\phi}_N$ and $\tilde{\phi}_s,N$ are uniformly consistent in a weighted sup-norm.

**Proposition 5.3.** Suppose that assumptions (AX1), (AX2), (AS) and (AK) are fulfilled and the bandwidth $h_N$ satisfies $h_N^{-1} = o(N)$. Let $w$ be a positive Lipschitz function on $\mathbb{R}_+$ such that

$$(20) \quad 0 < w(z) \leq \min\{1, z^{-4}\}, \quad z \in \mathbb{R}_+.$$ 

Then for any fixed $s \in S$ and $x \in D$

$$\sup_{u \in \mathbb{R}^d} \left[ w(\|u\|) \hat{\phi}_N(u|s, x) - \phi(u|s, x) \right] = O_{a.s.} \left( \sqrt{\frac{\log(1+r)N}{Nh_N} + h_N^2} \right),$$

$$(21) \quad \sup_{u \in \mathbb{R}^d} \left[ w(\|u\|) \tilde{\phi}_s,N(u|s, x) - \partial_s \phi(u|s, x) \right] = O_{a.s.} \left( \sqrt{\frac{\log(1+r)N}{Nh_N^2} + h_N} \right),$$

with some $r > 0$. 

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Discussion  The right-hand sides of (21) and (22) are sums of two terms corresponding to “variance” and “bias” respectively. The optimal choice of the bandwidth parameter $h_N$ which minimizes (21) and (22) is given by

$$h_N = \left[ N^{-1} \log^{1+r}(N) \right]^{1/5}$$

in which case uniform convergence rates are given by

$$\sup_{u \in \mathbb{R}^d} \left[ w(||u||) \tilde{\phi}_N(u,s,x) - \phi(u,s,x) \right] = O_{\text{a.s.}} \left( \left( N^{-1} \log^{1+r}(N) \right)^{2/5} \right),$$

$$\sup_{u \in \mathbb{R}^d} \left[ w(||u||) \tilde{\phi}_{s,N}(u,s,x) - \partial_s \phi(u,s,x) \right] = O_{\text{a.s.}} \left( \left( N^{-1} \log^{1+r}(N) \right)^{1/5} \right).$$

Remark 5.4. The condition (20) on the decay of weighting function $w$ cannot be, in general, weakened. For example, in the case of a one-dimensional Brownian motion with volatility $\sigma^2$, the simplest affine process, we get $\partial_s \phi(u,s,x)|_{s=0} = \sigma^4 u^4/4$. This means that approximation errors of local linear estimates in (18) and (19) at the point 0 are of order $h^2 \sigma^4 u^4/8$ and $h \sigma^4 u^4/8$ respectively. So in order to be able to prove uniform consistency in $u$ we have to assume (20).

6 Numerical example

Let us consider a class of stochastic volatility models of the type

\begin{align*}
    dX(t) &= -\frac{1}{2} V(t) \, dt + \sqrt{V(t)} \, dW^S(t) + dZ_t, \\
    dV(t) &= \lambda(\theta - V(t)) \, dt + \zeta \sqrt{V(t)} \, dW^V(t),
\end{align*}

where $X(t) = \log(S(t))$ is the log-price process, $W^S$ and $W^V$ are two independent Brownian motions, $\lambda, \theta, \zeta$ are positive constants and $Z_t$ is a pure-jump Lévy process with Lévy density $\nu(x)$. This is a special type of the model introduced in Bates (2005). In our numerical example we take $Z(t)$ to be $\alpha$-stable Lévy process with stability index $\alpha < 1$, i.e.

$$\nu(x) = C/|x|^{1+\alpha}, \quad \rho(x) = 2 \left( 1 - \sin \frac{x}{x} \right) \nu(x)$$

for some constant $C > 0$. For the sake of simplicity we consider a fixed design and simulate a set of i.i.d pairs

\begin{align*}
    (X^{(n)}(0), X^{(n)}(\Delta)), \quad n = 1, \ldots, N
\end{align*}

with some fixed $\Delta > 0$, where

\begin{align*}
    (X^{(1)}(0), V^{(1)}(0)) = \ldots = (X^{(N)}(0), V^{(N)}(0)).
\end{align*}

Without loss of generality we set $(X^{(1)}(0), V^{(1)}(0)) = (0, 1)$. Our aim is to reconstruct $\rho$ using sample (21). First compute

$$\hat{\psi}_{s,N}(u|0) = \hat{\phi}_{s,N}(u|0) := \frac{1}{N \Delta} \sum_{n=1}^{N} \left[ \exp \left( i u^\top X^{(n)}(\Delta) \right) - \exp \left( i u^\top X^{(n)}(0) \right) \right]$$
and
\[ \hat{\Psi}_N(u) := \int_{-1}^{1} \left( \hat{\psi}_{s,N}(u|0) - \hat{\psi}_{s,N}(u + w|0) \right) dw. \]

**Remark 6.1.** In the case of high-frequency data observations are usually available for different frequency scales \( \Delta \) and the choice of an appropriate frequency for estimation procedure should be done depending on \( N \), the number of points available for the given frequency scale. If \( \Delta \) is too small then the variance of \( \hat{\phi}_{s,N}(u|0) \) explodes. On the other hand, if \( \Delta \) is too large than the approximation error of \( \hat{\phi}_{s,N}(u|0) \) becomes large.

Next define a parametric family of functions
\[ \tilde{\rho}^{(0)}_{N}(x; U) = \text{Re} \left\{ \frac{1}{2\pi} \int_{-U}^{U} e^{-iux} \left[ \hat{\Psi}_N(u) - \Psi_N(U) \right] du \right\}, \quad U > 0 \]
and find \( U \) by solving the following minimization problem
\[ \hat{U} = \arg\inf_U \left\{ \int_{|u|>U} \left| \hat{\Psi}_N(u) - \hat{\Psi}_N(U) \right|^2 du + \pi \int \left| \partial_{xx} \tilde{\rho}^{(0)}_{N}(x; U) \right| dx \right\}, \]
where \( \pi > 0 \) is a regularization parameter. In fact, this approach for choosing \( U \) employs additional information about smoothness of \( \rho \) and turns out to be rather efficient in practice. In Figure 2 typical results of estimation based on \( N = 1000 \) samples \( \Delta = 0.1 \) are shown for two specifications of the process \( Z_t \). As can be seen the overall quality of estimation is good taking into account severely ill-posedness of the underlying estimation problem. However, the behavior of the transformed Lévy density \( \rho \) at zero is not captured by estimation method. In order to correct \( \rho(x) \) at \( x = 0 \) we separately estimate the stability index \( \alpha \) using a modification of the spectral algorithm proposed in Belomestny (2009) for Lévy processes. Motivated by relations

![Figure 2](image_url)
and (3), we define for any \( a \in (0, 1) \)

\[
O(a) := \min_{(l_0, l_1, l_2, l_3)} \int_0^\infty \left( \psi_{s,N}(u|0) - l_3 u^a - l_2 u^2 - l_1 u - l_0 \right)^2 du
\]

and estimate \( \alpha \) via \( \tilde{\alpha} := \arg\min_{a \in (0, 1)} O(a) \). In Figure 3 functions \( O(a) \) based on the same samples as in Figure 2 are shown. The resulting estimates for \( \alpha \) are \( \tilde{\alpha} = 0.451 \) and \( \tilde{\alpha} = 0.783 \).

![Figure 3](image_url)

Figure 3: Function \( O(a) \) in the case of symmetric stable process \( Z_t \) with stability indexes 0.5 (left) and 0.8 (right) respectively.

respectively. Now we correct estimate \( \tilde{\rho}_N^{(0)}(x; \hat{U}) \) by setting

\[
\tilde{\rho}_N^{(0)}(x; \hat{U}) = \begin{cases} 
  c(\varepsilon) \left( 1 - \sin^2 \frac{x}{2} \right) |x|^{-(1+\tilde{\alpha})}, & |x| \leq \varepsilon, \\
  \tilde{\rho}_N^{(0)}(x; \hat{U}), & |x| > \varepsilon.
\end{cases}
\]

where for any \( \varepsilon > 0 \) the constant \( c(\varepsilon) \) is chosen in such a way that function \( \tilde{\rho}_N^{(0)}(x; \hat{U}) \) is continuous. Finally, we find small enough \( \varepsilon > 0 \) which minimizes the integral \( \int |\partial_x \tilde{\rho}_N^{(0)}(x; \hat{U})| dx \).

Here again the smoothness of \( \rho \) is used. A corrected estimate \( \tilde{\rho}_N^{(0)}(x; \hat{U}) \) is shown in Figure 4.

7 Proofs

7.1 Proof of Lemma 2.2

Due to (6) all derivatives of functions \( F_0(z) \) and \( F_1(z) \) up to order \( k \) exist for \( \text{Re } z \in \mathbb{R}^m \times \{0\} \times \ldots \times \{0\} \). Fix some \( s \in \mathcal{S} \) and \( x \in \mathcal{D} \). Using (2) and (4), we get for \( 1 \leq l \leq k + 1 \)

\[
\frac{\partial^l \phi(u|s,x)}{\partial s^l} = H \left( \sum_{l=0}^{k-1} (\psi_{l}(u,s)), \sum_{l=0}^{k-1} (\psi_{l}(u,s)), \ldots, \sum_{l=0}^{k-1} (\psi_{l}(u,s)), \psi_{k}(u,s), \ldots, \psi_{k}(u,s) \right) \phi(u|s,x).
\]
Figure 4: Corrected estimates for transformed Lévy density $\rho$ (dashed black line) together with the true $\rho$ (solid red line) in the Bates stochastic volatility model with symmetric stable process $Z_t$ and different stability indexes $\alpha$.

where $H$ is a polynomial of order $l$ in $\mathbb{R}^{2l}$. Note that under admissibility conditions and functions $F_0$ and $F_1$ simplify to

$$F_0(z) = \sum_{i,j=m+1}^{d} \alpha_{ij}^{(0)} z_i z_j + (z, \beta^{(0)}) - \gamma^{(0)} + \int_D \left( e^{z^T u - 1 - (\chi(u), z)} \right) \nu^{(0)}(du)$$

$$F_{1,j}(z) = (\alpha_j^{(1)} z, z) + \sum_{i=1}^{m} z_i \beta_{j,i}^{(1)} - \gamma_j^{(1)}, \quad j = 1, \ldots, m,$$

$$F_{1,j}(z) = \sum_{i=m+1}^{d} z_i \beta_{j,i}^{(1)}, \quad j = m+1, \ldots, d.$$ Solving the system of linear equations

$$\frac{\partial \psi_{1,j}(u,s)}{\partial s} = F_{1,j}(\psi_1(u,s)), \quad \psi_{1,j}(u,s) = iu_j, \quad j = m+1, \ldots, d,$$

we get $(\psi_{1,m+1}(u,s), \ldots, \psi_{1,d}(u,s)) = e^{sR}(iu_{m+1}, \ldots, iu_d)^T$. Furthermore, it follows from the theory of Riccati ODE that the solution of a system

$$\frac{\partial \psi_{1,j}(u,s)}{\partial s} = F_{1,j}(\psi_1(u,s)), \quad \psi_{1,j}(u,s) = iu_j$$

can have at most linear growth in $u$. Since

$$\max\{|F_1^{(l)}(z)|, |F_0^{(l)}(z)|\} \lesssim \|z\|_2^2, \quad \|z\|_2 \to \infty, \quad l = 0, \ldots, k$$

it holds

$$\max\{|F_1^{(l)}(\psi_1(u,s))|, |F_0^{(l)}(\psi_1(u,s))|\} \lesssim \|\psi_1(u,s)\|_2^2 \lesssim \|u\|_2^2, \quad \|u\|_2 \to \infty, \quad l = 0, \ldots, k + 1.$$ Combining the last inequality with (24), we get (17).
7.2 Proof of Proposition 3.3

The analysis of the system of Riccati ODEs (2) and (3) performed in the proof of Lemma 2.2 leads to bounds

\[ \text{Re}[\psi_0(u,s)] = \int_0^s \text{Re}[F_0(\psi_1(u,t))] \, dt \geq -\|u\|_2^2 \lambda_{\text{max}}(\mathfrak{A}) \int_0^s \lambda_{\text{max}}(e^{tB}) \, dt, \]

\[ \text{Re}[\psi_1(u,s)] = o\left(\|u\|_2^2\right), \quad \|u\|_2 \to \infty. \]

7.3 Proof of Lemma 4.1

We prove only a more involved bound (13). Since

\[ \partial_s \phi - \hat{\psi}_{s,N} \hat{\phi}_N = (\partial_s \phi - \hat{\psi}_{s,N} \hat{\phi}_N) + \hat{\psi}_{s,N}(\phi - \hat{\phi}_N), \]

we have

\[ \partial_s \phi - \hat{\psi}_{s,N} = \phi^{-1} \left[ (\partial_s \phi - \hat{\psi}_{s,N} \hat{\phi}_N) - \hat{\psi}_{s,N}(\phi - \hat{\phi}_N) \right] \]

and hence

\[ |\partial_s \phi - \hat{\psi}_{s,N}| \leq \varrho_0^{-1} \left[ |\partial_s \phi - \hat{\psi}_{s,N} \hat{\phi}_N| + \varrho_1 |\phi - \hat{\phi}_N| \right]. \]

Furthermore, it holds

\[ |\hat{\phi}_{s,N} - \hat{\psi}_{s,N} \hat{\phi}_N| \leq \left| \hat{\phi}_{s,N} - \varrho_1 |\hat{\phi}_N| \frac{\hat{\phi}_{s,N}}{|\hat{\phi}_N|} \right| 1(|\hat{\phi}_{s,N}|/|\hat{\phi}_N| > \varrho_1) \]

\[ = \left| |\hat{\phi}_{s,N}| - \varrho_1 |\hat{\phi}_N| \right| 1(|\hat{\phi}_{s,N}|/|\hat{\phi}_N| > \varrho_1) \]

\[ \leq \left| |\hat{\phi}_{s,N}| - |\partial_s \phi| \right| \frac{1}{|\hat{\phi}_N|} \]

(28)

Combining (27) and (28), we get

\[ |\partial_s \phi - \hat{\psi}_{s,N}| \leq \varrho_0^{-1}(u) \left[ 2\varrho_1(u) |\phi - \hat{\phi}_N| + |\hat{\phi}_{s,N} - \partial_s \phi| \right] \]

Now the assumptions (5) and (9) imply

(29) \[ \sup_{x \in X} \sup_{u \in \mathbb{R}^d} \left[ w_1(\|u\|)|\hat{\phi}_{s,N}(u|x,s) - \partial_s \phi(u|x,s)| \right] = O_{u,s}(\zeta_N). \]

with \( w_1(u) = \varrho_0(u)(2^{-1}\varrho_1^{-1}(u) \wedge 1)w(u). \)
7.4 Proof of Proposition 4.3

Lemma 4.1 implies

\[
\sup_{u \in \mathbb{R}^d} \left[ w_1(\|u\|) \left( \hat{\Psi}_N(u) - \Psi(u) \right) \right] = O_{a.s}(\zeta_N).
\]

Since

\[
\mathcal{L}_{U,N} - \mathcal{L}_0 = \int_{\mathbb{R}^d} \left[ \mathcal{K}^U(u_1) \times \mathcal{K}^U(u_d) \right] \left( \hat{\Psi}_N(u) - \Psi(u) \right) \, du
\]

and \( w_1(z) \) is monotone decreasing, we have for any \( U > 0 \)

\[
|\mathcal{L}_{U,N} - \mathcal{L}| \leq w_1^{-1}(U) \int_{\mathbb{R}^d} \left[ \mathcal{K}^U(u_1) \times \mathcal{K}^U(u_d) \right] \times \left[ w_1(\|u\|) \left( \hat{\Psi}_N(u) - \Psi(u) \right) \right] \, du
\]

\[
+ \int_{\mathbb{R}^d} \left[ \mathcal{K}^U(u_1) \times \mathcal{K}^U(u_d) \right] \mathcal{F}[\rho^{(0)}](u) \, du.
\]

Then conditions (14) and (15) imply

\[
\int_{\mathbb{R}^d} \left[ \mathcal{K}^U(u_1) \times \mathcal{K}^U(u_d) \right] \mathcal{F}[\rho^{(0)}](u) \, du \leq C U^{-d\varepsilon}
\]

with some constant \( C > 0 \). Combining (31) with (30), we get

\[
|\mathcal{L}_{U,N} - \mathcal{L}| = O_{a.s} \left( w_1^{-1}(U)\zeta_N + U^{-d\varepsilon} \right).
\]

Furthermore, using the Fourier inversion formula, we get

\[
\sup_{x \in \mathcal{D}} |\rho^{(0)}(x) - \hat{\rho}^N_N(x; U)| \lesssim U^d \left[ \sup_{\|u\| \leq U} \left( \hat{\Psi}_N(u) - \Psi(u) \right) + |\mathcal{L}_{U,N} - \mathcal{L}| \right]
\]

\[
+ \left| \int_{\{\|u\| > U\}} e^{-iu^\top x} \mathcal{F}[\rho^{(0)}](u) \, du \right|
\]

\[
\lesssim U^d \left[ w_1^{-1}(\|U\|) \sup_{\|u\| \leq U} \left( w_1(\|u\|) \left( \hat{\Psi}_N(u) - \Psi(u) \right) \right) + |\mathcal{L}_{U,N} - \mathcal{L}| \right]
\]

\[
+ U^{-(\kappa-1)}, \quad U \to \infty.
\]

Recalling definition of \( w_1 \) (see Lemma 3.1) and using Lemma 2.2, we get (16).
7.5 Proof of Proposition 4.4

In order to prove minimax lower bounds we apply general results from Tsybakov (2008). Let \( \Theta \) be a semi-parametric class of models. Consider a family \( \{ P_\theta, \theta \in \Theta \} \) of probability measures, indexed by \( \Theta \). For any \( \theta_1, \theta_2 \in \Theta \) let \( d(\theta_1, \theta_2) \) be a semi-distance between two models \( \theta_1 \) and \( \theta_2 \).

**Lemma 7.1.** Suppose that \( \Theta \) contains two elements \( \theta_1 \) and \( \theta_2 \) such that \( d(\theta_1, \theta_2) > 2s \) for some \( s > 0 \) and \( \chi^2(P_{\theta_1}^{\otimes N}, P_{\theta_2}^{\otimes N}) \leq \tau < 1/2 \), where

\[
\chi^2(P, Q) = \begin{cases} 
\int \left( \frac{dP}{dQ} - 1 \right)^2 dQ, & \text{if } P \ll Q \\
\infty, & \text{otherwise}
\end{cases}
\]

for any two measures \( P \) and \( Q \). Then

\[
\inf_{\hat{\theta}} \sup_{\theta \in \Theta} P_\theta(d(\hat{\theta}, \theta) \geq s) \geq c(\tau) > 0,
\]

where \( c(\tau) \) is constant depending on \( \tau \) and infimum is taken over all estimates \( \hat{\theta} \) of \( \theta \) based on \( N \) observations under \( P_\theta \).

Turn now to the construction of models \( \theta_1 \) and \( \theta_2 \) from the class \( \mathcal{A}(\Lambda, \kappa, R) \). Let us consider a symmetric stable Lévy model with a nonzero diffusion part \( (\sigma > 0) \)

\[
\psi(u) = i\mu u - \sigma^2 u^2/2 + \vartheta(u), \quad \vartheta(u) = -\eta|u|^{\alpha}, \quad 0 < \alpha \leq 1, \quad u \in \mathbb{R}.
\]

For any \( \delta \) satisfying \( 0 < \delta < \alpha \) and \( M > 0 \) define

\[
\psi_\delta(u) = i\mu u - \sigma^2 u^2/2 + \vartheta_\delta(u),
\]

with

\[
\vartheta_\delta(u) = -\eta|u|^{\alpha} 1_{|u| \leq M} - \eta M^{\delta} |u|^{\alpha - \delta} 1_{|u| > M}.
\]

Then \( \phi_\delta(u) = \exp(\psi_\delta(u)) \) is a characteristic function of some Lévy process and

\[
\phi_\delta(u) = \phi(u), \quad |u| \leq M,
\]

where \( \phi(u) = \exp(\psi(u)) \). Indeed, \( \vartheta_\delta(u) \) is continuous, non-positive, symmetric function which is convex on \( \mathbb{R}_+ \) for large enough \( M \). According to the well known Pólya criteria (see e.g. Ushakov (1994)), the function \( \exp(\xi \vartheta_\delta(u)) \) is the c. f. of some absolutely continuous distribution for any \( \xi > 0 \). In particular, for any natural \( n \) the function \( \exp(\vartheta_\delta(u)/n) \) is the c. f. of some absolutely continuous distribution. Hence, \( \exp(\vartheta_\delta(u)) \) is the c. f. of some infinitely divisible distribution. Define now two affine (in fact, Lévy) models \( \theta_1 \) and \( \theta_2 \) corresponding to the characteristic exponents \( \psi \) and \( \psi_\delta \) respectively. Let \( \nu_{\theta_1} \) and \( \nu_{\theta_2} \) be the corresponding Lévy measures. It holds

\[
\chi^2(P_{\theta_1}^{\otimes N}, P_{\theta_2}^{\otimes N}) = N \chi^2(p_{\theta_1}, p_{\theta_2}) = N \int_{\mathbb{R}} \frac{|p_{\theta_1}(y) - p_{\theta_2}(y)|^2}{p_{\theta_1}(y)} dy,
\]

where \( p_{\theta_1} \) and \( p_{\theta_2} \) are densities corresponding to c.f. \( \phi_{\theta_1} \) and \( \phi_{\theta_2} \) respectively. Using the asymptotic inequality

\[
p_{\theta_1}(y) \gtrsim |y|^{-(\alpha + 1)}, \quad |y| \to \infty
\]

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and the fact that the density of stable law does not vanish on any compact set in \( \mathbb{R} \), we derive

\[
N \chi^2(p_{\theta_1}, p_{\theta_2}) \leq N C_1 \int_{|y| \leq A} |p_{\theta_1}(y) - p_{\theta_2}(y)|^2 dy + N C_2 \int_{|y| > A} |y|^\alpha |p_{\theta_1}(y) - p_{\theta_2}(y)|^2 dy = NC_1 I_1 + NC_2 I_2
\]

for large enough \( A > 0 \) and some constants \( C_1, C_2 > 0 \). Parseval’s identity implies

\[
I_1 \leq \frac{1}{2\pi} \int_{\mathbb{R}} |\phi_{\theta_1}(u) - \phi_{\theta_2}(u)|^2 du \\
\leq \frac{1}{2\pi} \int_{|u| > M} e^{-\sigma^2 |u|^2} du \lesssim M^{-1} e^{-\sigma^2 M^2}, \quad M \to \infty
\]

\[
I_2 \leq \frac{1}{2\pi} \int_{|u| > M} |(\phi_{\theta_1}(u) - \phi_{\theta_2}(u))'|^2 du \\
\lesssim \int_{|u| > M} |u|^2 e^{-\sigma^2 |u|^2} du \lesssim M e^{-\sigma^2 M^2}, \quad M \to \infty.
\]

The choice \( M \approx [\sigma^{-2} \log (N \log^2 N)]^{1/2} \) with some \( \beta > 0 \) yields

\[
N \chi^2(p_{\theta_1}, p_{\theta_2}) < 1/2
\]

for large enough \( N \). On the other hand

\[
\bar{\vartheta}(u) - \bar{\vartheta}_\delta(u) = -\eta \int_{-1}^{1} \left[ |u|^\alpha 1_{\{|u| > M\}} - |u + w|^\alpha 1_{\{|u+w| > M\}} \right] dw \\
+ \eta M^\delta \int_{-1}^{1} \left[ |u|^{\alpha-\delta} 1_{\{|u| > M\}} - |u+w|^{\alpha-\delta} 1_{\{|u+w| > M\}} \right] dw
\]

with

\[
\bar{\vartheta}(u) := \int_{-1}^{1} \left[ \vartheta(u) - \vartheta(u + w) \right] dw,
\]

\[
\bar{\vartheta}_\delta(u) := \int_{-1}^{1} \left[ \vartheta_\delta(u) - \vartheta_\delta(u + w) \right] dw.
\]

Using the identity

\[
\int_{-1}^{1} \left[ |u|^\alpha 1_{\{|u| > M\}} - |u + w|^\alpha 1_{\{|u+w| > M\}} \right] dw = |u|^\alpha \int_{-1}^{1} \left[ 1 - |1 + w/u|^\alpha \right] dw = 2 \sum_{k=1}^{\infty} \binom{\alpha}{2k} \frac{|u|^\alpha - 2k}{2k + 1},
\]

which holds for any \( |u| > M + 1 \) and \( M > 1 \), we get

\[
\left| \int_{\mathbb{R}} [\bar{\vartheta}(u) - \bar{\vartheta}_\delta(u)] du \right| \geq M^{\alpha-1}, \quad M \to \infty
\]
for any $0 < \alpha < 1$. Denote
\[
\rho_{\theta_1}(x) := \int_{\mathbb{R}} e^{ixu} \bar{\vartheta}(u) \, du = \left(1 - \frac{\sin x}{x}\right) \nu_{\theta_1}(x),
\]
\[
\rho_{\theta_2}(x) := \int_{\mathbb{R}} e^{ixu} \bar{\vartheta}(u) \, du = \left(1 - \frac{\sin x}{x}\right) \nu_{\theta_2}(x),
\]
then the Fourier inversion formula implies that
\[
\sup_{x \in \mathbb{R}} |\rho_{\theta_1}(x) - \rho_{\theta_2}(x)| \geq \left| \int_{\mathbb{R}} [\bar{\vartheta}(u) - \bar{\vartheta}(\delta)(u)] \, du \right| \geq M^{\alpha-1}, \quad M \to \infty.
\]
Asymptotic expansion (32) shows that there is a constant $R$ depending on $\eta$ such that
\[
|u|^{2-\alpha} \bar{\vartheta}(u) \leq R, \quad |u|^{2-\alpha} \bar{\vartheta}(\delta)(u) \leq R, \quad u \in \mathbb{R}.
\]
Hence, taking $\Lambda = \sigma^2/2$, $\kappa = 2 - \alpha$, we conclude that both models $\theta_1$ and $\theta_2$ are in $A(\Lambda, \kappa, R)$.

7.6 Proof of Proposition 5.3

The smallest eigenvalue $\lambda_{\min}(\Gamma)$ of the matrix $\Gamma$ satisfies
\[
\lambda_{\min}(\Gamma) = \min_{\|W\|=1} W^\top \Gamma W \geq \min_{\|W\|=1} W^\top \bar{\Gamma} W + \min_{\|W\|=1} W^\top (\bar{\Gamma} - \Gamma) W \geq \min_{\|W\|=1} W^\top \bar{\Gamma} W - \sum_{1 \leq k,l \leq 2} |\Gamma_{k,l} - \bar{\Gamma}_{k,l}|.
\]
According to the assumption (AS) it holds $\min_{\|W\|=1} W^\top \bar{\Gamma} W \geq \gamma_0$. Fix some $s > 0$, $k, l \in \{0, 1\}$ and denote
\[
\Delta_n := \frac{1}{h_N} \left( \frac{\delta_n - s}{h_N} \right)^{k+l} K \left( \frac{\delta_n - s}{h_N} \right) - \int_{\mathbb{R}} z^{k+l} K(z)p_\delta(s + h_Nz) \, dz.
\]
We have $E[\Delta_n] = 0$,
\[
\Delta_n \leq h_N^{-1} \sup_{z \in \mathbb{R}} \left[(1 + |z|^2)K(z)\right] =: D_1 h_N^{-1}
\]
and
\[
E[\Delta_n]^2 \leq \int_{\mathbb{R}} z^{2l+2k} K^2(z)p_\delta(s + h_Nz) \, dz \leq \frac{p_{\max}}{h_N} \int_{\mathbb{R}} (1 + |z|^4) K^2(z) \, dz =: D_2 h_N^{-1}
\]
with $p_{\max} = \max_{s \in S} p_\delta(s)$ and some positive constant $D_1$ and $D_2$. Hence, due to the Bernstein inequality we get for any $\delta > 0$
\[
P \left( |\Gamma_{k,l} - \bar{\Gamma}_{k,l}| \geq \delta \right) = P \left( \frac{1}{N} \sum_{n=1}^{N} \Delta_n \geq \delta \right) \leq L \exp(-\delta^2 BN h_N)
\]
with some positive constants \( L \) and \( B \). Combining (33) and (34), we get

\[
P(\lambda_{\min}(\Gamma) \leq \gamma_0/2) \leq 4L \exp(-\gamma_0^2 BN h_N/4).
\]

Due to the definition of estimates \( \tilde{\phi}_N(u) \) and \( \tilde{\phi}_{s,N}(u) \)

\[
P \left( w(\|u\|)|\tilde{\phi}_N(u) - \phi(u)| \geq \delta \right) \leq P(\lambda_{\min}(\Gamma) \leq \gamma_0/2)
\]

\[
+ P \left( w(\|u\|)|\tilde{\phi}_N(u) - \phi(u)| \geq \delta, \lambda_{\min}(\Gamma) > \gamma_0/2 \right),
\]

\[
P \left( w(\|u\|)|\tilde{\phi}_{s,N}(u) - \partial_s\phi(u)| \geq \delta \right) \leq P(\lambda_{\min}(\Gamma) \leq \gamma_0/2)
\]

\[
+ P \left( w(\|u\|)|\tilde{\phi}_{s,N}(u) - \partial_s\phi(u)| \geq \delta, \lambda_{\min}(\Gamma) > \gamma_0/2 \right).
\]

Furthermore, the following representation holds on the set \( \{\lambda_{\min}(\Gamma) > \gamma_0/2\} \)

\[
(\tilde{\phi}_N(u,s,x) - \phi(u,s,x), h_N(\tilde{\phi}_{s,N}(u,s,x) - \partial_s\phi(u,s,x))) = \Gamma^{-1}\varepsilon_N(u)
\]

with

\[
\varepsilon_{N,k}(u) := \frac{1}{N} \sum_{n=1}^{N} \left[ \exp(iu^\top X(t_n + \delta_n)) - \phi(u,s,x) - \partial_s\phi(u,s,x)(s - \delta_n) \right] \pi_{n,k}(s)
\]

and

\[
\pi_{n,k}(s) = \frac{1}{h_N} \left( \frac{\delta_n - s}{h_N} \right)^k K \left( \frac{\delta_n - s}{h_N} \right), \quad k = 0, 1.
\]

We have \( |\pi_{n,k}| \leq \pi^*_1 h_N^{-1} \) and

\[
\mathbb{E} \left[ \pi_{n,k}^2 \log^{2(1+\varepsilon)} \pi_{n,k}^2 \right] \leq \pi^*_2 h_N^{-1} \log^{2(1+\varepsilon)}(h_N), \quad k = 0, 1, \ \varepsilon > 0,
\]

with some positive constants \( \pi^*_1 \) and \( \pi^*_2 \). The following lemma holds

**Lemma 7.2.** Fix some \((s, x) \in S \times D\) and denote

\[
W_{N,l}(u) := \frac{1}{N} \sum_{n=1}^{N} \pi_{n,l}(s) w(\|u\|) D_n(u), \quad l = 0, 1,
\]

where \( D_n(u) := \exp(iu^\top X_n) - e^{-iu^\top x} \phi(u|\delta_n, x) \). Let \( w \) be a Lipschitz continuous weighting function satisfying

\[
0 < w(z) \leq \min\{1, \log^{-1/2-\delta}(z)\}, \quad z \in \mathbb{R}_+
\]

for arbitrary small \( \delta > 0 \). Then for large enough \( \zeta > 0 \)

\[
P \left( \sup_{u \in \mathbb{R}^d} |W_{N,l}(u)| \geq \frac{\zeta}{2} \sqrt{\frac{\log^{1+r} N}{Nh_N}} \right) \lesssim \log^{(r-1)d/2}(N)N^{-\kappa}, \quad N \to \infty
\]

with some \( \kappa > 1, r > 0 \) and \( l = 0, 1 \).
Proof. Consider the sequence $A_k = e^k$, $k \in \mathbb{N}$ and cover each cube $[-A_k, A_k]^d$ by $M_k = ([(2d^{1/2})/\gamma] + 1)^d$ disjoint small cubes $A_{k,1}, \ldots, A_{k,M_k}$, the edges of each cube being of the length $\gamma/d^{1/2}$. Let $u_{k,1}, \ldots, u_{k,M_k}$ be the centers of these cubes. We have for any natural $K > 0$

$$
\max_{k=1,\ldots,K} \sup_{\|u\| \leq A_k} |W_{N,0}(u)| \leq \max_{k=1,\ldots,K} \max_{\|u\| > A_{k-1}} |W_{N,0}(u_{k,m})| + \max_{k=1,\ldots,K} \sup_{1 \leq m \leq M_k, u \in A_{k,m}} |W_{N,0}(u) - W_{N,0}(u_{k,m})|.
$$

Hence

$$
P \left( \max_{k=1,\ldots,K} \sup_{\|u\| \leq A_k} |W_{N,0}(u)| > \lambda \right) \leq \sum_{k=1}^K \sum_{\|u\| > A_{k-1}} P(|W_{N,0}(u_{k,m})| > \lambda/2) + P \left( \sup_{\|u-v\| < \gamma} |W_{N,0}(v) - W_{N,0}(u)| > \lambda/2 \right).
$$

(39)

It holds for any $u, v \in \mathbb{R}^d$

$$
\|W_{N,0}(v) - W_{N,0}(u)\| \leq 2\pi_1^1 h_N^{-1}\|w(\|v\|) - w(\|u\|)\| + \frac{\pi_1^1}{Nh_N} \sum_{n=1}^N \left| \exp(i\delta^T X_n) - \exp(iu^T X_n) \right|

+ \frac{\pi_1^1}{Nh_N} \sum_{n=1}^N \left| e^{-i\delta^T X_n} - e^{-iu^T X_n} \right|

\leq 2\pi_1^1 h_N^{-1}\|u - v\| \left[ L_w + \frac{1}{N} \sum_{n=1}^N \|\tilde{X}_n\| + \frac{1}{N} \sum_{n=1}^N \mathbb{E}\|\tilde{X}_n\| \right],
$$

(40)

where $L_w$ is the Lipschitz constant of $w$. The Markov inequality implies

$$
P \left( \frac{1}{N} \sum_{n=1}^N \|\tilde{X}_n\| - \mathbb{E}\|\tilde{X}_n\| > c \right) \leq e^{-c^p} N^{-p} \mathbb{E} \left( \sum_{n=1}^N \|\tilde{X}_n\| - \mathbb{E}\|\tilde{X}_n\| \right)^p
$$

for any $c > 0$. Using now Dedecker and Rio inequalities (see Dedecker and et al. [2007]) and taking into account assumptions (AX1)-(AX2), we get

$$
\mathbb{E} \left( \sum_{n=1}^N \|\tilde{X}_n\| - \mathbb{E}\|\tilde{X}_n\| \right)^p \leq C_p(\beta) N^{p/2},
$$

where $C_p(\beta)$ is some constant depending on $\beta$ and $p$ from assumptions (AX1) and (AX2) respectively. Hence, (41)

$$
P \left( \frac{1}{N} \sum_{n=1}^N \|\tilde{X}_n\| > 2\beta_{1,N} \right) \leq C_p(\beta) N^{-p/2}/\beta_{1,N}^p,
$$

where $\beta_{1,N} = \frac{1}{N} \sum_{n=1}^N \mathbb{E}\|\tilde{X}_n\|$. Note that since $X$ is ergodic it holds

$$
\beta_{1,N} \to \beta_1^* = \int \|x\| \pi(dx) < \infty, \quad N \to \infty,
$$

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where $\pi$ is a unique invariant distribution. Setting $\gamma = \lambda/(24\pi_1^* h_N^{-1} \max\{\beta_{1,N}, L_w\})$ and combining (40) with the inequality (41), we obtain

$$P \left( \sup_{\|u-v\| < \eta} |W_{N,0}(v) - W_{N,0}(u)| > \lambda/2 \right) \leq B_1 N^{-p/2}$$

with some constant $B_1$ not depending on $\lambda$ and $N$. Let us turn now to the first term on the right-hand side of (29). If $\|u_{k,m}\| > A_{k-1}$ then it follows from Theorem 8.1 and Corollary 8.2 (see Appendix)

$$P (|\Re [W_{N,0}(u_{k,m})]| > \lambda/4) \leq B_2 \exp \left( -\frac{B_3 \lambda^2 N}{4w^2(A_{k-1}) \log^{2(1+\varepsilon)}(h_N w(A_{k-1})) \pi_{2}^*/h_N + \lambda \log^2(N) w(A_{k-1}) \pi_1^*/h_N} \right),$$

$$P (|\Im [W_{N,0}(u_{k,m})]| > \lambda/4) \leq B_4 \exp \left( -\frac{B_3 \lambda^2 N}{4w^2(A_{k-1}) \log^{2(1+\varepsilon)}(h_N w(A_{k-1})) \pi_{2}^*/h_N + \lambda \log^2(N) w(A_{k-1}) \pi_1^*/h_N} \right),$$

with some constants $B_2$, $B_3$ and $B_4$ depending only on the characteristics of the process $X$. Taking $\lambda = \zeta h_N^{-1/2} \log^{1+\varepsilon} (h_N) N^{-1/2} \log^{1/2} N$ with $\zeta > 0$ and using the fact that $h_N^{-1} = O(N)$, we get

$$\sum_{\|u_{k,m}\| > A_{k-1}} P(|W_{N,0}(u_{k,m})| > \lambda/2) \leq \left( \left( 2d^{1/2} A_{k-1} \right) / \lambda \right)^d \times \exp \left( -\frac{B_3 \lambda^2 N}{4w^2(A_{k-1}) \log^{2(1+\varepsilon)}(h_N w(A_{k-1})) \pi_{2}^*/h_N + \lambda \log^2(N) w(A_{k-1}) \pi_1^*/h_N} \right) \lesssim A_{k-1}^{-d/2} N^{d/2} \exp \left( -\frac{B_3 \lambda^2 N}{4w^2(A_{k-1}) \log^{2(1+\varepsilon)}(w(A_{k-1}))} \right) \log^{(r-1)d/2} (N), \quad N \to \infty$$

with $r = 2(1+\varepsilon)$ and some constant $B > 0$. Fix $\theta > 0$ such that $B \theta > d$ and compute

$$\sum_{\|u_{k,m}\| > A_{k-1}} P(|W_{N,0}(u_{k,m})| > \lambda/2) \lesssim h_N^{-d/2} e^{kd-\theta B(k-1)} N^{d/2} \log^{(r-1)d/2} (N) e^{-B(k-1)(\zeta^2 \log N - \theta)} \lesssim e^{k(d-\theta B)} \log^{(r-1)d/2} (N) e^{-B(k-1)(\zeta^2 \log N - \theta) + d \log(N)}.$$

If $\zeta^2 \log N > \theta$ we get asymptotically

$$\sum_{k=2}^{K} \sum_{\|u_{k,m}\| > A_{k-1}} P(|W_{N,0}(u_{k,m})| > \lambda/2) \lesssim \log^{(r-1)d/2} (N) e^{-(B \zeta^2 - d) \log(N)}.$$

Taking large enough $\zeta > 0$, we get (33). \qed
Combining Lemma 7.2 with the inequality
\[ |\phi(u|\delta_n, x) - \phi(u|s, x) - \partial_s\phi(u|s, x)(s - \delta_n)| \leq h_N^2 \sup_{s \in S} |\partial_s\phi(u|s, x)|, \quad |s - \delta_n| \leq h_N \]
and using Lemma 2.2, we obtain for large enough \( \zeta > 0 \)
\[ P\left( \sup_{u \in \mathbb{R}^d} |w(\|u\|)| \varepsilon_k(u) \right) > \zeta \left[ \sqrt{\frac{\log^{1+\epsilon} N}{N h_N}} + h_N \right] \lesssim \log^{-d/2}(N) N^{-\kappa}, \quad N \to \infty, \quad k = 0, 1. \]

Finally, inequalities (35), (36) and (37) together with the Borel-Cantelli lemma entail (21) and (22).

8 Appendix

For any two \( \sigma \)-algebras \( \mathcal{A} \) and \( \mathcal{B} \), define the \( \alpha \)-mixing coefficient by
\[ \alpha_Z(\mathcal{A}, \mathcal{B}) = \sup_{A \in \mathcal{A}, B \in \mathcal{B}} |P(A \cap B) - P(A) P(B)|. \]

Let \((Z_k, k \geq 1)\) be a sequence of real random variables defined on \((\Omega, \mathcal{F}, P)\). This sequence is called strongly mixing if
\[ \alpha_Z(n) = \sup_{k \geq 1} \alpha(\mathcal{M}_k, \mathcal{G}_{k+n}) \to 0, \quad n \to \infty, \]
where \( \mathcal{M}_j = \sigma(Z_i, i \leq j) \) and \( \mathcal{G}_j = \sigma(Z_i, i \geq j) \) for \( j \geq 1 \). The following theorem can be found in [Merlevède, Peligrad and Rid 2009].

**Theorem 8.1.** Let \((Z_k, k \geq 1)\) be a strongly mixing sequence of centered real valued random variables on the probability space \((\Omega, \mathcal{F}, P)\) with the mixing coefficients satisfying
\[ (43) \quad \alpha(n) \leq \bar{\alpha} \exp(-cn), \quad n \geq 1, \quad \bar{\alpha} > 0, \quad c > 0. \]
Assume that \( \sup_{k \geq 1} |Z_k| \leq M \) a.s., then there is a positive constant \( C \) depending on \( c \) and \( \bar{\alpha} \) such that
\[ P\left\{ \sum_{i=1}^{N} Z_i \geq \zeta \right\} \leq \exp\left[ - \frac{C \zeta^2}{N v^2 + M^2 + M \zeta \log^2(N)} \right]. \]
for all \( \zeta > 0 \) and \( N \geq 4 \), where
\[ v^2 := \sup_i \left( \mathbb{E}[Z_i]^2 + 2 \sum_{j \geq 1} \text{Cov}(Z_i, Z_j) \right). \]

**Corollary 8.2.** Denote
\[ \rho_j = \mathbb{E}\left[ Z_j^2 \log^{2(1+\epsilon)}(|Z_j|^2) \right], \quad j = 1, 2, \ldots, \]
with some \( \varepsilon > 0 \) and suppose that all \( \rho_j \) are finite. Then
\[
\sum_{j \geq i} \text{Cov}(Z_i, Z_j) \leq C \max_i \rho_i
\]
for some constant \( C > 0 \) provided that (43) holds. Consequently the following inequality holds
\[
v^2 \leq \sup_i \left( \mathbb{E}[Z_i]^2 + C \max_i \rho_i \right).
\]

**Proof.** Due to Rio inequality
\[
|\text{Cov}(Z_i, Z_j)| \leq 2 \int_0^{\alpha(|j-i|)} Q_{Z_i}(u)Q_{Z_j}(u)du,
\]
where for any random variable \( X \) we denote by \( Q_X \) the quantile function of \( X \). Define
\[
\rho_X = \mathbb{E}\left[ X^2 \log^{2(1+\varepsilon)}(|X|^2) \right]
\]
for some \( \varepsilon > 0 \). Markov inequality implies for small enough \( u > 0 \)
\[
P\left( |X| > \frac{\rho_X^{1/2}}{u^{1/2}| \log(u)|^{(1+\varepsilon)}} \right) \leq \frac{\mathbb{E}\left[ X^2 \log^{2(1+\varepsilon)}(|X|^2) \right]}{u^{-1} \log^{-2(1+\varepsilon)}(u)} \frac{\rho_X^{-1}}{\log^{-2(1+\varepsilon)}(u)}
\times \log^{-2(1+\varepsilon)} \left( \frac{\rho_X}{u \log^{2(1+\varepsilon)}(u)} \right)
= u \log^{-2(1+\varepsilon)} \left( \rho_X \log^{-2(1+\varepsilon)}(u) \right) \leq u
\]
and therefore
\[
Q_X(u) \leq \frac{\rho_X^{1/2}}{u^{1/2}| \log(u)|^{(1+\varepsilon)}}.
\]
Hence
\[
|\text{Cov}(Z_i, Z_j)| \leq 2 \int_0^{\alpha(|j-i|)} \frac{\sqrt{\rho_Z \rho_Z}}{u \log^{2(1+\varepsilon)}(u)}du \leq 2 \sqrt{\rho_Z \rho_Z} \log^{-1-2\varepsilon}(\alpha(|j-i|))
\]
and
\[
\sum_{j \geq i} \text{Cov}(Z_i, Z_j) \leq C \sqrt{\rho_Z \rho_Z} \sum_{j > i} \frac{1}{|j-i|^{1+2\varepsilon}}
\]
with some constant \( C > 0 \) depending on \( \bar{\alpha} \).

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References

Aït-Sahalia, Y. and Jacod, J. (2006). Volatility estimators for discretely sampled Lévy processes. *Annals of Statistics*, **37**, 184–222.

Aït-Sahalia, Y. and Jacod, J. (2008). Estimating the degree of activity of jumps in high frequency financial data. *Annals of Statistics*, to appear.

Belomestny, D. (2009). Spectral estimation of the fractional order of a Lévy process, to appear in *Annals of Statistics*.

Bates, D. (2000). Post-'87 crash fears in the S&P 500 futures option market. *Journal of Econometrics*, **94**, 181–238.

Bates, D. (2005). Maximum Likelihood Estimation of Latent Affine Processes. *Review of Financial Studies*, 909–965.

Basawa, I. V. and Brockwell, P. J. (1982). Nonparametric estimation for nondecreasing Lévy processes, *J. Roy. Statist. Soc. Ser. B*, **44**, 262–269.

Cont, R., and Mancini C. (2004). Nonparametric Tests for Analyzing the Fine Structure of Price Fluctuations, SSRN Paper.

Dedecker, J., P. Doukhan, G. Lang, J. León, S. Louhichi and C. Prieur (2007). Weak Dpendence: With Examples and Applications. Lecture Notes in Statistics **190**, Springer.

Duffie, D., Pan, J. and Singleton, K. (2000). Transform analysis and asset pricing for affine jump diffusions. *Econometrica*, **68**, 1343–1376.

Duffie, D., Filipović, D. and Schachermayer, W. (2003). Affine processes and applications in finance. *Annals of Applied Prob.*, **13**, 984–1053.

Figueroa-López, J.E. (2004). Nonparametric estimation of Lévy processes with a view towards mathematical finance. PhD thesis, Georgia Institute of Technology, [http://etd.gatech.edu](http://etd.gatech.edu) No. etd-04072004-122020.

Figueroa-López, J.E. (2009). Nonparametric estimation of time-changed Lévy models under high-frequency data. Working paper.

Figueroa-López, J.E. (2009). Jump-diffusion models driven by Lévy processes. Invited paper to appear in the Handbook of Computational Finance, Jin-Chuan Duan, James E. Gentle, and Wolfgang Hardle (eds.). Springer.

Glasserman, P. and Kyoung-kuk Kim (2007). Moment Explosions and Stationary Distributions in Affine Diffusion Models, to appear in *Mathematical Finance*.

Jiang, G. and Oomen, R. (2007). Estimating Latent variables and jump diffusion models using high-frequency data. *Journal of Financial Econometrics*, **5**, 1–30.

Jongbloed, G., van der Meulen, F.H. and van der Vaart, A.W. (2005). Nonparametric inference for Lévy-driven Ornstein-Uhlenbeck processes. *Bernoulli*, **11**(5), 759–791.
Keller-Ressel, M. and Steiner, Th. (2008). Yield Curve Shapes and the Asymptotic Short Rate Distribution in Affine One-Factor Models, to appear in *Finance Stoch*.

Masuda, H. (2007). Ergodicity and exponential $\beta$-mixing bounds for multidimensional diffusions with jumps, *Stochastic Process. Appl.*, 117(1), 35-56.

Merlevéde, F., Peligrad, M. and Rio, E. (2009). Bernstein inequality and moderate deviation under strong mixing conditions. Working paper.

Neumann, M. and Reiß, M. (2007). Nonparametric estimation for Lévy processes from low-frequency observations, to appear in *Bernoulli*.

Rubin, H. and Tucker, H.G. (1959). Estimating the parameters of a differential process, *Ann. Math. Statist.*, 30, 641-658.

Sato, K. (1999). *Lévy Processes and Infinitely Divisible Distributions*. Cambridge University Press.

Singleton, K. (2001). Estimation of Affine Asset Pricing Models Using the Empirical Characteristic Function. *Journal of Econometrics*, 10, 111–141.

Tsybakov, A. (2008). *Introduction to Nonparametric Estimation*, Springer Series in Statistics, Springer.

Ushakov, N. (1999). Selected topics in characteristic functions. Modern Probability and Statistics. VSP, Utrecht.