Abstract
Nonlinear oscillators arise everywhere in engineering, and though there are many analytical methods available, a fast and accurate estimation of the frequency–amplitude relationship is much needed in practical applications. He’s frequency formulation meets this requirement, but the local points are chosen randomly. In this paper, the Gaussian interpolation points are adopted, making He’s method more mathematically rigorous and physically reliable. The cubic-quintic Duffing equation is used for comparison, and an excellent result is obtained.

Keywords
Frequency formulation, Duffing oscillator, nonlinear conservative oscillator, Gaussian interpolation

Introduction
Nonlinear oscillation arises everywhere in science and engineering.\(^1\)\(^-\)\(^4\) However, its exact solution might be too complex to be used for a practical application, or sometimes, it is very hard to obtain its exact solution. Because of this, there have been many analytical methods developed to overcome this difficulty, including the variational iteration method,\(^5\) the homotopy perturbation method,\(^6\) the max–min approach,\(^7\)\(^,\)\(^8\) the Chebyshev series expansion method,\(^9\) the rational variational approach,\(^10\) the amplitude frequency formulation,\(^11\) the global error minimization method,\(^12\) the energy balance method,\(^13\)\(^,\)\(^14\) and so forth. Besides these, some asymptotic methods for strongly nonlinear equations can be found in literature.\(^15\) The relationship between the frequency and its amplitude is very important to a nonlinear oscillator. The simplest method to estimate the relationship of frequency and amplitude can be found in literature.\(^16\) The approximation can be characterized by its relative error, which attracts our great interests.

Recently the literature\(^16\) proposed a frequency formulation to estimate the frequency–amplitude relationship of a nonlinear oscillator with extremely simple calculation and relatively high accuracy. Due to its simplicity and effectiveness, He’s frequency formulation has become an effective tool to various nonlinear vibrations.\(^17\) But we find that when \(y(0)\) is larger, the relative error goes outside of acceptable limits. Therefore, based on the method proposed in literature,\(^16\) we will use the Lagrange interpolation method\(^18\) to improve the approximation, especially for the larger \(y(0)\).

He’s frequency formulation
Now we consider the following general nonlinear oscillator as follows
\[
y'' + f(y) = 0, \quad y(0) = A, \quad y'(0) = 0
\] (1)
He’s frequency formulation predicts that\(^{19,20}\)
\[
\omega^2 = \frac{df(y)}{dy} \mid_{y=A/2}
\]
(2)
or
\[
\omega^2 = \frac{1}{N+1} \sum_{i=0}^{N} \frac{df(y_i)}{dy}
\]
(3)
where \(\omega\) is the frequency, \(y_i(i = N)\) are location points.

Now consider the cubic-quintic Duffing oscillator\(^{16,19}\)
\[
y'' + y + y^3 + y^5 = 0, \quad y(0) = A, \quad y'(0) = 0
\]
(4)
Here, \(f(y) = y + y^3 + y^5\). Using He’s frequency formulation, we have
\[
\omega = \sqrt{1 + \frac{3}{4} A^2 + \frac{5}{16} A^4}
\]
(5)
The frequency–amplitude relationship given in equation (5) has relatively high accuracy for all \(A > 0\). Higher accuracy can be obtained by choosing many points, \(y_i(i = N)\) and then an average value is used. But there is no rule to how to choose the location points, here we show how the Lagrange interpolation method\(^{18}\) with \(2n + 1\) order accuracy can be powerfully applied to He’s frequency formulation.

**Lagrange interpolation**

In He’s frequency formulation, the location points play an important role, generally we choose \(\pm A\), but other location points can be also chosen, for examples, \(\pm 4A\), \(\pm 3A\) and \(\pm 2A\), in order to make the method more mathematically rigorous, the Gaussian interpolation\(^{20}\) can be adopted
\[
\omega = \sum_{i=1}^{n} w_i \omega(\xi_i), \quad \sum_{i=1}^{n} w_i = 1
\]
(6)

| \(A\)  | Exact frequency | Numerical frequency | Relative error \((\text{num} - \text{exact}) \times 100\%\) |
|-------|-----------------|---------------------|-----------------------------------------------|
| 0.1   | 1.00377         | 1.0050272           | 0.12525                                       |
| 0.3   | 1.03554         | 1.04702801          | 1.10937                                       |
| 0.5   | 1.10654         | 1.13843296          | 2.88222                                       |
| 1     | 1.52359         | 1.62769702          | 6.83301                                       |
| 3     | 7.26863         | 7.47315892          | 2.8136                                        |
| 5     | 19.1815         | 19.3651528          | 0.95745                                       |
| 8     | 48.2946         | 48.4140497          | 0.24734                                       |
| 10    | 75.1774         | 75.2393147          | 0.08236                                       |
| 20    | 299.223         | 298.827203          | 0.13228                                       |
| 50    | 1867.57         | 1864.06361          | 0.18775                                       |
| 70    | 3659.98         | 3652.91665          | 0.19299                                       |
| 100   | 7468.83         | 7454.23147          | 0.19546                                       |
| 300   | 67215.57        | 67082.71            | 0.19766                                       |
| 500   | 186709.04       | 186339.67           | 0.19783                                       |
| 700   | 365949.25       | 365225.11           | 0.19788                                       |
| 1000  | 746834.69       | 745356.66           | 0.19791                                       |
where \( n \) is the number of interpolation points, \( \xi_i \) is the \( i \)th interpolation point, \( w_i \) is the corresponding weight coefficient, and \( \omega(\xi_i) = \sqrt{\omega_n} \).

We can take five Gaussian interpolation points in \([0, A]\) and obtain a better result as shown in Table 1. However, from Table 1, we find that the relative errors are relatively bigger when \( A \leq 1 \). Thus, by combining the method proposed by He,\(^{16}\) we set the approximation frequency \( \omega_{\text{num}} \) as

\[
\omega_{\text{num}} = \begin{cases} 
\omega(A/2) & A \leq 1 \\
\sum_{i=1}^{n} w_i \omega(\xi_i) & A > 1
\end{cases}
\]  

(7)

\[\text{Table 2. The approximate results of frequencies from formula (7) and literature.}^{19}\]

| \( A \) | \( \omega_{\text{exa}} \) | \( \omega_{\text{num}} \) | \( \frac{\mid \omega_{\text{exa}} - \omega_{\text{num}} \mid}{\omega_{\text{num}}} \times \% \) | Ref.\(^{19}\) | Relative error of Khan et al.\(^{19}\) (%) |
|---|---|---|---|---|---|
| 0.1 | 1.003770 | 1.003756160732 | 0.001139543189988 | 1.0038 | 0.00028542 |
| 0.3 | 1.03554 | 1.03442314842621 | 0.107852093959335 | 1.0377 | 0.0226153 |
| 0.5 | 1.10654 | 1.09864973945293 | 0.710569665592376 | 1.1083 | 0.1635 |
| 1 | 1.52359 | 1.4364066163451 | 5.73968970428349 | 1.5484 | 1.63324 |
| 3 | 7.26863 | 4.7315892210105 | 0.108385793610423 | 7.725 | 6.29895 |
| 5 | 19.1815 | 19.3651528311874 | 0.95744770319004 | 20.597 | 6.92444 |
| 8 | 48.2946 | 48.4731586922353 | 0.247335463251167 | 51.738 | 7.12965 |
| 10 | 75.1774 | 75.2393147363422 | 0.00823581724763 | 80.572 | 7.17588 |
| 20 | 299.223 | 299.827202669513 | 0.132275035883346 | 208 | 7.23675 |
| 50 | 365.98 | 1864.06361372916 | 0.187751263451695 | 2003.04 | 7.25363 |
| 70 | 1867.57 | 1864.916525597 | 0.192988689563997 | 3925.51 | 7.2552 |
| 100 | 7468.83 | 7454.2314743213 | 0.01549330858289 | 8010.77 | 7.25603 |
| 300 | 67,215.57 | 67,082.7102272643 | 0.197662197517157 | 72,093.2 | 7.25674 |
| 500 | 186,709.04 | 186,339.6689748777 | 0.197832426926328 | 200,258 | 7.2568 |
| 700 | 365,949.25 | 365,225.107160414 | 0.19788067323159 | 392,506 | 7.25682 |
| 1000 | 746,834.69 | 745,356.663327701 | 0.19790546584004 | 801,031 | 7.25682 |

\[\times 10^5\]

| \( A \) | Exact frequency | Numerical frequency | Relative error of Khan et al.\(^{19}\) (%) |
|---|---|---|---|
| 0 | 100 | 200 | 300 | 400 | 500 | 600 | 700 | 800 | 900 | 1000 |
| 0.1 | 1.003770 | 1.003756160732 | 0.001139543189988 | 1.0038 | 0.00028542 |
| 0.3 | 1.03554 | 1.03442314842621 | 0.107852093959335 | 1.0377 | 0.0226153 |
| 0.5 | 1.10654 | 1.09864973945293 | 0.710569665592376 | 1.1083 | 0.1635 |
| 1 | 1.52359 | 1.4364066163451 | 5.73968970428349 | 1.5484 | 1.63324 |
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Figure 1. The comparison of the exact frequency and numerical frequency.
Consequently, the approximate results are listed in Table 2, and the comparison of the solutions and relative
errors are shown in Figures 1 and 2, correspondingly.
As shown in Table 1, Table 2, Figure 1, and Figure 2, the present modification leads to a much higher accuracy
with almost same calculation process, especially for the case with a bigger $A$, making He’s frequency formulation
much more attractive and reliable.

Discussion and conclusion
He’s frequency formulation is extremely simple and relatively efficient. This paper suggests a general approach to
choice of the location points based on He’s frequency formulation (2). The value of $y$ not only can be chosen by
$A/2$, but also can be chosen with the different value in $[0, A]$ according to Gaussian interpolation points and the
high accuracy can be reached. This method is also suitable for other oscillators, especially for Fangzhu
oscillator.21

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ORCID iD
Yue Shen https://orcid.org/0000-0002-0463-4487

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