Hyperholomorphic bundles
over a hyperkähler manifold.

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0. Introduction.

The main object of this paper is the notion of a hyperholomorphic bundle (Definition 2.4) over a hyperkähler manifold $M$ (Definition 1.1). The hyperholomorphic bundle is a direct sum of holomorphic stable holomorphic bundles. The first Chern class of a hyperholomorphic bundle is of zero degree.

Roughly speaking, the hyperholomorphic bundle is a bundle which is holomorphic with respect to all complex structures induced by the hyperkähler structure on $M$. It was proven (Proposition 4.1 of [V]) that if $M$ is a complex hyperkähler surface ($K3$ or abelian surface) then any stable bundle with first Chern class of zero degree is hyperholomorphic.

Interesting properties of hyperholomorphic bundles include the analogue of $(p,q)$-decomposition, $\partial\bar{\partial}$-lemma and an analogue of the strong Lefshetz theorem on the holomorphic cohomology $H^*(B)$ of a hyperholomorphic $B$ with a parallel real structure (proven in the Section 4).

For a hyperkähler manifold, one can define the action of quaternions on its cohomology groups. The characteristic classes of a hyperholomorphic bundle are invariant under this action.

Conversely, if $B$ is a stable bundle with the first two Chern classes invariant under the quaternion action, the bundle $B$ is hyperholomorphic (Theorem 2.5).

We are describing a coarse moduli space of deformations of a hyperholomorphic bundle locally (Theorem 6.2). In particular, we show that there are no obstructions for a deformation besides Yoneda pairing (Definition 6.2).

This description is used to construct a hyperkähler structure on the space of deformations of a given hyperholomorphic bundle (Theorem 6.3), thus generalizing results of [M] and [Ko].

As an application, one can prove that the stable moduli space for the stable bundles with certain Chern classes do not depend on the choice of a base manifold in its deformation class (Proposition 10.3).

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Most of results of this paper can be generalized on projectively hyperholomorphic bundles, which stand in the same relation to hyperholomorphic ones as projectively flat bundles stand to flat bundles (Section 11). Over a hyperkähler surface (K3 or abelian sufrace), any stable Yang-Mills bundle is hyperholomorphic (Proposition 11.2).

Contents.

1. Hyperkähler manifolds.
2. Hyperholomorphic bundles.
3. Hermitian bundles: preliminary computations.
4. Hodge analysis and \((p,q)\)-decomposition on the holomorphic cohomology \(H^*(B)\) for a hyperholomorphic \(B\).
5. Proof of Theorem 2.5, which states that the stable bundle is hyperholomorphic if its first two Chern classes are invariant with respect to the isotropy group of the base hyperkähler manifold.
6. Deformation spaces for holomorphic and hyperholomorphic bundles.
7. Local deformations of a hyperholomorphic connection. The proof of Theorem 6.2, which states that the only obstruction to the deformation of a hyperholomorphic bundle is Yoneda product.
8. Comparing Laplacians.
9. The space of deformations of a hyperholomorphic bundle is hyperkähler (Theorem 6.3).
10. Some applications.
11. Projectively hyperholomorphic bundles.

1. Hyperkähler manifolds.

Definition 1.1 ([B], [Bes]) A hyperkähler manifold is a Riemannian manifold \(M\) endowed with three complex structures \(I, J\) and \(K\), such that the following holds.

(i) \(M\) is Kähler with respect to these structures and

(ii) \(I, J\) and \(K\), considered as endomorphisms of a real tangent bundle, satisfy the relation \(I \circ J = -J \circ I = K\).

This means that the hyperkähler manifold has the natural action of quaternions \(\mathbb{H}\) in its real tangent bundle. Therefore its complex dimension is even.
Let \( adI, \ adJ \) and \( adK \) be the operators on the bundles of differential forms over a hyperkähler manifold \( M \) which are defined as follows. Define \( adI \). Let this operator act as a complex structure operator \( I \) on the bundle of differential 1-forms. We extend it on \( i \)-forms for arbitrary \( i \) using Leibnitz formula: 
\[
adI(\alpha \wedge \beta) = \adI(\alpha) \wedge \beta + \alpha \wedge \adI(\beta).
\]
Since Leibnitz formula is true for a commutator in a Lie algebras, one can immediately obtain the following identities, which follow from the same identities in \( \mathbb{H} \):

\[
[\adI, \adJ] = 2\adK; \quad [\adJ, \adK] = 2\adI;
\]

\[
[\adK, \adI] = 2\adJ
\]

Therefore, the operators \( \adI, \adJ, \adK \) generate a Lie algebra \( \mathfrak{su}(2) \) acting on the bundle of differential forms. We can integrate this Lie algebra action to the action of a Lie group \( SU(2) \). In particular, operators \( I, J \) and \( K \), which act on differential forms by the formula 
\[
I(\alpha \wedge \beta) = I(\alpha) \wedge I(\beta),
\]
belong to this group.

**Proposition 1.1:** There is an action of the Lie group \( SU(2) \) and Lie algebra \( \mathfrak{su}(2) \) on the bundle of differential forms over a hyperkähler manifold. This action is parallel, and therefore it commutes with Laplace operator. ■

If \( M \) is compact, this implies that there is a canonical \( SU(2) \)-action on \( H^i(M, \mathbb{R}) \) (see [V1]).

Let \( M \) be a hyperkähler manifold with a Riemannian form \( \langle \cdot, \cdot \rangle \). Let the form \( \omega_I := \langle I(\cdot), \cdot \rangle \) be the usual Kähler form which is closed and parallel (with respect to the connection). Analogously defined forms \( \omega_J \) and \( \omega_K \) are also closed and parallel.

The simple linear algebraic consideration ([B]) shows that \( \omega_J + \sqrt{-1} \omega_K \) is of type \( (2, 0) \) and, being closed, this form is also holomorphic. It is called the **canonical holomorphic symplectic form of a manifold** \( M \). Conversely, if there is a parallel holomorphic symplectic form on a Kähler manifold \( M \), this manifold has a hyperkähler structure ([B]).

If some **compact** Kähler manifold \( M \) admits non-degenerate holomorphic symplectic form \( \Omega \), the Calabi-Yau ([Y]) theorem implies that \( M \) is hyperkähler (see [B]). This follows from the existence of a Kähler metric on \( M \) such that \( \Omega \) is parallel for the Levi-Civitta connection associated with this metric.
Let $M$ be a hyperkähler manifold with complex structures $I$, $J$ and $K$. For any real numbers $a$, $b$, $c$ such that $a^2 + b^2 + c^2 = 1$ the operator $L := aI + bJ + cK$ is also an almost complex structure: $L^2 = -1$. Clearly, $L$ is parallel with respect to a connection. This implies that $L$ is a complex structure, and that $M$ is Kähler with respect to $L$.

**Definition 1.2** If $M$ is a hyperkähler manifold, the complex structure $L$ is called **induced by a hyperkähler structure**, if $L = aI + bJ + cK$ for some real numbers $a, b, c$ such that $a^2 + b^2 + c^2 = 1$.

If $M$ is a hyperkähler manifold and $L$ is induced complex structure, we will denote $M$, considered as a complex manifold with respect to $L$, by $(M, L)$ or, sometimes, by $M_L$.

Consider the Lie algebra $g_M$ generated by $adL$ for all $L$ induced by a hyperkähler structure on $M$. One can easily see that $g_M = su(2)$. The Lie algebra $g_M$ is called **isotropy algebra** of $M$, and corresponding Lie group $G_M$ is called an **isotropy group** of $M$. By Proposition 1.1, the action of the group is parallel, and therefore it commutes with Laplace operator in differential forms. In particular, this implies that the action of the isotropy group $G_M$ preserves harmonic forms, and therefore this group canonically acts on cohomology of $M$.

**Proposition 1.2:** Let $\omega$ be a differential form over a hyperkähler manifold $M$. The form $\omega$ is $G_M$-invariant if and only if it is of Hodge type $(p, p)$ with respect to all induced complex structures on $M$.

**Proof:** Assume that $\omega$ is $G_M$-invariant. This implies that all elements of $g_M$ act trivially on $\omega$ and, in particular, that $adL(\omega) = 0$ for any induced complex structure $L$. On the other hand, $adL(\omega) = (p - q)\sqrt{-1}$ if $\omega$ is of Hodge type $(p, q)$. Therefore $\omega$ is of Hodge type $(p, p)$ with respect to any induced complex structure $L$.

Conversely, assume that $\omega$ is of type $(p, p)$ with respect to all induced $L$. Then $adL(\omega) = 0$ for any induced $L$. By definition, $g_M$ is generated by such $adL(\omega) = 0$, and therefore $g_M$ and $G_M$ act trivially on $\omega$. ■

2. Hyperholomorphic bundles.
Let $B$ be a Hermitian vector bundle over the complex manifold $M$. Let $\theta$ be a Hermitian connection on $B$ and $\Theta \in \Lambda^2 \otimes \text{End}(B)$ be its curvature. If $B$ is holomorphic, this connection is called **compatible with a holomorphic structure** if $\nabla_X(\zeta) = 0$ for any holomorphic section $\zeta$ and any antiholomorphic tangent vector $X$. If there exist a holomorphic structure compatible with the given Hermitian connection then this connection is called **integrable**.

One can define a **Hodge decomposition** in the space of differential forms with coefficients in any complex bundle, in particular, $\text{End}(B)$ (see [GH]).

**Theorem 2.1** (Newlander-Nirenberg) The connection $\theta$ in $B$ is integrable if and only if $\Theta \in \Lambda^{1,1} \otimes \text{End}(B)$, where $\Lambda^{1,1} \otimes \text{End}(B)$ denotes $(1,1)$-forms with respect to the Hodge decomposition. The holomorphic structure compatible with a given connection $\theta$ is unique.

**Proof:** This is Proposition 4.17 of [Ko], Chapter I.

**Definition 2.1** Let $B$ be a Hermitian bundle with a Hermitian connection $\theta$ over a hyperkähler manifold $M$. The connection $\theta$ is called **hyperholomorphic** if it is integrable with respect to any complex structure induced by a hyperkähler structure.

As follows from the Theorem 2.1, $\theta$ is hyperholomorphic if and only if its curvature $\Theta$ is of type $(1,1)$ with respect to any of complex structures induced by a hyperkähler structure.

As follows from Proposition 1.2, $\theta$ is hyperholomorphich if and only if $\Theta$ is a $G_M$-invariant differential form.

Let $M$ be a Kähler manifold with a Kähler form $\omega$. For differential forms with coefficients in any vector bundle there is Hodge operator $L : \eta \mapsto \omega \wedge \eta$. There is also fiberwise-adjoint Hodge operator $\Lambda$ (see [GH]).

**Definition 2.2** (see [UY]) Let $B$ be a holomorphic bundle over $M$ with a holomorphic Hermitian connection $\theta$ and a curvature $\Theta \in \Lambda^{1,1} \otimes \text{End}(B)$. The Hermitian metric on $B$ and the connection $\theta$ defined by this metric is called **Yang-Mills** if

$$\Lambda(\Theta) = \text{constant} \cdot \text{Id},$$
where $\Lambda$ is a Hodge operator and $Id$ is the identity endomorphism which is a section of $\text{End}(B)$.

**Definition 2.3** Let $F$ be a coherent sheaf over $n$-dimensional compact Kähler manifold $M$. We define $\text{deg}(B)$ as

$$\text{deg}(B) = \int_M c_1(F) \wedge \omega^{n-1}$$

and $\text{slope}(B)$ as

$$\text{slope}(B) = \frac{1}{\text{rank}(F)} \cdot \text{deg}(B)$$

The number $\text{slope}(F)$ is rational, and it depends only on a cohomology class of $c_1(F)$. The simple bundle $B$ is called stable if for every coherent subsheaf $B' \subset B$ with $\text{rank}(B') < \text{rank}(B)$

$$\text{slope}(B') < \text{slope}(B)$$

Later on, we will usually consider the bundles $B$ with $\text{deg}(B) = 0$.

**Proposition 2.1:** Let $M$ be a compact Kähler manifold, and $B$ be a Hermitian holomorphic bundle over $M$ with $\text{deg}(B) = 0$. The bundle $B$ is Yang-Mills if and only if $\Lambda(\Theta) = 0$.

**Proof:** One can see this using the following argument. The number

$$\text{deg}(B) = \int_M c_1(B) \wedge \omega^{n-1}$$

is equal by the Gauss-Bonnet theorem to the value of an integral

$$\int_M \frac{\Lambda(\text{Tr}(\Theta))}{\text{Vol}(M)}.$$ 

For a Yang-Mills bundle, the equation $\text{deg}(B) = 0$ implies that $\int_M \Lambda(\Theta) = 0$.

A holomorphic bundle is called **undecomposable** if it cannot be decomposed onto a direct sum of two or more holomorphic bundles.

There is an important theorem ([UY])
Theorem 2.2 (Uhlenbeck-Yau): Let $B$ be some undecomposable holomorphic bundle over a compact Kähler manifold. Then $B$ admits a Yang-Mills Hermitian metric if and only if it is stable, and this metric is unique (up to a constant).

Theorem 2.3 A hyperholomorphic connection in a bundle $B$ is Yang-Mills. Moreover, for such connection $\Lambda(\Theta) = 0$ where $\Theta$ is a curvature in $B$.

Proof: We will use the definition of a hyperholomorphic connection as one with $G_M$-invariant curvature. Theorem 2.3 follows from the

Lemma 2.1 Let $\Theta$ be a $G_M$-invariant section of $\Lambda^2(M) \otimes \text{End}(B)$. Then $\Lambda_L(\Theta) = 0$ for each induced complex structure $L$. By $\Lambda_L$ we understand the Hodge operator $\Lambda$ associated with the Kähler complex structure $L$.

Proof of Lemma 2.1 The action of $G_M$, as defined above, is orthogonal with respect to the standard metric on the space of differential forms. From the theory of group representations we know the following fact. Let $G$ be a group which acts orthogonally on the Hermitian space $V$. Let $V_{\text{inv}}$ be the space of $G$-invariants in $V$ and $V = V_{\text{inv}} \oplus V_1$ be a $G$-invariant decomposition. Then $V_1$ is orthogonal to $V_{\text{inv}}$. We will use this to prove Lemma 1.2.

Consider the bundle $R \subset \Lambda^2(M) \otimes \text{End}(B)$ spanned by the Kähler forms $\omega_L$, for all induced complex structures $L$ on $M$. The element of a group $G_M$ maps each of these forms into a linear combination of these forms. This could be proven by a direct computation. Therefore, $R$ is invariant with respect to the $G_M$-action. Moreover, no section of $R$ is $G_M$-invariant. This is proven by another direct computation. Both of these computations are based on the fact that $R$ is a trivial bundle over $M$ with a fiber $g_M$, and $G_M$ acts on $R$ by means of adjoint representation on $g_M$.

By the statement in the beginning of the proof, this bundle is orthogonal to the bundle of $G_M$-invariants.

Now, by the definition of the Hodge operator $\Lambda_L$, for $G_M$-invariant $\Theta$,

$$\Lambda_L(\Theta) = 0$$

because the Kähler form $\omega_L$ is orthogonal to $\Theta$. Theorem 2.3 is proven. □

From now on the base hyperkähler manifold $M$ is supposed to be compact.
**Theorem 2.4** Let $M$ be a hyperkähler surface, i.e., $K3$ or abelian surface. A stable holomorphic bundle $B$ with $\deg(B) = 0$ over $M$ always admits a hyperholomorphic connection, which is unique. ([V], Theorem 2.1)

**Proof:**

Take Yang-Mills Hermitian connection on $B$. We shall prove that it is hyperholomorphic. The uniqueness of a hyperholomorphic connection follows from Theorem 2.3, because it is Yang-Mills, and Yang-Mills Hermitian connection in a stable bundle is unique.

The space of complex 2-forms over $M$ has a trivial 3-dimensional subbundle $P$ spanned by Kähler form, holomorphic symplectic form and the form conjugated to the holomorphic symplectic form. Let $P^\perp$ be its orthogonal completion. One easily sees that if $M$ is a surface then $P^\perp$ is a bundle of $G_M$-invariant 2-forms. On the other hand, the curvature $\Theta$ of a Yang-Mills bundle of degree zero is orthogonal to $P$ for any hyperkähler manifold by the following reasons. First, $\Theta$ is of type $(1,1)$ and this is sufficient for the orthogonality to the holomorphic symplectic form and its conjugate. Second, $\Lambda(\Theta) = 0$ is equivalent to $\Theta$ being orthogonal to the Kähler form by definition of $\Lambda$.

One should note that in the proof above, $P$ is identical to $\Lambda^+(M)$ and $P^\perp$ is $\Lambda^-(M)$, where $\Lambda^2(M) = \Lambda^+(M) \oplus \Lambda^-(M)$ is a standard decomposition of 2-forms over a Riemannian 4-fold on autodual and anti-autodual forms.

For any stable holomorphic bundle there exist unique Hermitian Yang-Mills connection which, for some bundles, turns out to be hyperholomorphic. This observation gives us independent from connection notion of a stable hyperholomorphic bundle.

Let $M$ be a hyperkähler manifold, and let $I$ be an induced complex structure over $M$.

**Definition 2.4** The stable holomorphic bundle $B$ over $(M, I)$ is called **simple hyperholomorphic** if the unique Yang-Mills connection on $B$ is hyperholomorphic. Generally, $B$ is called **hyperholomorphic** if it could be decomposed onto a direct sum of simple hyperholomorphic stable bundles.

Further on, we will consider mostly simple hyperholomorphic bundles and omit the word “simple” when it is obvious from context.
For a hyperholomorphic bundle, its $p$-th Chern class is of type $(p,p)$ with respect to any of induced complex structures. By Proposition 1.2, this implies that Chern classes of a hyperholomorphic bundle are $G_M$-invariant. Conversely, there is a theorem of extreme importance, which is proven in the Section 5:

**Theorem 2.5:** Let $B$ be a stable bundle over $(M,I)$, where $M$ is a hyperkähler manifold and $I$ is an induced complex structure over $M$. If $c_1(B)$ and $c_2(B)$ are $G_M$-invariant, the stable bundle $B$ is hyperholomorphic.

One knows that $c_i(B)$ are $G_M$-invariant for any hyperholomorphic bundle (see the paragraph above the theorem). One also knows that the form is $G_M$-invariant if and only if it is of Hodge type $(p,p)$ with respect to all induced complex structures (Proposition 1.2).

### 3. Hermitian bundles: preliminary observations.

Throughout this section, $M$ is a Kähler manifold and $E$ is a Hermitian holomorphic bundle over $M$ with a Hermitian holomorphic connection $\nabla$ and curvature $\Theta$. Consider the Hodge decomposition for $\nabla$ ([GH])

\begin{equation}
\nabla = \partial + \bar{\partial}
\end{equation}

(traditionally, the holomorphic component of $\nabla$ is denoted by $\nabla'$ while antiholomorphic component is denoted by $\bar{\nabla}$). We will denote the fiberwise-adjoint operators by $\partial^*$ and $\bar{\nabla}^*$.

This notation is different from a traditional one.

Our major tool are famous Kodaira’s identities (see [GH]):

$$[\Lambda, \partial] = \sqrt{-1} \bar{\partial}^* \text{ and } [\Lambda, \bar{\partial}] = -\sqrt{-1} \partial^*$$

where $\Lambda$ is the Hodge operator.

Consider four Laplace operators:

$$\Delta_\partial = \partial \partial^* + \partial^* \partial; \quad \Delta_{\bar{\partial}} = \bar{\partial} \bar{\partial}^* + \bar{\partial}^* \bar{\partial}; \quad \Delta_d = \nabla \nabla^* + \nabla^* \nabla; \quad \Delta_{d^c} = I \circ \Delta_d \circ I^{-1}$$

where $I$ is a complex structure operator acting on differential forms by multiplicativity.
If $B$ is trivial, $\Delta_\theta = \Delta_{\bar{\theta}} = 1/2 \Delta_d = 1/2 \Delta_{d^c}$ (see [GH]). In case of nontrivial Hermitian holomorphic bundle $B$, Kodaira identities imply the following.

**Proposition 3.1**

a) $\Delta_\theta + \Delta_{\bar{\theta}} = \Delta_d$

b) $\Delta_\theta - \Delta_{\bar{\theta}} = \sqrt{-1} \lfloor \Lambda; \nabla^2 \rfloor$

c) $\Delta_d = \Delta_{d^c}$,

where $\Lambda$ is a Hodge operator.

**Proof**

a) By (3.1), $\Delta_d = \Delta_\theta + \Delta_{\bar{\theta}} + (\partial \bar{\partial} + \bar{\partial} \partial) + (\partial^* \bar{\partial} + \bar{\partial} \partial^*)$ We need only to prove that the last two terms are zero. By the Kodaira identity,

$$\Lambda \partial - \partial \Lambda = \sqrt{-1} \partial \bar{\partial}$$

and therefore

$$\frac{\partial \bar{\partial} + \bar{\partial} \partial}{\sqrt{-1}} = \partial \Lambda - \partial^2 \Lambda + \Lambda \partial^2 - \partial \Lambda \bar{\partial} = 0$$

Vanishing of the last term is analogous.

b) By Kodaira identities, $\sqrt{-1} \Delta_{\bar{\theta}} = \bar{\partial}(\Lambda \partial - \partial \Lambda) + (\Lambda \partial - \partial \Lambda) \bar{\partial}$, while $-\sqrt{-1} \Delta_\theta = \partial(\Lambda \bar{\partial} - \bar{\partial} \Lambda) + (\Lambda \bar{\partial} - \bar{\partial} \Lambda) \partial$.

The summation of these equations yields

$$\sqrt{-1} (\Delta_{\bar{\theta}} - \Delta_\theta) = - (\partial \bar{\partial} + \bar{\partial} \partial) \Lambda + \Lambda (\partial \bar{\partial} + \bar{\partial} \partial) = \lfloor \Lambda; \nabla^2 \rfloor.$$  

c) Let us take $d^c = I \circ \nabla \circ I^{-1}$. For symmetry, denote $\nabla$ by $d$. Kodaira identities state that $[\Lambda, d] = d^c$ and $[\Lambda, d^c] = -d^c$. Analogously to the case b),

$$\Delta_{d^c} = d^c(\Lambda d - d \Lambda) + (\Lambda d - d \Lambda) d^c$$

while

$$-\Delta_d = d(\Lambda d^c - d^c \Lambda) + (\Lambda d^c - d^c \Lambda) d.$$  

Summing up, we obtain $\Delta_{d^c} - \Delta_d = [\Lambda, d^c d + d d^c]$. Clearly,

$$d^c d + d d^c = \frac{\partial - \bar{\partial}}{\sqrt{-1}}(\theta + \bar{\theta}) + (\theta + \bar{\theta}) \frac{\partial - \bar{\partial}}{\sqrt{-1}} = -\sqrt{-1} (\theta \bar{\partial} - \bar{\theta} \partial + \bar{\theta} \partial - \theta \bar{\partial}) = 0 \blacksquare$$
One knows that $\nabla^2$ is a linear operator on forms which maps the form $\eta$ into $\Theta \wedge \eta$. Therefore the Proposition 3.1 implies that the difference between the Laplacians $\nabla_\theta$, $\nabla_\bar{\theta}$ and $\nabla_d$ is a differential operator of degree zero, i.e. the linear operator. We will use analogous computations in the section 4.

The rest of this section is dedicated to the real structures in Hermitian bundles.

The real structure on the complex vector bundle is anticomplex involution. In other words, the real structure on a complex bundle $B$ is an operator $T$ such that $T^2 = Id$ and $T(\lambda k) = \lambda x$ for $\lambda \in \mathbb{C}$.

Suppose that $B$ is a bundle with connection. We will call a real structure $T$ parallel if $T$ is parallel as a section of $\text{End}_R(B, B)$ with respect to a connection in $\text{End}_R(B, B)$ induced by a connection in $B$.

For a real structure $T$ on a bundle $B$, let $B_R := \{ b \in B \mid T(b) = b \}$ be the space of $T$-invariant vectors. Clearly, $B_R$ is endowed with a unique connection such that $B = B_R \otimes \mathbb{C}$ as a bundle with connection.

For a real structure $T$, the operator $-T$ also defines a real structure.

If $E$ is a Hermitian bundle, the bundle $\text{End}(E)$ is endowed with a canonical real structure $T$. This real structure maps an endomorphism $\alpha$ of $E$ to an endomorphism $-\alpha^\perp$. By $\alpha^\perp$ we denote the endomorphism which is adjoined to $\alpha$ with respect to the Hermitian product on $E$.

Consider the connection in $\text{End}(E)$ which is induced by a connection in $E$. The constructed above real structure operator is parallel with respect to this connection.

One should note that the connection in $\text{End}(E)$ is Yang-Mills if it is induced by a Yang-Mills connection in $E$.

Generally speaking, one can easily tell whether a Yang-Mills bundle $B$ admits a parallel real structure:

**Proposition 3.2** Let $B$ be a Yang-Mills bundle over a Kähler manifold $M$. The bundle $B$ is endowed with a parallel real structure if and only if $B$, taken as a holomorphic bundle, is isomorphic to its dual bundle, and this isomorphism is defined by a symmetric non-degenerate bilinear form. If $B$ is nondecomposable, this real structure is unique up to a multiplication by $-1$.

**Proof:** The bundle $B$ is always canonically isomorphic to $B^*$ as a real bundle with connection. The isomorphism is given by the real part of the
Hermitian form on $B$. Let $f : B \to B^*$ be the isomorphism operator. Clearly, $I \circ f = -f \circ I^*$, where $I$ denotes the complex structure operator (multiplication by $\sqrt{-1}$) on $B$ and $I^*$ denotes the complex structure operator on $B^*$. Now, if $B$ has a parallel real structure $T$, the operator $T \circ f$ commute with the complex structure operator because

$$I \circ T \circ f = -T \circ I \circ f = T \circ f \circ I^*.$$  

Since this $T \circ f$ is parallel, it defines an isomorphism of Hermitian bundles $B$ and $B^*$. The converse claim follows from the uniqueness of Yang-Mills metric.

If $B$ is endowed with a parallel real structure, the bundle $\Lambda^* (M, B)$ of differential forms with coefficients in $B$ is also endowed with a parallel real structure. It could be most easily seen on the following way. Let $\Lambda^i (M, \mathbb{R})$ be the bundle of real differential $i$-forms over $M$. If $B$ has a real structure, $B = B_{\mathbb{R}} \otimes \mathbb{C}$. Therefore,

$$\Lambda^i (M, \mathbb{R}) \otimes_{\mathbb{R}} B_{\mathbb{R}} \otimes_{\mathbb{R}} \mathbb{C} = \Lambda^i (M, B).$$ 

Consider the operator $T$ which acts as a complex conjugation in $\mathbb{C}$ and trivially on the first two components of a tensor product $\Lambda^i (M, \mathbb{R}) \otimes B_{\mathbb{R}} \otimes \mathbb{C}$. Obviously, the operator $T$ defines a parallel real structure on $\Lambda^i (M, B)$.

One can easily prove that $T$ maps $\Lambda^{pq} (M, B)$ into $\Lambda^{qp} (M, B)$.

4. Hodge analysis and (p,q)-decomposition on the holomorphic cohomology $H^* (B)$ for a hyperholomorphic $B$.

In this section we will prove several theorems about differential $(p,0)$-forms with coefficients in a hyperholomorphic bundle $B$. These theorems are direct analogues of Kodaira identities, strong Lefshetz theorem, $(p,q)$-decomposition and $dd^c$-lemma for forms with trivial coefficients over a Kähler manifold (see [GH]).

The difference between the traditional and our cases is following. Traditionally, these theorems are being proven for the space of (topological) cohomology $H^* (M, \mathbb{C})$ of a compact Kähler manifold. In our case, their analogues exist for the space $H^* (M, B)$ of holomorphic cohomology of $B$. 

12
Let $B$ be a holomorphic Hermitian bundle with a connection and a parallel real structure over a hyperkähler manifold $(M, I)$. Let $I$, $J$ and $K$ be complex structure operators in a real cotangent bundle over $M$, as usually, $I \circ J = -J \circ I = K$. These operators could be extended on the bundles of differential forms over $M$ by multiplicativity, i.e., by the formulas $J(\alpha \wedge \beta) = J(\alpha) \wedge J(\beta)$ etc. Since $I \circ J = -J \circ I$, the operator $J$ maps $(1,0)$-forms on $(0,1)$-forms, and therefore it maps $(p,0)$-forms on $(0,p)$-forms.

Consider the operator $\bar{J} : \Lambda^{(p,0)}(M, B) \rightarrow \Lambda^{(p,0)}(M, B)$ which is a composition of $J$ and a real structure operator. Let $\partial^j = \bar{J} \circ \partial \circ \bar{J}^{-1}$, where $\partial : \Lambda^{(p,0)}(M, B) \rightarrow \Lambda^{(p+1,0)}(M, B)$ is a $(1,0)$-component of a connection.

**Proposition 4.1** The following equations hold:

$$
(\partial^j)^2 = 0
$$

$$
\bar{\partial}^j \partial + \partial \bar{\partial}^j = 0
$$

**Proof:** The first formula is an obvious consequence of the relation $\partial^2 = 0$. Let $\Lambda \Lambda^p(B) = \Lambda(M, B) \otimes \mathbb{R} \mathbb{C}$. Since $J^2 = -1$, the bundle $\Lambda \Lambda^1(B)$ can be decomposed onto the direct sum

$$
\Lambda \Lambda^1(B) = \Lambda \Lambda^{1,0}(B) \oplus \Lambda \Lambda^{0,1}(B),
$$

where $\Lambda \Lambda^{1,0}(B)$ corresponds to the eigenspace of an eigenvalue $\sqrt{-1}$ for $\bar{J}$ and $\Lambda \Lambda^{0,1}(B)$ to eigenvalue $(-\sqrt{-1})$. Analogously, let us by multiplicativity decompose

$$
\Lambda \Lambda^n(B) = \oplus_{p+q=n} \Lambda \Lambda^{p,q}(B)
$$

like it is being done in a usual Hodge theory.

Let $\delta = \frac{\partial + \sqrt{-1} \partial^j}{2}$, and $\bar{\delta} = \frac{\partial - \sqrt{-1} \partial^j}{2}$ be the $(1,0)$ and $(0,1)$ components of $\partial$ with respect to $(p,q)$-decomposition on $\Lambda \Lambda^*(B)$. By definition, $\delta^2$ is $(2,0)$-component of $\partial^2$ with respect to the $\Lambda \Lambda$-decomposition. Analogously, $\bar{\delta}^2$ is $(0,2)$-component of $\partial^2$. Since $\partial^2 = 0$, we see that $\delta^2 = \bar{\delta}^2 = 0$. On the other hand,
\[ \delta^2 = \partial^2 + (\partial^j)^2 + \sqrt{-1} \partial \partial^j + \sqrt{-1} \bar{\partial} \bar{\partial} \]

and therefore

\[ \partial^j \partial + \partial \partial^j = 0. \]

Let \( L_I \) be the usual Hodge operator \( L \) in the space of differential forms \( \Lambda^*(M, B) \) (exterior multiplication by the Kähler form \( \omega_I \)). \( L_I \) and \( L_K \) be analogous operators associated with complex structures \( J \) and \( K \) on \( M \).

**Proposition 4.2** The following equation holds:

\[ [L_J, \partial^*] = \partial^j. \]

**Proof:** Let \( (\cdot)^J \) denote \( J \circ (\cdot) \circ J^{-1} \) where \( (\cdot) \) is some operator on differential forms, \( (\cdot)^I \) and \( (\cdot)^K \) be analogously defined operators. Let \( \nabla \) denote a connection in \( B \), and \( \nabla^* \) denote the adjoint operator. By definition, \( \partial = \nabla + \sqrt{-1} \nabla^I \), and therefore \( \partial^j = \frac{\nabla^J - \sqrt{-1} \nabla^K}{2} \). On the other hand,

\[ [L_J, \partial^*] = \left[ L_J, \frac{\nabla^* + \sqrt{-1} (\nabla^*)^I}{2} \right] = \frac{\nabla^J + \sqrt{-1} [L_J^{-1}, \nabla^*]^I}{2} \]

because by Kodaira identities \([L_R, \nabla^*] = \nabla^R \) for \( R = I, J \) or \( K \). We can apply Kodaira’s identity because \( B \) is hyperholomorphic and therefore \( \nabla \) is a holomorphic Hermitian connection with respect to \( I, J \) and \( K \). Now, the simple calculation shows that \( L_J^{-1} = -L_J \), and therefore

\[ [L_J, \partial^*] = \frac{\nabla^J + \sqrt{-1} \nabla^J \nabla^I}{2} = \frac{\nabla^J - \sqrt{-1} \nabla^K}{2}. \]

**Corollary 4.1** The following is also true:

\[ [L_J, \delta^*] = \sqrt{-1} \delta \]

\[ [L_J, \bar{\delta}^*] = -\sqrt{-1} \bar{\delta}. \]

**Proposition 4.3**

a). The operators \( \partial^* \) and \( \partial^j \) anticommute, i. e., \( \partial^* \partial^j + \partial^j \partial^* = 0 \). Analogously, the pairs \((\partial^j)^* \) and \( \partial^j \delta^* \) and \( \delta \), \( \delta^* \) and \( \bar{\delta} \) anticommute.

b) Operators \( \partial, \partial^j, \delta \) and \( \bar{\delta} \) pairwise anticommute.
**Proof** a) For instance, let us prove anticommutation of the operators in the first pair. By Proposition 4.2,

\[ \partial^* \partial^j + \partial^j \partial^* = \partial^* L_j \partial^* - \partial^* \partial^* L_j + L_j \partial^* \partial^* - \partial^* L_j \partial^* = 0 \]

b). The operator \( \delta \tilde{\delta} + \tilde{\delta} \delta = 1/2 \cdot (\partial \partial^j + \partial^j \partial) \) is just \((1,1)\) component of an operator \( \partial^2 \) for a \((p, q)\)-decomposition on \(\Lambda \Lambda(B)\) (see proof of Proposition 4.1). Since the connection in \(B\) is holomorphic, \(\partial^2 = 0\). Therefore the \((1,1)\) component of \(\partial^2\), namely \(\delta \tilde{\delta} + \tilde{\delta} \delta\), is also zero. The anticommutation of other pairs is analogous. \(\blacksquare\)

Let \( \Delta_{\partial^j} \) be the Laplacian associated with \( \partial^j \). Let \( \Delta_\delta \) and \( \Delta_{\tilde{\delta}} \) be Laplacians on the complex \(\Lambda \Lambda^*(B)\), associated with \(\delta\) and \(\tilde{\delta}\) respectively. The operators \( \Delta_{\partial^j} \) and \( \Delta_{\partial^j} \) could also be considered as operators on \(\Lambda \Lambda^*(B) = \Lambda^p,0(M, B) \otimes_\mathbb{R} \mathbb{C} \).

**Theorem 4.1** The following Laplace operators are equal (or proportional):

\[ \Delta_{\partial^j} = \Delta_\delta = 2 \Delta_\delta = 2 \Delta_{\tilde{\delta}} \]

**Proof**: By definition of \(\delta\) and \(\tilde{\delta}\),

\[ \Delta_{\partial^j} = \Delta_\delta + \Delta_{\tilde{\delta}} + (\delta \tilde{\delta}^* + \tilde{\delta}^* \delta) + (\delta^* \tilde{\delta} + \tilde{\delta}^* \delta^*) = \Delta_\delta + \Delta_{\tilde{\delta}} \]

since, by the Proposition 4.3a, \((\delta \tilde{\delta}^* + \tilde{\delta}^* \delta) = (\delta^* \tilde{\delta} + \tilde{\delta}^* \delta^*) = 0\).

By Corollary 4.1, \(\sqrt{-1} \Delta_{\tilde{\delta}} = \tilde{\delta}^*(L_j \delta^* - \delta^* L_j) + (L_j \delta^* - \delta^* L_j)\delta^*\),

while \(-\sqrt{-1} \Delta_\delta^* = \delta^*(\Lambda^j \tilde{\delta}^* - \tilde{\delta}^* \Lambda) + (\Lambda^j \tilde{\delta}^* - \tilde{\delta}^* \Lambda)\delta^*\).

The summation of these equations yields

\[ \sqrt{-1} (\Delta_{\tilde{\delta}} - \Delta_\delta) = -(\delta^* \delta^* + \delta^* \delta^*)L_j + L_j(\delta^* \tilde{\delta} + \tilde{\delta}^* \delta^*) = 0, \]

since \((\delta^* \delta^* + \delta^* \delta^*)\) is equal zero by the Proposition 4.3(b). Therefore, \(\Delta_{\tilde{\delta}} = \Delta_\delta\) and \(\Delta_{\partial^j} = 2 \Delta_\delta\). It remains only to prove that \(\Delta_\delta = \Delta_{\partial^j}\). This follows from equations

\[ \Delta_\delta = \partial \partial^* + \partial^* \partial = \partial[L_j \partial^j] + [L_j \partial^j] \partial = \partial L_j \partial^j - \partial^j L_j \partial \]

and

\[ \Delta_{\partial^j} = \partial^j \partial^* + \partial^* \partial^j = -\partial^j [L_j \partial] - [L_j \partial] \partial^j = \partial L_j \partial^j - \partial^j L_j \partial. \]
Corollary 4.2 (The \(p,q\)-decomposition on cohomologies.) The cohomology space \(H^n(B) \otimes_{\mathbb{R}} \mathbb{C}\) admits \((p,q)\)-decomposition:

\[
H^n(B) \otimes_{\mathbb{R}} \mathbb{C} =: \bigoplus_{p+q=n} H^{p,q}_{hyp}(B).
\]

Proof Laplacian \(\Delta\) preserves \((p,q)\)-decomposition defined on \(\Lambda^n(B)\), but its kernel is \(H^n(B) \otimes_{\mathbb{R}} \mathbb{C}\) by the Theorem 4.1.

Corollary 4.3 Let \(B\) be a hyperholomorphic bundle with a parallel real structure. There is a canonical action of the Lie group \(SU(2)\) on the holomorphic cohomology \(H^*(B)\) of \(B\) as on \(\mathbb{R}\)-linear space.

Proof: Consider the action of operators \(\bar{J}\) and \(I\) on \(\Lambda^p,0(M,B)\). Clearly, \(I \circ \bar{J} = -\bar{J} \circ I\). Let \(K := I \circ \bar{J}\). One can easily check that the set of operators

\[
\{aI + b\bar{J} + cK + d \cdot \text{Id} \mid a^2 + b^2 + c^2 + d^2 = 1\}
\]

form a group isomorphic to \(SU(2)\). By Theorem 4.1 the action of this group maps \(\Delta_{\partial}\)-harmonic forms from \(\Lambda^{p,0}(M,B)\) into \(\Delta_{\partial}\)-harmonic ones. This defines the action of \(SU(2)\) on the space of \(\Delta_{\partial}\)-harmonic forms from \(\Lambda^{p,0}(M,B)\), which is, by Hodge theory ([GH]), canonically isomorphic to the space of holomorphic cohomology \(H^p(B)\).

One notes that the \(SU(2)\)-action on \(H^1(B)\) extends to a quaternion action.

Let \(L_c = L_J + \sqrt{-1}L_K\) be an operator of type \((2,0)\) (see section 1) mapping the \(\Lambda^{p,0}(M,B)\) into \(\Lambda^{p+2,0}(M,B)\), and \(\Lambda_c = \Lambda_J + \sqrt{-1}\Lambda_K\) be its adjoint operator. Proposition 4.2 together with the Theorem 4.1 shows that \(\Lambda_c\) preserves harmonicity of forms, and therefore we can correctly define the action of \(L_c\) and \(\Lambda_c\) on the space of holomorphic cohomologies \(H^*(B)\). A simple computation shows that

\[
H = [L_c, \Lambda_c] \mid_{H^i(B)} = (n-2i),
\]

where \((n-2i)\) means multiplication by a scalar \(n-2i\), and \(n\) is a dimension of \(M\). There is an Hermitian metric on \(H^i(M,B)\) as on the space of \(\Delta_{\partial}\)-harmonic sections of the Hermitian bundle \(\Lambda^{i,0}(M,B)\).

Theorem 4.2 (analog of a strong Lefshetz theorem): Thus defined operators \(L_c, \Lambda_c\) and \(H\) generate a Lie algebra \(sl(2)\) which acts on \(H^*(M,B)\)
The standard Hodge-Riemann relations between the metric, Hodge decomposition and \( \mathfrak{sl}(2) \)-action also hold.

**Proof** The proof is completely analogous to the proof of Lefshetz theorem and Hodge-Riemann relations, see [GH].

The generalization of this theorem could be proven for any hyperholomorphic bundle \( B \) regardless of the existence of a parallel real structure. Consider the obvious map \( E \otimes \mathcal{O}_M \rightarrow E \), where \( E \) is a sheaf over \( M \) and \( \mathcal{O}_M \) is a structure sheaf. This map defines a Künneth map of cohomology

\[
H^i(E) \times H^j(\mathcal{O}_M) \rightarrow H^{i+j}(E).
\]

One knows that for a hyperkähler manifold of complex dimension \( 2n \) the ring \( H^*(\mathcal{O}_M) \) contains the sub ring of truncated polynomials of one variable:

\[
H^*(\mathcal{O}_M) = \mathbb{C}[x]/\{x^{n+1} = 0\}.
\]

There, \( x \) is a generator of \( H^2(\mathcal{O}_M) \) corresponding to a canonical holomorphic symplectic form over \( M \) (see [B]). Consider the multiplication by \( x \) as a map \( L_c : H^i(E) \rightarrow H^{i+2}(E) \). This map is defined for any sheaf \( E \) over a hyperkähler manifold \( M \).

**Theorem 4.2.A** Let \( E \) be a hyperholomorphic bundle. For \( i \leq n \) the map

\[
L_c : H^i(E) \rightarrow H^{i+2}(E)
\]

is injection. The map

\[
L_c^{n-i} : H^i(E) \rightarrow H^{2n-i}(E)
\]

is isomorphism.

**Proof:** Let \( B = E \oplus E^* \), where \( E^* \) is a dual bundle. The bundle \( B \) has a canonical real structure as following. Consider the Hermitian form as the anticomplex operator from \( E \) to \( E^* \). Denote this operator by \( \tau \). The real structure \( T \) in \( B \) is defined by the formula

\[
T(\alpha, \beta) = (\tau^{-1}(\beta), \tau(\alpha))
\]

Now, applying Theorem 4.2 to the bundle \( B \), we obtain the statement of Theorem 4.2.A.
Theorem 4.3 (\( \partial \bar{\partial}^j \)-lemma) Let \( \omega \in \Lambda^{p,0}(M, B) \) be some \( \partial \)- and \( \partial^j \)-closed form. Suppose that \( \omega \) is also either \( \partial \)- or \( \partial^j \)-exact. Then there is a form \( \kappa \) such that \( \partial \partial^j(\kappa) = \omega \).

Proof

One should keep in mind that by virtue of Proposition 4.3(a), \( \partial \partial^j(\eta) = -\partial^j \partial(\eta) \) for any differential form \( \eta \in \Lambda^i(M, B) \). By the same argument, the Laplacian \( \Delta \partial^j \) commutes with \( \partial \) and \( \partial^j \).

Let \( G_\partial \) be the Green operator associated with the Laplacian \( \Delta_\partial \), and \( G_{\partial^j} \) be one associated with \( \Delta_{\partial^j} \). By Theorem 4.1, \( G_\partial = G_{\partial^j} \). Let us denote this operator by \( G \). Obviously, \( G \) commutes with \( \partial \) and \( \partial^j \). Since \( \omega \) is \( \partial \)- (or \( \partial^j \)-) exact, \( G \Delta_\partial \omega = \omega \), and therefore its orthogonal projection on the space of harmonic forms is zero. This proves that \( \omega \) is both \( \partial \)- and \( \partial^j \)-exact.

On the other hand, \( \Delta_{\partial^j} \omega = \partial \partial^* \omega \) because \( \omega \) is closed. Therefore, \( \omega = G\Delta_{\partial^j} \omega = \partial \partial^* G \omega \). Let \( \omega = \partial^j \gamma \) for some \( \gamma \). Since \( \partial^j \) anticommutes with \( \partial \) and \( \partial^* \) (Proposition 4.3),

\[
\omega = G\Delta_\omega = \partial \partial^* G \omega = \partial \partial^* G \partial^j \gamma = -\partial \partial^j \partial^* G \gamma.
\]

5. Proof of Theorem 2.5, which states that the stable bundle is hyperholomorphic if its first two Chern classes are invariant with respect to the isotropy group of the base hyperkähler manifold.

Let \( B \) be a holomorphic stable bundle over \((M, J)\). Take the a Yang-Mills connection in \( B \). Assume that \( c_1(B) \) and \( c_2(B) \) are \( G_M \)-invariant. Let \( \Theta \) be a curvature of \( B \); we need to prove that \( \Theta \) is of type \((1,1)\) with respect to all induced complex structures on \( M \), or, what is the same, that \( \Theta \) is \( G_M \)-invariant.

Let \( I \) be an induced complex structure over \( M \), such that \( I \circ J = -J \circ I \) and \( I \circ J \) is another induced complex structure. One can easily show that the Lie algebra \( g_M \) is generated by \( ad I \) for all such \( I \). Therefore it is sufficient to prove that \( \Theta \) is of type \((1,1)\) with respect to \( I \).

Let \( \Theta = \Theta_{2,0} + \Theta_{1,1} + \Theta_{0,2} \) be a Hodge decomposition of \( \Theta \) with respect to \( I \). Let \( \Lambda_c = \Lambda_j + \sqrt{-1} \Lambda_K \), as in Section 4. This is an operator of Hodge type \((-2,0)\). For each point \( x \in M \) one can define a constant

\[
Tr(\Lambda^2_c(\Theta_{2,0} \wedge \Theta_{2,0})).
\]
In the end of this section, we will prove the analogue of Bogomolov-Gieseker ([Ko], [S]) unequality

\[ Tr(\Lambda_c^2(\Theta_{2,0} \wedge \Theta_{2,0})) < 0 \]

for point \( x \in M \) such that \( \Theta_{2,0} \neq 0 \) as a section of \( \Lambda^{2,0}(\text{End}(B)) \) at this point.

Let us show that

\[ \int_M Tr\Lambda_c^2(\Theta_{2,0} \wedge \Theta_{2,0}) = \int_M Tr\Lambda_c^2(\Theta \wedge \Theta). \]

This follows from the equation

\[ \Lambda_c^2(\Theta_{2,0} \wedge \Theta_{2,0}) = \Lambda_c^2(\Theta \wedge \Theta). \]

This equation holds because \( \Lambda_c \) maps the forms of Hodge type \((p, q)\) into the forms of Hodge type \((p - 2, q)\). Therefore for a 4-form \( \eta \) the operator \( \Lambda_c^2 \) acts trivially on all its Hodge components except the \((4,0)\)-component \( \eta_{4,0} \):

\[ \Lambda_c^2(\eta) = \Lambda_c^2(\eta_{4,0}). \]

Finally, one knows that \( \Theta_{2,0} \wedge \Theta_{2,0} \) is \((4,0)\)-component of \( \Theta \wedge \Theta \). This proves equations (5.3) and (5.2).

We will show now how (5.1) and (5.2) imply the statement of Theorem 2.5. One can deduce from the Chern identities that for any complex bundle \( B \) with a connection and a curvature \( \Theta \)

\[ (5.1A) \quad \left(2c_2(B) - \frac{r-1}{r}c_1(B)^2\right) = [Tr(\Theta \wedge \Theta)], \]

where \([Tr(\Theta \wedge \Theta)]\) means the cohomology class of \( \mathbb{C}\)-valued 4-form \( Tr(\Theta \wedge \Theta) \) (see [Ko]).

By \( r \) we denote the rank of \( B \) and by \([\alpha]\) the cohomology class in \( H^*(M, \mathbb{C}) \) corresponding to the \( \mathbb{C}\)-valued differential form \( \alpha \).

In our case \( c_1(B) = 0 \), and therefore

\[ \int_M \Lambda_c^2(\Theta \wedge \Theta) = 2\Lambda_c^2(c_2(B)) \]

This statement together with formula (5.1) imply that \( \Lambda_c^2(c_2(B)) \leq 0 \) and \( \Lambda_c^2(c_2(B)) = 0 \) only if \( \Theta_{2,0} = 0 \) everywhere. Therefore, if \( \Lambda_c^2(c_2(B)) = 0 \), the form \( \Theta_{2,0} = 0 \). Since \( B \) is a Hermitian bundle, \( \Theta_{0,2} = \Theta_{2,0} \). This implies that for any induced complex structure \( I \), the operator, \( adI \) acts trivially on \( \Theta(= \Theta_{1,1}) \). This implies that \( \Theta \) is \( G_M \)-invariant, and the bundle \( B \) is hyperholomorphic. Theorem 2.5 is proven (modulo the unequality (5.1)).

We will proceed to the proof of the unequality (5.1).
Since $B$ is Yang-Mills with respect to $J$, we have $\Lambda_J(\Theta) = 0$ by Proposition 2.1. Since the operator $\Lambda_K + \sqrt{-1} \Lambda_J$ is of type $(2,0)$ with respect to $J$, we have $\Lambda_K + \sqrt{-1} \Lambda_J(\Theta) = 0$. Since $\Theta$ is a real form with respect to a real structure on $\Lambda^2(\text{End}(B))$, we have also $\Lambda_J(\Theta) = \Lambda_K(\Theta) = 0$ as well.

Let $x_1, x_1', x_2, x_2', \ldots, x_n, x_n'$ be coordinates in the bundle $\Lambda^{1,0}(M)$ for some open set $U \subset M$ such that $J(x_i) = x_i'$ and $\omega_J + \sqrt{-1} \omega_K = \sum_{i=1}^{n} x_i \wedge x_i'$ (where $\omega_J$ and $\omega_K$ are Kähler forms associated with complex structures $J$ and $K$ respectively). Since $\Theta$ is of type $(1,1)$ with respect to $J$, the operator $J$ preserves $\Theta$, i.e., $J(\Theta) = \Theta$. Therefore, $J(z_i \wedge z_j \wedge \ldots) := J(z_i) \wedge J(z_j) \wedge \ldots$. One can see that the operator $J$ maps $(2,0)$ forms in $(0,2)$ forms, where Hodge decomposition is taken with respect to $I$. Therefore $J(\Theta_{2,0}) = \Theta_{0,2}$ and $\Theta_{2,0}$ belongs to $\Lambda \Lambda^{1,1}(\text{End}(B))$ in the sense of Section 4.

Let us represent

$$\Theta_{2,0} = \sum_{i,j=1}^{n} A_{ij} x_i \wedge x'_j$$

where $A_{ij}$ is a section of $\text{End}(B)$ over $U$. This representation exists because $\Theta_{2,0}$ is a section of $\Lambda \Lambda^{1,1}(\text{End}(B))$.

Since $J(\Theta_{2,0}) = \Theta_{2,0}$, and the real structure in $\text{End}(B)$ is defined by $A \mapsto -A^\perp$, we have $\sum A_{ij} x_i \wedge x'_j = \sum -A_{ij}^\perp x'_i \wedge x_j$. Therefore,

$$A_{ij} = A_{ij}^\perp. \tag{5.4}$$

On the other hand, for any form $\sum B_{ij} x_i \wedge x'_j$,

$$\Lambda_c \left( \sum B_{ij} x_i \wedge x'_j \right) = \sum_{i=1}^{n} B_{ii}$$

because the canonical holomorphic symplectic form is equal to $\sum_{i=1}^{n} x_i \wedge x'_i$. We have proved earlier that $\Lambda_c \left( \sum A_{ij} x_i \wedge x'_j \right)$ is a zero section of $\text{End}(B)$. Therefore

$$\sum A_{ii} = 0 \text{ at each point of } M. \tag{5.5}$$

An easy calculation shows that

$$\Lambda^2(\Theta_{2,0} \wedge \Theta_{2,0}) = \sum_{i \neq j} -A_{ij} A_{ji} + \sum_{i \neq j} A_{ii} A_{jj}. \tag{5.6}$$

Since $\text{Tr}(AA^\perp) > 0$ for every non-zero matrix $A$, we have that

$$\text{Tr} \sum_{i \neq j} -A_{ij} A_{ji} \overset{(5.4)}{=} \text{Tr} \sum_{i \neq j} -A_{ij} A_{ij}^\perp < 0$$

20
if any of $A_{ij}$, $i \neq j$ is non-zero.

On the other hand, let us show that $Tr \sum_{i \neq j} A_{ii} A_{jj} < 0$, if not all matrices $A_{ii}$, $i = 1...n$ are zero. First of all,

$$2 \sum_{i \neq j} A_{ii} A_{jj} = (\sum A_{ii})^2 - \sum A_{ii}^2.$$

By (5.5), $\sum A_{ii} = 0$ and therefore

$$2 \sum_{i \neq j} A_{ii} A_{jj} = - \sum A_{ii}^2.$$

By the previous argument,

$$Tr \sum_{i=1...n} -A_{ii}^2 \xrightarrow{(5.4)} Tr \sum_{i \neq j} -A_{ii} A_{ii}^\perp \leq 0$$

Therefore

$$Tr(A_C^2(\Theta_{2,0} \wedge \Theta_{2,0})) = Tr \sum_{i \neq j} -A_{ii} A_{ii}^\perp + Tr \sum_{i \neq j} -A_{ij} A_{ij}^\perp < 0$$

if $\Theta_{2,0} \neq 0$. This proves inequality (5.1).

6. Deformation spaces of holomorphic and hyperholomorphic bundles.

Let $F$ be a stable bundle over a compact smooth complex manifold $M$.

**Definition 6.1** The stable deformation $(X, x_0, F)$ of the bundle $F$ over $M$ is the analytic space $X$ with a marked point $x_0$ and a bundle $\mathcal{F}$ over $X \times M$ such that the following holds. The restriction of $\mathcal{F}$ on $\{x_0\} \times M$ is isomorphic to $F$ and the restriction of $\mathcal{F}$ on $\{x\} \times M$ is stable for each point $x \in X$.

For any $z \in X$ we will denote the restriction of $\mathcal{F}$ on $\{z\} \times M$ by $[z]$. This is a bundle over $M$, since $\{x\} \times M$ is canonically isomorphic to $M$.

**Definition 6.2** Let $F$ be some sheaf over a complex manifold $M$. Let $\cup$ be a standard $\cup$-multiplication:

$$\cup : H^1(\text{End}F) \times H^1(\text{End}F) \rightarrow H^2(\text{End}F \otimes \text{End}F).$$

21
Let $\bar{k}$ be a commutator map $\text{End}(F) \times \text{End}(F) \to \text{End}(F)$. Obviously, $\bar{k}$ induces a natural map $k : H^2(\text{End}F \times \text{End}F) \to H^2(\text{End}F)$. The composition $k \circ \cup$ is called \textbf{Yoneda pairing}. Since $H^i(\text{End}F) = \text{Ext}^i(F, F)$, Yoneda pairing is a map

$$\tau : \text{Ext}^1(E, E) \times \text{Ext}^1(E, E) \to \text{Ext}^2(E, E).$$

This map is bilinear and symmetric.

The following proposition is proven in the Section 16 of [KS].

\textbf{Proposition 6.1} The Yoneda pairing on $\text{Ext}^1(F, F)$ is an obstruction to the existence of a deformation of $F$ in the following sense. Suppose that $\rho \in \text{Ext}^1(F, F)$ has nonzero Yoneda square. Then there is no deformation $(D, x_0, F)$ of $F$ where $D$ is a disc in $\mathbb{C}$ such that the image of Kodaira-Spencer map $T_{x_0}D = \mathbb{C} \to \text{Ext}^1(F, F)$ is proportional to $\rho$.  

It turns out that in hyperholomorphic case Yoneda pairing is the only obstruction to the existence of the deformation. We will prove this in section 7.

\textbf{Definition 6.3} ([G]) The marked analytic space $\text{Spl}^{\mathfrak{g}}(F), [F]$ is called \textbf{coarse moduli space of deformations}, if there is a simple bundle $F'$ over $M$ associated to each point $[F'] \in \text{Spl}^{\mathfrak{g}}(F)$ and for any variation $^{\mathfrak{g}}(X, x_0, F)$ of $F$ with a connected $X'$ there is a unique map $f : (X, x_0) \to (\text{Spl}^{\mathfrak{g}}(F), [F])$ such that for each point $x \in X$ the restriction of $F$ on $x \times M$ is isomorphic to the bundle, associated with $f(x) \in \text{Spl}(F)$.

Later on, we will usually omit the word coarse.

There is a theorem of existence of a coarse moduli space of deformations:

\textbf{Theorem 6.1} (see [M], [Ko]) For any stable bundle $F$ there exists a coarse moduli space $\text{Spl}^{\mathfrak{g}}(F)$ of the deformations of this sheaf. If the $[F'] \in \text{Spl}(F)$ is the point which corresponds to the sheaf $[F']$ over $M$, then for the Zariski tangent space at this point the Kodaira-Spencer map $T\text{Spl}^{\mathfrak{g}}(F)|_{[F']} \to \text{Ext}^1(F', F')$ is isomorphism.

Note that the space $\text{Spl}^{\mathfrak{g}}(F)$ is by no means separated or reduced. We are chiefly interested in its reduction, denoted by $\text{Spl}(F)$. For this space, the Kodaira-Spencer map $T\text{Spl}(F)|_{[F']} \to \text{Ext}^1(F', F')$ is embedding.
The following theorem, describing $Spl(F)$ locally, is proven in the Section 7:

**Theorem 6.2** If $F$ is a hyperholomorphic bundle, then the space $Spl(F)$ locally in the neighbourhood of the point $[F]$ is isomorphic to an intersection of an open ball in $Ext^1(F, F)$ with a quadratic cone

$$\{ \rho \in Ext^1(F, F) \mid \iota(\rho, \rho) = 0 \}.$$

In other words, Theorem 6.2 says that there are no obstructions to a deformation of a hyperholomorphic bundle except the Yoneda pairing.

By Theorem 2.5, the stable deformation of a hyperholomorphic bundle is also hyperholomorphic. Therefore, one can define the hyperkähler structure in the coarse deformation space (Section 9). Unfortunately, the deformation space is not smooth, so the definition of a hyperkähler manifold must be modified to include singular manifolds.

First, we have to make some preliminary definitions. Let $R$ be an $\mathbb{R}$-vector space with a quaternion action and a Euclidean metric $(\cdot, \cdot)$. For each $I \in \mathbb{H}$ with $I^2 = -1$ the action $I$ on $R$ defines a complex structure on $R$. Such complex structure is called **induced by quaternion action**. The space $R$ is called **quaternionic Hermitian** if the metric on $R$ is Hermitian for any induced complex structure $I$. For any such $I$, take $\omega_I = (\cdot, I \cdot)$. This is a real symplectic form on $R$. Now, take $J \in \mathbb{H}$ with $J^2 = -1$ and $K = I \circ J$. Of course, $J$ and $K$ define induced complex structures on $R$. One can prove that the 2-form $\omega_J + \sqrt{-1} \omega_K$ does not depend on the choice of $J, K$ as long as these quaternions satisfy the conditions above. This form is called **canonical symplectic form** associated with $R$ and $I$.

**Definition 6.5** The (possibly singular) real analytic variety $S$ is called **(singularly) hyperkähler** if it is endowed with the following structures.

(i) There is an action of an algebra of quaternions $\mathbb{H}$ on the real Zariski tangent space $TS$ to $S$.

(ii) For each $\bar{I} \in \mathbb{H}$ with $\bar{I}^2 = -1$ there is a complex structure $I$ over $S$ such that the complex structure operator on $TS$ associated with $I$ is equal to $\bar{I}$. This complex structure is called **induced by the quaternion action**.
(iii) For each $x \in S$ the fiber $TS_x$ of the sheaf $TS$ in $x$ has a Euclidean metric, which defines a quaternionic Hermitian structure on $TS_x$.

Consider $S$ as a complex variety with an induced complex structure $I$. Consider a coherent sheaf $Hom_{O_S}(TS \otimes TS, O_S)$ of bilinear forms over $S$. Of course, the fiber of $Hom_{O_S}(TS \otimes TS, O_S)$ at each point $x \in S$ is canonically embedded in a space of bilinear $\mathbb{C}$-valued forms over $TS_x$.

(iv) There is a holomorphic section $\Omega$ of $Hom_{O_S}(TS \otimes TS, O_S)$, which, being restricted to a fiber of $TS$ in $x \in S$, gives a canonical symplectic form associated with $TS_x$ and $I$.

Note that this definition is compatible with a usual one: a smooth manifold is hyperkähler in traditional sense if and only if it is hyperkähler in the sense of this definition.

The following theorem is to be proven in Section 9:

**Theorem 6.3.** The space $Spl(B)$ is (singularly) hyperkähler for a hyperholomorphic $B$.

This result is a generalization of the theorem by Kobayashi:

**Theorem 6.4.** ([Ko2]) Let $B$ be a simple Yang-Mills bundle $B$ over a hyperkähler manifold $M$ such that $H^2(End(B)) = \mathbb{C}$. Let $Spl_{ns}(B)$ be the open subspace in $Spl(B)$ consisting of all nonsingular points $[B']$ such that $H^2(End(B')) = \mathbb{C}$. The space $Spl_{ns}(B)$ has a canonical hyperkähler structure.

In the case when $M$ is a surface, $Spl(B)$ is smooth (Mukai [M]) and hyperkähler (Mukai [M] and Itoh [I]).

7. Local deformations of a hyperholomorphic connection. The proof Theorem 6.2, which states that the only obstruction to the deformation of a hyperholomorphic bundle is Yoneda pairing.

Let $B$ be a Hermitian bundle of degree zero over a Kähler manifold $M$. 24
By the virtue of Uhlenbeck-Yau theorem (see Section 1), the stable holomorphic structures on $B$ are in the one-to-one correspondence to the Hermitian connections $\nabla$ with a curvature $\Theta$ such that

\begin{align*}
(7.1) \quad & \Lambda(\Theta) = 0 \quad \text{(where $\Lambda$ is the Hodge operator)} \\
(7.2) \quad & \Theta \in \Lambda^{1,1} \otimes \text{End}(B), \text{ and } B \text{ is indecomposable with respect to } \nabla.
\end{align*}

Let $B$ be the undecomposable Hermitian vector bundle with Yang-Mills connection $\nabla$ over a Kähler manifold $M$. We are interested in local deformations of $\nabla$ which preserve conditions (7.1) and (7.2). One can see that the space of such deformations is a local deformation moduli space for $B$ as for a stable holomorphic bundle.

Let $\hat{\rho} \in \Lambda^1_\mathbb{R}(M) \otimes u(B)$, where $u(B)$ is the bundle of skew-adjoint endomorphisms of $B$ and $\Lambda^1_\mathbb{R}$ is a space of real differential 1-forms. Obviously, the connection $\nabla_1 = \nabla + \hat{\rho}$ is Hermitian, but in general, neither Yang-Mills nor holomorphic.

The bundle $E = \text{End}(B) = u(B) \otimes \mathbb{C}$ has a unique real structure such that $u(B)$ is a real subbundle. Therefore we can decompose $\hat{\rho}$ into the sum of $(1,0)$ and $(0,1)$ parts: $\hat{\rho} := \rho + \overline{\rho}$ (see Section 3).

The curvature $\Theta_1$ of the connection $\nabla_1$ is equal to $\Theta + \nabla(\hat{\rho}) + \hat{\rho} \wedge \hat{\rho}$.

The following proposition partially answers this question.

**Proposition 7.1.** Let a form $\hat{\rho} \in \Lambda^1_\mathbb{R}(M) \otimes u(B)$ be decomposed as above: $\hat{\rho} = \rho + \overline{\rho}$. If $\rho$ is $\Delta_g$-harmonic, then $\nabla(\hat{\rho})$ satisfies conditions (7.1) and (7.2). The condition (7.1) means that $\Lambda(\nabla \hat{\rho}) = 0$ and (7.2) means that $\nabla \hat{\rho} \in \Lambda^{1,1}(M) \otimes \text{End}(B)$.

**Proof:** Proving (7.1): the form $\nabla \hat{\rho}$ is of type $(1,1)$ because its $(2,0)$-part $\partial \rho$ and its $(0,2)$-part $\overline{\partial} \rho$ both vanish.

Proving (7.2): the condition (7.1) implies that $\nabla \hat{\rho} = \overline{\partial} \rho + \partial \rho$. Therefore, by Kodaira’s identities,

$$
\Lambda \nabla \hat{\rho} = \Lambda(\overline{\partial} \rho + \partial \rho) = \overline{\partial^*} \rho - \partial^* \overline{\partial} \rho + \nabla \Lambda \hat{\rho}
$$

25
The last term vanishes because $\Lambda(\alpha) = 0$ for any 1-form $\alpha$, and $\partial^* \rho = \bar{\partial}^* \rho = 0$ because $\rho$ is $\partial$-harmonic. Therefore $\Lambda \nabla \hat{\rho} = 0$.

Note that the space of $\Delta_\partial$-harmonic forms in $\Lambda^{1,0}(M) \otimes \text{End}(B)$ is canonically conjugate to the space $H^1(EndB) \cong \text{Ext}^1(B, B)$.

Denote the standard Hermitian norm on a space of differential forms with coefficients in the Hermitian bundle as $\| \cdot \|$.

Last part of this section is dedicated to the proof of the following theorem:

**Theorem 7.1** For any given hyperholomorphic bundle $B$ with a hyperholomorphic (in particular, Yang-Mills) connection $\nabla$ and a curvature $\Theta$ there exists a constant $\varepsilon$ with the following property.

For each $\Delta_\partial$-harmonic form

$$\rho \in \Lambda^{1,0}(M, \text{End}(B)) \text{ with } \| \rho \| < \varepsilon, \quad \iota(\rho, \rho) = 0$$

there exist a form $\eta \in \Lambda^{1,0}(M, \text{End}(B))$ with $\| \eta \| < 1/4 \| \rho \|$ such that a connection

$$\nabla_{\rho} = \nabla + \eta + \bar{\eta} + \rho + \bar{\rho}$$

has a curvature

$$\nabla(\rho + \bar{\rho}) + \Theta$$

There $\Theta$ denotes the curvature of $B$. Thus constructed by $\rho$ form $\eta$ holomorphically depends on $\rho$. In the statement above $\iota$ denotes Yoneda pairing, see Definition 2.2.

The Theorem 7.1 will be proven later in this section. Now we will demonstrate some of its implications.

**Proposition 7.2.** The connection $\nabla_{\rho}$, supplied by Theorem 7.1, is Yang-Mills.

**Proof.** Let $\hat{\rho} := \rho + \bar{\rho}$ as in Proposition 7.1. Since $\rho$ is $\partial$-harmonic, $\partial \rho = \bar{\partial} \rho = 0$. This implies that $\nabla \hat{\rho}$ is $(1,1)$ form. Therefore $\Theta + \nabla \hat{\rho}$ is $(1,1)$-form and by Newlander-Nierenberg theorem (Theorem 2.1) the connection $\nabla_{\rho}$ is holomorphic. To see that it is Yang-Mills we need to prove that $\Lambda(\Theta + \nabla_{\rho}) = 0$. By our construction, $\Lambda(\Theta) = 0$, while $\Lambda(\nabla \hat{\rho}) = 0$ by Proposition 7.1.
Let us show how Theorem 7.1 imples Theorem 6.2. Denote the intersection of a quadratic cone
\[{\rho \in \text{Ext}^1(B, B) \mid \iota(\rho, \rho) = 0}\]
with an open disc of radius \(\varepsilon\) in
\[\ker \Delta_{\theta} \mid _{\Lambda^{1,0}(M) \otimes \text{End}(B)} = \text{Ext}^1(B, B)\]
as \(S\). Choose a complex structure on \(S\) conjugate to the standard one. Consider the bundle \(B\) over \(S \times M\) which is, as a vector bundle, isomorphic to a pullback of \(B\) from \(M\) to \(S \times M\) by the projection map. Choose a connection \(\nabla_1\) on \(B\) which is (a) trivial in the direction of \(S\) and (b) at each point of \(\rho_0 \times M\) is isomorphic to \(\nabla_{\rho_0}\) in the direction of \(M\). Let \(\Theta_{\rho}\) be a curvature of a connection \(\nabla_{\rho}\). Let \(\Xi\) be the 2-form over \(S \times M\) with coefficients in \(\text{End}B\), which is glued of the forms \(\Theta_{\rho}\) on the following way. The forms \(\Theta_{\rho}\), \(\rho \in S\) are defined over \(M \times \{\rho\}\), where \(M \times \{\rho\}\) is the fibre over \(\rho\) of the projection \(S \times M \to S\). Let \(\Theta_1\) be the curvature of \(\nabla_1\).

Let us decompose \(\nabla\) into the sum of two components: \(\nabla = \nabla_S + \nabla_M\). The component \(\nabla_S\) corresponds to the derivation in the direction of \(S\), while the component \(\nabla_M\) corresponds to the derivation in the direction of \(M\). Analogously, let us decompose the (1,0)-component \(\partial\) of the connection \(\nabla: \partial = \partial_S + \partial_M\).

In this notation, one can easily see that \(\Theta_1 - \Xi = \nabla_{S\hat{\rho}}\).

The form \(\Xi\) is of type (1,1) by the construction. Since \(\eta\) holomorphically depends on \(\rho\) in the standard complex structure, it depends on \(\rho\) antiholomorphically in the opposite one. Therefore \(\partial_S \eta = 0\) in this structure and \(\nabla_{S\hat{\rho}}\) is a form of type (1,1). Therefore, the form \(\Theta\) is of type (1,1) and the connection \(\nabla_1\) is holomorphic.

This consideration shows how Theorem 7.1 gives a construction of a holomorphic local variation \((S, B)\) of Yang-Mills connections for a hyperholomorphic \(B\). By the Uhlenbeck-Yau theorem, this is the same as the construction of local variation of a stable bundle. Since \(\|\eta\| < 1/4 \|\rho\|\) and \(\rho\) is represented by the classes in \(\text{Ext}^1(B, B)\), the differential of an obvious map \(K: S \to \text{Ext}^1(B, B)\) is the Kodaira-Spencer map.

We have constructed a local variation of \(B\) with the base \(S\) and the following properties. First, the Kodaira map \(K: S \to \text{Ext}^1(B, B)\) is imbedding. Second, its image coincides with the set of vectors \(\kappa \in \text{Ext}^1(B, B)\)
such that $\ell(\kappa, \kappa) = 0$. These two properties are sufficient for the variation to be locally universal (see [KS]), and the Theorem 6.2 is proven.

**Corollary 7.1** Theorem 7.1 yields a construction of a universal local variation of a stable bundle $B$, if $B$ is hyperholomorphic.

**Proof of Theorem 7.1.**

As usually, we denote $\hat{\eta} := \eta + \bar{\eta}$ and $\hat{\rho} := \rho + \bar{\rho}$.

**Lemma 7.1.** The form $\eta$ suffices the statement of Theorem 7.1 if the following equation holds:

$$-\nabla \hat{\eta} = \hat{\rho} \wedge \hat{\rho} + \hat{\rho} \wedge \hat{\eta} + \hat{\eta} \wedge \hat{\rho} + \hat{\eta} \wedge \hat{\eta}$$

(7.3)

**Proof:** By definition, $\nabla \rho = \nabla + \hat{\rho} + \hat{\eta}$. Here, as elsewhere, the forms $\hat{\rho}$ and $\hat{\eta}$ are considered as operators as follows: $\hat{\eta}(\alpha) = \hat{\eta} \wedge \alpha$. In this notation, $\nabla \circ \hat{\rho} + \hat{\rho} \circ \nabla = \nabla(\rho)$. Therefore,

$$\Theta_\rho := \nabla^2 = \nabla(\hat{\eta}) + \nabla(\hat{\rho}) + \hat{\rho} \wedge \hat{\rho} + \hat{\rho} \wedge \hat{\eta} + \hat{\eta} \wedge \hat{\rho} + \hat{\eta} \wedge \hat{\eta} + \nabla^2$$

$\Theta = \Theta_\rho = \Theta + \nabla(\hat{\rho})$, and by (7) this is equivalent to $\nabla \hat{\eta} + \hat{\rho} \wedge \hat{\rho} + \hat{\rho} \wedge \hat{\eta} + \hat{\eta} \wedge \hat{\rho} + \hat{\eta} \wedge \hat{\eta} = 0$. ■

We will try to find a solution of (7.3) in the form of a Taylor serie. Let us make the following change of notation. We shall redenote $\hat{\rho}$ by $t \hat{\eta}_1$ and $\hat{\eta}$ by $(t^2 \hat{\eta}_2 + t^3 \hat{\eta}_3 + ...)$.

Then the equation (7.3) could be rewritten as a system of equations:

$$-\nabla \hat{\eta}_2 = \hat{\eta}_1 \wedge \hat{\eta}_1$$
$$-\nabla \hat{\eta}_3 = \hat{\eta}_1 \wedge \hat{\eta}_2 + \hat{\eta}_2 \wedge \hat{\eta}_1$$

(7.4)
$$-\nabla \hat{\eta}_n = \sum_{j + j = n} \hat{\eta}_j \wedge \hat{\eta}_j$$

We will find $\eta_2,$ .... $\eta_n,$ .... which satisfy (7.4) and show that for $t$ small enough the power series

$$\hat{\eta} = t^2 \hat{\eta}_2 + t^3 \hat{\eta}_3 + ...$$

converges. Further on, we will as usually denote the (1,0) component of $\hat{\eta}_i$ as $\eta_i$ and the (0,1) component - as $\bar{\eta}_i$.  

28
Let us solve the first equation
\[ -\nabla \hat{\eta}_2 = \hat{\eta}_1 \wedge \hat{\eta}_1 (= \lambda^2 \hat{\rho} \wedge \hat{\rho}). \]

By our assumptions, for Yoneda pairing
\[ \iota : H^1(\text{End}B) \times H^1(\text{End}B) \rightarrow H^2(\text{End}B). \]
\( \iota(\rho, \rho) = 0. \) One can easily prove that (see [KS]) that the cohomology class
of \( \rho \wedge \rho \) in \( H^2(\text{End}(B)) \) coincides with \( \iota(\rho, \rho) \). Therefore the form \( \rho \wedge \rho \),
being obviously \( \partial \)-closed, is also \( \partial \)-exact.

Let \( G_\partial \) be the Green operator associated with the Laplacian \( \Delta_\partial \), and
\( \Gamma = \partial^* \circ G_\partial \). Clearly, \( G_\partial \Delta_\partial \tau = \tau \) and \( \Delta_\partial \tau = \partial \partial^* \tau \) for \( \partial \)-exact \( \tau \). Since
\( G\partial \partial^* = \partial \partial^* G \), we see that \( \partial \partial^* G \tau = \tau \) for \( \partial \)-exact \( \tau \).

We have proven
**Lemma 7.2.** For each \( \partial \)-exact form \( \tau \) with coefficients in a holomorphic
Hermitian bundle over arbitrary Kähler manifold the following equation holds:
\[ \partial \circ \Gamma(\tau) = \tau. \]

As a corollary, we obtain the following proposition: Let \( \eta_2 = -\Gamma(\eta_1 \wedge \eta_1) \).
Then
\[ (7.5) \quad \partial \eta_2 = -\eta_1 \wedge \eta_1 \]
and
\[ (7.5') \quad \bar{\partial} \bar{\eta}_2 = -\bar{\eta}_1 \wedge \bar{\eta}_1. \]

The bundle \( \Lambda^1(M, \text{End}(B)) \) in endowed with a canonical real structure \( T \)
(see Section 3). The operator \( T \) interchanges subbundles \( \Lambda^{0,1}(M, \text{End}(B)) \)
and \( \Lambda^{1,0}(M, \text{End}(B)) \). Consider the derivation \( \tilde{T} \) of the algebra of \( \text{End}(B) \)-valued differential forms which acts on \( \Lambda^1(M, \text{End}(B)) \) as \( \tilde{T} \) and on \( \Lambda^i(M) \otimes \text{End}(B) \) for \( i > 1 \) by Leibnitz formula
\[ \tilde{T}(\alpha \wedge \beta) = \tilde{T}(\alpha) \wedge \beta + \alpha \wedge \tilde{T}(\beta). \]

**Lemma 7.3** For any two \( \text{End}(B) \)-valued (1,0)-forms \( \lambda \) and \( \mu \), the follow-

\[ \tilde{T}(\lambda \wedge \mu) = \bar{\lambda} \wedge \mu + \lambda \wedge \bar{\mu} \]
\[ T(\partial \lambda + \bar{\partial} \bar{\lambda}) = 2\bar{\partial} \lambda + 2\bar{\partial} \bar{\lambda} \]

The first property is just a part of definition of \( T \). Let \( \hat{\lambda} := \lambda + \bar{\lambda} \). Let, as usualy, \( \nabla := \partial + \bar{\partial} \). We will say that the form is real if it is preserved by \( T \). It is very easy to see that for a real \( p \)-form \( \eta \), \( \tilde{T}(\eta) = p\eta \). Now, \( \tilde{T}(\nabla(\hat{\lambda})) = 2\nabla(\hat{\lambda}) \) because \( \nabla(\hat{\lambda}) \) is a real form. Therefore

\[ \tilde{T}(\bar{\partial} \bar{\lambda} + \partial \bar{\lambda} + \partial \lambda + \bar{\partial} \lambda) = 2(\bar{\partial} \bar{\lambda} + \partial \bar{\lambda} + \partial \lambda + \bar{\partial} \lambda) \]

The operator \( \tilde{T} \) interchanges \( \Lambda^{1,1}(M, \text{End}(B)) \) and \( \Lambda^{2,0}(M, \text{End}(B)) \oplus \Lambda^{0,2}(M, \text{End}(B)) \). Therefore

\[ T(\partial \lambda + \bar{\partial} \bar{\lambda}) = 2\bar{\partial} \lambda + 2\bar{\partial} \bar{\lambda}. \]

Applying \( \tilde{T} \) to the sum of equations (7.5) and (7.5') one obtains that

(7.6) \[ \partial \bar{\eta}_2 + \bar{\partial} \eta_2 = -\eta_1 \wedge \bar{\eta}_1 - \bar{\eta}_1 \wedge \eta_1 \]

The summation of (7.5), (7.5') and (7.6) gives us a solution \( \bar{\eta}_2 = \eta_2 + \bar{\eta}_2 \) of an equation

\[ -\nabla \bar{\eta}_2 = \bar{\eta}_1 \wedge \eta_1. \]

The same consideration proves the following lemma:

**Lemma 7.4** Let \( \xi_1, ..., \xi_n \) be (1,0)-forms with coefficients in \( \text{End}(B) \). Let \( \bar{\xi}_i \) be conjugate forms with respect to the real structure on \( \text{End}(B) \). Let \( \xi_i = \xi + \bar{\xi} \). Suppose that \( \sum_{i+j=n} \xi_i \wedge \xi_j \) is \( \partial \)-exact. Let \( \Gamma \) be the inverse operator to \( \partial \) defined above, and \( \sigma = \Gamma(\sum_{i+j=n} \xi_i \wedge \xi_j) \). Let \( \tilde{\sigma} = \sigma + \bar{\sigma} \). Then \( \nabla \tilde{\sigma} = \sum_{i+j=n} \hat{\xi}_i \wedge \hat{\xi}_j \)

**Proof:** By our assumption, \( \sum_{i+j=n} \xi_i \wedge \xi_j \) is \( \partial \)-exact. The Lemma 7.2 implies that in this case

\[ \partial \sigma = \sum_{i+j=n} \xi_i \wedge \xi_j. \]

Applying \( T \) to this equation one obtains

\[ \bar{\partial} \sigma = \sum_{i+j=n} \hat{\xi}_i \wedge \hat{\xi}_j, \]
and applying $\tilde{T}$ to the sum of these two equations one obtains

$$\partial \sigma + \partial \tilde{\sigma} = \sum_{i+j=n} (\xi_i \wedge \xi_j + \tilde{\xi}_i \wedge \tilde{\xi}_j).$$

Finally, summing up these three equations one obtains

$$\nabla \tilde{\sigma} = \sum_{i+j=n} \tilde{\xi}_i \wedge \tilde{\xi}_j.$$  

Now, let us define the following recursive sequence:

$$\eta_1 = \rho/t, \; \eta_2 = \Gamma(\eta_1 \wedge \eta_1), \; \ldots, \; \eta_m = \Gamma(\sum_{i+j=n} \eta_i \wedge \eta_j), \; \ldots.$$ 

By Lemma 7.4, if each step yields the $\partial$-exact form $\sum_{i+j=n} \eta_i \wedge \eta_j$, then

$$\nabla \eta_n = \sum_{i+j=n} \eta_i \wedge \eta_j$$

and we obtained solutions of (7.4) we looked for. Let us prove that the form $\tau_n := \sum_{i+j=n} \eta_i \wedge \eta_j$ is $\partial$-exact. First of all, we show that this form is $\partial$-closed. Since

$$\partial \eta_n = \sum_{i+j=n} \eta_i \wedge \eta_j$$

on each inductive step of our construction, we obtain that

$$\kappa := \partial(\sum_{i+j=n} \eta_i \wedge \eta_j) = \sum_{i+j+k=n} \eta_i \wedge \eta_j \wedge \eta_k.$$ 

It is easy to see that each 3-vector in $\Lambda^{3,0}$ is mapped by $\kappa$ to the element of $\text{End}(B)$ which is equal to a sum of several terms like $[A[B, C]] + [B, C][A] + [[C, A][B]]$. By the Jacoby identity, $\kappa$ vanishes, and the form $\sum_{i+j=n} \eta_i \wedge \eta_j$ is closed.

To prove exactness of $\tau$ we use $\partial \partial^{j}$-lemma (Theorem 4.3). First of all, the form $\rho$ is $\Delta_{\tilde{\partial}}$-harmonic and by Theorem 4.1 it is also $\partial^{j}$-closed (even $\partial^{j}$-harmonic). By Proposition 4.4, the operator $\partial^{j}$ anticommutes with $\partial$ and $\partial^{*}$. Therefore $\partial^{j}$ commutes with $\Delta_{\tilde{\partial}}, G_{\tilde{\partial}}$ and anticommutes with $\Gamma := \partial^{*} \circ G_{\tilde{\partial}}$. This implies that each form $\eta_i$ constructed recursively above is $\partial^{j}$-closed.
Let us prove inductively that the forms $\eta_i \mid i > 1$ are $\partial\bar{\partial}$-exact. Suppose that $\eta_i$ is $\partial\bar{\partial}$-exact for $1 < i < n$. The form

$$\tau_n = \sum_{i+j=n} \eta_i \wedge \eta_j$$

is $\partial\bar{\partial}$-exact because it is a sum of several summands, each of those is a product of two $\partial\bar{\partial}$-closed forms, at least one of which is $\partial\bar{\partial}$-exact. By Proposition 5.3, $\Gamma(\tau_n) = \eta_n$ also $\partial\bar{\partial}$-exact. Finally, since $\eta_n$ is also $\partial$-closed, by $\partial\partial\bar{\partial}$-lemma (Theorem 5.3) it is $\partial$-exact.

This proves that the system (7.4) will have solutions of the following form:

$$\eta_2 = \Gamma(\eta_1 \wedge \eta_1), \ldots, \eta_n = \Gamma\left(\sum_{i+j=n} \eta_i \wedge \eta_j\right), \ldots$$

One knows that the Green operator $G$ is compact. Therefore $\Gamma$ is also compact. Let $r$ be the norm of $\Gamma$. Obviously,

$$\|\eta_n\| \leq r \cdot n \cdot \sup_{i<n} \|\eta_i\|^2 \leq n! \cdot \|\eta_1\|^{2n} r^{2n} = n! \cdot t^{-2n} \|\rho\|^{2n} r^{2n}$$

If $\|\rho\| < \frac{e}{4r}$ the serie

$$\eta = \sum_{i \geq 2} \eta_i t^i$$

will converge for $t = 1$ and the result will be sufficiently small: $\|\eta\| < 1/4\|\rho\|$. Finally, $\eta$ holomorphically depends on $\rho$ since $\Gamma$ is linear and the map

$$\{\eta_1, \ldots, \eta_n \rightarrow \sum_{i+j=n} \eta_i \wedge \eta_j\}$$

is holomorphic. Theorem 7.1 is proven. ■

Section 8. Comparing Laplacians.

Let $B$ be a hyperholomorphic bundle over a hyperkähler manifold $M$. Let $\Delta_{\partial L}$ be the Laplace operator $\Delta_{\partial}$ associated with arbitrary induced complex structure $L$. 32
Theorem 8.1 If \( I, \ L \) are two induced complex structures over \( M \), we have an identity:

\[
(\Delta_\partial I)^L = (\Delta_\partial L').
\]

We use there the notation of the Section 4, where \(( )^L\) denotes \(L \circ () \circ L^{-1}\).

**Proof:** Let \( d \) denote the connection operator in \( B \) (denoted by \( \nabla \) elsewhere). Proposition 3.1 immediately implies that

\[
\Delta_\partial I = 1/2(\Delta_d + \sqrt{-1} [\Lambda_I, (d^2)])
\]

Since \( B \) is hyperholomorphic, its curvature operator \( d^2 \) is \( L \)-invariant with respect to an induced complex structure operator \( L \). Therefore \( (d^2)^L = d^2 \). By Proposition 3.1(c), \( \Delta^L_d = \Delta_d \). Therefore

\[
(\Delta_\partial I)^L = 1/2(\Delta_d^L + \sqrt{-1} [\Lambda^L_I, (d^2)^L]) = 1/2(\Delta_d + \sqrt{-1} [\Lambda^L_I, d^2]).
\]

Finally, the identity \( \Lambda^L_I = \Lambda_{L \circ I \circ L^{-1}} \) is proven by a simple computation with quaternions. \( \blacksquare \)

Theorem 8.1 implies that the group of unitary quaternions acts transitively on the set of Laplace operators \( \Delta_\partial L \) where \( L \) runs through the set of induced complex structures (see Lemma 8.1). Let \( H^i_I(B) \) denote the space of holomorphic cohomology of a hyperholomorphic bundle \( B \), taken with respect to the induced complex structure \( L \).

**Corollary 8.1:** The space \( H^i_I(B) \) does not depend on the choice of an induced complex structure \( I \).

**Proof:** The space \( H^i_I(B) \) (to be precise, its complex conjugate) can be canonically identified with the space of \( \Delta_\partial L \)-harmonic sections of \( \Lambda^1_I(B) \), where \( \Lambda^1_I(B) \) means that Hodge grading is taken with respect to \( I \). One can easily see that the action of \( L \) on \( \Lambda^1(M, B) \) produces an isomorphism between \( \Lambda^1_I(B) \subset \Lambda^1(M, B) \) and \( \Lambda^1_{I'}(B) \subset \Lambda^1(M, B) \). By Theorem 8.1, under this isomorphism \( \Delta_\partial I \) goes into \( \Delta_\partial I' \). Therefore \( L \) maps \( H^i_I(B) \) into \( H^i_{I'}(B) \). Finally, Corollary 8.1 is implied by the following lemma:

**Lemma 8.1:** For each pair of induced complex structures \( L \) and \( L' \) over \( M \) there is an induced \( R(L, L') \) such that \( L^R(L, L') = L' \).

**Proof:** This is an easy linear algebraic computation.

Let \( G_M \) be the isotropy group for \( M \). This group is isomorphic to \( SU(2) \). There is a unique element \( r \neq 1 \) in \( SU(2) \) such that \( r^2 = 1 \). This element
is defined by the matrix $-1$. The set of induced complex structures is isomorphic to the set of elements $a \in G_M$ such that $a^2 = r$.

The group $G_M$ is isomorphic to the group of unitary quaternions $\{h \in \mathbb{H} \mid h\bar{h} = 1\}$. The element $r$ is just $-1 \in \mathbb{H}$. The unitary quaternion $s$ has the square $-1$ if $s = aI + bJ + cK$ for real $a, b, c$ such that $a^2 + b^2 + c^2 = 1$. Therefore the set $S$ of such quaternions is a unit sphere in $\mathbb{R}^3$. Clearly,

$$I \circ (aI + bJ + cK) \circ I^{-1} = aI - bJ - cK$$

This implies that for induced complex structure $B \in S$, the map $A \mapsto A^B$ from $S$ to $S$ is a symmetry ($180^\circ$ angle rotation) of $S$ around the line going through $B$ and $-B$. Now, for each $L, L'$ in $S$ there is an element $R(L, L')$ in $S$ such that the symmetry around the line which goes through $R(L, L')$ and $-R(L, L')$ maps $L$ into $L'$. Simply, connect $L$ and $L'$ with a big circle. Take the point on this circle which equally divides the interval between $L$ and $L'$. This is $R(L, L')$.

The Lemma 8.1, and therefore Corollary 8.1 are proven.

By $Ext^i_I(B, B)$ we mean $Ext^i(B, B)$ where $B$ is taken as a holomorphic bundle over $(M, I)$.

**Corollary 8.2:** The graded rings $\oplus_i Ext^i_I(B, B)$ are isomorphic for all induced $I$ over $M$.

**Proof:** For each $i$, $Ext^i_I(B, B) \cong H^i_I(\text{End}(B))$, and $\text{End}(B)$ is hyperholomorphic for hyperholomorphic $B$. The isomorphism constructed in Corollary 8.1 is clearly multiplicative.

9. The space of deformations of a hyperholomorphic bundle is hyperkähler (Theorem 6.3).

This proof is completely analogous to the proof of the same theorem for the case when $M$ is a surface ([M]) and for the case when $H^2(\text{End}(B)) = \mathbb{C}$ (Theorem 6.4, proven in [Ko2]).

We need to show that $\text{Spl}(B)$ suffices points (i)-(iv) of Definition 6.5.

(i)-(ii): Corollary 8.2 together with Theorem 6.2 provide that the local deformation space $(S, I)$ of a hyperholomorphic bundle $B$ over $(M, I)$ is independent on the choice of an induced complex structure $I$. Since the $(S, I) =: S$ is a subset of an open ball in $Ext^1(B, B)$, one can easily check
the following, using Corollary 8.2. Let \( \tilde{I}, \tilde{J}, \) and \( \tilde{K} \) are complex structures on \( S \) defined by induced complex structures \( I, J, K \) over \( M \). Assume that \( I \circ J = -J \circ I = K \). Then \( \tilde{I} \circ \tilde{J} = -\tilde{J} \circ \tilde{I} = \tilde{K} \).

The last statement is sufficient to prove that \( \text{Spl}(B) \) satisfies (i)-(ii) from the Definition 6.5.

The quaternion action on the tangent space \( T_{[B']} \text{Spl}(B) \) in the point, corresponding to the bundle \( B' \), is compatible with the quaternion action on \( H^1(\text{End}(B')) \) constructed in Corollary 4.3.

(iii) Since \( T_{[B']} \text{Spl}(B) \) is a subspace of \( H^1_\mathbb{H}(\text{End}(B)) \), one can restrict the Hermitian metric on \( H^1_\mathbb{H}(\text{End}(B)) \) to \( T_{[B']} \text{Spl}(B) \). Now, the Hermitian metric on \( H^1_\mathbb{H}(\text{End}(B)) \) is defined as follows. Cohomology classes from \( H^1_\mathbb{H}(\text{End}(B)) \) can be canonically identified with \( \Delta_\partial \)-harmonical forms in \( \Lambda^1 \mathbb{H}_0(\text{End}(B)) \).

The bundle \( \Lambda^1 \mathbb{H}_0(\text{End}(B)) \) is Hermitian, so there is an Hermitian metric on the space of its continuous sections. To find the Hermitian product of two continuous sections of a Hermitian bundle over a compact oriented Riemann manifold \( M \), one takes their product pointwise, and then integrates over \( M \) the resulting \( \mathbb{C} \)-valued function times volume form of \( M \).

Take the real part of this Hermitian metric. For induced \( L \) and \( \text{End}(B) \)-differential forms \( \eta_1 \) and \( \eta_2 \) we have \( (\eta_1, \eta_2) = (\eta_1^L, \eta_2^L) \), where \( (\cdot, \cdot) \) means the point-wise scalar product. Combining this with the proof of Corollary 8.1, we obtain that this positively defined \( \mathbb{R} \)-valued scalar product is independent on the choice of \( I \). Since it is Hermitian with respect to induced complex structures by the construction, this metric is quaternionic Hermitian. The part (iii) of Definition 6.5 is proven.

(iv) Fix an induced complex structure \( I \) on \( M \). Let \( S \) be the neighbourhood of \( [B] \in \text{Spl}(B) \), constructed in Corollary 7.1. The space \( M \times S \), according to the construction preceding Corollary 7.1, is endowed with the classifying bundle \( B \) with the following property. Take a hyperholomorphic bundle \( B' \) over \( M \) such that \( [B'] \in S \subset \text{Spl}(B) \). The restriction of \( B \) on \( M \times [B'] \subset M \times S \) is isomorphic to \( B' \).

Consider the projection \( \pi : M \times S \to S \) and take \( E_1 := R^1 \pi_*(\text{End}(B)) \). One can immediately see that the fiber of \( E_1 \) at each point \( [B'] \in S \) is isomorphic to \( H^1(\text{End}(B)) = \text{Ext}^1(B, B) \). Consider the cup-product

\[ \cup : R^1 \pi_*(\text{End}(B)) \times R^1 \pi_*(\text{End}(B)) \to R^2 \pi_*(\text{End}(B)) \]

defined by the bilinear product on \( \text{End}(B) \). This is a bilinear map of coherent sheaves over \( S \). Denote \( R^2 \pi_*(\text{End}(B)) \) by \( E_2 \). The cup-product map, in this

35
notation, is a bilinear map $\cup$ from $E_1 \times E_1$ to $E_2$, or a coherent sheaves’ map from $E_1 \otimes \mathcal{O}_S E_1$ to $E_1$. By Theorem 6.2, the Zariski tangent sheaf $TS$ is imbedded in $E_1$. Moreover, $TS$ as a coherent subsheaf of $E_1$ is generated by all sections $\rho$ of $E_1$, such that $\cup(\rho, \rho) = 0$.

There is a trace map from $\text{End}(\mathcal{B})$ to $\mathcal{O}_{M \times S}$. By functoriality, this map defines the map

$$\rho : R^2\pi_*(\text{End}(\mathcal{B})) \rightarrow R^2\pi_*(\mathcal{O}_{M \times S}).$$

From projection formula, we see that that $R^2\pi_*(\mathcal{O}_{M \times S})$ is a trivial sheaf over $S$ with a fiber $H^2(M, \mathcal{O}_M)$. The space $H^2(M, \mathcal{O}_M)$ for a hyperkähler $M$ is endowed with a canonical linear form, which is defined by a canonical symplectic form over $M$. This form is equal to the operator $\Lambda_c$ from $H^2(M, \mathcal{O}_M)$ to $H^0(M, \mathcal{O}_M) = \mathbb{C}$ (see Theorem 4.2). Therefore there is a canonical map

$$\tau : R^2\pi_*(\mathcal{O}_{M \times S}) \rightarrow \mathcal{O}_S$$

Composing $\cup : E_1 \otimes \mathcal{O}_S E_1 \rightarrow E_2$ with $\rho : E_2 \rightarrow R^2\pi_*(\mathcal{O}_{M \times S})$ and $\tau : R^2\pi_*(\mathcal{O}_{M \times S}) \rightarrow \mathcal{O}_S$, one obtains the form $\tilde{\Omega}$ on $E_1$. Since $TS \subset E_1$, one can restrict this form to $TS$ to obtain the holomorphic section $\Omega_S$ of $\text{Hom}(TS \otimes \mathcal{O}_S TS, \mathcal{O}_S)$.

We shall prove that this section is one required by (iv) of Definition 6.5.

Consider the restriction of $\Omega_S$ on the fiber $TS_x$ of $TS$ in $x \in S$, where $x$ corresponds to a hyperholomorphic bundle $B$ over $M$. By Theorem 6.2, $TS_x$ is a subspace of $\text{Ext}^1(B, B) = H^1(\text{End}(B))$ spanned by all $\gamma$ with $\iota(\gamma, \gamma) = 0$. There is a trace map $\tilde{\rho} : H^2(\text{End}(B)) \rightarrow H^2(\mathcal{O}_M)$, the map $\tau = \Lambda_c$ from $H^2(\mathcal{O}_M)$ to $\mathbb{C}$ and the cup-product $\cup : H^1(\text{End}(B)) \times H^1(\text{End}(B)) \rightarrow H^2(\text{End}(B))$.

The restriction $\Omega$ of $\Omega_S$ to $TS_x \otimes TS_x$ is a composition of embedding $i : TS_x \otimes TS_x \hookrightarrow H^1(\text{End}(B)) \otimes H^1(\text{End}(B))$, then the cup-product, then the trace $\tilde{\rho}$ and then $\tilde{\tau}$:

$$\Omega := \Omega_{TS_x \otimes TS_x} = i \circ \cup \circ \tilde{\rho} \circ \tilde{\tau} : TS_x \otimes TS_x \rightarrow \mathbb{C}.$$

**Proposition 9.1:** The form $\Omega$ is equal to the canonical symplectic form constructed by the complex structure $I$ on the quaternionic Hermitian space $TS_x$. 

36
Proof: First of all, one can redescribe $\Omega$ in terms of harmonic forms as follows. Take two classes $[\alpha]$ and $[\beta]$ in $TS_x \subset H^1(End(B))$ and let $\alpha$, $\beta$ be the unique $\partial$-harmonic $(1,0)$-forms representing these classes. Obviously,

$$\Omega([\alpha],[\beta]) = \int_M Tr\Lambda_c(\alpha \wedge \beta)Vol(M).$$

There is an action of quaternions in $\Lambda^{1,0}(End(B))$, which commutes with Laplacian and induces the constructed above quaternion action on $H^1(End(B))$ (see Corollary 4.3). This action is Hermitian with respect to the metric on $\Lambda^{1,0}(End(B))$ arising from a metric on $B$ and on $\Lambda^{1,0}(M)$ which is identified with $\Lambda^1(M, \mathbb{R})$ as a real bundle. The canonical symplectic form on the fibres of this bundle is given by the formula

$$\gamma_1, \gamma_2 \mapsto Tr\Lambda_c(\gamma_1 \wedge \gamma_2).$$

This statement immediately follows from the definition of Hermitian form in $\Lambda^{1,0}(End(B))$, definition of $\Lambda_c$ and the definition of a canonical symplectic form. Since the Hermitian product on the space of $\partial$-harmonic forms is given by the integration of product in fibres along $M$, one sees that the canonical symplectic form on $TS_x \subset H^1(End(B)$ is given by the formula

$$\gamma_1, \gamma_2 \mapsto \int_M Tr\Lambda_c(\gamma_1 \wedge \gamma_2)Vol(M).$$

This is exactly the same formula as one for the restriction of $\Omega_S$ to $TS_x$. Proposition 9.1 and Theorem 6.3 are proven.

10. Some applications.

Let $I_1$ and $I_2$ be two complex structures induced by the same hyperkähler structure. Let $B$ be a bundle with connection. Assume that this connection is hyperholomorphic with respect to the hyperkähler structure. This connection is integrable over $(M, I_1)$ and $(M, I_2)$. Consider the holomorphic bundles $B_1$ over $(M, I_1)$ and $B_2$ over $(M, I_2)$ which are constructed by integrating this connection in $B$. Theorem 6.3 endows the manifolds $Spl(B_1; (M, I_1))$ and $Spl(B_2; (M, I_2))$ with a hyperkähler structure.

Corollary 10.1 Two manifolds $Spl(B_1; (M, I_1))$ and $Spl(B_2; (M, I_2))$ are isomorphic as hyperkähler manifolds. The complex structures on $Spl(B) := Spl(B_i; (M, I_i))$ which come from $(M, I_1)$ or from $(M, I_2)$ are induced by this hyperkähler structure.
From now till the end of the section we will suppose that the first Chern class of $B$ vanishes: $c_1(B) = 0$.

**Definition 10.1** The holomorphic bundle $B$ of degree zero is called **strongly simple** if it has no subsheaves of degree zero.

**Definition 10.2** The complex structure on a Kähler manifold $M$ is called of **general type** if $H^{11}(M) \cap H^2(M, \mathbb{Z}) = \{0\}$.

**Definition 10.3**: The hyperkähler structure $\mathcal{H}$ is called of **general type** if there is a complex structure $I$ of general type on $M$ such that $I$ is induced by $\mathcal{H}$.

**Proposition 10.1** If a complex structure on $M$ is of general type then the strong simplicity of a bundle $B$ over $M$ is equivalent to its stability.

**Proof** (see also [V], Proposition 3.1) We know that for any coherent sheaf $F$ over a Kähler manifold $M$, the cohomology class of $c_1(F)$ lies in group $H^{11}(M) \cup H^2(M, \mathbb{Z})$. Since $M$ is of general type, this group is zero. Therefore $c_1(F) = 0$ and $\text{deg}(F) = 0$ for any sheaf $F$ over $M$.

By definition of stability, this means that the bundle $B$ is stable iff it has no subsheaves $G \subset B$ with $\text{rank}(G) < \text{rank}(B)$.

The following proposition is an easy consequence of Calabi-Yau theorem ([Y]) and it was proven in [Tod].

**Proposition 10.2** The set $\mathcal{H}$ of hyperkähler structures which induce the given complex structure $I$ on $M$ is canonically isomorphic to a convex cone of signature $(+, -, -, ..., -)$ in $H^{11}(M) \cap H^2(M, \mathbb{R})$. The set of hyperkähler structures of general type is Zariski dense in $\mathcal{H}$.

We call two bundles $B_1$ and $B_2$ algebraically equivalent if there is a connected variety $S$ and a bundle $\mathcal{B}$ over $M \times S$ such that $\mathcal{B}$ restricted to $M \times \{s_1\}$ is isomorphic to $B_1$ and $\mathcal{B}$ restricted to $M \times \{s_2\}$ is isomorphic to $B_2$, where $s_1$ and $s_2$ are points in $S$.

Let $K_M$ be the set of equivalence classes of hyperholomorphic bundles with vanishing $c_1$ modulo algebraic equivalence. Let $\{B_i, i \in K_M\}$ be the
set of representatives of bundles in $K$, one in each component. Denote the disjoint union of all spaces $\{\text{Spl}(B_i), i \in K_M\}$ by $\mathcal{M}_M$.

The following proposition was proven in [V] for K3 or abelian surfaces:

**Proposition 10.3** (See also Corollary 3.1 from [V]) Suppose that $M_1$ and $M_2$ are hyperkähler manifolds of general type which belong to the same deformation class. The spaces $\mathcal{M}_{M_1}$ and $\mathcal{M}_{M_2}$ are diffeomorphic.

In other words, the classifying space of hyperholomorphic bundles over the generic hyperkähler $M$ does not depend on the deformation class of $M$.

**Proof:** We call the following operation on a complex manifold $(M, I)$ **standard**.

a) choose some hyperkähler structure $\mathcal{H}$ of general type such that $I$ is induced by $\mathcal{H}$.

b) choose another complex structure $L$ which is induced by $\mathcal{H}$.

The result of this operation will be the complex manifold $(M, L)$.

By the Corollary 10.1 and Proposition 10.1, the standard operation does not change the diffeomorphism class of $\mathcal{M}_{(M, I)}$ if $(M, I)$ was of general type. Therefore to prove Proposition 10.3 we should prove the following theorem. For a surface, this is Theorem 4.1 from [V], for arbitrary dimension see [Tod]:

**Theorem 10.1** Let $M_1$ and $M_2$ be two hyperkähler manifolds with a hyperkähler structure of general type. Suppose that $M_1$ and $M_2$ belong to the same deformation class. Then $M_1$ can be obtained from $M_2$ by the sequence of standard operations. ■

11. Projectively hyperholomorphic bundles.

Most facts about hyperholomorphic bundles with connection can be generalized to the case of projectively hyperholomorphic bundles. As usually, $M$ is a compact hyperkähler manifold, and $(M, I)$ is the same manifold, considered as a complex one with induced complex structure $I$.

**Definition 11.1.** Let $B$ be a bundle of rank $r$ with connection and a curvature $\Theta \in \Lambda^2(\text{End}(B))$. Take the 2-form $Tr(\Theta) \in \Lambda^2(\mathbb{C})$. The $\text{End}(B)$-valued 2-form

$$\Theta_{\text{tl}} := \Theta - \frac{1}{r} Tr(\Theta)$$
is called traceless curvature of $B$.

**Definition 11.2.** Let $B$ be a Hermitian holomorphic bundle over $(M, I)$ with a traceless curvature $\Theta_H$. The bundle $B$ is called projectively hyperholomorphic if $\Theta_H$ is $G_M$-invariant. By $G_M$ we mean the isotropy group of $M$, see Definition 1.2.

**Proposition 11.1.** The Hermitian holomorphic bundle $B$ over $M$ is projectively hyperholomorphic if and only if $\text{End}(B)$ is hyperholomorphic as a bundle with connection induced from $B$.

**Proof** (See [Ko] for analogous results:) The curvature $\Theta_1$ of $\text{End}(B)$ maps a $\text{End}(B)$-valued form $\alpha$ to $\alpha \wedge \Theta - \Theta \wedge \alpha$, where $\Theta$ is a curvature form in $B$. This implies $\Theta_1\alpha = \Theta \wedge \alpha - \alpha \wedge \Theta = \Theta_H \wedge \alpha - \alpha \wedge \Theta_H$, because the scalar 2-form $Tr(\Theta)$ commutes with $\alpha$. The last form is $G_M$-invariant if $B$ is projectively hyperholomorphic, and therefore $\text{End}(B)$ is hyperholomorphic if $B$ is projectively hyperholomorphic.

Conversely, assume that $\Theta_1$ is $G_M$-invariant, or, what is the same, that $\text{End}(B)$ is hyperholomorphic. Let $\Theta = \sum_i A_i \omega_i$, where $\omega_i \in \Lambda^2(M)$ and $A_i \in \text{End}(B)$. Take $A \in \Gamma(\text{End}(B))$ as the section of $\Lambda^0(M, \text{End}(B))$. Since $A$ and $\Theta_1$ are $G_M$-invariant differential forms, the form $\Theta_1(A) = \sum_i [A, A_i] \omega_i$ is also $G_M$-invariant. Therefore for any non-$G_M$-invariant $\omega_i$ the commutator $[A, A_i] = 0$ for any $A$. Therefore $A_i = f \cdot Id$ for such $i$, where $f$ is a scalar function, and $\Theta_H$ is $G_M$-invariant. ■

Now, let $B$ be Yang-Mills bundle over $M$. One can apply our version of Bogomolov-Gieseker inequality (5.1) and (5.1A) to $\text{End}(B)$ considered as a (not necessarily holomorphic) bundle with connection over $(M, J)$ and obtain the following result.

**Theorem 11.1** The Yang-Mills bundle $B$ of rank $r$ over $(M, I)$ is projectively hyperholomorphic if and only if the cohomology class $c_2(B) - \frac{r-1}{2r}c_1(B)^2$ is $G_M$-invariant.

This also follows from Theorem 2.5 applied to the bundle $\text{End}(B)$ and Proposition 11.1.

One can apply Proposition 11.1 to prove the following:
Proposition 11.2: If $B$ is Yang-Mills stable bundle over a hyperkähler surface (K3 or abelian surface), then $B$ is projectively hyperholomorphic.

Proof: This follows from Theorem 2.4, and Proposition 11.1. For stable Yang-Mills $B$ the bundle $\text{End}(B)$ satisfy the following. If $\Theta$ is its curvature, then $\Lambda(\Theta) = 0$. Applying the proof of Theorem 2.4 to $\text{End}(B)$, one sees that $\text{End}(B)$ is hyperholomorphic.

Since Theorem 6.2 depends only on $\partial\bar{\partial}$-lemma in cohomology of $\text{End}(B)$ (Theorem 4.3), one immediately obtains its generalization to projectively hyperholomorphic case:

Theorem 11.2: If $B$ is a projectively hyperholomorphic Yang-Mills bundle, the space $\text{Spl}(B)$ locally around the point $[B]$ is isomorphic to the intersection of an open ball in $\text{Ext}^1(B, B)$ with the quadratic cone $\{\rho \in \text{Ext}^1(F, F) | \iota(\rho, \rho) = 0\}$.

To prove the generalization of Theorem 6.3 to the projectively hyperholomorphic bundles, one does the following. Assume that $B$ is projectively hyperholomorphic. Proposition 4.3 supplies the construction of quaternionic action on $T_[B]\text{Spl}(B) \hookrightarrow H^1(\text{End}(B))$. This shows that $S = \text{Spl}(B)$ suffices (i) of Definition 6.5. The quaternionic Hermitian metric on $H^1(\text{End}(B))$ ((iii) of Definition 6.5) is being constructed exactly as in hyperholomorphic case. Moreover, the holomorphic section of $\text{Hom}(TS \otimes TS, \mathcal{O}_S)$ is constructed as in the proof of Theorem 6.3 (Section 9), and one can immediately see that $\text{Spl}(B)$ suffices (iv) of Definition 6.5.

The only possible difficulty one meets is to prove that the for any $L \in \mathbb{H} | L^2 = -1$ the action of $L$ on $TS$ is integrable, i.e., is induced by some complex structure on the variety $S$, considered as the real analytic space. To prove that, one should realize the open neighbourhood $U$ of $[B]$ as an intersection of of the open ball $H^1(\text{End}(B))$ and the quadratic cone $\iota(\eta, \eta) = 0$ as in Theorem 11.2. This cone is quaternionic invariant, so one can induce the quaternionic action on $U$ from $H^1(\text{End}(B))$. This latter action is obviously integrable. Moreover, as one can prove (see [Ko2]), this action is compatible with constructed above canonical symplectic form. This implies that this particular quaternionic action (a priori, dependent on $B$) coincides with the canonical quaternionic action constructed above.

We have proven the following theorem:

Theorem 11.3: For a projectively hyperholomorphic stable bundle $B$ over a hyperkähler manifold $M$, the space $\text{Spl}(B)$ is (singularly) hy-
perkähler.

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