STRICHARTZ INEQUALITY FOR ORTHONORMAL FUNCTIONS ASSOCIATED WITH DUNKL LAPLACIAN-SCHRÖDINGER OPERATOR

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Abstract. Strichartz inequality for the solutions of free Schrödinger equation associated with Dunkl Hermite operator $H_\kappa$ is generalized to any system of orthonormal functions with initial data. A relation between the kernels of Schrödinger equation solution operators associated with the operators Dunkl Hermite and Dunkl Laplacian is established using which corresponding Strichartz inequalities for orthonormal function associated with Dunkl Laplacian are obtained.

1. Introduction and main results

Consider the free Schrödinger equation

\[
\begin{cases}
i u_t(t, x) = -\Delta u(t, x) & x \in \mathbb{R}^d, t \in \mathbb{R} \\ u(0, x) = f(x)
\end{cases}
\]

where $\Delta = \sum_{j=1}^{d} \frac{\partial^2}{\partial x_j^2}$, Laplacian on $\mathbb{R}^d$. $e^{it\Delta} f$ is the unique solution to the initial value problem (1.1). The associated Strichartz inequality reads,

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} |e^{it\Delta} f |^{2q} dx \right)^{\frac{p}{q}} dt \leq C \left( \int_{\mathbb{R}^d} |f(x)|^{2q} dx \right)^{\frac{p}{q}},
\]

where $p, q \geq 1$ satisfy $(p, q, d) \neq (1, \infty, 2)$ and

\[
\frac{2}{p} + \frac{d}{q} = d,
\]

see [32, 33, 31, 29, 33]. Rupert L. Frank et al. in [12] and [13] generalized the above inequality for a system orthonormal functions and the authors proved the following:

Theorem 1.1. Let $d \geq 1$ and $p, q \geq 1$ be such that

\[
\frac{2}{p} + \frac{d}{q} = d, \quad 1 \leq q < \frac{d+1}{d-1}.
\]

Then for any infinite or finite orthonormal system $\{u_j\}$ in $L^2(\mathbb{R}^d)$ and for any sequence $\{n_j\}$ in $\mathbb{C}$, we have

\[
\int_{\mathbb{R}} \left( \int_{\mathbb{R}^d} \left| \sum_j n_j e^{it\Delta} u_j(x) \right|^2 dx \right)^{\frac{q}{p}} dt \leq C_{d,q}(\sum_j |n_j|^{\frac{2q}{p+1}})^{\frac{p(q+1)}{2q}},
\]

where $C_{d,q}$ is a universal constant which only depends on $d$ and $q$.

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In [12], the authors proved that if $q \geq \frac{2d+1}{d-1}$, then the inequality (1.3) doesn’t hold and the exponent on the right hand side of inequality (1.3) is optimal i.e., the exponent $\frac{2q}{q+1}$ can not be increased. It is mentioned in [12] that the functional inequalities involving a large number of orthonormal functions like the inequality (1.3) are very useful in the mathematical analysis of large quantum system. Our main aim of this article to prove the inequality analogues to the inequality (1.3) when we replace Laplacian by Dunkl Laplacian (for the definition, see Section 2). Let $e^{-it\Delta f}$ be the unique solution of the free Schrödinger equation associated with Dunkl Laplacian:

\[
\begin{align*}
  iu_t(t, x) &= -\Delta_\kappa u(t, x) \quad x \in \mathbb{R}^d, t \in \mathbb{R} \\
  u(0, x) &= f(x).
\end{align*}
\]

Then the authors in [2] proved the following Strichartz inequality for the case of Dunkl Laplacian.

\[
\left( \int_\mathbb{R} \left( \int_{\mathbb{R}^d} |(e^{it\Delta_\kappa f})|^2 dx \right)^{\frac{2}{q}} dt \right)^{\frac{q}{2}} \leq C \left( \int_{\mathbb{R}^d} |f(x)|^2 dx \right)^{\frac{p}{2}},
\]

where $d \geq 1$ and $p, q \geq 1$ are such that

\[
\frac{2}{p} + \frac{d + 2\gamma_\kappa}{q} = d + 2\gamma_\kappa.
\]

For all unexplained notations we refer Section 2. In the same spirit of [12] and [13], it is natural to ask the inequality analogues to (1.3) for the case Dunkl Laplacian. Out main result lies on this direction:

**Theorem 1.2** (Strichartz inequality for orthonormal functions for Dunkl Laplacian). Let $d \geq 1$ and $p, q \geq 1$ satisfy

\[
1 \leq q < \frac{d + 2\gamma_\kappa + 1}{d + 2\gamma_\kappa - 1} \quad \text{and} \quad \frac{2}{p} + \frac{d + 2\gamma_\kappa}{q} = d + 2\gamma_\kappa.
\]

Then for any (possible infinite) system $(u_j)$ of orthonormal functions in $L^2_\kappa(\mathbb{R}^d)$ and any coefficients $(n_j) \subset \mathbb{C}$, we have

\[
\left( \int_\mathbb{R} \left( \int_{\mathbb{R}^d} \left| \sum_j n_j (e^{it\Delta_\kappa u_j})(x) \right|^2 dx \right)^{\frac{q}{2}} dt \right)^{\frac{2}{q}} \leq C_{d,q} \left( \sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{p(q+1)}{2q}},
\]

where $C_{d,q}$ is a universal constant which only depends on $d$ and $q$.

**Remark 1.3.** For $q = 1$ and $p = \infty$, we have the bound

\[
\sup_{t \in \mathbb{R}} \left( \int_{\mathbb{R}^d} \left| \sum_j n_j (e^{it\Delta_\kappa f_j})(x) \right|^2 h^2_\kappa(x) dx \right) \leq \sum_j |n_j|
\]

which is an obvious consequences of the triangle inequality and of the fact that $e^{it\Delta_\kappa}$ is a unitary operator on $L^2_\kappa(\mathbb{R}^d)$, for any fixed $t \in \mathbb{R}$.

Further extension of inequality (1.2) has been made for the Schrödinger equation of the form $iu_t(t, x) + \Delta_\kappa u(t, x) - V(x)u(t, x) = 0$, for a suitable potential $V$ by several authors [15, 16, 19]. In particular, when $V(x) = |x|^2$ and $\kappa = 0$, the Strichartz inequalities have been studied in the literature see [20]. It is natural to ask such inequalities are true when
\[ V(x) = |x|^2 \] and \( \Delta \) is replaced by \( \Delta_\kappa \). In this case the initial value problem (1.1) turns out to be an initial value problem for the Schrödinger equation associated with the Dunkl Hermite operator \( H_\kappa = -\Delta_\kappa + |x|^2 \):  

(1.7) \[ \begin{cases} 
  iu_t(t, x) = H_\kappa u(t, x) & x \in \mathbb{R}^d, t \in \mathbb{R} \\
  u(0, x) = f(x). 
\end{cases} \]

The authors in [2] proved the Strichartz inequalities for the above problem. If \( f \in L^2_\kappa(\mathbb{R}^d) \), the solution of the initial value problem (1.7) is given by \( u(t, x) = e^{-itH_\kappa} f(x) \). The Strichartz inequality in this case reads as:

**Theorem 1.4.** [2] Let \( f \in L^2_\kappa(\mathbb{R}^d) \). If \( p, q \geq 1 \) satisfying 

\[ \left( \frac{d + 2\gamma_\kappa - 2}{d + 2\gamma_\kappa} \right) < \frac{1}{q} \leq 1 \text{ and } 1 \leq \frac{1}{p} \leq 2, \]

or

\[ 0 \leq \frac{1}{p} < 1 \text{ and } \frac{2}{p} + \frac{d + 2\gamma_\kappa}{q} \geq (d + 2\gamma_\kappa). \]

Then

(1.8) \[ \| e^{-itH_\kappa} f \|_{L^p_t L^q(\mathbb{T} \times \mathbb{R}^d)} \leq C \| f \|_{L^2_\kappa(\mathbb{R}^d)}. \]

We are interested to extend the inequality (1.8) for orthonormal family of functions. So we proved the following

**Theorem 1.5.** Let \( d \geq 1 \) and \( p, q \geq 1 \) satisfy

\[ 1 \leq q < \frac{d + 2\gamma_\kappa + 1}{d + 2\gamma_\kappa - 1} \text{ and } \frac{2}{p} + \frac{d + 2\gamma_\kappa}{q} = d + 2\gamma_\kappa. \]

Then for any (possible infinite) system \( (u_j) \) of orthonormal functions in \( L^2_\kappa(\mathbb{R}^d) \) and any coefficients \( (n_j) \subset \mathbb{C} \), we have

(1.9) \[ \int_{-\pi}^{\pi} \left( \int_{\mathbb{R}^d} \left| \sum_j n_j \left( e^{-itH_\kappa} u_j \right)(x) \right|^q dx \right)^{\frac{p}{q}} \, dt \leq C_{d, q} \left( \sum_j |n_j|^{\frac{2p}{q+1}} \right)^{\frac{q+1}{q+1}}, \]

where \( C_{d, q} \) is a universal constant which only depends on \( d \) and \( q \).

The idea of proving Theorem 1.5 is motivated by the famous works of Strichartz [32] and Frank-Sabin [13], where the Fourier restriction theorem for some quadratic surface is linked to space time decay estimates for certain evolution equations.

Let \( f \in L^1_\kappa(\mathbb{R}^d) \). Define the Dunkl Hermite transforms of \( f \) by

\[ \mathcal{H}_\kappa f(\mu) := \int_{\mathbb{R}^d} f(x) \phi_\mu^2 h_\mu(x) \, dx, \quad \mu \in \mathbb{N}_0^d \]

where \( \phi_\mu^2 \)'s are the d-dimensional Dunkl Hermite functions. If \( f \in L^2_\kappa(\mathbb{R}^d) \) then \( \{ \mathcal{H}_\kappa f(\mu) \} \in \ell^2(\mathbb{N}_0^d) \) and satisfies the plancheral formula

\[ \| f \|_{L^2(\mathbb{R}^d)}^2 = \sum_{\mu \in \mathbb{N}_0^d} |\mathcal{H}_\kappa f(\mu)|^2. \]
where \( \mathbb{N}_0 \) denotes the set of all non-negative integers. The inverse Dunkl Hermite transform is given by

\[
    f(x) = \sum_{\mu \in \mathbb{N}_0^d} \mathcal{H}_\kappa f(\mu) \phi^\mu_h(x).
\]

Given a discrete surface \( S \) in \( \mathbb{N}_0^d \), we define the restriction operator \( (\mathcal{R}_S f)(x) := \{ \mathcal{H}_\kappa f(\mu) \}_{\mu \in S} \) and the operator dual to \( \mathcal{R}_S \) is called the extension operator defined as \( \mathcal{E}_S(\{g(\mu)\}) := \sum_{\mu \in S} g(\mu) \phi^\mu_h(x) \) and ask the following question. For which exponents \( 1 \leq p \leq 2 \), the sequence of Dunkl Hermite transforms of a function \( f \in L^p_\kappa(\mathbb{R}^d) \) belongs to \( \ell^2(S) \)? This question can be reframed to the boundedness of the operator \( \mathcal{E}_S \) from \( \ell^2(S) \) to \( L^p_\kappa(\mathbb{R}^d) \), where \( p' \) is the conjugate exponent of \( p \) by a duality argument. Since \( \mathcal{E}_S \) is bounded from \( \ell^2(S) \) to \( L^p_\kappa(\mathbb{R}^d) \) if and only if \( T_S := \mathcal{E}_S(\mathcal{E}_S)^* \) is bounded from \( L^2_\kappa(\mathbb{R}^d) \) to \( L^p_\kappa(\mathbb{R}^d) \). This problem in the literature is known as restriction theorem.

Restriction theorems for the Fourier transforms for certain quadratic surfaces \( S \) of \( \mathbb{R}^d \) have been proved by Stein \cite{21} and Strichartz \cite{32} using the Stein’s interpolation theorem \cite{22}. As an application of Holder’s inequality, it can be verified that \( T_S \) is \( L^p - L^p \) bounded if and only if the operator \( W_1 T_S W_2 \) is bounded from \( L^2 \) to \( L^2 \) for any \( W_1, W_2 \in L^{2p'/p} \). Recently Frank and Sabin \cite{13} considered the similar problem for the case of Fourier transform and they proved that for any \( W_1, W_2 \in L^{2p'/p}(\mathbb{R}^d) \), the operator \( W_1 T_S W_2 \) is not only bounded operator on \( L^2 \), but also belongs to a Schatten class (for definitions see Subsection 2.3). The estimate that they proved is

\[
    \| W_1 T_S W_2 \|_{\mathcal{E}^\alpha(L^2(\mathbb{R}^d))} \leq C \| W_1 \|_{L^{2p'/p}(\mathbb{R}^d)} \| W_2 \|_{L^{2p'/p}(\mathbb{R}^d)}
\]

for some \( \alpha > 0 \), where \( C > 0 \) independent of \( W_1 \) and \( W_2 \). Using these kind of estimates along with duality argument (Lemma 3 in \cite{13}), they proved inequality \((1.3)\).

We consider a particular discrete surface \( S \subset \mathbb{N}_0^{d+1} \) with respect to counting measure \( \mu \) on \( S \). Let \( \mathcal{E}_S = e^{-itH_\kappa} \) and define an analytic family of operators \( (T_z) \) defined on the strip \( a \leq \text{Re} z \leq b \) in the complex plane such that \( T_z = T_c \) for some \( c \in (a, b) \). Then we show that the operator \( W_1 T_S W_2 \) belongs to a Schatten class for \( W_1, W_2 \) in suitable mixed norm spaces and applying the duality argument (see Lemma \( 2.1 \)) we obtain the Stricharz inequality \((1.9)\) for the system of orthonormal function for Dunkl Hermite operator \( H_\kappa \). In order to prove Theorem \( 1.2 \), we will establish a relation between Schrödinger kernels associated with Dunkl Laplacian and Dunkl Hermite operator using which along with Theorem \( 1.5 \) we obtain the inequality \((1.6)\).

The paper is organized as follows: We give introduction and main results in Section 1. In Section 2 we recall the basic ingredients in Dunkl setting including Dunkl Hermite functions and their generating function identity and Schatten class of operators. We also state duality statements. We will establish necessary theorems and proposition to give the proofs of our main results Theorem \( 1.2 \), \( 1.5 \) and Theorem \( 1.6 \) in Section 3. As an application of inhomogeneous Strichartz inequality, we prove the global well-posedness for the Dunkl Hermite-Hartree equation in Schatten spaces in Section 4. Finally, in Section 5 we will give an alternate proof of Theorem \( 1.2 \) for \( 1 \leq q < 1 + \frac{2}{d+2\gamma_\kappa} \).

1.1. Dual Strichartz inequality. In order to prove Theorem \( 1.5 \) we prove that \((1.9)\) or to prove Theorem \( 1.2 \) we prove the inequality \((1.6)\) is equivalent to a dual inequality and then we proved the dual inequality.
One can verify that the operator
\[ V \in L_t' L_{\kappa,x}^q(X \times \mathbb{R}^d) \mapsto \int_{\mathbb{R}} e^{itP}V(t, x)e^{-itP} dt \in S_{2q'} \]
is the dual of the operator \( \gamma \in S_{\frac{2d}{q'+1}} \mapsto \rho_{\gamma(t)}(x) \in L_t^2 L_{\kappa,x}^q(X \times \mathbb{R}^d) \), where \( X = [-\pi, \pi] \) when \( P = H_\kappa \) and \( X = \mathbb{R} \) when \( P = \Delta_\kappa \).

Therefore the duality argument shows that Theorem 1.5 and Theorem 1.2 is equivalent to the following:

**Theorem 1.6.** (Strichartz inequality in Schatten spaces, dual version) Assume that \( p', q', d \geq 1 \) satisfy
\[
1 + \frac{d + 2\gamma_\kappa}{2} \leq q < \infty \quad \text{and} \quad \frac{2}{p'} + \frac{d + 2\gamma_\kappa}{q'} = 2.
\]

We have
\[
(1.10) \quad \left\| \int_{\mathbb{R}} e^{itP}V(t, x)e^{-itP} dt \right\|_{S_{2q'}} \leq C_{d,q} \| V \|_{L_t^p L_{\kappa,x}^q(X \times \mathbb{R}^d)}
\]
where \( C_{d,q} \) is the same constant as in Theorem 1.2, \( X = [-\pi, \pi] \) when \( P = H_\kappa \) and \( X = \mathbb{R} \) when \( P = \Delta_\kappa \).

Remark 1.7. For \( q' = \infty \), and \( p' = 1 \), we have the inequality
\[
\left\| \int_{\mathbb{R}} e^{itP}V(t, x)e^{-itP} dt \right\| \leq \| V \|_{L_t^1 L_{\kappa,x}^\infty(X \times \mathbb{R}^d)}
\]

Using (1.10), we prove an inhomogeneous Strichartz inequality. Consider the equation given by
\[
(1.11) \quad \begin{cases}
    i\dot{\gamma}(t) = [-H_\kappa, \gamma(t)] + iR(t) \\
    \gamma(t_0) = 0,
\end{cases}
\]
where \( R(t) \) is a self adjoint operator on \( L^2(\mathbb{R}^d) \) and is bounded for almost every \( t \). the solution of the system (1.11) can be written as
\[
(1.12) \quad \gamma(t) = \int_{t_0}^t e^{i(t-s)H_\kappa} R(s)e^{i(s-t)H_\kappa} ds.
\]

We obtain the following inhomogeneous Strichartz inequality.

**Corollary 1.8.** (Inhomogeneous Strichartz inequality) Assume that \( p, q, d \geq 1 \) satisfy
\[
1 \leq q < \frac{d + 2\gamma_\kappa + 1}{d + 2\gamma_\kappa - 1} \quad \text{and} \quad \frac{2}{p} + \frac{d + 2\gamma_\kappa}{q} = d + 2\gamma_\kappa
\]
and let \( \gamma(t) \) be given by (1.12). Then
\[
\| \rho_{\gamma(t)} \|_{L_t^p L_{\kappa,x}^q(T \times \mathbb{R}^d)} \leq \left\| \int_{-\pi}^\pi e^{i(s)H_\kappa} |R(s)| e^{-i(s)H_\kappa} ds \right\|_{S_{\frac{2d}{q'+1}}}
\]
for a constant \( C \) which is independent of \( t_0 \).
Proof. Using the Duality argument, For a trace -class operator $\gamma$ and any bounded function $V$ of compact support,

$$\text{Tr}(V(x)\gamma) = \int_{\mathbb{R}^d} V(x)\rho_{\gamma}(x)dx,$$

where $V(x)$ is identified with the corresponding multiplication operator on $L^2(\mathbb{R}^d)$. And for a time-dependent potential $V(t, x) \in L^2([\pi, \pi] \times \mathbb{R}^d)$, we have

$$\left|\int_{t_0}^{t} \int_{\mathbb{R}^d} V(t, x)\rho_{\gamma}(t,x)dxdt\right| = \left|\int_{t_0}^{t} \int_{t_0}^{t} \text{Tr}(e^{itH_s}V(t, x)e^{-itH_s}e^{isH_s}R(s)e^{-isH_s})dsdt\right|$$

$$\leq \int_{t_0}^{t} \int_{t_0}^{t} \text{Tr}(e^{itH_s}|V(t, x)|e^{-itH_s}e^{isH_s}|R(s)e^{-isH_s})dsdt$$

$$\leq \text{Tr}\left(\left(\int_{t_0}^{t} e^{itH_s}|V(t, x)|e^{-itH_s}\right)\left(\int_{t_0}^{t} e^{isH_s}|R(s)e^{-isH_s}\right)\right).$$

In the first inequality we have used the fact that $|\text{Tr}(AB)| \leq Tr(|A||B|)$ for self-adjoint operators $A$ and $B$. Then applying the Hölder’s inequality for traces and (1.10) for the term involving $V(t, x)$. This completes the proof. □

2. Preliminaries

In this section we will define Dunkl operators, some other related operators and function spaces which will be used in this paper. Dunkl operators were introduced by Charles Dunkl (1989) to built a framework for a theory of special functions and integral transforms in several variables related to reflection groups. Such operators are relevant in physics, namely for the analysis of quantum many body systems of Calogero-Moser-Sutherland type (see [7, 30]). From the mathematical analysis point of view, the importance of Dunkl operators lies on the fact that they generalize the theory of symmetric spaces of Euclidean type. There are many developments in harmonic analysis of the operators which are defined in terms of Dunkl operators in different directions in recent years. There is a vast literature related to Dunkl operators, see for instance [2, 3, 8, 9, 11, 24, 25].

2.1. The general Dunkl setting. The basic ingredients in the theory of Dunkl operators are the root systems and finite reflection groups associated to them. For $\nu \in \mathbb{R}^d \setminus \{0\}$, we denote by $\sigma_{\nu}$ the reflection in the hyperplane perpendicular to $\nu$, i.e.,

$$\sigma_{\nu}(x) = x - 2\frac{\langle \nu, x \rangle}{|\nu|^2}\nu.$$

Let $O(d)$ be the group of orthogonal matrices acting on $\mathbb{R}^d$. Given a root system $R$ associate a finite subgroup $G \subset O(d)$ is the reflection group which is generated by the reflections $\{\sigma_{\nu} : \nu \in R\}$.

A function $\kappa : R \to \mathbb{C}$ is said to be a multiplicity function on $R$, if it is invariant under the natural action of $G$ on $R$, i.e. $\kappa(g\nu) = \kappa(\nu)$ for all $\nu \in R$ and $g \in G$.

Every root system can be written as a disjoint union $R = R_+ \cup (-R_+)$, where $R_+$ and $-R_+$ are separated by a hyperplane through the origin. Such $R_+$ is called the set of all positive roots in $R$. Of course its choice is not unique.
The weight function associated to the root system $R$ and the multiplicity function $\kappa$ is defined by

$$h_\kappa^2(x) := \prod_{\nu \in R_+} |\langle x, \nu \rangle|^{2\kappa(\nu)}.$$  

Note that $h_\kappa^2(x)$ is $G$-invariant and homogenous of degree $2\gamma_\kappa$ where, by definition,

$$\gamma_\kappa := \sum_{\nu \in R_+} \kappa(\nu).$$

We will assume throughout the article $2\gamma_\kappa$ is a non negative integer. Let $L^p_\kappa(\mathbb{R}^d)$, $1 \leq p < \infty$, stands for the space of $L^p$- functions with respect to the measure $h_\kappa^2(x)dx$. $L^p_\kappa L^q_\kappa(\mathbb{T} \times \mathbb{R}^d)$, $1 \leq p, q \leq \infty$ stands for the space of all measurable functions $h(t, x)$ on $\mathbb{T} \times \mathbb{R}^d$ for which

$$\|h\|_{L^p_\kappa L^q_\kappa(\mathbb{T} \times \mathbb{R}^d)} := \|h(t, \cdot)\|_{L^q(\mathbb{R}^d)} < \infty.$$  

And $L^p_\kappa L^q_\kappa(\mathbb{T} \times \mathbb{R}^d)$, $1 \leq p, q \leq \infty$ stands for the space of all measurable functions $h(t, x)$ on $\mathbb{T} \times \mathbb{R}^d$ for which

$$\|h\|_{L^p_\kappa L^q_\kappa(\mathbb{T} \times \mathbb{R}^d)} := \|h(t, \cdot)\|_{L^q(\mathbb{T} \times \mathbb{R}^d)} < \infty.$$  

We may consider $\mathbb{T} = (-\pi, \pi)$, $(-\pi/4, \pi/4)$, $(0, \pi/4)$, $(0, \infty)$, $\mathbb{R}$ or any interval in $\mathbb{R}$ with Lebesgue measure.

Now we define the difference-differential operators which were introduced and studied by C. F. Dunkl (for $\kappa \geq 0$) see [8, 9]. These operators are also called Dunkl operators and are analogues (generalization) of directional derivatives. We fix a root system $R$ with a positive subsystem $R_+$ and the associated reflection group $G$. We also fix a nonnegative multiplicity function $\kappa$ defined on $R$.

For $\xi \in \mathbb{R}^d$, the Dunkl operator $T_\xi := T_\xi(\kappa)$ is defined by

$$T_\xi f(x) = \partial_\xi f(x) + \sum_{\nu \in R_+} \kappa(\nu) \langle \nu, \xi \rangle f(x) - f(\sigma_\nu x) \langle \nu, x \rangle$$

for smooth functions $f$ on $\mathbb{R}^d$. Here $\partial_\xi$ denotes the directional derivative along $\xi$. For the standard coordinate vectors $\xi = e_j$ of $\mathbb{R}^d$ we use the abbreviation $T_j = T_{e_j}$.

Let $P$ be the space of all polynomials with complex coefficients in $d$-variables and $P_m$ the subspace of homogeneous polynomials of degree $m$. The Dunkl-operators $T_\xi$ and directional derivatives $\partial_\xi$ are closely related and intertwined by an isomorphism on $P$. Indeed, if the multiplicity function $\kappa$ is nonnegative then by Theorem 2.3 and Proposition 2.3 in Rösler [25], there exist a unique linear isomorphism (intertwining operator) $V_\kappa$ of $P$ such that $V_\kappa(P_m) = P_m$, $V_\kappa|_{P_0} = id$ and $T_\xi V_\kappa = V_\kappa \partial_\xi$ for all $\xi \in \mathbb{R}^d$. It can be checked that $V_\kappa g = g \circ V_\kappa$ for all $g \in G$.

For $y \in \mathbb{C}^d$, define

$$E_\kappa(x, y) := V_\kappa \left( e^{-(\cdot, y)} \right)(x), \ x \in \mathbb{R}^d.$$  

The function $E_\kappa$ is called the Dunkl-kernel, or $\kappa$-exponential kernel, associated with $G$ and $\kappa$, see [10].

The Dunkl-Laplacian is the second order operator defined by

$$\Delta_\kappa = \sum_{j=1}^d T_j^2$$

which can be explicitly calculated, see Theorem 4.4.9 in Dunkl-Xu [11].
It can be seen that $\Delta_\kappa = \sum_{j=1}^d T_{\xi_j}^2$ for any orthonormal basis $\{\xi_1, \xi_2, \ldots, \xi_d\}$ of $\mathbb{R}^d$, see [9]. We also recall the definition of heat kernel, see [26]. We are considering the initial value problem for heat equation associated to the Dunkl Laplacian,

$$u_t(t, x) = \Delta_\kappa u(t, x), \quad u(0, x) = f(x), \quad t > 0, \quad x \in \mathbb{R}^d.$$ 

For $f \in L_p^0(\mathbb{R}^d)$ the solution of this equation is given by

$$u(t, x) = \int_{\mathbb{R}^d} \Gamma_\kappa(t, x, y) f(y) h_\kappa^2(y) dy$$

Where the heat kernel $\Gamma_\kappa$ associated to the Dunkl Laplacian is explicitly given by

$$\Gamma_\kappa(t, x, y) = \frac{M_\kappa}{(2t)^{\gamma_\kappa + d/2}} e^{-\frac{|x|^2 + |y|^2}{4t}} E_\kappa\left(\frac{x}{2t}, y\right),$$

where $M_\kappa = \left(\int_{\mathbb{R}^d} e^{-\frac{|x|^2}{4} h_\kappa^2(x) dx}\right)^{-1}$. The kernel of the operator $e^{-it\Delta_\kappa}$ is given by

$$L_{it, \kappa}(x, y) = \frac{M_\kappa}{(2it)^{\gamma_\kappa + d/2}} e^{-\frac{\frac{d}{2} |x|^2 + |y|^2}{4it}} E_\kappa\left(\frac{x}{2it}, y\right).$$

2.2. Dunkl harmonic oscillator. The Dunkl harmonic oscillator (which we also call the Dunkel Hermite operator) is defined by

$$H_\kappa = -\Delta_\kappa + |x|^2.$$

Let $\mathbb{N}_0$ be the set of all non-negative integers and for the multi index $\mu = (\mu_1, ..., \mu_d) \in \mathbb{N}_0^d$ Rósler introduced eigenfunctions $\phi_\mu^\kappa$ of Dunkl harmonic oscillator $H_\kappa$ in [26] with eigenvalue $(2|\mu| + d + 2\gamma_\kappa)$, where $|\mu| = \mu_1 + \cdots + \mu_d$, i. e.,

$$H_\kappa \phi_\mu^\kappa = (2|\mu| + d + 2\gamma_\kappa) \phi_\mu^\kappa,$$

and they form a complete orthonormal basis in $L^2(\mathbb{R}^d, h_\kappa^2(x) dx)$.

These functions $\phi_\mu^\kappa$ are called the Dunkel Hermite functions (also called generalized Hermite functions) and they satisfy the following generating function identity:

$$\sum_{\mu \in \mathbb{N}_0^d} \phi_\mu^\kappa(x) \phi_\mu^\kappa(y) u^{\mu_1} = 2^{d+\gamma_\kappa} M_\kappa (1 - w^2)^{-(d+\gamma_\kappa)/2} e^{-\frac{1}{2} \left(\frac{1+w^2}{1-w^2}\right) (|x|^2 + |y|^2)} E_\kappa\left(\frac{2wx}{1-w^2}, y\right)$$

For the $Re(z) > 0$, we consider the kernel,

$$K_z(x, y) = \sum_{\mu \in \mathbb{N}_0^d} e^{-z(2|\mu| + d + 2\gamma_\kappa)} \phi_\mu^\kappa(x) \phi_\mu^\kappa(y)$$

$$= e^{-(d+2\gamma_\kappa)z} \sum_{\mu \in \mathbb{N}_0^d} e^{-2|\mu|z} \phi_\mu^\kappa(x) \phi_\mu^\kappa(y).$$

In the view of Meheler’s formula (2.3) for $w = e^{-z}$, we will get

$$K_z(x, y) = \frac{M_\kappa}{(\sinh 2z)^{d+\gamma_\kappa}} e^{-\frac{1}{2} \coth 2z (|x|^2 + |y|^2)} E_\kappa\left(\frac{x}{\sinh 2z}, y\right)$$
Let $z = r + it$ with $r \to 0^+$, we will get the kernel of the operator $e^{-itH_{\kappa}}$:

$$K_{it}(x, y) = \frac{M_{\kappa}}{(i \sin 2t)^{\frac{d}{2} + \gamma_{\kappa}}} e^{\frac{1}{2} \cot 2t(|x|^2 + |y|^2)} E_{\kappa} \left( \frac{x}{i \sin 2t}, y \right)$$

(2.4)

It can be easily verified that the following:

(2.5) $K_{-it}(x, y) = \overline{K_{it}(x, y)}$ and $K_{i(t+\frac{\pi}{2})}(x, y) = (-1)^{\frac{d}{2} + \gamma_{\kappa}} K_{it}(-x, y)$.

for all $t \in \mathbb{R} \setminus \left( \frac{\pi}{2} \right) \mathbb{Z}$.

### 2.3. Schatten Spaces.

Let $\gamma$ be a compact operator on $L^2_{\kappa}(\mathbb{R}^d)$. We say that $\gamma \in \mathcal{S}^p = \mathcal{S}^p(L^2_{\kappa}(\mathbb{R}^d))$, Schatten space, for $1 \leq p < \infty$, if $Tr|\gamma|^p < \infty$, where $|\gamma| = \sqrt{\gamma^{*} \gamma}$ and $Tr(\gamma)$ is trace of $\gamma$. For $\gamma \in \mathcal{S}^p$, the Schatten $p$-norm of $\gamma$ is defined by

$$\|\gamma\|_{\mathcal{S}^p} = (Tr|\gamma|^p)^{\frac{1}{p}}.$$ 

It can be verified that $\|\gamma\|_{\mathcal{S}^p} = \left( \sum_j |\lambda_j|^p \right)^{\frac{1}{p}}$, where $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n \geq \cdots \geq 0$ are the singular values of $\gamma$, that is the eigenvalues of $|\gamma| = \sqrt{\gamma^{*} \gamma}$.

When $p = 2$, the Schatten $p$-norm coincides with the Hilbert-Schmidt norm and when the operator is given by an integral kernel, this coincides with the $L^2_{\kappa}(\mathbb{R}^d \times \mathbb{R}^d)$ norm of the kernel. Also when $p = \infty$, we define $\|\gamma\|_{\mathcal{S}\infty}$ to be the operator norm of $\gamma$ on $L^2_{\kappa}(\mathbb{R}^d)$. The following lemma plays an important role in proving the main results.

**Lemma 2.1.** *(Duality principle)* Let $p, q \geq 1$ and $\alpha \geq 1$. Let $A f (x, t) = e^{-itH_{\kappa}} f(x)$. Then the following statements are equivalent.

1. There is a constant $C > 0$ such that

$$\|W A A^* W\|_{\mathcal{S}^\alpha \left( L^2_{\kappa, x}(\mathbb{T} \times \mathbb{R}^d) \right)} \leq C \|W\|^2_{L^p_{\kappa} L^{q'}_{\kappa, x}(\mathbb{T} \times \mathbb{R}^d)}$$

(2.6)

for all $W \in L^2_{\kappa} \left( (\pi, \pi), L^2_{\kappa}(\mathbb{R}^d) \right)$, where the function $W$ is interpreted as an operator which acts by multiplication.

2. For any orthonormal system $(f_j)_{j \in J}$ in $\mathcal{H}$ and any sequence $(n_j)_{j \in J} \subset \mathbb{C},$

$$\left\| \sum_{j \in J} n_j |A f_j| \right\|_{L^p_{\kappa} L^q_{\kappa, x}(\mathbb{T} \times \mathbb{R}^d)} \leq C' \left( \sum_{j \in J} |n_j|^{\alpha'} \right)^{1/\alpha'},$$

(2.7)

where $C'$ is a constant.

Let $S$ be the discrete surface $S = \{(\mu, \nu) \in \mathbb{N}_0^d \times \mathbb{N}_0 : \nu = 2|\mu| + d + 2\gamma_{\kappa}\}$ with respect to the counting measure. Then for all $f \in L^1(S)$ and for all $(x, t) \in \mathbb{R}^d \times [-\pi, \pi]$, the extension operator can be written as

$$\mathcal{E}_S f(x, t) = \sum_{\mu, \nu \in S} \hat{f}(\mu, \nu) \phi^\kappa_{\mu}(x) e^{-it\nu},$$

(2.8)

where $\hat{f}(\mu, \nu) = \int_{\mathbb{R}^d} \int_{\mathbb{T}} f(x, t) \phi^\kappa_{\mu}(x) e^{ixt} d\mu e^{ith^2_{\kappa}(x)} dx$. Choosing

$$\hat{f}(\mu, \nu) = \begin{cases} \hat{u}(\mu) & \text{if } \nu = 2|\mu| + d + 2\gamma_{\kappa}, \\ 0 & \text{otherwise}, \end{cases}$$

we have

$$\mathcal{E}_S f(x, t) = \sum_{\mu, \nu \in S} \hat{u}(\mu) \phi^\kappa_{\mu}(x) e^{-it\nu}.$$
for any suitable $u : \mathbb{R}^d \to \mathbb{C}$. We will get

$$E_\mathcal{S}f(x,t) = \sum_{\mu, \nu \in S} \hat{f}(\mu, \nu) \phi^\mathcal{S}_\mu(x)e^{-it\nu}$$

$$= \int_{\mathbb{R}^d} \left( \sum_{\mu} \phi^\mathcal{S}_\mu(x) \phi^\mathcal{S}_\mu(y)e^{-i(u|\mu|+d+2\gamma_\alpha)} \right) u(y)h^2_\mathcal{S}(y)dy$$

$$= e^{-itH_\kappa}u(x).$$

For any compact operator $\gamma$ on $L^2(\mathbb{R}^d)$ we define

$$\gamma(t) := e^{-itP}\gamma e^{itP}, \ t \in \mathbb{R},$$

where $P = -\Delta_\kappa$ or $H_\kappa$.

If the operator $\gamma$ is of the form $\gamma := \sum_j n_j |f_j\rangle\langle f_j|$ associated with a given orthonormal system $(f_j)_j$, where $|f\rangle\langle g|$ is Dirac’s notation for the rank one operator $\phi \mapsto \langle \phi, g \rangle f$, for such $\gamma$ one may check that

$$\rho_\gamma(x) := \sum_j n_j |e^{-itP}f_j(x)|^2$$

Using this relation, the inequality (1.6) in Theorem 1.2 or the inequality (1.9) in Theorem 1.5 can also be written as

$$(2.9) \quad \|\rho_\gamma(x)\|_{L^p} \leq C_{d,q} \|\gamma\|_{\mathcal{S}_{\mathcal{S}}}$$

where $\|\gamma\|_{\mathcal{S}_{\mathcal{S}}} = \left( \sum_j |n_j|^{\frac{2q}{q+1}} \right)^{\frac{q+1}{q}}$, in the case, $X = [-\pi, \pi]$, then $P$ is $H_\kappa$ and $X = \mathbb{R}$, then $P$ is $\Delta_\kappa$. By Lemma 2.1 (2.6) and (2.7) are equivalent to the following bound: for any $\gamma \in \mathcal{S}_\alpha(L^2_{\mathcal{S},x}(\mathbb{R}^d))$ we have

$$\|\rho_\gamma\|_{L^p} \leq C_{\alpha} \|\gamma\|_{\mathcal{S}_\alpha(L^2_{\mathcal{S},x}(\mathbb{R}^d))}.$$}

### 3. Proofs

**3.1. Proof of Theorem 1.2.** In this subsection we will give the proof of Theorem 1.2 by assuming Theorem 1.3 and we will prove Theorem 1.3 in Subsection 3.2. In order to prove Theorem 1.2 we first establish the relation between the kernels of $e^{it\Delta_\kappa}$ and $e^{-itH_\kappa}$ then the proof follows from Theorem 1.3.

**Lemma 3.1.** Let $K_{it}(x,y)$ and $L_{it}(x,y)$ be the same as in (2.4) and (2.2) which are the kernels of $e^{it\Delta_\kappa}$ and $e^{-itH_\kappa}$ respectively. For $v > 0$ we have

$$(3.1) \quad K_{\frac{\tan^{-1}v}{2}}(x,y) = (1 + v^2)^{\frac{d+2\gamma_\alpha}{2}} e^{-\frac{iv}{2} |x|^2} L_{it/2}(x\sqrt{1 + v^2}, y).$$

**Proof.** Let $v = \tan 2t$ for $0 < t < \frac{\pi}{4}$ then $t = \frac{1}{2} \tan^{-1}v$ and $\cot(2t) = \frac{1}{v}$ and $\sin(2t) = \frac{v}{\sqrt{1+v^2}}$ which implies

$$(3.2) \quad K_{\frac{\tan^{-1}v}{2}}(x,y) = M_\kappa \left( \frac{\sqrt{1+v^2}}{iv} \right)^{\frac{d+2\gamma_\alpha}{2}} e^{\frac{iv}{2} |x|^2} E_\kappa \left( \frac{x\sqrt{1+v^2}}{iv}, y \right)$$

and

$$(3.3) \quad L_{\frac{t}{2}}(x\sqrt{1+v^2}, y) = M_\kappa \left( \frac{1}{iv} \right)^{\frac{d+2\gamma_\alpha}{2}} e^{\frac{iv}{2} |x|^2} E_\kappa \left( \frac{x\sqrt{1+v^2}}{iv}, y \right).$$
In view of (3.2) and (3.3) we get our required identity which completes the proof of the lemma. \(\square\)

Now we are ready to prove Theorem 1.2. It can be verified using Lemma 3.1 that for \(p, q \geq 1\)
\[
(3.4) \quad \left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|^p_{L^p_t L^q_s((0,\infty),\mathbb{R}^d)} = \left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|^p_{L^p_t L^q_s((0,\infty),\mathbb{R}^d)}.
\]
For any orthonormal family of functions \(\{u_j\}\) in \(L^2_\kappa(\mathbb{R}^d)\) and \((n_j) \subset \mathbb{C}\), let us define
\[
\varphi_{\{u_j\}}(t) := \left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|_{L^p_t L^q_s(\mathbb{R}^d)},
\]

then the identity (3.4) can be written as
\[
(3.5) \quad \int_0^{\pi/2} |\varphi_{\{u_j\}}(t)|^p dt = \left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|^p_{L^p_t L^q_s((0,\infty),\mathbb{R}^d)}.
\]
Using the properties of kernel \(K_{it}(x,y)\) given in (2.5) one can show that \(\varphi_{\{u_j\}}\) satisfies the following:
\[
(3.6) \quad \varphi_{\{u_j\}}(t + \pi/2) = \varphi_{\{u_j\}}(t) \quad \text{and} \quad \varphi_{\{u_j\}}(-t) = \varphi_{\{\pi_j\}}(t).
\]
Using (3.6) and (3.5), it can be easily proved that
\[
(3.7) \quad \left( \int_{-\pi}^{\pi} |\varphi_{\{u_j\}}(t)|^p dt \right) = 4 \int_{-\pi/4}^{\pi/4} |\varphi_{\{u_j\}}(t)|^p dt = \int_{-\infty}^{\infty} \left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|^p_{L^p_t L^q_s(\mathbb{R}^d)} dt.
\]
Thus we establish
\[
\left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|^p_{L^p_t L^q_s((-\pi,\pi),\mathbb{R}^d)} = \left\| \sum_j n_j |e^{-it\Delta} u_j|^2 \right\|^p_{L^p_t L^q_s(\mathbb{R}^d)}.
\]
In view of the above identity Theorem 1.2 follows from Theorem 1.5.

3.2. Proof of Theorem 1.5 and Theorem 1.6. Our strategy of the proof of the Theorem 1.5 is as follows: We first prove the estimate (1.10) in Theorem 1.6 for the exponents \((p', q') = (1, \infty)\) and \((p', q') = (1 + \frac{d+2\gamma}{2}, 1 + \frac{d+2\gamma}{2})\) (diagonal case) and then apply complex interpolation theorem to complete the proof of Theorem 1.6. See Theorem 3.3 (diagonal case) for proof of the estimate (1.10) for the exponents \((p', q') = (1 + \frac{d+2\gamma}{2}, 1 + \frac{d+2\gamma}{2})\).

Using the duality argument we get the inequality (1.9) from Theorem 1.6 for the range of the exponent \(q, 1 < q \leq 1 + \frac{d+2\gamma}{2}\). Then we make use of Theorem 3.3 and Lemma 2.1 to prove the inequality (1.9) for the remaining range of the exponent \(q\) i.e., \(1 + \frac{2}{d+2\gamma} < q < \frac{d+2\gamma+1}{d+2\gamma-1}\) which would complete the proof of Theorem 1.5.

The Schatten norm estimates for the operator \(W_1 T_5 W_2\) which are useful in proving Theorem 3.3 and Theorem 3.4 are proved using the following proposition. For the proof of the following proposition we refer the reader to see Proposition 1 in [13] with appropriate modifications.
Proposition 3.2. Let \((T_z)\) be an analytic family of operators on \(\mathbb{T} \times \mathbb{R}^d\) in the sense of Stein defined on the strip \(-\lambda_0 \leq \text{Re} z \leq 0\). Assume that we have the following bounds
\[
\|T_{iz}\|_{L^2(\mathbb{T} \times \mathbb{R}^d) \to L^2(\mathbb{T} \times \mathbb{R}^d)} \leq M_0 e^{a|z|},
\]
for all \(s \in \mathbb{R}\), for some \(a, b \geq 0\) and for some \(M_0, M_1 \geq 0\). Then, for all \(W_1, W_2 \in L^{2\lambda_0}(\mathbb{T} \times \mathbb{R}^d, \mathbb{C})\) the operator \(W_1 T_{-1} W_2\) belongs to \(\mathcal{S}^{2\lambda_0}(L^2(\mathbb{T} \times \mathbb{R}^d))\) and we have the estimate
\[
\|W_1 T_{-1} W_2\|_{\mathcal{S}^{2\lambda_0}(L^2(\mathbb{T} \times \mathbb{R}^d))} \geq M_0^{1-\frac{1}{\lambda_0}} M_1^{\frac{1}{\lambda_0}} \|W_1\|_{L^2(\mathbb{T} \times \mathbb{R}^d)} \|W_2\|_{L^2(\mathbb{T} \times \mathbb{R}^d)}.
\]

To define the analytic family of operators, we first consider the generalized functions \(\kappa(x, y, t)\) and \(\phi(\mu, \nu)\). Assume that we have the following
\[
G_z(\mu, \nu) = \frac{1}{\Gamma(z + 1)} (\nu - (2|\mu| + d + 2\gamma_\kappa))^z, \quad \text{where } x^z_+ = \begin{cases} x^z & \text{for } x > 0 \\ 0 & \text{for } x \leq 0. \end{cases}
\]
For Schwartz class functions \(\phi \) on \(\mathbb{N}^{d+1}_0\), we have \((G_z, \phi) = \lim_{\nu \to 0} \frac{1}{\Gamma(z + 1)} \sum_{\mu, \nu} \phi(\mu, \nu)(\nu - (2|\mu| + d + 2\gamma_\kappa))^z = \sum_{(\mu, \nu) \in \mathbb{C}^{d+1}} \phi(\mu, \nu).\) For the distributional calculus (see [14]) of \((\nu - (2|\mu| + d + 2\gamma_\kappa))^z\) we have
\[
G_{-1} = \delta_S. \quad \text{Define the analytic family of operators } T_z \text{ by}
\]
\[
T_z g(x, t) = \sum_{\mu, \nu} \hat{g}(\mu, \nu) G_z(\mu, \nu) \phi^\kappa(\mu) e^{-it\nu},
\]
Then
\[
T_z g(x, t) = \int_{\mathbb{R}^d} (K_z(x, y, \cdot) \ast g(y, \cdot))(t) dy,
\]
where \(K_z(x, y, t) = \sum_{\mu, \nu} \phi^\kappa(\mu) \phi^\kappa(\nu) G_z(\mu, \nu) e^{-it\nu} \ast \phi^\kappa(\mu) \phi^\kappa(\nu)\). Using the definition of \(G_z(\mu, \nu)\) we have
\[
K_z(x, y, t) = \frac{1}{\Gamma(z + 1)} \sum_{\mu} \phi^\kappa(\mu) \phi^\kappa(\nu) e^{-it(2|\mu| + d + 2\gamma_\kappa)} \sum_{k=0}^{\infty} k^z_+ e^{-itk}.
\]
By using (2.4), we get
\[
K_z(x, y, t) = \frac{1}{\Gamma(z + 1)} K_{it}(x, y) \sum_{k=0}^{\infty} k^z_+ e^{-itk},
\]
where \(\sum_{k=0}^{\infty} k^z_+ e^{-itk} \sim (z + 1)(it)^{-z-1} + b(t)\) for some \(b \in C^\infty[-\pi, \pi]\), see Proposition 4.1 in [20].

Theorem 3.3. (Diagonal Case) Let \(d \geq 1\). Then for any (possibly infinite) system \(\{u_j\}\) of orthonormal functions in \(L^2(\mathbb{R}^d)\) and any \(\{u_j\} \subset \mathbb{C}\), we have
\[
\left\| \sum_j |n_j| e^{-itH_{2\gamma_\kappa} u_j|^2} \right\|_{L^{1+\frac{2}{d+2\gamma_\kappa}}\mathbb{R}^d} \leq C \left( \sum_j |n_j|^{\frac{d+2\gamma_\kappa+2}{d+2\gamma_\kappa+2}} \right)^\frac{d+2\gamma_\kappa+1}{d+2\gamma_\kappa+2}.
\]

Proof. To prove (3.15), in view of (3.7), it is enough to prove that
\[
\left\| \sum_j |n_j| e^{-itH_{2\gamma_\kappa} u_j|^2} \right\|_{L^{1+\frac{2}{d+2\gamma_\kappa}}\mathbb{R}^d} \leq C \left( \sum_j |n_j|^{\frac{d+2\gamma_\kappa+2}{d+2\gamma_\kappa+2}} \right)^\frac{d+2\gamma_\kappa+1}{d+2\gamma_\kappa+2}.
\]
In order to prove \((3.16)\), using Lemma 2.1, it is enough to prove that
\[
(3.17) \quad \|W_1T_{\lambda}W_2\|_{L_t^{d+2\gamma_\kappa+2}(\mathbb{R}^d)} \leq C\|W_1\|_{L_t^{d+2\gamma_\kappa+2}(\mathbb{R}^d)}\|W_2\|_{L_t^{d+2\gamma_\kappa+2}(\mathbb{R}^d)}
\]
for all \(W_1, W_2 \in L_t^{d+2\gamma_\kappa+2}(\mathbb{R}^d)\), where \(S = \{((\mu, \nu) \in \mathbb{N}_0^d \times \mathbb{N}_0^d : \nu = 2|\mu| + d + 2\gamma_\kappa\}\). We now show that the family of operator \((T_{\lambda})\) defined in \((3.11)\) satisfies \((3.8)\). When \(Re(z) = 0,\) in view of \((3.10)\), we have
\[
(3.18) \quad \|T_{\lambda}g\|_{L_t^2((\mathbb{R}^d))} \leq \|g\|_{L_t^2((\mathbb{R}^d))}, \quad \text{for all } t \in (-\pi, \pi).
\]
When \(z = -\lambda_0 + is\), in view of definition of \(T_{\lambda}\), see \((3.11)\), we have
\[
|T_{\lambda}g(x, y, t)| \leq \left(\sup_{t \in (-\pi, \pi)} |K_{\lambda}(x, y, t)|\right)\|g\|_{L_t^2((\mathbb{R}^d))}, \quad \text{for all } t \in (-\pi, \pi).
\]
Therefore, \(T_{\lambda}\) is bounded from \(L_t^1((\mathbb{R}^d))\) to \(L_t^\infty((\mathbb{R}^d))\) if and only if \(|K_{\lambda}(x, y, t)|\) is bounded on \((\mathbb{R}^d)\) and in this case
\[
(3.19) \quad \|T_{-\lambda_0 + is}\|_{L_t^1((\mathbb{R}^d))} \leq \sup_{(x, y, t) \in (\mathbb{R}^d) \times (-\frac{\pi}{4}, \frac{\pi}{4})} |K_{-\lambda_0 + is}(x, y, t)|.
\]
By \((3.14)\) and \((2.4)\), we get
\[
(3.20) \quad |K_{\lambda}(x, y, t)| \sim C\frac{|e^{iz}\pi/2|}{|t|Re(z + 1 + \frac{d + 2\gamma_\kappa}{2})}
\]
for \(t \in (-\frac{\pi}{4}, \frac{\pi}{4})\). So \(K_{\lambda}(x, y, t)\) is bounded if and only if \(Re(z) = -\left(\frac{d + 2\gamma_\kappa + 2}{2}\right)\), in which case we will have
\[
\|T_{-\lambda_0 + is}\|_{L_t^1((\mathbb{R}^d))} \leq \sup_{(x, y, t) \in (\mathbb{R}^d) \times (-\frac{\pi}{4}, \frac{\pi}{4})} |K_{-\lambda_0 + is}(x, y, t)| \leq C e^{\pi|s|/2}.
\]
In view of Proposition \((3.2)\) and \(T_{-1} = T_S\) we get the inequality \((3.17)\), which completes proof of the theorem.

\[\square\]

**Theorem 3.4.** Let \(S\) be the discrete surface \(S = \{((\mu, \nu) \in \mathbb{N}_0^d \times \mathbb{N}_0^d : \nu = 2|\mu| + d + 2\gamma_\kappa\}\) with respect to the counting measure and \(\mathbb{T} = (-\frac{\pi}{4}, \frac{\pi}{4})\) with Lebesgue measure. Then for all exponents \(p, q \geq 1\) satisfying
\[
\frac{2}{p} + \frac{d + 2\gamma_\kappa}{q} = d + \gamma_\kappa, \quad \text{and} \quad \frac{d + 2\gamma_\kappa + 2}{d + 2\gamma_\kappa} < q < \frac{d + 2\gamma_\kappa + 1}{d + 2\gamma_\kappa - 1}
\]
we have
\[
\|W_1T_{\lambda}W_2\|_{L_t^{2p'} L_{x,y}^{2q'}(\mathbb{T} \times \mathbb{R}^d)} \leq \|W_1\|_{L_t^{2p'} L_{x,y}^{2q'}(\mathbb{T} \times \mathbb{R}^d)}\|W_2\|_{L_t^{2p'} L_{x,y}^{2q'}(\mathbb{T} \times \mathbb{R}^d)}
\]
with \(C > 0\) independent of \(W_1, W_2\).

**Proof.** For \((\mu, \nu) \in \mathbb{N}_0^d \times \mathbb{N}_0\), the family of generalized functions \(G_z\) coincides with the operators \(T_{\lambda}\) when \(z = -1\). Notice that the operator \(T_{-\lambda_0 + is}\) is an integral operator with kernel \(K_{-\lambda_0 + is}(x, x', t - t')\) defined in \((3.12)\). An application of Hardy-Littlewood-Sobolev inequality along with \((3.18)\) and \((3.20)\) yields
\[
\|W_1^\lambda T_{-\lambda_0 + is} W_2^\lambda - is\|_{L_t^2}^2
\]
\[ = \int_{T} \int_{\mathbb{R}^{2d}} W_1(t, x_0) 2^{2\lambda_0} |K_{-\lambda_0 + s}(x, x', t - t')|^2 W_2(t', x_0') 2^{2\lambda_0} h_{\kappa}^2(x) h_{\kappa}^2(x') dx dx' dt dt' \]

\[ \leq C_1 \int_{T} \int_{\mathbb{R}^{2d}} \frac{W_1(t, x_0) 2^{2\lambda_0} W_2(t', x_0') 2^{2\lambda_0}}{|t - t'|^{d + 2\gamma + 2 - 2\lambda_0}} h_{\kappa}^2(x) h_{\kappa}^2(x') dx dx' dt dt' \]

\[ \leq C_1 e^{\pi |s|} \int_{T} \int_{\mathbb{R}^{2d}} \frac{\|W_1(t)\|^{2\lambda_0} L_{(r)}^{4\lambda_0} L_{(R)}^{2\lambda_0} \|W_2(t')\|^{2\lambda_0}}{|t - t'|^{d + 2\gamma + 2 - 2\lambda_0}} dt dt' \]

\[ \leq C_1 e^{\pi |s|} \|W_1(t)\|^{2\lambda_0} L_{(r)}^{4\lambda_0} L_{(R)}^{2\lambda_0} \|W_1(t')\|^{2\lambda_0} \]

provided we have \(0 \leq d + 2\gamma + 2 - 2\lambda_0 < 1\), that is \((d + 2\gamma + 1) < \lambda_0 \leq d + 2\gamma + 2\). By Theorem 2.9 of \([23]\) we have

\[ \|W_1 W_2\|_{L^q_0(L^2_2(T \times \mathbb{R}^d))} \leq \|W_1(t)\|^{2\lambda_0} L_{(r)}^{4\lambda_0} L_{(R)}^{2\lambda_0} \|W_1(t')\|^{2\lambda_0} \]

for \(d + 2\gamma + 1 < \lambda_0 \leq d + 2\gamma + 2\). Since \(\frac{d + 2\gamma + 2}{d + 2\gamma} < q < \frac{d + 2\gamma + 1}{d + 2\gamma - 1}\), we have \(\frac{d + 2\gamma + 1}{2} < q' < \frac{d + 2\gamma + 2}{2}\).

Thus the theorem is proved by choosing \(\lambda_0 = q'\).

**Remark 3.5.** In future work we will prove the following:

1. We will try to prove that the range of the exponents given in Theorem 1.2 and Theorem 1.5 are necessary to satisfy the estimates (1.6) and (1.9).
2. We will try to prove the Schetzen exponent \(\alpha = \frac{2d}{q+1}\) appear in the right hand side of the estimates (1.6) and (1.9) is optimal.

4. **Applications of Strichartz estimates: Global well-posedness for the Dunkl Hermite-Hartree equation in Schatten spaces**

In this section we show the well-posedness results in the sprit of \([17, 18]\) for a system of infinitely many equations (with out a trace class assumption) in Schatten spaces for the Dunkl Hermite-Hartree equation by applying our orthonormal Strichartz inequalities for Hermite operator. Recall that \(f \ast_{\kappa} g\) denotes the generalized (Dunkl) convolution of functions \(f\) and \(g\) (for the definition see in \([34]\)).

**Theorem 4.1.** Let

\[ 1 \leq q < 1 + \frac{2}{d + 2\gamma - 1} \quad \text{and} \quad \frac{2 + d + 2\gamma}{p} = d + 2\gamma \]

and \(f \mapsto w \ast_{\kappa} f\) is bounded operator from \(L^p_0(\mathbb{R}^d) \to L^\infty_{\kappa}(\mathbb{R}^d)\). Then for any \(\gamma_0 \in \mathbb{S}^{2\gamma}_{n+1}\), there exist a unique \(\gamma \in C^0([0, T], \mathbb{S}^{2\gamma}_{n+1})\) satisfying \(\rho_\gamma \in L^p_0 L^q_{\kappa,x}([0, T] \times \mathbb{R}^d)\) and

\[ i \partial_t \gamma = [H_{\kappa} + w \ast_{\kappa} \rho_\gamma, \gamma] \]

\[ \gamma|_{t=0} = \gamma_0 \]

**Proof.** Let \(R > 0\) such that \(\|\gamma_0\|_{\mathbb{S}^{2\gamma}_{n+1}} = R < \infty\). Let \(T = T(R)\). \(\leq 1\)to choose later)\.

Consider the space

\[ X_T = \{(\gamma, \rho) \in C^0([0, T], \mathbb{S}^{2\gamma}_{n+1}) \times L^p_0 L^q_{\kappa,x}([0, T] \times \mathbb{R}^d) : \|(\gamma, \rho)\|_{X_T} \leq 4 \max\{1, C_{\text{Str}}\} R\}, \]
where the norm $\|(\gamma, \rho)\|_{X_T}$ is defined by

$$\|(\gamma, \rho)\|_{X_T} := \|\gamma\|_{C^0_T \cap L^2_{\kappa,T}} + \|\rho\|_{L^p_t L^q_{\kappa,x}([0,T] \times \mathbb{R}^d)}.$$ 

Consider the map

$$\Phi_1(\gamma, \rho)(t) = e^{-itH_\kappa} \gamma_0 e^{itH_\kappa} - i \int_0^t e^{i(s-t)H_\kappa} [w * \rho_\gamma(s), \gamma(s)] e^{-i(s-t)H_\kappa} ds$$

Using the above map, we define the contraction map $\Phi$ by

$$\Phi(\gamma, \rho) = (\Phi_1(\gamma, \rho), \rho[\Phi_1(\gamma, \rho)])$$

where we have used the notation $\rho[\gamma] = \rho_\gamma$. Now

$$\|\Phi_1(\gamma, \rho)\|_{C^0_T \cap L^2_{\kappa,T}} \leq R + 2 \int_0^T \|w * \rho(s)\|_{L^\infty_{\kappa,x}\|\gamma(s)\|_{C^0_T \cap L^2_{\kappa,T}}} ds$$

$$\leq R + 2T^{1/p'} C_w\|\rho\|_{L^p_t L^{2\infty}_{\kappa,x}} \|\gamma\|_{C^0_T \cap L^2_{\kappa,T}}$$

$$\leq R + 8T^{1/p'} C_w \max(1, C_{\text{Stri}}) R^2$$

and from Corollary 1.8 we also have

$$\|\rho[\Phi_1(\gamma, \rho)]\|_{L^1_t L^{2\infty}_{\kappa,x}} \leq C_{\text{Stri}} R + 8C_{\text{Stri}} T^{1/p'} C_w \max(1, C_{\text{Stri}}^2) R^2.$$ 

Choosing $T \leq 1$ small enough so that

$$8C_{\text{Stri}} T^{1/p'} C_w \max(1, C_{\text{Stri}}^2) R^2 \leq C_{\text{Stri}} R$$

and $\Phi$ maps $X$ to itself. Thus $\Phi$ is a contraction mapping and has a unique fixed point on $X$ which is a solution to the Hatree equation on $[0,T]$. $\square$

5. Alternative proof of Theorem 1.2

5.0.1. Proof of the main inequality: (Theorem 1.2 and 1.6)

Proof. The duality argument shows that Theorem 1.2 is equivalent to Theorem 1.6 has already discussed before. By assuming $V \in L^\infty_{c}(\mathbb{R} \times \mathbb{R}^d)$ and that $\gamma$ is finite rank, we have to show that the operator

$$L^{p'}(\mathbb{R}, L^{q'}_\kappa(\mathbb{R}^d)) \ni V \mapsto \int_\mathbb{R} e^{-it\Delta_\kappa} V(t,x) e^{it\Delta_\kappa} dt \in \mathbb{S}^{2p'}$$

is bounded. By using complex interpolation method [1, Chap. 4], it is enough to prove this fact by proving for the two points $(p', q') = (1, \infty)$ and $(p', q') = (1 + \frac{d+2\gamma_\kappa}{2}, 1 + \frac{d+2\gamma_\kappa}{2})$.

For the point $(p', q') = (1, \infty)$, the argument is well known. We simply bound the operator norm by

$$\left\| \int_\mathbb{R} e^{-it\Delta_\kappa} V(t,x) e^{it\Delta_\kappa} dt \right\| \leq \int_\mathbb{R} \left\| e^{-it\Delta_\kappa} V(t,x) e^{it\Delta_\kappa} \right\| dt = \int_\mathbb{R} \| V(t, \cdot) \|_{L^\infty_{\kappa}(\mathbb{R}^d)} dt,$$

Now it is time to go for the case $p' = q' = 1 + \frac{d+2\gamma_\kappa}{2}$. Without any loss of generality, we may assume that $V \geq 0$. Then we have $e^{-it\Delta_\kappa} V(t,x) e^{it\Delta_\kappa} \geq 0$ as an operator on $L^{q'}_\kappa(\mathbb{R}^d)$, for all $t$ in $\mathbb{R}$. We can also assume that $V \in L^\infty_{c}(\mathbb{R} \times \mathbb{R}^d)$. We recall that $e^{-it\Delta_\kappa} x_j e^{it\Delta_\kappa} = x_j - 2it\nabla_j$ for each $j = 1, 2, ..., d$ And it can be written as

$$e^{-it\Delta_\kappa} x_j e^{it\Delta_\kappa} = x - 2it\nabla_j.$$
where $\nabla_\kappa = (T_1, T_2, \ldots, T_d)$ and $x$ is identified with the multiplication operator. The multiplication operator can be seen by differentiating with respect to $t$. By the functional calculus

$$e^{-it\Delta_\kappa} f(x)e^{it\Delta_\kappa} = f(x + 2tp),$$

where $p := -i\nabla_\kappa$. From this we can get

$$e^{-it\Delta_\kappa} V(t, x)e^{it\Delta_\kappa} = V(t, x + 2tp).$$

Using the fact $V \geq 0$, we can write the Schatten norm as

$$\left\| \int_\mathbb{R} e^{-it\Delta_\kappa} V(t, x)e^{it\Delta_\kappa} dt \right\|_{\mathfrak{S}^{d+2\gamma+2}}^{d+2\gamma+2} = Tr \left( \int_\mathbb{R} e^{-it\Delta_\kappa} V(t, x)e^{it\Delta_\kappa} dt \right)^{d+2\gamma+2} = Tr \left( \int_\mathbb{R} V(t, x + 2tp) dt \right)^{d+2\gamma+2} = Tr \left( \int_\mathbb{R} \cdots \int_\mathbb{R} V(t_1, x + 2t_1p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) dt_1 \cdots dt_{d+2\gamma+2} \right) dt_1 \cdots dt_{d+2\gamma+2}$$

To exchange the trace and the integral, we need to prove that

$$\int_\mathbb{R} \cdots \int_\mathbb{R} \left\| V(t_1, x + 2t_1p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^1} dt_1 \cdots dt_{d+2\gamma+2} \leq \infty$$

(We are assuming $V \in L^\infty_c(\mathbb{R} \times \mathbb{R}^d)$ throughout). In order to estimate the Schatten norm in the integral, we make use of the following

**Lemma 5.1.** Let $\alpha, \beta, \gamma, \delta \in \mathbb{R}$. We have

$$\left\| f(\alpha x + \beta p)g(\gamma x + \delta p) \right\|_{\mathfrak{S}^r} \leq \frac{\|f\|_{L^r_c(\mathbb{R}^d)} \|g\|_{L^r_c(\mathbb{R}^d)}}{c^2 |\alpha \delta - \beta \gamma|^{(d+2\gamma)/r}}$$

for all $r \geq 2$.

The generalization (5.2) implicitly appears in [1], sec. 2.1. And for a value $\alpha = \delta = 1$ and $\beta = \gamma = 0$, the estimate is the well-known Kato-Seele-Simon inequality ([28] and [27], Thm. 4.1).

We postpone the proof of the lemma and going with the proof of estimate to the trace norm (5.1). Now by using the fact $V \geq 0$ and H"older's inequality in Schatten space, we can write

\[
\left\| V(t_1, x + 2t_1p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^1} \\
= \left\| V(t_1, x + 2t_1p) \sqrt{V(t_2, x + 2t_2p)} \sqrt{V(t_2, x + 2t_2p)} \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\times \cdots \times \sqrt{V(t_{d+2\gamma+1}, x + 2t_{d+2\gamma+1}p)} V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\leq \left\| V(t_1, x + 2t_1p) \sqrt{V(t_2, x + 2t_2p)} \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\left\| V(t_2, x + 2t_2p) \sqrt{V(t_3, x + 2t_3p)} \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\times \cdots \times \left\| \sqrt{V(t_{d+2\gamma+1}, x + 2t_{d+2\gamma+1}p)} V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\times \cdots \times \sqrt{V(t_{d+2\gamma+1}, x + 2t_{d+2\gamma+1}p)} V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\times \cdots \times \left\| \sqrt{V(t_{d+2\gamma+1}, x + 2t_{d+2\gamma+1}p)} V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^{d+2\gamma+1}} \\
\times \cdots \times \left\| \sqrt{V(t_{d+2\gamma+1}, x + 2t_{d+2\gamma+1}p)} V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2}p) \right\|_{\mathfrak{S}^{d+2\gamma+1}} \]

Using (5.2) and the fact $V \in L^\infty_c(\mathbb{R} \times \mathbb{R}^d)$, we get
\[ \|V(t_1, x + 2t_1 p) \cdots V(t_{d+2\gamma +2}, x + 2t_{d+2\gamma +2} p)\|_{\mathfrak{S}} \]
\[ \leq \frac{\|V(t_1, \cdot)\|_{L_{\infty}^{d+2\gamma+1}} \|V(t_2, \cdot)\|_{L_{\infty}^{(d+2\gamma+1)/2}} \cdots \|V(t_{d+2\gamma+1}, \cdot)\|_{L_{\infty}^{(d+2\gamma+1)/2}} \|V(t_{d+2\gamma+2}, \cdot)\|_{L_{\infty}^{d+2\gamma+1}}}{c_k^{d+2\gamma} \prod_{j=1}^{\gamma} \left| t_1 - t_2 \right|^{d+2\gamma+1} \cdots \left| t_d + 2\gamma +1 - t_{d+2\gamma+2} \right|^{d+2\gamma+1}} \]
\[ \leq C' \frac{c_k^{d+2\gamma} \prod_{j=1}^{\gamma} \left| t_1 - t_2 \right|^{d+2\gamma+1} \cdots \left| t_d + 2\gamma +1 - t_{d+2\gamma+2} \right|^{d+2\gamma+1}}{1(a \leq t_j \leq b)} \]
where \((a, b)\) is the support of \(V\) in the time variable. Now we will use the multilinear Hardy-Littlewood-Sobolev inequality.

**Theorem 5.2** (Multilinear Hardy-Littlewood-Sobolev inequality (See [5], Thm. 6)). Assume that \((\beta_{ij})_{1 \leq i, j \leq N}\) and \((r_k)_{1 \leq k \leq N}\) are real-numbers such that

\[ \beta_{ii} = 0, \quad 0 \leq \beta_{ij} = \beta_{ji} < 1, \quad r_k > 1, \quad \sum_{k=1}^{N} \frac{1}{r_k} > 1, \quad \sum_{i=1}^{N} \beta_{ik} = \frac{2(r_k - 1)}{r_k}. \]

Then there exist a constant \(C\) such that

\[ (5.3) \quad \left| \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} f_1(t_1) \cdots f_N(t_N) \prod_{i<j} |t_i - t_j|^{\beta_{ij}} dt_1 \cdots dt_N \right| \leq C \prod_{k=1}^{N} \|f_k\|_{L^{r_k}(\mathbb{R})} \]

for all \(f_k \in L^{r_k}\).

For the particular case of Multilinear Hardy-Littlewood-Sobolev inequality where all the \(\beta_{ij}\) and the \(r_k\) are identical, in [6], Prop. 2.2. Applying (5.3) with taking \(r_1 = r_{d+2\gamma+2} = \frac{2(d+2\gamma+1)}{d+2\gamma+2}\) and \(r_2 = \cdots = r_{d+2\gamma+1} = d + 2\gamma + 1\), then we will get (5.1). Hence now we can write

\[ T r \left( \int_{\mathbb{R}} e^{-it\Delta} V(t, x) e^{it\Delta} dt \right)^{d+2\gamma+2} = \int_{\mathbb{R}} \cdots \int_{\mathbb{R}} T r (V(t_1, x + 2t_1 p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p)) dt_1 \cdots dt_{d+2\gamma+2} \]

By the previous argument we will now obtain a more symmetric estimate of the trace in the integral. Simply using the cyclicity of the trace we get,

\[ |T r (V(t_1, x + 2t_1 p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p))| \]
\[ = \left| T r \left( \sqrt{V(t_1, x + 2t_1 p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p) \sqrt{V(t_1, x + 2t_1 p)}} \right) \right| \]
\[ \leq \left\| \sqrt{V(t_1, x + 2t_1 p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p)} \right\|_{\mathfrak{S}^{d+2\gamma+2}} \]
\[ \times \cdots \times \left\| \sqrt{V(t_{d+2\gamma+1}, x + 2t_{d+2\gamma+1} p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p)} \right\|_{\mathfrak{S}^{d+2\gamma+2}} \]
\[ \times \left\| \sqrt{V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p) \cdots V(t_{d+2\gamma+2}, x + 2t_{d+2\gamma+2} p)} \right\|_{\mathfrak{S}^{d+2\gamma+2}} \]

Again using of (5.2) and multilinear Hardy-Littlewood-Sobolev inequality (5.3) with \(r_1 = \cdots = r_{d+2\gamma+2} = 1 + \frac{d+2\gamma+2}{2}\) we will get our required inequality.
\[
Tr \left( \int_{\mathbb{R}} e^{-it\Delta} V(t, x) e^{it\Delta} dt \right)^{d+2\gamma_n+2} = C \| V \|_{L^1(\mathbb{R}, L^{1+d+2\gamma_n}(\mathbb{R}^d))}^{d+2\gamma_n+2}
\]

5.0.2. Proof of the lemma [5.7]

Proof. We prove the inequality (5.2) for \( r = 2 \) and \( r = \infty \), then the general case follows from the complex interpolation. For \( r = \infty \), by noting that the Schatten \( \infty \)-norm is the operator norm on \( L^\infty(\mathbb{R}^d) \), the inequality (5.2) can be seen easily.

For \( r = 2 \), we get

\[
\| f(\alpha x + \beta p) g(\gamma x + \delta p) \|_{L^2} = Tr[ |f(\alpha x + \beta p)|^2 |g(\gamma x + \delta p)|^2] = \frac{1}{c_n^2} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |f(\alpha x)|^2 |g(\gamma x)|^2 h_{\kappa}(\xi) h_{\kappa}(\xi) d\xi dx
\]

The inequality in \( \mathcal{S}^r \) now follows from the complex interpolation.

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