Weighted Particle Monte Carlo Method for Effective Simulation of Electromagnetic Cascades

Nina Elkina
Ludwig-Maximilians Universität München, 80539, Germany
E-mail: N.Elkina@gsi.de

Abstract. We present a new version of the event generator for simulation of quantum electromagnetic cascades generated in plasma irradiated by ultra-intense laser field. The new code is capable to account for very soft photons as well as for polarization effects in nonlinear vacuum. Numerical implementation differs significantly from the original version aiming to provide a guaranteed accuracy of computations. The code is benchmarked against the semi-analytical solution describing electromagnetic cascade in a rotating field.

1. Introduction
The Extreme Light Infrastructure (ELI) is anticipated to provide an experimental access to study intense field quantum electrodynamic (QED) effects in laser driven plasmas. Since analytical results for relativistic plasma with QED effects are rather limited, numerical simulation has become the only tool to explore various scenarios of such phenomena. Thus it is important to verify numerical accuracy of numerical plasma codes which are applied to study the QED regime. For this purpose one can think of a unified testbed case to benchmark different codes. The QED cascade can serve this purpose since in particular field configurations it can be treated analytically. The example is provided in [1], where analytical theory of electromagnetic cascade was developed for homogeneous rotating electric field. This analytical theory was later validated by the numerical simulation presented in [2]. Although good qualitative agreement has been observed, the difference between theoretical and numerical predictions for electron-positron yields remained unexplained. In this contribution we report on a benchmark study of the new version of our code and compare it with the original one [2].

2. Basic equations
The mathematical basis for study of QED cascades is provided by a set of semi-classical transport equations for $\gamma e^\pm$ plasma. For example, equation for electron distribution function $f(r, p, t)$ in the phase space reads

$$\frac{df(r, p, t)}{dt} = \frac{\partial f(r, p, t)}{\partial t} + v \cdot \nabla f(r, p, t) - e \{E(r, t) + v \times B(r, t)\} \cdot \frac{\partial f(r, p, t)}{\partial p}$$

$$= \int w_{rad}(r, t, p + k \rightarrow k)f(r, p + k, t) d^3k - f(r, p, t) \int w_{rad}(r, t, p \rightarrow k) d^3k,$$

where $v = p/\varepsilon$ and $\varepsilon(p) = \sqrt{p^2 + m^2}$ are the velocity and energy for given momentum and $\hbar = c = 1$. By definition, $w_{rad}(p \rightarrow k)$ is the differential probability rate for emission of a
Figure 1. Left: growth of the number of pairs as a function of time with and without soft photons cutoff; right: hard and soft photon spectra, together with spread of emitted photons in $\chi_\gamma$.

While the probability rate $w_e$ of electron-positron pair production

$$w_e = \frac{dW_{e \pm}}{d\varepsilon_e} = \frac{\alpha m^2}{\varepsilon_e^{2/3}} \left\{ A_1(x) + \left[ \frac{g(\phi)}{x} - \chi_\gamma \sqrt{x} \right] A'_1(x) \right\}, \quad x = \left( \frac{\chi_\gamma}{\chi_e (\chi_\gamma - \chi_e)} \right)^{2/3},$$

is relatively easy for computations, the photon emission one

$$w_{\text{rad}} = \frac{dW_{\text{rad}}}{d\varepsilon_\gamma} = -\frac{\alpha m^2}{\varepsilon_e^{2/3}} \left[ A_1(x) + \left[ \frac{g(\phi)}{x} + \chi_\gamma \sqrt{x} \right] A'_1(x) \right], \quad x = \left( \frac{\chi_\gamma}{\chi_e (\chi_\gamma - \chi_e)} \right)^{2/3},$$

may be problematic at small energies, possessing a weak singularity $w_{\text{rad}}(\varepsilon_\gamma) = \mathcal{O}(\varepsilon_\gamma^{-2/3})$ at $\varepsilon_\gamma \to 0$. In the previous version [2] of our event generator the soft photon singularity was isolated by putting a cutoff energy below which photons were not sampled. In the new version the impact of soft photons can be estimated by splitting of the rate integrals into singular and nonsingular parts as follows

$$W_{\text{rad}} = \int_0^{\varepsilon_e} \frac{dW_{\text{rad}}}{d\varepsilon_\gamma} d\varepsilon_\gamma = \int_0^{\varepsilon_{\text{min}}} + \int_{\varepsilon_{\text{min}}}^{\varepsilon_e}.$$

The aforementioned singularity can then be integrated using an appropriate substitution. The results of simulation runs with and without soft photons sampling are presented in the left panel of Fig. 1, where the growth of the number of pairs is shown as a function of time. It is clear from the picture that soft photons do not play important role in strong cascading regime. However, the increment of growth turns out to be smaller than the one reported in [2]. This can be attributed
Figure 2. Angular distribution of photons produced via nonlinear Compton scattering of energetic electrons with $\gamma = 10^3$ on counter-propagating laser pulse with $a_0 = 10^2$, with and without cutoff.

to more accurate numerics used to calculate integrals in the sampling equation. The results are in good agreement with theoretical estimate for the growth rate $\Gamma_{th} = \alpha \mu^{1/4} \sqrt{m_e c^2 \omega_0 / \hbar}$, where $\mu = \frac{\hbar \omega_0}{\alpha m_e c^2 a_0}$. Thus for high $a_0$ one can neglect soft photons by imposing an appropriate energy cut off. More insights on formation of the photon spectra is presented in the right panel of Fig. 1, where each dot of the scatter plot represents a photon emitted with energy $\hbar \omega$ and quantum parameter $\gamma$. The spectrum curve $dN/d\hbar \omega$ splits into the soft photons (red) and hard photons one (black). One can see that the number of really soft photons below 0.1 MeV is not significant enough to interfere with low-frequency plasma or laser waves. Although impact of soft photons turns out to be unimportant at higher field intensity, their influence can be still significant at less severe parameters. For example, this is the case when photon beam is produced via nonlinear Compton scattering of electrons counter-propagating laser pulse with intensity insufficient for pair photoproduction. The difference in angular distributions of emitted photons computed with and without imposing a soft photons cutoff is presented in Fig. 2. This resulting photon beam has been used in [5] to probe the vacuum birefringence effect.

In order to describe briefly the basic functionality of the code, let us consider the method of sampling of emission of a polarized photon as an example. Over each time step we first calculate the total probability rate $W^u_{\text{tot}}$ for electron to produce an unpolarized photon. The probability for a particle to generate photon during the time step $\Delta t$ if given by $\Delta t W^u_{\text{tot}}$. A new photon is produced if the condition $\Delta t W^u_{\text{tot}} > r_1$ holds, where $0 < r_1 < 1$ is a random number. Using two other random numbers $0 < r_2, r_3 < 1$ we sample the polarization state and energy of secondary photon as following. The discrete polarization state $p \in \{0, \pi/2\}$ is picked up according to the relative ratio between the corresponding total probabilities $W^{u0}_{\text{tot}} / (W^{\pi/2}_{\text{tot}} + W^{0}_{\text{tot}}) \geq r_2$. Finally, the energy $\varepsilon_\gamma$ of polarized photon is selected by solving the sampling equation

$$\frac{1}{W^p_{\text{tot}}} \int_0^{\varepsilon_\gamma} dW^p (\hbar \omega') d\hbar \omega' = r_3.$$  \hspace{1cm} (4)

In order to speed up the nonlinear sampling equation solver we adopt a modified false position
method instead of the bisection root finding algorithm which was previously applied in the original version of our code. However, in rare cases of slow convergence the code can still call back the bisection method. In this case the root is finally located in the interval returned by the false position method after it hits the maximal allowed number of iterations (usually $N \simeq 10$). The code also provides two options to compute integrals in the sampling equation by either by the Gauss-Kronrod [6] or Double Exponentiation rule [7]. Each method is supplemented with built in error control allowing to preserve numerical accuracy in the course of computation. The code is further optimized by gradual creation of a sparse 2d array containing the values of both total and differential probability rates for variables ($\varepsilon, \chi$) on a grid, with further interpolation to the actually required values of that variables.

4. Summary and conclusions
In conclusion, we have carried out benchmark study of the new version of event generator. The main task of this work was to determine the impact of soft photons, which was neglected in the earlier version of the code. Our results suggest that soft photons do not affect significantly the plasma dynamics in a very strong laser field capable to support exponentially growing cascades. Moreover, the photon spectrum is very sparsely populated at low frequencies with almost no photon emission at plasma and laser wave scales. This suggests that semi-classical modeling of QED effects is justified at least for very high intensities ($I > 10^{24}$ W/cm$^2$). However, soft photons should be taken into account at moderate intensities, for example in various scenarios with electron beams scattering in laser field, where the photon flux is generated via nonlinear Compton scattering. If soft photons are included, the resulting photon flux exhibits noticeable angular spread compared to simulations with imposing a cutoff.

In summary, we have presented a list of new features of the event generator which is a part of the PIC-ANTARES code under development. The main difference with previous version is that the impact of soft photons below 1 MeV can be now taken into account if needed. Polarization effects are now also implemented in the code, thus allowing to study various nonlinear effects like vacuum birefringence. Finally, the built in automatic accuracy control is now implemented in each part of the code.

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References
[1] Fedotov A M, Narochny N B, Mourou G and Korn G 2010 Phys. Rev. Lett. 105(8) 080402
[2] Elkina N V, Fedotov A M, Kostyukov I Y, Legkov M V, Narochny N B, Nerush E N and Ruhl H 2011 Physical Review Special Topics Accelerators and Beams 14 054401
[3] Birdsall C and Langdon A 2004 Plasma Physics via Computer Simulation Series in Plasma Physics and Fluid Dynamics (Taylor & Francis)
[4] Elkina N V, Fedotov A M, Herzing C and Ruhl H 2014 Physical Review E 89 053315
[5] King B and Elkina N 2016 Physical Review A 94 062102
[6] Kronrod A 1965 Nodes and weights of quadrature formulas: sixteen-place tables (Consultants Bureau)
[7] Mori M and Sugihara M 2001 Journal of Computational and Applied Mathematics 127 287 – 296