Two-loop heavy top effects
on the $m_Z$–$m_W$ interdependence

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Abstract

The $O(\alpha^2 m_t^2/m_W^2)$ correction to the relation between $G_\mu$ and the vector boson masses is computed in the $\overline{MS}$ scheme, and the results are used to investigate the magnitude of the effect on the theoretical prediction of $m_W$ and $\sin^2 \theta_W(m_Z^2)$ from $\alpha$, $G_\mu$, and $m_Z$. 
The interdependence between \( m_w \), \( m_z \), and \( G_\mu \) has been studied for a long time. The original one-loop calculation of \( \Delta r \) [1] has subsequently been augmented by the inclusion of higher order corrections related to mass singularity contributions and heavy top effects. The inclusion of the leading logarithms of \( O(\alpha \ln(m_z/m_f)) \) (where \( m_f \) is a generic fermion mass) in \( \Delta r \) was investigated in Refs. [2,3], while Consoli, Hollik, and Jegerlehner [4] showed how to take into account, in the On-Shell (OS) scheme, the leading two-loop contribution of a heavy top, namely the term that scales as \( m_t^4 \). A similar analysis concerning the leading top-mass power correction was performed in the \( \overline{\text{MS}} \) framework in Ref. [5]. More recently, the full Higgs dependence of the leading \( m_t^4 \) contribution was calculated by several groups [6,7]. Our knowledge of mass singularity contributions to \( \Delta r \) goes actually beyond the two-loop leading effects, as the incorporation of the \( O(\alpha^2 \ln(m_z/m_f)) \) terms was presented in Ref. [8]. Concerning the two-loop top corrections, however, a discussion of the \( O(\alpha^2 m_t^2/m_w^2) \) correction is still missing.

Indeed, the uncertainty coming from the unknown higher order contributions can be ascribed mainly to our ignorance of the \( O(\alpha^2 m_t^2/m_w^2) \), as two and three-loop QCD corrections seem to be well under control [8], and two-loop heavy Higgs effects have been shown to be negligible [9]. A first investigation of the potential magnitude of higher order corrections of electroweak origin was carried out by the Working Group on Precision Calculations (WGPC) at CERN [10]. The results of five different computer codes for the evaluation of radiative corrections were compared; the codes were based on different renormalization frameworks, and allowed various resummation options, all equivalent at the order of known contributions, and differing precisely at \( O(\alpha^2 m_t^2/m_w^2) \). Therefore, in comparing the results obtained by choosing different options, one could have an indication on the importance of the higher-order corrections which have not yet been explicitly calculated. Although in most cases the experimental precision is well above the uncertainty obtained in this way, one of the conclusions of the WGPC report was that a full calculation of \( O(\alpha^2 m_t^2/m_w^2) \) would greatly reduce the theoretical error originating from higher order effects.

In this respect, it is interesting to note that the estimate of the WGPC for the theoretical uncertainty on observables like \( m_w \) and the effective mixing angle measured at LEP is in very good agreement with Ref. [11], where the result of a complete calculation of the \( O(\alpha^2 m_t^2/m_w^2) \) effects in the \( \rho \) parameter for \( \nu_\mu - e \) scattering was used as the basis for an extrapolation to \( \Delta r \). In general, the result was that \( O(\alpha^2 m_t^2/m_w^2) \) could be as large as the leading \( m_t^4 \) contributions. In particular, the estimate of Ref. [11] was that neglected \( O(\alpha^2 m_t^2/m_w^2) \) effects could shift the theoretical prediction of the mass of the W boson by up to about 23 MeV, depending on the Higgs boson and top masses. More interestingly, the prediction of the effective sine measured on the Z peak at LEP and SLC, \( \sin^2 \theta_{\text{eff}}^{\text{lep}} \), could be shifted by up to \( 1.4 \times 10^{-4} \), closer to the present experimental accuracy of \( 3 \times 10^{-4} \). Also in view of the prospects of improving the experimental accuracy on these observables, we feel that a complete calculation of this kind of effects on the interdependence between \( m_w \), \( m_z \), and \( G_\mu \) is indeed timely. It is therefore the aim of this paper to provide explicit analytical expressions for the \( O(\alpha^2 m_t^2/m_w^2) \) contributions to \( \Delta r \) and to investigate their magnitude.

To begin our discussion of the electroweak corrections of \( O(\alpha^2 m_t^2/m_w^2) \) to \( \Delta r \) we write the relation between the \( \mu \)-decay constant and the charged current amplitude expressed in
terms of bare quantities. At the two-loop level we have

\[
\frac{G_\mu}{\sqrt{2}} = \frac{e_0^2}{8 s_0^2 m_{w_0}^2} \left\{ 1 + \Delta r_0 \right\} \tag{1a}
\]

\[
\Delta r_0 = \left\{ - \frac{A_{wW}(0)}{m_{w_0}^2} + V_w + m_{w_0}^2 B_w + \frac{A_{wW}(0)^2}{m_w^4} - \frac{A_{wW}(0)V_w}{m_w^2} \right\}, \tag{1b}
\]

where \(e\) and \(m_w\) are the electric charge and the \(W\) mass, respectively, \(s^2 \equiv \sin^2 \theta_w\), \(\theta_w\) being the weak interaction mixing angle, \(A_{wW}\) is the transverse part of the \(W\) self-energy, and \(V_w\) and \(B_w\) represent the relevant vertex and box corrections. In Eq. (1) the subscript 0 indicates that we are dealing with unrenormalized quantities. To express \(G_\mu\) in terms of renormalized parameters, we insert in Eq. (1) \(e_0^2 = e^2 - \delta e^2\), \(s_0^2 = s^2 - \delta s^2\), \(m_{w_0}^2 = m_w^2 - \delta m_w^2 = m_w^2 - \text{Re} A_{wW}(m_w^2)\). Our choice of \(\delta m_w^2\) identifies \(m_w\) as the physical mass while, for the moment, we do not specify the renormalized parameters \(e^2\) and \(s^2\), but assume that the counterterms \(\delta e^2\) \(\delta s^2\) can contain mass singularity and \(m_t\)-power corrections. After some simple algebra, and using the fact that at one-loop \((\text{Re} A_{wW}(m_w^2) - A_{wW}(0))/m_w^2\) and \(V_w\) do not contain mass singularities or \(m_t\)-corrections, Eq. (1) becomes

\[
\frac{G_\mu}{\sqrt{2}} = \frac{e^2}{8 s^2 m_w^2} \left\{ 1 + \frac{\text{Re} A_{wW}(m_w^2)}{m_w^2} - \frac{A_{wW}(0)}{m_w^2} + V_w + m_w^2 B_w - \frac{\delta e^2}{e^2} + \frac{\delta s^2}{s^2} \right. \left. - 2 \left( \frac{\delta e^2}{e^2} - \frac{\delta s^2}{s^2} \right) \left[ \frac{\text{Re} A_{wW}(m_w^2)}{m_w^2} - \frac{A_{wW}(0)}{m_w^2} + V_w + m_w^2 B_w \right] \right. \right.
\]

\[
+ \left( \frac{\delta e^2}{e^2} \right)^2 + 2 \left( \frac{\delta s^2}{s^2} \right)^2 - 2 \frac{\delta e^2}{e^2} \frac{\delta s^2}{s^2} \right\}, \tag{2}
\]

with the understanding that the one-loop contribution is now written in terms of the renormalized parameters \(e^2\) and \(s^2\). In Eq. (2), the second and third lines take into account explicitly the expansion of the overall coupling \(e_0^2/s_0^2\) in the lowest order contribution, while mass counterterm effects and shifts in additional \(s_0^2\) are included by definition in the two-loop terms. From Eq. (2) it is easy to see that the replacement in Eq. (1a) [3]

\[
1 + \Delta r \rightarrow \frac{1}{1 - \Delta r} \tag{3}
\]

takes correctly into account the \(\ln(m_Z/m_t)\) terms contained in \(\delta e^2/e^2\), to \(O(\alpha^2 \ln(m_Z/m_t))\), once the renormalized parameter \(e\) is identified with the electric charge at zero momentum transfer. However, as emphasized by Consoli, Hollik, and Jegerlehner [4], there is a mismatch in the iteration of the one-loop \(\delta s^2/s^2\) term. This contribution in the \(\overline{MS}\) scheme contains finite corrections proportional to \(m_t^2\) and therefore the replacement (3) does not take into account correctly the reducible contribution of \(O(\alpha^2 m_t^4/m_0^4)\) and \(O(\alpha^2 m_t^2/m_0^2)\). A way to circumvent this problem is to use an \(\overline{MS}\) subtraction for the parameter \(s\), namely to choose the counterterm \(\delta s^2\) to subtract just the terms proportional to \(\delta = (n - 4)^{-1} + [\gamma - \ln(4\pi)]/2\) [4]. As \(\delta s^2\) does not contain any finite part, this procedure automatically takes into account all reducible contributions.
The above discussion tells us that the simplest way to take into account the $O(\alpha^2 m_t^2/m_W^2)$ corrections to the relation between the $\mu$-decay constant and the W mass is through an $\overline{MS}$ subtraction for the weak interaction angle. The relations between the $\overline{MS}$ and the OS frameworks were worked out in Ref. [5]. Here we just recall the basic corrections of the $\overline{MS}$ framework that enter into the $m_W$–$m_Z$ interdependence. They are $\Delta \hat{\rho}_W$, that relates $G_\mu$ to the $\overline{MS}$ weak interaction angle defined at the scale $m_Z$, $\sin^2 \hat{\theta}_W(m_Z^2)$ [12], henceforth abbreviated as $\hat{s}^2$, with $\hat{c}^2 \equiv 1 - \hat{s}^2$,

$$\frac{G_\mu}{\sqrt{2}} = \frac{\pi \alpha}{2 \hat{s}^2 m_W^2} \frac{1}{1 - \Delta \hat{\rho}_W},$$

and $\hat{\rho}$

$$\hat{\rho} = \frac{m_W^2}{m_Z^2 \hat{c}^2} \equiv \frac{c^2}{\hat{c}^2},$$

that is given explicitly by

$$\hat{\rho} = \frac{1}{1 - \Delta \hat{\rho}} = \frac{1}{1 - Y_{\overline{MS}}}$$

with

$$Y_{\overline{MS}} = \frac{1}{\hat{\rho} m_Z^2} \Re \left[ \frac{A_{WW}(m_W^2)}{\hat{c}^2} - A_{ZZ}(m_Z^2) \right]_{\overline{MS}}.$$  

In Eq. (5a) $A_{ZZ}(m_Z^2)$ is the transverse $Z$ self-energy evaluated at the physical $Z$ mass, the subscript $\overline{MS}$ indicates both the $\overline{MS}$ subtraction and the choice $\mu = m_Z$ for the 't Hooft mass scale, and we have neglected small contributions proportional to the $\gamma Z$ mixing in the $Z$ mass counterterm that do not contain mass singularities or terms proportional to $m_t^2$. In terms of these corrections, the $m_W$–$m_Z$ interdependence can be expressed as

$$\frac{m_W^2}{m_Z^2} = \hat{\rho} \left\{ 1 + \left[ 1 - \frac{4 A^2}{m_Z^2 \hat{\rho}(1 - \Delta \hat{\rho}_W)} \right]^{1/2} \right\}.$$  

where $A = \left( \frac{\pi \alpha}{\sqrt{2} G_\mu} \right)^{1/2} = (37.2803 \pm 0.0003)$ GeV.

We now start discussing the $O(\alpha^2 m_t^2/m_W^2)$ corrections to $\Delta \hat{\rho}_W$. Comparing Eq. (4) and Eq. (5), and keeping in mind the replacement (3), we see that contributions of this order come not only from the W and photon two-point functions (the latter is included in $\delta e^2/e^2$), but also from vertex and box diagrams. As explained in Ref. [11], by writing the one-loop result in terms of $\overline{MS}$ coupling and physical $W$ and $Z$ masses, one automatically takes into account the $O(\alpha^2 m_t^2/m_W^2)$ corrections coming from the box diagrams and, to a large extent, the similar contribution coming from the vertices. Only the vertex diagrams involving a mixing between vector bosons and unphysical scalars through a fermionic blob have to be explicitly calculated. They can be easily expressed, however, in terms of two-loop self-energy integrals at zero momentum transfer, and therefore computed on the same footing as the self-energy contribution.
Because of the presence of the two-loop $W$ mass counterterm, the calculation of $\Delta \hat{r}_W$ involves the evaluation of two-loop self-energy integrals both at $q^2 = 0$, and at $q^2 = m_W^2$, where $q$ is the momentum transfer. Similarly, Eqs. (5) show that $\hat{\rho}$ entails two-point functions evaluated at non-zero momentum transfer. Two-loop self-energy diagrams with non-vanishing masses and momenta cannot in general be expressed in terms of known functions like polylogarithms. However, the extraction of the leading $m_t^4$ and next-to-leading $m_t^2$ contributions from a two-loop self-energy diagram at non-zero $q^2$ can be performed through an asymptotic expansion of the corresponding integrals in inverse powers of the top mass [13]. The $q^2 = 0$ self-energy integrals, instead, can be exactly solved for any mass, expressed in a closed form [14], and then expanded in top mass powers. The zero momentum transfer contributions of the $W$ and $Z$ self-energies have been explicitly checked with the Ward identities of the theory, which were derived up to $O(\alpha^2 m_t^2/m_W^2)$ by current algebra methods [8,14]. In the calculation we consider the Higgs mass as a free parameter. Therefore, before performing the heavy top expansion, we need to specify whether $m_H$ can be considered light with respect to $m_t$, or $m_H, m_t \gg m_Z$, with an arbitrary ratio $m_H/m_t$. Consequently, we derive expressions for the two-loop corrections in the two regions.

The identification of the $m_t^2$ two-loop contribution to $\Delta \hat{r}_W$ and $\Delta \hat{\rho}$ requires a precise specification of the one-loop term. Our one-loop contribution coincides with the analytic expressions reported in Ref. [1], written in terms of physical masses and couplings $\alpha$ and $\hat{c}^2$ ($\Delta \hat{r}_W$), and $\hat{\alpha}$, the $\overline{MS}$ e.m. running coupling, and $\hat{c}^2$ ($\Delta \hat{\rho}$).

With this convention for the one-loop term, we find for the two-loop $O(\alpha^2 m_t^2/m_W^2)$ contribution to $\Delta \hat{r}_W$, in units $N_c (\alpha/(4\pi \hat{s}^2))^2 m_t^2/m_W^2$, with the colour factor $N_c = 3$,

$$
\Delta \hat{r}_W^{(2)} = -\frac{13}{144} - \frac{1}{48 \hat{c}^4} - \frac{41}{96 \hat{c}^2} + \frac{61 \hat{c}^2}{72} + \frac{7 - 16 \hat{c}^2}{27} \pi \sqrt{ht} - \frac{\pi^2}{36} - \frac{5 ht^2}{144 \hat{c}^4 z t^2} + \frac{35 ht}{288 \hat{c}^2 z t} \\
+ \frac{5}{12} \left( 1 + \frac{ht^2}{12 \hat{c}^4 z t^2} - \frac{ht}{3 \hat{c}^2 z t} \right) B0[m_w^2, m_t^2, m_w^2] + \frac{1 + 20 \hat{c}^2 - 24 \hat{c}^4}{48 \hat{c}^4} B0[m_w^2, m_w^2, m_z^2] \\
- \left( 5 \hat{s}^2 ht^2 + 3 ht z t + 48 \hat{c}^2 ht z t - 60 \hat{c}^4 ht z t - 3 \hat{c}^2 z^2 t - 8 \hat{c}^4 z^2 t + 20 \hat{c}^6 z^2 t \right) \ln \hat{c}^2 \\
+ \frac{5 ht \left( ht^2 - 4 \hat{c}^2 ht z t + 12 \hat{c}^4 z^2 t \right) \ln ht}{144 \hat{c}^4 z^2 \left( ht - \hat{c}^2 z t \right)} + \left( \frac{17}{36} - \frac{13 \hat{c}^2}{18} \right) \ln \frac{m_t^2}{\mu^2} \\
- \left( 5 \hat{c}^2 ht^2 - 3 ht z t - 60 \hat{c}^2 ht z t + 60 \hat{c}^4 ht z t + (3 \hat{c}^2 + 60 \hat{c}^4 - 20 \hat{c}^6) z^2 t \right) \ln z t \\
- \frac{144 \hat{c}^4 z t \left( ht - \hat{c}^2 z t \right)}{144 \hat{c}^4 z t \left( ht - \hat{c}^2 z t \right)}, \tag{7a}
$$

for a light Higgs expansion, $m_H \ll m_t$, while in the region $m_H \gg m_Z$ we obtain

$$
\Delta \hat{r}_W^{(2)} = -\frac{121}{288} - \frac{1}{48 \hat{c}^4} - \frac{41}{96 \hat{c}^2} + \frac{77 \hat{c}^2}{12} + \frac{19}{72} \tfrac{ht}{\hat{c}^2} + \left( \frac{41}{216} - \frac{4 \hat{c}^2}{27} \right) \frac{ht}{\hat{c}^2} - \frac{(19 + 21 ht) \pi^2}{432 \hat{c}^2 z^2 t^2} \\
- \left( \frac{1}{2} - \frac{1}{48 \hat{c}^4} - \frac{5}{12 \hat{c}^2} \right) B0[m_w^2, m_w^2, m_z^2] + \frac{16 \hat{c}^2 - 7}{216} (ht - 4) \sqrt{ht} g(ht) \\
- \left( \frac{1}{12} - \frac{1}{3 ht} \right) \Lambda(ht) + \left( \frac{19 + 21 ht - 12 ht^2 - 31 ht^3 + 9 ht^4}{72 \hat{c}^2 z^2 t^2} \right) \text{Li}_2(1 - ht),
$$

for a heavy Higgs expansion. In both cases, the leading $m_t^4$ contribution is $\Delta \hat{r}_W^{(2)} = (1 - \hat{c}^2 / 3) \Delta \hat{r}_W^{(1)}$. The contribution to $\Delta \hat{\rho}$ is [14,15].
\[
- \frac{(1 + 21 \hat{c}^2 - 25 \hat{c}^4) \ln \hat{c}^2}{48 \hat{c}^2 s^2} + \left( \frac{17}{36} - \frac{13 \hat{c}^2}{18} \right) \ln \frac{m_t^2}{\mu^2} + \frac{(1 + 20 \hat{c}^2 - 25 \hat{c}^4) \ln zt}{48 \hat{c}^4} \\
+ \frac{372 + (96\hat{c}^2 - 213)ht + (432\hat{c}^2 - 318)ht^2 + (97 - 160\hat{c}^2)ht^3 - (7 - 16\hat{c}^2)ht^4}{216(ht - 4)ht} \ln ht \\
+ \frac{96 - (384 - 64\hat{c}^2)ht - (2 + 64\hat{c}^2)ht^2 + 231 ht^3 - 85 ht^4 + 9ht^5}{144(ht - 4)ht^2} \phi \left( \frac{ht}{4} \right).
\]

In Eqs. (7), \( ht \equiv (m_u/m_t)^2, zt \equiv (m_z/m_t)^2 \),

\[
g(x) = \begin{cases} 
\sqrt{4 - x} \left( \pi - 2 \arcsin \sqrt{x/4} \right) & 0 < x \leq 4 \\
2\sqrt{x/4 - 1} \ln \left( \frac{1 - \sqrt{1 - 4x}}{1 + \sqrt{1 - 4x}} \right) & x > 4,
\end{cases} \tag{8a}
\]

\[
\Lambda(x) = \begin{cases} 
-\frac{1}{\sqrt{x}} \frac{1}{2} g(x) + \frac{\pi}{2} \sqrt{4/x - 1} & 0 < x \leq 4 \\
-\frac{1}{\sqrt{x}} \frac{1}{2} g(x) & x > 4,
\end{cases} \tag{8b}
\]

we have indicated the dilogarithmic function as \( \text{Li}_2(x) = -\int_0^x dt \frac{\ln(1-t)}{t} \), and introduced

\[
\phi(z) = \begin{cases} 
4\sqrt{1-z} \text{Cl}_2(2 \arcsin \sqrt{z}) & 0 < z \leq 1 \\
\frac{1}{\chi} \left[ -4 \text{Li}_2(1-\lambda) + 2 \ln^2(1-\lambda) - 2 \ln(4z) + \pi^2/3 \right] & z > 1,
\end{cases} \tag{8c}
\]

where \( \text{Cl}_2(x) = \text{Im} \text{Li}_2(e^{ix}) \) is the Clausen function with \( \lambda = \sqrt{1 - \frac{1}{z}} \). The function \( B_0[q^2, m_1^2, m_2^2] \) is defined through the one-loop integral \( \epsilon = (4 - n)/2 \)

\[
-i m_t^2 e^{\gamma \epsilon} \int \frac{d^nk}{\pi^{n/2}} \frac{1}{[k^2 - m_1^2][|k - q|^2 - m_2^2]} = \frac{1}{\epsilon} + B_0[q^2, m_1^2, m_2^2] + O(\epsilon), \tag{9}
\]

whose analytic form is well known (see for example Ref. [9]). It is interesting to note that the \( O(\epsilon) \) part of one-loop integrals like the one in Eqs. (7) cancel exactly in the final two-loop expression.

The two-loop contribution to \( \Delta \hat{r}_W \) is quite small over the whole range of realistic top and Higgs mass values. For instance, using \( m_t = 180 \text{ GeV} \) and \( \hat{s}^2 = 0.2314 \), we find that \( \Delta \hat{r}_W^{(2)} \) has an absolute maximum at \( m_u = 0, +5.7 \times 10^{-5} \), then decreases very rapidly for increasing \( m_u \). The two expansions Eqs. (7) meet at \( m_u \approx 0.3 m_t \), and for the whole range \( 65 \text{GeV} < m_u < 1 \text{ TeV}, |\Delta \hat{r}_W^{(2)}| < 1 \times 10^{-5} \). The same happens for different but realistic values of \( m_t \). This is indeed a quite small effect, comparable in size to routinely neglected contributions.

The calculation of \( \Delta \hat{\rho}^{(2)} \) yields, in units \( N_c [\hat{\alpha} m_t^2/(16\pi \hat{s}^2 \hat{c}^2 m_2^2)]^2 \approx N_c x_t^2 (x_t = G_N m_t^2/(8\pi^2 \sqrt{2})) \):

\[
\Delta \hat{\rho}^{(2)} = 19 - \frac{53}{3} \frac{ht}{3} + \frac{3}{2} \frac{ht^2 \pi}{2} + \frac{8}{9} \frac{ht^2}{zt} - \frac{5}{9} \frac{ht^2}{c^2 zt} + \left( \frac{845}{27} - \frac{1}{3} c^2 + \frac{427}{27} - \frac{122}{9} \right) c^4 zt
\]
for a light Higgs \( m_h \) completely independent of the gauge sector of the theory. Indeed this part can be computed

\[
\Delta \rho^{(2)} = 25 - 4 h t + \left( \frac{1}{2} - \frac{1}{h t} \right) \pi^2 + \frac{(h t - 4) \sqrt{h t} g(h t)}{2} + \left( -6 - 6 h t + \frac{h t^2}{4} \right) \ln h t
\]

\[
+ \left( \frac{6}{h t} - 15 + 12 h t - 3 h t^2 \right) \Pi_2(1 - h t) + \frac{3}{2} (-10 + 6 h t - h t^2) \phi(\frac{h t}{4})
\]

\[
+ z t \left[ \frac{1}{54c^2 (h t - 4) h t} \left( -1776 \dot{c}^4 + (72 - 6250 \dot{c}^2 - 3056 \dot{c}^4 + 3696 \dot{c}^6) h t \right)
\]

\[
+ (-18 + 1283 \dot{c}^2 + 1371 \dot{c}^4 - 1436 \dot{c}^6) h t^2 + (68 \dot{c}^2 - 124 \dot{c}^4 + 128 \dot{c}^6) h t^3)
\]

\[
+ \frac{(6 \dot{c}^2 h t - 37 \dot{c}^2 - 119 h t^2 + 56 \dot{c}^2 h t^2) \pi^2}{27 h t^2} + \left( \frac{32 \dot{c}^4}{3} - \frac{3}{3} - 12 \dot{c}^2 \right) B0[m_w^2, m_w^2, m_w^2]
\]

\[
+ \left( \frac{20}{3} + \frac{1}{3 \dot{c}^2 - 8 \dot{c}^2} \right) B0[m_w^2, m_w^2, m_w^2] + \frac{(17 - 58 \dot{c}^2 + 32 \dot{c}^4) (4 - h t) \sqrt{h t} g(h t)}{27}
\]

\[
- \frac{40s^2 (4 - h t) \Lambda(h t)}{3 h t} + \frac{2 \dot{c}^2 \left( 37 - 6 h t - 12 h t^2 - 22 h t^3 + 9 h t^4 \right) \Pi_2(1 - h t)}{9 h t^2}
\]

\[
- \frac{(1 + 14 \dot{c}^2 + 16 \dot{c}^4) \ln \dot{c}^2}{3} + \left( 11520 - 15072 \dot{c}^2 - (7170 - 8928 \dot{c}^2 - 768 \dot{c}^4) h t \right)
\]

\[
+ (3411 - 7062 \dot{c}^2 + 3264 \dot{c}^4) h t^2 - (1259 - 3547 \dot{c}^2 + 2144 \dot{c}^4) h t^3
\]

\[
+ (238 - 758 \dot{c}^2 + 448 \dot{c}^4) h t^4 - (17 - 58 \dot{c}^2 + 32 \dot{c}^4) h t^5 \right) \frac{\ln h t}{27(h t - 4)^2 h t}
\]

\[
+ \frac{8}{9} \left( 4 - 26 \dot{c}^2 - 5 \dot{c}^4 \right) \ln \frac{m_t^2}{\mu^2} + \frac{(3 + 5 \dot{c}^2 - 26 \dot{c}^4 - 48 \dot{c}^6) \ln z t}{9 \dot{c}^2}
\]

\[
+ \left( 3840 s^2 - (4310 - 4224 \dot{c}^2 - 256 \dot{c}^4) h t + (1706 - 1312 \dot{c}^2 - 320 \dot{c}^4) h t^2 \right)
\]

\[
-(315 + 476 \dot{c}^2 - 64 \dot{c}^4) h t^3 + (24 + 454 \dot{c}^2) h t^4 - 112 \dot{c}^2 h t^5 + 9 \dot{c}^2 h t^6 \right) \frac{\phi(h t)}{9(h t - 4)^2 h t^2} \right].
\]

The first two lines of Eqs. \([10b]\) represent the leading \( O(G^2/m_t^4) \) result \([1, 7]\), which is completely independent of the gauge sector of the theory. Indeed this part can be computed
in the framework of a pure Yukawa theory, obtained from the SM in the limit of vanishing gauge coupling constants. The rest of Eq. (10a) is proportional to $zt = m^2_t / m^2_W$ and represents the first correction to the Yukawa limit.

We note that the $O(\alpha^2 m^4_t / m^4_W)$ contribution to Eqs. (10) is much more relevant than the one to $\Delta \rho$, following a pattern very similar to the one for the $\rho$ parameter in $\nu_\mu - e$ scattering [4,11]. In Fig.1 we show the behaviour of the two expansions in Eqs. (10) as functions of $m_H$, for $m_t = 180\text{GeV}$ and $s^2 = 0.2314$. We also show the leading $m^4_t$ contribution (first two lines of Eq. (10b)), and a simple interpolation curve which reproduces the light and heavy higgs expansions with very good accuracy in their expected ranges of validity.

For $160\text{GeV} < m_H < 200\text{GeV}$, $m_H < 3.8 m_t$ and $s^2 \approx 0.2314$, this interpolation function takes the form:

$$f(h, m_t) = -15.642 + 0.036382 m_t + \sqrt{h}(2.301 - 0.01343 m_t) + h(0.01809 m_t - 9.953) + h^2(5.687 - 0.01568 m_t) + h^3(0.005369 m_t - 1.647) + h^4(0.1852 - 0.000646 m_t) \ (h = m_H/m_t).$$

The figure clearly shows the magnitude of the new $O(\alpha^2 m^2_t / m^2_W)$ correction: leaving aside the very low Higgs region, where large cancellations conspire to make the leading $O(\alpha^2 m^4_t / m^4_W)$ correction particularly small, we observe that in most of the allowed Higgs range the $O(\alpha^2 m^2_t / m^2_W)$ term is roughly as large as the leading correction. With the purpose of investigating the magnitude of the $O(\alpha^2 m^2_t / m^2_W)$ corrections to physical observables, we have calculated the theoretical predictions for $m_W$ and $\sin^2 \theta_W (m^2_Z)$ for different values of $m_t$ and $m_H$, solving iteratively Eqs. (4) and (6). In Table 1 we report the shifts induced in $m_W$ and $s^2$ by the new $O(\alpha^2 m^2_t / m^2_W)$ contributions with respect to the inclusion of the

\[\Delta \hat{\rho}^{(2)}(N_c x_t^2) = \frac{(m^2_t/(4m^2_Z))^2}{2}.\]
leading $O(\alpha^2 m_t^4/m_W^4)$ term only. We emphasize that because of the presence of irreducible $O(\alpha^2 m_t^2/m_W^2)$ effects in the $\gamma - Z$ mixing on the $Z$-peak, the shifts obtained for $s^2$ cannot be simply related [17] to the effective sine measured at LEP and SLC.

In conclusion, we have presented the results of a complete analytic calculation of the $O(\alpha^2 m_t^2/m_W^2)$ effects on the interdependence of $m_W$, $m_Z$, and $G_\mu$. We find that for most of the relevant $m_\mu$ values the $O(\alpha^2 m_t^2/m_W^2)$ correction is of the order of the leading $O(\alpha^2 m_t^4/m_w^4)$ term, and that the effect on the prediction for $m_W$ from $\alpha$, $G_\mu$, and $m_Z$, can be as large as 20MeV, depending on the top and Higgs masses (see Table 1). Details of this calculation will be presented in a forthcoming communication.

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| $m_H$ | $m_t$ | $\delta m_w$ (MeV) | $\delta \hat{s}^2$ ($10^{-4}$) | $\delta \hat{\rho}^{(2)}$ ($10^{-4}$) |
|-------|-------|---------------------|-----------------------------|-----------------------------------|
| 160   | 180   | -10.1               | 0.59                        | -1.72                             |
| 65    | 180   | -15.2               | 0.88                        | -2.57                             |
|       | 200   | -21.2               | 1.23                        | -3.62                             |
| 160   | 180   | -9.6                | 0.55                        | -1.63                             |
|       | 100   | -14.3               | 0.82                        | -2.45                             |
|       | 200   | -20.2               | 1.15                        | -3.46                             |
| 160   | 180   | -8.3                | 0.46                        | -1.41                             |
| 300   | 180   | -12.5               | 0.70                        | -2.14                             |
|       | 200   | -17.7               | 0.98                        | -3.05                             |
| 160   | 180   | -7.0                | 0.39                        | -1.16                             |
| 600   | 180   | -11.0               | 0.62                        | -1.87                             |
|       | 200   | -16.2               | 0.90                        | -2.76                             |

Table 1: Shifts induced by the complete $O(\alpha^2 m_t^2 / m_W^2)$ corrections in $m_w$, $\sin^2 \hat{\theta}_{\text{MS}}(m_Z^2)$, and $\hat{\rho}$ with respect to the inclusion of the leading $O(\alpha^2 m_t^4 / m_W^4)$ term alone. Top and Higgs masses are expressed in GeV.