$CPT$–conserving Hamiltonians and their nonlinear supersymmetrization using differential charge-operators $C$

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Abstract

A brief overview is given of recent developments and fresh ideas at the intersection of $\mathcal{PT}$– and/or $\mathcal{CPT}$–symmetric quantum mechanics with supersymmetric quantum mechanics (SUSY QM). Within the framework of the resulting supersymmetric version of $\mathcal{CPT}$–symmetric quantum mechanics we study the consequences of the assumption that the “charge” operator $\mathcal{C}$ is represented in a differential-operator form of the second or higher order. Besides the freedom allowed by the Hermiticity constraint for the operator $\mathcal{CP}$, encouraging results are obtained in the second-order case. In particular, the integrability of intertwining relations proves to match the closure of our nonlinear ($\textit{viz.}$, polynomial) SUSY algebra. In a particular illustration, our form of $\mathcal{CPT}$–symmetric SUSY QM leads to a new class of non-Hermitian polynomial oscillators with real spectrum which turn out to be $\mathcal{PT}$–asymmetric.

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1 Introduction

The recent growth of interest in the possibility of working with non-Hermitian observables in quantum theory (cf. the concise review papers [1]) is mainly due to the influential Bender’s and Boettcher’s letter [2] where its authors observed that the spectrum of certain Hamiltonians $H \neq H^\dagger$ seems real and discrete and bounded below.

They conjectured that such an observation may find a firmer mathematical background and explanation in the symmetry of their models with respect to the combined action of the parity $P$ and time-reversal (i.e., complex conjugation) $T$. This inspiring idea has been further developed and re-formulated as proposals of the so called $\mathcal{PT}$—symmetric quantum mechanics [3], pseudo-Hermitian quantum mechanics [4] and $\mathcal{CPT}$—symmetric quantum mechanics [5]. They all deal with more or less the same class of the specific non-Hermitian models characterized, in the language of the latter reference, by another symmetry operator $C$ which is very conveniently called ”charge”.

There exists an extensive literature on $\mathcal{PT}$-symmetric quantum mechanics [6, 7]. In particular, in a number of papers [8, 9, 10, 11], unexpected consequences of the non-Hermiticity of Hamiltonians have been noticed to emerge after its supersymmetrization a la Witten [12]. In terms of local models $H = p^2 + V(x)$ on the real line ($x \in \mathbb{R}$) where $V(x) = V^*(−x)$, these Hamiltonians satisfy the intertwining relation

$$H^\dagger P = P H.$$  \hspace{1cm} (1)

Such $\mathcal{PT}$-symmetric Hamiltonians may have either complex, or real spectra. When the $\mathcal{PT}$ symmetry remains spontaneously unbroken and all the spectrum is real [2], one has elaborated the concept of quasi-Hermiticity of the Hamiltonian [13, 14]. This means that the intertwining relation (1) holds also with $P$ replaced by a positive-definite operator $\Theta = \Theta^\dagger > 0$ which plays the role of a metric operator. The physical interpretation of such models is standard [15]. When the spectrum is complex, relation (1) can still be written with $P$ replaced by a pseudo-metric [16, 17, 18].

We shall now generalize the previous considerations to a new type of symmetry. The framework of our constructions proposed in our recent letter [19] will incorporate Hamiltonians with both real and complex spectra. Correspondingly, we shall also deal, in general, with non-positive metric (i.e., pseudo-metric). As for the case of $\mathcal{PT}$ symmetry, the interpretation of this type of quantum mechanics can be disputable [14] and might require some innovation. However, we stress that, in our framework, we can find models which have real spectra, where, in particular, a non-Hermitian Hamiltonian is related by similarity transformations not only to a Hermitian operator, but, more specifically, to a Hermitian Schrödinger operator. Thus, for these
cases, we recover the standard quantum mechanics, after similarity. Therefore, these cases are not disputable in their interpretation. From a conservative point of view, one might restrict the interest of our supersymmetric approach insofar as one takes it instrumentally as a strategy to find complex Hamiltonians with real spectrum that do not satisfy $\mathcal{PT}$ invariance (see Section 4 below).

1.1 SUSY intertwining relations

In the same spirit as in Ref. [19], we shall study the intertwining relations

$\mathcal{F} H^\dagger = H \mathcal{F} \tag{2}$

mediated by the Hermitian operator

$\mathcal{F} = \mathcal{C} \mathcal{P} \quad (\mathcal{F} = \mathcal{F}^\dagger), \tag{3}$

where $\mathcal{P}$ is the parity operator, and $\mathcal{C}$ a generalized ”charge” operator, assumed to be a polynomial in the differential operator $d/dx$. For any Hamiltonian $H$, Eq. (2) is equivalent to $\mathcal{CPT}$ conservation, with $\mathcal{T}$ the time reversal operator,

$\mathcal{CPT} H = H \mathcal{CPT}. \tag{4}$

In this paper we shall not discuss in detail the metric interpretation for $\mathcal{F}$, but only stress the fact that, if $\mathcal{F}$ and $H$ satisfy Eq. (2), then also $\mathcal{F}^{-1}$ (if it exists) and $H$ meet an intertwining

$H^\dagger \mathcal{F}^{-1} = \mathcal{F}^{-1} H \tag{5}$

which means that $H$ is pseudo-Hermitian with respect to $\mathcal{F}^{-1}$, i.e., $\mathcal{F}^{-1}$—pseudo-Hermitian [4].

This observation may be useful for implementing the metric based on $\mathcal{F}^{-1}$, when $\mathcal{F}^{-1}$ has a better behavior than $\mathcal{F}$, e.g. with respect to boundedness. Nevertheless, in our text we also use for a $\mathcal{F}$ satisfying Eqs. (3), (2), the word “metric” operator. In fact, Eq. (2) implies that $H \mathcal{F}$ is Hermitian. As a consequence of (5), if $| \phi \rangle$ and $| \psi \rangle$ are two arbitrary vectors of the Hilbert space $L^2(\mathbb{R})$, we have

$\int \phi^*(x) (\mathcal{F}^{-1} H \psi) (x) dx = \int \psi^*(x) (\mathcal{F}^{-1} H \phi) (x) dx. \tag{3}$

This can be interpreted as a Hermiticity condition for $H$ provided the scalar product is defined as

$\langle \phi | \psi \rangle_{\mathcal{F}^{-1}} = \int \phi^*(x) (\mathcal{F}^{-1} \psi) (x) dx; \quad \langle \psi | \phi \rangle_{\mathcal{F}^{-1}} = \int \psi^*(x) (\mathcal{F}^{-1} \phi) (x) dx. \tag{4}$
It is worthwhile to point out, however, that, in absence of additional constraints, neither $\mathcal{F}^{-1}$ nor $\mathcal{F}$ is necessarily positive definite so that, for instance, the equation $\mathcal{F} | \phi \rangle = 0$ might have a non-trivial solution different from $| \phi \rangle = 0$. At this level, $\langle \phi | \phi \rangle_{\mathcal{F}^{-1}}$ does not define a true norm but merely a pseudo-norm [17, 20].

It is evident that solving Eq. (2) amounts to analyzing the compatibility between $\mathcal{C}$ and $\mathcal{H}$; in other words, $\mathcal{C}$ and $\mathcal{H}$ are to be found contextually. Once Eq. (2) is formally solved, one can investigate its supersymmetrization [21]. By this we mean the construction of super-charges

$$Q = \begin{pmatrix} 0 & \mathcal{F} \\ 0 & 0 \end{pmatrix}, \quad \tilde{Q} = \begin{pmatrix} 0 & 0 \\ \mathcal{F}^* & 0 \end{pmatrix}$$

(6)

with anti-commutator

$$K \equiv \{ Q, \tilde{Q} \} = \begin{pmatrix} \mathcal{F} \mathcal{F}^* & 0 \\ 0 & \mathcal{F}^* \mathcal{F} \end{pmatrix},$$

(7)

and a polynomial formulae

$$\mathcal{F} \mathcal{F}^* = \sum_{k=0}^{n} a_k H^k, \quad \mathcal{F}^* \mathcal{F} = \sum_{k=0}^{n} a^*_k (H^*)^k$$

(8)

with the final goal to elucidate the conditions leading to such a type of the closure of the algebra [22].

### 1.2 Plan of the paper

In Section 2 we elaborate a particular solution to our problem inspired by the specific second-order supersymmetry (SSUSY) results of ref. [11]. Our solution of Eq. (8) will have the form

$$\mathcal{F} \mathcal{F}^* = h_1^2 - \frac{c^2}{4}, \quad \mathcal{F}^* \mathcal{F} = h_2^2 - \frac{c^2}{4},$$

(9)

where $h_1$ is naturally related to $h_2$ by Hermitian conjugation,

$$h_1 = h_2^\dagger,$$

(10)

if $c^2$ is real. Our explicit solution to the problem is rendered possible by a SSUSY inspired gluing constraint [21, 22]. We show that

$$\mathcal{F} h_2 = h_1 \mathcal{F},$$

(11)

which, because of Eq. (10), is now equivalent to Eq. (2). This amounts to

$$\mathcal{CPT} h_1 = h_1 \mathcal{CPT}.$$
Explicit analytic examples of $\mathcal{PT}$–asymmetric models are expressed in terms of circular or hyperbolic functions.

In Section 3 we perform a detailed investigation of eq. (2) for a charge operator which is of the second order in derivatives,

$$\mathcal{C} = \frac{d^2}{dx^2} + G(x) \frac{d}{dx} + D(x),$$

and where $G(x)$ and $D(x)$ are complex functions of the real coordinate $x$:

$$G(x) = G_R(x) + iG_I(x),$$

$$D(x) = D_R(x) + iD_I(x).$$

We further derive the polynomial algebra of Eq. (8). In order to show explicitly that our formalism allows to generate $\mathcal{PT}$–asymmetric models with real spectrum, we discuss in Section 4 a particular polynomial oscillator model.

In Section 5 we generalize the postulate (13) and derive the general form of the charge operator $\mathcal{C}$ of any finite order in the derivative $d/dx$ such that $\mathcal{F} \equiv CP$ is Hermitian. At the very end, in section 6 we give some perspectives on the impact of our results on a variety of fields where the use of similar $\mathcal{F}$ might play significant role.

## 2 SUSY gluing constraint

Starting with a second-order $\mathcal{C}$ of the form (13) we have to guarantee, first of all, the Hermiticity of $\mathcal{F} = CP$ and $\mathcal{F}^{-1} = PC^{-1}$. It is easy to show (see also section 5 below for an exhaustive discussion of these important conditions for polynomial charges) that the latter Hermiticity condition forces us to impose the necessary and sufficient requirements

$$D_R(x) = D_R(-x) + \frac{d}{dx}G_R(x), \quad D_I(x) = -D_I(-x) + \frac{d}{dx}G_I(x)$$

where $G_R(x) = G_R(-x)$ is even while $G_I(x) = -G_I(-x)$ must be odd.

### 2.1 Factorization

In the subsequent step of our considerations we factorize our second-order charge operator $\mathcal{C}$ as follows,

$$\mathcal{C} = q_1 q_2, \quad q_1 = \frac{d}{dx} + U(x), \quad q_2 = \frac{d}{dx} + W(x),$$

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$$q_1 = \frac{d}{dx} + U(x), \quad q_2 = \frac{d}{dx} + W(x),$$
where
\[ U(x) + W(x) = G(x), \quad \frac{d}{dx} W(x) + U(x) W(x) = D(x). \tag{15} \]

In order to simplify the problem at the start we impose the following "gluing" constraint on \( q_1 \) and \( q_2 \),
\[ q_2(q_2^\dagger)^* = (q_1^\dagger)^* q_1 + c, \tag{16} \]
where \( c \) is a complex number. By inserting Eqs. (14) into Eq. (16), we obtain
\[ \left( \frac{d}{dx} + W \right) \left( - \frac{d}{dx} + W \right) = \left( - \frac{d}{dx} + U \right) \left( \frac{d}{dx} + U \right) + c, \]
whence
\[ \frac{d}{dx} W(x) + W^2(x) = - \frac{d}{dx} U(x) + U^2(x) + c. \tag{17} \]

We find the following representation for \( \mathcal{F}\mathcal{F}^* \) and \( \mathcal{F}^*\mathcal{F} \) (Eq. (3))
\[
\mathcal{F}\mathcal{F}^* = \mathcal{F}\left(\mathcal{F}^\dagger\right)^* = (q_1 q_2 \mathcal{P}) \cdot \left( \mathcal{P} q_2^\dagger q_1^\dagger \right)^* = q_1 q_2 \left( \mathcal{P} \right)^2 \left( q_2^\dagger \right)^* \left( q_1^\dagger \right)^*,
\]
which, taking Eq. (16) into account, becomes
\[
\mathcal{F}\mathcal{F}^* = q_1 \left[ (q_1^\dagger)^* q_1 + \frac{c}{2} + \frac{c}{2} \right] \left( q_1^\dagger \right)^* = \left[ q_1 \left( q_1^\dagger \right)^* + \frac{c}{2} + \frac{c}{2} \right] \cdot \left[ q_1 \left( q_1^\dagger \right)^* + \frac{c}{2} - \frac{c}{2} \right].
\]
Correspondingly,
\[
\mathcal{F}^*\mathcal{F} = \left( \mathcal{F}^\dagger \right)^* \mathcal{F} = \left[ \mathcal{P} \left( q_2^\dagger \right)^* q_2 \mathcal{P} - \frac{c}{2} - \frac{c}{2} \right] \cdot \left[ \mathcal{P} \left( q_2^\dagger \right)^* q_2 \mathcal{P} - \frac{c}{2} + \frac{c}{2} \right].
\]

Defining the Hamiltonian operators
\[
h_1 = q_1 (q_1^\dagger)^* + \frac{c}{2} = \left( \frac{d}{dx} + U \right) \left( - \frac{d}{dx} + U \right) + \frac{c}{2} = - \frac{d^2}{dx^2} + \frac{d}{dx} U(x) + U^2(x) + \frac{c}{2},
\]
6
and
\[
\begin{align*}
    h_2 &= \mathcal{P}(q_1^\dagger)q_2\mathcal{P} - \frac{c}{2} \\
    &= \mathcal{P}\left(-\frac{d}{dx} + W(x)\right)\left(\frac{d}{dx} + W(x)\right)\mathcal{P} - \frac{c}{2} \\
    &= -\frac{d^2}{dx^2} - \frac{d}{dx}W(-x) + W^2(-x) - \frac{c}{2},
\end{align*}
\]
equation (7) provides the following representation for \( K \)
\[
K = \mathcal{H}^2 - \frac{c^2}{4}, \quad \mathcal{H} = \begin{pmatrix} h_1 & 0 \\ 0 & h_2 \end{pmatrix}.
\]
Comparing with Section 3.3 below, where \( \mathcal{F}\mathcal{F}^* = H^2 + \alpha H + \gamma \), and setting \( h_1 = H \), we get in the present case \( \alpha = 0 \) and, correspondingly, \( V_0 = 0 \), according to Eq. (53) below, as well as \( \gamma = -c^2/4 \). This shows explicitly how the present model can be derived from the general results of section 3.

### 2.2 Hamiltonians

Remembering the first of Eqs. (15), Eq. (17) becomes
\[
\frac{d}{dx}G(x) + W^2(x) - (G(x) - W(x))^2 = c,
\]
i.e.,
\[
\frac{d}{dx}G(x) - G^2(x) + 2G(x)W(x) = c,
\]
or
\[
G(x)W(x) = \frac{1}{2}\left(G^2(x) - \frac{d}{dx}G(x) + c\right). \quad \text{(18)}
\]
We immediately deduce that
\[
\begin{align*}
    W(x) &= \frac{G^2(x) - \frac{d}{dx}G(x) + c}{2G(x)} \\
    U(x) &= G(x) - W(x) = \frac{G^2(x) + \frac{d}{dx}G(x) - c}{2G(x)}.
\end{align*}
\]
(19)
Thus
\[
h_1 = -\frac{d^2}{dx^2} + V(x),
\]
with
\[
V(x) = G'(x) - \frac{(G'(x))^2}{4G^2(x)} + \frac{G''(x)}{2G(x)} + \frac{G^2(x)}{4} + \frac{c^2}{4G^2(x)}. \quad \text{(20)}
\]
From Eq. (18) we also get that at the zeros $\bar{x}$ of $G$, we must have

$$ \left. \frac{dG}{dx} \right|_{x=\bar{x}} = c,$$

which is a constraint on $G$, too. In fact, the method would fail if $G$ had several zeros with non-identical values of the first derivative at each of them.

An important comment must be made here since even if a function does not vanish on the real axis, one can investigate its zeros in the complex $x$ plane. For instance, if

$$G(x) = G_0(x) \equiv z(x) = \frac{1 + i \sinh(\alpha x)}{2}, \quad \alpha \in \mathbb{R}, \quad (21)$$

it is immediate to check that $z(x_n) = 0$ at $x_n = -i (2n + 3/2)\pi/\alpha$, $n = 0, \pm 1, \ldots$. This would mean that $dG_0(x_n)/dx = (i\alpha/2) \cosh(\alpha x_n) = 0$, thus implying that we must put $c = 0$ in this case.

In the similar spirit, we may consider the whole class of functions which depend on $x$ only via $z(x)$ of Eq. (21) in an arbitrary nonlinear manner, $G_m(x) \equiv G(z(x))$, since, as a function of $x$, $z$ is $\mathcal{PT}$-symmetric, and any real function of $z$ is $\mathcal{PT}$-symmetric, too, and is an acceptable candidate for $G$.

It becomes convenient to change variables and express the Hamiltonian, $H = -d^2/dx^2 + V(x)$, with $V(x)$ given by formula (20), as a function of $z$, by observing that

$$\frac{d}{dx} = \frac{dz}{dx} \frac{d}{dz}, \quad \frac{d^2}{dx^2} = \left( \frac{dz}{dx} \frac{d}{dz} \right)^2 = -\alpha^2 \left( \frac{1}{2} - z \right) \frac{d}{dz} - \alpha^2 z(1 - z) \frac{d^2}{dz^2},$$

and

$$V(z) = i\alpha \sqrt{z(1-z)} \frac{d}{dz} G + \alpha^2 z(1-z) \frac{d}{dz} \left( \frac{d}{dz} G \right)^2 - \alpha^2 \frac{1-2z}{4G^2} \frac{d}{dz} G^n + \frac{c^2}{4G^2}. \quad (22)$$

### 2.3 Consistency

We prove now an important constraint on the complex number $c = c_R + ic_I$. From the second of Eqs. (15), we have

$$D(x) = \frac{d}{dx} W(x) + U(x) W(x)$$

$$= \frac{1}{2G^2(x)} \left[ \left( 2G \frac{d}{dx} G - \frac{d^2}{dx^2} G \right) G - \frac{d}{dx} G \left( G^2 - \frac{d}{dx} G + c \right) \right]$$

$$+ \frac{1}{4G^2} \left[ G^4 - \left( \frac{d}{dx} G \right)^2 - c^2 + 2c \frac{d}{dx} G \right].$$


or

\[
D(x) = \frac{1}{4G^2} \left[ 2G^2 \frac{d}{dx} G - 2G \frac{d^2}{dx^2} G + \left( \frac{d}{dx} G \right)^2 + G^4 - c^2 \right], \tag{23}
\]

\[
D^*(-x) = \frac{1}{4G^2} \left[ -2G^2 \frac{d}{dx} G - 2G \frac{d^2}{dx^2} G + \left( \frac{d}{dx} G \right)^2 + G^4 - (c^*)^2 \right], \tag{24}
\]

where the functions on the right-hand-sides of Eqs. (23) and (24) are all computed at \( x \).

In deriving Eq. (24), use has been made of the fact that \( G \) and \( \frac{d^2 G}{dx^2} \) are \( \mathcal{PT} \)-symmetric, while \( \frac{dG}{dx} \) is \( \mathcal{PT} \)-antisymmetric, i.e., \( (\frac{dG}{dx}(-x))^* = -\frac{dG}{dx}(x) \). Eq. (23) is obviously consistent with the general form of \( D \) as a function of \( G \) given by Eqs. (48), (50), with \( c^2/4 = -I_0 - D(x_0)G^2(x_0) \).

Subtracting Eq. (24) from Eq. (23) side by side, we obtain

\[
D(x) - D^*(-x) = \frac{d}{dx} \left( \frac{d}{dx} G + \frac{c^2}{2G} \right) + \frac{c^2}{4G^2}.
\]

(25)

Combining Eq. (25) with Eq. (33), we obtain the important result

\[
(c^*)^2 - c^2 = 0 \quad \rightarrow \quad \Im(c^2) = 0 \quad \rightarrow \quad c_R c_I = 0.
\]

From Eq. (19) we easily obtain

\[
U(x) = W^*(-x) - \frac{c_R}{G(x)},
\]

whence

\[
\left( \frac{d}{dx} U(x) \right)^* = -\frac{d}{dx} W(-x) + \frac{c_R}{(G^*(x))^2} \frac{d}{dx} G^*, \tag{26}
\]

and

\[
(U^*(x))^2 = W^2(-x) + \frac{c_R^2}{(G^*(x))^2} - 2c_R \frac{W(-x)}{G^*(x)}. \tag{27}
\]

Thus

\[
h_1^\dagger - h_2 = \left( \frac{d}{dx} U(x) + U^2(x) \right)^* + \frac{d}{dx} W(-x) - W^2(-x) + c_R
\]

\[
= c_R \left[ \frac{1}{(G^2(x))^*} \left( \frac{d}{dx} G^* + c_R \right) - 2 \frac{W(-x)}{G^*(x)} + 1 \right].
\]

Using Eq. (19) to replace \( W(-x) \), we obtain

\[
h_1^\dagger - h_2 = c_R \left[ \frac{1}{(G^2)^*} \left( \frac{d}{dx} G^* + c_R \right) - \frac{1}{(G^2)^*} \left( (G^2)^* + \frac{d}{dx} G^* + c \right) + 1 \right]
\]

\[
= c_R \frac{c_R - c}{(G^2)^*} = -i c_R c_I \frac{G^2}{|G|^4}. \tag{28}
\]

Therefore

\[
h_1^\dagger = h_2 \quad \Leftrightarrow \quad c_R c_I = 0.
\]
2.4 Periodic potential

Let us now give an example which generalizes $\mathcal{PT}$-symmetric periodic potentials [8, 23]:

$$G(x) = e^{i\alpha x} + r, \quad \alpha \in \mathbb{R}, \quad r \in \mathbb{R}, \quad r \neq \pm 1.$$ 

In this case we have, for all $x \in \mathbb{R}$,

$$U(x) = \frac{1}{2} \left( e^{i\alpha x} + r \right) + \frac{1}{2} \left( i\alpha e^{i\alpha x} - c \right) / \left( e^{i\alpha x} + r \right)$$

and

$$W(x) = \frac{1}{2} \left( e^{i\alpha x} + r \right) - \frac{1}{2} \left( i\alpha e^{i\alpha x} - c \right) / \left( e^{i\alpha x} + r \right).$$

Since $G$ never vanishes, we do not have any constraint on the value of $c$, in addition to the one which requires that $c$ be either real, or imaginary. The spectral analysis of the corresponding Schrödinger operators $h_1$ and $h_2$ with periodic potentials can be performed as a generalization to the non-$\mathcal{PT}$-symmetric case of the investigation done by in Ref. [24].

We now examine the invertibility of $C$ and the boundedness of $C^{-1}$. First notice that $C$ can be written in the following form

$$C = C_1 C_2, \quad C_1 = C_U + \frac{r}{2}, \quad C_2 = C_U + \frac{r}{2}$$

(29)

where

$$C_U = \frac{d}{dx} + U_1, \quad U_1 = U - \frac{r}{2}$$

and

$$C_W = \frac{d}{dx} + W_1, \quad W_1 = W - \frac{r}{2}.$$ 

We will discuss the invertibility of each factor in (29) separately. As for $C_1$ we first observe that the numerical range $\{ z = \langle C_U \psi, \psi \rangle : \psi \in H^1(\mathbb{R}) \}$ of $C_U$ is contained in the strip $\{ z : |Re z| \leq a \}$ where $a = \max_{x \in \mathbb{R}} |U_1(x)|$. Hence, if $|r| > 2a$ then $-r/2$ is in the resolvent set of $C_U$ [25] and, therefore, $C_1$ is invertible with bounded inverse on $L^2(\mathbb{R})$. A similar argument holds for $C_W$. Thus, for sufficiently large values of $|r|$, operator $C$ is invertible and $C^{-1}$ is bounded on $L^2(\mathbb{R})$.

3 Second-order charge operator $C$

3.1 Re-construction of the potential

We already noticed that in the second-order charge operator (13), the notation of Section 5 below implies that we have the correspondences $\gamma_2(x) = 1$, $\gamma_1(x) = G(x)$ and $\gamma_0(x) = D(x)$, so that the Hermiticity constraints on
the real and imaginary parts of $\gamma_\ell(x)$, Eqs. (70) and (71), with $\omega = 2$ and $\ell = 0, 1$, immediately give

\[
G_R(x) - G_R(-x) = 0; \quad G_I(x) + G_I(-x) = 0; \tag{30}
\]

\[
D_R(x) - D_R(-x) = \frac{d}{dx}G_R(x); \quad D_I(x) + D_I(-x) = \frac{d}{dx}G_I(x). \tag{31}
\]

As a consequence of Eq. (30), $G$ is $\mathcal{PT}$-symmetric

\[
G(x) = G^*(-x), \tag{32}
\]

while Eq. (31) yields

\[
D(x) - D^*(-x) = \frac{d}{dx}G(x). \tag{33}
\]

We assume that $F$ and $H$ satisfy the intertwining condition (2) and that $H$ depends on a local complex potential, $V(x)$:

\[
H = -\frac{d^2}{dx^2} + V(x), \tag{34}
\]

with $V(x) = V_R(x) + iV_I(x)$. In turn, $V_R(x)$ and $V_I(x)$ are conveniently decomposed into their even and odd parts:

\[
V_R(x) = V_R^E(x) + V_R^O(x), \quad V_I(x) = V_I^E(x) + V_I^O(x),
\]

with $V_R^E(x) = V_R^E(-x)$ and $V_R^O(x) = -V_R^O(-x)$, ($K = R, I$). We write now condition (2) explicitly and obtain three non-trivial equations by imposing that the coefficients of $(d/dx)^2$, $d/dx$ and $(d/dx)^0$ vanish,

\[
-2 \left( V_R^O + iV_I^E \right) + 2 \frac{d}{dx} (G_R + iG_I) = 0, \quad (35)
\]

\[
2 \frac{d}{dx} \left( V_R^E + iV_I^O \right) - 2 \frac{d}{dx} \left( V_R^O + iV_I^E \right) + 2 \frac{d}{dx} (D_R + iD_I) + \frac{d^2}{dx^2} (G_R + iG_I) - 2 \left( V_R^O + iV_I^E \right) (G_R + iG_I) = 0, \quad (36)
\]

\[
\frac{d^3}{dx^3} \left( V_R^E + iV_I^O \right) - \frac{d^2}{dx^2} \left( V_R^O + iV_I^E \right) + \frac{d}{dx} (D_R + iD_I) + (G_R + iG_I) \frac{d}{dx} \left( V_R^E(x) + iV_I^O(x) \right) - (G_R + iG_I) \frac{d}{dx} \left( V_R^O + iV_I^E \right) - 2 (D_R + iD_I) \left( V_R^O + iV_I^E \right) = 0 \tag{37}
\]

while the coefficients of $(d/dx)^4$ and $(d/dx)^3$ are identically zero.
3.2 Integrability

The first of the above equations (35) yields

\[ V_R^O = \frac{d}{dx} G_R; \quad V_I^E = \frac{d}{dx} G_I. \]  (38)

The second equation (36) yields

\[
\frac{d}{dx} V_R^E - \frac{d}{dx} V_R^O G_R + V_I^E G_I + \frac{d}{dx} D_R + \frac{1}{2} \frac{d^2}{dx^2} G_R = 0; \\
\frac{d}{dx} V_I^O - \frac{d}{dx} V_I^E G_R - V_R^O G_I + \frac{d}{dx} D_I + \frac{1}{2} \frac{d^2}{dx^2} G_I = 0,
\]

and is easily integrated for the other two components of the potential, \( V_R^E (x) \) and \( V_I^O (x) \), as functions of \( G_R (x) \), \( G_I (x) \), \( D_R (x) \) and \( D_I (x) \), by replacing \( V_R^O (x) \) and \( V_I^E (x) \) with their expressions (38):

\[
V_R^E (x) = \frac{1}{2} \frac{d}{dx} G_R (x) + \frac{1}{2} (G_R (x))^2 - \frac{1}{2} (G_I (x))^2 - D_R (x) + V_0 \quad (39) \\
V_I^O (x) = \frac{1}{2} \frac{d}{dx} G_I (x) + G_R (x) G_I (x) - D_I (x). \quad (40)
\]

Here, \( V_0 \) is a real integration constant. The corresponding integration constant in the equation for \( V_I^O (x) \) must be zero, because the function is odd.

Both equations can be recombined as

\[ V(x) = \frac{3}{2} \frac{d}{dx} G(x) + \frac{1}{2} G^2 (x) - D(x) + V_0. \quad (41) \]

Finally, the third equation (37) allows us to express the \( G_J (x) \)'s, \( (J = R, I) \), as functions of the \( D_K (x) \)'s, \( (K = R, I) \), or, more conveniently, vice versa.

\[
\begin{align*}
- \frac{1}{2} \frac{d^3}{dx^3} G_R + \frac{G_R}{2} \frac{d^2}{dx^2} G_R + & \left( \frac{d}{dx} G_R \right)^2 + \left( G_R^2 - G_I^2 - 2D_R \right) \frac{d}{dx} G_R \\
- \frac{G_I}{2} \frac{d^2}{dx^2} G_I - & \left( \frac{d}{dx} G_I \right)^2 + 2 (D_I - G_I G_R) \frac{d}{dx} G_I - G_R \frac{d}{dx} D_R + G_I \frac{d}{dx} D_I \\
= & 0 \quad (42)
\end{align*}
\]

Eqs. (42) can be recombined in the following first-order linear equation expressing the unknown function \( D(x) \) in terms of the known function \( G(x) \)
and its derivatives up to third order

\[
\frac{1}{2} \frac{d^3}{dx^3} G - \frac{1}{2} \frac{d^2}{dx^2} G - \left( \frac{d}{dx} G \right)^2 - G^2 \frac{d}{dx} G + 2 \left( \frac{d}{dx} G \right) D + G \frac{d}{dx} D = 0. \tag{43}
\]

Eq. (43) is easily solved by direct integration. Let us define the auxiliary functions

\[
\begin{align*}
g(x) &\equiv 2 \frac{d}{dx} G, \\
f(x) &\equiv - \frac{1}{2} \frac{d^3}{dx^3} G + \frac{1}{2} G \frac{d^2}{dx^2} G + \left( \frac{d}{dx} G \right)^2 + G^2 \frac{d}{dx} G, \\
\frac{1}{p(x)} \frac{d}{dx} p(x) &\equiv \frac{g(x)}{G(x)}. \tag{46}
\end{align*}
\]

Eq. (46) is promptly integrated by use of definition (44) to

\[
p(x) = \exp \left( 2 \int_{x_0}^{x} d \ln G(x') \right) = \frac{G^2(x)}{G^2(x_0)}, \tag{47}
\]

where \(x_0\) is an initial point where \(G\) is different from zero. It is now easy to check that the general solution to Eq. (43) can be written in the form

\[
p(x)D(x) = \int_{x_0}^{x} dx' \frac{p(x') f(x')}{G(x')} + p(x_0) D(x_0),
\]

or

\[
D(x) = \frac{1}{G^2(x)} \int_{x_0}^{x} dx' G(x') f(x') + \frac{D(x_0) G^2(x_0)}{G^2(x)}. \tag{48}
\]

The integral on the right-hand side of Eq. (48) is computed by elementary methods in the form

\[
\int_{x_0}^{x} dx' G(x') f(x') = \frac{G^4(x)}{4} + \frac{G^2(x) G''(x)}{2} - \frac{G(x) G'''(x)}{2} + \frac{(G'(x))^2}{4} + I_0, \tag{49}
\]

with

\[
I_0 \equiv - \frac{G^4(x_0)}{4} - \frac{G^2(x_0) G'(x_0)}{2} + \frac{G(x_0) G'''(x_0)}{2} - \frac{(G'(x_0))^2}{4}, \tag{50}
\]

where \(G' \equiv dG/dx\), and so on, thus providing the most general expression of \(D\) as a function of \(G\) and of its derivatives.
3.3 SSUSY algebra

Assuming a charge operator, $\mathcal{C}(x)$, of the form (13), we now verify that the operator

$$\mathcal{F}(x)\mathcal{F}^*(x) = \mathcal{C}(x)\mathcal{P}\mathcal{C}^*(x)\mathcal{P} = \mathcal{C}(x)\mathcal{C}^*(-x)$$

can be written as a particular case of formula (8)

$$\mathcal{F}(x)\mathcal{F}^*(x) = H^2 + \alpha H + \gamma,$$

where $\alpha$ and $\gamma$ are constants to be determined and $H$ is Hamiltonian (34) with $V$ given in (41). In fact, we have

$$\mathcal{C}(x)\mathcal{C}^*(-x) = \left(\frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x)\right) \cdot \left(\frac{d^2}{dx^2} - G^*(-x)\frac{d}{dx} + D^*(-x)\right)$$

$$= \left(\frac{d^2}{dx^2} + G(x)\frac{d}{dx} + D(x)\right) \cdot \left(\frac{d^2}{dx^2} - G(x)\frac{d}{dx} + D(x) - G'(x)\right)$$

where use has been made of relations (30), (31) stemming from Hermiticity of $\mathcal{C}(x)$. After some algebra, the right-hand side of the above expression is brought to the form

$$\mathcal{C}(x)\mathcal{C}^*(-x) = \frac{d^4}{dx^4} + \left(2D(x) - G^2(x) - 3G'(x)\right)\frac{d^2}{dx^2} + \left(2V(x) + \alpha\right)\frac{d}{dx} + V^2(x) - V''(x) + \alpha V(x) + \gamma,$$

and is to be compared with

$$H^2 + \alpha H + \gamma = \left(-\frac{d^2}{dx^2} + V(x)\right)^2 + \alpha \left(-\frac{d^2}{dx^2} + V(x)\right) + \gamma$$

$$= \frac{d^4}{dx^4} - (2V(x) + \alpha)\frac{d^2}{dx^2} - 2V'(x)\frac{d}{dx} + V^2(x) - V''(x) + \alpha V(x) + \gamma,$$

where $V(x)$ may be expressed as a function of $D(x)$ and $G(x)$ according to Eq. (41). Direct comparison of the right-hand sides of the above formulae allows us to determine the $\alpha$ constant as

$$\alpha = -2V_0.$$  

The value of $\gamma$ expresses the compatibility between $\mathcal{C}$ and the polynomial algebra through the equation

$$V^2(x) - V''(x) + \alpha V(x) + \gamma =$$
\[ D''(x) - G'''(x) + G(x)D'(x) - G(x)G''(x) + D^2(x) - D(x)G'(x) \]

Here, we insert the expressions of \( V(x) \) and \( V''(x) \) in terms of \( G(x) \), \( D(x) \) and of their derivatives obtained from formula (41), and making use of Eq. (43), as well as of its general solution (48), (49), we obtain the final result

\[ \gamma = V_0^2 + I_0 + D(x_0)G^2(x_0), \]

where \( I_0 \) is defined in Eq. (50). This makes it possible to interpret \( \gamma \) as a kind of integration constant. Thus, \( CPT \) invariance leads to the SSUSY polynomial algebra, Eqs. (7), (8).

### 4 Polynomial oscillators

The simplest factorization of \( C \) reads

\[ C(x) = \left( \frac{d}{dx} + \frac{G(x)}{2} \right) \cdot \left( \frac{d}{dx} + \frac{G(x)}{2} \right), \]

so that, correspondingly,

\[ D(x) = \frac{G'(x)}{2} + \frac{G^2(x)}{4}. \]

In this case, Eq. (43) yields \( G'''(x) = 0 \), i.e.,

\[ G(x) = ax^2 + ibx + c \]

where \( a, b \) and \( c \) are real numbers, owing to the fact that \( G(x) \) is \( PT \)-symmetric. From Eq. (41) we obtain:

\[ V(x) = \frac{1}{4}G^2(x) + G'(x) + V_0 \]

\[ = \frac{1}{4}a^2x^4 + \frac{1}{4}(b^2 - 2ac)x^2 + \frac{1}{2}iabx^3 + \frac{1}{2}x(4bc + 4a) + ib + \frac{c^2}{4} + V_0. \]

If we make the additional assumption \( c = 0 \), for the sake of simplicity, the polynomial algebra provides the constraint

\[ \gamma = V_0^2 + \frac{b^2}{4} \]

on \( \gamma \) [Eq. (54)].
4.1 The problem of invertibility

We will now make the spectral analysis for $H$ and study the invertibility of $F$ in the case $c = 0$. Then

$$V(x) = \frac{1}{4}a^2 x^4 - \frac{1}{4}b^2 x^2 + \frac{1}{2}i ab x^3 + 2ax + ib + V_0. \quad (59)$$

Setting $\mu^2 = \frac{a^2}{4}$ and $\nu^2 = \frac{b^2}{4}$, we obtain an expression for the Schrödinger operator $H$ of the same type as that presented in Eqs. (22), (23) of Ref. [19], namely

$$H = -\frac{d^2}{dx^2} + \mu^2 x^4 - \nu^2 x^2 + 2i \mu \nu x^3 + 4 \mu x + 2i \nu + V_0 \quad (60)$$

and $D(H) = H^2(\mathbb{R}) \cap D(x^4)$, $\forall \mu, \nu \in \mathbb{R}, \mu \neq 0$. As in Ref. [19], $H$ has discrete spectrum, i.e., the spectrum consists of a sequence of isolated eigenvalues with finite multiplicity.

In order to prove the reality of the spectrum of $H$, we first notice that $H$ can be rewritten as

$$H = -\frac{d^2}{dx^2} + x^2(\mu x + i \nu)^2 + 4 \mu x + 2i \nu + V_0. \quad (61)$$

Let us now perform the complex translation $x \rightarrow x - \frac{i \nu}{2 \mu}$. Then $H = S^{-1} H_1 S$ where $S \psi(x) = \psi(x - \frac{i \nu}{2 \mu})$ on a dense set of functions $\psi \in L^2(\mathbb{R})$ and

$$H_1 = -\frac{d^2}{dx^2} + \left(x - \frac{i \nu}{2 \mu}\right)^2 \left(\mu x - \frac{i \nu}{2} + i \nu\right)^2 + 4 \mu x - 2i \nu + 2i \nu + V_0$$

$$= -\frac{d^2}{dx^2} + \mu^2 \left(x - \frac{i \nu}{2 \mu}\right)^2 \left(x + \frac{i \nu}{2 \mu}\right)^2 + 4 \mu x + V_0$$

$$= -\frac{d^2}{dx^2} + \mu^2 \left(x^2 + \frac{\nu^2}{4 \mu^2}\right)^2 + 4 \mu x + V_0 \quad (62)$$

Hence $H$ has the same spectrum of $H_1$. In turn $H_1$ is selfadjoint on $D(H_1) = D(H) = H^2(\mathbb{R}) \cap D(x^4)$, thus it has real spectrum for all $\mu, \nu, V_0 \in \mathbb{R}, \mu \neq 0$.

We may stress that Hamiltonian (60) is not $\mathcal{PT}$–invariant but has still a real spectrum because it is related by explicit similarity to the standard self-adjoint anharmonic oscillator. In our opinion this is an exceptional example since in general the proof of the reality of the spectra of non-Hermitian Hamiltonians cannot proceed in such a straightforward manner and, generically, the necessary maps are non-local [26]. Moreover, by our construction, the reality of the spectrum is robust insofar as its $\mathcal{CPT}$–symmetry cannot be spontaneously broken. In this sense, our example (60) may be perceived as a $\mathcal{PT}$–asymmetric parallel to the $\mathcal{PT}$–symmetric quartic oscillator of Buslaev and Grecchi [27].
4.2 The problem of boundedness

Let us now turn to the operator $F = CP$. In order to prove the invertibility of $F$ and the boundedness of $F^{-1}$ on $L^2(\mathbb{R})$ it is enough to demonstrate the same facts for $C$. Factorization (55) implies that it will suffice to prove that $C_1 = (\frac{d}{dx} + \frac{G}{2})$ is invertible and that $C_1^{-1}$ is bounded on $L^2(\mathbb{R})$ if $G$ is given by (57). Indeed, we have

$$C_1 = \frac{d}{dx} + \frac{1}{2}a x^2 + \frac{i}{2} b x$$

(63)

and we now proceed as in Ref. [19]. More precisely

$$C_1 = \frac{d}{dx} + a\left( x + \frac{ib}{2a} \right)^2 + \frac{b^2}{8a}$$

(64)

is similar to

$$C_2 = \frac{d}{dx} + \frac{a}{2} x^2 + \frac{b^2}{8a}$$

(65)

via the complex translation $x \rightarrow x - \frac{ib}{2a}$. Hence $C_1$ has the same spectrum as $C_2$. In turn $C_2$ is unitarily equivalent, via the Fourier transformation, to

$$C_3 = -\frac{a}{2} \frac{d^2}{dx^2} + i x + \frac{b^2}{8a}$$

(66)

Therefore $C_1$ has the same spectrum as $C_3$. Finally, we perform the unitary dilation $(U\psi)(x) = (a/2)^{1/6}\psi[(a/2)^{1/3}x]$ and obtain that $C_1$ has the same spectrum as

$$C_4 = UC_3U^{-1} = \left( \frac{a}{2} \right)^{1/3} \left[ -\frac{d^2}{dx^2} + i x + \left( \frac{a}{2} \right)^{-1/3} \frac{b^2}{8a} \right].$$

(67)

Now, since the Schrödinger operator $-\frac{d^2}{dx^2} + ix$ has an empty spectrum (see Ref. [28]), so does $C_1$. In particular $z = 0$ belongs to resolvent set of $C_1$, so that $C_1$ is invertible and its inverse is bounded and defined on the whole of $L^2(\mathbb{R})$.

5 Towards operators $C$ of any finite order

We shall postulate that the charge-operator component $C$ of the pseudo-metric $CP$, where $P$ denotes parity, is a polynomial of any finite degree $\omega = 0, 1, \ldots$ in the momentum operator $p$,

$$C = \sum_{k=0}^{\omega} \gamma_k(x) \frac{d^k}{dx^k}, \quad \gamma_k(x) = \gamma^R_k(x) + i \gamma^I_k(x).$$

(68)

The functions $\gamma^R_k(x)$ and $\gamma^I_k(x)$ are both assumed real, and our main task here is just to guarantee, at any integer $\omega$, that the operator candidate for the metric $CP$ is Hermitian.
5.1 The metric $\mathcal{CP}$ in differential form

From

$$C^\dagger = \sum_{k=0}^{\omega} (-1)^k \sum_{\ell=0}^{k} \binom{k}{\ell} \left[ \frac{d^{(k-\ell)}}{dx^{(k-\ell)}} \gamma_k^*(x) \right] \frac{d^\ell}{dx^\ell} = \left( \omega \sum_{\ell=0}^{\omega-\ell} \sum_{m=0}^{\omega-\ell} (-1)^m \binom{\ell + m}{\ell} \left[ \gamma_{\ell+m}^R(x) - i \gamma_{\ell+m}^I(x) \right] \right) \frac{d^\ell}{dx^\ell},$$

where the superscripts $(m)$ at the functions $\gamma^R$ and $\gamma^I$ indicate their $m$-tuple differentiation, one obtains that the Hermiticity condition $\mathcal{CP} = \mathcal{PC}^\dagger$ is equivalent to the $(\omega + 1)$-plet of relations

$$\mathcal{P} \gamma_\ell \mathcal{P} = \gamma_\ell^R(-x) + i \gamma_\ell^I(-x) = \sum_{m=0}^{\omega-\ell} (-1)^m \binom{\ell + m}{\ell} \left[ \gamma_{\ell+m}^R(x) - i \gamma_{\ell+m}^I(x) \right]$$

with a trivial decoupling into its real and imaginary parts

$$\gamma_\ell^R(-x) - \gamma_\ell^R(+x) = \sum_{m=1}^{\omega-\ell} (-1)^m \binom{\ell + m}{\ell} \gamma_{\ell+m}^R(x) \quad (70)$$

and

$$\gamma_\ell^I(-x) + \gamma_\ell^I(+x) = -\sum_{m=1}^{\omega-\ell} (-1)^m \binom{\ell + m}{\ell} \gamma_{\ell+m}^I(x), \quad (71)$$

respectively, with $\ell = \omega - k = 0, 1, \ldots, \omega$.

5.2 Functional freedom in complex coefficients $\gamma_k(x)$

At the first few $k = 0, 1, \ldots$ the above Hermiticity constraints degenerate to the comparatively elementary relations,

$$\gamma_\omega^R(x) - \gamma_\omega^R(-x) = 0, \quad k = 0,$$

$$\gamma_{\omega-1}^R(x) - \gamma_{\omega-1}^R(-x) = \left( \frac{\omega}{1} \right) \gamma_{\omega}^{R(1)}(x), \quad k = 1,$$

$$\gamma_{\omega-2}^R(x) - \gamma_{\omega-2}^R(-x) = \left( \frac{\omega - 1}{1} \right) \gamma_{\omega-1}^{R(1)}(x) - \left( \frac{\omega}{2} \right) \gamma_{\omega}^{R(2)}(x), \quad k = 2,$$

etc, or, in parallel,

$$\gamma_\omega^I(x) + \gamma_\omega^I(-x) = 0, \quad k = 0,$$

$$\gamma_{\omega-1}^I(x) + \gamma_{\omega-1}^I(-x) = \left( \frac{\omega}{1} \right) \gamma_{\omega}^{I(1)}(x), \quad k = 1,$$

$$\gamma_{\omega-2}^I(x) + \gamma_{\omega-2}^I(-x) = \left( \frac{\omega - 1}{1} \right) \gamma_{\omega-1}^{I(1)}(x) - \left( \frac{\omega}{2} \right) \gamma_{\omega}^{I(2)}(x), \quad k = 2,$$
etc. This means that the symmetric parts $H_\ell(x) = H_\ell(-x)$ of all $\gamma_\ell^R(x)$ are arbitrary functions while, in parallel, the antisymmetric parts $h_\ell(x) = -h_\ell(-x)$ of all $\gamma_\ell^I(x)$ are also arbitrary. We may conjecture that the remaining components $R_\ell(x) = \gamma_\ell^R(x) - H_\ell(x) = -R_\ell(-x)$ and $r_\ell(x) = \gamma_\ell^I(x) - h_\ell(x) = r_\ell(-x)$ obey the rules

$$R_\omega = 0, \quad R_{\omega-1}(x) = \frac{\omega}{2}H_\omega^{(1)}(x), \quad R_{\omega-2}(x) = \frac{\omega-1}{2}H_{\omega-1}^{(1)}(x), \quad \ldots \quad (72)$$

while

$$r_\omega = 0, \quad r_{\omega-1}(x) = \frac{\omega}{2}h_\omega^{(1)}(x), \quad r_{\omega-2}(x) = \frac{\omega-1}{2}h_{\omega-1}^{(1)}(x), \quad \ldots \quad (73)$$

and are fully determined by the respective recurrent relations (70) and (71).

5.3 Proof

We see that both the sequences $R_{\omega-k}(x)$ and $r_{\omega-k}(x)$ have precisely the same structure so that just the sequence of $R_{\omega-k}(x)$ may be considered without any loss of generality. Its elements should be evaluated in the recurrent manner with respect to the growing $k$. The appropriate Ansätze may be written in the finite-series form where, formally, $H_{\omega+1}(x) = H_{\omega+2}(x) = \cdots = 0$ and $h_{\omega+1}(x) = h_{\omega+2}(x) = \cdots = 0$,

$$\gamma_{\omega-k}^R(x) = H_{\omega-k}(x) + \sum_{m=1}^{k} c_m \frac{(\omega - k + m)!}{(\omega - k)!} H_{\omega-k+m}^{(m)}(x), \quad (74)$$

$$\gamma_{\omega-k}^I(x) = h_{\omega-k}(x) + \sum_{m=1}^{k} c_m \frac{(\omega - k + m)!}{(\omega - k)!} h_{\omega-k+m}^{(m)}(x). \quad (75)$$

With an auxiliary $c_0 = 1$ these Ansätze describe all the $\omega$—dependence of our functions $\gamma = \gamma^R + i\gamma^I$ in closed form.

As already stated above, the first term and the subsequent sum are of an opposite parity in both these formulae since $c_{2n} = 0$ at all $n = 1, 2, \ldots$. This observation is easily proved since after the insertion of the latter two Ansätze, the complicated recurrences (70) are replaced by their simplified version

$$2c_1 = \frac{c_0}{1!}, \quad 2c_2 = \frac{c_1}{1!} - \frac{c_0}{2!}, \quad \ldots \quad ,$$

i.e.,

$$2c_k = \sum_{m=0}^{k-1} (-1)^{k-m-1} \frac{c_m}{(k-m)!}. \quad (76)$$

It is worthwhile to point out that the $c_k$ coefficients with odd $k$ can be written in terms of Bernoulli numbers (see, e.g., Ref. [29])

$$c_{2n-1} = \frac{2(2^{2n} - 1)}{(2n)!} B_{2n} \quad (n > 0). \quad (77)$$
The key idea of an explicit solution of these recurrences is that the generating function \( f(x) = \sum c_k x^k \) of the coefficients \( c_m \) must satisfy the functional equation \( f(x) - 2 = -f(x)/e^x \) which is, in its turn, easily solvable. In this way we arrive at the solution of recurrences (76) in the following compact form,

\[
f(x) = c_0 + c_1 x + c_2 x^2 + \ldots = \frac{2}{1 + \exp(-x)} = 1 + \tanh \frac{x}{2} = (78)
\]

\[
= 1 + \sum_{n=1}^{\infty} \frac{2(2^{2n} - 1)}{(2n)!} B_{2n} x^{2n-1}.
\]

Obviously, all the possible parity-violating terms in the right-hand side of our Hermiticity conditions (70) vanish. This makes the form of our polynomial charge \( C \) extremely flexible and confirms the consistency of its present construction.

We may conclude that the requirement of Hermiticity of the metric \( \mathcal{CP} \) defines all the antisymmetric components \( R_\ell(x) \) and their spatially symmetric partners \( r_\ell(x) \). It does not impose any additional constraint either upon the real and spatially symmetric coefficient functions \( H_\ell(x) \) or upon their purely imaginary spatially antisymmetric partners \( h_\ell(x) \).

6 Outlook

We now sketch some possible applications of our methods to a variety of problems where quasi-Hermitian or pseudo-Hermitian operators are involved.

In the context of the Klein-Gordon description of the free motion of a spinless particle in the “usual” Hilbert space \( \mathcal{H} \) the relativistic evolution is generated by the Feshbach-Villars [30] “Hamiltonian” \( H_{(FV)} \) which proves non-Hermitian,

\[
|\psi(t)\rangle = e^{-iH_{(FV)}(t-t_0)}|\psi(t_0)\rangle, \quad H_{(FV)} = -\frac{1}{2}\left( \begin{array}{cc} 1 - \Delta & -\Delta \\ \Delta & \Delta - 1 \end{array} \right) (79)
\]

(cf., e.g., p. 341 in ref. [31]). One should notice that this model works with the differential pseudo-Hermitian operator with structure which strongly resembles the usual Schrödinger operators in the simplest non-trivial, two-dimensional coupled-channel case. Thus, we may expect that the methods described in our previous study might find an immediate extension to the similar problems.

The idea may also find applications in a broader context, say, of the boson mappings in nuclear physics which were comprehensively discussed in the paper [13]. It is shown there that a consistent quantum mechanical framework,
and in particular a viable variational calculation for non-hermitian Hamiltonians, can indeed be constructed after the introduction of a non-trivial metric. In the context of Holstein-Primakoff type mappings this freedom defines the link with so-called Dyson-Maleev type mappings (see [32]). In practical computations, a puzzling non-Hermiticity of observables proved more then compensated by the advantages, as has been amply demonstrated in applications of generalized Dyson-Maleev mappings (see [33] and references cited therein).

All the technical conditions imposed upon the “true physical metric” $\Theta$ in review [13] are important, especially if one tries to work within a truly infinite-dimensional Hilbert space. This has been emphasized by Kretschmer and Szymanowski [14] who showed that the use of the toy metric operators might require a careful scrutiny because these operators remain unbounded. In this context, ref. [19] as well as our present paper demonstrated persuasively that a switch to the use of the differential operators $C$ might be understood as an important new idea.

All the similar observations must be perceived as individual steps of a systematic improvement of the mathematically correct understanding of the use of the differential operators in connections with many applications of the quasi-Hermitian observables which seems to range, at present, from the elementary descriptions of the localization transitions in solid state physics [34] up to many ambitious $\mathcal{P}\mathcal{T}$-symmetric models in quantum field theory [35].

The experience gained during our study of the simple Schrödinger equations might equally well find applications on the very boundary of quantum mechanics (like, say, in cosmology [36]) or even in the domain of the classical model-building (e.g., in the magneto-dynamics of fluids [37]) and in the various physical models of different origin characterized by the simple matrix structure of their description (see a number of their most elementary samples mentioned in the short and nice review [38]) where the eigenvalues coalesce or almost coalesce in the manner which contradicts the standard and robust finite-dimensional Hermitian-matrix mathematics. Of course, all these mathematical problems and not entirely standard physical situations may impose new and challenging tasks and motivate a deeper future analysis of the questions outlined in our present paper.
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