On Algebraic-Geometry Approach to Ribaucour Transformations

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Abstract
We are developing the idea of using an algebraic-geometry approach to classical differential geometry problems. We introduce an algebraic-geometric construction to obtain an orthogonal net and any number of smooth orthogonal nets that are Ribaucour transformations of the initial orthogonal net.

1 Introduction

Krichever in his work [1] introduced an approach for constructing curvilinear orthogonal coordinates in Euclidian space according to algebraic-geometric data. This method was extended to construct orthogonal coordinates in various Riemannian spaces in the subsequent works [2], [3], [4]. In this article we study transformations of initial algebraic-geometric data that result in Ribaucour transformations of orthogonal nets.

A smooth map \( x(u) : U \subset \mathbb{R}^n \rightarrow \mathbb{R}^{n+N}, u = (u^1, \ldots, u^n), x = (x^1, \ldots, x^{n+N}) \) is called \( n \)-dimensional orthogonal net if

1. \( \partial_i \partial_j x(u) = c_{ij}^l(u) \partial_l x(u) + c_{ij}^l(u) \partial_j x(u), \quad i \neq j, \)
2. \( \partial_i x(u) \cdot \partial_j x(u) = 0, \quad i \neq j, \)

where \( x \cdot y \) is standard Euclidian scalar product in \( \mathbb{R}^{n+N} \).

If \( N = 0 \) then orthogonal nets are curvilinear orthogonal coordinates in \( \mathbb{R}^n \). Krichever was the first who proposed algebraic-geometry methods for constructing orthogonal coordinate systems. In his work [1] such systems were described in terms of the \( \theta \)-functions of smooth algebraic curves.

If \( N = 1 \) then orthogonal nets are hypersurfaces that parametrized by curvature line coordinates. An important example of such hypersurfaces is a constant curvature hypersurface with orthogonal coordinates on it. An extension of Krichever’s method to this case was developed in the work [3].
For orthogonal nets of an arbitrary codimension, a generalization of Krichever’s construction was proposed in [4]. The geometry of an orthogonal net normal bundle according to algebraic-geometric data was also studied in this work.

All the formulae for the described constructions are expressed in terms of the \( \theta \)-functions of a smooth algebraic curve. One has a certain difficulty to deal with such formulae. It was solved by Mironov and Taimanov in [2]. They constructed curvilinear orthogonal coordinates by algebraic-geometry methods considering a singular algebraic curve that results in elementary functions description.

The geometry of orthogonal nets is closely connected to the theory of integrable systems. Integrability of Lamé equations (that describe flat diagonal metrics) via the dressing method was proved by Zakharov in [5]. Further application of diagonal metrics includes, for example, works on hydrodynamic type systems [6], [7], and Frobenius manifolds [8], [9], [10].

We study Ribaucour transformations of orthogonal nets that are obtained by algebraic-geometry methods. The theory of Ribaucour transformations was developed by Bianchi for 2-dimensional surfaces of codimension 1, and it has various extensions (see [11], [12]). We choose an analytical description for Ribaucour transformations of orthogonal nets.

A pair of \( n \)-dimensional orthogonal nets \( x(u), \tilde{x}(u) \) is called a Ribaucour pair if for some functions \( \lambda_1(u), \ldots, \lambda_n(u) \) we have

\[
\partial_i \tilde{x}(u) = \lambda_i(u) \left( \partial_i x(u) - 2 \frac{\partial_i x(u) \cdot \delta x(u)}{\delta x(u) \cdot \delta x(u)} \delta x(u) \right),
\]

where \( \delta x(u) = \tilde{x}(u) - x(u) \). The orthogonal net \( \tilde{x}(u) \) is called a Ribaucour transformation of \( x(u) \).

Ribaucour transformations possess an important permutability property. It states that for any orthogonal net \( x(u) \) and its two Ribaucour transformations \( x_1(u) \) and \( x_2(u) \) there exists a fourth orthogonal net \( y(u) \) that is a Ribaucour transformation of \( x_1(u) \) and \( x_2(u) \) (see [13]). A set of all four orthogonal nets is called Bianchi quadrilateral.

Permutability property has extensions to a greater number of initial Ribaucour transformations [14]. It plays a key role in the theory of discrete orthogonal nets [15]. Let us also note that there is an algebraic-geometric solution for discrete orthogonal nets [16].

2 Algebraic-Geometry Approach to Orthogonal Nets

We define here \( n \)-dimensional algebraic-geometric orthogonal nets of codimension \( N \). The construction is based on [4] and is a natural development of Krichever construction [1].

We take the genus of a smooth complex curve \( \Gamma \) as \( g \), fix \( l \in \mathbb{N} \) and three divisors on the curve: \( P_1 + \cdots + P_n, \gamma = \gamma_1 + \cdots + \gamma_{g+l+N-1}, R = R_1 + \cdots + R_{l+N} \), and introduce a local parameter \( k_j^{-1} \) in the neighborhood of every point \( P_j \),
\[ k_j^{-1}(P_j) = 0. \] Denote all the data by \( S \):

\[
S = \{ \Gamma, g; \{ P_j, k_j^{-1} \}_{j=1}^{n}; R_1, \ldots, R_{l+N}; \gamma_1, \ldots, \gamma_{g+l+N-1} \}
\]

A function \( \psi(u^1, \ldots, u^n; Q | d) \), \( Q \in \Gamma, \) \( d = (d_1, \ldots, d_{l+N}) \in \mathbb{R}^{l+N} \), with the following analytical properties

1. in the neighborhood of every point \( P_j \) the function has an essential singularity:

\[
\psi(u; Q) = e^{k_j u^j} \left( \xi_0^j(u) + \xi_1^j(u) \cdot k_j^{-1} + \ldots \right);
\]

2. the function is meromorphic outside \( P_1 + \cdots + P_n \) and has simple poles only at the point of the divisor \( \gamma \);

3. \( \psi(u; R_\alpha) = d_\alpha \in \mathbb{R}, \alpha = 1, \ldots, l + N; \)

is called \( n \)-point Baker–Akhiezer function.

A Baker–Akhiezer function exists and unique, so it vanishes if the vector \( d \) is equal to zero. One can express Baker–Akhiezer function in terms of the \( \theta \)-function of the curve \( \Gamma \) (see [1]).

We consider the curve \( \Gamma \) only if it has an holomorphic involution \( \sigma : \Gamma \to \Gamma, \sigma^2 = id \) such that

1. the local parameters \( k_i \) are odd: \( k_i(\sigma(P)) = -k_i(P) \);

2. the involution \( \sigma \) has only \( 2(n+N) \) fixed points: \( P_1, \ldots, P_n, R_{l+1}, \ldots, R_{l+N} \), and we denote the remaining points by \( Q_1, \ldots, Q_{n+N} \).

We consider the divisors \( \gamma \) and \( R \) only if there is an even (with respect to the involution \( \sigma \)) meromorphic differential \( \Omega \) on \( \Gamma \) such that

1. it equals to zero at \( n+2(g+N+l-1) \) points: \( (\Omega)_0 = P_1 + \cdots + P_n + \gamma + \sigma(\gamma) \);

2. it has \( n + 2N + 2l \) simple poles: \( (\Omega)_\infty = Q_1 + \cdots + Q_{n+N} + R_1 + \cdots + R_{l+N} + \sigma(R_1) + \cdots + \sigma(R_l) \);

3. all the residues of \( \Omega \) are equal to 1: \( \text{Res}_{Q_1} \Omega = \cdots = \text{Res}_{Q_{n+N}} \Omega = 1 \).

We denote the residues of \( \Omega \) at the points \( R_1, \sigma R_1, \ldots, R_l, \sigma R_l \) by \( r_1, \ldots, r_l \):

\[
r_\alpha = \text{Res}_{R_\alpha} \Omega = \text{Res}_{\sigma R_\alpha} \Omega, \quad \alpha = 1, \ldots, l.
\]  

(2)

The notion is correct since \( \Omega \) is even.

Let us introduce reality conditions to get real-valued functions.

**Lemma 2.1.** Let \( \Gamma \) be a curve with antiholomorphic involution \( \tau : \Gamma \to \Gamma \) such that the points \( P_1, \ldots, P_n, Q_1, \ldots, Q_{n+N}, R_1, \ldots, R_{l+N} \) are fixed points of the involution \( \tau \), the divisor \( \gamma \) is iniant under \( \tau \): \( \tau(\gamma) = \gamma \), and local parameters are antiholomorphic: \( k_i(\tau(P)) = k_i(P) \).

Then the Baker-Akhiezer function satisfies the relation

\[
\psi(u, Q | d) = \overline{\psi(u, \tau(Q) | d)}.
\]
For the proper choice of algebraic-geometric data that is described above, we have the following theorem.

**Theorem 2.1.** Functions \( x^k(u) = \psi(u, Q_k), \ k = 1, \ldots, n + N \) define a real \( n \)-dimensional orthogonal net in \((n + N)\)-dimensional Euclidian space.

This theorem provides us with an \( n \)-dimensional orthogonal net in \( \mathbb{R}^{n+N} \), that is \( x(u) \).

### 3 Algebraic-Geometry Approach to Ribaucour Transformations

We choose now \( d = (1, \ldots, 1) \) for the simplicity and obtain the orthogonal net \( x(u) \) from the theorem 2.1:

\[
S \rightarrow \psi(u, Q) \rightarrow x(u).
\]

For every \( \alpha = 1, \ldots, l \) the following transformation of algebraic geometrical data

\[
S_\alpha = \{ \Gamma, g; \{ P_j, k_j^{-1}\}_{j=1}^n; R_1, \ldots, R_{\alpha-1}, \sigma R_\alpha, R_{\alpha+1}, \ldots, R_{t+N}; \gamma_1, \ldots, \gamma_{g+t+N-1} \}
\]

defines another orthogonal net \( x_\alpha(u) \):

\[
S_\alpha \rightarrow \psi_\alpha(u, Q) \rightarrow x_\alpha(u).
\]

**Theorem 3.1.** A pair of the orthogonal nets \( x(u) \) and \( x_\alpha(u) \) is a Ribaucour pair.

**Proof.** To prove the theorem we need two lemmas.

Let us denote the first terms of the function \( \psi_\alpha(u, Q) \) expansion at \( P_j \) by \( \xi_{0, \alpha}^j(u) \), i.e. the expansion in the neighborhood of \( P_j \) has the following form:

\[
\psi_\alpha(u, Q) = e^{k_j u_j} \left( \xi_{0, \alpha}^j(u) + \frac{\xi_{1, \alpha}^j(u)}{k_j} + \ldots \right).
\]

We consider a function

\[
\Phi_{i, \alpha}(u, Q) = \xi_{0, \alpha}^i(u) \partial_i \psi_\alpha(u, Q) - \xi_{0, \alpha}^i(u) \partial_i \psi(u, Q).
\]

It is straightforward that \( \Phi_{i, \alpha}(u, Q) \) is an \( n \)-point Baker–Akhiezer function with non-constant values at the points of the divisor \( R \). The values are

\[
\Phi_{i, \alpha}(u, R_k) = 0; \ k \neq \alpha; \quad (3)
\]

\[
\Phi_{i, \alpha}(u, R_\alpha) = -\xi_{0, \alpha}^i(u) \partial_i \phi_\alpha(u); \ \Phi_{i, \alpha}(u, \sigma R_\alpha) = \xi_{0}^i(u) \partial_i \phi_{\alpha, \alpha}(u), \quad (4)
\]

where \( \phi_\alpha(u) = \psi(u, \sigma R_\alpha), \phi_{\alpha, \alpha}(u) = \psi_\alpha(u, R_\alpha) \).
Lemma 3.1. The function $\Phi_{i, \alpha}(u, Q)$ on the curve $\Gamma$ is proportional to the function $\psi_{\alpha}(u, Q) - \psi(u, Q)$:

$$\Phi_{i, \alpha}(u, Q) = \xi^i_{0, \alpha}(u) \cdot \frac{\partial \phi_{\alpha}(u)}{\phi_{\alpha}(u)} \left( \psi_{\alpha}(u, Q) - \psi(u, Q) \right).$$

(5)

Proof. We calculate the values of the right side at the points in the divisor $R$:

$$\xi^i_{0, \alpha}(u) \cdot \frac{\partial \phi_{\alpha}(u)}{\phi_{\alpha}(u)} \left( \psi_{\alpha}(u, K) - \psi(u, K) \right) = 0, \ k \neq \alpha;$$

$$\xi^i_{0, \alpha}(u) \cdot \frac{\partial \phi_{\alpha}(u)}{\phi_{\alpha}(u)} \left( \psi_{\alpha}(u, \alpha) - \psi(u, \alpha) \right) = -\xi^i_{0, \alpha}(u) \partial \phi_{\alpha}(u).$$

Thus, the difference of two sides of the equality (5) is an $n$-point Baker–Akhiezer function with all the constants $d_i$ equal to zero, which means that the difference equals to zero.

To express the coefficient on the right side of (5) we prove the following lemma.

Lemma 3.2. The function $\phi_{\alpha}(u)$ relates to the embedding functions $x(u)$ and $x_{\alpha}(u)$ in the following way

$$\frac{\partial_i \phi_{\alpha}}{\phi_{\alpha} - 1} = -2 \frac{\partial_i x \cdot \delta x}{\delta x \cdot \delta x}.$$  

(6)

Proof. Let us consider a differential form $\partial_i \psi \cdot (\psi_{\alpha} - \psi)^{\sigma} \cdot \Omega$, where $(\psi_{\alpha}(u, Q) - \psi(u, Q))^{\sigma} = \psi_{\alpha}(u, \sigma Q) - \psi(u, \sigma Q)$. One can show that all the essential singularities of function are not ones for the differential. Then the differential is meromorphic and it has only simple poles $Q_1, \ldots, Q_n, \sigma R_{\alpha}$.

All the residues of the differential sum up to zero:

$$\sum_{s=1}^{n+N} \partial_i x^s(u) \cdot (x^s_{\alpha}(u) - x^s(u)) + r_{\alpha} \partial_i \phi_{\alpha}(u) (\phi_{\alpha, \alpha}(u) - 1) = 0,$$

where $r_{\alpha}$ as in (2). We get

$$\partial_i x \cdot \delta x = -r_{\alpha} \partial_i \phi_{\alpha} (\phi_{\alpha, \alpha} - 1).$$

(6)

Let us consider a differential $(\psi_{\alpha} - \psi) \cdot (\psi_{\alpha} - \psi)^{\sigma} \cdot \Omega$. It is also meromorphic and has only simple poles $Q_1, \ldots, Q_n, R_{\alpha}, \sigma R_{\alpha}$.

All the residues of the differential sum up to zero:

$$\sum_{s=1}^{n+N} (x^s_{\alpha}(u) - x^s(u))^2 + 2r_{\alpha} (\phi_{\alpha, \alpha}(u) - 1) (1 - \phi_{\alpha}(u)) = 0.$$  

We get

$$\delta x \cdot \delta x = 2r_{\alpha} (\phi_{\alpha} - 1) (\phi_{\alpha, \alpha} - 1).$$

(7)

Dividing (6) by (7) we prove the lemma. \qed
To prove the main theorem we consider the values of (5) at the point $Q_k$, $k = 1, \ldots, n + N$:

$$
\xi_0^i \partial_i x^k - \xi_{\alpha}^i \partial_\alpha x^k = \xi_{\alpha}^i \frac{\partial \phi_\alpha}{\phi_\alpha} - 1 \left( x^k_\alpha - x^k \right), \quad k = 1, \ldots, n + N.
$$

In the vector form:

$$
\xi_0^i \partial_i x_\alpha = \xi_{\alpha}^i \frac{\partial \phi_\alpha}{\phi_\alpha} - 1 \delta x.
$$

Using lemma 3.2 we obtain (1) with $\lambda_i(u) = \frac{\xi_{\alpha}^i(u)}{\xi_0^i(u)}$ and prove the theorem.

If $l > 1$ then we construct two different Ribaucour transformation of $x(u)$: $x_\alpha(u)$ and $x_\beta(u)$. We transform transformed algebraic-geometric data and obtain corresponding orthogonal net in the following way:

$$
S_{\alpha \beta} = (S_\alpha)_{\beta} = (S_\beta)_\alpha \rightarrow x_{\alpha \beta}(u).
$$

Thus, we have Bianchi quadrilaterals:

$$
\begin{array}{c c c c}
S_\beta & \rightarrow & S_{\alpha \beta} & \quad x_\beta(u) & \rightarrow & x_{\alpha \beta}(u) \\
\uparrow & \uparrow & \uparrow & \uparrow & \uparrow \\
S & \rightarrow & S_\alpha & \quad x(u) & \rightarrow & x_\alpha(u)
\end{array}
$$

One can obtain $2^l$ orthogonal nets corresponding to the initial data $S$. Ribaucour pairs among them form the structure of $l$-dimensional cube with 2-faces described above.

### 4 Examples

For $l \geq 2$ the construction is complicated and involves the $\theta$-function of the smooth curve $\Gamma$. However, if $l = 1$ the construction is described by the following theorem.

**Theorem 4.1.** If $l = 1$ and $d_2 = \cdots = d_{N+1} = 0$ we have the only transformation $x_1(u)$ and

$$
x_1(u) = \frac{c}{x(u) \cdot x(u)} x(u),
$$

where $c \in \mathbb{R}$ defined by $S$.

**Proof.** The first step is to prove that

$$
\psi_1(u, Q) = \frac{1}{\phi_1(u)} \cdot \psi(u, Q),
$$

where $\phi_1(u) = \psi(u, \sigma R_1)$.
The function $\frac{1}{\phi_1(u)} \cdot \psi(u, Q)$ is a Baker–Akhiezer function and it equals to 1 at the point $\sigma R_1$. Since a Baker–Akhiezer function for data $S_1$ is unique we have the equality (8).

Thus, the orthogonal nets $x(u)$ and $x_1(u)$ are proportional:

$$x_1(u) = \frac{1}{\phi_1(u)} \cdot x(u).$$

We prove now that

$$\phi_1(u) = -\frac{x(u) \cdot x(u)}{2r_1}.$$

Let us consider a differential form $\psi(u, Q)\psi(u, \sigma Q)\cdot \Omega(Q)$. It is meromorphic and has only simple poles $Q_1, \ldots, Q_n, R_1, \sigma R_1$. The sum of all the residues vanishes:

$$\sum_{s=1}^{n+N} (x^k)^2 + 2r_1\phi_1 = 0.$$  

That proves the theorem. \hfill \Box

Another way to get the explicit examples is to choose a singular curve $\Gamma$ as in [2]. When the curve is reducible and all its irreducible components are rational curves, all the formulae are expressed in elementary functions. For the proper choice of the initial data, one needs to solve a system of linear equations to get $2^l$ orthogonal nets.

We do not show calculations here since the formulae are hard to read. However, it is an interesting question, how to choose the initial data to obtain a specific orthogonal net.

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