A LOWER BOUND FOR THE DOUBLE SLICE GENUS

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Abstract. In this paper, we develop a lower bound for the double slice genus of a knot using Casson-Gordon invariants. As an application, we show that the double slice genus can be arbitrarily larger than twice the slice genus. As an analogue to the double slice genus, we also define the superslice genus of a knot, and give both an upper bound and a lower bound in the topological category.

1. Introduction

A surface smoothly embedded in the 4-sphere is unknotted if it bounds a handlebody. Regarding the 3-sphere $S^3$ as the equator of the 4-sphere, a knot $K \subset S^3$ is smoothly doubly slice if it is the intersection of an unknotted 2-sphere with the equator $S^3$. Obviously not every knot is doubly slice, for there exists knots which are not even slice, i.e. bounding properly embedded disks in the 4-ball. Moreover, not every slice knot is doubly slice. In fact, about six decades ago Fox posed a challenging question: determine which slice knots are doubly slice (cf. Problem 39 of [9]). Since then this question has been the center of the study of double sliceness, and many obstructions to double sliceness were found (e.g. [9, 11, 15, 16, 17, 18, 21]). Recently Livingston and Meier introduced a notion called the double slice genus of a knot, which sets this topic in larger context [16]. We recall the definition below.

Definition 1.1. Given a knot $K \subset S^3$, its double slice genus is defined as

$$g_{ds}(K) = \min_S \{g(S) | S \text{ is an unknotted surface in } S^4, S \cap S^3 = K\},$$

where we view $S^3$ as the equator of $S^4$.

Note $g_{ds}(K)$ is defined, for a surface $S$ satisfying the above requirements exists for every $K$. To see this, let $F$ be a surface obtained by pushing the interior of some Seifert surface of $K$ into the 4-ball. Then the double of $F$ clearly bounds a 3-manifold homeomorphic to $F \times I$. Furthermore, this observation also implies the double slice genus is bounded above by twice the Seifert genus. On the other hand, it is straightforward to see $g_{ds}(K)$ is bounded below by twice the slice genus of $K$. In summary, we have

$$2g_4(K) \leq g_{ds}(K) \leq 2g_3(K).$$

Along the lines of Fox’s question to tell sliceness and double sliceness apart, a natural question in this context is: can the double slice genus be arbitrarily larger than twice the slice genus? Answering this question requires a lower bound for the double slice genus. Note while many knot invariants give lower bounds for $g_4$, there previously are no algebraic invariants that improve on the lower bound $2g_4$ for $g_{ds}$. By using Casson-Gordon invariants of the two-fold branched cover in conjunction

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with another algebraic invariant that we define, we develop the first such lower bound in this paper. As a primary application, we prove

**Theorem 1.2.** There exist ribbon knots $K_n$, $n \in \mathbb{N}$, such that
\[
\lim_{n \to \infty} g_{ds}(K_n) = \infty.
\]

Closely related to the double sliceness is a notion called supersliceness. Recall that a knot $K$ is called *superslice* if there is a slice disk $D$ whose double along $K$ produces an unknotted 2-sphere in $S^4$. As an analogue to the double slice genus, we define the superslice genus of a knot.

**Definition 1.3.** Given a knot $K \subset S^3$, its superslice genus is defined as
\[
g^s(K) = \min_{F} \{ g(F) \mid F \text{ is properly smoothly embedded in } D^4, \partial F = K \text{ and the double of } F \text{ bounds a handlebody in } S^4 \}.
\]

It is easy to see $g_{ds}(K) \leq 2g^s(K)$, and hence the lower bounds for the double slice genus hold for the superslice genus as well. However, greater rigidity encoded in the definition of the superslice genus compared to that of the double slice genus allows us to obtain a much more accessible bound.

**Theorem 1.4.** Let $K$ be a knot in $S^3$ and $\Sigma$ be the two-fold branched cover of $S^3$ along $K$. Then the minimum number of generators of $H_1(\Sigma; \mathbb{Z})$ is a lower bound for $2g^s(K)$.

In fact, the idea contained in the proof of Theorem 1.4 may serve as a prototype for lower bounds for the double slice genus. Compare Subsection 2.1 and Subsection 3.1.

In addition to lower bounds, it is also natural to ask if one can give upper bounds for $g^s$ or $g_{ds}$. In this paper, we pursue this direction in the topological category, i.e. the surfaces used in Definition 1.1 and Definition 1.3 are allowed to be topologically embedded and locally flat, and denote the corresponding quantities by $g^s_{top}(K)$ and $g_{ds}^{top}(K)$. We remark that the topological category and the smooth category are different [17, 20], and the lower bounds constructed in this paper also hold in the topological category.

We give upper bounds in terms of the Alexander polynomial. Freedman proved knots with trivial Alexander polynomial are topologically slice [7, 8]. Recently, Feller generalized this theorem: the degree of the Alexander polynomial is an upper bound for twice the topological slice genus [3]. Here the degree of the Alexander polynomial is the breadth of the polynomial. In the context of superslice genus, two results of Freedman imply that knots with trivial Alexander polynomial are exactly the topologically superslice knots; see [16] and [17], or see the discussion in Subsection 3.2. So it is natural to wonder if one can bound the topological superslice genus by the degree of the Alexander polynomial. Indeed, we have the following result.

**Theorem 1.5.** The degree of the Alexander polynomial of a knot is an upper bound for twice its topological superslice genus.

This theorem has the following immediate corollary.

**Corollary 1.6.** If the degree of the Alexander polynomial of a knot $K$ is 2, then $g^s_{top}(K) = 1$. 
Remark 1.7. Similar inequalities as in Theorem 1.4 and Theorem 1.5 also appeared in the context of Z-slice genus of a knot, i.e. minimal genus of surfaces in the 4-ball whose boundary is the given knot and whose complement has fundamental group isomorphic to Z (cf. Theorem 1 and Proposition 12 of [4]). In fact, the proof of Theorem 1.5 together with Theorem 1.1 of [5] imply the topological superslice genus is equal to the Z-slice genus. This was pointed out to the author by Peter Feller and Lukas Lewark. We refer the interested readers to their papers [4, 5] for a more systematic study of the Z-slice genus.

The rest of the paper is organized as follows: the lower bounds for the double slice genus are constructed in Subsection 2.1–2.3. Theorem 1.2 is proved in Subsection 2.4, modulo a technical lemma which is proved in the appendix. Theorem 1.4 is proved in Subsection 3.1. Theorem 1.5 and Corollary 1.6 are proved in Subsection 3.2.

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2. Construction of the lower bound

In this section, we construct a lower bound for the double slice genus and use it prove Theorem 1.2. This bound comes from studying Σ(K), the two-fold branched cover of S^3 along a knot K (we expect similar bounds can be defined using higher order branched covers): we first derive a lower bound in terms of the singular homology of Σ(K) in Subsection 2.1, and then study the Casson-Gordon invariants of Σ(K) to give other lower bounds in Subsection 2.2, finally we combine the bounds in Subsection 2.1-2.2 to give the desired lower bound in Subsection 2.3. In Subsection 2.4, we prove Theorem 1.2.

2.1. Double slice genus and the singular homology of Σ(K). Regard S^3 as the equator of S^4 = D^4_1 ∪ S^3 ∪ D^4_2, where the spaces D^4_i’s are two copies of the 4-ball. Let K be a knot in S^3 and let F_i be properly embedded surfaces in D^4_i such that ∂F_i = K, i = 1, 2, and F = F_1 ∪ F_2 bounds a handlebody in S^4. Let g_i denote the genus of F_i, let g = g_1 + g_2, and let W_i be the two-fold branched cover of D^4_i along F_i, i = 1, 2. Then ∂W_1 = −∂W_2 = Σ(K), and W = W_1 ∪_Σ(K) W_2 is diffeomorphic to # g S^2 × S^2, the connected sum of g copies of S^2 × S^2. In the set up, the genus of F is captured by the homology groups of the 4-manifolds. Indeed, we have the fact that b_2(W_i) = 2g_i (see Proposition 2.1 (i) for a proof).

We use the homology groups of Σ(K) to give lower bounds for b_2(W_i) by examining various long exact sequences relating these spaces. For convenience, we write Σ for Σ(K) hereafter. Throughout this paper we use integer coefficient for the singular homology groups unless otherwise specified. Note Σ is a rational homology sphere. From the Mayer-Vietoris sequence for W_1 ∪_Σ W_2, we have

(2.1) \[ 0 \to H_2(W_1) \oplus H_2(W_2) \to H_2(W) \to H_1(Σ) \to H_1(W_1) \oplus H_1(W_2) \to 0. \]
The long exact sequence for the pair \((W, \Sigma)\) gives rise to

\[
0 \to H_2(W) \to H_2(W, \Sigma) \to H_1(\Sigma) \to 0.
\]

From the long exact sequence for \((W_i, \Sigma), i = 1, 2\), we obtain

\[
0 \to H_2(W_i) \to H_2(W_i, \Sigma) \to H_1(\Sigma) \to H_1(W_i) \to 0,
\]

where surjectivity of the map \(H_1(\Sigma) \to H_1(W_i)\) is derived from (2.1).

Finally, the long exact sequence for \((W, W_i), i = 1, 2\) shows

\[
0 \to H_2(W_i) \to H_2(W) \to H_2(W, W_i) \to H_1(W_i) \to 0.
\]

Correspondingly, we can deduce the following properties of the homology groups involved in the above exact sequences.

**Proposition 2.1.** **In the notation established above, we have the following properties.**

(i) \(H_2(W_i) \cong \mathbb{Z}^{2g_i}, i = 1, 2\).

(ii) \(|H_1(W_i)|^2 | |H_1(\Sigma)|, i = 1, 2\).

(iii) There exists \(a_j^i \in H_1(W_i) \oplus \mathbb{Z}^{2g_i}\) for \(i = 1, 2, \ldots, 2g\) such that

(a) \(\left(\frac{H_1(W_i) \oplus \mathbb{Z}^{2g_i}}{\langle a_j^i \mid j = 1, \ldots, 2g_i \rangle}\right) \cong H_1(W_{i+1})\) for \(i = 1, 2\), where we let \(W_3 = W_1\).

(b) \(\frac{H_1(W_1) \oplus \mathbb{Z}^{2g_1} \oplus (H_1(W_2) \oplus \mathbb{Z}^{2g_2})}{\langle \{a_j^1, a_j^2 \mid j = 1, \ldots, 2g\} \rangle} \cong H_1(\Sigma)\).

Here \(\langle \{a_j^i \mid j = 1, \ldots, 2g_i \} \rangle\) stands for the subgroup generated by the set \(\{a_j^i \mid j = 1, \ldots, 2g_i \}\).

**Proof.**

(i) The long exact sequence (2.1) implies \(H_2(W_i)\) is free abelian, since it is mapped injectively into \(H_2(W) \cong \mathbb{Z}^{2g}\). The statement would then follow from the claim that \(b_2(W_i) = 2g_i\). To see this claim, note \(\chi(W_i) = 2\chi(D^4 - (F_i \times D^2)) + \chi(F_i \times D^2) - \chi(F_i \times S^3) = 1 + 2g_i\), and we also have \(\chi(W_i) = 1 - b_1(W_i) + b_2(W_i) - b_3(W_i)\). Long exact sequence (2.3) implies \(b_1(W_i) = 0\). As we have \(H_1(W_i) \to H_1(W_i, \Sigma) \to 0\) from the long exact sequence for the pair \((W_i, \Sigma)\), we have \(b_3(W_i) = \text{rk}H^3(W_i) = \text{rk}H_1(W_i, \Sigma) \leq \text{rk}H_1(W_i) = b_1(W_i) = 0\). Therefore, \(b_2(W_i) = \chi(W_i) - 1 - 2g_i\).

(ii) Consider the long exact sequence (2.3). First we claim \(|\text{Coker}\{H_2(W_i) \to H_2(W_i, \Sigma)\}|\) is divisible by \(|H_1(W_i)|\). To see the claim, note \(H_2(W_i, \Sigma) \cong H^2(W_i) \cong H_2(W_i) \oplus H_1(W_i)\) by Poincaré duality and the universal coefficient theorem. Assume \(H_1(W_i) = \mathbb{Z}_{m_1} \oplus \cdots \oplus \mathbb{Z}_{m_k}\). Then \(H_2(W_i) \oplus H_1(W_i)\) has a presentation matrix of the form

\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 & \cdots & 0 \\
p & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & m_k & 0 & \cdots & 0
\end{bmatrix}
\]

where \(\cdots\) represents \(\mathbb{Z}^{2g_i}\).
Then \( \text{Coker}\{H_2(W_i) \to H_2(W_i, \Sigma)\} \) has a presentation matrix
\[
\begin{bmatrix}
m_1 & 0 & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & m_k & 0 & \cdots & 0 \\
* & * & * & * & \cdots & * \\
\vdots & \ddots & \vdots & \ddots & \cdots & \vdots \\
* & * & * & * & \cdots & *
\end{bmatrix}
\]

Note \( |\text{Coker}\{H_2(W_i) \to H_2(W_i, \Sigma)\}| \) is equal to the absolute value of the determinant of the above matrix, and hence is divisible by \( m_1m_2 \cdots m_k = |H_2(W_i)| \). With this claim in mind, the statement then follows easily from (2.3).

(iii) Note there are isomorphisms
\[
H_2(W, W_i) \cong H_2(W_{i+1}, \Sigma) \cong H_2(W_{i+1}) \oplus H_1(W_{i+1}) \cong H_1(W_{i+1}) \oplus \mathbb{Z}^{2g_1+1}.
\]

Here the first isomorphism follows from excision, and the second isomorphism is obtained by Poincaré duality and the universal coefficient theorem. Now (a) follows from this and long exact sequence (2.4).

Identify a regular neighborhood of \( \Sigma \) in \( W \) as \( \Sigma \times [-1, 1] \). Let \( W_1' = W_1 - \Sigma \times (-1, 0) \) and let \( W_2' = W_2 - \Sigma \times [0, 1] \), which are obviously deformation retracts of \( W_1 \) and \( W_2 \) respectively. Then
\[
H_2(W, \Sigma) \cong H_2(W, \Sigma \times [-1, 1])
\]
\[
\cong H_2(W_1' \cup W_2', \Sigma \times \{0\} \cup \Sigma \times \{1\}) \cong H_2(W_1, \Sigma) \oplus H_2(W_2, \Sigma).
\]

Here the second isomorphism is obtained by excision. With this in mind, we have
\[
H_2(W, \Sigma) \cong H_2(W_1, \Sigma) \oplus H_2(W_2, \Sigma) \cong H_1(W_1) \oplus \mathbb{Z}^{2g_1} \oplus H_1(W_2) \oplus \mathbb{Z}^{2g_2}.
\]

Then (b) follows from this observation and long exact sequences (2.3).

These observations motivate the following definitions.

**Definition 2.2.** Let \( G \) be a finite abelian group. A pair of finite abelian groups \( (G_1, G_2) \) is said to be admissible for \( G \) if
\[
(i) \quad |G_i|^2 ||G_i|, \ i = 1, 2.
\]
\[
(ii) \quad \text{There exist } n_1, n_2 \in \mathbb{N} \text{ and } a_j^i \in G_i \oplus \mathbb{Z}^{2n_i} \text{ for } i = 1, 2 \text{ and } j = 1, 2, \ldots, 2(n_1 + n_2), \text{ such that the following requirements are satisfied.}
\]
\[
(1) \quad (a_j^i)_{j=1 \ldots 2(n_1+n_2)} \cong G_{i+1} \text{ for } i = 1, 2, \text{ where we let } G_3 \cong G_1.
\]
\[
(2) \quad (a_j^1 \oplus a_j^2)_{j=1 \ldots 2(n_1+n_2)} \cong G.
\]

**Definition 2.3.** Let \( G \) be a finite abelian group and let \( (G_1, G_2) \) be an admissible pair for \( G \). We define a numerical invariant
\[
\theta_1(G, G_1, G_2) = \min\{n_1 + n_2 \mid n_1, n_2 \text{ are as in Definition 2.2}\}.
\]

It follows from Proposition 2.1 that
\[
\theta_1(H_1(\Sigma), H_1(W_1), H_1(W_2)) \leq gds(K).
\]
However, $\theta_1$ has an apparent drawback in that $H_1(W_1)$ cannot be inferred from the knot $K$, and taking a further minimum over all possible admissible pairs would lead to rather trivial bounds.

### 2.2. Double slice genus and Casson-Gordon invariants

In this subsection, we use Casson-Gordon invariants to construct a lower bound for the double slice genus. First we recall the relevant facts about Casson-Gordon invariants below. Detailed information may be found in [2][10][19].

We begin with recalling the definition of Casson-Gordon invariants. Let $M$ be an oriented three manifold equipped with a character $\phi : H_1(M) \to \mathbb{Z}_d$, where $d$ is a non-negative integer. By bordism theory, there exists some positive integer $r$ such that $r \cdot (M, \phi) = \partial(V, \phi')$, where $V$ is a compact 4-manifold and $\phi' : H_1(V) \to \mathbb{Z}_d$ is a character that restricts to $\phi$ on the boundary. $\phi'$ determines a cyclic cover $\tilde{V} \to V$ with a preferred covering transformation $T : \tilde{V} \to \tilde{V}$. Let $T_* : H_2(\tilde{V}; \mathbb{C}) \to H_2(\tilde{V}; \mathbb{C})$ be the induced automorphism and let $\tilde{H}_2(V, \phi')$ be the $e^{2\pi i/d}$ - eigenspace of $T_*$. Note that the intersection form on $H_2(\tilde{V}; \mathbb{Z})$ extends naturally to a Hermitian pairing $\langle , \rangle$ on $H_2(\tilde{V}; \mathbb{C}) = H_2(\tilde{V}; \mathbb{Z}) \otimes \mathbb{C}$. Let $\tilde{\sigma}(V, \phi')$ denote the signature of this Hermitian pairing restricted to $H_2(V, \phi')$. Then define the Casson-Gordon invariant associated to $(M, \phi)$ as

$$\sigma(M, \phi) = \frac{1}{r}(\tilde{\sigma}(V, \phi') - \sigma(V)).$$

Here $\sigma(V)$ denotes the usual signature invariant of $V$.

The key fact that relates Casson-Gordon invariants and the double slice genus is the following proposition due to Gilmer.

**Proposition 2.4** (Proposition 1.4 of [10]). **If** $\phi'$ **is of prime power order, i.e.** $d = p^n$ **for some prime** $p$ **and positive integer** $n$, **let** $\bar{b}_2(V) := \dim_{\mathbb{C}} H_2(V, \phi')$ **and** $b_2(V, \mathbb{Z}_p) := \dim_{\mathbb{Z}_p} H_2(\tilde{V}; \mathbb{Z}_p)$. **Then**

$$\bar{b}_2(V) \leq b_2(V, \mathbb{Z}_p).$$

With the above preparation on Casson-Gordon invariants, we return to the double slice genus. Let $K$, $W_i$ for $i = 1, 2$, $W$ and $\Sigma$ be as in Subsection 2.1. Let $p$ be a prime, and for a finite abelian group $G$, define $\xi_p(G) := \dim_{\mathbb{Z}_p} G \otimes \mathbb{Z}_p$.

**Theorem 2.5.** **Let** $d = p^n$ **and** $\phi_i : H_1(\Sigma) \to \mathbb{Z}_d$ **be a character that factors through** $H_1(W_i)$ **along the inclusion induced homomorphism** $\iota_i : H_1(\Sigma) \to H_1(W_i)$, **i = 1, 2**. **Then**

(i) $|\sigma(\Sigma, \phi_1) - \sigma(\Sigma, \phi_2)| - \xi_p(H_1(W_1) \oplus H_1(W_2)) \leq 2g_{ds}(K)$.

(ii) $|\sigma(\Sigma, \phi_1) + \sigma(\Sigma, \phi_2)| - \xi_p(H_1(W_i)) \leq 2g_i$.

**Proof.**

(i) **Note** $\Sigma = \partial W_1 = -\partial W_2$, **hence**

$$|\sigma(\Sigma, \phi_1) - \sigma(\Sigma, \phi_2)| = |\tilde{\sigma}(W_1, \phi_1) + \tilde{\sigma}(W_2, \phi_2) - \sigma(W_1) - \sigma(W_2)|$$

$$= |\tilde{\sigma}(W_1, \phi_1) + \tilde{\sigma}(W_2, \phi_2) - \sigma(W)|$$

$$= |\tilde{\sigma}(W_1, \phi_1) + \tilde{\sigma}(W_2, \phi_2)|$$

$$\leq \bar{b}_2(W_1) + \bar{b}_2(W_2)$$

$$\leq b_2(W_1; \mathbb{Z}_p) + b_2(W_2; \mathbb{Z}_p)$$

$$= 2g_1 + \xi_p(H_1(W_1)) + 2g_2 + \xi_p(H_1(W_2)).$$
Here we used Novikov additivity for the second equality, $\sigma(W) = \sigma(\#pS^2 \times S^2) = 0$ for the third equality, and the universal coefficient theorem for the last equality. Finally, note $g_{ds}(K) = g_1 + g_2$ and $\xi_p(H_1(W_1) \oplus H_1(W_2)) = \xi_p(H_1(W_1)) + \xi_p(H_1(W_2))$, hence the statement follows.

(ii) Note that $\sigma(K) = \sigma(W_1) = -\sigma(W_2)$ by Theorem 3.1 of [14]. Similar to the argument in (i) above we have

$$|\sigma(\Sigma, \phi_1) + \sigma(K)| = |\sigma(\Sigma, \phi_1)|$$

$$\leq \bar{b}_2(W_i)$$

$$\leq b_2(W_i; \mathbb{Z}_p)$$

$$= 2g_i + \xi_p(H_1(W_i)).$$

The statement readily follows.

The following definition is motivated by the above theorem.

**Definition 2.6.** Let $(G_1, G_2)$ be an admissible pair for $H_1(\Sigma)$. Define

$$\theta_2(\Sigma, G_1, G_2) = \frac{1}{2} \min_{(i_1, i_2)} \max_{(\phi_1, \phi_2, p)} \{ |\sigma(\Sigma, \phi_1) - \sigma(\Sigma, \phi_2)| - \xi_p(G_1 \oplus G_2) \}$$

and

$$\theta_3(K, G_1, G_2) = \frac{1}{2} \min_{(i_1, i_2)} \max_{(\phi_1, \phi_2)} \{ \max(0, |\sigma(\Sigma, \phi_1) + \sigma(K)| - \xi_p(G_1)) + \max(0, |\sigma(\Sigma, \phi_2) + \sigma(K)| - \xi_p(G_2)) \}$$

where in both equations the maximum is taken over all primes $p$ and characters $\phi_1$ and $\phi_2$ satisfying the constraints, and the minimum is taken over all surjective homomorphisms $\iota_i : H_1(\Sigma) \rightarrow G_i$ for $i = 1, 2$.

In view of Theorem 3.3 of [26] we clearly have $\theta_2(\Sigma, H_1(W_1), H_1(W_2)) \leq g_{ds}(K)$ and $\theta_3(K, H_1(W_1), H_1(W_2)) \leq g_{ds}(K)$. However, like $\theta_1$, these invariants are difficult to utilize since one has little control of $H_1(W_i)$ for $i = 1, 2$.

2.3. **Combining $\theta_i$.** So far we have defined various $\theta_i$'s, all of which require the input of an admissible pair that cannot be deduced from the knot. One obvious remedy is to take a minimum over all the admissible pairs, which unfortunately does not lead to a useful lower bound if one uses a single $\theta_i$. However, this can be overcome by combining these invariants. First note that $\theta_1$ and $\theta_2$ are well defined if we replace $\Sigma$ with an arbitrary rational homology sphere. This allows us to make the following definition.

**Definition 2.7.** (i) Given a rational homology 3-sphere $Y$, define

$$\delta(Y) = \min_{(G_1, G_2)} \max\{ \theta_1(H_1(Y), G_1, G_2), \theta_2(Y, G_1, G_2) \}.$$  

Here the minimum is taken over all admissible pairs $(G_1, G_2)$ for $H_1(Y)$.

(ii) Let $K$ be a knot in $S^3$, and let $\Sigma$ be the two-fold branched cover of $S^3$ along $K$. Define

$$\delta(K) = \delta(\Sigma).$$  


Recall every closed, orientable 3-manifold \( Y \) embeds in \( \#_n S^2 \times S^2 \) for sufficiently large \( n \), and the minimum such \( n \) is defined to be the embedding number \( \epsilon(Y) \) \([1]\).

We have the following theorem on the \( \delta \)-invariant.

**Theorem 2.8.** Let \( Y \) be a rational homology 3-sphere. Then

\[
\delta(Y) \leq \epsilon(Y).
\]

In particular, for a knot \( K \subset S^3 \), we have \( \delta(K) \leq \epsilon(\Sigma) \leq g_{ds}(K) \).

*Proof.* Embedding \( Y \) in \( \#_{\epsilon(Y)} S^2 \times S^2 \) separates \( \#_{\epsilon(Y)} S^2 \times S^2 \) into two 4-manifolds \( V_1 \) and \( V_2 \). The proofs of Theorem 2.4 and Theorem 2.5 carry over verbatim with \( \Sigma, W_1 \) and \( W_2 \) replaced by \( Y, V_1 \) and \( V_2 \). Therefore,

\[
\delta(Y) \leq \max\{\theta_1(H_1(Y), H_1(V_1), H_1(V_2)), \theta_2(Y, H_1(V_1), H_1(V_2))\}
\]

\[
\leq \frac{1}{2} b_2(V_1 \cup_Y V_2) = \frac{1}{2} b_2(\#_{\epsilon(Y)} S^2 \times S^2)
\]

\[
= \epsilon(Y).
\]

Given a knot \( K \), note the two-fold branched cover \( \Sigma \) embeds in \( \#_{g_{ds}(K)} S^2 \times S^2 \) (see the first paragraph of Subsection 2.1). Therefore, we have \( \epsilon(\Sigma) \leq g_{ds}(K) \). \( \square \)

When only interested in knots, we can use \( \theta_3 \) instead of \( \theta_2 \) to give a better lower bound for the double slice genus.

**Definition 2.9.** Let \( K \) be a knot in the 3-sphere. Define

\[
\theta(K) = \min_{(G_1, G_2)} \max\{\theta_1(H_1(\Sigma(K)), G_1, G_2), \theta_3(K, G_1, G_2)\}.
\]

Here the minimum is taken over all admissible pairs \( (G_1, G_2) \) for \( H_1(\Sigma(K)) \).

**Theorem 2.10.** For any knot \( K \subset S^3 \), \( \theta(K) \leq g_{ds}(K) \).

*Proof.* Let \( W_1, W_2 \) and \( W \) be as in Subsection 2.1. Then in view of Proposition 2.4 and Theorem 2.5, we have

\[
\theta(K) \leq \max\{\theta_1(H_1(\Sigma(K)), H_1(W_1), H_1(W_2)), \theta_3(K, H_1(W_1), H_1(W_2))\}
\]

\[
\leq \frac{1}{2} b_2(W_1) + \frac{1}{2} b_2(W_2) = \frac{1}{2} b_2(W) = g_{ds}(K).
\]

\( \square \)

2.4. **Proof of Theorem 1.2** In this subsection we prove Theorem 1.2 using the \( \theta \)-invariant of Definition 2.9.

We begin by constructing the knots. Take \( J \) to be the two-bridge knot corresponding to \( \frac{2}{5} \), which is known to be ribbon (e.g. see \([2]\)). Let \( K_n = J \# \cdots \# J \).

Note \( \Sigma(J) = L(9, 4) \) and hence \( \Sigma(K_n) = L(9, 4) \# \cdots \# L(9, 4) \).

To estimate \( \theta(K_n) \), we need to understand the behavior of Casson-Gordon invariants of \( \Sigma(K_n) \). This is addressed in the following technical proposition.

**Proposition 2.11.** Let \( m \) be a nonnegative integer and \( s : H_1(\Sigma(K_n)) \rightarrow \mathbb{Z}_9 \oplus \cdots \oplus \mathbb{Z}_9 \) be a surjective map, then there exists a map \( j : \mathbb{Z}_9 \oplus \cdots \oplus \mathbb{Z}_9 \rightarrow \mathbb{Z}_9 \) such that

\[
\sigma(\Sigma(K_n), j \circ s) \geq \frac{10}{3} m.
\]
A LOWER BOUND FOR THE DOUBLE SLICE GENUS

The proof of this proposition appears in Appendix A.

Theorem 1.2 follows from the next theorem.

**Theorem 2.12.** \( \theta(K_{110n}) \geq n \), and hence \( g_{ds}(K_{110n}) \geq n \).

**Proof.** Let \((G_1, G_2)\) be an admissible pair for \( H_1(\Sigma(K_{110n})) \). Since \( G_1 \oplus G_2 \) is a quotient group of \( H_1(\Sigma(K_{110n})) = \oplus_{110n} \mathbb{Z}_9 \), we may write \( G_1 \oplus G_2 = \mathbb{Z}_9 \oplus \cdots \oplus \mathbb{Z}_9 \oplus \mathbb{Z}_3 \oplus \cdots \oplus \mathbb{Z}_3 \) for some \( l \). We will prove \( \theta(K_{110n}) \geq n \) by considering the possible values of \( l \) in two cases.

First, if \( 110n - l \geq 2n \), then we must have \( \theta_1(H_1(\Sigma), G_1, G_2) \geq n \) in view of (ii)-(2) of Definition 2.2, since one must employ at least another \( 110n - l \) generators of order 9 to get to \( H_1(\Sigma) \).

Second, if \( 110n - l < 2n \), then \( l > 108n \). Then at least one of \( G_i \), say \( G_1 \), has \( l' \) many \( \mathbb{Z}_9 \) summands with \( l' \geq 54n \). By Proposition 2.11 we have a character \( \phi_1 : H_1(\Sigma(K_{110n})) \to \mathbb{Z}_9 \) that factors through \( \iota_1 : H_1(\Sigma(K_{110n})) \to G_1 \), such that \( \sigma_3(K_{110n}, \phi_1) \geq \frac{10}{9} l' \). Also note that \( \xi_3(G_1) \leq l' + 110n - l \) and \( \sigma(K_{110n}) = 0 \). Therefore,
\[
\theta_3(K_{110n}, G_1, G_2) \geq \frac{1}{2} \left( \frac{10}{9} l' - (l' + 110n - l) \right) \\
\geq \frac{1}{2} \left( \frac{10}{9} l' - 2n \right) \\
\geq \frac{1}{2} \left( \frac{54n}{9} - 2n \right) \\
\geq n.
\]

Therefore, \( \theta(K) \geq n \) in either case. \( \square \)

**Remark 2.13.** For any given nonnegative integer \( m \), one can similarly prove there is a family of knots whose slice genera are all equal to \( m \) and double slice genera grow arbitrarily large. In fact, taking the connected sum of \( m \) copies of the trefoil knot and \( K_n \) produces such examples.

However, the author is not able to prove the embedding numbers of \( \Sigma(K_n) \) grow arbitrarily large by the \( \delta \)-invariant defined in Subsection 2.3. The lower bounds for the embedding number in \( \mathbb{II} \) come from lower bounds for the 2-nd betti number of spin 4-manifolds bounded by the given 3-manifold, and cannot be applied here since our 3-manifolds already bound rational homology balls. It seems natural to ask the following question.

**Question 2.14.** Can one find a family of rational homology spheres that are boundaries of spin rational homology balls and whose embedding numbers can grow arbitrarily large?

Moreover, the author wonders if one can find examples so that the \( \delta \)-invariant can be applied to answer the above question.

3. BOUNDS FOR THE SUPERSLICE GENUS

3.1. A lower bound for the superslice genus. We prove Theorem 1.4 in this subsection. As a preparation we begin with two lemmas.

**Lemma 3.1.** Let \( F \) be a surface properly embedded in \( D^4 \) whose double is a closed surface that bounds a handlebody in \( S^4 \). Then \( \pi_1(D^4 - F) \cong \mathbb{Z} \).
Proof. Write the double of $F$ as $F_+ \cup F_-$. Since $F_+ \cup F_-$ bounds a handlebody in $S^4$, $\pi_1(S^4 - (F_+ \cup F_-)) \cong \mathbb{Z}$. Applying Van Kampen’s theorem, we have the following pushout diagram. Using the universal property we see there is a surjective map $\pi_1(S^4 - (F_+ \cup F_-)) \cong \mathbb{Z} \rightarrow \pi_1(D^4 - F)$. Therefore $\pi_1(D^4 - F)$ is a cyclic group and must be isomorphic to $\mathbb{Z}$, since $H_1(D^4 - F) \cong \mathbb{Z}$ by Alexander duality.

\begin{equation}
\begin{tikzcd}
\pi_1(D^4 - F_+) & \pi_1(D^4 - F) \\
\pi_1(S^3 - K) & \pi_1(S^4 - (F_+ \cup F_-)) \ar[r, dashed] & \pi_1(D^4 - F) \\
\pi_1(D^4 - F_-) \ar[u, Id] & \ar[u, Id] \ar[u, Id]
\end{tikzcd}
\end{equation}

Lemma 3.2. Let $F$ be as in the previous lemma, and let $W$ be the two-fold branched cover of $D^4$ along $F$. Then $H_1(W) = 0$.

Proof. Let $\tilde{F} \subset W$ be the lift of $F$. Then by Lemma 3.1, $\pi_1(W - \tilde{F}) \cong \mathbb{Z}$. Therefore $H_1(W - \tilde{F}) \cong \mathbb{Z}$, generated by the homology class of a meridian of $\tilde{F}$. Gluing $\tilde{F}$ back annihilates this and hence $H_1(W) = 0$.

Proof of Theorem 1.4. Assume $F$ is a surface achieving the minimal superslice genus of the knot $K$, and $W$ is the two-fold branched cover of $D^4$ along $F$. Furthermore, let $\Sigma = \partial W$ denote the two-fold branched cover of $S^3$ along $K$. Note that $H_2(\Sigma) = 0$, and $H_1(W) = 0$ by the previous lemma. Then the long exact sequence associated to the pair $(W, \Sigma)$ gives

\[ 0 \to H_2(W) \to H_2(W, \Sigma) \to H_1(\Sigma) \to 0. \]

Note $H_2(W, \Sigma) \cong H^2(W)$, and $H^2(W)$ is free abelian by the universal coefficient theorem and the fact that $H_1(W) = 0$. Since $b_2(W) = 2g(F)$ by Proposition 2.1, (i) and $g(F) = g^*(K)$ by our assumption, we have a presentation for $H_1(\Sigma)$:

\[ 0 \to \mathbb{Z}^{2g^*(K)} \to \mathbb{Z}^{2g^*(K)} \to H_1(\Sigma) \to 0. \]

Hence the theorem follows.

\section{3.2. An upper bound for the topological superslice genus.}

In this subsection we prove Theorem 1.5. We first recall three non-trivial results.

Theorem 3.3 (Theorem 7 of [7], Theorem 11.7B of [8]). Let $K$ be a knot in $S^3$ such that $\Delta_K(t) = 1$, then $K$ bounds a locally flat, topologically embedded disk $D \subset D^4$ such that $\pi_1(D^4 - D) \cong \mathbb{Z}$.

Theorem 3.4 (Theorem 6 of [7], Theorem 11.7A of [8]). A locally flat embedding $f : S^2 \to S^4$ is unknotted if and only if $\pi_1(S^4 - f(S^2)) \cong \mathbb{Z}$.

Proposition 3.5 (Proposition 2 of [8]). Let $K$ be a knot. Every Seifert surface $F_K$ of $K$ contains a simple closed curve $J$ separating $F_K$ into two subsurfaces $C_{K,J}$ and $F_J$ such that:

(i) The Alexander polynomial of $J$ is trivial.

(ii) $F_J$ is a Seifert surface for $J$ with $2g(F_J) = 2g(F_K) - \deg(\Delta_K(t))$. 

We move on to apply these results to prove Theorem 1.5. First, note Theorem 3.3 and Theorem 3.4 together implies

**Proposition 3.6** (Theorem 4.5 of [16], Corollary 3.3 of [17]). Knots with trivial Alexander polynomial are topologically superslice.

**Proof.** Doubling a slice disk asserted as in Theorem 3.3 produces a 2-knot satisfying the condition of Theorem 3.4.

We set up some terminology for convenience. Given a superslice knot $J$, a 3-ball (locally flatly) embedded in $S^4$ is called a **superslice ball** for $J$ if its boundary is the double of some superslice disk for $J$. Given a submanifold $N$ (possibly with non-empty boundary) of a manifold $M$, by a normal bundle of $N$ we mean a disk bundle $E$ over $N$ together with an embedding of $E$ into $M$ such that $N$ is identified with the 0-section of $E$. In this subsection, the normal bundles we will encounter are always trivial. By abusing notation, we denote a trivial normal bundle $\phi : N \times D^k \to S^n$ of $N \subset S^n$ by $N \times D^k$ for appropriate $n$ and $k$, with $\ast \times D^k$ understood as $\phi(\ast) \times D^k$.

Let $F$ be an orientable surface (possibly with boundary) in $S^3$ and $\alpha : [0,1] \to S^3$ be embedded arc. We use the letter $\alpha$ to denote the map interchangeably with its image $\alpha([0,1])$ by abusing notation. Assume $\alpha \cap F = \{\alpha(0), \alpha(1)\}$ and $\alpha$ approaches both ends from the same side in a normal direction of $F$; we can perform a stabilization of $F$ along $\alpha$: Take a normal bundle $\alpha \times D^2 \subset S^3$ of $\alpha$ such that $(\alpha \times D^2) \cap F = \{\alpha(0), \alpha(1)\} \times D^2$, then a stabilization of $F$ along $\alpha$ is defined to be the surface $F' = (F - \{\alpha(0), \alpha(1)\} \times D^2) \cup (\alpha \times \partial D^2)$.

Given a knot $K$, the idea for proving Theorem 1.5 is patching together a superslice ball for $J$ and a thickened Seifert surface for $K$, where $J$ is a knot as in Proposition 3.3. To achieve this, one must understand the cross-section of a superslice ball at the equator $S^3$. The key lemma towards this goal is the following.

**Lemma 3.7.** Let $B_J$ be a superslice ball for $J$. Assume $B_J$ intersects $S^3$ transversally and denote $B_J \cap S^3 = S_0 \cup S_1 \cup \cdots \cup S_n$, where $S_0$ is a Seifert surface for $J$ and $S_1, \ldots, S_n$ are closed orientable surfaces. Let $S_i'$ be a stabilization of $S_i$ along some arc whose interior is disjoint from all the $S_i$’s. Then there is a boundary-fixing isotopy of $B_J$ to $B_J'$ such that $B_J' \cap S^3 = S_0 \cup \cdots \cup S_i' \cup \cdots \cup S_n \cup \partial(\text{nb}(L))$, where $L$ is a link in the complement of $S_0 \cup \cdots \cup S_i' \cup \cdots \cup S_n$ and $\text{nb}(L)$ is a regular neighborhood of $L$.

**Proof.** Denote the arc along which we stabilize $S_i$ by $\alpha$. We first pick an embedded arc $\beta$ in $B_J$ such that $\beta \cap (S_0 \cup \cdots \cup S_n) = \{\alpha(0), \alpha(1)\}$. Roughly, $\beta$ is obtained by pushing the interior of a path in $S_i$ connecting $\alpha(0)$ and $\alpha(1)$ into $B_J \setminus S_i$. For the sake of a clear discussion, we fix a specific choice of $\beta$. First, parametrize a closed neighborhood of $S^3$ by $S^3 \times [-1,1]$, where we identify the equator $S^2$ as $S^3 \times \{0\}$ and require $S^3 \times [-1,1]$ to be thin enough so that $B_J \cap (S^3 \times [-1,1]) = (S_0 \cup S_1 \cup \cdots \cup S_n) \times [-1,1]$. Let $\beta' : [0,1] \to S_i$ be an embedded path connecting $\alpha(0)$ and $\alpha(1)$, and let $\phi : [0,1] \to [0,1]$ be the function $t \mapsto \sqrt{\frac{1}{4} - (t - \frac{1}{2})^2} + \frac{1}{4}$.

We then set

$$
\beta : [0,1] \to S_i \times [0,1],
\begin{align*}
t &\mapsto \begin{cases}
(\alpha(0), t), & 0 \leq t \leq \frac{1}{3}; \\
(\beta'(3t - 1), \phi(3t - 1)), & \frac{1}{3} \leq t \leq \frac{2}{3}; \\
(\alpha(1), 1 - t), & \frac{2}{3} \leq t \leq 1.
\end{cases}
\end{align*}
$$
The idea for proving this lemma is to find an embedded disk \( W \) bounded by \( \alpha \cup \beta \) which only intersects \( B_J \) at \( \beta \), and then push \( B_J \) across this disk. A schematic picture is shown in Figure 1.

![Figure 1](image1.png)

**Figure 1.** A one dimension lower schematic picture for pushing \( B_J \) across a disk to realize a stabilization of the cross-section.

We set this up more carefully. Let \( \beta \times D^2 \subset B_J \) be a normal bundle of \( \beta \) such that \((\beta \times D^2) \cap (S_0 \cup \cdots \cup S_n) = \{\beta(0), \beta(1)\} \times D^2\). Similarly, let \( \alpha \times D^2 \subset S^3 \) be a normal bundle of \( \alpha \) such that \( \partial \alpha \times D^2 \) is identified with \( \partial \beta \times D^2 \). We want an embedded disk \( W \) such that \( W \cap (B_J \cup \alpha) = \beta \cup \alpha \), and \( W \) admits a normal bundle \( W \times D^2 \) such that \( \partial W \times D^2 = (\alpha \times D^2) \cup (\beta \times D^2) \).

To find such a \( W \), we first construct a collar of \( \partial W \) carefully. Let \( B_J \times [-\epsilon, \epsilon] \) be a normal bundle of \( B_J \) in \( S^3 \). Note \( \alpha \cap (B_J \times [-\epsilon, \epsilon]) = \{\alpha(0), \alpha(1)\} \times [0, \epsilon] \), for \((B_J \times [-\epsilon, \epsilon]) \cap S^3\) is a normal bundle of the cross-section in \( S^3 \). Let \((\alpha \times D^2) \times [-t_0, t_0]\) be a normal bundle of \( \alpha \times D^2 \) in \( S^4 \) obtained by taking a product with a short interval in the 4-th dimension, where \( t_0 < 1/3 \). Then \( ((\alpha \times D^2) \times [-t_0, t_0]) \cap (B_J \times [-\epsilon, \epsilon]) = (\partial \alpha \times D^2) \times [0, \epsilon] \times [-t_0, t_0] \). (See Figure 2 and Figure 3.

![Figure 2](image2.png)

**Figure 2.** A schematic picture for the intersection of \( B_J \times [-\epsilon, \epsilon] \) and \((\alpha \times D^2) \times [-t_0, t_0]\) in \( S^3 \). The entire intersection can be viewed as the product of the intersection in \( S^3 \) with \([-t_0, t_0]\).

Let \( \tilde{\alpha} = \alpha - \{\alpha(0), \alpha(1)\} \times [0, \epsilon] \) and let \( A = (\beta \times [0, \epsilon]) \cup (\tilde{\alpha} \times [0, t_0]) \). Then \( A \) is an annulus with one of the boundary components being \( \alpha \cup \beta \) (Figure 3). Clearly, \( A \) admits a normal bundle that restricts to \((\alpha \cup \beta) \times D^2\) as desired. Denote the other component of \( \partial A \) by \( l \), note \( l \subset \partial((B_J \times [-\epsilon, \epsilon]) \cup (\alpha \times D^2 \times [-t_0, t_0])) \). Note \((B_J \times [-\epsilon, \epsilon]) \cup (\alpha \times D^2 \times [-t_0, t_0])\) is homeomorphic to \( S^1 \times D^3 \) and \( l \) is isotopic
to $S^1 \times \{pt\} \subset S^1 \times \partial D^3$. As any embedded $S^1$ is unknotted in $S^2 \times D^2$, $l$ bounds a disk $D$ in the closed complement of $(B_J \times [-\epsilon, \epsilon]) \cup (\alpha \times D^2 \times [-t_0, t_0])$. The desired disk is then $W = A \cup D$ and has a normal bundle extending the one on $A$ ([8] Theorem 9.3A).

\begin{figure}[h]
\centering
\includegraphics[width=0.8\textwidth]{figure3}
\caption{Schematic picture of $(B_J \times [-\epsilon, \epsilon]) \cup (\alpha \times D^2 \times [-t_0, t_0])$ (left) and the annulus $A$ (right).}
\end{figure}

Note by construction $A \cap S^3 = \alpha \subset \partial W$. Since $W = A \cup D$, by transversality the interior of $W$ intersects $S^3$ at a collection of circles contained in $D$, which is a link $L \subset S^3$ in the complement of the $S'_i$s. Modify $B_J$ by pushing $\beta \times D^2$ across $W \times D^2$. Equivalently, this is same as removing the interior of $\beta \times D^2$ and glue back $\partial(W \times D^2 \setminus (\beta \times D^2))$. Call the resulting 3-ball $\tilde{B}_J$, note $\tilde{B}_J \cap S^3 = S_0 \cup \cdots \cup S_n \cup (\alpha \times D^2) \cup \partial(nb(L))$. Finally, push $\alpha \times \text{Int}(D^2)$ slightly into the 4-th dimension to yield the desired ball $B''_J$ with a cross-section as stated in the lemma. \hfill $\square$

With Lemma 3.7 at hand, we can modify an arbitrary superslice ball to obtain favorable cross-sections. More concretely, we have the following proposition.

**Proposition 3.8.** For any knot $K$ there exists some Seifert surface $F_K$ such that

1. there exists a knot $J$ contained in $F_K$ satisfying the conclusion of Proposition 3.6.
2. $J$ has a superslice ball $B_J$ such that $B_J \cap F_K = F_J$.

**Proof.** Let $F'_K$ be an arbitrary Seifert surface for $K$. Apply Proposition 3.6 to get $F'_K = C_{J,K} \cup J F'$, where $F'$ is a Seifert surface for $J$. According to Proposition 3.6, $J$ is superslice. Let $B'_J$ be a superslice ball for $J$ such that $B'_J \cap S^3 = F''_J \cup S_1 \cup \cdots \cup S_n$, where $F''_J$ is some Seifert Surface for $J$ and $S_1, \ldots, S_n$ are closed orientable surfaces. Let $F'_J$ be a common stabilization of $F'_J$ and $F''_J$. Our goal will be achieved by stabilizing both $F'_K$ and $B'_J \cap S^3$ properly. We divide the procedure into three steps.

**Step 1.** Stabilize $F'_K$ to get $F_K = C_{K,J} \cup J$. Note this is possible, for if an arc we use to stabilize $F'_J$ intersects $C_{K,J}$, then we can push the arc off $C_{K,J}$ along a path from the intersection point to the $K$-boundary. See Figure 4.

**Step 2.** Isotope $B'_J$ to $B''_J$ so that the Seifert surface appeared in the cross-section $B''_J \cap S^3$ is $F_J$. We explain how to achieve this by assuming $F_J$ can be
obtained from $F''_J$ by a single stabilization along an arc $\alpha$. The general case easily follows by repeating the argument.

We induct on the number of intersection points of $\alpha \cap (S_1 \cup \cdots \cup S_n)$. Denote this number by $2k$. Suppose there are no intersection points between $\alpha$ and the $S_i$'s, then Step 2 can be accomplished by applying Lemma 3.7. Assume Step 2 can be accomplished when there are fewer than $2k$ intersection points. Now suppose there are $2k$ intersection points with $k > 0$. Then the $\alpha$ arc is divided into subarcs by these points. There must be a subarc $\alpha'$ such that the end points of $\alpha'$ lie on some surface $S_i$ and the interior of $\alpha'$ does not intersect any of the surfaces. To see this, assume the arc $\alpha$ intersects surfaces $S_{m_1}, \ldots, S_{m_l}$ for some $l \leq n$. There must be a surface $S_{m_i}$ which bounds a region that does not contain the end points of $\alpha$ and any other surface $S_{m_j}, j \neq i$. Note $\alpha$ must intersect such a surface consecutively at two points, and we take $\alpha'$ to be the subarc between two such intersection points. Apply Lemma 3.7 to $\alpha'$. Note $\alpha$ does not intersect the newly appeared tori $\partial (\nu b (L))$ as we may assume $\alpha \cap L = \emptyset$ by a general position argument, and we can arrange $\alpha$ to go through the newly appeared tunnel gained from the stabilization. Therefore, we have two fewer intersection points between $\alpha$ and the new cross-section, and the claim follows from the inductive hypothesis. See Figure 5.

Step 3. Isotope $B''_J$ to $B_J$ so that the closed components in the cross-section do not intersect $C_{K,J}$. To see this, note to build $F_K$ from $F_J$ one needs to add a number of bands, and $C_{K,J}$ is isotopic to the union of a collar of $J$ and the bands. It is clear that if the core of a band does not intersect the $S_i$'s, then the whole band does not intersect the $S_i$'s. If not, suppose the core of the band does intersect the
surfaces, then a stabilization arrangement as in Step 2 can be employed to remove
the intersections. Therefore, after possibly stabilizing the $S_i$’s and introducing more
tori, we may assume these closed surfaces do not intersect $C_{K,J}$.

Theorem 1.5 now follows readily from Proposition 3.8.

Proof of Theorem 1.5. Given a knot $K$, let $F_K$ and $B_J$ be a Seifert surface for $K$
and a superslice ball for $J$ respectively as in Proposition 3.8. Thicken $F_K$ slightly to
$F_K \times [-\epsilon, \epsilon]$ using the 4-th dimension. By transversality we may assume $B_J \cap (F_K \times [-\epsilon, \epsilon]) = F_J \times [-\epsilon, \epsilon]$ provided $\epsilon$ is small enough. Now construct a 3-dimensional
handlebody $H = (F_K \times [-\epsilon, \epsilon]) \cup B_J$. It is easier to see $H$ is a handlebody by
viewing $H = (C_{K,J} \times [-\epsilon, \epsilon]) \cup D_J$. Denote by $D_J$ the superslice disk that
doubles to $\partial B_J$. Then $\partial(H)$ is the double of a surface isotopic to $C_{K,J} \cup D_J$ relative
to $K$ in the 4-ball. Finally, note $2g(C_{K,J} \cup D_J) = \deg(\Delta_K(t))$.

As we mentioned in the introduction, this theorem has an immediate corollary
which says that if a knot $K$ has $\deg(\Delta_K(t)) = 2$, then $g_{\text{top}}(K) = 1$.

Proof of Corollary 1.6. It is understood $g_{\text{top}}(K) = 0$ if and only if $\Delta_K(t) = 1$ ([10]
[12]). Therefore $\deg(\Delta_K(t)) = 2$ implies $g_{\text{top}}(K) > 0$. In view of Theorem 1.5
$g_{\text{top}}(K) \leq 1$. Hence we have $g_{\text{top}}(K) = 1$.

APPENDIX A.

In this appendix we prove Proposition 2.11. We begin by computing the Casson-
Gordon invariants of $L(9,4)$. The author found it convenient to use the strategy in
[10] to describe the characters and compute the invariants.

Note that $L(9,4)$ admits a surgery diagram as shown in Figure 6.

![Figure 6. A surgery diagram for $L(9,4)$.

We fix an isomorphism $H_1(L(9,4)) \cong \mathbb{Z}_9$ by taking the homology class generated
by the meridian (with an arbitrarily chosen orientation) of the $-3$-framed unknot
to be $1 \in \mathbb{Z}_9$. Let $\chi_a : H_1(L(9,4)) \to \mathbb{Z}_9$ be the character sending $1$ to $a$.

Applying a formula in [10] (on page 370) one gets $\sigma(L(9,4), \chi_a)$ as shown in the table:

| a     | 0   | 1   | 2   | 3   | 4   | 5   | 6   | 7   | 8   |
|-------|-----|-----|-----|-----|-----|-----|-----|-----|-----|
| $\sigma(L(9,4), \chi_a)$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $-\frac{1}{3}$ | $-\frac{2}{3}$ | $\frac{1}{3}$ | $\frac{2}{3}$ | $\frac{1}{3}$ | $-\frac{2}{3}$ |

We need the following lemma before proving Proposition 2.11.

Lemma A.1. Let $m$, $n$ be nonnegative integers and let $s : H_1(\#_n L(9,4)) \to \oplus_m \mathbb{Z}_9$
be a surjective homomorphism. Identify $H_1(\#_n L(9,4)) \cong \oplus_n \mathbb{Z}_9$ using the isomorphism described above. Then one can choose a basis for $\oplus_m \mathbb{Z}_9$ and reorder the basis
Write for now the basis for \(\oplus_n \mathbb{Z}_9\) if necessary, so that in terms of these two bases, \(s\) can be represented by a matrix of the form
\[
\begin{bmatrix}
1 & \cdots & 0 & a_1^{m+1} & \cdots & a_1^n \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & a_m^{m+1} & \cdots & a_m^n
\end{bmatrix}.
\]

Here the entries of the matrix are counted mod 9.

**Proof.** Start with an arbitrary basis for \(\oplus_n \mathbb{Z}_9\). Write \(s\) as a matrix
\[
\begin{bmatrix}
a_1^1 & \cdots & a_1^m & a_1^{m+1} & \cdots & a_1^n \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
a_m^1 & \cdots & a_m^m & a_m^{m+1} & \cdots & a_m^n
\end{bmatrix}.
\]

Since \(s\) is surjective, then the first row represents a surjective map \(\oplus_n \mathbb{Z}_9\) to \(\mathbb{Z}_9\). Therefore, there must exist \(a_i^1\) for some \(i\) such that \(3 \nmid a_i^1\). So \(a_i^1\) is a generator in \(\mathbb{Z}_9\), and we may alter the first basis element of \(\oplus_n \mathbb{Z}_9\) by a proper multiplication, so that under the new basis, \(a_i^1\) becomes 1. Reorder the basis for \(H_1(\#_nL(9,4))\) if necessary to achieve \(a_i^1 = 1\).

Then under the new bases, \(s\) has the following matrix representation (by abusing notation we still denote the undetermined values by \(a_i^j\)):
\[
\begin{bmatrix}
1 & \cdots & a_1^m & a_1^{m+1} & \cdots & a_1^n \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
a_m^1 & \cdots & a_m^m & a_m^{m+1} & \cdots & a_m^n
\end{bmatrix}.
\]

Write for now the basis for \(\oplus_n \mathbb{Z}_9\) as \(e_1, \ldots, e_m\). Make a change of basis: \(e_i' = e_i + \sum_{j=2}^m a_j^i e_j\), \(e_i' = e_i\) for \(i = 2, 3, \ldots, m\). Then under the new basis \(\{e_i'\}\), (abusing notation again) \(s\) has the following matrix representation:
\[
\begin{bmatrix}
1 & a_2^1 & \cdots & a_m^1 & a_1^{m+1} & \cdots & a_1^n \\
0 & a_2^2 & \cdots & a_m^2 & a_1^{m+1} & \cdots & a_1^n \\
\vdots & \vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & a_m^2 & \cdots & a_m^m & a_1^{m+1} & \cdots & a_1^n
\end{bmatrix}.
\]

Repeating this process for each row leads to a matrix of the desired form. \(\square\)

We are ready to prove Proposition 2.11.

**Proof of Proposition 2.11** By Lemma A.1 we may assume \(s\) is of the form
\[
\begin{bmatrix}
1 & \cdots & 0 & a_1^{m+1} & \cdots & a_1^n \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 1 & a_m^{m+1} & \cdots & a_m^n
\end{bmatrix}.
\]

We want to choose a map \(j : \oplus_n \mathbb{Z}_9 \rightarrow \mathbb{Z}_9\) so that the corresponding Casson-Gordon invariant is big enough. Write \(j\) as
\[
\begin{bmatrix}
j_1 & j_2 & \cdots & j_m
\end{bmatrix}.
\]

Using the additivity of Casson-Gordon invariants (e.g. [13]), we have
\[
\sigma(\#_n L(9,4) \circ s) = \sigma(L(9,4), \chi_{j_1}) + \ldots + \sigma(L(9,4), \chi_{j_m}) + \sigma(L(9,4), \chi_{j_1 a_1^{m+1} + \ldots + j_m a_m^{m+1}}) + \ldots + \sigma(L(9,4), \chi_{j_1 a_1^n + \ldots + j_m a_m^n}).
\]
We encode the last \( n - m \) terms of the above equation into a vector
\[
H(j_1, j_2, \ldots, j_m) = [\sigma(L(9, 4), \chi_{j_1}a_1^{n_1+1} + \cdots + j_m a_{m}^{n_m+1}), \ldots, \sigma(L(9, 4), \chi_{j_1}a_1^{n_1} + \cdots + j_m a_{m}^{n_m})].
\]

Consider \( H(2, 2, \ldots, 2) \). If \( H(2, 2, \ldots, 2) \) has less than \( m \) entries of value \(-\frac{1}{9}\), let \( j = [2, 2, \ldots, 2] \). Since all other values in the table are nonnegative, we have
\[
\sigma(\#_n L(9, 4), j \circ s) = \sigma(L(9, 4), \chi_2) + \cdots + \sigma(L(9, 4), \chi_2)
+ \sigma(L(9, 4), \chi_{2a_1^{n_1+1} + \cdots + 2a_m^{n_m+1}}) + \cdots + \sigma(L(9, 4), \chi_{2a_1^{n_1} + \cdots + 2a_m^{n_m}})
\geq \frac{11}{9} m - \frac{1}{9} m
= \frac{10}{9} m.
\]

If \( H(2, 2, \ldots, 2) \) has at least \( m \) entries of value \(-\frac{1}{9}\), then choose \( j = [6, 6, \ldots, 6] \).
Using the table one sees the entries of value \(-\frac{1}{9}\) in \( H(2, 2, \ldots, 2) \) all become 1 in \( H(6, 6, \ldots, 6) \). Also note that there will not be any negative values in \( H(6, 6, \ldots, 6) \) since \( 3(6a_1^i + \cdots + 6a_m^i) \) for any \( i \) and such character always corresponds to value 1 or 0. Therefore, in this case
\[
\sigma(\#_n L(9, 4), j \circ s) = \sigma(L(9, 4), \chi_6) + \cdots + \sigma(L(9, 4), \chi_6)
+ \sigma(L(9, 4), \chi_{6a_1^{n_1+1} + \cdots + 6a_m^{n_m+1}}) + \cdots + \sigma(L(9, 4), \chi_{6a_1^{n_1} + \cdots + 6a_m^{n_m}})
\geq m + m
\geq \frac{10}{9} m.
\]

\[ \square \]

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