NEW GENERAL INTEGRAL INEQUALITIES FOR
$(\alpha, m)$–GA-CONVEX FUNCTIONS VIA HADAMARD
FRACTIONAL INTEGRALS

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Abstract. In this paper, the authors gives a new identity for Hadamard fractional integrals. By using of this identity, the authors obtains new estimates on generalization of Hadamard, Ostrowski and Simpson type inequalities for $(\alpha, m)$-GA-convex function via Hadamard fractional integral.

1. Introduction

Let a real function $f$ be defined on some nonempty interval $I$ of real line $\mathbb{R}$. The function $f$ is said to be convex on $I$ if inequality

$$f(tx + (1-t)y) \leq tf(x) + (1-t)f(y)$$

holds for all $x, y \in I$ and $t \in [0,1]$.

Following inequalities are well known in the literature as Hermite-Hadamard inequality, Ostrowski inequality and Simpson inequality respectively:

Theorem 1. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a convex function defined on the interval $I$ of real numbers and $a, b \in I$ with $a < b$. The following double inequality holds:

$$f\left(\frac{a+b}{2}\right) \leq \frac{1}{b-a} \int_a^b f(x)dx \leq \frac{f(a) + f(b)}{2}.$$ 

Theorem 2. Let $f : I \subseteq \mathbb{R} \to \mathbb{R}$ be a mapping differentiable in $I^\circ$, the interior of $I$, and let $a, b \in I^\circ$ with $a < b$. If $|f'(x)| \leq M$, $x \in [a, b]$, then we the following inequality holds:

$$\left|f(x) - \frac{1}{b-a} \int_a^b f(t)dt\right| \leq \frac{M}{b-a} \left(\frac{(x-a)^2 + (b-x)^2}{2}\right)$$

for all $x \in [a, b]$. The constant $\frac{1}{4}$ is the best possible in the sense that it cannot be replaced by a smaller one.

Theorem 3. Let $f : [a, b] \to \mathbb{R}$ be a four times continuously differentiable mapping on $(a, b)$ and $\|f^{(4)}\|_\infty = \sup_{x \in (a, b)} |f^{(4)}(x)| < \infty$. Then the following inequality holds:

$$\left|\frac{1}{3} \left[\frac{f(a) + f(b)}{2} + 2f\left(\frac{a+b}{2}\right)\right] - \frac{1}{b-a} \int_a^b f(x)dx\right| \leq \frac{1}{2880} \|f^{(4)}\|_\infty (b-a)^4.$$ 

The following definitions are well known in the literature.

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Definition 1. [11] A function $f : I \subseteq (0, \infty) \to \mathbb{R}$ is said to be GA-convex (geometric-arithmetically convex) if
\[
f(x^ty^{1-t}) \leq tf(x) + (1-t)f(y)
\]
for all $x, y \in I$ and $t \in [0, 1]$.

Definition 2. [8] Let $f : (0, b] \to \mathbb{R}$, $b > 0$, and $(\alpha, m) \in (0, 1)^2$. If
\[
f(x^ty^{m(1-t)}) \leq t^\alpha f(x) + m(1-t^\alpha)f(y)
\]
for all $x, y \in (0, b]$ and $t \in [0, 1]$, then $f$ is said to be a $(\alpha, m)$-GA-convex function.

Note that $(\alpha, m) \in \{(1, m), (1, 1), (\alpha, 1)\}$ one obtains the following classes of functions: $m$-GA-convex, GA-convex, $\alpha$-GA-convex (or GA-$s$-convex in the first sense, if we take $s$ instead of $\alpha$ (see [19])).

We will now give definitions of the right-sided and left-sided Hadamard fractional integrals which are used throughout this paper.

Definition 3. [4] Let $f \in L [a, b]$. The right-sided and left-sided Hadamard fractional integrals $J^\theta_{a+}f$ and $J^\theta_{b-}f$ of order $\theta > 0$ with $b > a \geq 0$ are defined by
\[
J^\theta_{a+}f(x) = \frac{1}{\Gamma(\theta)} \int_a^x \left( \ln \frac{x}{t} \right)^{\theta-1} f(t) \frac{dt}{t}, \quad a < x < b
\]
and
\[
J^\theta_{b-}f(x) = \frac{1}{\Gamma(\theta)} \int_x^b \left( \ln \frac{t}{x} \right)^{\theta-1} f(t) \frac{dt}{t}, \quad a < x < b
\]
respectively, where $\Gamma(\theta)$ is the Gamma function defined by $\Gamma(\theta) = \int_0^\infty e^{-t}t^{\theta-1}dt$.

In [20], İcan represented Hermite-Hadamard’s inequalities for GA-convex functions in fractional integral forms as follows:

Theorem 4. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a function such that $f \in L[a, b]$, where $a, b \in I$ with $a < b$. If $f$ is a GA-convex function on $[a, b]$, then the following inequalities for fractional integrals hold:
\[
f\left(\sqrt{ab}\right) \leq \frac{\Gamma(\theta + 1)}{2 \left( \ln \frac{a}{b} \right)^\theta} \left\{ J^\theta_{a+}f(b) + J^\theta_{b-}f(a) \right\} \leq \frac{f(a) + f(b)}{2}
\]
with $\alpha > 0$.

In [20], İcan gave the following identity for differentiable functions.

Lemma 1. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a, b]$, where $a, b \in I$ with $a < b$. Then for all $x \in [a, b]$, $\lambda \in [0, 1]$ and $\alpha > 0$
we have:

\[ I_f (x, \lambda, \theta, a, b) = (1 - \lambda) \left[ \ln^\theta \frac{x}{a} + \ln^\theta \frac{b}{x} \right] f(x) + \lambda \left[ f(a) \ln^\theta \frac{x}{a} + f(b) \ln^\theta \frac{b}{x} \right] \]

\[ - \Gamma(\theta + 1) \left[ J_{\alpha}^\theta f(a) + J_{\alpha}^\theta f(b) \right] \]

\[ = a \left( \ln \frac{x}{a} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left( \frac{x}{a} \right)^t f'(x^t a^{1-t}) dt \]

\[ - b \left( \ln \frac{b}{x} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left( \frac{x}{b} \right)^t f'(x^t b^{1-t}) dt. \]

In recent years, many authors have studied errors estimations for Hermite-Hadamard, Ostrowski and Simpson inequalities; for refinements, counterparts, generalization see [1, 2, 3, 5, 6, 7, 12, 13, 15, 16, 17, 18].

In this paper, new identity for fractional integrals have been defined. By using of this identity, we obtained a generalization of Hadamard, Ostrowski and Simpson type inequalities for \((\alpha, m)\)-GA-convex functions via Hadamard fractional integrals.

2. Main results

Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^0 \), the interior of \( I \), throughout this section we will take

\[ K_f (\lambda, \theta, x^m, a^m, b^m) = (1 - \lambda) m^\theta \left[ \ln^\theta \frac{x}{a} + \ln^\theta \frac{b}{x} \right] f(x^m) \]

\[ + \lambda m^\theta \left[ f(a^m) \ln^\theta \frac{x}{a} + f(b^m) \ln^\theta \frac{b}{x} \right] \]

\[ - \Gamma(\theta + 1) \left[ J_{\alpha}^\theta f(a^m) + J_{\alpha}^\theta f(b^m) \right] \]

where \( a, b \in I \) with \( a < b \), \( x \in [a, b] \), \( \lambda \in [0, 1] \), \( \theta > 0 \) and \( \Gamma \) is Euler Gamma function.

Similarly to Lemma 1, we can prove the following lemma.

**Lemma 2.** Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^0 \) such that \( f' \in L[a^m, b^m] \), where \( a^m, b \in I \) with \( a < b \) and \( m \in (0, 1) \). Then for all \( x \in [a, b] \), \( \lambda \in [0, 1] \) and \( \theta > 0 \) we have:

\[ K_f (\lambda, \theta, x^m, a^m, b^m) = m^{\theta+1} a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left( \frac{x}{a} \right)^{mt} f' \left( x^{mt} a^{(1-t)} \right) dt \]

\[ - m^{\theta+1} b^m \left( \ln \frac{b}{x} \right)^{\theta+1} \int_0^1 (t^\theta - \lambda) \left( \frac{x}{b} \right)^{mt} f' \left( x^{mt} b^{(1-t)} \right) dt. \]

**Theorem 5.** Let \( f : I \subseteq (0, \infty) \rightarrow \mathbb{R} \) be a differentiable function on \( I^0 \) such that \( f' \in L[a^m, b^m] \), where \( a^m, b \in I^0 \) with \( a < b \) and \( m \in (0, 1) \). If \( |f'|^q \) is \((\alpha, m)\)-GA-convex on \([a^m, b]\) for some fixed \( q \geq 1 \), \( x \in [a, b] \), \( \lambda \in [0, 1] \) and \( \theta > 0 \) then the
following inequality for fractional integrals holds

\[ |K_f (\lambda, \theta, x^m, a^m, b^m)| \leq m^{\alpha+1} C_\alpha (\theta, \lambda) \left( 1 - \frac{1}{\theta} \right) \]

\times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \left( \int_0^1 |t^\theta - \lambda| \left( \frac{x}{a} \right)^{\alpha m t^\theta} \right)^{\frac{q}{q+m}} \right. \\
+ b^m \left( \ln \frac{b}{x} \right)^{\theta+1} \left( \int_0^1 |t^\theta - \lambda| \left( \frac{x}{b} \right)^{\alpha p m t^\theta} \right)^{\frac{q}{q+p}} \right\} \quad (2.1) \]

where

\[ C_\alpha (\theta, \lambda) = \frac{2\theta \lambda^{1+\frac{1}{\theta}} + 1 - \lambda}{\theta + 1} \]

\[ C_1 (x, \theta, \lambda, q, m, \alpha) = \int_0^1 |t^\theta - \lambda| \left( \frac{x}{a} \right)^{\alpha m t^\theta} \right)^{q m t^\theta} dt, \]

\[ C_2 (x, \theta, \lambda, q, m, \alpha) = \int_0^1 |t^\theta - \lambda| \left( \frac{x}{a} \right)^{\alpha m t^\theta} (1 - t^\alpha) \right)^{q m t^\theta} dt, \]

\[ C_3 (x, \theta, \lambda, q, m, \alpha) = \int_0^1 |t^\theta - \lambda| \left( \frac{x}{b} \right)^{\alpha m t^\theta} \right)^{q m t^\theta} dt, \]

\[ C_4 (x, \theta, \lambda, q, m, \alpha) = \int_0^1 |t^\theta - \lambda| \left( \frac{x}{b} \right)^{\alpha m t^\theta} (1 - t^\alpha) \right)^{q m t^\theta} dt. \]

Proof. From Lemma 2 property of the modulus and using the power-mean inequality we have

\[ |f|^q \text{ is } (\alpha, m)\text{-GA-convex on } [a^m, b], \text{ for all } t \in [0, 1] \]

\[ |f' \left( x^{m t} a^{m(1-t)} \right) \right|^q \leq t^{\alpha} |f' (x^m)|^q + m \left( 1 - t^\alpha \right) |f' (a)|^q, \quad (2.3) \]

\[ |f' \left( x^{m t} b^{m(1-t)} \right) \right|^q \leq t^{\alpha} |f' (x^m)|^q + m \left( 1 - t^\alpha \right) |f' (b)|^q. \quad (2.4) \]

By a simple computation

\[ \int_0^1 |t^\theta - \lambda| \right) dt = \int_0^1 (\lambda - t^\theta) dt + \int_{\lambda^1/\theta}^1 (t^\theta - \lambda) dt \]

\[ = \frac{2\theta \lambda^{1+\frac{1}{\theta}} + 1 - \lambda}{\theta + 1}. \quad (2.5) \]
If we use (2.3), (2.4) and (2.5) in (2.2), we obtain (2.1). This completes the proof. □

**Corollary 1.** Under the assumptions of Theorem 5 with \( q = 1 \), the inequality (2.1) reduced to the following inequality

\[
K_f (\lambda, \theta, x^m, a^m, b^m) \leq m^{\theta+1} \times \left\{ a^m \left[ (\ln \frac{x}{a})^{\theta+1} \left( |f'(x^m)| C_1 (x, \theta, \lambda, 1, m, \alpha) + m |f'(a)| C_2 (x, \theta, \lambda, 1, m, \alpha) \right) \right] + b^m \left[ (\ln \frac{b}{x})^{\theta+1} \left( |f'(x^m)| C_3 (x, \theta, \lambda, 1, m, \alpha) + m |f'(b)| C_4 (x, \theta, \lambda, 1, m, \alpha) \right) \right] \right\}
\]

**Corollary 2.** Under the assumptions of Theorem 5 with \( x = \sqrt{ab}, \lambda = \frac{1}{2} \) from the inequality (2.1) we get the following Simpson type inequality for fractional integrals

\[
\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left( \frac{1}{3}, \theta, \left( \sqrt{ab} \right)^m, a^m, b^m \right) \right| = \left| f(a^m) + 4 f \left( \left( \sqrt{ab} \right)^m + f(b^m) \right) \right| \leq \frac{m \ln \frac{b}{a}}{4} \Gamma \left( \frac{1}{3} \right)
\]

\[
\times \left\{ a^m \left[ |f' \left( \left( \sqrt{ab} \right)^m \right)|^q C_1 (\sqrt{ab}, \theta, \frac{1}{2}, q, m, \alpha) + m |f' (a)|^q C_2 (\sqrt{ab}, \theta, \frac{1}{2}, q, m, \alpha) \right] + b^m \left[ |f' \left( \left( \sqrt{ab} \right)^m \right)|^q C_3 (\sqrt{ab}, \theta, \frac{1}{2}, q, m, \alpha) + m |f' (b)|^q C_4 (\sqrt{ab}, \theta, \frac{1}{2}, q, m, \alpha) \right] \right\}
\]

**Corollary 3.** Under the assumptions of Theorem 5 with \( x = \sqrt{ab}, \lambda = 0 \) from the inequality (2.1) we get the following midpoint-type inequality for fractional integrals

\[
\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left( 0, \theta, \left( \sqrt{ab} \right)^m, a^m, b^m \right) \right| = \left| f \left( \left( \sqrt{ab} \right)^m \right) - \frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left[ J^m_{(\sqrt{ab})^m} f(a^m) + J^m_{(\sqrt{ab})^m} f(b^m) \right] \right| \leq \frac{m \ln \frac{b}{a}}{4} \left( \frac{1}{\theta + 1} \right)^{\frac{1}{2}} \left\{ a^m \left[ |f' \left( \left( \sqrt{ab} \right)^m \right)|^q C_1 (\sqrt{ab}, \theta, 0, q, m, \alpha) + m |f' (a)|^q C_2 (\sqrt{ab}, \theta, 0, q, m, \alpha) \right] + b^m \left[ |f' \left( \left( \sqrt{ab} \right)^m \right)|^q C_3 (\sqrt{ab}, \theta, 0, q, m, \alpha) + m |f' (b)|^q C_4 (\sqrt{ab}, \theta, 0, q, m, \alpha) \right] \right\}
\]

**Remark 1.** If we take \( \theta = 1, m = 1 \) in Corollary 3 we have the following midpoint-type inequality for \( \alpha \)-GA-convex function (or \( GA \)-s-convex function in the first
inequality (2 sense), which is the same with the inequality (9) of Theorem 3.4.b. in [10].

\[
\left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \ln \frac{b}{a} \left( \frac{1}{2} \right)^{3-\frac{q}{2}} \left\{ a \left[ \left| f' \left( \sqrt{ab} \right) \right|^q C_1 \left( \sqrt{ab}, 1, 0, q, 1, \alpha \right) \right] + \left| f'(a) \right|^q C_2 \left( \sqrt{ab}, 1, 0, q, 1, \alpha \right) \right\} ^{\frac{1}{q}} + b \left[ \left| f' \left( \sqrt{ab} \right) \right|^q C_3 \left( \sqrt{ab}, 1, 0, q, 1, \alpha \right) \right] ^{\frac{1}{q}} + b \left[ \left| f'(b) \right|^q C_4 \left( \sqrt{ab}, 1, 0, q, 1, \alpha \right) \right] ^{\frac{1}{q}} \right\} .
\]

Remark 2. If we take \( \theta = 1, m = 1, \alpha = 1 \) in Corollary 3 we have the following midpoint-type inequality for GA-convex function, which is the same with the inequality (13) of Corollary 3.5 in [10].

\[
\left| f \left( \sqrt{ab} \right) - \frac{1}{\ln b - \ln a} \int_{a}^{b} \frac{f(x)}{x} \, dx \right|
\]

\[
\leq \ln \frac{b}{a} \left( \frac{1}{2} \right)^{3-\frac{q}{2}} \left\{ a \left[ \left| f' \left( \sqrt{ab} \right) \right|^q C_1 \left( \sqrt{ab}, 1, 0, q, 1, 1 \right) \right] + \left| f'(a) \right|^q C_2 \left( \sqrt{ab}, 1, 0, q, 1, 1 \right) \right\} ^{\frac{1}{q}} + b \left[ \left| f' \left( \sqrt{ab} \right) \right|^q C_3 \left( \sqrt{ab}, 1, 0, q, 1, 1 \right) \right] ^{\frac{1}{q}} + b \left[ \left| f'(b) \right|^q C_4 \left( \sqrt{ab}, 1, 0, q, 1, 1 \right) \right] ^{\frac{1}{q}} \right\} .
\]

Corollary 4. Under the assumptions of Theorem 5 with \( x = \sqrt{ab} \), \( \lambda = 1 \) from the inequality (2.1) we get the following trapezoid-type inequality for fractional integrals

\[
\frac{2^{\theta-1}}{(m \ln \frac{1}{2})^\theta} \left| K_{f} \left( 1, \theta, \left( \sqrt{ab} \right)^{m}, a_{m}, b_{m} \right) \right|
\]

\[
= \left| \frac{f(a_{m}) + f(b_{m})}{2} - \frac{2^{\theta-1}(\theta + 1)}{(m \ln \frac{1}{2})^\theta} \left[ J^{\theta}_{\sqrt{ab}} f(a_{m}) + J^{\theta}_{\sqrt{ab}} f(b_{m}) \right] \right|
\]

\[
\leq \frac{m \ln \frac{\theta}{\theta + 1}}{4} \left( \frac{\theta}{\theta + 1} \right)^{\frac{1}{q}} \left\{ a_{m} \left[ \left| f' \left( \sqrt{ab} \right) \right|^m C_1 \left( \sqrt{ab}, \theta, 1, q, m, \alpha \right) \right] + b_{m} \left[ \left| f' \left( \sqrt{ab} \right) \right|^m C_3 \left( \sqrt{ab}, \theta, 1, q, m, \alpha \right) \right] \right\} \right\} .
\]

Corollary 5. Let the assumptions of Theorem 5 hold. If \( |f'(u)| \leq M \) for all \( u \in [a_{m}, b_{m}] \) and \( \lambda = 0 \), then from the inequality (2.1) we get the following Ostrowski
Theorem 6. Let $x$ for each type inequality for fractional integrals
\[
\left[ \left( \frac{\ln x}{a} \right)^{\theta} + \left( \frac{b}{x} \right)^{\theta} \right] f(x^m) - \frac{\Gamma(\theta + 1)}{m^{\theta}} [J_{x^m}^m f(a^m) + J_{x^m}^m f(b^m)] \\
\leq \frac{mM}{(\theta + 1)^{\frac{1}{\theta}}} \left\{ a^m \left( \frac{\ln x}{a} \right)^{\theta+1} \left( C_1(x, \theta, 0, q, m, \alpha) + mC_2(x, \theta, 0, q, m, \alpha) \right) + b^m \left( \frac{\ln x}{b} \right)^{\theta+1} \left( C_3(x, \theta, \lambda, q, m, \alpha) + mC_4(x, \theta, \lambda, q, m, \alpha) \right) \right\}^{\frac{1}{\theta}}
\]
for each $x \in [a, b]$.

Theorem 6. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in [0, 1]$. If $|f'|^q$ is $(\alpha, m)$-GA-convex on $[a^m, b]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the following inequality for fractional integrals holds
\[
|Kf(\lambda, \theta, x^m, a^m, b^m)| \leq m^q R_0(\theta, \lambda, p) \times \left\{ a^m \left( \frac{\ln x}{a} \right)^{\theta+1} \left( |f'(x^m)|^q R_1(x, q, m, \alpha) + m |f'(a)|^q R_2(x, q, m, \alpha) \right) + b^m \left( \frac{\ln x}{b} \right)^{\theta+1} \left( |f'(x^m)|^q R_3(x, q, m, \alpha) + m |f'(b)|^q R_4(x, q, m, \alpha) \right) \right\}^{\frac{1}{\theta}}
\]
where
\[
R_0(\theta, \lambda, p) = \frac{1}{0} |t^\theta - \lambda|^p dt
\]
\[
= \left\{ \begin{array}{ll}
\frac{1}{\theta+1} \beta \left( \frac{1}{\theta}, p + 1 \right) + \frac{(1-\lambda)^{p+1}}{\theta(p+1)} & , \quad \lambda = 0 \\
\frac{1}{\theta} \left( \frac{1}{\theta}, 1; p+1; 1-\lambda \right) & , \quad 0 < \lambda < 1 \\
\frac{1}{\theta} \left( \frac{1}{\theta}, p + 1 \right) & , \quad \lambda = 1
\end{array} \right.
\]
\[
R_1(x, q, m, \alpha) = \int_0^1 \left( \frac{x}{a} \right)^{mq} t^\alpha dt, \\
R_2(x, q, m, \alpha) = \int_0^1 \left( \frac{x}{a} \right)^{mq} (1 - t^\alpha) dt, \\
R_3(x, q, m, \alpha) = \int_0^1 \left( \frac{x}{b} \right)^{mq} t^\alpha dt, \\
R_4(x, q, m, \alpha) = \int_0^1 \left( \frac{x}{b} \right)^{mq} (1 - t^\alpha) dt,
\]
$2F_1$ is hypergeometrical function defined by
\[
2F_1(a; b; c; z) = \frac{1}{\beta(b, c - b)} \int_0^1 t^{b-1} (1 - t)^{c-b-1} (1 - zt)^{-a} dt, \\
c > b > 0, |z| < 1 (\text{see } \mathbb{H}),
\]
\( \beta \) is beta function defined by
\[
\beta(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = \int_0^1 t^{x-1} (1-t)^{y-1} \, dt, \quad x, y > 0,
\]
and \( \frac{1}{p} + \frac{1}{q} = 1. \)

**Proof.** From Lemma \[2\] property of the modulus and using the Hölder inequality we have
\[
|K_f(\lambda, \theta, x^n, a^m, b^m)| \leq m^{\theta+1} \left( \int_0^1 |t^{\theta} - \lambda|^p \, dt \right)^{\frac{1}{p}}
\]
\[
\times \left\{ \begin{array}{ll}
\frac{\lambda^{(p+1)/p}}{p} \beta \left( \frac{1}{p}, p+1 \right) + \frac{(1-\lambda)^{p+1}}{p(p+1)} \\
\times 2\beta (1 - \frac{1}{p}, p+1, 1-p) \end{array} \right\}, \quad 0 < \lambda < 1.
\]
(2.8)

Since \( |f'|^q \) is \((\alpha, m)\)-GA-convex on \([a, b] \), for all \( t \in [0, 1] \), if we use \[2.3\], \[2.4\] and \[2.8\] in \[2.7\], we obtain \[2.6\]. This completes the proof. \( \square \)

**Corollary 6.** Under the assumptions of Theorem \[7\] with \( x = \sqrt{ab}, \lambda = \frac{1}{3} \) from the inequality \[2.6\] we get the following Simpson type inequality for fractional integrals
\[
\frac{2^{\theta-1}}{(m \ln b)^{\theta}} K_f \left( \frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m \right) = \frac{1}{6} \left[ f(a^m) + 4f \left( \sqrt{ab}^m \right) + f(b^m) \right]
\]
\[
-2^{\theta-1} \Gamma(\theta + 1) \left( \frac{2^\theta}{(m \ln b)^{\theta}} \right)^{\frac{1}{\theta}} \left[ f(a^m) + 4f \left( \sqrt{ab}^m \right) + f(b^m) \right] \leq \frac{m \ln \frac{b}{a} R_0 \left( \frac{1}{3}, \frac{1}{3}, p \right)}{4}
\]
\[
\times \left\{ \begin{array}{ll}
a^m \left[ f' \left( \sqrt{ab}^m \right) \right]^q R_1 \left( \sqrt{ab}, q, m, \alpha \right) \\
+ m |f'|^q R_2 \left( \sqrt{ab}, q, m, \alpha \right) \end{array} \right\}^{\frac{1}{q}} + b^m \left[ f' \left( \sqrt{ab}^m \right) \right]^q R_3 \left( \sqrt{ab}, q, m, \alpha \right)
\]
\[
+ m |f'|^q R_4 \left( \sqrt{ab}, q, m, \alpha \right) \right\}^{\frac{1}{q}},
\]
(2.9)

**Corollary 7.** Under the assumptions of Theorem \[8\] with \( x = \sqrt{ab}, \lambda = 0 \) from the inequality \[2.6\] we get the following midpoint-type inequality for fractional integrals
\[
\frac{2^{\theta-1}}{(m \ln b)^{\theta}} K_f \left( \frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m \right) = \frac{1}{6} \left[ f(a^m) + 4f \left( \sqrt{ab}^m \right) + f(b^m) \right]
\]
\[
-2^{\theta-1} \Gamma(\theta + 1) \left( \frac{2^\theta}{(m \ln b)^{\theta}} \right)^{\frac{1}{\theta}} \left[ f(a^m) + 4f \left( \sqrt{ab}^m \right) + f(b^m) \right] \leq \frac{m \ln \frac{b}{a} R_0 \left( \frac{1}{3}, \frac{1}{3}, p \right)}{4}
\]
\[
\times \left\{ \begin{array}{ll}
a^m \left[ f' \left( \sqrt{ab}^m \right) \right]^q R_1 \left( \sqrt{ab}, q, m, \alpha \right) \\
+ m |f'|^q R_2 \left( \sqrt{ab}, q, m, \alpha \right) \end{array} \right\}^{\frac{1}{q}} + b^m \left[ f' \left( \sqrt{ab}^m \right) \right]^q R_3 \left( \sqrt{ab}, q, m, \alpha \right)
\]
\[
+ m |f'|^q R_4 \left( \sqrt{ab}, q, m, \alpha \right) \right\}^{\frac{1}{q}},
\]
If we take $\theta = 1, m = 1, p = \frac{a}{q - 1}$ in Corollary [19] we have the following midpoint-type inequality for $\alpha$-GA-convex function (or GA-s-convex function in the first sense), which is the same with the inequality (17) of Theorem 3.7.b. in [19].

\[
\frac{2^{q-1}}{(m \ln \frac{b}{a})} \left| K_f \left( 0, \theta, \left( \sqrt{ab} \right), a^{m}, b^{m} \right) \right|
\]

\[
= \left| f \left( \left( \sqrt{ab} \right)^{m} \right) - \frac{2^{q-1}(\theta + 1)}{(m \ln \frac{b}{a})} \left[ f^{(\sqrt{ab})^{m}} - f(a^{m}) + f^{(\sqrt{ab})^{m}} + f(b^{m}) \right] \right|
\]

\[
\leq \frac{\ln b}{4} \left( \frac{1}{\theta p + 1} \right)^{\frac{1}{p}} \left\{ a^{m} \left[ \left| f' \left( \left( \sqrt{ab} \right)^{m} \right) \right| R_{1} \left( \sqrt{ab}, q, m, \alpha \right) \right]^{\frac{1}{q}} \right.
\]

\[
+ b^{m} \left[ \left| f' \left( \left( \sqrt{ab} \right)^{m} \right) \right| R_{3} \left( \sqrt{ab}, q, m, \alpha \right) \right]^{\frac{1}{q}} \}
\]

**Remark 3.** If we take $\theta = 1, m = 1, p = \frac{a}{q - 1}$ in Corollary [19] we have the following midpoint-type inequality for $\alpha$-GA-convex function (or GA-s-convex function in the first sense), which is the same with the inequality (17) of Theorem 3.7.b. in [19].

\[
\frac{\ln b}{4} \left( \frac{1}{\theta p + 1} \right)^{\frac{1}{p}} \left\{ a \left[ \left| f' \left( \left( \sqrt{ab} \right)^{m} \right) \right| R_{1} \left( \sqrt{ab}, q, 1, \alpha \right) \right]^{\frac{1}{q}} \right.
\]

\[
+ b \left[ \left| f' \left( \left( \sqrt{ab} \right)^{m} \right) \right| R_{3} \left( \sqrt{ab}, q, 1, \alpha \right) \right]^{\frac{1}{q}} \}
\]

**Remark 4.** If we take $\theta = 1, m = 1, \alpha = 1, p = \frac{a}{q - 1}$ in Corollary [19] we have the following midpoint-type inequality for GA-convex function, which is the same with the inequality (21) of Corollary 3.8 in [19].

\[
\frac{\ln b}{4} \left( \frac{1}{\theta p + 1} \right)^{\frac{1}{p}} \left\{ a \left[ \left| f' \left( \left( \sqrt{ab} \right)^{m} \right) \right| R_{1} \left( \sqrt{ab}, q, 1, 1 \right) \right]^{\frac{1}{q}} \right.
\]

\[
+ b \left[ \left| f' \left( \left( \sqrt{ab} \right)^{m} \right) \right| R_{3} \left( \sqrt{ab}, q, 1, 1 \right) \right]^{\frac{1}{q}} \}
\]
Corollary 8. Under the assumptions of Theorem 6 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.9), we get the following trapezoid-type inequality for fractional integrals

$$\frac{2^{\frac{q-1}{2}}}{(m \ln \frac{b}{a})^q} \left| K_f \left( 1, \theta, \left( \sqrt{ab} \right)^m, a^m, b^m \right) \right|$$

$$= \frac{m a^m + f(b^m)}{2} - \frac{2^{\frac{q-1}{2}} \Gamma(\theta + 1)}{(m \ln \frac{b}{a})^q} \left[ J^{\theta} (\sqrt{ab})_m f(a^m) + J^{\theta} (\sqrt{ab})_m f(b^m) \right]$$

$$\leq \frac{m \ln \frac{b}{a}}{4} \left( \frac{1}{\theta} \left( \frac{1}{\theta} + 1 \right) \right)^{\frac{1}{q}} \left\{ a^m \left\{ f' \left( \left( \sqrt{ab} \right)^m \right) \right\}^q R_1 \left( \sqrt{ab}, q, m, \alpha \right) + m |f'(a)|^q R_2 \left( \sqrt{ab}, q, m, \alpha \right) \right\}$$

$$+ b^m \left\{ m |f'(b)|^q R_3 \left( \sqrt{ab}, q, m, \alpha \right) \right\}$$

Corollary 9. Let the assumptions of Theorem 6 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b^m]$ and $\lambda = 0$, then from the inequality (2.9), we get the following Ostrowski type inequality for fractional integrals

$$\left| \left( \ln \frac{x}{a} \right)^\theta + \left( \ln \frac{b}{x} \right)^\theta \right| f(x^m) \left[ J^{\theta} \left( \frac{a^m}{x^m} \right) f(a^m) + J^{\theta} \left( \frac{b^m}{x^m} \right) f(b^m) \right]$$

$$\leq \frac{m M}{(\theta p + 1)^{\frac{1}{q}}} \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \left( R_1 (x, q, m, \alpha) + m R_2 (x, q, m, \alpha) \right) \right\}$$

$$+ b^m \left( \ln \frac{b}{x} \right)^{\theta+1} \left( R_3 (x, q, m, \alpha) + R_4 (x, q, m, \alpha) \right)$$

for each $x \in [a, b]$.

Theorem 7. Let $f : I \subset (0, \infty) \rightarrow \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a^m, b^m]$, where $a^m, b \in \mathbb{R}^+ \cap I^\circ$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is $(\alpha, m)$-GA-convex on $[a^m, b^m]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the following inequality for fractional integrals holds

$$|K_f \left( \lambda, \theta, x^m, a^m, b^m \right)| \leq m^{\theta+1}$$

$$\times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} T_1 (x, \theta, \lambda, p, m) \left( \left| f'(x^m) \right|^q + m \left( \alpha + 1 \right) \left| f'(a) \right|^q \right) \right\}$$

$$+ b^m \left( \ln \frac{b}{x} \right)^{\theta+1} T_2 (x, \theta, \lambda, p, m) \left( \left| f'(x^m) \right|^q + m \left( \alpha + 1 \right) \left| f'(b) \right|^q \right)$$

where

$$T_1 (x, \theta, \lambda, p, m) = \int_0^1 \left| t^\theta - \lambda \right|^p \left( \frac{x}{a} \right)^{\lambda^p} dt,$$

$$T_2 (x, \theta, \lambda, p, m) = \int_0^1 \left| t^\theta - \lambda \right|^p \left( \frac{x}{b} \right)^{\lambda^p} dt,$$

and $\frac{1}{p} + \frac{1}{q} = 1$. 
Proof. Since $|f'|^q$ is $(\alpha, m)$-GA-convex on $[a^m, b^m]$, for all $t \in [0, 1]$, if we use (2.3), (2.4)

$$
\int_0^1 |f' \left( x^{mt} a^{m(1-t)} \right) |^q \, dt \leq \int_0^1 t^\alpha |f' (x^m)|^q + m (1-t^\alpha) |f' (a)|^q \, dt
$$

$$
= \frac{|f' (x^m)|^q + m \alpha |f' (a)|^q}{\alpha + 1}, \tag{2.10}
$$

$$
\int_0^1 |f' \left( x^{mt} b^{m(1-t)} \right) |^q \, dt \leq \int_0^1 t^\alpha |f' (x^m)|^q + m (1-t^\alpha) |f' (b)|^q \, dt
$$

$$
= \frac{|f' (x^m)|^q + m \alpha |f' (b)|^q}{\alpha + 1}. \tag{2.11}
$$

From Lemma 2, property of the modulus, (2.10), (2.11) and using the Hölder inequality, we have

$$
|K_f (\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1}
$$

$$
\times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \left( \int_0^1 |t^\theta - \lambda|^p \left( \frac{x}{a} \right)^{\text{mpt}} \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f' \left( x^{mt} a^{m(1-t)} \right) |^q \, dt \right)^{\frac{1}{q}} + b^m \left( \ln \frac{b}{x} \right)^{\theta+1} \left( \int_0^1 |t^\theta - \lambda|^p \left( \frac{x}{b} \right)^{\text{mpt}} \, dt \right)^{\frac{1}{p}} \left( \int_0^1 |f' \left( x^{mt} b^{m(1-t)} \right) |^q \, dt \right)^{\frac{1}{q}} \right\}
$$

$$
\leq m^{\theta+1} \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \left( \int_0^1 |t^\theta - \lambda|^p \left( \frac{x}{a} \right)^{\text{mpt}} \, dt \right)^{\frac{1}{p}} \left( \frac{|f' (x^m)|^q + m \alpha |f' (a)|^q}{\alpha + 1} \right)^{\frac{1}{q}} + b^m \left( \ln \frac{b}{x} \right)^{\theta+1} \left( \int_0^1 |t^\theta - \lambda|^p \left( \frac{x}{b} \right)^{\text{mpt}} \, dt \right)^{\frac{1}{p}} \left( \frac{|f' (x^m)|^q + m \alpha |f' (b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}
$$

This completes the proof. \qed

Corollary 10. Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = \frac{1}{3}$ from the inequality (2.9) we get the following Simpson type inequality for fractional integrals

$$
\frac{2^{q-1} \Gamma (\theta+1)}{(m \ln \frac{1}{a})^q} \left| K_f \left( \frac{1}{3}, \theta, \sqrt{ab} \right)^m, a^m, b^m \right| = \left| \frac{1}{6} \left[ f(a^m) + 4 f \left( \sqrt{ab} \right)^m + f(b^m) \right] \right|
$$

$$
- \frac{2^{q-1} \Gamma (\theta+1)}{(m \ln \frac{1}{a})^q} \left\{ |f(\sqrt{ab})| - f(a^m) + f(\sqrt{ab})^m + f(b^m) \right\} \leq \frac{m \ln \frac{b}{a}}{4}
$$

$$
\times \left\{ a^m T_1^{\frac{1}{p}} \left( \sqrt{ab}, \theta, \frac{1}{3}, p, m \right) \left( \frac{|f' \left( \left( \sqrt{ab} \right)^m \right) |^q + m \alpha |f' (a)|^q}{\alpha + 1} \right)^{\frac{1}{q}} + b^m T_2^{\frac{1}{p}} \left( \sqrt{ab}, \theta, \frac{1}{3}, p, m \right) \left( \frac{|f' \left( \left( \sqrt{ab} \right)^m \right) |^q + m \alpha |f' (b)|^q}{\alpha + 1} \right)^{\frac{1}{q}} \right\}
$$
Corollary 11. Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = 0$ from the inequality \((2,9)\) we get the following midpoint-type inequality for fractional integrals

$$\frac{2^{\theta-1}}{(m \ln \frac{a}{x})^\theta} \left| K_f \left(0, \theta, \left(\sqrt{ab}\right)^m, a^m, b^m\right) \right|$$

$$= \left| f \left(\left(\sqrt{ab}\right)^m\right) - \frac{2^{\theta-1}\Gamma (\theta + 1)}{m \ln \frac{a}{x}} \left[ J^\theta (\sqrt{ab}) \left( a^m + b^m \right) \right] \right|$$

$$\leq m \ln \frac{\frac{b}{x}}{4} \left\{ a^m T_1^{\frac{1}{m}} \left( a^m, \theta, 0, p, m \right) \left( \frac{f' \left( \left(\sqrt{ab}\right)^m \right) + m \alpha |f'(a)|^q}{\alpha + 1} \right) \right\}$$

$$+ b^m T_2^{\frac{1}{m}} \left( a^m, \theta, 0, p, m \right) \left( \frac{f' \left( \left(\sqrt{ab}\right)^m \right) + m \alpha |f'(b)|^q}{\alpha + 1} \right)$$

Corollary 12. Under the assumptions of Theorem 7 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality \((2,9)\) we get the following trapezoid-type inequality for fractional integrals

$$\frac{2^{\theta-1}}{(m \ln \frac{a}{x})^\theta} \left| K_f \left(1, \theta, \left(\sqrt{ab}\right)^m, a^m, b^m\right) \right|$$

$$= \left(\ln \frac{b}{x}\right) - \frac{2^{\theta-1}\Gamma (\theta + 1)}{m \ln \frac{a}{x}} \left[ J^\theta (\sqrt{ab}) \left( a^m + b^m \right) \right]$$

$$\leq m \ln \frac{\frac{b}{x}}{4} \left\{ a^m T_1^{\frac{1}{m}} \left( a^m, \theta, 1, p, m \right) \left( \frac{f' \left( \left(\sqrt{ab}\right)^m \right) + m \alpha |f'(a)|^q}{\alpha + 1} \right) \right\}$$

$$+ b^m T_2^{\frac{1}{m}} \left( a^m, \theta, 1, p, m \right) \left( \frac{f' \left( \left(\sqrt{ab}\right)^m \right) + m \alpha |f'(b)|^q}{\alpha + 1} \right)$$

Corollary 13. Let the assumptions of Theorem 7 hold. If $|f'(u)| \leq M$ for all $u \in [a^m, b^m]$ and $\lambda = 0$, then from the inequality \((2,9)\) we get the following Ostrowski type inequality for fractional integrals

$$\left[ \ln \frac{a}{x} \right] \left[ \ln \frac{b}{x} \right] \left( f \left( x^m \right) - \frac{\Gamma (\theta + 1)}{m \theta} \left[ J^\theta (a^m + b^m) \right] \right)$$

$$\leq mM \left( 1 + \frac{m \alpha}{\alpha + 1} \right)$$

$$\times \left\{ a^m \left( \ln \frac{\frac{b}{x}}{a} \right)^{\theta+1} T_1^{\frac{1}{m}} \left( x, \theta, 0, p, m \right) + b^m \left( \ln \frac{\frac{b}{x}}{x} \right)^{\theta+1} T_2^{\frac{1}{m}} \left( x, \theta, 0, p, m \right) \right\}$$

for each $x \in [a, b]$

Theorem 8. Let $f : I \subseteq (0, \infty) \to \mathbb{R}$ be a differentiable function on $I^\circ$ such that $f' \in L[a^m, b^m]$, where $a^m, b \in I^\circ$ with $a < b$ and $m \in (0, 1]$. If $|f'|^q$ is $(\alpha, m)$-GA-convex on $[a^m, b]$ for some fixed $q > 1$, $x \in [a, b]$, $\lambda \in [0, 1]$ and $\theta > 0$ then the
following inequality for fractional integrals holds

\[ |K_f (\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} \]

\[ \times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \frac{\int_0^1 \left( \int_0^{x^{\lambda p m t_t}} dt \right)^\frac{\theta}{p+1} \left( \int_0^1 |t^\theta - \lambda|^q \left| f'(x^{\lambda p m t_t}) \right|^q dt \right)^\frac{1}{q} }{V_3^\beta} \right\} + b^m \left( \int_0^1 \left( \int_0^{x^{\lambda p m t_t}} dt \right)^\frac{\theta+1}{p+1} \left( \int_0^1 |t^\theta - \lambda|^q \left| f'(x^{\lambda p m t_t}) \right|^q dt \right)^\frac{1}{q} \right) \]

\[ \text{where} \]

\[ V_1 (\theta, \lambda, \alpha, q) = \int_0^1 |t^\theta - \lambda|^q t^\alpha dt \]

\[ = \left\{ \begin{array}{ll}
\frac{1}{\lambda^{(\theta q + \alpha + 1)/\sigma}} \beta \left( \frac{\sigma}{\sigma}, q + 1 \right) & , \quad \lambda = 0 \\
\frac{1}{\lambda^{(\theta q + \alpha + 1)/\sigma}} \beta \left( \frac{\sigma}{\sigma}, q + 1 \right) + \frac{1-\lambda}{\lambda^{(\theta q + \alpha + 1)/\sigma}} \beta \left( \frac{\sigma}{\sigma}, q + 1 \right) & , \quad 0 < \lambda < 1 \\
\frac{1}{\beta (\frac{\sigma}{\sigma}, q + 1)} & , \quad \lambda = 1
\end{array} \right. \]

\[ V_2 (\theta, \lambda, \alpha, q) = \int_0^1 |t^\theta - \lambda|^q (1 - t^\alpha) dt \]

\[ = \left\{ \begin{array}{ll}
\frac{1}{\lambda^{(\theta q + \alpha + 1)/\sigma}} \beta \left( \frac{\sigma}{\sigma}, q + 1 \right) & , \quad \lambda = 0 \\
\frac{1}{\lambda^{(\theta q + \alpha + 1)/\sigma}} \beta \left( \frac{\sigma}{\sigma}, q + 1 \right) + \frac{1-\lambda}{\lambda^{(\theta q + \alpha + 1)/\sigma}} \beta \left( \frac{\sigma}{\sigma}, q + 1 \right) & , \quad 0 < \lambda < 1 \\
\frac{1}{\beta (\frac{\sigma}{\sigma}, q + 1)} & , \quad \lambda = 1
\end{array} \right. \]

\[ V_3 = \int_0^1 \left( \frac{x}{a} \right)^{\lambda p m t_t} dt = \left\{ \begin{array}{ll}
\frac{1}{\ln \left( \frac{x}{a} \right)^{\lambda p m t_t}} & , \quad x \neq a \\
1 & , \quad \text{otherwise}
\end{array} \right. \]

\[ V_4 = \int_0^1 \left( \frac{x}{b} \right)^{\lambda p m t_t} dt = \left\{ \begin{array}{ll}
\frac{1}{\ln \left( \frac{x}{b} \right)^{\lambda p m t_t}} & , \quad x \neq b \\
1 & , \quad \text{otherwise}
\end{array} \right. \]

and \( \frac{1}{\mu} + \frac{1}{\nu} = 1. \)

Proof. From Lemma 2, property of the modulus, the Hölder inequality and by using (2.3), (2.4), (2.15), and (2.16) we have

\[ |K_f (\lambda, \theta, x^m, a^m, b^m)| \leq m^{\theta+1} \]

\[ \times \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} \left( \int_0^1 \left( \int_0^{x^{\lambda p m t_t}} dt \right)^\frac{\theta}{p+1} \left( \int_0^1 |t^\theta - \lambda|^q \left| f'(x^{\lambda p m t_t}) \right|^q dt \right)^\frac{1}{q} \right) \right\} + b^m \left( \int_0^1 \left( \int_0^{x^{\lambda p m t_t}} dt \right)^\frac{\theta+1}{p+1} \left( \int_0^1 |t^\theta - \lambda|^q \left| f'(x^{\lambda p m t_t}) \right|^q dt \right)^\frac{1}{q} \right) \]
Corollary 16. Under the assumptions of Theorem 8 with $x = \sqrt[12]{ab}$, $\lambda = \frac{1}{12}$ from the inequality (2.12) we get the following Simpson type inequality for fractional integrals

$$\leq m^{\theta+1} \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta+1} V_3^b \times \left( \int_0^1 |t^\theta - \lambda|^q \left[ +m(1-t^\theta) |f'(a)|^q \right] dt \right)^{\frac{1}{q}} + b^m \left( \ln \frac{b}{x} \right)^{\theta+1} V_4^b \times \left( \int_0^1 |t^\theta - \lambda|^q \left[ +m(1-t^\theta) |f'(b)|^q \right] dt \right)^{\frac{1}{q}} \right\}$$

(2.17)

By a simple computation we verify (2.13) and (2.14). If we use (2.13), (2.14), (2.15) and (2.16) in (2.17) we obtain (2.12). This completes the proof. □

Corollary 14. Under the assumptions of Theorem 8 with $x = \sqrt{ab}$, $\lambda = \frac{1}{2}$ from the inequality (2.12) we get the following mid-point-type inequality for fractional integrals

$$\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left( \frac{1}{3}, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right| = \left| \frac{1}{3} \left[ f(a^m) + 4f \left( \sqrt{ab}^m \right) + f(b^m) \right] \right|$$

$$- \frac{2^{\theta-1} \Gamma (\theta + 1)}{(m \ln \frac{b}{a})^\theta} \left[ J^\theta (\sqrt{ab})^m - f(a^m) + J^\theta (\sqrt{ab})^m + f(b^m) \right] \leq \frac{m \ln \frac{b}{a}}{4}$$

$$\times \left\{ a^m \left( \frac{\ln (\frac{b}{a})}{\ln (\frac{x}{a})} - 1 \right)^{\frac{1}{q}} \left[ V_1 \left( \theta, \frac{1}{3}, \alpha, q \right) |f'(x^m)|^q \right] + mV_2 \left( \theta, \frac{1}{3}, \alpha, q \right) |f'(a)|^q \right\} + b^m \left( \frac{\ln (\frac{b}{a})}{\ln (\frac{x}{a})} - 1 \right)^{\frac{1}{q}} \left[ V_1 \left( \theta, \frac{1}{3}, \alpha, q \right) |f'(x^m)|^q \right] + mV_2 \left( \theta, \frac{1}{3}, \alpha, q \right) |f'(b)|^q \right\}$$

Corollary 15. Under the assumptions of Theorem 8 with $x = \sqrt{ab}$, $\lambda = 0$ from the inequality (2.12) we get the following trepezoid-type inequality for fractional integrals

$$\frac{2^{\theta-1}}{(m \ln \frac{b}{a})^\theta} \left| K_f \left( 0, \theta, (\sqrt{ab})^m, a^m, b^m \right) \right|$$

$$= \left| f \left( (\sqrt{ab})^m \right) - \frac{2^{\theta-1} \Gamma (\theta + 1)}{(m \ln \frac{b}{a})^\theta} \left[ J^\theta (\sqrt{ab})^m - f(a^m) + J^\theta (\sqrt{ab})^m + f(b^m) \right] \right|$$

$$\leq \frac{m \ln \frac{b}{a}}{4} \left\{ a^m \left( \frac{\ln (\frac{b}{a})}{\ln (\frac{x}{a})} - 1 \right)^{\frac{1}{q}} \left[ \left( \frac{1}{q} + \alpha + 1 \right) \frac{m}{q+\alpha+1} |f'(x^m)|^q \right]^{\frac{1}{q}} + mV_2 \left( \theta, \frac{1}{3}, \alpha, q \right) |f'(a)|^q \right\}$$

$$+ b^m \left( \frac{\ln (\frac{b}{a})}{\ln (\frac{x}{a})} - 1 \right)^{\frac{1}{q}} \left[ \left( \frac{1}{q} + \alpha + 1 \right) \frac{m}{q+\alpha+1} |f'(x^m)|^q \right]^{\frac{1}{q}}$$

Corollary 16. Under the assumptions of Theorem 8 with $x = \sqrt{ab}$, $\lambda = 1$ from the inequality (2.12) we get the following trepezoid-type inequality for fractional integrals
Corollary 17. Let the assumptions of Theorem 7 hold. If \(|f'(a)| \leq M\) for all \(a \in [a^m, b]\) and \(\lambda = 0\), then from the inequality \([2,12]\) we get the following Ostrowski type inequality for fractional integrals

\[
\left| \left[ \left( \ln \frac{x}{a} \right)^{\theta} + \left( \ln \frac{b}{x} \right)^{\theta} \right] f(x^m) - \frac{\Gamma(\theta + 1)}{m^\theta} \left[ J_{a^m}^\theta f(a^m) + J_{b^m}^\theta f(b^m) \right] \right| \\
\leq mM \left\{ a^m \left( \ln \frac{x}{a} \right)^{\theta + 1} \left( \frac{\theta + 1}{m} \right) \right\}^{\frac{1}{\theta}} \left\{ b^m \left( \ln \frac{b}{x} \right)^{\theta + 1} \left( \frac{\theta + 1}{m} \right) \right\}^{\frac{1}{\theta}}
\]

for each \(x \in [a, b]\)

References

[1] M. Alomaria, M. Darus, S.S. Dragomir, P. Cerone, Ostrowski type inequalities for functions whose derivatives are \(s\)-convex in the second sense, Applied Mathematics Letters 23 (2010) 1071–1076.

[2] M. Avci, H. Kavurmaci and M.E. Ozdemir, New inequalities of Hermite-Hadamard type via \(s\)-convex functions in the second sense with applications, Appl. Math. Comput., 217 (2011) 5171–5176.

[3] Z. Dahmani, On Minkowski and Hermite-Hadamard integral inequalities via fractional via fractional integration, Ann. Funct. Anal. 1 (1) (2010), 51-58

[4] A. A. Kilbas, H. M. Srivastava, J. J. Trujillo, Theory and Applications of Fractional Differential Equations, Elsevier, Amsterdam, 2006.

[5] I. Iscan, A new generalization of some integral inequalities for \((\alpha, m)\)-convex functions, Mathematical Sciences, 2013, doi:10.1186/2225-1746-7-22.

[6] I. Iscan, New estimates on generalization of some integral inequalities for \((\alpha, m)\)-convex functions, Contemporary Analysis and Applied Mathematics, 1 (2013) 253-264.

[7] I. Iscan, New estimates on generalization of some integral inequalities for \(s\)-convex functions and their applications, International Journal of Pure and Applied Mathematics, 86 (4) (2013) accepted.

[8] A.P. Ji, T.Y. Zhang and F. Qi, Integral inequalities of Hermite-Hadamard type for \((\alpha, m)\)-GA-convex functions, arXiv:1306.0852 Available online at http://arxiv.org/abs/1306.0852

[9] Y. Shuang, H.-P. Yin, and F. Qi, Hermite-Hadamard type integral inequalities for geometric-arithmetically \(s\)-convex functions, Analysis (Munich) 33 (2) (2013), 197-208. Available online at http://dx.doi.org/10.1524/anly.2013.1192

[10] C. P. Niculescu, Convexity according to the geometric mean, Math. Inequal. Appl. 3 (2) (2000), 155-167. Available online at http://dx.doi.org/10.7155/mia-03-19

[11] C. P. Niculescu, Convexity according to means, Math. Inequal. Appl. 6 (4) (2003), 571-579. Available online at http://dx.doi.org/10.7155/mia-06-53
[12] M.E. Ozdemir, M. Avei, H. Kavurmaci, Hermite-Hadamard type inequalities for s-convex and s-concave functions via fractional integrals, arXiv:1202.0380v1.

[13] J. Park, Generalization of some Simpson-like type inequalities via differentiable s-convex mappings in the second sense, International Journal of Mathematics and Mathematical Sciences, vol. 2011, Article ID 493531, 13 pages, doi:10.1155/493531.

[14] T-Y. Zhang, A.-P. Ji and F. Qi, On integral inequalities of Hermite-Hadamard Type for s-Geometrically Convex Functions, Abstract and Applied Analysis, 2012 (2012), Article ID 560586, 14 pages, doi:10.1155/2012/560586.

[15] M.Z. Sarıkaya and N. Aktan, On the generalization of some integral inequalities and their applications, Mathematical and Computer Modelling, 54 (2011) 2175-2182.

[16] E. Set, New inequalities of Ostrowski type for mapping whose derivatives are s-convex in the second sense via fractional integrals, Computers and Math. with Appl. 63 (2011) 2147-2154.

[17] M.Z. Sarıkaya and H. Ogunmez, On new inequalities via Riemann-Liouville fractional integration, Abstract an Applied Analysis, vol. 2012, Article ID 428983, 10 pages, doi:10.1155/2012/428983.

[18] E. Set, M.E. Ozdemir and M.Z. Sarıkaya, On new inequalities of Simpson’s type for quasi-convex functions with applications, Tamkang Journal of Mathematics, 43 (3) (2012) 357-364.

[19] I. Iscan, Hermite-Hadamard type inequalities for GA-s-convex functions, Le Matematiche, LXIX (2014) Fasc. II, 129-146.

[20] I. Iscan, New general integral inequalities for quasi-geometrically convex functions via fractional integrals, Journal of Inequalities and Applications, 2013, (2013) 491.

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