Flatness of CR Submanifolds in a Sphere

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Dedicated to Professor Yang, Lo in the Occasion of his 70th Birthday

1 Introduction

The Cartan-Janet theorem asserted that for any analytic Riemannian manifold \((M^n, g)\), there exist local isometric embeddings of \(M^n\) into Euclidean space \(\mathbb{E}^N\) as \(N\) is sufficiently large. The CR analogue of Cartan-Janet theorem is not true in general. In fact, Forstneric \cite{F086} and Faran \cite{Fa88} proved the existence of real analytic strictly pseudoconvex hypersurfaces \(M^{2n+1} \subset \mathbb{C}^{n+1}\) which do not admit any germ of holomorphic mapping taking \(M\) into sphere \(\partial \mathbb{B}^{N+1}\) for any \(N\).

There are recent progress on CR submanifolds in sphere \(\partial \mathbb{B}^{N+1}\). Zaitsev \cite{Za08} constructed explicit examples for the Forstneric and Faran phenomenon above. Ebenfelt, Huang and Zaitsev \cite{EHZ04} proved rigidity of CR embeddings of general \(M^{2n+1}\) into spheres with CR co-dimension \(< \frac{n}{2}\), which generalizes a result of Webster that was for the case of co-dimension 1 \cite{We79}. S.-Y. Kim and J.-W. Oh \cite{KO06} gave a necessary and sufficient condition for local embeddability into a sphere \(\partial \mathbb{B}^{N+1}\) of a generic strictly pseudoconvex psuedo-hermitian CR manifold \((M^{2n+1}, \theta)\) in terms of its Chern-Moser curvature tensors and their derivatives.

In Euclidean geometry, for a real submanifold \(M^n \subset \mathbb{E}^{n+a}\), \(M\) is a piece of \(\mathbb{E}^n\) if and only if its second fundamental form \(II_M \equiv 0\). In projective geometry, for a complex submanifold \(M^n \subset \mathbb{C}P^{n+a}\), \(M\) is a piece of \(\mathbb{C}P^n\) if and only if its projective second fundamental form \(II_M \equiv 0\) (c.f. \cite{IL03}, p.81). In CR geometry, we prove the CR analogue of this fact in this paper as follows:

**Theorem 1.1** Let \(H : M' \to \partial \mathbb{B}^{N+1}\) be a smooth CR-embedding of a strictly pseudoconvex CR real hypersurface \(M' \subset \mathbb{C}^{n+1}\). Denote \(M := H(M')\). If its CR second fundamental
form $II_M \equiv 0$, then $M \subset F(\partial B^{n+1}) \subset \partial B^{N+1}$ where $F : B^{n+1} \rightarrow B^{N+1}$ is a certain linear fractional proper holomorphic map.

Previously, it was proved by P. Ebenfelt, X. Huang and D. Zaitsev ([EHZ04], corollary 5.5), under the above same hypothesis, that $M'$ and hence $M$ are locally CR-equivalent to the unit sphere $\partial B^{n+1}$ in $\mathbb{C}^{n+1}$.

There are several definitions of the CR second fundamental forms $II_M$ of $M$ (see Section 3, 4, 5, and 6). The result in [EHZ04] used Definition 1 or 2. However, to prove Theorem 1.1, we need to use Definitions 3 and 4. We’ll prove in Section 4 that $II_M \equiv 0$ by any one of the four definitions will imply $II_M \equiv 0$ for all other three definitions. One of the ingredients for our proof of Theorem 1.1 is the result of Ebenfelt-Huang-Zaitsev [EHZ04] so that $M$ can be regarded as the image of a rational CR map $F : \partial \mathbb{H}^{n+1} \rightarrow M \subset \partial \mathbb{H}^{N+1}$. Another ingredient is a theorem of Huang ([Hu99]) that such a map $F$ is linear if and only if its geometric rank $\kappa_0$ is zero. The third one is a result from [HJY09] about a special lift for maps between spheres.

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## 2 Preliminaries

- **Maps between balls** We denote by $\text{Prop}(B^n, B^N)$ the space of all proper holomorphic maps from the unit ball $B^n \subset \mathbb{C}^n$ to $B^N$, denote by $\text{Prop}_k(B^n, B^N)$ the space $\text{Prop}(B^n, B^N) \cap C^k(B^n)$, and denote by $\text{Rat}(B^n, B^N)$ the space $\text{Prop}(B^n, B^N) \cap \{\text{rational maps}\}$. We say that $F$ and $G \in \text{Prop}(B^n, B^N)$ are equivalent if there are automorphisms $\sigma \in \text{Aut}(B^n)$ and $\tau \in \text{Aut}(B^N)$ such that $F = \tau \circ G \circ \sigma$.

Write $\mathbb{H}^n := \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) > |z|^2\}$ for the Siegel upper-half space. Similarly, we can define the space $\text{Prop}(\mathbb{H}^n, \mathbb{H}^N)$, $\text{Prop}_k(\mathbb{H}^n, \mathbb{H}^N)$ and $\text{Rat}(\mathbb{H}^n, \mathbb{H}^N)$ similarly. By the Cayley transformation $\rho_n : \mathbb{H}^n \rightarrow B^n$, $\rho_n(z, w) = (\frac{2z}{1-w^2}, \frac{1+iw}{1-w^2})$, we can identify a map $F \in \text{Prop}_k(\mathbb{H}^n, B^N)$ or $\text{Rat}(\mathbb{H}^n, B^N)$ with $\rho_N^{-1} \circ F \circ \rho_n$ in the space $\text{Prop}_k(\mathbb{H}^n, \mathbb{H}^N)$ or $\text{Rat}(\mathbb{H}^n, \mathbb{H}^N)$, respectively. We say that $F$ and $G \in \text{Prop}(\mathbb{H}^n, \mathbb{H}^N)$ are equivalent if there are automorphisms $\sigma \in \text{Aut}(\mathbb{H}^n)$ and $\tau \in \text{Aut}(\mathbb{H}^N)$ such that $F = \tau \circ G \circ \sigma$.

We denote by $\partial \mathbb{H}^n = \{(z, w) \in \mathbb{C}^{n-1} \times \mathbb{C} : \text{Im}(w) = |z|^2\}$ for the Heisenberg hypersurface. For any map $F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$, by restricting on $\partial \mathbb{H}^n$, we can regard $F$ as a $C^2$ CR map from $\partial \mathbb{H}^n$ to $\partial \mathbb{H}^N$, and we denote it as $F \in \text{Prop}_2(\partial \mathbb{H}^n, \partial \mathbb{H}^N)$. We say that $F$ and $G \in \text{Prop}_2(\partial \mathbb{H}^n, \partial \mathbb{H}^N)$ are equivalent if there are automorphisms $\sigma \in \text{Aut}(\partial \mathbb{H}^n) = \text{Aut}(\mathbb{H}^n)$ and $\tau \in \text{Aut}(\partial \mathbb{H}^N) = \text{Aut}(\mathbb{H}^N)$ such that $F = \tau \circ G \circ \sigma$. 


We can parametrize $\partial \mathbb{H}^n$ by $(z, \overline{z}, u)$ through the map $(z, \overline{z}, u) \to (z, u + i|z|^2)$. In what follows, we will assign the weight of $z$ and $u$ to be 1 and 2, respectively. For a non-negative integer $m$, a function $h(z, \overline{z}, u)$ defined over a small ball $U_0$ in $\partial \mathbb{H}^n$ is said to be of quantity $o_{\text{wt}}(m)$ if $\frac{h(z, \overline{z}, u)}{|z|^m} \to 0$ uniformly for $(z, u)$ on any compact subset of $U$ as $t(\in \mathbb{R}) \to 0$.

- **Partial normalization of $F$** Let $F = (f, \phi, g) = (\tilde{f}, g) = (f_1, \cdots, f_n-1, \phi_1, \cdots, \phi_{N-n}, g)$ be a non-constant map in Prop$_2(\partial \mathbb{H}^n, \partial \mathbb{H}^N)$ with $F(0) = 0$. For each $p \in \partial \mathbb{H}^n$, we write $\sigma_p^0 \in \text{Aut}(\mathbb{H}^n)$ with $\sigma_p^0(0) = p$ and $\tau_p^F \in \text{Aut}(\mathbb{H}^N)$ with $\tau_p^F(F(p)) = 0$ for the maps

\[
\begin{align*}
\sigma_p^0(z, w) &= (z + z_0, w + w_0 + 2i(z, \overline{z}_0)), \\
\tau_p^F(z^*, w^*) &= (z^* - \tilde{f}(z_0, w_0), w^* - g(z_0, w_0) - 2i(z^*, \overline{\tilde{f}(z_0, w_0)})).
\end{align*}
\]

$F$ is equivalent to $F_p = \tau_p^F \circ F \circ \sigma_p^0 = (f_p, \phi_p, g_p)$. Notice that $F_0 = F$ and $F_p(0) = 0$. The following is basic for the understanding of the geometric properties of $F$.

**Lemma 2.1** ([82, Lemma 5.3, Hu99], [Lemma 2.0, Hu03]): Let $F$ be a non-constant map in Prop$_2(\partial \mathbb{H}^n, \partial \mathbb{H}^N)$, $2 \leq n \leq N$ with $F(0) = 0$. For each $p \in \partial \mathbb{H}^n$, there is an automorphism $\tau_p^\ast \in \text{Aut}_0(\mathbb{H}^N)$ such that $F_p^\ast := \tau_p^\ast \circ F_p$ satisfies the following normalization:

\[
\begin{align*}
f_p^\ast &= z + \frac{i}{2} \partial_p^\ast(1)(z)w + o_{\text{wt}}(3), \\
\phi_p^\ast &= \phi_p^\ast(2)(z) + o_{\text{wt}}(2), \\
g_p^\ast &= w + o_{\text{wt}}(4), \\
\langle \overline{z}, \overline{a_p^\ast(1)}(z) \rangle |z|^2 &= |\phi_p^\ast(2)(z)|^2.
\end{align*}
\]

Let $A(p) = -2i(\partial^2(f_p^\ast)^\ast)_{1 \leq i, j \leq n-1}$, $\partial_p^\ast(1)(z)w = \partial_p^\ast(2)(z) + o_{\text{wt}}(2)$. We call the rank of $A(p)$, which we denote by $Rk_F(p)$, the geometric rank of $F$ at $p$. $Rk_F(p)$ depends only on $p$ and $F$, and is a lower semi-continuous function on $p$. We define the geometric rank of $F$ to be $\kappa_0(F) = \max_{p \in \partial \mathbb{H}^n} Rk_F(p)$. Notice that we always have $0 \leq \kappa_0 \leq n - 1$. We define the geometric rank of $F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$ to be the one for the map $\rho_n^{-1} \circ F \circ \rho_n \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$.

**Lemma 2.2** (ct. [Hu99], theorem 4.3) $F \in \text{Prop}_2(\mathbb{H}^n, \mathbb{H}^N)$ has geometric rank 0 if and only if $F$ is equivalent to a linear map.

Denote by $S_0 = \{(j, l) : 1 \leq j \leq \kappa_0, 1 \leq l \leq (n - 1), j \leq l\}$ and write $S := \{(j, l) : (j, l) \in S_0, \text{ or } j = \kappa_0 + 1, l \in \{\kappa_0 + 1, \cdots, \kappa_0 + N - n - \frac{(2n - \kappa_0 - 1)\kappa_0}{2}\}\}$.
Lemma 2.3 ([Lemma 3.2, Hu03]): Let $F$ be a $C^2$-smooth CR map from an open piece $M \subset \partial \mathbb{H}^n$ into $\partial \mathbb{H}^N$ with $F(0) = 0$ and $Rk_F(0) = \kappa_0$. Let $P(n, \kappa_0) = \frac{\kappa_0(2n - \kappa_0 - 1)}{2}$. Then $N \geq n + P(n, \kappa_0)$ and there are $\sigma \in Aut_0(\partial \mathbb{H}^n)$ and $\tau \in Aut_0(\partial \mathbb{H}^N)$ such that $F^{***} = \tau \circ F \circ \sigma := (f, \varphi, g)$ satisfies the following normalization conditions:

\[
\begin{align*}
    f_j &= z_j + \frac{i\mu_j}{2} z_j w + o_w(3), \quad \frac{\partial^2 f_j}{\partial w^2}(0) = 0, \quad j = 1, \ldots, \kappa_0, \quad \mu_j > 0, \\
    f_j &= z_j + o_w(3), \quad j = \kappa_0 + 1, \ldots, n - 1 \\
    g &= w + o_w(4), \\
    \phi_{jl} &= \mu_j z_j z_l + o_w(2), \text{ where } (j, l) \in S \text{ with } \mu_{jl} > 0 \text{ for } (j, l) \in S_0 \text{ and } \mu_{jl} = 0 \text{ otherwise}
\end{align*}
\]

where $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \leq \kappa_0$ if $j \neq l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l > \kappa_0$ or if $j = l \leq \kappa_0$.

- Pseudohermitian metric and Webster connection  
Let $M$ be a $C^2$ smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote by $T^*M = TM \cap iTM \subset TM$ its maximal complex tangent bundle with the complex structure $J : T^cM \rightarrow T^cM$. Here $J(\frac{\partial}{\partial x_j}) = \frac{\partial}{\partial y_j}$ and $J(\frac{\partial}{\partial y_j}) = -\frac{\partial}{\partial x_j}$ in terms of holomorphic coordinates. We denote by $\mathcal{V} = T_{0,1}M = \{X + iJX \mid X \in T^cM\} \subset \mathbb{C}TM := TM \otimes \mathbb{C}$ the CR bundle. We also denote $T^{1,0}M = \overline{\mathcal{V}}$. All $T^cM$, $\mathcal{V}$ and $\overline{\mathcal{V}}$ are complex rank $n$ vector bundles.

Write $T^0M := (T^{1,0}M \oplus T^{0,1}M)\perp \subset CT^*M$ for its rank one subbundle. Write $T'M := T^{0,1\perp} \subset \mathbb{C}T^*M$ for its rank $n + 1$ holomorphic or $(1,0)$ cotangent bundle of $M$. Here $T^0 \subset T'M$.

A real nonvanishing 1-form $\theta$ over $M$ is called a contact form if $\theta \wedge (d\theta)^n \neq 0$. Let $M$ be as above given by a defining function $r$. Then the 1-form $\theta = i\partial r$ is a contact form of $M$.

We say that $(M, \theta)$ is strictly pseudoconvex if the Levi-form $L_\theta$ is positive definite for all $z \in M$. Here the Levi-form $L_\theta$ with respect to $\theta$ is defined by

\[L_\theta(\overline{\nu}, \overline{v}) := -i\partial(\nu \wedge \overline{v}), \quad \forall \nu, \overline{v} \in T^{1,0}_p(M), \forall p \in M.\]

Associated with a contact form $\theta$ one has the Reeb vector field $R_\theta$, defined by the equations: (i) $d\theta(R_\theta, \cdot) \equiv 0$, (ii) $\theta(R_\theta) \equiv 1$. As a skew-symmetric form of maximal rank $2n$, the form $d\theta|_{T_pM}$ has a 1- dimensional kernel for each $p \in M^{2n+1}$. Hence equation (i) defines a unique line field $\langle R_\theta \rangle$ on $M$. The contact condition $\theta \wedge (d\theta)^n \neq 0$ implies that $\theta$ is non-trivial on that line field, so the unique real vector field is defined by the normalization condition (ii).
According to Tanaka [T75] and Webster [We78], \( (M, \theta) \) is called a strictly pseudoconvex pseudohermitian manifold if there are \( n \) complex 1-forms \( \theta^\alpha \) so that \( \{ \theta^1, ..., \theta^n \} \) forms a local basis for holomorphic cotangent bundle \( H^\ast (M) \) and

\[
d\theta = i \sum_{\alpha, \beta = 1}^n h_{\alpha \beta} \theta^\alpha \wedge \theta^\beta
\]

where \( (h_{\alpha \beta}) \), called the Levi form matrix, is positive definite. Such \( \theta^\alpha \) may not be unique. Following Webster (1978), a coframe \( (\theta, \theta^\alpha) \) is called admissible if (5) holds. The admissible coframes are determined up to transformations \( \tilde{\theta}^\alpha = u^\alpha_\beta \theta^\beta \) where \( (u^\alpha_\beta) \in GL(\mathbb{C}^n) \).

**Theorem 2.4 (Webster, 1978)** Let \( (M^{2n+1}, \theta) \) be a strictly pseudoconvex pseudohermitian manifold and let \( \theta^j \) be as in (5). Then there are unique way to write

\[
d\theta^\alpha = \sum_{\gamma = 1}^n \theta^\gamma \wedge \omega^\alpha_\gamma + \theta \wedge \tau^\alpha,
\]

where \( \tau^\alpha \) are \((0,1)\)-forms over \( M \) that are linear combination of \( \theta^\alpha \) and \( \omega^\beta_\alpha \) are 1-forms over \( M \) such that

\[
0 = dh_{\alpha \beta} - h_{\gamma \beta} \omega^\gamma_\alpha - h_{\alpha \gamma} \omega^\gamma_\beta,
\]

We may denote \( \omega_{\alpha \beta} = h_{\gamma \beta} \omega^\gamma_\alpha \) and \( \overline{\omega_{\alpha \beta}} = h_{\alpha \gamma} \omega^\gamma_\beta \). In particular, if

\[
h_{\alpha \beta} = \delta_{\alpha \beta},
\]

the identity in (7) becomes \( 0 = -\omega_{\alpha \beta} - \overline{\omega_{\alpha \beta}} \), i.e.,

\[
0 = \omega^\alpha_\beta + \omega^\beta_\alpha.
\]

The condition on \( \tau^\beta \) means:

\[
\tau^\beta = A^\beta_\gamma \theta^\gamma, \quad A^\alpha_\beta = A^\beta_\alpha,
\]

which holds automatically. The curvature is given by

\[
d\omega^\alpha_\beta - \omega^\gamma \wedge \omega^\beta_\gamma = R^\alpha_\beta \mu \nu \theta^\mu \wedge \theta^\nu + W^\alpha_\beta \mu \nu \theta^\mu \wedge \theta - W^\beta_\alpha \mu \nu \theta^\nu \wedge \theta + i \theta^\alpha \wedge \tau^\beta - i \tau^\alpha \wedge \theta^\beta
\]

where the functions \( R^\alpha_\beta \mu \nu \) and \( W^\alpha_\beta \mu \) represent the pseudohermitian curvature of \( (M, \theta) \).
3 CR second fundamental forms —– Definition 1

We are going to survey four definitions of the CR second fundamental forms $II_M$ of $M$ in $\partial \mathbb{H}^{N+1}$. We start with Definition 1 which is the intrinsic one in terms of a coframe.

Lemma 3.1 ([EHZ04], corollary 4.2) Let $M$ and $\tilde{M}$ be strictly pseudoconvex CR-manifolds of dimensions $2n+1$ and $2\tilde{n}+1$ respectively, and of CR dimensions $n$ and $\tilde{n}$ respectively. Let $F: M \rightarrow \tilde{M}$ be a smooth CR-embedding. If $(\theta, \theta^{\alpha})$ is an admissible coframe on $M$, then in a neighborhood of a point $\tilde{p} \in F(M)$ in $\tilde{M}$ there exists an admissible coframe $(\tilde{\theta}, \tilde{\theta}^{\alpha})$ on $\tilde{M}$ with $F^*(\tilde{\theta}, \tilde{\theta}^{\alpha}, 0) = (\theta, \theta^{\alpha}, 0)$. In particular, the Reeb vector field $\tilde{R}$ is tangent to $F(M)$. If we choose the Levi form matrix of $M$ such that the functions $h_{\alpha\beta}$ in (5) with respect to $(\theta, \theta^{\alpha})$ to be $\delta_{\alpha\beta}$, then $(\tilde{\theta}, \tilde{\theta}^{\alpha})$ can be chosen such that the Levi form matrix of $\tilde{M}$ relative to it is also $\delta_{AB}$. With this additional property, the coframe $(\tilde{\theta}, \tilde{\theta}^{\alpha})$ is uniquely determined along $M$ up to unitary transformations in $U(n) \times U(\tilde{n} - n)$.

If $(\theta, \theta^{\alpha})$ and $(\tilde{\theta}, \tilde{\theta}^{\alpha})$ are as above such that the condition on the Levi form matrices in Lemma 3.1 are satisfied, we say that the coframe $(\tilde{\theta}, \tilde{\theta}^{\alpha})$ is adapted to the coframe $(\theta, \theta^{\alpha})$. In this case, by (9), we have

$$d\theta^{\alpha} = \sum_{\gamma=1}^{n} \theta^{\gamma} \wedge \omega^{\alpha}_{\gamma} + \theta \wedge \tau^{\alpha}, \quad 0 = \omega^{\beta}_{\alpha} + \omega^{\alpha}_{\beta}, \quad \forall 1 \leq \alpha, \beta \leq n,$$

and

$$d\tilde{\theta}^{A} = \sum_{B=1}^{\tilde{n}} \tilde{\theta}^{C} \wedge \tilde{\omega}^{A}_{C} + \tilde{\theta} \wedge \tilde{\tau}^{A}, \quad 0 = \tilde{\omega}^{B}_{A} + \tilde{\omega}^{A}_{B}, \quad \forall 1 \leq A, B \leq N.$$

For simplicity, we may denote $F^*\tilde{\omega}^{A}_{B}$ by $\omega^{A}_{B}$. We also denote $F^*\tilde{\omega}^{A}_{AB}$ by $\omega^{A}_{AB}$ where $\omega^{A}_{AB} = \omega^{AB}_{A}$.

Write $\omega^{\mu}_{\alpha \beta} = \omega^{\mu \beta}_{\alpha}$. The matrix of $(\omega^{\mu}_{\alpha \beta})$, $1 \leq \alpha, \beta \leq n$, $n+1 \leq \mu \leq 2n$, defines the CR second fundamental form of $M$. It was used in [We79] and [Fa90].

4 CR second fundamental forms —– Definition 2

Definition 2 introduced in [EHZ04] will be the extrinsic one in terms of defining function.

Let $F: M \rightarrow \tilde{M}$ be a smooth CR-embedding between $M \subset \mathbb{C}^{n+1}$ and $\tilde{M} \subset \mathbb{C}^{N+1}$ where $M$ and $\tilde{M}$ are real strictly pseudoconvex hypersurfaces of dimensions $2n+1$ and $2\tilde{n}+1$, and
CR dimensions $n$ and $\tilde{n}$, respectively. Let $p \in M$ and $\tilde{p} = F(p) \in \tilde{M}$ be points. Let $\tilde{\rho}$ be a local defining function for $\tilde{M}$ near the point $\tilde{p}$. Let 

$$E_k(p) := \text{span}_\mathbb{C}\{L^J(\tilde{\rho}_Z \circ F)(p) \mid J \in (\mathbb{Z}^n)^n, 0 \leq |J| \leq k\} \subset T^{1,0}_p\mathbb{C}^{n+1},$$

where $\tilde{\rho}_Z := \partial_{\tilde{\rho}}$ is the complex gradient (i.e., represented by vectors in $\mathbb{C}^{n+1}$ in some local coordinate system $Z'$ near $\tilde{p}$). Here we use multi-index notation $L^J = L_{1}^{\alpha} \cdots L_{n}^{\alpha}$ and $|J| = J_1 + \cdots + J_n$. It was shown in [La01] that $E_k(p)$ is independent of the choice of local defining function $\tilde{\rho}$, coordinates $Z'$ and the choice of basis of the CR vector fields $L_T, \ldots, L_N$.

The CR second fundamental form $H_M$ of $M$ is defined by (cf. [EHZ04], §2)

$$IH_M(X_p, Y_p) := \pi(XY(\tilde{\rho}_Z \circ f)(p)) \in \tilde{T}_pM/E_1(p) \quad (12)$$

where $\tilde{\rho}_Z = \partial_{\tilde{\rho}}$ is represented by vectors in $\mathbb{C}^{n+1}$ in some local coordinate system $Z'$ near $\tilde{p}$, $X, Y$ are any $(1,0)$ vector fields on $M$ extending given vectors $X_p, Y_p \in T^{1,0}_p(M)$, and $\pi : \tilde{T}_pM \to \tilde{T}_pM/E_1(p)$ is the projection map.

Since $\tilde{M}$ and $M$ are strictly pseudoconvex, the Levi form of $\tilde{M}$ (at $\tilde{p}$) with respect to $\tilde{\rho}$ defines an isomorphism

$$\tilde{T}^{1,0}_p\tilde{M}/E_1(p) \cong T^{1,0}_p\tilde{M}/F_1(T^{1,0}_pM)$$

and the CR second fundamental form can be viewed as an $\mathbb{C}$-linear symmetric form

$$IH_{M,p} : T^{1,0}_pM \times T^{1,0}_pM \to T^{1,0}_p\tilde{M}/F_1(T^{1,0}_pM) \quad (13)$$

that does not depend on the choice of $\tilde{\rho}$ (cf.[EHZ04], §2).

The relation between Definition 1 and Definition 2 was discussed in [EHZ04]. Let $(M, \tilde{M}), (\theta, \theta^\alpha), (\tilde{\theta}, \tilde{\theta}^\alpha)$ be as in Lemma 3.1, and we abuse the structure bundle $(\theta, \theta^\alpha)$ on $M$ with the structure bundle $(\tilde{\theta}, \tilde{\theta}^\alpha)$ on $\tilde{M}$. We can choose a defining function $\tilde{\rho}$ of $\tilde{M}$ near a point $\tilde{p} = F(p) \in \tilde{M}$ where $p \in M$ such that $\theta = i\tilde{\rho}$ on $\tilde{M}$, i.e., in local coordinates $Z'$ in $\mathbb{C}^{n+1}$, we have

$$\theta = i \sum_{k=1}^{N+1} \frac{\partial \tilde{\rho}}{\partial \tilde{Z}_k} d\tilde{Z}_k,$$

where we pull back the forms $d\tilde{Z}_1, \ldots, d\tilde{Z}_{N+1}$ to $\tilde{M}$. Then we consider the coframe $(\theta, \theta^\alpha) = (F^*\tilde{\theta}, F^*\tilde{\theta}^\alpha)$ on $M$ near $p$ with $F(p) = \tilde{p}$. We take its dual frame $(T, L_A)$ of $(\theta, \theta^A)$ and have

$$L_\beta(\tilde{\rho}_Z \circ F)^\alpha = -iL_\beta \omega d\theta = g_{\beta\gamma}^\alpha d\theta^\gamma = g_{\beta\gamma}^\alpha \theta^\gamma.$$ 

(14)
Here we used the definition of the construction, (5) and the dual relationship \( \langle L_\beta, \theta^\alpha \rangle = \delta^\alpha_\beta \) and also notice that \( g_{\beta \gamma} = \delta_{\beta \gamma} \). Applying \( L_\alpha \) to both sides of (14), we obtain
\[
L_\alpha L_\beta (\tilde{\rho} Z \circ F) = g_{\beta \gamma} L_\alpha \omega_{\alpha} \theta^\gamma \mod (\theta, \theta^\gamma)
\]
which implies
\[
II_M(L_\alpha, L_\beta) = \omega_{\alpha} \beta, n + 1 \leq \mu \leq N.
\]
This identity gives the equivalent relation of the intrinsic and extrinsic definitions of \( II_M \).

Notice that we need a right choice of \((\theta, \theta^A), (T, L_A)\) and \(\tilde{\rho}\).

By using \((\omega_{\alpha}^b \beta)\) and (15), as in (13), we can also define
\[
II_{M,p} : T_{p}^{1,0} M \times T_{p}^{1,0} M \to T_{p}^{1,0} \tilde{M}/F_{p}(T_{p}^{1,0} M)
\]
which is independent of the choice of the adapted coframe \((\theta, \theta^A)\) in case \(\tilde{M}\) is locally CR embeddable in \(\mathbb{C}^{N+1}\) (cf. [EHZ04], § 4).

5 CR second fundamental forms —— Definition 3

Definition 3 will be the one as a tensor with respect to the group \( GL^Q(\mathbb{C}^{N+2}) \).

The bundle \( GL^Q(\mathbb{C}^{N+2}) \) over \( \partial \mathbb{H}^{N+1} \)

We consider a real hypersurface \( Q \) in \( \mathbb{C}^{N+2} \) defined by the homogeneous equation
\[
\langle Z, Z \rangle := \sum A Z^A Z^A + \frac{i}{2} (Z^0 Z^{N+1} - Z^0 Z^{N+1}) = 0,
\]
where \( Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2} \). Let
\[
\pi_0 : \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1}, \quad (z_0, ..., z_{N+1}) \mapsto [z_0 : ... : z_{N+1}],
\]
be the standard projection. For any point \( x \in \mathbb{CP}^{N+1} \), \( \pi_0^{-1}(x) \) is a complex line in \( \mathbb{C}^{N+2} - \{0\} \). For any point \( v \in \mathbb{C}^{N+2} - \{0\} \), \( \pi_0(v) \in \mathbb{CP}^{N+1} \) is a point. The image \( \pi_0(Q - \{0\}) \) is the Heisenberg hypersurface \( \partial \mathbb{H}^{N+1} \subset \mathbb{CP}^{N+1} \).

For any element \( A \in GL(\mathbb{C}^{N+2}) \):
\[
A = (a_0, ..., a_{N+1}) = \begin{bmatrix} a_0^{(0)} & a_1^{(0)} & \cdots & a_{N+1}^{(0)} \\ a_0^{(1)} & a_1^{(1)} & \cdots & a_{N+1}^{(1)} \\ \vdots & \vdots & \ddots & \vdots \\ a_0^{(N+1)} & a_1^{(N+1)} & \cdots & a_{N+1}^{(N+1)} \end{bmatrix} \in GL(\mathbb{C}^{N+2}),
\]
where each $a_j$ is a column vector in $\mathbb{C}^{N+2}$, $0 \leq j \leq N + 1$. This $A$ is associated to an automorphism $A^* \in Aut(\mathbb{C}P^{N+1})$ given by

$$A^*\left([z_0 : z_1 : \ldots : z_{N+1}]\right) = \left[\sum_{j=0}^{N+1} a_j^{(0)} z_j : \sum_{j=0}^{N+1} a_j^{(1)} z_j : \ldots : \sum_{j=0}^{N+1} a_j^{(N+1)} z_j\right]. \quad (20)$$

When $a_0^{(0)} \neq 0$, in terms of the non-homogeneous coordinates $(w_1, \ldots, w_n)$, $A^*$ is a linear fractional from $\mathbb{C}^{N+1}$ which is holomorphic near $(0, \ldots, 0)$:

$$A^*(w_1, \ldots, w_{N+1}) = \left(\sum_{j=0}^{N+1} a_j^{(1)} w_j, \ldots, \sum_{j=0}^{N+1} a_j^{(N+1)} w_j\right), \quad \text{where } w_j = \frac{z_j}{z_0}. \quad (21)$$

We denote $A \in GL^{Q}(\mathbb{C}^{N+2})$ if $A$ satisfies $A(Q) \subseteq Q$ where we regard $A$ as a linear transformation of $\mathbb{C}^{N+2}$. If $A \in GL^{Q}(\mathbb{C}^{N+2})$, we must have $A^*(\partial \mathbb{H}^{N+1}) \subseteq \partial \mathbb{H}^{N+1}$, so that $A^* \in Aut(\partial \mathbb{H}^{N+1})$. Conversely, if $A^* \in Aut(\partial \mathbb{H}^{N+1})$, then $A \in GL^{Q}(\mathbb{C}^{N+2})$.

We define a bundle map:

$$\pi : \quad GL(\mathbb{C}^{N+2}) \rightarrow \mathbb{C}P^{N+1} \\
A = (a_0, a_1, \ldots, a_{N+1}) \mapsto \pi_0(a_0).$$

Then by (20), for any map $A \in GL(\mathbb{C}^{N+2})$, $A \in \pi^{-1}(\pi_0(a_0)) \iff A^*([1 : 0 : \ldots : 0]) = \pi_0(a_0)$. In particular, by the restriction, we consider a map

$$\pi : \quad GL^{Q}(\mathbb{C}^{N+2}) \rightarrow \partial \mathbb{H}^{N+1} \\
A = (a_0, a_1, \ldots, a_{N+1}) \mapsto \pi_0(a_0). \quad (22)$$

We get $\partial \mathbb{H}^{N+1} \simeq GL^{Q}(\mathbb{C}^{N+2})/P_1$ where $P_1$ is the isotropy subgroup of $GL^{Q}(\mathbb{C}^{N+2})$. Then by (20), for any map $A \in GL^{Q}(\mathbb{C}^{n+2})$,

$$A \in \pi^{-1}(\pi_0(a_0)) \iff A^*([1 : 0 : \ldots : 0]) = \pi_0(a_0). \quad (23)$$

**CR submanifolds of $\partial \mathbb{H}^{N+1}$** Let $H : M' \rightarrow \partial \mathbb{H}^{N+1}$ be a CR smooth embedding where $M'$ is a strictly pseudoconvex smooth real hypersurface in $\mathbb{C}^{n+1}$. We denote $M = H(M')$.

Let $R_{M'}$ be the Reeb vector field of $M'$ with respect to a fixed contact form on $M'$. Then the real vector $R_{M'}$ generates a real line bundle over $M'$, denoted by $\mathcal{R}_{M'}$. Since we can regard the rank $n$ complex vector bundle $T^{1,0}M'$ as the rank $2n$ real vector bundle, over the real number field $\mathbb{R}$ we have:

$$TM' = T^cM' \oplus \mathcal{R}_{M'} \simeq T^{1,0}M' \oplus \mathcal{R}_{M'}. \quad (24)$$
given by
\[(a_j \frac{\partial}{\partial x_j}, b_j \frac{\partial}{\partial y_j}) + cR_{M'} \mapsto (a_j + ib_j) \frac{\partial}{\partial z_j} + cR_{M'}, \; \forall a_j, b_j, c \in \mathbb{R}. \tag{25}\]

Since \( H \) is a CR embedding, we have
\[H_*(T^{1,0}M') = T^{1,0}M \subset T^{1,0}(\partial \mathbb{H}^{N+1}), \; TM \simeq H_*(T^{1,0}M') \oplus H_*(\mathcal{R}_{M'}) \subset T(\partial \mathbb{H}^{N+1}). \tag{26}\]

**Lifts of the CR submanifolds** Let \( M = H(M') \subset \partial \mathbb{H}^{N+1} \) be as above. Consider the commutative diagram
\[\begin{array}{ccc}
GL^Q(\mathbb{C}^{N+2}) & \xrightarrow{e} & M \\
\downarrow \pi & & \leftarrow \downarrow \partial \mathbb{H}^{N+1} \\
\end{array}\]

Any map \( e \) satisfying \( \pi \circ e = Id \) is called a lift of \( M \) to \( GL^Q(\mathbb{C}^{N+2}) \).

In order to define a more specific lifts, we need to give some relationship between geometry on \( \partial \mathbb{H}^{N+1} \) and on \( \mathbb{C}^{N+2} \) as follows. For any subset \( X \in \partial \mathbb{H}^{N+1} \), we denote \( \hat{X} := \pi_0^{-1}(X) \) where \( \pi_0: \mathbb{C}^{N+2} - \{0\} \to \mathbb{CP}^{N+1} \) is the standard projection map \( (18) \). In particular, for any \( x \in M, \hat{x} \) is a complex line and for the real submanifold \( M^{2n+1} \), the real submanifold \( M^{2n+3} \) is of dimension \( 2n + 3 \).

For any \( x \in M \), we take \( v \in \hat{x} = \pi_0^{-1}(x) \subset \mathbb{C}^{N+2} - \{0\} \), and we define
\[\hat{T}_x M = T_v \hat{M}, \; \hat{T}_x^{1,0} M = T_v^{1,0} \hat{M}, \; \hat{R}_{M,x} := \hat{R}_{\hat{M},v}\]
where \( \hat{R}_{\hat{M}} = \cup_{v \in \hat{M}} \hat{R}_{\hat{M},v} \). These definitions are independent of choice of \( v \).

A lift \( e = (e_0, e_\alpha, e_\mu, e_{N+1}) \) of \( M \) into \( GL^Q(\mathbb{C}^{N+2}) \), where \( 1 \leq \alpha \leq n \) and \( n + 1 \leq \mu \leq N \), is called a first-order adapted lift if it satisfies the conditions:
\[e_0(x) \in \pi_0^{-1}(x), \; \text{span}_\mathbb{C}(e_0, e_\alpha)(x) = \hat{T}_x^{1,0} M, \; \text{span}(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0} M \oplus \hat{R}_{M,x} \tag{27}\]
where
\[\text{span}(e_0, e_\alpha, e_{N+1})(x) := \{c_0 e_0 + c_\alpha e_\alpha + c_{N+1} e_{N+1} \mid c_0, c_\alpha \in \mathbb{C}, \; c_{N+1} \in \mathbb{R}\}. \tag{28}\]

Here we used \( (25) \) and the fact that the Reeb vector is real. Locally first-order adapted lifts always exist (see Theorem \( 7.1 \) below).

We have the restriction bundle \( \mathcal{F}^0_M := GL^Q(\mathbb{C}^{N+2})|_M \) over \( M \). The subbundle \( \pi: \mathcal{F}^1_M \to M \) of \( \mathcal{F}^0_M \) is defined by
\[\mathcal{F}^1_M = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}^0_M \mid [e_0] \in M, \; (27) \text{ are satisfied}\}. \]
Local sections of $\mathcal{F}_M^1$ are exactly all local first-order adapted lifts of $M$.

For two first-order adapted lifts $s = (e_0, e_j, e_\mu, e_{N+1})$ and $\tilde{s} = (\tilde{e}_0, \tilde{e}_j, \tilde{e}_\mu, \tilde{e}_{N+1})$, by (27), we have

\[
\begin{align*}
\tilde{e}_0 &= g_0^0 e_0, \\
\tilde{e}_j &= g_j^0 e_0 + g_j^k e_k, \\
\tilde{e}_\mu &= g_\mu^0 e_0 + g_\mu^j e_j + g_\mu^{N+1} e_{N+1}, \\
\tilde{e}_{N+1} &= g_{N+1}^0 e_0 + g_{N+1}^j e_j + g_{N+1}^{N+1} e_{N+1}, \\
\end{align*}
\]

(29)

Notice that by (25), $g_{N+1}^{N+1}$ is some real-valued function, while other are complex-valued functions. In other words, $\tilde{s} = s \cdot g$ where

\[
g = (g_0, g_j, g_\mu, g_{N+1}) = \begin{pmatrix} g_0^0 & g_0^k & g_0^{N+1} \\
g_j^0 & g_j^k & g_j^{N+1} \\
g_\mu^0 & g_\mu^j & g_\mu^{N+1} \\
g_{N+1}^0 & g_{N+1}^j & g_{N+1}^{N+1} \end{pmatrix}
\]

(30)

is a smooth map from $M$ into $GL^Q(\mathbb{C}^{N+2})$. Then the fiber of $\pi : \mathcal{F}_M^1 \to M$ over a point is isomorphic to the group

\[
G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_0^k & g_0^{N+1} \\
g_j^0 & g_j^k & g_j^{N+1} \\
g_\mu^0 & g_\mu^j & g_\mu^{N+1} \\
g_{N+1}^0 & g_{N+1}^j & g_{N+1}^{N+1} \end{pmatrix} \in GL^Q(\mathbb{C}^{N+2}) \right\},
\]

where we use the index ranges $1 \leq \alpha, \beta \leq n$ and $n + 1 \leq \mu, \nu \leq N$.

We pull back the Maurer-Cartan form from $GL^Q(\mathbb{C}^{N+2})$ to $\mathcal{F}_M^1$ by a first-order adapted lift $e$ of $M$ as

\[
\omega = \begin{pmatrix} \omega_0^0 & \omega_0^\alpha & \omega_0^\mu & \omega_0^{N+1} \\
\omega_\beta^0 & \omega_\beta^\alpha & \omega_\beta^\mu & \omega_\beta^{N+1} \\
\omega_\mu^0 & \omega_\mu^\alpha & \omega_\mu^\mu & \omega_\mu^{N+1} \\
\omega_{N+1}^0 & \omega_{N+1}^\alpha & \omega_{N+1}^\mu & \omega_{N+1}^{N+1} \end{pmatrix}.
\]

Since $\omega = e^{-1} de$, i.e., $e \omega = de$. Then we have

\[
\begin{align*}
d e_0 &= e_0 \omega_0^0 + e_\alpha \omega_0^\alpha + e_\mu \omega_0^\mu + e_{N+1} \omega_0^{N+1}. \\
\end{align*}
\]

(31)

On the other hand, we claim:

\[
\begin{align*}
d e_0 &= e_0 \omega_0^0 + e_\alpha \omega_0^\alpha + e_{N+1} \omega_0^{N+1}. \\
\end{align*}
\]

(32)
In fact, take local coordinates systems \((x_1, \ldots, x_{2n+1})\) for the real manifold \(M\), and \((y_1, y_2, x_1, \ldots, x_{2n+1})\) for the real manifold \(\hat{M}\) where \((y_1, y_2)\) is the coordinates for fibers. By the first condition in (27), fixing \(x_1, \ldots, x_{j-1}, x_{j+1}, \ldots, x_{2n+1}\), \(e_0(\ldots, x_j, \ldots)\) is a curve into \(\hat{M}\) with parameter \(x_j\). Then \(\frac{\partial e_0}{\partial x_j} \in T_M\) is a tangent vector to this curve. Since \(\text{span}(e_0, e_\alpha, e_{N+1})(x) = \hat{T}_x^{1,0} M \oplus \hat{R}_{M,x}\) in (27) and \(T\hat{M} \cong T^{1,0} \hat{M} \oplus R_M\), we obtain

\[
\frac{\partial e_0}{\partial x_j} = b_0^j e_0 + b_\alpha^j e_\alpha + b_{N+1}^j e_{N+1}, \quad 1 \leq j \leq 2n + 1
\]  

(33)

for some functions \(b_0^j, b_\alpha^j\) and \(b_{N+1}^j\). We also have

\[
\frac{\partial e_0}{\partial y_i} = 0, \quad \text{for} \ i = 1, 2,
\]

(34)

because \((y_1, y_2)\) are the coordinates for fibers. From (33) and (34), we get

\[
de_0 = \frac{\partial e_0}{\partial y_1} dy_1 + \frac{\partial e_0}{\partial y_2} dy_2 + \sum_j \frac{\partial e_0}{\partial x_j} dx_j = \sum_j (b_0^j e_0 + b_\alpha^j e_\alpha + b_{N+1}^j e_{N+1}) dx_j
\]

\[
= (\sum_j b_0^j dx_j) e_0 + (\sum_j b_\alpha^j dx_j) e_\alpha + (\sum_j b_{N+1}^j dx_j) e_{N+1}.
\]

(35)

Since the 1-forms \(\omega_0^\alpha, \omega_0^\alpha, \omega_{N+1}^\alpha\) in (31) are unique, from (35), it proves Claim (32).

By (31) and (32), we conclude \(\omega_0^\alpha = 0, \forall \mu\). By the Maurer-Cartan equation \(d\omega = -\omega \wedge \omega\), one gets \(0 = d\omega_\mu = -\omega_\mu^\nu \wedge \omega_\nu^\alpha - \omega_\nu^{\alpha+1} \wedge \omega_0^{N+1}\), i.e., \(0 = -\omega_\mu^\nu \wedge \omega_\nu^\alpha \mod(\omega_0^{N+1})\). Then by Cartan’s lemma,

\[
\omega_\beta^\mu = q_\alpha^\beta \omega_\nu^\alpha \mod(\omega_0^{N+1}),
\]

for some functions \(q_\alpha^\beta = q_\beta^\alpha\).

**The CR second fundamental form**  In order to define the CR second fundamental form \(\Pi_M = \Pi^*_M = q_\alpha^\beta \omega_\alpha^\mu \omega_\beta^\nu \otimes e_\mu \mod(\omega_0^{N+1})\), let us define \(e_\alpha\) as follows.

For any first-order adapted lift \(e = (e_0, e_\alpha, e_\nu, e_{N+1})\) with \(\pi_0(e_0) = x\), we have \(e_\alpha \in \hat{T}_x^{1,0} M\). Recall \(T_E G(k, V) \simeq E^* \otimes (V/E)\) where \(G(k, V)\) is the Grassmannian of \(k\)-planes that pass through the origin in a vector space \(V\) over \(\mathbb{R}\) or \(\mathbb{C}\) and \(E \in G(k, V)\) ([IL03], p.73). Then \(T_x M \simeq (\hat{x})^* \otimes (\hat{T}_x M/\hat{x})\) and hence the vector \(e_\alpha\) induces \(e_\alpha \in T_x^{1,0} M\) by

\[
e_\alpha = e_0 \otimes (e_\alpha \mod(e_0)),
\]

where we denote by \((e_0, e_\alpha, e_\nu, e_{N+1})\) the dual basis of \((\mathbb{C}^{N+2})^*\). Similarly, we let

\[
e_\mu = e_0 \otimes (e_\mu \mod(\hat{T}_x^{1,0} M)) \in N_x^{1,0} M,
\]

(36)
where $N^{1,0}M$ is the CR normal bundle of $M$ defined by $N^{1,0}_x M = T_x^{1,0} (\partial \mathbb{H}^{N+1}) / T_x^{1,0} M$.

By direct computation, we obtain a tensor

$$II_M = II_M^\ast = q^\mu_{\alpha \beta} \psi^0_{\alpha} \psi^0_{\beta} \otimes e^\mu \in \Gamma (M, S^2 T^{1,0}_{\pi_0(e_0)} M \otimes N^{1,0}_{\pi_0(e_0)} M) \mod (\omega^0_{N+1}).$$

(37)

The tensor $II_M$ is called the CR second fundamental form of $M$.

Pulling back a lift

Let $M \subset \partial \mathbb{H}^{N+1}$ be as above with a point $Q_0 \in M$. Let $A \in GL^Q (\mathbb{C}^{N+2})$, $A^* \in Aut (\partial \mathbb{H}^{N+1})$ with $A^*(Q_0) = P_0$ and $\tilde{M} = A^* (M)$. Let $\tilde{s} : \tilde{M} \rightarrow GL^Q (\mathbb{C}^{N+2})$ be a lift. We claim:

$$s := A^{-1} \cdot \tilde{s} \circ A^*,$$

(38)

is also a lift from $M$ into $GL^Q (\mathbb{C}^{N+2})$. In fact, in order to prove that $s$ is a lift, it suffices to prove: $\pi s = Id$, i.e., for any point $Q \in M$ near $Q_0$, $\pi s (Q) = Q$. In fact,

$$\pi s (Q) = \pi (A^{-1} \cdot \tilde{s} \circ A^*) (Q) = \pi (A^{-1} \cdot \tilde{s} (P)) = (A^*)^{-1} (\pi \tilde{s} (P)) = (A^*)^{-1} (P) = Q.$$

so that our claim is proved.

If, in addition, $\tilde{s}$ is a first-order adapted lift of $\tilde{M}$ into $GL^Q (\mathbb{C}^{N+2})$, $s$ is also a first-order adapted lift of $M$ into $GL^Q (\mathbb{C}^{N+2})$.

Let $\Omega$ be the Maurer-Cartan form over $GL^Q (\mathbb{C}^{N+2})$. Then by the invariant property $A^* \Omega = \Omega$, we have $s^* \Omega = (A^{-1} \cdot \tilde{s} \circ A^*)^* \Omega = (A^*)^* (\tilde{s}^*)^* (A^{-1})^* \Omega = (A^*)^* (\tilde{s})^* \Omega$, i.e., it holds on $M$ that

$$\omega = (A^*)^* \tilde{\omega}$$

(39)

where $\omega = s^* \Omega$ and $\tilde{\omega} = \tilde{s}^* \Omega$ so that $\omega^\alpha_\alpha = (A^*)^* \tilde{\omega}^\alpha_\alpha$ and $\omega^\mu_\beta = (A^*)^* \tilde{\omega}^\mu_\beta$. The last equality yields

$$q^\mu_{\alpha \beta} = \tilde{q}^\mu_{\alpha \beta} \circ A^*.$$

(40)

6 CR second fundamental forms —— Definition 4

Definition 4 will be the one as a tensor with respect to the group $SU(N+1,1)$.

As for Definition 3, we consider the real hypersurface $Q$ in $\mathbb{C}^{N+2}$ defined by the homogeneous equation

$$\langle Z, Z \rangle := \sum_A Z^A Z^A + \frac{i}{2} (Z^{N+1} \overline{Z^0} - \overline{Z^0} Z^{N+1}) = 0,$$

(41)
where \( Z = (Z^0, Z^A, Z^{N+1})^t \in \mathbb{C}^{N+2} \). This can be extended to the scalar product
\[
\langle Z, Z' \rangle := \sum_A Z^A \overline{Z'^A} + \frac{i}{2}(Z^{N+1}\overline{Z^0} - Z^0\overline{Z^{N+1}}),
\]
for any \( Z = (Z^0, Z^A, Z^{N+1})^t, Z' = (Z'^0, Z'^A, Z'^{N+1})^t \in \mathbb{C}^{N+2} \). This product has the properties: \( \langle Z, Z' \rangle \) is linear in \( Z \) and anti-linear in \( Z' \); \( \langle Z, Z \rangle = \langle Z', Z \rangle \); and \( Q \) is defined by \( \langle Z, Z \rangle = 0 \).

Let \( SU(N+1, 1) \) be the group of unimodular linear transformations of \( \mathbb{C}^{N+2} \) that leave the form \( \langle Z, Z \rangle \) invariant (cf. [CM74]).

By a \( Q \)-frame is meant an element \( E = (E_0, E_A, E_{N+1}) \in GL(\mathbb{C}^{N+2}) \) satisfying (cf. [CM74, (1.10)])
\[
\begin{align*}
det(E) &= 1, \\
\langle E_A, E_B \rangle &= \delta_{AB}, \\
\langle E_0, E_{N+1} \rangle &= -\langle E_{N+1}, E_0 \rangle = -\frac{i}{2},
\end{align*}
\]
while all other products are zero.

There is exactly one transformation of \( SU(N+1, 1) \) which maps a given \( Q \)-frame into another. By fixing one \( Q \)-frame as reference, the group \( SU(N+1, 1) \) can be identified with the space of all \( Q \)-frames. Then \( SU(N+1, 1) \subset GL^Q(\mathbb{C}^{N+1}) \) is a subgroup with the composition operation. By \([22]\) and the restriction, we have the projection
\[
\pi : SU(N+1, 1) \to \partial\mathbb{H}^{N+1}, \quad (Z_0, Z_A, Z_{N+1}) \mapsto \text{span}(Z_0).
\]
which is called a \( Q \)-frames bundle. We get \( \partial\mathbb{H}^{N+1} \cong SU(N+1, 1)/P_2 \) where \( P_2 \) is the isotropy subgroup of \( SU(N+1, 1) \). \( SU(N+1, 1) \) acts on \( \partial\mathbb{H}^{N+1} \) effectively.

Consider \( E = (E_0, E_A, E_{N+1}) \in SU(N+1, 1) \) as a local lift. Then the Maurer-Cartan form \( \Theta \) on \( SU(N+1, 1) \) is defined by \( dE = (dE_0, dE_A, dE_{N+1}) = E\Theta \), or \( \Theta = E^{-1} \cdot dE \), i.e.,
\[
d\begin{pmatrix} E_0 & E_A & E_{N+1} \end{pmatrix} = \begin{pmatrix} E_0 & E_B & E_{N+1} \end{pmatrix} \begin{pmatrix} \Theta_0^0 & \Theta_0^A & \Theta_0^{N+1} \\ \Theta_B^0 & \Theta_B^A & \Theta_B^{N+1} \\ \Theta_{N+1}^0 & \Theta_{N+1}^A & \Theta_{N+1}^{N+1} \end{pmatrix},
\]
where \( \Theta_A^B \) are 1-forms on \( SU(N+1, 1) \). By \([13]\) and \([15]\), the Maurer-Cartan form \( \Theta \) satisfies
\[
\begin{align*}
\Theta_0^0 + \Theta_{N+1}^{N+1} &= 0, & \Theta_0^{N+1} &= \overline{\Theta_0^{N+1}}, & \Theta_0^0 &= \overline{\Theta_0^{N+1}}, \\
\Theta_{N+1}^0 &= 2i\Theta_A^A, & \Theta_{N+1}^A &= -\frac{i}{2}\Theta_A^A, & \Theta_B^0 + \Theta_B^A &= 0, & \Theta_0^0 + \Theta_A^A + \Theta_{N+1}^{N+1} &= 0.
\end{align*}
\]
where $1 \leq A \leq N$. For example, from $\langle E_A, E_B \rangle = \delta_{AB}$, by taking differentiation, we obtain
\[ \langle dE_A, E_B \rangle + \langle E_A, dE_B \rangle = 0. \]

By (45), we have
\[
\begin{align*}
\langle E_0 \Theta_A^0 + E_B \Theta_B^0 + E_{N+1} \Theta_{N+1}^0, E_B \rangle &+ \langle E_A, E_0 \Theta_A^0 + E_B \Theta_B^0 + E_{N+1} \Theta_{N+1}^0 \rangle = 0,
\end{align*}
\]
which implies $\Theta_A^0 + \Theta_B^0 = 0$. In particular, from (46), $\Theta_A^0 = -2i \Theta_A^0$. $\Theta$ satisfies
\[ d\Theta = -\Theta \wedge \Theta. \quad (47) \]

Let $M \hookrightarrow \partial \mathbb{H}^{N+1}$ be the image of $H : M' \to \partial \mathbb{H}^{N+1}$ where $M' \subset \mathbb{C}^{n+1}$ is a CR strictly pseudoconvex smooth hypersurface. Consider the inclusion map $M \hookrightarrow \partial \mathbb{H}^{N+1}$ and a lift $e = (e_0, e_1, ..., e_{N+1}) = (e_0, e_\alpha, e_\nu, e_{N+1})$ of $M$ where $1 \leq \alpha \leq n$ and $n + 1 \leq \nu \leq N$
\[
\begin{array}{c}
\pi_0(e_0(x)) = x, \quad \text{span}_C(e_0, e_\alpha)(x) = \tilde{T}^{1,0}_x M, \quad \text{span}_C(e_0, e_\alpha, e_{N+1})(x) = \tilde{T}_x^{1,0} M \oplus \hat{R}_{M,x}.
\end{array}
\]
(48)
Locally first-order adapted lifts always exist (see Theorem 7.1 below). We have the restriction bundle $\mathcal{F}^0_M := SU(N+1,1)|_M$ over $M$. The subbundle $\pi : \mathcal{F}^1_M \to M$ of $\mathcal{F}^0_M$ is defined by
\[ \mathcal{F}^1_M = \{(e_0, e_j, e_\mu, e_{N+1}) \in \mathcal{F}^0_M \mid [e_0] \in M, \ (48) \text{ are satisfied} \}. \]
Local sections of $\mathcal{F}^1_M$ are exactly all local first-order adapted lifts of $M$. The fiber of $\pi : \mathcal{F}^1_M \to M$ over a point is isomorphic to the group
\[ G_1 = \left\{ g = \begin{pmatrix} g_0^0 & g_0^\beta & g_0^\nu & g_0^{N+1} \\ g_\beta^0 & g_\beta^\alpha & g_\beta^\nu & g_\beta^{N+1} \\ g_\nu^0 & g_\nu^\alpha & g_\nu^\nu & g_\nu^{N+1} \\ 0 & 0 & 0 & g_\nu^{N+1} \end{pmatrix} \in SU(N+1, 1) \right\}, \]
where we use the index ranges $1 \leq \alpha, \beta \leq n$ and $n + 1 \leq \mu, \nu \leq N$.

By the remark below (29), $g_{N+1}^{N+1}$ is real-valued. By (43), we have $\langle g_0, g_{N+1} \rangle = -\tau$, it implies $g_0 \cdot g_{N+1} = 1$. In particular, both $g_{N+1}^{N+1}$ and $g_0^0$ are real. Since $\langle g_0, g_\mu \rangle = 0$ and $g_0^0 \neq 0$, it implies $g_{N+1}^\mu = 0$. Since $\langle g_\alpha, g_\beta \rangle = \delta_{\alpha\beta}$, it implies that the matrix $(g_\alpha^\beta)$ is unitary. Since $\text{deg}(g) = 1$, it implies $g_0^0 \cdot \det(g_\alpha^\beta) \cdot \det(g_\mu^\nu) \cdot g_{N+1}^{N+1} = 1$, i.e., $\det(g_\alpha^\beta) \cdot \det(g_\mu^\nu) = 1$.

By considering all first-order adapted lifts from $M$ into $SU(N + 1, 1)$, as the definition of $II_M$ in Definition 3, we can defined CR second fundamental form $II_M$ as in (37):

$$II_M = II_M^s = g_{\alpha\beta}^\mu \omega_\alpha^\mu \omega_\alpha^\beta \otimes \varepsilon_\mu \in \Gamma(M, S^2T^1_{\pi_0(e_0)} M \otimes N^{1,0}_{\pi_0(e_0)} M), \text{ mod}(\omega_0^{N+1}),$$

which is a well-defined tensor, and is called the CR second fundamental form of $M$.

We remark that the notion of $II_M$ in Definition 4 was introduced in a paper by S.H. Wang [Wa06].

**Pulling back a lift**

Let $M \subset \partial H^{N+1}$ be as above with a point $Q_0 \in M$. Let $A \in SU(N + 1, 1)$, $A^* \in \text{Aut}(\partial H^{N+1})$ with $A^*(Q_0) = P_0$ and $\tilde{M} = A^*(M)$. Let $\tilde{s} : \tilde{M} \to SU(N + 1, 1)$ be a lift. We claim:

$$s := A^{-1} \circ \tilde{s} \circ A^*,$$

is also a lift from $M$ into $SU(N + 1, 1)$. Similarly as in (39) and (40), we have

$$\omega = (A^*)^* \tilde{\omega}$$

and

$$g_\alpha^\mu = g_\alpha^\beta \circ A^*.$$

where $\omega = s^* \Omega$, $\tilde{\omega} = \tilde{s}^* \Omega$ and $\Omega$ is the Maurer-Cartan form over $SU(N + 1, 1)$.

**Example**

Consider the maps in (11) and (2):

$$\sigma_p^0(z, w) = (z + z_0, w + w_0 + 2i\langle z, \overline{z_0} \rangle),$$

$$\tau_p^F(z^*, w^*) = (z^* - \tilde{f}(z_0, w_0), w^* - g(z_0, w_0) - 2i\langle z^*, \tilde{f}(z_0, w_0) \rangle)$$

where $p = (z_0, w_0)$, $z = \mathbb{C}^n$, $w = z_{n+1}$, $\sigma_p^0 \in \text{Aut}(\partial H^{N+1})$, and $\tau_p^F \in \text{Aut}(\partial H^{N+1})$.

By (19) and (21), these two maps correspond to two matrices:

$$A_{\sigma_p^0} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ z_0 & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ z_0 & 0 & \cdots & 1 & 0 \\ w_0 & 2iz_0 & \cdots & 2iz_0 & 1 \end{bmatrix} \in SU(n + 1, 1)$$

(53)
and

$$A_{gF} = \begin{bmatrix} 1 & 0 & \cdots & 0 & 0 \\ -\widetilde{f}_{01} & 1 & \cdots & 0 & 0 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ -\widetilde{f}_{0N-n} & 0 & \cdots & 1 & 0 \\ -g(z_0, w) & -2i\bar{f}_1(z_0, w_0) & \cdots & -2i\bar{f}_{N-n}(z_0, w_0) & 1 \end{bmatrix} \in SU(N+1, 1) \quad (54)$$

where $z_0 = (z_{01}, \ldots, z_{0n})$ and $w_0 = z_{0n+1}$.

[Example] Consider the map $F_{\lambda, r, \bar{a}, U} = (f, g) \in Aut_0(\partial \mathbb{H}^{n+1})$

$$f(z) = \frac{\lambda(z + \bar{a}w)}{1 - 2i\langle z, \bar{a} \rangle - (r + i\|\bar{a}\|^2)w}, \quad g(z) = \frac{\lambda^2 w}{1 - 2i\langle z, \bar{a} \rangle - (r + i\|\bar{a}\|^2)w}$$

where $\lambda > 0, r \in \mathbb{R}, \bar{a} \in \mathbb{C}^n$ and $U = (u_{\alpha\beta})$ is an $(n-1) \times (n-1)$ unitary matrix. By (19) and (21), its corresponding matrix,

$$A_{F_{\lambda, r, \bar{a}, U}} = \begin{bmatrix} 1 & -2i\bar{a}_1 & \cdots & -2i\bar{a}_n & -(r + i\|\bar{a}\|^2) \\ 0 & \lambda u_{11} & \cdots & \lambda u_{1n} & \lambda a_1 \\ \vdots & \vdots & \ddots & \vdots & \vdots \\ 0 & \lambda u_{n1} & \cdots & \lambda u_{nn} & \lambda a_n \\ 0 & 0 & \cdots & 0 & \lambda^2 \end{bmatrix}, \quad (55)$$

is not in $SU(n+1, 1)$ in general. In fact, we can write

$$F_{\lambda, r, \bar{a}, U} = F_{\lambda, 0,0, Id} \circ F_{1,0,0, U} \circ F_{1,r, \bar{a}, Id} \quad (56)$$

or $A_{F_{\lambda, r, \bar{a}, U}} = A_{F_{\lambda, 0,0, Id}} \cdot A_{F_{1,0,0, U}} \cdot A_{F_{1,r, \bar{a}, Id}}$. Here $A_{F_{1,0,0, U}}$ and $A_{F_{1,r, \bar{a}, Id}}$ are in $SU(N+1, 1)$; while $A_{F_{\lambda, 0,0, Id}}$ is in $SU(N+1, 1)$ if and only if $\lambda = 1$. Therefore

$$A_{F_{\lambda, r, \bar{a}, U}} \text{ is in } SU(n+1, 1) \text{ if and only if } \lambda = 1. \quad (57)$$

7 Existence of First-order Adapted Lifts from $M$ into $SU(N+1, 1)$ or into $GL^Q(\mathbb{C}^{N+2})$

Existence of first-order adapted lifts. Let $(M', 0)$ be a germ of smooth real hypersurface in $\mathbb{C}^{n+1}$ defined by the defining function

$$r = \sum_{j=1}^{n} z_j\bar{z}_j + \frac{i}{2}(w - \bar{w}) + o(2). \quad (58)$$
We take
\[ \theta = i \partial r = i \left( \sum_{j=1}^{n} z_j \, dz_j - \frac{1}{2} \, dw \right) + o(1). \]
as a contact form of \( M' \).

Write \( w = u + iv \). Here \( v = \sum_{j=1}^{n} |z_j|^2 + o(2) \). Take \( (z_j, u) \) as a coordinates system of \( M' \). By considering the coordinate map: \( h : \mathbb{C}^n \times \mathbb{R} \to M', (z_j, u) \mapsto (z_j, u + i|z_j|^2 + o(2)) \), we get the pushforward
\[ h_*(\frac{\partial}{\partial z_j}) = L_j := \frac{\partial}{\partial z_j} + i(z_j + o(1)) \frac{\partial}{\partial u}, \quad h_*(\frac{\partial}{\partial u}) = R_{M'} := (1 + o(1)) \frac{\partial}{\partial u} \]
for \( j = 1, 2, \ldots, n \). Then \( \{L_j\}_{1 \leq j \leq n} \) form a basis of the complex tangent bundle \( T^{1,0}M' \) of \( M' \). Since \( d\alpha = -i \sum_{j=1}^{n} dz_j \wedge d\overline{z}_j \), we see that \( R \) is the Reeb vector field of \( M' \). In particular, as the restriction at 0, we have
\[ L_j|_0 = \frac{\partial}{\partial z_j}|_0, \quad R_{M'}|_0 = \frac{\partial}{\partial u}|_0. \] (59)

**Theorem 7.1** Let \( M \hookrightarrow \partial \mathbb{H}^{N+1} \) be the image of \( H : M' \to \partial \mathbb{H}^{N+1} \) where \( M' \subset \mathbb{C}^{n+1} \) is a smooth strictly pseudoconvex CR-hypersurface. Then for any point in \( M \), the first-order adapted lift \( E = (E_0, E_\alpha, E_\mu, E_{N+1}) \) of \( M \) into \( SU(N+1,1) \) (hence into \( GL^Q(\mathbb{C}^{N+2}) \)) exists in some neighborhood of the point in \( M \).

**Proof:** **Step 1.** Without of loss of generality, we assume that \( 0 \in M \) so that it suffices to construct a lift \( E = (E_0, E_\alpha, E_\mu, E_{N+1}) \) in a neighborhood of the point 0. Here we denote \([1 : 0 : \ldots : 0]\) by 0.

Assume that \( M' \) is defined by the equation \( \text{Im} \, w = |z|^2 + o(|z|^2) \) in \( (z, w) \in \mathbb{C}^n \times \mathbb{C} \) where \( w = u + iv \). Assume that \( H = (1, f_\alpha, \phi_\mu, g) \) is the smooth CR embedding of \( M' \) into \( \partial \mathbb{H}^{N+1} \) with \( H(0) = 0 \) and
\[ f = z + O(|(z, w)|^2), \quad \phi = O(|(z, w)|^2), \quad g = w + O(|(z, w)|^2). \] (60)
Let \( L_\alpha, \alpha = 1, 2, \ldots, n \) be a basis of the CR vector fields and \( R \) is the Reeb vector field on \( M' \). Then as in (59) with (60), we have
\[ L_\alpha|_0 = \frac{\partial}{\partial z_j}|_0, \quad \text{and} \quad R|_0 = \frac{\partial}{\partial u}|_0. \] (61)
It follows that $\bar{L}_\alpha H = 0$ as $H$ is a CR map. By the Lewy extension theorem, $H$ extends holomorphically to one side of $M'$, denoted by $D$, where $D$ is obtained by attaching the holomorphic discs. By applying the maximum principle and the Hopf lemma to the sub-harmonic function $\sum |f_\alpha|^2 + \sum |\phi_\mu|^2 + \frac{i}{2} (g - \bar{g})$ on $D$, it follows that $\frac{\partial Im g}{\partial u}(0) \neq 0$. Since $\frac{\partial g}{\partial w} = 0$ and $\frac{\partial Im g}{\partial u}(0) = 0$, we have $Rg(0) = \frac{\partial g}{\partial u}(0) = \frac{\partial Im g}{\partial u}(0) \neq 0$.

**Step 2. Direct construction of $E_0, E_\alpha$ and $E_{N+1}$**

We define

$$E_0 := \begin{bmatrix} 1 \\ f_\alpha(z, w) \\ \phi_\mu(z, w) \\ g(z, w) \end{bmatrix}$$

which can be regarded as a point in $\partial \mathbb{H}^{N+1}$. Then $\langle E_0, E_0 \rangle = 0$ holds:

$$\sum f_\alpha \bar{f}_\alpha + \sum \phi_\mu \bar{\phi}_\mu + \frac{i}{2} (g - \bar{g}) = 0, \text{ on } M. \quad (63)$$

Apply the CR vector field $L_\beta$ to $E_0$, we define

$$\tilde{E}_\beta = (0, L_\beta f_\alpha, L_\beta \phi_\mu, L_\beta g)^t,$$

which form the basis of the complex tangent bundle $T^{1,0}_{\pi_0(E_0)}(M)$. Then in a neighborhood of 0 in $M$, we have

$$Span_C(E_0, \tilde{E}_\alpha) = \hat{T}^{(1,0)}_{\pi_0(E_0)}M.$$

Now, we have $\langle E_0, \tilde{E}_\alpha \rangle = 0$ by applying $L_\beta$ to $\langle E_0, E_0 \rangle = 0$:

$$\sum \bar{f}_\alpha L_\beta f_\alpha + \sum \bar{\phi}_\mu L_\beta \phi_\mu + \frac{i}{2} L_\beta g = 0. \quad (64)$$

By the Gram-Schmid orthonormalization procedure, we can obtain, from $\{\tilde{E}_\beta\}$, an orthonormal set with respect to the usual Hermitian inner product $\langle , \rangle_0$; we denote it by $\{E_\beta\}$. By the definition (62), we notice that for any $Z = (Z^0, Z^A, Z^{N+1})$ and $Z' = (Z^0', Z^A', Z'^{N+1})$,

$$\langle Z, Z' \rangle = \left\langle \left(\frac{i}{2} Z^{N+1}, Z^A, -\frac{i}{2} Z^0 \right), \left( Z^0', Z^A', Z'^{N+1} \right) \right\rangle_0 = \langle \hat{Z}, Z' \rangle_0, \quad (65)$$

where $\langle , \rangle_0$ is the usual Hermitian inner product and $\hat{Z} := (\frac{i}{2} Z^{N+1}, Z^A, -\frac{i}{2} Z^0)$. Then we see from (64) that

$$\langle E_0, E_\beta \rangle = \left\langle \left(\frac{i}{2} g, f_\alpha, \phi_\mu, -\frac{i}{2} \right), \left(0, L_\beta f_\alpha, L_\beta \phi_\mu, L_\beta g \right) \right\rangle_0 = 0.$$
Also we observe \( \langle E_\alpha, E_\beta \rangle = \langle E_\alpha, E_\beta \rangle_0 = \delta_{\alpha\beta} \). Then \( \langle E_0, E_0 \rangle = 0, \langle E_0, E_\beta \rangle = 0 \) and \( \langle E_\alpha, E_\beta \rangle = \delta_{\alpha\beta} \) hold.

Applying the Reeb vector field \( R \), we define another vector
\[
\tilde{E}_{n+1} := (0, R f_\alpha, R \phi, R g)^t
\]
over a neighborhood of 0 in \( M \) such that
\[
\text{span}(E_0, E_\alpha, \tilde{E}_{n+1}) = \hat{T}_{\pi_0(E_0)} M.
\]

We want to construct
\[
E_{n+1} = AE_0 + B_\alpha E_\alpha + C \tilde{E}_{n+1}
\]
such that
\[
\langle E_{n+1}, E_0 \rangle = \frac{i}{2}, \langle E_\alpha, E_{n+1} \rangle = 0, \text{ and } \langle E_{n+1}, E_{n+1} \rangle = 0.
\]

From \( \langle E_{n+1}, E_0 \rangle = \frac{i}{2} \), we get \( \langle AE_0 + B_\alpha E_\alpha + C \tilde{E}_{n+1}, E_0 \rangle = \frac{i}{2} \) so that
\[
C = \frac{i}{2 \langle \tilde{E}_{n+1}, E_0 \rangle}.
\] (66)

By (61), we notice that
\[
\langle \tilde{E}_{n+1}, E_0 \rangle |_0 = \sum \frac{\partial f_\alpha}{\partial u} |_0 \tilde{f}_\alpha(0) + \sum \frac{\partial \phi_\mu}{\partial u} |_0 \phi_\mu(0) + \frac{i}{2} \frac{\partial g}{\partial u} |_0
\]
and therefore \( \langle \tilde{E}_{n+1}, E_0 \rangle |_0 = \frac{i}{2} R g(0) \neq 0 \).

From \( \langle E_{n+1}, E_\alpha \rangle = 0 \), we get \( \langle AE_0 + B_\beta E_\beta + C \tilde{E}_{n+1}, E_\alpha \rangle = 0 \) so that
\[
B_\alpha = -C \delta_{\alpha \beta} \langle \tilde{E}_{n+1}, E_\beta \rangle = -C \langle \tilde{E}_{n+1}, E_\alpha \rangle.
\] (67)

From \( \langle E_{n+1}, E_{n+1} \rangle = 0 \), we get \( \langle AE_0 + B_\beta E_\beta + C \tilde{E}_{n+1}, AE_0 + B_\beta E_\beta + C \tilde{E}_{n+1} \rangle = 0 \).

Since \( C \langle \tilde{E}_{n+1}, E_0 \rangle = \frac{i}{2}, C \langle E_0, \tilde{E}_{n+1} \rangle = -\frac{i}{2}, B_\alpha = -C \langle \tilde{E}_{n+1}, E_\alpha \rangle \) and \( B_\alpha = -C \langle E_\alpha, \tilde{E}_{n+1} \rangle \) by (66) and (67), we obtain
\[
-\frac{i}{2} A + \frac{i}{2} A - \sum_\alpha |B_\alpha|^2 + |C|^2 \langle E_{n+1}, E_{n+1} \rangle = 0,
\]
so that
\[
\text{Im}(A) = \sum_\alpha |B_\alpha|^2 - |C|^2 \langle E_{n+1}, E_{n+1} \rangle.
\] (68)
Therefore $E_{N+1}$ is determined.

So far we have $\langle E_0, E_0 \rangle = \langle E_{N+1}, E_{N+1} \rangle = \langle E_0, E_0 \rangle = \langle E_{N+1}, E_{N+1} \rangle = 0$, $\langle E_0, E_0 \rangle = \delta_{\alpha\beta}$ and $\langle E_0, E_{N+1} \rangle = -\frac{i}{2}$ hold.

**Step 3. Construction of $E$** From Step 2, at the point 0, we have vectors

\[
E_0|_0 = [1 : 0 : \ldots : 0], \quad E_1|_0 = [0 : 1 : 0 : \ldots : 0], \ldots, E_n|_0 = [0 : 0 : \ldots : 1 : 0 : \ldots : 0],
\]

and

\[
E_{N+1}|_0 = [0 : 0 : \ldots : 0 : 1].
\]

Therefore we can define $E$ at the point 0 by

\[
E(0) := Id \in SU(N + 1, 1).
\]

For any other point $P$ in a small neighborhood of 0 in $M$, we are going to define $E(P) \in SU(N + 1, 1)$ as follows.

Write $H(p) = P$ for some $p \in M'$. Then we take a map $\Psi_P \in SU(N + 1, 1)$ such that

\[
\Psi_P^*(P) = 0, \quad T_0^{1,0}\Psi_0(M) = \text{span}_C(E_0|_0, E_0|_0), \quad \text{and} \quad T_0\Psi_0(M) = \text{span}(E_0|_0, E_0|_0, E_{N+1}|_0).
\]

where $E_0|_0, E_0|_0$ and $E_{N+1}|_0$ are defined in (59) and (70). The map $\Psi_P$ can be defined as $A_{F_1,\sigma,\alpha} \circ A_{\sigma,\beta}$ where $A_{\sigma,\beta} \in SU(N + 1, 1)$ as in (54) and $A_{F_1,\sigma,\alpha} \in SU(N + 1, 1)$ as in (55).

Notice in the construction of the normalization $F^{**}$ and $F^{***}$, we can always choose $\lambda = 1$ so that (50) can be used. $\Psi_P$ is smooth as $P$ varies. Then we define

\[
E(P) := (\Psi_P)^*E(0) = (\Psi_P)^{-1}E(0).
\]

This definition is the same as in (50). Since $\Psi_P$ is invariant for the Hermitian scalar product $\langle \cdot, \cdot \rangle$ defined in (12) and $E(0)$ satisfies the identities (13), it implies that $E(P)$ satisfies the identities (13), i.e., $E(p) \in SU(N + 1, 1)$.

As a matrix, we denote $E(P) = (\hat{E}_0, \hat{E}_a, \hat{E}_{N+1})$. Since the map $\Psi_P$ preserves the CR structures and the tangent vector spaces of $M$ and $\Psi_P(M)$, we have

\[
\text{span}_C(\hat{E}_0, \hat{E}_a) = \text{span}_C(E_0, E_a)|_P, \quad \text{span}(\hat{E}_0, \hat{E}_a, \hat{E}_{N+1}) = \text{span}(E_0, E_a, E_{N+1})|_P.
\]

where $E_0, E_a$ and $E_{N+1}$ are constructed in Step 2. We remark that we can replace $(\hat{E}_0, \hat{E}_a, \hat{E}_{N+1})$ by $(E_0, E_a, E_{N+1})$. □

**Existence of a more special first-order adapted lifts when $M$ is spherical** When $M = F(\mathbb{H}^{N+1})$ where $F \in Prop_2(\mathbb{H}^{N+1}, \mathbb{H}^{N+1})$, we can construct a more special first-order adapted lift of $M$ into $SU(N + 1, 1)$ as follows (cf. [HJY09]).
Let $F = (f, \phi, g) \in \text{Prop}_2(\partial \mathbb{H}^{n+1}, \partial \mathbb{H}^{N+1})$ be any map with $F = F_p^{**}$. Then $F(0) = 0$. We introduce a local biholomorphic map near the origin

$$F_{fg} := (f, g) : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \ (z, z_{N+1}) \mapsto (f, g) = (\hat{z}, \hat{z}_{N+1})$$

with its inverse

$$F_{fg}^{-1} : \mathbb{C}^{n+1} \to \mathbb{C}^{n+1}, \ (\hat{z}, \hat{z}_{N+1}) \mapsto ((F_{fg}^{-1})^{(1)}, \ldots, (F_{fg}^{-1})^{(n)}(F_{fg}^{-1})^{(N+1)}) = (z, z_{N+1}).$$

Here we use $(\hat{z}, \hat{z}_{N+1})$ as a coordinates system of $M = F(\partial \mathbb{H}^{n+1})$ near $F(0) = 0$. Denote $\text{Proj}_{fg} : \mathbb{C}^{N+1} \to \mathbb{C}^{n+1}, (\hat{z}, \hat{z}_\mu, \hat{z}_{N+1}) \mapsto (\hat{z}, \hat{z}_{N+1})$. Then we have $\text{Proj}_{fg} \circ F = F_{fg}$:

$$F : \partial \mathbb{H}^{n+1} \to M$$

$$\downarrow \ F_{fg} \downarrow \text{Proj}_{fg}$$

$$\mathbb{C}^{n+1}$$

We also have a pair of inverse maps $F : \partial \mathbb{H}^{n+1} \to M$ and $(F_{fg}^{-1}) \circ \text{Proj}_{fg} : M \to \partial \mathbb{H}^{n+1}$.

Locally we can regard $M$ as a graph: $F \circ F_{fg}^{-1} : \mathbb{C}^{n+1} \to M \subset \mathbb{C}^{N+2}$:

$$(\hat{z}, \hat{z}_{N+1}) \mapsto (\hat{z}, \phi((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1})), \hat{z}_{N+1})$$

Now let us define a lift of $M$ into $SU(N + 1, 1)$

$$e = (e_0, e_\alpha, e_\mu, e_{N+1}) \in SU(N + 1, 1), \ 1 \leq \alpha \leq n, \ n + 1 \leq \mu \leq N \quad (73)$$

as follows.

We define $e_0 : M \hookrightarrow \mathbb{C}^{N+2}$ be the inclusion:

$$e_0(\hat{z}, \hat{z}_{N+1}) = F \circ F_{fg}^{-1}(\hat{z}, \hat{z}_{N+1}) = \left[ 1 : \hat{z} : \phi((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1})) : \hat{z}_{N+1} \right]^t \quad (74)$$

$\forall (\hat{z}, \hat{z}_{N+1}) \in \mathbb{C}^{n+1}$. We define $e_\alpha : M \to \mathbb{C}^{N+2}$ for $1 \leq \alpha \leq n$:

$$e_\alpha := \frac{1}{\sqrt{|L_\alpha f|^2 + |L_\alpha \phi|^2}} [0 : L_\alpha f : L_\alpha \phi : L_\alpha g]^t \circ F_{fg}^{-1}. \quad (75)$$

where $L_\alpha = \frac{\partial}{\partial z^\alpha} + 2i z^\alpha \frac{\partial}{\partial z_{N+1}}$. By the definition (12), we have $\langle e_0, e_0 \rangle = 0$ because $f \cdot \overline{f} + \phi \cdot \overline{\phi} - \frac{1}{2}(g - \overline{g}) = \hat{z} \cdot \overline{\hat{z}} + \phi((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1}))\overline{\phi((F_{fg})^{-1}(\hat{z}, \hat{z}_{N+1}))} + \frac{i}{2}(\hat{z}_{N+1} - \overline{\hat{z}_{N+1}}) = 0$ holds on $\partial \mathbb{H}^{n+1}$, and $\langle e_\alpha, e_\alpha \rangle = 0$ because $L_\alpha f \cdot \overline{f} + L_\alpha \phi \cdot \overline{\phi} + \frac{i}{2} L_\alpha g = 0$ holds on $\partial \mathbb{H}^{n+1}$, and $\langle e_\alpha, e_\beta \rangle = \delta_{\alpha \beta}$ because $L_\alpha f \cdot L_\beta \overline{f} + L_\alpha \phi \cdot L_\beta \overline{\phi} = 0$ holds on $\partial \mathbb{H}^{n+1}$ for $\alpha \neq \beta$.  

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If we define \( \tilde{e}_{N+1} := (0, T f, T \phi, T g)^t \circ F^{-1}_f \) where \( T = \frac{\partial}{\partial u} \) with \( z^{N+1} = u + iv \), then
\[
\text{span}(e_0, e_\alpha, \tilde{e}_{N+1}) = \hat{T} \pi_0(e_0) M.
\]
We then find coefficient functions \( A, B, C \) such that
\[
e_{N+1} = A e_0 + \sum B_\alpha e_\alpha + C \tilde{e}_{N+1}
\]
satisfies
\[
\langle e_0, e_{N+1} \rangle = -\frac{i}{2}, \quad \langle e_\alpha, e_{N+1} \rangle = 0, \quad \langle e_{N+1}, e_{N+1} \rangle = 0.
\]

(76)

8 Relationship among four definitions of \( II_M \)

Lemma 8.1 Let \( H : M' \to \partial \mathbb{H}^{N+1} \) be a CR smooth embedding where \( M' \) is a strictly pseudoconvex smooth real hypersurface in \( \mathbb{C}^{n+1} \). We denote \( M = H(M') \). Then the following statements are equivalent:

(i) The CR second fundamental form \( II_M \) by Definition 1 identically vanishes.
(ii) The CR second fundamental form \( II_M \) by Definition 2 identically vanishes.
(iii) The CR second fundamental form \( II_M \) by Definition 3 identically vanishes.
(iv) The CR second fundamental form \( II_M \) by Definition 4 identically vanishes.

Proof (i) \( \iff \) (ii) by \([15]\).

(iii) \( \iff \) (iv) The equivalence follows by the facts that, for Definition 3 and 4, \( II_M^e \equiv 0 \) for one first-order adapted lift \( e \) if and only if \( II_M^s \equiv 0 \) for any first-order adapted lift \( s \), that a first-order adapted lift from \( M \) to \( SU(N+1,1) \) must be a first-order adapted lift from \( M \) to \( GL_Q(\mathbb{C}^{N+2}) \).

(iv) \( \implies \) (i): Let \( M \subset \partial \mathbb{H}^{N+1} \) be a \((2n+1)\) dimensional CR submanifold with CR dimension \( n \) that admits a first-order adapted lift \( e \) into \( SU(N+1,1) \). Consider the pull-backed Maurer-Cartan form over \( M \) by \( e \)
\[
\omega = \begin{pmatrix}
\omega_0^0 & \omega_0^\alpha & \omega_0^{N+1} \\
\omega_\alpha^0 & \omega_\alpha^\alpha & \omega_\alpha^{N+1} \\
0 & \omega_\beta^\mu & \omega_\beta^{N+1} \\
\omega_0^{N+1} & \omega_\beta^{N+1} & 0 & \omega_\beta^{N+1}
\end{pmatrix}.
\]
with
\[
\omega_0^0 + \omega_{N+1} = 0, \quad \omega_0^{N+1} = \omega_0^0, \quad \omega_\alpha^{N+1} = \omega_\alpha^\alpha, \quad \omega_{N+1} = \omega_{N+1}^\alpha.
\]
\[
\omega_A^{N+1} = 2i \omega_A^0, \quad \omega_{N+1}^A = -\frac{i}{2} \omega_A^0, \quad \omega_B^A + \omega_A^B = 0, \quad \omega_0^0 + \omega_A^A + \omega_{N+1}^N = 0,
\]
where \( 1 \leq A \leq N \).

(77)
Let \( \theta = \omega_0^{N+1} \) which is a real 1-form by (77). By \( d\omega = -\omega \wedge \omega \) and (77), we obtain
\[
d\theta = -\omega^{N+1}_0 \wedge \omega_0^\alpha - \omega_0^{N+1} \wedge \omega_0^N = 2i\omega_0^\alpha \wedge \overline{\omega}_0^\alpha - \theta \wedge (\omega_0^0 + \overline{\omega}_0^0) = i\theta^\alpha \wedge \overline{\theta}^\alpha,
\]
where we denote
\[
\theta^\alpha = \sqrt{2}\omega_0^\alpha + c^\alpha \theta
\]
for some functions \( c^\alpha \). Therefore, (8) holds and hence \( M \) is a strictly pseudoconvex pseudoholomorphic manifold with an admissible coframe \((\theta, \theta^\alpha)\). Hence Definition 4 of \( II_M \equiv 0 \) implies Definition 1 of \( III_M \equiv 0 \).

(i) \( \Rightarrow \) (iv): Definition 1 of \( III_M \) gives a coframe \((\theta, \theta^\alpha)\) which corresponds to Definition 2 of \( III_M \) with respect to a defining function \( \rho \) of \( M \) in \( \partial H^{N+1} \).

Now take a first-order adapted lift \( e \) from \( M \) into \( SU(N+1, 1) \). By (78), it corresponds to a coframe \((\theta, \theta^\alpha)\) on \( M \) and by (16), it corresponds Definition 2 of \( III_M \) by some choice of the defining function \( \hat{\rho} \) of \( M \) in \( \partial H^{N+1} \).

The above \( \rho \) and \( \hat{\rho} \) may not be the same. But Definition 2 of \( III_M \equiv 0 \) is independent of choice of defining functions, which gives (i) \( \Rightarrow \) (iv).  

9 Proof of Theorem 1.1

Lemma 9.1 (cf. [EHZ04], corollary 5.5) Let \( H : M' \to M \hookrightarrow \partial H^{N+1} \) be a smooth CR embedding of a strictly pseudoconvex smooth real hypersurface \( M \subset C^{n+1} \). Denote by \((\omega^\mu_{\alpha \beta})\) the CR second fundamental form matrix of \( H \) relative to an admissible coframe \((\theta, \theta^A)\) on \( \partial H^{N+1} \) adapted to \( M \). If \( \omega^\mu_{\alpha \beta} \equiv 0 \) for all \( \alpha, \beta \) and \( \mu \), then \( M' \) is locally CR-equivalent to \( \partial H^{n+1} \).

Proof of Theorem 1.1 Step 1. Reduction to a problem for geometric rank By Lemma 8.1 and Lemma 9.1 and the hypothesis that the CR second fundamental form identically vanishes, we know that \( M \) is locally CR equivalent to \( \partial H^{n+1} \).

Then \( M \) is the image of a local smooth CR map \( F : U \subset \partial H^{n+1} \to M \subset \partial H^{N+1} \) where \( U \) is an open set in \( \partial H^{n+1} \). By a result of Forstneric [Fo89], the map \( F \) must be a rational map. It suffices to prove that \( F \) is equivalent to a linear map. By Lemma 2.2 it is sufficient to prove that the geometric rank of \( F \) is zero: \( \kappa_0 = 0 \).

Suppose \( \kappa_0 > 0 \) and we seek a contradiction.

Step 2. Reduction to a lift of \((H \circ \tau_p^F)(M), 0\) Take any point \( p \in U \subset \partial H^{n+1} \) with \( \kappa_0 = \kappa_0(p) > 0 \), and consider the associated map (see Lemma 2.1)
\[
F_{p}^{***} = H \circ \tau_p^F \circ F \circ \sigma_p^0 \circ G : \partial H^{n+1} \to \partial H^{N+1}, \quad F_{p}^{***}(0) = 0,
\]
where $\sigma^0_p$ is defined in (1), $\tau^F_p$ is defined in (2), $G \in Aut_0(\mathbb{H}^{n+1})$ and $H \in Aut_0(\mathbb{H}^{N+1})$ are automorphisms. By Theorem 2.3, $F_{p}^{\ast\ast\ast} = (f, \phi, g)$ satisfies the following normalization conditions:

$$
\begin{align*}
\begin{cases}
  f_j = z_j + i \mu_j \frac{z_j w + o_{wt}(3)}{2}, & \frac{\partial^2 f_j}{\partial w^2}(0) = 0, \quad j = 1 \cdots, \kappa_0, \quad \mu_j > 0, \\
  f_j = z_j + o_{wt}(3), & j = \kappa_0 + 1, \cdots, n - 1 \\
  g = w + o_{wt}(4), \\
  \phi_{jl} = \mu_{jl} z_j z_l + o_{wt}(2), & (j, l) \in S \text{ with } \mu_{jl} > 0 \text{ for } (j, l) \in S_0 \\
  \mu_{jl} = 0 \text{ otherwise}
\end{cases}
\end{align*}
$$

(80)

where $\mu_{jl} = \sqrt{\mu_j + \mu_l}$ for $j, l \leq \kappa_0$ and $j \neq l$, $\mu_{jl} = \sqrt{\mu_j}$ if $j \leq \kappa_0$ and $l > \kappa_0$ or if $j = l \leq \kappa_0$.

Here the assumption that $\kappa_0 > 0$ is used.

From (79) we obtain

$$
\begin{align*}
\begin{array}{c}
(M, F(p)) & \xrightarrow{H \circ \tau^F_p} & (H \circ \tau^F_p(M), 0) \\
\uparrow F & & \uparrow F_{p}^{\ast\ast\ast} \\
(\partial \mathbb{H}^{n+1}, p) & \xleftarrow{\sigma^0_p \circ G} & (\partial \mathbb{H}^{n+1}, 0)
\end{array}
\end{align*}
$$

If we can show that there exists a first-order adapted lift $e$ from the submanifold $H \circ \tau^F_p(M)$ near 0 into $SU(N + 1, 1)$ such that the corresponding CR second fundamental form

$$
II^e_{H \circ \tau^F_p(M)} \neq 0 \text{ at } 0,
$$

(81)

then we obtain a first-order adapted lift $\tilde{e} := (H \circ \tau^F_p)^{-1} \circ e \circ H \circ \tau^F_p$ from the submanifold $M$ near $F(p)$ into $GL^Q(\mathbb{C}^{N+1})$ such that the corresponding CR second fundamental form

$$
II^{\tilde{e}}_M \neq 0 \text{ at } F(p).
$$

(82)

Notice that the map $H \circ \tau^F_p \in GL^Q(\mathbb{C}^{N+2})$ but $H \circ \tau^F_p \notin SU(N + 1, 1)$, so that the lift $\tilde{e}$ is not from $M$ into $SU(N + 1, 1)$. This is why we have to introduce Definition 3.

Since we take arbitrary $p \in \partial \mathbb{H}^{n+1}$, from (82) it concludes that $II_M \neq 0$, but this is a desired contradiction.

**Step 3. Calculation of the second fundamental form**

It remains to prove existence of the lift $e$ such that (81) holds.

The lift $e$ constructed in the second half of Section 7 is a first-order adapted lift from $H \circ \tau^F_p(M)$ near 0 into $SU(N + 1, 1)$ which defines a CR second fundamental form as a
tensor $II_{H_{0\tau F_p}(M)} = q^\mu_{\alpha\beta}\omega^\alpha\omega^\beta \otimes (e_\mu)$ in (49). If we can show $q^\mu_{\alpha\beta}(0) = \partial^2 \phi_\mu \partial z_\alpha \partial z_\beta|_0$, where $F_{\ast\ast\ast} = (f, \phi, g) = (f_\alpha, \phi_\mu, g)$. Since we assume that $\kappa_0 > 0$, by (80) and (83), it implies $q^\mu_{\alpha\beta}(0) \neq 0, \forall \alpha, \beta$ and $\mu$, i.e., $II_{H_{0\tau F_p}(M)} \neq 0$. This proves (81).

Let $E = (e_0, e_\alpha, \hat{E}_\mu, e_{N+1})$ be the lift constructed in Theorem 7.1 (see the remark at the end of the proof of Theorem 7.1) and in (74), (75) and (76). Since $E|_0 = Id$, we have $\omega|_0 = (E^{-1}|_0)(dE)|_0 = dE|_0$ so that

$$
\omega|_0 = \begin{bmatrix}
0 & * & ... & *
\end{bmatrix}
\begin{bmatrix}
dz_1 & * & ... & * \\
\vdots & \vdots & \ddots & \vdots \\
dz_n & * & ... & * \\
* & * & ... & * \\
\vdots & \vdots & \ddots & \vdots \\
* & * & ... & *
\end{bmatrix}.
$$

Hence $\omega^1|_0 = dz_1, \ldots, \omega^n|_0 = dz_n, \omega^{N+1}|_0 = dz_{N+1}$. Then by applying the chain rule, we obtain

$$
\omega^\mu_j|_0 = dE^\mu_j|_0 = d((L_j\phi_\mu) \circ (F_{fg})^{-1})|_0 = \partial \partial z_k (L_j\phi_\mu) \circ (F_{fg})^{-1})|_0 dz_k = \partial^2 \phi_\mu \partial z_k \partial z_j|_0 \omega^k|_0,
$$

for any $j, k \in \{1, 2, \ldots, n, N+1\}$, $n+1 \leq \mu \leq N$. Hence (83) is proved. The proof of Theorem 1.1 is complete. \(\square\)

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