Four-loop critical properties of polymerized membranes

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Abstract – We calculate four-loop order corrections to the critical exponent \( \eta \) in the two-field model of flat phase membranes. Obtained results show better agreement with the other calculation methods and confirm the validity of the perturbative approach to the considered problem.

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Introduction. – The description of many real-life systems can be reduced to the models of polymerized \( D \)-dimensional membranes in their low-temperature flat phase embedded in the \( d \)-dimensional Euclidean space. Using renormalization group methods, one can describe the flow of the theory between non-interacting UV-stable Gaussian fixed point and IR-attractive flat phase fixed point. Many different approaches describe the model near the IR fixed point, non-perturbative such as NPRG [1], self-consistent screening approximation (SCSA) [2–4], numerical SCSA [5,6] and Monte Carlo simulations [7–9]. Also, perturbative methods based on small epsilon expansion have been successfully applied to the problem starting from pioneering works [10,11] and recently extended to three-loop order in the series of papers [12–14].

The main object of interest is to estimate critical exponents controlling power-law behavior of the phonon-phonon and flexural-flexural correlation functions near the critical point.

The recent three-loop calculation [14] provides us with a new value of critical exponent very close to the results from other methods. Moreover, it demonstrates an apparent convergence of the perturbative series in the proper direction. The present paper aims to perform four-loop calculations and check perturbative series behavior for the critical exponent at the next loop order.

We carry out all our calculations in the two-field model described in detail in [11] with action given by

\[
S = \frac{1}{2} \int d^D x \left[ (\partial^2 h)^2 + 2\mu u_{ij}^2 + \lambda u_{ii}^2 \right],
\]

\[
u_{ij} = \frac{1}{2} (\partial_i u_j + \partial_j u_i + \partial_i h \cdot \partial_j h).
\]

Here we keep only the relevant part of \( u_{ij} \) and appropriately rescaled fields \( u, h \) and Lamé coefficients \( \lambda, \mu \). Our goal is to calculate renormalization constants of fields and couplings entering (1) as expansion about the upper critical dimension \( D_{uc} = 4 \) in a small parameter \( \varepsilon = 2 - D/2 \). Zeroes of beta functions derived from the renormalization constant provide us with the coordinates of the set of fixed points \( \{\mu^*, \lambda^*\} \) which after substitution into the field anomalous dimension lead to the final answer for the critical exponent of interest.

Our main result is a new value of \( \eta \) for the \( d_c = 1 \) case corresponding to the \( D = 2 \) membrane embedded into the \( (d = 3) \)-dimensional space at the four-loop order:

\[
\eta = 0.8670.
\]

In the following section, we provide details of our four-loop calculation and in the third section we make a comparison with available results and present analytical expressions for obtained results together with cross-checks.

Calculation details. – The possibility of highly non-trivial four-loop calculations described below is based on two essential features of the considered model:

- In all our calculations, we are allowed to consider massless diagrams only.
- Thanks to the Ward identities, all needed renormalization constants can be derived from two-point functions only.

Recent two-loop [12] and three-loop [14] calculations successfully adopted both of these facts. Table 1 provides us with the number of diagrams that need to be calculated.

To simplify notations, we use the model in such a form that fields and Lamé coefficients are normalized by \( \kappa \) and use simply \( \lambda \) and \( \mu \) for quantities divided by \( \kappa \).
up to the four-loop level, which is an order of magnitude higher than in the three-loop problem considered before [14]. Thus, our main improvement consists of applying modern tools of multi-loop calculations and enhanced renormalization strategy compared to [12, 14].

Feynman rules for the model (1) used in our calculation can be found in [15]. We generate all diagrams with DIANA [16] and calculate all massless integrals up to the four loops with FORCER [17]. Applying appropriate projectors we obtain bare results for the sum of one-particle irreducible (1PI) diagrams. In the case of the h-field the projector is trivial

\[ \Pi_{h,h} = \delta_{a,b} \Pi_{h,h}, \quad a, b = 1 \ldots d_c, \]  
(4)

with \( d_c = d - D \) but in the case of the u-field we split result of the calculation into transverse and longitudinal parts:

\[ \Pi_{u,u}^{\mu} = \left( \delta_{i,j} - \frac{Q_i \cdot Q_j}{Q^2} \right) \Pi_{u,u}^T + \frac{Q_i \cdot Q_j}{Q^2} \Pi_{u,u}^L, \]
(5)

where \( i, j = 1 \ldots D \). Bare quantities\(^2\) are related to renormalized ones by \( h_B = \sqrt{Z} h, u_B = Z u, \lambda_B = Z \lambda, \mu_B = Z \mu \).

Renormalizability of the theory allows us to greatly simplify renormalization of the theory and the calculation of constants \( Z, Z_u, Z_h \). Instead of an explicit account of diagrams with counterterms insertions or any other ways of subtraction of UV divergencies like BPHZ, it is possible to renormalize couplings in the bare result for the calculated Green’s functions and multiply it with an appropriate renormalization constant. A similar approach was pioneered in the three-loop QCD renormalization [18] and proved to be especially useful for renormalization of the Standard Model [19].

Our starting point is the calculation of the sum of bare one-loop diagrams up to the four-loop order denoted as \( \Pi_{h,h} \) for the h-field and \( \Pi_{u,u}^T, \Pi_{u,u}^L \) for transverse and longitudinal parts in the case of the u-field, respectively. Here it is important to keep the terms in \( \varepsilon \)-expansion up to the order \( \varepsilon^3-L \) in the calculation of the L-loop diagrams. The sum of 1PI diagrams with an appropriate tree term after replacement of bare couplings with renormalized ones and multiplication by the overall renormalization constant is finite and we have three equations to fix three renormalization constants \( Z, Z_u, Z_h \):

\[ Z^2 \left( \mu Z_u - \Pi_{u,u}^T (\mu B \rightarrow Z u, \lambda B \rightarrow Z \lambda) \right) = O(\varepsilon^0), \]
\[ Z^2 \left( 2 \mu Z_u + \lambda Z \Pi_{u,u}^L (\mu B \rightarrow Z u, \lambda B \rightarrow Z \lambda) \right) = O(\varepsilon^0), \]
\[ Z \left( 1 - \Pi_{h,h} (\mu B \rightarrow Z u, \lambda B \rightarrow Z \lambda) \right) = O(\varepsilon^0), \]
(6)

Due to the complicated dependence of \( Z_i \) on couplings \( \mu, \lambda \), it is useful to introduce the loop counting parameter and solve eq. (6) perturbatively order by order to get four-loop renormalization constants \( Z, Z_u, Z_h \) (see footnote\(^3\)).

Beta functions \( \beta_X = \frac{\partial X}{\partial \log m} \) and field anomalous dimension \( \gamma = \frac{\partial \log \mu}{\partial \log \mu} \) are defined as logarithmic derivatives in \( \overline{\text{MS}} \) scale parameter \( m \). From calculated \( Z, Z_u, Z_h \) we can find the beta functions:

\[ \beta_\mu = \frac{2 \varepsilon \partial \log \mu \log Z}{\det \begin{pmatrix} \partial \log \mu \log Z \\ \partial \log \mu \log Z \end{pmatrix}^{\mu} \lambda}, \]
(7)

\[ \beta_\lambda = \frac{2 \varepsilon \partial \log \mu \log Z}{\det \begin{pmatrix} \partial \log \mu \log Z \\ \partial \log \mu \log Z \end{pmatrix}^{\mu} \lambda}, \]
(8)

and the field anomalous dimension

\[ \gamma = \beta_\mu \partial \log Z + \beta_\lambda \partial \log \mu Z. \]
(9)

The absence of \( \varepsilon \) poles in (7), (8) and (9) is a strong indication of the validity of the obtained results.

\textbf{Results.} – From the set of equations \( \beta_\mu (\mu^*, \lambda^*) = 0, \beta_\lambda (\mu^*, \lambda^*) = 0 \) we have found four different solutions \( (\mu^*, \lambda^*) \) corresponding to four different fixed points. We adopt the same notation as in [11], where the point \( P_1 \) is the Gaussian one, and \( P_2 \) is the IR attractive one we are interested in. In addition, we consider the unstable fixed point \( P_3 \), since it allows comparing the result of the calculations with 1\( \mu_c \) results available in the literature. To fix the notation and simplify comparison with [12, 14], we provide one-loop coordinates of the points \( P_3 \) and \( P_4 \):

\[ \mu_3^* = \frac{12}{20 + d_c} \varepsilon + O(\varepsilon^2), \quad \lambda_3^* = -\frac{6}{20 + d_c} \varepsilon + O(\varepsilon^2), \]
(10)

\[ \mu_4^* = \frac{12}{24 + d_c} \varepsilon + O(\varepsilon^2), \quad \lambda_4^* = -\frac{4}{24 + d_c} \varepsilon + O(\varepsilon^2). \]
(11)

Substituting four-loop results for fixed-point coordinates into four-loop field anomalous dimension (10), we obtain critical exponents \( \eta = \gamma (\mu^*, \lambda^*) \) for two selected fixed points.

Full analytic four-loop results for the critical exponents \( \eta_3 \) (fixed point \( P_3 \)) and \( \eta_4 \) (fixed point \( P_4 \)) can be found in the appendix in eq. (A.1) and eq. (A.2) respectively. For the case \( d_c = 1 \) our result reads

\[ \eta_3 = 0.95 \varepsilon - 0.071 \varepsilon^2 - 0.069 \varepsilon^3 - 0.075 \varepsilon^4 + O(\varepsilon^5), \]
(12)

\[ \eta_4 = 0.96 \varepsilon - 0.0461 \varepsilon^2 - 0.0267 \varepsilon^3 - 0.02 \varepsilon^4 + O(\varepsilon^5). \]
(13)

\footnote{\textsuperscript{3}All results are available in a computer-readable form in the supplementary files README.dms, blam41.dms, bm41.dms, et41.dms, fp3subs4l.dms, and fp4subs4l.dms.}
\[ \eta_3 = \frac{20\varepsilon}{[20 + d_c]} + \left( \frac{2800}{[20 + d_c]^3} + \frac{1060}{3[20 + d_c]^2} - \frac{74}{3[20 + d_c]} \right) \varepsilon^2 + \left( \frac{784000}{[20 + d_c]^5} - \frac{40(615553 - 591624\zeta_3)}{27[20 + d_c]^4} + \frac{2(1024193 - 1006344\zeta_3)}{27[20 + d_c]^3} - \frac{2(17105 - 20736\zeta_3)}{27[20 + d_c]^2} - \frac{155}{9[20 + d_c]} \right) \varepsilon^3 + \left( \frac{274400000}{[20 + d_c]^7} - \frac{2800(648943 - 591624\zeta_3)}{27[20 + d_c]^6} - \frac{40(63897618439 + 174575927736\zeta_3 - 26395162880\zeta_3)}{243[20 + d_c]^5} \right) \varepsilon^4 + O(\varepsilon^5), \] (A.1)

\[ \eta_4 = \frac{24\varepsilon}{[24 + d_c]} + \left( \frac{2880}{[24 + d_c]^3} + \frac{456}{[24 + d_c]^2} - \frac{24}{24 + d_c} \right) \varepsilon^2 + \left( \frac{691200}{[24 + d_c]^5} - \frac{576(234137 - 192096\zeta_3)}{125[24 + d_c]^4} + \frac{8(1031777 - 923616\zeta_3)}{125[24 + d_c]^3} - \frac{4(39029 - 86832\zeta_3)}{375[24 + d_c]^2} - \frac{64}{3[24 + d_c]} \right) \varepsilon^3 + \left( \frac{207360000}{[24 + d_c]^7} - \frac{165888(20501 - 1600\zeta_3)}{5[24 + d_c]^6} - \frac{32(1174399340197 + 3188610294336\zeta_3 - 4827670269120\zeta_5)}{1875[24 + d_c]^5} \right) \varepsilon^4 + O(\varepsilon^5). \] (A.2)

The three-loop parts of (12) and (13) coincide with result of three-loop calculation [14] and the four-loop term is new. For the important case \( D = 2 \), substituting \( \varepsilon = 1 \) into (12) and (13) we obtain our final result for the critical exponent \( \eta \). Comparison with the earlier two-loop [12] and three-loop [14] calculations and also with the non-perturbative results obtained with the NPRG technique [1], SCSA [2] and from MC simulation [9] are summarized in table 2. The obtained result provides additional support for apparent convergence of the series for \( \eta \) near \( D = 2 \) without any additional resummation. As the trivial exercise, we also construct a [2/2] Padé approximant for the 4-loop result \( \eta_{(2/2)} = 0.806 \), which provides us with a slightly different result, and indicates the possible need for more careful resummation of the series. The growth of the last expansion term in (12) compared to the three-loop one implies the possibility of the divergence of the series and motivates for further more careful resummation of obtained series.

Another check of the validity of the obtained perturbative series stems from the comparison with known \( 1/d_c \) expansions in the vicinity of the chosen critical points. Our main interest is the expression for \( \eta_4 \) but the result for \( \eta_3 \) is also important, since its leading order expansion in \( 1/d_c \) can also be verified with results available in the literature.

In [2], the following result for the leading term in \( 1/d_c \) expansion corresponding to the fixed point \( P_4 \) was derived:

\[ \eta(D, d_c) = \frac{8}{d_c} \frac{D - 1}{2} \frac{\Gamma(D)}{\Gamma(D/2)^2} \frac{\Gamma(2 - D/2)}{\Gamma(2 - D/2)} + O\left(\frac{1}{d_c^2}\right). \] (14)

After expansion in \( \varepsilon = 2 - D/2 \) up to the \( \varepsilon^4 \), it perfectly matches the leading term of (18) expansion in \( 1/d_c \) and
reads

\[ \eta_4 = \frac{1}{d_c} \left( 24\varepsilon - 24\varepsilon^2 - \frac{64}{3}\varepsilon^3 \right) \]
\[ - \frac{16}{9} (13 - 27\zeta_3)\varepsilon^4 + \mathcal{O}(\varepsilon^5) \] + \mathcal{O}\left( \frac{1}{d_c^2} \right). \quad (15) \]

For the fixed point \( P_3 \) according to [4] the leading term of the \( 1/d_c \) expansion is determined by \( \eta \left( \frac{d_c^{D-2}(D+1)}{D+2}d_c \right) \), and after expansion in \( \varepsilon \) becomes equal to the \( 1/d_c \) expansion of (17):

\[ \eta_3 = \frac{1}{d_c} \left( 20\varepsilon - \frac{74}{3}\varepsilon^2 - \frac{155}{9}\varepsilon^3 \right) \]
\[ - \frac{1}{54} (769 - 2160\zeta_3)\varepsilon^4 + \mathcal{O}(\varepsilon^5) \] + \mathcal{O}\left( \frac{1}{d_c^2} \right). \quad (16) \]

**Conclusion.** — We have calculated four-loop beta functions and field anomalous dimension in the two-field model of polymerized membranes. From the obtained perturbative results, we have found a set of fixed points and derived the value of the critical exponent \( \eta \) for the \( D = 2 \) case. The obtained results are in good agreement with other calculation methods and demonstrate apparent convergence of the perturbative series. The validity of the result is confirmed by comparison with known results in the \( 1/d_c \) expansion.

**Appendix: analytical four-loop results for critical exponents.** — Here we provide explicit four-loop results:

\[ \text{see eqs. (A.1) and (A.2) on top of the previous page} \]

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