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SPIN 1/2 PARTICLE IN THE FIELD OF THE DIRAC STRING ON THE BACKGROUND OF ANTI DE SITTER SPACE–TIME
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Abstract

The Dirac monopole string is specified for anti de Sitter cosmological model. Dirac equation for spin 1/2 particle in presence of this monopole has been examined on the background of anti de Sitter space-time in static coordinates. Instead of spinor monopole harmonics, the technique of Wigner $D$-functions is used. After separation of the variables radial equations have been solved exactly in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\epsilon,j,m,\lambda}(t,r,\theta,\phi)$ has been constructed, the most attention is given to treating the states of minimal values for total moment quantum number $j_{min}$. At all values of $j$, the energy spectrum is discrete.

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1 Introduction

De Sitter and anti de Sitter geometrical models are given steady attention in the context of developing quantum theory in a curved space-time – for instance, see in [1]. In particular, the problem of description of the particles with different spins on these curved backgrounds has a long history – see [2–34]. Here we will be interested mostly in treating the Dirac equation in de Sitter model. In the present paper, the influence of the Dirac monopole string on the spin 1/2 particle in the anti de Sitter cosmological model is investigated. Instead of spinor monopole harmonics, the technique of Wigner $D$-functions is used. After separation of the variables radial equation have been solved exactly in terms of hypergeometric functions. The complete set of spinor wave solutions $\Psi_{\epsilon,j,m,\lambda}(t,r,\theta,\phi)$ has been constructed. Special attention is given to treating the states of minimal values for total moment quantum number $j_{min}$, these states turn to be much more complicated than in the flat Minkowski space. At all values of $j$, the energy spectrum is discrete.

2 Dirac particle in the anti de Sitter space

The Dirac equation (the notation according to [39] is used)

$$\left[ i\gamma^c (\epsilon^\alpha_{(c)} \partial_\alpha + \frac{1}{2} \sigma^{ab} \gamma_{abc}) - M \right] \Psi = 0$$

(1)

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in static coordinates and tetrad of the anti de Sitter space-time

\[ ds^2 = \Phi \, dt^2 - \frac{dr^2}{\Phi} - r^2(d\theta^2 + \sin^2 \theta d\phi^2) , \quad \Phi = 1 + r^2 , \]

\[ e^{\alpha}_{(0)} = \left( \frac{1}{\sqrt{\Phi}}, 0, 0, 0 \right) , \quad e^{\alpha}_{(3)} = \left( 0, \sqrt{\Phi}, 0, 0 \right) , \]

\[ e^{\alpha}_{(1)} = \left( 0, 0, \frac{1}{r}, 0 \right) , \quad e^{\alpha}_{(2)} = \left( 1, 0, 0, \frac{1}{r \sin \theta} \right) , \]

\[ \gamma^{030} = \frac{\Phi'}{2\sqrt{\Phi}} \, , \quad \gamma^{311} = \sqrt{\Phi} \frac{1}{r} \, , \quad \gamma^{322} = \sqrt{\Phi} \frac{1}{r} \, , \quad \gamma^{122} = \frac{\cos \theta}{r \sin \theta} \] , \tag{2}

takes the form

\[ \left[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \left( \gamma^3 \partial_r + \frac{\gamma^1 \sigma^{31} + \gamma^2 \sigma^{32}}{r} + \frac{\Phi'}{2\Phi \gamma^0 \sigma^{03}} \right) + \frac{1}{r} \Sigma_{\theta,\phi} - M \right] \Psi(x) = 0 \] , \tag{3}

where

\[ \Sigma_{\theta,\phi} = i \gamma^1 \partial_\theta + \gamma^2 i \partial + i \sigma^{12} \cos \theta \sin \theta . \]

Eq. (3) reads

\[ \left[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \left( \partial_r + \frac{1}{r} + \frac{\Phi'}{4\Phi} \right) + \frac{1}{r} \Sigma_{\theta,\phi} - M \right] \Psi(x) = 0 \] . \tag{4}

From (4), with the substitution \( \Psi(x) = r^{-1/4} \Phi^{-1/4} \psi(x) \), we get

\[ \left( i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \partial_r + \frac{1}{r} \Sigma_{\theta,\phi} - M \right) \psi(x) = 0 . \] \tag{5}

Below the spinor basis will be used

\[ \gamma^0 = \begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix} \, , \quad \gamma^j = \begin{bmatrix} 0 & -\sigma_j \\ \sigma_j & 0 \end{bmatrix} \, , \quad i \sigma^{12} = \begin{bmatrix} \sigma_3 & 0 \\ 0 & \sigma_3 \end{bmatrix} \, . \]

### 3  Separation of the variables

Let us start with the monopole Abelian potential in the Schwinger’s form \[35\] in flat Minkowski space

\[ A^a(x) = (A^0, A^i) = \left( 0, g \frac{(\vec{r} \times \vec{n})}{r} \left( \frac{\vec{r} \times \vec{n}}{r^2 - (\vec{r} \times \vec{n})^2} \right) \right) . \] \tag{6}

Specifying \( \vec{n} = (0, 0, 1) \) and translating the \( A_\alpha \) to the spherical coordinates, we get

\[ A_0 = 0 \, , \quad A_r = 0 \, , \quad A_\theta = 0 \, , \quad A_\phi = g \cos \theta . \] \tag{7}
It is easily verified that this potential \( A_\phi \) obeys Maxwell equations in anti de Sitter space

\[
\frac{1}{\sqrt{-g}} \frac{\partial}{\partial x^\alpha} \sqrt{-g} F^{\alpha \beta} = 0, \quad \sqrt{-g} = r^2 \sin \theta ,
\]

\[
F_{\phi \theta} = g \sin \theta , \quad \frac{1}{r^2 \sin \theta} \frac{\partial}{\partial \theta} r^2 \sin \theta \frac{1}{r^2 \sin^2 \theta} g \sin \theta = 0 . \quad (8)
\]

Correspondingly, the Dirac equation in this electromagnetic field takes the form

\[
\left[ i \frac{\gamma^0}{\sqrt{\Phi}} \partial_t + i \sqrt{\Phi} \gamma^3 \partial_r + \frac{1}{r} \Sigma_{\theta, \phi}^k - M \right] \psi(x) = 0 , \quad (9)
\]

where

\[
\Sigma_{\theta, \phi}^k = i \gamma^1 \partial_\theta + \gamma^2 \frac{i \partial_\phi + (i \sigma^{12} - k) \cos \theta}{\sin \theta} , \quad (10)
\]

and \( k \equiv eg/hc \). As readily verified, the wave operator in (9) commutes with the following three ones

\[
J_1^k = l_1 + \frac{(i \sigma^{12} - k) \cos \phi}{\sin \theta} , \quad J_2^k = l_2 + \frac{(i \sigma^{12} - k) \sin \phi}{\sin \theta} , \quad J_3^k = l_3 , \quad (11)
\]

which obey the \( \mathfrak{su}(2) \) Lie algebra. Clearly, this monopole situation come entirely under the Schrödinger [36] and Pauli [37] approach (detailed treatment of the method was given in [40]). Correspondingly to diagonalizing the \( J_2^k \) and \( J_3^k \), the function \( \psi \) is to be taken as \( (D_{\sigma} \equiv D_{-m, \sigma}(\phi, \theta, 0) \) stands for Wigner functions [38])

\[
\psi^k_{\epsilon jm}(t, r, \theta, \phi) = e^{-i ct} \begin{vmatrix}
  f_1 & D_{k-1/2} \\
  f_2 & D_{k+1/2} \\
  f_3 & D_{k-1/2} \\
  f_4 & D_{k+1/2}
\end{vmatrix} . \quad (12)
\]

Further, with the help of recursive relations [38]

\[
\partial_\theta D_{k+1/2} = a D_{k-1/2} - b D_{k+3/2} , \quad \partial_\theta D_{k-1/2} = c D_{k-3/2} - a D_{k+1/2} ,
\]

\[
\sin^{-1} \theta \left[ -m - (k + 1/2) \cos \theta \right] D_{k+1/2} = (-a D_{k-1/2} - b D_{k+3/2}) ,
\]

\[
\sin^{-1} \theta \left[ -m - (k - 1/2) \cos \theta \right] D_{k-1/2} = (-c D_{k-3/2} - a D_{k+1/2}) ,
\]

\[
b = \frac{\sqrt{(j - k - 1/2)(j + k + 3/2)}}{2} ,
\]

\[
c = \frac{\sqrt{(j + k - 1/2)(j - k + 3/2)}}{2} ,
\]

\[
a = \frac{1}{2} \sqrt{(j + 1/2)^2 - k^2}.
\]
we find how the $\Sigma_{\theta,\phi}^k$ acts on $\psi_{\epsilon jm}^k$

$$\Sigma_{\theta,\phi}^k \psi_{\epsilon jm}^k = i \sqrt{(j + 1/2)^2 - k^2} e^{-i\epsilon t} \begin{vmatrix} -f_4 & D_{k-1/2} \\ f_3 & D_{k+1/2} \\ +f_2 & D_{k-1/2} \\ -f_1 & D_{k+1/2} \end{vmatrix};$$  \hspace{1cm} (13)

hereafter the factor $\sqrt{(j + 1/2)^2 - k^2}$ will be denoted by $\nu$. For the $f_i(r)$, the radial system derived is

$$\frac{\epsilon}{\sqrt{\Phi}} f_3 - i \frac{\epsilon}{\sqrt{\Phi}} \frac{d}{dr} f_3 - i \frac{\nu}{r} f_4 - M f_1 = 0,$$
$$\frac{\epsilon}{\sqrt{\Phi}} f_4 + i \frac{\epsilon}{\sqrt{\Phi}} \frac{d}{dr} f_4 + i \frac{\nu}{r} f_3 - M f_2 = 0,$$
$$\frac{\epsilon}{\sqrt{\Phi}} f_1 + i \frac{\epsilon}{\sqrt{\Phi}} \frac{d}{dr} f_1 + i \frac{\nu}{r} f_2 - M f_3 = 0,$$
$$\frac{\epsilon}{\sqrt{\Phi}} f_2 - i \frac{\epsilon}{\sqrt{\Phi}} \frac{d}{dr} f_2 - i \frac{\nu}{r} f_1 - M f_4 = 0. \hspace{1cm} (14)$$

Else one operator can be diagonalized together with $i\partial_t, \vec{J}_k^2, J_k^3$; namely, a generalized Dirac operator

$$\hat{K}^k = -i \gamma^0 \gamma^3 \Sigma_{\theta,\phi}^k. \hspace{1cm} (15)$$

From the equation $\hat{K}^k \psi_{\epsilon jm} = \lambda \psi_{\epsilon jm}$ we find two possible eigenvalues and restrictions on $f_i(r)$

$$f_4 = \delta f_1, \hspace{0.5cm} f_3 = \delta f_2, \hspace{0.5cm} \lambda = -\delta \sqrt{(j + 1/2)^2 - k^2}. \hspace{1cm} (16)$$

Correspondingly, the system (14) reduces to

$$\left( \sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r} \right) f + \left( \frac{\epsilon}{\sqrt{\Phi}} + \delta M \right) g = 0, \hspace{0.5cm} (17)$$

$$\left( \sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r} \right) g - \left( \frac{\epsilon}{\sqrt{\Phi}} - \delta M \right) f = 0,$$

where

$$f = \frac{f_1 + f_2}{\sqrt{2}}, \hspace{0.5cm} g = \frac{f_1 - f_2}{\sqrt{2}i}.$$

It is known that quantization of $k = eg/hc$ and $j$ is given by

$$eg/hc = \pm 1/2, \hspace{0.1cm} \pm 1, \hspace{0.1cm} \pm 3/2, \ldots$$

$$j = |k| -1/2, \hspace{0.1cm} |k| +1/2, \hspace{0.1cm} |k| +3/2, \ldots \hspace{1cm} (18)$$

The case of minimal value $j_{min} = |k| -1/2$ must be treated separately in a special way. For example, let $k = +1/2$, then to the minimal value $j = 0$ there corresponds the wave function in terms of only $(t, r)$-dependent quantities

$$\psi_{\epsilon jm}^{(j=0)}(x) = e^{-i\epsilon t} \begin{vmatrix} f_1(r) \\ 0 \\ f_3(r) \\ 0 \end{vmatrix}. \hspace{1cm} (19)$$
At $k = -1/2$, we have

$$\psi_{k=-1/2}^{(j=0)}(x) = e^{-i ct} \left| \begin{array}{c} 0 \\ f_2(r) \\ 0 \\ f_4(r) \end{array} \right|. \quad (20)$$

Thus, if $k = \pm 1/2$, then to the minimal values $j_{\min}$ there correspond the function substitutions which do not depend at all on the angular variables $(\theta, \phi)$; at this point there exists some formal analogy between these electron-monopole states and $S$-states (with $l = 0$) for a boson field of spin zero: $\Phi_{l=0} = \Phi(r, t)$. However, it would be unwise to attach too much significance to this formal similarity because that $(\theta, \phi)$-independence of $(e - g)$-states is not a fact invariant under tetrad gauge transformations.

In contrast, the relation below (let $k = +1/2$)

$$\Sigma_{\theta, \phi}^{+1/2} \psi_{k=+1/2}^{(j=0)}(x) = \gamma^2 \cot \theta (i \sigma^{12} - 1/2) \psi_{k=+1/2}^{(j=0)} \equiv 0 \quad (21)$$

is invariant under arbitrary tetrad gauge transformations. Correspondingly, the matter equation (9) takes on the form

$$\left( i \frac{\gamma^0}{\sqrt{\Phi}} \frac{\partial}{\partial t} + i \gamma^3 \frac{\partial}{\partial r} - M \right) \psi^{(j=0)} = 0. \quad (22)$$

It is readily verified that both (19) and (20) representations are directly extended to $(e - g)$-states with $j = j_{\min}$ at all the other $k = \pm 1, \pm 3/2, \ldots$. Indeed,

$$k = +1, +3/2, +2, \ldots, \quad \psi_{j_{\min}^{k>0}}^{k>0}(x) = e^{-i ct} \left| \begin{array}{c} f_1(r) D_{k-1/2} \\ 0 \\ f_3(r) D_{k-1/2} \\ 0 \end{array} \right|; \quad (23)$$

$$k = -1, -3/2, -2, \ldots, \quad \psi_{j_{\min}^{k<0}}^{k<0}(x) = e^{-i ct} \left| \begin{array}{c} 0 \\ f_2(r) D_{k+1/2} \\ 0 \\ f_4(r) D_{k+1/2} \end{array} \right|, \quad (24)$$

and the relation $\Sigma_{\theta, \phi} \psi_{j_{\min}} = 0$ still holds. For instance, let us consider in more detail the case of positive $k$. Using the recursive relations

$$\partial_\theta D_{k-1/2} = \frac{1}{2} \sqrt{2k - 1} D_{k-3/2},$$

$$\sin^{-1} \theta \left[ -m - (k - 1/2) \cos \theta \right] D_{k-1/2} = -\frac{1}{2} \sqrt{2k - 1} D_{k-3/2},$$

we get

$$i \gamma^1 \partial_\theta \left| \begin{array}{c} f_1(r) D_{k-1/2} \\ 0 \\ f_3(r) D_{k-1/2} \\ 0 \end{array} \right| = \frac{i}{2} \sqrt{2k - 1} \left| \begin{array}{c} 0 \\ -f_3(r) D_{k-3/2} \\ 0 \\ +f_1(r) D_{k-3/2} \end{array} \right|,$$

$$\gamma^2 \frac{i \partial_\theta + (i \sigma^{12} - k) \cos \theta}{\sin \theta} \left| \begin{array}{c} f_1(r) D_{k-1/2} \\ 0 \\ f_3(r) D_{k-1/2} \\ 0 \end{array} \right| = \frac{i}{2} \sqrt{2k - 1} \left| \begin{array}{c} 0 \\ +f_3(r) D_{k-3/2} \\ 0 \\ -f_1(r) D_{k-3/2} \end{array} \right|. \quad (20)$$
in a sequence, the identity $\Sigma_{\theta, \phi} \psi_{j_{\min}} \equiv 0$ holds. The case of negative $k$ can be considered in the same way. Thus, at every $k$, the $j_{\min}$-state equation has the same unique form

$$
\left( i \frac{\gamma^0}{\sqrt{\Phi}} \frac{\partial}{\partial t} + i \gamma^3 \sqrt{\Phi} \frac{\partial}{\partial r} - M \right) \psi_{j_{\min}} = 0 ;
$$

which leads to the same unique radial system

$k = +1/2, +1, \ldots$

$$
\begin{align*}
\frac{e}{\sqrt{\Phi}} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - M f_1 &= 0 , \\
\frac{e}{\sqrt{\Phi}} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 - M f_3 &= 0 ;
\end{align*}
$$

$k = -1/2, -1, \ldots$

$$
\begin{align*}
\frac{e}{\sqrt{\Phi}} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 - M f_2 &= 0 , \\
\frac{e}{\sqrt{\Phi}} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - M f_4 &= 0 .
\end{align*}
$$

In the limit of flat space–time, these equations are equivalent respectively to $k = +1/2, +1, \ldots$

$$
\left( \frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_1 = 0 , \quad f_3 = \frac{1}{m} \left( \epsilon + i \frac{d}{dr} \right) f_1 ;
$$

$k = -1/2, -1, \ldots$

$$
\left( \frac{d^2}{dr^2} + \epsilon^2 - m^2 \right) f_4 = 0 , \quad f_2 = \frac{1}{m} \left( \epsilon + i \frac{d}{dr} \right) f_4 .
$$

These equation both lead us to the functions $f = \exp(\pm \sqrt{m^2 - \epsilon^2} r)$. In particular, at $\epsilon < m$, we have a solution

$$
\exp \left( - \sqrt{m^2 - \epsilon^2} r \right) ,
$$

which seems to be appropriate to describe bound states in the electron-monopole system.

4 Solution of the radial equations

Let us turn back to the system (17) and (for definiteness) consider equations at $\delta = +1$ (formally the second case $\delta = -1$ corresponds to the change $M \rightarrow -M$)

$$
\begin{align*}
(\sqrt{\Phi} \frac{d}{dr} + \frac{\nu}{r} ) f + (\frac{e}{\sqrt{\Phi}} + M) g &= 0 , \\
(\sqrt{\Phi} \frac{d}{dr} - \frac{\nu}{r} ) g - (\frac{e}{\sqrt{\Phi}} - M) f &= 0 .
\end{align*}
$$
Here we see additional singularities at the points
\[ \epsilon + \sqrt{\Phi} M = 0 \quad \text{or} \quad \epsilon - \sqrt{\Phi} M = 0 . \]

For instance, the equation for \( f(r) \) has the form
\[
\frac{d^2}{dr^2} f + \left( \frac{2r}{1 + r^2} - \frac{Mr}{\sqrt{1 + r^2}(\epsilon + M\sqrt{1 + r^2})} \right) \frac{d}{dr} f + \left( \frac{\epsilon^2}{(1 + r^2)^2} - \frac{M^2}{1 + r^2} \right.
\]
\[
- \frac{\nu^2}{r^2(1 + r^2)} - \frac{\nu}{r^2(1 + r^2)^{3/2}} - \frac{M\nu}{(1 + r^2)(\epsilon + M\sqrt{1 + r^2})} \bigg) f = 0 .
\]

However, there exists possibility to move these singularities away through a special transformation of the functions \( f(r), g(r) \) \[24\]. To this end, let us introduce a new variable \( r = \sinh \rho \), eqs. \[31\] look simpler
\[
\frac{d}{d\rho} + \frac{\nu}{\sinh \rho} f + \left( \frac{\epsilon}{\cosh \rho} + M \right) g = 0 ,
\]
\[
\frac{d}{d\rho} - \frac{\nu}{\sinh \rho} g - \left( \frac{\epsilon}{\cosh \rho} - M \right) f = 0 .
\]

Summing and subtracting two last equations, we get
\[
\frac{d}{d\rho}(f + g) + \frac{\nu}{\sinh \rho}(f - g) - \frac{\epsilon}{\cosh \rho}(f - g) + M(f + g) = 0 ,
\]
\[
\frac{d}{d\rho}(f - g) + \frac{\nu}{\sinh \rho}(f + g) + \frac{\epsilon}{\cosh \rho}(f + g) - M(f - g) = 0 .
\]

Introducing two new functions
\[
f + g = e^{-\rho/2}(F + G) , \quad f - g = e^{+\rho/2}(F - G) ,
\]

or in matrix form
\[
\begin{pmatrix} G \\ H \end{pmatrix} = \begin{pmatrix} \cosh \rho/2 & -\sinh \rho/2 \\ -\sinh \rho/2 & \cosh \rho/2 \end{pmatrix} \begin{pmatrix} g \\ h \end{pmatrix} ,
\]

where (see definition of the variable \( z \) below)
\[
\cosh \frac{\rho}{2} = \sqrt{1 - \frac{z + 1}{2}} , \quad \sinh \frac{\rho}{2} = \sqrt{1 - \frac{z - 1}{2}} ,
\]

one transforms \[33\] into
\[
\frac{d}{d\rho} e^{-\rho/2}(F + G) + \frac{\nu}{\sinh \rho} e^{+\rho/2}(F - G)
\]
\[
- \frac{\epsilon}{\cosh \rho} e^{+\rho/2}(F - G) + M e^{-\rho/2}(F + G) = 0 ,
\]
\[
\frac{d}{d\rho} e^{+\rho/2}(F - G) + \frac{\nu}{\sinh \rho} e^{-\rho/2}(F + G)
\]
\[
+ \frac{\epsilon}{\cosh \rho} e^{-\rho/2}(F + G) - M e^{+\rho/2}(F - G) = 0 ,
\]
\[
\frac{d}{d\rho}(F + G) - \frac{1}{2}(F + G) + \frac{\nu}{\sinh \rho} (\cosh \rho + \sinh \rho)(F - G)
- \frac{\epsilon}{\cosh \rho} (\cosh \rho + \sinh \rho)(F - G) + M(F + G) = 0 ,
\]

\[
\frac{d}{d\rho}(F - G) + \frac{1}{2}(F - G) + \frac{\nu}{\sinh \rho} (\cosh \rho - \sinh \rho)(F + G)
+ \frac{\epsilon}{\cosh \rho} (\cosh \rho - \sinh \rho)(F + G) - M(F - G) = 0 .
\]

Now summing and subtracting two last equations, we obtain

\[
\left( \frac{d}{d\rho} + \nu \frac{\cosh \rho}{\sinh \rho} - \epsilon \frac{\sinh \rho}{\cosh \rho} \right) F + \left( \epsilon + M - \nu - \frac{1}{2} \right) G = 0 ,
\]

\[
\left( \frac{d}{d\rho} - \nu \frac{\cosh \rho}{\sinh \rho} + \epsilon \frac{\sinh \rho}{\cosh \rho} \right) G + \left( -\epsilon + M + \nu - \frac{1}{2} \right) F = 0 .
\]

Let us translate eqs. (37) to the variable \( z \):

\[
r^2 = \sinh^2 \rho = -z, \quad \frac{d}{d\rho} = 2\sqrt{-z(1-z)} \frac{d}{dz} ,
\]

\[
\left( 2\sqrt{-z(1-z)} \frac{d}{dz} + \nu \sqrt{\frac{1-z}{-z}} - \epsilon \frac{\sqrt{-z}}{\sqrt{1-z}} \right) F
+ (\epsilon + M - \nu - \frac{1}{2}) G = 0 ,
\]

\[
\left( 2\sqrt{-z(1-z)} \frac{d}{dz} - \nu \sqrt{\frac{1-z}{-z}} + \epsilon \frac{\sqrt{-z}}{\sqrt{1-z}} \right) G
+ (-\epsilon + M + \nu - \frac{1}{2}) F = 0 .
\]

From (38) it follow two 2-nd order differential equations for \( F \) and \( G \) respectively

\[
z(1-z) \frac{d^2 F}{dz^2} + \left( \frac{1}{2} - z \right) \frac{dF}{dz}
+ \left[ \frac{1}{4} \left( M - \frac{1}{2} \right)^2 - \frac{\epsilon(\epsilon - 1)}{4(1-z)} - \frac{\nu(\nu + 1)}{4z} \right] F = 0 ,
\]

\[
z(1-z) \frac{d^2 G}{dz^2} + \left( \frac{1}{2} - z \right) \frac{dG}{dz}
+ \left[ \frac{1}{4} \left( M - \frac{1}{2} \right)^2 - \frac{\epsilon(\epsilon + 1)}{4(1-z)} - \frac{\nu(\nu - 1)}{4z} \right] G = 0 .
\]

With the use of substitutions

\[ F = z^A (1-z)^B \tilde{F}(z) , \quad G = z^K (1-z)^L \tilde{G}(z) , \]
eqs. (39) take the form

\[
z(1-z) \frac{d^2 \bar{F}}{dz^2} + \left[ 2A + \frac{1}{2} - (2A + 2B + 1)z \right] \frac{d\bar{F}}{dz} \\
+ \left[ \frac{1}{4} \left( M - \frac{1}{2} \right)^2 - (A + B)^2 - \frac{\epsilon(\epsilon - 1) + 2B(1 - 2B)}{4(1 - z)} \\
- \frac{\nu(\nu + 1) - 2A(2A - 1)}{4z} \right] \bar{F} = 0 ,
\]  
\tag{40}

\[
z(1-z) \frac{d^2 \bar{G}}{dz^2} + \left[ 2K + \frac{1}{2} - (2K + 2L + 1)z \right] \frac{d\bar{G}}{dz} \\
+ \left[ \frac{1}{4} \left( M - \frac{1}{2} \right)^2 - (K + L)^2 - \frac{\epsilon(\epsilon + 1) + 2L(1 - 2L)}{4(1 - z)} \\
- \frac{\nu(\nu - 1) - 2K(2K - 1)}{4z} \right] \bar{G} = 0 .
\]  
\tag{41}

First let us consider eq. (40); at \( A \) and \( B \) taken accordingly

\[
A = \frac{1 + \nu}{2}, \quad -\frac{\nu}{2}, \quad B = \frac{\epsilon}{2}, \quad \frac{1 - \epsilon}{2}
\]  
\tag{42}

it becomes simpler

\[
z(1-z) \frac{d^2 f}{dz^2} + \left[ 2A + \frac{1}{2} - (2A + 2B + 1)z \right] \frac{df}{dz} \\
+ \left[ \frac{1}{4} \left( M - \frac{1}{2} \right)^2 - (A + B)^2 \right] f = 0 ,
\]  
\tag{43}

which is of hypergeometric type with parameters

\[
a = \frac{M}{2} - \frac{1}{4} + A + B , \quad b = -\frac{M}{2} + \frac{1}{4} + A + B , \quad c = 2A + 1/2 .
\]

To construct functions appropriate to describe bound states we must choose

\[
A = \frac{1 + \nu}{2} > 0 , \quad B = \frac{1 - \epsilon}{2} < 0 , \quad c = \nu + 3/2 ;
\]  
\tag{44}

polynomial solutions will arise with the quantization rule imposed

\[
a = -n , \quad \epsilon_n = M + 2n + \nu + \frac{3}{2} , \\
b = -n - M - 1/2 , \quad c = \nu + 3/2 .
\]  
\tag{45}

Now let us turn to eq. (41). At \( A, \ B \) chosen according to

\[
K = \frac{1 + \nu}{2} , \quad \frac{\nu}{2} , \quad L = -\frac{\epsilon}{2} , \quad \frac{1 + \epsilon}{2}
\]  
\tag{46}
it will be simpler
\[
z(1 - z) \frac{d^2 g}{dz^2} + \left[ 2K + \frac{1}{2} - (2K + 2L + 1)z \right] \frac{dg}{dz}
+ \left[ \frac{1}{4} \left( M - \frac{1}{2} \right)^2 - (K + L)^2 \right] g = 0,
\] (47)

which is of hypergeometric type
\[
\alpha = \frac{M}{2} - \frac{1}{4} + K + L,
\beta = -\frac{M}{2} + \frac{1}{4} + K + L,
\gamma = 2K + \frac{1}{2}.
\]

Again, to get bound states we choose the values
\[
K = \frac{\nu}{2} > 0, \quad L = -\frac{\epsilon}{2} < 0,
\] (48)
then the quantization rule arises
\[
\alpha = -N, \quad \epsilon_N = M + 2N + \nu - \frac{1}{2}.
\] (49)

It can be noted that \(\epsilon_N = \epsilon_n\), when \(N = n + 1\).

Let us calculate relative coefficient between functions \(F(z)\) and \(G(z)\). These being taken in the form
\[
F(z) = F_0 z^{(1+\nu)/2} (1 - z)^{(1-\epsilon)/2} \bar{F}(a, b, c; z), \quad c = \frac{3}{2} + \nu,
\]
\[
a = \frac{M}{2} + \frac{3}{4} + \frac{\nu}{2} - \frac{\epsilon}{2}, \quad b = -\frac{M}{2} + \frac{5}{4} + \frac{\nu}{2} - \frac{\epsilon}{2};
\] (50)

and
\[
G(z) = G_0 z^{\nu/2} (1 - z)^{-\epsilon/2} \bar{G}(\alpha, \beta, \gamma; z), \quad \gamma = \frac{1}{2} + \nu = c - 1,
\]
\[
\alpha = \frac{M}{2} - \frac{1}{4} + \frac{\nu}{2} - \frac{\epsilon}{2} = a - 1, \quad \beta = -\frac{M}{2} + \frac{1}{4} + \frac{\nu}{2} - \frac{\epsilon}{2} = b - 1,
\] (51)

must obey the following system
\[
\begin{align*}
\left( 2\sqrt{-z(1 - z)} \frac{d}{dz} + \nu \frac{\sqrt{1 - z}}{\sqrt{-z}} - \epsilon \frac{\sqrt{-z}}{\sqrt{1 - z}} \right) F + (+\epsilon + M - \nu - \frac{1}{2}) G &= 0, \\
\left( 2\sqrt{-z(1 - z)} \frac{d}{dz} - \nu \frac{\sqrt{1 - z}}{\sqrt{-z}} + \epsilon \frac{\sqrt{-z}}{\sqrt{1 - z}} \right) G + (-\epsilon + M + \nu - \frac{1}{2}) F &= 0.
\end{align*}
\]

To find a relative factor, it is convenient to use the second equation
\[
\begin{align*}
\left( -2\sqrt{-z(1 - z)} \frac{d}{dz} - \nu \frac{\sqrt{1 - z}}{\sqrt{-z}} + \epsilon \frac{\sqrt{-z}}{\sqrt{1 - z}} \right) G \\
+ (-\epsilon + M + \nu - \frac{1}{2}) F &= 0.
\end{align*}
\]
Substituting expressions for $F$ and $G$, after simple calculation we get to

$$2 i G_0 \frac{d \tilde{G}}{dz} = F_0 (\epsilon + M + \nu - \frac{1}{2}) \tilde{F}.$$  

Allowing for the known rule for differentiating hypergeometric functions

$$\frac{d}{dz} G(z) = \frac{d}{dz} F(a - 1, b - 1, c - 1; z) = \frac{(a - 1)(b - 1)}{c - 1} F(a, b, c; z),$$

we obtain

$$2 i G_0 \frac{(a - 1)(b - 1)}{c - 1} = F_0 (\epsilon + M + \nu - \frac{1}{2}).$$

Ultimately, we arrive at the formula

$$F_0 = i \frac{M - 1/2 + N}{2} G_0,$$

remembering that $\epsilon N = M - 1/2 + 2N + \nu$.

5 Radial equations in the case $j_{\min}$

Let us turn back to the case of the minimal value of $j$:

$$k = +1/2, +1, \ldots$$

$$\epsilon \sqrt{\Phi} f_3 - i \sqrt{\Phi} \frac{d}{dr} f_3 - M f_1 = 0,$$

$$\epsilon \sqrt{\Phi} f_1 + i \sqrt{\Phi} \frac{d}{dr} f_1 - M f_3 = 0; \quad (53)$$

from where for new functions

$$H = \frac{f_1 + f_3}{\sqrt{2}}, \quad G = \frac{f_1 - f_3}{i \sqrt{2}}$$

we derive

$$k = +1/2, +1, \ldots$$

$$\sqrt{\Phi} \frac{d}{dr} H + \left( \frac{\epsilon}{\sqrt{\Phi}} + M \right) G = 0,$$

$$\sqrt{\Phi} \frac{d}{dr} G - \left( \frac{\epsilon}{\sqrt{\Phi}} - M \right) H = 0. \quad (54)$$

And in the same manner for another case we have

$$k = -1/2, -1, \ldots$$

$$\epsilon \sqrt{\Phi} f_4 + i \sqrt{\Phi} \frac{d}{dr} f_4 - M f_2 = 0,$$

$$\epsilon \sqrt{\Phi} f_2 - i \sqrt{\Phi} \frac{d}{dr} f_2 - M f_4 = 0; \quad (55)$$

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from whence for new functions

\[ H = \frac{f_2 + f_4}{\sqrt{2}} , \quad G = \frac{f_2 - f_4}{i\sqrt{2}} \]

we obtain

\[
\sqrt{\Phi} \frac{d}{dr} G + \left( \frac{\epsilon}{\sqrt{\Phi}} - M \right) H = 0 , \\
\sqrt{\Phi} \frac{d}{dr} H - \left( \frac{\epsilon}{\sqrt{\Phi}} + M \right) G = 0 .
\]  

(56)

We can use the above method to eliminate nonphysical singular points. Let us perform special transformation on the functions

\[ G + H = e^{-\rho/2}(g + h) , \quad G - H = e^{+\rho/2}(g - h) . \]  

(57)

After simple calculation we arrive at

instead of (54)

\[
\left( \frac{d}{d\rho} + \epsilon \frac{\sinh \rho}{\cosh \rho} \right) g + (-\epsilon + M - 1/2) h = 0 , \\
\left( \frac{d}{d\rho} - \epsilon \frac{\sinh \rho}{\cosh \rho} \right) h + (+\epsilon + M - 1/2) g = 0 ;
\]  

(58)

instead of (56)

\[
\left( \frac{d}{d\rho} + \epsilon \frac{\sinh \rho}{\cosh \rho} \right) h + (-\epsilon - M - 1/2) g = 0 , \\
\left( \frac{d}{d\rho} - \epsilon \frac{\sinh \rho}{\cosh \rho} \right) g + (+\epsilon - M - 1/2) h = 0 .
\]  

(59)

In the variable \( z \)

\[ r = \sinh \rho = \sqrt{-z} \]

the system (58) takes the form

\[
\sqrt{-z}(1-z) \left( \frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) g - \frac{(-\epsilon + M - 1/2)}{2} h = 0 , \\
\sqrt{-z}(1-z) \left( \frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) h - \frac{(+\epsilon + M - 1/2)}{2} g = 0 .
\]  

(60)

Note that the system is symmetric with respect to changes

\[ f \leftrightarrow h , \quad \epsilon \leftrightarrow -\epsilon . \]  

(61)
After excluding the function \( h \) from (60) we get
\[
\begin{align*}
    h &= \frac{2}{(-\epsilon + M - 1/2)} \sqrt{(-z)(1-z)} \left( \frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) g, \\
    \sqrt{(-z)(1-z)} \left( \frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) \sqrt{(-z)(1-z)} \left( \frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) g \\
    &\quad - \frac{(M - 1/2)^2 - \epsilon^2}{4} g = 0 .
\end{align*}
\] (62)

Ultimately, an equation for \( g(z) \) reads
\[
\begin{align*}
    z(1-z) \frac{d^2g}{dz^2} + (1/2 - z) \frac{dg}{dz} + \left( \frac{(M - 1/2)^2}{4} - \frac{\epsilon^2 + \epsilon}{4} \frac{1}{1-z} \right) g &= 0 . \quad \text{(63)}
\end{align*}
\]

In the same manner we get a second order differential equation for \( h \) after exclusion of \( g \):
\[
\begin{align*}
    g &= \frac{2}{(\epsilon + M - 1/2)} \sqrt{(-z)(1-z)} \left( \frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) h, \\
    \sqrt{(-z)(1-z)} \left( \frac{d}{dz} - \frac{\epsilon/2}{1-z} \right) \sqrt{(-z)(1-z)} \left( \frac{d}{dz} + \frac{\epsilon/2}{1-z} \right) h \\
    &\quad - \frac{(M - 1/2)^2 - \epsilon^2}{4} h = 0 ,
\end{align*}
\] (64)

and ultimately
\[
\begin{align*}
    z(1-z) \frac{d^2h}{dz^2} + (1/2 - z) \frac{dh}{dz} + \left( \frac{(M - 1/2)^2}{4} - \frac{\epsilon^2 - \epsilon}{4} \frac{1}{1-z} \right) h &= 0 . \quad \text{(65)}
\end{align*}
\]

Equations (65) and (63) differ only in the sign at the parameter \( \epsilon \).

### 6 Solutions of radial equations in the case \( j_{\text{min}} \)

With the use of substitution \( g = (1-z)^A \varphi(z) \), from (63) we produce for \( \varphi \)
\[
\begin{align*}
    z(1-z) \varphi'' + \left[ \frac{1}{2} - (1 + 2A)z \right] \varphi' \\
    + \left[ \left( A^2 - \frac{A}{2} - \frac{\epsilon^2 + \epsilon}{4} \right) \frac{1}{1-z} - A^2 + \frac{(M - 1/2)^2}{4} \right] \varphi.
\end{align*}
\] (66)

Requiring
\[
A^2 - \frac{A}{2} - \frac{\epsilon^2 + \epsilon}{4} = 0 \quad \implies \quad 2A = \epsilon + 1, -\epsilon
\]
one gets
\[
\begin{align*}
    z(1-z) \varphi'' + \left[ \frac{1}{2} - (1 + 2A)z \right] \varphi' - \frac{4A^2 - (M - 1/2)^2}{4} \varphi &= 0 , \\
    \varphi &= F(a,b,c,z) , \quad c = \frac{1}{2} , \quad a + b = 2A , \quad ab = \frac{4A^2 - (M - 1/2)^2}{4} ,
\end{align*}
\] (67)
that is
\[ a = \frac{2A + (M - 1/2)}{2}, \quad b = \frac{2A - (M - 1/2)}{2}. \]  

(68)

Below we will use negative values for \( A \)
\[ A = -\epsilon/2, \quad g(z) = (1 - z)^{-\epsilon/2}\varphi(z); \]  

(69)

so that
\[ a = -\frac{\epsilon + (M - 1/2)}{2}, \quad b = -\frac{\epsilon - (M - 1/2)}{2}. \]  

(70)

Any 2-nd order differential equation has two linearly independent solutions; here they are
\[ \varphi_1 = U_1(z) = F(a, b, c; z), \]
\[ \varphi_2 = U_5(z) = z^{1-c}F(a+1-c, b+1-c, 2-c; z). \]  

(71)

Similar analysis can be performed for eq. (65)
\[ z(1-z)\frac{d^2h}{dz^2} + (1/2 - z)\frac{dh}{dz} + \left(\frac{(M - 1/2)^2}{4} - \frac{\epsilon^2 - \epsilon}{4}\frac{1}{1-z}\right)h = 0. \]

(72)

With the use of substitution \( h(z) = (1-z)^L\eta(z), \) for \( \eta(z) \) we produce
\[ z(1-z)\eta'' + \left[\frac{1}{2} - (1 + 2L)z\right]\eta' + \left[\left(L^2 - \frac{L}{2} - \frac{\epsilon^2 - \epsilon}{4}\right)\frac{1}{1-z} - L^2 + \frac{(M - 1/2)^2}{4}\right]\eta = 0. \]  

(73)

Requiring
\[ L^2 - \frac{L}{2} - \frac{\epsilon^2 - \epsilon}{4} = 0 \quad \Rightarrow \quad 2L = +\epsilon, -\epsilon + 1 \]

one gets
\[ z(1-z)\eta'' + \left[\frac{1}{2} - (1 + 2L)z\right]\eta' - \frac{4L^2 - (M - 1/2)^2}{4}\eta = 0, \]
\[ \eta = F(\alpha, \beta, \gamma, z), \quad \gamma = \frac{1}{2}, \]
\[ \alpha + \beta = 2L, \quad \alpha\beta = \frac{4L^2 - (M - 1/2)^2}{4}, \]  

(74)

that is
\[ \alpha = \frac{2L - (M - 1/2)}{2}, \quad \beta = \frac{2L + (M - 1/2)}{2}. \]  

(75)
Below we will use negative values for $L$

$$L = (-\epsilon + 1)/2 < 0, \quad h(z) = (1 - z)^{(-\epsilon+1)/2} \eta(z),$$

(76)

so that

$$\alpha = -\epsilon + 1 + (M - 1/2) \quad \frac{2}{2}, \quad \beta = -\epsilon + 1 - (M - 1/2) \quad \frac{2}{2}.$$

(77)

Functions $g(z)$ and $h(z)$ must obey the above system of first order differential equations. To verify that, let us start with the functions

$$g = G_0(1 - z)^A \varphi_1(z) \quad \text{and} \quad h = H_0(1 - z)^L \eta_2(z) \quad \text{and} \quad \text{for} \quad a = \frac{\alpha + 1 - \gamma = -\epsilon + 2 + (M - 1/2)}{2}, \quad b = \frac{\beta + 1 - \gamma = -\epsilon + 2 + (M - 1/2)}{2}.$$

(78)

After simple calculations we obtain

$$G_0 \frac{d}{dz} F(a, b, c, z) = H_0 \frac{(-\epsilon + M - 1/2)}{2} F(a + 1, b + 1, c + 1, z),$$

from whence it follows

$$G_0 a b c = H_0 \frac{(-\epsilon + M - 1/2)}{2},$$

that is

$$H_0 = i (-\epsilon - M + 1/2) G_0.$$

To get polynomial solutions we must require

$$a = -n \quad \Rightarrow \quad \epsilon_n = M + 2n - 1/2 , \quad b = -n - M + 1/2 , \quad c = 1/2 , \quad g(z) = (1 - z)^{(-\epsilon_n+1)/2} F(a, b, c, z).$$

(80)

note that

$$g(z) = (1 - z)^{-n-(M-1/2)/2} F(-n, -n - M + \frac{1}{2}, \frac{1}{2}, z);$$

(81)
therefore as \( z = -r^2 \to -\infty \) the function \( g(z) \) tends to zero

\[
g(z) \to 0, \quad \text{only if} \quad M > \frac{1}{2}.
\]

In usual units, that condition for existence of bound states consistent with anti de Sitter geometry structure, the inequality \( M > \frac{1}{2} \), looks as

\[
\rho > \frac{1}{2} \frac{\hbar}{M c} = \frac{1}{2} \lambda_e = 1.213 \times 10^{-12} \text{ metre}
\]

so it can be broken only in a very strong anti de Sitter gravitation background, the latter is beyond of our treatment.

Let us write down several energy levels (in usual units)

\[
\epsilon_0 = M c^2 - \frac{1}{2} c \hbar \frac{\rho}{\rho}, \quad \epsilon_1 = M c^2 + \frac{3}{2} c \hbar \frac{\rho}{\rho}, \quad \epsilon_2 = M c^2 + \frac{5}{2} c \hbar \frac{\rho}{\rho}, \ldots
\]

or

\[
\epsilon_0 = M c^2 (1 - \frac{1}{2} \frac{\lambda_e}{\rho}), \quad \epsilon_1 = M c^2 (1 + \frac{3}{2} \frac{\lambda_e}{\rho}), \quad \epsilon_2 = M c^2 (1 + \frac{5}{2} \frac{\lambda_e}{\rho}), \ldots
\]

If one mentally increases the curvature radius \( \rho \), the energy levels will become denser and the minimal level tends to the value \( M c^2 \)

\[
\epsilon_0 = M c^2 (1 - \frac{1}{2} \frac{\lambda_e}{\rho}) \to M c^2.
\]

7 Conclusions and discussion

To understand better results, let us discuss the case of minimal \( j_{\text{min}} \) in the limit of vanishing curvature. To this end, let us specify in more detail solutions for minimal values \( j_{\text{min}} \) in Minkowski space:

\[
k = +1/2, +1, \ldots
\]

\[
\epsilon f_3 - i \frac{d}{dr} f_3 - M f_1 = 0,
\]

\[
\epsilon f_1 + i \frac{d}{dr} f_1 - M f_3 = 0;
\]

\[
k = -1/2, -1, \ldots
\]

\[
\epsilon f_4 + i \frac{d}{dr} f_4 - M f_2 = 0,
\]

\[
\epsilon f_2 - i \frac{d}{dr} f_2 - M f_4 = 0.
\]
Let detail the case of positive \( k = +1/2, +1, \ldots \). Let it be
\[
\frac{f_1 + f_3}{\sqrt{2}} = h(r), \quad \frac{f_1 - f_3}{i\sqrt{2}} = g(r)
\] (87)
relevant equations are
\[
\frac{d}{dr} h + (\epsilon + M) g = 0, \quad \frac{d}{dr} g - (\epsilon - M) h = 0.
\] (88)
With the substitutions
\[
h(r) = H e^{\gamma r}, \quad g(r) = G e^{\gamma r}
\] (89)
we get (first let it be \((\epsilon^2 - M^2) > 0)\)
\[
\gamma^2 = - (\epsilon^2 - M^2) \equiv = -p^2, \quad \gamma = +ip, -ip.
\]
\[
H\gamma + (\epsilon + M) G = 0 \quad \text{or} \quad G\gamma - (\epsilon - M) H = 0.
\] (90)
Thus we have two linearly independent solutions
\[
h_1(r) = H_1 e^{+ipr}, \quad g_1(r) = G_1 e^{+ipr}, \quad G_1 = \frac{\epsilon - M}{ip} H_1;
\]
\[
h_2(r) = H_2 e^{-ipr}, \quad g_2(r) = G_2 e^{-ipr}, \quad G_2 = \frac{\epsilon - M}{-ip} H_2.
\] (91)
Below, we take \( H_1 = H_2 = 1 \). We can introduce two linear combinations of these solutions the first
\[
\frac{h_1(r) + h_2(r)}{2} = \cos pr, \quad \frac{g_1(r) + g_2(r)}{2} = \frac{\epsilon - M}{p} \sin pr;
\] (92)
the second
\[
\frac{h_1(r) - h_2(r)}{2i} = \sin pr, \quad \frac{g_1(r) - g_2(r)}{2i} = \frac{\epsilon - M}{-p} \cos pr.
\] (93)
Now let us specify the case \((\epsilon^2 - M^2) < 0)\)
\[
\gamma^2 = - (\epsilon^2 - M^2) \equiv = +q^2, \quad \gamma = +q, -q.
\]
\[
H\gamma + (\epsilon + M) G = 0 \quad \text{or} \quad G\gamma - (\epsilon - M) H = 0.
\] (94)
Thus we have two linearly independent solutions
\[
h_1(r) = H_1 e^{+qr}, \quad g_1(r) = G_1 e^{+qr}, \quad G_1 = \frac{\epsilon - M}{q} H_1;
\]
\[
h_2(r) = H_2 e^{-qr}, \quad g_2(r) = G_2 e^{-qr}, \quad G_2 = \frac{\epsilon - M}{-q} H_2.
\] (95)
Below, we take $H_1 = H_2 = 1$. We can introduce two linear combinations of these solutions the first

\[
\frac{h_1(r) + h_2(r)}{2} = \cosh qr ,
\]
\[
\frac{g_1(r) + g_2(r)}{2} = \frac{\epsilon - M}{q} \sinh qr
\]

(96)

the second

\[
\frac{h_1(r) - h_2(r)}{2} = \sinh qr ,
\]
\[
\frac{g_1(r) - g_2(r)}{2} = \frac{\epsilon - M}{q} \cosh qr .
\]

(97)

Evidently, above constructed solutions in de Sitter model provide us with generalizations of these of Minkowski space. It may be verified additionally by direct limiting process when $\rho \to \infty$. To this end, let us translate solutions in de Sitter space to usual units

\[
g_1(R) = \left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2\hbar}} F(a, b, c; -\frac{R^2}{\rho^2}) , \quad c = 1/2 ,
\]
\[
g_2(R) = R \left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2\hbar}} F(a + 1 - c, b + 1 - c, 2 - c; -\frac{R^2}{\rho^2}) ,
\]
\[
h_1(R) = \left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2\hbar}+1/2} F(\alpha, \beta, \gamma; -\frac{R^2}{\rho^2}) , \quad \gamma = 1/2 ,
\]
\[
h_2(R) = R \left(1 + \frac{R^2}{\rho^2}\right)^{-\frac{E\rho}{2\hbar}+1/2} F(\alpha + 1 - \gamma, \beta + 1 - \gamma, 2 - \gamma; -\frac{R^2}{\rho^2}) ,
\]

Parameters of hypergeometric functions are given by

\[
a = \frac{1}{2} \left(\frac{E\rho}{\hbar} + (\frac{mcp}{\hbar} - \frac{1}{2})\right) , \quad b = \frac{1}{2} \left(\frac{E\rho}{\hbar} - (\frac{mcp}{\hbar} - \frac{1}{2})\right) ,
\]
\[
\alpha = \frac{1}{2} \left(-\frac{E\rho}{\hbar} + 1 + (\frac{mcp}{\hbar} - \frac{1}{2})\right) , \quad \beta = \frac{1}{2} \left(-\frac{E\rho}{\hbar} + 1 - (\frac{mcp}{\hbar} - \frac{1}{2})\right) .
\]

Let us examine the limiting procedure at $\rho \to \infty$ in $F(a, b, c; -\frac{R^2}{\rho^2})$. Because

\[
\frac{1}{2!} \frac{ab}{c} \left(-\frac{R^2}{\rho^2}\right) \to \frac{1}{2!} \left(\frac{m^2 c^2}{\hbar^2} - \frac{E^2}{\hbar^2} c^2\right) R^2 = \frac{1}{2!} (pR)^2 ,
\]
\[
\frac{1}{2!} \frac{a(a + 1)b(b + 1)}{c(c + 1)} \left(-\frac{R^2}{\rho^2}\right) \to \frac{(pR)^4}{4!} ,
\]
\[
\frac{1}{3!} \frac{a(a + 1)(a + 2)b(b + 1)(b + 2)}{c(c + 1)(c + 2)} \left(-\frac{R^2}{\rho^2}\right) \to -\frac{(pR)^6}{6!} ,
\]

and so on, we obtain the following limiting relation

\[
\lim_{\rho \to \infty} F(a, b, c; -\frac{R^2}{\rho^2}) = \cos pr \quad \implies \quad \lim_{\rho \to \infty} g_1(R) = \cos pr .
\]
Similarly, we get
\[
\lim_{\rho \to \infty} h_1(R) = \cos pr .
\] (98)

In the same manner, we arrive at two limiting relationships
\[
\lim_{\rho \to \infty} pR g_2(R) = \sin pR , \quad \lim_{\rho \to \infty} pR h_2(R) = \sin pR .
\] (99)

To rationalize how the finite sums (polynomials of \(n\)-order) may approximate the functions \(\cos pR\) and \(\sin pR\) (infinite series), we should take into account the quantization condition
\[
\alpha = -n \implies E = Mc^2 + \left(2n - \frac{1}{2}\right) \frac{ch}{\rho}
\]

At any fixed \(E\), as \(\rho\) increases the number \(n\) also must increase. This means, that the finite sums of \(n\) terms when \(\rho\) increases will approximate infinite series.

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