Third gravitational wave polarization mode in Rastall theory and analogy with $f(R)$ theories

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Abstract

The recent starting of the gravitational wave (GW) astronomy with the events GW150914, GW151226, GW170104, and the very recent GW170814 and GW170817 seems to be fundamental not only in order to obtain new intriguing astrophysical information from our surrounding Universe, but also in order to discriminate among Einstein’s general theory of relativity (GTR) and alternative gravitational theories. At the present time, despite the cited events, and in particular the last ones, which are the events GW170814 and GW170817, have put very strong constraint on the GTR, extended theories of gravity have not been completely ruled out. Here we discuss, in our knowledge for the first time in the literature, GWs in the Rastall theory of gravity. In fact, the Rastall theory recently obtained a renovated interest in the literature. We show that there is a profound analogy between GWs in $f(R)$ theories of gravity and GWs in the Rastall theory. This will permit us to linearize the Rastall field equations and to find the corresponding GWs. We will also study the motion of the test masses due to GWs in this theory which could help, in principle, to discriminate between the GTR, $f(R)$ theories and the Rastall theory of gravity.
1 Introduction

The observations of GWs from binary black hole (BH) mergers, i.e. the events GW150914 [1], GW151226 [2], GW170104 [3] and the recent GW170814 [48], and the very recent observation of GWs from a neutron star (NS) merger, i.e. the event GW170817 [49], represented the starting of the era of the GW astronomy. These remarkable GW detections are unanimously considered a cornerstone for science and for gravitational physics in particular. On one hand, the great result is to discover new, intriguing information on the Universe. On the other hand, the nascent GW astronomy could be useful in order to discriminate, in an ultimate way, among the GTR and potential alternative theories [4]. Extended gravity [4 - 7] is indeed an useful and popular tool to attempt to understand the big puzzles in the standard model of cosmology like the well known dark energy [8, 9] and dark matter [10, 11] problems. In this framework, it is important stressing that all of the potential alternatives to the GTR must be viable. In other words, such alternatives must be metric theories in order to be in agreement with the Einstein’s equivalence principle, which is today supported by a very strong empirical evidence [5]. In addition, as they must pass the solar system tests, deviation from the standard GTR must be weak [4].

Among various different proposals and attempting to extend the physical framework of the GTR, the theory proposed by P. Rastall in 1972 [12] recently gained a renewed interest in the literature [13 - 16]. The Rastall theory has indeed some good behavior. It shows a good agreement with observational data on the Universe age and on the Hubble parameter [17]. It can, in principle, provide an alternative description for the matter dominated era with respect to the GTR [18]. It is in agreement with observational data from the helium nucleosynthesis [19]. All these evidences have motivated physicists to study the various cosmic eras in this framework [20 - 24]. In addition, it seems to do not suffer from the entropy and age problems which appear in the framework of standard cosmology [25]. It seems also consistent with the gravitational lensing phenomena [26, 27]. Further details on the Rastall theory can be found in [29 - 33] and references within.

A key point on the Rastall theory is that it is a theory which considers a non-divergence-free energy-momentum [12 - 33]. The curvature-matter theory of gravity [34 - 38] works in a similar way. This theory is indeed similar to the Rastall theory in a way that the matter and geometry are coupled to each other in a non-minimal way [34 - 38]. Hence, the standard energy-momentum conservation law does not work [34 - 38].

In the following we discuss GWs in the Rastall theory also finding the motion of the test masses due to such GWs. This could help, in principle, to discriminate between the GTR and the Rastall theory in the developing of the GW astronomy. It is important to stress that despite the various GW detections, and in particular the last ones, that are the events GW170814 [48] and GW170817 [49], have put very strong constraint on the GTR, extended gravity theories have not been completely ruled out. In fact, despite current tests are strongly in favor of the purely GTR polarizations against scalar polarizations.
which are present in extended theories \cite{48}, for example in $f(R)$ gravity \cite{51}, binary BH systems are not at all promising for studying such scalar polarizations because of a consequence of the no-hair theorems for BHs, see \cite{40, 50}. BHs indeed radiate away any scalar field, so that a binary BH system in $f(R)$ gravity behaves as in the GTR. We show in this paper that the situation is analogous in the Rastall theory, where a scalar GW component is present. Similarly, binary NS systems, like the event GW170817, are also not effective testing grounds for scalar radiation \cite{40, 50}. This is because NS masses tend to cluster around the Chandrasekhar limit of $1.4M_{\odot}$, being $M_{\odot}$ the solar mass, and the sensitivity of NSs is not a strong function of mass for a given equation of state \cite{40, 50}. Thus, in systems like the binary NSs, scalar radiation is naturally suppressed by symmetry, and the bound achievable cannot compete with those from the solar system \cite{40, 50}. Hence the most promising binary systems are mixed: BH-NS, BH-WD or NS-WD \cite{40, 50}. GWs from those mixed systems have not been yet detected by the ground based GW interferometers.

\section{The Rastall theory of gravity}

In the Rastall theory, the ordinary energy-momentum conservation law is modified as \cite{12}

\begin{equation}
T^{\mu\nu} = \lambda R^{\cdot\nu},
\end{equation}

where $\lambda$ and $R$ are the Rastall constant parameter and the Ricci scalar, respectively. Taking into account the Bianchi identity, one gets the Rastall field equations as \cite{12}

\begin{equation}
G_{\mu\nu} + \kappa \lambda g_{\mu\nu} R = \kappa T_{\mu\nu},
\end{equation}

where $\kappa$ is the Rastall gravitational coupling constant. Combining eqs. \eqref{2} and \eqref{1} one gets $R(4\kappa \lambda - 1) = \kappa T$ and

\begin{equation}
T^{\mu\nu} = \frac{\kappa \lambda}{4\kappa \lambda - 1} T^{\cdot \nu},
\end{equation}

respectively. Thus, for traceless solutions, that is $T = R = 0$, the Rastall field equations reduce to the standard GRT field equations.

Now, let us consider a congruence of geodesics of parameter $\tau$, which is distinguished by the parameter $\Lambda$ in a way that its tangent vector field ($v^\alpha$) and the separation vector ($\xi^\alpha$) between the geodesics curves are evaluated as

\begin{equation}
v^\alpha = \frac{dx^\alpha}{d\tau},
\end{equation}

and

\begin{equation}\xi^\alpha = \frac{dx^\alpha}{d\Lambda},\end{equation}
respectively [39]. In this situation, the geodesic equation reads [39, 40]

\[
\frac{D^2 \xi^\alpha}{D\tau^2} = R^\alpha_{\beta\mu\nu} v^\beta v^\mu \xi^\nu,
\]

(6)

where \( R^\alpha_{\beta\mu\nu} \) is the Reimann tensor and [39, 40]

\[
\frac{D^2 \xi^\alpha}{D\tau^2} \equiv \frac{d^2 \xi^\alpha}{d\tau^2} + \Gamma^\alpha_{\mu\nu} \frac{d\xi^\mu}{d\tau} \frac{d\xi^\nu}{d\tau},
\]

(7)

Eq. (6) shows that the geodesics are bent by the space-time curvature. The Reimann tensor can be expanded as

\[
R_{\alpha\beta\gamma\delta} = \left( g_{\alpha\gamma} R_{\delta\beta} - g_{\alpha\delta} R_{\gamma\beta} + g_{\beta\delta} R_{\gamma\alpha} - g_{\beta\gamma} R_{\delta\alpha} \right) - \frac{R}{6} (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}) + C_{\alpha\beta\gamma\delta},
\]

(8)

where \( C_{\alpha\beta\gamma\delta} \) is the Weyl tensor [39]. In addition, if one uses Eq. (2), one gets

\[
R_{\alpha\beta} = \kappa [E_{\alpha\beta} + \frac{\kappa \lambda}{4\kappa \lambda - 1} T g_{\alpha\beta}],
\]

(9)

where

\[
E_{\alpha\beta} = T_{\alpha\beta} - \frac{1}{2} T g_{\alpha\beta}
\]

(10)

is the Einsteinian part. This equation covers the Einsteinian results at the \( \lambda \to 0 \) limit. Finally, for a space-time of metric \( g_{\alpha\beta} \) filled by an energy-momentum source \( T_{\alpha\beta} \), Eq. (8) can be rewritten as

\[
R_{\alpha\beta\gamma\delta} = \kappa R_{\alpha\beta\gamma\delta} + C_{\alpha\beta\gamma\delta} + \kappa \lambda \tilde{R}_{\alpha\beta\gamma\delta}
\]

(11)

where

\[
R_{\alpha\beta\gamma\delta} = \left( g_{\alpha\gamma} E_{\delta\beta} - g_{\alpha\delta} E_{\gamma\beta} + g_{\beta\delta} E_{\gamma\alpha} - g_{\beta\gamma} E_{\delta\alpha} \right) \frac{2}{2}
\]

\[
+ \frac{T}{6} (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}),
\]

(12)

and \( T = \frac{4\kappa \lambda - 1}{\kappa} R \). Moreover,

\[
\kappa \lambda \tilde{R}_{\alpha\beta\gamma\delta} = \kappa \lambda \left[ \frac{T}{6} (g_{\alpha\gamma} g_{\delta\beta} - g_{\alpha\delta} g_{\gamma\beta}) + \frac{4\kappa \lambda - 1}{\kappa} R \right]
\]

(13)
is the correction term which directly comes from the Rastall hypothesis. In the above equations

\[ \tilde{E}_{\alpha\beta} = \frac{\kappa}{4\kappa\lambda - 1} T g_{\alpha\beta}, \]

\[ \tilde{T} = \tilde{E}_{\alpha}^{\alpha} = \frac{4\kappa}{4\kappa\lambda - 1} T = 4R. \] (14)

Hence, for the traceless sources, such as the radiation field, where \( T = 0 \), we have

\[ \tilde{R}_{\alpha\beta\gamma\delta} = 0. \] (15)

In the Rastall theory, \( \kappa \) differs from the Einstein gravitational coupling (\( \kappa_E \)) [12, 17, 18]. Thus, we have

\[ \kappa = \frac{4\gamma - 1}{6\gamma - 1} \kappa_E, \] (16)

where \( \gamma \equiv \kappa\lambda \) is the dimensionless Rastall parameter [13]. Inserting all of the above results in Eq. (6), one gets

\[ \frac{D^2 \xi^\alpha}{D\tau^2} = (\frac{D^2 \xi^\alpha}{D\tau^2})_E + (\frac{D^2 \xi^\alpha}{D\tau^2})_R, \] (17)

where

\[ (\frac{D^2 \xi^\alpha}{D\tau^2})_E = [\kappa_E \bar{R}_{\beta\mu\nu}^\alpha + C_{\beta\mu\nu}^\alpha] u^\beta v^\mu \xi^\nu, \] (18)

and

\[ (\frac{D^2 \xi^\alpha}{D\tau^2})_R = \gamma [\bar{R}_{\beta\mu\nu}^\alpha + \frac{2}{1 - 6\gamma} T_{\beta\mu\nu}] u^\beta v^\mu \xi^\nu, \] (19)

are the geodesic equations in the Einstein framework and the correction term to the geodesic equation due to the Rastall hypothesis, respectively. Clearly, the results of the GTR are obtainable in the appropriate limit of \( \lambda \to 0 \) parallel to the \( \gamma = 0 \) limit. From now, we work in units setting \( \kappa_E = 1 \). Thus, \( \kappa = \frac{4\gamma - 1}{6\gamma - 1} \).

### 3 Linearized Rastall field equations

In this Section, GWs in the Rastall framework will be analysed. For the sake of simplicity, we work with \( G = 1 \), \( c = 1 \) and \( \hbar = 1 \) (natural units) in the following. We consider a weak field situation where the observer is too far away from the energy-momentum source. In other words, we investigate a space-time in which the metric \( g_{\mu\nu} \) can be expanded as [11]

\[ g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}, \] (20)
where $\eta_{\mu\nu}$ is the Minkowski metric and $h_{\mu\nu}$ denote the deviation from the Minkowski space-time due to the curvature carried by the GW. Applying Eq. (20) to Eq. (2) and following the approach of [41, 42], one finds

$$h^\alpha_{\nu,\alpha\mu} + h^\alpha_{\mu,\alpha\nu} - h_{\mu\nu} - \Box h_{\mu\nu} - (1 - 2\gamma)\eta_{\mu\nu}(h^\alpha_{\nu,\alpha\beta} - \Box h) = 2\kappa T_{\mu\nu}. \tag{21}$$

Let us simplify Eq. (21), in similarity to the case of the GTR. One defines [41, 42]

$$\bar{h}_{\mu\nu} = h_{\mu\nu} - \frac{1}{2} \eta_{\mu\nu} h. \tag{22}$$

Combining Eq. (22) with Eq. (21) one gets

$$2\kappa T_{\mu\nu} = \bar{h}^\alpha_{\nu,\alpha\mu} + \bar{h}^\alpha_{\mu,\alpha\nu} - \Box \bar{h}_{\mu\nu} + \gamma \eta_{\mu\nu} \Box h + (1 - 2\gamma)\eta_{\mu\nu} \bar{h}^\alpha_{\nu,\alpha\beta}, \tag{23}$$

which leads to

$$2\kappa T_{\mu\nu} = \gamma \eta_{\mu\nu} \Box h - \Box \bar{h}_{\mu\nu} \tag{24}$$

Considering the Lorenz condition [43] $\bar{h}^\alpha_{\nu,\alpha} = 0$, for either a vacuum spacetime or a long distance from the source we have

$$\Box \bar{h}_{\mu\nu} = \gamma \eta_{\mu\nu} \Box h. \tag{25}$$

One can also check that, at the $\gamma \to 0$ limit, the Einstein linearized field equations are re-obtained [41, 42].

Now, let us define

$$\bar{h}_{\mu\nu} - \gamma \eta_{\mu\nu} h \equiv \bar{H}_{\mu\nu}. \tag{26}$$

Combining Eq. (26) with Eq. (22) one gets

$$h_{\mu\nu} = \bar{H}_{\mu\nu} + \frac{(1 - 2\gamma)\bar{H}}{2(4\gamma - 1)} \eta_{\mu\nu}. \tag{27}$$

where $\bar{H}$ is the trace of $\bar{H}_{\mu\nu}$. Inserting Eq. (26) into Eq. (25), one easily finds

$$\Box \bar{H}_{\mu\nu} = 0. \tag{28}$$
It is also interesting to mention here that, based on Eqs. \((27), (26)\) and \((22)\), it seems that the traceless solutions of Eq. \((28)\) are not affected by the mutual non-minimal coupling between geometry and matter fields (or equally by the \(\gamma\) parameter). This is due to the fact that the \(\gamma\) parameter only appears with the trace of solutions and thus perturbed metric. Therefore, it seems that the transverse and traceless solutions of this equations are exactly the same as those of the GTR case \([41, 42]\).

In the absence of energy-momentum source, the linearized approximation of Eq. (1) leads to

\[
h^{\alpha\beta}_{\alpha\beta} - \Box h = C, \tag{29}\]

where \(C\) is an integration constant. On the other hand, the trace of Eq. (2) implies \(C = 0\) in the absence of energy-momentum source. Bearing the Lorenz gauge in mind \([43]\) and using Eq. (27), one immediately finds

\[
\Box H = 0, \tag{30}\]

in full agreement with Eq. (28). One may also use the Bianchi identity as well as Eqs. (1) and (22) to obtain

\[
\left(\frac{1}{2} [h_{\mu\nu} - h_{,\nu\mu}] - \Box h_{\mu\nu}\right)^{\nu} = 0, \tag{31}\]

which is compatible with Eqs. (28) and (30). In fact, since \(h_{\mu\nu} = h_{,\nu\mu}\), one can use Eq. (27) to get

\[
\Box h_{\mu\nu} = \Box H_{\mu\nu} + \frac{(1 - 2\gamma)}{2(4\gamma - 1)} \eta_{\mu\nu} \Box H = 0. \tag{32}\]

This indicates that Eq. (31) does not give us anything more than Eqs. (28) and (30).

4 Gravitational waves

It is straightforward to check that the plane waves

\[
\tilde{H}_{\mu\nu} = Q_{\mu\nu} \exp(i k_{\alpha} x^{\alpha}) + c.c., \tag{33}\]

where \(k_{\alpha} k^{\alpha} = 0\), are solutions of Eq. (28). Then, for the null wave-vector \(k_{\alpha}\), one gets

\[
k^{\alpha} = (\omega, k^{1}, k^{2}, k^{3}). \tag{34}\]
Here, $\omega^2 \equiv k^i k^i$ is the wave frequency observed by an observer with four-velocity $U^\alpha$, i.e. $\omega = -k_\alpha U^\alpha$.

Now, combining Eqs. (33), (26) and (27), one finds

\[ \bar{h}_{\mu\nu} = A_{\mu\nu} \exp(ik_\alpha x^\alpha) + c.c., \]  

and

\[ h_{\mu\nu} = a_{\mu\nu} \exp(ik_\alpha x^\alpha) + c.c., \]  

where $A_{\mu\nu} = Q_{\mu\nu} + \gamma Q_{4\mu\nu} - \eta_{\mu\nu}$ and $a_{\mu\nu} = Q_{\mu\nu} + \frac{1 - 2\gamma}{2(4\gamma - 1)} Q_{\mu\nu}$, respectively. Since $h_{\mu\nu}$ is symmetric, $Q_{\mu\nu}$, $A_{\mu\nu}$ and $a_{\mu\nu}$ have 10 independent components. In addition, bearing the Lorenz gauge ($h_{\alpha\beta} = k^\alpha A_{\alpha\beta} = 0$) in mind \[43\], we get

\[ k^\mu Q_{\mu\nu} = \frac{\gamma Q}{1 - 4\gamma} k_\nu. \]  

(37)

In the above equations, $Q(\equiv Q_\mu^\mu)$ is the trace of $Q_{\mu\nu}$. Since $k_\alpha$ is a null vector, Eq. (37) leads to

\[ k^\nu k^\mu Q_{\mu\nu} = 0. \]  

(38)

It is interesting to note here that this result can also be obtained if one uses Eq. (29). In fact, as we considered an empty spacetime, we have $R \simeq h_{\mu\nu} - \Box h = 0$ at the weak field limit \[41, 42\] which leads to $k^\mu k_\nu a^{\mu\nu} = 0$ and so the above result. The Lorenz condition ($k^\mu A^{\mu\nu} = 0$) \[45\] also reduces the independent components of $A_{\mu\nu}$ to 6 components, and therefore, $Q_{\mu\nu}$ and $a_{\mu\nu}$ will also have 6 independent components, a result in agreement with Eqs. (37) and (38). This implies that the Rastall theory should have 4 additional GW polarizations with respect to the two standard polarizations of the GTR case \[41, 42\].

Now, inserting $\omega = -k_\alpha U^\alpha$ into Eq. (37), it is also easy to obtain

\[ U^\nu k^\mu Q_{\mu\nu} = \frac{Q\gamma\omega}{4\gamma - 1}, \]  

(39)

which recovers the GTR result in the appropriate limit of $\gamma = 0$. Theoretically, this equation should also reduce the number of independent components of $Q_{\mu\nu}$ to 2. But, as $U^\alpha$ and the wave-vector field are in the mutual relation $\omega = -k_\alpha U^\alpha$, one component of the $U^\alpha$ vector field is not arbitrary and free. Thus, Eq. (39) may reduce independent components of $Q_{\mu\nu}$, $A_{\mu\nu}$ and $a_{\mu\nu}$ to maximum 3 components. In fact, Eqs. (38) and (39) specify relations between the wave amplitude, the wave-vector and the four-velocity of the observer. Thus, the Rastall theory has one additional GW polarization with respect to the 2...
standard polarizations of the GTR case \[41, 42\]. We notice that there is a strong analogy with GWs in \( f(R) \) theories of gravity. In fact, in that case the linearized field equations are \[51\]

\[\Box \bar{h}_{\mu\nu} = 0\] (40)

\[\Box h_f = m^2 h_f\]

where \( h_f \) is a massive effective scalar field representing the third GW polarization in \( f(R) \) theories of gravity. \( h_f \) is due to the presence of curvature high order terms in the \( f(R) \) action and is obtained by taking the trace of the field equations. In the current case of the Rastall theory, if one gathers Eq. (28) with Eq. (30) one gets

\[\Box \bar{H}_{\mu\nu} = 0\] (41)

\[\Box \bar{H} = 0\]

and we can interpret \( \bar{H} \) as being a massless effective scalar field representing the third GW polarization in the Rastall theory. In fact, also in Rastall theory \( \bar{H} \) is obtained by taking the trace of the field equations and its physical origin arises from the curvature term in Eq. (2). Thus, in order to further go ahead in the linearization process, we can follow the analysis in \[51\], but keeping in mind that now the effective scalar field is massless rather than massive.

The solutions of the first of Eqs. (41) are the plane waves \[53\]. One must add the solutions of the second of Eqs. (41) obtaining

\[\bar{H}_{\mu\nu} = Q_{\mu\nu}(\vec{k}) \exp(ik^\alpha x_\alpha) + c.c.\] (42)

\[\bar{H} = Q(\vec{k}) \exp(ik^\alpha x_\alpha) + c.c.\] (43)

The solutions \[42\] and \[43\] take the conditions \[57\]. One considers a GW propagating in the positive \( z \) direction with

\[k^\mu = (k, 0, 0, k).\] (44)

Eqs. \[57\] imply

\[Q_{0\nu} = -Q_{3\nu}\]

\[Q_{\nu 0} = -Q_{\nu 3}\]

\[Q_{00} = -Q_{30} + Q_{33}.\] (45)

One recalls that the freedom degrees of \( Q_{\mu\nu} \) are 3. In fact, we started with 10 components (\( Q_{\mu\nu} \) is a symmetric tensor). As it has been stressed above, the Lorenz condition \( k_\mu A^{\mu\nu} = 0 \) \[43\] reduces the components to 6. Then, we take \( Q_{00}, Q_{11}, Q_{22}, Q_{21}, Q_{31}, Q_{32} \) like independent components. The condition \[39\] sets to zero 3 more components. Now, one takes
\[ \epsilon_\mu = \epsilon_\mu(-\mathbf{k}) \exp(i k^\alpha x_\alpha) + c.c. \] (46)

\[ k^\mu \epsilon_\mu = 0, \]

with the condition \( \Box \epsilon_\mu = \partial^\mu \bar{H}_\mu \) for the parameter \( \epsilon_\mu \). Then, the transform law for \( Q_\mu \) reads (one considers again the Lorenz condition \[43\] and Eq. (42))

\[ Q_\mu \rightarrow Q'_\mu = Q_\mu - 2ik(\mu \bar{\epsilon}_\nu). \] (47)

Hence, one can write down the six components of interest as

\[
\begin{align*}
Q_{00} &\rightarrow Q_{00} + 2ik\bar{\epsilon}_0 \\
Q_{11} &\rightarrow Q_{11} \\
Q_{22} &\rightarrow Q_{22} \\
Q_{21} &\rightarrow Q_{21} \\
Q_{31} &\rightarrow Q_{31} - ik\bar{\epsilon}_1 \\
Q_{32} &\rightarrow Q_{32} - ik\bar{\epsilon}_2.
\end{align*}
\] (48)

Clearly, the components of \( Q_\mu \) having physical meaning are the gauge-invariants \( Q_{11}, Q_{22} \) and \( Q_{21} \). Thus, one chooses \( \bar{\epsilon}_\nu \) to set equal to zero the others. The massless effective scalar field is obtained as

\[ \bar{H} = \bar{H}_{11} + \bar{H}_{22}. \] (49)

Now, defining \( h_{Rast} \equiv -\bar{H} \), the total GW perturbation propagating in the \( z + \) direction reads

\[ h_\mu^\nu(t - z) = Q_+^\nu(t - z)\epsilon_\mu^{(+)} + Q_\times^\nu(t - z)\epsilon_\mu^{(\times)} + h_{Rast}(t - z)\epsilon_\mu^{(Rast)}. \] (50)

The term \( Q_+(t - z)\epsilon_\mu^{(+)} + Q_\times(t - z)\epsilon_\mu^{(\times)} \) represents the two standard GW polarizations of the GTR in the transverse-traceless gauge \[41\] while the term \( h_{Rast}(t - z)\epsilon_\mu^{(Rast)} \) is the third additional polarization due to the curvature term in Eq. (2).

5 Geodesic deviation equation and test masses motion

From Eq. (50) one gets the total GW line-element as

\[ ds^2 = dt^2 - dz^2 - (1 + Q_+ + h_{Rast})dx^2 - (1 - Q_+ + h_{Rast})dy^2 - 2Q_\times dx dy. \] (51)

As the GW astronomy is performed in a laboratory environment on Earth, one usually uses the coordinate system in which the space-time is locally flat \[41\]. In that case, the distance between any two points and/or test masses is given simply by the difference in their coordinates in the sense of Newtonian
In this gauge, called the gauge of the local observer, GWs manifest themselves by exerting tidal forces on the test masses, which are the mirror and the beam-splitter in the case of an interferometer like LIGO. A complete analysis of gauge of the local observer is given in [41]. Here we limit ourselves to recall only the more important behaviors of this gauge:

1. The proper time of the observer O is given by the time coordinate $x_0$.
2. Spatial axes are centered in O.
3. If both of acceleration and rotation are null, then the spatial coordinates $x_j$ are the proper distances along the axes. In that case the gauge of the local observer reduces to a local Lorentz gauge and the metric is
   \[ ds^2 = (-dx^0)^2 + \delta_{ij}dx^i dx^j + O(|dx^j|^2)dx^a dx^b; \] (52)
4. The effect of GWs on test masses is described by the geodesic deviation equation
   \[ \ddot{x}^i = -\tilde{R}^i_{0k0}x^k, \] (53)
   where $\tilde{R}^i_{0k0}$ the linearized Riemann tensor [41].

The effect on test masses due to the two standard GTR polarizations $Q_+$ and $Q_\times$ is well known [41]. Thus, we will consider only the effect on test masses by the additional polarization $h_{\text{Rast}}$. In that case, the line element (51) reduces to
\[ ds^2 = dt^2 - dz^2 - (1 + h_{\text{Rast}})dx^2 - (1 + h_{\text{Rast}})dy^2. \] (54)

In order to further stress the analogy with GWs in $f(R)$ theories of gravity, one makes the following coordinate transformation
\[
\begin{align*}
    dx' &= dx \\
    dy' &= dy \\
    dz' &= (1 + h_{\text{Rast}})dz - \frac{1}{2}h_{\text{Rast}}dt \\
    cd't' &= (1 + \frac{1}{2}h_{\text{Rast}})dt - \frac{1}{2}h_{\text{Rast}}dz.
\end{align*}
\] (55)

Then, the new line element is the conformally flat one
\[ ds^2 = [1 + h_{\text{Rast}}(t - z)](-dt^2 + dz^2 + dx^2 + dy^2). \] (56)

The new gauge of Eq. (56) is very similar to the one which is usually used to discuss GWs in $f(R)$ theories of gravity, that is [51, 53]
\[ ds^2 = [1 + h_f(t - v_G z)](-dt^2 + dz^2 + dx^2 + dy^2). \] (57)

In fact, we recall that the third GW polarization in $f(R)$ theories of gravity is interpreted in terms of a massive field which can be discussed like a wave-packet [51, 53]. The group-velocity of a wave-packet of $h_f$ centered in $\vec{p}$ is
\[ \vec{v}_g = \frac{\vec{p}}{\omega}. \] (58)
which is exactly the velocity of a massive particle with mass $m$ and momentum $\vec{p}$. The group-velocity results \[51, \, 53\]

$$ v_G = \frac{\sqrt{\omega^2 - m^2}}{\omega}, \quad (59) $$

and, if one wants a constant speed of the wave-packet, one gets \[51, \, 53\]

$$ m = \sqrt{(1 - v_G^2)\omega}. \quad (60) $$

Thus, from Eq. \[50\], one sees that the speed of the third additional massless GW mode in the Rastall theory is exactly the speed of light, while, from Eq. \[57\] one sees that the speed of the third additional massive GW mode in $f(R)$ theories of gravity is the group-velocity $v_G < c$, as one expects.

Now, we want to study the test mass motion in the presence of the third GW polarization in the Rastall theory of gravity. In order to achieve this, one could derive the coordinates transformation from the line element \[50\] to the local Lorentz line element \[52\]. This is, in principle, possible, because an analogous coordinates transformation is well known in the standard GTR case, see for example \[52\]. On the other hand, this is not necessary, because the linearized Riemann tensor is invariant under gauge transformations \[41, \, 53\]. Hence, it can be directly computed from Eq. \[50\]. Again, we can use the analogy with GWs in $f(R)$ theories of gravity and perform a computation similar to the one in \[52\], but keeping in mind the the field is now massless rather than massive. From \[41, \, 53\] one gets

$$ \tilde{R}_{\mu\nu\rho\sigma} = \frac{1}{2} \{ \partial_\mu \partial_\beta h_{\alpha\nu} + \partial_\nu \partial_\alpha h_{\mu\beta} - \partial_\alpha \partial_\beta h_{\mu\nu} - \partial_\mu \partial_\nu h_{\alpha\beta} \}, \quad (61) $$

that, in the case eq. \[50\], reads

$$ \tilde{R}_{0\gamma0} = \frac{1}{2} \{ \partial^\alpha \partial_0 h_{Rast}^\eta_{\eta\gamma} + \partial_0 \partial_\gamma h_{Rast}^\delta_{00} - \partial^\alpha \partial_\gamma h_{Rast}^\delta_{00} - \partial_0 \partial_0 h_{Rast}^\delta_{00} \}. \quad (62) $$

One writes the different elements as (only the non zero ones will be considered)

$$ \partial^\alpha \partial_0 h_{Rast}^\eta_{\eta\gamma} = \begin{cases} \partial_0^2 h_{Rast} & \text{for } \alpha = \gamma = 0 \\ -\partial_z \partial_0 h_{Rast} & \text{for } \alpha = 3; \gamma = 0 \end{cases} \quad (63) $$

$$ \partial_0 \partial_\gamma h_{Rast}^\delta_{00} = \begin{cases} \partial_0^2 h_{Rast} & \text{for } \alpha = \gamma = 0 \\ \partial_\ell \partial_\gamma h_{Rast} & \text{for } \alpha = 0; \gamma = 3 \end{cases} \quad (64) $$
\[-\partial^\alpha \partial_\gamma h_{\text{Rast}} \eta_{00} = \partial^\alpha \partial_\gamma \Phi = \begin{cases} -\partial_2^2 h_{\text{Rast}} & \text{for } \alpha = \gamma = 0 \\ \partial_2^2 h_{\text{Rast}} & \text{for } \alpha = \gamma = 3 \\ -\partial_0 \partial_2 h_{\text{Rast}} & \text{for } \alpha = 0; \gamma = 3 \\ \partial_2 \partial_0 h_{\text{Rast}} & \text{for } \alpha = 3; \gamma = 0 \end{cases} \] (65)

\[-\partial_0 \partial_0 h_{\text{Rast}} \delta^\alpha_\gamma = -\partial_2^2 h_{\text{Rast}} \text{ for } \alpha = \gamma. \] (66)

Now, putting these results in eq. (62) one obtains
\[
\tilde{R}^1_{010} = -\frac{1}{2} \ddot{h}_{\text{Rast}} \\
\tilde{R}^2_{010} = -\frac{1}{2} \ddot{h}_{\text{Rast}} \\
\tilde{R}^3_{030} = 0.
\] (67)

We recall that in the case of $f(R)$ theories of gravity one gets \[53\]
\[
\tilde{R}^1_{010} = -\frac{1}{2} \ddot{h}_f \\
\tilde{R}^2_{010} = -\frac{1}{2} \ddot{h}_f \\
\tilde{R}^3_{030} = \frac{1}{2} m^2 h_f,
\] (68)

instead. In fact, in the case of $f(R)$ theories of gravity the field is massive and not transverse, see \[51, 53\] for details.

Using Eqs. (67) and (63) one gets
\[
\ddot{x} = \frac{1}{2} \ddot{h}_{\text{Rast}} x, \] (69)
\[
\ddot{y} = \frac{1}{2} \ddot{h}_{\text{Rast}} y, \] (70)

which means that in the current Rastall case the field is massless and transverse. Thus, we consider a test mass which is free to move in the plane $z = 0$. Equations (69) and (70) give the tidal acceleration of our test mass caused by the third polarization of the Rastall GW in the $x$ direction and in the $y$ direction respectively. Following \[41\] one can equivalently say that there is a gravitational potential given by
\[
V(\vec{r}, t) = -\frac{1}{4} \ddot{h}_{\text{Rast}} t [x^2 + y^2]. \] (71)

generating the tidal forces. Therefore, the motion of the test mass is governed by the Newtonian equation
One finds the solution of Eqs. (69) and (70) through the perturbation method [41].

To first order in the amplitude $h_{Rast}$ the displacements of the test mass due to the third polarization of the Rastall GW are given by

$$\delta x(t) = \frac{1}{2} x_0 h_{Rast}(t)$$

(73)

and

$$\delta y(t) = \frac{1}{2} y_0 h_{Rast}(t),$$

(74)

where $x_0$ and $y_0$ are the initial coordinates of the test mass, i.e. the coordinates of the test mass before the arrival of the Rastall GW. If one considers again the analogy with GWs in $f(R)$ theories of gravity, one recalls that the displacements of the test mass due to the third polarization of the $f(R)$ GW are instead given by [53]

$$\delta x(t) = \frac{1}{2} x_0 h_f(t)$$

$$\delta y(t) = \frac{1}{2} y_0 h_f(t)$$

$$\delta z(t) = \frac{1}{2} \left(1 - \frac{v^2}{c^2}\right) z_0 h_f(t).$$

(75)

Thus, one see that, differently from the current case of the Rastall theory, in $f(R)$ theories of gravity a longitudinal component is present. In fact, in that case, the effect of the mass is the generation of a *longitudinal* force in addition to the transverse one, see [53].

Hence, one sees that the total displacements of the test mass due to a GW in Rastall theory are different with respect to the total displacements of the test mass due to a GW in the standard GTR and with respect to the total displacements of the test mass due to a GW in $f(R)$ theories of gravity. In the case of the GTR one indeed obtains [41]

$$\delta x(t) = \frac{1}{2} [x_0 Q_+(t) - y_0 Q_\times(t)]$$

$$\delta y(t) = -\frac{1}{2} [y_0 Q_+(t) + x_0 Q_\times(t)]$$

(76)

$$\delta z(t) = 0.$$

In the case of $f(R)$ theories of gravity one gets [53]

$$\delta x(t) = \frac{1}{2} [x_0 Q_+(t) - y_0 Q_\times(t)] + \frac{1}{2} x_0 h_f(t)$$

$$\delta y(t) = -\frac{1}{2} [y_0 Q_+(t) + x_0 Q_\times(t)] + \frac{1}{2} y_0 h_f(t)$$

$$\delta z(t) = \frac{1}{2} \left(1 - \frac{v^2}{c^2}\right) z_0 h_f(t).$$

(77)
Finally, in the case of the Rastall theory of gravity one obtains

\[
\delta x(t) = \frac{1}{2}[x_0 Q_+ (t) - y_0 Q_\times (t)] + \frac{1}{2}x_0 h_{Rast}(t)
\]

\[
\delta y(t) = -\frac{1}{2}[y_0 Q_+ (t) + x_0 Q_\times (t)] + \frac{1}{2}y_0 h_{Rast}(t)
\]

\[
\delta z(t) = 0.
\]

Thus, Eqs. (76), (77) and (78) can, in principle, be used in order to discriminate among the GTR, \( f(R) \) theories of gravity and the Rastall theory of gravity. On the other hand, at the present time, the sensitivity of the current ground based GW interferometers is not sufficiently high to determine if the total displacements of the test mass are governed by Eqs. (76), or if they are governed by Eqs. (77) or by Eqs. (78). A network including various interferometers in addition to LIGO and Virgo with different orientations is indeed required. In fact, one hopes that future advancements in ground-based projects and space-based projects will have a sufficiently high sensitivity to determine, with absolute precision, the direction of the GW propagation and the motion of the various involved test masses. In other words, in the nascent GW astronomy we hope not only to obtain new, precious astrophysical information, but we also hope to be able to discriminate between Eqs. (76), Eqs. (77) and Eqs. (78).

6 Conclusion remarks

The era of the GW astronomy which recently started with the events GW150914 [1], GW151226 [2], GW170104 [3], GW170814 [8] and GW170817 [9] is considered a cornerstone for science and for gravitational physics in particular. On one hand, GW astronomy permits to obtain new fundamental astrophysical information from the Universe. On the other hand, it could ultimately discriminate among the GTR and alternative gravitational theories. At the present time, despite the cited events have put strong constrains on the GTR, alternative gravitational theories are still viable. In this paper we focused on the Rastall theory of gravity, a particular extended theory of gravity which recently obtained an increasing interest in the literature. Following a profound analogy between GWs in \( f(R) \) theories and GWs in the Rastall theory, we linearized the Rastall field equations and found the corresponding GWs. After that, the motion of the test masses due to GWs in this theory has been also studied. This could help, in principle, to discriminate between the GTR, \( f(R) \) theories and the Rastall theory of gravity. In fact, the main result of this paper is the system of equations (78) which governs the total displacements of a test mass due to a GW in Rastall theory. Despite the sensitivity of the current ground based GW interferometers is not sufficiently high to determine if the total displacements of the test mass are governed by such equations or by Eqs. (76), which are the traditional equations governing the total displacements of the test mass due to a GW in the standard GTR, or by Eqs. (77), which are the traditional equations...
governing the total displacements of the test mass due to a GW in $f(R)$ theories, we hope in future advancements in ground-based projects and space-based projects.

As a final remark we start to discuss an intriguing issue, which could be, in principle, developed in the future. Following the analogy between $f(R)$ theories and the Rastall theory of gravity, in this paper we have shown that also the Rastall theory admits a third GW polarization like the case of $f(R)$ theories of gravity. The difference is that in $f(R)$ theories such a third mode is massive while in the Rastall theory it is massless. We have also seen that in both of the cases the existence of the third polarization is due to the presence of additional curvature terms in the gravitational action. Remarkably, we know that, in general, all the extended theories of gravity should have additional polarizations with respect to the standard $+$ and $\times$ polarizations of the GTR, see for example [54]. Thus, one can ask which is the physical meaning of those additional polarizations. Is there any general principle behind the existence of such additional polarizations in extended gravity? On one hand, we think that, in some way, the solution could be in the vacuum re-definition. It could be necessary to give up the notion of “perfect vacuum” and this could enable a plurality of gauge conditions which must be compatible each other. On the other hand, if one considers the particular case of the Rastall theory one can see this theory as a relativistic rewriting of Newtonian theory. Then, the two standard $+$ and $\times$ polarizations depends on the field structure of the GTR within the Maxwellian approach, see [33], and the third mode should arise from the Newtonian background. Developing these thoughts in a rigorous way could be not banal and could require further profound analysis, maybe searching analogies also with the Einstein-Cartan theory [55 - 58]. In that case, the algebraic connection between the Rastall and Einstein-Cartan theories should be the "freedom degree" respect to the GTR, and it should be physically equivalent to "double face" vacuum. Then, also the Einstein-Cartan theory might have a third polarization model.

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