The statistic thermodynamical entropy of de Sitter space and it’s 1-loop correction

Bo-Bo Wang* and Liao Liu†

Department of Physics, Beijing Normal University, Beijing 100875, China.

Abstract

From the partition function of canonical ensemble we derive the entropy of the de Sitter space by anti-Wick rotation. And then from the one-loop bubble $S^2 \times S^2$ created from vacuum fluctuation in de Sitter background space, we obtain the one-loop quantum correction to the entropy of de Sitter space.

PACS numbers: 04.70.Dy.

*E-mail: bbwang@mail.263.net.cn
†E-mail: liaoliu@bnu.edu.cn
I. THE ENTROPY OF THE DE SITTER SPACE

Gibbons and Hawking was the first one who argued that the de Sitter space has entropy which equals to the fourth of the area of cosmological horizon of the de Sitter space \[1\]. However a derivation based on canonical ensemble in quantum statistic thermodynamics is wanted.

As in order to cancel the coordinate singularity and to reveal the imaginary period of time variable of the static de Sitter space

\[
ds^2 = - \left( 1 - \frac{\Lambda r^2}{3} \right) dt^2 + \left( 1 - \frac{\Lambda r^2}{3} \right)^{-1} dr^2 + r^2 d\Omega^2, \tag{1}
\]

where \(\Lambda\) is the cosmological constant, \(d\Omega^2 = d\theta^2 + \sin^2 \theta d\varphi^2\) is the unit 2-sphere. One should introduce a Kruskal-like coordinate \(U, V, \theta, \varphi\) by

\[
r = \sqrt{3 \frac{UV + 1}{\Lambda} - UV}, \tag{2}
\]

\[
\exp \left( 2 \sqrt{\frac{\Lambda}{3}} t \right) = -VU^{-1}, \tag{3}
\]

then Eq.(1) becomes

\[
ds^2 = 3\Lambda^{-1} (UV - 1)^{-2} \left[ -4dUdV + (UV + 1)^2 d\Omega^2 \right]. \tag{4}
\]

From Eqs.(2) and (3), the space-time has clearly an imaginary period of

\[
\beta = 2\pi \sqrt{\frac{3}{\Lambda}}. \tag{5}
\]

As is known, for any system including the gravity, if time has an imaginary period then by using Schrödinger picture, the partition function of the system will be the partition function of the canonical ensemble \[2\], i.e.

\[
Z = \sum_n \exp (-E_n \beta) = \int D[g] \exp \left( -\hat{I} [g] \right) \tag{6}
\]

and
\[ \ln Z = -\hat{I}, \quad (7) \]

where the summation is over the different quantum state, \( \hat{I} \) is the Euclidean action, Eq.(7) holds in saddle point approximation. From

\[ S = -\sum_n P_n \ln P_n, \quad (8) \]

and

\[ P_n = Z^{-1} \exp(-\beta E_n) \quad (9) \]

we get the entropy \( S \) and the energy \( \langle E \rangle \) of the gravitating system

\[ \langle E \rangle = -\frac{\partial}{\partial \beta} \ln Z \quad (10) \]

\[ S = \beta \langle E \rangle + \ln Z. \quad (11) \]

Eqs.(6)-(11) are the so-called quantum gravitational statistic thermodynamics in canonical ensemble.

Now the problem is how to define the Euclidean action of the above manifold.

As is known, the Euclidean section \( S^4 \) of the de Sitter manifold has no boundary, so the contribution to the action of the de Sitter space comes only from the volume term.

\[ I = \text{volume term} = \frac{1}{16\pi} \int \sqrt{-g} (R - 2\Lambda) d^4x \]

\[ = \frac{1}{16\pi} \int \sqrt{-g} (R - 2\Lambda) dt d^3x. \quad (12) \]

In order that the path-integral

\[ \int D[g] \exp(iI) \quad (13) \]

be convergent, we usually put

\[ -iI \equiv \hat{I}, \quad \text{or} \int D[g] \exp(-\hat{I}) \quad (14) \]
wherein the wick rotation of the time variable $t \to -i\tau$ in $I$ is used, $\hat{I}$ is then the so-called Euclidean action. As is known from positive action theorem, the Euclidean action $\hat{I}$ obtained by such way is always positive for matter field and certain gravitational field (e.g. Schwarzschild spacetime) but it is not for de Sitter space-time. As can be easily shown from the above Euclideanization (14) of (12) 

\[
\hat{I} = -\frac{1}{16\pi} \int \sqrt{g} (R - 2\Lambda) d\tau d^3x = -3\pi\Lambda^{-1} < 0
\]  

(15)

for Euclidean de Sitter space (1). However now Eq.(14) can’t be convergent for the “wrong sign –” in Eq.(15). Instead of (14), if we put

\[-iI = \hat{I}\]

(16)

but anti-Wick rotation

\[t \to i\tau\]

(17)

is used, then

\[
\hat{I} = \frac{1}{16\pi} \int \sqrt{g} (R - 2\Lambda) d\tau d^3x = 3\pi\Lambda^{-1} > 0.
\]  

(18)

The “wrong sign” in Eq.(13) disappears and Eq.(14) becomes

\[
\int D[g] \exp (-\hat{I}), \quad \hat{I} > 0
\]  

(19)

which is convergent.

We don’t wish to be involved in the debate of choosing Wick rotation or anti-Wick rotation in field theory including gravitation raised by Hawking and Linde several years ago \[3\]. Any way we believe that in de Sitter case the only right choice seems to be anti-Wick rotation.

From Eqs.(3) and (14), we can easily get for de Sitter space

\[
\hat{I} = \frac{1}{\pi} \beta^2, \quad \langle E \rangle = \frac{1}{2\pi} \beta = \sqrt{\frac{3}{\Lambda}}, \quad S = \frac{1}{4\pi} \beta^2 = \frac{3\pi}{\Lambda} = \frac{1}{4} A_c,
\]  

(20)

(21)

(22)
where $A_c = 12\pi \Lambda^{-1}$ is the area of the cosmological horizon of de Sitter space and the average energy $\langle E \rangle$ of the de Sitter space in canonical ensemble is two times the classical vacuum energy $E_{vac} = (1/2) \sqrt{3/\Lambda} = \beta/(2\pi)$ of de Sitter space. The above calculation shows one can equally well using partition function of the canonical ensemble to get the entropy of de Sitter space if anti-Wick rotation is used.

II. ONE LOOP QUANTUM CORRECTION OF THE ENTROPY OF DE SITTER SPACE

Hawking conjectured many years ago that the space-time foam may be formed from $S^2 \times S^2$ bubbles [2,4]. However a proof of this conjecture is wanted. Recently in Ref. [5] it is shown that the $S^2 \times S^2$ bubbles can be created really from vacuum fluctuation in one-loop approximation of both steady state universe

$$ds^2 = -dt^2 + \exp\left(\frac{2t}{\alpha}\right) \sum_{i=1}^{3} (dx^i)^2, \quad \alpha \equiv \sqrt{\frac{3}{\Lambda}}$$

(23)

and closed de Sitter universe

$$ds^2 = -dt^2 + \alpha^2 \cosh^2\left(\frac{t}{\alpha}\right) \left(d\chi^2 + \sin^2\chi d\Omega^2\right).$$

(24)

A brief account of the proof is given as follows.

The space-time metrics (23) and (24) are conformal flat and their conformal vacuum is Minkovsky vacuum $|0\rangle_M$. In such conformal trivial case the renormalized vacuum energy-stress tensor from vacuum fluctuation of space-time (23) and (24) in one-loop approximation is given in signature $+2$ by [8]

$$\langle 0 \mid T^\nu_\mu \mid 0 \rangle_{ren} = -\frac{q(s)}{960\pi^2\alpha^2}g^\nu_\mu, \quad \alpha = \sqrt{\frac{3}{\Lambda}},$$

(25)

where $s$ is the spin of the matter field,

$$q(0) = 1, q\left(\frac{1}{2}\right) = \frac{11}{2}, q(1) = 62.$$  

(26)

Henceforth we put $q(s) = q(0) = 1$.  

5
For obtaining the foam structure of certain background spacetime, we should consider the loop expansion of the effective action. In the semiclassical Einstein gravitational theory, the loop expansion of the effective action $\Gamma_m [g]$ of a quantum matter field can be expressed as

$$\Gamma_m [g, \varphi] = \frac{1}{\hbar} \Gamma_m^{(0)} [g, \varphi] + \Gamma_m^{(1)} [g, \varphi] + \hbar \Gamma_m^{(2)} [g, \varphi] + \cdots,$$

(27)

where $\hbar$ is the Planck constant, $\Gamma_m^{(0)} [g, \varphi]$ is the classical matter action or 0-loop approximation of quantum matter field, $\Gamma_m^{(1)} [g, \varphi]$ is the renormalized 1-loop contribution to the effective action of quantum matter field, $\Gamma_m^{(2)} [g, \varphi]$ is the renormalized 2-loop contribution to the effective action of quantum matter field, and so on.

The semiclassical Einstein gravitational field equation then reads

$$\delta [I_g [g] + \Gamma_m [g, \varphi]] = 0,$$

(28)

where $I_g [g]$ is the classical Einstein-Hilbert action. Substituting Eq.(27) into Eq.(28), one can obtain the so called self-consistent solution of Eq.(28), where the back reaction of quantum matter field on the classical background geometry is included. However in this calculation no separation of spacetime foam-like structure from the classical background geometry structure is possible, so the idea of spacetime foam-like picture is rather ambiguous. But if one notes that the deviation of the classical background geometry caused by n-th-loop contribution of the quantum matter field, say $\Gamma_m^{(n)} [g, \varphi]$, should be the same order of magnitude as $\Gamma_m^{(n)} [g, \varphi]$ and so on, we may also give a similar “loop-expansion” of $I_g [g]$ as

$$I_g [g] = \frac{1}{\hbar} I_g^{(0)} [g] + I_g^{(1)} [g] + \hbar I_g^{(2)} [g] + \cdots,$$

(29)

though the gravitational field now is not quantized. Inserting Eqs.(27) and (29) into Eq.(28), just by counting powers of $\hbar$, we can obtain a series of gravitational field equation

$$\delta (I_g^{(0)} + \Gamma_m^{(0)}) = 0,$$

(30)

$$\delta (I_g^{(1)} + \Gamma_m^{(1)}) = 0,$$

(31)

$$\delta (I_g^{(2)} + \Gamma_m^{(2)}) = 0,$$

(32)

$$\cdots \cdots,$$
where the solution of equation (30) is just the classical background geometry. The solution of equation (31) is the geometry created from 1-loop contribution of quantum matter vacuum fluctuation, and so on.

The advantage of the above loop expansion Eqs. (27) and (29) is that now we have a clear separation of the background geometry from the spacetime foam geometry created from the fluctuation of the vacuum matter field of the classical background geometry, so as a result, a concrete picture of spacetime foam-like structure is obtained.

According to Hawking’s conjecture the spacetime foam-like structure might be consisted of $S^2 \times S^2$ bubble, which is a solution (Nariai solution) of Einstein gravitational field equation, so we may further assume that all “loop actions” of different order are of Einstein-Hilbert type.

Now if the bubbles are really created from one-loop vacuum fluctuation of the background space-time in signature (+2), then Eq.(31) should be

$$G_{\nu}^{\mu} = 8\pi \langle 0 \left| T_{\mu}^{\nu} \right| 0 \rangle_{ren}, \quad (33)$$

which is satisfied by the renormalized vacuum energy-stress tensor (25) and the Einstein tensor $G_{\mu}^{\nu}$ of the bubble $S^2 \times S^2$.

As is known the metric of $S^2 \times S^2$ is given by Nariai, Bousso and Hawking

$$ds^2 = \frac{1}{\lambda} \left( \sin^2 \chi d\Psi^2 + d\chi^2 + d\Omega^2 \right) \quad (34)$$

in Euclidean signature, and

$$ds^2 = \frac{1}{\lambda} \left( -\sin^2 \chi d\Psi^2 + d\chi^2 + d\Omega^2 \right) \quad (35)$$

in Lorentzian signature of (+2), where $\lambda^{-1/2}$ is the radius of the 2-sphere $S^2$.

It is straightforward to show that the Einstein tensor $G_{\mu}^{\nu}$ of Eq.(35) is

$$G_{\mu}^{\nu} = -\lambda g_{\mu}^{\nu}. \quad (36)$$

So Eq.(33) is satisfied if
\[ \lambda = \frac{8\pi}{9 \times 960\pi^2} \Lambda^2 = \frac{1}{120\pi\alpha^4} = \frac{\pi^3}{120\beta^{-4}}, \quad \beta = \pi\alpha. \] (37)

Now we would like to study the one-loop correction of the entropy of de Sitter space from \( S^2 \times S^2 \) bubbles.

For de Sitter space of metric (1), (23) and (24), we have

\[ \hat{I} = 3\pi\Lambda^{-1} = \frac{1}{4\pi} \beta^2. \] (38)

However the vacuum energy-stress tensor (25) of vacuum fluctuation in one-loop approximation is the same only for steady state de Sitter space (23) and closed de Sitter space (24), but different for static de Sitter space (1), the latter is [6]

\[ \langle 0 \left| T^\mu_\nu \right| 0 \rangle_{\text{static dS}} = \langle 0 \left| T^\mu_\nu \right| 0 \rangle_{(25)} + \langle 0 \left| T^\mu_\nu \right| 0 \rangle_{\text{Rindler}} \]

\[ = -\frac{q(s)}{960\pi^2\alpha^4} \delta_\mu^\nu - \frac{p(s)}{2\pi^2} \left( \alpha^2 - r^2 \right)^{-2} \text{diag.} \left( 1, -\frac{1}{3}, -\frac{1}{3}, -\frac{1}{3} \right), \] (39)

for the conformal vacuum of (1) is Rindler vacuum. Here \( \langle 0 \left| T^\mu_\nu \right| 0 \rangle_{(25)} \) means the renormalized vacuum energy-stress tensor of Eq.(25). However their trace is the same, i.e.,

\[ \langle 0 \left| T^\mu_\mu \right| 0 \rangle_{\text{static dS}} = \langle 0 \left| T^\mu_\mu \right| 0 \rangle_{(25)} + 0. \] (40)

So from \( R = 8\pi \langle T \rangle \) the scalar curvature of the bubble created from the vacuum fluctuation of the background static de Sitter space (1) is the same as that from background de Sitter spaces (23) and (24). This implies that the action \( \hat{I}_{\text{one-loop}} \) of \( S^2 \times S^2 \) bubble for the above three different de Sitter spaces (1), (23) and (24) are the same. It is easy to find [2,7]

\[ \hat{I}_{\text{one-loop}} = \hat{I} \left[ S^2 \times S^2 \right] = 2\pi\lambda^{-1} = \frac{15}{\pi^2} \beta^4, \] (41)

where the anti-Wick rotation and the relation (37) are used.

Then the one-loop correction of the action and entropy of the de Sitter space are

\[ \hat{I} = \frac{1}{4\pi} \beta^2 + \frac{15}{\pi^2} \beta^4 \] (42)

and
\[ S = -\beta \frac{\partial}{\partial \beta} \ln Z + \ln Z = \frac{1}{4\pi} \beta^2 + \frac{45}{\pi^2} \beta^4 \] (43)

respectively.

Obviously, the above formulas only hold when \( \beta^4 << \beta^2 \), or \( \Lambda >> L_p^{-2} \sim 10^{66} cm^{-2} \), where \( L_p \) is the Planck length. This is the case for very early inflating universe of size \( \sqrt{3/\Lambda} \ll L_p \). The results of (43) implies, that if the de Sitter space could be looked upon as a canonical ensemble, then the bubbles created from vacuum fluctuation in 1-loop approximation will induce an entropy increase of \( 45\beta^4/\pi^2 \) to the de Sitter space. It is reasonable, for such an increase of the entropy implies an increase of disorder when vacuum fluctuation is considered.

ACKNOWLEDGMENTS

We thank Dr. Chao-Guang Huang for his helpful discussion. This work is supported by National Natural Sciences Foundation of China under grant 19473005.
REFERENCES

[1] G. Gibbons and S. W. Hawking, *Phy. Rev.* D 15, 2738 (1977).

[2] S. W. Hawking, in *General Relativity: An Einstein centenary Survey*, ed by S. W. Hawking and W. Israel, (Cambridge University Press 1979).

[3] A. Linde, gr-qc/9802038; S. W. Hawking and N. Turok, gr-qc/9802062.

[4] S. W. Hawking, *Phys. Rev.* D 53, 3099 (1996).

[5] to be published by *Chinese Physics Letters* (2001).

[6] N. C. Birrell and P. C. W. Davies, *Quantum fields in curved space*, Cambridge university Press (1982).

[7] R. Bousso, S. W. Hawking, *Phys. Rev.* D 54, 6312 (1996).

[8] M. V. Fischetti, J. B. Hartle and B. L. Hu, *Phys. Rev.* D 20, 1757 (1979).