Couplings for Compactification

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ABSTRACT
A general formula is obtained for Yukawa couplings in compactification on Landau-Ginzburg orbifolds and corresponding Calabi-Yau spaces. Up to the kinetic term normalizations, this equates the classical Koszul ring structure with the Landau-Ginzburg orbifold chiral ring structure and the true superconformal field theory ring structure.
1. Introduction, Results and Synopsis

In any Kaluża-Klein type compactification on a compact ‘internal space’ \( \mathcal{M} \), the coupling parameters of the effective spacetime field theory depend on the geometry of this ‘internal space’ and can often be determined from it \([1,2]\). Even when a geometrical description of this ‘internal sector’ eludes us, valuable selection rules and sometimes even exact and complete solutions can be obtained; see for example, Refs. \([3]\).

Consider superstring Calabi-Yau compactifications, that is, superstring models with an ‘internal sector’ \( \mathcal{M} \) which is known to correspond to a compact, complex 3-fold with trivial canonical class. Superstring propagation is governed by a \( \sigma \)-model which may be viewed as (a) describing the world sheet \( \Sigma \) immersed in \( \mathcal{M} \) or (b) a 2-dimensional quantum field theory on \( \Sigma \). The geometry of maps \( \Sigma \rightarrow \mathcal{M} \) is then promisingly linked to the quantum field theory aspects of such models as one can use the methods in one field to obtain results in the other; correlation functions of the 2-dimensional field theory are couplings of the compactified effective spacetime field theory.

Clearly, the analysis presented here may also be regarded as a toy model: the geometry of the target \( \mathcal{M} \) and the configuration space \( \{ \Sigma \rightarrow \mathcal{M} \} \) of any \( \sigma \)-model may be employed to advance our understanding of quantum field theory. In this sense, our results pertain to the algebra of observables in a wide class of quantum field theories. Since the algebraic geometry which we use remains valid regardless of the triviality of the canonical class (criticality), the relations and applications to field theory can be made equally general, although perhaps not as easily interpreted. For sake of immediate application, we do focus on (2,2)-supersymmetric, that is, Calabi-Yau compactifications only and will confine our cross-dictionary accordingly.

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Take, for instance, \( \mathcal{M} \) to be a Calabi-Yau complete intersection of hypersurfaces in a product of flag-spaces \( \mathbb{F}_{\vec{n}} \overset{\text{def}}{=} U(N)/(\prod_i U(n_i)) \), where \( N = \sum_i n_i \). Already the simplest case involving products of projective spaces \( \mathbb{P}^n = U(n+1)/(U(1) \times U(n)) \) provides several thousand examples \([4]\). The weighted (quasi-homogeneous) variants are obtained as quotients \( \mathbb{F}_{\vec{n}}/\Delta_{\vec{m}} \), with quasi-homogeneous coordinates \( y_i \overset{\text{def}}{=} x_i^{m_i} \), and where \( \Delta_{\vec{m}} = \prod_i \mathbb{Z}_{m_i} \) is the cyclic group which leaves the \( y_i \) invariant \([5]\). Again, even just simple hypersurfaces in a single weighted projective space provide several thousand examples \([4]\).

For all such \( \mathcal{M} \) (in fact, regardless of being Calabi-Yau), all desired cohomology may be computed using equivariant (co)homological algebra based on the Koszul sequence and the Bott-Borel-Weil Theorem \([3,7,8]\); call this simply ‘Koszul computation’.

A favorable subset of such manifolds define Landau-Ginzburg orbifolds and thereby superconformal field theories \([9,10]\) as the limit of their renormalization flow; see however also Refs. \([11,12]\). The 27 and 27\(^*\) massless fields in such models may be found by semiclassical methods \([13]\) and correspond to charge-(1,1) and -(−1,1) states in the (2,2)-supersymmetric field theory. Note that each flag-space may be identified with a quotient space other than in the definition above: for example, \( \mathbb{P}^n = U(n+1)/(U(1) \times U(n)) \) also equals the quotient \( \mathbb{P}^n = \mathbb{C}^{n+1}/\mathbb{C}^* \), where \( \mathbb{C}^* \) is the multiplicative group by non-zero
complex numbers. The defining polynomial of the hypersurface \( f(x) = 0 \) in \( \mathbb{P}^n \) becomes the superpotential of the Landau-Ginzburg field theory, with the fields spanning the affine space \( \mathbb{C}^{n+1} \); the quotient thereof is the Landau-Ginzburg orbifold \(^1\) which corresponds to the projective space \( \mathcal{M} \hookrightarrow \mathbb{P}^n \). Thus, we write \( \mathbb{P}^d(w_1,...,w_5)[d] \) for the family of degree-\( d \) hypersurfaces in the weighted projective 4-space with weights \( w_i \), but \( \mathbb{C}_5^d(w_1,...,w_5)[d] \) for the affine configuration space of the corresponding Landau-Ginzburg field theory.

It was proved \(^1\) that beyond the final agreement of the spectra as found, respectively, by the Koszul and the Landau-Ginzburg orbifold calculations, the two approaches display a remarkably precise correspondence in many details. This is most easily stated as being a (somewhat formal) isomorphism between the ‘Chiral Ring’ structure of Landau-Ginzburg orbifolds \(^1\) and the Jacobian ring structure of the cohomology as obtained by the Koszul computation \(^8,12,15\).

The purpose of this article is twofold. (a) The general formula for the Yukawa couplings in the Koszul computation \(^8\) is herein shown to agree with the standard Landau-Ginzburg orbifold analysis. (b) Some computational short-cuts and identities are discussed which facilitate toggling between the Koszul and the Landau-Ginzburg orbifold description. In addition, a preliminary comparison is discussed with those exactly soluble superconformal field theories \(^16\) where both a Koszul and a Landau-Ginzburg orbifold description are also known, suggesting an equivalence of these three ring structures.

The underlying reason for the existence of such a relation is the common notion of the function ring—in fact algebra\(^2\). It is practically a tautology that a (quantum) field theory is specified by its algebra of observables, that is, an additive group of operators with the ring (algebra) structure specified through the correlation functions; no Lagrangian or action functionals need be known. Similarly, the ultimate point of view of algebraic geometry is that a variety itself may be ignored and that ‘all one needs to know’ is the function ring \(^7\). In Landau-Ginzburg orbifolds alike, the superpotential serves for computations, but the desired information lies in the Chiral ring.

Besides these general arguments, sample computations are also included supporting the above claims and to demonstrate that the normalization of the Chiral (Jacobian) ring elements remains free to adjust for agreement with exactly soluble superconformal field theory models.

Roughly, our strategy is to establish a 1–1 correspondence between the Koszul \(^7,8\), the Landau-Ginzburg orbifold \(^13\), working ultimately towards the Gepner-type \(^3,16,18\) computations. Since the last of these is most exact but least general, the idea is to “analytically continue” (in moduli space) these exact results of Gepner-type models to all Koszul computations and so extend the work reported in Ref. \(^12\). Following Refs. \(^2,8,12\)

\(^1\) To find the spectrum, passing to a quotient by a finite subgroup of \( \mathbb{C}^\ast \) suffices.

\(^2\) A ring is a commutative group with respect to addition, equipped with a multiplication that obeys the usual distributive laws. Actually, most of the commonly encountered rings in physics here are in fact algebras: the ring elements may be multiplied freely by complex scalars, which form the ground ring, moreover the ground field, \( \mathbb{C} \).
and others, a few examples will be presented in detail hoping that the general case and limitations will be clear.

The article is organized as follows. In section 2, we describe the typical results of the Koszul and the Landau-Ginzburg orbifold calculations, that is, their respective descriptions of the spectrum of massless fields. In section 3, we focus on the \((c, c)\)-sector only and study a simple model and its ‘ineffectively split’ version. The \((a, c)\)-sector is then studied, in section 4, by applying our \((c, c)\)-sector results to the mirror of a well known model which was used to obtain a 3-generation theory \([19]\). Section 5 contains a brief discussion of a comparison with exactly soluble models and some additional comments. Appendix A presents a gauge theoretic introduction to the Koszul computation while the equality of the Koszul ideal and the ideal of the Chiral Ring structure is proven in Appendix B.

2. Koszul vs. Landau-Ginzburg Orbifold Spectra

At the risk of overlapping with the recent literature, a telegraphic inventory of the standard toolboxes for algebraic geometry and for \((2,2)\)-supersymmetric effective field theory is presented here. Refs. \([5,7,8]\) and \([13]\), respectively, are recommended for further information and details.

2.1. Koszul field representatives

The quickest (and perhaps conceptually simplest) way of introducing this framework begins by starting with the embedding spaces of our choice, which are products of flag varieties \(F_{n} \overset{\text{def}}{=} U(N)/(\prod_{i} U(n_{i}))\). The Bott-Borel-Weil theorem ensures that (homogeneous, holomorphic) bundles over such spaces are classified as representations of \(\prod_{i} U(n_{i})\), while cohomology valued in these bundles furnishes representations of \(U(N)\), where \(N = \sum_{i} n_{i}\). Representations of \(SU(n)\) groups are rather well known to physicists and suffice it here to note that \(U(n) \approx U(1) \times SU(n)\), so that representations of \(U(n)\) are specified as those of \(SU(n)\), with an additional \(U(1)\) charge. Appendix A provides more details about the application of the Bott-Borel-Weil theorem.

As is well known, \(SU(n)\) representations are easily assigned tensorial variables. Irreducible \(U(n)\)-tensors have their indices (anti)symmetrized and all possible traces are subtracted. Note that traces are taken with the only invariant tensor of \(U(n)\): the Kronecker \(\delta^{b}_{a}\) symbol; the totally antisymmetric symbol \(\epsilon^{a_{1} \ldots a_{n}}\) is an \(SU(n)\), but not a \(U(n)\) invariant—its \(U(1)\)-charge is \(n\).
Now, the spaces under study are not the entire flag spaces, but their subspaces defined as the zero-set of a system of holomorphic homogeneous polynomial constraint equations—complete intersections. Each one of these equations may be written as \( f(x) = 0 \), where \( f(x) \) is the defining polynomial and may be written as

\[
f(x) = f_{(ab...c)} x_a x_b \ldots x_c , \quad \text{deg}(f) = d ,
\]

in terms of suitable homogeneous coordinates \( x_a \). This is equally well represented by \( f_{(ab...c)} \), which we call the defining tensor (coefficient).

**Notation:** Tensor coefficient will be indexed according to the standard convention: subscripts for co-variant and superscripts for contra-variant vectors. However, with a little forethought, coordinates are indexed by subscripts although they are actually contra-variant. Thus, \( \phi_a \) and \( \gamma^a \) denote tensor coefficients of a co- and contra-variant vector, so that \( \phi_a \gamma^a \) is invariant; owing to this exceptional indexing of coordinates, \( \phi_a \) can be contracted with \( x_a \), but \( \gamma^a \) and \( x_a \) cannot; they transform alike.

It then follows that all tensor–valued cohomology on a complete intersection submanifold is representable in terms of \( U(N) \)-tenors, which are however further reduced by taking traces with the (dual) defining tensors \([5]\).

In other words, the tensor algebra on a flag space is generated by the variously symmetrized traceless tensors, their usual products and contractions with the aid of the Kronecker \( \delta^b_a \). However, on the subspace \( \mathcal{M} \) where \( f(x) = f_{a...c} x_a \ldots x_c \) vanishes, contractions with the aid of \( f_{a...c} \) are also invariant and so tensors on \( \mathcal{M} \) are irreducible only upon also subtracting \( f_{a...c} \)-traces.

In this approach, therefore, the required field representatives are obtained as various (components of) tensor coefficients. Typically\(^3\), these will be totally symmetric tensors or tensors which factorize into products of the totally antisymmetric symbol \( \varepsilon^{a_1 \ldots a_n} \) and totally symmetric tensors.

### 2.2. Landau-Ginzburg field representatives

A Landau-Ginzburg orbifold model is given in terms of a set of chiral superfields \( X_a \) and a superpotential \( P(X) \); the kinetic term is typically taken to be the flat one, \( \int d^2 \theta d^2 \bar{\theta} \|X\|^2 \) and may be ignored for most part of the analysis. Corresponding to projective spaces in the (quasi)homogeneous flag-spaces, the superpotential \( P(X) \) is homogeneous. Such field theories fall in the general category of Wess-Zumino models, which have been studied extensively in the last two decades. A number of simple results can be derived about the correlation functions, such as the fact that the chiral \( n \)-point functions \( \langle \prod_i (X_{a_i}(z_i))^{k_i} \rangle \) do not depend on the positions \( z_i \) and are largely determined by the PCAC theorem\([20]\).

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\(^3\) This is so for complete intersections of hypersurfaces. More general tensors arise for intersections of higher codimension subspaces, i.e. when more constraint equations are required globally than is the difference between the dimension of the embedding and the embedded space; see Ref. \([3]\).
Refs. [13] provide a well-adopted technique for listing the states in the \((c,c)\)- and \((a,c)\)-rings of a Landau-Ginzburg orbifold. Firstly, by (quasi)homogeneity of the superpotential,

\[ \lambda P(X_a, Y_a) = P(\lambda^{q_a} X_a, \lambda^{q_a} Y_a), \]

with \(q_a = n_a/d\) and \(q_\alpha = n_\alpha/d\), where \(d\), the total degree, and \(n_a, n_\alpha\) are integers. One then considers the general cyclic symmetry \(\Theta\) with a \(d\)-diagonal action, \(X_a, Y_\alpha \mapsto (e^{2i\pi \theta_a} X_a, e^{2i\pi \theta_\alpha} Y_\alpha)\). When \(\theta_a = q_a\) and \(\theta_\alpha = q_\alpha\), \(\Theta\) is the \(U(1)\)-current \(J_0\). Clearly, as charges of a cyclic symmetry, \(\theta_a\) and \(\theta_\alpha\) are defined only up to integers. Upon passing to the quotient by \(\Theta\), the Hilbert space of the field theory decomposes into sectors, one for each element of \(\Theta\). As each element of the cyclic group \(\Theta\) is obtained as a power (denoted \(\ell\)) of a chosen generator, this conveniently labels the sectors. The general formula for the charges of the Ramond vacuum in the \(\ell\)-twisted sector is [13]:

\[ \frac{J_0}{J_0} |0\rangle^\ell_R = \left\{ \pm \left[ \sum_{\Theta_i(\ell) \notin \mathbb{Z}} (\Theta_i(\ell) - \lfloor \Theta_i(\ell) \rfloor - \frac{1}{2}) \right] \right\} |0\rangle^\ell_R, \]

where \(\Theta_i(\ell)\), typically \(\Theta_i(\ell) = \ell \theta_i\), is the twisting angle of the \(i\)th field in the \(\ell\)th sector. The matching \(|0\rangle^\ell_{(c,c)}\) and \(|0\rangle^\ell_{(a,c)}\) vacua are obtained by spectral flows \(U(1/2,1/2)\) and \(U(-1/2,1/2)\), of charges \((\frac{3}{2}, \frac{3}{2})\) and \((-\frac{3}{2}, \frac{3}{2})\), respectively; note the \(\ell \rightarrow \ell + 1\) shift in the Ramond \(\rightarrow\) (a,c) flow.

Upon obtaining a full list of \((c,c)\)- and \((a,c)\)-vacua for each sector, one looks for charge-(1,1) and charge-(-1,1) states of the form \(\Phi |0\rangle^\ell\) where \(\Phi |0\rangle^\ell\) is left invariant by the quotient group \(\Theta\). Here, \(\Phi\) is a polynomial (possibly just the identity) in those fields which are left invariant by the \(\Theta_i(\ell)\) action and \(|0\rangle^\ell\) is the \(\ell\)-twisted vacuum. Clearly, the charges of the polynomial \(\Phi\) and the vacuum \(|0\rangle^\ell\) ought to add up to \((\pm 1, 1)\) for a marginal operator (which corresponds to a massless field in the spacetime effective field theory). Finally, two polynomials are considered equivalent if they differ by a multiple of a gradient of the superpotential. This is because, modulo the equations of motion, \(\partial P/\partial X\) is proportional to \(D^2 \overline{X}\), which is a descendant field and yields no new information.

In addition, from general conformal field theory consideration, we know that (in the conformal field theory limit) each twisted vacuum may be obtained from the untwisted vacuum by multiplication with an appropriate twist-field.

2.3. Koszul vs. Landau-Ginzburg dictionary

By straightforward contraction, tensors are dual to formal polynomials in \(x_a\), \(dx_a\) (\(dx_a\) is replaced by spinors \(\psi^a\) in a corresponding field theory) and their formal duals. Corresponding to the fact that irreducible tensors on the hypersurface \(f(x) = 0\) have no \(f(x)\)-traces, these formal polynomials are taken modulo multiples of \(f(x)\).

In fact, \(U(N)\)-linear transformations of the global homogeneous coordinates on the embedding flag space cannot have any intrinsic effect on the flag space or any of its manifolds. Since

\[ x_i \lambda_i^j \partial_j f(x) = \lambda_{(i}^a f_{a|b...c}) x_i x_b \ldots x_c, \]

(2.4)
is the $U(N)$-linear transformation of $f(x)$, irreducible symmetric tensors (dual to polynomials in $x_a$) on the hypersurface $f(x) = 0$ vanish upon contraction with all but one index of $f(a\ldots c)$.

Thus, we naturally expect the Koszul and the Landau-Ginzburg orbifold field representatives to be each other's duals. Indeed, in the untwisted sector of any Landau-Ginzburg orbifold this duality checks out readily. All Landau-Ginzburg orbifold field representatives there are always polynomials of the same degree as the superpotential; they produce polynomial deformations of the superpotential:

$$\{\phi(ab\ldots c)/\lambda(a|ij|b\ldots c)\} \sim \{\phi(x)/x_a\lambda_a^i\partial_i f(x)\} .$$

The latter, in turn, always occurs as a subsector of result of the Koszul computation, a subsector which is represented by totally symmetric tensors precisely dual to the polynomial deformations $[5,8,12]$.

Ref. [12] has verified that this dual $1\to1$ correspondence goes beyond the untwisted sector: all the Koszul representatives do have a $1\to1$ counterpart among the complete set of untwisted and twisted states. Moreover, just as the Landau-Ginzburg orbifold representatives are of the form $\Phi(X|\text{vacuum})^\ell$, the Koszul representatives factor into a totally symmetric tensor, which is dual to $\Phi(X)$, and a (product of) totally antisymmetric symbol(s) which correspond (at least formally) to the twist fields. This correspondence certainly covers the obvious properties such as the various charges. To further this relation, we will propose a “polynomial” equivalent of the antisymmetric symbol $\epsilon^{a_1\ldots a_n}$ at least in the sense that it fits the Koszul vs. Landau-Ginzburg orbifold correspondence.

Beyond the general discussion presented above, it is a different (and rather more involved) issue to determine precisely which tensors are required to span completely a given cohomology group on $\mathcal{M}$. In Calabi-Yau compactifications of heterotic superstrings, elements of $H^1(T\mathcal{M})$ and $H^1(T\mathcal{M}^*)$ correspond, respectively, to the charge-$(-1,1)$ and charge-$(1,1)$ states of the corresponding Landau-Ginzburg orbifold, and we will use the classical geometry and Landau-Ginzburg orbifold notation interchangeably. For Calabi-Yau complete intersections in products of projective spaces, $H^1(T\mathcal{M})$ is not infrequently well described by deformations of the defining polynomials [2]. In general (product of any flag-spaces and $\mathcal{M}$ not necessarily Calabi-Yau), the fully fledged Koszul computation [5,7,8] is necessary to compute $H^1(T\mathcal{M}); H^1(T\mathcal{M}^*)$ is also obtained by the same method. We now focus on these two sectors in turn.

3. The $(c,c)$-Ring

Rather than attempt to directly deal with the general case and unnecessarily clutter the notation, we consider a simple example where the $(c,c)$-sector contains both twisted and untwisted states; computational details are included in order to facilitate future applications of the present results to other models.
Before doing so, however, a remark is in order. As noted below, we shall use a trick to extend the Yukawa coupling formula of Ref. [2] to the twisted \((c,c)\)-states. Undeniably, this will be possible only in a subclass of models. However, the obtained formula agrees in all detail with the general Koszul computation [5,8]. Since the correspondence is equally general [12], we conclude that the extension of the Yukawa coupling will be valid even when the calculational trick employed below will not be available. On the other hand, this also proves that the Koszul calculations of Yukawa couplings [5,8] in fact are as exact as those obtainable in Landau-Ginzburg orbifolds, when those are available.

3.1. A sample model

Consider the Calabi-Yau manifold of the type

\[ Q \in \begin{bmatrix} 4 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix}^{2,86}_{-168} : \begin{cases} f(x) = f_{abcd} x_a x_b x_c x_d = 0 \\ g(x,y) = g_{a \alpha \beta} x_a y_\alpha y_\beta = 0 \end{cases} \]

where \((x_0, \ldots, x_4)\) and \((y_0, y_1)\) are homogeneous coordinates on \(\mathbb{P}^4 \times \mathbb{P}^1\), denoted by the ‘bra’ part, \(\begin{bmatrix} 4 \\ 1 \end{bmatrix}\). Obviously, \(f_{abcd}\) is totally symmetric in all its indices, while \(g_{a \alpha \beta} = g_{a \beta \alpha}\). The two columns in the ‘ket’ part of the matrices above denote the homogeneity of \(f(x)\) and \(g(x,y)\). The matrix represents an entire family of complex manifolds, of the same topological type but with the complex structure parametrized by the various choices of the coefficients in \(f(x)\) and \(g(x,y)\) and \(Q\) is a member of this family. Generic \(^4\) choices of \(f(x)\) and \(g(x,y)\) produce smooth \(Q\) and the subscripts on the matrices above denote the Euler characteristic while the superscripts denote \(b_{1,1}\) and \(b_{2,1}\) of such a generic manifold.

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The Landau-Ginzburg orbifold is obtained by assigning chiral superfields \(X_a\) and \(Y_\alpha\) to the coordinates \(x_a\) and \(y_\alpha\) and using \(P(X,Y) \defeq f(X) + g(X,Y)\) as the superpotential [3]. For a generic choice of \(f(X)\) and \(g(X,Y)\), the superpotential is non-degenerate. We easily find:

\[ Q : \theta_X = q_X = \frac{1}{4}, \quad \theta_Y = q_Y = \frac{3}{8}, \quad \Theta = \mathbb{Z}_8. \]

The formula (2.3) now produces the charges of all eight Ramond ground states and spectral flow then provides the Neveu-Schwarz vacua. For completeness, Table 1 lists the five classes of vacua.

3.2. Field representatives for \(Q\)

The Koszul computation provides a full complement:

\[ H^1(Q, T_Q) \sim \left[ \{ \phi_{(abcd)} \}/\{ f_{e(abc) \lambda_d}^e \} \right]_{45}, \]

\[ \oplus \left[ \{ \varphi_{a(\beta \gamma)} \}/\{ g_d \beta \gamma \lambda_a^d \oplus g_a \delta(\beta \gamma) \delta \} \right]_{11}, \]

\[ \oplus \left[ \{ \epsilon^{\alpha \beta} \varphi_{(abc)} \}/\{ \epsilon^{\alpha \beta} f_{abcd} \lambda_d \} \right]_{30}, \]

\(^4\) “Generic” means “all except a subset of strictly smaller dimension”.

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of 86 independent tensor components, as counted by the subscripts; \(a, b, \ldots = 1, \ldots, 5\) and \(\alpha, \beta, \ldots = 1, 2\). Here \(\lambda_{ab}^b\), \(\lambda_{\alpha\beta}^\alpha\) and \(\lambda^a\) are reparametrization degrees of freedom which can be used to “gauge away” 25, 4 and 5 components, respectively, of the \(\phi, \varphi\) and \(\vartheta\). Note that the \(\lambda_{ab}^b\)-“gauge equivalence” occurs in two places; the above subscripts imply that all of it was used up to gauge away components of \(\phi_{(abcd)}\), rather then of \(\phi_a(\beta\gamma)\). Clearly, this choice is subject to the practitioner’s whim, provides equivalent representatives for the elements of \(H^1(Q, T_Q)\) and—most importantly—relates many Yukawa couplings.

Upon contracting the first 56 tensors with appropriate coordinates, the Reader will promptly recognize the deformations of the defining polynomials \(f(x)\) and \(g(x, y)\), taken modulo linear reparametrizations \([2]\). The last 30 elements however do not have such a simple interpretation and cannot be found by polynomial deformations \([21]\). Note the anti-symmetric symbol \(\epsilon^{\alpha\beta}\) occurring in these.

A quick glance at Table 1 verifies that only the \((c, c)\)-vacua with \(\ell = 0, 4\) may be used to construct charge-(1, 1) states.

For \(\ell = 0\), all \(X_a, Y_\alpha\) fields are left invariant and since the vacuum charges are zero, we need polynomials of charge (1, 1). In view of Eq. (3.2), the untwisted \((c, c)\)-sector is spanned by polynomials of bi-degree (4,0) and (1,2) in \(X_a, Y_\alpha\)—precisely the polynomial deformations of \(f(X)\) and \(g(X, Y)\) and dual to the first 56 representatives in (3.3). The number of independent such polynomials, taken modulo the ideal \(\mathcal{I}[\partial P]\), is found \([13]\) as the coefficient of the \(t^8\)-term in

\[
P(t^8) = \left( \frac{1 - t^8(1 - q_x)}{1 - t^8 q_x} \right)^5 \left( \frac{1 - t^8(1 - q_y)}{1 - t^8 q_y} \right)^2 = 1 + \ldots + 56 t^8 + \ldots \quad (3.4)
\]

It is not a least bit obvious that the ideal generated by the gradients of \(P(X, Y) = f(X) + g(X, Y)\) is the same as the one dictated by the Koszul computation, but this is nevertheless true \([12]\).

Now, for \(\ell = 4\), only the \(X_a\) fields are left invariant and since the vacuum charges are both \(\frac{1}{4}\), we need polynomials of charge \((\frac{3}{4}, \frac{3}{4})\). The twisted \((c, c)\)-sector is therefore spanned by cubic polynomials in \(X_a\)—corresponding to the last 30 representatives in (3.3). Here, the Landau-Ginzburg orbifold and the Koszul ideal are more easily shown to be the same. What remains is to identify, at least formally, the anti-symmetric symbol \(\epsilon^{\alpha\beta}\) with the twist field which creates \(\left| \frac{1}{4}, \frac{1}{4} \right|_4^4\) from \(|0, 0|_0^0\). The calculation of the Yukawa coupling (below) will support this correspondence in rather more detail.

Suffice it here to note that \(\epsilon^{\alpha\beta}\) transforms the same as \((ydy)\), which has charge \((\frac{3}{4}, \frac{3}{4})\) and is therefore dual to \(\left| \frac{1}{4}, \frac{1}{4} \right|_4^4\) just as \(\theta_{abc}\) is dual to the monomial \(x_a, x_b, x_c\). A related fact is that all tensorial representatives (3.3) have an integral charge under the \(\mathbb{Z}_8\) symmetry (3.2).
It is rather well known that the polynomial deformation method for computing the $27^3$ Yukawa couplings applies \textit{verbatim} to the untwisted $(c,c)$-sector of Landau-Ginzburg orbifolds. Refs. \cite{16} shows how to (a) translate this method to the Koszul computation and (b) extend to all of $H^1(T)$, whether corresponding to twisted or untwisted $(c,c)$-states. The resulting ring structure will here be called ‘Jacobian’, borrowing from the standard case of a single hypersurface (see Refs. \cite{15} and references therein).

Given the above 1–1 correspondence between Koszul and Landau-Ginzburg orbifold field representatives, it is of course a simple matter to translate this result into the Landau-Ginzburg orbifold formalism. However, we presently re-derive it from the Landau-Ginzburg orbifold framework itself, using a little trick for which we need a model which is closely related to $Q$.

### 3.3. The ‘ineffectively split’ model

For many complete intersection manifolds, there is a sequence of related manifolds obtained by so-called ‘ineffective splitting’, first discussed and named in the third article in Refs. \cite{4}; see also Refs. \cite{5,11}.

To be precise, \textit{every} complete intersection can be split, but this may or may not change the manifold; a ‘split’ is called ‘ineffective’ when the manifold does not change although the embedding is changed through ‘splitting’. If the complete intersection can be written as a hypersurface in a product of a complex 1-dimensional and a complex 3-dimensional space, the simple form of ‘ineffective splitting’ as described below is possible. More general forms of ‘splitting’, by introducing more than two new variables, will provide ‘ineffective splits’ of many other examples (see for example p. 279 of Ref. \cite{5}). To the best of our understanding, the precise limitation of this process is not known in general.

Consider the following ‘ineffectively split’ version of $Q$:

$$
\begin{bmatrix}
4 & 4 & 1 & 0 \\
1 & 0 & 0 & 2 \\
1 & 0 & 1 & 1
\end{bmatrix}
\overrightarrow{\text{2.86}}
\begin{bmatrix}
f(x) = f_{abcd} x_a x_b x_c x_d = 0 , \\
m(x, z) = m_{aB} x_a z_B = 0 , \\
h(y, z) = h_{\alpha\beta C} y_\alpha y_\beta z_C = 0 .
\end{bmatrix}
$$

The relation between $Q$ and $\tilde{Q}$ is given by $g(x, y) = \det \left( \frac{\partial(m, h)}{\partial(z_1, z_2)} \right)$. Since the topological numbers remained the same, $\tilde{Q}$ and $Q$ are identical, except that not all polynomial deformations of the complex structure in one family can be attained by polynomial deformations in the other and \textit{vice versa}.

---

\textsuperscript{5)} Given that this technique is quite effective, this appears to be somewhat of a misnomer.
Without much ado, the Koszul computation produces
\[ H^1(\widetilde{Q}, T_{\widetilde{Q}}) \sim \left\{ \frac{\phi_{(abcd)}}{f_{(abc)\lambda_d}^e} \right\}_{45}, \]
\[ \oplus \left[ \left\{ \mu_{aB} \right\}/\left\{ m_{cB} \lambda_a^c \oplus m_a C \lambda_B^C \right\} \right]_6, \]
\[ \oplus \left[ \left\{ \eta_{(\alpha\beta)C} \right\}/\left\{ h_{\delta(\alpha|C) \lambda_\beta}^\delta \oplus h_{\alpha\beta D} \lambda_C^D \right\} \right]_2, \]
\[ \oplus \left[ \left\{ \epsilon^{\alpha\beta} \epsilon_{AB} \partial_{(abc)} \right\}/\left\{ \epsilon^{\alpha\beta} \epsilon_{AB} f_{abcd} \lambda_d \right\} \right]_{30}, \]
\[ \oplus \left[ \left\{ \epsilon^{\alpha\beta} \varphi_c \right\}/\left\{ \epsilon^{\alpha\beta} m_{cB} \lambda_B^B \right\} \right]_5. \] (3.6)

Similarly to \( Q \), the \( \widetilde{Q} \) Landau-Ginzburg orbifold is constructed with the superpotential
\[ P(X, Y, Z) = f(X) + m(X, Z) + h(Y, Z), \] charges
\[ \widetilde{Q} : \theta_X = q_X = \frac{1}{4}, \quad \theta_Y = q_Y = \frac{1}{8}, \quad \theta_Z = q_Z = \frac{3}{4}, \quad \Theta = \mathbb{Z}_8, \] (3.7)
and the vacua listed in Table 2. Again, the charge-(1,1) states are found amongst the untwisted and \( \ell = 4 \) twisted states. This time, however, the \( \ell = 4 \) twisted vacuum has both charges equal to \( \frac{3}{4} \), whence the charge-(1,1) state is obtained by multiplying this only by linear combinations of \( X_a \). Of course, owing to more variables, there are now more untwisted states, so the net count is the same—86.

To simplify the presentation and also for later convenience, we now shift to the special case, where
\[ f_{abcd} = \begin{cases} 1 & a, b, c, d \text{ all equal}, \\ 0 & \text{otherwise}, \end{cases} \quad g_{a\beta\gamma} = \begin{cases} 1 & a, \beta, \gamma \text{ all equal}, \\ 0 & \text{otherwise}, \end{cases} \] (3.8)
for \( Q \), and
\[ f_{abcd} = \begin{cases} 1 & a, b, c, d \text{ all equal}, \\ 0 & \text{otherwise}, \end{cases} \quad m_{aB} = \begin{cases} -\epsilon_{aB} & a, B = 1, 2, \\ 0 & \text{otherwise}, \end{cases} \]
\[ h_{\alpha\beta C} = \begin{cases} 1 & \alpha, \beta, C \text{ all equal}, \\ 0 & \text{otherwise}, \end{cases} \] (3.9)
for \( \widetilde{Q} \). The respective Landau-Ginzburg orbifold superpotentials become
\[ P_Q(X, Y) = \sum_{\alpha=1}^{2} (X_\alpha^4 + X_\alpha Y_\alpha^2) + X_3^4 + X_4^4 + X_5^4, \] (3.10)
and (upon a convenient rescaling \( Z_1 \rightarrow Z_2, Z_2 \rightarrow -Z_1 \))
\[ P_{\widetilde{Q}}(X, Y, Z) = \sum_{A=1}^{2} (X_A^4 + X_A Z_A + Y_A^2 Z_A) + X_3^4 + X_4^4 + X_5^4. \] (3.11)

It is of course convenient to use a monomial basis for the states and Table 3 presents a comparative list of the 86 field representatives of \( Q, \widetilde{Q}, \) both the Koszul and the Landau-Ginzburg orbifold version.
First of all, all Koszul representatives have integral charges under the $\mathbb{Z}_8$ (are left invariant by it); also, the total scaling charge of the holomorphic volume form $(xdx)(ydy)(zdz)$ is 3. Now, the Koszul representatives which correspond to twisted states in fact have zero scaling charge, while those corresponding to untwisted states have non-zero scaling charges, equal here to $\pm 1$. We will see below that this is also true of the $(a,c)$-sector, taken modulo the charge of the full (holomorphic) volume-form.

Next, notice that the Koszul representative $\epsilon^{\alpha\beta}\epsilon^{AB}\vartheta_{(abc)}$ in $H^1(\tilde{Q},\mathcal{T}_{\tilde{Q}})$ corresponds to an untwisted state although it contains antisymmetric symbols. Since $X_a X_b X_c Y_\alpha Y_\beta |0,0\rangle \sim \epsilon^{\alpha\beta} \epsilon^{AB} \vartheta_{(abc)}$, we conclude that $\epsilon^{\alpha\beta} \epsilon^{AB}$ ought to be dual to $Y_\alpha Y_\beta$. Indeed, this is not hard to prove, since these two quantities can be contracted using the defining tensors:

$$\epsilon^{\beta\gamma} \epsilon^{AB} h_{(\alpha\beta)A} h_{(\gamma\delta)B} Y_\alpha Y_\gamma$$

is invariant; recall that coordinates are indexed by sub-scripts although they are contravariant. Thus, $\epsilon^{\beta\gamma} \epsilon^{AB}$ is dual to $Y_\alpha Y_\gamma (= Y_1 Y_2)$, much like $\vartheta_{(abc)}$ is dual to $X_a X_b X_c$.

It only remains to interpret the single $\epsilon$'s. To this end we consider the Yukawa couplings in the $(c,c)$-sector.

### 3.4. Yukawa couplings

The 86 field representatives in Table 3 are easily divided into three groups: (1) those which are untwisted in both $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$, denoted $\mathbf{U}$, (2) those which are untwisted in $\mathcal{Q}$ but twisted in $\tilde{\mathcal{Q}}$, denoted $\mathbf{V}$, and (3) those which are twisted in $\mathcal{Q}$ but untwisted in $\tilde{\mathcal{Q}}$, denoted $\mathbf{W}$.

The Yukawa coupling is a scalar product of the type

$$\kappa_{ijk} = \langle 0,0|-3,0\rangle^1 \langle 0,-3\rangle^7 |1,1\rangle_i^{\ell_i} |1,1\rangle_j^{\ell_j} |1,1\rangle_k^{\ell_k} ,$$

where $|1,1\rangle_i^{\ell_i}$ are the charge-$(1,1)$ states (Landau-Ginzburg orbifold field representatives) from Table 3 and of course

$$\ell_i + \ell_j + \ell_k \equiv 0 \pmod{d} ,$$

with $d = 8$ here, is the twist-number selection rule in the $(c,c)$-sector, and the $J_0, \mathcal{J}_0$-charges have been balanced already. Note that the $\mathbb{Z}_8$ selection rule (3.15), $\ell_i + \ell_j + \ell_k \equiv 0 \pmod{8}$, applied both for $\mathcal{Q}$ and $\tilde{\mathcal{Q}}$, guarantees that

$$\langle \mathbf{U}, \mathbf{U}, \mathbf{U} \rangle , \quad \langle \mathbf{U}, \mathbf{V}, \mathbf{V} \rangle , \quad \langle \mathbf{U}, \mathbf{W}, \mathbf{W} \rangle$$

are the only non-zero types of Yukawa couplings.
The first of these types is entirely in the untwisted \((c, c)\)-sector of both \(Q\) and \(\tilde{Q}\). Therefore, the polynomial deformation formula of Ref. [2] applies and for a concrete value we only need to calculate \(Q \overset{\text{def}}{=} \det \left[ \partial^2 P \right] \), the ‘top’ element of the Chiral (Jacobian) ring. Straightforwardly, we have

\[
Q_Q \approx 2^{12} 3^{5^2} X_1^3 X_2^3 X_3^2 X_4^2 X_5^2 , \tag{3.17a}
\]

\[
Q_{\tilde{Q}} \approx 2^{12} 3^{2^7} X_1^3 X_2^3 X_3^2 X_4^2 X_5^2 . \tag{3.17b}
\]

each of which has to be understood as representative of an equivalence class, up to the ideal generated by the partials

\[
\Im[\partial P_Q] \sim \bigoplus_{\alpha=1,2} \left\{ (4X_\alpha^3 + Y_\alpha^2), (2X_\alpha Y_\alpha) \right\} \oplus \bigoplus_{a=3,4,5} \left\{ 4X_a^3 \right\} , \tag{3.18}
\]

and

\[
\Im[\partial P_{\tilde{Q}}] \sim \bigoplus_{\alpha=1,2} \left\{ (4X_A^3 + Z_A), (2Y_A Z_A), (X_A + Y_A^2) \right\} \oplus \bigoplus_{a=3,4,5} \left\{ 4X_a^3 \right\} , \tag{3.19}
\]

respectively. The overall numerical factors \(2^{12} 3^{5^2}\) and \(2^{12} 3^{2^7}\), respectively, in Eqs. \((3.17a)\) and \((3.17b)\) are easily absorbed in an overall rescaling of the superpotential and will be ignored hereafter. Thus up to wave function renormalization, the Yukawa coupling is given by

\[
\kappa_{uuu} = \frac{U^3}{Q_Q} \tag{3.20}
\]

and similarly for \(\tilde{Q}\).

The remaining \(\langle U, V, W \rangle\)-type couplings are straightforward to evaluate in \(Q\), since both \(U\) and \(V\) are untwisted there. Similarly, the \(\langle U, W, W \rangle\)-type couplings are easily evaluated in \(\tilde{Q}\). Once we know the values of these couplings in the version where they are untwisted, we require that the value of the coupling in the twisted version be the same. Thus, for example,

\[
\langle 0, 0 | -3, 0 \rangle^1 |0, -3\rangle^7 (X_3 Y_1 Y_2 |0, 0\rangle^0 )^2 (X_4^2 X_5^2 |0, 0\rangle^0 ) \tag{3.21}
\]

is the untwisted version, in \(Q\), of the Yukawa coupling

\[
\langle 0, 0 | -3, 0 \rangle^1 |0, -3\rangle^7 (X_3 |\frac{3}{4}, \frac{3}{4}\rangle^4 )^2 (X_4^2 X_5^2 |0, 0\rangle^0 ) \tag{3.22}
\]

in the twisted sector of \(\tilde{Q}\). The former of these easily produces \(Q_Q\) upon using the ideal \((3.18)\): the Yukawa coupling is 16 (up to the irrelevant overall numerical coefficient). The latter evaluates to

\[
(X_3^2 X_4^2 X_5^2) \langle 0, 0 | -3, 0 \rangle^1 |0, -3\rangle^7 |\frac{3}{4}, \frac{3}{4}\rangle^4 |\frac{3}{4}, \frac{3}{4}\rangle^4 |0, 0\rangle^0 \tag{3.23}
\]

Comparing with Eq. \((3.17b)\), it must be that

\[
\left[ |\frac{3}{4}, \frac{3}{4}\rangle^4 \right]^{2} = 16 (X_3^3 X_5^3) |0, 0\rangle^0 \simeq (Z_1 Z_2) |0, 0\rangle^0 , \tag{3.24}
\]

\[\boxed{12}\]
in \( \tilde{Q} \), at least in the sense that this fits the Yukawa couplings and hence the ring structure. A similar comparison of the \( \langle U, W, W \rangle \)-type couplings, we derive that, in \( Q \),

\[
\left[ \left| \frac{1}{4}, \frac{1}{4} \right|^4 \right]^2 = (X_1 X_2) |0, 0\rangle^0. 
\] (3.25)

Formally, we can also set

\[
\left| \frac{3}{4}, \frac{3}{4} \right|^4 = \sqrt{Z_1 Z_2} |0, 0\rangle^0, \quad \text{in } \tilde{Q},
\] (3.26)

and

\[
\left| \frac{1}{4}, \frac{1}{4} \right|^4 = \sqrt{X_1 X_2} |0, 0\rangle^0, \quad \text{in } Q.
\] (3.27)

Because of the square-root, the fields in the superconformal field theory corresponding to \( \sqrt{Z_1 Z_2} \) and \( \sqrt{X_1 X_2} \) must have a branch-cut which is a well-known property of twist-fields. This supports our identification of \( \sqrt{Z_1 Z_2} \) as the twist-field that creates \( \left| \frac{3}{4}, \frac{3}{4} \right|^4 \) from \( |0, 0\rangle^0 \) in \( \tilde{Q} \) and \( \sqrt{X_1 X_2} \) as the one that creates \( \left| \frac{1}{4}, \frac{1}{4} \right|^4 \) from \( |0, 0\rangle^0 \) in \( Q \). The \( J_0, J_0 \)-charges of \( \sqrt{Z_1 Z_2} \) and \( \sqrt{X_1 X_2} \) clearly satisfy this identification.

---

Looking back at the Koszul field representatives, the quadratic relation (3.24) implies that, on \( \tilde{Q} \), \( \epsilon^{\alpha\beta} \) is dual to \( \sqrt{Z_1 Z_2} \). Indeed, this is easy to prove, since the product of the squares of these objects is easily contracted into a scalar:

\[
\left[ \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} h_{\alpha\beta A} h_{\gamma\delta B} Z_A Z_B \right] \text{ is invariant on } \tilde{Q}. 
\] (3.28)

Similarly, on \( Q \), \( \epsilon^{\alpha\beta} \) is dual to \( \sqrt{X_1 X_2} \); indeed, this again is easy to prove:

\[
\left[ \epsilon^{\alpha\gamma} \epsilon^{\beta\delta} g_{\alpha\beta a} g_{\gamma\delta b} X_a X_b \right] \text{ is invariant on } Q. 
\] (3.29)

Clearly, the same calculation can be repeated for any other non-degenerate choice of the superpotential; the choice (3.8) and (3.9) merely simplified the derivation.

---

Thus, we have derived the same “polynomial” representative for the twist fields

(a) by comparing the Yukawa couplings in the \((c, c)\)-sector of a Landau-Ginzburg orbifold and its ‘ineffectively split’ version, and

(b) by identifying the twist-fields with \( \epsilon \)'s in the Koszul representative and contracting with the defining tensor coefficients. Note that such relations, Eqs. (3.13), (3.26), (3.27), (3.28) and (3.29), hold precisely because of the way the tensor algebra on the submanifold was defined in Section 2.1.

With such a “polynomial” representative, the Yukawa couplings can be computed for all \((c, c)\)-states, untwisted and twisted.

It is a straightforward matter to verify that the resulting numerical values of the couplings are identical to those calculated entirely within the Koszul computation framework [3,8]. In view of the non-renormalization theorem of Ref. [22], this should not come as a surprise. This identifies the Koszul ring structure with the Chiral ring structure—up to the normalization of the field representatives. In other words, the foregoing discussion of Yukawa couplings pertains to the so called “un-normalized” couplings, with the kinetic terms of the various fields left undetermined. We will return to this later.
3.5. The general case

With the above we have shown that indeed the Koszul computation agrees with the Landau-Ginzburg ring calculation. Unfortunately, in a number of cases we know of no non-degenerate Landau-Ginzburg orbifold corresponding to the manifold. Yet, our technique may equally well be applied in those situations.

Let $\mathcal{M}$ be a family of manifolds, each defined as a hypersurface in a product of $N$ projective spaces $\mathbb{P}^{n_1} \times \ldots \times \mathbb{P}^{n_N}$ where $x^i_j$ are the coordinates on the $\mathbb{P}^{n_i}$. The hypersurface is given by $M$ constraints of the form $f_{a \ldots b}^{(m)} x_a^{(i)} \ldots x_b^{(j)} = 0$, $m = 1, \ldots, M$, $i, j = 1, \ldots, N$.

The polynomial deformations will as usual be spanned by polynomials taken modulo the Jacobian ideal. In order to find a representative for the various $\epsilon \ldots$ we consider the square, $\epsilon^2$ contracted with the appropriate number of the defining tensors, $f_{ \ldots }^{(m)}$ and coordinates, $x_j^{(i)}$, to make the expression invariant. Just as in the above example, $\epsilon \ldots$ will be the dual of the square root of the product of the corresponding $x_j^{(i)}$‘s. By using the Koszul computation and the above identification, we can write explicit monomial representatives for all complex deformations—although we do not know the corresponding Landau-Ginzburg orbifold.

With the $(c, c)$ ring at hand, we then go on to compute the Yukawa couplings using the usual technique. In fact, the knowledge of the ring structure may help us in finding the underlying conformal field theory.

4. The $(a, c)$-Ring

Having extended the polynomial deformation formula for the Yukawa couplings to the complete $(c, c)$-sector of a Landau-Ginzburg orbifold model, the extension of this formula to the $(a, c)$-sector is now easily obtained by using mirror symmetry. Again, as a calculational trick, we will use the ‘mirror map’.

Although the list of models for which the ‘mirror model’ is known is growing, it will not be known in the general case. As with the $(c, c)$-ring, we will again show that the results of the Koszul computation agree with those obtained by extending the Yukawa coupling formula with the aid of the mirror map. It is important to realize at this point an inherent limitation of the Koszul computation: while the choice of the complex structure is controlled, the choice of the Kähler class is not, and so a generic choice is implied. This means that certain parameters (and so Yukawa couplings) will remain undetermined by the Koszul computation although they can be calculated by other geometrical means. Our aim is to show that these can be chosen so as to agree with the Landau-Ginzburg orbifold calculations. With this limitation in mind, the Koszul-calculated Yukawa couplings will be show to be equal to the Landau-Ginzburg orbifold results. We present two examples in detail to facilitate the application to other models.
4.1. An almost trivial example

Let \( \hat{Q} \) be the mirror model of \( Q \). Then, the \((a,c)\)-sector of \( Q \) maps 1–1 to the \((c,c)\)-sector of \( \hat{Q} \). Thus, we employ the full \((c,c)\)-extension of the Yukawa coupling formula [2] to the \((c,c)\)-sector of \( \hat{Q} \) and then map the results back to the \((a,c)\)-sector of \( Q \).

As seen from Table 1, there are only two charge-\((-1,1)\) states in the \((a,c)\)-sector of \( Q \): two \( (a,c) \)-vacua in fact: \([-1,1]^3 \) and \([-1,1]^4 \). The Koszul computation (and also a straightforward application of the Lefschetz hyperplane theorem) assures us that these must correspond to the pull-backs of the Kähler forms of \( \mathbb{P}^4 \) and \( \mathbb{P}^1 \). Their (geometrical) Yukawa couplings are easily calculated as they are isomorphic to generic hyperplanes in \( \mathbb{P}^4 \) and \( \mathbb{P}^1 \), respectively [3]. In the matrix notation of (3.1), we have \( J_x \approx [1 \ 0] \) and \( J_y \approx [0 \ 1] \). The Yukawa couplings are then simply the number of intersection points:

\[
\langle J_x, J_x, J_x \rangle = \chi_E \begin{bmatrix} 4 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 1 \\ 0 & 0 & 0 \end{bmatrix} = 8 ,
\]

\[
\langle J_x, J_x, J_y \rangle = \chi_E \begin{bmatrix} 4 & 4 & 1 \\ 1 & 0 & 2 \end{bmatrix} \begin{bmatrix} 1 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix} = 4 ,
\]

all others being zero. These results are however subject to world-sheet instanton corrections [25], which are insignificant only in the large size limit. That is, the corrections to the above results come as a formal power series in powers of the smallest characteristic size of the ‘internal’ manifold, in units of Planck length. This smallest characteristic size is simply the size of minimal \( \mathbb{P}^1 \)'s embedded in the Calabi-Yau manifold and is unrelated to the overall size. For example, orbifolds have singular points, and the minimal \( \mathbb{P}^1 \)'s are in fact these singular points themselves, regardless of the overall size of the orbifold.

Clearly, therefore, the above (geometric) result is unreliable unless an independent argument ensures that world-sheet instanton effects are negligible. No such argument is available for Landau-Ginzburg orbifolds, and in fact one expects that at least some of the characteristic sizes are small.

However, our strategy is to relate the Yukawa couplings in the \((a,c)\)-sector of \( Q \) to those in the \((c,c)\)-sector of the mirror model, \( \hat{Q} \). The latter couplings are protected by the non-renormalization theorem of Ref. [22], whereupon the inferred Yukawa couplings in the \((a,c)\)-sector of \( Q \) will also be exact.

For the specific choice of superpotential (3.10), the mirror model is easily constructed [23]. Note that the Landau-Ginzburg orbifold model \( Q \) consist of five “building blocks”, uncoupled up to the \( \mathbb{Z}_8 \) GSO-like projection: two copies of a \( D_5 \)-type model, \( p_\alpha = X_\alpha^4 + X_\alpha Y_\alpha^2 \), \( \alpha = 1, 2 \) and three copies of a \( A_3 \)-type model, \( p_\alpha = X_\alpha^4 , \alpha = 3, 4, 5 \). The mirror of \( A_k \)-type models has the same superpotential, but is a \( \mathbb{Z}_{k+2} \) quotient thereof. One way to construct the mirror of \( D_k \)-type models is to ‘transpose’ the superpotential and then divide by an extra \( \mathbb{Z}_2 \) [23]. All in all, the mirror model of (3.10) will have the superpotential

\[
P_\hat{Q}(X, \hat{Y}) = \sum_{\alpha=1}^{2}(X_\alpha^4 \hat{Y}_\alpha + \hat{Y}_\alpha^2) + \hat{X}_3 + \hat{X}_4 + \hat{X}_5^4 ,
\]

\[−15−\]
This dictates that
\[ q(\hat{X}_\alpha) = \frac{1}{8} , \quad q(\hat{Y}_\alpha) = \frac{1}{2} , \quad \alpha = 1, 2 , \quad q(\hat{X}_a) = \frac{1}{4} , \quad a = 3, 4, 5 . \] (4.3)

With these variables, we have
\[ \vartheta_x \stackrel{\text{def}}{=} \hat{X}_1 \hat{X}_2 \hat{X}_3 \hat{X}_4 \hat{X}_5 , \quad \vartheta_y \stackrel{\text{def}}{=} \hat{Y}_1 \hat{Y}_2 , \] (4.4)
as the mirror images of \( J_x \) and \( J_y \); actually, the exact identification as to which \( \vartheta \) is the mirror of which \( J \) will be clear from the analysis below. The general results of Refs. [15] then imply that these two representatives will be valid for a generic choice of the superpotential \( P_Q \), not just the special case (3.10).

For the superpotential (4.2), the Jacobian ideal is generated by
\[ \Im[\partial P_Q] \sim \bigoplus_{\alpha=1,2} \{ (4 \hat{X}_\alpha^3 \hat{Y}_\alpha) , \ (\hat{X}_\alpha^4 + 2 \hat{Y}_\alpha) \} \oplus \bigoplus_{a=3,4,5} \{ 4 \hat{X}_a^3 \} , \] (4.5)
and
\[ Q_\hat{\varphi} = 2^{10} 3^3 \hat{X}_1^6 \hat{X}_2^6 \hat{X}_3^2 \hat{X}_4^2 \hat{X}_5^2 \sim 2^{12} 3^3 \hat{X}_1^2 \hat{Y}_1 \hat{X}_2^2 \hat{Y}_2 \hat{X}_3^2 \hat{X}_4^2 \hat{X}_5^2 . \] (4.6)
It is easy to show that \( \vartheta_x^3 \simeq 0 \) and \( \vartheta_y^2 \simeq 0 \), whence \( \langle \vartheta_x, \vartheta_y \rangle \) remains the only non-zero Yukawa coupling.

---

Consider now the \((a,c)\)-sector of the Landau-Ginzburg orbifold \( Q \). For a product of three charge-\((-1,1)\) states, the Yukawa coupling is
\[ \kappa_{ijk} = \langle 0, 0 | +3, 0 \rangle^7 | 0, -3 \rangle^7 \langle -1, 1 \rangle^{\ell_i} | -1, 1 \rangle^{\ell_j} | -1, 1 \rangle^{\ell_k} , \] (4.7)
While the \( J_0 \)-charges are automatically balanced, the twist-number selection rule in the \((a,c)\)-sector requires
\[ \ell_i + \ell_j + \ell_k \equiv 2 \pmod{d} , \] (4.8)
with \( d = 8 \) here. Therefore, the only non-zero Yukawa coupling is
\[ \langle 0, 0 | +3, 0 \rangle^7 | 0, -3 \rangle^7 \langle -1, 1 \rangle^3 | -1, 1 \rangle^3 | -1, 1 \rangle^4 . \] (4.9)
This implies that
\[ | -1, 1 \rangle^3 \overset{M}{\simeq} \vartheta_x , \quad | -1, 1 \rangle^4 \overset{M}{\simeq} \vartheta_y \] (4.10)
are two mirror-pairs.

Next, comparing with the geometrical results in (4.1a-d), we see that \( | -1, 1 \rangle^3 \sim J_x \) while \( | -1, 1 \rangle^4 \sim J_y \) is the only possible identification. The fact that \( \langle J_x, J_x, J_x \rangle \neq 0 \) in the geometrical calculation shows that the world-sheet instanton corrections indeed are not negligible: they in fact cancel out the classical (tree-level, topological) contribution to \( \langle J_x, J_x, J_x \rangle \).

The fact that the instanton effects conspire so as to cancel the ‘topological’ contribution to some of the Yukawa couplings is easily seen to be the consequence of a quantum symmetry, that is, a special (and possibly singular) choice of the Kähler class for which
the symmetry used for the GSO-type projection is in fact an isometry. Such cancellations were to be expected, much as symmetries of the superpotential relate Yukawa couplings in the \((c,c)\)-sector and often cause some of them to vanish.

Note that there does exist a singular limit in which the geometrical Yukawa coupling \(\langle J_x, J_x, J_x \rangle\) vanishes but \(\langle J_x, J_x, J_y \rangle\) remains non-zero. We observe that \(\langle J_x, J_x, J_x \rangle\) may be identified with the real 6-volume of the projection of the Calabi-Yau 3-fold \(\mathcal{Q}\) on \(\mathbb{P}^4\), while the projection of \(\langle J_x, J_x, J_y \rangle\) on \(\mathbb{P}^4\) has the interpretation of a 4-volume. Thus, for \(\langle J_x, J_x, J_x \rangle\) but not \(\langle J_x, J_x, J_y \rangle\) to vanish, it would suffice to collapse \(\mathbb{P}^4\) in one (complex) dimension only. It seems possible to think of the world-sheet instanton contributions as effectively distorting the originally homogeneous geometry of \(\mathbb{P}^4\) in such an amusing fashion.

Finally, suffice it here to state that the Koszul representatives for \(J_x\) and \(J_y\) are simply two scalars, consistent with the fact that \(J_x\) and \(J_y\) are invariant under any holomorphic symmetry of \(\mathcal{Q}\). (To prove this invariance, simply use the Fubini-Study Kähler forms: \(\partial \bar{\partial} \log \|X\|^2\) and \(\partial \bar{\partial} \log \|Y\|^2\).) Thus, the Koszul computation by itself does not provide enough information about the pull-backs of the Kähler forms of the embedding \(\mathbb{P}^n\)'s. We therefore rely on the geometrical calculation of Yukawa couplings and take an appropriate limit to recover the Landau-Ginzburg orbifold result; we will call this the ‘Landau-Ginzburg orbifold limit of the Koszul (geometrical) calculation’.

This degree of freedom simply means that, unlike the Landau-Ginzburg orbifold computation, the Koszul computation does not refer to a particular but rather to a general choice of the Kähler class on the manifold \([5,8]\). Consequently, agreement with the geometrical (large size) or Landau-Ginzburg orbifold (‘small’ size) results is obtained only upon fixing by hand this degree of freedom of the Koszul computation.

Alternatively, the Kähler class variation which takes us away from the limit in which \(\langle J_x, J_x, J_x \rangle = 0\) may easily be studied in the Landau-Ginzburg orbifold framework, as a deformation of the mirror model (4.2). For example, a multiple of a general quartic polynomial in \(\hat{X}_3, \hat{X}_4, \hat{X}_5\) may be added to \(P_{\mathcal{Q}}(\hat{X}, \hat{Y})\). Thereupon, the result \(\vartheta_y^2 \simeq 0\) will still hold, but \(\vartheta_x^3\) will no longer vanish in general. Of course, other deformations will result in a more general change and render all Yukawa couplings nonzero. In this way we can interpolate between the Landau-Ginzburg orbifold results, where some of the characteristic sizes are small, and the geometric results where all characteristic sizes are sufficiently big for the world-sheet instanton effects to be negligible.

4.2. Another model

The first sample model had a somewhat atypical \((a,c)\)-sector, in that the only field representatives were actually two \((a,c)\)-vacua. The next model will have a richer \((a,c)\)-sector and in fact will typify the general case.

Consider the family of Calabi-Yau 3-folds one member of which was used in Ref. [19] to construct a 3-generation manifold:

\[
\mathcal{M} \in \begin{bmatrix} 3 & 3 & 1 \end{bmatrix}^{8.35}_{3} \mathbb{Z}^{54} : \begin{cases} f(x) = f_{abc} x_a x_b x_c & = 0, \\ g(x, y) = g_{a \alpha \beta \gamma} x_a y_\alpha y_\beta y_\gamma & = 0. \end{cases}
\]  (4.11)
The most general variations of \( f(x) \) and \( g(x, y) \) are spanned by 60 monomials. Linear reparametrizations on \( \mathbb{P}^3 \times \mathbb{P}^2 \) are elements of \( PGL(4; \mathbb{C}) \times PGL(3; \mathbb{C}) \) which has \( 15 + 8 = 23 \) generators. Taking into account two overall rescalings of the two homogeneous equations (4.11), the 60 monomials form \( 60 - 25 = 35 \) independent classes; in fact, all of \( H^1(T) \) is represented by deformations of the defining polynomials \( f(x) \) and \( g(x, y) \) \[21\]. So, the \((c, c)\)-sector here is simpler than in the first example in that there are only untwisted charge-\((1, 1)\) states. Of course, the Chiral and the Jacobian ring structures agree, as was well known.

For generic choices of \( f(x) \) and \( g(x, y) \), the associated Landau-Ginzburg orbifold is again well defined simply by setting the superpotential to be \( P_M = f(X) + g(X, Y) \) \[9\]. We again easily find:

\[
\mathcal{M} : \theta_X = q_X = \frac{1}{3}, \quad \theta_Y = q_Y = \frac{2}{9}, \quad \Theta = \mathbb{Z}_9. \tag{4.12}
\]

The charges of all 9 Ramond ground states are again found using (2.3); spectral flow then provides the Neveu-Schwarz vacua. For completeness, Table 4 lists all five classes of vacua.

Again, there exists a special choice of such a superpotential,

\[
P_M = \sum_{\alpha=1}^{3} (X_\alpha ^3 + X_\alpha Y_\alpha ^3) + X_0^3, \tag{4.13}
\]

and the limit of the renormalization flow of this Landau-Ginzburg orbifold results in a tensor product \( (E_7)^3(A_2) \) of minimal models \[16\]. This choice corresponds to setting

\[
f_{abc} \overset{\text{def}}{=} \begin{cases} 1 & a = b = c, \\ 0 & \text{otherwise}, \end{cases} \quad g_{\alpha \alpha \beta \gamma} \overset{\text{def}}{=} \begin{cases} 1 & a = \alpha = \beta = \gamma, \\ 0 & \text{otherwise}, \end{cases} \tag{4.14}
\]

in Eq. (4.11).

4.3. Field representatives

For \( H^1(\mathcal{M}, T^{*}_\mathcal{M}) \), the Koszul calculation produces

\[
H^1(T^{*}) \sim \{J_x\} \oplus \{J_y\} \oplus \{e^{abcd}g^{(ij)} : f_{ijk}g^{(ij)} = 0\}_6. \tag{4.15}
\]

This time, besides the pull-backs of the Kähler forms \( J_x \) and \( J_y \) of \( \mathbb{P}^3 \) and \( \mathbb{P}^2 \), represented by two scalars, we also obtain 6 tensors where the totally antisymmetric symbol \( e^{abcd} \) is readily factored out. Note that the scaling charge of the holomorphic volume form \((xd^3x)(yd^2y)\) is 2. The scaling charge of the two scalars, \( J_x \) and \( J_y \), is of course zero, while the scaling charge of \( e^{abcd}g^{(ij)} \) is \( 6\frac{1}{3} = 2 \), which again is zero modulo the scaling charge of the holomorphic volume form. This was expected, as all these field representatives correspond to twisted states in the Landau-Ginzburg orbifold framework.

The dual cohomology, \( H^2(T) \), is then spanned by two copies of the holomorphic volume element on \( \mathbb{P}^3 \times \mathbb{P}^2 \):

\[
(xd^3x)(yd^2y) \overset{\text{def}}{=} (x_0dx_1dx_2dx_3)(y_1dy_2dy_3), \tag{4.16}
\]
and six independent tensor coefficients in
\[
\{ \epsilon^{\alpha\beta\gamma} \vartheta_{(ij)} \simeq \epsilon^{\alpha\beta\gamma} \vartheta_{(ij)} + \epsilon^{\alpha\beta\gamma} \chi^k f_{ijk} \}^6.
\]  
(4.17)

Note again that the scaling charges of all these eight field representatives are zero, modulo the scaling charge of the total holomorphic volume form.

The latter tensor coefficients, \( \vartheta_{(ij)} \), are themselves naturally dual to monomials \( X_i X_j \), which transform like the original tensor coefficients \( \vartheta^{(ij)} \). Thus we have
\[
\vartheta^{(ij)} \sim \vartheta_{(ij)} \sim X_i X_j , \quad \implies \quad \vartheta^{(ij)} \approx X_i X_j .
\]  
(4.18)

So, again, we are looking for a “polynomial” representative for \( \epsilon^{abcd} \), that is, one which is dual to \( \epsilon^{\alpha\beta\gamma} \).

---

As remarked above and which is now obvious from Table 4, the only integral charge-(1,1) states come from the untwisted (c,c)-sector. In the (a,c)-sector, however there are two charge-(-1,1) vacua, with \( \ell = 3,5 \) and there is also the vacuum \( [-\frac{5}{3}, \frac{1}{3}]^4 \), which becomes a charge-(-1,1) state when multiplied with quadrics in \( X_a \). Indeed, \( \ell = 4 \) in the (a,c)-sector stems from \( \ell = 3 \) in the Ramond sector and there the \( X_a \) but not the \( Y_\alpha \) are left invariant by the action of \( \Theta = \mathbb{Z}_9 \). Reducing furthermore modulo the ideal generated by the gradients of the superpotentials, this yields six states of the form
\[
\vartheta (X) \big| [-\frac{5}{3}, \frac{1}{3}]^4 , \quad \vartheta (X) \cong \vartheta (X) + \lambda^a \partial_a f (X) .
\]  
(4.19)

It is immediate that such quadrics are dual to the rank-2 tensor \( \vartheta_{(ij)} \) in (4.17), that is, the quadrics \( \vartheta (X) \) here may be identified with the tensors \( \vartheta^{(ij)} \) in (1.13). Again, twisted vacuum \( [-\frac{5}{3}, \frac{1}{3}]^4 \) is tentatively identified with the \( \epsilon^{abcd} \) symbol and we seek a “polynomial” representative for it.

---

By switching to \( \hat{\mathcal{M}} \), the mirror model of \( \mathcal{M} \), the eight charge-(1,1) states will all acquire charge-(1,1) counterparts. For the special choice of the superpotential \( P_M \) as in (4.13), the superpotential is self-transposed and mirror is obtained simply as a quotient of \( \mathcal{M} \) itself \[23\]. As this is not the case in general, we shall place a ‘hat’ on the coordinates when they refer to \( \hat{\mathcal{M}} \). It is easy to check that (the pull-backs of) the Kähler forms \( J_x \) and \( J_y \) have the following mirrors:
\[
\vartheta_x \overset{\text{def}}{=} \hat{X}_0 \hat{Y}_1 \hat{Y}_2 \hat{Y}_3 , \quad \vartheta_y \overset{\text{def}}{=} \hat{X}_1 \hat{X}_2 \hat{X}_3 ,
\]  
(4.20)

where again the choice of the labeling involved a little forethought. In addition, the mirrors of \( X_a X_b \big| [-\frac{5}{3}, \frac{1}{3}]^4 \) will be twisted (c,c)-states, however, only consisting of twisted (c,c)-vacua \[6\].

Note that the above two representatives, \( \vartheta_x \) and \( \vartheta_y \), are the two factors of the ‘fundamental monomial’ \( \prod_\alpha \hat{X}_\alpha \prod_\alpha \hat{Y}_\alpha \) (see Ref. \[15\]) which are invariant under the \( \mathbb{Z}_3 \times \mathbb{Z}_9 \)

\[6\] We thank M. Kreuzer for pointing out an error in an early version of the paper.
symmetry—the action of which we must divide out to obtain the mirror \( \hat{\mathcal{M}} \). Also, \( \vartheta_x \) and \( \vartheta_y \) are nontrivial deformations of \( \mathcal{M} \) (upon un-hatting the coordinate fields) as well as of the mirror model (with the hats on), and will remain so also away from the specially symmetric choice (4.13) \([15]\).

4.4. Yukawa couplings

The twist-number selection rule (4.8) alone dictates that

\[
\langle J_x, J_y, J_y \rangle \sim \langle 0, 0 | +3, 0 \rangle^7 \langle 0, -3 \rangle^7 | -1, 1 \rangle^3 | -1, 1 \rangle^5 | -1, 1 \rangle^5,
\]

\[
\langle \epsilon^{\cdots ij}, \epsilon^{\cdots pq}, J_y \rangle \sim \langle 0, 0 | +3, 0 \rangle^7 \langle 0, -3 \rangle^7 (X_i X_j \mid -\frac{5}{3}, \frac{1}{3})^4)(X_p X_q \mid -\frac{5}{3}, \frac{1}{3})^4 | -1, 1 \rangle^5
\]

are the only non-vanishing \((a, c)\)-sector Yukawa couplings in the Landau-Ginzburg orbifold limit.

As in the previous case, away from the Landau-Ginzburg orbifold limit, the geometrical computation also yields \( \langle J_x, J_x, J_y \rangle \neq 0 \). Again, the \( \langle J_x, J_x, J_y \rangle \neq 0 \) result can be obtained also in the Landau-Ginzburg orbifold framework, by calculating \( \langle \vartheta_x, \vartheta_x, \vartheta_y \rangle \neq 0 \) for a deformed model of \( \hat{\mathcal{M}} \), e.g., one in which a multiple of \( \vartheta_x \) has been added to \( P_{\hat{\mathcal{M}}} \). This happens in exact parallel to the previous example and we obtain that

\[
| -1, 1 \rangle^3 \overset{M}{\sim} \vartheta_x , \quad | -1, 1 \rangle^5 \overset{M}{\sim} \vartheta_y
\]

are mirror-duals. In fact, since both \( X_0 Y_1 Y_2 Y_3 \) and \( X_1 X_2 X_3 \) are invariant under the \( \mathbb{Z}_3 \times \mathbb{Z}_3 \) quotient which gives the mirror, they are present both in the original and the mirror model, so that \( | -1, 1 \rangle^3 \) and \( | -1, 1 \rangle^5 \) may be identified with \( X_0 Y_1 Y_2 Y_3 \) and \( X_1 X_2 X_3 \) respectively as their monomial representatives. The ‘top element’ \( Q_\mathcal{M} = 2 \cdot 3^7 \cdot 7^3 X_0 X_1^2 Y_1 X_2^2 Y_2 X_3^2 Y_3 \) is inherited by the mirror model, since \( \hat{\mathcal{M}} = \mathcal{M} / \Delta \) and only those quotients are allowed where \( \Delta \) leaves \( Q_\mathcal{M} \) invariant. Clearly, \( \vartheta_x \vartheta_y^2 \propto Q_{\hat{\mathcal{M}}} = 2 \cdot 3^7 \cdot 7^3 \hat{X}_0 \hat{X}_1^2 \hat{Y}_1 \hat{X}_2^2 \hat{Y}_2 \hat{X}_3^2 \hat{Y}_3 \), which in fact equals \( Q_\mathcal{M} \).

The type of the Yukawa coupling (4.21b) did not occur in the previous model and provides new information here. From (4.21b) and the discussion in the above paragraph we find that

\[
\left( \left| -\frac{5}{3}, \frac{1}{3} \right|^4 \right)^2 = Y_1 Y_2 Y_3 \left( | 0, 0 \rangle^0 \right)^2
\]

must hold. In other words, \( \sqrt{Y_1 Y_2 Y_3} \) may be regarded as the ‘polynomial’ representative of the twist-field producing \( | -\frac{5}{3}, \frac{1}{3} \rangle^4 \) from \( | 0, 0 \rangle^0 \).

Turning back to the Koszul computation, the analogue of this result is not hard to prove. That is, the factor \( \epsilon^{abcd} \) in the field representative (4.13) may be assigned the ‘polynomial’ representative \( \sqrt{Y_1 Y_2 Y_3} \). To see this, consider the square of the \( \epsilon^{\alpha \beta \gamma} \)

\( \text{[Footnote 7]: The analysis can also be carried out by considering the \((c, c)\)-ring of the mirror } \hat{\mathcal{M}} \text{ only. By using the selection rules and the ringstructure, in particular the existence of the top element } Q_{\hat{\mathcal{M}}}, \text{ one finds that the six twisted \('\text{non-polynomial}'\) \((1, 1)\) states are given by } \hat{X}_a \hat{X}_b \sqrt{Y_1 Y_2 Y_3} \text{ for } a \neq b. \)
from the six dual representatives in (4.17). Through contraction with the defining tensors $g^a_{\alpha\mu\lambda}g^b_{\beta\nu\kappa}g^c_{\gamma\rho\sigma}$, the square of the antisymmetric symbol, $\epsilon^{a\beta\gamma}\epsilon^{\mu\nu\rho}$ is dual to $\check{\vartheta}_y \cdot Y_1 Y_2 Y_3$, (4.14)

where the result on the r.h.s. of the arrow in (4.24) has been obtained with the simple choice of the defining tensors (4.14), and $\check{\vartheta}_y = X_1 X_2 X_3$ is the un-hatted version of $\vartheta_y$.

Recall finally that in the Koszul computation, the pull-backs of the Kähler forms enter as scalars, rather than tensors dual to $\vartheta_x = X_0 Y_1 Y_2$ or $\vartheta_y = X_1 X_2 X_3$. Therefore, the overall factor of $\check{\vartheta}_y$ in Eq. (4.24) was necessary just so as to make the Yukawa coupling (4.21b) non-zero in the Koszul framework.

It is then clear that the Landau-Ginzburg orbifold and the $\langle J_x, J_x, J_y \rangle \to 0$ limit of the Koszul calculation (as discussed in the previous example) predict the same Yukawa couplings to vanish. We may now address the issue of the ratio between all the non-zero Yukawa couplings (4.21a) and (4.21b). In the Landau-Ginzburg orbifold framework, this is a definite number, equal to 1 with the current definitions of the field representatives. In the Koszul computation, the coupling (4.21a) is simply a product of scalars and is therefore already a scalar quantity, which can be set equal to the value of (4.21b).

The ratio of the Yukawa couplings is therefore in both frameworks a matter of normalization of the field representatives and can clearly be made to agree. Of course, in doing so, we had to take the Landau-Ginzburg orbifold limit of the Koszul results, whence $\langle J_x, J_x, J_y \rangle$ was made to vanish as discussed in detail in the previous example.

4.5. The general case

As for the $(c, c)$ ring, we would like to extend the results for the $(a, c)$ ring to the situation when there is no (known) Landau-Ginzburg orbifold corresponding to the complete intersection manifold description.

Let $\mathcal{M}$ be defined as in section 3.5. The tensor representatives for the elements in $H^1(\mathcal{M}, T_\mathcal{M})$ are of the form $\epsilon^{\cdots}\phi^{\cdots}$. Except for some exceptional cases [3, 21], here are also $N$ scalars $J_i$ each one being the pull-back of the corresponding Kähler form on $\mathbb{P}^{n_i}$. In order to derive the monomial representatives for the various $\epsilon^{\cdots}\phi^{\cdots}$s, we consider the dual cohomology $H^2(\mathcal{T})$. The dual of the $\epsilon^{\cdots}\phi^{\cdots}$ are given by $\tilde{\epsilon}^{\cdots}\psi^{\cdots}$, where $\epsilon^{\cdots}$ and $\tilde{\epsilon}^{\cdots}$ are dual in the sense that their product is the total $\epsilon$-symbol for the whole field space. As for the $(c, c)$ ring then, we find the dual monomial representatives of the $\tilde{\epsilon}^{\cdots}$ by considering the square, $\tilde{\epsilon}^2$, and contract with the appropriate $f^{(m)}_{\cdots}$ and $x^{(i)}_j$. The square root of the so entered product of $x^{(i)}_j$’s is the ‘monomial’ representative for $\epsilon^{\cdots}$.

We also need to identify the $J_i$ with a certain ‘canonical’ product of coordinates. To that effect we have to consider the ‘canonical’ deformations of the mirror manifold $\mathcal{W}$. However, only in a very few cases do we know what the mirror is. As for the class of weighted projective spaces one may guess that $\mathcal{W}$ will be a quotient of some other complete intersection $\mathcal{M}$. For a large class of models of the former type, it has been argued that the
mirror model is obtained as a quotient of a model based on the transpose of the defining polynomial. It is tempting to say that the same will be true for the whole class of complete intersection Calabi-Yau manifolds. Thus the mirror manifold would be $M^T/H$ where $M^T$ is the transposed complete intersection and $H$ is the quotient group necessary to produce $W$. By $M^T$ we mean to take the transpose of the degree matrix, given by the defining equations. In order for the transposed theory to be a Calabi-Yau manifold, one will have to consider the hypersurface to be embedded in a different product of weighted projective spaces such that the condition of vanishing first Chern class is satisfied in each of the projective spaces. We hope to return shortly with a proof of the above mirror hypothesis.

From each defining equation (there are now $N$ of them) we have a ‘fundamental deformation’ much in the same way as in the case of just one (weighted) projective space. These $N$ monomials we identify as the $J_i$. At this point we have a complete set of coordinate representatives for the $H^1(M, T^*_M)$. The computation of the Yukawa couplings follows in the same way as in the $(c,c)$ ring.

5. Concluding Remarks

To complete proving the equivalence of the various computations of the Yukawa couplings, up to the unspecified Kähler class in the Koszul computation, one has to address the issue of field representative normalizations, i.e., the kinetic terms.

Besides studying this problem by explicit computation, model by model, we are aware of two general approaches. On one hand, a powerful machinery is currently being developed for the calculation of so-called periods $\int_\gamma \Omega$ of the holomorphic 3-form. These quantities turn out to completely determine both the ‘un-normalized’ Yukawa couplings and also the kinetic terms for the effective $27$ and $27^*$ fields. This study will be reported elsewhere.

Another method is based on the analysis of Ref. [20], where it was proven— for Fermat type polynomials, i.e., tensor products of $A_k$-type minimal models—that the Landau-Ginzburg orbifold analysis can be extended to provide the kinetic terms identical to those of the exact superconformal models. The key point is that the Witten vacuum wave functions can be completely determined by solving certain ‘Schrödinger equations’, and that the usual normalization of these wave-functions precisely reproduces the kinetic terms dictated by the exact superconformal field theory. Of course, the general case involves more than just the $A_k$-type minimal models. Classifications of admissible superpotentials for Landau-Ginzburg orbifolds enable us to isolate, in addition to the Fermat type, two more classes of polynomials and the corresponding Wess-Zumino models and their analysis will be reported shortly.

In summary, we have extended the results of Ref. [12], and demonstrated a complete agreement between the Koszul computation and the Landau-Ginzburg orbifold computation of the Yukawa couplings. This agreement extends over both twisted and untwisted and both $(c,c)$ and $(a,c)$-sectors. To demonstrate this, we have used ‘ineffective splitting’

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8) We thank P. Candelas and collaborators for sharing their results prior to publication.
and the mirror map. Next, the ‘twist fields’ which create twisted vacua from untwisted ones have successfully been identified with the $\epsilon^-$ symbols of the Koszul computation. Moreover, these have been assigned representatives which are square roots of polynomials and which allow a straightforward generalization of the well-known Yukawa coupling formula $[2]$—for all $27$’s and $27^*$’s. In fact, we find (modulo our mirror conjecture for the complete intersection Calabi-Yau manifolds) that the Koszul computation can be translated into the existence of a polynomial $(c/a, c)$ ring even when the underlying $N = 2$ superconformal field theory is not known.

Finally, we wish to emphasize that neither the Koszul computation, nor the Landau-Ginzburg orbifold techniques depend on criticality (the Calabi-Yau condition), although the relation between the two will become more involved off-criticality. Also, these methods may be extended to field theories in higher dimensions, since they pertain to the analysis of supersymmetric vacuum wave-functions which exhibit a high degree of universality $[21]$.

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Appendix A. Gauge-Theoretic Introduction to Koszul Computation

The Koszul computation may be formulated as an extension of the familiar idea of gauge transformation and gauge invariance. In fact, some treatments of (second class) gauge constraints in the physics literature are remarkably similar to the standard treatment of the Koszul complex in the mathematical literature. The modest basic facts are reviewed here and we refer to Refs. [5,7,8] for more details and a more complete account.

The Koszul computation in general refers to the task of computing certain cohomology groups of a manifold \( M \), which is given as a submanifold in a better known \( X \) as the vanishing set of a system of constraints. To this end, consider the simple case in which \( M \) is a hypersurface in \( X \), given globally as the zero-set of a single constraint equation \( f(x) = 0 \). It is rather standard that local functions \( \phi(x) \) on \( M \) are obtained as a restriction to \( x \in M \) of local functions \( \Phi(x) \) on \( X \). In doing so, however, \( \phi(x) \) will be well defined only up to appropriate \( \lambda(x) \)-multiples of \( f(x) \):

\[
\phi(x) \cong \Phi(x) + \lambda(x) \cdot f(x) , \quad x \in M ,
\]

simply because on \( M \), \( f(x) \) vanishes and so do the \( \lambda(x) \)-multiples of \( f(x) \). Of course, the functions \( \lambda(x) \) are chosen so that the product \( \lambda(x) \cdot f(x) \) would transform just as \( \phi(x) \) does. In other words, we consider the gauge-transformation

\[
\delta \Phi(x) = \lambda(x) \cdot f(x)
\]

in the space of functions on \( X \). All such gauge-transforms of a function will agree on \( M \subset X \) and form the equivalence class which represents a single function on \( M \).

The same principle applies equally well to tensors, provided we take into account the fact that a component of a vector is (anyway) well defined only up to linear reparametrizations. Since a tangent vector of \( M \) is also a tangent vector of \( X \), it is clear that the former is undefined up to multiples of gradients of the defining function.

Finally, when looking for various harmonic forms on \( M \), it must be realized that the differential conditions of closedness and exactness, \( d\omega = 0 \) and \( \omega = d\alpha \) respectively, have to be appropriately restricted from \( X \) to \( M \).

The Koszul computation offers a purely algebraic analogue to all these requirements; one never has to solve any differential equation. In a sense, the Koszul computation is like an integral transform, which turns a system of differential and algebraic equations into a purely algebraic system.

There are three basic stages of the Koszul computation.

1. A tensor-valued form on \( M \) has to be specified in terms of tensor-valued forms on \( X \), restricted however as a function to \( M \subset X \);
2. The restrictions to \( M \) of tensor-valued forms on \( X \) have to be specified in terms of unrestricted forms on \( X \);
3. Finally, all required forms on \( X \) have to be specified in a fashion which will be convenient for calculations.

For this last stage, the Bott-Borel-Weil theorem guarantees that all required forms can be given in terms of \( U(N) \)-representations if the embedding space is chosen to be a product of flag-spaces of the form \( U(N)/ \prod_i U(n_i), \sum_i n_i = N \).
For the first stage, one uses the well-known fact that the gradient of $f(x)$ provides a normal to the hypersurface $M \subset X$. Thus, at any particular point of $M \subset X$, a tangent vector of $X$ decomposes into a vector tangent to $M$ and a multiple of $\nabla f(x)$. This relation is then iterated for tensors of higher rank and/or when $M$ is an intersection of two or more hypersurfaces, that is, the simultaneous zero-set of a system of constraints rather than a single one.

In the second stage, the components of the various tensors are treated as functions and are taken modulo appropriate multiples of (gradients of) the defining function. If the subspace $M$ is defined as the simultaneous zero-set of more than one constraint, this passage to gauge-equivalent classes can be done taking one constraint at a time. However, there is a subtlety in doing so and this is the basic reason for the growing complexity of the Koszul computation. Suffice it here to consider a case with two constraints, $f(x) = 0$ and $g(x) = 0$.

Since both $f(x)$ and $g(x)$ are set to zero on $M \subset X$, a function on $M$ is given as a class of function on $X$, defined up to the gauge-transformation

$$\delta \Phi(x) = \lambda_f(x) \cdot f(x) + \lambda_g(x) \cdot g(x), \quad x \in M,$$

(A.3)

where $\lambda_f(x)$ and $\lambda_g(x)$ are suitably transforming functions. It would thus appear that these two functions are arbitrary and independent and that the space of functions is a quotient

$$\{ \phi(x) \} = \left\{ \{ F(x) \} \big/ \{ \lambda_f(x) \cdot f(x) \oplus \lambda_g(x) \cdot g(x) \} \right\}.$$

(A.4)

Notice, however, that by choosing

$$\lambda_f(x) = -\mu(x) g(x), \quad \lambda_g(x) = +\mu(x) f(x),$$

(A.5)

the gauge-transformation of $\Phi(x)$ vanishes trivially, even away from $M$. Therefore, too much has been divided out in the quotient (A.4); the equivalence relation in (A.3) is too big. The gauge transformation (A.3) is itself subject to a second-level gauge-transformation

$$\delta \lambda_f(x) = -\mu(x) g(x), \quad \delta \lambda_g(x) = +\mu(x) f(x).$$

(A.6)

In a field theory where the constraints $f(x) = 0$ and $g(x) = 0$ are implemented by means of Lagrange multipliers, the second-level gauge-transformation (A.6) would act on the Lagrange multipliers of $f(x)$ and $g(x)$. Just as in a BRST treatment there would be introduced ghost fields of $\lambda_f(x)$ and $\lambda_g(x)$, there would now also be introduced ghost-for-ghost of $\mu(x)$.

Clearly, with more than two constraints, there will be gauge invariances of higher levels, the total level number equaling the total number of independent constraints. In the Koszul computation, these hierarchies of gauge-transformations are systematically and effectively taken care by the so-called spectral sequence, and we refer the interested Reader to Refs. [5,7,8] for details of this technique.
Appendix B. On Ideals

It was shown in Ref. [12] that the Koszul ideal and the ideal of the Chiral ring structure are in fact equal. For the sake of completeness and also to illustrate several simplifications which take place when a Landau-Ginzburg orbifold formulation is possible, we compute this for the particular example studied in section 2.

For simplicity let us only consider the ideal that occurs in the Chiral ring. The superpotential is defined to be the sum

\[ P_Q \overset{\text{def}}{=} f(X) + g(X,Y) \, . \]  

(B.1)

Clearly, for this to make sense, one has to be able to assign scaling charges to the chiral fields \( X_a \) and \( Y_\alpha \) such that \( P_Q \) scales quasihomogeneously; in general, this may be an important obstruction to construct Landau-Ginzburg orbifolds corresponding to a given Calabi-Yau manifold [11,12].

Thereupon, the ideal in the Chiral ring is generated by the gradients of \( P_Q \):

\[ \mathcal{I}[\partial P_Q] \sim \left\{ (\partial_a f + \partial_a g), (\partial_\alpha g) \right\} , \]  

(B.2)

where, of course, \( \partial_a \overset{\text{def}}{=} \frac{\partial}{\partial X_a} \) and \( \partial_\alpha \overset{\text{def}}{=} \frac{\partial}{\partial Y_\alpha} \).

Since the forms we are interested in are all (co)tangent-valued, they will be well defined up to addition of multiples of gradients of the defining functions \( f(x) \) and \( g(x,y) \) in (3.1). In fact, since these functions are homogeneous, the gauge-transformation of the type (A.3) will also be generated by addition of multiples of the gradients of \( f(x) \) and \( g(x,y) \). Naïvely, therefore, one would take that the various tangent-valued forms are defined up to additive elements of the ideal

\[ \mathcal{I}[\partial f, \partial g] \sim \left\{ (\partial_a f), (\partial_a g), (\partial_\alpha g) \right\} , \]  

(B.3)

consisting of all multiples of the indicated gradients. Comparing the ideal (B.3) with (B.2), it is quite obvious that this naïve ideal is too big. In fact, we have the same problem as with the quotient (A.4). The complete Koszul computation takes care of the second-level gauge-equivalence \( \partial_a (f - g) \). In the present case, \( \partial_\alpha f = 0 \) trivially, so that \( \partial_\alpha (f - g) \) does not generate a redundant relation in the Koszul ideal. Since

\[ \left\{ \lambda^a \cdot \partial_a f \oplus \lambda^\alpha_\cdot \partial_\alpha g \right\}_{\mu} \approx \left\{ \nu \cdot [\partial_a f + \partial_a g] \right\} , \]  

(B.4)

we have proven that the Koszul ideal equals the Landau-Ginzburg one, occurring in the Chiral ring of the Landau-Ginzburg orbifold.

With more than two defining functions for a Calabi-Yau manifold—if a Landau-Ginzburg orbifold can be defined—the same equality will hold, although proving it will be rather more involved technically. A simple example with three defining functions is presented in Ref. [12].
### Table 1: The Ramond and Neveu-Schwarz vacua of the Landau-Ginzburg orbifold $Q$, defined in Eq. (3.4).

| $(a, a)$ | $(c, c)$ | Ramond | $(a, c)$ | $(c, a)$ |
|----------|----------|---------|----------|----------|
| $|{-3, -3}\rangle^0$ | $|0, 0\rangle^0$ | $|{-\frac{3}{2}, -\frac{3}{2}}\rangle^0$ | $|{-3, 0}\rangle^1$ | $|0, {-3}\rangle^7$ |
| $|{-3, 0}\rangle^1$ | $|0, +3\rangle^1$ | $|{-\frac{3}{2}, +\frac{3}{2}}\rangle^1$ | $|{-3, +3}\rangle^2$ | $|0, 0\rangle^0$ |
| $|{-1, -2}\rangle^2$ | $|+2, +1\rangle^2$ | $|{+\frac{1}{2}, -\frac{1}{2}}\rangle^2$ | $|{-1, +1}\rangle^3$ | $|+2, -2\rangle^1$ |
| $|{-1, -2}\rangle^3$ | $|+2, +1\rangle^3$ | $|{+\frac{1}{2}, -\frac{1}{2}}\rangle^3$ | $|{-1, +1}\rangle^4$ | $|+2, -2\rangle^2$ |
| $|{-\frac{11}{4}, -\frac{11}{4}}\rangle^4$ | $|{+\frac{1}{4}, +\frac{1}{4}}\rangle^4$ | $|{-\frac{5}{4}, -\frac{5}{4}}\rangle^4$ | $|{-\frac{11}{4}, +\frac{1}{4}}\rangle^5$ | $|{+\frac{1}{4}, -\frac{11}{4}}\rangle^3$ |
| $|{-2, -1}\rangle^5$ | $|+1, +2\rangle^5$ | $|{-\frac{1}{2}, +\frac{1}{2}}\rangle^5$ | $|{-2, +2}\rangle^6$ | $|+1, -1\rangle^4$ |
| $|{-2, -1}\rangle^6$ | $|+1, +2\rangle^6$ | $|{-\frac{1}{2}, +\frac{1}{2}}\rangle^6$ | $|{-2, +2}\rangle^7$ | $|+1, -1\rangle^5$ |
| $|0, {-3}\rangle^7$ | $|+3, +0\rangle^7$ | $|{+\frac{3}{2}, -\frac{3}{2}}\rangle^7$ | $|0, 0\rangle^0$ | $|+3, -3\rangle^6$ |

### Table 2: The Ramond and Neveu-Schwarz vacua of the Landau-Ginzburg orbifold $\tilde{Q}$, defined in Eq. (3.6).
### Table 3: The Koszul and the Landau-Ginzburg orbifold rendition of the \((c,c)\)-states for the models \(\mathcal{Q}\) and \(\widetilde{\mathcal{Q}}\). Recall, \(a, b, . = 1, \ldots, 5\) and \(\alpha, \beta, A, B = 1, 2\); different labels mean different values \((a \neq b\ etc.)\).

| \((a, a)\) | \((c, c)\) | Ramond | \((a, c)\) | \((c, a)\) |
|-----------|-----------|---------|-----------|-----------|
| \((-3, -3)^0\) | \(0, 0)^0\) | \(-\frac{3}{2}, -\frac{3}{2})^0\) | \(-3, 0)^1\) | \(0, -3)^7\) |
| \((-3, 0)^1\) | \(0, +3)^1\) | \(-\frac{3}{2}, +\frac{3}{2})^1\) | \(-3, +3)^2\) | \(0, 0)^0\) |
| \((-1, -2)^2\) | \(+2, +1)^2\) | \(+\frac{1}{2}, -\frac{1}{2})^2\) | \(-1, +1)^3\) | \(+2, -2)^1\) |
| \((-\frac{5}{3}, -\frac{8}{3})^3\) | \(+\frac{4}{3}, +\frac{13}{3})^3\) | \(-\frac{1}{6}, -\frac{7}{6})^3\) | \(-\frac{5}{3}, +\frac{13}{3})^4\) | \(+\frac{4}{3}, -\frac{8}{3})^2\) |
| \((-1, -2)^4\) | \(+2, +1)^4\) | \(+\frac{1}{2}, -\frac{1}{2})^4\) | \(-1, +1)^5\) | \(+2, -2)^3\) |
| \((-2, -1)^5\) | \(+1, +2)^5\) | \(-\frac{1}{2}, +\frac{1}{2})^5\) | \(-2, +2)^6\) | \(+1, -1)^4\) |
| \((-\frac{8}{9}, -\frac{5}{9})^6\) | \(+\frac{1}{3}, +\frac{4}{3})^6\) | \(-\frac{7}{6}, -\frac{1}{6})^6\) | \(-\frac{8}{9}, +\frac{4}{3})^7\) | \(+\frac{1}{3}, -\frac{5}{3})^5\) |
| \((-2, -1)^7\) | \(+1, +2)^7\) | \(-\frac{1}{2}, +\frac{1}{2})^7\) | \(-2, +2)^8\) | \(+1, -1)^6\) |
| \((0, -3)^8\) | \(+3, +0)^8\) | \(+\frac{3}{2}, -\frac{3}{2})^8\) | \(0, 0)^0\) | \(+3, -3)^7\) |

Table 4: The Ramond and Neveu-Schwarz vacua of the Landau-Ginzburg orbifold \(\mathcal{M}\).
References

[1] P. Candelas and S. Weinberg: *Nucl. Phys.* **B237** (1984)397.
[2] P. Candelas: *Nucl. Phys.* **B298** (1988)458.
[3] J. Distler and B.R. Greene: *Nucl. Phys.* **B309** (1988)295;
    B.R. Greene, C.A. Lütken and G.G. Ross: *Nucl. Phys.* **B325** (1989)101;
    S.F. Cordes and Y. Kikuchi: Correlation Functions and Selection Rules in Minimal
    N=2 String Compactifications. *University of Texas A&M report* CTP-TAMU-92/88
    (1988, unpublished), *Mod. Phys. Lett.* **A4** (1989)1365;
    R. Schimmrigk: *Phys. Lett.* **229B** (1989)227.
[4] T. Hübsch: *Commun. Math. Phys.* **108** (1987)291;
    P. Green and T. Hübsch: *Commun. Math. Phys.* **109** (1987)99;
    P. Candelas, A.M. Dale, C.A. Lütken and R. Schimmrigk: *Nucl. Phys.* **B298**
    (1988)493.
[5] T. Hübsch: *Calabi-Yau Manifolds—A Bestiary for Physicists*
    (World Scientific, Singapore, 1992).
[6] P. Candelas, M. Lynker and R. Schimmrigk: *Nucl. Phys.* **B341** (1990)383;
    A. Klemm and R. Schimmrigk : Landau-Ginzburg String Vacua, CERN preprint
    CERN-TH 6459/92;
    M. Kreuzer and H. Skarke: No Mirror Symmetry in Landau-Ginzburg Spectra,
    CERN preprint CERN-TH 6461/92.
[7] M.G. Eastwood: *Math. Proc. Camb. Phil. Soc.* **97**(1985)165;
    M.G. Eastwood and T. Hübsch: *Commun. Math. Phys.* **132** (1990)383.
[8] P. Berglund, T. Hübsch and L. Parkes: *Mod. Phys. Lett.* **A5** (1990)1485,
    *Commun. Math. Phys.* **148** (1992)57.
[9] B.R. Greene, C. Vafa and N.P. Warner: *Nucl. Phys.* **B324** (1989)371.
[10] B.R. Greene: *Commun. Math. Phys.* **130** (1990)335.
[11] T. Hübsch: *Class. Quant. Grav.* **8** (1991)L31.
[12] P. Berglund, B. Greene and T. Hübsch: *Mod. Phys. Lett.* **A7** (1992)1855.
[13] C. Vafa: *Mod. Phys. Lett.* **A4** (1989)1169;
    K. Intrilligator and C. Vafa: *Nucl. Phys.* **B339** (1990)95.
[14] C. Vafa and N. Warner: *Phys. Lett.* **218B** (1989)51;
    W. Lerche, C. Vafa and N. Warner: *Nucl. Phys.* **B324** (1989)427.
[15] T. Hübsch and S.-T. Yau: *Mod. Phys. Lett.* **A7** (1992)3277; see also
    *Essays on Mirror Manifolds* p.372–387, ed. S.-T. Yau, (International Press, Hong Kong, 1992).
[16] D. Gepner: String Theory on Calabi-Yau Manifolds: the Three Generations Case.
    *Princeton University report* (December 1987, unpublished).
[17] R. Hartshorne: *Algebraic Geometry* (Springer-Verlag, New York, 1977).
[18] D. Gepner: *Phys. Lett.* **199B** (1987)380.
[19] R. Schimmrigk: *Phys. Lett.* **193B** (1987)175.
[20] S. Cecotti, L. Girardello and A. Pasquinucci: *Nucl. Phys.* **B328** (1989)701.
[21] P. Green and T. Hübsch: *Commun. Math. Phys.* **113** (1987)505.
[22] J. Distler and B.R. Greene: *Nucl. Phys.* **B309** (1988)295.
[23] P. Berglund and T. Hübsch: *Nucl. Phys.* **B** (in press), also in
    *Essays on Mirror Manifolds* p.388–407, ed. S.-T. Yau, (International Press, Hong Kong, 1992).
[24] J. Distler, B. Greene, K. Kirklin and P. Miron: *Phys. Lett.* **195B** (1987)41;
    P. Green and T. Hübsch: *Class. Quant. Grav.* **6** (1989)311.
[25] M. Dine, N. Seiberg, X.G. Wen and E. Witten: *Nucl. Phys. B278* (1986) 769, *ibid. B289* (1987) 319.

[26] M. Kreuzer and H. Skarke: On the Classification of Quasihomogeneous Functions, CERN preprint CERN-TH-6373/92.