REALIZATION OF HOMOGENEOUS TRIEBEL-LIZORKIN SPACES WITH $p = \infty$ AND CHARACTERIZATIONS VIA DIFFERENCES

M. BENALLIA, M. MOUSSAI

Abstract. In this paper, via the decomposition of Littlewood-Paley, the homogeneous Triebel-Lizorkin space $\dot{F}^s_{\infty,q}$ is defined on $\mathbb{R}^n$ by distributions modulo polynomials in the sense that $\|f\| = 0$ if and only if $f$ is a polynomial on $\mathbb{R}^n$. We consider this space as a set of “true” distributions and we are lead to examine the convergence of the Littlewood-Paley sequence of each element in $\dot{F}^s_{\infty,q}$. First we use the realizations and then we obtain the realized space $\tilde{\dot{F}}^s_{\infty,q}$ of $\dot{F}^s_{\infty,q}$. After that, we study the commuting translations and dilations of realizations in $\dot{F}^s_{\infty,q}$ and employing distributions vanishing at infinity in the weak sense, we construct $\tilde{\dot{F}}^s_{\infty,q}$. Then, as another possible definition of $\dot{F}^s_{\infty,q}$, in the case $s > 0$, we make use of the differences and describe $\tilde{\dot{F}}^s_{\infty,q}$ as $s > \max(n/q - n, 0)$.

Keywords: Triebel-Lizorkin spaces, Littlewood-Paley decomposition, realizations.

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1. Introduction

In this paper we study a realization of homogeneous Triebel-Lizorkin spaces $\dot{F}^s_{\infty,q}$ on $\mathbb{R}^n$. The spaces $\dot{F}^s_{\infty,q}$ are defined by distributions modulo polynomials in the sense that $\|f\|_{\dot{F}^s_{\infty,q}} = 0$ if and only if $f$ is a polynomial on $\mathbb{R}^n$. Some of their properties can be found in [12], [22].

The basic definition of $\dot{F}^s_{\infty,q}$ is given via the Littlewood-Paley decomposition (abbreviated as LP decomposition). To recall this, we introduce some notations.

By $\rho$ we denote an infinitely differentiable radial function obeying the estimates $0 \leq \rho \leq 1$ such that

$$
\rho(\xi) = 1 \quad \text{as} \quad |\xi| \leq 1, \quad \rho(\xi) = 0 \quad \text{as} \quad |\xi| \geq \frac{3}{2}.
$$

We denote $\gamma(\xi) := \rho(\xi) - \rho(2\xi)$. This function is supported in the annulus $\frac{1}{2} \leq |\xi| \leq \frac{3}{2}$, and

$$
\gamma(\xi) = 1 \quad \text{as} \quad \frac{3}{4} \leq |\xi| \leq 1, \quad \sum_{j \in \mathbb{Z}} \gamma(2^j \xi) = 1 \quad \text{as} \quad \xi \neq 0.
$$

For $m \in \mathbb{N}$, the symbol $\mathcal{P}_m$ stands for the set of all polynomials on $\mathbb{R}^n$ of degree less than $m$ obeying $\mathcal{P}_0 = \{0\}$. By $\mathcal{P}_\infty$ we denote the set of all polynomials. For $m \in \mathbb{N}_0 \cup \{\infty\}$, the set $\mathcal{S}_m'$ of the tempered distributions modulo polynomials is the dual space of $\mathcal{S}_m$, which is the orthogonal space of $\mathcal{P}_m$ in $\mathcal{S}$, that is, $\mathcal{S}_m$ is the set of all $f \in \mathcal{S}$ such that $\langle u, f \rangle = 0$ for all $u \in \mathcal{P}_m$. For a tempered distributions $f \in \mathcal{S}'$, the symbol $[f]_m$ denotes the equivalence class of $f$ modulo $\mathcal{P}_m$.

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We define the operators \( Q_j \) by the formula
\[
Q_j f := \gamma(2^{-j}(-\cdot)) f, \quad j \in \mathbb{Z}.
\]
These operators are defined on \( S' \) as well as on \( S'_m \) since \( Q_j f = 0 \) if and only if \( f \in \mathcal{P}_m \). For instance, we have \( Q_j(S) \subset S_{\infty} \). All these operators take values in the space of analytical functions of exponential type, see the Paley-Wiener theorem. Finally, we adopt the following convention: for \( f \in S'_m \), we define \( Q_j f := Q_j f_1 \) for all \( f_1 \in S' \) such that \([f_1]_m = f\).

We turn to the LP decomposition; for all \( f \in S_{\infty} \) (or \( S'_\infty \)) the identity
\[
f = \sum_{j \in \mathbb{Z}} Q_j f \quad \text{in} \quad S_{\infty} \quad \text{(or} \quad S'_\infty) \tag{1}
\]
holds; this is an easy application of Lemma [7] below. However, once we work in \( \hat{F}^s_{\infty,q} \), it is possible to obtain the convergence of the series of the LP decomposition in \( S'_\mu \) for some integer \( \mu \), see (7) below. This leads us to the need to realize \( \hat{F}^s_{\infty,q} \) and to obtain the realized spaces by using the notion of realization. For a quasi-Banach distribution space \( E \hookrightarrow S'_\infty \), we need to find a continuous linear mapping \( \sigma : E \rightarrow S'_\mu \) such that \([\sigma(f)]_m \) coincides with \( f \) modulo polynomials in \( \mathcal{P}_m \) for all \( f \in E \), cf. Definition [4] below. If in addition, \( E \) is a translation or a dilation invariant, that is,
\[
\|\tau_a f\|_E = \|f\|_E \quad \text{or} \quad \|h_\lambda f\|_E = \lambda^r \|f\|_E
\]
with \( r \in \mathbb{R} \), where \( \tau_a f(x) := f(x-a) \) and \( h_\lambda f(x) := f(x/\lambda) \) for all \( x,a \in \mathbb{R}^n \) and all \( \lambda > 0 \), the existence of such a \( \sigma \) commuting with translation or dilation operators, that is, obeying
\[
\tau_a \circ \sigma = \sigma \circ \tau_a \quad \text{or} \quad h_\lambda \circ \sigma = \sigma \circ h_\lambda,
\]
is nontrivial.

We note that the realizations have been introduced by G. Bourdaud [3] for the homogeneous Besov spaces \( \hat{B}^s_{p,q} \); the corresponding integer \( \mu \) was defined in [7]. In the same way, we know the realizations of both the homogeneous Triebel-Lizorkin spaces \( \hat{F}^s_{p,q} \) with \( p < \infty \) and the homogeneous Sobolev spaces \( \dot{W}^m \), and some of their properties, see, for instance, [2], [5], [6], [7], [10], [21]. Also, nowadays there are various papers presenting applications of the realizations to Navier-Stokes equations, pseudodifferential operators, wavelet, etc., see, for instance, [9], [15], [20] and in particular, a comment in [11].

On the other hand, the distributions vanishing at infinity play an important role to characterize such realization. We recall this notion.

**Definition 1.** We say that a distribution \( f \in S' \) vanishes at infinity if
\[
\lim_{\lambda \to 0} h_\lambda f = 0 \quad \text{in} \quad S'.
\]
The set of all such distributions is denoted by \( \tilde{C}_0 \).

For instance, we have \( f \in \tilde{C}_0 \) if \( f \in L_p \) (1 \( \leq p < \infty \)). If either \( f \in L_\infty \) or \( f \in \tilde{C}_0 \) then \( \partial_j f \in \tilde{C}_0 \) (\( j = 1, \ldots, n \)). An easy statement is given by identity \( \tilde{C}_0 \cap \mathcal{P}_\infty = \{0\} \) (see, for instance, [3]).

As usually, \( \mathbb{N} \) stands for the natural numbers \( \{1, 2, \ldots\} \) and \( \mathbb{N}_0 := \mathbb{N} \cup \{0\} \). All function spaces occurring in the paper are defined in the Euclidean space \( \mathbb{R}^n \). By \( \| \cdot \|_p \) we denote the \( L_p \) quasi-norm for \( 0 < p \leq \infty \). For \( s \in \mathbb{R} \), the symbol \([s]\) denotes the integer part of \( s \). For all \( m \in \mathbb{N}_0 \), the standard norms in \( S \) are given by
\[
\zeta_m(f) := \sup_{x \in \mathbb{R}^n} \sup_{|\alpha| \leq m} (1 + |x|)^m |f^{(\alpha)}(x)|.
\]
The Fourier transform for a function \( f \in L_1 \) is defined as
\[
\mathcal{F} f(\xi) = \hat{f}(\xi) := \int_{\mathbb{R}^n} e^{-ix \cdot \xi} f(x) \, dx, \quad \xi \in \mathbb{R}^n.
\]
The operator \( \mathcal{F} \) can be extended to the whole \( S' \) in the usual way. In the same way we define the inverse Fourier transform \( \mathcal{F}^{-1} \),
\[
\mathcal{F}^{-1} f(x) := (2\pi)^{-n} \hat{f}(-x).
\]
For an arbitrary function \( f \), we define the difference operators as
\[
\Delta_h f = \Delta_h^1 f := \tau_h f - f, \quad \Delta_h^m f := \Delta_h(\Delta_h^{m-1} f), \quad h \in \mathbb{R}^n, \ m = 2, 3, \ldots
\]
The constants \( c, c_1, \ldots \) are strictly positive and depend only on the fixed parameters as \( n, s, \) and probably on auxiliary functions, their values may vary from line to line. The notation \( A \lesssim B \) means that \( A \leq cB \). The symbol \( E \hookrightarrow F \) denotes that we have the embedding \( E \subseteq F \) and the natural mapping \( E \to F \) is continuous. Throughout the paper, the real numbers \( s, q \) satisfy as \( s \in \mathbb{R} \) and \( 0 < q \leq \infty \) unless otherwise is stated.

The paper is organized as follows. In Section 2 we recall the definitions and some properties of homogeneous Triebel-Lizorkin spaces \( \dot{F}^{s}_{\infty, q} \) and of inhomogeneous ones \( F^{s}_{\infty, q} \). Section 3 is devoted to the realizations of \( \dot{F}^{s}_{\infty, q} \). In Section 4 by means of the differences, we characterize the realized spaces of \( \dot{F}^{s}_{\infty, q} \) in the case \( s > \max(n/q - n, 0) \).

2. Preliminaries

2.1. Homogeneous spaces \( \dot{F}^{s}_{\infty, q} \). By \( P_{k, \nu} \) \((k \in \mathbb{Z}, \nu \in \mathbb{Z}^n)\) we denote the dyadic cube with side length \( 2^{-k} \), left lower corner in the point \( 2^{-k}\nu \) and sides parallel to the coordinate axes, that is,
\[
P_{k, \nu} := \{ x \in \mathbb{R}^n : 2^{-k}\nu_j \leq x_j < 2^{-k}(\nu_j + 1), \ j = 1, 2, \ldots, n \}.
\]
The definition of \( \dot{F}^{s}_{\infty, q} \) was given by Frazier and Jawerth [12] as follows.

**Definition 2.** Let \( q \in [0, \infty[ \). The space \( \dot{F}^{s}_{\infty, q} \) is the set of \( f \in S'_\infty \) such that
\[
\|f\|_{\dot{F}^{s}_{\infty, q}} := \sup_{k \in \mathbb{Z}} \sup_{\nu \in \mathbb{Z}^n} \left( 2^{kn} \int_{P_{k, \nu}} \sum_{j \geq k} 2^{jsq}|Q_j f(x)|^q \, dx \right)^{1/q} < \infty.
\]

**Remark 1.** For \( q = \infty \), the set \( \dot{F}^{s}_{\infty, \infty} \) coincides with the H"older space \( \dot{B}^{s}_{\infty, \infty} \), see [14, Eq. (1.3)] and Lemma 3 below. We let
\[
\|f\|_{\dot{F}^{s}_{\infty, \infty}} := \sup_{j \in \mathbb{Z}} 2^{js} \|Q_j f\|_{\infty} < \infty.
\]

The space \( \dot{F}^{s}_{\infty, q} \) becomes a quasi-Banach with the above defined quasi-seminorm. On the one hand, its definition is independent of the choice of \( \gamma \), see [12, Cor. 5.3]. On the other hand, by [1] and Lemma 7 below, we have \( S_\infty \hookrightarrow \dot{F}^{s}_{\infty, q} \hookrightarrow S'_\infty \). We also have the following statements.

**Lemma 1.** There exist two constants \( c_1, c_2 > 0 \) such that the inequalities
\[
c_1 \|f\|_{\dot{F}^{s}_{\infty, q}} \leq \lambda \|h_\lambda f\|_{\dot{F}^{s}_{\infty, q}} \leq c_2 \|f\|_{\dot{F}^{s}_{\infty, q}}
\]
(2) holds for all \( f \in \dot{F}^{s}_{\infty, q} \) and all \( \lambda > 0 \).

**Proof.** At the first step, we prove (2) with \( \lambda := 2^N, \ N \in \mathbb{Z} \). Here by using the identity
\[
Q_j(h_{2^N} f) = Q_{j+N} f(2^{-N}(\cdot)),
\]
we obtain easily that
\[
\|h_{2^N} f\|_{\dot{F}^{s}_{\infty, q}} = 2^{-Ns} \|f\|_{\dot{F}^{s}_{\infty, q}}.
\]
Then it is not difficult to prove that

\[ \|f(\lambda \cdot)\|_{F_\infty,q} = 2^{Ns}\|f(2^{-N}\lambda \cdot)\|_{F_\infty,q}. \]

Then it is not difficult to prove that

\[ c_1\|f\|_{F_\infty,q} \leq \|f(2^{-N}\lambda \cdot)\|_{F_\infty,q} \leq c_2\|f\|_{F_\infty,q} \]

for some positive constants \( c_1 \) and \( c_2 \) independent of \( N, \lambda \) and \( f \). This completes the proof. \( \square \)

The next lemma was proved in [11].

**Lemma 2.** There exists a constant \( c > 0 \) such that

\[ \sup_{x \in P_{j,\nu}} |\varphi(x)| \leq c2^{jn/q} \sup_{\eta \in \mathbb{Z}^n} \|\varphi\|_{L_q(P_{j,\nu})} \]

holds for all \( j \in \mathbb{Z}, \nu \in \mathbb{Z}^n \), and \( \varphi \in S' \) with \( \text{supp} \hat{\varphi} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j+1} \} \).

**Proof.** The identity is known, see, for instance, [12] and here we provide a proof of the embedding for more clarity.

Let \( f \in F_{\infty,q} \). By Lemma 2 we have

\[ |Q_j f(x)|^q \leq c_1 2^{jn} \sup_{\eta \in \mathbb{Z}^n} \int_{P_{j,\nu}} |Q_j f(y)|^q \, dy \quad \text{for all} \quad x \in P_{j,\nu}, \]

which is bounded by

\[ c_1 2^{-js} 2^{jn} \sup_{\eta \in \mathbb{Z}^n} \int_{P_{j,\nu}} \sum_{l \geq j} 2^{lsq}|Q_l f(y)|^q \, dy, \]

where the constant \( c_1 \) is independent of \( f, j \) and \( \nu \). This inequality implies that

\[ |Q_j f(x)| \lesssim 2^{-js}\|f\|_{F_\infty,q} \quad (\forall x \in P_{j,\nu}). \]

Then we get

\[ \|f\|_{F_{\infty,\infty}} = \sup_{\eta \in \mathbb{Z}^n} \sup_{k \geq j} \sup_{z \in P_{j,\nu}} 2^{ks}|Q_k f(z)| \lesssim \|f\|_{F_{\infty,q}}. \]

The proof is complete. \( \square \)

**Remark 2.** An inequality opposite to (3) can be easily proved, and for this, the assumption \( \text{supp} \hat{\varphi} \subset \{ \xi \in \mathbb{R}^n : |\xi| \leq 2^{j+1} \} \) is not needed.

**Remark 3.** In case \( 1 < q < \infty \), the space \( \dot{F}_{\infty,q}^s \) has another definition introduced by Triebel [19], which is compatible with the one of Frazier and Jawerth, see a comment in [12].

### 2.2. Inhomogeneous spaces \( F_{\infty,q}^s \)

For each \( f \in S \) (or \( f \in S' \)), we use the inhomogeneous LP decomposition \( f = \mathcal{F}^{-1}\rho * f + \sum_{j > 0} Q_j f \) in \( S \) (or \( S' \)) and we obtain the inhomogeneous Triebel-Lizorkin spaces \( F_{\infty,q}^s \) as introduced in [12].

**Definition 3.** The space \( F_{\infty,q}^s \) is the set of \( f \in S' \) such that

\[ \|f\|_{F_{\infty,q}^s} := \|\mathcal{F}^{-1}\rho * f\|_{\infty} + \sup_{k \in \mathbb{N}} \sup_{\eta \in \mathbb{Z}^n} \left( \int_{P_{k,\nu}} \sum_{j \geq k} 2^{jsq}|Q_j f(x)|^q \, dx \right)^{1/q} < \infty. \]
Also as above,
\[ \|f\|_{F^{s}_{\infty, \infty}} = \|f\|_{B^{s}_{\infty, \infty}} := \|\mathcal{F}^{-1} \rho \ast f\|_{\infty} + \sup_{j \geq 0} 2^{js}\|Q_j f\|_{\infty} < \infty, \]

cf. Lemma 3 and see also [19, Sect. 2.3.4, Rem. 3].

For some properties of \( F^{s}_{\infty, q} \), we refer to [12]. The case \( s > 0 \) is related with the case of the homogeneous space.

**Lemma 4.** Let \( s > 0 \). Then

(i) \( F^{s}_{\infty, q} \hookrightarrow L_{\infty} \),

(ii) \( F^{s}_{\infty, q} \) is the set of \( f \in L_{\infty} \) such that \( [f]_{\infty} \in \hat{F}^{s}_{\infty, q} \). The expression \( \|f\|_{\infty} + \|[f]_{\infty}\|_{F^{s}_{\infty, q}} \) is an equivalent quasi-norm in \( F^{s}_{\infty, q} \).

**Proof.** Proof of (i). This embedding can be found in [22], see in particular, Statement (iii) in Propositions 2.4 and Proposition 2.6 in the cited work as well as Remark 8 below.

Proof of (ii). Let \( f \in L_{\infty} \) be such that \( [f]_{\infty} \in \hat{F}^{s}_{\infty, q} \). Thanks to the convolution inequality
\[ \|\mathcal{F}^{-1} \rho \ast f\|_{\infty} \leq \|\mathcal{F}^{-1} \rho\|_{1} \|f\|_{\infty}, \]
we have
\[ \|f\|_{F^{s}_{\infty, q}} \leq \|f\|_{\infty} + \|[f]_{\infty}\|_{F^{s}_{\infty, q}}. \]

For the opposite inequality, let \( f \in F^{s}_{\infty, q} \). By (i), we first have \( \|f\|_{\infty} \leq \|f\|_{F^{s}_{\infty, q}} \). Then for all \( k \leq 0 \) and all \( \nu \in \mathbb{Z}^{n} \), we obtain
\[ 2^{kn} \int_{P_{k, \nu}} \sum_{j \geq k} 2^{jsq}|Q_j f|^q dx = 2^{kn} \int \left( \sum_{k \leq j \leq 0} + \sum_{j \geq 1} \right) 2^{jsq}|Q_j f|^q dx \leq \|f\|_{\infty}^q \sum_{j \geq 0} 2^{jsq} + 2^{kn} \int \sum_{j \geq 1} 2^{jsq}|Q_j f|^q dx. \]

On the one hand, denoting by \( E(x) \) the vector \( ([x_1], \ldots, [x_n]) \in \mathbb{Z}^{n} \) for \( x \in \mathbb{R}^{n} \), we get an elementary inequality
\[ |2^{1-k} \nu_j| \leq 2x_j < |2^{1-k} \nu_j| + 1 + 2^{1-k}, \quad x \in P_{k, \nu}, \quad k \leq 0, \ j = 1, \ldots, n, \]
and this yields
\[ x \in P_{k, \nu} \Rightarrow x \in \bigcup_{r=0}^{1+2^{1-k}} P_{1, E(2^{1-k} \nu)+rw_0}, \]
where \( w_0 := (1, 1, \ldots, 1) \in \mathbb{Z}^{n} \). We then obtain
\[ \int \sum_{P_{k, \nu}} \sum_{j \geq 1} 2^{jsq}|Q_j f|^q dx \leq \sum_{r=0}^{1+2^{1-k}} \int_{P_{1, E(2^{1-k} \nu)+rw_0}} \sum_{j \geq 1} 2^{jsq}|Q_j f|^q dx \leq (2 + 2^{1-k}) \sup_{\nu \in \mathbb{Z}^{n}} \int_{P_{1, \nu}} \sum_{j \geq 1} 2^{jsq}|Q_j f|^q dx \leq (2 + 2^{1-k}) \sup_{\eta \in \mathbb{Z}^{n}} \sum_{r=0}^{2^{rn}} \int_{P_{\nu, \eta}} \sum_{j \geq r} 2^{jsq}|Q_j f|^q dx \leq (2 + 2^{1-k}) \|f\|_{F^{s}_{\infty, q}}^q. \]
Finally, by inserting this inequality into (4), and taking into account that $2^{kn}(2 + 2^{-k}) \leq 4$ for $k \leq 0$, we get

$$2^{kn} \int_{P_{k,v}} \sum_{j \geq k} 2^{jsq} |Q_j f|^q dx \lesssim \|f||q_\infty + \|f||F_{\infty,q}^q \lesssim \|f||F_{\infty,q}^q, \quad k \leq 0. \quad (5)$$

On the other hand, clearly for all $k \in \mathbb{N}$,

$$2^{kn} \int_{P_{k,v}} \sum_{j \geq k} 2^{jsq} |Q_j f|^q dx \leq \sup_{r \in \mathbb{N}} 2^m \int_{P_{r,v}} \sum_{j \geq r} 2^{jsq} |Q_j f|^q dx \leq \|f||F_{\infty,q}^q.$$ 

Then this estimate and (5) yield the desired result. The proof is complete. \hfill \Box

The space $F_{\infty,q}^s$ can be described via differences. We recall the following statement.

**Lemma 5.** Let $m \in \mathbb{N}$ be such that

$$\max (n/q - n, 0) < s < m. \quad (6)$$

Then

(i) A function $f$ belongs to $F_{\infty,q}^s$ if and only if $f \in L_\infty$ and

$$\mathcal{N}_{\infty,q}^{s,m,1}(f) := \sup_{k \in \mathbb{N}, v \in \mathbb{Z}^n} \left(2^{2^{1-k}} \int_{0}^1 t^{-sq} \sup_{t/2 \leq |h| < t} \int_{P_{k,v}} |\Delta^m_h f(x)|^q dx \frac{dt}{t} \right)^{\frac{1}{q}} < \infty.$$ 

Moreover, the expression $\|f||\infty + \mathcal{N}_{\infty,q}^{s,m,1}(f)$ is an equivalent quasi-seminorm in $F_{\infty,q}^s$.

(ii) The same conclusion holds by replacing in (i) the term $\mathcal{N}_{\infty,q}^{s,m,1}(f)$ by

$$\mathcal{N}_{\infty,q}^{s,m,2}(f) := \sup_{k \in \mathbb{N}, v \in \mathbb{Z}^n} \left(2^{2^{1-k}} \int_{0}^1 t^{-sq} \int_{P_{k,v}} (t^{-n} \int_{t/2 \leq |h| < t} |\Delta^m_h f(x)|^q dx \frac{dt}{t})^\frac{1}{q} \right),$$

or

$$\mathcal{N}_{\infty,q}^{s,m,3}(f) := \sup_{k \in \mathbb{N}, v \in \mathbb{Z}^n} \left(2^{2^{1-k}} \int_{0}^1 t^{-sq} \int_{P_{k,v}} t^{-n} \int_{t/2 \leq |h| < t} |\Delta^m_h f(x)|^q dh dx \frac{dt}{t} \right)^{\frac{1}{q}}.$$ 

**Proof.** We refer to [22, Rem. 4.8] if $0 < q < \infty$, and to [22, Cor. 4.3] as $q = \infty$, in which the statement was proved for the Besov-type spaces $B_{\infty,\infty}^{s,\tau}$, but $B_{\infty,\infty}^{s,0} = B_{\infty,\infty}^s$. \hfill \Box

### 2.3. Definition of realizations.

**Definition 4.** Let $m \in \mathbb{N}_0 \cup \{\infty\}$ and $k \in \{0, \ldots, m\}$. Let $E$ be a vector subspace of $S'_m$ endowed with a quasi-norm such that the continuous embedding $E \hookrightarrow S'_m$ holds. A realization of $E$ into $S'_k$ is a continuous linear mapping $\sigma : E \rightarrow S'_k$ such that $[\sigma(f)]_m = f$ for all $f \in E$. The image set $\sigma(E)$ is called the realized space of $E$ with respect to $\sigma$.

**Remark 4.** In case $k = m$ the identity is the unique realization.

If a realization is known, then it generates other realizations. We recall the following statement, see [6, Prop. 1].

**Lemma 6.** Let $\sigma_0 : E \rightarrow S'_k$ be a realization. For all finite families $(\mathcal{L}_\alpha)_{k \leq |\alpha| \leq N}$ of continuous linear functionals on $E$, the following formula defines a realization of $E$ in $S'_k$:

$$\sigma(f)(x) := \sigma_0(f)(x) + \sum_{k \leq |\alpha| \leq N} \mathcal{L}_\alpha(f) x^\alpha \quad (\text{modulo } \mathcal{P}_k).$$

And vice versa, each realization of $E$ modulo $\mathcal{P}_k$ is given in such a way.
3. Realizations of $\dot{F}^s_{\infty,q}$

In what follows, to any space $\dot{F}^s_{\infty,q}$, we associate a number $\mu \in \mathbb{N}_0$ defined by:

$$\mu := \max(0, [s] + 1).$$

We shall employ the following lemma, a classical consequence of Taylor formula, see, for instance, [16, Prop. 2.5].

**Lemma 7.** Let $0 < p \leq \infty$ and $N \in \mathbb{N}_0$. There exist $c_1, c_2 > 0$ and $m_1, m_2 \in \mathbb{N}_0$ such that

(i) $\|Q_j \varphi\|_p \leq c_1 2^{-jN} \zeta_{m_1}(\mathcal{F}^{-1} \gamma) \zeta_{m_1}(\varphi)$ for all $\varphi \in \mathcal{S}$ and all $j \in \mathbb{N}_0$.

(ii) $\|Q_j \varphi\|_p \leq c_2 2^{-jN} \zeta_{m_2}(\mathcal{F}^{-1} \gamma) \zeta_{m_2}(\varphi)$ for all $\varphi \in \mathcal{S}_N$ and all $j \in \mathbb{Z} \setminus \mathbb{N}$.

Our main aim is to prove the following result.

**Theorem 1.** Let $f \in \dot{F}^s_{\infty,q}$. Then the series $\sum_{j \in \mathbb{Z}} Q_j f$ converges in $S'_\mu$. Let us define $\sigma(f)$ as the sum belonging to $S'_\mu$. Then the mapping $\sigma : \dot{F}^s_{\infty,q} \rightarrow S'_\mu$ is a translation and a dilation commuting realization of $\dot{F}^s_{\infty,q}$ into $S'_\mu$. The element $\sigma(f)$ is the unique representative of $f$ in $S'_\mu$ satisfying $[\sigma(f)]_\infty = f$ in $S'_\infty$ and $\partial^\alpha \sigma(f) \in \bar{C}_0$ for all $|\alpha| = \mu$. Moreover,

$$\|\sigma(f)\|_\infty \|\dot{F}^s_{\infty,q} = \|f\|_{\dot{F}^s_{\infty,q}}.$$ 

**Proof.** Step 1. Let $f \in \dot{F}^s_{\infty,q}$. We introduce a radial and positive function $\tilde{\gamma} \in \mathcal{D}(\mathbb{R}^n \setminus \{0\})$ such that $\gamma \tilde{\gamma} = \gamma$. Then we define a sequence of operators $(\tilde{Q}_j)$ as $(Q_j)$ by taking $\tilde{\gamma}$ instead of $\gamma$.

Let $g \in S'_\mu$. We begin with the inequality

$$\|\langle Q_j f, \tilde{Q}_j g \rangle\| \leq 2^{j_s} \|Q_j f\|_\infty 2^{-j_s} \|\tilde{Q}_j g\|_1.$$ 

Then by Lemma 7 with $p = 1$, $\varphi := g$ and an arbitrary $N$ and $\dot{F}^s_{\infty,q} \hookrightarrow \dot{B}^s_{\infty,\infty}$ we get:

$$\|\langle Q_j f, \tilde{Q}_j g \rangle\| \leq 2^{-j_s} \min(2^{-jN}, 2^{j_s}) \zeta_m(g) \|f\|_{\dot{F}^s_{\infty,q}}$$ 

where an integer $m$ depends only on $N$ and $\mu$. We choose $N$ such that $N + s > 0$, and by the definition of $\mu$ we have $\mu - s > 0$. Then by the identity $\langle Q_j f, g \rangle = \langle Q_j f, \tilde{Q}_j g \rangle$ we get

$$\sum_{j \in \mathbb{Z}} \|\langle Q_j f, g \rangle\| \lesssim \zeta_m(g) \|f\|_{\dot{F}^s_{\infty,q}}.$$ 

**Step 2.** Inequality 3 yields

$$\sup_{g \in S'_\mu, \zeta_m(g) \leq 1} \|\sigma(f), g\| \lesssim \|f\|_{\dot{F}^s_{\infty,q}}$$

for all $f \in \dot{F}^s_{\infty,q}$. Then $\sigma$ is a realization of $\dot{F}^s_{\infty,q}$ into $S'_\mu$.

**Step 3.** The identity $[\sigma(f)]_\infty = f$ in $S'_\infty$ is implied by 1.

**Step 4.** Let $|\alpha| = \mu$, $\lambda > 0$ and $g \in \mathcal{S}$. We introduce an integer $r$ such that $2^{-r-1} < \lambda \leq 2^{-r}$. Then $\sup \mathcal{F}(h_\lambda(Q_{j-r} f^{(\alpha)}))$ is contained in the annulus $\lambda^{-1} \leq |\xi| \leq 3 \cdot 2^j$, and

$$\mathcal{F}(Q_k h_\lambda(Q_{j-r} f^{(\alpha)})) = 0 \text{ for } k - j \geq 3 \text{ or } k - j \leq -2.$$ 

Hence,

$$\langle h_\lambda(Q_{j-r} f^{(\alpha)}), g \rangle = \sum_{k=-2}^{3} \langle h_\lambda(Q_{j-r} f^{(\alpha)}), Q_{j+k} g \rangle.$$ 

By Bernstein inequality we have

$$\|h_\lambda(Q_{j-r} f^{(\alpha)})\|_\infty \lesssim 2^{j-r}|\alpha| \|Q_{j-r} f\|_\infty \lesssim 2^{j-m-s} \lambda^{\mu-s} \|f\|_{\dot{B}^s_{\infty,\infty}}.$$
on the one hand. On the other hand, by Lemma 7(i) and the fact that \(\|Q_{j+k}g\|_1 \lesssim \|g\|_1\), for some \(N \in \mathbb{N}_0\) and \(m := m(N) \in \mathbb{N}_0\) we have

\[
\langle h_\lambda (\partial^\alpha \sigma(f)), g \rangle \lesssim \lambda^{\mu-s} \|f\|_{F^s_{\infty,q}} \left( \sum_{j=0}^{\infty} 2^{j(\mu-s-N)} \|g\|_1 \sum_{j<0} 2^{j(\mu-s)} \right).
\]

Choosing \(N\) such that \(N + s - \mu > 0\), and taking into account that \(\mu - s > 0\) for all \(s \in \mathbb{R}\), we pass to limit as \(\lambda\) tends to 0 and arrive at \(\partial^\alpha \sigma(f) \in \tilde{C}_0\).

Step 5. Let \(f_i \in S'_\mu\), \(i = 1, 2\), satisfy the identity \([f_1]_\infty = [f_2]_\infty = f\) and \(\partial^\alpha f_i \in \tilde{C}_0\) for all \(|\alpha| = \mu\). Then

\[
f_1 - f_2 \in \mathcal{P}_\infty \quad \text{and} \quad \partial^\alpha (f_1 - f_2) \in \tilde{C}_0 \cap \mathcal{P}_\infty = \{0\} \quad \text{for all} \quad |\alpha| \geq \mu.
\]

Hence, \(f_1 - f_2 \in \mathcal{P}_\mu\).

Step 6. Since each operator \(Q_j\) commutes with the mapping \(\tau_a\) for all \(a \in \mathbb{R}^n\), the realization \(\sigma\) commutes also with \(\tau_a\).

Let \(\lambda > 0\). Since \(F^s_{\infty,q}\) is dilation invariant, that is, \(h_\lambda f \in F^s_{\infty,q}\); see Lemma 1, it follows that \(\sigma(h_\lambda f) = \sum_{j \in \mathbb{Z}} Q_j (h_\lambda f) \in S'_\mu\). We define the operators \(Q_{j,\lambda}\) as \(Q_j\) replacing \(\gamma\) by \(h_\lambda \gamma\). It is easy to see that \(Q_j (h_\lambda f) = h_\lambda Q_j,\lambda f\) in \(S'\) since \(Q_j \varphi(\lambda(\cdot)) = Q_{j,\lambda}(h_\lambda \varphi)\) for all \(\varphi \in \mathcal{S}\); recall that \(Q_j(S) \subset S_\infty\). We now define the realization \(\sigma_{\lambda}(f) := \sum_{j \in \mathbb{Z}} Q_{j,\lambda} f\) of \(F^s_{\infty,q}\) into \(S'_\mu\). Then

\[
\langle \sigma(h_\lambda f), \varphi \rangle = \sum_{j \in \mathbb{Z}} \langle h_\lambda Q_{j,\lambda} f, \varphi \rangle = \lambda^n \sum_{j \in \mathbb{Z}} \langle Q_j f, \varphi(\lambda(\cdot)) \rangle = \lambda^n \langle \sigma(f), \varphi(\lambda(\cdot)) \rangle
\]

for all \(\varphi \in S_\mu\). Hence,

\[
\sigma(h_\lambda f) = h_\lambda \sigma(f) \quad \text{in} \quad S'_\mu. \tag{10}
\]

As above, we also obtain that for \(\sigma_{\lambda}\), the arguing in Steps 1–5 hold true. Then

\[
[\sigma(f)]_\infty = [\sigma_{\lambda}(f)]_\infty = f,
\]

and \(\sigma(f) - \sigma_{\lambda}(f) \in \mathcal{P}_\infty\). But \(\partial^\alpha (\sigma(f) - \sigma_{\lambda}(f)) \in \tilde{C}_0 \cap \mathcal{P}_\infty = \{0\}\) if \(|\alpha| \geq \mu\), and hence, \(\sigma(f) - \sigma_{\lambda}(f) \in \mathcal{P}_\mu\). This implies \(h_\lambda (\sigma(f) - \sigma_{\lambda}(f)) \in \mathcal{P}_\mu\). Therefore,

\[
h_\lambda \sigma(f) = h_\lambda \sigma_{\lambda}(f) \quad \text{in} \quad S'_\mu. \tag{11}
\]

Now, by (10) and (11) we obtain that \(\sigma(h_\lambda f) = h_\lambda \sigma(f)\) in \(S'_\mu\).

Step 7. It is clear that \(Q_j Q_j f = 0\) as \(|j - r| \geq 2\). Then

\[
\| [\sigma(f)]_\infty \|_{F^s_{\infty,q}} = \sup_{l \in \mathbb{Z}} \sup_{\nu \in \mathbb{Z}^n} \left( 2^{ln} \int_{P_{l,\nu}} \left| \sum_{j \geq 1} 2^{jsq} \sum_{j-1 \leq r \leq j+1} Q_j,\lambda Q_j f \right|^q dx \right)^{1/q}
\]

\[
= \sup_{l \in \mathbb{Z}} \sup_{\nu \in \mathbb{Z}^n} \left( 2^{ln} \int_{P_{l,\nu}} \sum_{j \geq 1} 2^{jsq} \left| \sum_{m=-1}^{\infty} Q_{m+j} f \right|^q dx \right)^{1/q}. \tag{12}
\]

We let

\[
\tilde{\gamma}_1 := \sum_{m=-1}^{1} \gamma(2^{-m} \cdot \gamma),
\]

and define the operators \(\tilde{Q}_{j,1}\) as

\[
\tilde{Q}_{j,1} f := \tilde{\gamma}_1(2^{-j}(\cdot)) \hat{f}.
\]

Then we get

\[
\sum_{m=-1}^{1} Q_{m+j} Q_j = \tilde{Q}_{j,1} \quad \text{for all} \quad j \in \mathbb{Z}. \tag{13}
\]
We have
\[ \sup \bar{\gamma}_1 \subset \left\{ \xi \in \mathbb{R}^n : \frac{1}{2} \leq |\xi| \leq \frac{3}{2} \right\} \quad \text{and} \quad \bar{\gamma}_1(\xi) \geq 1 \quad \text{as} \quad \frac{3}{4} \leq |\xi| \leq 1 \]

since \( \bar{\gamma}_1(\xi) \geq \gamma^2(\xi) \), see the definition of \( \gamma \) in Section [1]. Then \( \bar{\gamma}_1 \) satisfies equations (2.1)–(2.3) in [12] and owing to equation (5.1) and Corollary 5.3 in [12], we can replace the operators \( Q_j \) by \( \tilde{Q}_{j,1} \) in Definition 2 to obtain
\[ \|f\|_{F_{\infty,q}^s} \lesssim \sup_{l \in \mathbb{Z}} \sup_{\nu \in \mathbb{Z}^n} \left( \sum_{m=1}^{2^n} \sum_{j \geq l} 2^{jsq} \left| \sum_{m=-1}^{1} Q_{m+j}Q_j f \right|^q dx \right)^{1/q} \lesssim \|f\|_{F_{\infty,q}^s}. \]

Hence, it follows from [12] that \( \|\sigma(f)\|_{F_{\infty,q}^s} = \|f\|_{F_{\infty,q}^s} \).

Finally, for this identity for quasi-seminorms, we can add the following observation. Let \( f_1 \in \mathcal{S}' \) be such that \( [f_1]_{\infty} = [\sigma(f)]_{\infty} \). We have
\[ \|\sigma(f)\|_{F_{\infty,q}^s} = \|[f_1]_{\infty}\|_{F_{\infty,q}^s}. \]

Let \( f_2 \in \mathcal{S}' \) be such that \( [f_2]_{\infty} = f \). By Step 5, \( f_1 - f_2 \) is a polynomial; we denote \( f_1 - f_2 =: \tilde{f}. \)

But \( Q_j([\sigma(f)]_{\infty}) = Q_j f_1 = Q_j f_2 \) since \( Q_j \tilde{f} = 0 \); we also have \( Q_j f_1 = Q_j f_2 \) in the sense of functions, since both \( Q_j f_1 \) and \( Q_j f_2 \) are smooth functions of exponential type, see Paley-Wiener theorem [13, Thm. 1.7.7]). We again arrive at the desired identity. The proof is complete. \( \square \)

**Remark 5.** For all \( s \in \mathbb{R} \), if \( f \in \dot{F}_{\infty,q}^s \), the series \( \sum_{j \geq 0} Q_j f \) converges in \( \mathcal{S}' \). Indeed, the inequality [8] becomes
\[ |\langle Q_j f, \tilde{Q}_j g \rangle| \lesssim 2^{-j(N+s)} \zeta_m(g) \|f\|_{F_{\infty,q}^s} \]
for all \( g \in \mathcal{S} \) and all \( j \in \mathbb{N}_0 \); here \( \tilde{Q}_j \) is the same as in Step 1 in the proof of Theorem [4].

The next lemma characterizes the number \( \mu \); the proof of this lemma is similar to that given by G. Bourdaud for Besov spaces [4, Prop. 2.2.1].

**Lemma 8.** Let \( s \geq 0 \). Then there exists a function \( f \in \dot{F}_{\infty,q}^s \) such that the series \( \sum_{j \leq 0} Q_j f \) diverges in \( \mathcal{S}_{\mu-1}' \).

**Proof.** We briefly outline the proof, since in case \( q < \infty \) we do not have the same spaces as in [4]. We denote \( m := \mu - 1 = [s] \). Let \( \varphi \in \mathcal{D} \) be such that \( \int \varphi(x)dx = 1 \). As \( \partial_1^m \varphi \in \mathcal{S}_m \), we split the sum \( \sum_{j \leq 0} \langle Q_j f, \partial_1^m \varphi \rangle \) into \( I_1 + I_2 \), where
\[ I_1 := (-1)^m \sum_{j \leq 0} \int_{\mathbb{R}^n} (\partial_1^m Q_j f(x) - \partial_1^m Q_j f(0)) \varphi(x)dx, \quad I_2 := (-1)^m \sum_{j \leq 0} \partial_1^m Q_j f(0). \]

It is sufficient to construct a function \( f \in \dot{F}_{\infty,q}^s \) such that \( |I_1| < \infty \) and \( |I_2| = \infty \). For this purpose, let \( g \in \mathcal{S} \) be such that
\[ \hat{g} \in \mathcal{D}, \quad \hat{g} \geq 0, \quad \supp \hat{g} \subset \left\{ \xi : \frac{3}{4} \leq |\xi| \leq 1, \xi_1 \geq 0 \right\}. \]

We let
\[ f(x) := \sum_{k \geq 0} 2^{k(s+m)/2} g(2^{-k} x). \]

Clearly, we have
\[ Q_j f(x) = 2^{-j(s+m)/2} g(2^j x) \quad \text{if} \quad j \leq 0, \quad Q_j f(x) = 0 \quad \text{if} \quad j \geq 1. \]
since \( \gamma(2^{-j}k)\widehat{g}(2^k\xi) = 0 \) if \( k \neq j \) and \( \gamma\widehat{g} = \widehat{g} \); we recall that \( \gamma(\xi) = 1 \) as \( \frac{3}{4} \leq |\xi| \leq 1 \). It is also clear that for all \( j \leq 0 \) the identities hold:

\[
\partial_1^m Q_j(f)(0) = (2\pi)^{-n}i^m(2^{j(m-s)/2})\int_{\mathbb{R}^n} \xi_1^m\widehat{g}(\xi) \, d\xi,
\]

\[
|\partial_1^m Q_j f(x) - \partial_1^m Q_j f(0)| \leq (2\pi)^{-n}2^{j(m-s+2)/2}\sum_{k=1}^{n} |x_k| \int_{\mathbb{R}^n} |\xi_k| \xi_1^m\widehat{g}(\xi) \, d\xi.
\]

Then

\[
|\sum_{j \leq 0} \partial_1^m Q_j f(0)| = \infty, \quad \sum_{j \leq 0} \|\nabla\partial_1^m Q_j f\|_\infty < \infty.
\]

It remains to prove that \( [f]_\infty \in \dot{F}^s_{\infty,q} \). Since

\[
\int_{F_{k,\nu}} |g(2^j x)|^q \, dx \leq 2^{-jn} \|g\|_1^q
\]

and \( s - m \geq 0 \), that is, \( 2^{j(m-s)/2} \leq 1 \) for all \( j \leq 0 \), we first have

\[
2^{kn} \int_{F_{k,\nu}} \sum_{0 \geq j \geq k} 2^{j(s-m)/2}|g(2^j x)|^q \, dx \leq \|g\|_1^q \sum_{0 \geq j \geq k} 2^{(k-j)n} \lesssim \|g\|_1^q
\]

for all \( k \in \mathbb{Z} \setminus \mathbb{N} \). Therefore, by taking the supremum over \( k \in \mathbb{Z} \setminus \mathbb{N} \) and \( \nu \in \mathbb{Z}^n \) in (14), we get

\[
\|[f]_\infty\|_{\dot{F}^s_{\infty,q}} \leq 1.
\]

The proof is complete. \( \square \)

Without use the LP decomposition, we define the realized space of \( \dot{F}^s_{\infty,q} \).

**Definition 5.** The realized space of \( \dot{F}^s_{\infty,q} \) denoted by \( \dot{F}^s_{\infty,q} \) is the set of all \( f \in \mathcal{S}' \) such that \( [f]_\infty \in \dot{F}^s_{\infty,q} \) and \( f^{(\alpha)} \in \dot{C}_0 \) for all \( |\alpha| = \mu \).

We should be sure of the identity \( \sigma(\dot{F}^s_{\infty,q}) = \dot{F}^s_{\infty,q} \), where the mapping \( \sigma \) was defined in Theorem 1. The direct embedding is by the definition; let us prove the opposite one.

Let \( f \in \dot{F}^s_{\infty,q} \), then \( f - \sigma([f]_\infty) \) is a polynomial. Since \( \dot{C}_0 \cap \mathcal{P}_\infty = \{0\} \) and \( f^{(\alpha)} - \partial^{\alpha}\sigma([f]_\infty) \in \dot{C}_0 \) for all \( |\alpha| \geq \mu \), we conclude \( f - \sigma([f]_\infty) \in \mathcal{P}_\mu \), that is, \( f = \sigma([f]_\infty) \) in \( \mathcal{S}' \).

The space \( \dot{F}^s_{\infty,q} \) is equipped with a quasi-seminorm defined as

\[
\|f\|_{\dot{F}^s_{\infty,q}} := \|[f]_\infty\|_{\dot{F}^s_{\infty,q}}.
\]

Of course, one has to justify this definition. If \( [f]_\mu = [f_1]_\mu \) and \( [f]_\infty = [f_2]_\infty \), then \( f_1 - f_2 \in \mathcal{P}_\infty \), but \( Q_j(f_1 - f_2) = 0 \), which is a sufficient argument. In the case \( s \geq 0 \), \( \dot{F}^s_{\infty,q} \) can be characterized in \( \mathcal{S}' \). This is done in the next lemma; for the case \( s = 0 \) see Remark 6 below.

**Lemma 9.** Let \( s > 0 \). Then \( \dot{F}^s_{\infty,q} \) is the set of \( f \in \mathcal{S}' \) such that \( [f]_\infty \in \dot{F}^s_{\infty,q} \), and \( f^{(\alpha)} \in \dot{C}_0 \) for all \( |\alpha| = \mu \), and moreover:

(i) If \( s \notin \mathbb{N} \), then \( f \in \mathcal{C}^{n-1} \) and \( f^{(\alpha)}(0) = 0 \) for all \( |\alpha| \leq \mu - 1 \),

(ii) If \( s \in \mathbb{N} \), then \( f \in \mathcal{C}^{n-2} \) and \( f^{(\alpha)}(0) = 0 \) for all \( |\alpha| \leq \mu - 2 \) with \( \mu = s + 1 \geq 2 \).

**Proof.** The proof is similar to the proofs of Proposition 4.8 in [7] and of Theorem 4.5 in [16] thanks to the embedding \( \dot{F}^s_{\infty,q} \to \dot{B}^s_{\infty,\infty} \); let us briefly outline this.
Proof of (i). We first define $\tilde{F}^{s}_{\infty,q}$ in $S'$ by replacing each $Q_{j} f$ by a polynomial of degree less than $\mu$ in $\sigma(f)$, see Theorem 1. Then we get a realization denoted $\sigma_{1}$. Since any realization on $\tilde{F}^{s}_{\infty,q}$ is a surjective mapping, then if $f \in \tilde{F}^{s}_{\infty,q}$, there exists $g \in \tilde{F}^{s}_{\infty,q}$ such that $[f]_{\mu} = g$, and it is sufficient to take $f := \sigma_{1}(g)$.

Construction of $\sigma_{1}$. Let $g \in \tilde{F}^{s}_{\infty,q}$. Then the series

$$\sigma_{1}(g) := \sum_{j \in \mathbb{Z}} (Q_{j} g - \sum_{|\alpha| < \mu} (Q_{j} g)^{(\alpha)}(0) \frac{x^{\alpha}}{|\alpha|!})$$

converges in $S'$. The mapping $\sigma_{1} : \tilde{F}^{s}_{\infty,q} \rightarrow S'$ is a realization of $\tilde{F}^{s}_{\infty,q}$ into $S'$, where $\sigma_{1}(f)$ is the unique representative of $g$ in $S'$, of class $C^{\mu-1}$, $\partial^{\alpha} \sigma_{1}(g)(0) = 0$ for all $|\alpha| \leq \mu - 1$, $\partial^{\alpha} \sigma_{1}(g) \in \tilde{C}_{0}$ for all $|\alpha| = \mu$ and $\|\sigma_{1}(g)\|_{\tilde{F}^{s}_{\infty,q}} = \|g\|_{\tilde{F}^{s}_{\infty,q}}$.

We now present the role of the assumption $s \notin \mathbb{N}$: by the Bernstein inequality

$$\|(Q_{j} g)^{(\alpha)}\|_{\infty} \lesssim 2^{|\alpha|} \|Q_{j} g\|_{\infty} \lesssim 2^{j(|\alpha|-s)} \|g\|_{B^{s}_{\infty,\infty}},$$

we get

$$\left| Q_{j} g(x) - \sum_{|\alpha| < \mu} (Q_{j} g)^{(\alpha)}(0) \frac{x^{\alpha}}{|\alpha|!} \right| \lesssim \|Q_{j} g\|_{\infty} + \sum_{|\alpha| \leq \mu - 1} \frac{|x|^{\alpha}}{|\alpha|!} \|(Q_{j} g)^{(\alpha)}\|_{\infty} \lesssim (2^{-j} + 2^{j(|\mu-1|-s)}(1 + |x|)^{\mu-1}) \|g\|_{B^{\mu}_{\infty,\infty}}, x \in \mathbb{R}^{n}, j \in \mathbb{N}_{0}.$$ 

On the other hand, by the Taylor formula we have

$$\left| Q_{j} g(x) - \sum_{|\alpha| < \mu} (Q_{j} g)^{(\alpha)}(0) \frac{x^{\alpha}}{|\alpha|!} \right| \leq \mu \sum_{|\alpha| = \mu} \frac{|x|^{\alpha}}{|\alpha|!} \int_{0}^{1} (1-t)^{\mu-1}|(Q_{j} g)^{(\alpha)}(tx)| \, dt \lesssim 2^{j(\mu-s)} |x|^\mu \|g\|_{B^{\mu}_{\infty,\infty}}.$$ 

Therefore,

$$|\sigma_{1}(g)(x)| \lesssim \left\{ \sum_{j \geq 0} \left( 2^{-j} + 2^{j(\mu-1-s)}(1 + |x|)^{\mu-1} \right) + \sum_{j < 0} 2^{j(\mu-s)} |x|^\mu \right\} \|g\|_{\tilde{F}^{s}_{\infty,q}}.$$ 

Thus, thanks to assumption $s \in \mathbb{R}^{+}\setminus\mathbb{N}_{0}$, we get the convergence of above series with $\mu - 1 - s = [s] - s < 0$ and $\mu - s > 0$.

Proof of (ii). As in the previous step, we consider the mapping:

$$\sigma_{2}(g) := \sum_{j \geq 0} Q_{j} g + \sum_{j < 0} \left( Q_{j} g - \sum_{|\alpha| < \mu} (Q_{j} g)^{(\alpha)}(0) \frac{x^{\alpha}}{|\alpha|!} \right) \text{ for all } g \in \tilde{F}^{s}_{\infty,q}, \quad (15)$$

where $\sigma_{2}(g)$ is the unique representative of $g$ in $S'$, and $\sigma_{2}$ is also a realization of $\tilde{F}^{s}_{\infty,q}$ into $S'$ satisfying $\partial^{\alpha} \sigma_{2}(g) \in \tilde{C}_{0}$ for all $|\alpha| = \mu$ and $\|\sigma_{2}(g)\|_{\tilde{F}^{s}_{\infty,q}} = \|g\|_{\tilde{F}^{s}_{\infty,q}}$. If in addition $s > 0$, then $\sigma_{2}(g)$ is of class $C^{\mu-2}$.

Owing to Lemma 6, if $f \in \tilde{F}^{s}_{\infty,q}$, there exists $g \in \tilde{F}^{s}_{\infty,q}$ such that $[f]_{\mu} = g$ and it is sufficient to take

$$f := \sigma_{2}(g) - \sum_{|\beta| \leq \mu - 2} \left( \sum_{j \geq 0} (Q_{j} g)^{(\beta)}(0) \frac{x^{\beta}}{\beta!} \right).$$ 

For the realization $\sigma_{2}$ we refer to [7, Rem. 4.9]. In case $s > 0$, for $|\beta| \leq \mu - 2$, we have $|\beta| - s \leq \mu - 2 - s = -1$, and then

$$\sum_{j \geq 0} \|(Q_{j} g)^{(\beta)}\|_{\infty} \lesssim \|g\|_{\tilde{F}^{s}_{\infty,q}} \sum_{j \geq 0} 2^{(|\beta|-s)j} \lesssim \|g\|_{\tilde{F}^{s}_{\infty,q}};$$
the estimate for the sum
\[
\sum_{j<0} |\partial^2 \{Q_j g - \sum_{|\alpha|<\mu} (Q_j g)^{(\alpha)}(0) \frac{x^\alpha}{\alpha!}\}|
\]
can be obtained as in [10]. The proof is complete. \(\square\)

**Remark 6.** If \(f \in \hat{\mathcal{F}}_{\infty,q}^s\) then \(f = \sigma_2(g)\), where \(\sigma_2(g)\) is defined in the above proof, see (15).

**Remark 7.** Clearly, we cannot identify \(\hat{\mathcal{F}}_{\infty,2}^0\) with \(\text{BMO}\), where the space \(\text{BMO}\) is as defined in [10], since \(\|f\|_{\mathcal{F}_{\infty,q}^s} = 0\) for all polynomials, while one can easily find a polynomial \(f \notin \mathcal{P}_1\) such that \(\int_{\mathbb{R}^n} (1 + |x|^n)^{-1} |f(x)|dx = \infty\), see [10].

4. Characterizations by differences

We now present a characterization of realized spaces \(\hat{\mathcal{F}}_{\infty,q}^s\) by means of differences. In view of Lemmata 4 and 5 one could think that the scales \(\mathcal{N}_{\infty,q}^{s,m,i}(f)\), \(i = 1, 2, 3\), are other equivalent quasi-seminorms in \(\hat{\mathcal{F}}_{\infty,q}^s\). But this is not the case since for any polynomial \(f\) of degree \(m\) we can have \(\mathcal{N}_{\infty,q}^{s,m,i}(f) \neq 0\), while \(\|f\|_{\mathcal{F}_{\infty,q}^s} = 0\); for instance \(f(x) := x_1^m\), then \(\Delta_h^m f(x) = m! h_1^m\) and \(\mathcal{N}_{\infty,q}^{s,m,1}(f) = m! 2^{m-s}(q(m-s))^{-1/q}\), which tends to infinity as \(s \uparrow m\); the kernel of \(\Delta_h^m\) is \(\mathcal{P}_m\).

**Lemma 10.** Let (6) be satisfied. Then there exists a constant \(c > 0\) such that the inequality \(\mathcal{N}(f) \leq c\|f\|_{\mathcal{F}_{\infty,q}^s}\) holds for all \(f \in \mathcal{F}_{\infty,q}^s\), where \(\mathcal{N} := \mathcal{N}_{\infty,q}^{s,m,1}\). The same holds if we replace \(\mathcal{N}_{\infty,q}^{s,m,1}\) by \(\mathcal{N}_{\infty,q}^{s,m,i}\) with \(i = 2, 3\).

**Proof.** Lemmata 4 and 5 we have
\[
\mathcal{N}(f) \lesssim \|f\|_\infty + \|f\|_{\mathcal{F}_{\infty,q}^s}
\]
for all \(f \in \mathcal{F}_{\infty,q}^s\). Replacing \(f\) by \(f_\lambda := f(\lambda(\cdot))\) arbitrary \(\lambda > 0\) in this inequality and using Lemma 1 we obtain:
\[
\lim_{\lambda \to \infty} \lambda^{-s} \mathcal{N}(f_\lambda) \leq c\|f\|_{\mathcal{F}_{\infty,q}^s} \quad \text{for all} \quad f \in \mathcal{F}_{\infty,q}^s.
\]
(16)

Let now \(\lambda > 1\) and \(N \in \mathbb{N}\) be such that \(2^N \leq \lambda < 2^{N+1}\). By the elementary inequality
\[
\forall x \in P_{k,\nu} : [2^N \lambda^{-1} \nu_j] \leq 2^{k+N} \lambda^{-1} x_j < [2^N \lambda^{-1} \nu_j] + 2, \quad j = 1, \ldots, n
\]
recall that \(2^{-1} < 2^N \lambda^{-1} \leq 1\), we obtain
\[
x \in P_{k,\nu} \Rightarrow \lambda^{-1} x \in P_{k+N,E(2^N \lambda^{-1} \nu)+w_0} \cup P_{k+N,E(2^N \lambda^{-1} \nu)+w_0},
\]
where \(w_0 := (1, 1, \ldots, 1) \in \mathbb{Z}^n\) and we have employed the notation \(E(x) = ([x_1], \ldots, [x_n]) \in \mathbb{Z}^n, \quad x \in \mathbb{R}^n\). As \(\Delta_h^m f(x) = \Delta_h^m f_\lambda(\lambda^{-1} x)\), with the change of variables \(y := \lambda^{-1} x, r := \lambda^{-1} t\) and \(u := \lambda^{-1} h\), we get:
\[
2^{kn} \int_0^{2^{1-k}} \sup_{\frac{1}{2} \leq |h| < t} \int_{P_{k,\nu}} |\Delta_h^m f(x)|^q \ dx \frac{dt}{t}
\]
\[
\lesssim \lambda^{-sq} \sum_{l=0}^{1} 2^{(k+N)n} \int_0^{r^{-sq}} \sup_{\frac{1}{2} \leq |u| < r} \int_{P_{k+N,E(2^N \lambda^{-1} \nu)+w_0}} |\Delta_u f_\lambda(y)|^q \ dy \frac{dr}{r}.
\]
(17)
We assume that $k \in \mathbb{N}_0$ and this allows us to bound last term in (17) by
\[
c\lambda^{-\nu} \sup_{j \in \mathbb{N}_0} \sup_{\nu \in \mathbb{Z}^n} 2^{m} \int_0^{2^{1-j}} \sup_{r/2 \leq |u| < r} \int_{P_{j,\nu}} |\Delta_{u}^m f\lambda(y)|^q \frac{dy}{r},
\]
where $c$ is independent of $k$. Calculating the supremum over $k \in \mathbb{N}_0$ and $\nu \in \mathbb{Z}^n$ in (17), and taking (18) into consideration, we obtain $\mathcal{N}(f) \leq c\lambda^{-\nu} \mathcal{N}(f\lambda)$. Finally by (16), we complete the proof.

Here our second main result is as follows.

**Theorem 2.** Let $m \in \mathbb{N}$ be such that \([6]\) is satisfied. Then $\mathcal{N}_{\infty,q}^{s,m,i}(f)$, $i = 1, 2, 3$, define equivalent quasi-seminorms in $\hat{F}^{s}_{\infty,q}$.

**Proof.** We consider only $\mathcal{N}_{\infty,q}^{s,m,1}(f)$, since the estimates of $\mathcal{N}_{\infty,q}^{s,m,i}(f)$, $i = 2, 3$, can be obtained in the same way. To simplify the notations, in the proof we write $\mathcal{N}(f)$ instead of $\mathcal{N}_{\infty,q}^{s,m,1}(f)$.

The proof of $\|\|f\|_{\infty} \|\hat{F}^{s}_{\infty,q} \leq c\mathcal{N}(f)$, for all regular tempered distribution $f$ obeying $\mathcal{N}(f) < \infty$ can be done as in \([18]\) Subs. 4.1 and we omit the details.

The opposite inequality is similar to that given in \([18]\), and we present only the necessary changes. Let $f \in \hat{F}^{s}_{\infty,q}$. We denote $f_k := \sum_{-k \leq j \leq k} Q_j f$, where $k \in \mathbb{N}_0$. We also define $k_s := 0$ as $s \in \mathbb{N}$ and $k_s = k$ as $s \in \mathbb{R}^+ \setminus \mathbb{N}$. Then the function $f_k$ belongs to $\hat{F}^{s}_{\infty,q}$. Indeed, the inequality $\|f_k\|_{\infty} \leq c\|f\|_{\infty} \|\hat{F}^{s}_{\infty,q}$ with a constant $c := c(k) > 0$, can be obtained by the assumption on $s$ and the following estimate:
\[
|Q_j f(x)| \leq c 2^{-js} \|f\|_{\hat{F}^{s}_{\infty,q}}, \quad j \in \mathbb{Z}, \quad x \in \mathbb{R}^n.
\]

In order to prove (19), it is sufficient to employ the embedding $\hat{F}^{s}_{\infty,q} \hookrightarrow \hat{F}^{s}_{\infty,\infty} = \hat{B}^{s}_{\infty,\infty}$.

Now we are going to prove that
\[
\|f_k\|_{\infty} \|\hat{F}^{s}_{\infty,q} \leq c\|f\|_{\infty} \|\hat{F}^{s}_{\infty,q}.
\]

with a constant independent of $f$ and $k$. We proceed as in Step 7 in the proof of Theorem 1. Then similar to (12) recalling that $Q_r Q_j f = 0$ as $|j - r| \geq 2$, we get
\[
\|f_k\|_{\infty} \|\hat{F}^{s}_{\infty,q} = \sup_{l \in \mathbb{Z}} \sup_{\nu \in \mathbb{Z}^n} \left(2^l \int_{P_{l,\nu}} \sum_{j \geq 1} \sum_{-k \leq l \leq k_s} Q_r Q_j f \right)^q 2^{lsq} dx \right)^{1/q}
\]
\[
= \sup_{l \in \mathbb{Z}} \sup_{\nu \in \mathbb{Z}^n} \left(2^{l-N} \int_{P_{l-N,\nu}} \sum_{j \geq l-N} \sum_{-k \leq l \leq k_s} Q_r Q_j f \right)^q 2^{lsq} dx \right)^{1/q},
\]
for all $N \in \mathbb{Z}$. Since here the supremum is taken over all $l \in \mathbb{Z}$, it is translation invariant in $\mathbb{Z}$. The last identity is trivial but is useful for the next computation. On the one hand, in the sum $\sum_{|r-j| \leq 1} \ldots$ we have at most three terms corresponding to $r \in \{j - 1, j, j + 1\}$, and hence
\[
\left| \sum_{-k \leq r \leq k_s, |r-j| \leq 1} Q_r Q_j f \right|^q \leq 2^{2q-1} \sum_{-k \leq r \leq k_s, |r-j| \leq 1} |Q_r Q_j f|^q.
\]

On the other hand, by the following elementary inequalities
\[
\text{if } -k \leq r \leq k_s \text{ and } |r-j| \leq 1 \Rightarrow -k - 1 \leq j \leq k_s + 1,
\]
\[
\text{if } -k - 1 \leq j \leq k_s + 1 \text{ and } |r-j| \leq 1 \Rightarrow -k - 2 \leq r \leq k_s + 2,
\]
by the fact that
\[
\{r \in \mathbb{Z} : -k \leq r \leq k_s\} \subset \{r \in \mathbb{Z} : -k - 2 \leq r \leq k_s + 2\},
\]
and by using (22), we obtain
\[
\sum_{j \geq l - N} \left| \sum_{-k \leq r < k, |r - j| \leq 1} Q_r Q_j f \right|^q 2^{jq} \leq c \sum_{j \geq l - N} \left| \sum_{-k \leq r < k, |r - j| \leq 1} Q_r Q_j f \right|^q 2^{jq} \\
\leq c \sum_{j \geq l - N, -k - 1 \leq j \leq k, |r - j| \leq 1} \left| \sum_{-k \leq r < k} Q_r Q_j f \right|^q 2^{jq}.
\]
Choosing the integer \( N \) such that \(-k - 1 \geq l - N\), we bound the last term in (23) as follows:
\[
c \sum_{j \geq l - N, |m| \leq 1} \left| Q_{j + m} Q_j f \right|^q 2^{jq} \quad \text{with} \quad m := r - j.
\]
Substituting this bound into (21), letting \( \ell := l - N \), and taking the supremum over all \( \ell \in \mathbb{Z} \), we get
\[
\| [f_k]_{\infty} \|_{\mathcal{F}^{\infty,q}} \leq c \sum_{|m| \leq 1} \sup_{\ell \in \mathbb{Z}} \left( 2^{tn} \int_{\mathcal{P}_{\ell,\nu}} \left| Q_{j + m} Q_j f \right|^q 2^{jq} \, dx \right)^{1/q}
\]
for all \( k \in \mathbb{N}_0 \). We continue by letting \( \tilde{\gamma}_m := \gamma(2^{-m}(\cdot)) \), and this function possesses the following properties:
\[
supp \tilde{\gamma}_0 \subset \left\{ \xi \in \mathbb{R}^n \colon \frac{1}{2} \leq \left| \xi \right| \leq \frac{3}{2} \right\}, \quad \tilde{\gamma}_0(\xi) \geq 1 \quad \text{as} \quad \frac{3}{4} \leq \left| \xi \right| \leq 1,
\]
\[
supp \tilde{\gamma}_1 \subset \left\{ \xi \in \mathbb{R}^n \colon \frac{1}{2} \leq \left| \xi \right| \leq \frac{3}{4} \right\}, \quad \tilde{\gamma}_1(\xi) > 0 \quad \text{as} \quad \frac{9}{16} \leq \left| \xi \right| \leq \frac{11}{16}.
\]
Hence,
\[
\tilde{\gamma}_1(\xi) \geq c > 0 \quad \text{on} \quad \left\{ \xi \in \mathbb{R}^n : \frac{9}{16} \leq \left| \xi \right| \leq \frac{11}{16} \right\}, \quad c := \min_{\frac{9}{16} \leq |\eta| \leq \frac{11}{16}} \gamma(2\eta) \gamma(\eta).
\]
The next property is
\[
supp \tilde{\gamma}_1 \subset \left\{ \xi \in \mathbb{R}^n : 1 \leq \left| \xi \right| \leq \frac{3}{2} \right\}, \quad \tilde{\gamma}_1(\xi) > 0 \quad \text{as} \quad \frac{9}{8} \leq \left| \xi \right| \leq \frac{11}{8},
\]
and hence,
\[
\tilde{\gamma}_1(\xi) \geq c > 0 \quad \text{on} \quad \left\{ \xi \in \mathbb{R}^n : \frac{9}{8} \leq \left| \xi \right| \leq \frac{11}{8} \right\}, \quad c := \min_{\frac{9}{8} \leq |\eta| \leq \frac{11}{8}} \gamma \left( \frac{\eta}{2} \right) \gamma(\eta).
\]
Then we define the operators \( \tilde{Q}_{j,m} \) as \( \tilde{Q}_{j,m} \) and as in (13), this yields
\[
Q_{m+j} Q_j = \tilde{Q}_{j,m} \quad \text{for all} \quad j \in \mathbb{Z}.
\]
We replace the operators \( Q_j \) by \( \tilde{Q}_{j,m} \) with \( m \in \{-1, 0, 1\} \) in Definition 2 and we denote by \( \| \cdot \|_{\mathcal{F}^{\infty,q}} \) the associated quasi-seminorms. By [12, Cor. 5.3], we have:
\[
\| [f]_{\infty} \|_{\mathcal{F}^{\infty,q}} \leq c \| [f]_{\infty} \|_{\mathcal{F}^{\infty,q}},
\]
where \( c \) is independent of \( f \). But from (24), we also have
\[
\| [f_k]_{\infty} \|_{\mathcal{F}^{\infty,q}} \leq c \sum_{m=-1}^{1} \| [f]_{\infty} \|_{\mathcal{F}^{\infty,q}} \quad \text{for all} \quad k \in \mathbb{Z}.
\]
This proves estimate (20).
Applying now Lemma \([10]\) to \(f_k\), we obtain
\[
\mathcal{N}(f_k) \leq c \|f\|_{\dot{F}^{s,q}_\infty} \quad \text{for all} \quad k \in \mathbb{N}_0,
\] (25)
the constant \(c\) is independent of \(k\), see \([20]\). On the other hand, letting
\[
r_j(x) := \sum_{|\alpha| < \mu} (Q_j f)(\alpha) \frac{x^\alpha}{\alpha!}
\]
and recalling that \(\mu = [s] + 1\), cf. \([7]\), we obtain that the sequence \((f_k - \sum_{-k \leq j \leq k} r_j)\) \(k \geq 0\) converges uniformly on each compact subset of \(\mathbb{R}^n\) to a limit denoted \(v\), see \([18]\) (22), Subs. 2.2] for \(B^s_{\infty,\infty}\). At the same time, \(\dot{F}^s_{\infty,q} \hookrightarrow B^s_{\infty,\infty}\) cf. Lemma \([3]\). By applying twice the Fatou lemma in \([25]\), we get
\[
\mathcal{N}(v) \leq c \|f\|_{\dot{F}^{s,q}_\infty}.
\] (26)
In case \(s \in \mathbb{N}\), we add the following inequality:
\[
\mathcal{N}\left(\sum_{j \geq 0} Q_j f\right) \leq c \|f\|_{\dot{F}^{s,q}_\infty},
\] (27)
that is, \(\sum_{j \geq 0} Q_j f \in F^{s,q}_\infty\). The latter can be obtained by Lemma \([10]\) since we can apply \([19]\), thanks to \(s > 0\), see \([6]\), and to obtain
\[
\|\sum_{j \geq 0} Q_j f\|_{\infty} \lesssim \|f\|_{\dot{F}^{s,q}_\infty},
\]
and similar to Step 7 in the proof of Theorem \([1]\) we also have
\[
\|\sum_{j \geq 0} Q_j f\|_{F^{s,q}_\infty} \lesssim \|f\|_{F^{s,q}_\infty}.
\]

We let \(g := v + \sum_{j \geq 0} Q_j f\) if \(s \in \mathbb{N}\) and \(g := v\) if \(s \in \mathbb{R}^+ \setminus \mathbb{N}\). We have \(f - g \in \mathcal{P}_\mu\) and \(\mathcal{N}(\mathcal{P}_\mu) = \{0\}\); recall that \(\Delta^n_\alpha(x) = 0\) for all \(|\alpha| < m\), and by assumption \(m \geq \mu > s\). Then it follows from \([26]\) and \([27]\) that
\[
\mathcal{N}(f) \leq \mathcal{N}(f - g) + \mathcal{N}(g) \lesssim \|f\|_{\dot{F}^{s,q}_\infty}.
\]
The proof is complete. \(\square\)

**Remark 8.** Of course, the statement of Lemma \([4]\) is certainly known and in particular (i) is classical, but now this can be deduced from Theorem \([2]\) at least for \(q \geq 1\). Indeed, the difficult part in the proof of Lemma \([4]\) is \(\|f\|_{\dot{F}^{s,q}_\infty} \lesssim \|f\|_{F^{s,q}_\infty}\), where now, we get
\[
\|f\|_{\dot{F}^{s,q}_\infty} \lesssim \mathcal{N}^{s,m,1}_{\infty,q}(f) \lesssim \mathcal{N}^{s,m,1}_{\infty,q}(f) + \|f\|_{\infty} \lesssim \|f\|_{F^{s,q}_\infty}
\]
if \(q \geq 1\) and \(m \in \mathbb{N}\) is such that \(0 < s < m\).

**Conclusion**

The realized spaces \(\dot{F}^{s,q}_{\infty,q}\) of the homogeneous Triebel-Lizorkin spaces \(\dot{F}^{s,q}_{\infty,q}\) are now characterized by quasi-seminorms in discrete and continuous (if \(s > 0\)) forms. Our next step will be the extension of the study on \(\dot{F}^{s,q}_{\infty,q}\) to:

- the pointwise multiplication as in e.g. \([2]\),
- the composition operators as in case of the realized homogeneous Besov spaces, see e.g. \([8]\) Thm. 4] or \([17]\) Thm. 5.1],
- the pseudodifferential operators as in e.g. \([15]\).
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