Computing Greeks for Lévy Models: The Fourier Transform Approach.

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Abstract

The computation of Greeks for exponential Lévy models are usually approached by Malliavin Calculus and other methods, as the Likelihood Ratio and the finite difference method. In this paper we obtain exact formulas for Greeks of European options based on the Lewis formula for the option value. Therefore, it is possible to obtain accurate approximations using Fast Fourier Transform. We will present an exhaustive development of Greeks for Call options. The error is shown for all Greeks in the Black-Scholes model, where Greeks can be exactly computed. Other models used in the literature are compared, such as the Merton and Variance Gamma models. The presented formulas can reach desired accuracy because our approach generates error only by approximation of the integral.

Keywords: Greeks; exponential Lévy models.

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1 Introduction

We consider a Lévy process \( X = \{X_t\}_{t \geq 0} \) defined on a probability space \((\Omega, \mathcal{F}, Q)\), a financial market model with two assets, a deterministic savings account \( B = \{B_t\}_{t \geq 0} \), given by

\[
B_t = B_0 e^{rt},
\]

with \( r \geq 0 \) and \( B_0 > 0 \), and a stock \( S = \{S_t\}_{t \geq 0} \), given by

\[
S_t = S_0 e^{rt + X_t},
\]

with \( S_0 > 0 \), where \( X = \{X_t\}_{t \geq 0} \) is a Lévy process. When the process \( X \) has continuous paths, we obtain the classical Black-Scholes model (Merton (1973)). For general reference on the subject we refer to (Kyprianou (2006)) or (Cont and Tankov (2004)).

The aim of this paper is the computation of the price partial derivatives of an European option with general payoff with respect to any parameter of interest. These derivatives are usually named as “Greeks”, and consequently we use the term Greek to refer to any price parcial derivative of the option (of any order and with respect to any parameter).

Our approach departs from the subtle observation by Cont and Tankov see (Cont and Tankov (2004), p. 365):

“Contrary to the classical Black-Scholes case, in exponential-Lévy models there are no explicit formulae for call option prices, because the probability density of a Lévy process is typically not known in closed form. However, the characteristic function of this density can be expressed in terms of elementary functions for the majority of Lévy processes discussed in the literature. This has led to the development of Fourier-based option pricing methods for exponential-Lévy models. In these methods, one needs to evaluate one Fourier transform numerically but since they simultaneously give option prices for a range of strikes and the Fourier transform can be efficiently computed using the FFT algorithm, the overall complexity of the algorithm per option price is comparable to that of evaluating the Black-Scholes formula.”

In other words, in the need of computation of a range of option prices, from a practical point of view, the Lewis formula works as a closed formula, as it can be implemented and computed with approximately the same precision and in the same time as the Black Scholes formula.

Some papers have addressed this problem. Eberlein, Glau and Papapantoleon (Eberlein, Glau, and Papapantoleon (2009)) obtained a formula
similar to the Lewis one, and derived delta ($\Delta$) and gamma ($\Gamma$), the price partial derivatives with respect to the initial value $S_t$ of first and second order, for an European payoff function. The assumptions are similar to the ones we require.

Takahashi and Yamazaki (Takahashi and Yamazaki (2008)) also obtain these Greeks in the case of Call options, based on the Carr and Madan approach. The advantage of the Lewis formula is that it gives option prices for general European payoffs, while Carr-Madam only price European vanilla options.

Other works deal with the problem of Greeks computation for more general payoff functions, including path dependent options, for example see (Chen and Glasserman (2007)), (Glasserman and Liu (2007)), (Glasserman and Liu (2008)), (Kienitz (2008)), (Boyarchenko and Levendorskiı (2009)), (Jeannin and Pistorius (2010)). These works are based on different techniques, such as simulation or finite differences introducing a method error, that has to be analyzed whereas our approach does not.

In the present paper we obtain closed formulas for Greeks based on the Lewis formula, that computes efficiently and with arbitrary precision (as exposed in (Cont and Tankov (2004))), for arbitrary payoff European options in the Lévy models with respect to any parameter and arbitrary order. As an example we analyze the case of Call options.

2 Greeks for General European Options in Exponential Lévy Models

In this paper we do not address the interesting problem of the determination of the pricing measure, see for instance (Cont and Tankov (2004)), assuming then that the given measure $Q$ is the risk-neutral pricing measure. In other words, we assume that the martingale condition is satisfied under $Q$, that in view of (1), stands for the condition

$$E e^{X_t} = 1. \quad (2)$$

Furthermore, by the Lévy-Khinchine Theorem, we obtain that $E e^{izX_t} = e^{t\Psi(z)}$, where the characteristic exponent is

$$\Psi(z) = -iz(1 - iz)\frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^{iy} - 1 - iz(e^y - 1))\nu(dy), \quad (3)$$
with $\sigma \geq 0$ is the standard deviation of the gaussian part of the Lévy process, and $\nu$ its jump measure.

Regarding the payoff, following (Lewis (2001)), denote $s = \ln S_T$ and consider a payoff $w(s)$. For instance, if $K$ is a strike price,

$$w(s) = (e^s - K)^+$$

(4)

is the call option payoff. Then, being $\widehat{w}(z)$ the Fourier Transform of $w(s)$, the Lewis formula (Lewis (2001)) for the European options, valued at time $t$, and denoting $\tau = T - t$ the time to maturity, is:

$$V_t = e^{-r\tau} \int_{iv+\mathbb{R}} e^{-iz(ln(S_T)+r\tau)} e^{r\Psi(-z)} \widehat{w}(z) dz,$$

(5)

where $z \in S_V = \{u + iv: u \in \mathbb{R}\}$ and $v$ must be chosen depending on the payoff function (Lewis (2001)). In this context, it is simple to obtain some general formulas for the Greeks.

In order to differentiate under the integral sign, we present the following classical result.

**Lemma 2.1.** Let $\Theta \subset \mathbb{R}$ an interval and $\mathcal{I} = iv + \mathbb{R}$. Let $h : \mathcal{I} \times \Theta \rightarrow \mathbb{C}$ and $g : \mathcal{I} \rightarrow \mathbb{C}$ such that

- $h(\cdot, \theta)g(\cdot)$ is integrable for all $\theta \in \Theta$ and $g$ is integrable.
- $h(z, \cdot)$ is differentiable in $\Theta$ for all $z \in \mathcal{I}$ and $\frac{\partial h}{\partial \theta}$ is bounded.

Then, $\int_{\mathcal{I}} h(x, \theta)g(x)dx$ is differentiable and

$$\frac{\partial}{\partial \theta} \int_{\mathcal{I}} h(x, \theta)g(x)dx = \int_{\mathcal{I}} \frac{\partial h(x, \theta)}{\partial \theta} g(x)dx \quad \forall \theta \in \Theta.$$

**Proof.** We observe that $|\frac{\partial h(z, \theta)g(z)}{\partial \theta}| \leq C|g(z)|$ for all $z \in \mathcal{I}$, $\theta \in \Theta$. The result is obtained from Theorem 2.27 in (Folland (1999)).

In consequence, in what follows, we will always assume that the conditions in Lemma 2.1 are satisfied for the real part of the integrand because the price imaginary part integrate is zero.
2.1 First Order Greeks

We introduce the auxiliary function
\[ \vartheta(z) = e^{-iz\ln(S_t) + rt}e^{\tau \Psi(-z)}\hat{w}(z). \]

Departing from (5) and (3), by differentiation under the integral sign we obtain
\[
\Delta_t = \frac{\partial V_t}{\partial S_t} = -\frac{1}{S_t}e^{-rt} \int_{iv+R} iz\vartheta(z)dz,
\]
\[
\rho_t = \frac{\partial V_t}{\partial \tau} = -\frac{r}{2\pi} \int_{iv+R} (1 + iz)\vartheta(z)dz,
\]
\[
\nu_t = \frac{\partial V_t}{\partial \sigma} = \tau \sigma e^{-rt} \int_{iv+R} z(1 + iz)\vartheta(z)dz,
\]
\[
\Theta_t = \frac{\partial V_t}{\partial \tau} = \frac{e^{-rt}}{2\pi} \int_{iv+R} \left[ \Psi(-z) - (1 + iz)r \right] \vartheta(z)dz.
\]

Usually, the Lévy models used in the literature depend on a set of parameters that specify the jump measure. Therefore we denote \( \nu(dy) = \nu_\theta(dy) \) and \( \Psi(z) = \Psi_\theta(z) \), then:
\[
\frac{\partial V_t}{\partial \theta} = \frac{e^{-rt}}{2\pi} \int_{iv+R} \frac{\partial \Psi_\theta(-z)}{\partial \theta} \vartheta(z)dz.
\]

2.2 Second Order Greeks

Similarly, we obtain
\[
\Gamma_t = \frac{\partial^2 V_t}{\partial S_t^2} = \frac{1}{S_t^2} e^{-rt} \int_{iv+R} iz(1 + iz)\vartheta(z)dz,
\]
\[
\text{Vanna}_t = \frac{\partial^2 V_t}{\partial \sigma \partial S_t} = \tau \sigma e^{-rt} \int_{iv+R} z^2(1 + iz)\vartheta(z)dz,
\]
\[
\text{Vomma}_t = \frac{\partial^2 V_t}{\partial \sigma^2} = \tau e^{-rt} \int_{iv+R} \left[ 1 + \tau \sigma^2 i(z(1 + iz)) \right] iz(1 + iz)\vartheta(z)dz,
\]
\[
\text{Charm}_t = \frac{\partial^2 V_t}{\partial S_t \partial \tau} = -\frac{1}{S_t} e^{-rt} \int_{iv+R} iz\left[ \Psi(-z) - (1 + iz)r \right] \vartheta(z)dz.
\]
\[
\begin{align*}
Veta_t &= \frac{\partial^2 V_t}{\partial \sigma \partial \tau} = \frac{e^{-r \tau}}{2\pi} \int_{iv+\mathbb{R}} iz(1 + iz)[\tau \Psi(-z) - (iz + 1)r \tau + 1] \vartheta(z) dz, \\
Verat &= \frac{\partial^2 V_t}{\partial \sigma \partial r} = -\tau^2 \frac{e^{-r \tau}}{2\pi} \int_{iv+\mathbb{R}} iz(iz + 1)^2 \vartheta(z) dz.
\end{align*}
\]

Other derivatives can be obtained analogously. In next section we will focus in the case of Call options. This allows to obtain more explicit formulas.

### 3 Greeks for Call Options in Exponential Lévy Models

In order to exploit the particular payoff function, we exhaustively develop the Greeks for Call options. The Put option corresponding formulas can be obtained immediately via Put-Call parity. For other payoff the procedure to obtain the Greeks is analogous.

When the strike \(K\) is fixed, \(x = \ln(K/S_t) - r \tau\) is variable in terms of \(S_t, r\) and \(\tau\). Then, we must consider this for the computation of Greeks \(\Delta, \Gamma, \rho\) and others.

**Lemma 3.1.** Let \(X_\tau\) be a Lévy process with triplet \((\gamma, \sigma, \nu)\) and characteristic exponent \(\Psi(z)\) such that \(\Psi(-i) = 0\) and \(\int_{|y| > 1} e^{\nu y} \nu(dy) < \infty\) with \(v \geq 0\). Then, if \(z \in iv + \mathbb{R}\)

\[
|\Psi_j(-z)| \leq (|z|^2 + |z|) \frac{e^{v}}{2} \int_{|y| \leq 1} y^2 \nu(dy) + 2 \int_{|y| > 1} (e^{\nu y} + 1) \nu(dy)
\]

and

\[
|\Psi(-z)| \leq (|z|^2 + |z|) \left( \frac{e^{v}}{2} \int_{|y| \leq 1} y^2 \nu(dy) + \frac{\sigma^2}{2} \right) + 2 \int_{|y| > 1} (e^{\nu y} + 1) \nu(dy), (6)
\]

where \(\Psi_j(z) = \int_{\mathbb{R}} [e^{iz y} - 1 - iz(e^{y} - 1)] \nu(dy)\).

**Proof.** Let \(I(z) = \int_{\mathbb{R}} [e^{iz y} - 1 - izy1_{\{|y| \leq 1\}}] \nu(dy)\). Applying Taylor’s expansion with Lagrange error form at point \(y = 0\), there exists \(\theta_y\) with \(|\theta_y| \leq |y|\) such that
\[ e^{izy} - 1 - izy1_{\{|y| \leq 1\}} = izy1_{\{|y| > 1\}} - z^2 y^2 e^{iz\theta_y} \]
\[ = -z^2 y^2 e^{iz\theta_y} \frac{1}{2} 1_{\{|y| \leq 1\}} + (izy - z^2 y^2 e^{iz\theta_y})1_{\{|y| > 1\}} \]
\[ = -z^2 y^2 e^{iz\theta_y} \frac{1}{2} 1_{\{|y| \leq 1\}} + (e^{izy} - 1)1_{\{|y| > 1\}}. \]

Then
\[ |I(-z)| \leq \int_{\mathbb{R}} \left| -z^2 y^2 e^{-iz\theta_y} \frac{1}{2} 1_{\{|y| \leq 1\}} + (e^{-izy} - 1)1_{\{|y| > 1\}} \right| \nu(dy) \]
\[ \leq |z|^2 \int_{|y| \leq 1} \frac{e^{i\theta_y}}{2} y^2 \nu(dy) + \int_{|y| > 1} (e^{iy} + 1)\nu(dy) \]
\[ \leq |z|^2 \frac{e^v}{2} \int_{|y| \leq 1} y^2 \nu(dy) + \int_{|y| > 1} (e^{iy} + 1)\nu(dy). \quad (7) \]

Using (7) we have
\[ |\Psi_J(-z)| = |I(-z) + izI(-i)| \]
\[ \leq (|z|^2 + |z|) \frac{e^v}{2} \int_{|y| \leq 1} y^2 \nu(dy) + 2 \int_{|y| > 1} (e^{iy} + 1)\nu(dy). \]

For the continuous part, let \( \Psi_C(-z) = (iz - z^2) \frac{\sigma^2}{2} \), thus
\[ |\Psi(-z)| \leq |\Psi_C(-z)| + |\Psi_J(-z)| \]
\[ \leq (|z|^2 + |z|) \left( \frac{e^v}{2} \int_{|y| \leq 1} y^2 \nu(dy) + \frac{\sigma^2}{2} \right) + 2 \int_{|y| > 1} (e^{iy} + 1)\nu(dy). \]

\[ \square \]

**Lemma 3.2.** Let \( \{X_t\}_{t \geq 0} \) be a Lévy process with triplet \( (\gamma, \sigma, \nu) \) and characteristic exponent \( \Psi(z) \), such that \( \Psi(-i) = 0 \) and \( E[e^{\alpha X_t}] < \infty \) with \( \alpha > 0 \).

1. If \( \int_{\mathbb{R}} |z|^{-1} |e^{-\Psi(z)}| dz < \infty \) then
\[ P(X_t > x) = -\frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{e^{ixz}}{iz} e^{-\Psi(-z)} dz, \quad (8) \]
\[ E(e^{X_t}1_{\{X_t > x\}}) = -\frac{1}{2\pi} \int_{\mathbb{R}^+} \frac{e^{(1+iz)x}}{1+iz} e^{-\Psi(z)} dz. \quad (9) \]
2. If \( \int_{iv+\mathbb{R}} |z|^n |e^{r\Psi(-z)}| dz < \infty \) for some \( n \in \mathbb{Z} \), then \( X_\tau \) has a density of class \( C^n \) and
\[
\frac{\partial^n f(x)}{\partial x^n} = \frac{1}{2\pi} \int_{iv+\mathbb{R}} (iz)^n e^{izx} e^{r\Psi(-z)} dz. \tag{10}
\]

**Proof.** For a Call option the Fourier Transform of the payoff function is \( \hat{w}(z) = \frac{e^{iz(\log(S_t)/\tau + r\tau)} - e^{ix}}{iz(1+iz)} \). Then from the option value (5) we have, with \( x = \log(K/S_t) - r\tau \),
\[
C_t(x) = S_t e^x \int_{iv+\mathbb{R}} \frac{e^{izx} e^{r\Psi(-z)}}{iz(iiz+1)} dz. \tag{11}
\]

Then, being \( x \in [\alpha, \beta] \) and \( C_1 = \max_{\alpha \leq x \leq \beta} e^{(1-v)x} \)
\[
\left| \frac{\partial e^{(1+iz)x} e^{r\Psi(-z)} [iz(iiz+1)]^{-1}}{\partial x} \right| \leq C_1 |z|^{-1} |e^{r\Psi(-z)}| \in L^1(iv + \mathbb{R}),
\]
and by Theorem 2.27 in (Folland (1999)) we can differentiate under the integral sign. Therefore, with \( S_t = 1 \)
\[
\mathbf{P}(X_\tau > x) = -e^{-x} \frac{\partial}{\partial x} \int_{x}^{\infty} (e^s - e^x) F(ds) = -e^{-x} \frac{\partial C_t(x)}{\partial x} = -\frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{izx} e^{r\Psi(-z)}}{iz} dz.
\]

On the other hand, with \( S_t = 1 \)
\[
\mathbf{E}(e^{X_\tau 1_{\{X_\tau>x\}}}) = \int_{x}^{\infty} e^s F(ds) = -e^x \frac{\partial}{\partial x} \int_{x}^{\infty} (e^s - e^x) F(ds) = -e^x \frac{\partial e^{-x} C_t(x)}{\partial x} = C_t(x) - \frac{\partial C_t(x)}{\partial x} = \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{iz(1+iz)} e^{r\Psi(-z)} dz - \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{iz} e^{r\Psi(-z)} dz = -\frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(1+iz)x}}{1+iz} e^{r\Psi(-z)} dz.
\]

For the second part, observe in (8) that if \( x \in [\alpha, \beta] \) and \( C_2 = \max_{\alpha \leq x \leq \beta} e^{-vx} \)
\[
\left| \frac{\partial^{n+1} e^{izx} e^{r\Psi(-z)}}{\partial x^{n+1}} \right| \leq C_2 |z|^n |e^{r\Psi(-z)}| \in L^1(iv + \mathbb{R}).
\]

The result is obtained from Theorem 2.27 in (Folland (1999)). \( \square \)
3.1 Generalized Black-Scholes Formula for Lévy Processes.

We consider by now that $X = \{X_t\}_{t \geq 0}$ in (1) is an arbitrary stochastic process satisfying the martingale condition (2), and we introduce the measure $\tilde{Q}$ by the equation
\[
\frac{d\tilde{Q}}{dQ} = e^{X_T}.
\]

This new measure $\tilde{Q}$ is the Esscher Transform of $Q$ with parameter $\theta = 1$, and it was baptized by Shiryaev et al. (Shiryaev, Kabanov, Kramkov, and Melnikov (1994)) as the dual martingale measure.

We consider a call option with payoff (4), and denote the log forward moneyness\(^1\) by $x = \ln(K/S_0) - r\tau$. Its price in the model we consider can be transformed as
\[
C_t(x) = e^{-r\tau} E (S_t e^{r\tau + X_T} - S_t e^{r\tau + x})^+ = S_t E (e^{X_T} - e^{x})^+
\]
\[
= S_t E (e^{X_T} - e^{x}) 1_{\{X_T > x\}} = S_t \left( E e^{X_T} 1_{\{X_T > x\}} - e^x E 1_{\{X_T > x\}} \right)
\]
\[
= S_t \left( \tilde{Q}(X_T > x) - e^x Q(X_T > x) \right).
\]

Then, we have a closed formula in terms of the probability $Q$ and $\tilde{Q}$. This formula is obtained in (Tankov (2010), p. 68) and is a generalization of the Black-Scholes formula when the underlying asset $X_T$ is a normal random variable. Furthermore we observe that the first term to be computed
\[
E e^{X_T} 1_{\{X_T > x\}} = \tilde{Q}(X_T > x)
\]
is the price of an asset or nothing option, while the second term
\[
E 1_{\{X_T > x\}} = Q(X_T > x)
\]
is the price of a digital option.

In the case that $X = \{X_t\}_{t \geq 0}$ is a Lévy process under $Q$, we obtain the characteristic triplet $(\tilde{\sigma}^2, \tilde{\nu}, \tilde{\gamma})$ under $\tilde{Q}$ by the formulas
\[
\tilde{\sigma} = \sigma,
\]
\[
\tilde{\nu}(dx) = e^x \nu(dx),
\]
\[
\tilde{\gamma} = \frac{\sigma^2}{2} + \int_{\mathbb{R}} (e^{-y} - 1 + h(y)) \tilde{\nu}(dy).
\]

\(^1\)This seems to be the standard definition, although in (Cont and Tankov (2004)) is defined as the opposite quantity.
Furthermore, if $X_t$ has a density $f_t(x)$, by (12), we obtain the density $\tilde{f}_t$ of $X_t$ under $\tilde{Q}$, given by

$$\tilde{f}_t(s) = e^s f_t(s).$$

In order to obtain Greeks in terms of the risk neutral measure, we replace $P$ by $Q$ in (8) and consequently (9), (10) and (11) are related to the probability measure $Q$.

### 3.2 First Order Greeks for Call Options

In this section we do not assume general requirements. We specify the requirements in each case.

#### Delta

Assume that $\int_{iv+R} |z|^{-1} |e^{\tau \Psi(-z)}| dz < \infty$ and $S_t \in [A, B]$. From (11) we obtain

$$\Delta_t^L = \frac{\partial C_t(x(S_t))}{\partial S_t} = \frac{\partial}{\partial S_t} S_t \frac{1}{2\pi} \int_{iv+R} \frac{e^{(1+iz)x(S_t)}}{iz(1+iz)} e^{\tau \Psi(-z)} dz = \frac{1}{2\pi} \int_{iv+R} \frac{e^{(1+iz)x}}{iz(1+iz)} e^{\tau \Psi(-z)} dz - \frac{1}{2\pi} \int_{iv+R} \frac{e^{(1+iz)x}}{iz} e^{\tau \Psi(-z)} dz$$

$$= - \frac{1}{2\pi} \int_{iv+R} \frac{e^{(1+iz)x}}{1+iz} e^{\tau \Psi(-z)} dz = \tilde{Q}(X_\tau > x).$$

#### Rho

Denote now $x = \ln(K/S_t) - r\tau$, that depends on the interest rate $r$. Assume that $\int_{iv+R} |z|^{-1} |e^{\tau \Psi(-z)}| dz < \infty$ and $r \in [R_1, R_2]$. Then

$$\rho_t^L = \frac{\partial C_t(x(r))}{\partial r} = S_t \frac{1}{2\pi} \int_{iv+R} -\tau \frac{e^{(1+iz)x}}{iz} e^{\tau \Psi(-z)} dz = \tau S_t e^x Q(X_\tau > x).$$
Vega

In Black-Scholes, Vega shows the change in variance of the log-price. In expec-
Lévy models, the derivative of $C_t(x)$ w.r.t. $\sigma$ does not give exactly the same
information. We assume that $X_{\tau}$ has density $f$, $\sigma \in [\Sigma_1, \Sigma_2]$ with $\Sigma_1 > 0$
and $z \in i\nu + \mathbb{R}$. Let

$$h(z, \sigma) = e^{\tau iz(1+iz)^2}, \quad g(z) = \frac{e^{izx+i\tau f_k(\psi(y-1)\nu(dy))}}{iz(1+iz)}.$$  

Thus $\frac{\partial h(z,\sigma)}{\partial \sigma}$ is bounded. On the other hand $\int_{i\nu + \mathbb{R}} |g(z)|dz < \infty$ because
$|\mathbb{E}(e^{-i\nu \cdot J_\tau})| \leq \mathbb{E}(e^{J_\tau}) < \infty$, where $J_\tau$ is the jump part of $X_{\tau}$. By Lemma
2.1 we can differentiate under the integral sign.

Then,

$$\mathbb{V}^L_t = \frac{\partial C_t(x)}{\partial \sigma} = S_t \frac{e^{x^2}}{2\pi} \int_{i\nu + \mathbb{R}} \frac{e^{izx+i\tau \psi(y-1)}}{iz(1+iz)} \sigma iz(1+iz)dz$$

$$= S_t \tau \sigma e^{x^2} f_\tau (x).$$

In order to complete the information provided by vega we can calculate
the derivative with respect to jumps intensity.

We assume that $\int_{i\nu + \mathbb{R}} |e^{\tau \psi(y-1)}|dz < \infty$. Let $\nu(dy) = \lambda \tilde{\nu}(dy)$ with $\lambda 
\in [\lambda_1, \lambda_2]$, then let

$$h(z, \lambda) = \frac{e^{\tau \lambda} [e^{-izy-1}+iz(e^{-1}+1)] \tilde{\nu}(dy)}{\tau \int_{\mathbb{R}} [e^{-izy-1}+iz(e^{-1}+1)] \tilde{\nu}(dy)},$$

$$g(z) = \frac{e^{(iz+1)x} e^{\tau \psi(y+iz)}}{iz(1+iz)} \tau \int_{\mathbb{R}} [e^{-izy-1}+iz(e^{-1}+1)] \tilde{\nu}(dy),$$

where $\frac{\partial h(z,\lambda)}{\partial \lambda}$ is bounded and from Lemma 3.1 $\int_{i\nu + \mathbb{R}} |g(z)|dz < \infty$, then by
Lemma 2.1

$$\frac{\partial \tau C_t(x)}{\partial \lambda} = \frac{\partial}{\partial \lambda} S_t \frac{1}{2\pi} \int_{i\nu + \mathbb{R}} \frac{e^{izx+i\tau \psi(y-1)}}{iz(1+iz)} dz$$

$$= \tau S_t \frac{1}{2\pi} \int_{i\nu + \mathbb{R}} \frac{e^{izx+i\tau \psi(y-1)}}{iz(1+iz)} \tilde{\nu}(dy) dz,$$
with $\Psi_J(-z) = \int_{\mathbb{R}} \left[ e^{-izy} - 1 + iz(e^y - 1) \right] \bar{\nu}(dy)$. Using Fubini’s Theorem we obtain

$$\frac{\partial \tau}{\partial \lambda} C_t(x) = \frac{\partial}{\partial \lambda} S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau \Psi(-z)}}{iz(iz+1)} dz$$

$$= \tau S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau \Psi(-z)}}{iz(iz+1)} \left[ e^{-izy} - 1 + iz(e^y - 1) \right] \bar{\nu}(dy) dz$$

$$= \tau S_t \left[ \int_{\mathbb{R}} \left( e^y e^{x-y} \int_{iv+\mathbb{R}} e^{iz(x-y)} \frac{e^{\tau \Psi(-z)}}{iz(1+iz)} dz \right) \frac{-e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau \Psi(-z)}}{iz(1+iz)} dz \right.$$

$$\left. + \left( e^y - 1 \right) \frac{e^x}{2\pi} \int_{iv+\mathbb{R}} e^{izx} \frac{e^{\tau \Psi(-z)}}{1+iz} dz \right] \bar{\nu}(dy)$$

$$= \tau \left[ \int_{\mathbb{R}} \left( e^y C_t(x-y) - C_t(x) - S_t(e^y - 1) \bar{Q}(X_\tau > x) \right) \bar{\nu}(dy) \right].$$

The use of Fubini’s Theorem is justified by (7) and the additional hypothesis $\int_{iv+\mathbb{R}} |e^{\tau \Psi(-z)}| dz < \infty$.

**Theta**

We assume that $\int_{iv+\mathbb{R}} |z^2 e^{\tau \Psi(-z)}| dz < \infty$, $\tau \in [T_1, T_2]$ and $z \in iv + \mathbb{R}$, let

$$h(z, \tau) = e^{(iz+1)x} e^{\tau \Psi(-z)}, \quad g(z) = \frac{1}{iz(1+iz)}.$$  

Then, $\int_{iv+\mathbb{R}} |g(z)| dz < \infty$, moreover, from (6) and $\int_{iv+\mathbb{R}} |z^2 e^{\tau \Psi(-z)}| dz < \infty$

$$\frac{\partial h(z, \tau)}{\partial \tau} = e^{(iz+1)x} e^{\tau \Psi(-z)} \left( - r(1+iz) + \Psi(-z) \right)$$

is bounded and by Lemma 2.1,

$$\Theta^\tau_t = \frac{\partial \tau}{\partial \tau} C_t(x, \tau) = \frac{\partial}{\partial \tau} S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau \Psi(-z)}}{iz(iz+1)} dz$$

$$= S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x} e^{\tau \Psi(-z)}}{iz(1+iz)} \left( \Psi(-z) - r(1+iz) \right) dz.$$
Using Fubini’s Theorem we obtain

$$\Theta_L = S_t \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x \tau} e^{r\Psi(-z)}}{iz(1+iz)} (\Psi(-z) - r(1+iz)) \, dz$$

$$= S_t \left[ -\frac{r}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x \tau} e^{r\Psi(-z)}}{iz} \, dz + \frac{1}{2\pi} \int_{iv+\mathbb{R}} \frac{e^{(iz+1)x \tau} e^{r\Psi(-z)}}{iz(iz+1)} \left( iz(1+iz) \frac{\sigma^2}{2} \right. \right.$$ 

$$\left. + \int_{\mathbb{R}} [e^{-izy} - 1 + iz(e^y - 1)] \nu(dy) \right) \, dz$$

$$= S_t \left[ re^{x \tau} Q(X_\tau > x_\tau) + \frac{\sigma^2}{2} e^{x \tau} f_\tau(x_\tau) \right. \right.$$ 

$$\left. + \int_{\mathbb{R}} \left( e^y \frac{e^{x-y}}{2\pi} \int_{iv+\mathbb{R}} e^{iz(x-y)} e^{r\Psi(-z)} \frac{dz}{iz(1+iz)} + \frac{e^{x-y}}{2\pi} \int_{iv+\mathbb{R}} e^{izx} e^{r\Psi(-z)} \frac{dz}{iz(1+iz)} \right) \nu(dy) \right]$$

$$= S_t \left[ re^{x \tau} Q(X_\tau > x_\tau) + \frac{\sigma^2}{2} e^{x \tau} f_\tau(x_\tau) \right. \right.$$ 

$$\left. + \int_{\mathbb{R}} \left( e^y C_\tau(x_\tau - y) - C_\tau(x_\tau) - S_t(e^y - 1) \tilde{Q}(X_\tau > x_\tau) \right) \nu(dy) \right].$$

The use of Fubini’s Theorem is justified by (7) and the additional hypothesis

$$\int_{iv+\mathbb{R}} |z^2 e^{r\Psi(-z)}| \, dz < \infty.$$
Vanna

We assume that \( \int_{iv + \mathbb{R}} |ze^{\tau \Psi(-z)}|dz < \infty \) and \( 0 < \Sigma_1 \leq \sigma \leq \Sigma_2 \), then

\[
\frac{\partial^2 C_t(x)}{\partial \sigma \partial S_t} = \frac{\partial V_t^L}{\partial S_t} = \tau \sigma e^{x(S_t)} f_t(x(S_t)) - \tau \sigma e^{x(S_t)} \left( f_t(x(S_t)) + f_t'(x(S_t)) \right)
= -\tau \sigma e^{x} f_t'(x).
\]

Vomma

We assume that \( \int_{iv + \mathbb{R}} |z^2 e^{\tau \Psi(-z)}|dz < \infty \) and \( 0 < \Sigma_1 \leq \sigma \leq \Sigma_2 \), let \( z \in iv + \mathbb{R} \) and denote

\[
h(z, \sigma) = z^2 e^{\tau iz(1+iz)} e^{\tau \Psi(-z)} \frac{\nu(dy)}{z^2}, \quad g(z) = e^{izx + \int_{iv+\mathbb{R}} (e^{izy} - 1 - iz(e^y - 1)) \nu(dy)}.
\]

Thus \( \frac{\partial h(z, \sigma)}{\partial \sigma} \) is bounded and \( \int_{iv + \mathbb{R}} |g(z)|dz < \infty \), because \( |E(e^{-izJ})| \leq E(e^{J}) < \infty \), where \( J \) is the jump part of \( X_t \). By Lemma 2.1 we can differentiate under the integral sign.

\[
\frac{\partial^2 C_t(x)}{\partial \sigma^2} = \frac{\partial V_t^L}{\partial \sigma} = S_t \tau e^{x} f_t(x) + S_t \tau \sigma \frac{e^{x}}{2\pi} \int_{iv+\mathbb{R}} e^{izx + \tau \Psi(-z)} \tau \sigma (iz - z^2)dz
= S_t \tau e^{x} \left( f_t(x) + \tau \sigma^2 [f_t'(x) + f_t''(x)] \right).
\]

Charm

We assume that \( \int_{iv + \mathbb{R}} |z^3 e^{\tau \Psi(-z)}|dz < \infty \), \( \tau \in [T_1, T_2] \) and \( z \in iv + \mathbb{R} \), let

\[
h(z, \tau) = ze^{(iz+1)x} e^{\tau \Psi(-z)}, \quad g(z) = \frac{1}{z(1 + iz)}.
\]

Then, \( \int_{iv + \mathbb{R}} |g(x)|dx < \infty \) and by Lemma 3.1

\[
\frac{\partial h(z, \tau)}{\partial \tau} = ze^{(iz+1)x} e^{\tau \Psi(-z)} \left( -r(1 + iz) + \Psi(-z) \right)
\]

is bounded. By Lemma 2.1,
\[
\frac{\partial^2 C_t(x)}{\partial \tau \partial S_t} = \frac{\partial \tilde{Q}(X_\tau > x_\tau)}{\partial \tau} = \frac{\partial}{\partial \tau} \frac{-1}{2\pi} \int_{iv+\mathbb{R}} e^{(iz+1)x_\tau} e^{\tau \Psi(-z)} \frac{dz}{1 + iz} = \frac{1}{2\pi} \int_{iv+\mathbb{R}} e^{(iz+1)x_\tau} \frac{dz}{1 + iz} \left( r(1 + iz) - \Psi(-z) \right) dz.
\]

(13)

Using Fubini’s Theorem we obtain

\[
\frac{\partial^2 C_t(x)}{\partial \tau \partial S_t} = -r e^{x_\tau} f_\tau(x_\tau) + \frac{\sigma^2}{2} e^{x_\tau} f_\tau'(x_\tau) + \int_{\mathbb{R}} \left[ -e^y \tilde{Q}(X_\tau > x_\tau - y) + \tilde{Q}(X_\tau > x_\tau) \right] \nu(dy)
\]

\[
+ (e^y - 1) \left\{ e^{x_\tau} f_\tau(x_\tau) + \frac{\sigma^2}{2} e^{x_\tau} f_\tau'(x_\tau) \right\} \nu(dy)
\]

\[
= -r e^{x_\tau} f_\tau(x_\tau) + \frac{\sigma^2}{2} e^{x_\tau} f_\tau'(x_\tau)
\]

\[
- \int_{\mathbb{R}} \left[ e^y \left( \tilde{Q}(X_\tau > x_\tau) - \tilde{Q}(X_\tau > x_\tau - y) \right) \right] \nu(dy)
\]

\[
+ (e^y - 1) e^{x_\tau} f_\tau(x_\tau) \nu(dy).
\]

(14)

The use of Fubini’s Theorem is justified by (7) and the additional hypothesis

\[
\int_{iv+\mathbb{R}} |z^3 e^{\tau \Psi(-z)}| dz < \infty.
\]
We assume that \( \int_{iv+R} |z^4 e^{\Psi(-z)}|dz < \infty \). Similar to \( \text{Charm}_t \), we assume that \( \tau \in [\mathcal{T}_1, \mathcal{T}_2] \) and \( z \in iv + \mathbb{R} \), and denote

\[
    h(z, \tau) = z^2 e^{(iz+1)x_r} e^{\tau \Psi(-z)}, \quad g(z) = \frac{1}{z^2}.
\]

Then, \( \int_{iv+R} |g(z)|dz < \infty \) and by Lemma 3.1

\[
    \frac{\partial h(z, \tau)}{\partial \tau} = z^2 e^{(iz+1)x_r} e^{\tau \Psi(-z)} \left( -r(1 + iz) + \Psi(-z) \right)
\]

is bounded. By Lemma 2.1 we can differentiate under the integral sign,

\[
    \frac{\partial^2 C_t(x_r)}{\partial \sigma \partial \tau} = \frac{\partial V^L_t}{\partial \tau} = \frac{\partial S_t \tau \sigma e^{x_r f_t(x_r)}}{\partial \tau} + \frac{\tau e^{x_r}}{2\pi} \int_{iv+R} \frac{\partial}{\partial \tau} e^{izx_r} e^{\tau \Psi(-z)} dz
\]

\[
    = S_t \sigma \left[ e^{x_r f_t(x_r)} - r \tau e^{x_r f_t(x_r)} + \frac{\tau e^{x_r}}{2\pi} \int_{iv+R} e^{izx_r} e^{\tau \Psi(-z)} \left( \Psi(-z) - riz \right) dz \right].
\]

Using Fubini’s Theorem, we obtain

\[
    \frac{\partial^2 C_t(x_r)}{\partial \sigma \partial \tau} = S_t \sigma \left[ e^{x_r f_t(x_r)} - r \tau e^{x_r f_t(x_r)} \right.
\]

\[
    + \frac{\tau e^{x_r}}{2\pi} \int_{iv+R} e^{izx_r} e^{\tau \Psi(-z)} \left( \Psi(-z) - riz \right) dz
\]

\[
    = S_t \sigma e^{x_r} \left[ f_t(x_r) - r \tau [f_t(x_r) + f_t'(x_r)] \right]
\]

\[
    + \frac{\tau}{2\pi} \int_{iv+R} e^{izx_r} e^{\tau \Psi(-z)} \left\{ \frac{\sigma^2}{2} (iz - z^2) \right\}
\]

\[
    + \int_{\mathbb{R}} (e^{-izy} - 1 + iz(e^y - 1)) \nu(dy) \right\} dz.
\]

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\[-S_t \sigma e^{x_r} \left[ f_{\tau}(x_\tau) - r\tau \left[ f_{\tau}(x_\tau) + f_{\tau}'(x_\tau) \right] + \tau \frac{\sigma^2}{2} \left[ f_{\tau}'(x_\tau) + f_{\tau}''(\tau) \right] \\
+ \tau \int_{\mathbb{R}} \left[ f_{\tau}(x_\tau - y) - f_{\tau}(x_\tau) + (e^y - 1)f_{\tau}'(x_\tau) \right] \nu(dy) \right]. \]

(16)

The use of Fubini’s Theorem is justified by (7) and the additional hypothesis \( \int_{iv+\mathbb{R}} |z^4 e^{\tau \Psi(-z)}| \, dz < \infty \).

Vera

Assuming that \( \int_{iv+\mathbb{R}} |z^4 e^{\tau \Psi(-z)}| \, dz < \infty \), and \( 0 \leq \sigma \leq \Sigma_2 \),

\[
\frac{\partial^2 C_t(x_r)}{\partial \sigma \partial r} = \frac{\partial V^L_t}{\partial r} = S_t \tau \sigma e^{x_r} \left( -\tau f_{\tau}(x_\tau) - \tau f_{\tau}'(x_\tau) \right) \\
= -S_t \tau^2 \sigma e^{x_r} \left( f_{\tau}(x_\tau) + f_{\tau}'(x_\tau) \right).
\]

3.4 Third Order Greeks for Call Options

Vera

We assume that \( \int_{iv+\mathbb{R}} |z^4 e^{\tau \Psi(-z)}| \, dz < \infty \), \( \tau \in \mathcal{T}_1, \mathcal{T}_2 \). Let \( z \in iv + \mathbb{R} \) and

\[
h(z, \tau) = z^2 e^{(iz+1)x_\tau} e^{\tau \Psi(-z)} \\
g(z) = \frac{1}{z^2}.
\]

Then, \( \int_{iv+\mathbb{R}} |g(x)| \, dx < \infty \) and by Lemma 3.1

\[
\frac{\partial h(z, \tau)}{\partial \tau} = z^2 e^{(iz+1)x_\tau} e^{\tau \Psi(-z)} \left( -r(1+iz) + \Psi(-z) \right)
\]

is bounded. By Lemma 2.1 we can differentiate under the integral sign.

Thus,

\[
\frac{\partial^3 C_t(x)}{\partial S_t^2 \partial \tau} = \frac{\partial^3 \Gamma_t^L}{\partial \tau} = \frac{1}{S_t^2 \pi} \int_{iv+\mathbb{R}} e^{(iz+1)x_\tau} e^{\tau \Psi(-z)} \left( -r(iz+1) + \Psi(-z) \right) \, dz.
\]

(17)
Using Fubini’s Theorem we obtain

\[
\frac{\partial^3 C_t(x)}{\partial S_t^2 \partial \tau} = \frac{1}{S_t^2} \frac{\partial^3 \Gamma_L}{\partial S_t^3} = \frac{e^x}{S_t} \left[ \frac{1}{2\pi} \int_{i\nu+R} e^{izx} e^{\tau \Psi(-z)} \int_{\mathbb{R}} e^{-izy} - 1 + iz(e^y - 1) \nu(dy)dz \right]
\]

Fubini is justified by (7) and the hypothesis \(\int_{i\nu+R} |z^4 e^{\tau \Psi(-z)}| dz < \infty\).

**Speed**

Assuming that \(\int_{i\nu+R} |z e^{\tau \Psi(-z)}| dz < \infty\),

\[
\frac{\partial^3 C_t(x_r)}{\partial S_t^3} = \frac{\partial \Gamma_L}{\partial S_t} = \frac{e^x(S_i) \left( -\frac{1}{S_i} f_r(x(S_i)) - \frac{1}{S_i} f'_r(x(S_i)) \right) S_t - e^x(S_i) f_r(x(S_i))}{S_t^2}
\]

\[= - \frac{e^x}{S_t} \left( 2 f_r(x) + f'_r(x) \right).\]

**Ultima**

We assume that \(\int_{i\nu+R} |z^6 e^{\tau \Psi(-z)}| dz < \infty\). First we calculate \(\frac{\partial f_r^{(n)}(x)}{\partial \sigma} \) for \(n = 0, 1, 2\). For \(0 < \Sigma_1 \leq \sigma \leq \Sigma_2\) and \(z \in i\nu + \mathbb{R}\), and denote

\[h_n(z, \sigma) = (iz)^{n+2} e^{\tau \Psi(-z)}, \quad g(z) = -\frac{e^{izx}}{z^2}.
\]

Thus, \(\int_{i\nu+R} |g(z)| dz < \infty\) and \(\frac{\partial h_n(z, \sigma)}{\partial \sigma}\) is bounded for \(n = 0, 1, 2\). By Lemma 2.1 we can differentiate under the integral sign. Then,
\[ \frac{\partial^n f_\tau(x)}{\partial \sigma} = \frac{\partial}{\partial \sigma} \frac{1}{2\pi} \int_{i\mathbb{R}} (iz)^n e^{ixz} e^{\tau \Psi(-z)} dz \]
\[ = \tau \sigma \frac{1}{2\pi} \int_{i\mathbb{R}} [(iz)^{n+1} - (iz)^{n+2}] e^{ixz} e^{\tau \Psi(-z)} dz \]
\[ = \tau \sigma \left(f^{(n+1)}_\tau(x) + f^{(n+2)}_\tau(x)\right). \quad (19) \]

Now, we have
\[ \frac{\partial^3 C_t(x)}{\partial \sigma^3} = \frac{\partial S_t \tau e^x \left(f_\tau(x) + \tau \sigma^2 \left[f'_\tau(x) + f''_\tau(x)\right]\right)}{\partial \sigma} \]
\[ = S_t \tau^2 \sigma e^x \left(3f'_\tau(x) + f''_\tau(x) + \tau \sigma^2 \left[f''_\tau(x) + 2f''_\tau(x) + f^{(iv)}_\tau(x)\right]\right). \]

**Zomma**

We assume that \( \int_{i\mathbb{R}} z^2 e^{\tau \Psi(-z)} dz < \infty \) and \( 0 < \Sigma_1 \leq \sigma \leq \Sigma_2 \). Then,
\[ \frac{\partial^3 C_t(x)}{\partial S_t^2 \partial \sigma} = \frac{\partial Vanna_t^L}{\partial S_t} = \frac{\partial \tau \sigma e^x(S_t f'_\tau(x(S_t)))}{\partial S_t} \]
\[ = - \frac{\tau \sigma e^x}{S_t} \left(f'_\tau(x) + f''_\tau(x)\right). \]

**4 Examples**

**4.1 The Black-Scholes Model**

If we assume that the gaussian distribution and density are exactly computed in R software, we can compare the Greeks for Black-Scholes model using Lewis representation.

To approximate the Fourier Transform we cut the integral between \(-A/2\) and \(A/2\) and take a uniform partition of \([-A/2, A/2]\) of size \(N\):
\[ \int_{\mathbb{R}} e^{izx} g(z) dz \approx \int_{-A/2}^{A/2} e^{izx} g(z) dz \approx \frac{A}{N} \sum_{k=0}^{N-1} w_k e^{iz_k x} g(z_k), \]
where \( z_k = -\frac{4}{2} + k \frac{A}{N-1} \) and \( w_k \) are weights that correspond to the integration numerical rule.

Table 1 shows the \( \ell_\infty \)-errors in Black-Scholes model via Lewis representation and Fast Fourier Transform using: \( S_t = 1, \ r = 0.05, \ T = 1, \ \sigma = 0.1, \ A = 300 \) and \( N = 2^{22} \). The \( \ell_\infty \)-errors are

\[
\ell_\infty\text{-error}(GL) = \max_{x \in [-0.7, 0.7]} |GL - G|,
\]

for \( x = \ln(K/S_t) - r\tau \).

| Greek | Expression | \( \ell_\infty \)-error |
|-------|------------|------------------------|
| Call  | \( C = S E(e^{x_1} - e^x)^+ \) | 1.2e-07 |
| Delta | \( \partial_s C(x) \) | 2.4e-07 |
| Rho   | \( \partial_r C_t(x) \) | 1.9e-07 |
| Vega  | \( \partial_s C(x) \) | 9.5e-08 |
| Theta | \( \partial_\tau C(x) \) | 1.2e-08 |
| Gamma | \( \partial^2_{ss} C(x) \) | 9.5e-07 |
| Vanna | \( \partial^2_{ss} C(x) \) | 6.3e-07 |
| Vomma | \( \partial^2_{ss} C(x) \) | 7.5e-07 |
| Charm | \( \partial^2_{ss} C(x) \) | 6.8e-08 |
| Veta  | \( \partial^2_{sr} C(x) \) | 8.9e-08 |
| Vera  | \( \partial^2_{sr} C(x) \) | 5.8e-07 |
| Color | \( \partial^3_{ssr} C(x) \) | 5.6e-07 |
| Speed | \( \partial^3_{sss} C(x) \) | 6.3e-06 |
| Ultima| \( \partial^3_{ssr} C(x) \) | 1.2e-05 |
| Zomma | \( \partial^3_{sss} C(x) \) | 9.5e-06 |

Table 1: \( \ell_\infty \)-errors in Black-Scholes model via Lewis representation and Fast Fourier Transform using: \( S_t = 1, \ r = 0.05, \ T = 1, \ \sigma = 0.1, \ A = 300 \) and \( N = 2^{22} \).

4.2 The Merton model

In this section we show some results for the Merton model. The Merton model has four parameters \( (\sigma, \mu_J, \sigma_J, \lambda) \) where \( \sigma \) is the diffusion parameter, \( \lambda \) is the jump intensity, \( \mu_J \) and \( \sigma_J \) are the mean and standard deviation of the jump which are gaussianly distributed. The characteristic function for the Merton model is:
\[ E(e^{izX_T}) = \exp \left\{ iz \left[ \frac{\sigma^2}{2} - \lambda (e^{\mu J + \frac{\sigma^2}{2}} - 1) \right] + z^2 \frac{\sigma^2}{2} + \lambda (e^{iz\mu J - z^2 \frac{\sigma^2}{2}} - 1) \right\}. \]

(20)

All Greeks for \textit{At The Money} \((K = S_0 e^{-rT})\) are shown in Table 2 following section 3. Here we took \(A = 500, N = 2^{20}\) and \(A = 500, N = 2^{22}\), the \(\ell_\infty\)-error for \(x \in [-0.7, 0.7]\) is in all Greeks lower than \(10^{-5}\). In Figure 1 the curves are shown in terms of \(x = \ln(K/S_0) - rT\) for all Greeks with the comparison of the Black-Scholes model with volatility equal to implied volatility \textit{At The Money}.

|          | \(A = 500, N = 2^{20}\) | \(A = 500, N = 2^{21}\) | error     |
|----------|--------------------------|--------------------------|-----------|
| Call     | 0.0547129                | 0.0547129                | 2.6e-08   |
| Delta    | 0.5273560                | 0.5273562                | 2.5e-07   |
| Rho      | 0.4726431                | 0.4726433                | 2.2e-07   |
| Vega     | 0.3077754                | 0.3077755                | 1.5e-07   |
| Theta    | 0.0524286                | 0.0524286                | 2.5e-08   |
| Gamma    | 3.0777536                | 3.0777550                | 1.5e-06   |
| Vanna    | 0.1538877                | 0.1538878                | 7.3e-08   |
| Vomma    | 0.9091776                | 0.9091780                | 4.3e-07   |
| Charm    | 0.1682859                | 0.1682860                | 8.1e-08   |
| Veta     | 0.1222075                | 0.1222076                | 5.8e-08   |
| Vera     | -0.1538877               | -0.1538878               | 7.3e-08   |
| Color    | 1.8556786                | 1.8556795                | 8.8e-07   |
| Speed    | -4.6166303               | -4.6166325               | 2.2e-06   |
| Ultima   | -11.5390901              | -11.5390956              | 5.5e-06   |
| Zomma    | -21.6857596              | -21.6857699              | 1.0e-05   |

Table 2: Greeks in Merton model with: \(S_0 = 1, r = 0.05, x = 0, T = 1, \sigma = 0.1, \mu_J = -0.005, \sigma_J = 0.1, \lambda = 1\).

The characteristic function in this case is (20). To compute sensitivities w.r.t. \(\mu_J, \sigma_J\) and \(\lambda\) we only need to differentiate the characteristic exponent with respect to these parameters:

\[
\frac{\partial \Psi(-z)}{\partial \mu_J} = \lambda i z \left[ e^{\mu J + \sigma_J^2/2} - e^{-i z \mu J - z^2 \sigma_J^2/2} \right],
\]
Figure 1: Greeks in terms of $x = \ln(K/S_0) - rT$ for the Merton Model with parameters equal to Table 2 (continuous line). Discontinuous line: Black-Scholes Model with volatility equal to implied volatility in $x = 0$ ($\sigma_{imp}(0) \approx 0.137$).
\begin{align*}
\frac{\partial \Psi(-z)}{\partial \sigma_j} &= \lambda \sigma_j \left[ ie^{\mu_j + \sigma_j^2/2} - z^2 e^{-iz\mu_j - z^2\sigma_j^2/2} \right], \\
\frac{\partial \Psi(-z)}{\partial \lambda_j} &= iz \left[ e^{\mu_j + \sigma_j^2/2} - 1 \right] + e^{-iz\mu_j - z^2\sigma_j^2/2} - 1,
\end{align*}

and for \( \theta = \mu_J, \sigma_J, \lambda, \)

\[
\frac{\partial C_\theta(x)}{\partial \theta} = \tau S_t \frac{e^x}{2\pi \int_{iv+R} e^{-izx} e^{-iz\Psi(-z)} dz},
\]

The differentiation under the integral sign is justified as above.

Using the same parameters presented in Table 2 we obtain the sensitivities for ATM given in Table 3.

| \( A = 500, \ N = 2^{20} \) | \( A = 1000, \ N = 2^{22} \) | error |
|---|---|---|
| \( \mu_J \)-sensitivity | 0.006703850 | 0.006703855 | 4.7e-09 |
| \( \sigma_J \)-sensitivity | 0.239001059 | 0.239001230 | 1.7e-07 |
| \( \lambda \)-sensitivity | 0.013407701 | 0.013407711 | 9.6e-09 |

Table 3: Sensitivities for Merton model with: \( S_0 = 1, \ r = 0.05, \ x = 0, \ T = 1, \) \( \sigma = 0.1, \ \mu_J = -0.005, \ \sigma_J = 0.1, \ \lambda = 1. \)

In Figure 2 we show the Greeks in terms of \( x = \ln(K/S_0) - rT. \)

In (Kienitz (2008)) are shown some results for a Digital Option in the Merton model, which were obtained by applying finite difference approximations to the formula for the option prices in (Madan, Carr, and Chang (1998)). Now we will deduce Delta, Gamma and Vega for a Digital Option and thus we will compare the results.

A Digital Option has a payoff given by:

\[
1_{\{S_T - K > 0\}} = 1_{\{X_T - x > 0\}}.
\]

Using Lewis representation, the value for a Digital Option is:

\[
D(x) = Q(X_T > x) = -\frac{1}{2\pi \int_{iv+R} e^{-izx} e^{-iz\Psi(-z)} dz},
\]

where \( x = \ln(K/S_t) - rT. \) A direct differentiation leads to:
Figure 2: Sensitivities in terms of $x = \ln(K/S_0) - rT$ for Merton Model with parameters equal to Table 3.

\[
\frac{\partial D(x)}{\partial S_t} = \frac{1}{S_t} f'_\tau(x), \quad (22)
\]

\[
\frac{\partial^2 D(x)}{\partial S_t^2} = -\frac{1}{S_t^2} \left( f_\tau(x) + f'_\tau(x) \right), \quad (23)
\]

\[
\frac{\partial D(x)}{\partial \sigma} = -\tau \sigma \left( f_\tau(x) + f'_\tau(x) \right). \quad (24)
\]

Observe that the formulas (21)-(24) are valid in general for Digital Call options with $\int_{\mathbb{R}^+} |z e^{\nu\xi}^{-u} - u| |dz| < \infty$. In (24), differentiation under integral sign is similar to (19).

Then, our results via FFT are shown in Table 4. In (Kienitz (2008)) this values are (by finite difference): $D= 0.531270$, $D-Delta= 0.016610$, $D-Gamma= -2.800324 \times 10^{-4}$, $D-Vega= -0.560070$. To obtain a given strike we define $\delta = 2\pi N^{-1} \times \frac{1}{N\Lambda}$.

4.3 The Variance Gamma Model

In this section we will compare some results from the literature. As an example, in (Glasserman and Liu (2007)) some results are shown for the
D-Call  D-Delta  D-Gamma  D-Vega
N = 2^20  0.531269863  0.016610445  -0.000280032  -0.560064360
N = 2^22  0.531270245  0.016610457  -0.000280032  -0.560064763
error    3.8e-07  1.2e-08  2.0e-10  4.0e-07

Table 4: Digital Option and Greeks in Merton model with: \( \delta = 0.01 \), \( S_0 = 100 \), \( K = 100 \), \( T = 1 \), \( r = 0.07 \), \( \sigma = 0.2 \), \( \mu_J = 0.05 \), \( \sigma_J = 0.15 \) and \( \lambda = 0.5 \).

Variance Gamma model with parameters \( (\rho, \nu, \theta) \) where the characteristic function is:

\[
E[e^{izX_T}] = \exp \left\{ \frac{T}{\nu} \left[ iz \ln \left( 1 - \theta \nu - \frac{\rho^2 \nu}{2} \right) - \ln \left( 1 - iz \theta \nu + \frac{z^2 \rho^2 \nu}{2} \right) \right] \right\}.
\]

To obtain a given strike we define \( \delta = 2\pi \frac{N-1}{NA} \). Thus, in Table 5 we present two results for \( N = 2^{20} \) and \( N = 2^{22} \) with \( \delta = 0.01 \). The error shows the convergence of the complex integral. In (Glasserman and Liu (2007)) these results are obtained by applying finite difference approximations to the formula for the option prices in (Madan, Carr, and Chang (1998)):

\[
\text{Call} = 11.2669, \text{Delta} = 0.7282 \text{ and } \rho\text{-derivative} = 23.0434 \text{ and in general with LRM method, the error is worse than } 10^{-2}.
\]

|         | Call     | Delta    | Gamma    | \( \frac{\partial \text{Call}}{\partial \rho} \) |
|---------|----------|----------|----------|---------------------------------|
| N = 2^{20}, \( \delta = 0.01 \) | 11.26689113 | 0.72818427 | 0.01427437 | 23.04334371 |
| N = 2^{22}, \( \delta = 0.01 \) | 11.26689919 | 0.72818479 | 0.01427438 | 23.04336021 |
| err<    | 8.1e-06  | 5.2e-07  | 1.0e-08  | 1.6e-05          |

Table 5: Greeks and \( \rho \)-sensitivity for Variance Gamma model with: \( (\rho, \nu, \theta) = (0.2, 1, -0.15) \), \( r = 0.05 \), \( T = 1 \), \( S_0 = K = 100 \) (\( x = -0.05 \)).

## 5 Conclusions

Greeks are an important input for market makers in risk management. A lot of options are path dependent and they don’t have explicit formula. However, for the European options in the exponential Lévy models we have the Lewis formula, which allows us to obtain closed formulas for Greeks, many of which
We observe that, if the density of \( X^t \) is known, then many of the Greeks to time to maturity. Thus, the Greeks for call options can be calculated through

\[
\frac{\partial C}{\partial C} = \text{call Greeks}
\]

For a fix strike \( K \), we consider \( \text{ln} \frac{S}{Y} = \text{ln}x \).

Greeks approximations are computed through the finite difference technique.

In order to estimate the accuracy of their peak payoff functions. However, in order to obtain the accuracy of their

Table 6: Greeks in exponential Lévy models in terms of \( \text{ln}x \) from a family of\( \text{ln}x \) to S.\( \text{ln}x \) is a simple integral, can be approximated with high accuracy because they are a simple integral.

| Order | Formula | Greek |
|-------|---------|-------|
| First order | \( (x)^\frac{\partial f}{\partial x} + (x)^\frac{\partial x}{\partial x} \) | Speed |
| Second order | \( (x)^\frac{\partial^2 f}{\partial x^2} + (x)^\frac{\partial x^2}{\partial x^2} \) | Gamma |
| Third order | \( (x)^\frac{\partial^3 f}{\partial x^3} + (x)^\frac{\partial x^3}{\partial x^3} \) | Vomma |
| | \( (x)^\frac{\partial^4 f}{\partial x^4} + (x)^\frac{\partial x^4}{\partial x^4} \) | Ultima |

\[ \lambda = \frac{\partial}{\partial \lambda} \]

\[ \nu = \frac{\partial}{\partial \nu} \]

\[ \rho = \frac{\partial}{\partial \rho} \]
can be exactly obtained. Some examples of these are: Normal Inverse Gaussian, Variance Gamma, Generalized Hyperbolic, Meixner and others.

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