Optimal Galaxy Distance Estimators

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Abstract

The statistical properties of galaxy distance estimators corresponding to the Tully-Fisher and $D_n - \sigma$ relations are studied, and a rigorous framework for identifying and removing the effects of Malmquist bias due to observational selection is developed. The prescription of Schechter (1980) for defining unbiased distance estimators is verified and extended to more general – and more realistic – cases. Finally, the derivation of 'optimal' unbiased estimators of minimum dispersion, by utilising information from additional suitably correlated observables, is discussed and the results applied to a calibrating sample from the Fornax cluster, as used in the Mathewson spiral galaxy redshift survey. The optimal distance estimator derived from apparent magnitude, diameter and 21cm line width has an intrinsic scatter which is 25\% smaller than that of the Tully-Fisher relation for this calibrating sample.
1 INTRODUCTION

In recent years the analysis of redshift surveys of galaxies has made a significant contribution to our emerging understanding of the formation and evolution of large scale structure in the universe. A crucial element in this analysis is the accurate estimation of galaxy distances, and an important feature of many recent surveys has been the availability of redshift independent distance indicators which allow one to determine directly an estimate of the radial peculiar velocity of each galaxy in the survey. By far the most prevalent examples of such distance indicators are the Tully-Fisher and $D_n - \sigma$ relations. These have been used by a number of authors in attempts to reconstruct, from various redshift surveys, the full 3-dimensional peculiar velocity and density contrast fields (c.f. [14], [1], [6], [17]). This work has been at the forefront of a mounting body of evidence in support of galaxy clustering and coherent streaming motions on scales of the order of 100 Mpc; evidence which, nevertheless, has attracted considerable controversy in the literature – not least because of the difficulties which it presents for currently popular theories of structure formation. Much of this debate has focussed upon the statistical properties of the Tully-Fisher and $D_n - \sigma$ relations, and the extent to which detections of galaxy streaming might be a statistical artefact of the distance indicators.

The aim of this paper is to address and clarify several statistical issues relating to the use of redshift independent distance indicators, particularly with respect to the systematic biases which arise in surveys subject to observational selection. These systematic effects have been referred to generically in the literature as ‘Malmquist bias’, although there exists a lack of consensus as to precisely what is meant by this term – and consequently some disagreement over how one should best deal with its effects in analyses of galaxy redshift surveys. In this paper we identify Malmquist bias and examine its effects on redshift independent distance indicators by following the statistical formalism which we adopted previously in this context in [10] (hereafter HS). In particular we examine in what circumstances Malmquist bias may be eliminated completely from redshift independent distance indicators, thus defining what one might regard as an ‘optimal’ galaxy distance estimator. We will also consider the statistical basis of other approaches to Malmquist bias which have been adopted in the literature (c.f. [17], [13]), and clarify the important differences between these approaches and the formalism which we adopt here. In a concurrent paper [18] we examine in detail the consequences of using biased distance indicators for reconstructing the large scale velocity and density fields – particularly with respect to the POTENT method [1], [3].

The Tully-Fisher and $D_n - \sigma$ relations are both derived empirically, by fitting a power law to the relationship between two intrinsic physical characteristics of galaxies: the luminosity and the width of the HI 21cm line of spirals in the case of Tully-Fisher, and the intrinsic diameter and central velocity dispersion of ellipticals in the case of $D_n - \sigma$. Both relations are generally expressed in terms of log quantities, and are thus fitted to be linear in form – e.g. for Tully-Fisher
we have an expression of the form:

\[ M = a \log W + b \]  

(1)

where \( M \) is the absolute magnitude and \( W \) the 21cm line width. The constants \( a \) and \( b \), the slope and zero-point of the relation, are determined empirically – usually with a calibrating sample of reference galaxies the distances of which have been measured independently.\(^1\) To apply the relation one simply measures the line width of a given galaxy, and infers from equation (1) an estimate of its absolute magnitude. This can then be combined with the galaxy’s observed apparent magnitude to obtain an estimate of its distance.

Finding the ‘best’ values of the constants \( a \) and \( b \) has been a thorny issue in the literature for a number of years. The straight line relationship given by equation (1) is generally fitted by performing a linear regression on the calibrating sample. The question of which linear regression is most appropriate is non-trivial, however, particularly when the one’s survey is subject to observational selection effects – a fact which has been widely recognised (c.f. [23], [27], [28], [12], [2]). We can illustrate this with the following simple example. Figure (1) represents schematically the typical scatter of the Tully-Fisher relation, assuming that absolute magnitude and log line width are random variables whose joint distribution is bivariate normal. (More precisely, the ellipse shown in Figure (1) is an isoprobability contour enclosing a given confidence region for magnitude and log line width). The solid and dotted lines indicate the linear relationships obtained by regressing line widths on magnitudes and magnitudes on line widths respectively. Thus the dotted line is the mean, or expected, value of absolute magnitude conditional upon log line width. Conversely the solid line is the expected log line width conditional upon absolute magnitude. Since in practice one wishes to infer the value of \( M \) from the measured line width, the regression of magnitudes on line widths is generally referred to as the ‘direct’ Tully-Fisher relation, while regressing line widths on magnitudes is often termed the ‘inverse’ Tully-Fisher relation. Introducing \( P \) as a shorthand for log line width (c.f. [27], [12], [2]), the following equations define the direct and inverse regression lines for the bivariate normal case:

\[
E(M|P) = M_0 + \rho \frac{\sigma_M}{\sigma_P} (M - M_0) 
\]

(2)

\[
E(P|M) = P_0 + \rho \frac{\sigma_P}{\sigma_M} (P - P_0) 
\]

(3)

where \( M_0, P_0, \sigma_M, \sigma_P \) and \( \rho \) denote the means, dispersions and correlation coefficient of the bivariate normal distribution of magnitudes and log line widths. Note that we have also adopted the standard statistical convention of denoting random variables by bold face characters.

It follows from equations (2) and (3) that both the direct and inverse regression lines can be used to infer an estimate of the absolute magnitude of a given galaxy.

\(^1\)In some analyses of redshift surveys, c.f. [8], the slope and zero point are fitted simultaneously with the parameters of a specific velocity field model, using all of the survey galaxies. We will consider the specific statistical issues raised by this contrasting approach elsewhere.
which is a linear function of its measured log line width, although the constants $a$ and $b$ in equation 1 will clearly be different in each case. Moreover the definition of the estimate of $M$ inferred from each regression line is also subtly different. For the direct regression line the estimate of $M$ is the mean absolute magnitude at the observed log line width – i.e. $E(M|P = P_{\text{obs}})$. For the inverse regression line, on the other hand, the estimated absolute magnitude is the value of $M$ such that the mean log line width conditional upon $M$ is equal to its observed value – i.e. $E(P|M) = P_{\text{obs}}$. Consequently – as indeed is apparent from their slopes – the two lines give rise to markedly different distance estimators.

The situation becomes more complex when we include the effects of observational selection. Figure (2) shows schematically the distribution of $M$ and $P$ for observable galaxies in a sample subject to a sharp cut-off in absolute magnitude – as would be the case for e.g. a more distant cluster observed in an apparent magnitude limited survey. We can see that in this case the expected value of $M$ conditional on $P$ is dramatically different from the direct regression line for the complete sample: in fact $E(M|P)$ is no longer linear in $P$ but curves sharply as $M$ approaches the magnitude limit.

This means that if one calibrates the Tully-Fisher relation in the nearby cluster using the direct regression line, and then applies this relation to estimate the distance of a more distant cluster (or indeed a distant field galaxy), one will systematically underestimate its distance because the expected value of $M$ given $P$ in the more distant sample is systematically brighter than the value predicted by the direct regression line. It is essentially this systematic error or bias in the inferred distance which we identify as ‘Malmquist bias’, although we will define more rigorously what we mean by the bias of a distance estimator in section 2 below. The bias is precisely analogous to the effect identified by [15] in considering the mean absolute magnitude of observable ‘standard candles’ brighter than some given apparent magnitude limit. The effect of Malmquist bias upon the Tully-Fisher relation has been illustrated in a similar manner to Figures (1) and (2) by a number of different authors (c.f. [14], [28], [29]).

For the case where one’s sample is subject only to luminosity selection [23] recognised that the slope of the inverse regression line is unchanged, irrespective of the magnitude-completeness of one’s sample. In other words this regression line is free from the Malmquist effect, and it may therefore be used to provide an unbiased galaxy distance estimate. Although the unbiased property of the ‘Schechter’ inverse regression line has been generally recognised, its ramifications for estimating galaxy distances have not – it would seem to us – been fully appreciated, and the application of Schechter’s ideas to more realistic situations has not been fully explored. Such an extension forms the central aim of this paper. We set out to place the Schechter result on a rigorous statistical footing, following the same formalism previously developed in HS, in order to confirm its range of validity, examine the assumptions upon which it depends and consider to what extent those assumptions may be generalised.

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2Schechter’s original treatment was for the Faber Jackson relation between luminosity and velocity dispersion for ellipticals, although he noted that precisely the same principle held for Tully-Fisher.
To this end, in section 2 we study the properties of a general distance estimator formed from an arbitrary linear combination of apparent magnitude and log line width. It is easy to see that whichever linear regression one adopts the corresponding distance estimator will take this simple linear form, since one will always infer the absolute magnitude of a given galaxy as a linear function of log(line width). Our analysis is carried out in the first instance for the Tully-Fisher case, with the corresponding results for $D_n - \sigma$ indicated where appropriate.

2 PROPERTIES OF GENERAL LINEAR ESTIMATOR

2.1 WHAT DO WE MEAN BY MALMQUIST BIAS?

The approach which we adopt here is a natural extension of the formalism developed in HS to study the properties of distance estimators which are functions of only one observable – apparent magnitude. Before we proceed in earnest we recall from HS a rigorous definition of what we mean by the bias of an estimator, and clarify the differences between this approach and other treatments of Malmquist bias in the literature. The contrasting approaches can be classed as belonging to one of two categories: ‘frequentist’ and ‘Bayesian’. These terms reflect the fact that at the heart of the difference between the two approaches lies the long-standing dichotomy between a Bayesian and frequentist view of the nature of probability.

The frequentist picture is essentially based on the intuitively familiar concept that the probability of an event measures the relative frequency of that event occurring in a large number of repeated experiments or trials. In the limit as the number of trials tends to infinity a histogram of relative frequencies tends to the probability density function (pdf) of a random variable – in this case our galaxy distance estimator. Crucial to the frequentist approach is the idea that the true distance of the galaxy in every trial is a fixed, though of course unknown, parameter – an ‘unknown state of nature’ in the usual statistical terminology.

We can state these ideas more rigorously as follows, taking as an illustration the case of an estimator of log distance since we have seen in section 1 that such an estimator arises naturally from the Tully-Fisher and $D_n - \sigma$ relations. Estimators of log distance will be the focus of our analysis for most of this paper, although similar remarks will clearly also apply to an estimator of distance or any other parameter.

Suppose that $w_0$ is the true log distance of a given galaxy. Let $\hat{w}$ denote an estimator of $w_0$. (Following the standard convention we denote an estimator of a parameter by a caret). Let $p(\hat{w} | w_0)$ denote the pdf of $\hat{w}$, given the true value of $w_0$. One defines $\hat{w}$ to be unbiased if the expected value of $\hat{w}$ is equal to $w_0$. In general the bias, $B$, of $\hat{w}$ at true log distance $w_0$ is given by:

$$B(\hat{w}, w_0) = \int \hat{w} p(\hat{w} | w_0) d\hat{w} - w_0$$

More correctly, an estimator of log distance – a point to which we return presently.
Another important quantity which one can introduce is the mean square error or risk, $R$, of an estimator, defined by:

$$R(\hat{w}, w_0) = \int (\hat{w} - w_0)^2 p(\hat{w}|w_0) d\hat{w}$$

(c.f. eqs. (16) and (17) of HS). Note that for an unbiased estimator, the risk is identically equal to the variance. Note also that both the bias and risk are in general functions of the true log distance, $w_0$. This fact indicates the essential difficulty of completely removing Malmquist bias from galaxy distance estimators: the magnitude of the bias for any given galaxy in general depends upon its true distance, which is unknown.

The definition of the bias of an estimator given by equation 5 differs from that adopted in those treatments of Malmquist bias which we may categorise as Bayesian – most notably the derivation of ‘Malmquist corrections’ in [14] (hereafter LB) and [13] (hereafter LS). In the Bayesian picture one regards the true log distance of the sampled galaxies itself as a random variable, to which one can ascribe some prior probability distribution, $p(w_0)$, based upon an assumed spatial density distribution and selection function. (note that $w_0$ is now written in bold face). Following the measurement of the log distance estimator, $\hat{w}$, for each galaxy one can define a posterior distribution, $p(w_0|\hat{w})$, for $w_0$ conditional upon $\hat{w}$ which will differ from the prior. It is the properties of this posterior distribution which LB and LS consider in defining an estimator as unbiased. By applying Bayes’ theorem one can derive an expression for $p(w_0|\hat{w})$, viz:-

$$p(w_0|\hat{w}) = \frac{p(\hat{w}|w_0)p(w_0)}{\int p(\hat{w}|w_0)p(w_0)dw_0}$$

where the likelihood function, $p(\hat{w}|w_0)$, is simply the pdf of $\hat{w}$ conditional on true log distance $w_0$, as in equation 5 above.

LB and LS define $\hat{w}$ as unbiased if the expected value of $w_0$ with respect to the posterior distribution, $p(w_0|\hat{w})$, is equal to $\hat{w}$. In general the bias of $\hat{w}$ is defined by:-

$$B(\hat{w}, w_0) = \int w_0 p(w_0|\hat{w}) dw_0 - \hat{w}$$

By assuming a posterior distribution and likelihood function LB and LS derive a Malmquist correction to remove the bias of their ‘raw’ log distance estimator (which they denote by $l_a$), so that the corrected estimator is unbiased.

The question of which approach one should take to the definition (to say nothing of the elimination!) of Malmquist bias is far from clear-cut, and depends strongly upon the context in which galaxy distance estimators are being used. For example [15] develop their POTENT distance error analysis from the Bayesian viewpoint and apply Malmquist corrections to their raw distance estimates. They argue that this approach is essential to their analysis due to the nature of the smoothing procedures carried out in POTENT.

We study the effects of biased distance estimators on POTENT in [18], and give a full discussion of the broader statistical issues relating to the merits of the frequentist and Bayesian descriptions elsewhere (c.f. [25], [14]). Although we
concentrate on the frequentist description of Malmquist bias for the remainder of this paper, our results nevertheless have crucial implications for the Bayesian approach. This is because the Malmquist corrections defined in LB and LS are derived on the assumption of a raw log distance estimator, $l_e$, for each galaxy which is normally distributed with mean value equal to the true log distance. If this condition is not met then the Malmquist corrections derived from $l_e$ will not eliminate Malmquist bias [19].

In short, then, LB and LS make the crucial assumption that $l_e$ is unbiased in the frequentist sense, in order to define a corrected estimator which is unbiased in the Bayesian sense. It is this fact which makes our discussion of (frequentist) unbiased estimators in this paper extremely important for both frequentist and Bayesian approaches to Malmquist bias.

Rather than assuming a pdf for our log distance estimator as in LB and LS, in section 2.2 we now derive the pdf in terms of the intrinsic joint distribution of absolute magnitude and line widths, and the observational selection effects.

2.2 THE OBSERVED DISTRIBUTION OF M AND P

Let the absolute magnitude, $M$, and spatial position, $r$, of a galaxy be random variables. Suppose we now introduce a third random variable, $P$, which denotes some intrinsic physical characteristic of the galaxy such that the measured value of $P$ in general provides information on the value of $M$—i.e., $M$ and $P$ are correlated. It is convenient to identify $P$ explicitly as log line width, as we have been doing up until now, although one should bear in mind that the formalism holds more generally for any suitably correlated physical variable.

Suppose next that neither $M$ nor $P$ is correlated with $r$, so that we may meaningfully introduce $\Psi(M, P)$, the intrinsic joint distribution of $M$ and $P$, which is independent of spatial position. Let $N(M, P, r)dMdPdV$ denote the actual number of galaxies in volume element $dV$ at spatial position $r$ with absolute magnitude in the range $M$ to $M + dM$ and log line width in the range $P$ to $P + dP$. It then follows that:

$$N(M, P, r)dMdPdV = \Psi(M, P)n(r)dMdPdV$$

(8)

where $n(r)$ is the number density of galaxies at $r$.

Consider now the joint distribution, $\rho(M, P, r)$, of $M$, $P$, and $r$, for observable galaxies in a sample subject to observational selection effects. We characterise the selection effects by a selection function, $S(M, P, r)$, defined as the probability that a galaxy of absolute magnitude, $M$, and log line width, $P$, at spatial position $r$, would be observable.

An expression for $\rho(M, P, r)$ in terms of $\Psi(M, P)n(r)dMdPdV$ and $n(r)$ now follows easily:

$$\rho(M, P, r) = \frac{\Psi(M, P)n(r)S(M, P, r)}{\int\int\int \Psi(M, P)n(r)S(M, P, r)dMdPdV}$$

(9)

Note that the selection function, $S(M, P, r)$, does not measure the probability that a galaxy would actually be observed: clearly this would depend on the
true local number density of galaxies, \( n(r) \), which will in general be unknown. \( S(M, P, r) \) as defined here will be independent of \( n(r) \) and, moreover, will also be independent of direction provided that one has corrected for the directional dependence of galactic extinction. A number of standard observational methods exist for carrying out these corrections (c.f. [22], [3]).

Because \( S(M, P, r) \) is defined independently of direction it is meaningful to consider the distribution, \( \phi(M, P|w_0) \), of absolute magnitude and log line width for observable galaxies conditional on true distance, \( r_0 \), or equivalently on true log distance, \( w_0 \). It follows from equation 9 that \( \phi(M, P|w_0) \) is given by:

\[
\phi(M, P|w_0) = \frac{\Psi(M, P)S(M, P, w_0)}{\iint \Psi(M, P)S(M, P, w_0)dMdP} \tag{10}
\]

Note that this distribution is independent of the local number density, \( n(r) \), of galaxies. Although this useful property of conditional distributions was pointed out by [20], it would seem that its relevance to Tully-Fisher type relations has not been widely appreciated. The joint distribution of \( M \) and \( P \) at given distance is generally derived on the assumption of a uniform spatial number density (c.f. [27], [2]). We see from equation 10 that such an assumption is in fact unnecessary, and in particular \( \phi(M, P|w_0) \) is identical for both field and cluster galaxies – provided of course that one can assume the intrinsic joint distribution \( \Psi(M, P) \) to be independent of environment.

### 2.3 Bias of General Linear Estimator

Ignoring absorption and cosmological effects, the following equation relates the apparent and absolute magnitudes of a galaxy at given true log distance, \( w_0 \):

\[
m = M + 5w_0 + \kappa \tag{11}
\]

Here \( \kappa \) is a constant which depends upon our units of distance. e.g. if distances are measured in Mpc then \( \kappa = 25 \). If distances are measured in kms\(^{-1}\) by tying the calibration of one’s distance estimator to a cluster at some assumed redshift distance (as is commonly the case in the literature) then \( \kappa = 15 - 5 \log h \).

A sensible form for a general linear estimator, \( \hat{w}_{GL} \), of \( w_0 \) is now clearly given by:

\[
\hat{w}_{GL} = 0.2(m - \hat{M} - \kappa) = 0.2(m - aP - b - \kappa) \tag{12}
\]

where \( \hat{M} = aP + b \), and \( a \) and \( b \) are constants. (c.f. eq. [1] above).

By combining equations [11] and [12] we can determine the joint distribution function of \( m \) and \( P \) for observable galaxies – and from that the pdf of \( \hat{w}_{GL} \) – conditional on \( w_0 \). There is a somewhat more direct route to the same result, however. Substituting equation [11] back into equation [12] and rearranging we obtain:

\[
\hat{w}_{GL} - w_0 = 0.2(M - aP - b) \tag{13}
\]

Equation [13] is of little practical use in defining \( \hat{w}_{GL} \) since both \( M \) and \( w_0 \) are unknown. However an expression for the bias of \( \hat{w}_{GL} \) now follows directly, viz:-

\[
B(\hat{w}_{GL}, w_0) = 0.2 \left(E(M|w_0) - aE(P|w_0) - b \right) \tag{14}
\]
where the expected values of $M$ and $P$ are with respect to the joint distribution function for observable galaxies given by equation 10.

Equation 14 is valid for a completely general selection function of $M$, $P$ and $w_0$. Consequently, the expected values of $M$ and $P$ are both, in general, functions of $w_0$ and it is this fact which makes the complete elimination of Malmquist bias from a linear estimator impossible in the general case: one cannot choose values of the constants $a$ and $b$ which define the distance estimator so that equation 14 is identically zero for all true distances. To make any further progress towards identifying an unbiased distance estimator requires making some assumptions about the nature of the distribution function, $\phi(M,P|w_0)$.

2.4 SCHECHTER’S SOLUTION FOR AN UNBIASED ESTIMATOR

We can rewrite the intrinsic joint distribution function, $\Psi(M,P)$, of $M$ and $P$ as follows:-

$$\Psi(M,P) = \Psi(M)\Psi(P|M)$$

$\Psi(M)$ is just the galaxy luminosity function, well described by e.g. a Schechter function or a gaussian, but regarded as an arbitrary function for the moment. Note that this factorisation does not require any assumption about in which variable lies the scatter in the Tully-Fisher relation, but is valid in the completely general (and more realistic!) case of scatter in both variables. Taking as our lead the approach of [23], suppose we now make the following two crucial assumptions:-

1. the selection function is independent of $P$

2. the conditional expectation of $P$ given $M$ is linear in $M$, i.e.:

$$E(P|M) = \alpha M + \beta$$

where $\alpha$ and $\beta$ are constants, equal to the slope and zero point of the regression line of $P$ upon $M$. With these two assumptions equation 14 reduces to:-

$$B(\hat{w}_{GL}, w_0) = 0.2 \left( (1 - \alpha a)E(M|w_0) - b - a\beta \right)$$

from which one sees that if $a = \alpha^{-1}$ and $b = -\beta\alpha^{-1}$ in equation 17, then the bias of $\hat{w}_{GL}$ is zero for all values of $w_0$. In other words this solution identifies an unbiased log distance estimator, $\hat{w}_I$, viz:-

$$\hat{w}_I = 0.2 \left( m - \alpha^{-1}(P - \beta) - \kappa \right)$$

We use the subscript ‘$I$’ since this unbiased solution corresponds exactly to estimator one obtains from applying the inverse Tully-Fisher relation – i.e. regressing line widths on magnitudes – in complete concordance with Schechter’s result.

To fix these ideas with a specific example, consider again the case where $M$ and $P$ are jointly normally distributed. This case certainly satisfies the assumption that the conditional expectation of $P$ given $M$ be linear in $M$. Comparing
equations 3 and 16 we see that \( \alpha = \rho \sigma_p \) and \( \beta = P_0 - \rho \sigma_p M \), which implies the following expression for the unbiased ‘inverse’ estimator:-

\[
\hat{w}_I = 0.2 \left( m - M_0 - \frac{\sigma_M}{\rho \sigma_p} (P - P_0) - \kappa \right)
\]  

(19)

It is instructive to compare \( \hat{w}_I \) with the ‘direct’ log distance estimator, \( \hat{w}_D \), corresponding to the direct regression of magnitudes on line widths. The values of the constants \( a \) and \( b \) for this case follow from equation 2, and give:-

\[
\hat{w}_D = 0.2 \left( m - M_0 - \frac{\rho \sigma_M}{\sigma_p} (P - P_0) - \kappa \right)
\]  

(20)

which differs from equation 19 only in the switching of the correlation coefficient, \( \rho \), from denominator to numerator, reflecting the different slope of the direct regression line (c.f. Figures (1) and (2). The bias of \( \hat{w}_D \) now follows from equation 17, after a little reduction:-

\[
B(\hat{w}_D, w_0) = 0.2 \left( 1 - \rho^2 \right) \left\{ E(M|w_0) - M_0 \right\}
\]  

(21)

Several points emerge from this equation. Firstly notice that when \( \rho = 0 \), i.e. when \( P \) and \( M \) are uncorrelated, then the bias of \( \hat{w}_D \) reduces to the bias of the ‘naive’ estimator, \( \hat{w}_n \), of log distance defined in HS and [12] by:-

\[
\hat{w}_n = 0.2 \left( m - M_0 - \kappa \right)
\]  

(22)

i.e. assuming that all galaxies are standard candles of absolute magnitude, \( M_0 \), and ignoring the effects of Malmquist bias. This is not surprising, since when \( \rho = 0 \) the measured log line width provides no additional information about the value of \( M \). The second point to note is that as \( |\rho| \) tends to unity, on the other hand, the bias of \( \hat{w}_D \) tends to zero at all true distances. Again this follows automatically from the fact that as \( |\rho| \to 1 \) the direct and inverse regression lines become collinear, and \( \hat{w}_D \) and \( \hat{w}_I \) are identical. Lastly note that if there are no magnitude selection effects then \( \hat{w}_D \) is again unbiased at all true distances, simply because we then have \( E(M|w_0) = M_0 \) for all \( w_0 \). It is easy to see that this result is true for an arbitrary joint intrinsic distribution function, \( \Psi(M, P) \), in the absence of selection effects.

Finally we consider the risk and the higher moments of the \( \hat{w}_{GL} \) distribution. We can do this most easily by introducing a new random variable \( t = P - (\alpha M + \beta) \). This allows us to rewrite equation 13 as follows:-

\[
\hat{w}_{GL} - w_0 = 0.2 \left[ (1 - \alpha A)M - B - \beta A - At \right]
\]  

(23)

For the unbiased inverse estimator we see that all but the final term of the right hand side vanishes. It follows immediately from this that the moments of \( \hat{w}_I - w_0 \) are equal simply to a constant multiple of the moments of, \( t \), independent of the true log distance! Moreover, since we are assuming that \( E(P|M) = \alpha M + \beta \), it follows that the probability distribution of \( \hat{w}_I \) is identical in shape to the intrinsic conditional distribution, \( \Psi(P|M) \). This latter distribution is generally modelled
to be gaussian (c.f. [27], [2]), thus implying that the inverse estimator is normally
distributed, unbiased, and of constant variance at all true log distances.

As we recalled in section 2.2, these are precisely the properties assumed for
the raw log distance estimator, \( l_e \), in LB and LS. Our results confirm, therefore,
that \( \hat{\omega}_I \) is the correct raw log distance estimator to use in defining Malmquist
corrections.

It follows from equation 23, on the other hand, that \( \hat{\omega}_D \) will not be normally
distributed for all \( w_0 \), and in fact will lead to incorrect Malmquist corrections if
these are derived on the assumption of a normal raw estimator. Notwithstanding
this important result, to our knowledge a direct linear regression has been
used exclusively to date in the derivation both homogeneous and inhomogeneous
Malmquist corrections in the literature (c.f. [14], [1], [6], [4]). We examine the
consequences of this incorrect choice of raw distance estimator in [18] and [19].

2.5 PROPERTIES OF THE UNBIASED ‘INVERSE’ ESTIMATOR

It is instructive to summarise the properties of the inverse estimator, \( \hat{\omega}_I \), which
we have thus far confirmed or determined, and add several further results which
follow easily from them.

1. In a sample subject to observational selection effects, provided that the
measurements of line width are selection-free and the conditional expectation of line width at given absolute magnitude is linear in \( M \), then it is possible to define a general linear estimator of log distance which is unbiased at all true distances, and the appropriate linear combination corresponds exactly to the estimator derived from a regression of (log) line widths upon magnitudes, as prescribed in [23]. This result is valid in the general case where one accounts for intrinsic and observational scatter in both variables, and does not require the assumption that the scatter lies only in line widths.

2. The ‘inverse’ estimator thus defined is the only unbiased linear estimator
of log distance. Any other linear combination of magnitude and log line
width, and in particular any other regression line, yields an estimator which
is biased at all true distances for a magnitude selection function. Examples
of biased regression lines in this case include, therefore, not only the direct regression used by e.g. [14] (in its equivalent form for the \( D_n - \sigma \) relation), but also the orthogonal regression (accounting for residuals on both observables – c.f. [7]); ‘bisector’ regression (i.e. the line which bisects the direct and inverse regression lines – c.f. [21]) and mean (i.e. the line whose slope is the arithmetic mean of the direct and inverse lines – c.f. [17]) regression lines.

3. The shape of the pdf, and hence in particular the risk (or equivalently variance), of the inverse estimator is constant at all true distances. It follows from this property that confidence intervals derived from the inverse estimator, following the method outlined in HS, are of constant width. For
any other general linear estimator, on the other hand, the shape of the pdf is severely distorted at large true distances as luminosity selection effects become significant.

4. The pdf of the inverse estimator is, in fact, identical in shape to the intrinsic pdf of log line width conditional upon absolute magnitude. If the latter distribution is normal and of constant variance, as is commonly assumed, then so too will be the pdf of the inverse estimator. It is therefore the correct choice of ‘raw’ log distance estimator for the derivation of Malmquist corrections.

5. The unbiased property of the inverse estimator is true for an arbitrary luminosity function and magnitude selection function, and is independent of the true number density distribution of galaxies. This is a particularly useful property, since it follows from equations 10 and 14 that the bias of any other linear estimator will depend explicitly upon the form of the luminosity function and magnitude selection effects, so that any attempt to correct for or reduce the bias would necessarily be model dependent. Indeed, [4] shows that the magnitude of the bias of the direct regression line is substantially different for gaussian and Schechter luminosity functions.

6. One may also define an unbiased log distance estimator for other distance indicators, including the $D_n - \sigma$ and magnitude-colour relations, subject to the same condition that there be one observable free from selection, but not requiring one observable to be distance-independent. In a diameter-complete survey, for example, one may construct an unbiased distance estimator from the observed angular diameter and apparent magnitude. As above, it is straightforward to show that this unbiased estimator corresponds exactly to the regression of the selection-free observable upon the other observable.

7. The inverse estimator is an unbiased estimator of log distance: consequently the corresponding distance estimator is biased. It is a simple matter, however, to define a corresponding unbiased distance estimator, particularly in the case where $\hat{w}_i$ is normally distributed (c.f. 14).

2.6 UNBIASED ESTIMATORS IN MORE REALISTIC CASES

Although we have striven to show in this paper that the definition of unbiased estimators following the prescription of [23] rests on few assumptions and is otherwise a very general result, one must nevertheless accept that even these modest assumptions may not be met in most practical situations. In particular, if neither observable is free from selection effects then an unbiased estimator formally cannot be defined as a simple linear combination of the observables.

In the context of both the Tully-Fisher and $D_n - \sigma$ relations, however, the problem of $P$ selection is somewhat less important than one might expect. Most surveys will be subject to a lower selection limit on line width or velocity dispersion: e.g. it will not be possible to measure accurately velocity dispersions of
the order of 150\,kms$^{-1}$ [14]. The interesting – and very useful – property of this selection limit is, however, that it becomes increasingly less important at larger distances. This is easy to understand, since at large distances only intrinsically brighter (or larger), and thus sufficiently large line width, galaxies will be observable. In other words, at large distances the galaxies which are ‘lost’ to the survey due to their small velocity widths would have been unobservable in any case, owing to their faint luminosity.

As an illustrative example, Figure (3) shows the bias of the inverse log distance estimator derived from the combined Virgo and Ursa Major calibrating sample of [21], and assuming a sharp I-band magnitude limit at $I_{\text{lim}} = 14$. The curves show the bias of $\hat{w}_i$ as a function of true distance (expressed in kms$^{-1}$) for three different line width selection limits. Note that the effect of $P$ selection is to introduce a positive bias – in contrast to the negative Malmquist bias caused by an upper limit on observable apparent magnitude. The effect is clearly very small, however. A bias of 0.01 in $\hat{w}_i$ corresponds to a systematic distance error of $\sim 2\%$. Hence, one sees that the effect of a line width limit as large as $P_{\text{lim}} = 200$ kms$^{-1}$ is negligible, and even with a limit of $P_{\text{lim}} = 250$ kms$^{-1}$ the effect can still be ignored at cosmologically interesting distances in this case.

In the event that selection effects on $P$ are large enough to be significant – or, for example, if the line width selection cannot be well described by a sharp limit, independent of distance and morphological type – one can adopt an iterative method to reduce Malmquist bias – although such an approach will necessarily be model dependent. We discussed this method in HS, for the case of an estimator which is a function of apparent magnitude only – so that Schechter’s ideas are inapplicable. The extension to estimators of Tully-Fisher type is straightforward, however. Let $\hat{w}(\mathbf{m}, P)$ denote an estimator of log distance as before. Rearranging equation 4 observe that we may write:-

$$E(\hat{w}(\mathbf{m}, P)|w_0) = w_0 + B(\hat{w}_i, w_0)$$  \hspace{1cm} (24)

This is essentially equation (19) of HS, in the equivalent form for an estimator of log distance.

Although we cannot use equation 24 to remove the bias of $\hat{w}(\mathbf{m}, P)$ exactly, since the true log distance $w_0$ is unknown, suppose we form a new estimator, $\hat{w}_1(\mathbf{m}, P)$, defined by:-

$$\hat{w}_1(\mathbf{m}, P) = \hat{w}(\mathbf{m}, P) - B(\hat{w}(\mathbf{m}, P), w_0 = \hat{w}(\mathbf{m}, P))$$  \hspace{1cm} (25)

In other words for each $\mathbf{m}$ and $P$ we subtract from $\hat{w}(\mathbf{m}, P)$ the bias of the estimator assuming that the true log distance is equal to its estimated value. (c.f. eq. (20) of HS). One can then compute the bias of the new estimator, $\hat{w}_1(\mathbf{m}, P)$, apply equation 25 again to define $\hat{w}_2(\mathbf{m}, P)$ in terms of $\hat{w}_1$, and so on.

It is not obvious that the above iterative scheme will in all cases converge to an unbiased estimator. In fact we have shown [12] that this is not the case for estimators which are functions of apparent magnitude only. Numerical studies indicate that convergence is achieved for the Tully-Fisher case with selection on both observables, however, provided that the scatter in the intrinsic joint distribution of $\mathbf{M}$ and $P$ is not too large.
Perhaps a more serious problem in defining unbiased distance estimators lies in the calibration of the distance relation itself. In order to define the inverse estimator (or indeed the direct estimator), one must determine the parameters of the joint distribution of $M$ and $P$ – e.g. the five parameters $M_0$, $P_0$, $\sigma_M$, $\sigma_P$, and $\rho$ in the bivariate normal case. It is obviously of great importance, therefore, to ensure that the estimates of these parameters obtained from one’s calibrating sample accurately reflect their true intrinsic values. It has been suggested (c.f. [26]) that the scatter measured in distance relations underestimates the true scatter – leading one to suppose a less serious contribution from Malmquist bias – simply because the number of calibrating galaxies is insufficient to accurately determine the slope and zero point of the relation.

We have addressed this question in some quantitative detail, carrying out numerical experiments on artificial cluster samples of a range of different sizes and true parameters, in order to determine how many calibrators are required to achieve a given level of accuracy in the fitted Tully-Fisher slope. As an illustration, Figure (4) shows the results of Monte Carlo simulations carried out assuming a bivariate normal model for the distribution of $M$ and $P$ and adopting as true parameter values those given by the Fornax cluster used in the calibration of the Mathewson galaxy survey (c.f. [16]). The bold and dotted lines show the true inverse and direct regression line slopes respectively, while the two curves show $1\sigma$ confidence limits for the estimated inverse regression line slope as a function of the number of galaxies in the calibrating sample.

One can see from Figure (4) that for calibrating samples containing less than $\sim 40$ galaxies, the dispersion of the estimated slope of the inverse regression line is greater than the difference between the true slopes of the inverse and direct regression lines. Hence one requires a calibrating sample of over 40 galaxies in order that the scatter in the slope of the inverse regression line due to sampling error be smaller than the difference between the slopes of the two lines.

Putting this another way, with a considerably smaller sample of calibrators there is a strong possibility that the bias in the (supposedly unbiased!) inverse estimator due to incorrect determination of the estimator slope will be larger than the Malmquist bias of the direct estimator.

Clearly, then, it is important to use as large a calibrating sample as possible to minimise this problem. One solution is to combine data from several different clusters, as in [21] and [17], combining two samples from the Virgo and Ursa Major clusters, whose distance moduli have been found to be equal. [4]) discuss two different methods of tackling the problem of combining calibration data from clusters at different distances, and obtaining optimal estimates of the slope and zero point of the distance relation simultaneously with relative distances to each cluster.

Of course another way in which the problems of sampling error can be reduced is by identifying distance relations of intrinsically smaller scatter. In section 3 we consider how one might achieve this by defining estimators which are functions of more than two observables.
3 ESTIMATORS OF DISTANCE USING THREE OR MORE OBSERVABLES

In this section we briefly discuss the properties of distance estimators which are defined as a function of apparent magnitude and two other observable quantities, such as one might define in extending the Tully-Fisher relation to include the observed angular diameter of spiral galaxies. Of particular interest is the question of whether one may still define unbiased estimators in this case, analogous to the P on M estimator of the previous section, and if so whether one may construct unbiased estimators which have a smaller risk than their two-variable counterparts.

One can carry out an analysis which follows closely the formulation adopted in section 2; i.e. first one derives the joint distribution at given true log distance, $w_0$, after accounting for observational selection effects, of the random variables – $M$, $P$ and $D$ say, denoting for example absolute magnitude, log line width and log of absolute diameter – in terms of their intrinsic joint distribution and selection function to obtain an expression analogous to equation 10, viz:

$$\phi(M, P, D | w_0) = \frac{\Psi(M, P, D) S(M, P, D, w_0)}{\iiint \Psi(M, P, D) S(M, P, D, w_0) dMdPdD} \quad (26)$$

One can then determine, for a general linear combination of the observables, the distribution, bias and risk of this ‘general linear’ estimator and, as before, identify for which values the estimator is unbiased. The details of these calculations are somewhat tedious and add little to the previous analysis for two variables. We present, therefore, a summary of the main results for the 3-variable case.

We considered two cases: firstly where only one of the three observables is free from observational selection, and secondly where two observables are selection-free. In both cases it was possible to define an unbiased estimator of log distance by appropriate linear combination of the observables. The values of the coefficients corresponding to the unbiased solution were given in terms of the parameters of the intrinsic distribution function, $\Psi(M, P, D)$, as in the two variable case. To take a specific example, if $M$, $P$ and $D$ were jointly normally distributed, then the coefficients depend solely upon the mean values, dispersions and correlation coefficients of the trivariate normal distribution.

In the first case where only one observable is selection-free, we found that an unbiased estimator can, in general, be defined only as a linear combination of all three observables. This has important consequences for our earlier results. In the case of the Tully-Fisher relation, for example, if one’s sample is subject to both diameter and magnitude selection then the inverse estimator defined in section 2 using only apparent magnitude and log line width will no longer be unbiased. This is because the selection on diameters affects the joint distribution of $m$ and $P$, since the galaxy diameter is correlated with these variables. A similar effect is discussed in [17], where selection on diameter and surface brightness ‘pollutes’ the distribution of $m$ and $P$ and affects the bias of the Tully-Fisher relation. Clearly, therefore, great care must be taken in ensuring no additional observables introduce selection ‘by proxy’ into one’s samples. The fact that an
observable does not appear in the definition of one’s distance estimator does not imply that it can have no effect on the bias of that estimator.

In the second case, where two observables are free from selection, a rather different picture emerges. Taking again the example of magnitude, line width and diameter to fix ideas, we found that in this case the inverse estimator defined in section 2 is still unbiased at all true distances, so that Schechter’s prescription is still valid. The inverse estimator is, however, no longer the only unbiased estimator of log distance - although it is still the only unbiased estimator formed from a linear combination of magnitude and line width alone. By forming an estimator from three observables, we have sufficient freedom to define an unbiased estimator of minimum variance, and one may show that the variance of this optimal 3-variable estimator is always less than or equal to that of the inverse estimator defined by magnitude and line width alone.

The precise factor, \( \Delta \), by which the addition of a third observable, \( D \), reduces the variance of the inverse estimator depends only upon the values of the correlation coefficients between the three observables (c.f. \([12]\)). As an illustration, consider the specific case where \( M \), \( P \) and \( D \) are jointly normally distributed, with correlation coefficients denoted by \( \rho_{MP} \), \( \rho_{MD} \), and \( \rho_{PD} \). In this case \( \Delta \) is given by the following expression:

\[
\Delta = \frac{\rho_{MP}^2}{(1 - \rho_{MP}^2)[\rho_{MP}^2 - 2\rho_{MP}\rho_{MD}\rho_{PD} + \rho_{MD}^2]}
\]  

Figures (5), (6) and (7) show respectively scatter diagrams for the I-band magnitude versus log line width, magnitude versus log diameter and log diameter versus log line width relations for the Fornax cluster, determined from the Mathewson galaxy redshift survey. It is clear from these figures that a very good correlation exists between all three observables, and the correlation coefficients for this calibrating sample were found to be \( \rho_{MP} = -0.985 \), \( \rho_{MD} = -0.963 \) and \( \rho_{PD} = 0.942 \). Notwithstanding the fact that the Fornax cluster is a rather small calibrating sample, in the light of our remarks in section 2.6, if we assume these correlation coefficients to be equal to the intrinsic values for the magnitude – diameter – line width relation then substituting in equation (27) gives a value of \( \Delta = 0.64 \). In other words the variance of the 3 variable estimator is more than 35% smaller than that of the corresponding P on M estimator. This corresponds to a reduction in the mean distance error dispersion from \( \sim 20\% \) to around 15%.

It would seem clear, therefore, that utilising the measurements of a third observable can offer a means of significantly reducing the dispersion of unbiased distance estimators, and thus obtaining more reliable distance estimates. When such an observable is available – as is the case in the above example of the magnitude – diameter – linewidth relation, its use would seem to be strongly advised.

4 CONCLUSIONS

In this paper we have studied the properties of galaxy distance estimators derived from combining measurements of two or more observables, as is the case
for the Tully-Fisher and $D_n - \sigma$ relations. We have considered the effects of observational selection upon the distribution, bias and risk of these estimators and have established that, subject to modest but crucial assumptions, it is possible to define estimators which are unbiased at all true distances, in confirmation of the results of [23]. We have shown that these results are more general than is often assumed in the literature: in particular, that one can define unbiased distance estimators independently of the form of the magnitude selection function and the local number density of galaxies, and almost independently of the intrinsic joint distribution of magnitude and line width. Moreover, the results are derived in the general case of observational and intrinsic scatter on both correlated variables.

We have compared our treatment of Malmquist bias with other approaches which have been adopted in the literature, and shown how the differences between them can be understood as fundamentally different interpretations of the nature of probability. Moreover, we have shown that when the distribution of log line widths conditional on magnitudes is normal, then so too is the pdf of the unbiased inverse estimator. It is therefore the only appropriate choice of raw log distance estimator which is consistent with the assumptions made in deriving homogeneous and inhomogeneous Malmquist corrections in the literature.

Finally, we have also considered how one can define unbiased estimators of smaller variance by utilising additional, suitably correlated, observables. In future work we will apply these multivariate estimators to the analysis of real galaxy surveys, in order to extend and improve the optimal techniques for smoothing and recovery of the peculiar velocity field described in [18] and [24].

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References

[1] Bertschinger, E., Dekel, A., Faber, S., Dressler, A., Burstein, D., 1990, ApJ, 364, 370
[2] Bicknell, G., 1992, ApJ, 399, 1
[3] Burstein, D., Heiles, C., 1982, AJ, 87, 1165
[4] Courteau, S. M., 1993, in Bouchet, F., Lachieze-Rey, M., eds, Cosmic Velocity Fields, Editions Frontieres, in press
[5] Dekel, A., Bertschinger, E., Faber, S., 1990, ApJ, 364, 349
[6] Dekel, A., Bertschinger, E., Yahil, A., Strauss, M. A., Davis, M., Huchra, J. P., 1993, ApJ, 412, 1
[7] Giraud, E., 1987, A&A, 174, 23
[8] Han, M., Mould, J., 1990, ApJ, 360, 448
[9] Hendry, M. A., Lattimer, T. R. B., 1993, in preparation
[10] Hendry, M. A., Simmons, J. F. L., 1990, A&A, 237, 275
[11] Hendry, M. A., Simmons, J. F. L., Newsam, A. M., 1993, in preparation
[12] Hendry, M. A., 1992, PhD thesis, University of Glasgow, UK
[13] Landy, S., Szalay, A., 1992, ApJ, 391, 494
[14] Lynden-Bell, D., Faber, S. M., Burstein, D., Davies, R., Dressler, A., Terlevich, R., Wegner, G., 1988, ApJ, 326, 19
[15] Malmquist, K., 1920, Medd Lund Astr Obs, 20, 1
[16] Mathewson, D., Ford, V., Buchhorn, M., 1992, ApJ. (supp.), 81, 413
[17] Mould, J., Akeson, R., Bothun, G., Han, M., Huchra, J., Roth, J., Schommer, R., 1993, ApJ, 409, 14
[18] Newsam, A. M., Simmons, J. F. L., Hendry, M. A., 1993a, preprint
[19] Newsam, A. M., Simmons, J. F. L., Hendry, M. A., 1993b, in Bouchet, F., Lachieze-Rey, M., eds, Cosmic Velocity Fields, Editions Frontieres, in press
[20] Neyman, J., Scott, P., 1952, ApJ, 412, 1
[21] Pierce, M., Tully, R., 1988, ApJ, 330, 579
[22] Sandage, A., Tammann, G., 1981, Revised Shapely-Ames Catalog of Bright Galaxies. Carnegie Institution, Washington
[23] Schechter, P., 1980, AJ, 85, 801
[24] Simmons, J. F. L., Newsam, A. M., Hendry, M. A., 1993, preprint
[25] Simmons, J. F. L., 1993, apj, submitted
[26] Tammann, G., 1987, in Hewitt, A., ed, Observational Cosmology - Proceedings of IAU Colloquium 130. D. Reidel, Dordrecht-Holland, p. 326
[27] Teerikorpi, P., 1984, A&A, 141, 407
[28] Tully, R., 1988, Nature, 334, 209
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Figure 1. Schematic Tully-Fisher relations, derived by applying a direct and inverse linear regression to a complete calibrating sample – e.g. a nearby cluster.

Figure 2. The expected value of absolute magnitude conditional upon log line width, and log line width conditional upon magnitude, in a distribution subject to a sharp selection limit on absolute magnitude – e.g. a distant cluster. The shaded region represents unobservable galaxies.

Figure 3. Bias of the inverse estimator with line width selection effects, assuming a bivariate normal distribution for M and P with distribution parameters taken from the Virgo and Ursa Major composite calibrating sample of [21].

Figure 4. 1σ confidence limits for the sample estimate of the slope of the inverse regression line as a function of the number of galaxies in the calibrating sample. Distribution parameters are taken from the Fornax cluster – as determined in the Mathewson galaxy survey.

Figure 5. Scatter plot of the Tully-Fisher, I-band magnitude versus log line width, relation for the Fornax cluster, derived from the Mathewson redshift survey.

Figure 6. Scatter plot of the I-band magnitude versus log diameter relation for the Fornax cluster, derived from the Mathewson redshift survey.

Figure 7. Scatter plot of the log diameter versus log line width relation for the Fornax cluster, derived from the Mathewson redshift survey.