FINITE TYPE INVARIANTS AND MILNOR INVARIANTS
FOR BRUNNIAN LINKS

KAZUO HABIRO AND JEAN-BAPTISTE MEILHAN

Abstract. A link \( L \) in the 3-sphere is called \( \)Brunnian if every proper sublink of \( L \) is trivial. In a previous paper, the first author proved that the restriction to Brunnian links of any Goussarov-Vassiliev finite type invariant of \((n+1)\)-component links of degree \(<2n\) is trivial. The purpose of this paper is to study the first nontrivial case. We show that the restriction of an invariant of degree \(2n\) to \((n+1)\)-component Brunnian links can be expressed as a quadratic form on the Milnor link-homotopy invariants of length \(n+1\).

1. Introduction

The notion of Goussarov-Vassiliev finite type link invariants \([7, 8, 28]\) enables us to understand the various quantum invariants from a unifying viewpoint, see e.g. \([1, 26]\). The theory involves a descending filtration \( \mathbb{Z}L(m) = J_0(m) \supset J_1(m) \supset \ldots \) of the free abelian group \( \mathbb{Z}L(m) \) generated by the set \( L(m) \) of the ambient isotopy classes of \( m \)-component, oriented, ordered links in \( S^3 \). Here each \( J_n(m) \) is generated by alternating sums of links over \( n \) independent crossing changes. A homomorphism from \( \mathbb{Z}L(m) \) to an abelian group \( A \) is said to be a Goussarov-Vassiliev invariant of degree \( n \) if it vanishes on \( J_{n+1}(m) \). Thus, for \( L, L' \in L(m) \), we have \( L - L' \in J_n(m) \) if and only if \( L \) and \( L' \) have the same values of Goussarov-Vassiliev invariants of degree \( \leq n \) with values in any abelian group.

It is natural to ask what kind of informations a Goussarov-Vassiliev link invariants can contain and what is the topological meaning of the unitrivalent diagrams. Calculus of claspers, introduced by Goussarov and the first author \([9,10,15]\), answers these questions. (We will recall the definition of claspers in Section 2.) A special type of claspers, called graph claspers, can be regarded as topological realizations of unitrivalent diagrams. For knots, claspers enables us to give a complete topological characterization of the informations that can be contained by Goussarov-Vassiliev invariants of degree \(<n\) \([10,15]\): The difference of two knots is in \( J_n \) if and only if these two knots are \( C_n \)-equivalent. Here \( C_n \)-equivalence is generated by a certain type of local moves, called \( C_n \)-moves (called \((n-1)\)-variations by Goussarov), which is defined as surgeries along certain tree claspers.

For links with more than 1 components, the above-mentioned properties of Goussarov-Vassiliev invariants does not hold. It is true that if \( L, L' \in L(m) \) are \( C_n \)-equivalent, then we have \( L - L' \in J_n(m) \), but the converse does not hold in general. A counterexample is Milnor’s link \( L_{n+1} \) of \( n+1 \) components depicted in Figure 1.
If \( n \geq 2 \), \( L_n \) is \( C_n \)-equivalent but not \( C_{n+1} \)-equivalent to the \((n+1)\)-component unlink \( U \), while we have \( L_{n+1} - U \in J_{2n}(n+1) \) (but \( L_{n+1} - U \notin J_{2n+1}(n+1) \)), see [15, Proposition 7.4]. (This fact is contrasting to the case of string links: Conjecturally [15, Conjecture 6.13], two string links \( L, L' \) of the same number of components are \( C_n \)-equivalent if and only if \( L - L' \in J_n \).

Milnor’s links are typical examples of Brunnian links. Recall that a link in an oriented, connected 3-manifold is said to be Brunnian if every proper sublink of it is an unlink. In some sense, an \( n \)-component Brunnian link is a ‘pure \( n \)-component linking’. Thus studying the behavior of Goussarov-Vassiliev invariants on Brunnian links would be a first step in understanding the Goussarov-Vassiliev invariants for links.

The first author generalized a part of the above-mentioned properties of Milnor’s links to Brunnian links:

**Theorem 1.1** ([16]). Let \( L \) be an \((n+1)\)-component Brunnian link in a connected, oriented 3-manifold \( M \) (\( n \geq 1 \)), and let \( U \) be an \((n+1)\)-component unlink in \( M \). Then we have the following.

1. \( L \) and \( U \) are \( C_n \)-equivalent.
2. If \( n \geq 2 \), then we have \( L - U \in J_{2n}(n+1) \). Hence \( L \) and \( U \) are not distinguished by any Goussarov-Vassiliev invariants of degree \( < 2n \).

The case \( M = S^3 \) of Theorem 1.1 was announced in [15], and was later proved also by Miyazawa and Yasuhara [24], independently to [16].

The purpose of the present paper is to study the restrictions of Goussarov-Vassiliev invariants of degree \( 2n \) to \((n+1)\)-component Brunnian links in \( S^3 \), which is the first nontrivial case according to Theorem 1.1. The main result in the present paper expresses any such restriction as a quadratic form of Milnor \( \bar{\mu} \) link-homotopy invariants of length \( n + 1 \):

**Theorem 1.2.** Let \( f \) be any \( \mathbb{Z} \)-valued Goussarov-Vassiliev link invariant of degree \( 2n \). Then there are (non-unique) integers \( f_{\sigma,\sigma'} \) for elements \( \sigma, \sigma' \) of the symmetric group \( S_{n-1} \) on the set \( \{1, \ldots, n-1\} \) such that, for any \((n+1)\)-component Brunnian link \( L \), we have

\[
f(L) - f(U) = \sum_{\sigma, \sigma' \in S_{n-1}} f_{\sigma,\sigma'} \bar{\mu}_\sigma(L) \bar{\mu}_{\sigma'}(L).
\]

Here, \( U \) is an \((n+1)\)-component unlink, and we set, for \( \sigma \in S_{n-1} \),

\[
\bar{\mu}_\sigma(L) = \bar{\mu}_{\sigma(1), \sigma(2), \ldots, \sigma(n-1), n, n+1}(L) \in \mathbb{Z}.
\]

A possible choice for the integers \( f_{\sigma,\sigma'} \) is given in terms of tree claspers in Section 7.3.

**Remark 1.3.** The proof of Theorem 1.2 involves calculus of claspers. The first preprint version of the present paper ([arxiv:math.GT/0510534v1]) contained a one-page sketch of an alternative proof of Theorem 1.2 using the Kontsevich integral. This alternative proof has been separated from the present paper, and has been published in [17]. Though shorter than the clasper-based proof below, the proof in
[17] relies heavily on the properties of unitrivalent diagrams, and the topological meaning of the steps in the proof are therefore not always very clear. The present proof gives a better understanding of Theorem 1.2 from that point of view.

Recall that Milnor invariants of length $n+1$ for string links are Goussarov-Vassiliev invariants of degree $\leq n$ [2, 13] (see also [14]). As is well-known, Milnor’s invariants is not well-defined for all links, and hence it does not make sense to ask whether Milnor invariants of length $n+1$ is of degree $\leq n$ or not. However, as Theorem 1.2 indicates, a quadratic expression in such Milnor invariants, which is well-defined at least for $(n+1)$-component Brunnian links, may extend to a link invariant of degree $\leq 2n$.

In the study of Milnor’s invariants, tree claspers seem at least as useful as Cochran’s construction [3]. For the use of claspers in the study of the Milnor invariants, see also [6, 11, 22]. For other relationships between finite type invariants and the Milnor invariants, see [2, 19, 14, 13, 20].

We organize the rest of the paper as follows.

In Section 2, we recall some definitions from clasper calculus.

In Section 3, we recall the notion of $C^n_k$-equivalence for links, studied in [16]. If a link $L$ is $C^n_k$-equivalent (for any $k$) to a Brunnian link, then $L$ also is a Brunnian link.

In Section 4, we study the group $\overline{BSL}_{n+1}$ of $C^n_{n+1}$-equivalence classes of $(n+1)$-component string links. We establish an isomorphism

$$\theta_n: T_{n+1} \cong BSL_{n+1}$$

from an abelian group $T_{n+1}$ of certain tree diagrams. This map is essentially the inverse to the Milnor link-homotopy invariants of length $n+1$.

In Section 5, we apply the results in Section 4 to Brunnian links. The operation of closing string links induces a bijection

$$\bar{c}_{n+1}: BSL_{n+1} \cong B_{n+1},$$

where $B_{n+1}$ is the set of $C^n_{n+1}$-equivalence classes of $(n+1)$-component Brunnian links. As a byproduct, we obtain another proof of a result of Miyazawa and Yasuhara [24].

In Section 6, we recall the definition of the Goussarov-Vassiliev filtration for links using claspers.

In Section 7, we study the behavior of Goussarov-Vassiliev invariants of degree $2n$ for $(n+1)$-component Brunnian links. We first show that two $C^n_{n+1}$-equivalent, $(n+1)$-component Brunnian links cannot be distinguished by Goussarov-Vassiliev invariants of degree $2n$. We have a quadratic map

$$\kappa_{n+1}: B_{n+1} \rightarrow \overline{J}_{2n}(n+1)$$

defined by $\kappa_{n+1}([L]_{C^n_{n+1}}) = [L - U]_{J_{2n+1}}$. We prove Theorem 1.2 using $\kappa_{n+1}$.

Acknowledgments. The authors wish to thank Akira Yasuhara for helpful conversations.

2. Claspers

In this section, we recall some definitions from calculus of claspers. For the details, we refer the reader to [15].

A clasper in an oriented 3-manifold $M$ is a compact, possibly unorientable, embedded surface $G$ in $\text{int} M$ equipped with a decomposition into connected subsurfaces called leaves, disk-leaves, nodes, boxes, and edges. Two distinct non-edge subsurfaces are disjoint. Edges are disjoint bands which connect two subsurfaces
of the other types. A connected component of the intersection of one edge $E$ and another subsurface $F$ (of different type), which is an arc in $\partial E \cap \partial F$, is called an attaching region of $F$.

- A leaf is an annulus with one attaching region.
- A disk-leaf is a disk with one attaching region.
- A node is a disk with three attaching regions. (Usually, a node is incident to three edges, but it is allowed that the two ends of one edge are attached to a node.)
- A box is a disk with three attaching regions. (The same remark as that for node applies here, too.) Moreover, one attaching region is distinguished with the other two. (This distinction is done by drawing a box as a rectangle, see [15].)

A clasper $G$ for a link $L$ in $M$ is a clasper in $M$ such that the intersection $G \cap L$ consists of finitely many transverse double points and is contained in the interior of the union of disk-leaves.

We often use the drawing convention for claspsers as described in [15].

Surgery along a clasper $G$ is defined to be surgery along the associated framed link $L_G$ to $G$. Here $L_G$ is obtained from $G$ by the rules described in Figure 2.1.

A tree clasper is a connected clasper $T$ without boxes, such that the union of edges and nodes of $T$ is simply connected. A tree clasper $T$ is called strict if each component of $T$ has no leaves and at least one disk-leaf. Surgery along a strict tree clasper $T$ is tame in the sense of [15, Section 2.3], i.e., the result of surgery along $T$ preserves the 3-manifold and the surgery may be regarded as a move on a link.

A tree clasper $T$ for a link $L$ is simple (with respect to $L$) if each disk-leaf of $T$ has exactly one intersection point with $L$.

The degree of a strict tree clasper $G$ is defined to be the number of nodes of $T$ plus 1. For $n \geq 1$, a $C_n$-tree is a strict tree clasper of degree $n$. A (simple) $C_n$-move is a local move on links defined as surgery along a (simple) $C_n$-tree. For example, a simple $C_1$-move is a crossing change, and a simple $C_2$-move is a delta move [21, 23]. The $C_n$-equivalence is the equivalence relation on links generated by $C_n$-moves. This equivalence relation is also generated by simple $C_n$-moves. The $C_n$-equivalence becomes finer as $n$ increases.
3. \( C_k^n \)-equivalence

We recall from [16] the definition of the \( C_k^n \)-equivalence.

**Definition 3.1.** Let \( L \) be an \( m \)-component link in a 3-manifold \( M \). For \( k \geq m - 1 \), a \( C_k^n \)-tree for \( L \) in \( M \) is a \( C_k \)-tree \( T \) for \( L \) in \( M \), such that

1. for each disk-leaf \( A \) of \( T \), all the strands intersecting \( A \) are contained in one component of \( L \), and
2. each component of \( L \) intersects at least one disk-leaf of \( T \), i.e., \( T \) intersects all the components of \( L \).

Note that the condition (1) is vacuous if \( T \) is simple.

A \( C_k^n \)-move on a link is surgery along a \( C_k^n \)-tree. The \( C_k^n \)-equivalence is the equivalence relation on links generated by \( C_k^n \)-moves. A \( C_k^n \)-forest is a clasper consisting only of \( C_k^n \)-trees.

Clearly, the above notions are defined also for tangles, particularly for string links.

What makes the notion of \( C_k^n \)-equivalence useful in the study of Brunnian links is the fact that a link which is \( C_k^n \)-equivalent (for any \( k \)) to a Brunnian link is again a Brunnian link ([16 Proposition 5]).

Note that the \( C_k^n \)-equivalence is generated by simple \( C_k^n \)-moves, i.e., surgeries along simple \( C_k^n \)-trees [16]. In the following, we use technical lemmas from [16].

**Lemma 3.2** ([16] Lemma 7, \( C^n \)-version of [15 Theorem 3.17]). For two tangles \( \beta \) and \( \beta' \) in a 3-manifold \( M \), and an integer \( k \geq 1 \), the following conditions are equivalent.

1. \( \beta \) and \( \beta' \) are \( C_k^n \)-equivalent.
2. There is a simple \( C_k^n \)-forest \( F \) for \( \beta \) in \( M \) such that \( \beta_F \approx \beta' \).

**Lemma 3.3** ([16] Lemma 8, \( C^n \)-version of [15 Proposition 4.5]). Let \( \beta \) be a tangle in a 3-manifold \( M \), and let \( \beta_0 \) be a component of \( \beta \). Let \( T_1 \) and \( T_2 \) be \( C_k \)-trees for a tangle \( \beta \) in \( M \), differing from each other by a crossing change of an edge with the component \( \beta_0 \). Suppose that \( T_1 \) and \( T_2 \) are \( C_k^n \)-trees for either \( \beta \) or \( \beta \ \setminus \beta_0 \). Then \( \beta_{T_1} \) and \( \beta_{T_2} \) are related by one \( C_{k+1}^n \)-move.

4. The group \( BSL_{n+1} \)

4.1. The monoids \( BSL_{n+1} \) and \( BSL_{n+1} \). Let us recall the definition of string links. (For the details, see e.g. [12] [15].) Let \( x_1, \ldots, x_{n+1} \in \text{int} \, D^2 \) be distinct points. An \((n+1)\)-component string link \( \beta = \beta_1 \cup \cdots \cup \beta_{n+1} \) is a tangle in the cylinder \( D^2 \times [0,1] \), consisting of arc components \( \beta_1, \ldots, \beta_{n+1} \) such that \( \partial \beta_i = \{ x_i \} \times \{ 0,1 \} \) for each \( i \). Let \( SL_{n+1} \) denote the set of \((n+1)\)-component string links up to ambient isotopy fixing endpoints. There is a natural, well-known monoid structure for \( SL_{n+1} \) with multiplication given by ‘stacking’ of string links. The identity string link is denoted by \( 1 = 1_{n+1} \).

Let \( BSL_{n+1} \) denote the submonoid of \( SL_{n+1} \) consisting of Brunnian string links. Here a string link \( \beta \) is said to be Brunnian if every proper subtangle of \( \beta \) is the identity string link.

We have the following characterization of Brunnian string links.

**Theorem 4.1** ([16 Theorem 9], [24 Proposition 4.1]). An \((n+1)\)-component link (resp. string link) is Brunnian if and only if it is \( C_n^n \)-trivial, i.e., it is \( C_n^n \)-equivalent to the unlink (resp. the identity string link).

Set

\[ BSL_{n+1} = BSL_{n+1} / (C_{n+1}^n \text{-equivalence}). \]
By Theorem 4.1, $\text{BSL}_{n+1}$ can be regarded as the monoid of $C^n_{n+1}$-equivalence classes of $C^n_n$-trivial, $(n+1)$-component string links (in $D^2 \times [0,1]$).

In the rest of this section, we will describe the structure of $\text{BSL}_{n+1}$.

4.2. The group $\text{BSL}_{n+1}$ and the surgery map $\theta_n : T_{n+1} \to \text{BSL}_{n+1}$.

Proposition 4.2. $\text{BSL}_{n+1}$ is a finitely generated abelian group.

Proof. The assertion is obtained by adapting the proof of [15, Lemma 5.5, Corollary 5.6] into the $C^n$ setting. □

Let $n \geq 1$. By a (labeled) unitrivalent tree of degree $n$ we mean a vertex-oriented, unitrivalent graph $t$ such that the $n+1$ univalent vertices of $t$ are labeled by distinct elements from $\{1, 2, \ldots, n+1\}$. In figures, the counterclockwise vertex-orientation is assumed at each vertex.

Let $T_{n+1}$ denote the free abelian group generated by unitrivalent trees of degree $n$, modulo the well-known IHX and AS relations.

For a unitrivalent tree $t$, let $T_t$ denote a $C^n_n$-tree for 1 such that the tree shape and the labeling of $T_t$ is induced by those of $t$, and such that after choosing an orientation of $T_t$, for each $i = 1, \ldots, n+1$, the sign of the intersection of the $i$th string of 1 and the disk-leaf of $T_t$ corresponding to the univalent vertex of $t$ colored $i$ is positive. See for example Figure 4.1.

Proposition 4.3. There is a unique isomorphism

$$\theta_{n+1} : T_{n+1} \xrightarrow{\cong} \text{BSL}_{n+1},$$

such that $\theta_{n+1}(t) = [1]_{T_t C^n_{n+1}}$ for each unitrivalent tree $t$, where $T_t$ is as above.

Proof. Let $T'_{n+1}$ be the free abelian group generated by unitrivalent trees of degree $n$, modulo the AS relations. By adapting the proof of [15, Theorem 4.7] into the $C^n$ setting, we see that there is a unique surjective homomorphism

$$\theta'_{n+1} : T'_{n+1} \to \text{BSL}_{n+1}.$$ 

To see that $\theta'_{n+1}$ factors through the projection $T'_{n+1} \to T_n$, it suffices to see that the IHX relation is valid in $\text{BSL}_{n+1}$, i.e., $t_i - t_H + t_X \in T'_{n+1}$ is mapped to 0, where $t_i, t_H, t_X$ locally differs as in the definition of the IHX relation. This can be checked by adapting the IHX relation for tree claspers (see e.g. [10, 5, 4]) into the $C^n$ setting.

Let

$$\theta_{n+1} : T_{n+1} \to \text{BSL}_{n+1}$$

be the surjective homomorphism induced by $\theta'_{n+1}$. As in the statement of Theorem 1.2 for $\sigma \in S_{n-1}$ and $L \in B(n+1)$, we set

$$\mu_{\sigma}(T) = \mu_{\sigma(1), \sigma(2), \ldots, \sigma(n-1), n, n+1}(T),$$
FINITE TYPE INVARIANTS AND MILNOR INVARIANTS FOR BRUNNIAN LINKS

where \( \mu_{\sigma(1), \sigma(2), \ldots, \sigma(n-1), n, n+1}(T) \in \mathbb{Z} \) is the Milnor string link invariant of \( T \). Let \( t_\sigma \) denote the unitrivalent tree as depicted in Figure 4.2. The \( t_\sigma \) for \( \sigma \in S_{n-1} \) form a basis of \( \mathcal{T}_{n+1} \). Define a homomorphism

\[ \mu_{n+1} : \text{BSL}_{n+1} \rightarrow \mathcal{T}_{n+1} \]

by

\[ \mu_{n+1}(L) = \sum_{\sigma \in S_{n-1}} \mu_{\sigma}(L)t_\sigma, \]

By [15, Theorem 7.2], \( \mu_{n+1} \) is well defined.

To show that \( \mu_{n+1} \) is left inverse to \( \theta_{n+1} \), it suffices to prove that \( \mu_{n+1}\theta_{n+1}(t_\sigma) = \sigma \). Let \( L_\sigma \) denote the closure of \( T_\sigma \), which is Milnor’s link as depicted in Figure 1.1. Milnor [23] proved that for \( \tau \in S_{n-1} \)

\[ \mu_{\tau}(L_\sigma) = \begin{cases} 1 & \text{if } \tau = \sigma, \\ 0 & \text{otherwise}. \end{cases} \]  

Hence we have

\[ \mu_{n+1}\theta_{n+1}(t_\sigma) = \sum_{\tau \in S_{n-1}} \mu_{\tau}(L_\sigma)t_\tau = t_\sigma. \]

This completes the proof. \( \square \)

Corollary 4.4. For two Brunnian \( (n+1) \)-component string links \( T, T' \in \text{BSL}_{n+1} \), the following conditions are equivalent.

1. \( T \) and \( T' \) are \( C_{n+1} \)-equivalent.
2. \( T \) and \( T' \) have the same Milnor invariants of length \( n+1 \).
3. \( T \) and \( T' \) are link-homotopic.

Proof. The equivalence (2) \( \iff \) (3) is due to Milnor [23]. The equivalence (1) \( \iff \) (2) follows from the proof of Proposition 4.3. \( \square \)

Remark 4.5. Miyazawa and Yasuhara [24] prove a similar result for Brunnian links. It seems that their proof can be applied to the case of string links. See also the Remark 5.4 below.

5. The group \( \mathcal{B}_{n+1} \)

5.1. The set \( B_{n+1} \). Let \( B_{n+1} \) denote the set of the ambient isotopy classes of \( (n+1) \)-component Brunnian links. Let

\[ c_{n+1} : \text{BSL}_{n+1} \rightarrow B_{n+1} \]

denote the map such that \( c_{n+1}(\beta) \) is obtained from \( \beta \in \text{BSL}_{n+1} \) by closing each component in the well-known manner.

Proposition 5.1. The map \( c_{n+1} \) is onto.

Proof. This is an immediate consequence of [16, Proposition 12]. \( \square \)
5.2. The isomorphism $\bar{e}_{n+1} : \overline{BSL_{n+1}} \to \overline{B_{n+1}}$. Set

$$\overline{B_{n+1}} = B_{n+1}/(C_{n+1}^a\text{-equivalence}),$$

and let

$$\bar{e}_{n+1} : \overline{BSL_{n+1}} \to \overline{B_{n+1}}$$

denote the map induced by $e_{n+1}$, which is onto by Proposition 5.1.

Proposition 5.2. $\bar{e}_{n+1}$ is one-to-one.

Proof. It suffices to prove that there is a map $\overline{B_{n+1}} \to \mathcal{T}_{n+1}$ which is inverse to $\bar{e}_{n+1}\theta_{n+1} : \mathcal{T}_{n+1} \to \overline{B_{n+1}}$. This is proved similarly as in the proof of Proposition 4.3.

Proposition 5.2 provides the set $\overline{B_{n+1}}$ the well-known abelian group structure, with multiplication induced by band sums of Brunnian links.

As a corollary, we obtain another proof of a result of Miyazawa and Yasuhara [24].

Corollary 5.3 ([24, Theorem 1.2]). Let $L$ and $L'$ be two $(n + 1)$-component Brunnian links in $S^3$. Then the following conditions are equivalent.

1. $L$ and $L'$ are $C_{n+1}^a$-equivalent.
2. $L$ and $L'$ are $C_{n+1}$-equivalent.
3. $L$ and $L'$ are link-homotopic.

Proof. The result follows immediately from Propositions 4.3 and 5.2.

Remark 5.4. Miyazawa and Yasuhara [24] do not explicitly state the equivalence of (1) and others, but this equivalence follows from their proof.

Note that, unlike the $C_{n+1}^a$-equivalence, neither the $C_{n+1}$-equivalence nor the link-homotopy are closed for Brunnian links.

Remark 5.5. It is possible to show directly that $\mathcal{T}_{n+1}$ is isomorphic to $\overline{B_{n+1}}$, without using string links and the closure map $\bar{e}_{n+1}$. The proof uses Milnor’s $\mu$-invariants and the above result of Miyazawa and Yasuhara. Our approach provides an alternative proof of the latter (instead of using it).

5.3. Trees and the Milnor invariants. In this subsection, we fix some notations which are used in later sections. (Some has appeared in the proof of Proposition 4.3.)

For $\sigma \in S_{n+1}$, let $t_\sigma$ denote the unitrivalent tree as depicted in Figure 4.2. The $t_\sigma$ for $\sigma \in S_{n+1}$ form a basis of $\mathcal{T}_{n+1}$. Let $T_\sigma$ denote the corresponding $C_n^a$-tree for the $(n + 1)$-component unlink $U = U_1 \cup \cdots \cup U_{n+1}$, see Figure 5.1.

For $i_1, \ldots, i_{n+1}$ with $\{i_1, \ldots, i_{n+1}\} = \{1, \ldots, n + 1\}$, let

$$\mu_{i_1, \ldots, i_{n+1}} : B_{n+1} \to \mathbb{Z}$$
Finite Type Invariants and Milnor Invariants for Brunnian Links

CROSSED EDGE NOTATION

\[ \begin{array}{c|c|c} 
\times & \circ & \circ \\
\end{array} \]

crossed edge

Figure 6.1. The crossed edge notation

denote the Milnor invariant, which is additive under connected sum \[23\] (see also \[9, 27, 13\]). For \( \sigma \in S_{n-1} \), we set
\[
\bar{\mu}_\sigma = \bar{\mu}_{\sigma(1), \sigma(2), \ldots, \sigma(n-1), n, n+1}: B_{n+1} \to \mathbb{Z}.
\]

It is well known \[23\] that for \( \rho \in S_{n-1} \)
\[
\bar{\mu}_\rho(U_\tau) = \begin{cases} 1 & \text{if } \rho = \sigma, \\ 0 & \text{otherwise.} \end{cases}
\]

6. THE GOUSSAROW-VASSILIEV FILTRATION FOR LINKS

In this section, we briefly recall the formulation using claspers of the Goussarov-Vassiliev filtrations for links. See \[15\] for details.

6.1. Forest schemes and Goussarov-Vassiliev filtration.

A forest scheme of degree \( k \) for a link \( L \) in a 3-manifold \( M \) will mean a collection \( S = \{G_1, \ldots, G_l\} \) of disjoint (strict) tree claspers \( G_1, \ldots, G_l \) for \( L \) such that \( \sum_{i=1}^k \deg G_i = k \). A forest scheme \( S \) is said to be simple if every element of \( S \) is simple.

For \( n \geq 0 \), let \( L(M, n) \) denote the set of ambient isotopy classes of oriented, ordered links in \( M \).

For a forest scheme \( S = \{G_1, \ldots, G_l\} \) for a link \( L \) in \( M \), we set
\[
\langle L, S \rangle = \langle L; G_1, \ldots, G_l \rangle = \sum_{S' \subset S} (-1)^{|S'|} L_{S \cup S'} \in \mathbb{Z}L(M, n),
\]
where the sum is over all subsets \( S' \) of \( S \), and \( |S'| \) denote the number of elements of \( S' \).

For \( k \geq 0 \), let \( J_k(M, n) \) (sometimes denoted simply by \( J_k \)) denote the \( \mathbb{Z} \)-submodule of \( \mathbb{Z}L(M, n) \) generated by the elements of the form \( \langle L, S \rangle \), where \( L \in L(M, n) \) and \( S \) is a forest scheme for \( L \) of degree \( k \). We have
\[
\mathbb{Z}L(M, n) = J_0(M, n) \subset J_1(M, n) \subset \cdots ,
\]
which coincides with the Goussarov-Vassiliev filtration using alternating sums of links determined by singular links, see \[15\] Section 6.

6.2. Crossed edge notation.

It is useful to introduce a notation for depicting certain linear combinations of surgery along claspers, which we call crossed edge notation.

Let \( G \) be a clasper for a link \( L \) in a 3-manifold \( M \). Let \( E \) be an edge of \( G \). By putting a cross on the edge \( E \) in a figure, we mean the difference \( L_G - L_{G_0} \), where \( G_0 \) is obtained from \( G \) by inserting two trivial leaves into \( E \). See Figure 6.1. If we put several crosses on the edges of \( G \), then we understand it in a multilinear way. I.e., a clasper with several crosses is an alternating sum of the result of surgery along claspers obtained from \( G \) by inserting pairs of trivial, unlinked leaves into the crossed edges. We will freely use the identities depicted in Figure 6.2, which can be easily verified. The second identity implies that if \( G' \) is a connected graph
clasper contained in \( G \) and there are several crosses on \( G' \), then one can safely replace these crosses by just one cross on one edge in \( G' \). This properties can be generalized to the case where \( G' \) is a connected subsurface of \( G \) consisting only of nodes, edges, leaves and disk-leaves. Note also that if \( S = \{ G_1, \ldots, G_l \} \), is a forest scheme for \( L \), then \( [L, S] \) can be expressed by the clasper \( G_1 \cup \cdots \cup G_l \) with one cross on each component \( G_i \).

7. GOUSSAROV-VASSILIEV INVARIANTS OF BRUNNIAN LINKS

Throughout this section, let \( U = U_1 \cup U_2 \cup \cdots \cup U_{n+1} \) be the \((n+1)\)-component unlink.

7.1. The map \( \kappa_{n+1}: \overline{\mathcal{I}}_{n+1} \to \overline{J}_{2n}(n+1) \).

Proposition 7.1. Let \( n \geq 2 \). Let \( L \) and \( L' \) be two \((n+1)\)-component Brunnian links in an oriented, connected 3-manifold \( M \). If \( L \) and \( L' \) are \( C_{n+1}^a \)-equivalent (or link-homotopic), then we have \( L' - L \in J_{2n+1} \).

Proposition 7.1 implies the following.

Corollary 7.2. The restriction of any Goussarov-Vassiliev invariant of degree \( 2n \) to \((n+1)\)-component Brunnian links is a link-homotopy invariant.

Proof of Proposition 7.1. First, we consider the case \( L = U \). By using the same arguments as in the proof of \cite[Lemma 14]{16}, we see that there is a clasper \( G \) for \( U \) consisting of \( C_l \)-claspers with \( n+1 \leq l < 2n+1 \), such that \( U \) bounds \( n+1 \) disjoint disks which are disjoint from the edges and the nodes of \( G \), and such that \( U_G \sim_{C_{2n+1}} U_T \). The latter implies that \( U_G - U_T \in J_{2n+1} \). We use the equality \( U_G = \sum_{G \subset G}[U, G'] \). Clearly \([U, G'] \in J_{2n+1}\) for \(|G'| > 1\), so we may safely assume that \( G \) has only one component. We then have \( U_G - U \in J_{2n+1} \) as a direct application of \cite[Lemma 16]{16}. This completes the proof of the case \( L = U \).

Now consider the general case. We may assume that \( L' \) is obtained from \( L \) by one simple \( C_{n+1}^a \)-move. Since \( L \) is an \((n+1)\)-component Brunnian link, it follows from Theorem 5.11 and Lemma 5.22 that there exists a simple \( C_{n+1}^a \)-forest \( F \) for \( U \) such that \( L = U_F \). Also, there exists a simple \( C_{n+1}^a \)-tree \( T \) for \( L \) such that \( L' = L_T \). We may assume that \( T \) is a simple \( C_{n+1}^a \)-tree for \( U \) disjoint from \( F \) such that \( L' = U_{L,F} \). Let \( S \) be the forest scheme consisting of the trees \( T_1, \ldots, T_l \) of \( F \). We have \( L = \sum_{S \subset S'}[U, S'] \) and \( L' = \sum_{S \subset S'}[U_T, S'] \). Hence we have

\[
L' - L = \sum_{S \subset S'} [U, S' \cup \{T\}].
\]

Since \( \deg T = n+1 \) and \( \deg T_i = n \) for all \( i \), the term in the above sum is contained in \( J_{2n+1} \) unless \( S' = \emptyset \). Hence we have

\[
L' - L \equiv [U, \bar{T}] \equiv 0 \quad (\text{mod } J_{2n+1}),
\]

where the second congruence follows from the first case.
Quadraticity of $\kappa_{n+1}$. Let $n \geq 2$. In this subsection, we establish the following commutative diagram.

\[
\begin{array}{ccc}
\mathcal{T}_{n+1} & \xrightarrow{\psi_{n+1}} & \mathcal{B}_{n+1} \\
\downarrow{q_{n+1}} & \cong & \downarrow{\kappa_{n+1}} \\
\tilde{\text{Sym}}^2 \mathcal{T}_{n+1} & \xrightarrow{\delta_{n+1}} & \tilde{J}_{2n}(n+1)
\end{array}
\]

(7.1)

Definitions of $\psi_{n+1}$, $\tilde{\text{Sym}}^2 \mathcal{T}_{n+1}$, $q_{n+1}$ and $\delta_{n+1}$ are in order.

The isomorphism $\psi_{n+1}$ is the composition of \[\mathcal{T}_{n+1} \xrightarrow{\theta_n} \mathcal{B}_{n+1} \xrightarrow{\kappa_{n+1}} \mathcal{B}_{n+1}.\]

Let $\tilde{\text{Sym}}^2 \mathcal{T}_{n+1}$ denote the symmetric product of two copies of $\mathcal{T}_{n+1} := \mathcal{T}_{n+1} \otimes \mathbb{Q}$, and let $\tilde{\text{Sym}}^2 \mathcal{T}_{n+1}$ denote the $\mathbb{Z}$-submodule of $\tilde{\text{Sym}}^2 \mathcal{T}_{n+1}$ generated by $\frac{1}{2}a^2$, $a \in \mathcal{T}_{n+1}$. One can easily verify that $\tilde{\text{Sym}}^2 \mathcal{T}_{n+1}$ is $\mathbb{Z}$-spanned by the elements $\frac{1}{2}t_\sigma^2$ for $\sigma \in S_{n-1}$ and $t_\sigma t_{\sigma'}$ for $\sigma, \sigma' \in S_{n-1}$. (Of course we have $t_\sigma t_{\sigma'} = t_{\sigma'} t_\sigma$. Thus $\tilde{\text{Sym}}^2 \mathcal{T}_{n+1}$ is a free abelian group of rank $\frac{1}{2}(n-1)!((n-1)!+1)$.)

The arrow $q_{n+1}$ is the quadratic map defined by $q_{n+1}(x) = \frac{1}{2}x^2$ for $x \in \mathcal{T}_{n+1}$.

The arrow $\delta_{n+1}$ is the homomorphism defined as follows. For $\sigma, \sigma' \in S_{n-1}$, let $T_\sigma$ and $T_{\sigma'}$ be the corresponding simple $C_n^\sigma$-trees for $U$ as in Section 5.3. Let $\tilde{T}_{\sigma'}$ denote a simple $C_n^{\sigma'}$-tree obtained from $T_{\sigma'}$ by a small isotopy if necessary so that $\tilde{T}_{\sigma'}$ is disjoint from $T_\sigma$. Set

$$
\delta_{n+1}(t_\sigma t_{\sigma'}) = [U; T_\sigma, \tilde{T}_{\sigma'}]_{J_{2n+1}} \in J_{2n}(n+1),
$$

which does not depend on how we obtained $\tilde{T}_{\sigma'}$ from $T_\sigma$, since crossing changes between an edge of $T_\sigma$ and an edge of $\tilde{T}_{\sigma'}$ preserves the right-hand side. (This can be verified by using a ‘$C^n$-version’ of [13, Proposition 4.6].) For the case of $\frac{1}{2}l_\sigma^2$, we modify the above definition with $\sigma' = \sigma$ as follows. Let $T_\sigma$ and $\tilde{T}_\sigma$ be as above. See Figure 7.1. Let $T'$ be the $C_n-1$-tree obtained from $\tilde{T}_\sigma$ by first removing the disk-leaf $D$ intersecting $U_{n+1}$, the edge $E$ incident to $D$, and the node $N$ incident to $E$, and then gluing the ends of the two edges which were attached to $N$. Moreover, let $C$ be a $C_1$-tree which intersects $U_{n+1}$ and $U_{\sigma(1)}$ as depicted. Set

$$
\delta_{n+1}(\frac{1}{2}l_\sigma^2) = [U; T_\sigma, T', C].
$$

Lemma 7.3. We have

\[
(7.2) \quad [U; T_\sigma, \tilde{T}_\sigma] \equiv 2[U; T_\sigma, T', C] \pmod{J_{2n+1}}.
\]
Figure 7.2.

\[
\begin{array}{c}
\text{Figure 7.3.}
\end{array}
\]

**Proof.** By [15, Section 8.2], it suffices to prove the identity in the space of untrivalent diagram depicted in Figure 7.2, which can be easily verified using the STU relation several times. \(\square\)

It follows from Lemma 7.3 that \(\delta_{n+1}\) is a well-defined homomorphism. Set

\[
\frac{1}{2}[U;T_\sigma,\tilde{T}_\sigma]_{j_{2n+1}} = [U;T_\sigma,T',C]_{j_{2n+1}}.
\]

We have

\[
\delta_{n+1}\frac{1}{2}(\sigma) = \frac{1}{2}[U;T_\sigma,\tilde{T}_\sigma]_{j_{2n+1}}.
\]

**Theorem 7.4.** The diagram (7.1) commutes. In particular, \(\kappa_{n+1}\) is a quadratic map.

We need the following lemma before proving Theorem 7.4.

**Lemma 7.5.** Let \(C\) be a clasper for a link \(L\) such that there is a disk-leaf \(D\) of \(T\) which ‘monopolizes’ a component \(K\) of \(L\) in the sense of [16, Definition 15], and such that \(D\) is adjacent to a node. That is, \(T\) and \(L\) looks as depicted in the left hand side of Figure 7.3. Then we have the identity as depicted in the figure.

**Proof.** The identity is easily verified and left to the reader. (Note that Lemma 7.5 is essentially the same as [16, (4.4)].) \(\square\)

**Proof of Theorem 7.4.** Let \(\sigma \in S_{n-1}\). We must show that

\[
[U;T_\sigma]_{j_{2n+1}} = \frac{1}{2}[U;T_\sigma,\tilde{T}_\sigma]_{j_{2n+1}}.
\]

For \(i = 1, \ldots, n+1\), let \(D_i\) denote the disk-leaf of \(T_\sigma\) intersecting \(L_i\), and let \(E_i\) denote the incident edge. For \(i = 1, \ldots, n-1\), let \(N_i\) denote the node incident to \(E_i\).

By applying Lemma 7.5 to the edge of \(T_\sigma\) which is incident to \(N_{\sigma(1)}\) but not to \(D_{n+1}\) or \(D_{\sigma(1)}\), we obtain the identity depicted in Figure 7.4. Let \(B\) be the box and \(E\) be the edge as depicted. Let \(G\) be the clasper in the right hand side. By zip construction [15 Section 3.3] at \(E\), we obtain a crossed clasper depicted in Figure 7.5 which consists of two components \(T_\sigma\) and \(P\). The component \(P\) has \(n-2\) (non-disk) leaves.
We claim that we can unlink the leaves of $P$ from $T_\sigma$ without changing the class in $\bar{J}_{2n}(n+1)$. To see this, it suffices to show that
\begin{equation}
(U; T_\sigma, P) \equiv [U; T_\sigma, T'] \pmod{J_{2n+1}},
\end{equation}
where $T'$ is obtained from $P$ by the unlinking operation. Note that each unlinking is performed by a sequence of crossing changes between an edge of the $C_n$-tree $T_\sigma$ and a link component (after performing surgery along $P$ in the regular neighborhood of $P$), and thus can be performed by $C_{n+1}$-moves. Since all the links appearing in this sequence is Brunnian, we have (7.3) by Proposition 7.1. This completes the proof of the claim.

By the above claim, it follows that
\begin{equation}
[U; T_\sigma, P] = [U; T_\sigma, T'] \pmod{J_{2n+1}},
\end{equation}
where $T'$ is obtained from $P$ by removing the leaves, the incident edges, and the boxes, and then smoothing the open edges, see the left hand side of Figure 7.6 which is equal to the right hand side by Lemma 7.5. The result is related to the desired clasper defining $[U; T_\sigma, T', C]$ by half twists of two edges and homotopy with respect to $U$, and hence equivalent modulo $J_{2n+1}$ to $[U; T_\sigma, T', C]$. This completes the proof.

\section{Proof of Theorem 1.2}
In this subsection we prove Theorem 1.2.
Let \( L \in B_{n+1} \). We have
\[
[L]c^a_{n+1} = \sum_{\sigma \in S_{n-1}} \mu_\sigma(L)[U_{T_\sigma}]c^a_{n+1}
\]
in \( B_{n+1} \). (Recall that the sum is induced by band-sum in \( B_{n+1} \).) Hence we have by the commutativity of (7.1)
\[
[L - U]_{J_{2n+1}} = \kappa_{n+1}([L]c^a_{n+1})
\]
\[
= \delta_{n+1}q_{n+1}^{-1} \left( \sum_{\sigma \in S_{n-1}} \mu_\sigma(L)[U_{T_\sigma}]c^a_{n+1} \right)
\]
\[
= \delta_{n+1}q_{n+1}^{-1} \left( \sum_{\sigma \in S_{n-1}} \mu_\sigma(L)t_\sigma \right)
\]
\[
= \delta_{n+1} \left( \sum_{\sigma \in S_{n-1}} \mu_\sigma(L)t_\sigma \right) \cdot (L)
\]
\[
= \delta_{n+1} \left( \sum_{\sigma, \sigma' \in S_{n-1}} \mu_\sigma(L)\mu_{\sigma'}(L)t_\sigma t_{\sigma'} \right)
\]
\[
= \frac{1}{2} \sum_{\sigma, \sigma' \in S_{n-1}} \mu_\sigma(L)\mu_{\sigma'}(L)[U; T_\sigma, \tilde{T}_{\sigma'}].
\]
Hence we have
\[
(7.4) \quad f(L) - f(U) = \frac{1}{2} \sum_{\sigma, \sigma' \in S_{n-1}} \mu_\sigma(L)\mu_{\sigma'}(L)f([U; T_\sigma, \tilde{T}_{\sigma'}]).
\]

We give any total order on the set \( S_{n-1} \). Then we have
\[
f(L) - f(U) = \sum_{\sigma \in S_{n-1}} \left( \frac{1}{2} f([U; T_\sigma, \tilde{T}_{\sigma}]) \mu_\sigma(L) \mu_{\sigma'}(L) \right)
\]
\[
+ \sum_{\sigma < \sigma'} f([U; T_\sigma, \tilde{T}_{\sigma'}]) \mu_\sigma(L) \mu_{\sigma'}(L).
\]

Note that \( \frac{1}{2} f([U; T_\sigma, \tilde{T}_{\sigma}]) \in \mathbb{Z} \) and \( f([U; T_\sigma, \tilde{T}_{\sigma'}]) \in \mathbb{Z} \). Hence we have (7.1) by setting
\[
f_{\sigma, \sigma'} = \begin{cases} 
\frac{1}{2} f([U; T_\sigma, \tilde{T}_{\sigma}]) & \text{if } \sigma = \sigma', \\
\frac{1}{2} f([U; T_\sigma, \tilde{T}_{\sigma'}]) & \text{if } \sigma < \sigma', \\
0 & \text{if } \sigma > \sigma'.
\end{cases}
\]

This completes the proof of Theorem 1.4.

References

[1] D. Bar-Natan, On the Vassiliev knot invariants, Topology 34 (1995), no. 2, 423–472.
[2] D. Bar-Natan, Vassiliev homology string link invariants, J. Knot Theory Ramifications 4 (1995), no. 1, 13–32.
[3] T.D. Cochran, Derivatives of link: Milnor’s concordance invariants and Massey’s products, Mem. Amer. Math. Soc. 84 (1990), No. 427.
[4] J. Conant, R. Schneiderman and P. Teichner, Jacobi identities in low-dimensional topology, to appear in Compositio Math.
[5] J. Conant and P. Teichner, Grope cobordisms and Feynman diagrams, Math. Ann. 328 (2004), no. 1-2, 135–171.
[6] S. Garoufalidis, Links with trivial Alexander module and nontrivial Milnor invariants, preprint [math.GT/0206196].
[7] M.N. Gusarov, A new form of the Conway-Jones polynomial of oriented links. (Russian), Zap. Nauchn. Sem. Leningrad. Otdel. Mat. Inst. Steklov. (LOMI) 193 (1991), Geom. i Topol. 1, 4–9, 161; translation in “Topology of manifolds and varieties”, 167–172, Adv. Soviet Math., 18, Amer. Math. Soc., Providence, RI, 1994.
[8] M. Gusarov, On n-equivalence of knots and invariants of finite degree, “Topology of manifolds and varieties”, 173–192, Adv. Soviet Math., 18, Amer. Math. Soc., Providence, RI, 1994.
[9] M. Goussarov (Gusarov), Finite type invariants and n-equivalence of 3-manifolds, C. R. Acad. Sci. Paris Sér. I Math. 329 (1999), no. 6, 517–522.
[10] M.N. Gusarov, Variations of knotted graphs. The geometric technique of n-equivalence. (Russian), Algebra i Analiz 12 (2000), no. 4, 79–125; translation in St. Petersburg Math. J. 12 (2001), no. 4, 569–604.
[11] N. Habegger, Milnor, Johnson, and tree level perturbative invariants, preprint.
[12] N. Habegger and X.S. Lin, The classification of links up to link-homotopy, J. Amer. Math. Soc. 3 (1990), 389–419.
[13] N. Habegger and K.E. Orr, Milnor link invariants and quantum 3-manifold invariants, Comment. Math. Helv. 74 (1999), no. 2, 322–344.
[14] N. Habegger and G. Masbaum, The Kontsevich integral and Milnor’s invariants, Topology 39 (2000), no. 6, 1253–1289.
[15] K. Habiro, Claspers and finite type invariants of links, Geom. Topol. 4 (2000), 1–83.
[16] K. Habiro, Brunnian links, claspers, and Goussarov-Vassiliev finite type invariants, to appear in Math. Proc. Camb. Phil. Soc.
[17] K. Habiro, J.-B. Meilhan, On the Kontsevich integral of Brunnian links, Alg. Geom. Topol. 6 (2006), 1399–1412.
[18] V. S. Krushkal, Additivity properties of Milnor’s µ-invariants, J. Knot Theory Ramifications 7 (1998), no. 5, 625–637.
[19] X.S. Lin, Power series expansions and invariants of links, in “Geometric topology”, AMS/IP Stud. Adv. Math. 2.1, Amer. Math. Soc. Providence, RI (1997) 184–202.
[20] G. Masbaum and A. Vaintrob, Milnor numbers, spanning trees, and the Alexander-Conway polynomial, Adv. Math. 180 (2003), no. 2, 765–797.
[21] S.V. Matveev, Generalized surgeries of three-dimensional manifolds and representations of homology spheres. (Russian), Mat. Zametki 42 (1987), 268–278, 345; translation in Math. Notes 42 (1987), no. 1-2, 651–656.
[22] J.-B. Meilhan, Goussarov-Habiro theory for string links and the Milnor-Johnson correspondence, Topology Appl. 153 (2006), no. 14, 2709–2729.
[23] J. Milnor, Link groups, Ann. of Math. (2) 59 (1954), 177–195.
[24] H. A. Miyazawa, A. Yasuhara, Classification of n-component Brunnian links up to Cn-move, Topology Appl. 153 (2006), 1453–1550.
[25] H. Murakami and Y. Nakanishi, On a certain move generating link-homology, Math. Ann. 284 (1989), no. 1, 75–89.
[26] T. Ohtsuki, Quantum invariants. A study of knots, 3-manifolds, and their sets, Series on Knots and Everything, 29. World Scientific Publishing Co., Inc., River Edge, 2002.
[27] K.E. Orr, Homotopy invariants of links, Invent. Math. 95 (1989), no. 2, 379–394.
[28] V.A. Vassiliev, Cohomology of knot spaces, “Theory of singularities and its applications”, 23–69, Adv. Soviet Math., 1, Amer. Math. Soc., Providence, RI, 1990.