A new inverse formula for the Laplace's transformation.

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In the article is proved, that the complex part of the analytical continuation of the

\[ r(p) = \mathcal{L}\mathcal{L}Z(x) = \int_0^\infty e^{-pt}dt \int_0^\infty e^{-tx} Z(x)dx, p \in \{p : \text{Im } p \geq 0\}, \]

equals to \(-\pi Z(x), x \in (0, \infty)\), if \( p = s = -x \in (-\infty, 0)\) for a wide class of functions \( Z(x) \):

It is proved, that the odd functions

\[ Z(x) = \sum_{k=1}^l \gamma_k e^{\lambda_k x}, \quad \gamma_k = \text{res}_{p=\lambda_k} \frac{Q_p(p)}{P_p(p)}, \quad \lambda_k = -\alpha_k + \beta_k i, \quad \alpha_k > 0, \quad k = 1, \ldots, l, \]
satisfy to all the conditions of the theorems of the article.

THE TRANSFORMS OF FOURIER, THE TRANSFORM OF LAPLACE, THE NEW FORMULA OF TRANSFORMATION

**Introduction.**

In the article we prove (a theorem 2), that for the negative variable \( p = -x, x \in (0, \infty) \) a complex part of the analytical continuation \( r(p) = \mathcal{L}\mathcal{L}Z(x) \) equals to \(-\pi Z(x), x \in (0, \infty)\) for a wide class of functions \( Z(x) \):

\[ \pi Z(s) = -\text{Im } r_{An}(-s), \quad s \in (0, \infty), \quad r_{An}(s) \equiv r(s), s \in (0, +\infty), \]

\[ r(s) = \mathcal{L}\mathcal{L}(Z(x)), \quad s \in (0, \infty), \quad \mathcal{L}(Z(x)) = \int_0^\infty e^{-sx} Z(x)dx, \quad s \in (0, \infty), \]

where, by definition, \( r_{An}(p), p \in \mathbb{C} \), is the analytical continuation of the function \( r(p), p \in D = \{p : \text{Re } p > 0\} \), and the function \( r_{An}(p) \) is regular (analytic) in \( \mathbb{C} \setminus P, \quad P = \{z_j : z_j \in \mathbb{C}, z_j \notin (-\infty, 0), j = 1, \ldots, m\} \), the set \( P \) may be \( \emptyset \).

All results of the article are true, for instance, if

\[ Z(x) = \sum_{k=1}^{2N} \gamma_k e^{\lambda_k x} = \sum_{k=1}^N \gamma_k e^{-\alpha_k} \cos \beta_k x = \text{Re } Z(x), \quad \lambda_k = -\alpha_k + \beta_k i, \quad \alpha_k \in (0, \infty), \quad \gamma_k, \beta_k \in (-\infty, \infty), \]

\[ \lambda_{2j+1} = -\alpha_{2j+1} + \beta_{2j+1} i, \lambda_{2j+2} = -\alpha_{2j+1} - \beta_{2j+1} i, \quad j = 1, \ldots, N - 1, \quad \lambda_j \neq \lambda_i \text{ for all } i \neq j. \]

The equality \( r(s) = -i\mathcal{F}_0 \mathcal{F}_0(Z(x)) = \mathcal{L}\mathcal{L}(Z(x)), s \in (0, +\infty) \), is considered in the theorem 1, where, by definition,

\[ \mathcal{F}_0(Z(x)) = \int_0^\infty e^{i\rho x} Z(x)dx, \quad y \in (-\infty, \infty). \]

The main result of the article (the theorem 2) follows from the theorem 1.

All the conditions of the theorems 1 and 2 are checking in the remark 1 for the above functions \( Z(x) \).
1. The main result.
We shall use a designations \( f(p) \in \mathcal{A}nG \), if the function \( f(p) \) is regular in the open domain \( G \in \mathbb{C} \).

Theorem 1.
If for the function \( Z(p) \), we have

\[
R(p) = \int_{0}^{\infty} e^{-px} dx \int_{0}^{\infty} e^{itx} Z(t) dt \in \mathcal{A}nD_{*} = \{ p : -\pi/2 < \arg p < \varphi_{0} \},
\]

\[
F(p) = \int_{0}^{\infty} e^{-x_{1}} \frac{dx_{1}}{p} \int_{0}^{\infty} e^{itx_{1}/p} Z(t) dt \in \mathcal{A}nD_{*},
\]

for a variable \( \varphi_{0} : 0 < \varphi_{0} < \pi/2 \); if for all \( p = -is, s \in (0, \infty) \), the functions are continuous.

Then

\[
\mathcal{L}L(Z(x)) = -i\mathcal{F}_0\mathcal{F}_0(Z(x)) = Im \int_{0}^{\infty} e^{isx} dx \int_{0}^{\infty} e^{itx} Z(t) dt, s \in (0, \infty).
\]

Proof.
We can replace \( ux = x_{1}, p = s \in (0, \infty) \), in the integral of \( R(p) \), if \( p = u \in (0, \infty) \). We obtain \( R(u) = F(u), s \in (0, \infty) \). The functions are regular in \( D_{*}\{ p : -\pi/2 < \arg p < \varphi_{0} \} \), it is the open domain and \( (0, \infty) \in D_{*} \), then \( R(p) = F(p), p \in D \). The function are continuous for \( p = is, s \in (-\infty, 0) \), then \( R(-is) = F(-is), s \in (0, \infty) \), and after the inverse replace \( x_{1}/s = x, s \in (0, \infty), x_{1} \in [0, \infty) \) in the integral of \( F(-is) \) we obtain the theorem 1..

Theorem 2.
Let the function \( \mathcal{F}_0\mathcal{F}_0(Z(x)) = r_{F}(p) \) is regular in the domain \( D_{*} = \{ p : Im p > 0 \} \) and is continuous for all real \( p = s \in (-\infty, \infty) \).
Let all the conditions of the theorem 1 are holds.
Let the function \( r(s) = \mathcal{L}L(Z(x)), s \in (0, \infty) \), can be analytical continued in the left part of the plane : \( D_{-} = \{ p : Re p < 0 \} \setminus \{ p_{j}, j = 1, \ldots, m \} \) \( ( m \) can be 0).

then

\[
-\pi Z(x) = Im r_{A_{n}}(-x), x \in (0, \infty), r_{A_{n}}(p) = r(p), p \in D = \{ p : Re p > 0 \}.
\]

Proof.
Let

\[
-ir_{F}(p) = -i\mathcal{F}_0\mathcal{F}_0(Z(x)), r_{L}(p) = r_{A_{n}}(p), r_{L}(p) = \mathcal{L}L(Z(x)), p \in s \in (0, \infty).
\]

We can see, that the real part of \( r_{F}(s) \), \( s \in (-\infty, \infty) \), is equal to

\[
Re r_{F}(s) = \int_{0}^{\infty} \cos sx \, dx \int_{0}^{\infty} \cos tx \, Z(t) \, dt - \int_{0}^{\infty} \sin sx \, dx \int_{0}^{\infty} \sin tx \, Z(t) \, dt =
\]

\[
= (1/4) \int_{-\infty}^{\infty} e^{isx} \, dx \int_{-\infty}^{\infty} e^{itx} [Z_{od}(t) + Z_{ev}(t)] \, dt = (1/4) \int_{-\infty}^{\infty} e^{isx} \, dx \int_{-\infty}^{\infty} e^{itx} Z_{+}(t) \, dt = (\pi/2)Z_{+}(-s),
\]

\[
= (1/4) \int_{-\infty}^{\infty} e^{isx} \, dx \int_{-\infty}^{\infty} e^{itx} Z_{+}(t) \, dt = (\pi/2)Z_{+}(-s),
\]
\( s \in (-\infty, \infty) \), where
\[
Z_{od}(x) \equiv Z_{ev}(x) \equiv Z_+(x)/2 = Z(x), x \in (0, +\infty);
\]
\[
Z_{od}(-x) \equiv Z_{od}(x), Z_{ev}(-x) \equiv -Z(x), Z_+(-x) \equiv 0, -x \in (-\infty, 0).
\]

We obtain
\[
Im(-ir_F(-s)) \equiv (-\pi/2)Z_+((-s)) = -\pi Z(s), s \in (0, \infty)
\] (1.1).

From the theorem 1 we have
\[
r_L(s) = -ir_F(s) = r(s), s \in (0, +\infty).
\]

We shall prove, that the functions are equal for all the complex \( p : Im\, p \geq 0 \). Let \( p = z_j \) is a point, where the \( r_L(p) \) or the \( r_F(p) \) functions are not regular; let \( Q = \{ p = z_j, j = 1, \ldots, M \} \) is the set of all the points.

All the values of the function \( ir_F(p) \) are real, if \( p = s \in (0, \infty) \) - it follows from the theorem 1, where
\[
-ir_F(p) \equiv Re(-ir_F(p)) \equiv \mathcal{L}\mathcal{L}(Z(x)) = r(s), p = s \in (0, \infty).
\]

Then, we can use the Reaman's theorem about the symmetrical continuation ([2]), and the function \(-ir_F(p) = -i\mathcal{F}_0\mathcal{F}_0(Z(x)), Im -ir_F(s) \equiv 0, s \in (0, \infty)\), can be analytical continued from the domain \( D_0 = \{ p : Im\, p > 0 \} \) in the the low part of the complex plane in the domain \( D_1 \), such that the real axis \((0, \infty) \in D_1 : (0, \infty) \in D_1, -ir_{An}(p) \in AND_1, D_0 \in D_1, -ir_{An} \equiv -ir_F(p), p \in D_0\).

Now, from \(-ir_{An}(p) = r_L(p), p \in (0, \infty) \in D_1\), it follows \(-ir_{An}(p) = r_L(p), p \in D_1 \setminus Q\).

As the result we have \(-ir_{An}(p) = -ir_F(p), p \in D_0 \setminus Q \in D_1\). (We use, that \(-ir_F(p) \in AND_0 \setminus Q\).

Therefore \( r_L(p) = -ir_{An}(p) = -ir_F(p), p \in D_0 \setminus Q \).

The function \( r_F(p) \) is continuous, if \( p = s \in (-\infty, \infty) \). Then,
\[
-ir_F(-s) = \lim_{p \to s}(-ir_F(p)) = \lim_{p \to s} r_L(p) = r_L(-s), s \in (0, \infty).
\]

From the equality (1.1) we obtain
\[
-\pi Z(s) = Im(-ir_F(-s)) = Im r_L(-s) = Im r_{An}(-s), s \in (0, \infty).
\]

The theorem 2 is proved.

Remark 1.

For all \( l > n + 2 > 1, l, n \in 1, 2, \ldots \), we have (it is a well-known formula)
\[
\frac{Q_n(p)}{P_l(p)} = \frac{q_0 + q_1 p + \ldots + q_n p^n}{\prod_{k=1}^l (p - \lambda_k)} = 2\pi i \int_0^\infty \left[ \sum_{k=1}^l res_{p=\lambda_k} e^{px} \frac{Q_n(p)}{P_l(p)} \right] e^{-px}\, dx, \quad Re\, p \geq 0,
\]
if \( \lambda_k \neq 0, k = 1, \ldots, l \).

For all complex \( \lambda_k = -\alpha_k + \beta_k i, \alpha_m \in (0, \infty), \beta_k \in (-\infty, \infty) \), the function
\[
Z(x) = \sum_{k=1}^l \gamma_k e^{\lambda_k x}, \quad \gamma_k = res_{p=\lambda_k} \frac{Q_n(p)}{P_l(p)}, k = 1, \ldots, l
\]
satisfy to all the conditions of the theorems 1 and 2, if, for instance, \( \lambda_i \neq \lambda_j \) for all \( i \neq j \).

(In the conditions of introduction the function \( Z(x) \) is equal to \( Z_{re}(x) \).

Proof.

1. It is obviously, the function \( R(p), p = x + iy, \) is regular for all \( x > 0, y \in (-\infty, \infty) \) and continuous for all \( p = iy, y \in (-\infty, \infty) \)

\[
R(p) = \int_0^\infty \frac{Q_n(t)}{P(t)} e^{-(x+iy)t} dt.
\]

2. If \( p \in D_\ast = \{p = x + iy : -\pi/2 < \arg p < \varphi_0 > 0\}, \) and \( 0 < \varphi_0 < \min_{0 < \kappa < 1} |\arctan \alpha_k/\beta_k| \), then

\[
\int e^{-x_1} \frac{Q_n(-ix_1/(x+iy))}{\prod_{k=1}^l ((-ix_1/(x+iy)) + \alpha_k - \beta_k i)} dx_1
\]

is continuous for all \( p \in D_\ast; \) we use

\[
[x_1/\sqrt{x^2 + y^2}](-i)(x - iy) = T(-ix - y) \neq -\alpha_k + \beta_k i, \text{ for all } T \in (0, \infty),
\]

if, other \( p \in D_1 = \{x + iy : x > 0, y \leq 0\}, (\alpha > 0), \) or \( p \in D_2 = \{x + iy : y > 0, x > 0\} \)

\[
\bigcap -\pi/2 - \arg(-\alpha_k - |\beta_k i|) < \arg(-ix - y) < -\pi/2, k = 1, \ldots, l \}
\]

where \( D_2 = \{x + iy : x > 0, y > 0\} \)

\[
\bigcap 0 < \arg(x + iy) < \min_{l=1,\ldots,l} (\arctan|\alpha_k/\beta_k|), \quad D_1 \cup D_2 = D_\ast.
\]

Then

\[
F(p) = (1/p) \int e^{-x_1} \frac{Q_n(-ix_1/(x+iy))}{\prod_{k=1}^l ((-ix_1/(x+iy)) + \alpha_k - \beta_k i)} dx_1 \in A_n D,
\]

3. To prove, that \( r_{Am(p)} \) is continuous for all \( p = x \in (-\infty, 0) \) and \( r_{Am}(p) = r(p), \) if \( p \in D_3 = \{p : Im p > 0\}, \) we consider an equality

\[
r(p) = \mathcal{L}(Z(x)) = \int_0^\infty Z(x)[1/(p + x)] dx \in A_n \mathbb{C} \setminus (-\infty, 0).
\]

We can write

\[
r(p) = \mathcal{L}(Z(x)) = \int_0^\infty Z(x)[1/(p + x)] dx \in A_n \mathbb{C} \setminus (-\infty, 0).
\]

Let

\[
p = p_0 t, t \in (0, \infty), p_0 = \text{const.}, \arg p_0 = \pi - \varepsilon, 0 < \varepsilon, \ll 1,
\]

and \( x/(tp_0) = z, x \in [0, \infty), t \in (0, \infty). \) We have

\[
r_1(p_0 t) = \int_{l(p_0)} Z(p_0 tz)[1/(1 + z)] dz, t \in (0, \infty), l(p_0) = \{z : z = \tau/p_0, \tau \in [0, \infty)\},
\]

4
\( l(p_0) = \{ z : \arg z = \arg(1/p_0) \} \), where \( r_1(p) = r(p), p = p_0 t, t \in (0, \infty) \); and \( r_1(p) \in \mathcal{A} \mathcal{D} \mathcal{E} = \{ p : -2\varepsilon < \arg p/p_0 < 2\varepsilon \wedge p \neq 0 \} \), while

\[
\begin{align*}
    r_1(p) &= \int_{l(p_0)} Z(pz)[1/(1+z)]dz = \int_0^\infty Z((p/p_0)\tau)[1/(1+(\tau/p_0))]d\tau/p_0,
\end{align*}
\]

and, if \( p \in D_\varepsilon \), we obtain

\[
\frac{dr_1(p)}{dp} = \int_0^\infty \frac{dZ((p/p_0)\tau)}{dp}[1/(1+(\tau/p_0))]d\tau/p_0,
\]

is continuous for all \( p \in D_\varepsilon, (Re (p/p_0) > \text{const.} > 0) \).

We obtain, that \( r_1(p) = r(p), p \in l(p_0) \), and \( r_1(p) = r_{\mathcal{A} \mathcal{D} \mathcal{E}}(p) = r(p), p \in D_\varepsilon, (-\infty, 0) \in D_\varepsilon \), where the functions are regular in the upper part of the \( \mathbb{C} \).

4. The function \( F_0 F_0(Z(x)) \) is regular in the upper part of \( \mathbb{C} \) for all \( p : \{ Im p > 0 \} \), and it is continuous for all \( p = x \in (-\infty, 0) \), (if \( 1 < l > n + 2 \)). We use \( |e^{ipt}| \leq e^{-y}, p = x + iy, y > 0 \).

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