ON A FRACTIONAL MONGE-AMPERE OPERATOR

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Abstract. In this paper we consider a fractional analogue of the Monge-Ampère operator. Our operator is a concave envelope of fractional linear operators of the form \( \inf_{\lambda \in A} L_A u \), where the set of operators corresponds to all affine transformations of determinant one of a given multiple of the fractional Laplacian.

We set up a relatively simple framework of global solutions prescribing data at infinity and global barriers. In our key estimate, we show that the operator that realizes the infimum remains strictly elliptic, which allows to deduce an Evans-Krylov regularity result and therefore that solutions are classical.

1. Introduction

The classical Monge-Ampère equation \( \det D^2 u = f \) arises in many areas of analysis, geometry, and applied mathematics. Standard boundary value problems are the Dirichlet problem and optimal transportation problem, where we prescribe the image of a domain by the gradient map.

In the Dirichlet problem, one prescribes a smooth domain \( \Omega \subset \mathbb{R}^n \), boundary data \( g(x) \) on \( \partial \Omega \), and a right-hand side \( f(x) \) in \( \Omega \), and studies existence and regularity of a function \( u \) such that

\[
\begin{cases}
\det D^2 u = f & \text{in } \Omega \\
u = g & \text{on } \partial \Omega.
\end{cases}
\]

For the problem to fit in the framework of the theory of fully nonlinear elliptic equations, one must seek convex solutions \( u \) to ensure that \( \det D^2 u \) is indeed a monotone function of \( D^2 u \). Thus, we must require \( f(x) \) positive and \( \Omega \) convex. The convexity of \( \Omega \) is required in order to construct appropriate smooth subsolutions that act as lower barriers, see [2].

In that case, there is considerable work in the literature establishing the existence, uniqueness and regularity of solutions to (1.1), see [1, 2, 6] and the references therein. The main ingredients entering the theory are, roughly speaking, the following:

(a) The Monge-Ampère equation is a concave fully nonlinear equation. For a convex solution

\[
\det D^2 u = f
\]

is equivalent to

\[
\inf_{\lambda \in \mathcal{A}} L u = f,
\]

where \( \mathcal{A} \) is the family of linear operators \( L u = \text{trace} (A D^2 u) \) for \( A > 0 \) with eigenvalues \( \lambda_j(A) \) that satisfy \( \prod_{j=1}^n \lambda_j(A) = n^{-n} f^{n-1} \). Furthermore, if we take \( nA \) equal to the matrix of cofactors of \( D^2 u \), then

\[
n^n \prod_{j=1}^n \lambda_j(A) = (\det D^2 u)^{n-1} = f^{n-1},
\]
the infimum is realized and the equation satisfied. Moreover, from the concavity of \( \det^{1/n}(\cdot) \)
any other choice of the eigenvalues would give a larger value than the prescribed \( f \), making \( u \) a subsolution.

In other words, the Monge-Ampère equation can be thought of as the infimum of a family of linear operators that consists of all affine transformations of determinant one of a given multiple of the Laplacian.

(b) The fact that \( \det D^2 u \) can be represented as a concave fully nonlinear equation implies that pure second derivatives are subsolutions of an equation with bounded measurable coefficients and as such, are bounded from above.

Indeed, if we consider the second-order incremental quotient in the direction \( e \in \partial B_1(0) \),

\[
\delta(u, x_0, y) = u(x_0 + he) + u(x_0 - he) - 2u(x_0)
\]

and choose

\[
L u = \text{trace} \left( AD^2 v \right)
\]

with \( nA \) the matrix of cofactors of \( D^2 u(x_0) \), we have that \( Lu(x_0) = f(x_0) \) while on the other hand, the matrix

\[
B = \left[ \frac{f(x_0 + he)}{f(x_0)} \right]^{\frac{n-1}{n}} A
\]

satisfies

\[
\det B = n^{-n} f(x_0 + he)^{n-1},
\]

which makes it eligible to compete for the minimum of \( \text{trace} (ND^2 u(x_0 + he)) \). This implies,

\[
\left[ \frac{f(x_0 + he)}{f(x_0)} \right]^{\frac{n-1}{n}} Lu(x_0 + he) \geq f(x_0 + he)
\]

or equivalently,

\[
Lu(x_0 + he) \geq f(x_0 + he)^{\frac{1}{n}} f(x_0)^{\frac{n-1}{n}}.
\]

We deduce that at a maximum of a second derivative \( D^2_{ee} u \) the function \( f \) must satisfy,

\[
f(x_0) \frac{n-1}{n} D^2_{ee} f^{1/n}(x_0) \leq 0.
\]

If \( D^2_{ee} f \) is bounded and we have an appropriate barrier, plus control of the second derivatives of \( u \) at the boundary of \( \Omega \), we deduce that \( u \) is not only convex but also semiconcave. For that purpose, the boundary and data must be smooth and the domain strictly convex. This allows for the construction of appropriate subsolutions as barriers.

(c) Then, the last ingredient of the theory is that the equation \( \prod_{j=1}^n \lambda_j = f \) with \( f \) strictly positive implies that all \( \lambda_j \) are strictly positive. This implies that the operators involved with the minimization can be restricted to a uniformly elliptic family and the corresponding general theory applies. In particular, Evans-Krylov theorem implies that solutions are \( C^{2,\alpha} \) and from there, as smooth as two derivatives better than \( f \).

The discussion above suggests that one could carry out a similar program for a non-local or fractional Monge-Ampère equation of the form

\[
\inf L_A u = f
\]

where the set of operators \( L_A \) corresponds to that of all affine transformations of determinant one of a given multiple of the fractional Laplacian. In fact, one may consider any concave function of the Hessian as in [2] as an infimum of affine transformations of the Laplacian, the affine transformations corresponding now to the different linearization coefficients of the function \( F(\lambda_1, \ldots, \lambda_n) \) and consider the corresponding nonlocal operator.
One can take \( \inf_{A \in \mathcal{A}} L_A u = f \), where \( \mathcal{A} \) corresponds to a family of symmetric positive matrices with determinant bounded from above and below,

\[
0 < \lambda \leq \det A \leq \Lambda,
\]

and

\[
L_A u(x) = \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|A^{-1}y|^{n+2s}} \, dy.
\]

The kernel under consideration does not need to be necessarily \(|A^{-1}x|^{-(n+2s)}\), it could be a more general kernel \(K(Ax)\). In fact, the geometry of the domain is an important issue for the “inherited from the boundary” regularity theory for degenerate operators depending on the eigenvalues of the Hessian, see [2].

In this article we shall set up a relatively simple framework of global solutions prescribing data at infinity and global barriers to avoid having to deal with the technical issues inherited from boundary data, which is rather complex for non-local equations. As in the second order case, we intend to prove:

(a) Existence of solutions.
(b) Solutions are semiconcave, i.e. second derivatives are bounded from above.
(c) Along each line, the fractional Laplacian is bounded from above and strictly positive.
(d) The operator that realizes the infimum remains strictly elliptic.
(e) The non-local fully nonlinear theory developed in [3, 4] applies, in particular the nonlocal Evans-Krylov theorem, and solutions are “classical”.

To be more precise, let us introduce the non-local Monge-Ampère operator \(D_s\) that we are going to consider in the sequel, given by

\[
D_s u(x) = \inf \left\{ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|A^{-1}(y-x)|^{n+2s}} \, dy \right. \mid A > 0, \, \det A = 1 \}
\]

\[
(1.2) = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|A^{-1}y|^{n+2s}} \, dy \right. \mid A > 0, \, \det A = 1 \}.
\]

We shall always use the definition that is most suitable to each case. Let us mention that, with either definition, if \(u\) is convex, asymptotically linear, and \(1/2 < s < 1\), then

\[
\lim_{s \to 1} \left( (1 - s) D_s u(x) \right) = \det(D^2 u(x))^{1/n},
\]

up to a constant factor that depends only on the dimension \(n\) (see the appendix for a proof of this fact).

We shall study the following Dirichlet problem,

\[
(1.3) \begin{cases}
D_s u(x) = g(x, u(x)) & \text{in } \mathbb{R}^n \\
(u - \phi)(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]

where \(1/2 < s < 1\) and we prescribe boundary data at infinity \(\phi(x)\) (that, at the same time, acts as a smooth lower barrier). Let us describe the precise hypothesis that \(g\) and \(\phi\) must satisfy.

First, \(\phi \in C^{2,\alpha}(\mathbb{R}^n)\) is strictly convex in compact sets and \(\phi \leq \Gamma + \eta\) near infinity, with \(\Gamma(x)\) a cone and

\[
|\eta(x)| \leq a|x|^{-\epsilon}, \quad |\nabla \eta(x)| \leq a|x|^{-(1+\epsilon)}, \quad \text{and} \quad |D^2 \eta(x)| \leq a|x|^{-(2+\epsilon)}
\]

for some constants \(a > 0\) and \(0 < \epsilon < n\). In particular, as \(|x| \to \infty\),

\[
-(-\Delta)^s \eta(x) = O\left(|x|^{-(2s+\epsilon)}\right)
\]
(see Section 2 for the definition of the fractional Laplacian) and
\[ c_1 |x|^{1-2s} \leq -(-\Delta)^s \Gamma(x) \leq c_2 |x|^{1-2s} \]
from homogeneity, where \( c_1, c_2 \) are some positive constants depending on the strict convexity of the section of \( \Gamma \). We normalize \( \phi \) so that \( \phi(0) = 0, \nabla \phi(0) = 0 \).

The model problem that we consider is \( g(x, u(x)) = u(x) - \phi(x) \). On the other hand, the results below can be proved under more general hypothesis, for instance that \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) satisfies
\[ g \text{ is globally semiconvex with constant } C, \]
\[ x \mapsto g(x, t) \text{ is Lipschitz continuous with constant } \text{Lip}(g), \]
and, there exists \( \mu > 0 \) such that
\[ g(x, t_1) - g(x, t_2) \geq \mu (t_1 - t_2) \quad \forall t_1, t_2 \in \mathbb{R} \text{ uniformly in } x. \]

We would like to point out that hypothesis (1.4) implies that the function \( g \) is locally uniformly elliptic and thus the known theory for uniformly elliptic nonlocal operators applies (see for instance [3] Proposition 2.1.7). In particular,
\[ \frac{|g(x, t) - g(y, t)|}{|x - y|} \leq \frac{2 \text{osc}(g(\cdot, t), B_{R/2}(x_0))}{R} + CR \quad \forall x, y \in B_{R/2}(x_0), \]
for any \( R > 0 \). Therefore, hypothesis (1.5) can be replaced, for instance, by the following
\[ x \mapsto g(x, t) \text{ is Lipschitz continuous in } \mathbb{R}^n \setminus B_{R_0}(0) \text{ for some radius } R_0 > 0 \]
with constant \( \text{Lip}(g, \mathbb{R}^n \setminus B_{R_0}(0)) \), uniformly in \( t \),
and
\[ \text{osc}(g(\cdot, t), B_{R_0/2}(x_0)) \text{ bounded in } t. \]

In the sequel, we shall assume (1.5) to simplify the presentation.

The paper is organized as follows. In Section 2 we present the notation to be used, the notion of solution, and some preliminary results. In Section 3 we prove a comparison principle for problem (1.3), that yields uniqueness of solutions and existence via Perron’s method in Section 4. In Section 5 we prove Lipschitz continuity and semiconcavity of solutions to problem (1.3). Finally, in Section 6 we prove the main result of the paper, namely, that the infimum in (1.2) cannot be realized by matrices that are too degenerate, effectively proving that the fractional Monge-Ampère operator is locally uniformly elliptic and thus the known theory for uniformly elliptic nonlocal operators applies (see for instance [3] and the references therein).

2. Notation and preliminaries

In this section we are going to state notations and recall some basic results and definitions.

For square matrices, \( A > 0 \) means positive definite and \( A \geq 0 \) positive semidefinite. We shall denote \( \lambda_1(A) \) the eigenvalues of \( A \), in particular \( \lambda_{\min}(A) \) and \( \lambda_{\max}(A) \) are the smallest and largest eigenvalues, respectively.

We shall denote the \( k \)-dimensional ball of radius 1 and center 0 by \( B_1^k(0) = \{ x \in \mathbb{R}^k : \sum_{i=1}^k x_i^2 \leq 1 \} \) and \( \partial B_1^k(0) \) will the corresponding \( k \)-dimensional sphere \( \{ x \in \mathbb{R}^k : \sum_{i=1}^k x_i^2 = 1 \} \). Whenever \( k \) is clear from context, we shall simply write \( B_1(0) \) and \( \partial B_1(0) \). \( \mathcal{H}^k \) stands for the \( k \)-dimensional Hausdorff measure. We shall denote \( \omega_k = \mathcal{H}^{k-1}(\partial B_1^k(0)) \).

Given a function \( u \), we shall denote the second-order increment of \( u \) at \( x \) in the direction of \( y \) as \( \delta(u, x, y) = u(x + y) + u(x - y) - 2u(x) \).
Let $A \subset \mathbb{R}^n$ be an open set. We say that a function $u : A \to \mathbb{R}$ is semiconcave if it is continuous in $A$ and there exists $C \geq 0$ such that $\delta(u, x, y) \leq C|y|^2$ for all $x, y \in \mathbb{R}^n$ such that $[x - y, x + y] \subset A$. The constant $C$ is called a semiconcavity constant for $u$ in $A$.

Alternatively, a function $u$ is semiconcave in $A$ with constant $C$ if $u(x) - \frac{C}{2}|x|^2$ is concave in $A$. Geometrically, this means that the graph of $u$ can be touched from above at every point by a paraboloid of the type $a + \langle b, x \rangle + \frac{C}{2}|x|^2$.

A function $u$ is called semiconcave in $A$ if $-u$ is semiconcave.

Let us mention here for the reader’s convenience the definition of the fractional Laplacian, 

$$-(-\Delta)^s u(x) = c_{n,s} \text{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|y - x|^{n+2s}} \, dy = \frac{c_{n,s}}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|y|^{n+2s}} \, dy$$

where $c_{n,s}$ is a normalization constant. Notice that $-c_{n,s}^{-1} (-\Delta)^s u(x)$ belongs to the class of operators over which the infimum in the definition of $\mathcal{D}_s u(x)$ is taken.

We recall from [3] the notion of viscosity solution that we are going to use in the sequel.

**Definition 2.1.** A function $u : \mathbb{R}^n \to \mathbb{R}$, upper (resp. lower) semicontinuous in $\overline{\Omega}$, is said to be a subsolution (supersolution) to $\mathcal{D}_s u = f$, and we write $\mathcal{D}_s u \geq f$ (resp. $\mathcal{D}_s u \leq f$), if every time all the following happen,

- $x$ is a point in $\Omega$,
- $N$ is an open neighborhood of $x$ in $\Omega$,
- $\psi$ is some $C^2$ function in $\overline{N}$,
- $\psi(x) = u(x)$,
- $\psi(y) > u(y)$ (resp. $\psi(y) < u(y)$) for every $y \in N \setminus \{x\}$,

and if we let

$$v := \begin{cases} \psi & \text{in } N \\ u & \text{in } \mathbb{R}^n \setminus N \end{cases},$$

then we have $\mathcal{D}_s v(x) \geq f(x)$ (resp. $\mathcal{D}_s v(x) \leq f(x)$). A solution is a function $u$ that is both a subsolution and a supersolution.

The following lemma states that $\mathcal{D}_s u$ can be evaluated classically at those points $x$ where $u$ can be touched by a paraboloid.

**Lemma 2.2.** Let $1/2 < s < 1$ and $u : \mathbb{R}^n \to \mathbb{R}$ with asymptotically linear growth. If we have $\mathcal{D}_s u \geq f$ in $\mathbb{R}^n$ (resp. $\mathcal{D}_s u \leq f$) in the viscosity sense and $\psi$ is a $C^2$ function that touches $u$ from above (below) at a point $x$, then $\mathcal{D}_s u(x)$ is defined in the classical sense and $\mathcal{D}_s u(x) \geq f(x)$ (resp. $\mathcal{D}_s u(x) \leq f(x)$).

**Proof.** Let us deal first with the subsolution case, that is, assume first that $\psi \in C^2$ touches $u$ from above at a point $x$. Define for $r > 0$,

$$v_r(y) = \begin{cases} \psi(y) & \text{in } B_r(x) \\ u(y) & \text{in } \mathbb{R}^n \setminus B_r(x). \end{cases}$$

Then, we have that

$$-c_{n,s}^{-1} (-\Delta)^s v_r(x) \geq \mathcal{D}_s v_r(x) \geq f(x)$$

and then the arguments in the proof of [3] Lemma 3.3] yield that $\delta(u, x, y)/|y|^{n+2s}$ is integrable. Therefore, $-(-\Delta)^s u(x)$ is defined in the classical sense and $\mathcal{D}_s u(x) < +\infty$. Notice that,

$$\lambda_{\min}(A)^{n+2s} \frac{\delta(u, x, y)}{|y|^{n+2s}} \leq \frac{\delta(u, x, y)}{\|A^{-1}y\|^{n+2s}} \leq \lambda_{\max}(A)^{n+2s} \frac{\delta(u, x, y)}{|y|^{n+2s}}.$$
Thus, \( \delta(u, x, y)/|A^{-1}y|^{n+2s} \) is integrable and

\[
L_A u(x) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u, x, y)}{|A^{-1}y|^{n+2s}} dy
\]

(2.1)
is also defined in the classical sense. By definition of viscosity solution, we have

\[
L_A u(x) + L_A (v_r - u)(x) \geq \mathcal{D}_x v_r(x) \geq f(x).
\]

But then, \( 0 \leq \delta(v_r - u, x, y) \leq \delta(v_{r_0} - u, x, y) \) for all \( r < r_0 \), \( \delta(v_{r_0} - u, x, y)/|A^{-1}y|^{n+2s} \) is integrable and \( \delta(v_r - u, x, y) \to 0 \) as \( r \to 0 \). Hence, by the dominated convergence theorem, \( L_A (v_r - u)(x) \to 0 \) as \( r \to 0 \). We conclude \( L_A u(x) \geq f(x) \) in the classical sense. Since the matrix \( A \) is arbitrary and we could pick any matrix \( A > 0 \) with \( \det A = 1 \), we have that \( \mathcal{D}_x u(x) \geq f(x) \) in the classical sense.

In the supersolution case, that is, when \( \psi \in \mathcal{C}^2 \) touches \( u \) from below at \( x \), some modifications are required. Fix \( \epsilon > 0 \), arbitrary, and let \( A_\epsilon > 0 \) with \( \det A_\epsilon = 1 \) such that

\[
L_{A_\epsilon} v_r(x) \leq f(x) + \epsilon.
\]

It is easy to see that \( \delta(v_r, x, y) \) is non-decreasing in \( r \) and \( \delta(v_r, x, y) \to \delta(u, x, y) \) as \( r \to 0 \). By the monotone convergence theorem, \( \delta(u, x, y)/|A_\epsilon^{-1}y|^{n+2s} \) is integrable and \( L_{A_\epsilon} u(x) \leq f(x) + \epsilon \) in the classical sense. We find that

\[
\mathcal{D}_x u(x) \leq L_{A_\epsilon} u(x) \leq f(x) + \epsilon,
\]

and we conclude letting \( \epsilon \to 0 \), since it is arbitrary. \( \square \)

### 3. Comparison and Uniqueness

Next, we prove a comparison principle that yields uniqueness for problem (1.3). In fact, the same proof applies to

\[
\begin{cases}
\mathcal{D}_A w = g(x, w) & \text{in } \mathbb{R}^n \\
(w - \phi)(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]

where

\[
\mathcal{D}_A w(x) = \inf \left\{ \int_{\mathbb{R}^n} \delta(w, x, y)K(y) dy \mid K \in \mathcal{A} \right\}
\]

(3.1)

for \( \mathcal{A} \) any class of positive, symmetric kernels for which the infimum is well-defined. Notice that the family \( \mathcal{A} \) can be very degenerate, as we do not require uniform ellipticity constants.

**Theorem 3.1.** Assume \( 1/2 < s < 1 \), and let \( g: \mathbb{R}^{n+1} \to \mathbb{R} \) a continuous function satisfying (1.6). Consider \( \phi \in \mathcal{C}^{2,\alpha}(\mathbb{R}^n) \), and \( u \in \text{USC} \) and \( v \in \text{LSC} \) such that

\[
\begin{cases}
\mathcal{D}_s u(x) \geq g(x, u) & \text{in } \mathbb{R}^n \\
(u - \phi)(x) \to 0 & \text{as } |x| \to \infty
\end{cases}
\]

and

\[
\begin{cases}
\mathcal{D}_s v(x) \leq g(x, v) & \text{in } \mathbb{R}^n \\
(v - \phi)(x) \to 0 & \text{as } |x| \to \infty
\end{cases}
\]

in the viscosity sense. Then, \( u \leq v \) in \( \mathbb{R}^n \).

**Remark 3.2.** Some of the hypotheses can be weakened. In particular, it is enough to assume \( t \mapsto g(x, t) \) strictly increasing for any \( x \in \mathbb{R}^n \) instead of (1.6) to derive a contradiction in (3.13).

**Proof.** Let us first present the ideas of the proof in the case when \( u, v \) are a classical sub- and supersolution, then we shall consider the viscosity counterparts.

Since we seek to prove \( u \leq v \), let us assume to the contrary that \( \sup_{\mathbb{R}^n} (u - v) > 0 \). As \( (u - v)(x) \to 0 \) as \( |x| \to \infty \), there exists \( x_0 \in \mathbb{R}^n \) such that

\[
(u - v)(x_0) = \sup_{\mathbb{R}^n} (u - v) > 0.
\]
Fix $\delta > 0$, arbitrary, and let $A_\delta > 0$ with $\det A_\delta = 1$, such that

$$L_{A_\delta}v(x_0) \leq D_s v(x_0) + \delta \leq g(x_0, v(x_0)) + \delta,$$

for $L_{A_\delta}$ defined as in (2.1). On the other hand, for the same matrix,

$$L_{A_\delta}u(x_0) \geq D_s u(x_0) \geq g(x_0, u(x_0)).$$

At a maximum point $\delta(u - v, x_0, y) \leq 0$, and

$$0 \geq L_{A_\delta}(u - v)(x_0) \geq g(x_0, u(x_0)) - g(x_0, v(x_0)) - \delta.$$

Therefore, since $\delta$ is arbitrary, we can let $\delta \to 0$ and get

$$g(x_0, v(x_0)) \geq g(x_0, u(x_0)),$$

a contradiction with the fact that $g(x_0, \cdot)$ is strictly increasing.

In the general case, we cannot be certain that $L_{A_\delta}u(x_0)$ and $L_{A_\delta}v(x_0)$ above are well defined, since $u$ and $v$ may not have the necessary regularity. To remedy that we shall use sup- and inf-convolutions and work with regularized functions. However, we shall rather apply the regularizations to the functions $\bar{u} = u - \phi$ and $\bar{v} = v - \phi$, since they are bounded above and below respectively (notice that $\bar{u} \in USC$, $\bar{v} \in LSC$, and $\bar{u}(x), \bar{v}(x) \to 0$ as $|x| \to \infty$ imply that $\bar{u}, \bar{v}$ have respectively a maximum and a minimum).

Consider the sup- and inf-convolution of $\bar{u}, \bar{v}$, respectively,

$$\bar{u}^\epsilon(x) = \sup_y \left\{ \bar{u}(y) - \frac{|x - y|}{\epsilon} \right\},$$

and

$$\bar{v}_\epsilon(x) = \inf_y \left\{ \bar{v}(y) + \frac{|x - y|}{\epsilon} \right\}.$$

Before we start with the proof, let us recall for the reader’s convenience two properties of $\bar{u}^\epsilon$ that we shall use in the sequel. Analogous properties hold for $\bar{v}_\epsilon$, noticing that $\bar{v}_\epsilon = -(-\bar{v})^\epsilon$.

1. $\bar{u}^\epsilon$ is bounded above. Since $\bar{u}$ is bounded above by some constant $C$, we have

$$\bar{u}^\epsilon(x) \leq \sup_y \left\{ C - \frac{|x - y|}{\epsilon} \right\} = C.$$

2. The supremum in the definition of (3.2) is achieved. In fact,

$$\bar{u}^\epsilon(x) = \sup_{|y - x|^2 \leq 2\|\bar{u}\|_{\infty}} \left\{ \bar{u}(y) - \frac{|x - y|}{\epsilon} \right\} = \bar{u}(x^*) - \frac{|x - x^*|^2}{\epsilon}$$

for some $x^*$ such that

$$|x - x^*|^2 \leq 2\|\bar{u}\|_{\infty}$$

(here we are slightly abusing notation for the sake of brevity since, as $\bar{u} \in USC$, we should write $\sup \bar{u}$ instead of $\|\bar{u}\|_{\infty}$). To see this, first notice that since $\bar{u}^\epsilon$ is bounded above, for any given $\delta > 0$ there exists $x_\delta$ such that,

$$\bar{u}^\epsilon(x) = \sup_y \left\{ \bar{u}(y) - \frac{|x - y|^2}{\epsilon} \right\} \leq \bar{u}(x_\delta) - \frac{|x - x_\delta|^2}{\epsilon} + \delta.$$

Since $\bar{u}(x) \leq \bar{u}^\epsilon(x)$ (pick $y = x$ in the definition of $\bar{u}^\epsilon(x)$), we conclude that $|x - x_\delta|^2 \leq (2\|\bar{u}\|_{\infty} + 1)\epsilon$, assuming $\delta < 1$. Therefore,

$$\bar{u}^\epsilon(x) \leq \sup_{|y - x|^2 \leq (2\|\bar{u}\|_{\infty} + 1)\epsilon} \left\{ \bar{u}(y) - \frac{|x - y|^2}{\epsilon} \right\} + \delta.$$
Since \( \delta \) is arbitrary, we can let \( \delta \to 0 \) and conclude that the supremum in the definition of (3.2) is achieved,

\[
\bar{u}^\epsilon(x) = \sup_{|y-x|^2 \leq (2\|u\|_\infty + 1)\epsilon} \left\{ \bar{u}(y) - \frac{|x-y|^2}{\epsilon} \right\}.
\]

At this point, we can repeat the previous argument with \( \delta = 0 \) and get formula (3.3).

Now, again for the sake of contradiction, assume \( \sup_{\mathbb{R}^n}(u - v) > 0 \). Notice that \( \bar{u}^\epsilon(x) - \bar{v}(x) \geq \bar{u}(x) - \bar{v}(x) \) (pick \( y = x \) in the definitions of \( \bar{u}^\epsilon(x), \bar{v}(x) \)), and therefore,

\[
(3.5) \quad \sup_{\mathbb{R}^n}(\bar{u}^\epsilon - \bar{v}) \geq \sup_{\mathbb{R}^n}(\bar{u} - \bar{v}) = \sup_{\mathbb{R}^n}(u - v) > 0.
\]

Moreover, \( (\bar{u}^\epsilon - \bar{v})(x) \to 0 \) as \( |x| \to \infty \). To see this, notice that

\[
\bar{u}(x) - \bar{v}(x) \leq \bar{u}^\epsilon(x) - \bar{v}(x)
\]

\[
= \sup_{|y|^2 \leq 2\|u\|_\infty \epsilon} \left\{ \bar{u}(x+y) - \frac{|y|^2}{\epsilon} \right\} - \inf_{|y|^2 \leq 2\|u\|_\infty \epsilon} \left\{ \bar{v}(x+y) + \frac{|y|^2}{\epsilon} \right\}
\]

\[
\leq \sup_{|y|^2 \leq 2\|u\|_\infty \epsilon} \bar{u}(x+y) - \inf_{|y|^2 \leq 2\|u\|_\infty \epsilon} \bar{v}(x+y),
\]

and \( \bar{u}(x) - \bar{v}(x), \sup_{|y|^2 \leq 2\|u\|_\infty \epsilon} \bar{u}(x+y), \text{ and } \inf_{|y|^2 \leq 2\|u\|_\infty \epsilon} \bar{v}(x+y) \) converge to 0 as \( |x| \to \infty \).

Thus, there exists \( x_\epsilon \) such that

\[
(3.6) \quad (\bar{u}^\epsilon - \bar{v})(x_\epsilon) = \sup_{\mathbb{R}^n}(\bar{u}^\epsilon - \bar{v}).
\]

An important point in the sequel is that both functions \( \bar{u}^\epsilon \) and \( \bar{v} \) are \( C^{1,1} \) at \( x_\epsilon \), so that the integrals in the operators appearing in the subsequent computations are well-defined. This follows from the following three facts:

- The paraboloid

\[
P(x) = \bar{u}(x_\epsilon^*) - \frac{|x-x_\epsilon^*|^2}{\epsilon}
\]

touches \( \bar{u}^\epsilon \) from below at \( x_\epsilon \) for \( x_\epsilon^* \) such that \( \bar{u}^\epsilon(x_\epsilon) = \bar{u}(x_\epsilon^*) - \frac{|x_\epsilon-x_\epsilon^*|^2}{\epsilon} \).

- The paraboloid

\[
Q(x) = \bar{v}(x_\epsilon^*) + \frac{|x-x_\epsilon^*|^2}{\epsilon}
\]

touches \( \bar{v} \) from above at \( x_\epsilon \) for \( x_\epsilon^* \) such that \( \bar{v}(x_\epsilon) = \bar{v}(x_\epsilon^*) + \frac{|x_\epsilon-x_\epsilon^*|^2}{\epsilon} \).

- Since \( x_\epsilon \) is a maximum point of \( \bar{u}^\epsilon - \bar{v} \), the function \( \bar{v}(x_\epsilon) - \bar{v}(x_\epsilon) + \bar{u}^\epsilon(x_\epsilon) \) touches \( \bar{u}^\epsilon \) from above at \( x_\epsilon \).

We conclude from these three facts that the paraboloids \( Q(x) - \bar{v}(x_\epsilon) + \bar{u}^\epsilon(x_\epsilon) \) and \( P(x) + \bar{v}(x_\epsilon) - \bar{u}^\epsilon(x_\epsilon) \) touch respectively \( \bar{u}^\epsilon \) from above and \( \bar{v} \) from below at the point \( x_\epsilon \). Therefore, both \( \bar{u}^\epsilon \) and \( \bar{v} \) can be touched from above and below by a paraboloid at \( x_\epsilon \) and they are \( C^{1,1} \) at \( x_\epsilon \).

The fact that both \( \bar{u}^\epsilon \) and \( \bar{v} \) are \( C^{1,1} \) at \( x_\epsilon \) is crucial to make rigorous the formal argument described at the beginning of the proof. Since \( \bar{u} \in C^{1,1} \) at \( x_\epsilon \), there exists a paraboloid \( P(x) \) that touches \( \bar{u}^\epsilon \) from above at \( x_\epsilon \). Then, the function

\[
\tilde{P}(x) = P(x + x_\epsilon - x_\epsilon^*) + \frac{|x_\epsilon - x_\epsilon^*|^2}{\epsilon} + \phi(x)
\]

touches \( u \) from above at \( x_\epsilon^* \). On the other hand, there exists a paraboloid \( Q(x) \) that touches \( \bar{v} \) from below at \( x_\epsilon \) and then, the function

\[
\tilde{Q}(x) = Q(x + x_\epsilon - x_\epsilon^*) - \frac{|x_\epsilon - x_\epsilon^*|^2}{\epsilon} + \phi(x)
\]
touches \( v \) from below at \( x_{\epsilon,*} \). By Lemma 2.2 we have

\[
\mathcal{D}_s u(x^*_\epsilon) \geq g(x^*_\epsilon, u(x^*_\epsilon)), \quad \text{and} \quad \mathcal{D}_s v(x_{\epsilon,*}) \leq g(x_{\epsilon,*}, v(x_{\epsilon,*}))
\]

in the classical sense.

Fix \( \eta > 0 \), arbitrary, and let \( A_\eta > 0 \) with \( \det A_\eta = 1 \) such that

\[
L_{A_\eta} v(x_{\epsilon,*}) \leq g(x_{\epsilon,*}, v(x_{\epsilon,*})) + \eta,
\]

and

\[
L_{A_\eta} u(x^*_\epsilon) \geq \mathcal{D}_s u(x^*_\epsilon) \geq g(x^*_\epsilon, u(x^*_\epsilon)),
\]

with \( L_{A_\eta} \) defined as in (2.1). Subtracting, we get

\[
(3.7) \quad L_{A_\eta} u(x^*_\epsilon) - L_{A_\eta} v(x_{\epsilon,*}) \geq g(x^*_\epsilon, u(x^*_\epsilon)) - g(x_{\epsilon,*}, v(x_{\epsilon,*})) - \eta.
\]

The rest of the proof is devoted to derive a contradiction from the previous inequality by showing that, for \( \epsilon \) small enough, the left-hand side is strictly smaller than the right-hand side.

Let us prove first that,

\[
(3.8) \quad \lim_{\epsilon \to \infty} (L_{A_\eta} u(x^*_\epsilon) - L_{A_\eta} v(x_{\epsilon,*})) \leq 0.
\]

By definition of the operator \( L_{A_\eta} \), we have

\[
(3.9) \quad L_{A_\eta} u(x^*_\epsilon) - L_{A_\eta} v(x_{\epsilon,*}) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u, x^*_\epsilon, y) - \delta(v, x_{\epsilon,*}, y)}{|A_\eta^{-1} y|^{n+2s}} dy.
\]

Notice that

\[
(3.10) \quad \delta(\bar{u}^\epsilon, x_\epsilon, y) \geq \delta(\bar{u}, x_\epsilon, y) \quad \text{and} \quad \delta(\bar{v}, x_\epsilon, y) \leq \delta(\bar{v}, x_{\epsilon,*}, y).
\]

Since the proof of both inequalities is analogous, let us show how to obtain the first one. As we have seen,

\[
\bar{u}'(x^*_\epsilon) = \bar{u}(x^*_\epsilon) - \frac{|x^*_\epsilon - x^*_\epsilon|^2}{\epsilon},
\]

On the other hand, picking \( z = x^*_\epsilon - x_\epsilon \),

\[
\bar{u}^\epsilon(x^*_\epsilon \pm y) = \sup_z \left\{ \bar{u}(x^*_\epsilon \pm y + z) - \frac{|z|^2}{\epsilon} \right\} \geq \bar{u}(x^*_\epsilon \pm y) - \frac{|x^*_\epsilon - x^*_\epsilon|^2}{\epsilon}.
\]

From these two expressions, we get (3.10).

Now, using (3.10), we have that

\[
\delta(u, x^*_\epsilon, y) - \delta(v, x_{\epsilon,*}, y) = \delta(\bar{u}, x^*_\epsilon, y) - \delta(\bar{v}, x_{\epsilon,*}, y) + \delta(\phi, x^*_\epsilon, y) - \delta(\phi, x_{\epsilon,*}, y)
\]

\[
\leq \delta(\bar{u}', x^*_\epsilon, y) - \delta(\bar{v}, x_{\epsilon,*}, y) + \delta(\phi, x^*_\epsilon, y) - \delta(\phi, x_{\epsilon,*}, y)
\]

\[
= \delta(\bar{u}' - \bar{v}, x_{\epsilon,*}, y) + \delta(\phi, x^*_\epsilon, y) - \delta(\phi, x_{\epsilon,*}, y).
\]

Observe that \( x^*_\epsilon \) is a maximum point of \( \bar{u}' - \bar{v} \) and therefore \( \delta(\bar{u}' - \bar{v}, x_{\epsilon,*}, y) \leq 0 \). We conclude

\[
(3.11) \quad \delta(u, x^*_\epsilon, y) - \delta(v, x_{\epsilon,*}, y) \leq \delta(\phi, x^*_\epsilon, y) - \delta(\phi, x_{\epsilon,*}, y).
\]

From (3.7), (3.9) and (3.11), we get

\[
L_{A_\eta} \phi(x^*_\epsilon) - L_{A_\eta} \phi(x_{\epsilon,*}) = \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(\phi, x^*_\epsilon, y) - \delta(\phi, x_{\epsilon,*}, y)}{|A_\eta^{-1} y|^{n+2s}} dy
\]

\[
\geq \frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u, x^*_\epsilon, y) - \delta(v, x_{\epsilon,*}, y)}{|A_\eta^{-1} y|^{n+2s}} dy = L_{A_\eta} u(x^*_\epsilon) - L_{A_\eta} v(x_{\epsilon,*})
\]

\[
\geq g(x^*_\epsilon, u(x^*_\epsilon)) - g(x^*_\epsilon, v(x_{\epsilon,*})) + g(x^*_\epsilon, v(x_{\epsilon,*})) - g(x_{\epsilon,*}, v(x_{\epsilon,*})) - \eta.
\]
Recall from (3.5) and (3.6) that
\[ (\bar{u}^\epsilon - \bar{v})_\epsilon(x_\epsilon) = \sup_{\mathbb{R}^n} (\bar{u}^\epsilon - \bar{v})_\epsilon \geq \sup_{\mathbb{R}^n} (\bar{u} - \bar{v}) > 0. \]
Therefore,
\[ \bar{u}(x^*_\epsilon) - \bar{v}(x^*_\epsilon) \geq \sup_{\mathbb{R}^n} (\bar{u} - \bar{v}) + \frac{|x_\epsilon - x^*_\epsilon|^2}{\epsilon} \]
or equivalently,
\[ u(x^*_\epsilon) - v(x^*_\epsilon) \geq \sup_{\mathbb{R}^n} (\bar{u} - \bar{v}) + (\phi(x^*_\epsilon) - \phi(x^*_\epsilon)) + \frac{|x_\epsilon - x^*_\epsilon|^2}{\epsilon}, \]
Notice that from estimate (3.4) and its analogous for the inf-convolution, we have
\[ |x_\epsilon - x^*_\epsilon|^2 \leq 2\|\bar{u}\|_\infty \epsilon \quad \text{and} \quad |x_\epsilon - x^*_\epsilon|^2 \leq 2\|\bar{v}\|_\infty \epsilon. \]
Thus, by the continuity of \( \phi \), we have that for \( \epsilon \) small enough,
\[ u(x^*_\epsilon) - v(x^*_\epsilon) \geq \frac{1}{2} \sup_{\mathbb{R}^n} (\bar{u} - \bar{v}) > 0. \]
Since \( \phi \in C^{2,\alpha} \), in particular \( L_{A_0} \phi(x) \) is a continuous function and, for \( \epsilon \) small enough
\[ (L_{A_0} \phi(x^*_\epsilon) - L_{A_0} \phi(x^*_\epsilon)) \leq \eta. \]
By the continuity of \( g \), we can also assume that \( g(x^*_\epsilon, v(x^*_\epsilon)) - g(x^*_\epsilon, v(x^*_\epsilon)) \geq -\eta \). Then, we have from (3.12) and (1.6) that
\[ 3\eta \geq g(x^*_\epsilon, u(x^*_\epsilon)) - g(x^*_\epsilon, v(x^*_\epsilon)) \geq \mu (u(x^*_\epsilon) - v(x^*_\epsilon)) \geq \frac{\mu}{2} \sup_{\mathbb{R}^n} (\bar{u} - \bar{v}) > 0. \]
Since \( \eta \) is arbitrary, we can choose \( \eta \leq \frac{\mu}{12} \sup_{\mathbb{R}^n} (\bar{u} - \bar{v}) \) and get a contradiction. \( \square \)

4. Existence of solutions

In this section we prove existence of solutions to the problem
\[
\begin{cases}
D_A w = w - \phi & \text{in } \mathbb{R}^n \\
(w - \phi)(x) \to 0 & \text{as } |x| \to \infty,
\end{cases}
\]
where
\[ D_A w(x) = \inf \left\{ \int_{\mathbb{R}^n} \delta(w, x, y) K(y) dy \left| K \in \mathcal{A} \right. \right\} \]
for \( \mathcal{A} \) any class of positive, symmetric kernels that contains \( K(y) = |y|^{-(n+2s)} \) for which the infimum is well-defined. Notice that the family \( \mathcal{A} \) can be very degenerate, as we do not require uniform ellipticity constants.

Lemma 4.1. Denote \( g(x) = \min\{1, |x|^{-(2s+\tau)}\} \) and \( u_F(x) = C_F \cdot |x|^{2s-n} \), the fundamental solution of \((-\Delta)^s\) for an appropriate constant \( C_F \). Then, there exist constants \( 0 < \tau < \min\{2s - 1, n - 2s\} \) and \( M > 0 \) such that \( u = \phi + M \cdot (u_F * g) \) satisfies
\[
\begin{cases}
-(-\Delta)^s u \leq c_{n,s} (u - \phi) & \text{in } \mathbb{R}^n \\
(u - \phi)(x) \to 0 & \text{as } |x| \to \infty.
\end{cases}
\]
Proof. We construct an upper barrier of the form \( u = \phi + w \) as a potential. We start the construction of \( w \) with \( w_0 = u_F \ast g_0 \) where \( g_0(x) = |x|^{-(2s+\tau)} \) for some small \( 0 < \tau < n-2s \). Since both \( n-2s < n \) and \( 2s+\tau < n \) while \( (n-2s) + (2s+\tau) > n \), there are constants \( a_0, a_1 > 0 \) such that
\[
|a_0|x^{-\tau} \leq w_0(x) \leq a_1|x|^{-\tau} \quad \text{as } |x| \to \infty.
\]
Also, by construction,
\[
(-\Delta)^s w_0(x) = g_0(x) = |x|^{-(2s+\tau)}.
\]
Notice that \( w_0 \) decays at infinity (and therefore \( u - \phi = w_0 \to 0 \) as \( |x| \to \infty \)) but it is not bounded at 0. Consequently, we truncate \( g_0 \) and define \( g_1 = \min\{1, g_0\} \) and \( w_1 = u_F \ast g_1 \).

The function \( w_1 \) is bounded, still radially decreasing, and has the same decay as \( w_0 \). To prove the last assertion, first notice that
\[
w_1 = u_F \ast g_1 = w_0 - u_F \ast (g_0 - g_1).
\]
The function \( g_0 - g_1 \) is supported in the ball of radius one, therefore
\[
(u_F \ast (g_0 - g_1))(x) = C_F \int_{B_1(0)} \left(|y|^{-(2s+\tau)} - 1\right)|x-y|^{2s-n} \, dy.
\]
For \( |y| \leq 1 \) and \( |x| > 2 \) we have \(|y| < |x|/2\) and from there we deduce
\[
\left(\frac{2}{3}\right)^{n-2s} |x|^{2s-n} \leq |x-y|^{2s-n} \leq 2^{n-2s} |x|^{2s-n}.
\]
On the other hand, \( \tau < n-2s \) implies that
\[
\int_{B_1(0)} \left(|y|^{-(2s+\tau)} - 1\right) \, dy
\]
is constant and therefore
\[
b_0|x|^{2s-n} \leq (u_F \ast (g_1 - g_0))(x) \leq b_1|x|^{2s-n} \quad \text{as } |x| \to \infty
\]
form some constants \( b_0, b_1 > 0 \). Again, since \( \tau < n-2s \) \( \langle 12 \rangle \), \( \langle 13 \rangle \), and \( \langle 14 \rangle \) prove that \( w_1 \) has the same decay as \( w_0 \) as claimed.

Moreover, there exist constants \( A_0, A_1 > 0 \) such that
\[
A_0 \min\{1, |x|^{-\tau}\} \leq w_1(x) \leq A_1 \min\{1, |x|^{-\tau}\}.
\]
Also, by construction, we have
\[
(-\Delta)^s w_1(x) = g_1(x) = \min\{1, |x|^{-(2s+\tau)}\}.
\]
As a third and final step, we dilate \( w_1 \) to our final \( w \). To this aim, define \( w(x) = M \cdot w_1(x) \) for some large constant \( M \) to be chosen. Then,
\[
(-\Delta)^s w = M \cdot g_1(x) = M \min\{1, |x|^{-(2s+\tau)}\}
\]
and
\[
w(x) \geq MA_0 \min\{1, |x|^{-\tau}\}.
\]

We are ready to check that for an appropriate \( M \) the function \( u = \phi + w \) satisfies
\[
-(\Delta)^s u \leq c_{n,s}(u - \phi).
\]
Indeed,
\[
c_{n,s}(u - \phi) = c_{n,s}w \geq c_{n,s}MA_0 \min\{1, |x|^{-\tau}\}
\]
where \( c_{n,s}, A_0 \) are given and \( M \) is to be chosen. On the other hand,
\[
-(\Delta)^s u(x) = -(\Delta)^s w(x) - (\Delta)^s \phi(x)
\]
\[
= -M \min\{1, |x|^{-(2s+\tau)}\} - (\Delta)^s \phi(x) \leq -(\Delta)^s \phi(x).
\]
From our hypotheses on $\phi$,

$$-(-\Delta)^s \phi(x) = -(\Delta)^s \Gamma(x) - (-\Delta)^s \eta(x) \leq C(|x|^{1-2s} + |x|^{-(2s+\epsilon)}) \leq C|x|^{1-2s}$$

for $|x|$ large, while $-(-\Delta)^s \phi(x)$ is bounded in every neighborhood of the origin. Therefore

$$\tag{4.7} - (-\Delta)^s \phi(x) \leq C \cdot \min\{1, |x|^{1-2s}\}$$

for some constant $C > 0$.

In view of (4.5), (4.6), and (4.7), we only have to control $C \cdot \min\{1, |x|^{1-2s}\}$ by $c_{n,s} MA_0 \min\{1, |x|^{-\tau}\}$ to conclude. If $\tau < 2s - 1$, a large value of $M$ does it.

**Proposition 4.2. (Existence of solutions)** There exists a unique solution of

$$\tag{4.8} \begin{cases} D_A u = u - \phi & \text{in } \mathbb{R}^n \\ (u - \phi)(x) \to 0 & \text{as } |x| \to \infty. \end{cases}$$

**Proof.** First, observe that $\phi$ is a subsolution to the problem, since by convexity $\delta(\phi, x, y) \geq 0$, and therefore $D_A u(x) \geq 0$.

The idea to find a supersolution is that, by the definition of $D_A$ as an infimum of linear operators, it is enough to have the appropriate inequality for just one of them. In particular, $-c^{-1}_{n,s} (-\Delta)^s$ is one of the operators that compete for the infimum and we know from Lemma 4.1 that there is a function $\bar{u}$ such that $-c^{-1}_{n,s} (-\Delta)^s \bar{u} \leq \bar{u} - \phi$ with the right “boundary data at infinity”, that is, $(\bar{u} - \phi) \to 0$ as $|x| \to \infty$. We have,

$$D_A \bar{u}(x) \leq -c^{-1}_{n,s} (-\Delta)^s \bar{u}(x) \leq \bar{u} - \phi.$$  

By comparison, see Theorem 3.1 $\phi \leq \bar{u}$ and then Perron’s method implies existence of a unique solution $u$ to problem (4.8). \qed

### 5. Some Regularity Questions

In this section, we prove Lipschitz continuity and semiconcavity of solutions to

$$\tag{5.1} \begin{cases} D_s v(x) = g(x, v(x)) & \text{in } \mathbb{R}^n \\ (v - \phi)(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

with $\phi$ under the hypothesis of Section 4. These results will be needed in Section 6 to prove the main result of the paper.

Let us point out that the proofs can be used for the more general operators $D_A$ defined in (3.1). We start with the particular case when $g(x, v(x)) = v(x) - \phi(x)$ to illustrate the key ideas.

**Proposition 5.1.** Assume $\phi$ is semiconcave and Lipschitz continuous and let $v$ be the solution of

$$\tag{5.2} \begin{cases} D_s v(x) = v(x) - \phi(x) & \text{in } \mathbb{R}^n \\ (v - \phi)(x) \to 0 & \text{as } |x| \to \infty \end{cases}$$

Then, $v$ is Lipschitz continuous and semiconcave with the same constants as $\phi$.

**Proof.** In the following proof, we assume for clarity of presentation that $v$ is a classical solution to (5.2) and all the equations hold pointwise. The argument can be made rigorous using a regularization argument (similar to the one in the proof of Theorem 3.1) that is explained in detail in the proofs of the more general results Propositions 5.2 and 5.3 below, so we shall skip it here.

1. For the proof of Lipschitz continuity, fix $e \in \mathbb{R}^n$ and consider the first-order incremental quotient $v(x + e) - v(x)$. Observe that

$$v(x + e) - v(x) = (v - \phi)(x + e) - (v - \phi)(x) + \phi(x + e) - \phi(x) \leq o(1) + \text{Lip}(\phi) |e|$$
as $|x| \to \infty$, and therefore $v(x+e) - v(x)$ is bounded above. Furthermore, we can assume that
\[ \sup_{x \in \mathbb{R}^n} (v(x+e) - v(x)) > \text{Lip}(\phi) |e|, \]
since we are done otherwise. Then, there exists some $x_0$ such that
\[ v(x_0 + e) - v(x_0) = \sup_{x \in \mathbb{R}^n} (v(x+e) - v(x)). \]

Fix $\eta > 0$, arbitrary, and let $A_\eta > 0$ such that
\[ L_{A_\eta} v(x_0) \leq v(x_0) - \phi(x_0) + \eta, \]
and
\[ L_{A_\eta} v(x_0 + e) \geq D_s v(x_0 + e) \geq v(x_0 + e) - \phi(x_0 + e), \]
with $L_{A_\eta}$ defined as in (2.1). We have from the above expressions that
\[ L_{A_\eta} (v(x_0 + e) - v(x_0)) \geq (v(x_0 + e) - v(x_0)) - (\phi(x_0 + e) - \phi(x_0)) - \eta. \]
Notice that $\delta(v(x_0 + e) - v(x_0), y) \leq 0$, and therefore $L_{A_\eta} (v(x_0 + e) - v(x_0)) \leq 0$. Consequently,
\[ \sup_{x \in \mathbb{R}^n} (v(x_e) - v(x)) = v(x_0 + e) - v(x_0) \leq \text{Lip}(\phi) |e| + \eta \]
and we conclude letting $\eta \to 0$.

A symmetric argument, where $x_0$ is a point such that
\[ v(x_0 + e) - v(x_0) = \inf_{x \in \mathbb{R}^n} (v(x + e) - v(x)) < -\text{Lip}(\phi) |e|, \]
and the operator $L_{A_\eta}$ is such that,
\[ L_{A_\eta} v(x_0 + e) \leq v(x_0 + e) - \phi(x_0 + e) + \eta, \]
and
\[ L_{A_\eta} v(x_0) \geq v(x_0) - \phi(x_0) \]
yields
\[ \inf_{x \in \mathbb{R}^n} (v(x + e) - v(x)) \geq -\text{Lip}(\phi) |e|. \]

2. For the proof of semiconcavity, consider the second-order incremental quotient $\delta(v, x, e) = v(x+e) + v(x-e) - 2v(x)$. Denote by $SC(\phi)$ the semiconcavity constant of $\phi$, and notice that
\[ \delta(v, x, e) = \delta(v - \phi, x, e) + \delta(\phi, x, e) \leq o(1) + SC(\phi) |e|^2 \quad \text{as } |x| \to \infty \]
so $\delta(v, x, e)$ is bounded above. Furthermore, we can assume that
\[ \sup_{x \in \mathbb{R}^n} \delta(v, x, e) > SC(\phi) |e|^2 \]
since we are done otherwise. Then, there exists some $x_0$ such that
\[ \delta(v, x_0, e) = \sup_{x \in \mathbb{R}^n} \delta(v, x, e). \]
As before, fix $\eta > 0$ arbitrary, and let $A_\eta > 0$ such that
\[ L_{A_\eta} v(x_0) \leq v(x_0) - \phi(x_0) + \eta, \]
and
\[ L_{A_\eta} v(x_0 + e) \geq D_s v(x_0 + e) \geq v(x_0 + e) - \phi(x_0 + e), \]
with $L_{A_\eta}$ defined as in (2.1). We have from the above expressions that
\[ L_{A_\eta} \delta(v, x_0, e) \geq \delta(v, x_0, e) - \delta(\phi, x_0, e) - 2\eta. \]
Notice that \( \delta(\delta(v, e), x_0, z) \leq 0 \), and therefore \( L_{A_\eta} \delta(v, x_0, e) \leq 0 \). Consequently,
\[
\delta(v, x, e) \leq \delta(v, x_0, e) \leq \delta(\phi, x_0, e) + 2\eta \leq SC|e|^2 + 2\eta.
\]
We conclude letting \( \eta \to 0 \). \( \square \)

In the next result we prove that solutions to (5.1) are Lipschitz continuous whenever \( g \) on the right-hand side satisfies (1.5) and (1.6).

**Proposition 5.2 (Lipschitz continuity of the solution).** Let \( g : \mathbb{R}^{n+1} \to \mathbb{R} \) satisfy (1.5) and (1.6). Then, \( v \), the solution to (5.1), is uniformly Lipschitz continuous, namely, for every \( x, y \in \mathbb{R}^n \),
\[
\frac{|v(x) - v(y)|}{|x - y|} \leq \max \left\{ \frac{\text{Lip}(g)}{\mu}, \text{Lip}(\phi) \right\}.
\]

**Proof.** The following proof uses a regularization process similar to the proof of Theorem 3.1. For the sake of clarity, let us present first the main ideas assuming that \( v \) is a classical solution.

Fix \( e \in \mathbb{R}^n \) and consider the first-order incremental quotient \( v(x + e) - v(x) \). Observe that
\[
v(x + e) - v(x) = (v - \phi)(x + e) - (v - \phi)(x) + \phi(x + e) - \phi(x) \leq o(1) + \text{Lip}(\phi)|e|
\]
as \( |x| \to \infty \), and therefore \( v(x + e) - v(x) \) is bounded above. Furthermore, we can assume that
\[
\sup_{x \in \mathbb{R}^n} (v(x + e) - v(x)) > \text{Lip}(\phi)|e|,
\]
since we are done otherwise. Then, there exists some \( x_0 \) such that
\[
v(x_0 + e) - v(x_0) = \sup_{x \in \mathbb{R}^n} (v(x + e) - v(x)).
\]

Fix \( \eta > 0 \), arbitrary, and let \( A_\eta > 0 \) with \( \text{det} A_\eta = 1 \) such that
\[
L_{A_\eta} v(x_0) \leq g(x_0, v(x_0)) + \eta,
\]
and
\[
L_{A_\eta} v(x_0 + e) \geq D_s v(x_0 + e) \geq g(x_0 + e, v(x_0 + e)),
\]
with \( L_{A_\eta} \) defined as in (2.1).

We have from the above expressions that
\[
L_{A_\eta} v(x_0 + e) - L_{A_\eta} v(x_0) \geq g(x_0 + e, v(x_0 + e)) - g(x_0, v(x_0)) - \eta.
\]
Notice that \( \delta(v(\cdot + e) - v, x_0, y) \leq 0 \), and therefore \( L_{A_\eta} (v(x_0 + e) - v(x_0)) \leq 0 \). Consequently,
\[
g(x_0 + e, v(x_0 + e)) - g(x_0, v(x_0)) \leq g(x_0 + e, v(x_0)) \pm g(x_0 + e, v(x_0)) \leq \eta.
\]
At this point we can let \( \eta \to 0 \) and, using (1.5) and (1.6), get
\[
(5.3) \quad v(x + e) - v(x) \leq v(x_0 + e) - v(x_0) \leq \frac{\text{Lip}(g)}{\mu} |e|.
\]

A symmetric argument, where \( x_0 \) is a point such that
\[
v(x_0 + e) - v(x_0) = \inf_{x \in \mathbb{R}^n} (v(x + e) - v(x)) < -\text{Lip}(\phi)|e|,
\]
and the operator \( L_{A_\eta} \) is such that,
\[
L_{A_\eta} v(x_0 + e) \leq g(x_0 + e, v(x_0 + e)) + \eta,
\]
and
\[
L_{A_\eta} v(x_0) \geq g(x_0, v(x_0))
\]
yields
\[
g(x_0, v(x_0)) - g(x_0 + e, v(x_0 + e)) \pm g(x_0 + e, v(x_0)) \leq 0.
\]

\[14\]
and from there,
\[-\frac{\text{Lip}(g)}{\mu}|e| \leq v(x_0 + e) - v(x_0) \leq v(x + e) - v(x)\].

In general, in the above argument we cannot guarantee that \(v\) is regular enough so that both \(L_{A_0}v(x_0 + e)\) and \(L_{A_0}v(x_0)\) are well-defined and the corresponding equations hold in the classical sense.

To complete the argument, we are going to use a regularization process similar to the one in the proof of Theorem 3.1. Let us show the details in the proof of (5.3).

To simplify the notation in the sequel, let us denote \(u(x) = v(x + e)\) and consider the sup- and inf-convolution of \(u\), and \(v\), respectively,
\[u^\varepsilon(x) = \sup_y \left\{ u(y) - \frac{|x - y|^2}{\varepsilon} \right\} = \sup_y \left\{ v(y + e) - \frac{|x - y|^2}{\varepsilon} \right\}\]
and
\[v_\varepsilon(x) = \inf_y \left\{ v(y) + \frac{|x - y|^2}{\varepsilon} \right\}.

In the proof of Theorem 3.1 we were dealing with the regularization of \(v - \phi\), a bounded function. In our case, \(v\) is not bounded but its growth at infinity is controlled by \(\phi\), which allows to prove the following:

1. \(u^\varepsilon(x)\) is bounded above. Specifically, there exists a constant \(C > 0\) depending only on \(\phi\) and \(\|v - \phi\|_\infty\) such that \(u^\varepsilon(x) \leq C(1 + |x + e|)\). To see this, notice that by our hypotheses on \(\phi\),
   \[\phi(x) \leq a|x|^{-\varepsilon} + \Gamma(x) \leq a|x|^{-\varepsilon} + b|x| \leq a + b|x|\]
   for \(|x|\) large enough, where \(b\) depends on the convexity of the sections of \(\Gamma\). Since \(\phi\) is bounded near 0, we conclude that \(\phi(x) \leq a + b|x|\) for all \(x\), maybe for a different constant \(a\). Since \(v - \phi\) is bounded,
   \[u^\varepsilon(x) = \sup_y \left\{ (v - \phi)(y + e) + \phi(y + e) - \frac{|x - y|^2}{\varepsilon} \right\} \leq \sup_y \left\{ \|v - \phi\|_\infty + a + b|y + e| - \frac{|x - y|^2}{\varepsilon} \right\} \leq \|v - \phi\|_\infty + a + b|x + e| + \sup_y \left\{ b|x - y| - \frac{|x - y|^2}{\varepsilon} \right\} \leq \|v - \phi\|_\infty + a + b|x + e| + b^2 \varepsilon \leq C(1 + |x + e|).

2. As a consequence, the supremum in the definition of \(u^\varepsilon(x)\) is finite, and for any given \(\delta > 0\) there exists \(x_\delta\) such that,
   \[u^\varepsilon(x) = \sup_y \left\{ u(y) - \frac{|x - y|^2}{\varepsilon} \right\} \leq u(x_\delta) - \frac{|x - x_\delta|^2}{\varepsilon} + \delta.

3. The supremum in the definition of \(u^\varepsilon\) is achieved. In fact,
   \[u^\varepsilon(x) \leq \sup_{|y - x| \leq \sqrt{\varepsilon}R} \left\{ u(y) - \frac{|x - y|^2}{\varepsilon} \right\} = u(x) - \frac{|x - x^*|^2}{\varepsilon}\]
   for some \(x^*\) such that \(|x - x^*| \leq \sqrt{\varepsilon}R\), where \(R\) depends on Lip(\(\phi\)) and \(\|v - \phi\|_\infty\) but can be chosen independent of \(\varepsilon\) and \(x\).
To see this, fix $\delta < 1$ and notice that $u(x) \leq u^\epsilon(x) \leq u(x_\delta) - \frac{|x-x_\delta|^2}{\epsilon} + \delta$. We conclude
\[
\frac{|x-x_\delta|^2}{\epsilon} \leq (v-\phi)(x_\delta + e) - (v-\phi)(x + e) + \text{Lip}(\phi) \mid x_\delta - x \mid + \delta
\]
\[
\leq 2\|v-\phi\|_\infty + \text{Lip}(\phi) \mid x_\delta - x \mid + 1.
\]
From this expression, it follows that $|x-x_\delta| < \sqrt[3]{R}$ for some $R$ as before ($\sqrt[3]{R}$ is basically the larger root of the quadratic polynomial in $|x-x_\delta|$). Therefore,
\[
u^\epsilon(x) \leq \sup_{|y-x| \leq \sqrt[3]{R}} \left\{u(y) - \frac{|x-y|^2}{\epsilon} + \delta\right\}.
\]
Since $\delta$ is arbitrary, we can let $\delta \to 0$ and conclude that the supremum in the definition of $u^\epsilon$ is achieved.

(4) Analogous properties hold for $v_\epsilon$. Notice that property (1) is simpler,
\[
v^\epsilon(x) = \inf_y \left\{v(y) + \frac{|x-y|^2}{\epsilon}\right\} = \inf_y \left\{(v-\phi)(y) + \phi(y) + \frac{|x-y|^2}{\epsilon}\right\} \geq \inf(v-\phi) > -\infty.
\]

We are ready now to complete the proof. Following the formal argument above, we can assume that there exists $x_0$ such that
\[
v(x_0 + e) - v(x_0) = \sup_{x \in \mathbb{R}^n} (v(x + e) - v(x)) > \text{Lip}(\phi) \mid e \mid.
\]
First, we need to prove that there exists $x^\epsilon$ such that
\[
(u^\epsilon - v_\epsilon)(x^\epsilon) = \sup_{x \in \mathbb{R}^n} (u^\epsilon - v_\epsilon).
\]
To see this, observe that
\[
\sup_{x \in \mathbb{R}^n} (u^\epsilon - v_\epsilon) \geq (u^\epsilon - v_\epsilon)(x_0) \geq (u - v)(x_0) = \sup_{x \in \mathbb{R}^n} (v(x + e) - v(x)) > \text{Lip}(\phi) \mid e \mid.
\]
On the other hand,
\[
(u^\epsilon - v_\epsilon)(x) = u(x^\epsilon) - \frac{|x-x^\epsilon|^2}{\epsilon} - v(x_\epsilon) - \frac{|x-x_\epsilon|^2}{\epsilon}
\]
\[
\leq (v-\phi)(x^\epsilon + e) - (v-\phi)(x_\epsilon) + \text{Lip}(\phi)(2\sqrt[3]{\epsilon}R + \mid e \mid).
\]
Therefore, for $\epsilon$ small enough, $\text{Lip}(\phi)(2\sqrt[3]{\epsilon}R + \mid e \mid) < \sup_{x \in \mathbb{R}^n} (u^\epsilon - v_\epsilon)$ and
\[
(u^\epsilon - v_\epsilon)(x) < o(1) + \sup_{x \in \mathbb{R}^n} (u^\epsilon - v_\epsilon) \quad \text{as} \mid x \mid \to \infty.
\]
Following the proof of Theorem 3.1, we can prove that both $u^\epsilon$ and $v_\epsilon$ are $C^{1,1}$ at $x_\epsilon$, so that the integrals in the subsequent computations are well defined. The idea is that the paraboloids
\[
\frac{|x-x_{\epsilon,\epsilon}|^2}{\epsilon} + v(x_{\epsilon,\epsilon}) - v_\epsilon(x_\epsilon) + u_\epsilon(x_\epsilon)
\]
and
\[
-\frac{|x-x_\epsilon|^2}{\epsilon} + v_\epsilon(x_\epsilon) + u(x_\epsilon) - u_\epsilon(x_\epsilon)
\]
touch respectively $u^\epsilon$ from above and $v_\epsilon$ from below at the point $x_\epsilon$. Therefore, $u^\epsilon$ and $v_\epsilon$ can both be touched from above and below by a paraboloid at $x_\epsilon$ and they are $C^{1,1}$ at $x_\epsilon$.

Since $u^\epsilon \in C^{1,1}$ at $x_\epsilon$, there exists a paraboloid $P(x)$ that touches $u^\epsilon$ from above at $x_\epsilon$. Then
\[
P(x + x_\epsilon - x_{\epsilon,\epsilon}) + \frac{|x_\epsilon - x_{\epsilon,\epsilon}|^2}{\epsilon}
\]
Observe that, ultimately, we seek to prove in Section 6 that the infimum in the definition of $D$ attained and $L$ in our context if semiconcavity is expected from the solutions. Since $L$ with $\epsilon$ the result follows letting $\epsilon \to 0$.

In the next result we show that solutions to (5.1) are semiconcave, informally, that second derivatives of solutions to (5.1) are bounded from above, under certain conditions on the right-hand side $g$. Before stating the result, let us identify heuristically the natural hypotheses on $g$ in our context if semiconcavity is expected from the solutions.

To simplify, consider instead of $D_s$ a linear operator $L_A$ (defined as in (2.1)) such that

$$L_A v(x) = g(x, v(x)).$$

Observe that, ultimately, we seek to prove in Section 6 that the infimum in the definition of $D_s$ is attained and $D_s$ coincides with some linear operator at each point.
Formally, we have that $D^2_{ee}v(x_0)$ satisfies

$$L_{A_n}D^2_{ee}v(x_0) = \sum_{1 \leq i,j \leq n} \partial^2_{x,ix,j}g(x_0, v(x_0))e_ie_j,$$

where $\sum_{1 \leq i,j \leq n} \partial^2_{x,ix,j}g(x_0, v(x_0))e_ie_j$ is the second derivative in the direction $e$, at the point $x_0$, of the composite function $x \mapsto g(x, v(x))$. Now, if $x_0$ is a maximum point of $D^2_{ee}v$ we get

$$\sum_{1 \leq i,j \leq n} \partial^2_{x,ix,j}g(x_0, v(x_0))e_ie_j = L_{A_n}D^2_{ee}v(x_0) \leq 0.$$

It can be checked that

$$[\partial^2_{x,ix,j}g(x, v(x))]_{1 \leq i,j \leq n} = \begin{bmatrix} I_{n \times n} \nabla v(x)^t \end{bmatrix}_{n \times (n+1)} [\partial^2_{i,j}g(x, v(x))]_{1 \leq i,j \leq n+1} \begin{bmatrix} I_{n \times n} \nabla v(x) \end{bmatrix}_{(n+1) \times n}$$

$$+ \partial_{n+1}g(x, v(x)) D^2 v(x),$$

where $\partial^2_{i,j}g(x, v(x))$ and $\partial_{n+1}g(x, v(x))$ denote derivatives of $g$ as a function of $n+1$ variables evaluated at the point $(x, v(x))$. Writing $\xi = (e^t, (\nabla v(x), e))^t$ for convenience, we have

$$0 \geq \sum_{1 \leq i,j \leq n} \partial^2_{x,ix,j}g(x_0, v(x_0))e_ie_j = \sum_{1 \leq i,j \leq n+1} \partial^2_{i,j}g(x_0, v(x_0))\xi_i\xi_j + \partial_{n+1}g(x_0, v(x_0)) D^2_{ee}v(x_0)$$

or equivalently,

$$\partial_{n+1}g(x_0, v(x_0)) D^2_{ee}v(x_0) \leq - \sum_{1 \leq i,j \leq n+1} \partial^2_{i,j}g(x_0, v(x_0))\xi_i\xi_j.$$

This inequality suggests that in order to get an upper bound on $D^2_{ee}v(x_0)$ it is natural to require $D^2 g \geq -CId$ and $\partial_{n+1}g(x_0, v(x_0)) > \mu > 0$, namely hypotheses (1.4) and (1.6), since then

$$\mu D^2_{ee}v(x_0) \leq - \sum_{1 \leq i,j \leq n+1} \partial^2_{i,j}g(x_0, v(x_0))\xi_i\xi_j \leq C |\xi|^2 \leq C (1 + |\nabla v(x_0)|^2).$$

From here we have the desired estimate as long as we can guarantee that $v$ is Lipschitz. In Proposition 5.2 we proved that this is actually the case provided hypotheses (1.5), and (1.6) hold true.

In the following result we justify the heuristic argument above.

**Proposition 5.3 (Semiconcavity of the solution).** Let $g : \mathbb{R}^{n+1} \to \mathbb{R}$ satisfy (1.4), (1.5), and (1.6). Then, the solution to (5.1) is semiconcave, that is, for every $x \in \mathbb{R}^n$,

$$\delta(v, x, y) \leq \frac{C}{\mu} \left( 1 + \max \left\{ \frac{\text{Lip}(g)}{\mu}, \text{Lip}(\phi)^2 \right\} \right) |y|^2.$$

**Proof.** Let $v$ be the solution to problem (5.1), $e \in \mathbb{R}^n$ fixed, and assume that

$$\sup_{x \in \mathbb{R}^n} \delta(v, x, e) > 0,$$

as the result is trivial otherwise. We observe that $\delta(v, x, e) \to 0$ as $|x| \to \infty$. To see this, notice first that $\delta(v, x, e) = \delta(v - \phi, x, e) + \delta(\phi, x, e) = o(1) + \delta(\phi, x, e)$ as $|x| \to \infty$. Also, by our hypotheses on $\phi$, we have that

$$\frac{\delta(\phi, x, e)}{|e|^2} = O \left( \frac{1}{|x|} \right) \text{ as } |x| \to \infty.$$

Therefore, there is some $x_0$ such that

$$\delta(v, x_0, e) = \sup_{x \in \mathbb{R}^n} \delta(v, x, e) > 0.$$
To complete the proof we need a regularization process as in the proof of Proposition 5.2. Again, let us present the ideas first assuming that \( v \) is a classical solution and all the equations hold pointwise.

Fix \( \eta > 0 \) arbitrary, and let \( A_\eta > 0 \) such that
\[
L_{A_\eta} v(x_0) \leq g(x_0, v(x_0)) + \eta,
\]
and
\[
L_{A_\eta} v(x_0 \pm \varepsilon) \geq D_s v(x_0 \pm \varepsilon) \geq g(x_0 \pm \varepsilon, v(x_0 \pm \varepsilon)),
\]
with \( L_{A_\eta} \) defined as in (2.1). We have from the above expressions that
\[
L_{A_\eta} \delta(v, x_0, e) \geq g(x_0 + e, v(x_0 + e)) + g(x_0 - e, v(x_0 - e)) - 2g(x_0, v(x_0)) - 2\eta.
\]
Notice that \( \delta(v, \cdot, e), x_0, z) \leq 0 \), and therefore \( L_{A_\eta} \delta(v, x_0, e) \leq 0 \). Consequently,
\[
g(x_0 + e, v(x_0 + e)) + g(x_0 - e, v(x_0 - e)) - 2g(x_0, v(x_0)) \leq 2\eta.
\]
At this point we can let \( \eta \to 0 \) and rewrite the resulting expression as
\[
g((x_0, v(x_0)) + \theta_1) - g((x_0, v(x_0)) - \theta_1) \leq 2g((x_0, v(x_0)) - g((x_0, v(x_0)) + \theta_1) - g((x_0, v(x_0)) - \theta_1)
\]
for \( \theta_1 = (e, v(x_0 + e) - v(x_0)) \) and \( \theta_2 = (-e, v(x_0 - e) - v(x_0)) \). Then, by (1.4) and (1.6) we have
\[
\mu \delta(v, x_0, e) \leq g(x_0 - e, v(x_0 - e)) - g(x_0 - e, 2v(x_0) - v(x_0 + e))
\]
\[
= g((x_0, v(x_0)) + \theta_2) - g((x_0, v(x_0)) - \theta_1)
\]
\[
\leq 2g((x_0, v(x_0)) - g((x_0, v(x_0)) + \theta_1) - g((x_0, v(x_0)) - \theta_1) \leq C|\theta_1|^2
\]
and therefore, for any \( x \in \mathbb{R}^n \),
\[
\delta(v, x_0, e) \leq C \mu \left( 1 + \left( \frac{v(x_0 + e) - v(x_0)}{|e|} \right)^2 \right) |e|^2.
\]
The result follows applying Proposition 5.2

To complete the proof in the general case, let us sketch the regularization procedure. The details follow the lines of the proof of Proposition 5.2. To simplify the notation, let us denote \( u(x) = v(x + e), \ w(x) = v(x - e) \) and consider the sup-convolution of \( u, w \) and the inf-convolution of \( v \), namely,
\[
u^\epsilon(x) = \sup_y \left\{ u(y) - \frac{|x - y|^2}{\epsilon} \right\} = \sup_y \left\{ v(y + e) - \frac{|x - y|^2}{\epsilon} \right\} = v(x_\epsilon + e) - \frac{|x - x_\epsilon|^2}{\epsilon},
\]
\[
u^\epsilon(x) = \sup_y \left\{ w(y) - \frac{|x - y|^2}{\epsilon} \right\} = \sup_y \left\{ v(y - e) - \frac{|x - y|^2}{\epsilon} \right\} = v(x_\epsilon^* - e) - \frac{|x - x_\epsilon^*|^2}{\epsilon},
\]
and
\[
u_\epsilon(x) = \inf_y \left\{ v(y) + \frac{|x - y|^2}{\epsilon} \right\} = v(x_\epsilon) + \frac{|x - x_\epsilon|^2}{\epsilon}.
\]
for some points \( x_\epsilon, x_\epsilon^* \), and \( x_\epsilon \) within a distance \( \sqrt{\epsilon} R \) from \( x \) (see property 3 in the proof of Proposition 5.2).

Assume \( 5.4 \). Then, on the one hand, we have that
\[
\sup_{\mathbb{R}^n} u^\epsilon + w^\epsilon - 2v_\epsilon \geq u^\epsilon(x_0) + w^\epsilon(x_0) - 2v_\epsilon(x_0) \geq \delta(v, x_0, e) = \sup_{x \in \mathbb{R}^n} \delta(v, x, e) > 0.
\]
On the other hand,
\[
  u^\varepsilon(x) + w^\varepsilon(x) - 2v^\varepsilon(x) \leq (v - \phi)(x^* + e) + (v - \phi)(x^{**} - e) - 2(v - \phi)(x_*) + (\phi(x^* + e) - \phi(x + e)) + (\phi(x^{**} - e) - \phi(x - e)) - 2(\phi(x_*) - \phi(x)) + \delta(\phi, x, e)
\]
\[
  \leq o(1) + 4\text{Lip}(\phi)\sqrt{|R|} + O\left(\frac{1}{|x|}\right)
\]
as \(|x| \to \infty\). Therefore, for \(\varepsilon\) small enough, there exists \(x_\varepsilon\) such that
\[
u^\varepsilon(x_\varepsilon) + w^\varepsilon(x_\varepsilon) - 2v^\varepsilon(x_\varepsilon) = \sup_{\mathbb{R}^n}(u^\varepsilon + w^\varepsilon - 2v^\varepsilon).
\]
Now, consider the following three paraboloids:
\[
P(x) = u(x^*) - \frac{|x - x^*_2|^2}{\epsilon}, \quad Q(x) = v(x_{\varepsilon,*}) + \frac{|x - x_{\varepsilon,*}|^2}{\epsilon},
\]
and
\[
R(x) = w(x^{**}) - \frac{|x - x^{**}_2|^2}{\epsilon}
\]
Then, all three \(u^\varepsilon, w^\varepsilon,\) and \(v^\varepsilon\) are \(C^{1,1}\) at \(x_\varepsilon\). To see this, notice that
- \(P(x)\) touches \(u^\varepsilon\) from below at \(x_\varepsilon\) and
  \[
  2Q(x) - R(x) + u^\varepsilon(x_\varepsilon) + w^\varepsilon(x_\varepsilon) - 2v^\varepsilon(x_\varepsilon)
  \]
touches from above.
- \(Q(x)\) touches \(v^\varepsilon\) from above at \(x_\varepsilon\) and
  \[
  \frac{1}{2}P(x) + \frac{1}{2}R(x) + v^\varepsilon(x_\varepsilon) - \frac{1}{2}u^\varepsilon(x_\varepsilon) - \frac{1}{2}w^\varepsilon(x_\varepsilon)
  \]
touches from below.
- \(R(x)\) touches \(w^\varepsilon\) from below at \(x_\varepsilon\) and
  \[
  2Q(x) - P(x) + u^\varepsilon(x_\varepsilon) + w^\varepsilon(x_\varepsilon) - 2v^\varepsilon(x_\varepsilon)
  \]
touches from above.
Then, there are three paraboloids that touch \(v\) from above at \(x^*_2 + e\) and \(x^{**}_2 - e\), and from below at \(x_{\varepsilon,*}\). By Lemma 2.2, we have
\[
D_x v(x^*_2 + e) \geq g(x^*_2 + e, v(x^*_2 + e)), \quad D_x v(x^{**}_2 - e) \geq g(x^{**}_2 - e, v(x^{**}_2 - e)),
\]
and
\[
D_x v(x_{\varepsilon,*}) \leq g(x_{\varepsilon,*}, v(x_{\varepsilon,*}))
\]
in the classical sense. Fix \(\eta > 0\), arbitrary, and let \(A_\eta > 0\) such that
\[
L_{A_\eta} v(x_{\varepsilon,*}) \leq g(x_{\varepsilon,*}, v(x_{\varepsilon,*})) + \eta
\]
with \(L_{A_\eta}\) defined as in (2.1). Then
\[
L_{A_\eta} u(x^*_2) + L_{A_\eta} w(x^{**}_2) - 2L_{A_\eta} v(x_{\varepsilon,*})
\]
\[
\geq g(x^*_2 + e, v(x^*_2 + e)) + g(x^{**}_2 - e, v(x^{**}_2 - e)) - 2g(x_{\varepsilon,*}, v(x_{\varepsilon,*})) - 2\eta.
\]
As in the proof of Theorem 3.1,
\[
\delta(u, x^*_2, y) + \delta(w, x^{**}_2, y) - 2\delta(v, x_{\varepsilon,*}, y) \leq \delta(u^\varepsilon + w^\varepsilon - 2v^\varepsilon, x_\varepsilon, y) \leq 0.
\]
Since \(\eta\) is arbitrary, we conclude
\[
g(x^*_2 + e, v(x^*_2 + e)) + g(x^{**}_2 - e, v(x^{**}_2 - e)) - 2g(x_{\varepsilon,*}, v(x_{\varepsilon,*})) \leq 0.
\]
Rearranging terms, we get,
\[ g(x^* - e, v(x^* - e)) \pm g(x^* - e, 2v(x_{*,s}) - v(x^* + e)) - g(2x_{*,s} - x^* - e, 2v(x_{*,s}) - v(x^* + e)) \leq 2g(x_{*,s}, v(x_{*,s})), \]
where in the last step we have used (1.4). Therefore,
\[ \theta \]
Let us analyze the left-hand side of the inequality first. By (1.6) and (1.5), we have
\[ \text{locally uniformly elliptic.} \]
Now we can let \( \theta \to 0 \) and apply Proposition 5.2 to conclude.

\[ \text{(6.1)} \]

Therefore, if we denote \( \theta = (x^* + e - x_{*,s}, v(x^* + e) - v(x_{*,s})) \) the right-hand side becomes
\[ 2g(x_{*,s}, v(x_{*,s})) - g(x^* + e, v(x^* + e)) - g(2x_{*,s} - x^* - e, 2v(x_{*,s}) - v(x^* + e)) = 2g(x_{*,s}, v(x_{*,s})) - g((x_{*,s}, v(x_{*,s})), \theta) \leq C|\theta|^2 \]
where in the last step we have used (1.4). Therefore,
\[ \mu(v(x^* + e) + v(x^* - e) - 2v(x_{*,s})) \leq C|\theta|^2 + \text{Lip}(g)|x^* + x^* - 2x_{*,s}| \]

Observe that
\[ v(x^* + e) + v(x^* - e) - 2v(x_{*,s}) \geq (u^e + w^e - 2v_e)(x_e) \]
\[ \geq (u^e + w^e - 2v_e)(x_0) \geq (u + w - 2v)(x_0) = \delta(v, x_0, e) = \sup_{x \in \mathbb{R}^n} \delta(v, x, e). \]

On the other hand,
\[ |\theta|^2 = |x^* + e - x_{*,s}|^2 + |v(x^* + e) - v(x_{*,s})|^2 \leq (1 + \text{Lip}(v)^2)|x^* + e - x_{*,s}|^2 \leq (1 + \text{Lip}(v)^2)(|x^* - x_{*,s}| + |e|)^2 \]
Finally, recall that \( |x^* - x_{*,s}| \leq 2\sqrt{\epsilon}R \) and \( |x^* + x^* - 2x_{*,s}| \leq 4\sqrt{\epsilon}R \). All the above together yields,
\[ \mu \sup_{x \in \mathbb{R}^n} \delta(v, x, e) \leq C(1 + \text{Lip}(v)^2)(2\sqrt{\epsilon}R + |e|)^2 + 4\text{Lip}(g)\sqrt{\epsilon}R. \]

Now we can let \( \epsilon \to 0 \) and apply Proposition 5.2 to conclude. \( \square \)

6. Local uniform ellipticity of the fractional Monge-Ampère equation

In this section we shall prove that the infimum in the definition of (1.2) cannot be realized by matrices that are too degenerate, effectively proving that the fractional Monge-Ampère operator is locally uniformly elliptic.

To this aim, consider the following approximate, non-degenerate operator,
\[ \mathcal{D}_s^\theta u(x) = \inf \left\{ \text{P.V.} \int_{\mathbb{R}^n} \frac{u(y) - u(x)}{|A^{-1}(y - x)|^{n+2s}} \, dy \bigg| A > 0, \ \det A = 1, \ \lambda_{\min}(A) \geq \theta \right\} \]
(6.1)

\[ = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|A^{-1}y|^{n+2s}} \, dy \bigg| A > 0, \ \det A = 1, \ \lambda_{\min}(A) \geq \theta \right\}. \]

Let us point out that the conditions \( \det A = 1, \ \lambda_{\min}(A) \geq \theta \) imply \( \lambda_{\max}(A) \leq \theta^{1-n} \) and this bound is realized by matrices with eigenvalues \( \theta \) (simple) and \( \theta^{1-n} \) (multiplicity \( n-1 \)). Therefore, \( \mathcal{D}_s^\theta \) belongs to the class of uniformly elliptic, nonlocal operators with extremal Pucci operators
\[ \mathcal{M}_{\theta, \theta^{1-n}}^+ u(x) = \sup \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|A^{-1}y|^{n+2s}} \, dy \bigg| \theta I \leq A \leq \theta^{1-n}I \right\} \]
and
\[ \mathcal{M}_{\theta, \theta^{1-n}}^- u(x) = \inf \left\{ \frac{1}{2} \int_{\mathbb{R}^n} \frac{u(x + y) + u(x - y) - 2u(x)}{|A^{-1}y|^{n+2s}} \, dy \bigg| \theta I \leq A \leq \theta^{1-n}I \right\}. \]
Observe that in general $M'_{\theta, \beta^{-\alpha}} u(x) < D^\theta_s u(x)$, as the class of matrices over which the infimum is taken is broader for the Pucci operator.

The main result of this section and of the paper is the following.

**Theorem 6.1.** Consider $\frac{1}{2} < s < 1$ and let $u$ be Lipschitz continuous and semiconcave (with constants $L$ and $C$ respectively) and such that

\begin{equation}
D_s u(x) \geq \eta_0 \quad \forall x \in \Omega
\end{equation}

in the viscosity sense for some constant $\eta_0 > 0$ and $\Omega \subset \mathbb{R}^n$. Then,

\begin{equation}
D_s u(x) = D^\theta_s u(x) \quad \forall x \in \Omega
\end{equation}

in the classical sense, for $D^\theta_s$ is the approximate operator defined by (6.1) and

$$\theta < \left( \frac{C_3}{\mu_1 \omega_n} \right)^{\frac{n-1}{2s}}$$

with $\mu_1, C_3$ defined in (6.5) and (6.6) below.

**Remark 6.2.** It can be checked that

$$\frac{C_3}{\mu_1} = O(\eta_0^n (1-s)^n (2s-1)^n)$$

as $s \to 1$, $s \to 1/2$, or $\eta_0 \to 0$.

For simplicity, we shall assume that $0 \in \Omega$ and then prove (6.3) for $x = 0$. Note for the sequel that since $u$ is semiconcave, Lemma 2.2 implies that $D_s u(x)$ is defined in the classical sense for all $x \in \Omega$ and (6.2) holds pointwise.

The proof of Theorem 6.1 has two parts. In the first part we prove that the (one-dimensional) fractional Laplacian of the restriction of $u$ to any line is positive and bounded from above. Then, in the second part, we shall use this fact to prove that there exists $C > 0$ such that

$$\int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} dy \geq C \lambda_{\min}(A) \frac{2s}{n-1}$$

and therefore the infimum in the fractional Monge-Ampère operator cannot be achieved for matrices that are too degenerate.

The two parts we have mentioned correspond to the following two results.

**Proposition 6.3.** Assume the same hypotheses of Theorem 6.1. Then, for every $e \in \partial B_1(0)$,

$$0 < \mu_0 \leq -(-\Delta)^s u(0) = \int_{\mathbb{R}} \frac{u(te) - u(0)}{|t|^{1+2s}} dt \leq \mu_1,$$

with

\begin{equation}
\mu_0 = C_1^{-n} C_2^{-1} \left( \frac{\eta_0}{2} \right)^n
\end{equation}

for $C_1, C_2$ defined in (6.9) and (6.10), and

\begin{equation}
\mu_1 = \max \{ C/2, L \} \left( 1 - s \right) \left( 2s - 1 \right).
\end{equation}

**Proposition 6.4.** Assume $\epsilon_1, \ldots, \epsilon_n$ are positive constants such that $\prod_{j=1}^n \epsilon_j = 1$. Then, in the same hypotheses of Theorem 6.1, we have,

$$\int_{\mathbb{R}^n} \frac{u(y) - u(0)}{\left( \sum_{j=1}^n \epsilon_j^2 y_j^2 \right)^{s+1/2}} dy \geq C_3 \epsilon^{-2s}_{\min}.$$
where

\begin{equation}
(6.6) \quad C_3 = \frac{\eta_0^n}{2C_1^{n-1}C_2^{n+2s}},
\end{equation}

with \( C_1, C_2 \) defined in \((6.9)\) and \((6.10)\).

Propositions \(6.3\) and \(6.4\) (that we prove below) allow to prove the main result of this section, Theorem \(6.1\).

Proof of Theorem \(6.1\) Consider a symmetric matrix \( A > 0 \) with \( \det A = 1 \) and \( \lambda_{\min}(A) < \frac{1}{k} \). We can write \( A = PJP^t \), and denote \( \tilde{u}(y) = u(Py) \). Observe that then Proposition \(6.4\) implies

\[
\int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy = \int_{\mathbb{R}^n} \frac{u(Py) - u(0)}{|J^{-1}y|^{n+2s}} \, dy = \int_{\mathbb{R}^n} \frac{\tilde{u}(y) - \tilde{u}(0)}{\left(\sum_{j=1}^{n} \frac{\epsilon_j^2 y_j^2}{\sum_{j=1}^{n} \epsilon_j^2 y_j^2}\right)^{\frac{n+2s}{2}}} \, dy > C_3 k^{\frac{2s}{n-1}}
\]

and we get the estimate

\begin{equation}
(6.7) \quad \inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1, \lambda_{\min}(A) < \frac{1}{k} \right\} \geq C_3 k^{\frac{2s}{n-1}}.
\end{equation}

Observe now, that choosing \( A = I \), Proposition \(6.3\) yields

\begin{equation}
(6.8) \quad \inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1 \right\} \leq \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|y|^{n+2s}} \, dy = \int_{\partial B_1(0)} \int_0^\infty \frac{u(re) - u(0)}{r^{n+2s}} \, dr \, d\mathcal{H}^{n-1}(e) \leq \mu_1 \omega_n.
\end{equation}

Therefore, from \((6.7)\) and \((6.8)\) we have that whenever \( k > (\mu_1 \omega_n C_3^{-1})^{\frac{n+1}{2s}} \)

\[
\inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1, \lambda_{\min}(A) < \frac{1}{k} \right\} > \inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1 \right\}.
\]

This implies \((6.3)\), since

\[
\inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1 \right\} = \min \left\{ \inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1, \lambda_{\min}(A) < \frac{1}{k} \right\}, \inf \left\{ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{|A^{-1}y|^{n+2s}} \, dy \mid A > 0, \det A = 1, \lambda_{\min}(A) \geq \frac{1}{k} \right\} \right\}.
\]

\[\square\]

The rest of this section is devoted to the proof of Propositions \(6.3\) and \(6.4\).
Lemma 6.6. We have, and the result follows noticing that both integrals on the right-hand side are constant.

A change of variables

\[ u(x) = C \int_{\mathbb{R}^n} \frac{v(y_1, 0)}{(\epsilon^2 y_1^2 + \epsilon^{-n-1} [y^2]^{n+2s})} \ dy \]

with \( C = \frac{\omega_{n-1} \cdot \Gamma \left( \frac{n-1}{2} \right) \Gamma \left( \frac{s + 1}{2} \right)}{2 \Gamma \left( \frac{n}{2} + s \right)} \).

Proof. A change of variables \( z_1 = y_1, z_j = \frac{y_j}{\epsilon y_1}, j = 2, \ldots, n \) yields,

\[
\int_{\mathbb{R}^n} \frac{\min \{ |y|, |y^2| \}}{(\epsilon^2 y_1^2 + \epsilon^{-n-1} [y^2]^{n+2s})} \ dy = \int_{\mathbb{R}} (1 + z_1^2)^{-\frac{n+2s}{2}} \ dz_1 \int_{\mathbb{R}^{n-1}} \frac{\min \{ |\bar{z}|, |\bar{z}^2| \}}{|\bar{z}|^{n-1+2s}} \ d\bar{z},
\]

and the result follows noticing that both integrals on the right-hand side are constant.

Lemma 6.5. Let \( \epsilon > 0 \) and assume the same hypotheses of Theorem 6.1. Then,

\[
\int_{\mathbb{R}^n} \frac{u(y_1, \bar{y}) - u(y_1, 0)}{(\epsilon^2 y_1^2 + \epsilon^{-n-1} [y^2]^{n+2s})} \ dy \leq C_1 \cdot \epsilon^{2s}
\]

with

\[
C_1 = \max \left\{ \frac{C}{2}, L \right\} \cdot \frac{\omega_{n-1}}{2(1-s)(2s-1)} \cdot \frac{\sqrt{\pi} \cdot \Gamma \left( \frac{n-1}{2} + s \right)}{\Gamma \left( \frac{n}{2} + s \right)}.
\]

Proof. Since \( u \) is Lipschitz and semiconcave, we have

\[
\int_{\mathbb{R}^n} \frac{u(y_1, \bar{y}) - u(y_1, 0)}{(\epsilon^2 y_1^2 + \epsilon^{-n-1} [y^2]^{n+2s})} \ dy \leq \max \left\{ \frac{C}{2}, L \right\} \int_{\mathbb{R}^n} \frac{\min \{ |\bar{y}|, |\bar{y}^2| \}}{(\epsilon^2 y_1^2 + \epsilon^{-n-1} [y^2]^{n+2s})} \ dy.
\]

A change of variables

\( z_1 = \epsilon y_1 \), \( z_j = y_j \), \( j = 2, \ldots, n \), yields,

\[
\int_{\mathbb{R}^n} \frac{\min \{ |\bar{y}|, |\bar{y}^2| \}}{(\epsilon^2 y_1^2 + \epsilon^{-n-1} [y^2]^{n+2s})} \ dy = \epsilon^{2s} \int_{\mathbb{R}^n} \frac{v(y_1, 0)}{|y_1|^{1+2s}} \ dy.
\]

Lemma 6.7. Under the same hypotheses of Theorem 6.1 we have

\[
\int_{\mathbb{R}} \frac{u(y_1, 0) - u(0)}{|y_1|^{1+2s}} \ dy \geq \mu_0,
\]

where \( \mu_0 \) is given by 6.1.
Proof. From Lemmas 6.5 and 6.6 we have that
\[ C_1 \cdot \epsilon^{\frac{2\alpha}{n-1}} \geq \int_{\mathbb{R}^n} \frac{v(y_1, \bar{y})}{(\epsilon^2 y_1^2 + \epsilon^{\frac{-2}{n-1}} |\bar{y}|^2)^{\frac{n+2\alpha}{4}}} dy - C_2 \epsilon^{-2s} \int_{\mathbb{R}} v(y_1, \bar{0}) |y_1|^{1+2s} dy. \]

Then, by (6.2) and the definition of \( D \), we get
\[ \int_{\mathbb{R}^n} \frac{v(y_1, \bar{y})}{(\epsilon^2 y_1^2 + \epsilon^{\frac{-2}{n-1}} |\bar{y}|^2)^{\frac{n+2\alpha}{4}}} dy \geq \inf \left\{ \int_{\mathbb{R}^n} u(y) - u(0) |A^{-1}y|^{n+2s} dy \right\} A > 0, \det A = 1 \geq \eta_0 > 0. \]

Therefore,
\[ C_1 \cdot \epsilon^{\frac{2\alpha}{n-1}} \geq \eta_0 - C_2 \epsilon^{-2s} \int_{\mathbb{R}} v(y_1, \bar{0}) |y_1|^{1+2s} dy. \]

We get the result from this expression by choosing \( \epsilon = \left( \frac{\eta_0}{2C_1} \right)^{\frac{n-1}{2s}} \). \( \square \)

From Lemma 6.7 we can finally prove Proposition 6.3.

Proof of Proposition 6.3. First, we are going to prove that the one-dimensional fractional Laplacian of the restriction of \( u \) to any line is bounded above. Indeed, from the Lipschitz continuity and semiconcavity of \( u \),
\[ \int_{\mathbb{R}} \frac{u(te) - u(0)}{|t|^{1+2s}} dt \leq \int_{\mathbb{R}} \frac{1}{2} u(te) + \frac{1}{2} u(-te) - u(0) \frac{\min \{ 2L |t|, C |t|^2 \} }{|t|^{1+2s}} dt = \mu_1, \]
where \( \mu_1 \) is given by (6.5).

Now, fix \( e \in \partial B_1(0) \), and choose \( P \) such that \( e \) is its first column and the rest of columns complete an orthonormal basis of \( \mathbb{R}^n \). Notice that \( \tilde{u}(x) = u(Px) \) is in the hypotheses of Theorem 6.1. Hence, we can apply Lemma 6.7 to \( \tilde{u} \) and get
\[ \int_{\mathbb{R}} \frac{\tilde{u}(y_1, \bar{0}) - \tilde{u}(0)}{|y_1|^{1+2s}} dy_1 \geq \mu_0, \]
but then, \( \tilde{u}(y_1, \bar{0}) = \tilde{u}(y_1e_1) = u(y_1 Pe_1) = u(y_1 e) \) by definition of \( P \). \( \square \)

Next, we provide the proof of Proposition 6.4 that uses Proposition 6.3.

Proof of Proposition 6.4. Our aim is to prove that the infimum in the fractional Monge-Ampère operator is not realized by matrices that are very degenerate. Assume \( \epsilon_1 = \epsilon_{\min} \).
\[ \int_{\mathbb{R}^n} \frac{u(y) - u(0)}{(\sum_{j=1}^{n} \epsilon_j^2 y_j^2)^{\frac{n+2s}{2}}} dy = \int_{\partial B_1(0)} \int_0^{\infty} \frac{u(re) - u(0)}{r^{1+2s}} dr \frac{1}{\left( \sum_{j=1}^{n} \epsilon_j^2 \cos^2 \theta_j \right)^{\frac{n+2s}{2}}} d\mathcal{H}^{n-1}(e) \]
where \( \cos \theta_j \) are the director cosines of \( e \). From Proposition 6.3, we have
\[ \int_{\partial B_1(0)} \int_0^{\infty} \frac{u(re) - u(0)}{r^{1+2s}} dr \frac{1}{\left( \sum_{j=1}^{n} \epsilon_j^2 \cos^2 \theta_j \right)^{\frac{n+2s}{2}}} d\mathcal{H}^{n-1}(e) \geq \mu_0 \int_{\partial B_1(0)} \frac{1}{\left( \sum_{j=1}^{n} \epsilon_j^2 \cos^2 \theta_j \right)^{\frac{n+2s}{2}}} d\mathcal{H}^{n-1}(e). \]
Denoting \( E = \prod_{j=1}^{n} \{ e \in \partial B_1(0) : \epsilon_j^2 \cos^2 \theta_j \leq \epsilon_1^2 \} \), we have the estimate,
\[
\int_{\partial B_1(0)} \frac{1}{\left( \sum_{j=1}^{n} \epsilon_j^2 \cos^2 \theta_j \right)^{n+2s}} \, dH^{n-1}(e) \geq \frac{\mathcal{H}^{n-1}(E)}{n^{n+2s} \epsilon_1^{n+2s}} \geq \frac{\prod_{j=2}^{n} \frac{2\epsilon_j}{\epsilon_1}}{n^{n+2s} \epsilon_1^{n+2s}} \geq \frac{2^{n-1}}{n^{n+2s} \epsilon_1^{2s}},
\]
where we have used that \( \prod_{j=1}^{n} \epsilon_j = 1 \). This completes the proof. \( \square \)

**APPENDIX A.**

In this appendix we include for the reader’s convenience the proof of the following fact, mentioned in the introduction.

**Lemma A.1.** If \( u \) is convex, asymptotically linear and \( 1/2 < s < 1 \), then
\[
\lim_{s \to 1} ((1-s) D_s u(x)) = \frac{\omega_n}{4} \cdot \det(D^2 u(x))^{1/n}
\]
in the viscosity sense.

The proof of Lemma [A.1] is a direct consequence of the following two results.

**Lemma A.2.** If \( u \) is asymptotically linear and \( 1/2 < s < 1 \), then
\[
\lim_{s \to 1} ((1-s) D_s u(x)) = \frac{\omega_n}{4n} \cdot \inf \left\{ \text{trace}(AA^t D^2 u(x)) \mid A > 0, \det A = 1 \right\}
\]
in the viscosity sense.

**Lemma A.3.** Let \( B \) be a symmetric and positive semidefinite matrix. Then,
\[
n \det(B)^{1/n} = \inf \{ \text{trace}(AA^t B) \mid \det A = 1 \}.
\]

We devote the rest of this appendix to the proof of Lemmas [A.2] and [A.3] (Lemma [A.3] is well-known, but we include a proof for completeness).

**Proof of Lemma [A.2].** We shall first consider the case when \( u \in C^2(\mathbb{R}^n) \) and then show how to adapt the arguments to the viscosity setting. To this aim, let \( A > 0 \) such that \( \det A = 1 \) and \( 0 < \rho < R \) to be chosen later on. Then,
\[
\begin{align*}
\int_{\mathbb{R}^n} \delta(u, x, z) \frac{d\nu}{|A^{-1}z|^{n+2s}} &= \int_{B_\rho(0)} \frac{\langle D^2 u(x) Ay, Ay \rangle}{|y|^{n+2s}} \, dy + \int_{B_\rho(0)} \frac{\delta(u, x, Ay) - \langle D^2 u(x) Ay, Ay \rangle}{|y|^{n+2s}} \, dy \\
&\quad + \int_{B_R(0) \setminus B_\rho(0)} \frac{\delta(u, x, Ay)}{|y|^{n+2s}} \, dy + \int_{\mathbb{R}^n \setminus B_R(0)} \frac{\delta(u, x, Ay)}{|y|^{n+2s}} \, dy.
\end{align*}
\]

Now, we are going to analyze each term on the right-hand side of (A.1). First, notice that
\[
\int_{B_\rho(0)} \frac{\langle D^2 u(x) Ay, Ay \rangle}{|y|^{n+2s}} \, dy = \frac{\rho^{2-2s}}{2-2s} \int_{\partial B_1(0)} \langle D^2 u(x) Ay, Ay \rangle \, dH^{n-1}(y)
\]
\[
= \frac{\rho^{2-2s}}{2-2s} |B_1(0)| \text{trace}(AA^t D^2 u(x)).
\]

Fix \( \epsilon > 0 \), small. Since \( \delta(u, x, Ay) = \langle D^2 u(x) Ay, Ay \rangle + o(|y|^2) \) as \( |y| \to 0 \), we have that \( |\delta(u, x, Ay) - \langle D^2 u(x) Ay, Ay \rangle| \leq \epsilon |y|^2 \) if \( |y| \leq \rho \) with \( \rho \) sufficiently small. Thus,
\[
\int_{B_\rho(0)} \frac{\delta(u, x, Ay) - \langle D^2 u(x) Ay, Ay \rangle}{|y|^{n+2s}} \, dy \leq \frac{\epsilon \omega_n}{2-2s} \rho^{2-2s}.
\]
On the other hand,

\[(A.4) \quad \left| \int_{B_R(0) \setminus B_{\rho}(0)} \frac{\delta(u, x, Ay)}{|y|^{n+2s}} \, dy \right| \leq \frac{4\omega_n}{2s} \|u\|_L^\infty_{n, B_R(0) \setminus B_{\rho}(0)}(\rho^{-2s} - R^{-2s}).\]

As for the last term in (A.1), since \(u\) is asymptotically linear, for \(R > 0\) large enough, there exists some constant \(L > 0\) such that \(|\delta(u, x, Ay)| \leq 2L|Ay| \leq 2L\lambda_{\text{max}}^{1/2}(AA^t)|y|\). Therefore,

\[(A.5) \quad \left| \int_{\mathbb{R}^n \setminus B_R(0)} \frac{\delta(u, x, Ay)}{|y|^{n+2s}} \, dy \right| \leq \frac{2L\lambda_{\text{max}}^{1/2}(AA^t)\omega_n}{(2s-1)} R^{1-2s}.\]

Collecting (A.1)-(A.5), we get

\[2(1-s)D_s u(x) \leq \frac{\omega_n}{2n} \cdot \text{trace}(AA^t D^2 u(x)) \rho^{2-2s} + \frac{\epsilon \omega_n}{2} \rho^{2-2s} \]

\[+ \left( 1 - \frac{s}{2s} \right) 4\omega_n \|u\|_L^\infty_{n, B_R(0) \setminus B_{\rho}(0)}(\rho^{-2s} - R^{-2s}) + \left( 1 - \frac{s}{2s-1} \right) 2L \lambda_{\text{max}}^{1/2}(AA^t)\omega_n R^{1-2s},\]

and then,

\[\lim_{s \to 1} ((1-s)D_s u(x)) \leq \frac{\omega_n}{4n} \cdot \text{trace}(AA^t D^2 u(x)) + \frac{\epsilon \omega_n}{4}.\]

Since both \(A\) and \(\epsilon\) are arbitrary, we have

\[(A.6) \quad \lim_{s \to 1} ((1-s)D_s u(x)) \leq \frac{\omega_n}{4n} \cdot \text{trace}(AA^t D^2 u(x)) \mid A > 0, \det A = 1.\]

On the other hand, from (A.1)-(A.5) we also get

\[\frac{\omega_n}{2n} \cdot \rho^{2-2s} \cdot \text{inf} \left\{ \text{trace}(AA^t D^2 u(x)) \mid A > 0, \det A = 1 \right\} \]

\[\leq (1-s) \int_{\mathbb{R}^n} \frac{\delta(u, x, z)}{|A^{-1} z|^{n+2s}} \, dz + \frac{\epsilon \omega_n}{2} \rho^{2-2s} \]

\[+ \left( 1 - \frac{s}{2s} \right) 4\omega_n \|u\|_L^\infty_{n, B_R(0) \setminus B_{\rho}(0)}(\rho^{-2s} - R^{-2s}) + \left( 1 - \frac{s}{2s-1} \right) 2L \lambda_{\text{max}}^{1/2}(AA^t)\omega_n R^{1-2s},\]

Let \(A_\epsilon > 0\) with \(\det A_\epsilon = 1\) such that

\[\frac{1}{2} \int_{\mathbb{R}^n} \frac{\delta(u, x, z)}{|A^{-1} z|^{n+2s}} \, dy \leq D_s u(x) + \epsilon.\]

Then,

\[\frac{\omega_n}{4n} \cdot \rho^{2-2s} \cdot \text{inf} \left\{ \text{trace}(AA^t D^2 u(x)) \mid A > 0, \det A = 1 \right\} \]

\[\leq (1-s)D_s u(x) + (1-s)\epsilon + \frac{\epsilon \omega_n}{4} \rho^{2-2s} \]

\[+ \left( 1 - \frac{s}{s} \right) \omega_n \|u\|_L^\infty_{n, B_R(0) \setminus B_{\rho}(0)}(\rho^{-2s} - R^{-2s}) + \left( 1 - \frac{s}{2s-1} \right) L \lambda_{\text{max}}^{1/2}(AA^t_\epsilon) \omega_n R^{1-2s}.\]

Finally, letting first \(s \to 1\) and then \(\epsilon \to 0\), we conclude

\[(A.7) \quad \lim_{s \to 1} ((1-s)D_s u(x)) \geq \frac{\omega_n}{4n} \cdot \text{inf} \left\{ \text{trace}(AA^t D^2 u(x)) \mid A > 0, \det A = 1 \right\}\]

and therefore, the equality.

To conclude, let us show how to adapt this argument to the viscosity setting. According to Definition 2.3, whenever a function \(\psi\) touches \(u\) (from above or from below) at a point \(x\) in the sense that

- \(\psi(x) = u(x)\),
that 0 is an eigenvalue of the matrix B definite. Hence, the inequality between the arithmetic and geometric means yields, Proof of Lemma A.3. Let A with det A = 1. Then, the matrix AA^t is symmetric and positive definite. Hence, the inequality between the arithmetic and geometric means yields,

\[ n \det(B)^{1/n} \leq \text{trace}(AA^t B). \]

As this is true for any A with det A = 1, we deduce,

\[ n \det(B)^{1/n} \leq \inf \{ \text{trace}(AA^t B) \mid \det A = 1 \}. \]

To derive the converse inequality, assume first that B > 0. If we choose

\[ A = P_B \text{ diag } \left[ \frac{\det(B)^{1/(2n)}}{\lambda_i(B)^{1/2}} \right] P_B^t \]

with \( P_B \) such that \( B = P_B \text{ diag}(\lambda_i(B)) P_B^t \) and \( P_B P_B^t = I \), we get

\[ n \det(B)^{1/n} = \text{trace}(AA^t B) = \min \{ \text{trace}(AA^t B) \mid \det A = 1 \}. \]

Let us now consider the case when B ≥ 0. Since the result is trivial when B = 0, we can assume that 0 is an eigenvalue of the matrix B with multiplicity \( n - k < n \), that is,

\[
B = P_B \begin{pmatrix}
\lambda_1(B) & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\
0 & \cdots & \lambda_k(B) & 0 & \cdots & 0 \\
0 & \cdots & 0 & 0 & \cdots & 0 \\
\vdots & \ddots & \vdots & \ddots & \ddots & \vdots \\
0 & \cdots & 0 & 0 & \cdots & 0
\end{pmatrix} P_B^t
\]

with \( P_B P_B^t = I \). Fix \( \epsilon > 0 \) and define

\[ A_\epsilon = P_B \text{ diag}(\lambda_i(A_\epsilon)) P_B^t \]
with $P_B$ the same as before and

$$\lambda_i(A_\epsilon) = \begin{cases} \left( \frac{\epsilon}{\text{trace}(B)} \right)^{\frac{1}{2}} & \text{for } i = 1, \ldots, k \\ \left( \frac{\epsilon}{\text{trace}(B)} \right)^{\frac{k}{2(n-k)}} & \text{for } i = k + 1, \ldots, n. \end{cases}$$

In this way, $A_\epsilon$ is positive definite with $\det(A_\epsilon) = 1$, and

$$\text{trace}(A_\epsilon A_\epsilon^t B) = \sum_{i=1}^{k} \lambda_i(A_\epsilon)^2 \lambda_i(B) = \epsilon.$$ 

Since $\epsilon > 0$ is arbitrary, we conclude that

$$\inf\{\text{trace}(AA^t B) \mid \det A = 1\} = 0 = n \det(B)^{1/n}.$$ 

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