Dual subgradient method for constrained convex optimization problems

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Abstract

This paper considers a general convex constrained problem setting where functions are not assumed to be differentiable nor Lipschitz continuous. Our motivation is in finding a simple first-order method for solving a wide range of convex optimization problems with minimal requirements. We study the method of weighted dual averages (Nesterov, 2009) in this setting and prove that it is an optimal method.

Keywords: convex optimization; subgradient method; non-smooth optimization; iteration complexity; constrained optimization

1 Introduction

In this work we are interested in constrained minimization problems,

\[
\min_{x \in \mathbb{R}^d} f(x) \tag{1}
\]

s.t. \( f_i(x) \leq 0 \quad i = 1, \ldots, n \)

\( h_i(x) = 0 \quad i = 1, \ldots, p, \)

where \( f(x) \) and all \( f_i(x) \) are convex functions and all \( h_i(x) \) are affine functions from \( \mathbb{R}^d \) to \( \mathbb{R} \). Our goal is to develop an algorithm with a proven convergence rate to the constrained minimum of \( f(x) \) without any further assumptions besides standard regularity conditions. In particular, we do not assume that \( f(x) \) or the \( f_i(x) \) are differentiable nor Lipschitz continuous, and we do not assume a priori that algorithm iterates are constrained to any bounded set. All that is required is that an optimal primal and dual solution exist and that strong
duality holds.

Convex functions which are neither differentiable nor Lipschitz continuous arise naturally in optimization problems, such as the maximum of a set of quadratic functions and the unconstrained soft-margin SVM formulation (Shalev-Shwartz and Ben-David, 2014, Ch. 15). Our work’s main motivation though is in finding a general algorithm which can be applied to a wide range of applications without the need for detailed function properties or problem specific parameter tuning.

In terms of non-asymptotic convergence guarantees for constrained non-smooth convex optimization problems, there exists deterministic algorithms, such as the subgradient method (Nesterov, 2004, Theorem 3.2.3) and its extension using mirror descent (Beck et al., 2010), as well as algorithms for stochastic optimization settings such as the cooperative stochastic approximation algorithm (Lan and Zhou, 2020), which can be seen as a stochastic extension of the subgradient method, and the primal-dual stochastic gradient method of Xu (2020), which is based on analysis of the augmented Lagrangian. After $K$ iterations, all of the algorithms discussed have a proven rate of convergence of $O\left(\frac{1}{\sqrt{K}}\right)$ towards an optimal solution, which is the best rate achievable using a first-order method, in the sense of matching the lower complexity bound for the unconstrained version of our problem setting (Nesterov, 2004, Section 3.2.1). All of these algorithms’ convergence results rely on some combination of a compact feasible region, bounds on the subgradients, or bounds on the constraint functions though.

If we consider the unconstrained problem with $n = p = 0$, recent works include that of Grimmer (2019) which proved that the convergence rate of the subgradient method holds under the more relaxed assumption compared to Lipschitz continuity, that $f(x) - f(x^*) \leq D(\|x - x^*\|^2)$ holds where $x^*$ is an optimal solution and $D(\cdot)$ is a non-negative non-decreasing function. The convergence rate of $O\left(\frac{1}{\sqrt{K}}\right)$ for the general unconstrained convex optimization problem was solved earlier though by Nesterov (2009), with the method of weighted dual averages. The iterates of the algorithm can be shown to be bounded for a range of convex optimization problems, including unconstrained minimization without the assumption of a global Lipschitz parameter. The path taken in this paper is to apply the method of weighted dual averages, presented as Algorithm 1 to the general convex constrained problem and establish the same rate of convergence as previous works under our more relaxed assumptions.

2 Preliminaries

We define the Lagrangian function as

$$L(x, \mu, \theta) := f(x) + \sum_{i=1}^{n} \mu_i f_i(x) + \sum_{i=1}^{p} \theta_i h_i(x),$$
and the dual problem as
\[
\max_{\mu \geq \mathbb{R}^n_{+}} \min_{x \in \mathbb{R}^d} L(x, \mu, \theta).
\]

The following assumptions are sufficient for \( \Pi \) to be a convex optimization problem with an optimal primal and dual solution with strong duality.

**Assumptions 1.**

1. \( f(x) \) and \( f_i(x) \) for \( i = 1, \ldots, n \) are convex functions, and \( h_i(x) \) for \( i = 1, \ldots, p \) are affine functions over \( \mathbb{R}^d \).
2. Slater’s condition holds: there exists an \( \hat{x} \in \mathbb{R}^d \) such that \( f_i(\hat{x}) < 0 \) for \( i = 1, \ldots, n \) and \( h_i(\hat{x}) = 0 \) for \( i = 1, \ldots, p \).
3. There exists an optimal solution, denoted \( x^* \).

These assumptions are sufficient since the optimal objective value of \( f(x^*) \) is finite by the continuity of \( f(x) \), and given that Slater’s condition holds, strong duality holds and there exists at least one dual optimal solution \((\mu^*, \theta^*)\) \cite[Prop. 5.3.5]{Bertsekas:2009}.

Strong duality holds if and only if \((x^*, \mu^*, \theta^*)\) is a saddle point of \( L(x, \mu, \theta) \) \cite[Prop. 3.4.1]{Bertsekas:2009}, i.e. \( \forall x \in \mathbb{R}^d, \mu \geq \mathbb{R}^n_{+}, \theta \in \mathbb{R}^p \),
\[
L(x^*, \mu^*, \theta^*) \leq L(x^*, \mu, \theta) \leq L(x, \mu^*, \theta^*).
\]

We will work with an unconstrained version of \( \Pi \), written as
\[
\min_{x \in \mathbb{R}^d} \max_{\lambda \geq 0} F(x, \lambda) := f(x) + \lambda \overline{f}(x),
\]

where
\[
\overline{f}(x) := \max(f_1(x), f_2(x), \ldots, f_n(x), |h_1(x)|, |h_2(x)|, \ldots, |h_p(x)|).
\]

Let \( \lambda^* = \sum_{i=1}^n \mu_i^* + \sum_{i=1}^p |\theta_i^*| \). By the fact that \( \overline{f}(x^*) = 0 \) and complementary slackness, \( \sum_{i=1}^n \mu_i^* f_i(x^*) = 0 \),
\[
F(x^*, \lambda) = f(x^*) = L(x^*, \mu^*, \theta^*) \forall \lambda,
\]
and
\[
L(x, \mu^*, \theta^*) = f(x) + \sum_{i=1}^n \mu_i^* f_i(x) + \sum_{i=1}^p \theta_i^* h_i(x)
\leq f(x) + \sum_{i=1}^n \mu_i^* f_i(x) + \sum_{i=1}^p |\theta_i^*| |h_i(x)|
\leq f(x) + \lambda^* \overline{f}(x)
= F(x, \lambda^*),
\]
hence from (2), for all \((x, \lambda) \in \mathbb{R}^{d+1}\),
\[
F(x^*, \lambda) \leq F(x, \lambda^*). \tag{3}
\]

We will use the following notation for the subgradients needed of \(F(x, \lambda)\),
\[
g(x) \in \partial f(x) \\
g_i(x) \in \partial f_i(x) \\
\overline{\mathcal{F}}(x) \in \partial \overline{\mathcal{F}}(x) = \text{Conv}\{\partial f_i(x) : f_i(x) = \overline{f}(x)\} \cup \{\partial |h_i(x)| : |h_i(x)| = \overline{f}(x)\}
\]
\[
G_x(x, \lambda) \in \partial x F(x, \lambda) = \partial f(x) + \lambda \partial f(x) \\
G_{\lambda}(x, \lambda) = \nabla_{\lambda} F(x, \lambda) = \overline{f}(x) \\
G(x, \lambda) \in \partial F(x, \lambda),
\]
where for subdifferential of \(\partial f(x)\), see for example (Nesterov, 2004, Lemma 3.1.10). Following the standard measure of convergence to a primal solution, with an optimal solution \(x^*\), we define an algorithm’s output \(\bar{x}\) as an \((\epsilon_1, \epsilon_2)\)-optimal solution if
\[
f(\bar{x}) - f(x^*) \leq \epsilon_1 \quad \text{and} \quad \overline{f}(\bar{x}) \leq \epsilon_2.
\]

### 3 Weighted dual method

Convergence to an optimal solution is proven using the method of weighted dual averages of Nesterov (2009) presented as Algorithm 1. When convenient we will use the column vector \(w := [\bar{x}; \lambda] := [\bar{x}^T, \lambda^T]^T\), and the notation \(G_k := \frac{[G_x(w_k); -G_{\lambda}(w_k)]}{\|G(w_k)\|_2}\) (note that \(\|G_k\|_2 = 1\)).

**Algorithm 1 Method of weighted dual averages**

**Input:** \(w_0 = [x_0 \in \mathbb{R}^d; \lambda_0 \geq 0]; s_0 = \hat{s}_0 = \hat{x}_0 = 0; \beta_0 = 1\)

for \(k = 0, 1, ..., K - 1\) do

Compute \(G(w_k) \in \partial F(w_k)\)
\[
s_{k+1} = s_k + \frac{[G_x(w_k); -G_{\lambda}(w_k)]}{\|G(w_k)\|_2} \\
w_{k+1} = w_0 - \frac{s_{k+1}}{\beta_k} \\
\beta_{k+1} = \beta_k + \frac{1}{\beta_k} \\
\hat{s}_{k+1} = \hat{s}_k + \frac{1}{\|G(w_k)\|_2} \\
\hat{x}_{k+1} = \hat{x}_k + \frac{\hat{s}_{k+1}}{\|G(w_k)\|_2}
\]

end for

\(\hat{s}_{K+1} = \hat{s}_K + \frac{1}{\|G(w_K)\|_2} \quad \hat{x}_{K+1} = \hat{x}_K + \frac{\hat{s}_{K+1}}{\|G(w_K)\|_2}\)

return \(\overline{x}_{K+1} = s_{K+1}^{-1} \hat{x}_{K+1}\)

In each iteration \(w_{k+1} = w_0 - \frac{s_{k+1}}{\beta_k}\) is the maximizer of
\[
U_\beta^s(w) := -\langle s, w - w_0 \rangle - \frac{\beta}{2} \|w - w_0\|_2^2 \tag{4}
\]
for $s = s_{k+1}$ and $\beta = \beta_k$, with

$$U^{s_{k+1}}_{\beta_k}(w_{k+1}) = \frac{\|s_{k+1}\|_2^2}{2\beta_k}. \tag{5}$$

In addition, $U^s_\beta(w)$ is strongly concave in $w$ with parameter $\beta$,

$$U^s_\beta(w) \leq U^s_\beta(w') + \langle \nabla U^s_\beta(w'), w - w' \rangle - \frac{\beta}{2}\|w - w'\|_2^2. \tag{6}$$

Given that $G_\lambda(w_k) \geq 0$, it holds that $\lambda_{k+1} \geq \lambda_k$, with the $\lambda_k$ iterates always remaining feasible. The following property examines the case Algorithm 1 crashes due to $\|G(w_k)\|_2 = 0$.

**Property 2.** If $\|G(w_k)\|_2 = 0$, then $x_k$ is an optimal solution to (1).

**Proof.** If $\|G(w_k)\|_2 = 0$, this implies that $G_\lambda(w_k) = \overline{f}(x_k) = 0$ and hence $x_k$ is a feasible solution. From $\|G_x(w_k)\|_2 = 0$, $0 \in \partial_x F(x_k, \lambda_k)$, and as $F(x, \lambda_k)$ is convex in $x$, $x_k$ is a minimizer of $F(x, \lambda_k)$. It follows that for all $x \in \mathbb{R}^d$ feasible in (1),

$$f(x_k) = f(x) + \lambda_k \overline{f}(x_k) \leq f(x) + \lambda_k \overline{f}(x) = f(x).$$

A key property of Algorithm 1 is that by redefining $G(w_k)$ appropriately, the iterates are bounded for quite general convex optimization problems. In particular, all that is required is that (11) in the proof below holds for the iterates to be bounded using [Nesterov, 2009, Theorem 3]. For the sake of completeness we present the full proof for our application.

**Property 3.** For all iterates of Algorithm 1, it holds that

$$\|w_k - w^*\|_2 \leq \|w_0 - w^*\|_2 + 1,$$

with the inequality being strict when $w_0 \neq w^*$.

**Proof.** From (5),

$$U^{s_{k+1}}_{\beta_k}(w_{k+1}) = \frac{\beta_{k-1}}{\beta_k} \frac{\|s_{k+1}\|_2^2}{2\beta_{k-1}} = \frac{\beta_{k-1}}{\beta_k} \frac{\|s_k + G_k\|_2^2}{2\beta_{k-1}} = \frac{\beta_{k-1}}{\beta_k} \left( \frac{\|s_k\|_2^2}{2\beta_{k-1}} + \frac{1}{\beta_{k-1}}(s_k, G_k) + \frac{\|G_k\|_2^2}{2\beta_{k-1}} \right) = \frac{\beta_{k-1}}{\beta_k} \left( U^{s_k}_{\beta_{k-1}}(w_k) + \frac{1}{\beta_{k-1}}(s_k, G_k) + \frac{1}{2\beta_{k-1}} \right) = \frac{\beta_{k-1}}{\beta_k} \left( U^{s_k}_{\beta_{k-1}}(w_k) + \langle w_0 - w_k, G_k \rangle + \frac{1}{2\beta_{k-1}} \right).$$

5
Rearranging,
\[
\langle w_k - w_0, \overline{G}_k \rangle = U_{\beta_{k-1}}^{s_k}(w_k) - \frac{\beta_k}{\beta_{k-1}} U_{\beta_k}^{s_{k+1}}(w_{k+1}) + \frac{1}{2\beta_{k-1}} \leq U_{\beta_{k-1}}^{s_k}(w_k) - U_{\beta_k}^{s_{k+1}}(w_{k+1}) + \frac{1}{2\beta_{k-1}},
\]
since $\beta_k$ is increasing. Telescoping these inequalities for $k = 1, .., K$, and using the fact that $\|s_1\|_2^2 = 1$,
\[
\sum_{k=1}^{K} \langle w_k - w_0, \overline{G}_k \rangle \leq U_{\beta_0}^{s_1}(w_1) - U_{\beta_K}^{s_{K+1}}(w_{K+1}) + \sum_{k=1}^{K} \frac{1}{2\beta_{k-1}} \tag{7}
\]
\[
= \frac{1}{2\beta_0} - U_{\beta_K}^{s_{K+1}}(w_{K+1}) + \sum_{k=0}^{K-1} \frac{1}{2\beta_k}
\]
\[
= -U_{\beta_K}^{s_{K+1}}(w_{K+1}) + \frac{1}{2} \left( \sum_{k=0}^{K-1} \frac{1}{\beta_k} \right) + \beta_0. \tag{8}
\]

Expanding the recursion $\beta_k = \frac{1}{\beta_{k-1}} + \beta_{k-1}$,
\[
\sum_{k=1}^{K} \langle w_k - w_0, \overline{G}_k \rangle \leq -U_{\beta_K}^{s_{K+1}}(w_{K+1}) + \frac{\beta_K}{2}. \tag{8}
\]

Given the convexity of $F(x, \lambda)$ in $x$ and linearity in $\lambda$,
\[
F(x^*, \lambda_k) \geq F(x_k, \lambda_k) + \langle G_x(x_k, \lambda_k), x^* - x_k \rangle \tag{9}
\]
\[
F(x_k, \lambda^*) = F(x_k, \lambda_k) + \langle G_\lambda(x_k, \lambda_k), \lambda^* - \lambda_k \rangle. \tag{10}
\]

Subtracting (9) from (10) and using (3),
\[
0 \leq \langle [G_x(w_k); -G_\lambda(w_k)], w_k - w^* \rangle. \tag{11}
\]

It follows that
\[
0 \leq \sum_{k=1}^{K} \langle w_k - w^*, \overline{G}_k \rangle \\
= \sum_{k=1}^{K} \langle w_0 - w^*, \overline{G}_k \rangle + \sum_{k=1}^{K} \langle w_k - w_0, \overline{G}_k \rangle \\
\leq \sum_{k=1}^{K} \langle w_0 - w^*, \overline{G}_k \rangle - U_{\beta_K}^{s_{K+1}}(w_{K+1}) + \frac{\beta_K}{2} \\
= \langle w_0 - w^*, s_{K+1} \rangle - U_{\beta_K}^{s_{K+1}}(w_{K+1}) + \frac{\beta_K}{2}, \tag{12}
\]
where the second inequality uses (8), and the second equality follows since $s_{k+1} = s_k + G_k$.

Considering inequality (6) with $s = s_{K+1}$, $\beta = \beta_k$, $w = w^*$, and $w' = w_{K+1}$,

\[
U_{\beta_k}^{s_{K+1}}(w^*) \leq U_{\beta_k}^{s_{K+1}}(w_{K+1}) + \langle \nabla U_{\beta_k}^{s_{K+1}}(w_{K+1}), w^* - w_{K+1} \rangle - \frac{\beta_k}{2} \|w^* - w_{K+1}\|_2^2
\]

\[
= U_{\beta_k}^{s_{K+1}}(w_{K+1}) - \frac{\beta_k}{2} \|w^* - w_{K+1}\|_2^2,
\]

given that $w_{K+1}$ is the maximum of $U_{\beta_k}^{s_{K+1}}(w)$. Applying this inequality in (12),

\[
0 \leq \langle w_0 - w^*, s_{K+1} \rangle - U_{\beta_k}^{s_{K+1}}(w^*) - \frac{\beta_k}{2} \|w^* - w_{K+1}\|_2^2 + \frac{\beta_k}{2} \|w^* - w_0\|_2^2 - \frac{\beta_k}{2} \|w^* - w_{K+1}\|_2^2 + \frac{\beta_k}{2} \|w^* - w_0\|_2^2.
\]

where the first equality uses the definition of $U_{\beta_k}^{s_{K+1}}(w^*)$ (11). Rearranging,

\[
\|w^* - w_{K+1}\|_2^2 \leq \|w^* - w_0\|_2^2 + 1.
\] (13)

As $K \geq 1$ from (7), this implies that (13) holds for $k \geq 2$. Considering now when $k = 1$,

\[
\|w_1 - w^*\|_2^2 = \|w_0 - w^* - G_0\|_2^2
\]

\[
= \|w_0 - w^*\|_2^2 - 2\langle w_0 - w^*, G_0 \rangle + 1
\]

\[
\leq \|w_0 - w^*\|_2^2 + 1,
\]

where the last line uses (11). Now for all $k$,

\[
\left(\|w^* - w_0\|_2 + 1\right)^2 = \|w^* - w_0\|_2^2 + 2\|w^* - w_0\| + 1
\]

\[
\geq \|w^* - w_k\|_2^2 + 2\|w^* - w_0\|,
\]

so that

\[
\|w^* - w_0\|_2 + 1 \geq \|w^* - w_k\|_2,
\] (14)

with (14) being strict when $w^* \neq w_0$.

In order to prove the convergence result of Algorithm 1, we require bounding the norm of the subgradients $G(w_k)$.

**Property 4.** For all $k$ there exists a constant $L$ such that

\[
\|G(w_k)\|_2 \leq L(\|w_0 - w^*\|_2 + \lambda^* + 3).
\]

**Proof.** Recall that $g(x) \in \partial f(x)$ and $\overline{g}(x) \in \partial \overline{f}(x),$

\[
\|G(w_k)\|_2 = \|[G_x(w_k); G_\lambda(w_k)]\|_2
\]

\[
= \|[g(x_k) + \lambda_k \overline{g}(x_k); \overline{f}(x_k)]\|_2
\]

\[
\leq \|g(x_k)\|_2 + \lambda_k \|\overline{g}(x_k)\|_2 + \|\overline{f}(x_k)\|.
\] (15)
The iterates of Algorithm 1 are bounded in a convex compact region, \( w_k \in D := \{ w : \| w - w^* \|_2 \leq \| w_0 - w^* \|_2 + 1 \} \). This implies that \( x_k \in D_x := \{ x : \| x - x^* \|_2 \leq \| w_0 - w^* \|_2 + 1 \} \) and \( \lambda_k \in D_{\lambda} := \{ \lambda : |\lambda - \lambda^*| \leq \| w_0 - w^* \|_2 + 1 \} \). It follows that there exists an \( L_1 \geq 0 \) such that \( f(x) \) is \( L_1 \)-Lipschitz continuous on \( D_x \) [Hiriart-Urruty and Lemaréchal, 1996, Theorem IV.3.1.2],

\[
|f(x) - f(x')| \leq L_1 \| x - x' \|_2, \tag{16}
\]

for all \( x, x' \in D_x \). Assuming that \( w_0 \neq w^*, x_k \in \text{Int } D_x \). For any \( x \in \text{Int } D_x \), taking \( \theta > 0 \) small enough such that \( x' = x + \theta \frac{g(x)}{\| g(x) \|_2} \in D_x \),

\[
\langle g(x), x' - x \rangle \leq f(x') - f(x) \iff \langle g(x), x' - x \rangle \leq L_1 \| x' - x \|_2
\]

\[
\iff \langle g(x), \theta \frac{g(x)}{\| g(x) \|_2} \rangle \leq L_1 \theta
\]

\[
\iff \| g(x) \|_2 \leq L_1. \tag{17}
\]

If \( w_0 = w^*, x_k \in \text{Int } D_x^\delta := \{ x : \| x - x^* \|_2 \leq \delta + 1 \} \) for any \( \delta > 0 \), and \( L_1 \) can be increased such that (16) holds over \( D_x^\delta \) so that (17) holds for all \( x \in D_x \). Similarly, there exists an \( L_2 \geq 0 \) such that \( |\overline{f}(x) - \overline{f}(x')| \leq L_2 \| x - x' \|_2 \) and \( \| g(x) \|_2 \leq L_2 \) for all \( x, x' \in D_x \). In addition,

\[
\overline{f}(x_k) = |\overline{f}(x_k) - \overline{f}(x^*)|
\]

\[
\leq L_2 \| x_k - x^* \|_2
\]

\[
\leq L_2(\| w_0 - w^* \|_2 + 1),
\]

and \( \lambda_k \leq \| w_0 - w^* \|_2 + 1 + \lambda^* \) from the definition of \( D_{\lambda} \). Combining these bounds in (15) and taking \( L = \max(L_1, L_2) \),

\[
\| G(w_K) \|_2 \leq \| g(x_k) \|_2 + \lambda_k \| g(x_k) \|_2 + \overline{f}(x_k)
\]

\[
\leq L_1 + (\| w_0 - w^* \|_2 + 1 + \lambda^*)L_2 + L_2(\| w_0 - w^* \|_2 + 1)
\]

\[
\leq L(2\| w_0 - w^* \|_2 + \lambda^* + 3).
\]

For all \( k \) the value of \( \beta_k \) can be bounded as follows by induction.

**Property 5.** [Nesterov, 2009, Lemma 3]

\[
\beta_k \leq \frac{1}{1 + \sqrt{3} + \sqrt{2k + 1}}
\]

We can now prove a convergence rate of \( O(\frac{1}{\sqrt{K}}) \) to an optimal solution of problem (1).

**Theorem 6.** Running Algorithm 1 for \( K \) iterations,

\[
f(\bar{x}_{K+1}) - f(x^*) \leq \frac{C(\| w_0 - w^* \|_2 + 1)}{2(K + 1)} \left( \frac{1}{1 + \sqrt{3} + \sqrt{2K + 1}} \right)
\]
\[
\bar{f}(\hat{x}_{K+1}) \leq \frac{C(4\|w_0 - w^*\|_2 + 1)^2 + 1}{2(K + 1)} \left( \frac{1}{1 + \sqrt{3}} + \sqrt{2K + 1} \right),
\]

where \(C := L(2\|w_0 - w^*\|_2 + \lambda^* + 3)\).

**Proof.** Using equation (8), and recalling that \(w_{K+1}\) maximizes \(U_{\beta_K}^{\bar{s}_{K+1}}(w)\) (9),

\[
\frac{\beta_K}{2} \geq \sum_{k=1}^{K} (w_k - w_0, G_k) + U_{\beta_K}^{\bar{s}_{K+1}}(w_{K+1})
\]

\[
= \sum_{k=1}^{K} (w_k - w_0, G_k) + \max_{w^* \in \mathbb{R}^{d+1}} \{- \sum_{k=0}^{K} G_k, w - w_0\} - \frac{\beta_K}{2} \|w - w_0\|^2_2
\]

\[
= \max_{w^* \in \mathbb{R}^{d+1}} \left\{ \sum_{k=0}^{K} - (G_k, w - w_k) - \frac{\beta_K}{2} \|w - w_0\|^2_2 \right\}. \tag{18}
\]

Like \(\overline{\pi}_{K+1}\), let \(\overline{\mu}_{K+1} := \hat{s}_{K+1}^{-1} \sum_{k=0}^{K} \frac{w_k}{\|G(w_k)\|_2}\) and \(\overline{\lambda}_{K+1} := \hat{s}_{K+1}^{-1} \sum_{k=0}^{K} \frac{\lambda_k}{\|G(w_k)\|_2}\). Multiplying both sides of (18) by \(\hat{s}_{K+1}^{-1}\),

\[
\hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \geq \hat{s}_{K+1}^{-1} \max_{w^* \in \mathbb{R}^{d+1}} \left\{ \sum_{k=0}^{K} - (G_k, w - w_k) - \frac{\beta_K}{2} \|w - w_0\|^2_2 \right\}
\]

\[
= \hat{s}_{K+1}^{-1} \max_{w^* \in \mathbb{R}^{d+1}} \left\{ \sum_{k=0}^{K} - \frac{G_x(w_k)}{\|G(w_k)\|_2}, x - x_k \right\} + \left\{ \frac{G_\lambda(w_k)}{\|G(w_k)\|_2}, \lambda - \lambda_k \right\} - \frac{\beta_K}{2} \|w - w_0\|^2_2
\]

\[
\geq \hat{s}_{K+1}^{-1} \max_{w^* \in \mathbb{R}^{d+1}} \left\{ \sum_{k=0}^{K} \frac{F(x_k, \lambda_k) - F(x, \lambda_k)}{\|G(w_k)\|_2} + \frac{F(x_k, \lambda) - F(x, \lambda)}{\|G(w_k)\|_2} - \frac{\beta_K}{2} \|w - w_0\|^2_2 \right\}
\]

\[
= \hat{s}_{K+1}^{-1} \max_{w^* \in \mathbb{R}^{d+1}} \left\{ \sum_{k=0}^{K} \frac{F(x_k, \lambda) - F(x, \lambda_k)}{\|G(w_k)\|_2} - \frac{\beta_K}{2} \|w - w_0\|^2_2 \right\}
\]

\[
\geq \max_{w^* \in \mathbb{R}^{d+1}} \left\{ F(\pi_{K+1}, \lambda) - F(x, \overline{\lambda}_{K+1}) - \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \|w - w_0\|^2_2 \right\}
\]

\[
= \max_{w^* \in \mathbb{R}^{d+1}} \left\{ f(\pi_{K+1}) + \lambda \bar{f}(\pi_{K+1}) - f(x) - \overline{\lambda}_{K+1} \bar{f}(x) - \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \|w - w_0\|^2_2 \right\}, \tag{19}
\]

where the third inequality uses Jensen’s inequality. Given the maximum function, the inequality holds for any choice of \(w\). We consider two cases, the first being \(x = \pi_{K+1}\) and
\( \lambda = \lambda_{K+1} + 1 \). From (19),

\[
\hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \geq f(\overline{x}_{K+1}) + (\lambda_{K+1} + 1)f(\overline{x}_{K+1}) - f(x_{K+1}) - \lambda_{K+1}f(x_{K+1})
- \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \|\overline{x}_{K+1}; \lambda_{K+1} + 1 - w_0\|^2
falling to
\]

\[
= f(\overline{x}_{K+1}) - \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \|\overline{x}_{K+1}; \lambda_{K+1} + 1 - w_0\|^2.
\] (20)

Further,

\[
\|\overline{x}_{K+1}; \lambda_{K+1} + 1 - w_0\|_2 \leq \|\overline{x}_{K+1} - w_0\|_2 + 1
= \|\overline{x}_{K+1} - w^* + w^* - w_0\|_2 + 1
\leq \|\overline{x}_{K+1} - w^*\|_2 + \|w^* - w_0\|_2 + 1
\leq \hat{s}_{K+1}^{-1} \sum_{k=0}^{K} \|w_k - w^*\|_2 + \|w^* - w_0\|_2 + 1
\leq \hat{s}_{K+1}^{-1} \sum_{k=0}^{K} \|w_0 - w^*\|_2 + \|w^* - w_0\|_2 + 1
\leq 2(\|w_0 - w^*\|_2 + 1),
\] (21)

where the third inequality uses Jensen's inequality and the fourth inequality uses Property 3. Combining (20) and (21),

\[
\overline{f}(\overline{x}_{K+1}) \leq \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} (4(\|w_0 - w^*\|_2 + 1)^2 + 1).
\]

The second case will use \( w = w^* \). Starting from (19),

\[
\hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \geq f(\overline{x}_{K+1}) + \lambda^* f(\overline{x}_{K+1}) - f(x^*) - \lambda_{K+1}f(x^*) - \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \|w^* - w_0\|^2
\geq f(\overline{x}_{K+1}) - f(x^*) - \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \|w^* - w_0\|^2,
\]
since \( f(x^*) = 0 \) and \( \lambda^* f(\overline{x}_{K+1}) \geq 0 \). Rearranging,

\[
f(\overline{x}_{K+1}) - f(x^*) \leq \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} (\|w_0 - w^*\|_2^2 + 1).
\]

Using Properties 4 and 5, \( \hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \) can be bounded as follows.

\[
\hat{s}_{K+1}^{-1} \frac{\beta_K}{2} \leq \frac{1}{2} \left( \sum_{k=0}^{K} \frac{1}{\|G(w_k)\|_2} \right)^{-1} \left( \frac{1}{1 + \sqrt{3}} + \sqrt{2k + 1} \right)
\leq \frac{1}{2} \left( \sum_{k=0}^{K} \frac{1}{C} \right)^{-1} \left( \frac{1}{1 + \sqrt{3}} + \sqrt{2k + 1} \right)
\leq \frac{C}{2(K + 1)} \left( \frac{1}{1 + \sqrt{3}} + \sqrt{2k + 1} \right).
\]
Algorithm 1 is an optimal method as its convergence rate of $O\left(\frac{1}{\sqrt{K}}\right)$ from Theorem 6 matches the lower complexity bound for minimizing the unconstrained version of (1) as discussed in the introduction. The following corollary establishes the $O\left(\min(\epsilon_1, \epsilon_2)^{-2}\right)$ iteration complexity required to achieve an $(\epsilon_1, \epsilon_2)$-optimal solution.

**Corollary 7.** An $(\epsilon_1, \epsilon_2)$ optimal solution is obtained after running Algorithm 1 for

$$K \geq \alpha \max \left(\frac{C_1}{\epsilon_1}, \frac{C_2}{\epsilon_2}\right)^2$$

iterations, where $C_1 = C(\|w_0 - w^*\|^2_2 + 1)$, $C_2 = C(4(\|w_0 - w^*\|^2_2 + 1)^2 + 1)$, and $\alpha = \frac{1}{2}\left(\frac{1}{\sqrt{8(1+\sqrt{3})} + 1}\right)^2$.

**Proof.** From Theorem 6 for $i = 1, 2$, we need to compute a lower bound on $K$ which ensures that

$$\frac{C_i}{2(K+1)}\left(\frac{1}{1 + \sqrt{3}} + \sqrt{2K + 1}\right) \leq \epsilon_i.$$

Since for $K \geq 1$,

$$\frac{\sqrt{K}}{2(K+1)}\left(\frac{1}{1 + \sqrt{3}} + \sqrt{2K + 1}\right) = \frac{\sqrt{K}}{2(1 + \sqrt{3})(K + 1)} + \frac{\sqrt{2K^2 + K}}{2(K + 1)} \leq \frac{1}{4(1 + \sqrt{3})} + \frac{1}{\sqrt{2}} \leq \frac{1}{\sqrt{2}}\left(\frac{1}{\sqrt{8(1+\sqrt{3})} + 1}\right),$$

it holds that

$$\frac{C_i}{2(K+1)}\left(\frac{1}{1 + \sqrt{3}} + \sqrt{2K + 1}\right) \leq \frac{\sqrt{\alpha C_i}}{\sqrt{K}}.$$

To ensure convergence within $\epsilon_i$, it is sufficient for $\epsilon_i \geq \sqrt{\frac{\alpha C_i}{\sqrt{K}}}$, or that $K \geq \alpha (\frac{C_i}{\epsilon_i})^2$. Taking the maximum over $i$ gives the result.

### 4 Conclusion

In this paper we have established the existence of a simple first-order method for the general convex constrained optimization problem without the need for differentiability nor Lipschitz continuity. We see this as a general use algorithm for practitioners since it requires minimal knowledge of the problem, with no parameter tuning for its implementation, while still achieving the optimal convergence rate for first-order methods.
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References

Amir Beck, Aharon Ben-Tal, Nili Guittman-Beck, and Luba Tetruashvili. The CoMirror algorithm for solving nonsmooth constrained convex problems. *Operations Research Letters*, 38(6):493–498, 2010.

Dimitri P. Bertsekas. *Convex Optimization Theory*. Athena Scientific, 2009.

Benjamin Grimmer. Convergence Rates for Deterministic and Stochastic Subgradient Methods without Lipschitz Continuity. *SIAM Journal on Optimization*, 29(2):1350–1365, 2019.

Jean-Baptiste Hiriart-Urruty and Claude Lemaréchal. *Convex Analysis and Minimization Algorithms I: Fundamentals*. Springer-Verlag, 1996.

Guanghui Lan and Zhiqiang Zhou. Algorithms for stochastic optimization with functional or expectation constraints. *Computational Optimization and Applications*, 76:461–498, 2020.

Yurii Nesterov. *Introductory Lectures on Convex Optimization: A Basic Course*. Springer Science+Business Media, 2004.

Yurii Nesterov. Primal-dual subgradient methods for convex problems. *Mathematical Programming*, 120(1):221–259, 2009.

Shai Shalev-Shwartz and Shai Ben-David. *Understanding Machine Learning: From Theory to Algorithms*. Cambridge University Press, 2014.

Yangyang Xu. Primal-Dual Stochastic Gradient Method for Convex Programs with Many Functional Constraints. *SIAM Journal on Optimization*, 30(2):1664–1692, 2020.