Nonnegative curvature, symmetry and fundamental group

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Abstract

We prove a result on equivariant deformations of flat bundles, and as a corollary, we obtain two “splitting in a finite cover” theorems for isometric group actions on Riemannian manifolds with infinite fundamental groups, where the manifolds are either compact of Ric ≥ 0, or complete of sec ≥ 0.

1 Introduction

J. Cheeger and D. Gromoll [CG71] showed, as an application of their Splitting Theorem, that if $M$ is a compact manifold with Ric ≥ 0 and infinite $\pi_1(M)$, then $M$ has a structure of a flat (orbifold) bundle whose generic fiber is a simply-connected manifold $C$ with Ric ≥ 0, the base is a flat orbifold, and the holonomy group lies in Iso($C$). In particular, $M$ has a finite cover diffeomorphic to the Riemannian product of a flat torus $T$ and a simply-connected manifold $C$ with Ric ≥ 0. The cover $C \times T \to M$ is Riemannian (i.e. a local isometry) precisely when the holonomy group of the flat bundle is finite. For example, if $M$ is Ricci flat, the cover is Riemannian, because in this case $C$ is Ricci flat and so Iso($C$) is finite by Bochner’s theorem. Now if the cover $C \times T \to M$ is Riemannian, then one can understand the group Iso($M$) by relating it to Iso($C \times T$) $\cong$ Iso($C$) $\times$ Iso($T$); this turns out to be true even for non-Riemannian covers, as follows.

Theorem 1.1. Let $(M,g)$ be a compact Riemannian manifold of Ric ≥ 0 with $\pi_1(M)$ infinite, and let $H$ be a compact subgroup of Iso($M,g$). Then there exists a smooth deformation of metrics $g_t$ on $M$ such that $g_0 = g$, $H \leq$ Iso($M,g_t$), the universal Riemannian covers of $(M,g_t)$ are all isometric, and a finitely-sheeted Riemannian cover of $(M,g_1)$ is isometric to the Riemannian product of a flat torus $T$ and a simply-connected manifold $C$ of Ric ≥ 0.

Theorem 1.1 generalizes a result of B. Wilking [Wi00] who proved it for trivial $H$. Theorem 1.1 follows from a general result on equivariant deformations of flat bundles (see

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Section 2 for details). Essentially, we show that any flat bundle of this kind can be equiv-
ariantly deformed, through flat bundles, to a flat bundle with finite holonomy. This also
gives a version of Theorem 1.1 for isometric actions on complete manifolds of sec \( \geq 0 \).
Since the isometries of \( T \) are well-understood, Theorem 1.1 roughly means that studying
the isometries \( M \) of can be reduced to studying the isometries of \( C \). We now list some
sample corollaries of Theorem 1.1.

- If \( H \) is connected and simply-connected, then the \( H \)-action on \( (M,g_1) \) lifts uniquely
to an isometric \( H \)-action on \( C \times T \) such that the \( H \)-action on the \( T \)-factor is trivial.
- If \( H \) has a fixed point \( x \) and the induced \( H \)-action on \( \pi_1(M,x) \) is trivial, then the
\( H \)-action on \( M \) lifts to an isometric \( H \)-action on \( C \times T \) such that the \( H \)-action on
the \( T \)-factor is trivial.
- If \( H \) is connected, then the \( H \)-action on \( M \) lifts to an isometric action of a finite cover
\( \tilde{H} \) of \( H \) on \( C \times T \). In this case the \( \tilde{H} \)-action on the \( T \)-factor is also well-understood
(see Appendix B).

Given a compact Lie group \( G \) acting smoothly and effectively on a closed manifold \( M \), it
is natural to ask whether \( M \) admits a \( G \)-invariant metric satisfying \( \text{Ric} \geq 0 \) or \( \text{Ric} > 0 \).
It is well-known that if \( G \) acts transitively, then it preserves a metric of \( \text{Ric} \geq 0 \), which
has \( \text{Ric} > 0 \) if and only if \( \pi_1(M) \) is finite. It was shown in [GZ02] that the same is true
for cohomogeneity one actions. By [GPT98], \( M \) admits a \( G \)-invariant metric of \( \text{Ric} > 0 \) if
the action is free, \( G \) is connected, \( M/G \) admits a metric of \( \text{Ric} > 0 \), and \( \pi_1(M) \) is finite.
However, for general actions little is known. Theorem 1.1 gives rise to many examples of
group actions on compact manifolds that are not isometric in any metric of \( \text{Ric} \geq 0 \). For
instance, we prove the following.

**Theorem 1.2.** Let \( M \) be a compact Riemannian manifold with \( \pi_1(M) \) infinite, and let \( H \)
be a closed subgroup of \( \text{Iso}(M) \). Suppose the set of the fixed points of \( H \) has a component
\( F \) of dimension \( \geq 3 \) and for some \( x \in F \), the \( H \)-action on \( \pi_1(M,x) \) is trivial. Then
there are infinitely many smooth \( H \)-actions on \( M \) that are non-isometric in any metric of
\( \text{Ric} \geq 0 \), and such that if \( \text{scat}(M) > 0 \) and \( \dim(F) \geq 5 \), then the actions are isometric in
some metrics of \( \text{scat} > 0 \).

To date, there seem to be no examples of smooth \( G \)-action on a compact manifold \( M \) with
finite \( \pi_1(M) \) such that the \( G \)-action preserves a metric of \( \text{scat} \geq 0 \) and preserves no metric
of \( \text{Ric} \geq 0 \). This problem is well-known for trivial \( G \), and one wonders whether it could
become easier for nontrivial \( G \). If the assumption \( \text{Ric} \geq 0 \) is replaced by \( \text{sec} \geq 0 \), we
construct such examples as follows.

**Theorem 1.3.** Let \( M \) be a complete Riemannian manifold and let \( H \) be a compact subgroup
of \( \text{Iso}(M) \). Suppose the set of the fixed points of \( H \) has a component \( F \) of dimension \( \geq 3 \),
and for some \( x \in F \), the \( H \)-action on \( \pi_1(M,x) \) is trivial. Then there are infinitely many
smooth \( H \)-actions on \( M \) that are non-isometric in any complete metric of \( \text{sec} \geq 0 \), and
such that if \( \text{scat}(M) > 0 \) and \( \dim(F) \geq 5 \), then the actions are isometric in some complete
metrics of \( \text{scat} > 0 \).
Note that the question whether a given $G$-action on $M$ preserves a metric of $\text{scal} > 0$ is also wide open, except for the case of free actions: by O’Neill’s formula, if $G$ acts freely, and either $M/G$ admits a metric of $\text{scal} > 0$, or $G$ has a bi-invariant metric of $\text{scal} > 0$, then $M$ has a $G$-invariant metric of $\text{scal} > 0$. See also [Lot00] for obstructions in the case of semifree circle actions.

The $H$-actions constructed in Theorems 1.2–1.3 are obtained from the original action by taking an equivariant connected sum at a fixed point with certain $H$-actions on the standard sphere. There are other ways to produce $H$-actions with the same properties as in Theorems 1.2–1.3 but for the sake of brevity we do not consider them here, and leave it to the interested reader to pursue.

This paper grew out of the author’s joint work [BK01] with Vitali Kapovitch. I am grateful to Slawomir Kwasik for Remark 2.11 and to Reinhard Schultz for the idea of using equivariant connected sums in Theorems 1.2–1.3.

2 Equivariant splitting in a finite cover

Cheeger and Gromoll proved [CG71] that a closed Riemannian manifold $(M, g)$ of $\text{Ric} \geq 0$ has a finite cover diffeomorphic to the product of flat torus and a simply-connected manifold of $\text{Ric} \geq 0$. In general, the pullback of $g$ to the finite cover is not a product metric, e.g. the metric on $S^2 \times S^1$ obtained by pushing down the standard product metric on $S^2 \times \mathbb{R}$ to $(S^2 \times \mathbb{R})/\mathbb{Z}$ with the $\mathbb{Z}$-action given by $n \cdot (v, t) = (e^{in}v, t + n)$ has sec $\geq 0$ and is not a product metric in any finite cover, as was noted in [CG72].

However, Wilking showed in [Wil00] the metric $g$ can be deformed, through metrics of $\text{Ric} \geq 0$, to a metric $g_1$ such that a finitely-sheeted Riemannian cover of $(M, g_1)$ is isometric to the Riemannian product of a flat torus and a simply-connected manifold with $\text{Ric} \geq 0$. A similar conclusion holds in some other situations e.g. when $M$ is a complete manifolds of sec $\geq 0$ [Wil00, BK01].

In this section we show, by a modification of Wilking’s argument, that one can choose the deformation from $g$ to $g_1$ to be equivariant with respect to the isometry group of $(M, g)$. Our exposition is self-contained yet for the reader’s convenience we mostly follow Wilking’s notations. Like in [Wil00], we actually prove a much more general result with no curvature assumptions, stated in the following definition.

**Definition 2.1.** If $M$ be a complete Riemannian manifold and $H$ is a compact subgroup of the isometry group of $M$, then the pair $(M, H)$ is said to satisfy $(\ast)$ if there exists a regular Riemannian covering $\tilde{M} \to M$ with the infinite group of covering transformation $\Pi$ such that

(i) any element of $H$ lifts to an isometry of $\tilde{M}$, and
(ii) if $N \times \mathbb{R}^n$ is the de Rham decomposition of $\tilde{M}$ [EH98], where $N$ is the product of the non-Euclidean factors, then the $\Pi$-action on $N$, which is the composition of the deck-transformation action $\rho: \Pi \to \text{Iso}(\tilde{M}) \cong \text{Iso}(N) \times \text{Iso}(-n)$ with the projection onto the $\text{Iso}(N)$-factor, has a precompact image.
If \((M, H)\) satisfies \((*)\), we use the following notations. We denote the metrics on \(M\), \(\tilde{M}\) by \(g\), \(\tilde{g}\), respectively. We write the isomorphism \(\text{Iso}(N \times \mathbb{R}^n) \cong \text{Iso}(N) \times \text{Iso}(\mathbb{R}^n)\) as \(k \rightarrow (k_n, k_f)\) for \(k \in \text{Iso}(N \times \mathbb{R}^n)\), and given a subgroup \(K\) of \(\text{Iso}(N \times \mathbb{R}^n)\), we denote by \(K_n\), \(K_f\) the projections of \(K\) into \(\text{Iso}(N)\), \(\text{Iso}(\mathbb{R}^n)\), respectively. Thus, condition (ii) of Definition 2.1 means that \(\rho(\Pi)_n\) is precompact in \(\text{Iso}(N)\). Let \(\tilde{H}\) be the set of all the lifts of elements of \(H\). By [Bre72 I. Theorem 9.3], \(\tilde{H}\) is a Lie subgroup of \(\text{Iso}(M)\) and the map \(\tilde{H} \rightarrow H\) sending \(\tilde{h}\) to \(h\), where \(\tilde{h}\) is a lift of \(h\), is a surjection of Lie groups with kernel \(\Pi\).

**Remark 2.2.** Since \(\Pi\) is infinite and discrete, (ii) implies that \(n > 0\). Also (i) is a purely topological condition on the \(H\)-action on \(M\), which for example is true automatically if \(M\) is the universal cover of \(\tilde{M}\).

**Example 2.3.** \((M, H)\) satisfies \((*)\) if \(\tilde{M}\) is the universal cover of \(M\), \(\pi_1(M)\) is infinite, \(H\) is a compact subgroup of \(\text{Iso}(M)\), and \(M\) satisfies one of the conditions below. 
1. \(M\) is a compact of \(\text{Ric} \geq 0\). Here \(\rho(\Pi)_n\) is precompact since \(N\) is compact [CG71]; 
2. \(M\) is a complete of \(\text{sec} \geq 0\). Here \(\rho(\Pi)_n\) is precompact since \(\text{Iso}(N)\) is compact [CG72]; 
3. \(M\) is complete of \(\text{Ric} \geq 0\) and \(M\) is the normal bundle of a compact submanifold \(S\) such that either \(S\) is totally convex, or there exists a distance nonincreasing retraction \(N \rightarrow S\). Here \(\text{Iso}(N)\) may be noncompact, yet \(\rho(\Pi)_n\) is precompact as is shown in [BK01 Proposition 2.2], where a more general result is proved.

**Remark 2.4.** By [CG72, Per92], if \(M\) satisfies \((*)\), then it satisfies \((**)(2)\).

**Remark 2.5.** Topologically, if \((M, H)\) satisfies \((*)\), then \(M\) is a flat (orbifold) bundle with holonomy group \(\rho(\Pi)_n\), the generic fiber \(N\), and the base \(\mathbb{R}^n/\rho(\Pi)_r\). The \(H\)-action on \(M\) takes fibers to fibers. Theorem 2.10, the main result of this section, gives an \(H\)-equivariant deformation of such flat bundles that starts with the given bundle and ends in a bundle with a finite holonomy group.

**Remark 2.6.** The main difference between the proof in [Wil00] and the arguments of this section is that Wilking deforms the deck-transformation group \(\Pi\) in \(\text{Iso}(\tilde{M})\), while we deform the whole group \(\tilde{H}\), which surjects onto \(H\) with kernel \(\Pi\). The compactness of \(H\) comes into the proof on several occasions.

**Lemma 2.7.** If \((M, H)\) satisfies \((*)\), then there is an \(d\)-dimensional affine subspace \(V^d \subset \mathbb{R}^n\) with \(d \in [1, n]\), such that \(\tilde{H}\) stabilizes \(V^d\) and \(\Pi\) acts on \(V^d\) with finite kernel and the image is a discrete cocompact subgroup of \(\text{Iso}(V^d) \cong \text{Iso}(\mathbb{R}^d)\). In particular, \(\Pi\) has a finite index normal subgroup isomorphic to \(\mathbb{Z}^d\).

**Proof.** Since \(\rho(\Pi)_n\) is precompact in \(\text{Iso}(N)\), the homomorphism \(\Pi \rightarrow \rho(\Pi)_r\), which is the composition of \(\rho\) with the projection to \(\text{Iso}(\mathbb{R}^n)\), has finite kernel and \(\rho(\Pi)_r\) is a discrete subgroup of \(\text{Iso}(\mathbb{R}^n)\). In particular, \(\Pi\) has a normal finite index subgroup \(\Pi_1\) isomorphic to \(\mathbb{Z}^d\) for some \(d \in [1, n]\), and furthermore, we arrange the subgroup to be invariant under conjugation in \(\tilde{H}\). (The conjugation by elements of \(\tilde{H}\) defines a homomorphism \(\tilde{H} \rightarrow \text{Aut}(\Pi)\), and since \(\Pi\) is finitely generated, any finite index subgroup of \(\Pi\) contains a finite index subgroup that is invariant under all the endomorphisms of \(\Pi\) [KM79 Exercise 15.2.3]).
The group \( \Pi_1 \) acts on \( \mathbb{R}^n \) by freely, hence by the Soul Theorem, which in this case can be found in [Wolf 1984, Theorem 3.3.3], \( \Pi_1 \) acts cocompactly on an affine \( d \)-dimensional subspace \( W^d \subset \mathbb{R}^n \). Since \( \Pi_1 \trianglelefteq \tilde{H} \), for each \( \gamma \in \Pi_1 \) and \( \tilde{h} \in \tilde{H} \), the transformation \( \gamma \tilde{h} \) stabilizes \( \tilde{h}_r(W^d) \). Hence \( \tilde{h}_r(W^d) \) is an affine subspace parallel to \( W^d \). Therefore, the \( \tilde{H} \)-action on \( \mathbb{R}^n \) descends to an isometric \( \tilde{H} \)-action on \( \mathbb{R}^n/W^d \), which is isometric to \( \mathbb{R}^{n-d} \), and \( \Pi_1 \) lies in the kernel of the action. Now \( \tilde{H}/\Pi_1 \) is compact and any isometric action of a compact Lie group on a Euclidean space has a fixed point. Thus, \( \tilde{H} \) stabilizes an affine subspace \( V^d \) of \( \mathbb{R}^n \) parallel to \( W^d \). Note that \( \Pi \) acts on \( V^d \) with finite kernel and the image is a discrete cocompact subgroup of \( \text{Iso}(V^d) \cong \text{Iso}(\mathbb{R}^d) \).

The stabilizer of \( V^d \) in \( \text{Iso}(\mathbb{R}^n) \) can be identified to \( \text{O}(n-d) \times \text{Iso}(\mathbb{R}^d) \), so by Lemma 2.7, we can view \( \tilde{H}_n \) as a subgroup of \( \text{O}(n-d) \times \text{Iso}(\mathbb{R}^d) \). Since \( \tilde{H} \) is compact and \( \rho(\Pi)_n \) is precompact, \( \tilde{H}_n \) is also precompact. Then the closure of \( \tilde{H}_n \times \text{O}(n-d) \) in \( \text{Iso}(\tilde{N}) \times \text{O}(n-d) \) is a compact Lie subgroup, which we denote by \( G \). Thus, \( \tilde{H} \) can be viewed as a subgroup of \( G \times \text{Iso}(\mathbb{R}^d) \), and we denote the inclusion by \( (\psi, \epsilon): \tilde{H} \to G \times \text{Iso}(\mathbb{R}^d) \).

**Theorem 2.8.** If \((M, H)\) satisfies \((*)\), then there exists a smooth proper map \( \bar{H} \times [0, 1] \to G \times \text{Iso}(\mathbb{R}^d) \), given by \((\bar{h}, t) \to (\psi_t(\bar{h}), \epsilon(\bar{h}))\), such that

(i) \( (\psi_0, \epsilon) = (\psi, \epsilon) \),

(ii) \( \psi_t: \bar{H} \to G \) is a Lie group homomorphism, for each \( t \in [0, 1] \),

(iii) \( \psi_1(\Pi) \) is finite,

(iv) \( (\psi_t, \epsilon)(\Pi) \) is a discrete subgroup of \( G \times \text{Iso}(\mathbb{R}^d) \) that acts freely on \( \bar{M} \),

(v) \( (\psi_t, \epsilon)(\Pi) \) is injective for each \( t \in [0, 1] \).

**Proof.** We first construct \( \psi_t: \bar{H} \to G \). Since \( \Pi_1 \) is free abelian, the closure of \( \psi(\Pi_1) \) is a compact abelian group, which is normalized by \( \psi(\bar{H}) \) because \( \Pi_1 \trianglelefteq \bar{H} \). Let \( T \) be the identity component of the compact abelian group and let \( t \) be the Lie algebra of \( T \). Also \( \psi(\bar{H}) \) normalizes \( T \) so \( \Pi_2 = \Pi_1 \cap \psi^{-1}(T) \) is a normal subgroup of \( \bar{H} \).

Let \( \bar{H}_Z \) be the centralizer of \( \Pi_2 \) in \( \bar{H} \). Since \( \Pi_2 \trianglelefteq \bar{H} \), we deduce that \( \bar{H}_Z \trianglelefteq \bar{H} \). Furthermore, \( \bar{H}/\bar{H}_Z \) is finite. Indeed, \( \bar{H}_Z \) is the kernel of the \( \bar{H} \)-action on \( \Pi_2 \) be conjugation. Since \( \Pi_2 \subset \bar{H}_Z \) and \( \Pi_2 \) is abelian, the \( \bar{H} \)-action on \( \Pi_2 \) be conjugation descends to a Lie group homomorphism \( \bar{H}/\Pi_2 \to \text{Aut}(\Pi_2) \), where the domain is compact and the target is discrete, so that the homomorphism has finite image.

Let \( k = |\bar{H}/\bar{H}_Z| \). Let \( A = \{\gamma^k : \gamma \in \Pi_2\} \); this is a normal subgroup of \( \bar{H} \). Choose a homomorphism \( f: \Pi_2 \to t \) with \( \exp \circ f = \psi_{|\Pi_2} \), and define the map \( \phi: A \to t \) by

\[
\phi(\gamma^k) = \sum_{\tilde{h} \in \bar{H}/\bar{H}_Z} \text{Ad}_{\psi(\tilde{h})}(f(\tilde{h}^{-1} \gamma \tilde{h})).
\]

It is straightforward to check that \( \phi \) is a homomorphism satisfying \( \exp \circ \phi = \psi_{|\Pi_2} \) and \( \phi(\tilde{h}^a) = \text{Ad}_{\psi(\tilde{h})}(\phi(a)) \) for \( a \in A \). Fix an identification of \( A \) with the standard lattice \( \mathbb{Z}^d \subset \mathbb{R}^d \), and use the \( \bar{H} \)-action on \( A \) by conjugation to define the \( \bar{H} \)-action on \( \mathbb{R}^d \) via

\[
\alpha: \bar{H} \to \text{Aut}(\Pi) \cong \text{GL}_d(\mathbb{Z}) \subset \text{GL}_d(\mathbb{R}).
\]
Form the semidirect product $\mathbb{R}^d \rtimes_\alpha \tilde{H}$ with multiplication $(v,g) \cdot (w,h) = (v + \alpha(g)w, gh)$ where $g,h \in \tilde{H}$ and $v,w \in \mathbb{R}^d$. We identify $\tilde{H}$ with the subgroup $\{(0,h)\}$ of $\mathbb{R}^d \rtimes_\alpha \tilde{H}$. Define a map $\Psi: \mathbb{R}^d \rtimes_\alpha \tilde{H} \to G$ by

$$\Psi \left( \sum \lambda_i e_i, \tilde{h} \right) = \exp \left( \sum \lambda_i \phi(e_i) \right) \psi(\tilde{h}),$$

where $\{e_i\}$ is the standard basis in $\mathbb{Z}^d \subset \mathbb{R}^d$, thought of as sitting in $A$, $\lambda_i \in \mathbb{R}$, and $\tilde{h} \in \tilde{H}$. Using that $\phi$ is a homomorphism and $\phi(hah^{-1}) = \text{Ad}_{\psi(h)} \phi(a)$ one checks that $\Psi$ is Lie group homomorphism. Also $\Psi|_\tilde{H} = \psi$.

Note that $N = \{-a,a\} \in \mathbb{R}^d \rtimes_\alpha \tilde{H}$, and $N \subset \ker(\Psi)$. Hence $\Psi$ descends to a Lie group homomorphism $(\mathbb{R}^d \rtimes_\alpha \tilde{H})/N \to G$. Since the subgroups $\mathbb{R}^d$, $\tilde{H}$ have trivial intersections with $N$, they projects isomorphically into $(\mathbb{R}^d \rtimes_\alpha \tilde{H})/N$. Furthermore, $\mathbb{R}^d$ projects to a normal subgroup whose quotient is isomorphic to the compact Lie group $\tilde{H}/A$. Thus, we get an extension of Lie groups

$$1 \to \mathbb{R}^d \to (\mathbb{R}^d \rtimes_\alpha \tilde{H})/N \to \tilde{H}/A \to 1.$$

By [Seg70] Proposition 4.3 such extensions, with the fixed $\tilde{H}/A$-action on $\mathbb{R}^d$, are classified by the continuous cohomology group $H^2_\text{c}(\tilde{H}/A, \mathbb{R}^d)$. Since $\tilde{H}/A$ is compact, the cohomology group vanishes [Gui80 section III.2.1], hence there is exactly one such an extension for each $\tilde{H}/A$-action on $\mathbb{R}^d$. Thus the extension coincides with $(\mathbb{R}^d \rtimes_\alpha \tilde{H})/N \cong \mathbb{R}^d \rtimes_\alpha \tilde{H}/A$ is an injective Lie group homomorphism.

We write elements of $\mathbb{R}^d \rtimes_\alpha \tilde{H}/A$ as $(v(\tilde{h}), hA)$, where $v(\tilde{h}) \in \mathbb{R}^d$ and $hA \in \tilde{H}/A$. Define $I_t: \tilde{H} \to \mathbb{R}^d \rtimes_\alpha \tilde{H}/A$ by $I_t(\tilde{h}) = ((1 - t)v(\tilde{h}), hA)$. This is the deformation of $I_0$ to the homomorphism $I_1: \tilde{h} \to (0,hA)$, whose image is isomorphic to the compact group $\tilde{H}/A$. Note that $I_t(\Pi) = \Pi/A$ is a finite group.

Then $\psi_t(\tilde{h}) = \Psi \circ I_t$ satisfies (i)–(iii). To prove (iv) and (v) note that discreteness of $\Pi_t := (\psi_t, \epsilon)(\Pi)$ follows from that fact that $\Pi$ is discrete in $\tilde{H}$ and $(\psi_t, \epsilon)$ is a proper map. Also $\Pi_t$ acts isometrically, and hence properly, on $\tilde{M}$. So no infinite order elements of $\Pi_t$ can fix a point. If $(\psi_t, \epsilon)(\gamma)$ has finite order for $\gamma \in \Pi$, then $(\psi_t, \epsilon)(\gamma)$ is conjugate to $(\psi_0, \epsilon)(\gamma)$ in $G \rtimes \text{Iso}(\mathbb{R}^d)$ (see e.g. [Wil00] Lemma 6.7 or [CF64] Lemma 38.1), so since $\Pi_0$ acts freely, so does $\Pi_t$. Since $\Pi_t$ acts freely, $(\psi_t, \epsilon)$ is injective.

**Remark 2.9.** Similarly to [Wil00], one can get an explicit bound on $|\Pi : A|$. We leave it to the interested reader to work out. Note that our choice of $T$ does not allow to control $|\Pi_1 : \Pi_2|$, however this can be done by a more delicate choice of $T$ as in [Wil00].

**Theorem 2.10.** If $(M, H)$ satisfies (\*), then there exists a smooth family $q_t$ of complete Riemannian metrics on $M$ such that

(i) $g_0 = g$,
(ii) $H \leq \text{Iso}(M, g_t)$ for each $t$,
(iii) there is a smooth family of Riemannian covering maps $p_t: (\bar{M}, \bar{g}) \to (M, g_t)$,
(iv) there exists a finite index normal subgroup $A$ of the group of deck-transformations of $p_1$, such that the Riemannian covering space $(\tilde{M}/A, \tilde{g}_1)$ of $(M, g_1)$ is isometric to the Riemannian product of $N$, $\mathbb{R}^{n-d}$, and a flat $d$-torus, for some $1 \leq d \leq n$.

**Proof.** Since $\Pi_t \leq \tilde{H}_t$, every element $\tilde{h}_t := \Psi(I_t(\tilde{h}))$ of $\tilde{H}_t$ is an $\Pi_t$-equivariant self-diffeomorphism of $\tilde{M}$, hence it descends to a self-diffeomorphism $h_t$ of $M/\Pi_t$. If $\tilde{h}_t \in \Pi_t$, then $h_t = \text{id}$, so the smooth $\tilde{H}$-action on $\tilde{M}/\Pi_t$ given by $\tilde{h} \rightarrow h_t$ factors through the surjection $\tilde{H} \rightarrow \tilde{H}/\Pi = H$, and hence descends to an $H$-action on $M/\Pi_t$.

We now look at the $\tilde{H}$-action on $\tilde{M} \times \{0,1\}$ given by $\tilde{h} \cdot (\tilde{m}, t) = (\tilde{h}_t(\tilde{m}), t)$ and the corresponding covering map $\tilde{M} \times \{0,1\} \rightarrow (\tilde{M} \times \{0,1\})/\Pi$. The projection $\tilde{M} \times \{0,1\} \rightarrow [0,1]$ onto the second factor descends to the surjection $(\tilde{M} \times \{0,1\})/\Pi \rightarrow [0,1]$. By Morse theory, $(\tilde{M} \times \{0,1\})/\Pi \rightarrow [0,1]$ is a trivial $M$-bundle over $[0,1]$. By the argument in the previous paragraph, the $\tilde{H}$-action on $\tilde{M} \times \{0,1\}$ descends to an $H$-action on the bundle $(\tilde{M} \times \{0,1\})/\Pi$ by fiber preserving diffeomorphisms. Since $\tilde{H}$ is compact, the equivariant covering homotopy theorem [Bie73] implies that the bundle $(\tilde{M} \times \{0,1\})/\Pi \rightarrow [0,1]$ is $H$-equivariantly smoothly isomorphic to the trivial $M$-bundle over $[0,1]$, where $H$ acts on $\tilde{M}$ as in (*), and acts trivially on $[0,1]$. In particular, for every $t$, the $\tilde{H}$-action on $\tilde{M}/\Pi_t$ given by $\tilde{h} \rightarrow h_t$ is smoothly equivalent to the $\tilde{H}$-action on $\tilde{M}/\Pi_0$ given by the surjection $\tilde{H} \rightarrow H$ with kernel $\Pi$.

The composition $\tilde{M} \times \{0,1\} \rightarrow (\tilde{M} \times \{0,1\})/\Pi \cong M \times \{0,1\}$ is a covering map, which we write as $(\tilde{m}, t) \rightarrow (p_t(\tilde{m}), t)$. Then $p_t : \tilde{M} \rightarrow M$ is a covering map with $\Pi_t$ as the group of covering transformations. The map $p_t$ is equivariant under the surjection $\tilde{H} \rightarrow H$ given by $\tilde{h} \rightarrow h_t \rightarrow h_0$, which is simply the surjection with kernel $\Pi$.

Since $\Pi_t$ acts isometrically on $\tilde{M}$, there is a unique Riemannian metric $g_t$ on $M$ that makes $p_t$ a Riemannian covering. Thus $p_t$ is a smooth family of Riemannian coverings and $g_t$ is a smooth family of Riemannian metrics. Also $H \leq \text{Iso}(M, g_t)$, and since the group $\psi_1(A)$ is trivial, $(M, g_t)$ has a finite Riemannian cover isometric to $\tilde{M}/A$, which is the Riemannian product of $N$, $\mathbb{R}^{n-d}$, and a flat $d$-torus.

**Remark 2.11.** Let $M$ be a compact manifold with infinite fundamental group that admits a metric of $\text{Ric} \geq 0$, so that $\pi_1(M)$ is virtually-$\mathbb{Z}^n$ and the universal cover of $M$ is isometric to $C \times \mathbb{R}^n$ with $\text{Ric}(C) \geq 0$. The choice of a metric of $\text{Ric} \geq 0$ on $M$ uniquely specifies the isometry type of $C$, and hence the diffeomorphism type of $C$. It is natural to ask whether $M$ could admit a different metric, with the universal cover of $M$ isometric to $C' \times \mathbb{R}^n$, for which $C'$ and $C$ are non-diffeomorphic.

It turns out that the $s$-cobordism theorem rules out this possibility if $\dim(C) \geq 5$. Namely, a finite cover of $M$ has to be diffeomorphic to $C \times T^n$ as well as to $C' \times T^n$. The diffeomorphism $\phi : C \times T^n \rightarrow C' \times T^n$, defines an automorphism of $\pi_1(T^n)$, and hence a self-homotopy equivalence of $T^n$. Since any self-homotopy equivalence of $T^n$ is homotopic to a diffeomorphism, we can precompose $\phi$ by this diffeomorphism, so that we can assume that $\phi$ induces the trivial homomorphism of the fundamental groups. If $n = 1$, then the lift of $\phi$ to the universal covers defines an $h$-cobordism between $C$ and $C'$, so they are diffeomorphic. If $n > 1$, we can proceed by induction splitting off one circle at a time, and getting an $h$-cobordism between $C \times T^k$ and $C' \times T^k$, which is trivial because $\mathbb{Z}^k$ has
trivial Whitehead torsion. After splitting all the circle factors, we again conclude that $C$ and $C'$ are diffeomorphic.

Note that there do exist simply-connected 4-manifolds $C$, $C'$ that become diffeomorphic after taking products with $S^1$. For example, by Seiberg-Witten theory, the $K3$-surface has infinitely many smooth structures, but each compact manifold of dimension $\geq 5$ has only finitely many smooth structures. It it unclear however if such examples can admit $\text{Ric} \geq 0$.

3 Equivariant connected sum and positive scalar curvature

Let $M$, $N$ be manifolds of the same dimension each equipped with a smooth action of a compact Lie group $G$ that has a nonempty set of the fixed points. Assume that at some fixed points $m \in M$, $n \in N$ the isotropy representations of $G$ are equivalent. If the $G$-actions are isometric in some Riemannian metrics on $M$, $N$, then there are small balls around $m$, $n$ which are $G$-equivariantly diffeomorphic. The equivariant connected sum of $M$ and $N$ at the points $m$, $n$ is a $G$-manifold obtained by removing the above balls and gluing their complements by a $G$-equivariant diffeomorphism of the boundary spheres. For the purposes of this paper we ignore the issue of orientability for $M \# N$, so we need not put any restrictions on the diffeomorphism of the boundary spheres.

It follows easily from [GL80] that positivity of scalar curvature is preserved under equivariant connected sums. Namely, let $(M, g)$ be a complete manifold of $\text{scal} > 0$ of dimension $m \geq 3$, and let $G$ be a compact subgroup of $\text{Iso}(M)$ that fixes a point $x \in M$. It was shown in [GL80] that $M \setminus \{x\}$ has a metric $g'$ of positive scalar curvature for which there exist small $R > r > 0$ such that outside the ball $B_g(x, R)$ we have $g' = g$, and on $B_g(x, r) \setminus \{x\}$ the metric $g'$ is the product $\mathbb{R} \times S^{m-1}$, where $S^{m-1}$ is round sphere of a small fixed radius. Furthermore, the metric $g'$ on $B_g(x, R) \setminus \{x\}$ is rotationally symmetric, in particular, the $G$-action on $M \setminus \{x\}$ is isometric with respect to $g'$. Thus, one can keep the scalar curvature positive while taking equivariant connected sums.

In our applications we take equivariant connected sums of a fixed $G$-action on $M$ and a certain $G$-action on the standard sphere $S^m$, whose set of the fixed points is a homology sphere $S$. Thus, the equivariant connected sum $M \# S^m$ is diffeomorphic to $M$, while the $G$-action changes considerably. Namely, if $F$ is the component of the fixed point set of the original $G$-action on $M$ that contains the point at which we take the equivariant connected sum, then the corresponding component of the fixed point set of the $G$-action on $M \# S^m$ is diffeomorphic to $F \# S$.

First, we need background on homology spheres that bound (smooth) contractible manifolds. In each dimension $n \geq 3$ there are non-simply-connected homology $n$-spheres that bound contractible manifolds. In fact if $n \geq 5$, then after changing a smooth structure any homology sphere bounds a contractible manifold [Ker69]. Also any homology 4-sphere bounds a contractible manifold [Ker69]. Some homology 3-spheres bound contractible manifolds [Gla67, Gor75] while some do not [Ker69]. (Note that any homology 3-sphere bounds a topological contractible manifold [Pre82], but we need to work in smooth category). In our application, $\partial(D^k \times C)$ must be diffeomorphic to $S^{k+n}$, where $D^k$ denotes the closed $k$-disk with $k > 0$. It is easy to see that $\partial(D^k \times C)$ is simply-connected, so if $\dim(D^k \times C) \geq 6$
then by the $s$-cobordism theorem applied to $D^k \times C$ with a small open ball in $\text{Int}(D^k \times C)$ removed, $\partial(D^k \times C)$ is diffeomorphic to $S^{k+n}$. If $D^k \times C$ is a 5-manifold, so that $k = 1$, $\dim(C) = 4$, the above argument does not work, because the smooth Poincaré conjecture is still open. So we have to restrict ourselves only to those $C$'s for which $\partial(D^k \times C)$ is diffeomorphic to $S^4$. This is true in examples of \cite{Gla67} (note that $\partial(D^k \times C)$ is the double of $C$).

Now fix a homology $n$-sphere $S$ that bounds a (smooth) contractible manifold $C$, and if $n = 3$ we assume that the double of $C$ is diffeomorphic to $S^4$. Let $G$ be a compact Lie group acting on $\mathbb{R}^k$ via an embedding $\rho: G \hookrightarrow O(k)$, and suppose that the only fixed point of $\rho(G)$ is the origin. Consider the closed unit ball $D^k$ in $\mathbb{R}^k$ centered at the origin with the induced $G$-action. Also assume that $G$ acts trivially on $C$. Given the data, we next construct a $G$-action on $S^m$, with $m = n + k$, such that the fixed point set is $S$ and the isotropy representation at any fixed point is equivalent to the direct sum of $\rho$ and the trivial $n$-dimensional representation.

Look at the product of the $G$-actions on on $D^k \times C$. By equivariantly smoothing the corners we get a smooth $G$-action on the manifold $\partial(D^k \times C) = (D^k \times S) \cup (\partial D^k \times C)$ which is diffeomorphic to $S^m$. Since the origin is the only fixed point of $\rho(G)$, the set of the fixed points for the $G$-action on $\partial(D^k \times C)$ is the homology sphere $\{0\} \times S$, and the isotropy representation at any fixed point is equivalent to the direct sum of $\rho$ and the trivial $n$-dimensional representation.

By taking equivariant connected sums of $q$ copies of the $G$-action on $S^m$, we get the $G$-action on $S^m$ with the same isotropy representation at a fixed point and such that the set of the fixed points is the connected sum $\#_q S$ of $q$ copies of $S$. These actions are topologically inequivalent for $\#_q S$ and $\#_p S$ are not homeomorphic if $q \neq p$, because by Grushko’s theorem \cite{Sta71} the rank (i.e. the minimum number of generators) of $\pi_1(\#_q S) \cong *_q \pi_1(S)$ is $q \cdot \text{rk}(\pi_1(S))$.

Finally, we show that if $n \geq 5$, then these actions on $S^m$ can be arranged to preserve some metrics of scal > 0. By Lemmas below we can assume that $C$ has a metric of scal > 0 that is the product metric near the boundary. Equip $D^k$ with an $O(k)$-invariant metric of scal ≥ 0 which is the product metric near the boundary. Then the product metric on $D^k \times C$ has scal > 0 away from the boundary. By smoothing the metric at the corners as in \cite{Gaj87}, we can assume that $\partial(D^k \times C)$ has a $G$-invariant metric of scal > 0.

\textbf{Lemma 3.1.} Let $\pi$ be a finitely presented group with $H_1(\pi) = 0 = H_2(\pi)$ and let $n \geq 5$ be an integer. Then there exists a homology $n$-sphere with fundamental group $\pi$ and of scal > 0.

\textit{Proof.} By \cite[p.68]{Ker69} the desired homology $n$-sphere can be constructed from a connected sum of several copies of $S^1 \times S^{n-1}$ by surgeries on circles and 2-spheres. Thus the surgeries have codimension ≥ 3 and the homology sphere have scal > 0 by \cite{SY79, CL80}.

\textbf{Lemma 3.2.} If $n \geq 5$, then after possibly changing a smooth structure, each homology $n$-sphere $M$ of scal$(M) > 0$ bounds a contractible manifold with scal > 0 so that the metric is a product near the boundary.
Proof. We argue as in [Ker69, pp.70–71]. First by doing surgery on circles and 2-spheres we turn $M$ into a homotopy sphere $\Sigma$ of scalar $>0$. By [Gaj87] the trace of this surgery (a cobordism $W$ between $M$ and $\Sigma$) also has scalar $>0$ so that the metric is the product near the boundary. We think of $W$ as $M \times I$ with some handles attached to $M \times \{1\}$, where $I = [0,1]$. For some $m \in M$, the interval $m \times I$ touches none of the previous surgeries and meets orthogonally each copy of $\partial W \times \{t\}$ near the boundary, where the product in $\partial W \times \{t\}$ refers to the product metric. Let $N(m \times I)$ be a small open tubular neighborhood of $m \times I$. Then following [Gaj87], one can smooth the metric at the corners of $W' = W \setminus N(m \times I)$ to a metric that has scalar $>0$ and is a product near the boundary. As in [Ker69, p.71], $W'$ is a contractible manifold with boundary $M \# \Sigma$.

4 Constrains on isometric actions

Part (iv) of Theorem 2.10 yields a nontrivial restriction on the topology $M$, namely, a finite cover of $M$ splits off a torus factor. This idea was explored in [BK01, BK03], to show, for example, that many vector bundles admit no complete metric of sec $\geq 0$.

The purpose of this section is to show that Theorem 2.10 also gives a lot of information information on compact subgroups of $\text{Iso}(M)$. This leads, in particular, to numerous examples of group actions on compact manifolds that are non-isometric in any metric of Ric $\geq 0$, as well as examples of group actions on manifolds that are non-isometric in any complete metric of sec $\geq 0$.

Our first goal is to lift the isometric $H$-action on $(M, g_1)$, constructed in Theorem 2.10, to the covering $\tilde{M}/A \to M$. This is generally impossible, e.g. the circle action on itself by left translations does not lift to any cover. However, by the standard results on lifting groups actions to covering spaces, which are reviewed in Appendix A, the $H$-action on $M$ always lifts to an $\tilde{H}/A$-action on $\tilde{M}/A$, and furthermore, the following is true.

(1) If $H$ fixes a point $x \in M$, then the $H$-action on $M$ lifts to an $\tilde{H}$-action on $\tilde{M}/A$.

(2) If $H$ is connected, the $H$-action on $M$ lifts to an $\tilde{H}$-action on $\tilde{M}/A$, where $\tilde{H}$ is a connected Lie group that surjects onto $H$ and the kernel of the surjection is a subgroup of $\Pi/A$, which is a finite group.

(3) If $H$ is connected and simply-connected, then any $H$-action on $M$ lifts to an $\tilde{H}$-action on $\tilde{M}/A$.

Since $(\tilde{M}/A, \tilde{g}_1)$ is isometric to $N \times \mathbb{R}^{n-d} \times T^d$, where $T^d$ is a flat torus, in the cases (1), (3), $H$ acts on $\tilde{M}/A$ via an embedding into $\text{Iso}(N) \times \text{Iso}(\mathbb{R}^{n-d}) \times \text{Iso}(T^d)$. In particular, the $H$-action preserves the decomposition of $\tilde{M}/A$ into the product of $N$, $\mathbb{R}^{n-d}$, and $T^d$. Of course, isometric actions of compact Lie groups on $\mathbb{R}^{n-d}$ and $T^d$ are well-understood.

Using the results on actions on tori, stated in Section B, we see that in the cases (1), (3), the $H$-action on the $T^d$-factor is as follows.

(1)' Assume $H$ fixes a point $x \in M$, so the lifted $H$-action on $\tilde{M}/A$ also fixes a point, and let $t$ be the fixed point of the $H$-action on the $T^d$-factor. Then if $h \in H$ acts
nontrivially on the $T^d$-factor, then the induced action of $h$ on $\pi_1(T^d, t)$ is nontrivial. In particular, if $H$ acts trivially on $\pi_1(M, x)$, then the $H$-action on the $T^d$-factor is trivial.

(3)’ If $H$ is connected and simply-connected, then the $H$-action on the $T^d$-factor is trivial.

**Remark 4.1.** The fact that $H$ acts trivially on the $T^d$-factor implies in particular that each component of the fixed points set has a direct $T^d$-factor. This is the key point in Theorem 4.2 below.

**Theorem 4.2.** Let $H$ be a compact Lie group acting smoothly and effectively on a manifold $M$. Suppose the set of the fixed points of $H$ has a component $F$ of dimension $s \geq 3$ such that for some $x \in F$, the $H$-action on $\pi_1(M, x)$ is trivial. Then there are infinitely many smooth $H$-actions on $M$ for which $(M, H)$ does not satisfy (**).

**Proof.** Following Section 3 we construct a smooth $H$-action on the standard sphere $S^m$, where $m = \dim(M)$, such that the fixed point set is a non-simply-connected integral homology $s$-dimensional sphere $S$, and the isotropy actions at $x \in M$ and at some fixed point of $S^m$ are equivalent. By replacing $(S^m, H)$ with the equivariant connected sum at $x$ of the two copies of $(S^m, H)$, we can assume that $S$ is a connected sum of two non-simply-connected homology spheres. In particular, $\pi_1(S)$ has infinitely many ends [Sta71, 4.A.6.6]. Then the equivariant connected sum of $M$ and $S^m$ at the fixed points is a smooth $H$-action on $M$ such that one of the components $F'$ of the fixed point set is diffeomorphic $F \# S$. Also the induced $H$-action on $\pi_1(M, y)$ is trivial for $y \in F'$.

Arguing by contradiction, assume the new $H$-action on $M$ satisfies (**), for some $\tilde{M}$, II, etc. Now apply Theorem 2.8 to find $A$ and $g_t$, and let $q: \tilde{M}/A \to M$ be the corresponding covering. Pick $\tilde{x} \in \tilde{M}/A$ with $q(\tilde{x}) = x$, and let $\tilde{F}$ be the component of $q^{-1}(F')$ that contains $\tilde{x}$. The $H$-action on $M$ lifts uniquely to an $H$-action on $\tilde{M}/A$ that fixes $\tilde{x}$. Then the set of the fixed points of the $H$-action on $\tilde{M}/A$ has $\tilde{F}$ as a component. Note that $q|_{\tilde{F}}: \tilde{F} \to F'$ is a finitely-sheeted covering. By (3)’ above, $\tilde{F}$ is the product of $T^d$ and a submanifold of $N$, and therefore, $\pi_1(\tilde{F})$ has a direct $\mathbb{Z}^d$-factor.

Hence the group $\pi_1(F')$ has 1 or 2 ends [Sta71, 4.4.6.1, 4.4.6.3], where $\pi_1(F')$ has 2 ends precisely when $d = 1$ and $\pi_1(\tilde{F})/\mathbb{Z}^d$ is finite. On the other hand, $\pi_1(F') = \pi_1(F) \ast \pi_1(S)$ so $\pi_1(F')$ has infinitely many ends [Sta71, 4.4.6.6].

To get infinitely many examples of such $H$-actions we proceed as follows. Fix a homology $s$-sphere $S$ that bounds a contractible manifold, whose double is $S^4$ if $s = 3$. For each $s$-dimensional connected component $F$ of the fixed point set of $(M, H)$, we build an $H$-action on $S^m$ with fixed point set $S$ and the isotropy representation at a fixed point equivalent to the isotropy representation at a point $f$ of $F$. For every positive integer $q$ and every $F$, we take the equivariant connected sum of $M$ at $f$ with $q$ copies of $(S^m, H)$, so that for this new $H$-action $p_q$ on $M$, each $F$ turns into $F_q := F\# qS$. By Grushko’s theorem [Sta71], the rank (i.e. the minimum number of generators) of $\pi_1(F_q)$ is $\text{rk}(\pi_1(F)) + q \cdot \text{rk}(\pi_1(S))$.

Let $r_q$ be the minimum of the numbers $\text{rk}(\pi_1(F_q))$ over all $s$-dimensional components of the fixed point set of $\rho_q$. The minimum is of course realized on some component, so if $q \neq p$, then $r_q \neq r_p$, therefore, $\rho_q$ and $\rho_p$ are topologically inequivalent.
Proof of Theorem 1.2. We construct the $H$-actions on $M$ as in the proof of Theorem 4.2 in particular, the actions preserve no metrics of $Ric \geq 0$. If $\text{scal}(M) > 0$ and $\dim(F) \geq 5$, then by Section 3 the $H$-action obtained by the equivariant connected sum as above preserves a metric of $\text{scal} > 0$. □

Proof of Theorem 4.3. The argument here is very similar to the proof of Theorem 4.2 in fact we only need to replace the second paragraph in that proof. Namely, we need another argument explaining that some finite index subgroup of $\pi_1(F')$ has a $\mathbb{Z}^d$-factor for $d \geq 1$. Since $S$ is a nontrivial connected sum, $\pi_1(S)$ is infinite which ensures that $\pi_1(F') \cong \pi_1(F) * \pi_1(S)$ is infinite. Arguing by contradiction assume that the new $H$-action preserves a complete metric of $\text{sec} \geq 0$. Then $F'$ is a totally geodesic submanifold, and hence $\text{sec}(F') \geq 0$. Therefore, a finite index subgroup of $\pi_1(F')$ has a $\mathbb{Z}^d$-factor for $d \geq 1$.

If $\text{scal}(M) > 0$ and $\dim(F) \geq 5$, then by Section 3 the $H$-action obtained by the equivariant connected sum as above preserves a metric of $\text{scal} > 0$. □

Remark 4.3. Theorem 4.3 can be extended to complete manifolds of nonnegative $k$-Ricci curvature (see [Wu87] for a definition), provided $\dim(F) \geq k+1$, because these assumptions force $F$ to have nonnegative Ricci curvature, so that $\pi_1(F)$ is virtually free abelian. The only $k$-Ricci curvature for which the extension is void is the Ricci curvature itself, i.e. the $k$-Ricci curvature with $k = \dim(M) - 1$; indeed, $k = \dim(M) - 1$ implies $M = F$. In general, little is known about the totally geodesic submanifolds in manifolds of $Ric \geq 0$, which is one of the obstacles to understanding simply-connected manifolds with $Ric \geq 0$.

A Lifting group actions to covering spaces.

If $p: \tilde{X} \to X$ is a covering and $\pi: \tilde{G} \to G$ is a surjection of Lie groups, then the $\tilde{G}$-action $\tilde{\theta}: \tilde{G} \times \tilde{X} \to \tilde{X}$ on $\tilde{X}$ is said to cover the $G$-action $\theta: G \times X \to X$ on $X$ if $p \circ \tilde{\theta} = \theta \circ (\pi \times p)$. We also say that $\theta$ lifts to $\tilde{\theta}$.

It is shown in [Bre72] section I.9 that if $p: \tilde{X} \to X$ is a regular covering with deck transformation group $\Delta$, and if the Lie group $G$ acts effectively on $X$, then the set of all lifts of elements of $G$ to self-homeomorphisms of $\tilde{X}$ is a Lie group $\tilde{G}$ that surjects onto $G$ with kernel $\Delta$, so that the $\tilde{G}$-action on $\tilde{X}$ covers the $G$-action on $X$. In fact, in many cases one can find a much smaller Lie subgroup $\tilde{G}$ of $\tilde{G}$ such that the $\tilde{G}$-action on $\tilde{X}$ still covers the $G$-action on $X$. The kernel of the surjection $\tilde{G} \to G$ is a subgroup of $\Delta$.

For example, if $G$ fixes a point $x \in X$ and the induced $G$-action on $\pi_1(X,x)$ preserves the subgroup $p_*(\pi_1(X,x))$ for $\tilde{x} \in p^{-1}(x)$, then we can take $\tilde{G} = G$, because then, any $g \in G$ lifts uniquely to $\tilde{g} \in \tilde{G}$ with $\tilde{g}(\tilde{x}) = \tilde{x}$ [Bre72] section I.9].

If $G$ is connected, then one can choose $\tilde{G}$ to be connected [Bre72] section I.9], so that $\tilde{G}$ is a covering space of $G$. In particular, if $G$ is also simply-connected, then we can take $\tilde{G} = G$, in other words, any action of a connected simply-connected Lie group $G$ lifts to a $G$ action on $\tilde{X}$. Also if $G$ is torus and $p: \tilde{X} \to X$ is a finite cover, then $\tilde{G}$ is also a torus and the surjection $\tilde{G} \to G$ is a finite cover.
B  Actions on tori.

If a connected compact Lie group $K$ acts effectively on a torus $T^m$, then $K$ is a torus, the action is free, and in fact $T^m$ is a trivial principal $K$-bundle [Bre72, section IV.9]. It seems the base $T^m/K$ of the bundle must also be diffeomorphic to a torus, but I do know how to prove it. The result is clear if the $K$-action preserves a flat metric, because in this case, $T^m/K$ carries a Riemannian submersion metric of sec $\geq 0$, and by the homotopy sequence of the fibration $T^m/K$ has contractible universal cover and by above $\pi_1(T^m/K)$ is free abelian, so the metric on $T^m/K$ is flat [CG72], and hence $T^m/K$ is a torus [Wol84].

If a compact Lie group $K$ acting effectively on a torus $T^m$ fixes a point $t$, then the induced $K$-action on $\pi_1(T^m, t)$ is effective [Bre72, section IV.9]. Here is an example of an involution on $T^m$ that is not isometric in any metric of scal $\geq 0$. Consider an isometric involution of $T^m$ whose fixed-point-set has a component $F$ of dimension $\geq 3$. Then by taking an equivariant connected sum at a point of $F$ as in Section 3 we get an involution on $T^m$ such that a component of the fixed point set is diffeomorphic to $F\# S$ for a homology sphere $S$. Now if the action is isometric in a flat metric, then $F\# S$ is totally geodesic, hence flat, so $\pi_1(F\# S)$ is virtually abelian, but $\pi_1(F\# S) \cong \pi_1(F) \ast \pi_1(S)$ is not virtually abelian unless $S$ is simply-connected. Since any metric of scal $\geq 0$ on a torus is flat [LM89], we have constructed an involution on a torus that is not isometric in any metric of scal $\geq 0$.

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