On the strong Markov property for stochastic differential equations driven by $G$-Brownian motion

Mingshang Hu *    Xiaojun Ji †    Guomin Liu ‡

Abstract. The objective of this paper is to study the strong Markov property for the stochastic differential equations driven by $G$-Brownian motion ($G$-SDEs for short). We first extend the deterministic-time conditional $G$-expectation to optional times. The strong Markov property for $G$-SDEs is then obtained by Kolmogorov’s criterion for tightness. In particular, for any given optional time $\tau$ and $G$-Brownian motion $B$, the reflection principle for $B$ holds and $(B_{\tau+t} - B_{\tau})_{t \geq 0}$ is still a $G$-Brownian motion.

Key words: $G$-expectation, Strong Markov property, Stochastic differential equations, $G$-Brownian motion, Reflection principle.

AMS 2010 subject classifications: 60H10, 60H30

1 Introduction

The strong Markov property for stochastic differential equations (SDEs) is one of the most fundamental results in the theory of classical stochastic processes. It claims that for any given optional time $\tau$ we have

$$E_P[\varphi(X_{\tau+t_1}^x, \cdots, X_{\tau+t_m}^x)|\mathcal{F}_\tau] = E_P[\varphi(X_{t_1}^y, \cdots, X_{t_m}^y)|y = X_\tau^x]$$  \quad (1.1)

for SDEs $(X_t^x)_{t \geq 0}$ with initial value $x$. Here $E_P$ and $E_P[\cdot|\mathcal{F}_\tau]$ stands for the expectation and conditional expectation, respectively, related to a probability measure $P$. It was obtained by K. Itô in his pioneering work [11], and since then, it has been widely applied to stochastic control, mathematical finance and probabilistic method for partial differential equations (PDEs); see, e.g., [1, 4, 19].

Recently, motivated by probabilistic interpretations for fully nonlinear PDEs and financial problems with model uncertainty, Peng [20–22] systematically introduced the notion of nonlinear $G$-expectation $\hat{E}[\cdot]$ by stochastic control and PDE methods. Under the $G$-expectation framework, a new kind of Brownian motion, called $G$-Brownian motion, was constructed. The corresponding stochastic calculus of Itô’s type was also established. Furthermore, by the contracting mapping theorem, Peng obtained the existence and

*Zhongtai Securities Institute for Financial Studies, Shandong University, humingshang@sdu.edu.cn. Research supported by NSF (No. 11671231) and Young Scholars Program of Shandong University (No. 2016WLJH10)
†School of Mathematics, Shandong University, xiaojunji@163.com
‡Zhongtai Securities Institute for Financial Studies, Shandong University, gmliusdu@163.com (Corresponding author). Research supported by NSF (11601282) and Shandong Province NSF (No. ZR2016AQ10). Hu, Ji and Liu’s research was partially supported by NSF (No. 11526205 and No. 11626247) and the 111 Project (No. B12023).
uniqueness of the solution of G-SDEs:

\[
\begin{aligned}
&dx_t = b(x_t)dt + \sum_{i,j=1}^d h_{ij}(x_t)d\langle B^i, B^j \rangle_t + \sum_{j=1}^d \sigma_j(x_t)dB^j_t, \quad t \in [0,T], \\
&x_0 = x,
\end{aligned}
\]

where \( B = (B^1, \ldots, B^d) \) is G-Brownian motion and \( \langle B^i, B^j \rangle \) is its cross-variation process, which is not deterministic unlike the classical case.

A very interesting problem is whether, for G-SDEs, the following generalized strong Markov property is true:

\[
\hat{E}_{\tau} [\phi(x_{\tau+t_1}, \ldots, x_{\tau+t_m})] = \hat{E} [\phi(x_{\tau}, \ldots, x_{\tau}) | y = x_{\tau}] 
\]

In this paper, we first construct the conditional G-expectation \( \hat{E}_{\tau} [\cdot] \) for any given optional time \( \tau \) by extending the definition of conditional G-expectation \( \hat{E}_t [\cdot] \) to optional times. The main tools in this construction are a universal continuity estimate for \( \hat{E}_t [\cdot] \) (see Lemma 3.3) and a new kind of consistency property (see Proposition 3.9). We also show that \( \hat{E}_{\tau} [\cdot] \) can preserve most useful properties of classical conditional expectations except the linearity. Based on the conditional expectation \( \hat{E}_{\tau} [\cdot] \), we then further obtain the strong Markov property (1.3) for G-SDEs by adapting the standard discretization method. In contrast to the linear case, the main difficulty is that in the nonlinear expectation context the dominated convergence theorem does not hold in general. We tackle this problem by using Kolmogorov’s criterion for tightness and the properties of \( \hat{E}_{\tau} [\cdot] \). In particular, for G-Brownian motion \( B \), we obtain that the reflection principle for \( B_{\tau+t} - B_\tau \geq 0 \) is still a G-Brownian motion. Finally, with the help of the strong Markov property, the level set of G-Brownian motion is also investigated.

We note that problem of constructing \( \hat{E}_{\tau} [\cdot] \) was first considered in [18], where \( \hat{E}_{\tau} [\cdot] \) is defined for all upper semianalytic (more general than Borel-measurable) functions by the analytic sets theory. But the corresponding conditional expectation is also upper semianalytic and when the usual Borel-measurability can be attained remains unknown. In our paper, by a completely different approach, our construction focuses on a large class of Borel functions to obtain more regularity properties for \( \hat{E}_{\tau} [\cdot] \), among which is its measurability with respect to \( F_{\tau+} \). Moreover, some of these properties are important for the derivation of strong Markov property for G-SDEs.

This paper is organized as follows. In Section 2, we recall some basic notions of G-expectation, G-Brownian motion and G-SDEs. Section 3 is devoted to the construction of the conditional G-expectation \( \hat{E}_{\tau} [\cdot] \) and the investigation of its properties. Then, in Section 4, we study the strong Markov property for G-SDEs. Finally, in Section 5, we use the strong Markov property to prove that the level set of G-Brownian motion has no isolated point.

2 Preliminaries

In this section, we review some basic notions and results of G-expectation. More relevant details can be found in [6, 15–17, 20–23]
2.1 G-expectation space

Let \( \Omega \) be a given nonempty set and \( \mathcal{H} \) be a linear space of real-valued functions on \( \Omega \) such that if \( X_1, \ldots, X_d \in \mathcal{H} \), then \( \varphi(X_1, X_2, \ldots, X_d) \in \mathcal{H} \) for each \( \varphi \in C_{b,Lip}(\mathbb{R}^d) \), where \( C_{b,Lip}(\mathbb{R}^d) \) is the space of bounded, Lipschitz functions on \( \mathbb{R}^d \). \( \mathcal{H} \) is considered as the space of random variables.

**Definition 2.1** A sublinear expectation \( \hat{\mathbb{E}} \) on \( \mathcal{H} \) is a functional \( \hat{\mathbb{E}} : \mathcal{H} \to \mathbb{R} \) satisfying the following properties:

(i) Monotonicity: \( \hat{\mathbb{E}}[X] \geq \hat{\mathbb{E}}[Y] \) if \( X \geq Y \);

(ii) Constant preserving: \( \hat{\mathbb{E}}[c] = c \) for \( c \in \mathbb{R} \);

(iii) Sub-additivity: \( \hat{\mathbb{E}}[X + Y] \leq \hat{\mathbb{E}}[X] + \hat{\mathbb{E}}[Y] \);

(iv) Positive homogeneity: \( \hat{\mathbb{E}}[\lambda X] = \lambda \hat{\mathbb{E}}[X] \) for \( \lambda \geq 0 \).

The triple \( (\Omega, \mathcal{H}, \hat{\mathbb{E}}) \) is called a sublinear expectation space.

**Definition 2.2** Two \( d \)-dimensional random vectors \( X_1 \) and \( X_2 \) defined respectively on sublinear expectation spaces \( (\Omega_1, \mathcal{H}_1, \hat{\mathbb{E}}_1) \) and \( (\Omega_2, \mathcal{H}_2, \hat{\mathbb{E}}_2) \) are called identically distributed, denoted by \( X_1 \overset{d}{=} X_2 \), if

\[
\hat{\mathbb{E}}_1[\varphi(X_1)] = \hat{\mathbb{E}}_2[\varphi(X_2)], \quad \text{for each } \varphi \in C_{b,Lip}(\mathbb{R}^d).
\]

**Definition 2.3** On the sublinear expectation space \( (\Omega, \mathcal{H}, \hat{\mathbb{E}}) \), an \( n \)-dimensional random vector \( Y \) is said to be independent from a \( d \)-dimensional random vector \( X \), denoted by \( Y \perp X \), if

\[
\hat{\mathbb{E}}[\varphi(X,Y)] = \hat{\mathbb{E}}[\hat{\mathbb{E}}[\varphi(x,Y)]_{x=X}], \quad \text{for each } \varphi \in C_{b,Lip}(\mathbb{R}^{d+n}).
\]

A \( d \)-dimensional random vector \( \bar{X} \) is said to be an independent copy of \( X \) if \( \bar{X} \overset{d}{=} X \) and \( \bar{X} \perp X \).

**Definition 2.4** (G-normal distribution) A \( d \)-dimensional random vector \( X \) defined on \( (\Omega, \mathcal{H}, \hat{\mathbb{E}}) \) is called \( G \)-normally distributed if for any \( a, b \geq 0 \),

\[
aX + b\bar{X} \overset{d}{=} \sqrt{a^2 + b^2}X,
\]

where \( \bar{X} \) is an independent copy of \( X \). Here the letter \( G \) denotes the function \( G(A) := \frac{1}{\sqrt{2}}\hat{\mathbb{E}}[\langle AX, X \rangle] \) for \( A \in \mathbb{S}(d) \), where \( \mathbb{S}(d) \) denotes the space of all \( d \times d \) symmetric matrices.

In the rest of this paper, we denote by \( \Omega := C([0, \infty); \mathbb{R}^d) \) the space of all \( \mathbb{R}^d \)-valued continuous paths \( (\omega_t)_{t \geq 0} \), equipped with the distance

\[
\rho_d(\omega^1, \omega^2) := \sum_{i=1}^{\infty} \frac{1}{2^i} \langle \|\omega^1 - \omega^2\|_{C^i[0,1]} \wedge 1 \rangle,
\]

where \( \|\omega^1 - \omega^2\|_{C^i[0,T]} := \max_{t \in [0,T]} |\omega^1_t - \omega^2_t| \) for \( T > 0 \). Given any \( T > 0 \), we also define \( \Omega_T := \{ (\omega_{t \wedge T})_{t \geq 0} : \omega \in \Omega \} \).

Let \( B_t(\omega) := \omega_t \) for \( \omega \in \Omega \), \( t \geq 0 \) be the canonical process. We set

\[
L_{ip}(\Omega_T) := \{ \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}}) : n \in \mathbb{N}, 0 \leq t_1 < t_2 \cdots < t_n \leq T, \varphi \in C_{b,Lip}(\mathbb{R}^{d \times n}) \}.
\]
as well as
\[ L_{tp}(\Omega) := \bigcup_{m=1}^{\infty} L_{tp}(\Omega_m). \] (2.1)

Let \( G : \mathbb{S}(d) \to \mathbb{R} \) be a given monotonic and sublinear function. The \( G \)-expectation on \( L_{tp}(\Omega) \) is defined by
\[ \hat{E}[X] := \hat{E}[\varphi(\sqrt{t_1}\xi_1, \sqrt{t_2-t_1}\xi_2, \cdots, \sqrt{t_n-t_{n-1}}\xi_n)], \]
for all \( X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}), 0 \leq t_1 < \cdots < t_n < \infty, \) where \( \{\xi_i\}_{i=1}^n \) are \( d \)-dimensional identically distributed random vectors on a sublinear expectation space \( (\hat{\Omega}, \hat{\mathcal{F}}, \hat{\mathbb{E}}) \) such that \( \xi_i \) is \( G \)-normal distributed and \( \xi_{i+1} \) is independent from \( (\xi_1, \cdots, \xi_i) \) for \( i = 1, \cdots, n-1 \). Then under \( \hat{\mathbb{E}} \), the canonical process \( B_t = (B^1_t, \cdots, B^n_t) \) is a \( d \)-dimensional \( G \)-Brownian motion in the sense that:

(i) \( B_0 = 0; \)

(ii) For each \( t, s \geq 0 \), the increments \( B_{t+s} - B_t \) is independent from \( (B_{t_1}, \cdots, B_{t_n}) \) for each \( n \in \mathbb{N} \) and \( 0 \leq t_1 \leq \cdots \leq t_n \leq t; \)

(iii) \( B_{t+s} - B_t \overset{d}{=} \sqrt{s}\xi \) for \( t, s \geq 0 \), where \( \xi \) is \( G \)-normal distributed.

**Remark 2.5** (i) It is easy to check that \( G \)-Brownian motion is symmetric, i.e., \( (-B_t)_{t \geq 0} \) is also a \( G \)-Brownian motion.

(ii) If specially \( G(A) = \frac{1}{2} \text{tr}(A) \), then the \( G \)-expectation is a linear expectation which corresponds to the Wiener measure \( P \), i.e., \( \hat{\mathbb{E}} = E_P \).

The conditional \( G \)-expectation for \( X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}) \) at \( t = t_j, 1 \leq j \leq n \) is defined by
\[ \hat{E}_{t_j}[X] := \phi(B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_j} - B_{t_{j-1}}), \]
where \( \phi(x_1, \cdots, x_j) = \hat{E}[\varphi(x_1, \cdots, x_j, B_{t_{j+1}} - B_{t_j}, \cdots, B_{t_n} - B_{t_{n-1}})]. \)

For each \( p \geq 1 \), we denote by \( L^p_G(\Omega_t) \) (\( L^p_G(\Omega) \) resp.) the completion of \( L_{tp}(\Omega_t) \) (\( L_{tp}(\Omega) \) resp.) under the norm \( ||X||_p := (\hat{E}[|X|^p])^{1/p} \). The conditional \( G \)-expectation \( \hat{E}_{t_j}[\cdot] \) can be extended continuously to \( L^1_G(\Omega) \) and satisfies the following proposition.

**Proposition 2.6** For \( X, Y \in L^1_G(\Omega), t, s \geq 0, \)

(i) \( \hat{E}_t[X] \leq \hat{E}_t[Y] \) for \( X \leq Y; \)

(ii) \( \hat{E}_t[\eta] = \eta \) for \( \eta \in L^1_G(\Omega_t); \)

(iii) \( \hat{E}_t[X + Y] \leq \hat{E}_t[X] + \hat{E}_t[Y]; \)

(iv) If \( \eta \in L^1_G(\Omega_t) \) and is bounded, then \( \hat{E}_t[\eta X] = \eta^+ \hat{E}_t[X] + \eta^- \hat{E}_t[-X]; \)

(v) \( \hat{E}_t[\varphi(\eta, X)] = \hat{E}_t[\varphi(p, X)]_{p=\eta}, \) for each \( \eta \in L^1_G(\Omega_t; \mathbb{R}^d), X \in L^1_G(\Omega; \mathbb{R}^n) \) and \( \varphi \in C_{b, Lip}(\mathbb{R}^{d+n}); \)

(vi) \( \hat{E}_t[X] = \hat{E}_{t+s}[X]. \)
We define
\[ F_t := \sigma(B_s : s \leq t) \quad \text{and} \quad \mathcal{F} := \bigvee_{t \geq 0} F_t \]
as well as
\[ L^0(\mathcal{F}_t) := \{ X : X \text{ is } \mathcal{F}_t\text{-measurable} \} \quad \text{and} \quad L^0(\mathcal{F}) := \{ X : X \text{ is } \mathcal{F}\text{-measurable} \} \]
The following is the representation theorem.

**Theorem 2.7** ([3, 9]) There exists a family \( \mathcal{P} \) of weakly compact probability measures on \((\Omega, \mathcal{F})\) such that
\[ \hat{E}[X] = \sup_{P \in \mathcal{P}} E_P[X], \quad \text{for each } X \in L^1_G(\Omega). \]
\( \mathcal{P} \) is called a set that represents \( \hat{E} \).

**Remark 2.8** Under each \( P \in \mathcal{P} \), the \( G \)-Brownian motion \( B \) is a martingale.

Given \( P \) that represents \( \hat{E} \), we define the capacity
\[ c(A) := \sup_{P \in \mathcal{P}} P(A), \quad \text{for each } A \in \mathcal{F}. \]
A set \( A \in \mathcal{B}(\Omega) \) is said to be polar if \( c(A) = 0 \). A property is said to holds “quasi-surely” (q.s.) if it holds outside a polar set. In the following, we do not distinguish two random variables \( X \) and \( Y \) if \( X = Y \) q.s.

**Lemma 2.9** Let \( \{A_n\}_{n=1}^\infty \) be a sequence in \( \mathcal{B}(\Omega) \) such that \( A_n \uparrow A \). Then \( c(A_n) \uparrow c(A) \).

For each \( p \geq 1 \), we set
\[ L^p(\Omega) := \{ X \in L^0(\mathcal{F}) : \sup_{P \in \mathcal{P}} E_P[|X|^p] < \infty \} \]
and the larger space
\[ \mathcal{L}(\Omega) := \{ X \in L^0(\mathcal{F}) : E_P[X] \text{ exists for each } P \in \mathcal{P} \}. \]
We extend the \( G \)-expectation to \( \mathcal{L}(\Omega) \), still denote it by \( \hat{E} \), by setting
\[ \hat{E}[X] := \sup_{P \in \mathcal{P}} E_P[X], \quad \text{for } X \in \mathcal{L}(\Omega). \]
From [3], we know that \( L^p(\Omega) \) is a Banach space under the norm \( \| \cdot \|_p := (\hat{E}[|\cdot|^p])^{1/p} \) and \( L^p_G(\Omega) \subset L^p(\Omega) \).

For \( \{X_n\}_{n=1}^\infty \subset L^p(\Omega), X \in L^p(\Omega), \) we say that \( X_n \to X \) in \( L^p \), denoted by \( X = \lim_n \text{-} \lim_{n \to \infty} X_n \), if \( \lim_{n \to \infty} \hat{E}[|X_n - X|^p] = 0 \).

**Lemma 2.10** Let \( X_n \in \mathcal{L}(\Omega) \) be a sequence such that \( X_n \uparrow X \) q.s. and \(-\hat{E}[-X_1] > -\infty \). Then
\[ \hat{E}[X_n] \uparrow \hat{E}[X]. \]

For each \( T > 0 \) and \( p \geq 1 \), we define
\[ M^{p,0}_G(0,T) := \{ \eta = \sum_{j=0}^{N-1} \xi_j(\omega)I_{[t_j,t_{j+1})}(t) : N \in \mathbb{N}, 0 \leq t_0 \leq t_1 \leq \cdots \leq t_N \leq T, \]
\[ \xi_j \in L^p_G(\Omega_{t_j}), \ j = 0,1, \ldots, N \}. \]
For each $\eta \in M^p_G(0, T)$, set the norm $\|\eta\|_{M^p_G} := (\hat{E}[\int^T_0 |\eta|^p dt])^{\frac{1}{p}}$ and denote by $M^p_G(0, T)$ the completion of $M^p_G(0, T)$ under $\| \cdot \|_{M^p_G}$.

According to [14, 22], we can define $\int^t_0 \eta_s dB^i_s$, $\int^t_0 \xi_s d(B^i, B^j)_s$ and $\int^t_0 \xi_s ds$ for $\eta \in M^2_G(0, T)$ and $\xi \in M^2_G(0, T)$, where $(B^i, B^j)$ denotes the cross-variation process, for $1 \leq i, j \leq d$.

### 2.2 Stochastic differential equations driven by $G$-Brownian motion

We consider the following $G$-SDEs: for each given $0 \leq t \leq T < \infty$,

$$
\begin{aligned}
\left\{ 
&dX^i_t = b(X^i_t) dt + \sum_{i,j=1}^d h_{ij}(X^i_t) d(B^j_s) + \sum_{i,j=1}^d \sigma_j(X^i_t) dB^i_s, \quad s \in [t, T], \\
&X^i_t = \xi,
\end{aligned}
$$

(2.2)

where $\xi \in L^p_G(\Omega; \mathbb{R}^n)$, $p \geq 2$ and $b, h_{ij}, \sigma_j : \mathbb{R}^n \rightarrow \mathbb{R}^n$ are given deterministic functions satisfying the following assumptions:

(H1) Symmetry: $h_{ij} = h_{ji}, 1 \leq i, j \leq d$;

(H2) Lipschitz continuity: there exists a constant $L$ such that for each $x, x' \in \mathbb{R}^n$,

$$
|b(x) - b(x')| + \sum_{i,j=1}^d |h_{ij}(x) - h_{ij}(x')| + \sum_{j=1}^d |\sigma_j(x) - \sigma_j(x')| \leq L|x - x'|.
$$

For simplicity, $X^{0,x}_t$ will be denoted by $X^x_t$ for $x \in \mathbb{R}^n$. We have the following estimates for $G$-SDE (2.2) which can be found in [5, 22].

**Lemma 2.11** Assume that the conditions (H1) and (H2) hold. Then $G$-SDE (2.2) has a unique solution $(X^x_t)_{x \in [t,T]} \in M^2_G(t, T; \mathbb{R}^n)$. Moreover, there exists a constant $C$ depending on $p, T, L, G$ such that for any $x, y \in \mathbb{R}^n$, $t, t' \in [0, T]$,

$$
\hat{E}[\sup_{x \in [0,T]} |X^x_t|^p] \leq C(1 + |x|^p),
$$

(2.3)

$$
\hat{E}[|X^y_{t'} - X^y_t|^p] \leq C(|x - y|^p + (1 + |x|^p)|t - t'|^{p/2}).
$$

(2.4)

Noting that $X^x_t = X^{t,X^x_t}_s$ for $s \geq t$, we see from Theorem 4.4 in [8] that

**Lemma 2.12** For each given $\varphi \in C_{b, Lip}(\mathbb{R}^n)$ and $0 \leq t \leq T$, we have

$$
\hat{E}_t[\varphi(X^x_{t+s})] = \hat{E}_t[\varphi(X^{t,x+s}_{t+s})]_{y=X^x_t}, \quad \text{for } s \in [0, T - t].
$$

### 3 Construction of the conditional $G$-expectation $\hat{E}_{\tau^+}$

In this section, we provide a construction of the conditional $G$-expectation $\hat{E}_{\tau^+}$ for any optional time $\tau$ and study its properties. This notion is needed in the derivation of strong Markov property for $G$-SDEs in Section 4. We shall also give an application on the reflection principle for $G$-Brownian motion at the end of this section.
3.1 The construction of conditional $G$-expectation $\hat{E}_{\tau+}$ on $L_{G}^{1,\tau+}(\Omega)$

The mapping $\tau : \Omega \to [0, \infty)$ is called a stopping time if $\{\tau \leq t\} \in \mathcal{F}_t$ for each $t \geq 0$ and an optional time if $\{\tau < t\} \in \mathcal{F}_t$ for each $t \geq 0$. A stopping time is an optional time but the converse may not hold.

For each optional time $\tau$, we define the $\sigma$-field

$$\mathcal{F}_{\tau+} := \{A \in \mathcal{F} : A \cap \{\tau < t\} \in \mathcal{F}_t, \forall t \geq 0\},$$

where $\mathcal{F}_t = \cap_{s>t} \mathcal{F}_s$. If $\tau$ is a stopping time, we also define

$$\mathcal{F}_\tau := \{A \in \mathcal{F} : A \cap \{\tau \leq t\} \in \mathcal{F}_t, \forall t \geq 0\}.$$

Let $\tau$ be an optional time. For each $p \geq 1$, we set

$$L_{G}^{0,p,\tau+}(\Omega) = \{X = \sum_{i=1}^{n} \xi_i I_{A_i} : n \in \mathbb{N}, \{A_i\}_{i=1}^{n} \text{ is an } \mathcal{F}_{\tau+}-\text{partition of } \Omega, \xi_i \in L_{G}^{p}(\Omega), i = 1, \cdots, n\}$$

and denote by $L_{G}^{1,\tau+}(\Omega)$ the completion of $L_{G}^{0,p,\tau+}(\Omega)$ under the norm $|| \cdot ||_p$. In this subsection, we want to define the conditional $G$-expectation

$$\hat{E}_{\tau+} : L_{G}^{1,\tau+}(\Omega) \to L_{G}^{1,\tau+}(\Omega) \cap L_{0}^{0}(\mathcal{F}_{\tau+}).$$

This will be accomplished in three stages by progressively constructing the conditional expectation on $L_{ip}(\Omega)$, $L_{G}^{1}(\Omega)$ and finally $L_{G}^{1,\tau+}(\Omega)$.

**Remark 3.1** According to Theorem 25 in [3], for $X \in L_{G}^{1}(\Omega)$, we have

$$\hat{E}[|X|1_{(|X|>N)}] \to 0, \quad \text{as } N \to \infty.$$

This, together with a direct calculation, implies that (3.1) still holds for $X \in L_{G}^{1,\tau+}(\Omega)$.

In the following, unless stated otherwise, we shall always assume that the optional time $\tau$ satisfying the following assumption:

(H3) $c(\{\tau > T\}) \to 0, \quad \text{as } T \to \infty.$

**Stage one: $\hat{E}_{\tau+}$ on $L_{ip}(\Omega)$**

Let $X \in L_{ip}(\Omega)$. The construction of

$$\hat{E}_{\tau+} : L_{ip}(\Omega) \to L_{G}^{1,\tau+}(\Omega) \cap L_{0}^{0}(\mathcal{F}_{\tau+})$$

consists of two steps.

**Step 1.** For any given simple discrete stopping time $\tau$ taking values in $\{t_i : i \geq 1\}$, we define

$$\hat{E}_{\tau+}[X] := \sum_{i=1}^{\infty} \hat{E}_{t_i}[X]1_{\{\tau=t_i\}}, \quad (3.2)$$

7
where a discrete stopping (or optional) time is *simple* means that $t_i \uparrow \infty$, as $i \to \infty$. Here we employ the convention that $t_{n+1} := t_n + i, i \geq 1, if \tau is a discrete stopping (or optional) time taking finitely many values \{t_i : i \leq n\} with $t_i \leq t_{i+1}$.

**Step 2.** For a general optional time $\tau$, let $\tau_n$ be a sequence of simple discrete stopping times such that $\tau_n \to \tau$ uniformly. We define

$$\hat{E}_{\tau_n}[X] := \mathbb{L}^{1}_{-} \lim_{n \to \infty} \hat{E}_{\tau_n +}[X].$$

(3.3)

**Proposition 3.2** The conditional expectation $\hat{E}_{\tau +} : L_{ip}(\Omega) \to L^1_{G}(\Omega) \cap L^0(F_{\tau +})$ is well-defined.

In the following, for notation simplicity, we always use $C_X$ to denote the bound of $X$ for any bounded function $X : \Omega \to \mathbb{R}$. Similarly, for any given bounded, Lipschitz function $\varphi : \mathbb{R}^n \to \mathbb{R}$, we always use $C_\varphi$ and $L_\varphi$ to denote its bound and Lipschitz constant respectively.

The proof relies on the following lemmas. We set

$$\Lambda_{d,T} := \{(u_1, u_2) : 0 \leq u_1, u_2 \leq T, |u_1 - u_2| \leq \delta\}.$$

The first three lemmas concern the continuity properties of conditional expectation $\hat{E}_t$ on $L_{ip}(\Omega)$.

**Lemma 3.3** Let $X = \varphi(B_{t_1}, B_{t_2} - B_{t_1}, \ldots, B_{t_n} - B_{t_{n-1}})$ for $\varphi \in C_b, L_{ip}(\mathbb{R}^{n \times d})$ with $0 \leq t_1 < t_2 < \cdots < t_n < \infty$. Then for any $T \geq 0$ and $0 \leq s_1 \leq s_2 \leq T$, we have

$$|\hat{E}_{s_2}[X] - \hat{E}_{s_1}[X]| \leq C \sup_{(u_1, u_2) \in \Lambda_{2s-1,T}} (|B_{u_2} - B_{u_1}| \wedge 1) + \sqrt{s_2 - s_1}$$

(3.4)

where $C$ is a constant depending only on $X$ and $G$.

**Proof.** First suppose $s_1, s_2 \in [t_i, t_{i+1}]$ for some $0 \leq i \leq n$ with the convention that $t_0 = 0, t_{n+1} = \infty$. By the definition of conditional $G$-expectation on $L_{ip}(\Omega)$, we have

$$\hat{E}_{s_j}[X] = \psi_j(B_{t_1}, \ldots, B_{t_i} - B_{t_{i-1}}, B_{s_j} - B_{t_i}), \quad \text{for } j = 1, 2,$$

(3.5)

where

$$\psi_j(x_1, \ldots, x_i, x_{i+1}) = \hat{E}[\varphi(x_1, \ldots, x_i, x_{i+1} + B_{t_{i+1}} - B_{s_j}, \ldots, B_{t_n} - B_{t_{n-1}})].$$

From the sub-additivity of $\hat{E}$,

$$|\psi(x_1, \ldots, x_i, x_{i+1}) - \psi(x_1', \ldots, x_i', x_{i+1}')| \leq (L_\varphi \left( \sum_{j=1}^{i+1} |x_j - x_j'| + \hat{E}[|B_{s_2} - B_{s_1}|] \right) \wedge (2C_\varphi)$$

$$\leq C_1 \left( \sum_{j=1}^{i+1} |x_j - x_j'| \wedge 1 + \sqrt{s_2 - s_1} \right),$$

where $C_1 = (L_\varphi (1 \vee \hat{E}[|B_1|]) \cup (2C_\varphi)$. Combining this with (3.5), we obtain

$$|\hat{E}_{s_2}[X] - \hat{E}_{s_1}[X]| \leq C_1 (|B_{s_2} - B_{s_1}| \wedge 1 + \sqrt{s_2 - s_1}).$$

(3.6)
Next, suppose $s_1 \in [t_i, t_i + 1], s_2 \in [t_j, t_j + 1]$ for some $j \geq i$. Applying estimate (3.6), we have

$$|\hat{E}_{s_2}[X] - \hat{E}_{s_1}[X]| \leq |\hat{E}_{s_2}[X] - \hat{E}_{t_j}[X]| + |\hat{E}_{t_j}[X] - \hat{E}_{t_{i-1}}[X]| + \cdots + |\hat{E}_{t_{i+1}}[X] - \hat{E}_{s_1}[X]|$$

$$\leq C_1(|B_{s_2} - B_{t_j}| + 1 + \cdots + |B_{t_{i+1}} - B_{s_1}| + 1) + C_1(\sqrt{t_2 - t_j} + \cdots + \sqrt{t_{i+1} - s_1})$$

$$\leq C\left(\sup_{(u_1, u_2) \in \Lambda_{s_2 - s_1, T}} (|B_{u_2} - B_{u_1}| + 1) + \sqrt{s_2 - s_1}\right),$$

where $C = (n + 1)C_1$. ■

Note that the estimate in the above lemma is universal: the right-hand side of estimate (3.4) depends only on the difference $s_2 - s_1$ instead of the values of $s_1$ and $s_2$. Then we can easily get the following discrete stopping time version. A more general form is given in Lemma 3.18.

**Lemma 3.4** Let $X \in L_{\text{lip}}(\Omega)$. Then for any $T, \delta > 0$ and discrete stopping times $\tau, \sigma \leq T$ taking finitely many values such that $|\tau - \sigma| \leq \delta$, we have

$$|\hat{E}_{\tau + \delta}[X] - \hat{E}_{\sigma + \delta}[X]| \leq C\left(\sup_{(u_1, u_2) \in \Lambda_{\delta, T}} (|B_{u_2} - B_{u_1}| + 1) + \sqrt{\delta}\right),$$

where $C$ is a constant depending only on $X$ and $G$.

**Proof.** Assume $\tau = \sum_{i=1}^{n} t_i I_{\{\tau = t_i\}}, \sigma = \sum_{i=1}^{m} s_i I_{\{\sigma = s_i\}}$. By the definition (3.2), we have

$$|\hat{E}_{\tau + \delta}[X] - \hat{E}_{\sigma + \delta}[X]| = \left|\sum_{i=1}^{n} \hat{E}_{t_i}[X] I_{\{\tau = t_i\}} - \sum_{j=1}^{m} \hat{E}_{s_j}[X] I_{\{\sigma = s_j\}}\right|$$

$$\leq \sum_{i=1}^{n} \sum_{j=1}^{m} |\hat{E}_{t_i}[X] - \hat{E}_{s_j}[X]| I_{\{\tau = t_i\}} \cap \{\sigma = s_j\}.$$

Then by Lemma 3.3, there exists a constant $C$ depending on $X$ and $G$ such that

$$|\hat{E}_{\tau + \delta}[X] - \hat{E}_{\sigma + \delta}[X]| \leq \sum_{i=1}^{n} \sum_{j=1}^{m} C\left(\sup_{(u_1, u_2) \in \Lambda_{t_i - s_j, T}} (|B_{u_2} - B_{u_1}| + 1) + \sqrt{|t_i - s_j|}\right) I_{\{\tau = t_i\}} \cap \{\sigma = s_j\}$$

$$\leq C\left(\sup_{(u_1, u_2) \in \Lambda_{\delta, T}} (|B_{u_2} - B_{u_1}| + 1) + \sqrt{\delta}\right).$$

The proof is complete. ■

**Lemma 3.5** Let $T > 0$ be a given constant. Then

$$\hat{E}\left[\sup_{(u_1, u_2) \in \Lambda_{\delta, T}} (|B_{u_2} - B_{u_1}| + 1)| \downarrow 0, \quad \text{as} \quad \delta \downarrow 0.\right.$$

**Proof.** Given any $\varepsilon > 0$, by the tightness of $\mathcal{P}$, we may pick a compact set $K \subset \Omega_T$ such that $c(K^c) < \varepsilon$. Then by the Arzelà-Ascoli theorem, there exists a $\delta > 0$ such that $|B_{u_1}(\omega) - B_{u_2}(\omega)| \leq \varepsilon$ for $\omega \in K$ and $|u_1 - u_2| \leq \delta, 0 \leq u_1, u_2 \leq T$. Consequently,

$$\hat{E}\left[\sup_{(u_1, u_2) \in \Lambda_{\delta, T}} (|B_{u_2} - B_{u_1}| + 1)\right] \leq \hat{E}\left[\sup_{(u_1, u_2) \in \Lambda_{\delta, T}} |B_{u_2} - B_{u_1}| I_K + c(K^c)\right] \leq 2\varepsilon.$$

Since $\varepsilon$ can be arbitrarily small, we obtain the lemma. ■
Remark 3.6 From the proof, we know that the above lemma is still true for a more general case that \( \hat{\mathbb{E}} \) is the upper expectation of a tight family of probability measures. To be precise, for any fixed \( T \), let \( \Omega_T \) be defined as in Section 2, \( (B_t)_{0 \leq t \leq T} \) be the canonical process and \( \hat{\mathbb{E}} = \sup_{P \in \mathcal{P}'} E_P \), where \( \mathcal{P}' \) is a tight family of probability measures on \( \Omega_T \), then (3.8) holds. This generalization will be used in the next section.

The following lemma is analogous to the classical one.

**Lemma 3.7** Let \( X \in L_{ip}(\Omega) \) and \( \tau, \sigma \) be two simple discrete stopping times. Then \( \hat{\mathbb{E}}_{(\tau \wedge \sigma)^+}[X] = \hat{\mathbb{E}}_{\tau^+}[X] \) on \( \{ \tau \leq \sigma \} \).

**Proof.** Assume \( \tau, \sigma \) taking values in \( \{ t_i : i \geq 1 \} \) and \( \{ s_i : i \geq 1 \} \). Then

\[
\hat{\mathbb{E}}_{(\tau \wedge \sigma)^+}[X] = \sum_{i,j=1}^{\infty} \hat{\mathbb{E}}_{(t_i \wedge s_j)}[X]I_{\{ \tau = t_i, \sigma = s_j \}}.
\]

Multiplying \( I_{\{ \tau \leq \sigma \}} \) on both sides, since \( t_i \leq s_j \) on \( \{ \tau = t_i, \sigma = s_j \} \cap \{ \tau \leq \sigma \} \), it follows that

\[
I_{\{ \tau \leq \sigma \}} \hat{\mathbb{E}}_{(\tau \wedge \sigma)^+}[X] = \sum_{i,j=1}^{\infty} \hat{\mathbb{E}}_{t_i}[X]I_{\{ \tau \leq \sigma \}}I_{\{ \tau = t_i, \sigma = s_j \}} = \sum_{i=1}^{\infty} \hat{\mathbb{E}}_{t_i}[X]I_{\{ \tau = t_i \}}I_{\{ \tau \leq \sigma \}} = I_{\{ \tau \leq \sigma \}} \hat{\mathbb{E}}_{\tau^+}[X],
\]

which is the desired conclusion. \( \blacksquare \)

**Proof of Proposition 3.2.** Assume \( X \in L_{ip}(\Omega) \). Let \( \tau_n \) be a sequence of simple discrete stopping times such that \( \tau_n \to \tau \) uniformly. We need to show that \( \hat{\mathbb{E}}_{\tau_n^+}[X] \) is a Cauchy sequence in \( L^1 \) and the limit is independent of the choice of the approximation sequence \( \tau_n \). Assume \( \tau_n = \sum_{i=1}^{\infty} t_i^n I_{\{ \tau_n = t_i^n \}} \) and \( |\tau_n - \tau| \leq \delta_n \to 0 \), as \( n \to \infty \). We can take \( n_0 \) large enough such that \( \delta_n \leq 1 \) for \( n \geq n_0 \), and hence \( \{ \tau \leq T \} \subset \{ \tau_n \leq T+1 \} \) and \( \{ \tau \leq T \} \subset \{ \tau_m \leq T+1 \} \), for \( m, n \geq n_0 \). Then it follows from Lemma 3.7 that

\[
|\hat{\mathbb{E}}_{\tau_n^+}[X] - \hat{\mathbb{E}}_{\tau_m^+}[X]| = |\hat{\mathbb{E}}_{\tau_n^+}[X] - \hat{\mathbb{E}}_{\tau_m^+}[X]|I_{\{ \tau \leq T \}} + |\hat{\mathbb{E}}_{\tau_n^+}[X] - \hat{\mathbb{E}}_{\tau_m^+}[X]|I_{\{ \tau > T \}} \\
\leq |\hat{\mathbb{E}}_{(\tau_n \wedge (T+1))^{+}}[X] - \hat{\mathbb{E}}_{(\tau_m \wedge (T+1))^{+}}[X]|I_{\{ \tau \leq T \}} + 2C_X \| I_{\{ \tau > T \}} \|.
\]

(3.9)

For any \( \varepsilon > 0 \), we choose \( T \) large enough such that \( c(\{ \tau > T \}) \leq \varepsilon \) by (H3). Taking expectation on both sides of (3.9) and letting \( n, m \to \infty \), we then obtain by Lemma 3.4 and Lemma 3.5

\[
\lim_{n,m \to \infty} \hat{\mathbb{E}} \| \hat{\mathbb{E}}_{\tau_n^+}[X] - \hat{\mathbb{E}}_{\tau_m^+}[X] \| \leq 2C_X c(\{ \tau > T \}) \leq 2C_X \varepsilon.
\]

Since \( \varepsilon \) can be arbitrarily small, this implies

\[
\lim_{n,m \to \infty} \hat{\mathbb{E}} \| \hat{\mathbb{E}}_{\tau_n^+}[X] - \hat{\mathbb{E}}_{\tau_m^+}[X] \| = 0.
\]

Similar argument shows that if there exists another simple discrete sequences \( \tau'_n \) such that \( \tau'_n \to \tau \) uniformly, we have

\[
\lim_{n \to \infty} \hat{\mathbb{E}} \| \hat{\mathbb{E}}_{\tau_n^+}[X] - \hat{\mathbb{E}}_{\tau'_n^+}[X] \| = 0.
\]

Next, for each \( n \geq 1 \), we set

\[
\tau_n := f_n(\tau) := \sum_{i=1}^{\infty} t_i^n I_{\{ t_i^n-1 \leq \tau < t_i^n \}}, \quad \text{where } t_i^n := \frac{i}{2^n}, \ i \geq 0.
\]

(3.10)
Then we deduce $\hat{E}_{\tau_n}[X] \in L_G^{1,\tau^+}(\Omega) \cap L^0(\mathcal{F}_{\tau^+})$ by the observation that
\[
\sum_{i=1}^{m} \hat{E}_t I_{(\tau_n=t_i^\circ)} \in L_G^{0,1,\tau^+}(\Omega) \cap L^0(\mathcal{F}_{\tau^+}), \quad \text{for each } m \geq 1
\]
and
\[
\begin{align*}
\hat{E}[\sum_{i=1}^{\infty} \hat{E}_t I_{(\tau_n=t_i^\circ)}] & = \hat{E}[\sum_{i=m+1}^{\infty} |\hat{E}_t I_{(\tau_n=t_i^\circ)}|] \\
& \leq C \hat{E}[\sum_{i=m+1}^{\infty} I_{(\tau_n=t_i^\circ)}] \\
& = C \hat{E}(|\tau \geq t_{m}^\circ|) \to 0, \quad \text{as } n \to \infty.
\end{align*}
\]
By the definition (3.3), this implies $\hat{E}_{\tau^+}[X] \in L_G^{1,\tau^+}(\Omega) \cap L^0(\mathcal{F}_{\tau^+})$.

Finally, if $\tau$ is itself a simple discrete stopping time, then $\hat{E}_{\tau^+}$ defined by (3.3) coincides with the one defined by (3.2) since we can take the approximation sequence $\tau_n \equiv \tau, n \geq 1$ in Step 2.

Now we give three fundamental properties which are important for the extension of $\hat{E}_{\tau^+}$ to $L_G^1(\Omega)$.

**Proposition 3.8** The conditional expectation $\hat{E}_{\tau^+}$ satisfies the following properties: for $X, Y \in L_{ip}(\Omega)$,

(i) $\hat{E}_{\tau^+}[X] \leq \hat{E}_{\tau^+}[Y], \quad \text{for } X \leq Y$;

(ii) $\hat{E}_{\tau^+}[X + Y] \leq \hat{E}_{\tau^+}[X] + \hat{E}_{\tau^+}[Y]$;

(iii) $\hat{E}[\hat{E}_{\tau^+}[X]] = \hat{E}[X]$.

In order to prove (iii), we need the following proposition. It is a generalized version of Proposition 2.5 (vi) in [7].

**Proposition 3.9** Let $A_i \in \mathcal{F}_{t_i}, i \leq n$ for $0 \leq t_1 \leq \cdots \leq t_n$ such that $\sqcup_{i=1}^{n} A_i = \Omega$ and $A_i \cap A_j = \emptyset$ for $i \neq j$. Then for each $\xi_i \in L_G^1(\Omega), i \leq n$, we have
\[
\hat{E}[\sum_{i=1}^{n} \xi I_{A_i}] = \hat{E}\left[\sum_{i=1}^{n} \hat{E}_{t_i}[\xi_i I_{A_i}]\right].
\]

**Proof.** *Step 1.* Suppose first $\xi_i \geq 0, i = 1, \cdots, n$. For any $P \in \mathcal{P}$, by Lemma 17 in [10], we have
\[
E_P[\xi_i | \mathcal{F}_{t_i}] \leq \hat{E}_{t_i}[\xi_i] \quad \text{P-a.s.}
\]
Then
\[
E_P[\sum_{i=1}^{n} \xi_i I_{A_i}] = E_P[\sum_{i=1}^{n} E_P[\xi_i | \mathcal{F}_{t_i}] I_{A_i}] \leq E_P[\sum_{i=1}^{n} \hat{E}_{t_i}[\xi_i I_{A_i}] \leq \hat{E}[\sum_{i=1}^{n} \hat{E}_{t_i}[\xi_i I_{A_i}].
\]
This implies
\[
\hat{E}[\sum_{i=1}^{n} \xi_i I_{A_i}] = \sup_{P \in \mathcal{P}} E_P[\sum_{i=1}^{n} \xi_i I_{A_i}] \leq \hat{E}[\sum_{i=1}^{n} \hat{E}_{t_i}[\xi_i I_{A_i}]
\]

11
Now we prove the reverse inequality. We only need to show that, for each $P \in \mathcal{P}$,

$$E_P \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i I_{A_i}] \right] \leq \hat{E} \left[ \sum_{i=1}^{n} \xi_i I_{A_i} \right].$$  \hspace{1cm} (3.12)

Let $P \in \mathcal{P}$ be given. For $i \leq n$, noting that $A_i, A_i^c \in \mathcal{F}_t$, we can choose a sequence of increasing compact sets $K_m^i \subset A_i$, $m \geq 1$ such that $P(A_i \setminus K_m^i) \downarrow 0$, as $m \uparrow \infty$ and a sequence of increasing compact sets $\tilde{K}_m^i \subset A_i^c$, $m \geq 1$ such that $P(A_i^c \setminus \tilde{K}_m^i) \downarrow 0$, as $m \uparrow \infty$. Moreover, since $K_m^i \cap \tilde{K}_m^i = \emptyset$ and $K_m^i, \tilde{K}_m^i$ are compact sets, we have

$$\rho_d(K_m^i, \tilde{K}_m^i) > 0. \hspace{1cm} (3.13)$$

For each $i, m$, by Theorem 1.2 in [2] and (3.13), there exist two sequences $\{ \varphi_i^{i,m} \}_{i=1}^{\infty}, \{ \tilde{\varphi}_i^{i,m} \}_{i=1}^{\infty} \subset C_b(\Omega_t)$ such that $\varphi_i^{i,m} \downarrow I_{K_m^i}, \tilde{\varphi}_i^{i,m} \downarrow I_{\tilde{K}_m^i}$, as $i \to \infty$ and

$$\varphi_i^{i,m} \cdot \tilde{\varphi}_i^{i,m} = 0, \text{ for all } i \geq 1. \hspace{1cm} (3.14)$$

Applying the classical monotone convergence theorem under $P$, we have

$$E_P \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i I_{A_i}] \right] = E_P \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i I_{A_i} \prod_{j=1}^{i-1} I_{A_j}] \right]
= \lim_{m \to \infty} E_P \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i I_{K_m^i} \prod_{j=1}^{i-1} I_{\tilde{K}_m^j}] \right]
= \lim_{m \to \infty} \lim_{l \to \infty} E_P \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] \right]
\leq \lim_{m \to \infty} \lim_{l \to \infty} \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] \right]. \hspace{1cm} (3.15)$$

For any fixed $m, l$, by (vi), (ii), (iv) of Proposition 2.6, we have

$$\hat{E}_t \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] \right] = \hat{E} \left[ \hat{E}_{t-1} \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] \right] \right]
= \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] + \hat{E}_{t-1} [\xi_n \varphi_n^{n,m} \prod_{j=1}^{n-1} \tilde{\varphi}_j^{j,m}] \right].$$

By (3.14) and Proposition 2.6 (iv), we note that

$$\hat{E}_{t-1} [\xi_n \varphi_n^{n-1,m}] + \hat{E}_{t-1} [\xi_n \varphi_n^{n,m} \varphi_i^{i-1,m}] = \hat{E}_{t-1} [\xi_n \varphi_n^{n-1,m} + \xi_n \varphi_n^{n,m} \varphi_i^{i-1,m}].$$

We thus obtain

$$\hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] \right] = \hat{E} \left[ \sum_{i=1}^{n-2} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] + \hat{E}_{t-1} [\xi_{n-1} \varphi_{n-1,m} + \xi_n \varphi_n^{n,m} \varphi_i^{i-1,m}] \prod_{j=1}^{n-1} \tilde{\varphi}_j^{j,m}] \right].$$

Repeating this procedure, we conclude that

$$\hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t [\xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m}] \right] = \hat{E} \left[ \hat{E}_{t-1} \left[ \sum_{i=1}^{n} \xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m} \right] \right] = \hat{E} \left[ \sum_{i=1}^{n} \xi_i \varphi_i^{i,m} \prod_{j=1}^{i-1} \tilde{\varphi}_j^{j,m} \right]. \hspace{1cm} (3.16)$$
Substituting (3.16) into (3.15), we arrive at the inequality

\[ E_P \left[ \sum_{i=1}^{n} \hat{E}_t_i [\xi_i | I_{A_i}] \right] \leq \lim_{m \to \infty} \lim_{l \to \infty} \hat{E} \left[ \sum_{i=1}^{n} \xi_i \phi_i^{l,m} \prod_{j=1}^{i-1} \hat{\phi}_{j,m} \right]. \]  

(3.17)

By Theorem 1.31 in Chap VI of [22], we note that

\[ \lim_{l \to \infty} \hat{E} \left[ \sum_{i=1}^{n} \xi_i \phi_i^{l,m} \prod_{j=1}^{i-1} \hat{\phi}_{j,m} \right] = \hat{E} \left[ \sum_{i=1}^{n} \xi_i I_{K_m} \prod_{j=1}^{i-1} I_{K_m} \right] \]

\[ \leq \hat{E} \left[ \sum_{i=1}^{n} \xi_i I_{A_i} \right] \]

Thus (3.12) is proved.

**Step 2.** Consider now the general case. We define \( \xi_i^N = \xi_i \vee (-N) \) for constant \( N > 0 \). By Step 1,

\[ \hat{E} \left[ \sum_{i=1}^{n} (\xi_i^N + N) I_{A_i} \right] = \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t_i [\xi_i^N + N | I_{A_i}] \right]. \]  

(3.18)

Note that

\[ \hat{E} \left[ \sum_{i=1}^{n} (\xi_i^N + N) I_{A_i} \right] = \hat{E} \left[ \sum_{i=1}^{n} \xi_i^N I_{A_i} \right] + N \]

and

\[ \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t_i [\xi_i^N] I_{A_i} \right] = \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t_i [\xi_i^N] I_{A_i} \right] + N. \]

Subtracting \( N \) from both sides of (3.18), we obtain

\[ \hat{E} \left[ \sum_{i=1}^{n} \xi_i^N I_{A_i} \right] = \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_t_i [\xi_i^N] I_{A_i} \right]. \]

Letting \( N \to \infty \) yields (3.11) by (3.1) \( \Box \)

**Proof of Proposition 3.8.** (i), (ii) follows immediately from the definition of \( \hat{E}_{\tau+} \) and Proposition 2.6 (i), (iii). We just need to prove (iii).

First suppose \( \tau \) is a simple discrete stopping time. By Proposition 3.9, noting that \( \{ \tau = t_i \} \in \mathcal{F}_{t_i}, i \geq 1 \), we have,

\[ \hat{E} \left[ \hat{E}_{\tau+} [X] \right] = \hat{E} \left[ \sum_{i=1}^{\infty} \hat{E}_{t_i} [X | \tau = t_i] \right] = \lim_{n \to \infty} \hat{E} \left[ \sum_{i=1}^{n} \hat{E}_{t_i} [X | \tau = t_i] \right] = \lim_{n \to \infty} \hat{E} \left[ \sum_{i=1}^{n} X I_{\{\tau = t_i\}} \right] = \hat{E} [X]. \]

Now we consider the general optional time \( \tau \). Taking a simple discrete stopping time sequence \( \tau_n \to \tau \) uniformly, we obtain

\[ \hat{E} \left[ \hat{E}_{\tau+} [X] \right] = \hat{E} [L^1 - \lim_{n \to \infty} \hat{E}_{\tau_n+} [X]] = \lim_{n \to \infty} \hat{E} [\hat{E}_{\tau_n+} [X]] = \hat{E} [X], \]

which is the desired result. \( \Box \)
Stage two: \( \hat{E}_{\tau^+} \) on \( L^1_G(\Omega) \)

We proceed to define
\[
\hat{E}_{\tau^+} : L^1_G(\Omega) \to L^1_G(\Omega) \cap L^0(\mathcal{F}_{\tau^+}).
\]
Let \( X \in L^1_G(\Omega) \). Then there exists a sequence \( \{X_n\}_{n=1}^{\infty} \subset L_{ip}(\Omega) \) such that \( X_n \to X \) in \( L^1 \). We define
\[
\hat{E}_{\tau^+}[X] := \lim_{n \to \infty} \hat{E}_{\tau^+}[X_n].
\]
This extension of \( \hat{E}_{\tau^+} \) also satisfies the basic properties in Proposition 3.8.

Proposition 3.10 The conditional expectation \( \hat{E}_{\tau^+} : L^1_G(\Omega) \to L^1_G(\Omega) \cap L^0(\mathcal{F}_{\tau^+}) \) is well-defined and satisfies: for \( X,Y \in L^1_G(\Omega) \),

(i) \( \hat{E}_{\tau^+}[X] \leq \hat{E}_{\tau^+}[Y] \), for \( X \leq Y \);

(ii) \( \hat{E}_{\tau^+}[X + Y] \leq \hat{E}_{\tau^+}[Y] + \hat{E}_{\tau^+}[Y] \);

(iii) \( \hat{E}[\hat{E}_{\tau^+}[X]] = \hat{E}[X] \).

Proof. (i)-(iii) are obvious by the definition and Proposition 3.8. We just show that \( \hat{E}_{\tau^+} \) is well-defined on \( L^1_G(\Omega) \).

Let \( X \in L^1_G(\Omega) \). Take any \( \{X_n\}_{n=1}^{\infty} \subset L_{ip}(\Omega) \) such that \( X_n \to X \) in \( L^1 \). By (i), (ii), (iii) of Proposition 3.8, we have
\[
\hat{E}[|\hat{E}_{\tau^+}[X_n] - \hat{E}_{\tau^+}[X_m]|] \leq \hat{E}[|X_n - X_m|] = \hat{E}[|X_n - X_m|] \to 0, \quad \text{as } n,m \to \infty.
\]
Moreover, a similar argument shows that the limit is independent of the choice of the approximation sequence \( \{X_n\}_{n=1}^{\infty} \). \( \blacksquare \)

Stage three: \( \hat{E}_{\tau^+} \) on \( L^{1,\tau^+}_G(\Omega) \)

Finally, we define
\[
\hat{E}_{\tau^+} : L^{1,\tau^+}_G(\Omega) \to L^{1,\tau^+}_G(\Omega) \cap L^0(\mathcal{F}_{\tau^+})
\]
by two steps.

Step 1. Let \( X = \sum_{i=1}^{n} \xi_i I_{A_i} \in L^{0,1,\tau^+}_G(\Omega) \), where \( \xi_i \in L^1_G(\Omega) \) and \( \{A_i\}_{i=1}^{n} \) is an \( \mathcal{F}_{\tau^+} \)-partition of \( \Omega \). We define
\[
\hat{E}_{\tau^+}[X] := \sum_{i=1}^{n} \hat{E}_{\tau^+}[\xi_i] I_{A_i}.
\]
Then \( \hat{E}_{\tau^+} \) is well-defined by the following lemma.

Lemma 3.11 Let \( A \in \mathcal{F}_{\tau^+} \) and \( \xi, \eta \in L^1_G(\Omega) \). Then \( \xi A \geq \eta I_A \) implies
\[
I_A \hat{E}_{\tau^+}[\xi] \geq I_A \hat{E}_{\tau^+}[\eta]. \quad (3.19)
\]
Proof. By approximation, we may assume that $\xi, \eta \in L_{ip}(\Omega)$.

We first prove the case that $\tau$ is a simple discrete stopping time taking values in $\{t_i : i \geq 1\}$ and $A \in \mathcal{F}_{\tau}$. Applying Lemma 2.4 in [7], we have

$$I_A \hat{E}_{\tau^+}[\xi] = \sum_{i=1}^{\infty} \hat{E}_{t_i}[\xi I_{A \cap \{\tau = t_i\}}] \geq \sum_{i=1}^{\infty} \hat{E}_{t_i}[\eta I_{A \cap \{\tau = t_i\}}] = I_A \hat{E}_{\tau^+}[\eta].$$

Now for the general $\tau$, take $\tau_n$ as (3.10). Since $A \in \mathcal{F}_{\tau^+} \subset \mathcal{F}_{\tau_n}$, we have

$$I_A \hat{E}_{\tau^+}[\xi] = \lim_{n \to \infty} I_A \hat{E}_{\tau_n^+}[\xi] \geq \lim_{n \to \infty} I_A \hat{E}_{\tau_n^+}[\eta] = I_A \hat{E}_{\tau^+}[\eta].$$

This proves the lemma. $\blacksquare$

**Proposition 3.12** The conditional expectation $\hat{E}_{\tau^+} : L_G^{0,1,\tau^+}(\Omega) \to L_G^{1,\tau^+}(\Omega) \cap L^0(\mathcal{F}_{\tau^+})$ satisfies: for $X, Y \in L_G^{0,1,\tau^+}(\Omega)$,

(i) $\hat{E}_{\tau^+}[X] \leq \hat{E}_{\tau^+}[Y]$, for $X \leq Y$;

(ii) $\hat{E}_{\tau^+}[X + Y] \leq \hat{E}_{\tau^+}[X] + \hat{E}_{\tau^+}[Y]$;

(iii) $\hat{E}[^{\hat{E}_{\tau^+}[X]}] = \hat{E}[X]$.

**Proof.** We just prove (iii). The proof for (i), (ii) is trivial.

First assume that $\tau$ is a simple discrete stopping time taking values in $\{t_j : j \geq 1\}$ and $X = \sum_{i=1}^{n} \xi_i I_{A_i}$, where $\xi_i \in L_{ip}(\Omega)$ and $\{A_i\}_{i=1}^{n}$ is an $\mathcal{F}_{\tau^+}$-partition of $\Omega$. By Proposition 3.9,

$$\hat{E}[\hat{E}_{\tau^+}[X]] = \hat{E}[\sum_{i=1}^{n} \hat{E}_{\tau^+}[\xi_i I_{A_i}]] = \lim_{m \to \infty} \hat{E}[\sum_{i=1}^{m} \hat{E}_{t_{j_i}}[\xi_i I_{A_i \cap \{\tau = t_{j_i}\}}]] = \lim_{m \to \infty} \hat{E}[\sum_{i=1}^{m} \xi_i I_{A_i \cap \{\tau = t_{j_i}\}}] = \hat{E}[X].$$

Next suppose that $\tau$ is an optional time and $X = \sum_{i=1}^{n} \xi_i I_{A_i}$, where $\xi_i \in L_{ip}(\Omega)$ and $\{A_i\}_{i=1}^{n}$ is an $\mathcal{F}_{\tau^+}$-partition of $\Omega$. Taking $\tau_m$ as (3.10), then we derive that

$$\hat{E}[\hat{E}_{\tau^+}[X]] = \hat{E}[^{\sum_{i=1}^{n} \hat{E}_{\tau_m^+}[\xi_i I_{A_i}]}] = \lim_{m \to \infty} \hat{E}[\sum_{i=1}^{n} \hat{E}_{\tau_m^+}[\xi_i I_{A_i}]] = \lim_{m \to \infty} \hat{E}[\hat{E}_{\tau^+}[\sum_{i=1}^{n} \xi_i I_{A_i}]] = \lim_{m \to \infty} \hat{E}[X] = \hat{E}[X].$$

Consider finally the general case that $\tau$ is an optional time and $X = \sum_{i=1}^{n} \xi_i I_{A_i}$, where $\xi_i \in L_{ip}(\Omega)$ and $\{A_i\}_{i=1}^{n}$ is an $\mathcal{F}_{\tau^+}$-partition of $\Omega$. We can take sequences $\xi^k_i \in L_{ip}(\Omega)$ such that $\xi^k_i \to \xi_i$ in $L^1$, $i \leq n$ to conclude that

$$\hat{E}[\hat{E}_{\tau^+}[X]] = \hat{E}[\sum_{i=1}^{n} \hat{E}_{\tau^+}[\xi^k_i I_{A_i}]] = \lim_{k \to \infty} \hat{E}[\sum_{i=1}^{n} \hat{E}_{\tau^+}[\xi^k_i I_{A_i}]] = \lim_{k \to \infty} \hat{E}[\sum_{i=1}^{n} \xi^k_i I_{A_i}] = \hat{E}[X],$$

as desired. $\blacksquare$

**Step 2.** Let $X \in L_G^{1,\tau^+}(\Omega)$. Then there exists a sequence $\{X_n\}_{n=1}^{\infty} \subset L_G^{0,1,\tau^+}(\Omega)$ such that $X_n \to X$ in $L^1$. We define

$$\hat{E}_{\tau^+}[X] := \lim_{n \to \infty} \hat{E}_{\tau^+}[X_n].$$

**Proposition 3.13** The conditional expectation $\hat{E}_{\tau^+} : L_G^{0,1,\tau^+}(\Omega) \to L_G^{1,\tau^+}(\Omega) \cap L^0(\mathcal{F}_{\tau^+})$ is well-defined and satisfies the following properties: for $X, Y \in L_G^{0,1,\tau^+}(\Omega)$,
(i) $\hat{E}_{\tau^+}[X] \leq \hat{E}_{\tau^+}[Y]$, for $X \leq Y$;
(ii) $\hat{E}_{\tau^+}[X + Y] \leq \hat{E}_{\tau^+}[Y] + \hat{E}_{\tau^+}[Y]$;
(iii) $\hat{E}[\hat{E}_{\tau^+}[X]] = \hat{E}[X]$.

**Proof.** It is immediate from the definition of $\hat{E}_{\tau^+}$ on $L^{1,\tau^+}_{G}(\Omega)$ and Proposition 3.12. □

**Remark 3.14** If $G(A) = \frac{1}{2}\text{tr}(A)$, we have $L^{1,\tau^+}_{G}(\Omega) = L^{1,\tau^+}_{p}(\Omega) = L^{1}_{p}(\Omega)$ for the Wiener measure $P$, where $L^{1}_{p}(\Omega) := \{X \in \mathcal{F} : E_{P}[|X|] < \infty\}$. Moreover, $\hat{E}_{\tau^+}[\cdot]$ is just the classical conditional expectation $E_{P}[\cdot | \mathcal{F}_{\tau^+}]$.

**Remark 3.15** Let $\tau$ be a stopping time satisfying (H3).

(i) We define $L^{1,\tau^+}_{G}(\Omega)$ as $L^{1,\tau^+}_{G}(\Omega)$ with $\mathcal{F}_{\tau}$ in place of $\mathcal{F}_{\tau^+}$. By a similar manner, we can define the conditional expectation at $\tau$ 

$$\hat{E}_{\tau} : L^{1,\tau^+}_{G}(\Omega) \to L^{1,\tau^+}_{G}(\Omega) \cap L^{0}(\mathcal{F}_{\tau}),$$

and analogous properties (throughout this paper) hold for $\hat{E}_{\tau}$ and $L^{1,\tau}_{G}(\Omega)$. For convenience of readers, we sketch the construction.

**Stage one.** Let $X \in L^{1}_{ip}(\Omega)$. First for a simple discrete stopping time $\tau$ taking values in $\{t_{i} : i \geq 1\}$, we define 

$$\hat{E}_{\tau}[X] := \sum_{i=1}^{\infty} \hat{E}_{t_{i}}[X] I_{\{\tau = t_{i}\}}.$$ 

Then for the general $\tau$, we take a sequence of simple discrete stopping times $\tau_{n}$ such that $\tau_{n} \to \tau$ uniformly and define

$$\hat{E}_{\tau}[X] := L^{1}_{-}\lim_{n \to \infty} \hat{E}_{\tau_{n}}[X].$$

**Stage two.** Let $X \in L^{1}_{G}(\Omega)$. Then there exists a sequence $\{X_{n}\}_{n=1}^{\infty} \subset L^{1}_{ip}(\Omega)$ such that $X_{n} \to X$ in $L^{1}$. We define

$$\hat{E}_{\tau}[X] := L^{1}_{-}\lim_{n \to \infty} \hat{E}_{\tau}[X_{n}].$$

**Stage three.** First for $X = \sum_{i=1}^{n} \xi_{i} I_{A_{i}} \in L^{0,1,\tau}_{G}(\Omega)$, where $\xi_{i} \in L^{0}_{G}(\Omega)$ and $\{A_{i}\}_{i=1}^{n}$ is an $\mathcal{F}_{\tau}$-partition of $\Omega$, we define

$$\hat{E}_{\tau}[X] := \sum_{i=1}^{n} \hat{E}_{\tau}[\xi_{i}] I_{A_{i}}.$$ 

For $X \in L^{1,\tau}_{G}(\Omega)$, there exists a sequence $\{X_{n}\}_{n=1}^{\infty} \subset L^{0,1,\tau}_{G}(\Omega)$ such that $X_{n} \to X$ in $L^{1}$. We define

$$\hat{E}_{\tau}[X] := L^{1}_{-}\lim_{n \to \infty} \hat{E}_{\tau}[X_{n}].$$

(ii) If $\tau \equiv t$ for some constant $t \geq 0$, then $\hat{E}_{\tau}$ and $L^{1,\tau}_{G}(\Omega)$ reduce to $\hat{E}_{t}$ and $L^{1}_{t}(\Omega)$ defined in [7].

(iii) In the case that $\tau$ is a stopping time, both $\hat{E}_{\tau^+}$ and $\hat{E}_{\tau}$ are defined. From the definitions of $\hat{E}_{\tau^+}$ and $\hat{E}_{\tau}$, it is easy to see that

$$\hat{E}_{\tau^+}[X] = \hat{E}_{\tau}[X], \quad \text{for } X \in L^{1,\tau}_{G}(\Omega).$$

If $G(A) = \frac{1}{2}\text{tr}(A)$, then $L^{1}_{G}(\Omega) = L^{1,\tau}_{G}(\Omega) = L^{1}_{p}(\Omega)$ and $\hat{E}_{\tau}[\cdot]$ reduces to the classical conditional expectation $E_{P}[\cdot | \mathcal{F}_{\tau}]$, where $P$ is the Wiener measure.
3.2 Some further properties of $\hat{E}_{r+}$ on $L^{1,τ+}_{G}(\Omega)$

Let $\tau$ be an optional time satisfying (H3). In this subsection, we describe several interesting properties enjoyed by the conditional expectation $\hat{E}_{r+}$ on $L^{1,τ+}_{G}(\Omega)$. We begin by observing the following four significant statements.

Proposition 3.16 The conditional expectation $\hat{E}_{r+} : L^{1,τ+}_{G}(\Omega) \to L^{1,τ+}_{G}(\Omega) \cap L^{0}(\mathcal{F}_{r+})$ satisfies the following properties:

(i) If $X_i \in L^{1,τ+}_{G}(\Omega)$, $i = 1, \ldots, n$ and $\{A_i\}_{i=1}^n$ is an $\mathcal{F}_{r+}$-partition of $\Omega$, then $\hat{E}_{r+}[\sum_{i=1}^n X_i I_{A_i}] = \sum_{i=1}^n \hat{E}_{r+}[X_i]I_{A_i}$;

(ii) If $r$ and $\sigma$ are two optional times and $X \in L^{1,τ+}_{G}(\Omega)$, then $\hat{E}_{r+}[X]Q_{\{r \leq \sigma\}} = \hat{E}_{\{r \wedge \sigma\}}[X Q_{\{r \leq \sigma\}}]$;

(iii) If $X \in L^{1,τ+}_{G}(\Omega)$, then $\hat{E}_{\{r \wedge \sigma\}+}[X Q_{\{r \leq \sigma\}}] \to \hat{E}_{r+}[X]$ in $L^1$, as $T \to \infty$;

(iv) If $\{\tau_n\}_{n=1}^\infty$ are optional times such that $\tau_n \to \tau$ uniformly, as $n \to \infty$ and $X \in L^{1,τ_{0n}}_{G}(\Omega)$, where $\tau_0 := \tau \wedge (\wedge_{n=1}^\infty \tau_n)$, then $\hat{E}_{\tau_n}[X] \to \hat{E}_{\tau}[X]$ in $L^1$, as $n \to \infty$; in particular, if $\tau_n \downarrow \tau$ uniformly, as $n \to \infty$ and $X \in L^{1,τ}_{G}(\Omega)$, then $\hat{E}_{\tau_n}[X] \to \hat{E}_{\tau}[X]$ in $L^1$, as $n \to \infty$.

Remark 3.17 For two optional times $r$ and $\sigma$, since $A \cap \{r \leq \sigma\}, A \cap \{r = \sigma\} \in \mathcal{F}_{\{r \wedge \sigma\}+} \subset \mathcal{F}_{\sigma+}$ for $A \in \mathcal{F}_{r+}$, we have $X I_{\{r \leq \sigma\}}, X I_{\{r = \sigma\}} \in L^{1,τ+}_{G}(\Omega) \cap L^{0}_{G}(\Omega)$ for $X \in L^{1,τ+}_{G}(\Omega)$. Hence the conditional expectations $\hat{E}_{(r \wedge \sigma)+}[X I_{\{r \leq \sigma\}}]$, $\hat{E}_{(r \wedge \sigma)+}[X I_{\{r = \sigma\}}]$, $\hat{E}_{\{r \wedge \sigma\}+}[X I_{\{r \leq \sigma\}}]$ and $\hat{E}_{\{r \wedge \sigma\}+}[X I_{\{r = \sigma\}}]$ are all meaningful.

The following generalization of Lemma 3.4 is needed for the proof of Proposition 3.16 (iv).

Lemma 3.18 Let $X \in L^p_{\mathcal{G}}(\Omega)$. Then there exists a constant $C$ depending on $X$ and $G$ such that

$$|\hat{E}_{r+}[X] - \hat{E}_{\sigma+}[X]| \leq C \left\{ \sup_{(u_1, u_2) \in \Lambda_{k, n}} ((B_{u_2} - B_{u_1}) \wedge 1) + \sqrt{\delta} \right\},$$

for any $T, \delta > 0$ and optional times $r, \sigma \leq T$ such that $|r - \sigma| \leq \delta$.

Proof. Let $\tau_n, \sigma_n \leq T + 1$ be two sequences of discrete stopping times taking finitely many values such that $\tau_n \to \tau, \sigma_n \to \sigma$ uniformly, as $n \to \infty$. For any $\varepsilon > 0$, we have $|\tau_n - \sigma_n| \leq \delta + \varepsilon$ when $n$ large enough. Then by Lemma 3.4, there exists a constant $C$ depending on $X, G$ such that

$$|\hat{E}_{\tau_n}[X] - \hat{E}_{\sigma_n}[X]| \leq C \left\{ \sup_{(u_1, u_2) \in \Lambda_{k+1, n}} ((B_{u_2} - B_{u_1}) \wedge 1) + \sqrt{\delta + \varepsilon} \right\}.$$

First letting $n \to \infty$ and then letting $\varepsilon \downarrow 0$, we get the desired conclusion. ■

Proof of Proposition 3.16. (i) Let $X_i = \sum_{j=1}^m \zeta_{j} I_{B_j}$, where $\zeta_{j} \in L^1_{G}(\Omega)$ and $\{B_i\}_{j=1}^m$ is an $\mathcal{F}_{r+}$-partition of $\Omega$. By the definition of $\hat{E}_{r+}$ on $L^{1,τ+}_{G}(\Omega)$, we have

$$\hat{E}_{r+}[\sum_{i=1}^n X_i I_{A_i}] = \hat{E}_{r+}[\sum_{i=1}^n \sum_{j=1}^m \zeta_{j} I_{B_j} I_{A_i}]$$

$$= \hat{E}_{r+}[\sum_{i=1}^n \sum_{j=1}^m \zeta_{j} I_{A_i \cap B_j}]$$

$$= \sum_{i=1}^n \sum_{j=1}^m \hat{E}_{r+}[\zeta_{j} I_{A_i \cap B_j}].$$
Using the definition of $\hat{E}_{\tau+}$ again, this can be further written as

$$
\sum_{i=1}^{n} \left( \sum_{j=1}^{m} \hat{E}_{\tau+}[\xi]_{I_{B_j}} \right) I_{A_i} = \sum_{i=1}^{n} \hat{E}_{\tau+}[X] I_{A_i}.
$$

Now the result for the general case of $X_i \in L_{1G}^{\tau+}(\Omega)$ follows from a direct limit argument.

(ii) First assume $X \in L_{1\text{lip}}(\Omega)$. Let $\tau_n := f_n(\tau), \sigma_n := f_n(\sigma)$ be as in (3.10). Since $\{\tau \leq \sigma\} \subset \{\tau_n \leq \sigma_n\}$, by Lemma 3.7, we have

$$
I_{\{\tau \leq \sigma\}} \hat{E}_{(\tau_n \wedge \sigma_n)+}[X] = I_{\{\tau \leq \sigma\}} \hat{E}_{\tau_n+[X]}.
$$

Letting $n \to \infty$, we obtain

$$
I_{\{\tau \leq \sigma\}} \hat{E}_{(\tau \wedge \sigma)+}[X] = I_{\{\tau \leq \sigma\}} \hat{E}_{\tau+[X]}.
$$

Then by a simple approximation, we get for $X \in L_{1G}^{1}(\Omega)$

$$
I_{\{\tau \leq \sigma\}} \hat{E}_{(\tau \wedge \sigma)+}[X] = I_{\{\tau \leq \sigma\}} \hat{E}_{\tau+[X]}.
$$

Now it follows from (i) that

$$
\hat{E}_{(\tau \wedge \sigma)+}[XI_{\{\tau \leq \sigma\}}] = I_{\{\tau \leq \sigma\}} \hat{E}_{(\tau \wedge \sigma)+}[X] = I_{\{\tau \leq \sigma\}} \hat{E}_{\tau+[X]}.
$$

Next we consider the case $X = \sum_{i=1}^{n} \xi_i I_{A_i}$, where $\xi_i \in L_{1G}^{1}(\Omega)$ and $\{A_i\}_{i=1}^{n}$ is an $\mathcal{F}_{\tau+}$-partition of $\Omega$. We have

$$
\hat{E}_{(\tau \wedge \sigma)+}[X I_{\{\tau \leq \sigma\}}] = \hat{E}_{(\tau \wedge \sigma)+}\left[ \sum_{i=1}^{n} \xi_i I_{A_i \wedge (\tau \leq \sigma)} \right]
$$

$$
= \sum_{i=1}^{n} \hat{E}_{(\tau \wedge \sigma)+}[\xi_i] I_{A_i \wedge (\tau \leq \sigma)}
$$

$$
= \sum_{i=1}^{n} \hat{E}_{(\tau \wedge \sigma)+}[\xi_i] I_{\{\tau \leq \sigma\}} I_{A_i}.
$$

Since $\hat{E}_{(\tau \wedge \sigma)+}[\xi_i] I_{\{\tau \leq \sigma\}} = \hat{E}_{\tau+[\xi_i]} I_{\{\tau \leq \sigma\}}$, it follows that

$$
\hat{E}_{(\tau \wedge \sigma)+}[X I_{\{\tau \leq \sigma\}}] = \sum_{i=1}^{n} \hat{E}_{\tau+[\xi_i]} I_{\{\tau \leq \sigma\}} I_{A_i} = \hat{E}_{\tau+[X]} I_{\{\tau \leq \sigma\}}.
$$

Finally, we obtain the the conclusion for $X \in L_{1G}^{1,\tau+}(\Omega)$ after an approximation.

(iii) We first assume that $X$ is bounded. By (i) and (ii),

$$
\hat{E}_{(\tau \wedge T)+}[XI_{\{\tau \leq T\}}] I_{\{\tau \leq T\}} = \hat{E}_{(\tau \wedge T)+}[XI_{\{\tau \leq T\}}] = \hat{E}_{\tau+[X]} I_{\{\tau \leq T\}}.
$$

Then we directly calculate

$$
\|\hat{E}_{\tau+[X]} - \hat{E}_{(\tau \wedge T)+}[XI_{\{\tau \leq T\}}]\| = \|\hat{E}_{\tau+[X]} - \hat{E}_{(\tau \wedge T)+}[XI_{\{\tau \leq T\}}] I_{\{\tau > T\}}\| 
$$

$$
\leq C_{\chi}(\{\tau > T\}) \to 0, \quad \text{as } T \to \infty.
$$
To pass to the case of general \( X \) we may argue as follows. Set \( X_N := (X \wedge N) \vee (-N) \) for constant \( N > 0 \). For any \( \varepsilon > 0 \), by Remark 3.1, we can take \( N \) large enough such that
\[
\hat{E}[|X - X_N|] \leq \varepsilon.
\]
Then
\[
\hat{E}[|\hat{E}_{\tau+}[X] - \hat{E}_{(\tau \wedge T) +}[XI_{(\tau \leq T)}]|] \leq \hat{E}[|\hat{E}_{\tau+}[X] - \hat{E}_{\tau+}[X_N]|] + \hat{E}[|\hat{E}_{\tau+}[X_N] - \hat{E}_{(\tau \wedge T) +}[XN I_{(\tau \leq T)}]|] + \hat{E}[|\hat{E}_{(\tau \wedge T) +}[XN I_{(\tau \leq T)}] - \hat{E}_{(\tau \wedge T) +}[X I_{(\tau \leq T)}]|]
\leq 2\varepsilon + \hat{E}[|\hat{E}_{\tau+}[X] - \hat{E}_{(\tau \wedge T) +}[XN I_{(\tau \leq T)}]|].
\]
Letting \( T \to \infty \), we get
\[
\limsup_{T \to \infty} \hat{E}[|\hat{E}_{\tau+}[X] - \hat{E}_{(\tau \wedge T) +}[X I_{(\tau \leq T)}]|] \leq 2\varepsilon,
\]
which implies, since \( \varepsilon \) can be arbitrarily small,
\[
\hat{E}[|\hat{E}_{\tau+}[X] - \hat{E}_{(\tau \wedge T) +}[X I_{(\tau \leq T)}]|] \to 0, \quad \text{as } T \to \infty.
\]
(iv) Step 1. Suppose that \( \tau_n, \tau \leq T \). We first assume \( X \in L_G^1(\Omega) \). For any given \( \varepsilon > 0 \), there exists an \( \tilde{X} \in L_{lip}(\Omega) \) such that
\[
\hat{E}[|\tilde{X} - X|] \leq \varepsilon.
\]
Then
\[
\hat{E}[|\hat{E}_{\tau_n+}[X] - \hat{E}_{\tau+}[X]|] \leq \hat{E}[|\hat{E}_{\tau_n+}[X] - \hat{E}_{\tau_n+}[\tilde{X}]|] + \hat{E}[|\hat{E}_{\tau_n+}[\tilde{X}] - \hat{E}_{\tau+}[\tilde{X}]|] + \hat{E}[|\hat{E}_{\tau+}[\tilde{X}] - \hat{E}_{\tau+}[X]|] 
\leq 2\varepsilon + \hat{E}[|\hat{E}_{\tau_n+}[\tilde{X}] - \hat{E}_{\tau+}[\tilde{X}]|].
\]
We now let \( n \to \infty \) and use Lemma 3.18 and Lemma 3.5 to obtain
\[
\limsup_{n \to \infty} \hat{E}[|\hat{E}_{\tau_n+}[X] - \hat{E}_{\tau+}[X]|] \leq 2\varepsilon,
\]
which implies
\[
\hat{E}_{\tau_n+}[X] \to \hat{E}_{\tau+}[X] \quad \text{in } L^1.
\] (3.20)
Next, for \( X = \sum_{i=1}^k X_i I_{A_i} \), where \( X_i \in L_G^1(\Omega) \) and \( \{A_i\}_{i=1}^k \) is an \( \mathcal{F}_{\tau_0+} \)-partition of \( \Omega \), the conclusion follows from (3.20) and the observation that
\[
\hat{E}[|\hat{E}_{\tau_n+}[X] - \hat{E}_{\tau+}[X]|] \leq \sum_{i=1}^k \hat{E}[|\hat{E}_{\tau_n+}[X_i] - \hat{E}_{\tau+}[X_i]|].
\]
Finally, for \( X \in L_G^{1,\tau_0+}(\Omega) \), we can find an \( \tilde{X} \in L_G^{0,1,\tau_0+}(\Omega) \) such that
\[
\hat{E}[|\tilde{X} - X|] \leq \varepsilon.
\]
Following the argument for the case of \( X \in L_G^1(\Omega) \) we can then obtain the conclusion for \( L_G^{1,\tau_0+}(\Omega) \).

Step 2. We now consider the case that \( \tau \) is not bounded. Without loss of generality, we can assume \( 0 \leq \tau \vee (\vee_{n=1}^\infty \tau_n) - \tau_0 \leq 1 \). For any \( T > 0 \), by (ii),
\[
\hat{E}_{\tau_n+}[X]I_{(\tau_n \leq T+1)} = \hat{E}_{(\tau_n \wedge (T+1)) +}[X I_{(\tau_n \leq T+1)}].
\]
Multiplying $I_{\{\tau_0 \leq T\}}$, (i) implies

$$\hat{E}_{\tau_0}[X]I_{\{\tau_0 \leq T\}} = \hat{E}_{(\tau_0 \wedge (T+1))}[XI_{\{\tau_0 \leq T\}}].$$

Similarly, we have

$$\hat{E}_{\tau_1}[X]I_{\{\tau_0 \leq T\}} = \hat{E}_{(\tau_1 \wedge (T+1))}[XI_{\{\tau_1 \leq T\}}].$$

Let first $X$ be bounded. We have

$$|\hat{E}_{\tau_0}[X] - \hat{E}_{\tau_1}[X]| = |\hat{E}_{\tau_0}[X] - \hat{E}_{\tau_1}[X]|I_{\{\tau_0 \leq T\}} + |\hat{E}_{\tau_0}[X] - \hat{E}_{\tau_1}[X]|I_{\{\tau_0 > T\}}$$

$$\leq |\hat{E}_{(\tau_0 \wedge (T+1))}[XI_{\{\tau_0 \leq T\}}] - \hat{E}_{(\tau_1 \wedge (T+1))}[XI_{\{\tau_0 \leq T\}}]| + 2C_XI_{\{\tau_0 > T\}}.$$  \(3.21\)

For any $\varepsilon > 0$, we choose $T$ large enough such that $c(\{\tau_0 > T\}) \leq c(\{\tau > T\}) \leq \varepsilon$. Taking expectation $\hat{E}$ on both sides of (3.21) and letting $n \to \infty$, we then obtain

$$\hat{E}[|\hat{E}_{\tau_0}[X] - \hat{E}_{\tau_1}[X]|] \leq 2C_X\varepsilon,$$

which implies the conclusion. If $X$ is not necessarily bounded, we obtain the same conclusion by a similar truncation technique as in (iii).

The next result concerns the pull-out properties.

**Proposition 3.19** The conditional expectation $\hat{E}_{\tau_1}$ satisfies:

(i) If $X \in L_G^{1,\tau_{+}}(\Omega)$ and $\eta, Y \in L_G^{1,\tau_{+}}(\Omega) \cap L^0(\mathcal{F}_{\tau_1})$ such that $\eta$ is bounded, then $\hat{E}_{\tau_1}[\eta X + Y] = \eta^+ \hat{E}_{\tau_1}[X] + \eta^- \hat{E}_{\tau_1}[-X] + Y$;

(ii) If $\eta \in L_G^{1,\tau_{+}}(\Omega; \mathbb{R}^d) \cap L^0(\mathcal{F}_{\tau_1}; \mathbb{R}^d)$, $X \in L_G^{1,\tau_{+}}(\Omega; \mathbb{R}^n)$ and $\varphi \in C_bL^p(\mathbb{R}^{d+n})$, then $\hat{E}_{\tau_1}[\varphi(\eta, X)] = \hat{E}_{\tau_1}[\varphi(\eta, X)]_{p=\eta}.$

In the proof of Proposition 3.19, we shall need the following lemmas. We first study the local property of $\hat{E}_{\tau_1}$.

**Lemma 3.20** Let $X \in L_G^{1,\tau_{+}}(\Omega)$ for two optional times $\tau$ and $\sigma$. Then

$$\hat{E}_{\tau_{+}}[X]I_{\{\tau = \sigma\}} = \hat{E}_{\sigma_{+}}[XI_{\{\tau = \sigma\}}]. \hspace{1cm} (3.22)$$

**Proof.** By Proposition 3.16 (ii),

$$\hat{E}_{(\tau \wedge \sigma_{+})}[XI_{\{\tau \leq \sigma\}}] = \hat{E}_{\sigma_{+}}[XI_{\{\tau \leq \sigma\}}].$$

Multiplying $I_{\{\tau = \sigma\}}$ on both sides, we see from Proposition 3.16 (i) that

$$\hat{E}_{(\tau \wedge \sigma_{+})}[XI_{\{\tau = \sigma\}}] = \hat{E}_{\tau_{+}}[XI_{\{\tau = \sigma\}}]. \hspace{1cm} (3.23)$$

Noting that $XI_{\{\tau = \sigma\}} \in L_G^{1,\sigma_{+}}(\Omega)$, we can apply a similar argument to $\hat{X} = XI_{\{\tau = \sigma\}}, \hat{\sigma} = \tau, \hat{\tau} = \sigma$ to obtain

$$\hat{E}_{(\tau \wedge \sigma_{+})}[XI_{\{\tau = \sigma\}}] = \hat{E}_{\sigma_{+}}[XI_{\{\tau = \sigma\}}].$$

Combining this with (3.23), we obtain the lemma.
Lemma 3.21 Let \( X \in L_{G}^{1,\tau^+}(\Omega) \) for a simple optional time \( \tau \) taking values in \( \{t_i : i \geq 1\} \). Then

\[
\hat{\mathbb{E}}_{\tau^+}[X] = \sum_{i=1}^{\infty} \hat{\mathbb{E}}_{t_i^+}[X_{(\tau=t_i)}].
\]

Proof. Note that \( \{\tau = t_i\} \in \mathcal{F}_{\tau^+} \). Applying Lemma 3.20, we have

\[
\hat{\mathbb{E}}_{\tau^+}[X] = \sum_{i=1}^{\infty} \hat{\mathbb{E}}_{t_i^+}[X_{(\tau=t_i)}] = \sum_{i=1}^{\infty} \hat{\mathbb{E}}_{t_i^+}[X_{(\tau=t_i)}].
\]

The following deterministic-time version of Proposition 3.19 is also needed.

Lemma 3.22 For each \( t \geq 0 \), the conditional expectation \( \hat{\mathbb{E}}_t \) satisfies the following properties:

(i) If \( X \in L_{G}^{1,t}(\Omega) \) and \( \eta, Y \in L_{G}^{1,\ell}(\Omega) \cap L^0(\mathcal{F}_t) \) such that \( \eta \) is bounded, then \( \hat{\mathbb{E}}_t[\eta X + Y] = \eta^+ \hat{\mathbb{E}}_t[X] + \eta^- \hat{\mathbb{E}}_t[-X] + Y \);

(ii) If \( \eta \in L_{G}^{1,\ell}(\Omega; \mathbb{R}^d) \cap L^0(\mathcal{F}_t; \mathbb{R}^d), X \in L_{G}^{1,\ell}(\Omega; \mathbb{R}^n) \), then \( \hat{\mathbb{E}}_t[\varphi(\eta, X)] = \hat{\mathbb{E}}_t[\varphi(\rho, X)]_{\rho = \eta}, \) for each \( \varphi \in C_{b,Lip}(\mathbb{R}^{d+n}) \).

Proof. We just prove (i). Statement (ii) can be proved similarly.

Step 1. We first assume

\[
\eta = \sum_{i=1}^{n} \eta_i I_{A_i}, \quad Y = \sum_{i=1}^{n} Y_i I_{A_i}, \quad X = \sum_{i=1}^{n} X_i I_{A_i},
\]

where \( \eta_i, Y_i \in L_{G}^{1}(\Omega), X_i \in L_{G}^{1}(\Omega) \) such that \( \eta_i \) is bounded and \( \{A_i\}_{i=1}^{n} \) is an \( \mathcal{F}_t \)-partition of \( \Omega \). By the definition of \( \hat{\mathbb{E}}_t \) on \( L_{G}^{0,1,t}(\Omega) \) (see Remark 3.15) and properties (ii), (iv) of Proposition 2.6, we have

\[
\hat{\mathbb{E}}_t[\eta X + Y] = \hat{\mathbb{E}}_t[\sum_{i=1}^{n} (\eta_i X_i + Y_i) I_{A_i}]
= \sum_{i=1}^{n} \hat{\mathbb{E}}_t[\eta_i X_i + Y_i] I_{A_i}
= \sum_{i=1}^{n} (\eta_i^+ \hat{\mathbb{E}}_t[X_i] + \eta_i^- \hat{\mathbb{E}}_t[-X_i] + Y_i) I_{A_i}
= \eta^+ \hat{\mathbb{E}}_t[X] + \eta^- \hat{\mathbb{E}}_t[-X] + Y.
\]

Step 2. Now we consider the general case. We take a sequence \( \{X_n\}_{n=1}^{\infty} \subset L_{G}^{0,1,t}(\Omega) \) such that

\[
X_n \to X \quad \text{in} \quad L_1.
\]

Moreover, we define

\[
\eta_n := \sum_{-2^n}^{2^n} \frac{k C_{n}}{2^n} I_{\left( \frac{k C_{n}}{2^n} \leq \eta < \frac{(k+1) C_{n}}{2^n} \right)}, \quad (3.24)
\]

and

\[
Y_n := \sum_{-n2^n}^{n2^n - 1} \frac{k}{2^n} I_{\left( \frac{k}{2^n} \leq Y < \frac{k+1}{2^n} \right)} + nI_{\{Y \geq n\}} - nI_{\{Y < -n\}}. \quad (3.25)
\]
Then

$$|\eta_n - \eta| \leq \frac{C_n}{2^n} \quad \text{and} \quad Y_n \to Y \quad \text{in} \ L^1, \ \text{as} \ n \to \infty$$

since

$$\hat{E}[|Y_n - Y|] \leq \hat{E}[|Y_n - Y|I_{\{-n \leq Y < n\}}] + \hat{E}[|Y_n - Y|I_{\{|Y| \geq n\}}] \leq \frac{1}{2^n} + \hat{E}[|Y|I_{\{|Y| \geq n\}}] \to 0, \quad \text{as} \ n \to \infty$$

because of Remark 3.1. Applying Step 1, we have

$$\hat{E}_t[\eta_n X_n + Y_n] = \eta_n^+ \hat{E}_t[X_n] + \eta_n^- \hat{E}_t[-X_n] + Y_n. \quad (3.26)$$

We note that

$$\hat{E}[|\eta_n X_n + Y_n - \eta X - Y|] \leq \hat{E}[|\eta_n X_n - \eta_n X|] + \hat{E}[|X||\eta_n - \eta|] + \hat{E}[|Y_n - Y|]$$

$$\leq C_n \hat{E}[|X_n - X|] + \frac{C_n}{2^n} \hat{E}[|X|] + \hat{E}[|Y_n - Y|]$$

$$\to 0, \quad \text{as} \ n \to \infty$$

and similarly,

$$\hat{E}[|\eta_n^+ \hat{E}_t[X_n] + \eta_n^- \hat{E}_t[-X_n] + Y_n - (\eta^+ \hat{E}_t[X] + \eta^- \hat{E}_t[-X] + Y)|] \to 0, \quad \text{as} \ n \to \infty.$$

Thus the left-hand side (right-hand side resp.) of (3.26) converges to the left-hand side (right-hand side resp.) of

$$\hat{E}_t[\eta X + Y] = \eta^+ \hat{E}_t[X] + \eta^- \hat{E}_t[-X] + Y,$$

which completes the proof. ■

**Proof of Proposition 3.19.** We define $\tau_n$ as (3.10). Since $\mathcal{F}_{\tau+} \subset \mathcal{F}_{\tau_n}$, we have $L^{1,\tau+}(\Omega) \subset L^{1,\tau_n}(\Omega)$.

Thus for any $Z \in L^{1,\tau_n}(\Omega)$, we have $ZI_{\{\tau_n = t_n\}} \in L^{1,t_n}(\Omega)$, and hence $\hat{E}_{t_n^+}[ZI_{\{\tau_n = t_n\}}] = \hat{E}_{t_n^+}[ZI_{\{\tau_n = t_n\}}]$ according to Remark 3.15 (ii). Then by Proposition 3.16 (iv) and Lemma 3.21,

$$\hat{E}_{t_n^+}[\eta X + Y] = \lim_{n \to \infty} \hat{E}_{\tau_n^+}[\eta X + Y]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{\infty} \hat{E}_{t_n^+}[(\eta X + Y)I_{\{\tau_n = t_n\}}]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{\infty} \hat{E}_{t_n^+}[(\eta X + Y)I_{\{\tau_n = t_n\}}]$$

$$= \lim_{n \to \infty} \sum_{i=1}^{\infty} \hat{E}_{t_n^+}[(\eta I_{\{\tau_n = t_n\}})XI_{\{\tau_n = t_n\}} + YI_{\{\tau_n = t_n\}}].$$

By Lemma 3.22 (i), we note that

$$\hat{E}_{t_n^+}[(\eta I_{\{\tau_n = t_n\}})XI_{\{\tau_n = t_n\}} + YI_{\{\tau_n = t_n\}}]$$

$$= \eta^+ I_{\{\tau_n = t_n\}} \hat{E}_{t_n^+}[XI_{\{\tau_n = t_n\}}] + \eta^- I_{\{\tau_n = t_n\}} \hat{E}_{t_n^+}[-XI_{\{\tau_n = t_n\}}] + YI_{\{\tau_n = t_n\}}$$

$$= \eta^+ \hat{E}_{t_n^+}[XI_{\{\tau_n = t_n\}}] + \eta^- \hat{E}_{t_n^+}[-XI_{\{\tau_n = t_n\}}] + YI_{\{\tau_n = t_n\}}.$$
We thus have
\[
\hat{E}_\tau[\eta X + Y] = \mathbb{L}^{1,-} \lim_{n \to \infty} (\eta^+ \hat{E}_{\tau_n} + [X] + \eta^- \hat{E}_{\tau_n} + [-X]) + Y \\
= \eta^+ \hat{E}_\tau + [X] + \eta^- \hat{E}_\tau + [-X] + Y.
\]

The property (ii) is proved similarly. ■

3.3 Extension from below

For a sequence \( \{X_n\}_{n=1}^\infty \) in \( L^{1,\tau+}_G(\Omega) \) such that \( X_n \uparrow X \) q.s., we can not expect \( X \in L^{1,\tau+}_G(\Omega) \) (e.g., \( X_n := n, n \geq 1 \)). So it is necessary to introduce the extension of \( \hat{E}_{\tau+} \) from below as follows to guarantee the upward monotone convergence.

Let \( \tau \) be a given optional time and recall the convention (H3). We set
\[
L^{1,\tau+,\ast}_G(\Omega) := \{ \eta \in L_0^0(F) : \text{there exists } X_n \in L^{1,\tau+}_G(\Omega) \text{ such that } X_n \uparrow X \text{ q.s.} \}
\]

For \( X \in L^{1,\tau+,\ast}_G(\Omega) \), let \( \{X_n\}_{n=1}^\infty \subset L^{1,\tau+}_G(\Omega) \) such that \( X_n \uparrow X \) q.s. We define
\[
\hat{E}_\tau[X] := \lim_{n \to \infty} \hat{E}_\tau[X_n].
\]

**Proposition 3.23** The conditional expectation \( \hat{E}_{\tau+} : L^{1,\tau+,\ast}_G(\Omega) \to L^{0,F_{\tau+}}_G(\Omega) \) is well-defined and satisfies: for \( X,Y \in L^{1,\tau+,\ast}_G(\Omega) \),

(i) \( \hat{E}_{\tau+}[X] \leq \hat{E}_{\tau+}[Y] \), for \( X \leq Y \);

(ii) \( \hat{E}_{\tau+}[X + Y] \leq \hat{E}_{\tau+}[Y] + \hat{E}_{\tau+}[Y] \);

(iii) \( \hat{E}[\hat{E}_{\tau+}[X]] = \hat{E}[X] \).

We need the following lemmas for the proof of the above proposition.

**Lemma 3.24** Let \( X_n, X \in L^{1,\tau+}_G(\Omega) \) such that \( X_n \uparrow X \) q.s. Then \( \hat{E}_{\tau+}[X_n] \uparrow \hat{E}_{\tau+}[X] \) q.s.

**Proof.** Since \( X_n \leq X \) implies \( \hat{E}_{\tau+}[X_n] \leq \hat{E}_{\tau+}[X] \) by Proposition 3.13 (i), we have
\[
\lim_{n \to \infty} \hat{E}_{\tau+}[X_n] \leq \hat{E}_{\tau+}[X].
\]

Then it suffices to prove the reverse inequality. Assume on the contrary that \( \eta := \lim_{n \to \infty} \hat{E}_{\tau+}[X_n] \geq \hat{E}_{\tau+}[X] \) q.s. does not hold, i.e.,
\[
c(\{\eta < \hat{E}_{\tau+}[X]\}) > 0.
\]

Since
\[
D_k := \{\eta + \frac{1}{k} \leq \hat{E}_{\tau+}[X]\} \cap \{|\eta| \leq k\} \uparrow \{\eta < \hat{E}_{\tau+}[X]\},
\]
we can take \( k \) large enough such that, by Lemma 2.9,
\[
c(D_k) > 0.
\]
Then by Lemma 2.10, Proposition 3.13 (iii), Proposition 3.16 (i) and Proposition 3.19 (i), we have

\[ \tilde{E}[(X + k)I_{D_k}] = \lim_{n \to \infty} \tilde{E}[X_n + k] \geq \tilde{E}[(\tilde{E}_\tau[X] + k)I_{D_k}] = \tilde{E}[(\eta + k)I_{D_k}]. \]

But

\[ \tilde{E}[(X + k)I_{D_k}] = \tilde{E}[(E_\tau[X] + k)I_{D_k}] \geq \tilde{E}[(\eta + \frac{1}{k} + k)I_{D_k}], \]

which is a contradiction by Proposition 29 in [10].

**Proof of Proposition 3.23.** Let \( X \in L^{1,r+\sigma}_G(\Omega) \). For any \( X_n \in L^{1,r+\sigma}_G(\Omega) \) such that \( X_n \uparrow X \) q.s., obviously \( \lim_{n \to \infty} \tilde{E}_\tau[|X_n|] \) exists. We now show that if moreover there is another sequence \( \tilde{X}_n \in L^{1,r+\sigma}_G(\Omega) \) such that \( \tilde{X}_n \uparrow X \) q.s., it holds

\[ \lim_{n \to \infty} \tilde{E}_\tau[X_n] = \lim_{n \to \infty} \tilde{E}_\tau[\tilde{X}_n] \quad q.s. \]

Noting that \( X_n \wedge \tilde{X}_m \uparrow X_n \), as \( m \to \infty \), by Lemma 3.24, we have

\[ \tilde{E}_\tau[X_n] = \lim_{m \to \infty} \tilde{E}_\tau[X_n \wedge \tilde{X}_m] \leq \lim_{m \to \infty} \tilde{E}_\tau[\tilde{X}_m]. \]

This follows

\[ \lim_{n \to \infty} \tilde{E}_\tau[X_n] \leq \lim_{n \to \infty} \tilde{E}_\tau[\tilde{X}_n] \]

Exchanging \( X_n, \tilde{X}_n \), we get the reverse

\[ \lim_{n \to \infty} \tilde{E}_\tau[X_n] \geq \lim_{n \to \infty} \tilde{E}_\tau[\tilde{X}_n]. \]

Thus

\[ \lim_{n \to \infty} \tilde{E}_\tau[X_n] = \lim_{n \to \infty} \tilde{E}_\tau[\tilde{X}_n]. \]

Therefore, \( \tilde{E}_\tau \) is well-defined.

Given the definition of \( \tilde{E}_\tau \) on \( L^{1,r+\sigma}_G(\Omega) \) and Proposition 3.13, the proof for properties (i), (ii), (iii) is straightforward. We shall just omit it.

**Proposition 3.25** The conditional expectation \( \tilde{E}_\tau \) on \( L^{1,r+\sigma}_G(\Omega) \) satisfies the following properties:

(i) If \( X_i \in L^{1,r+\sigma}_G(\Omega), i = 1, \ldots, n \) and \( \{A_i\}_{i=1}^n \) is an \( \mathcal{F}_\tau \)-partition of \( \Omega \), then \( \tilde{E}_\tau[\sum_{i=1}^n X_i I_{A_i}] = \sum_{i=1}^n \tilde{E}_\tau[X_i] I_{A_i}; \)

(ii) If \( \tau, \sigma \) are two optional times and \( X \in L^{1,r+\sigma}_G(\Omega) \), then \( \tilde{E}_\tau[X I_{\{\tau \leq \sigma\}}] = \tilde{E}_\tau[X \wedge \sigma] I_{\{\tau \leq \sigma\}}; \)

(iii) If \( X \in L^{1,r+\sigma}_G(\Omega) \) and \( \eta, Y \in L^{1,r+\sigma}_G(\Omega) \cap L^0(\mathcal{F}_\tau) \) such that \( \eta, X \) is nonnegative, then \( \tilde{E}_\tau[\eta X + Y] = \eta \tilde{E}_\tau[X] + Y; \)

(iv) If \( X_n \in L^{1,r+\sigma}_G(\Omega) \) such that \( X_n \uparrow X \) q.s., then \( X \in L^{1,r+\sigma}_G(\Omega) \) and \( \tilde{E}_\tau[X_n] \uparrow \tilde{E}_\tau[X] \) q.s.

**Proof.** Statements (i), (ii), (iii) follow directly from Proposition 3.16 (i), (ii), Proposition 3.19 (i) and the definition of \( \tilde{E}_\tau \) on \( L^{1,r+\sigma}_G(\Omega) \).

(iv) By Proposition 3.23 (i), we have

\[ \tilde{E}_\tau[X] \geq \lim_{n \to \infty} \tilde{E}_\tau[X_n]. \]
To prove the reverse inequality, for each \( X_n \), we take a sequence \( X_m \in L_G^{1,\tau^+}(\Omega) \) such that \( X_n \uparrow X_m \), as \( m \to \infty \). We define \( \tilde{X}_m := \vee_{n=1}^m X_n \in L_G^{1,\tau^+}(\Omega) \). Then

\[
\tilde{X}_m \leq \vee_{n=1}^m X_n = X_m \quad \text{and} \quad \tilde{X}_m \uparrow X, \quad \text{as } m \to \infty.
\]

It follows from the definition of \( \tilde{\mathbb{E}}_{\tau^+} \) on \( L_G^{1,\tau^+}(\Omega) \) that

\[
\tilde{\mathbb{E}}_{\tau^+}[X] = \lim_{m \to \infty} \tilde{\mathbb{E}}_{\tau^+}[\tilde{X}_m] \leq \lim_{m \to \infty} \tilde{\mathbb{E}}_{\tau^+}[X_m],
\]

as desired. ■

**Remark 3.26** Let \( \tau \) be a stopping time satisfying \((H3)\). We define \( L_G^{1,\tau^+}(\Omega) \) as \( L_G^{1,\tau^+}(\Omega) \) with \( \mathcal{F}_\tau \) replacing \( \mathcal{F}_{\tau^+} \). We can similarly extend \( \tilde{\mathbb{E}}_{\tau} \) from below to \( L_G^{1,\tau^+}(\Omega) \) and similar properties also hold for \( \tilde{\mathbb{E}}_{\tau} \) on \( L_G^{1,\tau^+}(\Omega) \). Moreover,

\[
\tilde{\mathbb{E}}_{\tau^+}[X] = \tilde{\mathbb{E}}_{\tau}[X], \quad \text{for } X \in L_G^{1,\tau^+}(\Omega).
\]

### 3.4 The reflection principle for G-Brownian motion

As an application, we give the following reflection principle for \( G \)-Brownian motion.

**Theorem 3.27** Let \( \tau \) be an optional time (without the assumption that \( \tau \) satisfies \((H3)\)). Then

\[
\tilde{B}_t := 2B_{t \wedge \tau} - B_t = B_t - (B_t - B_\tau)I_{\{t \geq \tau\}}, \quad \text{for } t \geq 0,
\]

is still a \( G \)-Brownian motion.

**Proof.** It suffices to prove that the two processes have the same finite-dimensional distributions, i.e., for any \( 0 \leq t_1 < t_2 < \cdots < t_n \leq T < \infty \), we have

\[
(\tilde{B}_{t_1}, \tilde{B}_{t_2} - \tilde{B}_{t_1}, \cdots, \tilde{B}_{t_n} - \tilde{B}_{t_{n-1}}) \overset{d}{=} (B_{t_1}, B_{t_2} - B_{t_1}, \cdots, B_{t_n} - B_{t_{n-1}}).
\]  

(3.27)

Moreover, by replacing \( \tau \) with \( \tau \wedge T \) we may assume without loss of generality that \( \tau \leq T \).

Suppose first that \( \tau \) is a stopping time taking finitely many values. We may assume that \( \tau \) also takes values in \( \{t_i : i \leq n\} \) since we can refine the partition in (3.27). Then by the version of Lemma 3.21 for \( \tilde{\mathbb{E}}_{\tau} \), we have

\[
\tilde{\mathbb{E}}_{\tau}[\varphi(\tilde{B}_{t_1}, \tilde{B}_{t_2} - \tilde{B}_{t_1}, \cdots, \tilde{B}_{t_n} - \tilde{B}_{t_{n-1}})]
\]

\[
= \tilde{\mathbb{E}}_{\tau}[\varphi(2B_{t_1 \wedge \tau} - B_{t_1}, \cdots, 2B_{t_n \wedge \tau} - B_{t_n} - (2B_{t_{n-1} \wedge \tau} - B_{t_{n-1}}))]
\]

\[
= \sum_{i=1}^n \tilde{\mathbb{E}}_{t_i}[\varphi(2B_{t_1 \wedge t_i} - B_{t_1}, \cdots, 2B_{t_n \wedge t_i} - B_{t_n} - (2B_{t_{n-1} \wedge t_i} - B_{t_{n-1}}))I_{\{\tau = t_i\}}]
\]

\[
= \sum_{i=1}^n \tilde{\mathbb{E}}_{t_i}[\varphi(2B_{t_1 \wedge t_i} - B_{t_1}, \cdots, 2B_{t_n \wedge t_i} - B_{t_n} - (2B_{t_{n-1} \wedge t_i} - B_{t_{n-1}}))I_{\{\tau = t_i\}}]
\]

Note that, for \( k \leq i \),

\[
2B_{t_k \wedge t_i} - B_{t_k} - (2B_{t_{k-1} \wedge t_i} - B_{t_{k-1}}) = B_{t_k} - B_{t_{k-1}},
\]

25
and for $k > i$,

$$2B_{t_k \wedge t_i} - B_{t_k} - (2B_{t_{k-1} \wedge t_i} - B_{t_{k-1}}) = -(B_{t_k} - B_{t_{k-1}}) \leq B_{t_k} - B_{t_{k-1}}$$

because of the symmetry of $G$-Brownian motion. We see from the definition of conditional expectation $\hat{E}_{t_i}$ on $Lip(\Omega)$ that

$$\hat{E}_{t_i}[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})] = \hat{E}_{t_i}[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})] = \hat{E}_{t_i}[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})].$$

Therefore,

$$\hat{E}_t[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})] = \sum_{i=1}^{n} \hat{E}_{t_i}[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})]I_{\{t_i \leq t\}} = \hat{E}_t[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})].$$

Taking expectation $\hat{E}$ on both sides, we have

$$\hat{E}[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})] = \hat{E}[(2B_{t_{i+1}} - B_{t_i}) - (2B_{t_{i-1}} - B_{t_{i-1}})].$$

Turning to the general optional time $\tau \leq T$, we take a sequence of stopping times $\tau_k \leq T + 1$ with finitely many values such that $0 \leq \tau_k - \tau \leq \frac{1}{k} \downarrow 0$. Then

$$\hat{E}[(2B_{t_{i+1} \wedge \tau_k} - B_{t_i}, \cdots, 2B_{t_{n+1} \wedge \tau_k} - B_{t_n} - (2B_{t_{n-1} \wedge \tau_k} - B_{t_{n-1}})] = \hat{E}[(2B_{t_{i+1}} - B_{t_i}, \cdots, B_{t_n} - B_{t_{n-1}})]. \quad (3.28)$$

By a similar analysis as in the first paragraph in the proof of Lemma 3.3, we have for some constant $C$ depending on $\varphi$

$$\hat{E}[(2B_{t_{i+1} \wedge \tau_k} - B_{t_i}, \cdots, 2B_{t_{n+1} \wedge \tau_k} - B_{t_n} - (2B_{t_{n-1} \wedge \tau_k} - B_{t_{n-1}})] - \varphi(2B_{t_{i+1} \wedge \tau} - B_{t_i}, \cdots, 2B_{t_{n+1} \wedge \tau} - B_{t_n} - (2B_{t_{n-1} \wedge \tau} - B_{t_{n-1}})]] \leq C\hat{E}[(B_{u_2} - B_{u_1} \wedge \tau) \downarrow 0, \text{ as } k \to \infty].$$

Thus (3.27) follows from letting $k \to \infty$ in (3.28). □

## 4 Strong Markov Property for G-SDEs

With the notion of conditional expectation $\hat{E}_{t+}$ in hand, we now turn our attention to the strong Markov property for G-SDEs. We first state the Markov property for G-SDEs.

**Lemma 4.1** For $\varphi \in C_b(Lip(\mathbb{R}^{m \times n}), \ 0 \leq t_1 < t_2 < \cdots < t_m < \infty \text{ and } t \geq 0$, we have

$$\hat{E}_t[\varphi(X_{t_1}^x, X_{t_2}^x, \cdots, X_{t_m}^x)] = \hat{E}[(\varphi(X_{t_1}^y, X_{t_2}^y, \cdots, X_{t_m}^y)|_{y=X_t^x}].$$

26
Proof. Since \((B_{t+s} - B_t)_{s \geq 0}\) is still a \(G\)-Brownian motion and the coefficients \(b, h_{ij}, \sigma_j\) in G-SDE (2.2) are independent of the time variable, we have, for any \(s \geq 0\), \(y \in \mathbb{R}^n\),

\[
X^{t,y}_{t+s} = X^{y}_s.
\]

This implies, for \(\tilde{\varphi} \in C_{b,Lip}(\mathbb{R}^n)\),

\[
\mathbb{E}[\tilde{\varphi}(X^{t,y}_{t+s})]_{y=X^\xi_t} = \mathbb{E}[\tilde{\varphi}(X^y_s)]_{y=X^\xi_t}.
\]

Hence by Lemma 2.12,

\[
\mathbb{E}_t[\tilde{\varphi}(X^x_s)] = \mathbb{E}[\tilde{\varphi}(X^y_s)]_{y=X^\xi_t}.
\] (4.1)

For \(\varphi \in C_{b,Lip}(\mathbb{R}^{m \times n})\), by Proposition 2.6 (vi) and (v), we have

\[
\mathbb{E}_t[\varphi(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}})] = \mathbb{E}_t[\mathbb{E}_{t+t_{m-1}}[\varphi(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}})]]
\]

where

\[
\mathbb{E}_{t+t_{m-1}}(y_1, \cdots, y_{m-1}) := \mathbb{E}_{t+t_{m-1}}[\varphi(y_1, y_2, \cdots, y_{m-1}, X^{x}_{t+t_{m-1}})], \quad (y_1, \cdots, y_{m-1}) \in \mathbb{R}^{(m-1) \times n}.
\]

We note that

\[
\mathbb{E}_{t+t_{m-1}}(y_1, \cdots, y_{m-1}) = \mathbb{E}[\varphi(y_1, y_2, \cdots, y_{m-1}, X^{x}_{t+t_{m-1}})]_{y_{m-1}=X^{x}_{t+t_{m-1}}}
\]

by (4.1). Then

\[
\mathbb{E}_{t+t_{m-1}}(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}}) = \varphi_{m-1}(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}}),
\]

where

\[
\varphi_{m-1}(y_1, \cdots, y_{m-1}) := \mathbb{E}[\varphi(y_1, y_2, \cdots, y_{m-1}, X^{y_{m-1}}_{t+t_{m-1}})], \quad (y_1, \cdots, y_{m-1}) \in \mathbb{R}^{(m-1) \times n}.
\]

Thus we have

\[
\mathbb{E}_t[\varphi(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}})] = \mathbb{E}_t[\varphi_{m-1}(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}})].
\]

Repeating this procedure, we get

\[
\mathbb{E}_t[\varphi(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_i})] = \mathbb{E}_t[\varphi_{m-i}(X^{x}_{t+t_1}, X^{x}_{t+t_2}, \cdots, X^{x}_{t+t_{m-1}})]
\]

\[
= \mathbb{E}_t[\varphi(X^{x}_{t+t_1})]
\]

\[
= \mathbb{E}[\varphi(X^y_s)]_{y=X^\xi_t},
\]

where

\[
\varphi_{m-i}(y_1, \cdots, y_{m-i}) := \mathbb{E}[\varphi_{m-(i-1)}(y_1, y_2, \cdots, y_{m-i}, X^{y_{m-i}}_{t+(m-i-1)t_{m-1}})], \quad 1 \leq i \leq m-1.
\]

27
Taking $t = 0$, $x = y$ in (4.2), we obtain
\[ \hat{E}[\varphi(X_{t_1}^y, X_{t_2}^y, \ldots, X_{t_m}^y)] = \hat{E}[\varphi_1(X_{t_1}^y)], \quad \text{for any } y \in \mathbb{R}^n. \] (4.3)

This, combining with (4.2), proves the corollary.

We now give the strong Markov property for $G$-SDEs. It generalizes the well-known strong Markov property for classical SDEs to $G$-SDEs in the framework of nonlinear $G$-expectation. We set $\Omega' := C([0, \infty); \mathbb{R}^n)$ with the distance $\rho_n$ and denote by $B'$ the corresponding canonical process. Recall that we always assume that the optional time $\tau$ satisfies (H3).

**Theorem 4.2** Let $(X^0_t)_{t \geq 0}$ be the solution of $G$-SDE (2.2) satisfying (H1), (H2) and $\tau$ be an optional time. Then for each $\varphi \in C_{\text{b,Lip}}(\mathbb{R}^{m \times n})$ and $0 \leq t_1 \leq \cdots \leq t_m =: T' < \infty$, we have
\[ \hat{E}_{\tau +}[\varphi(X_{t_1}^x, \ldots, X_{t_m}^x)] = \hat{E}[\varphi(X_{t_1}^y, \ldots, X_{t_m}^y)]_{y = X^\tau}. \] (4.4)

We first need the following lemma to justify that the conditional expectation on the left-hand side of (4.4) is meaningful. We denote the paths for a process $Y$ by $Y := (Y_t)_{t \geq 0}$.

**Lemma 4.3** We have
\[ \varphi(X_{\tau + t_1}^x, \ldots, X_{\tau + t_m}^x) \in L^1_G(\Omega). \] (4.5)

**Proof.** Step 1. First assume $\tau \leq T$. Take discrete stopping time $\tau_n \leq T + 1$ as (3.10). By the definition of $L^0_G(\Omega)$, we have
\[ \varphi(X_{\tau_n + t_1}^x, \ldots, X_{\tau_n + t_m}^x) \in L^0_G(\Omega). \]

Then it suffices to show that
\[ \hat{E}[\varphi(X_{\tau_n + t_1}^x, \ldots, X_{\tau_n + t_m}^x) - \varphi(X_{\tau + t_1}^x, \ldots, X_{\tau + t_m}^x)] \to 0, \quad \text{as } n \to \infty. \] (4.6)

Consider now the mapping $\Omega \xrightarrow{X^\tau} \Omega'$. By Lemma 2.11 (2.4), for each $T_1 \geq 0$, there exists a constant $C_{T_1}$ (depending on $T_1$) such that for each $t, s \leq T_1$,
\[ E_P[|X^x_t - X^y_t|^4] \leq \hat{E}[|X^x_t - X^y_T|^4] \leq C_{T_1}|t - s|^2, \quad \text{for each } P \in \mathcal{P}. \]

Then we can apply the well-known Kolmogorov’s moment criterion for tightness (see, e.g., Problem 2.4.11 in [13]) to conclude that the induced probability family \{ $P \circ (X^x)^{-1}$ : $P \in \mathcal{P}$ \} is tight on $\Omega'$. We denote the induced capacity by $c^x_2 := \sup_{P \in \mathcal{P}} P \circ (X^x)^{-1}$ and the induced sublinear expectation by $\hat{E}^x_2 := \sup_{P \in \mathcal{P}} E_{P_0(X^x)^{-1}}$. Then
\[ \hat{E}[|\varphi(X_{\tau_n + t_1}^x, \ldots, X_{\tau_n + t_m}^x) - \varphi(X_{\tau + t_1}^x, \ldots, X_{\tau + t_m}^x)|] \]
\[ \leq \hat{E} \left[ \sup_{s, s' \in \Lambda_{-n, T_1+1}} |\varphi(X_{s + t_1}^x, \ldots, X_{s + t_m}^x) - \varphi(X_{s' + t_1}^x, \ldots, X_{s' + t_m}^x)| \right] \]
\[ = \hat{E}^x_2 \left[ \sup_{s, s' \in \Lambda_{-n, T_1+1}} |\varphi(B_{s + t_1}^x, \ldots, B_{s + t_m}^x) - \varphi(B_{s' + t_1}^x, \ldots, B_{s' + t_m}^x)| \right]. \]

Proceeding similarly to the first paragraph in proof of Lemma 3.3, we obtain for some constant $C$ depending on $\varphi$
\[ \hat{E}^x_2 \left[ \sup_{s, s' \in \Lambda_{-n, T_1+1}} |\varphi(B_{s' + t_1}^x, \ldots, B_{s' + t_m}^x) - \varphi(B_{s + t_1}^x, \ldots, B_{s + t_m}^x)| \right] \leq C \hat{E}^x_2 [\sup_{s, s' \in \Lambda_{-n, T_1+1+T'}} (|B_{s}^x - B_{s'}^x| \wedge 1)], \]

28
which converges to 0 as \( n \to \infty \) by Remark 3.6.

**Step 2.** For the general case, by Step 1, we have

\[
\varphi(X_{T+\tau+t_1}, \ldots, X_{T+\tau+t_m}) \in L_G^{1,(\tau+T)}(\Omega) \subset L_G^{1,\tau+}(\Omega).
\]

Note that

\[
\hat{E}[\varphi(X_{T+\tau+t_1}, \ldots, X_{T+\tau+t_m}) - \varphi(X_{\tau+t_1}, \ldots, X_{\tau+t_m})] \leq 2C_\varphi \epsilon(\{\tau > T\}) \to 0, \quad \text{as } T \to \infty.
\]

The result now follows. \( \blacksquare \)

**Proof of Theorem 4.2.** Let \( \tau \leq T \). We define \( \tau_n \) as (3.10). Then \( \tau_n \leq T + 1 \) takes finitely values \( \{t^y_i : i \leq d_n\} \) with \( d_n := \lfloor 2^n T \rfloor + 1 \). By (4.6) and Proposition 3.16 (iv), we have

\[
\hat{E}[\hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] - \hat{E}_{\tau_n}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] - \hat{E}_{\tau_n}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] + \hat{E}_{\tau_n}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] - \hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] - \hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] + \hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] - \hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] \to 0, \quad \text{as } n \to \infty.
\]

Moreover, since \( \varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m}) \in L_G^{1,\tau_n}(\Omega) \), by Remark 3.15, we have

\[
\hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] = \hat{E}_{\tau_n}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] = \hat{E}_{\tau_n}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})].
\]

Combining these with the version of Lemma 3.21 for \( \hat{E}_{\tau_n} \), we have

\[
\hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] = \lim_{n \to \infty} \hat{E}_{\tau_n}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] = \lim_{n \to \infty} \sum_{i=1}^{d_n} \widetilde{E}_{t^y_i}[\varphi(X_{t^y_i+t_1}, \ldots, X_{t^y_i+t_m})] I\{\tau_n = t^y_i\}.
\]

Note that from Lemma 4.1

\[
\hat{E}_{t^y_i}[\varphi(X_{t^y_i+t_1}, \ldots, X_{t^y_i+t_m})] = \hat{E}[\varphi(X^y_{t_1}, \ldots, X^y_{t_m})]_{y=X^y_{t^y_i}}.
\]

We thus obtain

\[
\hat{E}_{\tau_n+}[\varphi(X_{\tau_n+t_1}, \ldots, X_{\tau_n+t_m})] = \lim_{n \to \infty} \sum_{i=1}^{d_n} \hat{E}[\varphi(X^y_{t_1}, \ldots, X^y_{t_m})]_{y=X^y_{t^y_i}} I\{\tau_n = t^y_i\} = \lim_{n \to \infty} \hat{E}[\varphi(X^y_{t_1}, \ldots, X^y_{t_m})]_{y=X^y_{\tau_n}} = \hat{E}[\varphi(X^y_{t_1}, \ldots, X^y_{t_m})]_{y=X^y_{\tau}},
\]

where the last equality is derived from a proof similar to that of Lemma 4.3 by using (2.4) of Lemma 2.11 for spatial variables.

Now for the general \( \tau \), applying Step 1, we have

\[
\hat{E}_{(\tau+T)+}[\varphi(X_{\tau+T+t_1}, \ldots, X_{\tau+T+t_m})] = \hat{E}[\varphi(X^y_{t_1}, \ldots, X^y_{t_m})]_{y=X^y_{\tau+T}}.
\]
Since \( \varphi(X^x_{\tau+t_1}, \ldots, X^x_{\tau+t_m}) \in L^{1+}_G(\Omega) \) by Lemma 4.3, we can apply Proposition 3.16 (iii) to obtain
\[
\hat{E}[\hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})] - \hat{E}_{t}[\varphi(X^x_{\tau+t_1}, \ldots, X^x_{\tau+t_m})]]
\]
\[
\leq \hat{E}[\hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})] - \hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})]] + \hat{E}[\hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})] - \hat{E}_{t}[\varphi(X^x_{\tau+t_1}, \ldots, X^x_{\tau+t_m})]]
\]
\[
\leq \hat{E}[\hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})] - \hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})]] + \hat{E}[\hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})] - \hat{E}_{t}[\varphi(X^x_{\tau+T}, \ldots, X^x_{\tau+T+t_m})]]
\]
\[
\leq C_\varphi c(\tau > T) + \hat{E}[\hat{E}_{(T\wedge t)}[\varphi(X^x_{\tau+t_1}, \ldots, X^x_{\tau+t_m})] - \hat{E}_{t}[\varphi(X^x_{\tau+t_1}, \ldots, X^x_{\tau+t_m})]]
\]
\[
\to 0, \quad \text{as } T \to \infty.
\]
Thus letting \( T \to \infty \) in (4.7) yields (4.4). □

Next we consider an extension of Theorem 4.2 in which the cylinder functions \( \varphi \) is replaced by (lower semi-) continuous functions \( \tilde{\varphi} \) depending on the whole paths of \( G \)-SDEs. It maybe useful in the following work.

**Theorem 4.4** Let \( \varphi \in C_b(\Omega') \). Then
\[
\hat{E}_{t}[\varphi(X^x_{\tau+})] = \hat{E}[\varphi(X^y_{\tau+})] = \hat{E}[\varphi(X^y)]_{y = X^x}.
\]
(4.8)

The conditional expectation on the left-hand side of (4.8) is meaningful by the following two lemmas.

**Lemma 4.5** Assume \( \varphi \in C_b(\Omega') \) and there exists a constant \( \mu > 0 \) such that for some \( T' > 0 \),
\[
|\varphi(\omega^1)| - |\varphi(\omega^2)| \leq \mu|\omega^1 - \omega^2|_{C^0[0, T']} \quad \text{for each } \omega^1, \omega^2 \in \Omega'.
\]
Then
\[
\varphi(X^x_{\tau+}) \in L^{1+}_G(\Omega).
\]
(4.10)

**Remark 4.6** Note that (4.9) implies that \( \varphi \) only depends on the path of \( \omega \in \Omega' \) on \([0, T']\).

**Proof.** As in the Step 2 of the proof of Lemma 4.3, it suffices to suppose that \( \tau \leq T \) for some \( T > 0 \). Consider for each \( m \in \mathbb{N} \) the function from \( \mathbb{R}^{(m+1)\times n} \) to \( \Omega' \) defined by
\[
\phi_m(x_0, x_1, x_2, \ldots, x_m)(t) = \sum_{k=0}^{m-1} \frac{(t_k^m - t)x_k + (t - t_k^m)x_{k+1}}{t_k^m - t_k} + x_m|_{t_m^\infty},
\]
where \( t_k^m = \frac{kT'}{m}, k = 0, 1, \ldots, m \). Since \( \varphi \circ \phi_m \) is a bounded, Lipschitz function from \( \mathbb{R}^{(m+1)\times n} \) to \( \mathbb{R} \), by Lemma 4.3, we have
\[
\varphi(\phi_m(X^x_{\tau+t_0^m}, X^x_{\tau+t_1^m}, X^x_{\tau+t_2^m}, \ldots, X^x_{\tau+t_m^m})) \in L^{1+}_G(\Omega).
\]
We employ the notation in the proof of Lemma 4.3 and proceed similarly to obtain some constant \( C \geq 0 \).
depending on \( \varphi \) such that
\[
\mathbb{E}[|\varphi(\phi_m(X^x_{\tau+m}, X^x_{\tau+r}, \ldots, X^x_{\tau+T}) - \varphi(X^x_{\tau+T})|] \\
\leq \mathbb{E}[\sup_{0 \leq t \leq T} |\varphi(\phi_m(X^x_{\tau+t}, X^x_{\tau+t}, \ldots, X^x_{\tau+T}) - \varphi(X^x_{\tau+T})|] \\
= \mathbb{E}[\sup_{0 \leq t \leq T} |\varphi(\phi_m(B^x_{t+1}, B^x_{t+1}, \ldots, B^x_{t+T}) - \varphi(B^x_{t+T})|] \\
\leq C\mathbb{E}[\sup_{s,s' \in [0, \tau+T]} (|B^x_s - B^x_{s'}| + 1)] \\
\to 0, \quad \text{as } m \to \infty,
\]
This completes the proof. ■

**Lemma 4.7** Let \( \varphi \in C_b(\Omega') \). Then
\[
\varphi(X^x_{\tau+T}) \in L^{1,\tau+}_G(\Omega).
\]  
(4.11)

**Proof.** Let
\[
\varphi_m(\omega) := \inf_{\omega' \in \Omega} \{ \varphi(\omega') + m||\omega - \omega'||_{C^n[0,m]} \}, \quad \text{for } \omega \in \Omega'.
\]
Then by Lemma 3.1 in Chap VI of \([22]\), \( \varphi_m \in C_b(\Omega') \) satisfies
\begin{enumerate}[\text{(i)}]
\item \( |\varphi_m(\omega^1) - \varphi_m(\omega^2)| \leq m||\omega^1 - \omega^2||_{C^n[0,m]}, \) for \( \omega^1, \omega^2 \in \Omega' \); \n\item \( \varphi_m \uparrow \varphi; \)
\item \( |\varphi_m| \leq C_\varphi. \)
\end{enumerate}
Thus we have \( \varphi_m(\tau^x_{\tau+T}) \in L^{1,\tau+}_G(\Omega) \) by Lemma 4.5.

As discussed in the proof of Lemma 4.5, it suffices to prove the result for \( \tau \leq T \). Let \( \mathbb{E}^x_2 \) and \( c^x_2 \) be defined as in the proof of Lemma 4.3. We have
\[
\mathbb{E}[|\varphi_m(X^x_{\tau+T}) - \varphi(X^x_{\tau+T})|] \leq \mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_m(X^x_{\tau+T}) - \varphi(X^x_{\tau+T})|] \\
= \mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_m(B^x_{t+1}) - \varphi(B^x_{t+1})|].
\]
Given any \( \varepsilon > 0 \), since \( c^x_2 \) is tight on \( \Omega' \), we can pick a compact set \( K \subset \Omega' \) such that \( c^x_2(K) < \varepsilon \). Note that \( K \times [0, T] \) is still compact and \( (\omega, t) \mapsto \varphi_m(B^x_{t+1}), \varphi(B^x_{t+1}) \) are continuous functions such that \( \varphi_m(B^x_{t+1}) \uparrow \varphi(B^x_{t+1}) \). We have by Dini’s theorem
\[
\varphi_m(B^x_{t+1}) \uparrow \varphi(B^x_{t+1}) \quad \text{uniformly on } K \times [0, T].
\]
Hence, we can choose \( m \) large enough such that
\[
|\varphi_m(B^x_{t+1}) - \varphi(B^x_{t+1})| \leq \varepsilon \quad \text{on } K \times [0, T].
\]
Then
\[
\mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_m(B^x_{t+1}) - \varphi(B^x_{t+1})|] \\
\leq \mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_m(B^x_{t+1}) - \varphi(B^x_{t+1})|I_K] + \mathbb{E}[\sup_{0 \leq t \leq T} |\varphi_m(B^x_{t+1}) - \varphi(B^x_{t+1})|I_{K^C}] \\
\leq \varepsilon + 2\varepsilon C_\varphi.
\]
Since ε can be arbitrarily small, we obtain
\[ \hat{E}[|\varphi_m(X^y_{\tau^+}) - \varphi(X^y_{\tau^+})|] \to 0, \quad \text{as } m \to \infty. \]
This proves the lemma. \[ \blacksquare \]

**Proof of Theorem 4.4.** Step 1. Suppose τ ≤ T for some T > 0 and φ ∈ C_b(Ω') such that (4.9) holds for some T'' > 0.

For each m ∈ N, we define φ_m as in the proof of Lemma 4.5. Then Theorem 4.2 gives
\[ \hat{E}_\tau[|\varphi(\phi_m(X^x_T, X^x_{t_1}, X^x_{t_2}, \cdots, X^x_{t_n})]|) = \hat{E}[|\varphi(\phi_m(X^{y_0}_T, X^{y_0}_{t_1}, X^{y_0}_{t_2}, \cdots, X^{y_0}_{t_n})]|)_{y = x^*}. \quad (4.12) \]

According to the proof of Lemma 4.5,
\[ \varphi(\phi_m(X^x_T, X^x_{t_1}, X^x_{t_2}, \cdots, X^x_{t_n})) \to \varphi(X^x_T) \quad \text{in } L^1, \quad \text{as } m \to \infty. \]
Consequently,
\[ \hat{E}_\tau[|\varphi(\phi_m(X^x_T, X^x_{t_1}, X^x_{t_2}, \cdots, X^x_{t_n})]|) \to \hat{E}_\tau[|\varphi(X^x_T)|] \quad \text{in } L^1, \quad \text{as } m \to \infty. \]

It remains to consider the right side of (4.12). For any fixed R > 0, by Kolmogorov’s criterion for tightness, the family \( \mathcal{P}_R := \bigcup_{0 \leq t \leq B_R(0)} \{ P \circ (X^y)^{-1} : P \in \mathcal{P} \} \) is tight on Ω', where \( B_R(0) \) is an open ball with center 0 and radius R in \( \mathbb{R}^n \) and \( B_R(0) \) is its closure. We denote the corresponding sublinear expectation by \( \hat{E}^R := \sup_{P \in \mathcal{P}, y \in \overline{B_R(0)} E_P(\varphi(X^y)^{-1})^R \). We may apply a similar analysis as in the proof of Lemma 4.5 to obtain for some constant C depending on φ
\[ \hat{E}[|\varphi(\phi_m(X^{y_0}_T, X^{y_0}_{t_1}, \cdots, X^{y_0}_{t_n}))|) - \varphi(X^y)|) = \hat{E}[|\varphi(\phi_m(B^{y_0}_T, B^{y_0}_{t_1}, \cdots, B^{y_0}_{t_n}) - \varphi(B')|) \leq \hat{E}^R[|\varphi(\phi_m(B^{y_0}_T, B^{y_0}_{t_1}, \cdots, B^{y_0}_{t_n}) - \varphi(B')|) \leq C \hat{E}^R \sup_{s, s' \in \Lambda_{m-\tau', T}} (|B_s^y - B_{s'}^y|)^1 \]
\[ \to 0, \quad \text{as } m \to \infty, \quad \text{for any } y \in \overline{B_R(0)}. \]

where \( \hat{E}^y := \sup_{P \in \mathcal{P}} E_P(\varphi(X^y)^{-1}). \) That is,
\[ \hat{E}[|\varphi(\phi_m(X^{y_0}_T, X^{y_0}_{t_1}, \cdots, X^{y_0}_{t_n}))|) - \varphi(X^y)|) \to 0, \quad \text{as } m \to \infty, \quad \text{uniformly for } y \in \overline{B_R(0)}. \quad (4.13) \]
For any fixed ε > 0, we can first choose R large enough such that by Lemma 2.11 (2.3)
\[ c(\{ X^y_T > R \}) \leq \frac{\hat{E}[|X^y_T|]}{R} \leq \frac{\hat{E}[\sup_{0 \leq t \leq T} |X^y_T|]}{R} \leq \varepsilon \]
and then choose m large enough such that by (4.13)
\[ \hat{E}[|\varphi(\phi_m(X^{y_0}_T, X^{y_0}_{t_1}, \cdots, X^{y_0}_{t_n}))|) - \varphi(X^y)|) \leq \varepsilon, \quad \text{for all } y \in \overline{B_R(0)}. \]
Thus we have
\[ \hat{E}[|\hat{E}[\varphi(\phi_m(X^{y_0}_T, X^{y_0}_{t_1}, X^{y_0}_{t_2}, \cdots, X^{y_0}_{t_n})])_{y = x^*} - \hat{E}[\varphi(X^y)]_{y = x^*} |) \leq \varepsilon + 2C_\varphi c(\{ |X^y_T| > R \}) \]
\[ \leq \varepsilon + 2C_\varphi \varepsilon, \]
which implies
\[ \hat{E}[\varphi_m(X^n_0, X^n_1, X^n_2, \cdots, X^n_m)]|_{y=X^n_t} = \hat{E}[\varphi(X^n)|_{y=X^n_t}] \rightarrow 0, \quad \text{as } m \rightarrow \infty. \]

Therefore, letting \( m \rightarrow \infty \) in (4.12), we obtain
\[ \mathbb{E}_{\tau^+}[\varphi(X_{t+}^y)] = \hat{E}[\varphi(X^y)|_{y=X^t}]. \]

**Step 2.** Assume \( \tau \leq T \) and \( \varphi \in C_b(\Omega) \). Define \( \varphi_m \) as in the proof of Lemma 4.7. According to Step 1,
\[ \mathbb{E}_{\tau^+}[\varphi_m(X_{t+}^y)] = \hat{E}[\varphi_m(X^y)|_{y=X^t}]. \] (4.14)

Letting \( m \rightarrow \infty \), from the proof of Lemma 4.7, we obtain that
\[ \mathbb{E}_{\tau^+}[\varphi(X_{t+}^y)] = \hat{E}[\varphi(X^y)|_{y=X^t}], \]
where the convergence of right-hand side is obtained by a similar analysis as in Step 1 and the proof of Lemma 4.7.

**Step 3.** We proceed as in the last paragraph of the proof of Theorem 4.2 to obtain the result for the general case that \( \tau \) is an optional time and \( \varphi \in C_b(\Omega) \).

**Corollary 4.8** Let \( \varphi \) be lower semi-continuous on \( \Omega' \) and bounded from below, i.e., \( \varphi \geq c \) for some constant \( c \). Then \( \varphi(X_{t+}^y) \in L^{1,\tau^+,*}_G(\Omega) \) and
\[ \mathbb{E}_{\tau^+}[\varphi(X_{t+}^y)] = \hat{E}[\varphi(X^y)|_{y=X^t}]. \]

**Proof.** We pick a sequence \( \varphi_m \in C_b(\Omega') \) such that \( \varphi_m \uparrow \varphi \). Then the conclusion follows from Theorem 4.4, Lemma 2.10 and Proposition 3.25 (iv).

Assuming \( n = d, \ x = 0, \ b = h_{ij} = 0, \ \sigma := (\sigma_1, \cdots, \sigma_d) = I_{d \times d} \) in Corollary 4.8, we immediately have the strong Markov property for \( G \)-Brownian motion.

**Corollary 4.9** Let \( \varphi \) be lower semi-continuous, bounded from below on \( \Omega \) and \( \tau \) be an optional time. Then
\[ \mathbb{E}_{\tau^+}[\varphi(B_{t+})] = \hat{E}[\varphi(B^y)|_{y=B_t}], \] (4.15)
where \( B^y_t := y + B_t, \ t \geq 0 \) for \( y \in \mathbb{R}^d \). In particular, for each \( \phi \in C_{b,Lip}(\mathbb{R}^{m \times d}) \) and \( 0 \leq t_1 \leq \cdots \leq t_m < \infty \),
\[ \mathbb{E}_{\tau^+}[\phi(B_{t_1}, \cdots, B_{t_m})] = \hat{E}[\phi(B_{t_1}, \cdots, B_{t_m})|_{y=B_t}]. \]

The following result says that \( G \)-Brownian motion starts afresh at an optional time, i.e., \( \overline{B}_t := (B_{t+t} - B_t)_{t \geq 0} \) is still a \( G \)-Brownian motion.

**Corollary 4.10** Let \( \tau, \varphi \) be assumed as in the above Corollary. Then
\[ \mathbb{E}_{\tau^+}[\varphi(B_{t+} - B_t)] = \hat{E}[\varphi(B_{t+} - B_t)] = \hat{E}[\varphi(B)] \] (4.16)
In particular, for each \( \phi \in C_{b,Lip}(\mathbb{R}^{m \times d}) \), \( 0 \leq t_1 \leq \cdots \leq t_m < +\infty \), \( m \in \mathbb{N} \), we have
\[ \mathbb{E}_{\tau^+}[\phi(B_{t_1}, \cdots, B_{t_m} - B_t)] = \hat{E}[\phi(B_{t_1}, \cdots, B_{t_m} - B_t)] = \hat{E}[\phi(B_{t_1}, \cdots, B_{t_m})]. \]
Proof. We only need to prove the first one, which implies the second one as a special case. Setting 
\[ \tilde{\varphi}(\omega) := \varphi((\omega t - \omega_0)_{t \geq 0}) \] in (4.15), we have
\[ \hat{E}_\tau[\varphi(B_{\tau+} - B_\tau)] = \hat{E}[\varphi(B_\tau)]. \]
Taking expectation on both sides, by Proposition 3.13, we then obtain
\[ \hat{E}[\varphi(B_{\tau+} - B_\tau)] = \hat{E}[\varphi(B_\tau)]. \]

5 An application

Let \((B_t)_{t \geq 0}\) be a 1-dimensional \(G\)-Brownian motion such that \(\sigma^2 := -\hat{E}[-B_1^2] > 0\) (non-degeneracy). Let \(a \in \mathbb{R}\) be given. For each \(\omega \in \Omega\), define the level set
\[ \mathcal{L}_\omega(a) := \{ t \geq 0 : B_t(\omega) = a \}. \] (5.1)
It is proved in [26] that \(\mathcal{L}_\omega(a)\) is q.s. closed and has zero Lebesgue measure. Using the strong Markov property for \(G\)-Brownian motion, we can obtain the following theorem.

Theorem 5.1 For q.s. \(\omega \in \Omega\), the level set \(\mathcal{L}_\omega(a)\) has no isolated point in \([0, \infty)\).

To prove Theorem 5.1, we need the following two lemmas.

Lemma 5.2 For q.s. \(\omega\), \(G\)-Brownian motion \((B_t)_{t \geq 0}\) changes sign infinitely many times in \([0, \varepsilon]\), for any \(\varepsilon > 0\).

Proof. Define \(\tau_1 := \inf\{ t > 0 : B_t > 0 \}\). Then \(\tau_1\) is an optional time by Lemma 7.6 in Chap 7 of [12]. Let \(P \in \mathcal{P}\) and \(t \geq 0\) be given. Since \(B\) is a martingale, we can apply the classical optional sampling theorem to obtain \(E_P[-B_{\tau_1 \wedge t}] = 0\). Thus \(\hat{E}[-B_{\tau_1 \wedge t}] = 0\). Noting that \(-B_{\tau_1 \wedge t} \geq 0\), we then have \(-B_{\tau_1 \wedge t} = 0\) q.s., i.e., \(B_{\tau_1 \wedge t} = 0\) q.s. Similar analysis for \(-B\) shows \(B_{\tau_2 \wedge t} = 0\) q.s., for \(\tau_2 := \{ t > 0 : B_t < 0 \}\). Therefore, \(B_{\tau_0 \wedge t} = 0\) q.s., for \(\tau_0 := \tau_1 \vee \tau_2\). This implies \(B_{\tau_0 \wedge t} = 0\) for each \(t \geq 0\), q.s.

Applying Proposition 1.13 in Chap IV of [24] under each \(P \in \mathcal{P}\), we then have \((B)_{\tau_0 \wedge t} = 0\) for each \(t \geq 0\), q.s. But from Corollary 5.4 in Chap III of [22] that \((B)_t + s - (B)_t \geq \sigma^2 s > 0\) for each \(s > 0\), we must have \(\tau_0 = 0\) q.s. Hence, \(\tau_1 = 0\) and \(\tau_2 = 0\), q.s., which imply the desired result. ■

Lemma 5.3 We have
\[ \sup_{0 \leq t < \infty} B_t = +\infty \text{ and } \inf_{0 \leq t < \infty} B_t = -\infty, \text{ q.s.} \] (5.2)

Proof. We only prove the first equality, from which the second one follows by the symmetry of \(G\)-Brownian motion.

Define \(\tau_t = \inf\{ s \geq 0 : (B)_s > t \}\). Under each \(P \in \mathcal{P}\), \(B\) is a martingale. Then by Theorem 1.6 in Chap V of [24], \((B_{\tau_t})_{t \geq 0}\) is a classical Brownian motion. Applying Lemma 3.6 in Chap I of [25], we have
\[ \sup_{0 \leq t < \infty} B_{\tau_t} = +\infty \text{ P-a.s.} \]
Since \( \{ \tau_t : t \in [0, \infty) \} = [0, \infty) \), we then obtain

\[
\sup_{0 \leq t < \infty} B_t = +\infty \quad P\text{-a.s.}
\]

Therefore,

\[
\sup_{0 \leq t < \infty} B_t = +\infty \quad \text{q.s.}
\]

\[\square\]

**Remark 5.4** This lemma implies that \( \mathcal{L}_\omega(a) \) is q.s. unbounded.

**Proof of Theorem 5.1.** Let \( t \geq 0 \). Define the optional time after \( t \)

\[
\tau_t = \inf \{ s > t : B_s = a \}.
\]

By Lemma 5.3 (see also Remark 5.4), \( \tau_t \) is q.s. finite. Now we are going to show that

\[
\tau_{\tau_t} = \inf \{ s > \tau_t : B_s = a \} = \tau_t \quad \text{q.s.} \tag{5.3}
\]

For any \( n \geq 1 \), since \( \tau_t \wedge n \) satisfies (H3), then Corollary 4.10 implies that \( (B_{\tau_t \wedge n + s} - B_{\tau_t \wedge n})_{s \geq 0} \) is still a \( G \)-Brownian motion. Hence, by Lemma 5.2, there exists a set \( \Omega_n \subset \Omega \) such that \( c(\Omega_n^c) = 0 \) and on \( \Omega_n \),

\((B_{\tau_t \wedge n + s} - B_{\tau_t \wedge n})_{s \geq 0}\) changes its sign infinitely many times on any \([0, \varepsilon]\).

Let

\[
\Omega_0 := \bigcup_{n=1}^{\infty} (\Omega_n \cap \{ \tau_t \leq n \}).
\]

For any \( P \in \mathcal{P} \), we have

\[
P(\Omega_0^c) = P\left( \bigcap_{n=1}^{\infty} (\Omega_n^c \cup \{ \tau_t \geq n \}) \right) \leq P(\Omega_0^c \cup \{ \tau_t > n \}) = P(\{ \tau_t > n \}) \to P(\{ \tau_t = \infty \}) = 0, \quad \text{as } n \to \infty.
\]

Thus

\[
c(\Omega_0^c) = 0.
\]

For any fixed \( \omega \in \Omega_0 \), there exists an \( n \) such that \( \omega \in \Omega_n \cap \{ \tau_t \leq n \} \). Since \( \tau_t(\omega) \wedge n = \tau_t(\omega) \), then

\((B_{\tau_t + s} - B_{\tau_t})(\omega))_{s \geq 0}\) changes its sign infinitely many times on any \([0, \varepsilon]\). Therefore,

\[
\tau_{\tau_t}(\omega) = \tau_t(\omega),
\]

which proves (5.3).

Note that, for any fixed \( p < q \),

\[
\Lambda_{p,q} := \{ \omega \in \Omega : \text{there is only one } s \in (p, q) \text{ such that } B_s(\omega) = a \} \subset \{ \omega \in \Omega : \tau_p < q, \tau_p \geq q \}.
\]

We must have \( c(\Lambda_{p,q}) = 0 \). Thus the set

\[
\{ \omega \in \Omega : \mathcal{L}_\omega(a) \text{ has isolated point} \} = \bigcup_{0 \leq p < q \in \mathbb{Q}} \Lambda_{p,q}
\]

is a zero capacity set. \[\square\]
References

[1] Bensoussan, A., Lions, J L., Applications of variational inequalities in stochastic control. Elsevier, 2011.

[2] Billingsley, P., Convergence of probability measures. John Wiley & Sons, 1968.

[3] Denis, L., Hu, M., Peng, S., Function spaces and capacity related to a sublinear expectation: application to $G$-Brownian motion paths. Potential Analysis, 2011, 34: 139-161.

[4] Friedman, A., Stochastic differential equations and applications, Volume 1. Academic Press, 1975.

[5] Gao, F., Pathwise properties and homomorphic flows for stochastic differential equations driven by $G$-Brownian motion. Stochastic Processes and their Applications, 2009, 119, 3356-3382.

[6] Gao, F., Jiang, H., Large deviations for stochastic differential equations driven by $G$-Brownian motion. Stochastic Processes and their Applications, 2010, 120(11): 2212-2240.

[7] Hu, M., Ji, S., Peng, S., Song, Y., Backward stochastic differential equations driven by $G$-Brownian motion. Stochastic Processes and their Applications, 2014, 124(1): 759-784.

[8] Hu, M., Ji, S., Peng, S., Song, Y., Comparison theorem, Feynman-Kac formula and Girsanov transformation for BSDEs driven by $G$-Brownian motion. Stochastic Processes and their Applications, 2014, 124(2): 1170-1195.

[9] Hu, M., Peng, S., On representation theorem of $G$-expectations and paths of $G$-Brownian motion. Acta Mathematicae Applicatae Sinica (English Series), 2009, 25(3): 539-546.

[10] Hu, M., Peng, S., Extended conditional $G$-expectations and related stopping times. arXiv: 1309.3829v1, 2013.

[11] Itô, K., Differential equations determining a Markov process (in Japanese). J. Pan-Japan Math. Coll., 1942, 1077: 1352-1400.

[12] Kallenberg, O., Foundations of modern probability. Springer Science & Business Media, 2006.

[13] Karatzas, I., Shreve, S., Brownian motion and stochastic calculus. Springer Science & Business Media, 2012.

[14] Li, X., Peng, S., Stopping times and related Itô’s calculus with $G$-Brownian motion. Stochastic Processes and their Applications, 2011, 121(7): 1492-1508.

[15] Lin, Q., Some properties of stochastic differential equations driven by the $G$-Brownian motion. Acta Mathematica Sinica, English Series, 2013, 29(5): 923-942.

[16] Lin, Y., Stochastic differential equations driven by $G$-Brownian motion with reflecting boundary conditions. Electronic Journal of Probability, 2013, 18.

[17] Luo, P., Wang, F., Stochastic differential equations driven by $G$-Brownian motion and ordinary differential equations. Stochastic Processes and their applications, 2014, 124(11): 3869-3885.
[18] Nutz, M., Van Handel, R., Constructing sublinear expectations on path space. Stochastic Processes and their Applications, 2013, 123(8): 3100-3121.

[19] Øksendal, B., Stochastic differential equations. Springer Berlin Heidelberg, 2003: 65-84.

[20] Peng, S., G-expectation, G-Brownian motion and related stochastic calculus of Itô type. Stochastic Analysis and Applications, 2006, 2(4):541-567.

[21] Peng, S., Multi-dimensional G-Brownian motion and related stochastic calculus under G-expectation. Stochastic Processes and their Applications, 2008, 118(12): 2223-2253.

[22] Peng, S., Nonlinear expectations and stochastic calculus under uncertainty. arXiv:1002.4546v1, 2010.

[23] Peng, S., Backward stochastic differential equation, nonlinear expectation and their applications. Proceedings of the International Congress of Mathematicians. 2010, 1: 393-432.

[24] Revuz, D., Yor, M., Continuous martingales and Brownian motion. Springer Science & Business Media, 2013.

[25] Rogers, L. C. G. and Williams, D., Diffusions, Markov processes and martingales. Vol. 1. John Wiley & Sons, 1994.

[26] Wang, F., Zheng, G., Some sample path properties of G-Brownian motion. arXiv:1407.0211, 2014.