VECTOR-VALUED SEPARATION FUNCTIONS AND CONSTRAINED VECTOR OPTIMIZATION PROBLEMS: OPTIMALITY AND SADDLE POINTS

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Abstract. In this paper, we consider a class of constrained vector optimization problems by using image space analysis. A class of vector-valued separation functions and a $\mathcal{C}$-solution notion are proposed for the constrained vector optimization problems, respectively. Moreover, existence of a saddle point for the vector-valued separation function is characterized by the (regular) separation of two suitable subsets of the image space. By employing the separation function, we introduce a class of generalized vector-valued Lagrangian functions without involving any elements of the feasible set of constrained vector optimization problems. The relationships between the type-I/II saddle points of the generalized Lagrangian functions and that of the function corresponding to the separation function are also established. Finally, optimality conditions for $\mathcal{C}$-solutions of constrained vector optimization problems are derived by the saddle-point conditions.

1. Introduction. It is well-known that vector optimization problems (shortly, VOP), which arise from economics, decision theory, optimal control and game theory, are very important models in mathematics; see [5, 19, 23]. For the past decades, there exist various kinds of solution notions of vector optimization. Particularly, Pareto/weak efficient solution notions of vector optimization have been extensively studied.
studied; see [1, 2, 6, 3, 32, 27]. In recent years, optimality conditions for optimization problems are widely studied since they play a crucial role in duality theory and algorithm design; see [1, 7, 8, 9, 10, 28, 33, 34]. Basically, all optimality conditions are derived using theorem of separation and theorem of alternative or an adequate substitute such as Ekeland’s principle; see [6, 7, 34, 11, 12, 13]. In [14], Giannessi proposed the image space analysis (shortly, ISA) for studying any kind of constrained optimization problems, such as constrained extremum problems, variational inequalities and equilibrium problems, that can be equivalently expressed as the inconsistency of a parametric system. The inconsistency of such a system is reduced to the separation of two suitable subsets of the image space. The separation between the two suitable subsets is proved by showing that they lie in two disjoint level sets of a separating functional. Since then, ISA is extensively applied to establish separation theorems and theorems of alternative of constrained extremum problems by various kinds of separation functions such as linear and nonlinear cases, and then, Kuhn-Tucker type, Fritz-John type and Lagrangian-type optimality conditions of the constrained extremum problems are established by the corresponding theorems of separation and theorems of alternative; see [6, 28, 33, 11, 12, 13, 14, 15, 20, 21, 29, 16, 30].

Compared with the existing other approaches, a prominent advantage of ISA is that it can be applied to deal with nonconvex, nonsmooth and even discontinuous constrained optimization problems from the perspective of geometry.

Recently, ISA was applied to deal with optimality conditions and duality of VOP and vector variational inequalities (shortly, VVI) in [7]. Thereafter, the ISA received an extensive attention in the optimality, duality, penalty and regularity fields for VOP, VVI and related constrained vector problems; see [9, 33, 16, 22, 18, 24, 25, 4]. In [20], the linear separation of the image set of a constrained VVI and a corresponding convex cone was characterized by a real-valued linear separation function under the convexity assumptions. The Lagrangian type optimality conditions, gap functions and weak sharpness were obtained for VVI by a theorem of separation. In [31], an error bound and gap function for weak VVI and its lower semicontinuity were derived by using the ISA and the nonlinear scalarization function. It is noticed that the theorems of separation and theorems of alternative are closely related with the separation functions; see [9, 12, 13, 14, 15, 20, 21, 29, 22, 18, 24, 25, 31, 26]. However, theorems of separation, theorems of alternative, optimality conditions and duality theorems for VOP and related constrained vector problems were studied under the convexity conditions of the involved functions and the constraints; see [7, 9, 10, 13, 20, 21, 22, 18, 24]. Moreover, the Lagrangian functions of VOP and VVI in [7, 13, 20, 21, 29, 22, 18, 24, 31] rely on the element of their feasible sets. So, it is interesting to study the relationship between the classic Lagrangian functions and the Lagrangian functions introduced by corresponding separation functions.

On the other hand, the Lagrangian duality for scalar constrained extremum problem was discussed via real-valued separation functions, and the relationships between Wolfe and Mond-Weir dualities were analyzed in the image space under suitable generalized convexity assumptions; see [8, 9, 10]. A nonlinear separation scheme was proposed in the image space associated with an infinite-dimensional constrained extremum problem in the scalar case, applied to nonlinear Lagrangian duality and exact and inexact exterior penalty methods, and stated that the existence of a regular nonlinear separation is equivalent to a saddle point condition for a generalized Lagrangian function associated with the given problem; see [26]. Very recently, a unified duality theory for a constrained extremum problem was
introduced via its real-valued regular weak separation function in the image space. Further, the Lagrangian type duality, Wolfe duality and Mond-Weir duality were discussed as special duality schemes in a unified interpretation; see [33, 34]. It is not difficult to see that the dual problems in [8, 9, 10, 33, 34, 26] were constructed via the real-valued separation functions, and discussed by ISA. However, the objectives in many practical problems have different dimension, so that the scalarization method may be not suitable to these problems. Besides, the optimal value of the primal VOP is vector-valued, and the optimal value of its dual problems constructed by the real-valued separation functions is real-valued, i.e., the strong duality and converse duality between the primal VOP and its dual problems usually do not hold. That is why the real-valued separation functions may be not applied to such practical problems. So, it is necessary to study VOP by the vector-valued separation functions. Note that theorems of separation, saddle point conditions as well as Ekeland’s principle play a crucial role in the duality filed. To the best of our knowledge, there are very few results on the saddle point conditions of vector-valued Lagrangian function for constrained vector optimization problems (shortly, CVOP) in image space analysis except for [7]. Therefore it is fascinating and necessary to consider the separation and saddle-point type optimality conditions of CVOP by vector-valued separation functions and ISA.

Motivated and inspired by the works [7, 8, 9, 10, 33, 34, 29, 26], this work aims to study a class of CVOP by using ISA and introduce a class of vector-valued separation functions. The existence of a saddle point for the vector-valued separation function is characterized by the (regular) separation of two suitable subsets of the image space. By the vector-valued separation function, we introduce a class of generalized vector-valued Lagrangian functions without involving any elements of the feasible set of CVOP. The relationships between the type-I (II) saddle points of the generalized Lagrangian functions and that of the functions corresponding to the separation functions are also presented. Finally, optimality conditions for C-solutions of CVOP are obtained by the saddle-point conditions.

This paper is organized as follows. We recall some basic notions in Sect.2; In Sect.3, we propose a class of general vector-valued separation functions which includes almost existing linear and nonlinear separation functions; Optimality conditions and saddle point conditions are established for CVOP in Sect.4; Finally, we give the conclusions.

2. Preliminaries and ISA for CVOP. Throughout this paper, without special statements, let $X$ be a nonempty subset of the $n$-dimensional Euclidean space $\mathbb{R}^n$, $\mathbb{C} \subseteq \mathbb{R}^f$ be a closed, convex and pointed cone with nonempty interior, and $D \subseteq \mathbb{R}^m$ be a nonempty, closed and convex cone. For any set $\mathcal{U} \subseteq \mathbb{R}^\ell$, denote the interior, closure and boundary of $\mathcal{U}$ by $\text{int}\mathcal{U}, \overline{\mathcal{U}}$ and $\partial\mathcal{U}$, respectively. Let $\mathcal{C}_0 := \mathcal{C} \setminus \{0\}$ and $\mathcal{C}^0 := \text{int}\mathcal{C}$. Set $\mathfrak{C} \in \{\mathcal{C}_0, \mathcal{C}_0\}$. Let $D^* := \{\mu \in \mathbb{R}^m : \langle \mu, z \rangle \geq 0, \ \forall z \in D\}$ be the dual cone of a convex cone $D$, and let $Q^* := \{\xi \in \mathbb{R}^{\ell \times m} : \xi q \geq_C 0, \ \forall q \in Q\}$ be the vector dual cone of a cone $Q \subseteq \mathbb{R}^m$ with respect to $C$, where $\mathbb{R}^{\ell \times m}$ denotes the set of matrices with real entries and of order $\ell \times m$, and where the inequality $\xi q \geq_C 0$ means $\xi q \in C$. Let $f : \mathbb{R}^n \rightarrow \mathbb{R}^\ell$, $g : \mathbb{R}^n \rightarrow \mathbb{R}^m$ and $\omega : \mathbb{R}^{\ell \times m} \times \Pi \rightarrow \mathbb{R}^f$ be vector-valued functions, where $\Pi$ is a set of parameters to be specified case by case and $\mathbb{R}^{\ell \times m} := \mathbb{R}^\ell \times \mathbb{R}^m$. For each $\pi \in \Pi$ and for each set $S \subseteq \mathbb{R}^\ell$, the set

$$\text{lev}_S \omega(\cdot; \pi) := \{z = (u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : \omega((u, v); \pi) \in S\}$$

is called level set of $\omega$ with respect to $S$. 
In this paper, we consider the following constrained vector optimization problem (shortly, CVOP):

\[
\min_{\mathcal{C}} f(x), \quad \text{s.t.} \quad x \in \mathfrak{X} := \{x \in X : g(x) \in D\},
\]

where \(\min_{\mathcal{C}}\) marks \(\mathcal{C}\)-minimum: \(\bar{x} \in \mathfrak{X}\) is a \(\mathcal{C}\)-solution of CVOP iff,

\[
f(x) - f(\bar{x}) \notin -\mathcal{C}, \quad \forall \ x \in \mathfrak{X}.
\]

Clearly, the notion of \(\mathcal{C}\)-solution of CVOP includes the notions of weak efficient solution and Pareto efficient solution of CVOP as special cases. It is trivial to note that \(\bar{x}\) is a \(\mathcal{C}\)-solution of CVOP iff, the system (in the unknown \(x\)):

\[
f(x) - f(\bar{x}) \in -\mathcal{C}, \quad g(x) \in D, \quad x \in X,
\]

is infeasible.

Motivated by [7], we firstly recall the main features of the ISA for CVOP. Let \(\bar{x} \in X\) be a parameter. Define the mapping \(A_{\bar{x}} : X \to \mathbb{R}^\ell \times \mathbb{R}^m\) by

\[
A_{\bar{x}}(x) := (f(\bar{x}) - f(x), g(x)),
\]

and set \(H := \mathcal{C} \times D\),

\[
K(\bar{x}) := \{(u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m : (u, v) = A_{\bar{x}}(x), \ x \in X\}.
\]

\(K(\bar{x})\) is called the image of CVOP under \(A_{\bar{x}}\), while the space \(\mathbb{R}^\ell \times \mathbb{R}^m\) is the image space. Then the system (2) can be equivalently reformulated as the system:

\[
A_{\bar{x}}(x) \in H, \quad x \in X.
\]

So, the infeasibility of the system (3) is characterized by

\[
K(\bar{x}) \cap H = \emptyset.
\]

It is easy to see that (4) holds iff

\[
[K(\bar{x}) - \tilde{H}] \cap H = \emptyset,
\]

where the difference is meant in vector sense. Indeed, if (4) does not hold, it follows from \(0 \in \tilde{H}\) that (5) does not hold. Conversely, if (5) is false, then there exists \((u, v) \in \mathbb{R}^\ell \times \mathbb{R}^m\) such that

\[
(u, v) \in [K(\bar{x}) - \tilde{H}] \cap H.
\]

So, there exist \((u_1, v_1) \in K(\bar{x})\) and \((u_2, v_2) \in \tilde{H}\) such that \((u, v) = (u_1, v_1) - (u_2, v_2)\), and then

\[
(u_1, v_1) = (u, v) + (u_2, v_2) \in H + \tilde{H} \subseteq H.
\]

Consequently, (4) does not hold.

\(E(\bar{x}) := K(\bar{x}) - H\) is called conic extension of the image \(K(\bar{x})\). The conic extension \(E(\bar{x})\) often enjoys more properties than the image \(K(\bar{x})\), such as convexity; see [7].

So, the \(\mathcal{C}\)-solution of CVOP can be characterized by the separation of sets \(K(\bar{x})\) (or, \(E(\bar{x})\)) and \(H\). Here \(\bar{x}\) is reviewed as a parameter. From the above, the following results hold.

**Proposition 1.** Let \(\bar{x} \in \mathfrak{X}\). The following statements are equivalent:

(i) \(\bar{x}\) is a \(\mathcal{C}\)-solution of CVOP.

(ii) the system (2) is infeasible.

(iii) \(K(\bar{x}) \cap H = \emptyset\).

(iv) \(E(\bar{x}) \cap H = \emptyset\).
The following results, which are mostly well-known and which will be used later, are recalled here for the reader’s convenience.

**Lemma 2.1.** [5, 4] Let $C \subseteq \mathbb{R}^\ell$ be a closed, convex and pointed cone with nonempty interior. Then

(i) $y \in C \iff \langle y^*, y \rangle \geq 0, \forall y^* \in C^*$.
(ii) $y \in C^0 \iff \langle y^*, y \rangle > 0, \forall y^* \in C^* \setminus \{0\}$.
(iii) $C^* = (C_0)^* = (C^0)^*$.

It is worth noting that Lemma 2.1 (i), which is the famous Bipolar Theorem, indeed holds when $C$ is a nonempty, closed and convex cone.

3. Vector-valued separation functions. It is well-known that separation functions play a vital role in studying optimality and duality for various kinds of optimization problems, variational inequalities and equilibrium problems; see [7, 9, 33, 12, 14, 20, 21, 22]. Moreover, Lagrangian functions for optimization problems are closely related with separation functions. As we know, Lagrangian type dual problem and Wolfe type dual problem of vector optimization problems can be constructed by their vector-valued Lagrangian functions. So, it is interesting to study the vector-valued separation functions.

We now recall the definitions of weak separation functions and regular weak separation functions, which were introduced by Giannessi[13].

**Definition 3.1.** [13] The class of all the functions $\omega : \mathbb{R}^{\ell+m} \times \Pi \rightarrow \mathbb{R}^\ell$, such that

(i) $\bigcap_{\pi \in \Pi} \text{lev}_{C_\omega} (\cdot ; \pi) \supseteq H, \forall \pi \in \Pi$,
(ii) $\bigcap_{\pi \in \Pi} \text{lev}_{C_\omega} (\cdot ; \pi) \subseteq H$,

is called the class of weak separation functions and is denoted by $W(\Pi)$, where $\Pi$ is a set of parameters, and $H = \mathcal{E} \times D$, where $\mathcal{E} \in \{C_0, C^0\}$.

**Definition 3.2.** [13] The class of all the functions $\omega : \mathbb{R}^{\ell+m} \times \Pi \rightarrow \mathbb{R}^\ell$, such that

$$\bigcap_{\pi \in \Pi} \text{lev}_{\mathcal{E}} \omega (\cdot ; \pi) = H,$$

is called the class of regular weak separation functions and is denoted by $W_R(\Pi)$, where $\Pi$ is a set of parameters, and $H = \mathcal{E} \times D$, where $\mathcal{E} \in \{C_0, C^0\}$.

Let $P_1 \subseteq \mathbb{R}^{\ell \times \ell}$, $P_2 \subseteq \mathbb{R}^{\ell \times m}$ be two sets of parameters. We propose the following vector-valued function $\omega : \mathbb{R}^\ell \times \mathbb{R}^m \times P_1 \times P_2 \rightarrow \mathbb{R}^\ell$:

$$\omega(u, v; \theta, \lambda) = \varphi(u; \theta) + \omega(v; \lambda), \theta \in P_1, \lambda \in P_2,$$

with

$$\forall \theta \in P_1 \setminus O, \forall \lambda \in P_2 \text{ s.t. } \omega(\cdot, \cdot, \theta, \lambda) \neq 0,$$

where $O := \{\theta \in P_1 : \varphi(u; \theta) = 0, \forall u \in \mathbb{R}^\ell\}, \varphi : \mathbb{R}^\ell \times P_1 \rightarrow \mathbb{R}^\ell$ and $\omega : \mathbb{R}^m \times P_2 \rightarrow \mathbb{R}^\ell$ satisfy the following conditions:

$$\bigcap_{\theta \in P_1} \text{lev}_{C_\varphi} \varphi (\cdot ; \theta) = C,$$

$$\bigcap_{\theta \in P_1 \setminus O} \text{lev}_{\mathcal{E}} \varphi (\cdot ; \theta) = \mathcal{E}.$$
where
\[
\begin{align*}
\Theta := \left\{ \theta \in P_1 : \varpi(u; \theta) = 0, \forall u \in \mathbb{R}^\ell, \quad \text{if } \mathcal{C} = C_0, \right. \\
\left. \theta \in P_1 : \varpi(u; \theta) \in \partial C, \forall u \in \mathbb{R}^\ell, \quad \text{if } \mathcal{C} = C^0, \right. \\
\forall \theta \in P_1, \forall \alpha \in \mathbb{R}_+, \exists \theta_\alpha \in P_1 \text{ s.t. } \alpha \varpi(u; \theta) = \varpi(u; \theta_\alpha), \forall u \in \mathbb{R}^\ell
\end{align*}
\]
and
\[
\bigcap_{\lambda \in \mathcal{P}_2} \text{lev}_C \varpi(\cdot; \lambda) = \emptyset,
\]
\[
\forall \lambda \in \mathcal{P}_2, \forall \alpha \in \mathbb{R}_+, \exists \lambda_\alpha \in \mathcal{P}_2 \text{ s.t. } \alpha \varpi(v; \lambda) = \varpi(v; \lambda_\alpha), \forall v \in \mathbb{R}^m.
\]
According to the definition of the vector-valued function \( \omega : \mathbb{R}^\ell \times \mathbb{R}^m \times P_1 \times \mathcal{P}_2 \to \mathbb{R}^\ell \), we deduce from (7), (8) and (9) that
\[
\forall u \notin \mathcal{C}, \exists \theta \in P_1 \text{ s.t. } \varpi(u; \theta) \notin \mathcal{C}, \quad \text{(12)}
\]
\[
\forall u \notin \mathcal{C}, \exists \theta \in P_1 \setminus \Theta \text{ s.t. } \varpi(u; \theta) \notin \mathcal{C}, \quad \text{(13)}
\]
and
\[
\exists \bar{\theta} \in P_1 \text{ s.t. } \varpi(\cdot; \bar{\theta}) \equiv 0, \quad \text{(14)}
\]
where (7) \( \Rightarrow \) (12), (8) \( \Rightarrow \) (13), and where (9) \( \Rightarrow \) (14) at \( \alpha = 0 \). Moreover, from (7) and (8), one has
\[
\begin{cases}
\varpi(0; \theta) = 0, \forall \theta \in P_1, & \text{if } \mathcal{C} = C_0, \\
\varpi(0; \theta) \in \partial C, \forall \theta \in P_1, & \text{if } \mathcal{C} = C^0.
\end{cases}
\]
Since (7) \( \Rightarrow \) \( \varpi(0; \theta) \in C \), \( \forall \theta \in P_1 \setminus \Theta \) and (8) \( \Rightarrow \) \( \varpi(u; \theta) \in \mathcal{C}, \forall u \in \mathcal{C}, \theta \in P_1 \setminus \Theta \), these relationships show that (15) is true.

Besides, (10) and (11) imply that
\[
\forall v \notin D, \exists \lambda \in \mathcal{P}_2 \text{ s.t. } \varpi(v; \lambda) \notin \mathcal{C}, \quad \text{(16)}
\]
\[
\exists \bar{\lambda} \in \mathcal{P}_2 \text{ s.t. } \varpi(\cdot; \bar{\lambda}) \equiv 0, \quad \text{(17)}
\]
where (10) \( \Rightarrow \) (16) and where (11) \( \Rightarrow \) (17) at \( \alpha = 0 \). Consequently, (14) and (17) yield that \( \Theta \supseteq \mathcal{O} \neq \emptyset \) and \( \Lambda \(=\) \{ \lambda \in \mathcal{P}_2 : \varpi(v; \lambda) = 0, \forall u \in \mathbb{R}^m \} \neq \emptyset \).

**Remark 1.** Under the different sets of parameters \( P_1 \) and \( P_2 \), there are many particular cases of the class (6), which are useful for the applications. Some special cases are listed as follows:

(i) If \( D = \mathbb{R}_+^m, P_1 = \mathcal{C}_\mathcal{E}^c \subseteq \mathbb{R}^{\ell \times \ell}, P_2 = D_\mathcal{C}^* \subseteq \mathbb{R}^{\ell \times m}, \varpi(u; \theta) = \theta u \) and \( \varpi(v; \lambda) = \lambda v \), then
\[
\omega(u, v; \theta, \lambda) = \theta u + \lambda v, \quad \theta \in \mathcal{C}_\mathcal{E}^c, \lambda \in D_\mathcal{C}^*, \quad \text{(18)}
\]
which was studied in [7, p.160 and p.170]. It easily follows from the definition of vector dual cone that \( \varpi(u; \theta) = \theta u \) and \( \varpi(v; \lambda) = \lambda v \) satisfy the conditions (7), (8), (9) and (10), (11), respectively. Particularly, if \( C = \mathbb{R}_+^\ell \),
\[
D_\mathcal{R}_+^* = \left\{ \lambda = \begin{pmatrix} d_1 \\ \vdots \\ d_l \end{pmatrix} \in \mathbb{R}^{\ell \times m} : d_i \in \mathbb{R}_+^m, i = 1, 2, \cdots, l \right\},
\]

(ii) If \( \mathbb{R}_+^\ell = ]-\infty, +\infty[ \), \( C = P_1 = \mathbb{R}_+ = [0, +\infty[, C^0 = C_0 = ]0, +\infty[ \) and \( P_2 = D^* \), then
According to (14) and (17), we have

\[
\omega(u; \theta) = \theta u + \omega(v; \lambda),
\]

which was proposed in [29];

(b) if \( \omega(u; \theta) = \theta u \), then \( \omega(u, v; \theta, \lambda) = \theta u + \omega(v; \lambda) \), which was proposed in [13] and studied in [28]; Particularly, if \( P_2 = \{ (\rho_1, \rho_2, \cdots, \rho_m) \in \mathbb{R}_n^m \} \) and \( \omega(v; \lambda) = -\rho \Delta_D(v) \) for all \( \lambda \in P_2 \), where \( \Delta_D(v) := d_D(v) - d_{D_\lambda}(v) \) and \( d_D(v) := \inf_{x \in D} \| v - x \| \) and \( d_\theta(v) := +\infty \), then \( \omega(u, v; \theta, \lambda) = \theta u + \omega(v; \lambda) \), which was studied; see [34, 21, 30];

(c) if \( \omega(u; \theta) = \theta u \) and \( \omega(v; \lambda) = (\lambda, v) \), then \( \omega(u, v; \lambda) = \theta u + (\lambda, v) \), which has been widely studied; see [7, 9, 12, 13, 14, 20, 21, 22].

The following proposition shows that the vector-valued function \( \omega \) is a weak separation function under the different parameter set \( \Pi \).

**Proposition 2.** (i) The vector-valued function \( \omega \) defined by (6) is a weak separation function, i.e., the class (6) is a subset of \( \mathcal{W}(\Pi) \) with \( \Pi = P_1 \times P_2 \).

(ii) The vector-valued function \( \omega \) defined by (6) is a regular weak separation function if \( \theta \in P_1 \setminus \Theta \), i.e., the class (6) is a subclass of \( \mathcal{W}_R(\Pi) \) with \( \Pi = (P_1 \setminus \Theta) \times P_2 \).

**Proof.** (i) (8) and (11) imply that

\[
\omega(u; \theta) \in C, \quad \forall u \in C, \theta \in P_1
\]

and

\[
\omega(v; \lambda) \in C, \quad \forall v \in D, \theta \in P_2.
\]

So, one has

\[
\omega(u, v; \theta, \lambda) = \omega(u; \theta) + \omega(v; \lambda) \in C + C \subseteq C,
\]

for all \( (u, v) \in \mathcal{H} = \mathcal{C} \times D, (\theta, \lambda) \in P_1 \times P_2 \). Therefore, we get

\[
\text{lev}_C \omega(\cdot; \cdot, \theta, \lambda) = \mathcal{H}, \quad \forall (\theta, \lambda) \in P_1 \times P_2.
\]

On the other hand, note that

\[
\bigcap_{\theta \in P_1} \text{lev}_C \omega(\cdot; \theta) \subseteq \bigcap_{\theta \in P_1 \setminus \Theta} \text{lev}_C \omega(\cdot; \theta) = \mathcal{C}
\]

and

\[
\bigcap_{\lambda \in P_2} \text{lev}_C \omega(\cdot; \lambda) \subseteq \bigcap_{\lambda \in P_2} \text{lev}_C \omega(\cdot; \lambda) = D.
\]

According to (14) and (17), we have

\[
\bigcap_{(\theta, \lambda) \in P_1 \times P_2} \text{lev}_C \omega(\cdot; \theta, \lambda)
\]

\[
\subseteq \left[ \bigcap_{\theta \in P_1 \setminus \Theta} \text{lev}_C \omega(\cdot; \theta, \lambda) \right] \cap \left[ \bigcap_{\lambda \in P_2} \text{lev}_C \omega(\cdot; \theta, \lambda) \right]
\]

\[
= \left[ \bigcap_{\theta \in P_1 \setminus \Theta} \text{lev}_C \omega(\cdot; \theta) + \omega(\cdot; \lambda) \right] \cap \left[ \bigcap_{\lambda \in P_2} \text{lev}_C \omega(\cdot; \theta) + \omega(\cdot; \lambda) \right]
\]
Then there exists \( \tilde{\omega} \) such that
\[
(17) \quad \text{yields that}
\]
\[
\text{which contradicts (23).}
\]

Again, from (8), one deduces that \( \overline{\omega}(u; \theta) \in \mathcal{C} \) for all \( \theta \in P_1 \setminus \Theta \). Therefore,

\[
(8) \quad \text{implies that}
\]
\[
\text{Conversely, suppose to the contrary that}
\]
\[
\text{Then there exists} \ (\tilde{u}, \tilde{v}) \in \mathbb{R}_x \times \mathbb{R}_y \text{ such that}
\]
\[
\text{and} \ (\tilde{u}, \tilde{v}) \notin \mathcal{H}. \text{ Clearly,} \ (\tilde{u}, \tilde{v}) \notin \mathcal{H} \text{ is equivalent to that} \ \tilde{u} \notin \mathcal{C} \text{ or,} \ \tilde{v} \notin \mathcal{D}.
\]

If \( \tilde{u} \notin \mathcal{C} \), then there exist \( \tilde{\theta} \in P_1 \setminus \Theta \) such that \( \overline{\omega}(\tilde{u}; \tilde{\theta}) \notin \mathcal{C} \). This together with (17) yields that

\[
\omega(\tilde{u}, \tilde{v}; \tilde{\theta}, \tilde{\lambda}) = \overline{\omega}(\tilde{u}; \tilde{\theta}) + \omega(\tilde{v}; \tilde{\lambda}) = \overline{\omega}(\tilde{u}; \tilde{\theta}) \notin \mathcal{C},
\]

which contradicts (23).

If \( \tilde{v} \notin \mathcal{D} \), we only need consider the case \( \tilde{v} \notin \mathcal{D} \) and \( \tilde{u} \in \mathcal{C} \), then there exists \( \tilde{\lambda} \in P_2 \) such that \( \omega(\tilde{v}; \tilde{\lambda}) \notin \mathcal{C} \). By (11), for any \( \alpha \in \mathbb{R}_+ \setminus \{0\} \), there exists \( \tilde{\lambda}_\alpha \in P_2 \) such that

\[
\omega(\tilde{v}; \tilde{\lambda}_\alpha) = \alpha \omega(\tilde{v}; \tilde{\lambda}) \notin \mathcal{C}.
\]

Then there exists \( c^* \in C^* \) such that

\[
\langle c^*, \omega(\tilde{v}; \tilde{\lambda}_\alpha) \rangle = \langle c^*, \alpha \omega(\tilde{v}; \tilde{\lambda}) \rangle = \alpha \langle c^*, \omega(\tilde{v}; \tilde{\lambda}) \rangle < 0.
\]

Again, from (8), one deduces that \( \overline{\omega}(\tilde{u}; \theta) \in \mathcal{C} \) for all \( \theta \in P_1 \setminus \Theta \). Therefore,

\[
\langle c^*, \overline{\omega}(\tilde{u}; \theta) \rangle \geq 0, \ \forall \theta \in P_1 \setminus \Theta.
\]

Observe that \( \omega(\tilde{u}, \tilde{v}; \tilde{\lambda}_\alpha) = \overline{\omega}(\tilde{u}; \tilde{\theta}) + \omega(\tilde{v}; \tilde{\lambda}_\alpha) = \overline{\omega}(\tilde{u}; \tilde{\theta}) + \alpha \omega(\tilde{v}; \tilde{\lambda}) \) for all \( \tilde{\theta} \in P_1 \setminus \Theta \). Then, one has

\[
\langle c^*, \omega(\tilde{u}, \tilde{v}; \tilde{\theta}, \tilde{\lambda}_\alpha) \rangle = \langle c^*, \overline{\omega}(\tilde{u}; \tilde{\theta}) + \alpha \omega(\tilde{v}; \tilde{\lambda}) \rangle
\]

\[
= \langle c^*, \overline{\omega}(\tilde{u}; \tilde{\theta}) \rangle + \langle c^*, \alpha \omega(\tilde{v}; \tilde{\lambda}) \rangle
\]

\[
= \langle c^*, \overline{\omega}(\tilde{u}; \tilde{\theta}) \rangle + \alpha \langle c^*, \omega(\tilde{v}; \tilde{\lambda}) \rangle.
\]

It follows from (24) and (25) that there exists \( \tilde{\alpha} \in \mathbb{R}_+ \) such that

\[
\langle c^*, \omega(\tilde{u}, \tilde{v}; \tilde{\theta}, \tilde{\lambda}_\alpha) \rangle = \langle c^*, \overline{\omega}(\tilde{u}; \tilde{\theta}) \rangle + \tilde{\alpha} \langle c^*, \omega(\tilde{v}; \tilde{\lambda}) \rangle < 0.
\]
Combined with Lemma 2.1(i) yields \( \omega(\tilde{u}, \tilde{v}; \tilde{\theta}, \tilde{\lambda}) \notin \mathcal{C} \), which contradicts (23).

Altogether, (22) does not hold. Thus, one has

\[
\mathcal{H} \supseteq \bigcap_{(\theta, \lambda) \in (P_1 \setminus \Theta) \times P_2} \text{lev}_{\mathcal{E}} \omega(\cdot, \theta, \lambda).
\]  

(26)

Consequently, we deduce from (21) and (26) that \( \omega \) defined by (6) is a regular weak separation function with \( \Pi = (P_1 \setminus \Theta) \times P_2 \), i.e., \( \omega \in \mathcal{W}_R((P_1 \setminus \Theta) \times P_2) \).

\[\square\]

**Remark 2.** Owing to \( P_1 \setminus \Theta = \mathcal{C}_e \) and Remark 1, Proposition 2 improves Proposition 1 and Proposition 4 of [7] and Proposition 3.1 of [29], and the proof of Proposition 2 is distinct from that of Proposition 1 and Proposition 4 in [7].

We next propose a concrete nonlinear vector-valued weak separation function \( \tilde{\omega} \):

\[
\tilde{\omega}(u, v; \theta, \lambda) = \theta u + \lambda P_\mathcal{D}(v), \ \forall u \in \mathbb{R}^\ell, v \in \mathbb{R}^m, \theta \in C^*_C, \lambda \in D^*_C,
\]  

(27)

where \( \overline{\varphi}(u; \theta) := \theta u \) and \( \overline{\omega}(v; \lambda) := \lambda \gamma(v) \) for all \( u \in \mathbb{R}^\ell, v \in \mathbb{R}^m, \theta \in P_1 := C^*_C, \lambda \in P_2 := D^*_C \), and \( P_\mathcal{D}(v) \) marks the metric projection of \( v \) onto the set \( D \), i.e.,

\[
\|v - P_\mathcal{D}(v)\| = \min\{\|v - z\| : z \in D\}. \text{ It is well-known that the metric projection is usually nonlinear. After verification, } \overline{\varphi}(u; \theta) \text{ and } \overline{\omega}(v; \lambda) \text{ satisfy the conditions (7),(8), (9) and (10),(11), respectively.}
\]

The following results are directly derived from Proposition 2.

**Corollary 1.** (i) The function \( \tilde{\varphi} \) defined by (27) is a weak separation function, i.e., the class (27) is a subclass of \( \mathcal{W}(\Pi) \) with \( \Pi = C^*_C \times D^*_C \).

(ii) The function \( \tilde{\omega} \) defined by (27) is a regular weak separation function if \( \theta \in \mathcal{C}_e \), i.e., the class (27) is a subclass of \( \mathcal{W}_R(\Pi) \) with \( \Pi = \mathcal{C}_e \times D^*_C \).

4. **Optimality and saddle points for CVOP.** In this section, we investigate optimality conditions and saddle point conditions for CVOP by the vector-valued separation function \( \omega \) defined by (6) in the image space.

Given that it is not easy to prove (4) directly, we introduce the following separation notion for the sets \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) via the class (6).

**Definition 4.1.** Let \( \bar{x} \in X \). The sets \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) admit:

(i) a separation with respect to \( \omega \) defined by (6) iff, there exists \( (\tilde{\theta}, \tilde{\lambda}) \in P_1 \times P_2 \) such that

\[
\omega(A_\bar{x}(x); \tilde{\theta}, \tilde{\lambda}) \notin \mathcal{C}, \ \forall x \in X.
\]  

(28)

(ii) a regular separation iff, \( \tilde{\theta} \in P_1 \setminus \Theta \) in (28).

By the definition of the image \( \mathcal{K}(\bar{x}) \), (28) is equivalent to the following inequality:

\[
\omega(u, v; \tilde{\theta}, \tilde{\lambda}) = \overline{\varphi}(u; \tilde{\theta}) + \overline{\omega}(v; \tilde{\lambda}) \notin \mathcal{C}, \ \forall (u, v) \in \mathcal{K}(\bar{x})
\]

We now present sufficient optimality conditions of CVOP via the separation approach.

**Theorem 4.2.** Let \( \bar{x} \in \mathfrak{S} \). Assume that any one of the following conditions hold:

(i) \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) admit a regular separation with respect to \( \omega \) defined by (6);

(ii) there exists \( (\tilde{\theta}, \tilde{\lambda}) \in P_1 \times P_2 \) such that

\[
\overline{\varphi}(f(\bar{x}) - f(x); \tilde{\theta}) + \overline{\omega}(g(x); \tilde{\lambda}) \notin C, \ \forall x \in X.
\]  

(29)

Then \( \bar{x} \) is a \( \mathcal{C} \)-solution of CVOP.
Theorem 4.3. Let $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular separation with respect to $\omega$, then there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that
\[
\omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) = \mathcal{P}(f(\bar{x}) - f(x); \bar{\theta}) + \omega(g(x); \bar{\lambda}) \notin \mathcal{C}, \ \forall x \in X.
\]
Therefore, one has
\[
\{A_{\bar{x}}(x) : x \in X\} \cap \text{lev}_{C}\omega(\cdot; \bar{\theta}, \bar{\lambda}) = \emptyset.
\]
Note that $\mathcal{K}(\bar{x}) = \{A_{\bar{x}}(x) : x \in X\}$. It follows from Proposition 2 (ii) and Definition 3.2 that
\[
\mathcal{K}(\bar{x}) \cap \mathcal{H} = \emptyset.
\]
This together with Proposition 1 yields that $\bar{x}$ is a $\mathcal{C}$-solution of CVOP.

Remark 3. Theorem 4.2 extends [29, Theorem 4.2] from the scalar case to the vector case and unifies [7, Theorem 1, p.161 and Theorem 4, p.170]. In Theorem 4.2, we characterize the sufficient optimality conditions of CVOP via the general vector-valued separation function $\omega$ defined by (6); Theorem 4.2 of [29] gives the sufficient optimality conditions for constrained scalar extremum problems by a real-valued separation function and Theorems 1 and 4 of [7] give the sufficient optimality conditions for CVOP by the vector-valued linear separation function $\omega(u, v; \theta, \lambda) = \theta u + \lambda v$, $\theta \in P_1, \lambda \in P_2$, where $P_1 = (C \setminus \{0\})_{c\setminus (a)}^*$ or, $P_1 = (\text{int } C)_{\text{int } C}^*$ and $P_2 = (\mathbb{R}^m)_{c\setminus (a)}^*$.

It is noted that the separation of $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ can be characterized by the saddle point conditions of some separation functions; see [7, 21, 29, 18, 4]. In view of this, we give two saddle point conditions to ensure the separation of $\mathcal{K}(\bar{x})$ and $\mathcal{H}$.

Theorem 4.3. Let $\bar{x} \in \mathfrak{X}$. The following statements hold:

(i) if $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a separation with respect to $\omega$ defined by (6), then there exists $(\bar{\theta}, \bar{\lambda}) \in P_1 \times P_2$ such that $(\bar{x}, \bar{\theta}, \bar{\lambda})$ is a saddle point of the function $\omega(A_{\bar{x}}(x); \theta, \lambda)$ on $X \times P_1 \times P_2$, i.e., for all $x \in X$ and $(\theta, \lambda) \in P_1 \times P_2$, $\omega(A_{\bar{x}}(x); \theta, \bar{\lambda}) - \omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) \notin \mathcal{C}$ and
\[
\omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \omega(A_{\bar{x}}(x); \theta, \lambda) \notin \mathcal{C}.
\]
(ii) if there exists $(\bar{\theta}, \bar{\lambda}) \in P_1 \times P_2$ such that for all $x \in X$ and $(\theta, \lambda) \in P_1 \times P_2$,
\[
\begin{cases}
\omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \omega(A_{\bar{x}}(x); \bar{\beta}, \bar{\lambda}) \notin \mathcal{C}, & \text{if } \mathcal{C} = C_0, \\
\omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) \notin \mathcal{C}, & \text{if } \mathcal{C} = C_0,
\end{cases}
\]
and (31) hold, then $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a separation with respect to $\omega$ defined by (6).

(iii) if $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular separation with respect to $\omega$ defined by (6), then there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that $(\bar{x}, \bar{\theta}, \bar{\lambda})$ is a saddle point of the function $\omega(A_{\bar{x}}(x); \theta, \lambda)$ on $X \times P_1 \times P_2$.

(iv) if there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that (31) and (32) hold, then $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular separation with respect to $\omega$ defined by (6).
Proof. We only give the proof of (i) and (ii) since the proof of (iii) and (iv) are similar to that of (i) and (ii), respectively.

(i) Let $K(x)$ and $H$ admit a separation with respect to $\omega$ defined by (6). Then there exists $(\tilde{\theta}, \tilde{\lambda}) \in P_1 \times P_2$ such that

$$\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \not\in C, \ \forall x \in X,$$

and $g(x) \in D$. Furthermore, one has $\omega(g(x); \lambda) \in C$ for all $\lambda \in P_2$. Recall that $\varpi(0; \theta) \in \partial C$ for all $\theta \in P_1$. Therefore, we have

$$\omega(A_x(x); \theta, \lambda) \in C, \ \forall (\theta, \lambda) \in P_1 \times P_2.$$  

Using (33) yields

$$\begin{cases} 
\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) = 0, & \text{if } C = C_0, \\
\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in \partial C, & \text{if } C = C^0.
\end{cases}$$

If $\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) - \omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in C$ for all $x \in X$, then

$$\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in \omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) + C \subseteq \partial C + C \subseteq C, \ \forall x \in X,$$

which contradicts (33).

If $\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) - \omega(A_x(x); \theta, \lambda) \in C$ for all $(\theta, \lambda) \in P_1 \times P_2$, from (34), we get

$$\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in \omega(A_x(x); \theta, \lambda) + C \subseteq C,$$

which contradicts (35).

Consequently, (31) holds, namely, $(\tilde{x}, \tilde{\theta}, \tilde{\lambda})$ is a saddle point of $\omega(A_x(x); \theta, \lambda)$ on $X \times P_1 \times P_2$.

(ii) Assume that there exists $(\hat{\theta}, \hat{\lambda}) \in P_1 \times P_2$ such that (31) and (32) hold. Due to $\tilde{x} \in \tilde{\mathfrak{X}}$, we have

$$\omega(A_x(x); \hat{\theta}, \hat{\lambda}) = \varpi(0; \hat{\theta}) + \omega(g(x); \hat{\lambda}) \in C.$$

This together with (14)(17) and (31) yields that

$$\begin{cases} 
\omega(A_x(x); \hat{\theta}, \hat{\lambda}) = 0, & \text{if } C = C_0, \\
\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in \partial C, & \text{if } C = C^0.
\end{cases}$$

We next split the rest proof of (ii) into the following two cases:

If $C = C^0$, we have $\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in \partial C$. By Lemma 2.1(ii), there exists $\tilde{\zeta} \in C^* \setminus \{0\}$ such that $\langle \tilde{\zeta}, \omega(A_x(x); \hat{\theta}, \hat{\lambda}) \rangle = 0$. Suppose that there exists $\tilde{x} \in X$ such that

$$\omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \in C.$$

It then follows from Lemma 2.1 that $\langle \tilde{\zeta}, \omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \rangle > 0$ and so,

$$\begin{align*}
0 &\geq \langle \tilde{\zeta}, \omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) - \omega(A_x(x); \hat{\theta}, \hat{\lambda}) \rangle \\
&= \langle \tilde{\zeta}, \omega(A_x(x); \hat{\theta}, \hat{\lambda}) \rangle - \langle \tilde{\zeta}, \omega(A_x(x); \tilde{\theta}, \tilde{\lambda}) \rangle \\
&> 0,
\end{align*}$$

which is a contradiction. Therefore, we have

$$\omega(A_x(x); \hat{\theta}, \hat{\lambda}) \in C, \ \forall x \in X.$$  

By Definition 4.1, $K(x)$ and $H$ admit a separation with respect to $\omega$. If $C = C_0$, then (32) implies the separation of $K(x)$ and $H$ with respect to $\omega$.  \[\square\]
We next give an example to show Theorems 4.2 and 4.3 by the separation function \( \tilde{\omega} \) defined by (27).

**Example 1.** Let \( \mathbb{R}^n = \mathbb{R}^\ell = \mathbb{R}^m := \mathbb{R}^2, C = D := \mathbb{R}^2_+, P_2 := D^*_C, \mathcal{C} := \mathbb{R}^2_+ \setminus \{0\}, P_1 := C^*_C, X := \{x = (x_1, x_2)^T \in \mathbb{R}^2_+: x_1 + x_2 \geq 1\} \), and let

\[
    f(x) := \left( \ln(x_1 + x_2 + 1) \right), \quad g(x) := \left( \frac{\sqrt{x_1 + x_2} - 1}{10 - e^{x_1 + x_2}} \right).
\]

After verification, \( \mathcal{G} = \{x = (x_1, x_2)^T \in \mathbb{R}^2_+ : 1 \leq x_1 + x_2 \leq \ln 10\}, P_1 \setminus \Theta = \mathcal{C}_\epsilon \) and \( \mathcal{S} := \{x \in X : x_1 + x_2 = 1\} \) is the set of \( \mathcal{C} \)-solutions of CVOP. Set \( \varpi(u; \theta) := \theta u \) and \( \omega(v; \lambda) := \lambda P_D(v) \) for all \( u, v \in \mathbb{R}^2, \theta \in C^*_C, \lambda \in D^*_C \), where \( P_D(v) \) is the metric projection of \( v \) onto \( D \). Then

\[
    \omega(u, v; \theta, \lambda) = \theta u + \lambda P_D(v), \quad \forall u, v \in \mathbb{R}^2, \theta \in C^*_C, \lambda \in D^*_C.
\]

Let \( \bar{x} = (\bar{x}_1, \bar{x}_2)^T \in \mathcal{G} \), assume that \( \mathcal{K}(\bar{x}) \) and \( \mathcal{H} \) admit a regular separation with \( \bar{\theta} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) \in \mathcal{C}_\epsilon \) and \( \bar{\lambda} = \left( \begin{array}{c} 0 \\ 0 \end{array} \right) \in D^*_C \). After calculation, we have \( \bar{x}_1 + \bar{x}_2 = 1 \) and so, \( \bar{x} \in \mathcal{S} \). Due to

\[
    \omega(A_\bar{x}(x); \bar{\theta}, \bar{\lambda}) \neq \omega(A_\bar{x}(\bar{x}); \bar{\theta}, \bar{\lambda})
\]

and

\[
    \omega(A_\bar{x}(x); \bar{\theta}, \bar{\lambda}) \neq \omega(A_\bar{x}(\bar{x}); \theta, \lambda)
\]

Consequently, \( (\bar{x}, \bar{\theta}, \bar{\lambda}) \) is a saddle point of \( \tilde{\omega}(A_\bar{x}(x); \theta, \lambda) \) on \( X \times P_1 \times P_2 \).

Conversely, let \( \bar{x} \in \mathcal{G} \) and assume that there exist \( \bar{\theta} = \left( \begin{array}{cc} 1 & 1 \\ 0 & 0 \end{array} \right) \in \mathcal{C}_\epsilon \) and \( \bar{\lambda} = \left( \begin{array}{cc} 0 & 0 \\ 1 & 0 \end{array} \right) \in D^*_C \) such that \( (\bar{x}, \bar{\theta}, \bar{\lambda}) \) is a saddle point of \( \tilde{\omega}(A_\bar{x}(x); \theta, \lambda) \) on \( X \times P_1 \times P_2 \). Then

\[
    \omega(A_\bar{x}(\bar{x}); \bar{\theta}, \bar{\lambda}) \neq \omega(A_\bar{x}(\bar{x}); \theta, \lambda) \neq \omega(A_\bar{x}(x); \bar{\theta}, \bar{\lambda}) \neq \omega(A_\bar{x}(x); \theta, \lambda) \neq \omega(A_\bar{x}(x); \theta, \lambda)
\]

Taking \( \lambda = 0 \in P_2 \) in (36), we have

\[
    \tilde{\omega}(A_\bar{x}(x); \bar{\theta}, \bar{\lambda}) = \tilde{\omega}(A_\bar{x}(x); \bar{\theta}, \bar{\lambda}) \neq \omega(A_\bar{x}(x); \bar{\theta}, \bar{\lambda}) \neq \omega(A_\bar{x}(x); \theta, \lambda) \neq \omega(A_\bar{x}(x); \theta, \lambda) \neq \omega(A_\bar{x}(x); \theta, \lambda)
\]
i.e., $\bar{x}_1 + \bar{x}_2 \leq 1$ and so, $\bar{x}_1 + \bar{x}_2 = 1$. Moreover, one has
\[
\bar{\omega}(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \bar{\lambda}) = \bar{\omega}(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \bar{\lambda}) \notin \mathcal{C}, \forall x \in X.
\]
Therefore $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular separation with $\bar{\theta}$ and $\bar{\lambda}$. Motivated by [7], we present the following saddle point conditions for the separations of $\mathcal{K}(\bar{x})$ and $\mathcal{H}$.

**Theorem 4.4.** Let $\bar{x} \in \mathfrak{F}$ and $\mathcal{C} = C_0$. Then the following statements hold:

(i) $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a separation with respect to $\omega$ defined by (6) if and only if there exists $(\bar{\theta}, \bar{\lambda}) \in P_1 \times P_2$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the function $\omega(A_{\bar{x}}(x); \bar{\theta}, \lambda)$ on $X \times P_2$, i.e., for all $x \in X$ and $\lambda \in P_2$, $\omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \omega(A_{\bar{x}}(x); \bar{\theta}, \lambda) \notin \mathcal{C}$ and
\[
\omega(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \omega(A_{\bar{x}}(x); \bar{\theta}, \lambda) \notin \mathcal{C}. \tag{37}
\]

(ii) $\mathcal{K}(\bar{x})$ and $\mathcal{H}$ admit a regular separation with respect to $\omega$ defined by (6) if and only if there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the function $\omega(A_{\bar{x}}(x); \theta, \lambda)$ on $X \times P_2$.

**Proof.** Inspect to the proof of Theorem 4.3. □

From Theorem 4.2, Theorem 4.3 and Theorem 4.4, we obtain the following results, which give the saddle point type optimality conditions for CVOP.

**Corollary 2.** Let $\bar{x} \in \mathfrak{F}$. If there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that $(\bar{x}, \bar{\theta}, \bar{\lambda})$ satisfies (31) and (32), then $\bar{x}$ is a $\mathcal{C}$-solution of CVOP.

**Corollary 3.** Let $\bar{x} \in \mathfrak{F}$ and $\mathcal{C} = C_0$. If there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that $(\bar{x}, \bar{\lambda})$ is a saddle point of the function $\omega(A_{\bar{x}}(x); \theta, \lambda)$ on $X \times P_2$, then $\bar{x}$ is a $\mathcal{C}$-solution of CVOP.

We next apply Example 1 to show Theorem 4.4 and Corollaries 2-3.

**Example 2.** Let us consider Example 1 again. Let $\bar{x} \in \mathfrak{F}$. Assume that $(\bar{x}, \bar{\lambda})$ is a saddle point of the function $\bar{\omega}(A_{\bar{x}}(x); \bar{\theta}, \lambda)$ on $X \times P_2$, where $\bar{\theta}$ and $\bar{\lambda}$ are the same as Example 1. Then
\[
\bar{\omega}(A_{\bar{x}}(x); \bar{\theta}, \lambda) - \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \lambda) \notin \mathcal{C}, \forall x \in X \tag{38}
\]
and
\[
\bar{\omega}(A_{\bar{x}}(x); \bar{\theta}, \lambda) - \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \lambda)
= \left( \begin{array}{cc} 1 - \lambda_1 & -\lambda_2 \\ -\lambda_3 & -\lambda_4 \end{array} \right) \left( \begin{array}{c} \sqrt{x_1 + x_2} - 1 \\ 10 - e^{x_1 + x_2} \end{array} \right) \notin \mathcal{C}, \forall \lambda = \left( \begin{array}{c} \lambda_1 \\ \lambda_3 \\ \lambda_2 \\ \lambda_4 \end{array} \right) \in P_2.
\]

Let $\lambda = 0 \in P_2$ in the above formula, one has
\[
\left( \begin{array}{cc} 1 & 0 \\ 0 & 0 \end{array} \right) \left( \begin{array}{c} \sqrt{x_1 + x_2} - 1 \\ 10 - e^{x_1 + x_2} \end{array} \right) = \left( \begin{array}{c} \sqrt{x_1 + x_2} - 1 \\ 0 \end{array} \right) \notin \mathcal{C}.
\]
This implies that $\sqrt{x_1 + x_2} \leq 1$ and so, $\bar{x}_1 + \bar{x}_2 = 1$. Hence $\bar{x}$ is a $\mathcal{C}$-solution of CVOP. As a matter of fact, (38) $\Rightarrow$ (32), and for any $(\theta, \lambda) \in P_1 \times P_2$,
\[
\bar{\omega}(A_{\bar{x}}(x); \bar{\theta}, \bar{\lambda}) - \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \lambda) = \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \bar{\lambda}) - \bar{\omega}(A_{\bar{x}}(\bar{x}); \bar{\theta}, \lambda).
\]
So, \((\bar{x}, \bar{\theta}, \bar{\lambda})\) satisfies (31) and (32). Note that
\[
\omega(\mathcal{A}_x(x); \bar{\theta}, \bar{\lambda})
= \hat{\theta}(f(\bar{x}) - f(x)) + \hat{\lambda}P_D(g(x))
\]
\[
= \begin{pmatrix} 1 & 1 \\ 0 & 0 \end{pmatrix} \left( \frac{\ln 2 - \ln(x_1 + x_2 + 1)}{1 - \sqrt{1 + x_2}} \right) + \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \left( \frac{\sqrt{x_1 + x_2} - 1}{\max\{0, 10 - e^{x_1 + x_2}\}} \right)
\]
\[
= \begin{pmatrix} \ln 2 - \ln(x_1 + x_2 + 1) \\ 0 \end{pmatrix} \notin \mathcal{C}, \quad \forall x \in X.
\]
Therefore \(K(\bar{x})\) and \(\mathcal{H}\) admit a regular separation.

Conversely, assume that \(K(\bar{x})\) and \(\mathcal{H}\) admit a regular separation with \(\bar{\theta}\) and \(\bar{\lambda}\), where \(\bar{\theta}\) and \(\bar{\lambda}\) are the same as Example 1. Then, \(\bar{x}_1 + \bar{x}_2 = 1\), i.e., \(\bar{x}\) is a solution of \(\mathcal{C}\)-solution of CVOP. It is easy to verify that \(\omega(\mathcal{A}_x(x); \bar{\theta}, \bar{\lambda}) = 0\). So, (38) holds and
\[
\omega(\mathcal{A}_x(x); \bar{\theta}, \bar{\lambda}) - \omega(\mathcal{A}_x(x); \bar{\theta}, \bar{\lambda})
= \begin{pmatrix} (e - 10)\lambda_2 \\ (e - 10)\lambda_4 \end{pmatrix} \notin \mathcal{C}, \quad \forall \lambda = \begin{pmatrix} \lambda_1 & \lambda_2 \\ \lambda_3 & \lambda_4 \end{pmatrix} \in P_2.
\]
Consequently, \((\bar{x}, \bar{\lambda})\) is a saddle point of the function \(\omega(\mathcal{A}_x(x); \bar{\theta}, \bar{\lambda})\) on \(X \times P_2\), where \(\bar{\theta}\) and \(\bar{\lambda}\) are the same as Example 1.

Particularly, if \(\mathbb{R}^l = \mathbb{R}, C = \mathbb{R}_+ = [0, +\infty]\) and \(f : \mathbb{R}^n \to \mathbb{R}\), the following results show that saddle point conditions are equivalent to the separations of \(K(\bar{x})\) and \(\mathcal{H}\) and \(\bar{x} \in \mathfrak{F}\).

**Theorem 4.5.** Let \(C = \mathbb{R}_+, f : \mathbb{R}^n \to \mathbb{R}\) and \(\bar{x} \in X\). Then, the following hold:

(i) \(K(\bar{x})\) and \(\mathcal{H}\) admit a separation with respect to \(\omega\) defined by (6) and \(\bar{x} \in \mathfrak{F}\) if and only if there exists \((\theta, \lambda) \in P_1 \times P_2\) such that \((\bar{x}, \bar{\theta}, \bar{\lambda})\) is a saddle point of the function \(\omega(\mathcal{A}_x(x); \theta, \lambda)\) on \(X \times P_1 \times P_2\), i.e.,
\[
\omega(\mathcal{A}_x(x); \bar{\theta}, \bar{\lambda}) \leq \omega(\mathcal{A}_x(x); \bar{\theta}, \lambda) \leq \omega(\mathcal{A}_x(x); \theta, \lambda), \quad \forall (x, \theta, \lambda) \in X \times P_1 \times P_2.
\]

(ii) \(K(\bar{x})\) and \(\mathcal{H}\) admit a regular separation with respect to \(\omega\) defined by (6) and \(\bar{x} \in \mathfrak{F}\) if and only if there exists \((\theta, \lambda) \in (P_1 \setminus \Theta) \times P_2\) such that \((\bar{x}, \bar{\theta}, \bar{\lambda})\) is a saddle point of the function \(\omega(\mathcal{A}_x(x); \theta, \lambda)\) on \(X \times P_1 \times P_2\).

**Proof.** (i) From Theorem 4.3(i)(ii), we only need to prove that (39) implies \(\bar{x} \in \mathfrak{F}\), i.e., \(g(\bar{x}) \in D\). Let \(\theta = \bar{\theta}\) in (39). Then
\[
\omega(g(\bar{x}); \bar{\lambda}) - \omega(g(\bar{x}); \lambda) \notin \mathcal{C} = [0, +\infty[\), \(\forall \lambda \in P_2,
\]
i.e.,
\[
\omega(g(\bar{x}); \bar{\lambda}) \leq \omega(g(\bar{x}); \lambda), \quad \forall \lambda \in P_2.
\]
Suppose that \(g(\bar{x}) \notin D\). By (10), there exists \(\bar{\lambda} \in P_2\) such that
\[
\omega(g(\bar{x}); \bar{\lambda}) < 0.
\]
It follows from (11) that for any \(\alpha \geq 0\), there exists \(\bar{\lambda}_\alpha \in P_2\) such that
\[
\omega(g(\bar{x}); \bar{\lambda}) \leq \omega(g(\bar{x}); \bar{\lambda}_\alpha) = \alpha \omega(g(\bar{x}); \bar{\lambda}) \to -\infty \text{ as } \alpha \to +\infty,
\]
which is a contradiction. Consequently, we have \(\bar{x} \in \mathfrak{F}\).

(ii) Similar to the proof of (i).
Remark 4. Compared with [29, Theorem 4.3], Theorem 4.5(i) does not require that the condition $\omega(0, \theta) = 0$ for any $\theta \in P_1$.

Corollary 4. Let $C = \mathbb{R}_+, f : \mathbb{R}^n \to \mathbb{R}$ and $\bar{x} \in X$. If there exists $(\bar{\theta}, \bar{\lambda}) \in (P_1 \setminus \Theta) \times P_2$ such that $(\bar{x}, \bar{\theta}, \bar{\lambda})$ satisfies (39), then $\bar{x}$ is a solution of (1).

In the following, we propose a class of generalized vector-valued Lagrangian functions of CVOP via the class of separation functions (6):

$$\mathcal{L}(x; \theta, \lambda) = \varphi(f(x); \theta) - \omega(A_x(x); \theta, \lambda), \quad \forall (x, \theta, \lambda) \in X \times P_1 \times P_2,$$

(40)

where $\bar{x} \in X, \varphi : \mathbb{R}^\ell \times P_1 \to \mathbb{R}^\ell$ is additive with respect to the first argument, i.e., for each $\theta \in P_1$,

$$\varphi(u + v; \theta) = \varphi(u; \theta) + \varphi(v; \theta), \quad \forall u, v \in \mathbb{R}^\ell.$$

(41)

It follows from (41) that for each $\theta \in P_1, u \in \mathbb{R}^\ell$, $\varphi(0; \theta) = 0$ and $\varphi(-u; \theta) = -\varphi(u; \theta)$. Moreover, one has

$$\mathcal{L}(x; \theta, \lambda) = \varphi(f(x); \theta) - \omega(g(x); \lambda), \quad \forall (x, \theta, \lambda) \in X \times P_1 \times P_2.$$

This shows that $\mathcal{L}(x; \theta, \lambda)$ is independent on $\bar{x} \in X$. Specially, if $\omega(g(x); \lambda) := \lambda g(x)$ and $\varphi(f(x); \theta) := \theta f(x)$ or, $\varphi(f(x); \theta) := f(x)$, then the generalized Lagrangian function $\mathcal{L}(x; \theta, \lambda)$ coincides with the classic Lagrangian function. Besides, the generalized Lagrangian function $\mathcal{L}(x; \theta, \lambda)$ is distinct with that of [8, 20, 21, 29, 30, 22] even in the real-valued case.

Definition 4.6. $(\bar{x}, \bar{\theta}, \bar{\lambda}) \in X \times P_1 \times P_2$ is said to be:

(i) a type-I saddle point of $\mathcal{L}(x; \theta, \lambda)$ iff, for any $(x, \theta, \lambda) \in X \times P_1 \times P_2$,

$$\mathcal{L}(x; \bar{\theta}, \bar{\lambda}) - \mathcal{L}(x; \bar{\theta}, \bar{\lambda}) \not\in -\mathcal{C} \quad \text{and} \quad \mathcal{L}(\bar{x}; \bar{\theta}, \bar{\lambda}) - \mathcal{L}(x; \bar{\theta}, \bar{\lambda}) \not\in -\mathcal{C}.$$

(ii) a type-II saddle point of $\mathcal{L}(x; \theta, \lambda)$ iff, for any $(x, \lambda) \in X \times P_2$,

$$\mathcal{L}(x; \bar{\theta}, \bar{\lambda}) - \mathcal{L}(\bar{x}; \bar{\theta}, \bar{\lambda}) \not\in -\mathcal{C} \quad \text{and} \quad \mathcal{L}(\bar{x}; \bar{\theta}, \bar{\lambda}) - \mathcal{L}(x; \bar{\theta}, \bar{\lambda}) \not\in -\mathcal{C}.$$

It is easy to see that every type-I saddle point of $\mathcal{L}(x; \theta, \lambda)$ is its type-II saddle point. We next show the relations between the saddle points of $\mathcal{L}(x; \theta, \lambda)$ and that of $\omega(A_x(x); \theta, \lambda)$.

Proposition 3. $(\bar{x}, \bar{\theta}, \bar{\lambda}) \in X \times P_1 \times P_2$ is a type-II saddle point of $\mathcal{L}(x; \theta, \lambda)$ if and only if $(\bar{x}, \bar{\theta}, \bar{\lambda})$ is a saddle point of $\omega(A_x(x); \theta, \lambda)$ on $X \times P_1 \times P_2$.

Proof. By the definition of $\mathcal{L}(x; \theta, \lambda)$, one has

$$\mathcal{L}(x; \bar{\theta}, \bar{\lambda}) - \mathcal{L}(\bar{x}; \bar{\theta}, \bar{\lambda})$$

$$= \varphi(f(x); \bar{\theta}) - \omega(g(x); \bar{\lambda}) - \left[\varphi(f(\bar{x}); \bar{\theta}) - \omega(g(\bar{x}); \bar{\lambda})\right]$$

$$= -\left[\varphi(f(\bar{x}); \bar{\theta}) - \varphi(f(x); \theta)\right] - \omega(g(x); \bar{\lambda}) + \omega(g(\bar{x}); \bar{\lambda})$$

$$= -\left[\varphi(f(\bar{x}) - f(x); \bar{\theta}) + \omega(g(x); \bar{\lambda})\right] + \omega(g(\bar{x}); \bar{\lambda})$$

$$= \varphi(f(x) - f(\bar{x}); \bar{\theta}) + \omega(g(x); \bar{\lambda}) - \left[\varphi(f(\bar{x}) - f(x); \bar{\theta}) + \omega(g(x); \bar{\lambda})\right]$$

$$= \omega(A_x(x); \bar{\theta}, \bar{\lambda}) - \omega(A_x(x); \bar{\theta}, \bar{\lambda}), \quad \forall x \in X$$

\(\square\)
Proposition 5. \( (\bar{x}, \tilde{\theta}, \tilde{\lambda}) \in X \times P_1 \times P_2 \) is a type-II saddle point of \( \mathcal{L}(x; \theta, \lambda) \) if and only if \( (\bar{x}, \lambda) \) is a saddle point of \( \omega(A_{\bar{x}}(x); \tilde{\theta}, \tilde{\lambda}) \) on \( X \times P_2 \).

Proof. Inspect the proof of Proposition 3.

Corollary 5. (i) If \( (\bar{x}, \tilde{\theta}, \tilde{\lambda}) \in X \times P_1 \times P_2 \) is a type-I saddle point of \( \mathcal{L}(x; \theta, \lambda) \), then \( (\bar{x}, \tilde{\theta}, \tilde{\lambda}) \) is a saddle point of \( \omega(A_{\bar{x}}(x); \theta, \lambda) \) on \( X \times P_1 \times P_2 \).

(ii) If \( (\bar{x}, \tilde{\theta}, \tilde{\lambda}) \in X \times P_1 \times P_2 \) is a type-I saddle point of \( \mathcal{L}(x; \theta, \lambda) \), then \( (\bar{x}, \tilde{\lambda}) \) is a saddle point of \( \omega(A_{\bar{x}}(x); \theta, \lambda) \) on \( X \times P_2 \).

Proof. It follows readily from Propositions 3 and 4.

Remark 5. Proposition 3 improves [7, Theorem 8, p. 178]. It follows from (41) that for each \( \theta \in P_1 \), \( \overline{\mathcal{L}}(\cdot; \theta) \) has the following form:

\[
\overline{\mathcal{L}}(u; \theta) = \phi(\theta)u, \quad \forall \ u \in \mathbb{R}^\ell,
\]

where the function \( \phi : P_1 \to \mathbb{R}^{\ell \times \ell} \) satisfies that \( \phi(\theta) \in C^*_C \) for all \( \theta \in P_1 \) and \( \phi(\theta) \neq 0 \) for all \( \theta \in P_1 \setminus \Theta \) or, \( \phi : P_1 \to \mathbb{R} \) satisfies that \( \phi(\theta) \geq 0 \) for all \( \theta \in P_1 \) and \( \phi(\theta) \neq 0 \) for all \( \theta \in P_1 \setminus \Theta \). Particularly, let \( \mathcal{C} := C_0, D := \mathbb{R}^m_+ \) and \( P_1 := C^*_C \) and \( P_2 := D^*_C \). If \( \overline{\mathcal{L}}(u; \theta) := \theta u, \omega(v; \lambda) := \lambda v \), then Proposition 3 reduces to [7, Theorem 8, p. 178].

Now we present the sufficient optimality conditions for CVOP via the saddle point conditions.

Proposition 5. Let \( \bar{x} \in \mathcal{F} \). If there exists \( (\tilde{\theta}, \tilde{\lambda}) \in (P_1 \setminus \Theta) \times P_2 \) such that \( (\bar{x}, \tilde{\theta}, \tilde{\lambda}) \) is a type-I (II) saddle point of \( \mathcal{L}(x; \theta, \lambda) \), then \( \bar{x} \) is a \( \mathcal{C} \)-solution of CVOP.

Proof. Combining Theorem 4.3, Corollaries 2.3 and 5 with Propositions 3 and 4.

Remark 6. Since the cone \( D \) does not require \( \text{int} \ D \neq \emptyset \), it is possible that \( \text{int} \ H = \text{int} (\mathcal{C} \times D) = \emptyset \). So, the nonconvex separation theorem [17, Theorem 2.3.6, p. 44] can not be applied to the CVOP even in the scalar sense. Even if the nonconvex separation theorem [17, Theorem 2.3.6, p. 44] holds by adding extra assumptions such as \( \text{int} \ H = \text{int} \mathcal{H} = \text{int} (\mathcal{C} \times D) \neq \emptyset \) and there exists \( k^0 \in \mathbb{R}^{\ell + m} \setminus \{0\} \) such that \( \mathcal{H} + [0, +\infty) \cdot k^0 \subseteq \mathcal{H}, \mathcal{H} + (0, +\infty) \cdot k^0 \subseteq \text{int} \mathcal{H}, \mathcal{H} \) does not contain lines parallel to \( k^0 \) and \( \mathbb{R} \cdot k^0 - \mathcal{H} = \mathbb{R}^{\ell + m} \), we can only obtain that the separation function \( \varphi_{\mathcal{H}, \mathcal{K}_0} \) defined by (2.33) in [17, p. 39] is weak separation function by [17, Theorem 2.3.1, p. 40], but it is not regular weak separation function.
5. **Conclusions.** In this paper, a general vector-valued separation function for CVOP, which includes the linear vector-valued separation function, is proposed. Further, the existence of a saddle point for the vector-valued separation is characterized by the (regular) separation of the subsets $K(\bar{x})$ and $H$ in the image space. By the vector-valued separation function, a class of generalized vector-valued Lagrangian functions is introduced for CVOP without involving any elements of its feasible set. The relationships between the saddle points of the generalized Lagrangian functions and that of $\omega(A_\bar{x}(\cdot);\cdot)$ are also established via ISA. Lastly, optimality conditions for CVOP are derived by the saddle-point conditions.

It is well-known that optimality conditions of VOP and saddle points conditions of its Lagrangian functions play a crucial role in studying its duality such as Wolfe duality, Mond-Weir duality, Lagrangian duality and mixed duality. As far as we know, there are very few results on duality of VOP with data uncertainty from the image space point of view. So the presented results in this paper will stimulate the duality of VOP with data uncertainty/free-uncertainty by the ISA.

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