Fast Polarization and Finite-Length Scaling for Non-Stationary Channels

Hessam Mahdavifar
Department of Electrical and Computer Engineering, University of Michigan Ann Arbor, USA
Email: hessam@umich.edu

Abstract—We consider the problem of polar coding for transmission over a non-stationary sequence of independent binary-input memoryless symmetric (BMS) channels \( \{W_i\}_{i=1}^\infty \), where the \( i \)-th encoded bit is transmitted over \( W_i \). We show, for the first time, a polar coding scheme that achieves the effective average symmetric capacity
\[
T((W_i)_{i=1}^\infty) \defeq \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N I(W_i),
\]
assuming that the limit exists. The polar coding scheme is constructed using Arıkan’s channel polarization transformation in combination with certain permutations at each polarization level and certain skipped operations. This guarantees a fast polarization process that results in polar coding schemes with block lengths upper bounded by a polynomial of \( 1/\epsilon \), where \( \epsilon \) is the gap to the average capacity. More specifically, given an arbitrary sequence of BMS channels \( \{W_i\}_{i=1}^N \) and \( P_e \), where \( 0 < P_e < 1 \), we construct a polar code of length \( N \) and rate \( R \) guaranteeing a block error probability of at most \( P_e \) for transmission over \( \{W_i\}_{i=1}^N \) such that
\[
N \leq \frac{\kappa}{(\tilde{T}_N - R)^\mu},
\]
where \( \mu \) is a constant, \( \kappa \) is a constant depending on \( P_e \) and \( \mu_e \), and \( \tilde{T}_N \) is the average of the symmetric capacities \( I(W_i) \), for \( i = 1, 2, \ldots, N \). We further show a numerical upper bound on \( \mu_e \) that is: \( \mu_e \leq 10.78 \). The encoding and decoding complexities of the constructed polar code preserves \( O(N \log N) \) complexity of Arıkan’s polar codes.

Index Terms—Polar codes, finite-length scaling, non-stationary channel, polarization

I. INTRODUCTION

Polar codes were introduced by Arıkan in the seminal work of [1]. Polar codes are the first family of codes for the class of binary-input symmetric discrete memoryless channels that are provable to be capacity-achieving with low encoding and decoding complexity [1]. Polar codes and polarization phenomenon have been successfully applied to various problems such as wiretap channels [2], data compression [3], [4], broadcast channels [5], and multiple access channels [6], [7]. The error exponent and finite-length scaling behavior of polar codes are also very well-studied [8]–[13].

A problem of interest in polar coding is the following: what is the most general set-up in a single user point-to-point communication in which achievability can be established by polar coding? In this paper, we consider an extension of classical communication over binary memoryless channels where the underlying channel changes arbitrarily at each time instance. In this set-up, a binary code of length \( N \) is transmitted over a non-stationary sequence of channels \( \{W_i\}_{i=1}^N \), where \( i \)-th bit is transmitted over \( W_i \). The sequence of channels is assumed to be known at both the transmitter and the receiver prior to communication. The channel polarization problem for this set-up was first considered in [14], where it is shown that the polarization happens by applying Arıkan’s polarization transform. This result does not conclude an achievability scheme as it requires a fast polarization in order to show that, in an asymptotic sense, the probability of error can be made arbitrarily small while the rate is arbitrarily close to the average symmetric capacity.

In this paper we show how a fast polarization is possible for an arbitrary sequence of non-stationary BMS channels. To this end we modify Arıkan’s channel polarization transform as the polarization process evolves. Certain permutations are applied to each sub-block of bit-channels at each polarization level before channel combining operations are applied. Also, the channel combining operations are skipped whenever it can not be guaranteed they improve a certain measure of polarization. The speed of polarization is defined and bounds on the speed of the resulting polarization process are derived which show a fast polarization. Furthermore, a two stage polarization process is shown which results in construction of polar coding schemes that perform arbitrarily close to the average capacity of the channels. In particular, given \( N = 2^n \) and \( P_e > 0 \), we construct a polar code of length \( N \) with the gap to the average capacity \( \epsilon \) and the probability of block error upper bounded by \( P_e \) such that \( N \leq \frac{\kappa}{\epsilon \mu} \), where \( \mu \) is a constant and \( \kappa \) is a constant that depends on \( \mu \) and \( P_e \). Furthermore, a numerical upper bound on the parameter \( \mu \) is established. As a result, as the length of the constructed polar code grows large, we achieve the limit of the average capacities, assuming that the limit exists.

The rest of this paper is organized as follows. In Section II some background on polar codes and a summary of prior work on the subject are provided. In Section III a fast polarization scheme is shown by specifying certain permutations that we apply at each polarization level. In Section IV the speed of polarization is defined and it is shown that the polarization scheme of Section III in combination with another modification results in a fast polarization. In Section V it is shown how polar coding schemes can be constructed via a two stage polarization in order to achieve the average capacity. Furthermore, the finite-length scaling behavior of the constructed codes is bounded. Finally, the paper is concluded in Section VI.

II. BACKGROUND AND PRIOR WORK

The channel polarization phenomenon was discovered by Arıkan [1]. The basic Arıkan’s polarization transform takes two independent copies of a binary-input discrete memoryless channel (B-DMC) \( W : \{0, 1\} \to \mathcal{Y} \) and turns them into
\((W^-, W^+)\). It is proved that by repeating the same process recursively, almost all the resulting bit-channels \(W^+\), for \(s^n \in \{+, -\}^n\), become either \textit{almost} noiseless with capacity very close to 1 or \textit{almost} noisy, with capacity very close to 0 \([1]\).

The same basic polarization transform can be applied when the two underlying channels \(W_1 : \{0, 1\} \rightarrow \mathcal{B}_1\) and \(W_2 : \{0, 1\} \rightarrow \mathcal{B}_2\) are independent but not identical. The channel combination operations are defined as follows:

\[
W_1 \boxplus W_2(y_1, y_2|u_1) = \frac{1}{2} \sum_{u_2 \in \{0, 1\}} W_1(y_1|u_1 \oplus u_2)W_2(y_2|u_2),
\]

\[
W_1 \otimes W_2(y_1, y_2, u_1|u_2) = \frac{1}{2} W_1(y_1|u_1 \oplus u_2)W_2(y_2|u_2),
\]

where \(y_1 \in \mathcal{B}_1, y_2 \in \mathcal{B}_2\), and \(u_1, u_2 \in \{0, 1\}\). In fact, when \(W_1 = W_2 = W\), \(W \boxplus W = W^-\) and \(W \otimes W = W^+\).

The problem of polarizing an arbitrary non-stationary sequence of channels \(\{W_i\}_{i=1}^{\infty}\) was first considered in \([14]\). Let \(W_{n,i}\) denote the \(i\)-th bit-channel after \(n\) level of polarization of the sequence \(\{W_i\}_{i=1}^{N}\), where \(N = 2^n\). The parameters \(N\) and \(n\), where \(N = 2^n\), are always used in this paper to specify the length of the polarization block and the constructed code. Let \([N]\) denote the set of positive integers less than or equal to \(N\).

It is proved in \([14]\) that the fraction of non-polarized channels approach zero, i.e., for every \(0 < a \leq b < 1\),

\[
\liminf_{N \to \infty} \frac{1}{N} \{i \in [N] : I(W_{n,i}) \in [a, b]\} = 0 \tag{3}
\]

Note that this result does not conclude that a polar code can be constructed that approaches the average symmetric capacity of \(\{W_i\}_{i=1}^{\infty}\). In order to establish capacity achieving property, one needs to prove \([7]\) while \(Na\) and \(N(1 - b)\) also approach 0 as \(N \to \infty\). Also, in order to study the finite-length scaling behavior, one needs to characterize how fast the expression in \([5]\) approaches zero.

In the conventional set-up, where the underlying channels are identical copies of the same channel \(W\), it is shown that the required block length \(N\) to guarantee a block error probability of \(P_e\) is upper bounded by \(c/e^H\), where \(c\) is the gap to the symmetric capacity, and \(\mu\) and \(c\) are constants \([9, 10]\).

Furthermore, it is shown in \([9]\) how to upper bound and lower bound this finite-length scaling behavior for different classes of channels. In other words, assuming that \(N\) scales as \(1/e^H\), upper bounds and lower bounds on the scaling exponent \(\mu\) are derived in \([9]\). In a sense, in this paper we extend these results to the case where the channels are independent but not identical by modifying Arıkan’s channel polarization transform in many important respects.

Arıkan used the Bhattacharyya parameter of \(W\), denoted by \(Z(W)\), to measure the reliability of \(W\) and to upper bound the block error probability of polar codes. The Bhattacharyya parameter is defined as

\[
Z(W) \coloneqq \sum_{y \in \mathcal{Y}} \sqrt{W(y|0)W(y|1)}. \tag{4}
\]

It is known \([11, 15]\) that

\[
Z(W_1 \otimes W_2) = Z(W_1)Z(W_2) \tag{4}
\]

and

\[
Z(W_1 \boxplus W_2) \leq Z(W_1) + Z(W_2) - Z(W_1Z(W_2)). \tag{5}
\]

III. A FAST POLARIZATION SCHEME

In this section we show a polarization process for non-stationary channels which combines Arıkan’s polarization transform with proper sorting of the bit-channels at each polarization level. In the next section it will be shown how this polarization scheme can be used, in combination with other modifications, to establish a fast polarization.

Suppose that a sequence of \(N\) channels \(\{W_i\}_{i=1}^{N}\), where \(N = 2^n\), is given. A straightforward method of applying Arıkan’s polarization transform results in the following process. In the first level of polarization, for \(i \in \mathbb{N}/2\), \(W_{2i-1}\) and \(W_{2i}\) are combined and

\[
W_{1,i} = W_{2i-1} \boxplus W_{2i}, \quad W_{1,N/2+i} = W_{2i-1} \otimes W_{2i}. \tag{6}
\]

In the second level of polarization, the same procedure is applied to \(W_{1,i} \boxplus W_{2i}\) and \(W_{1,i} \otimes W_{2i}\) in parallel. In general, in the \(j\)-th level of polarization, and for \(i \in \mathbb{N}/2\), let

\[
i = b2^{n-j} + r,
\]

where \(r \in \{0, 2^{n-j}\} \) and \(0 \leq l = 2^{j-1} - 1\). Then we have

\[
W_{j,(2j)2^{n-j}+i} = W_{j-1,2^{j-1}+i} \boxplus W_{j-1,2^{j-1}+i}, \quad W_{j,(2j+1)2^{n-j}+i} = W_{j-1,2^{j-1}+i} \otimes W_{j-1,2^{j-1}+i}, \tag{6}
\]

where the \textit{initial} channels at the level zero of polarization are \(W_{0,i} = W_i\).

In the original channel polarization set-up of \([1]\), where all initial channels are identical, all the channels in each sub-block of length \(2^n-j\), in the \(j\)-th level of polarization, are also identical. However, in our set-up, where the initial channels are not identical, combining channels that are \textit{very different}, in terms of either the symmetric capacity or the Bhattacharyya parameter, does not lead to a good polarization. Therefore, roughly speaking, in order to minimize the effect of combining non-identical channels, we sort the channels in each sub-block, according to their Bhattacharyya parameter, before combining them in the next polarization level. A deterministic process with certain permutations at each polarization is defined next followed by specifying the permutations.

Definition 1: \textit{(Deterministic Polarization I)} Consider a polarization level \(j, j = 0, 1, \ldots, n - 1\), with the sequence of channels \(\{W_{j,i}\}, i \in \mathbb{N}\). This sequence is split into \(2^j\) sub-blocks of equal length \(2^n-j\) of consecutively indexed channels. Consider a sub-block \(\{W_{j,l2^{n-j}+i}\}, \) for \(i \in \{2^n-j\}\), where \(0 \leq l \leq 2^j - 1\). Let \(\pi_{j,i} : \{2^n-j\} \rightarrow \{2^n-j\}\) denote a permutation which is determined according to a certain criteria.

Then \(W_{j,l2^{n-j}+i}\) is replaced by \(W_{j,l2^{n-j}+\pi_{j,i}(i)}\) after which the channel combining operations of \([6]\) are applied to compute the channels of the next polarization level \(j + 1\). We refer to this polarization process, which is a combination of Arıkan’s polarization process and certain permutations, as deterministic polarization \(1\). For the sake of brevity, we often drop the permutations \(\pi_{j,i}\) from the description of the deterministic polarization \(1\) assuming that they are known and are applied, as described above, unless otherwise is mentioned.
In order to guarantee a fast polarization, the permutations $\pi_{j,t}$ are set in such a way that when they are applied to sub-blocks of channels, the sequence of Bhattacharyya parameters within each sub-block becomes decreasing, i.e.,

$$Z(W_{j,2^n-j+i}+\pi_{j,t}(i)) \geq Z(W_{j,2^n-j+i}+\pi_{j,t}(i'))$$

if and only if $\pi_{j,t}(i) \leq \pi_{j,t}(i')$, where $i, i' \in [2^n-j]$.

Next we describe how low complexity encoding and decoding of original polar codes [1] can be adopted to accommodate the deterministic polarization 1 described above. Note that a memory of size $O(N \log N)$ must be added in order to save the permutations $\pi_{j,t}$ at both the encoder and the decoder prior to communication begins. Since an efficient implementation of the low complexity successive cancellation (SC) decoder of polar codes requires only $O(N)$ space complexity, this means that the space complexity is increased to $O(N \log N)$ which is still quasi-linear with $N$.

The decoding process of polar codes consists of computations of likelihood ratios (LRs) through an $(n+1) \times N$ trellis reflecting the channel polarization operations [1]. The LRs at level zero are the channel reliabilities. At level $j$ of the decoding trellis, for $j \in [n]$, the LR combination operations are done within $2^j$ sub-blocks of length $2^{n-j}$ each. The hard decisions on information bits are decided at the level $n+1$ of the trellis, where these hard decisions are propagated back through the trellis. The only modification that the deterministic polarization 1 adds on top of this architecture is to apply the permutation $\pi_{j,t}$ to the $l$-th sub-block at level $j$ before processing that sub-block. Also, the inverse permutation $\pi_{j,t}^{-1}$ is applied to the hard decisions that are propagated back through the trellis. The resulting complexity of these changes will be negligible compared to $O(N \log N)$ decoding complexity of polar codes. Also, since applying a permutation to a sequence can be done in one unit of time, the decoding latency $O(N)$ of polar codes is not affected by this modification. Similarly, the encoding process will be modified by invoking $\pi_{j,t}^{-1}$’s accordingly without affecting the total complexity or latency of the encoder.

IV. Speed of Polarization

Consider a sequence of binary memoryless symmetric (BMS) channels $\{W_i\}_{i=1}^N$. A metric for measuring polarization within this sequence of channels can be defined as follows:

$$E(\{W_i\}_{i=1}^N) \overset{\text{def}}{=} \frac{1}{N} \sum z_i(1-z_i) \quad (7)$$

where $z_i = Z(W_i)$. If $E$ is close to zero, it implies that most of the channels have Bhattacharyya parameter either close to 0 or close to 1. Therefore, one wants to have a smaller $E$ in order to have a better polarization. This measure is actually used in [1] in order to prove polarization, where the convergence of the expected value $E \{Z_n(1-Z_n)\}$ is analyzed for a certain random process $\{Z_n\}$. However, since the probability space of the random variable $Z_n$ is finite and the distribution is uniform, this expected value is equivalent to the average in (7) measured over the $2^n$ bit-channels after $n$ levels of polarization denoted by $E_n$. In order to characterize the finite-length behavior of polar codes one could study how fast $E_n$ goes to zero as $n$ grows large.

For simplicity, consider the case that initial $W_i$’s are identical and are binary erasure channel (BEC) with erasure probability $p$, for some $p \in (0, 1)$. For a BEC $W$, let $z = Z(W)$, $z^- = Z(W^-)$ and $z^+ = Z(W^+)$. Then

$$\frac{z^+(1-z^+)+z^-(1-z^-)}{2z(1-z)} = 1 - z(1-z).$$

Note that $\sup z \in (0, 1), 1 - z(1-z) = 1$. This means that, intuitively, as the polarization process evolves, the speed of convergence of $E_n$ reduces and one can not bound it away from 0. In order to resolve this problem, functions of the type $z^b(1-z)^b$ are used in [9] to replace quantities $z_i(1-z_i)$ in (7).

In this paper, we take a similar approach and define

$$f(z) \overset{\text{def}}{=} z^b(1-z)^b \quad (8)$$

where the value of $b$ $\in (0, 1)$ is specified later. Then the definition in (7) is modified as follows:

$$E(\{W_i\}_{i=1}^N) \overset{\text{def}}{=} \frac{1}{N} \sum f(z_i) \quad (9)$$

where $z_i = Z(W_i)$.

Given the initial channels $W_{0,i} = W_i$, for $i \in [N]$, $E_{j,n}$, for $j = 0, 1, \ldots, n$, is defined as follows:

$$E_{j,n} = E(\{W_{j,i}\}_{i=1}^N) \quad (10)$$

where $\{W_{j,i}\}_{i=1}^N$ is the sequence of polarized channels in the $j$-th level of polarization according to the procedure explained in the previous section.

Definition 2: Given the initial channels $\{W_i\}_{i=1}^N$, the speed of polarization at level $j$, where $j \in [n]$, of the polarization process is defined as

$$\eta_{j,n} \overset{\text{def}}{=} - \log E_{j,n}/E_{j-1,n} \quad (11)$$

and the average speed of polarization at level $n$ is defined as

$$\bar{\eta}_n \overset{\text{def}}{=} - \frac{1}{n} \log E_{n,n} \quad (12)$$

We call the polarization fast if $\lim \inf_{n \to \infty} \bar{\eta}_n > 0$ for a given sequence of channels $\{W_i\}_{i=1}^\infty$.

The goal is to find a lower bound for the speed of polarization $\eta_n$ and consequently for $\bar{\eta}_n$. This will require upper bounding expressions of the following form away from 1:

$$\Delta_f(W, W') \overset{\text{def}}{=} \frac{f(Z(W) \oplus W')} {f(Z(W)) + f(Z(W'))} \quad (13)$$

Motivated by [4, 5], and Lemma 9 on relations between Bhattacharyya parameters of combined channels, $g(z_1, z_2)$, for any $z_1, z_2 \in (0, 1)$, is defined as follows:

$$g(z_1, z_2) \overset{\text{def}}{=} \sup_{z \in [z_1, z_2]} \frac{f(z_1) + f(z_2)}{f(z_1) + f(z_2)} \quad (14)$$

Then we have

$$\Delta_f(W, W') \leq g(Z(W), Z(W')) \quad (15)$$
Therefore, one can first bound $g(Z(W), Z(W'))$ and then use it in combination with (15) in order to upper bound $\Delta_f(W, W')$.

Note that since $f(.)$ is a convex function over $(0, 1)$ with the maximum at $\frac{1}{2}$, the supremum in (14) happens at $z^- = \frac{1}{2}$, if $\frac{1}{2} \in [\sqrt{z_1^2 + z_2^2} - z_1 z_2, z_1 + z_2 - z_1 z_2]$ and otherwise, it happens at either of the extreme cases of $\sqrt{z_1^2 + z_2^2} - z_1 z_2$ or $z_1 + z_2 - z_1 z_2$.

If we do not impose any restriction on $z_1$ and $z_2$, the supremum of $g(z_1, z_2)$ over all $z_1, z_2 \in (0, 1)$ is at least 1. This can be observed by setting $z_1 = p, z_2 = 1 - p$, and letting $p \to 0$. Similarly, if $W'$ is BEC($p$) and $W$'s BEC($1 - p$), then the supremum of $\Delta_f(W, W')$, defined in (13), is also at least 1. In order to resolve this problem, we quantize $(0, 1)$ into a certain number of sub-intervals and consider the expression in (13) only when both $Z(W_1)$ and $Z(W_2)$ belong to the same sub-interval. In other words, the expression in (13) is considered only when $Z(W_1)/Z(W_2)$ and/or $Z(W_1) - Z(W_2)$ are less than a certain threshold thereby making it possible to bound away the supremum of $g(Z(W_1), Z(W_2))$ from 1. This idea is elaborated through the rest of this section.

**Definition 3:** (Fine Quantization of (0, 1)) The $[0, 1]$ interval is quantized into roughly $c_1 \log N + c_2$ sub-intervals as follows. The interval $[c, c + 1 - c]$ is quantized into equal subintervals of length $\lambda$, for some $c > 0$. Then $[0, c]$ is quantized into $[0, c/2m], [c/2m, c/2m - 1], \ldots, [c/4, c/2], [c/2, c]$, where $c/2m \leq N^{-\tau}$ and $\tau$ is specified later. Therefore, we set
\[
m = \left\lfloor \log c + \tau \log N \right\rfloor.
\]
Similarly, $[1 - c, 1]$ is quantized into $[1 - c, 1 - c/2], \ldots, [1 - c/2m - 1, c/2m]$, $[1 - c/2m, 1]$.

For numerical evaluation we let $\lambda = \epsilon = 0.1$. These specific choices of parameters slightly affect the speed of polarization that we will compute numerically. In total, we have
\[
2(m + 1) + \frac{1 - 2c}{\lambda} \leq 2\tau \log N + 12 + 2 \log c
\]
(16)
sub-intervals, where $c_1 = 2\tau$ and $c_2 = 12 - 2 \log 10 < 6$.

Motivated by the quantization of [0, 1] described in Definition 3, the function $h(z)$ for $z \in (0, 1)$ is defined as follows:

\[
h(z) = \begin{cases}
    \sup_{z \in [z_1, z_2]} g(z, z') & \text{if } z < c, \\
    \sup_{z \in [z, \min(z + \lambda, 1 - c)]} g(z, z') & \text{if } c \leq z \leq 1 - c, \\
    \sup_{z \in [z, 1 + z/2]} g(z, z') & \text{if } z > 1 - c,
\end{cases}
\]
(17)
where $g(z_1, z_2)$ is defined in (14) and $\lambda = \epsilon = 0.1$ is used for numerical evaluation purposes. It can be shown that $h(z)$ is continuous and bounded while its derivative is also bounded.

Note that the right and the left derivatives may not be the same for $z = c$ and $z = 1 - c$, however, it does not affect the argument. Then,
\[
\eta \triangleq -\log \sup_{z \in (0, 1)} h(z)
\]
(18)
can be numerically estimated by quantizing $(0, 1)$ with fine enough resolution. Also, note that $\eta$ depends on the parameter $b$ in the definition of $f(z) = z^b(1 - z)^b$ and can be optimized by picking a proper $b$. Intuitively, larger $\eta$ leads to a larger speed of polarization. Using a numerical evaluation, we pick $b = 0.72$. Then $\eta$ is estimated to be 0.1391. For illustration purposes, $h(z)$ is shown in Figure 1 for $b = 0.72$.

**Lemma 1:** Let $W$ and $W'$ be two BMS channels with $z_1 = Z(W)$ and $z_2 = Z(W')$. If $c/2m \leq z_1, z_2 \leq 1 - c/2m$, and $z_1, z_2$ belong to the same sub-interval of the fine quantization, described in Definition 3 then
\[
\Delta_f(W, W') \leq 2^{-n},
\]
where $\Delta_f(\cdot, \cdot)$ is defined in (13) and $\eta$ is given in (18).

**Proof:** By (14), (5), (12), and the definition of $g(.)$ in (14) we have $\Delta_f(W, W') \leq g(z_1, z_2)$. Suppose that $z_1 \leq z_2$. By the certain structure of the quantization and the definition of $h$ in (17), $g(z_1, z_2) \leq h(z_1) \leq \sup_{z \in (0, 1)} h(z)$ which completes the proof.

Now, we are almost ready to state and prove the main theorem of this section. Before that we discuss another modification on top of the deterministic polarization 1, defined in Definition 1. We modify deterministic polarization 1 by skipping certain channel combining operations. In particular, when we cannot guarantee that $\Delta_f(W_{j, 2i-1}, W_{j, 2i}) < 1$, the combining operations of $W_{j, 2i-1}$ and $W_{j, 2i}$ are skipped. The indices for skipped operations are saved in a certain matrix $T$ specified next and will be used in both encoding and decoding of the constructed polar code, described in the next section, accordingly.

**Definition 4:** The $n \times N/2$ matrix $T$, representing skipped channel combining operations, is defined as follows. For a pair of channels $W_{j, -2i-1}$ and $W_{j, -2i}, j \in \{n\}, i \in \{N/2\}$, let $z_1 = Z(W_{j, -2i-1})$ and $z_2 = Z(W_{j, -2i})$.

Then $T(j, i) = 0$ if $c/2m \leq z_1, z_2 \leq 1 - c/2m$ and $z_1, z_2$ are in the same sub-interval. Otherwise, $T(j, i) = 1$.

**Definition 5:** (Deterministic Polarization 2) Consider the deterministic polarization 1 defined in Definition 1 and a binary $n \times N/2$ matrix $T$. For each pair of channels $W_{j, -2i-1}$ and $W_{j, -2i}$ to be combined according to (6), we proceed with the...
channel combining operations only if $T(j, i) = 0$. Otherwise, 
\[ W_{j,(2^j)2^{n-j+i}} = W_{j-1,2i-1}, W_{j,(2^j+1)2^{n-j+i}} = W_{j-1,2i}. \]
In other words, the channel combining operations are skipped for this particular pair of channels. We refer to this process as the deterministic polarization 2.

Remark. For the rest of this section, we consider the deterministic polarization 2 with the matrix $T$, representing indices of skipped operations, as defined in Definition 4. Note that both $T$ and the polarization process are evolved jointly, i.e., when we are at the $j$-th level of the polarization process, the $j$-th row of $T$ is determined followed by performing $j$-th polarization level. Depending on the resulting bit-channels in the $j+1$-th level, the $j+1$-th row of $T$ is determined etc. However, the definitions of deterministic polarization 2 and the matrix $T$ are separated since in the next section we will use deterministic polarization 2 in combination with other matrices $W$.

Remark. Note that the skipped operations in the deterministic polarization 2 only reduce the complexity of both encoding and decoding as the corresponding ILR combining operations, related to the skipped channel combining operations, will be just transparent. The decoder needs to save the indices of the skipped operations which requires a memory of size at most $O(N \log N)$.

Lemma 2: In the deterministic polarization 2, defined in Definition 5, we have $E_{j,n} \leq E_{j-1,n}$, for $j \in \{n\}$.

Proof: According to the structure of the deterministic polarization 2, when two channels $W_{j-1,2i-1}$ and $W_{j-1,2i}$ are combined, they satisfy the conditions of Lemma 4 and hence, $\Delta_j(W_{j-1,2i-1}, W_{j-1,2i}) \leq 2^{-\eta} < 1$. If the combining operations are skipped, then the channels are in the same level $j$ and their Bhattacharyya parameters remain the same. Let $z_{j,i}$ denote $Z(W_{j,i})$. Then the lemma follows by summing over all $f(z_{j,i})$’s and using the definition of $E_{j,n}$ in (10).

The following theorem summarizes the main result of this section.

Theorem 3: For $\rho < \frac{\eta}{\rho + \tau}$, where $\eta$ is defined in (18), the parameters of the deterministic polarization 2, defined in Definition 5, can be set such that for any $N = 2^n$ and initial sequence of channels $\{W_{i,j}\}_{j=1}^{\infty}$, we have
\[ \eta_n = -\frac{1}{n} \log E_{n,n} > \rho - \frac{e_p}{n} \]
for a constant $e_p > 0$ depending only on $\rho$.

Proof: Let $\rho < \frac{\eta}{\rho + \tau}$ and the sequence of channels $\{W_{i,j}\}_{j=1}^{\infty}$ be given. Let $\tau = \rho/b$, where the parameter $b$ is used in the definition of $f(.)$ in (5), in the fine quantization of $(0, 1)$ described in Definition 2. Consider the deterministic polarization 2 applied to the sequence of initial channels $\{W_{j,i}\}_{j=1}^{\infty}$. Let $z_{j,i} = Z(W_{j,i})$, for $i \in \{N\}$ and $j = 0, 1, \ldots, n$, where $W_{j,i}$’s are assumed to be sorted within each sub-block. Then define
\[ D_j = \frac{1}{N} \sum_{z_{j,i} \leq c + 2^m} f(z_{j,i}). \]
Note that $D_j \leq N^{-\rho b} = 2^{-\rho n}$. Let $s = c_1 \log N + c_2$ denote the number of sub-intervals of the fine quantization of $(0, 1)$. At level $j < n$, let $m_j$ denote the number of pairs of channels $(W_{j,2i-1}, W_{j,2i})$ such that $z_{j,2i-1}$ and $z_{j,2i}$ belong to different sub-intervals of the quantization and hence, the combining operation is skipped. Since $z_{j,i}$’s are decreasing within each sub-block and the number of sub-blocks is $2^j$, we have
\[ m_j \leq s2^j. \]

Then by applying Lemma 4 to all the pairs of channels $(W_{j,2i-1}, W_{j,2i})$ and noting that $D_j$ is increasing with $j$ we get
\[ E_{j+1,n} - D_{j+1} \leq E_{j+1,n} - D_j \leq \frac{m_j}{N} + (E_{j,n} - D_j)2^{-\eta}. \]

For $k \in \{n\}$, by combining (20) for $j = 0, 1, \ldots, k - 1$ and using (19) we get
\[ E_{k,n} \leq D_k + s2^{-n+k} + E_{0,n}2^{-\eta k} \leq 2^{-\rho n} + s2^{-n+k} + 2^{-\eta k} \]

Let $k = \lfloor n/(\eta + 1) \rfloor$. We choose
\[ c_\rho = \log (2 + \max(c_1 n + c_2)2^{\rho n + \lfloor n/(\eta + 1) \rfloor}) \]
which is well-defined because $\rho + 1/(\eta + 1) - 1 < 0$. Then (21) can be simplified as
\[ E_{k,n} \leq 2^{-\rho n + c_\rho}. \]

By Lemma 4, $E_{n,n} \leq E_{k,n}$ which completes the proof.

V. Code Construction and Finite-Length Analysis

In this section we show how capacity achieving polar coding schemes can be constructed for the non-stationary channels and study their finite-length behavior.

In order to find explicit upper bounds on the Bhattacharyya parameters of the polarized channels, we define an extremal deterministic process. Roughly speaking, when two channels $W_1$ and $W_2$, with $z_1 = Z(W_1)$ and $z_2 = Z(W_2)$, are combined in the polarization process, we simply replace $Z(W_1 \oplus W_2)$ by $z_1 + z_2$, which, by (5), is an upper bound on $Z(W_1 \oplus W_2)$. Due to having a simpler structure this process simplifies characterizing certain bounds in the finite-length regime. A more precise definition is provided next.

Definition 6: Let $x_{0,i} \in [0, 1]$, for $i \in \{N\}$. A recursive process of combining $x_{j,i}$’s is defined as follows. For $j \in \{n\}$, the sequence $\{x_{j-1,i}\}_{i=1}^{N}$ is split into $2^{j-1}$ equal sub-blocks, each containing $N/2^{j-1}$ consecutive elements. Then $x_{j-1,i}$’s are permuted in such a way that they become sorted in a decreasing order within each sub-block, before being combined for the $j$-th level as described next. For $i \in \{N/2^{j}\}$, let
\[ u = x_{j-1,2i-1}, \quad v = x_{j-1,2i}. \]
Also, let $i = (2^{n-j} + r$, where $r \in \{2^{n-j}\}$ and $0 \leq l \leq 2^{l-1} - 1$. If, $u, v \leq 1$ or $u, v \geq 1$, we let
\[ x_{j,2l}2^{n-j} + i = u + v, \quad x_{j,2l+1}2^{n-j} + i = uv, \]
and otherwise, i.e., when $u > v > 1$, we let
\[ x_{j,2l}2^{n-j} + i = u, \quad x_{j,2l+1}2^{n-j} + i = v, \]
In other words, the combining operation is skipped in this case. This finite deterministic process is referred to as the extremal deterministic process associated with $\{x_{0,i}\}_{i=1}^N$.

The function $q: \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ is defined as follows:

$$q(x) = \begin{cases} x(2-x) & \text{if } x \leq 1, \\ 1 & \text{if } x > 1. \end{cases} \quad (25)$$

In the next lemma we show that $q(.)$ can be used as a potential function in an extremal deterministic process.

**Lemma 4:** In an extremal deterministic process, defined in Definition 6 we have

$$\sum_{i=1}^N q(x_{j,i}) \leq \sum_{i=1}^N q(x_{j-1,i}).$$

for $j \in \{1\}$. 

**Proof:** For $u, v \leq 1$, if $u + v \leq 1$, we have

$$q(u) + q(v) - q(uv) - q(u + v) = u^2 v^2 \geq 0, \quad (26)$$

and if $u + v > 1$, $q(u + v) = 1$ and

$$q(u) + q(v) - q(u) - q(v) = (1 - u)(1 - v)(u + v + uv - 1) \geq 0. \quad (27)$$

If $u, v \geq 1$, then

$$q(u) + q(v) - q(uv) - q(u + v) = 0. \quad (28)$$

Note that if $u > 1 > v$, a similar inequality does not always hold but according to (26), the combining operations are skipped in this case. Therefore, the lemma follows by using (26), (27), and (28) together with definitions of combining operations in (25) and (33) in an extremal deterministic process. 

**Corollary 5:** In an extremal deterministic process, defined in Definition 6 we have

$$\left\{ i : i \in \mathbb{N}, x_{j,i} \geq 1 \right\} \leq \sum_{i=1}^N q(x_{0,i}) \quad (29)$$

for $j \in \{1\}$. 

**Proof:** Since $q(x) = 1$ for $x \geq 1$ and by Lemma 4, we have

$$\left\{ i : i \in \mathbb{N}, x_{j,i} \geq 1 \right\} \leq \sum_{i=1}^N q(x_{j,i}) \leq \sum_{i=1}^N q(x_{0,i}).$$

For a non-negative integer $i$, let $s(i)$ denote the number of 1’s in the binary representation of $i - 1$. In other words, let $b_1 b_2 \ldots b_n$ denote the binary representation of $i - 1$. Then

$$s(i) = \sum_{i=1}^n b_i. \quad (30)$$

Also, assuming $n$ is fixed, $s_j(i)$ denote the number of 1’s in the leftmost $j$ bits in the $n$-bit binary representation of $i - 1$, i.e.,

$$s_j(i) = \sum_{i=1}^j b_i. \quad (31)$$

**Remark.** It can be observed that in the evolution of a bit-channel $W_{j,i}$, for $i \in \{\mathbb{N}\}$, the number of polarization levels where the operation $\otimes$ is applied is $s_j(i)$ and in the rest of $j - s_j(i)$ levels the operation $\oplus$ is applied. Therefore, it is expected that the quality of the bit-channels $W_{n,i}$’s, e.g., their Bhattacharyya parameters, can be well bounded in terms of $s(i)$. Similarly, in an extremal deterministic process, $x_{n,i}$’s can be bounded in terms of $s(i)$ and the initial parameters. The following lemma provides such a bound, which is one of the key elements in establishing upper bounds on the probability of error of constructed polar coding schemes.

**Lemma 6:** Let $x_i = x_{0,i}$, for $i \in \{\mathbb{N}\}$. Consider an extremal deterministic process, defined in Definition 6 associated with $x_i = x_{0,i}$. Then the criteria for permutations and skipped operations in the process can be set such that for $j \in \{1\}$

$$\left\{ i : i \in \mathbb{N}, x_{j,i} \leq 2^{2s_j(i)+a_j,i} \right\} \geq N - 4 \sum_{i=1}^N x_i(1 - x_i) \quad (32)$$

where $s_j(i)$ is defined in (31) and $\sum_{i=1}^N a_j,i \leq 3j$. 

**Proof:** Let $y_{l,i} = y_{0,i} = 2x_i < 1$, for $i \in \mathbb{N}$. Consider the extremal deterministic processes $\{x_{j,i}\}$ and $\{y_{j,i}\}$. Suppose that there were no skipped operations, i.e., the processes evolve only according to (25). Then it can be observed that at the $j$-th level of the process we would have

$$y_{j,i} = 2^{2s_j(i)} x_{j,i}, \quad (33)$$

where $s_j(i)$ is defined in (31).

Next, the effect of skipped operations on (33) is characterized. We modify the criteria for skipped operations in $\{x_{j,i}\}$ such that it follows the pattern of skipped operations of the process $\{y_{j,i}\}$, i.e., the operations in (25) are applied if both $y_{j-1,2i-1}$ and $y_{j-1,2i}$ are less than $1$ or greater than $1$ and otherwise, (24) is applied. Also, the same permutations that are applied to the sub-blocks of $\{y_{j,i}\}$ to make them decreasing within each sub-block of length $2^{n-j}$ are applied to the process $\{x_{j,i}\}$.

The parameters $a_{j,i}$, for $j = 0, 1, \ldots, n$ and $i \in \{\mathbb{N}\}$, are recursively derived such that

$$y_{j,i} \geq 2^{2s_j(i)-a_j,i} x_{j,i}. \quad (34)$$

We confirm that (34) holds by induction on $j$. For the base of induction $j = 0$, we define $a_{0,i} = 0$ and (34) holds. In level $j$, consider $u = y_{j,2i-1}$ and $v = y_{j,2i}$ to be combined for the next level. Let $i = 2^{n-j-1} + r$, where $r \in \{\mathbb{N}\}$ and $0 \leq l \leq 2j - 1$, and

$$i_1 = (2l)2^{n-j-1} + i, \quad i_2 = (2l + 1)2^{n-j-1} + i$$

denote the indices of the combined $u$ and $v$ in the next level, i.e., if $u, v \geq 1$ or $u, v \leq 1$, then

$$y_{j+1,i_1} = u + v, \quad \text{and } y_{j+1,i_2} = uv, \quad (35)$$

In this case, we set

$$a_{j+1,i_1} = \max(a_{j,2i-1}, a_{j,2i}), \quad \text{and } a_{j+1,i_2} = a_{j,2i-1} + a_{j,2i}. \quad (36)$$

By induction hypothesis (34) holds for $(j, 2i - 1)$ and $(j, 2i)$.

Also, note that

$$s_{j+1}(i_1) = s_j(2i - 1) = s_j(2i) = s_{j+1}(i_2) - 1.$$
This together with the induction hypothesis, \(35\), and \(36\) imply that \(34\) holds for \((j + 1, i_1)\) and \((j + 1, i_2)\).

If \(u > 1 > v\), then

\[ y_{j+1,i_1} = u, \text{ and } y_{j+1,i_2} = v. \]

In this case we set

\[ a_{j+1,i_1} = a_{j,2i-1}, \text{ and } a_{j+1,i_2} = a_{j,2i} + 2s_j(2i). \tag{37} \]

Then since \(s_j+1(i_1) = s_j(2i - 1)\), induction hypothesis holds for \((j + 1, i_1)\). Also, we have

\[ 2s_{j+1}(i_2) - a_{j+1,i_2} = 2s_j(2i) + 1 - a_{j,2i} - 2s_j(2i) = 2s_j(2i) - a_{j,2i}, \]

and therefore, induction hypothesis holds for \((j + 1, i_2)\), too.

What remains is to upper bound \(\sum_{i=1}^{N_j} a_{j,i}\). Let

\[ A_j \overset{\text{def}}{=} \sum_{i=1}^{N_j} a_{j,i}. \]

Note that at level \(j\), there are \(2^j\) sub-blocks of length \(2^{n-j}\) each. Also, \(s_j(i)\) is the same for all indices \(i\) in one sub-block. Let \(k\) denote the number of skipped operations at level \(j\). Since \(y_{j,i}\)'s are sorted within each sub-block, there is at most one skipped operation within each sub-block and hence, we have \(k \leq 2^j\). Let \(i_1, \ldots, i_k\) denote the even indices of the pairs in skipped operations. Then we have

\[ \sum_{i=1}^{k} 2^{s_j(i)} \leq \sum_{i=0}^{j} \binom{j}{i} 2^i = 3^j \tag{38} \]

By \(36\), \(37\), and \(38\) we have

\[ A_{j+1} \leq 2A_j + 3^j. \]

And by induction on \(j\), \(A_j \leq 3^j\). This together with \(34\), Corollary 5 applied to the process \(\{y_{j,i}\}\), and noting that \(q(y_i) = 4x_i(1-x_i)\) lead to the proof of lemma.

Next, we state the main result of this section. In the proof of the next theorem, a construction of polar codes along with a bound on its finite-length behavior is shown. The construction is based on a two-stage deterministic polarization. In the first stage, deterministic polarization 2, defined in Definition 5 is invoked to have a sufficiently close to the average capacity fraction of almost good bit-channels by exploiting Theorem 3.

In the second stage of the polarization, only the almost good bit-channels of the first stage are combined following a certain extremal deterministic process, defined in Definition 6. At the end of the second polarization stage, an additional negligible fraction of bit-channels are removed and it is shown that the rest have Bhattacharyya parameters bounded by \(P_e/N\), where \(P_e\) is the target block error probability and \(N\) is the block length. Therefore, a polar code constructed according to certain good bit-channels, after the two polarization stages, guarantees a block error probability bounded by \(P_e\).

**Theorem 7:** For a sequence of BMS channels \(\{W_i\}_{i=1}^{N}\), target block error probability \(P_e \in (0, 1)\), and a constant \(\mu\) such that

\[ \mu > 2 + \log 3 + \frac{1}{\eta}, \tag{39} \]

where \(\eta\) is defined in \((18)\), we construct a polar code of length \(N = 2^n\) and rate \(R\) guaranteeing a block error probability of at most \(P_e\) for transmission over \(\{W_i\}_{i=1}^{N}\) such that

\[ N \leq \frac{\kappa}{(T_N - R)^{\mu}}, \]

where \(\kappa\) is a constant depending on \(P_e\) and \(\mu\), and \(T_N\) is the average of the symmetric capacities \(f(W_i)\), for \(i \in [N]\).

**Proof:** The parameter \(\rho \in \mathbb{R}^+\) is picked such that

\[ \mu > 1 + \log 3 + \frac{1}{\rho} > 2 + \log 3 + \frac{1}{\eta}. \tag{40} \]

In particular, we pick

\[ \rho = \frac{2}{\mu + \frac{1}{\eta} - \log 3}. \tag{41} \]

Therefore, \(\rho < \eta/(\eta + 1)\). Let

\[ n_1 = \left[ \frac{1}{1 + \rho (1 + \log 3)} \right] \tag{42} \]

and

\[ n_2 = n - n_1 > \frac{\rho (1 + \log 3)}{1 + \rho (1 + \log 3)} n - 1. \tag{43} \]

Let also \(N_1 = 2^{n_1}\) and \(N_2 = 2^{n_2}\). We exploit Lemma 13 in Appendix to partition the set of channels \(\{W_i : i \in [N]\}\) into \(N_2\) subsets \(A_1, A_2, \ldots, A_{N_2}\) of size \(N_1\) such that

\[ T_{N_1,k} \geq T_N - 2^{-n_1}, \tag{44} \]

where \(T_{N_1,k}\) is the average of symmetric capacities of the channels in the set \(A_k\), for \(k \in [N_2]\).

In the first stage of polarization the deterministic polarization 2, defined in Definition 5 is applied to each of the sets \(A_k\) separately. The permutations \(\pi_{j,l}\), for \(j \in [N_1], k \in [N_2]\) and \(0 \leq l \leq 2^j - 1\) are set in such a way that the sequence of Bhattacharyya parameters of bit-channels within the \(l\)-th sub-block of length \(2^{n_1-j}\) in the \(j\)-th level of polarization of the set \(A_k\) becomes decreasing. Also, the sets \(T_{l}\) representing the indices of the skipped operations in the polarization of \(A_k\) are set according to Definition 4. Let \(W_{j,i}^{(k)}\) denote the \(i\)-th bit-channel in the \(j\)-th polarization level of the set \(A_k\) and \(z_{j,i}^{(k)}\) denote its Bhattacharyya parameter.

By Theorem 3 there exists a constant \(c_{\mu}\) such that

\[ \frac{1}{N_1} \sum_{i=1}^{N_1} f(z_{n_1,i}^{(k)}) \leq 2^{-n_1 \rho + c_{\mu}}. \tag{45} \]

Then by applying Corollary 12 to \(\{W_{j,i}^{(k)}\}_{i=1}^{N_1}\) with the choice of \(t < \frac{1}{2}\) we have

\[ \frac{1}{N_1} \sum_{i=1}^{N_1} f(z_{n_1,i}^{(k)}) \geq T_{N_1,k} - \frac{2}{t N_1} \sum_{i=1}^{N_1} f(z_{n_1,i}^{(k)}) \tag{46} \]

\[ \geq T_N - 2^{-n_1} - \frac{1}{t} 2^{-n_1 \rho + c_{\mu} + 1} \tag{47} \]

\[ \geq T_N - d_1 2^{-n_1 \rho} \tag{48} \]

\[ \geq T_N - d_1 N^{-1/\mu}. \tag{49} \]
where \(46\) is by Corollary \(12\) \(47\) is by \(44\) and \(45\), \(48\) is by \(d_1\) being defined as
\[
d_1 \overset{\text{def}}{=} \frac{1}{2} 2^r + 1, \tag{50}
\]
and \(49\) is by \(40\) and \(42\).

Let
\[
M = \left[ N_1 (\bar{T}_N - d_1 N^{-1/\mu}) \right]. \tag{51}
\]

Without loss of generality, we can assume that
\[
z_{n_1,i}^{(k)} \leq \frac{1}{2}
\]
for \(k \in [N_2]\) and \(i \in [M]\). In the second stage of polarization, we will apply a polarization process to \(\left\{W^{(k)} \right\}_{k=0}^{N_2}\), for \(i \in [M]\). Note that the actual indices of these good bit-channels within each of the polarization blocks, corresponding to sets \(A_k\), does not matter as long as they are in increasing order. In other words, roughly speaking, the \(i\)-th good bit-channels in the polarization of the sets \(A_k\) will be combined in the second stage of polarization. All the other \(N_2 (N_1 - M)\) bit-channels will not be further polarized and will be frozen to zeros.

Fix \(i_0 \in [M]\) and for ease of notation let \(W_k = W_{n_1,i_0}^{(k)}\).

Let \(x_{0,k} = Z(W_k)\) and \(y_{0,k} = 2x_{0,k}\). Consider an extremal deterministic process \(\{y_{j,k}\}\) for \(j = 0, 1, \ldots, n_2\) and \(k \in [N_2]\), defined in Definition \(6\) with the initial values \(\{y_{0,k}\}\).

Also, consider another extremal deterministic process \(\{x_{j,k}\}\) with the initial values \(\{x_{0,k}\}\), where permutations and indices of skipped operations follow the process \(\{y_{j,k}\}\). By Lemma \(6\) for any \(j \in [n_2]\),
\[
\left\{ k : k \in [N_2], x_{j,k} \leq 2^{-2^j(k) + a_{j,k}} \right\} \tag{52}
\]
\[
\geq N_2 - 4 \sum_{k=1}^{N} Z(W_k) (1 - Z(W_k)),
\]
where \(a_{j,k}\) is defined in \(31\) and \(\sum_{k=1}^{N} a_{j,k} \leq 3^j\). Let
\[
l = \left\lfloor \frac{n_2}{1 + \log 3} \right\rfloor \tag{53}
\]
and
\[
\alpha = 1 + \frac{1}{\rho (1 + \log 3)}, \quad \beta = -\log P_e + \alpha \tag{54}
\]

By Lemma \(14\) we have
\[
\log \left\{ k : k \in [N_2], 2q(k) - a_{i,k} \leq \alpha n_2 + \beta \right\} \tag{55}
\]
\[
\leq n_2 - l + p(n_2, \alpha, \beta),
\]
where \(p(n, \alpha, \beta)\) is defined in \(76\). Let \(\gamma = 1 + \log 3\) and define
\[
q(n) \overset{\text{def}}{=} n \left( 1 - \frac{\rho}{1 + \gamma \rho} \right) + 1 + \frac{1}{\gamma} + p \left( \frac{\rho n}{1 + \gamma \rho}, \alpha, \beta \right) \tag{56}
\]
where \(\alpha\) and \(\beta\) are defined in \(54\). Note that by \(40\), the coefficient of \(n\) in the definition of \(q(n)\) is negative. Also, \(p\) is bounded by a polynomial of \(\log n\). Therefore,
\[
\lim_{n \to \infty} q(n) = -\infty
\]
and
\[
d_2 = \sup_{n \in \mathbb{N}} q(n) \tag{57}
\]
is a well-defined constant. Therefore, by the choice of \(l\) in \(53\), we have
\[
l - p(n_2, \alpha, \beta) \geq \frac{n_2}{\mu} - d_2 \tag{58}
\]
Also, by the choice of \(\alpha\) and \(\beta\) in \(54\) we have
\[
2^{-\alpha n_2 - \beta} \leq P_e \tag{59}
\]

Then we have
\[
\left\{ k : k \in [N_2], x_{i,k} \leq \frac{P_e}{N} \right\} \tag{60}
\]
\[
\geq N_2 - 4 \sum_{i=1}^{N} Z(W_k) (1 - Z(W_k)) - N_2 2^{-d_2 - l + p(n_2, \alpha, \beta)} \tag{61}
\]
\[
\geq N_2 - 4 \sum_{i=1}^{N} Z(W_k) (1 - Z(W_k)) - N_2 2^{-\frac{3}{4} - d_2} \tag{62}
\]
\[
\geq N_2 - 4 \sum_{i=1}^{N} Z(W_k) (1 - Z(W_k)) - N_2 2^{-\frac{3}{4} + d_2} \tag{63}
\]
where \(60\) is by \(59\), \(61\) is by the union bound, \(62\) is by \(52\) and \(55\), and \(63\) is by \(50\).

Recall that \(63\) holds for any \(i_0 \in [M]\), where we dropped the index \(i_0\) from the derivations for notation convenience.

Let \(x_{j,k}\) be re-indexed by \(x_{j,k+N_2(i_0-1)}\), for \(i_0 \in [M]\). By summing \(63\) for all \(i_0\) we have
\[
\left\{ i : i \in [MN_2], x_{1,i} \leq \frac{P_e}{N} \right\} \tag{64}
\]
\[
\geq MN_2 - 4 \sum_{i=0}^{N} Z(x_{i,k}^{(k)} (1 - Z(x_{i,k}^{(k)})) - MN_2 2^{-\frac{3}{4} + d_2} \tag{65}
\]
\[
\geq MN_2 - 4 \sum_{i=0}^{N} Z(x_{i,k}^{(k)} (1 - Z(x_{i,k}^{(k)})) - MN_2 2^{-\frac{3}{4} + d_2} \tag{66}
\]
\[
\geq MN_2 - 2^{-\frac{3}{4} + d_2} N^{1 - \frac{1}{\mu}} \tag{67}
\]
where \(66\) and \(67\) were also used \(M \leq N_1\). By plugging \(M\) from \(51\) in \(66\) and normalizing both sides by \(N\) we get
\[
\frac{1}{N} \left\{ i : i \in [MN_2], x_{1,i} \leq \frac{P_e}{N} \right\} \geq \frac{1}{N} T_N - d_3 N^{-\frac{1}{\mu}} \tag{67}
\]
where \(d_3\) is defined as
\[
d_3 \overset{\text{def}}{=} d_1 + 2^{\frac{3}{4} + d_2} + 2d_2 + 1, \tag{68}
\]
where \(d_1\) is defined in \(50\), \(d_2\) is defined in \(57\), \(\rho\) is defined in \(41\), and \(c_\rho\), as a function of \(\rho\), is defined in \(22\). Note that \(d_3\) is a constant that depends only on \(\mu\) and \(P_e\).
In the second stage of polarization, deterministic polarization 2, defined in Definition 4, is applied to \( \{ W_n^{(k)} \}_{k=1}^{N_2} \) for \( i_0 \in [M] \), where the channels are combined according to the index \( k \). Let the bit-channels at the \( j \)-th level of the second stage be denoted by \( \{ W_{n_1+j,i_0}^{(k)} \}_{k=1}^{N_2} \). The permutations and the indices for skipped operations follow the extremal process \( \{ y_{j,k} \} \) described earlier in the proof for the fixed \( i_0 \). The polarization stops at level \( l \), where \( l \) is given in (53). One can alternatively assume that the process continues but all the operations for \( l < j \leq n_2 \) are skipped. To simplify the notation let the bit-channels \( W_{n_1+j,i_0}^{(k)} \) be re-indexed by \( W_{n_1+i_0+N_2(k-1)}^{(k)} \). The polar code is constructed by picking all the bit-channels at the final level \( n = n_1 + n_2 \) with the Bhattacharyya parameter bounded by \( \mu / N \). The rate \( R \) of the code is given by

\[
R = \frac{1}{N} \left\{ \frac{1}{i \in [M,N_2]}, Z(W_{n,i}) \leq \frac{P_e}{N} \right\}
\]

(69)

By Lemma 15

\[
Z(W_{j,i}) \leq x_{j,i}
\]

for \( j \in [n_2] \) and \( i \in [M,N_2] \). Using this together with (69) and (67) we have

\[
R \geq \frac{1}{N} \left\{ \frac{1}{i \in [M,N_2]}, x_{l,i} \leq \frac{P_e}{N} \right\}
\]

\[
\geq T_N - d_3 N^{-\frac{1}{d}}.
\]

(70) can be re-written as

\[
N \leq \frac{\kappa}{(\frac{1}{N} - R)^\mu},
\]

where

\[
\kappa \overset{\text{def}}{=} d_3^\mu
\]

and \( d_3 \) is defined in (68). Note that \( \kappa \) depends only on \( \mu \) and \( P_e \). Also, the probability of block error, under successive cancellation decoding, is bounded by the sum of Bhattacharyya parameters of the selected bit-channels which is bounded by \( P_e / N \), similar to what is shown for original polar codes [1].

**Remark.** The low complexity \( O(N \log N) \) architecture of successive cancellation decoding of original polar codes can be adopted for the polar code constructed via a two stage polarization explained in the proof of Theorem 7. The decoding trellis can be also separated into two stages to reflect the two stages of polarization. The first stage of the decoding trellis consists of \( N_1 \) trellises, each of size \( (n_2 + 1) \times N_2 \). These trellises are run in parallel while the permutations \( \pi_{j,i}^{(k)} \) are applied accordingly as described in Section III Also, the indices for the skipped operations, saved in \( T_k \)'s, are taken into account and the the corresponding LR combination operations are skipped. The second stage of decoding trellis consists of \( M \) trellises, each of size \( (n_1 + 1) \times N_1 \). These trellises are run successively while the permutations and skipped operations, derived from the extremal deterministic process as described in the proof of Theorem 7, are also applied. When the decoding process in the first stage trellises arrive at the \( i \)-th selected index in the level \( n_1 + 1 \), for \( i \in [M] \), the \( i \)-th trellis of the second stage is run. When it is done, the hard decisions are passed to the first stage trellises and they continue to run until they arrive at the \( i + 1 \)-th selected index and so on. The complexity of the total decoding process is \( O(N \log N) \).

**Theorem 8:** Given a non-stationary sequence of independent BMS channels \( \{ W_i \}_{i=1}^\infty \) and assuming the average capacity

\[
T_i(\{ W_i \}_{i=1}^\infty) = \lim_{N \to \infty} \frac{1}{N} \sum_{i=1}^N I(W_i)
\]

is well-defined, there exists a sequence of polar codes \( \{ C_i \}_{i=1}^\infty \), with length \( N_i \), rate \( R_i \) and probability of block error \( P_i \) under a low complexity \( O(N_i \mu \log N_i) \) SC decoder, such that

\[
\lim_{i \to \infty} R_i = T(\{ W_i \}_{i=1}^\infty), \quad \text{and} \quad \lim_{i \to \infty} P_i = 0
\]

when the \( j \)-th bit of \( C_i \) is transmitted over \( W_j \), for \( j \in [N_i] \).

**Proof:** Consider two arbitrary sequences of \( \{ \epsilon_i \}_{i=1}^\infty \) and \( \{ \overline{P}_i \}_{i=1}^\infty \) such that

\[
\lim_{i \to \infty} \epsilon_i = 0, \quad \text{and} \quad \lim_{i \to \infty} \overline{P}_i = 0
\]

Fix \( \mu \) such that

\[
\mu > 2 + \log 3 + \frac{1}{\eta},
\]

where \( \eta \) is given in (18). For every \( i \in \mathbb{N} \), let

\[
n_i = [ \log \kappa - \mu \log \epsilon_i ]
\]

where \( \kappa \) is the constant given in Theorem 7 as a function \( \overline{P}_i \) and \( \mu \). Then by Theorem 7 we construct a polar code \( C_i \) of length \( N_i = 2^{n_i} \) and rate \( R_i \) such that

\[
R_i > T_{N_i} - \epsilon_i,
\]

(71)

where

\[
T_{N_i} = \frac{1}{N_i} \sum_{j=1}^{N_i} I(W_j)
\]

and the probability of block error \( P_i \) of \( C_i \) under SC decoder when transmitted over \( \{ W_j \}_{j=1}^{N_i} \) is bounded by \( \overline{P}_i \). Note that

\[
\lim_{i \to \infty} T_{N_i} = T(\{ W_i \}_{i=1}^\infty)
\]

and

\[
\lim_{i \to \infty} (T_{N_i} - \epsilon_i) = \overline{T}(\{ W_i \}_{i=1}^\infty) - \lim_{i \to \infty} \epsilon_i = \overline{T}(\{ W_i \}_{i=1}^\infty).
\]

Therefore, by (71) and since \( R_i < T_{N_i} \), we have

\[
\lim_{i \to \infty} R_i = \overline{T}(\{ W_i \}_{i=1}^\infty)
\]

Also, since \( P_i \leq \overline{P}_i \),

\[
\lim_{i \to \infty} P_i = 0
\]

which completes the proof.

**Remark.** The value of \( \eta \), defined in (18), and used in Theorem 3 in Theorem 7, and Theorem 8 can be estimated with arbitrarily fine resolution as described in Section IV. In particular, it can be numerically guaranteed that

\[
\eta > \mu \log \frac{1}{0.139} \approx 10.7792
\]

Combining this with (39) implies that

\[
\mu > 2 + \log 3 + \frac{1}{0.139} \approx 10.7792
\]
is sufficient for Theorem[7]. For instance, we pick $\mu = 10.78$. Clearly this is not the lowest possible $\mu$ for which Theorem[7] holds. This value can be possibly lowered by a better partitioning of the interval $[0, 1]$, described in Section[V] or by other techniques that yield a larger $\eta$, or by reducing the constant term $2 + \log 3$. Therefore, $10.78$ is just an upper bound on the scaling behavior of polar codes constructed for non-stationary channels.

VI. DISCUSSIONS AND CONCLUSION

In this paper we considered the problem of polar coding for a non-stationary sequence of channels where the channels are independent but not identical. It is shown how to modify Arikan’s channel polarization transform in several important respects in order to polarize any arbitrary set of channels while lower bounding the speed of polarization. It is shown how the resulting fast polarization scheme in combination with another polarization stage can be used in order to construct polar coding schemes that achieve the average symmetric capacity of channels. Furthermore, it is shown that the block length $N$ of the constructed code is upper bounded by a polynomial of the inverse of the gap to the capacity of the code.

There are several directions for future research. Improving the upper bound on the finite-length scaling of constructed polar codes is an interesting problem. For instance, if one can avoid the skipped channel combining operations using an alternative technique, that can immediately improve the upper bound. Speed of polarization for the family of binary erasure channels is characterized in [9] and the question is does the speed of polarization remains the same if all the channels in the non-stationary set-up are binary erasure channels, perhaps, with different erasure probabilities? In other words, can one construct an example for a sequence of binary erasure channels such that their speed of polarization, no matter what permutations are applied during the polarization process, is smaller than the speed of polarization of the stationary binary erasure channels? The answer to this question can potentially shed some light on whether there is a fundamental difference between the stationary and non-stationary set-ups in terms of their finite-length behavior. Also, studying the error exponent of polar codes over non-stationary channels, as an extension of [8] to the non-stationary case, and unified scaling laws for non-stationary channels, as extension of the results of [11] in the stationary case, are other interesting problems.

REFERENCES

[1] E. Arikan, “Channel polarization: A method for constructing capacity-achieving codes for symmetric binary-input memoryless channels,” IEEE Transactions on Information Theory, vol. 55, no. 7, pp. 3051–3073, 2009.
[2] H. Mahdavifar and A. Vardy, “Achieving the secrecy capacity of wiretap channels using polar codes,” IEEE Transactions on Information Theory, vol. 57, no. 10, pp. 6426–6443, 2011.
[3] E. Arikan, “Source polarization,” Proceedings of IEEE International Symposium on Information Theory, pp. 899–903, 2010.
[4] E. Abbe, “Randomness and dependencies extraction via polarization,” Proceedings of Information Theory and Applications Workshop (ITA), pp. 1–7, 2011.
[5] M. Mondelli, S. H. Hassani, I. Sason, and R. Urbanke, “Achieving marton’s region for broadcast channels using polar codes,” Proceedings of IEEE International Symposium on Information Theory, pp. 306–310, 2014.
[6] E. Şaşoğlu, E. Telatar, and E. Yeh, “Polar codes for the two-user binary-input multiple-access channel,” IEEE Transactions on Information Theory, vol. 59, no. 10, pp. 6583–6592, 2013.
[7] H. Mahdavifar, M. El-Khamy, J. Lee, and I. Kang, “Achieving the uniform rate region of general multiple access channels by polar coding,” IEEE Transactions on Communications, vol. 64, no. 2, pp. 467–478, 2016.
[8] E. Arikan and E. Telatar, “On the rate of channel polarization,” Proceedings of IEEE International Symposium on Information Theory, pp. 1493–1495, 2009.
[9] S. H. Hassani, K. Alishahi, and R. L. Urbanke, “Finite-length scaling for polar codes,” IEEE Transactions on Information Theory, vol. 60, no. 10, pp. 5875–5898, 2014.
[10] V. Guruswami and P. Xia, “Polar codes: Speed of polarization and polynomial gap to capacity,” in IEEE 54th Annual Symposium on Foundations of Computer Science (FOCS). IEEE, 2013, pp. 310–319.
[11] M. Mondelli, R. Urbanke, and S. H. Hassani, “Unified scaling of polar codes: Error exponent, scaling exponent, moderate deviations, and error floors,” in 2015 IEEE International Symposium on Information Theory (ISIT). IEEE, 2015, pp. 1422–1426.
[12] D. Goldin and D. Burshtein, “Improved bounds on the finite length scaling of polar codes,” IEEE Transactions on Information Theory, vol. 60, no. 11, pp. 6966–6978, 2014.
[13] A. Fazeli and A. Vardy, “On the scaling exponent of binary polarization kernels,” in Communication, Control, and Computing (Allerton), 2014 52nd Annual Allerton Conference on. IEEE, 2014, pp. 797–804.
[14] M. Aisian and E. Telatar, “A simple proof of polarization and polarization for non-stationary channels,” IEEE Transactions on Information Theory, vol. 62, no. 9, pp. 4873–4878, 2016.
[15] S. B. Korada, “Polar codes for channel and source coding,” Ph.D. dissertation, École Polytechnique Fédérale De Lausanne, 2009.

APPENDIX

Lemma 9: For any two BMS channels $W_1$ and $W_2$, we have

$$Z(W_1 \boxplus W_2) \geq \sqrt{z_1^2 + z_2^2 - z_1 z_2}$$

(72)

where $z_1 = Z(W_1)$ and $z_2 = Z(W_2)$.

Proof: It can be shown that (72) turns to equality when both $W_1$ and $W_2$ are binary symmetric channels (BSC). It is well-known that any BMS channels can be decomposed into a convex combination of multiple BSCs. Also, note that $\sqrt{z_1^2 + z_2^2 - z_1 z_2}$ is convex in both $z_1$ and $z_2$ when the other parameter is fixed. Therefore, the lemma follows.

Lemma 10: For any BMS channel $W$, we have

$$I(W) \leq 1 - Z(W)^2.$$ 

Proof: We use the fact any BMS channel can be decomposed into a convex combination of multiple BSCs, same as in the proof of Lemma[9] Since $-z^2$ is a concave function, it is then sufficient to prove the lemma for a BSC $W$. If $W$ is BSC($p$), then we have

$$I(W) = 1 + p \log p + (1 - p) \log(1 - p)$$

and

$$Z(W) = 2 \sqrt{p(1 - p)}$$

Therefore, it suffices to show that

$$g(p) \overset{\text{def}}{=} - p \log p - (1 - p) \log(1 - p) - 4 p(1 - p) \geq 0. \quad (73)$$

Since $g(p) = g(1 - p)$, it is enough to show (73) holds for $p \in (0, 0.5]$ (for $p = 0, g(p) = 0$). Note that

$$g'(p) = \log \frac{1 - p}{p} + 8p - 4$$

and

$$g''(p) = - \frac{\log e}{p(1 - p)} + 8,$$
where $g'$ and $g''$ are also continuous over $(0, 0.5]$. Since $g''(p)$ is increasing over $(0, 0.5]$, it is zero for at most one $p \in (0, 0.5]$ and hence, $g'(p) = 0$ has at most two solutions over $(0, 0.5]$ which are

$$p_1 \approx 0.096, \text{ and } p_2 = 0.5.$$ 

Note that $g(p)$ is increasing for $p \in [0, p_1]$ and $g(p_2) = 0$. Therefore, $\min_{p \in [0,0.5]} g(p) = 0$ and the lemma follows.

**Lemma 11:** Let $b, t \in (0, 1)$, and $f(z) = z^b(1 - z)^b$. Then for any sequence of BMS channels $\{W_i\}_{i=1}^N$

$$\sum_{z_i > t} I(W_i) \leq \frac{2}{t} \sum_{i=1}^{N} f(z_i),$$

where $z_i = Z(W_i)$.

**Proof:** Using Lemma 10 we have

$$\sum_{z_i > t} I(W_i) \leq \frac{2}{t} \sum_{z_i > t} z_i(1 - z_i) \leq \frac{2}{t} \sum_{z_i > t} f(z_i).$$

**Corollary 12:** Under the same assumptions in Lemma 11

$$\left| \{i \in \mathbb{N} : z_i \leq t \} \right| \geq \sum_{z_i \leq t} I(W_i) = \sum_{i=1}^{N} I(W_i) - \sum_{z_i > t} I(W_i)$$

followed by using Lemma 11.

**Lemma 13:** Let $N = MK$, where $N, M, K \in \mathbb{N}$, and let $A = \{x_1, x_2, \ldots, x_N\}$, where $x_i \in [0, 1]$, for $i \in \mathbb{N}$, and

$$\lambda = \frac{1}{N} \sum_{i=1}^{N} x_i.$$ 

Then one can partition $A$ into $K$ subsets $A_1, A_2, \ldots, A_K$ of size $M$ such that

$$\min_{j \in [K]} \lambda_j \geq \lambda - \frac{1}{M},$$ 

where $\lambda_j$ is the average of the elements in $A_j$.

**Proof:** Since the number of possible partitions is finite, one can find the partition such that

$$\min_{j \in [K]} \lambda_j$$

is maximum among all the possible partitions. We show that this partition must satisfy (74). Let

$$j_0 = \arg\min_{j \in [K]} \lambda_j$$

Assume to the contrary that (74) does not hold, i.e.,

$$\lambda_{j_0} < \lambda - \frac{1}{M}.$$ 

Note that there exists $j_1 \in [K]$ such that

$$\lambda_{j_1} > \lambda$$

Let $x_{i_0}$ be the minimum element in $A_{j_0}$ and $x_{j_1}$ be the maximum element in $A_{j_1}$. We make a new partition by swapping the elements $x_{i_0}$ and $x_{j_1}$ in $A_{j_0}$ and $A_{j_1}$. Note that

$$0 < x_{j_1} - x_{i_0} \leq 1$$

Therefore, the average of $A_{j_0}$ is strictly increased in the new partition, while the average of $A_{j_1}$ is decreased by at most $1/M$, which means it is still greater than the old average of $A_{j_0}$. Hence,

$$\min_{j \in [K]} \lambda_j$$

is strictly increased in the new partition which is a contradiction. This proves the lemma.

**Lemma 14:** Let $N = 2^n$, $2 \leq j \leq n/(1 + \log 3)$, $\alpha, \beta > 0$ and $a_1, a_2, \ldots, a_N \in \mathbb{N}$ such that $\sum_{i=1}^{N} a_i \leq 3^j$. Then we have

$$\log \left( \left| \{ i \in \mathbb{N} : 2^{\gamma} - a_i \leq \alpha n + \beta \} \right| ight) \leq n - j + p(n, \alpha, \beta)$$

where $s_j(i)$ is defined in (31) and

$$p(n, \alpha, \beta) \equiv (2 - \log \gamma + \log n + \log(\alpha n + \beta))(\log n - \log \gamma) + \log(\beta - \log \gamma + \log n + \log(\alpha n + \beta)),$$

where $\gamma = 1 + \log 3$.

**Proof:** Let

$$A = \{ i \in \mathbb{N} : 2^{\gamma} - a_i \leq \alpha n + \beta \}$$

and $r = |A|$. Note that $r \leq N = 2^n$. Let $t$ denote the smallest integer such that $t \geq 2$ and

$$r \leq 2^{n-j} \sum_{l=0}^{t} \binom{j}{l}. \quad (77)$$

Note that $l \leq j$. Then we have

$$2^{n-j} \sum_{l=0}^{t-1} \binom{j}{l} 2^l \leq \sum_{i \in A} 2^{s_j(i)}. \quad (78)$$

By definition of $A$ and (77) we have

$$\sum_{i \in A} 2^{s_j(i)} \leq (\alpha n + \beta) \sum_{i=1}^{N} a_i \leq (\alpha n + \beta) 2^{n-j} \sum_{l=0}^{t} \binom{j}{l} + 3^j. \quad (79)$$

By combining (78) and (79) we get

$$2^{n-j} \sum_{l=0}^{t} \binom{j}{l} 2^l \leq (\alpha n + \beta) 2^{n-j} \sum_{l=0}^{t} \binom{j}{l} + 3^j. \quad (80)$$

Since $j \leq n/(1 + \log 3)$, we have $2^{n-j} \geq 3^j$. Combining this with (80) we get

$$\left( \frac{j}{t-1} \right) 2^{t-1} \leq (\alpha n + \beta) \sum_{l=0}^{t} \binom{j}{l} \leq (\alpha n + \beta)(t+1) \binom{j}{t} \quad (81)$$

and therefore,

$$2^{t-1} \leq 2(\alpha n + \beta) j \quad (82)$$
or equivalently
\[ t \leq 2 + \log(\alpha n + \beta) + \log j \\]
\[ \leq 2 - \log(1 + \log 3) + \log n + \log(\alpha n + \beta) \] \hspace{1cm} (83)

By (77) and (83) we have
\[ r \leq 2^{n-j} \sum_{t=0}^{j} \binom{j}{t} \leq 2^{n-j}(t+1)j, \]
and the lemma follows by taking logarithm of both sides and the bound on \( t \) given in (83).

**Lemma 15:** Given a sequence of channels \( \{W_i\}_{i=1}^N \), where \( N = 2^n \), let \( x_{0,i} = Z(W_i) \) and consider the extremal deterministic process \( \{x_{j,i}\} \), for \( j = 0, 1, \ldots, n \) and \( i \in [N] \), defined in Definition 6. Let also \( \{W_{j,i}\} \) denote the deterministic polarization 2, defined in Definition 5, with the set of initial channels \( W_{0,i} = W_i \) such that the permutations and the indices of skipped operations follow the extremal deterministic process \( \{x_{j,i}\} \). Then we have
\[ Z(W_{j,i}) \leq x_{j,i}, \]
for \( i \in [N] \) and \( j = 0, 1, \ldots, n. \)

**Proof:** The proof is by induction on \( j \) and using (4) and (5) on the Bhattacharyya parameters of the combined channels and the definition of extremal deterministic process in Definition 5.