We combine numerical and analytical methods to study two dimensional active crystals formed by permanently linked swimmers and with two distinct alignment interactions. The system admits a stationary phase with quasi long range translational order, as well as a moving phase with quasi-long range velocity order. The translational order in the moving phase is significantly influenced by alignment interaction. For Vicsek-like alignment, the translational order is short-ranged, whereas the bond-orientational order is quasi-long ranged, implying a moving hexatic phase. For elasticity-based alignment, the translational order is quasi-long ranged parallel to the motion and short-ranged in perpendicular direction, whereas the bond orientational order is long-ranged. We also generalize these results to higher dimensions.

**Introduction** One of the most interesting and fundamental issues about active systems [1–6] is the stability of orders. According to the Mermin-Wagner theorem [7], two-dimensional (2D) equilibrium systems with continuous symmetry and short range interaction cannot exhibit not long range order (LRO). However, LRO was discovered in 2D polar active fluid both in simulation [8] and in hydrodynamic theory [9–11, 13]. This LRO is accompanied by super-diffusion and giant number fluctuations [2, 6, 11, 12], neither of which is seen in equilibrium systems with short range interactions. Many variants of Vicsek models with different particle polarity, alignments and exclusion [12–26] have been studied, a variety of novel phenomena have been discovered.

Dense active systems with repulsive interactions may also exhibit translational orders. Solid phases as well as fluid-solid phase separations have been repeatedly observed in active colloidal systems both experimentally [28–30] and numerically [27, 31–34]. In most of these works, there is no alignment interaction, and no visible collective motion. More recently, Weber et al. [20] simulated a model active crystal with Vicsek-type alignment, and discovered a stationary phase with quasi-long range (QLR) translational order, as well as a phase of moving crystal domains separated by grain boundaries. Very recently, Maitra et al. [35] studied an active generalization of nematic elastomer [38] with spontaneous breaking of rotational symmetry [36, 37], and found QLR translational orders in 2D. Their elastic energy contains a hidden rotational symmetry (and its resulting Goldstone modes) involving both shear deformation and orientational order, which are difficult to realize experimentally.

Regardless of many previous studies, it is not clear whether there exists a moving phase with certain translational order in active systems alignment interactions, if the soft-mode in Ref. [35] does not come into play. To address this interesting question, here we combine analytic and numerical approaches to study a model system of active crystal consisting of 2D triangular array of swimmers linked permanently by springs. We introduce alignment interaction between neighboring swimmers that is either Vicsek-like (AD-I) or elasticity-based [39, 40] (AD-II). In the strong noise/weak alignment regime, we find a stationary phase with QLR translational order, which was also seen in Ref. [20]. In the weak noise/strong alignment regime we find a moving phase with QLR velocity order, and with the nature of translational order depending on the alignment. For Vicsek-like alignment (AD-I), the moving phase exhibits only short-range (SR) translational order and QLR bond orientational order, and hence should be identified with moving hexatic phase. For elasticity-based alignment (AD-II), the translational order is QLR along the moving direction, and SR in the perpendicular direction, whereas the bond-orientational order is LR. We generalize the model to higher dimensions, and show in active systems velocity alignment tends to destabilize crystal orders.

**Simulation model** As schematized in Fig. 1, our simulation model consists of a triangular array of swimmers connected permanently by harmonic springs. Each swimmer moves under the influences of elastic force, friction and noise, as well as active force. The position $\mathbf{r}_k(t)$ of $k$-th swimmer in the lab frame then obeys the following
over-damped Langevin equation:
\[
\gamma \dot{r}_k(t) = b \hat{n}(\theta_k) + F_k(t) + \gamma \sqrt{2D} \xi_k(t), \tag{1}
\]
where \(\gamma\) is the friction coefficient, \(\hat{n}(\theta_k) = (\cos \theta_k, \sin \theta_k)\) is the director of active force. The magnitude of active force \(b\) is assumed to be fixed in our model. The elastic force is
\[
F_k = \sum_{j \neq n,n.k} \kappa (|r_k - r_j| - a_0) \frac{r_k - r_j}{|r_k - r_j|}, \tag{2}
\]
where the summation is over six nearest neighbors of swimmer \(k\), whilst \(\kappa\) and \(a_0\) are respectively the elastic constant and natural length of the springs. The last term of Eq. (1) is the random force, with \(\xi_k(t)\) the unit Gaussian white noise.

We consider two distinct alignment dynamics for the active forces. The first (AD-I) is Vicsek-like \([41]\), with each swimmer trying to align its director of active force with its neighbors \([53]\), subject to an internal noise:
\[
\theta_k(t) = \dot{d}(\hat{n}(\theta_k) \times \langle \hat{n}_j \rangle_k) \cdot \hat{z} + \sqrt{2D\eta} \eta_k(t), \tag{AD-I}
\]
where \(\hat{z}\) is the unit normal to the plane, \(\langle \hat{n}_j \rangle_k \equiv \sum_{j \neq n,n,k} \hat{n}(\theta_j) / 6\) the average director of all six nearest neighbors, and \(\eta_k(t)\) is a unit Gaussian white noise.

The second alignment dynamics (AD-II) is elasticity-based \([39, 40, 42]\), with each swimmer aligning its active force with the local elastic force, so as to reduce the local elastic energy:
\[
\dot{\theta}_k(t) = c (\hat{n}(\theta_k) \times \langle F_k \rangle) \cdot \hat{z} + \sqrt{2D\theta} \eta_k(t). \tag{AD-II}
\]

We use a rhombic cell with periodic boundary condition (c.f. Fig. 1), and numerically integrate the dynamic equations, Eqs. (1) and (AD-I) or (AD-II), using the first-order Euler-Maruyama scheme \([43]\), with the time step \(\Delta t = 10^{-3}\). Simulation details as well as the definitions of all dimensionless parameters, are given in Supplementary Information (SI) Sec. I.

We first determine the phase diagram by computing the velocity order parameter, defined as the steady state time average of magnitude of system-averaged active force \(P = \langle |\hat{u}_x| \rangle_t\), which is proportional to the velocity of collective motion. As shown in Fig. 2(a), for weak alignment/strong active noise (dark blue in upper left) there is a stationary phase where \(P\) is approximately zero \([54]\), whereas for strong alignment/weak active noise (bright yellow in lower right) there is a collectively moving phase where \(P\) is finite. As shown in Fig. 2(b), fitting of \(P\) as a function of alignment strength suggests that these two phases are separated by a line of second order phase transitions.

To study the translational order in the stationary phase, we carry out a larger simulation with system size \(256 \times 256\). The total number of time steps is \(2 \times 10^6\) and simulation samples are collected every 2000 steps in the steady state. We Fourier transform the phonon field, and compute averages of their norm squared: \(\langle |\hat{u}_x| \rangle^2\), \(\langle |\hat{u}_y| \rangle^2\), which are often called as phonon correlation functions (in momentum space). The technical details of numerical computation are presented in SI Sec. II. In Fig. 2(c) we plot \(\langle \hat{u}_x(k) \rangle^2\) and \(\langle \hat{u}_y(k) \rangle^2\) along \(k_x\) and \(k_y\) directions in \(k\) space, and both for AD-I and AD-II in log-log scale. For comparison we also plot the corresponding result for the model without any alignment (AD-0), which corresponds to \(\hat{d} = \hat{c} = 0\) in Eqs. (AD-I) and (AD-II). It is remarkable that all curves collapses onto each other, and exhibit \(k^{-4}\) scaling in the intermediate length scale \((0.1 < k < 1)\). This signify anomalously large structure fluctuations in the length scales...
FIG. 3: (a)-(d) Translational correlation functions; (e) & (f) velocity correlation functions; (g) & (h) bond-orientational correlation functions. 

Left column: AD-I, right volume: AD-II.

The parameters are $D_4 \equiv 0.05$, $(\tilde{\kappa}, \tilde{d}) = (100,10)$ for AD-I and $(\tilde{\kappa}, \tilde{c}) = (20,1)$ for AD-II. We start from an initial state with perfect triangular lattice and $\theta_k = 0$. Data are collected after the system reaches a steady state with $P \approx 0.9915$ in AD-I and $P \approx 0.9830$ in AD-II.

6$a_0 < \ell < 60a_0$ ($a_0$ the lattice constant) which are caused by the fluctuations of active forces. These results strongly suggest that alignment does not play a role in the stationary phase. For smaller $k$ the phonon correlation functions crossover to $k^{-2}$ scaling, indicating a QLR translation order, in consistent with the result obtained in Ref. [20].

We are however most interested in the collective moving phase. For this purpose, we study the translational correlation function $g_{\hat{r}}(\hat{r})$, velocity correlation function $\langle \tilde{v}(0) \tilde{v}(\hat{r}) \rangle_z$, and the bond-orientational correlation function $g_{\theta}(\hat{r})$, all of which are defined in SI Sec. III. In particular, $g_{\hat{r}}(\hat{r})$, defined in Eq. (12) of SI Sec. III, characterizes the correlation of translational order with Bragg vector $\mathbf{q}$ at two regions separated by a dimensionless distance $\hat{r} \equiv r/a_0$. As demonstrated in Ref. [44], because of sample-to-sample fluctuations of crystal structure, the Bragg peak $\mathbf{q}$ used for $g_{\hat{r}}(\hat{r})$ must be identified carefully for each sample state. In Fig. 3(a) & (c) we plot $g_{\hat{r}}(\hat{r})$ for AD-I along $\hat{x}$ and $\hat{y}$ axes for a typical state of the moving phase, which clearly decays faster than power law. Hence there is only short-range (SR) translational order in the moving phase of AD-I. The corresponding results for AD-II are shown in Fig. 3(b) & (d), where it is seen that $g_{\hat{r}}(\hat{r})$ decays algebraically along the $\hat{x}$ axis, and decays faster along the $\hat{y}$ axis. In Fig. 3(e) & (f) we show that velocity correlation functions decay in power-law, which indicate QLRO of velocity field for both AD-I and AD-II. As displayed in Fig. 4(c) & (d) of SI Sec. III, we also find QLRO for the director correlation of active forces. Furthermore, it appears that the exponent of all these power-law scalings are very close to unity. In Fig. 3(g) & (h), we show that the bond-orientational correlation function $g_{\theta}(\hat{r})$ decays in power law for AD-I and converges to a finite limit for AD-II. Hence the bond orientational order is quasi-long ranged for AD-I and long-ranged for AD-II.

In summary, the moving phase of AD-I exhibits QLRO both in velocity and in bond-orientation, but only has SR translational order. Hence it should be categorized as a moving hexatic phase. By contrast, the moving phase of AD-II exhibits QLRO in velocity and LRO in bond-orientation, yet the translation order is quasi-long ranged along the moving direction and short ranged in the other direction. This resembles the active smectic phase [45, 46], even though there is no visible layer structures in our system. These numerical results indicate that there is no enhancement of velocity order due to translation order. On the other hand, alignment interactions tend to destabilize translational order in active crystals, and that the destabilizing effect is stronger for Vicsek-like alignment (AD-I) than for elasticity-based alignment (AD-II).

Analytic treatment Our lattice model of active crystal can be coarse-grained to yield a continuous theory in terms of phonon fields $\mathbf{u}(\mathbf{r}, t)$ and director field $\theta(\mathbf{r}, t)$. Detailed analyses is given by SI Sec. IV. Further linearizing, the elastic force Eq. (2) becomes:

$$\begin{align*}
F_x &= (\lambda + 2\mu)\partial^2_x u_x + \mu\partial^2_y u_y + (\lambda + \mu)\partial_x \partial_y u_y, \\
F_y &= (\lambda + 2\mu)\partial^2_y u_y + \mu\partial^2_x u_x + (\lambda + \mu)\partial_x \partial_y u_x,
\end{align*}$$

where $\lambda, \mu$ are two Lamé coefficients characterizing the solid elasticity. The dynamics of AD-I is described by:

$$\begin{align*}
\gamma \dot{u}_x &= F_x + \sqrt{2\gamma T} \xi_x, \\
\gamma \dot{u}_y &= b\theta + F_y + \gamma \sqrt{2D} \xi_y, \\
\dot{\theta} &= d\Delta \theta + \sqrt{2D} \eta,
\end{align*}$$

where $\xi_x(\mathbf{r}, t), \xi_y(\mathbf{r}, t), \eta(\mathbf{r}, t)$ are all unit Gaussian white noises. We Fourier transform Eqs. (4), and calculate the steady state correlation function for $\mathbf{u}(\mathbf{k}, t)$ and $\theta(\mathbf{k}, t)$:

$$\langle |\hat{u}_x(\mathbf{k})|^2 \rangle = \frac{W_1(\alpha)}{k^6} + \frac{W_2(\alpha)}{k^2},$$
\[ |\hat{u}_y(k)|^2 = \frac{W_3(\alpha)}{k^6} + \frac{W_4(\alpha)}{k^2}, \quad (5b) \]
\[ |\hat{\theta}(k)|^2 = \frac{D_\theta}{k^2}, \quad (5c) \]
where \( \alpha = \tan^{-1}(k_y/k_x) \) is the polar angle of \( \mathbf{k} \), and the functions \( W_i(\alpha), i = 1, 2, 3, 4 \) are defined in Eqs. (28) of SI Sec. IV. While \( W_2(\alpha), W_3(\alpha), W_4(\alpha) \) are all positive, \( W_1(\alpha) = \frac{b^2 D_\theta (d_\gamma + 4\lambda) \sin^2 2\alpha}{12d\lambda (d^2\gamma^2 + 4d\gamma\lambda + 3\lambda^2)} \).

vanishes along \( \hat{k}_x \) and \( \hat{k}_y \) axes, where \( \alpha = 0, \pi/2, \pi, 3\pi/2 \) respectively. Hence along these axes, \( |\hat{u}_y(k)|^2 \) scales as \( k^{-2} \) instead of \( k^{-6} \) along other directions.

The real space fluctuations of phonon fields and director field can be obtained by integrating Eqs. (5) over \( \mathbf{k} \). Integrating Eq. (5c) we see that \( |\hat{\theta}(k)|^2 \) diverges logarithmically with system size, and hence the active force exhibits QLRO. Since the velocity is massively coupled to the active force, it should also exhibit QLRO, as demonstrated by our numerical simulation. On the other hand, Integrating Eqs. (5a), (5b) we see that \( |\hat{u}_x(k)|^2 \) and \( |\hat{u}_y(k)|^2 \) diverge in power law with system size, indicating SR translation orders. These results again agree with our numerical observations shown in Fig. 3.

The dynamics of AD-II in the linearized continuous theory can be similarly obtained:

\[
\begin{align*}
\gamma \dot{u}_x &= F_x + \gamma \sqrt{2D}\xi_x, \\
\gamma \dot{u}_y &= b \theta + F_y + \gamma \sqrt{2D}\xi_y, \\
\dot{\theta} &= c F_y + \sqrt{2D_{\theta}}\eta.
\end{align*}
\]

The steady state correlation functions, calculated in detail in SI Sec. IV, turn out to be more complicated. Here we only display the leading order terms for small \( k \):

\[
\begin{align*}
|\hat{u}_x(k)|^2 &= \frac{W_5(\alpha)}{k^6} + O(k^0), \quad (8a) \\
|\hat{u}_y(k)|^2 &= \frac{W_6(\alpha)}{k^4} + O(k^{-2}), \quad (8b) \\
|\hat{\theta}(k)|^2 &= \frac{W_7(\alpha)}{k^2} + O(k^0), \quad (8c)
\end{align*}
\]

where \( W_5(\alpha), W_6(\alpha), W_7(\alpha) \) are all positive definite. Since Eq. (8c) scales the same as Eq. (5c) we see that the velocity order is again quasi-long ranged. Integrating Eqs. (8a), (8b) we see that that translational order for AD-II is QLR along \( x \) axis and SR along \( y \) axis, again consistent with the numerical results displayed in Fig. 3. In SI Sec. V we compare the contour plots of analytical and numerical correlation functions in \( \mathbf{k} \) plane for both active dynamics, see quantitative agreements.

To achieve better understanding of our model of active solids, we can generalize the analytic theory to arbitrary \( d \) dimensions. The elasticity of a \( d \) dimensional isotropic solid can be obtained by adapting Eq. (3), if we replace \( u_y \) and \( \partial_y \) by \( u_\perp \) and \( \nabla_\perp \) where \( \perp \) denotes the subspace perpendicular to the moving direction \( \hat{x} \), replace \( \theta \) by \( \delta \hat{n}_\perp \), the fluctuation of active force director in the perpendicular subspace. Equations (4) and (7) can be similarly generalized to \( d \) dimensions. In SI Sec. VI, an explicit derivation is given for the case \( d = 3 \). The correlation functions for the phonon fields and for the director fluctuation \( \delta \hat{n}_\perp \) can be similarly computed, and the results are still given by Eqs. (5) and (8), as long as \( u_y \) are replaced by \( u_\perp \) and \( \theta \) by \( \delta \hat{n}_\perp \). Hence we conclude that for Vicsek-like alignment (AD-I), the translational order of active crystal has critical dimension \( d^c_{\parallel} = 6 \), whilst the velocity order has critical dimension \( d^c_{\perp} = 2 \). By contrast, for elasticity-based alignment (AD-II), the translational order along the moving direction has critical dimension \( d^c_{\parallel} = 2 \), and that perpendicular to the moving direction has critical dimension \( d^c_{\perp} = 4 \), whilst the velocity order has critical dimension \( d^c_{\perp} = 2 \). The nature of bond-orientational order however can not be easily determined from our analytic theory. Nonetheless, given our 2D results, we deduce the bond orientational order is long-range above two dimensions both for AD-I and AD-II. The nature of various orders for 2d and 3d cases are summarized in Table I.

| Stationary Phase | Moving Phase |
|------------------|--------------|
| 2D               | LRO SRO QLRO |
| 3D               | LRO SRO LRO |

TABLE I: Stability of translational order, bond-orientational orders, and velocity order in active crystals in 2D and 3D. In the stationary phase, bond-orientational order is always long-ranged whereas velocity is always short-ranged.
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[52] Note however in this work the dynamic equations contain no noise term.

[53] Note that in the original Vicsek model [41], swimmers control their velocities instead of active forces. In the over-damped regime, this difference is essential.

[54] It is never strictly zero because the system is finite and there are always instantaneous fluctuations at each time.

[55] Here the orientation of real space displacement is averaged over. The directionality of \( g_\mathbf{q}(\mathbf{r}) \) is carried by the Bragg vector \( \mathbf{q} \). For details, see SI Sec. III.
Supplementary Information

Alignment Destabilizes Crystal Orders in Active Systems

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I. DIMENSIONLESS FORM OF THE EQUATIONS OF MOTION AND THE CHOICE OF PARAMETERS

We derive in this section the dimensionless equations of the model and specify the parameters used in simulation. To distinguish the coefficients used in simulation with their counterparts in analytical results, we temporarily add a subscript or superscript \( L \) (meaning “Lattice”) in this section.

In the simulation model, the space and time units are chosen as \( a_0 \) (lattice constant) and \( \tau_0 \) (the value of \( \tau_0 \) would be determined later). The dimensionless control parameters are:

\[
\tilde{b}_L \equiv b\tau_0/(\gamma a_0), \quad \tilde{\kappa} \equiv \kappa\tau_0/\gamma, \quad \tilde{D}_L \equiv D\tau_0/a_0^2, \quad \tilde{D}_0 \equiv D_\theta\tau_0, \quad \tilde{d}_L \equiv d\tau_0, \quad \tilde{c}_L \equiv c\gamma a_0, \\
\tilde{F}_k \equiv F_k\tau_0/(\gamma a_0), \quad \tilde{\eta}_k \equiv \eta\sqrt{\tau_0}, \quad \tilde{\xi}_k \equiv \hat{\xi}\sqrt{\tau_0}.
\] (1)

The dimensionless equation of Eq. (1) is,

\[
\dot{\tilde{r}}_k(t) = \tilde{b}_L \hat{n}(\theta_k) + \tilde{F}_k(t) + \sqrt{2\tilde{D}_L}\tilde{\xi}_k(t),
\] (2)

The dimensionless equations of the two alignment dynamics become:

\[
\dot{\theta}_k(t) = \tilde{d}_L(\tilde{n}(\theta_k) \times \langle \hat{n}\rangle_k) \cdot \dot{\tilde{z}} + \sqrt{2\tilde{D}_0}\tilde{\eta}_k, \quad \text{(AD-\( \tilde{\Pi} \))}
\]

\[
\dot{\tilde{z}}_k(t) = \tilde{c}_L(\tilde{n}(\theta_k) \times \tilde{F}_k) \cdot \dot{\tilde{z}} + \sqrt{2\tilde{D}_0}\tilde{\eta}_k, \quad \text{(AD-\( \tilde{\Pi} \))}
\]

We numerically integrate the above dynamic equations using the first-order Euler-Maruyama scheme [1]. We choose \( \tau_0 \equiv 0.01a_0^2/D_L \) to ensure the perturbation assumption such that the magnitude of the noise \( \sqrt{\tilde{D}_L} = 0.1 \) is small compared to the active force and the elastic force. The other parameters are chosen as:

\[
\tilde{b}_L = 2, \quad \tilde{\kappa} = 100 \text{ for AD-I, } \tilde{\kappa} = 20 \text{ for AD-II,}
\] (5)

In determining the phase diagram, we use a relatively small lattice of size 32 \( \times \) 32 to save computational resources, start from a perfect triangular lattice with all active forces orienting randomly.

In studying the translational order and other correlation functions, we use a larger lattice of size 256 \( \times \) 256. For the stationary phase, we choose the parameters \( \tilde{D}_0 = 0.3, \tilde{d} = 0.1 \) for both AD-I and AD-II, and start from a perfect triangular lattice and all active forces orienting randomly. For the collective moving phase, we use the parameter \( \tilde{D}_0 = 0.05 \) (Note
that by our choice, the magnitude of the angular noise also satisfies the small perturbation condition $\sqrt{D_{\theta}} < 1$), $\tilde{d} = 10$ for AD-I and $\tilde{d} = 1$ for AD-II, and also start from a perfect lattice, but with all active forces orienting toward the $\hat{x}$ axis. Note that for any finite-size lattice, the average velocity of the collectively moving crystal would slowly change, so we redirect it back toward the $\hat{x}$ axis right after each sample has been collected but before the subsequent simulation steps start.

In the coarse-grained continuous model, using the space and time units $a_0$ and $\tau_0$, we obtain the reduced quantities:

$$\tilde{b} \equiv \frac{b\tau_0}{\gamma a_0}, \quad \tilde{F} \equiv \frac{F\tau_0}{\gamma a_0}, \quad \begin{pmatrix} \tilde{\mu} \\ \tilde{\lambda} \end{pmatrix} \equiv \frac{\tau_0}{\gamma a_0^2} \begin{pmatrix} \mu \\ \lambda \end{pmatrix}, \quad \tilde{D} \equiv \frac{D\tau_0}{a_0^2}, \quad \tilde{D}_{\theta} \equiv D_{\theta}\tau_0,$$

$$\tilde{d} \equiv \frac{d\tau_0}{a_0^2}, \quad \tilde{c} \equiv c\gamma a_0, \quad \begin{pmatrix} \tilde{\eta} \\ \tilde{\xi} \end{pmatrix} \equiv \sqrt{\tau_0} \begin{pmatrix} \eta \\ \xi \end{pmatrix}. \quad (6)$$

The dimensionless form of Eq. (4) is:

$$\dot{\tilde{u}}(\tilde{r}, \tilde{t}) = \tilde{b}\theta \tilde{y} + \tilde{F}(\tilde{r}, \tilde{t}) + \sqrt{2}\tilde{D}\tilde{\xi}(\tilde{r}, \tilde{t}). \quad (7)$$

The two coarse-grained dimensionless alignment dynamics are, respectively,

$$\dot{\theta}(\tilde{r}, \tilde{t}) = \tilde{d}\Delta \theta + \sqrt{2}\tilde{D}_{\theta}\tilde{\eta}(\tilde{r}, \tilde{t}), \quad (AD-I') \quad (8)$$

$$\dot{\theta}(\tilde{r}, \tilde{t}) = \tilde{c}(\tilde{n} \times \tilde{F}) \cdot \tilde{z} + \sqrt{2}\tilde{D}_{\theta}\tilde{\eta}(\tilde{r}, \tilde{t}). \quad (AD-II') \quad (9)$$

where $\tilde{\Delta} \equiv a_0^2\Delta$. Since the linearized elasticity theory in the continuous model also applies to the two-dimensional hexagonal crystal used in our simulation [2], we may relate the Lamé constants and the elastic constant via $\tilde{\lambda} = \tilde{\mu} = \tilde{\kappa}\sqrt{3}/4$ [3] in order to compare the two models.

II. FOURIER TRANSFORM ON NON-ORTHOGONAL LATTICE

The simulations use a triangular lattice with rhomb-like boundary, where the lattice points are shown in Fig. 1(a) as the solid black points. We wish to implement the Fourier Transform (FT) to the data (e.g., the displacement $u_x, u_y$, and the angle $\theta$) sampled on
this triangular lattice into the reciprocal $k$-space. Although it is easy to implement FT on a square lattice by using the Fast Fourier Transform (FFT) techniques, no direct method exists for FT on a triangular lattice. However, the triangular lattice can be deemed as originating from a linear transform of the square lattice. Since FT is also a linear transform, we expect a combination of the two linear transforms may serve our purpose. Below is a derivation of this process.

Suppose we sample a complex function $f : \mathbb{R}^n \rightarrow \mathbb{C}$ on a set of $n$-dimensional points $\Lambda \in \mathbb{R}^n$ in real space. Let $\Phi : \mathbb{R}^n \rightarrow \mathbb{R}^n$ be a linear isomorphic map under which $n$-dimensional integers are mapped to $n$-dimensional real values such that $\mathbb{Z}^n \rightarrow \Lambda$. We may obtain the FT of $f$ in terms of the FT of $(f \circ \Phi)$, which is a function on $\mathbb{Z}^n$. The FT of $(f \circ \Phi)$ is,

$$\mathcal{F}(f \circ \Phi)(s) \equiv \int f \circ \Phi(x)e^{-2\pi i(s,x)}dx$$

$$= \frac{1}{|\det \Phi|} \int f(y)e^{-2\pi i(\Phi^{-T}s,y)}dy$$

$$= \frac{1}{|\det \Phi|} \mathcal{F}(f)(\Phi^{-T}s)$$

Here a substitution $y = \Phi x$ was made, and it was used that $\langle s, \Phi^{-1}y \rangle = \langle \Phi^{-T}s, y \rangle$. Note that the conventions with respect to the sign of the exponent and factors of $2\pi$ may differ.

We now discuss the meaning of the above formula. The function $f \circ \Phi$ is the value of the sample points of $f$ arranged in an $n$-dimensional array lying on a standard grid (equal-axis orthogonal grid). Its Fourier transform gives an array of the same shape, but the interpretation is different: the value at index $s$ in reality is the value at $\Phi^{-T}s \in \mathbb{R}^n$ in $k$-space, differed by a multiplier $|\det \Phi|$.

In our simulation’s setup, $n = 2$ and $\Lambda \in \mathbb{R}^2$ is the set of the triangular lattice points in real space. The function $f$ whose FT we want to compute, can be the displacement field $u_x, u_y$ or the orientation field $\theta$. The linear map $y = \Phi x$ with $\Phi = \begin{pmatrix} 1 & 1 \\ 0 & \sqrt{3} \end{pmatrix}$ transforms a square regular lattice with lattice constant $a_0$ into the desired triangular lattice with the same lattice constant. The transverse of its inverse is $\Phi^{-T} = \begin{pmatrix} 1 & 0 \\ -1/\sqrt{3} & 2/\sqrt{3} \end{pmatrix}$. The function $f \circ \Phi$ takes the value of the sample points of $f$ on $\Lambda$, but its arguments lie on the corresponding
square lattice points. We apply FFT on this real space square lattice (standard grid), obtain its FT values on a $k$-space square lattice (standard grid) indexed by $s$, and assign the FT values to its true coordinate in $k$-space by left multiplying $\Phi^{-T}$ to $s$. Suppose the standard grid in real space has a linear size $L \times L$ and each dimension is divided into equally spaced $n$ points with distance of unity $a_0 = 1$, then there are $n \times n$ total sample points in both real and $k$-space.

**FIG. 1:** A lattice with lattice constant $a_0 = 1$ and $L = 8$. The black points denote (a) Sample points in real space and (b) the corresponding $k$-space points. The red triangles are the standard grid in each space respectively. The dashed lines denote the boundary of the simulation box.

In Fig. 1, a lattice of $L = 8$ and $n = 8$ is shown. The lattice constant of the standard grid in real space is $a_0 = 1$ and that in $k$-space is $\Delta k = 2\pi/L = \pi/4$. The $k$-space vector of the standard grid with the minimal magnitude $s_1 = (0, \frac{2\pi}{L})$ transforms to the true $k$-space coordinate, $g_1 = (0, \frac{2\pi}{L} \frac{2}{\sqrt{3}})$, while $s_2 = (\frac{2\pi}{L}, 0)$ transforms to $g_2 = (\frac{2\pi}{L}, -\frac{1}{\sqrt{3}} \frac{2\pi}{L})$ with both $|g_i| = \frac{2\pi}{L} \frac{2}{\sqrt{3}}$.

**III. TRANSLATIONAL CORRELATION, PAIR CORRELATION, AND BOND-ORIENTATIONAL CORRELATION**

In this section, we discuss the definition and evaluation of the translational correlation, pair correlation of velocity and director of active force, and bond-orientational correlation.
in AD-I and AD-II dynamics. For simplicity, the quantities such as \( r \) and \( q \) in this section all refer to dimensionless variables.

We first follow the steps in Ref. [4] by defining the translational correlation function as:

\[
g_q(r) = \frac{1}{2\pi r \Delta r \rho N} \sum_{j \neq k} \zeta (r - |r_j - r_k|) \text{Re}(e^{i\mathbf{q} \cdot (r_j - r_k)}). \tag{12}
\]

where \( \zeta = 1 \) if \( |r_j - r_k| \) is in the region \( r \sim r + \Delta r \), otherwise \( \zeta = 0 \). The number density \( \rho = (\sqrt{3}/2)^{-1} \) is the inverse of the area of a unit cell in the triangular lattice of unit length. We choose \( q \) by evaluating the Bragg peaks of the structure factor

\[
Q(q_x, q_y) = \frac{1}{N} \langle \rho(q_x, q_y) \rho(-q_x, -q_y) \rangle, \tag{13}
\]

where

\[
\rho(q_x, q_y) = \sum_{j=1}^{N} \exp \left[ i (q_x x_j + q_y y_j) \right], \tag{14}
\]

with \( x_j \) and \( y_j \) the Cartesian coordinates of the \( j \)th particle. The choice of \( q \) should be carefully decided from simulation results, i.e., the true peak value is numerically evaluated around the Bragg peak of a perfect hexagonal lattice \( q_x = (4\pi, 0) \) along the \( \hat{x} \) axis, \( q_y = (0, 2\pi \cdot (\sqrt{3}/2)^{-1}) \) along the \( \hat{y} \) axis, and \( q_t = 2\pi (1, 1/\sqrt{3}) \) along a tilted direction.

In Fig. 2 and Fig. 3, we plot \( Q(q_x, q_y) \) (the first column) and the corresponding translational correlation function (the second column in Log-Log scale and the third in Log-Linear scale) of typical sample configurations, for AD-I with \((\tilde{\kappa}, \tilde{d}_L) = (100, 10)\) and AD-II with \((\bar{\kappa}, \bar{c}_L) = (20, 1)\), respectively. Data is collected after the system has reached a steady state of collective moving, with \( P \approx 0.9915 \) in AD-I and \( P \approx 0.9830 \) in AD-II. In AD-I, the translational correlation function decays faster than power law along the wave vector \( q_x, q_y \) and \( q_t \), indicating short-range order. However, in AD-II, it has a power law decay along the \( \hat{x} \) axis, and exhibits short-range order along \( q_y \) and \( q_t \).

In the first row of Fig. 4, we show the correlation function of the director of active force in k-space \( \langle |\hat{\theta}|^2 \rangle \). We observe a \( k^{-2} \) decay in AD-I, and a \( k^{-2} \) decay followed by a constant in AD-II. These scaling properties match with the analytical results Eqn. (35) and (41) given in Appendix D.

In the second row of Fig. 4, we plot 10 independent samples at different times of the the pair correlation function of the director of active force,

\[
\langle \hat{n}(0) \cdot \hat{n}(r) \rangle = \frac{1}{2\pi r \Delta r \rho N} \sum_{j \neq k} \zeta (r - |r_j - r_k|) (\hat{n}_j \cdot \hat{n}_k). \tag{15}
\]
FIG. 2: Bragg peak of $Q(q_x, q_y)$ and the corresponding translational correlation function in AD-I
with (a)-(c) $q_x = (12.5172, -0.0292)$; (d)-(f) $q_y = (0.0196, 7.2905)$; (g)-(i) $q_z = (6.2998, 3.6363)$. The second column is in Log-Log scale, while the third column is in Log-Linear scale.

Note we have subtracted the asymptotic value of $\langle \hat{n}(0) \cdot \hat{n}(r) \rangle$ at large enough distance $\langle \hat{n}(0) \cdot \hat{n}(\infty) \rangle$ in the steady state, i.e., $\langle \hat{n}(0) \cdot \hat{n}(r) \rangle_c \equiv \langle \hat{n}(0) \cdot \hat{n}(r) \rangle - \langle \hat{n}(0) \cdot \hat{n}(\infty) \rangle$. We find a power-law decay with exponent being approximately $-1$ indicating quasi-long range order.

A similar treatment is applied to the velocity with

$$\langle \hat{v}(0) \cdot \hat{v}(r) \rangle = \frac{1}{2\pi r \Delta r \rho N} \sum_{j \neq k} \zeta (r - |\mathbf{r}_j - \mathbf{r}_k|) (\hat{v}_j \cdot \hat{v}_k).$$

where the velocity $\mathbf{v}_i$ of each particle is defined by subtracting the coordinates in the two configurations separated by time interval $100\Delta t$, i.e., $\mathbf{v}_i(\tilde{t}) \equiv [\mathbf{r}_i(\tilde{t}) - \mathbf{r}_i(\tilde{t} - 100\Delta t)]/(100\Delta t)$ and $\hat{v}_i \equiv \mathbf{v}_i/|\mathbf{v}_i|$. We also define and plot the value $\langle \hat{v}(0) \cdot \hat{v}(r) \rangle_c \equiv \langle \hat{v}(0) \cdot \hat{v}(r) \rangle - \langle \hat{v}(0) \cdot \hat{v}(\infty) \rangle$ as shown in Fig. 3(c)-(f) in the main text. The result gives a power-law decay of similar
FIG. 3: Bragg peak of $Q(q_x, q_y)$ and the corresponding translational correlation function in AD-II with (a)-(c) $q_x = (12.5667, -0.0015)$; (d)-(f) $q_y = (0.0166, 7.2272)$; (g)-(i) $q_x = (6.2847, 3.6213)$. The second column is in Log-Log scale, while the third column is in Log-Linear scale.

exponents with the director of active force.

In Fig. 4(e)-(f), we show the bond-orientational correlation function [4]:

$$g_6(r) = \text{Re} \langle \psi_6(\vec{r}_i) \psi_6^*(\vec{r}_j) \rangle = \frac{1}{2\pi r \Delta r \rho N} \sum_{j \neq k} \zeta(r - |\vec{r}_j - \vec{r}_k|) \text{Re}(\psi_6(\vec{r}_i) \psi_6^*(\vec{r}_j)),$$

(17)

where $r = |\vec{r}_i - \vec{r}_j|$. The local-bond orientational parameter is defined as $\psi_6(\vec{r}_j) = \frac{1}{n} \sum_m^n \exp i(6\theta_m^j)$ where the sum runs over the $n$ Voronoi neighbors of the particle, and $\theta_m^j$ is the angle of $(\vec{r}_m - \vec{r}_j)$ relative to any fixed axis. In Fig. 4(e)-(f), $g_6(r)$ in AD-I has a power law decay with a non-zero but small exponent $-0.0284$. This exponent is obtained from linear fitting of the average of 10 independent sample curves at different simulation steps, after the system has reached the steady state. The value of $g_6(r)$ in AD-II approaches
a constant at large distances. From the numerical results we may conclude that the bond-orientational order in AD-I is QLRO and that in AD-II is LRO. We leave the verification of these predictions and the evaluation of a more accurate power-law exponent to future studies.

**IV. ANALYTICAL SOLUTION OF THE CORRELATION FUNCTIONS**

To obtain a more thorough understanding of these numerical results, and also to understand analogous systems in higher dimensions, we resort to analytical approach. Our lattice model of active crystal can be coarse-grained to yield a continuous theory that is amenable for analytic study. Let \( \mathbf{r} \) be the Lagrange coordinate for a generic swimmer which denotes its position in the initial reference state, \( \mathbf{u}(\mathbf{r}, t) \) the phonon field characterizing its displacement from the uniformly moving state, the swimmer’s position at time \( t \) is then \( \mathbf{r} + \mathbf{u}(\mathbf{r}, t) + \mathbf{v}_0 t \), where \( \mathbf{v}_0 = b/\gamma \) is the mean velocity as determined from Eq. (1) in the main text (Hence our analytic results apply to the moving phase). The elastic force \( \mathbf{F}(\mathbf{r}, t) \) acting on the swimmer at \( \mathbf{r} \) can be expressed in terms of \( \mathbf{u}(\mathbf{r}, t) \) using the linearized elasticity theory [5, 6]:

\[
\mathbf{F} = \mu \nabla^2 \mathbf{u} + (\lambda + \mu) \nabla (\nabla \cdot \mathbf{u}),
\]

(18a)

where \( \lambda \) and \( \mu \) are two Lamé coefficients. This result is the coarse-grained version of Eq. (2) in the main text. In terms of components we have

\[
\begin{align*}
F_x &= (\lambda + 2\mu) \partial_x^2 u_x + \mu \partial_y^2 u_x + (\lambda + \mu) \partial_x \partial_y u_y, \\
F_y &= (\lambda + 2\mu) \partial_y^2 u_y + \mu \partial_x^2 u_y + (\lambda + \mu) \partial_x \partial_y u_x.
\end{align*}
\]

(18b)

The Langevin equations are coarse-grained into stochastic PDEs, which can be further linearized. In particular Eq. (1) in the main text becomes

\[
\gamma \dot{\mathbf{u}}(\mathbf{r}, t) = b \theta(\mathbf{r}, t) \hat{\mathbf{y}} + \mathbf{F}(\mathbf{r}, t) + \gamma \sqrt{2D} \hat{\xi}(\mathbf{r}, t).
\]

(19)

where \( \theta(\mathbf{r}, t) \) is the coarse-grained version of \( \theta_k(t) \), and \( \hat{\xi}(\mathbf{r}, t) \) is a 2d Gaussian white noise satisfying \( \langle \xi_i(\mathbf{r}, t) \rangle = 0, \langle \xi_i(\mathbf{r}, t) \xi_j(\mathbf{r}', t') \rangle = \delta_{ij} \delta(\mathbf{r} - \mathbf{r}') \delta(t - t') \). The stochastic PDE for the phonon field can be obtained by substituting Eq. (18b) into Eq. (19). Here in Eq. (19) and also below in Eqs. (20), (21), the parameters \( b, c, d, \gamma, D, D_\theta \) are in general different from
FIG. 4: (a)-(b) The correlation functions $\langle|\hat{\theta}|\rangle$ along $\hat{k}_x$ and $\hat{k}_y$ axis (Note that the $k_y$ value in AD-II is rescaled for comparison according to Eq. (41).), (c)-(d) the pair correlation functions of director of active force, and (e)-(f) the bond-orientational correlation functions of AD-I (the first column) and AD-II (the second column).

their lattice versions in Eqs. (1), (AD-I) and (AD-II) in the main text. We however need not
to be concerned with these differences since we are only interested in the scaling behaviors of correlation functions in the analytic theory.

Coarse-graining and linearizing Eq. (AD-I), we obtain the following set of linearized stochastic PDEs for AD-I:

\[
\begin{align*}
\gamma \dot{u}_x &= F_x + \sqrt{2\gamma} \xi_x \\
\gamma \dot{u}_y &= b \theta + F_y + \gamma \sqrt{2\bar{D}} \xi_y \\
\dot{\theta} &= d \Delta \theta + \sqrt{2\bar{D}_\theta} \eta
\end{align*}
\]

(20)

where \( \eta = \eta(r, t) \) is a Gaussian white noise satisfying \( \langle \eta(r, t) \rangle = 0 \), \( \langle \eta(r, t) \eta(r', t') \rangle = \delta(r - r')\delta(t - t') \). Further carrying out the spatial Fourier transform of Eqs. (20), we obtain linear Langevin equations for \( u(k, t) \) and \( \theta(k, t) \).

The linearized theory of AD-II can be similarly obtained:

\[
\begin{align*}
\gamma \dot{u}_x &= F_x + \gamma \sqrt{2\bar{D}} \xi_x \\
\gamma \dot{u}_y &= b \theta + F_y + \gamma \sqrt{2\bar{D}} \xi_y \\
\dot{\theta} &= c F_y + \sqrt{2\bar{D}_\theta} \eta
\end{align*}
\]

(21)

We are now ready to derive the steady state correlation function solution for \( u(k, t) \) and \( \theta(k, t) \). But before that, let us first consider the solution of a general multidimensional Langevin equation:

\[
\dot{x}(k, t) + \Gamma x(k, t) = \zeta(k, t).
\]

(22)

where \( \langle \zeta_i(k, t) \zeta_j^\dagger(k', t') \rangle = \langle \zeta_i(k, t) \zeta_j(-k', t') \rangle = 2B_{ij} \delta(k - k') \delta(t - t') \). The matrices \( B \) and \( \Gamma \) are real, and \( x(k, t) \) and \( \zeta(k, t) \) are column vectors. Define \( y = e^{\Gamma t} x \), it is easy to prove \( dy/dt = e^{\Gamma t} \zeta \), and we have the formal solution:

\[
x(k, t) = \int_{-\infty}^{t} e^{-(\Gamma(t-\tau))} \zeta(k, \tau) d\tau.
\]

(23)

The correlation matrix

\[
\langle x(k, t)x^\dagger(k', t') \rangle = \int_{-\infty}^{t} \int_{-\infty}^{t'} e^{-(\Gamma(t-\tau))} 2B \delta(k - k') \delta(\tau - \tau') e^{-(\Gamma(t'-\tau'))} d\tau d\tau'
\]

(24)

\[
= \int_{-\infty}^{\min(t, t')} e^{-(\Gamma(t-\tau))} 2B \delta(k - k') e^{-(\Gamma(t'-\tau))} d\tau
\]

(25)
where we used $\Gamma^\dagger = \Gamma^T$ in the last equation. Define the autocorrelation matrix as

$$M \equiv \langle x(k,t)x^\dagger(k,t) \rangle = \int_{-\infty}^{t} e^{-\Gamma(t-\tau)} 2Be^{-\Gamma^T(t-\tau)}d\tau$$

(26)

we have

$$\Gamma M + M\Gamma^T = \int_{-\infty}^{t} e^{-\Gamma(t-\tau)} 2(\Gamma B + B\Gamma^T) e^{-\Gamma^T(t-\tau)} d\tau$$

$$= 2 e^{-\Gamma t} \left[ \int_{-\infty}^{t} \frac{d}{d\tau} \left( e^{\Gamma \tau} Be^{\Gamma^T \tau} \right) d\tau \right] e^{-\Gamma t} = 2B$$

(27)

(28)

This is a Lyapunov equation [7] that can be solved analytically.

In AD-I, after implementing Fourier transform for the spatial component, we obtain the Lyapunov equation:

$$\begin{pmatrix}
\dot{\hat{u}}_x \\
\dot{\hat{u}}_y \\
\dot{\hat{\theta}}
\end{pmatrix} + \Gamma
\begin{pmatrix}
\hat{u}_x \\
\hat{u}_y \\
\hat{\theta}
\end{pmatrix} =
\begin{pmatrix}
\sqrt{2D}\hat{\xi}_x \\
\sqrt{2D}\hat{\xi}_y \\
\sqrt{2D}\hat{\eta}
\end{pmatrix}$$

(29)

where

$$\Gamma = \frac{1}{\gamma}
\begin{pmatrix}
\lambda(3k_x^2 + k_y^2) & 2\lambda k_x k_y & 0 \\
2\lambda k_x k_y & \lambda(3k_y^2 + k_x^2) & -b \\
0 & 0 & \gamma d(k_x^2 + k_y^2)
\end{pmatrix}$$

(30)

and

$$M =
\begin{pmatrix}
\langle |\hat{u}_x|^2 \rangle & \langle \hat{u}_x \hat{u}_x^* \rangle & \langle \hat{u}_x \hat{\theta}^* \rangle \\
\langle \hat{u}_y \hat{u}_x^* \rangle & \langle |\hat{u}_y|^2 \rangle & \langle \hat{u}_y \hat{\theta}^* \rangle \\
\langle \hat{\theta} \hat{u}_x^* \rangle & \langle \hat{\theta} \hat{u}_y^* \rangle & \langle |\hat{\theta}|^2 \rangle
\end{pmatrix},
B =
\begin{pmatrix}
D & 0 & 0 \\
0 & D & 0 \\
0 & 0 & D\theta
\end{pmatrix}$$

(31)
The solution is

\[
\begin{align*}
\langle |\hat{u}_x(k)|^2 \rangle &= \frac{D(k_x^2 + 3k_y^2)}{3k^4\lambda} \gamma + \frac{b^2 D_\theta k_x^2 k_y^2 (d\gamma + 4\lambda)}{3k^{10}\lambda d(d^2\gamma^2 + 4d\gamma\lambda + 3\lambda^2)} \\
&= W_1(\alpha) + W_2(\alpha) \\
\langle |\hat{u}_y(k)|^2 \rangle &= \frac{D(3k_x^2 + k_y^2)}{3k^4\lambda} \gamma + \frac{b^2 D_\theta \left[d(3k_x^4 + 3k_x^2 k_y^2 + k_y^4)\gamma + (3k_x^2 + k_y^2)^2\lambda\right]}{3k^{10}\lambda d(d^2\gamma^2 + 4d\gamma\lambda + 3\lambda^2)} \\
&= W_3(\alpha) + W_4(\alpha) \\
\langle |\hat{\theta}(k)|^2 \rangle &= \frac{D_\theta}{k^2} 
\end{align*}
\]

where \( k = \sqrt{k_x^2 + k_y^2} \) and \( \alpha \) is the polar angle such that \((k_x, k_y) = k(\cos \alpha, \sin \alpha)\). The constants are:

\[
\begin{align*}
W_1(\alpha) &= \frac{b^2 D_\theta (d\gamma + 4\lambda) \sin^2 2\alpha}{12d\lambda d(d^2\gamma^2 + 4d\gamma\lambda + 3\lambda^2)} \\
W_2(\alpha) &= \frac{\gamma D(2 - \cos 2\alpha)}{3\lambda} \\
W_3(\alpha) &= \frac{b^2 D_\theta \left[d\gamma(15 + 8\cos 2\alpha + \cos 4\alpha) + \lambda(36 + 32\cos 2\alpha + 4\cos 4\alpha)\right]}{24d\lambda d(d^2\gamma^2 + 4d\gamma\lambda + 3\lambda^2)} \\
W_4(\alpha) &= \frac{\gamma D(2 + \cos 2\alpha)}{3\lambda}.
\end{align*}
\]

\[ (32) \]

**FIG. 5:** The functions \( W_3(\alpha) \) and \( W_5(\alpha) \) are always positive in the interval \([0, 2\pi]\).

It is obvious that all the above coefficients \( W_1(\alpha), W_2(\alpha), W_3(\alpha) \) (See Fig. 5), \( W_4(\alpha) \) are positive, except that \( W_1(\alpha) \) might be zero at \( \alpha = 0, \pi/2, \pi, 3\pi/2 \). We may conclude that as \( k \to 0 \) in AD-I, \( \langle |\hat{u}_x|^2 \rangle \) scales as \( k^{-6} \), \( \langle |\hat{\theta}|^2 \rangle \) scales as \( k^{-2} \), and \( \langle |\hat{u}_y|^2 \rangle \) scales as \( k^{-4} \). However, along the two special orthogonal axis \( \hat{k}_x \) (\( \alpha = 0 \) or \( \pi \)) and \( \hat{k}_y \) (\( \alpha = \pi/2 \) or \( 3\pi/2 \)), \( \langle |\hat{u}_x|^2 \rangle \) scales as \( k^{-2} \). To be more specific, along the two orthogonal axis \( \hat{k}_x \) and \( \hat{k}_y \) the correlation functions are,
\[ \langle \hat{u}_x(k_x, 0) \rangle^2 = \frac{\gamma D}{3 \lambda k_x^2}, \quad \langle \hat{u}_x(0, k_y) \rangle^2 = \frac{\gamma D}{\lambda k_y^2}, \quad (33) \]
\[ \langle \hat{u}_y(k_x, 0) \rangle^2 = \frac{b^2 D_\theta}{d \lambda (d \gamma + \lambda) k_x^6} + \frac{\gamma D}{\lambda k_x^2}, \quad \langle \hat{u}_y(0, k_y) \rangle^2 = \frac{b^2 D_\theta}{3 d \lambda (d \gamma + 3 \lambda) k_y^6} + \frac{\gamma D}{3 \lambda k_y^2}, \quad (34) \]
\[ \langle \hat{\theta}(k_x, 0) \rangle^2 = \frac{D_\theta}{d k_x^2}, \quad \langle \hat{\theta}(0, k_y) \rangle^2 = \frac{D_\theta}{d k_y^2}. \quad (35) \]

In AD-II, after implementing Fourier Transform to the spatial component, we obtain

\[
\begin{pmatrix}
\dot{\hat{u}}_x \\
\dot{\hat{u}}_y \\
\dot{\hat{\theta}}
\end{pmatrix}
+ \Gamma
\begin{pmatrix}
\hat{u}_x \\
\hat{u}_y \\
\hat{\theta}
\end{pmatrix}
= \begin{pmatrix}
\sqrt{2D} \xi_x \\
\sqrt{2D} \xi_y \\
\sqrt{2D} \eta
\end{pmatrix}.
\quad (36)
\]

where

\[
\Gamma = \frac{1}{\gamma}
\begin{pmatrix}
\lambda(3k_x^2 + k_y^2) & 2\lambda k_x k_y & 0 \\
2\lambda k_x k_y & \lambda(3k_y^2 + k_x^2) & -b \\
2\gamma c k_x k_y & \gamma c(3k_y^2 + k_x^2) & 0
\end{pmatrix}.
\quad (37)
\]

The solution is

\[
\begin{align*}
\langle |\hat{u}_x(k) |^2 \rangle &= \frac{b \left[ 16 D_\theta k_x^2 k_y^2 + c^2 D_\theta (k_x^4 + 12k_x^2 k_y^2 + 27k_y^4) \gamma^2 \right] + 12 c D k^4 (k_x^2 + 3 k_y^2) \gamma \lambda }{3 c k^4 \lambda \left[ b c (k_x^2 + 9 k_y^2) \gamma + 12 k^4 \lambda \right]} \\
&\approx \frac{W_5(\alpha)}{k^2} + O(1), \\
\langle |\hat{u}_y(k) |^2 \rangle &= \frac{36^2 c D_\theta k^2 \gamma^2 + b (3k_x^2 + k_y^2) \left[ 4 D_\theta (3k_x^2 + k_y^2) + c^2 D (k_x^2 + 9 k_y^2) \gamma^2 \right] \lambda + 12 c D k^4 (3 k_x^2 + k_y^2) \gamma \lambda^2 }{3 c k^2 \lambda \left[ b c (k_x^2 + 9 k_y^2) \gamma + 12 k^4 \lambda \right]} \\
&\approx \frac{W_6(\alpha)}{k^4} + O\left(\frac{1}{k^2}\right), \\
\langle |\hat{\theta}(k) |^2 \rangle &= \frac{b^2 c^2 D_\theta k_x^2 + 3k_x^2) \gamma^2 + b c k^2 \gamma \left[ 13 D_\theta k_x^2 + c^2 D (k_x^2 + 9 k_y^2) \gamma^2 \right] \lambda + 12 k^6 (D_\theta + c^2 \gamma^2 D) \lambda^2 }{b c k^2 \lambda \left[ b c (k_x^2 + 9 k_y^2) \gamma + 12 k^4 \lambda \right]} \\
&\approx \frac{W_7(\alpha)}{k^2} + O(1).
\end{align*}
\]

where

\[
W_5(\alpha) = \frac{c^2 \gamma^2 D(12 + 2 \cos 4\alpha + 13 \cos 2\alpha) + D_\theta(2 - 2 \cos 4\alpha)}{3 c^2 \gamma \lambda (5 - 4 \cos 2\alpha)},
\]
\[
W_6(\alpha) = \frac{b D_\theta}{c \lambda^2 (5 - 4 \cos 2\alpha)},
\]
\[
W_7(\alpha) = \frac{\gamma D_\theta (2 - \cos 2\alpha)}{\lambda (5 - 4 \cos 2\alpha)}. \quad (38)
\]
It is obvious that all the coefficients $W_5(\alpha)$ (See Fig. 5), $W_6(\alpha)$ and $W_7(\alpha)$ are always positive. We may conclude that as $k \to 0$ in AD-II, $\langle |\hat{u}_x|^2 \rangle$ and $\langle |\hat{\theta}|^2 \rangle$ scale as $k^{-2}$, while $\langle |\hat{u}_y|^2 \rangle$ scales as $k^{-4}$. Specially, along the two orthogonal axis $\hat{k}_x$ ($\alpha = 0$ or $\pi$) and $\hat{k}_y$ ($\alpha = \pi/2$ or $3\pi/2$) the phonon correlation functions are,

$$\langle |\hat{u}_x(k_x, 0)|^2 \rangle = \frac{\gamma D}{3\lambda k_x^2},$$
$$\langle |\hat{u}_x(0, k_y)|^2 \rangle = \frac{\gamma D}{\lambda k_y^2},$$

(39)

$$\langle |\hat{u}_y(k_x, 0)|^2 \rangle = \frac{bD_\theta}{c\lambda^2 k_x^4} + \frac{\gamma D}{\lambda k_x^2},$$
$$\langle |\hat{u}_y(0, k_y)|^2 \rangle = \frac{bD_\theta}{9c\lambda^2 k_y^4} + \frac{\gamma D}{3\lambda k_y^2},$$

(40)

$$\langle |\hat{\theta}(k_x, 0)|^2 \rangle = \frac{\gamma D_\theta}{\lambda k_x^2} + \frac{D_\theta + c^2 \gamma^2 D}{bc},$$
$$\langle |\hat{\theta}(0, k_y)|^2 \rangle = \frac{\gamma D_\theta}{3\lambda k_y^2} + \frac{D_\theta + c^2 \gamma^2 D}{bc}.$$ 

(41)

In the limit of $b = 0$, the phonon correlation functions in both AD-I and AD-II reduce to the same form, which is exactly the phonon correlation of passive crystals:

$$\langle |\hat{u}_x(k)|^2 \rangle = \frac{\gamma D(2 - \cos 2\alpha)}{3\lambda k^2},$$

(42)

$$\langle |\hat{u}_y(k)|^2 \rangle = \frac{\gamma D(2 + \cos 2\alpha)}{3\lambda k^2}.$$  

(43)

and along the $\hat{k}_x$ and $\hat{k}_y$ axis,

$$\langle |\hat{u}_x(k_x, 0)|^2 \rangle = \frac{\gamma D}{3\lambda k_x^2},$$  
$$\langle |\hat{u}_x(0, k_y)|^2 \rangle = \frac{\gamma D}{\lambda k_y^2};$$  

(44)

$$\langle |\hat{u}_y(k_x, 0)|^2 \rangle = \frac{\gamma D}{\lambda k_x^2},$$  
$$\langle |\hat{u}_y(0, k_y)|^2 \rangle = \frac{\gamma D}{3\lambda k_y^2}.$$ 

(45)

These results are consistent with the displacement correlation functions of isotropic solids [2]:

$$\langle |\hat{u}_i(k)\hat{u}_j(-k)| \rangle = \frac{T\hat{k}_i\hat{k}_j}{3(\lambda + 2\mu)k^2} + \frac{T(1 - \hat{k}_i\hat{k}_j)}{\mu k^2}.$$ 

(46)

by choosing $\lambda = \mu$ and $i = j$.

V. CONTOUR PLOTS OF THE PHONON CORRELATION FUNCTIONS IN THE COLLECTIVE MOVING PHASE

In Fig. 6 we plot the contour curves of the phonon correlation functions in the collective moving phase in $k_x$-$k_y$ plane. The parameters are: $\tilde{d}_L = \tilde{d} = 10$ and $\tilde{c} = 100$ for AD-I, $\tilde{c}_L = \tilde{c} = 1$ and $\tilde{c} = 20$ for AD-II, $\tilde{b}_L = \tilde{b} = 2$, $\tilde{D}_\theta^L = \tilde{D}_\theta = 0.05$, $\tilde{\lambda} = \tilde{\mu} = \tilde{\kappa}\sqrt{3}/4$. Note that we let $\tilde{d}_L = \tilde{d}$ even though there is no direct correspondence between these two coefficients. The numerical results are given in the first row within each subplot of Fig. 6.
FIG. 6: Contour plot of the logarithm of the phonon correlation functions \( \langle |\hat{u}_x(k_x, k_y)|^2 \rangle \), \( \langle |\hat{u}_y(k_x, k_y)|^2 \rangle \) and \( \langle |\hat{\theta}(k_x, k_y)|^2 \rangle \) in \( k \)-space (arbitrary unit) in the Collective Moving phase, obtained from both simulation (first row in each subplot) and theoretical (second row in each subplot) results. (a) AD-I with \( \tilde{d} = \tilde{d}_L = 10 \). (b) AD-II with \( \tilde{c} = \tilde{c}_L = 1 \).

and the analytical results in the second row for comparison. The absolute value between the simulation and analytical curves might not be exactly equal, since they may differ by some constants such as the Fourier coefficients during FFT, which however are not of importance for our purpose. It can be seen that the scaling behaviors predicted by theory is consistent with the simulation results. Besides, \( \langle |\hat{\theta}|^2 \rangle \) in AD-I is rotationally invariant, which originates from the rotational invariance of the active dynamics of \( \theta \) in AD-I.
VI. GENERALIZATION OF THE EQUATIONS OF MOTION INTO THREE DIMENSIONS

In this section we discuss the analytical treatment of Eq. (3a) in three dimensions, which can be written as

\[ F_i = \mu \sum_{j=1}^{3} \partial_j^2 u_i + (\lambda + \mu) \sum_{j=1}^{3} \partial_i \partial_j u_j \]  

(47)

or more explicitly as

\[
\begin{cases}
F_x = \mu(\partial_x^2 u_x + \partial_y^2 u_x + \partial_z^2 u_x) + (\lambda + \mu)(\partial_x^2 u_x + \partial_y u_y + \partial_z u_z), \\
F_y = \mu(\partial_x^2 u_y + \partial_y^2 u_y + \partial_z^2 u_y) + (\lambda + \mu)(\partial_y \partial_x u_x + \partial_y^2 u_y + \partial_z u_z), \\
F_z = \mu(\partial_x^2 u_z + \partial_y^2 u_z + \partial_z^2 u_z) + (\lambda + \mu)(\partial_z \partial_x u_x + \partial_z \partial_y u_y + \partial_z^2 u_z).
\end{cases}
\]  

(48)

Under the assumption of \( \theta \ll 1 \), we may decompose the active force direction into one parallel to the collective moving direction \( \hat{x} \) and the other perpendicular to it, i.e., \( \hat{n} = \hat{n}_\parallel + \hat{n}_\perp \) where \( |\hat{n}_\perp| \ll 1 \). We can also decompose the elastic force \( F = F_\parallel + F_\perp \). The overdamped dynamics of the displacement field is,

\[ \gamma \dot{u}(r, t) = b \hat{n}_\perp + F(r, t) + \gamma \sqrt{2D} \hat{\xi}(r, t). \]  

(49)

where \( \hat{\xi}(r, t) \) is a vector Gaussian white noise satisfying \( \langle \xi_i(r, t) \rangle = 0, \langle \xi_i(r, t) \xi_j(r', t') \rangle = \delta_{ij} \delta(r - r') \delta(t - t') \), and it can be decomposed as \( \hat{\xi} = \xi_x \hat{x} + \xi_\perp \). The continuum limit of (AD-I) and (AD-II) are respectively,

\[
\begin{align*}
\dot{\hat{n}}_\perp(r, t) &= d \Delta \hat{n}_\perp + \sqrt{2D_\theta} \hat{\eta}_\perp(r, t), \quad \text{(AD-I')} \\
\dot{\hat{n}}_\perp(r, t) &= c (\hat{n} \times F) \times \hat{n} + \sqrt{2D_\theta} \hat{\eta}_\perp(r, t) \\
&\approx c F_\perp + \sqrt{2D_\theta} \hat{\eta}_\perp(r, t). \quad \text{(AD-II')}
\end{align*}
\]

where \( \hat{\eta}_\perp(r, t) \) is a 2d vector Gaussian white noise in the \( \hat{y}-\hat{z} \) plane satisfying \( \langle \eta_{ij}(r, t) \rangle = 0, \langle \eta_{ij}(r, t) \eta_{ij}(r', t') \rangle = \delta_{ij} \delta(r - r') \delta(t - t') \).

We can substitute Eq.(48) into Eq.(49) together with Eq.(AD-I'), and obtain the dynamic
equations for AD-I:

\[
\begin{align*}
\gamma \dot{u}_x &= F_x + \gamma \sqrt{2D} \xi_x \\
\gamma \dot{u}_\perp &= b \hat{n}_\perp + F_\perp + \gamma \sqrt{2D} \hat{\xi}_\perp \\
\dot{\hat{n}}_\perp &= d \Delta \hat{n}_\perp + \sqrt{2D_\theta} \hat{\eta}_\perp
\end{align*}
\]  

(50)

For AD-II, the dynamic equations are given by,

\[
\begin{align*}
\gamma \dot{u}_x &= F_x + \gamma \sqrt{2D} \xi_x \\
\gamma \dot{u}_\perp &= b \hat{n}_\perp + F_\perp + \gamma \sqrt{2D} \hat{\xi}_\perp \\
\dot{\hat{n}}_\perp &= c F_\perp + \sqrt{2D_\theta} \hat{\eta}_\perp
\end{align*}
\]  

(51)

It can be easily seen that Eqs. (50) and (51) are direct generalizations of Eqs. (5) and (8) to the 3d case. The above arguments can be easily extended to higher dimensions.

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