Lax Operator for the Quantised Orthosymplectic Superalgebra $U_q[osp(m|n)]$

K.A. Dancer, M.D. Gould and J. Links

Centre for Mathematical Physics, School of Physical Sciences, The University of Queensland, Brisbane 4072, Australia.

Abstract

Representations of quantum superalgebras provide a natural framework in which to model supersymmetric quantum systems. Each quantum superalgebra, belonging to the class of quasi-triangular Hopf superalgebras, contains a universal $R$-matrix which automatically satisfies the Yang–Baxter equation. Applying the vector representation $\pi$, which acts on the vector module $V$, to the left-hand side of a universal $R$-matrix gives a Lax operator. In this Communication a Lax operator is constructed for the quantised orthosymplectic superalgebras $U_q[osp(m|n)]$ for all $m > 2, n \geq 0$ where $n$ is even. This can then be used to find a solution to the Yang–Baxter equation acting on $V \otimes V \otimes W$, where $W$ is an arbitrary $U_q[osp(m|n)]$ module. The case $W = V$ is studied as an example.

1 Introduction

Solutions to the (spectral parameter dependent) Yang-Baxter equation (YBE) lie at the core of the Quantum Inverse Scattering Method for the construction of integrable quantum systems, and underlie the applicability of Bethe ansatz methods for deriving their exact solutions (e.g. see [1,2]). Integrable systems with both bosonic and fermionic degrees of freedom may be constructed through a $\mathbb{Z}_2$-graded, or supersymmetric, analogue of the YBE. Several examples exist for models constructed in this manner which can be used to describe systems of strongly correlated electronic [3–10] and also integrable supersymmetric field theories [11–13].

A systematic approach to solve the YBE is provided by the Quantum Double construction [14] (see [15] for the $\mathbb{Z}_2$-graded extension). This gives a prescription for embedding a Hopf algebra and its dual into a quasi-triangular Hopf algebra $A$, which possesses a universal $R$-matrix $R \in A \otimes A$ providing an algebraic solution for the YBE. Each tensor product matrix representation of the quasi-triangular Hopf algebra then yields a matrix solution for the YBE. The most well-known examples of quasi-triangular Hopf superalgebras are the quantum superalgebras [16] (denoted $U_q[g]$ where $g$ is a Lie superalgebra), which are deformations of the universal enveloping algebras of Lie superalgebras.

In cases where $g$ is affine, loop representations of $U_q[g]$ provide supersymmetric matrix solutions of the YBE dependent on a spectral parameter $u$. The limit $u \to \infty$ gives a solution associated with a non-affine subsuperalgebra of $g$. It is possible however to start with a solution for the non-affine case and then introduce a spectral parameter, a process known as Baxterisation. A systematic way to perform Baxterisation in the case of quantum superalgebras, through the use of tensor product graph methods, is described in [17,18]. This result then reduces the problem to determining the explicit form for $R$-matrices associated with non-affine quantum superalgebras.

In practice however, it is still a difficult technical challenge to explicitly compute $R$ for a given non-affine $U_q[g]$. Our approach here is to simplify the problem by instead looking to determine the Lax operator $R = (\pi \otimes \text{id})R$ where $\pi$ denotes the vector representation of $U_q[g]$. Explicit forms for the Lax
operator are known in the case of $U_q[gl(m)]$ [19] and $U_q[gl(m|n)]$ [20] but even in the non-graded cases of $U_q[o(m)]$ and $U_q[sp(n)]$ the Lax operator is not known. In this Communication a Lax operator is constructed for the quantised orthosymplectic superalgebras $U_q[osp(m|n)]$ for all $m > 2, n \geq 0$ where $n$ is even. The case with $n = 0$ corresponds to constructing a Lax operator for $U_q[o(m)]$. For technical reasons, the cases $m = 1$ and $m = 2$ do not fit into the general framework that we will employ here. This issue will be discussed in more detail in Sect. 3. We mention here that in the case $m = 1$ the Lax operator can be deduced from an isomorphism given in [21], while the Lax operator for the case $m = 2$ is derived elsewhere [22]. One immediate application of the Lax operator is that it affords a means to construct the Casimir invariants of the associated quantum superalgebra. This construction for $U_q[osp(m|n)]$, as well as a derivation of the eigenvalues of the Casimir invariants when acting on irreducible highest weight modules, can be found in [23].

We begin in Sect. 2 with a construction for $osp(m|n)$ which is necessary to establish our notational conventions, and in particular the choice of root system. Sect. 3 discusses the $q$-deformation of the universal enveloping algebra and the associated quasi-triangular Hopf superalgebra structure. In Sect. 4 we describe the construction of the Lax operator by choosing a particular ansatz and imposing that the Lax operator intertwines the co-product. It will be shown that, for this choice of ansatz, there is a unique solution satisfying the intertwining property. We will further show that this solution satisfies all other properties which follow from the quasi-triangular structure of the quantum orthosymplectic superalgebras. Through use of the Lax operator we derive the $R$-matrix for the vector representation in Sect. 5, and concluding remarks are given in Sect. 6.

## 2 The Construction of $osp(m|n)$

To construct the quantised orthosymplectic superalgebra $U_q[osp(m|n)]$ we closely follow the method used in [18,24], but provide details here in order to establish our notational conventions. We begin by developing $osp(m|n)$ as a graded subalgebra of $gl(m|n)$. The enveloping algebra of $osp(m|n)$ is then deformed to yield $U_q[osp(m|n)]$, which reduces to the original enveloping superalgebra as $q \to 1$. The construction of $osp(m|n)$ starts with the standard generators $e^a_b$ of $gl(m|n)$, the $(m + n) \times (m + n)$-dimensional general linear superalgebra, whose even part is given by $gl(m) \oplus gl(n)$. The commutator for a $\mathbb{Z}_2$-graded algebra satisfies the relation

$$[A, B] = -(-1)^{|A||B|}[B, A],$$

where $A, B$ are homogeneous operators and $|A| \in \mathbb{Z}_2$ is the grading of $A$. In particular, the generators of $gl(m|n)$ satisfy the graded commutation relations

$$[e^a_b, e^c_d] = \delta^c_b e^a_d - (-1)^{(|a|+|b|)(|c|+|d|)} \delta^a_b e^c_d$$

where

$$|a| \begin{cases} 0, & a = i, \ 1 \leq i \leq m, \\ 1, & a = \mu, \ 1 \leq \mu \leq n. \end{cases}$$

Throughout this Communication we use Greek letters $\mu, \nu$ etc. to denote odd indices and Latin letters $i, j$ etc. for even indices. If the grading is unknown, the usual $a, b, c$ etc. are used. Which convention applies will be clear from the context. We will only ever consider the homogeneous elements, but all results can be extended to the inhomogeneous elements by linearity.

The orthosymplectic superalgebra $osp(m|n)$ is a subsuperalgebra of $gl(m|n)$ with even part equal to $o(m) \oplus sp(n)$, where $o(m)$ is the orthogonal Lie algebra of rank $\frac{m(m - 1)}{2}$ and $sp(n)$ is the symplectic Lie algebra of rank $\frac{n(n - 1)}{2}$. The latter only exists if $n$ is even, so we set $n = 2k$. We also set $l = \frac{m}{2}$, so $m = 2l$ or $m = 2l + 1$. To construct $osp(m|n)$ we require an even non-degenerate supersymmetric metric $g_{ab}$. Any can be used, but for the sake of simplicity we choose $g_{ab} = \xi_a \delta^a_b$, with inverse metric $g^{ba} = \xi_b \delta^a_b$. Here
\[\varpi = \begin{cases} m + 1 - a, & [a] = 0, \\ n + 1 - a, & [a] = 1, \end{cases}\]

and \(\xi_a = \begin{cases} 1, & [a] = 0, \\ (-1)^a, & [a] = 1. \end{cases}\)

The \(\mathbb{Z}_2\)-graded subalgebra \(osp(m|n)\) actually arises naturally from considering the automorphism \(\omega\) of \(gl(m|n = 2k)\) given by:

\[\omega(e_a^b) = \begin{cases} -(-1)^{[a][b]}\xi_a\xi_b, & [a] = 0, \\ (-1)^{[a][b]}\xi_a\xi_b. & [a] = 1 \end{cases}\]

This is clearly of degree 2, with eigenvalues \(\pm 1\), so it gives a decomposition of \(gl(m|n)\):

\[gl(m|n) = S \oplus T, \quad \text{with } [S,S] \subset S, \quad [T,T] \subset S \quad \text{and } [S,T] \subset T,\]

where

\[\omega(x) = x \quad \forall x \in S,\]

\[\omega(x) = -x \quad \forall x \in T.\]

Here \(T\) is generated by operators

\[T_{ab} = g_{ac}e_c^b + (-1)^{[a][b]}g_{bc}e_a^c = (-1)^{[a][b]}T_{ba},\]

while \(S\) is generated by

\[\sigma_{ab} = g_{ac}e_c^b - (-1)^{[a][b]}g_{bc}e_a^c = (-1)^{[a][b]}\sigma_{ba}.\]

The fixed-point \(\mathbb{Z}_2\)-graded subalgebra \(S\) is the orthosymplectic superalgebra \(osp(m|n)\), with the operators \(\sigma_{ab}\) providing a basis. They satisfy the commutation relations

\[[\sigma_{ab}, \sigma_{cd}] = g_{cb}\sigma_{ad} - (-1)^{[a][b][c][d]}g_{ad}\sigma_{cb} = \begin{cases} (-1)^{[a][b][c][d]}(g_{db}\sigma_{ac} - (-1)^{[a][b][c][d]}g_{ac}\sigma_{db}), & \text{if } 1 \leq i, j \leq l, 1 \leq \mu, \nu \leq k, \end{cases}\]

As a more convenient basis for \(osp(m|n)\) we choose the set of Cartan-Weyl generators, given by:

\[\sigma_a^b = g^{ac}\sigma_{cb} = e_b^a - (-1)^{[a][b]}\xi_a\xi_b,\]

(1)

Then the Cartan subalgebra \(H\) is generated by the diagonal operators

\[\sigma_a^a = e_a^a = e_{\varpi},\]

which satisfy

\[[\sigma_a^a, \sigma_b^b] = 0, \quad \forall a, b.\]

As a weight system, we take the set \(\{\varepsilon_i, 1 \leq i \leq m\} \cup \{\delta_\mu, 1 \leq \mu \leq n\}\), where \(\varepsilon_i = -\varepsilon_i\) and \(\delta_\mu = -\delta_\mu\). Conveniently, when \(m = 2l + 1\) this implies \(\varepsilon_i + 1 = -\varepsilon_i + 1 = 0\). Acting on these weights, we have the invariant bilinear form defined by:

\[(\varepsilon_i, \varepsilon_j) = \delta_j^i, \quad (\delta_\mu, \delta_\nu) = -\delta_\nu^\mu, \quad (\varepsilon_i, \delta_\mu) = 0, \quad 1 \leq i, j \leq l, \quad 1 \leq \mu, \nu \leq k.\]

When describing an object with unknown grading indexed by \(a\) the weight will be described generically as \(\varepsilon_a\). This should not be assumed to be an even weight.
The even positive roots of $osp(m|n)$ are composed entirely of the usual positive roots of $o(m)$ together with those of $sp(n)$, namely:

$$
\varepsilon_i \pm \varepsilon_j, \quad 1 \leq i < j \leq l,
\varepsilon_i, \quad 1 \leq i \leq l \quad \text{when } m = 2l + 1,
\delta_\mu + \delta_\nu, \quad 1 \leq \mu, \nu \leq k,
\delta_\mu - \delta_\nu, \quad 1 \leq \mu < \nu \leq k.
$$

The root system also contains a set of odd positive roots, which are:

$$\delta_\mu + \varepsilon_i, \quad 1 \leq \mu \leq k, \ 1 \leq i \leq m.$$

Throughout we choose to use the following set of simple roots:

$$\alpha_i = \varepsilon_i - \varepsilon_{i+1}, \quad 1 \leq i < l,
\alpha_l = \begin{cases} 
\varepsilon_l + \varepsilon_{l-1}, & m = 2l, \\
\varepsilon_l, & m = 2l + 1,
\end{cases}
\alpha_\mu = \delta_\mu - \delta_{\mu+1}, \quad 1 \leq \mu < k,
\alpha_s = \delta_k - \varepsilon_1.
$$

Note this choice is only valid for $m > 2$.

Corresponding to these simple roots we have raising generators $e_a$, lowering generators $f_a$ and Cartan elements $h_a$ given by:

$$
e_i = \sigma_i^{i+1}, \quad f_i = \sigma_i^{-1}, \quad h_i = \sigma_i^i - \sigma_i^{i+1}, \quad 1 \leq i < l,
\varepsilon_l = \sigma_l^l, \quad f_l = \sigma_l^{-1}, \quad h_l = \sigma_l^l, \quad m = 2l,
\varepsilon_l = \sigma_{l+1}^l, \quad f_l = \sigma_l^l + 1, \quad h_l = \sigma_l^l, \quad m = 2l + 1,
e_\mu = \sigma_\mu^{\mu+1}, \quad f_\mu = -\sigma_\mu^{-1}, \quad h_\mu = \sigma_\mu^{\mu+1} - \sigma_\mu^\mu, \quad 1 \leq \mu < k,
e_s = \sigma_s^s, \quad f_s = -\sigma_s^{-1}, \quad h_s = -\sigma_s^{s+1} - \sigma_s^s.
$$

These automatically satisfy the defining relations of a Lie superalgebra, which for the case at hand are:

$$
[h_a, e_b] = (\alpha_a, \alpha_b)e_b,
[h_a, f_b] = -(\alpha_a, \alpha_b)f_b,
[h_a, h_b] = 0,
[e_a, f_b] = \delta_b^a h_a,
[e_a, e_b] = [f_a, f_b] = 0 \quad \text{for } (\alpha_a, \alpha_a) = 0,
(ad e_b)^{1-a_{bc}}e_c = 0 \quad \text{for } b \neq c, \ (\alpha_b, \alpha_b) \neq 0, \quad (2)
(ad f_b)^{1-a_{bc}}f_c = 0 \quad \text{for } b \neq c, \ (\alpha_b, \alpha_b) \neq 0, \quad (3)
$$

where the $a_{bc}$ are the entries of the corresponding Cartan matrix,

$$
a_{bc} = \begin{cases} 
\frac{2(\alpha_b, \alpha_c)}{(\alpha_b, \alpha_c)}, & (\alpha_b, \alpha_b) \neq 0, \\
(\alpha_b, \alpha_c), & (\alpha_b, \alpha_b) = 0,
\end{cases}
$$
and $ad$ represents the adjoint action

$$ad x \circ y = [x, y].$$  \hfill (4)

The relations (4) and (3) are known as the Serre relations [25]. Superalgebras also have higher order defining relations, not included here, which are known as the extra Serre relations. They are dependent on the structure of the chosen simple root system [16].

3 The $q$-Deformation: $U_q[osp(m|n)]$

A quantum superalgebra is a generalised version of a classical super algebra involving a complex parameter $q$, which reduces to the classical case as $q \to 1$. In particular, we construct $U_q[osp(m|n)]$ by $q$-deforming the original enveloping algebra of $osp(m|n)$ so that the generators remain unchanged, but are now related by a deformation of the defining relations. Throughout this Communication $q$ is assumed not to be a root of unity.

First note that in the enveloping algebra of $osp(m|n)$ the graded commutator is realised by

$$[A, B] = AB - (-1)^{|A||B|} BA.$$  

With this operation, we then have:

**Definition 3.1** The defining relations for $U_q[osp(m|n)]$ are:

$$[h_a, e_b] = (\alpha_a, \alpha_b)e_b,$$

$$[h_a, f_b] = -(\alpha_a, \alpha_b)f_b,$$

$$[h_a, h_b] = 0,$$

$$[e_a, f_b] = \delta_b^a \frac{(q^{h_a} - q^{-h_a})}{(q - q^{-1})},$$

$$[e_a, e_a] = [f_a, f_a] = 0 \quad \text{for} \ (\alpha_a, \alpha_a) = 0,$$

$$(ad e_b \circ)^{1-a \alpha} e_c = 0 \quad \text{for} \ b \neq c, \ (\alpha_b, \alpha_b) \neq 0, \quad \text{(5)}$$

$$(ad f_b \circ)^{1-a \alpha} f_c = 0 \quad \text{for} \ b \neq c, \ (\alpha_b, \alpha_b) \neq 0. \quad \text{(6)}$$

The relations (5) and (6) are called the $q$-Serre relations. Again, there are also extra $q$-Serre relations which are not included here. A complete list of them, including those for affine superalgebras, can be found in [16]. Both the standard and extra $q$-Serre relations depend on the adjoint action, which is no longer simply the graded commutator. To define the adjoint action for a quantum superalgebra, we first need some new operations.

The coproduct, $\Delta : U_q[osp(m|n)] \otimes 2 \to U_q[osp(m|n)] \otimes 2$, is the superalgebra homomorphism given by:

$$\Delta(e_a) = q^{\frac{1}{2}h_a} \otimes e_a + e_a \otimes q^{-\frac{1}{2}h_a},$$

$$\Delta(f_a) = q^{\frac{1}{2}h_a} \otimes f_a + f_a \otimes q^{-\frac{1}{2}h_a},$$

$$\Delta(q^{\pm \frac{1}{2}h_a}) = q^{\pm \frac{1}{2}h_a} \otimes q^{\mp \frac{1}{2}h_a},$$

$$\Delta(ab) = \Delta(a)\Delta(b). \quad \text{(7)}$$

Note that in a $\mathbb{Z}_2$-graded algebra, multiplying tensor products induces a grading term, according to

$$(a \otimes b)(c \otimes d) = (-1)^{|b||c|}(ac \otimes bd).$$
We also require the antipode, \( S : U_q[osp(m|n)] \to U_q[osp(m|n)] \), a superalgebra anti-homomorphism defined by:

\[
\begin{align*}
S(e_a) &= -q^{\frac{1}{2}(\alpha_a, \alpha_a)}e_a, \\
S(f_a) &= -q^{\frac{1}{2}(\alpha_a, \alpha_a)}f_a, \\
S(q^\pm h_a) &= q^\mp h_a, \\
S(ab) &= (-1)^{|a||b|}S(b)S(a).
\end{align*}
\]

It can be shown that both the coproduct and antipode are consistent with the defining relations of the superalgebra. These mappings are necessary to define the adjoint action for a quantum superalgebra, as it can no longer be written simply in terms of the graded commutator. If we adopt Sweedler’s notation for the coproduct \( \Delta \), the twist map composed with the coproduct, is denoted \( \Delta^T \). Then a universal \( R \)-matrix, \( \mathcal{R} \), is an even, non-singular element of \( U_q[osp(m|n)] \otimes^2 \) satisfying the following properties:

\[
\mathcal{R} = \mathcal{R}^{-1} = \mathcal{R}^* = \mathcal{R}^{op}.
\]

### 3.1 \( U_q[osp(m|n)] \) as a Quasi-Triangular Hopf Superalgebra

A quantum superalgebra is actually a specific type of quasi-triangular Hopf superalgebra. This guarantees the existence of a universal \( R \)-matrix, which provides a solution to the quantum Yang–Baxter equation. Before elaborating, we need to introduce the graded twist map.

The graded twist map \( T : U_q[osp(m|n)] \otimes^2 \to U_q[osp(m|n)] \otimes^2 \) is given by

\[
T(a \otimes b) = (-1)^{|a||b|}(b \otimes a).
\]

For convenience \( T \circ \Delta \), the twist map composed with the coproduct, is denoted \( \Delta^T \).
$\mathcal{R}\Delta(a) = \Delta^T(a)\mathcal{R}, \quad \forall a \in U_q[osp(m|n)]$,

$(\text{id} \otimes \Delta)\mathcal{R} = \mathcal{R}_{13}R_{12}$,

$(\Delta \otimes \text{id})\mathcal{R} = \mathcal{R}_{13}\mathcal{R}_{23}$.

Here $\mathcal{R}_{ab}$ represents a copy of $\mathcal{R}$ acting on the $a$ and $b$ components respectively of $U_1 \otimes U_2 \otimes U_3$, where each $U$ is a copy of the quantum superalgebra $U_q[osp(m|n)]$. When $a > b$ the usual grading term from the twist map is included, so for example $\mathcal{R}_{21} = [\mathcal{R}^T]_{12}$, where $\mathcal{R}^T = T(\mathcal{R})$ is the opposite universal $R$-matrix.

One of the reasons $R$-matrices are significant is that as a consequence of (9) they satisfy the YBE, which is prominent in the study of integrable systems [1]:

$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}$

A superalgebra may contain many different universal $R$-matrices, but there is always a unique one belonging to $U_q[osp(m|n)]^- \otimes U_q[osp(m|n)]^+$, and its opposite $R$-matrix in $U_q[osp(m|n)]^+ \otimes U_q[osp(m|n)]^-$. Here $U_q[osp(m|n)]^+$ is the Hopf subsuperalgebra generated by the lowering generators and Cartan elements, while $U_q[osp(m|n)]^+$ is generated by the raising generators and Cartan elements. These particular $R$-matrices arise out of Drinfeld’s double construction [14]. In this Communication we consider the universal $R$-matrix belonging to $U_q[osp(m|n)]^- \otimes U_q[osp(m|n)]^+$.

## 4 An Ansatz for the Lax Operator

Now we have the necessary information for the construction of a Lax operator for $U_q[osp(m|n)]$. Previously this had only been done in the superalgebra case for $U_q[gl(m|n)]$ [21]. Before defining a Lax operator, however, we need to introduce the vector representation.

Let $\text{End } V$ be the set of endomorphisms of $V$, an $(m + n)$-dimensional vector space. Then the irreducible vector representation $\pi : U_q[osp(m|n)] \rightarrow \text{End } V$ is left undeformed from the classical vector representation of $osp(m|n)$, which acts on the Cartan-Weyl generators given in equation (4) according to:

$$\pi(\sigma^a_b) = E^a_b - (-1)^{|a||b|}E^b_a,$$

where $E^a_b$ is the $(m + n) \times (m + n)$-dimensional elementary matrix with $(a, b)$ entry 1 and zeroes elsewhere.

Now let $\mathcal{R}$ be a universal $R$-matrix of $U_q[osp(m|n)]$ and $\pi$ the vector representation. The Lax operator associated with $\mathcal{R}$ is given by

$$R = (\pi \otimes \text{id})\mathcal{R} \in (\text{End } V) \otimes U_q[osp(m|n)]$$

and the $R$-matrix in the vector representation $\mathcal{R}$ is given by:

$$\mathcal{R} = (\pi \otimes \pi)\mathcal{R} = (\text{id} \otimes \pi)R \in (\text{End } V) \otimes (\text{End } V).$$

Then the Yang-Baxter equation reduces to:

$$\mathcal{R}_{12}R_{13}\mathcal{R}_{23} = R_{23}R_{13}\mathcal{R}_{12}$$

acting on the space $V \otimes V \otimes U_q[osp(m|n)]$.

Previously an $R$-matrix in the vector representation, $\mathcal{R}$, has been calculated for both $U_q[osp(m|n)]$ and its affine extension [27–29], however in general the Lax operator is still unknown. The Lax operator is significant because we can use it to calculate solutions to the quantum Yang-Baxter equation for an arbitrary finite-dimensional representation.
In the following sections we also sometimes make use of the bra and ket notation. The set \(\{ |a\rangle, a = 1, ..., m + n \}\) is a basis for \(V\) satisfying the property
\[
E_{b}^{a}|c\rangle = \delta_{b}^{c}|a\rangle.
\]
The set \(\{ \langle a|, a = 1, ..., m + n \}\) is the dual basis such that
\[
\langle c|E_{b}^{a} = \delta_{c}^{a}\langle b| \quad \text{and} \quad \langle a|b\rangle = \delta_{a}^{b}.
\]
As we wish to find the Lax operator belonging to \(\pi(U_{q}[osp(m|n)]^{-}) \otimes U_{q}[osp(m|n)]^{+}\), we adopt the following ansatz for \(R\):
\[
R \equiv \sum_{\varepsilon_{a} < \varepsilon_{b}} q^{\varepsilon_{b} \otimes h_{a}} \left[ I \otimes I + (q - q^{-1}) \sum_{\varepsilon_{a} < \varepsilon_{b}} (-1)^{|b|} E_{b}^{a} \otimes \hat{\sigma}_{ba} \right].
\]
Here \(\{ h_{a} \}\) is the basis for the Cartan subalgebra such that \(h_{a} = h_{e_{a}}\), and \(\{ h^{a}\}\) the dual basis, so \(h^{a} = (-1)^{|a|} h_{e_{a}}\). The \(\hat{\sigma}_{ba}\) are unknown operators for which we are trying to solve. Throughout the remainder of this Communication, when working in the vector representation, we simply use \(h_{a}\) rather than \(\pi(h_{a})\), and \(e_{a}\) rather than \(\pi(e_{a})\).

### 4.1 Constraints Arising from the Defining Relations

The Lax operator \(R\) must be consistent with the defining relations for the \(R\)-matrix, which were given as equation \((9)\). In particular, it must satisfy the intertwining property for the raising generators,
\[
R \Delta (e_{c}) = \Delta^{T}(e_{c})R.
\]
To apply this, recall that
\[
\Delta (e_{c}) = q^{\frac{1}{2} h_{e_{c}}} \otimes e_{c} + e_{c} \otimes q^{-\frac{1}{2} h_{e_{c}}}
\]
Then, from the defining relations, we have
\[
\Delta^{T}(e_{c}) q^{\sum_{\varepsilon_{a} < \varepsilon_{b}} h_{a} \otimes h^{a}} = (q^{\frac{1}{2} h_{e_{c}}} \otimes e_{c} + e_{c} \otimes q^{-\frac{1}{2} h_{e_{c}}}) q^{\sum_{\varepsilon_{a} < \varepsilon_{b}} h_{a} \otimes h^{a}}
\]
\[
= q^{\sum_{\varepsilon_{a} < \varepsilon_{b}} h_{a} \otimes h^{a}} (e_{c} \otimes q^{-\frac{1}{2} h_{e_{c}}} + q^{-\frac{1}{2} h_{e_{c}}} \otimes e_{c})
\]
Using this, we see
\[
\Delta^{T}(e_{c}) R = q^{\sum_{\varepsilon_{a} < \varepsilon_{b}} h_{a} \otimes h^{a}} \left( e_{c} \otimes q^{-\frac{1}{2} h_{e_{c}}} + q^{-\frac{1}{2} h_{e_{c}}} \otimes e_{c} \right)
\]
\[
\times \left[ I \otimes I + (q - q^{-1}) \sum_{\varepsilon_{a} < \varepsilon_{b}} (-1)^{|b|} E_{b}^{a} \otimes \hat{\sigma}_{ba} \right]
\]
\[
= q^{\sum_{\varepsilon_{a} < \varepsilon_{b}} h_{a} \otimes h^{a}} \left\{ e_{c} \otimes q^{-\frac{1}{2} h_{e_{c}}} + q^{-\frac{1}{2} h_{e_{c}}} \otimes e_{c} \right. \\
\left. + (q - q^{-1}) \sum_{\varepsilon_{a} < \varepsilon_{b}} (-1)^{|b|} e_{c} E_{b}^{a} \otimes q^{-\frac{1}{2} h_{e_{c}}} \hat{\sigma}_{ba} \right. \\
\left. + (-1)^{|a|+|b|} c \frac{1}{2} (\alpha_{c}, e_{a}) E_{b}^{a} \otimes e_{c} \hat{\sigma}_{ba} \right\}. \quad (10)
\]
Also,
\[
R \Delta(e_c) = \sum_a h_a \otimes h^a \left\{ q^{\frac{1}{2}} h_c \otimes e_c + e_c \otimes q^{-\frac{1}{2}} h_c \right. \\
+ (q - q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{|b|} \left[ q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} E^a_b \otimes \hat{\sigma}_{ba} e_c \\
+ (-1)^{(\lceil a \rceil + |b|) \langle c \rangle} E^a_b e_c \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2}} h_c \right] \right\},
\]

(11)

Hence to apply the intertwining property we simply equate (10) and (11). First note that \( R \) is weightless, so \( \hat{\sigma}_{ba} \) has weight \( \varepsilon_b - \varepsilon_a \), and thus

\[
q^{-\frac{1}{2} h_c} \hat{\sigma}_{ba} = q^{-\frac{1}{2}(\alpha_c, \varepsilon_b - \varepsilon_a)} \hat{\sigma}_{ba} q^{-\frac{1}{2} h_c}.
\]

Then, equating those terms with zero weight in the first element of the tensor product, we obtain

\[
(q^{\frac{1}{2} h_c} - q^{-\frac{1}{2} h_c}) \otimes e_c
\]

\[
= (q - q^{-1}) \sum_{\varepsilon_b - \varepsilon_a = \alpha_c} (-1)^{|b|} \left[ q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} E^a_b e_c - (q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} e_c E^a_b - (q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} e_c \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2} h_c}. \right.
\]

(12)

Comparing the remaining terms, we also find

\[
\sum_{\varepsilon_a \leq \varepsilon_b} (-1)^{|b|} \left[ q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} E^a_b e_c - (q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} e_c \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2} h_c}
\]

\[
= \sum_{\varepsilon_a \leq \varepsilon_b} (-1)^{|b|} E^a_b \otimes \left( q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} \hat{\sigma}_{ba} e_c - (q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} e_c \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2} h_c} \right).
\]

(13)

From the first of these equations we can deduce the simple values of \( \hat{\sigma}_{ba} \), namely those for which \( \varepsilon_b - \varepsilon_a \) is a simple root; from the second, relations involving all the \( \hat{\sigma}_{ba} \). Before doing so, however, it is convenient to define a new set, \( \overrightarrow{\Phi}^+ \).

**Definition 4.1** The extended system of positive roots, \( \overrightarrow{\Phi}^+ \), is defined by

\[
\overrightarrow{\Phi}^+ \equiv \{ \varepsilon_b - \varepsilon_a \varepsilon_b > \varepsilon_a \} = \Phi^+ \cup \{ 2\varepsilon_i | 1 \leq i \leq l \}
\]

where \( \Phi^+ \) is the usual system of positive roots.

Now consider equation (13). In the case when \( \varepsilon_b - \varepsilon_a + \alpha_c \notin \overrightarrow{\Phi}^+ \), by collecting the terms of weight \( \varepsilon_b - \varepsilon_a + \alpha_c \) in the second half of the tensor product we find:

\[
q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} \hat{\sigma}_{ba} e_c - (q^{\frac{1}{2}(\alpha_c, \varepsilon_b)} e_c \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2} h_c} = 0.
\]

(14)

Similarly, when \( \varepsilon_b > \varepsilon_a \) and \( \varepsilon_b - \varepsilon_a + \alpha_c = \varepsilon_{b'} - \varepsilon_{a'} \in \overrightarrow{\Phi}^+ \) we find:

\[
\sum_{\varepsilon_a \leq \varepsilon_b} (-1)^{|b'|} \left[ q^{\frac{1}{2}(\alpha_c, \varepsilon_{b'})} e_c E^a_{b'} - (q^{\frac{1}{2}(\alpha_c, \varepsilon_{b'})} e_c \otimes \hat{\sigma}_{a'b'} q^{-\frac{1}{2} h_c}
\]

\[
= (-1)^{|b'|} E^a_{b'} \otimes \left( q^{\frac{1}{2}(\alpha_c, \varepsilon_{b'})} \hat{\sigma}_{ba} e_c - (q^{\frac{1}{2}(\alpha_c, \varepsilon_{b'})} e_c \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2} h_c} \right).
\]
However $e_aE_{a'}^q$ and $E_a^a$ are linearly independent unless $b = b'$, as are $E_{a'}^q e_c$ and $E_b^a$ for $a \neq a'$, and thus this equation reduces to

\[ q^{-\frac{1}{2}(\alpha_c \varepsilon_b - \varepsilon_a + \alpha_c)} e_c E_{a'}^q \otimes \hat{\sigma}_{ba'} q^{-\frac{1}{2} h_c} \bigg|_{\varepsilon_{a'} = \varepsilon_a - \alpha} \\
- (-1)^{|[a|+[b]|]}|c| q^{\frac{1}{2}(\alpha_c \varepsilon_b)} (a'|c|b') \hat{\sigma}_{ba} q^{-\frac{1}{2} h_a} \bigg|_{\varepsilon_{b'} = \varepsilon_b + \alpha_c} = E_b^a \otimes (q^{\frac{1}{2}(\alpha_c \varepsilon_b)} \hat{\sigma}_{ba} e_c - (-1)^{|[a|+[b]|]}|c| q^{-\frac{1}{2}(\alpha_c \varepsilon_b)} e_c \hat{\sigma}_{ba}) \]

for $\varepsilon_b > \varepsilon_a$. This further simplifies to

\[ q^{-\frac{1}{2}(\alpha_c \varepsilon_b - \varepsilon_a)} (a|c|d') \hat{\sigma}_{ba} - (-1)^{|[a|+[b]|]}|c| q^{\frac{1}{2}(\alpha_c \varepsilon_b)} (b'|c|b) \hat{\sigma}_{ba} = q^{\alpha_c \varepsilon_b} \hat{\sigma}_{ba} e_c q^{\frac{1}{2} h_c} - (-1)^{|[a|+[b]|]}|c| q^{-(\alpha_c \varepsilon_b)} e_c q^{\frac{1}{2} h_a} \hat{\sigma}_{ba} \]

for $\varepsilon_b > \varepsilon_a$. All the necessary information is contained within equations (12) and (15). To construct the Lax operator $R = (\pi \otimes 1) R$ first we use equation (12) to find the solutions for $\hat{\sigma}_{ba}$ associated with the simple roots $\alpha_c$. Then we apply the recursion relations arising from (13) to find the remaining values of $\hat{\sigma}_{ba}$.

### 4.2 The Simple Operators

In this section we solve equation (12), rewritten below, to find the simple values of $\hat{\sigma}_{ba}$.

\[ (q^{\frac{1}{2} h_c} - q^{-\frac{1}{2} h_c}) \otimes e_c = (q - q^{-1}) \sum_{\varepsilon_b - \varepsilon_a = \alpha_c} (-1)^{|b|} q^{\frac{1}{2}(\alpha_c \varepsilon_b)} (a'|c|b) \hat{\sigma}_{ba} \]

To solve this we must consider the various simple roots individually. Consider the simple roots $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, $1 \leq i \leq l$. In the vector representation we know

\[ e_i = E_{i+1}^i - E_{i+1}^i, \quad h_i = E_{i+1}^i - E_{i+1}^i + E_{i+1}^i = E_{i+1}^i. \]

Hence the left-hand side of (12) becomes:

\[ \text{LHS} = (q^{\frac{1}{2} h_i} - q^{-\frac{1}{2} h_i}) \otimes e_i = \left\{ (q^{\frac{1}{2}} - q^{-\frac{1}{2}}) (E_{i+1}^i + E_{i+1}^i + E_{i+1}^i + E_{i+1}^i) \right\} \otimes e_i \]

whereas the right-hand side is:

\[ \text{RHS} = (q - q^{-1}) \sum_{\varepsilon_b - \varepsilon_a = \alpha_i} (q^{-1} e_i E_b^a - E_b^a e_i) \otimes \hat{\sigma}_{ba} q^{-\frac{1}{2} h_i} \\
= (q - q^{-1}) \left\{ (q^{-1} E_{i+1}^i - E_{i+1}^i) \otimes \hat{\sigma}_{i+1} q^{-\frac{1}{2} h_i} \\
- (q^{-1} E_{i+1}^i - E_{i+1}^i) \otimes \hat{\sigma}_{i+1} q^{-\frac{1}{2} h_i} \right\}. \]
Equating these gives

$$\hat{\sigma}_{i+1} = -\hat{\sigma}_{i+1} = q^{\frac{1}{2}} e_i q^{\frac{1}{2} h_i}, \quad 1 \leq i < l.$$ 

By performing similar calculations for the other simple roots we obtain the simple operators given below in Table 1. These values for $\hat{\sigma}_{ba}$ form the basis for finding $R$, as from these all the others can be explicitly determined in any given representation.

| Simple Root | Corresponding $\hat{\sigma}_{ba}$ |
|-------------|-----------------------------------|
| $\alpha_i = \varepsilon_i - \varepsilon_{i+1}, i < l$ | $\hat{\sigma}_{i+1} = -\hat{\sigma}_{i+1} = q^{\frac{1}{2}} e_i q^{\frac{1}{2} h_i}$ |
| $\alpha_l = \varepsilon_{l-1} + \varepsilon_l, m = 2l$ | $\hat{\sigma}_{l-1} = -\hat{\sigma}_{l-1} = q^{\frac{1}{2}} e_l q^{\frac{1}{2} h_l}$ |
| $\alpha_l = \varepsilon_l, m = 2l + 1$ | $\hat{\sigma}_{l+1} = -\hat{\sigma}_{l+1} = q^{\frac{1}{2}} e_l q^{\frac{1}{2} h_l}$ |
| $\alpha_{\mu} = \delta_{\mu} - \delta_{\mu+1}, \mu < k$ | $\hat{\sigma}_{\mu+1} = \delta_{\mu+1}$ |
| $\alpha_s = \delta_k - \varepsilon_1$ | $\hat{\sigma}_{\mu=1} = (-1)^{h_s} q e_1 q^{\frac{1}{2} h_s}$ |

### 4.3 Constructing the Non-Simple Operators

Now we develop the recurrence relations required to calculate the remaining values of $\hat{\sigma}_{ba}$. Recall that for $\varepsilon_b > \varepsilon_a$,

$$q^{-\frac{1}{2}(\alpha_{a}, \varepsilon_a) - (\alpha_{a}, c)} \langle a | e_c | a' \rangle \hat{\sigma}_{ba} - (-1)^{|(a)|+(b)} | c \rangle q^{-\frac{1}{2}(\alpha_{a}, c)} \langle b | e_c | b' \rangle \hat{\sigma}_{ba}$$

$$= q^{(\alpha_{a}, c)} e_b q^{\frac{1}{2} h_c} - (-1)^{|(a)|+(b)} | c \rangle q^{-(\alpha_{a}, c)} e_c q^{\frac{1}{2} h_c} \hat{\sigma}_{ba}. \quad (16)$$

To extract the recurrence relations to be applied to the simple values of $\hat{\sigma}_{ba}$, we must again consider the simple roots individually. We begin with the case $\alpha_i = \varepsilon_i - \varepsilon_{i+1}$, so $e_i = \sigma_{i+1} \equiv E_{i+1} - E_i$. Now

$$\langle a | e_i = \delta_{ai} (i+1) - \delta_{a_{i+1}} (i), \quad e_i | b \rangle = \delta_{bi+1} (i) - \delta_{b_{i+1}} (i+1).$$

We then apply this to equation (16) to obtain:

$$q^{-\frac{1}{2}(\alpha_{a}, \alpha_i)} \left\{ \delta_{ai} q^{\frac{1}{2}(\alpha_{a}, \varepsilon_a)} \hat{\sigma}_{bi+1} - \delta_{a_{i+1}} q^{-\frac{1}{2}(\alpha_{a}, \varepsilon_{i+1})} \hat{\sigma}_{bi} \right\}$$

$$- \left\{ \delta_{bi+1} q^{\frac{1}{2}(\alpha_{a}, \varepsilon_{i+1})} \hat{\sigma}_{ia} - \delta_{bi} q^{-\frac{1}{2}(\alpha_{a}, \varepsilon_i)} \hat{\sigma}_{i+1} \right\}$$

$$= q^{(\alpha_{a}, c)} \hat{\sigma}_{ba} e_i q^{\frac{1}{2} h_i} - q^{-(\alpha_{a}, c)} e_i q^{\frac{1}{2} h_i} \hat{\sigma}_{ba}$$

for $\varepsilon_b > \varepsilon_a$. Recalling that $\hat{\sigma}_{i+1} = -\hat{\sigma}_{i+1} = q^{\frac{1}{2}} e_i q^{\frac{1}{2} h_i}$, $1 \leq i < l$, the above simplifies to

$$\delta_{ai} \hat{\sigma}_{bi+1} - \delta_{a_{i+1}} \hat{\sigma}_{bi} = \delta_{bi+1} \hat{\sigma}_{ia} + \delta_{bi} \hat{\sigma}_{i+1} \hat{a} = q^{(\alpha_{a}, c)} \hat{\sigma}_{ba} \hat{\sigma}_{i+1} - q^{-(\alpha_{a}, c)} \hat{\sigma}_{i+1} \hat{\sigma}_{ba}$$

$$= q^{-(\alpha_{a}, c)} \hat{\sigma}_{i+1} \hat{\sigma}_{ba} - q^{(\alpha_{a}, c)} \hat{\sigma}_{ba} \hat{\sigma}_{i+1}.$$
In the case \( \sigma \), there is a unique operator-valued matrix\( R \in (\text{End } V) \otimes U_q(\text{osp}(m|n))^{	ext{+}} \) of the form
\[
R = q^a b^a \otimes h^n \left[ I \otimes I + (q - q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{|b|} E_0^a \otimes \hat{\sigma}_{ba} \right],
\]
satisfying \( R \Delta(e_c) = \Delta_T(e_c) R \). The simple operators for that matrix are given by:

\[
\begin{align*}
\hat{\sigma}_{b_i+1} &= \hat{\sigma}_b \hat{\sigma}_{i+1} - q^{-1} \hat{\sigma}_{i+1} \hat{\sigma}_b, & \varepsilon_b < \varepsilon_i, \\
\hat{\sigma}_{i+1} &= \hat{\sigma}_a \hat{\sigma}_{i+1} - q^{-1} \hat{\sigma}_{i+1} \hat{\sigma}_a, & \varepsilon_a < -\varepsilon_i, \\
\hat{\sigma}_{b_i} &= q^{(\alpha_i, \epsilon_b)} \hat{\sigma}_{b_i} \hat{\sigma}_{i+1} - q^{-1} \hat{\sigma}_{i+1} \hat{\sigma}_{b_i}, & \varepsilon_b > -\varepsilon_{i+1}, b \neq i + 1, \\
\hat{\sigma}_{i+1} &= q^{-(\alpha_i, \epsilon_a)} \hat{\sigma}_{i+1} \hat{\sigma}_{i+1} - q^{-1} \hat{\sigma}_{i+1} a \hat{\sigma}_{i+1}, & \varepsilon_a < \varepsilon_{i+1}, a \neq i + 1, \\
\hat{\sigma}_{i+1} + \hat{\sigma}_{i+1} &= q^{-1} [\hat{\sigma}_{i+1}, \hat{\sigma}_{i+1}], \\
q^{(\alpha_i, \epsilon_b)} \hat{\sigma}_{ba} \hat{\sigma}_{i+1} - q^{-(\alpha_i, \epsilon_a)} \hat{\sigma}_{i+1} \hat{\sigma}_{ba} &= 0, & \varepsilon_b > \varepsilon_a, a \neq i, i + 1, b \neq i + 1.
\end{align*}
\]
\[ \hat{\sigma}_{i+1} = q^{\frac{1}{2}} e_i q^{\frac{1}{2}} h_i, \quad 1 \leq i < l, \]
\[ \hat{\sigma}_{l+1} = q^{\frac{1}{2}} e_l q^{\frac{1}{2}} h_l, \quad m = 2l, \]
\[ \hat{\sigma}_l = 0, \quad m = 2l, \]
\[ \hat{\sigma}_{l+1} = q^{-\frac{1}{2}} e_l q^{\frac{1}{2}} h_l, \quad m = 2l + 1, \]
\[ \hat{\sigma}_{\mu+1} = q^{\frac{1}{2}} e_{\mu} q^{\frac{1}{2}} h_{\mu}, \quad 1 \leq \mu < k, \]
\[ \hat{\sigma}_{k+1} = (-1)^k q^2 e_{s} q^{\frac{1}{2}} h_{s}, \quad \text{with } \hat{\sigma}_{ba} \text{ as given before.} \]

and the remaining values can be calculated using

(i) the \( q \)-commutation relations

\[ q^{(\alpha_c, \varepsilon_a)} \hat{\sigma}_{ba} e_q h_{c} - (-1)^{[a]+[b][c]} q^{-\alpha_c, \varepsilon_a} e_q h_{c} \hat{\sigma}_{ba} = 0, \quad \varepsilon_b > \varepsilon_a, \] (22)

where neither \( \varepsilon_a - \alpha_c \) nor \( \varepsilon_b + \alpha_c \) equals any \( \varepsilon_x \); and

(ii) the induction relations

\[ \hat{\sigma}_{ba} = q^{-\varepsilon_a, \varepsilon_c} \hat{\sigma}_{bc} \hat{\sigma}_{ca} - q^{-\varepsilon_c, \varepsilon_c} (-1)^{[b]+[c]} [(a) + [c]] \hat{\sigma}_{ca} \hat{\sigma}_{bc}, \quad \varepsilon_b > \varepsilon_c > \varepsilon_a, \] (23)

where \( c \neq b \) or \( a \).

We have found a set of simple operators and relations which uniquely define the unknowns \( \hat{\sigma}_{ba} \). Theoretically the resultant matrix \( R \) must be a Lax operator, as we know there is one of the given form. We choose to confirm this, however, by verifying that \( R \) satisfies the remaining \( R \)-matrix properties. These are

\[(\text{id} \otimes \Delta)R = R_{13} R_{12} \] (24)

and the intertwining property for the remaining generators,

\[ R\Delta(a) = \Delta^T(a)R, \quad \forall a \in U_q[osp(m|n)]. \]

We also calculate the opposite Lax operator \( R^T \), and briefly examine whether the defining relations for the \( \hat{\sigma}_{ba} \) incorporate the \( q \)-Serre relations for \( U_q[osp(m|n)] \).

### 4.4 Calculating the Coproduct

We begin by considering the first of these defining properties, equation (24). In order to evaluate \((\text{id} \otimes \Delta)R\), however, we need to know \( \Delta(\hat{\sigma}_{ba}) \). First note an alternative way of writing \( R \) is

\[ R = \sum_a E_a^a \otimes q^{h_a} + (q - q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{[b]} E_a^a \otimes q^{h_a} \hat{\sigma}_{ba}, \]

with the \( \hat{\sigma}_{ba} \) as given before. Using this form, we find
By considering each of those cases separately, substituting in the expressions for $\Delta(\hat{\sigma}_{ba})$ we obtain
\[ R_{13}R_{12} = \left( \sum_a E_a \otimes I \otimes q^{h_{ea}} \right) + (q - q^{-1}) \sum_{e_b > e_a} (-1)^{|b|} E_b \otimes q^{h_{ea}} \hat{\sigma}_{ba} \]
\[ + \left( \sum_c E_c \otimes q^{h_{ec}} \otimes I \right) + (q - q^{-1}) \sum_{e_d > e_c} (-1)^{|d|} E_d \otimes q^{h_{ec}} \hat{\sigma}_{dc} \otimes I \]
\[ = \sum_a E_a \otimes q^{h_{ea}} \otimes q^{h_{ea}} \]
\[ + (q - q^{-1}) \sum_{e_b > e_a} (-1)^{|b|} E_b \otimes q^{h_{ea}} \hat{\sigma}_{ba} \otimes q^{h_{ea}} + q^{h_{eb}} \otimes q^{h_{ea}} \hat{\sigma}_{ba} \]
\[ + (q - q^{-1})^2 \sum_{e_b > e_c > e_a} (-1)^{|b|+|c|} E_b \otimes q^{h_{ea}} \hat{\sigma}_{bc} \otimes q^{h_{ea}} \hat{\sigma}_{ca}. \]

Also, the coproduct properties (7) imply
\[ (id \otimes \Delta)R = \sum_a E_a \otimes q^{h_{ea}} \otimes q^{h_{ea}} \]
\[ + (q - q^{-1}) \sum_{e_b > e_a} (-1)^{|b|} E_b \otimes (q^{h_{ea}} \otimes q^{h_{ea}}) \Delta(\hat{\sigma}_{ba}). \]

Hence $R$ will satisfy equation (24) if and only if $\Delta(\hat{\sigma}_{ba})$ is given by:
\[ \Delta(\hat{\sigma}_{ba}) = \hat{\sigma}_{ba} \otimes I + q^{h_{ea}} \hat{\sigma}_{bc} \hat{\sigma}_{ca} + (q - q^{-1}) \sum_{e_b > e_c > e_a} (-1)^{|c|} q^{h_{ec} - h_{ea}} \hat{\sigma}_{bc} \otimes \hat{\sigma}_{ca}. \]

The simple values of $\hat{\sigma}_{ba}$ and the inductive relations can be used to calculate $\Delta(\hat{\sigma}_{ba})$, and show that it is indeed of this form. First the coproduct of each of the simple operators given in equation (21) is directly calculated, and found to be of the form required. To find the coproduct for the remaining values of $\hat{\sigma}_{ba}$ we use the inductive relations (29):
\[ \hat{\sigma}_{ba} = q^{-(e_b, e_a)} \hat{\sigma}_{bc} \hat{\sigma}_{ca} - q^{-(e_c, e_a)} (-1)^{|b|+|c|+|a|} \hat{\sigma}_{ca} \hat{\sigma}_{bc}, \quad e_b > e_c > e_a, \]
where $c \neq b$ or $\bar{b}$. We assume our formula for the coproduct holds for $\hat{\sigma}_{bc}$ and $\hat{\sigma}_{ca}$, where $e_b > e_c > e_a$, and then show it is also true for $\hat{\sigma}_{ba}$.

The coproduct is an algebra homomorphism, so for $e_b > e_c > e_a$, $c \neq \bar{a}$ or $\bar{b}$, we have
\[ \Delta(\hat{\sigma}_{ba}) = q^{-(e_b, e_a)} \Delta(\hat{\sigma}_{bc}) \Delta(\hat{\sigma}_{ca}) - q^{-(e_c, e_a)} (-1)^{|b|+|c|+|a|} \Delta(\hat{\sigma}_{ca}) \Delta(\hat{\sigma}_{bc}). \]

We can always choose $c$ satisfying the conditions such that either $e_b - e_c$ or $e_c - e_a$ is a simple root. By considering each of those cases separately, substituting in the expressions for $\Delta(\hat{\sigma}_{bc})$ and $\Delta(\hat{\sigma}_{ca})$ and using the relations (22) and (28) to simplify, after lengthy but straightforward calculations given in (30) we obtain
\[ \Delta(\hat{\sigma}_{ba}) = \hat{\sigma}_{ba} \otimes I + q^{h_{ea}} \hat{\sigma}_{bc} \otimes \hat{\sigma}_{ca} + (q - q^{-1}) \sum_{e_b > e_c > e_a} (-1)^{|c|} q^{h_{ea}} \hat{\sigma}_{bc} \otimes \hat{\sigma}_{ca} \]
as required. As all the relations in Tables 3 and 5 have either $e_b - e_c$ or $e_c - e_a$ as a simple root, this is sufficient to prove the formula for $\Delta(\hat{\sigma}_{ba})$ for all $e_b > e_a$. As a check, however, we also confirmed that this is consistent with the $q$-commutation relations (22), although again the calculations are tedious and omitted. Hence we have proven that the matrix $R$ satisfies the property
\[ (id \otimes \Delta)R = R_{13}R_{12}. \]
4.5 The Intertwining Property

To confirm that we have a Lax operator we need to check one last relation, namely the intertwining property for the other generators.

\[ R\Delta(a) = \Delta^T(a)R, \quad \forall a \in U_q[osp(m|n)]. \]  

(25)

Now \( R \) is weightless, so it commutes with all the Cartan elements. Moreover, \( \Delta(q^{h_a}) = \Delta^T(q^{h_a}), \forall h_a \in H \), so the Cartan elements will automatically satisfy equation (25). Thus it remains only to verify the intertwining property for the lowering generators, \( f_a \). Unfortunately, knowing the raising generators satisfy the intertwining property does not appear helpful. Instead, we start by assuming the form of the Lax operator and that it satisfies the intertwining property for the lowering generators, and then proceed as in Section 4.1. Again two different equations are obtained, this time

\[ f_c \otimes (q^{\frac{1}{2}h_c} - q^{-\frac{1}{2}h_c}) = (q - q^{-1}) \sum_{\varepsilon_b - \varepsilon_a = \alpha_c} (-1)^{[b]} E_b^a \otimes (q^{\frac{1}{2}(\varepsilon_c, \varepsilon_b)} \tilde{\sigma}_{ba} f_c - (-1)^{[c]} q^{\frac{1}{2}(\varepsilon_c, \varepsilon_a)} f_c \tilde{\sigma}_{ba}) \]  

(26)

and

\[ q^{\frac{1}{2}(\varepsilon_c - \varepsilon_a, \alpha_c)} \langle a | f_c | a' \rangle \tilde{\sigma}_{ba} q^{\frac{1}{2}h_c} - (-1)^{([a]+[b]) ([a]+[b])} f_c | b \rangle \tilde{\sigma}_{ba} f_c q^{\frac{1}{2}h_c} = q^{\frac{1}{2}(\varepsilon_c, \varepsilon_b)} \tilde{\sigma}_{ba} f_c - (q^{-1})^{([a]+[b]) ([a]+[b])} q^{\frac{1}{2}(\varepsilon_c, \varepsilon_a)} f_c \tilde{\sigma}_{ba}, \]  

(27)

By considering the simple roots individually and using the \( U_q[osp(m|n)] \) defining relations, equation (26) can be shown to be consistent with the simple values of \( \tilde{\sigma}_{ba} \) given in Table 4.[30], as expected. The various relations that can be deduced from equation (27) are no longer inductive in form, so cannot be used to directly construct the \( \tilde{\sigma}_{ba} \) for direct comparison. They can, however, be shown to be consistent with relations (21) and (22), as demonstrated in [30]. Thus the conditions obtained by considering the intertwining property for the lowering generators,

\[ R\Delta(f_c) = \Delta^T(f_c)R \]

are satisfied by the matrix \( R \), and hence \( R \) satisfies the intertwining property for all elements \( a \in U_q[osp(m|n)] \).

4.6 The Lax Operator

We have now proven, as expected, that the matrix \( R \) found earlier satisfies both the intertwining property and \( (id \otimes \Delta)R = R_{13}R_{12} \). The other \( R \)-matrix property, containing \( (\Delta \otimes id)R \), is clearly not applicable here. It is not necessary, however, as we know there is a Lax operator belonging to \( \pi(U_q[osp(m|n)])^- \otimes U_q[osp(m|n)]^+ \), and we have shown there is only one such possibility. Thus the work in the previous two sections confirms the following proposition:

Proposition 4.2 The Lax operator, \( R = (\pi \otimes id)R \) for the quantum superalgebra \( U_q[osp(m|n)] \), where \( R \in U_q[osp(m|n)]^- \otimes U_q[osp(m|n)]^+ \) and \( m > 2 \), is given by

\[ R = q^{h_c \otimes h^+} \left[ I \otimes I + (q - q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{[b]} E_b^a \otimes \tilde{\sigma}_{ba} \right] \]

\[ = \sum_a E_a^a \otimes q^{h_{\varepsilon_a}} + (q - q^{-1}) \sum_{\varepsilon_a < \varepsilon_b} (-1)^{[b]} E_b^a \otimes q^{h_{\varepsilon_a}} \tilde{\sigma}_{ba}, \]

where the operators \( \tilde{\sigma}_{ba} \) satisfy equations (21), (22) and (23).
As an aside, the two properties verified directly are sufficient to prove $R$ satisfies the Yang-Baxter equation. For using only those, we see

\[
R_{23}R_{13}R_{12} = R_{23}(\text{id} \otimes \Delta)R
= [(\text{id} \otimes \Delta^T)R]R_{23}
= [(\text{id} \otimes T)(\text{id} \otimes \Delta)R]R_{23}
= [(\text{id} \otimes T)R_{13}R_{12}]R_{23}
= R_{12}R_{13}R_{23}
\]

as required. It is very surprising that there is a unique solution to

\[
R\Delta(e_c) = \Delta^T(e_c)R,
\]
given we only considered elements of \(\pi(U_q[osp(m|n)]^-) \otimes U_q[osp(m|n)]^+\).

### 4.7 The Opposite Lax Operator

Having found the Lax operator \(R = (\pi \otimes \text{id})R\), we wish to use that result to find its opposite \(R^T = (\pi \otimes \text{id})R^T\), where \(R^T\) is the opposite universal \(R\)-matrix of \(U_q[osp(m|n)]\). We begin by showing that \(R^T\) is in fact equal to \(R^\dagger\), where \(\dagger\) represents graded conjugation, defined below.

A graded conjugation on \(U_q[osp(m|n)]\) is defined on the simple generators by:

\[
\begin{align*}
\sigma_a^\dagger &= (-1)^{|a|+|b|}\sigma_b^a, \\
(ab)^\dagger &= (-1)^{|a||b|}b^a^\dagger, \\
(a \otimes b)^\dagger &= a^\dagger \otimes b^\dagger, \\
\Delta(a)^\dagger &= \Delta(a^\dagger).
\end{align*}
\]

It is consistent with the coproduct and extends naturally to all remaining elements of \(U_q[osp(m|n)]\), satisfying the properties:

\[
\begin{align*}
\sigma_a^\dagger &= (-1)^{|a|+|b|}\sigma_b^a, \\
(ab)^\dagger &= (-1)^{|a||b|}b^a^\dagger, \\
(a \otimes b)^\dagger &= a^\dagger \otimes b^\dagger, \\
\Delta(a)^\dagger &= \Delta(a^\dagger).
\end{align*}
\]

Returning to the universal \(R\)-matrix \(R\), we know

\[
\begin{align*}
R\Delta(a) &= \Delta^T(a)R, \\
\Rightarrow \Delta(a)^\dagger R^\dagger &= R^\dagger \Delta^T(a)^\dagger \\
\Rightarrow \Delta(a)^\dagger R^\dagger &= R^\dagger \Delta^T(a^\dagger) \\
\Rightarrow \Delta(a)R^\dagger &= R^\dagger \Delta^T(a),
\end{align*}
\]

\(\forall a \in U_q[osp(m|n)]\).

Similarly, \(R^\dagger\) satisfies the other \(R\)-matrix properties. As there is a unique universal \(R\)-matrix belonging to \(U_q[osp(m|n)]^+ \otimes U_q[osp(m|n)]^-\), the only possibility is \(R^T = R^\dagger\).

Now it is known that the vector representation is superunitary. A discussion of superunitary representations is given in [31], where they are called grade star representations, but here we need only note this implies

\[
\pi(a^\dagger) = \pi(a)^\dagger, \quad \forall a \in U_q[osp(m|n)].
\]

Hence
\[ RT = (\pi \otimes \text{id}) R^\dagger \]
\[ = ((\pi \otimes \text{id}) R)^\dagger \]
\[ = R^\dagger. \]

Thus we can find the opposite Lax operator \( RT \) simply by using the usual rules for graded conjugation. As \( R \) is given by
\[ R = \sum_a E_a^a \otimes q^{h_{\varepsilon_a}} + (q - q^{-1}) \sum_{\varepsilon_b > \varepsilon_a} (-1)^{|b|} E_b^a \otimes q^{h_{\varepsilon_a}} \hat{\sigma}_{ba}, \]
we find the opposite Lax operator \( RT \) can be written as
\[ RT = \sum_a E_a^a \otimes q^{h_{\varepsilon_a}} + (q - q^{-1}) \sum_{\varepsilon_b > \varepsilon_a} (-1)^{|a|} E_a^b \hat{\sigma}_{ab} q^{h_{\varepsilon_a}}, \] (28)
where
\[ \hat{\sigma}_{ab} = (-1)^{|b||a|} \hat{\sigma}_{ba}, \quad \varepsilon_b > \varepsilon_a. \]

### 4.8 q-Serre Relations

Having shown the relations found in Section 4.3 define a Lax operator, we also wish to see if they incorporate the \( q \)-Serre relations. It is too space-consuming to list all of these, so we will merely provide a couple of examples, including the extra \( q \)-Serre relations.

First recall that if \( \varepsilon_b - \varepsilon_a \) is a simple root, then \( \hat{\sigma}_{ba} \propto e_c q^{\frac{1}{2}h_c} \) for either \( c = b \) or \( c = \overline{b} \). Then setting \( E_a = e_a q^{\frac{1}{2}h_a} \), we see from Definition 3.1 that:
\[ \Delta(E_a) = q^{h_a} \otimes E_a + E_a \otimes 1 \]
\[ S(E_a) = -q^{-\frac{1}{2}(\alpha_{a} \cdot \alpha_{a})} q^{-\frac{1}{2}h_a} E_a \]
\[ = -q^{-h_a} E_a \]
\[ \therefore \quad ad E_a \circ b = -(1)^{|a||b|} q^{h_{\varepsilon_a}} b q^{-h_a} E_a + E_a b \]
\[ = E_a b - (-1)^{|a||b|} q^{(\alpha_{a} \cdot \varepsilon_b)} b E_a. \] (29)

Now consider the simple generators \( \hat{\sigma}_{i+1}^{i+1} \) and \( \hat{\sigma}_{i+1}^{i+2} \).
\[ (ad \hat{\sigma}_{i+1}^{i+1} \circ \hat{\sigma}_{i+1}^{i+2}) (ad \hat{\sigma}_{i+1}^{i+1} \circ \hat{\sigma}_{i+1}^{i+2} - q^{-1} \hat{\sigma}_{i+1}^{i+2} \hat{\sigma}_{i+1}^{i+1}) \]
\[ = ad \hat{\sigma}_{i+1}^{i+1} \circ \hat{\sigma}_{i+1}^{i+2} \]
\[ = \hat{\sigma}_{i+1}^{i+1} \hat{\sigma}_{i+1}^{i+2} - q \hat{\sigma}_{i+1}^{i+2} \hat{\sigma}_{i+1}^{i+1} \]
\[ = 0 \]
from \( \text{22} \). This is equivalent to the \( q \)-Serre relation \( (ad e_b \circ)^{1-a} = e_a = 0 \) for this pair of simple operators. In a similar way, we can verify this relation for any \( b \neq c, (\varepsilon_b, \varepsilon_c) \neq 0 \). The defining relations for the \( \hat{\sigma}_{ba} \), therefore, incorporate all the standard \( q \)-Serre relations for raising generators.

This still leaves the extra \( q \)-Serre relations, which involve the odd root. There are only two of these for our choice of simple roots \([16]\). Explicitly, taking into account the different conventions between \([16]\) and here, the relevant extra \( q \)-Serre relations for \( U_q[osp(m|n)] \) can be written as
The Lax operator can be used to explicitly calculate an $R$-matrix for any tensor product representation $\pi \otimes \phi$, where $\phi$ is an arbitrary representation. In particular, it provides a more straightforward method of calculating $R$ for the tensor product of the vector representation, $\pi \otimes \pi$, than using the tensor product graph method [29].

By specifically constructing the $R$-matrix for the vector representation, we also illustrate concretely the way the recursion relations can be applied to find the $R$-matrix for an arbitrary representation. Although the values for $\hat{\sigma}_{ba}$ obtained will change for each representation, they can always be constructed by applying the same equations in the same order. We could choose to use only the relations listed in the tables in the appendix, but using the general form of the inductive relations shortens and simplifies the process. Only the method of calculation is included here, but the full calculations can be found in [30].

First the vector representation is applied to the simple operators given in Table 11 with the results written below in Table 2.

| Simple Root | Corresponding $\hat{\sigma}_{ba}$ |
|-------------|----------------------------------|
| $\alpha_\ell = \epsilon_\ell - \epsilon_{\ell+1}$, $\ell < l$ | $\hat{\sigma}_{i+1} = -\hat{\sigma}_{i+1}$ | $E_{l+1}^1 - E_{l+1}^1$ |
| $\alpha_\ell = \epsilon_{\ell-1} + \epsilon_\ell$, $m = 2l$ | $\hat{\sigma}_{l+1} = -\hat{\sigma}_{l+1}$ | $E_{l}^{l+1} - E_{l}^{l+1}$ |
| $\alpha_\ell = \epsilon_\ell$, $m = 2l + 1$ | $\hat{\sigma}_{l+1} = -q^{-\frac{1}{2}}\hat{\sigma}_{l+1}$ | $E_{l+1}^1 - q^{-\frac{1}{2}}E_{l+1}^1$ |
| $\epsilon_\mu = \delta_{\mu} - \delta_{\mu+1}$, $\mu < k$ | $\hat{\sigma}_{\mu+1} = \hat{\sigma}_{\mu+1}$ | $E_{\mu+1}^\mu + E_{\mu+1}^\mu$ |
| $\alpha_\ell = \delta_k - \epsilon_1$ | $\hat{\sigma}_{\mu} = (1)^q\hat{\sigma}_{\mu}$ | $E_{\mu+1}^\mu + (1)^{\mu + 1}E_{\mu+1}^\mu$ |

Then the inductive relations [23] are applied to these simple operators to find the remaining values of $\hat{\sigma}_{ba}$. One of many equivalent ways of doing this is given below. Construct
1. \( \hat{\sigma}_{ji} \), \( \hat{\sigma}_{ij} \) for \( 1 \leq j < i \leq [\frac{m}{2}] \), using \( \hat{\sigma}_{i+1} \), \( \hat{\sigma}_{m+i} \) for \( 1 \leq i < l \) and \( \hat{\sigma}_{l+1} \), \( \hat{\sigma}_{l+1} \) when \( m = 2l + 1 \)

2. \( \hat{\sigma}_{\mu \nu} \), \( \hat{\sigma}_{m+n} \) for \( 1 \leq \nu < \mu \leq k \), using \( \hat{\sigma}_{\mu k = i - 1} \) and \( \hat{\sigma}_{\mu \nu} \) together with the values calculated in steps 1 and 2.

3. \( \hat{\sigma}_{\mu \nu} \), \( \hat{\sigma}_{m+n} \) for \( 1 \leq i \leq [\frac{m}{2}] \), \( 1 \leq \mu \leq k \), using \( \hat{\sigma}_{\mu k = i - 1} \) and \( \hat{\sigma}_{\mu \nu} \) together with the results calculated in steps 1 and 2.

4. \( \hat{\sigma}_{ij} \) for \( 1 \leq i, j \leq l \), using \( \hat{\sigma}_{l+1} \) and \( \hat{\sigma}_{l+1} \) when \( m = 2l + 1 \) or \( \hat{\sigma}_{l-1} \), \( \hat{\sigma}_{l+1} \) and \( \hat{\sigma}_{l} = 0 \) when \( m = 2l \), together with the results from step 1.

5. \( \hat{\sigma}_{i, \mu} \), \( \hat{\sigma}_{\mu \nu} \) for \( 1 \leq i \leq l, 1 \leq \mu \leq k \), using the results from steps 3 and 4.

6. \( \hat{\sigma}_{\mu \nu} \) for \( 1 \leq \mu, \nu \leq k \), using the results from steps 3 and 4.

Following this procedure in the vector representation, we find the following form for the operators \( \hat{\sigma}_{ba} \), \( \varepsilon_b > \varepsilon_a \):

\[
\hat{\sigma}_{ba} = q^{- (\varepsilon_a, \varepsilon_b)} E_a^b - (-1)^{[b]} q^{(\varepsilon_a, \varepsilon_a)} q^{(\varepsilon_a, \varepsilon_a)} E_a^{\varepsilon_b}.
\]

Thus we have shown the R-matrix for the vector representation, \( \mathcal{R} = (\pi \otimes \pi) \mathcal{R} \), is given by

\[
\mathcal{R} = q^{(\varepsilon_a, \varepsilon_b)} \left[ I \otimes I + (q - q^{-1}) \sum_{\varepsilon_b > \varepsilon_a} (-1)^{[b]} E_a^b \otimes \hat{\sigma}_{ba} \right].
\]

where

\[
\hat{\sigma}_{ba} = q^{- (\varepsilon_a, \varepsilon_b)} E_a^b - (-1)^{[b]} q^{(\varepsilon_a, \varepsilon_a)} q^{(\varepsilon_a, \varepsilon_a)} E_a^{\varepsilon_b}.
\]

This can be written in a more elegant form, namely

\[
\mathcal{R} = \sum_{a,b} q^{(\varepsilon_a, \varepsilon_b)} E_a^b \otimes E_b^b + (q - q^{-1}) \sum_{\varepsilon_b > \varepsilon_a} (-1)^{[b]} E_a^b \otimes \hat{\sigma}_{ba},
\]

where \( \hat{\sigma}_{ba} = q^{(\varepsilon_a, \varepsilon_b)} \). Hence we have the following result:

**Proposition 5.1** The R-matrix for the vector representation, \( \mathcal{R} = (\pi \otimes \pi) \mathcal{R} \), is given by

\[
\mathcal{R} = \sum_{a,b} q^{(\varepsilon_a, \varepsilon_b)} E_a^b \otimes E_b^b + (q - q^{-1}) \sum_{\varepsilon_b > \varepsilon_a} (-1)^{[b]} E_a^b \otimes \hat{\sigma}_{ba},
\]

where

\[
\hat{\sigma}_{ba} = E_a^b - (-1)^{[b]} q^{(\varepsilon_a, \varepsilon_a)} E_a^{\varepsilon_b},
\]

\( \varepsilon_b > \varepsilon_a \).

We can also explicitly find the opposite R-matrix \( \mathcal{R}^T \) as given in equation (28), using

\[
(E_a^b)^\dagger = (-1)^{[a]} q^{(\varepsilon_a, \varepsilon_b)} E_a^b.
\]

We obtain this result:

**Proposition 5.2** The opposite R-matrix for the vector representation, \( \mathcal{R}^T = (\pi \otimes \pi) \mathcal{R}^T \), is given by

\[
\mathcal{R}^T = \sum_{a,b} q^{(\varepsilon_a, \varepsilon_b)} E_a^b \otimes E_b^b + (q - q^{-1}) \sum_{\varepsilon_b > \varepsilon_a} (-1)^{[b]} E_a^b \otimes \hat{\sigma}_{ba},
\]

where

\[
\hat{\sigma}_{ba} = E_b^a - (-1)^{[a]} q^{(\varepsilon_a, \varepsilon_b)} E_a^{\varepsilon_b}.
\]
These formulae for $\mathcal{R}$ and $\mathcal{R}^T$ on the vector representation agree with those given in [29]. In that thesis the $R$-matrix for the vector representation was calculated using projection operators onto invariant submodules of the tensor product. The greatest advantage of the current method is it gives a straightforward way of constructing a solution to the Yang-Baxter Equation in an arbitrary representation of $U_q[osp(m|n)]$.

Applying the tensor product graph method to $\mathcal{R}$, it was shown in [29] that the spectral dependent $R$-matrix for the vector representation of the quantum affine superalgebra $U_q[osp(m|n)^{(1)}]$ is

\[
\mathcal{R}(z) = \frac{(q - q^{-1})}{(q - z q^{-1})} P - \frac{(q - q^{-1})z(z - 1)}{(q - q^{-1})z(q^m - n - 2)} \sum_{a,b} (-1)^{[a][b]} \xi_a \xi_b q^{\rho + \epsilon_a - \epsilon_b} E_a \otimes E_b
\]

\[
- \frac{(z - 1)}{(q - z q^{-1})} \left\{ I + (q^{1/2} - q^{-1/2}) \sum_a (-1)^{[a]} E_a^a \otimes \tilde{\sigma}_a^a + (q - q^{-1}) \sum_{\epsilon_a < \epsilon_b} (-1)^{[b]} E_b^b \otimes \tilde{\sigma}_a^b \right\}
\]

and the spectral dependent $R$-matrix for the vector representation of the quantum twisted affine superalgebra $U_q[gl(m|n)^{(2)}]$ is

\[
\mathcal{R}(z) = \frac{(q - q^{-1})}{(q - z q^{-1})} P - \frac{(q - q^{-1})z(z - 1)}{(q - q^{-1})z(q^m - n - 2)} \sum_{a,b} (-1)^{[a][b]} \xi_a \xi_b q^{\rho + \epsilon_a - \epsilon_b} E_a \otimes E_b
\]

\[
- \frac{(z - 1)}{(q - z q^{-1})} \left\{ I + (q^{1/2} - q^{-1/2}) \sum_a (-1)^{[a]} E_a^a \otimes \tilde{\sigma}_a^a + (q - q^{-1}) \sum_{\epsilon_a < \epsilon_b} (-1)^{[b]} E_b^b \otimes \tilde{\sigma}_a^b \right\}.
\]

In a similar way the tensor product graph can be applied to the Lax Operator to find the spectral dependent $R$-matrix for other affinisable representations of the form $\pi \otimes \sigma$ applied to $U_q[osp(m|n)^{(1)}]$ or $U_q[gl(m|n)^{(2)}]$.

6 Conclusion

In this Communication a Lax operator was constructed for the $B$ and $D$ series of quantum superalgebras. A general ansatz for the Lax operator in terms of unknown elements of $U_q[osp(m|n)]$ was assumed. Formulae identifying the simple operators were found, and then a set of inductive and $q$-commutative relations developed that can be used to calculate the remaining non-simple operators. This result is universal and thus the Lax operator generates a solution of the Yang-Baxter equation on the space $V \otimes V \otimes W$ for any module $W$. A specific example was given in Section 4 where the $R$-matrix for the vector representation was calculated from the Lax operator. Together with the results of [20] for the $A$ series and [22] for the $C$ series, this completes the construction of Lax operators for all non-exceptional quantum superalgebras.

Acknowledgements – We gratefully acknowledge financial support from the Australian Research Council.
A The Relations Governing the Operators \( \hat{\sigma}_{ba} \)

| Relation | Conditions |
|----------|-------------|
| \( \hat{\sigma}_{ba} = \hat{\sigma}_{ba} \) | \( i < l, \varepsilon_b > \varepsilon_1 \) |
| \( \hat{\sigma}_{ba} + \hat{\sigma}_{a,b} = q^{-1}[\hat{\sigma}_{i+1, \mu+1} \hat{\sigma}_{\mu+1} a - q \hat{\sigma}_{\mu+1} a \hat{\sigma}_{\mu+1}, \hat{\sigma}_{a,b} = 0, \) | \( i < l, \varepsilon_a < \varepsilon_i \) |
| \( \hat{\sigma}_{\mu+1, \mu+1} = \hat{\sigma}_{\mu+1, \mu+1} - q \hat{\sigma}_{\mu+1, \mu+1}, \hat{\sigma}_{\mu+1, \mu+1} = \hat{\sigma}_{\mu+1, \mu+1} - q \hat{\sigma}_{\mu+1, \mu+1}, \) | \( i < l, \varepsilon_b > \varepsilon_{i+1}, \mu \neq i+1 \) |
| \( \hat{\sigma}_{\mu+1, \mu+1} = \hat{\sigma}_{\mu+1, \mu+1} - q \hat{\sigma}_{\mu+1, \mu+1}, \hat{\sigma}_{\mu+1, \mu+1} = \hat{\sigma}_{\mu+1, \mu+1} - q \hat{\sigma}_{\mu+1, \mu+1}, \) | \( b \neq i+1 \) |
| \( \hat{\sigma}_{\mu+1, \nu} = \hat{\sigma}_{\mu+1, \nu} - q \hat{\sigma}_{\mu+1, \nu}, \hat{\sigma}_{\mu+1, \nu} = \hat{\sigma}_{\mu+1, \nu} - q \hat{\sigma}_{\mu+1, \nu}, \) | \( i < l, \) \( \varepsilon_a < \varepsilon_i, \mu < k \) |
| \( \hat{\sigma}_{\mu+1, \nu} = \hat{\sigma}_{\mu+1, \nu} - q \hat{\sigma}_{\mu+1, \nu}, \hat{\sigma}_{\mu+1, \nu} = \hat{\sigma}_{\mu+1, \nu} - q \hat{\sigma}_{\mu+1, \nu}, \) | \( i < l, \) \( \varepsilon_b > \varepsilon_1 \) |

Table 3: The relations for the operators \( \hat{\sigma}_{ba} \) common to all values of \( m \)
Table 4: The relations for the operators $\hat{\sigma}_{ba}$ that hold only for even $m$

| Relation                                                                 | Conditions |
|-------------------------------------------------------------------------|-------------|
| $\hat{\sigma}_{bl+1} = q^{(\epsilon_a, \epsilon_b)} \hat{\sigma}_{bl} \hat{\sigma}_{l+1} - q^{-1} \hat{\sigma}_{l+1} \hat{\sigma}_{bl}$, | $e_b > e_l$ |
| $\hat{\sigma}_{bl} = \hat{\sigma}_{bl-1} \hat{\sigma}_{l-1} \hat{\sigma}_{bl-1} - q^{-1} \hat{\sigma}_{l-1} \hat{\sigma}_{bl-1}$, | $e_b > e_{l-1}$ |
| $\hat{\sigma}_{la} = \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} a - q^{-1} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} a \hat{\sigma}_{l-1}$, | $e_a < -e_{l-1}$ |
| $\hat{\sigma}_{l-1} a = q^{-(\epsilon_a, \epsilon_a)} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} a - q^{-1} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} a$, | $e_a < -e_l$ |
| $q^{(\epsilon_a, \epsilon_a)} \hat{\sigma}_{ba} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} a - q^{-(\epsilon_a, \epsilon_a)} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} \hat{\sigma}_{l-1} a = 0$, | $e_b > e_a; a \neq l, l-1; b \neq l-1, l$ |

Table 5: The relations for the operators $\hat{\sigma}_{ba}$ that hold only for odd $m$

| Relation                                                                 | Conditions |
|-------------------------------------------------------------------------|-------------|
| $\hat{\sigma}_{bl+1} = \hat{\sigma}_{bl} \hat{\sigma}_{l+1} - q^{-1} \hat{\sigma}_{l+1} \hat{\sigma}_{bl}$, | $e_b > e_l$ |
| $\hat{\sigma}_{bl} = q^{(\epsilon_a, \epsilon_b)} \hat{\sigma}_{bl+1} \hat{\sigma}_{l+1} - q^{-1} \hat{\sigma}_{l+1} \hat{\sigma}_{bl+1}$, | $e_b > 0$ |
| $\hat{\sigma}_{la} = \hat{\sigma}_{l+1} \hat{\sigma}_{l+1} \hat{\sigma}_{l+1} a - \hat{\sigma}_{l+1} a \hat{\sigma}_{l+1}$, | $e_a < 0$ |
| $\hat{\sigma}_{l+1} a = \hat{\sigma}_{l+1} \hat{\sigma}_{l+1} a - q^{-1} \hat{\sigma}_{l+1} a \hat{\sigma}_{l+1}$, | $e_a < -e_l$ |
| $q^{(\epsilon_a, \epsilon_b)} \hat{\sigma}_{ba} \hat{\sigma}_{l+1} - q^{-(\epsilon_a, \epsilon_b)} \hat{\sigma}_{l+1} \hat{\sigma}_{ba} = 0$, | $e_b > e_a; a \neq l, l+1; b \neq l+1, l$ |

References

[1] Baxter, R.J.: *Exactly solved models in statistical mechanics* London: Academic Press, 1982

[2] Jimbo M. (ed.): *Yang-Baxter equations in integrable systems* Singapore: World Scientific, 1990

[3] Foerster A. and Karowski M.: Algebraic properties of the Bethe ansatz for an $osp(2,1)$ supersymmetric $t - J$ model. Nucl. Phys. B **396**, 611–638 (1993)

[4] Essler, F.H.L. and Korepin, V.E.: Higher conservation-laws and algebraic Bethe ansätze for the supersymmetric $t - J$ model. Physical Review B **46**, 9147–9162 (1992)

[5] Essler, F.H.L., Korepin, V.E. and Schoutens, K.: Electronic model for superconductivity. Phys. Rev. Lett. **70**, 73–76 (1993)

[6] Foerster, A. and Karowski M.: The supersymmetric $t - J$ model with quantum group invariance. Nucl. Phys. B **408**, 512–534 (1993)

[7] Bracken, A.J., Gould, M.D., Links, J.R. and Zhang, Y.-Z.: New exactly solvable and supersymmetric model of correlated electrons. Phys. Rev. Lett. **74**, 2768–2771 (1995)

[8] Ramos, P.B. and Martins M.J.: One-parameter family of an integrable $sp(2/1)$ vertex model: Algebraic Bethe ansatz and ground state structure. Nucl. Phys. B **474**, 678–714 (1996)

[9] Gould, M.D., Links, J.R., Zhang, Y.-Z. and Tsohantjis, I.: Twisted quantum affine superalgebra $U_q[sl(2[2]^0)]$, $U_q[osp(2)]$ invariant $R$-matrices and a new integrable electronic model. J. Phys. A: Math. Gen. **30**, 4313–4325 (1997)

[10] Martins, M.J. and Ramos P.B.: Solution of a supersymmetric model of correlated electrons. Phys. Rev. B **56**, 6376–6379 (1997)

[11] Saleur H.: The continuum limit of $sl(N/K)$ integrable super spin chains. Nucl. Phys. B **578**, 552–576 (2000)

[12] Salier, H. and Wehefritz-Kaufmann, B.: Integrable quantum field theories with $OSP(m/2n)$ symmetries. Nucl. Phys. B **628**, 407–441 (2002)
[13] Saluer, H. and Wehefritz-Kaufmann, B.: Integrable quantum field theories with supergroup symmetries: the OSP(1/2) case. Nucl. Phys. B 663, 443-466 (2003)

[14] Drinfeld, V.G.: Quantum groups. In: Proc. Int. Congress of Mathematicians 1,2, 798–820 Providence, RI: Amer. Math. Soc., 1987

[15] Gould, M.D., Zhang, R.B. and Bracken A.J.: Quantum double construction for graded Hopf-algebras. Bull. Aust. Math. Soc. 47, 353–375 (1993)

[16] Yamane, H.: On Defining Relations of the affine Lie superalgebras and their quantized universal enveloping superalgebras. Publ. Res. Inst. Math. Sci. 35, 321–390 (1999)

[17] Delius, G.W., Gould, M.D., Links, J.R. and Zhang, Y.-Z.: On type I quantum affine superalgebras. Int. J. Mod. Phys. A 10, 3259–3282 (1995)

[18] Gould, M.D. and Zhang, Y.-Z.: Twisted Quantum Affine Superalgebra $U_q[gl(m|n)^{(2)}]$ and New $U_q[osp(m|n)]$ Invariant R-matrices. Nucl. Phys. B 566, 529–546 (2000)

[19] Jimbo, M.: A $q$-analog of $U(GL(N + 1))$, Hecke algebra, and the Yang-Baxter equation. Lett. Math. Phys. 11, 247–252 (1986)

[20] Zhang, R.B.: Universal $L$ operator and invariants of the quantum supergroup $U_q(gl(m/n))$. J. Math. Phys. 33, 1970–1979 (1992)

[21] Zhang, R.B.: Finite dimensional representations of $U_q(osp(1/2n))$ and its connection with quantum so(2n + 1). Lett. Math. Phys. 25, 317–325 (1992)

[22] Dancer, K.A., Gould, M.D. and Links, J: Lax Operator for the Quantiﬁed Orthosymplectic Superalgebra $U_q[osp(2|n)]$. In preparation.

[23] Dancer, K.A., Gould, M.D. and Links, J: Eigenvalues of Casimir Invariants for $U_q[osp(m|n)]$. In preparation.

[24] Gould, M.D. and Zhang, Y.-Z.: Quasispin graded-fermion formalism and $gl(m|n) \downarrow osp(m|n)$ branching rules. J. Math. Phys. 40, 5371–5386 (1999)

[25] Serre, J.-P.: Algèbres de Lie semi-simples complexes New York: W.A. Benjamin, 1966

[26] Sweedler, M.E. Hopf algebras (W.A. Benjamin, New York 1969)

[27] Scheunert, M.: The $R$-matrix of the symplectic-orthogonal quantum superalgebra $U_q(spo(2n|2m))$ in the vector representation preprint [math.QA/0004032]

[28] Gallegos, W. and Martins, M. J.: $R$-matrices and spectrum of vertex models based on superalgebras. Nucl. Phys. B 699, 455-486 (2004)

[29] Mehta, M.: New Solutions of the Yang–Baxter Equation Associated with Quantised Orthosymplectic Lie Superalgebras PhD Thesis, The University of Queensland, 2003

[30] Dancer, K.A.: Solutions to the Yang-Baxter Equation and Casimir Invariants for the Quantised Orthosymplectic Superalgebra PhD Thesis, The University of Queensland 2005

[31] Links, J.R. and Gould, M.D.: Classification of unitary and grade star irreps for $U_q(osp(2|2n))$. J. Math. Phys. 36, 531–545 (1995)