Local Estimation of a Multivariate Density and its Derivatives

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Abstract

We present methods for estimating the multivariate probability density (or the log-density) and its first and second order derivatives simultaneously. Two methods, local log-likelihood and Hyvärinen score estimation, are in terms of weighted scoring rules with local polynomials. A third approach is matching of local moments. Consistency and asymptotic convergence results are shown and compared with corresponding results for kernel density estimators.

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1 Introduction

Suppose that \( f : \mathbb{R}^d \to \mathbb{R} \) is a probability density function and the random sample \( X_1, X_2, \ldots, X_n \) is drawn from a distribution with density function \( f \). Our goal is to estimate \( f \) or \( \log f \) and its derivatives of orders one and two. The four different estimators considered here are all based on a kernel function \( K \) and a
bandwidth parameter $h$. One estimator we consider is the well known kernel density estimator (KDE) for $f$
defined as

$$ \hat{f}_{n,KDE}(x) := \frac{1}{nh} \sum_{i=1}^{n} K_h(x - X_i) $$

with $K_h(z) := h^{-d}K(z/h)$; see Section 3.6. The derivatives of $f$ are estimated by taking the corresponding
derivatives of $\hat{f}_{n,KDE}(x)$; see also Chacón et al. (2011).

The other three estimators are functionals of local sample moments, i.e. of

$$ s_{n,h}^\alpha(x) := \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^\alpha, $$

where $\alpha$ is a $d$-dimensional multi-index. With a slightly different normalization, Müller and Yan (2001)
studied local sample moments in the special case of a uniform kernel function. We discuss local moments
more generally in Section 3.1.

The local moment matching estimator (LMME) in Section 3.5 matches the empirical local moments with
the Taylor expansion of their theoretical counterparts. This rather simple approach leads to a system of
linear equations and the asymptotic properties are interesting for comparison.

The other two estimators are both based on proper scoring rules which are localized by a kernel function;
see Gneiting and Raftery (2007), Holzmann and Klar (2017) and Section 3.2. We consider the well-known
logarithmic score (Good (1952)) as well as the Hyvärinen score (Hyvärinen (2005, 2007)). Minimizing those
empirical localized scores with respect to the class of functions whose logarithm is a polynomial of degree two,
leads to a proportionally locally proper solution; see Sections 3.3, 3.4 and the paper by Holzmann and Klar
(2017). In simple words, the solution in the neighborhood of the considered point is close to the underlying
density up to a factor, hence the shape of the estimated density is similar to the true one.

The local log-likelihood estimator (LLLE) in Section 3.3 is based on the logarithmic score. This estimator
was already considered by Loader (1996) and he proved an asymptotic result in the case of kernel functions
with compact support. In general, the estimator can only be calculated by numerical integration.

The local Hyvärinen score estimator (LHSE) in Section 3.4 is based on the Hyvärinen score. The estimator
is the solution of a system of linear equations. In case of the Gaussian kernel this method coincides with the
LLLE; see Section 4.1.

The KDE and LMME enable estimating the density and derivatives to any desired order, whereas the
LHSE only allows estimating the first and second order derivatives of the log-density and the LLLE allows
for estimating the log-density and the derivatives thereof of order one and two.

The manuscript is structured as follows. Notation and assumptions are given in Section 2. Section
3 starts with some general results on the sample local moments followed by explaining how to obtain a
proportionally locally proper weighted scoring rule from a proper scoring rule. After that the different
estimators are explained in more detail, and consistency and asymptotic normality results are shown. In
Section 4, we compare the estimators using the Gaussian kernel. All proofs and some technical results are
given in the Appendix.

2 Notation and Assumptions

Let $\gamma = (\gamma_1, \ldots, \gamma_d) \in \mathbb{N}_0^d$ be a $d$-dimensional multi-index with $|\gamma| := \sum_{j=1}^{d} \gamma_j$, $\gamma! := \prod_{j=1}^{d} \gamma_j!$, and for
$y \in \mathbb{R}^d$ let $y^{\gamma} := \prod_{j=1}^{d} y_j^{\gamma_j}$ be the $\gamma$-power of $y$. We define $\gamma^- := \sum_{j=1}^{d} \mathbb{1}\{\gamma_j \text{ is odd}\} e_j$ and $\gamma^+ := \gamma + \gamma^-$,
where $e_1, \ldots, e_d$ denotes the standard basis of $\mathbb{R}^d$, i.e. all components of $\gamma^+$ are rounded up to an even
number. This notation will be useful for the Taylor expansions of the local moments.

For a function $f : \mathbb{R}^d \to \mathbb{R}$ we write $f \in C^L(\mathbb{R}^d)$ if all partial derivatives up to order $L$ of $f$ exist and are
continuous, and we define

$$ \|f\|_{\infty,L} := \max_{|\gamma|=L} \sup_{z \in \mathbb{R}^d} |f^{(\gamma)}(z)|, \text{ where } f^{(\gamma)}(z) := \frac{\partial^{\gamma_1}}{\partial z_1^{\gamma_1}} \cdots \frac{\partial^{\gamma_d}}{\partial z_d^{\gamma_d}} f(z). $$

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Thus, \( \|f\|_{\infty,L} < \infty \) means that all partial derivatives of order \( L \) are uniformly bounded. Furthermore, we define
\[
\|f\|_{\infty,L} = \max_{0 \leq j \leq L} \|f\|_{\infty,j}.
\]
Thus, all partial derivatives up to order \( L \) are uniformly bounded. For two matrices \( A = (a_{ij})_{i,j=1}^{p,q} \in \mathbb{R}^{p \times q} \), \( B \in \mathbb{R}^{r \times s} \) we define the Kronecker product by
\[
A \otimes B := \begin{pmatrix} a_{11}B & \cdots & a_{1q}B \\ \vdots & \ddots & \vdots \\ a_{p1}B & \cdots & a_{pq}B \end{pmatrix} \in \mathbb{R}^{pr \times qs};
\]
see Magnus and Neudecker (1999), and for \( y \in \mathbb{R}^{d} \) and \( r \in \mathbb{N}_{0} \) we write
\[
y^{\otimes r} := y \otimes \cdots \otimes y \in \mathbb{R}^{d^r}
\]
for the \( r \)-th Kronecker power of \( y \) with the convention \( y^{\otimes 1} = y \) and \( y^{\otimes 0} = 1 \). The entries of the \( r \)-th Kronecker power are \( y^{\gamma} \) with \( |\gamma| = r \). Applying the Kronecker power to the operator \( D = \partial/\partial x = (\partial/\partial x_1, \partial/\partial x_2, \ldots, \partial/\partial x_d)^{\top} \) leads to
\[
D^{\otimes r}f(x) := \frac{\partial^{r} f}{(\partial x)^{\otimes r}}(x) \in \mathbb{R}^{d^r}.
\]
This vector contains all partial derivatives of order \( r \) of \( f \) at \( x \). We refrain from reducing this redundancy due to better tractability. The \text{vec}() and \text{mat}() operator are used to transform matrizes into vectors and vice versa. Let \( A = [A_1, A_2, \ldots, A_s] \in \mathbb{R}^{s \times s} \) be a matrix with columns \( A_1, A_2, \ldots, A_s \in \mathbb{R}^{s} \) and \( y \in \mathbb{R}^{s^2} \) be a vector, then
\[
\text{vec}(A) := \begin{pmatrix} A_1 \\ A_2 \\ \vdots \\ A_d \end{pmatrix} \quad \text{and} \quad \text{mat}(y) := \begin{pmatrix} y_1 & y_{s+1} & \cdots & y_{s(s-1)+1} \\ y_2 & y_{s+2} & \cdots & y_{s(s-1)+2} \\ \vdots & \vdots & \ddots & \vdots \\ y_s & y_{2s} & \cdots & y_{s^2} \end{pmatrix}.
\]
For a symmetric matrix \( A = (a_{ij})_{i,j=1}^{s,s} \in \mathbb{R}^{s \times s} \) it is useful to consider the half-vectorization of the lower-triangular matrix
\[
\text{vech}(A) := (a_{11}, a_{21}, \ldots, a_{d1}, a_{22}, a_{32}, \ldots, a_{d2}, a_{33}, a_{43}, \ldots, a_{d,d-1}, a_{dd})^{\top} \in \mathbb{R}^{\frac{d(d+1)}{2}}.
\]
We denote for \( r \in \mathbb{N}_{0} \)
\[
\Omega^{r} := \begin{pmatrix} \Omega^{\otimes 0} \\ \Omega^{\otimes 1} \\ \vdots \\ \Omega^{\otimes r} \end{pmatrix} \in \mathbb{R}^{1+d+\ldots+d^r}, \quad \Omega^{r} := \begin{pmatrix} \Omega^{\otimes 1} \\ \Omega^{\otimes 2} \\ \vdots \\ \Omega^{\otimes r} \end{pmatrix} \in \mathbb{R}^{d+d^2+\ldots+d^r},
\]
where \( \Omega \) is either a vector in \( \mathbb{R}^{d} \) or the operator \( D \). This allows for writing several powers or orders of derivatives in one vector. For any function \( F : \mathbb{R}^{d} \to \mathbb{R}^{s} \) with components \( F_i \in \mathcal{L}^{2}(\mathbb{R}^{d}) \), define
\[
R(F) := \int F(z)F(z)^{\top} \, dz \in \mathbb{R}^{s \times s}. \tag{1}
\]
Suppose (K0) holds, we write for $h > 0$

$$B_h^r := \text{diag}(1, h 1_d, h^2 1_{d^2}, \ldots, h^r 1_{d^r}), \quad B_h^r := \text{diag}(1, h^2 1_{d^2}, \ldots, h^r 1_{d^r}) \quad \text{and} \quad B_h := \begin{bmatrix} I_d & 0 \\ 0 & h I_{d^2} \end{bmatrix},$$

where $\text{diag}(\cdot)$ is a quadratic matrix with the elements of the argument on the diagonal and zero otherwise, and $1_d$ denotes the $d$-dimensional vector with all entries equal to 1. Furthermore, we will use

$$J^r := \text{diag}(1, 1_d, (2!)^{-1} 1_{d^2}, \ldots, (r!)^{-1} 1_{d^r}) \quad \text{and} \quad J^{-} := \text{diag}(1_d, (2!)^{-1} 1_{d^2}, \ldots, (r!)^{-1} 1_{d^r}). \quad (2)$$

All considered estimators need certain assumptions on the data generating probability density function $f$, the kernel function $K$ and the bandwidth $h$. These assumptions are listed below.

**Assumption 1.** For a kernel function $K : \mathbb{R}^d \to \mathbb{R}_{\geq 0}$ with $U := \text{supp}(K) \subset \mathbb{R}^d$ open and convex, we define for all $\gamma \in \mathbb{N}_0^d$ such that $\int K(z)\|z\|^\gamma \, dz < \infty$,

$$\mu_\gamma(K) = \mu_\gamma = \int K(z) z^\gamma \, dz,$$

the $\gamma$-moment of $K$.

**K0** Suppose $K$ is bounded, symmetric and permutation invariant, that means

$$K(z_1, z_2, \ldots, z_d) = K(\xi_0 z_{\sigma(1)}, \xi_2 z_{\sigma(2)}, \ldots, \xi_d z_{\sigma(d)})$$

for all $z \in \mathbb{R}^d$, $\xi \in \{-1, 1\}^d$ and $\sigma \in S_d$, the set of permutations on $\{1, 2, \ldots, d\}$. Furthermore, $K$ is a probability density function and the corresponding covariance matrix is the identity, hence

$$\mu_0 := \mu_0 = 1 \quad \text{and} \quad \mu_2 := \mu_2 = 1.$$

**LLLE K1** Suppose (K0) holds and $\int K(z) \exp(\varepsilon \|z\|^2) \, dz < \infty$ for some $\varepsilon > 0$.

**LHSE K1** Suppose (K0) holds and $K \in C^2(U)$, $K(z) \exp(\varepsilon \|z\|^2) \to 0$, $\|DK(z)\| \exp(\varepsilon \|z\|^2) \to 0$ as $z \to \partial U$,

$$\int K(z) \exp(\varepsilon \|z\|^2) \, dz < \infty \quad \text{and} \quad \int \|DK(z)\| \exp(\varepsilon \|z\|^2) \, dz < \infty$$

for some $\varepsilon > 0$.

**LMME K1** Suppose (K0) holds and $\int K(z) \|z\|^4 \, dz < \infty$.

**LMME K2** Suppose (K0) holds and $\int K(z) \|z\|^5 \, dz < \infty$.

**LMME K3** Suppose (K0) holds and $\int K(z) \|z\|^6 \, dz < \infty$.

**KDE K1** Suppose (K0) holds, $K \in C^2(U)$, $\int K(z) \|z\|^4 \, dz < \infty$ and $\int \|D^2 K(z)\|^2 \, dz < \infty$.

**KDE K2** Suppose (K0) holds, $K \in C^2(U)$, $\int K(z) \|z\|^5 \, dz < \infty$ and $\int \|D^2 K(z)\|^{2+\delta} \, dz < \infty$ for some $\delta > 0$.

**KDE K3** Suppose (K0) holds, $K \in C^2(U)$, $\int K(z) \|z\|^6 \, dz < \infty$ and $\int \|D^2 K(z)\|^2 \, dz < \infty$.

(K0) implies $\mu_\gamma = 0$ if $\gamma$ has at least one odd component and $\mu_\gamma = \mu_\gamma$ if $\gamma$ is a permutation of $\gamma$. Therefore, we neglect zeros in the notation, e.g. for $\mu_{20} = \mu_{02}$ we may write $\mu_2$. Recall that $K_h(z) := h^{-d} K(z/h)$ and hence

$$D^{\otimes r} K_h(z) = h^{-r} (D^{\otimes r} K)_h(z) = h^{-(d+r)} D^{\otimes r} K \left( \frac{z}{h} \right),$$

where

$$(D^{\otimes r} K)_h(z) := h^{-d} (D^{\otimes r} K) \left( \frac{z}{h} \right),$$

whenever $K \in C^r(U)$. 

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Assumption 2. Let \( f \) be the data generating density function.

(A0) The data generating density function \( f : \mathbb{R}^d \rightarrow \mathbb{R}_{>0} \) fulfills \( f \in C^2(\mathbb{R}^d) \), \( \| f \|_{\infty} < \infty \) and \( f(x) > 0 \) for all \( x \in \mathbb{R}^d \).

(A1) Suppose (A0) holds and \( \| f \|_{\infty,2} < \infty \).

(A2) Suppose (A0) holds and \( f \in C^3(\mathbb{R}^d) \) with \( \| f \|_{\infty,3} < \infty \).

(A3) Suppose (A0) holds and \( f \in C^4(\mathbb{R}^d) \) with \( \| f \|_{\infty,4} < \infty \).

(LLLE A1) Suppose (A0) holds and \( f \in C^3(\mathbb{R}^d) \) with \( \| \log f \|_{\infty,3} < \infty \).

(LLLE A2) Suppose (A0), (A1) hold and \( f \in C^4(\mathbb{R}^d) \) with \( \| \log f \|_{\infty,4} < \infty \).

Note that for the probability density function \( f \in C^L(\mathbb{R}^d) \) the condition \( \| f \|_{\infty,L} < \infty \) implies \( f^{(\gamma)}(x) \to 0 \) as \( \| x \| \to \infty \) for any multi-index \( \gamma \) with \( |\gamma| < L \).

In practice we observe data \( X_1, X_2, \ldots, X_n \) for a fixed number \( n \) and choose a suitably small bandwidth \( h > 0 \). For the asymptotic results we assume \( X_1, X_2, X_3, \ldots \) as a sequence of independent random vectors in \( \mathbb{R}^d \) drawn from \( f \) and let \( h \) tend to zero as \( n \) goes to infinity. The order of magnitude of \( h \) for the estimators should be such that \( nh^{d+1} \to \infty \) to ensure consistency and \( nh^d \to \infty \), \( nh^{d+6} \to \lambda^2 \) for some \( \lambda \geq 0 \) to ensure asymptotic normality. The mean of the asymptotic law is equal to zero if \( \lambda \) is equal to zero. Convergence in probability is denoted as \( \stackrel{p}{\rightarrow} \) and convergence in distribution as \( \stackrel{d}{\rightarrow} \).

The LHSE and LLLE estimate the log-density and its derivatives, whereas the KDE and LMME estimate the density and its derivatives. For comparability of the estimators we use a transformation of the latter. Let \( \phi : \mathbb{R}_{>0} \times \mathbb{R}^d \times \mathbb{R}^{d^2} \rightarrow \mathbb{R} \times \mathbb{R}^d \times \mathbb{R}^{d^2} \) be the mapping

\[
\xi = \begin{pmatrix}
\xi_0 \\
\xi_1 \\
\xi_2
\end{pmatrix} \mapsto \begin{pmatrix}
\log \xi_0 \\
\xi_1 \\
\xi_2 - \xi_0 \xi_2 \xi_0 \\
\xi_0
\end{pmatrix},
\]

then

\[
\phi(D^2 f(x)) = D^2 \log f(x)
\]

with Jacobian matrix

\[
J_\phi(\xi) = \begin{bmatrix}
1/\xi_0 & 0_{1 \times d} & 0_{1 \times d^2} \\
-\xi_1 & I_d/\xi_0 & 0_{d \times d^2} \\
\xi_1 & -I_d \otimes I_d \otimes I_d \otimes I_d/\xi_0 & I_{d^2}/\xi_0
\end{bmatrix}
\]

and let \( \tilde{\phi} : \mathbb{R}_{>0} \times \mathbb{R}^d \times \mathbb{R}^{d^2} \rightarrow \mathbb{R}^d \times \mathbb{R}^{d^2} \) be the mapping

\[
\xi = \begin{pmatrix}
\xi_0 \\
\xi_1
\end{pmatrix} \mapsto \begin{pmatrix}
\xi_1 \\
\xi_2 - \xi_0 \xi_2 \xi_0 \\
\xi_0
\end{pmatrix},
\]

then

\[
\phi(D^2 f(x)) = D^2 \log f(x)
\]

with Jacobian matrix

\[
J_{\tilde{\phi}}(\xi) = \begin{bmatrix}
I_d/\xi_0 & 0_{d \times d^2} \\
-I_d \otimes I_d \otimes I_d \otimes I_d/\xi_0 & I_{d^2}/\xi_0
\end{bmatrix}.
\]
3 Local Density Derivative Estimation

3.1 Local Moments

The concept of local moments appears first in Müller and Yan (2001). However, they use a slightly different definition. They localize by choosing a rectangular window rather than a general kernel function. In this subsection, we first present our approach and give some connections to the approach in Müller and Yan (2001) afterwards.

**Definition 3.1.** The local sample $\alpha$-moment at $x$ for a multi-index $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ is

$$s_{n,h}^\alpha(x) := s_n^\alpha(x) := \frac{1}{n} \sum_{i=1}^n K_h(x_i - x) \left( \frac{X_i - x}{h} \right)^\alpha.$$  

The vector version is

$$S_{n,h}^\Xi(x) := S_n^\Xi(x) := \frac{1}{n} \sum_{i=1}^n K_h(x_i - x) \left( \frac{X_i - x}{h} \right)^\Xi,$$

where $\Xi$ is a placeholder for either $\otimes r$, $\tau$ or $r$ with $r \in \mathbb{N}$. The first choice corresponds to the vector containing all local sample moments of order $r$, the other two choices correspond to the vector with all local sample moments up to order $r$, where in the last one the 0-moment is excluded. Furthermore, let

$$s_n := s_n^0(x), \quad s_n := S_n^1(x) \quad \text{and} \quad S_n := \text{mat}(S_n^2(x)).$$

**Definition 3.2.** The local moments are the expectations of the local sample moments and denoted by suppressing the subscript $n$. Hence, the local $\alpha$-moment at $x$ for a multi-index $\alpha \in \mathbb{N}_0^d$ and $x \in \mathbb{R}^d$ is

$$s_h^\alpha(x) := \mathbb{E}(s_{n,h}^\alpha(x)) = \int K(z)z^\alpha f(x + hz) \, dz$$

and the vector version is

$$S_h^\Xi(x) := \mathbb{E}(S_{n,h}^\Xi(x)) = \int K(z)z^\Xi f(x + hz) \, dz.$$

Furthermore, we write

$$s_h := \mathbb{E}(s_n) = s_h^0(x), \quad s_h := \mathbb{E}(s_n) = S_h^1(x) \quad \text{and} \quad S_h := \mathbb{E}(S_n) = \text{mat}(S_h^2(x)).$$

The first lemma reveals the structure of the local moments and hints at their usefulness for estimating density derivatives.

**Lemma 3.3.** Assume $f \in \mathcal{C}^L(\mathbb{R}^d)$ with $\|f\|_{\infty,L} < \infty$. For a function $F : \mathbb{R}^d \to \mathbb{R}$ with $\int |F(z)||z|^L \, dz < \infty$ we have

$$\int F(z)f(x + hz) \, dz = \sum_{|\gamma| \leq L} \frac{h^{|\gamma|}}{\gamma!} f^{(\gamma)}(x) \int F(z)z^\gamma \, dz + o(h^L)$$

as $h \to 0$. In particular, if $\int K(z)||z||^{\alpha + L} \, dz < \infty$, then, for $x \in \mathbb{R}^d$ and as $h \to 0$, we have

$$s_h^\alpha(x) = \sum_{|\gamma| \leq L} h^{|\gamma|} \frac{\mu_{\alpha + \gamma}}{\gamma!} f^{(\gamma)}(x) + o(h^L).$$

Consequently, for $L = |\alpha^-| + 2$ is

$$s_h^\alpha(x) = h^{\alpha^-} f^{(\alpha^-)}(x) \mu_{\alpha^+} + \sum_{j=1}^d \frac{h^{\alpha^- + 2e_j}}{(\alpha^- + 2e_j)!} f^{(\alpha^- + 2e_j)}(x) \mu_{\alpha^+ + 2e_j} + o(h^L).$$

The next results are about the asymptotic behavior of the local sample moments.
Theorem 3.4. Assume $f \in C(\mathbb{R}^d)$ and $L \in \mathbb{N}_0$. For a multi-index $\alpha$ such that $\int K(z)\|z\|^{2|\alpha|}dz < \infty$, we have
\[ h^{-L}\mathbb{E}|s_{n,h}^\alpha(x) - s_h^\alpha(x)| \rightarrow 0 \]
as $n \rightarrow \infty$, $h \rightarrow 0$ and $nh^{d+2L} \rightarrow \infty$. Assume furthermore that $f \in C(\alpha^-)(\mathbb{R}^d)$ and $\|f\|_{\infty,|\alpha^-|} < \infty$, then
\[ \mathbb{E}\left| \frac{s_{n,h}^\alpha(x)}{h^{\alpha^-}} - \frac{\mu_{\alpha^+}}{(\alpha^-)!} f^{(\alpha^-)}(x) \right| \rightarrow 0 \]
as $n \rightarrow \infty$, $h \rightarrow 0$ and $nh^{d+2|\alpha^-|} \rightarrow \infty$.

Theorem 3.5. Let $f \in C(\mathbb{R}^d)$ and $r \in \mathbb{N}$ with
\[ \int K(z)\|z\|^{2r}dz < \infty. \]
Assume that $h \rightarrow 0$ as $n \rightarrow \infty$, then
\[ \text{Var}(\sqrt{n}h^dS_n(x)) \rightarrow f(x)R(K(\cdot)^r), \]
where
\[ R(K(\cdot)^r) = \int K(z)z^r(z^r)^\top dz \]
as defined in (1). If for some number $\delta > 0$ such that
\[ \int K(z)\|z\|^{2r+\delta}dz < \infty \]
and $h \rightarrow 0$, $nh^d \rightarrow \infty$ as $n \rightarrow \infty$, then
\[ \sqrt{n}h^d(S_{n,h}^\alpha(x) - S_h^\alpha(x)) \xrightarrow{d} N(0, f(x)R(K(\cdot)^\alpha)) \]

Corollary 3.6. Assume that $f \in C(\mathbb{R}^d)$ and $\alpha \in \mathbb{N}_0^d$ is a multi-index, such that $\int K(z)\|z\|^{2|\alpha|+\delta}dz < \infty$ for some $\delta > 0$. Then,
\[ \sqrt{n}h^d(s_{n,h}^\alpha(x) - s_h^\alpha(x)) \xrightarrow{d} N(0, f(x)R(K(\cdot)^\alpha)) \]
as $n \rightarrow \infty$, $h \rightarrow 0$ and $nh^d \rightarrow \infty$. In addition, if $f \in C^L(\mathbb{R}^d), \|f\|_{\infty,L} < \infty$ for $L = |\alpha^-| + 2$ and $nh^{d+2L} \rightarrow \lambda^2$ for some constant $\lambda \geq 0$ as $n \rightarrow \infty$, then,
\[ \sqrt{n}h^d\left(s_{n,h}^\alpha(x) - s_h^\alpha(x)\right)f^{(\alpha^-)}(x)\mu_{\alpha^+} \xrightarrow{d} \mathcal{N}\left(\lambda \sum_{j=1}^d f^{(\alpha^-)+2e_j}(x)\frac{\mu_{\alpha^+}+2e_j}{(\alpha^-+2e_j)!}, f(x)R(K(\cdot)^\alpha)\right) \]

Müller and Yan (2001) use a more restrictive definition of local moments. Namely,
\[ m^\alpha(x) := \lim_{h \rightarrow 0} \frac{1}{h^{\alpha^+}} \mathbb{E}(X - z)^\alpha | X \in U_h(x)) = \lim_{h \rightarrow 0} \frac{1}{h^{\alpha^+}} \int \mathbb{1}\{y \in U_h(x)\}(y-x)^\alpha f(y)dz, \]
where $U_h(x) = \prod_{j=1}^d[x_j - h, x_j + h]$ is a cubic window around $x$ with length $2h$. The local sample moment is
\[ m_{n,h}^\alpha(x) := \sum_{i=1}^n \mathbb{1}\{X_i \in U_h(x)\}(X_i - x)^\alpha, \]
This corresponds to the kernel function $K(z) := \mathbb{1}\{z_j \in [-1, 1] : 1 \leq j \leq d\}$ and we may write
\[ m^\alpha(x) = \lim_{h \rightarrow 0} \frac{h^{\alpha}}{h^{\alpha^+}} \int K(z)x^\alpha f(x + hz)dz = \lim_{h \rightarrow 0} \frac{s_n^\alpha(x)}{s_h^\alpha(x)} = \frac{\mu_{\alpha^+} f^{(\alpha^-)}(x)}{f(x)} \]
and 
\[ m_{\alpha,h}(x) = \frac{\sum_{i=1}^n K_h(X_i - x)(X_i - x)^\alpha}{\sum_{i=1}^n K_h(X_i - x)} = \frac{s_{\alpha,h}(x)}{h^\alpha} \]

Note that for the kernel \( K \) in Müller and Yan (2001), we have \( \mu_0 = 2d \) and \( \mu_2 = 2d/3 \) but the kernel is permutation invariant and symmetric in all components and Lemma 3.3 remains valid. The local moment \( m_{\alpha}(x) \) in (7) generalizes Müller and Yan (2001, Theorem 2.1) to different kernels and Corollary 3.6 generalizes the asymptotic law stated in Müller and Yan (2001, Theorem 3.1).

### 3.2 Weighted Scoring Rules with Local Polynomials

Let \( \mathcal{P} \) be a class of absolutely continuous probability measures on \( \mathbb{R}^d \) represented by their density functions. A **scoring rule** is a map \( S : \mathcal{P} \times \mathbb{R}^d \to \mathbb{R} \) such that \( S(q,p) := \int S(q,y)p(y)dy \) exists for all \( p, q \in \mathcal{P} \) with \( S(p,p) < \infty \). \( S \) is (strictly) proper with respect to \( \mathcal{P} \) if for any \( p \in \mathcal{P} \),

\[ S(p,p) \leq S(q,p) \]

for all \( q \in \mathcal{P} \) (with equality if and only if \( q = p \)).

Let \( \mathcal{W} \) be a class of weight functions, that is, it consists of bounded, non-negative, measurable functions on \( \mathbb{R}^d \) and let \( \mathcal{H} \) be a class of non-negative, continuous functions on \( \mathbb{R}^d \). A **weighted scoring rule** is a map \( \tilde{S} : \mathcal{H} \times \mathbb{R}^d \times \mathcal{W} \to \mathbb{R} \) such that, for each \( w \in \mathcal{W} \), \( \tilde{S}(h,p,w) := \int \tilde{S}(h,y,w)p(y)dy \) exists for all \( h \in \mathcal{H} \), \( p \in \mathcal{P} \).

A weighted scoring rule \( \tilde{S} \) is **localizing** if, for each \( w \in \mathcal{W} \),

\[ \tilde{S}(h,y,w) = \tilde{S}(g,y,w) \quad \text{for almost all } y \in \mathbb{R}^d \text{ if } h = g \text{ on } \{ w > 0 \}. \]

This implies for each \( w \in \mathcal{W} \) and \( p \in \mathcal{P} \),

\[ \tilde{S}(g,p,w) = \tilde{S}(h,p,w) \quad \text{whenever } g, h \in \mathcal{H} \text{ and } g = h \text{ on } \{ w > 0 \}. \]

A localizing weighted scoring rule is called **locally proper** with respect to \( (\mathcal{H}, \mathcal{P}) \) if, for each \( w \in \mathcal{W} \) and \( p \in \mathcal{P} \),

\[ \tilde{S}(g,p,w) \leq \tilde{S}(h,p,w) \quad \text{whenever } g, h \in \mathcal{H} \text{ and } g = p \text{ on } \{ w > 0 \}. \]

A locally proper weighted scoring rule \( \tilde{S} \) is called **strictly locally proper** with respect to \( (\mathcal{H}, \mathcal{P}) \) if it is locally proper with respect to \( (\mathcal{H}, \mathcal{P}) \) and for each \( w \in \mathcal{W} \) and \( p \in \mathcal{P} \),

\[ \tilde{S}(g,p,w) = \tilde{S}(h,p,w) \quad \text{whenever } g, h \in \mathcal{H} \text{ and } g = p \text{ on } \{ w > 0 \} \quad \text{implies} \quad h = p \text{ on } \{ w > 0 \}. \]

A localizing weighted scoring rule is called **proportionally locally proper** with respect to \( (\mathcal{H}, \mathcal{P}) \) if, for each \( w \in \mathcal{W} \) and \( p \in \mathcal{P} \),

\[ \tilde{S}(g,p,w) \leq \tilde{S}(h,p,w) \quad \text{whenever } g, h \in \mathcal{H} \text{ and } g \propto p \text{ on } \{ w > 0 \} \quad \text{with equality, if and only if, } h \propto p \text{ on } \{ w > 0 \}, \]

where “\( \propto \)” denotes two proportional functions.

The above definitions are a modification of those in Holzmann and Klar (2017). Here, the values of the weight functions are not restricted to \([0,1]\), allowing for kernel functions with arbitrary small bandwidth \( h > 0 \) as weight functions. Furthermore, the first argument of a weighted scoring rule is not necessarily a probability density function. The idea is to look for a function that is identical or proportional to the underlying probability density function \( p \in \mathcal{P} \) only on the support of the weight function. So, there is no need that the function is a probability density function on \( \mathbb{R}^d \) or that it is integrable.

Holzmann and Klar (2017, Theorem 1) show how to turn proper scoring rules into locally proper scoring rules and strictly proper scoring rules into proportionally locally proper scoring rules. The latter can then be made strictly locally proper; see Holzmann and Klar (2017, Theorem 2). These statements hold also for the definitions given above and are stated in Theorems 3.7 and 3.8.
Theorem 3.7. For any weight function $w \in W$, we define

$$\mathcal{H}_w := \left\{ h \in \mathcal{H} : 0 < \int w(y) h(y) \, dy < \infty \right\} \quad \text{and} \quad \hat{\mathcal{H}}_w := \left\{ y \mapsto h_w(y) := \frac{w(y)h(y)}{\int w(z)h(z) \, dz} : h \in \mathcal{H}_w \right\}.$$  

Let $\mathcal{P}$ be such that $\hat{\mathcal{H}}_w \subseteq \mathcal{P}$ for all $w \in W$ and $S : \mathcal{P} \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}$ be a proper scoring rule with respect to $\mathcal{P}$. Then

$$\tilde{S} : \mathcal{H} \times \mathbb{R}^d \times W \to \mathbb{R} \cup \{\infty\}, \quad \tilde{S}(h, y, w) := \begin{cases} w(y)S(h_w, y), & \text{if } h \in \mathcal{H}_w, \\ \infty, & \text{if } h \notin \mathcal{H}_w, \end{cases}$$

is a locally proper scoring rule with respect to $(\mathcal{H}, \mathcal{P})$. Further, if $S$ is strictly proper with respect to $\mathcal{P}$, then $\tilde{S}$ is proportionally locally proper with respect to $(\mathcal{H}, \mathcal{P})$.

Remark. The assumptions on the scoring rule $S$ can be relaxed. It is sufficient to have strict propriety for restrictions of $S$ onto $\mathcal{H}_w$. Namely, if for each $w \in W$,

$$S|_{\mathcal{H}_w} : \mathcal{H}_w \times \mathbb{R}^d \to \mathbb{R} \cup \{\infty\}, \quad S|_{\mathcal{H}_w}(h, y) := S(h, y),$$

is a strictly proper scoring rule with respect to $\mathcal{H}_w$, then Theorem 3.7 continues to hold.

Theorem 3.8. Let $\mathcal{H}_w$ be as in Theorem 3.7 and $Q : \mathbb{R}_{>0} \times \{0, 1\} \to \mathbb{R} \cup \{\infty\}$ be a scoring rule for a binary outcome, such that for all $\beta \in (0, 1]$

$$Q(\beta, \beta) \leq Q(\alpha, \beta) \quad \text{for all } \alpha \in \mathbb{R}_{>0}, \quad \text{(with equality if and only if } \alpha = \beta),$$

where

$$Q(\alpha, \beta) := \beta Q(\alpha, 1) + (1 - \beta) Q(\alpha, 0).$$

Then, $S_Q : \mathcal{H} \times \mathbb{R}^d \times W \to \mathbb{R} \cup \{\infty\}$,

$$S_Q(h, y, w) := \begin{cases} w(y)Q\left(\frac{\int w(z)h(z) \, dz}{m_w}, 1\right) + (m_w - w(y))Q\left(\frac{\int w(z)h(z) \, dz}{m_w}, 0\right), & \text{if } h \in \mathcal{H}_w, \\ \infty, & \text{if } h \notin \mathcal{H}_w, \end{cases}$$

is locally proper with respect to $(\mathcal{H}, \mathcal{P})$, where $m_w := \sup_{y \in \mathbb{R}^d} w(y)$. Furthermore, if $\tilde{S}$ is proportionally locally proper with respect to $(\mathcal{H}, \mathcal{P})$, then

$$\tilde{S}(h, y, w) := S_Q(h, y, w) + \tilde{S}(h, y, w)$$

is strictly locally proper with respect to $(\mathcal{H}, \mathcal{P})$.

Remark. There are other possible choices for $m_w$. It is only needed, that for each $w \in W$ and $p \in \mathcal{P}$

$$\int w(z)p(z) \, dz \leq m_w.$$  

This ensures that for all $h \in \mathcal{H}$, such that there exists $p \in \mathcal{P}$ with $h = p$ on $\{w > 0\}$, we have

$$\int w(z)h(z) \, dz < \infty.$$  

The scoring rule $Q$ in Theorem 3.8 is similar to a strictly proper scoring rule for binary events. The only difference is, that $\alpha$ is not restricted to $(0, 1)$. Therefore $S_Q$ can be defined for all $h \in \mathcal{H}_w$. 

9
Suppose \( X_1, X_2, \ldots, X_n \) is a random sample drawn from a distribution with density function \( f \in \mathcal{P} \). Let \( \hat{S} \) be a strictly/proportionally locally proper scoring rule with respect to \((\mathcal{H}, \mathcal{P})\). For some weight function \( w \in \mathcal{W} \), we are interested in the function \( h \in \mathcal{H} \) with minimal score, so

\[
  h \in \arg\min_{g \in \mathcal{H}} \hat{S}(g, f, w) = \arg\min_{g \in \mathcal{H}} \mathbb{E}\hat{S}(g, X, w),
\]

where \( X \) is a random vector with distribution according to the density function \( f \). Since \( f \) is unknown, we replace the distribution of \( X \) with the empirical distribution of \( X_1, X_2, \ldots, X_n \). Thus, an estimator for \( h \in \mathcal{H} \) is

\[
  \hat{h} \in \arg\min_{g \in \mathcal{H}} \frac{1}{n} \sum_{i=1}^{n} \hat{S}(g, X_i, w).
\]

Since we are interested in the first and second order derivatives of the log-density at \( x \in \mathbb{R}^d \) we use the Taylor approximation for the true log-density \( \log f \) at \( x \) and write

\[
  \log f(y) = \log f(x) + \nabla \log f(x)^\top (y-x) + \frac{1}{2} (y-x)^\top H_{\log f}(x) (y-x) + o(\|y-x\|^2)
\]

as \( y \to x \). Thus, in a small neighborhood of \( x \), \( f \) may be approximated by a function \( f_{c,b,A}(\cdot - x) \) with

\[
  f_{c,b,A} \in \mathcal{F} := \left\{ y \mapsto \exp\left( c + b^\top y + \frac{1}{2} y^\top A y \right) : c \in \mathbb{R}, b \in \mathbb{R}^d, A \in \mathbb{R}^{d \times d} \right\}.
\]

The parameters \( c, b \) and \( A \) correspond to the log-density, its gradient and Hessian at \( x \) of \( f \), respectively. The functions in \( \mathcal{F} \) are generally not probability density functions and they are unbounded whenever \( A \) has a positive eigenvalue. For the class of weight functions we choose

\[
  \mathcal{W} := \{ K_h \}_{h>0}
\]

for a kernel \( K \) with properties given in Assumption 1; see Section 2. We identify each \( w \in \mathcal{W} \) with its bandwidth \( h > 0 \) and define, for each \( h > 0 \),

\[
  \mathcal{F}_h := \left\{ f_{c,b,A} \in \mathcal{F} : \int K_h(z) f_{c,b,A}(z) \, dz < \infty \right\}
\]

and

\[
  \hat{\mathcal{F}}_h = \left\{ y \mapsto \hat{f}_{b,A,h}(y) := \frac{K_h(y) f_{c,b,A}(y)}{\int K_h(z) f_{c,b,A}(z) \, dz} : f_{c,b,A} \in \mathcal{F} \right\}.
\]

With

\[
  \hat{f}_{b,A,h}(y) = \frac{K_h(y) \exp\left( b^\top y + \frac{1}{2} y^\top A y \right)}{\int K_h(z) \exp\left( b^\top z + \frac{1}{2} z^\top A z \right) \, dz}
\]

in \( \hat{\mathcal{F}}_h \) being a probability density functions, no longer depending on the parameter \( c \) of the function class \( \mathcal{F}_h \). In some situations it may be more convenient to use a different parametrization. Instead of parametrizing the gradient as a vector and the Hessian as a matrix separately, we parametrize them altogether in a vector \( \theta \in \Theta \subset \mathbb{R}^{d+d^2} \), corresponding to \( D \log f(x) \), where

\[
  \Theta := \left\{ \begin{pmatrix} \theta_1 \\ \theta_2 \end{pmatrix} \in \mathbb{R}^d \times \mathbb{R}^{d^2} : \text{mat}(\theta_2) = \text{mat}(\theta_2)^\top \right\}.
\]

For \( \theta_1 = b \) and \( \theta_2 = \text{vec}(A) \), the functions in \( \mathcal{F} \) are in the vector parametrization given by

\[
  f_{\theta}(y) = \exp\left( c + \theta_1^\top y + \frac{1}{2} \theta_2^\top y \otimes 2 \right) = \exp\left( c + \theta^\top J^2 y^2 \right),
\]
where $J^2 = \text{diag}(1_d, 1_d/2)$ defined in (2) and, for each $h > 0$, the functions in $\tilde{F}_h$ are

$$
\tilde{f}_{\theta,h}(y) = \frac{K_h(y) \exp(\theta^\top J^2 y^2)}{\int K_h(z) \exp(\theta^\top J^2 z^2) \, dz}.
$$

Whenever we have a parametric class of (weight) functions we may use the parameters as arguments of the scoring rule instead of the functions. In case of $H = \mathcal{F}$ and $W = \{K_h\}_{h>0}$, we write

$$
\tilde{S}(c, b, A, y, h) \quad \text{or} \quad \tilde{S}(c, \theta, y, h),
$$

if $\tilde{S}$ is a strictly locally proper scoring rule with respect to $(\mathcal{F}, \mathcal{P})$, and

$$
\tilde{S}(b, A, y, h) \quad \text{or} \quad \tilde{S}(\theta, y, h),
$$

if $\tilde{S}$ is a proportionally locally proper scoring rule with respect to $(\mathcal{F}, \mathcal{P})$. Note that we neglect the parameter $c$, because a proportionally locally proper scoring rule is constant in $c$. Estimators for the log-density, its gradient and Hessian at $x$ are

$$(\hat{c}, \hat{b}, \hat{A}) \in \arg \min_{c \in \mathbb{R}, b \in \mathbb{R}^n, \vec{A} \in \mathbb{R}^d_{d \times d}} \frac{1}{n} \sum_{i=1}^n \tilde{S}(c, b, A_i, x - x, h) \quad \text{or} \quad (\hat{c}, \hat{\theta}) \in \arg \min_{c \in \mathbb{R}, \theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \tilde{S}(c, \theta, x_i - x, h),
$$

and estimators for the gradient and Hessian of the log-density at $x$ are

$$(\hat{b}, \hat{A}) \in \arg \min_{b \in \mathbb{R}^d, \vec{A} \in \mathbb{R}^d_{d \times d}} \frac{1}{n} \sum_{i=1}^n \tilde{S}(b, A_i, x_i - x, h) \quad \text{or} \quad \hat{\theta} \in \arg \min_{\theta \in \Theta} \frac{1}{n} \sum_{i=1}^n \tilde{S}(\theta, x_i - x, h).
$$

In context of the vector parametrization and the fact that we minimize over the space $\Theta$ and not $\mathbb{R}^{d+d^2}$ some notations and properties from matrix calculus are useful. Let $D_d \in \mathbb{R}^{d^2 \times \frac{d(d+1)}{2}}$ be the duplication matrix with the property

$$
D_d \text{vec}(A) = \text{vec}(A) \quad \text{for all} \quad A \in \mathbb{R}^{d \times d},
$$

where $\text{vec}(A) \in \mathbb{R}^{d^2}$ is the vector obtained from $\text{vec}(A)$ by eliminating all superdiagonal elements of $A$, and for $p = d + \frac{d(d+1)}{2}$. The extended duplication matrices are defined as

$$
E_d := \begin{bmatrix} I_d & 0 \\ 0 & D_d \end{bmatrix} \in \mathbb{R}^{d^2+d \times d^2} \quad \text{and} \quad \tilde{E}_d := \begin{bmatrix} I_{d+1} & 0 \\ 0 & D_d \end{bmatrix} \in \mathbb{R}^{1+d+d^2 \times d^2},
$$

with Moore-Penrose inverse denoted as $E^+_d$ and $\tilde{E}^+_d$, respectively; see Magnus and Neudecker (1999, Chapter 2). The properties of $D_d$ are discussed in Magnus and Neudecker (1999, Chapter 3) and translate easily to $E_d$ and $\tilde{E}_d$. Note that

$$
D^2 f(x) = E_d \begin{bmatrix} Df(x) \\ \text{vech(mat}(D^{\otimes 2} f(x)))] \end{bmatrix}, \quad D^2 \tilde{f}(x) = \tilde{E}_d \begin{bmatrix} f(x) \\ Df(x) \\ \text{vech(mat}(D^{\otimes 2} f(x)))] \end{bmatrix}
$$

and

$$
\begin{bmatrix} Df(x) \\ \text{vech(mat}(D^{\otimes 2} f(x)))] \end{bmatrix} = E^+_d D^2 f(x), \quad \begin{bmatrix} f(x) \\ Df(x) \\ \text{vech(mat}(D^{\otimes 2} f(x)))] \end{bmatrix} = \tilde{E}^+_d D^2 \tilde{f}(x).
$$

Further properties are stated in Lemma A.1.
3.3 Local Log-Likelihood Estimator

In this section we consider the negative log-likelihood function proposed by Good (1952) and given by

\[ S : \mathcal{P} \times \mathbb{R}^d \rightarrow \mathbb{R} \cup \{\infty\}, \quad S(p, y) = -\log p(y). \]

This is a strictly proper scoring rule with respect to any class of absolutely continuous probability measures \( \mathcal{P} \) and for any density function \( p \in \mathcal{P} \) it holds that

\[ -\int \log p(z) \cdot p(z) \, dz > -\infty. \]

If \( \mathcal{H} \) and \( \mathcal{W} \) are such that the conditions in Theorem 3.7 are fulfilled, then the weighted scoring rule

\[ \hat{S} : \mathcal{H} \times \mathbb{R}^d \times \mathcal{W} \rightarrow \mathbb{R} \cup \{\infty\}, \quad \hat{S}(h, y, w) := \begin{cases} -w(y) \log h_w(y), & \text{if } h \in \mathcal{H}_w, \\ \infty, & \text{if } h \notin \mathcal{H}_w, \end{cases} \]

is proportionally locally proper with respect to \((\mathcal{H}, \mathcal{P})\). Therefore, for each \( w \in \mathcal{W} \), we have

\[ \hat{S}(h, y, w) = -w(y) \log h(y) + w(y) \log \left( \int w(z)h(z) \, dz \right) - w(y) \log w(y), \]

which is infinite whenever \( h \notin \mathcal{H}_w \). It is surprising that even with evaluation of the integral \( \int w(z)h(z) \, dz \), the score is only proportionally locally proper. However, applying Theorem 3.8 with the scoring rule

\[ Q : \mathbb{R}^2 \rightarrow \mathbb{R}, \quad Q(\alpha, z) = -z(\log(\alpha) + 1) + \alpha \]

for binary events leads to a strictly locally proper scoring rule; see Holzmann and Klar (2017). For \( h \in \mathcal{H}_w \),

\[ S_Q(h, y, w) = -w(y) \log \left( \frac{\int w(z)h(z) \, dz}{m_w} \right) - w(y) + w(y) \int \frac{w(z)h(z) \, dz}{m_w} + (m_w - w(y)) \int \frac{w(z)h(z) \, dz}{m_w} \]

and for \( h \notin \mathcal{H}_w \), we have \( S_Q(h, y, w) = \infty \). Hence,

\[ \hat{S}(h, y, w) = -w(y) \log h(y) + \int w(z)h(z) \, dz - w(y) \left( 1 + \log \left( \frac{w(y)}{m_w} \right) \right) \]

is a strictly locally proper scoring rule with respect to \((\mathcal{H}, \mathcal{P})\). This score, without the term \(-w(y) \log (w(y)/m_w)\), is known as penalized weighted likelihood rule and studied in detail by Peleinis (2014).

We neglect all terms not depending on \( h \in \mathcal{H} \) and work with the equivalent score

\[ \hat{S}(h, y, w) = -w(y) \log h(y) + \int w(z)h(z) \, dz. \]

If we chose \( \mathcal{H} = \mathcal{F} \) defined in (8) and \( \mathcal{W} = \{K_h\}_{h>0} \) parameterized with \( c \in \mathbb{R}, b \in \mathbb{R}^d, A \in \mathbb{R}^{d \times d} \) or \( c \in \mathbb{R}, \theta \in \mathbb{R}^{d+d^2} \) and bandwidth \( h > 0 \), respectively, the conditions in Theorem 3.7 and 3.8 are fulfilled, whenever (LLLE K1) holds. This leads to the score

\[ \hat{S}(c, b, A, y, h) = -K_h(y) \left( c + b^\top y + \frac{1}{2} y^\top A y \right) + \int K(z) \exp \left( c + h b^\top z + \frac{h^2}{2} z^\top A z \right) \, dz \]

in the usual parametrization or

\[ \hat{S}(c, \theta, y, h) = -K_h(y) \left( c + \theta^\top J^2 y^2 \right) + \int K(z) \exp \left( c + \theta^\top J^2 (h z)^2 \right) \, dz \]
in the vector parametrization, respectively. In case of a random sample \( X_1, X_2, \ldots, X_n \) drawn from a distribution with density function \( f \in \mathcal{P} \), we minimize the empirical score

\[
\frac{1}{n} \sum_{i=1}^{n} \tilde{S}(c, b, A, X_i - x) = \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - x) (c + b^\top (X_i - x) + \frac{1}{2} (X_i - x)^\top A (X_i - x)) \\
+ \int K(z) \exp \left( c + h b^\top z + \frac{h^2}{2} z^\top A z \right) dz
\]

or

\[
\frac{1}{n} \sum_{i=1}^{n} \tilde{S}(c, \theta, X_i - x, h) = \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - x) \left( c + \theta^\top J^2(X_i - x) \theta \right) + \int K(z) \exp \left( c + \theta^\top J^2(h z) \theta \right) dz,
\]

to obtain an estimator for the log-density and its first and second order partial derivatives of \( f \) at \( x \). The functionals in (11), (12) are convex, so we set the gradient with respect to the parameters to zero and multiply it by the inverse of the matrix \( B_\theta^2 J^2 \). Thus, the minimizers of (11) or (12) are solutions of the equation systems

\[
\frac{1}{n} \sum_{i=1}^{n} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^\top = \int K(z) z^\top \exp \left( c + h b^\top z + \frac{h^2}{2} z^\top A z \right) dz
\]

for the usual parametrization, and

\[
\frac{1}{n} \sum_{i=1}^{n} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^\top = \int K(z) z^\top \exp \left( c + \theta^\top J^2(h z) \theta \right) dz
\]

for the vector parametrization, respectively. Such solutions exist whenever \( d + 1 \) observations receive nonzero weights; see Loader (1996). Furthermore, \( A \) or \( \theta \) can be uniquely chosen, such that \( A \) is symmetric and \( \theta \in \Theta \), respectively. In general a closed form solution does not exist, however, in the case of a Gaussian kernel a closed form solution is obtained in Section 4.1. Theorem 3.9 shows consistency and the asymptotic structure and Theorem 3.10 the asymptotic law of the estimator for \( D^2 \log f(x) \) and \( D^2 \log f(x) \). In these results we only consider the vector parametrization.

This weighted scoring rule was studied in Loader (1996) and asymptotic results are given there in a slightly different setting.

**Theorem 3.9.** Suppose (LLLE K1), (LLLE A1) hold and \( h \to 0 \), \( nh^{d+4} \to \infty \) as \( n \to \infty \). Then the local log-likelihood estimator fulfills

\[
\frac{c_n}{\theta_n} \xrightarrow{D} D^2 \log f(x).
\]

For if also (LLLE A2) hold and \( nh^{d+6} = O(1) \), then

\[
\begin{align*}
\frac{c_n}{\theta_n} &= \log f(x) + h^4 b^{\text{LLLE}}_0 + O(h^4) + O_p((nh^d)^{-1/2}), \\
\theta_n &= D^2 \log f(x) + h^4 \left( b^{\text{LLLE}}_2 \right) + B^{-1}_h O_p((nh^{d+2})^{-1/2})
\end{align*}
\]
Theorem 3.10. Suppose (LLE K1), (LLE A1) hold and \( h \to 0, nh^d \to \infty, nh^{d+6} \to \lambda^2 \geq 0 \) as \( n \to \infty \). The local log-likelihood estimator fulfills

\[
\sqrt{nh^d} B_h^T \left( \frac{\hat{\theta}^{\text{LLE}}_n}{\hat{\theta}^{\text{LLE}}_n} - D^2 \log f(x) \right) \xrightarrow{d} \mathcal{N} \left( \lambda \left( \begin{bmatrix} b_1^{\text{LLE}} \\ 0 \end{bmatrix}, f(x)^{-1} \right), \lambda^2 \mathbf{V}^{\text{LLE}} \right)
\]

In particular

\[
\sqrt{nh^{d+2}} B_h (\hat{\theta}^{\text{LLE}}_n - D^2 \log f(x)) \xrightarrow{d} \mathcal{N} \left( \lambda \left( \begin{bmatrix} b_1^{\text{LLE}} \\ 0 \end{bmatrix}, f(x)^{-1} \mathbf{V}^{\text{LLE}} \right) \right)
\]

with

\[
\mathbf{V}^{\text{LLE}} := (J_2^2)^{-1} E_d [0_p, I_p] R(\sqrt{K} \tilde{E}_d^+ (\cdot) 1^T) R(\sqrt{K} \tilde{E}_d^+ (\cdot) 1^T) R(\sqrt{K} \tilde{E}_d^+ (\cdot) 1^T) [0_p, I_p]^T E_d^+ (J_2^2)^{-1}.
\]

Corollary 3.11. Suppose (LLE K1), (LLE A1) hold and \( h \to 0, nh^d \to \infty, nh^{d+6} \to \lambda^2 \geq 0 \) as \( n \to \infty \). The local log-likelihood estimator

\[
\hat{\theta}^{\text{LLE}}_n = \begin{pmatrix} \hat{\theta}^{\text{LLE}}_{1,n} \\ \hat{\theta}^{\text{LLE}}_{2,n} \end{pmatrix}
\]

fulfills

\[
\sqrt{nh^{d+2}} (\hat{\theta}^{\text{LLE}}_{1,n} - D \log f(x)) \xrightarrow{d} \mathcal{N} \left( \lambda b_1^{\text{LLE}}, f(x)^{-1} R(K(\cdot) 1^T) \right),
\]

and

\[
\sqrt{nh^{d+4}} (\hat{\theta}^{\text{LLE}}_{2,n} - D^2 \log f(x)) \xrightarrow{d} \mathcal{N} \left( 0, 4f(x)^{-1} D_d R(\sqrt{K} D_d^+ (\cdot)^{\otimes 2})^{-1} D_d^T R(K(\cdot)^{\otimes 2} - \text{vec}(I_d)) D_d R(\sqrt{K} D_d^+ (\cdot)^{\otimes 2})^{-1} D_d^T \right).
\]
3.4 Local Hyvärinen Score Estimator

In this section we consider the Hyvärinen Score proposed by Hyvärinen (2005) and given by

\[ S : \mathcal{P} \times \mathbb{R}^d \to \mathbb{R}, \quad S(p, y) := \frac{1}{2} \| \nabla \log p(y) \|^2 + \Delta \log p(y). \tag{13} \]

Whenever the conditions in Theorem 3.7 hold the Hyvärinen Score leads to a proportionally locally proper scoring rule \( \tilde{S} : \mathcal{H} \times \mathbb{R}^d \times \mathcal{W} \to \mathbb{R} \cup \{ \infty \} \),

\[ \tilde{S}(h, y, w) = w(y)S(hw, y) = \frac{w(y)}{2} \| \nabla \log hw(y) \|^2 + w(y)\Delta \log hw(y) \]
\[ = \frac{w(y)}{2} \| \nabla \log w(y) + \nabla \log h(y) \|^2 + w(y)\Delta \log h(y) + w(y)\Delta \log w(y) \]
\[ = w(y)S(h, y) + w(y)\nabla \log w(y), \nabla \log h(y)) + w(y)S(w, y) \]
\[ = w(y)S(h, y) + (\nabla w(y), \nabla \log h(y)) + \kappa(w, y), \]

where \( \kappa(w, y) := w(x)S(w, y) \).

**Remark.** Applying Theorem 3.8 leads to a strictly locally proper scoring rule, allowing to estimate also \( \log f(x) \) in addition to \( D \log f(x) \). For the Hyvärinen score we refrain from applying Theorem 3.8 to avoid adding an integral term to the score. This makes the score much simpler to minimize. For the log-score we apply Theorem 3.8, because it does not change the complexity of the score and adds the benefit for an estimator of \( \log f(x) \).

We choose \( \mathcal{H} = \mathcal{F} \) defined in (8) and \( \mathcal{W} = \{ K_h \}_{h > 0} \) parametrized with \( c \in \mathbb{R}, b \in \mathbb{R}^d, A \in \mathbb{R}^{d \times d} \) or \( c \in \mathbb{R}, \theta \in \mathbb{R}^{d+d^2} \) and \( h > 0 \), respectively, the conditions in Theorem 3.7 are fulfilled whenever (LHSE K1) holds. This leads to the score

\[ \tilde{S}(b, A, y, h) = \frac{1}{2} K_h(y)(b + Ay)^\top (b + Ay) + K_h(y) \mathrm{tr}(A) + \nabla (K_h(y))^\top (b + Ay) + \kappa(K_h, y) \]

in the usual parametrization and

\[ \tilde{S}(\theta, y, h) = \frac{1}{2} K_h(y)(\nabla_y \theta^\top J^\top y \cdot 2^\top + \nabla_y \theta^\top J^\top y \cdot 2^\top) + K_h(y)\Delta_y \theta^\top J^\top y \cdot 2^\top 
\]
\[ + \nabla_y (K_h(y))^\top (\nabla_y \theta^\top J^\top y \cdot 2^\top) + \kappa(K_h, y) \]
\[ = \frac{1}{2} K_h(y)\theta^\top E_d E_d^\top \left( \begin{bmatrix} 1 & y^\top \nabla_y \theta^\top J^\top y \cdot 2^\top + \kappa(K_h, y) \right) \right) + \kappa(K_h, y) \]

in the vector parametrization. From here on, we only consider the vector parametrization and neglect the parts not depending on \( \theta \). For a random sample \( X_1, X_2, \ldots, X_n \) drawn from \( f \), we minimize the empirical score

\[ L_n^{\mathrm{LHSE}}(\theta) := \frac{1}{n} \sum_{i=1}^{n} \tilde{S}(\theta, X_i - x, h) = \frac{1}{2} \theta^\top E_d E_d^\top B_h(S^*_n \otimes I_d) B_h E_d E_d^\top \theta + \theta E_d E_d^\top B_h v_n \tag{14} \]

and

\[ Q_n^{\infty}(x) := \frac{1}{n} \sum_{i=1}^{n} \left( \frac{X_i - x}{h} \right)^{\Xi} \otimes ((\nabla K)_h(X_i - x)) = \frac{1}{n} \sum_{i=1}^{n} \left( \frac{X_i - x}{h} \right)^{\Xi} \otimes \nabla (K_h(X_i - x)), \tag{15} \]

with \( \Xi \) being a placeholder for either \( \mathcal{P}, \mathcal{L} \) or \( \otimes r \).
Then the local Hyvärinen score estimator fulfills
\[ \sum_{i=1}^{n} 1\{K_{h}(X_{i} - x) > 0\} \geq d + 1. \]

Furthermore, if \(-B_{h}^{-1}(S_{n}^{*} \otimes I_{d})^{-1}v_{n} \in \Theta\), then \(\hat{\theta}_{n}^{\text{LHSE}} = -B_{h}^{-1}(S_{n}^{*} \otimes I_{d})^{-1}v_{n}\).

Theorem 3.13. Suppose (LHSE K1), (A1) hold and \(h \to 0\), \(nh^{d+1} \to \infty\) as \(n \to \infty\). Then the local Hyvärinen score estimator fulfills
\[ \hat{\theta}_{n}^{\text{LHSE}} \xrightarrow{p} D\hat{\theta} \log f(x) \]
and if also (A3) holds, then
\[ \hat{\theta}_{n}^{\text{LHSE}} = D\hat{\theta}f(x) + h^{2}b^{\text{LHSE}} + B_{h}^{-1}O(h^{3}) + B_{h}^{-1}O_{p}((nh^{d+2})^{-1/2}) \]
with
\[ b^{\text{LHSE}} := \begin{pmatrix} b_{1}^{\text{LHSE}} \\ b_{2}^{\text{LHSE}} \end{pmatrix}, \]
where
\[ b_{1}^{\text{LHSE}} := (I_{d} \otimes \text{vec}(I_{d})\mathbb{T})D^{\otimes 3} \log f(x), \]
\[ b_{2}^{\text{LHSE}} := D_{d}D_{d}^{\top}(I_{d^{2}} \otimes \text{vec}(I_{d})\mathbb{T}) \left( -\frac{1}{2} \frac{Df(x)}{f(x)} \otimes \frac{D^{\otimes 3}f(x)}{f(x)} - \frac{1}{2} \frac{D^{\otimes 2}f(x)}{f(x)} \right) \]
\[ + D_{d}D_{d}^{\top} \left( \int K(z)I_{d} \otimes z(\mathbb{Z} \otimes 3)^{\top} dz \right) \left( \frac{1}{6} \frac{D^{\otimes 4}f(x)}{f(x)} - \frac{1}{2} \frac{D^{\otimes 2}f(x)}{f(x)} \right). \]

Theorem 3.14. Suppose (LHSE K1), (A2) hold and \(h \to 0\), \(nh^{d+1} \to \infty\), \(nh^{d+6} \to \lambda^{2} \geq 0\) as \(n \to \infty\). Then the local Hyvärinen score estimator fulfills
\[ \sqrt{nh^{d+2}}B_{h}(\hat{\theta}_{n}^{\text{LHSE}} - D\hat{\theta} \log f(x)) \xrightarrow{d} N\left(\begin{pmatrix} \lambda_{1}^{\text{LHSE}} \\ 0 \end{pmatrix}, (f(x)^{-1}R(E_{d}^{+}(\cdot) \otimes DK)E_{d}^{\top})\right). \]

Corollary 3.15. Suppose (LHSE K1), (A2) hold and \(h \to 0\), \(nh^{d+1} \to \infty\), \(nh^{d+6} \to \lambda^{2} \geq 0\) as \(n \to \infty\). Then the local Hyvärinen score estimator
\[ \hat{\theta}_{n}^{\text{LLE}} = \begin{pmatrix} \hat{\theta}_{1,n}^{\text{LHSE}} \\ \hat{\theta}_{2,n}^{\text{LHSE}} \end{pmatrix} \]
fulfills
\[ \sqrt{nh^{d+2}}(\hat{\theta}_{1,n}^{\text{LHSE}} - D\log f(x)) \xrightarrow{d} N(\lambda b_{1}^{\text{LHSE}}, f(x)^{-1}R(\cdot) \otimes DK), \]
and
\[ \sqrt{nh^{d+4}}(\hat{\theta}_{2,n}^{\text{LHSE}} - D^{\otimes 2} \log f(x)) \xrightarrow{d} N(0, f(x)^{-1}D_{d}D_{d}^{+}R((\cdot) \otimes DK)(D_{d}D_{d}^{+})\mathbb{T}). \]
3.5 Local Moment Matching Estimator

The idea of local moment matching appears in Müller and Yan (2001) in combination with a uniform kernel function. As the name suggests, one matches the local sample moments with their theoretical counterparts. The second order Taylor expansion leads to the system of equations

\[ s_{n,h}^{\alpha}(x) = \frac{1}{n} \sum_{i=1}^{n} K_h(X_i - x) \left( \frac{X_i - x}{h} \right) = \sum_{\gamma \leq 2} \frac{h^{\gamma}}{\gamma!} \mu_{\alpha+\gamma} f_{n}(x) \]

for all \(|\alpha| \leq 2\) or equivalently

\[ S_{n}^{\alpha}(x) = R(\sqrt{K} ; \mathcal{F}) J^{T} B_{n}^{T} \mathcal{B}^{T} f_{n}(x). \]

Equivalent transformations lead to the following estimator:

\[ \tilde{S}_{n,h}^{\alpha}(x) = R(\sqrt{K} ; \mathcal{F}) J^{T} B_{n}^{T} \mathcal{B}^{T} f_{n}(x) \]

or for \(1 \leq i \leq d\)

\[ \tilde{f}_{n}(x) = \frac{\left( \mu_{4} + (d-1)\mu_{22} \right) s_{n}(x) - \sum_{j=1}^{d} 2 \mu_{2j}^{e_{j}}(x) }{\mu_{4} + (d-1)\mu_{22} - d}, \]

\[ \tilde{f}_{n}(e_{i})(x) = \frac{s_{n}(x)}{h}, \]

\[ \tilde{f}_{n}(e_{i} + e_{k})(x) = \frac{s_{n}(x)}{h^{2}} \]

if \(i \neq k \in \{1, 2, \ldots, d\},\)

\[ \tilde{f}_{n}(2e_{i})(x) = \frac{2}{h^{2}} \left( \frac{\left( \mu_{4} + (d-1)\mu_{22} \right) s_{n}(x) - \sum_{j=1}^{d} 2 \mu_{2j}^{e_{j}}(x) }{\mu_{4} - \mu_{22} - 1} \right) + \frac{\mu_{22} - 1}{\mu_{4} - \mu_{22}}. \]

The estimator for the log-density and its derivatives up to order 2 is

\[ \left( \hat{\varphi}_{n}^{LME}, \hat{\theta}_{n}^{LME} \right) := \phi(\tilde{D}^{2} f_{n}(x)), \]

where \(\phi\) is defined in (3). The following two theorems show the consistency, asymptotic structure and normality of the local moment matching estimator (LMME).

**Theorem 3.16.** Suppose (LMME K1), (A1) hold and \(h \to 0, nh^{d+1} \to \infty\) as \(n \to \infty\), then

\[ \tilde{D}^{2} f_{n}(x) \overset{p}{\to} D^{2} f(x) \]

and

\[ \frac{\hat{\varphi}_{n}^{LME}}{\sqrt{n}} \overset{p}{\to} \log f(x), \]

\[ \frac{\hat{\theta}_{n}^{LME}}{\sqrt{n}} \overset{D}{\to} \log f(x). \]
For if also (LMME K3), (A3) holds, then

\[
\hat{f}_n(x) = \log f(x) + h^{4b_0^{\text{LME}}} + o(h^4) + O_p((nh^d)^{-1/2})
\]

\[
\hat{D}^2f_n(x) = D^2f(x) + h^2\left(b_1^{\text{LME}} + b_2^{\text{LME}}\right) + B_h^{-1}o(h^3) + B_h^{-1}O_p((nh^{d+2})^{-1/2}),
\]

with

\[b_0^{\text{LME}} := \frac{1}{12} \text{vec}(I_d)D_dR\left(\sqrt{K}D_d^+(\cdot)^{\otimes 2}\right)^{-1}D_d^+ \int K(z)z^{\otimes 2}(z^{\otimes 2})^\top dz D^{\otimes 2}f(x),\]

\[b_1^{\text{LME}} := \frac{1}{6} \int K(z)z(z^{\otimes 3})^\top dz D^{\otimes 3}f(x),\]

\[b_2^{\text{LME}} := \frac{1}{12}(D_d^+)^\top R(\sqrt{K}D_d^+(\cdot)^{\otimes 2})^{-1}D_d^+ \int K(z)z^{\otimes 2}(z^{\otimes 4})^\top dz D^{\otimes 4}f(x).
\]

and if in addition \(nh^{d+6} = O(1)\), then

\[
\hat{\theta}_n^{\text{LME}} = \log f(x) + h^4b_0^{\text{LME}} + o(h^4) + O_p((nh^d)^{-1/2}),
\]

\[\hat{\theta}_n^{\text{LME}} = D^2\log f(x) + h^2\left(b_1^{\text{LME}} + b_2^{\text{LME}}\right) + o(h^4) + B_h^{-1}O_p((nh^{d+2})^{-1/2})
\]

with

\[b_0^{\text{LME}} := \frac{1}{12} \text{vec}(I_d)D_dR\left(\sqrt{K}D_d^+(\cdot)^{\otimes 2}\right)^{-1}D_d^+ \int K(z)z^{\otimes 2}(z^{\otimes 2})^\top dz \frac{D^{\otimes 4}f(x)}{f(x)},\]

\[b_1^{\text{LME}} := \frac{1}{6} \int K(z)z(z^{\otimes 3})^\top dz \frac{D^{\otimes 3}f(x)}{f(x)},\]

\[b_2^{\text{LME}} := \frac{1}{12}(D_d^+)^\top R(\sqrt{K}D_d^+(\cdot)^{\otimes 2})^{-1}D_d^+ \int K(z)z^{\otimes 2}(z^{\otimes 4})^\top dz \frac{D^{\otimes 4}f(x)}{f(x)}.
\]

Theorem 3.17. Suppose (LMME K2), (A2) hold and \(h \to 0\), \(nh^{d+6} \to \lambda^2 \geq 0\) as \(n \to \infty\), then

\[
\sqrt{nh^d}B_h^{\text{T}}(\hat{D}^2f_n(x) - D^2f(x)) \xrightarrow{d} \mathcal{N}\left(\lambda\left(b_0^{\text{LME}} \right), f(x)(J^2)^{-1}(\hat{E}_d^+)^\top R(\sqrt{K}\hat{E}_d^+(\cdot)^{\otimes 2})^{-1}R(K\hat{E}_d^+(\cdot)^{\otimes 2})R(\sqrt{K}\hat{E}_d^+(\cdot)^{\otimes 2})^{-1}\hat{E}_d^+(J^2)^{-1}\right)
\]

and

\[
\sqrt{nh^d}B_h^{\text{T}}\left(\hat{\theta}_n^{\text{LME}} - D^2\log f(x)\right) \xrightarrow{d} \mathcal{N}\left(\lambda\left(b_0^{\text{LME}} \right), f(x)(J^2)^{-1}(\hat{E}_d^+)^\top R(\sqrt{K}\hat{E}_d^+(\cdot)^{\otimes 2})^{-1}R(K\hat{E}_d^+(\cdot)^{\otimes 2})R(\sqrt{K}\hat{E}_d^+(\cdot)^{\otimes 2})^{-1}\hat{E}_d^+(J^2)^{-1}\right).
\]

In particular,

\[
\sqrt{nh^{d+2}}B_h^{\text{T}}\left(\hat{\theta}_n^{\text{LME}} - D^2\log f(x)\right) \xrightarrow{d} \mathcal{N}\left(\lambda\left(b_0^{\text{LME}} \right), f(x)^{-1}\lambda\left(b_0^{\text{LME}} \right), f(x)^{-1}V^{\text{LME}}\right)
\]

with

\[V^{\text{LME}} = [0 \ I_{d+d^2}](J^2)^{-1}(\hat{E}_d^+)^\top R(\sqrt{K}\hat{E}_d^+(\cdot)^{\otimes 2})^{-1}R(K\hat{E}_d^+(\cdot)^{\otimes 2})R(\sqrt{K}\hat{E}_d^+(\cdot)^{\otimes 2})^{-1}\hat{E}_d^+(J^2)^{-1}[0 \ I_{d+d^2}])^\top.
\]
**Corollary 3.18.** Suppose (LMME K2), (A2) hold and \( h \to 0, \ n h^{d+6} \to \lambda^2 \geq 0 \) as \( n \to \infty \), then the local moment matching estimator

\[
\hat{\theta}_{n}^{\text{LMME}} = \begin{pmatrix} \hat{\theta}_{1,n}^{\text{LMME}} \\ \hat{\theta}_{2,n}^{\text{LMME}} \end{pmatrix}
\]

fulfills

\[
\sqrt{nh^{d+2}}(\hat{\theta}_{1,n}^{\text{LMME}} - D \log f(x)) \xrightarrow{d} \mathcal{N}(\lambda b_{1,n}^{\text{LMME}}, f(x)^{-1}R(K(\cdot)^{\otimes 1}))
\]

and

\[
\sqrt{nh^{d+4}}(\hat{\theta}_{2,n}^{\text{LMME}} - D \circ 2 \log f(x)) \xrightarrow{d} \mathcal{N}\left(0, 4f(x)^{-1}(D_{d}^{+})^\top R(\sqrt{K}D_{d}^{+}(\cdot)^{\otimes 2})^{-1}D_{d}^{+}R(K((\cdot)^{\otimes 2} - \text{vec}(I_d)))D_{d}R(\sqrt{K}D_{d}^{+}(\cdot)^{\otimes 2})^{-1}D_{d}^{+}\right).
\]

### 3.6 Kernel Density Estimation

In this section we consider the well known kernel density estimator proposed by Rosenblatt (1956), Parzen (1962). The kernel density estimator for estimating \( f(\alpha)(x) \) with \( \alpha \in \mathbb{N}_0 \) is

\[
\hat{f}_{n}(\alpha)(x) = \frac{1}{n} \sum_{i=1}^{n} (K_{h}(x - X_{i}))^{(\alpha)} = \frac{1}{nh^{d+|\alpha|}} \sum_{i=1}^{n} K^{(\alpha)} \left( \frac{x - X_{i}}{h} \right) = \frac{1}{nh^{\alpha}} \sum_{i=1}^{n} (K^{(\alpha)})_{h}(x - X_{i}).
\]

Therefore, the density \( f(x) \) is estimated by

\[
\hat{f}_{n}(x) = \frac{1}{n} \sum_{i=1}^{n} K_{h}(x - X_{i}) = s_{n}^{0}(x),
\]

the gradient \( Df(x) \) by

\[
\hat{Df}_{n}(x) = \frac{1}{n} \sum_{i=1}^{n} D(K_{h}(x - X_{i})) = \frac{1}{nh} \sum_{i=1}^{n} (DK)_{h}(x - X_{i}) = Q_{n}^{0}(x),
\]

where the last equality follows from Equation (15), and the vector of second order derivatives \( D \circ 2 f(x) \) by

\[
\hat{D^{\circ 2}f}_{n}(x) = \frac{1}{n} \sum_{i=1}^{n} D^{\circ 2}(K_{h}(x - X_{i})) = \frac{1}{nh^{2}} \sum_{i=1}^{n} (D^{\circ 2}K)_{h}(x - X_{i}).
\]

Stacked in one vector this is

\[
\hat{D^2f}_{n}(x) = (B_{h}^{2})^{-1} \frac{1}{n} \sum_{i=1}^{n} (D^{2}K)_{h}(x - X_{i}).
\]

An estimator for \( D^2 \log f(x) \) is the given by

\[
\left( \begin{array}{c} \hat{\phi}^{\text{c}} \\ \hat{\theta}^{\text{c}} \end{array} \right) := \phi(\hat{D^2f}_{n}(x))
\]

where \( \phi \) is defined in (3). The following results show the consistency, asymptotic structure and normality of the KDE. Parts of the results were developed in Chacón et al. (2011) and reformulated here.

**Theorem 3.19.** Suppose (KDE K1), (A1) hold and \( h \to 0, nh^{d+4} \to \infty \) as \( n \to \infty \), then

\[
\hat{D^2f}_{n}(x) \overset{p}{\to} D^2f(x)
\]

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Suppose \((\text{KDE K3})\) and \((A3)\) hold, then

\[
\sqrt{n} \tilde{c}_n \xrightarrow{P} \log f(x), \\
\tilde{\theta}^\text{KDE}_n \xrightarrow{P} D^2 \log f(x).
\]

Suppose \((\text{KDE K2}), (A2)\) hold and \(\sqrt{n}h^{d+6} = O(1)\), then

\[
D^2 f_n(x) = D^2 f(x) + h^2 \tilde{b}^\text{KDE} + \left( \frac{O(h^4)}{o(h^3)} \right) + (B_h^2)^{-1} O_p((nh^d)^{-1/2})
\]

with

\[
\tilde{b}^\text{KDE} := \begin{pmatrix}
\tilde{b}^\text{KDE}_0 \\
\tilde{b}^\text{KDE}_1 \\
\tilde{b}^\text{KDE}_2
\end{pmatrix} := \begin{pmatrix}
\frac{\text{vec}(I_d^\top) \partial^2 f(x)}{f(x)} \\
(I_d \otimes \text{vec}(I_d^\top)) \left( \frac{\partial^2 f(x)}{f(x)} \right) + \frac{\partial^3 f(x)}{f(x)} \\
(I_d \otimes \text{vec}(I_d^\top)) \left( 2 \left( \frac{\partial^2 f(x)}{f(x)} \right)^{\otimes 2} \otimes \frac{\partial^2 f(x)}{f(x)} - \left( \frac{\partial^2 f(x)}{f(x)} \right)^{\otimes 2} + \frac{\partial^4 f(x)}{f(x)} \right)
\end{pmatrix}.
\]

Theorem 3.20. Suppose \((\text{KDE K2}), (A2)\) hold and \(h \to 0\), \(nh^{d+6} \to \lambda^2 \geq 0\) as \(n \to \infty\), then

\[
\sqrt{nh^{d+2}} B_h (D^2 f_n(x) - D^2 f(x)) \xrightarrow{d} \mathcal{N} \left( \lambda \left( \frac{1}{2} (I_d \otimes \text{vec}(I_d^\top)) D^3 f(x) \right), f(x) R(D^2 K) \right)
\]

and

\[
\sqrt{nh^{d+2}} B_h (\tilde{\theta}^\text{KDE}_n - D^2 \log f(x)) \xrightarrow{d} \mathcal{N} \left( \lambda \left( \tilde{b}^\text{KDE}_2 \right), f(x)^{-1} R(D^2 K) \right).
\]

Corollary 3.21. Suppose \((\text{KDE K2}), (A2)\) hold and \(h \to 0\), \(nh^{d+6} \to \lambda^2 \geq 0\) as \(n \to \infty\), then the kernel density estimator

\[
\tilde{\theta}^\text{KDE}_n = \begin{pmatrix}
\tilde{\theta}^\text{KDE}_{1,n} \\
\tilde{\theta}^\text{KDE}_{2,n}
\end{pmatrix}
\]

fulfills

\[
\sqrt{nh^{d+2}} (\tilde{\theta}^\text{KDE}_{1,n} - D \log f(x)) \xrightarrow{d} \mathcal{N}(f(x)^{-1} R(K))
\]

and

\[
\sqrt{nh^{d+2}} (\tilde{\theta}^\text{KDE}_{2,n} - D^2 \log f(x)) \xrightarrow{d} \mathcal{N}(0, f(x)^{-1} R(D^2 K)).
\]
4 Comparison

4.1 Gaussian Kernel

In this section we compare the different estimators using the Gaussian kernel. The Gaussian kernel is

\[ K(z) := (2\pi)^{-d/2} \exp(-\|z\|/2) \quad \text{and} \quad DK(z) = -zK(z). \]

This special structure of the gradient implies that for \( r \in \{0, 1\} \)

\[
Q_{n,h}^{(r)}(x) = \frac{1}{nh} \sum_{i=1}^{n} (DK)_h(X_i - x) \left( \frac{X_i - x}{h} \right)^{\otimes r}
\]

\[
= -\frac{1}{nh} \sum_{i=1}^{n} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^{(r+1)}
\]

\[
= \frac{1}{h} S_{n,h}^{(r+1)}(x).
\]

The LHSE is then

\[
\theta_n^{\text{LHSE}} = -B^{-1}_h (S_n^* \otimes I_d)^{-1} \left( Q_n^h(x) + \frac{s_n(x)}{h} \left( \frac{0_d}{\text{vec}(I_d)} \right) \right)
\]

\[
= -B^{-1}_h s_n^{-1} \left[ \begin{array}{c}
\frac{x^\top}{s_n} \hat{S}_n^{-1} \frac{x}{s_n} - \frac{x^\top}{s_n} \hat{S}_n^{-1} \\
- \frac{1}{s_n} \frac{x^\top}{s_n} \hat{S}_n^{-1} \frac{x}{s_n}
\end{array} \right] \otimes I_d \times \left( \frac{\hat{x}}{\text{vec}(I_d)} \right)
\]

\[
= (B^2_h)^{-1} \left[ \begin{array}{c}
\frac{x^\top}{s_n} \hat{S}_n^{-1} \frac{x}{s_n} - \frac{x^\top}{s_n} \hat{S}_n^{-1} \\
- \frac{1}{s_n} \frac{x^\top}{s_n} \hat{S}_n^{-1} \frac{x}{s_n}
\end{array} \right] \otimes I_d \times \left( \frac{\hat{x}}{\text{vec}(I_d)} \right)
\]

For the LLLE we get exactly the same estimator. To see this we minimize the functional in (12)

\[ L_{x,h}(c, \theta) = -c - \theta^\top J^2 B_h^T S_{n,h}^2(x) + \int K(z) \exp(c + \theta^\top J^2 B_h^T z^2). \]
where \( c \in \mathbb{R} \) and \( \theta = (\theta_1^T, \theta_2^T)^T \in \mathbb{R}^d \times \mathbb{R}^{d^2} \). For the Gaussian kernel we get then for the integral part

\[
\int K(z) \exp(c + h\theta_1^T z + \frac{h^2}{2} z^\top \text{mat}(\theta_2) z) \, dz
\]

\[
= (2\pi)^{-d/2} \exp(c) \int \exp\left(h\theta_1^T z + \frac{h^2}{2} z^\top (h^2 \text{mat}(\theta_2) - I_d) z\right) \, dz
\]

\[
= (2\pi)^{-d/2} \exp(c) \exp\left(\frac{h^2}{2} \theta_1^T (I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1\right)
\]

\[
\times \int \exp\left(-\frac{1}{2} \left(z - h(I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1\right)^\top (I_d - h^2 \text{mat}(\theta_2)) \left(z - h(I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1\right)\right) \, dz
\]

\[
= \exp(c) \det(I_d - h^2 \text{mat}(\theta_2))^{-1/2} \exp\left(\frac{h^2}{2} \theta_1^T (I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1\right),
\]

where the last equality follows from

\[
\int \exp(-\frac{1}{2} y^\top \Sigma y) \, dy = (2\pi)^{d/2} \det(\Sigma)^{1/2}
\]

for any symmetric positive definite matrix \( \Sigma \in \mathbb{R}^{d \times d} \). Hence,

\[
L_{x,h}(c, \theta) = -\theta^\top J^T \overline{D}_n^2 S_n(x) + \exp(c) \det(I_d - h^2 \text{mat}(\theta_2))^{-1/2} \exp\left(\frac{h^2}{2} \theta_1^T (I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1\right)
\]

and by differentiation with respect to \( c \) we see that this is minimal for

\[
\hat{c} = -\frac{h^2}{2} \theta_1^T (I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1 + \log \det(I_d - h^2 \text{mat}(\theta_2))^{1/2} + \log(s_0^2).
\]

Neglecting the terms not depending on \( \theta \), this leads to

\[
L_{x,h}(\theta_1, \theta_2 | \hat{c}) = -h \theta_1^\top S_n - \frac{h^2}{2} \theta_2^\top S_n^\otimes 2(x) - \frac{s_n}{2} \log \det(I_d - h^2 \text{mat}(\theta_2)) + \frac{h^2 s_0^2}{2} \theta_1^\top (I_d - h^2 \text{mat}(\theta_2))^{-1} \theta_1
\]

Setting the gradient with respect to \( \theta_1 \) to zero we have

\[
\hat{\theta}_1 = h^{-1} (I_d - h^2 \text{mat}(\theta_2)) \frac{S_n}{s_n}
\]

and

\[
L_{x,h}(\theta_2 | \hat{c}, \hat{\theta}_1) = s_n \left( -\frac{1}{2} \log \det(I_d - h^2 \text{mat}(\theta_2)) - \frac{1}{2} \frac{s_n}{s_n} (I_d - h^2 \text{mat}(\theta_2)) \frac{S_n^\top}{s_n} - \frac{h^2}{2} \frac{\theta_1^\top S_n^\otimes 2}{s_n} \right)
\]

\[
= \frac{s_n}{2} \left( \text{tr} \left( (I_d - h^2 \text{mat}(\theta_2)) \hat{S}_n \right) - \log \det \left( (I_d - h^2 \text{mat}(\theta_2)) \hat{S}_n \right) - \text{tr}(\hat{S}_n) + \log \det(\hat{S}_n) \right). \tag{16}
\]

Let \( \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_d \) be the eigenvalues of the matrix \( (I_d - h^2 \text{mat}(\theta_2)) \hat{S}_n \), then (16) is minimal if, and only if,

\[
\sum_{i=1}^d (\lambda_i - \log(\lambda_i))
\]

is minimal. Since the function \( x - \log(x) \) is convex with unique minimum at \( x = 1 \), (16) is minimal if

\[
(I_d - h^2 \text{mat}(\theta_2)) \hat{S}_n = I_d
\]

and

\[
\hat{\theta}_2 = \frac{1}{h^2} \text{vec}(I_d - \hat{S}_n^{-1}).
\]

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Finally the estimator for $D\hat{\log} f(x)$ is

$$\hat{\theta}_{n}^{LLE} = (B_{h}^{2})^{-1} \left( \hat{S}_{n}^{-1} \frac{\hat{\theta}_{n}}{\text{vec}(I_{d} - \hat{S}_{n}^{-1})} \right).$$

For the calculation of the KDE with the Gaussian kernel note that

$$D^{\otimes 2}K(z) = -D(zK(z)) = -K(z) \text{vec}(I_{d}) + K(z) \text{vec}(zz^{\top}) = K(z)(\text{vec}(zz^{\top}) - \text{vec}(I_{d})).$$

Recall that

$$\hat{f}_{n}(x) = \frac{1}{n} \sum_{i=1}^{n} K_{h}(x - X_{i}) = s_{n},$$

$$\hat{Df}_{n}(x) = Q_{n}^{0}(x) = -\frac{1}{h}s_{n},$$

$$\hat{D^{\otimes 2}f}_{n}(x) = \frac{1}{h^{2}} \sum_{i=1}^{n} (D^{\otimes K})_{h}(x - X_{i}) = \frac{1}{h^{2}} \sum_{i=1}^{n} K_{h}(x - X_{i}) \left( \text{vec} \left( \frac{X_{i} - x}{h} \left( \frac{X_{i} - x}{h} \right)^{\top} - \text{vec}(I_{d}) \right) \right) = h^{-2} \text{vec}(S_{n}) - h^{-2} s_{n} \text{vec}(I_{d}).$$

Applying $\hat{\phi}$ defined in (5) on $\hat{D^{\otimes 2}f}(x)$ leads to

$$\hat{D^{\otimes 2}\log f}(x) = \left( \frac{\hat{Df}_{n}(x)^{\top}}{\text{vec}(\hat{Df}_{n}(x))} \right) = (B_{h}^{2})^{-1} \left( \frac{\hat{\theta}_{n}}{\text{vec}(S_{n} - I_{d})} \right).$$

In case of the Gaussian kernel we have

$$\hat{\theta}_{n}^{LHSE} = \hat{\theta}_{n}^{LLE} = (I_{d+1} \otimes \hat{S}_{n}^{-1})^{\text{KDE}} \hat{\theta}_{n}^{LLE}.$$ 

### 4.2 Summary of the Asymptotic Results

Here is a summary of the asymptotic properties of the four estimators. In what follows the greek letter $\Xi$ stands for one of the estimators with the corresponding assumptions listed in Table 1. The bias and variance terms are listed below.

**Consistency**

$$\hat{D^{\otimes f}}_{n}(x) \xrightarrow{p} D^{\otimes f}(x)$$ (P1)

if $h \to 0$, $nh^{d+4} \to \infty$ as $n \to \infty$.

$$\hat{\theta}_{n} \xrightarrow{p} \log f(x),$$ (P2)

$$\hat{\theta}_{n} \xrightarrow{p} D^{2}f(x)$$ (P3)

if $h \to 0$, $nh^{d+4} \to \infty$, $nh^{d+6} \to O(1)$ as $n \to \infty$. (For the LHSE the condition $nh^{d+6} = O(1)$ is not needed)

**Taylor Expansion**

$$\hat{f}_{n}(x) = f(x) + nh^{4}h_{0}^{2} + o(h^{4}) + O_{p}((nh^{d})^{-1/2}),$$ (P4)

$$\hat{f}_{n}(x) = f(x) + nh^{2}h_{0}^{2} + O(h^{4}) + O_{p}((nh^{d})^{-1/2}),$$ (P5)

$$\hat{D^{\otimes 2}f}_{n}(x) = D^{2}f(x) + h^{2}b_{h}^{2} + B_{h}^{1}o(h^{3}) + B_{h}^{1}O_{p}((nh^{d+2})^{-1/2}),$$ (P6)
Asymptotic Normality

if \( nh \to 0, n \to \infty \),

\[
\hat{c}_n^\Xi = \log f(x) + h^2 \hat{b}_0^\Xi + o(h^4) + O_p((nh^d)^{-1/2}),
\]

\[
\hat{\theta}_n^\Xi = D^2 \log f(x) + h^2 \left( \hat{b}_2^\Xi \right) + B_h^{-1}o(h^3) + B_h^{-1}O_p((nh^{d+2})^{-1}),
\]

if \( nh^{d+4} \to \infty, nh^{d+6} = O(1) \) as \( n \to \infty \).

### Asymptotic Normality

\[
\sqrt{nh^d} B_h^D \left( \frac{\hat{D}^2 f_n}{D^2 f(x)} \right) \xrightarrow{d} N\left( \lambda \left( \frac{0}{\hat{b}_0^\Xi} \right), f(x) V^\Xi \right),
\]

\[
\sqrt{nh^{d+2}} B_h \left( \frac{\hat{D^2} f_n}{D^2 f(x)} \right) \xrightarrow{d} N\left( \lambda \left( \frac{0}{\hat{b}_0^\Xi} \right), f(x) V^\Xi \right)
\]

if \( h \to 0, nh^{d+4} \to \infty, nh^{d+6} \to \lambda^2 \geq 0 \) as \( n \to \infty \) and

\[
\sqrt{nh^d} B_h^D \left( \frac{\hat{c}_n^\Xi}{\hat{\theta}_n^\Xi} - \frac{\hat{D}^2 f_n}{D^2 f(x)} \right) \xrightarrow{d} N\left( \lambda \left( \frac{0}{\hat{b}_0^\Xi} \right), f(x)^{-1} V^\Xi \right),
\]

\[
\sqrt{nh^{d+2}} B_h \left( \hat{\theta}_n^\Xi - \frac{\hat{D^2} f_n}{D^2 f(x)} \right) \xrightarrow{d} N\left( \lambda \left( \frac{0}{\hat{b}_0^\Xi} \right), f(x)^{-1} V^\Xi \right)
\]

if \( h \to 0, nh^{d+4} \to \infty, nh^{d+6} \to \lambda^2 \geq 0 \) as \( n \to \infty \).

### Bias

\[
\hat{b}_0^\Xi = \frac{1}{12} \text{vec}(I_d) D_d R \left( \sqrt{K} D_d^+ (\cdot)^{\otimes 2} \right)^{-1} D_d^+ \int K(z) z^{\otimes 2} (z^{\otimes 4})^T dz D^{\otimes 4} f(x)
\]

for LMME, for KDE,

\[
\hat{b}_1^\Xi = \frac{1}{6} \int K(z) z (z^{\otimes 3})^T dz D^{\otimes 3} f(x)
\]

for LMME, for KDE,

\[
\hat{b}_2^\Xi = \frac{1}{72} (D_d^+)^T R \left( \sqrt{K} D_d^{+} (\cdot)^{\otimes 2} \right)^{-1} D_d^{+} \int K(z) z^{\otimes 2} (z^{\otimes 4})^T dz D^{\otimes 4} f(x)
\]

for LMME, for KDE,
\[
\begin{align*}
\mathbb{E}_b &= \left\{ \frac{1}{T^2} \int K(z)(z^{\otimes 4})^T \, dz D^{\otimes 4} \log f(x) + \frac{1}{T} \int K(z) \left( \frac{Df(x)}{f(x)} - z \cdot D^{\otimes 3} \log f(x)^{\top} z^{\otimes 3} \right) \, dz \right. \\
&\quad + \frac{1}{12} \text{vec}(I_d)^T D_d R(\sqrt{K} D_d^\perp(\cdot)^{\otimes 2})^{-1} D_d^+ \\
&\quad \left. \times \int K(z) \left( \text{vec}(I_d) - z^{\otimes 3} \right) \left( (z^{\otimes 4})^T D^{\otimes 4} \log f(x) + 2D \log f(x)^{\top} z \cdot D^{\otimes 3} \log f(x)^{\top} z^{\otimes 3} \right) \, dz \right. \\
&\quad \left. + \frac{1}{72} \text{vec}(I_d) D_d R(\sqrt{K} D_d^\perp(\cdot)^{\otimes 2})^{-1} D_d^+ \int K(z) z^{\otimes 2}(z^{\otimes 4})^T \, dz \frac{D^{\otimes 4}f(x)}{f(x)} \right. \\
&\quad \left. \text{for } \text{LLLE, } \text{LMME, } \text{KDE,} \right.
\end{align*}
\]

\[
\begin{align*}
\mathbb{E}_b &= \left\{ \frac{1}{T^2} \int K(z)(z^{\otimes 3})^T \, dz D^{\otimes 3} \log f(x) \right. \\
&\quad \left. + \frac{1}{24} \text{vec}(I_d)^T D_d R(\sqrt{K} D_d^\perp(\cdot)^{\otimes 2})^{-1} D_d^+ \\
&\quad \left. \times \int K(z) \left( \text{vec}(I_d) - z^{\otimes 3} \right) \left( (z^{\otimes 4})^T D^{\otimes 4} \log f(x) + 2D \log f(x)^{\top} z \cdot D^{\otimes 3} \log f(x)^{\top} z^{\otimes 3} \right) \, dz \right. \\
&\quad \left. + \frac{1}{72} \text{vec}(I_d) D_d R(\sqrt{K} D_d^\perp(\cdot)^{\otimes 2})^{-1} D_d^+ \int K(z) z^{\otimes 2}(z^{\otimes 4})^T \, dz \frac{D^{\otimes 4}f(x)}{f(x)} \right. \\
&\quad \left. \text{for } \text{LLLE, } \text{LMME, } \text{KDE,} \right.
\end{align*}
\]

Variance

\[
\mathbb{V} = \left\{ \begin{array}{ll}
\mathbb{V}_{\text{LLLE}} &= \left( J_T^2 \right)^{-1} \mathbb{E}_d R(\sqrt{K} E_d^\perp(\cdot)^{\otimes 2})^{-1} R(\sqrt{K} E_d^\perp(\cdot)^{\otimes 2})^{-1} \mathbb{E}_d^\perp J_T^{-1} \mathbb{I}_d^{\otimes 2} \\
\mathbb{V}_{\text{LMME}} &= \left( J_T^2 \right)^{-1} \mathbb{E}_d R(\sqrt{K} E_d^\perp(\cdot)^{\otimes 2})^{-1} R(\sqrt{K} E_d^\perp(\cdot)^{\otimes 2})^{-1} \mathbb{E}_d^\perp J_T^{-1} \mathbb{I}_d^{\otimes 2} \\
\mathbb{V}_{\text{KDE}} &= \left[ 0 \mathbb{I}_{d+2} \right] \mathbb{V}_{\text{LLLE}} \left[ 0 \mathbb{I}_{d+2} \right]^\top \text{ for } \text{LLLE, } \\
&\quad \mathbb{E}_d R(\sqrt{K} E_d^\perp(\cdot)^{\otimes 2} \otimes DK) E_d^\perp \text{ for } \text{LMME, } \\
&\quad \left[ 0 \mathbb{I}_{d+2} \right] \mathbb{V}_{\text{LLLE}} \left[ 0 \mathbb{I}_{d+2} \right]^\top \text{ for } \text{KDE.} \\
\end{array} \right.
\]

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### Appendix

#### A.1 Auxiliary Results

**Lemma A.1.** Let \( p = d + d(d + 1)/2 \).

1. The Moore-Penrose inverse of \( D_d \) is given by

\[
D_d^+ = (D_d^T D_d)^{-1} D_d^T \in \mathbb{R}^{d(d+1) \times d^2} \quad \text{and} \quad D_d D_d^+ a = \frac{1}{2} (a + \text{vec}(\text{mat}(a)^T)) \quad \text{for any} \ a \in \mathbb{R}^{d^2}.
\]

2. Let \( x, y \in \mathbb{R}^d \), then

\[
D_d D_d^+(x \otimes y) = D_d D_d^+(y \otimes x).
\]

3. The Moore-Penrose inverse of \( E_d \) and \( \tilde{E}_d \) are given by

\[
E_d^+ = (E_d^T E_d)^{-1} E_d^T \in \mathbb{R}^{p \times d^2 + d^2} \quad \text{and} \quad \tilde{E}_d^+ = (\tilde{E}_d^T \tilde{E}_d)^{-1} \tilde{E}_d^T \in \mathbb{R}^{p \times 1 + d^2 + d^2},
\]

respectively, and

\[
E_d E_d^+(b \ a) = \left( \frac{1}{2}(a + \text{vec}(\text{mat}(a)^T)) \right) \quad \text{for any} \ \begin{bmatrix} b \\ a \end{bmatrix} \in \mathbb{R}^{d+d^2},
\]

\[
\tilde{E}_d \tilde{E}_d^+(c \ b \ a) = \left( \frac{1}{2}(a + \text{vec}(\text{mat}(a)^T)) \right) \quad \text{for any} \ \begin{bmatrix} c \\ b \\ a \end{bmatrix} \in \mathbb{R}^{1+d+d^2}.
\]

In particular, \( E_d E_d^+ \) is the orthogonal projection from \( \mathbb{R}^{d+d^2} \) onto \( \Theta \) and \( \tilde{E}_d \tilde{E}_d^+ \) is the orthogonal projection from \( \mathbb{R}^{1+d+d^2} \) onto \( \mathbb{R} \times \Theta \).
4. Left-multiplication by $E_d$ defines a isomorphism from $\mathbb{R}^p$ onto $\Theta$ and left-multiplication by $E_d^+$ is its inverse. Left-multiplication by $\tilde{E}_d$ defines a isomorphism from $\mathbb{R}^{1+p}$ onto $\mathbb{R} \times \Theta$ and left-multiplication by $\tilde{E}_d^+$ is its inverse.

**Proof.** The first part is shown in Magnus and Neudecker (1999, Chapter 3, Theorem 12). The second part is a consequence of part 1. Part 3 follows from the first part and the definition of $E_d$ and $\tilde{F}_d$. The fourth part follows from part 2 and $E_d^+E_d = I_p$. □

**Lemma A.2.** Suppose $(K0)$ holds and $\int K(z)||z||^4\,dz < \infty$, then

$$R\left(\sqrt{K}E_d^+(\cdot)^\top\right)^{-1}$$

\[
= \begin{bmatrix}
1 + \text{vec}(I_d)^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top \text{vec}(I_d) & 0 & -\text{vec}(I_d)^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} \\
0 & I_d & 0 \\
-R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top \text{vec}(I_d) & 0 & R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1}
\end{bmatrix}
\]

and

$$(E_d^+)^\top R\left(\sqrt{K}E_d^+(\cdot)^\top\right)^{-1} E_d^+$$

\[
= \begin{bmatrix}
1 + \text{vec}(I_d)^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top \text{vec}(I_d) & 0 & -\text{vec}(I_d)^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^+ \\
0 & I_d & 0 \\
-(D_d^+)^\top R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top \text{vec}(I_d) & 0 & (D_d^+)^\top R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^+
\end{bmatrix}.
\]

**Proof.** We have

$$R\left(\sqrt{K}E_d^+(\cdot)^\top\right) = \begin{bmatrix}
I_{d+1} & V^\top D_d^+ \\
(D_d^+)^\top V & R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)
\end{bmatrix} = \begin{bmatrix}
I_{d+1} & V^\top D_d \\
D_d^\top V & R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)
\end{bmatrix}$$

with

$$V := [\text{vec}(I_d) \quad 0_{d^2 \times d}].$$

Applying the Schur-complement leads

$$R\left(\sqrt{K}E_d^+(\cdot)^\top\right)^{-1} = \begin{bmatrix}
I_{d+1} + V^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top V & -V^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} \\
-R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top V & R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1}
\end{bmatrix}$$

with

$$\left(V^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top V\right)_{i,j} = \begin{cases}
\text{vec}(I_d)^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} D_d^\top \text{vec}(I_d) & \text{if } i, j = 1, \\
0 & \text{else},
\end{cases}$$

for $1 \leq i, j \leq d + 1$ and

$$V^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} = \begin{bmatrix}
\text{vec}(I_d)^\top D_d R\left(\sqrt{K}D_d^+(\cdot)^\otimes 2\right)^{-1} \\
0
\end{bmatrix}.$$

Hence the claim follows. □

**Lemma A.3.** Let $A_1, B_1 \in \mathbb{R}_{>0}$ and $A_2, B_2 \in \mathbb{R}$. Then is

$$\frac{A_1 + h^2 A_2 + o(h^2)}{B_1 + h^2 B_2 + O(h^4)} = \frac{A_1}{B_1} + h^2 \left(\frac{A_2}{B_1} - \frac{A_1 B_2}{B_1^2}\right) + o(h^2).$$
Proof. We have
\[
\left(1 + h^2 \frac{B_2}{B_1} + o(h^2)\right)^{-1} = 1 - h^2 \frac{B_2}{B_1} + o(h^2)
\]
and therefore
\[
\frac{A_1 + h^2 A_2 + o(h^2)}{B_1 + h^2 B_2 + o(h^2)} = \left(\frac{A_1}{B_1} + h^2 \frac{A_2}{B_1} + o(h^2)\right) \left(1 - h^2 \frac{B_2}{B_1} + o(h^2)\right)
\]
\[
= \frac{A_1}{B_1} + h^2 \left(\frac{A_2}{B_1} - \frac{A_1 B_2}{B_1} \right) + o(h^2).
\]

\[\square\]

Lemma A.4. Let \( B, \Gamma_h \in \mathbb{R}^{d \times d} \) and \( h_0 > 0 \) be such that \( A_h := I_d + h^2 B + \Gamma_h \) is invertible for all \( 0 < h < h_0 \) and \( \| \Gamma_h \|_F = o(h^2) \) as \( h \to 0 \). For all \( 0 < h < h_0 \), we can write
\[
A_h^{-1} = I_d - h^2 B - \Lambda_h,
\]
where \( \Lambda_h \in \mathbb{R}^{d \times d} \) with \( \| \Lambda_h \|_F = o(h^2) \) as \( h \to 0 \).

Furthermore, if \( A_h \in \mathbb{R}^{d \times d} \) for all \( 0 < h < h_0 \), then
\[
B, \Gamma_h, \Lambda_h \in \mathbb{R}^{d \times d}_{\text{sym}}.
\]

Proof. We have
\[
(I_d + h^2 B + \Gamma_h)(I_d - h^2 B - \Lambda_h) = I_d + h^2 B - h^2 B - A_h \Lambda_h + \Gamma_h (I_d - h^2 B) - h^4 B^2
\]
and taking
\[
\Lambda_h := A_h^{-1} (\Gamma_h (I_d - h^2 B) - h^4 B^2)
\]
leads that (17) is the identity matrix \( I_d \) and
\[
A_h^{-1} = I_d - h^2 B - \Lambda_h.
\]
Since the Frobenius norm is submultiplicative, we get
\[
\| \Lambda_h \|_F \leq \| A_h^{-1} \|_F \| \Gamma_h \|_F \| I_d - h^2 B \|_F + h^4 \| A_h^{-1} \|_F \| B^2 \|_F = o(h^2) \quad \text{as} \quad h \to 0.
\]
If \( A_h \) is symmetric for \( 0 < h < h_0 \), then is \( B + h^{-2} \Gamma_h \) symmetric and equivalently \( B, \Gamma_h \) are both symmetric or both asymmetric. We assume that \( B = (b_{ij})_{i,j=1}^d, \Gamma_h = (\gamma(h)_{ij})_{i,j=1}^d \) are both asymmetric. Thus, there exist indices \( 1 \leq i < j \leq d \) such that
\[
\delta := |b_{ij} - b_{ji}| = h^{-2} |\gamma(h)_{ij} - \gamma(h)_{ji}| > 0.
\]
But because \( \| \Gamma_h \|_F = o(h^2) \) we have
\[
\delta = h^{-2} |\gamma(h)_{ij} - \gamma(h)_{ji}| \to 0 \quad \text{as} \quad h \to 0,
\]
which is a contradiction to \( \delta > 0 \). So, \( B, \Gamma_h \) are symmetric for all \( 0 < h < h_0 \). Finally, \( \Lambda_h \) is symmetric, because \( A_h^{-1} \) and \( B \) are symmetric.

\[\square\]

Corollary A.5. Let \( M, B, \Gamma_h \in \mathbb{R}^{d \times d} \) and \( h_0 > 0 \) be such that \( A_h := M + h^2 B + \Gamma_h \) and \( M \) are invertible for all \( 0 < h < h_0 \) and \( \| \Gamma_h \|_F = o(h^2) \) as \( h \to 0 \). For all \( 0 < h < h_0 \), we can write
\[
A_h^{-1} = M^{-1} - h^2 M^{-1} BM^{-1} - \Lambda_h,
\]
where \( \Lambda_h \in \mathbb{R}^{d \times d} \) with \( \| \Lambda_h \|_F = o(h^2) \) as \( h \to 0 \).

Furthermore, if \( A_h, M \in \mathbb{R}^{d \times d}_{\text{sym}} \) for all \( 0 < h < h_0 \), then
\[
B, \Gamma_h, \Lambda_h \in \mathbb{R}^{d \times d}_{\text{sym}}.
\]
Proof. Apply Lemma A.4 to the matrix $M^{-1}A_k$. \hfill \Box

**Lemma A.6** (Weak law of large numbers). For $n \in \mathbb{N}$ let $Z_{n1}, Z_{n2}, \ldots, Z_{nn}$ be independent random vectors with values in a real Hilbert space $(\mathbb{H}, (\cdot, \cdot), \| \cdot \|)$ such that $\mathbb{E}\|Z_{ni}\| < \infty$ for $1 \leq i \leq n$. Further let $\mu_{ni} := \mathbb{E}(Z_{ni}) \in \mathbb{H}$. Suppose that
\[
\sum_{i=1}^{n} \mathbb{E}(\|Z_{ni}\| \min(\|Z_{ni}\|, 1)) \to 0
\]
as $n \to \infty$. Then
\[
\mathbb{E}\left\| \sum_{i=1}^{n} (Z_{ni} - \mu_{ni}) \right\| \to 0
\]
as $n \to \infty$.

**Lemma A.7** (Continuous mapping theorem). Let $(\mathcal{X}, d_{\mathcal{X}}), (\mathcal{Y}, d_{\mathcal{Y}})$ be metric spaces and $g : \mathcal{X} \to \mathcal{Y}$ a continuous function in $x \in \mathcal{X}$. Let $(Z_n)_{n \in \mathbb{N}}$ be a sequence of random variables in $\mathcal{X}$ and $(x_n)_{n \in \mathbb{N}}$ a sequence in $\mathcal{X}$, such that $d_{\mathcal{X}}(x_n, x)$ converges to 0 and $d_{\mathcal{X}}(Z_n, x_n)$ converges to 0 in probability as $n \to \infty$. Then,
\[
d_{\mathcal{Y}}(g(Z_n), g(x_n)) \xrightarrow{P} 0
\]
as $n \to \infty$.

Proof. For any $\epsilon > 0$ exists a $\delta > 0$, such that any $\tilde{x} \in \mathcal{X}$ fulfilling $d_{\mathcal{Y}}(g(\tilde{x}), g(x)) \geq \epsilon/2$ also fulfills $d_{\mathcal{X}}(\tilde{x}, x) \geq \delta$ by the continuity of $g$. Therefore,
\[
\left\{ d_{\mathcal{Y}}(g(Z_n), g(x_n)) \geq \epsilon \right\} \subset \left\{ d_{\mathcal{Y}}(g(Z_n), g(x)) + d_{\mathcal{Y}}(g(x_n), g(x)) \geq \epsilon \right\}
\]
\[
\subset \left\{ d_{\mathcal{Y}}(g(Z_n), g(x)) \geq \epsilon/2 \right\} \cup \left\{ d_{\mathcal{Y}}(g(x_n), g(x)) \geq \epsilon/2 \right\}
\]
\[
\subset \left\{ d_{\mathcal{X}}(Z_n, x) \geq \delta \right\} \cup \left\{ d_{\mathcal{X}}(x_n, x) \geq \delta \right\}
\]
\[
\subset \left\{ d_{\mathcal{X}}(Z_n, x_n) + d_{\mathcal{X}}(x_n, x) \geq \delta \right\} \cup \left\{ d_{\mathcal{X}}(x_n, x) \geq \delta \right\}
\]
\[
\subset \left\{ d_{\mathcal{X}}(Z_n, x_n) \geq \delta/2 \right\} \cup \left\{ d_{\mathcal{X}}(x_n, x) \geq \delta/2 \right\}
\]
and we obtain
\[
\mathbb{P}\left( d_{\mathcal{Y}}(g(Z_n), g(x_n)) \geq \epsilon \right) \leq \mathbb{P}(d_{\mathcal{X}}(Z_n, x_n) \geq \delta/2) + \mathbb{P}(d_{\mathcal{X}}(x_n, x) \geq \delta/2) \to 0
\]
as $n \to \infty$. \hfill \Box

**Lemma A.8.** Let $a \geq 1$ and $\int K(z) \exp(\epsilon \|z\|^a) \, dz < \infty$ for some $\epsilon > 0$, then for any number $0 \leq \delta < \epsilon$ and $b \geq 1$
\[
\int K(z) \|z\|^b \exp(\delta \|z\|^a) \, dz \leq C \int K(z) \exp(\epsilon \|z\|^a) \, dz
\]
with $C = \left( \frac{b}{e\alpha(\epsilon - \delta)} \right)^{b/a}$. In particular,
\[
\int K(z) \|z\|^b \exp(\delta \|z\|^a) \, dz < \infty
\]
and
\[
\int K(z) \max\{1, \|z\|^b\} \exp(\delta \|z\|^a) \, dz < \infty.
\]
Proof. We show first that for $z \geq 0$, $0 \leq \delta < \epsilon$ and $a, b \geq 1$

$$z^b \exp(\delta z^a) \leq \left(\frac{b}{ea(\epsilon - \delta)}\right)^{b/a} \exp(\epsilon z^a). \quad (18)$$

Obtain that

$$\sup_{z \geq 0} \frac{z^b \exp(\delta z^a)}{\exp(\epsilon z^a)} = \sup_{z \geq 0} z^b \exp(-\epsilon z^a)$$

$$= \sup_{t \geq 0} \left(\frac{t}{\epsilon - \delta}\right)^{b/a} \exp(-t)$$

$$= \left(\frac{b/a}{\epsilon - \delta}\right)^{b/a} \exp(-b/a) = \left(\frac{b}{ea(\epsilon - \delta)}\right)^{b/a}$$

because for $t > 0$

$$\frac{\partial}{\partial t} t^{b/a} \exp(-t) = \left((b/a)t^{b/a-1} - t^{b/a}\right) \exp(-t) = t^{b/a} \exp(-t) \left(b/a - t\right) \begin{cases} > 0, & t < b/a, \\ < 0, & t > b/a. \end{cases}$$

Hence, (18) holds and we get

$$\int K(z) \|z\|^b \exp(\delta \|z\|^a) \, dz \leq C \int K(z) \exp(\epsilon \|z\|^a) \, dz.$$

□

Theorem A.9 (Modified Delta Method). Let $F : \mathbb{R}^p \rightarrow \mathbb{R}^p$ be a continuous differentiable map in a neighborhood of $x$ and let the Jacobian matrix of $F$ at $x$, denoted as $J_F(x)$, have full rank. Furthermore, let $X_n$ be a sequence of random vector and $Z$ be a random vector in $\mathbb{R}^p$, such that

$$r_n \left(F(X_n) - F(x_n)\right) \overset{d}{\rightarrow} Z$$

for some sequence $x_n \rightarrow x$ and numbers $r_n \rightarrow \infty$ as $n \rightarrow \infty$. Then,

$$r_n(X_n - x_n) \overset{d}{\rightarrow} J_F(x)^{-1}Z$$

and

$$r_n \|X_n - x_n - J_F(x)^{-1}(F(X_n) - F(x_n))\| \overset{p}{\rightarrow} 0.$$

Proof. By the inverse function theorem exist a neighborhood $U$ of $x$ and neighborhood $V$ of $F(x)$ such that

$$F|_U : U \rightarrow V$$

is a diffeomorphism with inverse function

$$G : V \rightarrow U$$

and for all $u \in U$ is

$$J_G(v) = J_F(u)^{-1} \quad \text{for} \quad v = F(u).$$

Define $y := F(x)$, $y_n := F(x_n)$ and $Y_n := F(X_n)$ and let $\delta_0 > 0$ be such that $B_{\delta_0}(y) \subset V$. Let $n$ be sufficiently large such that $\|y - y_n\| < \delta_0/2$, and for $1 \leq j \leq p$ let $G_j : \mathbb{R}^p \rightarrow \mathbb{R}$ be the $j$-th component of the function $G$. Thus,

$$T_n(h) := \|G(y_n + h) - G(y_n) - J_G(y)h\|^2$$

$$= \sum_{j=1}^{p} |G_j(y_n + h) - G_j(y_n) - \nabla G_j(y)^\top h|^2$$

$$= \sum_{j=1}^{p} \left|\left(\nabla G_j(y_n + \xi_j h) - \nabla G_j(y)\right)^\top h\right|^2$$
for some $0 < \xi_0, \xi_2, \ldots, \xi_p < 1$ by the mean value theorem. By continuity of $\nabla G_j$ exists for all $\epsilon > 0$ a $0 < \delta_j \leq \delta_0$, such that

$$\|\nabla G_j(z) - \nabla G_j(y)\| < \epsilon / \sqrt{p} \quad \text{for all } z \text{ with } \|z - y\| < \delta_j.$$  

Thus, for $\delta := \min_j \{\delta_j\}$ is

$$T_n(h) \leq \sum_{j=1}^p \|\nabla G_j(y_n + \xi_j h) - \nabla G_j(y)\|^2 \|h\|^2 < \epsilon^2 \|h\|^2 \quad \text{for all } h \text{ with } \|h\| < \delta_0 / 2.$$  

Hence,

$$\|G(y_n + h) - G(y_n) - J_G(y)h\| < \epsilon \|h\| \quad \text{for all } h \text{ with } \|h\| < \delta_0 / 2.$$  

We show the second statement from the theorem. For $n$ sufficiently large and $\eta > 0$,

$$\mathbb{P}(r_n \|X_n - x_n - J_F(x)^{-1}(F(X_n) - F(x_n))\| \geq \eta)$$
$$\leq \mathbb{P}(\|Y_n - y_n\| \geq \delta / 2) + \mathbb{P}(\|Y_n - y_n\| < \delta / 2, r_n \|G(Y_n) - G(y_n) - J_G(y)(Y_n - y_n)\| \geq \eta)$$
$$\leq \mathbb{P}(\|Y_n - y_n\| \geq \delta / 2) + \mathbb{P}(\|Y_n - y_n\| < \delta / 2, r_n \|Y_n - y_n\| > \eta / \epsilon).$$

The first probability goes to zero as $n \to \infty$, because $r_n(Y_n - y_n) \xrightarrow{d} Z$ and so $r_n(Y_n - y_n)$ is tight. The second probability can be made arbitrarily small by choosing $\epsilon$ small. The first claim of the theorem follows immediately from the second claim. 

**Theorem A.10.** Let $F : \mathbb{R}^p \to \mathbb{R}^q$ be a continuously differentiable map in a neighborhood of $x_o$. Furthermore, let $X_n$ be a sequence of random vectors in $\mathbb{R}^p$, such that

$$J_F(x_o)(X_n - x_n - b_n) = O_p(r_n^{-1})$$

for some sequences $x_n \to x_o$, $J_F(x_o)b_n = O(\tilde{r}_n)$ in $\mathbb{R}^p$ and numbers $r_n \to \infty$, $\tilde{r}_n \to 0$ such that $r_n\tilde{r}_n = O(1)$ as $n \to \infty$. Then,

$$F(X_n) - F(x_n) - J_F(x_o)b_n = O_p(r_n^{-1}).$$

**Proof.** For $M, n$ sufficiently large we have

$$\mathbb{P}(r_n \|F(X_n) - F(x_n) - J_F(x_o)b_n\| > M)$$
$$\leq \mathbb{P}(r_n \|F(X_n) - F(x_n) - J_F(x_o)(X_n - x_n)\| > M / 2) + \mathbb{P}(r_n \|J_F(x_o)(X_n - x_n - b_n)\| > M / 2)$$
$$\leq \mathbb{P}(r_n \|J_F(x_o)(X_n - x_n)\| \geq M / 2)$$
$$+ \mathbb{P}(r_n \|J_F(x_o)(X_n - x_n)\| < M / 2, r_n \|F(X_n) - F(x_n) - J_F(x_o)(X_n - x_n)\| > M / 2) + o(1)$$

with

$$\mathbb{P}(r_n \|J_F(x_o)(X_n - x_n)\| \geq M / 2)$$
$$\leq \mathbb{P}(r_n \|J_F(x_o)(X_n - x_n - b_n)\| \geq M / 4) + \mathbb{P}(r_n\tilde{r}_n \|J_F(x_o)b_n\| / \tilde{r}_n \geq M / 4) = o(1)$$

where both probabilities on the right hand side go to zero by assumption. By Taylor’s Theorem we have for all $\epsilon > 0$, there exists $\delta > 0$ such that

$$\|F(x_n + h) - F(x_n) - J_F(x_o)h\| < \epsilon \|h\| \quad \text{for all } h \text{ with } \|h\| < \delta.$$  

Hence, for sufficiently large $n$ such that $M/(2r_n) < \delta$ and $\epsilon < 1$,

$$\mathbb{P}(r_n \|X_n - x_n\| < M / 2, r_n \|F(X_n) - F(x_n) - J_F(x_o)(X_n - x_n)\| > M / 2)$$
$$\leq \mathbb{P}(\|X_n - x_n\| < M/(2r_n), \|X_n - x_n\| > \epsilon^{-1} M/(2r_n)) = 0.$$
Theorem A.11. Let $F : \mathbb{R}^p \to \mathbb{R}^p$ be a continuously differentiable map in a neighborhood of $x$ and let the Jacobian matrix of $F$ at $x$ have full rank. Furthermore, let $X_n$ be a sequence of random vector in $\mathbb{R}^p$, such that

$$F(X_n) - F(x_n) - b_n = o_p(r_n^{-1})$$

for some sequences $x_n \to x$, $b_n = O(\tilde{r}_n)$ in $\mathbb{R}^p$ and numbers $r_n \to \infty$, $\tilde{r}_n \to 0$ such that $r_n\tilde{r}_n = O(1)$ as $n \to \infty$. Then,

$$X_n - x_n - J_F(x)^{-1}b_n = o_p(r_n^{-1}).$$

**Proof.** Define $y := F(x)$, $y_n := F(x_n)$, $Y_n := F(X_n)$ and let $G$ be the local inverse of $F$ at $x$ as defined in the proof of Theorem A.9. For sufficiently large $M, n$ we have

$$\mathbb{P}(\|X_n - x_n - J_F(x)^{-1}b_n\| > M)$$

$$\leq \mathbb{P}(r_n\|G(Y_n) - G(y_n) - J_G(y)(Y_n - y_n)\| > M/2) + \mathbb{P}(r_n\|J_G(y)(Y_n - y_n - b_n)\| > M/2)$$

$$\leq \mathbb{P}(r_n\|Y_n - y_n\| \geq M/2) + \mathbb{P}(r_n\|Y_n - y_n\| < M/2, r_n\|G(Y_n) - G(y_n) - J_G(y)(Y_n - y_n)\| > M/2)$$

$$+ o(1),$$

with

$$\mathbb{P}(r_n\|Y_n - y_n\| \geq M/2) \leq \mathbb{P}(r_n\|Y_n - y_n - b_n\| \geq M/4) + \mathbb{P}(r_n\|b_n\| \geq M/4)$$

where the first probability goes to zero by assumption and for the second we have

$$\mathbb{P}(r_n\|b_n\| \geq M/4) = \mathbb{P}(r_n\tilde{r}_n\|b_n\|/\tilde{r}_n \geq M/4) \to 0 \quad \text{as } n \to \infty,$$

because $r_n\tilde{r}_n = O(1)$ and $\|b_n\| = O(\tilde{r}_n)$. By the proof of Theorem A.9 we have for all $\varepsilon > 0$, there exists $\delta > 0$ such that

$$\|G(y_n + h) - G(y_n) - J_G(y)h\| < \varepsilon\|h\|$$

for all $h$ with $\|h\| < \delta/2$.

Hence, for sufficiently large $n$ such that $M/r_n < \delta$ and $\varepsilon < 1$,

$$\mathbb{P}(r_n\|Y_n - y_n\| < M/2, r_n\|G(Y_n) - G(y_n) - J_G(y)(Y_n - y_n)\| > M/2)$$

$$\leq \mathbb{P}(r_n\|Y_n - y_n\| < M/2, r_n\varepsilon\|Y_n - y_n\| > M/2) = 0.$$

\[ \square \]

**A.2 Proofs of the Results in Section 3.1**

**Proof of Lemma 3.3.** By Taylor’s Theorem we know that for $x, z \in \mathbb{R}^d$ and $h \geq 0$ there exist $\xi(z) \in [0, h]$ such that

$$f(x + hz) = \sum_{|\gamma| < L} \frac{h^{|\gamma|}}{\gamma!} f^{(\gamma)}(x)z^\gamma + \sum_{|\gamma| = L} \frac{h^L}{\gamma!} f^{(\gamma)}(x + \xi(z)z)z^\gamma$$

$$= \sum_{|\gamma| \leq L} \frac{h^{|\gamma|}}{\gamma!} f^{(\gamma)}(x)z^\gamma + h^L \sum_{|\gamma| = L} \frac{1}{\gamma!} (f^{(\gamma)}(x + \xi(z)z) - f^{(\gamma)}(x))z^\gamma.$$ 

Thus,

$$\int F(z)f(x + hz)dz = \sum_{|\gamma| \leq L} \frac{h^{|\gamma|}}{\gamma!} f^{(\gamma)}(x) \int F(z)z^\gamma dz + h^L \sum_{|\gamma| = L} \frac{1}{\gamma!} \int (f^{(\gamma)}(x + \xi(z)z) - f^{(\gamma)}(x))F(z)z^\gamma dz.$$
It remains to show, that each summand in the second expression converges to 0 as $h \to 0$. For all $B > 0$ and $|\gamma| = L$

$$\left| \int (f^{(\gamma)}(x + \xi(z)z) - f^{(\gamma)}(x)) F(z)z^\gamma dz \right|$$

\begin{align*}
\leq \sup_{\|\tilde{z}\| \leq hB} \left| f^{(\gamma)}(x + \tilde{z}) - f^{(\gamma)}(x) \right| \int |F(z)||z|^L dz + 2 \sup_{y \in \mathbb{R}^d} |f^{(\gamma)}(y)| \int |F(z)||z|^L dz.
\end{align*}

The first summand goes to zero by the continuity of $f^{(\gamma)}$ as $h \to 0$ for any $B > 0$ and the second summand goes to zero as $B \to \infty$. In particular, if $F(z) = K(z)z^n$ the second claim follows.

**Proof of Theorem 3.4.** We define a triangular array of random variables for $n \in \mathbb{N}$ and $1 \leq i \leq n$ as

$$Z_{ni} = \frac{1}{nhL} K_h \left( x_i - x \right) \left( \frac{X_i - x}{h} \right)^\alpha,$$

such that

$$\frac{s_{n,h}^\alpha(x)}{h^L} = \frac{1}{nhL} \sum_{i=1}^n K_h \left( x_i - x \right) \left( \frac{X_i - x}{h} \right)^\alpha = \sum_{i=1}^n Z_{ni}.$$

Then

$$\sum_{i=1}^n \mathbb{E}(|Z_{ni}| \min\{|Z_{ni}|, 1\}) \leq \sum_{i=1}^n \mathbb{E}(|Z_{ni}|^2)$$

\begin{align*}
= \frac{1}{nh^{d+2L}} \int |K(z)z^n|^2 f(x + hz) dz \leq \frac{1}{nh^{d+2L}} \|f\|_{\infty}\|K\|_{\infty} \int K(z)||z||^{2\alpha} dz.
\end{align*}

The first claim follows from the WLLN, see Lemma A.6, because

$$\sum_{i=1}^n \mathbb{E}(|Z_{ni}| \min\{|Z_{ni}|, 1\}) \to 0$$

as $n \to \infty$ and $nh^{d+2L} \to \infty$. Furthermore,

$$\mathbb{E} \left| \frac{s_{n,h}^\alpha(x)}{h^{|\alpha|}} - \frac{\mu_{\alpha+}}{(\alpha+)!} f^{(\alpha)}(x) \right| \leq \mathbb{E} \left| \frac{s_{n,h}^\alpha(x)}{h^{|\alpha|}} - \frac{s_h^\alpha(x)}{h^{|\alpha|}} \right| + \left| \frac{s_h^\alpha(x)}{h^{|\alpha|}} - \frac{\mu_{\alpha+}}{(\alpha+)!} f^{(\alpha)}(x) \right| \to 0$$

as $n \to \infty$, $h \to 0$ and $nh^{d+2|\alpha^-|} \to \infty$, because the first term converges to zero by the first claim and the second term converges to zero by Lemma 3.3.

**Proof of Theorem 3.5.** For

$$Y_{ni} = n^{-1/2} h^{d/2} K_h(X_i - x) \left( \frac{X_i - x}{h} \right)^\top,$$

we have

$$\mathbb{E} Y_{ni} = n^{-1/2} h^{d/2} S_h^\top(x)$$

and, using Lemma 3.3,

$$\text{Var} Y_{ni} = n^{-1} h^d \left( \mathbb{E} \left( K_h^2(X_i - x) \left( \frac{X_i - x}{h} \right)^\top \left( \frac{X_i - x}{h} \right)^\top \right) - S_h^\top(x) S_h^\top(x)^\top \right)$$

\begin{align*}
= n^{-1} f(x) R(K(\cdot)^\top) + O(n^{-1} h).
\end{align*}
The Lyapunov condition is fulfilled as \( n \rightarrow \infty \) and

\[
\operatorname{Var}(\sqrt{nh^d} S_n(x)) = \operatorname{Var}\left(\sum_{i=1}^{n} Y_{ni}\right) \rightarrow f(x)R(K(\cdot)^\top)
\]

Applying Lindeberg’s Central Limit Theorem shows the claim.

Thus,

\[
\sum_{i=1}^{n} \mathbb{E}\|Y_{ni}\|^{2+\delta} = n(n^{-1/2}h^{d/2})^{2+\delta} \int K_h(y - x)^{2+\delta} \left\| \left( \frac{y - x}{h} \right)^\top \right\|^{2+\delta} f(y) \, dy
\]

\[
= n(n^{-1/2}h^{d/2})^{2+\delta} \int K(z)^{2+\delta} \left\| z \right\|^{2+\delta} f(x + z) \, dz = O((nh^d)^{-\delta/2}).
\]

The statement in the remark holds, because for each \( \hat{S} \) is localizing. To see that \( \hat{S} \) is proportionally locally proper. The statement in the remark holds, because for each \( w \in \mathcal{W} \) we use the strict propriety with respect to \( \mathcal{H}_w \subset \mathcal{P} \).

\section{Proofs of the Results in Section 3.2}

\textbf{Proof of Theorem 3.7.} Let \( w \in \mathcal{W} \) be a weight function. For \( h, g \in \mathcal{H} \) with \( h = g \) on \( \{ w > 0 \} \) either it holds that \( h, g \in \mathcal{H}_w \) with \( h_w = g_w \) and therefore \( \hat{S}(h, y, w) = \hat{S}(g, y, w) \) on \( \mathbb{R}^d \), or \( h, g \in \mathcal{H} \setminus \mathcal{H}_w \) and both scores are infinite by definition. Thus, \( \hat{S} \) is a localizing weighted scoring rule.

Let be \( p \in \mathcal{P} \), such that there exists \( g \in \mathcal{H} \) with \( g = p \) on \( \{ w > 0 \} \). For \( h \in \mathcal{H}_w \) we have

\[
\hat{S}(g, p, w) = \int w(y)S(g_w, y)p(y) \, dy = \int S(p_w, y)p_w(y) \, dy \int w(z)p(z) \, dz
\]

\[
\leq \int S(h_w, y)p_w(y) \, dy \int w(z)p(z) \, dz = \int w(y)S(h, y)p(y) \, dy = \hat{S}(h, p, w) \quad (19)
\]

by propriety of \( S \) and \( g_w = p_w \). For \( h \in \mathcal{H} \setminus \mathcal{H}_w \) we have

\[
\hat{S}(g, p, w) \leq \hat{S}(h, p, w) = \infty \quad (20)
\]

by definition of \( \hat{S} \). Equation (19) and (20) show that \( \hat{S} \) is locally proper if \( S \) is proper. Further, if \( S \) is strictly proper and \( g \in \mathcal{H}, p \in \mathcal{P} \) such that \( g \propto p \) on \( \{ w > 0 \} \), then is \( g_w = p_w \). By Equation (19) and (20) we obtain

\[
\hat{S}(g, p, w) \leq \hat{S}(h, p, w) \quad \text{for all} \quad h \in \mathcal{H}
\]

with equality if, and only if, \( g_w = p_w = h_w \) or equivalently \( g \propto p \propto h \) on \( \{ w > 0 \} \). Therefore, \( \hat{S} \) is proportionally locally proper. The statement in the remark holds, because for each \( w \in \mathcal{W} \) we use the strict propriety with respect to \( \mathcal{H}_w \subset \mathcal{P} \).

\textbf{Proof of Theorem 3.8.} The weighted scoring rule \( S_Q \) depends on \( h \in \mathcal{H} \) only through \( \int w(z)h(z) \, dz \), whence \( S_Q \) is localizing. To see that \( S_Q \) is proper, let \( w \in \mathcal{W} \) and \( p \in \mathcal{P} \), such that there exists \( g \in \mathcal{H} \) with \( g = p \) on
\{w > 0\}$. For $h \in \mathcal{H}_w$, we obtain

\[
\frac{S_Q(g, p, w)}{m_w} = Q \left( \frac{\int w(y)g(y) dy}{m_w}, 1 \right) \int w(y)p(y) dy m_w + Q \left( \frac{\int w(y)g(y) dy}{m_w}, 0 \right) \left( 1 - \frac{\int w(y)p(y) dy}{m_w} \right)
\]

\[
= Q \left( \frac{\int w(y)g(y) dy}{m_w}, \int w(y)p(y) dy \right)
\]

\[
= Q \left( \frac{\int w(y)h(y) dy}{m_w}, \int w(y)p(y) dy \right)
\]

\[
\leq Q \left( \frac{\int w(y)h(y) dy}{m_w}, \int w(y)p(y) dy \right)
\]

\[
= Q \left( \frac{\int w(y)h(y) dy}{m_w}, 1 \right) \int w(y)p(y) dy m_w + Q \left( \frac{\int w(y)h(y) dy}{m_w}, 0 \right) \left( 1 - \frac{\int w(y)p(y) dy}{m_w} \right)
\]

\[
= S_Q(h, p, w).
\]

For $h \in \mathcal{H}_w \setminus \mathcal{H}_w$ we have $S_Q(g, p, w) < S_Q(h, q, w) = \infty$.

As a sum of two localizing weighted scoring rules is $\hat{S}$ a localizing weighted scoring rule, too. For each $w \in \mathcal{W}$, let be $p \in \mathcal{P}$, such that there exists $g \in \mathcal{H}$ with $g = p$ on $\{w > 0\}$. By the propriety of $S_Q$ and $\hat{S}$ we have

\[
\hat{S}(g, p, w) = S_Q(g, p, w) + \hat{S}(g, p, w) \leq S_Q(h, p, w) + \hat{S}(h, p, w) = \hat{S}(h, p, w) \quad \text{for all} \quad h \in \mathcal{H}.
\]

To see that $\hat{S}$ is strictly locally proper, suppose that the above inequality is indeed an equality. The propriety of $S_Q$ and the proportional propriety of $\hat{S}$ imply, that $\hat{S}(g, p, w) = \hat{S}(h, p, w)$ and $S_Q(g, p, w) = S_Q(h, p, w)$. The first identity implies $h \propto g \propto p$ on $\{w > 0\}$, by the proportional propriety of $\hat{S}$. The second identity implies

\[
\int w(z)g(z) dz = \int w(z)p(z) dz = \int w(z)h(z) dz,
\]

by the strict propriety of $Q$ and Equation (21). Both statements together imply $g = p = h$ on $\{w > 0\}$. \qed

### A.4 Proofs of the Results in Section 3.3

**Lemma A.12.** Suppose (LLLE K1) holds, then

\[
S^2_h(x) = \int K(z)z^T \exp(D^T f(x) J^T z^T) dz = \begin{cases} \frac{h^3}{b_0} + o(h^3) & \text{if (LLLE A1) holds,} \\ \frac{h^3}{b_1} + o(h^4) & \text{if (LLLE A2) holds,} \end{cases}
\]

with

\[
b_0 = \frac{1}{12} \int K(z)z^{(4)} dz D^{(4)} \log f(x) \cdot f(x) + \frac{1}{6} \int K(z) (D f(x)^T z \cdot D^{(3)} \log f(x)^T z^{(3)}) dz,
\]

\[
b_1 = \frac{1}{6} \int K(z)z^{(3)} dz D^{(3)} \log f(x) \cdot f(x),
\]

\[
b_2 = \frac{1}{12} \int K(z)z^{(2)}(z^{(4)}) dz D^{(4)} \log f(x) \cdot f(x) + \frac{1}{6} \int K(z)z^{(2)}(D f(x)^T z \cdot D^{(3)} \log f(x)^T z^{(3)}) dz.
\]

**Proof.** Let

\[
\log f(x + hz) = D^T \log f(x)^T J^T (hz) + h^3 R(z)
\]
where 
\[ R(z) := \frac{1}{h^3} (\log f(x + h z) - D^2 \log f(x)^\top J^T B_h^T z z^\top). \]

By Taylor’s Theorem we have 
\[ R(z) = \left\{ \begin{array}{ll} 
\frac{1}{6} D^{\odot 3} \log f(x)^\top z^{\odot 3} + \frac{1}{6} \left( D^{\odot 3} \log f(x + h \xi_2(z) z) - D^{\odot 3} \log f(x) \right)^\top z^{\odot 3} \\
\frac{1}{12} D^{\odot 3} \log f(x)^\top z^{\odot 3} + h \frac{1}{6} D^{\odot 4} \log f(x)^\top z^{\odot 4} + \frac{h}{12} \left( D^{\odot 4} \log f(x + h \xi_3(z) z) z - D^{\odot 4} \log f(x) \right)^\top z^{\odot 4} 
\end{array} \right. \]

for some numbers \( \xi_2(z), \xi_3(z) \in [0, 1] \). Furthermore, 
\[ 1 - \exp(-t) = t - \frac{1}{2} \exp(\xi_4(t)t^2) \]

for some numbers \( t \in \mathbb{R} \) and \( \xi_4(t) \in [0, 1] \). Thus, 
\[ S_{h}^T(x) = \int K(z) z^T \exp(D^T f(x)^\top B_h^T J^T B_h^T z^\top) \, dz \]
\[ = \int K(z) z^T \left( f(x + h z) - \exp(D^T \log f(x)^\top J^T B_h^T z z^\top) \right) \, dz \]
\[ = \int K(z) z^T f(x + h z) (1 - \exp(-h^3 R(z))) \, dz \]
\[ = h^3 \int K(z) z^T f(x + h z) R(z) \, dz - \frac{h^6}{2} \int K(z) z^T \exp(h^3 \xi_4(z, h) R(z)) R(z)^2 \, dz. \]

For any multi-index \( \gamma \in \mathbb{N}_0^n \) with \(|\gamma| \leq 2\) we have by Lemma A.8 
\[ \int K(z) z^\gamma \exp(h^3 \xi_4(z, h) R(z)) R(z)^2 \, dz \]
\[ \leq \frac{1}{h} \int K(z) \max \{ 1, ||z||^8 \} \, \log f \|_{2\infty,2} \exp(h^2 \| \log f \|_{2\infty,2} \| z \|^2) = O(h^{-1}) \]

for \( h \) sufficiently small, provided \( \int K(z) \exp(\varepsilon \| z \|^2) \, dz < \infty \) for some number \( \varepsilon > 0 \). Using 
\[ f(x + h z) = f(x) + h D f(x + h \xi_5(z) z)^\top z \]

for some number \( \xi_5(z) \in [0, 1] \) and the same arguments as in Lemma 3.3 lead to 
\[ \int K(z) z^T f(x + h z) R(z) \, dz = \frac{1}{6} \int K(z) z^T (z^{\odot 3})^\top dz D^{\odot 3} f(x) \cdot f(x) + o(1) \]
if (LLLE A1) holds and 
\[ \int K(z) z^T f(x + h z) R(z) \, dz = \frac{1}{6} \int K(z) z^T (z^{\odot 3})^\top dz D^{\odot 3} \log f(x) \cdot f(x) \]
\[ + \frac{h}{12} \int K(z) z^T (z^{\odot 4})^\top dz D^{\odot 4} \log f(x) \cdot f(x) + \frac{h}{6} \int K(z) z^T (D f(x)^\top z \cdot D^{\odot 3} \log f(x)^\top z^{\odot 3}) \, dz + o(h) \]
if (LLLE A2) holds. The above considerations and (K0) show the claims. 

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Proof of Theorem 3.9. Let

\[ F : \mathbb{R}^{1+p} \to \mathbb{R}^{1+p} ; \psi \mapsto \tilde{E}_d^+ \int K(z)z^\top \exp(\psi^\top \tilde{E}_d^+ z^\top) dz \]

with positive definite Jacobian matrix

\[ J_F(\psi) = \tilde{E}_d^+ \int K(z)z^\top (z^\top)^\top \exp(\psi^\top \tilde{E}_d^+ z^\top) dz (\tilde{E}_d^+)^\top . \]

We apply Theorem A.11 with

\[
\begin{align*}
&\quad r_n := \sqrt{nh^d}, \\
&\hat{r}_n := h^3, \\
&X_n := \tilde{E}_d^T B_h^T J^T \left( \frac{\hat{\theta}^{\text{MLE}}}{\hat{\theta}^{\text{MLE}}} \right), \\
x_n := \tilde{E}_d^T B_h^T J^T D^2 \log f(x) \to x_0 := (\log f(x), 0, \ldots, 0)^\top \in \mathbb{R}^{1+p}, \text{ as } n \to \infty, \\
J_F(x_0) = f(x) R(\sqrt{K} \tilde{E}_d^+ (\cdot)^\top), \\
b_n := \tilde{E}_d^+ b_n = \tilde{E}_d^+ S_h^2(x) - F(x_n) = O(h^3).
\end{align*}
\]

By Theorem 3.5 we have

\[ F(X_n) - F(x_n) - b_n = \tilde{E}_d^+ (S_h^2(x) - S_h^2(x)) = O_p((nh^d)^{-1/2}) \]

and hence

\[ \tilde{E}_d^+ J^T B_h^T \left( \frac{\hat{\theta}^{\text{MLE}}}{\hat{\theta}^{\text{MLE}}} \right) = \tilde{E}_d^+ J^T B_h^T D^2 \log f(x) + f(x)^{-1} R(\sqrt{K} \tilde{E}_d^+ (\cdot)^\top)^{-1} b_n + O_p((nh^d)^{-1/2}). \]

We multiply this equation by \((B_h^T)^{-1}(J^T)^{-1}\tilde{E}_d^+ \) from the left and note that

\[
(J^T)^{-1} f(x)^{-1} \tilde{E}_d R(\sqrt{K} \tilde{E}_d^+ (\cdot)^\top)^{-1} b_n = \begin{cases} 
0 & \text{if (LLLE A1) holds}, \\
\begin{pmatrix} 0 \\ h^3 b_1^{\text{MLE}} \\ 0 \\ h^4 b_0^{\text{MLE}} \\ h^3 b_1^{\text{MLE}} \\ h^4 b_2^{\text{MLE}} \end{pmatrix} & \text{if (LLLE A2) holds},
\end{cases}
\]

where we used Lemma A.2. Thus,

\[
\begin{pmatrix} \frac{\hat{\theta}_1^{\text{MLE}}}{\hat{\theta}_n^{\text{MLE}}} \\ \frac{\hat{\theta}_2^{\text{MLE}}}{\hat{\theta}_n^{\text{MLE}}} \end{pmatrix} = D^2 \log f(x) + \begin{pmatrix} 0 \\ h^2 b_1^{\text{MLE}} \\ 0 \end{pmatrix} + (B_h^T)^{-1} o(h^3) + (B_h^T)^{-1} O_p((nh^d)^{-1/2}) \quad \text{if (LLLE A1) holds},
\]

leading to the consistency result if \(nh^{d+4} \to \infty\) and

\[
\begin{pmatrix} \frac{\hat{\theta}_1^{\text{MLE}}}{\hat{\theta}_n^{\text{MLE}}} \\ \frac{\hat{\theta}_2^{\text{MLE}}}{\hat{\theta}_n^{\text{MLE}}} \end{pmatrix} = D^2 \log f(x) + \begin{pmatrix} h^4 b_1^{\text{MLE}} \\ h^2 b_1^{\text{MLE}} \\ h^4 b_2^{\text{MLE}} \end{pmatrix} + (B_h^T)^{-1} o(h^4) + (B_h^T)^{-1} O_p((nh^d)^{-1/2}) \quad \text{if (LLLE A2) holds}.
\]

\[ \square \]
Proof of Theorem 3.10. Let

\[ F : \mathbb{R}^{1+p} \to \mathbb{R}^{1+p}; \psi \mapsto \hat{E}_d^+ \int K(z)z^\top \exp(\psi^\top \hat{E}_d^+ z^\top) \, dz \]

with positive definite Jacobian matrix

\[ J_F(\psi) = \hat{E}_d^+ \int K(z)z^\top (z^\top z)^\top \exp(\psi^\top \hat{E}_d^+ z^\top) \, dz \, (\hat{E}_d^+)^\top. \]

For

\[ r_n := \sqrt{nh^d}, \]
\[ X_n := \hat{E}_d^+ B_h^\top J^\top \left( \frac{\sigma^2_{\text{LLE}}}{\hat{\theta}_n} \right), \]
\[ x_n := \hat{E}_d^+ B_h^\top J^\top D^\top \log f(x) \to x_o := \left( \log f(x), 0, \ldots, 0 \right)^\top \in \mathbb{R}^{1+p}, \text{ as } n \to \infty, \]

we observe by Lemma A.12 that

\[ r_n (\hat{E}_d^+ S_h^\top(x) - F(x_n)) = \sqrt{nh^d} \hat{E}_d^+ \left( S_h^\top(x) - \int K(z)z^\top \exp(D^\top \log f(x)^\top J^\top B_h^\top z^\top) \, dz \right) \to \lambda \left( \begin{array}{c} 0 \\ \tilde{b}_{\text{LSE}} \end{array} \right) \]

and by Theorem 3.5

\[ \sqrt{nh^d} \hat{E}_d^+ (S_h^\top(x) - S_h^\top(x)) \xrightarrow{d} \mathcal{N}(0, f(x)R(K \hat{E}_d^+ (\cdot)^\top)) \].

Hence, by Slutsky’s Lemma we have

\[ r_n (F(X_n) - F(x_n)) = \sqrt{nh^d} \hat{E}_d^+ \left( \int K(z)z^\top (\tilde{c}_n + \hat{\theta}_n^\top J^\top (hz)^\top) \, dz - \int K(z)z^\top \exp(D^\top \log f(x)^\top J^\top (hz)^\top) \, dz \right) \]

\[ = \sqrt{nh^d} \hat{E}_d^+ \left( S_h^\top(x) - S_h^\top(x) + S_h^\top(x) - \int K(z)z^\top \exp(D^\top \log f(x)^\top J^\top (hz)^\top) \, dz \right) \]

\[ \xrightarrow{d} \mathcal{N} \left( \lambda \left( \begin{array}{c} 0 \\ \tilde{b}_{\text{LSE}} \end{array} \right), f(x)R(K \hat{E}_d^+ (\cdot)^\top) \right) \].

For the asymptotic law of the LMME we apply the modified delta method (Lemma A.9) with \( J_F(x_o)^{-1} = f(x)^{-1}R(\sqrt{K \hat{E}_d^+ (\cdot)^\top})^{-1} \). Thus

\[ \sqrt{nh^d} \hat{E}_d^+ B_h^\top J^\top \left( \left( \frac{\sigma^2_{\text{LLE}}}{\hat{\theta}_n} \right) - D^\top \log f(x) \right) = r_n (X_n - x_n) \]

\[ \xrightarrow{d} \mathcal{N} \left( \lambda \left( \begin{array}{c} 1 \\ \int K(z)z^\top \, dz \end{array} \right)^{-1} d^\top D^\top \log f(x), f(x)^{-1}R(\sqrt{K \hat{E}_d^+ (\cdot)^\top})^{-1} R(K \hat{E}_d^+ (\cdot)^\top) R(\sqrt{K \hat{E}_d^+ (\cdot)^\top})^{-1} \right) \].

The second claim follows directly from the first claim.

\[ \square \]

A.5 Proofs of the Results in Section 3.4

Proof of Theorem 3.12. Denote

\[ \hat{\theta}_n := \arg \min_{\theta \in \Theta} L_n^{\text{LSE}}(\theta) \quad \text{and} \quad \hat{\psi}_n := \arg \min_{\psi \in \mathbb{R}^p} L_n^{\text{LSE}}(E_d \psi). \]
By Lemma A.1 is \( \hat{\psi} = E_d^\top \hat{\theta} \) and \( \hat{\theta} = E_d \hat{\psi} \). Thus,

\[
L_n^{\text{LHSE}}(E_d \hat{\psi}) = \frac{1}{2} \hat{\psi}^\top E_d^\top B_h E_d E_d^\top (S_n^* \otimes I_d)(E_d^\top)^\top E_d^\top B_h E_d \hat{\psi} + \psi^\top E_d B_h E_d E_d^\top v_n
\]

and

\[
\hat{\psi}_n = -(E_d^\top B_h E_d)^{-1} (E_d^\top (S_n^* \otimes I_d)(E_d^\top)^\top)^{-1} E_d^\top v_n.
\]

Note that

\[
E_d (E_d^\top B_h E_d)^{-1} = \begin{bmatrix} I_d & 0 \\ 0 & h^{-1} D_d (D_d^\top D_d)^{-1} \end{bmatrix} = \begin{bmatrix} I_d & 0 \\ 0 & h^{-1} (D_d^\top)^\top \end{bmatrix} = B_h^{-1} (E_d^\top)^\top
\]

and we obtain

\[
\hat{\theta}_n = E_d \hat{\psi}_n = -B_h^{-1} (E_d^\top)^\top (E_d^\top (S_n^* \otimes I_d)(E_d^\top)^\top)^{-1} E_d^\top v_n.
\]

For the last claim, consider the quadratic function

\[
L_n^{\text{LHSE}}(\theta) := \frac{1}{2} \theta^\top B_h (S_n^* \otimes I_d) B_h \theta + \theta^\top B_h v_n
\]

with unique minimizer \(-B_h^{-1}(S_n^* \otimes I_d)^{-1} v_n\). Since \( L_n^{\text{LHSE}} = \hat{L}_n^{\text{LHSE}} \) on \( \Theta \) it is also a minimizer of \( L_n^{\text{LHSE}} \).

In the proofs of Theorem 3.13 and 3.14 we need the results stated in Lemma A.13 – A.16.

**Lemma A.13.** \( S_n^* \) is positive definite and almost surely strictly positive definite, if and only if,

\[
\sum_{i=1}^n \mathbb{I}\{K_h(X_i - x) > 0\} \geq d + 1.
\]

Moreover, when the inverse of \( S_n^* \) exists, we have

\[
(S_n^*)^{-1} = \begin{bmatrix} s_n^{-1} + s_n^{-2} s_n | S_n^{-1} | s_n & -s_n^{-2} s_n | S_n^{-1} | s_n \\ -s_n^{-2} s_n | S_n^{-1} | s_n & s_n^{-1} | S_n^{-1} | s_n \end{bmatrix} = s_n^{-1} \left[ 1 + \frac{s_n}{s_n^{-1}} \frac{S_n}{s_n} - \frac{s_n}{s_n^{-1}} \frac{S_n}{s_n} \right],
\]

where

\[
\tilde{S}_n = \frac{S_n}{s_n} - \frac{s_n s_n^\top}{s_n^2}.
\]

**Proof.** For all \( y \in \mathbb{R}_\neq 0_n^{d+1} \) is

\[
y^\top S_n^* y = \frac{1}{n} \sum_{i=1}^n K_h(X_i - x) y^\top \begin{bmatrix} 1 \\ (X_i - x)^\top \end{bmatrix}(X_i - x)(X_i - x)^\top y = \frac{1}{n} \sum_{i=1}^n K_h(X_i - x) \left\| y^\top \begin{bmatrix} 1 \\ (X_i - x) \end{bmatrix} \right\|^2 \geq 0
\]

with strict inequality, if and only if, for all \( y \in \mathbb{R}_\neq 0_n^{d+1} \) exists \( 1 \leq i \leq n \) with \( K_h(X_i - x) > 0 \) such that \( \|y^\top(X_i - x)^\top\|^2 > 0 \), or equivalently

\[
\text{span}\left\{ \begin{bmatrix} 1 \\ X_i - x \end{bmatrix} ; 1 \leq i \leq n \text{ with } K_h(X_i - x) > 0 \right\} = \mathbb{R}_n^{d+1}.
\]

The last statement holds almost surely, whenever \( K_h(X_i - x) > 0 \) for at least \( d+1 \) observations. The second statement is an application of the Schur-complement. \( \square \)
Lemma A.14. Suppose (LHSE K1), (A1) and for if $h \to 0$, $nh^d \to \infty$ as $n \to \infty$, we have

$$
(E_d^+ (S_n^* \otimes I_d) (E_d^+)\top)^{-1} = f(x)^{-1} E_d^+ E_d + f(x)^{-1} E_d^+ \begin{bmatrix} h^2 M_1 & -h(Df(x)^\top \otimes I_d) \\ -h(Df(x)^\top \otimes I_d)^\top & h^2 M_2 \end{bmatrix} E_d = o(h^2) + o_p(1)
$$

and for if $h \to 0 nh^{d+2} \to \infty$ as $n \to \infty$

$$
B_h^{-1} (E_d^+ (E_d^+ (S_n^* \otimes I_d) (E_d^+)\top)^{-1} E_d^+) B_h
$$

$$
= f(x)^{-1} E_d E_d^+ \begin{bmatrix} I_d & 0 \\ -Df(x)^\top \otimes I_d & I_d \end{bmatrix} E_d E_d^+ + f(x)^{-1} E_d E_d^+ \begin{bmatrix} h^2 M_1 + o(h^2) & -h^2(Df(x)^\top \otimes I_d) + o(h^3) \\ o(h) & h^2 M_2 + o(h^2) \end{bmatrix} E_d E_d^+ = o_p(1),
$$

with

$$
\begin{align*}
M_1 &= \frac{1}{2} \frac{Df(x)^\top Df(x)}{f(x)^2} I_d + \frac{1}{2} \frac{Df(x)Df(x)^\top}{f(x)^2}, \\
M_1 &= \bar{M}_1 - \frac{1}{2} \frac{D^2 f(x)^\top}{f(x)} \text{vec}(I_d) I_d, \\
\bar{M}_2 &= -\frac{1}{2} \left( \int K(z)zz^\top (I_d \otimes \text{vec}(I_d)^\top) dz - (I_d \otimes \text{vec}(I_d)^\top) \left( I_d \otimes \frac{D^2 f(x)^\top}{f(x)} \right) \right) + \frac{Df(x)Df(x)^\top}{f(x)^2}, \\
M_2 &= \bar{M}_2 \otimes I_d - \left( \frac{D^2 f(x)^\top}{f(x)} \text{vec}(I_d) \right) \text{vec}(I_d).
\end{align*}
$$

Furthermore,

$$
E^{-2} \left( (E_d^+ (S_n^* \otimes I_d) (E_d^+)\top)^{-1} - (E_d^+ (S_n^* \otimes I_d) (E_d^+)\top)^{-1} \right) \xrightarrow{p} 0
$$

if $h \to 0$, $nh^{d+4} \to \infty$ as $n \to \infty$.

Proof. Let be

$$
S_n^* := E(S_n^*) = \begin{bmatrix} s_{h} & s_{\bar{h}}^T \\ s_h & S_h \end{bmatrix}
$$

and

$$
\bar{S}_h := \frac{S_h}{s_h} - \frac{s_h s_{\bar{h}}^T}{s_h^2}.
$$

By the Schur-complement is

$$
(E_d^+ (S_n^* \otimes I_d) (E_d^+)\top)^{-1} = \begin{bmatrix} s_h I_d & (s_{h}^\top \otimes I_d) (D_d^+)\top \\ D_d^+ (s_h \otimes I_d) & D_d^+ (S_h \otimes I_d) D_d^+ \end{bmatrix}^{-1}
$$

$$
= \frac{1}{s_h} \begin{bmatrix} I_d + (s_{\bar{h}}^T \otimes I_d) (D_d^+)\top \bar{S}_h^{-1} D_d^+ (s_h \otimes I_d) \\ -\bar{S}_h^{-1} D_d^+ (s_{\bar{h}}^T \otimes I_d) \end{bmatrix},
$$

where

$$
\bar{S}_h := D_d^+ (S_h \otimes I_d) (D_d^+)\top.
$$

By Lemma 3.3 is

$$
s_h = f(x) + h^\frac{1}{2} \int K(z)(z \otimes 2)^\top dz + o(h^2) = f(x) + h^\frac{1}{2} D^2 f(x)^\top \text{vec}(I_d) + o(h^2),
$$

$$
s_{\bar{h}} = hDf(x) + o(h^2)
$$

$$
S_h = f(x)I_d + h^\frac{1}{2} \int K(z)zz^\top (I_d \otimes D^2 f(x)) dz = f(x)I_d + o(h^2),
$$

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and by Lemma A.3 we observe
\[ s_h^{-1} = \frac{1}{f(x)} - \frac{h^2}{2} \frac{D^{\otimes 2} f(x)^\top}{f(x)^2} \text{vec}(I_d) + o(h^2), \]
\[ \frac{s_h}{s_h'} = h \frac{D f(x)}{f(x)} + o(h^2), \]
\[ \frac{S_h}{s_h} = I_d + \frac{h^2}{2} \left( \int K(z) (zz^\top - I_d) \otimes (z^{\otimes 2})^\top dz \right) \left( I_d \otimes \frac{D^{\otimes 2} f(x)}{f(x)} \right) + o(h^2). \]

Thus,
\[ \left( D_d^+ (S_h \otimes I_d) (D_d^+)\top \right)^{-1} = D_d^+ D_d + h^2 D_d^+ (\tilde{M}_2 \otimes I_d) D_d + o(h^2), \]
where we used that
\[ \left( D_d^+ (D_d^+)\top \right)^{-1} = ((D_d^+ D_d)^{-1} D_d^+ D_d (D_d^+ D_d)^{-1})^{-1} = D_d^+ D_d \]
and
\[ \left( D_d^+ (D_d^+)\top \right)^{-1} D_d^+ = D_d^+ D_d (D_d^+ D_d)^{-1} D_d^+ = D_d^+. \]

Furthermore,
\[ \left( \frac{s_h}{s_h} \otimes I_d \right) (D_d^+)^\top (D_d^+ (S_h \otimes I_d) (D_d^+)\top)^{-1} D_d^+ \left( \frac{s_h}{s_h} \otimes I_d \right) \]
\[ = h^2 \left( \frac{D f(x)}{f(x)^\top} \otimes I_d \right) (D_d^+)^\top D_d^+ D_d \left( \frac{D f(x)}{f(x)} \otimes I_d \right) + o(h^2) \]
\[ = h^2 \frac{D f(x)}{f(x)^\top} \otimes I_d \left( D_d^+ \left( \frac{D f(x)}{f(x)} \otimes I_d \right) \right) + o(h^2) \]
\[ = \frac{h^2}{2} \frac{D f(x)}{f(x)^2} I_d + \frac{h^2}{2} \frac{D f(x)}{f(x)^2} D_d^+ I_d + o(h^2) \]
\[ = h^2 M_1 + o(h^2), \]
and
\[ \left( \frac{s_h}{s_h} \otimes I_d \right) (D_d^+)^\top (D_d^+ (S_h \otimes I_d) (D_d^+)\top)^{-1} \]
\[ = h \left( \frac{D f(x)}{f(x)} \otimes I_d \right) (D_d^+)^\top D_d^+ D_d + o(h^2) \]
\[ = h \left( \frac{D f(x)}{f(x)} \otimes I_d \right) D_d + o(h^2). \]
Thus,
\[
(E_d^+ (S_h^* \otimes I_d) (E_d^+)^\top)^{-1} = \left( \frac{1}{f(x)} - h^2 \frac{D \otimes 2f(x) \otimes \text{vec}(I_d) + o(h^2)}{2f(x)^2} \right) 
\times 
\left[ I_d + M_1 \right] 
\left[ I_d - h \left( \frac{D(f(x)) \otimes I_d}{f(x)} \right) D_d \right] 
\left[ I_d - h D_d^\top \left( \frac{D(f(x)) \otimes I_d}{f(x)} \right) D_d + h^2 D_d^\top (M_2 \otimes I_d) D_d \right] + o(h^2)
\]
\[
= \frac{1}{f(x)} E_d^\top \left[ I_d + h^2 M_1 \right] \left[ I_d - h \left( \frac{D(f(x)) \otimes I_d}{f(x)} \right) D_d \right] E_d + o(h^2).
\]
Let
\[
N_n := E_d^+ (S_h^* \otimes I_d) (E_d^+)^\top \quad \text{and} \quad N_h := \mathbb{E}(N_n) = E_d^+ (S_h^* \otimes I_d) (E_d^+)^\top.
\]
By Theorem 3.4 we obtain
\[
N_n = N_h + o_p(1) \quad \text{with} \quad N_h \rightarrow f(x)(E_d^+ E_d)^{-1}
\]
if \( h \rightarrow 0 \), \( nh^d \rightarrow \infty \) as \( n \rightarrow \infty \) and
\[
E_d^\top B_h^{-1} E_d N_n E_d^\top B_h E_d = E_d^\top B_h^{-1} E_d N_h E_d^\top B_h E_d + o_p(1)
\]
with
\[
E_d^\top B_h^{-1} E_d N_h E_d^\top B_h E_d \rightarrow E_d^\top \left[ \begin{array}{c} f(x)I_d \\ (Df(x) \otimes I_d) \\ f(x) \end{array} \right] E_d
\]
if \( h \rightarrow 0 \), \( nh^{d+2} \rightarrow \infty \) as \( n \rightarrow \infty \). By the continuous mapping theorem (Lemma A.7), this leads to
\[
N_n^{-1} = N_h^{-1} + o_p(1)
\]
if \( h \rightarrow 0 \), \( nh^d \rightarrow \infty \) as \( n \rightarrow \infty \) and
\[
B_h^{-1} (E_d^+)^\top N_n^{-1} E_d^\top B_h = E_d (E_d^\top B_h E_d)^{-1} N_n^{-1} (E_d^\top B_h E_d)^{-1} E_d^\top
\]
\[
= E_d (E_d^\top B_h^{-1} E_d N_n E_d^\top B_h E_d)^{-1} E_d^\top
\]
\[
= E_d (E_d^\top B_h^{-1} E_d N_h E_d^\top B_h E_d)^{-1} E_d^\top
\]
\[
= B_h^{-1} (E_d^+)^\top N_h^{-1} (E_d^+)^\top B_h + o_p(1)
\]
if \( h \rightarrow 0 \), \( nh^{d+2} \rightarrow \infty \) as \( n \rightarrow \infty \). Hence the first two claims follow. For the last claim we use the WLLN (Lemma A.6) with
\[
Z_{ni} := (nh^2)^{-1} K_h(X_i - x) N_h^{-1} \left[ \begin{array}{c} 1 \\ X_i - x \\ X_i - x \end{array} \right] \left[ \begin{array}{c} X_i - x \\ X_i - x \end{array} \right]^\top
\]
leading to
\[
\sum_{i=1}^{n} Z_{ni} = h^{-2} N_h^{-1} N_n \quad \text{and} \quad \sum_{i=1}^{n} E Z_{ni} = h^{-2} I_p.
\]
Furthnummer,
\[
\sum_{i=1}^{n} E \|Z_{ni}\|^2 = (nh^{d+4})^{-1} \int K(z)^2 \left\| N_h^{-1} \left( \begin{array}{c} 1 \\ z \end{array} \right) \left( \begin{array}{c} z^\top \\ z \end{array} \right) \otimes I_d \right\|^2 f(x + h z) dz
\]
\[
\leq (nh^{d+4})^{-1} \|K\|_\infty \|f\|_\infty \|N_h^{-1}\|^2 d^2 (d + 1)^2 \max_{0 \leq z \leq 4} \left\{ \int K(z) \|z\|^2 \right\} \rightarrow 0
\]
if \( h \rightarrow 0 \), \( nh^{d+4} \rightarrow 0 \) as \( n \rightarrow \infty \), provided \( \int K(z) \|z\|^4 dz < \infty \), and the WLLN shows the claim. \( \square \)
Lemma A.15. Suppose (LHSE K1) holds, then

\[-E_dE_d^+B_h^{-1}v_h = D^2f(x) + \begin{cases} B_h^{-1}o(h) & \text{if (A1) holds}, \\
  h^2\left(b_1^{LHSE} + B_h^{-1}o(h^2)\right) & \text{if (A2) holds}, \\
  h^2\left(b_2^{LHSE} + B_h^{-1}o(h^3)\right) & \text{if (A3) holds}, \end{cases}\]

with

\[
\begin{align*}
\tilde{b}_1^{LHSE} &= \frac{1}{2}(I_d \otimes \text{vec}(I_d)^\top)D^\otimes 3 f(x), \\
\tilde{b}_2^{LHSE} &= D_dD_d^+ \frac{1}{6} \int K(z)(I_d \otimes z(\otimes^3)^\top)dzD^\otimes 4 f(x).
\end{align*}
\]

Proof. Note first that by partial integration and Lemma 3.3

\[
E_dE_d^+Q_h^\top(x) := E_dE_d^+E(Q_h^\top(x)) = \frac{1}{h} E_dE_d^+ \int \left(\frac{y-x}{h}\right)^\top (DK)_h(y-x)f(y) \, dy
\]

\[
= \frac{1}{h} \int (f(x+hz)z^\top) \otimes DK(z) \, dz
\]

\[
= -E_dE_d^+ \int K(z) \left( (z \otimes Df(x+hz)) + h^{-1} \text{vec}(I_d)f(x+hz) \right) \, dz
\]

\[
= -E_dE_d^+ \int K(z) \left( Df(x+hz) \right) \, dz - h^{-1}s_h \left( 0_d \text{vec}(I_d) \right).
\]

Hence, by Lemma 3.3 and the Taylor expansion we obtain

\[
\int K(z)Df(x+hz) \otimes z^\otimes^{\ell} \, dz = \int Df(x) \otimes z^\otimes^{\ell} + \frac{h^j}{j!} \int K(z)(I_d \otimes z^\otimes^{\ell}(z^\otimes^j)^\top)D^\otimes(j+1) f(x) \, dz + o(h^{j})
\]

for \( \ell \in \{0, 1\} \) and \( L = 1, 2, 3 \) if (A1), (A2), (A3) holds, respectively. Thus,

\[
-E_dE_d^+B_h^{-1}v_h = -E_dE_d^+B_h^{-1}Q_h^\top(x) + \frac{s_h}{h} B_h^{-1} \left( 0_d \text{vec}(I_d) \right)
\]

\[
= E_dE_d^+B_h^{-1} \int K(z) \left( Df(x+hz) \right) \, dz
\]

\[
= \int D^2f(x) + \begin{cases} B_h^{-1}o(h) & \text{if (A1) holds}, \\
  h^2\left(b_1^{LHSE} + B_h^{-1}o(h^2)\right) & \text{if (A2) holds}, \\
  h^2\left(b_2^{LHSE} + B_h^{-1}o(h^3)\right) & \text{if (A3) holds}, \end{cases}
\]

Lemma A.16. Suppose (LHSE K1) and \( h \to 0 \) as \( n \to \infty \), then

\[
\text{Var}(\sqrt{nh^{\ell+2}}v_n) \to f(x)R((\cdot)^\top \otimes DK).
\]
In particular,

\[ v_n = v_h + O_p\left((nh^{d+2})^{-1/2}\right) \]

and

\[ \sqrt{nh^{d+2}}(v_n - v_h) \xrightarrow{d} N\left(0, f(x)R(\cdot)^T \otimes DK\right) \]

as \( n \to \infty \) and \( nh^d \to \infty \), \( h \to 0 \).

**Proof.** For

\[ Z_{ni} := n^{-1/2}h^{d/2}\left(\frac{X_i - x}{h}\right) \otimes (DK)_h(X_i - x) \otimes + K_h(X_i - x) \left(\frac{0_d}{vec(I_d)}\right) \]

we have

\[ \sqrt{nh^{d+2}}v_n = \sum_{i=1}^{n} Z_{ni}, \quad \mathbb{E}Z_{ni} = n^{-1/2}h^{d/2+1}v_h \quad \text{and} \quad \sum_{i=1}^{n} Z_{ni} = \sqrt{nh^{d+2}}v_h. \]

In order to calculate the variance we first consider

\[
\begin{align*}
\mathbb{E}(Z_{ni}Z_{ni}^\top) &:= n^{-1} \int \left(z^T \otimes DK(z) + K(z) \left(\frac{0_d}{vec(I_d)}\right)\right) \left(z^T \otimes DK(z) + K(z) \left(\frac{0_d}{vec(I_d)}\right)\right)^\top f(x + hz) \, dz \\
&= n^{-1} f(x)R(\cdot)^T \otimes DK + O(n^{-1}h^d) \\
&\quad + n^{-1} f(x) \left(2 \int K(z)(z^T \otimes DK(z)) \, dz \left(\frac{0_d}{vec(I_d)}\right)^\top + \int K(z)^2 \, dz \left(\frac{0_d}{vec(I_d)}\right) \left(\frac{0_d}{vec(I_d)}\right)^\top\right).
\end{align*}
\]

Since by partial integration

\[ 2 \int K(z)(z^T \otimes DK(z)) \, dz = - \int K(z)^2 \, dz \left(\frac{0_d}{vec(I_d)}\right) + o(1) \]

we have

\[ \text{Var}(Z_{ni}) = \mathbb{E}(Z_{ni}Z_{ni}^\top) - \mathbb{E}Z_{ni}\mathbb{E}Z_{ni}^\top = n^{-1} f(x)R(\cdot)^T \otimes DK + o(n^{-1}) \]

and

\[ \text{Var}\left(\sum_{i=1}^{n} Z_{ni}\right) = f(x)R(\cdot)^T \otimes DK + o(1) \rightarrow f(x)R(\cdot)^T \otimes DK. \]

The Lyapunov condition

\[
\sum_{i=1}^{n} \mathbb{E}\|Z_{ni}\|^2+\delta = n(n^{-1/2}h^{-d/2})^{2+\delta} h^d \int \left\|z^T \otimes DK(z) + K(z) \left(\frac{0_d}{vec(I_d)}\right)\right\|^{2+\delta} f(x + hz) \, dz
\]

\[ = n^{-\delta/2}h^{-d\delta/2}(f(x)O(1) + o(1)) = O((nh^d)^{-\delta/2}) \to 0 \]

is fulfilled, whence we apply the central limit theorem and observe

\[ \sqrt{nh^{d+2}}(v_n - v_h) \xrightarrow{d} N\left(0, f(x)R(\cdot)^T \otimes DK\right). \]
\textbf{Proof of Theorem 3.13.} By Lemma A.14, A.15, A.16 and Slutsky's Theorem is

\[ \hat{\theta}_n^{\text{LHSE}} = -B_h^{-1}E_d^+ (E_d^+ (S_n^* \otimes I_d)(E_d^+)\T)^{-1} E_d^+ B_h E_d E_d^+ B_h^{-1} v_n \]

\[ \xrightarrow{p} \frac{1}{f(x)} E_d E_d^+ \left[ \begin{array}{c} I_d \ \ 0 \end{array} \right] E_d E_d^+ D^2 f(x) = \left( \frac{D f(x)}{f(x)} \otimes I_d - \frac{D f(x)^2}{f(x)^2} \right) = D \hat{\Delta} \log f(x). \]

if (A1) holds and \( h \to 0, nh^{d+4} \to \infty \) as \( n \to \infty \), and if for (A3) we have

\[ \hat{\theta}_n^{\text{LHSE}} = D \hat{\Delta} \log f(x) + h^2 b_1^{\text{LHSE}} + B_h^{-1} o(h^3) + B_h^{-1} O_p((nh^{d+2})^{-1/2}) \]

with

\[ b_1^{\text{LHSE}} := \left( b_1^{\text{LHSE}} \ b_2^{\text{LHSE}} \right), \]

where

\[ b_1^{\text{LHSE}} := M_1 \frac{D f(x)}{f(x)} \otimes I_d \frac{D \otimes 2 f(x)}{f(x)} + \frac{1}{2} \left( I_d \otimes \text{vec}(I_d)^\T \right) \frac{D \otimes 3 f(x)}{f(x)} \]

\[ = \frac{1}{2} \left( I_d \otimes \text{vec}(I_d)^\T \right) D \otimes 3 \log f(x) \]

\[ b_2^{\text{LHSE}} := -D_d D_d^+ \left( \frac{D f(x)}{f(x)} \otimes I_d \right) \frac{1}{2} \left( I_d \otimes \text{vec}(I_d)^\T \right) \frac{D \otimes 3 f(x)}{f(x)} \]

\[ + D_d D_d^+ \int K(z) \left( I_d \otimes z (z \otimes 3)^\T \right) dz \frac{D \otimes 4 f(x)}{f(x)} + \frac{1}{2} \left( \frac{D \otimes 2 f(x)}{f(x)} \otimes 2 + \frac{D f(x)}{f(x)} \otimes 2 + \frac{D \otimes 2 f(x)}{f(x)} \right), \]

\[ + D_d D_d^+ \int K(z) I_d \otimes z (z \otimes 3)^\T dz \left( \frac{1}{6} \frac{D \otimes 4 f(x)}{f(x)} - \frac{1}{2} \left( \frac{D \otimes 2 f(x)}{f(x)} \otimes 2 \right) \right). \]

The formulae for the estimator and the bias terms follows from long but simple calculations.

\textbf{Proof of Theorem 3.14.} We write

\[ \sqrt{nh^{d+2}} B_h (\hat{\theta}_n^{\text{LHSE}} - D \hat{\Delta} \log f(x)) \]

\[ = -\left( E_d^+ \right)^\T \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} E_d^+ \sqrt{nh^{d+2}} (v_n - \nu_h) \]

\[ + h^{-2} \left( E_d^+ \right)^\T \left( \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} - \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} \right) E_d^+ \sqrt{nh^{d+6}} \nu_h \]

\[ - \sqrt{nh^{d+2}} \left( \left( E_d^+ \right)^\T \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} - \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} \right) E_d^+ \nu_h + B_h D \hat{\Delta} \log f(x) \].

Note that by Lemma A.14 \( (E_d^+)^\T (E_d^+ (S_n^* \otimes I_d)(E_d^+)\T)^{-1} E_d^+ \xrightarrow{p} f(x)^{-1} E_d E_d^+ \). Applying Lemma A.16 and Slutsky's Theorem shows that (22) converges in distribution to

\[ \mathcal{N} \left( 0, f(x)^{-1} E_d R(E_d^+ \cdot)^\T \otimes DK \right) E_d^+ \]

The second summand (23) converges to zero in probability, because \( \sqrt{nh^{d+6}} \nu_h = O(1) \) and

\[ h^{-2} \left( E_d^+ \right)^\T \left( \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} - \left( E_d^+ (S_n^* \otimes I_d)(E_d^+)\T \right)^{-1} \right) E_d^+ = o_p(1) \]
In case (LMME K3), (A3) we have

\[
-\sqrt{nh^d} \left( (E_d^+)^T (E_d^+ (S_d^T \otimes I_d)(E_d^+)^T)^{-1} E_d^+ v_h + B_h D \log f(x) \right)
\]

\[
= -\sqrt{nh^d+2} B_h \left( -D \log f(x) - h^2 \left( b_{\text{LIM}} \right) + o(h^2) + D \log f(x) \right) \]

\[
= \sqrt{nh^d+6} B_h \left( b_{\text{LIM}} \right) + B_h o(\sqrt{nh^d+6}) \rightarrow \lambda \left( b_{\text{LIM}} \right)
\]

if \( nh^d+6 \rightarrow \lambda^2 \) as \( n \rightarrow \infty \). This three results together show the asymptotic law of the LHSE. \( \square \)

A.6 Proofs of the Results in Section 3.5

Lemma A.17. The estimator

\[
\hat{D}^2 f_n(x) := (B_h^T)^{-1} (J_d^T)^{-1} M^{-1} S_h^n(x) = (B_h^T)^{-1} (J_d^T)^{-1} (\hat{E}_d^T)^{-1} R(\sqrt{K} \hat{E}_d^T) S_h^n(x)
\]

fulfills

\[
E(\hat{D}^2 f_n(x)) - D^2 f(x) = \begin{cases} (B_h^T)^{-1} o(h^2) & \text{if (LMME K1), (A1) hold,} \\ h^2 \left( \frac{z_{\text{LIM}}}{b_1} \right) + (B_h^T)^{-1} o(h^3) & \text{if (LMME K2), (A2) hold,} \\ h^4 \left( \frac{z_{\text{LIM}}}{b_2} \right) + (B_h^T)^{-1} o(h^4) & \text{if (LMME K3), (A3) hold.} \end{cases}
\]

Proof. By Lemma 3.3 we have

\[
S_h^n(x) = \left\{ \begin{array}{ll} R(\sqrt{K} \cdot)^T J_d^T B_h^T D^2 f(x) + o(h^2) & \text{if (LMME K1), (A1) hold,} \\ R(\sqrt{K} \cdot)^T \int K(z) z^T (z^T)^T \ dz \ J_d^T B_h^T D^2 f(x) + o(h^3) & \text{if (LMME K2), (A2) hold,} \\ R(\sqrt{K} \cdot)^T \int K(z) z^T (z^T)^T \ dz \ J_d^T B_h^T D^2 f(x) + o(h^4) & \text{if (LMME K3), (A3) hold.} \end{array} \right.
\]

In case (LMME K3), (A3) we have

\[
E(\hat{D}^2 f_n(x)) = (B_h^T)^{-1} (J_d^T)^{-1} (\hat{E}_d^T)^T R(\sqrt{K} \hat{E}_d^T) S_h^n(x)
\]

\[
= (B_h^T)^{-1} (J_d^T)^{-1} (\hat{E}_d^T)^T R(\sqrt{K} \hat{E}_d^T) S_h^n(x)
\]

\[
\times \left[ R(\sqrt{K} \cdot)^T \int K(z) z^T (z^T)^T \ dz \ J_d^T B_h^T D^2 f(x) + o(h^4) \right]
\]

\[
= (B_h^T)^{-1} (J_d^T)^{-1} (\hat{E}_d^T)^T R(\sqrt{K} \hat{E}_d^T) S_h^n(x)
\]

\[
= D^2 f(x) + \left( \frac{h^2}{3!} K(z)^T (z^T)^T D^3 f(x) + \frac{h^4}{4!} (B_h^T)^{-1} (J_d^T)^{-1} (\hat{E}_d^T)^T R(\sqrt{K} \hat{E}_d^T) S_h^n(x) + \frac{h^2}{2!} (B_h^T)^{-1} o(h^4) \right)
\]

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By Lemma A.17 we obtain

we have

Part 1 of Theorem 3.5 implies

\[ \hat{S}_n^*(x) = S_n^*(x) + O_p((nh^d)^{-1/2}), \]

hence

\[ \hat{D}^2 f_n(x) - E \hat{D}^2 f_n(x) = (B_h^*)^{-1} O_p((nh^d)^{-1/2}). \]

By Lemma A.17 we obtain

\[
\hat{D}^2 f_n(x) - D^2 f(x) = E(\hat{D}^2 f_n(x)) - D^2 f(x) + \hat{D}^2 f_n(x) - E(\hat{D}^2 f_n(x)) = (B_h^*)^{-1} \begin{cases} 
\begin{align*}
O(h^2) & \quad \text{if (LMME K1), (A1) hold and } nh^{d+4} \to \infty, \\
O(h^4) & \quad \text{if (LMME K3), (A3) hold.}
\end{align*}
\end{cases}
\]

The first case implies the consistency of \( \hat{c}_n^\text{LMLE} \) and \( \hat{\theta}_n^\text{LMLE} \). For

\[
\begin{align*}
 r_n & := \sqrt{nh^d} \\
 \hat{r}_n & := h^3 \\
 X_n & := B_h^* \hat{D}^2 f(x)_n \\
 x_n & := B_h^* D^2 f(x) \\
 x_o & := \lim_{n \to \infty} x_n = (f(x), 0, \ldots, 0) ^\top \\
 b_n & := B_h^*(E(\hat{D}^2 f(x)_n) - D^2 f(x))
\end{align*}
\]

we have

\[ X_n - x_n - b_n = B_h^*(D^2 f(x)_n - E(\hat{D}^2 f(x)_n)) = O_p(r_n^{-1}) \]

and

\[ b_n = \begin{pmatrix} h^4 \text{LMLE} \\
 h^3 b_0^\text{LMLE} \\
 h^2 b_2^\text{LMLE} \end{pmatrix} + O(h^4) = O(\hat{r}_n). \]

Note that \( \phi(B_h^* \xi) = B_h^* \phi(\xi) \) and \( J(\phi(x_o)) = f(x)^{-1} I_{1+d+2d}. \) Applying Theorem A.10 leads to

\[
\begin{align*}
\hat{c}_n^\text{LMLE} & = \log f(x) + h^4 b_0^\text{LMLE} + O(h^4) + O_p((nh^d)^{-1/2}), \\
\hat{\theta}_n^\text{LMLE} & = D^2 \log f(x) + h^2 (b^\text{LMLE}_2) + B_h^{-1} o(h^3) + B_h^{-1} O_p((nh^{d+2})^{-1/2}),
\end{align*}
\]

provided \( nh^{d+6} = O(1) \) as \( n \to \infty. \)
Proof of Theorem 3.17. By Lemma A.17 and Theorem 3.5 we obtain

\[ \sqrt{nh^d}B_h^n(D^2 f_n(x) - D^2 f(x)) = \sqrt{nh^d}B_h^n(\mathbb{E}D^2 f_n(x) - D^2 f(x)) + (J^T)^{-1}(\hat{E}_d^+)^\top R(\sqrt{K}E_d^+)^{-1}\hat{E}_d^+\sqrt{nh^d}(S_n^-(x) - S_n^-(x)) \]

\[ \xrightarrow{d} N\left(\lambda\left(\begin{array}{c} \delta \varepsilon_{\text{i}} \\ \delta \varepsilon_{\text{b}} \end{array}\right), f(x)(J^T)^{-1}(\hat{E}_d^+)^\top R(\sqrt{K}E_d^+)^{-1}R(\sqrt{K}E_d^+)^{-1}\hat{E}_d^+(J^T)^{-1}\right). \]

Furthermore, we have

\[ B_h^n D^2 f(x) \rightarrow x_o := \left(\begin{array}{c} f(x) \\ 0_{d^2} \end{array}\right), \quad \text{and} \quad J\phi(x_o) = f(x)^{-1}I_{1+d^2}. \]

By the delta method; see van der Vaart (1998, Theorem 3.8) and Lemma A.2 we get

\[ \sqrt{nh^d}B_h^n\left(\left(\begin{array}{c} \delta \varepsilon_{\text{i}} \\ \delta \varepsilon_{\text{b}} \end{array}\right) - D^2 \log f(x)\right) = \sqrt{nh^d}\left(\phi\left(B_h^n D^2 f_n(x) - \phi\left(B_h^n D^2 f(x)\right)\right)\right) \]

\[ \xrightarrow{d} N\left(\lambda\left(\begin{array}{c} \delta \varepsilon_{\text{i}} \\ \delta \varepsilon_{\text{b}} \end{array}\right), f(x)^{-1}(J^T)^{-1}(\hat{E}_d^+)^\top R(\sqrt{K}E_d^+)^{-1}R(\sqrt{K}E_d^+)^{-1}\hat{E}_d^+(J^T)^{-1}\right). \]

\[ \square \]

A.7 Proofs of the Results in Section 3.6

Lemma A.18. The kernel density estimator fulfills

\[ \mathbb{E}(\hat{f}^2_n(x)) - D^2 f(x) = \begin{cases} o(1) & \text{if (KDE K1), (A1) hold,} \\ h^2/B_h^n + (B_h^n)^{-1}o(h^3) & \text{if (KDE K2), (A2) hold,} \\ h^2b_{\text{KDE}} + h^4\left(\int K(z)(z^{\otimes 4})^\top dzD^{\otimes 4}f(x)\right) + (B_h^n)^{-1}o(h^4) & \text{if (KDE K3), (A3) hold.} \end{cases} \]

Proof. By Lemma 3.3 and the Taylor expansion

\[ D^{\otimes r} f(x - hz) = D^{\otimes r} f(x) + \sum_{j=1}^L (-h_j)^j I_{d^r} \otimes (z^{\otimes j})^\top D^{\otimes (r+j)} f(x) + o(h^L), \]

for if \( f \in C^{r+L}(\mathbb{R}^d) \) with \( \|f\|_{r+L} < \infty \) and \( \int K(z)\|z\|^L < \infty \), we obtain

\[ \mathbb{E}(\hat{f}^2_n(x)) = (B_h^n)^{-1} \frac{1}{h^d} \int (D^2 K)(\frac{x-y}{h}) f(y) dy \]

\[ = \frac{1}{h^d} \int K\left(\frac{x-y}{h}\right) D^2 f(y) dy \]

\[ = \int K(z) D^2 f(x - hz) dz \]

\[ = D^2 f(x) + \begin{cases} o(1) & \text{if (KDE K1), (A1) hold,} \\ h^2/B_h^n + (B_h^n)^{-1}o(h^3) & \text{if (KDE K2), (A2) hold,} \\ h^2b_{\text{KDE}} + \left(\int K(z)(z^{\otimes 4})^\top dzD^{\otimes 4}f(x)\right) + (B_h^n)^{-1}o(h^4) & \text{if (KDE K3), (A3) hold.} \end{cases} \]

\[ \square \]
Proof of Theorem 3.19. We have

\[
\text{Var}(\hat{D^2f}_n(x)) = \frac{1}{nh^{2d}} \text{Var}\left((B_h^\top)^{-1}(D^2K)(\frac{x-X}{h})\right)
\]

\[
= \frac{1}{nh^{2d}}(B_h^\top)^{-1} \int (D^2K)\left(\frac{x-y}{h}\right)\left((D^2K)\left(\frac{x-y}{h}\right)\right)^\top f(y) dy
\]

\[-\frac{1}{nh^{2d}}(B_h^\top)^{-1} \int (D^2K)\left(\frac{x-y}{h}\right)f(y) dy \left(\int (D^2K)\left(\frac{x-y}{h}\right)f(y) dy\right)^\top (B_h^\top)^{-1}
\]

\[
= \frac{1}{nh^{d}}(B_h^\top)^{-1}(R(D^2K)f(x) + o(1))(B_h^\top)^{-1} - \frac{1}{n}(D^2K)(D^2f(x)^\top + o(1))
\]

\[
= \frac{1}{nh^{d}}(B_h^\top)^{-1}R(D^2K)f(x)(B_h^\top) + (B_h^\top)o((nh^d)^{-1})(B_h^\top)^{-1}
\]

if \(h \to 0\) as \(n \to \infty\). Hence,

\[
\hat{D^2f}_n(x) = (B_h)^{-1}O_p((nh^d)^{-1/2})
\]

if \(h \to 0\) as \(n \to \infty\). By Lemma A.18 we obtain

\[
\hat{D^2f}_n(x) - D^2f(x) = E(\hat{D^2f}_n(x)) - D^2f(x) + D^2f_n(x) - E\left(D^2f_n(x)\right)
\]

\[
= \begin{cases} 
  o(1) + (B_h^\top)^{-1}O_p((nh^d)^{-1/2}) = o_p(1) & \text{if (KDE K1), (A1) hold and } nh^{d+4} \to \infty, \\
  h^2b_{\text{KDE}} + (\frac{O(h^4)}{o(h^2)}) + (B_h^\top)^{-1}O_p((nh^d)^{-1/2}) & \text{if (KDE K3), (A3) hold.}
\end{cases}
\]

The consistency results follows from the continuous mapping theorem. For

\[
r_n := \sqrt{nh^d}
\]

\[
\hat{r}_n := h^3
\]

\[
X_n := B_h^\top \hat{D^2f}(x)_n
\]

\[
x_n := B_h^\top \hat{D^2f}(x) + \left(h^2 \text{vec}(I_d)\right)^\top D^\otimes 2 f(x)\]

\[
x_o := \lim_{n \to \infty} x_n = (f(x), 0, \ldots, 0)^\top
\]

\[
b_n := B_h^\top (\hat{E}_n \hat{D^2f}(x)_n - \hat{D^2f}(x)) - \left(h^2 \text{vec}(I_d)\right)^\top D^\otimes 2 f(x)
\]

we have

\[
X_n - x_n - b_n = B_h^2(D\hat{\omega}(x)_n - E\hat{\omega}(x)_n) = O_p(r_n)
\]

and

\[
b_n = \begin{pmatrix} O(h^4) \\
h^3b_{\text{KDE}} + o(h^4) \\
h^4b_{\text{KDE}} + o(h^4)
\end{pmatrix}.
\]
Note that \( \phi(B_h^T \xi) = B_h^T \phi(\xi) \). Applying Theorem A.10 to
\[
\left( \frac{c_{\text{KDE}}}{\theta_n} \right) = \phi(x_n) + (B_h^T)^{-1} J_\phi(x_n) b_n + (B_h^T)^{-1} O_p((nh^d)^{-1/2})
\]

and
\[
= \left( \begin{array}{c}
\frac{\log(f(x)) + h^2 \text{vec}(I_d)^T D^{\otimes 2} f(x)}{f(x)} + O(h^4) \\
\frac{Df(x)}{f(x)} + h^2 b_{\text{KDE}}^2 f(x)^{-1} + o(h^3)
\end{array} \right) + (B_h^T)^{-1} O_p((nh^d)^{-1/2})
\]

with
\[
b_{\text{KDE}} = \left( \begin{array}{c}
\text{vec}(I_d)^T D^{\otimes 2} f(x) \\
\frac{Df(x)}{f(x)} + \left( D^{\otimes 2} f(x) \otimes D^{\otimes 2} f(x) + (I_d \otimes \text{vec}(I_d)^T) D^{\otimes 3} f(x) \right)
\end{array} \right).
\]

The last equality can be seen from the Taylor expansion of \( \log(\cdot) \) and Lemma A.3.

**Proof of Theorem 3.20.** We have
\[
\sqrt{nh^d} B_n^2 D^2 f_n(x) = \sqrt{\frac{h^d}{n}} B_n^2 \sum_{i=1}^n (B_n^2)^{-1} (D^2 K)_h(X_i - x) = \sum_{i=1}^n Z_{ni}
\]

with
\[
Z_{ni} := \sqrt{\frac{h^d}{n}} (D^2 K)_h(X_i - x).
\]

The bias of the asymptotic law is then
\[
\sqrt{nh^d} B_n^2 \left( \mathbb{E}(D^2 f_n(x)) - D^2 f(x) \right) = \sqrt{nh^d} B_n^2 \frac{1}{2} \left( h^2 I_d \otimes \text{vec}(I_d)^T D^{\otimes 3} f(x) + o(1) \right) \frac{1}{O(h)} \rightarrow \frac{1}{2} \left( I_d \otimes \text{vec}(I_d)^T D^{\otimes 3} f(x) \right)
\]
as \( n \to \infty \) and the variance
\[
\text{Var} \left( \sum_{i=1}^n Z_{ni} \right) = f(x) R(D^2 K) + o(1),
\]

by the same calculation as in the proof of Theorem 3.19. The Lyapunov-condition is fulfilled, because
\[
\sum_{i=1}^n \mathbb{E}(\|Z_{ni}\|^{2+\delta}) = n(n^{-1/2} h^{-d/2})^{2+\delta} h^d \int \|D^2 K(z)\|^{2+\delta} f(x - hz) \, dz = O((nh^d)^{-\delta/2}) \to 0
\]

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as \( n \to \infty \) and we can apply the central limit theorem and Slutsky’s Theorem to see the first claim. To prove the second claim we show first

\[
\sqrt{nh^d} B_h^2 \left( \frac{\hat{D^2} f_n(x)}{\hat{f}_n(x)} - \frac{D^2 f(x)}{f(x)} \right) \overset{d}{\to} \mathcal{N} \left( \lambda \left( \begin{bmatrix} b \\ 0 \end{bmatrix} - f(x) R(D \hat{\mathbf{2}} f) \right) \right). \tag{25}
\]

To see this we write

\[
\sqrt{nh^d} B_h^2 \left( \frac{\hat{D^2} f_n(x)}{\hat{f}_n(x)} - \frac{D^2 f(x)}{f(x)} \right) = \frac{1}{\hat{f}_n(x)} \sqrt{nh^d} B_h^2 (\hat{D^2} f_n(x) - D^2 f(x)) + \frac{1}{\hat{f}_n(x)} \sqrt{nh^d} B_h^2 (f(x) - \hat{f}_n(x)) \frac{D^2 f(x)}{f(x)}.
\]

The first summand converges in distribution to

\[
\mathcal{N} \left( \lambda f(x)^{-1} \frac{1}{2} \left( (I_d \otimes \text{vec}(I_d)^\top) D \otimes f(x) \right), f(x)^{-1} R(D \mathbf{2}) \right)
\]

by the fact that \( \hat{f}_n(x)^{-1} \overset{p}{\rightarrow} f(x)^{-1} \), the first claim and Slutsky’s Theorem. The second summand convergence in probability to

\[-\lambda \frac{1}{2} \left( \text{vec}(I_d)^\top \left( (D f(x))^\top \frac{D f(x)}{f(x)} \right) \right),
\]

because

\[
\sqrt{nh^d} B_h^2 (f(x) - \hat{f}_n(x)) \overset{p}{\rightarrow} \frac{1}{2} \left( \text{vec}(I_d)^\top D \otimes f(x) \right)
\]

by Lemma 3.3, Theorem 3.4 and again the fact that \( \hat{f}_n(x)^{-1} \overset{p}{\rightarrow} f(x)^{-1} \). Applying Slutsky’s Theorem ones more leads to the claim in (25). Now we apply the delta method with the continuous differentiable function \( \hat{\phi} : \mathbb{R}^{d+d^2} \to \mathbb{R}^{d+d^2} \)

\[
\left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) \mapsto \left( \begin{array}{c} \xi_1 \\ \xi_2 - \xi_1 \otimes \xi_1 \end{array} \right)
\]

with Jacobian matrix

\[
J_{\hat{\phi}} \left( \begin{array}{c} \xi_1 \\ \xi_2 \end{array} \right) = \left[ -I_d \otimes \xi_1 - \xi_1 \otimes I_d \right] \begin{bmatrix} I_d \\ 0 \\ I_{d^2} \end{bmatrix}.
\]

We have

\[
\hat{\phi} \left( B_h^2 f(x) \frac{D^2 \hat{f}}{f(x)} \right) = B_h^2 \left( \frac{D \hat{f}}{f(x)} - \text{vec} \left( \left( D f(x) \right)^\top \frac{D f(x)}{f(x)^2} \right) \right) = B_h^2 D \hat{\mathbf{2}} \log f(x) \to 0
\]

as \( n \to \infty \). Because \( J_{\hat{\phi}}(0) = I_{d+d^2} \) the delta method leads

\[
\sqrt{nh^d} B_h^2 \left( \hat{\theta}_{\text{MLE}}^\text{MLE} - D \log f(x) \right) = \sqrt{nh^d} \left( \hat{\phi} \left( B_h^2 \frac{D \hat{f}}{f_n(x)} \right) - \hat{\phi} \left( B_h^2 \frac{D f(x)}{f(x)} \right) \right) \overset{d}{\to} \mathcal{N} \left( \lambda \left( \begin{bmatrix} \hat{b}_{\text{MLE}}^\text{MLE} \\ 0_{d^2} \end{bmatrix} \right), f(x)^{-1} R(D \hat{\mathbf{2}} K) \right).
\]

\[
\square
\]

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