Gyroscope precession in cylindrically symmetric spacetimes

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Abstract. We present calculations of gyroscope precession in spacetimes described by Levi-Civita and Lewis metrics, under different circumstances. By doing so we are able to establish a link between the parameters of the metrics and observable quantities, thereby providing a physical interpretation for those parameters, without specifying the source of the field.

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1. Introduction

As stressed by Bonnor [1] in his review on the physical interpretation of vacuum solutions of Einstein’s field equations, relativists have not been diligent in interpreting such solutions. One way to alleviate this deficiency of the physical content of the theory consists in providing a link between the characteristic parameters of the solutions and quantities measured from well defined and physically reasonable experiments, thereby providing a physical interpretation for those parameters.

It is the purpose of this work to establish such a link for cylindrically symmetric spacetimes. The general form of the metric in this case was given by Lewis [2, 3] and describes a stationary spacetime. This metric can be split into two families called the Weyl and Lewis class. Here we shall restrict our study to the Weyl class where all parameters appearing in the metric are real. For the Lewis class these parameters can be complex. The corresponding static limit was obtained by Levi-Civita [4].

The motivation for this choice is provided, on one hand, by the fact that the physical interpretation of the parameters of these metrics is still a matter of discussion (see [1]), and on the other hand, by the fact that some of the parameters of these metrics are related to topological defects not entering into the expression of the physical components of curvature tensor. The physical (gedanken) experiment proposed here consists in observing the precession of a gyroscope under different conditions in such spacetimes. Specifically, we calculate the rate of rotation of a gyroscope at rest in the frame in which the metric is presented, and also, the total precession per revolution of a gyroscope circumventing the symmetry axis, along
a circular path (geodesic or not). By doing so, the four parameters of the Weyl class, from the Lewis metric, become measurable, in the sense that they are expressed through quantities obtained from well defined and physically reasonable experiments (we are, of course, not discussing the actual technical feasibility of such experiments).

All calculations are carried out using the method proposed by Rindler and Perlick [5], and a very brief summary of this is given in the next section together with the notation and the specification of the spacetime under consideration. In section 3 we calculate the rate of precession of a gyroscope at rest in the original lattice, and in section 4 we obtain the precession per revolution relative to the original frame of a gyroscope rotating round the axis of symmetry. The results are discussed in the final section.

2. The spacetime and the Rindler–Perlick method

2.1. The Lewis metric

The Lewis metric [2, 3] can be written as

\[ ds^2 = -f \, dt^2 + 2k \, dt \, d\phi + e^\mu (dr^2 + dz^2) + l \, d\phi^2, \]  

where

\[ f = a r^{1-n} - \frac{c^2}{na} r^{1+n}, \]  

\[ k = -Af, \]  

\[ l = \frac{r^2}{f} - A^2 f, \]  

\[ e^\mu = r^{(n^2-1)/2}, \]  

with

\[ A = \frac{cr^{1+n}}{na} + b. \]  

Observe that taking \( ds \) in (1) to have a dimension of length \( L \) then

\[ [t] = [L]^{2n/(1+n)} \]  

\[ [r] = [L]^{2/(1+n)} \]  

and

\[ [b] = [L]^{2n/(1+n)} \]  

\[ [c] = [L]^{-2n/(1+n)} \]  

whereas \( n \) and \( a \) are dimensionless and \( e^\mu \) is multiplied by a unit constant with dimensions

\[ [L]^{(2n-n^2-1)/(n+1)}. \]

The four parameters \( n, a, b \) and \( c \) can be either real or complex, and the corresponding solutions belong to the Weyl or Lewis classes, respectively. Here we restrict our study to the Weyl class (not to confuse with Weyl metrics representing static and axially symmetric spacetimes).
The transformation [6]
\[ d\tau = \sqrt{a(dt + b\,d\phi)}, \]
\[ d\tilde{\phi} = \frac{1}{n}[c - dt + (n - bc)\,d\phi], \]
casts the Weyl class of the Lewis metric into the Levi-Civita metric (for recent discussions on these metrics see [7–14] and references therein). However, the transformation above is not valid globally, and therefore both metrics are equivalent only locally; a fact that can be verified by calculating the corresponding Cartan scalars [7]. In order to globally transform the Weyl class of the Lewis metric into the static Levi-Civita metric, we have to make \( b = 0 \). Indeed, if \( b = 0 \) and \( c \) is different from zero, equation (12) gives an admissible transformation for the time coordinate and (13) represents a transformation to a rotating frame. However, since rotating frames (as in special relativity) are not expected to cover the whole spacetime and, furthermore, since the new angle coordinate ranges from \(-\infty \) to \( \infty \), it has been argued in the past [7] that both \( b \) and \( c \) have to vanish for (12) and (13) to be globally valid. This point of view is also reinforced by the fact that, assuming that only \( b \) has to vanish in order to globally cast (1) into Levi-Civita form, we are led to the intriguing result that there is no dragging outside rotating cylinders (see comments at the end of section 3). We shall recall this question later.

2.2. The Rindler–Perlick method

This method consists in transforming the angular coordinate \( \phi \) by
\[ \phi = \phi' + \omega t, \]
where \( \omega \) is a constant. Then the original frame is replaced by a rotating frame. The transformed metric is written in a canonical form,
\[ ds^2 = -e^{2\Psi}(dt - \omega_i\,dx^i)^2 + h_{ij}\,dx^i\,dx^j, \]
with Latin indices running from 1 to 3 and \( \Psi, \omega_i \) and \( h_{ij} \) depending on the spatial coordinate \( x^i \) only (we are omitting primes). Then, it may be shown that the 4-acceleration \( A_\mu \) and the rotation 3-vector \( \Omega^i \) of the congruence of worldlines \( x^i = \text{constant} \) are given by [5],
\[ A_\mu = (0, \Psi, \omega_i), \]
\[ \Omega^i = \frac{1}{2}e^\Psi(\det h_{mn})^{-1/2}e^{ijk}\omega_{k,j}, \]
where the comma denotes a partial derivative. It is clear from the above that if \( \Psi = 0 \), then particles at rest in the rotating frame follow a circular geodesic. On the other hand, since \( \Omega^i \) describes the rate of rotation with respect to the proper time at any point at rest in the rotating frame, relative to the local compass of inertia, then \( -\Omega^i \) describes the rotation of the compass of inertia (the gyroscope) with respect to the rotating frame. Applying (14) to the original frame of (1), with \( t = t', r = r' \) and \( z = z' \), we cast (1) into the canonical form (15), where
\[ e^{2\Psi} = f - \omega^2 l - 2ok, \]
\[ \omega_i = (0, 0, \omega_\phi), \]
\[ \omega_\phi = e^{-2\Psi}(cal + k), \]
\[ h_{rr} = h_{zz} = e^{\alpha}, \]
\[ h_{\phi\phi} = l + e^{2\Psi}\omega_\phi^2. \]
From (12)–(17) and
\[ \Omega \equiv (h_{ij}\Omega^i\Omega^j)^{1/2}, \]
we obtain that
\[ \Omega = \sqrt{[h_{zz}(\Omega^2)^2]}^{1/2}, \] (24)

where
\[ (\Omega^2)^2 = \frac{e^{4\Psi} \omega^2}{4e^{2\omega}(fl + k^2)}. \] (25)

### 2.3. Circular geodesics

From (18) we obtain with the condition \( \Psi, r = 0 \) that
\[ \omega = \frac{-k_r \pm (f_r l_r + k_r^2)^{1/2}}{l_r}, \] (26)

which yields the expression for the angular velocity of a particle on a circular geodesic in the Lewis metric (see (49) in [14]). Now substituting (2)–(5) into (26) we obtain [16]
\[ \omega = \left( \pm \omega_0 + \frac{c}{n} \right) \left[ 1 - b \left( \pm \omega_0 + \frac{c}{n} \right) \right]^{-1/2}, \] (27)

where \( \omega_0 \) is the angular velocity when the spacetime is static, \( b = c = 0 \), given by Levi-Civita’s metric,
\[ \omega_0^2 = \frac{1 - n}{1 + n} a^2 r^{-2n}. \] (28)

We can calculate the tangential velocity \( W \) of the circular geodesic particles (see (53) in [14]),
\[ W = \frac{\omega(f l + k^2)^{1/2}}{f - \omega k}. \] (29)

Substituting (2)–(5) and (28) into (29), we obtain
\[ W = \left( \pm \omega_0 + \frac{c}{n} \right) \left( a r^{-n} \pm \frac{c}{n a} \omega r^n \right)^{-1}. \] (30)

It is worth noticing the fact, which follows from (27) and (30), that \( b \) and \( c \) affect the angular velocity \( \omega \), while for the tangential velocity \( W \) only \( c \) plays a role.

### 3. Precession of a gyroscope at rest in the original lattice

To calculate the precession in this case, we only have to put \( \omega = 0 \) in (17)–(24) (see [17] for a similar case). Then we obtain, after simple calculations,
\[ \Omega = \frac{e^{-\mu/2} |k f_r - f k_r|}{2f (fl + k^2)^{1/2}}, \] (31)

or, using (2)–(5) in (31),
\[ \Omega = c e^{(1-n)/4} \left( a r^{1-n} - \frac{c^2}{n^2 a} r^{1+4n} \right)^{-1}. \] (32)

Thus, the parameter \( c \) appears to be essential in the precession of a gyroscope at rest in the frame of (1), whereas \( n \) and \( a \) just modify its absolute value. This fact reinforces the interpretation of \( c \), already given in [7, 14], in the sense that it represents the vorticity of the source, when described by a rigidly rotating anisotropic cylinder [7] and that it provides the
dragging correction to the angular velocity of a particle in circular orbits in Lewis spacetime [14]. However, here there was no need to specify the source that produces the field and, on the other hand, although some kind of frame dragging effect may also be related to $b$ (see (60) in [14]), this last parameter does not play any role in the precession under consideration.

Another interesting point about (32) is that if $c$ is small, it becomes

$$\Omega \approx \frac{c}{a} e^{\frac{n-3}{4}},$$

and we observe that $n = 3$ produces a constant $\Omega$, independent of $r$. It is known that when $b = c = 0$ and $n = 3$, the metric becomes locally Taub’s plane metric [18–20]. This fact suggests that the gravitational potential becomes constant, not modifying the precession with respect to its distance.

Next, introducing

$$\beta \equiv \frac{c}{na},$$

we can write (32) as

$$\Omega = \frac{\beta nr^{(1-n)(n-3)/4}}{1 - \beta r^n}.$$ (35)

Then performing a series of measurements of $\Omega$ for different values of $r$, we can in principle obtain $n$ and $\beta$ by adjusting these parameters to the obtained curve $\Omega = \Omega(r)$, which in turn allows us to obtain the value of $c/a$.

In the Levi-Civita metric, $b = c = 0$, the gyroscope at rest will not precess, as expected for a vacuum static spacetime (for the electrovacuum case, however, this may change [17]).

All these comments above (after equation (32)), are valid as long as we adopt the point of view that equations (12) and (13) globally transform (1) into the Levi-Civita spacetime, if and only if $b = c = 0$. However, if we adopt the point of view that only $b$ has to vanish for that transformation to be globally valid, then the precession given by (32) is just a coordinate effect, implying that there is no frame dragging in the Lewis spacetime (Weyl class), since the precession of the gyroscope is due to the fact that the original frame of (1) is rotating itself. This is quite a surprising result, if we recall that material sources for (1) consist of steadily rotating fluids (see [7, 15]). Furthermore, the vorticity of the source given in [7] is, at the boundary surface, proportional to $c$ (not $b$).

4. Precession of a gyroscope moving in a circle around the axis of symmetry

According to the meaning of $\Omega$ given above, it is clear that the orientation of the gyroscope moving around the axis of symmetry, after one revolution, changes by

$$\Delta \phi' = -\Omega \Delta \tau,$$ (36)

where $\Delta \tau$ is the proper time interval corresponding to one period. Then from (15),

$$\Delta \phi' = -2\pi \frac{\Omega e^\phi}{\omega},$$ (37)

as measured in the rotating frame. In the original system, we have

$$\Delta \phi = 2\pi \left(1 - \frac{\Omega e^\phi}{\omega}\right).$$ (38)
To calculate (24) and (38) for the metric (1) we first obtain from (18) and (19), using (2)–(5),
\[ e^{2\phi} = aM^2 r^{1-n} - \frac{N^2}{n^2 a} r^{1+n}, \]  
\[ \omega_{\phi,r} = 2MN e^{-4\phi} r, \]  
where
\[ M = 1 + b \omega, \]  
\[ N = n \omega - c(1 + b \omega). \]  
Now substituting (39), (40) and (25) into (24) and (38), we obtain
\[ \Omega = MN r \left( 1 - \frac{n^2}{4} \right) / \left( M^2 a r^{1-n} - \frac{N^2}{n^2 a} r^{1+n} \right)^{-1}, \]  
\[ \Delta \phi = 2\pi \left[ 1 - \frac{MN \omega^{1-n^2}/4}{\omega} \left( M^2 a r^{1-n} - \frac{N^2}{n^2 a} r^{1+n} \right)^{-1/2} \right]. \]  
When a particle follows a circular geodesic around the axis, then the angular velocity is (27) and then (43) and (44) become
\[ \Omega ≈ \frac{1}{2} (1 - n^2)^{1/2} r^{-(3n^2)/4}, \]  
\[ \Delta \phi ≈ 2\pi \left[ 1 - \frac{n(1-n)^{1/2} a r^{-n}}{n(1-n)^{1/2} a r^{-n} + (1+n)^{1/2} c} \left( \frac{n(1+n)}{2a} \right)^{1/2} r^{-(1-n^2)/4} \right]. \]  
Surprisingly, neither \( b \) nor \( c \) enter into expression (45) for \( \Omega \). Also, expression (46) is unaffected by \( b \). If \( \omega_0 r ≪ 1 \) and \( c ≪ \omega_0 \), we have from (46),
\[ \Delta \phi ≈ \delta + 3\pi \frac{\omega_0^2 r^2}{a^{3/2}} + 2\pi \frac{c}{\sqrt{a} \omega_0}, \]  
and if \( a = 1 \) we have \( \Delta \phi ≈ 3\pi \omega_0^2 r^2 / 2 + 2\pi c / \omega_0 \), which coincides with the Schiff precession [26] in the Kerr spacetime, if we identify the Kerr parameter with \( c \). We shall now apply (43)–(46) to some specific cases.

### 4.1. Levi-Civita spacetime case

When in (1) \( b = c = 0 \) we have the static Levi-Civita spacetime then (43) and (44) reduce to
\[ \Omega = \frac{n a \omega r^{1-n}(n-3)/4}{a^2 - \omega^2 r^{2n}}, \]  
\[ \Delta \phi = 2\pi \left[ 1 - \frac{n \sqrt{a} r^{-(1-n^2)/4}}{(a^2 - \omega^2 r^{2n})^{1/2}} \right]. \]  
Let us now assume that the trajectory of the gyroscope is a geodesic, then \( \omega = \omega_0 \) given by (28), we have from (48) and (49),
\[ \Omega ≈ \frac{1}{2} (1 - n^2)^{1/2} r^{-(3n^2)/4}, \]  
\[ \Delta \phi ≈ 2\pi \left[ 1 - \left( \frac{n(1+n)}{2a} \right)^{1/2} r^{-(1-n^2)/4} \right]. \]  
In the case \( \omega_0 r ≪ 1 \), equation (51) becomes
\[ \Delta \phi ≈ \delta + 3\pi \frac{\omega_0^2 r^2}{a^{3/2}}. \]
If $a = 1$, we have $\Delta \phi \approx 3\pi \omega_0^2 r^2$, which coincides with the Fokker–de Sitter precession [23, 24] in the Schwarzschild spacetime. It is interesting to note that if $n = 0$, which corresponds to the null circular geodesics [1, 14] as can be seen from (30) when $W = 1$, we have from (49) that $\Delta \phi = 2\pi$. This behaviour means that the precession becomes so large that, independently of $\omega$, the orientation of the gyroscope is locked to the lattice of the rotating frame. Exactly the same behaviour appears in the Schwarzschild spacetime.

4.2. Flat spacetime case

From the Cartan scalars we see that only $n$ helps to curve the spacetime [7]. When $n = 1$ the spacetime becomes flat and (43) and (44) become

$$\Omega = \frac{a\dot{\omega}}{a^2 - \omega^2 r^2}, \quad \Delta \phi = 2\pi \left[ 1 - \frac{(1 + bc)\sqrt{a}\dot{\omega}}{(\omega + c)(a^2 - \omega^2 r^2)^{1/2}} \right],$$

where $\dot{\omega}$ is the effective coordinate rate of rotation,

$$\dot{\omega} = \frac{\omega}{1 + b\omega} - c.$$

From (54) we see that $b$ and $c$ affect $\dot{\omega}$ by increasing (decreasing) it when they are in the same (opposite) sense of rotation with respect to $\omega$. Let us first consider the case $b = c = 0$, then (48) and (49) become

$$\Omega = \frac{a\omega}{a^2 - \omega^2 r^2}, \quad \Delta \phi = 2\pi \left[ 1 - \frac{\sqrt{a}}{(a^2 - \omega^2 r^2)^{1/2}} \right].$$

These expressions, (56) and (57), demonstrate the influence of $a$ on the Thomas precession of a gyroscope moving around a string with linear energy density $\lambda$ given by [19, 21]

$$\lambda = \frac{1}{4} \left( 1 - \frac{1}{\sqrt{a}} \right).$$

We recall that $a$ changes the topological structure of the spacetime, giving rise to an angular deficit $\delta$ equal to [22]

$$\delta = 2\pi \left( 1 - \frac{1}{\sqrt{a}} \right).$$

In the case $\omega r \ll 1$, equation (57) becomes

$$\Delta \phi \approx \delta - \pi \omega^2 r^2 \frac{a^2}{a^2 - \omega^2 r^2},$$

and if $a = 1$ we have the usual Thomas precession $\Delta \phi \approx -\pi \omega^2 r^2$. If $b = 0$ and $c \ll \omega$ equation (54) becomes

$$\Delta \phi \approx 2\pi \left[ 1 - \frac{\sqrt{a}}{(a^2 - \omega^2 r^2)^{1/2}} + \frac{c}{\omega} \frac{a^2 + \omega^2 r^2}{(a^2 - \omega^2 r^2)^{3/2}} \right],$$

and if $c = 0$ and $b\omega \ll 1$ we have from (54)

$$\Delta \phi \approx 2\pi \left[ 1 - \frac{\sqrt{a}}{(a^2 - \omega^2 r^2)^{1/2}} + b\omega \frac{\omega^2 r^2}{(a^2 - \omega^2 r^2)^{3/2}} \right].$$
We see from (60) and (62) that if the order of magnitude of \( c/\omega \) and \( b\omega \) are equal, \( O(c/\omega) = O(b\omega) \), then the contribution of \( c \) is larger than \( b \) to the precession. The expression (62) exhibits the modifications on the Thomas precession, associated with the topological defect created by \( b \). It is worth noting that a quantum scalar particle moving around a spinning cosmic string exhibits a phase factor proportional to \( b\sqrt{a} \), an evident reminder of the Aharonov–Bohm effect [25].

5. Conclusions

We have been able to establish a set of expressions linking the parameters of the Lewis metric to quantities obtained from the observation of a gyroscope’s precession. In the particular case of a gyroscope at rest in the original lattice, the relevance of the parameter \( c \) is clearly illustrated. Curiously enough, the parameter \( b \) does not enter into the expression of the angular velocity of precession (see the above discussion on this point). In the case of a gyroscope rotating around the axis of symmetry, we obtain that in the Levi-Civita case the precession vanishes at the photon orbit. A similar result is known to happen for the Schwarzschild [5] and the Ernst [27] spacetimes. However, in the general case, \( b \neq 0 \) and \( c \neq 0 \), the same result is observed, which is different from previous results found in stationary spacetimes [28, 29]. This probably happens due to the fact already mentioned, that the Weyl class of the Lewis metric is a rather sui generis class of stationary metrics, since it is locally static. For the special cases with \( n = 1 \), we found how different parameters affect the Thomas precession, providing at the same time a tool for their measurement. We would like to conclude with the following comment. Except for \( n \), neither of the parameters \( a \), \( b \) and \( c \) of the Lewis metric enter into the expressions for the physical components of the Riemann tensor [7]. This implies that they cannot be measured by means of tidal force observations. Therefore, gyroscope precession experiments (i.e. experiments leading to a measurement of the rate of precession of a gyroscope at rest in the frame of a given metric and/or the total precession per revolution of a gyroscope circumventing the source of such a metric) provide a good alternative for observing those aspects of gravitation which are not directly related to the curvature. We have in mind not only topological defects, as is the case here, but other issues appearing in the study of gravity and which are not directly related to the value of the physical components of the Riemann tensor (see, for example, [17, 30–32] and references therein). In the case of real experiments which are now being contemplated as the GP-B in the solar system, it might in principle (we are completely ignorant about the accuracy of such experiment) help to determine what is, among all stationary solutions, the spacetime associated with a rotating source (which we expect to be the Kerr metric).

References

[1] Bonnor W B 1992 Gen. Rel. Grav. 24 551
[2] Lewis T 1932 Proc. R. Soc. 136 176
[3] Kramer D, Stephani H, MacCallum M A H and Herlt E 1980 Exact Solutions of Einstein’s Field Equations (Cambridge: Cambridge University Press)
[4] Levi-Civita T 1917 Rend. Acc. Lincei 26 307
[5] Rindler W and Perlick V 1990 Gen. Rel. Grav. 22 1067
[6] Stachel J 1982 Phys. Rev. D 26 1281
[7] da Silva M F A, Herrera L, Paiva F M and Santos N O 1995 Gen. Rel. Grav. 27 859
[8] da Silva M F A, Herrera L, Paiva F M and Santos N O 1995 J. Math. Phys. 36 3625
[9] da Silva M F A, Herrera L, Paiva F M and Santos N O 1995 Class. Quantum Grav. 12 111
[10] MacCallum M A H 1998 Gen. Rel. Grav. 30 131
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[11] Philbin T 1996 Class. Quantum Grav. 13 1217
[12] Banerjee A, Das A and Basu M 1996 Astrophys. Space Sci. 239 141
[13] Haggag S and Desokey F 1996 Class. Quantum Grav. 13 3221
[14] Herrera L and Santos N O 1998 J. Math. Phys. 39 3817
[15] Bonnor W B 1980 J. Phys. A: Math. Gen. 13 2121
[16] Mashhoon B and Santos N O 1999 Rotating cylindrical systems and gravitomagnetism Preprint
[17] Bonnor W B 1991 Phys. Lett. A 158 23
[18] Taub A H 1951 Ann. Math. 53 472
[19] Wang A Z, da Silva M F A and Santos N O 1997 Class. Quantum Grav. 14 2417
[20] da Silva M F A, Wang A and Santos N O 1998 Phys. Lett. A 244 462
[21] Linet B 1985 Gen. Rel. Grav. 17 1109
[22] Dowker J S 1967 Nuovo Cimento B 52 129
[23] Fokker A D 1920 Proc. K. Akad. Wet. Amsterdam 23 729
[24] de Sitter W 1920 Mon. Not. R. Astron. Soc. 77 155
[25] Jensen B and Kučera J 1993 J. Math. Phys. 34 4975
[26] Schiff L I 1960 Phys. Rev. Lett. 4 215
[27] Rajesh Nayak K and Vishveshwara C V 1997 Gen. Rel. Grav. 29 291
[28] Rajesh Nayak K and Vishveshwara C V 1996 Class. Quantum Grav. 13 1183
[29] Rajesh Nayak K and Vishveshwara C V 1998 Gen. Rel. Grav. 30 593
[30] de Felice F and Usseglio-Tomasset S 1993 Class. Quantum Grav. 10 353
[31] de Felice F 1994 Class. Quantum Grav. 11 1283
[32] Semerak O 1997 Gen. Rel. Grav. 29 153