Newton-Hooke spacetimes, Hpp-waves
and the cosmological constant

G.W. Gibbons* and C.E. Patricot**

D.A.M.T.P.,
Cambridge University,
Wilberforce Road,
Cambridge CB3 0WA,
U.K.

October 26, 2018

Abstract

We show explicitly how the Newton-Hooke groups $N_{10}^\pm$ act as symmetries of the equations of motion of non-relativistic cosmological models with a cosmological constant. We give the action on the associated non-relativistic spacetimes $M_4^\pm$ and show how these may be obtained from a null reduction of 5-dimensional homogeneous pp-wave Lorentzian spacetimes $M_5^\pm$. This allows us to realize the Newton-Hooke groups and their Bargmann type central extensions as subgroups of the isometry groups of $M_5^\pm$. The extended Schrödinger type conformal group is identified and its action on the equations of motion given. The non-relativistic conformal symmetries also have applications to time-dependent harmonic oscillators. Finally we comment on a possible application to Gao’s generalization of the matrix model.

* g.w.gibbons@damtp.cam.ac.uk
** c.e.patricot@damtp.cam.ac.uk
1 Introduction

Recent observations of the Cosmic Microwave Background support the idea that the motion of the universe was dominated by a large positive cosmological term during a period of primordial inflation in the past. These and observations of type Ia supernovae also suggest that the universe is presently entering another phase of exponential expansion due to a much smaller positive value of cosmological constant $\Lambda$. Reconciling these facts with fundamental theory such as M or String theory or indeed with the most elementary notions of quantum field theory is not easy. On the other hand, a negative cosmological term plays an essential role in the AdS/CFT correspondence and attempts to establish whether some sort of Holographic Principle holds in Quantum Gravity.

This suggests that we still lack an adequate understanding of the basic physics associated with the cosmological constant and that it is worthwhile examining it from all possible angles. Moreover, if the cosmological constant really is non-zero at present, then some processes at least are going on right now in which its effects are decisive. A standard general strategem for understanding any physical process is to consider a limiting situation and see whether any simplifications occur, for example whether the symmetries of the problem change or possibly become enhanced. In the case of the cosmological constant the relevant symmetry groups are the de-Sitter or Anti-de-Sitter groups which are relativistic symmetries involving the velocity of light $c$. One possible limit is the non-relativistic one in which $c \to \infty$ and $\Lambda \to 0$ but keeping $c^2 \Lambda$ finite, in which the de-Sitter or Anti-de-Sitter groups become what are called the Newton-Hooke groups [1], the analogues of the Galilei group in the presence of a universal cosmological repulsion or attraction. It is this limit which we propose studying in this paper. We shall begin by setting up the basic equations for non-relativistic cosmology with a cosmological term and then exhibit the action on the solutions of these equations of the Newton-Hooke groups and their central and conformal extensions, the analogues of the Bargmann and Schrödinger groups. The Newton-Hooke groups act on a non-relativistic spacetime, the analogue of Newton-Cartan spacetime for the Galilei group but the geometrical structures involved are complicated. The picture simplifies dramatically if one regards this generalized Newton-Cartan spacetime as a Kaluza-type null reduction of a five-dimensional spacetime with a conventional Lorentzian structure. In the case of the Galilei group the

\[1\text{It is important to take $\Lambda$ to zero as $c$ goes to infinity, otherwise, as we shall see later, one looses the boost symmetry}\]
five-dimensional spacetime is flat; in the case of the Newton-Hooke groups it turns out to be a homogeneous pp-wave of the same general type that have been at the centre of attention recently in connection with Penrose limits of the AdS/CFT.

The suggestion that the Newton-Hooke algebras could have an application to non-relativistic cosmology is not new, it goes back to their very beginnings in [1] and it was developed to some extent in [2]. More recently it was revived in [3], motivated precisely by observations of Type Ia supernovae. Within M/String theory, and with similar motivations, Gao has given a modification of the Matrix model using the Newton-Hooke group [4]. It has also been argued recently that the Carrollian contraction of the Poincaré group in which $c \downarrow 0$ is relevant to the problem of tachyon condensation [5]. Another possible limit that has been considered is that of a very large cosmological constant [6].

Quite apart from these rather formal considerations, it is possible that our work may prove useful in the study of large scale structure since the equations of motion we study appear there under the guise of the Dimitriev-Zel’dovich\(^2\) equations [7] and we provide a complete account of their symmetries.

The plan of the paper is as follows. In section 2 we derive the equations of motion of self-gravitating non-relativistic particles in a universe with cosmological constant, and show that the relevant limit, in order to preserve boost symmetries, is taking $c \to \infty$ and $\Lambda \to 0$ keeping $\Lambda c^2$ fixed. We relate the equations to the Dimitriev-Zel’dovich equation, of which we give a derivation in the Appendix. We present the Newton-Hooke groups $N_{10}^\pm$ in section 3 and review the initial motivation of Lévy-Leblond and Bacry [11] who realized these groups could be obtained by an Inönü-Wigner contraction of the de-Sitter and Anti-de-Sitter groups. We then define in section 4 their four-dimensional cosets $M_4^\pm$, or associated Newton-Hooke spacetimes. We show that the action of $N_{10}^\pm$ on $M_4^\pm$ precisely corresponds to changes of inertial frames of a non-relativistic cosmological model with cosmological constant. In other words, the equations of motion of such models are left invariant under the action of the Newton-Hooke groups, in the same way as Newton’s equations are invariant under Galilean transforms. However, as their Galilean counterpart the Newton-Cartan spacetime, the Newton-Hooke spacetimes $M_4^\pm$ do not admit invariant metrics. Burdet, Duval, Perrin and Künzle [8] gave an elegant description of Newton-Cartan spacetime in terms of Bargmann

\(^2\)The name is not quite standard. We have adopted it because no other suitable name is in general use and these authors appear to have been the first to write them down
structures. We give a similar description of $M^\pm_4$ in section 4 and it turns out the relevant 5-dimensional manifolds $M^\pm_5$ are homogeneous plane-waves. As one expects the Heisenberg isometry groups of the plane-waves correspond to the centrally extended Newton-Hooke groups $N^\pm_{11}$, where the central extension represents the mass of particles in $M^\pm_4$. Whereas $M^-_5$ admits a causal Killing vector, $M^+_5$ does not: this difference is reminiscent of that between their relativistic counterparts Anti-de-Sitter and de-Sitter space. In section 6, motivated by [9] and [10] we find the Bargmann conformal groups or extended Schrödinger groups of $M^\pm_5$. We show how these 13-dimensional groups act on the cosmological equations. They send solutions with a given cosmological constant and gravitational coupling to solutions with the same cosmological constant but with a possibly time-varying gravitational coupling, in an analogous way to the Lynden-Bell transformations [11] in Newtonian theory. The symmetries we exhibit also have applications in the theory of time-dependent harmonic oscillators. In section 7 we explain that Gao’s modification of the Matrix model admits Newton-Hooke symmetries, and conclude in section 8.

2 Non-relativistic cosmological constant

In this section we shall derive some of the equations governing a non-relativistic cosmological model with a cosmological constant. Let us begin by considering the case of a single non-relativistic particle moving in a spacetime with a cosmological constant. The effect of the cosmological constant is to provide a repulsive ($\Lambda > 0$) or attractive ($\Lambda < 0$) force proportional to the distance from an arbitrary centre leading to the equation of motion

$$\frac{d^2 q}{dt^2} - \frac{c^2 \Lambda}{3} q = 0. \quad (1)$$

This can be shown for example by looking at the geodesics in de-Sitter space. The metric inside the cosmological horizon $r < \sqrt{\Lambda/3}$ is given by

$$ds^2 = -c^2(1 - \frac{\Lambda r^2}{3})dt^2 + \frac{dr^2}{(1 - \frac{\Lambda r^2}{3})} + r^2(d\theta^2 + \sin^2 \theta d\phi^2). \quad (2)$$

The Hamilton-Jacobi function $S(t, r)$ of a particle of mass $m$ with no angular momentum about the sphere satisfies

$$g^{\mu\nu} \partial_\mu S \partial_\nu S = -m^2 c^2. \quad (3)$$

Then $\partial_t S$ is conserved and we have

$$\partial_t S = E = mc^2 + \epsilon.$$
where $\epsilon$ is the non-relativistic energy of the particle, and (3) becomes
\[
(1 - \frac{\Lambda r^2}{3})(\partial_r S)^2 = \frac{(mc^2 + \epsilon)^2}{c^2(1 - \frac{\Lambda r^2}{3})} - m^2c^2.
\] (4)

In the low velocity limit, when $\epsilon \ll mc^2$ this becomes
\[
(1 - \frac{\Lambda r^2}{3})^2(\partial_r S)^2 = m^2c^2\frac{\Lambda r^2}{3} + 2mc\epsilon,
\]
and in the weak field limit $\Lambda r^2 \ll 1$, if we call $\partial_r S \equiv p$, this yields
\[
\epsilon = \frac{p^2}{2m} - \frac{m\Lambda c^2}{2}\frac{r^2}{3}.
\] (5)

It is important to notice that this limit can also be obtained directly from (4) by simultaneously taking $c \rightarrow \infty$ and $\Lambda \rightarrow 0$ provided,
\[
\lim c^2\Lambda = \frac{1}{\tau^2}.
\] (6)

When $\Lambda$ is taken to have the dimensions $[\text{Length}]^{-2}$, the non-relativistic limit $c \rightarrow \infty$ yields an acceptable non-relativistic motion only if $\Lambda \rightarrow 0$ and (6) is satisfied. As a consequence (5) remains valid for large $r \ll 1/\sqrt{\Lambda}$. From this we readily obtain (4), and also the free particle limit when $\tau \rightarrow \infty$.

One might ask what would have happened if we had merely taken $c$ to infinity keeping $\Lambda$ fixed. The corresponding theory would not have a 10-dimensional kinematical group. It could still be isotropic, but there would be no boost symmetry. In detail, the de-Sitter algebras contain the bracket relation
\[
[H, P_i] = \frac{\epsilon^2\Lambda}{3}K_i,
\] (7)
where $H$ generates time translations, $P_i$ space translations and $K_i$ boosts. If we were to take the limit $c \uparrow \infty$ but keep $\Lambda$ fixed, we would have to delete the boost from the algebra. On the other hand the translations have the bracket
\[
[P_i, P_i] = \frac{\Lambda}{3}\epsilon_{ijk}L_k,
\] (8)
where the $L_i$ generate rotations. The corresponding cosmological models would be of the same type that were developed in the late nineteenth century, before the advent of either Special or General Relativity, such as that of Schwarzschild [12] in which the geometry of space was taken to be of constant curvature and independent of time ($\mathbb{R}P^3$ in his case) or more daringly, in the case hinted at by Calinon [13] in which the curvature was allowed to
vary with time. In such models Galilei invariance is broken by the curvature of space. The limit we take is precisely that needed to get a consistent $c \to \infty$ In"on"u-Wigner contraction of the de Sitter groups to the Newton-Cartan groups.

We could consider more than one particle, a finite number of point particles, which not only experience the cosmological attraction or repulsion, but also suffer mutual gravitational attractions. Thus if $m_a$ is the mass of particle $a$ we have (suspending the summation convention for $a$ and $b$)

$$m_a \frac{d^2 q_a}{dt^2} - m_a \frac{c^2 \Lambda}{3} q_a = \sum_{b \neq a} G m_a m_b \frac{(q_b - q_a)}{|q_a - q_b|^3}. \quad (9)$$

$G$ is of course Newton’s constant. Again, this can be derived from the low velocity and weak limits in de-Sitter or Anti-de-Sitter spaces with point particles. Equations (11) and (11) are in fact particular cases of the Dimitriev-Zel’dovich equation [7] (see also [14]) which determines the non-relativistic motion of a group of point particles in an expanding homogeneous isotropic universe with cosmic time $t$. The proper distances of the particles to the origin are written $r_a(t) = A(t) x_a(t)$, where $A(t)$ is the scale factor for an F-R-W universe, and $x_a(t)$ are comoving coordinates. The Dimitriev-Zel’dovich equation reads:

$$\frac{d}{dt} \left( A(t)^2 \frac{dx_a}{dt} \right) = \frac{G}{A(t)} \sum_{b \neq a} m_b \frac{(x_b - x_a)}{|x_a - x_b|^3}. \quad (10)$$

For the readers convenience, we give a derivation of this equation from Newtonian theory in Appendix A. If one lets $\mathbf{x} = a(t) \mathbf{q}$ then (10) becomes

$$\frac{d^2 q_a}{dt^2} - \frac{c^2 \Lambda(t)}{3} q_a = G \sum_{b \neq a} m_b \frac{(q_b - q_a)}{|q_a - q_b|^3} \quad (11)$$

provided that $a(t) = 1/A(t)$, and with

$$\Lambda(t) = -\frac{3}{c^2 a^2} (\ddot{a} a - 2 \dot{a}^2) = \frac{3}{c^2} \frac{\ddot{A}}{A}$$

Thus (10) describes what we could call non-relativistic motion of particles in a universe with time-dependent cosmological 'constants' $\Lambda(t)$. In fact, as the equation

$$\ddot{A} - \frac{c^2 \Lambda(t)}{3} A = 0 \quad (12)$$

always has solutions $A(t)$ given $\Lambda(t)$, (11) and (11) are equivalent. One obtains $\Lambda = 0$ for $A(t) = A_o + B_0 t$, and the linear dependence in the scale factor is reminiscent of the Galilean invariance of (10) in this case. For de-Sitter space with $\Lambda > 0$ constant, one recovers the
scale factor \( A(t) = A_0 \cosh(t/\tau) + B_0 \sinh(t/\tau) \), with \( \tau = \sqrt{3/\Lambda c^2} \) the Hubble time.

As we shall see in section 6, the conformal geometry of Newton-Hooke spacetimes and their associated homogeneous plane-waves, together with that of certain time-dependent plane-waves, provides a description of the symmetries of (10) and (11).

3 The Newton-Hooke groups

The two Newton-Hooke groups \( N^\pm_{10} \) appear to have first surfaced in the work of Lévy-Leblond and Bacry [1] (see also [15]) who classified the possible ten-dimensional kinematic Lie algebras. The commutation relations of their Lie algebras \( \mathfrak{n}^\pm_{10} \) are

\[
\begin{align*}
[J_i, J_j] &= \epsilon_{ijk} J_k, \\
[J_i, P_j] &= \epsilon_{ijk} P_k, \\
[J_i, K_j] &= \epsilon_{ijk} K_k, \\
[P_i, P_j] &= 0, \\
[K_i, K_j] &= 0, \\
[K_i, P_j] &= 0, \\
[H, J_i] &= 0, \\
[H, P_i] &= \pm \frac{1}{\tau^2} K_i, \\
[H, K_i] &= P_i,
\end{align*}
\]

where the latin indices run from 1 to 3. Thus the \( J_i \) generate rotations, the \( P_i \) are to be thought of as generating (commuting) space translations, and the \( K_i \) as generating (commuting) boosts, which also commute with space translations. Finally \( H \) should be thought of as generating time translations and commutes neither with boosts nor with space translations. One description of the groups \( N^+_{10} \) and \( N^-_{10} \) is as the semi-direct products \( (SO(3) \times SO(2)) \otimes_L \mathbb{R}^6 \) and \( (SO(3) \times SO(1,1)) \otimes_L \mathbb{R}^6 \) respectively, where \( SO(3) \times SO(2) \) and \( SO(3) \times SO(1,1) \) are the subgroups generated by the \( J_i \) and \( H \) acting by automorphisms on the abelian subgroup \( \mathbb{R}^6 \) of boosts and translations generated by the \( K_i \) and \( P_i \).

Lévy-Leblond and Bacry recognized that \( \mathfrak{n}^-_{10} \) and \( \mathfrak{n}^+_{10} \) can be respectively obtained as Inönü-Wigner contractions [16] of the Anti-de-Sitter algebra \( \mathfrak{so}(3,2) \) and the de-Sitter algebra \( \mathfrak{so}(4,1) \), in a similar way to how one obtains the Galilei algebra \( \mathfrak{gal}(3,1) \) as a contraction of the Poincaré algebra \( \mathfrak{e}(3,1) \) in the limit that the speed of light goes to infinity. However one must also rescale the cosmological constant in the limit in order to get a finite parameter \( \tau \) which turns out to be

\[
\frac{1}{\tau^2} = \pm \frac{c^2 \Lambda}{3},
\]

This is completely analogous to the low velocity and weak field limits to obtain (10). Thus the Newton-Hooke algebras \( \mathfrak{n}^{\pm}_{10} \) depend upon the real parameter \( \tau \) (taken positive) which has
the dimensions of time, and when this is taken to infinity, one obtains the Galilei algebra as is easily seen looking at the commutation relations above. Since the Poincaré algebra is also a contraction of the de-Sitter or anti-de-Sitter algebras as \( \Lambda \rightarrow 0 \), one obtains a commutative diagram of group contractions, with \( \tau \) defined by \( N \):

\[
SO(3, 2) \xrightarrow{\Lambda \uparrow 0} E(3, 1) \xleftarrow{\Lambda \downarrow 0} SO(4, 1)
\]

One sees that the Newton-Hooke groups are to the de-Sitter groups what the Galilei group is to the Poincaré group, and are to the Galilei group what the de-Sitter groups are to the Poincaré group. Hence, as was suggested by Lévy-Leblond and Bacry, they can be thought of as the kinematical groups of non-relativistic cosmological models. This will be justified in the next section. Of course one can define the Newton-Hooke groups for \( n + 1 \) dimensional isotropic spacetimes in the obvious way, and get \( (SO(n) \times SO(2)) \otimes_L \mathbb{R}^{2n} \) and \( (SO(n) \times SO(1, 1)) \otimes_L \mathbb{R}^{2n} \). These groups have the same dimension as the standard kinematical groups of \( n + 1 \) dimensional maximally symmetric spacetimes, \( SO(n, 2) \), \( SO(n + 1, 1) \), \( E(n, 1) \) and \( \text{Gal}(n, 1) \). One obtains the same contraction diagram.

Lévy-Leblond and Bacry also recognized the connection of \( N^{-10}_1 \), obtained by contraction of \( SO(3, 2) \), with systems of oscillators. This is already clear from the fact that for \( N^{-10}_1 \), the Hamiltonian \( H \) generates a periodic time translation of period \( 2\pi \tau \): indeed using the Baker-Campbell-Hausdorff formula one gets

\[
\exp(tH)P_i \exp(-tH) = \cos(t/\tau)P_i - \frac{1}{\tau} \sin(t/\tau)K_i
\]

\[
\exp(tH)K_i \exp(-tH) = \cos(t/\tau)K_i + \tau \sin(t/\tau)P_i.
\]

Similarly in \( N^{+10}_1 \) one gets

\[
\exp(tH)P_i \exp(-tH) = \cosh(t/\tau)P_i + \frac{1}{\tau} \sinh(t/\tau)K_i
\]

\[
\exp(tH)K_i \exp(-tH) = \cosh(t/\tau)K_i + \tau \sin(t/\tau)P_i.
\]

so that the Hamiltonian \( H \) generates a motion which is periodic in imaginary time, which is indicative of physics at non-zero temperature. In fact the way we have set things up is such that it suffices to take \( \tau \) to \( i\tau \) in the formulae concerning \( N^{-10}_1 \) to get the corresponding formulae for \( N^{+10}_1 \).
4 Newton-Hooke spacetimes

The cosmological interpretation of $N_{10}^\pm$ and their connection with oscillators can be made rather more concrete by using the full group composition law obtained by exponentiating the entire algebra and exhibiting its action on the Newton-Hooke spacetimes $M_4^\pm$. These are defined as the four-dimensional coset spaces $\exp(tH + q_i P_i)$ of the Newton-Hooke groups, i.e as the quotients of $N_{10}^\pm$ by the subgroup $SO(3) \otimes \mathbb{R}^3$ generated by the $J_i$ and the $K_i$. This is analogous to defining Minkowski space as the coset $E(3,1)/SO(3,1)$; in fact Bacry and Nuyts [15] have shown that all 10-dimensional kinematical groups have a four dimensional space-time interpretation.

Under the action (by left translation) of a group element

$$(t_0, a_i, v_i, R) = \exp(t_0 H) \exp(a_i P_i) \exp(v_i K_i) R$$

with $R = \exp(n_i J_i) \in SO(3)$, the coordinates of a space-time point $(t, q_i)$ are transformed to, in the case of $N_{10}^-$:

$$t \to t + t_0,$$

$$q_i \to (Rq)_i + v_i \tau \sin(t/\tau) + a_i \cos(t/\tau).$$

(16)

In the case of $N_{10}^+$ one must replace the trigonometric functions by hyperbolic-trigonometric functions, or $\tau$ by $i\tau$. The action of $N_{10}^\pm$ on $M_4^\pm$ represents a change of coordinates between two Newton-Hooke 'inertial' frames, the second being obtained from the first by successively making a rotation $R$, a Newton-Hooke boost of velocity $v_i$, a space translation by $a_i$, and a time translation by $t_0$. The parameters are given in the first frame. Clearly in the limit $\tau \to \infty$, for both $M_4^+$ and $M_4^-$, we obtain the standard formulae for Galilei transformations acting on Newton-Cartan space-time $M_4^0$. Note also that $t$ is an absolute time in $M_4^\pm$.

The Newton-Hooke transformations mathematically define the changes between 'inertial' frames in the space-times $M_4^\pm$, but we can see now that these transformations indeed represent what one would define as inertial transformations of a non-relativistic cosmological model with cosmological constant. The equation of motion (11) is clearly invariant under the action of the Newton-Hooke group (16). Furthermore, it is easy to see that (9) is also invariant under the action (16). In fact we could replace Newton’s law of gravitational attraction by any other central force law and still obtain an equation of motion invariant under the action of the Newton-Hooke group. Thus if $\{q_a(t)\}$ is a set of trajectories describing the
motion of non-relativistic particles of mass \( m \) experiencing mutual gravitational attraction and cosmological attraction (or repulsion), then so is its image under any Newton-Hooke transformation. In other words, the Newton-Hooke coordinate transformations are indeed inertial transformations, where inertial now refers to the laws of physics described by equations such as (9).

We have constructed the Newton-Hooke spacetimes \( M_{4}^{\pm} \) as the homogeneous spaces \( N_{10}^{\pm}/(SO(3) \otimes L \mathbb{R}^{3}) \); in fact they are symmetric spaces. Indeed, if \( \mathfrak{h} \) is the Lie subalgebra of \( n_{10}^{\pm} \) spanned by the \( J_{i} \) and \( K_{i} \), and \( \mathfrak{q} \) the vector space spanned by the \( P_{i} \) and \( H \), then we have:

\[
 n_{10}^{\pm} = \mathfrak{h} \oplus \mathfrak{q}, \quad [\mathfrak{h}, \mathfrak{h}] \subset \mathfrak{h}, \quad [\mathfrak{h}, \mathfrak{q}] \subset \mathfrak{q}, \quad \text{and} \quad [\mathfrak{q}, \mathfrak{q}] \subset \mathfrak{h},
\]

so that \( n_{10}^{\pm} = \mathfrak{h} \oplus \mathfrak{q} \) is a symmetric split. Since \( SO(3) \otimes L \mathbb{R}^{3} \) is not a normal subgroup, \( M_{4}^{\pm} \) are not group manifolds. One can represent the infinitesimal generators of the group action (16) by vector fields on \( M_{4}^{+} \)

\[
 H = \partial_{t}, \quad P_{i} = \cos(t/\tau)\partial_{i},
\]

\[
 K_{i} = \tau \sin(t/\tau)\partial_{i}, \quad J_{i} = \epsilon_{ijk}q^{k}\partial_{j},
\]

and correspondingly for \( M_{4}^{-} \). Newton-Hooke spacetimes \( M_{4}^{\pm} \), like their limit Newton-Cartan spacetime \( M_{4}^{0} \), do not admit an invariant non-degenerate metric: indeed, the co-metric given at the origin by a symmetric tensor

\[
 L = g_{tt}H^{2} + g_{ij}HP^{i} + g_{jt}P^{j}H + g_{kl}P^{k}P^{l},
\]

needs to be invariant under the adjoint action of \( SO(3) \otimes L \mathbb{R}^{3} \) for the metric to be well-defined on the coset \( M_{4}^{\pm} \). Since \( [K^{i}, L] = 0 \) implies that \( g_{tt} = g_{jt} = 0 \), \( L \) is necessarily a degenerate co-metric, and one cannot define the associated metric. Nevertheless \( M_{4}^{\pm} \) do have a well-defined geometric structure which may be described in a covariant fashion. One description proceeds by endowing \( M_{4}^{\pm} \) with degenerate co-metrics (or contravariant metrics) and an affine connection. The co-metrics can be obtained group theoretically \( (L = \delta_{ij}P^{i}P^{j}) \) or by taking the limits \( c \to \infty \) and \( \Lambda \to 0 \) of the Anti-de-Sitter and de-Sitter metrics (see (2)). However, just as in the case of Newton-Cartan spacetime, it proves more convenient to regard Newton-Hooke spacetimes as a Kaluza type null reduction of an ordinary 5-dimensional Lorentzian spacetime \( M_{5}^{\pm} \). We shall see in the next section that \( M_{5}^{\pm} \) are in fact homogeneous plane-waves.
5 Bargmann Structures

Duval, Burdet, Künzle and Perrin [8] have given an elegant construction of Newton-Cartan spacetime $M_5^0$ as the null reduction of a certain five-dimensional Lorentzian spacetime $M_5$ equipped with what they called a Bargmann structure. This is essentially a covariantly constant null Killing vector field $V$ generating an $\mathbb{R}$, or possibly $S^1$, action which we shall call $G_{\text{null}}$. The idea was further developed in [9] where $M_5$ was shown to be a Brinkmann or pp-wave spacetime. The merit of the approach is that it exhibits the action of the Galilei group, its central extension the Bargmann group, and the non-relativistic conformal or Schrödinger group as subgroups of the isometry group, or conformal group of $M_5$ which commute with the projection $\pi : M_5 \rightarrow M_5^0 \equiv M_5/G_{\text{null}}$. One can apply these ideas to Newtonian Cosmology and the Newtonian N-body problem [9].

We shall now adapt this construction to the case of Newton-Hooke spacetimes. Consider the five dimensional plane-wave space-times $M_5^\pm$ with Lorentz metrics

$$ ds_\pm^2 = dq_i dq_i + 2 dt dv \pm \frac{1}{\tau^2} q_i q_i dt^2. $$

(18)

In addition to the obvious action of the rotation group $SO(3)$, the metric is easily seen to be invariant under the eight-parameter group whose action is

$$ t \rightarrow t + t_0, \\
q_i \rightarrow q_i + F_i(t), \\
v \rightarrow v + q_i G_i(t) + F(t), $$

(19)
as long as

$$ \ddot{F}_i + G_i = 0, \quad \dot{G}_i \pm \frac{1}{\tau^2} F_i = 0, \quad 2\ddot{F}_i + \dot{F}_i \dot{F}_i \pm \frac{1}{\tau^2} F_i F_i = 0. $$

Thus

$$ \ddot{F}_i + \frac{1}{\tau^2} F_i = 0, $$

and we may we take as its solution in the case of $M_5^-$

$$ F_i = \tau v_i \sin(t/\tau) + a_i \cos(t/\tau), $$

where $v_i$ and $a_i$ are 2 constant 3-vectors (and take $\tau \rightarrow i\tau$ for $M_5^+$). The function $F(t)$ is determined only up to a constant of integration $v_0$, so that with $t_0$, the motion (19) has 8
parameters. We explicitly introduce $v_0$ as corresponding to the action of $G_{\text{null}}$,

$$v \rightarrow v + v_0,$$

with associated covariantly constant null Killing vector field

$$V = \partial_v.$$

The Killing vector fields $P_i$ and $K_i$ of $M^\pm_4$ given in (17), when lifted to $M^\pm_5$, are modified to

$$P_i \rightarrow P_i + q_i \frac{1}{\tau} \sin(t/\tau) \partial_v, \quad K_i \rightarrow K_i - q_i \cos(t/\tau) \partial_v,$$

and their Lie bracket becomes

$$[P_i, K_j] = -\delta_{ij} V. \quad (21)$$

The other commutation relations given in (13) remain the same, so that $V$ is a central element. Thus the isometry groups $N^\pm_{11}$ of $M^\pm_5$ are non-trivial central extensions of the Newton-Hooke groups $N^\pm_{10}$ by $G_{\text{null}}$. The informed reader will recognize (21) as defining the Heisenberg subalgebra of the isometry algebra of Hpp-waves, with $H = \partial_t$ and $J_i = \epsilon_{ijk} q^k \partial_j$ acting on it as outer automorphisms. In other words the groups $N^\pm_{11}$ have the structure of the semi-direct products $(SO(3) \times SO(2)) \otimes L \text{H}(7)$ and $(SO(3) \times SO(1,1)) \otimes L \text{H}(7)$, where $\text{H}(7)$ is the Heisenberg group generated by $P_i$, $K_i$ and $V$.

It is clear that all the isometries of $M^\pm_5$ commute with the projection $\pi : M^\pm_5 \rightarrow M^\pm_4$ given by $(v, t, q_i) \mapsto (t, q_i)$. This is completely analogous to the usual Bargmann group which is the central extension of the Galilei group, or symmetry group of Newton-Cartan space-time, and indeed our formulae reduce to that case in the limit $\tau \rightarrow \infty$. Therefore we propose calling $N^\pm_{11}$ the Bargmann extensions of the Newton-Hooke groups, or the Bargmann-Newton-Hooke groups for short. The groups $N^\pm_{11}$ have Casimirs

$$c_1 = V,$$

$$c_2 = 2VH + P_i^2 \mp \frac{1}{\tau^2} K_i^2,$$

$$c_4 = (VJ_i - \epsilon_{ijk} K_j P_k)^2.$$

To pass to the Newton-Hooke group one sets $V = -m$, where $m$ is the mass of the system one is considering; $H$ is then its Hamiltonian, $c_2$ its internal energy and $c_4/m^2$ the square of
its spin vector.

It has long been known that while the Anti-de-Sitter group $SO(3, 2)$ admits a well defined notion of positive energy, no such notion is possible for the de-Sitter group $SO(4, 1)$. This is connected with the existence of Killing vector fields on Anti-de-Sitter spacetime $AdS_4$ which are everywhere timelike. The metric is globally static with respect to these Killing fields and there are no Killing horizons. By contrast in the case of de-Sitter spacetime $dS_4$, there are only locally time-like Killing fields with Killing horizons beyond which the Killing field becomes spacelike. In fact the Killing fields generate $SO(2)$ subgroups of $SO(3, 2)$ or $SO(1, 1)$ subgroups of $SO(4, 1)$ respectively. Note that this is does not mean that de-Sitter space does not admit a global time function; it does of course.

This dichotomy is reflected on the Hpp-wave spacetimes. In the case of $M_5^+$ the Killing field $H = \frac{\partial}{\partial t} = H^\alpha \frac{\partial}{\partial x^\alpha}$ is everywhere causal, becoming null on the timelike two-surface $q_i = 0$. By contrast in the case of $M_5^-$ the Killing field $H = \frac{\partial}{\partial t} = H^\alpha \frac{\partial}{\partial x^\alpha}$ is almost everywhere spacelike, becoming null on the timelike two-surface $q_i = 0$. Moreover, the Killing field $K = \frac{\partial}{\partial t} - \mu \frac{\partial}{\partial v}$ has an ergo-region: it is causal inside the cylinder $q_i q_i \leq 2\mu \tau^2$, and becomes spacelike beyond the timelike hypersurface $q_i q_i = 2\mu \tau^2$. This substantial difference occurs although both Newton-Hooke spacetimes and their Bargmann manifolds have an absolute 'time' coordinate, as we see from (16) which remains true in $M_5^\pm$.

A freely falling particle of mass $m$ moving on a geodesic in $M_5^\pm$ has a conserved energy $E = -mg_{\alpha\beta} \frac{dx^\alpha}{d\sigma} H^\beta$. If the geodesic is future directed and causal, then in $M_5^+$ the energy $E$ can never be negative. By contrast in the case of $M_5^-$ the energy can take either sign. Of course when we consider null geodesics this statement is equivalent to saying that the energy of a non-relativistic particle in an upside-down potential is unbounded.

6 Non-relativistic conformal symmetries

Following the ideas of Burdet, Perrin and Duval [10] [17] on the relation between the ”chrono-projective geometry” of Bargmann structures and the Schrödinger equation, further applied to Newtonian gravity in [9], we now take a closer look at the conformal symmetries of $M_5^\pm$.

As we shall see, the Bargmann structure (Hpp-wave) introduced to define Newton-Hooke space-times enables us to find other symmetries of the equations of Newtonian cosmology
with cosmological constant $\Omega$ and $\Omega$. Consider a plane-wave space-time with the following metric in Brinkmann coordinates

$$ds^2 = 2dt dv + \alpha(t)q^i q^j dt^2 + dq^i dq^j.$$  

(22)

Since this metric is conformally flat, its conformal group is locally isomorphic to $SO(5,2)$. As a side remark, this means that the centrally extended Newton-Hooke groups $N^\pm_{11}$, realized as the isometry groups of $M^\pm_5$, are subgroups of $SO(5,2)$, in a similar way as is the central extension of the Galilei group. We find the conformal symmetries of (22) by going to conformally flat coordinates, which means going to Rosen coordinates first, and then redefining $t \to U$ so that the metric scales properly: let $A(t)$ a solution of

$$\ddot{A} - \alpha(t) A = 0$$

(23)

and let

$$t = t, \quad v = V - \frac{1}{2} \dot{A}(t) A(t) X^i X^i, \quad q^j = A(t) X^j.$$  

(24)

Further define

$$U = \int^t dt' \frac{dt'}{A(t')^2} \equiv f(t)$$

(25)

and the metric (22) becomes

$$ds^2 = A(f^{-1}(U))^2 (2dU dV + dX^i dX^i).$$

(26)

For $M^-_5$, we can take $A(t) = \cos(t/\tau)$ and $U = \tau \tan(t/\tau)$, and for $M^+_5$, $A(t) = \cosh(t/\tau)$ and $U = \tau \tanh(t/\tau)$.

It is easy to see that the null geodesics of (22) can be chosen to have affine parameter $t$, in which case they satisfy

$$\dot{q} - \alpha(t) q = 0, \quad \dot{v} = m$$

(27)

so that they describe the non-relativistic motion of particles of mass $m$ in a time-dependent harmonic potential $-\frac{1}{2} \alpha(t) q^2$. Since conformally equivalent metrics have the same null geodesics up to change of coordinates, the null geodesics of flat space yield all the solutions of (27); however we need to solve (23) before hand to find conformally flat coordinates. We will show in fact that the action of the conformal group $SO(5,2)$ on (26) provides additional symmetries of (27) and (11).

Symmetries of time-dependent harmonic oscillators have been studied in the past, and extensively used in string theory on time dependent plane-wave backgrounds. Lewis and
Riesenfeld [18] developed a theory of invariants of these systems (time independent quantum operators), which Blau and O’Loughlin [19] explained geometrically in the setting of plane-wave space-times. Essentially the isometries of these spaces (Heisenberg algebra) can be used to construct the invariants. The Bargmann conformal structure yields additional symmetries.

Burdet, Duval and Perrin define the Bargmann conformal group of a pp-metric, or more generally of a Bargmann structure, to be the subgroup of the group conformal transformations of the metric which leaves the covariantly constant null Killing vector invariant. In other words, if \( g \) denotes the metric and \( \xi \) the Killing vector, a local diffeomorphism \( D \) is a Bargmann conformal transformation if

\[
D^* g = \Omega^2 g, \quad \text{and} \quad D^*_* \xi = \xi,
\]

where \( \Omega^2 \) depends on the coordinates. In our case \( \xi = \partial_v \). As a consequence, a null geodesic of (22) with mass parameter \( \dot{v} = m \) is mapped to a null geodesic with the same mass \( m \): the Bargmann conformal symmetries of (22) leave (27) invariant.

In practice, \( D : (t, v, q^i) \to (t^*, v^*, (q^i)^*) \) satisfies (28) if and only if \( D \) is conformal and commutes with the projection \( \Pi : (t, v, q^i) \to (t, q^i) \), and \( \frac{\partial v^*}{\partial v} = 1 \). The conformal transformations \((U, V, X^i) \to (U^*, V^*, (X^i)^*)\) of

\[
ds^2 = dUdV + dX^idX^i
\]

correspond to those of (22). Requiring that the associated transforms \( t \to t^* \) and \( q^i \to (q^i)^* \) do not depend on \( v \) and \( \frac{\partial v^*}{\partial v} = 1 \) is then equivalent to

\[
\frac{\partial U^*}{\partial V} = \frac{\partial (X^i)^*}{\partial V} = 0, \quad \frac{\partial V^*}{\partial V} = 1.
\]

This follows immediately from the change of coordinates (24) and (25). The Bargmann conformal transformations of (29) are easily found to be [17] [9]:

\[
U^* = \frac{dU + e}{aU + b},
\]
\[
V^* = V + \frac{a}{2} \frac{(AX + bU + c)^2}{aU + b} - \langle b, AX \rangle - \frac{U}{2} b^2 + h,
\]
\[
X^* = \frac{AX + bU + c}{aU + b},
\]

where \( A \in SO(3), b, c \in \mathbb{R}^3; d, e, a, b, h \in \mathbb{R} \) with \( db - ea = 1 \). These transformations define via coordinate transform the 13-dimensional Bargmann conformal group of the plane-wave
metrics \[22\]. It can be described as the semi-direct product \( (SO(3) \times SL(2, \mathbb{R})) \otimes_L H(7) \).

The conformal factor for \[26\] is easily seen to be:

\[
\Omega^2(U) = \frac{A(f^{-1}(U^*))^2}{A(f^{-1}(U))^2} \frac{\partial U^*}{\partial U} = \frac{A(f^{-1}(U^*))^2}{A(f^{-1}(U))^2} \frac{1}{(aU + b)^2}.
\]

(32)

In terms of the initial coordinates \((t, v, q)\), using \(f'(t) = 1/A(t)^2\), this becomes:

\[
\Omega^2(t) = \frac{\partial t^*}{\partial t}, \quad \text{where} \quad t^* = f^{-1}\left(\frac{df(t) + e}{af(t) + b}\right).
\]

(33)

We now focus on the specific non-relativistic conformal transformations given by \(A = Id, b = c = 0\), which generate a subgroup isomorphic to \(SL(2, \mathbb{R})\). We have:

\[
t^* = f^{-1}\left(\frac{df(t) + e}{af(t) + b}\right),
\]

\[
q^* = \frac{A(t^*)}{A(t)} \frac{q}{(af(t) + b)}.
\]

(34)

with \(db - ea = 1\). By construction, these symmetries leave \[27\] invariant. Thus a set of solutions \((q_a(t), m_a)\) of the time-dependent oscillator equation is mapped by \[34\] to another set of solutions \((q^*_a(t^*), m_a)\). When these symmetries have non-trivial conformal factors \[33\], they are ‘additional’: they do not derive from the isometries of the metric \[22\]. Since the isometry group of \[22\] is at most 11-dimensional for \(\alpha(t)\) non-trivial, \[34\] does provide extra symmetries.

Consider now a solution \((q_a(t), m_a)\) of Newton’s equations \[11\] with cosmological ‘constant’ \(\Lambda(t) = 3\alpha(t)/c^2\) and gravitational coupling constant \(G_o\). We have:

\[
\frac{d^2q_a}{dt^*^2} = \frac{c^2\Lambda(t^*)}{3} q_a = \left(\frac{A(t)}{A(t^*)} (af(t) + b)\right)^3 \frac{d^2q_a}{dt^2} - \frac{c^2\Lambda(t)}{3} q_a,
\]

\[
= \frac{A(t)}{A(t^*)} (af(t) + b)G_o \sum_{b \neq a} m_b \frac{(q^*_b - q^*_a)}{|q^*_b - q^*_a|^3},
\]

(35)

so that \((q_a(t), m_a)\) is taken by \[34\] to a solution \((q^*_a(t^*), m_a)\) with cosmological term \(\Lambda(t^*)\) but with a time-dependent gravitational constant

\[
G(t^*) = \frac{A(t)}{A(t^*)} (af(t) + b)G_o.
\]

(36)

When \(\Lambda \equiv 0\) \[22\] is \[29\], we can take \(A \equiv 1\): this is the case studied in \[9\], and we recover the Lynden-Bell symmetries of Newton’s equations \[11\]. For \(\Lambda < 0\) constant, we have

\[
t^* = \tau \arctan\left(\frac{d \tan(t/\tau) + e/\tau}{a \tau \tan(t/\tau) + b}\right)
\]

\[
q^* = -\cos(t^*/\tau) \sqrt{(-d + a \tau \tan(t^*/\tau))^2 + \frac{1}{\tau^2}(-e + b \tau \tan(t^*/\tau))^2} q
\]

\[
G(t^*) = -\left(\cos(t^*/\tau) \sqrt{(-d + a \tau \tan(t^*/\tau))^2 + \frac{1}{\tau^2}(-e + b \tau \tan(t^*/\tau))^2}\right)^{-1}
\]

(37)
For $\Lambda > 0$, replace tan and arctan by tanh and arctanh.

7 The Matrix model

We recall here the modification proposed by Gao \cite{4} for incorporating de-Sitter physics into the standard matrix model. The latter is essentially a non-relativistic model of a system with non-commuting coordinates. Since it is formulated in the light-cone gauge it admits Galilei symmetry, and so Gao, to incorporate some cosmological features, constructed a model admitting Newton-Hooke symmetry. The equations of motion are

$$\frac{d^2 q_a}{dt^2} + \frac{1}{\tau^2} q_a = \sum_b [q_b, [q_a, q_b]],$$  \hspace{1cm} (38)

where now the $q_i$ are $N \times N$ hermitian matrices. The standard matrix model may be obtained by a dimensional reduction of one spacetime dimension of non-abelian Yang-Mills theory with group $U(N)$. One picks a gauge in which the $u(N)$ valued connection one form is $A_\mu = (0, q_a(t))$. One could obtain Gao’s modification by breaking gauge invariance by adding a mass term

$$\frac{1}{2\tau^2} \text{Tr} A_\mu A^\mu$$  \hspace{1cm} (39)

The mass term is “tachyonic” for $N_{10}^+$, i.e for positive cosmological constant. The equations of motion (38) are invariant under (16) as long as $v_i$ and $a_i$ are multiples of the unit $N \times N$ matrix. The unit matrices are associated to the centre of mass motion and are commutative. They do not contribute to the right hand side of (38) and moreover the Newton-Hooke transformations act only on the unit matrices, and so one can say that it is a symmetry only of the centre of mass motion, just as in the case of self-gravitating particles (19).

We have seen earlier how to lift the commutative equations (19) to a higher dimensional (mildly) curved spacetime with commutative coordinates. This suggests we can use Gao’s suggestion to construct a higher-dimensional curved spacetime with non-commuting coordinates.

8 Conclusion

We have shown that the Newton-Hooke groups are indeed the relevant symmetry groups of non-relativistic cosmological models with cosmological constant. Though we have only considered Newton’s equations in such models, the non-relativistic Schrödinger equation, in
its Bargmann formulation [10], admits the same symmetries. It is worth noticing that these models, depending on whether $\Lambda$ is negative or positive, share many common features with their relativistic counterparts the anti-de-Sitter and de-Sitter cases: compactness or non-compactness of the time generator $H$, definiteness or indefiniteness of the energy of particles, existence or non-existence of causal Killing fields (not all of which are independent). It seems therefore that they provide an interesting geometrical setting for a better understanding of the cosmological constant.

Our results, including the action of the Bargmann conformal group of $M^+_5$ on the cosmological equations of motion, might also have applications to string theories and Matrix models which have the same symmetries. Moreover, the fact that the metric of $M^-_5$ is similar to that obtained [20] by taking a Penrose limit of $AdS \times S$ spaces raises questions. The former consists of a non-relativistic limit of $AdS$ lifted to a spacetime with one extra dimension, while the latter essentially describes the spacetime (with many non-relativistic features) viewed by an observer reaching the speed of light in $AdS \times S$. In fact, different Inönü-Wigner contractions of the symmetry groups involved occur, and yield the same result up to a central extension, the extended Newton-Hooke group $N^-_{11}$.

A  A Derivation of the Dimitriev-Zel’ dovich Equations from Newton’s equations

We start with the exact equations of motion for a large but finite number of particles:

$$m_a \ddot{x}_a = \sum_{b \neq a} \frac{G m_a m_b (x_b - x_a)}{|x_a - x_b|^3} \tag{40}$$

and assume that the particles fall into two classes, with $a = i, j, k \ldots$ and $a = I, J, K, \ldots$. The second set form a cosmological background and we make the approximation that their motion is unaffected by the first class of particles, galaxies, whose motion is however affected both by the background particles and their mutual attractions.

Thus the equations of motion (40) split into two sets

$$m_I \ddot{x}_I = \sum_{J \neq I} \frac{G m_I m_J (x_I - x_J)}{|x_I - x_J|^3} \tag{41}$$
and
\[ m_i \ddot{x}_i = \sum_{j \neq i} \frac{G m_i m_j (x_j - x_i)}{|x_i - x_j|^3} + \sum_j \frac{G m_i m_j (x_J - x_i)}{|x_i - x_J|^3}. \] (42)

We now assume that the background particles form a central configuration
\[ x_f = a(t) r_f. \] (43)

Thus the deviation of the first set of particles from this mean Hubble flow is given by
\[ m_i \ddot{x}_i = \sum_{j \neq i} \frac{G m_i m_j (x_j - x_i)}{|x_i - x_j|^3} + \sum_j \frac{G m_i m_j (a(t) r_f - x_i)}{|x_i - a(t) r_f|^3}. \] (44)

We replace the absolute positions of the galaxies by the co-moving positions \( x_i = a(t) r_i \) and obtain
\[ m_i \ddot{r}_i + 2a(t) \dot{r}_i + a(t) \ddot{r}_i = \frac{1}{a^2(t)} \sum_{j \neq i} \frac{G m_i m_j (r_j - r_i)}{|r_i - r_j|^3} + \frac{1}{a^2(t)} \sum_j \frac{G m_i m_j (r_f - r_i)}{|r_i - r_f|^3}. \] (45)

The second term on the right hand side of (45) is the force \( F_i \) acting on the \( i \)'th galaxies by the background particles. The numerical work in [21] provided very good evidence that for a large number of background particles, the central configuration is to a very good approximation statistically spherically symmetric and homogeneous. It follows that the force exerted by the background is radial
\[ \sum_j \frac{G m_i m_j (r_f - r_i)}{|r_i - r_f|^3} = -cm_i r_i, \] (46)

where the constant \( c \) and scale factor \( a(t) \) satisfy
\[ a^2 \ddot{a} = -c \] (47)

It follows that the force \( F_i \) on the right hand side of (45) cancels the third term on the left hand side. We are left with
\[ m_i (a(t) \ddot{r}_i + 2a(t) \dot{r}_i + \ddot{a}(t) r_i) = \frac{1}{a^2(t)} \sum_{j \neq i} \frac{G m_i m_j (r_j - r_i)}{|r_i - r_j|^3}, \] (48)

or
\[ \frac{d(a(t)^2 \dot{r}_i)}{dt} = \frac{1}{a(t)} \sum_{j \neq i} \frac{G m_j r_j - r_i)}{|r_i - r_j|^3}, \] (49)

which is the Dimitriev-Zel’dovich equation.
References

[1] Bacry H and Lévy-Leblond JM 1967 Possible kinematics *J. Math. Phys.* 9 1605

[2] Derome J-R and Dubois J-G 1972 Hooke’s symmetries and nonrelativistic cosmological kinematics -I *Nuovo Cimento* 9B 351
   Dubois J-G 1973 Hooke’s symmetries and nonrelativistic cosmological kinematics -II *Nuovo Cimento* 15B 1

[3] Aldrovandi R, Barbosa A L, Crispino L C B, Pereira J G 1999 Non-relativistic spacetimes with cosmological constant *Class. Quantum Grav.* 16 495

[4] Gao Y-h 2001 Symmetries, matrices, and de Sitter gravity *Preprint* hep-th/0107067

[5] Gibbons G W, Hashimoto K and Piljin Y 2002 Tachyon Condensates, Carrolian Contraction of Lorentz Group and Fundamental Strings, *JHEP* 0209 061 *Preprint* hep-th/0209034

[6] Aldrovandi R, Barbosa A L, Calcada M, Pereira JG 2003 Kinematics of a Spacetime with an Infinite Cosmological Constant *Found Phys Lett* 33 613 *Preprint* gr-qc/0105068

[7] Dmitriev N A and Zel’dovich Ya B 1964 The energy of accidental motions in the expanding universe *Sov. Phys. JETP* 18 793

[8] Duval C, Burdet G, Künzle H P and Perrin M 1985 Bargmann structures and Newton-Cartan theory *Phys. Rev. D* 31 1841

[9] Duval C, Gibbons G and Horváthy P 1991 Celestial mechanics, conformal structures, and gravitational waves *Phys. Rev. D* 43 3907

[10] Burdet G, Duval C and Perrin M 1985 Time dependent quantum systems and chronoprojective geometry *Lett. Math. Phys.* 10 255

[11] Lynden-Bell D 1988 *Observatory* 102 86

[12] Schwarzschild K 1900 Über das zulässige Krümmungsmass des Raumes, *Vierteljahrsschrift d. Astronom. Gesellschaft* 35 337, translated as On the permissable curvature of space 1998 *Class. Quant. Grav.* 15 2539

[13] Calinon A 1893 Les Espaces Géométriques *Revue Philosophique* 36 595
[14] Peebles P J E 1980 The Large-scale structure of the universe *Princeton University Press, Princeton, NJ*

[15] Bacry H and Nuyts J 1986 Classification of ten-dimensional kinematical groups with space isotropy *J. Math. Phys.* 27 2455

[16] Inönü E and Wigner 1953 On the Contraction of groups and their representations *Proc. Natl Acad. Sci. USA* 39 510

[17] Perrin M, Burdet G and Duval C 1986 Chronoprojective invariance of the five-dimensional Schrödinger formalism *Class. Quantum Grav.* 3 461

[18] Lewis H R 1968 Class of exact invariants for classical and quantum time dependent harmonic oscillators *J. Math. Phys* 9 1976 Lewis H R and Riesenfeld W B 1969 An exact quantum theory of the time dependent harmonic oscillator and of a charged particle in a time dependent electromagnetic field *J. Math. Phys.* 10 1458

[19] Blau M and O’Loughlin M 2003 Homogeneous plane waves *Nucl. Phys. B* 654 135

[20] Blau M, Figueroa-O’Farill J and Papadopoulos G 2002 Penrose limits, supergravity and brane dynamics *Class. Quantum Grav.* 19 4753 *Preprint* [hep-th/0202021](http://arxiv.org/abs/hep-th/0202021)

[21] Battye R, Gibbons G W, Sutcliffe P M 2003 Central Configurations in Three Dimensions *Proc Roy Soc A* 459 911 *Preprint* [hep-th/0201101](http://arxiv.org/abs/hep-th/0201101)