The large-$N$ limit for two-dimensional Yang–Mills theory

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Abstract

The analysis of the large-$N$ limit of $U(N)$ Yang–Mills theory on a surface proceeds in two stages, the analysis of a the Wilson loop functional for a simple closed curve and the reduction of more general loops to a simple closed curve. We give a rigorous treatment of the second stage of analysis in the case of 2-sphere. Specifically, we assume that the large-$N$ limit of the Wilson loop functional for a simple closed curve in $S^2$ exists and that the associated variance goes to zero. Under this assumption, we establish the existence of the limit and the vanishing of the variance for arbitrary loops with simple crossings. The proof is based on the Makeenko–Migdal equation for the Yang–Mills measure on surfaces, as established rigorously by Driver, Gabriel, Hall, and Kemp, together with an explicit procedure for reducing a general loop in $S^2$ to a simple closed curve.

The methods used here also give a new proof of these results in the plane case, as an alternative to the methods used by Lévy. In the plane case, the proof is not dependent on any unproven assumptions. Finally, we consider the case of an arbitrary surface. We obtain similar results in this setting for homotopically trivial loops that satisfy a certain “smallness” assumption.

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1 Introduction and main results

1.1 The Makeenko–Migdal equation in two dimensions

Let us fix a compact Lie group $K$ together with a $\operatorname{Ad}$-invariant inner product on its Lie algebra, $\mathfrak{k}$. The path-integral for Euclidean Yang–Mills theory over a manifold $M$ is supposed to describe a probability measure on the space of connections for a principle $K$-bundle over $M$. One of the main objects of study in such a theory is the Wilson loop functional, namely the expectation value of the trace (in some fixed representation of $K$) of the holonomy of the connection around a loop. The Makeenko–Migdal equation is an identity for the variation of Wilson loop functionals with respect to a variation in the loop. The original version of this equation, in any number of dimensions, was proposed by Makeenko and Migdal in [MM]. A version specific to the two-dimensional case was then developed by Kazakov and Kostov in [KK, Eq. (24)]. (See also [K, Eq. (9)] and [GG, Eq. (6.4)].)

A special feature of the two-dimensional Yang–Mills measure its invariance under area-preserving diffeomorphisms. Suppose we fix the topological type of a loop $L$ in a surface $\Sigma$ and consider the faces of $L$, that is, the connected components of the complement of $L$ in $\Sigma$. Then the Wilson loop functional depends only on the areas of the faces of $L$. Let us now take $K = U(N)$ with
the inner product on the Lie algebra $u(N)$ given by the scaled Hilbert–Schmidt inner product,
\[ \langle X, Y \rangle := N \text{Trace}(X^*Y). \]  
(1)

It is then convenient to express the Wilson loop functionals in terms of the normalized trace,
\[ \text{tr}(X) := \frac{1}{N} \text{Trace}(X). \]  
(2)

We now consider a loop $L$ with simple crossings, and we let $v$ be one such crossing. We label the four faces of $L$ adjacent to the crossing in cyclic order as $F_1, \ldots, F_4$, with $F_1$ denoting the face whose boundary contains the two outgoing edges of $L$. We then let $t_1, \ldots, t_4$ denote the areas of these faces. (See Figure 1.) We also let $L_1$ denote the loop from the beginning to the first return to $v$ and let $L_2$ denote the loop from the first return to the end. (See Figure 2.)

The two-dimensional version of the Makeenko–Migdal equation, in the $U(N)$ case, is then as follows:
\[
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \mathbb{E}\{\text{tr}(\text{hol}(L))\} = \mathbb{E}\{\text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2))\},
\]  
(3)

where $\text{hol}(\cdot)$ denotes the holonomy. Although the curves $L_1$ and $L_2$ occurring on the right-hand side of (3) are simpler than the loop $L$, the right-hand side of (3) involves the expectation of the product of the traces, rather than the product of the expectations. Thus, even if one has already computed the Wilson loop functionals $\mathbb{E}\{\text{tr}(\text{hol}(L_1))\}$ and $\mathbb{E}\{\text{tr}(\text{hol}(L_2))\}$, the right-hand side of (3) cannot be regarded as a known quantity. As we will see, however, in the large-$N$ limit, the Makeenko–Migdal equation becomes an effective tool for inductive computation of Wilson loop functionals.

The original argument of Makeenko and Migdal for the equation that bears their names was based on heuristic manipulations of the path integral. In the plane case, Lévy then gave a rigorous proof of the Makeenko–Migdal equation.
in [Lév2]. (See Eq. (117) in Proposition 6.24 of [Lév2].) Subsequent proofs of the planar Makeenko–Migdal equation were then provided by Dahlqvist [Dahl] and Driver–Hall–Kemp [DHK2].

Meanwhile, in [DGHK], Driver, Gabriel, Hall, and Kemp gave a rigorous derivation of the Makeenko–Migdal equation for $U(N)$ Yang–Mills theory over an arbitrary surface. Actually, the proof given in [DHK2] in the plane case extends with minor modifications to the case of a general surface.

1.2 The master field in two dimensions

In the paper [’t H], ’t Hooft proposed that Yang–Mills theory for $U(N)$ in any dimension should simplify in the limit as $N \to \infty$. In particular, it is expected that in this limit, the path integral should concentrate onto a single connection (modulo gauge transformations), known as the master field. The concentration phenomenon for the Yang–Mills measure has an important implication for the form of the two-dimensional Makeenko–Migdal equation. In the limit, there is no difference between the expectation of a product of traces and the product of the associated expectations: both $\mathbb{E}\{fg\}$ and $\mathbb{E}\{f\} \mathbb{E}\{g\}$ should become $f(M_0)g(M_0)$, where $M_0$ is the master field.

If, therefore, the large-$N$ limit of $U(N)$ Yang–Mills theory exists on a surface $\Sigma$, we expect it to satisfy a Makeenko–Migdal equation of the form

$$
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) w(L) = w(L_1)w(L_2),
$$

(4)

where $w(L)$ is the limiting value of $\mathbb{E}\{\text{tr}(\text{hol}(L))\}$. Note that the loops $L_1$ and $L_2$ on the right-hand side of (4) have fewer crossings than $L$, since neither $L_1$ nor $L_2$ has a crossing at $v$. Thus, one may hope that the large-$N$ Makeenko–Migdal equation may allow one to reduce computations of Wilson loop functionals for general curves to simpler ones, until one eventually reaches a simple closed curve.
Of course, since a simple closed curve has no crossings, the Makeenko–Migdal equation gives no information about the Wilson loop for such a curve.

In the plane case, the structure of the master field was worked out by Singer [Si], Gopakumar and Gross [GG, Gop], Xu [Xu], Sengupta [Sen4], Anshelevic and Sengupta [AS], and then in greater detail by Lévy [Lév2]. In particular, the expected concentration phenomenon was verified in detail in the plane case in [Lév2]. (See the explicit variance estimate in Theorem 5.6 of [Lév2].)

In [Lév2], Lévy shows that the large-\(N\) limit of the Wilson loop functional for a loop in the plane with simple crossings is completely determined by (4), together with another, simpler condition. This simpler condition—given as Axiom \(\Phi_4\) in Section 0.5 of [Lév2] and called the “unbounded face condition” in [DHK2, Theorem 2.3]—gives a simple formula for the derivative of the Wilson loop functional with respect to the area of any face of \(L\) that adjoins the unbounded face.

### 1.3 The master field on the sphere

The existence of a large-\(N\) limit of Yang–Mills theory on a general surface \(\Sigma\) is currently unknown. There has, however, been much interest in the problem because of connections with string theory, as developed by Gross and Taylor [Gr, GT1, GT2].

The \(S^2\) case, meanwhile, has been extensively studied at varying levels of rigor. The analysis proceeds in two stages. First, one studies the large-\(N\) limit of the Wilson loop functional for a simple closed curve. Second, one attempts to use the large-\(N\) Makeenko–Migdal equation to reduce Wilson loop functionals for all other loops with simple crossings to the simple closed curve.

In the first stage of analysis, a formula has been proposed for the Wilson loop functional for a simple closed curve. (See Section 1.5 for more information.) A notable feature of this formula is the presence of a phase transition. If the total area of the sphere is less than \(\pi^2\), the Wilson loop for a simple closed curve is expressible in terms of the semicircular distribution from random matrix theory. If, however, the total area is greater than \(\pi^2\), the Wilson loop is much more complicated. In addition to the proposed formula for the limiting Wilson loop functional, it is expected that the limit should be deterministic, in keeping with the idea of the master field. This brings us to the following conjecture.

**Conjecture 1** If \(L\) is a simple closed curve on \(S^2\) then the limit

\[
\lim_{N \to \infty} \mathbb{E} \{ \text{tr} \text{hol}(L) \} \quad (5)
\]

exists and depends continuously on the areas of the two faces of \(L\). Furthermore, the associated variance tends to zero:

\[
\lim_{N \to \infty} \text{Var} \{ \text{tr} \text{hol}(L) \} = 0. 
\]

(6)

Although this result is widely expected in the physics literature, it does not appear that a rigorous proof is currently available.
Notation 2. We denote the (conjectural) large-\(N\) limit of the Wilson loop functional for a simple closed curve by \(w_1\):

\[
    w_1(a, b) = \lim_{N \to \infty} \mathbb{E}\{\text{tr(hol}(L))\},
\]

where \(L\) is a simple closed curve and where \(a\) and \(b\) are the areas of the faces of \(L\).

In the second stage of analysis, it has been claimed by Daul and Kazakov that, “All averages for self-intersecting loops can be reproduced from the average for a simple (non-self-intersecting) loop by means of loop equations.” (See the abstract of [DaK]. The loop equations referred to are the large-\(N\) Makeenko–Migdal equation (4).) It should be noted, however, that Daul and Kazakov analyze only two examples, and it is not obvious how to extend their analysis to general loops; see Section 3. Furthermore, they assume that the large-\(N\) limit exists and satisfies the large-\(N\) Makeenko–Migdal equation.

1.4 New results

In this paper, we give a rigorous treatment of the second stage of the analysis of the large-\(N\) limit for Yang–Mills theory on \(S^2\). Specifically, assuming Conjecture 1, we establish the following results: (1) the existence of the large-\(N\) limit of Wilson loop functionals for arbitrary loops with simple crossings; (2) the vanishing of the associated variance; and (3) the large-\(N\) Makeenko–Migdal equation for the limiting theory. In particular, we give a concrete procedure for reducing the Wilson loop functional for general loops in \(S^2\) to the Wilson loop functional for a simple closed curve.

Here are some notable features of our approach.

- We do not assume the existence of the large-\(N\) limit ahead of time, except for a simple closed curve.

- We do not assume ahead of time that the limiting theory satisfies the large-\(N\) Makeenko–Migdal equation. Rather, we assume only the finite-\(N\) Makeenko–Migdal equation in (3), as established rigorously in [DGHK]. We then prove that the limiting theory satisfies a large-\(N\) version of the equation.

- We give a constructive procedure for reducing the Wilson loop functional an arbitrary loop in \(S^2\) with simple crossings inductively to that for a simple closed curve. Specifically, we show that any loop can first be reduced to one that winds \(n\) times around a simple closed curve, which can then be reduced to a simple closed curve.

Our main results may be stated as follows.
Figure 3: A checkerboard variation of the areas

**Theorem 3** Let $L$ be a closed curve traced out on a graph in $S^2$ and having only simple crossings. Assuming Conjecture 1, we establish the following results. First, the limit

$$w(L) := \lim_{N \to \infty} E \{\text{tr(hol}(L))\}$$

exists and depends continuously on the areas of the faces of $L$. Second, the associated variance goes to zero:

$$\lim_{N \to \infty} \text{Var} \{\text{tr(hol}(L))\} = 0.$$  \hfill (8)

Third, the limiting expectation values satisfy the following large-$N$ Makeenko–Migdal equation. Let us vary the areas of the faces surrounding a crossing $v$ in a checkerboard pattern as in Figure 3, resulting in a family of curves $L(t)$. Then

$$\frac{d}{dt} w(L(t)) = w(L_1(t))w(L_2(t)),$$  \hfill (9)

where $L_1(t)$ and $L_2(t)$ are derived from $L(t)$ in the usual way.

In Figure 3, we do not assume the four faces are distinct. If, say, the two faces labeled as $+t$ are the same, we are then increasing the area of that face by $2t$.

The reason for stating the Makeenko–Migdal equation in the form in (9) is that we have not established the differentiability of the large-$N$ Wilson loop functional $w(L)$ with respect to the area of an individual face. If this differentiability property turns out to hold, we can then apply the chain rule to express the derivative on the left-hand side of (9) in the usual form as an alternating sum of such derivatives. This issue is of little consequence, since the result in (9) is the way one applies the Makeenko–Migdal equation in all applications.

We also provide a new proof of Theorem 3 in the plane case, as an alternative to the methods used by Lévy in [Lév2]. In the plane case, our result is not
dependent on any unproven assumption, since Conjecture 1 is easily established in the plane case. See Section 5.1.

Finally, we consider the Wilson loop functional for a loop $L$ with respect to the Yang–Mills measure on an arbitrary compact surface $\Sigma$. We consider the case in which $L$ is contained in a topological disk $U \subset \Sigma$ and satisfies a certain “smallness” assumption. Then, assuming Conjecture 1 holds for homotopically trivial simple closed curves in $\Sigma$, we are able to establish Theorem 3 for $L$. See Section 5.3.

1.5 The Wilson loop for simple closed curve in $S^2$

In this section, we describe what is known (rigorously and nonrigorously) about the Wilson loop functional for a simple closed curve in the sphere. If $L$ is a simple closed curve on $S^2$ and the areas of the two faces of $L$ are $a$ and $b$, Sengupta’s formula reads

$$\mathbb{E}\{\text{tr} (\text{hol}(L))\} = \frac{1}{Z} \int_{U(N)} \text{tr}(U) \rho_a(U) \rho_b(U) \, dU,$$

where $Z = \rho_{a+b} (\text{id})$ is a normalization factor. The probability measure

$$\frac{1}{Z} \rho_a(U) \rho_b(U) \, dU$$

can be interpreted as the distribution of a Brownian bridge on $U(N)$, starting at the origin and returning to the origin at time $a + b$. The large-$N$ behavior of the Wilson loop functional in (10) has been analyzed, with varying degrees of rigor, by three different methods.

First, one may write the heat kernels as sums over the characters of the irreducible representations of $U(N)$. In the large-$N$ limit, one attempts to find the “most probable representation,” that is, the one whose character contributes the most to the sum. The representations, meanwhile, are labeled by certain diagrams; the objective is then to determine the limiting shape of the diagram for the most probable representation. Using this method, physicists have found different shapes in the small-area phase (namely $a + b < \pi^2$) and the large-area phase (namely $a + b > \pi^2$). (See works by Douglas and Kazakov [DoK] and Boulatov [Bou].) At a rigorous level, Lévy and Maida [LM] have analyzed the partition function (i.e., the normalization factor $Z = \rho_{a+b} (\text{id})$) by this method and confirmed the existence of a phase transition at $a + b = \pi^2$.

Second, one may write the heat kernels in (10) as a sum over all geodesics connecting the identity to $U$. When $a$ and $b$ are small, the contribution of the shortest geodesic dominates. Recall that we are using the scaled Hilbert–Schmidt inner product (1) on the Lie algebra $u(N)$. Since the Laplacian scales oppositely to the inner product, the Laplacian on $U(N)$ is scaled by a factor of $1/N$ compared to the Laplacian for the unscaled Hilbert–Schmidt inner product. Thus, at a heuristic level, the large-$N$ limit ought to be pushing us toward the small-time regime for the heat kernels $\rho_a$ and $\rho_b$. It is therefore possible that in
the large-$N$ limit, one can simply “neglect the winding terms,” that is, include only the contribution from the shortest geodesic.

The contribution of the shortest geodesic, meanwhile, is a Gaussian integral of the sort that arises in the Gaussian unitary ensemble (GUE) in random matrix theory. Thus, if it is valid to keep only the contribution from the shortest geodesic, the Wilson loop functional may be computed using results from GUE theory. (See the work of Daul and Kazakov in [DaK].) On the other hand, a consistency argument indicates that it neglecting the winding terms can only be valid in the small area phase. Little work has been done, however, in estimating the size of the winding terms.

Third, one may, as we have noted, interpret the probability measure in (11) as the distribution of a Brownian bridge on $U(N)$. Forrester, Majumdar, and Schehr have then developed a method [FMS] to represent the partition function for Yang–Mills theory in terms of a collection of $N$ nonintersecting Brownian bridges on the unit circle. In fact, the distribution of the eigenvalues of the Brownian bridge in $U(N)$ is precisely the distribution of these nonintersecting Brownian bridges. (This point is explained in the notes [Ha]. The claim is analogous to the well-known result that the eigenvalues of a Brownian motion in the space of $N \times N$ Hermitian matrices are described by the “Dyson Brownian motion” [Dys] in $\mathbb{R}^N$.) Thus, not just the partition function, but also the Wilson loop functional for a simple closed curve can be expressed in terms of nonintersecting Brownian bridges.

Liechty and Wang [LW1, LW2] have obtained various rigorous results about the large-$N$ behavior of the nonintersecting Brownian bridges in $S^1$. In particular, they confirm the existence of a phase transition: When the lifetime $a + b$ of the bridge is less than $\pi^2$, the nonintersecting Brownian motions do not wind around the circle, whereas for lifetime greater than $\pi^2$ they do. It is possible that one could establish Conjecture 1 rigorously in the small-area phase using results from [LW1]. (Theorem 1.2 of [LW1] would be relevant.) In the large-area phase, however, [LW1] does not provide information about the distribution of eigenvalues when $t$ is close to half the lifetime of the bridge. (See the restrictions on $\theta$ in Theorem 1.5(a) of [LW1].)

2 Tools for the proof

In this section, we review some prior results that will allow us to prove our main theorem. Our main tool is the Makeenko–Migdal equation for $U(N)$ Yang–Mills theory on compact surfaces, which was established at a rigorous level in [DGHK]. More precisely, we require not only the standard Makeenko–Migdal equation in (3), but also an “abstract” Makeenko–Migdal equation, which allows us to compute the alternating sum of derivatives of expectation values of more general functions. We also require an estimate on the variance of the product of two bounded random variables, as described in Section 2.2.
2.1 Variation of the Wilson loop and of the variance

Rigorous constructions of the two-dimensional Yang–Mills measure with structure group $K$ from a continuum perspective were given in the plane case by Gross, King, and Sengupta [GKS] and by Driver [Dr], and in the case of a compact surface, possibly with boundary, by Sengupta [Sen1, Sen2, Sen3]. (See also [Lév1].) In particular, suppose $G$ is an “admissible” graph in a surface $\Sigma$, meaning that $G$ contains the boundary of $\Sigma$ and that each face of $G$ is a topological disk. Let $e$ denote the number of unoriented edges of $G$ and let $g$ be a gauge-invariant function of the connection that depends only on the parallel transports $x_1, \ldots, x_e$ along the edges of $G$. Then Driver (in the plane case) and Sengupta (in the general case) give a formula for the expectation value of $g$ with respect to the Yang–Mills measure. The formulas of Driver and Sengupta correspond to what is known as the heat kernel action in the physics literature, as developed by Menotti and Onofri [MO] and others. Explicitly, we have

$$E\{g\} = \frac{1}{Z} \int_{K^e} g(x_1, \ldots, x_e) \prod_{i=1} |F_i| \langle \text{hol}(F_i) \rangle \ dx_1 \cdots dx_e,$$

(12)

where $dx_i$ denotes the normalized Haar measure on $K$, $|F_i|$ is the area of the $i$th face, and $\text{hol}(F_i)$ is the product of edge variables going around the boundary of $F_i$. Here $Z$ is a normalization constant. If the boundary of $\Sigma$ is nonempty, it is possible to incorporate into (12) constraints on the holonomies around the boundary components; the proof of the Makeenko–Migdal equation in [DGHK] holds in this more general context.

Remark 4 In the rest of the paper, when we speak about “varying the areas” of the faces of graph, we mean more precisely that replace the numbers $\{|F_i|\}$ by some other collection of positive real numbers $\{t_i\}$ in Sengupta’s formula (12). If the sum of the $t_i$’s equals the sum of the $|F_i|$’s, it may be possible to implement this variation geometrically, by continuously deforming the graph, but this is not necessary. In particular, the Makeenko–Migdal equation (3) was proved under such an “analytic” (i.e., not necessarily geometric) variation of the area.

Suppose now that $L$ is a loop that can be traced out on an oriented graph in $\Sigma$ and let $G$ be a minimal graph on which $L$ can be traced. We now explain what it means for $L$ to have a simple crossing at a vertex $v$. First, we assume that $G$ has four edges incident to $v$, where we count an edge $e$ twice if both the initial and final vertices of $e$ are equal to $v$. Second, we assume that $L$, when viewed as a map of the circle into the plane, passes through $v$ exactly twice. Third, we assume that each time $L$ passes through $v$, it comes in along one edge and passes “straight across” to the cyclically opposite edge. Last, we assume that $L$ traverses two of the edges on one pass through $v$ and the remaining two edges on the other pass through $v$.

Under these assumptions, Theorem 1 of [DGHK] gives a rigorous derivation of the Makeenko–Migdal equation in (3). We now restate the Makeenko–Migdal equation for $U(N)$, in the $S^2$ case, in a way that facilitates the large-$N$ limit. In addition, we derive a similar result for the variation of the variance of $\text{tr}(\text{hol}(L))$. 
Proposition 5 Let $L$ be a loop traced out on a graph in $S^2$ and having only simple crossing. Let $v$ be one such crossing and let $L_1$ and $L_2$ be obtained from $L$ as usual in the Makeenko–Migdal equation. Then we have

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \mathbb{E}\{\text{tr}(\text{hol}(L))\}$$

$$= \mathbb{E}\{\text{tr}(\text{hol}(L_1))\} \mathbb{E}\{\text{tr}(\text{hol}(L_2))\} + \text{Cov}\{\text{tr}(\text{hol}(L_1)), \text{tr}(\text{hol}(L_2))\}$$

(13)

and

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \text{Var}\{\text{tr}(\text{hol}(L))\}$$

$$= 2\text{Cov}\{\text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2)), \text{tr}(\text{hol}(L))\}$$

$$+ \frac{2}{N^2} \mathbb{E}\{\text{tr}(\text{hol}(L_3))\},$$

(14)

where $L_3$ is the composite curve $L_1 \circ L_2 \circ L_2 \circ L_1$. (Thus $\text{hol}(L_3)$ is equal to $\text{hol}(L_1)\text{hol}(L_2)^2\text{hol}(L_1)$.) Here Cov denotes the covariance, defined as $\text{Cov}\{f, g\} = \mathbb{E}\{fg\} - \mathbb{E}\{f\} \mathbb{E}\{g\}$.

Proof. Equation (13) is simply the Makeenko–Migdal equation (3) rewritten using the definition of the covariance. To establish (14) we need to use the abstract Makeenko–Migdal equation established in Theorem 2 of [DGHK]. (This result generalizes the abstract Makeenko–Migdal equation formulated and proved for the plane case by Lévy in [Lévy2, Proposition 6.22].) Following the argument in Section 2.3 of [DHK2], we express the loop $L$ as

$$L = e_1 A e_4^{-1} e_2 B e_3^{-1},$$

where $A$ and $B$ are words in edges other than $e_1, \ldots, e_4$. Then $L_1 = e_1 A e_4^{-1}$ and $L_2 = e_2 B e_3^{-1}$. If $a_1, \ldots, a_4$ are the edge variables corresponding to $e_1, \ldots, e_4$, we then have (following the convention that parallel transport is order reversing)

$$\text{hol}(L) = a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1,$$

where $\alpha$ and $\beta$ are words in the edge variables other than $a_1, \ldots, a_4$.

Now, the abstract Makeenko–Migdal equation reads

$$\left(\frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4}\right) \mathbb{E}\{f\} = -\mathbb{E}\{\nabla^{a_1} \cdot \nabla^{a_2} f\},$$

(15)

whenever $f$ has “extended gauge invariance” at the vertex $v$. When the edges $e_1, \ldots, e_4$ are distinct, extended gauge invariance at $v$ means that

$$f(a_1 x, a_2, a_3 x, a_4, b) = f(a_1, a_2 x, a_3, a_4 x, b) = f(a_1, a_2, a_3, a_4, b)$$

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for all \( x \in K \), where \( b \) is the collection of edge variables other than \( a_1, \ldots, a_4 \). (See Section 4 of [DHK2] for a discussion of extended gauge invariance when the edges are not distinct.) We apply (15) to the function
\[
f(a_1, a_2, a_3, a_4, b) = |\text{tr}(\text{hol}(L))|^2 = |\text{tr}(a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1)|^2,
\]
which has extended gauge invariance at \( v \).

A straightforward computation then shows that
\[
\nabla a_1 \cdot \nabla a_2 f = \sum X \{2\text{tr}(a_3^{-1} \beta a_2 X a_4^{-1} \alpha a_1) \text{tr}(a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1) + 2\text{tr}(a_3^{-1} \beta a_2 X a_4^{-1} \alpha a_1) \text{tr}(a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1 X)\},
\]
where the sum is over an arbitrary orthonormal basis of \( u(N) \) with respect to the inner product in (1). We then use the following elementary matrix identities (e.g., Proposition 3.1 in [DHK1]):
\[
\sum X XAX = -\sum X \text{tr}(A)I \sum X \text{tr}(AX)\text{tr}(BX) = -\frac{1}{N^2} \text{tr}(AB).
\]
The result is that
\[
\nabla a_1 \cdot \nabla a_2 f = 2\text{tr}(a_3^{-1} \beta a_2)\text{tr}(a_4^{-1} \alpha a_1)\text{tr}(a_3^{-1} \beta a_2 a_4^{-1} \alpha a_1) + \frac{2}{N^2} \text{tr}(a_4^{-1} \alpha a_1 a_3^{-1} \beta a_2 a_4^{-1} \beta a_2 a_4^{-1} \alpha a_1)
\]
\[
= 2\text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2))\text{tr}(\text{hol}(L)) + \frac{2}{N^2} \text{tr}(\text{hol}(L_3)).
\]
Thus, (15) takes the form
\[
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \mathbb{E} \{ |\text{tr}(\text{hol}(L))|^2 \} 
= 2\mathbb{E} \{ \text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2))\text{tr}(\text{hol}(L)) \}
+ \frac{2}{N^2} \mathbb{E} \{ \text{tr}(\text{hol}(L_3)) \},
\]
whereas (3) tells us that
\[
\left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) (\mathbb{E} \{ \text{tr}(\text{hol}(L)) \})^2
= 2\mathbb{E} \{ \text{tr}(\text{hol}(L)) \} \left( \frac{\partial}{\partial t_1} - \frac{\partial}{\partial t_2} + \frac{\partial}{\partial t_3} - \frac{\partial}{\partial t_4} \right) \mathbb{E} \{ \text{tr}(\text{hol}(L)) \}
= 2\mathbb{E} \{ \text{tr}(\text{hol}(L)) \} \mathbb{E} \{ \text{tr}(\text{hol}(L_1))\text{tr}(\text{hol}(L_2)) \}.
\]
The claimed formula (14) then follows easily from (16) and (17).
2.2 Variance estimates

For a complex-valued random variable $X$, we define the variance of $X$ by

$$\text{Var}(X) := \mathbb{E}\{|X - \mathbb{E}\{X\}|^2\} = \mathbb{E}\{|X|^2\} - |\mathbb{E}\{X\}|^2.$$ 

In particular,

$$\text{Var}(X) \leq \mathbb{E}\{|X|^2\}. \quad (18)$$

We then define the standard deviation of $X$, denoted $\sigma_X$, by

$$\sigma_X = \sqrt{\text{Var}(X)}.$$ 

We observe that for any two random variables $X$ and $Y$, we have

$$\sigma_{X+Y} \leq \sigma_X + \sigma_Y, \quad (19)$$

and similarly for any number of random variables. (It is harmless to assume that the expectation values of $X$ and $Y$—and therefore $X + Y$—are zero, in which case (19) is the triangle inequality for the $L^2$ norm.) We also recall the definition of the covariance of two random variables

$$\text{Cov}\{X,Y\} = \mathbb{E}\{(X - \mathbb{E}\{X\})(Y - \mathbb{E}\{Y\})\}$$

and record the elementary inequality

$$|\text{Cov}\{X,Y\}| \leq \sigma_X \sigma_Y, \quad (21)$$

which follows from the Cauchy–Schwarz inequality.

We now establish a simple estimate on the standard deviation of the product of two bounded random variables.

**Proposition 6** Suppose $X$ and $Y$ are two complex-valued random variables satisfying $|X| \leq c_1$ and $|Y| \leq c_2$ for some constants $c_1$ and $c_2$. Then

$$\sigma_{XY} \leq 2\sqrt{c_1c_2}\sigma_X\sigma_Y + c_1\sigma_Y + c_2\sigma_X.$$ 

**Proof.** For simplicity of notation, let $\bar{X} = \mathbb{E}\{X\}$ and let $\bar{X} = X - \bar{X}$. Thus, $\text{Var}(X) = \mathbb{E}\{|\bar{X}|^2\}$. Then since $|X| \leq c_1$, we have $|\bar{X}| \leq c_1$ and $|\bar{X}| \leq 2c_1$. Then by (18) and the Cauchy–Schwarz inequality, we have

$$\text{Var}(\bar{X}\bar{Y}) \leq \mathbb{E}\{|\bar{X}|^2|\bar{Y}|^2\}$$

$$\leq \sqrt{\mathbb{E}\{|\bar{X}|^4\}\mathbb{E}\{|\bar{Y}|^4\}}$$

$$\leq 4c_1c_2\sigma_X\sigma_Y,$$

since $|\bar{X}|^4 = |\bar{X}|^2|\bar{X}|^2 \leq 4c_1^2|\bar{X}|^2$ and similarly for $|\bar{Y}|^4$. Thus,

$$\text{Var}(XY) = \text{Var}(\bar{X}\bar{Y} + \bar{X}\bar{Y} + \bar{X}\bar{Y} + \bar{X}\bar{Y})$$

$$= \text{Var}(\bar{X}\bar{Y} + \bar{X}\bar{Y} + \bar{X}\bar{Y})$$

and so, by (19),

$$\sigma_{XY} \leq \sigma_{\bar{X}\bar{Y}} + |\bar{X}|\sigma_{\bar{Y}} + |\bar{Y}|\sigma_{\bar{X}}$$

$$\leq 2\sqrt{c_1c_2}\sigma_X\sigma_Y + c_1\sigma_Y + c_2\sigma_X,$$

as claimed. ■
3 Examples

Before developing a general method for analyzing a general loop in $S^2$ with simple crossings, we consider two illustrative examples, the same two that are considered in [DaK].

3.1 The figure eight

Although we consider loops on $S^2$, we can draw them as a loops on the plane by picking a face and placing a puncture in that face, so that what is left of $S^2$ is identifiable with $\mathbb{R}^2$. We need to keep in mind, however, that the “unbounded” face in such a drawing is actually a bounded face (with finite area) on $S^2$. Furthermore, by placing the puncture in different faces, the same loop on $S^2$ can give inequivalent loops on $\mathbb{R}^2$. As a simple example, consider the loop in Figure 4, which we refer to as the figure eight. The figure gives two different views of this loop coming from puncturing two different faces.

We write the holonomy around the figure eight as

$$\text{hol}_L(a, b, c)$$

to indicate the dependence of the Wilson loop functional on the areas. In this case, the loops $L_1$ and $L_2$ occurring on the right-hand side of the Makeenko–Migdal equation are both simple closed curves, with the areas of the faces being $a + b$ and $c$ in the one case and $b + c$ and $a$ in the other case.

**Theorem 7** Assuming Conjecture 1, the limit

$$w_L(a, b, c) := \lim_{N \to \infty} \mathbb{E} \{\text{tr}(\text{hol}_L(a, b, c))\}$$

exists and satisfies the large-$N$ Makeenko–Migdal equation in the form

$$\frac{d}{dt} w_L(a - t, b + 2t, c - t) \bigg|_{t=0} = w_1(a + b, c)w_1(a, b + c),$$

where $w_1$ is as in Notation 2. Furthermore, we have

$$\lim_{N \to \infty} \text{Var} \{\text{tr}(\text{hol}_L(a, b, c))\} = 0.$$
If the partial derivatives of $w_L(a, b, c)$ with respect $a$, $b$, and $c$ exist and are continuous, it follows from the chain rule that

$$\frac{d}{dt}w_L(a-t, b+2t, c-t) \bigg|_{t=0} = \left(2\frac{\partial}{\partial b} - \frac{\partial}{\partial a} - \frac{\partial}{\partial c}\right)w_L(a, b, c).$$

The following proof, however, does not establish the existence or continuity of the partial derivatives of $w_L(a, b, c)$.

**Proof.** We denote the holonomies for the two loops $L_1$ and $L_2$ as $\text{hol}_{L_1}(a+b, c)$ and $\text{hol}_{L_2}(a, b+c)$. The face labeled as $F_1$ should be the one bounded by the two outgoing edges of $L$ at $v$, which is the face with area $b$. Then $F_3$ coincides with $F_1$, while $F_2$ and $F_4$ are the faces with areas $a$ and $c$ (in either order). We assume $a \leq c$, with the case $c < a$ being entirely similar.

Proposition 5 takes the form

$$\frac{d}{dt}E\{\text{tr} (\text{hol}_{L}(a-t, b+2t, c-t))\} = \left(2\frac{\partial}{\partial b} - \frac{\partial}{\partial a} - \frac{\partial}{\partial c}\right)E\{\text{tr} (\text{hol}_{L}(a-t, b+2t, c-t))\} = E\{\text{tr} (\text{hol}_{L_1}(a+b+t, c-t))\} E\{\text{tr} (\text{hol}_{L_1}(a-t, b+c+t))\} + \text{Cov}\{\text{tr} (\text{hol}_{L_1}(a+b+t, c-t)), \text{tr} (\text{hol}_{L_1}(a-t, b+c+t))\}. \quad (22)$$

Let us denote $E\{\text{tr} (\text{hol}_{L}(a-t, b+2t, c-t))\}$ by $F(t)$ (with $a$, $b$, and $c$ fixed). We then write $F(0) = F(a-\varepsilon) - \int_0^{a-\varepsilon} F'(t) \, dt$; that is,

$$E\{\text{tr} (\text{hol}_{L}(a, b, c))\} = E\{\text{tr} (\text{hol}_{L}(\varepsilon, 2a+b-2\varepsilon, c-a+\varepsilon))\}$$

$$- \int_0^{a-\varepsilon} E\{\text{tr} (\text{hol}_{L_1}(a+b+t, c-t))\} E\{\text{tr} (\text{hol}_{L_1}(a-t, b+c+t))\} \, dt$$

$$- \int_0^{a-\varepsilon} \text{Cov}\{\text{tr} (\text{hol}_{L_1}(a+b+t, c-t)), \text{tr} (\text{hol}_{L_1}(a-t, b+c+t))\} \, dt. \quad (23)$$

Now, it should be clear geometrically that if we let the area $a$ in the figure eight tend to zero, the result is a simple closed curve. That is to say, we expect that

$$\lim_{\varepsilon \to 0} E\{\text{tr} (\text{hol}_{L}(\varepsilon, 2a+b-2\varepsilon, c-a+\varepsilon))\} = E\{\text{tr} (\text{hol}_{L_0}(2a+b, c-a))\}, \quad (24)$$

where $L_0(\alpha, \beta)$ is a simple closed curve on $S^2$ enclosing areas $\alpha$ and $\beta$. Analytically, (24) follows easily from Sengupta’s formula, using that $\rho_a(\cdot)$ converges to
a $\delta$-measure on $K$ as $a$ tends to zero. Thus, letting $\varepsilon$ tend to zero, we obtain

$$
E \{ \text{tr}(\text{hol}_L(a, b, c)) \} \\
= E \{ \text{tr}(\text{hol}_{L_0}(2a + b, c - a)) \} \\
- \int_0^a E \{ \text{tr}(\text{hol}_{L_1}(a + b + t, c - t)) \} E \{ \text{tr}(\text{hol}_{L_1}(a - t, b + c + t)) \} \, dt \\
- \int_0^a \text{Cov} \{ \text{tr}(\text{hol}_{L_1}(a + b + t, c - t)), \text{tr}(\text{hol}_{L_1}(a - t, b + c + t)) \} \, dt. 
$$

Note that all holonomies on the right-hand side of (25) are of simple closed curves. If we use (6) in Conjecture 1 together with the inequality (21) and dominated convergence, we find that the last term in (25) tends to zero as $N$ tends to infinity. Then using (5) in Conjecture 1 along with dominated convergence, we may let $N \to \infty$ to obtain

$$
\lim_{N \to \infty} E \{ \text{tr}(\text{hol}_L(a, b, c)) \} = w_1(2a + b, c - a) \\
- \int_0^a w_1(a + b + t, c - t) w_1(a - t, b + c + t) \, dt, 
$$

for $a \leq c$. (Note that the normalized trace defined in (2) satisfies $|\text{tr}(U)| \leq 1$ for all $U \in U(N)$, so that dominated convergence applies in both integrals in (25).) This result establishes the first claim in the theorem.

If we now subtract the value of (26) at $(a - s, b + 2s, c - s)$ and the value at $(a, b, c)$, we obtain

$$
w_L(a - s, b + 2s, c - s) - w_L(a, b, c) \\
= \int_0^s w_1(a + b + t, c - t) w_1(a - t, b + c + t) \, dt. 
$$

Dividing this relation by $s$ and letting $s$ tend to zero gives

$$
\left. \frac{\partial}{\partial s} w_L(a - s, b + 2s, c - s) \right|_{s=0} = w_1(a + b + t, c - t) w_1(a - t, b + c + t) 
$$

by the continuity of $w_1$. This relation is the desired large-$N$ Makeenko–Migdal equation.

Meanwhile, by the second relation in Proposition 5, we have

$$
\text{Var} \{ \text{tr}(\text{hol}_L(a, b, c)) \} = \text{Var} \{ \text{tr}(\text{hol}_{L_0}(2a + 2b, c - a - b)) \} \\
- 2 \int_0^a \text{Cov} \{ \text{tr}(\text{hol}(L_1)) \text{tr}(\text{hol}(L_2)), \text{tr}(\text{hol}(L)) \} \, dt \\
- \frac{2}{N^2} \int_0^a E \{ \text{tr}(\text{hol}(L_3)) \} \, dt. 
$$

The first term on the right-hand side tends to zero as $N$ tends to infinity, by Conjecture 1. In the second term on the right-hand side, $L_1$ and $L_2$ are simple
closed curves. Furthermore, the normalized trace satisfies $|\text{tr}(U)| \leq 1$ for all $U \in U(N)$. Thus, using (21) and Proposition 6 together with Conjecture 1, we see that the second term on the right-hand side of (27) tends to zero. Finally, since $|\text{tr}(U)| \leq 1$, the last term on the right-hand side manifestly goes to zero.

3.2 The trefoil

In this section, we briefly outline an analysis of the trefoil loop by a method similar to the one in the previous subsection. Later we will develop a systematic method for analyzing any loop; this will provide an alternative analysis of the trefoil. We label the areas of the faces as in Figure 5, where we may assume without loss of generality that $c \leq b$. We now perform a Makeenko–Migdal variation at the vertex in the top middle of the figure. In this case, the loops $L_1$ and $L_2$ turn out to be simple closed curves.

Let us denote by $L'$ the loop on the right-hand side of Figure 5. Then $L'$ is the limit as $\varepsilon$ tends to zero of the loops $L''_\varepsilon$ in Figure 6. Specifically, in Figure 6, we let $\varepsilon$ tend to zero, while keeping all of the areas of the faces fixed to the values indicated. Thus, $E \{\text{tr(hol}(L''_\varepsilon))\}$ is independent of $\varepsilon$ and we conclude that
$$\mathbb{E} \{ \text{tr}(\text{hol}(L')) \} = \mathbb{E} \{ \text{tr}(\text{hol}(L'')) \}.$$ But since the loop $L''$ is of the type analyzed in Section 3, we may already know the large-$N$ behavior of $\mathbb{E} \{ \text{tr}(\text{hol}(L')) \}$. The argument then proceeds much as in Section 3.1; since we will develop later a systematic method for analyzing arbitrary loops, we omit the details of this analysis.

Other examples are not quite so easy to simplify by using the Makeenko–Migdal equation at a single vertex. In the loop in Figure 7, for example, it is not evident how shrinking any one of the faces to zero simplifies the problem.

4 Analysis of a general loop

4.1 The strategy

Given an arbitrary loop $L$ on $S^2$ with simple crossings, we will consider a linear combination of Makeenko–Migdal variations of the areas over all the vertices of $L$. We will show that it is possible to make such a variation, depending on a parameter $t$, in such a way that as $t$ tends to 1, all but two of the areas of the faces tend to zero. Thus, in the $t \to 1$ limit, we effectively have a loop with only two faces. Indeed, we will show that the $t \to 1$ limit of the Wilson loop functional is the Wilson loop functional for a loop $L^a$ that winds $n$ times around a simple closed curve. Here $n$ is an integer determined by the topology of the original loop $L$.

Consider, for example, the trefoil loop of Section 3.2. If vary the areas by the amounts shown in Figure 8, the net effect on the areas is:

\[
\begin{align*}
    b &\mapsto b - tb \\
    a &\mapsto a + t(b + c + d)/2 \\
    c &\mapsto c - tc \\
    e &\mapsto e + t(b + c + d)/2 \\
    d &\mapsto d - td
\end{align*}
\]
Thus, as \( t \) varies from 0 to 1, the areas \( b, c, \) and \( d \) shrink simultaneously to zero, while the areas of the two remaining faces increase. The limiting curve is shown in Figure 9. If the areas of the three lobes are zero, the curve becomes a circle traversed twice.

Let \( L(t) \) denote the trefoil with areas varying as above. If we differentiate \( \mathbb{E}\{\text{tr}(\text{hol}(L(t)))\} \) with respect to \( t \), then by the chain rule, we will get a linear combination of terms of the form

\[
\mathbb{E}\{\text{tr}(\text{hol}(L_{1,j}(t)))\} \mathbb{E}\{\text{tr}(\text{hol}(L_{2,j}(t)))\},
\]

where \( L_{1,j}(t) \) and \( L_{2,j}(t) \) represent the loop \( L(t) \) cut at the \( j \)th vertex of the trefoil \( (j = 1, 2, 3) \), along with some covariance terms. Each \( L_{1,j}(t) \) and \( L_{2,j}(t) \) is actually a simple closed curve, but in any case, these curves have fewer crossings than the original trefoil. It is then a straightforward matter to let \( N \) tend to infinity to get the limiting Wilson loop functional, and similarly for the variance.

For an arbitrary loop \( L \) with \( k \) crossings, we will show that we can deform \( L \) into a loop \( L_n \) that winds \( n \) times around a simple closed curve. The variation of the Wilson loop functional along this path will be a linear combination of products of Wilson loop functionals for curves with at most \( k - 1 \) crossings. In an inductive argument then, it remains only to analyze \( L_n \), which we do by another induction, this time reducing \( L_n \) to \( L_{n-1} \), and so on.

### 4.2 Winding numbers

We consider \( S^2 \) with a fixed orientation. We then consider a loop \( L \) traced out on a graph in \( S^2 \) and having with only simple crossings. We consider the faces of \( L \), that is, the connected components of the complement of \( L \) in \( S^2 \). If we
pick a face $F_0$, we can puncture $F_0$, thus turning $S^2$ topologically into $\mathbb{R}^2$. The orientation on $S^2$ gives an orientation on $\mathbb{R}^2$. Thus, for each face $F$, we may speak about the winding number of $L$ around $F$. Since this winding number depends on which face $F_0$ we puncture, we denote it thus:

$$w_{F_0}(F).$$

It is important to understand how the winding number changes if the location of the puncture changes.

**Proposition 8** If $F_0$, $F'_0$, and $F$ are faces, then

$$w_{F_0}(F) = w_{F'_0}(F) + w_{F_0}(F'_0).$$

In particular, the difference between $w_{F_0}(F)$ and $w_{F'_0}(F)$ does not depend on $F$.

That is to say, if we change the location of the puncture, all the winding numbers change by the same amount.

**Proof.** Let us fix points $x$, $y$, and $z$ in $F_0$, $F'_0$, and $F$, respectively. Let us put our puncture initially in $x$, regarding $S^2 \setminus \{x\}$ as the plane. Let us then assume that $y$ is at the origin and $z$ is at the point $(2, 0)$. We may then regard $L$ as an element of $\pi_1(\mathbb{R}^2 \setminus \{y, z\})$, which is a free group on two generators $e_1$ and $e_2$. These generators may be identified with circles of radius one centered at $y$ and $z$, respectively, traversed in the counter-clockwise direction. Then $w_{F_0}(F)$ is the number of times of $L$ winds around $z$, which is the number of occurrences of the generator $e_2$ in the representation of $L$ as a word in $e_1$ and $e_2$.

Suppose we now shift our puncture from $x$ to $y$. This shift amounts to composing $L$ with the inversion map in the complex plane, $\mathbb{C} = \mathbb{R}^2$, that is, the map $z \mapsto 1/z$. After this process, the generator $e_1$ traverses the unit circle in the opposite direction, while the generator $e_2$ is now inside the unit circle (Figure 10). Thus, $w_{F'_0}(F)$ is the number of occurrences of $e_2$ minus the number of occurrences of $e_1$, giving the claimed result. 

---

Figure 9: The trefoil with the lobes shrunk to area zero
4.3 The span of the Makeenko–Migdal vectors

Let $L$ be a loop traced out on a graph in $S^2$ and having only simple crossings. We consider vectors assigning real numbers to the faces of $L$. For each vertex $v$ of $L$, we define the Makeenko–Migdal vector associated to $v$, denoted $\text{MM}_v$, as

$$\text{MM}_v = \sum_{i=1}^{4} (-1)^{i+1} \delta_{F_i}.$$  

If, for example, $F_1, \ldots, F_4$ are distinct, then $\text{MM}_v$ is the vector that assigns the numbers $1, -1, 1, -1$ to $F_1, \ldots, F_4$, respectively, and zero to all other faces.

Theorem 9 (T. Lévy) Fix a face $F_0$ of $L$. Let $u$ be a vector assigning a real number to each face of $L$. Then $u$ belongs to the span of the Makeenko–Migdal vectors if and only if (1) $u$ is orthogonal to the constant vector $1 := (1, 1, \ldots, 1)$, and (2) $u$ is orthogonal to the winding-number vector $w_{F_0} \cdot$.

This result is the $r = 1$ case of Lemma 6.28 in [Lév2]. Note that by Proposition 8, the winding number vector $w_{F_0}$ associated to some other face $F_0'$ differs from $w_{F_0}$ by a constant multiple of the constant vector $1$. Thus, if $u$ is orthogonal to $1$, then $u$ is orthogonal to $w_{F_0} \cdot$ if and only if $u$ is orthogonal to $w_{F_0} \cdot$. Thus, the condition in the theorem is independent of the choice of $F_0$.

4.4 Shrinking all but two of the faces

Let $f$ denote the number of faces of $L$. We now choose one face arbitrarily and denote it by $F_0$. We will show that there is another face $F_1$ such that we can perform a Makeenko–Migdal variation of the areas in which the areas of the faces $F_2, \ldots, F_{f-1}$ shrink simultaneously to zero, while the area of $F_0$ remains positive. In the generic case, the area of $F_1$ also remains positive, while in a certain borderline case, the area of $F_1$ tends to zero as well.

Proposition 10 Assume $L$ has at least three faces. Choose one face $F_0$ arbitrarily and let all winding numbers be computed relative to a puncture in $F_0$, so that $w(F_0) = 0$. Let $a = (a_0, a_1, \ldots, a_{f-1})$ denote the vector of areas of the faces and let $w = (0, w_1, \ldots, w_{f-1})$ be the vector of winding numbers. Suppose
\(a \cdot w \neq 0\) and adjust the labeling of \(F_1, \ldots, F_{f-1}\) so that \(w_1\) is maximal among the winding numbers if \(a \cdot w > 0\) and \(w_1\) is minimal if \(a \cdot w < 0\). Then there exists \(b\) in the span of the Makeenko–Migdal vectors such that \(a' := a + b\) has the form \(a' = (a'_0, a'_1, 0, \ldots, 0)\) with \(a'_0 \geq a_0\) and \(a'_1 > 0\). Meanwhile, if \(a \cdot w = 0\), there exists \(b\) in the span of the Makeenko–Migdal vectors such that \(a' := a + b\) has the form \(a' = (a'_0, 0, \ldots, 0)\) with \(a'_0 > 0\).

In the case of the trefoil loop, for example, suppose we take \(F_0\) to be the “unbounded” face in Figure 8 and we orient the loop in the counter-clockwise direction. Then the winding numbers are 1 for the three lobes of the trefoil and 2 for the central region. Since all winding numbers are positive in this case, we always have \(a \cdot w > 0\), in which case we take \(F_1\) to the central region. Figure 8 then illustrates Proposition 10.

**Proof.** Assume first that \(a \cdot w > 0\), in which case, the maximal winding number \(w_1\) must be positive. In light of Theorem 9, two vectors \(a\) and \(a'\) differ by a vector in the span of the Makeenko–Migdal vectors if and only if \(a\) and \(a'\) have the same inner products with the constant vector \(1\) and the winding number vector \(w\). To achieve these conditions, we first set \(a'_1\) equal to \(a \cdot w / w_1\) and \(a'_0, \ldots, a'_{f-1}\) equal to 0. Since \(w_0 = 0\), we then have \(a' \cdot w = a \cdot w\), regardless of the value of \(a'_0\). Next, we choose \(a'_0 = -a'_1 + a_0 + a_1 + \cdots + a_{f-1}\) to achieve the condition \(a' \cdot 1 = a \cdot 1\). Now, \(a' \cdot w = a_1 w_1 = a \cdot w\). Then since \(w_1\) is maximal, we have

\[
a'_1 w_1 = a_1 w_1 + \cdots + a_{f-1} w_{f-1} \\
\leq w_1(a_1 + \cdots + a_{f-1})
\]

Thus, \(a'_1 \leq a_1 + \cdots + a_{f-1}\), which shows that \(a'_0 \geq a_0\). (In particular, \(a'_0 > 0\).) The analysis of the case \(a \cdot w < 0\) is similar.

Finally, if \(a \cdot w = 0\), we may set \(a'_0 = a_0 + a_1 + \cdots + a_{f-1}\) and all other entries of \(a'\) equal to zero. In that case, \(a' \cdot w = a \cdot w = 0\) and \(a' \cdot 1 = a'_0 = a \cdot 1\), showing that \(a'\) differs from \(a\) by a vector in the span of the Makeenko–Migdal vectors.

**4.5 Analyzing the limiting case**

As a consequence of Proposition 10, we may start with an arbitrary loop \(L\) and perform a linear combination of Makeenko–Migdal variations at each vertex, obtaining a family \(L(t), 0 \leq t < 1,\) of loops with the same topological type with all but two of the areas shrinking to zero as \(t \to 1\). We now analyze the behavior of the Wilson loop functional in the limit \(t \to 1\).

We will analyze the limit “analytically,” using Sengupta’s formula for the finite-\(N\) case. For this reason, the structure group can be an arbitrary connected compact Lie group \(K\).

**Theorem 11** Let \(L\) be a loop traced out on a graph in \(S^2\) and having only simple crossings. Denote the number of faces of \(L\) by \(f\) and label the faces as \(F_0, F_1, F_2, \ldots, F_{f-1}\). Suppose we vary the areas of the faces as a function of \(a\)
parameter $t \in [0, 1)$ in such a way that as $t \to 1$, the areas of $F_2, \ldots, F_{f-1}$ tend to zero, while the areas of $F_0$ and $F_1$ approach positive real numbers $a$ and $c$, respectively. Then

$$\lim_{t \to 1} \mathbb{E}\{\text{tr}(\text{hol}(L))\} = \frac{1}{Z} \int_K \text{tr}(h^n) \rho_a(h) \rho_c(h) \, dh,$$

(28)

where $n = w_{F_0}(F_1)$ is the winding number of $L$ around $F_1$, relative to a puncture in $F_0$, and $Z$ is a normalization constant. Meanwhile, if the area of $F_1$ also tends to zero in the $t \to 1$ limit, then

$$\lim_{t \to 1} \mathbb{E}\{\text{tr}(\text{hol}(L))\} = 1.$$

The integral on the right-hand side of (28) is just the Wilson loop functional for a loop $L_n$ that winds $n$ times around a simple closed curve enclosing areas $|F_0|$ and $|F_1|$. Note that winding number of $L_n$ around $F_0$ is the same as the winding number of the original loop $L$ around $F_0$. Note also that by Theorem 9, the value of $a \cdot w$ for the loop $L_t$ is independent of $t$ for $t \in [0, 1)$. Since $w_0 = 0$ and $a_2$ through $a_{f-1}$ are tending to zero, this means that $a_1$ must be tending to the value $a \cdot w / n$, where $n = w_1$ is the winding number of $L$ around $F_1$. It then follows that the limiting loop $L_n$ has the same value of $a \cdot w$, even though $L_n$ is has a different topological type from $L$.

**Proof.** Let $G$ be a minimal oriented graph on which $L$ can be traced. We think of $G$ as a graph in the plane, by placing a puncture into $F_0$. If $e$ denotes the number of edges of $G$, then we may consider two different measures on $K^e$: the Yang–Mills measure $\mu^G_{\text{plane}}$ for $G$ viewed as a graph in the plane and the Yang–Mills measure $\mu^G_{\text{sphere}}$ for $G$ viewed as a graph in the sphere. By comparing Sengupta’s formula [Sen1, Sect. 5] in the sphere case to Driver’s formula [Dr, Theorem 6.4] in the plane case, we see that

$$d\mu^G_{\text{sphere}}(x) = \frac{1}{Z} \rho_{F_0}(\text{hol}_{F_0}(x)) \, d\mu^G_{\text{plane}}(x),$$

where $x \in K^e$ is the collection of edge variables, $|F_0|$ is the area of $F_0$ as a face in $S^2$, and $\text{hol}_{F_0}$ is the product of edge variables around the boundary of $F_0$. Here $Z$ is a normalization constant.

We may rewrite both of the Yang–Mills measures using “loop variables” as follows. Let us fix a vertex $v$ and a spanning tree $T$ for $G$. Then in Section 4.2 of [Lév2], Lévy gives a procedure associating to the faces $F_1, \ldots, F_{f-1}$ certain loops $L_1, \ldots, L_{f-1}$ in $G$ that constitute free generators for $\pi_1(G)$. (Note that there is no generator associated to the face $F_0$.) Each $L_i$ is a word in the edges of $G$. We may then associate to each bounded face $F_i$ of $G$ the associated loop variable, which is the product of edge variables (in the reverse order, since parallel transport reverses order). The map sending the collection of edge variables to the collection of loop variables defines a map $\Gamma : K^e \to K^{f-1}$, where $e$ is the number of edges of $G$.

According to Proposition 4.4 of [Lév2], the loop variables are independent heat-kernel distributed random variables with respect to $\mu^G_{\text{plane}}$. That is to say,
the push-forward of $\mu^G_{\text{plane}}$ under $\Gamma$ is just the product of heat kernels, at times equal to the areas of the faces. Although there is no generator associated to the face $F_0$, the $L_1, \ldots, L_{f-1}$ generate $\pi_1(G)$. Suppose, therefore, that $L_0$ is a loop that starts at $v$, travels along a path $P$ to a vertex in $\partial F_0$, then around the boundary of $F_0$, and then back to $v$ along $P^{-1}$. Then $L_0$ is expressible as a word in $L_1, \ldots, L_{f-1}$ and therefore $\text{hol}_{F_0}(x)$ is expressible as a word in the loop variables: $\text{hol}_{F_0}(x) = w_0(\Gamma(x))$ for some function $w_0$ on $K^{f-1}$. (Although the expression $\text{hol}_{F_0}(x)$ is not well defined, since it depends on the choice of $v$ and $P$, the invariance of the heat kernel under conjugation guarantees that $\rho_{|F_0|}(\text{hol}_{F_0}(x))$ is well defined.) It then follows from the measure-theoretic change of variables theorem that the push-forward of $\mu^G_{\text{sphere}}$ is given by

$$d\Gamma_{*}(\mu^G_{\text{sphere}})(h_1, \ldots, h_{f-1}) = \frac{1}{Z} \rho_{|F_0|}(w_0(h_1, \ldots, h_{f-1})) \left( \prod_{i=1}^{f-1} \rho_{|F_i|}(h_i) \right) dh_1 \ldots dh_{f-1}. $$

Meanwhile, the loop $L$ (whose Wilson loop functional we are considering) is also expressible as a word in the generators $L_1, \ldots, L_{f-1}$. (See Figure 11.) Thus,

$$\int_{K^*} \text{tr}(\text{hol}(L)) \ d\mu^G_{\text{sphere}} = \frac{1}{Z} \int_{K^{f-1}} \text{tr}(w_1(h_1, \ldots, h_{f-1}))$$

$$\times \rho_{|F_0|}(w_0(h_1, \ldots, h_{f-1})) \left( \prod_{i=1}^{f-1} \rho_{|F_i|}(h_i) \right) dh_1 \ldots dh_{f-1}. \quad (29)$$

It is now straightforward to take a limit in (29) as $t \to 1$, that is, as all areas other than $|F_0|$ and $|F_1|$ tend to zero. In this limit, each heat kernel associated to $h_i$, $i \geq 2$, becomes a $\delta$-measure, so we simply evaluate each such $h_i$ to the
conclude that this result in terms of the loop to the identity, the remaining generator will occur to the power 1.) Expressing theorem, this winding number is \( n \) of \( L \) in Figure 11 decomposes as \( n = \) the winding number of \( \partial F \) apply the just-mentioned homomorphism, the result will be \( L \). Let \( L \) of Wilson loop functionals for curves with fewer crossings, plus a covariance loop functional along this deformation will be a linear combination of products of Wilson loop functionals for curves with fewer crossings, plus a covariance term. We begin by recording a result for a loop that winds \( n \) times around a simple closed curve. Theorem 12 Let \( L_n(a, c) \) denote a loop that winds \( n \) times around a simple closed curve enclosing areas \( a \) and \( c \). Assuming Conjecture 1, we have that for

\[
\lim_{t \to 1} \int_{K^*} \text{tr}(\text{hol}(L)) \, d\rho_\text{sphere}^G = \frac{1}{Z} \int_K \text{tr}(w_1(h_1, \text{id}, \ldots, \text{id})) \rho_n(h_0(h_1, \text{id}, \ldots, \text{id})) \rho_c(h_1) \, dh_1. \tag{30}
\]

In the sphere case, the normalization factor is given by \( Z = \rho_A(\text{id}) \), where \( A \) is the total area of the sphere. Although \( Z \) may depend on \( t \) (since we do not currently assume that the area of the sphere is fixed), it has a limit as \( t \to 1 \).

If the limiting value \( a \) of \( |F_1| \) is also zero, then \( \rho_{|F_1|} \) becomes also becomes a \( \delta \)-function. Since the normalized trace of the identity matrix equals 1, the right-hand side of (30) becomes \( \rho_c(\text{id})/Z \). The total area of the sphere in this limit is just \( c \), so \( Z = \rho_c(\text{id}) \) and \( \rho_c(\text{id})/Z = 1 \).

Finally, if the limiting value \( a \) of \( |F_1| \) is nonzero, we must understand the effect on \( w_0 \) and \( w_1 \) of evaluating \( h_i \), \( i \geq 2 \), to the identity. Recall that the words \( w_0 \) and \( w_1 \) arise from representing the boundary of \( F_0 \) and the loop \( L \) as words in \( L_1, \ldots, L_{f-1} \), which are free generators for \( \pi_1(\mathbb{R}) \). Now, \( \pi_1(\mathbb{R}) \) is naturally isomorphic to \( \pi_1(\mathbb{R}^2 \setminus \{x_1, \ldots, x_{f-1}\}) \), where \( x_i \) is an arbitrarily chosen element of \( F_1 \). There is then a homomorphism from \( \pi_1(\mathbb{R}) \) to \( \pi_1(\mathbb{R}^2 \setminus \{x_1\}) \) induced by the inclusion of \( \mathbb{R}^2 \setminus \{x_1, \ldots, x_{f-1}\} \) into \( \mathbb{R}^2 \setminus \{x_1\} \). Since \( \pi_1(\mathbb{R}^2 \setminus \{x_1\}) \) is just a free group on the single generator \( L_1 \), this homomorphism is computed by mapping each of the generators \( L_2, \ldots, L_{f-1} \) to the identity, leaving only powers of \( L_1 \). Thus, if we write, say, \( \partial F_0 \) as a word in the generators \( L_1, \ldots, L_{f-1} \) and apply the just-mentioned homomorphism, the result will be \( L_1^{n_0} \), where \( n_0 \) is the winding number of \( \partial F_0 \) around \( x_1 \) (i.e., around \( F_1 \)). By the Jordan curve theorem, this winding number is 1, assuming that we traverse \( \partial F_0 \) in the counterclockwise direction. (In Figure 12, for example, the outer boundary of the loop in Figure 11 decomposes as \( L_2 L_3 L_4 L_1 \); if we set any three of the four generators to the identity, the remaining generator will occur to the power 1.) Expressing this result in terms of the loop variables, rather than the loops themselves, we conclude that \( w_0(h_1, \text{id}, \ldots, \text{id}) = h_1 \). Similarly, \( w_1(h_1, \text{id}, \ldots, \text{id}) = h_1^n \), where \( n \) is the winding number of \( L \) around \( F_1 \). ■

4.6 The induction

In this section, we prove Theorem 3 by induction on the number of crossings. Our strategy is to deform an arbitrary loop \( L \) with \( k \) crossings into a loop \( L_n \) that winds \( n \) times around a simple closed curve. The variation of the Wilson loop functional along this deformation will be a linear combination of products of Wilson loop functionals for curves with fewer crossings, plus a covariance term. We begin by recording a result for a loop that winds \( n \) times around a simple closed curve.

**Theorem 12** Let \( L_n(a, c) \) denote a loop that winds \( n \) times around a simple closed curve enclosing areas \( a \) and \( c \). Assuming Conjecture 1, we have that for
all $a$ and $c$, the limit

$$w_n(a,c) := \mathbb{E}\{\text{tr}(\text{hol}(L_n(a,c)))\}$$

exists and depends continuously on $a$ and $c$. Furthermore, the associated variance tends to zero:

$$\lim_{N \to \infty} \text{Var}\{\text{tr}(\text{hol}(L_n(a,c)))\} = 0.$$ 

The proof of this result is given in Section 4.7. We are now ready for the proof of our main result.

**Proof of Theorem 3.** Let $k$ denote the number of crossings of $L$. We will proceed by induction on $k$. When $k = 0$, the result is precisely Conjecture 1. Assume, then, that (7) and (8) and the continuity condition hold for all loops with $l < k$ crossings and consider a loop $L$ with $k$ crossings. Using Proposition 10, we may make a combination of Makeenko–Migdal variations at the vertices of $L$, giving loops $L(t)$ in which $L(0) = L$ and so that all but two of the faces shrink to zero as $t \to 1$. By Theorem 11, the limit as $t \to 1$ of the Wilson loop functional of $L(t)$ is the Wilson loop functional of a loop $L_n(a,c)$ that winds $n$ times around a simple closed curve enclosing areas $a$ and $b$. We may differentiate the Wilson loop functional of $L(t)$ using the chain rule and the first part of Proposition 5. Integrating the derivative then gives

$$\mathbb{E}\{\text{tr}(\text{hol}(L))\} = \mathbb{E}\{\text{tr}(\text{hol}(L_n(a,c)))\}$$

$$- \sum_j c_j \int_0^1 \mathbb{E}\{\text{tr}(\text{hol}(L_{1,j}(t)))\} \mathbb{E}\{\text{tr}(\text{hol}(L_{2,j}(t)))\} \, dt$$

$$- \sum_j c_j \text{Cov}\{\text{tr}(\text{hol}(L_{1,j}(t))), \text{tr}(\text{hol}(L_{2,j}(t)))\} \, dt.$$ 

(31)

Here $L_{1,j}(t)$ and $L_{2,j}(t)$ are the loops obtained by applying the Makeenko–Migdal equation at the vertex $v_j$ to the loop $L(t)$. In particular, since neither
of these loops has a crossing at \( v_j \), we see that \( L_{1,j}(t) \) and \( L_{2,j}(t) \) have at most \( k - 1 \) crossings.

By our induction hypothesis, the limit
\[
w(\text{hol}(L_{i,j}(t))) := \lim_{N \to \infty} E \{ \text{tr}(\text{hol}(L_{i,j}(t))) \}
\]
events for each \( t \in [0, 1] \), each \( i \in \{1, 2\} \), and each \( j \). Our induction hypothesis also tells us that the variance of \( \text{tr}(\text{hol}(L_{i,j}(t))) \) goes to zero as \( N \to \infty \); the inequality (21) then tells us that the covariances on the last line of (31) also tend to zero. Now, \( |\text{tr}(\text{hol}(U))| \leq 1 \) for all \( U \in U(N) \), from which it follows that \( |\text{Var}(\text{tr}(\text{hol}(U)))| \leq 1 \). Thus, we may apply dominated convergence to all integrals in (31), along with Theorem 12, to obtain
\[
\lim_{N \to \infty} E \{ \text{tr}(\text{hol}(L)) \} = w(L_n(a,c)) - \sum_j c_j \int_0^1 w(L_{1,j}(t))w(L_{2,j}(t)) \, dt,
\]
(32)
establishing the existence of the limit in (7) of Theorem 3.

We now establish the claimed continuity of \( w(L) \) with respect to the areas of the faces. When we deform our original loop \( L \) into a loop of the form \( L_n(a,b) \), the values of \( a \) and \( c \) depend continuously on the areas of the faces of \( L \); indeed, \( a = |F_0| + x \) and \( c = |F_1| + y \), where \( F_0 \) and \( F_1 \) are the chosen faces with minimal and maximal winding numbers, respectively, and where \( x \) and \( y \) are given explicitly in the proof of Proposition 10 in terms of the areas of the remaining faces of \( L \). (Note that although \( F_0 \) and \( F_1 \) are not necessarily unique, we may make a fixed choice of \( F_0 \) and \( F_1 \) once and for all for each topological type of loop \( L \).) Thus, \( w(L_n(a,c)) \) also depends continuously on the areas of the faces of \( L \), by the continuity claim in Theorem 12. Similarly, the areas in \( L_{1,j}(t) \) and \( L_{2,j}(t) \) depend continuously on the areas of \( L \). Thus, the continuity of the second term on the right-hand side of (32) follows from our induction hypothesis and dominated convergence.

To establish the second claim (8) of Theorem 3, we use the second part of Proposition 5. We then need to bound the covariance term appearing in (14). We first use (21) and then use Proposition 6 with \( c_1 = c_2 = 1 \) to bound the variance of the product of \( \text{tr}(\text{hol}(L_1)) \) and \( \text{tr}(\text{hol}(L_2)) \). The argument is then similar to the proof of (7).

Finally, we establish the large-\( N \) Makeenko–Migdal equation (9) for the limiting Wilson loop functionals. If we vary the areas in a checkerboard pattern along a path \( L(t) \) as in Figure 3, we have
\[
E \{ \text{tr}(\text{hol}(L(t))) \} = E \{ \text{tr}(\text{hol}(L(t_0))) \}
+ \int_{t_0}^t E \{ \text{tr}(\text{hol}(L_1(s))) \} E \{ \text{tr}(\text{hol}(L_2(s))) \} \, ds
+ \int_{t_0}^t \text{Cov}\{\text{tr}(\text{hol}(L_1(s))), \text{tr}(\text{hol}(L_2(s)))\} \, ds.
\]
Using the first two points (7) and (8) in Theorem 3, we can let $N$ tend to infinity to obtain
\[ w(L(t)) = w(L(t_0)) + \int_{t_0}^{t} w(L_1(s))w(L_2(s)) \, ds. \] (33)
Since we have shown that $w(L)$ depends continuously on the areas of $L$, we see that $w(L_1(s))$ and $w(L_2(s))$ depend continuously on $s$. Thus, we can apply the fundamental theorem of calculus to differentiate (33) with respect to $t$ to obtain the large-$N$ Makeenko–Migdal equation (9).

4.7 The $n$-fold circle

In this section, we analyze the Wilson loop functional for a loop that winds $n$ times around a simple closed curve, establishing Theorem 12. Assuming that the eigenvalue distribution of a Brownian bridge in $U(N)$ has a large-$N$ limit, the calculations here provide a recursive method for computing the higher moments of the limiting distribution, in terms of the first moment. (But there is no simple formula for the first moment!) A similar, but not identical, recursion for the moments of the free unitary Brownian motion—i.e., the large-$N$ limit of a Brownian motion in $U(N)$, rather than a Brownian bridge—was established by Biane. See the bottom of p. 16 of [Bi1] and p. 266 of [Bi2].

We approximate a loop that winds $n$ times around a simple closed curve by a loop with simple crossings, specifically, by a curve of the “loop within a loop within a loop” form, as in Figure 13. If there are total of $n$ loops, then there are $n - 1$ crossings and a total of $n + 1$ faces. The faces divide into the innermost circle, the outer region, and $n - 1$ annular regions. If we set the areas of the annular regions to zero, the curve will simply wind $n$ times around the outermost circle. Shifting the puncture from the innermost to outermost regions gives another curve of the same sort, with the order of the annular regions reversed.

The symmetry between the “inside” and “outside” of the circle is essential to our analysis. Suppose $L_n(a,c)$ denotes a loop that winds $n$ times around a simple closed curve enclosing areas $a$ and $c$. Suppose, for example, that $n = 3$ and we label the faces so that $a \leq c$. Then the induction procedure we will develop allows us to shrink the innermost annular region to zero, thus reducing $L_3(a,c)$ to $L_2(3a/2, c - a/2)$. The next stage of the process, however, depends on which of $3a/2$ and $c - a/2$ is larger. If $3a/2 > c - a/2$, then we must turn the circle inside out by identifying $L_2(3a/2, c - a/2)$ with $L_2(c - a/2, 3a/2)$ before proceeding.

Now, nothing prevents us from letting the areas of one or more of the annular regions equal zero. Specifically, if we make a Makeenko–Migdal variation at some or all of the vertices, both sides of the resulting formula will allow a limit as some of these areas tend to zero. Since we wish to remain as close as possible the curve that winds $n$ times around a simple closed curve, we set as many of these areas to zero as possible. We will show that it is possible to shrink the central region and the outer region, while increasing the innermost annular region and
keeping the outermost $n-2$ annular regions equal to zero, as in Figure 13. This observation motivates consideration of the following class of loops.

**Definition 13** Let $L_n(a,c)$ denote the loop that winds $n$ times around a simple closed curve enclosing areas $a$ and $c$. Now let $L_1(a+b,c)$ be a simple closed curve having two faces $F_1$ and $F_2$ with areas $a+b$ and $c$, respectively. For $n \geq 2$, let $L_n(a,b,c)$ denote the loop that winds $n-1$ times around $L_1(a+b,c)$ and then winds once around a region of area $a$ inside $F_1$.

When $n = 1$, we interpret $L_1(a,b,c)$ as being $L_1(a,b+c)$ (the loop that winds zero times around around $L_1(a+b,c)$ and then once around a region of area $a$). We now give an inductive procedure for computing the limiting Wilson loop functionals for loops of the form $L_n(a,c)$ and $L_n(a,b,c)$.

**Theorem 14** Assume Conjecture 1. Then for all $n \geq 1$ and all non-negative real numbers $a$, $b$, and $c$, the limits

$$w_n(a,c) := \lim_{N \to \infty} E\{\text{tr}(\text{hol}(L_n(a,c)))\}$$

(34)

and

$$W_n(a,b,c) := \lim_{N \to \infty} E\{\text{tr}(\text{hol}(L_N(a,b,c)))\}$$

(35)

exist, and the associated variances tend to zero. Furthermore, for some fixed
\[ n \geq 1, \text{ if } a \leq nc, \text{ we have the inductive formula} \]

\[
W_{n+1}(a, b, c) = w_n \left( \frac{n + 1}{n} a + b, c - \frac{a}{n} \right) \\
\quad - \sum_{k=1}^{n-1} k \int_0^{a/n} w_k(a + b + t, c - t) W_{n-k}(a - nt, b + (n + 1)t, c - t) \, dt \\
\quad - n \int_0^{a/n} w_n(a + b + t, c - t) w_1(a - nt, b + c + nt) \, dt, \quad (a \leq nc). \tag{36}
\]

Our main interest is in the quantity \( w_n(a, c) \), which is the same as \( W_n(a, 0, c) \), where by the symmetry of \( w_n(\cdot, \cdot) \), we may assume that \( a \leq c \). Note, however, that even if we put \( b = 0 \) on the left-hand side of (36), the right-hand side of (36) will still involve \( W_{n-k}(a', b', c') \) with nonzero values of \( b' \). On the other hand, the inductive procedure ultimately expresses \( W_n(a, b, c) \)—and thus, in particular, \( w_n(a, c) = W_n(a, 0, c) \)—in terms of \( w_1(\cdot, \cdot) \).

The restriction \( a \leq nc \) in (36) is needed to ensure that all areas on the right-hand side of the equation are non-negative. In particular, if \( a \leq nc \), then \( c-t \geq 0 \) for all \( t \) between 0 and \( a/n \). Note that if \( a \leq nc \), then \( a - nt \leq n(c-t) = nc - nt \), and thus \( a - nt \leq l(c - t) \) for all \( l \leq n \). We may, therefore, apply the induction procedure to the quantity \( W_{n-k}(a - nt, b + (n + 1)t, c - t) \) occurring in the second line of (36). On the other hand, in the expressions \( w_k(a + b + t, c - t) \) in the second and third lines of (36), the condition \( (a + b + t) \leq (k - 1)(c - t) \) may be violated. Fortunately, \( w_k(a, c) \) is symmetric in \( a \) and \( c \) and we may therefore write

\[
w_k(a + b + t, c - t) = w_k(\min(a + b + t, c - t), \max(a + b + t, c - t)), \tag{37}
\]

at which point we may apply the induction.

Suppose, for example, that we wish to compute \( w_3(a, c) = W_3(a, 0, c) \) in terms of \( w_1(\cdot, \cdot) \). Since \( w_3(a, c) \) is symmetric in \( a \) and \( c \), it is harmless to assume that \( a \leq c \). We first compute \( W_2(a, b, c) \) in terms of \( w_1(\cdot, \cdot) \), by applying (36) with \( n = 1 \), giving

\[
W_2(a, b, c) = w_1(2a + b, c - a) - \int_0^a w_1(a + b + t, c - t) w_1(a - t, b + c + t) \, dt, \tag{38}
\]

for \( a \leq c \). (This formula is just what we obtained in (26) in Section 3.1.) Next, we apply (36) with \( n = 2 \) and \( b = 0 \), giving

\[
w_3(a, c) = w_2(3a/2, c - a/2) \\
- \int_0^{a/2} w_1(a + t, c - t) W_2(a - 2t, 3t, c - t) \, dt \\
- 2 \int_0^{a/2} w_2(a + t, c - t) w_1(a - 2t, c + 2t) \, dt. \tag{39}
\]
Figure 14: The variation in the areas of $L_5(a, b, c)$

Last, we plug into (39) the values of $W_2(a - 2t, 3t, c - t)$ and $w_2(a + t, c - t)$ computed in (38). In the case of $W_2(a - 2t, 3t, c - t)$, the assumption that $a \leq c$ guarantees that $a - 2t \leq c - t$, so that we may directly apply (38). In the case of $w_2(a + t, c - t)$, we may need to make use of (37) for certain values of $t$.

Some explicit computations of these expressions in the plane case (i.e., the $c = \infty$ case) are given in Section 5.2.

**Proof of Theorem 14.** The argument is similar to the proof of Theorem 3, except with a different variation of the areas. We assume, inductively, that the limit in (34) exists for all $k \leq n$ and $a$ and $c$, and that the limit in (35) exists for all $k \leq n$ and all $a, b, c$ with $a \leq kc$. We also assume that the corresponding variances go to zero. (When $n = 1$, this claim is just Conjecture 1.) We then prove these results for $n + 1$, while also proving the formula (36).

We consider the loop $L_n(a, b, c)$, realized as the $\varepsilon \to 0$ limit of the $n$-fold “loop within a loop within a loop,” with the areas of the $n - 2$ outermost annular regions set equal to $\varepsilon$. We then order the $n - 1$ crossings from outermost to innermost. We then make a Makeenko–Migdal variation at the $k$th crossing, scaled by a factor of $k$. (See Figure 14.) The resulting change in the areas is

\[
\begin{align*}
  a &\mapsto a - (n - 1)t \\
  b &\mapsto b + nt \\
  c &\mapsto c - t
\end{align*}
\]

while the areas of the outer $n - 2$ annular regions remain unchanged.

If $a \leq nc$, the area of the outermost annular region will remain non-negative as the area of the innermost region shrinks to zero. Thus, we may apply the Makeenko–Migdal equation (13) with the $n - 2$ annular areas equal to $\varepsilon$ and then let $\varepsilon$ tend
to zero, giving a finite-$N$ version of (36):

\[
W_{n+1}^N(a, b, c) = w_n^N \left( \frac{n+1}{n} a + b, c - \frac{a}{n} \right) - \sum_{k=1}^{n-1} k \int_0^{a/n} w_k^N(a + b + t, c - t) W_{n-k}^N(a - nt, b + (n+1)t, c - t) \, dt \\
- n \int_0^{a/n} w_n^N(a + b + t, c - t) w_1^N(a - nt, b + c + nt) \, dt + \text{Cov.} \tag{40}
\]

Here $W_n^N(a, b, c) = \mathbb{E} \{ \text{tr}(\text{hol}(L_n(a, b, c))) \}$ and $w_n^N(a, b) = \mathbb{E} \{ \text{tr}(\text{hol}(L_n(a, b))) \}$ and “Cov” is a covariance term. Note that as we remarked following the statement of Theorem 14, if $a \leq nc$, then $a - nt \leq k(c - t)$ for all $k \leq n$. Thus, the factor of $W_{n-k}^N$ falls under our induction hypothesis.

Then, as in the proof of the first two points of Theorem 3, our induction hypothesis, together with (21) and Proposition 6, allows us to take the $N \to \infty$ limit. Thus, the large-$N$ limit of $W_{n+1}^N(a, b, c)$ exists for $a \leq nc$ and satisfies (36). The vanishing of the variance then follows by the same argument as in the proof of Theorem 3. Next, since $w_n^N(a, c)$ is symmetric in $a$ and $c$, we may assume $a \leq c$. Then the existence of the large-$N$ limit of $w_{n+1}^N(a, c) = W_{n+1}^N(a, 0, c)$ and the vanishing of the associated variance follows from the just-established result about $W_{n+1}^N$.

We have thus established the existence of the large-$N$ limit and vanishing of the variance for $W_n^N(a, b, c)$, provided $a \leq nc$. In light of the symmetry of $w_n^N(a, c)$ in $a$ and $c$, this result is sufficient to demonstrate the large-$N$ limit and vanishing variance of $w_n^N(a, c)$ for all $a$ and $c$. Thus, Theorem 12 is established. The proof of Theorem 3 is then complete, which means that actually the large-$N$ limit and vanishing variance hold for $W_n^N(a, b, c)$ for all $a$, $b$, and $c$. (But the inductive formula (36) for $W_{n+1}(a, b, c)$ does not even make sense unless $a \leq nc$.)

\section{The plane and compact surfaces}

\subsection{The plane case, revisited}

In [Lévy], Thierry Lévy establishes the large-$N$ limit of Wilson loop functionals for Yang–Mills theory in the plane, together with the vanishing of the associated variances. (See Theorems 5.1 and 5.6 in [Lévy].) The proofs of these results are based on a detailed analysis of the convergence of Brownian motion in $U(N)$ to Biane’s “free unitary Brownian motion.” The limiting expectation values for loops with simple crossings are then uniquely characterized in [Lévy] by the large-$N$ Makeenko–Migdal equation and another condition, labeled as Axiom
Φ₄ in Section 0.5 of [Lév2] and termed the unbounded face condition in Theorem 2.3 of [DHK2].

The methods of the present paper give a new proof of some of these results. Specifically, the proof of Theorem 3 applies with minor changes in the plane case. Furthermore, Conjecture 1 is easily established in the plane case, as we will demonstrate shortly. Thus, in the plane case, our proof of Theorem 3 is not conditional on any unproven results. We thus obtain a proof of the existence of the limiting expectation values and the vanishing of the variances that is conceptually different from the one in [Lév2]; Our proof is based on an analysis of the case of a simple closed curve and the $U(N)$ Makeenko–Migdal equation. On the other hand, the methods used here do not give the detailed variance estimates found in Theorem 5.6 of [Lév2].

**Theorem 15** Theorem 3 holds also in the plane case. The result is unconditional in this case, since we will give a proof of the plane case of Conjecture 1 in the plane case below.

**Proof.** We follow the same argument as in the sphere case, with a few minor modifications. First, in Section 4.4, we should choose $F_0$ to the unbounded face, so that the areas $a_1, \ldots, a_{f-1}$ are finite numbers. Theorem 11 still holds, with the same proof, in the plane case, provided we interpret $\rho_{|F_0|}$ as being the constant function 1. (The normalization factor $Z$ is then also equal to 1.) All other proofs go through without change—except that in Section 4.7, the condition $a \leq nc$ always holds because $c = \infty$. ■

We now supply the proof of the plane case of Conjecture 1. Of course, the claim follows from results in [Lév2], but we can give a elementary proof as follows.

**Proof of Conjecture 1 for $\mathbb{R}^2$.** The function $\text{tr}(U)$ is an eigenfunction for the Laplacian on $U(N)$—with respect to the scaled Hilbert–Schmidt metric in (1)—with eigenvalue $-1$. (See, e.g., Remark 3.4 in [DHK1].) It follows easily that

$$E\{\text{tr}(\text{hol}(L))\} = e^{-a/2}$$

for a simple closed curve $L$ in the plane enclosing area $a$. It is also possible to compute the variance of $\text{tr}(\text{hol}(L))$ explicitly. One way to do this is to use the elementary calculations in Example 3.5 of [DHK1], which show (after applying the normalized trace to all the functions there) that

$$e^{a\Delta/2}(\text{tr}(U))^2 = e^{-a}\cosh(a/N)(\text{tr}(U))^2 - e^{-a} \frac{a}{N^2} \frac{\sinh(a/N)}{a/N} \text{tr}(U^2)$$

where $\Delta$ is the Laplacian on $U(N)$. Evaluating at $U = \text{id}$ gives the expectation value

$$E\{(\text{tr}(U))^2\} = e^{-a}\cosh(a/N) - e^{-a} \frac{a}{N^2} \frac{\sinh(a/N)}{a/N}$$

from which it follows easily that the variance of $\text{tr}(U)$ tends to zero as $N$ tends to $\infty$. 33
More generally, Theorem 1.20 in [DHK1] identifies the leading-order term in the large-$N$ asymptotics of the $U(N)$ Laplacian, acting on “trace polynomials” (that is, sums of products of traces of powers of $U$). This leading term satisfies a \textit{first-order} product rule. Thus, the limiting heat operator is multiplicative, from which it follows that the variance of any trace polynomial vanishes as $N \to \infty$. (See the last displayed formula on p. 2620 of [DHK1].) ■

5.2 Example computations in the plane case

We now illustrate the computation of large-$N$ Wilson loop functionals in the plane case. We first compute the large-$N$ Wilson loop functionals $W_n(a,b,c)$ from Section 4.7 in the plane case, which corresponds to $c = \infty$, for $n \leq 3$. The condition $a \leq nc$ is always satisfied in that case, and, as noted in the previous subsection, we have $w_1(a,\infty) = e^{-a/2}$.

The \textbf{two-fold circle}. When $c = \infty$, (38) takes the following form:

$$W_2(a,b,\infty) = w_1(2a+b,\infty) - \int_0^a w_1(a+b+t,\infty)w_1(a-t,\infty) \, dt$$

which agrees with Lévy’s formula for the “loop within a loop” [Lévy2, Appendix B]. When $b = 0$, we get $w_2(a,\infty) = e^{-a}(1-a)$, which is the second moment of Biane’s distribution for the free unitary Brownian motion. (The moments may be found in Lemma 3 of [Bi1] or the Remark on p. 267 of [Bi2].)

The \textbf{three-fold circle}. When $n = 3$ and $c = \infty$, (39) becomes

$$W_3(a,b,\infty) = e^{-(b+3a/2)}(1-b-3a/2)$$

$$- \int_0^{a/2} e^{-(a+b+t)/2}e^{-(a-2t)-(b+3t)/2}(1-a+2t) \, dt$$

$$- 2 \int_0^a e^{-(a+b+t)}(1-a-b-t)e^{-(a-2t)/2} \, dt.$$
The trefoil. We now analyze the large-$N$ Wilson loop functional for the trefoil loop in the plane case, using the strategy outlined in Section 4.1. In this example, the loops $L_{1,j}$ and $L_{2,j}$ occurring on the right-hand side of the Makeenko–Migdal equation are simple closed curves for all $j = 1, 2, 3$, and the product of the Wilson loop functionals simplifies as

$$w(L_{1,j}(t))w(L_{2,j}(t)) = e^{-a-(b+c+d)/2},$$

where the dependence on $t$ and $j$ drops out. Thus, when we integrate from $t = 0$ to $t = 1$, we obtain simply the constant value of the integrand. Now, the signs in the Makeenko–Migdal variations in Figure 8 are reversed from usual labeling. After adjusting for this and noting that the coefficients of the variations in Figure 8 add to $(b+c+d)/2$, we obtain

$$w(L) = w_2(a + (b + c + d)/2, \infty) + \left(\frac{b + c + d}{2}\right) e^{-a-(b+c+d)/2}.$$  

Using the value for $w_2(a + (b + c + d)/2, \infty)$ computed above and simplifying gives

$$w(L) = e^{-a-(b+c+d)/2}(1 - a),$$

which agrees with the value of the master field for the trefoil in Appendix B of [Lév2].

5.3 Other surfaces

We now consider the Yang–Mills measure on a compact surface $\Sigma$, possibly with boundary. (See [Sen3] and also [Lév1].) If the boundary of $\Sigma$ is nonempty, we may optionally impose constraints on the holonomy around the boundary components. The Makeenko–Migdal equation in this setting was established rigorously in [DGHK].

For surfaces other than the plane, much less is known than in the cases of the sphere and the plane. For a general surface $\Sigma$, not all simple closed curves are the same—one at least must distinguish between those that are homotopically trivial and those that are not. Even in the homotopically trivial case, this author is not aware of any work toward establishing Conjecture 1 beyond the plane and sphere cases. Nevertheless, if we assume that Conjecture 1 holds for homotopically trivial loops in $\Sigma$, we can establish Theorem 3 for homotopically trivial loops in $\Sigma$ that satisfy a certain “smallness” assumption.

Conjecture 16 Consider a surface $\Sigma$ of area $\text{area}(\Sigma)$. Let $U \subset \Sigma$ be a topological disk and consider a simple closed curve $C$ in $U$. In Sengupta’s formula for the Yang–Mills measure on $\Sigma$, let us assign area $a$ to the interior of $C$ and area

$$c := \text{area}(\Sigma) - a$$

to the exterior of $C$, for any $a < \text{area}(\Sigma)$. Then

$$\lim_{N \to \infty} \mathbb{E}\{\text{tr} (\text{hol}(C))\}.$$
exists and depends continuously on \( a \) and \( c \). Furthermore,

\[
\lim_{N \to \infty} \text{Var}\{\text{tr}(\text{hol}(C))\} = 0.
\]

Assuming this claim, we will establish a version of Theorem 3 for loops in \( U \subset \Sigma \) satisfying an appropriate smallness condition. We start by establishing the correct smallness condition for a loop that winds \( n \) times around a simple closed curve.

**Theorem 17** Let the notation be as in Conjecture 16. For each nonzero integer \( n \), let \( L_n(a,c) \) denote the loop that winds \( n \) times around \( C \). Assuming Conjecture 16, Theorem 3 holds for \( L_n(a,c) \), provided that

\[
|n| a < \text{area} (\Sigma),
\]

(41)

**Proof.** It is harmless to assume \( n > 0 \). Following the proof of Theorem 12 in the sphere case, we deform \( L_n(a,c) \) into a loop of the form \( L_n(a,b,c) \). Let \( a = (a,b,c) \) denote the vector of areas of the faces of \( L_n(a,b,c) \) and let \( w = (w_1,w_2,w_3) \) be the associated vector of winding numbers, viewing \( L_n(a,b,c) \) as a loop in the disk \( U \), with \( w_3 = 0 \). We will actually prove Theorem 17 for the loops \( L_n(a,b,c) \) by induction on \( n > 0 \), under the assumption that

\[
a \cdot w < \text{area} (\Sigma),
\]

(42)

which reduces to (41) when \( n > 0 \) and \( b = 0 \). When \( n = 1 \), the loop \( L_1(a,b,c) \) is (by definition) the same as \( L_1(a,b+c) \), so that the desired result is just Conjecture 16. For \( n \geq 2 \), we may follow the same sort of inductive argument as in the sphere case, provided that we never shrink the area of the “\( c \)” region to zero.

Take \( n \geq 2 \) and assume that Theorem 17 holds for loops of the form \( L_k(a,b,c) \) satisfying (42), with \( k < n \). Consider a loop \( L_n(a,b,c) \) satisfying (42) and deform it into the loop \( L_n(a(t),b(t),c(t)) \) with \( 0 \leq t \leq a/(n-1) \), as in Section 4.7. As we vary the values of \( a \), \( b \), and \( c \), the values of area(\( \Sigma \)) and \( a \cdot w \) remains constant—as can be seen explicitly or as a consequence of Theorem 9. Thus, by (42), we have

\[
a \cdot w = na(t) + (n-1)b(t) < a(t) + b(t) + c(t).
\]

(43)

Since \( n \geq 2 \), (43) tells us that \( c(t) > (n-1)a(t) + (n-2)b(t) > 0 \).

Thus, \( c(t) \) remains positive as \( t \) approaches \( a/(n-1) \), when we obtain a loop of the form \( L_{n-1}(a',c') \), with

\[
a' = b(t)|_{t=a/(n-1)} = b + \frac{n}{n-1}a.
\]

We can then see explicitly that the value of \( a \cdot w \) for \( L_{n-1}(a',c') \) is the same as for \( L_n(a,b,c) \), namely \( na + (n-1)b \). Thus, by induction, Theorem 19 holds for \( L_{n-1}(a',c') \). Furthermore, the loops \( L_{1,j}(t) \) and \( L_{2,j}(t) \) obtained from the
Makeenko–Migdal equation will be \( L_k(a(t), b(t), c(t)) \) or \( L_{n-k}(a(t) + b(t), c(t)) \), with \( 1 \leq k < n \). It is then easy to see that the value of \( \mathbf{a} \cdot \mathbf{w} \) for these loops is no bigger than for \( L_n(a(t), b(t), c(t)) \), which is the same as for \( L_n(a, b, c) \). Thus, \( L_{1,j}(t) \) and \( L_{2,j}(t) \) satisfy (42) and by induction, Theorem 19 holds for these loops as well.

From this point, the argument is the same as in the sphere case. In particular, since we have ensured that \( c(t) \) remains positive as we deform the areas of \( L_n(a, b, c) \), we may apply (40) to compute the Wilson loop functional for \( L_n(a, b, c) \).

We now analyze a general loop \( L \) in \( U \subset \Sigma \) with simple crossings. Following the logic in Section 4, we first deform \( L \) into a loop of the form \( L_n(a, c) \). Since the area of \( F_0 \) increases during this process, there is no obstruction to carrying out this first step in the analysis of \( L \). Nevertheless, we require a smallness assumption on \( L \) that will ensure that the limiting loop \( L_n(a, c) \) will satisfy the hypothesis of Theorem 17. This smallness assumption must also be inherited by the loops \( L_{1,j}(t) \) and \( L_{2,j}(t) \) occurring on the right-hand side of the Makeenko–Migdal equation, so that these loops can be analyzed by induction on the number of crossings.

Now, we may cut \( L \) at a crossing \( v \) obtaining two loops \( L_1 \) and \( L_2 \). We may then cut either \( L_1 \) or \( L_2 \) at one of its crossings, and so on. We refer to any loop that can be obtained by a finite sequence of such cuts as a subloop of \( L \). In particular, \( L \) is a subloop of itself corresponding to making zero cuts. We label the faces of \( L \) as \( F_0, \ldots, F_{f-1} \) where \( F_0 \) is the face containing the complement \( U^c \) of \( U \).

Next, we define, for each face \( F_j \) of \( L \),

\[
|w|_{\max}(F_j) = \max_{L'} |w(L', F_j)|,
\]

where the maximum ranges over all subloops \( L' \) of \( L \) and where \( w(L', F) \) is the winding number of \( L' \) around \( F \). We then define

\[
|w|_{\max} = \max_j |w|_{\max}(F_j).
\]

Finally, if \( a_j \) is the area of \( F_j \), we define

\[
A = a_1 + a_2 + \cdots + a_{f-1} = \text{area}(\Sigma) - \text{area}(F_0).
\]

**Remark 18** One may bound \( |w|_{\max} \) as follows. Suppose for some \( k \), one can travel from any face of \( L \) to \( F_0 \) while crossing \( L \) at most \( k \) times. Then the same is true of any subloop \( L' \) of \( L \). Since the winding number of \( F_0 \) is zero and \( w(L', F) \) changes by one each time we cross \( L' \), we conclude that \( |w|_{\max} \) is at most \( k \).

**Theorem 19** Assume Conjecture 16. Let \( L \) be a loop traced out on a graph in a topological disk \( U \subset \Sigma \) with simple crossings. Then Theorem 3 holds, provided \( L \) satisfies the “smallness” assumption

\[
A|w|_{\max} < \text{area}(\Sigma).
\]
Note that since $L$ is contained in a disk, $L$ is certainly homotopically trivial in $\Sigma$. Thus, Theorem 19 does not tell us anything about homotopically nontrivial loops. Since $U$ is a topological disk, Theorem 9 applies in this context, provide we compute the winding numbers of the faces of $L$ by regarding $L$ as a loop in $U$. We start by extending Theorem 11 to the present context.

**Lemma 20** Consider the Yang–Mills measure on $\Sigma$ for an arbitrary connected compact Lie group $K$. Let $L$ be a loop traced out on a graph in $U \subset \Sigma$ and having only simple crossings. Denote the number of faces of $L$ by $f$ and label the faces as $F_0, F_1, F_2, \ldots, F_{f-1}$, where $F_0$ is the face containing $U^c$. Suppose we vary the areas of the faces as a function of a parameter $t \in [0, 1)$ in such a way that as $t \to 1$, the areas of $F_2, \ldots, F_{f-1}$ tend to zero, while the areas of $F_0$ and $F_1$ approach non-negative real numbers $a$ and $c$, respectively, with $c > 0$. Then

$$\lim_{t \to 1} \mathbb{E}\{\text{tr}(\text{hol}(L))\} = \mathbb{E}\{\text{tr}(\text{hol}(L_n(a, c)))\},$$

where $n$ is the winding number of $L$ around $F_1$.

**Proof.** Let $G$ be a minimal graph (necessarily connected) in which $L$ can be traced. Before we can apply Sengupta’s formula, we must embed $G$ into an admissible graph $G'$, that is, one that contains the boundary of $\Sigma$ and each of whose faces is a topological disk. Actually, by the Jordan curve theorem, all the faces of $G$ other than $F_0$ will automatically be disks. It is then possible to construct $G'$ by adding new edges entirely in the closure of $F_0$ (see Section 1.2 of [Lév1]). Thus, the faces of $G'$ may be chosen to be of the form $F'_0, F'_1, \ldots, F'_{f-1}$, where $F'_0$ is a subset of $F_0$ having the same area as $F_0$.

Let us divide the edge variables for $G'$ into the edge variables $x$ and the remaining edge variables $y$. Then integration with respect to the Yang–Mills measure for the graph $G'$ in $\Sigma$ may be written as

$$\int_{K^e} f(x, y) d\mu^G_{\Sigma} = \frac{1}{Z} \int_{K^{e'\to e}} \rho_{|F_0|}(\text{hol}_{F'_0}(x, y)) \int_{K^{e'}} f(x, y) d\mu^G_{\text{plane}}(x) dy,$$

where $e$ and $e'$ are the number of edges of $G$ and $G'$, respectively. Furthermore, we may write

$$\text{hol}_{F'_0}(x, y) = \text{hol}_{F_0}(x)g(y)$$

for some word $g(y)$ in the $y$ variables. Here, for notational simplicity, we consider the unconditional Yang–Mills measure, but a similar argument applies if there are constraints the holonomies around the boundary components of $\Sigma$.

In the case that $f$ is the trace of the holonomy of $L$, may imitate the proof of Theorem 11 to obtain

$$\mathbb{E}\{\text{tr}(\text{hol}(L))\} = \frac{1}{Z} \int_{K^{e'\to e}} \int_{K^{f-1}} \text{tr}(w_1(h_1, \ldots, h_{f-1}))$$

$$\times \rho_{|F_0|}(w_0(h_1, \ldots, h_{f-1})g(y)) \left( \prod_{i=1}^{f-1} \rho_{|F_i|}(h_i) \right) dh_1 \ldots dh_{f-1} dy.$$
As in that proof, if we let $t \to 1$, we obtain

$$
\lim_{t \to 1} \mathbb{E} \{ \text{tr}(\text{hol}(L)) \} = \frac{1}{Z} \int_{K^{t-\infty}} \int_{K^{t-1}} \text{tr}(h_1^\pi) \rho_a(h_1 g(y)) \rho_c(h_1) \, db_1 \, dy,
$$

where $n$ is the winding number of $L$ around $F_1$. But the right-hand side of (45) is just Sengupta’s formula for the Wilson loop functional for the loop that winds $n$ times around the boundary of $F_0$ (i.e., the outer boundary of $G$), enclosing areas $a$ and $c$.

**Proof of Theorem 19.** We proceed by induction on the number $k$ of crossings. If $k = 0$, the result is Conjecture 16. Assume, then, that the result holds for loops with fewer than $k$ crossings and consider a loop $L$ with $k$ crossings. As in Lemma 20, we deform $L$ into a loop $L_n(a, c)$. By the Makeenko–Migdal equation, the variation of the Wilson loop functional will involve loops of the form $L_{1,j}(t)$ and $L_{2,j}(t)$, all of which have fewer than $k$ crossings. We need to verify (1) that $L_n(a, c)$ satisfies $|n| a < \text{area}(\Sigma)$ and (2) that $L_{1,j}(t)$ and $L_{2,j}(t)$ both satisfy the smallness assumption (44). If so, we may apply Theorem 17 to the loops $L_n(a, c)$ and our induction hypothesis to $L_{1,j}(t)$ and $L_{2,j}(t)$ and the argument is then the same as in the sphere case.

For Point (1), we note that the value of $n$ is (Lemma 20) the winding number of $L$ around $F_1$, while the value of $a$ is the limiting value of $a_1(t)$ (the area of $F_1$) as $t$ approaches 1. Now, on the one hand, the value of $a \cdot w$ for $L(t)$ is independent of $t$, by Theorem 9. On the other hand, $a \cdot w$ approaches the value $na$ as $t$ approaches 1, since $a_1(t) \to a$ and $a_j(t) \to 0$ for $j \geq 2$. Thus, $na = a \cdot w$. But from the definitions of $A$ and $|w|_{\max}$, we have $|a \cdot w| \leq A |w|_{\max}$ and thus

$$
|n| a = |a \cdot w| \leq A |w|_{\max} < \text{area}(\Sigma).
$$

For Point (2), we note that for the loops $L(t)$, the area of $F_0$ is always increasing (Proposition 10), meaning that $A(t)$ is decreasing. Thus, for all $i$ and $j$, we have

$$
A_{L_{i,j}(t)} \leq A_{L(t)} \leq A.
$$

Furthermore, since every subloop of $L_{i,j}$ is also a subloop of $L$, we see that $|w_{L_{i,j}}|_{\max} \leq |w_L|_{\max}$. Thus, we have

$$
A_{L(t)} |w_{L_{i,j}}|_{\max} \leq A |w_L|_{\max} < \text{area}(\Sigma).
$$

Having verified these two points, the proof now proceeds as in the sphere case.

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