Chiral soliton models, large $N_c$ consistency and the $\Theta^+$ exotic baryon

Thomas D. Cohen

Department of Physics, University of Maryland, College Park, MD 20742-4111

Predictions for a light collective $\Theta^+$ baryon state (with strangeness +1) based on the collective quantization of chiral soliton models are shown to be inconsistent with large $N_c$ QCD. The lightest strangeness +1 state to emerge from the analysis has an excitation energy which at large $N_c$ scales as $N_c^0$ while collective quantization is legitimate only for excitations which go to zero as $N_c \to \infty$. This inconsistency strongly suggests that predictions for $\Theta^+$ properties based on collective quantization of chiral solitons are not valid.

There has been considerable recent excitement in hadronic physics. Several experimental groups have announced the identification of a narrow baryon resonance with a strangeness of +1 (i.e. containing one excess strange antiquark) \[1\]. Such a state is manifestly exotic in the sense of the quark model—it cannot be a simple three-quark state. This discovery has prompted considerable theoretical interest. Much of the theory has been in the context of generalized quark models in which the new baryon is identified as a pentaquark \[2, 3, 4, 5, 6, 7, 8, 9, 10\]. Unfortunately, the nature of this analysis is highly model dependent—there is no obvious way to see how phenomenological quark models in which the new baryon is identified as a pentaquark \[2, 3, 4, 5, 6, 7, 8, 9, 10\]. Unfortunately, the nature of this analysis is highly model dependent—there is no obvious way to see how phenomenological quark models emerge from QCD—and thus probably should be regarded presently as somewhat speculative. One theoretical approach to the problem clearly stands out—the analysis based on the SU(3) chiral soliton model treated with collective quantization \[11, 12, 13, 14, 15\]. This analysis has three obvious virtues: i) The calculation predates the observation \[11, 12\]; ii) it made a strikingly accurate prediction of the mass \[11, 12\] and has predicted a narrow width \[12\] consistent with those presently observed \[10\]; and iii) although apparently based on a particular model—the chiral soliton model—the analysis is completely insensitive to the details of the model such as the profile function which emerges from the detailed dynamics.

This third point is particularly important. There has been considerable experience over the years with relations in chiral soliton models which are independent of the dynamical details going back nearly twenty years \[17\]. Typically such relations are exactly satisfied in the large $N_c$ limit of QCD; the relations are derivable directly from QCD—and thus probably should be regarded presently as somewhat speculative. One theoretical approach to the problem clearly stands out—the analysis based on the SU(3) chiral soliton model treated with collective quantization \[11, 12, 13, 14, 15\]. This analysis has three obvious virtues: i) The calculation predates the observation \[11, 12\]; ii) it made a strikingly accurate prediction of the mass \[11, 12\] and has predicted a narrow width \[12\] consistent with those presently observed \[10\]; and iii) although apparently based on a particular model—the chiral soliton model—the analysis is completely insensitive to the details of the model such as the profile function which emerges from the detailed dynamics.

Let us begin by briefly reviewing the essential aspects of the analysis of refs. \[11, 12, 13, 14, 15\]. The starting point is a treatment of SU(3) chiral soliton models which was developed in the mid-1980s \[25\]. In this approach one finds a classical static “hedgehog” configuration in an SU(2) subspace (the u-d subspace). The details of the profile are model dependent but the general structure of the theory is not. If one neglects SU(3) symmetry breaking effects then there are eight collective (rotational) variables which are then quantized semi-classically using an SU(3) generalization \[25\] of the usual SU(2) collective quantization scheme \[26\]. The collective Hamiltonian is given by

$$H^{\text{rot}} = \frac{1}{2I_1} \sum_{A=1}^{3} j_A'^2 + \frac{1}{2I_2} \sum_{A=4}^{7} j_A'^2,$$

where $I_1$ ($I_2$) is the moment of inertia within (out of) the SU(2) subspace and $j_A'$ are generators of SU(3) in a body-fixed (co-rotating) frame. Again, the numerical
values of the moments of inertia are model dependent but the structure is not. There is an additional quantization constraint

\[ J'_{c} = \frac{N_{c}B}{2\sqrt{3}} , \]  

where \( B \) is the baryon number.

The explicit factor of \( N_{c} \) in eq. \ref{eq:constraint} plays a central role in this paper and it is useful to understand its origin. In Skyrme type models it follows directly from the Witten-Wess-Zumino term (which topology fixes to be an integer that can be identified with \( N_{c} \)). It can also be easily understood at the quark level. In a body-fixed frame the baryon number is associated with the SU(2) sub-manifold. There is also a body-fixed hypercharge associated with this sub-manifold which is related to the SU(3) generator in the usual manner: \( Y' = -2J's/\sqrt{3} \).

There is a general relation relating the baryon number, hypercharge and strangeness at large \( N_{c} \) which is valid at arbitrary \( N_{c} \):

\[ Y = \frac{N_{c}B}{3} + S ; \tag{3} \]

this only coincides with the familiar relation \( Y = B + S \) for \( N_{c} = 3 \). Equation \ref{eq:constraint2} follows from the fact that the hypercharge of up, down and strange quarks as being \( 1/3, 1/3 \) and \(-2/3 \), respectively. (These are the standard hypercharges of quarks in an \( SU(3) \) world. These hypercharge assignments must hold for general \( N_{c} \) and is expressed in terms of labels \( p, q \) which denote the SU(3) representation. The quantization condition in eq. \ref{eq:constraint2} greatly restricts the possible SU(3) representations: only SU(3) representations which contain hypercharge equal to \( N_{c}/3 \) are allowed: if the hypercharge in a body-fixed frame satisfies eq. \ref{eq:constraint2}, the representation will include a state with that hypercharge. Moreover, since in the SU(2) manifold \( I = J \) and \( S = 0 \), it follows that the number of angular momentum states associated with a representation, \( (2J+1) \), must equal the number of states in the representation with \( S = 0 \) or equivalently with \( Y = N_{c}/3 \).

There is an ambiguity in how one implements this quantization. One might choose to quantize the theory at large \( N_{c} \) and then systematically put in \( 1/N_{c} \) corrections. Alternatively, in implementing the quantization condition of eq. \ref{eq:constraint2} one can fix \( N_{c} = 3 \) at the outset. To the extent that \( N_{c} = 3 \) can be considered large it ought not make any difference which of these approaches is used, provided that one is studying states which are not large \( N_{c} \) artifacts. Historically the choice of taking \( N_{c} = 3 \) at the outset has been standard. Making this choice, it is straightforward to see that the lowest-lying states in this treatment are:

\[
\begin{align*}
J &= 1/2 \quad (p, q) = (1, 1) \quad \text{(octet)} \\
J &= 3/2 \quad (p, q) = (3, 0) \quad \text{(decuplet)} \\
J &= 1/2 \quad (p, q) = (0, 3) \quad \text{(anti-decuplet).} \tag{5}
\end{align*}
\]

The decuplet and the anti-decuplet can then be seen to have mass splittings relative to the octet given by:

\[
\begin{align*}
M_{10} - M_{8} &= \frac{3}{2I_{1}} , \tag{6} \\
M_{10} - M_{8} &= \frac{3}{2I_{2}} . \tag{7}
\end{align*}
\]

The preceding analysis is a variant of quite standard 1980’s vintage soliton physics. Note that this standard analysis of SU(3) solitons is only justified in the large \( N_{c} \) limit which plays an essential role in two ways. It justifies the use of the classical static hedgehog configurations; effects of quantum fluctuations around the hedgehogs are suppressed by \( 1/N_{c} \). It also justifies the semi-classical treatment in collective quantization; coupling between the collective motion and the internal structure of the hedgehog is also suppressed by \( 1/N_{c} \). It should be clear from the previous comment, however, that the validity of the collective approach depends on restricting its application to quantum collective modes. In order to track the \( N_{c} \) counting of various expressions we note that the moments of inertia \( I_{1,2} \) scale as \( N_{c} \).

The regime of validity of collective motion is critical to the analysis here, so it is useful to specify what it is and where it comes from. The key point is that a collective description is valid only for motion which is slow compared to the vibrational modes which are of order \( N_{c}^{0} \). The vibrational modes are computed against a backdrop of a static soliton. This is valid providing the physical scale of the vibration is fast compared to the scale over which the soliton rotates. If this is not true one cannot separate the collective from the vibrational motion; in such a case the energy of the vibrational and collective
motion are not additive and, indeed, it is a misnomer to refer to it as “collective” motion. Now the characteristic time scale of some type of quantized collective motion is given by the typical quantum mechanical result $\tau \sim (\Delta E)^{-1}$, where $\Delta E$ is the splitting between two neighboring collective levels. Thus collective motion is valid only for motion for which $\Delta E$ goes to zero in the large $N_c$ limit.

Conventional treatments of collectively quantized SU(3) solitons identify the octet and decuplet states with the physical $N_c=3$ octets and decuplets familiar from baryon spectroscopy, while the anti-decuplet has been dismissed as a large $N_c$ artifact in much the same way that $I=J=5/2$ baryons are generally dismissed as artifacts in SU(2) soliton models. The principal intellectual argument of ref. [12] is that the anti-decuplet for SU(3) solitons can be distinguished from the $J=I=5/2$ baryons in an essential way: the $J=I=5/2$ baryon width would be predicted to be so wide with real world parameters that the state could not be observed. In contrast, the anti-decuplet state might be expected to be narrow owing to suppressed phase space associated with the increased mass of kaons relative to pions. The fact that at the end of the calculation the predicted width of the $\Theta^+$ is seen to be small is taken as a self-consistent justification of this approach.

Before proceeding further, a brief remark about the calculation in ref. [12] is in order. Much of the detailed analysis concerns implementing SU(3) symmetry breaking effects in the calculation and how to find the resulting parameters from data. For the present purposes, however, these are side issues. The central question of principle is whether the predicted collective anti-decuplet states are physical.

There is a very general argument why quantum number exotic collective states in chiral soliton models are expected to be spurious. A modern view of such models is that they encode the predictions of large $N_c$ QCD relating the spin and flavor dependence of various observables. The detailed numbers emerging from the models—the values of the masses, coupling constants and the like—are not reliable even at large $N_c$ but the relations between them are. It is precisely because the analysis of refs. 11, 12, 13, 14, 15 does not depend on dynamical details but merely on the structure of the collective quantization, that one might believe that it correctly encodes the underlying QCD physics. However, there is an alternative method to deduce the spin-flavor properties of large $N_c$ baryons in a model independent way via the use of consistency conditions in describing meson-baryon scattering. The results are well known: a contracted SU($2N_f$) symmetry emerges in the large $N_c$ limit. Baryon states fall into multiplets of SU($2N_f$) and the low-lying states in these multiplets are split from the ground state by energies of order $1/N_c$—these excitations with the SU($2N_f$) multiplets are collective. Moreover, the multiplet of low-lying baryons has been explicitly constructed—it coincides exactly with the low spin states of a quark model with $N_c$ quarks confined to a single s-wave orbital. Thus, it is well known that there are no low-lying collective baryon states in large $N_c$ QCD with quantum numbers which are exotic for the large $N_c$ world. In particular, there are no collective states with strangeness +1 in large $N_c$ QCD. Any model which predicts such a collective state appears to be inconsistent with large $N_c$ QCD.

This general argument strongly suggests that any strangeness +1 state predicted via collective quantization of a chiral soliton must somehow be spurious. Yet, at first glance, the derivation of eq. 7 appears to be based on standard chiral soliton analysis. The issue is what, if anything, is wrong with the analysis? The answer lies in the collective quantization. Although the collective quantization of SU(3) solitons along the lines of 21 is the standard for the field, apparently, there has never been a careful study of the conditions for which the approach is consistent with large $N_c$ QCD. As will be shown below, the approach appears to give excitations consistent with large $N_c$ QCD for the lowest-lying $J=3/2$ states but not for the exotic strangeness +1 states.

As stressed previously, the standard semi-classical treatment for collectively quantizing the solitons can only be justified in the large $N_c$ approximation. The analysis outlined above appears to respect the underlying large $N_c$ dynamics, at least formally. After all, the mass splitting in eq. 7 goes as $1/I_2 \sim 1/N_c$. Thus, in the large $N_c$ limit the splitting appears to become small which seems to imply that the motion is collective. The semi-classical quantization approach thereby looks to be justified self-consistently.

However, this is misleading: one can only see this collective clearly in the large $N_c$ limit of the theory. Recall, however, that eq. 7 was not derived in the large $N_c$ limit. Its derivation depended on implementing the quantization condition in eq. 4 with $N_c=3$ at the outset. It was suggested above that making such a choice was innocuous, and indeed it is, provided the states being studied are not artifacts. However, since the entire question of relevance here is whether the states are spurious, we cannot start by using eq. 7 to see if the motion is truly collective. Rather, one must study the full theory in its large $N_c$ limit to see whether the motion turns out to be collective.

There are well-known peculiarities in studying SU(3) baryons in the large $N_c$ limit. First and foremost among these is the fact that the SU(3) representations which emerge are not the ones we are familiar with at $N_c=3$; indeed, as $N_c \to \infty$ all of these SU(3) representations be-
come infinite dimensional. However, this presents no insurmountable problem phenomenologically, one simply associates those states in the representation with isospin and strangeness quantum numbers that survive down to the \(N_c = 3\) with their real world analogs. The highly successful phenomenological study by Jenkins and Lebed of baryon masses based on large \(N_c\) scaling and SU(3) symmetry and its breaking was based precisely on this approach.

Consider the implementation of eqs. (2) and (4) for \(N_c\) arbitrary and large. To ensure that our baryons remain fermions we restrict our attention to \(N_c\) odd. The lowest-lying representation compatible with eq. (2) is easily seen to be \((p, q) = (1, \frac{N_c - 3}{2})\) with \(J = 1/2\) and is represented by the Young tableau a) in fig. (1). The states in this representation include those in the usual octet (and are thus taken to be their large \(N_c\) generalization): for convenience this representation will be denoted “8”. The quotation marks serve to remind us that this is not really an octet. The next representation is \((p, q) = (3, \frac{N_c - 3}{2})\) with \(J = 3/2\); it is represented by the Young tableau b) in fig. (1) and is denoted by “10”.

Note that there is an explicit \(N_c\) dependence in eq. (6). The significant point, of course, eq. (9) coincides with eq. (7) for the special case of \(N_c = 3\). However, unlike eq. (7), eq. (9) allows us to study the \(N_c\) scaling of the predicted splitting. Note that there is an explicit \(N_c\) in the numerator of the right-hand side while the denominator is proportional to \(I_2\) which scales as \(N_c\). Thus, the scaling at large \(N_c\) is given by

\[
M_{\text{10}} - M_{\text{8}} = \frac{3 + N_c}{4I_2} .
\]

Of course, eq. (9) coincides with eq. (7) for the special case of \(N_c = 3\). However, unlike eq. (7), eq. (9) allows one to study the \(N_c\) scaling of the predicted splitting. Note that there is an explicit \(N_c\) in the numerator of the right-hand side while the denominator is proportional to \(I_2\) which scales as \(N_c\). Thus, the scaling at large \(N_c\) is given by

\[
M_{\text{10}} - M_{\text{8}} \sim N_c^0 .
\]

In the large \(N_c\) limit this splitting does not go to zero: the excitation is not collective. Note that the scaling in eq. (10) is generic for states in large \(N_c\) QCD which are quantum number exotic in the sense that their quantum numbers cannot be obtained from \(N_c\) valence quarks. It is noteworthy that the only states whose excitation energies are of order \(N_c^{-1}\) are those whose Young tableau contains exactly \(N_c\) boxes; these are precisely the one seen in the general model independent analysis of ref. [21].

Recall that the energy of the exotic \(\Theta^+\) was obtained using the collective quantization which is only valid for collective modes. However, as seen in eq. (10), it is used to predict an excitation which is clearly not collective—its excitation energy remains finite at large \(N_c\). Thus, the prediction of the low-lying \(\Theta^+\) state is based on using collective quantization outside its domain of validity.

Let us now revisit the argument in ref. [12] based on the predicted hadronic widths that the predicted anti-decuplet state should not be regarded as spurious. Note this argument distinguished between the widths of the predicted anti-decuplet and the J=5/2 states (which are generally regarded as large \(N_c\) artifacts). From the perspective of this paper, it should be clear that these two states are entirely different beasts. The J = 5/2 states are collective modes whose properties one can safely predict in a large \(N_c\) world. The sole issue for the predicted J = 5/2 states is whether they survive in extrapolating back from large \(N_c\) to the real world at \(N_c = 3\). In contrast, the strangeness +1 exotic states are not collective.

FIG. 1: Young tableau for arbitrary but large \(N_c\); a) the “8” representation with \((p, q) = (1, \frac{N_c - 1}{2})\); b) the “10” representation with \((p, q) = (3, \frac{N_c - 3}{2})\); c) the “10” representation with \((p, q) = (0, \frac{N_c - 3}{2})\). The young tableau in a) and b) have \(N_c\) boxes; the tableau in c) has \(N_c + 3\) boxes.
even in the large $N_c$ limit; treating them using collective quantization will give rise to spuriously low energy modes. In short, the $J = 5/2$ state is spurious because its prediction depends on taking the large $N_c$ limit too seriously, while the collective $\Theta^+$ state is spurious because its prediction depends on not taking the large $N_c$ limit seriously enough. Thus, although the reasons for which one regards the $J=5/2$ state as spurious do not apply to the anti-decuplet, the anti-decuplet is spurious for entirely different reasons.

In summary, the predicted $\Theta^+$ baryon in ref. [11, 12, 15] was obtained using collective quantization in a regime where collective quantization does not apply. It was shown that quantum number exotic states in large $N_c$ QCD have excitation energies which are of order $N_c^0$ and thus are not collective. Accordingly, the prediction of the $\Theta^+$ as a collective excitation should be regarded as being invalid; the fact that the predicted mass was so near to the observed mass must be regarded as fortuitous.

Of course, none of the arguments presented here indicate that chiral soliton models are intrinsically incapable of describing exotic states or indeed of doing a reasonable phenomenological job in describing the $\Theta^+$ baryon. However, if exotic states do exist in this class of models, they must be obtained by methods which are suitable to describe excitations of order $N_c^0$ rather than $N_c^{-1}$. Such methods do exist. For example one can use linear response theory to describe mesons scattering from baryons. In principle, an exotic $\Theta^+$ state could emerge in such a picture as a resonant state of a kaon and an ordinary baryon. However, there is no general argument that an exotic resonance would be generated for all such models and the excitation energy of such a state, if it exists, is completely model dependent. This does not imply that such an analysis is useless. One important aspect of large $N_c$ QCD is that it correlates predictions. In particular, the existence of one light strangeness +1 resonant state implies the existence of other strangeness +1 resonant states which differ in energy from it by of order $1/N_c$. While the arguments presented in this paper show why the order $N_c^0$ splitting between the ground state and the exotic are unreliable, the order $1/N_c$ splittings between exotic states are reliable. These predicted new states are explored in ref. [20].

The author acknowledges Rich Lebed for several insightful remarks about this work. The support of the U.S. Department of Energy for this research under grant DE-FG02-93ER-40762 is gratefully acknowledged.

---

[1] T. Nakano et al., Phys. Rev. Lett. 91, 012002 (2003); V.V. Barmin et al., hep-ex/0304040; S. Stepanyan et al., hep-ex/0307018; J. Barth et al., hep-ex/0307083.
[2] F. Stancu, D. O. Riska hep-ph/0307010; [3] S. Capstick, P. R. Page and W. Roberts, hep-ph/0307019.
[4] B. G. Wybourne, hep-ph/0307170; [5] A. Hosaka, hep-ph/0307232; [6] R. L. Jaffe and F. Wilczek, hep-ph/0307341.
[7] C. E. Carlson, C. D. Carone, H. J. Kwee and V. Nazaryan, hep-ph/0307306.
[8] K. Cheung, hep-ph/0308176; [9] L. Ya. Glozman, hep-ph/0308232.
[10] R. D. Mathews, F. S. Navarra, M. Nielsen, R. Rodrigues da Silva and S. H. Lee, hep-ph/0309001.
[11] M. Praszalowicz, talk at “Workshop on Skyrmions and Anomalies”, M. Jezabek and M. Praszalowicz editors, World Scientific 1987, page 112. This work was not published in a journal at that time. An updated analysis based on this work may be found in hep-ph/0308114.
[12] D. Diakonov, V. Petrov and M. Polyakov, Z. Phys. A 359, 305 (1997).
[13] H. Walliser and V. B. Kopeliovich, hep-ph/0304058.
[14] D. Borisuk, M. Faber and A. Kobushkin, hep-ph/0307370.
[15] H.-C. Kim, hep-ph/0308242.
[16] It has been argued, however, that the width's reported in the experiments may be broadened by experimental issues associated with resolution. Comparison with previous data seems to suggest that the actual width may be much narrower. This argument is detailed in S. Nussinov, hep-ph/0307357.
[17] G. S. Adkins and C. R. Nappi, Nucl. Phys. B249 (1985)507.
[18] J-L. Gervais and B. Sakita Phys. Rev. Lett. 52, 87 (1984); Phys. Rev. D 30, 1795 (1984).
[19] R.F. Dashen and A.V. Manohar, Phys. Lett. 315B, 425 (1993); Phys. Lett. 315B, 438 (1993).
[20] E. Jenkins, Phys. Lett. 315B, 441 (1993).
[21] R.F. Dashen, E. Jenkins, and A.V. Manohar, Phys. Rev. 49, 4713 (1994); Phys. Rev. D 51, 3697 (1995).
[22] T. D. Cohen and W. B. Broniowski, Phys. Lett. B292, 5 (1992).
[23] T. D. Cohen, Phys. Lett. B359, 23 (1995); Rev. Mod. Phys. 68, 609 (1995).
[24] T. D. Cohen and R. F. Lebed, Phys. Rev. Lett. 91, 012001 (2003); Phys. Rev. D 67, 096008 (2002); hep-ph/0306102.
[25] E. Guadagnini, Nucl. Phys. B236, 35 (1984); P. O. Mazur, M. A. Nowak and M. Praszalowicz, Phys. Lett. B147,137 (1984); A. V. Manohar, Nucl. Phys. B248, 19 (1984); M. Chemtob, Nucl. Phys. B256, 600 (1985); S. Jain and S. R. Wadia, Nucl. Phys. B258, 713 (1985).
[26] G. Adkins, C. Nappi and E. Witten, Nucl. Phys. B228, 522 (1983).
[27] T. D. Cohen and D. K. Griegel, Phys. Rev. D43, 3089 (1991).
[28] E. Jenkins and R. F. Lebed, Phys. Rev. D52, 282 (1995).
[29] M. Karliner and M. P. Mattis, Phys. Rev. Lett. 56 (1986).
428; Phys. Rev. D34 (1986) 1991.
[30] T. D. Cohen and R. F. Lebed, hep-ph/0309150