Extension of “Renormalization of period doubling in symmetric four-dimensional volume-preserving maps”

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Abstract

We numerically reexamine the scaling behavior of period doublings in four-dimensional volume-preserving maps in order to resolve a discrepancy between numerical results on scaling of the coupling parameter and the approximate renormalization results reported by Mao and Greene [Phys. Rev. A 35, 3911 (1987)]. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor associated with coupling and confirm the approximate renormalization results.

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Universal scaling behavior of period doubling has been found in area-preserving maps [1–7]. As a nonlinearity parameter is varied, an initially stable periodic orbit may lose its stability and give rise to the birth of a stable period-doubled orbit. An infinite sequence of such bifurcations accumulates at a finite parameter value and exhibits a universal limiting behavior. However these limiting scaling behaviors are different from those for the one-dimensional dissipative case [8].

An interesting question is whether the scaling results of area-preserving maps carry over higher-dimensional volume-preserving maps. Thus period doubling in four-dimensional (4D) volume-preserving maps has been much studied in recent years [7,9–13]. It has been found in Refs. [11–13] that the critical scaling behaviors of period doublings for two symmetrically coupled area-preserving maps are much richer than those for the uncoupled area-preserving case. There exist an infinite number of critical points in the space of the nonlinearity and coupling parameters. It has been numerically found in [11,12] that the critical behaviors at those critical points are characterized by two scaling factors, $\delta_1$ and $\delta_2$. The value of $\delta_1$ associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor $\delta (= 8.721\ldots)$ for the area-preserving maps. However the values of $\delta_2$ associated with scaling of the coupling parameter vary depending on the type of bifurcation routes to the critical points.

The numerical results [11,12] agree well with an approximate analytic renormalization results obtained by Mao and Greene [13], except for the zero-coupling case in which the two area-preserving maps become uncoupled. Using an approximate renormalization method including truncation, they found three relevant eigenvalues, $\delta_1 = 8.9474$, $\delta_2 = -4.4510$ and $\delta_3 = 1.8762$ for the zero-coupling case [14]. However they believed that the third one $\delta_3$ is an artifact of the truncation, because only two relevant eigenvalues $\delta_1$ and $\delta_2$ could be indentified with the scaling factors numerically found.

In this Brief Report we numerically study the critical behavior at the zero-coupling point in two symmetrically coupled area-preserving maps and resolve the discrepancy between the numerical results on the scaling of the coupling parameter and the approximate renormaliza-
tion results for the zero-coupling case. In order to see the fine structure of period doublings, we extend the simple one-term scaling law to a two-term scaling law. Thus we find a new scaling factor $\delta_3 = 1.8505\ldots$ associated with coupling, in addition to the previously known coupling scaling factor $\delta_2 = -4.4038\ldots$. The numerical values of $\delta_2$ and $\delta_3$ are close to the renormalization results of the relevant coupling eigenvalues $\delta_2$ and $\delta_3$. Consequently the fixed map governing the critical behavior at the zero-coupling point has two relevant coupling eigenvalues $\delta_2$ and $\delta_3$ associated with coupling perturbations, unlike the cases of other critical points.

Consider a 4D volume-preserving map $T$ consisting of two symmetrically coupled area-preserving Hénon maps \cite{11,12},

$$
T: \begin{cases}
    x_1(t+1) = -y_1(t) + f(x_1(t)) + g(x_1(t), x_2(t)), \\
    y_1(t+1) = x_1(t), \\
    x_2(t+1) = -y_2(t) + f(x_2(t)) + g(x_2(t), x_1(t)), \\
    y_2(t+1) = x_2(t),
\end{cases}
$$

(1)

where $t$ denotes a discrete time, $f$ is the nonlinear function of the uncoupled Hénon’s quadratic map \cite{15}, i.e.,

$$f(x) = 1 - ax^2,
$$

(2)

and $g(x_1, x_2)$ is a coupling function obeying a condition

$$g(x, x) = 0 \text{ for any } x.
$$

(3)

The two-coupled map (1) is called a symmetric map \cite{11,12} because it is invariant under an exchange of coordinates such that $x_1 \leftrightarrow x_2$ and $y_1 \leftrightarrow y_2$. The set of all points, which are invariant under the exchange of coordinates, forms a symmetry plane on which $x_1 = x_2$ and $y_1 = y_2$. An orbit is called an in-phase orbit if it lies on the symmetry plane, i.e., it satisfies

$$x_1(t) = x_2(t) \equiv x(t), \quad y_1(t) = y_2(t) \equiv y(t) \text{ for all } t.
$$

(4)
Otherwise it is called an out-of-phase orbit. Here we study only in-phase orbits. They can be easily found from the uncoupled Hénon map because the coupling function $g$ satisfies the condition (3).

Stability analysis of an in-phase orbit can be conveniently carried out in a set of new coordinates $(X_1, Y_1, X_2, Y_2)$ defined by

$$X_1 = \frac{(x_1 + x_2)}{2}, \quad Y_1 = \frac{(y_1 + y_2)}{2},$$

$$X_2 = \frac{(x_1 - x_2)}{2}, \quad Y_2 = \frac{(y_1 - y_2)}{2}.$$  \hfill (5a)

Note that the in-phase orbit of the map (1) becomes the orbit of the new map (expressed in terms of new coordinates) with $X_2 = Y_2 = 0$. Moreover the new coordinates $X_1$ and $Y_1$ of the in-phase orbit also satisfy the the uncoupled Hénon map.

Linearizing the new map at an in-phase orbit point, we obtain the Jacobian matrix $J$ which decomposes into two $2 \times 2$ matrices:

$$J = \begin{pmatrix} J_1 & 0 \\ 0 & J_2 \end{pmatrix}. \hfill (6)$$

Here $0$ is the $2 \times 2$ null matrix, and

$$J_1 = \begin{pmatrix} f'(X_1) & -1 \\ 1 & 0 \end{pmatrix},$$

$$J_2 = \begin{pmatrix} f'(X_1) - 2G(X_1) & -1 \\ 1 & 0 \end{pmatrix},$$

$$\hfill (7)$$

$$\hfill (8)$$

where $f'(X) = \frac{df}{dX}$ and $G(X) \equiv \frac{\partial g(X_1, X_2)}{\partial X_2} \bigg|_{X_1=X_2=X}$. Hereafter the function $G(X)$ will be called the “reduced” coupling function of $g(X_1, X_2)$. Note also that the determinant of each $2 \times 2$ matrix $J_i$ ($i = 1, 2$) is one, i.e., $Det(J_i) = 1$. Hence they are area-preserving maps.

Stability of an in-phase orbit with period $q$ is then determined from the $q$-product $M_i$ of the $2 \times 2$ matrix $J_i$:

$$M_i \equiv \prod_{t=0}^{q-1} J_i(X_1(t)), \quad i = 1, 2. \hfill (9)$$
Since $\det(M_i) = 1$, each matrix $M_i$ has a reciprocal pair of eigenvalues, $\lambda_i$ and $\lambda_i^{-1}$. Associate with a pair of eigenvalues $(\lambda_i, \lambda_i^{-1})$ a stability index

$$\rho_i = \lambda_i + \lambda_i^{-1}, \quad i = 1, 2,$$

which is just the trace of $M_i$, i.e., $\rho_i = Tr(M_i)$. Since $M_i$ is a real matrix, $\rho_i$ is always real. Note that the first stability index $\rho_1$ is just that for the case of the uncoupled Hénon map and hence coupling affects only the second stability index $\rho_2$.

An in-phase orbit is stable only when the moduli of its stability indices are less than or equal to two, i.e., $|\rho_i| \leq 2$ for $i = 1$ and $2$. A period-doubling (tangent) bifurcation occurs when each stability index $\rho_i$ decreases (increases) through $-2$ (2). Hence the stable region of the in-phase orbit in the parameter plane is bounded by four bifurcation lines associated with tangent and period-doubling bifurcations (i.e., those curves determined by the equations $\rho_i = \pm 2$ for $i = 0, 1$). When the stability index $\rho_1$ decreases through $-2$, the in-phase orbit loses its stability via in-phase period-doubling bifurcation and gives rise to the birth of the period-doubled in-phase orbit. Here we are interested in scaling behaviors of such in-phase period-doubling bifurcations.

As an example we consider a linearly-coupled case in which the coupling function is

$$g(x_1, x_2) = \frac{c}{2}(x_2 - x_1).$$

Here $c$ is a coupling parameter. As previously observed in Refs. [11,12], each “mother” stability region bifurcates into two “daughter” stability regions successively in the parameter plane. Thus the stable regions of in-phase orbits of period $2^n$ ($n = 0, 1, 2, \cdots$) form a “bifurcation” tree in the parameter plane [17].

An infinite sequence of connected stability branches (with increasing period) in the bifurcation tree is called a bifurcation “route” [11,12]. Each bifurcation route can be represented by its address, which is an infinite sequence of two symbols (e.g., $L$ and $R$). A “self-similar” bifurcation “path” in a bifurcation route is formed by following a sequence of parameters $(a_n, c_n)$, at which the in-phase orbit of level $n$ (period $2^n$) has some given stability indices.
$\rho_1, \rho_2$ (e.g., $\rho_1 = -2$ and $\rho_2 = 2$). All bifurcation paths within a bifurcation route converge to an accumulation point $(a^*, c^*)$, where the value of $a^*$ is always the same as that of the accumulation point for the area-preserving case (i.e., $a^* = 4.136\,166\,803\,904\,\ldots$), but the value of $c^*$ varies depending on the bifurcation routes. Thus each bifurcation route ends at a critical point $(a^*, c^*)$ in the parameter plane.

It has been numerically found that scaling behaviors near a critical point are characterized by two scaling factors, $\delta_1$ and $\delta_2$. The value of $\delta_1$ associated with scaling of the nonlinearity parameter is always the same as that of the scaling factor $\delta (= 8.721\ldots)$ for the area-preserving case. However the values of $\delta_2$ associated with scaling of the coupling parameter vary depending on the type of bifurcation routes. These numerical results agree well with analytic renormalization results, except for the case of one specific bifurcation route, called the $E$ route. The address of the $E$ route is $[(L, R,)^{\infty}] (= [L, R, L, R, \ldots])$ and it ends at the zero-coupling critical point $(a^*, 0)$.

Using an approximate renormalization method including truncation, Mao and Greene obtained three relevant eigenvalues, $\delta_1 = 8.9474$, $\delta_2 = -4.4510$, and $\delta_3 = 1.8762$ for the zero-coupling case; hereafter the two eigenvalues $\delta_2$ and $\delta_3$ associated with coupling will be called the coupling eigenvalues (CE’s). The two eigenvalues $\delta_1$ and $\delta_2$ are close to the numerical results of the nonlinearity-parameter scaling factor $\delta_1 (= 8.721\ldots)$ and the coupling-parameter scaling factor $\delta_2 (= -4.403\ldots)$ for the $E$ route. However they believed that the second relevant CE $\delta_3$ is an artifact of the truncation, because it could not be identified with anything obtained by a direct numerical method.

In order to resolve the discrepancy between the numerical results and the renormalization results for the zero-coupling case, we numerically reexamine the scaling behavior associated with coupling. Extending the simple one-term scaling law to a two-term scaling law, we find a new scaling factor $\delta_3 = 1.8505\ldots$ associated with coupling in addition to the previously found coupling scaling factor $\delta_2 = -4.4038\ldots$, as will be seen below. The values of these two coupling scaling factors are close to the renormalization results of the relevant CE’s $\delta_2$ and $\delta_3$. 
We follow the in-phase orbits of period \(2^n\) up to level \(n = 14\) in the \(E\) route and obtain a self-similar sequence of parameters \((a_n, c_n)\), at which the pair of stability indices, \((\rho_{0,n}, \rho_{1,n})\), of the orbit of level \(n\) is \((-2, 2)\). The scalar sequences \(\{a_n\}\) and \(\{c_n\}\) converge geometrically to their limit values, \(a^*\) and 0, respectively. In order to see their convergence, define
\[
\delta_n \equiv \frac{\Delta a_{n+1}}{\Delta a_n} + 1 \quad \text{and} \quad \mu_n \equiv \frac{\Delta c_{n+1}}{\Delta c_n},
\]
where \(\Delta a_n = a_n - a_{n-1}\) and \(\Delta c_n = c_n - c_{n-1}\). Then they converge to their limit values \(\delta\) and \(\mu\) as \(n \to \infty\), respectively. Hence the two sequences \(\{\Delta a_n\}\) and \(\{\Delta c_n\}\) obey one-term scaling laws asymptotically:
\[
\Delta a_n = C^{(a)} \delta^{-n}, \quad \Delta c_n = C^{(c)} \mu^{-n} \quad \text{for large} \ n,
\]
where \(C^{(a)}\) and \(C^{(c)}\) are some constants, \(\delta = 8.721 \cdots\), and \(\mu = -4.403 \cdots\). The values of \(\delta\) and \(\mu\) are close to the renormalization results of the first and second relevant eigenvalues \(\delta_1\) and \(\delta_2\), respectively.

In order to take into account the effect of the second relevant CE \(\delta_3\) on the scaling of the sequence \(\{\Delta c_n\}\), we extend the simple one-term scaling law to a two-term scaling law:
\[
\Delta c_n = C_{11} \mu_{11}^{-n} + C_{12} \mu_{12}^{-n} \quad \text{for large} \ n,
\]
where \(|\mu_1| > |\mu_2|\). This is a kind of multiple scaling law. Eq. (13) gives
\[
\Delta c_n = t_1 \Delta c_{n+1} - t_2 \Delta c_{n+2},
\]
where \(t_1 = \mu_1 + \mu_2\) and \(t_2 = \mu_1 \mu_2\). Then \(\mu_1\) and \(\mu_2\) are solutions of the following quadratic equation,
\[
\mu^2 - t_1 \mu + t_2 = 0.
\]
To evaluate \(\mu_1\) and \(\mu_2\), we first obtain \(t_1\) and \(t_2\) from \(\Delta c_n\)'s using Eq. (14):
\[
t_1 = \frac{\Delta c_n \Delta c_{n+1} - \Delta c_{n-1} \Delta c_{n+2}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}, \quad (16a)
\]
\[
t_2 = \frac{\Delta c_n^2 - \Delta c_{n+1} \Delta c_{n-1}}{\Delta c_{n+1}^2 - \Delta c_n \Delta c_{n+2}}. \quad (16b)
\]
Note that Eqs. (13)-(16) hold only for large $n$. In fact the values of $t_i$‘s and $\mu_i$‘s ($i = 1, 2$) depend on the level $n$. Therefore we explicitly denote $t_i$‘s and $\mu_i$‘s by $t_{i,n}$‘s and $\mu_{i,n}$‘s, respectively. Then each of them converges to a constant as $n \to \infty$:

$$\lim_{n \to \infty} t_{i,n} = t_i, \quad \lim_{n \to \infty} \mu_{i,n} = \mu_i, \quad i = 1, 2. \quad (17)$$

Three sequences $\{\mu_{1,n}\}$, $\{\mu_{2,n}\}$, and $\{\mu_{1,n}^2/\mu_{2,n}\}$ are shown in Table I. The second column shows rapid convergence of $\mu_{1,n}$ to its limit values $\mu_1 (=-4.403 897 805)$, which is close to the renormalization result of the first relevant CE (i.e., $\delta_2 = -4.4510$). From the third and fourth columns, we also find that the second scaling factor $\mu_2$ is given by a product of two relevant CE’s $\delta_2$ and $\delta_3$,

$$\mu_2 = \frac{\delta_2^2}{\delta_3}, \quad (18)$$

where $\delta_2 = \mu_1$ and $\delta_3 = 1.85065$. It has been known that every scaling factor in the multiple-scaling expansion of a parameter is expressed by a product of the eigenvalues of a linearized renormalization operator [18]. Note that the value of $\delta_3$ is close to the renormalization result of the second relevant CE (i.e., $\delta_3 = 1.8762$).

We now study the coupling effect on the second stability index $\rho_{2,n}$ of the in-phase orbit of period $2^n$ near the zero-coupling critical point ($a^*, 0$). Figure I shows three plots of $\rho_{2,n}(a^*, c)$ versus $c$ for $n = 4, 5, \text{ and } 6$. For $c = 0$, $\rho_{2,n}$ converges to a constant $\rho_2^* (= -2.543 510 20\ldots)$, called the critical stability index [12], as $n \to \infty$. However, when $c$ is non-zero $\rho_{2,n}$ diverges as $n \to \infty$, i.e., its slope $S_n \equiv \frac{\partial \rho_{2,n}}{\partial c} \bigg|_{(a^*, 0)}$ at the zero-coupling critical point diverges as $n \to \infty$.

The sequence $\{S_n\}$ obeys a two-term scaling law,

$$S_n = D_1 \nu_1^n + D_2 \nu_2^n \quad \text{for large } n, \quad (19)$$

where $|\nu_1| > |\nu_2|$. This equation gives

$$S_{n+2} = r_1 S_{n+1} - r_2 S_n, \quad (20)$$
where \( r_1 = \nu_1 + \nu_2 \) and \( r_2 = \nu_1 \nu_2 \). As in the scaling for the coupling parameter, we first obtain \( r_1 \) and \( r_2 \) of level \( n \) from \( S_n \)'s:

\[
\begin{align*}
\nu_1,n &= \frac{S_{n+1}S_n - S_{n+2}S_{n-1}}{S_n^2 - S_{n+1}S_{n-1}}, \\
\nu_2,n &= \frac{S_{n+1}^2 - S_nS_{n+2}}{S_n^2 - S_{n+1}S_{n-1}}.
\end{align*}
\]

(21)

Then the scaling factors \( \nu_1,n \) and \( \nu_2,n \) of level \( n \) are given by the roots of the quadratic equation, \( \nu_2^2 - r_1,n\nu_2 + r_2,n = 0 \). They are listed in Table II and converge to constants \( \nu_1 ( = -4.403 897 805 09) \) and \( \nu_2 ( = 1.850 535) \) as \( n \to \infty \), whose accuracies are higher than those of the coupling-parameter scaling factors. Note that the values of \( \nu_1 \) and \( \nu_2 \) are also close to the renormalization results of the two relevant CE’s \( \delta_2 \) and \( \delta_3 \).

We have also studied several other coupling cases with the coupling function, \( g(x_1, x_2) = c \left( x_2^n - x_1^n \right) \) (\( n \) is a positive integer). In all cases studied (\( n = 2, 3, 4, 5 \)), the scaling factors of both the coupling parameter \( c \) and the slope of the second stability index \( \rho_2 \) are found to be the same as those for the above linearly-coupled case (\( n = 1 \)) within numerical accuracy. Hence universality also seems to be well obeyed.

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TABLE I. Scaling factors $\mu_{1,n}$ and $\mu_{2,n}$ in the two-term scaling for the coupling parameter are shown in the second and third columns, respectively. A product of them, $\frac{\mu_{1,n}^2}{\mu_{2,n}}$, is shown in the fourth column.

| $n$ | $\mu_{1,n}$     | $\mu_{2,n}$     | $\frac{\mu_{1,n}^2}{\mu_{2,n}}$ |
|-----|----------------|----------------|---------------------------------|
| 5   | -4.403908128   | 10.4374        | 1.85817                         |
| 6   | -4.403899694   | 10.4659        | 1.85309                         |
| 7   | -4.403898736   | 10.4582        | 1.85446                         |
| 8   | -4.403897867   | 10.4748        | 1.85152                         |
| 9   | -4.403897847   | 10.4739        | 1.85168                         |
| 10  | -4.403897806   | 10.4784        | 1.85089                         |
| 11  | -4.403897807   | 10.4786        | 1.85085                         |
| 12  | -4.403897805   | 10.4797        | 1.85065                         |

TABLE II. Scaling factors $\nu_{1,n}$ and $\nu_{2,n}$ in the two-term scaling for the slope of the second stability index are shown.

| $n$ | $\nu_{1,n}$     | $\nu_{2,n}$     |
|-----|----------------|----------------|
| 5   | -4.40389845359 | 1.8514335     |
| 6   | -4.40389773029 | 1.8507826     |
| 7   | -4.40389781385 | 1.8506036     |
| 8   | -4.40389780407 | 1.8505538     |
| 9   | -4.40389780521 | 1.8505400     |
| 10  | -4.40389780507 | 1.8505361     |
| 11  | -4.40389780509 | 1.8505350     |
| 12  | -4.40389780509 | 1.8505349     |
FIGURES

FIG. 1. Plots of the second stability index $\rho_{2,n}(a^*, c)$ versus $c$ for $n = 4, 5, 6$. 