SPECTRAL LIMITS OF SEMICLASSICAL COMMUTING SELF-ADJOINT OPERATORS

ÁLVARO PELAYO, SAN VŨ NGỌC

Dedicated to Professor J. M. Montesinos Amilibia, with admiration.

Abstract. Using an abstract notion of semiclassical quantization for self-adjoint operators, we prove that the joint spectrum of a collection of commuting semiclassical self-adjoint operators converges to the classical spectrum given by the joint image of the principal symbols, in the semiclassical limit. This includes Berezin-Toeplitz quantization and certain cases of \( \hbar \)-pseudodifferential quantization, for instance when the symbols are uniformly bounded, and extends a result by L. Polterovich and the authors. In the last part of the paper we review the recent solution to the inverse problem for quantum integrable systems with periodic Hamiltonians, and explain how it also follows from the main result in this paper.

1. Introduction

In inverse spectral problems one tries to recover geometric (or “classical”) information from the spectrum of a “quantum” operator. For instance, does the spectrum of the Laplacian on a bounded euclidean domain completely determine the geometry of the domain? The problem goes back to S. Bochner and H. Weyl \[41, 42\] in the late nineteenth and early twentieth century, and was made popular, in the context of Riemannian geometry, in M. Kac’s famous article on “can you hear the shape of a drum”, \[24\], who attributes the origin of the question to Bochner.

In this paper we will deal with a quite general setting of semiclassical self-adjoint operators. Roughly speaking, a quantum operator will be a family of operators \( (T_h) \) depending on a small real parameter \( h > 0 \) reminiscent of the Planck constant. To each such operator, one defines its “classical limit” to be a smooth function on a smooth manifold (the phase space), called the principal symbol of the operator. The semiclassical inverse problem is then the following.

Question 1. (Semiclassical Inverse Spectral Problem) Given the semiclassical joint spectrum

\[
(X_h)_{h>0} \subset \mathbb{R}^d
\]

of a quantum system of commuting semiclassical operators

\[
T_1 := (T_{1,h})_{h>0}, \ldots, T_d := (T_{d,h})_{h>0},
\]

how much can one recover about the classical system given by the principal symbols \( f_1, \ldots, f_d \) of \( T_1, \ldots, T_d \)?
Of course, a complete answer to this question would be to fully recover the principal symbols $f_1, \ldots, f_d$ themselves.

One can hope to obtain such general results by combining the use of microlocal and symplectic techniques in the spirit of Duistermaat, Helffer, Hörmander, Sjöstrand, etc. However, to date only a handful of results are known in this direction, see [6, 7, 39, 5, 25, 26]. If one restricts the class of operators to be Schrödinger operators, whose principal symbol is of the form $\xi^2 + V(x)$, then the question amounts to recovering the potential $V$; this has attracted a lot of mathematicians, and is still an active area of research, see [23, 8, 21].

In this paper we prove a general result giving a partial answer to Question 1, inspired by previous works of Colin de Verdiere [6, 7], Polterovich, and the authors [31], which says that even though we do not know how to recover the principal symbols themselves, we can recover the closure of their joint image, which is a subset of the affine space $\mathbb{R}^d$. This gives a rigorous proof of the quantum mechanical principle that says that: “in the high frequency limit $\hbar \to 0$, the spectrum of a quantum system converges to the numerical range of its associated classical system”.

**Theorem 2.** Let $I \subset (0, 1]$ be a set with a limit point at 0. Then the limit set of the joint spectrum of a family of pairwise commuting self-adjoint semiclassical operators

$$T_1 := (T_{1,h})_{h \in I}, \ldots, T_d := (T_{d,h})_{h \in I}$$

is the classical spectrum $S \subset \mathbb{R}^d$ of $T_1, \ldots, T_d$, that is, the closure of the joint image of the principal symbols of $T_1, \ldots, T_d$.

An illustration of the convergence statement in Theorem 2 is depicted in Figure 1, which shows the joint spectrum of the “normalized” Quantum Spherical Pendulum.\(^1\)

In [31] an analogous statement was proved, but taking the convex full on both the quantum and the classical spectrum. We achieve this improvement by introducing a new hypothesis, which takes the form of the following seemingly simple axiom for the abstract semiclassical quantization: for any symbol $f$, one should have

$$\|\text{Op}_h(f)^2 - \text{Op}_h(f^2)\| = O(h).$$

where $\text{Op}_h$ denotes the quantization operation. We refer to Theorem 17 for a detailed version of the above statement, to Definition 7 for the abstract notion of semiclassical operators we use, and the upcoming sections for the necessary preliminaries. The abstract notion we use, and hence the theorem, apply to Berezin-Toeplitz operators on compact manifolds, and certain

\(^1\)Instead of the standard energy and momentum operators, the energy is replaced by its square root, in order to obtain symbols which are both (asymptotically) homogeneous of degree one.
Figure 1. The dots in the figures form the semiclassical joint spectrum of the “normalized” Quantum Spherical Pendulum for the values of the Planck constant: $\hbar = 0.7, 0.5, 0.3, 0.05, 0.02$. As $\hbar \to 0$, the semiclassical joint spectrum fills the inside of the red curve, which is the boundary of the classical spectrum of the system; this gives an illustration of the convergence stated in Theorem 2.

classes of pseudodifferential operators (this is explained in Remark 11), for instance those with uniformly bounded derivatives.

As we will explain, Theorem 17 implies, in combination with a theorem of Atiyah-Guillemin-Sternberg and Delzant, a solution to the inverse problem for quantum toric integrable systems, which recovers a recent result of Charles and the authors [5]. This result was proved again shortly after by Polterovich and the authors [31] with a different method, which is in fact the one which serves as inspiration for Theorem 17 in combination with ideas introduced by Le Floch and the authors in [20].

2. Semiclassical operators and an abstract semiclassical quantization

We review Berezin-Toeplitz quantization, $h$-pseudodifferential quantization, and then introduce an abstract notion of semiclassical quantization which includes the former, and certain classes of the latter. This abstract notion is inspired by, and extends, a notion introduced by Polterovich and the authors [31] and as we will see in Section 4 it allows us to prove a stronger convergence result in certain cases.

2.1. Berezin-Toeplitz operators. The microlocal analysis of Toeplitz operators is rapidly evolving nowadays, see for instance [3, 9, 10, 11, 12, 28, 35] following the pioneer work of Boutet de Movel and Guillemin [4].

Let us recall the basic facts we need on connections of Hermitian line bundles (a good reference for this material are Duistermaat’s notes [17]). With the help of these facts we will introduce a fundamental notion in both geometry and analysis, that of a prequantum line bundle.

Let $M$ be a smooth manifold. Let $\mathcal{L} \to M$ be a Hermitian line bundle over $M$. That is, $\mathcal{L} \to M$ is a complex line bundle over $M$ which is endowed with a Hermitian metric. Denote by $C^\infty(M, \mathcal{L})$ the space of smooth sections of this bundle, and by $\Omega^1(M, \mathcal{L})$ the space of smooth $\mathcal{L}$-valued 1-forms. A connection of $\mathcal{L}$ is a linear operator $\nabla : C^\infty(M, \mathcal{L}) \to \Omega^1(M, \mathcal{L})$ which
satisfies Leibniz’s rule, that is,
\[ \nabla(fs) = df \otimes s + f \nabla s \]
for all smooth functions \( f \in C^\infty(M) \) and all smooth sections \( s \in C^\infty(M, \mathcal{L}) \).

Let \( X \) be a smooth vector field on \( M \). The covariant derivative of the section \( s \) of \( \mathcal{L} \) with respect to the vector field \( X \) is given by the formula \( \nabla_X s = \nabla s(X) \). The smooth 2-form \( R \) of \( M \) defined by the equation
\[ R(X, Y) = [\nabla_X, \nabla_Y] - \nabla_{[X,Y]} \]
for any vector fields \( X, Y \) of \( M \) is called the curvature of the connection.

Assume that \( \frac{1}{i} \omega \) is the curvature of a Hermitian line bundle connection. Then the cohomology class of the form \( \omega/2\pi \) is integral, that it, it lies in the image of the canonical homomorphism
\[ H^2(M, \mathbb{Z}) \to H^2(M, \mathbb{R}). \]
Conversely, for any smooth 2-form \( \omega \in \Omega^2(M, \mathbb{R}) \) for which \( [\omega]/2\pi \) is integral there is a Hermitian line bundle \( \mathcal{L} \to M \) endowed with a connection \( \nabla \) whose curvature is \( \frac{1}{i} \omega \). Moreover, the line bundle \( \mathcal{L} \) and \( \nabla \) are unique up to isomorphisms. For a proof of these results, we refer the reader to [17, Theorem 10.1] or [16, Section 15.3]. Now we are ready to recall the following essential definition.

**Definition 3.** Let \((M, \omega)\) be a symplectic manifold.

- A prequantum bundle on \( M \) is a Hermitian line bundle \( \mathcal{L} \to M \) with a connection of curvature \( \frac{1}{i} \omega \).
  In this case we say that the symplectic manifold \((M, \omega)\) is prequantizable.
- A prequantum bundle automorphism is a vector bundle automorphism of \( \mathcal{L} \to M \) which preserves both the metric and the connection.

Let us now consider a prequantum line bundle \( \mathcal{L} \to M \) over the symplectic manifold \( M \). Let \( G \) be a Lie group with Lie algebra \( \mathfrak{g} \). Assume that \( G \) acts on \( \mathcal{L} \) by prequantum bundle automorphisms, as defined above. This \( G \)-action lifts a \( G \)-action on \( M \). For \( X \in \mathfrak{g} \) let \( X^\sharp \) denote the infinitesimal action of \( X \) on \( M \). The latter \( G \)-action is Hamiltonian with momentum map \( F \) given by the following condition: the action induced by \( \mathfrak{g} \) on \( C^\infty (M, \mathcal{L}) \) is given by the Kostant-Souriau operators
\[ f \mapsto \nabla_{X^\sharp} f + i(F, X)f, \quad X \in \mathfrak{g}. \]
and \( \nabla \) denotes the covariant derivative of the prequantum bundle (cf. [16, Proposition 15.2]). If both the Lie group \( G \) and the manifold \( M \) are connected, then the \( G \)-action on \( \mathcal{L} \) is conversely determined by the action on \( M \) and by the momentum map \( F \). It is important to notice that we cannot obtain every momentum map generating a given action in this manner (such momentum maps correspond to the Lie algebra representations on the prequantum bundle by means of (1)).

Suppose that \((M, \omega)\) is a prequantizable closed (that is both compact and with no boundary) symplectic manifold. Let \( \mathcal{L} \to M \) be a prequantum line bundle, and assume that \( M \) admits a complex structure that is compatible with the symplectic form \( \omega \). In fact, \((M, \omega)\) is a Kähler manifold. The holomorphic structure of \( \mathcal{L} \to M \) is uniquely determined by the compatibility condition with the connection.

Consider a positive integer \( k = 1/h \) and let \( \mathcal{L}^k \) denote the \( k \)th tensor power of the line bundle \( \mathcal{L} \). We write

\[ \mathcal{H}_h := H^0(M, \mathcal{L}^k) \]

for the space of holomorphic sections of \( \mathcal{L}^k \). The space \( \mathcal{H}_h \) is a finite dimensional subspace of the Hilbert space \( L^2(M, \mathcal{L}^k) \) (this is because \( M \) is a closed manifold). If \( \lambda \) denotes the Liouville measure of \( M \), the scalar product is given by integration of the Hermitian (pointwise) scalar product of sections against \( \lambda \).

**Definition 4.** Let \( \Pi_h \) be the surjective orthogonal projector

\[ L^2(M, \mathcal{L}^k) \to \mathcal{H}_h. \]

- A semiclassical Berezin-Toeplitz operator is any sequence of the form

\[ T := (T_h := \Pi_h f(\cdot, k) : \mathcal{H}_h \to \mathcal{H}_h)_{h=1/k, k \in \mathbb{N}^*} \]

where the multiplication operator \( f(\cdot, k) \) is a sequence in \( C^\infty(M) \) with an asymptotic expansion

\[ f_0 + k^{-1}f_1 + k^{-2}f_2 + \cdots \]

for the \( C^\infty \) topology.

- The first coefficient \( f_0 \) is called the principal symbol of \( T \).

2.2. **Pseudodifferential operators.** \( h \) pseudodifferential operators acting on the Hilbert space \( \mathcal{H} = L^2(\mathbb{R}^n) \) give a semiclassical quantization of the manifold \( M = \mathbb{R}^{2n} \); this is a semiclassical version of the one given by homogeneous pseudodifferential operators, see for instance [14] or [13].

Let \( \mathcal{A}_0 \) be the Hörmander class whose elements are the functions \( f \) in the space \( C^\infty(\mathbb{R}^{2n}_{(x, \xi)}) \) such that the following holds: there is \( m \in \mathbb{R} \) for which

\[ |\partial^\alpha_{(x, \xi)} f| \leq C_\alpha \langle (x, \xi) \rangle^m \]
for every $\alpha \in \mathbb{N}^{2n}$ (here the notation $\langle z \rangle$ stands for $(1 + |z|^2)^{1/2}$). Symbolic calculus for pseudodifferential operators is known to hold in the case when the symbols of the operators are in $\mathcal{A}_0$ (for instance).

**Definition 5.** Let $f \in \mathcal{A}_0$. The Weyl quantization of $f$ is given on the Schwartz space $\mathcal{S}(\mathbb{R}^n)$ by the expression:

$$(\operatorname{Op}_\hbar(f)u)(x) := \frac{1}{(2\pi \hbar)^n} \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} e^{i \langle x-y, \xi \rangle} f\left(\frac{x+y}{2}, \xi\right) u(y) \, dy \, d\xi.$$ 

This definition is commonly used to define $\hbar$-pseudodifferential operators on $\mathbb{R}^n$. It can also give a semiclassical quantization of a cotangent bundle $M = T^*X$, where $X$ is a smooth $n$-dimensional closed manifold with a smooth density, as follows. Let $X$ be covered by a collection of smooth charts $\{U_1, \ldots, U_N\}$, where each $U_i$ is over a convex bounded domain of the Euclidean space $\mathbb{R}^n$ (equipped with the Lebesgue measure). By standard manifold theory, there exists a partition of unity $\chi_1^2, \ldots, \chi_N^2$ which is subordinated to the cover $\{U_1, \ldots, U_N\}$. In this case, $\mathcal{A}_0$ is the space of functions $f \in C^\infty(T^*X)$ satisfying, for all $(x, \xi) \in T^*X$, $\alpha \in \mathbb{N}^n$, and for some $m \in \mathbb{R}$, the condition:

$$\langle 3 \rangle \quad \left| \partial_x^\alpha \partial_\xi^\beta f(x, \xi) \right| \leq C_{\alpha}(\xi)^m |\beta|$$

Recall Definition 5 and let $\operatorname{Op}_\hbar^j(f)$ be the Weyl quantization in $U_j$. Define:

$$\operatorname{Op}_\hbar(f)u := \sum_{j=1}^N \chi_j \cdot \operatorname{Op}_\hbar^j(f)(\chi_j u), \quad u \in C^\infty(X),$$

which is a pseudodifferential operator on $X$. The principal symbol of this operator is the smooth function

$$f := \sum_{i=1}^N f \chi_j^2.$$

**Definition 6.** Let $X$ be either $\mathbb{R}^n$, or a closed manifold, as above. Let $(f_\hbar)_{\hbar \in (0, 1]}$ be a family of elements of $\mathcal{A}_0$ such that the estimate $\langle 2 \rangle$ (in the case $X = \mathbb{R}^n$) or $\langle 3 \rangle$ (if $X$ is a closed manifold) holds uniformly for $\hbar \in (0, 1]$. Then the family

$$T := (\operatorname{Op}_\hbar(f_\hbar))_{\hbar \in (0, 1]}$$

is called a semiclassical $\hbar$-pseudodifferential operator on $X$.

The above definition may also be made for a subset $I \subset (0, 1]$ which has a limit point at 0, for instance

$$I = \left\{ \frac{1}{k} \mid k \in \mathbb{N}^* \right\}.$$
2.3. **Abstract semiclassical quantization.** The results presented in this paper hold for both pseudodifferential and Berezin-Toeplitz quantization. In fact, they only require a few key properties, and it is interesting to state them in an abstract way, as follows.

Let $I \subset (0, 1]$ be a set that accumulates at 0. Suppose that $M$ is a connected manifold (closed or open) and let $\mathcal{A}_0$ be a subalgebra of the algebra of smooth functions $C^\infty(M; \mathbb{R})$ containing all constants as well as all compactly supported functions.

For a complex Hilbert space $\mathcal{H}$ we denote by $L(\mathcal{H})$ the set of all linear self-adjoint operators on $\mathcal{H}$ (bounded or unbounded). The following definition is essentially the same as in [31] with the exception of the new Axiom (Q5), which is needed for the proof of our main result.

**Definition 7.** A semiclassical quantization of the pair $(M, \mathcal{A}_0)$ is given by:

- a family of complex Hilbert spaces $\mathcal{H}_\hbar$, $\hbar \in I$, and
- a family of $\mathbb{R}$-linear maps $\text{Op}_\hbar : \mathcal{A}_0 \to L(\mathcal{H}_\hbar)$,

that satisfy the following axioms, where $f, g \in \mathcal{A}_0$:

(Q1) $\|\text{Op}_\hbar(1) - \text{Id}\| = O(\hbar)$ (normalization);
(Q2) for every function $f \geq 0$ there is a constant $C_f$ for which $\text{Op}_\hbar(f) \geq -C_f\hbar$ (quasi-positivity);
(Q3) if $f \in \mathcal{A}_0$ is such that $f \neq 0$ and also has compact support, then

$$\lim_{\hbar \to 0} \inf_{h} \|\text{Op}_\hbar(f)\| > 0$$

(non-degeneracy);
(Q4) if $g$ has compact support, then the operator $\text{Op}_\hbar(f) \circ \text{Op}_\hbar(g)$ is bounded for every $f$, and

$$\|\text{Op}_\hbar(f) \circ \text{Op}_\hbar(g) - \text{Op}_\hbar(fg)\| = O(\hbar),$$

(product formula);
(Q5) if $f \in \mathcal{A}_0$, then

$$\|\text{Op}_\hbar(f)^2 - \text{Op}_\hbar(f^2)\| = O(\hbar),$$

(square formula).

We say that a manifold is quantizable if it has a semiclassical quantization.

Let $\mathcal{A}_I$ be the algebra whose elements are collections $\vec{f} = (f_h)_{h \in I}$, $f_h \in \mathcal{A}_0$, that satisfy that for each $\vec{f}$ there is $f_0 \in \mathcal{A}_0$ such that

$$(4) \quad f_h = f_0 + hf_{1,h},$$

where $f_{1,h}$ is uniformly bounded in the parameter $h$ as well as supported in the same compact subset $K(\vec{f})$ of $M$.

**Definition 8.** A semiclassical operator is an element in the image of the map

$$\text{Op} : \mathcal{A}_I \to \prod_{h \in I} L(\mathcal{H}_h)$$
defined by 
\[ \vec{f} = (f_h) \mapsto (\text{Op}_h(f_h)) \].

**Definition 9.** Let \( \vec{f} \in A_I \). The function \( f_0 \in A_0 \) given by (4) is called the principal symbol of \( \text{Op}(\vec{f}) \).

It follows from the axioms in Definition 8 that the principal symbol in Definition 9 is uniquely defined, see [31].

Notice that in the definition of semiclassical operators, the manifold \( M \) is not required to be symplectic. The following proposition gives the two major examples of semiclassical operators, for which the phase space \( M \) is symplectic.

**Proposition 10.**
1. Semiclassical Berezin-Toeplitz operators satisfy the axioms (Q1–Q5).
2. Semiclassical pseudodifferential operators which mildly depends on \( \hbar \) satisfy the axioms (Q1–Q4). Here we say that \( f_\hbar \in A_I \) mildly depends on \( \hbar \) if \( f_\hbar \) can be written 
\[ f_\hbar(x,\xi) = f_0(x,\xi) + \hbar f_1(x,\xi), \]
where all \( f_1,\hbar(x,\xi) \) are both uniformly bounded in \( \hbar \) as well as compactly supported in the same set.
3. Semiclassical pseudodifferential operators which are uniformly bounded satisfy the axiom (Q5). More precisely, by uniformly bounded we mean that for any \( f_\hbar \in A_I \), and every \( \alpha \in \mathbb{N}^{2n} \), there is a constant \( C_\alpha \) such that 
\[ |\partial_\alpha (x,\xi) f_\hbar(x,\xi)| \leq C_\alpha \quad \forall \hbar \in (0,1]. \]

A proof of this can be found in most introductory papers or books on the subject. For instance, for Berezin-Toeplitz operators, one can refer to [3, 9, 10, 11, 12, 28, 35], and for pseudodifferential operators to the books [14] or [43]. Here we do not claim to have optimal hypothesis. For instance, the assumption on mild dependence on \( \hbar \) can certainly be weakened.

**Remark 11.** For pseudodifferential operators, axiom (Q5) (square formula) is more restrictive than the others; it does not hold for all classes of symbols. In order to use the results for differential operators like the Laplacian, one would need first a microlocalization estimate in order to truncate the original operators and hence transform them into uniformly bounded pseudodifferential operators. Such a truncation procedure is common in microlocal analysis (see for instance [14, Chapter 10]).

The following lemma is a consequence of the axioms (Q1–Q4):

**Lemma 12 ([31 Lemma 11]).** Take any \( \vec{f} = (f_h) \in A_I \) with principal part \( f_0 \), and let \( (\text{Op}_h(f_0)) \) be the corresponding semiclassical operator. Let \( \lambda_{\text{inf}}(h) \in [-\infty, +\infty) \) denote the infimum of the spectrum of \( \text{Op}_h(f_0) \). Then
\[ \lim_{h \to 0} \lambda_{\text{inf}}(h) = \inf_M f_0. \]
3. Joint Spectrum of a Family of Semiclassical Commuting Operators

We recall that to any self-adjoint operator $A$ on a Hilbert space, the spectral theorem associates a projector-valued measure $\mu_A$, called the spectral measure, such that

$$A = \int_\mathbb{R} td(\mu_A)(t),$$

and whose support is the spectrum of $A$. A similar theory holds for commuting operators. The self-adjoint operators $S_1, \ldots, S_d$ are said to be mutually commuting if their corresponding spectral measures $\mu_1, \ldots, \mu_d$ pairwise commute. In this case we may then define the joint spectral measure

$$\mu := \mu_1 \otimes \cdots \otimes \mu_d$$

on $\mathbb{R}^d$. The joint spectrum of $(S_1, \ldots, S_d)$ is the support of the joint spectral measure, that is:

$$c \in \text{JointSpec}(S_1, \ldots, S_d) \iff \forall \epsilon > 0, \quad \mu_1([c_1 - \epsilon, c_1 + \epsilon]) \circ \cdots \circ \mu_d([c_d - \epsilon, c_d + \epsilon]) \neq 0.$$

In this paper we are interested in the joint spectrum of semiclassical operators, which is defined as follows. For $j \in \{1, \ldots, d\}$ let

$$T_j = (T_{j,h})_{h \in I}$$

be semiclassical operators (as in Definitions 4 or 6) on Hilbert spaces $(H_h)_{h \in I}$. We assume that for any fixed $h \in I$, the self-adjoint operators $T_{1,h}, \ldots, T_{d,h}$ are mutually commuting. For fixed $h$, the joint spectrum of $(T_{1,h}, \ldots, T_{d,h})$ is as before the support of the joint spectral measure. For instance, if $H_h$ is finite dimensional (e.g. in the case of Berezin-Toeplitz quantization on a closed Kähler manifold), then

$$\text{JointSpec}(T_{1,h}, \ldots, T_{d,h})$$

is the set

$$\{ (\lambda_1, \ldots, \lambda_d) \in \mathbb{R}^d \mid \exists v \neq 0, \quad T_{j,h}v = \lambda_j v, \forall j = 1, \ldots, d \}.$$ 

We define the joint spectrum of the semiclassical operators $(T_1, \ldots, T_d)$ to be the collection of all joint spectra of $(T_{1,h}, \ldots, T_{d,h}), h \in I$.

4. The Inverse Problem for Commuting Operators

4.1. Convergence to classical spectrum. Following the physicists, we use the following definition.

**Definition 13.** We call classical spectrum of $(T_1, \ldots, T_d)$ the closure of the image $F(M) \subset \mathbb{R}^d$, where $F = (f_1, \ldots, f_d)$ is the map of principal symbols of $T_1, \ldots, T_d$.

In order to state the convergence results for the semiclassical spectrum of a collection of operators, we need to use a notion of limit for subsets of $\mathbb{R}^n$. 
Definition 14. Let \((A_h)_{h \in I}\) be a family of subsets of \(\mathbb{R}^n\), where \(I \subset (0, 1]\) is a set which accumulates at 0. The limit set of \((A_h)_{h \in I}\) is the subset \(A_0 \subset \mathbb{R}^n\) defined by
\[
a \in A_0 \iff \forall \epsilon > 0, \exists h_0 \in I, \exists h \leq h_0, \text{ such that } A_h \cap B(a, \epsilon) \neq \emptyset.
\]
Here \(B(a, \epsilon)\) is the euclidean ball around \(a\) of radius \(\epsilon\).

In the case of uniformly bounded subsets of \(\mathbb{R}^n\), the limit set is in fact a limit in the sense of the Hausdorff distance, which we recall now. Let \(\| \cdot \|\) be the euclidean norm in \(\mathbb{R}^n\). For any \(\epsilon > 0\) and any subset \(X\) of \(\mathbb{R}^n\), we denote by \(X_{\epsilon}\) the set
\[
\bigcup_{x \in X} \left\{ m \in \mathbb{R}^n \mid \|x - m\| \leq \epsilon \right\}.
\]
The Hausdorff distance between two subsets \(A\) and \(B\) of \(\mathbb{R}^n\) is the number
\[
\inf \left\{ \epsilon > 0 \mid A \subseteq B_{\epsilon} \text{ and } B \subseteq A_{\epsilon} \right\}.
\]
We denote it by \(d_H(A, B)\).

Definition 15. Let \((A_h)_{h \in I}\) and \((B_h)_{h \in I}\) be families of uniformly bounded subsets of \(\mathbb{R}^n\), where \(I \subset (0, 1]\) is a set which accumulates at 0.

- Fix \(N \in \mathbb{N}\). We say that
  \[
  A_h = B_h + O(h^N)
  \]
  if there exists a constant \(C > 0\) such that \(d_H(A_h, B_h) \leq C h^N\) for every \(h \in I\).

- We say that
  \[
  A_h = B_h + O(h^\infty)
  \]
  if \(d_H(A_h, B_h) = O(h^N)\) for every \(N \in \mathbb{N}\).

- Let \(A_0 \subset \mathbb{R}^n\). We say that \(A_0\) is a Hausdorff limit of \((A_h)_{h \in I}\) if
  \[
  \lim_{h \to 0} d_H(A_h, A_0) = 0.
  \]

Remark 16. Let \((A_h)_{h \in I}\) be a family of uniformly bounded subsets of \(\mathbb{R}^n\). The limit set of \((A_h)_{h \in I}\) is always a compact subset \(A_0 \subset \mathbb{R}^n\), and then \(A_0\) is a Hausdorff limit of \((A_h)_{h \in I}\). Conversely, if a compact set \(A_0\) is a Hausdorff limit of \((A_h)_{h \in I}\), then it coincides with the limit set of \((A_h)_{h \in I}\).

The following is the main theorem of this paper, which in some cases strengthens previously known theorems (for instance in the case of Berezin-Toeplitz operators, and certain classes of pseudodifferential operators).

Theorem 17. Let \((T_1, \ldots, T_d)\) be a family of pairwise commuting self-adjoint semiclassical operators in the sense of Definition 2.3. Then the limit set of the joint spectrum of \((T_1, \ldots, T_d)\) is the classical spectrum of \((T_1, \ldots, T_d)\).
Figure 2. The figure depicts the semiclassical joint spectrum of the Quantum Spherical Pendulum for the following values of the Planck constant: $\hbar = 0.7, 0.5, 0.3, 0.05, 0.02$. As $\hbar \to 0$, the semiclassical joint spectrum fills the inside of the red curve, which is the boundary of the classical spectrum of the system; this gives an illustration of the convergence stated in Theorem 17. Notice that, in this figure, the joint spectrum is quite well behaved, because the operators form a completely integrable system. In this case, the joint spectrum is locally diffeomorphic to a lattice, as predicted by the Bohr-Sommerfeld rules, see [40], and in fact much more than the classical spectrum can be recovered from the joint spectrum; see for instance [5, 26].

Proof. Let $\mu_j$ be the spectral measure of $T_j$, and let $\mu = \mu_1 \otimes \cdots \otimes \mu_d$ be the joint spectral measure of $(T_1, \ldots, T_d)$ on $\mathbb{R}^d$. For any $c = (c_1, \ldots, c_d) \in \mathbb{R}^d$, let $\varphi_c : \mathbb{R}^d \to \mathbb{R}$ be defined by

$$\varphi_c(x_1, \ldots, x_d) := \sum_{i=1}^d (x_i - c_i)^2.$$ 

Then

$$\varphi_c(T_1, \ldots, T_d) = \int_{\mathbb{R}^d} \varphi_c(x) d\mu(x) = \int_{\mathbb{R}} td(\varphi_c)_* \mu(t).$$ 

The last equality implies that the spectrum of $\varphi_c(T_1, \ldots, T_d)$ is the support of $(\varphi_c)_* \mu$.

Now from Axioms (Q1) and (Q5), the principal symbol of $\varphi_c(T_1, \ldots, T_d)$ is

$$\varphi_c(f_1, \ldots, f_d) = \|F - c\|^2$$

where $f_i$ is the principal symbol of $T_i$. We see that $c \in \overline{F(M)}$ if and only if $\inf \varphi_c(f_1, \ldots, f_d) = 0$. By Lemma 12

$$\inf_M \varphi_c(f_1, \ldots, f_d) = \lim_{\hbar \to 0} \inf \text{Spec}(\varphi_c(T_1, \ldots, T_d)).$$

We have $\text{Spec}(\varphi_c(T_1, \ldots, T_d)) = \text{supp}((\varphi_c)_* \mu)$. Since $\varphi_c$ is continuous,

$$\text{supp}((\varphi_c)_* \mu) = \varphi_c(\text{supp}(\mu)).$$

Assume that $c$ is not in the limit set of the joint spectrum of $(T_1, \ldots, T_d)$. Thus there is a small ball around $c$ which is disjoint from $\text{JointSpec}(T_1, \ldots, T_d)$
for $\hbar$ small enough, which implies that there is some constant $\epsilon > 0$ such that

$$\inf(\varphi_c(\text{JointSpec}(T_1, \ldots, T_d))) > \epsilon.$$ 

Since $\text{JointSpec}(T_1, \ldots, T_d) = \text{supp}(\mu)$, we get in view of (7) that

$$\inf(\text{supp}(\varphi_c^\ast \mu)) \geq \epsilon.$$ 

Therefore, by Equation (6), we get that

$$\inf_M \varphi_c(f_1, \ldots, f_d) \geq \epsilon.$$ 

Hence $c \not\in \overline{F(M)}$, which says that $\overline{F(M)}$ is contained in the limit set of the joint spectrum.

In fact, all converse implications hold true, which proves the reverse inclusion and hence the theorem.

Theorem 17 shows that the classical spectrum can be recovered from the quantum joint spectrum. In the case of Berezin-Toeplitz operators, this generalizes a result of [31], where the convexity of the classical spectrum was required. For classes of pseudodifferential operators for which Axiom (Q5) does not hold, we cannot apply Theorem 17; however, the convex case holds, as we recall below.

**Remark 18.** We want to emphasize that axiom (Q5), while seemingly simple and quite close indeed to axiom (Q4), gives in fact a great advantage in the form of a rudimentary (polynomial) functional calculus. We conjecture that the strong conclusion of Theorem 17 compared to Theorem 19 below, could not be obtained by axioms (Q1–Q4) alone.

In what follows we work with $h$-pseudodifferential operators which are not necessarily bounded (see Section 2.2).

**Theorem 19 ([31]).** Let $X$ be either $\mathbb{R}^n$, or a closed manifold. Let $(T_1, \ldots, T_d)$ be a family of pairwise commuting self-adjoint semiclassical $h$-pseudodifferential operators on $X$ whose symbols mildly depend on $h$. Let $\mathcal{S} \subset \mathbb{R}^d$ be the classical spectrum of $(T_1, \ldots, T_d)$. Suppose that $\mathcal{S}$ is a convex set. Then:

- from $\text{JointSpec}(T_1, \ldots, T_d)$ one can recover $\mathcal{S}$;
- if moreover each $T_i$, $1 \leq i \leq d$, is bounded, then $\overline{\mathcal{S}}$ is the Hausdorff limit, as $h \to 0$, of $\text{Convex Hull}(\text{JointSpec}(T_{1,h}, \ldots, T_{d,h}))$.

For further discussion on these results see [31, 30]. Notice that all the results presented in this paper strongly rely on the self-adjointness of the operators, which ensures a stable behaviour of the spectrum as $h \to 0$. For general non-selfadjoint operators, for which there is considerable recent interest (see [38, 36, 37]), similar results can probably be obtained for the semiclassical pseudo-spectrum instead of the spectrum, but to the authors
knowledge, this has never been studied in the case of commuting operators. On the other hand, for non-selfadjoint operators that are normal, the stability of the spectrum is expected to hold, see for instance [27].

4.2. The completely integrable case. Let \((M, \omega)\) be a \(2n\)-dimensional symplectic manifold. Given a smooth function \(f: M \to \mathbb{R}\), we define the Hamiltonian vector field \(X_f\) induced by \(f\) on \(M\) by
\[
\omega(X_f, \cdot) = -df.
\]
This differential equation (or rather, system of differential equations) is known as Hamilton’s equation.

**Definition 20.** A classical integrable system on \((M, \omega)\) is given by a smooth \(\mathbb{R}^n\)-valued map \(F := (f_1, \ldots, f_n): M \to \mathbb{R}^n\) such that each component function \(f_i\) is constant along the flow of the vector field \(X_{f_j}\) generated by the component \(f_j\), for all \(i, j\), and the vector fields \(X_{f_1}, \ldots, X_{f_n}\) are linearly independent almost everywhere.

The first of the conditions in Definition 20 can be rephrased as
\[
\{f_i, f_j\} = 0
\]
for all \(i, j\), where
\[
\{f_i, f_j\} := \omega(X_{f_i}, X_{f_j})
\]
are the so called Poisson brackets of \(f_i\) and \(f_j\); in this case we say that \(f_i\) and \(f_j\) are in involution. The integer \(n\) (half the dimension of \(M\)) in this definition is the largest integer for which the conditions of the definition hold: that is, there is no set of functions
\[
f_1, \ldots, f_k,
\]
with \(k > n\) which satisfies the conditions above and it is in this sense that the word “integrable system” is used.

**Remark 21.** The symplectic theory of finite dimensional integrable Hamiltonian systems relies on several fundamental results. Liouville-Mineur-Arnold’s action-angle theorem [29, 1] is one of the fundamental pieces of the modern theory of integrable systems. Duistermaat [15] described the obstruction to the existence of global-action coordinates in 1980, and this was the starting point of the global symplectic theory of integrable systems. Eliasson [19, 18] proved in the 1980s a major theorem on the linearization of smooth non-degenerate singularities of integrable systems, which continues to be one of the foundational and most useful results of the subjects; the majority of (but not all) results known to date about the general structure of integrable systems, assume that the singularities are non-degenerate.
In the 1980s the global classification of toric integrable systems of Atiyah, Guillemin-Sternberg, and Delzant opened up the doors and served as inspiration to many authors working on global symplectic invariants of integrable systems. Our next goal is to define toric integrable systems, and the natural transformations between them, and state the classification in the work of Atiyah, Guillemin-Sternberg and Delzant. Let \((M, \omega)\) be a 2n-dimensional symplectic manifold. A smooth map

\[
F := (f_1, \ldots, f_n): M \rightarrow \mathbb{R}^n
\]

on \((M, \omega)\) is a momentum map for a Hamiltonian n-torus action if each of the Hamiltonian flows

\[
t_j \mapsto \varphi^{t_j}_{f_j}
\]

of the vector fields \(X_{f_j}\) is periodic of period 1, and all of them pairwise commute, that is,

\[
\varphi^{t_j}_{f_j} \circ \varphi^{t_i}_{f_i} = \varphi^{t_i}_{f_i} \circ \varphi^{t_j}_{f_j}
\]

so that they define an action of the torus \(\mathbb{R}^n/\mathbb{Z}^n\).

**Definition 22.** We say that a momentum map for a Hamiltonian n-torus action is a toric integrable system, or simply a toric system, if in addition the following conditions hold:

- the manifold \(M\) is closed and connected;
- the action of the torus \(\mathbb{R}^n/\mathbb{Z}^n\) is effective.

The natural transformations between toric integrable systems preserve the toric and the symplectic structure simultaneously, they are precisely given by the following.

**Definition 23.** Two toric systems \((M, \omega, F)\) and \((M', \omega', F')\) are isomorphic if there exists a symplectomorphism \(\varphi: (M, \omega) \rightarrow (M', \omega')\) such that \(\varphi^* F' = F\).

Atiyah and Guillemin-Sternberg proved the following influential result (in fact their result applied to much more general momentum maps given by an \(m\)-tuple on a 2n-manifold, where \(n\) is not necessarily equal to \(m\) and the induced toral action is not necessarily effective):

**Theorem 24 (\cite{2, 22}).** The image \(F(M)\) of a toric system \(F: M \rightarrow \mathbb{R}^n\) is a convex polytope in \(\mathbb{R}^n\).

The set of fixed point (also called elliptic points) of the induced \(\mathbb{R}^n/\mathbb{Z}^n\)-action is a collection of symplectic submanifolds of \(M\), and its image under \(F\) gives a finite collection of points

\[
p_1, \ldots, p_k \in \mathbb{R}^n \quad k \geq 1.
\]

The convex polytope in Theorem 24 is precisely the set

\[
\Delta = \text{Convex Hull}(\{p_1, \ldots, p_k\}).
\]

Shortly after Atiyah and Guillemin-Sternberg proved their theorem, Delzant proved a converse type result, hence giving a classification of toric systems.
Theorem 25 ([13]). The image $F(M)$ of a toric system $F: M \to \mathbb{R}^n$ is a Delzant polytope (i.e. rational, simple, and smooth). Moreover, $(M, \omega, F)$ is classified, up to isomorphisms, by $F(M)$.

Theorem 25 was generalized in [32, 33] to a class of systems $f_1, f_2$ on four dimensional manifolds, called semitoric systems, in which only $f_1$ is required to generate a periodic flow.

Definition 26. A quantum integrable system is given by a collection of $n$ commuting semiclassical self-adjoint operators

$$T_1 := (T_{1, \hbar})_{\hbar \in I}, \ldots, T_n := (T_{n, \hbar})_{\hbar \in I}$$

whose principal symbols form a classical integrable system on $M$.

Definition 27. A quantum integrable system $T_1, \ldots, T_n$ on $(M, \omega)$ is toric if the principal symbols of $T_1, \ldots, T_n$ are a toric system.

Remark 28. It is known that not every symplectic manifold has a complex structure or a prequantum line bundle. In the case of toric integrable systems, the situation is better. A toric integrable system does admit a compatible complex structure, which is however not unique. Suppose that $F : M \to \mathbb{R}^n$ is the momentum map and let

$$\Delta := F(M)$$

be its image in $\mathbb{R}^n$, which is a convex polytope (by Theorem 24), say with vertices

$$p_1, \ldots, p_k.$$

Then the system is is prequantizable if and only if there is a constant $\ell \in \mathbb{R}^n$ such that

$$p_1 + \ell, \ldots, p_k + \ell \in 2\pi\mathbb{Z}^n.$$

If this holds, the prequantum line bundle is in fact is unique, up to isomorphisms.

In the case of Berezin-Toeplitz quantization, it is remarkable that the joint spectrum of a toric system can be completely described, as follows.

Theorem 29 ([5]). Let $T_1, \ldots, T_n$ be a Berezin-Toeplitz quantum toric system on a closed manifold $M$. Then

$$\text{JointSpec}(T_1, \ldots, T_n) = g \left( \Delta \cap \left( v + \frac{2\pi}{k} \mathbb{Z}^n \right); k \right) + O(k^{-\infty})$$

where:

- the set $\Delta \subset \mathbb{R}^n$ is the convex polytope $F(M)$ in the Atiyah-Guillemin-Sternberg theorem (Theorem 24);
- the point $v \in \mathbb{R}^n$ is any vertex of $\Delta$;
- $g(\cdot; k) : \mathbb{R}^n \to \mathbb{R}^n$ admits a $C^\infty$-asymptotic expansion

$$g(\cdot; k) = \text{Id} + k^{-1}g_1 + k^{-2}g_2 + \cdots$$

where each $g_j : \mathbb{R}^n \to \mathbb{R}^n$ is a smooth function.
Moreover, if the spectral parameter \( k \) is large enough then the multiplicity of the eigenvalues of

\[
\text{JointSpec}(T_1, \ldots, T_n)
\]

is precisely equal to 1, and and there is \( \delta > 0 \) such that if \( \nu \) is an eigenvalue then the ball \( B(\nu, \delta/k) \) centered at \( \nu \) or radius \( \delta/k \) contains precisely only the eigenvalue \( \nu \).

As an immediate consequence of this theorem, we have the following.

**Corollary 30.** Let \( T_1, \ldots, T_n \) be a quantum toric system on a closed manifold \( M \). Then the joint spectrum of \( T_1, \ldots, T_n \) modulo \( O(1/k) \) determines the classical integrable system given by principal symbols, up to isomorphisms.

A quicker alternative proof of Corollary 30 was given in [31], and it also follows (in the same way as therein) from Theorem 17. Indeed, Theorem 17 implies that from the joint spectrum one can recover \( F(M) \). But we know that \( F(M) = \Delta \) is the Delzant polytope of the system. Since, by Delzant’s theorem 25, the polytope is enough to reconstruct the manifold and the moment map \( F \), it follows that the joint spectrum completely determines the classical system.

To conclude, let us mention that an interesting consequence of the proofs of these results is that any classical toric system can be quantized.

**Corollary 31 ([5]).** There exists a quantization of any classical toric integrable system. That is, given a classical toric system there is a quantum toric system whose principal symbols are precisely those given by the classical toric system.

We do not know whether a similar statement holds for more general classes of completely integrable systems. In the analytic case, an algebraic obstruction was constructed in [20].
Acknowledgements. We thank Y. Le Floch and J. Palmer for comments on a preliminary version, and L. Polterovich for useful discussions. Part of this paper was written while the second author was holding the Lebesgue Chair at the Lebesgue Center, during the Thematic Semester in Analysis and PDEs.

A.P. is partially supported by NSF CAREER grant DMS-1518420, Lebesgue Chair 2015, and Severo Ochoa grant Sev-2011-0087.

V.N.S. is partially supported by the Institut Universitaire de France and the Lebesgue Center (ANR Labex LEBESGUE).

References

[1] V. I. Arnold. A theorem of Liouville concerning integrable problems of dynamics. *Siberian Math. J.* 4 (1963) 471-474.
[2] M. Atiyah. Convexity and commuting Hamiltonians. *Bull. London Math. Soc.* 14 (1982) 1–15.
[3] D. Borthwick, T. Paul and A. Uribe. Semiclassical spectral estimates for Toeplitz operators. *Ann. Inst. Fourier (Grenoble)* 48 (1998) 1189–1229.
[4] L. Boutet de Monvel and V. Guillemin. The Spectral Theory of Toeplitz Operators. Annals of Mathematics Studies, 99. Princeton University Press, Princeton, NJ; University of Tokyo Press, Tokyo, 1981. v+161 pp.
[5] L. Charles, A. Pelayo, and S. Vũ Ngọc. Isospectrality for quantum toric integrable systems (Dedicated to Peter Sarnak on his 60th Birthday), *Annales Sci. Ec. Norm. Sup.* 43 (2013) 815–849.
[6] Y. Colin de Verdière. Spectre conjoint d’opérateurs pseudo-différentiels qui commutent. II. Le cas intégrable, *Math. Z.* 171 (1980) 51–73.
[7] Y. Colin de Verdière. Spectre conjoint d’opérateurs pseudo-différentiels qui commutent. I. Le cas non intégrable, *Duke Math. J.* 46 (1979) 169–182.
[8] Y. Colin de Verdire and V. Guillemin: A semi-classical inverse problem I: Taylor expansions. Geometric aspects of analysis and mechanics, 81–95, Progr. Math., 292, Birkhäuser/Springer, New York, 2011.
[9] L. Charles. Berezin-Toeplitz operators, a semi-classical approach, *Comm. Math. Phys.* 239 (2003) 1–28.
[10] L. Charles. Symbolic calculus for Toeplitz operators with half-forms, *Journal of Symplectic Geometry* 4 (2006) 171–198.
[11] L. Charles. Toeplitz operators and Hamiltonian Torus Actions, *Journal of Functional Analysis* 236 (2006) 299–350.
[12] L. Charles. Semi-classical properties of geometric quantization with metaplectic correction, *Comm. Math. Phys.* 270 (2007) 445–480.
[13] T. Delzant. Hamiltoniens périodiques et image convexe de l’application moment. *Bull. Soc. Math. France* 116 (1988) 315–339.
[14] M. Dimassi, J. Sjöstrand. Spectral asymptotics in the semi-classical limit. London Mathematical Society Lecture Note Series, 268. Cambridge University Press, Cambridge, 1999. xii+227 pp.
[15] J.J. Duistermaat. On global action-angle variables. *Comm. Pure Appl. Math.* 33 (1980) 687–706.
[16] J.J. Duistermaat. The heat kernel Lefschetz fixed point formula for the spin-c Dirac operators. Reprint of the 1996 edition. Modern Birkhäuser Classics. Birkhäuser/Springer, New York, 2011.
[17] J.J. Duistermaat. Principal Fiber Bundles. Notes for *Spring School, June 17–22, 2004, Utrecht.*
[18] L.H. Eliasson. Normal forms for hamiltonian systems with Poisson commuting integrals – elliptic case. Comm. Math. Helv. 65 (1990) 4–35.
[19] L.H. Eliasson. Hamiltonian systems with Poisson commuting integrals, PhD thesis, University of Stockholm, 1984.
[20] M. Garay and D. van Straten: Classical and quantum integrability, Mosc. Math. J. 10 (2010) 519–545.
[21] V. Guillemin and T. Paul. Some Remarks about Semiclassical Trace Invariants and Quantum Normal Forms, Comm. Math. Phys. 294 (2009) 1–19.
[22] V. Guillemin and S. Sternberg. Convexity properties of the moment mapping. Invent. Math. 67 (1982) 491–513.
[23] H. Hezari. Inverse spectral problems for Schrödinger operators. Comm. Math. Phys. (2009)1061–1088.
[24] M. Kac. Can one hear the shape of a drum? (Polish) Translated from the English (Amer. Math. Monthly 73 (1966), no. 4, part II, 1–23). Wisdom. Mat. (2) 13 (1971) 11–35.
[25] Y. Le Floch. Inverse spectral theory for 1D Toeplitz operators. PhD Thesis, University Rennes 1, 2014.
[26] Y. Le Floch, Á. Pelayo and S. Vũ Ngöc. Semiclassical inverse spectral theory for Jaynes–Cummings type systems, Math. Annalen, in press
[27] Y. Le Floch and Á. Pelayo. Spectral asymptotics of semiclassical unitary operators, 42 pages, arXiv:1506:02873.
[28] X. Ma and G. Marinescu. Toeplitz operators on symplectic manifolds. J. Geom. Anal., 18 (2008) 565–611.
[29] H. Mineur: Sur les systèmes mécaniques dans lesquels figurent des paramètres fonctions du temps. Étude des systèmes admettant n intégrales premières uniformes en involution. Extension à ces systèmes des conditions de quantification de Bohr-Sommerfeld. J. Ecole Polytechn., III (1937) (Cahier 1, Fasc. 2 et 3):173–191, 237–270.
[30] Á. Pelayo: Symplectic spectral geometry of semiclassical operators. Bull. Belg. Math. Soc. Simon Stevin 20 (2013) 405–415.
[31] Á. Pelayo, L. Polterovich, and S. Vũ Ngöc. Semiclassical quantization and spectral limits of ℏ–pseudodifferential and Berezin-Toeplitz operators, Proc. Lond. Math. Soc. 109 (2014) 676–696.
[32] Á. Pelayo and S. Vũ Ngöc. Semitoric integrable systems on symplectic 4–manifolds. Invent. Math. 177 (2009) 571–597.
[33] Á. Pelayo and S. Vũ Ngöc. Constructing integrable systems of semitoric type. Acta Math. 206 (2011) 93–125.
[34] Á. Pelayo and S. Vũ Ngöc. Semiclassical inverse spectral theory for singularities of focus-focus type, Comm. Math. Phys. 329 (2014) 809–820.
[35] M. Schlichenmaier. Berezin-Toeplitz quantization for compact Kähler manifolds. A review of results. Adv. Math. Phys., pages Art. ID 927280, 38, 2010.
[36] J. Sjöstrand. Some results on nonselfadjoint operators: a survey. Further progress in analysis, 45–74, World Sci. Publ., Hackensack, NJ, 2009.
[37] J. Sjöstrand. Spectral properties of non–self–adjoint operators. arXiv:1001.4844v1, 118 pages, 2010.
[38] L. N. Trefethen and M. Embree. Spectra and pseudospectra. The behavior of nonnormal matrices and operators. Princeton University Press, 2005.
[39] S. Vũ Ngöc. Symplectic inverse spectral theory for pseudodifferential operators. Geometric aspects of analysis and mechanics, 353–372, Progr. Math., 292, 2011.
[40] S. Vũ Ngöc. Bohr-Sommerfeld conditions for integrable systems with critical manifolds of focus-focus type. Comm. Pure Appl. Math., vol. 53, no. 2, pp. 143–217, 2000.
[41] H. Weyl. Über die asymptotische Verteilung der Eigenwerte, Gott. Nach. (1911) 110–117.
[42] H. Weyl. Das asymptotische Verteilungsgesetz der Eigenwerte linearer partieller Differentialgleichungen, *Math. Ann.* **71** (1912) 441–479.

[43] M. Zworski. *Semiclassical analysis*, volume 138 of *Graduate Studies in Mathematics*. American Mathematical Society, Providence, RI, 2012.

Álvaro Pelayo  
Department of Mathematics  
University of California, San Diego  
9500 Gilman Drive #0112  
La Jolla, CA 92093-0112, USA  
alpelayo@math.ucsd.edu

San Vũ Ngọc  
Institut de Recherches Mathématiques de Rennes  
Université de Rennes 1  
Campus de Beaulieu  
F-35042 Rennes cedex, France  
san.vu-ngoc@univ-rennes1.fr