RESTRICTIONS OF HARMONIC FUNCTIONS AND
DIRICHLET EIGENFUNCTIONS OF THE HATA SET
TO THE INTERVAL

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ABSTRACT. In this paper we study the harmonic functions and the Dirichlet eigenfunctions of the Hata set, and their restrictions to the interval $[0, 1]$, its main edge. We prove that these restrictions of the harmonic functions are singular, i.e., monotone and with zero derivatives almost everywhere, and provide numerical evidence that this is also the case for the eigenfunctions.

1. Introduction

Interest in the study of analysis in fractals has increased since the publication of Kigami’s papers \cite{Kig89} and \cite{Kig93}. In particular, there has been interest in the explicit construction of harmonic functions and the eigenfunctions of the Laplacian of a postcritically finite (PCF) set. In \cite{DSV99}, Dalrymple, Strichartz and Vinson described algorithms for the construction of harmonic functions and the eigenfunctions in the Sierpinski gasket. The construction of harmonic functions is achieved by the general algorithm for PCF sets described in \cite{Kig01}, Chapter 3, while the construction of the eigenfunctions is achieved by decimation \cite{Shi91} (see also \cite{Str06}, Chapter 3 for a detailed explanation of the decimation method). The explicit construction of harmonic functions or eigenfunctions has been done in other fractals, as the Vicset set \cite{CSW11}, where one also has decimation, and the pentakun \cite{ASST03}, where one does not have decimation.

In \cite{DSV99}, the authors also described algorithms for the restriction of harmonic functions and the eigenfunctions to the edges of the Sierpinski gasket, allowing us to visualize them as functions on the interval $[0, 1]$. Demir, Dzhafarov, Koçak and Üreyen \cite{DDKU07} observed that these functions have zero derivatives on a dense subset of $[0, 1]$. Later De Amo, Díaz Carrillo and Fernández Sánchez \cite{DADCFS13} proved

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that such restrictions are singular functions whenever they are monotone, i.e. that have zero derivatives almost everywhere.

In this paper we construct the harmonic functions and the Dirichlet eigenfunctions for the harmonic structure of the Hata set \cite{Hat85}, and their restrictions to the interval \([0, 1]\), the longest edge contained in the set. Since the Hata set does not have decimation, we have to construct the eigenfunctions by the finite element method \cite{DSV99}. Moreover, the Hata set has a natural family of harmonic structures, so we study the properties of these restrictions with respect to the parameter of the family.

In section 2, we describe the family of harmonic structures of the Hata set, as well as explicitly calculate its Laplacians. In section 3 we construct the harmonic functions, its restrictions, and study whether these restrictions are singular functions. In section 4 we explicitly describe the Laplacian with respect to a self-similar homogenous measure, and in section 5 we study the Dirichlet eigenfunctions, as well as its restrictions, and give numerical evidence to decide whether they are also singular.

2. Harmonic structure

The Hata tree set is the unique compact set \(K \subset \mathbb{C}\) such that
\[
K = F_1(K) \cup F_2(K),
\]
where the functions \(F_1, F_2\) are given by
\[
F_1(z) = \alpha \bar{z} \quad \text{and} \quad F_2(z) = (1 - |\alpha|^2) \bar{z} + |\alpha|^2,
\]
and \(\alpha \in \mathbb{C}\) is such that \(0 < |\alpha|, |1 - \alpha| < 1\) \cite{Hat85, YHK97}. Observe that the points 0 and 1 are the fixed points of \(F_1\) and \(F_2\), respectively, \(\alpha = F_1(1)\), and
\[
|\alpha|^2 = F_1(\alpha) = F_2(0).
\]
Hence, the critical set is given by \(\mathcal{C} = \{|\alpha|^2\}\) and the post-critical set, its boundary, is
\[
V_0 = \{\alpha, 0, 1\}.
\]
(See Figure 1) We will denote the points \(\alpha, 0\) and 1 by \(p_0, p_1\) and \(p_2\), respectively.

We observe that \(K \setminus V_0\) is disconnected. We call \(L\) the closure of the connected component of \(K \setminus V_0\) that contains the interval \((0, 1)\), and \(M\) the closure of the component that contains the open segment from 0 to \(\alpha\).

If \(w \in W_m\), we define \(p_{wi} = F_w(p_i)\), for \(i = 0, 1, 2\). We note that \(p_0 = p_{12}\), and that
\[
|\alpha|^2 = p_{10} = p_{21}.\]
As points in $V_m$, $m \geq 1$, we see that $p_0 = p_{12\ldots 2}$, $p_1 = p_{1\ldots 1}$ and $p_2 = p_{2\ldots 2}$.

If $w \neq w'$ are in $W_m$ and $p \in K_w \cap K_{w'}$, then either
\[ p = p_{w1} = p_{w'0} \quad \text{or} \quad p = p_{w1} = p_{w'2}. \]

We have the former case if $p \in V_m \setminus V_{m-1}$, and the latter if $p \in V_{m-1}$. If $p \in K_w$, and in no other $K_{w'}$, then
\[ p = p_{w0} \quad \text{or} \quad p = p_{w2}. \]

Again, we have the former case if $p \in V_m \setminus V_{m-1}$, and the latter if $p \in V_{m-1}$ (and $p \neq p_1$).

A point in $V_m$ has only one adjacent vertex if it’s of the form $p_{w20}$ or $p_{w22}$; otherwise it has three adjacent vertices in $V_m$, except for $p_1$, which has only two\(^1\).

To construct a harmonic structure on the Hata set $K$, we need a Laplacian $D$ on $V_0$. Using the standard base $\{\chi_\alpha, \chi_0, \chi_1\}$, we set
\[ D = \begin{pmatrix} -h & h & 0 \\ h & -(h+1) & 1 \\ 0 & 1 & -1 \end{pmatrix}. \]

Then, if $r = (r_1, r_2)$, $(D, r)$ is a regular harmonic structure \cite{Kig01} for $K$ if $h > 1$,
\[ r_1 = \frac{1}{h} \quad \text{and} \quad r_2 = 1 - \frac{1}{h^2}. \]

\(^1\)We note that, since $p_0 = p_{12}$, $p_{w20} = p_{w212}$ and $p_{w10} = p_{w112}$. Thus, points of the form $p_{w12}$ may have one or three adjacent vertices, depending on the last term of $w$. 

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{Hata_tree.png}
\caption{Hata tree set, with $\alpha = 1/2 + \sqrt{3}i/6$. The critical set consists of the point $|\alpha|^2 = 1/3$.}
\end{figure}
Explicitly, the Laplacian $H_m$ at a point $p \in V_m \setminus V_0$ is given by, if $p = p_{w1} = p_{w'0}$,

$$H_m f(p) = \frac{1}{r_w} \left( f(p_{w2}) - f(p_{w1}) + h(f(p_{w0}) - f(p_{w1})) \right)$$

(2.1a)

$$+ \frac{h}{r_{w'}} \left( f(p_{w'1}) - f(p_{w'0}) \right);$$

if $p = p_{w1} = p_{w'2}$,

$$H_m f(p) = \frac{1}{r_w} \left( f(p_{w2}) - f(p_{w1}) + h(f(p_{w0}) - f(p_{w1})) \right)$$

(2.1b)

$$+ \frac{1}{r_{w'}} \left( f(p_{w'1}) - f(p_{w'2}) \right);$$

if $p = p_{w0}$ and in no other cell,

(2.1c)

$$H_m f(p) = \frac{h}{r_w} \left( f(p_{w1}) - f(p_{w0}) \right);$$

and, if $p = p_{w2}$ and in no other cell,

(2.1d)

$$H_m f(p) = \frac{1}{r_w} \left( f(p_{w1}) - f(p_{w2}) \right).$$

The Laplacian at the points $p_0$ and $p_2$ is given by formula (2.1d) for $m \geq 1$, while for $p = p_1$ is given by

$$H_m f(p) = \frac{1}{r_w} \left( f(p_{w2}) - f(p_{w1}) + h(f(p_{w0}) - f(p_{w1})) \right),$$

where $w = (1, 1, \ldots, 1) \in W_m$ is the word with $m$ ones. Note that $r_w = r_1^m$.

Observe that we have a family of harmonic structures for $K$, parameterized by $h$. For each $h > 1$, the Hausdorff dimension with respect to the effective resistance metric [Kig01] is the unique $d$ such that

$$\left( \frac{1}{h} \right)^d + \left( 1 - \frac{1}{h^2} \right)^d = 1.$$

We note that, since $F_1$ and $F_2$ are affine linear contractions with constants $|\alpha|$ and $1 - |\alpha|^2$, respectively, $d$ coincides with the Hausdorff dimension with respect to the Euclidean metric if $h = 1/|\alpha|$, by Hutchinson theorem [Hut81].

3. Harmonic functions

A function $u$ on $V_*$ is harmonic if $H_m u(p) = 0$ for any $p \in V_m \setminus V_0$. In this section we describe an algorithm to construct harmonic functions on $V_*$ from any boundary values on $V_0$. 

We say that \( T_w \subset V_m \) is a minimal cell in \( V_m \), if \( T_w \) a set of three vertices of the form \( p_{w0}, p_{w1} \) and \( p_{w2} \), with \( w \in W_m \). As points in \( V_{m+1} \), we have that

\[
T_w = \{p_{w12}, p_{w11}, p_{w22}\},
\]

and the “new points” in \( V_{m+1} \), contained in \( K_w \), are then \( p_{w10} = p_{w21} \) and \( p_{w20} \). In other words, \( K_w \cap V_{m+1} \) is the union of the minimal cells in \( V_{m+1} \)

\[
T_{w1} = \{p_{w10}, p_{w11}, p_{w12}\} \quad \text{and} \quad T_{w2} = \{p_{w20}, p_{w21}, p_{w22}\}
\]

Given a harmonic function on \( V_m \), we want to extend to a harmonic function on \( V_{m+1} \). The extension from \( T_w \) to \( T_{w1} \cup T_{w2} \) is given by the following algorithm.

**Algorithm 3.1.** Let \( u \) be a harmonic function on \( V_m \). If, for each \( w \in W_m \), \( T_w \) is a minimal cell in \( V_m \), and we extend \( u \) to \( T_{w1} \cup T_{w2} \) with

\[
\begin{align*}
(3.1a) \quad u(p_{w10}) &= \left(1 - \frac{1}{r_1^2}\right)u(p_{w1}) + \frac{1}{r_1}u(p_{w2}), \\
(3.1b) \quad u(p_{w20}) &= u(p_{w10}),
\end{align*}
\]

then \( u \) is harmonic in \( V_{m+1} \).

**Proof.** With respect to the basis \( \{\chi_{p_{12}}, \chi_{p_{11}}, \chi_{p_{22}}, \chi_{p_{10}}, \chi_{p_{20}}\} \), the matrix of \( H_1 \) is given by

\[
H_1 = \begin{pmatrix}
\frac{-1}{r_1} & \frac{1}{r_1} & 0 & 0 & 0 \\
\frac{1}{r_1} & \frac{1}{r_1 + h} & 0 & h & 0 \\
0 & \frac{1}{r_1} & \frac{1}{r_1} & 0 & 0 \\
0 & h & \frac{1}{r_1} & \frac{1}{r_1 + h} & 0 \\
0 & \frac{1}{r_1} & \frac{1}{r_1} & \frac{1}{r_2} & \frac{1}{r_2} \\
0 & 0 & 0 & \frac{1}{r_2} & \frac{1}{r_2}
\end{pmatrix}
\]

Writing the matrix as

\[
\begin{pmatrix}
T & J^t \\
J & X
\end{pmatrix},
\]

where \( T \) takes functions on \( V_0 \) to functions on \( V_0 \), \( J \) functions on \( V_0 \) to functions on \( V_1 \), and \( X \) functions on \( V_1 \) to functions on \( V_1 \), Theorem 2.1.6 in [Kig01] implies that, if \( u \) is harmonic, then

\[
u|_{V_1 \setminus V_0} = -X^{-1}J(u|_{V_0}).
\]

Multiplying the matrices, and by the compatibility of the sequence of \( H_m \) [Kig01], we obtain the result. \( \Box \)
Algorithm 3.1 allows us to construct harmonic functions on the Hata set with arbitrary precision, with any particular value of the parameter $h$. Figure 2 shows examples of harmonic functions, with distinct boundary values and distinct values of $h$.

![Harmonic functions on the Hata set.](image)

**Figure 2.** Harmonic functions on the Hata set. The three on the top correspond to $\alpha = 1/2 + \sqrt{3}i/6$ and $h = 2$, with boundary values $\chi_c$, $\chi_0$ and $\chi_1$, respectively. On the bottom, harmonic functions with boundary values $\chi_1$, with values of $h$ equal to $3/2$, $\sqrt{3}$ and $3$, respectively.

We observe that, since $K \setminus V_0$ is disconnected, we have harmonic functions supported in each one of the connected components $L$ and $M$ of $K \setminus V_0$, as we see in Figure 2 for harmonic functions with boundary values $\chi_1$ and $\chi_\alpha$, respectively.

Moreover, from equation (3.1a), the value of a harmonic function on each point $p$ in the line segment from 0 to 1 only depends on its values on the adjacent points to $p$ in the same line, from a previous iteration. Thus, one can easily construct restrictions of such harmonic functions to the line segment, following the work of [DSV99]. Figure 3 shows examples of such restrictions with boundary values $\chi_1$.

We note, as readily verified from equation (3.1a), that the restriction of a harmonic function to this line segment is a line if $h = 1/|\alpha|$, because the “middle” point in each iteration corresponds to the convex combination of its adjacent points in the line with weights $|\alpha|^2$ and $1 - |\alpha|^2$. However, if $h \neq 1/|\alpha|$, this restriction is a singular function, i.e. monotonic and with derivative 0 almost everywhere [YHK97].
RESTRICTIONS OF FUNCTIONS ON THE HATA SET

Figure 3. Restriction to $[0, 1]$ of harmonic functions on the Hata set, with $\alpha = 1/2 + \sqrt{3i}/6$, boundary values $\chi_1$ and values of $h$ equal to $3/2$, $\sqrt{3}$ and $3$, respectively. Observe that, in the case $h = 1/|\alpha|$, such restriction is a line. Otherwise, it is a singular function.

Theorem 3.2. Assume $h \neq 1/|\alpha|$ and let $u$ be a harmonic function on the Hata set with boundary values $u|_{V_0} = \chi_1$, and let $f$ be its restriction to the interval $[0, 1]$. Then

1. $f$ is increasing; and
2. $f$ is differentiable on $[0, 1] \setminus V_\ast$, with

$$f'(x) = 0$$

for every $x \in [0, 1] \setminus V_\ast$.

Proof. Since $|\alpha|^2 = p_{10} = F_1(F_1(1))$,

$$F_1(F_1(x)) = |\alpha|^2 x \quad \text{and} \quad F_2(x) = (1 - |\alpha|^2) x + |\alpha|^2,$$

we see that equation (3.1a) implies that $f$ satisfies the system of equations

$$
\begin{cases}
  f(x) = \frac{1}{h^2} f\left(\frac{1}{|\alpha|^2} x\right) & 0 \leq x \leq |\alpha|^2 \\
  f(x) = \left(1 - \frac{1}{h^2}\right) f\left(\frac{1}{|\alpha|^2} x - 1\right) + \frac{1}{h^2} |\alpha|^2 \leq x \leq 1.
\end{cases}
$$

Hence $f$ is essentially the same as Lebesgue’s singular function, and the result of the theorem follows as in [YHK97, Section 3.4].

We note that, if $h = 1/|\alpha|$, then we have $f(x) = x$. Recall that $d$ coincides with the Euclidean Hausdorff dimension precisely when $h = 1/|\alpha|$.
4. LAPLACIAN

In this section we calculate the Laplacian \( \Delta \) on the Hata set \( K \) with respect to the self-similar measure \( \mu \) with weights

\[
\mu_1 = \mu(F_1(K)) = r_1^d = \left( \frac{1}{h} \right)^d \quad \text{and} \quad \mu_2 = \mu(F_2(K)) = r_2^d = \left( 1 - \frac{1}{h^2} \right)^d.
\]

This measure is comparable to the Hausdorff measure with respect to the effective resistance metric [Kig94]. Moreover, \( \mu \) is homogenous with respect to this metric [Sae12].

As in [Kig01, Section 3.7], the domain \( D \) of the operator \( \Delta \) is given by the set of continuous functions \( u \) on \( K \) such that there exists a continuous function \( f \) with

\[
\lim_{m \to \infty} \max_{p \in V_m \setminus V_0} \left| \frac{1}{\mu_p^m} H_m u(p) - f(p) \right| = 0,
\]

where \( \mu_p^m = \int_K \psi_p^m d\mu \), the integral of the \( m \)-harmonic spline \( \psi_p^m \) that satisfies

\[
\psi_p^m(p) = 1 \quad \text{and} \quad \psi_p^m(q) = 0 \quad \text{for} \quad q \in V_m, q \neq p.
\]

If \( u \in D \) and \( f \) is as in (4.1), then we write \( \Delta u = f \). We can then approximate explicitly \( \Delta u \) once we calculate the normalizing numbers \( \mu_p^m \).

In order to calculate these numbers we observe that, by the self-similarity of \( \mu \),

\[
\mu_p^m = \int_K \psi_p^m d\mu = \sum_{w \in W_m} \int_{K_w} \psi_p^m d\mu
\]

\[
= \sum_{w \in W_m} \mu_w \int_K \psi_{F_w^{-1}(p)}^0 d\mu = \sum_{w \in W_m} \mu_w \mu_{F_w^{-1}(p)}^0,
\]

where \( \mu_w = \mu_{w_1} \cdots \mu_{w_m} \). Thus it is sufficient to calculate the three numbers \( \mu_0^\alpha, \mu_0^0 \) and \( \mu_0^1 \) corresponding to \( \alpha, 0 \) and 1, the points in \( V_0 \).

Again, using the self-similarity of \( \mu \), we observe that, by Algorithm 3.1

\[
\int_K \psi_0^\alpha d\mu = \mu_1 \int_K \psi_1^0 d\mu,
\]

\[
\int_K \psi_0^0 d\mu = \mu_1 \int_K \left( 1 - \frac{1}{h^2} \right) \psi_0^\alpha + \psi_0^0 \right) d\mu
\]

\[
+ \mu_2 \int_K \left( 1 - \frac{1}{h^2} \right) \psi_0^\alpha + \left( 1 - \frac{1}{h^2} \right) \psi_0^0 \right) d\mu,
\]

where...
and
\[ \int_K \psi_0 \, d\mu = \mu_1 \int_K \frac{1}{h^2} \psi_\alpha \, d\mu + \mu_2 \int_K \left( \frac{1}{h^2} \psi_0 + \frac{1}{h^2} \psi_\alpha + \psi_1 \right) \, d\mu, \]

Thus, \( \mu_c^0, \mu_0^0 \) and \( \mu_1^0 \) satisfy the system of equations
\[
\begin{align*}
\mu_\alpha^0 &= \mu_1 \mu_1^0 \\
\mu_0^0 &= \left( 1 - \frac{1}{h^2} \right) \mu_\alpha^0 + \mu_2 \left( 1 - \frac{1}{h^2} \right) \mu_0^0 \\
\mu_1^0 &= \frac{1}{h^2} \mu_\alpha^0 + \frac{\mu_2}{h^2} \mu_0^0 + \mu_2 \mu_1^0,
\end{align*}
\]

where we have already use the fact \( \mu_1 + \mu_2 = 1 \). Moreover, as the sum \( \psi_\alpha^0 + \psi_0^0 + \psi_1^0 \) is the constant function 1, we also have
\[
\mu_\alpha^0 + \mu_0^0 + \mu_1^0 = 1.
\]

Solving this system we obtain
\[
\mu_\alpha^0 = \frac{\mu_1 \mu_2}{\mu_1 \mu_2 + (h^2 - 1) \mu_1 + \mu_2}, \quad \mu_0^0 = \frac{(h^2 - 1) \mu_1}{\mu_1 \mu_2 + (h^2 - 1) \mu_1 + \mu_2},
\]

and
\[
\mu_1^0 = \frac{\mu_2}{\mu_1 \mu_2 + (h^2 - 1) \mu_1 + \mu_2}.
\]

5. Dirichlet Spectrum

We now proceed to study the Dirichlet spectrum of \( \Delta \). As there is no decimation on the Hata set, we have to use the finite element method in order to approximate the eigenvalues and eigenfunctions of \( \Delta \). We present a summary of the observations obtained numerically by solving the system of equations
\[
\begin{align*}
-\Delta_m u(x) &= \lambda u(x) \quad x \in V_m \setminus V_0 \\
u(p) &= 0 \quad p \in V_0,
\end{align*}
\]

where \( \Delta_m u(x) = \frac{1}{\mu_p} H_m u(x) \).

Recall that \( L \) is the closure of the connected component of \( K \setminus V_0 \) that contains the interval \((0, 1)\), and \( M \) the closure of the component that contains the open segment from 0 to \( \alpha \). Thus, linearly independent Dirichlet eigenfunctions are supported either in \( L \) or \( M \).

We observe numerically that the Dirichlet ground state is supported in \( L \), as observed in Figure 4 (for \( h = 2 \)), corresponding to \( \lambda_1 \approx 9.888 \).

For each eigenfunction \( \phi \) supported in \( L \), there is a corresponding eigenfunction supported in \( M \).
Figure 4. Dirichlet ground state $\phi$ for $h = 2$, $\lambda_1 \approx 9.888$, and its derived eigenfunction $\tilde{\phi} = \chi_{K_1} \cdot \phi \circ F_1^{-1}$, $\lambda_3 \approx 56.21$, approximated to the eigth iteration, with $\alpha = 1/2 + \sqrt{3}i/6$. Note that $\phi$ is supported in $L$ and $\tilde{\phi}$ is supported in $M$.

Proposition 5.1. Let $\phi$ be a Dirichlet eigenfunction of $\Delta$, supported in $L$, with respect to the eigenvalue $\lambda$. Then $\chi_{K_1} \cdot \phi \circ F_1^{-1}$ is an eigenfunction supported in $M$ with respect to the eigenvalue $\lambda / (r_1 \mu_1)$.

Proof. Let $\tilde{\phi} = \chi_{K_1} \cdot \phi \circ F_1^{-1}$. If $x \in L$, then $x \in K_2$ or $x \in L \cap K_1$. In the first case, clearly $\tilde{\phi}(x) = 0$ because $\chi_{K_1}(x) = 0$, unless $x = |\alpha|^2$, the critical point. But in that case $x \in L \cap K_1$, and $F_1^{-1}(x) \in M$, so $\phi \circ F_1^{-1}(x) = 0$ since $\phi$ is supported in $L$. Therefore $\tilde{\phi}$ is supported in $M$.

Now, for $u \in F_0$, by the self-similarity of the Dirichlet form $\mathcal{E}$,

$$\mathcal{E}(\tilde{\phi}, u) = \frac{1}{r_1} \mathcal{E}(\tilde{\phi} \circ F_1, u \circ F_1) + \frac{1}{r_2} \mathcal{E}(\tilde{\phi} \circ F_2, u \circ F_1)$$

$$= \frac{1}{r_1} \mathcal{E}(\phi, u \circ F_1) = \frac{\lambda}{r_1} \int_K \phi u \circ F_1 d\mu,$$

where we have used the fact that $\tilde{\phi}$ is supported in $K_1$ and $\phi$ is a Dirichlet eigenfunction with respect to $\lambda$. One the other hand, by the self-similarity of the measure $\mu$,

$$\int_K \tilde{\phi} ud\mu = \mu_1 \int_K \tilde{\phi} \circ F_1 u \circ F_1 d\mu + \mu_2 \int_K \tilde{\phi} \circ F_2 u \circ F_2 d\mu = \mu_1 \int_K \phi u \circ F_1 d\mu,$$

so we obtain

$$\mathcal{E}(\tilde{\phi}, u) = \frac{\lambda}{r_1 \mu_1} \int_K \phi ud\mu,$$

and thus we conclude $\Delta \tilde{\phi} = -\frac{\lambda}{r_1 \mu_1} \tilde{\phi}$.

Figure 4 (right) shows the eigenfunction corresponding to the eigenvalue $\lambda_3 = \lambda_1 / r_1 \mu_1 \approx 56.21$, where $\lambda_1$ is the first Dirichlet eigenvalue.
(\(h = 2\)). Proposition [5.1] lets us classify the Dirichlet eigenvalues (and eigenfunctions) in two classes, which we will call “primary” and “derived”. Table [1] shows the approximations to the first 20 Dirichlet eigenvalues, for \(h = 3/2, \sqrt{3}\) and 3. We note that the derived eigenvalues appear in different positions in the sequence \(\lambda_k\), depending on \(h\), and they seem to be more sparse as \(h\) increases. Figure [5] shows the first four Dirichlet eigenfunctions for those values of \(h\), where we can observe the appearance of the derived eigenfunctions corresponding to \(\lambda_2^{10} (h = 3/2)\) and \(\lambda_3^{10} (h = \sqrt{3})\).

More interestingly, we show in Figure [6] the restrictions of these eigenfunctions to the interval \([0, 1]\) (only the primary ones, as the derived are zero in \([0, 1]\)). One can ask whether these functions have singularity properties as in the case of the harmonic functions.
Figure 5. The first four Dirichlet eigenfunctions for the values $h = 3/2, \sqrt{3}$ and 3, respectively, approximated to the 10th iteration, with $\alpha = 1/2 + \sqrt{3}i/6$.

Figure 6. The restrictions to $[0, 1]$ for the first three primary Dirichlet eigenfunctions for the values $h = 3/2, \sqrt{3}$ and 3, respectively, approximated to the 10th iteration, with $\alpha = 1/2 + \sqrt{3}i/6$.

Recall that, in the case of the harmonic functions, if $q \in V_{m+1} \cap [0, 1]$ is the middle point between the points $x, y \in V_m \cap [0, 1]$, then the
harmonic function $u$ satisfies

\begin{align}
    u(q) = (1 - \theta)u(x) + \theta u(y),
\end{align}

where $\theta = 1/h^2$. In Figure 7, we show the values of $\theta$ for such middle

\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure7}
\caption{The values of $\theta$ that solve equation (5.1) for each middle point $q \in V_9 \cap [0,1]$ (top of each pair of graphs) and $q \in V_{10} \cap [0,1]$ (bottom), for the approximations at $m = 9$ and $m = 10$ to the restrictions to $[0,1]$ for the first three primary Dirichlet eigenfunctions for the values $h = 3/2, \sqrt{3}$ and 3, respectively, with $\alpha = 1/2 + \sqrt{3}i/6$. The line corresponds to $1/h^2$, and the plot range is $1/h^2 \pm 0.01$.}
\end{figure}

points $q \in V_9 \cap [0,1]$ and $q \in V_{10} \cap [0,1]$, with respect to their adjacent
points in $V_8$ and $V_9$, respectively, for the approximations at $m = 9$ and $m = 10$ to the restrictions to $[0,1]$ for the first three primary Dirichlet eigenfunctions for the values $h = 3/2, \sqrt{3}$ and 3.

We observe that they are closely equal to $1/h^2$, with better approximations as $h$ increases. We show the two iterations $m = 9$ and $m = 10$ in order to find out if the same proportions are preserved through two levels. As the $\theta$ are preserved through different iterations, we are lead to conjecture that the restrictions to $[0,1]$ of the Dirichlet eigenfunctions of the Laplacian with respect to $h \neq 1/|\alpha|$ are singular functions whenever they are monotone, as in the case of the harmonic functions.

6. Conclusions

We have studied harmonic and Dirichlet eigenfunctions for the family of harmonic structures of the Hata set $K$ parametrized by $h > 1$. The former can be constructed by means of the known algorithms for harmonic functions on PCF sets, and one observes that, when restricted to the interval $[0,1]$, the longest edge in $K$, one obtains singular functions for all but one value of the parameter $h$. In fact, we observed that the only value of $h$ for which these restrictions are not singular coincides with the value such that the Hausdorff dimensions of $K$ with respect to the Euclidean and effective resistance metrics are the same.

As is known, restrictions of harmonic functions to the edges of the Sierpinski gasket are also singular. This lead us to ask whether this behaviour is typical for harmonic functions on PCF sets. Moreover, since we know that in the case of the Hata set $K$ such functions are not singular for a particular embedding $K$ in the plane (given $h$, one can choose $\alpha$ such that $|\alpha| = 1/h$), one can also ask whether, for every PCF set, there exists an embedding such that these restrictions are not singular. In particular, is this true for the Sierpinski gasket?

The same questions can be asked for the eigenfunctions of a PCF set. We have numerical evidence for the case of the Hata set, and one can start by studying those PCF sets with decimation. In particular, do the eigenfunctions on the Sierpinski gasket have singular restrictions to the edges?

References

[ASST03] Bryant Adams, S. Alex Smith, Robert S. Strichartz, and Alexander Teplyaev, The spectrum of the Laplacian on the pentagasket, Fractals in Graz 2001, Trends Math., Birkhäuser, Basel, 2003, pp. 1–24. MR 2091699 (2006g:28017)
RESTRICTIONS OF FUNCTIONS ON THE HATA SET

[CSW11] Sarah Constantin, Robert S. Strichartz, and Miles Wheeler, *Analysis of the Laplacian and spectral operators on the Vicsek set*, Commun. Pure Appl. Anal. **10** (2011), no. 1, 1–44. MR 2746525 (2012b:28012)

[DADCFS13] Enrique De Amo, Manuel Díaz Carrillo, and Juan Fernández Sánchez, *Harmonic analysis on the Sierpiński gasket and singular functions*, To appear in Acta Math. Hungar. (2013).

[DDKÜ07] Bünyamin Demir, Vakif Dzhafarov, Şahin Koçak, and Mehmet Üreyen, *Derivatives of the restrictions of harmonic functions on the Sierpinski gasket to segments*, J. Math. Anal. Appl. **333** (2007), no. 2, 817–822. MR 2331696 (2009e:28031)

[DSV99] Kyallee Dalrymple, Robert S. Strichartz, and Jade P. Vinson, *Fractal differential equations on the sierpinski gasket*, Journal of Fourier Analysis and Applications **5** (1999), 203–284, 10.1007/BF01261610.

[Hat85] Masayoshi Hata, *On the structure of self-similar sets*, Japan J. Appl. Math. **2** (1985), no. 2, 381–414. MR 839336 (87g:58080)

[Hut81] John E. Hutchinson, *Fractals and self-similarity*, Indiana Univ. Math. J. **30** (1981), no. 5, 713–747. MR 625600 (82h:49026)

[Kig89] Jun Kigami, *A harmonic calculus on the Sierpiński spaces*, Japan J. Appl. Math. **6** (1989), no. 2, 259–290. MR 91g:31005

[Kig93, Kig94] Jun Kigami, *Harmonic calculus on p.c.f. self-similar sets*, Trans. Amer. Math. Soc. **335** (1993), no. 2, 721–755. MR 93d:39008

[Kig01] Jun Kigami, *Analysis on fractals*, Cambridge University Press, Cambridge, 2001. MR 1 840 042

[Sáe12] Ricardo A. Sáenz, *Nontangential limits and Fatou-type theorems on post-critically finite self-similar sets*, J. Fourier Anal. Appl. **18** (2012), no. 2, 240–265. MR 2898728

[Shi91] Tadashi Shima, *On eigenvalue problems for the random walks on the Sierpiński pre-gaskets*, Japan J. Indust. Appl. Math. **8** (1991), no. 1, 127–141. MR 1093832 (92g:60094)

[Str06] Robert S. Strichartz, *Differential equations on fractals*, Princeton University Press, Princeton, NJ, 2006, A tutorial. MR 2246975 (2007f:35003)

[YHK97] Masaya Yamaguti, Masayoshi Hata, and Jun Kigami, *Mathematics of fractals*, American Mathematical Society, Providence, RI, 1997, Translated from the 1993 Japanese original by Kiki Hudson. MR 98j:28006

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