Green function identities in Euclidean quantum field theory

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Abstract

Given a generic Lagrangian system of even and odd fields, we show that any infinitesimal transformation of its classical Lagrangian yields the identities which Euclidean Green functions of quantum fields satisfy.

1 Introduction

We aim to show that, obeying the first variational formula (5), any infinitesimal transformation of a classical Lagrangian field system yields the identities (12) which Euclidean Green functions of quantum fields satisfy.

A generic Lagrangian system of even and odd fields is considered [1-3]. It is degenerate, and must be brought into a nondegenerate Lagrangian system in order to be quantized in the framework of a perturbative QFT. An Euler–Lagrange operator of a degenerate Lagrangian system satisfies nontrivial Noether identities. They need not be independent, but obey the first-stage Noether identities, which in turn are subject to the second-stage ones, and so on. Being finitely generated, these Noether identities are parameterized by the modules of antifields. The Noether’s second theorem states the relation between these Noether identities and the reducible gauge supersymmetries of a degenerate Lagrangian system parameterized by ghosts. In the framework of the Batalin–Vilkovisky (BV) quantization of a degenerate Lagrangian system [4-7], its original Lagrangian is extended to the above mentioned ghosts and antifields in order to satisfy the so-called classical master equation. Replacing antifields with gauge fixed terms, one comes to a nondegenerate gauge-fixing Lagrangian which is quantized. Instead of a gauge symmetry of an original Lagrangian, this Lagrangian possesses the variational BRST supersymmetry.

Bearing in mind quantization, we consider a Lagrangian field system on $X = \mathbb{R}^n$, $n \geq 2$, coordinated by $(x^\lambda)$. It is described in algebraic terms of a graded commutative $C^\infty(X)$-algebra $\mathcal{P}^0$ with generating elements

\[ \{ s^a, s^a_{\lambda}, s^a_{\lambda_1 \lambda_2}, \ldots, s^a_{\lambda_1 \ldots \lambda_k}, \ldots \}, \]  

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and the bigraded differential algebra $\mathcal{P}^*$ of differential forms (the Chevalley–Eilenberg differential calculus) over $\mathcal{P}^0$ as an $\mathbb{R}$-algebra [1-3]. One can think of generating elements (1) of $\mathcal{P}^0$ as being sui generis coordinates of even and odd fields and their partial derivatives (jets) (Section 2). The symbol $[a] = [s^a] = [s^a_{\lambda_1...\lambda_k}]$ stands for their Grassmann parity. In fact, $\mathcal{P}^0$ is the algebra of polynomials in the graded elements (1) whose coefficients are real smooth functions on $X$. The graded commutative $\mathbb{R}$-algebra $\mathcal{P}^0$ is provided with the even graded derivations (called total derivatives)

$$d_A = \partial_\lambda + \sum_{0 \leq |\Lambda|} s^a_{\lambda+\Lambda} \partial^A_a, \quad d_A = d_{\lambda_1} \cdots d_{\lambda_k},$$

where $A = (\lambda_1...\lambda_k)$, $|\Lambda| = k$, and $\lambda + A = (\lambda, \lambda_1, \ldots, \lambda_k)$ are symmetric multi-indices. One can think of even elements

$$L = \mathcal{L}(x^\lambda, s^a_\lambda) d^n x, \quad \delta L = ds^a \wedge \mathcal{E}_a d^n x = \sum_{0 \leq |\Lambda|} (-1)^{|\Lambda|} ds^a \wedge d_A (\partial^A_a L) d^n x$$

of the differential algebra $\mathcal{P}^*$ as being a graded Lagrangian and its Euler–Lagrange operator, respectively. By infinitesimal transformations of a Lagrangian system are meant odd vertical contact graded derivations

$$\vartheta = v^a \partial_a + \sum_{0 < |\Lambda|} d_A v^a \partial^A_a$$

of the $\mathbb{R}$-algebra $\mathcal{P}^0$. The Lie derivative $L_\vartheta L$ of a Lagrangian $L$ (3) along such a derivation obeys the first variational formula

$$L_\vartheta L = v^a \mathcal{E}_a d^n x + d_A J^\Lambda d^n x,$$

where $J^\Lambda$ is a generalized Noether current. One says that $\vartheta$ (4) is a variational supersymmetry of a Lagrangian $L$ if the Lie derivative (5) is a divergence $L_\vartheta L = d_{\lambda} \sigma^\lambda d^n x$, e.g., it vanishes.

From now on, let $(\mathcal{P}^*, L)$ be either an original nondegenerate Lagrangian system or the nondegenerate Lagrangian system brought from an original degenerate one by means of the above mentioned BV procedure. Let us quantize this Lagrangian system in the framework of perturbative Euclidean QFT (Section 3). We suppose that $L$ is a Lagrangian of Euclidean fields on $X = \mathbb{R}^n$. The key point is that the algebra of Euclidean quantum fields $B_\Phi$ as like as $\mathcal{P}^0$ is graded commutative. It is generated by elements $\phi^a_{x\Lambda}, x \in X$. For any $x \in X$, there is a homomorphism

$$\gamma_x : f^{A_1...A_r}_{a_1...a_r} s^a_{\Lambda_1} \cdots s^a_{\Lambda_r} \mapsto f^{A_1...A_r}_{a_1...a_r}(x) \phi^a_{x\Lambda_1} \cdots \phi^a_{x\Lambda_r}, \quad f^{A_1...A_r}_{a_1...a_r} \in C^\infty(X),$$

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of the algebra $\mathcal{P}^0$ of classical fields to the algebra $B_\Phi$ which sends the basis elements $s_a^\Lambda \in \mathcal{P}^0$ to the elements $\phi_{x_\Lambda}^a \in B_\Phi$, and replaces coefficient functions $f$ of elements of $\mathcal{P}^0$ with their values $f(x)$ at a point $x$. Then a state $\langle . \rangle$ of $B_\Phi$ is given by symbolic functional integrals

$$
\langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \rangle = \frac{1}{\mathcal{N}} \int \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \exp \{- \int \mathcal{L}(\phi_{x_\Lambda}^a) d^n x \} \prod_x [d\phi_x^a],
$$

(7)

$$
\mathcal{N} = \int \exp \{- \int \mathcal{L}(\phi_{x_\Lambda}^a) d^n x \} \prod_x [d\phi_x^a],
$$

\begin{align*}
\mathcal{L}(\phi_{x_\Lambda}^a) &= \mathcal{L}(x, \gamma_x(s_a^\Lambda)),
\end{align*}

which restart complete Euclidean Green functions in the Feynman diagram technique.

Due to homomorphisms (6), any graded derivation $\tilde{\vartheta}$ (4) of $\mathcal{P}^0$ induces the graded derivation

$$
\tilde{\vartheta} : \phi_{x_\Lambda}^a \to (x, s_a^\Lambda) \to u_a^\Lambda(x, s_b^\Lambda) \to u_a^\Lambda(x, \gamma_x(s_b^\Lambda)) = \tilde{\vartheta}_x^a(\phi_{x_\Lambda}^a)
$$

(8)

of the algebra of quantum fields $B_\Phi$ (Section 4). With an odd parameter $\alpha$, let us consider the automorphism

$$
\tilde{U} = \exp \{ \alpha \tilde{\vartheta} \} = \text{Id} + \alpha \tilde{\vartheta}
$$

(9)

of the algebra $B_\Phi$. This automorphism yields a new state $\langle . \rangle'$ of $B_\Phi$ given by the equalities

$$
\langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \rangle' = \langle \tilde{U}(\phi_{x_1}^{a_1}) \cdots \tilde{U}(\phi_{x_k}^{a_k}) \rangle' = \frac{1}{\mathcal{N}'} \int \tilde{U}(\phi_{x_1}^{a_1}) \cdots \tilde{U}(\phi_{x_k}^{a_k}) \exp \{- \int \tilde{\mathcal{L}}(\tilde{U}(\phi_{x_\Lambda}^a)) d^n x \} \prod_x [d\tilde{U}(\phi_x^a)],
$$

(10)

$$
\mathcal{N}' = \int \exp \{- \int \tilde{\mathcal{L}}(\tilde{U}(\phi_{x_\Lambda}^a)) d^n x \} \prod_x [d\tilde{U}(\phi_x^a)].
$$

It follows from the first variational formula (5) that

$$
\int \tilde{\mathcal{L}}(\tilde{U}(\phi_{x_\Lambda}^a)) d^n x = \int (\mathcal{L}(\phi_{x_\Lambda}^a) + \alpha \tilde{\vartheta}_x^a \mathcal{E}_{xa}) d^n x,
$$

where $\mathcal{E}_{xa} = \gamma_x(\mathcal{E}_a)$ are the variational derivatives. It is a property of symbolic functional integrals that

$$
\prod_x [d\tilde{U}(\phi_x^a)] = (1 + \alpha \int \frac{\partial \tilde{\vartheta}_x^a}{\partial \phi_x^a} d^n x \prod_x [d\phi_x^a] = (1 + \alpha \text{Sp}(\tilde{\vartheta})) \prod_x [d\phi_x^a].
$$

(11)

Then the equalities (10) result in the identities

$$
\langle \tilde{\vartheta}(\phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k}) \rangle + \langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} (\text{Sp}(\tilde{\vartheta}) - \int \tilde{\vartheta}_x^a \mathcal{E}_{xa} d^n x) \rangle - \langle \phi_{x_1}^{a_1} \cdots \phi_{x_k}^{a_k} \rangle (\text{Sp}(\tilde{\vartheta}) - \int \tilde{\vartheta}_x^a \mathcal{E}_{xa} d^n x) = 0.
$$

(12)
for complete Euclidean Green functions (7).

In particular, if \( \vartheta \) is a variational supersymmetry of a Lagrangian \( L \) (e.g., the BRST supersymmetry of a gauge-fixing Lagrangian [8]), the identities (12) are the Ward identities

\[
\langle \hat{\vartheta}(\phi_{a_1} \cdots \phi_{a_k}) \rangle + \langle \phi_{a_1} \cdots \phi_{a_k} \text{Sp}(\hat{\vartheta}) \rangle - \langle \phi_{a_1} \cdots \phi_{a_k} \rangle \langle \text{Sp}(\hat{\vartheta}) \rangle = 0,
\]

generalizing the Ward (Slavnov–Taylor) identities in the Yang–Mills gauge theory [9] (see Section 5 for an example of supersymmetric Yang–Mills theory).

If \( \vartheta = c^a \partial_a, c^a = \text{const} \), the identities (12) take the form

\[
\sum_{r=1}^k (-1)^{[a_1]+\cdots+[a_{r-1}]} \langle \phi_{a_1}^{\alpha_1} \cdots \phi_{a_{r-1}}^{\alpha_{r-1}} \delta_{\alpha_r} \phi_{a_{r+1}}^{\alpha_{r+1}} \cdots \phi_{a_k}^{\alpha_k} \rangle - \langle \phi_{a_1}^{\alpha_1} \cdots \phi_{a_k}^{\alpha_k} \rangle \langle \int \mathcal{E}_{x_{a}} d^m x \rangle = 0.
\]

One can think of them as being equations for complete Euclidean Green functions, but they are not an Euclidean variant of the well-known Schwinger–Dyson equations [10]. For instance, they identically hold if a Lagrangian \( L \) is quadratic.

Clearly, the expressions (12)–(14) are singular, unless one follows regularization and renormalization procedures, which however can induce additional anomaly terms.

2 Lagrangian systems of even and odd fields

As was mentioned above, we consider a Lagrangian field system on \( X = \mathbb{R}^n \), coordinated by \((x^\lambda)\). Such a Lagrangian system is algebraically described in terms of the following bigraded differential algebra (henceforth BGDA) \( \mathcal{P}^* \) (16) [1-3].

Let \( Y \to X \) be an affine bundle coordinated by \((x^\lambda, y^i)\) whose sections are even classical fields, and let \( Q \to X \) be a vector bundle coordinated by \((x^\lambda, q^a)\) whose sections are odd ones. Let \( J^rY \to X \) and \( J^rQ \to X \), \( r = 1, \ldots \), be the corresponding \( r \)-order jet bundles, endowed with the adapted coordinates \((x^\lambda, y^i_\Lambda)\) and \((x^\lambda, q^a_\Lambda)\), respectively. The index \( r = 0 \) conventionally stands for \( Y \) and \( Q \). For each \( r = 0, \ldots \), we consider a graded manifold \((X, \mathcal{A}_{J^rQ})\), whose body is \( X \) and the algebra of graded functions consists of sections of the exterior bundle

\[
\wedge(J^rQ)^* = \mathbb{R} \oplus (J^rQ)^* \oplus 2 \mathbb{R} \wedge (J^rQ)^* \oplus \cdots,
\]

where \((J^rQ)^*\) is the dual of a vector bundle \( J^rQ \to X \). The global basis for \((X, \mathcal{A}_{J^rQ})\) is \(\{x^\lambda, c^a_\Lambda\}, |\Lambda| = 0, \ldots, r\). Let us consider the graded commutative \( C^\infty(X)\)-algebra \( \mathcal{P}^0 \) generated by the even elements \(y^i_\Lambda\) and the odd ones \(c^a_\Lambda, |\Lambda| \geq 0\). The collective symbols \(s^a_\Lambda\) stand for these elements together with the symbol \([a]\) for their Grassmann parity.
Let $\mathfrak{d}\mathcal{P}^0$ be the Lie superalgebra of graded derivations of the $\mathbb{R}$-algebra $\mathcal{P}^0$, i.e.,

$$u(f f') = u(f) f' + (-1)^{|u||f|} f u(f'), \quad f, f' \in \mathcal{P}^0, \quad u \in \mathfrak{d}\mathcal{P}^0.$$ 

Its elements take the form

$$u = u^\lambda \partial_\lambda + \sum_{0 \leq |\lambda|} u^a_\lambda \partial^a_\lambda, \quad u^\lambda, u^a_\lambda \in \mathcal{P}^0. \quad (15)$$

With the Lie superalgebra $\mathfrak{d}\mathcal{P}^0$, one can construct the minimal Chevalley–Eilenberg differential calculus

$$0 \to \mathbb{R} \to \mathcal{P}^0 \xrightarrow{d} \mathcal{P}^1 \xrightarrow{d} \cdots \mathcal{P}^2 \xrightarrow{d} \cdots$$

(16)

over the $\mathbb{R}$-algebra $\mathcal{P}^0$. It is the above mentioned BGDA $\mathcal{P}^*$ with the basis $\{s^a\}$. Its elements $\sigma \in \mathcal{P}^k$ are graded $\mathcal{P}^0$-linear $k$-forms

$$\sigma = \sum \sigma^{a_{\lambda_1} \cdots a_{\lambda_r}} \partial_{\lambda_1}^a \cdots \partial_{\lambda_r}^a dx^{\lambda_1} \wedge \cdots \wedge dx^{\lambda_r}$$
on $\mathfrak{d}\mathcal{P}^0$ with values in $\mathcal{P}^0$. The graded exterior product $\wedge$ and the graded exterior differential, obey the relations

$$\sigma \wedge \sigma' = (-1)^{|\sigma||\sigma'|+|\sigma||\sigma'|} \sigma' \wedge \sigma, \quad d(\sigma \wedge \sigma') = d\sigma \wedge \sigma' + (-1)^{|\sigma|} \sigma \wedge d\sigma',$$

where $|.|$ denotes the form degree. By $\mathcal{O}^*X$ is denoted the graded differential algebra of exterior forms on $X$. There is the natural monomorphism $\mathcal{O}^*X \to \mathcal{P}^*$.

Given a graded derivation $u$ (15) of the $\mathbb{R}$-algebra $\mathcal{P}^0$, the interior product $u \rfloor \sigma$ and the Lie derivative $\mathbf{L}_u \sigma$, $\sigma \in \mathcal{P}^*$, obey the relations

$$u \rfloor (\sigma \wedge \sigma') = (u \rfloor \sigma) \wedge \sigma' + (-1)^{|\sigma|+|\sigma'|} \sigma \wedge (u \rfloor \sigma'), \quad \sigma, \sigma' \in \mathcal{P}^*, \quad \mathbf{L}_u \sigma = u \rfloor d\sigma + d(u \rfloor \sigma), \quad \mathbf{L}_u (\sigma \wedge \sigma') = \mathbf{L}_u (\sigma) \wedge \sigma' + (-1)^{|\sigma|} \sigma \wedge \mathbf{L}_u (\sigma').$$

For instance, let us denote $d_\lambda \sigma = \mathbf{L}_d \sigma$

The BGDA $\mathcal{P}^*$ is decomposed into $\mathcal{P}^0$-modules $\mathcal{P}^{k,r}$ of $k$-contact and $r$-horizontal graded forms

$$\sigma = \sum_{0 \leq |\lambda_i|} \sigma^{a_{\lambda_1} \cdots a_{\lambda_k} \mu_1 \cdots \mu_r} \theta_{\lambda_1}^{a_1} \wedge \cdots \wedge \theta_{\lambda_k}^{a_k} \wedge dx^{\mu_1} \wedge \cdots \wedge dx^{\mu_r}, \quad \theta_{\lambda}^a = ds_{\lambda}^a - s_{\lambda+\Lambda}^a \wedge dx^\lambda.$$ 

Accordingly, the graded exterior differential on $\mathcal{P}^*$ falls into the sum $d = d_V + d_H$ of the vertical and total differentials where $d_H \sigma = dx^\lambda \wedge d_\lambda \sigma$. The differentials $d_H$ and $d_V$ and the graded variational operator $\delta$ split the BGDA $\mathcal{P}^*$ into the graded variational bicomplex $[1, 2, 11]$. Its elements $L \in \mathcal{P}^{0,n}$ and $\delta L \in \mathcal{P}^{1,n}$ (3) are a Lagrangian and its Euler–Lagrange operator, respectively.
A graded derivation \( u \) (15) is called contact if the Lie derivative \( L_u \) preserves the ideal of contact graded forms of the BGDA \( P^* \). Here, we restrict our consideration to vertical contact graded derivations, vanishing on \( O^*X \). Such a derivation takes the form (4), and is determined by its first summand \( u = u^a \partial_a \). The Lie derivative \( L_\vartheta L \) of a Lagrangian \( L \) along a vertical contact graded derivation \( \vartheta \) (4) admits the decomposition (5)

\[
L_\vartheta L = v^a \delta L + \sigma a.
\]

One says that an odd vertical contact graded derivation \( \vartheta \) (4) is a variational supersymmetry of a Lagrangian \( L \) if the Lie derivative \( L_\vartheta L \) is \( d_H \)-exact.

3 Quantization

Hereafter, let \( (P^*, L) \) be either an original nondegenerate Lagrangian system or the nondegenerate Lagrangian system derived from an original one by means of the above mentioned BV procedure. Let us quantize this nondegenerate Lagrangian system \( (P^*, L) \). Though our results lie in the framework of perturbative QFT, we start with algebraic QFT.

In algebraic QFT, a quantum field system is characterized by a topological *-algebra \( A \) and a continuous positive form \( f \) on \( A \) [12, 13]. For the sake of simplicity, let us consider even scalar fields on the Minkowski space \( X = \mathbb{R}^n \). One associates to them the Borchers algebra \( A_\Phi \) of tensor products of the nuclear Schwartz space \( \Phi = S(\mathbb{R}^n) \) of smooth complex functions of rapid decreasing at infinity on \( \mathbb{R}^n \). The topological dual of \( S(\mathbb{R}^n) \) is the space \( S'(\mathbb{R}^n) \) of tempered distributions [14, 15].

Let \( \mathbb{R}_n \) denote the dual of \( \mathbb{R}^n \) coordinated by \( (p_\lambda) \). The Fourier transform

\[
\psi(x) = \int \phi^F(p) e^{-ipx} dp, \quad d_n p = (2\pi)^{-n} d^np,
\]

yields an isomorphism between the spaces \( S(\mathbb{R}^n) \) and \( S(\mathbb{R}_n) \). The Fourier transform of distributions is defined by the condition

\[
\int \psi(x) \phi(x) d^n x = \int \psi^F(p) \phi^F(-p) d_n p,
\]

and is written in the form (18) – (19). It provides an isomorphism between the spaces of distributions \( S'(\mathbb{R}^n) \) and \( S'(\mathbb{R}_n) \).

Since \( \otimes S(\mathbb{R}^n) \) is dense in \( S(\mathbb{R}^{nk}) \), a state \( f \) of the Borchers algebra \( A_\Phi \) is represented by distributions

\[
f_k(\phi_1 \ldots \phi_k) = \int W_k(x_1, \ldots, x_k) \phi_1(x_1) \ldots \phi_k(x_k) d^n x_1 \ldots d^n x_k, \quad W_k \in S'(\mathbb{R}^{nk}).
\]
In particular, the $k$-point Wightman functions $W_k$ describe free fields in the Minkowski space. The complete Green functions characterize quantum fields created at some instant and annihilated at another one. They are given by the chronological functionals

$$f^c(\phi_1 \cdots \phi_k) = \int W^c_k(x_1, \ldots, x_k)\phi_1(x_1) \cdots \phi_k(x_k) d^n x_1 \ldots d^n x_k,$$

$$W^c_k(x_1, \ldots, x_k) = \sum_{(i_1 \ldots i_k)} \theta(x_{i_1}^0 - x_{i_2}^0) \cdots \theta(x_{i_k}^0 - x_{i_k}^0) W_k(x_1, \ldots, x_k), \quad W_k \in S'(\mathbb{R}^{nk}),$$

where $\theta$ is the step function, and the sum runs through all permutations $(i_1 \ldots i_k)$ of the numbers $1, \ldots, k$. However, the chronological functionals (20) need not be continuous and positive. At the same time, they issue from the Wick rotation of Euclidean states of the Borchers algebra $A_{\Phi}$ describing quantum fields in an interaction zone [16, 17]. Since the chronological functionals (20) are symmetric, these Euclidean states are states of the corresponding commutative tensor algebra $B_{\Phi}$. This is the enveloping algebra of the Lie algebra of the group $T(\Phi)$ of translations in $\Phi$. Therefore one can obtain a state of $B_{\Phi}$ as a vector form of a strong-continuous unitary cyclic representation of $T(\Phi)$ [18]. Such a representation is characterized by a positive-definite continuous generating function $Z$ on $\Phi$. By virtue of the Bochner theorem [18], this function is the Fourier transform

$$Z(\phi) = \int_{\Phi'} \exp[i\langle \phi, w \rangle] d\mu(w)$$

of a positive measure $\mu$ of total mass 1 on the topological dual $\Phi'$ of $\Phi$. If the function $\alpha \rightarrow Z(\alpha \phi)$ on $\mathbb{R}$ is analytic at 0 for each $\phi \in \Phi$, a state $F$ of $B_{\Phi}$ is given by the expression

$$F_k(\phi_1 \cdots \phi_k) = i^{-k} \frac{\partial}{\partial \alpha_1} \cdots \frac{\partial}{\partial \alpha_k} Z(\alpha^i \phi_i)|_{\alpha^i = 0} = \int \langle \phi_1, w \rangle \cdots \langle \phi_k, w \rangle d\mu(w).$$

Then one can regard $Z$ (21) as a generating functional of complete Euclidean Green functions $F_k$ (22). However, it is a problem is that, if a field Lagrangian is a polynomial of degree exceeding two, a generating functional $Z$ (21) and Green functions fail to be written in an explicit form.

Therefore, let us quantize the above mentioned nondegenerate Lagrangian system $(\mathcal{P}^*, L)$ in the framework of perturbative QFT. We assume that $L$ is a Lagrangian of Euclidean fields on $X = \mathbb{R}^n$. The key point is that an algebra of Euclidean quantum fields is graded commutative, and there are homomorphisms of the graded commutative algebra $\mathcal{P}^0$ of classical fields to this algebra.

Let $Q$ be the graded complex vector space whose basis is the basis $\{s^a\}$ for the BGDA $\mathcal{P}^*$. Let us consider the tensor product

$$\Phi = Q \otimes S'(\mathbb{R}^n)$$

(23)
of the graded vector space $Q$ and the space $S'(\mathbb{R}^n)$ of distributions on $\mathbb{R}^n$. One can think of elements $\Phi$ (23) as being $Q$-valued distributions on $\mathbb{R}^n$. Let $T(\mathbb{R}^n) \subset S'(\mathbb{R}^n)$ be a subspace of functions $\exp\{ipx\}$, $p \in \mathbb{R}_n$, which are generalized eigenvectors of translations in $\mathbb{R}^n$ acting on $S(\mathbb{R}^n)$. We denote $\phi^\circ_p = s^a \otimes \exp\{ipx\}$. Then any element $\phi$ of $\Phi$ can be written in the form

$$\phi(x') = s^a \otimes \phi_a(x') = \int \phi_a(p) \phi^\circ_p dp,$$

where $\phi_a(p) \in S'(\mathbb{R}_n)$ are the Fourier transforms of $\phi_a(-x')$. For instance, there are the $Q$-valued distributions

$$\phi^\circ_a(x') = \int \phi^\circ_a e^{-ipx} dp = s^a \otimes \delta(x-x'),$$

$$\phi^\circ_{xA}(x') = \int (-i)^k p\lambda_1 \cdots p\lambda_k \phi^\circ_p e^{-ipx} dnp.$$

In the framework of perturbative Euclidean QFT, we associate to a nondegenerate Lagrangian system $(P^*, L)$ the graded commutative tensor algebra $B_\Phi$ generated by elements of the graded vector space $\Phi$ (23) and the following state $\langle . \rangle$ of $B_\Phi$.

For any $x \in X$, there is a homomorphism $\gamma_x$ (6) of the algebra $\mathcal{P}^0$ of classical fields to the algebra $B_\Phi$ which sends the basis elements $s^a_\Lambda \in \mathcal{P}^0$ to the elements $\phi^a_\Lambda \in B_\Phi$, and replaces coefficient functions $f$ of elements of $\mathcal{P}^0$ with their values $f(x)$ at a point $x$. Then the above mentioned state $\langle . \rangle$ of $B_\Phi$ is given by symbolic functional integrals

$$\langle \phi_1 \cdots \phi_k \rangle = \frac{1}{N} \int \phi_1 \cdots \phi_k \exp\{- \int \mathcal{L}(\phi^a) d^n x \} \prod_p [d\phi^a_p],$$

$$N = \int \exp\{- \int \mathcal{L}(\phi^a) d^n x \} \prod_p [d\phi^a_p],$$

$$\mathcal{L}(\phi^a_p) = \mathcal{L}(\phi^\circ_{xA}) = \mathcal{L}(x, \gamma_x(s^a_\Lambda)),$$

where $\phi_i$ and $\gamma_x(s^a_\Lambda) = \phi^\circ_{xA}$ are given by the formulas (24) and (26), respectively. Clearly, the expression $\mathcal{L}(\phi^a_p)$ (29) is not local. The forms (27) are expressed both into the forms

$$\langle \phi^\circ_{p_1} \cdots \phi^\circ_{p_k} \rangle = \frac{1}{N} \int \phi^\circ_{p_1} \cdots \phi^\circ_{p_k} \exp\{- \int \mathcal{L}(\phi^a_p) d^n x \} \prod_p [d\phi^a_p],$$

and the forms $\langle \phi^\circ_{x_1} \cdots \phi^\circ_{x_k} \rangle$ (7) which provide Euclidean Green functions. It should be emphasized that, in contrast with a measure $\mu$ in the expression (21), the term $\prod_p [d\phi^a_p]$ in the formulas (27) – (28) fail to be a true measure on $T(\mathbb{R}^n)$ because the Lebesgue measure on infinite-dimensional vector spaces need not exist. Nevertheless, treated as generalization of Berezin’s finite-dimensional integrals [19], the functional integrals (7) and (30) restart Euclidean Green functions in the Feynman diagram technique. Certainly, these Green functions are singular, unless regularization and renormalization techniques are involved.
4 The Green function identities

Since a graded derivation $\hat{\vartheta} \ (4)$ of the algebra $\mathcal{P}^0$ is a $C^\infty(X)$-linear morphism over $\text{Id} X$, it induces the graded derivation

$$\hat{\vartheta}_x = \gamma_x \circ \hat{\vartheta} \circ \gamma_x^{-1} : \phi_{x\Lambda}^a \to (x, q_x^a) \to \hat{\vartheta}_\Lambda^a(x, q_x^a) \to \hat{\vartheta}_\Lambda^a(x, \gamma_x(q_x^a)) = \hat{\vartheta}_\Lambda^a(\phi_{x\Sigma})$$

(31)
of the range $\gamma_\mathcal{P}(\mathcal{P}^0) \subset B_\Phi$ of the homomorphism $\gamma_\mathcal{P} \ (6)$ for each $x \in \mathbb{R}^n$. The maps $\hat{\vartheta}_x \ (31)$ yield the maps

$$\hat{\vartheta}_p : \phi_p^a = \int \phi_p^a e^{ipx} d^nx \to \int \hat{\vartheta}_x(\phi_p^a) e^{ipx} d^nx \to \int \hat{\vartheta}_x(\phi_{x\Sigma}^b) e^{ipx} d^nx = \int \hat{\vartheta}_x(-i)^k p_{\sigma_1} \cdots p'_{\sigma_k} \hat{\vartheta}_p^a b \phi_{x\Sigma}^b e^{-ip'x} d_n p') e^{ipx} d^nx = \hat{\vartheta}_p^a, \quad p \in \mathbb{R}_n,$$

and, as a consequence, the graded derivation

$$\hat{\vartheta}(\phi) = \int \phi_a(p) \hat{\vartheta}(\phi_p^a) d_n p = \int \phi_a(p) \hat{\vartheta}_p^a d_n p$$
of the algebra $B_\Phi$. It can be written in the symbolic form

$$\hat{\vartheta} = \int u_p^a \frac{\partial}{\partial \phi_p^a} d_n p, \quad \frac{\partial \phi_{p'}^b}{\partial \phi_p^a} = \delta_a^b \delta(p' - p),$$

(32)

$$\hat{\vartheta} = \int u_p^a \frac{\partial}{\partial \phi_p^a} d_n x, \quad \frac{\partial \phi_{x'\Lambda}^b}{\partial \phi_x^a} = \delta_a^b \frac{\partial}{\partial x'\lambda_1} \cdots \frac{\partial}{\partial x'\lambda_k} \delta(x' - x).$$

(33)

Let $\alpha$ be an odd element. Then $\hat{U} \ (9)$ is an automorphism of the algebra $B_\Phi$, and can provide a change of variables depending on $\alpha$ as a parameter in the functional integrals (7) and (30) [19]. This automorphism yields a new state $\langle . \rangle'$ of $B_\Phi$ given by the relations (10) and

$$\langle \phi_1 \cdots \phi_k \rangle = \langle \hat{U}(\phi_1) \cdots \hat{U}(\phi_k) \rangle' = \frac{1}{\mathcal{N}'} \int \hat{U}(\phi_1) \cdots \hat{U}(\phi_k) \exp \{- \int \mathcal{L}_{GF}(\hat{U}(\phi_p^a)) d^nx \} \prod_p [d\hat{U}(\phi_p^a)],$$

$$\mathcal{N}' = \int \exp \{- \int \mathcal{L}_{GF}(\hat{U}(\phi_p^a)) d^nx \} \prod_p [d\hat{U}(\phi_p^a)].$$

Let us apply these relations to the Green functions (7) and (30).

Using the first variational formula (17), the equalities (11) and

$$\prod_x [d\hat{U}(\phi_x^a)] = (1 + \alpha \int \frac{\partial \hat{\vartheta}_x^a}{\partial \phi_x^a} d^nx) \prod_x [d\phi_x^a] = (1 + \alpha \text{Sp}(\hat{\vartheta})) \prod_x [d\phi_x^a],$$
one comes to the desired identities (12) and the similar identities for the Green functions $\langle \phi_{p_1}^a \cdots \phi_{p_k}^a \rangle$. If $\vartheta$ is a variational supersymmetry of $L$, we obtain the above mentioned Ward identities

$$\langle \vartheta(\phi_{p_1}^a \cdots \phi_{p_k}^a) \rangle + \langle \phi_{p_1}^a \cdots \phi_{p_k}^a \mathrm{Sp}(\vartheta) \rangle - \langle \phi_{p_1}^a \cdots \phi_{p_k}^a \rangle \langle \mathrm{Sp}(\vartheta) \rangle = 0,$$  \hspace{1cm} (34)$$

$$\sum_{i=1}^k (-1)^{[a_1] + \cdots + [a_{i-1}] - [a_{i+1}] + \cdots + [a_k]} \langle \phi_{p_1}^a \cdots \phi_{p_i}^a \cdots \phi_{p_{i+1}}^a \cdots \phi_{p_k}^a \rangle +$$

$$\langle \phi_{p_1}^a \cdots \phi_{p_k}^a \int \frac{\partial \hat{\vartheta}^a}{\partial \phi_p^a} d_n p \rangle - \langle \phi_{p_1}^a \cdots \phi_{p_k}^a \rangle \langle \int \frac{\partial \hat{\vartheta}^a}{\partial \phi_p^a} d_n p \rangle = 0,$$

$$\langle \vartheta(\phi_{x_1}^a \cdots \phi_{x_k}^a) \rangle + \langle \phi_{x_1}^a \cdots \phi_{x_k}^a \mathrm{Sp}(\vartheta) \rangle - \langle \phi_{x_1}^a \cdots \phi_{x_k}^a \rangle \langle \mathrm{Sp}(\vartheta) \rangle = 0,$$  \hspace{1cm} (35)$$

$$\sum_{i=1}^k (-1)^{[a_1] + \cdots + [a_{i-1}] - [a_{i+1}] + \cdots + [a_k]} \langle \phi_{x_1}^a \cdots \phi_{x_i}^a \cdots \phi_{x_{i+1}}^a \cdots \phi_{x_k}^a \rangle +$$

$$\langle \phi_{x_1}^a \cdots \phi_{x_k}^a \int \frac{\partial \hat{\vartheta}^a}{\partial \phi_{x}^a} d^n x \rangle - \langle \phi_{x_1}^a \cdots \phi_{x_k}^a \rangle \langle \int \frac{\partial \hat{\vartheta}^a}{\partial \phi_{x}^a} d^n x \rangle = 0.$$

A glance at the expressions (34) – (35) shows that these Ward identities generally contain anomaly because the measure terms of symbolic functional integrals need not be $\hat{\vartheta}$-invariant. If $\mathrm{Sp}(\vartheta)$ is either a finite or infinite number, the Ward identities

$$\langle \hat{\vartheta}(\phi_{p_1}^a \cdots \phi_{p_k}^a) \rangle = \sum_{i=1}^k (-1)^{[a_1] + \cdots + [a_{i-1}] - [a_{i+1}] + \cdots + [a_k]} \langle \phi_{p_1}^a \cdots \phi_{p_{i-1}}^a \hat{\vartheta}_{p_i}^a \phi_{p_{i+1}}^a \cdots \phi_{p_k}^a \rangle = 0,$$  \hspace{1cm} (36)$$

$$\langle \hat{\vartheta}(\phi_{x_1}^a \cdots \phi_{x_k}^a) \rangle = \sum_{i=1}^k (-1)^{[a_1] + \cdots + [a_{i-1}] - [a_{i+1}] + \cdots + [a_k]} \langle \phi_{x_1}^a \cdots \phi_{x_{i-1}}^a \hat{\vartheta}_{x_i}^a \phi_{x_{i+1}}^a \cdots \phi_{x_k}^a \rangle = 0$$  \hspace{1cm} (37)$$

are free of this anomaly.

5 Supersymmetric Yang–Mills theory

Let $G = G_0 \oplus G_1$ be a finite-dimensional real Lie superalgebra with a basis $\{ e_r \}$, $r = 1, \ldots, m$, and real structure constants $c_{ij}^r$. Further, the Grassmann parity of $e_r$ is denoted by $[r]$. Recall the standard relations

$$c_{ij}^r = -(-1)^{[i][j]} c_{ji}^r,$$

$$[r] = [i] + [j],$$

$$(-1)^{[i][b]} c_{ij}^r c_{ab} + (-1)^{[a][i]} c_{aj}^r c_{bi} + (-1)^{[b][a]} c_{bj}^r c_{ia} = 0.$$

Let us also introduce the modified structure constants

$$\bar{c}_{ij}^r = (-1)^{[i]} c_{ij}^r,$$

$$\bar{c}_{ij}^r = (-1)^{([i]+1)([j]+1)} \bar{c}_{ij}^r.$$
Given the universal enveloping algebra $\mathfrak{G}$ of $\mathcal{G}$, we assume that there is an invariant even quadratic element $h^{ij}e_i e_j$ of $\mathfrak{G}$ such that the matrix $h^{ij}$ is nondegenerate.

The Yang–Mills theory of gauge potentials on $X = \mathbb{R}^n$ associated to the Lie superalgebra $\mathcal{G}$ is described by the BGDA $\mathcal{P}^*$ where

$$Q = (X \times \mathcal{G}_1) \otimes_X T^*X, \quad Y = (X \times \mathcal{G}_0) \otimes_X T^*X.$$ 

Its basis is $\{a^r_\lambda\}, [a^r_\lambda] = [r]$. There is the canonical decomposition of the first jets of its elements

$$a^r_{\lambda \mu} = \frac{1}{2}(F^i_{\lambda \mu} + S^i_{\lambda \mu}) = \frac{1}{2}(a^r_{\lambda \mu} - a^r_{\mu \lambda} + c^r_{ij}a^i_\lambda a^j_\mu) + \frac{1}{2}(a^r_{\lambda \mu} + a^r_{\mu \lambda} - c^r_{ij}a^i_\lambda a^j_\mu).$$

Then the Euclidean Yang–Mills Lagrangian takes the form

$$L_{YM} = \frac{1}{4}h^{ij}\eta^{\lambda \mu}\eta^{\beta \nu}F^i_{\lambda \beta}F^j_{\mu \nu}d^n x,$$

where $\eta$ is the Euclidean metric on $\mathbb{R}^n$. It degenerate because its variational derivatives $E^\lambda_r$ obey the irreducible Noether identities

$$-c^r_{ij}a^i_\lambda E^\lambda_j - d^\mu E^\lambda_\mu = 0.$$

Therefore, the above mentioned BV procedure must be applied to this field model [8]. As a result, the original BGDA $\mathcal{P}^*$ is enlarged to the BGDA $\mathcal{P}^*$ whose basis

$$\{a^r_\lambda, c^r, c^*_r\}, \quad [c^r] = ([r] + 1)\text{mod }2, \quad [c^*_r] = [c^r],$$

consists of gauge potentials $a^r_\lambda$, ghosts $c^r$ and antighosts $c^*_r$. The final gauge-fixing Lagrangian reads

$$L_{GF} = L_{YM} + c^*_r M^r_j c^j d^n x + \frac{1}{8}h^{ij}\eta^{\lambda \mu}\eta^{\beta \nu}S^i_{\lambda \mu}S^j_{\beta \nu}d^n x =$$

$$L_{YM} + [(-1)^{[r]+1}\eta^{\lambda \mu}c^*_r d^\mu (-c^r_{ij}c^i_\lambda + c^r_\lambda) + \frac{1}{2}h^{ij}\eta^{\lambda \mu}\eta^{\beta \nu}a^i_{\lambda \mu}a^j_{\beta \nu}]d^n x,$$

where

$$M^r_j = (-1)^{[r]+1}\eta^{\lambda \mu}(-(-1)^{[i][j]+1}c^r_{ij}(a^i_{\mu \lambda} + a^i_{\lambda \mu}) + \delta^r_j d^\mu),$$

is a second order differential operator acting on the ghosts $c^j$. The Lagrangian $L_{GF}$ possesses the variational BRST symmetry

$$\vartheta = (-c^r_{ij}c^j_\lambda + c^r_\lambda)\frac{\partial}{\partial a^r_\lambda} - \frac{1}{2}c^r_{ij}c^i j\frac{\partial}{\partial c^r} + (-1)^{[j]}h^{ij}\eta^{\lambda \mu}a^i_{\lambda \mu} \frac{\partial}{\partial c^*_j}.$$
Quantizing this Lagrangian system in the framework of Euclidean perturbative QFT, we come to the graded commutative tensor algebra $B_\Phi$ generated by the elements \{\(a_{x\lambda}^r, c_{x\lambda}^r, c_{x\lambda}^*\}\). Its state \(\langle . \rangle\) is given by functional integrals

\[
\langle \phi \rangle = \frac{1}{N} \int \phi \exp\{-\int L_{GF}(a_{x\lambda\lambda}^r, c_{x\lambda}^r, c_{x\lambda}^* d^n x) \prod_x [da_{x\lambda}^r][dc_{x\lambda}^r][dc_{x\lambda}^*],
\]

\[
N = \int \exp\{-\int L_{GF}(a_{x\lambda\lambda}^r, c_{x\lambda}^r, c_{x\lambda}^* d^n x) \prod_x [da_{x\lambda}^r][dc_{x\lambda}^r][dc_{x\lambda}^*],
\]

\[
L_{GF} = L_{YM} + (-1)^{|r|+1}\eta^\lambda_{x\mu} c_{x\lambda}^* d_\mu (-c_{xij}^r c_{ij}^a a_{a\lambda}^r + c_{x\lambda}^r) + \frac{1}{2} h_{ij}\eta^\lambda_{x\mu} a_{a\lambda\mu}^i a_{a\lambda\mu}^j.
\]

Accordingly, the quantum BRST transformation (33) reads

\[
\hat{\vartheta} = \int \left[(-c_{ij}^r c_{ij}^a a_{a\lambda}^r + c_{x\lambda}^r) \frac{\partial}{\partial a_{x\lambda}^r} - \frac{1}{2} h_{ij} c_{ij}^a c_{ij}^a \frac{\partial}{\partial c_{x\lambda}^r} + (-1)^{|j|} h_{ij}\eta^\lambda_{x\mu} a_{a\lambda\mu}^i \frac{\partial}{\partial c_{x\lambda}^*} \right] d^n x.
\]

It is readily observed that \(\text{Sp}(\hat{\vartheta}) = 0\). Therefore, we obtain the Ward identities \(\langle \hat{\vartheta}(\phi) \rangle = 0\) (37) without anomaly.

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