Abstract

Let $f$ be an entire function of finite order $\rho \in (0, 1)$. The maximum modulus $M(r)$ of $f$ and the counting function of the zeros $N(r)$ are connected by the following best possible growth inequality known as Valiron's Theorem:

$$\limsup_{r \to \infty} \frac{N(r)}{\log M(r)} \geq \frac{\pi \rho}{\sin \pi \rho}.$$

For functions subharmonic in $\mathbb{R}^d$, Hayman obtained a corresponding result with a best possible constant involving the dimension $d$. For the special case of an entire function on $\mathbb{C}^d$, we obtain a corresponding dimension-free, best possible inequality.

1 Introduction

Let $f$ be a non-constant entire function. Let $M(r; f) = \sup_{|z|=r} |f(z)|$, the maximum modulus, and $N(r, \frac{1}{f})$ the smoothed counting function for the zeros of $f$. The order of $f$ is defined by

$$\rho = \limsup_{r \to \infty} \frac{\log \log M(r; f)}{\log r}. \quad (1)$$

The question of comparing the growth of the two real functions $\log M(r; f)$ and $N(r, \frac{1}{f})$ at infinity, in particular supplying a sharp lower bound in terms of $\rho$ for $\limsup_{r \to \infty} \frac{N(r, \frac{1}{f})}{\log M(r, f)}$, is a rather difficult question in the theory that remains largely unsolved at present. For recent developments see [1] and [2]. In the special case where $0 < \rho < 1$, a solution was supplied in the following
well-known result of Polya-Valiron [5], [7], [8]

\[
\limsup_{r \to \infty} \frac{N(r, \frac{1}{f})}{\log M(r; f)} \geq \frac{\sin \pi \rho}{\pi \rho}.
\]

Equality holds for entire functions of order \( \rho \in (0, 1) \) whose zeros are regularly distributed over one ray in the complex plane.

Similar sharp results are also known if \( \log M(r; f) \) is replaced by \( m_p(r; \log |f|) \), the \( L^p \) mean defined by

\[
m_p(r; \log |f|) = \left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\log |f(re^{i\theta})||^p d\theta \right\}^{1/p},
\]

provided the order \( \rho \in (0, 1) \). If we put

\[
\psi(\theta) = \frac{\pi \rho}{\sin \pi \rho} \cos \rho \theta, -\pi \leq \theta \leq \pi,
\]

then, for each \( 1 \leq p < \infty \), we have the sharp inequality

\[
\limsup_{r \to \infty} \frac{N(r, \frac{1}{f})}{m_p(r; \log |f|)} \geq \frac{1}{\left\{ \frac{1}{2\pi} \int_{-\pi}^{\pi} |\psi(\theta)|^p d\theta \right\}^{1/p}}.
\]

It should be pointed out that, in the special case \( p = 2 \), this later result holds true for meromorphic functions of any finite order by a well-known result of Miles-Shea.

The purpose of this paper is to obtain a dimension-free analogue of the above results for entire functions of several complex variables, and of order less than 1.

2 Statement of results

Let \( f : \mathbb{C}^d \to \mathbb{C} \) be an entire function and assume for simplicity that \( f(0) = 1 \). Denote by \( S = S^{2d-1} \) the unit sphere in \( \mathbb{C}^d \). For \( \zeta \in S \), the slice function \( f_\zeta : \mathbb{C} \to \mathbb{C} \), is defined by

\[
f_\zeta(z) = f(z\zeta).
\]

Clearly, \( f_\zeta \) is, for each fixed \( \zeta \), an entire function with \( f_\zeta(0) = 1 \), and the real functions \( M(r, f_\zeta), N(r, \frac{1}{f_\zeta}) \), and \( m_p(r; \log |f_\zeta|) \) are well-defined for \( r \geq 0 \). We now define the functions \( M(r; f), n_{\max}(r) \), and \( (m_p)_{\max} \) by

\[
M(r; f) = \sup_{\zeta \in S} M(r, f_\zeta), n_{\max}(r; f) = \sup_{\zeta \in S} n(r, \frac{1}{f_\zeta}), (m_p)_{\max}(r; f) = \sup_{\zeta \in S} m_p(r; \log |f_\zeta|),
\]

and introduce the two "smoothed" functions

\[
N_{\max}(r; f) = \sup_{\zeta \in S} N(r, \frac{1}{f_\zeta}), N(r, f) = \int_0^r \frac{n_{\max}(t; f)}{t} dt.
\]
Theorem 1

If \[ \frac{1}{n} \frac{1}{\pi} \int_{S} N(r, \frac{1}{r}) d\sigma \]
and it is easily seen that \( M(r; f) \) is the maximum modulus of \( f \) on the sphere in \( \mathbb{C}^d \) centered at \( O \) and having radius \( r \). But \( N(r; f) \geq N_{\text{max}}(r; f) \), and this later is larger than the more commonly used average with respect to surface area measure \([6]\), \[ \frac{1}{\pi} \int_{S} N(r, z) d\sigma \]
so that \( N_{\text{max}}(r; f) \) is less than \( \log M(r; f) \). In particular, the order of \( N_{\text{max}}(r; f) \) is less than or equal the order of \( f \). Jensen’s formula also gives us that \( \log M(2r; f) \) is a continuous increasing function of order less than or equal the order of \( f \).

Furthermore, these inequalities are best possible.

Proof. If \( X = \lim \sup_{r \to \infty} \frac{N_{\text{max}}(r)}{\log M(r; f)} \), then for \( \epsilon > 0 \), there is an \( R > 0 \) such that \( N_{\text{max}}(r) < (X + \epsilon) \log M(r; f) \) for all \( r \geq R \). Also, since \( \log M(r; f) \) is of order \( \rho \), there exists \([4]\) a slowly varying function \( L(r) \) and a sequence \( r_n \) increasing to infinity such that

\[
\begin{align*}
(i) \quad \log M(r; f) & \leq r^\lambda L(r), \ (r > 0); \\
(ii) \quad \log M(r_n; f) & = r_n^\lambda L(r_n).
\end{align*}
\]
Using \((i)\), and a well-known inequality for the maximum modulus of entire functions of order less than 1, we obtain, for sufficiently large \(n\), that

\[
\log M(r_n; f) = \sup_{\zeta \in S} \log M(r_n; f_{\zeta}) \leq \sup_{\zeta \in S} \int_0^\infty \frac{r_n N(t)}{(t + r_n)^2} dt \leq \int_0^\infty \frac{r_n N_{\text{max}}(t)}{(t + r_n)^2} dt
\]

\[
\leq \int_0^R \frac{r_n N_{\text{max}}(t)}{(t + r_n)^2} dt + (X + \epsilon) \int_R^\infty \frac{r_n \log M(t; f)}{(t + r_n)^2} dt
\]

\[
\leq \frac{R}{r_n + r_n} \log M(R; f) + (X + \epsilon) \int_R^\infty \frac{r_n t^\rho L(t)}{(t + r_n)^2} dt.
\]

Dividing both sides by \(\log M(r_n; f)\) and taking limits as \(r_n \to \infty\), we obtain, using properties of regularly varying functions,

\[
1 \leq (X + \epsilon) \lim \sup_{r_n \to \infty} \frac{1}{\log M(r_n; f)} \int_R^\infty \frac{r_n t^\rho L(t)}{(t + r_n)^2} dt = \lim_{r \to \infty} \frac{1}{r^\rho L(r)} \int_R^\infty \frac{r t^\rho L(t)}{(t + r)^2} dt = \frac{\pi \rho}{\sin \pi \rho}.
\]

We conclude that

\[
\lim \sup_{r \to \infty} \frac{N_{\text{max}}(r)}{\log M(r; f)} \geq \frac{\sin \pi \rho}{\pi \rho}.
\]

\[\blacksquare\]

3 The Best Possible Character of the result

It remains to demonstrate the sharpness of this result. For this purpose take any canonical product of order \(\rho \in (0, 1)\), whose zeros are regularly distributed along one ray. For example, we may take

\[
P(z) = \prod_{n=1}^\infty (1 + \frac{z}{r_n}), \quad (r_n = n^{1/\rho}, 0 < \rho < 1).
\]

Now, for \(z, \eta \in \mathbb{C}^d\), write \(z = (z_1, z_2, \ldots, z_d)\), \(\eta = (\eta_1, \eta_2, \ldots, \eta_d)\) and recall that the inner product and the norm are defined by \(z \cdot \eta = \sum_{j=1}^d z_j \eta_j\), and \(\| z \| = \sqrt{z \cdot z}\). Now let \(g(z) = P(z \cdot \eta)\) where \(\eta \in S\) is fixed throughout. Thus

\[
g(z) = \prod_{n=1}^\infty (1 + \frac{z \cdot \eta}{r_n}), \quad (r_n > 0, \eta \in S, z \in \mathbb{C}^d, 0 < \rho < 1).
\]

We shall calculate \(M(r; g)\) and \(N_{\text{max}}(r)\). If \(r > 0, z \in \mathbb{C}^d\), and \(\| z \| = r\), then, since \(|z \cdot \eta| \leq \| z \|\), we have that

\[
|g(z)| \leq \prod_{n=1}^\infty (1 + \frac{\| z \|}{r_n}) = P(r).
\]
On the other hand, $|g(r\eta)| = P(r)$. Hence $M(r; g) = P(r)$. Moving on to the counting functions, if $\zeta \in S$, and $\zeta \cdot \eta = 0$, then $N(r, \frac{1}{\zeta})$ is identically zero. If $\zeta \cdot \eta \neq 0$, then the slice function may be written in the form $g_\zeta(z) = \prod_{n=1}^{\infty} (1 + \frac{z}{r_n/\zeta \eta})$. Thus, if $z \in \mathbb{C}$ and $|z| = r$, then $N(r, \frac{1}{g_\zeta})$ is given by

$$N(r, \frac{1}{g_\zeta}) = \sum_{r_n \leq r | \zeta \eta|} \log \frac{r}{r_n | \zeta \cdot \eta|} \leq \sum_{r_n \leq r | \zeta \eta|} \log \frac{r}{r_n} \leq N(r, \frac{1}{g_\eta}) = N(r, \frac{1}{P}),$$

and it follows that $N_{\max}(r) = N(r, \frac{1}{P})$.

We now invoke a well-known abelian result to conclude that

$$\lim_{r \to \infty} \frac{N_{\max}(r)}{\log M(r; g)} = \lim_{r \to \infty} \frac{N(r, \frac{1}{P})}{\log P(r)} = \frac{\pi \rho}{\sin \pi \rho}.$$ 

This establishes the best possible character of the inequality in (2).

### 3.1 The $L_p$ result

In order to prove the second part of Theorem 1, we find it convenient to introduce the function $v$ defined by

$$v(z) = \text{Re} \int_0^{\infty} \frac{z}{t(t+z)} n_{\max}(t) dt; z = re^{i\theta}, r \geq 0, |\theta| < \pi.$$

Since the order of $n_{\max}$ is less than one, $v$ is well-defined and harmonic in the plane slit along the negative real axis. If now we put for $r > 0, 0 \leq \theta < \pi$

$$u(re^{i\theta}) = \frac{1}{\pi} \int_0^\theta v(re^{i\omega}) d\omega,$$

then

$$u(re^{i\theta}) = \frac{1}{\pi} \int_0^{\infty} \frac{r \sin \theta}{t^2 + 2tr \cos \theta + r^2} N(t; f) dt,$$

and it follows that $u$ vanishes on the positive real axis and has the boundary value $N(r; f)$ on the negative real axis. Accordingly, we set $u(-r) = N(r; f), r \geq 0$.

Fix $\theta \in (0, \pi)$ and let $E$ be a set of Lebesgue measure $2\theta$. Then for $\zeta \in S$, a well known computation [1] gives us that

$$\frac{1}{\pi} \int_E \log |f_\zeta(re^{i\omega})| d\omega \leq \frac{1}{\pi} \int_0^{\infty} \frac{r \sin \theta}{t^2 + 2tr \cos \theta + r^2} N(t; \frac{1}{f_\zeta}) dt$$

$$\leq \frac{1}{\pi} \int_0^{\infty} \frac{r \sin \theta}{t^2 + 2tr \cos \theta + r^2} N(t; f) dt = \frac{1}{\pi} \int_0^\theta v(re^{i\omega}) d\omega.$$

This inequality is also valid for $\theta = 0$ and $\theta = \pi$. But $v(re^{i\theta})$ is an even function of $\theta$ and it is non-increasing on $(0, \pi)$ as can be seen e.g. by writing it as a Stieltjes integral, and so the last inequality implies that

$$(\log |f_\zeta|)^\#(re^{i\theta}) = \sup_{|E| = 2\theta} \frac{1}{\pi} \int_E \log |f_\zeta(re^{i\omega})| d\omega \leq \sup_{|E| = 2\theta} \frac{1}{\pi} \int_E v(re^{i\omega}) d\omega = v^\#(re^{i\theta}),$$

5
where the sharp denotes the function introduced by Baernstein in his definition of the star function. Now using a result of Baernstein, we conclude that

\[ m_p(r, \log |f_\zeta|) \leq m_p(r, v), \quad (1 \leq p < \infty), \]

and hence that

\[ \sup_{\zeta \in S} m_p(r, \log |f_\zeta|) \leq m_p(r, v), \quad (1 \leq p < \infty). \]

It remains to study the growth of \( m_p(r, v) \) with respect to the function \( N(r; f) \), and this will be carried out at a special sequence \( r_n \) increasing to \( \infty \) and satisfying

\[ N(r_n; f) \leq r^\sigma L(r), \quad r > 0; \quad N(r_n; f) = r_n^\sigma L(r_n). \]

Here \( L \) is a slowly varying function in the sense of Karamata i.e. \( L \) is positive and \( L(\sigma r)/L(r) \to 1 \), as \( r \to \infty \), for each \( \sigma > 1 \). The existence of this sequence follows from the fact that \( N(r; f) \) is continuous, non-decreasing, and of order \( \rho \).

### 4 Extension To Vector-Valued Functions

The following extension of the result in section 2 to vector-valued functions is straightforward.

**Theorem 2** Let \( f : \mathbb{C}^m \to \mathbb{C}^n \) be an entire function of order \( \rho \in (0, 1) \) and satisfying \( f(0) = (1, 1, \ldots, 1) \). Write \( f = (f_1, f_2, \ldots, f_n) \) and put

\[
N_{\text{max}}(r) = \max_{1 \leq j \leq n} \{ \sup_{\zeta \in S} N(r, \frac{1}{f_j}) \}, \quad M_{\text{max}}(r) = \max_{1 \leq j \leq n} \{ \sup_{\zeta \in S} M(r, f_j) \},
\]

then

\[
\limsup_{r \to \infty} \frac{N_{\text{max}}(r)}{\log M_{\text{max}}(r; f)} \geq \frac{\sin \pi \rho}{\pi \rho}.
\]

Furthermore, this inequality is best possible.

### References

[1] F. Abi-Khuzam & B. Shayya, *On a Singular integral estimate for the maximum modulus of a canonical product*, Illinois J. Math. **44** (2000), 551-555.

[2] D. F. Shea and S. Wainger, *Growth problems for a class of entire functions via singular integral estimates*, Illinois J. Math. **25** (1981), 41-50.

[3] W. Hayman, *Subharmonic Functions*

[4] B. J. Levin, *Entire Functions*.
[5] G. Polya, *Bemerkung über unendlichen Folge und ganzen Functionen*, Math. Ann. **88** (1923), 169-183.

[6] W. Rudin, *Entire Functions of Several Complex Variables*, Springer

[7] G. Valiron, *Sur les fonctions entières d’ordre fini et ordre nul, et en particulier les fonctions a correspondence reguliere*, Ann. Fac. Toulouse (3) **5** (1913), 117-257.

[8] G. Valiron, *A propos d’un memoire de M. Polya*, Bull. Sci. Math. (2) **48** (1924), 9-12.