OPEN GROMOV-WITTEN INVARIANTS OF TORIC CALABI-YAU 3-FOLDS

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ABSTRACT. We present a proof of the mirror conjecture of Aganagic-Vafa and Aganagic-Klemm-Vafa on disk enumeration in toric Calabi-Yau 3-folds for all smooth semi-projective toric Calabi-Yau 3-folds. We consider both inner and outer branes, at arbitrary framing. In particular, we recover previous results on the conjecture for (i) an inner brane at zero framing in $K_{\mathbb{P}^1}$ (Graber-Zaslow [16]), (ii) an outer brane at arbitrary framing in the resolved conifold $\mathcal{O}_\mathbb{P}^1(-1) \oplus \mathcal{O}_\mathbb{P}^1(-1)$ (Zhou [41]), and (iii) an outer brane at zero framing in $K_{\mathbb{P}^2}$ (Brini [7]).

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1. Introduction

1.1. Open Gromov-Witten invariants. Let $X$ be a Kähler manifold (or more generally, an almost Kähler manifold) and let $L$ be a Lagrangian submanifold of $X$. Naively, open Gromov-Witten (GW) invariants of the pair $(X, L)$ count holomorphic maps (or more generally, $J$-holomorphic maps) from bordered Riemann surfaces to $X$ with boundaries mapped into $L$. Open GW theory can be viewed as a mathematical theory of A-model topological open strings. String dualities have produced many striking predictions on open GW invariants, especially when $X$ is a Calabi-Yau 3-fold and $L$ is a special Lagrangian submanifold (or more generally, when the Maslov class of $L$ is zero).

In this paper, we consider toric Calabi-Yau 3-folds and a particular class of Lagrangian submanifolds (called Aganagic-Vafa A-branes in this paper) introduced by Aganagic-Vafa in [3]. Let $X$ be a toric Calabi-Yau 3-fold and $L$ be an Aganagic-Vafa A-brane in $X$ (in particular, $L$ is diffeomorphic to $S^1 \times \mathbb{R}^2$). The open GW invariants of the pair $(X, L)$ depend on the following discrete data:

- topological type $(g, h)$ of the domain bordered Riemann surface $\Sigma$, where $g$ is the number of handles and $h$ is the number of holes when $\Sigma$ is a smooth bordered Riemann surface;
- topological type of the map $u : (\Sigma, \partial \Sigma = \bigcup_{i=1}^{h} R_i) \to (X, L)$, which is given by the degree $\beta' = u_*[\Sigma] \in H_2(X, L; \mathbb{Z})$ and winding numbers $\mu_i = u_*[R_i] \in H_1(L; \mathbb{Z}) = \mathbb{Z}$;
- framing $f \in \mathbb{Z}$ of the Aganagic-Vafa A-brane $L$.

More generally, $L$ can be a disjoint union of framed A-branes $(L_1, f_1), \ldots, (L_s, f_s)$, and the open GW invariants of $(X, L)$ are

$$N_{g, \beta', \mu^1, \ldots, \mu^s}^{X, L}(f_1, \ldots, f_s) \in \mathbb{Q}$$

where $\mu^i = (\mu^i_1, \ldots, \mu^i_{h_i})$ are the winding numbers associated to $L_i$, and $h = \sum_{i=1}^{s} h_i$.

When $L_1, \ldots, L_s$ are outer branes, $\mu^i_j$ are positive integers, and $\mu^i$ can be viewed as a (possibly empty) partition. In this case, the authors of [29] provided a mathematical definition of the open GW invariants in (1) as formal relative GW invariants of a relative formal toric Calabi-Yau (FTCY) 3-fold $(\tilde{Y}, D)$.

$$N_{g, \beta', \mu^1, \ldots, \mu^s}^{X, L}(f_1, \ldots, f_s) := \sum_{\pi(\tilde{\beta'}) = \beta'} N_{g, \tilde{\beta'}, \mu^1, \ldots, \mu^s}^{\tilde{Y}, \tilde{D}}$$

On the right hand side of (2), $\pi$ is a surjective group homomorphism $H_2(\tilde{Y}; \mathbb{Z}) \to H_2(X, L; \mathbb{Z})$, and the relative FTCY 3-fold $(\tilde{Y}, \tilde{D})$ is determined by the toric Calabi-Yau 3-fold $X$ and the framed Aganagic-Vafa A-branes $(L_1, f_1), \ldots, (L_s, f_s)$. More details will be given in Section 3.

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1In this paper, we use slightly different conventions. See [17].
1.2. **Large N duality: conjectures and results.** The large $N$ duality relates A-model topological string theory on Calabi-Yau 3-folds to Chern-Simons gauge theory on 3-manifolds. Motivated by the large $N$ duality, Aganagic-Klemm-Mariño-Vafa proposed the topological vertex [1], an algorithm of computing generating functions

$$
F_{\beta}(\lambda) = \sum_{g} N_{g,\beta}^{X} \lambda^{2g-2},
$$

(3)

$$
F_{\beta,\mu_{1},...,\mu_{s}}^{X}(\lambda; f_{1}, \ldots, f_{s}) = \sum_{g} N_{g,\beta,\mu_{1},...,\mu_{s}}^{X} (f_{1}, \ldots, f_{s}) \lambda^{2g-2+h},
$$

where $\beta \in H_{2}(X; \mathbb{Z})$ and $N_{g,\beta}^{X}$ is the genus $g$, degree $\beta$ (closed) GW invariants of $X$. Note that the generating functions in (3) are obtained by fixing a topological type of the map and summing over all possible topological types of the domain.

Let $Z_{X}^{\beta}$ and $Z_{2}^{\beta,\mu_{1},...,\mu_{s}}$ be the disconnected versions of $F_{\beta}^{X}$ and $F_{\beta,\mu_{1},...,\mu_{s}}^{X}$ respectively. The algorithm of AKMV consists of two steps:

O1. Explicit gluing formula which expresses $Z_{X}^{\beta}$, $Z_{X}^{\beta,\mu_{1},...,\mu_{s}}$, in terms of the topological vertex $C_{\mu_{1},\mu_{2},\mu_{3}}$, a generating function of open GW invariants of $\mathbb{C}^{3}$ relative to three Aganagic-Vafa A-branes.

O2. Evaluation of the topological vertex $C_{\mu_{1},\mu_{2},\mu_{3}}$ by relating it to the colored HOMFLY polynomial of a 3-component link in $S^{3}$.

In [29], Li-Liu-Liu-Zhou developed a mathematical theory of the topological vertex using algebraic relative GW theory developed by J. Li [26, 27].

R0. Open GW invariants of $(X, L)$ are defined as formal relative GW invariants of a relative FT CY 3-fold $(Y, D)$.

R1. The degeneration formula satisfied by these formal relative GW invariants agrees with the gluing formula in [1], with $C_{\mu_{1},\mu_{2},\mu_{3}}$ replaced by a generating function $\tilde{C}_{\mu_{1},\mu_{2},\mu_{3}}$ of invariants of an in-decomposable relative FT CY 3-fold.

R2. LLLZ evaluated $\tilde{C}_{\mu_{1},\mu_{2},\mu_{3}}$ using virtual localization and a formula of Hodge integrals provided in [30], and showed that it agrees with the formula of $C_{\mu_{1},\mu_{2},\mu_{3}}$ in [1] when one of the partitions is empty.

This proves the topological vertex algorithm up to 2-leg vertex. The validity of the full 3-leg case is a consequence of the proof of Gromov-Witten/Donaldson-Thomas correspondence of toric Calabi-Yau 3-folds by Maulik-Okounkov-Oblomkov-Pandharipande [35].

1.3. **Mirror Symmetry: conjectures and results.** The mirror symmetry relates A-model topological string theory on a Calabi-Yau 3-fold $X$ to the B-model topological string theory on the mirror Calabi-Yau 3-fold $X^\vee$.

Let $X$ be a smooth toric Calabi-Yau 3-fold. We assume that $X$ is semi-projective, i.e., $X$ has a torus fixed point and $X$ is projective over its affinization $\text{Spec}(H^{0}(X, O_{X}))$. By the results in [21, Section 2], a smooth toric variety is semi-projective if and only if it is equal to the GIT quotient of an affine space $C'$ by the action of a subtorus of $(\mathbb{C}^{*})^{r}$. Let $L$ be an Aganagic-Vafa A-brane in $X$. Aganagic-Vafa related a generating function of disk invariants of the pair $(X, L)$ to the Abel-Jacobi map of the mirror curve of $X$ [3]. It was clarified in [2] that the framing of $L$ corresponds to choice of flat coordinates in the B-model. The integrals in [2] are solutions to extended Picard-Fuchs equations [23, 24, 36, 37]. To our knowledge, the above mirror conjectures on disk invariants has been verified in the following cases: (i) $X$ is the total space of $K_{\mathbb{P}^{2}}$ and $L$ is an inner brane at zero framing (T. Graber and E. Zaslow [10]); (ii) $X$ is the resolved conifold (i.e. the total space of

\[\text{Symplectic relative GW theory was developed independently by Li-Ruan [25] and Ionel-Parker [18, 19].}\]
$O_p_1(-1) \oplus O_p_1(-1)$ and $L$ is an outer brane at arbitrary framing (J. Zhou [41]); (iii) $X$ is the total space of $K_{\mathcal{P}^2}$ and $L$ is an outer brane at zero framing (A. Brini [7, Section 5.3]).

Based on the work of Eynard-Orantin [10] and Maríño [34], Bouchard-Klemm-Maríño-Pasquetti [5] proposed a new formalism of the B-model topological strings on the mirrors of toric Calabi-Yau 3-folds, and provided an recursive algorithm, now known as the remodeling conjecture, which determines the generating functions

\[
F^X_g(Q) = \sum_{\beta \in H_2(X;\mathbb{Z})} N^X_{g,\beta} Q^\beta
\]

\[
F^{X,L}_{g,h}(Q, x_1, \ldots, x_h; f) = \sum_{\beta', \mu = (\mu_1, \ldots, \mu_h)} N_{\beta', \mu_1, \ldots, \mu_h}(f) Q^{\beta'} x_1^{\mu_1} \cdots x_h^{\mu_h}
\]

from the disk invariants $F^X_{0,1}$ and annulus invariants $F^{X,L}_{0,2}$. Note that the generating functions are obtained by fixing the topological type of the domain and summing over all possible topological types of the map. The remodeling conjecture has been proved for the framed 1-leg topological vertex by L. Chen [8] and by J. Zhou [39]. J. Zhou later proved the conjecture for the framed 3-leg topological vertex [40]. Recently, Bouchard-Catuneanu-Marchal-Sulkowski proved the remodeling conjecture for $F^C_3$ [4].

### 1.4. Summary of results

The goals of the present paper are twofold.

(i) We use localization to derive a formula of formal relative GW invariants of the relative FTCY 3-fold $(\hat{Y}, \hat{D})$ in terms of descendant GW invariants of the FTCY 3-fold $\hat{X} := \hat{Y} - \hat{D}$ (Proposition 3.1). As a consequence, we obtain a formula of open GW invariants of $(X, L)$ in terms of descendant GW invariants of $X$ when $L$ is a union of framed outer branes (Proposition 3.2). The formula allows us to extend the definition to a single framed inner brane (Proposition 3.4).

(ii) We use the approach in Graber-Zaslow [16] to prove the mirror conjecture on disk invariants in [3, 2, 23, 24, 36, 37] for any smooth semi-projective toric Calabi-Yau 3-folds. We consider both outer and inner branes, at arbitrary framing (Conjecture 4.3). We list explicit mirror formulae for the resolved conifold, local toric Fano surfaces, and toric crepant resolutions of $(O_p_1(-1) \oplus O_p_1(-1))/\mathbb{Z}_m$.

### 1.5. Outline of the paper

In Section 2 we review the geometry and topology of toric Calabi-Yau 3-folds and Aganagic-Vafa A-branes. In Section 3 we introduce and compare various GW invariants: formal relative GW invariants of $(\hat{Y}, \hat{D})$, descendant GW invariants of $\hat{X}$, open GW invariants of $(X, L)$, and descendant GW invariants of $X$. In Section 4 we review the geometry of mirrors of toric Calabi-Yau 3-folds and Aganagic-Vafa B-branes, and state the mirror conjecture on holomorphic disks (Conjecture 4.3). The proof of Conjecture 4.3 is given in Section 5. We list explicit mirror formulae in Section 6.

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### 2. Geometry and Topology of Toric Calabi-Yau 3-Folds

#### 2.1. Toric varieties as geometric quotients

We refer to [12] for the theory of toric varieties. In this subsection, we consider smooth, possibly noncompact toric varieties of any dimension.

Let $N \cong \mathbb{Z}^n$ be a free abelian group, and let $\Delta$ be a fan in $N$ (or in $N_R = N \otimes \mathbb{R} \cong \mathbb{R}^n$) of strongly convex rational polyhedral cones. Let $X = X(\Delta)$ be the toric variety associated to $\Delta$. In this paper, we assume that $X$ is smooth. We use the following notation:

- $\Delta(d)$ is the set of $d$-dimensional cones in $\Delta$. 

Given $\sigma \in \Delta(d)$, let $V(\sigma)$ denote the codimension $d$ orbit closure associated to $\sigma$.

- Let $\Delta(1) = \{\rho_1, \ldots, \rho_r\}$ be the set of 1-dimensional cones in $\Delta$, and let $v_i \in N$ be the unique generator of the semigroup $\rho_i \cap N$, so that $\rho_i \cap N = \mathbb{Z}_{\geq 0} v_i$.

- Let $M = \text{Hom}(M, \mathbb{Z})$ be the dual lattice of $N$.

There is a group homomorphism

$$\phi : \hat{N} := \bigoplus_{i=1}^{r} \mathbb{Z} \cdot \bar{v}_i \cong \mathbb{Z}^r \longrightarrow N \cong \mathbb{Z}^n, \quad \bar{v}_i \mapsto v_i.$$ 

We assume that $\phi$ is surjective. Let

$$l^{(a)} = (l_{1}^{(a)}, \ldots, l_{r}^{(a)}), \quad a = 1, \ldots, k,$$

be a $\mathbb{Z}$-basis of $\ker \phi \cong \mathbb{Z}^k$, where $k = r - n$. Let $\widetilde{M} = \text{Hom}(M; \mathbb{Z})$ be the dual lattice of $\hat{N}$. Since $\phi : \hat{N} \to N$ is surjective, the dual map $\phi^* : M \to \widetilde{M}$ is injective. Applying $\text{Hom}(-, \mathbb{C}^*)$ to the following short exact sequence

$$0 \to M \overset{\varphi^*}{\longrightarrow} \widetilde{M} \to A_{n-1}(X) \to 0,$$

we obtain another short exact sequence

$$(5) \quad 1 \to G \to \mathcal{T} \to \mathbb{T} \to 1,$$

where

$$G := \text{Hom}(A_{n-1}(X), \mathbb{C}^*) \cong (\mathbb{C}^*)^k,$$

$$\mathcal{T} := \text{Hom}(\widetilde{M}, \mathbb{C}^*) \cong (\mathbb{C}^*)^r,$$

$$\mathbb{T} := \text{Hom}(M, \mathbb{C}^*) \cong (\mathbb{C}^*)^n.$$

We also denote

$$\mathbb{L} = \ker \phi, \quad \mathbb{L}^\vee = A_{n-1}(X).$$

Notice that $\mathbb{L}^\vee \cong H^2(X; \mathbb{Z})$, and $\mathbb{L} \cong H_2(X; \mathbb{Z})$.

The torus $\mathcal{T} \cong (\mathbb{C}^*)^r$ acts on $\mathcal{C}^r = \text{Spec} \mathbb{C}[X_1, \ldots, X_r]$. Let $I(\Delta) \subset \mathbb{C}[X_1, \ldots, X_r]$ be the ideal generated by $\{\prod_{\rho_i \in \sigma} X_i \mid \sigma \in \Delta\}$, and let $Z(\Delta) \subset \mathcal{C}^r$ be the subvariety defined by $I(\Delta)$. Then $X$ can be identified with the geometric quotient:

$$(6) \quad X = (\mathcal{C}^r - Z(\Delta))/G,$$

where $G = (\mathbb{C}^*)^k$ acts on $\mathcal{C}^r$ by

$$(t_1, \ldots, t_k) \cdot (X_1, \ldots, X_r) = \left( \prod_{a=1}^{k} t_{a}^{i_{a}(e)} \cdot X_1, \ldots, \prod_{a=1}^{k} t_{a}^{i_{a}(e)} \cdot X_r \right).$$

The $\mathcal{T}$-divisor $\{X_i = 0\}$ in $\mathcal{C}^r - Z(\Delta)$ descends to a $\mathbb{T}$-divisor $D_i$ in $X$. When $X$ is semi-projective, the quotient (6) is also a quotient in the sense of GIT (geometric invariant theory) [21, Section 2].

### 2.2. Toric varieties as symplectic quotients

When $X$ is a smooth semi-projective toric variety, we may describe Kähler structures on $X$ as a symplectic quotient. Let $G_\mathbb{R} \cong U(1)^k$ be the maximal compact subgroup of $G \cong (\mathbb{C}^*)^k$. Then the dual $\mathfrak{g}_\mathbb{C}^*$ of the Lie algebra $\mathfrak{g}_\mathbb{R}$ of $G_\mathbb{R}$ can be canonically identified with $A_{n-1}(X) \otimes \mathbb{R} = H^{1,1}(X_\Sigma; \mathbb{R})$. Let $\mu : C^r \to \mathfrak{g}_\mathbb{C}^* \cong \mathbb{R}^k$ be the moment map of the Hamiltonian $G_\mathbb{R}$-action on $C^r$, equipped with the standard Kähler form

$$(7) \quad \frac{\sqrt{-1}}{2} \sum_{i=1}^{r} dX_i \wedge \overline{dX}_i.$$
Then
\[ \check{\mu}(X_1, \ldots, X_r) = (\sum_{i=1}^{r} t_i^{(1)}|X_i|^2, \ldots, \sum_{i=1}^{r} t_i^{(k)}|X_i|^2). \]

Let \((r_1, \ldots, r_k) \in H^{1,1}(X; \mathbb{R}) \cong \mathbb{R}^k\) be a Kähler class. Then
\[ X = \check{\mu}^{-1}(r_1, \ldots, r_k)/G_\mathbb{R}. \]
The standard Kähler form (7) on \(\mathbb{C}^r\) descends to a Kähler form \(\omega_{r_1, \ldots, r_k}\) on the quotient (8) in class \((r_1, \ldots, r_k) \in H^{1,1}(X; \mathbb{R})\). The real numbers \(r_1, \ldots, r_k\) are known as Kähler parameters of \(X\). Let \(T_a = -r_a + \sqrt{-1}\theta_a\) be complexified Kähler parameters of \(X\).

2.3. The Calabi-Yau condition. We say a toric variety \(X\) is Calabi-Yau if the canonical divisor \(K_X = -D_1 - \cdots - D_r\) is trivial. \(X\) is Calabi-Yau if and only if
\[ \sum_{i=1}^{r} t_i^{(a)} = 0, \quad a = 1, \ldots, k \]
When \(X\) is Calabi-Yau, we have a short exact sequence
\[ 1 \to T' \to T \to \mathbb{C}^r \to 1, \]
where \(T' \cong (\mathbb{C}^r)^{n-1}\) is the subtorus of \(T\) that acts trivially on \(\Lambda^n T_X \cong \mathcal{O}_X\).

2.4. Aganagic-Vafa A-branes. Let \(X = \check{\mu}^{-1}(r_1, \ldots, r_k)/G_\mathbb{R}\) be a smooth semi-projective toric Calabi-Yau 3-fold equipped with the Kähler form \(\omega_{r_1, \ldots, r_k}\). Here \(\check{\mu}^{-1}(r_1, \ldots, r_k)\) is defined by
\[ \sum_{i=1}^{k+3} t_i^{(a)}|X_i|^2 = r_a, \quad a = 1, \ldots, k. \]
Write \(X_i = \rho_i e^{\sqrt{-1}\phi_i}\), where \(\rho_i = |X_i|\). In [3], Aganagic-Vafa introduced Lagrangian submanifolds \(L\) of \(X\) defined by additional constraints:
\[ \sum_{i=1}^{k+3} t_i^{(a)} \in \mathbb{Z}, \quad \sum_{i=1}^{k+3} t_i^{(a)} = 0, \quad a = 1, 2. \]
Such a Lagrangian submanifold is diffeomorphic to \(S^1 \times \mathbb{C}\) and intersects a unique 1-dimensional orbit closure \(V(t_L)\), where \(t_L \in \Delta(2)\), along a circle. We say \(L\) is an outer brane if \(V(t_L) \cong \mathbb{C}\), and we say \(L\) is an inner brane if \(V(t_L) \cong \mathbb{P}^1\).

2.5. Framing. Let \(T'_R \cong U(1)^2\) be the maximal compact subgroup of \(T' \cong (\mathbb{C}^*)^2\). Then \(T'_R\) preserves any Aganagic-Vafa A-brane \(L\). The \(T'_R\)-action on \(L\) is given by
\[ (t_1, t_2) \cdot (z, u) = (t_1 z, t_2 u) \]
where \((t_1, t_2) \in T'_R = U(1) \times U(1), (z, u) \in L \cong S^1 \times \mathbb{C}\). Therefore \((X, L)\) is a \(T'_R\)-equivariant pair. The \(T'_R\)-action on \(X\) is Hamiltonian with respect to the symplectic form \(\omega_{r_1, \ldots, r_k}\). Let \(\mu^*: X \to (t'_R)^*\) be the moment map of the \(T'_R\)-action, where \((t'_R)^* \cong \mathbb{R}^2\) is the dual of the Lie algebra \(t'_R\) of \(T'_R\).

We define the 1-skeleton of \(X\) to be
\[ X^1 = \bigcup_{\tau \in \Delta(2)} V(\tau). \]
Then \( \mu'(X^1) \) is an immersed trivalent graph. \( \mu'(V(\tau)) \) is a line segment if \( V(\tau) \cong \mathbb{P}^1 \), and is a ray if \( V(\tau) \cong \mathbb{C} \). Suppose that \( \sigma \) is a 3-dimensional cone in \( \Delta \) and \( \tau_L \subset \sigma \). Then \( V(\sigma) \subset V(\tau_L) \), where \( V(\sigma) \) is the \( T \)-fixed point associated to \( \sigma \). Let

\[
u_L = (c_1)_{\tau_L}(T_{V(\sigma)}V(\tau_L)) \in H^2_\tau(V(\sigma); \mathbb{Z}) \cong H^2(BT'; \mathbb{Z}) \subset H^2(BT'; \mathbb{R}) \cong (\ell'_\mathbb{R})^*.
\]

Then \( u_L \) is tangent to \( \mu(V(\tau_L)) \). Let \( \tau'_L \in \Delta(2) \) be the 2-cone such that \( \tau'_L \subset \sigma \) and \( \mu'(V(\tau'_L)) \) is the first edge encountered in the counterclockwise direction from \( \mu'(V(\tau_L)) \). Define

\[
u_L = (c_1)_{\tau'_L}(T_{V(\sigma)}V(\tau'_L)) \in H^2_\tau(V(\sigma); \mathbb{Z}) \subset (\ell'_\mathbb{R})^*.
\]

Let \( \rho_1 \subset \sigma \) be the 1-cone that is not in \( \tau_L \), and let \( \rho_{i_2} \subset \sigma \) be the 1-cone that is not in \( \tau'_L \). We use \( \rho_{i_3} \) to denote the 1-cone in \( \sigma \) other than \( \rho_1, \rho_{i_2} \). If \( V(\tau_L) \cong \mathbb{P}^1 \), then there is another 3-cone \( \sigma' \in \Delta(3) \) other than \( \sigma \) of which \( \tau_L \) is a face. Let \( \rho_{i_4} \) be the 1-cone in \( \sigma' \) but not in \( \tau_L \).

**Figure 1.** images of \( D_{i_1}, D_{i_2}, D_{i_3}, D_{i_4}, V(\sigma), V(\sigma') \) under \( \mu' \).

Let \( L \) be an Aganagic-Vafa A-brane. Then \( \mu'(L) \) is a point. A framing of \( L \) is a choice of a vector \( f \in H^2(BT'; \mathbb{Z}) \) such that \( u_L \wedge f = u_L \wedge v_L \), i.e., \( f = v_L - fu_L \) for some \( f \in \mathbb{Z} \). We sometimes call the integer \( f \) the a framing of \( L \). See Figure 2.

**Figure 2.** framing of an A-brane

We may view \( f \) as an element in \( \text{Hom}(T', \mathbb{C}^*) \). Let \( T_{L,f} \cong \mathbb{C}^* \) be the kernel of \( f : T' \to \mathbb{C}^* \).

### 2.6. Topology

Suppose that \( p_1, \ldots, p_k \) is a \( \mathbb{Z} \)-basis of \( H^2(X; \mathbb{Z}) \cong \mathbb{L}^\vee \) dual to \( l^{(1)}, \ldots, l^{(k)} \in \mathbb{L} \). We choose appropriate \( \{l^{(a)}_{i_1}, \ldots, l^{(a)}_{i_k}\} \) and let \( \omega_{i_1}, \ldots, \omega_{i_k} \) be \( \omega_{i_1}, \ldots, \omega_{i_k} \) such that the class of \( \omega_{i_1}, \ldots, \omega_{i_k} \) is \( \sum_{a=1}^k r_a p_a \in H^2(X; \mathbb{R}) = H^{1,1}(X; \mathbb{R}) \). Let \( Q_a = e^{2\pi i \omega} \), which are formal Kähler parameters. Given \( \beta \in H^2(X; \mathbb{Z}) \), define

\[
Q^\beta := e^{(\beta, \sum_{a=1}^k r_a p_a)} = \prod_{a=1}^k Q_a^{d_a},
\]

where \( d_a = \langle \beta, p_a \rangle \), \( a = 1, \ldots, k \). For \( i = 1, \ldots, r \), we have

\[
D_i^* := c_1(O_X(D_i)) = \sum_{a=1}^k l^{(a)}_{i_a} p_a, \quad \langle D_i^*, \beta \rangle = \sum_{a=1}^k d_a l^{(a)}_{i_a}.
\]

Let \( \Delta(2)_{c} = \{ \tau \in \Delta(2) \mid V(\tau) \cong \mathbb{P}^1 \} \). The inclusion \( X^1 \hookrightarrow X \) induces a surjective group homomorphism

\[
H_2(X^1; \mathbb{Z}) = \bigoplus_{\tau \in \Delta(2)_{c}} V(\tau) \to H_2(X; \mathbb{Z}).
\]
Let
\[ L = L_1 \cup \cdots \cup L_s \]
be a disjoint union of \( s \) Aganagic-Vafa A-branes \( L_1, \ldots, L_s \) in \( X \). We have the following long exact sequence of relative homology groups of the pair \((X, L)\):
\[
\cdots \to H_2(L; \mathbb{Z}) \to H_2(X; \mathbb{Z}) \to H_2(X, L; \mathbb{Z}) \xrightarrow{\partial} H_1(L; \mathbb{Z}) \to H_1(X; \mathbb{Z}) \to \cdots
\]
where \( H_2(L; \mathbb{Z}) = 0 \) and \( H_1(X; \mathbb{Z}) = 0 \).

For \( i = 1, \ldots, s \), let \( V(\tau_i) \) be the unique 1-dimensional orbit closure that intersects \( L_i \). We assume that \( \tau_1, \ldots, \tau_s \) are distinct. We choose a \( \mathbb{T} \)-fixed point \( x_i \in V(\tau_i) \). When \( L_i \) is an outer brane, \( V(\tau_i) \cong \mathbb{C} \), and there is a unique such point; when \( L_i \) is an inner brane, there are exactly two such points, \( x_i \) and \( x_i^- \). Then \( V(\tau_i) - L_i \cap V(\tau_i) \) has two connected components, one of which is a holomorphic disk \( B_i \) containing \( x_i \). We orient \( B_i \) by the holomorphic structure. Then \( B_i \) represents a relative homology class \( b_i \in H_2(X, L; \mathbb{Z}) \). Let \( \gamma_i \in H_1(L_i; \mathbb{Z}) \) be the image of \( b_i \) under the map \( \partial \) in (10). Then
\[
H_1(L; \mathbb{Z}) = \bigoplus_{i=1}^{s} H_1(L_i; \mathbb{Z}) = \bigoplus_{i=1}^{s} \mathbb{Z} \gamma_i.
\]
Any element \( \beta' \in H_2(X, L; \mathbb{Z}) \) is of the form
\[
\beta' = \beta + \sum_{i=1}^{s} d_i b_i,
\]
where \( d_i \in \mathbb{Z} \) and \( \beta \in H_2(X; \mathbb{Z}) \).

Let \( \mathcal{A} \) be the collection of anti-cones
\[
\mathcal{A} = \{ J \subset \{1, \ldots, r\} | \sum_{j \notin J} \mathbb{R}_{\geq 0} v_j \text{ is a cone in } \Delta \}.
\]
The effective cone \( \mathbb{L}_{\text{eff}} \) is
\[
\mathbb{L}_{\text{eff}} = \{ \beta \in \mathbb{L} | \{ i \in \{1, \ldots, r\} | \langle D_i^*, \beta \rangle \in \mathbb{Z}_{\geq 0} \} \in \mathcal{A} \}.
\]
Usually the cone \( \mathbb{L}_{\text{eff}} \otimes \mathbb{Z} \mathbb{R} \) in the real vector space \( \mathbb{L} \otimes \mathbb{Z} \mathbb{R} \) is called the Mori cone. One says \( \beta \geq 0 \) if \( \beta \in \mathbb{L}_{\text{eff}} \) and \( \beta > 0 \) if \( \beta \) is also non-zero. The closure of the Kähler cone in \( \mathbb{L}^\vee \otimes \mathbb{Z} \mathbb{R} \) is
\[
C_X = \bigcap_{I \in \mathcal{A}} (\sum_{i \in I} \mathbb{R}_{>0} v_i).
\]
It is dual to the Mori cone.

3. Gromov-Witten Invariants

Let \( X \) be a toric Calabi-Yau 3-fold defined by a fan \( \Delta \). Let
\[
(L_1, f_1), \ldots, (L_s, f_s)
\]
be framed Aganagic-Vafa A-branes (see Section 2.4), and let \( \tau_1, \ldots, \tau_s \in \Delta(2) \) be defined as in Section 2.6. Let \( \sigma_1, \ldots, \sigma_s \in \Delta(3) \) be the top dimensional cones corresponding to \( \mathbb{T} \) fixed points \( x_1, \ldots, x_s \), respectively, where \( x_i \) are defined as in Section 2.6. Then
\[
\tau_i \subset \sigma_i, \quad V(\sigma_i) = \{x_i\}, \quad i = 1, \ldots, s.
\]
Define
\[
w_i^1 = u_{L_i}, \quad w_i^2 = v_{L_i}, \quad w_i^3 = -w_i^1 - w_i^2, \quad f_i = w_i^2 - f_i w_i^1.
\]
In Section 3.1 [3.4], we assume \( L_1, \ldots, L_s \) are outer branes. In Section 3.5 [3.6] \( L = L_1 \) is a single outer or inner brane.
3.1. The relative FTCY 3-fold \((\hat{Y}, \hat{D})\). We refer to [29] Section 3 for the definition of formal toric Calabi-Yau (FTCY) graphs and construction of relative FTCY 3-folds. Let \(\Gamma_X\) be the formal toric Calabi-Yau (FTCY) graph associated to the smooth toric 3-fold \(X\) (see [29] Section 3.1). We define a FTCY graph

\[ \Gamma_{\eta, t_1, \ldots, t_s,...} \] by replacing the noncompact edge \(\hat{e}_i\) in \(\Gamma_X\) associated to \(\tau_i\) by a compact edge \(\hat{e}_i'\) with framing \(f_i\). Figure 3.1 shows the construction near the edge \(\hat{e}_i\).

![Figure 3](image)

\[ \text{Figure 3. The FTCY graphs } \Gamma_X \text{ and } \Gamma_{\eta, t_1, \ldots, t_s,...}. \text{ The trivalent vertex } v_i \text{ corresponds to the cone } \sigma_i \in \Delta(3). \]

Let \((\hat{Y}, \hat{D})\) be the relative FTCY 3-fold associated to the FTCY graph \(\Gamma = \Gamma_{\eta, t_1, \ldots, t_s,...}\). Then \(\hat{D}\) is the disjoint union of its connected components \(\hat{D}_1, \ldots, \hat{D}_s\) which correspond to the \(s\) univalent vertices \(v'_1, \ldots, v'_s\) in \(\Gamma\). \(T = (\mathbb{C}^*)^3\) and \(T' = (\mathbb{C}^*)^2\) acts on \((\hat{Y}, \hat{D})\). Let \(E_c(\Gamma_X)\) and \(E_c(\Gamma)\) denote the set of compact edges in \(\Gamma_X\) and \(\Gamma\), respectively. Then \(E_c(\Gamma) = E_c(\Gamma_X) \cup \{\hat{e}_1', \ldots, \hat{e}_s'\}\).

Each compact edge \(\hat{e}\) corresponds to a torus invariant projective line \(C^{\hat{e}}\) in \(X\) or in \(\hat{Y}\). We have

\[ H_2(X^1; \mathbb{Z}) = \bigoplus_{\hat{e} \in E_c(\Gamma_X)} \mathbb{Z}[C^{\hat{e}}], \]

\[ H_2(\hat{Y}; \mathbb{Z}) = \bigoplus_{\hat{e} \in E_c(\Gamma)} \mathbb{Z}[C^{\hat{e}}] = H_2(X^1; \mathbb{Z}) \oplus \bigoplus_{i=1}^s \mathbb{Z}[C^{\hat{e}}], \]

where \(C^{\hat{e}_i} = V(\tau_i)\). There is a surjective group homomorphism

\[ \pi : H_2(\hat{Y}; \mathbb{Z}) = H_2(X^1; \mathbb{Z}) \oplus \bigoplus_{i=1}^s \mathbb{Z}[C^{\hat{e}_i}] \to H_2(X, L; \mathbb{Z}) = H_2(X; \mathbb{Z}) \oplus \bigoplus_{i=1}^s \mathbb{Z}b_i \]

given by

\[ \beta + \sum_{i=1}^s d_i[C^{\hat{e}_i}] \mapsto i_* \beta + \sum_{i=1}^s d_i b_i, \]

where \(\beta \in H_2(X^1; \mathbb{Z})\), and \(i_* : H_2(X^1; \mathbb{Z}) \to H_2(X; \mathbb{Z})\) is induced by the inclusion \(i : X^1 \to X\).

3.2. Formal relative GW invariants of \((\hat{Y}, \hat{D})\). We refer to [29] Section 4 for the precise definitions of relative stable morphisms to a relative FTCY 3-fold, and formal relative GW invariants. Here we briefly recall the definitions for our particular setting.

An effective class of \(\Gamma\) is a pair \((\vec{d}, \vec{\mu})\), where

- \(\vec{d} : E_c(\Gamma) \to \mathbb{Z}_{\geq 0}\) is a map from the set of compact edges in \(\Gamma\) to the set of nonnegative integers,
- \(\vec{\mu} : \{v'_1, \ldots, v'_s\} \to \mathcal{P}\) is a map from the set of univalent vertices in \(\Gamma\) to the set of partitions.

\(^{3}\text{In [29] all the noncompact edges are replaced by a compact edge ending at an univalent vertex. Here we only compactify } \hat{e}_1, \ldots, \hat{e}_s.\)
• $\bar{\mu}(v_i')$ is a partition of $\bar{d}(\bar{v}_i')$.

$\bar{\mu}$ can be viewed as an $s$-uple of partitions: $\bar{\mu} = (\mu^1, \ldots, \mu^s)$, where $\mu^i := \bar{\mu}(v_i')$.

Define

$$\ell(\bar{\mu}) = \sum_{i=1}^s \ell(\mu^i), \quad \text{Aut}(\bar{\mu}) = \prod_{i=1}^s \text{Aut}(\mu^i).$$

We also use $\bar{d}$ to denote the following effective curve class:

$$\sum_{\bar{v} \in \Gamma} \bar{d}(\bar{v})[C^\bar{v}] \in H_2(\bar{Y}; \mathbb{Z}).$$

A genus $g$, class $(\bar{d}, \bar{\mu})$ relative stable morphism to $(\bar{Y}, \bar{D})$ is an $\ell(\bar{\mu})$-pointed, genus $g$, degree $\bar{d}$ stable morphism $u : (C, q) \to \bar{Y}$, where

$$q = \{q_j^i | 1 \leq i \leq s, 1 \leq j \leq \ell(\mu^i)\}$$

are distinct smooth points on $C$, and

$$u^{-1}(\bar{D}^i) = \sum_{i=1}^s \mu^i q^i_j$$

as Cartier divisors. Let $\mathcal{M}_{g,d,\mu}(\bar{Y}, \bar{D})$ be the moduli space of genus $g$, class $(\bar{d}, \bar{\mu})$ relative stable morphisms to $(\bar{Y}, \bar{D})$. The stable compactification $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$ of $\mathcal{M}_{g,d,\mu}(\bar{Y}, \bar{D})$ contains relative morphisms to expanded targets $(\bar{Y}_m, \bar{D}_m)$. Then $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$ is a formal Deligne-Mumford stacks equipped with a perfect obstruction theory of virtual dimension $\ell(\bar{\mu})$. Moreover, $\mathbb{T}' = (\mathbb{C}^*)^2$ acts $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$, and the perfect obstruction theory are $\mathbb{T}'$-equivariant. The genus $g$, class $(\bar{d}, \bar{\mu})$ formal relative GW invariants of $(\bar{Y}, \bar{D})$ is defined by

$$(11) \quad F^\bar{Y},\bar{D}_{g,d,\mu}(\lambda) = \frac{1}{|\text{Aut}(\bar{\mu})|} \int_{\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})} \prod_{i=1}^s \prod_{j=1}^{\ell(\mu^i)} (e^j_i)^* (c_1(T(\mathcal{O}_{\bar{Y}_i}(L_i))))_{e^j_i(\bar{N}_{\vir})},$$

where $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$ is the $\mathbb{T}'$ fixed points substack in $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$, $e^j_i$ is the evaluation at the marked point $q^i_j$, $L^i$ is a $\mathbb{T}'$-equivariant divisor in $\bar{D}^i$, and $\bar{N}_{\vir}$ is the virtual normal bundle of $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$ in $\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$. Let $\{u, v\}$ be a $\mathbb{Z}$-basis of $H^2(\mathbb{T}'(\mathbb{Z}); \mathbb{Z})$, so that $H^*(BT'; \mathbb{Q}) = \mathbb{Q}[u, v]$. A priori the right hand side of $(\overline{\mathcal{M}}_{g,d,\mu}(\bar{Y}, \bar{D})$ is a rational function in $u, v$, homogeneous of degree 0, but it is indeed a rational number independent of $u, v$.

Introduce variables $p = \{p^i_j | 1 \leq i \leq s, j \in \mathbb{Z}_{\geq 0}\}$ and $Q = \{Q_{\bar{e}} | \bar{e} \in \mathbb{E}_c(\Gamma)\}$. We define generating functions of formal relative GW invariants of $(\bar{Y}, \bar{D})$:

$$(12) \quad F^\bar{Y},\bar{D}_{g,d,\mu}(\lambda) = \sum_{g} N^\bar{Y},\bar{D}_{g,d,\mu}(\lambda) \lambda^{2g-2+\ell(\bar{\mu})},$$

$$(13) \quad F^\bar{Y},\bar{D}(\lambda, Q, p) = \sum_{\mu \neq (\emptyset, \ldots, \emptyset)} F^\bar{Y},\bar{D}_{g,d,\mu}(\lambda) Q^\mu p_{\bar{\mu}}$$

where $Q^\mu = \prod_{\bar{e} \in \mathbb{E}_c(\Gamma)} Q_{\bar{e}}^{p^\mu(\bar{e})}$. Define generating functions $F^\star_{g,d,\mu}$ and $Z^\star_{g,d,\mu}$ by

$$(14) \quad Z^\bar{Y},\bar{D}(\lambda, Q, p) = \exp \left( F^\bar{Y},\bar{D}(\lambda, Q, p) \right) = \sum_{(\bar{d}, \bar{\mu})} F^\bar{Y},\bar{D}_{g,d,\mu}(\lambda) Q^\mu p_{\bar{\mu}}.$$
The surjective group homomorphism $H_2(\tilde{X}; \mathbb{Z}) = H_2(X^1; \mathbb{Z}) = \bigoplus_{\tilde{e} \in E_{c}(\Gamma X)} \mathbb{Z}[C^{\tilde{e}}].$

Let the character of $S$ be the formal completion of $X$ along the 1-skeleton $X^1$. Then $\tilde{X} = \tilde{Y} - \tilde{D}$ is a FTCY 3-fold, and

$$H_2(\tilde{X}; \mathbb{Z}) = H_2(X^1; \mathbb{Z}) = \bigoplus_{\tilde{e} \in E_{c}(\Gamma X)} \mathbb{Z}[C^{\tilde{e}}].$$

3.3. Descendant GW invariants of $\tilde{X}$. Let $\tilde{X}$ be the formal completion of $X$ along the 1-skeleton $X^1$. Then $\tilde{X} = \tilde{Y} - \tilde{D}$ is a FTCY 3-fold, and

$$H_2(\tilde{X}; \mathbb{Z}) = H_2(X^1; \mathbb{Z}) = \bigoplus_{\tilde{e} \in E_{c}(\Gamma X)} \mathbb{Z}[C^{\tilde{e}}].$$

The surjective group homomorphism $\pi : H_2(\tilde{Y}; \mathbb{Z}) \to H_2(X, L; \mathbb{Z})$ restricts to a surjective group homomorphism $H_2(\tilde{X}; \mathbb{Z}) \to H_2(X; \mathbb{Z}).$

An effective curve class of $\Gamma_X$ is a function $\tilde{d} : E_c(\Gamma X) \to \mathbb{Z}_{>0}$; we also use $\tilde{d}$ to denote the effective curve class

$$\sum_{\tilde{e} \in E_{c}(\Gamma X)} \tilde{d}(\tilde{e})[C^{\tilde{e}}] \in H_2(\tilde{X}; \mathbb{Z}).$$

Let $M_{g,n}(\tilde{X}, \tilde{d})$ be the moduli space of $n$-pointed, genus $g$, degree $\tilde{d}$ stable morphisms to $\tilde{X}$. Define

$$G_{\tilde{X}}(\tilde{d}, \tilde{\mu}(u, v) = \left( -\sqrt{-1} \right)^{\ell(\tilde{\mu})} \prod_{i=1}^{s} \prod_{j=1}^{\ell(\mu_i)} \left( \mu_i \cdot \nu \cdot \nu \right)^{-1} \cdot \prod_{i=1}^{s} \prod_{j=1}^{\ell(\mu_i)} \left( \nu \cdot \nu \cdot \nu \right)^{-1}.$$  

$$\int_{M_{g,n}(\tilde{X}, \tilde{d}, \tilde{\mu})} \prod_{i=1}^{s} \prod_{j=1}^{\ell(\mu_i)} \left( \nu \cdot \nu \cdot \nu \right)^{-1}$$

where $\phi_i \in H_{2g-2}(\tilde{X}; \mathbb{Z})$ is the $T$-equivariant Poincaré dual of the torus fixed point $x_i \in V(\tau_1)$, and

$$ev_i : M_{g,0}(\tilde{X}, \tilde{d}) \to \tilde{X}, \quad i = 1, \ldots, s, \quad j = 1, \ldots, \ell(\mu_i),$$

are evaluations at the marked points.

Define generating functions

$$G_{\tilde{X}}(\lambda; u, v) = \sum_{g} \chi_{2g-2+\ell(\tilde{\mu})} G_{\tilde{d}, \tilde{\mu}}(u, v)$$

$$G_{\tilde{X}}(\lambda, Q, p; u, v) = \sum_{\tilde{\mu} \neq (\emptyset)} G_{\tilde{d}, \tilde{\mu}}(\lambda; u, v) Q^\tilde{d} p_{\tilde{\mu}}$$

$$G_{\tilde{X}}(\lambda, Q, p; u, v) = \exp \left( G_{\tilde{X}}(\lambda, Q, p; u, v) \right) = \sum_{(\tilde{d}, \tilde{\mu})} G_{\tilde{d}, \tilde{\mu}}(\lambda; u, v) Q^\tilde{d} p_{\tilde{\mu}}$$

Given two partitions $\mu^+, \mu^-$ of $d$, define

$$\Phi_{\mu^+, \mu^-}(\lambda) = \sum_{\nu^+ \vdash d} e^{\kappa_{\nu} \lambda/2} \chi_{\nu}(\mu^+) \chi_{\nu}(\mu^-)$$

where $\kappa_{\mu} = \sum_{j=1}^{\ell(\mu)} \mu_j (\mu_j - 2j + 1)$, $z_{\mu} = \text{Aut}(\mu) \prod_{j=1}^{\ell(\mu)} \mu_j$, and $\chi_{\nu}(\nu)$ denotes the value of the irreducible character of $S_d$ associated to $\nu \vdash d$ at the conjugacy class of $S_d$ associated to $\mu \vdash d$. 

Proposition 3.1.

\[
\sqrt{-1}^{\ell(\vec{m})}(-1)^{\sum_{i=1}^s f_i(\mu^i) - 1} G_{\vec{d},\vec{m}}^{X.\vec{D}}(\lambda) \\
= \sum_{|\nu^i|=|\mu^i|} G_{\vec{d},\vec{m}}^{X}(\lambda; u, v) \prod_{i=1}^{s} z_{\nu_i} \Phi_{\nu^i, \mu^i}(\sqrt{-1} f_i - \frac{w_i}{w_1}) \\
= \sum_{|\nu^i|=|\mu^i|} G_{\vec{d},\vec{m}}^{X}(\lambda; u, v) \prod_{i=1}^{s} z_{\nu_i} \Phi_{\nu^i, \mu^i}(\sqrt{-1}(f_i - \frac{w_i}{w_1})),
\]

where \( \vec{d} : E_\mu(\Gamma) \to \mathbb{Z}_{\geq 0} \) is the restriction of \( \vec{d} : E_\mu(\Gamma) \to \mathbb{Z}_{\geq 0} \).

**Proof.** We use the notation in [29] Section 7, and set \( \nu^i = \nu_{\vec{e}^i} \). By [29] Proposition 7.10,

\[
F_{\vec{d},\vec{m}}^{Y,\vec{D}}(\lambda) = \sum_{|\nu^i|=|\mu^i|} \prod_{|\nu^i|=\vec{d}(\vec{e}) \in E(\Gamma)} (-1)^{n^e(\vec{e})} z_{\nu^e} \prod_{v \in V_\delta(\Gamma)} \sqrt{-1}^{l(\nu^i)} G_{\vec{d},\vec{m}}^{Y}(\lambda; w_v) \\
= \sum_{|\nu^i|=|\mu^i|} \left( \sum_{|\nu^i|=\vec{d}(\vec{e}) \in E(\Gamma_X)} (-1)^{n^e(\vec{e})} z_{\nu^e} \prod_{v \in V_\delta(\Gamma)} \sqrt{-1}^{l(\nu^i)} G_{\vec{d},\vec{m}}^{Y}(\lambda; w_v) \right) \\
\prod_{i=1}^{s} (-1)(f_i - 1) |\mu^i| (-\sqrt{-1})^{l(\nu^i) + l(\mu^i)} z_{\nu_i} \Phi_{\nu^i, \mu^i}(\sqrt{-1} f_i - \frac{w_i}{w_1} \lambda)
\]

by localization computations similar to those in [29]. Therefore,

\[
F_{\vec{d},\vec{m}}^{Y,\vec{D}}(\lambda) = (-\sqrt{-1})^{l(\vec{m})}(-1)^{\sum_{i=1}^s (f_i - 1) |\mu^i|} \sum_{|\nu^i|=|\mu^i|} G_{\vec{d},\vec{m}}^{X}(\lambda; u, v) z_{\nu_i} \Phi_{\nu^i, \mu^i}(\sqrt{-1} f_i - \frac{w_i}{w_1} \lambda)
\]

\(\square\)

3.4. **Open GW invariants of \((X, L)\): multiple outer branes.** The open GW invariants of \(X\) relative to the framed outer branes \((L_1, f_1), \ldots, (L_k, f_k)\) are defined to be

\[
N_{g, \beta, \mu}^{X,L}(f_1, \ldots, f_k) = (-1)^{\sum_{i=1}^s (f_i - 1) |\mu^i|} \sum_{\pi(\vec{D}) = \beta} \tilde{F}_{g,\vec{D}}^{Y,\vec{D}} \in \mathbb{Q},
\]

where \( \tilde{\mu}(e^i)_i = \mu^i \) for \( i = 1, \ldots, s \), and the sign \( (-1)^{\sum_{i=1}^s (f_i - 1) |\mu^i|} \) is included for convenience of comparison with the predictions from mirror symmetry.

Given \( \beta \in H_2(X; \mathbb{Z}) \) and \( \vec{m} = (\mu^1, \ldots, \mu^s) \), define

\[
G_{g, \beta, \vec{m}}^{X}(u, v) = \frac{(-\sqrt{-1})^{l(\vec{m})}}{|\text{Aut}(\vec{m})|} \prod_{i=1}^{s} \frac{\prod_{j=1}^{\ell(\mu^i)} \mu_j^i w_j + mw_j}{\mu_j^i \cdot \mu_j^i (w_j^i)^{l(\mu^i) - 1}} \\
\int_{[\tau_{g, \beta, \vec{m}}^{X}(X, \beta)]^{vir}} \frac{1}{e^g(\tau_{vir})} \prod_{i=1}^{s} \prod_{j=1}^{\ell(\mu^i)} \frac{\mu_j^i w_j}{w_j^i} z_{\nu_i} \Phi_{\nu^i, \mu^i}(\sqrt{-1} f_i - \frac{w_i}{w_1} \lambda)
\]

(18)
Then
\[ G_{g, \beta, \vec{\mu}}^X = \sum_{\pi(d) = \beta} G_{g, \vec{d}, \vec{\mu}}^X. \]

In particular, when \( X = \mathbb{C}^3 \), \( s = 3 \), we have
\[ w_1 = w_2^3 = u, \quad w_2^2 = w_2^2 = v, \quad w_2^3 = w_3 = -u - v, \]
\[ \mathcal{M}_{g, \ell(\vec{\mu})}(\mathbb{C}^3, 0)^{\vec{\nu}'} = \mathcal{M}_{g, \ell(\vec{\mu})}, \]
\[ \frac{1}{e_{\nu'}(N^\text{vir})} = \prod_{i=1}^{\nu} \frac{\Lambda^\nu_i(w_i)}{w_i} \in H^*(\mathcal{M}_{g, \ell(\vec{\mu})}; \mathbb{Q}) \otimes \mathbb{Q}(u, v), \]
where \( \Lambda^\nu_i(u) = u^9 - \lambda_1 u^8 + \ldots + (-1)^9 \lambda_9 \). Define \( w_1 = w_4 = u, \ w_2 = v, \) and \( w_3 = -u - v \). Then
\[ G_{g, 0, \vec{\mu}}^X(u, v) = \frac{(-\sqrt{-1})^{\ell(\vec{\mu})}}{|\text{Aut}(\vec{\mu})|} \prod_{i=1}^{\nu} \prod_{j=1}^{\mu_i} \frac{\mu_i^{j-1} w_i + m w_i}{\mu_i^{j} - \psi_j}, \]
The following proposition follows from Proposition 3.1 and definitions.

**Proposition 3.2** (multiple framed outer branes). Let \( \beta \in H_2(X; \mathbb{Z}) \) be an effective curve class, and let \( \beta' = \beta + \sum_{i=1}^{s} |\mu_i| b_i \in H_2(X; \mathbb{Z}) \). Then
\[ (\sqrt{-1})^{\ell(\vec{\mu})} (-1)^{\sum_{i=1}^{s} f_i |\mu_i|} F_{\beta', \vec{\mu}}^{X, L}(\lambda; f_1, \ldots, f_s) \]
(19)
\[ = \sum_{|\nu'| = |\mu'|} G_{\beta', \vec{\mu}}^X(\lambda; u, v) \prod_{i=1}^{s} z_{\nu_i} \Phi_{\nu_i, \mu_i}((\sqrt{-1} f_i \frac{w_i}{w_1}) \]
\[ = \sum_{|\nu'| = |\mu'|} G_{\beta', \vec{\mu}}^X(\lambda; u, v) \prod_{i=1}^{s} z_{\nu_i} \Phi_{\nu_i, \mu_i}((\sqrt{-1} (f_i - \frac{w_i}{w_1})). \]

In particular, when \( s = 1, \ |\mu| = \mu, \ f_1 = f, \ b_1 = b \), we have
\[ F_{\beta + |\mu| b, \mu}^{X, L}(\lambda; f) = \sqrt{-1}^{f |\mu|} (-1)^{|\mu|} \sum_{|\nu| = |\mu|} G_{\beta, \nu}^X(\lambda; u, v) z_{\nu} \Phi_{\nu, \mu}(\sqrt{-1} (f - \frac{v_L}{u_L})). \]

Let \( T_{L, f} \cong \mathbb{C}^* \) be the torus of \( \mathbb{T}' \) defined in Section 2.5. It is the kernel of the character \( v_L - f u_L \in \text{Hom}(\mathbb{T}', \mathbb{C}^*). \) The inclusion \( T_{L, f} \hookrightarrow \mathbb{T}' \) induces a surjective ring homomorphism
\[ H^*(\mathbb{T}'_f; \mathbb{Q}) = \mathbb{Q}[u, v] = \mathbb{Q}[u_L, v_L] \to H^*(\mathbb{T}_{L, f}; \mathbb{Q}) = \mathbb{Q}[u_L] \]
given by \( u_L \mapsto u_L \) and \( v_L \mapsto f u_L \).

Recall that
\[ \Phi_{\nu, \mu}(0) = \frac{\delta_{\nu, \mu}}{z_{\nu}}. \]
Therefore,

\[ N_{g,\beta + |\mu|b,\mu}(f) = \sqrt{-1}^{(\mu)}(-1)^{|\mu|} \sum_{|\sigma|=|\mu|} G_{\beta,\mu}^X(\lambda; u, v) \bigg|_{\nu_L=f_u} \]

which is equivalent to the following corollary.

**Corollary 3.3** (single framed outer brane). Let \((L, f)\) be a framed outer brane in \(X\), let \(z\) be the unique \(\mathbb{T}'\)-fixed point in \(V(\tau_L)\), and let \(\phi \in H^0_{\phi}(X; \mathbb{Z})\) be the \(\mathbb{T}'\)-equivariant Poincaré dual of \(z\). For any effective curve \(\beta \in H_2(X; \mathbb{Z})\) and any partition \(\mu = (\mu_1, \ldots, \mu_h)\), we have

\[ N_{g,\beta + |\mu|b,\mu}(f) = \frac{1}{|\text{Aut}(\mu)|} \prod_{j=1}^h (-1)^{f|\mu_j|} \frac{\prod_{m=1}^{\mu_j-1} (f\mu_j + m)}{\mu_j \cdot \mu_j!} \left( \left( \frac{1}{\int_{[\overline{\mathcal{M}}_{g,h}(X,\beta)]^\text{vir}} \prod_{j=1}^h \text{ev}_j^* \frac{1}{\ell^* (N^\text{vir})} \right) \bigg|_{\nu_L=f_u} \right) \]

3.5. **Open GW invariants of** \((X, L)\): **single outer or inner brane.** Let \((L, f)\) be a framed (inner or outer) Aganagic-Vafa A-brane. Then \(L = L_1\). Let

\[ b = b_1 \in H_2(X, L; \mathbb{Z}), \quad \gamma = \gamma_1 \in H_1(L; \mathbb{Z}), \quad x^+ = x_1, \]

where \(b_1, \gamma_1, x_1\) are defined as in Section 2.2. Let \(\phi_L^+ \in H^0_{\phi}(X; \mathbb{Z})\) be the \(\mathbb{T}'\)-equivariant Poincaré dual of the \(\mathbb{T}'\) fixed points \(z^+ \in V(\tau_L)\). When \(L\) is an inner brane, let \(x^- = x_1^-\) be the other \(\mathbb{T}'\) fixed point in \(V(\tau_L)\), and let \(\phi_L^-\) be the \(\mathbb{T}'\)-equivariant Poincaré dual of \(x^-\).

Given \(\beta' \in H_2(X, L; \mathbb{Z})\), and \(w = (w_1, \ldots, w_h)\), where \(w_j\) are nonzero integers, we define

\[ \overline{\mathcal{M}}_{g,\beta',w} := \overline{\mathcal{M}}_{g,h}(X, L \mid \beta', w_1\gamma, \ldots, w_h\gamma) \]

where the right hand side is the moduli space of stable maps \(u: (\Sigma, \partial \Sigma = \sqcup_{j=1}^h R_j) \to (X, L)\), where \(\Sigma\) is a prestable Riemann surface of type \((g; h)\), \(R_j \cong S^1\) are connected components of the boundary \(\partial \Sigma\) of \(\Sigma\), \(u_*[\Sigma] = \beta', u_*[R_j] = w_j\gamma\). (See [20] Section 4 for the detailed definitions.) Then \(\mathbb{T}'_k\) acts on \(\overline{\mathcal{M}}_{g,\beta',w}\). The integral

\[ \left( \int_{[\overline{\mathcal{M}}_{g,h}(X,\beta')^\text{vir}} \frac{1}{\ell^* (N^\text{vir})} \right) \bigg|_{\nu_L=f_u} \]

is a rational number depending on \(f \in \mathbb{Z}\). It is defined up to a sign depending on choice of orientation on the virtual tangent bundle of \(\overline{\mathcal{M}}_{g,\beta',w}\). Define

\[ \text{Aut}(w) = \{ \sigma \in S_h \mid (w_{\sigma(1)}, \ldots, w_{\sigma(h)}) = (w_1, \ldots, w_h)\} \]

**Proposition 3.4.** Suppose that \(\beta \in H_2(X; \mathbb{Z})\), and either

- \(L\) is an outer brane, \(w_1, \ldots, w_h\) are positive integers, or
- \(L\) is an inner brane, \(w_1, \ldots, w_h\) are nonzero integers.

When \(L\) is an inner brane, let \(\alpha = [V(\tau_L)] \in H_2(X; \mathbb{Z})\). \(\{1, \ldots, h\}\) is a disjoint union of \(J_+\) and \(J_-\), where \(J_+ = \{j \in \{1, \ldots, h\} \mid \pm w_j > 0\}\). (So \(J_-\) is empty when \(L\) is an outer brane.) Define \(\beta' \in H_2(X, L; \mathbb{Z})\) by

\[ \beta' = \begin{cases} \beta + \left( \sum_{j=1}^h w_j \right) b & \text{if } L \text{ is an outer brane}, \\ \beta + \left( \sum_{j \in J_+} w_j \right) b + \left( \sum_{j \in J_-} (-w_j) \right) (\alpha - b) & \text{if } L \text{ is an inner brane}. \end{cases} \]
Then
\[ \frac{1}{|\text{Aut}(\vec{w})|} \int_{\overline{\mathcal{M}}_{g,\beta',\vec{w}}^{\text{vir}}} \frac{1}{e_{\mathbb{T}^2}(N_{\text{vir}})} \bigg|_{\nu_L = fu_L} = \pm N_{g,\beta',\vec{w}}(f) \]
where
\[ N_{g,\beta',\vec{w}}(f) = \frac{1}{|\text{Aut}(\vec{w})|} \prod_{j \in J_+} (-1)^{fw_j} \prod_{m=1}^{w_j} \frac{(fw_j + m)}{w_j \cdot m!} \]
(21)
\[ \cdot \prod_{j \in J_-} (-1)^{(f + n)w_j} \prod_{m=1}^{w_j} \frac{(-w_j) \cdot m!}{(-w_j)!} \]
\[ \cdot \left( \int_{\overline{\mathcal{M}}_{g,h}(X,\beta)_{\nu_L}^{\text{vir}}} \frac{1}{e_{\mathbb{T}^2}(N_{\text{vir}})} \cdot \prod_{j \in J_+} \frac{\text{ev}_j^* \phi_j^* \prod_{J \in J_+} \text{ev}_j^* \phi_L^*}{\prod_{j=1}^{b} [u_j \cdot x_j - v_j]} \right) \bigg|_{\nu_L = fu_L} \]

**Remark 3.5.** The above formula (21) agrees with (20) when \( L \) is an outer brane. We use (21) to extend the definition of open GW invariants when \((L,f)\) is an inner brane.

**Proof of Proposition 3.4.** When all the \( w_j \)'s are positive, the computation for an inner brane is the same as that for an outer brane. From now on, we assume that \( L \) is an inner brane. \( V(\tau_L) \) is the union of two disks \( D_+ \) and \( D_- \) which contain the torus fixed points \( x^+ \) and \( x^- \), respectively. \( D_+ \cap D_- = L \cap V(\tau_L) \).

![Figure 4. weights of the action by \( T_{L,f} \)](image)

Suppose that \( u : (C,q_1,\ldots,q_h) \to X \) is a stable map which represents a \( \mathbb{T}^2 \) fixed point in \( \overline{\mathcal{M}}_{g,h}(X,\beta) \), and that \( f(q_j) = \pm x \) for \( j \in J_\pm \). Let \( \Sigma \) be the prestable bordered Riemann surface
\[ \Sigma = C \cup D_1 \cup \cdots \cup D_h, \]
where \( D_j \) intersects \( C \) at a node \( q_j \). If \( j \in J_\pm \) then we define \( u_j : D_j \to D_{\pm} \) by \( z \mapsto z^\pm w_j \). Then we obtain a stable map \( \hat{u} : (\Sigma, \partial \Sigma) \to (X, L) \) which represents a \( \mathbb{T}^2 \)-fixed point in \( \overline{\mathcal{M}}_{g,\beta',\vec{w}} \). Conversely, all \( \mathbb{T}^2 \) fixed points in \( \overline{\mathcal{M}}_{g,\beta',\vec{w}} \) arise in this way. Let
\[ \hat{B}_1 = \text{Aut}(\Sigma), \hat{B}_2 = \text{Def}(\hat{u}), \hat{B}_4 = \text{Def}(\Sigma), \hat{B}_5 = \text{Obs}(\hat{u}), \]
\[ B_1 = \text{Aut}(C,q_1,\ldots,q_h), B_2 = \text{Def}(u), \]
\[ B_4 = \text{Def}(C,q_1,\ldots,q_h), B_5 = \text{Obs}(u). \]
(See [20] Section 4 for the definitions of these real vector spaces.) Let \( \hat{T}^1 - \hat{T}^2 \) be the virtual tangent space to \( \overline{\mathcal{M}}_{g,\beta',\vec{w}} \) at the moduli point \( \hat{u} \), \( \Sigma, \partial \Sigma) \to (X, L) \), and let \( T^1 - T^2 \) be the virtual tangent space to \( \overline{\mathcal{M}}_{g,h}(X,\beta) \) at the moduli point \( u : (C,q_1,\ldots,q_h) \to X \). Let \( e = e_{\mathbb{T}^2} \) denote the \( \mathbb{T}^2 \)-equivariant Euler class. Then
\[ e(T^2,m) \quad e(T^1,m) \]
\[ \frac{e(T^2,m)}{e(T^1,m)} = \frac{e(B_1^m) e(B_2^m)}{e(B_4^m) e(B_5^m)} = \frac{e(T^2,m)}{e(T^1,m)} = \frac{e(B_1^m) e(B_2^m)}{e(B_4^m) e(B_5^m)} \]
We have
\[ \hat{B}_1^m = B_1^m, \quad \hat{B}_3^m = B_3^m \oplus \bigoplus_{j=1}^{h} T_{q_i} C \oplus T_{q_i} D_j. \]

Therefore
\[ \frac{e(\hat{B}_1^m)}{e(B_1^m)} = \frac{e(\hat{B}_3^m)}{e(B_3^m)} \prod_{j \in J_+} \left( \frac{u_j}{w_j} - \psi_j \right) \prod_{j \in J_-} \left( -\frac{u_j}{w_j} - \psi_j \right) = \frac{e(\hat{B}_1^m)}{e(B_3^m)} \prod_{j=1}^{h} \left( \frac{u_j}{w_j} - \psi_j \right) \]

We also have a long exact sequence
\[ 0 \to \hat{B}_2 \to B_2 \oplus \bigoplus_{j=1}^{h} H^0(D_j) \to \bigoplus_{j \in J_+} T_{x} + X \oplus \bigoplus_{j \in J_-} T_{x} - X \]
\[ \to \hat{B}_5 \to B_5 \oplus \bigoplus_{j=1}^{h} H^1(D_j) \to 0. \]

where
\[
H^p(D_j) = H^p(D_j, \partial D_j, u_j^* T X, (u_j|_{\partial D_j})^* T L), \quad p = 0, 1.
\]

We have
\[
ev^r(T_x + X) = u_L v_L(-u_L - v_L) = ev^r \phi^+_L \quad j \in J_+,
\]
\[
ev^r(T_x - X) = (-u_L)(-v_L - nu_L)(v_L + (n + 1)u) = ev^r \phi^-_L \quad j \in J_-
\]

Therefore
\[ \frac{e(\hat{B}_2^m)}{e(B_2^m)} = \frac{e(\hat{B}_3^m)}{e(B_3^m)} \prod_{j \in J_+} \frac{ev^r \phi^+_L}{u_L} \prod_{m=1}^{w_j-1} \frac{w_j v_L + mu_L}{mu_L} \prod_{j \in J_-} \frac{ev^r \phi^-_L}{-u_L} \prod_{m=1}^{w_j-1} \frac{w_j v_L + (n + m)u_L}{mu_L} \]

Finally,
\[ \text{Aut}(\hat{u}) = \text{Aut}(u) \times \prod_{j=1}^{h} (\mathbb{Z}/w_j \mathbb{Z}). \]

Combining (22), (23), and (24), we get
\[
N_{g, \beta, \psi, \omega}(f) = \pm \frac{1}{|\text{Aut}(\hat{u})|} \prod_{j \in J_+} \frac{\prod_{m=1}^{w_j-1} (f w_j + m)}{w_j \cdot w_j!} \prod_{j \in J_-} \frac{\prod_{m=1}^{w_j-1} ((f + n)(-w_j) + m)}{(-w_j) \cdot (-w_j)!}
\]
\[
\left( \int_{\mathcal{M}_{g,h}(X, \beta)_{vir}} \frac{1}{\text{ev}^r(\overline{N}_{vir})} \prod_{j \in J_+} \frac{ev^r \phi^+_L}{u_L} \prod_{j \in J_-} \frac{ev^r \phi^-_L}{-u_L} \left| \prod_{j=1}^{h} \left[ \frac{u_j}{w_j} \left( \frac{u_j}{w_j} - \psi_j \right) \right] \right|_{\nu_L=f u_L} \right)
\]
\[ \square \]
3.6. Generating functions for a fixed topological type of the domain. Any $\beta' \in H_2(X, L; \mathbb{Z})$ is of the form $\beta' = \beta + w b$, where $\beta \in H_2(X; \mathbb{Z})$ and $w \in \mathbb{Z}$. As in Section 2 we fix a choice of $\mathbb{Z}$-basis $l^{(1)}, \ldots, l^{(k)}$ and its dual $p_1, \ldots, p_k$. Define

$$Q_{\beta'} = Q_w^0 \prod_{a=1}^k Q^{d_a}_a, \quad d_a = \langle \beta, p_a \rangle.$$ 

We define generation functions

$$F_{g,w}(Q; f) = \sum_{\beta' \in H_2(X, L; \mathbb{Z})} N_{g, \beta', w}(f) Q_{\beta'},$$

$$F^{g,h}(Q, Y; f) = \sum_{w_j \in \mathbb{Z} - \{0\}} F_{g,w_1,\ldots,w_h}(Q, f) \prod_{j=1}^h Y_j^{w_j},$$

where

$$Q = (Q_0, Q_1, \ldots, Q_k), \quad Y = (Y_1, \ldots, Y_h).$$

Note that when $h = 1$, $w = w_1 \in \mathbb{Z}$ is determined by $\partial \beta' = w \gamma$, so we may omit the variable $Y_1$. In particular, we define

$$F(Q; f) := F^{0,1}(Q, Y_1 = 1; f) = \sum_{w \in \mathbb{Z}, w \neq 0, \beta \in H_2(X, L; \mathbb{Z})} N_{0, \beta + w b, w} Q_w^0 \prod_{a=1}^k Q^{d_a}_a,$$

where $d_a = \langle \beta, p_a \rangle$.

For later convenience, we introduce the following generating function for any nonzero integer $w$:

$$J^L_w(Q; f) = \begin{cases} 
1 + \sum_{\beta \geq 0} Q^\beta \left( \int_{[\mathcal{M}_{0,1}(X, \beta)^{vir}]^{\vir}} \frac{1}{e_T(\nu_w (w_j - \psi_j))} \right)_{\nu_L = f u_L} & w > 0 \\
1 + \sum_{\beta > 0} Q^\beta \left( \int_{[\mathcal{M}_{0,1}(X, \beta)^{vir}]^{\vir}} \frac{1}{e_T(\nu_w (w_j - \psi_j))} \right)_{\nu_L = f u_L} & w < 0 
\end{cases}$$

where $\beta \in H_2(X; \mathbb{Z})$. We say $\beta \geq 0$ when $\beta$ is effective, and $\beta > 0$ when it is also non-zero. The following is an immediate consequence of Proposition 3.3.

**Corollary 3.6.** (a) If $L$ is an outer brane then

$$F(Q; f) = \sum_{w > 0} Q_w^0 (-1)^w \prod_{m=1}^{w-1} \left( f w + m \right) \frac{1}{w \cdot w!} J^L_w(Q; f).$$

(b) If is an inner brane then

$$F(Q; f) = \sum_{w > 0} Q_w^0 (-1)^w \prod_{m=1}^{w-1} \left( f w + m \right) \frac{1}{w \cdot w!} J^L_w(Q; f)$$

$$+ \sum_{w < 0} Q_w^0 (-1)^{w+n} \prod_{m=1}^{-w-1} \left( (f + n)(-w) + m \right) \frac{1}{w \cdot (-w)!} J^L_w(Q; f)$$
4. The Mirror Conjecture on Holomorphic Disks

Aganagic-Vafa and Aganagic-Klemm-Vafa [3, 2] construct a B-model Landau-Ginzburg model together with the superpotential $W$ as the mirror to the A-model on the Calabi-Yau toric threefold $X$. The superpotential $W$ is conjectured, up to a mirror transform, to be equal to the single brane disk amplitude constructed in Section 3.6. As discussed in Section 2.5, we fix a framed Aganagic-Vafa A-brane $(L, f)$, and let $x^+$ be the $\mathbb{T}$-fixed point in $V(\tau_L)$ (when $L$ is an inner brane there is another $\mathbb{T}$-fixed point $x^-$). We fix indices $i_1, i_2, i_3$, and in case of inner brane $L$, also an index $i_4$ as in Section 2.5 (Figure 4).

4.1. The Hori-Vafa mirror. Applying $\text{Hom}(-, \mathbb{C}^*)$ to (5), we obtain

$$1 \to \mathbb{T}^\vee \to \tilde{T}^\vee \to \mathbb{C}^* \to 1,$$

where $\mathbb{C}^* \cong (\mathbb{C}^*)^n$, $\tilde{T}^\vee \cong (\mathbb{C}^*)^r$, $G^\vee \cong (\mathbb{C}^*)^k$. Applying $\text{Hom}(-, \mathbb{C}^*)$ to (9), we obtain

$$1 \to (\mathbb{C}^*)^\vee \to \mathbb{T}^\vee \to (\mathbb{T}')^\vee \to 1,$$

where $(\mathbb{T}')^\vee \cong (\mathbb{C}^*)^{n-1}$.

Given $q = (q_1, \ldots, q_k) \in G^\vee$, define

$$T^\vee_q := \phi^{-1}(q) = \{\tilde{x} = (x_1, \ldots, x_{k+3}) \in \tilde{T}^\vee \mid \prod_{i=1}^{k+3} x_i^{(a)} = q_a, \ a = 1, \ldots, k \} \cong (\mathbb{C}^*)^3.$$

Let $\mathbb{C}^*$ acts on $\tilde{T}^\vee$ by $\lambda \cdot (x_1, \ldots, x_{k+3}) = (\lambda x_1, \ldots, \lambda x_{k+3})$. Then the $\mathbb{C}^*$-action preserves $T^\vee_q$. Let $T^\vee_q := T^\vee_q / \mathbb{C}^* \cong (\mathbb{C}^*)^2$.

Let $H(\tilde{x}, q)$ be the restriction of $\sum_{i=1}^{k+3} x_i$ to $T^\vee_q$. There is a $\mathbb{C}^*$ action on the following Calabi-Yau 4-fold

$$\tilde{X}^\vee_q = \{(w^+, w^-, \tilde{x}) \in \mathbb{C}^2 \times T^\vee_q \mid w^+ w^- = H(\tilde{x}, q)\},$$

given by

$$\lambda \in \mathbb{C}^* : \quad w^- \mapsto \lambda w^-,$$

$$(x_1, \ldots, x_{k+3}) \mapsto (\lambda x_1, \ldots, \lambda x_{k+3}).$$

The mirror of $X$ is the quotient noncompact Calabi-Yau 3-fold $\tilde{X}^\vee_q / \mathbb{C}^*$, denoted by $X_q^{\vee}$. The curve in $X_q^{\vee}$ given by the equation $H(\tilde{x}, q) = 0$, $w^+ = w^- = 0$ is the mirror curve.

For $a = 1, \ldots, k$, we extend the charge vectors $l^{(a)} = (l_1^{(a)}, \ldots, l_{k+3}^{(a)})$ to

$$\tilde{l}^{(a)} = (l_1^{(a)}, \ldots, l_{k+3}^{(a)}, 0, 0).$$

There is an additional charge vector $\tilde{l}^{(0)} = (l_1^{(0)}, \ldots, l_{k+5}^{(0)})$ for $i = 1, \ldots, k + 5$ with

$$\sum_{i=1}^{k+5} l_i^{(0)} = 0$$

that characterizes the B-brane. Define the index sets

$$I = \{1, \ldots, r\}, \quad \bar{I} = \{1, \ldots, r + 2\}, \quad I_0 = \{i_1, i_2, i_3\}.$$

If the A-brane carries the framing $f$, we set the open charge vector $l^{(0)} = (l_1^{(0)}, \ldots, l_{k+3}^{(0)})$ to be

$$l_{i_1}^{(0)} = 1, \quad l_{i_2}^{(0)} = f, \quad l_{i_3}^{(0)} = -f - 1, \quad l_i^{(0)} = 0 \quad \text{for} \ i \in I \setminus I_0.$$

We introduce an extra charge vector

$$\tilde{l}^{(0)} = (l_1^{(0)}, \ldots, l_r^{(0)}, 1, -1).$$
The mirror curve can be written down in the following coordinate patch

\[ x_{i_1} = \bar{x}, \ x_{i_2} = \bar{y}, \ x_{i_3} = 1. \]

Other \( x_i \) for \( i \in I \) are determined from \( \bar{x}, \bar{y} \) through the equation

\[ \prod_{i=1}^{k+3} x_i^{l_i(a)} = q_a, \ a = 1, \ldots, k. \]

The mirror curve is an affine curve in \( \mathbb{C}^* \times \mathbb{C}^* \) given by a single equation

\[ H(\bar{x}, q) = -\bar{x} + \bar{y} + 1 + \sum_{i \in I \setminus I_0} x_i(\bar{x}, \bar{y}, q) = 0 \]

in coordinates \( \bar{x}, \bar{y} \).

4.2. Aganagic-Vafa B-branes. As shown in [3], the mirror B-brane of the A-brane defined in Section 2.4 above is 2-cycle in \( X^\vee_q \), given by the equations

\[ w^- = 0, \ H(\bar{x}, q) = 0, \]

where \( \bar{x} = \bar{x}(w^+) = \{ \bar{x} | H(\bar{x}, q) = 0 \} \) is a function of the coordinate \( w^+ \) on the brane. We fix \( \bar{x} \) at \( w^+ = \infty \) to be \( \bar{x}(\infty) = \bar{x}^\ast \). Following [23, 24], the framed modulus \( x \) of the B-brane is determined by the value of \( \bar{x} \) on \( w^+ = 0 \)

\[ x = q_0 = -\prod_{i=1}^r x_i(0)^{l_i(0)} = \bar{x} y^f. \]

The the B-brane modulus in framing 0 is given by \( \bar{x} \). We denote the corresponding B-brane by \( C_x \), and the reference B-brane by \( C_x^\ast \), with \( x^\ast = -\prod_{i=1}^{k+3} x_i^\ast(0)^{l_i(0)} \). The reference brane \( C_x^\ast \) could be chosen as a holomorphic curve with constant \( x(w^+) = x^\ast \). Other branes cannot be holomorphic, since \( x \) has different bounded values at \( w^+ = 0 \) and \( w^+ = \infty \).

The mirror curve in coordinates \( \bar{x}, \bar{y} \) can be further written as an equation in the framed coordinates \( x, y \)

\[ M(x, y, q) = H(\bar{x}, q) = -x y^f + y + 1 + \sum_{i \in I \setminus I_0} x_i(x y^f, y, q) = 0 \]

by a change of coordinates

\[ x = \bar{x} y^f, \ y = \bar{y}. \]

4.3. Picard-Fuchs equations and the mirror map. Let \( \Gamma(x) \) be a 3-chain in \( X^\vee_q \) such that \( \partial \Gamma(x) = C_x - C_x^\ast \), and \( \gamma \in H^3(X^\vee_q; \mathbb{Z}) \) is any 3-cycle. Mayr and Lerche-Mayr [34, 23] show that the periods

\[ \int_{\Gamma(x)} \Omega \]

are eliminated by GKZ-type operators

\[ \mathcal{D}_a = \prod_{l_i(a) > 0} \prod_{j=0}^{l_i(a)-1} (\sum_{b} l_i^{(b)} \theta_b - j) - q_a \prod_{l_i(a) < 0} \prod_{j=0}^{l_i(a)-1} (\sum_{b=1}^{k} l_i^{(b)} \theta_b - j), \]

where \( a = 0, \ldots, k, \ i = 1, \ldots, r + 2, \) and \( \theta_a = q^a \frac{d}{dq^a} \). The extended Picard-Fuchs equations are

\[ \mathcal{D}_a W(q_0, q_1, \ldots, q_k) = 0. \]
For any $\beta W^{4.4}$. It agrees with (36).

(36) where $T(35)$

$\Pi_{S}$ Lerche-Mayr in [23] gives the open mirror correction as

$$I \equiv \sum_{i \in I} a_i \langle \mathbf{p}_i \rangle$$

Recall that for $a = 1, \ldots, k$, $T_a = -r_a + \sqrt{-1} \theta_a$ are complexified Kähler parameters. The mirror map is

$$T_a = \Pi^1_4(q).$$

where $\Pi^1_4(q)$ is a solution to the Picard-Fuchs equations above, i.e. $D_b \Pi^1_4(q) = 0$ for $b = 0, \ldots, k$. It has the leading term behavior

(35) $T_a = \log(q_a) + S_a(q)$,

where $S_a(q)$ is a power series in $q_1, \ldots, q_k$.

Recall that $d_a = (p_a, \beta)$ for $\beta \in H_2(X; \mathbb{Z})$, and $\langle D^*_i, \beta \rangle = \sum_{a=1}^{k} d_a l_i^{(a)}$. Denote $q^a = \prod_{a=1}^{k} q_a^{d_a}$.

For any $\beta \in \mathbb{L}_{\text{eff}}$ and $i_0 \in I$, if $\langle D^*_{i_0}, \beta \rangle < 0$ and $\langle D^*_i, \beta \rangle \geq 0$ for $i \in I \setminus \{i_0\}$, we define

$$E_{i_0}(\beta) = \frac{(-1)^{\langle -(D^*_{i_0}, \beta) - 1 \rangle}(-\langle D^*_{i_0}, \beta \rangle - 1)!}{\prod_{i \in I \setminus \{i_0\}} \langle D^*_i, \beta \rangle!}.$$ 

otherwise we define $E_{i_0}(\beta) = 0$. Recall that $\beta \geq 0$ when $\beta$ is effective, and $\beta > 0$ when it is also non-zero. In particular, using the Frobenius method, as shown in [36, 23], the mirror correction is given by

(36) $S_a(q) = \sum_{i \in I} l_i^{(a)} A_i(q), \quad a = 0, \ldots, k,$

where $A(q)$ is the following series of $q_1, \ldots, q_k$:

(37) $A_i(q) = \sum_{\beta > 0} E_i(\beta)q^\beta$.

The above formula is a direct reformulation of the equations [23 (2.20, 2.21, 2.22)].

Remark 4.1. In [36, 23], Lerche-Mayr explicitly compute the open mirror correction $S_0(q)$. Let $I' \subset \{1, \ldots, r\}$ be the index set such that $A_i(q) = 0$ for $i \notin I'$. One chooses an index subset $K \subset \{1, \ldots, k\}$ such that $|K| = |I'|$ and $S_a$ are linearly independent. Denote the matrix $L = (l_i^{(a)})_{a \in K, i \in I'}$. Lerche-Mayr in [23] gives the open mirror correction as $S_0(q) = \sum_{i \in I'} l_i^{(0)} \left( \sum_{a \in K} (L^{-1})^{(a)} S_a(q) \right)$. It turns out that

$$S_0(q) = \sum_{i \in I'} l_i^{(0)} \left( \sum_{a \in K} (L^{-1})^{(a)} S_a(q) \right)$$

$$= \sum_{i \in I'} l_i^{(0)} \left( \sum_{a \in K} (L^{-1})^{(a)} \sum_{j \in I'} l_j^{(a)} A_j(q) \right)$$

$$= \sum_{i \in I'} l_i^{(0)} A_i(q).$$

It agrees with [36].

4.4. Superpotential and the mirror prediction of disk invariants. In [3], the superpotential $W(C_x)$ associated to the B-brane $C_x$ is the integral over the chain

$$W = \int_{\Gamma(x)} \Omega,$$
where the boundary of the chain \( \partial \Gamma(x) = C_x - C_{x'} \). Let \( \bar{\mathcal{L}} = \mathbb{Z} \times \mathcal{L} \cong H_2(X, L; \mathbb{Z}) \). For any \( \beta' = (w, \beta) \in \bar{\mathcal{L}} \), denote \( q^{\beta'} = q_0^w q^\beta \). We define the extended pairing

\[
\langle D_i^*, \beta' \rangle = w_i^{(0)} + \langle D_i^*, \beta \rangle = \sum_{a=0}^{k} d_a \bar{t}_i^{(a)}.
\]

Define the cone \( \bar{\mathcal{L}}_{\text{eff}}(L) \) in \( \mathbb{Z} \times \mathcal{L}_{\text{eff}} \subset \bar{\mathcal{L}} \)

\[
\bar{\mathcal{L}}_{\text{eff}}(L) = \{ \beta' = (w, \beta) \in \bar{\mathcal{L}} | w \neq 0, \langle D_i^*, \beta' \rangle \geq 0, i \in I \setminus \{i_2, i_3\} \}.
\]

Given the charge vectors \( \bar{t}_i^{(a)} \), \( a = 0, \ldots, k \), Lerche and Mayr \cite{Lerche} show that \( W \) is a double logarithm solution of the Picard-Fuchs equations. Precisely \( W \) consists of a double logarithm part and a power series part

\[
W = \sum_{i,j=0}^{k} c_{ij} \log q_i \log q_j + W_0,
\]

where \( W_0 \) is explicitly in the following form

(38) \[
W_0 = \sum_{\beta' \in \bar{\mathcal{L}}_{\text{eff}}(L)} C(\beta') q^{\beta'}.
\]

The coefficient \( C(\beta') \) is obtained by applying the Frobenius method

\[
C(\beta') = \begin{cases}
\prod_{i=1}^{r+3} (-1)^{-\langle D_i^*, \beta' \rangle - 1} (-\langle D_i^*, \beta' \rangle - 1)! \prod_{i \in I \setminus \{i_2, i_3\}} (D_i^*, \beta')!, & \langle D_i^*, \beta' \rangle < 0 \\
\prod_{i=1}^{r+3} (-1)^{-\langle D_i^*, \beta' \rangle - 1} (-\langle D_i^*, \beta' \rangle - 1)! \prod_{i \in I \setminus \{i_2, i_3\}} (D_i^*, \beta')!, & \langle D_i^*, \beta' \rangle < 0 \\
\prod_{i \in I \setminus \{i_2, i_3\}} (D_i^*, \beta')! & \langle D_i^*, \beta' \rangle < 0 \end{cases}
\]

In \cite{Morrison} and \cite{Lerche}, the superpotential \( W_0 \) is explicitly computed by solving the mirror curve \( M(x, y, q) = 0 \). Writing \( y = y(x, q) \), and in \cite{Morrison} it is shown that the superpotential is

\[
x \frac{\partial W_0}{\partial x} = -\log y(x, q) \quad \text{up to a function in } q.
\]

Remark 4.2. The above formula differs by a sign with Equation 4.5 of \cite{Morrison}. We choose this sign convention in order to match the sign convention on Gromov-Witten theory. One may simply apply a change of coordinate \( y \mapsto 1/y \) to get rid of this minus sign. We only consider invariants of non-zero winding numbers. Bouchard and Sułkowski discuss the constant contributions of the higher genus superpotential associated to a mirror curve \cite{Bouchard}.

Conjecture 4.3. After a change of variables by the mirror transform \cite{Bouchard},

\[
F(Q; f) = W_0(q; f).
\]

We prove this conjecture in the next section.
5. Proof of Conjecture 4.3

5.1. Equivariant mirror theorem. A toric manifold \( X \) is semi-Fano if \( c_1(T_X) \geq 0 \). In this subsection, we state an equivariant mirror theorem for a semi-projective semi-Fano toric manifold \( X \) (Theorem 5.1). When \( X \) is projective, or is the total space of a direct sum of negative line bundles over a projective toric manifold \( Z \), Theorem 5.1 follows from the results in [30] [33] (when \( X \) or \( Z \) is a projective space) and in [31] [14] (in the general case). When \( X \) is a general noncompact, semi-projective, semi-Fano toric manifold, Theorem 5.1 follows from the results in [9].

Let \( X \) be a semi-projective toric manifold. Let \( D_1, \ldots, D_r \) be the \( \mathbb{T} \)-divisor of \( X \). Let \( D_i^* = c_1(\mathcal{O}_X(D_i)) \in H^2(X; \mathbb{Z}) \), \( (D_i^*)^r = (c_1)(\mathcal{O}_X(D_i)) \in H^1_\ast(X; \mathbb{Z}) \).

Define an \( H^\ast_\ast(X; \mathbb{Z}) \)-valued function \( I(t, z^{-1}) \):

\[
I(t, z^{-1}) = e^{(t_0^r + \sum_{a=1}^r \lambda_a (t_a + f_a(q))) + \sum_{\beta > 0}^\{\text{1.} \} \text{ev}_1^\ast(e^{(T_0^r + \sum_{a=1}^r \lambda_a (t_a + f_a(q))) + \sum_{\beta > 0}^\{\text{1.} \} \text{ev}_1^\ast(1)} (z - \psi_1),)
\]

Note that the parameter \( t_0^r \) is different from the parameter \( t_0 = \log q_0 \) introduced in Section 4.3.

Define another \( H^\ast_\ast(X; \mathbb{Z}) \)-valued function \( J(T, z^{-1}) \) as follows:

\[
\langle \gamma, J \rangle = \int_X e^{(T_0^r + \sum_{a=1}^r \lambda_a (t_a + f_a(q))) + \sum_{\beta > 0}^\{\text{1.} \} \text{ev}_1^\ast(e^{(T_0^r + \sum_{a=1}^r \lambda_a (t_a + f_a(q))) + \sum_{\beta > 0}^\{\text{1.} \} \text{ev}_1^\ast(1)} (z - \psi_1),)
\]

or equivalently,

\[
J = e^{(T_0^r + \sum_{a=1}^r \lambda_a (t_a + f_a(q))) + \sum_{\beta > 0}^\{\text{1.} \} \text{ev}_1^\ast(e^{(T_0^r + \sum_{a=1}^r \lambda_a (t_a + f_a(q))) + \sum_{\beta > 0}^\{\text{1.} \} \text{ev}_1^\ast(1)} (z - \psi_1),)
\]

Assume that \( X \) is semi-Fano. Then the expansion of the \( I \) function in \( z^{-1} \) is of the following form

\[
I(t, z^{-1}) = 1 + z^{-1}(t_0^r + f_0(q)) + \sum_{i=1}^r \lambda_i g_i(q) + \sum_{a=1}^k p_a(t_a + f_a(q))) + O(z^{-2}),
\]

where \( \lambda_1, \ldots, \lambda_r \in H^2(BT) \) are the universal first Chern classes. The expansion of the \( J \) function in \( z^{-1} \) is

\[
J(T, z^{-1}) = 1 + z^{-1}(T_0^r + \sum_{a=1}^k p_a T_a) + O(z^{-2}).
\]

The mirror map is given by

\[
T_0^r = t_0^r + f_0(q) + \sum_{i=1}^r \lambda_i g_i(q), \quad T_a = t_a + f_a(q), \quad a = 1, \ldots, k.
\]

In particular,

\[
Q_a = q_a \exp(f_a(q)), \quad a = 1, \ldots, k.
\]

Theorem 5.1 below is a special case of the results in [9]. When \( X \) is the total space of a direct sum of negative line bundles over a projective toric manifold (such as the six toric Calabi-Yau 3-folds considered in Section 6.1.6.2), Theorem 5.1 is a special case of the results in [31] [14]. (It is stated as Corollary 4.3 in [14].)
Theorem 5.1 (Equivariant mirror theorem). With the above identification

\[ T'_0 = t'_0 + f_0(q) + \sum_{i=1}^r \lambda_i g_i(q), \quad T_a = t_a + f_a(q), \quad a = 1, \ldots, k, \]

the I and J functions are equal

\[ I(t, z^{-1}) = J(T, z^{-1}). \]

5.2. Open and closed mirror maps. We now restrict to the subtorus \( \mathbb{T}_{L,f} \). Let \( \iota_+: x^+ \to X \) be the inclusion map, and let \( \phi^+ \in H^*_{{\mathbb{T}_{L,f}}}(X; \mathbb{Z}) \) be the \( \mathbb{T}_{L,f} \)-equivariant Poincaré dual of \( x^+ \). Recall that \( J_w^L(Q; f) \) is defined by (20) and \( A_0(q) \) is defined by (37).

Proposition 5.2. Let \( w \) be a positive integer. Then

\[ J_w^L(Q; f) = \left( \frac{q_0}{Q_0} \right)^w \left( 1 + \sum_{\beta > 0} q^\beta \prod_{i=1}^r \frac{\Gamma_{m=-\infty}^0(w_i^{(0)} + m)}{\Gamma_{m=-\infty}^0(w_i^{(0)} + m)} \right). \]

where \( Q = (Q_0, Q_1, \ldots, Q_k) \) and \( q = (q_0, q_1, \ldots, q_k) \) are related by the following open and closed mirror maps

(39)

\[ Q_a = q_a \exp(\sum_{i \in I} f_i^{(0)} A_i(q)), \quad a = 0, 1, \ldots, k. \]

Proof. In this proof, \( u = u_L \). The parameters \( t'_0 \) and \( t_0 \) are different.

\[ \iota_+^* J(T, z^{-1}) = e^{(T'_0 + \sum_{a=1}^k T_a^* p_a)/z} \left( 1 + \sum_{\beta > 0} Q^\beta \int_{\mathbb{P}^1 \cup \mathbb{P}^1} \frac{\text{ev}_1^* \phi^+}{w(z - \psi)} \right). \]

\[ \iota_+^* J(T, \frac{w}{u}) = e^{(T'_0 + \sum_{a=1}^k T_a^* p_a)/w} \left( 1 + \sum_{\beta > 0} Q^\beta \int_{\mathbb{P}^1 \cup \mathbb{P}^1} \frac{\text{ev}_1^* \phi^+}{w(z - \psi)} \right) \]

\[ \iota_+^* J(t, \frac{w}{u}) = e^{(t'_0 + \sum_{a=1}^k t_a^* p_a)/w} \left( 1 + \sum_{\beta > 0} q^\beta \prod_{i=1}^r \frac{\Gamma_{m=-\infty}^0(t_i^{(0)} u + m \frac{u}{w})}{\Gamma_{m=-\infty}^0(t_i^{(0)} u + m \frac{u}{w})} \right). \]

where we used \( \sum_{i=1}^r D_i^* = 0 \). We conclude that

\[ J_w^L(Q; f) = \left( \frac{q_0}{Q_0} \right)^w \left( 1 + \sum_{\beta > 0} q^\beta \prod_{i=1}^r \frac{\Gamma_{m=-\infty}^0(w_i^{(0)} + m)}{\Gamma_{m=-\infty}^0(w_i^{(0)} + m)} \right). \]

where \( Q_0 \) and \( q_0 \) are related by the following open mirror map

\[ Q_0 = q_0 \exp\left( \frac{T_0' - t'_0}{u} + \sum_{a=1}^k (T_a - t_a) \frac{t_a^* p_a}{u} \right), \]

or equivalently,

(40)

\[ T_0 = t_0 + \frac{t_a^* ((T_0' - t'_0) + \sum_{a=1}^k (T_a - t_a) p_a)}{u}. \]
It remains to show that the open and closed mirror maps are given by

\[ T_a = t_a + \sum_{i \in I} l_i^{(a)} A_i(q), \quad a = 0, 1, \ldots, k. \]

To see this, we expand the \( I \) function from Section ??:

\[
I(t, z^{-1}) = (1 + t'_0 \frac{\sum a=1}{z} t_a p_a + O(z^{-2}))(1 + \sum \beta>0 q^\beta \prod_{B \in \Lambda} \prod_{m=-\infty}^{\infty} ((D_i^*)_m^T + mz))
\]

The \( z^{-1} \)-term in \( I(t, z^{-1}) \) is

\[
z^{-1} \text{-term} = z^{-1}((t'_0 + \sum a=1 t_a p_a) + \sum A_i(q)(D_i^*)_T)
\]

Therefore, the mirror maps satisfy

\[
(T_0' - t'_0) + \sum a=1 (T_a - t_a)p_a = \sum A_i(q)(D_i^*)_T
\]

(41)

To obtain the closed mirror map, we consider the nonequivariant version of (41):

\[
f_0(q) + \sum a=1 (T_a - t_a)p_a = \sum A_i(q)D_i^*
\]

where \( D_i^* = \sum_{a=1}^{k} l_i^{(a)} p_a \in H_2(X; \mathbb{Z}) \). So \( f_0(q) = 0 \), and the closed mirror map is given by

\[ T_a = t_a + \sum_{i \in I} l_i^{(a)} A_i(q), \quad a = 1, \ldots, k. \]

To obtain the open mirror map, we apply \( \iota_* \) to (41) and restrict to \( \mathbb{P}_{L,f} \):

\[
\iota_* \left( (T_0' - t'_0) + \sum a=1 (T_a - t_a)p_a \right) = \sum A_i(q) \iota_* (D_i^*)_T = \sum A_i(q) l_i^{(0)} u.
\]

Combing (40) and (42), we obtain the open mirror map:

\[ T_0 = t_0 + \sum l_i^{(0)} A_i(q). \]

(43)

Note that (39) implies

\[
Q^\beta = q^\beta \exp \left( \sum_{i \in I} \langle \beta, D_i^* \rangle A_i(q) \right).
\]

for all \( \beta \in H_2(X; \mathbb{Z}) \).

**Remark 5.3.** The mirror maps (39) are exactly prescribed by the logarithmic solution of the Picard-Fuchs equations (35), as expected.
Corollary 5.4. Let $w$ be a negative integer, and $L$ be a framed inner brane. Define $l^- = (l^t_1, \ldots, l^t_{k+3})$ to be

$$l^-_1 = f + n + 1, \quad l^-_3 = -(f + n), \quad l^-_4 = -1,$$

and

$$l^-_i = 0 \text{ if } i \in \{1, 2, \ldots, k + 3\} \setminus \{i_2, i_3, i_4\}.$$  

(In other words, $l^-_i = l^{t(0)}_i - \langle D^*_i, \alpha \rangle, \quad i = 1, \ldots, k + 3$.) Then

$$J_w^L(Q; f) = \left( \frac{q_0^\alpha / Q_0}{Q_0} \right)^w \left( 1 + \sum_{\beta > 0} q^\beta \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l^-_i + m)}{\prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (w l^-_i + m)} \right).$$

Proof. Apply Proposition 5.2 with respect to the fixed point $q$ and $L$, we have

$$J_w^L(Q; f) = \left( \frac{q_0^\alpha / Q_0}{Q_0} \right)^w \left( 1 + \sum_{\beta > 0} q^\beta \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l^-_i + m)}{\prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (w l^-_i + m)} \right),$$

where $q_0^-$ and $Q_0^-$ are open string parameters related by

$$Q_0^- = q_0^- \exp(\sum_{i \in L} l^-_i A_i(q)).$$

Since $l^-_i = l^{t(0)}_i - \langle \alpha, D^*_i \rangle$, we have

$$Q_0^- = q_0^- \exp(\sum_{i \in L} l^{t(0)}_i A_i(q) - \sum_{i \in L} \langle \alpha, D^*_i \rangle A_i(q)).$$

On the other hand,

$$\frac{Q_0^-}{q_0^\alpha / Q_0} = \exp(\sum_{i \in L} l^{t(0)}_i A_i(q) - \sum_{i \in L} \langle \alpha, D^*_i \rangle A_i(q))$$

and this implies the statement. \qed

5.3. Outer brane. By Corollary 3.3 (a) and Proposition 5.2

$$F(Q; f) = \sum_{w > 0} q_w^\alpha (-1)^f w ! \prod_{m=-\infty}^{w-1} (f w + m) \prod_{m=-\infty}^{0} (w l_i^t + m) \left( 1 + \sum_{\beta > 0} q^\beta \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l_i^t + m)}{\prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (w l_i^t + m)} \right)$$

$$= \sum_{(w, \beta) \in \tilde{L}_{\text{eff}}(L), w > 0} (-1)^f w ! \prod_{m=-\infty}^{w-1} (f w + m) \prod_{m=-\infty}^{0} (w l_i^t + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (f w + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (-\langle D^*_i + f + 1 \rangle w + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (f w + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (f w + m)$$

$$q_w^\alpha q^\beta \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (f w + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (-\langle D^*_i + f + 1 \rangle w + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (f w + m) \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} \prod_{m=-\infty}^{\langle D^*_i, \beta \rangle} (f w + m)$$

This agrees with Equation (38).

Remark 5.5. The condition $\beta' = (w, \beta) \in \tilde{L}_{\text{eff}}(L)$ implies

$$w \neq 0, \quad \langle D^*_i, \beta' \rangle = w + \sum_{a=1}^{k} d_a l^{t(a)}_{i_a} \geq 0.$$
When $L$ is an outer brane, it further implies $w > 0$. One may choose appropriate $\mathbb{Z}$-basis $\{l^{(a)}\}_{a=1, \ldots, k}$ in $\mathbb{L}$ and $\{p_a\}_{a=1, \ldots, k}$ in $\mathbb{L}^\vee$ with respect to the $T$-fixed point $x^+$. Since the toric variety $X = (\mathbb{C}^r - Z(\Delta))/G$ is smooth, the $G$ action on $\mathbb{C}^r - Z(\Delta)$ is free; in particular the $G$ action on

$$\{(X_1, \ldots, X_r) : X_{i_1} = X_{i_2} = X_{i_3} = 0, X_i \in \mathbb{C}^r \text{ for other } i \} \subset \mathbb{C}^r - Z(\Delta)$$

is free. It follows that the composition with the projection

$$\mathbb{L} \to \widetilde{N} = \bigoplus_{i=1}^r \mathbb{Z}\tilde{e}_i \to \bigoplus_{i \in I \setminus I_0} \mathbb{Z}\tilde{e}_i$$

is an isomorphism. Thus we choose the preimages of $\tilde{e}_i$, where $i \in I \setminus I_0$ as a basis $\{l^{(a)}\}$ of $\mathbb{L}$. We write $l^{(a)}$ to be the preimage of $\tilde{e}_i$ for $i \in I \setminus I_0$. It is obvious that the cone

$$\sum_{a=1}^k \mathbb{Z}_{\geq 0}l^{(a)} \subset \mathbb{L}_{\text{eff}};$$

while the pairing $(p_a, l^{(b)}) = \delta_{ab}$ for the dual basis $p_a \in \mathbb{L}^\vee$, and $p_a = D_i^*$. With such a choice of $l^{(a)}$ and $p_a$, the intersection $L \cap X$ is described by the following equations

$$l^{(a)}_{i_1}|X_{i_1}|^2 + \sum_{i \in I \setminus I_0} l^{(a)}_{i}|X_{i}|^2 = r_a, \quad a = 1, \ldots, k,$$

where $r_1, \ldots, r_k$ are Kähler parameters. Since $L$ is an outer brane, if one moves $L$ to infinity, i.e. let $|X_{i_1}| \to \infty$, we will not encounter a $T$-fixed point, i.e. $|X_{i_1}|$ cannot be zero for any $i \in I \setminus I_0$. By our choice of basis $\{l^{(a)}\}$ and $\{p_a\}$ in $\mathbb{L}$ and $\mathbb{L}^\vee$, $l^{(a)}_{i_1} = \delta_{ij}$ for $i, j \in I \setminus I_0$. Thus $l^{(a)}_{i_1} \leq 0$ for $a = 1, \ldots, k$, and $\beta' \in \mathbb{L}_{\text{eff}}(L)$ already implies $w > 0$.

This completes the proof of Conjecture 4.3 when $L$ is a framed outer brane.

5.4. Inner brane. By Corollary 3.6 (b), Proposition 5.2 and Corollary 5.4

$$F(Q; f) = I_+ + I_-,$$

where

$$I_+ = \sum_{w > 0} q^w \left\{ (-1)^w \frac{\prod_{m=1}^{w-1} (f + w + m)}{w \cdot w!} \left( 1 + \sum_{\beta > 0} q^{\beta} \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l^{(0)}_i + m)}{\prod_{m=-\infty}^{0} (w l^{(0)}_i + m)} \right) \right\}$$

$$I_- = \sum_{w < 0} \left( \frac{q_0}{q_0^w} \right) \left\{ (-1)^w \frac{\prod_{m=1}^{w-1} (f + w - m)}{w \cdot w!} \left( 1 + \sum_{\beta > 0} q^{\beta} \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l^{(0)}_i - m)}{\prod_{m=-\infty}^{0} (w l^{(0)}_i - m)} \right) \right\}.$$

So $I_-$ can be rewritten as

$$I_- = \sum_{w < 0} q^w \left\{ (-1)^w \frac{\prod_{m=1}^{w-1} (f + w - m)}{w \cdot w!} \left( 1 + \sum_{\beta > 0} q^{\beta} \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l^{(0)}_i - m)}{\prod_{m=-\infty}^{0} (w l^{(0)}_i - m)} \right) \right\}$$

$$= \sum_{w < 0} q^w \left\{ (-1)^w \frac{\prod_{m=1}^{w-1} (f + w - m)}{w \cdot w!} \left( 1 + \sum_{\beta > 0} q^{\beta} \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w l^{(0)}_i - m)}{\prod_{m=-\infty}^{0} (w l^{(0)}_i - m)} \right) \right\}.$$
The calculation of $I_+$ is identical to the outer brane case.

$$
I_+ = \sum_{w > 0} q_0^w (1 - f_w + m) \frac{\prod_{m=-\infty}^{w-1} (f_w + m)}{w \cdot w!} \left( 1 + \sum_{\beta > 0} q_0^\beta \prod_{i=1}^{k+3} \frac{\prod_{m=-\infty}^{0} (w_0^i + m)}{\prod_{m=-\infty}^{(\beta, \beta')} m} \right)
$$

while the other half of the contribution $I_-$ is the following

$$
I_- = \sum_{\beta' \in \Lambda_{eff}(L), w < 0} (1 - f_{w-n} + m) \frac{\prod_{m=-\infty}^{w-1} ((f + n + 1)w + m)}{(w-n) \cdot (w-n)!} \frac{1}{\prod_{i \in \Pi \setminus \{\theta \cup \{i_3\}\} (D_i^*, \beta')} \prod_{m=-\infty}^{w-n} m} \prod_{m=-\infty}^{(D_i^*, \beta')} m q^{\beta'}
$$

Therefore,

$$
F(Q; f) = I_+ + I_- = \sum_{\beta' \in \Lambda_{eff}(L)} (1 - f_{w-n} + m) \frac{\prod_{m=-\infty}^{w-n} ((f + n + 1)w + m)}{(w-n) \cdot (w-n)!} \frac{1}{\prod_{i \in \Pi \setminus \{\theta \cup \{i_3\}\} (D_i^*, \beta')} \prod_{m=-\infty}^{w-n} m} \prod_{m=-\infty}^{(D_i^*, \beta')} m q^{\beta'}
$$

It agrees with Equation \((38)\). This completes the proof of Conjecture \(4.3\) when $L$ is a framed inner brane.

6. Explicit Mirror Formulae

In this section we list explicit mirror formulae for several examples, whose charge vectors are listed in Table 1. The open and closed mirror maps are given by

$$
Q_a = q_a \exp \left( \sum_{i \in I} t_i^{(a)} A_i(q) \right), \quad a = 0, \ldots, k.
$$

The formulae for

$$
A_i(q) = \sum_{d_1, \ldots, d_k \in \mathbb{Z}} a_i^{d_1, \ldots, d_k} \prod_{a=1}^{k} q_a^{d_a}.
$$

are listed in Table 2 at the end of this section. In particular, there is no mirror correction when $X$ is the resolved conifold. Notice $A_i(q) = 0$ for $i \neq 1$ for $X$ other than the toric crepant resolution $Y_m$ of $(\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1))/\mathbb{Z}_m$.

6.1. $\mathcal{O}_{\mathbb{P}^1}(-1) \oplus \mathcal{O}_{\mathbb{P}^1}(-1)$.

\textbf{B-model: $T_q^\vee$ is}

$$
x_1 x_2^{-1} x_3 x_4 = q = e^t.
$$

The mirror curves are listed in the following table.

| Phase | Mirror curve |
|-------|--------------|
| I     | $y + 1 - qy - xy^{-f} = 0$ |
| II    | $-xy^{-f} + y + 1 - qxy^{-f+1} = 0$ |
\begin{center}
\begin{tabular}{|c|c|}
\hline
$S$ & $l^{(a)}$, $a = 1, \ldots, k$ \\
\hline
$\mathcal{O}_{\mathbb{P}^1}(-1)$ & $l^{(1)} = (-1, -1, 1, 1)$ \\
$\mathbb{P}^2$ & $l^{(1)} = (-3, 1, 1, 1)$ \\
$\mathbb{P}^1 \times \mathbb{P}^1$ & $l^{(1)} = (-2, 1, 1, 0, 0)$ \\
& $l^{(2)} = (-2, 0, 0, 1, 1)$ \\
d$P_1$ & $l^{(1)} = (-2, 1, 1, 0, 0)$ \\
& $l^{(2)} = (-1, 0, -1, 1, 1)$ \\
d$P_2$ & $l^{(1)} = (-2, 1, 1, 0, 0)$ \\
& $l^{(2)} = (-2, 0, 0, 1, 1)$ \\
& $l^{(3)} = (-3, 1, 0, 1, 0, 1)$ \\
d$P_3$ & $l^{(1)} = (-2, 1, 1, 0, 0, 0)$ \\
& $l^{(2)} = (-2, 0, 0, 1, 1, 0, 0)$ \\
& $l^{(3)} = (-3, 1, 0, 1, 0, 1, 0)$ \\
& $l^{(4)} = (-3, 0, 1, 0, 1, 0, 1)$ \\
$Y_m$ & $l^{(1)} = (1, 1, 0, -2, 0, 0, \ldots, 0)$ \\
& $l^{(2)} = (0, 0, 1, -2, 1, 0, \ldots, 0)$ \\
& $l^{(3)} = (0, 0, 0, 1, -2, 1, \ldots, 0)$ \\
& $\ldots$ \\
& $l^{(m)} = (0, 0, 0, \ldots, 0, 1, -2, 1)$ \\
\hline
\end{tabular}
\end{center}

\textbf{Table 1.} charge vectors

**A-model:**

\[-|X_1|^2 - |X_2|^2 + |X_3|^2 + |X_4|^2 = r,\]

where $r > 0$.

\[|X_4|^2 - |X_2|^2 = c_1, \quad |X_1|^2 - |X_2|^2 = c_2.\]

Therefore

\[(|X_1|^2, |X_2|^2, |X_3|^2, |X_4|^2) = (c_2, 0, r - c_1 + c_2, c_1) + s(1, 1, 1, 1), \quad s \geq 0.\]

See Figure 5

\begin{center}
\begin{figure}
\begin{center}
\begin{tikzpicture}
\draw[->] (0,0) -- (4,0) node[above] {$X_4 = 0$};
\draw[->] (0,0) -- (0,4) node[right] {$X_2 = 0$};
\draw[->] (0,0) -- (0,0) node[above] {$X_1 = 0$};
\draw[->] (0,0) -- (2,2) node[midway] {$X_3 = 0$};
\fill (2,2) circle (2pt) node[above] {$r - c_1$};
\fill (2,0) circle (2pt) node[above] {$c_1$};
\fill (0,2) circle (2pt) node[right] {$c_2$};
\fill (4,4) circle (2pt) node[above] {$X_1 = 0$};
\end{tikzpicture}
\end{center}
\end{figure}
\end{center}

\textbf{Figure 5.} Phase I: $r > c_1 > 0, c_2 = 0$. Phase II: $c_1 = 0, c_2 > 0.$

**Mirror Formula:**

\[F(Q; f) = W(q; f) = \sum_{w \neq 0} \sum_d n_{d,w}(f)q_0^w q_1^d.\]
6.2. $K_{P_2}$.

**B-model:** $T_q^\vee$ is

$$x_1^2 x_2 x_3 x_4 = q = e^t.$$

The mirror curves are listed in the following table.

| Phase | Mirror curve |
|-------|--------------|
| I     | $1 - xy^{-1} + y - qx^{-1} y^{d-1} = 0$ |
| II    | $y + 1 - xy^{-1} - qx^{-1} y^{d-1} = 0$ |
| III   | $-xy^{-1} + 1 - qx^{-1} y^{d-1} = 0$ |

**A-model:**

$$|X_2|^2 + |X_3|^2 + |X_4|^2 - 3|X_1|^2 = r,$$

where $r > 0$.

Therefore

$$|X_2|^2 - |X_1|^2 = c_1, \quad |X_3|^2 - |X_1|^2 = c_2.$$

See Figure 6

![Figure 6](image)

**Figure 6.** Phase I: $r > c_1 > 0, c_2 = 0$. Phase II: $c_1 = 0, r > c_2 > 0$. Phase III: $c_1 = c_2 < 0$.

**Mirror Formula:** $F(Q; f) = W(q; f) = \sum_{w \geq 0} \sum_d n_{d,w}(f) q_0^w q_1^d$

| Phase | $(i_1, i_2, i_3)$ | $n_{d,w}(f)$ | $\mathbb{I}_{\text{eff}}(L)$ |
|-------|------------------|--------------|---------------------|
| I     | (2, 3, 1)        | $(-1)^{f+w+d} \frac{\prod_{m=d+1}^{d+w+1} (fw+m)}{w \cdot (w+d)! d!}$ | $d, w + d \geq 0$ |
| II    | (3, 1, 2)        | $(-1)^{f+w+d} \frac{\prod_{m=d+1}^{d+w+1} (fw+m)}{w \cdot (w+d)! d!}$ | $d, w + d \geq 0$ |
| III   | (1, 2, 3)        | $(-1)^{f+w+d} \frac{\prod_{m=d+1}^{d+w+1} (fw+m)}{w \cdot (w-3d)! d!}$ | $d \geq 0, w \geq 3d$ |
In Phase I, when \( f = 0 \) we have
\[
\sum_{w \neq 0, d \geq 0, w + d \geq 0} N_{0, d + w, w} (f) Q_0^w Q_1^d = \sum_{w \neq 0, d \geq 0, w + d \geq 0} (-1)^d \frac{(w + 3d - 1)!}{w \cdot (w + d)! (d)!} q_0^w q_1^d,
\]
which agrees with the formula in [16, Section 4.3].

In Phase III, when \( f = 0 \) we have
\[
\sum_{w > 0, d \geq 0} N_{0, d + w, w} (f) Q_0^w Q_1^d = \sum_{w > 0, d \geq 0, w + 3d \geq 0} (-1)^d \frac{(w - d - 1)!}{w \cdot (d)! (w - 3d)!} q_0^w q_1^d,
\]
which agrees with [23, Equation A.3].

6.3. \( K_{\mathbb{P}^1 \times \mathbb{P}^1} = K_{\mathbb{P}^1} \).

**B-model:** The torus \( T_q \) is given by the following equations
\[
x_1^{-2} x_2 x_3 = q_1 = e_1, \\
x_1^{-2} x_4 x_5 = q_2 = e_2.
\]

The mirror curves are listed in the following table.

| Phase | Mirror curve |
|-------|--------------|
| I     | \( y - q_1 x^{-1} y^{l+2} - x y^{-l} + 1 + q_2 y^2 = 0 \) |
| II    | \( 1 + q_1 y^{-1} + y - x y^{-l} - q_2 x^{-1} y^l = 0 \) |
| III   | \( -x y^{-l} + q_1 x^{2} y^{-2l} + 1 + y + q_2 x^{2} y^{-2l-1} = 0 \) |

**A-model:**
\[
|X_2|^2 + |X_3|^2 - 2 |X_1|^2 = r_1, \quad |X_4|^2 + |X_5|^2 - 2 |X_1|^2 = r_2
\]
where \( r_1, r_2 > 0 \).
\[
|X_3|^2 - |X_1|^2 = c_1, \quad |X_4|^2 - |X_1|^2 = c_2.
\]

Therefore
\[
(|X_1|^2, |X_2|^2, \ldots, |X_5|^2) = (0, r_1 - c_1, c_1, c_2, r_2 - c_2) + s(1, 1, 1, 1, 1), \quad s \geq 0.
\]

See Figure 7.

Figure 7. Phase I: \( r_1 > c_1 > 0, c_2 = 0 \). Phase II: \( c_1 = 0, r_2 > c_2 > 0 \). Phase III: \( c_1 = c_2 < 0 \).
Mirror Formula: $F(Q; f) = W(q; f) = \sum_{w \neq 0} \sum_{d_1, d_2} n_{d_1, d_2, w}(f) q_0^{w_1} t_1^{d_1} t_2^{d_2}$.

| Phase | $(i_1, i_2, i_3)$ | $n_{d_1, d_2, w}(f)$ | $L_{aw}(L)$ |
|-------|----------------|---------------------|-------------|
| I     | (3, 1, 4)     | $(-1)^{f_{d_2}+d_2} \prod_{m=-2d_1-2d_2+1}^{d_2} (f w + m) \overline{w \cdot (w + d_1) d_1 d_2!}$ | $d_1, d_2, w + d_1 \geq 0$ |
| II    | (4, 3, 1)     | $(-1)^{f_{d_2}+d_2} \prod_{m=-2d_1-2d_2+1}^{d_2} (f w + m) \overline{w \cdot (w + d_1) d_1 d_2!}$ | $d_1, d_2, w + d_2 \geq 0$ |
| III   | (1, 4, 3)     | $(-1)^{f_{d_2}+d_2} \prod_{m=-2d_1-2d_2+1}^{d_2} (f w + m) \overline{w \cdot (w + d_1) d_1 d_2!}$ | $d_1, d_2 \geq 0, w \geq 2d_1 + 2d_2$ |

6.4. $K_{dP_1} = K_{\mathbb{F}_1}$.

**B-model:** The torus $\mathbb{T}_q$ is given by the following equations

\[
x_1^{-2} x_2 x_3 = q_1 = e^{t_1}, \]
\[
x_1^{-1} x_2^{-1} x_4 x_5 = q_2 = e^{t_2}.
\]

The mirror curves are listed in the following table.

| Phase | Mirror curve |
|-------|--------------|
| I     | $y + 1 + q_1 y^2 - xy^{-j} - q_1 q_2 x^{-1} y^{j+1} + 1 = 0$ |
| II    | $-x y^{-j} + q_1 x y^{-j} - y + q_1 q_2 x^{-1} y^{j+1} + 1 = 0$ |
| III   | $-x y^{-j} + q_1 x y^{-j} - y + q_1 q_2 x^{-1} y^{j+1} + 1 = 0$ |
| IV    | $-x y^{-j} + q_1 x y^{-j} - y + q_1 q_2 x^{-1} y^{j+1} + 1 = 0$ |

**A-model:**

\[
|X_2|^2 + |X_3|^2 - 2|X_1|^2 = r_1, \quad |X_4|^2 + |X_5|^2 - |X_1|^2 - |X_3|^2 = r_2
\]

where $r_1, r_2 > 0$.

\[
|X_4|^2 - |X_1|^2 = c_1, \quad |X_2|^2 - |X_1|^2 = c_2.
\]

Therefore

\[
(|X_1|^2, |X_2|^2, \ldots, |X_5|^2) = (0, c_2, r_1 - c_2, c_1, r_1 + r_2 - c_1 - c_2) + s(1, 1, 1, 1, 1), \quad s \geq 0.
\]

See Figure 8

![Figure 8](https://example.com/fig8.png)

**Figure 8.** Phase I: $r_1 + r_2 > c_1 > 0, c_2 = 0$. Phase II: $c_1 = 0, r_1 > c_2 > 0$. Phase III: $r_2 > c_1 > 0, c_2 = r_1$. Phase IV: $c_1 = c_2 < 0$. Phase V: $c_1 = r_1 - c_2 < 0$. 
Mirror Formula: \( F(Q; f) = W(q; f) = \sum_{w \neq 0} \sum_{d_1, d_2} n_{d_1, d_2, w}(f) q_0^{d_1} q_2^{d_2}. \)

| Phase | \((i_1, i_2, i_3)\) | \(n_{d_1, d_2, w}(f)\) | \(L_{\text{eff}}(L)\) |
|-------|------------------|-----------------|-----------------|
| I     | (4, 1, 2)        | \((-1)^{f_w+d_1} \prod_{m=d_1-1}^{d_1-w-1} (f_w + m)\) | \(w + d_2 \geq 0, d_1 \geq d_2 \geq 0\) |
| II    | (2, 4, 1)        | \((-1)^{f_w+d_1} \prod_{m=d_1-1}^{d_1-w-1} (f_w + m)\) | \(w + d_1 \geq 0, d_1 \geq d_2 \geq 0\) |
| III   | (4, 3, 1)        | \((-1)^{f_w+d_1} \prod_{m=d_1-1}^{d_1-w-1} (f_w + m)\) | \(d_1, d_2 \geq 0, w + d_2 \geq 0\) |
| IV    | (1, 2, 4)        | \((-1)^{f_w+d_1} \prod_{m=d_1-1}^{d_1-w-1} (f_w + m)\) | \(d_1 \geq d_2 \geq 0, w \geq 2d_2 + d_4\) |
| V     | (1, 4, 3)        | \((-1)^{f_w+d_1} \prod_{m=d_1-1}^{d_1-w-1} (f_w + m)\) | \(d_1, d_2 \geq 0, w \geq 2d_4 + d_2\) |

6.5. \(K_{dP_2}\).

**B-model**: The torus \(\mathbb{T}^3_q\) is given by

\[
x_1^{-2} x_2 x_3 = q_1 = e^{i_1},
\]

\[
x_1^{-2} x_4 x_5 = q_2 = e^{i_2},
\]

\[
x_1^{-3} x_2 x_4 x_6 = q_3 = e^{i_3}.
\]

The mirror curves are listed in the following table.

| Phase | Mirror curve |
|-------|--------------|
| I     | \(y + 1 + q_1 y^2 - q_2 x^{-1}y^{j+2} - xy^{-j} - (q_2)^{-1}q_3 x y^{j+1} = 0\) |
| II    | \(1 - xy^{-j} + q_1 x^2 y^{-j+2} + q_2 y^{-j} + y - (q_2)^{-1}q_3 x^{-1} y^{j+1} = 0\) |
| III   | \(1 - (q_2)^{-1}q_3 x y^{-j-1} - q_1 q_2 (q_3)^{-1}y^{j+1} - q_2 x^{-1}y^{j+1} - xy^{-j} + y = 0\) |
| IV    | \(-xy^{-j} + 1 + q_1 x^2 y^{-j+2} + q_2 x y^{-j+2} + q_2 x y^{-j+2} - q_2 x^{-1} y^{-j-1} = 0\) |
| V     | \(-xy^{-j} + y + q_1 x^2 y^{j-1} + q_2 x^2 y^{j-1} + q_2 x^2 y^{j-1} - q_2 x y^{-j-1} = 0\) |
| VI    | \(-xy^{-j} - q_1 q_2 x^2 y^{j+1} - q_1 q_2 q_3 x y^{-j-1} + q_2 x^2 y^{-j+1} + q_2 x^2 y^{-j+1} + y + 1 = 0\) |

**A-model**:

\[
|X_2|^2 + |X_3|^2 - 2|X_1|^2 = r_1,
\]

\[
|X_4|^2 + |X_3|^2 - 2|X_1|^2 = r_2,
\]

\[
|X_2|^2 + |X_4|^2 + |X_6|^2 - 3|X_1|^2 = r_3,
\]

where \(r_1, r_2, r_3 > 0\), and max\(r_1, r_2\) < \(r_3 < r_1 + r_2\).

\[
|X_2|^2 - |X_1|^2 = c_1, \quad |X_3|^2 - |X_1|^2 = c_2.
\]

Therefore

\[
(|X_1|^2, |X_2|^2, \ldots, |X_6|^2) = (0, c_1, r_1 - c_1, r_2 - c_2, c_2, r_3 - r_2 - c_1 + c_2) + t(1, 1, 1, 1, 1), \quad t \geq 0.
\]

See Figure 9.
The mirror curves are listed in the following table.

\[
\begin{align*}
\text{Phase} & \quad (i_1, i_2, i_3) & \quad n_{d_1,d_2,d_3}(f) & \quad \mathbb{L}_{\text{eff}}(L) \\
\hline
\text{I} & (5,1,2) & \frac{(-1)^{w+d_1+d_3} \prod_{m=d_2+1}^{d_2+d_3} (f w + m)}{w(w+d_2)!d_1!d_2!d_3!} & d_1, d_3, d_2 + d_3 \geq 0 \quad w + d_2 \geq 0 \\
\text{II} & (2,5,1) & \frac{(-1)^{w+d_1+d_2} \prod_{m=d_3+1}^{d_2+d_3} (f w + m)}{w(w+d_1)!d_1!d_2!d_3!} & d_1, d_3, d_2 + d_3 \geq 0 \quad w + d_1 + d_3 \geq 0 \\
\text{III} & (5,6,1) & \frac{(-1)^{w+d_1+d_2} \prod_{m=d_3+1}^{d_2+d_3} (f w + m)}{w(w+d_1)!d_1!(d_1+d_3)!(d_2+d_3)!} & d_1, d_2 + d_3 \geq 0 \quad w + d_2 \geq 0 \\
\text{IV} & (1,4,2) & \frac{(-1)^{w+d_1+d_3} \prod_{m=d_2+1}^{d_2+d_3} (f w + m)}{w(w-2d_1-2d_2-3d_3)!d_1!d_2!d_3!} & d_1, d_3, d_2 + d_3 \geq 0 \quad w \geq 2d_1 + 2d_2 + 3d_3 \\
\text{V} & (1,2,5) & \frac{(-1)^{w+d_1+d_3} \prod_{m=d_2+1}^{d_2+d_3} (f w + m)}{w(w-2d_1-2d_2-3d_3)!d_1!d_2!d_3!} & d_1, d_3, d_2 + d_3 \geq 0 \quad w \geq 2d_1 + 2d_2 + 3d_3 \\
\text{VI} & (1,5,6) & \frac{(-1)^{w+d_1+d_3} \prod_{m=d_2+1}^{d_2+d_3} (f w + m)}{w(w-2d_1-2d_2-3d_3)!d_1!d_2!d_3!} & d_1, d_3, d_2 + d_3 \geq 0 \quad w \geq 2d_1 + 2d_2 + 3d_3
\end{align*}
\]

6.6. \( K_{dP_3} \).

**B-model:** The torus \( \mathbb{T}^3_q \) is given by

\[
\begin{align*}
x_1^{-2} x_2 x_3 &= q_1 = e^{i_1} \\
x_1^{-1} x_4 x_5 &= q_2 = e^{i_2} \\
x_1^{-3} x_2 x_4 x_6 &= q_3 = e^{i_3} \\
x_1^{-3} x_2 x_5 x_7 &= q_4 = e^{i_4}
\end{align*}
\]

The mirror curves are listed in the following table.

\[
\begin{align*}
\text{Phase} & \quad \text{Mirror curve} \\
\hline
\text{I} & 1 + y + q_1 y^{-1} - q_2 x^{-1} y^1 - x y^{-1} - (q_1 q_2) q_3 x y^{-1} - (q_1 q_2 q_3 x y^{-1} + 1 = 0 \\
\text{II} & -x y^{-1} + 1 + q_1 x^2 y^{-1} + q_2 x^2 y^{-1} + y - (q_1 q_2 q_3 x y^{-1} - (q_1 q_2 q_3 x y^{-1} + 1 = 0
\end{align*}
\]
**A-model:**

\[
|X_2|^2 + |X_3|^2 - 2|X_1|^2 = r_1, \\
|X_4|^2 + |X_3|^2 - 2|X_1|^2 = r_2, \\
|X_2|^2 + |X_4|^2 + |X_6|^2 - 3|X_1|^2 = r_3, \\
|X_3|^2 + |X_5|^2 + |X_7|^2 - 3|X_1|^2 = r_4,
\]

where \( r_1, r_2 > 0 \), \( \max(r_1, r_2) < r_3 < r_1 + r_2 \), \( \max(r_1, r_2) < r_4 < r_1 + r_2 \).

Therefore

\[
(|X_1|^2, |X_2|^2, |X_3|^2, |X_4|^2, |X_5|^2, |X_6|^2, |X_7|^2) = (0, c_1, r_1 - c_2, c_2, r_3 - r_2 - c_1 + c_2, r_4 - r_1 + c_1 - c_2) + t(1, 1, 1, 1, 1), \quad t \geq 0.
\]

See Figure 10.

![Figure 10](image)

**Figure 10.** Phase I: \( 0 < c_1 < r_4 - r_1 \). Phase II: \( c_1 = c_2 > 0 \).

**Mirror Formula:**

\[
F(Q; f) = W(q; f) = \sum_{w \neq 0} \sum_{d_1, d_2, d_3, d_4} n_{d_1, d_2, d_3, d_4} w(f) q_0^w \prod_{a=1}^4 q_0^{d_a}.
\]

| Phase | \((i_1, i_2, i_3)\) | \(n_{d_1, d_2, d_3, d_4} w(f)\) | \(L_{\text{eff}}(L)\) |
|-------|-----------------|-----------------|-----------------|
| I     | (5, 2, 1)       | \((-1)^{w+d_3+d_4} \prod_{a=d_1+d_2+1}^{d_1+d_2+d_3+4} (f^{w+m}) w(w+d_2+d_4) ! d_3 ! d_4 ! (d_2+d_3) ! (d_1+d_4) !\) | \(d_3, d_4, d_1 + d_4, d_2 + d_3 \geq 0, w + d_2 + d_4 \geq 0\) |
| II    | (1, 5, 2)       | \((-1)^{w+d_2+d_4} \prod_{a=d_1+d_2+3}^{d_1+d_2+d_3+4} (f^{w+m}) w(w-2d_1-2d_2-3d_3-3d_4) d_3 ! d_4 ! (d_2+d_3) ! (d_1+d_4) !\) | \(d_3, d_4, d_1 + d_4, d_2 + d_3 \geq 0, w \geq 2d_1 + 2d_2 + 3d_3 + 3d_4\) |

6.7. **The toric crepant resolution** \(Y_m\) of \((\mathcal{O}_{\mathbb{P}_1}(-1) \oplus \mathcal{O}_{\mathbb{P}_1}(-1))/\mathbb{Z}_m\).

**B-model:** The torus \(\mathbb{T}_q^\vee\) is given by

\[
x_1 x_2 x_4^{-2} = q_1, \\
x_3 x_5 x_4^{-2} = q_2, \\
x_4 x_6 x_5^{-2} = q_3, \\
\ldots \ldots \\
x_{m+1} x_{m+3} x_{m+2}^{-2} = q_m.
\]
One can write down the equations of the framed mirror curves \(33\) prescribed in Section 4.1. We do not list them here because they are too long.

**A-model:**

\[
|X_1|^2 + |X_2|^2 - 2|X_4|^2 = r_1,
|X_3|^2 + |X_5|^2 - 2|X_6|^2 = r_2,
|X_4|^2 + |X_6|^2 - 2|X_5|^2 = r_3,
|X_5|^2 + |X_7|^2 - 2|X_6|^2 = r_4,
\ldots,
|X_{m+1}|^2 + |X_{m+3}|^2 - 2|X_{m+2}|^2 = r_m,
\]

where \(r_1, \ldots, r_m > 0\).

The Aganagic-Vafa A-branes in each phase are given as follows:

- **I\(0\):**
  \[
  |X_1|^2 - |X_4|^2 = |X_3|^2 - |X_4|^2 = -c, \quad c > 0
  \]

- **I\(b\), \(1 \leq b \leq m-1\):**
  \[
  |X_1|^2 - |X_{b+3}|^2 = 0, \quad |X_{b+2}|^2 - |X_{b+3}|^2 = c, \quad 0 < c < r_{b+1}.
  \]

- **I\(m\):**
  \[
  |X_1|^2 - |X_{m+3}|^2 = 0, \quad |X_{m+2}|^2 - |X_{m+3}|^2 = c, \quad c > 0.
  \]

- **\(\Pi\)\(b\), \(b = 1, 2\):**
  \[
  |X_1|^2 - |X_{b+3}|^2 = c, \quad |X_{b+2}|^2 - |X_{b+3}|^2 = 0, \quad 0 < c < r_1.
  \]

- **\(\Pi\)\(b\), \(2 \leq b \leq m\):**
  \[
  |X_1|^2 - |X_{b+3}|^2 = c, \quad |X_{b+2}|^2 - |X_{b+3}|^2 = 0,
  0 < c < r_1 + 2(r_3 + 2r_4 + \cdots + (b-2)r_b).
  \]

As an example, the image of the 1-skeleton \(X_5^1\) under the moment map \(\mu' : X \to (t'_R)^*\) is in Figure 11.
Mirror Formula: \( F(Q; f) = W(q; f) = \sum_{w \neq 0} \sum_{d_1, \ldots, d_m} n_{d_1, \ldots, d_m, w} f(\sum_{a=1}^{m} d_a^w). \)

The mirror formulae for some phases are listed in the following table.

| Phase | \((i_1, i_2, i_3)\) | \(n_{d_1, \ldots, d_m, w}(f)\) | \(L_{\text{eff}}(L)\) |
|-------|-------------------|-------------------|-------------------|
| \(I_0\) | \((4, 3, 1)\) | \((-1)^d \frac{1}{d_1!} \frac{1}{d_2!} \frac{1}{d_3!} \frac{1}{d_4!} \frac{1}{d_5!} \frac{1}{d_6!} \ldots \frac{1}{(d_{m-2} - 2d_{m-1} + d_m)!} \frac{1}{(d_{m-1} - 2d_m)!} \) | \(w \geq 2d_1 + 2d_2 - d_3, \) |
| \(3 \leq b \leq m - 2\) | \((b + 2, 1, b + 3)\) | \((-1)^d \frac{1}{d_1!} \frac{1}{d_2!} \frac{1}{d_3!} \frac{1}{d_4!} \frac{1}{d_5!} \frac{1}{d_6!} \ldots \frac{1}{(d_{m-2} - 2d_{m-1} + d_m)!} \frac{1}{(d_{m-1} - 2d_m)!} \) | \(d_1 \geq 0, \) |
| \(I_b\) | \((b + 3, b + 2)\) | \((-1)^d \frac{1}{d_1!} \frac{1}{d_2!} \frac{1}{d_3!} \frac{1}{d_4!} \frac{1}{d_5!} \frac{1}{d_6!} \ldots \frac{1}{(d_{m-2} - 2d_{m-1} + d_m)!} \frac{1}{(d_{m-1} - 2d_m)!} \) | \(d_2 - 2d_3 + d_4 \geq 0, \) |
| \(3 \leq b \leq m - 2\) | \((b + 3, b + 2)\) | \((-1)^d \frac{1}{d_1!} \frac{1}{d_2!} \frac{1}{d_3!} \frac{1}{d_4!} \frac{1}{d_5!} \frac{1}{d_6!} \ldots \frac{1}{(d_{m-2} - 2d_{m-1} + d_m)!} \frac{1}{(d_{m-1} - 2d_m)!} \) | \(d_3 - 2d_4 + d_5 \geq 0, \) |
| \(\Pi_b\) | \((b + 3, b + 2)\) | \((-1)^d \frac{1}{d_1!} \frac{1}{d_2!} \frac{1}{d_3!} \frac{1}{d_4!} \frac{1}{d_5!} \frac{1}{d_6!} \ldots \frac{1}{(d_{m-2} - 2d_{m-1} + d_m)!} \frac{1}{(d_{m-1} - 2d_m)!} \) | \(d_4 - 2d_5 + d_6 \geq 0, \) |
| \(3 \leq b \leq m - 2\) | \((b + 3, b + 2)\) | \((-1)^d \frac{1}{d_1!} \frac{1}{d_2!} \frac{1}{d_3!} \frac{1}{d_4!} \frac{1}{d_5!} \frac{1}{d_6!} \ldots \frac{1}{(d_{m-2} - 2d_{m-1} + d_m)!} \frac{1}{(d_{m-1} - 2d_m)!} \) | \(d_5 - 2d_6 + d_7 \geq 0, \) |
| $S$                      | $i$     | $a_{d_1,\ldots,d_k}$                                                                 | constraints on $d_n$ in the sum                                                                 |
|--------------------------|---------|--------------------------------------------------------------------------------------|-------------------------------------------------------------------------------------------------|
| $\mathcal{O}_{\mathbb{P}^1}(-1)$ | $i = 1,\ldots,4$ | $(-1)^{d_1-1}(3d_1 - 1)!$                                                           | $d_1 > 0$                                                                                       |
| $\mathbb{P}^2$            | 1       | $-(2d_1 + 2d_2 - 1)!$                                                               | $d_1, d_2 \geq 0$                                                                               |
|                          | 2, 3, 4  | $d_1!(d_1 - d_2)!(d_2)!$                                                             | $(d_1,d_2) \neq (0,0)$                                                                           |
| $\mathbb{P}^1 \times \mathbb{P}^1$ | 1       | $(-1)^{d_2-1}(2d_1 + d_2 - 1)!$                                                      | $d_1 \geq d_2 \geq 0$                                                                           |
|                          | 2, \ldots, 5 | $d_1!(d_1 - d_2)!(d_2)!$                                                             | $(d_1,d_2) \neq (0,0)$                                                                           |
| $dP_1$                   | 1       | $(-1)^{d_3-1}(2d_1 + 2d_2 + 3d_3 - 1)!$                                             | $d_1, d_2, d_3 \geq 0$                                                                           |
|                          | 2, \ldots, 5 | $d_1!(d_1 + d_2)!(d_2 + d_3)!$                                                       | $(d_1,d_2,d_3) \neq (0,0,0)$                                                                    |
| $dP_2$                   | 1       | $\frac{(-1)^{d_3+d_4-1}(2d_1 + 2d_2 + 3d_3 + 3d_4 - 1)!}{(d_1 + d_3)!(d_1 + d_4)! (d_2 + d_3)!(d_2 + d_4)! (d_3)!}$ | $d_1 + d_3, d_2 + d_4, d_3 + d_4 \geq 0$                                                       |
|                          | 2, \ldots, 7 | $d_1!(d_1 + d_2)!(d_2 + d_3)!(d_2 + d_4)!d_3!$                                       | $(d_1,d_2,d_3,d_4) \neq (0,0,0,0)$                                                               |
| $dP_3$                   | 1       | $\frac{(-1)^{d_3+d_4-1}(2d_1 + 2d_2 + 3d_3 + 3d_4 - 1)!}{(d_1 + d_3)!(d_1 + d_4)! (d_2 + d_3)!(d_2 + d_4)! (d_3)!}$ | $d_1 + d_3, d_2 + d_4, d_3 + d_4 \geq 0$                                                       |
|                          | 2, \ldots, 7 | $d_1!(d_1 + d_2)!(d_2 + d_3)!(d_2 + d_4)!d_3!$                                       | $(d_1,d_2,d_3,d_4) \neq (0,0,0,0)$                                                               |
| $Y_m$                    | 4       | $\frac{(-1)^{2d_1+2d_2+3d_3-5}(2d_1 + 2d_2 - d_3 - 1)!}{(d_1)^2d_2!d_3!(d_m - 1 - 2d_m)!d_m!}$ | each factorial is non-negative                                                                  |
|                          | 5, \ldots, $r - 2$ | $\frac{(-1)^{d_3+d_4-1}(2d_1 + 2d_2 + 3d_3 + 3d_4 - 1)!}{(d_1)^2d_2!d_3!(d_m - 1 - 2d_m)!d_m!}$ | $(d_1, \ldots, d_m) \neq 0$                                                                     |
|                          | $r - 1$ | $\frac{(-1)^{d_3+d_4-1}(2d_1 + 2d_2 + 3d_3 + 3d_4 - 1)!}{(d_1)^2d_2!d_3!(d_m - 1 - 2d_m)!d_m!}$ | $(d_1, \ldots, d_m) \neq 0$                                                                     |
|                          | $r$     | $\frac{(-1)^{d_3+d_4-1}(2d_1 + 2d_2 + 3d_3 + 3d_4 - 1)!}{(d_1)^2d_2!d_3!(d_m - 1 - 2d_m)!d_m!}$ | $(d_1, \ldots, d_m) \neq 0$                                                                     |

Table 2. Formulae for $A(q)$ (mirror maps)
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