The elementary 3-Kronecker modules

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Abstract. The 3-Kronecker quiver has two vertices, namely a sink and a source, and 3 arrows. A regular representation of a representation-infinite quiver such as the 3-Kronecker quiver is said to be elementary provided it is non-zero and not a proper extension of two regular representations. Of course, any regular representation has a filtration whose factors are elementary, thus the elementary representations may be considered as the building blocks for obtaining all the regular representations. We are going to determine the elementary 3-Kronecker modules. It turns out that all the elementary modules are combinatorially defined.

Let $k$ be an algebraically closed field and $Q = K(3)$ the 3-Kronecker quiver

\[ 1 \longrightarrow 2 \]

The dimension vector of a representation $M$ of $Q$ is the pair $(\dim M_1, \dim M_2)$.

We denote by $A$ the arrow space of $Q$, it is a three-dimensional vector space, thus $\Lambda = \left[ \begin{array}{cc} k & A \\ 0 & k \end{array} \right]$ is the path algebra of $Q$. Note that $\Lambda$ is a finite-dimensional $k$-algebra which is connected, hereditary and representation-infinite. The $\Lambda$-modules will be called 3-Kronecker modules. Of course, choosing a basis of $A$, the 3-Kronecker modules are just the representations of $K(3)$.

Elementary modules. In general, if $\Lambda$ is a finite-dimensional $k$-algebra, we denote by $\text{mod} \Lambda$ the category of all (finite-dimensional left) $\Lambda$-modules. We denote by $\tau$ the Auslander-Reiten translation in $\text{mod} \Lambda$.

Now let $\Lambda$ be the path algebra of a finite acyclic quiver. A $\Lambda$-module $M$ is said to be preprojective provided there are only finitely many isomorphism classes of indecomposable modules $X$ with $\text{Hom}(X, M) \neq 0$, or, equivalently, provided $\tau^t M = 0$ for some natural number $t$. Dually, $M$ is said to be preinjective provided there are only finitely many isomorphism classes of indecomposable modules $X$ with $\text{Hom}(M, X) \neq 0$, or, equivalently, provided $\tau^{-t} M = 0$ for some natural number $t$. A $\Lambda$-module $M$ is said to be regular provided it has no indecomposable direct summand which is preprojective or preinjective.

A regular $\Lambda$-module $M$ is said to be elementary provided there is no short exact sequence $0 \rightarrow M' \rightarrow M'' \rightarrow 0$ with $M', M''$ being non-zero regular modules (the definition is due to Crawley-Boevey, for basic results see Kerner and Lukas [L,KL,K]) and the appendix 1. Of course, any regular module has a filtration whose factors are elementary. If $M$ is elementary, then all the modules $\tau^t M$ with $t \in \mathbb{Z}$ are elementary.

The aim of this note is to determine the elementary 3-Kronecker modules. Let $\alpha, \beta, \gamma$ be a basis of $A$. Let $X(\alpha, \beta, \gamma)$ and $Y(\alpha, \beta, \gamma)$ be the $\Lambda$-module defined by the following
Here, we draw a corresponding coefficient quiver and require that all non-zero coefficients are equal to 1. Thus, for example $X(\alpha, \beta, \gamma) = (k^2, k^2; \alpha, \beta, \gamma)$ with $\alpha(a, b) = (a, b)$, $\beta(a, b) = (b, 0)$ and $\gamma(a, b) = (0, a)$ for $a, b \in k$.

**Theorem.** The dimension vectors of the elementary 3-Kronecker modules are the elements in the $\tau$-orbits of $(1, 1), (2, 1), (2, 2)$ and $(4, 2)$.

Any indecomposable representation with dimension vector in the $\tau$-orbit of $(1, 1)$ and $(2, 1)$ is elementary.

An indecomposable representation with dimension vector $(2, 2)$ or $(4, 2)$ is elementary if and only if it is of the form $X(\alpha, \beta, \gamma)$ or $Y(\alpha, \beta, \gamma)$, respectively for some basis $\alpha, \beta, \gamma$ of $A$.

The indecomposable representations with dimension vectors in the $\tau$-orbits of $(1, 1)$ and $(2, 1)$ have been studied in several papers. They are the even index Fibonacci modules, see [FR2,FR3,R4]. If $M$ is indecomposable and $\text{dim } M = (1, 1)$ or $(2, 1)$, then there is a basis $\alpha, \beta, \gamma$ of $A$ such that $M = B(\alpha)$ or $M = V(\beta, \gamma)$, respectively, defined as follows:

![Diagram](https://via.placeholder.com/150)

Note that $B(\alpha)$ is the unique indecomposable 3-Kronecker module of length 2 which is annihilated by $\beta$ and $\gamma$, whereas $V(\beta, \gamma)$ is the unique indecomposable 3-Kronecker module of length 3 with simple socle which is annihilated by $\alpha$.

The indecomposable modules with dimension vector $(1, 1)$ are called bristles in [R3]. The indecomposable representations with dimension vector $(2, 1)$ have been considered in [BR]: there, it has been shown that any arrow $\alpha$ of a quiver gives rise to an Auslander-Reiten sequence with indecomposable middle term say $M(\alpha)$; in this way, we obtain the sequence:

$$0 \to V(\beta, \gamma) \to M(\alpha) \to \tau^{-1}V(\beta, \gamma) \to 0.$$ 

The study of the $\tau$-orbits of the indecomposable 3-Kronecker modules with dimension vectors $(1, 1)$ and $(2, 1)$ in the papers [FR2,FR3,R4,R5] uses the universal covering $\tilde{K}(3)$ of the Kronecker quiver $K(3)$. The quiver $\tilde{K}(3)$ is the 3-regular tree with bipartite orientation. Since the 3-Kronecker modules $B(\alpha)$ and $V(\beta, \gamma)$ are cover-exceptional (they are push-downs of exceptional representations of $\tilde{K}(3)$), it follows that all the modules in the $\tau$-orbits of $B(\alpha)$ and $V(\beta, \gamma)$ are cover-exceptional, and therefore tree modules in the sense of [R2].
In general, one should modify the definition of a tree module as follows: Let $Q$ be any quiver. For any pair of vertices $x, y$ of $Q$, let $A(x, y)$ be the corresponding arrow space, this is the vector space with basis the arrows $x \to y$. If $\alpha(1), \ldots, \alpha(t)$ are the arrows $x \to y$ and $\beta = \sum a_i \alpha(i)$ with all $a_i \in k$ is an element of $A(x, y)$, we may consider for any representation $M = (M_x, M_{\alpha})_{x \in Q, \alpha \in Q_1}$ the linear combination $M_\beta = \sum a_i M_{\alpha(i)}$. Given a basis $B(x, y)$ of the arrow space $A(x, y)$, for all vertices $x, y$ of $Q$ as well as a basis $B(M, x)$ of the vector space $M_x$, for all vertices $x$ of $Q$, we may write the linear maps $b \in B(x, y)$ as matrices with respect to the bases $B(M, x), B(M, y)$. Looking at these matrices, we obtain a coefficient quiver $\Gamma(B(x, y), B(M, x))$ as in [R2]. A representation $M$ of the path algebra $kQ$ should be called a tree module provided $M$ is indecomposable and there are bases $B(x, y)$ of the arrow spaces $A(x, y)$ and $B(M, x)$ of the vector spaces $M_x$ such that $\Gamma(B(x, y), B(M, x))$ is a tree. Of course, in case $Q$ has no multiple arrows, this coincides with the definition given in [R2]. But in general, we now allow base changes in the arrow spaces. Note that such base changes in the arrow spaces do not effect the $kQ$-module $M$, but only its realization as the representation of a quiver. There is the following interesting consequence: Any indecomposable representation of the 2-Kronecker quiver is a tree module, see Appendix 2. Using this modified definition, we see immediately that all the indecomposable modules with dimension vector in the $\tau$-orbits of $(1, 1)$ and $(2, 1)$ are tree modules. On the other hand, the modules $X(\alpha, \beta, \gamma)$ (and also $Y(\alpha, \beta, \gamma)$) are not tree modules, see Lemma 4.2.

Whereas the modules in the $\tau$-orbits of the elementary 3-Kronecker modules with dimension vectors $(1, 1)$ and $(2, 1)$ are quite well understood, a similar study of those in the $\tau$-orbits of modules with dimension vectors $(2, 2)$ and $(4, 2)$ is missing. It seems that any module $M$ in these $\tau$-orbits has a coefficient quiver with a unique cycle. A first structure theorem for these modules is exhibited in section 5.

We say that an element $(x, y) \in K_0(\Lambda) = \mathbb{Z}^2$ is non-negative provided $x, y \geq 0$. The non-negative elements in $K_0(\Lambda)$ are just the possible dimension vectors of $\Lambda$-modules. Note that $K_0(\Lambda)$ is endowed with the quadratic form $q$ defined by $q(x, y) = x^2 + y^2 - 3xy$ (see for example [R1]). A dimension vector $d$ is said to be regular provided $q(d) < 0$. There are precisely two $\tau$-orbits of dimension vectors $d$ with $q(d) = -1$, namely the $\tau$-orbits of $(1, 1)$ and $(2, 1)$. Similarly, there are precisely two $\tau$-orbits of dimension vectors $d$ with $q(d) = -4$, namely the $\tau$-orbits of $(2, 2)$ and $(4, 2)$. The remaining regular dimension vectors $d$ satisfy $q(d) \leq -5$.

**Corollary.** Let $\Lambda = kK(3)$ and $(x, y)$ a dimension vector. There exists an elementary module $M$ with dimension vector $(x, y)$ if and only if $q(x, y)$ is equal to $-1$ or $-4$.

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1. **The BGP-shift $\sigma$.**

Let $\sigma$ denote the BGP-shift of $K_0(\Lambda) = \mathbb{Z}^2$ given by $\sigma(x, y) = (3x - y, x)$, and let $\tau = \sigma^2$. 

3
We denote by $\sigma, \sigma^-$ the BGP-shift functors for mod $\Lambda$ (they correspond to the reflection functors of Bernstein-Gelfand-Ponomarev in [BGP], but take into account that the opposite of the 3-Kronecker quiver is again the 3-Kronecker quiver). If $M = (M_1, M_2; \alpha, \beta, \gamma)$ is a representation of $Q$, we denote by $(\sigma M)_1$ the kernel of the map $[\alpha \beta \gamma]: M_1^3 \rightarrow M_2$ and put $(\sigma M)_2 = M_1$; the maps $\alpha, \beta, \gamma: (\sigma M)_1 \rightarrow (\sigma M)_2$ are given by the corresponding projections. Similarly, $(\sigma^- M)_2$ is the cokernel of the map $[\alpha \beta \gamma]: M_1 \rightarrow M_2^3$ and we put $(\sigma^- M)_1 = M_2$; now the maps $\alpha, \beta, \gamma: (\sigma^- M)_1 \rightarrow (\sigma^- M)_2$ are just the corresponding restrictions. Note that $\sigma^2$ is just the Auslander-Reiten translation $\tau$ (we should stress that this relies on the fact that we deal with a quiver without cyclic walks of odd length, see [G]).

**Remark.** The functors $\sigma$ and $\sigma^-$ depend on the choice of the basis $\alpha, \beta, \gamma$ of $A$, thus we should write $\sigma = \sigma_{\alpha, \beta, \gamma}$ and $\sigma^- = \sigma_{\alpha, \beta, \gamma}$.

If $N$ is an indecomposable representation of $K(3)$ different from $S(2)$, then $\dim \sigma N = \sigma \dim N$; similarly, if $N$ is indecomposable and different from $S(1)$, then $\dim \sigma^\ast N = \sigma \dim N$ (here, $S(1)$ and $S(2)$ are the simple representations of $K(3)$; they are defined by $\dim S(1) = (1, 0), \dim S(2) = (0, 1)$).

An indecomposable $\Lambda$-module $M$ is regular if and only if all the modules $\sigma^n N$ and $\sigma^\ast N$ with $n \in \mathbb{N}$ are nonzero. The restriction of $\sigma$ to the full subcategory of all regular modules is a self-equivalence with inverse $\sigma^-$ and a regular module $M$ is elementary if and only if $\sigma M$ is elementary. We say that an indecomposable representation $M$ of $K(3)$ is of \textit{$\sigma$-type} $(x, y)$ provided $\dim M$ belongs to the $\sigma$-orbit of $(x, y)$.

In terms of $\sigma$, the main result can be formulated as follows:

**Theorem.** The elementary $kK(3)$-modules are of $\sigma$-type $(1, 1)$ and $(2, 2)$. All the indecomposable representations of $\sigma$-type $(1, 1)$ are elementary and tree modules. An indecomposable representation of $\sigma$-type $(2, 2)$ is either elementary or else a tree module.

The tree modules with dimension vector $(2, 2)$ are precisely the representations of the form

\[
\begin{array}{ccc}
\bullet & \rightarrow & \bullet \\
\bullet & \rightarrow & \bullet \\
\alpha & \downarrow & \beta & \downarrow & \gamma
\end{array}
\]

for some basis $\alpha, \beta, \gamma$ of $A$.

**2. Reduction to the dimension vectors $(x, y)$ with $\frac{2}{3}x \leq y \leq x$.**

Let us denote by $R$ the set of regular dimension vectors. As we have mentioned, $\sigma$ maps $R$ onto $R$. There is the additional transformation $\delta$ on $K_0(\Lambda)$ defined by $\delta(x, y) = (y, x)$. Of course, it also sends $R$ onto $R$. If $M$ is a representation of $Q(3)$, then $\delta(\dim M) = \dim M^\ast$, where $M^\ast$ is the dual representation of $M$ (defined in the obvious way: $(M^\ast)_1$ is the $k$-dual of $M_2$, $(M^\ast)_2$ is the $k$-dual of $M_1$, the map $(M^\ast)_\alpha$ is the $k$-dual of $M_\alpha$, and so on).
Lemma. The subset

$$
F = \{(x, y) \mid \frac{2}{3}x \leq y \leq x\}
$$

is a fundamental domain for the action of $\sigma$ and $\delta$ on $\mathbb{R}$.

The proof is easy. Let us just mention that $\sigma(3, 2) = (2, 3)$ and that for $(x, y) \in \mathbb{R}$ with $\sigma(x, y) = (x', y')$, we have $\frac{y}{x} > \frac{y'}{x'}$ (this condition explains why we call $\sigma$ a shift). □.

It follows that for looking at an elementary module, we may use the shift $\sigma$ and duality in order to obtain an elementary module $M$ with $\dim M \in F$. Here is the set $F$:

![Diagram of F set](image)

In the next section 3, we first will consider the pairs $(x, y) \in F$ with $y \geq 4$, they are marked by a circle $\circ$. Then we deal with the three special pairs $(3, 2), (3, 3)$ and $(4, 3)$ marked by a star $\star$ (actually, instead of $(4, 3)$ and $(3, 2)$, we will look at $(3, 4)$ and $(2, 3)$, respectively). As we will see in section 3, all these pairs cannot occur as dimension vectors of elementary modules.

As a consequence, the only possible dimension vectors in $F$ which can occur as dimension vectors of elementary modules are $(1, 1)$ and $(2, 2)$; they are marked by a bullet $\bullet$ and will be studied in section 4.

3. Dimension vectors without elementary modules.

Lemma 3.1. Assume that $M$ is a regular module with a proper non-zero submodule $U$ such that both dimension vectors $\dim U$ and $\dim M/U$ are regular. Then $M$ is not elementary.

Proof. This is a direct consequence of the fact that $M$ is elementary if and only if for any submodule $U$ the submodule $U$ is preprojective or the factor module $M/U$ is preinjective, see the Appendix 1.

Lemma 3.2. A 3-Kronecker module $M$ with $\dim M = (x, y)$ such that $2 \leq y \leq x + 1$ has a submodule $U$ with dimension vector $(1, 2)$.

Proof. Let us show that there are non-zero elements $m \in M_1$ and $\alpha \in A$ such that $\alpha m = 0$. The multiplication map $A \otimes_k M_1 \to M_2$ is a linear map, let $W$ be its kernel. Since $\dim A = 3$, we see that $\dim A \otimes_k M_1 = 3x$. Since $\dim M_2 = y$, it follows that $\dim W \geq 3x - y$. The projective space $\mathbb{P}(A \otimes M_1)$ has dimension $3x - 1$, the decomposable tensors
in $A \otimes M_1$ form a closed subvariety $\mathcal{V}$ of $\mathbb{P}(A \otimes M_1)$ of dimension $(3 - 1) + (x - 1) = x + 1$. Since $\mathcal{W} = \mathbb{P}(V)$ is a closed subspace of $\mathbb{P}(A \otimes M_1)$ of dimension $3x - y - 1$, it follows that

$$\dim(\mathcal{V} \cap \mathcal{W}) \geq (x + 1) + (3x - y - 1) - (3x - 1) = x - y + 1.$$ 

By assumption, $x - y + 1 \geq 0$, thus $\mathcal{V} \cap \mathcal{W}$ is non-empty. As a consequence, we get non-zero elements $m \in V, \alpha \in A$ such that $\alpha m = 0$, as required.

Given non-zero elements $m \in M_1$ and $\alpha \in A$ such that $\alpha m = 0$, the element $m$ generates a submodule $U'$ which is annihilated by $\alpha$, thus $\dim U' = (1, u)$ with $0 \leq u \leq 2$. Since $y \geq 2$, there is a semi-simple submodule $U''$ of $M$ with dimension vector $(0, 2 - u)$ such that $U' \cap U'' = 0$. Let $U = U' \oplus U''$. This is a submodule of $M$ with dimension vector $\dim U = \dim U' \oplus U'' = (1, 2)$.

Remark. Under the stronger assumption $2 \leq y < x$, we can argue as follows: We have $\langle (1, 2), (x, y) \rangle = x + 2y - 3y = x - y > 0$, where $\langle - , - \rangle$ is the canonical bilinear form on $K_0(\Lambda)$ (see [R1]), thus $\operatorname{Hom}(N.M) \neq 0$ for any module $N$ with $\dim N = (1, 2)$ The image of any non-zero map $f: N \to M$ has dimension vector $(1, u)$ with $0 \leq u \leq 2$.

\textbf{Lemma 3.3.} If $(x, y) \in F$ and $y \geq 4$, then $(x - 1, y - 2)$ is a regular dimension vector.

Proof. Since $y \leq x$, we have $y - 2 \leq x - 1$. On the other hand, the inequalities $y \geq 4$ and $y \geq \frac{2}{3}x$ imply the inequality $y - 2 \geq \frac{2}{5}(x - 1)$. Thus $\frac{2}{5}(x - 1) \leq y - 2 \leq x - 1$. As a consequence, $(x - 1, y - 2)$ is a regular dimension vector.

We are now able to provide a proof for the first assertion of the Theorem: \textit{The elementary $kK(3)$-modules are of $\sigma$-type $(1, 1)$ and $(2, 2)$.}

Proof. Let $M$ be elementary with dimension vector $\dim M = (x, y) \in F$. First, assume that $y \geq 4$. According to Lemma 3.2, there is a submodule $U$ with the regular dimension vector $\dim U = (1, 2)$. The factor module $M/U$ has dimension vector $(x - 1, y - 2)$ and $(x - 1, y - 1)$. According to Lemma 3.3, also $(x - 1, y - 2)$ is a regular dimension vector. Using Lemma 3.1, we obtain a contradiction.

It remains to show that the dimension vectors $(3, 2), (3, 3), (4, 3)$ cannot occur. Using duality, we may instead deal with the dimension vectors $(2, 3), (3, 3), (3, 4)$. Thus, assume there is given an elementary module $N$ with dimension vector $(2, 3), (3, 3)$ or $(3, 4)$. According to Lemma 3.2, it has a submodule $U$ with dimension vector $(1, 2)$. The corresponding factor module $M/U$ has dimension vector $(1, 1), (2, 1), (2, 2)$, respectively. But all these dimension vectors are regular. Again Lemma 3.1 provides a contradiction. \qed

4. The indecomposable modules with dimension vector $(1, 1)$ and $(2, 2)$.

\textbf{Dimension vector $(1, 1)$.} \textit{Any indecomposable $\Lambda$-module $M$ with dimension vector $(1, 1)$ is of the form}

\[
\begin{array}{c}
\bullet \\
\alpha \\
\bullet
\end{array}
\]
for some basis \( \alpha, \beta, \gamma \) of \( A \), thus a tree module. Namely, \( M = P(1)/U \), where \( P(1) \) is the indecomposable projective module corresponding to the vertex 1 and \( U \) is a two-dimensional submodule of \( P(1) \). Actually, we may consider \( U \) as a two-dimensional subspace of \( P(1) \). Let \( \alpha, \beta, \gamma \) be a basis of \( A \) such that \( U = \langle \beta, \gamma \rangle \).

Of course, any indecomposable \( \Lambda \)-module with dimension vector \( (1, 1) \) is elementary.

The indecomposable \( \Lambda \)-modules with dimension vector \( (2, 2) \).

**Lemma 4.1.** An indecomposable module with dimension vector \( (2, 2) \) is elementary if and only if it is of the form \( X(\alpha, \beta, \gamma) \).

Proof. First we show: The modules \( M = X(\alpha, \beta, \gamma) \) are elementary. We have to verify that any non-zero element of \( M_1 \) generates a 3-dimensional submodule. We see this directly for the elements \( (1, 0) \) and \( (0, 1) \) of \( M_1 = k^2 \). If \( (a, b) \) with \( a \neq 0, b \neq 0 \), then \( \beta(a, b) = (b, 0) \) and \( \gamma(a, b) = (0, a) \) are linearly independent elements of \( M_2 = k^2 \). This completes the proof.

Conversely, let \( M \) be an elementary module with dimension vector \( (2, 2) \). Let us show that the restriction of \( M \) to any 2-Kronecker subalgebra has 2-dimensional endomorphism ring. Let \( \alpha, \beta, \gamma \) be a basis of the arrow space and consider the restriction \( M' \) of \( M \) to the subquiver \( K(2) \) with basis \( \beta, \gamma \). If \( M' \) has a simple injective direct summand, then either \( M' \) is annihilated by \( \alpha \), then \( M' \) is a simple injective submodule of \( M \), therefore \( M \) is not indecomposable, impossible. If \( M' \) is not annihilated by \( \alpha \), then \( M' + \alpha(M') \) is an indecomposable submodule of dimension 2, thus \( M \) is not elementary. Dually, \( M' \) has no simple projective direct summand. It remains to exclude the case that \( M' = R \oplus R \) for some simple regular representation \( R \) of \( K(2) \). Without loss of generality, we can assume that \( M' \) is annihilated by \( \gamma \). Since \( M \) is annihilated by \( \gamma \), it is just a regular representation of the 2-Kronecker quiver with arrow basis \( \alpha \) and \( \beta \). But any 4-dimensional regular representation of a 2-Kronecker quiver has a 2-dimensional regular submodule. This shows that \( M \) is not elementary. Altogether we have shown that the restriction of \( M \) to any 2-Kronecker subalgebra has 2-dimensional endomorphism ring.

If \( u \) is a non-zero element of \( M_1 \), then \( \Lambda u \) contains \( M_2 \) and \( \dim \Lambda u = 3 \). Namely, if \( \Lambda u \) is of dimension 1, then \( \Lambda u \) is simple injective, thus \( M \) cannot be indecomposable. If \( \Lambda u \) is of dimension 2, then \( \Lambda u \) is the arrow space and \( \Lambda u \) is non-elementary. It follows that \( \dim \Lambda u = 3 \) and that \( M_2 \subset \Lambda u \). Given any non-zero element \( u \in M_1 \), there is a non-zero element which annihilates \( u \), say \( 0 \neq \beta \in A \). No element in \( A \setminus \langle \beta \rangle \) annihilates \( u \), since otherwise the dimension of \( \Lambda u \) is at most 2. Let \( u, v \) be a basis of \( M_1 \). Let \( \beta, \gamma \) be non-zero elements of \( A \) with \( \beta(u) = 0, \gamma(v) = 0 \). Then the elements \( \beta, \gamma \) are linearly independent, since otherwise we would have \( \gamma(u) = 0 \), thus the submodule \( \Lambda u \) would be of dimension at most 2. The elements \( \beta(v), \gamma(u) \) must be linearly independent, since otherwise the restriction of \( M \) to \( \beta, \gamma \) would be the direct sum of a simple projective and an indecomposable injective. We take \( \beta(v), \gamma(u) \) as an ordered basis of \( M_2 = k^2 \), so that \( \beta(v) = (1, 0) \) and \( \gamma(u) = (0, 1) \). Choose an element \( \alpha \in A \setminus \langle \beta, \gamma \rangle \), thus \( \alpha, \beta, \gamma \) is a basis of \( A \). Let \( \alpha(u) = (\kappa, \lambda) \) and \( \alpha(v) = (\mu, \nu) \) with \( \kappa, \lambda, \mu, \nu \) in \( k \). Since \( \alpha(u) \) cannot be a multiple of \( \gamma(u) = (0, 1) \), we see that \( \kappa \neq 0 \). Since \( \alpha(v) \) cannot be a multiple of
\[ \beta(v) = (1, 0), \] we see that \( \nu \neq 0. \) Let \( \alpha' = \alpha - \mu \beta - \lambda \gamma. \) Then

\[
\begin{align*}
\alpha'(u) &= \alpha(u) - \mu \beta(u) - \lambda \gamma(u) = (\kappa, \lambda) - (0, 0) - \lambda(0, 1) = (\kappa, 0), \\
\alpha'(v) &= \alpha(v) - \mu \beta(v) - \lambda \gamma(v) = (\mu, \nu) - \mu(1, 0) - (0, 0) = (0, \nu).
\end{align*}
\]

Let \( \beta' = \kappa \beta \) and \( \gamma' = \nu \gamma. \) Then we have

\[
\begin{align*}
\beta'(u) &= (0, 0), & \beta'(v) &= (\kappa, 0), \\
\gamma'(u) &= (0, \nu), & \gamma'(v) &= (0, 0),
\end{align*}
\]

Altogether, we see that

\[
\begin{array}{ccc}
u & \gamma' & \beta' \\
\alpha' & \downarrow & \downarrow \\
(k, 0) & \alpha' & (0, \nu)
\end{array}
\]

Since both elements \( \kappa \) and \( \nu \) are non-zero, the elements \( (\kappa, 0) \) and \( (0, \nu) \) form a basis of \( k^2, \) and \( \alpha', \beta', \gamma' \) form a basis of \( A. \) Thus \( M \) is isomorphic to \( X(\alpha', \beta', \gamma'). \) This completes the proof.

**Lemma 4.2.** A tree module with dimension vector \((2, 2)\) cannot be elementary.

Proof. If \( M \) is a tree module with dimension vector \((2, 2)\), then the coefficient quiver has to be of the form

\[
\begin{array}{ccc}
\bullet & \bullet \\
\downarrow & \downarrow & \downarrow
\end{array}
\]

But then \( M \) has a submodule \( U \) such that both \( U \) and \( M/U \) have dimension vector \((1, 1)\).

**Lemma 4.3.** If \( M \) is indecomposable with dimension vector \((2, 2)\) and not elementary, then \( M \) is of one of the following forms

\[
\begin{array}{ccc}
\alpha & \beta & \gamma \\
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow
\end{array} \quad \begin{array}{ccc}
\alpha & \beta & \alpha \\
\bullet & \bullet & \bullet \\
\downarrow & \downarrow & \downarrow
\end{array}
\]

for some basis \( \alpha, \beta, \gamma \) of \( A. \)

Proof. Let \( M \) be indecomposable with dimension vector \((2, 2)\). If \( M \) is not faithful, say annihilated by \( 0 \neq \gamma \in A, \) then \( M \) is a \( K(2) \)-module and therefore as shown on the right.

Now assume that \( M \) is faithful, and not elementary. Since \( M \) is not elementary, there is an element \( 0 \neq u \in M_1 \) such that \( \Lambda u \) has dimension vector \((1, 1)\). The annihilator \( B \) of \( u \) is a 2-dimensional subspace of \( A. \) Let \( v \in M_1 \setminus \langle u \rangle. \) Since \( M \) is indecomposable, we see that \( \Lambda v \) has to be 3-dimensional and there is a non-zero element \( \alpha \in A \) with \( \alpha v = 0. \) Since \( M \) is faithful, \( \alpha(u) \neq 0. \) Also, since \( M \) is faithful, we have \( B v = M_2. \) Thus, there is \( \beta \in B \) with \( \beta(v) = \alpha(u). \) Let \( \gamma \in B \setminus \langle \beta \rangle. \) Then \( \alpha(u), \beta(\gamma) \) is a basis of \( M_2. \) With respect to the
basis $\alpha, \beta, \gamma$ of $A$, the basis $u, v$ of $M_1$ and the basis $\alpha(u), \beta(\gamma)$ of $M_2$, the module $M$ has the form as depicted on the left. \hfill \Box.

5. The structure of the modules $\sigma^t X(\alpha, \beta, \gamma)$.

The 3-Kroncker modules $I_i = \sigma^i S(2)$ are the preinjective modules, see [FR1].

**Proposition.** For $t \geq 1$, there is an exact sequence

$$0 \to X(\alpha, \beta, \gamma) \to \sigma^t X(\alpha, \beta, \gamma) \to \bigoplus_{0 \leq i < t} I_i^2 \to 0.$$ 

Proof. First we consider the case $t = 1$. There is an obvious embedding of $X(\alpha, \beta, \gamma)$ into $Y(\alpha, \beta, \gamma) = \sigma X(\alpha, \beta, \gamma)$, thus there is an exact sequence of the form

$$0 \to X(\alpha, \beta, \gamma) \to Y(\alpha, \beta, \gamma) \to S(2) \to 0.$$ 

Now we use induction. We start with the sequence

$$0 \to X(\alpha, \beta, \gamma) \to \sigma^t X(\alpha, \beta, \gamma) \to \bigoplus_{0 \leq i < t} I_i^2 \to 0,$$

for some $t \geq 1$ and apply $\sigma$. In this way, we obtain the sequence

$$0 \to \sigma X(\alpha, \beta, \gamma) \to \sigma^{t+1} X(\alpha, \beta, \gamma) \to \bigoplus_{1 \leq i \leq t} I_i^2 \to 0.$$ 

This shows that $M = \sigma^{t+1} X(\alpha, \beta, \gamma)$ has a submodule $U$ isomorphic to $\sigma X(\alpha, \beta, \gamma)$, with $M/U$ isomorphic to $\bigoplus_{1 \leq i \leq t} I_i^2$. But the case $t = 1$ shows that $U$ has a submodule $U'$ isomorphic to $X(\alpha, \beta, \gamma)$ with $U/U'$ isomorphic to $S(2) = I_0^2$. The embedding of $U/U'$ into $M/U'$ has to split, since $I_0$ is injective. This completes the proof.

Appendix 1. Elementary modules.

According to [K], Proposition 4.4, a regular representation $M$ is elementary if and only if for any nonzero regular submodule $U$ of $M$, the factor module $M/U$ is preinjective. Let us include the proof of a slight improvement of this criterion.

We deal with the general setting where $\Lambda$ is a hereditary artin algebra.

**Proposition.** Let $M$ be non-zero regular module $M$. Then $M$ is elementary if and only if given any submodule $U$ of $M$, the submodule $U$ is preprojective or the factor module $M/U$ is preinjective.

Proof. Let $M$ be non-zero and regular. First, assume that for any submodule $U$ of $M$, the submodule $U$ is preprojective or the factor module $M/U$ is preinjective. Then $M$ cannot be a proper extension of regular modules, thus $M$ is elementary.
Conversely, let $M$ be elementary. Let $U$ be a submodule which is not preprojective. Since $M$ has no non-zero preinjective submodules, we can write $U = U_1 \oplus U_2$ with $U_1$ preprojective and $U_2$ regular. Since $U$ is not preprojective, we know that $U_2$ is non-zero. Since $M$ has no non-zero preprojective factor modules, we decompose $M/U_2$ as a direct sum of a regular and a preinjective module: there are submodules $V_1, V_2$ of $M$ with $V_1 \cap V_2 = U, V_1 + V_2 = M$ (thus $M/U = V_1/U_2 \oplus V_2/U_2$) such that $V_1/U_2$ is regular, and $V_2/U_2$ is preinjective.

Consider $V_2$. First of all, $V_2 \neq 0$, since $U_2$ is a non-zero submodule of $V_2$. Second, we claim that $V_2$ is regular. Namely, $V_2$ is an extension of the regular module $U_2$ by the preinjective module $V_2/U_2$, thus is has no non-zero preprojective factor module. Thus, we can decompose $V_2 = W_1 \oplus W_2$ with $W_1$ regular, $W_2$ preinjective. But $W_2$ is a preinjective submodule of $M$, therefore $W_2 = 0$. This shows that $V_2 = W_1$ is regular.

On the other hand, $W/V_2$ is isomorphic to $V_1/U_2$, thus regular. But since $M$ is not a proper extension of regular modules, it follows that $W/V_2 = 0$, thus $V_2 = M$. Therefore $M/U_2 = V_2/U_2$ is preinjective. But $M/U = M/(U_1 + U_2)$ is a factor module of $M/U_2$, and a factor module of a preinjective module is preinjective. This shows that $M/U$ is preinjective.

\[ \square \]

The definition of an elementary module implies that any regular module has a filtration by elementary modules. But such filtrations are not at all unique. This is well-known, but we would like to mention that the 3-Kronecker modules provide examples which are easy to remember. Here is the first such example $M$:

\[ \begin{array}{c}
\circ \\
\gamma \\ \\
\beta \\ \\
\alpha \\ \\
\circ \\
\end{array} \]

On the right we see that $X(\alpha, \beta, \gamma)$ is a factor module, and the corresponding kernel is $B(\gamma)$ (it is generated by the first base vector of $M_1$). On the other hand, on the left we see that $V(\beta, \gamma)$ is a factor module, and the corresponding kernel has dimension vector $(1, 2)$ (it is generated by the last base vector of $M_1$).

Here is the second example $N$:

\[ \begin{array}{c}
\circ \\
\beta \\ \\
\alpha \\ \\
\circ \\
\circ \\
\end{array} \]

On the right we see again that $X(\alpha, \beta, \gamma)$ is a submodule, and the corresponding factor module is $V(\alpha, \beta)$ (generated by the first two base vectors of $N_1$). On the other hand, going from left to right, we see that the module has a filtration whose lowest two factors are of the form $B(\beta)$, whereas the upper factor is $V(\alpha, \gamma)$ (generated by the last two base vectors of $N_1$).
Appendix 2: The representations of the 2-Kronecker quiver.

**Proposition.** Any indecomposable $K(2)$-module is a tree module (with respect to some basis of the arrow space of $K(2)$), and its coefficient quiver is of type $A$.

Proof. The preprojective and the preinjective modules are exceptional modules, thus they are tree modules with respect to any basis. The remaining indecomposable representations of $K(2)$ are of the form $R[t]$ where $R$ is simple regular, and $R[t]$ denotes the indecomposable regular module of dimension $2t$ with regular socle $R$. We may choose a basis of the arrow space such that $R$ is isomorphic to $(k, k; 1, 0)$. Then $R[t]$ is a tree module such that the underlying graph of the coefficient quiver is of type $A_{2t}$.

References

[BGP] I. N. Bernstein, I. M. Gelfand, V. A. Ponomarev: Coxeter functors, and Gabriel’s theorem. Russian mathematical surveys 28 (2) (1973), 17–32.

[FR1] Ph. Fahr, C. M. Ringel: A partition formula for Fibonacci numbers. Journal of Integer Sequences, Vol. 11 (2008)

[FR2] Ph. Fahr, C. M. Ringel: Categorification of the Fibonacci Numbers Using Representations of Quivers. Journal of Integer Sequences. Vol. 15 (2012), Article12.2.1

[FR3] Ph. Fahr, C. M. Ringel: The Fibonacci partition triangles. Advances in Mathematics 230 (2012)

[G] P. Gabriel: Auslander-Reiten sequences and representation-finite algebras. Spinger LNM 831 (1980), 1–71.

[K] O. Kerner: Representations of wild quivers. CMS Conf. Proc., vol. 19. Amer. Math. Soc., Providence, RI (1996), 65--107

[KL] O. Kerner, F. Lukas: Elementary Modules, Math.Z. 223(1996),21–434.

[L] F. Lukas: Elementare Moduln über wilden erblichen Algebren. Dissertation, Düsseldorf (1992).

[R1] C. M. Ringel: Representations of $K$-species and bimodules. Representations of $K$-species and bimodules. J. Algebra 41 (1976), 269–302.

[R2] C. M. Ringel: Exceptional modules are tree modules. Lin. Alg. Appl. 5-276 (1998),471–493.

[R3] C. M. Ringel: Distinguished bases of exceptional modules In Algebras, quivers and representations. Proceedings of the Abel symposium 2011. Springer Series Abel Symposia Vol 8. (2013) 253-274.

[R4] C. M. Ringel: Kronecker modules generated by modules of length 2. arXiv:1612.07679.

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