$L^p$ ESTIMATES FOR THE BERGMAN PROJECTION ON SOME REINHARDT DOMAINS

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Abstract. We obtain $L^p$ regularity for the Bergman projection on some Reinhardt domains. We start with a bounded initial domain $\Omega$ with some symmetry properties and generate successor domains in higher dimensions. We prove: If the Bergman kernel on $\Omega$ satisfies appropriate estimates, then the Bergman projection on the successor is $L^p$ bounded. For example, the Bergman projection on successors of strictly pseudoconvex initial domains is bounded on $L^p$ for $1 < p < \infty$. The successor domains need not have smooth boundary nor be strictly pseudoconvex.

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1. Introduction

The purpose of this paper is to establish $L^p$ regularity for the Bergman projection on certain domains. In [Huo17], the author began with an initial domain with certain symmetry properties. From this initial domain the author constructed various successor domains and computed (explicitly) the Bergman kernel on them in terms of the Bergman kernel on the initial domain.

Let $\Omega$ be an initial domain in $\mathbb{C}^n$. We consider two kinds of estimates on the Bergman kernel $K_\Omega$. A first estimate implies $L^p$ regularity of the Bergman projection on $\Omega$. If, also, a second estimate holds, then we obtain $L^p$ regularity of the Bergman projection on the successor domain. See Theorem 1.2. We use a variant of Schur’s Lemma to establish $L^p$ regularity. We state the crucial estimates in Theorem 3.3 and give the proof in Section 4.

Let $\Omega \subseteq \mathbb{C}^n$ be a bounded domain. The Bergman projection is the orthogonal projection from $L^2(\Omega)$ onto the closed subspace of square-integrable holomorphic functions, and thus is bounded on $L^2$. It is natural to ask when this operator is bounded on $L^p$ for $p \neq 2$. Using known estimates for the Bergman kernel, various authors have obtained $L^p$ regularity results for $1 < p < \infty$ in the following settings:

1. $\Omega$ is bounded, smooth, and strongly pseudoconvex. See [Pef74, PS77].
2. $\Omega \subseteq \mathbb{C}^2$ is a domain of finite type. See [McN89, McN94a, NRSW88].
3. $\Omega \subseteq \mathbb{C}^n$ is a convex domain of finite type. See [McN94a, McN94b, MS94].
4. $\Omega \subseteq \mathbb{C}^n$ is a domain of finite type with locally diagonalizable Levi form. See [CD06].

Progress has also been made on some domains with weaker assumption on boundary regularity. In some cases, the Bergman projection is $L^p$ bounded for $1 < p < \infty$, See [EL08 LS12]. For other domains, the projection has only a finite range of mapping regularity. See [Zey13, CZ16, EM16, EM17, Che17]. There are also smooth bounded domains where the projection has limited $L^p$ range. See [BS12].

We start with a bounded complete Reinhardt domain $\Omega$ in $\mathbb{C}^n$ with a defining function $\rho$, and analyze the $L^p$ regularity of the Bergman projection on the successor domains $U^\alpha(\Omega)$.
defined by

\[ U^\alpha(\Omega) = \left\{ (z, w) \in \mathbb{C}^n \times \mathbb{B}^k : \left( \frac{z_1}{(1 - \|w\|^2)^{\alpha_1}}, \ldots, \frac{z_n}{(1 - \|w\|^2)^{\alpha_n}} \right) \in \Omega \right\}. \tag{1.1} \]

Here \( \mathbb{B}^k \) is the unit ball in \( \mathbb{C}^k \) and \( \alpha = (\alpha_1, \ldots, \alpha_n) \) with each \( \alpha_j \) greater than 0. We will often use \( U^\alpha \) to denote \( U^\alpha(\Omega) \).

For each multi-index \( \beta \), let \( D_\beta \) denote the differential operator \( \left( \frac{\partial}{\partial z_1} \right)^{\beta_1} \cdots \left( \frac{\partial}{\partial z_n} \right)^{\beta_n} \). Given functions of several variables \( f \) and \( g \), we use \( f \lesssim g \) to denote that \( f \leq Cg \) for a constant \( C \). If \( f \lesssim g \) and \( g \lesssim f \), then we say \( f \) is comparable to \( g \) and write \( f \simeq g \).

Next we introduce the estimates needed for the derivatives of the Bergman kernel on \( \Omega \).

**Definition 1.1.** Let \( \Omega \) be a domain in \( \mathbb{C}^n \). Let \( h \) be a positive function on \( \Omega \). A kernel \( K \) on \( \Omega \times \Omega \) is \( h \)-regular of type \( l \) if there exists \( a > 0 \) such that for all \( \varepsilon \in (0, a) \), we have

\[ \int_{\Omega} |K(z; \zeta)| h^{-\varepsilon}(\zeta) dV(\zeta) \lesssim h^{-\varepsilon - l}(z). \tag{1.2} \]

Now we are ready to state our main theorem:

**Theorem 1.2.** Let \( \rho \) be a defining function for \( \Omega \subseteq \mathbb{C}^n \) and let \( U^\alpha \subseteq \mathbb{C}^{n+k} \) be as in (1.1). Suppose the Bergman kernel \( K_\Omega \) satisfies the following two properties:

1. \( K_\Omega \) is \((-\rho)\)-regular of type 0.
2. \( D^\beta K_\Omega(z; \zeta) \) is \((-\rho)\)-regular of type \( |\beta| \) whenever \( |\beta| \leq k \).

Then the Bergman projection is bounded on \( L^p(U^\alpha) \) for \( p \in (1, \infty) \).

We note that Assumption (1) implies that the Bergman projection on \( \Omega \) is bounded in \( L^p \) for \( 1 < p < \infty \). See Schur’s lemma in Section 3. Using estimates for derivatives of the Bergman kernel from \([\text{McN94b}], [\text{McN89}], [\text{NRSW88}], [\text{PS77}], [\text{CD06}]\), one can show that \( D^\beta K_\Omega \) is \((-\rho)\)-regular of type \( |\beta| \) for all \( \beta \in \mathbb{N}^n \) in classes of domains previously mentioned. In Theorem 1.2, we only require \( D^\beta K_\Omega \) to be \((-\rho)\)-regular of type \( |\beta| \) for all \( \beta \) such that \( |\beta| \leq k \).

In Section 2, we recall the technique in \([\text{Huo17}]\) relating the Bergman kernels of initial domains to those of their successors. In Section 3, we discuss several lemmas and state Theorem 3.3. This result is used to prove Theorem 1.2 via Schur’s lemma. We prove Theorem 3.3 in Section 4.

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2. A formula for computing the Bergman kernel

In this section we recall a construction from \([\text{Huo17}]\), which produces the Bergman kernel of various higher dimensional successors of an initial domain. We start with an initial domain \( \Omega \) and construct a class of domains \( U^\alpha(\Omega) \) by introducing new parameters \( \alpha \) to \( \Omega \).

The technique in \([\text{Huo17}]\) consists of the following 4 steps:

1. start with the kernel function \( K_\Omega \) on the initial domain.
2. construct a function on \( U^\alpha(\Omega) \times U^\alpha(\Omega) \) by evaluating \( K_\Omega \) at a point off the diagonal.
3. define a specific differential operator (depending on \( \alpha \)).
4. apply the operator in Step (3) to the function in Step (2), obtaining \( K_{U^\alpha(\Omega)} \).

The point at which we evaluate in Step (2) and the operator in Step (3) are independent of the initial domain \( \Omega \), but they depend on the parameters \( \alpha \).
We recall in the definition below the notion of “complete Reinhardt” for the symmetry property the initial domain must satisfy.

**Definition 2.1.** A domain $\Omega \subseteq \mathbb{C}^n$ is called complete Reinhardt in $(z_1, \ldots, z_n)$ if the containment $(z_1, \ldots, z_n) \subseteq \Omega$ implies the containment $$\{(\lambda_1 z_1, \ldots, \lambda_n z_n) : |\lambda_j| \leq 1 \text{ for } 1 \leq j \leq n\} \subseteq \Omega.$$ 

Let $\Omega \subseteq \mathbb{C}^n$ be a complete Reinhardt domain in $(z_1, \ldots, z_n)$. For $\alpha \in \mathbb{R}_+^n$ and $w \in \mathbb{B}$, set $$f_\alpha(z, w) = \left(\frac{z_1}{1-\|w\|^2}^{\alpha_1}, \ldots, \frac{z_n}{1-\|w\|^2}^{\alpha_n}\right).$$ (2.1) 

The successor $U^\alpha(\Omega)$ is defined by $$U^\alpha(\Omega) = \{(z, w) \in \mathbb{C}^n \times \mathbb{B} : f_\alpha(z, w) \in \Omega, \|w\| < 1\}. \tag{2.2}$$ 

For fixed $w \in \mathbb{B}$, let $U^\alpha_w(\Omega)$ denote the slice domain \{$z \in \mathbb{C}^n : (z, w) \in U^\alpha$\} of $U^\alpha$. We will often write $U^\alpha_w$ to denote $U^\alpha_w(\Omega)$. Since the mapping $f_\alpha(\cdot, w) : z \mapsto f_\alpha(z, w)$ is a biholomorphism from $U^\alpha_w(\Omega)$ onto $\Omega$, the kernel on $U^\alpha_w(\Omega)$ can be obtained from $K_\Omega$. 

The main result in [Huo17] relates the Bergman kernel on $U^\alpha_w(\Omega)$ to $K_\Omega$. To state this result, we need a few more notational definitions. Let $I$ denote the identity operator. We define $D_{U^\alpha}$ to be the differential operator: $$D_{U^\alpha} = \frac{(1-\|\eta\|^2)^{\alpha_1}}{\pi^k(1-\langle w, \eta \rangle)^{1+k+\alpha_1}} \prod_{l=1}^k \left\{ I + \sum_{j=1}^n \alpha_j \left( I + z_j \frac{\partial}{\partial z_j} \right) \right\}. \tag{2.3}$$ 

Let $h(z, w, \eta)$ denote the following: $$h(z, w, \eta) = \left( z_1 \left( \frac{1-\|\eta\|^2}{1-\langle w, \eta \rangle} \right)^{\alpha_1}, \ldots, z_n \left( \frac{1-\|\eta\|^2}{1-\langle w, \eta \rangle} \right)^{\alpha_n} \right). \tag{2.4}$$ 

The formula for $K_{U^\alpha}$ in [Huo17] can be expressed as follows: 

**Theorem 2.2.** For $(z, w; \zeta, \eta) \in U^\alpha \times U^\alpha$, let $D_{U^\alpha}$ and $h(z, w, \eta)$ be as (2.3) and (2.4). Then $$K_{U^\alpha}(z, w; \zeta, \eta) = D_{U^\alpha} K_{U^\alpha}(h(z, w, \eta); \zeta). \tag{2.5}$$ 

3. **Lemmas and Theorem 3.3**

The proof of Theorem 1.2 uses the following variant of Schur’s lemma. See [EM16] for its proof.

**Lemma 3.1 (Schur’s Lemma).** Let $\Omega$ be a domain in $\mathbb{C}^n$ and let $K$ be a non-negative measurable function on $\Omega \times \Omega$. Let $K$ be the integral operator with kernel $K$. Suppose there exists a positive auxiliary function $h$ on $\Omega$, and a number $a > 0$ such that for all $\epsilon \in (0, a)$, the following two inequalities hold:

1. $K(h^{-\epsilon})(z) = \int_{\Omega} K(z, \zeta) h(\zeta)^{-\epsilon} dV(\zeta) \lesssim h^{-\epsilon}(z),$
2. $K(h^{-\epsilon})(\zeta) = \int_{\Omega} K(z, \zeta) h(z)^{-\epsilon} dV(z) \lesssim h^{-\epsilon}(\zeta).$

Then $K$ is a bounded operator on $L^p(\Omega)$, for all $p \in (1, \infty)$. 
We will take the function $K(z, \zeta)$ from Lemma 3.1 to be the absolute Bergman kernel $|K_\Omega(z; \zeta)|$. Inequalities (1) and (2) in the lemma are equivalent since $K_\Omega(z; \zeta) = K_\Omega(\zeta, z)$. The $L^p$ boundedness of the corresponding operator $K$ then implies the $L^p$ boundedness of the Bergman projection. To show that the Bergman projection on $\Omega$ is $L^p$ bounded for $p \in (1, \infty)$, it suffices to find an auxiliary function $h$ as in Lemma 3.1 and show that $K_\Omega$ is $h$-regular of type 0. In many cases, one can choose $h$ to be the distance function to the boundary.

From now on we let $\Omega$ be a smooth bounded complete Reinhardt domain in $\mathbb{C}^n$. On such a domain $\Omega$, a defining function with several useful symmetry properties can be chosen.

**Lemma 3.2.** Let $\Omega \subseteq \mathbb{C}^n$ be a smooth complete Reinhardt domain. Then there exists a defining function $\rho$ of $\Omega$ satisfying the following properties:

(a) $\rho$ is smooth in a neighborhood of the boundary $b\Omega$.

(b) If $|z_j| = |\zeta_j|$ for $1 \leq j \leq n$, then $\rho(z) = \rho(\zeta)$.

(c) If $|z_j| \leq |\zeta_j|$ for $1 \leq j \leq n$, then $\rho(z) \leq \rho(\zeta)$.

(d) For $1 \leq j \leq n$, $z_j\rho_{z_j}(z) \geq 0$.

(e) If $z \in b\Omega$, then $\sum_{j=1}^n z_j\rho_{z_j}(z) > 0$.

**Proof.** Set $\rho$ to be the function defined by the distance between $z$ and $b\Omega$:

$$
\rho(z) = \begin{cases} 
-\text{dist}(z, b\Omega) & z \in \Omega \\
\text{dist}(z, b\Omega) & z \notin \Omega.
\end{cases}
$$

Then property (a) is true for any domain $\Omega$ with smooth boundary. Properties (b) and (c) also hold since $\Omega$ is complete Reinhardt. Consider polar coordinates $z_j = t_je^{i\theta_j}$ for $1 \leq j \leq n$. Since $\rho$ is invariant under the rotation in each coordinates, we have:

$$
0 = \frac{\partial}{\partial \theta_j} \rho(t_1e^{i\theta_1}, \ldots, t_ne^{i\theta_n}) = i \left(z_j\rho_{z_j}(t_1e^{i\theta_1}, \ldots, t_ne^{i\theta_n}) - \bar{z}_j\rho_{\bar{z}_j}(t_1e^{i\theta_1}, \ldots, t_ne^{i\theta_n})\right). 
$$

The monotonicity of $\rho$ in $|z_j|$ implies:

$$
0 \leq t_j \frac{\partial}{\partial t_j} \rho(t_1e^{i\theta_1}, \ldots, t_ne^{i\theta_n}) = z_j\rho_{z_j}(t_1e^{i\theta_1}, \ldots, t_ne^{i\theta_n}) + \bar{z}_j\rho_{\bar{z}_j}(t_1e^{i\theta_1}, \ldots, t_ne^{i\theta_n}). 
$$

Combining these two formulas yields Property (d).

To prove Property (e), it suffices to show that $\sum_{j=1}^n z_j\rho_{z_j}(z) \neq 0$ on $b\Omega$. Suppose not. Then there exists some $z \in b\Omega$ such that $z_j\rho_{z_j}(z) = 0$ for all $j$. Let $A$ denote the set of indices $j$ such that $z_j = 0$ and let $B$ denote the complement of $A$ in $\{1, \ldots, n\}$. Then $\rho_{z_j}(z)$ equals 0 for all $j \in A$. Since the gradient of $\rho$ does not vanish on $b\Omega$, there exists an index $j_0 \in B$ such that $\rho_{z_{j_0}}(z) \neq 0$. Thus $z_{j_0}$ equals 0. The fact that $z_{j_0} = 0$ and Property (c) then imply that $z$ is a local min for $\rho(z)$ in the $z_{j_0}$ direction. This contradicts $\rho_{z_{j_0}}(z) \neq 0$. Therefore the sum $\sum_{j=1}^n z_j\rho_{z_j}(z)$ does not vanish on the boundary. □

The crucial estimates for Theorem 1.2 arise from the following theorem:

**Theorem 3.3.** Let $\Omega \subseteq \mathbb{C}^n$ be a smooth complete Reinhardt domain with a defining function $\rho$. For $\alpha \in \mathbb{R}_+^n$, let $f_\alpha$ and $U^\alpha$ be as (2.7) and (2.8). If $D_\zeta^\beta K_\Omega$ is $(-\rho)$-regular whenever $|\beta| \leq k$, then $K_{U^\alpha}$ is $(\|w\|^2(-\rho \circ f_\alpha))$-regular of type 0.
We give a proof for Theorem 3.3 in Section 4. Theorem 3.3 implies Theorem 1.2. Indeed, the kernel $K_{U^\alpha}$ being $((1 - \|\eta\|^2)(-\rho \circ f_\alpha))$-regular of type 0 implies that the Bergman projection on $U^\alpha$ is bounded in $L^p$ for $p \in (1, \infty)$.

We end this section by referencing several estimates needed in the proof of Theorem 3.3. See for example [Zhu05].

**Lemma 3.4.** Let $\sigma$ denote Lebesgue measure on the unit sphere $S^k \subset \mathbb{C}^k$. For $\epsilon < 1$ and $w \in \mathbb{B}^k$, let

$$a_{\epsilon, \delta}(w) = \int_{\mathbb{B}^k} \frac{(1 - \|\eta\|^2)^{-\epsilon}}{|1 - \langle w, \eta \rangle|^{1+k-\epsilon-\delta}} dV(\eta),$$

and let

$$b_{\delta}(w) = \int_{S^k} \frac{1}{|1 - \langle w, \eta \rangle|^{k-\delta}} d\sigma(\eta).$$

Then

1. for $\delta > 0$, both $a_{\epsilon, \delta}$ and $b_{\delta}$ are bounded on $\mathbb{B}^k$.
2. for $\delta = 0$, both $a_{\epsilon, \delta}(w)$ and $b_{\delta}(w)$ are comparable to the function $-\log(1 - \|w\|^2)$.
3. for $\delta < 0$, both $a_{\epsilon, \delta}(w)$ and $b_{\delta}(w)$ are comparable to the function $(1 - \|w\|^2)^\delta$.

4. **Proof of Theorem 3.3**

Proof of Theorem 3.3. Recall that for each multi-index $\beta$, $D^\beta_z$ is the differential operator $(\frac{\partial}{\partial z_1})^{\beta_1} \cdots (\frac{\partial}{\partial z_n})^{\beta_n}$. Then $D_{U^\alpha}$ in the previous section can be regarded as a sum of $D^\beta_z$:

$$D_{U^\alpha} = \frac{(1 - \|\eta\|^2)^{|\alpha|}}{\pi^k(1 - \langle w, \eta \rangle)^{1+k+|\alpha|}} \left( \sum_{|\beta| \leq k} c_\beta z^\beta D^\beta_z \right),$$

where $c_\beta$ are fixed constants.

The main goal in this proof is to show the following inequality:

$$\int_{U^\alpha} \left| K_{U^\alpha}(z, w; \tilde{\zeta}, \tilde{\eta}) \right| \left( -\rho \left(f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV \lesssim \left( -\rho \left(f_\alpha(z, w) \right) (1 - \|w\|^2) \right)^{-\epsilon}. \quad (4.2)$$

To estimate the integral

$$\int_{U^\alpha} \left| K_{U^\alpha}(z, w; \tilde{\zeta}, \tilde{\eta}) \right| \left( -\rho \left(f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV, \quad (4.3)$$

we use the formula in Theorem 2.2. Substituting (2.3) into the integral in (4.3) yields

$$\int_{U^\alpha} \left| K_{U^\alpha}(z, w; \tilde{\zeta}, \tilde{\eta}) \right| \left( -\rho \left(f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV = \int_{U^\alpha} \left| D_{U^\alpha} K_{U^\alpha}(h(z, w, \eta; \tilde{\zeta})) \right| \left( -\rho \left(f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV. \quad (4.4)$$

We set

$$I_\beta = \frac{c_\beta (1 - \|\eta\|^2)^{|\alpha|}}{(1 - \langle w, \eta \rangle)^{1+k+|\alpha|}} z^\beta D^\beta_z,$$

and

$$J_\beta = \int_{U^\alpha} \left| I_\beta K_{U^\alpha} \left( h(z, w, \eta; \tilde{\zeta}) \right) \right| \left( -\rho \left(f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV. \quad (4.5)$$
By the triangle inequality, we have

$$
\int_{U^n} |D_{U^n} K_{U^n} \left( h(z, w, \eta); \tilde{\zeta} \right) \left( -\rho \left( f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV \leq \sum_{|\beta| \leq k} J_{\beta}. 
$$

(4.6)

Therefore it suffices to prove that $J_{\beta} \lesssim ( -\rho \left( f_\alpha(z, w) \right) (1 - \|w\|^2) )^{-\epsilon}$ for each $\beta$.

The integral $J_{\beta}$ equals

$$
c_\beta \int_{U^n} \frac{(1 - \|\eta\|^2)^{|\alpha|}}{(1 - \langle w, \eta \rangle)^{1+k+|\alpha|}} z^\beta D_z^2 K_{U^n} \left( h(z, w, \eta); \tilde{\zeta} \right) \left( -\rho \left( f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV. 
$$

(4.7)

In order to use $(-\rho)$-regularity assumptions of $D^\beta K_\Omega$ for estimating (4.7), we need to write $D_z^2 K_{U^n}$ in (4.7) in terms of $D^\beta K_\Omega$ and transform (4.7) into an integral on $B^k \times \Omega$.

Recall the mapping $f_\alpha(\cdot, \eta)$ from $\mathbb{Z}_+^n$ defined by

$$
f_\alpha(\cdot, \eta) : z \mapsto \left( \frac{z_1}{(1 - \|\eta\|^2)^{\alpha_1/2}}, \ldots, \frac{z_n}{(1 - \|\eta\|^2)^{\alpha_n/2}} \right). 
$$

(4.8)

It is a biholomorphism from $U^n_{\alpha}$ onto $\Omega$. Hence we can write the kernel function $K_{U^n_{\eta}}$ in terms of $K_\Omega$ using the biholomorphic transformation formula:

$$
K_{U^n_{\eta}}(z; \tilde{\zeta}) = (1 - \|\eta\|^2)^{-|\alpha|} K_\Omega(f_\alpha(z, \eta), f_\alpha(\zeta, \eta)). 
$$

(4.9)

Applying (4.9) to (4.7) yields:

$$
J_{\beta} = c_\beta \int_{B^k} \int_{\Omega} \frac{z^\beta D^2_z K_\Omega \left( h'(z, w, \eta); f_\alpha(\zeta, \eta) \right)}{(1 - \langle w, \eta \rangle)^{1+k+|\alpha|}} \left( -\rho \left( f_\alpha(\zeta, \eta) \right) (1 - \|\eta\|^2) \right)^{-\epsilon} dV(t) dV(\eta). 
$$

(4.10)

where $h'(z, w, \eta) = \left( \frac{z_1}{1 - \langle w, \eta \rangle)^{\alpha_1/2}}, \ldots, \frac{z_n}{1 - \langle w, \eta \rangle)^{\alpha_n/2}} \right)$.

By Substituting $t_j = \frac{1}{(1 - \|\eta\|^2)^{\alpha_j/2}}$ for $1 \leq j \leq n$ to (4.10), we transform $J_{\beta}$ into an integral on $B^k \times \Omega$:

$$
J_{\beta} = c_\beta \int_{B^k} \int_{\Omega} \frac{z^\beta D^2_z K_\Omega \left( h'(z, w, \eta); \tilde{t} \right)}{(1 - \|\eta\|^2)^{1+k+|\alpha|}} \left( -\rho (t) \right)^{-\epsilon} dV(t) dV(\eta). 
$$

(4.11)

For $1 \leq j \leq n$, let $D_j$ denote the partial derivative $\frac{\partial}{\partial z_j}$. Since

$$
D_j K_\Omega \left( h'(z, w, \eta); \tilde{t} \right) = \frac{\partial h'_j}{\partial h_j} K_\Omega \left( h'(z, w, \eta); \tilde{t} \right) = \frac{(1 - \|\eta\|^2)^{\alpha_j/2}}{(1 - \langle w, \eta \rangle)^{\alpha_j}} \frac{\partial}{\partial h'_j} K_\Omega \left( h'(z, w, \eta); \tilde{t} \right), 
$$

(4.12)

applying the $(-\rho)$-regularity of $D^2_z K_\Omega$ to the inner integral in (4.11) yields

$$
J_{\beta} \lesssim \int_{B^k} \left| \frac{z^\beta (-\rho (h'(z, w, \eta)))^{-\epsilon - |\beta|}}{(1 - \|\eta\|^2)^{1+k+\alpha|1+k+|\alpha|}} \right| dV(\eta). 
$$

Here we use the notation $\alpha \cdot \beta$ to denote $\sum_{j=1}^n \alpha_j \beta_j$ and use the notation 1 to denote the multi-index $(1, \ldots, 1) \in \mathbb{N}^n$. When $\beta = 0$, we have

$$
J_0 \lesssim \int_{B^k} \left| \frac{(-\rho (h'(z, w, \eta)))^{-\epsilon}}{(1 - \|\eta\|^2)^{1+k+|\alpha|}} \right| dV(\eta). 
$$

(4.13)
Since $w, \eta \in \mathbb{B}^k$, the triangle inequality and Cauchy-Schwarz inequality imply
\[
\left| \frac{z_j (1 - \|\eta\|^2)^{\alpha_j/2}}{(1 - \langle w, \eta \rangle)^{\alpha_j}} \right| \leq \left| \frac{z_j (1 - \|\eta\|^2)^{\alpha_j/2}}{(1 - \|w\|^2)^{\alpha_j/2}(1 - \|\eta\|^2)^{\alpha_j/2}} \right| = \frac{z_j}{(1 - \|w\|^2)^{\alpha_j/2}}.
\]
Therefore, Property (c) in Lemma 3.2 implies:
\[
J_0 \lesssim \int_{\mathbb{B}^k} \left| \frac{(-\rho (h'(z, w, \eta)))^{-\epsilon}}{(1 - \|\eta\|^2)^{-\epsilon - |\alpha|}/1 - \langle w, \eta \rangle}^{1+k+|\alpha|} \right| dV(\eta)
\leq (-\rho(f_\alpha(z, w)))^{-\epsilon} \int_{\mathbb{B}^k} \frac{(1 - \|\eta\|^2)^{-\epsilon}}{(1 - \langle w, \eta \rangle)^{1+k+|\alpha|}} dV(\eta). \tag{4.15}
\]
For $w, \eta \in \mathbb{B}^k$, we have
\[
\frac{1 - \|\eta\|^2}{1 - \langle w, \eta \rangle} < \frac{1 - \|\eta\|^2}{1 - \|\eta\|} < 2. \tag{4.16}
\]
Applying this inequality and Lemma 3.4 to (4.15) yields the inequality we need for $J_0$:
\[
J_0 \lesssim (-\rho(f_\alpha(z, w)))^{-\epsilon} \int_{\mathbb{B}^k} \frac{(1 - \|\eta\|^2)^{-\epsilon}}{(1 - \langle w, \eta \rangle)^{1+k}} dV(\eta)
\lesssim (-\rho(f_\alpha(z, w)))^{-\epsilon} (1 - \|w\|^2)^{-\epsilon}. \tag{4.17}
\]
For the case $\beta \neq 0$, we recall the integral we need to estimate:
\[
\int_{\mathbb{B}^k} \left| \frac{z^\beta (-\rho (h'(z, w, \eta)))^{-\epsilon - |\beta|}}{(1 - \|\eta\|^2)^{-\epsilon - |\alpha| - \alpha \cdot \beta/2}/(1 - \langle w, \eta \rangle)^{1+k+\alpha - (1+\beta)}} \right| dV(\eta). \tag{4.18}
\]
After rewriting the integral in spherical coordinates $\eta = rt$ with $r \in [0, 1]$ and $t \in S^k$, we would like to write $(-\rho (h'(z, w, \eta)))^{-\epsilon - |\beta|}$ in terms of the $|\beta|$-th order derivative of $(-\rho (h'(z, w, rt)))^{-\epsilon}$ in $r$. These derivatives vanish at the point $\eta = w$ and hence are relatively small when compared with $(-\rho)^{-\epsilon - |\beta|}$. To deal with this problem, we need to move the vanishing point $\eta = w$ to the origin.

When $w = 0$, we keep (4.18) the same. When $w \in \mathbb{B}^k - \{0\}$, we set
\[
\varphi_w(z) = \frac{w - P_w(z) - s_w Q_w(z)}{1 - \langle z, w \rangle},
\]
where $s_w = \sqrt{1 - \|w\|^2}$, $P_w(z) = \frac{(z, w)}{\|w\|^2} w$ and $Q_w(z) = z - \frac{(z, w)}{\|w\|^2} w$. Then $\varphi_w$ is the automorphism of $\mathbb{B}^k$ that sends $0$ to $w$ and satisfies $\varphi_w \circ \varphi_w = id$. We use this $\varphi_w$ to send the point $\eta = w$ to the origin. Setting $\tau = \varphi_w(\eta)$, then we have
\[
\eta = \varphi_w(\tau), \tag{4.19}
\]
\[
1 - \langle \eta, w \rangle = \frac{1 - \|w\|^2}{1 - \langle \tau, w \rangle}, \tag{4.20}
\]
\[
1 - \|\eta\|^2 = \frac{(1 - \|w\|^2)(1 - \|\tau\|^2)}{1 - \langle \tau, w \rangle^2}, \tag{4.21}
\]
\[
dV(\eta) = \left( \frac{1 - \|w\|^2}{1 - \langle \tau, w \rangle^2} \right)^{k+1} dV(\tau). \tag{4.22}
\]
Substituting (4.19), (4.20), (4.21) and (4.22) into the integral (4.18) yields

\[
\int_{B_k} \left| \frac{z^\beta ( - \rho ( h'(z, w, \eta)) ) - \epsilon^{-|\beta|}}{(1 - \|\eta\|^2)^{-|\alpha| - \alpha - |\beta|/2} |1 - \langle w, \eta\rangle|^{1 + k + \alpha}} \right| dV(\eta)
\]

\[
= \int_{B_k} \frac{|z^\beta ( (1 - \|\eta\|^2)(1 - \|\tau\|^2))^{\alpha/2 - \epsilon - |\alpha|}}{(1 - \|\tau\|^2)^{1 + k + \alpha}} (-\rho (h'(z, w, \varphi_w(\tau))))^{-\epsilon - |\beta|} dV(\tau).
\]

Canceling terms in the integral gives

\[
\int_{B_k} \frac{|z^\beta (1 - \|\tau\|^2)^{\alpha/2 - \epsilon + |\alpha|}}{(1 - \|\tau\|^2)^{1 + k - 2\epsilon - |\alpha|}} (-\rho (h'(z, w, \varphi_w(\tau))))^{-\epsilon - |\beta|} dV(\tau),
\]

which is consistent with (4.18) when \(w = 0\). Applying inequality (4.16) to (4.24) and using the fact that \(\frac{|z^\beta|}{(1 - \|\tau\|^2)^{\alpha/2}}\) is bounded on \(\Omega\), we obtain the following inequality:

\[
\int_{B_k} \frac{|z^\beta (1 - \|\tau\|^2)^{\alpha/2 - \epsilon + |\alpha|}}{(1 - \|\tau\|^2)^{1 + k - 2\epsilon - |\alpha|}} (-\rho (h'(z, w, \varphi_w(\tau))))^{-\epsilon - |\beta|} dV(\tau)
\]

\[
\leq \int_{B_k} \frac{|z^\beta|}{(1 - \|\tau\|^2)^{1 + k - \epsilon}} (-\rho (h'(z, w, \varphi_w(\tau))))^{-\epsilon - |\beta|} dV(\tau).
\]

We set \(l(z, w, \tau) = (l_1(z, w, \tau), \ldots, l_n(z, w, \tau))\) where

\[
l_j(z, w, \tau) = |h_j'(z, w, \varphi_w(\tau))| = \left| \frac{z_j \left( \frac{(1 - \|w\|^2)(1 - \|\tau\|^2)}{(1 - \langle w, \tau\rangle)^2} \right)^{\alpha_j/2}}{(1 - \|\tau\|^2)^{\alpha_j/2}} \right| = \frac{|z_j| (1 - \|\tau\|^2)^{\alpha_j/2}}{(1 - \|\tau\|^2)^{\alpha_j/2}}.
\]

Then Lemma 3.2 implies that \(\rho(h'(z, w, \varphi_w(\tau))) = \rho(l(z, w, \tau))\), and the integral in the last line of (4.26) becomes

\[
\int_{B_k} \frac{(-\rho (l(z, w, \tau))^{-\epsilon - |\beta|}}{(1 - \langle w, \tau\rangle)^{1+k-\epsilon}(1 - \|\tau\|^2)^{\epsilon}} dV(\tau).
\]

Rewriting (4.26) using spherical coordinates \(\tau = rt\) with \(r \in [0, 1)\) and \(t \in S^k\) yields:

\[
c_k \int_0^1 r^{2k-1} \int_{S^k} \frac{(-\rho (l(z, w, rt))^{-\epsilon - |\beta|}}{(1 - \langle rt, w\rangle)^{1+k-\epsilon}(1 - \|w\|^2)^{\epsilon}} d\sigma(t) dr,
\]

where \(c_k\) is a constant depending on the dimension \(k\).

By Property (e) in Lemma 3.2, there exists an open neighborhood \(U\) of \(b\Omega\) such that for any \(z \in U\),

\[
\sum_{j=1}^n z_j \rho z_j (z) > c,
\]

for some positive \(c\). For \(\delta > 0\), let \(\Omega_\delta\) denote the set

\[
\{ z \in C^n : \rho((1 + \delta)^{\alpha_j/2}z_1, \ldots, (1 + \delta)^{\alpha_j/2}z_n) \leq 0 \}.
\]
Then there exists a constant $\delta_0 > 0$ such that $\Omega - \mathcal{U} \subseteq \bar{\Omega}_{\delta_0}$. Since $\bar{\Omega}_{\delta_0}$ is compact in $\Omega$, we have $(-\rho(z))^{-1} < C$ in $\bar{\Omega}_{\delta_0}$ for some constant $C$. Let $\mathcal{U}_0$ denote the set $\bar{\Omega}_{\delta_0}$, and let $\mathcal{U}_1$ denote the set $\Omega - \mathcal{U}_0$. Then on $\mathcal{U}_1$, inequality (4.28) still holds. For $t \in S^k$ and $j = 0, 1$, set

$$U_j = \{r \in [0, 1] : l(z, w, rt) \in \mathcal{U}_j\}.$$

Here $U_j$'s are well-defined for any $t \in S^k$: for fixed $z$ and $w$, the value of $l(z, w, rt)$ only depends on $r$ and $\|t\|$. For each $U_j$, we set

$$I_j^\beta = \int_{U_j} r^{2k-1} \int_{S^k} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - |\beta|}}{1 - \langle rt, w \rangle |1 - \|w\|^2\epsilon |} d\sigma(t) dr. \quad (4.30)$$

We claim that $I_j^\beta \lesssim ((-\rho)(l(z, w, t))(1 - |w|^2))^{-\epsilon}$ for each $j$. Then by having

$$J_\beta \lesssim I_0^\beta + I_1^\beta \lesssim ((-\rho)(l(z, w, t))(1 - |w|^2))^{-\epsilon}, \quad (4.31)$$

we complete the proof.

We first consider $I_0^\beta$. Since $(-\rho(l(z, w, rt)))^{-1} < C$ for $r \in U_0$, we have

$$I_0^\beta \lesssim \int_{U_0} r^{2k-1} \int_{S^k} \frac{1}{1 - \langle rt, w \rangle |1 - \|w\|^2\epsilon |} d\sigma(t) dr. \quad (4.32)$$

Applying Lemma 3.4 to the inner integral of (4.32) yields:

$$\int_{S^k} \frac{1}{1 - \langle rt, w \rangle |1 - \|w\|^2\epsilon |} d\sigma(t) \lesssim (1 - \|w\|^2)^{-\epsilon} (1 - r^2 \|w\|^2)^{\epsilon - 1}. \quad (4.33)$$

Then inequality (4.33) gives the desired estimate for $I_0^\beta$:

$$I_0^\beta \lesssim \int_{U_0} r^{2k-1} (1 - \|w\|^2)^{-\epsilon} (1 - r^2 \|w\|^2)^{\epsilon - 1} dr \lesssim \int_{U_0} r^{2k-1} (1 - \|w\|^2)^{-\epsilon} (1 - r^2)^{\epsilon - 1} dr \lesssim (1 - \|w\|^2)^{-\epsilon} \lesssim ((-\rho)(l(z, w, t))(1 - |w|^2))^{-\epsilon}. \quad (4.34)$$

Now we turn to $I_1^\beta$. When $r \in U_1$, we have $l(z, w, rt) \in \mathcal{U}_1$ and

$$\sum_{j=1}^n l_j(z, w, rt) \rho_{z_j}(l(z, w, rt)) > c. \quad (4.35)$$

For such an $r$, $\frac{\partial}{\partial r} ((-\rho)^{-\epsilon - |\beta|+1}(l(z, w, rt)))$ is controlled from below by $(\cdot)^{-\epsilon - |\beta|}(l(z, w, rt))$:

$$- \frac{\partial}{\partial r} ((-\rho(l(z, w, rt))))^{-\epsilon - |\beta|+1}$$

$$= 2(\epsilon + |\beta| - 1)(-\rho)^{-\epsilon - |\beta|}(l(z, w, rt)) \sum_{j=1}^n \alpha_j r^{|z_j|}(1 - r^2)^{\alpha_j/2 - 1} \rho_{z_j}(l(z, w, rt))$$

$$\gtrsim r(-\rho)^{-\epsilon - |\beta|}(l(z, w, rt)) \sum_{j=1}^n l_j(z, w, rt) \rho_{z_j}(l(z, w, rt))$$

$$\gtrsim r(-\rho)^{-\epsilon - |\beta|}(l(z, w, rt)) \frac{\sum_{j=1}^n l_j(z, w, rt) \rho_{z_j}(l(z, w, rt))}{(1 - r^2)}. \quad (4.36)$$
Applying (4.36), (4.16) and Lemma 3.4 to (4.30) then yields:

\[
I_1^\beta \lesssim - \int_{U_1} r^{2k-2} \int_{\Omega} \frac{(1 - r^2) \partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{|1 - \langle rt, w \rangle|^{1+k-\epsilon}(1 - \|w\|^2)^\epsilon - \|l(z, w, rt)\|^\epsilon} d\sigma(t) dr
\]

\[
\lesssim - \int_{U_1} r^{2k-2} \int_{\Omega} \frac{\partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{|1 - \langle rt, w \rangle|^{1+k-\epsilon}(1 - \|w\|^2)^\epsilon - \|l(z, w, rt)\|^\epsilon} d\sigma(t) dr
\]

\[
\lesssim - (1 - \|w\|^2)^{-\epsilon} \int_{U_1} r^{2k-2} \frac{\partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{(1 - \|w\|^2)^\epsilon - \|l(z, w, rt)\|^\epsilon} \bigg|_{0}^{r_0} dr
\]

(4.38)

Noting that \( k = 1 \) also implies \( -\epsilon - |\beta| + 1 \geq -\epsilon - k + 1 = -\epsilon \), we have

\[
- (1 - \|w\|^2)^{-\epsilon} \int_{U_1} \frac{\partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{(1 - \|w\|^2)^\epsilon} dr
\]

\[
\leq \frac{(-\rho \left( l(z, w, 0) \right))^{-\epsilon} + (-\rho \left( l(z, w, r_0 t) \right))^{-\epsilon}}{(1 - \|w\|^2)^\epsilon}.
\]

(4.39)

By its definition, the point \( l(z, w, r_0 t) \) is in \( U_0 \). Therefore \( (-\rho \left( l(z, w, r_0 t) \right))^{-\epsilon - |\beta| + 1} \leq 1 \) and the desired estimate follows:

\[
I_1^\beta = - \int_{U_1} \frac{\partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{(1 - \|w\|^2)^\epsilon} dr \lesssim (-\rho \left( f_\alpha(z, w) \right))^{-\epsilon} (1 - \|w\|^2)^{-\epsilon}.
\]

(4.40)

When \( k > 1 \), integrating the last line of (4.37) by parts yields

\[
- (1 - \|w\|^2)^{-\epsilon} \int_{U_1} r^{2k-2} \frac{\partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{(1 - \|w\|^2)^\epsilon - \|l(z, w, rt)\|^\epsilon} dr
\]

\[
= - (1 - \|w\|^2)^{-\epsilon} \int_{r_0}^{\infty} r^{2k-2} \frac{\partial}{\partial r} \left( -\rho \left( l(z, w, rt) \right) \right) \frac{-\epsilon - |\beta| + 1}{(1 - \|w\|^2)^\epsilon - \|l(z, w, rt)\|^\epsilon} dr
\]

\[
= - \frac{r^{2k-2}(-\rho \left( l(z, w, rt) \right))^{-\epsilon - |\beta| + 1} \bigg|_{0}^{r_0}}{(1 - \|w\|^2)^\epsilon} + \int_{r_0}^{\infty} (2k - 2)r^{2k-3}(-\rho \left( l(z, w, rt) \right))^{-\epsilon - |\beta| + 1} \frac{dr}{(1 - \|w\|^2)^\epsilon}. \quad (4.41)
\]

The numerator of the first term in the last line equals \( r_0^{2k-2}(-\rho(l(z, w, r_0 t)))^{-\epsilon - |\beta| + 1} \), which is also controlled by a constant. Thus it remains to show that

\[
\int_{r_0}^{\infty} r^{2k-3}(-\rho \left( l(z, w, rt) \right))^{-\epsilon - |\beta| + 1} \frac{dr}{(1 - \|w\|^2)^\epsilon} \lesssim (-\rho \left( f_\alpha(z, w) \right))^{-\epsilon} (1 - \|w\|^2)^{-\epsilon}.
\]

(4.42)
Applying (4.36) to the left hand side of (4.42) gives
\[ \int_{0}^{r_{0}} r^{2k-3} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - |\beta| + 1}}{(1 - \|w\|^2)\epsilon} dr \lesssim - \int_{0}^{r_{0}} r^{2k-4} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - |\beta| + 2}}{(1 - \|w\|^2)\epsilon} dr. \]
This together with (4.41) implies that for \( k > 1 \)
\[ - \int_{0}^{r_{0}} r^{2k-2} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - |\beta| + 1}}{(1 - \|w\|^2)\epsilon} dr \lesssim - \int_{0}^{r_{0}} r^{2k-4} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - |\beta| + 2}}{(1 - \|w\|^2)\epsilon} dr. \] (4.43)
Since (4.43) holds whenever \(|\beta| \leq k\), we have for \( 0 < s \leq k \)
\[ - \int_{0}^{r_{0}} r^{2k-2} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - s + 1}}{(1 - \|w\|^2)\epsilon} dr \lesssim - \int_{0}^{r_{0}} r^{2k-4} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - s + 2}}{(1 - \|w\|^2)\epsilon} dr. \] (4.44)
Repeated use of inequality (4.44) then gives
\[ - \int_{0}^{r_{0}} r^{2k-2|\beta|} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon}}{(1 - \|w\|^2)\epsilon} dr \lesssim - \int_{0}^{r_{0}} r^{2k-4|\beta|} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon}}{(1 - \|w\|^2)\epsilon} dr \]
\[ \vdots \]
\[ \lesssim - \int_{0}^{r_{0}} r^{2k-2|\beta|} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon}}{(1 - \|w\|^2)\epsilon} dr. \] (4.45)
Noting that \( r^{2k-2|\beta|} \) is bounded on \([0, r_{0}]\), we have
\[ - \int_{0}^{r_{0}} r^{2k-2|\beta|} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon}}{(1 - \|w\|^2)\epsilon} dr \leq - \int_{0}^{r_{0}} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon}}{(1 - \|w\|^2)\epsilon} dr. \] (4.46)
Applying inequality (4.40) to (4.46) then yields
\[ I^{\beta}_{1} = - \int_{0}^{r_{0}} r^{2k-2} \frac{\partial}{\partial r} \frac{(-\rho(l(z, w, rt)))^{-\epsilon - |\beta| + 1}}{(1 - \|w\|^2)\epsilon} dr \lesssim (-\rho(f_{\alpha}(z, w)))^{-\epsilon} (1 - \|w\|^{-2})^{-\epsilon}, \] (4.47)
which completes the proof. \( \square \)

**Remark.** As in the proof of Theorem 3.3, we can obtain an \( L^p \) regularity result for the Bergman projection on more generalized domains which are generated from \( \Omega \) by iterating the construction of \( U^{(n)} \) from (2.2).

Set \( \alpha = (\alpha^{(1)}, \ldots, \alpha^{(l)}) \in \mathbb{R}^{n}_{+} \times \cdots \times \mathbb{R}^{k_{l}}_{+} \) where each \( \alpha^{(j)} \) is in \( \mathbb{R}^{n}_{+} \). Let \( k_1, \ldots, k_l \) be \( l \) positive integers. The successor \( U^{(n)} \) is defined by
\[ U(\Omega) = \{(z, w_1, w_2, \ldots, w_l) \in \mathbb{C}^{n} \times \mathbb{B}^{k_1} \times \cdots \times \mathbb{B}^{k_l} : (f_{\alpha}(z, w_1, \ldots, w_l)) \in \Omega \}, \] (4.48)
where
\[
f_\alpha(z, w_1, \ldots, w_l) = \left( \frac{z_1}{\prod_{j=1}^l (1 - \|w_j\|^2)^{\alpha_j}}, \ldots, \frac{z_n}{\prod_{j=1}^l (1 - \|w_j\|^2)^{\alpha_n}} \right). \tag{4.49}\]

Suppose \(\Omega \subseteq \mathbb{C}^n\) is a smooth complete Reinhardt domain with defining function \(\rho\) and \(D_\zeta K_\Omega(z; \bar{\zeta})\) is \((-\rho)\)-regular of type \(|\beta|\) for \(0 \leq |\beta| \leq \sum_{j=1}^l k_j\). Then the Bergman projection on \(U(\Omega)\) is \(L^p\) bounded for all \(1 < p < \infty\). The proof of this statement is similar to the proof for the first successor. We omit it here.

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