Fractional forcing number of graphs

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Abstract

The notion of forcing sets for perfect matchings was introduced in [HKZ91] by Harary, Klein, and Živković. The application of this problem in chemistry, as well as its interesting theoretical aspects, made this subject very active. In this work, we introduce the notion of the forcing function of fractional perfect matchings, which is continuous analogous to forcing sets defined over the perfect matching polytope of graphs. We show that our defined object is a continuous and concave function extension of the integral forcing set. Then, we use our results about this extension to conclude new bounds and results about the integral case of forcing sets for the family of edge and vertex-transitive graphs, and in particular, hypercube graphs.

1 Introduction

The notion of defining set is an important concept in studying combinatorial structures. Roughly speaking, when we talk about a defining set for a particular object, we mean a part of it, which uniquely extends to the entire object. As an example, a defining set for a particular perfect matching $M$ of a graph (also known as a forcing set) is a subset of $M$ such that $M$ is the unique perfect matching of the graph containing it. The size of the smallest forcing sets of a perfect matching is called the forcing number of it. The smallest and the largest forcing numbers over all possible perfect matchings of a graph are, respectively, called the forcing number and the maximum forcing number of the graph.

This parameter is particularly important in the theory of resonance in Chemistry. In fact, in [KR87], Klein and Randić defined the notion of the innate degree of freedom of Kekulé structures. In the language of graph theory, Kekulé structure is equivalent to the notion of perfect matching and is defined for the graph representation of the molecules. Also, the innate degree of freedom is equivalent to the notion of forcing set. Forcing set and forcing number appeared in the graph theory literature in [HKZ91] for the first time. In that work, Harary et al. studied these parameters for a particular family of graphs.

Due to the application importance as well as theoretical connections to other mathematical subjects, such as cryptography or Latin squares (see [AMM04, CDS94]), forcing number is extensively studied in the last three decades. The
main problem is to find the exact value or upper and lower bound on the forcing and the maximum forcing number of families of graphs. Another related question is to find the spectrum of the forcing number of a particular graph, i.e. the set of forcing numbers of all perfect matchings of it. We briefly review these results in the following.

- **Grid** $P_m \times P_n$. Patcher and Kim [PK98], found the minimum and the maximum forcing number of $P_{2n} \times P_{2n}$. Afshani et al. [AHM04] proved that for every integer $r \in [n, n^2]$, there exists a perfect matching in $P_{2n} \times P_{2n}$ with forcing number $r$. They also found some upper bounds on the forcing number of $P_m \times P_n$ for particular values $m$ and $n$. Zhao and Zhang [ZZ18], studied the forcing number of $2 \times n$ and $3 \times 2n$ grids. In general, the problem of forcing number of $m \times n$ grids is an open problem.

- **Cylindrical grid** $P_m \times C_n$. Afshani et al. [AHM04], studied the maximum forcing number of Cylindrical grids and found the exact value of the maximum forcing number of $P_{2m} \times C_{2n}$. They also asked about the maximum forcing number of $P_{2m} \times C_{2n+1}$. Jiang and Zhang [JZ16], answered this question and found the exact value of the maximum forcing number of $P_{2m} \times C_{2n+1}$. In general, the forcing number and the forcing spectrum of $P_m \times C_n$ are unknown.

- **Torus** $C_m \times C_n$. Riddle [Rid02], studied the forcing number of $2m \times 2n$ torus and found the exact value of its forcing number. Afshani et al. [AHM04], provided an upper bound on the maximum forcing number of $2m \times 2n$ torus. Kleinerman [Kle06], found the exact value of the maximum forcing number of $C_{2m} \times C_{2n}$. The forcing spectrum of $C_{2m} \times C_{2n}$ is unknown.

- **Stop sign**. In [LP03], Lam and Patcher introduced a family of graphs called $(n, k)$-stop sign. They found a lower and upper bound on the forcing number of perfect matchings of these graphs and showed that these bounds are sharp.

- **Hypercube** $Q_n$. Patcher and Kim [PK98], conjectured that the forcing number of $n$-dimentional hypercube is equal to $2^{n-2}$. Riddle [Rid02], proved this conjecture for even $n$, using the concept of trailing vertices. Diwan [Diw19], used an elegant matrix completion method to resolve this conjecture for all $n$. Adams et al. [AMM04], showed that for every $r \in [2^{n-2}, 2^{n-2} + 2^{n-5}]$, there exists a perfect matching in $Q_n$ with forcing number $r$. Riddle [Rid02], found a lower bound on the maximum forcing number of $n$-dimentional hypercube and showed that for every constant $c < 1$, for large enough $n$, there exists a perfect matching in $Q_n$ with forcing number at least $c 2^{n-1}$. Adams et al. [AMM04] extend their result for any $d$-regular graph with at least one perfect matching. In this paper, as an application of our main result, we find the upper bound on the maximum forcing number of $Q_n$.

- **Other families of graphs**. The forcing number of some other families of graphs has been studied in the context of Chemistry. See [CG88, HZ94, JZ11, RV06, VGR06, VR05, WYZW08, XBXZ13, ZL96, ZL95, ZZ95, ZYS10].
Recently, some related quantities such as anti-forcing sets and numbers, global forcing sets, forcing polynomials, and anti-forcing polynomials are defined and have been studied. (See [CC18, CC11, Den07, Den08, DZ17a, DZ17b, DZ17c, KR14, LY16, SWZ17, SZ16, SZ17, VT07, VT08, YZ15, ZBV11] for anti-forcing, [CZ12, Doš07, ZC14] for global forcing and [ZZ16, ZZL15, ZZ18, ZZ19] for forcing and anti-forcing polynomials.)

In this work, we first introduce the concept of fractional forcing function in a general setting. Then, we restrict our attention to the case where the functions are associated with fractional matchings. More precisely, for a fixed graph $G$, the forcing number is a function that assigns a non-negative value to every perfect matching, while the fractional forcing number assigns a non-negative value to every fractional perfect matching (see Definition 2 for the definition of fractional perfect matching). Thus, the fractional forcing number is a function defined over a larger domain. Among other things, we show that our definition of the forcing number is indeed a function extension of the traditional forcing number. In other words, both functions agree on the intersection of their domains, i.e. the set of all perfect matchings of $G$.

Then, we further study the fractional forcing number as a real-valued function defined over the set of fractional perfect matchings of the underlying graph $G$. Our results are best explained in geometric language. In this view, every Euclidian perfect matching corresponds to a point in an appropriate space. The forcing number will correspond to an integral valued function over the set of these points.

In the bipartite graphs, fractional perfect matching can be viewed as a point in the convex hull of the perfect matching points, and the fractional forcing number is a function extension of the forcing number over the entire convex hull. In section 2.1, we define the notion of the fractional matching polytope of a graph in general.

Our main result is that the fractional forcing number is a concave and continuous function on its domain. This result has interesting consequences which can be translated to the discrete world of integral perfect matchings and their forcing numbers. In particular, for the class of edge and vertex-transitive graphs, we describe the point for which the fractional forcing number is maximized. This can be used to obtain an upper bound for the maximum forcing number of those graphs. As an example, we derive the first non-trivial upper bound for the maximum forcing number of hypercube graphs.

The structure of the paper is as follows. In the next section, we introduce the notations and overview the basic definitions and propositions which we use in the subsequent sections. The next section contains the formal description of the problem as well as the main result of this work. In section 4, we present an application of our result.

2 Preliminaries

A graph $G$ is a pair $(V(G), E(G))$ in which $V(G)$ is a finite set of elements, called the vertices of $G$ and $E(G)$ is a subset of 2-subsets of $V(G)$. Elements of $E(G)$ are called the edges of $G$. The vertices $x$ and $y$ of an edge $\{x, y\}$ are called the endpoints of the edge. If $\{v_i, v_j\} \in E(G)$, we write $v_i \sim v_j$ and say $v_i$ is a neighbor of $v_j$. $N(v_i) = \{v_j \in V(G) : v_i \sim v_j\}$ is the open neighborhood
of \(v_i\) and \(N[v_i] := N(v_i) \cup \{v_i\}\) is a close neighborhood of \(v_i\).

A path \(P\) in \(G\) is a sequence \(v_1, v_2, \ldots, v_n\) of distinct vertices, where for all \(j \in \{1, \ldots, k-1\}\), \(v_i \sim v_{i+1}\). The length of the path \(P\) is the number of edges in \(P\) (i.e. \(k-1\)). For every \(v, w \in V(G)\) define \(d_G(v, w)\) to be the length of the shortest path from \(v\) to \(w\).

For any two vectors \(\vec{a} = (a_1, \ldots, a_n)\) and \(\vec{b} = (b_1, \ldots, b_n)\), define the Hamming distance between \(\vec{a}\) and \(\vec{b}\), denoted by \(d_H(\vec{a}, \vec{b}) := \omega (\vec{a} - \vec{b}) = \sum |a_i - b_i|\).

The hypercube graph \(Q_n\) is the graph with the vertex set \(V(Q_n) = \{ (a_1, \ldots, a_n) : \forall i, a_i \in \{0, 1\} \} \) and the edge set \(E(Q_n) = \{ \{ v_i, v_j \} : d_H (v_i, v_j) = 1 \}\). It is easy to see that \(d_{Q_n}(v_i, v_j) = d_H(v_i, v_j)\).

Let \(G\) and \(H\) be any Graphs. A homomorphism from \(G\) to \(H\), written as \(f : G \to H\), is a mapping \(f : V(G) \to V(H)\) such that \(f(u)f(v) \in E(H)\) whenever \(uv \in E(G)\).

**Proposition 1** ([HN04, Proposition 1.3]). A mapping \(f : V(C) \to V(G)\) is a homomorphism of cycle \(C = (v_1, \ldots, v_k)\) to \(G\) if and only if \(f(v_1), \ldots, f(v_k)\) is a closed walk in \(G\).

Now, we define the notion of \(g\)-factor and partial \(g\)-factor.

**Definition 2.** Let \(g : V(G) \to \mathbb{R}^\geq 0\) be a function. The function \(\gamma : E(G) \to \mathbb{R}^\geq 0\) is called a partial \(g\)-factor if for every vertex \(v \in V(G)\), \(\sum_{e \in \partial G(v)} \gamma(e) \leq g(v)\).

\(\gamma\) is called a \(g\)-factor if for every vertex \(v \in V(G)\), \(\sum_{e \in \partial G(v)} \gamma(e) = g(v)\).

Denote the constant function \(1_G\) over the vertex set \(V(G)\) by \(1_G(v) = 1\). Any partial \(1_G\)-factor is called a fractional matching and any \(1_G\)-factor is called fractional perfect matching. Every fractional perfect matching \(\gamma\), with \(\gamma(e) \in \mathbb{Z}\), for all \(e \in E(G)\), is an integral perfect matching (or perfect matching for short).

The support of every perfect matching corresponds to a subset of the edge set of the graph such that no two of them share an endpoint. The converse is also true (see Figure 1).

**Lemma 3 (Convex Combination).** Suppose that \(\gamma\) and \(\gamma'\) are two \(g\)-factors. Then, every convex combination of \(\gamma\) and \(\gamma'\) is also a \(g\)-factor.

**Proof.** Suppose that \(v \in V(G)\) and \(S\) are the set of all the edges that are incident with \(v\). Then,

\[
\sum_{e \in S} (\lambda \gamma(e) + (1 - \lambda) \gamma'(e)) = \lambda \sum_{e \in S} \gamma(e) + (1 - \lambda) \sum_{e \in S} \gamma'(e) \\
= \lambda g(v) + (1 - \lambda) g(v) \\
= g(v)
\]

The positivity condition is trivially satisfied.

Let \(M\) be a perfect matching of a graph \(G\). A subset \(S \subseteq M\) is called a forcing set for \(M\) if \(M\) is the unique perfect matching of \(G\) containing \(S\).
Definition 4. Let $G$ be a graph and $M$ be any perfect matching of $G$. We define the quantities forcing number of $M$, $f(G, M)$ the forcing number of $G$, $f(G)$ the maximum forcing number of $G$, $F(G)$ and the spectrum of the forcing numbers of $G$, $\text{Spec}(G)$ as follows

$$f(G, M) := \min \{|S| : S \text{ is a forcing set for } M\}$$

$$f(G) := \min \{f(G, M) : M \text{ is a perfect matching of } G\}$$

$$F(G) := \max \{f(G, M) : M \text{ is a perfect matching of } G\}$$

$$\text{Spec}(G) := \{f(G, M) : M \text{ is a perfect matching of } G\}$$

Observe that $f(G) = \min_{x \in \text{Spec}(G)} x$, and $F(G) = \max_{x \in \text{Spec}(G)} x$.

Proposition 5 ([Rid02, Proposition 3]). For any $\alpha < 1$, if $n$ is sufficiently large, then there exists a perfect matching $M$ of $Q_n$ with the forcing number at least $\alpha \frac{2^n}{n} - 1$, i.e., $F(Q_n) \geq \alpha \frac{2^n}{n} - 1$.

2.1 Perfect matching and fractional perfect matching polytope

A subset $C$ of $\mathbb{R}^n$ is said to be convex if $\lambda x + (1 - \lambda)y$ belongs to $C$ for all $x, y \in C$ and $0 \leq \lambda \leq 1$. The convex hull of a set $X \subseteq \mathbb{R}^n$, denoted by $\text{Conv}(X)$, is the smallest convex set containing $X$. A subset $\mathcal{P}$ of $\mathbb{R}^n$ is called a polytope if it is the convex hull of finitely many points in $\mathbb{R}^n$.

Let $G = (V, E)$ be a graph. For every subset $M \subseteq E$, let $\chi^M$ to be a 0-1 valued function which is defined as follows

$$\chi^M(e) = \begin{cases} 1 & e \in M \\ 0 & e \notin M \end{cases}$$

Notice that, for every $M \subseteq E$, $\chi^M(e)$ can be visualized as a point in $\mathbb{R}^{|E|}$ with coordinates being 0 or 1. $\chi^M$ is called the characteristic function (or characteristic vector when we view $\chi^M$ as a vector in $\mathbb{R}^M$) of $M$.

The convex hull of all the characteristic vectors of the perfect matchings of $G$ is called the perfect matching polytope of $G$ i.e.

$$\mathcal{P}(G) := \text{Conv}\{\chi^M : M \text{ is a perfect matching of } G\}$$
It is straightforward to observe that fractional perfect matchings with integer coordinates are precisely the characteristic vectors of perfect matchings. The set of all fractional perfect matchings forms a polytope called fractional perfect matching polytope of $G$ and is denoted by $P_f(G)$ (see [Edm65]).

The vertices of $P_f(G)$ have the following structure:

**Proposition 6 ([Sch03, Theorem 30.2]).** Let $\alpha$ be a vertex of $P_f(G)$. Then,

\[
\alpha(e) = \begin{cases} 
1 & e \in K \\
\frac{1}{2} & e \in C \\
0 & \text{otherwise}
\end{cases}
\]

where $K$ is the union of some vertex-disjoint $K_2$ and $C$ is the union of some vertex-disjoint odd cycles.

Since a bipartite graph has no odd cycle, the following statement trivially holds.

**Proposition 7 ([Edm65, Theorem P]).** If $G$ is bipartite then, $P_f(G) = \mathcal{P}(G)$.

Let $g : V(G) \rightarrow \mathbb{R}$ be a function. The set of all $g$-factors is called the $g$-polytope. The following proposition is a consequence of Lemma 3.

**Proposition 8.** Let $g : V(G) \rightarrow \mathbb{R}^{\geq 0}$ be a function. Then, every $g$-polytope is a polytope.

### 3 Main Result

**Definition 9.** Let $G$ be a graph and $\alpha, \alpha' : E(G) \rightarrow \mathbb{R}^{\geq 0}$ are two functions. Define the relation "$\preceq$" as follows

$\alpha \preceq \alpha' \iff \forall e \in E(G) : \alpha(e) \leq \alpha'(e)$

One can easily observe that indeed, $\preceq$ is a partial order on the set $(\mathbb{R}^{\geq 0})^E$.

If $\alpha : E(G) \rightarrow \mathbb{R}$, define $|\alpha| := \sum_{e \in E(G)} \alpha(e)$.

Let $g : V(G) \rightarrow \mathbb{R}^{\geq 0}$ be a function, $\alpha$ be a partial $g$-factor and $\gamma$ be a $g$-factor in a graph $G$. We say $\alpha$ is $g$-extendable (or simply extendable if $g$ is clear from the context) to $\gamma$ if $\alpha \preceq \gamma$. In this case, we say $\gamma$ is a $g$-extension (or extension, when $g$ is clear from the context) of $\alpha$.

$\alpha$ is a forcing function for $\gamma$ if $\alpha$ is uniquely $g$-extendable to $\gamma$ (i.e. $\alpha \preceq \gamma$ and $\gamma$ is the unique extension of $\alpha$). In this situation, we write $\alpha \uparrow \gamma$. $\alpha$ is a minimal forcing function for $\gamma$ if $\alpha \uparrow \gamma$ and, if $\alpha' \preceq \alpha$ and $\alpha' \uparrow \gamma$ then, $\alpha = \alpha'$.

In this case, we write $\alpha \uparrow \gamma$.

**Observation 10.** If $\alpha \uparrow \gamma$ and $\alpha \preceq \beta \preceq \gamma$ then $\beta \uparrow \gamma$.

**Proof.** Since $\beta \preceq \gamma$, if $\beta$ does not uniquely extend to $\gamma$, then there exists another $g$-factor $\gamma'$ such that $\beta \preceq \gamma'$. But this implies that $\alpha \preceq \beta \preceq \gamma'$, which contradicts the assumption $\alpha \uparrow \gamma$. \qed
Definition 11. In an extention $\gamma$ of the partial $g$-factor $\alpha$, an edge $e$ is called saturated if $\alpha(e) = \gamma(e)$. The next lemma is a useful tool when we study the structure of minimal forcing functions of $g$-factors.

Lemma 12 (Saturated Edges). Let $G$ be a graph, $\gamma$ be a $g$-factor and $\alpha \uparrow \gamma$. Then, for every edge $e \in E(G)$, $\alpha(e) \in \{0, \gamma(e)\}$.

The above lemma implies that the only way one can extend a minimal uniquely extendable partial $g$-factor $\alpha$ to a $g$-factor $\gamma$ is by increasing the value of $\alpha$ on the edges whose values are 0; i.e. $\alpha(e) = 0$.

Proof of Lemma 12. Let $e \in E(G)$ be an arbitrary edge of $G$. First note that $\alpha \uparrow \gamma$ and therefore $\alpha(e) \leq \gamma(e)$. We claim that $\alpha(e) \in \{0, \gamma(e)\}$. In the contrary, suppose that $0 < \alpha(e) < \gamma(e)$. Let $\alpha'$ be a partial $g$-factor that agrees with $\alpha$ on every edge except $e$ and also $\alpha'(e) = 0$. Clearly $\alpha' < \alpha$. Thus, $\alpha'$ is also $g$-extendable to $\gamma$. On the other hand, since $\alpha \uparrow \gamma$ and $\alpha' < \alpha$, $\alpha'$ is not a forcing function for $\gamma$. That is, $\alpha'$ is also $g$-extendable to another $g$-factor, say $\gamma'$.

Now, we show that there exists a sufficiently small positive $t$ such that $\alpha$ is $g$-extendable to $t\gamma' + (1-t)\gamma$. Notice that, by Lemma 3, we know that $t\gamma' + (1-t)\gamma$ is a $g$-factor.

First, take $e'$ to be any edge in $E \backslash \{e\}$. By the definition of $\alpha'$ and the fact that $\alpha' \prec \gamma, \gamma'$ we have

$$\alpha(e') = \alpha'(e') = t\alpha'(e') + (1-t)\alpha'(e') \leq t\gamma'(e') + (1-t)\gamma(e').$$

For the edge $e$, since $\gamma'(e)$ is finite and $\alpha(e) < \gamma(e)$, there exists a sufficiently small $t$ such that $\alpha(e) < t\gamma'(e) + (1-t)\gamma(e)$. Therefore, for every edge $e' \in E$ if $t$ is small enough, then $\alpha(e') < t\gamma'(e') + (1-t)\gamma(e')$. This implies that $\alpha \prec t\gamma' + (1-t)\gamma \neq \gamma$. This is a contradiction to the assumption $\alpha \uparrow \gamma$. 

Proof of Theorem 13. From the definition, it is clear that $\alpha' \preceq \gamma'$. Suppose that $\alpha'$ is not uniquely $g$-extendable to $\gamma'$. Thus, there exists a $g$-factor $\gamma''$ such that $\alpha' \preceq \gamma''$. Next, we show that there exists an $\epsilon > 0$ such that the following function $\eta$ is a $g$-factor such that $\alpha \preceq \eta$. This contradicts the fact that $\alpha$ is uniquely $g$-extendable to $\gamma$. Let

$$\eta = \gamma + \epsilon(\gamma'' - \gamma')$$
If \( e \in \text{Supp}(\gamma'' - \gamma') \) then,

\[
\gamma'(e) = 0 \Rightarrow \gamma(e) = 0 \Rightarrow \alpha(e) = 0
\]

Thus, in this case, it is enough to show that \( \eta(e) \geq 0. \) Since \( \gamma''(e) \geq 0, \) for every \( \varepsilon > 0 \) we have

\[
\eta(e) = \gamma(e) + \varepsilon(\gamma''(e) - \gamma'(e)) = \varepsilon\gamma''(e) \geq 0
\]

Suppose that \( e \in \text{Supp}(\gamma'). \) Then, \( \gamma'(e) > 0, \) and since \( \text{Supp}(\gamma) = \text{Supp}(\gamma'), \) we have \( \gamma(e) > 0. \) If \( e \in \text{Supp}(\alpha) \) then, by Lemma 12, \( \alpha(e) = \gamma(e). \) Thus, by definition of \( \alpha', \) \( \alpha'(e) = \gamma'(e). \) Therefore,

\[
\gamma'(e) = \alpha'(e) \leq \gamma''(e) \Rightarrow \gamma'(e) \leq \gamma'(e) \Rightarrow \varepsilon(\gamma''(e) - \gamma'(e)) \geq 0
\]

\[
\eta(e) = \gamma(e) + \varepsilon(\gamma''(e) - \gamma'(e)) \geq \gamma(e) = \alpha(e)
\]

If \( e \notin \text{Supp}(\alpha), \) the value of \( \gamma''(e) - \gamma'(e) \) can be negative. On the other hand \( \gamma(e) = \gamma'(e) > 0. \) In this case, according to Archimedean property of numbers, there exists \( \varepsilon > 0 \) such that \( \gamma(e) > \varepsilon(\gamma''(e) - \gamma'(e)). \) Therefore,

\[
\eta(e) = \gamma(e) - \varepsilon(\gamma''(e) - \gamma'(e)) \geq 0 \Rightarrow \eta(e) \geq \alpha(e)
\]

Thus, \( \alpha \preceq \eta. \) This contradicts the fact that \( \alpha \) is uniquely \( g \)-extendable to \( \gamma. \)

Now, suppose that \( \alpha' \) is not minimal forcing for \( \gamma'. \) Thus, there exists a partial \( g \)-factor \( \alpha'', \) such that \( \alpha'' \uparrow \gamma' \) and \( \alpha'' \preceq \alpha'. \) Suppose also that \( \alpha'' \) is minimal. Define \( \theta \) as follows

\[
\theta(e) = \begin{cases} 
\gamma(e) & \alpha''(e) = \gamma'(e) \\
0 & \text{otherwise}
\end{cases}
\]

Then, \( \theta \preceq \gamma \) and \( \theta \preceq \alpha. \) By the same argument that we used to show that \( \alpha' \uparrow \gamma', \) we can conclude that \( \theta \uparrow \gamma. \) This contradicts the minimality of \( \alpha. \)

\( \square \)

Theorem 13 combined with Lemma 12 imply that the minimal forcing functions for every \( \gamma \in \mathcal{P}_f(G) \) can be obtained in a two-stage process. In the first stage, we only need to know the support of \( G. \) Having access only to \( \text{Supp}(G), \) the support of every minimal forcing function for \( G \) is specified in the sense that the set \( \mathcal{D}_\gamma := \{ \text{Supp}(\alpha) : \alpha \uparrow \gamma \}, \) only depends on \( \text{Supp}(\gamma) i.e. \) if \( \text{Supp}(\gamma) = \text{Supp}(\gamma') \) then \( \mathcal{D}_\gamma = \mathcal{D}_{\gamma}. \)

In the second stage, once we have access to \( \text{Supp}(\alpha) \) and the values of \( \gamma(e), \) we know by Lemma 12, that \( \alpha \) is uniquely identified.

This observation raises the following question.

**Question 14.** If \( G \) is a graph, \( \gamma \in \mathcal{P}_f(G) \) and \( S \subseteq E(G), \) is there a fractional matching \( \alpha \) such that \( \alpha \uparrow \gamma \) and \( \text{Supp}(\alpha) = S? \)

The next theorem answers this question for bipartite graphs.

**Theorem 15.** Let \( G \) be a bipartite graph, \( \gamma \in \mathcal{P}_f(G), \) \( S \subseteq E(G) \) and \( T = E(G) \setminus \text{Supp}(\gamma). \) There exists a fractional matching \( \alpha \) such that \( \alpha \uparrow \gamma \) and \( \text{Supp}(\alpha) = S \) if and only if the following conditions are satisfied.
1. $S \subseteq \text{Supp}(\gamma)$

2. For every cycle $C$ of $G$, with a proper 2-coloring of the edges of $C$, each color class intersects $T \cup S$.

Proof. First, we prove the necessity of the conditions. The first condition is necessary since $\alpha \uparrow \gamma$ implies that $\alpha \preceq \gamma$ and thus $\text{Supp}(\alpha) \subseteq \text{Supp}(\gamma)$.

For the second condition, we argue as follows. Suppose that there exists a function $\alpha$ such that $\alpha \uparrow \gamma$ and $\text{Supp}(\alpha) = S$. Let $C$ be a cycle of $G$. Consider a proper 2-coloring of the edges of $C$ with color classes $C_1, C_2 \subseteq E(C)$. We claim that for both $i \in \{1, 2\}$, $(S \cup T) \cap C_i \neq \emptyset$. For a contradiction, suppose that $(S \cup T) \cap C_1 = \emptyset$. Define $\gamma' \in \mathcal{P}_f(G)$ as follows

$$
\gamma'(e) = \begin{cases} 
\gamma(e) + \varepsilon & e \in C_2 \\
\gamma(e) - \varepsilon & e \in C_1 \\
\gamma(e) & e \in \text{Supp}(\gamma) \setminus E(C) \\
0 & e \notin \text{Supp}(\gamma) \cup E(C)
\end{cases}
$$

where $\varepsilon$ is a sufficiently small positive number such that $\gamma(e) - \varepsilon$ is non-negative for all $e \in C_1$. Since $C_1 \cap (S \cup T) = \emptyset$, such an $\varepsilon$ exists. For example, we may take $\varepsilon = \frac{1}{2} \min_{e \in C_1} \gamma(e)$. One can check that $\gamma'$ is also a fractional perfect matching with the same support as $\gamma$ and also $\alpha \preceq \gamma'$. This contradicts the assumption $\alpha \uparrow \gamma$.

Next, we show that if $S$ satisfies the conditions 1, 2 then, there exists $\alpha$ such that $\alpha \uparrow \gamma$ and $\text{Supp}(\alpha) = S$. Define $\alpha$ as follows

$$
\alpha(e) = \begin{cases} 
\gamma(e) & e \in S \\
0 & e \notin S
\end{cases}
$$

Clearly $\alpha \preceq \gamma$. To complete the proof we must show that $\gamma$ is the unique element of $\mathcal{P}_f(G)$ such that $\alpha \preceq \gamma$. For a contradiction, assume that there exists a fractional perfect matching $\gamma' \in \mathcal{P}_f(G)$ such that $\gamma \neq \gamma'$ and $\alpha \preceq \gamma'$.

Let $\gamma_{dif} := \gamma - \gamma'$. Since both $\gamma, \gamma' \in \mathcal{P}_f(G)$ we have

$$
\forall v \in V \sum_{e, e' \ni v} \gamma_{dif}(e) = 0 \tag{1}
$$

This equation guarantees that there exists a cycle of $G$ such that the value of $\gamma_{dif}$ on the edges of $C$ are alternatively positive and negative. The reason is that since $\gamma \neq \gamma'$, there exists an edge $e$ for which $\gamma(e) \neq \gamma'(e)$ and thus $\gamma_{dif}(e) \neq 0$. Without loss of generality assume that $\gamma_{dif} > 0$. Let $v$ be an endpoint of $e$. By equation (1), there exists another adjacent edge $e'$ to $v$ such that $\gamma_{dif}(e) < 0$.

We can repeat this argument until we reach a vertex for the second time. Thus, we obtain a cycle with alternating sign of $\gamma_{dif}$ on its edges except possibly the first and the last edges. But since $G$ is bipartite, $C$ is even and therefore, the first and the last traversed edges must have also different signs.

Let $C_1, C_2$ be the sets of the edges with positive and negative values of $\gamma_{dif}$, respectively. According to the second condition of the theorem, $C_1 \cap (S \cup T) \neq \emptyset$. Let $e \in C_1 \cap (S \cup T)$. Therefore, $\gamma(e) - \gamma'(e) = \gamma_{dif}(e) > 0$. This implies that $\gamma(e) > \gamma'(e) \geq 0$ and consequently $\gamma(e) > 0$. That is, $e \in \text{Supp}(\gamma)$.
or equivalently \( e \notin T \). Thus, \( e \in S \). Now, according to the definition of \( \alpha \), \( \alpha(e) = \gamma(e) \). We also assume that \( \alpha \preceq \gamma \). Thus, \( \alpha(e) \leq \gamma'(e) \) and therefore

\[
0 < \gamma_{diff}(e) = \gamma(e) - \gamma'(e) \leq \alpha(e) - \alpha(e) = 0
\]

Contradiction shows that \( \gamma' \) may not exists, i.e. \( \alpha \uparrow \gamma \).

Remark 16. In the previous theorem, we can always take \( \alpha \) to be the following function.

\[
\alpha(e) = \begin{cases} 
\gamma(e) & e \in S \\
0 & e \notin S 
\end{cases}
\]

This is a direct consequence of observation 10 and the fact that \( \alpha \) is the largest function with support \( S \) and \( \alpha \preceq \gamma \).

Lemma 17. Let \( G \) be a graph and \( \gamma_1, \gamma_2, \ldots, \gamma_n \in \mathcal{P}_f(G) \) such that \( \gamma \) is a convex combination of \( \gamma_i \)'s with coefficients \( \lambda_i \). Let \( \alpha \uparrow \gamma \). Then, there exists fractional matchings \( \alpha_1, \alpha_2, \ldots, \alpha_n \) such that \( \alpha = \sum \lambda_i \alpha_i \) and for every \( i, \alpha_i \uparrow \gamma_i \).

Proof. For every \( i \), define \( \alpha_i(e) \) as follows

\[
\alpha_i(e) = \begin{cases} 
\gamma_i(e) & \alpha(e) \neq 0 \\
0 & \alpha(e) = 0 
\end{cases}
\]

Then, for every edge \( e \), \( \alpha_i(e) \leq \gamma_i(e) \). Thus, \( \alpha_i \) is extendable to \( \gamma_i \). Now, we show that for every edge \( e \),

\[
\alpha(e) = \sum \lambda_i \alpha_i
\]

If \( \alpha(e) = 0 \) then, \( \alpha_i(e) = 0 \). In this case, \( \sum \lambda_i \alpha_i(e) = 0 \). If \( \alpha(e) \neq 0 \) then, \( \alpha_i(e) = \gamma_i(e) \). Thus, \( \gamma = \sum \lambda_i \gamma_i = \sum \lambda_i \alpha_i \). Since \( \alpha \) is minimal, by Lemma 12, \( e \) must be saturated. Thus, \( \alpha(e) = \gamma(e) \) and therefore \( \alpha(e) = \sum \lambda_i \alpha_i(e) \).

Finally, we need to show that \( \alpha_i \uparrow \gamma_i \). If not, there exists \( \gamma_i \neq \gamma'_i \in \mathcal{P}_f(G) \) with \( \alpha_i \preceq \gamma'_i \). Thus, \( \alpha = \sum \lambda_i \alpha_i \) is extendable to \( \sum \lambda_i \gamma'_i \). This contradicts that \( \alpha \) is forcing for \( \gamma \).

In the case of a bipartite graph, the \( \gamma_i \)'s in Lemma 17 are integral and for each \( i \), \( \alpha_i \) corresponds to a forcing set for \( \gamma_i \). Note that, the following example shows that these \( \alpha_i \)'s are not necessarily minimal.

Example 18. Consider the graph \( P_5 \square P_4 \). A fractional perfect matching \( \gamma \) for \( G \) is specified in the following picture.

![Graph](image)

It is easy to see that \( \gamma \) can be written as \( \gamma = \frac{1}{12} \gamma_1 + \frac{1}{6} \gamma_2 + \frac{1}{12} \gamma_3 \) where for every \( i, \gamma_i \in \mathcal{P}(G) \).
This contradicts the assumption such that \( \text{Supp}(\gamma) \neq M \in \mathcal{P}(G) \). Therefore, we have \( \alpha \leq 1_{\text{Supp}(\alpha)} \leq 1_{M} \). This contradicts the assumption \( \alpha \uparrow M \).

**Lemma 20.** If \( M \in \mathcal{P}(G) \) and \( \alpha \uparrow M \) then \( \text{Supp}(\alpha) \) is a forcing set for \( M \).

**Proof.** If \( \text{Supp}(\alpha) \) is not a forcing set for \( M \), then there exists \( M \neq M' \in \mathcal{P}(G) \) such that \( \text{Supp}(\alpha) \) also extends to \( M' \). Therefore, we have \( \alpha \leq 1_{\text{Supp}(\alpha)} \leq 1_{M'} \).

**Lemma 21.** For every perfect matching \( M \) of \( G \), \( f_f(G, M) \geq f(G, M) \). Furthermore, if \( G \) is bipartite then \( f_f(G, M) = f(G, M) \).

**Proof.** Let \( \alpha \) be a minimum forcing function for \( M \) in which \( M \) is considered as a fractional perfect matching. Since \( M \) is also an element of \( \mathcal{P}(G) \), for every \( e \in E(G) \) we have \( M(e) \in \{0, 1\} \).

On the other hand, by Lemma 12, we know that \( \alpha(e) \in \{0, M(e)\} \). This implies that \( \alpha(e) \in \{0, 1\} \). Thus, \( |\alpha| = |\text{Supp}(\alpha)| \). By Lemma 20, \( \text{Supp}(\alpha) \) is a forcing set for \( M \), meaning \( f(G, M) \leq |\text{Supp}(\alpha)| = |\alpha| = f_f(G, M) \).
Now, using Lemma 21, in the case of bipartite graphs, we can show that the forcing number of a graph is equal to its fractional forcing number.

**Theorem 22.** Let $G$ be a bipartite graph. Then, $f_f(G) = f(G)$.

**Proof.** By Lemma 21, for every integral perfect matching $M$ we have $f(G, M) = f_f(G, M)$. Since $f(G) = \min_{\gamma \in \mathcal{P}(G)} f(G, \gamma)$ and $f_f(G) = \min_{\gamma \in \mathcal{P}_f(G)} f_f(G, \gamma)$ and $\mathcal{P}(G) \subseteq \mathcal{P}_f(G)$, we conclude that $f_f(G) \leq f(G)$.

Now, by using the Lemma 17, we prove $f_f(G) \leq f(G)$. Suppose that $\alpha$ is a fractional matching for which the graph takes its fractional forcing number $f_f(G)$. Thus, $\alpha$ is minimal and uniquely extendable to a fractional perfect matching $\gamma$. Since $G$ is a bipartite graph, $\gamma$ can be written as $\gamma = \sum_{i} \lambda_i \gamma_i$ where for each $i$, $\gamma_i$ is integral and $\sum \lambda_i = 1$. By using Lemma 17, there exists $\alpha_1, \alpha_2, \ldots, \alpha_n$ such that $\alpha = \sum \lambda_i \alpha_i$ and each $\alpha_i$ is uniquely extendable to $\gamma_i$. Thus, if we consider $\alpha_i$ as a matching and $\gamma_i$ as a perfect matching then, each $\alpha_i$ is a forcing set for $\gamma_i$. Let $f(G) = r$ and $k_i$ be the minimum forcing set for $\gamma_i$. Then

$$\begin{align*}
|\alpha| &\geq |k_i| \\
|\alpha| &= |\sum \lambda_i \alpha_i| = \sum \lambda_i |\alpha_i| \geq \sum \lambda_i |k_i| \\
|\alpha| &\geq r \Rightarrow f_f(G) \geq f(G).
\end{align*}$$

In the case of non-bipartite graphs, the fractional forcing number of the graph is not necessarily equal to the forcing number of the graph.

**Example 23.** In the following graph, $f(G) = 1$ but $f_f(G) \leq \frac{1}{2}$. Corollary 31, conclude that $f_f(G)$ is exactly equal to $\frac{1}{2}$.

**Example 24.** In the following graph $f(G) = 0$ and $f_f(G) \leq \frac{1}{2}$. Corollary 31, conclude that $f_f(G)$ is exactly equal to $\frac{1}{2}$.

**Corollary 25.** For every integer $s$, there exists a graph $G$ such that the difference between the forcing number and fractional forcing number of the graph is $s$ and there exists a graph $G'$ such that this difference is $s + \frac{1}{2}$.
Proof. The disjoint union of graphs that showed in examples 23 and 24 gives the results.

Observation 26. For every graph $G$ we have
\[ F_f(G) \geq F(G) \]

Proof. First, notice that for every perfect matching $M$ of $G$ we have $f(G, M) \leq F_f(G, M)$, by Lemma 21. Since $F_f(G) = \max_{M \in \mathcal{P}_{f}(G)} f(G, M)$ and $F(G) = \max_{M \in \mathcal{P}(G)} f(G, M)$ and $\mathcal{P}(G) \subseteq \mathcal{P}_{f}(G)$, we conclude that $F_f(G) \geq F(G)$.

In what follows, we show that $\text{Spec}_{f}(G)$ contains every real number in the interval $[f_f(G), F_f(G)]$. The function $d : \mathcal{P}_{f}(G) \times \mathcal{P}_{f}(G) \to \mathbb{R}^{\geq 0}$ where $d(\gamma_1, \gamma_2) = \sum_{e \in E} |\gamma_1(e) - \gamma_2(e)|$, is a metric on $\mathcal{P}_{f}(G)$. Consider the function $f_f : \mathcal{P}_{f}(G) \to \mathbb{R}$ where $f_f(\gamma) = f_f(G, \gamma)$. We have the following theorem

Theorem 27. The function $f_f : \mathcal{P}_{f}(G) \to \mathbb{R}^{\geq 0}$ with the metric $d$ is continuous.

To prove the theorem, we use the following lemma

Lemma 28. Let $\gamma_1, \gamma_2 \in \mathcal{P}_{f}(G)$, $\alpha_1 \uparrow \gamma_1$ and $d(\gamma_1, \gamma_2) < \epsilon$. Then, there exists fractional matching $\alpha_2$ such that $\alpha_2 \uparrow \gamma_2$ and
\[ |\sum_{e \in E} \alpha_1(e) - \sum_{e \in E} \alpha_2(e)| < \epsilon \]

Proof. Define fractional matching $\alpha_2$ as follows
\[ \alpha_2(e) = \begin{cases} \gamma_2(e) & e \in \text{Supp}(\alpha_1) \\ \gamma_2(e) & e \in \text{Supp}(\gamma_2) \setminus \text{Supp}(\gamma_1) \\ 0 & e \in \text{Supp}(\gamma_1) \setminus \text{Supp}(\alpha_1) \end{cases} \]

It is clear that $\alpha_2 \leq \gamma_2$. Now, suppose that $\alpha_2$ is not a forcing function. Thus, there exists a fractional perfect matching $\gamma'$ such that $\alpha_2 \leq \gamma'$. First, we show that there exists $t$ such that $\alpha_1 \uparrow (\gamma_1 + t(\gamma' - \gamma_2))$. This contradicts the fact that $\alpha_1$ is a forcing function. Consider the following three cases

- **1.** $e \in \text{Supp}(\alpha_1)$.
  \[ \begin{align*} e \in \text{Supp}(\alpha_1) \Rightarrow & \begin{cases} \alpha_2(e) \leq \gamma'(e) \\ \alpha_2(e) = \gamma_2(e) \end{cases} \\ \Rightarrow & \gamma_2(e) \leq \gamma'(e) \Rightarrow \gamma'(e) - \gamma_2(e) \geq 0 \end{align*} \]
  In this case, for every $t > 0$ we have $\gamma_1(e) + t(\gamma'(e) - \gamma_2(e)) \geq 0$ and for every $e \in \text{Supp}(\alpha_1)$ we have $\alpha_1(e) \leq \gamma_1(e) + t(\gamma'(e) - \gamma_2(e))$.

- **2.** $e \in \text{Supp}(\gamma_2) \setminus \text{Supp}(\gamma_1)$.
  \[ \begin{align*} e \in \text{Supp}(\gamma_2) \setminus \text{Supp}(\gamma_1) \Rightarrow & \begin{cases} \alpha_2(e) \leq \gamma'(e) \\ \alpha_2(e) = \gamma_2(e) \end{cases} \\ \Rightarrow & \gamma_2(e) \leq \gamma'(e) \Rightarrow \gamma'(e) - \gamma_2(e) \geq 0 \end{align*} \]
  In this case, for every $t > 0$ we have $\gamma_1(e) + t(\gamma'(e) - \gamma_2(e)) \geq 0$ and for every $e \in \text{Supp}(\gamma_2) \setminus \text{Supp}(\gamma_1)$ we have $\alpha_1(e) \leq \gamma_1(e) + t(\gamma'(e) - \gamma_2(e))$.
• 3. \( e \in \text{Supp}(\gamma_1) \setminus \text{Supp}(\alpha_1) \).

In this case, \( \gamma_1(e) > 0 \). If \( \gamma'(e) - \gamma_2(e) \geq 0 \), then \( \gamma_1(e) + t(\gamma'(e) - \gamma_2(e)) \geq 0 \). If \( \gamma'(e) - \gamma_2(e) < 0 \), then by Archimedean property of numbers, there exists \( t \) such that \( \gamma_1(e) > t(\gamma_2(e) - \gamma'(e)) \). Thus, \( \gamma_1(e) + t(\gamma'(e) - \gamma_2(e)) > 0 \).

Since \( e \notin \text{Supp}(\alpha_1) \), we have \( \alpha_1(e) = 0 \). Therefore, \( \alpha_1(e) \leq \gamma_1(e) + t(\gamma'(e) - \gamma_2(e)) \).

Thus, \( \alpha_1 \uparrow (\gamma_1 + t(\gamma' - \gamma_2)) \).

Next, we show that \( |\sum_{e \in E} \alpha_1(e) - \sum_{e \in E} \alpha_2(e)| < \varepsilon \). Let \( \delta = \varepsilon, S_1 = \text{Supp}(\alpha_1) \), \( S_2 = \text{Supp}(\gamma_2) \setminus \text{Supp}(\gamma_1) \), and \( S_3 = \text{Supp}(\gamma_1) \setminus \text{Supp}(\alpha_1) \). Then,

\[
|\sum_{e \in S_1} \alpha_1(e) - \sum_{e \in S_2} \alpha_2(e)| = |\sum_{e \in S_1} \alpha_1(e) + \sum_{e \in E \setminus S_1} \alpha_1(e) - \sum_{e \in S_2} \alpha_2(e) - \sum_{e \in E \setminus S_2} \alpha_2(e)|
\]

\[
= |\sum_{e \in S_1} \gamma_1(e) - \sum_{e \in S_2} \gamma_2(e) + \sum_{e \in S_1} \gamma_1(e) - \sum_{e \in S_2} \gamma_2(e) + \sum_{e \in S_1} \gamma_1(e) - \sum_{e \in S_2} \gamma_2(e) - \sum_{e \in S_3} \gamma_2(e)|
\]

\[
\leq |\sum_{e \in S_1} \gamma_1(e) - \sum_{e \in S_2} \gamma_2(e)| + |\sum_{e \in S_1} \gamma_1(e) - \sum_{e \in S_2} \gamma_2(e)| + |\sum_{e \in S_1} \gamma_1(e) - \sum_{e \in S_2} \gamma_2(e)|
\]

\[
= \sum_{e \in E} |\gamma_1(e) - \gamma_2(e)| < \varepsilon
\]

Finally, we show that \( \gamma_1 + t(\gamma' - \gamma_2) \) is a fractional perfect matching,

\[
\sum_{e \in E} (\gamma_1(e) + t(\gamma'(e) - \gamma_2(e))) = \sum_{e \in E} \gamma_1(e) + t(\sum_{e \in E} \gamma'(e) - \sum_{e \in E} \gamma_2(e))
\]

\[
= 1 + t(1 - 1) = 1
\]

On the other hand, we showed that for every edge \( e \), \( \gamma_1 + t(\gamma' - \gamma) \geq 0 \). This completes the proof.

\[\square\]

Proof of Theorem 27. It is enough to show that

\[\forall \varepsilon > 0 \exists \delta > 0, d(\gamma_1, \gamma_2) < \delta \Rightarrow |f_f(\gamma_1) - f_f(\gamma_2)| < \varepsilon \]

Let \( \alpha_1 \uparrow \gamma_1 \) and \( \alpha_2 \uparrow \gamma_2 \). Then,

\[|f_f(\gamma_1) - f_f(\gamma_2)| = |\sum_{e \in E} \alpha_1(e) - \sum_{e \in E} \alpha_2(e)|\]

Suppose that \( \sum_{e \in E} \alpha_1(e) \geq \sum_{e \in E} \alpha_2(e) \). Then, Lemma 28 implies that there exists fractional matching \( \theta \) such that \( \theta \uparrow \gamma_1 \) and \( |\sum_{e \in E} \theta(e) - \sum_{e \in E} \alpha_2(e)| < \varepsilon \). Since \( \alpha_1 \) is minimal, thus \( \sum_{e \in E} \alpha_1(e) \leq \sum_{e \in E} \theta(e) \). Therefore,

\[|\sum_{e \in E} \alpha_1(e) - \sum_{e \in E} \alpha_2(e)| \leq |\sum_{e \in E} \theta(e) - \sum_{e \in E} \alpha_2(e)| < \varepsilon\]
In the case \( \sum_{e \in E} \alpha_1(e) \leq \sum_{e \in E} \alpha_2(e) \) the proof is similar.

\[ \square \]

**Corollary 29.** \( \text{Spec}_f(G) = [f_f(G), F_f(G)] \).

**Proof.** Since \( f_f \) is a continuous map on the convex-hence connected-domain \( \mathcal{P}_f(G) \), the range of \( f_f \) must also be a connected subset of \( \mathbb{R} \).

Now, we are ready to prove the main property of \( f_f \) in the following theorem.

**Theorem 30.** For every graph \( G \), \( f_f \) is a concave function on \( \mathcal{P}_f(G) \).

**Proof.** First, notice that \( f_f \) is defined over the convex set \( \mathcal{P}_f(G) \). Thus, we must show that for every \( \gamma_1, \gamma_2 \in \mathcal{P}_f(G) \) and every \( \lambda \in (0, 1) \) we have

\[ f_f(\lambda \gamma_1 + (1 - \lambda) \gamma_2) \geq \lambda f_f(\gamma_1) + (1 - \lambda) f_f(\gamma_2). \]

Let \( \gamma := \lambda \gamma_1 + (1 - \lambda) \gamma_2 \) and \( \alpha \) be a minimum forcing function for \( \gamma \). Using Lemma 17, there exists fractional matchings \( \alpha_i, i \in \{1, 2\} \) such that \( \alpha = \lambda \alpha_1 + (1 - \lambda) \alpha_2 \) and \( \alpha \uparrow \gamma \). Therefore, \( f_f(\gamma_i) \leq |\alpha_i| \). On the other hand, we have

\[ f_f(\gamma) = |\alpha| = |\lambda \alpha_1 + (1 - \lambda) \alpha_2| = \lambda |\alpha_1| + (1 - \lambda) |\alpha_2| \]

\[ \geq \lambda f_f(\gamma_1) + (1 - \lambda) f_f(\gamma_2). \]

\[ \square \]

**Corollary 31.** Let \( G \) be a graph. Then, \( f_f(G) \) is half-integer.

**Proof.** Let \( \gamma \) be a minimizer at which \( f_f(G) \) takes its minimum. Let \( \alpha \) be a minimum forcing function for \( \gamma \). Clearly \( f_f(\gamma) = |\alpha| \). Since \( f_f \) is a concave function on the polytope \( \mathcal{P}_f(G) \), \( \gamma \) must be a vertex of \( \mathcal{P}_f(G) \). Since \( \alpha \uparrow \gamma \) by Lemma 12, \( \alpha(e) \in \{0, \gamma(e)\} \). Since \( \gamma \) is a vertex of \( \mathcal{P}_f(G) \), by Proposition 6, \( \gamma(e) \) is a half-integer number. Thus, \( f_f(\gamma) = |\alpha| \) is also a summation of half integers, hence a half-integer number.

\[ \square \]

**4 Application**

The problem of finding \( F(Q_n) \) is an open problem. In [Rid02], a lower bound for this quantity is found. However, to the best of our knowledge, no trivial upper bound for it is known. In this section, we provide a technique to find a non-trivial upper bound for \( F(Q_n) \) for arbitrary \( n \). This result is presented in the Lemma 33 below. We must also emphasize that this machinery is not specific to the hypercube graph \( Q_n \). In fact, it can be used to find the upper bound for \( F(G) \) for every vertex-transitive graph.

We start with some notation. Let \( G \) be a graph, \( \sigma \in \text{Aut}(G) \), \( \gamma \) be a fractional perfect matching in \( G \) and \( \gamma(\sigma(e)) = \gamma(\sigma(e)) \). Then, we have the following lemma.

**Lemma 32.** If \( \sigma \in \text{Aut}(G) \), then for every \( \gamma \in \mathcal{P}_f(G) \), \( f_f(G, \gamma) = f_f(G, \gamma\sigma) \).
Proof. Let $\alpha$ be a minimum forcing function for $\gamma$. First, we show that $\alpha_{\sigma} \uparrow \gamma_{\sigma}$. Suppose that $\alpha_{\sigma}$ is not a forcing function for $\gamma_{\sigma}$. Then, there exists a fractional perfect matching $\gamma'$ such that $\alpha_{\sigma} \preceq \gamma'$. Then,

$$\forall e \in E, \alpha_{\sigma}(e) \leq \gamma'(e) \Rightarrow \forall e' \in E, \alpha(\sigma^{-1}(e')) \leq \gamma'(\sigma^{-1}(e'))$$

$$\Rightarrow \forall e' \in E, \alpha(e') \leq \gamma'_{\sigma-1}(e')$$

$$\Rightarrow \alpha \leq \gamma'_{\sigma-1}$$

$$\Rightarrow \gamma'_{\sigma-1} = \gamma$$

$$\Rightarrow \gamma' = \gamma_{\sigma}$$

Thus, $\alpha_{\sigma} \uparrow \gamma_{\sigma}$. Now, we have

$$|\alpha| = \sum_{e \in E} \alpha(e) = \sum_{e \in E} \alpha(\sigma(e)) = \sum_{e \in E} \alpha_{\sigma}(e) = |\alpha_{\sigma}|$$

$$\Rightarrow f_f(G, \gamma) \leq f_f(G, \gamma_{\sigma})$$

$f_f(G, \gamma_{\sigma}) \leq f_f(G, \gamma)$ is true by symmetry.

Lemma 33. Let $G$ be a graph, $S \subseteq \text{Aut}(G)$, and $\gamma$ be a fractional perfect matching in $G$ with maximum fractional forcing number. Let

$$\gamma_0 = \frac{1}{|S|} \sum_{\sigma \in S} \gamma_{\sigma}$$

Then,

1. $\gamma_0$ is a fractional perfect matching.

2. $f_f(G, \gamma) = f_f(G, \gamma_0)$

Proof. 1. For every vertex $v \in V(G)$ we have

$$\sum_{e \in \delta v} \gamma_0(e) \overset{(1)}{=} \sum_{e \in \delta v} \frac{1}{|S|} \sum_{\sigma \in S} \gamma_{\sigma}(e)$$

$$\overset{(2)}{=} \frac{1}{|S|} \sum_{e \in \delta v} \sum_{\sigma \in S} \gamma(\sigma(e))$$

$$\overset{(3)}{=} \frac{1}{|S|} \sum_{\sigma \in S} \sum_{e \in \delta v} \gamma(\sigma(e))$$

$$\overset{(4)}{=} \frac{1}{|S|} \sum_{\sigma \in S} \sum_{e \in \delta v} \gamma(\sigma(e))$$

$$\overset{(5)}{=} \frac{1}{|S|} \sum_{\sigma \in S} 1 = 1$$

In the above chain of qualities, (1) is by definition of $\gamma_0$, (2) is according to definition of $\gamma_{\sigma}$, (3) is concluded from the displacement of summations, (4) is due to the fact that $\sigma \in \text{Aut}(G)$, and (5) is by definition of fractional perfect matching.
2. Let \( \alpha_0 \) be the minimum forcing function for \( \gamma_0 \). Since \( \gamma_0 = \frac{1}{|S|} \sum_{\sigma \in S} \gamma_\sigma \), by Lemma 17, for every \( \gamma \in S \) there exists a forcing function \( \alpha_\sigma \) for \( \gamma_\sigma \) such that

\[
\alpha_0 = \frac{1}{|S|} \sum_{\sigma \in S} \alpha_\sigma
\]

Thus,

\[
\alpha_\sigma \uparrow \gamma_\sigma \Rightarrow |\alpha_\sigma| \geq f_f(G, \gamma_\sigma) = f_f(G, \gamma) \\
\Rightarrow f_f(G, \gamma_0) = |\alpha_0| \geq f_f(G, \gamma)
\]

The equality in the first line is by Lemma 32.

Corollary 34. Let \( G \) be a vertex and edge-transitive graph, and \( v \in V(G) \). Then, the fractional perfect matching that assigns the value \( \frac{1}{\deg(e)} \) to all edges, has the maximum fractional forcing number.

Proof. Since \( f_f \) is a continuous function on the compact set \( \mathcal{P}_f(G) \), therefore it achieves its maximum. Let \( \gamma \in \mathcal{P}_f(G) \) be a point in which \( f_f \) is maximized. Define \( \gamma_0 := \sum_{\sigma \in \text{Aut}(G)} \gamma_\sigma \). By Lemma 33, \( f_f(G, \gamma_0) = f_f(G, \gamma) \). Thus, \( \gamma_0 \) is also a maximizer for \( f_f \). We claim that \( \gamma_0(e) = \gamma_0(e') \) for every \( e, e' \in E(G) \). This implies that \( \gamma_0 \) is the constant function that takes the value \( \frac{1}{\deg(e)} \) on its domain.

Let \( e, e' \) be two arbitrary edges of \( G \). Then,

\[
\gamma_0(e) = \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} \gamma_\sigma(e) \overset{(\dagger)}{=} \frac{1}{|\text{Aut}(G)|} \sum_{\sigma \in \text{Aut}(G)} \gamma_\sigma(e') = \gamma_0(e')
\]

In the above, equality (1) is a direct consequence of the fact that \( G \) is edge-transitive, and the fact that \( \gamma_\sigma(e) = \gamma(\sigma(e)) \).

Theorem 35. \( F_f(Q_n) \leq \frac{n^{2n-1-5\times2^{n-3}}}{n} \).

Proof. To prove the theorem, we first consider the fractional perfect matching \( \gamma \) which assigns \( \frac{1}{2} \) to every edge of \( Q_n \). Since \( Q_n \) is an edge and vertex-transitive graph, Corollary 34 guarantees that \( \gamma \) has the maximum forcing number. Thus, it is sufficient to show that \( f_f(Q_n, \gamma) \leq \frac{n^{2n-1-5\times2^{n-3}}}{n} \).

To this end, we take advantage of Theorem 35 for the defined \( \gamma \). Clearly, \( \text{Supp}(\gamma) = E(Q_n) \). Therefore, if we find a subset \( B_n \) of the edges in such a way that for every cycle \( C \) of \( Q_n \) and any proper 2-edge-coloring of \( C \), every color class intersects \( B_n \), then Theorem 35 will provide an upper bound on \( f_f(Q_n, \gamma) \). In the rest of the proof, we will introduce one such subset \( B_n \), inductively. From now on, we call the elements of \( B_n \), blue edges, and the elements in \( E(Q_n) \setminus B_n \), red edges.

The main idea is as follows. First, we define \( B_4 \) or equivalently, we define the blue edges in \( Q_4 \). Then, having the blue edges of \( Q_{n-1} \), we define the blue edges of \( Q_n \) (i.e. \( B_n \)). Consider the following subset of edges in \( Q_4 \)

\[
A = \{ \{0101, 0100\}, \{0100, 0110\}, \{0110, 0010\}, \{0010, 0011\}, \{0011, 0001\}, \{0001, 0101\}, \{0101, 0111\}, \{1101, 1111\}, \{1111, 1110\}, \{1110, 1010\}, \{1010, 1000\}, \{1000, 1001\}, \{1001, 1110\}, \{1000, 1100\} \}
\]
Let $B_4 := A \cup \{0a_1a_2a_3, 1a_1a_2a_3 : a_1, a_2, a_3 \in \{0, 1\}\}$ (See Figure 2).

Given the subset $B_{n-1}$ of $E(Q_{n-1})$ for $n \geq 5$, we define the subset $B_n$ of $E(Q_n)$ as follows. Define the function $g_n : \{0, 1\}^n \rightarrow \{0, 1\}^{n-1}$ as follows

$$g_n(a_1, \ldots, a_n) = \begin{cases} (a_2, \ldots, a_n) & a_1 = 0 \\ (1 - a_2, a_3, \ldots, a_n) & a_1 = 1 \end{cases}$$

It is straightforward to see that $g_n$ is a graph homomorphism from $Q_n$ to $Q_{n-1}$.

For every $n \geq 5$, define $B_n := g_n^{-1}(B_{n-1})$, where $g_n^{-1}(B_{n-1}) = \{\{v, w\} \in E(Q_n) : \{g_n(v), g_n(w)\} \in B_{n-1}\}$.

The object is to prove that each color class of any 2-edge-coloring of the edges of any cycle intersects $B_n$. To this end, first, consider a 2 edge coloring of $C$.

We will prove this claim by induction on $n$. First, suppose that the claim is valid for $n = 4$. For a cycle $C = (v_1, v_2, \ldots, v_{2k})$ in a graph, define the parity of the edge $\{v_i, v_{i+1}\}$, with respect to the given labelling as the parity of $i$. Notice that the parity of an edge of $C$ depends on the starting vertex in the ordering of its vertices. However, if two edges of $C$ have the same parity with respect to one ordering, they have the same parity with respect to any other ordering.

Now, Consider the following graph $Q_4$.

Note that the narrow edges represent the blue edges, and the bold edges represent the remaining edges of $Q_4$ (i.e. the red edges).
Let $C$ be an arbitrary cycle in $Q_4$. We show that $C$ contains two blue edges with different parity. All of the edges incident with the vertices 3 and 8 are blue. Therefore, if $C$ passes through either of these two vertices, let’s say vertex 3, then one of the edges of $C$ incident to the vertex 3 has odd parity, and the other one has even parity. Therefore, $C$ can not pass through 3 and 8. If $C$ passes through either of the vertices 0 or 11, let’s say vertex 0, then it contains two adjacent red edges. Otherwise, $C$ must use the edge that is incident to the vertex 8, which can not happen as we argued above. Thus, $C$ contains two adjacent red edges incident to the vertex 0, let say $\{0, 1\}$ and $\{0, 4\}$. The edge in $C$ before $\{0, 1\}$, and the edge in $C$ after $\{0, 4\}$ are both blue with different parities. Therefore, $C$ can not also pass through 0 and 11.

Next, we prove that $C$ can not passes through either of 1, 2, 9 or 10. On the contrary, let’s assume that $C$ passes through vertex 1. Then, $C$ must use two edges $\{1, 5\}$ and $\{1, 9\}$. Both of these two edges are blue with different parity. So far, we have shown that $C$ can not pass through $0, 1, 2, 3, 8, 9, 10, 11$. It is also easy to see that, if $C$ uses any vertex from the set $\{4, 5, 6, 7, 12, 13, 14, 15\}$, then it contains two blue edges with different parity.

Now, suppose that the claim is valid for $n - 1$. Let $C = v_1, \ldots, v_m, v_1$ be an arbitrary cycle in $Q_n$. Consider $g_n(C) = g_n(v_1), \ldots, g_n(v_m), g_n(v_1)$. By Proposition 1, $g_n(C)$ is a close walk. First, suppose that this walk contains no cycles, therefore, it is a union of some close walk of length 2. This case is only happen when two edges of $C$ are of the form $e_1 = \{(1, 0, a_3, \ldots, a_n)\}$ and $e_2 = \{(0, 0, a_3, \ldots, a_n)\}$, or of the form $e_3 = \{(1, 1, a_3, \ldots, a_n)\}$ and $e_4 = \{(0, 1, a_3, \ldots, a_n)\}$. Both of these two edges are blue according to the coloring. Since, these two edges are adjacent in $C$, every 2-edge-coloring of $C$ intersects one of them.

Now, suppose that this walk contains at least one cycle $C'$. Since $g_n$ is a graph homomorphism, every 2-edge-coloring of $C$ induces a 2-edge-coloring of $g_n(C)$ and in particular a 2-edge-coloring for the cycle $C'$ in $Q_{n-1}$.

By the induction hypothesis, we know that $B_{n-1}$ intersects each color class of the induced coloring of $C'$. Since $g_n^{-1}(B_{n-1}) \subseteq B_n$, we can conclude that $B_n$ intersects both color classes of the coloring of $C$. This completes the induction.

\begin{corollary}
$F(Q_n) \leq \lfloor \frac{n^{2n-1} - 5 \times 2^{n-3}}{n} \rfloor$.
\end{corollary}

\begin{proof}
This corollary is a direct consequence of Observation 26, and Theorem 35.
\end{proof}

This new upper bound for the maximum forcing number of $Q_n$ is close to the lower bound in Proposition 5. However, the problem of finding the exact value of $F(Q_n)$ remained open.

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