ONE-SIDED CURVATURE ESTIMATES FOR H-DISKS

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ABSTRACT. In this paper we prove an extrinsic one-sided curvature estimate for disks embedded in $\mathbb{R}^3$ with constant mean curvature which is independent of the value of the constant mean curvature. We apply this extrinsic one-sided curvature estimate in [23] to prove a weak chord arc type result for these disks. In Section 4 we apply this weak chord arc result to obtain an intrinsic version of the one-sided curvature estimate for disks embedded in $\mathbb{R}^3$ with constant mean curvature. In a natural sense, these one-sided curvature estimates generalize respectively, the extrinsic and intrinsic one-sided curvature estimates for minimal disks embedded in $\mathbb{R}^3$ given by Colding and Minicozzi in Theorem 0.2 of [9] and in Corollary 0.8 of [10].

1. INTRODUCTION

In this paper we prove a one-sided curvature estimate for disks embedded in $\mathbb{R}^3$ with constant mean curvature. An important feature of this estimate is its independence on the value of the constant mean curvature. Throughout the paper an orientable surface $\Sigma$ embedded in $\mathbb{R}^3$ with constant mean curvature $H$ will be referred to as an $H$-surface; when $\Sigma$ is a compact disk, it will be called an $H$-disk. By appropriately orienting $\Sigma$ we will always assume that $H$ is non-negative.

The main result of this paper, which is Theorem 1.1 below, states that if $D$ is an $H$-disk which lies on one side of a plane $\Pi$, then the norm of the second fundamental form of $D$ cannot be arbitrarily large at points sufficiently far from the boundary of $D$ and sufficiently close to $\Pi$. This estimate provides a natural generalization of the Colding-Minicozzi One-sided Curvature Estimate for minimal disks embedded in $\mathbb{R}^3$, which is given in Theorem 0.2 in [9].

In the next theorem, $B(R)$ denotes the open ball in $\mathbb{R}^3$ centered at the origin of radius $R$ and for a point $p$ on a surface $\Sigma \subset \mathbb{R}^3$, $|A_\Sigma|(p)$ denotes the norm of the second fundamental form of $\Sigma$ at $p$. Note that when $\Sigma$ is an $H$-surface, by the Gauss equation, estimating the norm of the second fundamental form of $\Sigma$ is equivalent to estimating the Gaussian curvature of $\Sigma$. For this reason, we refer to the estimates for $|A_\Sigma|$ as curvature estimates.

Theorem 1.1 (One-sided curvature estimate for $H$-disks). There exist $\varepsilon \in (0, \frac{1}{2})$ and $C \geq 2\sqrt{2}$ such that for any $R > 0$, the following holds. Let $D$ be an $H$-disk such that $D \cap B(R) \cap \{x_3 = 0\} = \emptyset$ and $\partial D \cap B(R) \cap \{x_3 > 0\} = \emptyset$.

Then:

$$\sup_{x \in D \cap B(\varepsilon R) \cap \{x_3 > 0\}} |A_D|(x) \leq \frac{C}{R}.$$  

In particular, if $D \cap B(\varepsilon R) \cap \{x_3 > 0\} \neq \emptyset$, then $H \leq \frac{C}{R}$.  

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The one-sided curvature estimate in Theorem 1.1 depends upon and is inspired by the one-sided curvature estimate for 0-disks of Colding-Minicozzi in [9]. In contrast to the minimal case, the constant $C$ in equation (1) need not improve with smaller choices of $\varepsilon$. To see this, let $S$ be the sphere of radius $\frac{1}{2}$ centered at $(0, 0, \frac{1}{2})$. Each surface in the sequence $E_n = (S + (0, 0, \frac{1}{n})) \cap B(1)$ is a compact disk that satisfies the hypotheses of the theorem for $R = 1$, has $|A_{E_n}| = 2\sqrt{2}$ and, as $n$ tends to infinity, $E_n$ moves arbitrarily close to the origin. In particular these examples show that the constant $C$ in the above theorem must be at least $2\sqrt{2}$ no matter how small $\varepsilon$ is.

Theorem 1.1 plays an important role in deriving a weak chord arc property for $H$-disks in [23, 25], which is described in Section 4. This weak chord arc property was inspired by and gives a generalization of Proposition 1.1 in [10] for 0-disks to the case of $H$-disks; we next apply this chord arc property to obtain an intrinsic version of the one-sided curvature estimate in Theorem 1.1, which we describe in Theorem 4.5. In the case $H = 0$, the proof of this intrinsic one-sided curvature estimate follows from Corollary 0.8 in [10].

The paper is organized as follows. In Section 2 we cover notation, definitions, concepts and statements of results from some of our other papers that we will use in later sections. In Section 3 we prove Theorem 1.1. In Section 4 we present some useful consequences of Theorem 1.1, including an explanation of how the intrinsic one-sided curvature estimate given in Theorem 4.5 follows from Theorem 1.1 and the results in [23].

2. Preliminaries.

Throughout this paper, we use the following notation. Given $a, b, R > 0, p \in \mathbb{R}^3$ and $\Sigma$ a surface in $\mathbb{R}^3$:

- $B(p, R)$ is the open ball of radius $R$ centered at $p$.
- $\overline{B}(R) = B(\vec{0}, R)$, where $\vec{0} = (0, 0, 0)$.
- For $p \in \Sigma$, $B_{\Sigma}(p, R)$ denotes the closed intrinsic ball in $\Sigma$ of radius $R$.
- $C(a, b) = \{(x_1, x_2, x_3) \mid x_1^2 + x_2^2 \leq a^2, |x_3| \leq b\}$.
- $A(r_1, r_2) = \{(x_1, x_2, 0) \mid r_1^2 \leq x_1^2 + x_2^2 \leq r_2^2\}$.

We next recall several results from [24] which will be used in this paper. We first introduce the notion of multi-valued graph, see [7] for further discussion and Figure 1. Intuitively, an $N$-valued graph is a simply-connected embedded surface covering
an annulus such that over a neighborhood of each point of the annulus, the surface consists of \( N \) graphs. The stereotypical infinite multi-valued graph is half of the helicoid, i.e., half of an infinite double-spiral staircase.

**Definition 2.1** (Multi-valued graph). Let \( \mathcal{P} \) denote the universal cover of the punctured \((x_1, x_2)\)-plane, \( \{ (x_1, x_2) \mid (x_1, x_2) \neq (0, 0) \} \), with global coordinates \((\rho, \theta)\).

1. An \( N \)-valued graph over the annulus \( A(r_1, r_2) \) is a single valued graph \( u(\rho, \theta) \) over \( \{ (\rho, \theta) \mid r_2 \leq \rho \leq r_1, \ |\theta| \leq \pi \} \subset \mathcal{P} \), if \( N \) is odd, or over \( \{ (\rho, \theta) \mid r_2 \leq \rho \leq r_1, \ (-N + 1)\pi \leq \theta \leq \pi(N + 1) \} \subset \mathcal{P} \), if \( N \) is even.

2. An \( N \)-valued graph \( u(\rho, \theta) \) over the annulus \( A(r_1, r_2) \) is called right-handed [lefthanded] if whenever it makes sense, \( u(\rho, \theta) < u(\rho, \theta + 2\pi) \) \([u(\rho, \theta) > u(\rho, \theta + 2\pi)]\).

3. The set \( \{ (r_2, \theta, u(r_2, \theta)), \theta \in [-N\pi, N\pi] \} \) when \( N \) is odd (or \( \{ (r_2, \theta, u(r_2, \theta)), \theta \in [(-N + 1)\pi, (N + 1)\pi] \} \) when \( N \) is even) is the inner boundary of the \( N \)-valued graph.

From Theorem 2.24 in [24] one obtains the following, accurate geometric description of an \( H \)-disk with large norm of the second fundamental form at the origin. The precise meaning of certain statements below are made clear in [24] and we refer the reader to that paper for further details.

**Theorem 2.2.** Given \( \varepsilon, \tau > 0 \) there exist constants \( \Omega_x := \Omega(\varepsilon, \tau), \omega_x := \omega(\varepsilon, \tau) \) and \( G_x := G(\varepsilon, \tau) \) such that if \( M \) is an \( H \)-disk, \( H \in \left(0, \frac{1}{\tau}\right)\), \( \partial M \subset \partial B(\varepsilon), \vec{0} \in M \) and \( |A_M|/\vec{0} > G_x \) then the following geometric description of \( M \) holds:

On the scale of the norm of the second fundamental form \( M \) looks like one or two helicoids nearby the origin and, after a rotation that turns these helicoids into vertical helicoids, \( M \) contains a 3-valued graph \( u \) over \( A(\varepsilon/\Omega_x, \frac{\varepsilon}{A_M(0)}) \) with norm of the gradient less than \( \tau \) and the inner boundary of the 3-valued graph lies in \( B(10\frac{\varepsilon}{A_M(0)}) \).

Theorem 2.2 was inspired by the pioneering work of Colding and Minicozzi in the minimal case [6, 7, 8, 9]; however in the constant positive mean curvature setting this description has led to a remarkable and different conclusion, that is the existence of curvature and radius estimates stated below.

**Theorem 2.3** (Extrinsic Radius Estimates, Theorem 3.5 in [24]). There exists an \( R \geq \pi \) such that for any \( H \)-disk \( D \),

\[
\sup_{p \in D} \{d_{\mathbb{R}^3}(p, \partial D)\} \leq \frac{R}{H}.
\]

**Theorem 2.4** (Extrinsic Curvature Estimates, Theorem 3.6 in [24]). Given \( \delta, \mathcal{H} > 0 \), there exists a constant \( K_0(\delta, \mathcal{H}) \) such that for any \( H \)-disk \( D \) with \( H \geq \mathcal{H} \),

\[
\sup_{p \in D} \{d_{\mathbb{R}^3}(p, \partial D) \geq \delta\} |A_D| \leq K_0(\delta, \mathcal{H})
\]

Indeed since the plane and the helicoid are complete simply-connected minimal surfaces properly embedded in \( \mathbb{R}^3 \), a radius estimate does not hold in the minimal case. Moreover rescalings of a helicoid give a sequence of embedded minimal disks with arbitrarily large norm of the second fundamental form at points arbitrarily far from its boundary; therefore in the minimal setting, the curvature estimates do not hold either.

Next, we recall the notion of the CMC flux or flux of an \( H \)-surface; see for instance [14, 15, 28] for further discussions of this invariant.

**Definition 2.5.** Let \( \gamma \) be a 1-cycle in an \( H \)-surface \( M \). The CMC flux or flux of \( \gamma \) is \( \int_{\gamma} (H\gamma + \xi) \times \dot{\gamma} \), where \( \xi \) is the unit normal to \( M \) along \( \gamma \).

The flux is a homological invariant and we say that \( M \) has zero flux if the flux of any 1-cycle in \( M \) is zero; in particular, since the first homology group of a disk is zero, an \( H \)-disk has zero flux.

Finally, we also recall the following definition.
Definition 2.6. Let $U$ be an open set in $\mathbb{R}^3$. We say that a sequence of surfaces $\Sigma(n) \subset U$, has locally bounded norm of the second fundamental form in $U$ if for every compact subset $B$ in $U$, the norms of the second fundamental forms of the surfaces $\Sigma(n) \cap B$ are uniformly bounded.

3. THE PROOF OF THEOREM 1.1.

Proof of Theorem 1.1. After rescaling, it suffices to prove Theorem 1.1 for $H$-disks where the radius of the related balls is $R = 1$.

Arguing by contradiction, suppose that Theorem 1.1 fails. Then there exists a sequence of $H_n$-disks $E(n)$ and numbers $\varepsilon_n \to 0$, such that $E(n) \cap B(\varepsilon_n) \cap \{x_3 > 0\}$ contains points $\tilde{p}_n$ with $\varepsilon_n |A_{E(n)}|(|\tilde{p}_n|) \to \infty$ as $n \to \infty$. Since we may assume that $\partial B(1)$ is transverse to $E(n)$, then after replacing $E(n)$ by a sub-disk containing $\tilde{p}_n$, we may also assume that $\partial E(n) \subset \partial B(1) \cap \{x_3 > 0\}$; note that it might be the case that $E(n)$ is not contained in $B(1)$ or in $\{x_3 > 0\}$ when $H > 0$ because the convex hull property need not hold.

By the extrinsic curvature estimates for $H$-disks with $H > 0$ given in Theorem 2.4, the mean curvatures $H_n$ of the disks $E(n)$ are tending to zero. Also, note that for $n$ large, there exist points $p_k(n) \in E(n)$ with vertical tangent planes and with intrinsic distances $d_{E(n)}(p_k(n), \tilde{p}_n) \to 0$; otherwise small but fixed sized intrinsic balls centered at the points $\tilde{p}_n$ would be stable (the inner product of the unit normal to $M$ with the vector $(0, 0, 1)$ is a nonzero Jacobi function), thereby having uniform curvature estimates (see for instance [27] for these estimates) and contradicting our choices of the points $\tilde{p}_n$ with their norms of the second fundamental form becoming arbitrarily large.

To obtain a contradiction, we are going to analyze the behavior of the connected set of points $\alpha_n$ in $E(n)$ where the tangent plane is vertical and that contains $p_k(n)$; we will prove that, for $n$ large, $\alpha_n$ is a curve that moves downward at a much faster rate than it moves sideways and that it must cross the $(x_1, x_2)$-plane near the origin. The existence of such a curve $\alpha_n \subset E(n)$ will then contradict the fact $E(n)$ is disjoint from the $(x_1, x_2)$-plane.

The next proposition describes the geometry of $E(n)$ around points which are close to the $(x_1, x_2)$-plane and where the tangent plane is vertical. The proposition states that at such points the surface must look like a vertical helicoid. This proof relies heavily on Theorem 2.2 and on the uniqueness of the helicoid by Meeks and Rosenberg [21], see also [1].

Proposition 3.1. Consider a sequence of points $q_n \in E(n) \cap C(\frac{1}{2}, \frac{1}{2}) \cap \{x_3 > 0\}$ with $x_3(q_n)$ converging to zero where the tangent planes $T_{q_n}E(n)$ to $E(n)$ are vertical. Then the numbers $\lambda_n := |A_{E(n)}(q_n)|$ diverge to infinity and a subsequence of the surfaces $M(n) = \lambda_n(E(n) - q_n)$ converges on compact subsets of $\mathbb{R}^3$ to a vertical helicoid $\mathcal{H}$ containing the $x_3$-axis and with maximal absolute Gaussian curvature $\frac{1}{2}$ at the origin. Furthermore, the multiplicity of the convergence of the surfaces $M(n)$ to $\mathcal{H}$ is one or two.

Proof. The surface $E(n)$ is locally graphical around $q_n$ over its tangent plane $T_{q_n}E(n)$. Recall that each tangent plane at $q_n$ is vertical, the sequence of positive numbers $x_3(q_n)$ is converging to zero and $E(n)$ lies above the $(x_1, x_2)$-plane near $q_n$. Then $B_{E(n)}(q_n, 2x_3(q_n))$ cannot be a graph of gradient less than or equal to 1 over its orthogonal projection to $T_{q_n}E(n)$.

Let $r(n) \in (0, 2x_3(q_n))$ be the largest number such that $B_{E(n)}(q_n, r(n))$ is a graph of gradient at most 1 over its projection to $T_{q_n}E(n)$. By the previous discussion,

$$\lim_{n \to \infty} r(n) = 0.$$  

Consider the sequence of translated and scaled surfaces

$$\Sigma(n) = \frac{1}{r(n)}(E(n) - q_n).$$

We claim that it suffices to prove that a subsequence of the $\Sigma(n)$ converges with multiplicity one or two to a vertical helicoid containing the $x_3$-axis. Arguing by contradiction,
suppose that a subsequence \( \Sigma(n_i) \) of these surfaces converges with multiplicity one or two to a vertical helicoid \( \mathcal{H}' \) containing the \( x_3 \)-axis, then \( \lambda := |A_{\mathcal{H}'}(\mathbf{0})| \in (0, \infty) \) and
\[
\lambda = \lim_{i \to \infty} |A_{\Sigma(n_i)}(\mathbf{0})| = \lim_{i \to \infty} r(n_i)|A_{E(n_i)}(q_{n_i})| = \lim_{i \to \infty} r(n_i)\lambda_{n_i}.
\]
Since \( r(n_i) \) is going to zero, this implies that the numbers \( \lambda_{n_i} \) must diverge to infinity, the subsequence of surfaces
\[
M(n_i) = \lambda_{n_i}(E(n_i) - q_{n_i}) = \lambda_{n_i}r(n_i)\Sigma(n_i)
\]
converges with multiplicity one or two to \( \mathcal{H} = \lambda\mathcal{H}' \), and the proposition follows. Thus, it suffices to prove that a subsequence of the \( \Sigma(n) \) converges with multiplicity one or two to a vertical helicoid containing the \( x_3 \)-axis.

There are two cases to consider.

**Case A:** The sequence of surfaces \( \Sigma(n) \) has locally bounded norm of the second fundamental form in \( \mathbb{R}^3 \).

**Case B:** The sequence of surfaces \( \Sigma(n) \) does not have locally bounded norm of the second fundamental form in \( \mathbb{R}^3 \).

Suppose that Case A holds. In this case a standard compactness argument gives that a subsequence of the \( \Sigma(n) \) converges \( C^0 \) to a minimal lamination of \( \mathbb{R}^3 \); see for example [2, 9, 21, 29] for these standard arguments when the surfaces \( \Sigma(n) \) are minimal surfaces. After possibly replacing by a subsequence, we will assume that the original sequence \( \Sigma(n) \) converges to a minimal lamination \( \mathcal{L} \) of \( \mathbb{R}^3 \).

Let \( L \) be the leaf of \( \mathcal{L} \) which passes through the origin. Note that the leaf \( L \) has a vertical tangent plane at the origin. Since \( L \) is not a plane, because it is not a graph over this vertical tangent plane with gradient less than 1, then \( L \) is a non-flat leaf of \( \mathcal{L} \). Note that \( L \) also has bounded norm of the second fundamental form in compact subsets of \( \mathbb{R}^3 \). By Theorem 1.6 in [21], \( L \) is a nonlimit leaf of \( \mathcal{L} \) and one of the following must hold:

- \( L \) is properly embedded in \( \mathbb{R}^3 \).
- \( L \) is properly embedded in an open half-space of \( \mathbb{R}^3 \).
- \( L \) is properly embedded in an open slab of \( \mathbb{R}^3 \).

Since \( L \) is properly embedded in a simply-connected open set \( \mathcal{O} \) of \( \mathbb{R}^3 \), then it separates the open set and so it is orientable. Since \( L \) is nonflat and it is complete, then \( L \) is not stable [11, 12]. We claim that this instability of \( L \) implies that the multiplicity of convergence of domains on \( \Sigma(n) \) can be at most one or two. Otherwise, suppose that the multiplicity of convergence of portions of the surfaces \( \Sigma(n) \) to \( L \) is greater than two and let \( \Omega \subset L \) be a smooth compact unstable domain. A standard argument that we now sketch produces a contradiction to the existence of \( \Omega \). By separation properties, the uniform boundedness of the second fundamental forms of the surfaces \( \Sigma(n) \) in a small \( \epsilon \)-neighborhood of \( \Omega \) in \( \mathcal{O} \) and the properness of \( L \), we have that for \( n \) large there exist three pairwise disjoint compact domains \( \Omega_1(n), \Omega_2(n), \Omega_3(n) \) in \( \Sigma(n) \) that are converging to \( \Omega \) and that are normal graphs over \( \Omega \). Without loss of generality, we may assume that the unit normal vectors of \( \Omega_2(n) \) and of \( \Omega_3(n) \) at corresponding points of \( \Omega \) have positive inner products converging to 1 as \( n \to \infty \). Moreover, if we let
\[
f_2(n), f_3(n) : \Omega \to \mathbb{R}
\]
denote the related graphing functions, we can assume that \( f_2(n) - f_3(n) > 0 \). After renormalizing this difference as \( F(n) = \frac{f_2(n) - f_3(n)}{(f_2(n) - f_3(n))q} \) for some \( q \in \text{Int}(\Omega) \), elliptic PDE theory implies that a subsequence of the \( F(n) \) converges to a positive Jacobi function on \( \Omega \), which implies \( \Omega \) is stable. This contradiction implies that the multiplicity of convergence is one or two.

Since the multiplicity of convergence of portions of the \( \Sigma(n) \) to \( L \) is one or two, then for \( n \) large we can lift any simple closed curve \( \gamma \) on \( L \) to one or two pairwise disjoint normal
graphs over $\gamma$ and contained in $\Sigma(n)$. This gives either one or two simple closed lifted curves in $\Sigma(n)$, the number of such lifts depending on the multiplicity of the convergence. Hence, since the domains $\Sigma(n)$ have genus zero, it follows that any pair of transversely intersecting simple closed curves on $L$ cannot intersect in exactly one point; therefore, $L$ also has genus zero and by the properness of finite genus leaves of a minimal lamination in $[19]$, $L$ must be properly embedded in $\mathbb{R}^3$. In fact, this discussion shows that all the leaves of $L$ are proper. Since the leaf $L$ is not flat, then the strong halfspace theorem in [13] implies that $L$ is the only leaf in $L$. If $L$ has more than one end, then by [3] it has non-zero flux and so, by the nature of the convergence, the domains $\Sigma(n)$ must have non-zero flux as well. However, this leads to a contradiction since flux is a homological invariant and thus $\Sigma(n)$, being topologically a disk, has zero flux. Therefore $L$ must have genus zero and one end, which implies that it is simply-connected. By [21], see also [1], $L$ is a helicoid and it remains to show that it is a vertical helicoid.

The fact that $L$ must be a vertical helicoid follows by using Theorem 2.2. If the axis of $L$ made an angle with the vertical axis, then since $x_3(q_n)$ is going to zero, choosing $\tau$ in Theorem 2.2 sufficiently small would give a sequence of 3-valued graphs in $E(n)$ intersecting the portion of the $\{x_3 = 0\}$-plane inside of $B(1)$. This is a contradiction of our hypotheses on $E(n)$. Thus $L$ is a vertical helicoid and since the tangent plane to $L$ at the origin is vertical, $L$ contains the $x_3$-axis. This finishes the proof of the proposition when Case A holds. To complete the proof of Proposition 3.1, it suffices to demonstrate the following assertion, that is the difficult point in the proof of Proposition 3.1.

**Assertion 3.2.** *Case B does not occur.*

**Proof.** Some of the techniques used to eliminate Case B are motivated by results in [5, 17, 18] and their corresponding proofs.

Arguing by contradiction, assume that the sequence of surfaces $\Sigma(n)$ does not have locally bounded norm of the second fundamental form in $\mathbb{R}^3$. For the clarity of the exposition, we will replace the sequence $\{\Sigma(n)\}_{n \in \mathbb{N}}$ by a specific subsequence such that for some non-empty closed set $\chi$ in $\mathbb{R}^3$, different from $\mathbb{R}^3$, the sequence of surfaces $\Sigma(n)$ have locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \chi$, converge $C^\alpha$ to a non-empty minimal lamination $L_\chi$ of $\mathbb{R}^3 - \chi$ and no further subsequence has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \chi'$, where $\chi'$ is a proper closed subset of $\chi$. This reduction is explained in the next claim.

**Claim 3.3.** *After replacing by a subsequence, the sequence of surfaces $\{\Sigma(n)\}_{n \in \mathbb{N}}$ satisfies the following properties:

1. There exists a closed non-empty set $\chi \subset \mathbb{R}^3$ such that for every point $s \in \chi$ and for each $k \in \mathbb{N}$, there exists an $N(s, k) \in \mathbb{N}$ such that for each $j \geq N(s, k)$, there exists a point $p(j) \in \Sigma(j) \cap B(s, \frac{1}{k})$ with $|A_{\Sigma(j)}(p(j))| \geq k$.
2. The sequence $\{\Sigma(n)\}_n$ has locally bounded norm of the second fundamental form in $\mathbb{R}^3 - \chi$.
3. The sequence $\{\Sigma(n)\}_n$ converges $C^\alpha$ on compact subsets of $\mathbb{R}^3 - \chi$ to a non-empty minimal lamination $L_\chi$ of $\mathbb{R}^3 - \chi$.
4. There exists a maximal open horizontal slab or open half-space $W$ in $\mathbb{R}^3 - \chi$, $\bar{0} \in W$ and $L = L_\chi \cap W$ is a minimal lamination of $W$.
5. The leaf $L$ of $L$ which passes through the origin is nonflat and contains an intrinsic open disk $\Omega$ passing through the origin which is the limit of the surfaces $B_{\Sigma(n)}(\bar{0}, 1)$ and $\Omega$ is a graph over its projection to $T_{\bar{0}} \Omega$ which is a vertical plane, and with norm of the gradient of the graphing function at most one.

**Proof of Claim 3.3.** We begin by constructing the set $\chi$ and the related subsequence of surfaces $\{\Sigma(n)\}_{n \in \mathbb{N}}$ described in the claim. The assumption that the original sequence of surfaces does not have locally bounded norm of the second fundamental form in $\mathbb{R}^3$ implies...
there exists a point \( q(1) \in \mathbb{R}^3 \) such that, after replacing this sequence of surfaces by a subsequence, \( \Gamma_1 := \{ \Sigma(1, n) \}_{n \in \mathbb{N}} \), it satisfies the following property: For each \( k \in \mathbb{N} \), there is a point \( p(1, k) \in \Sigma(1, k) \cap B(q(1), \frac{1}{2}) \) with \( |A_{\Sigma(1,k)}|(p(1, k)) \geq k \).

Let \( Q, Q^+ \) denote the set of rational numbers and the subset of positive rational numbers, respectively. Consider the countable collection of balls

\[
B = \{ B(x, q) \mid x \in Q^3, q \in Q^+ \},
\]

and let \( B = \{ B_1, B_2, \ldots, B_n, \ldots \} \) be an ordered listing of the elements in \( B \) where \( q(1) \in B_1 \). If \( \Gamma_1 \) has locally bounded norm of the second fundamental form in \( \mathbb{R}^3 - \{q(1)\} \), then \( \chi = \{q(1)\} \) and we stop our construction of the set \( \chi \). Assume now that \( \Gamma_1 \) does not have locally bounded norm of the second fundamental form in \( \mathbb{R}^3 - \{q(1)\} \). Let \( B_{n(2)} \) be the first indexed ball in the ordered listing of \( B - \{B_1\} \), such that the following happens: there is a point \( q(2) \in B_{n(2)} - \{q(1)\} \), a subsequence \( \Gamma_2 := \{ \Sigma(2, 1), \Sigma(2, 2), \ldots, \Sigma(2, k), \ldots \} \) of \( \Gamma_1 \) together with points \( p(2, k) \in \Sigma(2, k) \cap B(q(2), \frac{1}{2}) \) where \( |A_{\Sigma(2,k)}|(p(2, k)) \geq k \). Note that \( B_{n(2)} \) is just the first ball in the list \( B - \{B_1\} \) that contains a point \( q \) different from \( q(1) \) such that the norms of the second fundamental forms of the surfaces in the sequence \( \Gamma_1 \) are not bounded in any neighborhood of \( q \), and after choosing such a point we label it as \( q(2) \).

If \( \Gamma_2 \) does not have locally bounded norm of the second fundamental form in \( \mathbb{R}^3 - \{q(1), q(2)\} \), then there exists a first ball \( B_{n(3)} \) in the ordered list \( B - \{B_1, B_{n(2)}\} \) such that there is a point \( q(3) \in B_{n(3)} - \{q(1), q(2)\} \) and such that after replacing \( \Gamma_2 \) by a subsequence \( \Gamma_3 := \{ \Sigma(3, 1), \ldots, \Sigma(3, k), \ldots \} \), there are points \( p(3, k) \in \Sigma(3, k) \cap B(q(3), \frac{1}{2}) \) with \( |A_{\Sigma(3,k)}|(p(3, k)) \geq k \).

Define \( n(1) = 1 \). Then, continuing the above construction inductively, we obtain at the \( m \)-th stage a subsequence \( \Gamma_m \subset \Gamma_{m-1} \), \( \Gamma_m = \{ \Sigma(m, 1), \ldots, \Sigma(m, k), \ldots \} \), distinct points \( \{q(1), q(2), \ldots, q(m)\} \) and related balls \( \{B_{n(1)}, B_{n(2)}, \ldots, B_{n(m)}\} \in B \) such that for each \( i \in \{1, \ldots, m\} \), each \( \Sigma(m, k) \) contains points \( p(m, k, i) \) in the balls \( B(q(i), \frac{1}{2}) \), where the norm of the second fundamental form of \( \Sigma(m, k) \) is at least \( k \). Note that \( n(i+1) > n(i) \) for all \( i \).

Let \( \Gamma_\infty = \{ \Sigma(1, 1), \Sigma(2, 2), \ldots, \Sigma(n, n), \ldots \} \) be the related diagonal sequence and let \( \chi \) be the closure of the set \( \{q(n)\}_{n \in \mathbb{N}} \). By the definitions of \( \chi \) and of \( \Gamma_\infty \), for each point \( s \in \chi \) and for each \( k \in \mathbb{N} \), there exists \( N(s, k) \in \mathbb{N} \) such that for \( j \geq N(s, k) \), there are points \( p(j) \in \Sigma(j, j) \cap B(s, \frac{1}{2}) \) with \( |A_{\Sigma(j,j)}|(p(j)) \geq k \).

We claim that the sequence \( \{\Sigma(n, n)\}_{n \in \mathbb{N}} = \Gamma_\infty \) has locally bounded norm of the second fundamental form in \( \mathbb{R}^3 - \chi \). To prove this, it suffices to check that given a point \( y \in \mathbb{R}^3 - \chi \), there exists a closed ball \( B \) such that \( y \in \text{Int}(B) \) and \( \Gamma_\infty \) has uniformly bounded norm of the second fundamental form on \( B \). Choose an \( r \in (0, \frac{1}{4}d_{B^2}(y, \chi)) \cap Q^+ \) and let \( x \in \mathbb{Q}^3 \) be a point of distance less than \( r \) from \( y \). By the triangle inequality, one has \( B(x, 2r) \subset \mathbb{R}^3 - \chi \) and suppose \( B(x, 2r) = B_j \subset B \), for some \( J \in \mathbb{N} \). We claim that \( \Gamma_\infty \) has uniformly bounded norm of the second fundamental form on \( B(y, r) \subset B_j \). Otherwise, by compactness of \( B(y, r) \), there exists a point \( q \in B(y, r) \subset B_j \) such that the norm of the second fundamental forms of the surfaces in the sequence \( \Gamma_\infty \) are not bounded in any neighborhood of \( q \). By the inductive construction then \( \{q(1), \ldots, q(J)\} \subset B_j \neq \emptyset \); however, \( B_j \subset \mathbb{R}^3 - \chi \) and \( \{q(1), \ldots, q(J)\} \subset \chi \). This contradiction implies that \( \Gamma_\infty \) has locally bounded norm of the second fundamental form in \( \mathbb{R}^3 - \chi \).

Now replace \( \Gamma_\infty \) by a subsequence, which after relabeling we denote by \( \Gamma = \{\Sigma(n)\}_{n \in \mathbb{N}} \), such that the surfaces \( \Sigma(n) \) converge \( C^\infty \) to a minimal lamination \( \mathcal{L}_\chi \) of \( \mathbb{R}^3 - \chi \). This discussion completes the proofs of the first two items of the claim. The proofs of the remaining items of the claim, such as the fact that \( \mathcal{L}_\chi \neq \emptyset \), follow easily from the discussion below and the next Claim 3.4.

Recall that the intrinsic open balls \( B_{\Sigma(n)}(\overline{0}, 1) \subset \Sigma(n) \) are graphs of functions with gradient at most one over their vertical tangent planes. Hence, after refining the subsequence further, assume that the graphical surfaces \( B_{\Sigma(n)}(\overline{0}, 1) \) converge smoothly to a graph \( \Omega \) over.
a vertical plane. In particular for some \( \delta \in (0, \frac{1}{4}) \) small, depending on curvature estimates for \( H \)-graphs, we can find arcs

\[
\gamma(n) \subset B_{\Sigma(n)}(\bar{0}, 1) \cap \{|x_3| \leq \delta\}
\]

with one end point in the plane \( \{x_3 = \delta\} \) and the other end point in \( \{x_3 = -\delta\} \) and the \( \delta \)-neighborhood of \( \gamma(n) \) in \( \Sigma(n) \) is contained in \( B_{\Sigma(n)}(\bar{0}, \frac{1}{4}) \). In particular, the curves \( \gamma(n) \) stay an intrinsic distance at least \( \delta \) from any points of \( \Sigma(n) \) with very large curvature.

**Claim 3.4.** Let \( p \in \chi \). Then, after choosing a subsequence, there is a horizontal plane \( Q(p) \) passing through \( p \) such that \( Q(p) - \{p\} \) is the limit of a sequence of 3-valued graphs \( G(n) \subset \Sigma(n) \) over the annulus \( A(n, \frac{1}{n}) \) with gradients less than \( \frac{1}{n} \).

**Proof.** Note that by item 1 of Claim 3.3, for each \( k \in \mathbb{N} \), there exists an \( N(k) \in \mathbb{N} \) such that for each \( n \geq N(k) \), there exists a point \( p_k(n) \in \Sigma(n) \cap B(p, \frac{1}{k}) \) with \( |A_{\Sigma(n)}(p_k(n))| \geq k \).

Recall that

\[
\Sigma(n) = \frac{1}{r(n)}(E(n) - q_n)
\]

with \( \lim_{n \to \infty} \frac{1}{r(n)} = 0 \). Thus \( \bar{p}_k(n) = r(n)p_k(n) + q_n \) are points in \( E(n) \) arbitrarily close to \( q_n \) and with arbitrarily large norm of the second fundamental form as \( k, n \to \infty \). In particular, for \( k, n \) large, \( B(\bar{p}_k(n), \frac{1}{k}) \subset B(1) \). Let \( \bar{E}(n) \) be the connected component of \( \mathbb{B}(\bar{0}, \frac{1}{4}) \cap \{E(n) - \bar{p}_k(n)\} \) containing the origin and note that \( |A_{\bar{E}(n)}(\bar{0})| \geq \frac{k}{r(n)} \). By Theorem 2.2 and the fact that \( E(n) \subset \mathbb{R}^3 - (\mathbb{B}(1) \cap \{x_3 \geq 0\}) \), given \( \tau > 0 \) there exist constants \( \Omega_{\tau}, \omega_{\tau} \) such that \( \bar{E}(n) \) contains a 3-valued graph \( u_n \) over \( A(\frac{1}{\Omega_{\tau}}, \frac{\omega_{\tau}}{|A_{\bar{E}(n)}(\bar{0})|}) \) with gradient less than \( \tau \) for \( n \) large. Rescaling by \( \frac{1}{r(n)} \) gives that \( \frac{1}{r(n)} \bar{E}(n) \) contains a 3-valued graph \( u_n' \) over \( A(\frac{1}{\Omega_{\tau}}, \frac{\omega_{\tau}}{|A_{\bar{E}(n)}(\bar{0})|}) \) with gradient less than \( \tau \) for \( n \) large. Note that \( \frac{1}{r(n)} \bar{E}(n) \) is just a translation of \( \Sigma(n) \). Since \( k \) and \( n \) can be chosen large, depending on \( \omega_{\tau} \) and \( \Omega_{\tau} \), the previous discussion implies for any \( \tau > 0, \Lambda > 0 \) and \( \delta > 0 \) there exists \( N(\tau, \Lambda, \delta) \in \mathbb{N} \) such that for any \( n \geq N(\tau, \Lambda, \delta) \), \( \frac{1}{r(n)} \bar{E}(n) \) contains a 3-valued graph over \( A(\Lambda, \delta) \) with gradient less than \( \tau \). After choosing a subsequence, the 3-valued graphs converge to the \((x_1, x_2)\)-plane minus the origin and this finishes the proof of the claim.

The previous claim gives that in the slab \( S = \{-\frac{\delta}{4} < x_3 < \frac{\delta}{4}\} \), the sequence of surfaces \( \Sigma(n) \cap S \) has locally bounded norm of the second fundamental form, or equivalently stated, on compact balls \( F \subset S \), the collection of surfaces \( \Sigma(n) \cap F \) have uniformly bounded norm of the second fundamental form. Otherwise there would be a point \( p \in \chi \cap S \) and a horizontal plane \( Q(p) \) passing through \( p \) such that \( Q(p) - \{p\} \) is the limit of 3-valued graphs \( G(n) \subset \Sigma(n) \). This is impossible since these 3-valued graphs would intersect the arcs \( \gamma(n) \) contradicting embeddedness of the surfaces \( \Sigma(n) \).

Let \( \mathcal{L} \) be the set of open horizontal slabs containing the origin and disjoint from \( \chi \). The set \( \mathcal{T} \) is non-empty as it contains \( S \). Since \( \chi \neq \emptyset \), the union \( W = \cup_{S \in \mathcal{L}} S \) is a largest open slab in \( \mathcal{T} \) or a largest open half-space containing \( S \) for which \( W \cap \chi = \emptyset \). The fact that \( \mathcal{L} = \mathcal{L} \cap W \) is a non-empty lamination of \( W \) is clear by definition of lamination, and the validity of item 5 is also clear. This completes the proof of Claim 3.3.

**Claim 3.5.** The leaf \( L \) given in item 5 of Claim 3.3 has at most one point of incompleteness on each component of \( \partial W \) and any such point lies in \( \chi \). Here we refer to a point \( p \in \mathbb{R}^3 \) as being a point of incompleteness of \( L \), if \( p \) is the limiting end point in \( \mathbb{R}^3 \) of some proper arc \( \alpha : [0, 1) \to L \) of finite length.

**Proof.** Since \( L \) is a leaf of a lamination of \( W \), a point of incompleteness of \( L \) must lie in \( \partial W \). Let \( P \) be one of the horizontal planes in \( \partial W \) and let \( y_1 \in P \subset \partial W \). Suppose that there exists some smooth, proper, finite length embedded arc \( \alpha_{y_1} : [0, 1) \to L \) with limiting end point \( y_1 \) and with beginning point \( \alpha_{y_1}(0) \) in \( L \); then such a point \( y_1 \) is a point of incompleteness
of the leaf $L$. Note that the half-open arc $\alpha_{y_1}$ can be taken to be a $C^1$-limit of smooth embedded compact arcs $\alpha_{y_1}^n \subset \Sigma(n) \cap W$ of lengths converging to the length of $\alpha_{y_1}$. Let $y_1(n) \in \alpha_{y_1}^n$ be the end points of $\alpha_{y_1}^n$ which are converging to $y_1$. We claim that there must be small positive numbers $\tau_n$ with $\tau_n \to 0$, such that each $B_{\Sigma(n)}(y_1(n), \tau_n)$ contains points of arbitrarily large curvature as $n \to \infty$. In particular, it would then follow that $y_1 \in \chi$ and that we could modify the choice of the curves $\alpha_{y_1}^n$, so that the endpoints $y_1(n)$ are points with arbitrarily large norm of the second fundamental form as $n \to \infty$. To see that this claim holds, suppose to the contrary that after choosing a subsequence, for some small fixed $\varepsilon > 0$, the norms of the second fundamental forms of the intrinsic balls $B_{\Sigma(n)}(y_1(n), \varepsilon)$ are bounded, and by choosing $\varepsilon$ smaller, we may assume that these intrinsic balls are disks that are graphs of gradient at most 1 over their projections to their tangent planes at $y_1(n)$. Then a subsequence of these intrinsic balls would converge to an open minimal disk $D$ with $y_1 \in D$ and $D \cap L$ contains a subarc of $\alpha_{y_1}$. By the maximum principle, $D$ contains points on both sides of $P$. Since the distance from $P$ to $\chi$ is zero, then Claim 3.4 implies that $P$ is also the limit of a sequence of 3-valued graphs $G(n) \subset \Sigma(n)$. But then for $n$ large, this sequence $G(n)$ must intersect $D$ transversally at some point $q_n \in D$. Since the disks $B_{\Sigma(n)}(y_1(n), \varepsilon)$ converge smoothly to $D$, then for $n$ sufficiently large, $G(n)$ also intersects $B_{\Sigma(n)}(y_1(n), \varepsilon)$ transversely at some point. This proves the claim that $y_1 \in \chi$ and that we could modify the choice of the curves $\alpha_{y_1}^n$ so that the endpoints $y_1(n)$ are points with arbitrarily large norm of the second fundamental form as $n \to \infty$.

Without loss of generality, assume that $W$ lies below $P$. Suppose that there is a point $y_2 \in P$, $y_2 \neq y_1$, of incompleteness for $L$ with related proper arc $\alpha_{y_2}$ beginning at $\alpha_{y_2}(0)$ with limiting end point $y_2$, and related approximating curves $\alpha_{y_2}^n \subset \Sigma(n)$ and end points $y_2(n)$ converging to $y_2$, where the norm of the second fundamental form is arbitrarily large; also assume that $\alpha_{y_1}$, $\alpha_{y_2}$ are sufficiently close to their different limiting end points so that they are contained in closed balls in $\mathbb{R}^3$ that are a positive distance from each other. By Claim 3.4 there exist sequences of 3-valued graphs $G_1(n)$, $G_2(n)$ leaving $\Sigma(n)$ near $y_1(n)$, $y_2(n)$, respectively, and these 3-valued graphs can be chosen so that they each collapse to the plane $P$. For $n$ large, $G_1(n)$ must lie “above” the point $\alpha_{y_2}^n(0)$ which implies that $G_1(n)$ must lie above $G_2(n)$ as well, otherwise, $G_1(n)$ would intersect the arc $\alpha_{y_2}^n$, which contradicts embeddedness. Reversing the roles of $G_1(n)$ and $G_2(n)$, we find that $G_2(n)$ must lie above $G_1(n)$. This contradiction shows that there is at most one point $y \in P \cap \chi$ which corresponds to a point of incompleteness of $L$.

So far we have shown that $L$ can have at most one point of incompleteness on each of the components of $\partial W$. Define $S$ to be the finite set of one or two points that are points of incompleteness of $L$. The next claim describes some consequences of this finiteness of $S$.

**Claim 3.6.** The leaf $L$ has genus zero and the closure $\overline{L}$ of $L$ in $\mathbb{R}^3 - S$ is a minimal lamination of $\mathbb{R}^3 - S$ that is contained in $\overline{W} - S$.

**Proof.** We first prove that $L$ is not a limit leaf of $\mathcal{L}$. If $L$ were a limit leaf of $\mathcal{L}$, then its oriented 1 or 2-sheeted cover would be stable by Theorem 4.3 in [20]. Stability gives curvature estimates on $L$ away from $S$ and thus the closure $\overline{L}$ of $L$ in $\mathbb{R}^3 - S$, is a minimal lamination of $\mathbb{R}^3 - S$. By Corollary 2.2 in [18], the closure of $\overline{L}$ in $\mathbb{R}^3$ is a minimal lamination of $\mathbb{R}^3$ and each leaf is stable; in particular, by stability the closure of $\overline{L}$ in $\mathbb{R}^3$ must be a collection of parallel planes. This is a contradiction because $L$ lies in a horizontal slab and its tangent plane at the origin is vertical.

Since $L$ is not a limit leaf of $\mathcal{L}$, it follows that $L$ is properly embedded in $W$. Recall that $W$ is either an open slab or an open half-space. Since $L$ is properly embedded in a simply-connected open set of $\mathbb{R}^3$, then it separates the open set and so it is orientable. As $L$ is not stable, the convergence of portions of $\Sigma(n)$ to $L$ has multiplicity one or two which implies that $L$ has genus zero and zero flux; see the discussion when Case A holds for further details on this multiplicity of convergence bound and genus-zero property.
It remains to prove that the closure $\overline{L}$ of $L$ in $\mathbb{R}^3 - S$ is a minimal lamination of $\mathbb{R}^3 - S$. By the minimal lamination closure theorem in [22], this desired result is equivalent to proving that the injectivity radius function of $L$ is bounded away from zero on compact subsets of $\mathbb{R}^3 - S$. Otherwise, after scaling on the scale of the injectivity radius and applying a rigid motion of $\mathbb{R}^3$, the local picture theorem on the scale of the topology in [16] implies that there exists a sequence of compact domains $\Delta_n \subset L$ such that the scaled domains $\tilde{\Delta}_n$ converge smoothly with multiplicity one to a properly embedded genus-zero minimal surface $L_\infty$ in $\mathbb{R}^3$ with injectivity radius one or there exist closed geodesics $\gamma_n \subset \tilde{\Delta}_n$ with nontrivial flux. The latter happens when the $\Delta_n$ converge smoothly away from two vertical lines $L_1$ and $L_2$ to a foliation $\mathcal{F}$ by horizontal planes of $\mathbb{R}^3$, and where these lines are the singular sets of convergence to $\mathcal{F}$. Such a picture is called a parking garage structure on $\mathbb{R}^3$ and in this case there exist closed geodesics $\gamma_n$ on the domains $\tilde{\Delta}_n$ that correspond to “connection” loops between the “columns” $L_1$ and $L_2$ (see also [4, 9, 30]); these geodesics are easily seen to have non-zero flux [19]. So, the latter cannot happen as $L$ has zero flux. On the other hand, if the former happens then, by results in [26], $L_\infty$ has non-zero flux and therefore it cannot be the limit with multiplicity one of domains in $L$. In either case, we have obtained a contradiction, which proves that the injectivity radius function of $L$ is bounded away from zero on compact subsets of $\mathbb{R}^3 - S$. This completes the proof of the claim. \hfill \Box

Finally we prove that Case B cannot occur. By Claim 3.6, the closure $\overline{L}$ of $L$ in $\mathbb{R}^3 - S$ is a minimal lamination; in the language of [18], $\overline{L}$ is a minimal lamination of $\mathbb{R}^3$ with a countable (in fact at most two) set of singular points and it contains a genus-zero leaf $L$. By item 6 of Theorem 1.3 in [18], the closure of $L$ in $\mathbb{R}^3$ must be a properly embedded minimal surface in $\mathbb{R}^3$. Since $L$ is nonflat and is contained in the open slab or half-space $W$, the half-space theorem in [13] gives a contradiction, which proves that Case B cannot occur. This completes the proof of Assertion 3.2. \hfill \Box

Since Case B does not occur, then Case A must occur. We have already proven Proposition 3.1 when Case A holds, thus, the proof of Proposition 3.1 is finished. \hfill \Box

The next corollary follows easily from Proposition 3.1.

**Corollary 3.7.** Let $E(n)$ be the sequence of disks in Proposition 3.1. Given $\varepsilon_1 > 0$, there exist $\varepsilon_2 \in (0, \varepsilon_1)$ and $n \in \mathbb{N}$ such that the following holds. Let $p_k(n) \in E(n) \cap B(\varepsilon_2)$, $n > N$, be a point where the tangent plane to $E(n)$ is vertical. Then there exists a vertical helicoid $\mathcal{H}$ with maximal absolute Gaussian curvature $\frac{1}{2}$ at the origin such that the connected component of $|A_{E_n}|(p_k(n))(E_n - p_k(n)) \cap B(1)$ containing the origin is a normal graph $u$ over its projection $\Omega \subset [B(1 + 2\varepsilon_1) \cap \mathcal{H}]$, where $\Omega \supset [B(1 - 2\varepsilon_1) \cap \mathcal{H}]$ and $\|u\|_{C^2} \leq \varepsilon_1$. Furthermore, $|A_{E_n}|(p_k(n))(E_n - p_k(n)) \cap B(1)$ consists of 1 or 2 components and if there are two components, then the component not passing through the origin is a normal graph $u'$ over its projection $\Omega' \subset [B(1 + 2\varepsilon_1) \cap \mathcal{H}]$, where $\Omega' \supset [B(1 - 2\varepsilon_1) \cap \mathcal{H}]$ and $\|u'\|_{C^2} \leq \varepsilon_1$.

As a consequence of Proposition 3.1 and Corollary 3.7, we obtain the next claim which states that under the hypotheses of the theorem, the $H$-disk considered cannot contain a point with vertical tangent plane nearby the origin.

**Claim 3.8.** There exists an $\varepsilon > 0$ such that the following holds. If $E$ is an $H$-disk satisfying the hypotheses of Theorem 1.1 for $R = 1$, then $E \cap B(\varepsilon)$ contains no vertical tangent planes.

**Proof.** Arguing by contradiction, let $E(n)$ be a sequence of $H_n$-disks satisfying the hypothesis of Theorem 1.1 together with a sequence of points $p_k(n)$ in $E(n) \cap B(\frac{1}{2})$ with vertical tangent planes. Let $\delta > 0$ to be fixed and let $\Gamma_n$ be the connected component of $E(n) \cap B(\delta) \cap N_n^{-1}\{x_1^2 + x_2^2 = 1\}$ containing $p_k(n)$, where $N_n$ is the Gauss map of $E(n)$. It follows from Corollary 3.7 that given $\rho > 0$, there exists $\delta > 0$ such that for $n$ large, $\Gamma_n$ is an analytic curve with $|\hat{\Gamma}_n(0, 0, 1)| > 1 - \rho$. By taking $\rho$ sufficiently small, this curve
must then cross the \((x_1, x_2)\)-plane nearby the origin. This is impossible because the disks \(E(n)\) are disjoint from such plane. This contradiction completes the proof of the claim. □

To finish the proof of Theorem 1.1, recall that at the beginning of the proof of Theorem 1.1, we showed that if the theorem fails, then there exists a sequence of \(H_n\)-disks \(E(n)\) with \(H_n \leq 1\) satisfying the hypotheses of Theorem 1.1 and points \(p_k(n) \in E(n)\) with vertical tangent plane and converging to the origin. This contradicts Claim 3.8 and the proof of Theorem 1.1 is completed. □

The next corollary follows immediately from Theorem 1.1 by a simple rescaling argument. It roughly states that we can replace the \((x_1, x_2)\)-plane by any surface that has a fixed uniform estimate on the norm of its second fundamental form.

**Corollary 3.9.** Given \(A_0 \geq 0\), there exist \(\varepsilon \in (0, \frac{1}{2})\) and \(C_n > 0\) such that for any \(R > 0\), the following holds. Let \(\Delta\) be a compact immersed surface in \(\mathbb{B}(R)\) with \(\partial \Delta \subset \partial \mathbb{B}(R)\), \(\vec{0} \in \Delta\) and satisfying \(|A_{\Delta}| \leq A_0/R\). Let \(D\) be an \(H\)-disk such that
\[ D \cap \mathbb{B}(R) \cap \Delta = \emptyset \quad \text{and} \quad \partial D \cap \mathbb{B}(R) = \emptyset. \]
Then:
\[ (2) \quad \sup_{x \in D \cap \mathbb{B}(\varepsilon R)} |A_D|(x) \leq \frac{C_n}{R}. \]
In particular, if \(D \cap \mathbb{B}(\varepsilon R) \neq \emptyset\), then \(H \leq \frac{C_n}{R} \).

4. **Consequences of the One-sided Curvature Estimate**

In this section we state several theorems that depend on the one-sided curvature estimate in Theorem 1.1. We begin by making the following definition.

**Definition 4.1.** Given a point \(p\) on a surface \(\Sigma \subset \mathbb{R}^3\), \(\Sigma(p, R)\) denotes the closure of the component of \(\Sigma \cap \mathbb{B}(p, R)\) passing through \(p\).

In [23], we apply the one-sided curvature estimate in Theorem 1.1 to prove a relation between intrinsic and extrinsic distances in an \(H\)-disk, which can be viewed as a weak chord arc estimate. This result was motivated by and generalizes a previous result, Proposition 1.1 in [10], by Colding-Minicozzi for 0-disks. More precisely, the statement is the following.

**Theorem 4.2** (Weak Chord Arc Estimate, Theorem 1.2 in [23]). There exists a \(\delta_1 \in (0, \frac{1}{2})\) such that the following holds.

Let \(\Sigma\) be an \(H\)-disk in \(\mathbb{R}^3\). Then for all intrinsic closed balls \(\overline{B}_\Sigma(x, R)\) in \(\Sigma - \partial \Sigma:\)

1. \(\Sigma(x, \delta_1 R)\) is a disk with \(\partial \Sigma(\vec{0}, \delta_1 R) \subset \partial \mathbb{B}(\delta_1 R)\).
2. \(\Sigma(x, \delta_1 R) \subset B_\Sigma(x, R/2)\).

Roughly speaking, among other things Theorem 4.2 says that an \(H\)-disk \(D\) intersects sufficiently small closed Euclidean balls centered at any of its points sufficiently far from \(\partial D\) in compact components that are disks having their boundaries in the boundaries of these balls, independently of the value \(H\). By this relation between extrinsic and intrinsic distances, the extrinsic radius and curvature estimates given in Theorems 2.4 and 2.3, that were used to prove Theorem 4.2, become radius and curvature estimates that depend on intrinsic distances; see [24] for details on the proofs of the two theorems below.

In the next theorem the radius of a compact Riemannian surface with boundary is the maximum intrinsic distance of points in the surface to its boundary; in the second theorem, \(d_\Sigma\) denotes the intrinsic distance function of \(\Sigma\).

**Theorem 4.3** (Radius Estimates, Theorem 1.2 in [24]). There exists an \(R \geq \pi\) such that any compact disk embedded in \(\mathbb{R}^3\) of constant mean curvature \(H > 0\) has radius less than \(\frac{R}{\pi}\).
Theorem 4.4 (Curvature Estimates, Theorem 1.3 in [24]). Given $\delta$, $\mathcal{H} > 0$, there exists a $K(\delta, \mathcal{H}) \geq \sqrt{2}\mathcal{H}$ such that any compact disk $M$ embedded in $\mathbb{R}^3$ with constant mean curvature $H \geq \mathcal{H}$ satisfies

$$\sup_{\{p \in M \mid d_M(p, \partial M) \geq \delta\}} |A_M| \leq K(\delta, \mathcal{H}).$$

An immediate consequence of the triangle inequality and Theorem 4.2 is the following intrinsic version of the one-sided curvature estimate given in Theorem 1.1. In the case that $H = 0$, the next theorem follows from Corollary 0.8 in [10].

Theorem 4.5 (Intrinsic one-sided curvature estimate for $H$-disks). There exist $\varepsilon \in (0, \frac{1}{2})$ and $C_\varepsilon \geq 2\varepsilon^2$ such that for any $R > 0$, the following holds. Let $D$ be an $H$-disk such that $D \cap B(R) \setminus \{x_3 = 0\} = \emptyset$ and $x \in D \cap B(\varepsilon R)$ where $d_D(x, \partial D) \geq R$. Then:

$$(3) \quad |A_D|(x) \leq \frac{C_\varepsilon}{R}.$$  

In particular, $H \leq \frac{\varepsilon}{R}.$

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