Classical Computation of Number of Lines in Projective Hypersurfaces: Origin of Mirror Transformation

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Abstract

In this paper, we discuss classical derivation of the residue integral representation of the $d = 1$ rational Gromov-Witten invariants of projective hypersurfaces that followed from localization technique.

1 Introduction

Geometrical origin of the mirror transformation in the mirror computation of the Gromov-Witten invariants is considered as an effect of the difference between the moduli space of Gauged linear Sigma Model and the one of Non-Linear Sigma Model. In this paper, we focus on this problem in the case of the $d = 1$ rational Gromov-Witten invariants of projective hypersurface, which is the simplest example of the effect of the mirror transformation.

In [1], we introduced the compactified moduli space of polynomial maps with two marked points $\tilde{M}_{0,2}(N, d)$, which is expected to play the role of the moduli space of Gauged Linear Sigma Model whose target space is $CP^{N-1}$. In the $d = 1$ case, $\tilde{M}_{0,2}(N, 1)$ turns out to be $CP^{N-1} \times CP^{N-1}$. We then introduced the two point function $w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_{0,1}$ which is considered as the correlation function of the Gauged Linear Sigma Model whose target space is the degree $k$ hypersurface in $CP^N$ (we denote this hypersurface by $M^k_N$). Here $h$ is the hyperplane class of $CP^{N-1}$. Explicitly, it is given by the following formula:

$$w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_{0,1} = \int_{CP^{N-1} \times CP^{N-1}} (h_1)^a \cdot \left( \prod_{j=0}^k (jh_1 + (k-j)h_2) \right) \cdot (h_2)^b,$$

where $h_1$ (resp. $h_2$) is the hyperplane class of the first (resp. second) $CP^{N-1}$. The r.h.s. of (1.1) can be written as the residue integral:

$$w(\mathcal{O}_{h^a} \mathcal{O}_{h^b})_{0,1} = \frac{1}{2\pi i} \oint_{C_0} dz_1 \oint_{C_0} dz_2 (z_1)^a \cdot \left( \prod_{j=0}^k (jz_1 + (k-j)z_2) \right) \cdot (z_2)^b,$$

where $\frac{1}{2\pi i} \oint_{C_0} dz$ represents the operation of taking a residue at $z = 0$. On the other hand, we derived in [2] the residue integral representation of the two point rational Gromov-Witten invariant $\langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0,1}$ of $M^k_N$ by taking the non-equivariant limit of the result of localization technique by Kontsevich [4]:

$$\langle \mathcal{O}_{h^a} \mathcal{O}_{h^b} \rangle_{0,1} = -\frac{1}{2} \cdot \frac{1}{2\pi i} \oint_{C_0} \oint_{C_0} \frac{dz_1}{(z_1)^N} \frac{dz_2}{(z_2)^N} \left( \prod_{j=0}^k (jz_1 + (k-j)z_2) \right) \cdot (z_1 - z_2)^2 \cdot \frac{(z_1)^a - (z_2)^a}{z_1 - z_2} \cdot \frac{(z_1)^b - (z_2)^b}{z_1 - z_2}.$$

(1.3)
From (1.2) and (1.3), we can derive the equality:
\[
\langle \mathcal{O}_{h^\alpha} \mathcal{O}_{h^\beta} \rangle_{0,1} = w(\mathcal{O}_{h^\alpha} \mathcal{O}_{h^\beta})_{0,1} - w(\mathcal{O}_{h^{\alpha+\beta}} \mathcal{O}_{1})_{0,1},
\]
(1.4)
which is the simplest example of the mirror transformation.

In this paper, we discuss another proof of (1.3) by using the classical cohomology theory of the Grassmann variety $G(2, N)$, which parametrizes 2-dimensional linear subspaces in $\mathbb{C}^N$. Let us introduce the following rational functions in $z$ and $w$ to state our result in this paper:
\[
e^k(z, w) := \prod_{j=0}^{k} (jz + (k - j)w),
\]
\[
w_\alpha(z, w) := \frac{z^\alpha - w^\alpha}{z - w}.
\]
(1.5)

With this set-up, we state the following theorem on the $d = 1$ rational Gromov-Witten invariants of $M^G_N$.

**Theorem 1**
\[
\langle \prod_{j=1}^{n} \mathcal{O}_{h^\alpha_j} \rangle_{0,1} = \frac{1}{(2\pi i)^2} \oint_{C_0} \frac{dz_1}{C_0(z_1)^N} \oint_{C_0} \frac{dz_2}{C_0(z_2)^N} (z_2 - z_1) \cdot e^k(z_1, z_2) \cdot (z_1)^{a_1} \prod_{j=2}^{n} w_{a_\alpha_j}(z_1, z_2),
\]
(1.6)
where $1/(2\pi i)^2 \oint_{C_0} dz$ represents the operation of taking a residue at $z = 0$.

If $n = 2$, (1.6) reduces to (1.3). To prove Theorem 1, we use the formula that represents $\langle \prod_{j=1}^{n} \mathcal{O}_{h^\alpha_j} \rangle_{0,1}$ in terms of cohomology classes of $G(2, N)$:
\[
\langle \prod_{j=1}^{n} \mathcal{O}_{h^\alpha_j} \rangle_{0,1} = \int_{G(2, N)} c_{top}(S^k(S^*_G)) \cdot \langle \prod_{j=1}^{n} \sigma_{a_j} \rangle_1,
\]
(1.7)
where $S_G$ is the tautological vector bundle of $G(2, N)$ and $\sigma_{a_j}$ is the Schubert class that will be introduced in Section 3. Then we consider the following sequence of maps:
\[
(CP^{N-1} \times CP^{N-1} - \Delta) \xrightarrow{f} P(\mathbb{C}^N/S_1) \xrightarrow{\eta} G(2, N)
\]
(1.8)
where $\Delta$ is the diagonal set of $CP^{N-1} \times CP^{N-1}$, $S_1$ is the tautological line bundle of $CP^{N-1}$ and $P(\mathbb{C}^N/S_1)$ is the projective bundle over $CP^{N-1}$ that will be introduced in Section 2. Let $U$ be a set of pairs of two linearly independent vectors in $\mathbb{C}^N$:
\[
U = \{ (a_1, a_2) \mid a_1, a_2 \in \mathbb{C}^N, a_1 \text{ and } a_2 \text{ are linearly independent} \}.
\]
(1.9)

Since
\[
\begin{bmatrix}
\alpha \\
\gamma \\
\beta
\end{bmatrix}
\in GL(2, C)
\text{ acts on } U \text{ by } (a_1, a_2) \rightarrow (\alpha a_1 + \gamma a_2, \beta a_1 + \delta a_2),
\]
we have,
\[
(CP^{N-1} \times CP^{N-1} - \Delta) = U/B_1, \quad P(\mathbb{C}^N/S_1) = U/B_2, \quad G(2, N) = U/GL(2, C),
\]
(1.10)
where $B_1$ and $B_2$ are the subgroups of $GL(2, C)$ given by,
\[
B_1 = \{ \begin{bmatrix}
\alpha \\
0 \\
0
\end{bmatrix} \mid \alpha \delta \neq 0 \},
\]
\[
B_2 = \{ \begin{bmatrix}
\alpha \\
0 \\
0
\end{bmatrix} \mid \alpha \delta \neq 0 \}.
\]
(1.11)

Therefore, $f$ and $\eta$ in (1.8) are natural projections. The main idea to prove Theorem 1 is to rewrite the r.h.s. of (1.7) as the intersection number of $P(\mathbb{C}^N/S_1)$ via the projection formula with respect to $\eta$. But the final result can be interpreted as the intersection number of $CP^{N-1} \times CP^{N-1}$ because the residue integral representation indicates that the r.h.s. of (1.6) is an integral of a cohomology element of $CP^{N-1} \times CP^{N-1}$.

This paper is organized as follows. In Section 2, we introduce the moduli space of lines in $CP^{N-1}$ with one marked point and show that it is identified with $P(\mathbb{C}^N/S_1)$. In Section 3, we introduce the equality (1.7) and rewrite the r.h.s. of it as an intersection number of $P(\mathbb{C}^N/S_1)$. In Section 4, we represent the intersection number of $P(\mathbb{C}^N/S_1)$ as a residue integral and prove Theorem 1.

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2 Moduli Space of Lines in $CP^{N-1}$ with One Marked Point

First, we introduce a line $l$ in $CP^{N-1}$.

$$l : CP^1 \rightarrow CP^{N-1}$$

$$l(s : t) = [a_1 s + a_2 t],$$

(2.12)

where $a_1$ and $a_2$ are two linearly independent vectors in $C^N$. In (2.12), $[s]$ is used to represent equivalence class under projective equivalence. In this section, we consider a line with one marked point $(1 : 0) \in CP^1$, and we introduce $PSL(2, C)$ transformation $\varphi$ that fixes the marked point:

$$\varphi : CP^1 \rightarrow CP^1$$

$$\varphi(s : t) = (\alpha s + \beta t : \delta s), \quad (\alpha \delta \neq 0).$$

(2.13)

The moduli space of lines in $CP^{N-1}$ with one marked point is given by the set of equivalence classes of $l$ under the equivalence relation:

$$l \circ \varphi \sim l \iff (a_1, a_2) \sim (\alpha a_1, \beta a_1 + \delta a_2).$$

(2.14)

From (2.14), we can see that the moduli space is identified with the projective bundle $P(C^N/S_1) \rightarrow CP^{N-1}$ where $S_1$ is the tautological line bundle of $CP^{N-1} = \{[a_1]\}$. From now on, we simply denote by $P(C^N/S_1)$ this projective bundle. Let $S_2$ be the tautological line bundle of $P(C^N/S_1)$. It is well-known that $H^*(P(C^N/S_1))$ is generated by,

$$h_1 := -c_1(S_1) = c_1(S_1^*), \quad h_2 := -c_1(S_2) = c_1(S_2^*).$$

(2.15)

Since the total Chern class of $C^N/S_1$ is given by,

$$c(C^N/S_1) = \frac{1}{1-th_1} = 1 + th_1 + t^2(h_1)^2 + \cdots + t^{N-1}(h_1)^{N-1},$$

(2.16)

$h_1$ and $h_2$ satisfy the following relation:

$$(h_1)^N = 0, \quad \sum_{j=0}^{N-1} (h_1)^j \cdot (h_2)^{N-1-j} = 0.$$  

(2.17)

The above relations completely determine the ring structure of $H^*(P(C^N/S_1))$.

3 Computation of Number of Lines

In this section, we denote by $G(2, N)$ the Grassmann variety that parametrizes 2-dimensional linear subspaces in $C^N$. By taking the projective equivalence, it is identified with the moduli space of lines in $CP^{N-1}$ with no marked points. $G(2, N)$ has the Schubert cycle $\sigma_a$ ($a$: non-negative integer) that is defined by

$$\sigma_a := \{A \in G(2, N) \mid \dim(A \cap V_{N-1-a}) \geq 1\},$$

(3.18)

where $V_{N-1-a}$ is a fixed $(N-1-a)$-dimensional linear subspace in $C^N$. By taking the projective equivalence, it consists of lines that intersect with the fixed codimension $(a+1)$ linear subspace in $CP^{N-1}$. $\sigma_a$ is the homology cycle of $G(2, N)$ of complex codimension $a$, and we denote by $\sigma_a^* \in H^*(G(2, N))$ the Poincare dual of $\sigma_a$. Let $S_G$ be the tautological rank 2 vector bundle on $G(2, N)$. We have the following natural projection:

$$\eta : P(C^N/S_1) \rightarrow G(2, N),$$

$$\eta([([a_1, a_2]]) = \langle a_1, a_2 \rangle_c.$$  

(3.19)

Here $([a_1, a_2])$ is the equivalence class of $(a_1, a_2)$ under (2.14). This projection map is equivalent to changing the equivalence relation of pairs of linear independent vectors $(a_1, a_2)$ from

$$(a_1, a_2) \sim (\alpha a_1, \beta a_1 + \delta a_2), \quad (\alpha \delta \neq 0),$$

(2.20)

into,

$$(a_1, a_2) \sim (\alpha a_1 + \gamma a_2, \beta a_1 + \delta a_2), \quad (\alpha \delta - \beta \gamma \neq 0).$$

(3.21)
The equivalence relation (3.20) is nothing but taking quotient by the subgroup $B_2$ of $PSL(2, \mathbb{C})(= \text{Aut}(CP^1))$ that fixes the marked point $(1 : 0)$. Therefore, the fiber of the map $\eta$ is given by $PSL(2, \mathbb{C})/B_2 = CP^1$ and it corresponds to the position of $l(1 : 0)$ in $l(CP^1)$.

Since $(\prod_{j=1}^{n} \mathcal{O}_{h^*})_{0,1}$ is the number of lines in $M^N$ that intersect with codimension $a_j$ linear subspace in $CP^{N-1}$ ($j = 1, \cdots, n$), it can be represented as the following intersection number of $G(2; N)$:

**Proposition 1**

$$
(\prod_{j=1}^{n} \mathcal{O}_{h^*})_{0,1} = \int_{G(2; N)} c_{\text{top}}(S^k(S_G^*)) \cdot (\prod_{j=1}^{n} \sigma_{a_j-1}^*),
$$

where $S^k(S_G^*)$ is the $k$-th symmetric product of the dual vector bundle of $S_G$.

On the other hand, $(h_1)^a \in H^*(\mathbb{P}(\mathbb{C}^N/S_1))$ imposes the condition that $l(1 : 0) \subset CP^{N-1}$ lies inside the codimension $a$ linear subspace of $CP^{N-1}$ and the fiber of $\eta$ is the position of $l(1 : 0)$ in $l(CP^1)$. Therefore, we have

$$
\eta_*(\langle h_1 \rangle^a) = \sigma_{a-1}^*
$$

where $\eta_*: H^*(\mathbb{P}(\mathbb{C}^N/S_1)) \to H^*(G(2; N))$ is the fiber integration. Therefore, we obtain from the projection formula,

**Proposition 2**

$$
(\prod_{j=1}^{n} \mathcal{O}_{h^*})_{0,1} = \int_{\mathbb{P}(\mathbb{C}^N/S_1)} \eta^*(c_{\text{top}}(S^k(S_G^*))) \cdot (h_1)^{a_1} \cdot (\prod_{j=2}^{n} \eta^*(\sigma_{a_j-1}^*)).
$$

**Proposition 3**

$$
\eta^*(c_{\text{top}}(S^k(S_G^*))) = \prod_{j=0}^{k}(jh_1 + (k-j)h_2) = c^k(h_1, h_2),
$$

$$
\eta^*(\sigma_a^*) = \sum_{j=0}^{a} (h_1)^j \cdot (h_2)^{a-j} = w_{a+1}(h_1, h_2).
$$

**proof** From the definition of $\eta$, we can easily see that,

$$
\eta^{-1}(S_G) = S_1 \oplus S_2.
$$

Therefore, the first equality in (3.25) follows from $\eta^*(c(S_G^*)) = c(S_1^*)c(S_2^*) = (1 + th_1)(1 + th_2)$. On the other hand, we have the following identity that can be found in the page 411 of [3]:

$$
c(\mathbb{C}^N/S_G) = \sum_{a=0}^{N-2} t^a \sigma_a^*.
$$

From these two equalities, we have,

$$
\eta^*(c(\mathbb{C}^N/S_G)) = \sum_{a=0}^{N-2} t^a \eta^*(\sigma_a^*) = \frac{1}{c(S_1)c(S_2)} = \frac{1}{(1-th_1)(1-th_2)}.
$$

(3.25) directly follows from this equality. $\square$

**4 Proof of Theorem 1**

**Proposition 4** Let $f(h_1, h_2) \in H^*(\mathbb{P}(\mathbb{C}^N/S_1))$ given as a polynomial in $h_1, h_2$. Then we have,

$$
\int_{\mathbb{P}(\mathbb{C}^N/S_1)} f(h_1, h_2) = \frac{1}{(2\pi \sqrt{-1})^2} \oint_{C_0} \oint_{C_0} dz_2 \oint_{C_0} dz_1 \frac{z_2 - z_1}{(z_2)^N \cdot ((z_2)^N - (z_1)^N)} \cdot f(z_1, z_2),
$$

where $\frac{1}{2\pi \sqrt{-1}} \oint_{C_0} dz$ represents the operation of taking a residue at $z = 0$.  

4
Proposition 5

With this set-up, Theorem 1 follows from Proposition 2, 3, 4 and the following equality:

\[
\{ z_1 \in \mathbb{C} \mid |z_1| = \epsilon_1 \}, \quad \{ z_2 \in \mathbb{C} \mid |z_2| = \epsilon_2 \} \quad (0 < \epsilon_1 < \epsilon_2).
\] (4.30)

Under the above condition, \( z_1 \neq z_2 \) and we have,

\[
\sum_{j=0}^{N-1} (z_1)^j (z_2)^{N-1-j} = \frac{(z_2)^N - (z_1)^N}{z_2 - z_1}.
\] (4.31)

From degree counting, the r.h.s. of (4.29) is non-zero only if \( f(z_1, z_2) \) is a homogeneous polynomial of degree \( 2N - 3 \) in \( z_1 \) and \( z_2 \). Even when it is a homogeneous polynomial of degree \( 2N - 3 \), the r.h.s. of (4.29) vanishes if it is divided by \( (z_1)^N \) or \( \frac{(z_2)^N - (z_1)^N}{z_2 - z_1} \). Therefore, it suffices for us to check,

\[
\frac{1}{(2\pi i)^2} \oint_{C_0} dz_2 \oint_{C_0} dz_1 \frac{z_2 - z_1}{(z_1)^N \cdot ((z_2)^N - (z_1)^N)} \cdot (z_1)^{N-1} (z_2)^{N-2} = 1.
\] (4.32)

But it is obvious because we first integrate \( z_1 \)-variable. \( \square \)

With this set-up, Theorem 1 follows from Proposition 2, 3, 4 and the following equality:

**Proposition 5** Let \( f(z_1, z_2) \) be a polynomial in \( z_1 \) and \( z_2 \). Then we have,

\[
\frac{1}{(2\pi i)^2} \oint_{C_0} dz_2 \oint_{C_0} dz_1 \frac{z_2 - z_1}{(z_1)^N \cdot ((z_2)^N - (z_1)^N)} \cdot f(z_1, z_2)
\]

\[
= \frac{1}{(2\pi i)^2} \oint_{C_0} dz_2 \oint_{C_0} dz_1 \frac{z_2 - z_1}{(z_1)^N \cdot ((z_2)^N - (z_1)^N)} \cdot f(z_1, z_2),
\] (4.33)

where integration paths are taken as given in (4.30).

**proof** Let us consider the following integral:

\[
\frac{1}{(2\pi i)^2} \oint_{C_0} dz_2 \oint_{C_0} dz_1 \frac{z_2 - z_1}{(z_1)^N \cdot ((z_2)^N - u(z_1)^N)} \cdot f(z_1, z_2), \quad (0 \leq u \leq 1).
\] (4.34)

It suffices for us to show that the above integral is independent of \( u \). Under the condition (4.30), \( |z_1| < 1 \) and we can expand the integrand around \( z_1 = 0 \):

\[
\frac{z_2 - z_1}{(z_1)^N \cdot ((z_2)^N - u(z_1)^N)} \cdot f(z_1, z_2) = \frac{z_2 - z_1}{(z_1)^N \cdot (z_2)^N} \cdot f(z_1, z_2) \cdot \left( \sum_{j=0}^{\infty} u^j \frac{z_1}{z_2} \right)^N.
\] (4.35)

But the terms with positive power of \( u \) have zero contributions to the residue at \( z_1 = 0 \) because it contains no negative power of \( z_1 \). \( \square \)

**References**

[1] M. Jinzenji, Mirror Map as Generating Function of Intersection Numbers: Toric Manifolds with Two Kähler Forms, Preprint, arXiv:1006.0607.

[2] M. Jinzenji, Direct Proof of the Mirror Theorem for Projective Hypersurfaces up to degree 3 Rational Curves, Journal of Geometry and Physics, Vol. 61, Issue 8, (2011) 1564-1573.

[3] P. Griffiths, J. Harris. Principles of Algebraic Geometry Wiley-Interscience Series of Texts, Monographs and Tracts, 1978.

[4] M. Kontsevich. Enumeration of Rational Curves via Torus Actions The moduli space of curves, R.Dijkgraaf, C.Faber, G.van der Geer (Eds.), Progress in Math., v.129, Birkhäuser, 1995, 335-368.