On the geometric pictures of the non-equivalent smallest images of \((k; n)\)-arcs for \(n = 2, 3\) in \(\text{PG}(2, 5)\)

Zainab Shehab Hamed

Department of Mathematics, Collage of Science, Mustansiriah University

Abstract

In this paper, the smallest images sets in the PGL(2, 5)-orbits of \((k; n)\)-arcs in the projective plane of order 5, for \(n = 2, 3\) are classified. Also, this paper presents the incident structure of the orbits for the non-equivalent lexicographically least sets of \(k\)-arcs and \((k; 3)\)-arcs. Then the geometric pictures for the most interesting group orbits of these images are given.

Keywords:

Projective spaces, \((k; n)\)-arcs, incident structure, lexicographically least set.

1 Introduction

The projective plane \(\text{PG}(2, q)\) over the finite field \(GF(q)\), is an incident structure of points and lines. It contains \(q^2 + q + 1\) points and lines, where there are \(q + 1\) points passing through each line and \(q + 1\) lines passing through each point. So, there 31 points and lines in \(\text{PG}(2, 5)\), each line contains six points incident and each point incident with six lines. Also, a \((k; n)\)-arc \(K\) is a set of \(k\) points such that no \(n + 1\) of which are on a line, but some \(n\) are on a line. Then, the smallest image set of \((k; n)\)-arc is the canonical set of \(K\).
2 Incidence structure

An incidence structure $I$ is a triple $(P,B,R)$. Here, $P$ is a set whose elements are called points, $B$ is a set whose elements are called blocks or lines, and $R$ is a symmetric incidence relation, such that $R \subseteq (P \times B) \cup (B \times P)$. If $(P,B) \in R$, we say that $P$ is incident with $B$, or $P$ lies on $B$, or $B$ is incident with $P$, that $B$ contains $P$. The points in an incidence structure that lie on the same line are collinear points. Also, The lines that pass through a point are concurrent lines. The dual of an incidence structure $(P,B,R)$ is also incidence structure denoted by $R^D = (B,P,R)$.

Note that, three points are collinear if they are contained in a line. Dually, we say that three lines are concurrent in a point. An isomorphism between incidence structures is a bijection between the point sets together with a bijection between the line sets such that preserved incidence. An automorphism of an incidence structure is an isomorphism from the incidence structure to itself. An incidence structure is said to be a self-dual if it is isomorphic to its dual.

3 Lexicographically smallest set

Given the sets $H = \{h_1,\ldots,h_r\}$ and $M = \{m_1,\ldots,m_r\}$ of integers, with $h_1 < h_2 < \cdots < h_r$ and $m_1 < m_2 < \cdots < m_r$. Then $H \leq M$ lexicographically if either $H = M$ or if, for some $a$ with $1 \leq a < r$, we have $h_1 = m_1,\ldots,h_a = m_a$, but $h_{a+1} < m_{a+1}$.

**Lemma 3.1.** Let $\beta_i$ indicates the number of $i$-secants of an $(k;n)$-arc $K$ in $\text{PG}(2,q)$, $\mu_i(O)$ indicates the number of $i$-secants through the point $O \in K$, and $\theta_i(O_1)$ indicates the number of $i$-secants through the point $(O_1) \in \text{PG}(2,q) \setminus K$. Then the following hold:

\begin{align*}
(1) & \quad \sum_{i=0}^r \beta_i = q^2 + q + 1 ; \\
(2) & \quad \sum_{i=1}^r i \beta_i = k(q + 1) ; \\
(3) & \quad \sum_{i=2}^r \frac{i(i-1)\beta_i}{2} = \frac{k(k-1)}{2} ;
\end{align*}
\[
\sum_{i=1}^{r} \mu_i(O) = (q + 1);
\]

**Definition 3.2.** Let \( L \) is a line of \( \text{PG}(2, q) \) and let \( K \) is a \((k;n)\)-arc in \( \text{PG}(2, q) \) such that \( |L \cap K| = j \), then \( L \) is an \( j \)-secant of \( K \).

**Theorem 3.3.** [Bose]

\[
m_{2}(2, q) = \begin{cases} 
q + 1, & \text{for } q \text{ odd;} \\
q + 2, & \text{for } q \text{ even.}
\end{cases}
\]

### 4 Non-equivalent least images of \( k \)-arcs

This section presents the classification of non-equivalent lexicographically least images of \( k \)-arcs in \( \text{PG}(2, 5) \) for \( k = 3, 4, 5, 6 \). In this classification, the number of the non-equivalent least images of \( k \)-arcs is established. The process starts by classifying the \( k \)-arcs for the value of \( k = 3, 4, 5, 6 \), then the smallest images of \( k \)-arcs up to projective non-equivalence for \( k = 3, 4, 5, 6 \) are computed. Table 1 introduces the statistics of the non-equivalent least images and the stabiliser types. The symbol \( \xi_i \) indicates the non-equivalent least images of \( k \)-arcs and the symbol \( \mathcal{S} \) indicates the stabiliser group for each image.

| Symbol | \( k \)-arc | \( \xi_i \) | \( \mathcal{S} \) |
|--------|-------------|-------------|-------------|
| \( \mathcal{A}_1 \) | 3-arc | \{1, 2, 3\} | \((Z4 \times Z4) \rtimes Z3) \rtimes Z2\) |
| \( \mathcal{A}_2 \) | 4-arc | \{1, 2, 3, 5\} | \(S_4\) |
| \( \mathcal{A}_3 \) | 5-arc | \{1, 2, 3, 5, 13\} | \(Z5 \rtimes Z4\) |
| \( \mathcal{A}_4 \) | 6-arc | \{1, 2, 3, 5, 13, 28\} | \(S_5\) |

### 5 Incidence structure of the non-equivalent least images of \( k \)-arcs

From Table 1, the stabiliser groups of the non-equivalent images of 3-arc, 4-arc, 5-arc, and 6-arc are the groups \((Z4 \times Z4) \rtimes Z3) \rtimes Z2\), \(S_4\), \(Z5 \rtimes Z4\), and \(S_5\). These groups partition the
associated non-equivalent images of \( k \)-arcs for \( k = 3, 4, 5, 6 \) in one orbit. They are \( \mathcal{O}(A_1) = \{1, 3, 2\}, \mathcal{O}(A_2) = \{1, 2, 3, 5\}, \mathcal{O}(A_3) = \{1, 2, 5, 13, 3\}, \mathcal{O}(A_4) = \{1, 2, 13, 5, 28, 3\} \). Also, the secant distributions \( \{\lambda_2, \lambda_1, \lambda_0\} \) of \( \mathcal{O}(A_1), \mathcal{O}(A_2), \mathcal{O}(A_3), \) and \( \mathcal{O}(A_4) \) are \( \{3, 12, 16\}, \{6, 12, 13\}, \{10, 10, 11\}, \{15, 6, 10\} \) respectively. Accordingly, the geometric configurations of these orbits are vertices of a triangle, vertices of a quadrangle, ten lines, any four are concurrent, fifteen lines, any five are concurrent. The geometric descriptions of these orbits are given in Table 2.

| Symbol | Group | \( \{\lambda_2, \lambda_1, \lambda_0\} \) | description |
|--------|-------|---------------------------------|-------------|
| \( \mathcal{O}(A_1) \) | \((Z_4 \times Z_4) \rtimes Z_2\) | \( \{3, 12, 16\} \) | vertices of a triangle |
| \( \mathcal{O}(A_2) \) | \( S_4 \) | \( \{6, 12, 13\} \) | vertices of a quadrangle |
| \( \mathcal{O}(A_3) \) | \( Z_5 \rtimes Z_4 \) | \( \{10, 10, 11\} \) | ten lines, any four are concurrent |
| \( \mathcal{O}(A_4) \) | \( S_5 \) | \( \{15, 6, 10\} \) | fifteen lines, any five are concurrent |

6 Geometric pictures of the non-equivalent least images of \( k \)-arcs

In this section, the geometric pictures of the orbits of the non-equivalent images \( A_1, A_2, A_3, A_4 \) that illustrated in Table 2 are shown in Figures 1, 2, 3, 4.

(i) From Table 2, the geometric picture of the incidence structure of the orbit of the non-equivalent image \( A_1 \) of the 3-arc is a vertices of a triangle as describe in Figure 1.

![Geometric picture of the orbit \( \mathcal{O}(A_1) \)](image)

Figure 1: Geometric picture of the orbit \( \mathcal{O}(A_1) \)

(ii) The geometric picture of the incidence structure of the orbit of the non-equivalent image \( A_2 \) of the 4-arc is a vertices of a quadrangle as describe in Figure 2.
Figure 2: Geometric picture of the orbit $Or(\mathcal{A}_2)$

(iii) The geometric picture of the incidence structure of the orbit of the non-equivalent image $\mathcal{A}_3$ of the 5-arc is ten lines where each four are concurrent with a point of the $Or(\mathcal{A}_3)$ as given in Figure 3.

Figure 3: Geometric picture of the orbit $Or(\mathcal{A}_3)$

(iv) The geometric picture of the incidence structure of the orbit of the non-equivalent image $\mathcal{A}_4$ of the 6-arc is fifteen lines, each five are concurrent with a point of the $Or(\mathcal{A}_4)$ as given in Figure 4.
In this section, the classification of non-equivalent least images of \((k;3)\)-arcs is made. Here, the classification starts to establish the \((k;3)\)-arcs for \(k \geq 4\) and then the smallest images of these arcs up to projective non-equivalence for \(k \geq 4\) are computed. The number of these images are 1, 2, 7, 13, 14, 5, 2 respectively. In addition, these images have one of the stabiliser groups \(\mathbb{Z}_2, \mathbb{Z}_3, \mathbb{Z}_4, \mathbb{Z}_2 \times \mathbb{Z}_2, \mathbb{S}_3, \mathbb{S}_4, \mathbb{S}_5, \mathbb{D}_4, \mathbb{D}_6, \mathbb{Z}_4 \times \mathbb{S}_3, \mathbb{Z}_5 \times \mathbb{Z}_4\). In Table 3, the statistics of the non-equivalent smallest images of \((k;3)\)-arcs in \(\text{PG}(2,5)\) are given. The symbol \(\eta_j\) indicates the number of the non-equivalent smallest images of \((k;3)\)-arcs for \(k = 4, 5, 6, 7, 8, 9, 10, 11\) and the symbol \(\mathcal{G}\) indicates the stabiliser group type of each arc. Also \(t\) is the number of these stabilisers.
Table 3: Non-equivalent smallest images of \((k;3)\)-arcs

| Symbol | \((k;3)\)-arcs   | \(n_j\) | \(\mathcal{G} : t\) |
|--------|------------------|---------|---------------------|
| \(B_1\) | \((4;3)\)-arcs   | 1       | \(Z4 \times S3 : 1\) |
| \(B_2\) | \((5;3)\)-arcs   | 2       | \(D4 : 1, Z2 \times Z2 : 1\) |
| \(B_3\) | \((6;3)\)-arcs   | 7       | \(Z2 : 1, Z3 : 1, Z4 : 1, S3 : 2, S4 : 1, D4 : 1\) |
| \(B_4\) | \((7;3)\)-arcs   | 13      | \(I : 1, Z2 \times Z2 : 1, Z2 : 3, Z3 : 1, Z4 : 2, S3 : 2, S4 : 1, D4 : 1, D6 : 1\) |
| \(B_5\) | \((8;3)\)-arcs   | 13      | \(I : 3, Z2 : 6, Z2 \times Z2 : 2, Z4 : 1, D4 : 1\) |
| \(B_6\) | \((9;3)\)-arcs   | 14      | \(I : 2, Z2 : 4, Z3 : 2, Z4 : 1, Z2 \times Z2 : 1, S3 : 2, D4 : 1, D6 : 1\) |
| \(B_7\) | \((10;3)\)-arcs  | 5       | \(Z2 : 2, Z4 : 1, S4 : 1, S5 : 1\) |
| \(B_8\) | \((11;3)\)-arcs  | 2       | \(D4 : 1, Z5 \times Z4 : 1\) |

8 Incidence structures and geometric pictures of the non-equivalent least images of \((k;3)\)-arcs

In this section, the incident structures and the geometric picture of the orbits of the stabiliser groups of order at least 3 are given. From Table 3, there is a one non-equivalent image \(B_1\) of \((4;3)\)-arcs which has the stabiliser group \(Z4 \times S3\). This group partitions \(B_1\) into two orbits. They are \(\{1,3,4\}\), and \(\{2\}\). These orbits have the secant distributions \(\{\lambda_3, \lambda_2, \lambda_1, \lambda_0\}\) \(\{1,0,15,15\}\), \(\{0,0,6,25\}\). Thus, the geometric configuration picture of the two orbits is three collinear points and a single point. The non-equivalent images \(B_2\) of \((5;3)\)-arcs have the stabiliser groups \(D4, Z2 \times Z2\) that divide the associated images into a number of orbits, where the group \(D4\) divides the associated image into two orbits. They are \(\{4\}\), \(\{1,2,5,3\}\) and the orbits of \(Z2 \times Z2\) are \(\{1,4\}\), \(\{2,6\}\), \(\{3\}\). Here the values of \(\{\lambda_3, \lambda_2, \lambda_1, \lambda_0\}\) of these orbits are \(\{0,0,6,25\}\), \(\{0,6,12,13\}\), \(\{0,1,10,20\}\), \(\{0,1,10,20\}\), \(\{0,0,6,25\}\). The pictures of these orbits are a single point, vertices of a quadrangle, two collinear points, two collinear points, and a single point. The large stabiliser groups of the non-equivalent images \(B_3\) of \((6;3)\)-arcs are the groups \(Z3, Z4, S3, S4, D4\), each group splits the associated image into a number of orbits. The orbits of the group \(Z3\) are \(\{1,2,6\}\), \(\{3,4,5\}\), the group \(Z4\) are \(\{1,2,3,5\}\), \(\{4\}\), \(\{13\}\), the group \(S3\) are \(\{1,5,8\}\), \(\{2,3,4\}\), the group \(S4\) are \(\{1,2,3,5,20,4\}\), and the group \(D4\)
are \{1,4,6,13\}, \{2,3\}. The secant values of the orbits are \{0,3,12,16\}, \{0,3,12,16\},
\{0,6,12,13\}, \{0,0,6,25\}, \{0,3,12,16\}, \{0,3,12,16\}, \{4,3,18,6\}, \{0,6,12,13\},
\{0,1,10,20\}. The geometric pictures of the orbits are vertices of a triangle, vertices of
a triangle, vertices of a quadrangle, single point, single point, vertices of a triangle, vertices
of a triangle, a quadrilatera, vertices of a quadrangle, and two collinear points respectively.
Also, the large groups of the non-equivalent images \(B_4\) of (7;3)-arcs are the groups \(S_4\), \(D_4\), \(D_6\). The action of these groups on the associated least images are the orbits \{1,5,8\}, \{2,23,4,3\}, \{1,2,5,3\}, \{4\}, \{13,28\}, \{1,3,16,29,5,2\}, \{4\}. The secant
values of the orbits are \{0,3,12,16\}, \{0,6,12,13\}, \{0,6,12,13\}, \{0,1,10,20\}, \{0,15,6,10\},
\{0,0,6,25\}. Here, the pictures of these orbits are vertices of a triangle, vertices of a quadrangle,
vertices of a quadrangle, two collinear points, a conic \(xy + xz − yz\), and a single point.
The descriptions of the large groups of \(B_5\) of (8;3)-arcs and \(B_6\) of (9;3)-arcs are the same as the
descriptions of the previous groups. However, there are two groups of the non-equivalent
images \(B_7\) of (10;3)-arcs and \(B_8\) of (11;3)-arcs, they are the group \(S_5\) that splits the associated
image into one orbit, that is, \{1,2,23,28,13,16,4,3,5,29\}. Here, the geometric picture
is the conic \(x^2 + 3xz − y^2 + 3yz\). Also, the group \(Z5 \times Z4\) that divides the associated image
into two orbits \{1,3,8,5,4,12,17,16,6,19\}, \{2\}. The picture of the first orbit is the conic
\(x^2 + xy + 3xz − y^2 − yz\) while the picture of second orbit is a single point.

**Remark**

The geometric pictures for the above incidence structures are given in the following figures.

![Figure 5: Geometric picture of the group orbit of Z4 \times S3](image)

![Figure 6: Geometric picture of the group orbit of D4](image)

Geometric picture of the group orbit of Z2 \times Z2
Geometric picture of the group orbit of $\mathbb{Z}_3$

Geometric picture of the group orbit of $\mathbb{Z}_4$

Geometric picture of the group orbit of $S_3$

Geometric picture of the group orbit of $S_4$

Geometric picture of the group orbit of $D_4$

Geometric picture of the group orbit of $S_4$
Geometric picture of the group orbit of $D_4$

References

[1] S. Ball, A. Blokhuis, and F. Mazzocca. Maximal arcs in desarguesian planes of odd order do not exist. *Combinatorica*, 17(1):31–41, 1997.

[2] K. Coolsaet and H. Sticker. The complete (k, 3)-arcs of $\text{pg}(2,q)$, $q \leq 13$. *Journal of Combinatorial Designs*, 20(2):89–111, 2012.

[3] A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco. Computer search in projective planes for the sizes of complete arcs. *Journal of Geometry*, 82(1):50–62, Aug 2005.

[4] A. A. Davydov, G. Faina, S. Marcugini, and F. Pambianco. On sizes of complete caps in projective spaces $\text{pg}(n, q)$ and arcs in planes $\text{pg}(2, q)$. *Journal of Geometry*, 94(1):31–58, Sep 2009.

[5] G. Faina and F. Pambianco. On the spectrum of the values $k$ for which a complete $k$-cap in $\text{pg}(n, q)$ exists. *Journal of Geometry*, 62(1):84–98, Jul 1998.

[6] The GAP Group. *GAP – Groups, Algorithms, and Programming, Version 4.7.8*, 2017.

[7] J. W. P. Hirschfeld and G. Korchmáros. Arcs and curves over a finite field. *Finite Fields Appl.*, 5(4):393–408, 1999.

[8] J.W.P. Hirschfeld. *Projective Geometries Over Finite Fields. 2nd edition*. Oxford University Press, 1998.

[9] J.W.P. Hirschfeld and J.A. Thas. Open problems in finite projective spaces. *Finite Fields And Their Applications*, 32:44–81, 2015.