We give a short proof of the sharp weighted bound for sparse operators that holds for all $p$, $1 < p < \infty$. By recent developments this implies the bounds hold for any Calderón-Zygmund operator. The novelty of our approach is that we avoid two techniques that are present in other proofs: two weight inequalities and extrapolation. Our techniques are applicable to fractional integral operators as well.

1. Introduction

In this note we give an accessible proof of the sharp weighted bound for Calderón-Zygmund operators of Hytönen [5]:

\[(1) \quad \|T\|_{L^p(w)} \leq c_{p,T}[w]_{A_p}^{\max(1, p')} \quad 1 < p < \infty.\]

Inequality (1) was originally termed the $A_2$ theorem because if it held for $p = 2$ then it held for all $p$, $1 < p < \infty$; a fact that followed from a sharp version of Rubio de Francia’s extrapolation theorem. Most known proofs of (1) follow this paradigm, beginning with $p = 2$ and then extrapolating [2, 5, 10, 13, 16, 19, 20]. Of these proofs, some of them use two weight testing inequalities [5, 10, 13] while others do not [2, 19, 20]. Some proofs avoid extrapolation, but use two weight testing inequalities [6, 7, 8]. Our proof of (1) holds for all $1 < p < \infty$ and does not use two weight testing inequalities or extrapolation. Moreover, we do not use Bellman functions as the original proofs in [19, 20] do. Our techniques can be thought of as an extension of those in [2] to $p \neq 2$ and were inspired by the mixed estimates in [14]. The methods also apply to fractional integral operators; as a result, we are able to give a
new proof of the bound from [12],

\[ \| I_\alpha \|_{B(L^p(w^p),L^q(w^q))} \leq c_{p,\alpha}[w]_{A_{p,q}}^{(1-\frac{\alpha}{n})\max(1,\frac{p'}{q})}, \]

for a range of \( p \) and \( q \) that satisfy \( \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \). Each of the proofs uses the same tools: dyadic operators, sparse families of cubes, and universal maximal function bounds.

2. Preliminaries

We will consider two central families of integral operators in harmonic analysis. A Calderón-Zygmund operator will be an \( L^2(\mathbb{R}^n) \) bounded linear operator, associated to a standard kernel \( K \) (see Grafakos [4, p. 171]), that has the representation

\[ Tf(x) := \int_{\mathbb{R}^n} K(x,y)f(y) \, dy \]

for \( f \in L^2(\mathbb{R}^n) \) and \( x \notin \text{supp } f \). The family of fractional integral operators or Riesz potentials is defined for \( 0 < \alpha < n \), by

\[ I_\alpha f(x) := \int_{\mathbb{R}^n} \frac{f(y)}{|x-y|^{n-\alpha}} \, dy. \]

Calderón-Zygmund operators are bounded on \( L^p(w) \) for \( 1 < p < \infty \) when \( w \in A_p \):

\[ [w]_{A_p} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x) \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-\frac{n}{\alpha}} \, dx \right)^{p-1} < \infty. \]

Meanwhile, when \( 1 < p < n/\alpha \) and \( 1/q = 1/p - \alpha/n \), the fractional integral operator, \( I_\alpha \), maps \( L^p(w^p) \) into \( L^q(w^q) \) exactly when \( w \in A_{p,q} \):

\[ [w]_{A_{p,q}} := \sup_Q \left( \frac{1}{|Q|} \int_Q w(x)^q \, dx \right) \left( \frac{1}{|Q|} \int_Q w(x)^{-p'} \, dx \right)^{\frac{p}{q}} < \infty. \]

A dyadic grid, denoted \( \mathcal{D} \), is a collection of cubes in \( \mathbb{R}^n \) with the following properties:

(i) each \( Q \in \mathcal{D} \) satisfies \( |Q| = 2^{nk} \) for some \( k \in \mathbb{Z} \);
(ii) if \( Q, P \in \mathcal{D} \) then \( Q \cap P = \emptyset, P \), or \( Q \);
(iii) for each \( k \in \mathbb{Z} \), the family \( \mathcal{D}_k = \{ Q \in \mathcal{D} : |Q| = 2^{nk} \} \) forms a partition of \( \mathbb{R}^n \).

We say that a family of dyadic cubes \( \mathcal{I} \subset \mathcal{D} \) is sparse if for each \( Q \in \mathcal{I} \),

\[ \left| \bigcup_{Q' \subseteq Q \atop Q' \in \mathcal{I}} Q' \right| \leq \frac{1}{2}|Q|. \]
Given a sparse family, $\mathcal{S}$, if we define

$$E(Q) := Q \setminus \bigcup_{Q' \in \mathcal{S}, Q' \subseteq Q} Q',$$

then the family $\{E(Q)\}_{Q \in \mathcal{S}}$ is pairwise disjoint, $E(Q) \subset Q$, and $|Q| \leq 2|E(Q)|$. Sparse families have long been used in Calderón-Zygmund theory, our definition can be found in [9].

We now introduce discrete versions of $T$ and $I_\alpha$ using sparse families of cubes. We begin with the simpler operator $I_\alpha$. Given a dyadic grid $\mathcal{D}$ and $0 < \alpha < n$, define the dyadic fractional integral operator as

$$I^{\mathcal{D}}_\alpha f := \sum_{Q \in \mathcal{D}} |Q|^\frac{\alpha}{n} \int_Q f \, dx \cdot \chi_Q.$$

In [3] it is proven that there exists a finite collection of dyadic grids $\mathcal{D}^1, \ldots, \mathcal{D}^N$ such that

$$(3) \quad I_\alpha f \simeq \max_{1 \leq t \leq N} I^{\mathcal{D}^t}_\alpha f.$$  

We may further decompose the operator $I^{\mathcal{D}}_\alpha$ using sparse families. If $f \geq 0$ is bounded with compact support, then there exists a sparse family depending on $f$, $\mathcal{S} = \mathcal{S}(f)$, such that

$$(4) \quad I^{\mathcal{D}}_\alpha f \simeq \sum_{Q \in \mathcal{S}} |Q|^\frac{\alpha}{n} \int_Q f \, dx \cdot \chi_Q := I^{\mathcal{S}}_\alpha f$$

where the implicit constants depend on the dimension and $\alpha$, but not on $f$ or $\mathcal{S}$. The equivalence (4) can be found in [3, 12, 18, 21]. By combining (3) with (4) we obtain the following theorem.

**Theorem 2.1.** If $X$ and $Y$ are Banach function spaces (see [1, Chapter 1]) then there exists a constant $c_\alpha$ such that

$$\|I_\alpha\|_{\mathcal{S}(X,Y)} \leq c_\alpha \sup_{\mathcal{S} \subseteq \mathcal{D}} \|I^{\mathcal{S}}_\alpha\|_{\mathcal{S}(X,Y)}.$$

To define a dyadic version of a Calderón-Zygmund operator it is not enough to simply take $\alpha = 0$ in the definition of $I^{\mathcal{D}}_\alpha$, because the defining series will not always converge. Dyadic versions of Calderón-Zygmund operators must capture the cancelation of the operator $T$; one way to accomplish this is by using the so called Haar shift operators [5]. Another way is to use families of sparse cubes. Let $\mathcal{S} \subset \mathcal{D}$ be a sparse family and define sparse Calderón-Zygmund operator

$$T^{\mathcal{S}} f := \sum_{Q \in \mathcal{S}} \int_Q f \, dx \cdot \chi_Q.$$
Lerner [15, 16] proved the corresponding version of Theorem 2.1 for Calderón-Zygmund operators.

**Theorem 2.2.** Suppose $X$ and $Y$ are Banach function spaces on $\mathbb{R}^n$ and $T$ is a Calderón-Zygmund operator, then there exists a constant $c_T$

$$
\|T\|_{\mathcal{B}(X,Y)} \leq c_T \sup_{\gamma \in \mathcal{G}} \|T^\gamma\|_{\mathcal{B}(X,Y)}.
$$

Finally we will need one more tool: universal maximal function bounds. Given a Borel measure $\mu$ on $\mathbb{R}^n$ define the universal fractional maximal operator

$$
M_{\alpha,\mu}^{Q} f(x) := \sup_{Q \in \mathcal{D}} \frac{1}{\mu(Q)^{1-\alpha/n}} \int_{Q} |f| \, d\mu \cdot \chi_{Q}(x) \quad 0 \leq \alpha < n.
$$

When $\alpha = 0$ we simply write $M_{\mu}^{Q} = M_{0,\mu}^{Q}$. We have the following theorem concerning the boundedness of $M_{\alpha,\mu}^{Q}$.

**Theorem 2.3.** If $0 \leq \alpha < n$, $1 < p \leq \frac{n}{n-\alpha}$, and $\frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n}$, then

$$
\|M_{\alpha,\mu}^{Q} f\|_{L^q(\mu)} \leq \left(1 + \frac{p'}{q}\right)^{1-\frac{\alpha}{n}} \|f\|_{L^p(\mu)}.
$$

Before we prove Theorem 2.3 we note that the boundedness $M_{\alpha,\mu}^{Q}$ from $L^p(\mu)$ to $L^q(\mu)$ was proven in [17]. However, the constant

$$
\left(1 + \frac{p'}{q}\right)^{1-\frac{\alpha}{n}} = (p')^{1-\frac{\alpha}{n}} \left(1 - \frac{\alpha}{n}\right)^{1-\frac{\alpha}{n}}
$$

seems to be new. When $\alpha = 0$ we get the well known sharp bound

$$
\|M_{\mu}^{Q} f\|_{L^p(\mu)} \leq p' \|f\|_{L^p(\mu)}.
$$

We do not know if the constant in Theorem 2.3 is sharp for $\alpha > 0$.

**Proof.** By the standard properties of dyadic cubes we get the inequality

$$
\mu(\{x : M_{\alpha,\mu}^{Q} f(x) > \lambda\}) \leq \left(\frac{1}{\lambda} \int_{\{M_{\alpha,\mu}^{Q} f > \lambda\}} |f(x)| \, d\mu(x)\right)^{\frac{n}{n-\alpha}}.
$$

Let $q_0 = \frac{n}{n-\alpha}$ and note that if $q$ is defined as in the statement of Theorem 2.3, then $q > q_0$. We have

$$
\int_{\mathbb{R}^n} M_{\alpha,\mu}^{Q} f(x)^q \, d\mu = q \int_{0}^{\infty} \lambda^{q-1} \mu(\{M_{\alpha,\mu}^{Q} f(x) > \lambda\}) \, d\lambda
$$

\[ \leq q \int_{0}^{\infty} \lambda^{q-1} \left(\frac{1}{\lambda} \int_{\{M_{\alpha,\mu}^{Q} f(x) > \lambda\}} |f(x)| \, d\mu(x)\right)^{q_0} \, d\lambda \]
\[
q \left( \int_{\mathbb{R}^n} |f(x)| \left( \int_0^{M_{\alpha,\mu} f(x)} \lambda^{q-q_0-1} \, d\lambda \right)^{1/q_0} \, d\mu(x) \right)^{q_0} \\
= \frac{q}{q-q_0} \left( \int_{\mathbb{R}^n} |f(x)| M_{\alpha,\mu} f(x)^{\frac{q}{p}} \, d\mu(x) \right)^{q_0} \\
\leq \frac{q}{q-q_0} \|f\|_{L^p(\mu)}^{q_0} \|M_{\alpha,\mu} f\|_{L^q(\mu)}^{q_0}
\]

where in the second inequality we used Minkowski’s integral inequality and Hölder’s inequality in the last. Using the fact that \( \frac{q}{q-q_0} = 1 + \frac{p'}{q} \), we see that this yields the desired inequality.

3. Main Results

We now prove rather precise weighted estimates for the sparse operators \( T^\mathcal{S} \) and \( I^\mathcal{S}_\alpha \). By the Theorems 2.1 and 2.2 we see that these bounds will imply inequalities (1) and (2). At the heart of our proof we will simply use the constant \( [w]_{A_p} \) in the form of its definition:

\[
[w]_{A_p} = \sup_Q \left( \int_Q w \, dx \right) \left( \int_Q w^{1-p'} \, dx \right)^{-p-1} = \sup_Q \frac{w(Q)\sigma(Q)^{p-1}}{|Q|^p}
\]

where \( \sigma = w^{1-p'} \). We also use the properties of the family \( \{E(Q)\}_{Q \in \mathcal{S}} \): disjointness, \( E(Q) \subset Q \), and \( |Q| \leq 2|E(Q)| \). Our main theorem is the following.

**Theorem 3.1.** Suppose \( \mathcal{D} \) is a dyadic grid, \( \mathcal{S} \subset \mathcal{D} \) is a sparse family, \( 1 < p < \infty \), and \( w \in A_p \). Then the following estimate holds

\[
\|T^\mathcal{S}\|_{\mathcal{B}(L^p(w))} \leq c_p[w]_{A_p}^{\max(1, \frac{p'}{p})}
\]

where

\[
c_p = pp'2^{\max(\frac{p}{p'}, \frac{p'}{p})}.
\]

**Proof.** Since \( T^\mathcal{S} \) is a positive operator we may assume \( f \geq 0 \). We first consider the case \( p \geq 2 \), and let \( \sigma = w^{1-p'} \). We will use the well known formulation

\[
\|T^\mathcal{S}\|_{\mathcal{B}(L^p(w))} = \|T^\mathcal{S}(\cdot, \sigma)\|_{\mathcal{B}(L^p(\cdot, L^p(\sigma), L^p(w)))}.
\]

If \( g \geq 0 \) belongs to \( L^{p'}(w) \), then by duality it suffices to estimate

\[
\int_{\mathbb{R}^n} T^\mathcal{S}(f \sigma) gw \, dx = \sum_{Q \in \mathcal{S}} \int_Q f \sigma \, dx \cdot \int_Q gw \, dx.
\]
Multiplying and dividing by the precursor to the $A_p$ constant, we have

$$\sum_{Q \in \mathcal{Q}} \int_Q f \sigma \, dx \cdot \int_Q gw \, dx$$

$$= \sum_{Q \in \mathcal{Q}} \frac{w(Q) \sigma(Q)^{p-1}}{|Q|^p w(Q) \sigma(Q)^{p-1}} \int_Q f \sigma \, dx \cdot \int_Q gw \, dx$$

$$\leq [w]_{A_p} \sum_{Q \in \mathcal{Q}} \frac{|Q|^{p-1}}{w(Q) \sigma(Q)^{p-1}} \int_Q f \sigma \, dx \cdot \int_Q gw \, dx$$

$$= [w]_{A_p} \sum_{Q \in \mathcal{Q}} \frac{1}{\sigma(Q)} \int_Q f \sigma \, dx \cdot \frac{1}{w(Q)} \int_Q gw \, dx \cdot |Q|^{p-1} \sigma(Q)^2 - p$$

$$\leq 2^{p-1} [w]_{A_p} \sum_{Q \in \mathcal{Q}} A_\sigma(f, Q) A_w(g, Q) \cdot |E(Q)|^{p-1} \sigma(Q)^2 - p$$

where

$$A_\sigma(f, Q) = \frac{1}{\sigma(Q)} \int_Q f \sigma \, dx \quad \text{and} \quad A_w(g, Q) = \frac{1}{w(Q)} \int_Q gw \, dx,$$

and in the last inequality we have used $|Q| \leq 2|E(Q)|$. At this point we have the correct power on the constant $[w]_{A_p}$, so we must estimate the sum in (5) without using the $A_p$ property of the weight $w$. Since $p \geq 2$ and $E(Q) \subset Q$ we have

$$\sigma(Q)^{2-p} \leq \sigma(E(Q))^{2-p},$$

(note: $|E(Q)| \geq |Q|/2 > 0$ so $\sigma(E(Q)) > 0$) which in turn yields

$$\int_{\mathbb{R}^n} T^\mathcal{Q}(f \sigma)gw \, dx$$

$$\leq 2^{p-1} [w]_{A_p} \sum_{Q \in \mathcal{Q}} A_\sigma(f, Q) A_w(g, Q) \cdot |E(Q)|^{p-1} \sigma(E(Q))^{2-p}. \quad (5)$$

By Hölder’s inequality we have

$$|E(Q)| \leq w(E(Q))^{\frac{1}{p}} \sigma(E(Q))^{\frac{1}{p^*}},$$

so

$$|E(Q)|^{p-1} \sigma(E(Q))^{2-p} \leq \sigma(E(Q))^{\frac{1}{p^*}} w(E(Q))^{\frac{1}{p^*}}. \quad (6)$$

Utilizing inequality (6) in the sum in (5), followed by a discrete Hölder inequality, followed by the maximal function bounds from Theorem 2.3, we arrive at the desired estimate:

$$\sum_{Q \in \mathcal{Q}} A_\sigma(f, Q) A_w(g, Q) \cdot |E(Q)|^{p-1} \sigma(E(Q))^{2-p}$$
\[ \leq \sum_{Q \in \mathcal{S}} A_\sigma(f, Q)A_w(g, Q)\sigma(E(Q))^{\frac{1}{p'}}w(E(Q))^{\frac{1}{p}} \]

\[ \leq \left( \sum_{Q \in \mathcal{S}} A_\sigma(f, Q)^p\sigma(E(Q)) \right)^{\frac{1}{p'}} \cdot \left( \sum_{Q \in \mathcal{S}} A_w(g, Q)^p'w(E(Q)) \right)^{\frac{1}{p'}} \]

\[ \leq \|M^\mathcal{S}_f f\|_{L^p(\sigma)}\|M^\mathcal{S}_w g\|_{L^p'(w)} \]

\[ \leq pp'\|f\|_{L^p(\sigma)}\|g\|_{L^p'(w)}. \]

The case \( 1 < p < 2 \) follows from duality, since \((T^\mathcal{S})^* = T^\mathcal{S}\), we have

\[ \|T^\mathcal{S}\|_{L^p(w)} = \|T^\mathcal{S}\|_{L^p'(\sigma)} \leq pp'2^{p'-1}[\sigma]_{A_{p'}} = pp'2^{p'-1}[w]_{A_{p'}}^{\frac{1}{p'-1}}. \]

\[ \Box \]

**Theorem 3.2.** Suppose \( D \) is a dyadic grid, \( \mathcal{S} \subset D \) is a sparse family, \( 0 < \alpha < n, 1 < p < n/\alpha, \frac{1}{q} = \frac{1}{p} - \frac{\alpha}{n} \), \( \min\left(\frac{p'}{q}, \frac{q'}{p}\right) \leq 1 - \frac{\alpha}{n} \), and \( w \in A_{p,q} \). Then the following estimate holds

\[ \|I^\mathcal{S}_\alpha\|_{\mathcal{B}(L^p(w^p), L^q(w^q))} \leq c_{p,\alpha}[w]_{A_{p,q}}^{\left(1 - \frac{\alpha}{n}\right)\max\left(1, \frac{p'}{q'}\right)} \]

where

\[ c_{p,\alpha} = p' \left(1 + \frac{q}{p'}\right)^{-\frac{\alpha}{n}} 2^{\left(1 - \frac{\alpha}{n}\right)\max\left(\frac{p}{p'}, \frac{q}{q'}\right)}. \]

Before we prove Theorem 3.2 we remark that we have a somewhat unnatural assumption:

\[ \min\left(\frac{p'}{q}, \frac{q'}{p}\right) \leq 1 - \frac{\alpha}{n}. \]

Because of this we do not obtain the full range of \( p \) and \( q \) for the fractional integral operator \( I_\alpha \). We do not know if these techniques can be extended to the full range. We do point out, however, that inequality (7) is always satisfied when \( \alpha = 0 \) since \( \min(x, 1/x) \leq 1 \) for \( x > 0 \). It is for this reason that we do not encounter this obstacle in Theorem 3.1.

**Proof.** Suppose \( \frac{p'}{q} \leq 1 - \frac{\alpha}{n} \). Let \( u = w^q \) and \( \sigma = w^{-p'} \), so that

\[ [w]_{A_{p,q}} = \sup_Q \frac{u(Q)\sigma(Q)^{\frac{3}{2}}}{|Q|^{1+\frac{\alpha}{n}}} = [u]_{A_{1+\frac{\alpha}{n}, q^{\frac{3}{2}}}}. \]

Define the exponent \( r = 1 + \frac{\alpha}{p'} \) and notice that \( r' = 1 + \frac{q'}{q} \). We again use the fact

\[ \|I^\mathcal{S}_\alpha\|_{\mathcal{B}(L^p(w^p), L^q(w^q))} = \|I^\mathcal{S}_\alpha(\cdot, \sigma)\|_{\mathcal{B}(L^p(\sigma), L^q(u))}. \]
For \( g \in L^q(u) \), we have

\[
\int_{\mathbb{R}^n} I_\alpha^g(f\sigma)gu \, dx = \sum_{Q \in \mathcal{S}} |Q|^\frac{\alpha}{p} \int_Q f\sigma \, dx \cdot \int_Q gu \, dx.
\]

Proceeding as above, we multiply and divide by the precursor to the \( A_{p,q} \) constant raised to the power \( 1 - \frac{\alpha}{n} \):

\[
\sum_{Q \in \mathcal{S}} \frac{u(Q)^{1-\frac{\alpha}{n}}\sigma(Q)^{\frac{\alpha}{p}(1-\frac{\alpha}{n})}}{|Q|^{(1+\frac{\alpha}{p})(1-\frac{\alpha}{n})}} \frac{|Q|^\frac{\alpha}{p}(1-\frac{\alpha}{n})}{u(Q)^{1-\frac{\alpha}{n}}\sigma(Q)^{\frac{\alpha}{p}(1-\frac{\alpha}{n})}} \int_Q f\sigma \, dx \cdot \int_Q gu \, dx
\]

\[
\leq [w]_{A_{p,q}}^{1-\frac{\alpha}{n}} \sum_{Q \in \mathcal{S}} \frac{1}{\sigma(Q)} \int_Q f\sigma \, dx \cdot \frac{1}{u(Q)^{1-\frac{\alpha}{n}}} \int_Q gu \, dx
\]

\[
\cdot |Q|^\frac{\alpha}{p}(1-\frac{\alpha}{n}) \sigma(Q)^{-\frac{\alpha}{p}(1-\frac{\alpha}{n})}
\]

\[
= [w]_{A_{p,q}}^{1-\frac{\alpha}{n}} \sum_{Q \in \mathcal{S}} A_\sigma(f, Q)A_{\alpha,u}(g, Q) \cdot |Q|^\frac{\alpha}{p}(1-\frac{\alpha}{n}) \sigma(Q)^{-\frac{\alpha}{p}(1-\frac{\alpha}{n})}
\]

where \( A_{\alpha,u}(g, Q) = u(Q)^{\frac{\alpha}{n}-1} \int_Q gu \, dx \). Once again, we have the correct power on the constant \([w]_{A_{p,q}}\), and so from this point we will not be able to use the \( A_{p,q} \) properties of the weight. Since \( \frac{\alpha'}{q} \leq 1 - \frac{\alpha}{n} \) (this is exactly where we use assumption (7)) and \( E(Q) \subset Q \) we have

\[
\sigma(Q)^{-\frac{\alpha}{p}(1-\frac{\alpha}{n})} \leq \sigma(E(Q))^{-\frac{\alpha}{p}(1-\frac{\alpha}{n})}
\]

which, along with \( |Q| \leq 2|E(Q)| \) yields the bound

\[
\int_{\mathbb{R}^n} I_\alpha^g(f\sigma)gu \, dx \leq 2^\frac{\alpha}{p'} [w]_{A_{p,q}}^{1-\frac{\alpha}{n}} \sum_{Q \in \mathcal{S}} A_\sigma(f, Q)A_{\alpha,u}(g, Q) \cdot |E(Q)|^{\frac{\alpha}{p'}(1-\frac{\alpha}{n})} \sigma(E(Q))^{1-\frac{\alpha}{p'}}.
\]

Moreover, by Hölder’s inequality with \( r \) and \( r' \) we have

\[
|E(Q)| \leq u(E(Q))^{\frac{1}{r'}} \sigma(E(Q))^{\frac{1}{p'}}.
\]

Now notice that

\[
\frac{q}{p'} \left(1 - \frac{\alpha}{n}\right) = \frac{q}{p'} \left(\frac{1}{q} + \frac{1}{p'}\right) = \frac{1}{p'} \left(1 + \frac{q}{p'}\right) = \frac{r}{p'}
\]

which yields

\[
|E(Q)|^{\frac{\alpha}{p'}(1-\frac{\alpha}{n})} \sigma(E(Q))^{1-\frac{\alpha}{p'}} = |E(Q)|^{\frac{\alpha}{p'} \sigma(E(Q))}^{1-\frac{\alpha}{p'}} \leq u(E(Q))^{\frac{1}{r'}} \sigma(E(Q))^{\frac{r}{p'}} \sigma(E(Q))^{1-\frac{\alpha}{p'}} = u(E(Q))^{\frac{1}{r'}} \sigma(E(Q))^{\frac{r}{p'}}.
\]

\[
\sqrt{u(E(Q))^{\frac{1}{r'}} \sigma(E(Q))^{\frac{r}{p'}}} \sigma(E(Q))^{1-\frac{\alpha}{p'}} = u(E(Q))^{\frac{1}{r'}} \sigma(E(Q))^{\frac{r}{p'}}.
\]
Using inequality (9) to estimate the sum in (8) we have
\[
\sum_{Q \in \mathcal{S}} A_\sigma(f, Q) A_{\alpha, u}(g, Q) \cdot |E(Q)|^{\frac{q}{p'}(1 - \frac{q}{n})} \sigma(E(Q))^{1 - \frac{q}{p'}(1 - \frac{q}{n})}
\]
\[
\leq \sum_{Q \in \mathcal{S}} A_\sigma(f, Q) A_{\alpha, u}(g, Q) \cdot \sigma(E(Q))^{\frac{1}{p}} u(E(Q))^{\frac{1}{p'}}
\]
\[
\leq \left( \sum_{Q \in \mathcal{S}} A_\sigma(f, Q)^{p} \sigma(E(Q)) \right)^{\frac{1}{p}} \cdot \left( \sum_{Q \in \mathcal{S}} A_{\alpha, u}(g, Q)^{p'} u(E(Q)) \right)^{-\frac{1}{p'}}
\]
\[
\leq \| M^{p}_\sigma f \|_{L^p(\sigma)} \| M^{p'}_{\alpha, u} g \|_{L^{p'}(u)}
\]
\[
\leq p' \left( 1 + \frac{q}{p'} \right)^{1 - \frac{q}{n}} \| f \|_{L^p(\sigma)} \| g \|_{L^{p'}(u)}.
\]
Where in the last line we have used Theorem 2.3 for the boundedness of $M^{p'}_{\alpha, u}$ from $L^{p'}(u)$ to $L^{p'}(u)$ (note: $\frac{1}{p'} = \frac{1}{q} - \frac{2}{n}$) and $M^p_\sigma$ from $L^p(\sigma)$ to $L^p(\sigma)$. The case $\frac{q'}{q} \geq (1 - \frac{q}{n})^{-1}$ again follows from duality; we omit the details. □

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