Formation of Space-Time Structure in a Forest-Fire Model

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Abstract

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I. INTRODUCTION

In nature, there exist many systems where some kind of activity propagates without being damped. They are called excitable systems and comprise such different phenomena as spreading of diseases, oscillating chemical reactions, propagation of electrical activity in neurons or heart muscles, and many more (For a review on excitable systems see e.g. [1,2]). All these systems exist essentially in three states which can be called quiescent, excited, and refractory. Excitation spreads from one place to its neighbors if they are quiescent. After excitation, a site needs some time to recover its quiescent state. In many of these systems spiral waves are observed. So far, these systems are mainly described by differential equations or deterministic computer models. In this paper, we present a stochastic forest-fire model which can be viewed as computer model for excitable systems. It is defined as follows: Each site of a $d$-dimensional hypercubic lattice of size $L^d$ is occupied by a tree, a burning tree, or it is empty. During one timestep, the system is parallely updated according to the following rules

- burning tree $\rightarrow$ empty site
- tree $\rightarrow$ burning tree with probability $1 - g$ if at least one nearest neighbor is burning
- tree $\rightarrow$ burning tree with probability $f \ll 1$ if no neighbor is burning
- empty site $\rightarrow$ tree with probability $p$.

In its original version, which has been introduced by P. Bak, K. Chen, and C. Tang, the forest-fire model contained only the tree growth parameter $p$ [3].

Starting with arbitrary initial conditions, this system approaches after a short transition period a steady state the properties of which depend on the parameter values. Throughout this paper, we assume that the system size $L$ is large enough that no finite-size effects occur. In the simulations, we always chose periodic boundary conditions. Let $\rho_e$, $\rho_t$, and $\rho_f$ be the mean density of empty sites, of trees, and of burning trees in the steady state. These densities are related by the equations
\[
\rho_e + \rho_t + \rho_f = 1 \tag{1.1}
\]

and

\[
\rho_f = p\rho_e. \tag{1.2}
\]

The second equation says that the mean number of growing trees equals the mean number of burning trees in the steady state.

The most interesting behavior in this model occurs for those regions in the parameter space where the fire density decreases to zero. Here a phase transition takes place from a steady state with fire to a steady state without fire, and large scale structures in space and time occur. Eq. (1.2) indicates that the fire density decreases to zero when \( p \) approaches zero. Large-scale structures can only be expected if additionally \( f \ll p \). Otherwise trees cannot live long enough to become part of large forests. In the case \( f = 0 \), the fire density decreases to zero not only for \( p \to 0 \) but also when the immunity \( g \) approaches a critical value \( g_C(p) \).

In this paper, we first present the mean-field theory of the general forest-fire model which gives a rough idea of the phase diagram. Then, we focus on the three parameter regions indicated above: In Sect. [II] we describe quasideterministic behavior which arises in the limit \( p \to 0 \) with \( f = g = 0 \). In Sect. [III], we report self-organized critical behavior which arises for \( (g = 0, p \to 0, f/p \to 0) \) under the condition that the time scales of tree growth and burning down of forest clusters are separated. In Sect. [IV] we investigate the percolation-like transition which takes place for \( g \to g_C(p) \). In the last section, we discuss and summarize the results.

**II. MEAN-FIELD THEORY**

The mean-field theory is only a rough approximation of the true behavior of a model, since spatial and temporal correlations are neglected. In mean-field theory, the dynamics of the forest-fire model is completely described in terms of the three densities \( \rho_e, \rho_t, \) and \( \rho_f \).
Each site is assumed to be in state $i$ with the probability $\rho_i$ independently of the state of the neighboring sites. Consequently, the change of the densities during one timestep is given by

\[
\Delta \rho_e = \rho_f - p\rho_e \\
\Delta \rho_t = p\rho_e - \rho_t (1 - g) \left( f + (1 - f)(1 - (1 - \rho_f)^{2d}) \right) \\
\Delta \rho_f = -\rho_f + \rho_t (1 - g) \left( f + (1 - f)(1 - (1 - \rho_f)^{2d}) \right).
\]

The first equation says, that all burning trees become empty sites and that the portion $p$ of all empty sites become populated by trees. The forest density changes when trees grow or catch fire. This is expressed in the second equation, where $(1 - (1 - \rho_f)^{2d})$ is the probability that a tree has at least one burning neighbor. The third equation results from the other two and equation (1.1). Without the lightning parameter $f$, these equations are also derived in [4].

Starting with arbitrary initial conditions, this model evolves to a steady state where the three densities are constant in time. The $\Delta \rho_i$ are zero in the steady state. Eqs. (2.1) together with Eqs. (1.1) and (1.2) then lead to an equation for the fire density

\[
\rho_f = (1 - g)(1 - \rho_f(1 + \frac{1}{p}))(1 - (1 - f)(1 - \rho_f)^{2d}).
\]

From this equation, we can deduce the phase diagram of the system. A steady state without fire, i.e. a solution of (2.2) with $\rho_f = 0$ exists only if $p = 0$ or $g = 1$ or $f = 0$. When $p \neq 0$ and $g = 1$ or $f = 0$, the steady state is a completely green forest. When $p = 0$, any state with $\rho_f = 0$ is stationary. In the rest of the parameter space, Eq. (2.2) has just one solution $\rho_f \neq 0$. In the plane $f = 0$, Eq. (2.2) also has a solution $\rho_f \neq 0$ as long as $g < g_C$ with the critical immunity

\[
g_C = 1 - 1/2d.
\]

In this region, there exists a phase without fire and a phase with fire. When the immunity $g$ approaches its critical value, the fire density in the second phase decreases to zero. At $g_C$, a continuous transition takes place from a region with two phases to a region with a single
phase. The phase diagram is shown in Fig. 1. Only the planes $p = 0$, $g = 1$ and $f = 0$ are visible. In the rest of the phase space, there is a state with nonvanishing fire density.

In this paper, we focus on phase transitions in regions where either $f = 0$ or $f/p$ and $p$ very small. Near the critical point $f = p = 0$, the fire density is

$$
\rho_f = p \left(1 - \frac{1}{2d(1-g)}\right) + O(p^2) + \frac{f}{2d(2d(1-g) - 1)}(1 + O(p)) + O(f^2),
$$

and near the critical point $g = g_C$, $f = 0$, it is

$$
\rho_f = 2d(g_C - g)/(d + 1/p + 1/2).
$$

The density of trees is at both critical points

$$
\rho_t = 1 - \rho_f - \rho_f/p = 1/(2d(1-g)),
$$

and a burning tree ignites a given neighbor with probability $(1 - g)\rho_t = 1/2d$. The fire therefore propagates in a forest where the density of burnable trees is at the percolation threshold, i.e. the critical points of the mean field forest-fire model are identical to a critical point in percolation theory.

Since the mean-field theory neglects correlations, the true behavior of the forest-fire model is much more complicated. The mean-field theory cannot distinguish between the limit $(\lim_{p \to 0} \lim_{f \to 0})$ where the model shows quasideterministic behavior and the limit $(\lim_{f/p \to 0})$ with $(p \ll f/p)$ where the model is self-organized critical. Another shortcoming of the mean-field theory is that it gives no dependence of $g_C$ on $p$.

### III. THE QUASIDETERMINISTIC STATE

In this section, we consider the case $f = g = 0$ with $p$ very small. Fire spreads from burning trees to their neighbors, but cannot occur spontaneously. This is the forest-fire model as originally introduced by P. Bak, K. Chen, and C. Tang [3], which is – in contrast to earlier assumptions – not self-organized critical [3]. When the initial conditions are chosen in such a way that the fire does not die during the first few timesteps, the system...
develops a steady state with nonvanishing fire density. With decreasing $p$, the fire fronts assume a more and more regular, spiral-shaped form. A snapshot of such a steady state is shown in Fig. 2. A quantitative analysis of the spatial and temporal correlations of the fire reveals that the system has a characteristic length scale and a characteristic time scale both of which are proportional to $1/p$ and become more distinct with decreasing $p$. The model therefore becomes more and more deterministic with decreasing $p$. The origin of the spirals is the following: Fire fronts separate empty areas from forest areas. They move into the forest, i.e. the empty area grows at the cost of the forest. At the end of a fire front, there is a motion in the opposite direction: since there is no fire, the forest grows into the empty area. So the fire front and the tree growth “front” wind around each other, and the end of the fire front becomes a spiral center. The distance between two windings of a spiral is of the order $1/p$ which is the time the forest needs to grow again after it has been burned. This is the origin of the time and length scales mentioned above. There even is an explanation for the increasing determinism for small $p$. When $p$ is decreased by a factor $b$, the length and time scale in the system are increased by the same factor. Between two spiral arms, there are $b$ times as many sites than before. On a length scale proportional to the inverse tree growth rate, the forest density in front of the fire varies less than before (since the average is taken over more sites), and the fire front therefore becomes increasingly smooth.

IV. THE SELF-ORGANIZED CRITICAL STATE

In this section again we set $g = 0$ and introduce the lightning probability $f$. The ratio $f/p$ determines how many trees grow between two lightnings. We consider the case where $f/p$ is small and choose for a given value of $f/p$ a tree growth rate $p$ which is so small that a forest cluster which is struck by lightning burns down before new trees grow at its edge. Under this condition large and small forest clusters burn down in the same way, i.e. large and small fires are similar. In this situation, there occur fires of any size in the system, since lightning might strike any forest cluster and since there exist very large forest clusters
when \( f/p \) is small. So the system is self-organized critical \[8\] in the sense of Bak, Tang, and Wiesenfeld \[7\]. Fig. 3 shows a snapshot of the steady state. The double time scale separation (tree growth occurs much more often than lightning, and burning down of forest clusters is much faster than tree growth) is the essential condition required for self-organized critical behavior in the forest-fire model. The system is self-organized because the steady state is independent of the initial conditions and independent of the exact values of the parameters as long as time scales are separated in the manner mentioned above. It is critical because there are power-law correlations over long distances and long time intervals.

In the following, we first present a scaling theory for the self-organized critical state (subsection IV A). Then we show results of computer simulations in two dimensions which confirm this scaling theory (subsection IV B). In one dimension, the self-organized critical forest-fire model is nontrivial, and yet exact results can be obtained. This is shown in subsection IV C.

A. Scaling theory

Let \( \bar{s} \) be the mean number of trees destroyed by a lightning stroke. In the steady state it equals the mean number of trees growing between two lightning strokes and is given by

\[
\bar{s} = p(1 - \rho_t)/f \rho_t. \tag{4.1}
\]

In the limit \( f/p \to 0 \), this number diverges proportionally to \( (f/p)^{-1} \), since \( \rho_t \) must approach a constant value for small \( f/p \). Eq. (4.1) represents a power law indicating a critical point in the limit \( f/p \to 0 \).

Let \( n(s) \) be the mean number of forest clusters per unit volume consisting of \( s \) trees. Then the mean forest density is

\[
\rho_t = \sum_{1}^{\infty} s n(s), \tag{4.2}
\]

and the mean number of trees destroyed by a lightning stroke is
\[ \bar{s} = \sum_{1}^{\infty} s^2 n(s) / \rho_t. \] (4.3)

Since \( \rho_t(f/p \to 0) \) is finite and \( \bar{s} \) diverges \( \propto (f/p)^{-1} \), these equations imply that \( n(s) \) decreases at least like \( s^{-2} \) but not faster than \( s^{-3} \). As long as the system is not exactly at the critical point \( f/p = 0 \), i.e. for finite \( f/p \), there must be a cutoff in the cluster size distribution for very large forest clusters. We conclude that \[ n(s) \propto s^{-\tau} C(s/s_{\text{max}}) \] (4.4)

with \( 2 \leq \tau \leq 3 \) and

\[ s_{\text{max}}(f/p) \propto (f/p)^{-\lambda} \propto \bar{s}^\lambda. \] (4.5)

The cutoff function \( C(x) \) is more or less constant for \( x \leq 1 \) and decreases to zero for large \( x \). Eqs. (4.3) – (4.7) yield \( \bar{s} \propto s_{\text{max}}^{3-\tau} \), which leads to the scaling relation

\[ \lambda = 1/(3 - \tau). \] (4.6)

In the case \( \tau = 2 \), the right-hand side of Eq. (1.4) acquires a factor \( 1 / \ln(s_{\text{max}}) \) since the forest density given by Eq. (1.2) must not diverge in the limit \( f/p \to 0 \). The mean number of forest clusters per unit volume \( \sum_{1}^{\infty} n(s) \), therefore, decreases to zero for \( f/p \to 0 \), and consequently the forest density approaches the value 1.

We also introduce the cluster radius \( R(s) \) which is the mean distance of the cluster trees from their center of mass. It is related to the cluster size \( s \) by

\[ s \propto R(s)^\mu \] (4.7)

with the fractal dimension \( \mu \). The correlation length \( \xi \) is defined by

\[ \xi^2 = \sum_{s=1}^{\infty} s^2 n(s) R^2(s) / \sum_{s=1}^{\infty} s^2 n(s) \]

\[ \propto (f/p) \int_{1}^{\infty} s^{2-\tau+2/\mu} C(s/s_{\text{max}}) \, ds \propto (f/p) s_{\text{max}}^{3-\tau+2/\mu} \int_{0}^{\infty} x^{2-\tau+2/\mu} C(x) \, dx \]

\[ \propto (f/p)^{-2\lambda/\mu}. \]
We conclude

\[ \xi \propto (f/p)^{-\nu} \] with \( \nu = \lambda/\mu. \) \hfill (4.8)

Another quantity of interest is the mean cluster radius

\[ \bar{R} = \sum_{s=1}^{\infty} sn(s)R(s)/\sum_{s=1}^{\infty} sn(s) \]

\[ \propto (f/p)^{-(\nu-\lambda+1)}, \text{ if } \nu - \lambda + 1 \geq 0; \]
\[ = \text{ const., if } \nu - \lambda + 1 < 0. \]

This leads to

\[ \bar{R} \propto (f/p)^{-\tilde{\nu}} \] with \( \tilde{\nu} = \min(0, \nu - (\lambda - 1)). \)
\hfill (4.9)

The mean forest density \( \rho_t \) approaches its critical value \( \rho_c^t = \lim_{f/p \to 0} \rho_t \) via a power law

\[ \rho_c^t - \rho_t \propto (f/p)^{1/\delta}. \] \hfill (4.10)

Finally, we introduce some exponents characterizing the temporal behavior of the fire. Let \( T(s) \) be the average time a cluster of size \( s \) needs to burn down when ignited, and \( N(T) \) the portion of fires that live exactly for \( T \) timesteps. Then the exponents \( b \) and \( \mu' \) are defined by

\[ s \propto (T(s))^{\mu'} \] and \( N(T) \propto T^{-b}. \)
\hfill (4.11)

From

\[ N(T)dT \propto sn(s)ds \]

follows the scaling relation

\[ b = \mu' (\tau - 2) + 1. \] \hfill (4.12)

The timescale of the system is set by

\[ T_{\max} = T(s_{\max}) \propto (f/p)^{-\nu'} \] \hfill (4.13)

with
\[ \nu' = \lambda / \mu'. \]  

(4.14)

The dynamical critical exponent \( z \) is defined by

\[ T_{\text{max}} \propto \xi^z, \]

which leads to

\[ z = \mu / \mu' = \nu' / \nu. \]  

(4.15)

The condition of time scale separation now can be expressed in terms of the critical exponents and reads

\[ (f/p)^{-\nu'} \ll p^{-1} \ll f^{-1}, \]  

(4.16)

or equivalently

\[ f \ll p \ll f^{\nu'/(1 + \nu')}. \]  

(4.17)

The average lifetime of fires is

\[ \bar{T} = \sum_{s=1}^{\infty} s n(s) T(s) / \sum_{s=1}^{\infty} s n(s) \propto (f/p)^{-\bar{\nu}'} \]  

(4.18)

with

\[ \bar{\nu}' = \text{Min}(0, \nu' - (\lambda - 1)). \]  

(4.19)

The average number \( N_s(t) \) of trees that burn \( t \) timesteps after a cluster of size \( s \) is struck by lightning enters the definition of the temporal fire-fire correlation function \( G(\tau) \)

\[ G(\tau) \propto \sum_{s=1}^{\infty} n(s) s \sum_{t=0}^{\infty} N_s(t) N_s(t + \tau). \]

The power spectrum is the Fourier transform of the fire-fire correlation function

\[ G(\omega) = 2 \int_0^{\infty} G(\tau) \cos(\omega \tau) d\tau \propto \omega^{-\alpha} \text{ for small } \omega. \]  

(4.20)
B. Critical exponents in two dimensions

In two dimensions, the values of the critical exponents were determined by computer simulations \cite{9} in systems of size \( \leq 16384^2 \) and with parameters \( f/p \geq 1/32000 \). Fig. 4 shows the distribution \( sn(s) \) as function of \( s \) which gives the critical exponent \( \tau \). The values of all critical exponents are given in Table 4. Except for \( z \) and \( b \), these exponents have directly been determined in simulations. They satisfy the scaling relations derived in subsection \( \text{IV A} \). The exponents \( \tau, \nu, \delta \) and \( \lambda \) have also been determined by P. Grassberger \cite{10}. The values of \( \tau, \nu \), and \( \delta \) agree with ours, but not the value of \( \lambda \). Our result for \( \lambda \) satisfies the above scaling relations (4.6), (4.8), and (4.9). The value of the critical forest density is \( \rho_c = 0.4081(7) \) in agreement with \cite{10}. In \cite{8} simulations on samples of size \( 500^2 \) seemed to suggest classical values of the critical exponents (\( \tau = \mu = 2 \)).

C. Critical exponents in one dimension

Even in one dimension, the self-organized critical forest-fire model is nontrivial. Since there exist no infinitely large forest clusters for \( \rho_t < 1 \), the forest density has to approach the value 1 in the critical limit \( f/p \to 0 \). From the consideration which follows Eq. (4.6) we conclude \( \tau = 2 \). The value of this and other critical exponents as well as other exact results can be determined analytically. A detailed calculation is given in \cite{11}. A more intuitive approach is presented in the following. Consider a string of \( n \) sites with \( n \ll s_{\text{max}} \). The probability that lightning strikes this string is very small. Therefore trees grow on it until it is covered by a completely dense forest. Just before this dense forest portion burns down, it is part of a large forest which will be struck by lightning somewhere. After the fire has passed our string, it will be completely empty. Let \( P_n(m) \) be the probability that the string is covered by \( m \) trees. Since trees grow randomly and since fire does not pass the string before all its trees are grown, the \( P_n(m) \) are related by

\[
np_n(0) = p(1 - \rho_t),
\]
\[ p(n-m)P_n(m) = p(n-m+1)P_n(m-1) \] for \( m \neq 0, n, \)

which lead to the result

\[
P_n(m) = (1 - \rho_t)/(n-m) \quad \text{for} \quad m < n,
\]

\[
P_n(n) = 1 - (1 - \rho_t) \sum_{m=0}^{n-1} 1/(n-m)
\]

\[
= 1 - (1 - \rho_t) \sum_{m=1}^{n} 1/m.
\] (4.21)

Eq. (4.21) is exact up to terms of order \( f/p \) and thus the more accurate the closer to the critical point. The size distribution of forest clusters is

\[
n(s) = P_{s+2}(01 \ldots 10) = \frac{P_{s+2}(s)}{\binom{s+2}{s}}
\]

\[
= \frac{1 - \rho_t}{(s+1)(s+2)} \simeq (1 - \rho_t)s^{-2}.
\] (4.22)

This is a power law with the critical exponent \( \tau = 2 \). Fig. 5 shows the size distribution \( sn(s) \) for fires as determined by computer simulations for \( f/p = 1/25000 \). The smooth curve is the analytic result (4.22) which fits the simulation perfectly as long as \( s < s_{\text{max}} \simeq 1000 \).

The size distribution \( n_e(s) \) of clusters of empty sites is

\[
n_e(s) = P_{s+2}(10 \ldots 01) = \frac{P_{s+2}(2)}{\binom{s+2}{2}} = \frac{2(1 - \rho_t)}{s(s+1)(s+2)},
\] (4.23)

which is also derived in [12]. The result (4.21) contains much more information beyond the cluster size distribution. So it can be shown that a mean number of \( s+2 \) trees is added to a forest cluster of size \( s \) when a tree grows at its edge, i.e. the growth speed of forest clusters is proportional to their size.

In the limit \( f/p \to 0 \), \( s_{\text{max}} \) diverges and \( n \) may become large. Nevertheless \( P_n \) must remain \( \geq 0 \). It follows

\[
(1 - \rho_t) \propto [\ln(s_{\text{max}})]^{-1},
\] (4.24)

i.e. the forest density approaches logarithmically the value one in the limit \( f/p \to 0 \), and the mean number of forest clusters per unit length decreases to zero.
We calculate $s_{\text{max}}$ from the condition that a string of size $n \leq s_{\text{max}}$ is not struck by lightning until all trees are grown. When a string of size $n$ is completely empty at time $t = 0$, it will be occupied by $n$ trees after

$$T(n) = (1/p) \sum_{m=1}^{n} 1/m \simeq \ln(n)/p$$

(4.25) timesteps on an average. The mean number of trees after $t$ timesteps follows from the growth equation

$$\dot{\rho}_t = p(1 - \rho_t)$$

and is

$$m(t) = n[1 - \exp(-pt)].$$

The probability that lightning strikes a string of size $n$ before all trees are grown is

$$f \sum_{t=1}^{T(n)} m(t) \simeq (f/p)n(\ln(n) - 1) \simeq (f/p)n \ln(n)$$

We conclude

$$s_{\text{max}} \ln(s_{\text{max}}) \propto p/f \text{ for large } p/f,$$

(4.26) and, to leading order in $\ln(f/p)$

$$s_{\text{max}} \propto \frac{p}{f \ln(p/f)}.$$  

(4.27) Consequently the critical exponent $\lambda$ is $\lambda = 1$. The remaining critical exponents can directly be derived using scaling relations or the defining equations given in subsection IV A. They are given in Table I.

V. THE PERCOLATION PHASE TRANSITION

We now consider the case $f = 0$ and $g > 0$. Watching a 2-dimensional system on a video screen, one realizes that the fire fronts present for $g = 0$ become more and more fuzzy with increasing immunity $g$ and that the forest grows denser.
At the critical value \( g = g_C(p) \), the fire density goes to zero, and the critical fire spreading resembles percolation. For \( g > g_C \), the steady state of the system is a completely green forest. Fig. 6 shows a snapshot of the system for a value of \( g \) just below \( g_C \). Fig. 7 shows the density of empty sites \( \rho_e = \rho_f/p \) as function of \( g \) for \( p = 0.1 \) as determined from computer simulations. The fire density depends linearly on \((g_C - g)\) near the critical line. This linear dependence can already be obtained from mean-field theory [4]. In the general case it can be understood by the following argument. Consider the stationary state near \( g_C \). Since \( \rho_f = pp_e \), there exist for each burning tree \( p^{-1} \) empty sites on an average. Most of these empty sites have been visited by the fire less than \( \simeq 1/p \) timesteps ago and are situated within a distance \( \leq 1/p \) of the burning trees. The fire spreading therefore is influenced to a great deal by empty sites (and even other burning sites). Let \( 3 - \alpha(g) \) be the mean number of trees that are nearest neighbors of burning trees, divided by the number of burning trees. Obviously \( \alpha(g) > 0 \), since each fire site has at least one empty neighbor namely the site where the fire has been one timestep before. The larger \( \alpha(g) \), the more is the fire spreading hindered by empty sites and other burning sites. The mean number of trees that are ignited by a burning tree is \((1 - g)(3 - \alpha(g))\). In the stationary state this has to be one tree on an average, i.e.

\[
\alpha(g) = 3 - 1/(1 - g).
\]

In mean-field theory (with \( 2d \) replaced by \((2d - 1)\)) \( g_C = 2/3 \) and \( \alpha(g_C) = 0 \). The fact that actually \( \alpha(g_C) > 0 \) for all values of \( p \) (since \( g_C < 2/3 \)) indicates that a fire is influenced by the presence of other fires even in the limit of zero fire density. A burning tree often ignites two or three trees even at \( g = g_C \), and these fires interfere with each other when they spread further. We now show that \( (\partial \rho_f(g)/\partial g)_{g=g_C} \) is finite and nonvanishing, i.e. that any change in \( g \) is accompanied by a change in \( \rho_f \) and vice versa. To this end we first consider the system exactly at the critical line. Then the fire density is zero, i.e. an infinitesimally small portion of all sites in an infinite system are burning sites. The fire spreads by a branching-and-death mechanism: A burning tree may ignite two or three neighbors (a branching process), one
neighbor, or no neighbor (it dies). These processes are in an equilibrium in the steady state since the fire density is constant in time. On an average a burning tree produces just another burning tree. We mentioned already above that these branching-and death processes always are influenced by other burning sites since $\alpha(g_C) > 0$. This is not only true for single burning sites. Groups of burning sites also interfere with each other. Otherwise it would be possible to fill the system with these groups up to a finite fire density - in contradiction with the observation that the fire density is zero. Now we decrease the immunity slightly from its critical value. Since less trees are immune, more branching processes take place than before, and the fire density increases until a new equilibrium is established. We just have seen that there always is an interaction between fire sites. Therefore the newly created fire sites cannot evolve independently of the rest of the system, but they interfere immediately with the fire sites that have been in the system before the decrease of $g$. As a consequence $\alpha$ increases proportionally to the fire density. There cannot be a change in the fire density without a corresponding change in $\alpha$. Since on the other hand, via Eq.(11), changes of $\alpha$ are related linearly to changes of $g$, the derivative $(\partial \rho_f/\partial g)_{g=g_C}$ is finite.

Fig. 8 shows the critical line $g_C(p)$ as obtained from computer simulations. The critical immunity $g_C$ increases with increasing $p$. This is plausible since with increasing $p$ the fire can sooner return to sites where it already has been. The main features of $g_C(p)$ can be derived by the following consideration [4]. If the immunity is at its critical value and if the initial state is an insolated burning tree in an infinitely large forest, the mean number of burning trees after a long time interval is again a single burning tree, i.e. the fire is just able to survive. We therefore started with an isolated burning tree with one empty neighbor (the site where the fire has been one timestep ago) and calculated the mean number $F(t, p, g)$ of burning trees after $t$ timesteps. An approximate value of $g_C$ is given by the condition $F(t, p, g_C) = 1$. The larger $t$, the more accurate is the result. Fig. 9 shows $g_C(p)$ for $t = 4$. This curve is linear for small $p$ and flat for $p \to 0$, as obtained from the simulations.

Finally, we comment on the nature of the percolation phenomenon occurring at $g_C$. From the simulations we obtained $\lim_{p \to 0} g_C(p) \simeq 0.467$. Had we introduced the immunity by
making sites or bonds between sites permanently immune, we would have obtained a value 
$g_C(p = 0) = 0.41$ or $0.5$, which is just one minus the percolation threshold for site or bond 
percolation. The critical immunity in this case were independent of $p$ since there could not 
exist an infinite path for the fire through the system for $p > 0$ if it did not for $p = 0$. In our 
model, sites are not permanently immune and the fire can take paths that it cannot take 
in site percolation. The percolation threshold $1 - g_C(p = 0)$ therefore is lower than in site 
percolation. We call the new kind of percolation observed in this model “fluctuating site 
percolation”.

VI. SUMMARY AND DISCUSSION

In this paper, we have shown that a simple stochastic forest-fire model shows a variety of 
different phenomena. Depending on the range of parameter values, quasideterministic spiral-
shaped fire fronts, self-organized critical behavior, or percolation-like behavior is observed. 
An essential condition for the occurrence of self-organized critical behavior in the forest-fire 
model is a double separation of time scales. When time scales are not separated, completely 
different phenomena can be observed. The origin of the mentioned phenomena was explained 
by a series of simple physical arguments. Computer simulation results were presented, and 
analytic calculations including a mean-field theory and a scaling theory were performed. For 
the one-dimensional self-organized critical forest-fire model exact results were derived.

The forest-fire model can be viewed as a computer model for excitable systems. Thus, it 
can be expected that the phenomena occurring in the forest-fire model should also be observed 
in excitable systems, when the appropriate range of parameter values is investigated. Spiral 
waves in two- or higher-dimensional excitable systems are quite familiar, but percolation 
and self-organized critical behavior have not been reported so far. We hope that this paper 
will stimulate research aimed at the detection of these phenomena in excitable systems.
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TABLES

TABLE I. Critical exponents in one and two dimensions. Quantities with * have logarithmic corrections.

| d | τ | μ | λ | ν | ν’ | μ’ | ν’ | z | b | 1/δ | α |
|---|---|---|---|---|----|----|----|---|---|-----|---|
| 1 | 2 | 1 | 1* | 1* | 1 | 1* | 1* | 1 | 1 | 0* | 2* |
| 2 | 2.14(3) | 1.96(1) | 1.15(3) | 0.58 | 0.43 | 1.89(3) | 0.61(3) | 0.44 | 1.04(1) | 1.27(7) | 0.48(2) | 1.72(5) |
FIGURES

FIG. 1. Phase diagram derived from mean-field theory. In the inner of the cube is a phase with nonvanishing fire density.

FIG. 2. Snapshot of the Bak et al forest-fire model in the steady state for $p = 0.005$ and $L = 800$. Trees are black, empty sites are white. Burning trees are too small to be seen. They are situated at the sharp black-white interface.

FIG. 3. Snapshot of the self-organized critical state for $f/p = 1/200$ and $L = 500$. Trees are black, empty sites are white.

FIG. 4. Size distribution $s_n(s)$ of the fires for $f/p = 1/16000$ and $L = 8192$ in two dimensions.

FIG. 5. Size distribution of the fires for $f/p = 2/50000$ and $L = 2^{20}$ in one dimension. The smooth line is the theoretical result, which is valid for cluster sizes $\leq s_{\text{max}}$.

FIG. 6. Snapshot of the forest-fire model near the critical immunity for $p = 0.1$, $g = 0.48$, $L = 200$. Trees are grey, empty sites are white, and burning trees are black.

FIG. 7. The density of empty sites as function of immunity for $p = 0.1$ and $L = 500$.

FIG. 8. The critical immunity $g_C(p)$ obtained from computer simulations.

FIG. 9. The critical immunity $g_C(p)$ as calculated for $t = 4$. 