Inverse scattering in guides

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Abstract. We present statements and a method of solution to the inverse scattering problem of reconstructing permittivity of a dielectric inclusion in a 2D or 3D waveguide from the transmission characteristics. The approach employs a volume singular integral equation (VSIE) method. The unique solvability of VSIE is established. The inverse problem is solved by the method of iterations applied to VSIE; the convergence of the method is proved.

1. Introduction

The methods of reconstructing the shape of the scatterer or its permittivity are developed by Colton and Kress [3] mainly for the cases when the obstacles are supposed to be perfectly conducting or dielectric bodies in two- or three-dimensional space. Among alternative techniques, note an approach proposed by Shestopalov and Lozhechko [11] for cylindrical scatterers whose (two-dimensional) cross sections are formed by domains with infinite noncompact boundaries. The uniqueness for such problems stated in the whole space or in the half-space is proved when the data in the inverse problem of finding the shape of the scatterer or permittivity of the inclusion consist of the far-field patterns of the scattered field given for the plane wave irradiating the obstacle from all directions, and are available for all frequencies varying in a certain interval.

The direct and inverse problems associated with the acoustic and electromagnetic wave propagation in parallel-plane guides were studied by many authors (see e.g. [2], [1], [4], [6], [7] and the bibliography therein). Most of the results were obtained for parallel-plane waveguides. However, when a dielectric body is situated in a waveguide of a bounded cross-section (as in [2] where the so called waveguide method is presented), similar results concerning the unique solvability and efficient solution techniques for the inverse scattering problems of reconstructing permittivity of the scatterer are not available. This fact becomes a driving force of our effort in developing a new approach to the solution of both forward and inverse scattering problems in waveguides. The present work is devoted to the development of the methods of reconstructing permittivity of the scatterer in a 2D-waveguide and a 3D-waveguide of rectangular cross section. We develop solution techniques elaborated in [10], [9], [5], [20], [12], [16], and [17] for the analysis of forward and inverse boundary value problems (BVPs) for Maxwell’s and Helmholtz equations associated with the wave propagation in waveguides with dielectric inclusions. The specific methods applied in this study were proposed and justified in [15] on the basis of volume singular integral equation (VSIE) method [8] recently developed in [14], [18], and [13].
2. 2D-guide

2.1. Forward (diffraction) problem

We consider a parallel-plane waveguide $S = \{(y, z) : -\pi < y < \pi, |z| < \infty\}$ containing a nonmagnetic, isotropic, and inhomogeneous dielectric inclusion having the cross section $D \subset Q = \{(y, z) : -\pi < y < \pi, -2\pi\delta < z < 2\pi\delta\}$ bounded by a piecewise smooth closed contour $\partial D$ (Figure 1), where $Q \subset S$ denotes the so-called transition domain. The permittivity function $\varepsilon = \varepsilon(y, z)$ is assumed to be such that $\operatorname{supp} m(y, z) \subset Q$, where $m(y, z) = 1 - \varepsilon(y, z)$. We also assume that $\varepsilon(y, z)$ is a complex-valued function of two real arguments $y, z$ piecewise continuously differentiable and bounded in $S$ and denote

$$
\varepsilon(y, z) = \varepsilon_1(y, z) \exp \left(i\varepsilon_2(y, z)\right) = g_1(y, z) + i g_2(y, z).
$$

In accordance with physical assumptions of the model, the real and imaginary parts of (1) are positive, piecewise continuously differentiable, and bounded satisfying $g_1(y, z) \geq 1$, so that the modulus $\varepsilon_1$ and argument $\varepsilon_2$ of the $\varepsilon(y, z)$ are also positive functions that are piecewise continuous and bounded on the line and satisfy $0 \leq \varepsilon_2(y, z) < \pi/2$ and $\varepsilon_1(y, z) \geq 1$.

We further introduce the complex magnitude of the stationary electric and magnetic field, $E(r, t)$ and $H(r, t)$, respectively, where $r = (x, y, z)$, and consider the problem of diffraction of the TE mode (assumed to be linearly polarized)

$$
E(r, t) = E(r) \exp(-i\omega t), \quad H(r, t) = H(r) \exp(-i\omega t),
$$

$$
E(r) = (E_x, 0, 0), \quad H(r) = \left(0, \frac{1}{i\omega\mu_0} \frac{\partial E_x}{\partial z}, -\frac{1}{i\omega\mu_0} \frac{\partial E_x}{\partial y}\right),
$$

by a dielectric inclusion $D$.

The total field $u(y, z) = E_x(y, z) = E^{\text{inc}}_x(y, z) + E^{\text{scat}}_x(y, z) = u^t(y, z) + u^s(y, z)$ of the diffraction by the $D$ of the unit-magnitude TE wave with the only nonzero component is the solution to the BVP [10]

$$
[\Delta + \kappa^2 \varepsilon(y, z)]u(y, z) = 0 \text{ in } S, \quad u(\pm\pi, z) = 0,
$$

$$
u(y, z) = u^t(y, z) + u^s(y, z), \quad u^s(y, z) = \sum_{n=1}^{\infty} a_n^\pm \exp(i\Gamma_n z) \sin(ny),
$$

Figure 1. TE-mode diffraction by a dielectric inclusion in a parallel-plane waveguide
where $\Delta = \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is the Laplace operator, superscripts + and − correspond, respectively, to the domains $z > 2\pi\delta$ and $z < -2\pi\delta$, $\omega = \kappa c$ is the dimensionless circular frequency, $\kappa = \omega/c = 2\pi/\lambda$ is the dimensionless frequency parameter ($\lambda$ is the free-space wavelength), $c = (\varepsilon_0\mu_0)^{-1/2}$ is the speed of light in vacuum, and $\Gamma_n = (\kappa^2 - n^2)^{1/2}$ is the transverse wavenumber satisfying the conditions

$$Im \Gamma_n \geq 0, \quad \Gamma_n = i|\Gamma_n|, \quad |\Gamma_n| = Im \Gamma_n = (n^2 - \kappa^2)^{1/2}, \quad n > \kappa. \quad (4)$$

It is also assumed that the series in (3) converges absolutely and uniformly and allows for double differentiation with respect to $y$ and $z$. If $\varepsilon(y, z)$ is a piecewise continuous function, the continuity and transmission conditions [3, Ch. 2], should be added on the discontinuity lines.

Note that $u^i(y, z)$ satisfies (2) in $S$, the boundary condition, and radiation condition (3) only in the positive direction, so that the electromagnetic field with the $x$-component $u^i(y, z)$ may be interpreted as a mode coming from the domain $z < -2\pi\delta$.

### 2.2. Integral equation

We solve the (forward) problem (2)—(4) in the transition domain $Q$. To this end, make use of Green’s function (see [9])

$$G_0(y, z; y_0, z_0) = \frac{1}{\pi} \sum_{n=1}^{\infty} \exp[i\Gamma_n|z - z_0|] \sin(ny) \sin(ny_0) \Gamma_n^{-1}. \quad (5)$$

of problem (2)—(4) defined at $\varepsilon = 1$ in the domain $S$.

The IE with respect to the sought for scattered field $u(y, z)$ has the form

$$u(y, z) = - \int_{-2\pi\delta}^{2\pi\delta} \int_{-\pi}^{\pi} G_0(y, z; y_0, z_0) [1 - \varepsilon(y_0, z_0)] u(y_0, z_0) dy_0 dz_0 + u^i(y, z), \quad (y, z) \in Q. \quad (5)$$

We have obtained a Lippmann—Schwinger IE which can be written in the operator form $(I - K)u = f$, where $I$ denotes the identity operator. The following statement [15] is valid.

**Theorem 1.** Suppose that $m(y, z) = 1 - \varepsilon(y, z) = 0$ for $\sqrt{y^2 + z^2} \geq a$ with some $a > 0$ (so that the inclusion $D$ is contained in a circle $B_a^O$ of radius $a$ centered at the origin) and $B_a^O \subset S$ and let $\kappa^2 < 2/Ma^2$, where $M = \max_{B_a^O} |m(y, z)|$. Then there exists a unique solution to IE (5).

### 2.3. Asymptotic representation of the field

If only one TE mode propagates in the waveguide without the inclusion, (that is, $1 < \kappa < 2$), then the solution to (2)—(4) can be represented as a superposition of the reflected (in the domain $z < -2\pi\delta$) and transmitted (in the domain $z > 2\pi\delta$) fields and exponentially decreasing terms.

This form of the solution is crucial for developing a method of solution to both direct and inverse scattering problems.

Namely, under the condition $1 < \kappa < 2$, the asymptotic representation of the total field has the form

$$u(y, z) = \begin{cases} 
[e^{i\Gamma_1 z} + R(\kappa) e^{-i\Gamma_1 z}] \sin(y) + O(e^{i\Gamma_2 z}) = u^i(y, z) + R(\kappa) u^i(y, -z) + O(e^{i\Gamma_2 z}), & z < -2\pi\delta, \\
T(\kappa) \sin(y) e^{i\Gamma_1 z} + O(e^{-i\Gamma_2 z}) = T(\kappa) u^i(y, z) + O(e^{-i\Gamma_2 z}), & z > 2\pi\delta,
\end{cases}$$
where $R(\kappa)$ and $T(\kappa)$ are, respectively, the reflection and transmission coefficients. For $z < -2\pi\delta$ the scattered field

$$u^s(y, z) = -\sin(y) \exp(-i\Gamma_1 z)u^{(1)}_{\infty,-}(\kappa) - \sum_{n=2}^{\infty} \frac{i}{\pi} \frac{\exp(|\Gamma_n|z)}{|\Gamma_n|} u^{(n)}_{\infty,-}(y; \kappa)$$

\textbf{The partial far field patterns}

$$u^{(n)}_{\infty,-}(y, \kappa) = \sin(ny)u^{(n)}_{\infty,-}(\kappa), \quad n = 2, 3, \ldots$$

where

$$u^{(n)}_{\infty,-}(\kappa) = \frac{1}{\mathcal{Q}} \int \int \exp(-|\Gamma_n|z_0) \sin(ny_0)m(y_0, z_0) u(y_0, z_0) dy_0 dz_0.$$

An asymptotic expression for the scattered field in terms of partial far field patterns has the form

$$u^s(y, z) = -\sin(y) \exp(i\Gamma_1 z)u^{(1)}_{\infty,+}(\kappa) + O(e^{-|\Gamma_2|z})$$

$$= -u^t(y, z)u^{(1)}_{\infty,+}(\kappa) + O(e^{-|\Gamma_2|z}), \quad z > 2\pi\delta,$$

$$u^s(y, z) = -\sin(y) \exp(-i\Gamma_1 z)u^{(1)}_{\infty,-}(\kappa) + O(e^{\Gamma_2|z|})$$

$$= -u^t(y, -z)u^{(1)}_{\infty,-}(\kappa) + O(e^{\Gamma_2|z|}), \quad z < -2\pi\delta.$$
for $z < -2\pi\delta$ for the transmitted and reflected fields generated by partial far field patterns $U_{\infty, \pm}^j(y, \kappa)$, it follows that in the single-mode (when the condition $1 < \kappa < 2$ holds) parallel-plane waveguide excited by a unit-amplitude principal mode, the knowledge of two complex numbers $R(\kappa) = u_{\infty, -}^{(1)}(\kappa)$ and $T(\kappa) = u_{\infty, +}^{(1)}(\kappa)$ is virtually sufficient to solve the inverse problem of reconstructing permittivity of an inhomogeneous dielectric inclusion in the guide.

Let us briefly describe the solution technique for the inverse problem under study in the case of a homogeneous inclusion when the permittivity $\varepsilon = \text{const}$. Using conditions at infinity (3) and asymptotic representations (6) and (7), we can write

$$
\begin{align*}
\hat{u}(y, z) &= u^i(y, z) + a^+_1 \exp(i\Gamma_1 z) \sin(y) + \sum_{n=2}^{\infty} \frac{a^+_n}{n} \exp(i\Gamma_n z) \sin(ny), \\
T(\kappa) &= u^i(y, z) + O(e^{-|\Gamma_2|z}).
\end{align*}
$$

(8)

(9)

Discarding the exponentially decaying terms in the limit $z \to \infty$, taking into account that $\text{supp} (1 - \varepsilon(y, z)) = D$ and

$$
\int_D \int \exp(-i\Gamma_1 z_0) \sin(y_0) u(y_0, z_0) dy_0 dz_0 = \langle u, \overline{u}^i \rangle,
$$

where brackets denote the inner product in the space $L^2(D)$ and the bar complex conjugation, and equating the left-hand sides of (8) and (9), we obtain

$$
(1 - \varepsilon)(u, \overline{u}^i) = \hat{T};
$$

(10)

here it is assumed that $\hat{T} = T - 1$ is a known (measured) quantity.

If solution $u$ were known, then (10) can be used for the direct computation of the sought-for $\varepsilon$. However, $u$ is a quantity that must be determined. To this end, taking into notice that the integral representation for the scattered field has the same form as IE (5),

$$
\int_{-2\pi \delta}^{2\pi \delta} G_0(y, z; y_0, z_0) \left[ 1 - \varepsilon(y_0, z_0) \right] u(y_0, z_0) dy_0 dz_0 + u^i(y, z), \quad (y, z) \in S \setminus Q,
$$

we substitute (10) into (11) to obtain a nonlinear volume IE with respect to $u$

$$
\langle u - u^i, \overline{u}^i \rangle = \hat{G}u, \quad \hat{G}u = \int_D G_0(y, z; y_0, z_0) u(y_0, z_0) dy_0 dz_0, \quad (y, z) \in D.
$$

(11)

(12)

Setting $\hat{u} = u - u^i$ and $\hat{T}_0 = \frac{\hat{T}}{||\hat{u}||^2}$, we can rewrite nonlinear volume IE (12) in the form suitable for the application of contraction mapping

$$
\hat{u} = B(\hat{u}),
$$

$$
B(\hat{u}) = -\frac{\hat{u}(\hat{u}, \overline{u}^i)}{||\hat{u}||^2} + \hat{T}_0 \hat{G} \hat{u} + \hat{T}_0 \hat{G} u^i.
$$

(13)
Equation (13) can be solved by the method of fixed-point iterations. The justification of the method, including the proof that operator $B$ is a contraction and the convergence of the fixed-point iterations based on equation (13), is performed for the 3D case in Section 3.5. In fact, the solution technique for the inverse problem considered in a 3D-guide is of practical importance. We show that when the inclusion is homogeneous (the permittivity is a complex constant), one can determine the permittivity solely from one complex number, the transmission coefficient.

We note that the 2D case is mainly of theoretical character and is studied as the first step towards the practically important 3D case considered in the next sections.

3. 3D-guide
3.1. Forward (diffraction) problem
Assume that a waveguide

$$P := \{ x : 0 < x_1 < a, 0 < x_2 < b, -\infty < x_3 < \infty \}$$

with the perfectly conducting boundary surface $\partial P$ is given in the cartesian coordinate system. A three-dimensional body $Q$ ($Q \subset P$ is a domain) with a constant magnetic permeability $\mu_0$ and a positive $(3 \times 3)$ matrix (tensor) permittivity $\hat{\epsilon}(x)$ is placed in the waveguide. The components of $\hat{\epsilon}$ are bounded functions in $\bar{Q}$, $\hat{\epsilon} \in L_\infty(Q)$, and $\hat{\epsilon}^{-1} \in L_\infty(Q)$. The boundary $\partial Q$ of $Q$ is piecewise smooth and $Q$ does not touch the walls of the waveguide.

We look for the electromagnetic field $\mathbf{E}, \mathbf{H} \in L^{loc}_\infty(P)$ that is induced in the waveguide by the external field with the time dependence $e^{-i\omega t}$. The external field is induced by the electric current $\mathbf{j}_E^0 \in L^{loc}_\infty(P)$. The differential operators grad, div, and curl are interpreted in the sense of distributions.

We will seek weak (generalized) solutions to Maxwell’s system of equations

$$\text{rot}\mathbf{H} = -i\omega\hat{\epsilon}\mathbf{E} + \mathbf{j}_E^0, \quad \text{rot}\mathbf{E} = i\omega\mu_0\mathbf{H},$$

$$E_\tau|_{\partial P} = 0, \quad H_\nu|_{\partial P} = 0,$$

admitting for $|x_3| > C$ and sufficiently large $C > 0$ the representations (+ corresponds to $+\infty$
\[ \mathbf{E} = \sum_p R_p^{(\pm)} e^{-i\gamma_p^{(1)} |x_3|} \left( \lambda_p^{(1)} \Pi_p e_3 - i\gamma_p^{(1)} \nabla_2 \Pi_p \right) + \sum_p Q_p^{(\pm)} e^{-i\gamma_p^{(2)} |x_3|} \left( \lambda_p^{(2)} \Psi_p e_3 - i\gamma_p^{(2)} \nabla_2 \Psi_p \right). \]

Here, \( \gamma_p^{(j)} = \sqrt{k_0^2 - \lambda_p^{(j)}} \), \( \text{Im} \gamma_p^{(j)} < 0 \) or \( \text{Im} \gamma_p^{(j)} = 0 \), and \( \lambda_p^{(1)}, \Pi_p(x_1, x_2) \) and \( \lambda_p^{(2)}, \Psi_p(x_1, x_2) \) \( (k_0^2 = \omega^2 \varepsilon_0 \mu_0) \) are the complete system of eigenvalues and orthogonal and normalized in \( L_2(\Pi) \) eigenfunctions of the two-dimensional Laplace operator \( -\Delta \) in the rectangle \( \Pi := \{ x' = (x_1, x_2) : 0 < x_1 < a, 0 < x_2 < b \} \) with the Dirichlet and the Neumann conditions, respectively; and \( \nabla_2 \equiv e_1 \partial/\partial x_1 + e_2 \partial/\partial x_2 \).

We assume that \( \mathbf{E}^0 \) and \( \mathbf{H}^0 \) are solutions of BVP under consideration in the absence of the nonhomogeneous body \( Q, \hat{e}(x) = \varepsilon_0 \hat{I}, x \in P \) (\( \hat{I} \) is the identity tensor):

\[ \text{rot} \mathbf{H}^0 = -i\omega \varepsilon_0 \mathbf{E}^0 + j_0^E, \quad \text{rot} \mathbf{E}^0 = i\omega \mu_0 \mathbf{H}^0, \]

\[ \mathbf{E}^0|_{\partial P} = 0, \quad \mathbf{H}^0|_{\partial P} = 0. \]

These solutions can be expressed in an analytical form in terms of \( j_0^E \) using Green’s tensor of domain \( P \). These solutions should not satisfy the conditions at infinity (16). For example, \( \mathbf{E}^0 \) and \( \mathbf{H}^0 \) can be TM- or TE-mode of this waveguide.

### 3.2. Green’s tensor function for the waveguide of rectangular cross-section

One can construct explicitly (see [13]) the diagonal Green’s tensor \( \hat{G}_E = \text{diag}(G_E^1, G_E^2, G_E^3) \) whose components are the fundamental solutions to the Helmholtz equation in \( P \) with the coefficient \( k_0^2 \) and satisfy the boundary conditions of the first or second kind on \( \partial P \), which ensure that the tangential components of the intensity of the electric field vanish on the walls of the waveguide. The components of this tensor are

\[ G_E^1 = \frac{2}{ab} \sum_{n=0}^{\infty} \sum_{m=1}^{\infty} \frac{1}{(1 + \delta_{nm}) \gamma_{nm}} \cos(\pi n / a) \sin(\pi m / b) x_1 \cos(\pi n / a) \sin(\pi m / b) y_1 \varepsilon^{-\kappa_{nm}|x_3 - y_3|}, \]

\[ G_E^2 = \frac{2}{ab} \sum_{n=1}^{\infty} \sum_{m=0}^{\infty} \frac{1}{(1 + \delta_{nm}) \gamma_{nm}} \sin(\pi n / a) \cos(\pi m / b) x_1 \sin(\pi n / a) \cos(\pi m / b) y_1 \varepsilon^{-\kappa_{nm}|x_3 - y_3|}, \]

\[ G_E^3 = \frac{2}{ab} \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} \frac{1}{\gamma_{nm}} \sin(\pi n / a) \sin(\pi m / b) x_1 \sin(\pi n / a) \sin(\pi m / b) y_1 \varepsilon^{-\kappa_{nm}|x_3 - y_3|}, \]

where \( \delta_{nm} \) is the Kronecker delta. Here, \( \gamma_{nm} = \sqrt{(\pi n / a)^2 + (\pi m / b)^2 - k_0^2} \) and the branch of the square root is chosen such that \( \text{Re} \gamma_{nm} \geq 0 \) and \( \text{Im} \gamma_{nm} \geq 0 \).

Write \( G_E^j \) in the form in which the singularity at \( x = y \) is singled out:

\[ G_E^j = \frac{1}{4\pi} e^{ik_0|x-y|} + g^j(x, y), \quad x, y \in P. \]
Here, the function $g^j \in C^\infty(\tilde{Q} \times \tilde{P})$ (see [5, p. 85]). Note that Green’s function is not connected with a particular domain $\tilde{Q}$; our requirement however is that $Q$ cannot touch the walls which yields the smoothness of $g^j$. Due to the symmetry of Green’s functions $G^j_E(x, y) = G^j_E(y, x)$ ($j = 1, 2, 3$), we have the following proposition (see [8], [19]).

**Proposition.** Green’s tensor $\hat{G}_E$ can be represented by

$$\hat{G}_E = \frac{1}{4\pi} \frac{e^{ik_0|x-y|}}{|x-y|} \hat{I} + \hat{g}(x, y), \quad x, y \in P,$$

where the matrix function (tensor) $\hat{g} \in C^\infty(\tilde{Q} \times \tilde{P})$ and $\hat{g} \in C^\infty(\tilde{P} \times \tilde{Q})$.

Note that Green’s functions have a single singularity of the form $\frac{1}{4\pi} \frac{e^{ik_0|x-y|}}{|x-y|}$ and have no other singularities due to the assumption that the body does not touch the waveguide walls, i.e., $\tilde{Q} \subset P$.

### 3.3. Volume singular integral equation

Let BVPs (14)–(15) and (17), (18) have unique solutions. Rewrite Eq. (14) in the equivalent form

$$\text{rot} \mathbf{H} = -i\omega \epsilon_0 \mathbf{E} + j_E, \quad \text{rot} \mathbf{E} = i\omega \mu_0 \mathbf{H},$$

(19)

where

$$j_E = \hat{j}_E^0 + \hat{j}_E^p.$$

(20)

In the last equation, $\hat{j}_E^p = -i\omega (\hat{e}(x) - \epsilon_0 \hat{I}) \mathbf{E}$ is the polarization current.

The solution of BVP (19), (15) is given by

$$\mathbf{E} = i\omega \mu_0 \mathbf{A}_E - \frac{1}{i\omega \epsilon_0} \text{grad} \text{div} \mathbf{A}_E, \quad \mathbf{H} = \text{rot} \mathbf{A}_E$$

(21)

where

$$\mathbf{A}_E = \int_P \hat{G}_E(r) \hat{j}_E(y) dy, \quad r := |x-y|,$$

(22)

is the vector potential of the electric current that satisfies the equation

$$\triangle \mathbf{A}_E + k_0^2 \mathbf{A}_E = -\mathbf{j}_E.$$

Formulas (21) do not give an explicit solution of problem (19),(15) because the current $\hat{j}_E$ depends on $\mathbf{E}$. Relations (20)–(22) for $\mathbf{E}$ imply the integro-differential equation which can be reduced to a VSIE

$$\mathbf{E}(x) + \frac{1}{3} \left[ \hat{e}(x) - \hat{I} \right] \mathbf{E}(x) - v.p. \int_Q \hat{\Gamma}_1(x, y) \left\{ \left[ \hat{e}(y) \hat{e}_0 - \hat{I} \right] \mathbf{E}(y) \right\} dy$$

$$- \int_Q \hat{\Gamma}_1(x, y) \left\{ \left[ \frac{\hat{e}(y)}{\epsilon_0} - \hat{I} \right] \mathbf{E}(y) \right\} dy - \int_Q \hat{\Gamma}_2(x, y) \left\{ \left[ \frac{\hat{e}(y)}{\epsilon_0} - \hat{I} \right] \mathbf{E}(y) \right\} dy = \mathbf{E}^0(x),$$

(23)

where the tensors $\hat{\Gamma}$, $\hat{\Gamma}_1$, and $\hat{\Gamma}_2$ are given by

$$\hat{\Gamma}(x, y) = k_0^2 \hat{G}_E(r) + (\cdot, \text{grad}) \text{grad} G_0(r), \quad \hat{\Gamma}_1(x, y) = (\cdot, \text{grad}) \text{grad} G_1(r),$$

$$(\hat{\Gamma}_2(x, y))_{ij} = \frac{\partial^2 g^j(r)}{\partial x_i \partial x_j}, \quad \hat{G}_E(r) = G_0(r) \hat{I} + G_1(r) \hat{I} + \hat{G}_2(r),$$

$$G_0(r) = \frac{e^{ik_0 r} - 1}{4\pi r}, \quad G_1(r) = \frac{1}{4\pi r}, \quad \hat{G}_2(r) = \text{diag} \{g^1, g^2, g^3\}.$$
The solvability of Eq. (23) and the equivalence of the boundary value diffraction problem and the singular integral equation are established by the following theorems (see [8], [19]).

**Theorem 2.** Suppose the body $Q$ has the piecewise smooth boundary $\partial Q$ and the positive permittivity tensor $\hat{\epsilon} \in L_\infty(Q)$ and $\hat{\epsilon}^{-1} \in L_\infty(Q)$. Suppose that $E, H$ and $E^0, H^0$ are unique solutions of BVPs (14), (15), (17), (18) respectively. Then, the integral equation (23) has a unique solution $E \in L_2(Q)$. Conversely, if $E \in L_2(Q)$ is a solution to the integral equation (23), then formulas (20)–(22) give a solution of BVP for Maxwell’s equation system (14) satisfying condition (15).

Rewrite integral equation (23) for the electric field in the form
\[(I + S - K) E = E^0,\]
where operators $S$ and $K$ are defined according to (23):
\[
(S \mathbf{E})(x) = \frac{1}{3} \left[ \hat{\epsilon}(x) - \hat{\epsilon}_0 - i \sum_{l,n=1}^3 \left[ \hat{\epsilon}(y) - \hat{\epsilon}_0 - i \right] E(y) \right] dy,
\]
\[
(K \mathbf{E})(x) = \int_Q \hat{\Gamma}(x, y) \left[ \left[ \hat{\epsilon}(y) - \hat{\epsilon}_0 - i \right] E(y) \right] dy + \int_Q \hat{\Gamma}_2(x, y) \left[ \left[ \hat{\epsilon}(y) - \hat{\epsilon}_0 - i \right] E(y) \right] dy.
\]

**Theorem 3.** Assume that homogeneous integral equation (24) has only a trivial solution and the permittivity tensor is such that
\[
es 
\text{ess sup} \left| \sum_{l,n=1}^3 \frac{\epsilon_{ln}(x)}{\epsilon_0} - \delta_{ln} \right|^2 \right|^{1/2} < \left( 1 + \sqrt{\frac{1}{2}} \right)^{-1}.
\]
Then equation (24) is uniquely solvable for any right-hand side $E^0 \in L_2(Q)$.

### 3.4 Statement of the inverse problem

**Assumptions.** Let $Q$ be an isotropic dielectric body and $\hat{\epsilon}(x) = \epsilon$, where the effective permittivity $\epsilon$ of the medium filling $Q$ is an unknown constant to be determined. Assume that
\[
\frac{\pi}{a} < k_0 < \frac{\pi}{b}.
\]

Then the only one wave (mode) can propagate in the waveguide because $\text{Im} \gamma^{(2)}_1 = 0$,
\[
\gamma^{(2)}_1 = \sqrt{k_0^2 - \frac{\pi^2}{a^2}} > 0,
\]
and $\text{Im} \gamma^{(j)}_p < 0$ for all $p, j$ except for $p = 1$ and $j = 2$. Assume that
\[
E^0(x) = e^{2A^{(+)}}_2 i\omega \mu_0 \pi a \sin \frac{\pi x_1}{a} e^{-i\gamma^{(2)}_1 x_3}.
\]
Here $A^{(+) \neq 0}$ is the (known) amplitude of the propagating wave and $\Psi_1 = \cos \frac{\pi x_1}{a}$ (see formulas (16)).

**Statement of inverse problem.** Assume that it is given:
(i) a waveguide $P$ of rectangular cross-section with the perfectly conducting boundary surface $\partial P$, numbers $a$, $b$, and $k_0 = \omega \sqrt{\mu_0 \varepsilon_0}$ satisfy the condition (25);
(ii) a homogeneous three-dimensional dielectric inclusion, a body $Q$, which is placed in the waveguide, does not touch its walls ($Q \subset P$ is a domain), and has a constant magnetic permeability $\mu_0$ (given) and a constant (unknown) permittivity $\varepsilon$;
(iii) a positive number $\omega$, frequency of the initial electromagnetic field $\mathbf{E}^0(x)$ and two (complex) numbers $Q^{(+)}_1$ and $A^{(+)}$, the amplitudes of transmitted and incident waves, respectively.

It is necessary to determine (complex) permittivity $\varepsilon$ of dielectric inclusion $Q$.

3.5. Solution to the inverse problem

From (25), it follows that $G^1_E \to 0$ and $G^3_E \to 0$ uniformly with respect to $y \in Q$ for $x_3 \to +\infty$. We have

$$G^2_E = \frac{1}{ab\gamma_0} \sin \frac{\pi x_1}{a} \sin \frac{\pi y_1}{a} e^{-i\gamma_1^2 |x_3-y_3|} \to 0$$

uniformly with respect to $y \in Q$ for $x_3 \to +\infty$. Also $\text{div} G_E \to 0$ uniformly with respect to $y \in Q$ for $x_3 \to +\infty$ because this condition holds for $\frac{\partial G^2_E}{\partial x_3}$.

Calculating the limit for $x_3 \to +\infty$ in the representation formula for the electric field

$$\mathbf{E}(x) = \mathbf{E}^0(x) + k_0^2 \int_Q G_E \left[ \frac{\hat{\varepsilon}(y)}{\varepsilon_0} - \hat{I} \right] \mathbf{E}(y)dy + \text{grad} \text{div} \int_Q G_E \left[ \frac{\hat{\varepsilon}(y)}{\varepsilon_0} - \hat{I} \right] \mathbf{E}(y)dy, \quad x \in P \setminus Q, \quad (26)$$

and taking into account the condition at infinity (16) we have

$$\mathbf{e}_2 Q^{(+)}_1 e^{-i\gamma_1^{(2)} x_3 i \omega \mu_0} \frac{\pi x_1}{a} \sin \frac{\pi y_1}{a} e^{-i\gamma_1^2 |x_3-y_3|} \mathbf{E}_2(y)dy, \quad (27)$$

which yields

$$Q^{(+)}_1 = A^{(+)} + k_0^2 \frac{\varepsilon}{\varepsilon_0 - 1} \frac{1}{b\gamma_0 i \pi \omega \mu_0} \int_Q \sin \frac{\pi x_1}{a} \sin \frac{\pi y_1}{a} e^{-i\gamma_1^{(2)} y_3} \mathbf{E}_2(y)dy. \quad (27)$$

We can rewrite (27) as

$$\frac{\varepsilon}{\varepsilon_0 - 1} = \frac{C}{(\mathbf{E}, \mathbf{f})}, \quad (28)$$

where

$$C = \frac{i \pi \omega \mu_0 b\gamma_0(Q^{(+)}_1 - A^{(+)}), \quad \mathbf{f} = \mathbf{e}_2 \sin \frac{\pi y_1}{a} e^{-i\gamma_1^{(2)} y_3}, \quad (29)$$

$$(\mathbf{E}, \mathbf{f}) = \int_Q \mathbf{E}(y)\mathbf{f}(y)dy.$$  

$Q^{(+)}_1$ is a known (measured) quantity. Therefore, $C$ is also a known quantity.
Substituting (28) and (29) into the representation formula (26) for the electric field (considered at $x \in Q$) we obtain a nonlinear volume IE

$$\frac{(E, f)}{C}(E(x) - E^0(x)) = k_0^2 \int_Q \hat{G}_E(x, y)E(y)dy + \text{grad div} \int_Q \hat{G}_E(x, y)E(y)dy, \quad x \in Q. \tag{30}$$

Introduce a linear integral operator

$$\mathcal{T}_0 E = k_0^2 \int_Q \hat{G}_E(x, y)E(y)dy + \text{grad div} \int_Q \hat{G}_E(x, y)E(y)dy \tag{31}$$

and use (31) to rewrite equation (30)

$$\frac{(E, f)}{C}(E - E^0) = \mathcal{T}_0 E.$$

Performing some normalizations, we can further rewrite (30) in the form

$$U = \mathcal{T}(U) \equiv (-U, \tilde{f})U + \tilde{f}_0 + \hat{A}_0 U, \quad U = \hat{E} - f. \tag{32}$$

Here

$$\tilde{f} = \frac{f}{||f||^2}, \quad \tilde{f}_0 = \frac{f_0}{||f||^2}, \quad f_0 = \hat{A}_0 f, \quad \hat{A}_0 = \frac{A_0}{||f||^2},$$

$$\tilde{A}_0 = A_0 \frac{C}{A^{(+)}}, \quad A^{(+)} = A^{(+)} i \omega \mu_0 \frac{\pi}{a}$$

Let $S_r(0)$ be a closed ball of a radius $r$ centered at 0 in the space $L_2(Q)$. Assume that $U \in S_r(0)$; then the following estimate is valid

$$||\mathcal{T}(U)|| \leq \left( r ||\tilde{f}|| + ||\tilde{A}_0|| \right) r + ||\tilde{f}_0||, \quad \forall U \in S_r(0).$$

If

$$\left( r ||\tilde{f}|| + ||\tilde{A}_0|| \right) r + ||\tilde{f}_0|| \leq r, \tag{33}$$

then operator $\mathcal{T}$ acts from the ball $S_r(0)$ to ball $S_r(0)$.

Now, assuming that $U, V \in S_r(0)$ and using the Cauchy–Bunyakovski inequality, we obtain

$$||\mathcal{T}(U) - \mathcal{T}(V)|| \leq \left( 2r ||\tilde{f}|| + ||\tilde{A}_0|| \right) ||U - V||, \quad \forall U, V \in S_r(0).$$

Thus, if

$$2r ||\tilde{f}|| + ||\tilde{A}_0|| < 1, \tag{34}$$

operator $\mathcal{T}(U)$ is a contraction.

Choose radius $r$ of the ball so that both conditions (33) and (34) hold (below we will show when it is possible). Taking into account the equivalence of transformations leading from
equation (30) to equation (32) and applying the contraction principle we arrive at the main results proved in [13] and [14].

**Theorem 4.** Assume that conditions (33) and (34) hold. Then equation (32) \( U = T(U) \) (and, respectively, equation (30)) is uniquely solvable. Also, there exists one and only one solution to the inverse problem given by formula (28),

\[
\epsilon = \epsilon_0 \left(1 + \frac{C}{(E, f)}\right).
\]

Approximate solutions to equation (32) can be determined using the iterations \( U_{n+1} = T(U_n) \) which converge for any initial approximation \( U_0 \in S_r(0) \) with the rate of a geometrical progression.

**Theorem 5.** Assume that the condition

\[
|\tilde{C}| < F = \|f\|^2 \frac{1 - 2r^*}{\|T_0\|}
\]

holds. Then VSIE (30) is uniquely solvable. Also, there exists one and only one solution to the inverse problem given by formula (35). Approximate solutions \( E_n \in S_{r_0}(E^0) \) to equation (30) can be determined using the iterations

\[
E_{n+1} = E_n - \frac{1}{A^{(+)}f^2} \left[ (E_n, f) (E_n - E^0) - C(T_0E_n) \right],
\]

which converge to exact solution \( E \in S_{r_0}(E^0) \) for any initial approximation \( E_0 \in S_{r_0}(E^0) \) with the rate of a geometrical progression, where \( r^0 = |A^{(+)}| r^* \) and \( r^* \) is determined according to the formula

\[
r^* = \left( \frac{\|T_0f\|^2}{\|T_0\|^2} + \frac{\|T_0f\|^2 \|f\|}{\|T_0\|^2} \right)^{1/2} - \|T_0f\|.
\]

Condition (36) is fulfilled if the quantity \( |Q_1^{(+)} - A_1^{(+)}| \) is sufficiently small. From the physical viewpoint, it means that the amplitude of the transmitted wave does differ substantially from the amplitude of the incident wave.

Consider the iteration procedure (37) for the solution of the inverse problem under study; that is, for the solution of integral equation (30). One has to calculate, at \( n = 0, 1, 2, \ldots \), the image of integral operator \( T_0 \) defined by (31); the calculation algorithm is proposed and justified by Shestopalov and Smirnov [13] and Shestopalov, Smirnov and Mironov [14] using the VSIE technique [8]. As an initial approximation, one can choose \( E_0 = E^0 \). After solving iteratively equation (30) by (37) with a prescribed accuracy, one can find the sought-for permittivity \( \epsilon \) by formulas (28), (29).

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