Discrete Symmetries for Spinor Field in de Sitter Space

S. Moradi¹, S. Rouhani² and M.V. Takook¹,² *

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¹ Department of Physics, Razi University, Kermanshah, IRAN
² Plasma Physics Research Center, Islamic Azad University, P.O.BOX 14835-157, Tehran, IRAN

Abstract

Discrete symmetries, parity, time reversal, antipodal, and charge conjugation transformations for spinor field in de Sitter space, are presented in the ambient space notation, i.e. in a coordinate independent way. The PT and PCT transformations are also discussed in this notation. The five-current density is studied and their transformation under the discrete symmetries is discussed.

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1 Introduction

The interest in the de Sitter space increased when it turned out that it could play a central role in the inflationary cosmological paradigm [1]. Very recently, the existence of a non-zero cosmological constant has been proposed to explain the luminosity observations on the farthest supernovas [2]. If this hypothesis is validated in the future, our ideas on the large-scale universe needs to be changed and the de Sitter metric plays a further important role.

All these developments make it more compelling than ever to find a formulation of de Sitter quantum field theory with the same level of completeness and rigor as for its Minkowskian counterpart. In Minkowski space, a unique Poincaré invariant vacuum can be fixed by imposing the positive energy condition. In curved space-time, however, a global time-like Killing vector field does not exist and therefore the positive energy condition cannot be imposed. Thus symmetry alone is not sufficient for determination of a suitable vacuum state. In de Sitter space, however, symmetries identify the vacuum only in relation to a two parameter ambiguity |α, β⟩, corresponding to family of distinct de Sitter invariant vacuum states for scalar field (see [3] and references there in) . Only the one parameter family |α, 0⟩, is invariant under the PCT

*e-mail: takook@razi.ac.ir
transformation [4, 5, 6, 7]. By imposing the condition that in the null curvature limit the Wightman two-point function becomes exactly the same as Minkowskian Wightman two-point function, the other parameter ($\alpha$) can be fixed as well. This vacuum, $|0, 0\rangle$, is called Euclidean vacuum or Bunch-Davies vacuum. It should be noticed that this condition is different with the Hadamard condition, which requires that the leading short distance singularity in the Hadamard function $G^{(1)}$ should take its flat space value. The leading singularity of the Hadamard function is $\cosh 2\alpha$ times its flat space value [8].

Bros et al. [9, 10] presented a QFT of scalar free field in de Sitter space that closely mimics QFT in Minkowski space. They introduced a new version of the Fourier-Bros transformation on the one-sheeted hyperboloid [11], which allows us to completely characterize the Hilbert space of “one-particle” states and the corresponding irreducible unitary representations of the de Sitter group. In this construction the correlation functions are boundary values of analytical functions.

We generalized the Bros construction to the quantization of the spinor free field in de Sitter space [12, 13, 14]. In the case of the interaction fields, the tree-level scattering amplitudes of the scalar field, with one graviton exchange, has been calculated in de Sitter space [15]. We have shown that the $U(1)$ gauge theory in 4-dimensional de Sitter space describe the interaction between the spinor field (“electron”) and the massless vector fields (“photon”) [16]. In this paper the discrete symmetries, parity, charge conjugation, time reversal and antipodal transformation, for spinor field are studied in the ambient space notation \textit{i.e.} in a coordinate independent way. In the next section, first we briefly recall the appropriate notation. In section 3, the discrete symmetries, parity, time reversal, antipodal and PT transformations are presented. In section 4, charge conjugation transformations for spinor field are discussed. The five-current density and Gordon decomposition are calculated in section 5. The transformation of the five-current density under the discrete symmetries has been discussed in this section as well. Finally, a brief conclusion and outlook is set forth in section 6. In the appendix, the discrete symmetries in intrinsic global coordinates and their relation to the ambient space notation are presented.

2 Notation

The de Sitter space is an elementary solution of the positive cosmological Einstein equation in vacuum. It is conveniently seen as a hyperboloid embedded in a five-dimensional Minkowski space

$$X_H = \{x \in \mathbb{R}^5; x^2 = \eta_{\alpha\beta}x^\alpha x^\beta = -H^{-2}\}, \quad \alpha, \beta = 0, 1, 2, 3, 4,$$

where $\eta_{\alpha\beta} = \text{diag}(1, -1, -1, -1, -1)$. The kinematical group of the de Sitter space is the 10-parameter group $SO_0(1, 4)$ and its contraction limit, $H = 0$, is the Poincaré group.

The spinor wave equation in de Sitter space-time has been originally deduced by Dirac in 1935 [17], and can be obtained by the eigenvalue equation for the second order Casimir operator [12, 13, 14]

$$(-i \not{\!\!\! \beta} \gamma x_{\alpha} \gamma x_{\bar{\beta}} + 2i + \nu)\psi(x) = 0,$$

where $\not{\!\!\! \beta} = \eta_{\alpha\beta} \gamma_{\alpha} x_{\bar{\beta}}$ and $\partial_{\alpha} = \partial_{\alpha} + H^2 x_{\alpha} x \cdot \partial$. In this notation we need the five $4 \times 4$ matrices $\gamma^\alpha$, which are the generators of the Clifford algebra based on the metric $\eta_{\alpha\beta}$:

$$\gamma^\alpha \gamma^\beta + \gamma^\beta \gamma^\alpha = 2\eta^{\alpha\beta} \mathbb{1}, \quad \gamma^\alpha \dagger = \gamma^0 \gamma^\alpha \gamma^0.$$
An explicit quaternion representation, which is suitable for symmetry consideration, is provided by [12, 13]

\[
\begin{align*}
\gamma^0 &= \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}, & \gamma^4 &= \begin{pmatrix} 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & -1 \end{pmatrix} \\
\gamma^1 &= \begin{pmatrix} 0 & i & 0 & 0 \\ -i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -1 & 0 \end{pmatrix}, & \gamma^2 &= \begin{pmatrix} 0 & 0 & -i & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 \\ 0 & 0 & 1 & 0 \end{pmatrix}, & \gamma^3 &= \begin{pmatrix} 0 & i & 0 & 0 \\ i & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ -i & 0 & 0 & 0 \end{pmatrix}
\end{align*}
\]

(3)

in terms of the $2 \times 2$ unit $I_1$ and Pauli matrices $\sigma^i$. Due to the de Sitter group covariance, the adjoint spinor is defined as follows [13, 14, 18]:

\[
\bar{\psi}(x) \equiv \psi^\dagger(x) \gamma^0 \gamma^4.
\]

(4)

dS-Dirac plane waves solutions are [12, 13, 14]

\[
\psi^\xi_V(x) = (H x . \xi)^{-2 + i \nu} V(x, \xi) \equiv (H x . \xi)^{-2 + i \nu} H \not k \xi \gamma^\lambda(\xi),
\]

\[
\psi^\xi_U(x) = (H x . \xi)^{-2 - i \nu} U^\lambda(\xi),
\]

where $V^\lambda$ and $U^\lambda$ are the polarization spinors and

\[
\xi \in \mathbb{C}^+ = \{ \xi : \eta_{\alpha\beta} \zeta^\alpha \zeta^\beta = (\zeta^0)^2 - \zeta . \bar{\zeta} - (\zeta^4)^2 = 0, \zeta^0 > 0 \}.
\]

Due to the singularity and the sign ambiguity in phase value, the solutions are defined in the complex de Sitter space [9],

\[
z \in X^{(c)}_H \equiv \{ z = x + iy \in \mathbb{Q}^5; \not \eta_{\alpha\beta} z^\alpha z^\beta = (z^0)^2 - \bar{z} . z - (z^4)^2 = -H^{-2} \},
\]

\[
\psi^\xi_V(z) = (H z . \xi)^{-2 + i \nu} V(z, \xi),
\]

\[
\psi^\xi_U(z) = (H z . \xi)^{-2 - i \nu} U(\xi).
\]

The spinor field operator is defined by the boundary value of complex solutions

\[
\psi(z) = \int_T \sum_{\lambda=1,2} \{ a_\lambda(\xi, \nu) U^\lambda(\xi)(z, \xi)^{-2 - i \nu} + \bar{d}_\lambda^\dagger(\xi, \nu) H \not k \xi Y^\lambda(\xi)(z, \xi)^{-2 - i \nu} \} d\mu_T(\xi),
\]

(5)

where $T$ denotes an orbital basis of $\mathbb{C}^+$ and $d\mu_T(\xi)$ is an invariant measure on $\mathbb{C}^+$ [10]. The vacuum state, which is fixed by imposing the condition that in the null curvature limit the Wightman two-point function becomes exactly the same as Minkowskian Wightman two-point function, is defined as follows [14]

\[
a_\lambda(\xi, \nu) |\Omega > = 0 = \bar{d}_\lambda(\xi, \nu) |\Omega >.
\]

This vacuum, $|\Omega >$, is equivalent to the Euclidean vacuum $|0, 0 >$. “One particle” and “one anti-particle” states are

\[
d_\lambda^\dagger(\xi, \nu) |\Omega > = |\xi, \lambda, \nu >, \quad a_\lambda^\dagger(\xi, \nu) |\Omega > = |\bar{\xi}, \lambda, \nu >.
\]

(6)
3 Discrete symmetries

Many authors have thoroughly discussed the Kinematic invariance of de Sitter space (de Sitter group), which explains the evolution of various spin free fields [19, 18, 20, 21]. The internal symmetry, which describes the interaction between the spinor field and the gauge boson field, has been studied recently [16]. Discrete symmetries for scalar field have been considered in [21] as well. At this stage we consider the discrete symmetries, parity, time reversal and antipodal transformation, for spinor field. The charge conjugation symmetry will be considered in the next section.

In accordance with the invariance principle, we shall verify that the form of dS-Dirac field equation remains unaltered in two frames, related by the above transformations. Let our system be described by the wave function $\psi$ in the first frame and by $\psi'$ in the transformed frame. Both must satisfy the dS-Dirac equation,

$$(-i \not \! x' \gamma \not \! \partial' + 2i + \nu)\psi'(x') = 0, \quad x' = \Lambda x. \quad (7)$$

There must be a local relation between $\psi$ and $\psi'$, so that the observer in the second frame may reconstruct $\psi'$ by $\psi$. We assume that this relation is linear:

$$\psi'(x') = S(\Lambda)\psi(x), \quad (8)$$

where $S(\Lambda)$ is a nonsingular $4 \times 4$ matrix. Equation (7) now reads

$$(-S^{-1}(\Lambda)i \not \! x' \gamma \not \! \partial S(\Lambda) + 2i + \nu)\psi(x) = 0. \quad (9)$$

If this equation is be equivalent to (2), for any $\psi$, we must have

$$S^{-1}(\Lambda) \not \! x' \gamma \not \! \partial S(\Lambda) = \not \! x \gamma \not \! \partial. \quad (10)$$

Construction of $S(\Lambda)$ will have to comply with the following transformations.

3.1 Parity

Space reflection or parity transformation, in de Sitter ambient space, is defined by the following relation [21]

$$x = (x^0, \vec{x}, x^4) \rightarrow x_p = (x^0, -\vec{x}, x^4) \equiv \Lambda_p x$$

where the matrix $\Lambda_p$ is

$$\Lambda_p = \begin{pmatrix}
1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.$$  

Since $S(\Lambda_p)$ should satisfy (10), by use of the relation $\not \! x_p = \gamma^0 \gamma^4 \not \! x \gamma^4 \gamma^0$, it is easily verified that the simple choice is the desired transformation

$$\psi'(x_p) = S(\Lambda_p)\psi(x) = \eta_p \gamma^0 \gamma^4 \psi(x), \quad (11)$$

where $\eta_p$ is the appropriate sign.
where \( \eta_p \) is an arbitrary, unobservable phase quantity. Since in the null curvature limit, we have [12, 14]

\[
(\gamma^\mu \gamma^4)_{\partial S} \equiv (\gamma^\mu)_M; \quad \mu = 0, 1, 2, 3,
\]

(12)

the parity is exactly the same as in the Minkowskian space. Therefore through the parity transformation in classical spinor field, we have

\[
S(\Lambda_p) = \eta_p \gamma^0 \gamma^4.
\]

(13)

From the quantum field point of view, one can find an operator \( U_p \) which satisfies the following:

\[
U_p \psi(x) U_p^\dagger = S(\Lambda_p) \psi(x_p).
\]

### 3.2 Time reversal

The time reversal transformation in de Sitter ambient space is defined by the following relation [21]

\[
x = (x^0, \vec{x}, x^4) \rightarrow x_t = (-x^0, \vec{x}, x^4).
\]

It could be shown by

\[
\Lambda_t = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

S(\( \Lambda_t \)) should satisfy (10). By the use of the relation \( \beta_t = -\gamma^0 \beta \gamma^0 \), it is easy to see that the simple choice is

\[
S(\Lambda_t) = \eta_t \gamma^0,
\]

(14)

where \( \eta_t \) is an arbitrary, unobservable phase value. Therefore \( S(\Lambda_t) \) must be the time reversal transformation for the classical spinor field. From the quantum field point of view, one can find an operator \( U_t \) satisfying

\[
U_t \psi(x) U_t^\dagger = S(\Lambda_t) \psi(x_t).
\]

\( U_t \) is antiunitary and antilinear operator, in the Minkowski space, which preserves the sign of energy under the time reversal transformation. In de Sitter space, however, the concept of energy is not defined globally and one can not impose the positive energy condition for obtaining an antiunitary and antilinear operator \( U_t \).

### 3.3 PT transformation

The PT transformation in de Sitter ambient space is defined by the following relation [21]

\[
x = (x^0, \vec{x}, x^4) \rightarrow x_{pt} = (-x^0, -\vec{x}, x^4) \equiv \Lambda_{pt} x
\]

where the matrix \( \Lambda_{pt} \) is

\[
\Lambda_{pt} = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 \\
0 & -1 & 0 & 0 & 0 \\
0 & 0 & -1 & 0 & 0 \\
0 & 0 & 0 & -1 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]
$S(\Lambda_{pt})$ should satisfy (10). By the use of the relation $\beta_{pt} = -\gamma^4 \beta^4$, it is easy to see that

$$\psi'(x_{pt}) = S(\Lambda_{pt})\psi(x) = \eta_{pt}i\gamma^4\psi(x)$$

(15)

is the desired transformation. Here $\eta_{pt}$ is an arbitrary, unobservable phase value.

### 3.4 Antipodal transformation

The Antipodal transformation in de Sitter ambient space is defined by the following relation [21]

$$x = (x^0, \vec{x}, x^4) \rightarrow x_a = (-x^0, -\vec{x}, -x^4).$$

It is presented by the matrix $\Lambda_a$,

$$\Lambda_a = \begin{pmatrix} -1 & 0 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & -1 \end{pmatrix} = -I,$$

where $I$ is the unite $4 \times 4$ matrix. $S(\Lambda_a)$ should satisfy (10). By the use of the relation $\beta_a = -\beta$, it is easy to see that

$$\psi'(x_a) = S(\Lambda_a)\psi(x) = \eta_a I\psi(x)$$

(16)

is the desired transformation. Here $\eta_a$ is an arbitrary, unobservable phase value. Therefore $S(\Lambda_a)$ is the antipodal transformation for the classical spinor field,

$$S(\Lambda_a) = \eta_a I \equiv (-1)^n I.$$

(17)

It clearly seen that $\psi(x)$ is a homogeneous function of $\mathbb{R}^5$-variables $x^a$, with homogeneity degree $n$ [17]. This transformation has no analogy in Minkowski space.

The full de Sitter group has four connected component: $SO_0(1, 4), \Lambda_p SO_0(1, 4), \Lambda_t SO_0(1, 4)$ and $\Lambda_{pt} SO_0(1, 4)$. Since $\Lambda_t SO_0(1, 4) \equiv \Lambda_a SO_0(1, 4)$, due to the fact that $\Lambda_a \Lambda_t \in SO_0(1, 4)$, $\Lambda_a$ can be obtained by some rotations of angle $\pi$ of the spacelike coordinate $\vec{x}, x^4$.Theses two discrete transformations $\Lambda_a$ and $\Lambda_t$ are closely linked together. $\Lambda_t$ violation in de Sitter invariant theories will imply violation of $\Lambda_a$. Thus the CP-violating effects in K-meson decay would bring a drastic additional symmetry breaking namely either violation of both $\Lambda_t$ and $\Lambda_a$ (where CPT is conserved,) or $\Lambda_t$ is conserved and consequently CP and CPT are violated [21]. The axiomatic approach to the scalar field quantization in de Sitter space and their consequences to the CPT theorem is discussed by Bros et al [10, 11].

### 4 Charge conjugation

The $U(1)$ local gauge invariant spinor field equation is obtained in a coordinate independent way notation [16] i.e.

$$(-i \not\beta - q \not\beta) \mathcal{K}(x) + 2i + \nu)\psi(x) = 0,$$

(18)
where $K(x)$ is a “massless” vector field (electromagnetic field) [22, 23, 24] and $q$ is a free parameter which can be identified with the electric charge in the null curvature limit. In Minkowski space, the Dirac equation has a symmetry corresponding to the “particle ↔ antiparticle” interchange. This transformation is known as the charge conjugation. In de Sitter space the concept of particle, antiparticle, and charge are not defined clearly.

Similar to the Minkowski space, we have defined the charge conjugation transformation for de Sitter ambient space by the following relation $\{ q \rightarrow -q, \nu \rightarrow -\nu \}$. From the point of view of the representation theory of de Sitter group, the representation $U^{\nu,s}$ and $U^{-\nu,s}$ are equivalent [19, 18, 20, 22]. There is no difference between classical spinor fields with opposite sign of $\nu$. But in the quantum field representation (5), one can see that the opposite sign of the $\nu$ corresponds to the particle and antiparticle interchange (6). In other words quantized fields may describe particles of opposite charge with identical spin and opposite sign of $\nu$.

We thus seek a transformation $\psi \rightarrow \psi^c$, reversing the “charge” such that

$$(-i \not{x} \not{D} + q \not{x} \not{K} + 2i - \nu)\psi^c(x) = 0.$$ (19)

We demand that this transformation should be local. To construct $\psi^c$ we conjugate and transpose the equation (18) and get

$$\bar{\psi}\gamma^4(i\not{\bar{D}} - q \not{K} - 2i + \nu) = 0,$$

$$(-i(\not{x})^T(\not{D})^T + q(\not{x})^T(\not{K})^T + 2i - \nu)(\gamma^4)^T(\bar{\psi})^T = 0,$$ (20)

where $(\bar{\psi})^T = (\gamma^4)^T(\gamma^0)^T\psi^*$. In any representation of the $\gamma$ algebra there must exist a matrix $C$ which satisfies

$$C(\gamma^\alpha)^T(\gamma^\beta)^T C^{-1} = \gamma^\alpha\gamma^\beta, \quad \text{or} \quad C\gamma^\alpha C^{-1} = \pm\gamma^\alpha.$$ (21)

In order to obtain the Minkowskian charge conjugation in the null curvature limit, the negative sign is chosen. We then identify $\psi^c$ as

$$\psi^c = \eta_c C(\gamma^4)^T(\bar{\psi})^T,$$ (22)

where $\eta_c$ is an arbitrary unobservable phase value, generally chosen to be unity. In the present framework charge conjugation is an antilinear transformation. In the $\gamma$ representation (3) we have:

$$C\gamma^0 C^{-1} = -\gamma^0, C\gamma^4 C^{-1} = -\gamma^4,$$

$$C\gamma^1 C^{-1} = -\gamma^1, C\gamma^3 C^{-1} = -\gamma^3, C\gamma^2 C^{-1} = \gamma^2.$$ (23)

In this representation $C$ commutes with $\gamma^2$ and anticommutes with other $\gamma$-matrix therefore the simple choice may be taken as $C = \gamma^2$. It satisfies

$$C = -C^{-1} = -C^T = -C^\dagger.$$ (24)

This clearly illustrates the non singularity of $C$. From the quantum field point of view one can find an operator $U_c$ satisfying

$$U_c \psi U_c^\dagger = \psi^c = \eta_c C(\gamma^4)^T(\bar{\psi})^T.$$
The adjoint spinor, which is defined by \( \bar{\psi}(x) \equiv \psi^\dagger(x)\gamma^0\gamma^4 \), transforms in a different way from \( \psi \), under de Sitter transformation [14]

\[
\psi'(x') = S(\Lambda)\psi(x), \quad \bar{\psi}'(x') = -\bar{\psi}(x)\gamma^4S^{-1}(\Lambda)\gamma^4,
\]

\( x' = \Lambda x, \quad \Lambda \in SO(1, 4), \quad S \in Sp(2, 2). \) \hfill (25)

On the contrary it is easy to show that \( \psi^c \) transforms in the same way as \( \psi \)

\[
\psi^c(x') = S(\Lambda)\psi^c(x),
\]

which is an important result. According to (19), the wave equation of \( \psi^c \), is different from the wave equation of \( \psi \) by the signs of the \( q \) and \( \nu \). Consequently if \( \psi \) describes the motion of a dS-Dirac "particle" with the charge \( q \), \( \psi^c \) represents the motion of a dS-Dirac "anti-particle" with the charge \((-q)\). In other words, \( \psi \) and \( \psi^c \) can described as particle and antiparticle functions.

A very interesting result of this representation is that \( \psi \) and \( \psi^c \) are charge conjugation of each other

\[
(\psi^c)^c = \gamma^0C\psi^c* = \psi. \hfill (26)
\]

A fundamental property of local quantum theory in Minkowski space, proved by Pauli, Zumino and Schwinger, states that in all cases, the action is invariant under \( \Theta \equiv U_pU_cU_t \) transformation. This is the famous PCT conservation theorem. The PCT transformation in de Sitter ambient space is defined by the following relation

\[
\begin{cases}
q \\ \nu
\end{cases}
\xrightarrow{\quad x = (x^0, \vec{x}, x^4) \quad}
\begin{cases}
-q \\ -\nu
\end{cases}
\xrightarrow{\quad x' = (-x^0, -\vec{x}, x^4). \quad} \hfill (27)
\]

The construction of the operator \( \Theta \) and the transformation of the two parameter spinor-vacuum \( |\alpha, \beta\rangle \) and the quantum spinor field will be considered in the forthcoming paper.

## 5 The five-current density

In order to discuss dS-interaction field theory in the ambient space notation, we need to know the five-current density and its different transformations under different symmetries. Recently we have obtained the five-current density by using the Noether’s theorem [16]. Alternatively we have obtained here, the five-current density by using the dS-Dirac equation and its adjoint. The dS-Dirac equation for \( \bar{\psi} \) is [14]:

\[
\bar{\psi}\gamma^4(i\bar{\partial} \cdot \vec{\kappa} - 2i + \nu) = 0. \hfill (28)
\]

Combining Eqs. (2) and (28) and using \( \bar{\psi}\gamma^4\psi = 0 \) ([14] Appendix C), lead to:

\[
\bar{\partial}_{\alpha}(\bar{\psi}\gamma^4\gamma^\alpha \cdot \kappa\psi) = 0 \hfill (29)
\]

We have, therefore, a candidate for the current

\[
J^\alpha = H\bar{\psi}\gamma^4\gamma^\alpha \cdot \kappa\psi \equiv -H\bar{\psi}\gamma^4 \cdot \kappa\gamma^\alpha\psi, \hfill (30)
\]

8
which is exactly the same as the conserved current [16]. It can be easily shown that \( x \cdot J = H \bar{\psi} \gamma^4 \psi = 0 \).

We now consider the transformation of the five-current density under different symmetries. The current density, \( J^\alpha \), is transformed as a five vector under de Sitter transformation:

\[
J^\alpha = H \bar{\psi}' \gamma^4 \gamma^\alpha \not{\!k} \not{\!p}' = H \bar{\psi}(-\gamma^4 S^{-1} \gamma^4)\gamma^4 \gamma^\alpha S \not{\!k} S^{-1} S \psi = \Lambda_\beta^\alpha J^\beta ,
\]

where we have used the relation \( S^{-1} \gamma^\alpha S = \Lambda_\beta^\alpha \gamma^\beta \) [14]. Therefore the current density is a vector field on the de Sitter hyperboloid. Its transformation under the charge conjugation is:

\[
J^\alpha = H \bar{\psi}^c \gamma^4 \gamma^\alpha \not{\!k} \not{\!p}^c = H \bar{\psi}^c \gamma^4 \not{\!k} \not{\!p} = J^\alpha ,
\]

where \( \psi^c = \gamma^0 C \psi^* \) and \( \bar{\psi}^c = \psi_T \gamma^4 C \). Under the time reversal, the transformation is

\[
J^\alpha_t(x_t) = H \psi'^T \gamma^4 \gamma^\alpha \not{\!k} \not{\!T} \psi' = (\Lambda_t)^\alpha_\beta J^\beta(x). \tag{33}
\]

Under the parity transformation, we have

\[
J^\alpha_p(x_p) = H \psi'^T \gamma^4 \gamma^\alpha \not{\!k} \not{\!p} \psi' = -(\Lambda_p)^\alpha_\beta J^\beta(x), \tag{34}
\]

where \( J^\alpha_p \) is a pseudo-vector.

It is easy to show that \( \phi(x) = \psi^\dagger \gamma^0 \not{\!x} \psi = -\bar{\psi} \gamma^4 \not{\!x} \psi \) is a scalar field under de Sitter transformation

\[
\bar{\psi}' \gamma^4 \not{\!k} \not{\!p}' = -\bar{\psi} \gamma^4 S^{-1} \gamma^4 \gamma^4 \not{\!k} S \psi = \bar{\psi} \gamma^4 S^{-1} \not{\!k} S \psi = \bar{\psi} \gamma^4 \not{\!k} \not{\!p}. \tag{35}
\]

The transformation under parity is

\[
\phi'(x_p) = \psi'^T \gamma^0 \not{\!k} \not{\!p} \psi' = \psi^\dagger \gamma^0 \gamma^4 \gamma^0 \not{\!k} \not{\!p} \gamma^4 \psi = \psi^\dagger \gamma^0 \not{\!k} \not{\!p} \psi. \tag{36}
\]

The behavior of the scalar density under the charge conjugation is

\[
\phi^c = \psi'^\dagger \gamma^0 \not{\!k} \not{\!p}^c = \psi'^T \gamma^0 \not{\!k}^T \psi = \psi^\dagger \not{\!k}^\dagger \gamma^0 \psi = \psi^\dagger \gamma^0 \not{\!k} \not{\!p}. \tag{37}
\]

Finally it is interesting to calculate the dS-Gordon decomposition. By multiplying the adjoint dS-Dirac equation from right by \( \not{\!k} \not{\!p} \psi \) and dS-Dirac equation from left by \( \bar{\psi} \gamma^4 \not{\!k} \not{\!p} \), we obtain

\[
\bar{\psi} \gamma^4 (i H^{-2} \not{\!k} \not{\!p} + 2 i \not{\!k} \not{\!p} + \nu \not{\!k} \not{\!p}) \psi = 0 \tag{38}
\]

\[
\bar{\psi} \gamma^4 (i H^{-2} \not{\!k} \not{\!p} + 2 i \not{\!p} \not{\!k} \not{\!p} - \nu \not{\!k} \not{\!p}) \psi = 0. \tag{39}
\]

Combining these equations we obtain:

\[
i H^{-2} \bar{\psi} \gamma^4 (\not{\!k} \not{\!p} + \frac{\gamma^4}{\not{\!k}} \not{\!p} \gamma^4 \not{\!k} \not{\!p} + 2 \nu \bar{\psi} \gamma^4 \not{\!k} \not{\!p} \psi = 0. \tag{40}
\]

By using \( \not{\!k} \not{\!p} = a \cdot b + 2 i a_\alpha b_\beta S^{\alpha \beta} \), where \( S^{\alpha \beta} = -\frac{1}{4}[\gamma_\alpha, \gamma_\beta] \) we can write

\[
\not{\!k} \not{\!p} = a \cdot b + 2 i a_\alpha \bar{\gamma}_\beta S^{\alpha \beta} \tag{41}
\]

\[
\frac{\gamma^4}{\not{\!k}} \not{\!p} = a \cdot \frac{\gamma^4}{\not{\!k}} \not{\!p} - 2 i a_\alpha \frac{\gamma^4}{\not{\!k}} \not{\!p} S^{\alpha \beta}. \tag{42}
\]

By eliminating \( a_\alpha \), we can write the five-current density in the following form

\[
J^\alpha = H \bar{\psi} \gamma^4 \gamma^\alpha \not{\!k} \not{\!p} = \frac{1}{H \nu} [\bar{\psi} \gamma^4 S^{\alpha \beta} (\not{\!p} \gamma^4 \not{\!k} \not{\!p} \gamma^4 \not{\!k} \not{\!p} \gamma^4 \not{\!k} \not{\!p}) - (\not{\!p} \gamma^4 \not{\!k} \not{\!p} \gamma^4 \not{\!k} \not{\!p} \gamma^4 \not{\!k} \not{\!p})]. \tag{43}
\]
6 Conclusion and outlook

The formalism of the quantum field theory in de Sitter universe, in ambient space notation, is very similar to the quantum field formalism in Minkowski space. An explicit form of the charge conjugation matrix $C$, associated with the universal covering group of $SO_0(1, 4)$, i.e. $Sp(2, 2)$, has been obtained. Equipped with the results of the present paper, various interaction Lagrangians between the fermions and bosons, preserving the de Sitter invariance and discrete symmetries can easily be defined. Using charge conjugation matrix $C$ one can study the supersymmetry algebra in de Sitter space as well. The importance of this formalism may be shown further by the consideration of the linear quantum gravity [25] and supergravity in de Sitter space, which lays a firm ground for further study of evolution of the universe.

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A Discrete symmetries in intrinsic global coordinate

In de Sitter space one can define four type of coordinate systems: global, flat, open and static coordinates [26]. It is the global type coordinate that cover the whole de Sitter manifold. Therefore, only in this type of coordinates, the discrete symmetries can be defined properly [12]. If we chose the global coordinates as:

$$
\begin{align*}
  x^0 &= H^{-1} \sinh H t \\
  x^1 &= H^{-1} \cosh H t \cos \chi \\
  x^2 &= H^{-1} \cosh H t \sin \chi \cos \theta \\
  x^3 &= H^{-1} \cosh H t \sin \chi \sin \theta \cos \phi \\
  x^4 &= H^{-1} \cosh H t \sin \chi \sin \theta \sin \phi
\end{align*}
$$

with the metric

$$
d s^2 = \eta_{\alpha \beta} dx^\alpha dx^\beta \big|_{x^2 = -H^2} = g_{\mu \nu} dX^\mu dX^\nu = dt^2 - H^{-2} \cosh^2 H t [d\chi^2 + \sin^2 \chi (d\theta^2 + \sin^2 \theta d\phi^2)],
$$

the discrete symmetries are defined by:

|                        | antipodal | time reversal | parity |
|------------------------|----------|--------------|--------|
| $X^0 = t \longrightarrow -t$ | $t \longrightarrow -t$ | $t \longrightarrow t$ |
| $X^1 = \chi \longrightarrow \pi - \chi$ | $\chi \longrightarrow \chi$ | $\chi \longrightarrow \pi - \chi$ |
| $X^2 = \theta \longrightarrow \pi - \theta$ | $\theta \longrightarrow \theta$ | $\theta \longrightarrow \pi - \theta$ |
| $X^3 = \phi \longrightarrow \pi + \phi$ | $\phi \longrightarrow \phi$ | $\phi \longrightarrow \pi - \phi$ |

The Dirac equation in the general curved space time is [27]

$$
(\gamma^\mu(X) \nabla_\mu - m) \Psi(X) = 0 = (\bar{\gamma}^a \nabla_a - m) \Psi(X),
$$

where

$$
\{\gamma^\mu(X), \gamma^\nu(X)\} = 2g^{\mu \nu}, \quad \{\bar{\gamma}^a, \bar{\gamma}^b\} = 2\eta^{ab}, \quad \mu, a = 0, 1, 2, 3.
$$
Here $\nabla$ is the spinor covariant derivative

$$\nabla_a \Psi(X) = e^\mu_a (\partial_\mu + \frac{i}{2} e^\nu \nabla_\mu e^{ab} \Sigma_{cb}) \Psi(X),$$

where $e^\mu_a$ is the local vierbein, $e^\mu_a e^\nu_b \eta_{ab} = g_{\mu\nu}$, and $\Sigma_{cb} = \frac{i}{4} [\gamma_c, \gamma_b]$ is the spinor representation of the generators of the Lorentz transformation. The Dirac equation under the discrete symmetries transform in the following form

$$(\gamma^\mu (X') \nabla'_\mu - m) \Psi'(X') = 0 = (\bar{\gamma}^a \nabla'_a - m) \Psi(X').$$

By using the linear relation $\Psi'(X') = S \Psi(X)$ we obtained the condition

$$S^{-1} \bar{\gamma}^a \nabla'_a S = \bar{\gamma}^a \nabla_a, \text{ or } S^{-1} \gamma^\mu (X') \nabla'_\mu S = \gamma^\mu (X) \nabla_\mu.$$

Through a direct calculation, the discrete symmetries in this notation can be obtained:

| time reversal | parity | antipodal |
|---------------|--------|-----------|
| $S(t)\bar{\gamma}^0 S(t)^{-1} = -\bar{\gamma}^0$ | $S(p)\bar{\gamma}^0 S(p)^{-1} = -\bar{\gamma}^0$ | $S(a)\bar{\gamma}^0 S(a)^{-1} = -\bar{\gamma}^0$ |
| $S(t)\bar{\gamma}^1 S(t)^{-1} = \bar{\gamma}^1$ | $S(p)\bar{\gamma}^1 S(p)^{-1} = \bar{\gamma}^1$ | $S(a)\bar{\gamma}^1 S(a)^{-1} = \bar{\gamma}^1$ |
| $S(t)\bar{\gamma}^2 S(t)^{-1} = \bar{\gamma}^2$ | $S(p)\bar{\gamma}^2 S(p)^{-1} = \bar{\gamma}^2$ | $S(a)\bar{\gamma}^2 S(a)^{-1} = \bar{\gamma}^2$ |
| $S(t)\bar{\gamma}^3 S(t)^{-1} = \bar{\gamma}^3$ | $S(p)\bar{\gamma}^3 S(p)^{-1} = \bar{\gamma}^3$ | $S(a)\bar{\gamma}^3 S(a)^{-1} = \bar{\gamma}^3$ |

where $\eta$'s are the arbitrary, unobservable phase values. In the next appendix we discuss the relation between the two formalisms.

B The relation between the intrinsic and ambient space notation

First we briefly recall here the relation between de Sitter-Dirac equation (2) and the usual Dirac equation for curved spacetimes obtained by the method of covariant derivative (44), which is established in the paper by Gürsey and Lee [28].

They introduced a set of coordinates \(\{y^\alpha\} \equiv \{y^\mu, y^4 = H^{-1}\} \) related to the \(\{x^\alpha\}\)’s by

$$x^\alpha = (Hy^4)f^\alpha(y^0, y^1, y^2, y^3),$$

where arbitrary functions $f^\alpha$ satisfying $f^\alpha f_\alpha = -H^{-2}$. Introducing five matrices $\beta^\alpha \equiv \left(\frac{\partial x^\alpha}{\partial y^\mu}\right) \gamma^\beta$, satisfy the anticommutation properties

$$\{\beta^\mu, \beta^\nu\} = 2g^{\mu\nu}, \{\beta^\mu, \beta^4\} = 0$$

with $g^{\mu\nu} = \eta^{\alpha\beta} \frac{\partial x^\alpha}{\partial x^\mu} \frac{\partial x^\beta}{\partial x^\nu}$, $\mu, \nu = 0, \ldots, 3$, and $\psi = (1 \pm i\beta^4)\chi$, $\chi$ satisfies the Gürsey–Lee equation

$$\left(\beta^\mu \frac{\partial}{\partial y^\mu} - 2H \beta^4 - m\right) \chi = 0, \quad (45)$$
where $m = H\nu$. Choosing at every point of de Sitter spacetime, a local vierbein $e_\mu^a$, and setting $\gamma^\mu(X) \equiv e^\mu_a \gamma^a$, there exists a transformation $V$ such that $\gamma^\mu(X) = V^\beta(y)V^{-1}$. Then, under the $V$ transformation, the Gürsey–Lee equation (45) becomes equation (44) with $\Psi(X) = V\chi(y)$. It is interesting to notice that the matrix $\beta^4 = \gamma_\alpha x^\alpha$ is related to the constant matrix $\gamma^4$ by $\gamma^4 = V\beta^4V^{-1}$ [28].

Now we can write the relation between the spinor field in two notations

$$\psi(x) = V^{-1}(1 \pm i\gamma^4)\Psi(X).$$

It is important to notice that the inverse of matrix $V^{-1}(1 \pm i\gamma^4)$ does not exist-similar to the relation between tensor field in the two notations [22]. The matrix that transforms the $\psi(x)$ to $\Psi(X)$ can be calculated as well. By using the above equation and the transformations of the local vierbein $e_\mu^a$, the relation between the discrete symmetries in the two formalisms can be obtained directly.

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