Lower bounds for artificial neural network approximations: A proof that shallow neural networks fail to overcome the curse of dimensionality

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Abstract

Artificial neural networks (ANNs) have become a very powerful tool in the approximation of high-dimensional functions. Especially, deep ANNs, consisting of a large number of hidden layers, have been very successfully used in a series of practical relevant computational problems involving high-dimensional input data ranging from classification tasks in supervised learning to optimal decision problems in reinforcement learning. There are also a number of mathematical results in the scientific literature which study the approximation capacities of ANNs in the context of high-dimensional target functions. In particular, there are a series of mathematical results in the scientific literature which show that sufficiently deep ANNs have the capacity to overcome the curse of dimensionality in the approximation of certain target function classes in the sense that the number of parameters of the approximating ANNs grows at most polynomially in the dimension $d \in \mathbb{N}$ of the target functions under considerations. In the proofs of several of such high-dimensional approximation results it is crucial that the involved ANNs are sufficiently deep and consist a sufficiently large number of hidden layers which grows in the dimension of the considered target functions. It is the topic of this work to look a bit more detailed to the deepness of the involved ANNs in the approximation of high-dimensional target functions. In particular, the main result of this work proves that there exists a concretely specified sequence of functions which can be approximated without the curse of dimensionality by sufficiently deep ANNs but which cannot be approximated without the curse of dimensionality if the involved ANNs are shallow or not deep enough.
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1 Introduction

Artificial neural networks (ANNs) have become a very powerful tool in the approximation of high-dimensional functions. Especially, deep ANNs, consisting of a large number of hidden layers, have been very successfully used in a series of practical relevant computational problems involving high-dimensional input data ranging from classification tasks in supervised learning to optimal decision problems in reinforcement learning.

There are also a large number of mathematical results in the scientific literature which study the approximation capacities of ANNs; see, e.g., Cybenko [10], Funahashi [18], Hornik et al. [29,30], Leshno et al. [44], Guliyev & Ismailov [27], Elbrächter et al. [16], and the references mentioned therein. Moreover, in the recent years a series of articles have appeared in the scientific literature which study the approximation capacities of ANNs in the context of high-dimensional target functions. In particular, the results in such articles show that deep ANNs have the capacity to overcome the curse of dimensionality in the approximation of certain target function classes in the sense that the number of parameters of the approximating ANNs grows at most polynomially in the dimension $d \in \mathbb{N}$ of the target functions under considerations. For example, we refer to Elbrächter et al. [15], Jentzen et al. [33], Gonon et al. [20,21], Grohs et al. [22,23,25], Kutyniok et al. [43], Reisinger & Zhang [49], Beneventano et al. [6], Berner et al. [7], Hornung et al. [31], Hutzenthaler et al. [32], and the overview articles Beck et al. [4] and E et al. [13] for such high-dimensional ANN approximation results in the numerical approximation of solutions of PDEs and we refer to Barron [1–3], Jones [34], Girosi & Anzellotti [19], Donahue et al. [12], Gurvits & Koiran [28], Kůrková et al. [39–42], Kainen et al. [35,36], Klusowski & Barron [38], Li et al. [45], and Cheridito et al. [9] for such high-dimensional ANN approximation results in the numerical approximation of certain specific target function classes independent of solutions of PDEs (cf., e.g., also Maiorov & Pinkus [46], Pinkus [48], Guliyev & Ismailov [26], Petersen & Voigtlaender [47], and Bölcskei et al. [8] for related results). In the proofs of several of the above named high-dimensional approximation results it is crucial that the involved ANNs are sufficiently deep and consist a sufficiently large number of hidden layers which grows in the dimension of the considered target functions.

It is the key topic of this work to look a bit more detailed to the deepness of the involved ANNs in the approximation of high-dimensional target functions. More specifically, Theorem 6.1 in Section 6 below, which is the main result of this work, proves that there exists a concretely specified sequence of high-dimensional functions which can be approximated without the curse of dimensionality by sufficiently deep ANNs but which cannot be approximated without the curse of dimensionality if the involved ANNs are shallow or not deep enough. In the scientific literature related ANN approximation results can also be found in Daniely [11], Eldan & Shamir [17], and Safran & Shamir [51]. One of the differences between the results in the above named references and the results in this work is, roughly speaking, that the considered target functions in the above named references can be approximated by ANNs with two hidden layers without the curse of dimensionality but not with ANNs with one hidden layer while in this work the considered target functions can only be approximated without the curse of dimensionality if the number of the hidden layers of the approximating ANN grows like the dimensions of the target functions.

To illustrate the findings of this work in more detail, we now present in the following result, Theorem 1.1 below, a special case of Theorem 6.1. Below Theorem 1.1 we also add some explanatory comments regarding the mathematical objects appearing in Theorem 1.1 and regarding the statement of Theorem 1.1.
Theorem 1.1. Let \( \varphi: (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \to \mathbb{R} \) and \( \mathfrak{R}: (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \to (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \) satisfy for all \( d \in \mathbb{N}, x = (x_1, \ldots, x_d) \in \mathbb{R}^d \) that \( \varphi(x) = (2\pi)^{-d/2} \exp\left(-\frac{1}{2}(\sum_{j=1}^{d}|x_j|^2)\right) \) and \( \mathfrak{R}(x) = (\max\{x_1, 0\}, \ldots, \max\{x_d, 0\}) \), let \( N = \cup_{L \in \mathbb{N}} \cup_{l_0, l_1, \ldots, l_L \in \mathbb{N}} (\times_{k=1}^{L}(\mathbb{R}^{k_{l_k-1}} \times \mathbb{R}^{l_k})) \), and let \( \mathfrak{R}: N \to (\cup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l)), \mathcal{H}: N \to \mathbb{N}_0, \mathcal{P}: N \to \mathbb{N}, \) and \( \|\cdot\|: N \to \mathbb{R} \) satisfy for all \( L \in \mathbb{N}, l_0, l_1, \ldots, l_L \in \mathbb{N}, v_0 \in \mathbb{R}^{l_0}, v_1 \in \mathbb{R}^{l_1}, \ldots, v_L \in \mathbb{R}^{l_L}, \Phi = ((W_1, B_1), \ldots, (W_L, B_L)) = (((W_{i,j}, (i,j)) \in (1, \ldots, l_0), (B_{i,j}) \in (1, \ldots, l_3), \ldots, (W_{L,i,j}, (i,j)) \in (1, \ldots, l_1), (B_{L,i}) \in (1, \ldots, l_L)) \in (\times_{k=1}^{L}(\mathbb{R}^{k_{l_{k-1}}} \times \mathbb{R}^{l_k})) \) with \( \forall k \in \{1, 2, \ldots, L\}: v_k = \mathfrak{R}(W_kv_{k-1} + B_k) \) that \( \mathfrak{R}(\Phi) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_1}), (\mathfrak{R}(\Phi))(v_0) = WLv_{L-1} + B_L, \mathcal{H}(\Phi) = L - 1, \mathcal{P}(\Phi) = \sum_{k=1}^{L} l_k(k_{l_{k-1}} + 1) \), and \( \|\Phi\| = \max_{1 \leq i \leq L} \max_{1 \leq j \leq l_i} \max_{1 \leq k \leq l_{k-1}} \max\{|W_{n,i,j}|, |B_{n,i}|\} \). Then there exist continuously differentiable \( \tilde{f}_d: \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N} \), such that for all \( \delta \in (0, 1], \varepsilon \in (0, 1/2] \) there exists \( \mathfrak{c} \in (0, \infty) \) such that

(i) it holds for all \( c \in [\mathfrak{c}, \infty), d \in \mathbb{N} \) that

\[
\min \left\{ p \in \mathbb{N}: \exists \Phi \in N: p = \mathcal{P}(\Phi), \|\Phi\| \leq cd^c, \quad d \leq \mathcal{H}(\Phi) \leq cd, \mathfrak{R}(\Phi) \in C(\mathbb{R}^d, \mathbb{R}), \right.
\]
\[
\left. \quad \left[ \int_{\mathbb{R}^d} |\mathcal{R}(\Phi)(x) - \tilde{f}_d(x)|^2 \varphi(x) \, dx \right]^{1/2} \leq \varepsilon \right\} \leq cd^3 \quad (1.1)
\]

and

(ii) it holds for all \( c \in [\mathfrak{c}, \infty), d \in \mathbb{N} \) that

\[
\min \left\{ p \in \mathbb{N}: \exists \Phi \in N: p = \mathcal{P}(\Phi), \|\Phi\| \leq cd^c, \quad \mathcal{H}(\Phi) \leq cd^{1-\delta}, \mathfrak{R}(\Phi) \in C(\mathbb{R}^d, \mathbb{R}), \right. \]
\[
\left. \quad \left[ \int_{\mathbb{R}^d} |\mathcal{R}(\Phi)(x) - \tilde{f}_d(x)|^2 \varphi(x) \, dx \right]^{1/2} \leq \varepsilon \right\} \geq (1 + c^{-3})^{(d^3)} \quad (1.2)
\]

Theorem 1.1 above is an immediate consequence of Corollary 6.2 in Subsection 6.2 below. Corollary 6.2, in turn, follows from Theorem 6.1 in Subsection 6.1 below, which is the main result of the article. In the following we provide some explanatory comments regarding the statement of Theorem 1.1 and regarding the mathematical objects appearing in Theorem 1.1.

In Theorem 1.1 we measure the error between the target function and the realization of the approximating ANN in the \( L^2 \)-sense on the whole \( \mathbb{R}^d, d \in \mathbb{N} \), with respect to standard normal distribution. In particular, we observe that the function \( \varphi: (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \to \mathbb{R} \) in Theorem 1.1 appears in the \( L^2 \)-errors in items (i) and (ii) in Theorem 1.1 and describes the densities of the standard normal distribution. More formally, note that for all \( d \in \mathbb{N} \) it holds that the function \( \mathbb{R}^d \ni x \mapsto \varphi(x) = (2\pi)^{-d/2} \exp\left(-\frac{1}{2}(\sum_{j=1}^{d}|x_j|^2)\right) \in \mathbb{R} \) is nothing else but the density of the \( d \)-dimensional standard normal distribution.

Theorem 1.1 is an approximation result for ANNs with the rectifier function as the activation function and the function \( \mathfrak{R}: (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \to (\cup_{d \in \mathbb{N}} \mathbb{R}^d) \) in Theorem 1.1 describes multidimensional versions of the rectifier function. More specifically, observe that for all \( d \in \mathbb{N} \) it holds that the function \( \mathbb{R}^d \ni x \mapsto \mathfrak{R}(x) = (\max\{x_1, 0\}, \ldots, \max\{x_d, 0\}) \in \mathbb{R}^d \) is the \( d \)-dimensional version of the rectifier activation function \( \mathbb{R} \ni x \mapsto \max\{x_0\} \in \mathbb{R} \).

The set \( N = \cup_{L \in \mathbb{N}} \cup_{l_0, l_1, \ldots, l_L \in \mathbb{N}} (\times_{k=1}^{L}(\mathbb{R}^{k_{l_{k-1}}} \times \mathbb{R}^{l_k})) \) in Theorem 1.1 represents the set of all ANNs and the function \( \mathfrak{R}: N \to (\cup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l)) \) in Theorem 1.1 assigns to each ANN in \( N \) its realization function. More formally, note that for every ANN \( \Phi \in N \) it holds that the function \( \mathfrak{R}(\Phi) \in (\cup_{k,l \in \mathbb{N}} C(\mathbb{R}^k, \mathbb{R}^l)) \) is the realization function associated to the ANN \( \Phi \).

The function \( \mathcal{H}: N \to \mathbb{N}_0 \) in Theorem 1.1 describes the number of hidden layers of the considered ANN, the function \( \mathcal{P}: N \to \mathbb{N} \) in Theorem 1.1 counts the number of parameters (the number of weights and biases) used to describe the considered ANN, and the function \( \|\cdot\|: N \to \mathbb{R} \) in Theorem 1.1 specifies the size of the absolute values of the parameters of the considered ANN. More specifically, observe that for every ANN \( \Phi \in N \) it holds that \( \mathcal{H}(\Phi) \)
is the number of hidden layers of the ANN $\Phi$, that $\mathcal{P}(\Phi)$ is the number of real parameters used to describe the ANN $\Phi$, and that $\|\Phi\|$ is the maximum of the absolute values of the real parameters used to describe the ANN $\Phi$.

Roughly speaking, Theorem 1.1 asserts that there exists a sequence of continuously differentiable target functions $f_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, such that for every arbitrarily small prescribed approximation accuracy $\varepsilon \in (0, 1/2)$ it holds that the class of all sufficiently deep ANNs can approximate the target functions $f_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, without the curse of dimensionality (with the number of ANN parameters growing at most cubically in the dimension $d \in \mathbb{N}$; see (1.1) in item (i) in Theorem 1.1) and that the class of all shallow ANNs can only approximate the target functions $f_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, with the curse of dimensionality (with the number of ANN parameters growing at least exponentially in the dimension $d \in \mathbb{N}$; see (1.2) in item (ii) in Theorem 1.1). In that sense Theorem 1.1 shows for a specific class of target functions that deep ANNs can overcome the curse of dimensionality but shallow ANNs fail to do so.

The remainder of this article is organized as follows. In Section 2 we briefly recall a few general concepts and results from the scientific literature to describe and operate on ANNs. In Section 3 we establish suitable upper bounds for certain weighted tails of standard normal distributions. In Section 4 we use the upper bounds for certain weighted tails of standard normal distributions from Section 3 to establish appropriate lower bounds for the number of parameters of ANNs that approximate certain high-dimensional target functions. In Section 5 we establish suitable upper bounds for the number of parameters of ANNs that approximate such high-dimensional target functions. In Section 6 we combine the lower bounds from Section 4 with the upper bounds from Section 5 to establish in Theorem 6.1 the main ANN approximation result of this work. Theorem 1.1 above is a direct consequence of Corollary 6.2 in Section 6, which, in turn, follows from Theorem 6.1 in Section 6.

2 Basics on artificial neural networks (ANNs)

In this section we briefly recall a few general concepts and results from the scientific literature to describe and operate on ANNs. All the notions and the results in this section are well-known in the scientific literature. In particular, regarding Definition 2.2 we refer, e.g., to [24, Definitions 2.1 and 2.3], regarding Definition 2.4 we refer, e.g., to [24, Definition 2.5], regarding Definition 2.7 we refer, e.g., to [24, Definition 2.10], regarding Definition 2.8 we refer, e.g., to [24, Definition 2.11], regarding Definition 2.10 we refer, e.g., to [24, Definition 2.17], regarding Definition 2.13 we refer, e.g., to [25, Definition 3.15], regarding Definition 2.15 we refer, e.g., to [25, Definitions 3.7 and 3.10], regarding Definition 2.17 we refer, e.g., to [25, Definition 3.13], regarding Definition 2.19 we refer, e.g., to [25, Definition 3.17], and regarding Definition 2.22 we refer, e.g., to [5, Definition 2.11]. Moreover, note that Proposition 2.5 is, e.g., proved as [24, Proposition 2.6], note that Lemma 2.6 is, e.g., proved as [24, Lemma 2.8], note that Lemma 2.9 is, e.g., proved as [24, Lemma 2.13], note that Proposition 2.11 is, e.g., proved as [24, Proposition 2.19], note that Proposition 2.12 is, e.g., proved as [24, Proposition 2.20], note that Lemma 2.14 is, e.g., proved as [25, Lemma 3.16], note that Lemma 2.18 is, e.g., proved as [25, Lemma 3.14], and note that Lemma 2.20 is, e.g., proved as [25, Lemma 3.18]. The proof of Lemma 2.16 is clear and therefore is omitted.

2.1 Structured description of ANNs

Definition 2.1. We denote by $\mathcal{R}: (\bigcup_{d \in \mathbb{N}} \mathbb{R}^d) \to (\bigcup_{d \in \mathbb{N}} \mathbb{R}^d)$ the function which satisfies for all $d \in \mathbb{N}$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that $\mathcal{R}(x) = (\max\{x_1, 0\}, \max\{x_2, 0\}, \ldots, \max\{x_d, 0\})$. 

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Definition 2.2. We denote by $\mathbb{N}$ the set given by
\[
\mathbb{N} = \bigcup_{L \in \mathbb{N}} \bigcup_{l_0, l_1, \ldots, l_L \in \mathbb{N}} \left( \times_{k=1}^{L} (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k}) \right)
\] (2.1)
and we denote by $\mathcal{R}: \mathbb{N} \to (\bigcup_{L\in\mathbb{N}} C(\mathbb{R}^{l_0}, \mathbb{R}^{l_L}))$, $\mathcal{P}: \mathbb{N} \to \mathcal{L}$, $\mathcal{N}: \mathbb{N} \to \mathbb{N}$, $\mathcal{T}: \mathbb{N} \to \mathbb{N}$, $\mathcal{O}: \mathbb{N} \to \mathbb{N}$, $\mathcal{H}: \mathbb{N} \to \mathbb{N}_0$, $\mathcal{D}: \mathbb{N} \to (\bigcup_{n=0}^{\infty} \mathbb{N}_L^n)$, and $\mathcal{D}_n: \mathbb{N} \to \mathbb{N}_0$, $n \in \mathbb{N}_0$, the functions which satisfy for all $L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L \in \mathbb{N}$, $\Phi = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in \left( \times_{k=1}^{L} (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k}) \right)$, $v_0 \in \mathbb{R}^{l_0}, v_1 \in \mathbb{R}^{l_1}, \ldots, v_L \in \mathbb{R}^{l_L}$, $n \in \mathbb{N}_0$ with $\forall k \in \{1, 2, \ldots, L\}$: $v_k = \mathcal{R}(W_kv_{k-1} + B_k)$ that $\mathcal{R}(\Phi) \in C(\mathbb{R}^{l_0}, \mathbb{R}^{l_L})$, $(\mathcal{R}(\Phi))(v_0) = W_Lv_{L-1} + B_L$, $\mathcal{P}(\Phi) = \sum_{k=1}^{L} l_k(l_k-1)$, $\mathcal{L}(\Phi) = L$, $\mathcal{I}(\Phi) = l_0$, $\mathcal{O}(\Phi) = l_L$, $\mathcal{H}(\Phi) = L - 1$, $\mathcal{D}(\Phi) = (l_0, l_1, \ldots, l_L)$, and
\[
\mathcal{D}_n(\Phi) = \begin{cases} 
  l_n & : n \leq L \\
  0 & : n > L 
\end{cases}
\] (2.2)
(cf. Definition 2.1).

Definition 2.3 (Neural network). We say that $\Phi$ is a neural network if and only if it holds that $\Phi \in \mathbb{N}$ (cf. Definition 2.2).

2.2 Compositions of ANNs

Definition 2.4 (Compositions of ANNs). We denote by $(\cdot) \bullet (\cdot): \{ (\Phi_1, \Phi_2) \in \mathbb{N} \times \mathbb{N}: \mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2) \} \to \mathbb{N}$ the function which satisfies for all $L, \mathcal{L} \in \mathbb{N}$, $l_0, l_1, \ldots, l_L, l_0, l_1, \ldots, l_{\mathcal{L}} \in \mathbb{N}$, $\Phi_1 = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in \left( \times_{k=1}^{L} (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k}) \right)$, $\Phi_2 = ((\mathcal{M}_1, \mathcal{B}_1), (\mathcal{M}_2, \mathcal{B}_2), \ldots, (\mathcal{M}_{\mathcal{L}}, \mathcal{B}_{\mathcal{L}})) \in \left( \times_{k=1}^{\mathcal{L}} (\mathbb{R}^{l_k} \times \mathbb{R}^{l_k}) \right)$ with $l_0 = \mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2) = l_{\mathcal{L}}$ that
\[
\Phi_1 \bullet \Phi_2 = \left\{ \begin{array}{ll}
((\mathcal{M}_1, \mathcal{B}_1), (\mathcal{M}_2, \mathcal{B}_2), \ldots, (\mathcal{M}_{\mathcal{L}-1}, \mathcal{B}_{\mathcal{L}-1}), (W_1 \mathcal{M}_{\mathcal{L}}, W_1 \mathcal{B}_L + B_1), & : L > 1 < \mathcal{L} \\
(W_2, B_2), (W_3, B_3), \ldots, (W_L, B_L)) & : L > 1 = \mathcal{L} \\
((W_1 \mathcal{M}_1, W_1 \mathcal{B}_1 + B_1), (W_2, B_2), (W_3, B_3), \ldots, (W_L, B_L)) & : L = 1 < \mathcal{L} \\
((W_1 \mathcal{M}_1, W_1 \mathcal{B}_1 + B_1)) & : L = 1 = \mathcal{L} 
\end{array} \right. 
\] (2.3)
(cf. Definition 2.2).

Proposition 2.5. Let $\Phi_1, \Phi_2 \in \mathbb{N}$ satisfy $\mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2)$ (cf. Definition 2.2). Then

(i) it holds that
\[
\mathcal{D}(\Phi_1 \bullet \Phi_2) = (\mathcal{D}_0(\Phi_2), \mathcal{D}_1(\Phi_2), \ldots, \mathcal{D}_{\mathcal{H}(\Phi_2)}(\Phi_2), \mathcal{D}_1(\Phi_1), \mathcal{D}_2(\Phi_1), \ldots, \mathcal{D}_{\mathcal{L}(\Phi_1)}(\Phi_1)),
\] (2.4)
(ii) it holds that $\mathcal{H}(\Phi_1 \bullet \Phi_2) = \mathcal{H}(\Phi_1) + \mathcal{H}(\Phi_2),$
(iii) it holds that $\mathcal{R}(\Phi_1 \bullet \Phi_2) \in C(\mathbb{R}^{\mathcal{I}(\Phi_2)}, \mathbb{R}^{\mathcal{O}(\Phi_1)}),$
(iv) it holds that $\mathcal{R}(\Phi_1 \bullet \Phi_2) = [\mathcal{R}(\Phi_1)] \circ [\mathcal{R}(\Phi_2)]$
(cf. Definition 2.4).

Lemma 2.6. Let $\Phi_1, \Phi_2, \Phi_3 \in \mathbb{N}$ satisfy $\mathcal{I}(\Phi_1) = \mathcal{O}(\Phi_2)$ and $\mathcal{I}(\Phi_2) = \mathcal{O}(\Phi_3)$ (cf. Definition 2.2). Then $(\Phi_1 \bullet \Phi_2) \bullet \Phi_3 = \Phi_1 \bullet (\Phi_2 \bullet \Phi_3)$ (cf. Definition 2.4).
2.3 Powers of ANNs

**Definition 2.7.** Let $n \in \mathbb{N}$. Then we denote by $I_n \in \mathbb{R}^{n \times n}$ the identity matrix in $\mathbb{R}^{n \times n}$.

**Definition 2.8.** We denote by $(\cdot)^n$: \{\(\Phi \in \mathbb{N}: \mathcal{I}(\Phi) = \mathcal{O}(\Phi)\} \rightarrow \mathbb{N}, n \in \mathbb{N}_0, \Phi \in \mathbb{N}$ with $\mathcal{I}(\Phi) = \mathcal{O}(\Phi)$ that

\[
\Phi^n = \begin{cases} 
(I_{\mathcal{O}(\Phi)} - (0, 0, \ldots, 0)) \in \mathbb{R}^{\mathcal{O}(\Phi) \times \mathcal{O}(\Phi)} : n = 0 \\
\Phi \circ (\Phi^{n-1}) & : n \in \mathbb{N} 
\end{cases}
\]  

(\text{cf. Definitions 2.2, 2.4, and 2.7}).

**Lemma 2.9.** Let $d, i \in \mathbb{N}$, $\Psi \in \mathbb{N}$ satisfy $\mathcal{D}(\Psi) = (d, i, d)$ (cf. Definition 2.2). Then it holds for all $n \in \mathbb{N}_0$ that $\mathcal{H}(\Psi^n) = n$, $\mathcal{D}(\Psi^n) \in \mathbb{N}^{n+2}$, and

\[
\mathcal{D}(\Psi^n) = \begin{cases} 
(d, d) & : n = 0 \\
(d, i, i, \ldots, i, d) & : n \in \mathbb{N} 
\end{cases}
\]  

(\text{cf. Definition 2.8}).

2.4 Parallelizations of ANNs

**Definition 2.10** (Parallelization of ANNs with the same length). Let $n \in \mathbb{N}$. Then we denote by

\[
P_n: \{\Phi_1, \Phi_2, \ldots, \Phi_n\} \in \mathbb{N}^n: \mathcal{L}(\Phi_1) = \mathcal{L}(\Phi_2) = \ldots = \mathcal{L}(\Phi_n) \rightarrow \mathbb{N}
\]  

the function which satisfies for all $L \in \mathbb{N}, (l_1, l_2, \ldots, l_k, l_{k-1}, \ldots, l_1) \in \mathbb{N}^{L+1}$, $\Phi_1 = (W_{1,1}, B_{1,1}), (W_{1,2}, B_{1,2}), \ldots, (W_{1,L}, B_{1,L}) \in (\times_{k=1}^L (\mathbb{R}^{l_{k-1} \times l_k} \times \mathbb{R}^{l_k}))$, $\Phi_2 = (W_{2,1}, B_{2,1}), (W_{2,2}, B_{2,2}), \ldots, (W_{2,L}, B_{2,L}) \in (\times_{k=1}^L (\mathbb{R}^{l_{k-1} \times l_k} \times \mathbb{R}^{l_k}))$, \ldots, $\Phi_n = (W_{n,1}, B_{n,1}), (W_{n,2}, B_{n,2}), \ldots, (W_{n,L}, B_{n,L}) \in (\times_{k=1}^L (\mathbb{R}^{l_{k-1} \times l_k} \times \mathbb{R}^{l_k}))$ that

\[
P_n(\Phi_1, \Phi_2, \ldots, \Phi_n) = 
\begin{pmatrix}
W_{1,1} & 0 & 0 & \cdots & 0 \\
0 & W_{2,1} & 0 & \cdots & 0 \\
0 & 0 & W_{3,1} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & W_{n,1} \\
B_{1,1} & B_{1,2} & B_{1,3} & \cdots \\
B_{2,1} & B_{2,2} & B_{2,3} & \cdots \\
B_{3,1} & B_{3,2} & B_{3,3} & \cdots \\
B_{n,1} & B_{n,2} & B_{n,3} & \cdots \\
\end{pmatrix}
\]  

(\text{cf. Definition 2.2}).

**Proposition 2.11.** Let $n \in \mathbb{N}$, $\Phi = (\Phi_1, \Phi_2, \ldots, \Phi_n) \in \mathbb{N}^n$ satisfy $\mathcal{L}(\Phi_1) = \mathcal{L}(\Phi_2) = \ldots = \mathcal{L}(\Phi_n)$ (cf. Definition 2.2). Then
(i) it holds that $\mathcal{R}(P_n(\Phi)) \in C(\mathbb{R}[\sum_{j=1}^n I(\Phi_j)],[\sum_{j=1}^n O(\Phi_j)])$ and

(ii) it holds for all $x_1 \in \mathbb{R}[I(\Phi_1)], x_2 \in \mathbb{R}[I(\Phi_2)], \ldots, x_n \in \mathbb{R}[I(\Phi_n)]$ that

$$
\left(\mathcal{R}(P_n(\Phi))(x_1, x_2, \ldots, x_n) = ((\mathcal{R}(\Phi_1))(x_1), (\mathcal{R}(\Phi_2))(x_2), \ldots, (\mathcal{R}(\Phi_n))(x_n)) \in \mathbb{R}[\sum_{j=1}^n O(\Phi_j)]
\right) 
$$

(2.9)

(cf. Definition 2.10).

Proposition 2.12. Let $n \in \mathbb{N}, \Phi_1, \Phi_2, \ldots, \Phi_n \in \mathbb{N}$ satisfy $L(\Phi_1) = L(\Phi_2) = \ldots = L(\Phi_n)$ (cf. Definition 2.2). Then

$$
\mathcal{D}(P_n(\Phi_1, \Phi_2, \ldots, \Phi_n)) = \left(\sum_{j=1}^n D_0(\Phi_j), \sum_{j=1}^n D_1(\Phi_j), \ldots, \sum_{j=1}^n D_L(\Phi_j)\right)
$$

(2.10)

(cf. Definition 2.10).

Definition 2.13. We denote by $\mathcal{I} = (\mathcal{I}_d)_{d \in \mathbb{N}}: \mathbb{N} \rightarrow \mathbb{N}$ the function which satisfies for all $d \in \mathbb{N}$ that

$$
\mathcal{I}_1 = \left(\left(\begin{array}{c} 1 \\ -1 \end{array}\right), \left(\begin{array}{c} 0 \\ 0 \end{array}\right)\right), \left(\begin{array}{c} 1 \\ -1 \end{array}\right), 0) \in (\mathbb{R}^{2 \times 1} \times \mathbb{R}^2) \times (\mathbb{R}^{1 \times 2} \times \mathbb{R}^1)
$$

(2.11)

and

$$
\mathcal{I}_d = P_d(\mathcal{I}_1, \mathcal{I}_1, \ldots, \mathcal{I}_1)
$$

(2.12)

(cf. Definitions 2.2 and 2.10).

Lemma 2.14. Let $d \in \mathbb{N}$. Then

(i) it holds that $\mathcal{D}(\mathcal{I}_d) = (d, 2d, d) \in \mathbb{N}^3$,

(ii) it holds that $\mathcal{R}(\mathcal{I}_d) \in C(\mathbb{R}^d, \mathbb{R}^d)$, and

(iii) it holds for all $x \in \mathbb{R}^d$ that $(\mathcal{R}(\mathcal{I}_d))(x) = x$

(cf. Definitions 2.2 and 2.13).

2.5 Linear transformations as ANNs

Definition 2.15 (Affine linear transformation NN). Let $m, n \in \mathbb{N}, W \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^m$. Then we denote by $A_{W,B} \in (\mathbb{R}^{m \times n} \times \mathbb{R}^m) \subseteq \mathbb{N}$ the neural network given by $A_{W,B} = (W, B)$ (cf. Definitions 2.2 and 2.3).

Lemma 2.16. Let $m, n \in \mathbb{N}, W \in \mathbb{R}^{m \times n}, B \in \mathbb{R}^m$. Then

(i) it holds that $\mathcal{D}(A_{W,B}) = (n, m) \in \mathbb{N}^2$,

(ii) it holds that $\mathcal{R}(A_{W,B}) \in C(\mathbb{R}^n, \mathbb{R}^m)$, and

(iii) it holds for all $x \in \mathbb{R}^n$ that $(\mathcal{R}(A_{W,B}))(x) = Wx + B$

(cf. Definitions 2.2 and 2.15).
2.6 Scalar multiplications of ANNs

**Definition 2.17** (Scalar multiplications of ANNs). We denote by \((\cdot) \oplus (\cdot): \mathbb{R} \times \Phi \times \mathbb{N} \rightarrow \Phi\) the function which satisfies for all \(\lambda \in \mathbb{R}\), \(\Phi \in \mathbb{N}\) that \(\lambda \oplus \Phi = A_{\lambda \Phi} \oplus \Phi\) (cf. Definitions 2.2, 2.4, 2.7, and 2.15).

**Lemma 2.18.** Let \(\lambda \in \mathbb{R}, \Phi \in \mathbb{N}\) (cf. Definition 2.2). Then

(i) it holds that \(D(\lambda \oplus \Phi) = D(\Phi)\),

(ii) it holds that \(R(\lambda \oplus \Phi) \in C(\mathbb{R}^{2(\Phi)}, \mathbb{R}^{2(\Phi)})\), and

(iii) it holds for all \(x \in \mathbb{R}^{2(\Phi)}\) that \(R(\lambda \oplus \Phi)(x) = \lambda(R(\Phi)(x))\) (cf. Definition 2.17).

2.7 Sums of ANNs

**Definition 2.19.** Let \(m, n \in \mathbb{N}\). Then we denote by \(S_{m,n} \in (\mathbb{R}^{m \times (m \times m)} \times \mathbb{R}^{m})\) the neural network given by \(S_{m,n} = A_{(1_{m}, 1_{m}, \ldots, 1_{m}), 0}\) (cf. Definitions 2.3, 2.7, and 2.15).

**Lemma 2.20.** Let \(m, n \in \mathbb{N}\). Then

(i) it holds that \(D(S_{m,n}) = (mn, m) \in \mathbb{N}^{2}\),

(ii) it holds that \(R(S_{m,n}) \in C(\mathbb{R}^{mn}, \mathbb{R}^{m})\), and

(iii) it holds for all \(x_1, x_2, \ldots, x_n \in \mathbb{R}^m\) that \(R(S_{m,n})(x_1, x_2, \ldots, x_n) = \sum_{k=1}^{n} x_k\) (cf. Definitions 2.2 and 2.19).

2.8 On the connection to the vectorized description of ANNs

**Definition 2.21** (p-norm). We denote by \(\|\cdot\|_p: (\bigcup_{d=1}^{\infty} \mathbb{R}^d) \rightarrow [0, \infty), \ p \in [1, \infty]\), the functions which satisfy for all \(p \in [1, \infty), d \in \mathbb{N}, \theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d\) that \(\|\theta\|_p = (\sum_{i=1}^{d} |\theta_i|^p)^{1/p}\) and \(\|\theta\|_\infty = \max_{i \in \{1, 2, \ldots, d\}} |\theta_i|\).

**Definition 2.22.** We denote by \(T: \mathbb{N} \rightarrow (\bigcup_{d \in \mathbb{N}} \mathbb{R}^d)\) the function which satisfies for all \(L, d \in \mathbb{N}, l_0, l_1, \ldots, l_L \in \mathbb{N}, \Phi = ((W_1, B_1),(W_2, B_2),\ldots,(W_L, B_L)) = (x_{m=1}^{L} (\mathbb{R}^{l_m \times l_{m-1}} \times \mathbb{R}^{l_m})), \theta = (\theta_1, \theta_2, \ldots, \theta_d) \in \mathbb{R}^d, k \in \{1, 2, \ldots, L\}\) with \(T(\Phi) = \theta\) that

\[
d = P(\Phi), \quad B_k = \begin{pmatrix}
\frac{\theta_1}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + l_{k-1} + 1}
\frac{\theta_2}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + l_{k-1} + 2}
\vdots
\frac{\theta_k}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + l_{k-1} + k}
\end{pmatrix},
\]

\[
W_k = \begin{pmatrix}
\begin{pmatrix}
\frac{\theta_1}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + 1}
\frac{\theta_2}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + 2}
\vdots
\frac{\theta_k}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + k}
\end{pmatrix}
\end{pmatrix}
\cdots
\begin{pmatrix}
\begin{pmatrix}
\frac{\theta_1}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + 1}
\frac{\theta_2}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + 2}
\vdots
\frac{\theta_k}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + k}
\end{pmatrix}
\end{pmatrix}
\cdots
\begin{pmatrix}
\begin{pmatrix}
\frac{\theta_1}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + 1}
\frac{\theta_2}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + 2}
\vdots
\frac{\theta_k}{(\sum_{i=1}^{k-1} l_i(l_i-1)) + k}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\end{pmatrix}
\]
Lemma 2.23. Let $L, \mathcal{E} \in \mathbb{N}, l_0, l_1, \ldots, l_L, l_0, l_1, \ldots, l_\mathcal{E} \in \mathbb{N}$, $\Phi_1 = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$, $\Phi_2 = ((\mathcal{W}_1, \mathcal{B}_1), (\mathcal{W}_2, \mathcal{B}_2), \ldots, (\mathcal{W}_\mathcal{E}, \mathcal{B}_\mathcal{E})) \in (\times_{k=1}^L (\mathbb{R}^{l_k \times l_{k-1}} \times \mathbb{R}^{l_k}))$. Then

$$
\|T(\Phi_1 \bullet \Phi_2)\|_\infty \leq \max\{\|T(\Phi_1)\|_\infty, \|T(\Phi_2)\|_\infty, \|T(((W_1 \mathcal{W}_1, W_1 \mathcal{B}_1 + B_1)))\|_\infty\} \tag{2.14}
$$
(cf. Definitions 2.4, 2.21, and 2.22).

Proof of Lemma 2.23. Observe that (2.3) and (2.13) establish (2.14). The proof of Lemma 2.23 is thus complete.

3 Upper bounds for weighted Gaussian tails

In this section we establish in Lemma 3.12 in Subsection 3.3 below suitable upper bounds for certain weighted tails of standard normal distributions. Our proof of Lemma 3.12 employs the Gaussian segment type estimate in Lemma 3.11 in Subsection 3.2 below, the elementary integration formula for certain radial symmetric functions in Lemma 3.10 in Subsection 3.2, and the Gaussian tail estimate in Corollary 3.9 in Subsection 3.2.

Lemma 3.10 is a direct consequence of the integral transformation theorem and only for completeness we include in Subsection 3.2 the detailed proof for Lemma 3.10. Our proof of Lemma 3.11 uses Lemma 3.10 and the elementary estimates for the Gamma function which we present in Corollary 3.9 in the scientific literature. Our proof of Lemma 3.12 employs well-known functional equations for the Gamma and the Beta function which we briefly recall in Lemma 3.1. Lemma 3.1 is, e.g., proved in Robbins [50]. The equality in (3.34) in the proof of Lemma 3.3 is also referred to as Wallis’s formula in the scientific literature. Our proof of Lemma 3.12 employs well-known functional equations for the Gamma and the Beta function which we briefly recall in Lemma 3.1 in Subsection 3.1. Lemma 3.1 is, e.g., proved in Egan [14]. Our proof of Corollary 3.9 uses the well-known Bernoulli inequality which we recall in Lemma 3.8 in Subsection 3.2 and the elementary Gaussian tail estimates in Lemma 3.6 and Corollary 3.7 in Subsection 3.2. Only for completeness we include in this section also the detailed proofs for Lemma 3.1, Corollary 3.2, Lemma 3.3, and Lemma 3.8.

3.1 Lower and upper bounds for evaluations of the Gamma function

Lemma 3.1. Let $\Gamma: (0, \infty) \to (0, \infty)$ and $\mathbb{B}: (0, \infty)^2 \to (0, \infty)$ satisfy for all $x, y \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt$ and $\mathbb{B}(x, y) = \int_0^1 t^{x-1}(1-t)^{y-1} \, dt$. Then

(i) it holds for all $x \in (0, \infty)$ that $\Gamma(x+1) = x \Gamma(x),$

(ii) it holds that $\Gamma(1/2) = \sqrt{\pi}$, and

(iii) it holds for all $x, y \in (0, \infty)$ that $\mathbb{B}(x, y) = \mathbb{B}(y, x) = \frac{\Gamma(x)\Gamma(y)}{\Gamma(x+y)}$.

Proof of Lemma 3.1. Throughout this proof let $\Phi: (0, \infty) \times (0, 1) \to (0, \infty)^2$ satisfy for all $u \in (0, \infty), v \in (0, 1)$ that $\Phi(u, v) = (u(1-v), uv)$ and let $f_{x,y}: (0, \infty)^2 \to (0, \infty)$, $x, y \in (0, \infty)$, satisfy for all $x, y, s, t \in (0, \infty)$ that $f_{x,y}(s, t) = s^{(x-1)}t^{(y-1)}e^{-(s+t)}$. Note that the integration by parts formula assures that for all $x \in (0, \infty)$ it holds that

\begin{align*}
\Gamma(x+1) & = \int_0^\infty t^{(x+1)-1}e^{-t} \, dt = -\int_0^\infty t^x[-e^{-t}] \, dt \\
& = -\left([t^x e^{-t}]_{t=0}^{t=\infty} - x\left[\int_0^\infty t^{(x-1)}e^{-t} \, dt\right]\right) = x\left[\int_0^\infty t^{(x-1)}e^{-t} \, dt\right] = x\Gamma(x). \tag{3.1}
\end{align*}
This establishes item (i). Next observe that the integral transformation theorem shows that
\[
\Gamma \left( \frac{1}{2} \right) = \int_0^\infty t^{-1/2}e^{-t} \, dt = \int_0^\infty t^{-1}e^{-t} \, 2t \, dt = 2 \left[ \int_0^\infty e^{-t^2} \, dt \right] = 2 \sqrt{\frac{\pi}{2}} \int_0^\infty e^{-t^2} \, dt = \sqrt{\pi}. \tag{3.2}
\]
This establishes item (ii). Moreover, note that the integral transformation theorem ensures that for all \(x, y \in (0, \infty)\) it holds that
\[
\mathbb{B}(x, y) = \int_1^\infty t(x-1) (1-t)(y-1) \, dt = \int_1^\infty \left[ \frac{1}{2} \right]^{(x-1)} \left[ 1 - \frac{1}{2} \right]^{(y-1)} \frac{1}{t} \, dt
\]
\[
= \int_1^\infty t^{(-x-1)} \left[ \frac{1}{t} \right]^{(y-1)} \, dt = \int_1^\infty t^{(-x-y)} (t-1)^{(y-1)} \, dt \tag{3.3}
\]
\[
= \int_0^\infty (t+1)^{(-x-y)} t^{(y-1)} \, dt = \int_0^\infty \frac{t^{(y-1)}}{(t+1)^{(x+y)}} \, dt.
\]
In addition, observe that Fubini's theorem shows that for all \(x, y \in (0, \infty)\) it holds that
\[
\Gamma(x)\Gamma(y) = \left[ \int_0^\infty t^{(x-1)} e^{-t} \, dt \right] \left[ \int_0^\infty t^{(y-1)} e^{-t} \, dt \right] = \left[ \int_0^\infty s^{(x-1)} e^{-s} \, ds \right] \left[ \int_0^\infty t^{(y-1)} e^{-t} \, dt \right] \tag{3.4}
\]
\[
= \int_0^\infty \int_0^\infty s^{(x-1)} t^{(y-1)} e^{-(s+t)} \, dt \, ds \] \[= \int_{(0,\infty)^2} f_{x,y}(s,t) \, d(s,t).
\]
Furthermore, note that for all \(u \in (0, \infty), v \in (0, 1)\) it holds that
\[
\Phi'(u, v) = \begin{pmatrix} 1 - v & -u \\ v & u \end{pmatrix}. \tag{3.5}
\]
Hence, we obtain that for all \(u \in (0, \infty), v \in (0, 1)\) it holds that
\[
\det(\Phi'(u, v)) = (1 - v)u - v(-u) = u - vu + vu = u \in (0, \infty). \tag{3.6}
\]
Combining this with (3.4) and the integral transformation theorem shows that for all \(x, y \in (0, \infty)\) it holds that
\[
\Gamma(x)\Gamma(y) = \int_{(0,\infty)^2} f_{x,y}(\Phi(u, v)) \, |\det(\Phi'(u, v))| \, d(u, v)
\]
\[
= \int_0^\infty \int_0^1 (u(1 - v))^{(x-1)} (uv)^{(y-1)} e^{-(u(1-v)+uv)} \, u \, dv \, du
\]
\[
= \int_0^\infty \int_0^1 u^{(x+y-1)} e^{-u} v^{(y-1)} (1 - v)^{(x-1)} \, dv \, du \tag{3.7}
\]
\[
= \left[ \int_0^\infty u^{(x+y-1)} e^{-u} \, du \right] \left[ \int_0^1 v^{(y-1)} (1 - v)^{(x-1)} \, dv \right]
\]
\[
= \left[ \int_0^\infty u^{(x+y-1)} e^{-u} \, du \right] \left[ \int_0^1 (1 - v)^{(y-1)} v^{(x-1)} \, dv \right]
\]
\[
= \Gamma(x+y) \mathbb{B}(x, y).
\]
This establishes item (iii). The proof of Lemma 3.1 is thus complete. \(\square\)

**Corollary 3.2.** Let \(\Gamma: (0, \infty) \to (0, \infty)\) satisfy for all \(x \in (0, \infty)\) that \(\Gamma(x) = \int_0^x t^{-1}e^{-t} \, dt\). Then
(i) it holds that \( \Gamma(1) = 1 \) and
(ii) it holds for all \( d \in \mathbb{N} \) that
\[
\Gamma\left(\frac{d}{2}\right) = \begin{cases} 
\left(\frac{d}{2} - 1\right)! & : \frac{d}{2} \in \mathbb{N} \\
(\frac{(d-1)\sqrt{\pi}}{(\frac{d}{2})^{\frac{d}{2}}})^{\frac{d}{2}} & : \frac{d}{2} \notin \mathbb{N}.
\end{cases}
\] (3.8)

**Proof of Corollary 3.2.** Observe that the assumption that for all \( x \in (0, \infty) \) it holds that \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt \) ensures that
\[
\Gamma(1) = \int_0^\infty e^{-t} \, dt = \int_0^\infty e^{-t} \, dt = [-e^{-t}]_{t=\infty}^{t=0} = 1.
\] (3.9)

This establishes item (i). Next note that Lemma 3.1, induction, and item (i) assure that for all \( I, m \in \mathbb{N} \) with \( I = 2m \) it holds that
\[
\Gamma\left(\frac{I}{2}\right) = \Gamma(m) = (m-1)! = \left(\frac{I}{2} - 1\right)!. \tag{3.10}
\]

Moreover, observe that Lemma 3.1 and induction show that for all \( I, m \in \mathbb{N} \) with \( I = 2m - 1 \) it holds that
\[
\Gamma\left(\frac{I}{2}\right) = \Gamma\left(m - \frac{1}{2}\right) = \Gamma\left(m - \frac{3}{2}\right) \Gamma\left(m - \frac{5}{2}\right) \cdots \frac{1}{2} \Gamma\left(\frac{1}{2}\right) = \frac{(2m-3)!\sqrt{\pi}}{2^{m-1}} = \frac{(2m-3)!(2m-2)!\sqrt{\pi}}{2^{m-1}(2m-2)!} = \frac{(2m-2)!\sqrt{\pi}}{4^{m-1}(m-1)!} = \frac{(1-1)!\sqrt{\pi}}{2^{1-1}\left(\frac{1}{2}\right)!}.
\] (3.11)

Combining this with (3.10) establishes item (ii). The proof of Corollary 3.2 is thus complete. \( \square \)

**Lemma 3.3.** Let \( n \in \mathbb{N} \). Then
\[
\sqrt{2\pi n}\left[\frac{n}{e}\right]^n e^{-\frac{1}{2}} < n! < \sqrt{2\pi n}\left[\frac{n}{e}\right]^n e^{-\frac{1}{2}}.
\] (3.12)

**Proof of Lemma 3.3.** Throughout this proof let \( a = (a_n)_{n \in \mathbb{N}}: \mathbb{N} \to \mathbb{R} \), \( b = (b_n)_{n \in \mathbb{N}}: \mathbb{N} \to \mathbb{R} \), \( c = (c_n)_{n \in \mathbb{N}}: \mathbb{N} \to \mathbb{R} \), \( r = (r_n)_{n \in \mathbb{N}}: \mathbb{N} \to [0, \infty] \), and \( S = (S_n)_{n \in \mathbb{N}}: \mathbb{N} \to \mathbb{R} \) satisfy for all \( n \in \mathbb{N} \) that
\[
a_n = \int_n^{n+1} \ln(x) \, dx, \quad b_n = \frac{1}{2}[\ln(n+1) - \ln(n)], \quad c_n = a_n - \frac{1}{2}[\ln(n+1) + \ln(n)],
\] (3.13)

\[r_n = \sum_{k=1}^{\infty} c_k, \quad S_n = \ln(n!), \quad C \in [0, \infty] \] satisfy \( C = r_1 = \sum_{k=1}^{\infty} |c_k| \), and let \( I: \mathbb{N}_0 \to \mathbb{R} \) satisfy for all \( n \in \mathbb{N}_0 \) that \( I(n) = \int_0^n [\sin(x)]^n \, dx \). Note that (3.13) ensures that for all \( n \in \mathbb{N} \) it holds that \( \ln(n+1) = a_n + b_n - c_n \) and \( S_n = \ln(n!) = \sum_{k=1}^{n-1} \ln(k+1) \). The fact that for all \( x \in (0, \infty) \) it holds that \( \lbrack x \ln(x) - x\rbrack' = \ln(x) \) therefore shows that for all \( n \in \mathbb{N} \) it holds that
\[
S_n = \sum_{k=1}^{n-1} (a_k + b_k - c_k) = \int_n^{n+1} \ln(x) \, dx + \frac{1}{2} \ln(n) - \sum_{k=1}^{n-1} c_k = [x \ln(x) - x|_x^{x+1} + \frac{1}{2} \ln(n) - \sum_{k=1}^{n-1} c_k = [n + \frac{1}{2}] \ln(n) - n + 1 - \sum_{k=1}^{n-1} c_k.
\] (3.14)

Next observe that the fact that for all \( x \in (0, \infty) \) it holds that \( [x \ln(x) - x]' = \ln(x) \) assures that for all \( n \in \mathbb{N} \) it holds that
\[
c_n = (\int_n^{n+1} \ln(x) \, dx - \frac{1}{2}[\ln(n+1) + \ln(n)] = [x \ln(x) - x|_x^{x+1} - \frac{1}{2} \ln(n+1) + \ln(n) = [n + \frac{1}{2}] \ln(n+1) - \ln(n)] - 1 = [n + \frac{1}{2}] \ln(1 + \frac{1}{n}) - 1.
\] (3.15)
Moreover, note that the fact that for all \(x \in (-1, 1)\) it holds that \(\ln(1-x) = -\left[\sum_{n=1}^{\infty} \frac{x^n}{n}\right]\) shows that for all \(x \in (-1, 1)\) it holds that

\[
\ln\left(\frac{1+x}{1-x}\right) = \ln(1+x) - \ln(1-x) = \left[-\sum_{n=1}^{\infty} \frac{(-x)^n}{n}\right] - \left[-\sum_{n=1}^{\infty} \frac{x^n}{n}\right] = 2\sum_{n=1}^{\infty} \frac{x^{2n-1}}{2n-1}.
\] (3.16)

This implies that for all \(n \in \mathbb{N}\) it holds that

\[
\left[n + \frac{1}{2}\right]\ln(1 + \frac{1}{n}) = \left[n + \frac{1}{2}\right] \ln\left(\frac{1+(2n+1)^{-1}}{1-(2n+1)^{-1}}\right) = (2n+1)\left[\sum_{p=1}^{\infty} \frac{(2n+1)^{1-2p}}{2p-1}\right] = \sum_{p=1}^{\infty} \frac{(2n+1)^{2-2p}}{2p-1} = 1 + \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{7(2n+1)^6} + \ldots.
\] (3.17)

Combining this with (3.15) ensures that for all \(n \in \mathbb{N}\) it holds that

\[
0 < c_n = \sum_{p=1}^{\infty} \frac{(2n+1)^{2-2p}}{2p-1} = \sum_{p=1}^{\infty} \frac{(2n+1)^{-2p}}{2p+1} = \frac{1}{3(2n+1)^2} + \frac{1}{5(2n+1)^4} + \frac{1}{7(2n+1)^6} + \ldots.
\] (3.18)

The fact that for all \(x \in (-1, 1)\) it holds that \(\sum_{n=0}^{\infty} x^n = (1-x)^{-1}\) therefore assures that for all \(n \in \mathbb{N}\) it holds that

\[
0 < c_n = \sum_{p=1}^{\infty} \frac{(2n+1)^{1-2p}}{2p+1} < \frac{1}{3} \left[\sum_{p=1}^{\infty} (2n+1)^{-2p}\right] = \frac{1}{3(2n+1)^2} \left[\sum_{p=0}^{\infty} [(2n+1)^{-2}]^p\right] = \frac{1}{3(2n+1)^2} \left(1 - \frac{(2n+1)^{-2}}{1 - \frac{1}{3(2n+1)^2}}\right) = \frac{1}{12} \sum_{n=1}^{\infty} \frac{1}{(n+1)^2} = \frac{1}{12} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{n+1}.
\] (3.19)

This implies that for all \(n \in \mathbb{N}\) it holds that

\[
r_n = \sum_{k=n}^{\infty} |c_k| = \sum_{k=n}^{\infty} c_k < \frac{1}{3} \left(\sum_{k=n}^{\infty} \frac{1}{k^2} - \frac{1}{k+1}\right) = \frac{1}{3n^2}.
\] (3.20)

Next observe that the fact that for all \(x \in (-1, 1)\) it holds that \(\sum_{n=0}^{\infty} x^n = (1-x)^{-1}\), the fact that for all \(x \in (1, \infty)\) it holds that \(3^x > 2x + 1\), and (3.18) show that for all \(n \in \mathbb{N}\) it holds that

\[
c_n = \sum_{p=1}^{\infty} \frac{(2n+1)^{1-2p}}{2p+1} > \sum_{p=1}^{\infty} \frac{(2n+1)^{-2p}}{2p+1} = \frac{1}{3(2n+1)^2} \left[\sum_{p=0}^{\infty} [(2n+1)^{-2}]^p\right] = \frac{1}{3(2n+1)^2} \left(1 - \frac{(2n+1)^{-2}}{1 - \frac{1}{3(2n+1)^2}}\right) \geq \frac{1}{12} \sum_{n=1}^{\infty} \frac{1}{n^2} - \frac{1}{n+1}.
\] (3.21)

This ensures that for all \(n \in \mathbb{N}\) it holds that

\[
r_n = \sum_{k=n}^{\infty} |c_k| = \sum_{k=n}^{\infty} c_k > \frac{1}{12} \left(\sum_{k=n}^{\infty} \frac{1}{k^2} - \frac{1}{k+1}\right) = \frac{1}{12n^2}.
\] (3.22)

In addition, note that the fact that for all \(n \in \mathbb{N}\) it holds that \(c_n > 0\) and (3.20) ensure that for all \(n \in \mathbb{N}\) it holds that \(0 < r_n \leq r_1 = C < \infty\). Combining this with (3.14) shows that for all \(n \in \mathbb{N}\) it holds that

\[
\ln(n!) = S_n = \left[n + \frac{1}{2}\right]\ln(n) - n + 1 - \sum_{k=1}^{n-1} c_k = \left[n + \frac{1}{2}\right]\ln(n) - n + 1 - C + r_n.
\] (3.23)

Therefore, we obtain that for all \(n \in \mathbb{N}\) it holds that

\[
n! = \sqrt{n} \left[\frac{n}{e}\right]^n e^{r_n} e^{1-C}.
\] (3.24)

Combining this with (3.20) and (3.22) ensures that for all \(n \in \mathbb{N}\) it holds that

\[
\sqrt{n} \left[\frac{n}{e}\right]^n e^{1/2n+1-C} < n! < \sqrt{n} \left[\frac{n}{e}\right]^n e^{1/2n e^{1-C}}.
\] (3.25)
This implies that for all \( n \in \mathbb{N} \) it holds that
\[
e^{\frac{1}{2n+1}} < n! \left\lceil \frac{e}{n} \right\rceil n^{-1/2} e^{C-1} < e^{\frac{1}{2n}}.
\] (3.26)

Hence, we obtain that
\[
\lim_{n \to \infty} \left[ n! \left\lceil \frac{e}{n} \right\rceil n^{-1/2} e^{C-1} \right] = e^0 = 1.
\] (3.27)

Moreover, observe that the integration by parts formula and the chain rule ensure that for all \( n \in \mathbb{N} \cap [2, \infty) \) it holds that
\[
I(n) = \int_0^\pi [\sin(x)]^n dx = \int_0^\pi [\sin(x)]^{n-1} [\sin(x)] dx = -\int_0^\pi [\sin(x)]^{n-1} \frac{d}{dx} \cos(x)\] \(dx = -(n-1) \int_0^\pi [\sin(x)]^{n-2} [\cos(x)]^2 dx = (n-1) \int_0^\pi [\sin(x)]^{n-2} [1 - (\sin(x))^2] dx = (n-1)[I(n-2) - I(n)].
\] (3.28)

This implies that for all \( n \in \mathbb{N} \cap [2, \infty) \) it holds that
\[
I(n) = \left\lfloor \frac{n-1}{n} \right\rfloor I(n-2).
\] (3.29)

Combining this with the fact that \( I(0) = \int_0^\pi dx = \pi \) assures that for all \( n \in \mathbb{N} \) it holds that
\[
I(2n) = \left\lfloor \frac{2n-1}{2n} \right\rfloor I(2n-2) = \ldots = \left\lfloor \frac{2n-1}{2n} \right\rfloor \left\lfloor \frac{2n-3}{2n-2} \right\rfloor \ldots \left\lfloor \frac{1}{2} \right\rfloor I(0) = \pi \prod_{k=1}^{n} \left\lfloor \frac{2k-1}{2k} \right\rfloor .
\] (3.30)

Next note that the fact that \( I(1) = \int_0^\pi \sin(x) dx = [-\cos(x)]_x=0^\pi = 2 \) and (3.29) demonstrate that for all \( n \in \mathbb{N} \) it holds that
\[
I(2n+1) = \left\lfloor \frac{2n+1}{2n+2} \right\rfloor I(2n-1) = \ldots = \left\lfloor \frac{2n+1}{2n+2} \right\rfloor \left\lfloor \frac{2n-1}{2n+1} \right\rfloor \ldots \left\lfloor \frac{1}{2} \right\rfloor I(1) = 2 \prod_{k=1}^{n} \left\lfloor \frac{2k}{2k+1} \right\rfloor .
\] (3.31)

Furthermore, observe that the fact that for all \( n \in \mathbb{N}, x \in (0, \pi) \) it holds that \( 0 < [\sin(x)]^{2n+1} \leq [\sin(x)]^{2n} \leq [\sin(x)]^{2n-1} \) ensures that for all \( n \in \mathbb{N} \) it holds that \( 0 < I(2n+1) \leq I(n) \leq I(2n) \). This and (3.29) imply that for all \( n \in \mathbb{N} \) it holds that
\[
1 \leq \frac{I(2n)}{I(2n+1)} \leq \frac{I(2n-1)}{I(2n+1)} = \frac{2n+1}{2n} = 1 + \frac{1}{2n}.
\] (3.32)

Combining this with (3.30) and (3.31) shows that
\[
1 = \lim_{n \to \infty} \left[ \frac{I(2n)}{I(2n+1)} \right] = \lim_{n \to \infty} \left[ \frac{1}{2} \prod_{k=1}^{n} \left( \frac{2k-1}{2k+1} \right) \right] = \frac{1}{2} \lim_{n \to \infty} \left[ \frac{[(2n-1)!!]^2 (2n+1)!}{[(2n-1)!!]^2 2n!} \right].
\] (3.33)

This demonstrates that
\[
\lim_{n \to \infty} \left[ \frac{(2n)!!}{(2n-1)!! \sqrt{2n}} \right] = \lim_{n \to \infty} \left[ \frac{(2n)!!}{(2n-1)!! \sqrt{2n+1}} \right] = \left[ \frac{\pi}{2} \right]^{1/2}.
\] (3.34)

Combining this with (3.27) establishes that
\[
\left[ \frac{\pi}{2} \right]^{1/2} = \lim_{n \to \infty} \left[ \frac{(2n)!!}{(2n-1)!! \sqrt{2n}} \right] = \lim_{n \to \infty} \left[ \frac{(2n)!!}{(2n)! \sqrt{2n} \sqrt{2n+1}} \right] = \lim_{n \to \infty} \left[ \frac{(2n+1)!!}{(2n)! \sqrt{2n+2} \sqrt{2n+3}} \right].
\] (3.35)
Hence, we obtain that \( e^{1-C} = \sqrt{2\pi} \). This and (3.25) show that for all \( n \in \mathbb{N} \) it holds that

\[
\sqrt{2\pi n} \left[ \frac{n}{e} \right]^n e^{\frac{1}{12+\epsilon}} < n! < \sqrt{2\pi n} \left[ \frac{n}{e} \right]^n e^{\frac{1}{12\pi}}. \tag{3.36}
\]

The proof of Lemma 3.3 is thus complete. \( \square \)

**Corollary 3.4.** Let \( m \in \mathbb{N} \cap [2, \infty) \). Then

(i) it holds that

\[
\sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} \leq (m-1)! \leq \sqrt{3\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} \tag{3.37}
\]

and

(ii) it holds that

\[
\sqrt{\pi} \left[ \frac{m-1}{e} \right]^{m-1} \leq \frac{(2m-2)!\sqrt{\pi}}{4^{m-1}(m-1)!} \leq \sqrt{2\pi} \left[ \frac{m-1}{e} \right]^{m-1}. \tag{3.38}
\]

**Proof of Corollary 3.4.** Note that Lemma 3.3 (applied with \( n \) \( \cap \) \( m-1 \) in the notation of Lemma 3.3) implies that

\[
\sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{12m-17}} \leq (m-1)! \leq \sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{12m-12}}. \tag{3.39}
\]

The fact that \( e \leq \left( \frac{3}{2} \right)^{6(m-6)} \) therefore assures that

\[
\sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} \leq \sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{12m-11}} \leq (m-1)!
\]

\[
\leq \sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{12m-12}} \leq \sqrt{3\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1}. \tag{3.40}
\]

This establishes item (i). Moreover, observe that Lemma 3.3 (applied with \( n \) \( \cap \) \( 2m-2 \) in the notation of Lemma 3.3) ensures that

\[
\sqrt{2\pi (2m-2)} \left[ \frac{2m-2}{e} \right]^{2m-2} e^{\frac{1}{24m-24}} \leq (2m-2)! \leq \sqrt{2\pi (2m-2)} \left[ \frac{2m-2}{e} \right]^{2m-2} e^{\frac{1}{24m-24}}. \tag{3.41}
\]

Combining this with (3.39) demonstrates that

\[
\sqrt{2\pi} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{24m-24}(12m-12)} \leq \frac{(2m-2)!\sqrt{\pi}}{4^{m-1}(m-1)!} \leq \sqrt{2\pi} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{24m-24}(12m-11)}. \tag{3.42}
\]

The fact that \( e^{(11-12m)} \geq 2^{(24m-23)}(6-6m) \) and the fact that for all \( x \in [1, \infty) \) it holds that \( x^{(-12m+13)} \leq 1 \) hence ensure that

\[
\sqrt{\pi} \left[ \frac{m-1}{e} \right]^{m-1} \leq \sqrt{2\pi} \left[ \frac{m-1}{e} \right]^{m-1} e^{\frac{1}{24m-24}(12m-12)} \leq \frac{(2m-2)!\sqrt{\pi}}{4^{m-1}(m-1)!} \leq \sqrt{2\pi} \left[ \frac{m-1}{e} \right]^{m-1}. \tag{3.43}
\]

This establishes item (ii). The proof of Corollary 3.4 is thus complete. \( \square \)
**Corollary 3.5.** Let $\Gamma : (0, \infty) \to (0, \infty)$ satisfy for all $x \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{x-1} e^{-t} \, dt$. Then

(i) it holds for all $m \in \mathbb{N} \cap [2, \infty)$ that

$$\sqrt{2\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} \leq \Gamma(m) \leq \sqrt{3\pi (m-1)} \left[ \frac{m-1}{e} \right]^{m-1} \quad (3.44)$$

and

(ii) it holds for all $m \in \mathbb{N} \cap [2, \infty)$ that

$$\sqrt{\pi} \left[ \frac{m-1}{e} \right]^{m-1} \leq \Gamma \left( m - \frac{1}{2} \right) \leq \sqrt{2\pi} \left[ \frac{m-1}{e} \right]^{m-1} \quad (3.45)$$

**Proof of Corollary 3.5.** Note that Corollary 3.4 and item (ii) in Corollary 3.2 establish items (i) and (ii). The proof of Corollary 3.5 is thus complete. \qed

### 3.2 Lower and upper bounds for Gaussian tails

**Lemma 3.6.** Let $\sigma, s \in (0, \infty)$. Then

(i) it holds that

$$\int_0^\infty e^{-\sigma x^2} \, dx = \frac{\sqrt{\pi}}{2\sqrt{\sigma}},$$

(ii) it holds that

$$\int_s^\infty e^{-\sigma x^2} \, dx \leq \left[ \frac{\sqrt{\pi}}{2\sqrt{\sigma}} \right] e^{-\sigma s^2}, \quad (3.46)$$

and

(iii) it holds that

$$\int_0^s e^{-\sigma x^2} \, dx \geq \left[ \frac{\sqrt{\pi}}{2\sqrt{\sigma}} \right] (1 - e^{-\sigma s^2}). \quad (3.47)$$

**Proof of Lemma 3.6.** Observe that the integral transformation theorem shows that

$$\int_0^\infty e^{-\sigma x^2} \, dx = \frac{1}{\sqrt{2\pi}} \int_0^\infty \exp(-\frac{x^2}{2}) \, dx = \frac{\sqrt{\pi}}{\sqrt{\sigma}} \int_0^\infty \exp(-\frac{x^2}{2\pi}) \, dx = \frac{\sqrt{\pi}}{2\sqrt{\sigma}} \quad (3.48)$$

This establishes item (i). Next note that the integral transformation theorem and (3.48) ensure that

$$\int_s^\infty e^{-\sigma x^2} \, dx = \int_0^\infty e^{-\sigma(x+s)^2} \, dx = \int_0^\infty \left( e^{-\sigma x^2 - 2\sigma sx - \sigma s^2} \right) \, dx$$

$$= e^{-\sigma s^2} \left[ \int_0^\infty e^{-\sigma x^2 - 2\sigma sx} \, dx \right] \leq e^{-\sigma s^2} \left[ \int_0^\infty e^{-\sigma x^2} \, dx \right] = \left[ \frac{\sqrt{\pi}}{2\sqrt{\sigma}} \right] e^{-\sigma s^2}. \quad (3.49)$$

This establishes item (ii). Next we combine (3.48) and (3.49) to obtain that

$$\int_0^s e^{-\sigma x^2} \, dx = \int_0^\infty e^{-\sigma x^2} \, dx - \int_s^\infty e^{-\sigma x^2} \, dx = \frac{\sqrt{\pi}}{\sqrt{\sigma}} - \int_s^\infty e^{-\sigma x^2} \, dx \geq \left[ \frac{\sqrt{\pi}}{2\sqrt{\sigma}} \right] (1 - e^{-\sigma s^2}). \quad (3.50)$$

This establishes item (iii). The proof of Lemma 3.6 is thus complete. \qed

**Corollary 3.7.** Let $d \in \mathbb{N}, \sigma, s \in (0, \infty)$. Then
(i) it holds that
\[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq s\}} e^{-\sigma \|x\|_2^2} dx \geq \left[ \frac{\pi}{\sigma} \right]^{d/2} \left[ 1 - e^{-\sigma x^2/4} \right]^d \] (3.51)
and
(ii) it holds that
\[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq s\}} e^{-\sigma \|x\|_2^2} dx \leq \left[ \frac{\pi}{\sigma} \right]^{d/2} \left( 1 - \left[ 1 - e^{-\sigma x^2/4} \right]^d \right) \] (3.52)
(cf. Definition 2.21).

Proof of Corollary 3.7. Observe that item (i) in Lemma 3.6 implies that
\[ \int_{\mathbb{R}^d} e^{-\sigma \|x\|_2^2} dx = \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} e^{-\sigma (|x_1|^2 + |x_2|^2 + \ldots + |x_d|^2)} dx_1 \ldots dx_2 dx_1 
= \left[ \int_{\mathbb{R}} e^{-\sigma x^2} dx \right]^d = \left[ \frac{\pi}{\sigma} \right]^{d/2} \] (3.53)
(cf. Definition 2.21). Next note that item (iii) in Lemma 3.6 (applied with \( \sigma \wedge \sigma, s \wedge d^{-1/2}s \) in the notation of Lemma 3.6) and the fact that
\[ \{y = (y_1, y_2, \ldots, y_d) \in \mathbb{R}^d : (\forall j \in \{1, 2, \ldots, d\} : |y_j| \leq d^{-1/2}s)\} \subseteq \{y \in \mathbb{R}^d : \|y\|_2 \leq s\} \] (3.54)
ensure that
\[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq s\}} e^{-\sigma \|x\|_2^2} dx \geq \prod_{j=1}^d \int_{-d^{-1/2}s}^{d^{-1/2}s} e^{-\sigma |x_j|^2} dx_j 
= \left[ 2 \int_0^{d^{-1/2}s} e^{-\sigma x^2} dx \right]^d \geq \left[ \frac{\pi}{\sigma} \right]^{d/2} \left[ 1 - e^{-\sigma x^2/4} \right]^d. \] (3.55)
Combining this with (3.53) establishes items (i) and (ii). The proof of Corollary 3.7 is thus complete.

Lemma 3.8. Let \( \alpha \in \mathbb{R} \setminus (0, 1) \). Then it holds for all \( x \in (-1, \infty) \) that \( (1 + x)^\alpha \geq 1 + \alpha x \).

Proof of Lemma 3.8. Throughout this proof let \( f : (-1, \infty) \to \mathbb{R} \) satisfy for all \( x \in (-1, \infty) \) that \( f(x) = (1 + x)^\alpha - 1 - \alpha x \). Observe that the chain rule ensures that for all \( x \in (-1, \infty) \) it holds that
\[ f'(x) = \alpha(1 + x)^{\alpha - 1} - \alpha = \alpha[(1 + x)^{\alpha - 1} - 1]. \] (3.56)
The assumption that \( \alpha \in \mathbb{R} \setminus (0, 1) \) hence ensures that for all \( x \in (-1, 0) \) it holds that \( f'(x) = \alpha[(1 + x)^{\alpha - 1} - 1] \leq 0 \). This implies that the function \( (-1, 0) \ni x \mapsto f(x) \in \mathbb{R} \) is non-increasing. Hence, we obtain that for all \( x \in (-1, 0) \) it holds that \( f(x) \geq f(0) = 0 \). Next note that (3.56) and the assumption that \( \alpha \in \mathbb{R} \setminus (0, 1) \) demonstrate that for all \( x \in [0, \infty) \) it holds that \( f'(x) = \alpha[(1 + x)^{\alpha - 1} - 1] \geq 0 \). This ensures that the function \( [0, \infty) \ni x \mapsto f(x) \in \mathbb{R} \) is non-decreasing. Therefore, we obtain that for all \( x \in [0, \infty) \) it holds that \( f(x) \geq f(0) = 0 \). The proof of Lemma 3.8 is thus complete.

Corollary 3.9. Let \( d \in \mathbb{N}, \sigma, s \in (0, \infty) \). Then
\[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq s\}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma \|x\|_2^2} dx \leq de^{-\sigma x^2/4} \] (3.57)
(cf. Definition 2.21).
**Proof of Corollary 3.9.** Observe that the fact that $-e^{-\sigma x^2/\rho} \in (-1, \infty)$ and Lemma 3.8 (applied with $\alpha \land d$ in the notation of Lemma 3.8) ensure that $(1 - e^{-\sigma x^2/\rho})^d \geq 1 - de^{-\sigma x^2/\rho}$. Combining this with item (ii) in Corollary 3.7 (applied with $d \land d$, $\sigma \land \sigma$, $s \land s$ in the notation of Corollary 3.7) implies that

$$\int_{\{y \in \mathbb{R}^d : \|y\| \geq s\}} \left[\sigma \right]^{d/2} e^{-\sigma \|x\|_2^2} dx \leq 1 - \left[1 - e^{-\sigma x^2/\rho}\right]^d \leq 1 - \left[1 - de^{-\sigma x^2/\rho}\right] = de^{-\sigma x^2/\rho} \quad (3.58)$$

(cf. Definition 2.21). The proof of Corollary 3.9 is thus complete. $\square$

**Lemma 3.10.** Let $d \in \mathbb{N}$, $\sigma \in (0, \infty)$, $\alpha, s \in [0, \infty)$ and let $\Gamma : (0, \infty) \to (0, \infty)$ satisfy for all $x \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{\sigma-1}e^{-t} dt$. Then

$$\int_{\{y \in \mathbb{R}^d : \|y\| \geq s\}} \left[\sigma \right]^{d/2} \|x\|_2^\alpha e^{-\sigma \|x\|_2^2} dx = \frac{2\sigma^{d/2}}{\Gamma(d/2)} \left[\int_s^\infty e^{-\sigma r^2 r^\alpha} d\rho \right] \quad (3.59)$$

(cf. Definition 2.21).

**Proof of Lemma 3.10.** Throughout this proof assume w.l.o.g. $d > 1$, let $B : (0, \infty)^2 \to (0, \infty)$ satisfy for all $x, y \in (0, \infty)$ that $B(x, y) = \int_0^1 t^{\sigma-1}(1 - t)^{\rho-1} dt$, let $D \subseteq \mathbb{R}^{d-1}$ satisfy

$$D = \begin{cases} (0, 2\pi) & : d = 2 \\ (0, \pi)^{d-2} \times (0, 2\pi) & : d > 2, \end{cases} \quad (3.60)$$

and let $\Psi : (0, \infty) \times D \to \mathbb{R}^d$ satisfy for all $r \in (0, \infty)$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{d-1}) \in D$ that

$$\Psi(r, \varphi) = \begin{pmatrix} r \cos(\varphi_1) & \ldots & r \cos(\varphi_{d-1}) \\ \prod_{k=1}^d \sin(\varphi_k) & \ldots & \prod_{k=1}^d \sin(\varphi_k) \end{pmatrix} \quad (3.61)$$

Note that (3.61) shows that for all $r \in (0, \infty)$, $\varphi = (\varphi_1, \varphi_2, \ldots, \varphi_{d-1}) \in D$ it holds that $\|\Psi(r, \varphi)\|_2 = r$ and

$$|\det(\Psi'(r, \varphi))| = r^{d-1} \prod_{k=1}^{d-2} |\sin(\varphi_k)|^{d-k-1} \quad (3.62)$$

(cf. Definition 2.21). The integral transformation theorem hence ensures that

$$\int_{\{y \in \mathbb{R}^d : \|y\| \geq s\}} \|x\|_2^\alpha e^{-\sigma \|x\|_2^2} dx = \int_{\mathbb{R}^d} \|x\|_2^\alpha e^{-\sigma \|x\|_2^2} \mathbb{1}_{[s, \infty)}(\|x\|_2) dx$$

$$= \int_{\|x\|_2 \geq s} \|x\|_2^\alpha e^{-\sigma \|x\|_2^2} \mathbb{1}_{[s, \infty)}(\|x\|_2) dx$$

$$= \int_{0}^\infty \int_{D} \|\Psi(r, \varphi)\|_2^\alpha e^{-\sigma \|\Psi(r, \varphi)\|_2^2} |\det(\Psi'(r, \varphi))| \mathbb{1}_{[s, \infty)}(\|\Psi(r, \varphi)\|_2) d\varphi dr$$

$$= \int_{s}^\infty \int_{D} \|\Psi(r, \varphi)\|_2^\alpha e^{-\sigma \|\Psi(r, \varphi)\|_2^2} |\det(\Psi'(r, \varphi))| d\varphi dr$$

$$= \int_{s}^\infty e^{-\sigma r^2} r^\alpha |\det(\Psi'(r, \varphi))| d\varphi dr$$

$$= 2\pi \prod_{k=1}^{d-2} \int_0^\pi |\sin(x)|^k dx \left[\int_s^\infty e^{-\sigma r^2} r^\alpha dr \right]. \quad (3.63)$$
Next observe that the chain rule assures that for all \( x \in (0, 1) \) it holds that
\[
1 = \frac{d}{dx}(\text{arcsin}(x)) = \frac{d}{dx}(\sin(\text{arcsin}(x))) = \cos(\text{arcsin}(x))[\text{arcsin}'(x)] = [\text{arcsin}'(x)]\sqrt{1-x^2}.
\]
This implies that for all \( x \in (0, 1) \) it holds that
\[
\text{arcsin}'(x) = (1-x^2)^{-1/2}.
\]

The integral transformation theorem and Lemma 3.1 hence show that for all \( k \in \mathbb{N} \) it holds that
\[
\int_0^\pi [\sin(x)]^k \, dx = 2 \int_0^{\pi/2} [\sin(x)]^k \, dx = 2 \int_0^1 \left[ \frac{x^k}{(1-x^2)^{1/2}} \right] \, dx = \int_0^1 x^{k-1}(1-x)^{-1/2} \, dx
\]
\[
= \mathbb{B} \left( \frac{k+1}{2}, \frac{1}{2} \right) = \frac{\Gamma(k+1)}{\Gamma(1/2)} = \frac{\Gamma(k+1)}{\sqrt{\pi}}.
\]
Combining this with (3.63) and item (i) in Corollary 3.2 demonstrates that
\[
\int_{\{y \in \mathbb{R}^d : \|y\| \geq s\}} \|x\|^\alpha e^{-\sigma\|y\|^2} \, dx = 2\pi \left[ \prod_{k=1}^{d-2} \int_0^{\pi} [\sin(x)]^k \, dx \int_s^\infty e^{-\sigma^2 r^{\alpha+d-1}} \, dr \right]
\]
\[
= 2\pi \left[ \prod_{k=1}^{d-2} \left( \frac{\Gamma(k+1)}{\Gamma(k/2)} \frac{\sqrt{\pi}}{2} \right) \right] \int_s^\infty e^{-\sigma^2 r^{\alpha+d-1}} \, dr
\]
\[
= \frac{2\pi^{d/2}}{\Gamma(\frac{d}{2})} \int_s^\infty e^{-\sigma^2 r^d} \, dr.
\]
Therefore, we obtain that
\[
\int_{\{y \in \mathbb{R}^d : \|y\| \geq s\}} \left[ \frac{\sigma}{\pi} \right]^{d/2} \|x\|^\alpha e^{-\sigma\|y\|^2} \, dx = \frac{2\sigma^{d/2}}{\Gamma(\frac{d}{2})} \int_s^\infty e^{-\sigma^2 r^{d-1}} \, dr.
\]

The proof of Lemma 3.10 is thus complete. \( \square \)

**Lemma 3.11.** Let \( d \in \mathbb{N} \cap [3, \infty) \), \( \beta, \sigma \in (0, \infty) \). Then
\[
\int_{\{y \in \mathbb{R}^d : \|y\| \geq \sqrt{\frac{2}{d} + \frac{1}{\sqrt{2} \sigma}} \}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma\|y\|^2} \, dx \leq d \left[ \frac{1 + \beta}{e^\sigma} \right]^{d/2}
\]
(cf. Definition 2.21).

**Proof of Lemma 3.11.** Throughout this proof let \( \Gamma : (0, \infty) \to (0, \infty) \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^x t^{x-1} e^{-t} \, dt \). Note that Lemma 3.10 (applied with \( d \land d, \sigma \land \sigma, \alpha \land 0, s \land (2\sigma)^{-1/2}(d(1+\beta))^{1/2} \) in the notation of Lemma 3.10) implies that
\[
\int_{\{y \in \mathbb{R}^d : \|y\| \geq \sqrt{\frac{2}{d} + \frac{1}{\sqrt{2} \sigma}} \}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma\|y\|^2} \, dx = \frac{2\sigma^{d/2}}{\Gamma(\frac{d}{2})} \int_{\sqrt{\frac{2}{d} + \frac{1}{\sqrt{2} \sigma}}}^\infty e^{-\sigma^2 r^{d-1}} \, dr
\]
(cf. Definition 2.21). Next observe that Lemma 3.10 (applied with \( d \land d, \sigma \land \sigma, \alpha \land 0, s \land (2\sigma)^{-1/2}(d(1+\beta))^{1/2} \) in the notation of Lemma 3.10) shows that
\[
\int_{\{y \in \mathbb{R}^d : \|y\| \geq \sqrt{\frac{2}{d} + \frac{1}{\sqrt{2} \sigma}} \}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma\|y\|^2} \, dx = \frac{2\sigma^{d/2}}{\Gamma(\frac{d}{2})} \int_{\sqrt{d(1+\beta)} \sqrt{2} \sigma}^\infty e^{-\sigma^2 r^{d-1}} \, dr.
\]
Combining this with (3.70) ensures that

\[
\int \left\{ y \in \mathbb{R}^d : \frac{\|y\|}{\sqrt{\sigma}} \leq \frac{\|y\|}{\sqrt{\pi}} \right\} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma \|x\|^2} \, dx
\]

\[
= \int \left\{ y \in \mathbb{R}^d : \|y\| \geq \frac{\|y\|}{\sqrt{\sigma}} \right\} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma \|x\|^2} \, dx - \int \left\{ y \in \mathbb{R}^d : \|y\| \geq \frac{\|y\|}{\sqrt{\pi}} \right\} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma \|x\|^2} \, dx
\]

\[
= \frac{2\sigma^{d/2}}{\Gamma(d/2)} \int_{\frac{\|y\|}{\sqrt{\sigma}}}^{\infty} e^{-\sigma \|x\|^2} \, dr - \frac{2\sigma^{d/2}}{\Gamma(d/2)} \int_{\frac{\|y\|}{\sqrt{\pi}}}^{\infty} e^{-\sigma \|x\|^2} \, dr
\]

(3.72)

Next note that the chain rule ensures that for all \( x \in [(2\sigma)^{-1/2}d^{1/2}, \infty) \) it holds that

\[
\left( e^{-\sigma x^2} x^{d-1} \right)' = e^{-\sigma x^2} x^{d-2} (d - 1 - 2\sigma x^2) \leq e^{-\sigma x^2} x^{d-2} (d - 1 - d) < 0.
\]

This ensures that the function \( [(2\sigma)^{-1/2}d^{1/2}, \infty) \ni x \mapsto e^{-\sigma x^2} x^{d-1} \in \mathbb{R} \) is strictly decreasing. Hence, we obtain that

\[
\frac{2\sigma^{d/2}}{\Gamma(d/2)} \int_{\frac{\|y\|}{\sqrt{\sigma}}}^{\infty} e^{-\sigma \|x\|^2} \, dr \leq \frac{2\sigma^{d/2}}{\Gamma(d/2)} \int_{\frac{\|y\|}{\sqrt{\sigma}}}^{\infty} e^{-\frac{d(1+\beta)}{2\sigma}} \left[ \frac{d(1+\beta)}{2\sigma} \right]^{d/2} \, dr
\]

(3.74)

\[
\leq \frac{2\sigma^{d/2}}{\Gamma(d/2)} \left[ e^{-\frac{d(1+\beta)}{2\sigma}} \right] \left[ \frac{d(1+\beta)}{2\sigma} \right]^{d/2} \left[ \frac{d+1}{2} \right] \left[ 1 + \beta \right]^{d/2}
\]

Next observe that item (i) in Corollary 3.5 and the fact that for all \( m \in \mathbb{N} \cap [2, \infty) \) it holds that

\[
\left[ 1 + \frac{1}{m-1} \right]^{m-1} \frac{1}{2m-2} \leq e \left[ 1 + \frac{1}{m-1} \right]^{m-1} \frac{1}{2m-2} = e \left[ 1 + \frac{1}{m-1} \right]^{1/2} \leq e^{2^{1/2}} \leq 3e/2
\]

(3.75)

assure that for all \( k, m \in \mathbb{N} \) with \( k = 2m \geq 4 \) it holds that

\[
\frac{2}{\Gamma(d/2)} \left[ e^{-\frac{d(1+\beta)}{2\sigma}} \right] \left[ \frac{d+1}{2} \right] \left[ 1 + \beta \right]^{d/2} \leq \frac{2}{\Gamma(m)} \left[ e^{-m(1+\beta)} \right] \left[ (2m)^{m+1/2} \right] \left[ 1 + \beta \right]^{m/2}
\]

(3.76)

\[
\leq \frac{2}{\sqrt{2\pi(m-1)}} \left[ \frac{e}{m-1} \right]^{m-1} \left[ e^{-m(1+\beta)} \right] \left[ (2m)^{m+1/2} \right] \left[ 1 + \beta \right]^{m/2}
\]

\[
= \frac{2m}{\sqrt{\pi}} \left[ e^{-1-m\beta} \right] \left[ 1 + \frac{1}{m-1} \right]^{m-1} \left[ 1 + \beta \right]^{m/2} \leq \frac{2m}{\sqrt{\pi}} \left[ e^{-1-m\beta} \right] \left[ 3e/2 \right] \left[ 1 + \beta \right]^{m/2}
\]

\[
= \frac{3k}{2\sqrt{\pi}} \left[ 1 + \beta \right]^{k/2} \leq k \left[ 1 + \beta \right]^{k/2}
\]

Next note that item (ii) in Corollary 3.5 and the fact that for all \( m \in \mathbb{N} \cap [2, \infty) \) it holds that
\[ (1 + (2m - 2)^{-1})^{2m - 2} \leq e \] show that for all \( k, m \in \mathbb{N} \) with \( k = 2m - 1 \geq 3 \) it holds that
\[
\left[ \frac{2}{\Gamma \left( \frac{3}{2} \right)} \right] e^{-\frac{k(1 - \frac{1}{\beta})}{2} \left( \frac{k + 1}{2} \right)} \left[ \frac{1 + \beta}{2} \right]^{k/2} = \left[ \frac{2}{\Gamma \left( m - \frac{1}{2} \right)} \right] e^{-\frac{k}{2} (1 + \beta) (2m - 1)^m} \left[ \frac{1 + \beta}{2} \right]^{m - \frac{1}{2}} 
\]
\[
\leq \frac{2}{\sqrt{\pi}} \left[ \frac{e}{m - 1} \right]^{m - 1} e^{-\frac{1}{2} (1 + \beta) (2m - 1)^m} \left[ \frac{1 + \beta}{2} \right]^{m - \frac{1}{2}} = \left[ \frac{2}{\pi} \right]^{1/2} (2m - 1) e^{-\frac{1}{2} \frac{1}{2} m \beta - 1} \left[ \frac{1 + \beta}{2} \right]^{m - \frac{1}{2}} \Gamma \left( \frac{d + k}{2} \right) \frac{d + k}{\Gamma \left( \frac{3}{2} \right)} \left[ \frac{1 + \beta}{e^{\beta}} \right]^{d/2} \leq k \left[ \frac{1 + \beta}{e^{\beta}} \right]^{d/2} \] \quad (3.77)

Combining this with (3.74) and (3.76) assures that
\[
\int_{y \in \mathbb{R}^d : \frac{d(1 + \beta)}{\sqrt{2} \sigma} \leq \|y\|_2 \leq \frac{d(1 + \beta)}{\sqrt{2} \sigma}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\frac{1}{2} \frac{1}{2} \|y\|_2^2} \|x\|_2^k e^{-\sigma \|x\|_2^2} dx \leq d \left[ \frac{1 + \beta}{e^{\beta}} \right]^{d/2} \] \quad (3.78)

This and (3.72) imply that
\[
\int_{y \in \mathbb{R}^d : \frac{d(1 + \beta)}{\sqrt{2} \sigma} \leq \|y\|_2 \leq \frac{d(1 + \beta)}{\sqrt{2} \sigma}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\frac{1}{2} \frac{1}{2} \|y\|_2^2} \|x\|_2^k e^{-\sigma \|x\|_2^2} dx \leq d \left[ \frac{1 + \beta}{e^{\beta}} \right]^{d/2} \] \quad (3.79)

The proof of Lemma 3.11 is thus complete. \( \square \)

### 3.3 Upper bounds for weighted Gaussian tails

**Lemma 3.12.** Let \( d \in \mathbb{N} \cap [3, \infty) \), \( \beta, \sigma \in (0, \infty) \), \( k \in \mathbb{N}_0 \) and let \( \Gamma : (0, \infty) \rightarrow (0, \infty) \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^\infty t^{x-1} e^{-t} dt \). Then
\[
\int_{y \in \mathbb{R}^d : \|y\|_2 \geq \frac{d(1 + \beta)}{\sqrt{2} \sigma}} \left[ \frac{\sigma}{\pi} \right]^{d/2} \|x\|_2^k e^{-\sigma \|x\|_2^2} dx \leq d^{1+k} \left[ \frac{1 + \beta}{2 \sigma} \right]^{k/2} \left[ \frac{1 + \beta}{e^{\beta}} \right]^{d/2} \] \quad (3.80)

(cf. Definition 2.21).

**Proof of Lemma 3.12.** Observe that Lemma 3.11 (applied with \( d \searrow d \), \( \beta \searrow \beta \), \( \sigma \searrow \sigma \) in the notation of Lemma 3.11) ensures that
\[
\int_{y \in \mathbb{R}^d : \frac{d(1 + \beta)}{\sqrt{2} \sigma} \leq \|y\|_2 \leq \frac{d(1 + \beta)}{\sqrt{2} \sigma}} \left[ \frac{\sigma}{\pi} \right]^{d/2} \|x\|_2^k e^{-\sigma \|x\|_2^2} dx \leq d^k \left[ \frac{1 + \beta}{2 \sigma} \right]^{k/2} \int_{y \in \mathbb{R}^d : \frac{d(1 + \beta)}{\sqrt{2} \sigma} \leq \|y\|_2 \leq \frac{d(1 + \beta)}{\sqrt{2} \sigma}} \left[ \frac{\sigma}{\pi} \right]^{d/2} e^{-\sigma \|x\|_2^2} dx \] \quad (3.81)
In this section we employ the upper bounds for certain weighted tails of standard normal distributions from Section 3 to establish in Corollary 4.9 in Subsection 4.5 below suitable lower bounds for the number of parameters of appropriate ANNs that approximate certain high-dimensional functions. Our proof of Corollary 4.9 employs appropriate lower bounds for the product of the number of ANN parameters and the maximum of the absolute values of the ANN parameters which we establish in Corollary 4.8 in Subsection 4.5 below. Our proof of Corollary 4.8, in turn, employs the lower bounds for general ANNs in Theorem 4.7. Our proof of Theorem 4.7 uses the elementary lower bounds for normalized $L^2$-scalar products in Lemma 4.6 in Subsection 4.3 below as well as the upper bounds for $L^2$-scalar products involving realizations of ANNs in Lemma 4.5 in Subsection 4.2 below. Our proof of Lemma 4.5 employs the priori estimates for realizations of ANNs in Lemma 4.2, Corollary 4.3, and Lemma 4.4 in Subsection 4.1 below. Our proofs of Lemma 4.2 and Corollary 4.3 use the well-known matrix norm estimates in Lemma 4.1 below. Only for the sake of completeness we include in this section also the detailed proofs for Lemma 4.1 and Lemma 4.6.
4.1 Upper bounds for realizations of ANNs

**Lemma 4.1.** Let \( m, n \in \mathbb{N}, A = (A_{i,j})_{i,j \in \{1,2,\ldots,m\} \times \{1,2,\ldots,n\}} \in \mathbb{R}^{m \times n}, B = (B_1, B_2, \ldots, B_m) \in \mathbb{R}^m, x \in \mathbb{R}^n \). Then

(i) it holds that
\[
\|Ax + B\|_\infty \leq \sqrt{n}
\left[ \max_{i \in \{1,2,\ldots,m\}} \max_{j \in \{1,2,\ldots,n\}} |A_{i,j}| \right] \|x\|_2 + \|B\|_\infty \tag{4.1}
\]
and

(ii) it holds that
\[
\|Ax + B\|_\infty \leq n \left[ \max_{i \in \{1,2,\ldots,m\}} \max_{j \in \{1,2,\ldots,n\}} |A_{i,j}| \right] \|x\|_\infty + \|B\|_\infty \tag{4.2}
\]

(cf. Definition 2.21).

**Proof of Lemma 4.1.** Throughout this proof let \( \alpha \in \mathbb{R} \) satisfy \( \alpha = \max_{i \in \{1,2,\ldots,m\}} \max_{j \in \{1,2,\ldots,n\}} |A_{i,j}| \) and let \( \beta \in \mathbb{R} \) satisfy \( \beta = \|B\|_\infty \) (cf. Definition 2.21). Observe that the triangle inequality and the fact that for all \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \) it holds that \( \sum_{j=1}^n v_j \leq \sqrt{n} \|v\|_2 \) ensure that for all \( v = (v_1, v_2, \ldots, v_n) \in \mathbb{R}^n \) it holds that
\[
\|Av + B\|_\infty = \max_{i \in \{1,2,\ldots,m\}} \left| B_i + \sum_{j=1}^n A_{i,j} v_j \right| \leq \max_{i \in \{1,2,\ldots,m\}} \left( |B_i| + \sum_{j=1}^n |A_{i,j} v_j| \right) \tag{4.3}
\]
\[
\leq \beta + \alpha \sum_{j=1}^n |v_j| \leq \beta + \alpha \sqrt{n} \|v\|_2.
\]
This establishes item (i). Moreover, note that the fact that for all \( v \in \mathbb{R}^n \) it holds that \( \sqrt{n} \|v\|_2 \leq n \|v\|_\infty \) and item (i) demonstrate that for all \( v \in \mathbb{R}^n \) it holds that
\[
\|Av + B\|_\infty \leq \beta + \alpha \sqrt{n} \|v\|_2 \leq \beta + \alpha n \|v\|_\infty. \tag{4.4}
\]
This establishes item (ii). The proof of Lemma 4.1 is thus complete. \( \square \)

**Lemma 4.2.** Let \( L \in \mathbb{N} \cap [2, \infty), l_0, l_1, \ldots, l_L \in \mathbb{N}, \Phi = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in (\times_{k=1}^L (\mathbb{R}^{l_{k-1} \times l_k} \times \mathbb{R}^{l_k})), x_0 \in \mathbb{R}^{l_0}, x_1 \in \mathbb{R}^{l_1}, \ldots, x_L \in \mathbb{R}^{l_L} \) satisfy for all \( k \in \{1,2,\ldots,L\} \) that \( x_k = \mathcal{R}(W_k x_{k-1} + B_k) \) (cf. Definition 2.1). Then

(i) it holds for all \( k \in \{1,2,\ldots,L\}, j \in \{1,2,\ldots,k\} \) that
\[
\|x_k\|_\infty \leq l_{k-1} l_{k-2} \cdots l_{j-1} (\max \{1, \|T(\Phi)\|_\infty \})^j (\|x_{j-1}\|_\infty + j) \tag{4.5}
\]
and

(ii) it holds that
\[
\|\mathcal{R}(\Phi)(x_0)\|_\infty \leq l_{L-1} l_{L-2} \cdots l_1 (\max \{1, \|T(\Phi)\|_\infty \})^{L-1} (\|x_1\|_\infty + L - 1) \tag{4.6}
\]
(cf. Definitions 2.2, 2.21, and 2.22).
\textbf{Proof of Lemma 4.2.} Throughout this proof let $\alpha = \max\{1, \|T(\Phi)\|_\infty\}$ (cf. Definitions 2.21 and 2.22). Observe that the fact that for all $x \in \mathbb{R}$ it holds that $|\max\{x, 0\}| \leq |x|$ and item (ii) in Lemma 4.1 (applied for every $k \in \{1, 2, \ldots, L\}$ with $m \cap l_k, n \cap l_{k-1}, A \cap W_k, B \cap B_k, x \cap x_{k-1}$ in the notation of Lemma 4.1) imply that for all $k \in \{1, 2, \ldots, L\}$ it holds that
\[
\|x_k\|_\infty = \|\mathcal{R}(W_k x_{k-1} + B_k)\|_\infty \leq \|W_k x_{k-1} + B_k\|_\infty \\
\leq \alpha l_{k-1}\|x_{k-1}\|_\infty + \alpha \leq \alpha l_{k-1}(\|x_{k-1}\|_\infty + 1). \tag{4.7}
\]
This demonstrates that for all $k \in \{2, 3, \ldots, L\}, i \in \{1, 2, \ldots, k-1\}$ with $\|x_k\|_\infty \leq l_{k-1} l_{k-2} \cdots l_{k-i} \alpha^i(\|x_{k-i}\|_\infty + i)$ it holds that
\[
\|x_k\|_\infty \leq l_{k-1} l_{k-2} \cdots l_{k-j} \alpha^j(\|x_{k-j}\|_\infty + j)
\]
This establishes item (i). Next note that item (ii) in Lemma 4.1 (applied with $m \cap l, n \cap l_{L-1}, A \cap W, B \cap B_L, x \cap x_{L-1}$ in the notation of Lemma 4.1) ensures that
\[
\|(\mathcal{R}(\Phi))(x_0)\|_\infty = \|W_L x_{L-1} + B_L\|_\infty \leq \alpha l_{L-1}\|x_{L-1}\|_\infty + \alpha \leq \alpha l_{L-1}(\|x_{L-1}\|_\infty + 1) \tag{4.10}
\]
(cf. Definition 2.2). This and item (i) demonstrate that
\[
\|(\mathcal{R}(\Phi))(x_0)\|_\infty \leq \alpha l_{L-1}(\|x_{L-1}\|_\infty + 1) \\
\leq \alpha l_{L-1}(\|L_{L-2} L_{L-3} \cdots l_1 \alpha^{L-2}(\|x_1\|_\infty + L - 2) + 1) \tag{4.11}
\]
This establishes item (ii). The proof of Lemma 4.2 is thus complete. \(\square\)

\textbf{Corollary 4.3.} It holds for all $\Phi \in \mathbb{N}$, $x \in \mathbb{R}^{|\Phi|}$ that
\[
\|(\mathcal{R}(\Phi))(x)\|_\infty \leq \frac{\mathcal{P}(\Phi) \max\{1, \|T(\Phi)\|_\infty\}}{2 \mathcal{L}(\Phi)} \left(\|x\|_2 + \mathcal{L}(\Phi)\right) \tag{4.12}
\]
(cf. Definitions 2.2, 2.21, and 2.22).

\textbf{Proof of Corollary 4.3.} Throughout this proof let $L \in \mathbb{N}$, $l_0, l_1, \ldots, l_L \in \mathbb{N}$, $\Phi = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L)) \in \bigtimes_{k=1}^L (\mathbb{R}^{l_k} \times l_{k-1} \times \mathbb{R}^{l_k})$, $\alpha = \max\{1, \|T(\Phi)\|_\infty\}$, $x_0 \in \mathbb{R}^{l_0}$, $x_1 \in \mathbb{R}^{l_1}$ satisfy $x_1 = \mathcal{R}(W_1 x_0 + B_1)$ (cf. Definitions 2.1, 2.21, and 2.22). Observe that item (i) in Lemma 4.1 (applied with $m \cap l_1, n \cap l_0, A \cap W_1, B \cap B_1, x \cap x_0$ in the notation of Lemma 4.1) ensures that
\[
\|W_1 x_0 + B_1\|_\infty \leq \alpha \sqrt{l_0} \|x_0\|_2 + \alpha \leq \alpha \sqrt{l_0}(\|x_0\|_2 + 1). \tag{4.13}
\]
In the following we distinguish between the case $\mathcal{L}(\Phi) = 1$ and the case $\mathcal{L}(\Phi) > 1$. We first prove (4.12) in the case $\mathcal{L}(\Phi) = 1$. Note that (4.13) demonstrates that
\[
\|(\mathcal{R}(\Phi))(x_0)\|_\infty = \|W_1 x_0 + B_1\|_\infty \leq \alpha \sqrt{l_0}(\|x_0\|_2 + 1) \\
\leq \frac{(l_0 + 1)\alpha}{2} (\|x_0\|_2 + 1) \leq \frac{l_1 (l_0 + 1)\alpha}{2} (\|x_0\|_2 + 1) \tag{4.14}
\]
\[
= \frac{\mathcal{P}(\Phi) \max\{1, \|T(\Phi)\|_\infty\}}{2 \mathcal{L}(\Phi)} \left(\|x_0\|_2 + \mathcal{L}(\Phi)\right).
\]
This proves (4.12) in case $L(\Phi) = 1$. We now prove (4.12) in the case $L(\Phi) > 1$. Observe that (4.13) and the fact that for all $x \in \mathbb{R}$ it holds that $|\max\{x, 0\}| \leq |x|$ show that
\[
\|x_1\|_\infty = \|\mathcal{R}(W_1x_0 + B_1)\|_\infty \leq \|W_1x_0 + B_1\|_\infty \leq \alpha \sqrt[\ell_0]{\|x_0\|^2 + 1}. \tag{4.15}
\]
This and item (ii) in Lemma 4.2 (applied with $L \cap L, I_0 \cap I_0, I_1 \cap I_1, \ldots, I_L \cap I_L, \Phi \cap \Phi, x_0 \cap x_0, x_1 \cap x_1$ in the notation of Lemma 4.2) ensure that
\[
\|(\mathcal{R}(\Phi))(x_0)\|_\infty \leq l_{L-1}l_{L-2} \cdots l_1\alpha^{L-1}(\|x_1\|_\infty + L - 1) \\
\leq l_{L-1}l_{L-2} \cdots l_1\alpha^{L-1}(\alpha \sqrt[\ell_0]{\|x_0\|_2 + 1} + L - 1) \tag{4.16}
\]
\[
\leq l_{L-1}l_{L-2} \cdots l_1\sqrt[\ell_0]{\alpha^L(\|x_0\|_2 + L)}.
\]

In the next step note that the inequality of arithmetic and geometric means assures that
\[
\mathcal{P}(\Phi) = \sum_{k=1}^{L} l_k(l_{k-1} + 1) = l_1 + l_2 + \ldots + l_L + l_0l_1 + l_1l_2 + \ldots + l_{L-1}l_L
\]
\[
\geq 2L[(l_1l_2 \cdots l_L)(l_0l_1l_2 \cdots l_{L-1}l_L)]^{1/2L} = 2L[l_0(l_1)^3(l_2)^3 \cdots (l_{L-1})^3(l_L)^2]^{1/2L}
\]
\[
\geq 2L[l_0(l_1)^2(l_2)^2 \cdots (l_{L-1})^2]^{1/2L}.
\]

Hence, we obtain that
\[
l_{L-1}l_{L-2} \cdots l_1\sqrt{l_0} \leq \left[\frac{\mathcal{P}(\Phi)}{2L}\right]^L. \tag{4.18}
\]

Combining this and (4.16) shows that
\[
\|(\mathcal{R}(\Phi))(x_0)\|_\infty \leq l_{L-1}l_{L-2} \cdots l_1\sqrt{l_0} \alpha^L(\|x_0\|_2 + L)
\]
\[
\leq \left[\frac{\mathcal{P}(\Phi)\alpha}{2L}\right]^L(\|x_0\|_2 + L)
\]
\[
= \left[\frac{\mathcal{P}(\Phi)\max\{1, \|T(\Phi)\|_\infty\}}{2L(\Phi)}\right]^L(\|x_0\|_2 + L(\Phi)). \tag{4.19}
\]

This proves (4.12) in the case $L(\Phi) > 1$. The proof of Corollary 4.3 is thus complete. \qed

**Lemma 4.4.** Let $d \in \mathbb{N} \cap [4, \infty)$, $\beta, \sigma \in (0, \infty)$, $\Phi \in \mathbb{N}$ satisfy $L(\Phi) = d$ and $O(\Phi) = 1$ and let $\varphi : \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that $\varphi(x) = (\sigma/x)^{d/2} \exp(-\sigma \|x\|_2^2)$ (cf. Definitions 2.2 and 2.21). Then
\[
\int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \frac{\sqrt[d+1]{d}}{\sqrt[d-1]{d}}\}} \|(\mathcal{R}(\Phi))(x)\|^2 \varphi(x) \, dx \\
\leq |L(\Phi)|^2 \left[\frac{\mathcal{P}(\Phi)\max\{1, \|T(\Phi)\|_\infty\}}{2L(\Phi)}\right]^{2L(\Phi)} \left[1 + \frac{\beta}{e^{\beta}}\right]^{-\frac{d^3(6 + 4\beta + \sigma)}{4\sigma}} \tag{4.20}
\]
(cf. Definition 2.22).

**Proof of Lemma 4.4.** Throughout this proof let $\mathcal{S} \in \mathbb{R}$ satisfy $\sqrt{2\pi}\mathcal{S} = \sqrt{d(1+\beta)}$. Observe that the fact that for all $a, b \in \mathbb{R}$ it holds that $(a + b)^2 \leq 2(a^2 + b^2)$ and Corollary 4.3 imply that for all $x \in \mathbb{R}^d$ it holds that
\[
\|(\mathcal{R}(\Phi))(x)\|^2 = \|(\mathcal{R}(\Phi))(x)\|_\infty^2 \leq \left[\frac{\mathcal{P}(\Phi)\max\{1, \|T(\Phi)\|_\infty\}}{2L(\Phi)}\right]^{2L(\Phi)}(\|x\|_2 + L(\Phi))^2
\]
\[
\leq 2 \left[\frac{\mathcal{P}(\Phi)\max\{1, \|T(\Phi)\|_\infty\}}{2L(\Phi)}\right]^{2L(\Phi)}(\|x\|_2^2 + L(\Phi)^2) \tag{4.21}
\]
(cf. Definition 2.22). Note that item (i) in Lemma 3.1 ensures that \( \Gamma(d/2 + 1) = \frac{d}{2} \Gamma(d/2) \). Combining this with Lemma 3.12 (applied with \( d \cap d, \beta \cap \beta, \sigma \cap \sigma, k \cap 0, k \cap 2 \) in the notation of Lemma 3.12) assures that

\[
\int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} \left[ |\mathcal{L}(\Phi)|^2 + \|x\|^2 \right] \varphi(x) \, dx \\
= |\mathcal{L}(\Phi)|^2 \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} \varphi(x) \, dx + \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} \|x\|^2 \varphi(x) \, dx \\
= |\mathcal{L}(\Phi)|^2 \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} \varphi(x) \, dx + \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} \|x\|^2 e^{-\sigma\|x\|^2} \, dx \\
\leq |\mathcal{L}(\Phi)|^2 \left[ d \left[ \frac{1 + \beta}{e^\beta} \right] + \frac{d^3(1 + \beta)}{2\sigma} \left[ \frac{1 + \beta}{e^\beta} \right] + \frac{d^3(1 + \beta)}{2\sigma} \left[ \frac{1 + \beta}{e^\beta} \right] + \frac{(d + 2)}{2\sigma} e^{-\frac{x^2(1 + \beta)}{2(\sigma + 1)}} \right].
\]

This, the fact that \( d^3(1 + \beta) + 4(d + 2) + 4d\sigma \leq d^3 \left( \frac{3}{2} + \beta + \frac{d}{4} \right) \), and the fact that

\[
\max \left\{ e^{-\frac{d(1 + \beta)}{2}}, e^{-\frac{x^2(1 + \beta)}{2(d + 2)}}, \left[ \frac{1 + \beta}{e^\beta} \right] \right\} \leq \left[ \frac{1 + \beta}{e^\beta} \right]^{d/3}
\]

imply that

\[
\int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} \left[ |\mathcal{L}(\Phi)|^2 + \|x\|^2 \right] \varphi(x) \, dx \leq |\mathcal{L}(\Phi)|^2 \left[ \frac{1 + \beta}{e^\beta} \right]^{d/3} \left[ \frac{d^3(1 + \beta) + 4(d + 2) + 4d\sigma}{2\sigma} \right] \\
\leq |\mathcal{L}(\Phi)|^2 \left[ \frac{1 + \beta}{e^\beta} \right]^{d/3} \left[ \frac{d^3(6 + 4\beta + \sigma)}{8\sigma} \right].
\]

Combining this with (4.21) demonstrates that

\[
\int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sqrt{\frac{\text{var}(\sigma)}{\text{var}(\Phi)}}\}} |(R(\Phi))(x)|^2 \varphi(x) \, dx = \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \sigma\}} |(R(\Phi))(x)|^2 \varphi(x) \, dx \\
\leq 2 \left[ \frac{\mathcal{P}(\Phi) \max \{1, \|T(\Phi)\|_\infty\}}{2L(\Phi)} \right]^{2L(\Phi)} \left[ \frac{d^3(1 + \beta) + 4(d + 2) + 4d\sigma}{2\sigma} \right] \left[ \frac{1 + \beta}{e^\beta} \right]^{d/3} \left[ \frac{d^3(6 + 4\beta + \sigma)}{4\sigma} \right].
\]

The proof of Lemma 4.4 is thus complete.

\[\square\]

4.2 Upper bounds for scalar products involving realizations of ANNs

**Lemma 4.5.** Let \( d \in \mathbb{N} \cap [4, \infty), \beta, \sigma \in (0, \infty), \Phi \in \mathbb{N} \) satisfy \( \mathcal{I}(\Phi) = d \) and \( \mathcal{O}(\Phi) = 1 \), let \( \varphi : \mathbb{R}^d \to \mathbb{R}, f : \mathbb{R}^d \to \mathbb{R}, \) and \( g : \mathbb{R}^d \to \mathbb{R} \) be measurable, and assume for all \( x \in \mathbb{R}^d \) that \( \varphi(x) = (\sigma/\pi)^{d/2} \exp(-\sigma\|x\|^2), \int_{\mathbb{R}^d} |(R(\Phi))(y)| \, dy > 0, \int_{\mathbb{R}^d} g(y)^2 \, dy = 1, \) and

\[
f(x) = \left[ \int_{\mathbb{R}^d} |(R(\Phi))(y)|^2 \varphi(y) \, dy \right]^{-1/2} (R(\Phi))(x) \left[ \varphi(x) \right]^{1/2}
\]

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(cf. Definitions 2.2 and 2.21). Then
\[
\int_{\mathbb{R}^d} |f(x)g(x)| dx \leq \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \sqrt{2} \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2} + L(\Phi) \left[ \frac{P(\Phi) \max\{1, \|T(\Phi)\|_{\infty}\}}{2L(\Phi)} \right]^{4/6} \left[ \frac{1 + \beta}{e^{\beta^3}} \right]^{2\sqrt{\sigma} \left[ \int_{\mathbb{R}^d} |(R(\Phi))(y)|^2 \varphi(y) dy \right]^{1/2}}
\] (4.27)

(cf. Definition 2.22).

**Proof of Lemma 4.5.** Throughout this proof let \( \Gamma : (0, \infty) \to (0, \infty) \) satisfy for all \( x \in (0, \infty) \) that \( \Gamma(x) = \int_0^x t^{-1}e^{-t} dt, \) let \( a \in \mathbb{R} \) satisfy \( a = \left[ \int_{\mathbb{R}^d} |(R(\Phi))(y)|^2 \varphi(y) dy \right]^{1/2}, \) and let \( \beta \in \mathbb{R} \) satisfy \( \sqrt{2\sigma} \beta = \sqrt{d(1 + \beta)}. \) Observe that \( a \in (0, \infty) \) and
\[
\int_{\mathbb{R}^d} |f(x)|^2 dx = \left[ \int_{\mathbb{R}^d} |(R(\Phi))(y)|^2 \varphi(y) dy \right]^{-1} \int_{\mathbb{R}^d} |(R(\Phi))(x)|^2 \varphi(x) dx = 1.
\] (4.28)

Combining this with the Hölder inequality shows that
\[
\int_{\mathbb{R}^d} |f(x)g(x)| dx = \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |f(x)g(x)| dx + \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |f(x)g(x)| dx
\leq \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |f(x)|^2 dx \right]^{1/2} \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2} + \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |f(x)|^2 dx \right]^{1/2} \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2}
\leq \left[ \int_{\mathbb{R}^d} |f(x)|^2 dx \right]^{1/2} \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2} + \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |f(x)|^2 dx \right]^{1/2} \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2}
= \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2} \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |f(x)|^2 dx \right]^{1/2}.
\] (4.29)

Next we obtain that Lemma 4.4 (applied with \( d \wedge d, \beta \wedge \beta, \sigma \wedge \sigma, \Phi \wedge \Phi, \varphi \wedge \varphi \) in the notation of Lemma 4.4) implies that
\[
\int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |f(x)|^2 dx = a^{-2} \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |(R(\Phi))(x)|^2 \varphi(x) dx
\leq a^{-2} L(\Phi)^2 \left[ \frac{P(\Phi) \max\{1, \|T(\Phi)\|_{\infty}\}}{2L(\Phi)} \right] \left[ \frac{1 + \beta}{e^{\beta^3}} \right] \left[ \frac{d^3 (6 + 4\beta + \sigma)}{4\sigma} \right].
\] (4.30)

This and (4.29) imply that
\[
\int_{\mathbb{R}^d} |f(x)g(x)| dx \leq \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2} + \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \geq \phi(\|T\|_{\infty}) \}} |f(x)|^2 dx \right]^{1/2}
\leq \left[ \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \phi(\|T\|_{\infty}) \}} |g(x)|^2 dx \right]^{1/2} + a^{-1} L(\Phi) \left[ \frac{P(\Phi) \max\{1, \|T(\Phi)\|_{\infty}\}}{2L(\Phi)} \right] \left[ \frac{1 + \beta}{e^{\beta^3}} \right] \left[ \frac{d^3 (6 + 4\beta + \sigma)}{2\sqrt{\sigma}} \right].
\] (4.31)

The proof of Lemma 4.5 is thus complete. \( \square \)
4.3 On the connection of distances and scalar products

**Lemma 4.6.** Let \( d \in \mathbb{N} \), \( \alpha \in \mathbb{R} \), let \( f : \mathbb{R}^d \to \mathbb{R} \) and \( g : \mathbb{R}^d \to \mathbb{R} \) be measurable, and assume \( \int_{\mathbb{R}^d} |f(x)|^2 \, dx = \int_{\mathbb{R}^d} |g(x)|^2 \, dx = 1 \). Then

\[
\int_{\mathbb{R}^d} |\alpha f(x) - g(x)|^2 \, dx \geq 1 - \int_{\mathbb{R}^d} |f(x)g(x)| \, dx.
\]  

(4.32)

**Proof of Lemma 4.6.** Note that the Hölder inequality implies that

\[
\int_{\mathbb{R}^d} |f(x)g(x)| \, dx \leq \left[ \int_{\mathbb{R}^d} |f(x)|^2 \, dx \right]^{1/2} \left[ \int_{\mathbb{R}^d} |g(x)|^2 \, dx \right]^{1/2} = 1.
\]  

(4.33)

Next observe that

\[
\int_{\mathbb{R}^d} |\alpha f(x) - g(x)|^2 \, dx = \alpha^2 + 1 - 2\alpha \int_{\mathbb{R}^d} f(x)g(x) \, dx
\]

\[
= \left[ \alpha - \int_{\mathbb{R}^d} f(x)g(x) \, dx \right]^2 + 1 - \left[ \int_{\mathbb{R}^d} f(x)g(x) \, dx \right]^2
\]

\[
\geq 1 - \left[ \int_{\mathbb{R}^d} f(x)g(x) \, dx \right]^2 \geq 1 - \int_{\mathbb{R}^d} |f(x)g(x)| \, dx.
\]  

(4.34)

This and (4.33) ensure that

\[
\int_{\mathbb{R}^d} |\alpha f(x) - g(x)|^2 \, dx \geq 1 - \int_{\mathbb{R}^d} |f(x)g(x)| \, dx \geq 1 - \int_{\mathbb{R}^d} |f(x)g(x)| \, dx.
\]  

(4.35)

The proof of Lemma 4.6 is thus complete. \( \square \)

4.4 ANN approximations for a class of general high-dimensional functions

**Theorem 4.7.** Let \( d \in \mathbb{N} \cap [4, \infty) \), \( \beta, \sigma \in (0, \infty) \), \( \Phi \in \mathbb{N} \) satisfy \( I(\Phi) = d \) and \( O(\Phi) = 1 \), let \( \varphi : \mathbb{R}^d \to \mathbb{R} \), \( g : \mathbb{R}^d \to \mathbb{R} \), and \( g : \mathbb{R}^d \to \mathbb{R} \) be measurable, and assume for all \( x \in \mathbb{R}^d \) that \( \varphi(x) = (\sigma x)^{\beta/2} \exp(-\sigma\|x\|_2^2) \), \( \int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) dy \in (0, \infty) \), \( \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(y)| dy > 0 \), and \( g(x) \geq \left[ \int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) dy \right]^{-1/2} g(x) \) (cf. Definitions 2.2 and 2.21). Then

\[
\mathcal{L}(\Phi) \left[ \frac{\mathcal{P}(\Phi) \max\{1, \|T(\Phi)\|_\infty\}}{2\mathcal{L}(\Phi)} \right] \geq \frac{e^\beta}{1 + \beta} \left[ \frac{2\sqrt{\sigma} \mathcal{I}_{d}(\mathcal{R}(\Phi))(x)^2 \varphi(x) dx}{d^{3/2}(6 + 4 \beta + \sigma)^{1/2}} \right]^{1/2}
\]

\[
\cdot \left[ 1 - \int_{\{y \in \mathbb{R}^d : \|y\|_2 \leq \sqrt{\frac{R(\Phi) + 1}{\sqrt{\sigma}}}, \varphi(x) dx \}} \right]^{1/2} - \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) dx.
\]  

(4.36)

(cf. Definition 2.22).

**Proof of Theorem 4.7.** Throughout this proof let \( f : \mathbb{R}^d \to \mathbb{R} \) and \( g : \mathbb{R}^d \to \mathbb{R} \) satisfy for all \( x \in \mathbb{R}^d \) that \( g(x) = g(x)\|\varphi(x)\|^{1/2} \) and \( f(x) = \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(y)|^2 \varphi(y) dy \right]^{-1/2} (\mathcal{R}(\Phi))(x)\|\varphi(x)\|^{1/2} \) and let \( \mathcal{R} \in \mathbb{R} \) satisfy \( \sqrt{2\sigma \mathcal{R}} = \sqrt{d(1 + \beta)} \). Note that \( \int_{\mathbb{R}^d} |f(x)|^2 \, dx = \int_{\mathbb{R}^d} |g(x)|^2 \, dx = 1. \)
Lemma 4.6 (applied with $d \bowtie d$, $\alpha \bowtie \frac{1}{2} \int_{\mathbb{R}^d} \|(\Phi)(y)\|^2 \varphi(y) \, dy \|^{1/2}$, $f \bowtie f$, $g \bowtie g$ in the notation of Lemma 4.6) hence ensures that

$$
\int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx = \int_{\mathbb{R}^d} f(x) \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(y)|^2 \varphi(y) \, dy \right]^{1/2} - g(x) \right] dx 
$$

\[ \left(4.37\right) \]

Combining this with Lemma 4.5 (applied with $d \bowtie d$, $\beta \bowtie \beta$, $\sigma \bowtie \sigma$, $\Phi \bowtie \Phi$, $\varphi \bowtie \varphi$, $f \bowtie f$, $g \bowtie g$ in the notation of Lemma 4.5) demonstrates that

$$
\int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx \geq 1 - \int_{\mathbb{R}^d} |f(x)g(x)| \, dx \geq 1 - \left[ \int_{\{y \in \mathbb{R}^d: \|y\|_2 \leq \|\#\} \}}|g(x)|^2 \varphi(x) \, dx \right]^{1/2} 
$$

$$
\left(4.38\right)
$$

\[
- \mathcal{L}(\Phi) \left[ \frac{\mathcal{P}(\Phi) \max \{1, \|T(\Phi)\|_{\infty} \}}{2 \mathcal{L}(\Phi)} \right] \geq \left[ \frac{\epsilon}{1 + \beta} \right]^{1/6} \left( \left. \frac{\mathcal{P}(\Phi) \max \{1, \|T(\Phi)\|_{\infty} \}}{2 \mathcal{L}(\Phi)} \right) \right] \left[ \frac{d^{1/2}(6 + 4\beta + \sigma)^{1/2}}{2\sqrt{\sigma} \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(y)|^2 \varphi(y) \, dy \right]^{1/2}} \right] \left(4.39\right)
\]

The proof of Theorem 4.7 is thus complete.

\[ \square \]

4.5 ANN approximations for certain specific high-dimensional functions

Corollary 4.8. Let $d \in \mathbb{N} \cap [4, \infty)$, $\varepsilon \in (0, 1/4]$, let $\varphi: \mathbb{R}^d \to \mathbb{R}$ and $g: \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that $\varphi(x) = (2\pi)^{-d/2} \exp(-\frac{1}{2} \|x\|^2)$ and $g(x) = \sum_{j=1}^d \max\{|x_j| - \sqrt{2d}, 0\}^2$, let $g: \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that $g(x) = \int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) \, dy^{1/2} g(x)$, and let $\Phi \in \mathbb{N}$ satisfy $\mathcal{I}(\Phi) = d$, $\mathcal{O}(\Phi) = 1$, and $\int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx \leq \varepsilon$ (cf. Definitions 2.2 and 2.21). Then

$$
\mathcal{P}(\Phi) \max \{1, \|T(\Phi)\|_{\infty} \} \geq \left[ \frac{d}{4} \right]^{3/2} \exp \left( \frac{d}{2\mathcal{L}(\Phi)} \right) \left(4.40\right)
$$

\[(4.40)\]

(cf. Definition 2.22).
Proof of Corollary 4.8. Observe that the triangle inequality ensures that
\[
\left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x)|^2 \varphi(x) \, dx \right]^{1/2} \\
\geq \left[ \int_{\mathbb{R}^d} |g(x)|^2 \varphi(x) \, dx \right]^{1/2} - \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx \right]^{1/2} \\
= 1 - \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx \right]^{1/2} \geq 1 - \varepsilon^{1/2} \\
\geq 1 - 4^{-1/2} = \frac{1}{2} > 0.
\]
Hence, we obtain that \( \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x)| \, dx > 0 \). Next note that for all \( x = (x_1, x_2, \ldots, x_d) \in \{ y \in \mathbb{R}^d : \|y\|_2 \leq \sqrt{2d} \} \) it holds that \( |x_j| \leq \|x\|_2 \leq \sqrt{2d} \). This ensures that for all \( x = (x_1, x_2, \ldots, x_d) \in \{ y \in \mathbb{R}^d : \|y\|_2 \leq \sqrt{2d} \} \) it holds that \( g(x) = g(x) = 0 \). Combining Theorem 4.7 (applied with \( d \land d, \beta \land 1, \sigma \land 1/2, \Phi \land \varphi \land \varphi \land g \land g \land \varphi \) in the notation of Theorem 4.7), the fact that \( \varepsilon/2 \geq e^{3/10} \), the fact that \( \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x)| \, dx > 0 \), and (4.41) therefore implies that
\[
\mathcal{L}(\Phi) \left[ \frac{\mathcal{P}(\Phi) \max \{ 1, \| \mathcal{T}(\Phi) \|_\infty \} }{2 \mathcal{L}(\Phi)} \right]^{\mathcal{L}(\Phi)} \geq \left[ \frac{1}{2} \right]^{\varepsilon/6} \left[ \frac{\sqrt{2} \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x)|^2 \varphi(x) \, dx \right]^{1/2} }{d^{1/2} (6 + 4 + 1/2)^{1/2}} \right] \\
\cdot \left[ 1 - \left[ \int_{\{ y \in \mathbb{R}^d : \|y\|_2 \leq \sqrt{2d} \}} |g(x)|^2 \varphi(x) \, dx \right]^{1/2} - \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx \right] \\
= \left[ \frac{1}{2} \right]^{\varepsilon/6} \left[ \frac{\sqrt{2} \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x)|^2 \varphi(x) \, dx \right]^{1/2} }{d^{1/2} (6 + 4 + 1/2)^{1/2}} \right] \\
\cdot \left[ 1 - \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx \right] \\
\geq \left[ (21)^{-1/2} \right] \left[ \frac{1}{2} \right]^{\varepsilon/6} d^{-1/2} (1 - \varepsilon) \geq \frac{e^{\varepsilon/20}}{7d^{1/2}}.
\]
(cf. Definition 2.22). Hence, we obtain that
\[
\mathcal{P}(\Phi) \max \{ 1, \| \mathcal{T}(\Phi) \|_\infty \} \geq 2 \mathcal{L}(\Phi) \left[ \frac{e^{\varepsilon/20}}{7d^{1/2} \mathcal{L}(\Phi)} \right]^{1/\mathcal{L}(\Phi)} \geq \left[ \frac{2}{7} \right] \frac{1}{1} \mathcal{L}(\Phi) \frac{d^{3/2}}{20 \mathcal{L}(\Phi)}. \] (4.43)
The proof of Corollary 4.8 is thus complete.

**Corollary 4.9.** Let \( \varphi_d : \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}, g_d : \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}, \) satisfy for all \( d \in \mathbb{N}, x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) that \( \varphi_d(x) = (2\pi)^{-d/2} \exp(-\frac{1}{2} \sum_{j=1}^{d} |x_j|^2) \) and \( g_d(x) = \sum_{j=1}^{d} \max \{ |x_j| - \sqrt{2d} \} \) \( d \), let \( g_d : \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}, \) satisfy for all \( d \in \mathbb{N}, x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) that \( g_d(x) = \left[ \int_{\mathbb{R}^d} |g_d(y)|^2 \varphi_d(y) \, dy \right]^{-1/2} g_d(x) \), and let \( \delta \in (0, 1], \mathcal{C} \in \{100(\delta \ln(1.03))^{-2}, \infty\} \) satisfy \( 2\mathcal{C}^{1/\delta} \leq (1.03)^{3/2} \). Then it holds for all \( \epsilon \in (0, \mathcal{C}], d \in \mathbb{N}, \epsilon \in (0, 1/2], \Phi \in \mathbb{N} with \mathcal{I}(\Phi) = d, \mathcal{O}(\Phi) = 1, \mathcal{H}(\Phi) \leq 1, \| \mathcal{T}(\Phi) \|_\infty \leq \epsilon d^\delta, \) and \( \left[ \int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g_d(x)|^2 \varphi_d(x) \, dx \right]^{1/2} \leq \epsilon \) that \( \mathcal{P}(\Phi) \geq (1 + e^{-3}(d^\delta) \) (cf. Definitions 2.22, 2.21, and 2.22).

Proof of Corollary 4.9. Observe that the assumption that \( \mathcal{C} \in [100(\delta \ln(1.03))^{-2}, \infty) \) and the chain rule ensure that for all \( x \in (\mathcal{C}, \infty) \) it holds that
\[
2^{-1}(1.03)^{\sqrt{\mathcal{C}} x^{-5/4}} = (1.03)^{\sqrt{\mathcal{C}}} \ln(1.03) \left[ \frac{1}{4\sqrt{x}} x^{-5/4} - (1.03)^{\sqrt{\mathcal{C}}} \left[ \frac{5}{2\delta x} x^{-5/4} \right] \right] \\
= (1.03)^{\sqrt{\mathcal{C}}} \left[ \frac{x^{-5/4}}{4x} \right] \ln(1.03) \left[ \sqrt{x} - 10(\delta \ln(1.03))^{-1} \right] \\
\geq (1.03)^{\sqrt{\mathcal{C}}} \left[ \frac{x^{-5/4}}{4x} \right] \ln(1.03) \left[ \sqrt{\mathcal{C}} - 10(\delta \ln(1.03))^{-1} \right] \geq 0.
\]
This implies that the function $[C, \infty) \ni x \mapsto 2^{-1}(1.03)^{\sqrt{c}x^{-5/8}} \in \mathbb{R}$ is non-decreasing. The assumption that $C \in [100(\delta \ln(1.03))^{-2}, \infty)$ and the assumption that $2C^{5/8} \leq (1.03)^{\sqrt{c}}$ therefore ensure that for all $c \in [C, \infty)$ it holds that $c \geq 100(\delta \ln(1.03))^{-2}$ and
\[
2^{-1}(1.03)^{\sqrt{c}x^{-5/8}} \geq 2^{-1}(1.03)^{\sqrt{c}c^{5/8}} \geq 1. \tag{4.45}
\]
The fact that for all $x \in (0, \infty)$ it holds that $(1+x^{-1})^e \leq e$ hence ensures that for all $c \in [C, \infty)$, $d \in \mathbb{N}$, $\Phi \in \mathbb{N}$ with $d \leq c^{5/(2\delta)}$ it holds that
\[
(1 + c^{-3})^{(d^\delta)} \leq (1 + c^{-3})^{(5/2)} = [(1 + c^{-3})^{(\epsilon^3)}]^{1/\epsilon^2} \leq e^{1/\epsilon^2} \leq 2 \leq \mathcal{P}(\Phi) \tag{4.46}
\]
(cf. Definition 2.2). Moreover, note that the chain rule and (4.45) show that for all $c \in [C, \infty)$, $x \in [c^{5/(2\delta)}, \infty)$ it holds that
\[
[(1.03)^{(x^3)/c} - 2c^{-1}]^e = (1.03)^{(x^3)/c} \ln(1.03) \left[ \frac{\delta}{c} \right] x^{-2c-1} - 2c(1.03)^{(x^3)/c}x^{-2c-1}
\]
\[
= (1.03)^{(x^3)/c} \ln(1.03) \left[ \frac{\delta}{c} \right] x^{-2c-1} - 2c(1.03)^{(x^3)/c}x^{-2c-1}
\]
\[
\geq (1.03)^{(x^3)/c} \ln(1.03) \left[ \frac{\delta}{c} \right] x^{-2c-1} - 2c(1.03)^{(x^3)/c}x^{-2c-1}
\]
\[
\geq (1.03)^{(x^3)/c} x^{-2c-1} 8c > 0. \tag{4.47}
\]
This implies for all $c \in [C, \infty)$ that the function $[c^{5/(2\delta)}, \infty) \ni x \mapsto (1.03)^{(x^3)/c}x^{-2c} \in \mathbb{R}$ is strictly increasing. The fact that $e^{1/\epsilon^2} > 1.03$, (4.45), and the fact that for all $c \in [C, \infty)$ it holds that $2^{5/8} \geq 7c$ therefore demonstrate that for all $c \in [C, \infty)$, $d \in \mathbb{N}$ with $d \geq c^{5/(2\delta)}$ it holds that
\[
e^{(d^\delta)/(30c)} d^{-2c} \geq (1.03)^{(d^3)/c} d^{-2c} \geq (1.03)^{(5/2)} c^{-5/8} = [(1.03)^{\sqrt{c}c^{-5/8}}]^e \geq 2^e \geq \left[ \frac{7}{2} \right] c. \tag{4.48}
\]
The fact that for all $c \in [C, \infty)$ it holds that $(25c)^{-1} \geq (30c)^{-1} + c^{-3}$, the fact that for all $x \in \mathbb{R}$ it holds that $e^x \geq 1 + x$, and Corollary 4.8 hence ensure that for all $c \in [C, \infty)$, $d \in \mathbb{N}$, $\epsilon \in (0, 1/2]$, $\Phi \in \mathbb{N}$ with $d \geq c^{5/(2\delta)}$, $\mathcal{I}(\Phi) = d$, $\mathcal{O}(\Phi) = 1$, $\mathcal{H}(\Phi) \leq cd^{1-\delta}$, $\|\mathcal{T}(\Phi)\|_{\infty} \leq cd^\delta$, and $\left[ \int_{\mathbb{R}^d}((\mathcal{R}(\Phi))(x) - \mathcal{g}_d(x))^2 \varphi_d(x) dx \right]^{1/2} \leq \epsilon$ it holds that
\[
\mathcal{P}(\Phi) \geq (\max\{1, \|\mathcal{T}(\Phi)\|_{\infty}\})^{-1}\left[ \frac{\delta}{d} \right] d^{3/2} \exp\left( \frac{d^\delta}{200(\delta \epsilon^2)} \right) \geq \left[ \frac{\delta}{d} \right] \exp\left( \frac{d^\delta}{200(\delta \epsilon^2)} \right) d^{-3/2}\epsilon^{-1}
\]
\[
\geq \left[ \frac{\delta}{d} \right] \exp\left( \frac{d^\delta}{200(\delta \epsilon^2)} \right) d^{-3/2}\epsilon^{-1} \exp\left( \frac{d^\delta}{200(\delta \epsilon^2)} \right) \geq \exp\left( \frac{d^\delta}{200(\delta \epsilon^2)} \right) \geq (1 + c^{-3})^{(d^\delta)} \tag{4.49}
\]
(cf. Definitions 2.21 and 2.22). Combining this with (4.46) assures that for all $c \in [C, \infty)$, $d \in \mathbb{N}$, $\epsilon \in (0, 1/2]$, $\Phi \in \mathbb{N}$ with $\mathcal{I}(\Phi) = d$, $\mathcal{O}(\Phi) = 1$, $\mathcal{H}(\Phi) \leq cd^{1-\delta}$, $\|\mathcal{T}(\Phi)\|_{\infty} \leq cd^\delta$, and $\left[ \int_{\mathbb{R}^d}((\mathcal{R}(\Phi))(x) - \mathcal{g}_d(x))^2 \varphi_d(x) dx \right]^{1/2} \leq \epsilon$ it holds that $\mathcal{P}(\Phi) \geq (1 + c^{-3})^{(d^\delta)}$. The proof of Corollary 4.9 is thus complete.

5 Upper bounds for the number of ANN parameters in the approximation of high-dimensional functions

In this section we establish in Corollary 5.12 in Subsection 5.6 below appropriate upper bounds for the number of parameters of suitable ANNs that approximate certain high-dimensional
target functions. Corollary 5.12 is a consequence of the ANN approximation result in Theorem 5.11 in Subsection 5.6 below. Our proof of Theorem 5.11 employs (i) the elementary ANN representation result for multiplications with powers of real numbers which we establish in Lemma 5.10 in Subsection 5.5 below, (ii) the lower and upper bounds for appropriate Gaussian integrals which we present in Lemma 5.9 in Subsection 5.4 below, and (iii) the ANN approximation result for appropriate shifted squared rectifier functions in Corollary 5.5 in Subsection 5.3 below.

Our proof of Lemma 5.9 employs the well-known Gaussian tail estimates in Lemmas 5.6 and 5.7 in Subsection 5.4 below. Lemma 5.6 is, e.g., proved as Lemma 22.2 in Klenke [37] and only for completeness we include in Subsection 5.4 also the detailed proofs for Lemmas 5.6 and 5.7. Our proof of Corollary 5.5 uses the elementary ANN representation result for compositions with shifted absolute value functions which we present in Lemma 5.4 in Subsection 5.3 below as well as the ANN approximation result for the squared rectifier function in Corollary 5.3 in Subsection 5.2 below.

Our proof of Corollary 5.3 uses the well-known ANN approximation result for the square function in Lemma 5.2 in Subsection 5.1 below. The proof of Lemma 5.2, in turn, employs the well-known ANN representation result in Lemma 5.1 in Subsection 5.1. Lemmas 5.1 and 5.2 and their proofs are strongly based on Yarotsky [32, Proposition 2]. In the current form Lemmas 5.1 and 5.2 and their proofs are slight extensions of, e.g., the statement and the proof of Proposition 3.3 in Grohs et al. [24] (cf., e.g., also Elbrächter et al. [15, Lemma 6.1]). Only for completeness we include in Subsection 5.1 also the detailed proofs for Lemmas 5.1 and 5.2.

5.1 ANN approximations for the square function

Lemma 5.1. Let \((A_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{4 \times 4}, \mathbb{B} \subseteq \mathbb{R}^{4 \times 1}, (c_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}\) satisfy for all \(k \in \mathbb{N}\) that

\[
A_k = \begin{pmatrix} 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ -c_k & 2c_k & -c_k & 1 \end{pmatrix}, \quad \mathbb{B} = \begin{pmatrix} 0 \\ -\frac{2}{3} \\ -1 \\ 0 \end{pmatrix}, \quad \text{and} \quad c_k = 2^{1 - 2k}, \tag{5.1}
\]

let \(g_n : \mathbb{R} \to [0, 1], n \in \mathbb{N},\) satisfy for all \(n \in \mathbb{N}, x \in \mathbb{R}\) that

\[
g_1(x) = \begin{cases} 2x & : x \in [0, \frac{1}{2}] \\ 2 - 2x & : x \in \left[\frac{1}{2}, 1\right] \\ 0 & : x \in \mathbb{R} \setminus [0, 1] \end{cases} \tag{5.2}
\]

and \(g_{n+1}(x) = g_1(g_n(x))\), let \(f_n : [0, 1] \to [0, 1], n \in \mathbb{N}_0,\) satisfy for all \(n \in \mathbb{N}_0, k \in \{0, 1, \ldots, 2^n - 1\}, x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right]\) that \(f_n(1) = 1\) and

\[
f_n(x) = \left[\frac{2k+1}{2^n}\right] x - \frac{(k^2 + k)}{2^n}, \tag{5.3}
\]

and let \(r_k = \left(r_{k,1}, r_{k,2}, r_{k,3}, r_{k,4}\right) : \mathbb{R} \to \mathbb{R}^4, k \in \mathbb{N},\) satisfy for all \(k \in \mathbb{N}, x \in \mathbb{R}\) that \(r_1(x) = \Re(x, x - \frac{1}{2}, x - 1, x)\) and \(r_{k+1}(x) = \Re(A_k r_k(x) + \mathbb{B})\) (cf. Definition 2.1). Then

(i) it holds for all \(k \in \mathbb{N}, x \in \mathbb{R}\) that

\[
2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x) = g_k(x) \tag{5.4}
\]

and
(ii) it holds for all \( k \in \mathbb{N} \), \( x \in \mathbb{R} \) that

\[
   r_{k,4}(x) = \begin{cases} \int f_{k-1}(x) & : x \in [0, 1] \\ \max\{x, 0\} & : x \in \mathbb{R}\setminus[0, 1]. \end{cases} \tag{5.5}
\]

Proof of Lemma 5.1. We prove (5.4) and (5.5) by induction on \( k \in \mathbb{N} \). Observe that (5.2) and the assumption that for all \( x \in \mathbb{R} \) it holds that \( r_1(x) = \mathcal{R}(x, x - \frac{1}{2}, x - 1, x) \) show that for all \( x \in \mathbb{R} \) it holds that

\[
   2r_{1,1}(x) - 4r_{1,2}(x) + 2r_{1,3}(x) = 2\mathcal{R}(x) - 4\mathcal{R}(x - \frac{1}{2}) + 2\mathcal{R}(x - 1)
   = 2\max\{x, 0\} - 4\max\{x - \frac{1}{2}, 0\} + 2\max\{x - 1, 0\} = g_1(x). \tag{5.6}
\]

Furthermore, note that the assumption that for all \( x \in \mathbb{R} \) it holds that \( r_1(x) = \mathcal{R}(x, x - \frac{1}{2}, x - 1, x) \) and the fact that for all \( x \in [0, 1] \) it holds that \( f_0(x) = x = \max\{x, 0\} \) imply that for all \( x \in \mathbb{R} \) it holds that

\[
   r_{1,4}(x) = \max\{x, 0\} = \begin{cases} f_0(x) & : x \in [0, 1] \\ \max\{x, 0\} & : x \in \mathbb{R}\setminus[0, 1]. \end{cases} \tag{5.7}
\]

Combining this with (5.6) proves (5.4) and (5.5) in the base case \( k = 1 \). For the induction step let \( k \in \mathbb{N} \) satisfy for all \( x \in \mathbb{R} \) that

\[
   2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x) = g_k(x) \tag{5.8}
\]

and

\[
   r_{k,4}(x) = \begin{cases} f_{k-1}(x) & : x \in [0, 1] \\ \max\{x, 0\} & : x \in \mathbb{R}\setminus[0, 1]. \end{cases} \tag{5.9}
\]

Observe that (5.1), (5.6), (5.8), and the assumption that for all \( n \in \mathbb{N}, x \in \mathbb{R} \) it holds that \( r_{n+1}(x) = \mathcal{R}(A_n, r_n(x) + B) \) ensure that for all \( x \in \mathbb{R} \) it holds that

\[
   g_{k+1}(x) = g_1(g_k(x)) = \mathcal{R}(2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x))
   = 2\mathcal{R}(2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x))
   - 4\mathcal{R}(2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x) - \frac{1}{2})
   + 2\mathcal{R}(2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x) - 1)
   = 2r_{k+1,1}(x) - 4r_{k+1,2}(x) + 2r_{k+1,3}(x). \tag{5.10}
\]

In addition, note that (5.1), (5.8), and the assumption that for all \( n \in \mathbb{N}, x \in \mathbb{R} \) it holds that \( r_{n+1}(x) = \mathcal{R}(A_n, r_n(x) + B) \) demonstrate that for all \( x \in \mathbb{R} \) it holds that

\[
   r_{k+1,4}(x) = \mathcal{R}(-c_k r_{k,1}(x) + 2c_k r_{k,2}(x) - c_k r_{k,3}(x) + r_{k,4}(x))
   = \mathcal{R}(-[2^{1-2k}]r_{k,1}(x) + [2^{-2k}]r_{k,2}(x) - [2^{1-2k}]r_{k,3}(x) + r_{k,4}(x))
   = \mathcal{R}(-[2^{-2k}][2r_{k,1}(x) - 4r_{k,2}(x) + 2r_{k,3}(x)] + r_{k,4}(x))
   = \mathcal{R}(-[2^{-2k}]g_k(x) + r_{k,4}(x)). \tag{5.11}
\]

Combining this with (5.9), [24, Lemma 3.2], and the fact that for all \( x \in [0, 1] \) it holds that \( f_k(x) \geq 0 \) shows that for all \( x \in [0, 1] \) it holds that

\[
   r_{k+1,4}(x) = \mathcal{R}(-[2^{-2k}]g_k(x) + r_{k,4}(x)) = \mathcal{R}(-[2^{-2k}]g_k(x) + f_{k-1}(x))
   = \mathcal{R}( - [2^{-2k}g_k(x)] + x - \left[ \sum_{j=1}^{k-1}[2^{-2j}g_j(x)] \right] )
   = \mathcal{R}(x - \left[ \sum_{j=1}^{k}[2^{-2j}g_j(x)] \right] ) = \mathcal{R}(f_k(x)) = f_k(x). \tag{5.12}
\]

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Next observe that (5.9), (5.11), and the fact that for all \( x \in \mathbb{R} \setminus [0,1] \) it holds that \( g_k(x) = 0 \) prove that for all \( x \in \mathbb{R} \setminus [0,1] \) it holds that
\[
 r_{k+1,4}(x) = \mathcal{R}(-[2^{-2k}]g_k(x) + r_{k,4}(x)) = \mathcal{R}(r_{k,4}(x)) = \mathcal{R}(\max\{x,0\}) = \max\{x,0\}. \tag{5.13}
\]
Combining (5.10) and (5.12) hence proves (5.4) and (5.5) in the case \( k + 1 \). Induction thus establishes items (i) and (ii). The proof of Lemma 5.1 is thus complete.

**Lemma 5.2.** Let \( M \in \mathbb{N} \), \( (A_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{4 \times 4}, \ A, B \in \mathbb{R}^{4 \times 1}, \ (C_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{1 \times 4}, \ (c_k)_{k \in \mathbb{N}} \subseteq \mathbb{R} \) satisfy for all \( k \in \mathbb{N} \) that
\[
 A_k = \begin{pmatrix} 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ -c_k & 2c_k & -c_k & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 \\ 1 \\ 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}, \quad C_k = (-c_k, 2c_k, -c_k, 1), \tag{5.14}
\]
and \( c_k = 2^{k-2k} \) and let \( \Phi \in \mathbb{N} \) satisfy
\[
 \Phi = \begin{cases} ((A, B), (C_1, 0)) & : M = 1 \\ ((A, B), (A_1, B), (A_2, B), \ldots, (A_{M-1}, B), (C_M, 0)) & : M > 1 \end{cases} \tag{5.15}
\]
(cf. Definition 2.2). Then
(i) it holds that \( \mathcal{R}(\Phi) \in C(\mathbb{R}, \mathbb{R}) \),
(ii) it holds for all \( x \in [0,1] \) that \( |x^2 - (\mathcal{R}(\Phi))(x)| \leq 4^{-M-1} \),
(iii) it holds for all \( x \in \mathbb{R} \setminus [0,1] \) that \( (\mathcal{R}(\Phi))(x) = \mathcal{R}(x) \),
(iv) it holds that \( \mathcal{C}(\Phi) = (1, 4, 4, \ldots, 4, 1) \in \mathbb{N}^{M+2} \),
(v) it holds that \( \|\mathcal{T}(\Phi)\|_{\infty} \leq 4 \),
(vi) it holds that \( \mathcal{H}(\Phi) = M \), and
(vii) it holds that \( \mathcal{P}(\Phi) = 20M - 7 \)
(cf. Definitions 2.1, 2.21, and 2.22).

**Proof of Lemma 5.2.** Throughout this proof let \( g_n : \mathbb{R} \to [0,1], \ n \in \mathbb{N} \), satisfy for all \( n \in \mathbb{N}, \ x \in \mathbb{R} \) that
\[
 g_1(x) = \begin{cases} 2x & : x \in [0, \frac{1}{2}) \\ 2 - 2x & : x \in [\frac{1}{2}, 1] \\ 0 & : x \in \mathbb{R} \setminus [0,1] \end{cases} \tag{5.16}
\]
and \( g_{n+1}(x) = g_1(g_n(x)) \), let \( f_n : [0,1] \to [0,1], \ n \in \mathbb{N}_0 \), satisfy for all \( n \in \mathbb{N}_0, \ k \in \{0,1, \ldots, 2^n - 1\}, \ x \in \left[\frac{k}{2^n}, \frac{k+1}{2^n}\right) \) that \( f_n(1) = 1 \) and
\[
 f_n(x) = \left[\frac{2^n+1}{2^n}\right]x - \frac{(k^2+k)}{2^n}, \tag{5.17}
\]
and let \( r_k = (r_{k,1}, r_{k,2}, r_{k,3}, r_{k,4}) : \mathbb{R} \to \mathbb{R}^4, \ k \in \mathbb{N} \), satisfy for all \( k \in \mathbb{N}, \ x \in \mathbb{R} \) that
\[
 r_1(x) = \mathcal{R}(x, x - \frac{1}{2}, x - 1, x) \tag{5.18}
\]
and
\[
 r_{k+1}(x) = \mathcal{R}(A_k r_k(x) + B) \tag{5.19}
\]
(cf. Definition 2.1). Note that item (i) in Lemma 5.1 (applied with $\mathbf{A}_k \in \mathbb{R}$) and (c) in the notation of Lemma 5.1, (5.14), (5.15), (5.18), and (5.19) assure that for all $x \in \mathbb{R}$ it holds that

\[
(R(\Phi))(x) = -c_M r_{M,1}(x) + 2c_M r_{M,2}(x) - c_M r_{M,3}(x) + r_{M,4}(x) \\
= -[2^{1-2M}r_{M,1}(x) + 2^{2-2M}r_{M,2}(x) - 2^{1-2M}r_{M,3}(x) + r_{M,4}(x) \\
= -[2^{1-2M}][2r_{M,1}(x) - 4r_{M,2}(x) + 2r_{M,3}(x)] + r_{M,4}(x) \\
= -2^{2-2M}g_M(x) + r_{M,4}(x).
\]

This establishes item (i). Moreover, observe that (5.20), [24, Lemma 3.2], and item (ii) in Lemma 5.1 (applied with $\mathbf{A}_k \in \mathbb{R}$) and (c) in the notation of Lemma 5.1) show that for all $x \in [0,1]$ it holds that

\[
(R(\Phi))(x) = -[2^{1-2M}g_M(x) + r_{M,4}(x)] = -[2^{1-2M}g_M(x)] + f_{M-1}(x) \\
= -[2^{1-2M}g_M(x)] + x - \sum_{j=1}^{M-1}[2^{-2j}g_j(x)] \\
= x = \sum_{j=1}^{M-1}[2^{-2j}g_j(x)] = f_M(x).
\]

This and [24, Lemma 3.2] imply that for all $x \in [0,1]$ it holds that

\[
|x^2 - (R(\Phi))(x)| = |x^2 - f_M(x)| \leq 2^{-2M-2} = 2^{-M-1}.
\]

This establishes item (ii). Furthermore, note that (5.20), the fact that for all $x \in \mathbb{R} \setminus [0,1]$ it holds that $g_M(x) = 0$, and item (ii) in Lemma 5.1 (applied with $\mathbf{A}_k \in \mathbb{R}$) and (c) in the notation of Lemma 5.1) establish that for all $x \in \mathbb{R} \setminus [0,1]$ it holds that

\[
(R(\Phi))(x) = -[2^{1-2M}]g_M(x) + r_{M,4}(x) = r_{M,4}(x) = \max\{x, 0\} = 0(x).
\]

This establishes items (iii). In addition, observe that (5.14) and (5.15) imply that $\mathcal{D}(\Phi) = (1, 4, 4, \ldots, 4, 1) \in \mathbb{N}^{M+2}$, $\|\mathcal{T}(\Phi)\|_\infty \leq 4$, $\mathcal{H}(\Phi) = M$, and

\[
\mathcal{P}(\Phi) = 4(1 + 1) + \sum_{j=2}^{M}4(4+1) + (4+1) = 8 + 20(M-1) + 5 = 20M - 7.
\]

This establishes items (iv), (v), (vi), and (vii). The proof of Lemma 5.2 is thus complete. 

\[\square\]

5.2 ANN approximations for the squared rectifier function

Corollary 5.3. Let $M \in \mathbb{N}$, $R \in [1, \infty)$, $q \in (2, \infty)$, $(\mathbf{A}_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{q \times 4}$, $(\mathbf{B}_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{4 \times 1}$, $(\mathbf{C}_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{1 \times 4}$, $(\mathbf{C}_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}$ satisfy for all $k \in \mathbb{N}$ that

\[
\mathbf{A}_k = \begin{pmatrix} 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ -c_k & 2c_k & -c_k & 1 \end{pmatrix}, \quad \mathbf{A} = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, \quad \mathbf{B} = \begin{pmatrix} 0 & -c_k & 2c_k & -c_k & 1 \end{pmatrix}, \quad \mathbf{C}_k = (-c_k & 2c_k & -c_k & 1),
\]

and $c_k = 2^{1-2k}$, and let $\Psi, \Phi \in \mathbb{N}$ satisfy

\[
\Psi = \begin{cases} (\mathbf{A}, \mathbf{B}, (C_1, 0)) & : M = 1 \\ (\mathbf{A}, \mathbf{B}, (A_1, \mathbf{B}), (A_2, \mathbf{B}), \ldots, (A_{M-1}, \mathbf{B}), (C_M, 0)) & : M > 1 \end{cases}
\]

and $\Phi = \mathbf{A}_{R_{2,0} \cdot \Psi \cdot \mathbf{A}_{R^{-1,0}}}$ (cf. Definitions 2.2, 2.4, and 2.15). Then
(i) it holds that \( R(\Phi) \in C(\mathbb{R}, \mathbb{R}) \),

(ii) it holds for all \( x \in (-\infty, 0] \) that \( ||R(x)||^2 - (R(\Phi))(x)|| = 0 \),

(iii) it holds for all \( x \in [0, R] \) that \( ||R(x)||^2 - (R(\Phi))(x)|| \leq 4^{-M}1R^2 \),

(iv) it holds for all \( x \in [R, \infty) \) that \( ||R(x)||^2 - (R(\Phi))(x)|| \leq |R(x)|^qR^{2-q} \),

(v) it holds that \( D(\Phi) = (1, 4, \ldots, 4, 1) \in \mathbb{N}^{M+2} \),

(vi) it holds that \( H(\Phi) = M \),

(vii) it holds for all \( x \in (-\infty, 0] \) that \( R \) (cf. Definitions \( 2.1 \), \( 2.21 \), and \( 2.22 \)).

Proof of Corollary 5.3. Note that Lemma 5.2 (applied with \( M \in M, (A_k)_{k \in N} \in (A_k)_{k \in N}; \ A \in A, B \in B, (C_k)_{k \in N} \in (C_k)_{k \in N}, (c_k)_{k \in N} \in (c_k)_{k \in N}, \Phi \in \Phi \) in the notation of Lemma 5.2) assures that

(I) it holds that \( R(\Phi) \in C(\mathbb{R}, \mathbb{R}) \),

(II) it holds for all \( x \in \mathbb{R}\setminus[0, 1] \) that \( (R(\Phi))(x) = R(x) \), and

(III) it holds for all \( x \in [0, 1] \) that \( |x^2 - (R(\Phi))(x)| \leq 4^{-M-1} \)

(cf. Definition 2.1). Next observe that Proposition 2.5 and Lemma 2.16 imply that for all \( x \in \mathbb{R} \) it holds that \( R(\Phi) \in C(\mathbb{R}, \mathbb{R}) \) and

\[
(R(\Phi))(x) = (R(A_{R^2,0} \bullet \Psi \bullet A_{R^{-1},0}))(x) = (R(A_{R^2,0}))( (R(\Phi))((R(A_{R^{-1},0}))(x)) \)
= (R(A_{R^2,0}))( (R(\Phi))(R^{-1}x)) = R^2 [(R(\Phi))(R^{-1}x)].
\]

(5.27)

This establishes item (i). Moreover, note that (5.27), item (I), item (II), and the fact that for all \( x \in (-\infty, 0] \) it holds that \( R^{-1}x \in (-\infty, 0] \) ensure that for all \( x \in (-\infty, 0] \) it holds that

\[
||R(x)||^2 - (R(\Phi))(x)|| = ||R(x)||^2 - R^2 [(R(\Phi))(R^{-1}x)]
= ||R(x)||^2 - R^2R(R^{-1}x) = 0.
\]

(5.28)

This establishes item (ii). In the next step we observe that item (II), (5.27), and the fact that for all \( x \in [R, \infty) \) it holds that \( R^{-1}x \in [1, \infty) \) demonstrate that for all \( x \in [R, \infty) \) it holds that

\[
0 \leq (R(\Phi))(x) = R^2 [(R(\Phi))(R^{-1}x)] = R^2R(R^{-1}x) = Rx \leq x^2 = |R(x)|^2.
\]

(5.29)

The triangle inequality and the assumption that \( q \in (2, \infty) \) therefore ensure that for all \( x \in [R, \infty) \) it holds that

\[
||R(x)||^2 - (R(\Phi))(x)|| = |R(x)|^2 - (R(\Phi))(x) \leq |R(x)|^2
= |x|^2 = |x|^q |x|^{2-q} \leq |x|^q R^{2-q} = |R(x)|^q R^{2-q}.
\]

(5.30)

This establishes item (iv). Next note that item (III), (5.27), and the fact that for all \( x \in [0, R] \) it holds that \( R^{-1}x \in [0, 1] \) demonstrate that for all \( x \in [0, R] \) it holds that

\[
||R(x)||^2 - (R(\Phi))(x)|| = |x^2 - R^2 [(R(\Phi))(R^{-1}x)]
= R^2 [R^{-1}x]^2 - (R(\Phi))(R^{-1}x) \leq 4^{-M-1} R^2.
\]

(5.31)
This establishes item (iii). Next observe that (5.25) and (5.26) show that
\[
R^2C_M = \begin{pmatrix} -2^{1-2M}R^2 & 2^{2-2M}R^2 & -2^{1-2M}R^2 & R^2 \end{pmatrix} \in \mathbb{R}^{1 \times 4}
\] (5.32)
and
\[
\Phi = A_{R^2,0} \cdot \Psi \cdot A_{R^{-1},0} = \begin{pmatrix} (R^{-1}A, B), (R^2C_1, 0) \\ (R^{-1}A, B), (A_1, B), \ldots, (A_{M-1}, B), (R^2C_M, 0) \end{pmatrix} : M = 1
\] (5.33)
and
\[
\text{Combining this with (5.25) implies that } D(\Phi) = (1, 4, \ldots, 4, 1) \in \mathbb{N}^{M+2}, H(\Phi) = M, P(\Phi) = 20M - 7, \text{ and } \|T(\Phi)\|_\infty \leq \max\{4, R^2\} \text{ (cf. Definitions 2.21 and 2.22). This establishes items (v), (vi), (vii), and (viii). The proof of Corollary 5.3 is thus complete. } \]

5.3 ANN approximations for shifted squared rectifier functions

Lemma 5.4. Let $a \in \mathbb{R}, J, \Phi, \Psi \in \mathbb{N}$ satisfy
\[
J = \begin{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, ((1, 1), (-a)) \in ((\mathbb{R}^{2 \times 1} \times \mathbb{R}^2) \times (\mathbb{R}^{1 \times 2} \times \mathbb{R}^1)),
\] (5.34)
\[
\mathcal{I}(\Phi) = 1, \text{ and } \Psi = \Phi \cdot J \text{ (cf. Definitions 2.2 and 2.4). Then}
\]
(i) it holds that $D(\Psi) = (1, 2, D_1(\Phi), \ldots, D_{L}(\Phi)(\Phi)) \in \mathbb{N}^{L(\Phi)+2}$,
(ii) it holds for all $x \in \mathbb{R}$ that $(\mathcal{R}(\Psi))(x) = |x| - a$,
(iii) it holds for all $x \in \mathbb{R}$ that $(\mathcal{R}(\Psi))(x) = (\mathcal{R}(\Phi))(|x| - a)$, and
(iv) it holds that $\|T(\Psi)\|_\infty \leq (|a| + 1) \max\{1, \|T(\Phi)\|_\infty\}$
(cf. Definitions 2.21 and 2.22).

Proof of Lemma 5.4. Throughout this proof let $L \in \mathbb{N}, l_0, l_1, \ldots, l_L \in \mathbb{N}$ satisfy $(l_0, l_1, \ldots, l_L) = D(\Phi)$ and let $W_k \in \mathbb{R}^{l_k \times l_{k-1}}, k \in \{1, 2, \ldots, L\}$, and $B_k \in \mathbb{R}^{l_k}, k \in \{1, 2, \ldots, L\}$, satisfy $\Phi = ((W_1, B_1), (W_2, B_2), \ldots, (W_L, B_L))$. Note that $D(\mathcal{J}) = (1, 2, 1) \in \mathbb{N}^3$. Proposition 2.5 therefore ensures that $D(\Psi) = D(\Phi \bullet J) = (1, 2, D_1(\Phi), \ldots, D_{L}(\Phi)(\Phi)) \in \mathbb{N}^{L(\Phi)+2}$. This establishes item (i). Next observe that for all $x \in \mathbb{R}$ it holds that
\[
(\mathcal{R}(\mathcal{J}))(x) = \begin{pmatrix} 1 \\ 1 \end{pmatrix} \begin{pmatrix} \mathcal{R}(x + 0) \\ \mathcal{R}(-x + 0) \end{pmatrix} - a = \mathcal{R}(x) + \mathcal{R}(-x) - a = |x| - a
\] (5.35)
(cf. Definition 2.1). This establishes item (ii). Moreover, note that (5.35) and Proposition 2.5 assure that for all $x \in \mathbb{R}$ it holds that
\[
(\mathcal{R}(\Psi))(x) = (\mathcal{R}(\Phi \bullet J))(x) = (\mathcal{R}(\Phi))((\mathcal{R}(\mathcal{J}))(x)) = (\mathcal{R}(\Phi))(|x| - a).
\] (5.36)
This establishes item (iii). In addition, observe that
\[
\Psi = \Phi \bullet J = \begin{pmatrix} \begin{pmatrix} 1 \\ -1 \end{pmatrix}, \begin{pmatrix} 0 \\ 0 \end{pmatrix} \end{pmatrix}, (W_1 (1, 1), W_1 (-a) + B_1), (W_2, B_2), \ldots, (W_L, B_L).
\] (5.37)
The fact that for all $\mathcal{W} = (w_i)_{i \in \{1, 2, \ldots, l_1\}} \in \mathbb{R}^{l_1 \times 1}, \mathcal{B} = (b_1, b_2, \ldots, b_l) \in \mathbb{R}^l$ it holds that
\[
\mathcal{W} (1, 1) = \begin{pmatrix} w_1 & w_1 \\ w_2 & w_2 \\ \vdots & \vdots \\ w_l & w_l \end{pmatrix} \in \mathbb{R}^{l_1 \times 2} \quad \text{and} \quad \mathcal{W} (-a) + \mathcal{B} = \begin{pmatrix} -aw_1 + b_1 \\ -aw_2 + b_2 \\ \vdots \\ -aw_l + b_l \end{pmatrix} \in \mathbb{R}^l
\] (5.38)

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hence demonstrates that
\[
\|\mathcal{T}(\Psi)\|_\infty \leq \max\{1, \|\mathcal{T}(\Phi)\|_\infty, (|a| + 1)\|\mathcal{T}(\Phi)\|_\infty\} = \max\{1, (|a| + 1)\|\mathcal{T}(\Phi)\|_\infty\} \\
\leq \max\{(|a| + 1), (|a| + 1)\|\mathcal{T}(\Phi)\|_\infty\} = (|a| + 1)\max\{1, \|\mathcal{T}(\Phi)\|_\infty\}
\]
(5.39)
(cf. Definitions 2.21 and 2.22). This establishes item (iv). The proof of Lemma 5.4 is thus complete.

\[\square\]

**Corollary 5.5.** Let \(a \in [0, \infty)\), \(M \in \mathbb{N} \cap [2, \infty)\), \(R \in [1, \infty)\), \(q \in (2, \infty)\), \((A_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{4 \times 4}\), \(A, B \in \mathbb{R}^{4 \times 1}\), \((C_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}^{4 \times 4}\), \((c_k)_{k \in \mathbb{N}} \subseteq \mathbb{R}\) satisfy for all \(k \in \mathbb{N}\) that
\[
A_k = \begin{pmatrix} 2 & -4 & 2 & 0 \\ 2 & -4 & 2 & 0 \\ -c_k & 2c_k & -c_k & 1 \end{pmatrix}, \quad A = \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ -\frac{1}{2} \\ -1 \\ 0 \end{pmatrix}, \quad C_k = (-c_k \ 2c_k \ -c_k \ 1),
\]
and \(c_k = 2^{1-2k}\) and let \(\Phi, \Psi, \in \mathbb{N}\) satisfy
\[
\Phi = \left((R^{-1}A, B), (A_1, B), \ldots, (A_{M-1}, B), (R^2C_M, 0)\right), \tag{5.41}
\]
\[
\Psi = \Phi \cdot \mathbb{J} \tag{5.42}
\]
and \(\Psi = \Phi \cdot \mathbb{J}\) (cf. Definitions 2.2 and 2.4). Then

(i) it holds that \(\mathcal{R}(\Psi) \in C(\mathbb{R}, \mathbb{R})\),

(ii) it holds that \(\mathcal{D}(\Psi) = (1, 2, 4, 4, \ldots, 4, 1) \in \mathbb{N}^{M+1}\),

(iii) it holds that \(\mathcal{H}(\Psi) = M + 1\),

(iv) it holds that \(\mathcal{P}(\Psi) = 20M + 1\),

(v) it holds that \(\|\mathcal{T}(\Psi)\|_\infty \leq (|a| + 1)\max\{4, R^2\}\),

(vi) it holds for all \(x \in \mathbb{R}\) that \(\mathcal{R}(\Psi)(x) = \mathcal{R}(\Psi)(-x)\),

(vii) it holds for all \(x \in \mathbb{R}\) with \(|x| \leq a\) that \(\|\mathcal{R}(|x| - a)\|^2 - \mathcal{R}(\Psi)(x) = 0\),

(viii) it holds for all \(x \in \mathbb{R}\) with \(a \leq |x| \leq R + a\) that \(\|\mathcal{R}(|x| - a)\|^2 - \mathcal{R}(\Psi)(x) \leq 4^{-M-1}R^2\),

and

(ix) it holds for all \(x \in \mathbb{R}\) with \(|x| \geq R + a\) that \(\|\mathcal{R}(|x| - a)\|^2 - \mathcal{R}(\Psi)(x) \leq |x| - a]^qR^{2-q}\)

(cf. Definitions 2.1, 2.21, and 2.22).

**Proof of Corollary 5.5.** Note that Corollary 5.3 (applied with \(M \subset M, R \subset R, q \subset q, (A_k)_{k \in \mathbb{N}} \subset (A_k)_{k \in \mathbb{N}}, A \subset A, B \subset B, (C_k)_{k \in \mathbb{N}} \subset (C_k)_{k \in \mathbb{N}}, (c_k)_{k \in \mathbb{N}} \subset (c_k)_{k \in \mathbb{N}}, \Phi \subset \Phi\) in the notation of Corollary 5.3) implies that

(I) it holds that \(\mathcal{R}(\Phi) \in C(\mathbb{R}, \mathbb{R})\),

(II) it holds for all \(x \in (-\infty, 0]\) that \(\|\mathcal{R}(x)\|^2 - \mathcal{R}(\Phi)(x) = 0\),

(III) it holds for all \(x \in [0, R]\) that \(\|\mathcal{R}(x)\|^2 - \mathcal{R}(\Phi)(x) \leq 4^{-M-1}R^2\),

(IV) it holds for all \(x \in [R, \infty)\) that \(\|\mathcal{R}(x)\|^2 - \mathcal{R}(\Phi)(x) \leq |\mathcal{R}(x)|^qR^{2-q}\),

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(V) it holds that \( D(\Phi) = (1, 4, \ldots, 4, 1) \in \mathbb{N}^{M+2} \), and

(VI) it holds that \( \|T(\Phi)\|_\infty \leq \max\{4, R^2\} \)

(cf. Definitions 2.1, 2.21, and 2.22). Next observe that Lemma 5.4 (applied with \( a \bowtie a, J \bowtie J, \Phi \bowtie \Phi, \Psi \bowtie \Psi \) in the notation of Lemma 5.4), item (V), and item (VI) ensure that

\[
D(\Psi) = (1, 2, D_1(\Phi), \ldots, D_{|\mathcal{L}(\Phi)|}(\Phi)) = (1, 2, 4, \ldots, 4, 1) \in \mathbb{N}^{M+3}, \quad \mathcal{H}(\Psi) = M + 1, \quad (5.43)
\]

\[
\mathcal{P}(\Psi) = 2(1 + 1) + 4(2 + 1) + 4(4 + 1) + \ldots + 4(4 + 1) + 1(4 + 1) = 20M + 1, \quad (5.44)
\]

and

\[
\|T(\Psi)\|_\infty \leq (|a| + 1) \max\{1, \|T(\Phi)\|_\infty\} \leq (|a| + 1) \max\{4, R^2\}. \quad (5.45)
\]

This establishes items (i), (ii), (iii), (iv), and (v). Next note that Lemma 5.4 (applied with \( a \bowtie a, J \bowtie J, \Phi \bowtie \Phi, \Psi \bowtie \Psi \) in the notation of Lemma 5.4) assures that for all \( x \in \mathbb{R} \) it holds that

\[
(\mathcal{R}(J))(x) = |x| - a \quad (5.46)
\]

and

\[
(\mathcal{R}(\Psi))(x) = (\mathcal{R}(\Phi))(|x| - a) = (\mathcal{R}(\Phi))(|-x| - a) = (\mathcal{R}(\Psi))(x). \quad (5.47)
\]

This establishes item (vi). Furthermore, observe that (5.46) shows that for all \( x \in [-a, a] \) it holds that \( (\mathcal{R}(J))(x) = |x| - a \leq 0 \). Combining this with item (II) proves that for all \( x \in [-a, a] \) it holds that

\[
|[\mathcal{R}(|x| - a)]^2 - (\mathcal{R}(\Psi))(x)| = |[\mathcal{R}((\mathcal{R}(J))(x))]^2 - (\mathcal{R}(\Phi))(\mathcal{R}(J))(x)| = 0. \quad (5.48)
\]

This establishes item (vii). Moreover, note that (5.46) demonstrates that for all \( x \in \mathbb{R} \) with \( a \leq |x| \leq R + a \) it holds that \( (\mathcal{R}(J))(x) = |x| - a \in [0, R] \). This and item (III) ensure that for all \( x \in \mathbb{R} \) with \( a \leq |x| \leq R + a \) it holds that

\[
|[\mathcal{R}(|x| - a)]^2 - (\mathcal{R}(\Psi))(x)| = |[\mathcal{R}((\mathcal{R}(J))(x))]^2 - (\mathcal{R}(\Phi))(\mathcal{R}(J))(x)| \leq 4^{-M-1}R^2. \quad (5.49)
\]

This establishes item (viii). In addition, observe that (5.46) proves that for all \( x \in \mathbb{R} \) with \( |x| \geq R + a \) it holds that \( (\mathcal{R}(J))(x) = |x| - a \in [R, \infty] \). Item (IV) hence shows that for all \( x \in \mathbb{R} \) with \( |x| \geq R + a \) it holds that

\[
|[\mathcal{R}(|x| - a)]^2 - (\mathcal{R}(\Psi))(x)| = |[\mathcal{R}((\mathcal{R}(J))(x))]^2 - (\mathcal{R}(\Phi))(\mathcal{R}(J))(x)| \leq |\mathcal{R}((\mathcal{R}(J))(x))|^q R^{2-q} = |\mathcal{R}(|x| - a)|^q R^{2-q}. \quad (5.50)
\]

This establishes item (ix). The proof of Corollary 5.5 is thus complete. \( \square \)

5.4 Lower and upper bounds for integrals of certain specific high-dimensional functions

Lemma 5.6. Let \( s \in (0, \infty) \). Then

\[
\int_s^\infty e^{-\frac{1}{2}x^2} \, dx \geq e^{-\frac{1}{2}x^2} s \quad (5.51)
\]
Proof of Lemma 5.6. Note that the integration by parts formula ensures that
\[
\int_s^\infty e^{-\frac{1}{2}x^2} \, dx = \int_s^\infty -x^{-1} [e^{-\frac{1}{2}x^2}]' \, dx = \lim_{T \to \infty} \left( [x^{-1}e^{-\frac{1}{2}T^2}]_{x=s}^{x=\infty} \right) - \int_s^\infty [x^{-2}e^{-\frac{1}{2}x^2}] \, dx \tag{5.52}
\]
\[
= s^{-1}e^{-\frac{1}{2}s^2} - \int_s^\infty \left[ x^{-2}e^{-\frac{1}{2}x^2} \right] \, dx \geq s^{-1}e^{-\frac{1}{2}s^2} - s^{-2} \int_s^\infty e^{-\frac{1}{2}x^2} \, dx.
\]
Hence, we obtain that
\[
\int_s^\infty e^{-\frac{1}{2}x^2} \, dx = \left[ \frac{s^2}{1+s^2} \right] \left[ 1 + \frac{1}{s^2} \right] \left[ \int_s^\infty e^{-\frac{1}{2}x^2} \, dx \right]
\]
\[
= \left[ \frac{s^2}{1+s^2} \right] \left[ \int_s^\infty e^{-\frac{1}{2}x^2} \, dx + \frac{1}{s^2} \int_s^\infty e^{-\frac{1}{2}x^2} \, dx \right]
\]
\[
\geq \left[ \frac{s^2}{1+s^2} \right] \frac{e^{-\frac{1}{2}s^2}}{s} = \frac{e^{-\frac{1}{2}s^2}}{s + s^{-1}}.
\]
The proof of Lemma 5.6 is thus complete. \hfill \square

Lemma 5.7. Let \( \sigma, s \in (0, \infty) \). Then
\[
\int_s^\infty e^{-\sigma x^2} \, dx \geq \frac{e^{-\sigma s^2}}{s^{-1} + 2\sigma s}. \tag{5.54}
\]

Proof of Lemma 5.7. Observe that the integral transformation theorem and Lemma 5.6 (applied with \( s \sim s \sqrt{2\sigma} \) in the notation of Lemma 5.6) ensure that
\[
\int_s^\infty e^{-\sigma x^2} \, dx = \frac{1}{\sqrt{2\sigma}} \int_{s \sqrt{2\sigma}}^\infty e^{-\frac{1}{2}x^2} \, dx \geq \frac{1}{\sqrt{2\sigma}} \left[ e^{-\frac{1}{2}(s \sqrt{2\sigma})^2} \right] = e^{-\sigma s^2} \frac{1}{s^{-1} + 2\sigma s}. \tag{5.55}
\]
The proof of Lemma 5.7 is thus complete. \hfill \square

Lemma 5.8. Let \( d \in \mathbb{N} \). Then
\[
\frac{\sqrt{2d}(2d+1)}{4d^2(4d^2 + 6d + 1)} \left[ \frac{2}{\pi} \right]^{1/2} e^{-\frac{1}{19}d} \geq 50^{-1} d^{-\gamma/2}. \tag{5.56}
\]

Proof of Lemma 5.8. Note that \( 48d^2 - 28d - 20d^2 \geq 13 \). This implies that \( 4d^2 + 6d + 1 \leq (2/13)(4d^2 + 2d) = (50/13)d(2d+1) \). The fact that \( 13 \geq 2\sqrt{\pi}e^{\gamma/4} \) and the fact that \( -1 - \frac{1}{19} \geq -\frac{5}{4} \) hence ensure that
\[
\frac{\sqrt{2d}(2d+1)}{4d^2(4d^2 + 6d + 1)} \left[ \frac{2}{\pi} \right]^{1/2} e^{-\frac{1}{19}d} \geq \frac{\sqrt{2d}}{4d^2} \left[ \frac{13}{50d} \right] \left[ \frac{2}{\pi} \right]^{1/2} e^{-\gamma/4} \geq \frac{\sqrt{2d}}{4d^2} \left[ \frac{2\sqrt{2}}{50d} \right] = 50^{-1} d^{-\gamma/2}. \tag{5.57}
\]
The proof of Lemma 5.8 is thus complete. \hfill \square

Lemma 5.9. Let \( d \in \mathbb{N} \) and let \( \varphi \colon \mathbb{R}^d \to \mathbb{R} \) and \( g \colon \mathbb{R}^d \to \mathbb{R} \) satisfy for all \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) that \( \varphi(x) = (2\pi)^{-d/2} e^{-d/2} \exp(-\frac{1}{2} \sum_{j=1}^d |x_j|^2) \) and \( g(x) = \sum_{j=1}^d \max\{|x_j| - \sqrt{2d, 0}\} \). Then
\[
(50)^{-1} d^{-\gamma/2} e^{-d} \leq \int_{\mathbb{R}^d} |g(x)|^2 \varphi(x) \, dx \leq 3d^2 e^{-d}. \tag{5.58}
\]
\textbf{Proof of Lemma 5.9.} Throughout this proof let $\Gamma : (0, \infty) \rightarrow (0, \infty)$ satisfy for all $x \in (0, \infty)$ that $\Gamma(x) = \int_0^\infty t^{x-1}e^{-t} \, dt$. Observe that the fact that for all $k \in \mathbb{N}$, $a_1, a_2, \ldots, a_k \in \mathbb{R}$ it holds that

$$|a_1|^2 + |a_2|^2 + \ldots + |a_k|^2 \leq (|a_1| + |a_2| + \ldots + |a_k|)^2 \leq k(|a_1|^2 + |a_2|^2 + \ldots + |a_k|^2) \quad (5.59)$$

ensures that for all $(x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ it holds that

$$\sum_{j=1}^d \left[ \Re(|x_j| - \sqrt{2d}) \right]^4 \leq \left[ \sum_{j=1}^d \Re(|x_j| - \sqrt{2d}) \right]^2 \leq d \left[ \sum_{j=1}^d \Re(|x_j| - \sqrt{2d}) \right]^4 \quad (5.60)$$

(cf. Definition 2.1). The fact that for all $k \in \mathbb{N}$ it holds that $\int_{\mathbb{R}^k} (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \|x\|^2} \, dx = 1$ therefore demonstrates that

$$d \int_{\mathbb{R}} \left[ \Re(|x| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} x^2} \, dx$$

$$= d \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left[ \Re(|x_1| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \sum_{j=1}^d |x_j|^2} \, dx_d \ldots dx_1$$

$$= \sum_{j=1}^d \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left[ \Re(|x_j| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \sum_{j=1}^d |x_j|^2} \, dx_d \ldots dx_1$$

$$= \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left[ \sum_{j=1}^d \Re(|x_j| - \sqrt{2d}) \right]^2 (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \sum_{j=1}^d |x_j|^2} \, dx_d \ldots dx_1$$

$$\leq \int_{\mathbb{R}^d} |g(x)|^2 \varphi(x) \, dx$$

and

$$d^2 \int_{\mathbb{R}} \left[ \Re(|x| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} x^2} \, dx$$

$$= d^2 \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left[ \Re(|x_1| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \sum_{j=1}^d |x_j|^2} \, dx_d \ldots dx_1$$

$$= d \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left[ \sum_{j=1}^d \Re(|x_j| - \sqrt{2d}) \right]^2 (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \sum_{j=1}^d |x_j|^2} \, dx_d \ldots dx_1$$

$$\geq \int_{\mathbb{R}} \ldots \int_{\mathbb{R}} \left\{ \sum_{j=1}^d \Re(|x_j| - \sqrt{2d}) \right\}^2 (2\pi)^{-\frac{k}{2}} e^{-\frac{1}{2} \sum_{j=1}^d |x_j|^2} \, dx_d \ldots dx_1$$

$$= \int_{\mathbb{R}^d} |g(x)|^2 \varphi(x) \, dx$$

(cf. Definition 2.21). Hence, we obtain that

$$d \int_{\mathbb{R}} \left[ \Re(|x| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} x^2} \, dx \leq \int_{\mathbb{R}^d} |g(x)|^2 \varphi(x) \, dx \leq d^2 \int_{\mathbb{R}} \left[ \Re(|x| - \sqrt{2d}) \right]^4 (2\pi)^{-\frac{1}{2}} e^{-\frac{1}{2} x^2} \, dx \quad (5.63)$$
Next note that Lemma 5.7 (applied with $\sigma \sim 1/2$, $s \sim (2d)^{1/2} + (2d)^{-1/2}$ in the notation of Lemma 5.7) and Lemma 5.8 (applied with $d \sim d$ in the notation of Lemma 5.8) ensure that

$$
\int_{\mathbb{R}} [\Re(|x| - \sqrt{2d})]^{4}(2\pi)^{-1/2} e^{-\frac{1}{2}x^2} \, dx
$$

$$
= 2 \left[ \int_{0}^{\infty} [\Re(|x| - \sqrt{2d})]^{4}(2\pi)^{-1/2} e^{-\frac{1}{2}x^2} \, dx \right] \left[ \int_{\sqrt{2d}}^{\infty} [x - \sqrt{2d}]^{4} e^{-\frac{1}{2}x^2} \, dx \right] 
$$

$$
\geq \frac{1}{4d^2} \left[ \int_{0}^{\infty} [\Re(|x| - \sqrt{2d})]^{4}(2\pi)^{-1/2} e^{-\frac{1}{2}x^2} \, dx \right] \left[ \int_{\sqrt{2d}}^{\infty} (2d)^{1/2} (2d)^{-1/2} e^{-\frac{1}{2}(2d+2+2d^{-1})} \, dx \right] 
$$

$$
= e^{-d} \left[ \frac{\sqrt{2d}(2d+1)}{4d^2(4d^2+6d+1)} \right]^{1/2} e^{-1-\frac{1}{4d}} \geq 50^{-1} d^{-3/2} e^{-d}.
$$

(5.64)

Moreover, observe that the integral transformation theorem and Lemma 3.1 demonstrate that

$$
\int_{\mathbb{R}} [\Re(|x| - \sqrt{2d})]^{4}(2\pi)^{-1/2} e^{-\frac{1}{2}x^2} \, dx
$$

$$
= 2 \left[ \int_{0}^{\infty} [\Re(|x| - \sqrt{2d})]^{4}(2\pi)^{-1/2} e^{-\frac{1}{2}x^2} \, dx \right] \left[ \int_{\sqrt{2d}}^{\infty} [x - \sqrt{2d}]^{4} e^{-\frac{1}{2}x^2} \, dx \right] 
$$

$$
= \frac{2}{\pi} \left[ \int_{0}^{\infty} x^4 e^{-\frac{1}{2}(x+\sqrt{2d})^2} \, dx \right] \left[ \int_{\sqrt{2d}}^{\infty} x^4 e^{-\frac{1}{2}(x^2+2\sqrt{2dx})} \, dx \right] 
$$

$$
\leq \frac{2}{\pi} \left[ \int_{0}^{\infty} x^4 e^{-\frac{1}{2}x^2} \, dx \right] \left[ \frac{4}{\sqrt{\pi}} \right] e^{-d} \left[ \int_{0}^{\infty} x^3 e^{-x} \, dx \right] \left[ \frac{4}{\sqrt{\pi}} \right] e^{-d} \Gamma \left( \frac{5}{2} \right) = 3e^{-d}.
$$

(5.65)

Combining this with (5.63) and (5.64) demonstrates that

$$
50^{-1} d^{-3/2} e^{-d} \leq \int_{\mathbb{R}^d} |g(x)|^2 \varphi(x) \, dx \leq 3d^2 e^{-d}.
$$

(5.66)

The proof of Lemma 5.9 is thus complete. \hfill \Box

5.5 ANN representations for multiplications with powers of real numbers

**Lemma 5.10.** Let $n \in \mathbb{N}$, $\lambda \in \mathbb{R}$, $\Phi, \mathcal{I}, \Psi \in \mathbb{N}$ satisfy $\mathcal{I} = (\lambda \otimes \mathcal{I}_{\lambda\Phi}) \bullet A_{\lambda\Phi_{\lambda\Phi}}$ and $\Psi = (\mathcal{I}_{\lambda\Phi}) \bullet \Phi$ (cf. Definitions 2.2, 2.4, 2.7, 2.8, 2.13, 2.15, and 2.17). Then

(i) it holds that $\mathcal{I} \Psi = \mathcal{I} \Phi$,

(ii) it holds that $\mathcal{H} \Psi = \mathcal{H} \Phi + n$,

(iii) it holds that $\mathcal{P} \Psi \leq 2\mathcal{P} \Phi + 6n|\mathcal{O}|^2$,

(iv) it holds that $\|\mathcal{T} \Psi\|_\infty \leq \max\{1, |\lambda|\} \max\{|\lambda|, \|\mathcal{T} \Phi\|_\infty\}$, and

(v) it holds for all $x \in \mathbb{R}^{\mathcal{I} \Psi}$ that $(\mathcal{R} \Psi)(x) = \lambda^{2n} (\mathcal{R} \Phi)(x)$

(cf. Definitions 2.21 and 2.22).
Proof of Lemma 5.10. Throughout this proof let \( d, l_0, l_1, l_2 \in \mathbb{N} \) satisfy \( l_0 = l_2 = d = O(\Phi) \) and \( l_1 = 2d \), let \( O_n \in \mathbb{R}^n \), \( n \in \mathbb{N} \), satisfy for all \( n \in \mathbb{N} \) that \( O_n = 0 \), and let \( W_k \in \mathbb{R}^{l_k \times l_{k-1}} \), \( k \in \{1, 2\} \), satisfy \( J_d = ((W_1, O_{2d}), (W_2, O_d)) \) (cf. Lemma 2.14). Note that Lemma 2.9, Proposition 2.5, and Lemma 2.18 show that

\[
D(J^*) = (d, 2d, 2d, \ldots, 2d, d) \in \mathbb{N}^{n+2}, \quad \mathcal{H}(J^*) = n, \quad \mathcal{H}(\Phi) = \mathcal{H}(\Phi) + n,
\]

and

\[
D(\Phi) = (\mathcal{D}_0(\Phi), \mathcal{D}_1(\Phi), \ldots, \mathcal{D}_{n}(\Phi), 2d, 2d, \ldots, 2d, d) \in \mathbb{N}^{\mathcal{L}(\Phi) + n+1}.
\]

Therefore, we obtain that

\[
\mathcal{P}(\Phi) = \mathcal{P}(\Phi) + \mathcal{D}_{\mathcal{L}(\Phi)}(\Phi) [\mathcal{D}_{\mathcal{H}(\Phi)}(\Phi) + 1] + 2d(2d + 1) + \ldots + 2d(2d + 1) + d(2d + 1)
\]

\[
\leq 2\mathcal{P}(\Phi) + 6d^2 = 2\mathcal{P}(\Phi) + 6n|\mathcal{O}(\Phi)|^2.
\]

Moreover, observe that (2.3) and the fact that for all \( n \in \mathbb{N} \) it holds that \( \alpha \odot \phi = A_{\alpha \odot \phi} \).

Next note that the fact that \( J_d = ((W_1, O_{2d}), (W_2, O_d)) \), (2.8), (2.11), and (2.12) demonstrate that \( \mathcal{T}((W_1, O_{2d})) = \mathcal{T}((W_2, O_d)) = 1 \) (cf. Definitions 2.21 and 2.22). Combining this with (5.69) establishes that

\[
||\mathcal{T}(\mathcal{J}^*)|| = \begin{cases} |\lambda| : n = 1 \\ |\lambda| \max\{1, |\lambda|\} : n > 1. \end{cases}
\]

Furthermore, observe that the fact that \( J_d = ((W_1, O_{2d}), (W_2, O_d)) \), (2.8), (2.11), and (2.12) show that for all \( k \in \mathbb{N}, \mathfrak{W} \in \mathbb{R}^{d \times k}, \mathfrak{B} \in \mathbb{R}^d \) it holds that

\[
||\mathcal{T}((W_1 \mathfrak{W}, W_1 \mathfrak{W} + O_{2d}))|| = |\lambda| ||\mathcal{T}((\mathfrak{W}, \mathfrak{B}))||.
\]

This, Lemma 2.23, (5.69), and (5.70) establish that

\[
||\mathcal{T}(\Phi)|| = ||\mathcal{T}(\mathcal{J}^*) \odot \Phi|| \leq \max\{|\lambda| \max\{1, |\lambda|\}, ||\mathcal{T}(\Phi)||, |\lambda| ||\mathcal{T}(\Phi)||\}
\]

\[
\leq \max\{|\lambda| \max\{1, |\lambda|\}, ||\mathcal{T}(\Phi)||, |\lambda| ||\mathcal{T}(\Phi)||\}
\]

In addition, note that Proposition 2.5, Lemma 2.14, Lemma 2.16, and Lemma 2.18 demonstrate that for all \( x \in \mathbb{R}^d \) it holds that

\[
(\mathcal{R}(\mathcal{J}))(x) = (\mathcal{R}(\lambda \odot J_d) \odot A_{\lambda \odot \Phi}))(x) = (\mathcal{R}(\lambda \odot J_d))(\mathcal{R}(A_{\lambda \odot \Phi}))(x)
\]

\[
= (\mathcal{R}(\lambda \odot J_d))(\lambda x) = \lambda[(\mathcal{R}(\mathcal{J}_d))(\lambda x)] = \lambda [\lambda x] = \lambda^2 x.
\]

Induction therefore shows that for all \( x \in \mathbb{R}^d \) it holds that \( \mathcal{R}(\mathcal{J}^*) = x = \lambda^{2n} x \). Hence, we obtain that for all \( x \in \mathbb{R}^d \) it holds that

\[
\mathcal{R}(\Phi)(x) = (\mathcal{R}(\mathcal{J}^*) \odot \Phi))(x) = (\mathcal{R}(\mathcal{J}^*) \odot (\mathcal{R}(\Phi)))(x) = \lambda^{2n}(\mathcal{R}(\Phi))(x).
\]

Combining this with (5.67), (5.68), and (5.72) establishes items (i), (ii), (iii), (iv), and (v). The proof of Lemma 5.10 is thus complete.
5.6 ANN approximations for certain specific high-dimensional functions

**Theorem 5.11.** Let $d \in \mathbb{N}$, $M \in \mathbb{N} \cap [2, \infty)$, $R \in [1, \infty)$, let $\varphi: \mathbb{R}^d \to \mathbb{R}$ and $g: \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that $\varphi(x) = (2\pi)^{-d/2} \exp(-\frac{1}{2} \sum_{j=1}^{d} |x_j|^2)$ and $g(x) = \sum_{j=1}^{d} \max\{|x_j| - \sqrt{2d}, 0\}^2$, and let $\mathbf{g}: \mathbb{R}^d \to \mathbb{R}$ satisfy for all $x \in \mathbb{R}^d$ that $\mathbf{g}(x) = \int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) dy^{-1/2} g(x)$. Then there exists $\Phi \in \mathbb{N}$ such that

(i) it holds that $\mathcal{R}(\Phi) \in C(\mathbb{R}^d, \mathbb{R})$,

(ii) it holds that $\mathcal{H}(\Phi) = d + M + 1$,

(iii) it holds that $\mathcal{P}(\Phi) \leq 42d^2 M + 6d$,

(iv) it holds that $\|T(\Phi)\|_{\infty} \leq 12d^{3/2} \max\{4, R^2\}$, and

(v) it holds that $\int_{\mathbb{R}^d} (\mathcal{R}(\Phi))(x) - g(x) |^2 \varphi(x) dx \leq 50d^{7/2} [16^{-M+1} R^4 + 105 R^{-4}]

(cf. Definitions 2.2, 2.21, and 2.22).

**Proof of Theorem 5.11.** Throughout this proof let $\Gamma: (0, \infty) \to (0, \infty)$ satisfy for all $x \in (0, \infty)$ that $\Gamma(x) = \int_{0}^{\infty} t x e^{-t} dt$, let $\psi \in \mathbb{N}$ satisfy that

(I) it holds that $\mathcal{R}(\psi) \in C(\mathbb{R}, \mathbb{R})$,

(II) it holds that $\mathcal{D}(\psi) = (1, 2, 4, \ldots, 4, 1) \in \mathbb{N}^{M+3}$,

(III) it holds that $\|T(\psi)\|_{\infty} \leq (\sqrt{2d} + 1) \max\{4, R^2\}$,

(IV) it holds for all $x \in \mathbb{R}$ that $(\mathcal{R}(\psi))(x) = (\mathcal{R}(\psi))(-x)$,

(V) it holds for all $x \in \mathbb{R}$ with $|x| \leq \sqrt{2d}$ that $|\mathcal{R}(|x| - \sqrt{2d})|^2 - (\mathcal{R}(\psi))(x) = 0$,

(VI) it holds for all $x \in \mathbb{R}$ with $\sqrt{2d} \leq |x| \leq R + \sqrt{2d}$ that

$$\|\mathcal{R}(|x| - \sqrt{2d})^2 - (\mathcal{R}(\psi))(x)\| \leq 4^{-M+1} R^2,$$

(5.75)

and

(VII) it holds for all $x \in \mathbb{R}$ with $|x| \geq R + \sqrt{2d}$ that

$$\|\mathcal{R}(|x| - \sqrt{2d})^2 - (\mathcal{R}(\psi))(x)\| \leq |x| - \sqrt{2d} |^4 R^{-2},$$

(5.76)

(cf. Corollary 5.5), let $\lambda \in \mathbb{R}$ satisfy $\lambda = [\int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) dy]^{-1/(4d)}$, and let $\mathcal{I}, \Psi, \Phi \in \mathbb{N}$ satisfy

$\mathcal{I} = (\lambda \otimes \mathcal{I}_1) \bullet \mathbf{A}_{\lambda, 0}, \Psi = \mathcal{G}_{1,d} \bullet \mathcal{P}_d(\psi, \psi, \ldots, \psi)$, and $\Phi = (\mathcal{I}^*)^\dagger \bullet \Psi$ (cf. Definitions 2.1, 2.2, 2.4, 2.8, 2.10, 2.13, 2.15, 2.17, 2.19, 2.21, and 2.22). Observe that Lemma 5.9 (applied with $d \otimes d, \varphi \otimes \varphi, g \otimes g$ in the notation of Lemma 5.9) implies that

$$0 < \lambda = \left[ \int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) dy \right]^{-1/2} \leq \left[ 50^{-1} d^{-3/2} e^{-d} \right]^{-1/2} = \left[ 50d^{3/2} e^d \right]^{1/2} \leq \left[ 64d^2 e^d \right]^{1/2} \leq [8d^2]^{1/2} \geq [16^d]^{1/2} = [4d]^{1/2} = 4.$$
This and Lemma 5.10 (applied with $n \cap d$, $\lambda \cap \lambda$, $\Phi \cap \Psi$, $\mathscr{I} \cap \mathscr{I}$, $\Psi \cap \Phi$ in the notation of Lemma 5.10) ensure that for all $x \in \mathbb{R}^X(\Phi)$ it holds that

$$
(\mathcal{R}(\Phi))(x) = \lambda^{2d}(\mathcal{R}(\Psi))(x) = \left[ \int_{\mathbb{R}^d} |g(y)|^2\varphi(y)\,dy \right]^{-1/2} (\mathcal{R}(\Psi))(x).
$$

Next note that item (I), Lemma 2.20 (applied with $m \cap 1$, $n \cap d$ in the notation of Lemma 2.20), and Proposition 2.11 (applied with $n \cap d$, $(\Phi_1, \Phi_2, \ldots, \Phi_n) \cap (\psi, \psi, \ldots, \psi)$ in the notation of Proposition 2.11) assure that for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ it holds that $\mathcal{R}(\Psi) \in C(\mathbb{R}^d, \mathbb{R})$ and

$$
(\mathcal{R}(\Psi))(x) = \sum_{j=1}^d (\mathcal{R}(\psi))(x_j).
$$

Combining this with (5.78) establishes item (i). In the next step observe that item (II), Lemma 2.20 (applied with $m \cap 1$, $n \cap d$ in the notation of Lemma 2.20), Proposition 2.12 (applied with $n \cap d$, $(\Phi_1, \Phi_2, \ldots, \Phi_n) \cap (\psi, \psi, \ldots, \psi)$ in the notation of Proposition 2.12), and Proposition 2.5 (applied with $\Phi_1 \cap \mathcal{S}_1.d$, $\Phi_2 \cap \mathcal{P}_d(\psi, \psi, \ldots, \psi)$ in the notation of Proposition 2.5) show that

$$
\mathcal{D}(\mathcal{P}_d(\psi, \ldots, \psi)) = (d, 2d, 4d, \ldots, 4d, d) \in \mathbb{N}^{M+3}
$$

and

$$
\mathcal{D}(\Psi) = (d, 2d, 4d, \ldots, 4d, 1) \in \mathbb{N}^{M+3}.
$$

Therefore, we obtain that $\mathcal{H}(\Psi) = M + 1$ and

$$
\mathcal{P}(\Psi) = 2d(d + 1) + 4d(2d + 1) + 4d(4d + 1) + \ldots + 4d(4d + 1) + (4d + 1)
$$

$$
= 10d^2 + 10d + 1 + (M - 1)(16d^2 + 4d) \leq 21d^2 M.
$$

Combining this with Lemma 5.10 (applied with $n \cap d$, $\lambda \cap \lambda$, $\Phi \cap \Psi$, $\mathscr{I} \cap \mathscr{I}$, $\Psi \cap \Phi$ in the notation of Lemma 5.10) ensures that $\mathcal{H}(\Phi) = \mathcal{H}(\Psi) + d = d + M + 1$ and $\mathcal{P}(\Phi) \leq 2\mathcal{P}(\Psi) + 6d\mathcal{O}(\Psi)^2 \leq 42d^2 M + 6d$. This establishes items (ii) and (iii). Next note that for all $\mathfrak{M} = (w_{i,j})_{(i,j) \in \{1,2,\ldots,d\} \times \{1,2,\ldots,4d\}} \in \mathbb{R}^{d \times 4d} \mathfrak{B} = (b_1, b_2, \ldots, b_d) \in \mathbb{R}^d$ it holds that

$$
(1 \ 1 \ \cdots \ 1) \mathfrak{M} = ([\sum_{i=1}^d w_{i,1}], [\sum_{i=1}^d w_{i,2}], \ldots, [\sum_{i=1}^d w_{i,4d}]) \in \mathbb{R}^{1 \times 4d}
$$

and

$$
(1 \ 1 \ \cdots \ 1) \mathfrak{B} + 0 = [\sum_{i=1}^d b_i] \in \mathbb{R}.
$$

The fact that $\|\mathcal{T}(\mathcal{P}_d(\psi, \psi, \ldots, \psi))\|_{\infty} = \|\mathcal{T}(\psi)\|_{\infty}$ therefore implies that

$$
\|\mathcal{T}(\Psi)\|_{\infty} = \|\mathcal{T}(\mathcal{S}_1.d \bullet \mathcal{P}_d(\psi, \psi, \ldots, \psi))\|_{\infty} \leq d \|\mathcal{T}(\mathcal{P}_d(\psi, \psi, \ldots, \psi))\|_{\infty} = d \|\mathcal{T}(\psi)\|_{\infty}.
$$

Combining this with item (III) assures that

$$
\|\mathcal{T}(\Psi)\|_{\infty} \leq d \|\mathcal{T}(\psi)\|_{\infty} \leq d(\sqrt{2d + 1}) \max\{4, R^2\} \leq 3d^{3/2} \max\{4, R^2\}.
$$
Lemma 5.10 (applied with $n \land d$, $\lambda \land \lambda$, $\Phi \land \Psi$, $\mathcal{J} \land \mathcal{J}$, $\Psi \land \Phi$ in the notation of Lemma 5.10) and (5.77) hence demonstrate that

$$\|T(\Phi)\|_\infty \leq \max\{1, |\lambda|\} \max\{|\lambda|, \|T(\Psi)\|_\infty\} \leq 4 \max\{4, 3d^{3/2}\max\{4, R^2\}\} = 12d^{3/2}\max\{4, R^2\}\}.$$  \hspace{1cm} (5.87)

This establishes item (iv). Moreover, observe that the fact that for all $a_1, a_2, \ldots, a_d \in \mathbb{R}$ it holds that $(a_1 + a_2 + \ldots + a_d)^2 \leq d(|a_1|^2 + |a_2|^2 + \ldots + |a_d|^2)$ and (5.79) ensure that for all $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ it holds that

$$|(\mathcal{R}(\Psi))(x) - g(x)|^2 = \sum_{j=1}^d \left[ (\mathcal{R}(\psi))(x_j) - \Re(|x_j| - \sqrt{2d}) \right]^2 \leq d \sum_{j=1}^d \left[ (\mathcal{R}(\psi))(x_j) - \Re(|x_j| - \sqrt{2d}) \right]^2.$$ \hspace{1cm} (5.88)

Combining this with the fact that for all $k \in \mathbb{N}$ it holds that $\int_{\mathbb{R}^d} (2\pi)^{-d/2} e^{-\frac{1}{2}||x||^2} dx = 1$, Lemma 5.9 (applied with $d \land d$, $\varphi \land \varphi$, $g \land g$ in the notation of Lemma 5.9), and (5.78) implies that

$$\int_{\mathbb{R}^d} |(\mathcal{R}(\Phi))(x) - g(x)|^2 \varphi(x) dx = \left[ \int_{\mathbb{R}^d} |g(y)|^2 \varphi(y) dy \right]^{-1} \int_{\mathbb{R}^d} |(\mathcal{R}(\Psi))(x) - g(x)|^2 \varphi(x) dx \leq 50d^{3/2}e^d \int_{\mathbb{R}^d} |(\mathcal{R}(\Psi))(x) - g(x)|^2 \varphi(x) dx \leq 50d^{3/2}e^d \int_{\mathbb{R}^d} \left[ \sum_{j=1}^d \left[ (\mathcal{R}(\psi))(x_j) - \Re(|x_j| - \sqrt{2d}) \right]^2 \right] \varphi(x_1, x_2, \ldots, x_d) d(x_1, x_2, \ldots, x_d) = 50d^{3/2}e^d \int_{\mathbb{R}^d} \left[ (\mathcal{R}(\psi))(x_1) - \Re(|x_1| - \sqrt{2d}) \right]^2 \varphi(x_1, x_2, \ldots, x_d) d(x_1, x_2, \ldots, x_d) = 50d^{3/2}e^d \int_{\mathbb{R}^d} \left[ (\mathcal{R}(\psi))(x) - \Re(|x| - \sqrt{2d}) \right]^2 e^{-\frac{1}{2}x^2} dx.$$ \hspace{1cm} (5.89)

The integral transformation theorem and items (IV), (V), (VI), and (VII) therefore demonstrate
that
\[
\int_{\mathbb{R}^d}|(\mathcal{Y}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx
\]
\[
\leq 25 \sqrt{\frac{2}{\pi}} \, d^{7/2} e^d \int_{\mathbb{R}} \left[ (\mathcal{Y}(\psi))(x) - \mathcal{R}(|x| - \sqrt{2d}) \right]^2 e^{-\frac{1}{2}x^2} \, dx
\]
\[
= 50 \sqrt{\frac{2}{\pi}} \, d^{7/2} e^d \int_{\mathbb{R}} \left[ (\mathcal{Y}(\psi))(x) - \mathcal{R}(|x| - \sqrt{2d}) \right]^2 e^{-\frac{1}{2}x^2} \, dx
\]
\[
= 50 \sqrt{\frac{2}{\pi}} \, d^{7/2} e^d \int_{\mathbb{R}} \left[ (\mathcal{Y}(\psi))(x) - \mathcal{R}(|x| - \sqrt{2d}) \right]^2 e^{-\frac{1}{2}x^2} \, dx
\]
\[
+ 50 \sqrt{\frac{2}{\pi}} \, d^{7/2} e^d \int_{\mathbb{R}} \left[ (\mathcal{Y}(\psi))(x) - \mathcal{R}(|x| - \sqrt{2d}) \right]^2 e^{-\frac{1}{2}x^2} \, dx
\]
\[
\leq 50 \sqrt{\frac{2}{\pi}} \, d^{7/2} e^d \left[ 4^{-2M-2} R^4 \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} \, dx + R^{-4} \int_{\mathbb{R}} x^2 e^{-\frac{1}{2}(x^2 + 2\sqrt{2d} + 2d)} \, dx \right]
\]
\[
= 50 \sqrt{\frac{2}{\pi}} \, d^{7/2} \left[ 16^{-M-1} R^4 \int_{\mathbb{R}} e^{-\frac{1}{2}(x^2 + 2\sqrt{2d})} \, dx + R^{-4} \int_{\mathbb{R}} x^2 e^{-\frac{1}{2}(x^2 + 2\sqrt{2d})} \, dx \right]
\]
\[
\leq 50 \sqrt{\frac{2}{\pi}} \, d^{7/2} \left[ 16^{-M-1} R^4 \int_{\mathbb{R}} e^{-\frac{1}{2}x^2} \, dx + R^{-4} \int_{\mathbb{R}} x^2 e^{-\frac{1}{2}x^2} \, dx \right].
\]

Next note that the integral transformation theorem and Lemma 3.1 ensure that
\[
\int_0^\infty x^2 e^{-\frac{1}{2}x^2} \, dx = 8\sqrt{2} \int_0^\infty x^2 e^{-x} \, dx = 8\sqrt{2} \Gamma \left( \frac{3}{2} \right)
\]
\[
= 8\sqrt{2} \left[ \frac{7}{2} \left[ \frac{5}{2} \right] \frac{3}{2} \frac{1}{2} \right] \Gamma \left( \frac{1}{2} \right) = 105\sqrt{\pi} / \sqrt{2}.
\]
The fact that \( \int_0^\infty e^{-\frac{1}{2}x^2} \, dx = \sqrt{\pi} / \sqrt{2} \) and (5.90) therefore assure that
\[
\int_{\mathbb{R}^d}|(\mathcal{Y}(\Phi))(x) - g(x)|^2 \varphi(x) \, dx
\]
\[
\leq 50 d^{7/2} \int_{\mathbb{R}^d} \left[ 16^{-M-1} R^4 \int_0^\infty e^{-\frac{1}{2}x^2} \, dx + R^{-4} \int_0^\infty x^2 e^{-\frac{1}{2}x^2} \, dx \right]
\]
\[
= 50 d^{7/2} \left[ 16^{-M-1} R^4 + 105R^{-4} \right].
\]

This establishes item (v). The proof of Theorem 5.11 is thus complete. \( \square \)

**Corollary 5.12.** Let \( \epsilon \in (0, 1] \), \( C \in [1000\epsilon^{-1}, \infty) \), \( c \in [C, \infty) \), \( d \in \mathbb{N} \), let \( \varphi : \mathbb{R}^d \to \mathbb{R} \) and \( g : \mathbb{R}^d \to \mathbb{R} \) satisfy for all \( x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) that \( \varphi(x) = (2\pi)^{-d/2} \exp\left( -\frac{1}{2} \sum_{j=1}^d |x_j|^2 \right) \) and \( g(x) = \sum_{j=1}^d \max\{0, x_j^2\} \), and let \( g : \mathbb{R}^d \to \mathbb{R} \) satisfy for all \( x \in \mathbb{R}^d \) that \( g(x) = \int_{\mathbb{R}} g(y)^2 \varphi(y) \, dy \). Then there exists \( \Phi \in \mathbb{N} \) such that \( I(\Phi) = d \), \( O(\Phi) = 1 \), \( d \leq H(\Phi) \leq cd \), \( \|\varphi\|_\infty \leq c d^3 \), \( P(\Phi) \leq cd^3 \), and \( \|\mathcal{Y}(\Phi)\|_\infty \leq c d^3 \), and \( \|\mathcal{Y}(\Phi)\|_{L^2} \leq c \) (cf. Definitions 2.2, 2.21, and 2.22).

**Proof of Corollary 5.12.** Throughout this proof let \( M \in \mathbb{N} \cap [2, \infty) \), \( R \in [1, \infty) \) satisfy \( M = \max((-\infty, R] \cap \mathbb{N}) \) and \( R = 9d \epsilon^{-1/2} \). Observe that Theorem 5.11 (applied with \( d \land d \), \( M \land M \), \( R \land R \), \( \varphi \land \varphi \), \( g \land g \), \( g \land g \) in the notation of Theorem 5.11) ensures that there exists \( \Phi \in \mathbb{N} \) which satisfies that
(I) it holds that \( \mathcal{R}(\Phi) \in C(\mathbb{R}^d, \mathbb{R}) \),

(II) it holds that \( \mathcal{H}(\Phi) = d + M + 1 \),

(III) it holds that \( \mathcal{P}(\Phi) \leq 42d^2M + 6d \),

(IV) it holds that \( \| \mathcal{T}(\Phi) \|_\infty \leq 12d^{3/2}\max\{4, R^2\} \), and

(V) it holds that \( \int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - g(x)\|^2 \varphi(x) \, dx \leq 50d^{7/2}\left[16^{-M-1}R^4 + 105R^{-4}\right] \)

(cf. Definitions 2.2, 2.21, and 2.22). Therefore, we obtain that \( \mathcal{I}(\Phi) = d, \mathcal{O}(\Phi) = 1, d \leq \mathcal{H}(\Phi) = d + M + 1 \leq d + R + 1 = d + 9e^{-1/2} + 1 \leq 11d^{-1/2} \leq \mathcal{C}d \leq cd, \| \mathcal{T}(\Phi) \|_\infty \leq 12d^{3/2}\max\{4, R^2\} = 972d^{7/2}e^{-1} \leq \mathcal{C}d^c \leq cd^c \), and \( \mathcal{P}(\Phi) \leq 42d^2M + 6d \leq 42d^2R + 6d = 378d^3e^{-1/2} + 6d \leq 384d^3e^{-1/2} \leq \mathcal{C}d^c \leq cd^c \). Moreover, note that the fact that for all \( x \in [4, \infty) \) it holds that \( x^2 \leq 2^x \), the assumption that \( M = \max((-\infty, R] \cap \mathbb{N}) \), and the assumption that \( R = 9d(e^{-1/2}) \) show that \( 16^{-M-1}R^4 + 105R^{-4} \leq 106R^{-4} = 106e^2(9d)^{-4} \leq (50d^{7/2})^{-1}e^2 \). Combining this with item (V) implies that \( \int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - g(x)\|^2 \varphi(x) \, dx \|^{1/2} \leq [50d^{7/2}[16^{-M-1}R^4 + 105R^{-4}]]^{1/4} \leq \varepsilon \). The proof of Corollary 5.12 is thus complete. \( \square \)

6 Lower and upper bounds for the number of ANN parameters in the approximation of high-dimensional functions

In Section 6 we combine the lower bounds for the number of parameters of certain ANNs from Section 4 with the upper bounds for the number of parameters of certain ANNs from Section 5 to establish in Theorem 6.1 in Subsection 6.1 below the main ANN approximation result of this article. Theorem 1.1 in the introduction is a direct consequence of Corollary 6.2 in Subsection 6.2 below. The proof of Corollary 6.2, in turn, is based on an application of Theorem 6.1.

6.1 ANN approximations with specifying the target functions

**Theorem 6.1.** Let \( \varphi_d: \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}, \) and \( f_d: \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}, \) satisfy for all \( d \in \mathbb{N}, x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d \) that \( \varphi_d(x) = (2\pi)^{-d/2}\exp(-\frac{1}{2}(\sum_{j=1}^{d}x_j^2)) \) and \( f_d(x) = \sum_{j=1}^{d}[\max\{|x_j| - \sqrt{2d}, 0\}]^2 \), let \( f_d: \mathbb{R}^d \to \mathbb{R}, d \in \mathbb{N}, \) satisfy for all \( d \in \mathbb{N}, x \in \mathbb{R}^d \) that \( f_d(x) = [\int_{\mathbb{R}^d} f_d(y)^2 \varphi_d(y) \, dy]^{-1/2} f_d(x) \), and let \( \delta \in (0, 1], \varepsilon \in (0, 1/2) \). Then there exists \( \mathcal{C} \in (0, \infty) \) such that

(i) it holds for all \( c \in (\mathcal{C}, \infty), d \in \mathbb{N} \) that

\[
\min \left\{ p \in \mathbb{N}: \begin{cases} \exists \Phi \in \mathbb{N}: p = \mathcal{P}(\Phi), \mathcal{I}(\Phi) = d, \mathcal{O}(\Phi) = 1, \\ d \leq \mathcal{H}(\Phi) \leq cd, \| \mathcal{T}(\Phi) \|_\infty \leq cd^c, \\ [\int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - f_d(x)\|^2 \varphi_d(x) \, dx]^{1/2} \leq \varepsilon \end{cases} \right\} \leq cd^\delta \quad (6.1)
\]

and

(ii) it holds for all \( c \in (\mathcal{C}, \infty), d \in \mathbb{N} \) that

\[
\min \left\{ p \in \mathbb{N}: \begin{cases} \exists \Phi \in \mathbb{N}: p = \mathcal{P}(\Phi), \mathcal{I}(\Phi) = d, \mathcal{O}(\Phi) = 1, \\ \mathcal{H}(\Phi) \leq cd^{1-\delta}, \| \mathcal{T}(\Phi) \|_\infty \leq cd^c, \\ [\int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - f_d(x)\|^2 \varphi_d(x) \, dx]^{1/2} \leq \varepsilon \end{cases} \right\} \geq (1 + c^{-3})(d^\delta) \quad (6.2)
\]

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(cf. Definitions 2.2, 2.21, and 2.22).

Proof of Theorem 6.1. Throughout this proof let $C \in [100(\delta \ln(1.03))^{-2}, \infty) \cap [1000\varepsilon^{-1}, \infty)$, $c \in [C, \infty)$, $d \in \mathbb{N}$ satisfy $2C^{3/5} \leq (1.03)^{7/10}$. Observe that Corollary 5.12 (applied with $\varepsilon \land \varepsilon$, $C \land C$, $d \land d$, $\varphi \land \varphi_d$, $g \land f_d$, $g \land f_\delta$ in the notation of Corollary 5.12) assures that there exists $\Phi \in \mathbb{N}$ such that $I(\Phi) = d$, $O(\Phi) = 1$, $d \leq \mathcal{H}(\Phi) \leq cd$, $\|T(\Phi)\|_\infty \leq cd^3$, and $[\int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - g_d(x)]^2 \varphi_d(x) \, dx]^{1/2} \leq \varepsilon$ (cf. Definitions 2.2, 2.21, and 2.22). This establishes item (i). Moreover, note that Corollary 4.9 (applied with $\varphi_d \land \varphi_d$, $g_d \land f_d$, $g \land \delta$, $\varphi \land \varphi_d$ in the notation of Corollary 4.9) ensures that for all $\Phi \in \mathbb{N}$ with $I(\Phi) = d$, $O(\Phi) = 1$, $\mathcal{H}(\Phi) \leq cd^{1-\delta}$, $\|T(\Phi)\|_\infty \leq cd^3$, and $[\int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - f_d(x)]^2 \varphi_d(x) \, dx]^{1/2} \leq \varepsilon$ it holds that $\mathcal{P}(\Phi) \geq (1 + c^{-3})d^3$. This establishes item (ii). The proof of Theorem 6.1 is thus complete. \hfill \blacksquare

6.2 ANN approximations without specifying the target functions

Corollary 6.2. Let $\varphi_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $\varphi_d(x) = (2\pi)^{-d/2} \exp(-\frac{1}{2}\|x\|^2)$ (cf. Definition 2.21). Then there exist continuously differentiable $f_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, such that for all $\delta \in (0, 1]$, $\varepsilon \in (0, 1/2]$ there exists $C \in (0, \infty)$ such that

(i) it holds for all $c \in [C, \infty)$, $d \in \mathbb{N}$ that

$$\min \left\{ p \in \mathbb{N} : \begin{array}{l} \exists \Phi \in \mathbb{N} : p = \mathcal{P}(\Phi), I(\Phi) = d, O(\Phi) = 1, \\ d \leq \mathcal{H}(\Phi) \leq cd, \|T(\Phi)\|_\infty \leq cd^3, \\ [\int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - f_d(x)]^2 \varphi_d(x) \, dx]^{1/2} \leq \varepsilon \end{array} \right\} \leq cd^3 \quad (6.3)$$

and

(ii) it holds for all $c \in [C, \infty)$, $d \in \mathbb{N}$ that

$$\min \left\{ p \in \mathbb{N} : \begin{array}{l} \exists \Phi \in \mathbb{N} : p = \mathcal{P}(\Phi), I(\Phi) = d, O(\Phi) = 1, \\ \mathcal{H}(\Phi) \leq cd^{1-\delta}, \|T(\Phi)\|_\infty \leq cd^3, \\ [\int_{\mathbb{R}^d}(\mathcal{R}(\Phi))(x) - f_d(x)]^2 \varphi_d(x) \, dx]^{1/2} \leq \varepsilon \end{array} \right\} \leq (1 + c^{-3})d^3 \quad (6.4)$$

(cf. Definitions 2.2 and 2.22).

Proof of Corollary 6.2. Throughout this proof let $f_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $x = (x_1, x_2, \ldots, x_d) \in \mathbb{R}^d$ that $f_d(x) = \sum_{j=1}^d \max\{|x_j| - \sqrt{2d}, 0\}^2$, let $f_d: \mathbb{R}^d \to \mathbb{R}$, $d \in \mathbb{N}$, satisfy for all $d \in \mathbb{N}$, $x \in \mathbb{R}^d$ that $f_d(x) = f_d(x)[\int_{\mathbb{R}^d}|f_d(y)|^2 \varphi_d(y) \, dy]^{-1/2}$, and let $\delta \in (0, 1]$, $\varepsilon \in (0, 1/2]$. Observe that Theorem 6.1 (applied with $(\varphi_d)_{d \in \mathbb{N}} \land (\varphi_d)_{d \in \mathbb{N}}$, $(f_d)_{d \in \mathbb{N}} \land (f_d)_{d \in \mathbb{N}}$, $(f_d)_{d \in \mathbb{N}} \land (f_d)_{d \in \mathbb{N}}$, $\delta \land \delta$, $\varepsilon \land \varepsilon$ in the notation of Theorem 6.1) establishes items (i) and (ii). The proof of Corollary 6.2 is thus complete. \hfill \blacksquare

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