SPIN REPRESENTATIONS OF WEYL GROUPS AND THE SPRINGER CORRESPONDENCE

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Abstract. We give a common framework for the classification of projective spin irreducible representations of a Weyl group, modeled after the Springer correspondence for ordinary representations.

1. Introduction

Let $\Phi = (V, R, V^\vee, R^\vee)$ be a semisimple crystallographic $\mathbb{R}$-root system (see §2) with Weyl group $W$ and a choice of positive roots $R^+$. Assume that $V$ is endowed with a $W$-invariant inner product $\langle \ , \ \rangle$, and define the dual inner product on $V^\vee$, denoted by $\langle \ , \ \rangle$ as well. The Weyl group $W$ is a finite subgroup of $O(V)$ and therefore, one can consider the double cover $\tilde{W}$ of $W$ in $Pin(V)$, a double cover of $O(V)$. A classical problem is to classify the irreducible genuine $\tilde{W}$-representations (i.e., the representations that do not factor through $W$). This is known case by case, and goes back to Schur ([21]), in the case of $\tilde{S}_n$, and was completed about 30 years ago for the other root systems by Morris, Read, Stembridge and others (see [16, 17, 19, 24] and the references therein). In this paper, we attempt to unify these classifications in a common framework, based on Springer theory ([23, 13]) for ordinary $W$-representations. Our point of view is motivated by the construction of the Dirac operator for graded affine Hecke algebras ([1]).

The group $\tilde{W}$ is generated by certain elements $\tilde{s}_\alpha$ of order 4, $\alpha \in R^+$, with relations similar to the Coxeter presentation of $W$ (see [2, 3]). Let $\tilde{\alpha} \in R^\vee$ denote the coroot corresponding to the root $\alpha \in R$. The starting observation is the existence of a remarkable central element $\Omega_{\tilde{W}} \in \mathbb{C}[\tilde{W}]$ (see (1.0.1)):

$$\Omega_{\tilde{W}} = \sum_{\alpha > 0, \beta > 0, s_{\alpha}(\beta) < 0} |\tilde{\alpha}| |\tilde{\beta}| \tilde{s}_\alpha \tilde{s}_\beta; \quad (1.0.1)$$

this element, rather surprisingly, behaves like an analogue of the Casimir element for a Lie algebra. Every irreducible $\tilde{W}$-representation $\tilde{\sigma}$ acts by a scalar $\tilde{\sigma}(\Omega_{\tilde{W}})$ on $\Omega_{\tilde{W}}$. For example, $\tilde{W}$ has one (when $\dim V$ is even) and two (when $\dim V$ is odd) distinguished irreducible representations, which we call spin modules (2.3). If $S$ is one such spin module, then $S(\Omega_{\tilde{W}}) = \langle 2\tilde{\rho}, 2\tilde{\rho} \rangle$, where $\tilde{\rho} = \frac{1}{2} \sum_{\alpha \in R^+} \tilde{\alpha}$.

Before stating the main result, we need to introduce more notation. Let $\mathfrak{g}$ be the complex semisimple Lie algebra with root system $\Phi$ and Cartan subalgebra $\mathfrak{h} = V^\vee \otimes_{\mathbb{R}} \mathbb{C}$, and let $G$ be the simply connected Lie group with Lie algebra $\mathfrak{g}$. Extend the inner product from $V^\vee$ to $\mathfrak{h}$. Let us denote by $\mathcal{T}(G)$ the set of
$G$-conjugacy classes of Jacobson-Morozov triples $(e, h, f)$ in $\mathfrak{g}$. We set:
\[
\mathcal{T}_0(G) = \{ ([e, h, f]) \in \mathcal{T}(G) : \text{the centralizer of } \{e, h, f\} \text{ in } \mathfrak{g} \text{ is a toral subalgebra} \}.
\]
(1.0.2)

For example, if $G = \text{SL}(n)$, the nilpotent elements $e$ that occur in $\mathcal{T}_0(G)$ are those whose Jordan canonical form has all parts distinct. In general, every distinguished nilpotent element $e$ (in the sense of Bala-Carter) appears in $\mathcal{T}_0(G)$.

For every class in $\mathcal{T}(G)$, we may (and will) choose a representative $(e, h, f)$ such that $h \in \mathfrak{h}$. For every nilpotent element $e$, let $A(e)$ denote the $A$-group in $G$, and let $\widetilde{A(e)}_0$ denote the set of representations of $A(e)$ of Springer type. For every $\phi \in \widetilde{A(e)}_0$, let $\sigma_{e, \phi}$ be the associated Springer representation (see §3.2). Normalize the Springer correspondence so that $\sigma_{0, \text{triv}} = \text{sgn}$.

There is an equivalence relation $\sim$ on the space $\widetilde{W}_{\text{gen}}$ of genuine irreducible $\widetilde{W}$-representations: $\tilde{\sigma} \sim \tilde{\sigma} \otimes \text{sgn}$; here, $\text{sgn}$ is the sign $W$-representation.

**Theorem 1.0.1.** There is a surjective map
\[
\Psi : \widetilde{W}_{\text{gen}} \longrightarrow \mathcal{T}_0(G),
\]
(1.0.3)
with the following properties:

1. If $\Psi(\tilde{\sigma}) = ([e, h, f])$, then we have
\[
\tilde{\sigma}(\Omega_W) = (h, h),
\]
where $\Omega_W$ is as in (1.0.1).

2. Let $(e, h, f) \in \mathcal{T}_0(G)$ be given. For every Springer representation $\sigma_{e, \phi}$, $\phi \in \widetilde{A(e)}_0$, and every spin $\widetilde{W}$-module $S$, there exists $\tilde{\sigma} \in \Psi^{-1}([e, h, f])$ such that $\tilde{\sigma}$ appears with nonzero multiplicity in the tensor product $\sigma_{e, \phi} \otimes S$. Conversely, for every $\tilde{\sigma} \in \Psi^{-1}([e, h, f])$, there exists a spin $\widetilde{W}$-module $S$ and a Springer representation $\sigma_{e, \phi}$, such that $\tilde{\sigma}$ is contained in $\sigma_{e, \phi} \otimes S$.

3. If $e$ is distinguished, then properties (1) and (2) above determine a bijection:
\[
\Psi^{-1}([e, h, f]) \sim \longleftrightarrow \{ \sigma_{e, \phi} : \phi \in \widetilde{A(e)}_0 \}.
\]
(1.0.5)

Since $\text{triv}(\Omega_W) = \text{sgn}(\Omega_W)$, Theorem 1.0.1 says in particular that any two associate genuine $W$-types $\tilde{\sigma}_1 \sim \tilde{\sigma}_2$ lie in the same fiber of $\Psi$. This is why we need to quotient by $\sim$ in (1.0.5). It is natural to ask if one could reformulate (2) in the theorem so that a bijection like (1.0.5) (with certain appropriate quotients in the right hand side) holds for non-distinguished $e \in \mathcal{N}_0(\mathfrak{g})$. This is almost always the case, but there are counterexamples, e.g., Remark 3.9.1.

We should make clear that while the main result may appear close to a Springer type classification for $\widetilde{W}$, we do not provide here a geometric construction for genuine representations of $\widetilde{W}$. As we explain in [3], this classification fits in the setting of elliptic representation theory of $W$ (Reeder [20]) and $\widetilde{W}$, and its connection with nilpotent orbits. A different relation, between elliptic conjugacy classes in $W$ and a family of nilpotent orbits (“basic”) is presented in a recent paper by Lusztig [14].

There are two directions in which one can generalize Theorem 1.0.1. Firstly, it is apparent that one can extend these results to the generalized Springer correspondence [13], by using a Casimir element $\Omega_{W,e}$ for an appropriate parameter function $c : R^+ \rightarrow \mathbb{Z}$. We present the details in §3.10.3 which is the exact analogue of Theorem 1.0.1.
Secondly, an analogous correspondence should hold, even in the absence of nilpotent orbits, for non-crystallographic root systems, and more generally, for complex reflection groups. There, one should be able to substitute the nilpotent orbits and Springer representations in the right hand side of the correspondence in Theorem 1.0.1 with the space of elliptic tempered modules (in the sense of [20, 18]) for the corresponding graded Hecke algebra and their “lowest $W$-types”. This problem will be considered elsewhere.

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2. Preliminaries

2.1. Root systems. We fix an $\mathbb{R}$-root system $\Phi = (V, R, V^\vee, R^\vee)$. This means that $V, V^\vee$ are finite dimensional $\mathbb{R}$-vector spaces, with a perfect bilinear pairing $(\; , \; ) : V \times V^\vee \to \mathbb{R}$, $R \subset V \setminus \{0\}$, $R^\vee \subset V^\vee \setminus \{0\}$ are finite subsets in bijection

$$R \leftrightarrow R^\vee, \alpha \leftrightarrow \check{\alpha}, \text{ such that } (\alpha, \check{\alpha}) = 2. \quad (2.1.1)$$

Moreover, the reflections

$$s_\alpha : V \to V, \quad s_\alpha(v) = v - (v, \alpha)\alpha, \quad s_\alpha : V^\vee \to V^\vee, \quad s_\alpha(v') = v' - (\alpha, v')\check{\alpha}, \quad \alpha \in R, \quad (2.1.2)$$

leave $R$ and $R^\vee$ invariant, respectively. Let $W$ be the subgroup of $GL(V)$ (respectively $GL(V^\vee)$) generated by $\{s_\alpha : \alpha \in R\}$.

We will assume that the root system $\Phi$ is reduced, meaning that $\alpha \in R$ implies $2\alpha \notin R$, and crystallographic, meaning that $(\alpha, \check{\alpha}) \in \mathbb{Z}$ for all $\alpha \in R$, $\check{\alpha} \in R^\vee$. We also assume that $R$ generates $V$. We will fix a choice of simple roots $\Pi \subset R$, and consequently, positive roots $R^+ \subset V^+$ and positive coroots $R^\vee_{\text{root}}$. Often, we will write $\alpha > 0$ or $\alpha < 0$ in place of $\alpha \in R^+$ or $\alpha \in (-R^+)$, respectively.

We fix a $W$-invariant inner product $(\; , \; )$ on $V$. Denote also by $(\; , \; )$ the dual inner product on $V^\vee$. If $v$ is a vector in $V$ or $V^\vee$, we denote $|v| := (v, v)^{1/2}$.

2.2. The Clifford algebra. A classical reference for the Clifford algebra is [4] (see also section II.6 in [2]). Denote by $C(V)$ the Clifford algebra defined by $V$ and the inner product $(\; , \; )$. More precisely, $C(V)$ is the quotient of the tensor algebra of $V$ by the ideal generated by

$$\omega \otimes \omega' + \omega' \otimes \omega + 2(\omega, \omega'), \quad \omega, \omega' \in V.$$

Equivalently, $C(V)$ is the associative algebra with unit generated by $V$ with relations:

$$\omega^2 = -2(\omega, \omega), \quad \omega \omega' + \omega' \omega = -2(\omega, \omega'). \quad (2.2.1)$$

Let $O(V)$ denote the group of orthogonal transformation of $V$ with respect to $(\; , \; )$. This acts by algebra automorphisms on $C(V)$, and the action of $-1 \in O(V)$ induces a grading

$$C(V) = C(V)_{\text{even}} + C(V)_{\text{odd}}. \quad (2.2.2)$$

Let $\epsilon$ be the automorphism of $C(V)$ which is $+1$ on $C(V)_{\text{even}}$ and $-1$ on $C(V)_{\text{odd}}$. Let $^t$ be the transpose automorphism of $C(V)$ characterized by

$$\omega^t = -\omega, \quad \omega \in V, \quad (ab)^t = b^t a^t, \quad a, b \in C(V). \quad (2.2.3)$$
The Pin group is
\[ \text{Pin}(V) = \{ a \in C(V) : \epsilon(a)V a^{-1} \subset V, \ a^t = a^{-1} \}. \] (2.2.4)

It sits in a short exact sequence
\[ 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \text{Pin}(V) \longrightarrow \mathcal{O}(V) \longrightarrow 1, \] (2.2.5)
where the projection \( p \) is given by \( p(a)(\omega) = \epsilon(a)\omega a^{-1} \).

2.3. The spin modules \( S \). If \( \dim V \) is even, the Clifford algebra \( C(V) \) has a unique (up to equivalence) complex simple module \( (\gamma, S) \) of \( C(V) \), of dimension \( 2^{\dim V/2} \), endowed with a positive definite Hermitian form \( \langle \cdot, \cdot \rangle_S \) such that
\[ \langle \gamma(a)s, s' \rangle_S = \langle s, \gamma(a^t)s' \rangle_S, \quad \text{for all } a \in C(V) \text{ and } s, s' \in S. \] (2.3.1)

When \( \dim V \) is odd, there are two simple inequivalent unitary modules \( (\gamma_+, S^+) \), \( (\gamma_-, S^-) \) of dimension \( 2^{[\dim V/2]} \). In order to simplify the formulation of the results, we will often refer to any one of \( S \), \( S^+ \), \( S^- \), as a spin module. When there is no possibility of confusion, we may also denote by \( S \) any one of \( S^+ \) or \( S^- \), in order to state results in a uniform way.

Via \( \langle 2.2.4 \rangle \), a spin module \( S \) is an irreducible unitary \( \text{Pin}(V) \) representation. It is well-known (e.g., section II.6 in [2]) that as \( \text{Pin}(V) \)-representations, we have:
\[ S \otimes S \cong \bigwedge V, \text{ when } \dim V \text{ is even, } S \otimes S \cong \bigoplus_{i=0}^{[\dim V/2]} 2i \bigwedge V, \text{ when } \dim V \text{ is odd.} \] (2.3.2)

2.4. The spin cover \( \widetilde{W} \). The Weyl group \( W \) acts by orthogonal transformations on \( V \), so one can embed \( W \) as a subgroup of \( \mathcal{O}(V) \). We define the group \( \widetilde{W} \) in \( \text{Pin}(V) \):
\[ \widetilde{W} := p^{-1}(\mathcal{O}(V)) \subset \text{Pin}(V), \text{ where } p \text{ is as in } \langle 2.2.4 \rangle. \] (2.4.1)
Therefore, \( \widetilde{W} \) is a central extension of \( W \):
\[ 1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow \widetilde{W} \longrightarrow \mathcal{O}(V) \longrightarrow 1. \] (2.4.2)

The group \( \widetilde{W} \) has a Coxeter presentation similar to that of \( W \). Recall that as a Coxeter group, \( W \) has a presentation:
\[ W = \langle s_\alpha, \alpha \in \Pi | s_\alpha^2 = 1, (s_\alpha s_\beta)^m(\alpha, \beta) = 1, \alpha \neq \beta \in \Pi \rangle, \] (2.4.3)
for certain positive integers \( m(\alpha, \beta) \). Theorem 3.2 in [10] gives:
\[ \widetilde{W} = \langle z, \bar{s}_\alpha, \alpha \in \Pi | z^2 = 1, \bar{s}_\alpha^2 = z, (\bar{s}_\alpha \bar{s}_\beta)^m(\alpha, \beta) = 1, \alpha \neq \beta \in \Pi \rangle. \] (2.4.4)

We will also need the explicit embedding of \( \widetilde{W} \) into \( \text{Pin}(V) \).

**Theorem 2.4.1** ([10] Theorem 3.2). *The assignment*
\[ \tau(z) = -1 \]
\[ \tau(s_\alpha) = f_\alpha := \alpha/|\alpha|, \quad \alpha \in \Pi, \] (2.4.5)
*extends to a group homomorphism* \( \tau : \widetilde{W} \rightarrow \text{Pin}(V) \). *Moreover, we have* \( \tau(\bar{s}_\beta) = f_\beta := \beta/|\beta|, \text{ for all } \beta \in R^+ \).
Definition 2.4.2. We call a representation \( \hat{\sigma} \) of \( \hat{W} \) genuine (resp. non-genuine) if \( \hat{\sigma}(z) = -1 \) (resp. \( \hat{\sigma}(z) = 1 \)). The non-genuine \( \hat{W} \)-representations are the ones that factor through \( W \).

We say that two genuine \( \hat{W} \)-types \( \sigma_1, \sigma_2 \) are associate if \( \sigma_1 \cong \sigma_2 \otimes \text{sgn} \).

2.5. Via the embedding \( \tau \), we can regard \( S \) if \( \text{dim} V \) is even (resp. \( S^\pm \) if \( \text{dim} V \) is odd) as unitary (genuine) \( \hat{W} \)-representations. Since \( R \) spans \( V \), they are irreducible representations ([16, Theorem 3.3]). When \( \text{dim} V \) is odd, \( S^+ \) and \( S^- \) associate in the sense of Definition 2.4.2 while if \( \text{dim} V \) is even, \( S \) is self-associate.

Let \( S \) be a spin \( W \)-module and \( (\sigma, U) \) a \( W \)-type. From (2.3.2), we see that \( \sigma \otimes S \) contains a spin \( \hat{W} \)-module if and only if \( \sigma \) appears as a constituent of \( \wedge^i V \). Moreover, it is known ([11, Theorem 5.1.4]), that if \( W \) is irreducible, then \( \wedge^i V \), \( 0 \leq i \leq \text{dim} V \), forms a set of irreducible, pairwise inequivalent \( W \)-representations.

2.6. The Casimir element of \( \hat{W} \). The notions in this subsection are motivated by the results of [11], where the element \( \Omega_{\hat{W}} \) that we define here appeared naturally in the context of the Dirac operator for the graded affine Hecke algebra.

Let \( c : R^+ \to \mathbb{R} \) be a \( W \)-invariant function.

Definition 2.6.1. Denote

\[
\Omega_{\hat{W},c} = \sum_{\alpha > 0, \beta > 0} c(\alpha)c(\beta)|\hat{\alpha}||\hat{\beta}| s_{\alpha}s_{\beta} = \sum_{\alpha > 0, \beta > 0} \frac{\langle \hat{\alpha}, \hat{\beta} \rangle}{\cos(\alpha, \beta)} c(\alpha)c(\beta) s_{\alpha}s_{\beta}. \tag{2.6.1}
\]

The equality holds because the contributions in the second sum of the pairs \( \{\alpha, \beta\} \) and \( \{s_\alpha(\beta), \alpha\} \) cancel out, whenever \( s_\alpha(\beta) > 0 \). If \( c \equiv 1 \), we write \( \Omega_{\hat{W}} \) for \( \Omega_{\hat{W},1} \).

If \( C_w \) is the \( W \)-conjugacy class of \( w \in W \), then there are two possibilities for \( p^{-1}(C_w) \subset \hat{W} \):

1. \( p^{-1}(C_w) \) is a single \( \hat{W} \)-conjugacy class, or
2. \( p^{-1}(C_w) \) splits into two conjugacy \( \hat{W} \)-classes \( \tilde{C}_w := \{w' : w' \in C_w\} \) and \( z\tilde{C}_w := \{zw' : w' \in C_w\} \).

One sees that if \( w = s_\alpha s_\beta \), then the second case holds ([16]). This implies that we have

\[
\Omega_{\hat{W},c} \in \mathbb{C}[\hat{W}]. \tag{2.6.2}
\]

In particular, every \( \hat{\sigma} \in \hat{W} \) acts on \( \Omega_{\hat{W},c} \) by a scalar, which we denote \( \hat{\sigma}(\Omega_{\hat{W},c}) \).

We will refer to \( \Omega_{\hat{W}} \) as the Casimir element of \( \hat{W} \). The justification for the name is given by Theorem 1.0.1(1). As a hint towards this result, let us recall (e.g., [16 p. 562]) that

\[
\text{tr}_S(s_\alpha s_\beta) = |\cos(\alpha, \beta)| \dim S, \quad \alpha, \beta \in R^+, \tag{2.6.3}
\]

for a spin module \( S \). This means that we have

\[
S(\Omega_{\hat{W}}) = \sum_{\alpha > 0, \beta > 0} \langle \hat{\alpha}, \hat{\beta} \rangle = \langle 2\hat{\rho}, 2\hat{\rho} \rangle, \tag{2.6.4}
\]

where \( \hat{\rho} = \frac{1}{2} \sum_{\alpha > 0} \hat{\alpha} \).
3. $\tilde{W}$-types

In this section, we prove our main results, Theorems 1.0.1 and 3.10.3. Before that, we recall certain elements from the theory of elliptic representations of a finite group. While these elements are not necessary for proving Theorems 1.0.1 and 3.10.3, they are useful for setting our result in the appropriate context.

For a finite group $\Gamma$, let $R(\Gamma)$ denote the representation theory ring of $\Gamma$, and let $\hat{\Gamma}$ denote the set of irreducible representations of $\Gamma$.

3.1. Elliptic representations of a finite group. The reference for most of the results in this and the next subsection is [20]. Assume first that $\Gamma$ is an arbitrary finite subgroup of $GL(V)$. An element $\gamma \in \Gamma$ is called elliptic (or anisotropic) if $V^\gamma = 0$. Let $\Gamma_{\text{ell}}$ denote the set of elliptic elements in $\Gamma$. This is closed under conjugation by $\Gamma$.

Let $L$ be the set of subgroups $L \subseteq \Gamma$, such that $V^L \neq 0$. For every $L \in L$, let $\text{Ind}_L^{\Gamma} : R(L) \to R(\Gamma)$ be the induction map, and denote

$$R_{\text{ind}}(\Gamma) = \sum_{L \in L} \text{Ind}_L^{\Gamma}(R(L)) \subseteq R(\Gamma),$$

$$R(\Gamma) = R(\Gamma)/R_{\text{ind}}(\Gamma).$$

(3.1.1) (3.1.2)

One calls $R(\Gamma)$ the space of elliptic representations of $\Gamma$.

Define a bilinear pairing, called the elliptic pairing on $\Gamma$:

$$e(\sigma, \sigma') = \sum_{i \geq 0} (-1)^i \dim \text{Hom}_{\Gamma}(i \bigwedge V \otimes \sigma, \sigma'), \quad \sigma, \sigma' \in R(\Gamma).$$

(3.1.3)

Proposition 2.2.2 in [20] shows, in particular, that the radical of $e$ is precisely $R_{\text{ind}}(\Gamma)$, and thus $e$ induces a nondegenerate bilinear form on $R(\Gamma)$. Moreover, if $C_{\text{ell}}(\Gamma)$ denotes the set of $\Gamma$-conjugacy classes in $\Gamma_{\text{ell}}$, we have

$$\dim R(\Gamma) = |C_{\text{ell}}(\Gamma)|.$$

(3.1.4)

Lemma 3.1.1. If $\sigma \in R(\Gamma)$, then $\sigma \otimes \bigwedge^{\dim V} V - (-1)^{\dim V} \sigma$ is in $R_{\text{ind}}(\Gamma)$.

Proof. We have $\bigwedge^i V \otimes \bigwedge^{\dim V} V \cong \bigwedge^{\dim V - i} V$, as $\Gamma$-representations, for all $0 \leq i \leq \dim V$. From this, it follows that $e(\sigma \otimes \bigwedge^{\dim V} V, \sigma') = (-1)^{\dim V} e(\sigma, \sigma')$, for all $\sigma, \sigma'$.

(3.2.1)

Let $R_{\text{red}}(\Gamma)$ denote the quotient of $R(\Gamma)$ by the subspace generated by $\sigma \otimes \bigwedge^{\dim V} V - (-1)^{\dim V} \sigma$, for all $\Gamma$-types $\sigma$. Lemma 3.1.1 implies that the natural (surjective) map

$$R_{\text{red}}(\Gamma) \to R(\Gamma)$$

(3.1.5)

is well-defined and preserves $e$.

3.2. Elliptic representations of $W$. We specialize to $\Gamma = W$ here acting on the reflection representation $V$. [20] analyzes the relation between $\overline{R}(W)$ and Springer representations.

Let $\mathfrak{g}$ be the complex Lie algebra determined by the root system $\Phi$, and let $G$ be the simply connected connected Lie group with Lie algebra $\mathfrak{g}$. For every $x \in \mathfrak{g}$, let $Z_G(x)$ denote the centralizer of $x$ in $G$, and let $Z_G(x)^0$ be the identity component. Define the A-group of $x$ in $G$ to be the quotient

$$A(x) = Z_G(x)/Z_G(x)^0 Z(G),$$

(3.2.1)
where $Z(G)$ is the center of $G$.

Specialize to the case when $x \in \mathcal{N}(\mathfrak{g})$, the set of nilpotent elements of $\mathfrak{g}$. By the Jacobson-Morozov theorem, there exists a Lie algebra homomorphism $\kappa : \mathfrak{sl}(2, \mathbb{C}) \to \mathfrak{g}$ such that $\kappa \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} = e$. Let $\mathfrak{s}_0$ be a the semisimple part of the Lie algebra of a maximal torus in $Z_G(\kappa(\mathfrak{sl}(2, \mathbb{C})))$. As explained in §3.2 of [20], the group $A(e)$ acts naturally on $\mathfrak{s}_0$, i.e., $A(e) \subset \mathbb{G}(\mathfrak{s}_0)$, and therefore we may define $\mathcal{R}(A(e))$ with respect to this action.

**Definition 3.2.1.** An element $e \in \mathcal{N}(\mathfrak{g})$ is called distinguished if $Z_G(e)$ contains no nontrivial torus. An element $e \in \mathcal{N}(\mathfrak{g})$ is called quasi-distinguished if there exists a semisimple element $t \in Z_G(e)$ such that $t \exp(e)$ centralizes no nontrivial torus in $G$. In particular, every distinguished $e$ is quasi-distinguished with $t = 1$.

Corollary 3.2.3 in [20] shows that $\mathcal{R}(A(e)) \neq \emptyset$ if and only if $e$ is quasi-distinguished in $\mathfrak{g}$.

From Springer theory, recall $\mathcal{B}_e$, the variety of Borel subalgebras of $\mathfrak{g}$ containing $e$. Let $d_e$ denote the complex dimension of $\mathcal{B}_e$. Since $Z_G(e)$ acts on $\mathcal{B}_e$, we get an action of $A(e)$ on the cohomology $H^\bullet(\mathcal{B}_e)$. Let $\widehat{A}(e)_0$ be the set of irreducible $A(e)$-representations that appear in this action, and let $\mathcal{R}_0(A(e))$ be the subspace of $\mathcal{R}(A(e))$ spanned by $\widehat{A}(e)_0$.

The Springer correspondence constructs an action of $W$ on $H^\bullet(\mathcal{B}_e)$ which commutes with the action of $A(e)$, and gives a map:
\[
G \backslash \{ (e, \phi) : e \in \mathcal{N}(\mathfrak{g}), \phi \in \widehat{A}(e)_0 \} \longrightarrow \widehat{W}, \quad (e, \phi) \mapsto \sigma_{(e, \phi)} := \text{Hom}_{A(e)}[\phi, H^{2d_e}(\mathcal{B}_e)],
\]
which is well-defined and bijective.

For every $\phi \in \widehat{A}(e)_0$, set
\[
H_e(\phi) := \text{Hom}_{A(e)}[\phi, H^\bullet(\mathcal{B}_e)],
\]
and regard it as an element of $\mathcal{R}(W)$. Let $\mathcal{R}_e(W)$ be the span in $\mathcal{R}(W)$ of $H_e(\phi)$, $\phi \in \widehat{A}(e)_0$. The space $\{ H_e(\phi) : e, \phi \}$ is basis of $\mathcal{R}(W)$, and therefore, we have a decomposition $\mathcal{R}(W) = \sum_e \mathcal{R}_e(W)$, which induces a decomposition $\widehat{\mathcal{R}}(W) = \sum_e \widehat{\mathcal{R}}_e(W)$.

**Theorem 3.2.2** ([20]). The map $H_e : \mathcal{R}_0(A(e)) \rightarrow \mathcal{R}_e(W)$ induces a vector space isomorphism $\overline{H}_e : \mathcal{R}_0(A(e)) \rightarrow \overline{\mathcal{R}}_e(W)$. Moreover, we have:

1. The isomorphism $\overline{H}_e$ is an isometry with respect to the elliptic pairings $e_W$ and $e_{A(e)}$;
2. The spaces $\mathcal{R}_0(A(e))$ and $\mathcal{R}_e(W)$ are nonzero if and only if $e$ is quasi-distinguished;
3. If $e$ is distinguished, then $\{ H_e(\phi) : \phi \in \widehat{A}(e)_0 \}$ is an orthonormal basis of $\overline{\mathcal{R}}_e(W)$ with respect to $e_W$.

3.3. **Elliptic representations of $\widehat{W}$**. We specialize now to $\Gamma = \widehat{W}$ acting also on the (nongenuine) reflection representation $V$. Let $\mathcal{R}_{\text{gen}}(\widehat{W})$ denote the subspace of $\mathcal{R}(\widehat{W})$ spanned by $\widehat{W}_{\text{gen}}$, the irreducible genuine $\widehat{W}$-types. Every nongenuine $\widehat{W}$-type is a pullback of a $W$-type, so we may regard $\mathcal{R}(W)$ naturally as a subspace...
of $\mathcal{R}(\tilde{W})$. Clearly, we have
\[ e_{\tilde{W}}(\bar{\sigma}, \sigma') = 0, \text{ if } \bar{\sigma} \in \mathcal{R}_{\text{gen}}(\tilde{W}), \sigma' \in \mathcal{R}(W), \tag{3.3.1} \]
therefore we have an orthogonal decomposition $\mathcal{R}(\tilde{W}) = \mathcal{R}_{\text{gen}}(\tilde{W}) \oplus \mathcal{R}(W)$. As before, define $\mathcal{R}(\tilde{W})$ to be the quotient by the radical of $e_{\tilde{W}}$, and let $\mathcal{R}_{\text{gen}}(\tilde{W})$ be the image of $\mathcal{R}_{\text{gen}}(W)$ in $\mathcal{R}(W)$. Consequently, there is an orthogonal decomposition
\[ \mathcal{R}(\tilde{W}) = \mathcal{R}_{\text{gen}}(\tilde{W}) \oplus \mathcal{R}(W). \tag{3.3.2} \]
From [8,3.1.3], we have $\dim \mathcal{R}_{\text{gen}}(W) = |\mathcal{C}_{\text{ell}}(W)| - |\mathcal{C}_{\text{ell}}(W)|$. Recall the projection $p : \tilde{W} \to W$ from (2.2.5). Since $V$ is a nongenuine representation of $W$, an element $\tilde{w} \in \tilde{W}$ is elliptic if and only if $p(\tilde{w}) \in W$ is elliptic. Recall that if $C_w \subset W$ is a conjugacy class, then $p^{-1}(C_w)$ is a single conjugacy class in $\tilde{W}$ or it splits into two conjugacy classes in $\tilde{W}$. Let $\mathcal{C}_{\text{0}}(W)$ denote the set of conjugacy classes of $W$ which split in $\tilde{W}$ and set $\mathcal{C}_{\text{0}}(W) = \mathcal{C}_{\text{0}}(W) \cap \mathcal{C}_{\text{ell}}(W)$. Then we have
\[ \dim \mathcal{R}_{\text{gen}}(\tilde{W}) = |\mathcal{C}_{\text{0}}(W)| \leq |\mathcal{C}_{\text{ell}}(W)| = \dim \mathcal{R}(W). \tag{3.3.3} \]
The dimension of the second space is as follows, see [5,20 §3.1]:
\[ A_n - 1 : 1, \quad B_n : \text{the number of partitions of } n, \]
\[ D_n : \text{the number of partitions of } n \text{ with even number of parts}, \quad \text{dim } C_2 = 3, \quad F_4 : 9, \quad E_6 : 5, \quad E_7 : 12, \quad E_8 : 30. \tag{3.3.4} \]
Since $\bar{\sigma}(z) = -1$ for every genuine $\tilde{W}$-type $\bar{\sigma}$, it is clear that if $C_w \notin \mathcal{C}_{\text{0}}(W)$, then $\tr_{\tilde{W}}(\tilde{w}) = 0$, for all $\tilde{w} \in p^{-1}(C_w)$ and all genuine $\tilde{W}$-types $\bar{\sigma}$.

**Lemma 3.3.1.** Let $S$ be a spin module for $\tilde{W}$.

1. If $\dim V$ is even, then $\tr_{\tilde{S}}(\tilde{w}) \neq 0$ if and only if $\det_V(1 + p(\tilde{w})) \neq 0$.
2. If $\dim V$ is odd, then $\tr_{\tilde{S}}(\tilde{w}) \neq 0$ if $\det_V(1 - p(\tilde{w})) \neq 0$ (i.e., if $\tilde{w}$ is elliptic).

In particular, $\mathcal{C}_{\text{ell}}^0(W) = \mathcal{C}_{\text{ell}}(W)$ in this case.

**Proof.** If $\dim V$ is even, and $S$ is the spin module, we see by (2.3.2) that $\tr_{\tilde{S}}(\tilde{w})^2 = \tr_{\Lambda^0 V}(p(\tilde{w})) = \det_V(1 + p(\tilde{w}))$, and this proves (1). If $\dim V$ is odd, and $S^+, S^-$ are the two spin modules, $\det_V(1 - p(\tilde{w})) \neq 0$ implies $\tr_{S^+}(\tilde{w}) \neq 0$, and this proves (2).

**Remark 3.3.2.** In type $A_n - 1$, there is a single elliptic conjugacy class, consisting of the $n$-cycles in $S_n$, and it is easy to check directly that it splits in $\tilde{S}_n$. Theorem 4.1 and Lemma 6.4 in [10] for $B_n$ and $D_n$ respectively, and sections 6–9 in [17] for the exceptional groups, show that if $\dim V$ is even, the split elliptic conjugacy classes are precisely the ones on which $S$ does not vanish. In terms of the classification of elliptic elements of $W$ from Carter [5], when $\dim V$ is even, the set $\mathcal{C}_{\text{ell}}^0(W) \neq \mathcal{C}_{\text{ell}}(W)$ is explicitly as follows:

1. In type $B_{2n}$, the elliptic conjugacy classes corresponding to partitions of $2n$ with only even parts;
2. In type $D_{2n}$, the elliptic conjugacy classes corresponding to partitions of $2n$ with only even parts, or partitions with only odd parts and multiplicity one;
(3) in $G_2$: \{$A_2, G_2$\};
(4) in $F_4$: \{$A_2 + A_2, D_4(a_1), B_4, F_4(a_1)$\};
(5) in $E_6$: \{$3A_2, E_6, E_6(a_1), E_6(a_3)$\};
(6) in $E_8$: \{$A_8, 2A_1, 4A_2, D_8(a_1), D_8(a_2), 2D_4(a_1), E_6(a_2) + A_2, E_6 + A_2, E_8,$
$E_8(a_1), E_8(a_2), E_8(a_3), E_8(a_4), E_8(a_5), E_8(a_6), E_8(a_7), E_8(a_8)$\}.

Let $S$ denote a spin $\tilde{W}$-module. One can consider the linear map:

$$\iota_S : \mathcal{R}(W) \to \mathcal{R}_{\text{gen}}(\tilde{W}), \quad \iota_S(\sigma) = \sigma \otimes S.$$  \hfill (3.3.5)

Since the genuine $\tilde{W}$-types are determined by their values on $p^{-1}(C^0(W))$, the
map $\iota_S$ is surjective if and only if $\iota_S$ does not vanish on any conjugacy class in
$p^{-1}(C^0(W))$. Using Lemma 3.3.3 and Remark 3.3.2 again, we see that:

**Lemma 3.3.3.** The map $\iota_S$ is surjective if and only if $W$ is of type $B_n$, $D_n$, $G_2$,
$F_4$, or $E_8$.

Since $\mathcal{R}_{\text{ind}}(W)$ (resp. $\mathcal{R}_{\text{gen,ind}}(\tilde{W})$) can be identified with the vector subspace of
virtual characters that vanish on $W_{\text{ell}}$ (resp. $\tilde{W}_{\text{ell}}$), we have

$$\iota_S(\mathcal{R}_{\text{ind}}(W)) \subseteq \mathcal{R}_{\text{gen,ind}}(\tilde{W}),$$
and so $\iota_S$ gives a linear map

$$\tau_S : \mathcal{R}(W) \to \mathcal{R}_{\text{gen}}(\tilde{W}).$$ \hfill (3.3.6)

**Proposition 3.3.4.** The map $\tau_S$ is surjective.

**Proof.** By Lemma 3.3.3 this is true when $\tilde{W}$ is of type $B_n, D_n, G_2, F_4, E_8$, since $\iota_S$
is surjective. The conclusion is implied if $\iota_S$ is nonzero on every conjugacy class in
$p^{-1}(C^0(W))$: this turns out to be the case for every irreducible $W$, by Lemma
3.3.3 and Remark 3.3.2. \hfill $\Box$

**Corollary 3.3.5.** The map $\tau_S$ is an isomorphism if and only if $\dim V$ is odd or $W$
is of type $A$.

**Proof.** This is immediate from Proposition 3.3.4 by comparing the dimension of the
two spaces as in \hfill $\Box$

3.4. Applying (3.1.5) to this setting, we get a surjective linear map $\mathcal{R}^{\text{red}}(\tilde{W}) \to$
$\mathcal{R}_{\text{gen}}(\tilde{W})$ which preserves the elliptic pairing. Thus we have constructed two maps:

$$\mathcal{R}(W) \to \mathcal{R}_{\text{gen}}(\tilde{W}) \leftarrow \mathcal{R}^{\text{red}}(\tilde{W}).$$ \hfill (3.4.1)

Via

$$\xi : \mathcal{R}_{\text{gen}}(\tilde{W}) \to \mathcal{R}_{\text{gen}}(\tilde{W}), \quad \sigma \mapsto \tilde{\sigma} \otimes \text{sgn} + (-1)^{\dim V} \tilde{\sigma},$$ \hfill (3.4.2)

we may identify $\mathcal{R}^{\text{red}}(\tilde{W})$ with the image of $\xi$, i.e., with the subspace of $\mathcal{R}_{\text{gen}}(\tilde{W})$
spanned by \{\$\tilde{\sigma} \otimes \text{sgn} + (-1)^{\dim V} \tilde{\sigma} \neq 0 : \tilde{\sigma} \in \tilde{W}/\sim$\} (here we think of $\tilde{W}/\sim$ as a system
of representatives for the symmetry classes). This shows that $\dim \mathcal{R}^{\text{red}}(\tilde{W}) =$
$|\tilde{W}/\sim|$ when $\dim V$ is even, and $|\tilde{W}/\sim| - |\{\tilde{\sigma} \in \tilde{W} : \tilde{\sigma} \otimes \text{sgn} \cong \tilde{\sigma}\}|$ when $\dim V$ is
odd. From [17, 19, 21], we see that the dimension of $|\tilde{W}/\sim|$ equals:

\[ A_n : \text{the number of partitions of } n \text{ into distinct parts}, \]
\[ B_n : \text{the number of partitions of } n, \]
\[ D_n : \text{the number of equivalence classes of partitions of } n \text{ under transposition}, \]
\[ G_2 : 3, \quad F_4 : 9, \quad E_6 : 6, \quad E_7 : 13, \quad E_8 : 30. \]

(3.4.3)

In addition, in types $B_{2n+1}, D_{2n+1}, E_7$, there are no self-associate $\tilde{W}$-types. In type $A_{2n−1}$, the number of self-associate $\tilde{S}_{2n}$-types equals the number of partitions of odd length of $2n$ into distinct parts. Comparing (3.4.3) with (3.3.4), we see that if $W$ is of type $B_n$, $G_2, F_4$, or $E_8$, then we have $\dim R(W) = \dim R^\text{red}(\tilde{W})$. (If the type is $B_{2n+1}$, then the maps in (3.4.1) are both isomorphisms.)

**Remark 3.4.1.** One can ask if there is a natural linear map $R(W) \to R^\text{red}(\tilde{W})$, having good properties with respect to the elliptic pairing. When $\dim V$ is odd, it is easy to check that such a map is $\sigma \mapsto \frac{1}{\sqrt{2}} \sigma \otimes (S^+ − S^-)$, and that this map is an injective metric with respect to the elliptic pairing on $R(W)$ and the standard pairing on $R^\text{gen}(\tilde{W})$. When $\dim V$ is even, a similar construction exists, but instead of $W$, one needs to consider $\tilde{W}_\text{even} = \{ \tilde{w} \in \tilde{W} : \text{sgn}(\tilde{w}) = 1 \}$. This fits naturally with the theory of the Dirac index for graded Hecke algebras and it is considered in [10].

3.5. **The classification of $\tilde{W}$-types.** The rest of the section is dedicated to the proof of Theorem 1.0.1.

By the Jacobson-Morozov theorem, we know that there is a one-to-one correspondence between $G$-orbits of nilpotent elements in $\mathfrak{g}$, and the set $T(G)$ of $G$-conjugacy classes of Lie triples in $\mathfrak{g}$:

\[ (e, h, f) : [h, e] = 2e, [h, f] = −2f, [e, f] = h. \]

(3.5.1)

**Definition 3.5.1.** Let $N_0(\mathfrak{g})$ denote the set of all nilpotent elements $e$ whose centralizer in $\mathfrak{g}$ is a solvable subalgebra. Let $\mathcal{T}_0(G)$ denote the set of $G$-conjugacy classes of triples $(e, h, f)$ such that $e \in N_0(\mathfrak{g})$.

This definition of $\mathcal{T}_0(G)$ agrees with the one from the introduction by Proposition 2.4 in [3]. Every quasi-distinguished nilpotent element is in $N_0(\mathfrak{g})$, but in types $A, D, E_6$, not all $e \in N_0(\mathfrak{g})$ are quasi-distinguished in the sense of Definition 3.2.1.

For example, in $sl(n)$, the only quasi-distinguished nilpotent orbit is the regular orbit, but $N_0(\mathfrak{g})$ contains every orbit whose Jordan form has all blocks of distinct sizes.

3.6. **Type A.** We begin the proof of Theorem 1.0.1. This is a case-by-case verification, combinatorially for classical root systems, and a direct computation for exceptional.

The starting point is type $A_{n−1}$, $\tilde{W}_n = \tilde{S}_n$. Let $P(n)$ be the set of all partitions of $n$, and let $DP(n)$ be the set of distinct partitions. If $\lambda = (\lambda_1, \lambda_2, \ldots, \lambda_m)$ is a partition of $n$, written in decreasing order, we denote the length of $\lambda$ by $\ell(\lambda) = m$. We say that $\lambda$ is even (resp. odd) if $n − \ell(\lambda)$ is even (resp. odd).

It is well-known that every partition $\lambda$ parameterizes a unique $S_n$-type $\sigma_\lambda$, and this gives a one-to-one correspondence between $P(n)$ and $\tilde{S}_n$. The first part of Theorem 1.0.1 for $\tilde{S}_n$ is a classical result of Schur.
Theorem 3.6.1 ([21]). There exists a one-to-one correspondence
\[ \tilde{S}_n/\sim \longleftrightarrow DP(n). \]

For every even \( \lambda \in DP(n) \), there exists a unique \( \tilde{\sigma}_\lambda \in \tilde{S}_n \), and for every odd \( \lambda \in DP(n) \), there exist two associate \( \tilde{\sigma}_\lambda^+ , \tilde{\sigma}_\lambda^- \in \tilde{S}_n \). The dimension of \( \tilde{\sigma}_\lambda \) or \( \tilde{\sigma}_\lambda^\pm \) is
\[ 2^{(\lambda,\lambda)} \prod_{1 \leq i < j \leq m} \frac{\lambda_i - \lambda_j}{\lambda_i + \lambda_j} \]   
(3.6.1)

Notice that \( DP(n) \) precisely parameterizes the set of quasi-distinguished orbits in type \( A_{n-1} \) (the only local systems of Springer type are the trivial ones here).

In order to simplify the formulas below, we write \( \tilde{\sigma}_\lambda := \tilde{\sigma}_\lambda^+ \oplus \tilde{\sigma}_\lambda^- \), if \( \lambda \) is an odd partition in \( DP(n) \).

The decomposition of the tensor product of an \( S_n \)-type \( \sigma_\mu \) with the spin representation \( S = \tilde{\sigma}_\mu \) is well-known (see [24] §9.3) for example). If \( \lambda \neq (n) \) (this case has been covered by [23] already), we have:
\[ \dim \text{Hom}_{\tilde{S}_n}[\tilde{\sigma}_\lambda, \sigma_\mu \otimes \tilde{\sigma}_\mu] = \frac{1}{\epsilon_\lambda \epsilon_\mu} 2^{(\lambda,\mu)} g_{\lambda,\mu}, \]
(3.6.2)
where \( \epsilon_\lambda = 1 \) (resp. \( \epsilon_\lambda = \sqrt{2} \)) if \( \lambda \) is even (resp. odd), and \( g_{\lambda,\mu} \) are certain Kostka type numbers ([24] §9.3)). In particular, \( g_{\lambda,\lambda} = 0 \), and this proves (2) in Theorem 1.0.1.

In order to verify claim (1) of Theorem 1.0.1 we need a formula for the character of \( \tilde{\sigma}_\lambda \) on the conjugacy class represented by the cycle (123) in \( \tilde{S}_n \). This may be well-known, but I could not find a reference, so I include a combinatorial proof.

Lemma 3.6.2. If \( \lambda = (\lambda_1, \ldots, \lambda_m) \) is a partition in \( DP(n) \), \( n \geq 3 \), then we have
\[ \frac{|C^\lambda_{(123)}|}{\dim \tilde{\sigma}_\lambda} = \sum_{i=1}^{m} \frac{\lambda_i(\lambda_i^2 - 1)}{6} - \left( \frac{n}{2} \right), \]
(3.6.3)
where \( C^\lambda_{(123)} \) denotes the conjugacy class of the cycle (123) in \( \tilde{S}_n \).

Proof. The proof is by induction on \( n \). One can immediately verify this for \( n = 3 \), and let us assume it holds for \( n - 1 \). By restriction to \( \tilde{S}_{n-1} \), one has
\[ \text{tr}_{\tilde{\sigma}_\lambda} = \sum_{i=1}^{m} \text{tr}_{\tilde{\sigma}_{\lambda^i}}, \]
where \( \lambda^i = (\lambda_1, \ldots, \lambda_i - 1, \ldots, \lambda_m) \). One discards \( \lambda^i \) if it is not in \( DP(n-1) \). Using the induction hypothesis, and the ratios \( |C^\lambda_{(123)}|/|C^{\lambda^i}_{(123)}| = \frac{\dim \tilde{\sigma}_\lambda}{\dim \tilde{\sigma}_{\lambda^i}} = \frac{\frac{\lambda^i}{\lambda}}{\prod_{j \neq i} \frac{(\lambda_i - 1)(\lambda_i + 1)}{(\lambda_i - \lambda_j)(\lambda_i + \lambda_j - 1)}} \), an elementary calculation leads to the following curious identity that we need to verify:
\[ \sum_{i=1}^{m} \lambda_i^2(\lambda_i - 1) \prod_{j \neq i} \frac{(\lambda_i - 1)(\lambda_i + 1)}{(\lambda_i - \lambda_j)(\lambda_i + \lambda_j - 1)} = \sum_{i=1}^{m} \lambda_i^2(\lambda_i - 1) - \sum_{i \neq j} \lambda_i \lambda_j. \]   
(3.6.4)
A similar identity arises when one proves Schur’s dimension formula by induction ([12] Proposition 10.4)). We consider the function
\[ f(x) = (x^2 - x) \prod_{i=1}^{m} \frac{(x - \lambda_i - 1)(x + \lambda_i)}{(x - \lambda_i)(x + \lambda_i - 1)}. \]   
(3.6.5)
which has the expansion
\[ f(x) = (x^2 - x) + \sum_{i=1}^{m} A_i - \sum_{i=1}^{n} \frac{A_i \lambda_i (\lambda_i - 1)}{(x - \lambda_i)(x + \lambda_i - 1)}, \]
where \( A_i = -2\lambda_i \prod_{j \neq i} (\frac{(\lambda_i - \lambda_j)(\lambda_i + \lambda_j)}{(\lambda_i - \lambda_j)(\lambda_i + \lambda_j)}). \) Notice that the coefficient of \( x^{-2} \) in the Laurent expansion of the right hand side of (3.6.6) is precisely \((-2\lambda_i)\) times the left hand side of (3.6.4). Then one verifies (3.6.4) easily, by computing the coefficient of \( x^{-2} \) in the Laurent expansion of (3.6.6). □

Using the formula in Lemma 3.6.2, we immediately check that \( \tilde{\sigma}_\lambda(\Omega_{S_n}) = \sum_{i=1}^{n} \frac{\lambda_i (\lambda_i^2 - 1)}{3}. \) Here, for simplicity, we assumed that the roots of type \( n \) are classified by \([19]\), starting with the classification for \( \tilde{S}. \) This realization is very convenient for computing the character of \( \tilde{\sigma}_\lambda \) (and similarly \( \tilde{\sigma}_\lambda^\pm \)). Using the character of \( S \) and the characters of type \( A_{n-1} \), we immediately see that:
\[
\frac{\text{tr} \tilde{\sigma}_\lambda(\tilde{s}_\alpha \tilde{s}_\beta)}{\dim \tilde{\sigma}_\lambda} = \cos(\alpha, \beta),
\]
\[
\frac{\text{tr} \tilde{\sigma}_\lambda((123))}{\dim \tilde{\sigma}_\lambda},
\]
\[
\frac{\text{tr} \tilde{\sigma}_\lambda((12))}{\dim \tilde{\sigma}_\lambda},
\] if \( \alpha, \beta \) form an \( A_2 \)
\[
\frac{\text{tr} \tilde{\sigma}_\lambda(1)}{\dim \tilde{\sigma}_\lambda},
\] if \( \alpha, \beta \) form an \( B_2/C_2 \).

3.7. Types \( B, C \). The nilpotent orbits in \( sp(2n) \) and \( so(m) \) are parameterized (via an analogue of the Jordan canonical form) by partitions of \( 2n \) (resp. \( n \)), where the odd (resp. even) parts occur with even multiplicity. Such an orbit is in \( N_0(sp(2n)) \) (resp. \( N_0(so(m)) \)) if and only if the associated partition has only even (resp. odd) parts, and all parts have multiplicity at most 2.

Let \( W_n \) denote the Weyl group of type \( B_n, C_n \). The group \( W_n \) is a semidirect product \( W_n = S_n \rtimes (\mathbb{Z}/2\mathbb{Z})^n \), and therefore, as it is well-known, its representations are obtained by Mackey theory. More precisely, let \( \chi_k = (\text{triv} \boxtimes \cdots \boxtimes \text{triv})^{n-k} \boxtimes (\text{sgn} \boxtimes \cdots \boxtimes \text{sgn})^k \) be a character of \( (\mathbb{Z}/2\mathbb{Z})^n \), and let \( S_{n-k} \times S_k \) be the isotropy group of \( \chi_k \) in \( S_n \). For every partitions \( \lambda \) of \( n-k \) and \( \mu \) of \( k \), one constructs an irreducible \( W_n \) representation \( \sigma_{\lambda,\mu} \) as
\[
\sigma_{\lambda,\mu} = \text{Ind}_{S_{n-k} \times S_k \times (\mathbb{Z}/2\mathbb{Z})^n}^{W_n}(\sigma_{\lambda} \boxtimes \sigma_{\mu} \boxtimes \chi_k).
\]
This gives a bijection
\[
\tilde{W}_n \leftrightarrow BP(n), \quad \sigma_{\lambda,\mu} \leftrightarrow (\lambda, \mu),
\]
where \( BP(n) \) is the set of bipartitions of \( n \). In particular, in this notation, if \( \lambda \in P(n) \), the representation \( \sigma_{\lambda,0} \) is obtained from the \( S_n \)-type \( \sigma_\lambda \), by letting the simple reflections of \( W_n \) not in \( S_n \) act by the identity.

Let \( \tilde{W}_n \) denote the spin cover of \( W_n \). The genuine representations of \( \tilde{W}_n \) were classified by \([19]\), starting with the classification for \( \tilde{S}_n \)-types.

**Theorem 3.7.1** ([19] Theorem 5.1]). There is a one-to-one correspondence
\[
\left( \tilde{W}_n \right)_{\text{gen}}/\sim \leftrightarrow P(n).
\]
For every \( \lambda \in P(n) \), there exist:

1. one irreducible \( \tilde{W}_n \)-type \( \tilde{\sigma}_\lambda = \sigma_{\lambda,0} \otimes S \), if \( n \) is even;
2. two associate \( \tilde{W}_n \)-types \( \tilde{\sigma}_\lambda^\pm = \sigma_{\lambda,0} \otimes S^\pm \), if \( n \) is odd.

This realization is very convenient for computing the character of \( \tilde{\sigma}_\lambda \) (and similarly \( \tilde{\sigma}_\lambda^\pm \)). Using the character of \( S \) and the characters of type \( A_{n-1} \), we immediately see that:
\[
\frac{\text{tr} \tilde{\sigma}_\lambda(\tilde{s}_\alpha \tilde{s}_\beta)}{\dim \tilde{\sigma}_\lambda} = |\cos(\alpha, \beta)|, \quad \frac{\text{tr} \tilde{\sigma}_\lambda((123))}{\dim \tilde{\sigma}_\lambda}, \quad \frac{\text{tr} \tilde{\sigma}_\lambda((12))}{\dim \tilde{\sigma}_\lambda},
\]
\[
\text{if } \alpha, \beta \text{ form an } A_2, \quad \text{if } \alpha, \beta \text{ form an } B_2/C_2.
\]

\( \Box \)
The relevant formulas for \( \text{tr} \sigma_{\lambda} \) in \( S_n \) go back to Frobenius. In the form that we need, they are:

\[
|C_{(123)}| \frac{\text{tr}_{\sigma_{\lambda}}((123))}{\dim \sigma_{\lambda}} = p_2(\lambda) - \left( \frac{n}{2} \right), \quad |C_{(12)}| \frac{\text{tr}_{\sigma_{\lambda}}((12))}{\dim \sigma_{\lambda}} = p_1(\lambda),
\]

(3.7.4)

where \( p_k(\lambda) \) is the sum of \( k \) powers of the content of \( \lambda \).

The formula for \( \tilde{\sigma}_{\lambda}(\Omega_{\tilde{W}}) \) becomes very explicit. Assume for simplicity of notation that we take the standard Bourbaki coordinates for roots for type \( B \) and \( C \). Set

\[
\epsilon = 1, \quad \text{if} \ \Phi = B_n, \quad \epsilon = \frac{1}{2}, \quad \text{if} \ \Phi = C_n.
\]

(3.7.5)

Then, we have

\[
\tilde{\sigma}_{\lambda}(\Omega_{\tilde{W}}) = 4p_2(\lambda + \epsilon),
\]

(3.7.6)

where \( p_2(\lambda + \epsilon) \) denotes the sum of squares of the \( \epsilon \)-content of \( \lambda \), i.e., the content of the \( (i, j) \)-box is \( j - i + \epsilon \).

The set of all contents of \( \lambda + \epsilon \) (with repetitions) represents the coordinates of one half of the middle element of a nilpotent orbit \( O_{\lambda, \epsilon} \) in \( \mathfrak{so}(2n + 1) \) (if \( \epsilon = 1 \)), respectively \( \mathfrak{sp}(2n) \) (if \( \epsilon = 1/2 \)). Moreover, the nilpotent orbit lies in fact in \( \mathcal{N}_0(g) \).

(See [9, §4.4] for the combinatorics needed to verify this claim.)

The following algorithm (due to Slooten) attaches to each partition \( \lambda \) with content \( \lambda + \epsilon \) a set of bipartitions. All the \( W_n \)-types parameterized by these bipartitions are Springer representations for the nilpotent orbit \( O_{\lambda, \epsilon} \). This can be seen by comparing Slooten’s algorithm with Lusztig’s algorithm with S-symbols for the Springer correspondence ([9, §4.4]). Moreover, the relation between this algorithm and the elliptic representation theory of \( W \) (via the elliptic representation theory of the graded affine Hecke algebra attached to \( \Phi \)) is part of [8].

**Algorithm** ([22]). Start with an empty bipartition \( (\mu, \mu') \), \( \mu = \emptyset, \mu' = \emptyset \). Locate the largest content in absolute value in \( \lambda + \epsilon \). This could appear in the last box of the first row or the last box of the first column. Assume first that these two entries are distinct. If the largest content is in the first row, remove the row from \( \lambda \) and put its length in \( \mu \). If the largest content is in the first column, remove the column from \( \lambda \) and put its length in \( \mu' \). Continue with the remaining (smaller) partition.

If at this step there was an ambiguity, namely the largest entry in absolute value appeared twice, start two cases, and proceed in each one of them separately as above.

If at the last step, we are left with a single box, treat it as a row if its content is nonnegative, and as a column if its content is negative.

It is apparent that the number of \( W \)-types that the algorithm returns is always a power of 2. Moreover, it is known that the algorithm returns a unique \( W_n \)-type if and only if the associated nilpotent orbit is distinguished.

Let \( \sigma_{(\mu, \mu')} \) be one of the \( W_n \)-types returned by the algorithm. It remains to check that the multiplicity of \( \tilde{\sigma}_{\lambda} \) in \( \sigma_{(\mu, \mu')} \otimes S \) is nonzero (claim (2) of Theorem 1.0.1). We have:

\[
\text{Hom}_{\tilde{W}}[\tilde{\sigma}_{\lambda}, \sigma_{(\mu, \mu')} \otimes S] = \text{Hom}_{\tilde{W}}[\sigma_{(\lambda, \emptyset)} \otimes S, \sigma_{(\mu, \mu')} \otimes S] = \text{Hom}_W[\bigwedge V, \sigma_{(\lambda, \emptyset)} \otimes \sigma_{(\mu, \mu')}].
\]

(3.7.7)
In bipartition notation, we have $\bigwedge^k V = \sigma_{(\ell^n, k^k)}$. We claim that $\bigwedge^k V$, where $k = |\mu|$, occurs in $\sigma_{(\lambda, \theta)} \otimes \sigma_{(\mu, \mu')}$. To see this, notice that (3.7.1) implies:

\[
\sigma_{(\lambda, \theta)} \otimes \sigma_{((n-k), k^k)} = \sigma_{(\lambda, \theta)} \otimes \text{Ind}_{S_{n-k} \times S_k \times (Z/2Z)^n}^{W_n} (\text{triv} \boxtimes \text{sgn} \boxtimes (\text{triv}^{n-k} \boxtimes \text{sgn}^k)) = \text{Ind}_{S_{n-k} \times S_k \times (Z/2Z)^n}^{W_n} (\sigma \lambda_{S_{n-k} \times S_k \otimes (\text{triv} \boxtimes \text{sgn}) \boxtimes (\text{triv}^{n-k} \boxtimes \text{sgn}^k)}).
\]

From the construction of $\mu$ (using the rows of $\lambda$) and $\mu'$ (using the columns of $\lambda$), and the induction/restriction rules in $S_n$, we have:

\[
\sigma_\mu \boxtimes \sigma_{\mu'^t} \hookrightarrow \sigma_\lambda|_{S_{n-k} \times S_k},
\]

where $\mu'^t$ denotes the transpose partition of $\mu'$. Since $\sigma_{\mu'^t} = \sigma_{\mu'} \otimes \text{sgn}$, as $S_k$-representations, the claim follows by applying (3.7.1) again.

3.8. **Type $D$.** Let $W(D_n)$ denote the Weyl group of type $D_n$. This is a subgroup of $W_n$ of index 2. An irreducible $W_n$-representation $\sigma_{(\lambda, \mu)}$ restricts to a representation of $W(D_n)$ which is:

- irreducible if $\lambda \neq \mu$;
- a sum of two inequivalent irreducible $W(D_n)$-types if $\lambda = \mu$.

Moreover, $\sigma_{(\lambda, \mu)}$ and $\sigma_{(\mu, \lambda)}$, $\lambda \neq \mu$, restrict to the same $W(D_n)$-representation. In our case, the only $W(D_n)$-types that we consider are those associated via Springer’s correspondence with nilpotent orbits in $N_0(D_n)$. In the Jordan form partition notation ([4]), these are the orbits indexed by partitions of $2n$ where all parts are odd and appear with multiplicity at most 2. It turns out that for every Springer representation attached to one of these orbits, the bipartition $(\lambda, \mu)$ has the property that $\lambda \neq \mu$.

The classification of $\widehat{W(D_n)}^\text{gen}$ is obtained from Theorem 3.8. We define the equivalence relation $\sim$ on $P(n)$: $\lambda \sim \lambda'$. This is the combinatorial equivalent of the relation $\sim_0$ on $\widehat{S}_n$: $\sigma_\lambda \sim \sigma_\lambda \otimes \text{sgn} = \sigma_{\lambda'}$.

**Theorem 3.8.1** ([19 §7,8]). *There is a one-to-one correspondence*

\[
\widehat{W(D_n)}^\text{gen} / \sim \leftrightarrow P(n) / \sim.
\]

*More precisely, recall the irreducible representations $\sigma_\lambda$ of $S_n$ and $\sigma_\lambda^\prime$ of $\widehat{W_n}$. We have:*

1. If $n$ is odd, every irreducible genuine $\widehat{W_n}$-representation $\sigma_\lambda$ restricts to an irreducible $\widehat{W(D_n)}$-representation, and this gives a complete set of inequivalent irreducible genuine $\widehat{W(D_n)}$-representations. Moreover, $\sigma_\lambda$ and $\sigma_\lambda^\prime$ are associate as $\widehat{W(D_n)}$-representations.
2. If $n$ is even, and if $\lambda = \lambda'$, then $\sigma_\lambda$ restricts to a sum of two associate irreducible $\widehat{W(D_n)}$-type; if $\lambda \neq \lambda'$, then $\sigma_\lambda$ restricts to an irreducible $\widehat{W(D_n)}$-representation.

The analysis of the map $\Psi$ is now completely analogous to the cases $B_n/C_n$. The same formulas and algorithm hold with the convention that $\epsilon = 0$. 
3.9. Exceptional root systems. The character tables for exceptional $\tilde{W}$ are in [17]. We reorganize them here so that the claims in Theorem 1.0.1 follow. The notation for $\tilde{W}$-types is borrowed from [17]. We put a * next to $\tilde{W}$-type to indicated that it has an associate $\tilde{W}$-type. We use Carter’s notation for $W$-types and nilpotent orbits ([6]). In each table, we give the correspondences (1), (2) in Theorem 1.0.1 between genuine $\tilde{W}$-types, nilpotent orbits, and Springer representations attached to nilpotent orbits. We also give the traces of the characters of the genuine $\tilde{W}$-types on elements of the form $\tilde{s}_\alpha \tilde{s}_\beta$, where $\alpha, \beta$ form an $A_2$, $B_2$, or $G_2$. Using these traces and the sizes of the corresponding conjugacy classes (which are listed in [5]), we computed $\tilde{\sigma}(\Omega_{\tilde{W}})$ and verified assertion (1) in Theorem 1.0.1. For tensor product decompositions, we used the package chevie in the computer algebra system GAP.

Table 1. $G_2$

| Nilpotent $e \in N_0$ | $\sigma_e, \varnothing \in \tilde{W}$ | $\tilde{\sigma} \in \tilde{W}_{\text{gen}}$ | $\text{tr}_2(A_2)$ | $\text{tr}_2(G_2)$ |
|----------------------|-----------------------------|-----------------------------|------------------|------------------|
| $G_2$                | (1, 0)                      | $2_s$                       | 1                | $\sqrt{3}$      |
| $G_2(a_1)$           | (2, 1)                      | $2_{sss}$                   | $-2$             | 0                |
|                      | (1, 3)'                     | $2_{ss}$                    | $1$              | $-\sqrt{3}$     |

Table 2. $F_4$

| Nilpotent $e \in N_0$ | $\sigma_e, \varnothing \in \tilde{W}$ | $\tilde{\sigma} \in \tilde{W}_{\text{gen}}$ | $\text{tr}_2(A_2)$ | $\text{tr}_2(A_2^\ast)$ | $\text{tr}_2(B_2)$ |
|----------------------|-----------------------------|-----------------------------|------------------|-------------------|------------------|
| $F_4$                | (1, 0)                      | $4_s$                       | 2                | 2                 | $2\sqrt{2}$      |
| $F_4(a_1)$           | (4, 1)'                     | $12_s$                      | 0                | 0                 | $2\sqrt{2}$      |
|                      | (2, 4)'                     | $8_{sss}$                   | 4                | $-2$             | 0                |
| $F_4(a_2)$           | (9, 2)                      | $24_s$                      | 0                | 0                 | 0                |
|                      | (2, 4)'                     | $8_{sss}$                   | $-2$             | 4                 | 0                |
| $F_4(a_3)$           | (12, 4)'                    | $8_{ss}$                    | 0                | 0                 | $-2\sqrt{2}$     |
|                      | (9, 6)'                     | $12_{ss}$                   | $-2$             | $-2$             | 0                |
|                      | (6, 6)'                     | $8_s$                       | $-2$             | $-2$             | 0                |
|                      | (1, 12)'                    | $4_{ss}$                    | 2                | 2                 | $-2\sqrt{2}$     |

Remark 3.9.1. In $E_6$, there is a nilpotent orbit $D_4(a_1) \subset N_0(E_6)$, such that $A_6 = S_3$. There are three Springer representations attached to $D_4(a_1)$: $\sigma_{e,(3)}$, $\sigma_{e,(21)}$, and $\sigma_{e,(111)}$. Since the rank is even, there is a single spin module $S$. The fiber $\Psi^{-1}(e)$ contains three $\tilde{W}$-types $\tilde{\sigma}_1$, $\tilde{\sigma}_2$, $\tilde{\sigma}_3$ of dimensions 40, 20, 20, respectively. Moreover, $\tilde{\sigma}_2$ and $\tilde{\sigma}_3$ are associate. We compute that in the decomposition $\tilde{\sigma}_1 \otimes S$ occur all three $\sigma_{e,(3)}$, $\sigma_{e,(21)}$, and $\sigma_{e,(111)}$; while in the decomposition of $\tilde{\sigma}_2 \otimes S$ (equivalently $\tilde{\sigma}_3 \otimes S$) only $\sigma_{e,(21)}$ occurs among the three Springer representations.

3.10. The generalized Springer correspondence. The references for the construction of the generalized Springer correspondence, and for the results we need to use are [13] [15].
Let $G$ be a simply connected complex simple group, with a fixed Borel subgroup $B$, and maximal torus $H \subset B$. The Lie algebras will be denoted by the corresponding Gothic letter. Fix a nondegenerate $G$-invariant symmetric bilinear form $\langle \ , \rangle$ on $\mathfrak{g}$. Denote also by $\langle \ , \rangle$ the dual form on $\mathfrak{g}^*$.  

**Definition 3.10.1.** A cuspidal triple for $G$ is a triple $(\mathcal{L}, \mathcal{C}, \mathcal{L})$, where $\mathcal{L}$ is a Levi subgroup of $G$, $\mathcal{C}$ is a nilpotent $\mathcal{L}$-orbit on the Lie algebra $\mathfrak{l}$, and $\mathcal{L}$ is an irreducible $G$-equivariant local system on $\mathcal{C}$ which is cuspidal in the sense of \cite{13, 15}.

Let us fix a cuspidal triple $(\mathcal{L}, \mathcal{C}, \mathcal{L})$, such that $H \subset \mathcal{L}$, and $P = LU \supset B$ is a parabolic subgroup. Let $T \subset L$ denote the identity component of the center of $L$, with Lie algebra $\mathfrak{t}$. Write an orthogonal decomposition with respect to $\langle \ , \rangle$, $\mathfrak{h} = \mathfrak{t} + \mathfrak{a}$; here $\mathfrak{a}$ is a Cartan subalgebra for the semisimple part of $\mathfrak{l}$. Let

\[ \text{pr}_C : \mathfrak{h} \to \mathfrak{t} \]  

(3.10.1)

denote the corresponding projection onto $\mathfrak{t}$.

Following \cite{15} §2, we attach to $(\mathcal{L}, \mathcal{C}, \mathcal{L})$ an $\mathbb{R}$-root system $\Phi = (V, R, V^\vee, R^\vee)$ and a $W$-invariant function $c : R^+ \to \mathbb{Z}$. Let $R \subset \mathfrak{t}^*$ be the reduced part of the root system given by the nonzero weights of $\text{ad}(\mathfrak{t})$ on $\mathfrak{g}$; it can be identified with

---

### Table 3. $E_6$

| Nilpotent $e \in \mathcal{N}_0$ | $\sigma_{e, \phi} \in \hat{W}$ | $\tilde{\sigma} \in \hat{W}_{\text{gen}}$ | $\text{tr}_{\tilde{\sigma}}(A_2)$ |
|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $E_6$                         | $(1, 0)$                      | $8_s$                         | 4                             |
| $E_6(a_1)$                    | $(6, 1)$                      | $40_s$                        | 8                             |
| $E_6(a_3)$                    | $(30, 3)$                     | $120_s$                       | 0                             |
|                               | $(15, 5)$                     | $72_s$                        | 0                             |
| $D_5$                         | $(20, 2)$                     | $60_s$                        | 6                             |
| $D_5(a_1)$                    | $(64, 4)$                     | $80_s$                        | $-2$                          |
| $A_4 + A_1$                   | $(60, 5)$                     | $64_s$                        | $-4$                          |
| $D_4(a_1)$                    | $(80, 7), (90, 8), (20, 10)$  | $40_{ss}$                     | $-4$                          |
|                               | $(90, 8)$                     | $20_{ss}$                     | $-2$                          |

### Table 4. $E_7$

| Nilpotent $e \in \mathcal{N}_0$ | $\sigma_{e, \phi} \in \hat{W}$ | $\tilde{\sigma} \in \hat{W}_{\text{gen}}$ | $\text{tr}_{\tilde{\sigma}}(A_2)$ |
|-------------------------------|-------------------------------|-------------------------------|-------------------------------|
| $E_7$                         | $(1, 0)$                      | $8_s$                         | 4                             |
| $E_7(a_1)$                    | $(7, 1)$                      | $48_s$                        | 12                            |
| $E_7(a_2)$                    | $(27, 2)$                     | $168_s$                       | 24                            |
| $E_7(a_3)$                    | $(56, 3)$                     | $280_s$                       | 20                            |
|                               | $(21, 6)$                     | $112_s$                       | 8                             |
| $E_7(a_4)$                    | $(189, 5)$                    | $720_s$                       | 0                             |
|                               | $(15, 7)$                     | $120_s$                       | 0                             |
| $E_7(a_5)$                    | $(315, 7)$                    | $448_s$                       | $-4$                          |
|                               | $(280, 9)$                    | $560_s$                       | $-20$                         |
|                               | $(35, 13)$                    | $112_{ss}$                    | $-16$                         |
| $E_6(a_1)$                    | $(120, 4), (105, 5)$          | $512_s$                       | 16                            |
| $A_4 + A_1$                   | $(512, 11), (512, 12)$        | $64_s$                        | $-4$                          |
|                               | $(512, 11), (512, 12)$        | $64_{ss}$                     | $-4$                          |
the root system of the reductive part of $Z_G(x)$, where $x \in C$. Let $V$ be the $\mathbb{R}$-span of $R$ in $t^*$. The Weyl group is

$$W = N_G(L)/L. \quad (3.10.2)$$

This is a Coxeter group due to the particular form $L$ must have to allow a cuspidal local system.

Let $R^+$ be the subset of $R$ for which the corresponding weight space lives in $u$. The simple roots $\Pi = \{\alpha_i : i \in I\}$ correspond to the Levi subgroups $L_i$ containing $L$ maximally: $\alpha_i$ is the unique element in $R^+$ which is trivial on the center of $L_i$. For every simple $\alpha_i$, $c(\alpha_i) \geq 2$ is defined to be the smallest integer such that

$$\text{ad}(x)^{c(\alpha_i^{-1})} : \mathfrak{l}_i \cap u \to \mathfrak{l}_i \cap u \text{ is zero.} \quad (3.10.3)$$

This is a $W$-invariant function on $\Pi$ and we extend it to $R^+$. 

| Nilpotent $e \in N_0$ | $\sigma_{e, \phi} \in \tilde{W}$ | $\tilde{\sigma} \in \tilde{W}_{\text{gen}}$ | $\text{tr}_2(A_2)$ |
|-----------------------|-----------------|-----------------|-----------------|
| $E_8$                 | (1, 0)          | 16_s            | 8               |
| $E_8(a_1)$            | (8, 1)          | 112_s           | 32              |
| $E_8(a_2)$            | (35, 2)         | 448_s_s         | 80              |
| $E_8(a_3)$            | (112, 3)        | 1344_s_s        | 168             |
|                       | (28, 8)         | 320_s_s         | 40              |
| $E_8(a_4)$            | (210, 4)        | 2016_s_s        | 144             |
|                       | (160, 7)        | 1680_s_s        | 120             |
| $E_8(a_5)$            | (700, 6)        | 5600_s_s        | 160             |
|                       | (300, 8)        | 2800_s_s        | 80              |
| $E_8(a_6)$            | (1400, 8)       | 6480_s_s        | 0               |
|                       | (1575, 10)      | 9072_s_s        | 0               |
|                       | (350, 14)       | 2592_s_s        | 0               |
| $E_8(a_7)$            | (4480, 16)      | 896_s_s         | −72             |
|                       | (5670, 18)      | 2016_s_s        | −72             |
|                       | (4536, 18)      | 2016_s_s        | −48             |
|                       | (1680, 22)      | 1344_s_s        | −40             |
|                       | (1400, 20)      | 1120_s_s        | −32             |
|                       | (70, 32)        | 224_s           | −8              |
| $E_8(b_4)$            | (560, 5)        | 5600_s_s        | 280             |
|                       | (50, 8)         | 800_s           | 40              |
| $E_8(b_5)$            | (1400, 7)       | 6720_s          | 128             |
|                       | (1008, 9)       | 7168_s          | 120             |
|                       | (56, 19)        | 448_s           | 8               |
| $E_8(b_6)$            | (2240, 10)      | 8400_s_s        | −120            |
|                       | (840, 13)       | 5600_s_s        | −80             |
|                       | (175, 12)       | 2800_s_s        | −40             |
| $D_5 + A_2$           | (4536, 13), (840, 13) | 4800_s_s | −120 |
| $D_7(a_1)$            | (3240, 9), (1050, 10) | 11200_s_s | −40 |
| $D_7(a_2)$            | (4200, 12), (3360, 13) | 7168_s_s | −160 |
| $E_6(a_1) + A_1$      | (4096, 11), (4096, 12) | 8192_s_s | −128 |
The explicit classification of cuspidal triples (when \( G \) is simple), along with the corresponding values for the parameters \( c(\alpha) \) can be found in the tables of \[15\] §2.13.

Define \( R^\perp = \{ x \in t : \alpha(x) = 0, \text{ for all } \alpha \in R \} \), and let \( t' \) be the orthogonal complement of \( R^\perp \) in \( t \). For every \( \alpha \in R \), define \( \tilde{\alpha} \in t \) to be the unique element of \( t' \) such that \( \alpha(\tilde{\alpha}) = 2 \). Let \( R'^\perp \) denote the set of all \( \tilde{\alpha} \), and let \( V'^\perp \) be the \( \mathbb{R} \)-span of \( R'^\perp \) in \( t' \).

Consider the variety

\[
\hat{g} = \{ (x, gP) : \text{Ad}(g^{-1})x \in \mathcal{C} + t + u \}, \tag{3.10.4}
\]
on which \( G \times \mathbb{C}^\times \) acts via \( (g_1, \lambda) : x \mapsto \lambda^{-2} \text{Ad}(g_1)x, x \in g \), and \( gP \mapsto g_1gP, g \in G \).

Let \( \pi_1(e) = Z_G(e)/Z_G(e)^0 \) be the fundamental group of \( G \cdot e \). The hypercohomology with compact support \( \hat{H}^\ast_{\text{c}}(\mathcal{P}_e, \hat{\mathcal{L}}) \) carries a \( \pi_1(e) \times W \) action (\[13\]), see also \[15\]). Let \( \hat{A}(e)_\mathcal{C} \) denote the set of irreducible representations of \( \pi_1(e) \) which appear in this way. Moreover, for \( \phi \in \hat{A}(e)_\mathcal{C} \) we have:

\[
\sigma(e, \phi) := \text{Hom}_{\pi_1(e)}[\phi, \hat{H}^2_{\text{c}}(\mathcal{P}_e, \hat{\mathcal{L}})] \quad \tag{3.10.5}
\]
is an irreducible \( W \)-representation. The correspondence \( \cup_{e \in G \backslash \mathcal{G}(g)} \hat{A}(e)_\mathcal{C} \rightarrow \hat{W}, (e, \phi) \mapsto \sigma(e, \phi) \) is the generalized Springer correspondence of \[13\], and it is a bijection. We normalize it so that \( \sigma(e, \phi) = \text{sgn} \), when \( e \in G \cdot \mathcal{C} \) (there is single \( \phi \) that appears in that case).

**Definition 3.10.2.** Denote \( \mathcal{T}_0(G, \mathcal{C}) = \{(e, h, f) \in \mathcal{T}_0 : \hat{A}(e)_\mathcal{C} \neq 0 \}. \)

**Theorem 3.10.3.** There is a surjective map

\[
\Psi : \hat{W}_{\text{gen}} \rightarrow \mathcal{T}_0(G, \mathcal{C}), \tag{3.10.6}
\]

\( \mathcal{T}_0(G, \mathcal{C}) \) is as in **Definition 3.10.2** with the following properties:

1. If \( \Psi(\tilde{\sigma}) = [(e, h, f)] \), then

\[
\tilde{\sigma}(\Omega_{\hat{W}, e}) = (\text{pr}_C(h), \text{pr}_C(h)). \tag{3.10.7}
\]

where \( \Omega_{\hat{W}, e} \) is as in \[2.6.1\] and \( \text{pr}_C \) is as in \[3.10.1\];

2. Let \( (e, h, f) \in \mathcal{T}_0(G, \mathcal{C}) \) be given. For every Springer representation \( \sigma(e, \phi) \), \( \phi \in \hat{A}(e)_\mathcal{C} \), and every spin \( \hat{W} \)-module \( S \), there exists \( \tilde{\sigma} \in \Psi^{-1}[(e, h, f)] \) such that \( \tilde{\sigma} \) appears with nonzero multiplicity in the tensor product \( \sigma(e, \phi) \otimes S \).

Conversely, for every \( \tilde{\sigma} \in \Psi^{-1}[(e, h, f)] \), there exists a spin \( \hat{W} \)-module \( S \) and a Springer representation \( \sigma(e, \phi) \), such that \( \tilde{\sigma} \) is contained in \( \sigma(e, \phi) \otimes S \).

3. If \( e \) is distinguished, then properties (1) and (2) above determine a bijection:

\[
\Psi^{-1}[(e, h, f)] / \sim \leftrightarrow \{ \sigma(e, \phi) : \phi \in \hat{A}(e)_\mathcal{C} \}. \tag{3.10.8}
\]
Proof. Again the proof amounts to verifying the assertions in every case. The interesting cases are when $R$ is of type $B_n/C_n$, $G_2$, or $F_4$.

For type $G_2$, the cuspidal local system appears for the Levi subgroup $L = 2A_2$ in $G = E_6$. Assuming that the simple roots of $R$ of type $G_2$ are $\alpha, \beta$ with $\alpha$ long and $\beta$ short, the parameter function is $c(\alpha) = 1$, $c(\beta) = 3$. To give an explicit formula for $\Omega_{W, c}$, let’s normalize $\alpha, \beta$ such that such that $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$ and $\langle \tilde{\beta}, \tilde{\beta} \rangle = 6$. Then we have

$$\frac{1}{4} \tilde{\sigma}(\Omega_{W, c}) = \frac{3}{2} c(\alpha)^2 + 3c(\beta)^2 + 4\sqrt{3c(\alpha)c(\beta)} \frac{\text{tr}_2(G_2)}{\dim \sigma} + (c(\alpha)^2 + 3c(\beta)^2) \frac{\text{tr}_2(A_2)}{\dim \sigma};$$

recall that the notation for conjugacy classes is as in [5]. Then one can verify easily part (1) of Theorem 3.10.3.

For type $F_4$, the cuspidal local system appears for the Levi subgroup $L = (3A_1)'$ in $G = E_7$. Assuming that the simple roots of $R$ of type $F_4$ are $\alpha, \beta$ with $\alpha$ short and $\beta$ long, the parameter function is $c(\alpha) = 2$, $c(\beta) = 1$. We normalize $\alpha, \beta$ so that $\langle \tilde{\alpha}, \tilde{\alpha} \rangle = 2$ and $\langle \tilde{\beta}, \tilde{\beta} \rangle = 1$. Then we have:

$$\frac{1}{4} \tilde{\sigma}(\Omega_{W, c}) = 3(2c(\alpha)^2 + c(\beta)^2) + 16c(\alpha)^2 \frac{\text{tr}_2(A_2)}{\dim \sigma} + 8c(\beta) \frac{\text{tr}_2(A_2)}{\dim \sigma} + 18\sqrt{2c(\alpha)c(\beta)} \frac{\text{tr}_2(B_2)}{\dim \sigma}.$$  

The correspondence from Theorem 3.10.3 for $G_2$ and $F_4$ is listed in Tables 6 and 7. The explicit values of $pr_C(h)$ in coordinates, for these cases, can be found in [7] Tables 1 and 2.

| Nilpotent $e \in \mathcal{N}_{0, C}$ | $\sigma_{e, \phi} \in \hat{W}$ | $\tilde{\sigma} \in \hat{W}_{\text{gen}}$ |
|-----------------------------------|----------------|------------------|
| $E_6$                             | $\sigma_0, \sigma_1, \sigma_2$ | 2s               |
| $E_6(a_1)$                        | $\sigma_3, \sigma_4$ | 2ss              |
| $E_6(a_3)$                        | $\sigma_5, \sigma_6$ | 2ssss            |

Table 6. $2A_2$ in $E_6$, $W = G_2$.

| Nilpotent $e \in \mathcal{N}_{0, C}$ | $\sigma_{e, \phi} \in \hat{W}$ | $\tilde{\sigma} \in \hat{W}_{\text{gen}}$ |
|-----------------------------------|----------------|------------------|
| $E_7$                             | $\sigma_0, \sigma_1, \sigma_2$ | 4s               |
| $E_7(a_1)$                        | $\sigma_3, \sigma_4$ | 8ssss            |
| $E_7(a_2)$                        | $\sigma_5, \sigma_6$ | 12s              |
| $E_7(a_3)$                        | $\sigma_7, \sigma_8$ | 24s              |
| $E_7(a_4)$                        | $\sigma_9, \sigma_{10}$ | 4ss              |
| $E_7(a_5)$                        | $\sigma_{11}, \sigma_{12}$ | 8ssss            |

Table 7. $(3A_1)'$ in $E_7$, $W = F_4$.

For types $B_n$ and $C_n$, the combinatorics is similar to that in the proof of Theorem 3.10.1. More precisely, assume the notation from Theorem 3.7.1 and the discussion following it. Assume also that the roots of type $B_n$ and $C_n$ are in the standard...
Bourbaki coordinates. For a partition \( \lambda \) of \( n \), viewed as a left justified Young tableau, define the content of the box \((i,j)\) with parameters \(c_1, c_2\) to be the number \(m_c(i,j) := c_1(i-j) + 2\). Let us denote by \(p_2(\lambda, c_1, c_2)\) the sum of contents of boxes for \( \lambda \) and parameters \(c_1, c_2\). The same computation as after Theorem 3.7.1 shows that

\[
\tilde{\sigma}_\lambda(\Omega_{\tilde{W},c}) = 4p_2(\lambda, c_1(c_1 - c_2)(c_\epsilon(\epsilon_n))), \text{ for type } B_n,
\]

\[
\tilde{\sigma}_\lambda(\Omega_{\tilde{W},c}) = 4p_2(\lambda, c_1(c_1 - c_2), 1/2 c_2(2c_\epsilon(\epsilon_n))), \text{ for type } C_n.
\]

Notice that \(\tilde{\sigma}_\lambda(\Omega_{\tilde{W},c})\) for \( C_n \) is identical with \(\tilde{\sigma}_\lambda(\Omega_{\tilde{W},c})\) for \( B_n \) if we set \(c_2(2c_\epsilon(\epsilon_n)) = 2c_\epsilon(\epsilon_n))\). The geometric values of the parameters are as follows (§2.13 in [14]):

1. \( g = sp(2k + 2n), l = sp(2k) \oplus \mathbb{C}^n, C = (2, 4, \ldots, 2p) \oplus 0, k = p(p + 1)/2, \Phi = B_n, c_1(c_1 - c_2) = 2, c_\epsilon(\epsilon_n) = 2p + 2; \)
2. \( g = so(k + 2n), l = so(k) \oplus \mathbb{C}^n, C = (1, 3, \ldots, 2p + 1) \oplus 0, k = p^2, \Phi = B_n, c_1(c_1 - c_2) = 2, c_\epsilon(\epsilon_n) = 2p + 2; \)
3. \( g = so(k + 4n), l = so(k) \oplus sl(2)^n \oplus \mathbb{C}^n, C = (1, 3, \ldots, 4p + 1) \oplus 2^n \oplus 0, k = (p + 1)(2p + 1), c_1(c_1 - c_2) = 4, c_\epsilon(\epsilon_n) = 4p + 3; \)
4. \( g = so(k + 4n), l = so(k) \oplus sl(2)^n \oplus \mathbb{C}^n, C = (3, 7, 11, \ldots, 4p + 3) \oplus 2^n \oplus 0, k = (p + 1)(2p + 3), c_1(c_1 - c_2) = 4, c_\epsilon(\epsilon_n) = 4p + 5. \)

For the partition \( \lambda \) with content \(m_c(i,j)\), Slooten’s algorithm is the same as in the case \(c \equiv 1\). Also the analysis of the tensor product decomposition works in the same way as in the proof of Theorem 1.0.1 for \( B_n \) and \(c \equiv 1\). (The argument there only uses Slooten’s algorithm and the Weyl group of type \(B_n\), and not the parameter function \(c\).) The algorithm giving the nilpotent element \(e \in g\) from the partition \( \lambda \) with content \(m_c(i,j)\), at geometric values of \(c\), is again very similar to the one in [9] §4.4], and we skip the details.

\[\square\]

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