Binary scalar products

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Abstract

Let $A, B \subseteq \mathbb{R}^d$ both span $\mathbb{R}^d$ such that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in A$, $b \in B$. We show that $|A| \cdot |B| \leq (d+1)^2d$. This allows us to settle a conjecture by Bohn, Faenza, Fiorini, Fisikopoulos, Macchia, and Pashkovich (2015) concerning 2-level polytopes. Such polytopes have the property that for every facet-defining hyperplane $H$ there is a parallel hyperplane $H'$ such that $H \cup H'$ contain all vertices. The authors conjectured that for every $d$-dimensional 2-level polytope $P$ the product of the number of vertices of $P$ and the number of facets of $P$ is at most $d2^{d+1}$, which we show to be true.

1 Introduction

For two vectors $a = (a_1, \ldots, a_d), b = (b_1, \ldots, b_d)$, let $\langle a, b \rangle = \sum_{i=1}^{d} a_i b_i$ their scalar product. Given two sets $A, B \subseteq \mathbb{R}^d$ that both linearly span $\mathbb{R}^d$ with the property that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in A$, $b \in B$, how many points can $A$ and $B$ contain? It is easy to see that each individual set cannot contain more than $2^d$ points. This bound is tight since we may choose $A = \{0, e_1, \ldots, e_d\}$ and $B = \{0, 1\}^d$. However, it turns out that $|A|$ and $|B|$ cannot be close to this bound simultaneously. In fact, in this paper we prove the following.

\textbf{Theorem 1.} Let $A, B \subseteq \mathbb{R}^d$ both linearly span $\mathbb{R}^d$ such that $\langle a, b \rangle \in \{0, 1\}$ holds for all $a \in A$, $b \in B$. Then we have $|A| \cdot |B| \leq (d+1)^2d$.

The previous example also shows that this bound is tight. We note that if one restricts her attention to the families of vectors coming from the Boolean cube $\{0, 1\}^d$ then questions of similar nature are studied in extremal set theory. In particular, see [12, Chapter 10]. Certain extremal set theory-type problems for families of vectors coming from $\{0, \pm 1\}^d$ were studied in [6, 7, 9, 10, 11].

Our main motivation for studying this question is its close relation to point configurations associated to 2-level polytopes. A polytope $P$ is said to be 2-level if for every facet-defining hyperplane $H$ there is a parallel hyperplane $H'$ such that $H \cup H'$ contains all vertices of $P$. Basic examples of 2-level polytopes are hypercubes, cross-polytopes, and simplices. Actually, 2-level polytopes generalize a variety of interesting polytopes such as Birkhoff, Hamner, and Hansen polytopes, order polytopes and chain polytopes of posets, stable matching polytopes, and stable set polytopes of perfect graphs [2]. Moreover, they arise in different areas of
mathematics, most notably in the field of extended formulations, which has received much attention during the past decade.

A fundamental result in polyhedral combinatorics states that $d$-dimensional stable set polytopes of perfect graphs admit subexponential (in $d$) size linear extended formulations, i.e., they are linear images of polytopes with subexponentially many facets [13]. It is a major open problem whether such polytopes have polynomial-size extended formulations. Moreover, the famous log-rank conjecture by Lovász and Saks [16] in the field of communication complexity would imply subexponential-size extended formulations for all 2-level polytopes. However, no non-trivial bound is known for this general case. In contrast, it is known that, among all $d$-dimensional polytopes, 2-level polytopes admit smallest possible semidefinite extended formulations [13]. Details on these connections and several recent studies on 2-level polytopes can be found in [2, 3, 4, 5, 8, 13, 14, 15].

Among them are extensive experimental studies by Bohn, Faenza, Fiorini, Fisikopoulos, Macchia, and Pashkovich [4, 8, 5], which led to a beautiful conjecture about the combinatorial structure of 2-level polytopes. It is easy to see that for a $d$-dimensional 2-level polytope $P$ the number of vertices $f_0(P)$ and the number of facets $f_{d-1}(P)$ are both bounded by $2d$.

Similar to the setting of Theorem 1, Bohn et al. observed that for small values of $d$, $f_0(P)$ and $f_{d-1}(P)$ cannot be close to this bound simultaneously. More specifically, they verified that every 2-level polytope of dimension $d \leq 7$ satisfies $f_0(P)f_{d-1}(P) \leq 2^{d+1}$, which was later also confirmed for $d = 8$ in [17]. In [4] it is asked whether this holds for all $d$, and in the journal version [5] this is posed as a conjecture.

Recently, Aprile, Cevallos, and Faenza [2, 1] showed that this conjecture holds for many families of 2-polytopes, including the ones mentioned above. We show that it is true for all 2-level polytopes:

**Theorem 2.** Every $d$-dimensional 2-level polytope $P$ satisfies $f_0(P)f_{d-1}(P) \leq 2^{d+1}$.

Note that this bound is tight by choosing $P = [0,1]^d$. Another simple consequence of Theorem 1 is the following. Let $V$ be a finite set and let $A, B$ be families of subsets of $V$ such that $|A \cap B| \leq 1$ holds for all $A \in A, B \in B$. Then we have $|A| \cdot |B| \leq (|V| + 1)^2 |V|$. For instance, this implies that for any $n$-node graph the number of its stable sets times the number of its cliques is bounded by $(n+1)^2 n$ (see also [2, Thm. 3.1]), which is attained for the empty and complete graph, respectively.

**Outline** The proof of Theorem 1 is presented in the next section, in which we make use of several claims, whose proofs are given in Section 3. In Section 4 we provide the proof of Theorem 2.

## 2 Main proof

In this section, we provide a proof for Theorem 1. In what follows, depending on the situation, we treat the sets $A, B$ either as sets of vectors or as sets of points. In particular, ‘span’ always stands for ‘linearly span’, while ‘dim’ stands for affine dimension.

Let $f(d)$ be the maximum of $|A| \cdot |B|$ over all $A, B \subseteq \mathbb{R}^d$ that both span $\mathbb{R}^d$ such that $\langle a, b \rangle \in \{0,1\}$ for all $a \in A, b \in B$. We show that $f(d) \leq (d+1)2^d$ holds by induction on $d \geq 0$. Note that $f(0) = 1$ and let $d \geq 1$. 

Let \( \mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d \) both span \( \mathbb{R}^d \) such that \( \langle a, b \rangle \in \{0, 1\} \) for all \( a \in \mathcal{A}, b \in \mathcal{B} \). We may assume that \( \mathcal{A} \) and \( \mathcal{B} \setminus \{\mathbf{0}\} \) are inclusion-wise maximal with respect to this property. Note that every nonzero vector \( x \in \mathbb{R}^d \) defines two faces of the convex hull of \( \mathcal{A} \), and let \( \varphi(x) \) denote the maximum of the dimensions of these two faces. Let us pick \( b_d \in \mathcal{B} \setminus \{\mathbf{0}\} \) such that \( \varphi(b_d) \geq \varphi(b) \) holds for every \( b \in \mathcal{B} \setminus \{\mathbf{0}\} \).

In what follows, we will invoke the induction hypothesis in the affine hull of one of the two faces that \( b_d \) defines. To this end, it will be convenient to slightly modify \( \mathcal{A} \) and \( \mathcal{B} \), which is done in the following claim. We say that a set \( X \subseteq \mathbb{R}^d \) does not contain opposite points if \( |X \cap \{x, -x\}| \leq 1 \) holds for all \( x \in \mathbb{R}^d \). Let

\[
U := \{x \in \mathbb{R}^d : \langle x, b_d \rangle = 0\}
\]

and consider the orthogonal projection \( \pi : \mathbb{R}^d \to U \) on \( U \).

**Claim 1.** We may translate \( \mathcal{A} \) and replace some points in \( \mathcal{B} \) by their negatives such that the following holds.

(i) We can write \( \mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \), where \( \mathcal{A}_i = \{a \in \mathcal{A} : \langle a, b_d \rangle = i\} \) for \( i = 0, 1 \) such that

\[
|\mathcal{A}_0| \geq |\mathcal{A}_1|. \tag{1}
\]

(ii) We still have

\[
\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}. \tag{2}
\]

(iii) The set \( \pi(\mathcal{B}) \) does not contain opposite points.

Note that, after this transformation we may (still) assume that \( \mathcal{A} \) contains \( \mathbf{0} \), and thus the transformation does not affect the property that \( \mathcal{A} \) and \( \mathcal{B} \) both span \( \mathbb{R}^d \). Also, this transformation does neither affect the choice of \( b_d \), nor the cardinalities of \( \mathcal{A}, \mathcal{B} \).

**Claim 2.** Every point in \( \pi(\mathcal{B}) \) has at most two preimages in \( \mathcal{B} \).

Let \( \mathcal{B}_* \subseteq \mathcal{B} \) denote the set of \( b \in \mathcal{B} \) for which \( \pi(b) \) has a unique preimage. Claim 2 yields \( |\mathcal{B} \setminus \mathcal{B}_*| = 2|\pi(\mathcal{B} \setminus \mathcal{B}_*)| \). We obtain

\[
|\mathcal{B}| = |\mathcal{A}_0||\mathcal{B} \setminus \mathcal{B}_*| + |\mathcal{A}_0||\mathcal{B}_*| + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*| + |\mathcal{A}_1||\mathcal{B}_*|
\leq |\mathcal{A}_0||\mathcal{B} \setminus \mathcal{B}_*| + 2|\mathcal{A}_0||\mathcal{B}_*| + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*|
= 2|\mathcal{A}_0|(|\pi(\mathcal{B} \setminus \mathcal{B}_*)| + |\mathcal{B}_*|) + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*|
= 2|\mathcal{A}_0||\pi(\mathcal{B})| + |\mathcal{A}_1||\mathcal{B} \setminus \mathcal{B}_*|. \tag{3}
\]

where the first inequality follows from (1) and the last one from the definition of \( \mathcal{B}_* \). We will bound the latter two terms separately.

Let us first provide a bound on the term \( |\mathcal{A}_0||\pi(\mathcal{B})| \). To this end, let \( U_0 \subseteq U \) denote the subspace spanned by \( \mathcal{A}_0 \), and let \( \tau : U \to U_0 \) be the orthogonal projection onto \( U_0 \). Note that \( \tau(\pi(\mathcal{B})) \) spans \( U_0 \) as \( \mathcal{B} \) spans \( \mathbb{R}^d \). Moreover, for each \( a \in \mathcal{A}_0 \) and each \( b \in \mathcal{B} \) we have

\[
\langle a, \tau(\pi(b)) \rangle = \langle a, \pi(b) \rangle = \langle a, b \rangle \in \{0, 1\},
\]

1 Clearly, we can always include \( \mathbf{0} \) in \( \mathcal{B} \). However, we emphasize that our proof only uses the inclusion-wise maximality of \( \mathcal{B} \setminus \{\mathbf{0}\} \), a detail that becomes relevant for our application to 2-level polytopes.
where the last equality is due to (2). Thus, we obtain $|A_0||\tau(\pi(B))| \leq f(\dim U_0)$ and hence the induction hypothesis yields

$$|A_0||\tau(\pi(B))| \leq (\dim U_0 + 1)2^{\dim U_0}. \quad (4)$$

Moreover, we have the following relation between the sizes of $\tau(\pi(B))$ and $\pi(B)$:

**Claim 3.** We have $|\pi(B)| \leq 2^{d-1-\dim U_0} |\tau(\pi(B))|.$

Combining (3) and (4) with the above claim we thus obtain

$$|A||B| \leq 2^{d-\dim U_0} |A_0||\tau(\pi(B))| + |A_1||B \setminus B_*| \leq (\dim U_0 + 1)2^d + |A_1||B \setminus B_*|. \quad (5)$$

In order to bound the second term $|A_1||B \setminus B_*|$, the following observation is useful.

**Claim 4.** For each $b \in B \setminus B_*$ we have $|\{ \langle a, b \rangle : a \in A_0 \}| = 1$ or $|\{ \langle a, b \rangle : a \in A_1 \}| = 1$.

The above claim implies that we can partition $B \setminus B_*$ into two sets $B_0, B_1$ where

$$|\{ \langle a, b \rangle : a \in A_0 \}| = 1 \text{ for all } b \in B_0,$$
$$|\{ \langle a, b \rangle : a \in A_1 \}| = 1 \text{ for all } b \in B_1.$$

Note that we may choose this partition such that $0, b_d \notin B_0$. By (5) and (1) we have

$$|A||B| \leq (\dim U_0 + 1)2^d + |A_1||B_0| + |A_1||B_1| \leq (\dim U_0 + 1)2^d + |A_0||B_0| + |A_1||B_1|. \quad (6)$$

Finally, we make use of the following observation.

**Claim 5.** For $i = 0, 1$ we have $|A_i||B_i| \leq 2^d$.

If $\dim U_0 \leq d - 2$, then (6) together with the above claim yields

$$|A||B| \leq (d-1)2^d + 2 \cdot 2^d = (d + 1)2^d,$$

as required. It remains to consider the case $\dim U_0 = d - 1$. In this case, the only nonzero point in $B$ that has constant scalar product with all points in $A_0$ is $b_d$. Since $0, b_d \notin B_0$, we have $B_0 = \emptyset$. Again by (6) and Claim 5 we conclude

$$|A||B| \leq d2^d + |A_1||B_1| \leq d2^d + 2^d = (d + 1)2^d. \quad \square$$

### 3 Proofs of claims

In this section, we provide the proofs of all previous claims, whose statements we repeat here. We begin by explaining how the initial transformation in the proof of Theorem 1 can be performed.

**Claim 1.** We may translate $A$ and replace some points in $B$ by their negatives such that the following holds.
(i) We can write \( \mathcal{A} = \mathcal{A}_0 \cup \mathcal{A}_1 \), where \( \mathcal{A}_i = \{ a \in \mathcal{A} : \langle a, b_d \rangle = i \} \) for \( i = 0, 1 \) such that
\[
|\mathcal{A}_0| \geq |\mathcal{A}_1|.
\]

(ii) We still have
\[
\langle a, b \rangle \in \{0, 1\} \text{ for each } a \in \mathcal{A}_0 \text{ and } b \in \mathcal{B}.
\]

(iii) The set \( \pi(\mathcal{B}) \) does not contain opposite points.

Proof. If \(|\{ a \in \mathcal{A} : \langle a, b_d \rangle = 0 \}| \leq |\{ a \in \mathcal{A} : \langle a, b_d \rangle = 1 \}|\), then we can choose any \( a_* \in \mathcal{A} \) with \( \langle a_*, b_d \rangle = 1 \) (which exists since \( \mathcal{A} \) spans \( \mathbb{R}^d \)) and replace \( \mathcal{A} \) by \( \mathcal{A} - a_* \), \( \mathcal{B} \) by \( (\mathcal{B} \setminus \{ b_d \}) \cup \{-b_d\} \), and \( b_d \) by \(-b_d\). This yields (i).

After this replacement, for each \( b \in \mathcal{B} \) there is some \( \varepsilon_b \in \{\pm 1\} \) such that \( \langle a, b \rangle \in \{0, \varepsilon_b\} \) holds for all \( a \in \mathcal{A} \). Each \( b \) with \( \{ \langle a, b \rangle : a \in \mathcal{A}_0 \} = \{0, -1\} \) is replaced by \(-b\), which yields (ii).

Let \( \mathcal{A}'_1 \) be a translate of \( \mathcal{A}_1 \) such that \( 0 \in \mathcal{A}'_1 \). Note that, for each \( b \in \mathcal{B} \) we now have \( \{ \langle a, b \rangle : a \in \mathcal{A}_0 \} = \{0, 1\} \) or \( \{ \langle a, b \rangle : a \in \mathcal{A}_0 \} = \{0\} \). In the second case, we replace \( b \) by \(-b\) if \( \{ \langle a, b \rangle : a \in \mathcal{A}'_1 \} = \{0, -1\} \), otherwise we leave it as it is.

It remains to show that \( \pi(\mathcal{B}) \) does not contain opposite points after this transformation. To this end, let \( b, b' \in \mathcal{B} \) such that \( \pi(b) = \beta \pi(b') \) for some \( \beta \neq 0 \), where \( \pi(b), \pi(b') \neq 0 \). We have to show that \( \beta = 1 \). Note that for every \( a \in \mathcal{A}_0 \cup \mathcal{A}'_1 \subseteq U \) we have
\[
\langle a, b \rangle = \langle a, \pi(b) \rangle = \beta \langle a, \pi(b') \rangle = \beta \langle a, b' \rangle.
\]

Suppose first that \( \{ \langle a, b \rangle : a \in \mathcal{A}_0 \} \neq \{0\} \). By (2) there exists some \( a \in \mathcal{A}_0 \) with \( 1 = \langle a, b \rangle = \beta \langle a, b' \rangle \). Thus, we have \( \langle a, b' \rangle \neq 0 \) and hence \( \langle a, b' \rangle = 1 \), again by (2). This yields \( \beta = 1 \).

Suppose now that \( \{ \langle a, b \rangle : a \in \mathcal{A}_0 \} = \{0\} \). Note that this implies \( \{ \langle a, b' \rangle : a \in \mathcal{A}_0 \} = \{0\} \).

As \( \mathcal{A}_0 \cup \mathcal{A}'_1 \) spans \( U \), we must have \( \{ \langle a, b \rangle : a \in \mathcal{A}'_1 \} \neq \{0\} \) and hence there is some \( a \in \mathcal{A}'_1 \) with \( \langle a, b \rangle = 1 \). Moreover, we have \( \beta \langle a, b' \rangle = 1 \), and in particular \( \langle a, b' \rangle \neq 0 \). This implies \( \langle a, b' \rangle = 1 \) and hence \( \beta = 1 \).

As in the previous proof, let \( \mathcal{A}'_1 \) be a translate of \( \mathcal{A}_1 \) such that \( 0 \in \mathcal{A}'_1 \). Note that for each \( b \in \mathcal{B} \) there are \( \varepsilon_b, \gamma_b \in \{\pm 1\} \) such that
\[
\begin{align*}
\langle a, b \rangle &\in \{0, \varepsilon_b\} \text{ for each } a \in \mathcal{A} \quad \text{(7)} \\
\langle a, b \rangle &\in \{0, \gamma_b\} \text{ for each } a \in \mathcal{A}'_1 \quad \text{(8)}
\end{align*}
\]

The proofs of the subsequent claims rely on the following two lemmas.

**Lemma 1.** Suppose that \( X \subseteq \{0, 1\}^d \cup \{0, -1\}^d \) does not contain opposite points. Then we have \( |X| \leq 2^{\dim X} \).

Proof. We prove the statement by induction on \( d \geq 1 \), and observe that it is true for \( d = 1 \). Now let \( d \geq 2 \). If \( \dim X = d \), then we are also done. It remains to consider to case where \( X \) is contained in an affine hyperplane \( H \subseteq \mathbb{R}^d \). Let \( c = (c_1, \ldots, c_d) \in \mathbb{R}^d, \delta \in \{0, 1\} \) such that
\[
H = \{ x \in \mathbb{R}^d : \langle c, x \rangle = \delta \}.
\]

For each \( i \in \{1, \ldots, d\} \) let \( \pi_i : H \to \mathbb{R}^{d-1} \) denote the projection that forgets the \( i \)-th coordinate, and let \( e_i \in \mathbb{R}^d \) denote the \( i \)-th standard unit vector. Note that \( \pi_i(X) \subseteq \{0, 1\}^{d-1} \cup \{0, -1\}^{d-1} \).

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Suppose there is some $i^* \in \{1, \ldots, d\}$ such that $\langle c, e_{i^*}\rangle \neq 0$ and $\pi_{i^*}(X)$ does not contain opposite points. By the induction hypothesis we obtain

$$|X| = |\pi_{i^*}(X)| \leq 2^{\dim \pi_{i^*}(X)} = 2^{\dim X},$$

where the first equality and the last inequality hold since $\pi_{i^*}$ is injective (due to $\langle c, e_{i^*}\rangle \neq 0$).

It remains to consider the case in which there is no such $i^*$. Consider any $i \in \{1, \ldots, d\}$. If $\langle c, e_i\rangle \neq 0$, then there exist $x = (x_1, \ldots, x_d), x' = (x'_1, \ldots, x'_d) \in X$, $x \neq x'$ such that $\pi_i(x) = -\pi_i(x')$. We may assume that $\pi_i(x) \in \{0, 1\}^{d-1}$ and hence $\pi_i(x') \in \{0, -1\}^{d-1}$. As $X$ does not contain opposite points, we must have $x_i = 1$ and $x'_i = 0$, or $x_i = 0$ and $x'_i = -1$. In the first case we obtain

$$2\delta = \langle c, x \rangle + \langle c, x' \rangle = [\langle \pi_i(c), \pi_i(x) \rangle + c_i x_i] + [\langle \pi_i(c), \pi_i(x') \rangle + c_i x'_i]$$

$$= [\langle \pi_i(c), \pi_i(x) \rangle + c_i] + [\langle \pi_i(c), \pi_i(x') \rangle]$$

$$= c_i.$$

Similarly, in the second case we obtain $2\delta = -c_i$.

If $\delta = 0$, this would imply that $c = 0$, a contradiction to the fact that $H \neq \mathbb{R}^d$. Otherwise, $\delta = 1$ and hence every nonzero coordinate of $c$ is $\pm 2$. Thus, for every $x \in \mathbb{Z}^d$ we see that $\langle c, x \rangle$ is an even number, in particular $\langle c, x \rangle \neq \delta$. This means that $X \subseteq \mathbb{Z}^d \cap H = \emptyset$, and we are done.

**Lemma 2.** Let $\mathcal{A}, \mathcal{B} \subseteq \mathbb{R}^d$ such that $\mathcal{A}$ spans $\mathbb{R}^d$, $\mathcal{B}$ does not contain opposite points, and for every $b \in \mathcal{B}$ there is some $\varepsilon_b \in \{\pm 1\}$ such that $\{\langle a, b \rangle : a \in \mathcal{A}\} \subseteq \{0, \varepsilon_b\}$. Then we have $|\mathcal{B}| \leq 2^{\dim \mathcal{B}}$.

**Proof.** Let $a_1, \ldots, a_d \in \mathcal{A}$ be a basis of $\mathbb{R}^d$, and let $M \in \mathbb{R}^{d \times d}$ such that $a_i = M^\top e_i$ for $i = 1, \ldots, d$. For every $b \in \mathcal{B}$ we obtain

$$\langle e_i, Mb \rangle = \langle a_i, M^{-1}Mb \rangle = \langle a_i, b \rangle \in \{0, \varepsilon_b\}$$

for $i = 1, \ldots, d$ and hence $Mb \in \{0, \varepsilon_b\}^d$. Thus, the set $X := \{Mb : b \in \mathcal{B}\}$ is contained in $\{0, 1\}^d \cup \{0, -1\}^d$. As $\mathcal{B}$ does not contain opposite points, we see that also $X$ does not contain opposite points. Lemma 1 now implies $|\mathcal{B}| = |X| \leq 2^{\dim X} = 2^{\dim \mathcal{B}}$.

We are ready to continue with the proofs of the remaining claims.

**Claim 2.** Every point in $\pi(\mathcal{B})$ has at most two preimages in $\mathcal{B}$.

**Proof.** Let $y := \pi(b)$ for some $b \in \mathcal{B}$ and observe that $\pi^{-1}(y) = \{x \in \mathbb{R}^d : \pi(x) = \pi(y)\}$ is a one-dimensional affine subspace. By (7) and Lemma 2 we obtain $|\mathcal{B} \cap \pi^{-1}(y)| \leq 2$.

**Claim 3.** We have $|\pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}|\tau(\pi(\mathcal{B}))|$.

**Proof.** Fix any $b \in \mathcal{B}$ and let $v := \pi(b^*)$. Consider the orthogonal complement $W \subseteq U$ of $U_0$ in $U$. As $\pi^{-1}(\tau(v)) = v + W$, it suffices to show that

$$|(v + W) \cap \pi(\mathcal{B})| \leq 2^{d-1-\dim U_0}$$

holds. To this end, consider the linear subspace $\Pi \subseteq U$ spanned by $v$ and $W$ and let $\sigma : U \to \Pi$ denote the orthogonal projection on $\Pi$. 

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First, suppose that $\sigma(A'_1)$ spans $\Pi$. For every $a \in A'_1 \subseteq U$ and every $b \in B$ with $\pi(b) \in v + W \subseteq \Pi$ we have
\[ \langle \sigma(a), \pi(b) \rangle = \langle a, \pi(b) \rangle = \langle a, b \rangle \in \{0, \gamma_b\} \]
by (5). Moreover, recall that $\pi(\mathcal{B})$ does not contain opposite points by Claim (i). Thus, the pair $\sigma(A'_1)$ and $(v + W) \cap \pi(\mathcal{B})$ satisfies the requirements of Lemma 2 (in $U$), and hence we obtain
\[ |(v + W) \cap \pi(\mathcal{B})| \leq 2^{\dim(v+W)} = 2^{\dim W} = 2^{\dim U - \dim U_0} = 2^{d-1-\dim U_0}. \]
It remains to consider the case in which $\sigma(A'_1)$ does not span $\Pi$. Unless $|(v + W) \cap \pi(\mathcal{B})| = 1$, we will identify points $b_1, b_2 \in \mathcal{B}$ with $\max\{\varphi(b_1), \varphi(b_2)\} > \varphi(b_d)$, a contradiction to the choice of $b_d$.

As $A_0 \cup A'_1$ spans $U$, we know that $\sigma(A_0 \cup A'_1)$ spans $\Pi$. Since $A_0$ is orthogonal to $W$, this means that $\sigma(A_0)$ spans a line, and $\sigma(A'_1)$ spans a hyperplane $H$ in $\Pi$. Note that we have $v \notin W$ (otherwise $W = \Pi$ and so $\sigma(A'_1)$ spans $\Pi$). Thus, every nonzero point in $\sigma(A_0)$ has nonzero scalar product with $v$. Moreover, for every $a \in A_0$ with $\sigma(a) \neq 0$ we have $\langle \sigma(a), v \rangle = \langle a, v \rangle = \langle a, b \rangle \in \{0, 1\}$ by (2). Thus, since the nonzero vectors in $\sigma(A_0)$ are collinear, we obtain
\[ \sigma(A_0) \subseteq \{0, \sigma(a_0)\} \]
for some $a_0 \in A_0$. Since $0 \in H$, we have $\sigma(A_0) \setminus H \subseteq \{\sigma(a_0)\}$ and further, since $\sigma(A_0 \cup A'_1)$ spans $\Pi$, we have $\sigma(A_0) \setminus H = \{\sigma(a_0)\}$. Let $c \in \Pi$ be a normal vector of $H$. As $\sigma(a_0) \notin H$, we may scale $c$ so that $\langle \sigma(a_0), c \rangle = 1$. Let $a_* \in A_1$ such that $A'_1 = A_1 - a_*$. We define
\[ b_1 := c - \delta_1 b_d \neq 0, \]
where $\delta_1 := \langle a_*, c \rangle$. For every $a \in A_0$ we have
\[ \langle a, b_1 \rangle = \langle a, c \rangle = \langle \sigma(a), c \rangle \in \{0, c, \langle \sigma(a_0), c \rangle \} = \{0, 1\}, \]
and for every $a \in A_1$ we have
\[ \langle a, b_1 \rangle = \langle a - a_*, b_1 \rangle + \langle a_*, b_1 \rangle = \langle a - a_*, c \rangle + \langle a_*, b_1 \rangle = \langle \sigma(a - a_*), c \rangle + \langle a_*, b_1 \rangle = \langle a_*, b_1 \rangle = \langle a_*, c \rangle - \delta_1 \langle a_*, b_d \rangle = \langle a_*, c \rangle - \delta_1 = 0. \]
Thus, by the maximality of $\mathcal{B}$, (a scaling of) the vector $b_1$ is contained in $\mathcal{B}$. Since we assumed $0 \in A_0$, we have $\varphi(b_1) \geq \dim(A_1) + 1$.

In order to construct $b_2$, let us suppose that there is another point $b' \in \mathcal{B}$ with $v' := \pi(b') \neq v$ and $v' \in (v + W)$. If there is no such point, then the statement of the claim is true. Recall that $\sigma(a_0)$ is orthogonal to $W$, and let
\[ \xi := \langle \sigma(a_0), v' \rangle = \langle \sigma(a_0), v - v' \rangle + \langle \sigma(a_0), v' \rangle = \langle \sigma(a_0), v' \rangle. \]
Choose $v'' \in \{v, v'\}$ such that $\xi c \neq v''$, and let $b'' \in \{b, b'\}$ such that $\pi(b'') = v''$. Define $\delta_2 := \langle a_*, v'' - \xi c \rangle$ and note that
\[ b_2 := v'' - \xi c - \delta_2 b_d. \]
is nonzero since $v'' - \xi c \in U \setminus \{0\}$. For every $a \in A_0$ we have
\[
\langle a, b_2 \rangle = \langle a, v'' - \xi c \rangle = \langle \sigma(a), v'' - \xi c \rangle,
\]
which is zero if $\sigma(a) = 0$. Otherwise, $\sigma(a) = \sigma(a_0)$ and we obtain
\[
\langle a, b_2 \rangle = \langle \sigma(a_0), v'' \rangle - \xi \langle \sigma(a_0), c \rangle = \langle \sigma(a_0), v'' \rangle - \xi = 0.
\]
Thus, $b_2$ is orthogonal to $A_0$. Moreover, note that
\[
\langle a, b_2 \rangle = \langle a, v'' - \xi c \rangle - \delta_2 \langle a, b_d \rangle = 0.
\]
Thus, for every $a \in A_1$ we have
\[
\langle a, b_2 \rangle = \langle a - a_s, b_2 \rangle + \langle a_s, b_2 \rangle = \langle a - a_s, b_2 \rangle = \langle a - a_s, v'' \rangle - \xi \langle a - a_s, c \rangle - \delta_2 \langle a - a_s, b_d \rangle = 0,
\]
\[
\langle a - a_s, v'' \rangle = \langle a - a_s, b'' \rangle \in \{0, \gamma v'' \}
\]
by (8). Thus, again by the maximality of $B$, (a scaling of) the vector $b_2$ is contained in $B$, and since $b_2$ is orthogonal to $A_0$ and $a_s \in A_1$, we have $\varphi(b_2) \geq \dim(A_0) + 1$. However, by the choice of $b_d$ we must have
\[
\max\{\dim(A_0), \dim(A_1)\} + 1 \leq \max\{\varphi(b_1), \varphi(b_2)\} \leq \varphi(b_d) = \max\{\dim(A_0), \dim(A_1)\},
\]
a contradiction.

\begin{claim}
For each $b \in B \setminus B_*$ we have $|\{\langle a, b \rangle : a \in A_0\}| = 1$ or $|\{\langle a, b \rangle : a \in A_1\}| = 1$.
\end{claim}

\begin{proof}
Let $b \in B \setminus B_*$ and, for the sake of contradiction, suppose that $|\{\langle a, b \rangle : a \in A_0\}| = |\{\langle a, b \rangle : a \in A_1\}| = 2$. Let $b' \in B \setminus \{b\}$ such that $\pi(b) = \pi(b')$. In other words, we have $b' = b + \gamma b_d$ for some $\gamma \neq 0$. Then, by (2) we have
\[
\{\langle a, b' \rangle : a \in A_0\} = \{\langle a, b \rangle : a \in A_0\} = \{0, 1\}
\]
and hence we obtain $\varepsilon_b = \varepsilon_{b'} = 1$ by (7). Again by (7) we see
\[
\{0, 1\} \supseteq \{\langle a, b' \rangle : a \in A_1\} = \{\langle a, b \rangle : a \in A_1\} + \gamma = \{0, 1\} + \gamma = \{\gamma, 1 + \gamma\},
\]
which implies $\gamma = 0$, a contradiction.
\end{proof}

\begin{claim}
For $i = 0, 1$ we have $|A_i|/|B_i| \leq 2^d$.
\end{claim}

\begin{proof}
First, note that by (7) there is an invertible matrix $M \in \mathbb{R}^{d \times d}$ such that $M(A) := \{Ma : a \in A\} \subseteq \{0, 1\}^d$. Now let $i \in \{0, 1\}$. Denote by $V$ the span of $B_i$ and let $k := \dim V$. Clearly, we have $B_i \subseteq B \cap V$ and hence by (7) and Lemma 2 we obtain $|B_i| \leq |B \cap V| \leq 2^k$. Recall that for each $b \in B_i$ there some $\xi_b$ such that $\langle a, b \rangle = \xi_b$ holds for all $a \in A_i$. Thus, $A_i$ is a subset of $A \cap V'$, where $V' := \{x \in \mathbb{R}^d : \langle x, b \rangle = \xi_b \text{ for all } b \in B_i\}$, and hence we obtain
\[
|A_i| = |M(A_i)| \leq |M(A \cap M(V'))| \leq |\{0, 1\}^d \cap M(V')| \leq 2^\dim M(V') = 2^d + \dim V' \leq 2^d - d \dim V = 2^{d-k},
\]
where the third inequality follows from Lemma 1. We conclude that $|A_i|/|B_i| \leq 2^{d-k}2^k = 2^d$.
\end{proof}
4 Application to 2-level polytopes

In this section, we provide a proof for our main application:

**Theorem 2.** Every $d$-dimensional 2-level polytope $P$ satisfies $f_0(P)f_{d-1}(P) \leq d2^{d+1}$.

**Proof.** Let $P \subseteq \mathbb{R}^d$ be a $d$-dimensional 2-level polytope. We may assume that $0$ is among the vertices of $P$. Thus, there exists a finite set $\mathcal{B} \subseteq \mathbb{R}^d \setminus \{0\}$ such that

$$P = \{x \in \mathbb{R}^d : 0 \leq \langle x, b \rangle \leq 1 \text{ for every } b \in \mathcal{B}\},$$

where for each $b$, at least one of the equations $\langle x, b \rangle = 0, \langle x, b \rangle = 1$ defines a facet of $P$. Let $\mathcal{A}$ consist of the vertices of $P$. As $P$ is $d$-dimensional and pointed, both $\mathcal{A}$ and $\mathcal{B}$ span $\mathbb{R}^d$ and hence $\mathcal{A}$ and $\mathcal{B}$ satisfy the assumptions of Theorem 1. If no vector in $\mathcal{B}$ defines two facets of $P$, we have

$$f_0(P)f_{d-1}(P) = |\mathcal{A}||\mathcal{B}| \leq (d+1)2^d \leq d2^{d+1}.$$

If there exists a vector that defines two facets of $P$, then in the proof of Theorem 1 we may choose $b_d$ as this vector. Indeed, using the notation of Section 2, $\mathcal{A}_0, \mathcal{A}_1$ are then the vertex sets of the respective facets and $\varphi(b_d) = \dim \mathcal{A}_0 = \dim \mathcal{A}_1 = \dim U_0 = d - 1$. Recall the inequality (6), which yields

$$f_0(P)f_{d-1}(P) \leq 2|\mathcal{A}||\mathcal{B}| \leq (\dim U_0 + 1)2^{d+1} + 2|\mathcal{A}_0||\mathcal{B}_0| + 2|\mathcal{A}_1||\mathcal{B}_1|$$

$$= d2^{d+1} + 2|\mathcal{A}_0||\mathcal{B}_0| + 2|\mathcal{A}_1||\mathcal{B}_1|.$$

We claim that $\mathcal{B}_0 = \mathcal{B}_1 = \emptyset$, in which case we are done. To this end, suppose there is some $b \in \mathcal{B}_i$. Recall that $b$ has constant scalar product with all points in $\mathcal{A}_i$. As $\mathcal{B}_i \subseteq \mathcal{B}$ and $0 \notin \mathcal{B}$, we obtain $b = b_d$. However, using $0 \notin \mathcal{B}$ again, we have $b_d \in \mathcal{B}_s$, contradicting the fact that $\mathcal{B}_s \subseteq \mathcal{B} \setminus \mathcal{B}_s$. \qed

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