EVALUATION OF THE CONVOLUTION SUMS \( W_{1,42}(n), W_{2,21}(n), \ W_{3,14}(n) \) AND \( W_{6,7}(n) \)

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Abstract. In this paper, we use a modular form approach to evaluate the convolution sums 
\[
W_{1,42}(n) = \sum_{l+42m=n} \sigma(l)\sigma(m), \quad W_{2,21}(n) = \sum_{2l+21m=n} \sigma(l)\sigma(m), \quad W_{3,14}(n) = \sum_{3l+14m=n} \sigma(l)\sigma(m) \quad \text{and} \quad W_{6,7}(n) = \sum_{6l+7m=n} \sigma(l)\sigma(m)
\]
for all positive integers \( n \), and then use their evaluations to determine the number of representation of a positive integer \( n \) by the quadratic form
\[
x_1^2 + x_2^2 + x_3^2 + x_4^2 + x_5^2 + x_6^2 + x_7^2 + x_8^2 + x_9^2 + 14(x_2^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2).
\]

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1. Introduction

Let \( \mathbb{N}, \mathbb{Z}, \mathbb{Q} \) and \( \mathbb{C} \) denote the set of positive integers, integers, rational numbers and complex numbers respectively and let \( \mathbb{N}_0 = \mathbb{N} \cup \{0\} \). For \( k \in \mathbb{N} \) and \( n \in \mathbb{Q} \) we set

\[
\sigma_k(n) = \begin{cases} 
\sum_{d|n} d^k, & \text{if } n \in \mathbb{N}, \\
0, & \text{if } n \in \mathbb{Q}, n \notin \mathbb{N}.
\end{cases}
\]

We write \( \sigma(n) \) for \( \sigma_1(n) \). Suppose that \( r, s \in \mathbb{N} \) with \( r \leq s \). We define the convolution sum \( W_{r,s}(n) \) as follows:

\[
W_{r,s}(n) := \sum_{(l,m) \in \mathbb{N}_0^2, rl+sm=n} \sigma(l)\sigma(m).
\]

The evaluation of convolution sums \( W_{r,s}(n) \) for some levels \( rs \) have been done. See Table \[1\] for a list of known convolution sums. In the present paper, we completed the evaluation of the convolution sums \( W_{r,s}(n) \) for \( (r,s) = (1,42), (2,21), (3,14) \) and \( (6,7) \) by using a modular form approach. We also evaluated the results for \( (1,14), (2,7), (1,7) \) which were firstly given in \[2\] and see that they are consistent.

Key words and phrases. Convolution sums, eta quotients, Eisenstein series, modular forms, cusp forms, quadratic forms, representation numbers.

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Table 1. A list of previously known convolution sums.

| Level $rs$ | Authors | References |
|------------|---------|------------|
| 1          | M. Besge & J.W.L. Glaisher & S. Ramanujan | [9][14][28] |
| 2, 3, 4    | J. G. Huard & Z. M. Ou & B. K. Spearman & K. S. Williams | [15] |
| 5, 7       | M. Lemire & K. S. Williams & S. Cooper & P. C. Toh | [13][21] |
| 6          | Ş. Alaca & K. S. Williams | [8] |
| 8, 9       | K. S. Williams | [32][33] |
| 10, 11, 13, 14 | E. Royer | [29] |
| 12, 16, 18, 24 | A. Alaca & Ş. Alaca & K. S. Williams | [1][8][9] |
| 15         | B. Ramakrishnan & B. Sabu | [27] |
| 10, 20     | S. Cooper & D. Ye | [12] |
| 23         | H. H. Chan & S. Cooper | [10] |
| 25         | E. X. W. Xia & X. L. Tian & O. X. M. Yao | [34] |
| 27, 32, 48, 54 | Ş. Alaca & Y. Kesicioğlu | [6][7] |
| 36         | D. Ye | [35] |
| 14, 26, 28, 30 | A. Alaca & Ş. Alaca & E. Ntienjem | [2][23] |
| 22, 44, 52 | E. Ntienjem | [25] |
| 27, 40, 55 | B. Kendirli | [17] |
| 33, 40, 56 | E. Ntienjem | [26] |
| 48, 64     | E. Ntienjem | [24] |
| 17, 34, 68 | B. Köklüce | [20] |

For $l \in \mathbb{N}$ and $n \in \mathbb{N}_0$ we let $N_l(n)$ denote the representation number of $n$ by the form $x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2 + l(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)$, that is

$$N_l(n) := \text{card}\left\{(x_1, \ldots, x_8) \in \mathbb{Z}^8 : n = x_1^2 + x_1x_2 + x_2^2 + x_3x_4 + x_4^2 + l(x_5^2 + x_5x_6 + x_6^2 + x_7^2 + x_7x_8 + x_8^2)\right\}. \quad (1.2)$$

Explicit formulae for $N_l(n)$ are obtained before for $l \leq 12$ and for $l = 16$ and 18. See for example, [1][8][22][23][27][32]. The author of this article also found many such representation number formulae for the direct sum of quadratic forms of this type, see for example, [18][19]. In this article, we use the convolutions sums $W_{3,14}(n)$ and $W_{1,42}(n)$ obtained here with the convolution sum $W_{1,14}(n)$ to find explicit formula for $N_{14}(n)$.

The rest of this paper is organized as follows. In Section 2, we give some preliminary results related to the eta products and modular forms. In Section 3, we give
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In Section 4, we give a formula for \( N_{14}(n) \) and then prove it.

2. Preliminary Results

For \( N \in \mathbb{N} \) and \( k \in \mathbb{Z} \) we write \( M_k(\Gamma_0(N)) \) to denote the space of modular forms of weight \( k \) (with trivial multiplier system) for the modular subgroup \( \Gamma_0(N) \) defined by

\[
\Gamma_0(N) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL_2(\mathbb{Z}) : c \equiv 0 (\text{mod} \ N) \right\}.
\]  

(2.1)

It is known (see, for example [30, p.83]) that

\[
M_k(\Gamma_0(N)) = E_k(\Gamma_0(N)) \oplus S_k(\Gamma_0(N))
\]  

(2.2)

where \( E_k(\Gamma_0(N)) \) and \( S_k(\Gamma_0(N)) \) are the corresponding subspaces of Eisenstein forms and cusp forms of weight \( k \) for the modular subgroup \( \Gamma_0(N) \).

The Dedekind eta function \( \eta(z) \) is the holomorphic function defined on the upper half plane \( \mathbb{H} = \{ z \in \mathbb{C} : \text{Im}(z) > 0 \} \) by the product formula

\[
\eta(z) = e^{\pi iz/12} \prod_{n=1}^{\infty} (1 - e^{2\pi inz}).
\]  

(2.3)

Through the remainder of the paper we take \( q = q(z) := e^{2\pi iz} \) with \( z \in \mathbb{H} \) and so by (2.3) we have

\[
\eta(z) = q^{1/24} \prod_{n=1}^{\infty} (1 - q^n).
\]  

(2.4)

An eta quotient is defined to be a finite product of the form

\[
f(z) = \prod_{\delta} \eta^{r_\delta}(\delta z),
\]  

(2.5)

where \( \delta \) runs through a finite set of positive integers and the exponents \( r_\delta \) are nonzero integers. By taking \( N \) to be the least common multiple of \( \delta \)'s we can write the eta quotient (2.5) as

\[
f(z) = \prod_{\delta \mid N} \eta^{r_\delta}(\delta z)
\]  

(2.6)

where some of the exponents \( r_\delta \) may be zero. When all of the exponents \( r_\delta \) are nonnegative, \( f(z) \) is said to be an eta product. Now we define the following 20 eta quotients

\[
C_1(q) = \frac{\eta^5(z)\eta^5(7z)}{\eta(2z)\eta(14z)}.
\]  

(2.7)
\[ C_2(q) = \eta^2(z)\eta^2(2z)\eta^2(7z)\eta^2(14z), \quad (2.8) \]

\[ C_3(q) = \frac{\eta^6(z)\eta^6(14z)}{\eta^2(2z)\eta^2(7z)}, \quad (2.9) \]

\[ C_4(q) = \frac{\eta^6(2z)\eta^6(7z)}{\eta^2(z)\eta^2(14z)}, \quad (2.10) \]

\[ C_5(q) = \eta^2(z)\eta^2(2z)\eta^2(3z)\eta^2(6z), \quad (2.11) \]

\[ C_6(q) = \eta(2z)\eta^3(3z)\eta(7z)\eta^3(42z), \quad (2.12) \]

\[ C_7(q) = \eta(z)\eta^3(6z)\eta(14z)\eta^3(21z), \quad (2.13) \]

\[ C_8(q) = \eta^2(7z)\eta^2(14z)\eta^2(21z)\eta^2(42z), \quad (2.14) \]

\[ C_9(q) = \eta(3z)\eta(6z)\eta^3(7z)\eta^3(14z), \quad (2.15) \]

\[ C_{10}(q) = \eta^2(2z)\eta^2(6z)\eta^2(7z)\eta^2(21z), \quad (2.16) \]

\[ C_{11}(q) = \eta(z)\eta(2z)\eta^3(21z)\eta^3(42z), \quad (2.17) \]

\[ C_{12}(q) = \eta^3(z)\eta(6z)\eta^3(14z)\eta(21z), \quad (2.18) \]

\[ C_{13}(q) = \eta^3(z)\eta^3(2z)\eta(21z)\eta(42z), \quad (2.19) \]

\[ C_{14}(q) = \frac{\eta^2(2z)\eta^2(14z)\eta^6(21z)}{\eta^2(7z)}, \quad (2.20) \]

\[ C_{15}(q) = \frac{\eta^6(2z)\eta^2(3z)\eta^2(21z)}{\eta^4(6z)}, \quad (2.21) \]

\[ C_{16}(q) = \frac{\eta^5(2z)\eta^5(21z)}{\eta(z)\eta(42z)}, \quad (2.22) \]

\[ C_{17}(q) = \frac{\eta^6(6z)\eta^6(21z)}{\eta^2(3z)\eta^2(42z)}, \quad (2.23) \]

\[ C_{18}(q) = \frac{\eta^3(6z)\eta(7z)\eta(14z)\eta^4(21z)}{\eta(3z)}, \quad (2.24) \]

\[ C_{19}(q) = \eta^2(3z)\eta^2(6z)\eta^2(21z)\eta^2(42z), \quad (2.25) \]
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\[ C_{20}(q) = \frac{\eta^4(3z)\eta^4(14z)\eta^2(21z)\eta^2(42z)}{\eta^2(6z)\eta^2(7z)}, \quad (2.26) \]

and the integers $c_k(n)$ $(n \in \mathbb{N})$ for $1 \leq k \leq 20$ by

\[ C_k(q) = \sum_{n=1}^{\infty} c_k(n)q^n. \quad (2.27) \]

The Eisenstein series $L(q)$ and $M(q)$ are defined by

\[ L(q) = E_2(q) = 1 - 24 \sum_{n=1}^{\infty} \sigma(n)q^n, \quad (2.28) \]

and

\[ M(q) = E_4(q) = 1 + 240 \sum_{n=1}^{\infty} \sigma_3(n)q^n. \quad (2.29) \]

We use the following theorem to determine if a given eta product is in $M_k(\Gamma_0(N))$, see [16, Theorem 5.7, p.99]. Note that the eta quotients given in (2.7)-(2.10) are the same first four eta quotients which are used in [2].

**Theorem 1.** Let $N$ be a positive integer and let $f(z) = \prod_{1 \leq \delta \mid N} \eta^{r_\delta}(\delta z)$ be an eta quotient which satisfies the following conditions:

(i) \( \sum_{1 \leq \delta \mid N} r_\delta \equiv 0 \pmod{24}, \)

(ii) \( \sum_{1 \leq \delta \mid N} \frac{N}{\delta} r_\delta \equiv 0 \pmod{24}, \)

(iii) \( \prod_{1 \leq \delta \mid N} \delta^{r_\delta} \) is the square of a rational number,

(iv) for each $d \mid N$, \( \sum_{1 \leq \delta \mid N} \frac{\gcd(d, \delta)^2 r_\delta}{\delta} \geq 0, \)

(v) the weight $k = \frac{1}{2} \sum_{1 \leq \delta \mid N} r_\delta$ is an even integer.

Then $f(z)$ is in $M_k(\Gamma_0(N))$. In addition to the above conditions if all the inequalities in (iv) hold strictly then $f(z)$ is in $S_k(\Gamma_0(N))$.

Note that the cusp forms $C_k(q)$ $(1 \leq k \leq 20)$ defined in (2.7)-(2.26) are constructed in a way that they satisfy the conditions of Theorem 1 from (i) to (v).

**Theorem 2.** (a) \{\(C_k(q)\) $(1 \leq k \leq 20)$\} is a basis for $S_4(\Gamma_0(42))$.

(b) $E_4(q^t)$ $(t = 1, 2, 3, 6, 7, 14, 21, 42)$ constitute a basis for $E_4(\Gamma_0(42))$.

(c) \{\(C_k(q)\)(1 \leq k \leq 20)\} together with $E_4(q^t)$ $(t = 1, 2, 3, 6, 7, 14, 21, 42)$ constitute a basis for $M_4(\Gamma_0(42))$. 

Proof. (a) It follows from (2.7)-(2.26), (2.4)-(2.6) and Theorem 1 that $C_k(q)(1 \leq k \leq 20)$ are in $S_4(\Gamma_0(42))$. By [16, Theorem 3.8, p.50], the dimension of $S_4(\Gamma_0(42))$ is 20. We use the Maple software to show that there is no linear relationship among $C_k(q)(1 \leq k \leq 20)$. Thus $C_k(q)(1 \leq k \leq 20)$ form a basis of $S_4(\Gamma_0(42))$.

(b) It can be shown by using the dimension formula for the Eisenstein space (see for example [16, Theorem 3.8, p.50]) that the dimension of $E_4(\Gamma_0(42))$ is 8. Thus it follows from [30, Theorem 5.9, p.88] that $E_4(q^t)$ ($t = 1, 2, 3, 6, 7, 14, 21, 42$) constitute a basis of $E_4(\Gamma_0(42))$.

(c) It follows from parts (a), (b) and (2.2) that the dimension of $M_4(\Gamma_0(42))$ is 28, and therefore $E_4(q^t)$ ($t = 1, 2, 3, 6, 7, 14, 21, 42$) together with $C_k(q)$ ($1 \leq k \leq 20$) constitute a basis for the space $M_4(\Gamma_0(42))$. □

The following theorem is given in [16, Proposition 2.12, p 23].

**Theorem 3.** Let $N$ be a positive integer. Then

$$M = [SL_2(\mathbb{Z}) : \Gamma_0(N)] = N \prod_{p|N} (1 + \frac{1}{p}).$$

We use Theorem 3 with Sturm’s bound theorem (see [16, Theorem 3.13, p.53] or [31]) to see if two modular forms in the same space are equal. We can restate the Sturm’s theorem for our case as follows.

**Theorem 4.** Let $\Gamma_0 \in SL_2(\mathbb{Z})$ be a congruence subgroup of index $M$ and let $f \in M_4(\Gamma_0(N))$ be a modular form. If

$$v_\infty(f) > \frac{M}{3} = S(N)$$

then $f$ is identically zero. Thus, if $f_1(z)$ and $f_2(z)$ are two modular forms in $M_4(\Gamma_0(N))$ with Fourier series expansions $f_1(z) = \sum_{n=1}^{\infty} a_n q^n$ and $f_2(z) = \sum_{n=1}^{\infty} b_n q^n$ such that $a_n = b_n$ for all $n \leq S(N)$ then $f(z) = g(z)$. By Theorem 4, it is clear that the Sturm bound for $M_4(\Gamma_0(42))$ is $S(42) = 32$. (2.30)
Theorem 5. We have,

\[
(L(q) - 42L(q^{42}))^2 = 604 \frac{M(q)}{625} - 84 \frac{M(q^2)}{625} - 189 \frac{M(q^3)}{625} - 756 \frac{M(q^6)}{625} - 1029 \frac{M(q^7)}{625} - 4116 \frac{M(q^{14})}{625} - 9261 \frac{M(q^{21})}{625} + \frac{1065456}{625}M(q^{42}) + \frac{6912}{5}C_1(q) + \frac{24624}{25}C_2(q) + \frac{1728}{5}C_4(q) + \frac{1008}{125}C_5(q) - \frac{296352}{5}C_6(q) + \frac{534084}{25}C_7(q) - \frac{2067408}{125}C_8(q) - \frac{34272}{5}C_9(q) + \frac{2346624}{25}C_{10}(q) - \frac{3284064}{5}C_{11}(q) + \frac{1024128}{25}C_{12}(q) + \frac{124992}{5}C_{13}(q) + \frac{653184}{5}C_{14}(q) + \frac{48384}{5}C_{15}(q) + \frac{520128}{5}C_{16}(q) + \frac{217728}{5}C_{18}(q) + \frac{27216}{25}C_{19}(q) + \frac{36288}{5}C_{20}(q)
\]

\[
(2L(q^2) - 21L(q^{21}))^2 = -21 \frac{M(q)}{625} + \frac{2416}{625}M(q^2) - \frac{189}{625}M(q^3) - \frac{756}{625}M(q^6) - 1029 \frac{M(q^7)}{625} - \frac{4116}{625}M(q^{14}) + \frac{26364}{625}M(q^{21}) - \frac{37044}{625}M(q^{42}) + \frac{3456}{5}C_1(q) + \frac{24624}{25}C_2(q) + 1728C_3(q) - \frac{3456}{5}C_4(q) + \frac{1008}{125}C_5(q) - 78624C_6(q) + \frac{4251744}{25}C_7(q) + \frac{4282992}{125}C_8(q) - \frac{82656}{5}C_9(q) + \frac{2346624}{25}C_{10}(q) - \frac{2292192}{5}C_{11}(q) + \frac{983808}{25}C_{12}(q) - \frac{133056}{5}C_{13}(q) + \frac{870912}{5}C_{14}(q) + \frac{24192}{5}C_{15}(q) - \frac{471744}{5}C_{16}(q) - \frac{72576}{5}C_{18}(q) + \frac{27216}{25}C_{19}(q) + \frac{108864}{5}C_{20}(q)
\]
\[(3L(q^3) - 14L(q^{14}))^2 = -\frac{21}{625} M(q) - \frac{84}{625} M(q^2) + \frac{5436}{625} M(q^3) - \frac{756}{625} M(q^6) \]
\[\quad - \frac{1029}{625} M(q^7) + \frac{118384}{625} M(q^{14}) - \frac{9261}{625} M(q^{21}) \]
\[\quad - \frac{6912}{625} M(q^{42}) - \frac{14}{5} C_1(q) + \frac{3024}{25} C_2(q) \]
\[\quad + \frac{6912}{5} C_4(q) + \frac{1008}{125} C_5(q) + \frac{223776}{5} C_6(q) \]
\[\quad - \frac{5062176}{125} C_7(q) + \frac{2166192}{125} C_8(q) + \frac{34272}{5} C_9(q) \]
\[\quad - \frac{2491776}{25} C_{10}(q) + \frac{3284064}{5} C_{11}(q) \]
\[\quad - \frac{1052352}{25} C_{12}(q) + \frac{124992}{5} C_{13}(q) - \frac{653184}{5} C_{14}(q) \]
\[\quad - \frac{48384}{5} C_{15}(q) + \frac{520128}{5} C_{16}(q) + 15552C_{17}(q) \]
\[\quad - \frac{217728}{5} C_{18}(q) + \frac{221616}{25} C_{19}(q) - \frac{36288}{5} C_{20}(q) \]

\[(6L(q^6) - 7L(q^7))^2 = \frac{21}{625} M(q) - \frac{84}{625} M(q^2) - \frac{189}{625} M(q^3) + \frac{21744}{625} M(q^6) \]
\[\quad + \frac{29396}{625} M(q^7) - \frac{4116}{625} M(q^{14}) - \frac{9261}{625} M(q^{21}) \]
\[\quad - \frac{37044}{625} M(q^{42}) - \frac{3456}{5} C_1(q) + \frac{175824}{25} C_2(q) \]
\[\quad - \frac{1728C_3(q)}{5} - \frac{5184}{125} C_4(q) + \frac{217008}{125} C_5(q) \]
\[\quad + \frac{320544}{5} C_6(q) - \frac{5960736}{125} C_7(q) + \frac{6399792}{125} C_8(q) \]
\[\quad + \frac{82656}{25} C_9(q) - \frac{763776}{25} C_{10}(q) + \frac{2992192}{5} C_{11}(q) \]
\[\quad - \frac{1674432}{25} C_{12}(q) + \frac{133056}{5} C_{13}(q) - \frac{870912}{5} C_{14}(q) \]
\[\quad - \frac{24192}{5} C_{15}(q) + \frac{471744}{5} C_{16}(q) - 15552C_{17}(q) \]
\[\quad + \frac{72576}{5} C_{18}(q) + \frac{1776816}{25} C_{19}(q) - \frac{108864}{5} C_{20}(q) \]

*Proof.* We only prove the case for \((L(q) - 42L(q^{42}))^2\) as the rest can be proven in a similar way. By [30, Theorem 5.8] we have \(L(q) - 42L(q^{42}) \in M_2(\Gamma_0(42))\), and so
Thus, appealing to Theorem 1(c), we can express \((L(q) - 42L(q^{42}))^2\) as a linear combination of \(E_t(q^t)\)  \((t = 1, 2, 3, 6, 7, 14, 21, 42)\) and \(C_k(q)(1 \leq k \leq 20)\). So, there exist coefficients \(x_t\) \((t \in \mathbb{N}, t \mid 42)\) and \(y_k\) \((k = 1, \ldots, 20)\) such that

\[
(L(q) - 42L(q^{42}))^2 = \sum_{t \in \mathbb{N}, t \mid 42} x_t E_t(q^t) + \sum_{k=1}^{20} y_k C_k(q).
\]  

(3.1)

Appealing to (2.30) and equating the coefficients of \(q^n\) for \(1 \leq n \leq 32\) on both sides of (3.1) we obtain required result. The following theorem can be given as the result of Theorem 5, equations (2.27) and (2.29).

\[\square\]

**Theorem 6.**

\[
(L(q) - 42L(q^{42}))^2 = 1681 + \sum_{n=1}^{\infty} \left( \frac{28992}{125} \sigma_3(n) - \frac{4032}{125} \sigma_3\left(\frac{n}{2}\right) - \frac{9072}{125} \sigma_3\left(\frac{n}{3}\right) \right) \\
- \frac{36288}{125} \sigma_3\left(\frac{n}{6}\right) - \frac{49392}{125} \sigma_3\left(\frac{n}{7}\right) - \frac{197568}{125} \sigma_3\left(\frac{n}{14}\right) \\
- \frac{44528}{125} \sigma_3\left(\frac{n}{21}\right) + \frac{51141888}{125} \sigma_3\left(\frac{n}{42}\right) + \frac{6912}{5} c_1(n) \\
+ \frac{24624}{25} c_2(n) + \frac{1728}{5} c_4(n) + \frac{1008}{125} c_5(n) - \frac{296352}{5} c_6(n) \\
+ \frac{5340384}{25} c_7(n) - \frac{2067408}{125} c_8(n) - \frac{34272}{5} c_9(n) \\
+ \frac{2346624}{25} c_{10}(n) - \frac{3284064}{5} c_{11}(n) + \frac{1024128}{25} c_{12}(n) \\
- \frac{124992}{5} c_{13}(n) + \frac{653184}{5} c_{14}(n) + \frac{48384}{5} c_{15}(n) \\
- \frac{520128}{5} c_{16}(n) + \frac{217728}{5} c_{18}(n) + \frac{27216}{25} c_{19}(n) \\
+ \frac{36288}{5} c_{20}(n)q^n
\]
\[
\begin{align*}
(2L(q^2) - 21L(q^{21}))^2 &= 361 + \sum_{n=1}^{\infty} \left( -\frac{1008}{125} \sigma_3(n) + \frac{115968}{125} \sigma_3\left(\frac{n}{2}\right) - \frac{9072}{125} \sigma_3\left(\frac{n}{3}\right) \\
&- \frac{36288}{125} \sigma_3\left(\frac{n}{6}\right) - \frac{49392}{125} \sigma_3\left(\frac{n}{7}\right) - \frac{197568}{125} \sigma_3\left(\frac{n}{14}\right) \\
&+ \frac{12785472}{125} \sigma_3\left(\frac{n}{21}\right) - \frac{1778112}{125} \sigma_3\left(\frac{n}{42}\right) + \frac{3456}{5} c_1(n) \\
&+ \frac{24624}{25} c_2(n) + 1728c_3(n) - \frac{3456}{5} c_4(n) + \frac{1008}{125} c_5(n) \\
&- \frac{78624c_6(n)}{25} + \frac{4251744}{25} c_7(n) + \frac{4282992}{125} c_8(n) \\
&- \frac{82656}{5} c_9(n) + \frac{2346624}{25} c_{10}(n) - \frac{2292192}{5} c_{11}(n) \\
&+ \frac{98308}{25} c_{12}(n) - \frac{133056}{5} c_{13}(n) + \frac{870912}{5} c_{14}(n) \\
&+ \frac{24192}{5} c_{15}(n) - \frac{471744}{5} c_{16}(n) - \frac{72576}{5} c_{18}(n) \\
&+ \frac{27216}{25} c_{19}(n) + \frac{108864}{5} c_{20}(n) q^n
\end{align*}
\]

\[
\begin{align*}
(3L(q^3) - 14L(q^{14}))^2 &= 121 + \sum_{n=1}^{\infty} \left( -\frac{1008}{125} \sigma_3(n) - \frac{4032}{125} \sigma_3\left(\frac{n}{2}\right) + \frac{269928}{125} \sigma_3\left(\frac{n}{3}\right) \\
&- \frac{36288}{125} \sigma_3\left(\frac{n}{6}\right) - \frac{49392}{125} \sigma_3\left(\frac{n}{7}\right) + \frac{5682432}{125} \sigma_3\left(\frac{n}{14}\right) \\
&- \frac{44528}{125} \sigma_3\left(\frac{n}{21}\right) - \frac{1778112}{125} \sigma_3\left(\frac{n}{42}\right) - \frac{6912}{5} c_1(n) \\
&+ \frac{3024}{25} c_2(n) + \frac{6912}{5} c_4(n) + \frac{1008}{125} c_5(n) \\
&+ \frac{223776}{25} c_6(n) - \frac{5062176}{25} c_7(n) + \frac{2166192}{125} c_8(n) \\
&+ \frac{34272}{5} c_9(n) - \frac{2491776}{25} c_{10}(n) + \frac{3284064}{5} c_{11}(n) \\
&- \frac{1052352}{25} c_{12}(n) + \frac{124992}{5} c_{13}(n) - \frac{653184}{5} c_{14}(n) \\
&- \frac{48384}{5} c_{15}(n) + \frac{520128}{5} c_{16}(n) + 15552c_{17}(n) \\
&- \frac{217728}{25} c_{18}(n) + \frac{221616}{5} c_{19}(n) - \frac{36288}{5} c_{20}(n) q^n
\end{align*}
\]
EVALUATION OF THE CONVOLUTION SUMS $W_{1,42}(n)$, $W_{2,21}(n)$, $W_{3,14}(n)$ AND $W_{6,7}(n)$

\[
(6Lq^6 - 7Lq^7)^2 = 1 + \sum_{n=1}^{\infty} \left( -\frac{1008}{125}\sigma_3(n) - \frac{4032}{125}\sigma_3\left(\frac{n}{2}\right) - \frac{9072}{125}\sigma_3\left(\frac{n}{3}\right) \\
+ \frac{1043712}{125}\sigma_3\left(\frac{n}{6}\right) + \frac{1420608}{125}\sigma_3\left(\frac{n}{7}\right) - \frac{197568}{125}\sigma_3\left(\frac{n}{14}\right) \\
- \frac{444528}{125}\sigma_3\left(\frac{n}{21}\right) - \frac{1778112}{125}\sigma_3\left(\frac{n}{42}\right) - \frac{3456}{5}c_1(n) \\
+ \frac{175824}{25}c_2(n) - 1728c_3 - \frac{5184}{5}c_4 + \frac{217008}{125}c_5 \\
+ \frac{320544}{5}c_6(n) - \frac{5960736}{25}c_7(n) + \frac{6399792}{125}c_8(n) \\
+ \frac{82656}{5}c_9(n) - \frac{763776}{25}c_{10}(n) + \frac{2292192}{5}c_{11}(n) \\
- \frac{1674432}{25}c_{12}(n) + \frac{133056}{5}c_{13}(n) - \frac{870912}{5}c_{14}(n) \\
- \frac{24192}{5}c_{15}(n) + \frac{471744}{5}c_{16}(n) - 15552c_{17}(n) \\
+ \frac{72576}{5}c_{18}(n) + \frac{1776816}{25}c_{19}(n) - \frac{108864}{5}c_{20}(n)q^n \right) q^n
\]

**Theorem 7.** Let $n \in \mathbb{N}$. Then

\[
W_{1,42}(n) = \frac{1}{6000}\sigma_3(n) + \frac{1}{1500}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{2000}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{500}\sigma_3\left(\frac{n}{6}\right) \\
+ \frac{49}{6000}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{1500}\sigma_3\left(\frac{n}{14}\right) + \frac{147}{2000}\sigma_3\left(\frac{n}{21}\right) \\
+ \frac{147}{500}\sigma_3\left(\frac{n}{42}\right) + \left( \frac{1}{24} - \frac{n}{168} \right)\sigma(n) + \left( \frac{1}{24} - \frac{n}{4} \right)\sigma\left(\frac{n}{42}\right) \\
- \frac{1}{35}c_1(n) - \frac{57}{2800}c_2(n) - \frac{1}{140}c_4(n) - \frac{1}{600}c_5(n) \\
+ \frac{49}{40}c_6(n) - \frac{883}{200}c_7(n) + \frac{695}{600}c_8(n) + \frac{17}{120}c_9(n) - \frac{97}{50}c_{10}(n) \\
+ \frac{543}{40}c_{11}(n) - \frac{127}{150}c_{12}(n) + \frac{31}{60}c_{13}(n) - \frac{27}{10}c_{14}(n) - \frac{1}{5}c_{15}(n) \\
+ \frac{43}{20}c_{16}(n) - \frac{9}{10}c_{18}(n) - \frac{9}{400}c_{19}(n) - \frac{3}{20}c_{20}(n)
\]
\[ W_{2,21}(n) = \frac{1}{6000}\sigma_3(n) + \frac{1}{1500}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{2000}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{500}\sigma_3\left(\frac{n}{6}\right) \] (3.3)

\[ + \frac{49}{6000}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{1500}\sigma_3\left(\frac{n}{14}\right) + \frac{147}{2000}\sigma_3\left(\frac{n}{21}\right) \]

\[ + \frac{1}{70}c_1(n) - \frac{57}{2800}c_2(n) - \frac{1}{28}c_3(n) + \frac{1}{70}c_4(n) - \frac{1}{6000}c_5(n) \]

\[ + \frac{13}{8}c_6(n) - \frac{703}{2000}c_7(n) - \frac{4249}{6000}c_8(n) + \frac{41}{120}c_9(n) - \frac{97}{50}c_{10}(n) \]

\[ + \frac{379}{40}c_{11}(n) - \frac{61}{75}c_{12}(n) + \frac{11}{20}c_{13}(n) - \frac{18}{5}c_{14}(n) - \frac{1}{10}c_{15}(n) \]

\[ + \frac{39}{20}c_{16}(n) + \frac{3}{10}c_{18}(n) - \frac{9}{400}c_{19}(n) - \frac{9}{20}c_{20}(n) \]

\[ W_{3,14}(n) = \frac{1}{6000}\sigma_3(n) + \frac{1}{1500}\sigma_3\left(\frac{n}{2}\right) + \frac{3}{2000}\sigma_3\left(\frac{n}{3}\right) + \frac{3}{500}\sigma_3\left(\frac{n}{6}\right) \] (3.4)

\[ + \frac{49}{6000}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{1500}\sigma_3\left(\frac{n}{14}\right) + \frac{147}{2000}\sigma_3\left(\frac{n}{21}\right) \]

\[ + \frac{1}{35}c_1(n) - \frac{1}{400}c_2(n) - \frac{1}{35}c_4(n) - \frac{1}{6000}c_5(n) - \frac{37}{40}c_6(n) \]

\[ + \frac{837}{2000}c_7(n) - \frac{2149}{6000}c_8(n) - \frac{17}{120}c_9(n) + \frac{103}{50}c_{10}(n) \]

\[ - \frac{543}{40}c_{11}(n) + \frac{87}{100}c_{12}(n) - \frac{31}{60}c_{13}(n) + \frac{27}{10}c_{14}(n) + \frac{1}{5}c_{15}(n) \]

\[ - \frac{43}{20}c_{16}(n) - \frac{9}{28}c_{17}(n) + \frac{9}{10}c_{18}(n) - \frac{513}{2800}c_{19}(n) + \frac{3}{20}c_{20}(n) \]
EVALUATION OF THE CONVOLUTION SUMS \( W_{1,42}(n), W_{2,21}(n), W_{3,14}(n) \) AND \( W_{6,7}(n) \)

\[
W_{6,7}(n) = \frac{1}{6000} \sigma_3(n) + \frac{1}{1500} \sigma_3\left(\frac{n}{2}\right) + \frac{3}{2000} \sigma_3\left(\frac{n}{3}\right) + \frac{3}{500} \sigma_3\left(\frac{n}{6}\right) \tag{3.5}
\]
\[
+ \frac{49}{6000} \sigma_3\left(\frac{n}{7}\right) + \frac{49}{1500} \sigma_3\left(\frac{n}{14}\right) + \frac{147}{2000} \sigma_3\left(\frac{n}{21}\right) + \frac{147}{500} \sigma_3\left(\frac{n}{42}\right)
\]
\[
+ \frac{1}{24} \left(1 - \frac{n}{28}\right) \sigma\left(\frac{n}{6}\right) + \frac{1}{24} \left(1 - \frac{n}{24}\right) \sigma\left(\frac{n}{7}\right)
\]
\[
+ \frac{53}{1400} c_7(n) - \frac{6349}{6000} c_8(n) - \frac{41}{120} c_9(n)
\]
\[
+ \frac{1}{70} c_1(n) - \frac{407}{2800} c_2(n) + \frac{1}{28} c_3(n) + \frac{3}{140} c_4(n) - \frac{1507}{42000} c_5(n)
\]
\[
+ \frac{53}{40} c_6(n) + \frac{379}{350} c_{10}(n) - \frac{969}{700} c_{12}(n) - \frac{11}{20} c_{13}(n)
\]
\[
+ \frac{18}{5} c_{14}(n) + \frac{1}{10} c_{15}(n) - \frac{39}{20} c_{16}(n) + \frac{9}{28} c_{17}(n) - \frac{3}{10} c_{18}(n)
\]
\[
+ \frac{221}{350} c_{10}(n) + \frac{9}{20} c_{20}(n)
\]

Proof. We prove the formula for only \( W_{1,42}(n) \) as the rest can be proven similarly. Glaisher [14] has proved the following identity

\[
L^2(q) = 1 + \sum_{n=1}^{\infty} \left(240 \sigma_3(n) - 288n\sigma(n)\right) q^n. \tag{3.6}
\]

Replacing \( q \) by \( q^{42} \) in (3.6) we have

\[
L^2(q^{42}) = 1 + \sum_{n=1}^{\infty} \left(240 \sigma_3\left(\frac{n}{42}\right) - \frac{48}{7} n\sigma\left(\frac{n}{42}\right)\right) q^n. \tag{3.7}
\]

By (2.28) we have

\[
L(q)L(q^{42}) = \left(1 - 24 \sum_{n=1}^{\infty} \sigma (n) q^n\right) \left(1 - 24 \sum_{n=1}^{\infty} \sigma (n) q^{42n}\right)
\]
\[
= 1 - 24 \sum_{n=1}^{\infty} \sigma (n) q^n - 24 \sum_{n=1}^{\infty} \sigma\left(\frac{n}{42}\right) q^n
\]
\[
+ 576 \sum_{n=1}^{\infty} W_{1,42}(n) q^n \tag{3.8}
\]
From (3.6)–(3.8) we obtain

\[
(L(q) - 42L(q^{42}))^2 = L^2(q) - 84L(q)L(q^{42}) + 1764L^2(q^{42}) = 1681 + \sum_{n=1}^{\infty} (240\sigma_3(n) + 423360\sigma_3(n^{42})) + 48384\left(\frac{1}{24} - \frac{n}{168}\right)\sigma(n) + 48384\left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{42}\right) - 48384W_{1,42}(n)q^n
\]  

Equating the coefficients of \(q^n\) on the right hand sides of first part of Theorem 6 and (3.9) we obtain

\[
\frac{28992}{125}\sigma_3(n) - \frac{4032}{125}\sigma_3\left(\frac{n}{2}\right) - \frac{9072}{125}\sigma_3\left(\frac{n}{3}\right) - \frac{36288}{125}\sigma_3\left(\frac{n}{6}\right) - \frac{49392}{125}\sigma_3\left(\frac{n}{7}\right) - \frac{197568}{125}\sigma_3\left(\frac{n}{14}\right) - \frac{444528}{125}\sigma_3\left(\frac{n}{21}\right) + \frac{51141888}{125}\sigma_3\left(\frac{n}{42}\right) + \frac{6912}{5}c_1(n) + \frac{24624}{25}c_2(n) + \frac{1728}{5}c_4(n) + \frac{1008}{125}c_5(n) - \frac{296352}{25}c_6(n) + \frac{5340384}{125}c_7(n) - \frac{2067408}{125}c_8(n) - \frac{34272}{5}c_9(n) + \frac{2346624}{25}c_{10}(n) - \frac{3284064}{125}c_{11}(n) + \frac{1024128}{25}c_{12}(n) - \frac{124992}{5}c_{13}(n) + \frac{653184}{25}c_{14}(n) + \frac{48384}{5}c_{15}(n) - \frac{520128}{5}c_{16}(n) + \frac{217728}{5}c_{18}(n) + \frac{27216}{25}c_{19}(n) + \frac{36288}{5}c_{20}(n) = 240\sigma_3(n) + 423360\sigma_3\left(\frac{n}{42}\right) + 48384\left(\frac{1}{24} - \frac{n}{168}\right)\sigma(n) + 48384\left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{42}\right) - 48384W_{1,42}(n). \]

Solving this equation for \(W_{1,42}(n)\) we obtain the asserted formula. \(\square\)
EVALUATION OF THE CONVOLUTION SUMS $W_{1,42}(n)$, $W_{2,21}(n)$, $W_{3,14}(n)$ AND $W_{6,7}(n)$

4. THE REPRESENTATION NUMBER FORMULA FOR

\[ x_1^2 + x_1 x_2 + x_2^2 + x_2 x_3 + x_3 x_4 + x_4^2 + 14(x_5^2 + x_5 x_6 + x_6^2 + x_7^2 + x_7 x_8 + x_8^2) \]

**Theorem 8.** Let $n \in \mathbb{N}$, then

\[
N_{14}(n) = \frac{12}{125} \sigma_3(n) + \frac{48}{125} \sigma_3\left(\frac{n}{2}\right) + \frac{108}{125} \sigma_3\left(\frac{n}{3}\right) + \frac{432}{125} \sigma_3\left(\frac{n}{6}\right) + \frac{588}{125} \sigma_3\left(\frac{n}{7}\right) + \frac{2352}{125} \sigma_3\left(\frac{n}{14}\right) + \frac{5292}{125} \sigma_3\left(\frac{n}{21}\right) + \frac{21168}{125} \sigma_3\left(\frac{n}{42}\right) + \frac{288}{25} c_2(n) - \frac{6}{25} c_3(n) + \frac{294}{25} c_4(n) + \frac{2592}{175} c_2\left(\frac{n}{3}\right) - \frac{54}{25} c_3\left(\frac{n}{3}\right) - \frac{5778}{175} c_4\left(\frac{n}{3}\right) + \frac{18}{175} c_5(n) - \frac{648}{25} c_6(n) + \frac{2484}{125} c_7(n) + \frac{882}{125} c_8(n) - \frac{1296}{25} c_{10}(n) - \frac{252}{25} c_{12}(n) + \frac{972}{7} c_{17}(n) + \frac{15552}{175} c_{19}(n) \]

**Proof.** For $l \in \mathbb{N}_0$ we set,

\[
r(l) = \text{card} \left\{ (x_1, \ldots, x_4) \in \mathbb{Z}^4 : l = x_1^2 + x_1 x_2 + x_2^2 + x_2 x_3 + x_3 x_4 + x_4^2 \right\} , \tag{4.2} \]

It is known (see for example [15, Theorem 13]) that

\[
r(l) = 12\sigma(l) - 36\sigma\left(\frac{l}{3}\right), \quad l \in \mathbb{N} \tag{4.3} \]

It is clear from (1.2), (4.2) and (4.3) that

\[
N_{14}(n) = \sum_{l, m \in \mathbb{N}} r(l)r(m)
\]

\[
= r(n)r(0) + r(0)r\left(\frac{n}{14}\right) + \sum_{l, m \in \mathbb{N}} r(l)m = n
\]

\[
= 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{14}\right) - 36\sigma\left(\frac{n}{42}\right)
\]

\[
+ \sum_{l, m \in \mathbb{N}} (12\sigma(l) - 36\sigma\left(\frac{l}{3}\right))(12\sigma(m) - 36\sigma\left(\frac{m}{3}\right))
\]

\[
= 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{14}\right) - 36\sigma\left(\frac{n}{42}\right) + 144\sigma(l)\sigma(m) - 432\sigma\left(\frac{l}{3}\right)\sigma\left(\frac{m}{3}\right)
\]

\[
= 12\sigma(n) - 36\sigma\left(\frac{n}{3}\right) + 12\sigma\left(\frac{n}{14}\right) - 36\sigma\left(\frac{n}{42}\right) + 144W_{1,14}(n)
\]

\[
- 432W_{3,14}(n) - 432W_{1,42}(n) + 1296W_{1,14}\left(\frac{n}{3}\right)
\]

Formulae for \(W_{3,14}(n)\) and \(W_{1,42}(n)\) are obtained in this article. \(W_{1,14}(n)\) was evaluated in [2]. We also check that it can be given as

\[
W_{1,14}(n) = \frac{1}{600}\sigma_3(n) + \frac{1}{150}\sigma_3\left(\frac{n}{2}\right) + \frac{49}{600}\sigma_3\left(\frac{n}{7}\right) + \frac{49}{150}\sigma_3\left(\frac{n}{14}\right)
\]

\[
+ \left(\frac{1}{24} - \frac{n}{56}\right)\sigma(n) + \left(\frac{1}{24} - \frac{n}{4}\right)\sigma\left(\frac{n}{14}\right) + \frac{2}{175}c_2(n)
\]

\[
- \frac{1}{600}c_3(n) - \frac{107}{4200}c_4(n).
\]

Using (3.2), (3.4) and (4.4) we obtained the desired formula. \(\square\)

For simplicity, taking \(u(n)\) as follows

\[
u(n) = 1680c_2(n) + 2160c_2\left(\frac{n}{3}\right) - 35c_3(n) - 315c_3\left(\frac{n}{3}\right) + 1715c_4(n)
\]

\[
- 4815c_4\left(\frac{n}{3}\right) + 21c_5(n) - 18900c_6(n) + 14490c_7(n) + 1029c_8(n)
\]

\[
- 7560c_{10}(n) - 1470c_{12} + 20250c_{17}(n) + 12960c_{19}(n)
\]
EVALUATION OF THE CONVOLUTION SUMS $W_{1,42}(n)$, $W_{2,21}(n)$, $W_{3,14}(n)$ AND $W_{6,7}(n)$

we may write

$$N_{14}(n) = \frac{12}{125}\sigma_3(n) + \frac{48}{125}\sigma_3\left(\frac{n}{2}\right) + \frac{108}{125}\sigma_3\left(\frac{n}{3}\right) + \frac{432}{125}\sigma_3\left(\frac{n}{6}\right) + \frac{588}{125}\sigma_3\left(\frac{n}{7}\right) + \frac{2352}{125}\sigma_3\left(\frac{n}{14}\right) + \frac{5292}{125}\sigma_3\left(\frac{n}{21}\right) + \frac{21168}{125}\sigma_3\left(\frac{n}{42}\right) + \frac{6875}{875}u(n).$$

(4.5)

We checked our result for some values of $n$ by using Pari GP. Denoting the right hand side of (4.5) by $S_{14}(n)$ we give the first ten values of $N_{14}(n)$ and $S_{14}(n)$ in Table 2 to illustrate the equations.

| $n$ | $N_{14}(n)$ | $\sigma_3(n)$ | $u(n)$ | $S_{14}(n)$ |
|-----|-------------|---------------|--------|-------------|
| 1   | 12          | 1             | 1736   | 12          |
| 2   | 36          | 9             | 5068   | 36          |
| 3   | 12          | 28            | 1232   | 12          |
| 4   | 84          | 73            | 10724  | 84          |
| 5   | 72          | 126           | 8736   | 72          |
| 6   | 36          | 252           | −1484  | 36          |
| 7   | 96          | 344           | 8498   | 96          |
| 8   | 180         | 585           | 13972  | 180         |
| 9   | 12          | 757           | −12376 | 12          |
| 10  | 216         | 1134          | 8568   | 216         |

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