A Bell’s Inequality Test with Entangled Atoms

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Previous work on Bell’s inequality realised in the laboratory has used entangled photons. Here we describe how entangled atoms can violate Bell’s inequality, and how these violations can be measured with a very high detection efficiency. We first discuss a simple scheme based on two-level atoms inside a cavity to prepare the entangled state. We then discuss a scheme using four-level atoms, which requires a parameter regime much easier to access experimentally using current technology. As opposed to other schemes, our proposal relies on the presence of finite decay rates and its implementation should therefore be much less demanding.

I. INTRODUCTION

Bell’s inequalities have a central role in tests of quantum mechanics and relate to the degree of entanglement between subsystems, an essential resource in quantum information processing. There are a number of Bell inequalities for two subsystems where each subsystem contains a qubit of information. For example, there exist the original spin 1/2, Clauser Horne (CH) and information theoretic Bell inequalities, to name but a few. The particular one considered generally depends on the system under consideration. A scheme may violate one Bell inequality but not another. Recently an overview of Bell’s inequalities has been given by Peres [5].

A number of experimental tests of Bell’s inequality have already been performed [6–12] using entangled photons. Here we propose an experimental test of Bell’s inequality on two macroscopically separated atoms. Each atom possesses a two-level system with the states |0⟩ and |1⟩. We describe a scheme which allows us to prepare the atoms in an arbitrary superposition of a maximally entangled state and a product state which is of the form

$$|\varphi\rangle = \frac{\alpha}{\sqrt{2}} (|10\rangle - |01\rangle) + \sqrt{1-|\alpha|^2} |00\rangle$$

in a deterministic way. To do so we make use of a recently proposed idea by Beige et al. [13] of how to manipulate the decoherence-free states of N atoms inside a cavity. Together with the control over the prepared state which can be obtained by following a measurement proposal by Cook [14,15] based on “electron shelving” this allows us to investigate, characterise and test Bell’s inequality with a very high precision and detection efficiency.

The success rate for the preparation of the initial atomic state (4) will be denoted by P0. If a photon is emitted in the preparation, the scheme fails. If these events are not detected and ignored this leads to a decrease of the observed violation of Bell’s inequality. On the other hand, if the scheme succeeds the fidelity of the prepared state is very close to unity. Therefore we estimate, that Bell’s inequality is violated as long as the preparation probability exceeds 71 %, if the scheme is intended to prepare the atoms in the maximally entangled state. In this paper we determine P0 and show that it can, in principle, be arbitrarily close to unity.

Other tests using atoms or ions have been proposed [16–20]. For instance an experiment, based on the proposal by Cirac and Zoller [16], to entangle two atoms in a cavity has been performed by Hagley et al. [21]. Four trapped ions, respectively, have been entangled experimentally in a deterministic fashion by Sackett et al. [22] following a proposal by Mølmer and Sørensen [23]. But a test of Bell’s inequality using atoms has yet to be realised. The main limiting factor in these experiments is dissipation [21,22]. As opposed to this, the scheme proposed here is based on the presence of finite decay rates and should therefore be less demanding experimentally.

The investigation we are examining here is not strictly a strong test of quantum mechanics versus local realism due to the limited spatial separation of the atoms. For a strict test the scheme would require separating the two atoms by a distance larger than the speed of light times the measurement time. However this atom based experiment closes the detection inefficiency loophole while the photon experiments close the causality loopholes [1]. In the scheme we propose, the observable which is expected to violate Bell’s inequality is measured in each run of the experiment and the state of the two atoms can be determined with almost unity efficiency and a very high precision [13]. Hence this proposed experiment should be seen as complementary to the photon experiments.

The paper is organised as follows. We begin in the next Section with a description of a simple scheme based on two two-level atoms inside a cavity that can be used to generate the entangled state (4). We describe the single qubit rotation and a way to measure the state of the atoms. The required parameter regime is, however, experimentally demanding. Therefore, in Section III, a scheme is introduced based on two four-level atoms. This system behaves exactly like in the two-level atom case described above and the discussion in Section II is used to obtain the same results. In Section IV we discuss how to
test Bell’s inequality and for which parameters a violation of the inequality is expected. A final discussion of the results can be found in Section V.

II. A SIMPLE SCHEME USING TWO-LEVEL ATOMS

To prepare two two-level atoms in the entangled state \(|0\rangle\) they are placed at fixed positions in a cavity which acts as a resonator for an electromagnetic field. The atoms (or ions) can be stored in the nodes of a standing light field or in a linear trap. In the following \(|0\rangle_i\) denotes the ground state and \(|1\rangle_i\) the excited state of atom \(i\), respectively, and we assume that the cavity field is in resonance with the atomic transition. We also assume that the coupling constant of each atom with the cavity field is the same and given by \(g\), which can be chosen to be real. The cavity should be non-ideal, that is spontaneous decay rate of each atom equals \(\Gamma\). The distance between the atoms inside the cavity should be much larger than an optical wavelength. This allows us to address each atom individually with a laser pulse. The Rabi frequency for atom \(i\) will be denoted by \(\Omega^{(i)}\) and is in general complex, because we have already chosen \(g\) to be real.

\[
\begin{align*}
\text{FIG. 1. Experimental setup for the preparation of state } |0\rangle. \text{ The system consists of two two-level atoms placed at fixed positions inside a cavity. Each atom couples to the cavity mode with a constant } g \text{ and its spontaneous decay rate is given by } \Gamma. \text{ The rate } \kappa \text{ corresponds to the leakage of photons through the cavity mirrors.}
\end{align*}
\]

To test Bell’s inequality the atoms have to be moved out of the cavity. This can be done by moving the optical lattice or by applying an electric field, respectively, if the atoms are inside a linear ion trap. Another possibility is to let the two atoms fly together through the cavity field during each run of the experiment.

In the experiment we propose, the probability for spontaneous emission of a photon or leakage of a photon through the cavity mirrors will be shown to be small. This immediately suggests that we use the quantum jump approach \([25-28]\). This method leads to a conditional Hamiltonian \(H_{\text{cond}}\) which gives the time evolution of the system under the condition of no photon emission. Due to the non-Hermiticity of \(H_{\text{cond}}\), the norm of the state vector

\[
|\psi^0(t)\rangle = e^{-iH_{\text{cond}}t/\hbar}|\psi_0\rangle
\]

decreases with time and the probability \(P_0\) for no photon emission up to time \(t\) is given by the squared norm

\[
P_0(t) = \|\psi^0(t)\|^2.
\]

If no photon is emitted, the state of the system at time \(t\) is the state \(|\psi_0\rangle\) normalised to unity.

A. The preparation of the entangled state

To prepare the atoms in state \(|0\rangle\) we will take advantage of the fact that two-level atoms inside a cavity possess trapped states \([29-32]\) which can also be used to obtain an example of a decoherence-free subspace \([13,33-35]\). If the atoms are in a trapped state they cannot transfer excitation into the resonator field, even if upper levels are populated. Therefore, if the cavity field is empty and spontaneous emission can be neglected no photon can be emitted by the system and the system is in a decoherence-free state.

To find the decoherence-free states of the system let us first assume that the two atoms are inside the cavity, but no laser field is applied. We choose the interaction picture in a way that the atoms and the cavity mode plus environment are considered as the free system. Then the conditional Hamiltonian equals, as in Ref. \([13,24]\),

\[
H_{\text{cond}} = i\hbar g \sum_{i=1}^{2} \left( b |1\rangle_i\langle 0| - h.c. \right) - i\hbar \Gamma \sum_{i=1}^{2} |1\rangle_i\langle 1| - i\hbar \kappa b^\dagger b,
\]

where the operator \(b\) is the annihilation operator for photons in the cavity mode.

Decoherence-free states arise if no interaction between the system and its environment of free radiation fields takes place. If we neglect spontaneous emissions (\(\Gamma = 0\)) this is exactly the case if the cavity mode is empty \([13]\) and it is \(|\psi\rangle = |0\rangle \otimes |\varphi\rangle \equiv |0\varphi\rangle\). In addition, the systems own time evolution due to the interaction between the atoms and the cavity mode should not move the state of the system out of the decoherence-free subspace. Using Eq. (4) this leads as in Ref. \([13]\) to the condition

\[
\sum_{i=1}^{2} |0\rangle_i\langle 1| \varphi = 0,
\]

where \(|\varphi\rangle\) is the state of the atoms only. From this condition we find that the decoherence-free states are the superpositions of the two atomic states \(|g\rangle \equiv |00\rangle\) and \(|a\rangle \equiv (|10\rangle - |01\rangle)/\sqrt{2}\)

\[
\text{while the cavity mode is empty.}
\]
Once prepared in a decoherence-free state the state of the system does not change in time with respect to the chosen interaction picture. The reason for this is \( H_{\text{cond}} | \psi \rangle = 0 \) which can be shown by using Eq. (6) and (9).

To prepare the atoms in state \( | \psi \rangle \) a weak laser pulse can be used. As in Ref. [13] we assume in the following \( \Omega^{(1)} \neq \Omega^{(2)} \) and for all non-vanishing Rabi frequencies \( \Gamma \ll |\Omega^{(i)}| \ll g \) and \( \kappa \sim g \).

This corresponds to a strong coupling between the atoms and the cavity mode, while \( g \) and \( \kappa \) are of the same order of magnitude. In this parameter regime we can make use of an effect which can easily be understood in terms of the quantum Zeno effect [35-38]. The reason for this is that the entangled state given in Eq. (1) corresponds to a decoherence-free state. We assume now that the system is initially in its ground state which is also decoherence-free. If now rapidly repeated measurements are performed on the system of whether the state of the system still belongs to the decoherence-free subspace or not, the laser interaction cannot move the state of the system out of this subspace. Only a time evolution inside the subspace is possible. Hence the laser pulse can introduce entanglement into the system which is not possible in the free atom case. Equivalently we can interpret this inhibition without invoking Zeno effects as a simple consequence of adiabatic elimination using the separation of the frequency scales in Eq. (7) [13].

Let us define \( \Delta T \) as the time in which a photon leaks out through the cavity mirrors with a probability very close to unity if the system is initially prepared in a state with no overlap with a decoherence-free state. On the other hand, a system in a decoherence-free state will definitely not emit a photon in \( \Delta T \). Therefore the observation of the free radiation field over a time interval \( \Delta T \) can be interpreted as a measurement of whether the system is decoherence-free or not. The outcome of the measurement is indicated by an emission or no emission of a photon. This interpretation also holds to a very good approximation in the presence of the laser field because the effect of the laser over a time interval \( \Delta T \) can be neglected, which is why condition (7) has been chosen. As it has been shown in Ref. [39], \( \Delta T \) is of the order \( 1/\kappa \) and \( \kappa/g^2 \) and much smaller than \( 1/(\Omega^{(\pm)}) \),

\[
\Omega^{(\pm)} = \left( \Omega^{(1)} \pm \Omega^{(2)} \right) / \sqrt{2},
\]

the typical time scale for the laser interaction. Here the system continuously interacts with its environment and the system behaves in a very good approximation like a system under continuous observation whose time evolution can easily be predicted with the help of the quantum Zeno effect [35].

Using the measurement interpretation one can easily show that the effect of the laser field on the atomic states can be described by the effective Hamiltonian \( H_{\text{eff}} \) which equals [13]

\[
H_{\text{eff}} = \mathbb{P}_{\text{DFS}} H_{\text{cond}} \mathbb{P}_{\text{DFS}}
\]  

and where \( \mathbb{P}_{\text{DFS}} \) is the projector on the decoherence-free subspace. To obtain the conditional Hamiltonian of the system in the presence of the laser field the Hamiltonian

\[
H_{\text{laser}} = \frac{\hbar}{2} \sum_{i=1}^{2} \left( \Omega^{(i)} |1\rangle_i \langle 0| + \text{h.c.} \right)
\]

has to be added to the right hand side of Eq. (4). If we neglect spontaneous emission (\( \Gamma = 0 \)) this leads to

\[
H_{\text{eff}} = \frac{\hbar}{2} \left( \Omega^{(-)} |0\rangle \langle 0| + \text{h.c.} \right).
\]

By solving the corresponding time evolution, one finds that a laser pulse of length \( T \) prepares the atoms in the state given in Eq. (1) with

\[
\alpha = -i \frac{\Omega^{(-)}}{|\Omega^{(-)}|} \sin \left( \frac{|\Omega^{(-)}| T}{2} \right).
\]

Varying the length of the laser pulse allows to change arbitrarily the value of \( |\alpha| \) and the amount of entanglement in the system.

The Hamiltonian in Eq. (1) is Hermitian. Therefore the norm of a vector developing with \( H_{\text{eff}} \) is not decreasing and in a first approximation, due to Eq. (8), the emission of photons can be neglected. To a very good approximation the cavity mode never does become populated and the success rate of the preparation scheme \( P_0 \) equals unity.

\[
\begin{align*}
\Gamma = 0 & \quad \text{---} \\
\Gamma = 0.001 g & \quad \text{---} \\
\Gamma = 0.005 g & \quad \text{---}
\end{align*}
\]

\[
\begin{align*}
P_0 & \quad \text{---} \\
0 & \quad \text{---} \\
0.05 & \quad \text{---} \\
0.1 & \quad \text{---} \\
0.15 & \quad \text{---} \\
0.2 & \quad \text{---}
\end{align*}
\]

\[
\begin{align*}
\Omega^{(1)}/g & \quad \text{---} \\
0 & \quad \text{---} \\
0.1 & \quad \text{---} \\
0.15 & \quad \text{---} \\
0.2 & \quad \text{---}
\end{align*}
\]

\[
\begin{align*}
\text{Fig. 2.} & \quad \text{---} \\
\text{The probability for no photon emission during the} & \quad \text{---} \\
\text{preparation of the maximally entangled state for} & \quad \text{---} \\
\text{different Rabi frequencies} |\Omega^{(1)}| & \quad \text{---} \\
\text{and} |\Omega^{(2)}| = -|\Omega^{(1)}|, \text{different} & \quad \text{---} \\
\text{spontaneous decay rates} \Gamma & \quad \text{---}
\end{align*}
\]

This agrees very well with
the approximative results given above. As an example, we assumed
\[ T = \pi/|\Omega^{(-)}|, \] (13)
which leads, due to Eq. (12), to the preparation of the maximally entangled state of both atoms. In addition we assumed \( \Omega^{(2)} = -\Omega^{(1)} \) \[ 44 \]. As expected, for \( \Gamma = 0 \) the success rate of the preparation scheme can at least in principle be arbitrarily close to 1. For \( \Gamma \neq 0 \) the probability \( P_0 \) reaches a maximum value for a certain Rabi frequency \( \Omega^{(1)} \), but is always smaller than 1. To improve the experiment one can surround the cavity by detectors and repeat it were a decay photon to be registered.

We also determined the state of the atoms at the end of the laser pulse numerically. The fidelity of the prepared state \( F \) in case of no photon emission is given by the overlap of the state of Eq. (3) after normalization with the state given in Eq. (3). For the parameters chosen in Fig. 4, \( F \) is found to be always higher than 95%.

B. Realisation of a single qubit rotation

In this subsection we describe how the single qubit rotation on atom \( i \), defined by the operator \( U_{\text{rot}}^{(i)} \),
\[ U_{\text{rot}}^{(i)}(\xi, \phi) \equiv \cos \xi - i \sin \xi \left( e^{i\phi} |0\rangle_{ii} \langle 1| + \text{h.c.} \right), \] (14)
can be realised, where \( \xi \) and \( \phi \) are arbitrary parameters. Thereby the same laser as in the previous subsection can be used. To avoid the situation that the time evolution of the system is restricted to changes inside the decoherence-free subspace, the atom should be moved out of the cavity.

If we neglect again spontaneous emission (\( \Gamma = 0 \)), the laser Hamiltonian which describes the time evolution of atom \( i \) is given by
\[ H_{\text{laser}}^{(i)} = \hbar \left( \Omega^{(i)} |1\rangle_{ii} \langle 0| + \text{h.c.} \right). \] (15)
Calculating the corresponding time evolution operator for a laser pulse length \( T \) leads to Eq. (14) with
\[ \xi = \frac{\langle \Omega^{(i)} \rangle}{2} T \quad \text{and} \quad e^{i\phi} = \frac{\Omega^{(i)}}{|\Omega^{(i)}|}. \] (16)
To change the phase \( \phi \), the phase of the Rabi frequency \( \Omega^{(i)} \) has to be chosen very carefully, while \( \xi \) can easily be varied by varying the length \( T \) of the pulse.

Again, for \( \Gamma \neq 0 \) a photon may be emitted spontaneously during the single qubit rotation which leads to a failure of the experiment and therefore to a further decrease of the success rate of the scheme to test Bell’s inequality proposed here.

C. State measurement on a single atom

Whether an atom \( i \) is in state \( 0 \rangle \) or \( 1 \rangle \) can be measured with a very high precision following a proposal by Cook \[ 44 \]. To do this, we make use of a short strong laser pulse and an auxiliary level 2. The probe pulse couples one of the states, for instance the state \( 0 \rangle \), to state \( 2 \rangle_{ii} \), and has the Rabi frequency \( \Omega_2 \). The spontaneous decay rate of the auxiliary level is \( \Gamma_2 \). If the length of the laser pulse, \( T \), fulfills a minimum length,
\[ T \gg \max \left\{ 1/\Gamma_2, \Gamma_2/\Omega_2^2 \right\}, \] (17)
the absence or occurrence of photons from the 0-2 transition indicates whether the atom is found in state \( 0 \rangle \) or \( 1 \rangle \), respectively. If the system is initially prepared in level 0 photons are emitted until the end of the pulse. If the atom is in \( 1 \rangle \), the laser has no effect on the atomic state and no photon emissions will occur. For an arbitrary state of the atom
\[ |\varphi \rangle = \alpha_0 |0\rangle_i + \alpha_1 |1\rangle_i \] (18)
it has been shown by Beige and Hegerfeldt \[ 44 \] that photons are emitted with probability \( |\alpha_0|^2 \) as predicted for an ideal measurement. The proposition for this scheme to work is that the laser pulse is long enough that an atom initially in state \( 0 \rangle \) emits definitively a photon which leads to condition (13). As discussed in Ref. \[ 44 \] the precision of this measurement can be very high, even if the efficiency of the detectors measuring the photons from the 0-2 transition is very low. The population difference between the two levels is given by
\[ 2 - 2 |\alpha_0|^2 \] (19)
averaged over many runs.

III. AN IMPROVED SCHEME USING FOUR-LEVEL ATOMS

To observe a violation of Bell’s inequality the preparation of the maximally entangled state \( |\alpha \rangle \) should succeed with a probability above 71 % in each run of the experiment. For this, as can be seen in Fig. 6 the coupling constant \( g \) has to be at least 100 times larger than the spontaneous decay rate \( \Gamma \). This is difficult to achieve experimentally using optical frequencies, and has only been realised in micro cavities with circular Rydberg atoms coupled to a microwave cavity \[ 41 \].

In the following we describe how this problem can be circumvented easily by making use of two additional atomic levels. They allow us to replace all transitions in the two-level system by Raman transitions. We show that the four-level atoms possess the same decoherence-free states as the two-level atoms described in Section IIIC and again a weak laser pulse can be used to create entanglement between the atoms. We describe how to perform a single qubit rotation and how to measure the state of an atom.
A. The preparation of the entangled state

![Diagram of cavity mode coupling](image)

FIG. 3. The cavity mode couples with coupling strength $g$ and detuning $\Delta_2$ to the 0-2 transition of each atom. The 1-2 and the 0-3 transition are both driven by a laser field with Rabi frequency $\Omega_1$ or $\Omega_0$ and detuning $\Delta_2$ or $\Delta_3$, respectively. A weak laser field couples, in addition, to the 1-3 transition of atom $i$ with Rabi frequency $\Omega^{(i)}$ and detuning $\Delta_3$.

We consider now two four-level atoms with a configuration as shown in Fig. 3. The states $|0\rangle_i$, $|1\rangle_i$, and $|3\rangle_i$ are the ground states of atom $i$ and couple to the excited states denoted by $|2\rangle_i$ and $|3\rangle_i$. Ground states and excited states could each be obtained from one degenerate level. To prepare the atoms in state $|1\rangle_i$ they have to be moved into a cavity as described in Section II. In the following, $h\omega_i$ denotes the energy of level $i$. The frequency $\omega_{cav}$ of the single cavity mode equals $\omega_{cav} = \omega_2 - \omega_1 - \Delta_2$, where $\Delta_2$ denotes a detuning. A laser field with the same detuning and frequency $\omega_{21} = \omega_2 - \omega_1 - \Delta_2$ excites the 1-2 transition of each atom with Rabi frequency $\Omega_1$ and another laser drives the 0-3 transition of both atoms with Rabi frequency $\Omega_0$ and has the frequency $\omega_{30} = \omega_3 - \omega_0 - \Delta_3$. In addition, at time $t = 0$ a weak laser pulse with frequency $\omega_{31} = \omega_3 - \omega_1 - \Delta_3$ and with Rabi frequency $\Omega^{(i)}$ is applied to the 1-3 transition of atom $i$.

To describe the time evolution of the system under the condition of no photon emission we use again the quantum jump approach [25]. Here we chose the interaction picture with respect to the sum of the atomic Hamiltonian

$$H_0 = \sum_{i=1}^{2} \sum_{j=0}^{3} \hbar \omega_j \langle j | i \rangle_{ii} \langle j | - \sum_{i=1}^{2} \sum_{j=2}^{3} \hbar \Delta_j \langle j | i \rangle_{ii} \langle j |$$

(20)

and the Hamiltonian describing the energy of the cavity mode and the free radiation fields forming the environment of the system. Then the conditional Hamiltonian becomes

$$H_{\text{cond}} = i\hbar \sum_{i=1}^{2} \left( b | 1 \rangle_{ii} \langle 0 | - \text{h.c.} \right)$$

$$+ \frac{\hbar}{2} \sum_{i=1}^{2} \left( \Omega^{(i)} | 3 \rangle_{ii} \langle 1 | + \text{h.c.} \right)$$

$$- \frac{i\hbar}{2} \sum_{i=1}^{2} \sum_{j=2}^{3} (\Gamma_j + i\Delta_j) | j \rangle_{ii} \langle j | - i\hbar \kappa b^\dagger b ,$$

(21)

where we assumed again that the coupling constant $g$ is for both atoms the same and $\Gamma_j$ denotes the spontaneous decay rate of level $j$.

In the following we assume that the detunings $\Delta_2$ and $\Delta_3$ are much larger than all other system parameters,

$$\Omega_0 , | \Omega_1 | , | \Omega^{(i)} | , g , \Gamma_j \ll \Delta_2 \sim \Delta_3 ,$$

(22)

and write the (unnormalized) state of the system under the condition of no photon emission as

$$| \psi_0 (t) \rangle = \sum_{n=0}^{\infty} \sum_{j_1,j_2=0}^{3} c_{n,j_1,j_2} (t) | n,j_1,j_2 \rangle .$$

Because we are only interested in the time evolution of the system on a time scale much longer than $1/\Delta_2$ and $1/\Delta_3$ level 2 and level 3 can be eliminated adiabatically by eliminating the fast varying coefficients. All coefficients with $j_1$ or $j_2$ equal to 2 or 3 adapt essentially immediately to the state of the other levels and we can set their derivatives in the Schrödinger equation corresponding to Eq. (22) equal to zero. This allows us to determine the fast varying coefficients analytically. Substituting the result into the differential equations for the remaining slowly varying coefficients we find that their time evolution under the condition of no photon emission is governed by the effective Hamiltonian $H_{\text{cond}}$ with

$$H_{\text{cond}} = i\hbar \sum_{i=1}^{2} \left( b | 1 \rangle_{ii} \langle 0 | - \text{h.c.} \right)$$

$$+ \frac{\hbar}{2} \sum_{i=1}^{2} \left( \Omega^{(i)} | 3 \rangle_{ii} \langle 1 | + \text{h.c.} \right)$$

$$- \frac{i\hbar}{2} \sum_{i=1}^{2} \sum_{j=2}^{3} (\Gamma_j + i\Delta_j) | j \rangle_{ii} \langle j | - i\hbar \kappa b^\dagger b ,$$

(24)

Here all terms of second and higher order in $1/\Delta_2$ and $1/\Delta_3$ have been neglected. The effective atom-cavity coupling constant $g_{\text{eff}}$ is given by

$$g_{\text{eff}} \equiv -g \cdot \Omega_1^2 / (2\Delta_2)$$

(25)

and the effective Rabi frequencies $\Omega^{(i)}$ equal

$$\Omega_{\text{eff}}^{(i)} \equiv -\Omega^{(i)} \cdot \Omega_0^2 / (2\Delta_3) .$$

(26)

The level shifts in Eq. (24), which are proportional $1/\Delta_2$ and $1/\Delta_3$, can be neglected if they are for all states the same or if they are much smaller than the parameters...
governing the time evolution of the corresponding transition. We assume therefore in the following

$$|\Omega_0| = |\Omega_1| \ , \ \Delta_2 = \Delta_3 \ , \ g \ll |\Omega_1| \ \text{and} \ |\Omega^{(i)}| \ll |\Omega_0| \ .$$

(27)

For this parameter choice, the Hamiltonian $\tilde{H}_{\text{cond}}$ resembles the conditional Hamiltonian (3) of the two-level atoms in Section II to a very good approximation. Despite the values of $g$ and $\Omega^{(i)}$ are now replaced by $g_{\text{eff}}$ and $\Omega_{\text{eff}}^{(i)}$ and the spontaneous emission rate $\Gamma$ equals zero.

To prepare the atoms in the entangled state $|\bar{a}\rangle$ therefore the same idea as in the previous section can be used. The decoherence-free states are exactly the same - the superpositions of the two states $|0g\rangle$ and $|0a\rangle$. In analogy to Section II, Eq. (6), we assume now $|\Omega_{\text{eff}}^{(i)}| \ll g_{\text{eff}}$ and $\kappa \sim g_{\text{eff}}$. This leads in addition to Eq. (27) to the condition

$$|\Omega^{(i)}| \ll g \ \text{and} \ \kappa \sim g\Omega_1/\Delta_2 \ .$$

(28)

If condition (27) and (28) are fulfilled we expect that the weak laser pulse with the Rabi frequencies $\Omega^{(i)}$ does not move the system out of the decoherence-free subspace, if the system is initially in the ground state $|000\rangle$. Its effect can again be described by the effective Hamiltonian $H_{\text{eff}}$ given in Eq. (11). One only has to replace the Rabi frequencies $\Omega^{(i)}$ by $\Omega_{\text{eff}}^{(i)}$.

$$H_{\text{cond}} = \frac{\hbar}{2} \left( \Omega_0 |1i\rangle \langle 0i| + \Omega_1 |3i\rangle \langle 1i| + \text{h.c.} \right)$$

$$-\frac{\hbar}{4} \left( \frac{|\Omega_0|^2}{\Delta_3} |0i\rangle \langle 0i| + \frac{|\Omega_1|^2}{\Delta_2} |1i\rangle \langle 1i| \right).$$

(29)

Eq. (29) allows us again to eliminate level 2 and level 3 adiabatically. Proceeding as in the previous subsection we find that the atom can effectively be described by the Hamiltonian

$$\tilde{H}_{\text{cond}} = \frac{\hbar}{2} \left( \Omega_{\text{eff}}^{(i)} |1i\rangle \langle 0i| + \text{h.c.} \right)$$

$$-\frac{\hbar}{4} \left( \frac{|\Omega_{\text{eff}}^{(i)}|^2}{\Delta_3} |0i\rangle \langle 0i| + \frac{|\Omega_1|^2}{\Delta_2} |1i\rangle \langle 1i| \right).$$

(30)

where Eq. (29) has been used. This Hamiltonian does not depend on $\Gamma_2$ and $\Gamma_3$ and spontaneous emission by the atom can be neglected. If the parameters fulfill, as in the previous subsection, condition (27), then the last term in Eq. (30) is negligible whilst the remaining level shifts are for all states the same and introduce an overall phase factor to the state of the atom. The time evolution operator corresponds therefore up to a total phase factor with the operator given in Eq. (14) and equals

$$U(T,0) = \exp \left( i \frac{\Omega_{\text{eff}}^{(i)} T}{4\Delta_3} \right) U_{\text{rot}}^{(i)}(\xi,\phi)$$

(31)

with

$$\xi = \frac{|\Omega_{\text{eff}}^{(i)}|}{4\Delta_3} |T| \ \text{and} \ e^{i\phi} = -\frac{\Omega_{\text{eff}}^{(i)}}{|\Omega_{\text{eff}}^{(i)}|}. \ \ \ \ \ \ \ \ \ \ \ (32)$$

We will see later that the additional phase factor does not affect the outcome of the Bell measurement described in the next section. We can therefore ignore this factor and use the Hamiltonian (29) to realise the single qubit rotation.
C. State measurement on a single atom

To measure whether atom $i$ is in state $|0\rangle_i$ or $|1\rangle_i$, respectively, the same scheme as described in Section IIC can be used.

IV. A TEST OF THE BELL INEQUALITY

Given that the state (1) can be generated, the next interesting question is whether such a state will violate one of Bell’s inequalities? For certain parameters it must but what physical measurements are necessary to characterize this disagreement with local realism?

A. The Bell inequality

The spin (or correlation function) Bell inequality may be written formally as

$$B_S = |E(\theta_1, \theta_2) - E(\theta_1, \theta'_2) + E(\theta'_1, \theta_2) + E(\theta'_1, \theta'_2)| \leq 2,$$  \hspace{2cm} (33)

where the correlation function $E(\theta_1, \theta_2)$ is given by

$$E(\theta_1, \theta_2) = \langle \sigma^{(1)}_{\theta_1} \sigma^{(2)}_{\theta_2} \rangle.$$  \hspace{2cm} (34)

Here $\theta_1$ and $\theta_2$ are real parameters. In the following the operator $\sigma^{(i)}_a$ with $a = x, y$ or $z$ is the $a$ Pauli spin operator for the two-level system of atom $i$ and the operator $\sigma^{(i)}_{\theta_i}$ is defined as

$$\sigma^{(i)}_{\theta_i} = \cos \theta_i \sigma^{(i)}_x + \sin \theta_i \sigma^{(i)}_y.$$  \hspace{2cm} (35)

We describe now how the inequality (33) could be tested experimentally.

B. Description of the experimental test

To test Bell’s inequality the atoms have to be prepared first in a state for which a violation of Bell’s inequality is expected. This can be done with the help of the scheme discussed in Section IIA by preparing the atoms in state (3). The parameter $\alpha$ can be varied by changing the length $T$ of the laser pulse.

For certain initial states and in certain cases (including here) the correlation function depends only on the difference between the angles $\theta_1$ and $\theta_2$ and we have

$$E(\theta_1, \theta_2) = E(\theta_1 - \theta_2, 0).$$  \hspace{2cm} (36)

This can be proven easily and holds because the state (11) is not populated. Populating (11) by the preparation schemes proposed here is not possible, because the time evolution of the system is restricted to decoherence-free states (14). As an example to test Bell’s inequality we choose $\vartheta = \theta_1 - \theta_2 = \theta_2 - \theta'_2 = \theta'_1 - \theta'_2$. This leads to $\theta_1 - \theta'_2 = 3\vartheta$. Using Eq. (36) the inequality (33) simplifies for this parameter choice to

$$B_S = |3E(\vartheta, 0) - E(3\vartheta, 0)| \leq 2.$$  \hspace{2cm} (37)

A violation of this inequality corresponds to $|B_S| > 2$.

To find a way to measure the correlation functions $E(\vartheta, 0)$ we make use of the relation

$$U^{(i)\dagger}_{\text{rot}}(\xi, \phi) \sigma^{(i)}_z U^{(i)\dagger}_{\text{rot}}(\xi, \phi) = \cos 2\xi \sigma^{(i)}_z - \sin 2\xi \left( \cos \phi \sigma^{(i)}_x + \sin \phi \sigma^{(i)}_x \right).$$  \hspace{2cm} (38)

This allows us to rewrite $\sigma^{(i)}_{\theta_i}$ in terms of $\sigma^{(i)}_z$. By choosing $\xi = \pi/4$ and by making use of some trigonometric relations one obtains from Eq. (33)

$$\sigma^{(i)}_{\theta_i} = U^{(i)\dagger}_{\text{rot}} \left( \frac{\pi}{4}, \frac{3\pi}{2} - \theta_i \right) \sigma^{(i)}_z U^{(i)\dagger}_{\text{rot}} \left( \frac{\pi}{4}, \frac{3\pi}{2} - \theta_i \right),$$  \hspace{2cm} (39)

where $U^{(i)\dagger}_{\text{rot}}$ is the single qubit rotation defined in Eq. (14). Using this, Eq. (64) and (65) one can show that

$$E(\vartheta, 0) = \left\langle U^{(1)\dagger}_{\text{rot}} \left( \frac{\pi}{4}, \frac{3\pi}{2} - \vartheta \right) \sigma^{(1)}_z U^{(1)\dagger}_{\text{rot}} \left( \frac{\pi}{4}, \frac{3\pi}{2} - \vartheta \right) \right\rangle \times U^{(2)\dagger}_{\text{rot}} \left( \frac{\pi}{4}, \frac{3\pi}{2} \right) \sigma^{(2)}_z U^{(2)\dagger}_{\text{rot}} \left( \frac{\pi}{4}, \frac{3\pi}{2} \right).$$  \hspace{2cm} (40)

This expectation value can be measured in the following way. First, the single qubit rotation described in Section IIIB has to be applied on both atoms with $\xi = \pi/4$ and $\phi = 3\pi/2 - \vartheta$ for atom 1 and $\xi = \pi/4$ and $\phi = 3\pi/2$ for atom 2. Afterwards the observables $\sigma^{(1)}_z$ and $\sigma^{(2)}_z$ have to be measured. This can be done by measuring whether the atoms are in their ground state or not as described in Section IIC or IIIC, respectively. In an analogous way $E(3\vartheta, 0)$ can be determined experimentally.

It is important to point out that the correlation function represents an ensemble average obtained by performing the measurements over many runs, each time repreparing the initial state.

C. Expected violation of Bill’s inequality

It is straightforward to show that the correlation function for the initial state (3) is given by

$$E(\vartheta, 0) = -|\alpha|^2 \cos \vartheta,$$  \hspace{2cm} (41)

and hence Eq. (37) can assume a maximum of $|B_S| = 2\sqrt{2} |\alpha|^2$ where we have chosen $\vartheta = \pi/4$. Therefore, a violation of the spin Bell inequality is possible for $|\alpha|^2 > 1/\sqrt{2}$. The quantity $|\alpha|^2$ can be expressed in terms of the fundamental system parameter $|\Omega|^{-1}$ only.
with the help of Eq. (12). In Fig (3) we plot $|B_S|$ versus $|\Omega(\phi)|T$ and $\phi$.

FIG. 5. Plot of $|B_S|$ versus $|\Omega(\phi)|T$ and $\phi$. A violation of the spin Bell’s inequality occurs for $|B_S| > 2$ and are displayed as Islands in the $|\Omega(\phi)|T$ - $\phi$ plane. The angles have been chosen so as to maximise the violation utilising the maximally entangled state.

A significant region of violation is observed with the maximum of $|B_S| = 2\sqrt{2}$ occurring at $|\Omega(\phi)|T = \pi$. The state of the atoms at such a time is a maximally entangled state. This test on Bell’s inequality should be feasible with current technology.

V. DISCUSSION

In this article we have made use of a recently proposed scheme [13] to prepare in a controlled way with a very high success rate two atoms in an arbitrary superposition of a maximally entangled state and a product state. We show how the spin Bell inequality [14] can be characterised, tested and violated closing the detection loophole. To do so we use the highly efficient measurement proposal by Cook [15] based on “electron shelving”. The system discussed here has the appeal that the atoms are massive particles compared with photons and hence our proposal tests quantum mechanics in an all new macroscopic regime. In addition, while the photon experiments close the casualty loophole, the proposed atom experiment would close the detection efficiency loophole.

To summarise, entanglement is a necessary quantum resource used in quantum information. While entangled photons have to date been the engine of much recent work, their ‘flying’ nature renders them inappropriate for the storage of information. We have discussed a means in which trapped ions or atoms become entangled in a controlled way using dissipation, and the degree to which the resulting entanglement can be measured through Bell correlations.

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[1] J. S. Bell, Physics (N.Y.) 1, 195 (1965).
[2] J. F. Clauser and M. A. Horne, Phys. Rev. D 10, 526 (1974).
[3] J. F. Clauser, M. A. Horne, A. Shimony and R. A. Holt, Phys. Rev. Lett. 23, 880 (1969).
[4] S. L. Braunstein and C. M. Caves, Phys. Rev. Lett. 61, 622 (1988).
[5] A. Peres, Found. Phys. 29, 589 (1999).
[6] J. F. Clauser and A. Shimony, Rep. Prog. Phys. 41, 1981 (1978) and references therein.
[7] A. Aspect, P. Grangier, and G. Roger, Phys. Rev. Lett. 49, 91 (1982); A. Aspect, J. Dalibard, and G. Roger, Phys. Rev. Lett. 49, 1804 (1982).
[8] P. G. Kwiat, K. Mattle, H. Weinfurter, A. Zeilinger, A. V. Sergienko, and Y. Shih, Phys. Rev. Lett. 75, 4337 (1995).
[9] W. Tittel, J. Brendel, H. Zbinden, and N. Gisin, Phys. Rev. Lett. 81, 3563 (1998).
[10] P. G. Kwiat, E. Waks, A. G. White, I. Appelbaum, and P. H. Eberhard, Phys. Rev. A 60, R773 (1999).
[11] G. Weihs, T. Jennewein, C. Simon, H. Weinfurter, and A. Zeilinger, Phys. Rev. Lett. 81, 5039 (1998).
[12] A. Kuzmich, I. A. Walmsley, and L. Mandel, Phys. Rev. Lett. 85, 1349 (2000).
[13] A. Beige, D. Braun, and P. L. Knight, New J. Phys. 2, 22 (2000).
[14] R. C. Cook, Physica Scr. T21, 49 (1998).
[15] A. Beige and G. C. Hegerfeldt, J. Mod. Phys. 44, 345 (1997).
[16] J. I. Cirac and P. Zoller, Phys. Rev. A 50, R2799 (1994).
[17] S. J. D. Phoenix and S. M. Barnett, J. Mod. Opt. 40, 979 (1993).
[18] I. K. Kudryavtsev and P. L. Knight, J. Mod. Opt. 40, 1673 (1993).
[19] E. S. Fry, T. Walther, and S. Li, Phys. Rev. A 52, 4381 (1995).
[20] C. C. Gerry, Phys. Rev. A 53, 2857 (1996).
[21] E. Hagley, X. Maitre, G. Nogues, C. Wunderlich, M. Brune, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 79, 1 (1997).
[22] C. A. Sackett, D. Kielpinski, B. E. King, C. Langer, V. Meyer, C. J. Myatt, M. Rowe, Q. A. Turchette, W. M. Itano, D. J. Wineland, and I. C. Monroe, Nature 404, 256 (2000).
[23] K. Mølmer and A. Sørensen, Phys. Rev. Lett. 82, 1835 (1999); K. Mølmer and A. Sørensen, Phys. Rev. A 62, 022311 (2000).
[24] By a strong test of quantum we mean a test with no auxiliary assumptions. A test in the spirit of Bell original description.
[25] G. C. Hegerfeldt and T. S. Wilser, in Classical and Quantum Systems, Proceedings of the Second International Wigner Symposium, July 1991, edited by H. D. Doebner, W. Scherer, and F. Schroeck (World Scientific, Singapore, 1992), p. 104; G. C. Hegerfeldt and D. G. Sondermann, Quantum Semiclass. Opt. 5, 121 (1996).
[26] J. Dalibard, Y. Castin, and K. Mølmer, Phys. Rev. Lett. 68, 580 (1992).
[27] H. Carmichael, An Open Systems Approach to Quantum Optics, Lecture Notes in Physics, Vol. 18 (Springer, Berlin, 1993).
[28] For a recent review see M. B. Plenio and P. L. Knight, Rev. Mod. Phys. 70, 101 (1998) and references therein.
[29] M. B. Plenio, S. F. Huelga, A. Beige, and P. L. Knight, Phys. Rev. A 59, 2468 (1999).
[30] P. M. Radmore and P. L. Knight, J. Phys. B 15, 561 (1982).
[31] G. M. Meyer and G. Yeoman, Phys. Rev. Lett. 79, 1650 (1997).
[32] G. J. Yang, O. Zobay, and P. Meystre, Phys. Rev. A 59, 4012 (1999).
[33] G. M. Palma, K. A. Suominen, and A. K. Ekert, Proc. Roy. Soc. London Ser. A 452, 567 (1996).
[34] P. Zanardi and M. Rasetti, Phys. Rev. Lett. 79, 3306 (1997).
[35] D. A. Lidar, I. L. Chuang, and K. B. Whaley, Phys. Rev. Lett. 81, 2594 (1998).
[36] B. Misra and E. C. G. Sudarshan, J. Math. Phys. 18, 756 (1977).
[37] W. M. Itano, D. J. Heinzen, J. J. Bollinger, and D. J. Wineland, Phys. Rev. A 41, 2295 (1990).
[38] A. Beige and G. C. Hegerfeldt, Found. Phys. 263, 1671 (1997).
[39] A. Beige, S. Bose, D. Braun, S. F. Huelga, P. L. Knight, M. B. Plenio, and V. Vedral, J. Mod. Opt. 47, 2583 (2000).
[40] The scheme works as well for arbitrary choices of the Rabi frequencies as long as one has $\Omega^{(1)} \neq \Omega^{(2)}$. It is also possible to apply a laser pulse on one atom only.
[41] M. Brune, F. Schmidt-Kaler, A. Maali, J. Dreyer, E. Hagley, J. M. Raimond, and S. Haroche, Phys. Rev. Lett. 76, 1800 (1996).
[42] K. Bergmann, H. Heuer, and B. W. Shore, Rev. Mod. Phys. 70, 1003 (1998).
[43] N. V. Vitanov and S. Stenholm, Phys. Rev. A 55, 648 (1997).
[44] If the state $|1\rangle$ would be populated its population vanishes immediately during the next time interval $\Delta T$ by leakage of a photon out of the cavity or due to the no-photon time evolution with $H_{\text{cond}}$ given in Eq. (3).