Traversable wormholes with charge and non-commutative geometry in the STEGR

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We consider symmetric teleparallel gravity (STEGR), in which gravitational Lagrangian is given by the arbitrary function of non-metricity scalar \( Q \) to study static and spherically symmetric charged traversable wormhole solutions with non-commutative background geometry. The matter source at the wormhole throat is acknowledged to be anisotropic, and the redshift function has a constant value (thus, our wormhole solution is non-tidal). We derived numerically suitable forms of wormhole shape functions in the linear \( f(Q) = \alpha Q + \beta \) and non-linear \( f(Q) = Q + mQ^n \) STEGR models directly from the modified Einstein Field Equations (EFE’s). Besides, we probed these models via Null, Dominant, and Strong energy conditions w.r.t. free MOG parameters \( \alpha, \beta, m, \) and \( n \). We also used Tolman-Oppenheimer-Vokloff (TOV) equation to investigate the stability of WH anisotropic matter in considered MOG. Finally, we plot the effective equation of state.

I. INTRODUCTION

It is well known that wormholes (WHs) are generally the tunnels connecting two widely separated regions in the universe or even two separated universes. Flamm [1] first realized this hypothetical connection in 1916. After that, Einstein and Rosen [2] used his concept and constructed a bridge so-called Einstein-Rosen bridge. Later, in 1957, the term wormhole was introduced by Wheeler and Misner [3]. This field of study has been very popular for the last few decades. On the wormholes there was written a large number of papers, like [4–12]. Among this and other numerous works, one is of special interest - work written by Morris & Thorne in 1988, which presents humanly traversable spherically symmetric wormholes (in relation to the Einstein-Rosen bridge, which is non-traversable). But, as it turned out, in the conjecture of Morris-Thorne wormhole, if we consider classical GR gravity, given by the Einstein-Hilbert action below:

\[
S = \int_M d^4x \sqrt{-g} R \tag{1.1}
\]

where \( R \) is the Ricci scalar, the so-called Null Energy Condition (NEC) \( T_{\mu\nu}k^\mu k^\nu \geq 0 \) will be violated (here, \( T_{\mu\nu} \) is the energy-momentum tensor and \( k^\mu \) is a null vector). In GR, Morris-Thorne (MT) wormhole solutions could not be obtained if we consider the non-exotic matter as the matter source. To overcome this issue, researchers used various methods, such as considering the MT wormhole systems where the quantum effect competes with the classical ones [13–16]. Also, we could use additional fields to solve the exotic matter problem [17–19]. Finally, to overpass the problem of NEC violation, one could assume the modified Einstein-Hilbert action (i.e., modified gravity) because if we modify the EH action, then Einstein Field Equations will differ; thus, stress-energy tensor will change. Consequently, it could be possible that in one of the viable modified gravity theories, NEC will be satisfied.

A. Modified gravity and wormholes

General relativity gravity is a good choice at the large scale in our universe because it could sufficiently describe universe evolution. But, as it was noticed during the analysis of recent cosmological observations, GR classical gravity could not describe essential processes, such as cosmological inflation (which occurs at the very early times) or late-time accelerated expansion. Then, it is beneficial to assume proper EH action modification to describe these processes. For example, one of the most popular choices of the MOG form is \( f(R) \) gravity, in which we replace Ricci scalar with the arbitrary function of the Ricci scalar \( f(R) \). Viable \( f(R) \) cosmologies coincide very well with the data.
obtained from the space telescopes, such as Planck. For example, Starobinsky model could describe cosmological inflation [20–22] due to presence of squared Ricci scalar in the $f(R)$. Also, with the exponential form of the MOG, we could create the universe with both inflationary and late-time acceleration phases. Finally, even the dark energy problem could be solved [23, 24].

In the area of traversable wormholes in modified theories of gravity, many different interesting works have also been studied, such as [25–29] in $f(R)$ gravity, [30–36] in $f(R, T)$ gravity, [37–39] in $f(T)$ gravity, [40–42] in noncommutative geometries and so on.

The concept of non-commutative geometry is an intrinsic characteristic of the manifold itself as stated in [43], and it can be introduced in GR by modifying the matter source. It is believed that by using non-commutative geometry, some viewpoints of quantum gravity can be studied mathematically more effectively. An exciting result of the string theory is that the spacetime coordinates evolve noncommuting operators on a D-brane [44, 45]. Such non-commutative operator are used to encrypted in the commutator $[x^µ, x^ν] = iθ^µν$, where $θ^µν$ is the the anti-symmetric matrix of dimension $(length)^2$ and it is used to defines the discretization of spacetime [46–48].

In recent years, non-commutative geometry has become the considerable interest among researchers. It is considered the crucial property of space-time geometry and shows a vital role in different areas. In [49], Nozaria and Mehdipoura studied ‘Parikh–Wilczek Tunneling from Noncommutative Higher Dimensional Black Holes’ under Lorentzian distribution. Sushkov discussed wormholes supported by phantom energy by employing Gaussian distribution in [50]. Rahaman et al. [51] discussed wormhole solutions by taking Gaussian distribution in the background and found that wormhole solutions exist in the four, as well as in five dimensions only. Moreover, the stability of a particular class of thin-shell wormholes in GR under non-commutative geometry has been studied in [52]. Also, the BTZ blackhole under non-commutative background has been investigated in [53].

Our study is focused on recently proposed modified symmetric teleparallel gravity, or so-called $f(Q)$ gravity [54]. In this kind of MOG, gravitational Lagrangian is described by an arbitrary non-metricity scalar $Q$ function. We focused on this theory because, in recent years, $f(Q)$ MOG gained interest in the community of cosmologists. A large number of works have been studied on this gravity in theoretical and observational directions. We quote, for instance, in [55–57] some cosmological features of $f(Q)$ gravity were investigated, Energy conditions in [58] and also wormhole solutions have been studied in $f(Q)$ gravity in Refs. [59, 60]. One may check [61, 62] for more applications of $f(Q)$ gravity.

It is worth noting that in the current paper, we investigate the traversable wormhole with noncommutative geometry (both Gaussian and Lorentzian distributions) in the presence of an additional electrostatic field (metric tensor is very similar to the one which describes Reissner-Nordström charged black hole).

B. Article organization

This article is organized as follows: in the section (I), we provide an introduction into the topic of traversable wormholes and different modified gravity theories, the viability of MOG. In the Section (II) we present the formalism of the symmetric teleparallel $f(Q)$ gravity. In the section (III), we specify the metric tensor line element of the charged spherically symmetric wormholes and derive modified Einstein Field Equations for such choice of $g_{µν}$. Furthermore, we also present noncommutative geometry (with both Gaussian and Lorentzian distributions) in this section. In the Section (IV) we probe the energy conditions of $f(Q)$ charged wormholes with different kinds of noncommutative geometries and different forms of $f(Q)$ function. Additionally, in the section (V) we show how the equation of state parameter $ω$ change with the change of radial coordinate $r$, in the section (VI) we derive the fair values of MOG parameters, for which wormhole is stable. Finally, in the last section (VII), we provide the concluding remarks about the key topics of our investigation.

II. FORMALISM OF THE $f(Q)$ GRAVITY

In the $f(Q)$ gravity, the total Einstein Hilbert action is given:

$$S[g_{µν}, Γ, Ψ] = S_g + S_m = \frac{1}{2κ} \int d^4x \sqrt{-g} \left[ f(Q) + 2κL_m[g_{µν}, Ψ_i] \right]$$  (2.1)

where $f(Q)$ is arbitrary function of non-metricity scalar $Q$, $κ$ is gravitational constant, further we will assume that $κ = 1$, and finally $L_m[g_{µν}, Ψ_i]$ is the Lagrangian density of all perfect fluid (or spinor, gauge boson) matter fields $Ψ_i$ coupled to gravity $g_{µν}$. Firstly, we obviously want to define non-metricity tensor [63]

$$Q_{αµν} = ∇_{α}g_{µν}$$  (2.2)
The non-metricity conjugate is \[ P^\alpha_{\mu \nu} = \frac{1}{4} \left[ -Q^\alpha_{\mu \nu} + 2Q_{(\mu} ^\alpha _{\nu)} + Q^\alpha g_{\mu \nu} - \tilde{Q}^\alpha g_{\mu \nu} - \delta^\alpha_{(\mu} Q_{\nu)} \right] , \] (2.4)

where \( Q_\alpha = Q_\alpha ^\mu \mu \) and \( \tilde{Q}_\alpha = Q^\mu _\alpha \mu \) are traces of non-metricity tensor.

Furthermore, we assume fluid to be anisotropic with following stress-energy tensor \[ 65 \]:

\[ T_{\mu \nu} = [\rho, p_r, p_t, p_t] \] (2.5)

Here, \( \rho \) is matter-energy density, \( p_r, p_t \) are radial and tangential pressures respectively. Then, while we already defined all of the necessary, we could proceed to the derivation of the Einstein Field Equations by varying the EH action integral w.r.t. metric tensor \( g_{\mu \nu} \):

\[ \frac{2}{\sqrt{-g}} \nabla_\gamma \left( \sqrt{-g} f_Q P^\gamma_{\mu \nu} \right) + \frac{1}{2} g_{\mu \nu} f + f_Q \left( P_{\mu \gamma i} Q_\nu ^\gamma i - 2 Q_{\gamma i \mu} P^\gamma_i _\nu \right) = - T_{\mu \nu} , \] (2.6)

where \( f_Q \equiv \frac{df}{dQ} \). Also, by varying the action w.r.t. the affine connection \( \Gamma^\alpha_{\mu \nu} \) we obtain:

\[ \nabla_\mu \nabla_\nu \left( \sqrt{-g} f_Q P^\gamma_{\mu \nu} \right) = 0 . \] (2.7)

Therefore, we could go ahead and present the traversable wormhole spacetime in the next section.

III. TRAVERSABLE WORMHOLES IN STEGR AND NON-COMMUTATIVE GEOMETRY

Firstly, as usual we want to present the spherically symmetric, static traversable wormhole spacetime preserving a charge \( Q [6, 66] \):

\[ ds^2 = - \left( 1 + \frac{Q^2}{r^2} \right) dt^2 + \left( 1 - \frac{b}{r} + \frac{Q^2}{r^2} \right)^{-1} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \] (3.1)

where \( b(r) \) is the shape function that determines the shape of the wormhole. Shape function defines the geometry of the traversable wormhole, and must obey following (in)equalities: i) \( b - r = 0 \) at the WH throat \( (r = r_0) \), ii) \( \frac{b - r}{b'} > 0 \), iii) \( b' < 1 \), iii) \( \lim_{r \to \infty} \frac{b}{r} = 0 \) (because of the asymptotically flat background). Also, because we want to obtain only horizonless and non-singular solutions, \( e^{2\Omega(r)} \) (here \( \Omega(r) \) is the redshift function) must always be finite, and also, from the \( 6 \), tidal forces of the wormholes must be bearably small. Because of that conditions, we could consider the Zero Tidal Forces (ZTF) kind of traversable wormhole. It is worth notice that the line element (3.1) connects Morris-Thorne SS spacetime and Reissner-Nordström spacetime, so if \( Q = 0 \), we will have Morris-Thorne wormhole without charge, and if \( b = 0 \), we will have Reissner-Nordström black hole (because of the fact that \( r > 0 \), we don’t have the singularity in the charged WH spacetime). Then, we could present static and spherically symmetric spacetime line element of form:

\[ ds^2 = -e^{\xi(r)} dt^2 + e^{\zeta(r)} dr^2 + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \] (3.2)

and the EFE’s for this line element are [67]:

\[ \rho = \frac{e^{\xi-\zeta}}{2r^2} \left[ 2r f_Q Q' (e^\zeta - 1) + f_Q ((e^\zeta - 1)(2 + r \xi') + (1 + e^\zeta)r \zeta') + fr^2 e^\xi \right] \] (3.3)

\[ p_r = \frac{1}{2r^2} \left[ 2r f_Q Q' (e^\zeta - 1) + f_Q ((e^\zeta - 1)(2 + r \xi' + r \zeta') - 2r \zeta') + fr^2 e^\xi \right] \] (3.4)

\[ p_t = \frac{r}{4e^\xi} \left[ -2r f_Q Q' + f_Q (2e^\xi (e^\zeta - 2) - r \xi'^2 + \zeta(2e^\zeta + r \xi') - 2r \xi'') + fr^2 e^\xi \right] \] (3.5)
And, there is present the transformation from this spacetime to the charged WH one:

\[ \xi(r) \rightarrow \log \left( \frac{Q^2 + r^2}{r^2} \right), \quad \zeta(r) \rightarrow \log \left( \frac{r^2}{-rb + Q^2 + r^2} \right) \tag{3.6} \]

Consequently, EFE’s for the charged WH are:

\[
\rho(r) = \left[ (1 - \frac{rb + Q^2}{r^2}) \left( -\left[ f_Q(rb + Q^2) ((Q^2 + r^2)(rb + Q^2 - 2r^2)(r^2(b - b') + Q^2(r + 1)) + 2Q^2 r(rb + Q^2 - r^2)^2 \\
-2r(Q^2 + r^2)(rb + Q^2 - r^2)^2 \right] \right) \right] \left[ r^2(Q^2 + r^2)(rb + Q^2 - r^2)^2 \right] - \left[ 2f_Q\rho(rb + Q^2)(r(Q^2 + r^2)(b'(r^4(Q^2 + r^2)b' \\
-9Q^2r^4 + Q^6) + Q^2r((r^4 - Q^4)b'' + b((7Q^4r^4 + 2Q^6r^2 - 5r^8)b' + (-3Q^4r^5 - 2Q^6r^3 + r^9)b'' - 17Q^6r^2 - 27Q^4r^4 \\
+ 25Q^2r^6 + 3Q^8) + rb^2(r^2(Q^2 + r^2)((Q^2 + 3r^2)b' - r(Q^2 + r^2)b'') - 4Q^4r^2 - 26Q^2r^4 + 6Q^6 + 4r^6) \\
+b^3(4Q^2r^4 + 3Q^4r^2 - 3r^6) - 8Q^4r(Q^4 - r^4)) \right] \left[ r^4(rb(Q^2 + r^2) + Q^4 - r^4)^2 \right] + f(r^3) \right] \left[ 2r^2(-b - \frac{Q^2}{r} + r) \right] \tag{3.7} \]

\[
p_r(r) = \left[ (1 - \frac{rb + Q^2}{r^2}) \left( f_Q(rb + Q^2 - r^2)((rb + Q^2)(r^2 - (Q^2 + r^2)b' + rb(Q^2 + 3r^2) + 2(2Q^2r^2 + Q^4 - r^4)) \\
+ 2r(Q^2 + r^2)(rb + Q^2 - r^2)^2 \times \log\left( \frac{Q^2}{r^2} + 1 \right) \right) \right] \left[ Q^2 + r^2 \right] - \left[ 2f_Q\rho(rb + Q^2)(r(Q^2 + r^2)(b'(r^4(Q^2 + r^2)b' \\
- 9Q^2r^4 + Q^6) + Q^2r((r^4 - Q^4)b'' + b((7Q^4r^4 + 2Q^6r^2 - 5r^8)b' + (-3Q^4r^5 - 2Q^6r^3 + r^9)b'' - 17Q^6r^2 - 27Q^4r^4 \\
+ 25Q^2r^6 + 3Q^8) + rb^2(r^2(Q^2 + r^2)((Q^2 + 3r^2)b' - r(Q^2 + r^2)b'') - 4Q^4r^2 - 26Q^2r^4 + 6Q^6 + 4r^6) \\
+b^3(4Q^2r^4 + 3Q^4r^2 - 3r^6) - 8Q^4r(Q^4 - r^4)) \right] \left[ r^3(Q^2 + r^2)^2 \right] + f(r^3)(rb + Q^2 - r^2)^2) \right] \left[ 2r^2(rb + Q^2 - r^2)^3 \right] \tag{3.8} \]

\[
p_t(r) = \left[ b(Q^2r^2(Q^2 + r^2)(b'(2f_Q(5Q^2r^2 + 2Q^4 - 5r^4) - f_Qr^4(Q^2 + r^2)) + 2f_Qr(r^2 - 2Q^2)(Q^2 + r^2)b'') \\
+ r^4(Q^2 + r^2)(f^4(Q^2 + r^2)^2 + f_Q(7Q^4r^2 - 11Q^2r^4 + 11Q^6 + r^6)) + f_Qr^5(4Q^2 - 3r^2)(Q^2 + r^2)^3 \log\left( \frac{Q^2}{r^2} + 1 \right) \\
+ f_Q(-34Q^2r^4 - 54Q^6r^4 + 50Q^4r^6 + 6Q^{10})) + r(Q^2 + r^2)(b'(2f_Q\rho Q^2r^4(Q^2 + r^2)b' - f_Qr^4(Q^2 + r^2)(Q^4 + r^4) \\
+ 2f_Q\rho(Q^6 - 9Q^4r^4)) + 2f_Q\rho Q^4r(r^4 - Q^4)b'' + f_Qr^3(Q^2 + r^2)^2(-3Q^2r^2 + 2Q^4 + r^4) \log\left( \frac{Q^2}{r^2} + 1 \right) \\
+ rb^2(2f_Q\rho Q^2r^2(Q^2 + r^2)((Q^2 + 3r^2)b' - r(Q^2 + r^2)b'') + Q^2(f_Qr^4(Q^2 + r^2)(5Q^2 + 7r^2) + 4f_Q(-2Q^4r^2 \\
- 13Q^2r^4 + 3Q^6 + 2r^6)) + f_Qr^5(Q^2 + r^2)^3 \times \log\left( \frac{Q^2}{r^2} + 1 \right) + 2f_Q\rho Q^2r^2b^3(4Q^2r^2 + 3Q^4 - 3r^4) + r^3(Q^2 + r^2) \\
(f^4(Q - r)(Q + r)(Q^2 + r^2)^2 + f_Q(-6Q^4r^4 + 8Q^2r^6 + 6Q^8)) - 16f_Q\rho Q^6r(Q^4 - 3r^4) \right] \right] \left[ 2r^7(Q^2 + r^2)^3(rb + Q^2 - r^2) \right] \tag{3.9} \]

Also, non-metricity scalar reads:

\[
Q = \frac{rb + Q^2}{r^3}(r(Q^2 + r^2)b' + b(Q + r)(Q + r) - 4Q^2r) \tag{3.10} \]

Finally, we will proceed to the charged wormhole non-commutative geometry behavior.
A. Non-commutative geometry

Usually, non-commutative geometry is used in GR for the replacement of point-like structures as the smeared object (which allows us to eliminate the divergencies). This smearing effect could be achieved by the replacement of the Gaussian distributions of minimal length $\sqrt{\vartheta}$ with the Dirac delta function. In [68], Schneide and DeBenedictis deeply examined the background of both non-commutative distributions. In the next sections, we shall discuss the physical analysis of wormhole solutions under non-commutative Gaussian and Lorentzian distributions. For this purpose, we consider the Gaussian and Lorentzian distributions of the energy densities for the point-like gravitational source are given below [69, 70]:

$$\rho(r) = \frac{M e^{-\frac{r^2}{8\pi^{3/2}\vartheta^{3/2}}}}{2\pi^{3/2}\vartheta^{3/2}}$$ \hspace{1cm} (3.11)$$

$$\rho(r) = \frac{\sqrt{\vartheta}M}{\pi^2(\vartheta + r^2)^2}$$ \hspace{1cm} (3.12)$$

where $\vartheta$ is the non-commutativity parameter. $M$ is the smearing mass distribution and it could be a diffused centralized object such as a wormhole [71].

IV. CONSTRAINING CHARGED WH’S FROM ENERGY CONDITIONS

A. Energy Conditions

We will probe the following energy conditions in the current paper:

- Null Energy Condition (NEC): $\rho + p_r \geq 0$ $\land$ $\rho + p_t \geq 0$
- Weak Energy Condition (WEC): $\rho \geq 0$ and $\rho + p_r \geq 0$ $\land$ $\rho + p_t \geq 0$
- Strong Energy Condition (SEC): $\rho + p_r + 2p_t \geq 0$
- Dominant Energy Condition (DEC): $\rho \geq |p_r| \land \rho \geq |p_t|

As we know, in the GR, if traversable wormholes exist, there always must present so-called exotic matter at the throat, which violates Null Energy Condition (minimal requirement of WEC and SEC). In this paper, we will investigate the energy conditions of the wormhole in the viable $f(Q)$ cosmologies in the presence of non-commutative geometry.

B. Gaussian distribution

In this section, we are going to probe the different energy conditions for our charged traversable wormhole with various $f(Q)$ models and with Gaussian distribution energy density.

1. Linear model $f(Q) = \alpha Q + \beta$

As for the first model of STEGR, we consider following simplest linear form of $f(Q)$ function:

$$f(Q) = \alpha Q + \beta$$ \hspace{1cm} (4.1)$$

With the Gaussian distribution energy density (3.11) and from the (modified) EFE’s, we could derive a suitable wormhole shape function. We derived it numerically because of the complicated form of the EFE’s and the presence of wormhole charge. Numerical solutions (we used initial conditions $b(2) = 2$ and $b’(2) = 0.5$) for the wormhole and it’s flaring-out condition with the varying charge $Q$ we depicted on the Figure (1). As one may notice, flaring-out condition is satisfied on the throat ($b’(r_0) < 1$) and on the asymptotically flat background ($\langle (b(r) - r b’(r))/b^2(r) > 0 \rangle$ for $Q > 2$. But for very large values of charge $Q$, it is possible that the flaring out condition won’t be satisfied on the charged wormhole (CWH) throat.

In addition, we probed the Null, Dominant, and Strong energy conditions on the Figure (2). Unfortunately, because of the non-commutative geometry, NEC is violated for both radial and tangential pressures even at the asymptotically flat background. DEC is violated at each point of spacetime as well as the SEC.
FIG. 1. Linear $f(Q)$ gravity charged wormhole shape function and the flaring-out condition with $M = 12$ and $\vartheta = 0.5$, also we assumed that $\alpha = \beta = 2$

2. Non-linear model $f(Q) = Q + mQ^n$

Throughout this subsection, we have used non-linear form of the MOG function $f(Q)$ [72]:

$$f(Q) = Q + mQ^n$$

As usual, we numerically solved equations (3.7) and (3.11) with initial conditions $b(0.01) = 0.01$ and $b'(0.01) = 0.5$. We illustrated shape function and flaring-out condition solutions with varying WH charge on the Figure (3). As we have noticed during the numerical analysis, the flaring-out condition for the shape function in the non-linear $f(Q)$ gravity is satisfied at the CWH throat, and for small values of charge, $Q$ also satisfied at the asymptotically flat CWH region.

As well, we have solved the various energy conditions. As it turned out, unfortunately, for every positive value of WH charge $Q$, NEC is violated for the radial pressure and validated for tangential one. The DEC situation was the same, and SEC was violated near the WH throat.

C. Lorentzian distribution

In this section, we will probe the different energy conditions for our charged traversable wormhole with various $f(Q)$ models and with Lorentzian distribution energy density.

1. Linear model $f(Q) = \alpha Q + \beta$

We could start the linear model $f(Q) = \alpha Q + \beta$. For this kind of $f(Q)$ MOG, as usual by using the modified field equations and the Lorentzian distribution from equation (3.12) we could numerically derive shape function with the initial conditions $b(2) = 2$ and $b'(2) = 0.5$. In the Figure (5), we have plotted the numerical solution to the wormhole shape function with charge and its flaring-out condition. It is obvious that the flaring out condition is satisfied at the WH throat ($b'(r_0) < 0$) and at the external CWH region (asymptotically flat background).

Furthermore, we numerically constrained CWH spacetime by the Null, Dominant, and Strong energy conditions on the Figure (6). As it was revealed, in relation to the Gaussian distribution, in the Lorentzian one with the linear $f(Q)$ model NEC was violated for the radial and tangential pressure with any $Q \geq 0$. Dominant Energy Condition, in turn, was violated for both pressure kinds. Finally, SEC was also violated even further from the charged wormhole throat (because of the smeared WH mass).

2. Non-linear model $f(Q) = Q + mQ^n$

Our final model is the non-linear $f(Q) = Q + mQ^n$ MOG with the Lorentzian distribution. Consequently, in the Figure (7), we present the charged traversable wormhole shape function and its flaring-out condition. It is necessary
FIG. 2. Linear $f(Q)$ gravity Null, Dominant, and Strong energy conditions for the charged traversable wormhole spacetime. For simplicity, we assumed that $M = 12$ and $\vartheta = 0.5$, and also we took $\alpha = \beta = 2$ to notice that the flaring-out condition for the CWH spacetime with the Lorentzian non-commutative geometry is violated near the throat but validated at $r_0$ for any value of $Q \geq 0$.

As it was revealed, for the non-linear $f(Q)$ charged wormhole with the Lorentzian distribution energy density, all energy conditions (NEC for radial, DEC for both radial and tangential pressures, and SEC) was violated for any $Q \geq 0$ (for more details and numerical representation, see Figure (8)).
V. EFFECTIVE EQUATION OF STATE

In this section, we shall discuss the behavior of EoS parameter $\omega$ with different charges $Q$. For this, we have considered the relation between energy density $\rho$ and radial pressure $p_r$ as follows

$$p_r = \omega \rho,$$

where $\omega$ is the EoS parameter.

We have represented the behaviors of EoS parameter $\omega$ for different charges $Q$ in Figures 9 and 10 for both distributions. For the linear case, one may notice that for $Q = 0.1$ and 1, the values of the parameter $\omega$ lies between $-1$ to 0, i.e., $(-1 < \omega < 0)$ referred to the quintessence region at WH throat for both distributions. But for the non-linear case under both sources for $Q = 0.1$, EoS parameter $\omega$ also represents in the quintessence region, whereas, for $Q = 1$, it is less than -1, i.e., $\omega < -1$ which is referred to as the phantom region.

VI. STABILITY FROM TOV

In order to discuss the equilibrium configuration for the wormhole geometry under noncommutative distributions, we shall use the generalized Tolman-Oppenheimer-Volkoff (TOV) equation of the form [73, 74]

$$-\frac{dp_r}{dr} - \frac{\Omega'(r)}{2}(\rho + p_r) + \frac{2}{r}(p_t - p_r) = 0,$$

The forces namely, hydrostatic ($F_H$), the gravitational ($F_G$) and anisotropic force ($F_A$) are represented by following expressions

$$F_H = -\frac{dp_r}{dr}, \quad F_A = \frac{2}{r}(p_t - p_r), \quad F_G = -\frac{\Omega'}{2}(\rho + p_r),$$

and thus Eq. (6.1) takes the form given by

$$F_A + F_G + F_H = 0.$$  (6.3)

In Fig. 11, we have numerically checked our obtained wormhole solution’s stability for both models. For the linear model under both distributions, we found that the gravitational $F_G$ effect vanishes while the $F_A$ and $F_H$ forces are identical but opposite, resulting in the equilibrium of the solutions. For non-linear cases, one may also find from Fig. 11 that the gravitational $F_G$ force effect vanishes while the anisotropic force $F_A$ shows positive behavior, hydrostatic force $F_H$ is showing negative behavior at the throat. After that, it shows positive behavior for $r$ increases and shows negative for large $r$. Hence, both models satisfy the Eq. (6.3) to hold the system in equilibrium. Therefore, we could conclude that our obtained wormhole solutions for both models are stable under the non-commutative framework. However, one can check that the linear model is more stable than the non-linear model for both distributions. Readers may visit the Refs. [75, 76] where the authors have been studied deeply on this topic.
FIG. 4. Non-linear \( f(Q) \) gravity Null, Dominant and Strong energy conditions for the charged traversable wormhole spacetime. For simplicity, we assumed that \( M = 12 \) and \( \vartheta = 0.5 \), and also we took \( m = 1 \) and \( n = 2 \) (Quadratic, Staroniskry-like STEGR).

VII. CONCLUDING REMARKS

A wormhole describes a shortcut distance to link different parts of the universe. To examine these solutions, the violation of NEC plays an essential role associated with the exotic matter. The usage of exotic matter would be minimized to obtain a realistic model in favor of the wormhole. This work studied the spherically symmetric static wormhole solutions in symmetric teleparallel gravity under two well-known non-commutative distributions: Gaussian and Lorentzian distributions. We have developed the corresponding field equations [67] for the spherically symmetric spacetime metric with the help of the transformations for a charged wormhole in \( f(Q) \) gravity. In this work, we have considered two WH models such as linear \( (f(Q) = \alpha Q + \beta) \) and non-linear \( (f(Q) = Q + mQ^n) \) models and studied the WH solutions under non-commutative backgrounds. Due to the high non-linearity of field equations, we have plotted the graph numerically. The graphical behaviors of our obtained solutions have been discussed below.
For the linear model under Gaussian distribution, we have presented the behaviors of shape functions in Figure 1. One may observe that the shape function is showing increasing behavior with varying $Q$ and flaring out condition ($b'(r_0) < 1$) is also satisfied for $Q > 2$ on the asymptotically flatness background, but it could be possible that for very large value of $Q$, flaring out conditions will not be satisfied on the CWH throat. Moreover, in Figure 2, we have illustrated the behavior of energy conditions (NEC, DEC, and SEC). We noticed that NEC is violated for both radial and tangential pressures even at the asymptotic background. Also, DEC and SEC are violated at each spacetime point under the Gaussian framework. Again for the non-linear model under Gaussian distribution, in Figures 3 and 4, we have depicted the behavior of shape functions and energy conditions. We have noticed during the numerical analysis that the flaring out condition is satisfied at the CWH throat for small values of charge $Q$ on the asymptotically flat CWH region. Also, for any positive values of charge $Q$, NEC is violated for the radial pressure and satisfied for the tangential one. SEC was also violated in the vicinity of WH throat. Violation of NEC confirms the presence of exotic matter at the WH throat, which is necessary for the traversability of WH.

Moving forward, for the linear model under Lorentzian source, we have numerically plotted the graphs for the shape functions and energy conditions presented in Figures 5 and 6. It is obvious that the flaring out condition is validated at WH throat and at the external CWH region. NEC and DEC are violated for both pressures with any charge $Q \geq 0$. SEC is also violated even further from the charged WH throat because of the smeared mass. Furthermore, for the non-linear model under Lorentzian distribution, we found that the flaring out condition is violated near the WH throat but validated at the throat for any $Q \geq 0$. Moreover, we observed that for the non-linear $f(Q)$ model, all the energy conditions were violated for any $Q \geq 0$. One may check the Figures 7 and 8 for more details.

Further, we have studied the behavior of EoS parameter $\omega$ with different charge $Q$. Our analysis shows that $\omega$ represents under quintessence region at WH throat for the linear model under both distributions. But for the non-linear model, for $Q = 0.1$, EoS $\omega$ is showing under quintessence region, whereas for $Q = 1$, it is represented under the phantom region for both frameworks. Lastly, we have used a tool called the TOV equation to check the stability of our obtained WH solutions for both models. From our numerically plotted graph (see Figure 11), we found that our obtained linear model solutions are more stable than the non-linear model under both distributions. Thus, it would be interesting to mention here that our obtained results are consistent in the non-commutative framework in the symmetric teleparallel gravity. Also, It would be more interesting to explore wormhole solutions this modified $f(Q)$ gravity by taking other matter sources into account.

[1] L. Flamm, Phys. Z 17, 448 (1916).
[2] A. Einstein and N. Rosen, Phys. Rev. 48.
[3] C. W. Misner and J. A. Wheeler, Annals of Physics (1957).
[4] H. Ellis, (1973).
[5] K. A. Bronnikov, Acta Phys. Polon. B 4, 251 (1973).
[6] M. S. Morris and K. S. Thorne, American Journal of Physics 56, 395 (1988).
[7] D. Hochberg and T. W. Kephart, Phys. Rev. Lett. 70.
[8] M. Visser, Phys. Rev. D 39 ().
FIG. 6. Linear $f(Q)$ gravity Null, Dominant, and Strong energy conditions for the charged traversable wormhole spacetime (with the Lorentzian distribution). For simplicity, we assumed that $M = 12$ and $\vartheta = 0.5$, and also we took $\alpha = \beta = 2$.

[9] M. Visser, Phys. Rev. D 55 (1).
[10] S.-W. Kim and H. Lee, Phys. Rev. D 63.
[11] N. Dadhich, S. Kar, S. Mukherjee, and M. Visser, Phys. Rev. D 65.
[12] P. K. F. Kuhfittig, Phys. Rev. D 67, 064015 (2003), arXiv:gr-qc/0401028.
[13] M. Visser, Lorentzian Wormholes: From Einstein to Hawking (American Inst. of Physics, 1995).
[14] P. Gao, D. L. Jafferis, and A. C. Wall, Journal of High Energy Physics 2017, 1 (2016).
[15] J. Maldacena and X.-I. Qi, (2018), arXiv:1804.00491.
[16] E. Caceres, A. Kundu, A. K. Patra, and S. Shashi, JHEP 02, 149 (2020), arXiv:1912.08793 [hep-th].
[17] K. A. Bronnikov and S. Grinyok, Grav. Cosmol. 7, 297 (2001), arXiv:gr-qc/0201083.
[18] C. Armendáriz-Picón, Phys. Rev. D 65.
[19] A. Nicolis, R. Rattazzi, and E. Trincherini, Journal of High Energy Physics 2010, 1 (2010).
[20] D. Brooker, S. Odintsov, and R. Woodard, (2016).
FIG. 7. Non-linear $f(Q)$ gravity charged wormhole shape function (Lorentzian distribution) and the flaring-out condition with $M = 12$ and $\vartheta = 0.5$, also we assumed that $m = 1$ and $n = 2$ (Starobinsky-like quadratic STEGR).
FIG. 8. Non-Linear \( f(Q) \) gravity Null, Dominant and Strong energy conditions for the charged traversable wormhole spacetime (with the Lorentzian distribution) and with \( M = 12 \) and \( \vartheta = 0.5 \), also we assumed that \( m = 1 \) and \( n = 2 \) (Starobinsky-like quadratic STEGR).

[62] Physics of the Dark Universe 30, 100616 (2020).
[63] J. Beltrán Jiménez, L. Heisenberg, and T. Koivisto, Phys. Rev. D 98, 10.1103/PhysRevD.98.044048, arXiv:1710.03116 [gr-qc].
[64] W. Khyllep, A. Paliathanasis, and J. Dutta, (2021).
[65] M. F. Shamir and I. Fayyaz, The European Physical Journal C (2020).
[66] S.-W. Kim and H. Lee, Physical Review D 63, 064014 (2001).
[67] R.-H. Lin and X.-H. Zhai, Phys. Rev. D 103, 124001 (2021), arXiv:2105.01484 [gr-qc].
[68] M. Schneider and A. DeBenedictis, Phys. Rev. D 102, 024030 (2020).
[69] B. J. Barros, T. Barreiro, T. Koivisto, and N. J. Nunes, Phys. Dark Univ. 30, 100616 (2020), arXiv:2004.07867 [gr-qc].
[70] Physics Letters B 632, 547 (2006).
[71] J. P. de Leon, General Relativity and Gravitation 35, 1365 (2003).
[72] S. H. Shekh, Phys. Dark Univ. 33, 100850 (2021).
FIG. 9. Behavior of EoS parameter $\omega$ for different $Q$ for linear (left) and nonlinear (right) models under Gaussian distributions with $M = 12$, $\vartheta = 0.5$, $m = 1$ and $n = \alpha = \beta = 2$.

FIG. 10. Behavior of EoS parameter $\omega$ for different $Q$ for linear (left) and nonlinear (right) models under Lorentzian distributions with $M = 12$, $\vartheta = 0.5$, $m = 1$ and $n = \alpha = \beta = 2$.

[73] (2014).
[74] P. K. F. Kuhfittig, Fund. J. Mod. Phys. 14, 23 (2020), arXiv:2009.11179 [gr-qc].
[75] S. Rani and A. Jawad, Advances in High Energy Physics 2016, 7815242 (2016).
[76] F. Rahaman, S. Karmakar, I. Karar, and S. Ray, Physics Letters B 746, 73 (2015).
FIG. 11. Shows the behaviors of $F_A$, $F_G$ and $F_H$ for linear and nonlinear models under Gaussian (left) and Lorentzian (right) distributions with $M = 12$, $\vartheta = 0.5$, $m = 1$, $n = \alpha = \beta = 2$ and $Q = 0.25$. 