Feynman–Kac Formulas for Dirichlet–Pauli–Fierz Operators with Singular Coefficients

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Abstract. We derive Feynman–Kac formulas for Dirichlet realizations of Pauli–Fierz operators generating the dynamics of nonrelativistic quantum mechanical matter particles, which are minimally coupled to both classical and quantized radiation fields and confined to an arbitrary open subset of the Euclidean space. Thanks to a suitable interpretation of the involved Stratonovich integrals, we are able to retain familiar formulas for the Feynman–Kac integrands merely assuming local square-integrability of the classical vector potential and the coupling function in the quantized vector potential. Allowing for fairly general coupling functions becomes relevant when the matter-radiation system is confined to cavities with inward pointing boundary singularities.

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1. Introduction and Main Results

1.1. General Introduction

The main objective of this article is to derive Feynman–Kac formulas for Dirichlet realizations on arbitrary open subsets $\Lambda \subset \mathbb{R}^\nu$ of Pauli–Fierz operators with possibly quite singular coefficients. Pauli–Fierz operators are selfadjoint operators generating the dynamics of nonrelativistic quantum mechanical matter particles confined to $\Lambda$ and interacting with a quantized radiation field. Let $\mathcal{F}$ denote the bosonic Fock space modelled over the one-boson Hilbert space

$$\mathcal{F} := L^2(\mathcal{K}, \mathcal{R}, \mu).$$  \hspace{2cm} (1.1)
We assume the measure space \((\mathcal{K}, \mathcal{A}, \mu)\) to be \(\sigma\)-finite and countably generated, which entails separability of \(\mathfrak{g}\). Define
\[
\mathcal{D}(\Lambda) := C_0^\infty(\Lambda), \quad \mathcal{D}(\Lambda, \mathcal{E}) := \text{span}_C \{ f \psi \mid f \in \mathcal{D}(\Lambda), \psi \in \mathcal{E} \},
\]
for any complex vector space \(\mathcal{E}\). Then the Dirichlet–Pauli–Fierz operator investigated here—we denote it by \(H_\Lambda\)—acts in the Hilbert space \(L^2(\Lambda, \mathcal{F})\) and represents the closure of the quadratic form given by
\[
\hat{h}_\Lambda[\Psi] := \frac{1}{2} \sum_{j=1}^\nu \int_\Lambda \| (\partial_{x_j} - iA_j(\vec{x}) - i\varphi(G_j, \vec{x}))\Psi(\vec{x}) \|_F^2 d\vec{x}
\]
\[
+ \int_\Lambda \| d\Gamma(\omega)^{1/2}\Psi(\vec{x}) \|_F^2 d\vec{x} + \int_\Lambda V(\vec{x})\|\Psi(\vec{x})\|_F^2 d\vec{x},
\]
for all \(\Psi \in \mathcal{D}(\hat{h}_\Lambda) := \mathcal{D}(\Lambda, \mathcal{Q}(d\Gamma(1 \lor \omega))).\)

In the above expressions, \(\omega \geq 0\), the boson dispersion relation, is a multiplication operator in \(\mathfrak{g}\), and \(d\Gamma(\omega)\), the radiation field energy, is its differential second quantization; \(\mathcal{D}(\cdot)\) denotes domains and \(\mathcal{Q}(\cdot)\) form domains. By coefficients in (1.3) we mean the triple comprised of the electrostatic potential\(^1\)
\[
V \in L^1_{\text{loc}}(\Lambda, \mathbb{R}), \quad V \geq 0,
\]
the classical vector potential
\[
A = (A_1, \ldots, A_\nu) \in L^2_{\text{loc}}(\Lambda, \mathbb{R}^\nu),
\]
and the coupling function
\[
G = (G_1, \ldots, G_\nu) \in L^2_{\text{loc}}(\Lambda, \mathfrak{t}^\nu),
\]
that determines the interaction between the matter particles and the radiation field. As usual \(\varphi(G_j, \vec{x})\) stands for the field operator corresponding to \(G_j, \vec{x} := G_j(\vec{x})\).

The present article actually continues our earlier study [27] of Dirichlet–Pauli–Fierz operators with singular coefficients where we determined the domain and found natural operator cores of these operators. While many technical results of [27] hold in greater generality, these main results were obtained under the assumption that \(G \in L^\infty(\Lambda, \mathcal{Q}(\omega^{-1} + \omega)\nu)\) with a weak divergence \(\text{div}G \in L^\infty(\Lambda, \mathcal{Q}(\omega^{-1}))\). This is more than enough to cover the standard model of nonrelativistic quantum electrodynamics on Euclidean space with an ultraviolet cutoff or ultraviolet regularized models of quantum optics in bounded cavities with smooth boundaries. Recall that, according to the general quantization scheme for the electromagnetic field found in physics textbooks (see, e.g., [5]), the coupling function has the following form in applications to quantum optics in bounded cavities with \(\nu = 3\):
\[
G_x = c \sum_{n=1}^\infty \frac{\chi(\omega(n))}{\sqrt{\omega(n)}} E_n(x).
\]

\(^1\)A negative part will be subtracted from \(V\) only in Corollary 1.4 and Remark 1.5.
Here $0 < \omega(1) \leq \omega(2) \leq \cdots$ are the strictly positive eigenfrequencies of the Maxwell operator on $\Lambda$ with perfect electric conductor boundary conditions. The normalized function $E_n$ is the electric component of the eigenvector of the Maxwell operator corresponding to the frequency $\omega(n)$. Furthermore, $\epsilon \in \mathbb{R}$ is a combination of physical constants, and the auxiliary, sufficiently fast decaying function $\chi : [0, \infty) \to [0, 1]$ implements the ultraviolet cutoff.

The boundary $\partial \Lambda$ of a cavity $\Lambda$ might, however, not always be smooth. If $\partial \Lambda$ has singularities, like polyhedral structures with inward pointing edges and corners for instance, then the functions $E_n$ in (1.8) are singular as well at the inward pointing boundary singularities; see, e.g., [3] and the references given there. In particular, the usual $L^\infty$-conditions imposed on $G$ in [27] (and in almost all other articles on Pauli–Fierz type operators, dipole approximations being one exception) might not be fulfilled in the presence of boundary singularities. This motivates keeping the assumptions on $G$ more general while studying basic qualitative features of Dirichlet–Pauli–Fierz operators.

In this article we choose to consider a situation where the individual terms in the quadratic form (1.3) are well-defined and finite for every $\Psi$ as in (1.4). Since $\Psi$ in (1.4) can be the product of any function in $\mathcal{D}(\Lambda)$ and the Fock space vacuum, this necessitates (1.6), (1.7), and the first condition in (1.5). We assume the second condition in (1.5), since it is often convenient to have it in our proofs and our main results extend by standard arguments to suitable electrostatic potentials that are unbounded from below; see Corollary 1.4. (Making sufficient effort, magnetic Schrödinger operators can actually be constructed even without assuming local square-integrability of the vector potential and local integrability of the electrostatic potential [24].)

A good part of this article is made up of analyzing quadratic forms and diamagnetic inequalities and here the condition (1.7) is in fact sufficient. The Feynman–Kac formulas will, however, only be valid when the operators $\varphi(G_{j,x})$ admit the interpretation as position observables of the radiation field. The latter is the case when

$$G = (G_1, \ldots, G_\nu) \in L^2_{\text{loc}}(\Lambda, \mathfrak{k}_R^\nu), \quad (1.9)$$

where $\mathfrak{k}_R$ is an arbitrary completely real subspace of $\mathfrak{k}$ satisfying $e^{-t\omega k_R} \subset \mathfrak{k}_R$, for all $t > 0$. As it turns out, it is possible under the conditions (1.5), (1.6), and (1.9) to derive Feynman–Kac formulas for Dirichlet–Pauli–Fierz operators given by familiar expressions, provided that the Stratonovich integrals involving $A$ and $G$ in these formulas are defined as in (1.11) and (1.12) below.

1.2. The Main Theorem

In the whole article $(\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})$ denotes a filtered probability space satisfying the usual assumptions of completeness and right-continuity of the filtration $(\mathfrak{F}_t)_{t \geq 0}$. The letter $\mathbb{E}$ denotes expectation with respect to $\mathbb{P}$. Furthermore, $B$ denotes a $\nu$-dimensional $(\mathfrak{F}_t)_{t \geq 0}$-Brownian motion starting in 0, and we put $B^x := x + B$, for all $x \in \mathbb{R}^\nu$. Pick some $t > 0$ and let

$$B^{t,x} := (B_{t-s}^x)_{s \in [0,t]} \quad (1.10)$$
denote the time-reversal of $B^x$ at $t$. This time-reversed process is a semi-martingale when the underlying probability space is equipped with a suitable new filtration as explained in more detail in Sect. 8.2; see [10, 31] for the general theory of time-reversed diffusion processes. With this we define\(^2\)

\[
S_t(x) := \int_0^t V(B^x_s)ds - \frac{i}{2} \int_0^t A(B^x_s)dB^x_s + \frac{i}{2} \int_0^t A(B^{t:x}_s)dB^{t:x}_s, \tag{1.11}
\]

\[
K_t(x) := \frac{1}{2} \int_0^t j_s G_B dB^x_s - \frac{1}{2} \int_0^t j_{t-s} G_B^{t:x} dB^{t:x}_s. \tag{1.12}
\]

In the second line, \(\{j_s\}_{s \in \mathbb{R}}\) is a strongly continuous family of isometries originally introduced by E. Nelson [30]. These isometries are defined on \(\mathfrak{f}\) and attain values in the new Hilbert space

\[
\hat{\mathfrak{f}} := L^2(\mathbb{R} \times \mathcal{K}, \mathfrak{B}(\mathbb{R}) \otimes \mathfrak{K}, \lambda \otimes \mu),
\]

with \(\lambda\) denoting the one-dimensional Lebesgue measure and \(\mathfrak{B}(\mathbb{R})\) the Borel subsets of \(\mathbb{R}\). They are given by the formulas

\[
(j_s f)(\kappa, k) := \frac{1}{\pi^{1/2}} \frac{(\omega(k))^{1/2}}{(\kappa^2 + \omega(k)^2)^{1/2}} e^{-is\kappa} f(k), \quad \text{a.e.} \ (\kappa, k) \in \mathbb{R} \times \mathcal{K},
\]

for all \(f \in \mathfrak{f}\) and \(s \in \mathbb{R}\). We apply \(j_s\) componentwise to an element of \(\mathfrak{f}'\).

The construction of the four stochastic integrals above under the conditions (1.6) and (1.9) requires a few simple comments which are given in Sects. 9.1 and 9.2; their existence is guaranteed for a.e. \(x \in \Lambda\) at least. Notice that the first and second stochastic integrals in both (1.11) and (1.12) are defined with respect to different filtrations; in each line the linear combination of the two \(\text{Itô}\) type integrals substitutes more common expressions for Stratonovich integrals.

Next, let \(b^{t:y,x}\) be the semimartingale realization of a Brownian bridge from \(y \in \mathbb{R}^\nu\) to \(x \in \mathbb{R}^\nu\) in time \(t\) introduced in more detail in Sect. 8.2. As verified in [8, Appendix 4], the relevant results of [10, 31] on time-reversed processes also apply to Brownian bridges. Putting

\[
\hat{b}^{t:x,y} := (b_s^{t:y,x})_{s \in [0,t]}, \tag{1.13}
\]

we thus obtain a semimartingale realization of a Brownian bridge from \(x \in \mathbb{R}^\nu\) to \(y \in \mathbb{R}^\nu\) in time \(t\), provided that the original filtration is replaced by a suitable new one; see again Sect. 9.2. Analogously to (1.11) and (1.12) we define

\[
S_t(x, y) := \int_0^t V(b_s^{t:y,x})ds - \frac{i}{2} \int_0^t A(b_s^{t:y,x})db_s^{t:y,x} + \frac{i}{2} \int_0^t A(\hat{b}_s^{t:x,y})d\hat{b}_s^{t:x,y}, \tag{1.14}
\]

\(^2\)Readers who are wondering about the signs in (1.11) should notice that the complex conjugate of \(S_t(x)\) appears in our Feynman–Kac formula; see (1.16) and the first equality in (1.19).
\[
K_t(x, y) := \frac{1}{2} \int_0^t j_s G_{b_t^s; y, x} \, dB^t_s; y, x - \frac{1}{2} \int_0^t j_{t-s} G_{b_t^{t-s}; y, x} \, dB_s^{t-s}; x, y. \tag{1.15}
\]

Again the existence of the four stochastic integrals appearing here is ensured by (1.6) and (1.9), for a.e. \((x, y) \in \mathbb{R}^\nu\) at least; see Sects. 9.1 and 9.2.

We finally list all remaining notation needed to formulate our main theorem:

- We abbreviate
  \[
  W_t(x) := e^{-S_t(x)}\Gamma(j_t)^* e^{i\varphi(K_t(x))} \Gamma(j_0), \tag{1.16}
  \]
  \[
  W_t(x, y) := e^{-S_t(x, y)}\Gamma(j_t)^* e^{i\varphi(K_t(x, y))} \Gamma(j_0), \tag{1.17}
  \]
  where \(\Gamma(j_s)\) denotes the second quantization of the isometry \(j_s\).

- The first exit time of \(B^x\) from \(\Lambda\) is denoted by
  \[
  \tau_\Lambda(x) := \inf\{s \geq 0\mid B_s^x \notin \Lambda\}. \tag{1.20}
  \]

- We always employ the common convention \(\inf\emptyset := \infty\).

- The first exit time of \(b_t^{t;y, x}\) from \(\Lambda\) is denoted by
  \[
  \tau_\Lambda(t; y, x) := \inf \{s \in [0, t] \mid b_s^{t;y, x} \notin \Lambda\}. \tag{1.21}
  \]

- The symbol \(1_A\) stands for the indicator function of a set \(A\).

- We denote the Euclidean heat kernel by
  \[
  p_t(x, y) := (2\pi t)^{-\nu/2} e^{-|x-y|^2/2t}, \quad x, y \in \mathbb{R}^\nu, \ t > 0. \tag{1.18}
  \]

**Theorem 1.1.** Assume (1.5), (1.6), and (1.9). Let \(t > 0\) and \(\Psi \in L^2(\Lambda, \mathcal{F})\). Then, for a.e. \(x \in \Lambda\), we have the following Feynman–Kac formulas for the Dirichlet–Pauli–Fierz operator \(H_\Lambda\) representing the closure of the form given by (1.3) and (1.4),

\[
(e^{-tH_\Lambda} \Psi)(x) = \mathbb{E} \left[ 1_{\{\tau_\Lambda(x) > t\}} W_t(x)^* \Psi(B_t^x) \right] = \int_\Lambda p_t(x, y) \mathbb{E} \left[ 1_{\{\tau_\Lambda(t; y, x) = \infty\}} W_t(x, y) \Psi(y) \right] \, dy. \tag{1.19}
\]

**Proof.** This theorem is proven in Sect. 9.4. \(\square\)

**Remark 1.2.** Manifestly, \(W_t(x)^*\) and \(W_t(x, y)\) are strongly measurable maps from \(\Omega\) to \(\mathcal{B}(\mathcal{F})\). Furthermore,

\[
\|W_t(x)\| \leq 1, \quad \|W_t(x, y)\| \leq 1, \tag{1.20}
\]

pointwise on \(\Omega\). In particular, the \(\mathcal{F}\)-valued expectations in (1.19) are well-defined.

**Remark 1.3.** Write \(Q(\omega^{-1})_\mathbb{R} := Q(\omega^{-1}) \cap \mathfrak{t}_\mathbb{R}\) and replace (1.9) by the stronger condition

\[
G \in L^2_{\text{loc}}(\Lambda, Q(\omega^{-1})_\mathbb{R}), \tag{1.21}
\]

which is typically fulfilled in physically relevant examples with ultraviolet regularized interaction terms. Pick some \(t > 0\) and \(x, y \in \mathbb{R}^\nu\) such that all integrals in (1.14) and (1.15) exist. According to [8, Remark 17.7] we then have the alternative formula

\[
W_t(x, y)
\]
\[ e^{-S_t(x,y)-\|K_t(x,y)\|^2/2}F_{t/2}(ij_t^* K_t(x,y))F_{t/2}(-ij_0^* K_t(x,y))^*, \]

where the Fock space operator-valued maps

\[ Q(\omega^{-1}) \ni g \mapsto F_{t/2}(g) := \sum_{n=0}^\infty a_t(g)^n e^{-tdf(\omega)/2} \in \mathcal{B}(\mathcal{F}), \]

are analytic [8, Lemma 17.4], thus separably valued as \( Q(\omega^{-1}) \) is separable. (Here \( a_t(g) \) is the bosonic creation operator in \( \mathcal{F} \) associated with \( g \); see, e.g., [32].) In particular, \( W_t(x,y) : \Omega \to \mathcal{B}(\mathcal{F}) \) is measurable, separably valued, and bounded, whence the \( \mathcal{B}(\mathcal{F}) \)-valued expectation in

\[ e^{-tH_\Lambda}(x,y) := p_t(x,y)\mathbb{E}[1_{\{\tau_\Lambda(t;x,y) = \infty\}} W_t(x,y)] \in \mathcal{B}(\mathcal{F}) \]

is well-defined. In view of (1.19), the operators in (1.23) thus define a \( \mathcal{B}(\mathcal{F}) \)-valued integral kernel of \( e^{-tH_\Lambda} \). The random function \( W_t(x) \) can be written in the form (1.22) as well, provided that we drop \( y \) on the right hand side, of course.

In the following corollary we subtract a negative part \( U \) from \( V \). The form \( \hat{h}_\Lambda^U \) appearing in its statement is defined on \( \mathcal{D}(\hat{h}_\Lambda) \) and obtained upon putting \( V - U \) in place of \( V \) in (1.3).

**Corollary 1.4.** Assume (1.5), (1.6), (1.9), and let \( U : \Lambda \to [0,\infty) \) be form bounded with respect to one-half times the Dirichlet-Laplacian on \( \Lambda \) with relative form bound \( b \leq 1 \). Then \( U\mathbb{1}_\mathcal{F} \) is form bounded with respect to \( H_\Lambda \) with relative form bound \( \leq b \) and, in particular, \( \hat{h}_\Lambda^U \) is semibounded. Assume in addition that \( \hat{h}_\Lambda^U \) is closable and denote the selfadjoint operator representing its closure by \( H_\Lambda^U \). Then (1.19) remains true, when \( H_\Lambda \) is replaced by \( H_\Lambda^U \) and \( V - U \) is put in place of \( V \) in (1.11) and (1.14). If (1.21) is satisfied, then Remark 1.3 is still valid under the same replacements.

Notice that the somewhat implicit assumption that \( \hat{h}_\Lambda^U \) be closable is satisfied when \( b < 1 \). It is also satisfied when \( b \leq 1, \mathcal{D}(\hat{h}_\Lambda) \subset \mathcal{D}(H_\Lambda), \) and \( U \) is locally square-integrable, in which case \( H_\Lambda^U \) is a Friedrichs extension.

In Schrödinger operator theory even more singular \( U \) than the ones considered here have been treated; see [2,24,37] and the references given therein.

**Proof.** Corollary 1.4 is proven at the end of Sect. 9.4. \( \Box \)

Our Feynman–Kac formulas have several immediate and by now well-known applications that we shall mention only very briefly:

**Remark 1.5.** Assume (1.5), (1.6), and (1.21).

Adopting the notion of positivity on \( \mathcal{F} \) induced by its \( Q \)-space representation, we find that the semigroup of \( H_\Lambda^U \) with \( U \) as in Corollary 1.4 is ergodic; compare [26, ß10], [28, ß8.1], and the references therein. If \( U \) has an extension to \( \mathbb{R}^\nu \) that belongs to the Kato class of \( \mathbb{R}^\nu \), then we obtain \( L^p(\Lambda,\mathcal{F}) \) to \( L^q(\Lambda,\mathcal{F}) \) estimates (with \( 1 \leq p \leq q \leq \infty \)) for the semigroup of \( H_\Lambda^U \) and Gaussian upper bounds on its operator-valued integral kernel;
see [26] for references and further extensions in the case $\Lambda = \mathbb{R}^\nu$ with regular coefficients. If $U$ is Kato in the above sense and $\omega$ has a strictly positive lower bound, then the semigroup is hypercontractive simultaneously in the $x$- and $Q$-space-variables; see [15, Theorem 1.9 and B3.1] for an analogous bound in the renormalized Nelson model. If the latter hypercontractivity bound is available and $\Lambda$ is bounded and connected, then the infimum of the spectrum of $H^U_\Lambda$ is a non-degenerate eigenvalue and the corresponding eigenvector can be chosen strictly positive; see again [15, B3.1].

1.3. Brief Remarks on Earlier Results
For $\Lambda = \mathbb{R}^\nu$, $A = 0$, and under stronger assumptions on $G$, the first identity in (1.19) has been proven earlier by F. Hiroshima [13], and the second equality in (1.19) has been shown in [8]. The idea to represent Feynman–Kac integrands in nonrelativistic quantum field theory in the form (1.16) is originally due to E. Nelson [30], who considered scalar matter particles that are linearly coupled to quantized radiations fields.

In [8, 14] different possibilities to account for spin degrees of freedom in Feynman–Kac formulas for the Pauli–Fierz model are considered. An extension of Theorem 1.1 to a situation where the matter particles may have spin would, however, by no means be trivial and require extra conditions on the magnetic fields generated by the classical and quantized vector potentials.

As any meaningful survey of the extensive literature on Feynman–Kac formulas for magnetic Schrödinger operators and their various generalizations and applications would go beyond the scope of the discussion, we kindly ask the interested reader to consult, e.g., the remarks and long reference lists in the relatively recent article [11] and the books [7, 25] for a start. Explicitly, we mention only a few articles dealing with possibly very singular classical vector potentials on open subsets of the Euclidean space:

In [1] local Kato class assumptions are imposed on $A^2$ and $\text{div}A$ to derive Feynman–Kac formulas. The most singular case where quadratic forms still make sense on $\mathcal{D}(\Lambda)$, that is, $A \in L^2_{\text{loc}}(\Lambda, \mathbb{R}^\nu)$, is treated in [33] in the special case where $\Lambda^c$ has zero Lebesgue measure. Since the Feynman–Kac integrands are constructed with the help of compactness arguments in [33], they are, however, not given by explicit formulas there.

For every $A \in L^2_{\text{loc}}(\Lambda, \mathbb{R}^\nu)$, we actually find some $A_c \in L^2_{\text{loc}}(\Lambda, \mathbb{R}^\nu)$, having the same curl in distribution sense as $A$ and satisfying the Coulomb gauge condition $\text{div}A_c = 0$ in the weak sense, as well as some gauge potential $\gamma \in W^{1,2}_{\text{loc}}(\Lambda)$ such that $A = A_c + \nabla \gamma$; see [21, Lemma 1.1]. Exploiting the gauge invariance of Schrödinger operators [21, (Proof of) Theorem 1.2], we can thus derive a Feynman–Kac formula for the Schrödinger operator with vector potential $A_c$ containing only one stochastic integral in Itô’s sense, and obtain a Feynman–Kac type formula for $A$ by adding a $\gamma$-dependent term to the complex action. This strategy to find Feynman–Kac formulas for Dirichlet realizations of Schrödinger operators with highly singular vector potentials is treated as well-known in the more recent literature at least in the case where $A$ has a locally square-integrable extension to the whole $\mathbb{R}^\nu$ (see, e.g., [16]), and probably also in greater generality.
1.4. Organization, Proof Strategies, and Further Results

- In Sect. 2 we recall some Fock space calculus and provide precise definitions of the most important quadratic forms and operators considered in this article.
- Our general strategy is to infer Feynman–Kac formulas for proper open subsets \( \Lambda \subset \mathbb{R}^\nu \) from corresponding formulas in the case \( \Lambda = \mathbb{R}^\nu \). To this end we employ a procedure originally used for Schrödinger semigroups in [35] and later on for magnetic Schrödinger semigroups in [1]. In Sect. 3 we recall this procedure in a suitably abstracted version that applies to the quantum field theoretic models we are interested in here and in the recent work [15].
- A crucial ingredient for the proof procedure alluded to in the previous item are results on approximations with respect to the form norms of certain maximal Pauli–Fierz forms. (The closure of the form defined in (1.3) and (1.4) is the minimal Pauli–Fierz form.) These approximation results, which are non-trivial and possibly of independent interest, are obtained in Sect. 5. A Leibniz rule for vector-valued weak derivatives needed here is derived first in Sect. 4. As a byproduct we shall also see that the maximal and minimal Pauli–Fierz forms agree when \( \Lambda = \mathbb{R}^\nu \), as it is the case for Schrödinger operators [36].
- Also in the case \( \Lambda = \mathbb{R}^\nu \) our Feynman–Kac formulas are obtained by approximation. Here we depart from Feynman–Kac formulas for Pauli–Fierz operators with regularized coefficients. In Sect. 7 we therefore study strong resolvent convergence of Pauli–Fierz operators on \( \mathbb{R}^\nu \) when \( A \) and \( G \) are approximated in \( L^2_{\text{loc}} \) by more regular quantities. In doing so we employ a diamagnetic inequality for resolvents of Pauli–Fierz operators that we derive first in Sect. 6, more generally for Dirichlet–Pauli–Fierz operators on general open \( \Lambda \subset \mathbb{R}^\nu \). In its full generality this diamagnetic inequality is new even when \( \Lambda = \mathbb{R}^\nu \).
- For regular coefficients and \( \Lambda = \mathbb{R}^\nu \), we derive our Feynman–Kac formulas in Sect. 8, employing the stochastic differential equations associated with the Pauli–Fierz model analyzed in [8]. We shall push forward some results of [8] to non-vanishing \( A \). Eventually, we prove an associated strong Markov property (employing a “useful rule” for vector-valued conditional expectations verified in “Appendix A”) and show that the “probabilistic” right hand sides of the Feynman–Kac formulas give rise to a strongly continuous semigroup of bounded self-adjoint operators. The Pauli–Fierz operator finally turns out to be the generator of this semigroup, which proves the Feynman–Kac formulas for regular coefficients.
- The only technical obstacle remaining after the above preliminary results is to show convergence of the probabilistic sides of the Feynman–Kac formulas for \( \Lambda = \mathbb{R}^\nu \), when singular coefficients are approximated by regular ones. This is done in Sect. 9. Apart from that, we give a
detailed discussion of the Feynman–Kac integrands for singular coefficients and eventually complete the proofs of Theorem 1.1 and Corollary 1.4 in this final section.

2. Basic Definitions

In this section we collect the most important functional analytic definitions employed throughout the article. In the following subsections we shall, respectively, recall some Fock space calculus, define vector-valued weak derivatives, covariant derivatives, and finally introduce our Dirichlet–Pauli–Fierz operators.

In the whole article \( \Lambda \) denotes an arbitrary open subset of \( \mathbb{R}^\nu \); variables in \( \Lambda \) will most of the time be denoted by \( \mathbf{x} = (x_1, \ldots, x_\nu) \) or \( \mathbf{y} = (y_1, \ldots, y_\nu) \).

If \( T \) is a linear operator in some Hilbert space then its domain \( \mathcal{D}(T) \) is equipped with the graph norm

\[
\|\phi\|_{\mathcal{D}(T)} := \left( \|\phi\|^2 + \|T\phi\|^2 \right)^{1/2}, \quad \phi \in \mathcal{D}(T).
\]

If \( T \) is nonnegative and selfadjoint, then its form domain \( \mathcal{Q}(T) \) is equipped with the form norm

\[
\|\phi\|_{\mathcal{Q}(T)} := \left( \|\phi\|^2 + \|T^{1/2}\phi\|^2 \right)^{1/2}, \quad \phi \in \mathcal{Q}(T).
\]

2.1. Operators in the Bosonic Fock Space

Here we briefly recall some standard facts on the Weyl representation on bosonic Fock space. For a detailed textbook exposition of these matters we recommend [32].

Recall that the by assumption separable \( L^2 \)-space \( \mathcal{F} \) has been introduced in (1.1). The bosonic Fock space modeled over \( \mathcal{F} \) is given by the direct orthogonal sum

\[
\mathcal{F} := \mathbb{C} \oplus \bigoplus_{n=1}^{\infty} L^2(\mathcal{K}, \mathcal{R}, \mu^\otimes n),
\]

where \( \mathcal{R}^\otimes n \) is the \( n \)-fold product of the \( \sigma \)-algebra \( \mathcal{R} \) with itself and \( \mu^\otimes n \) is the \( n \)-fold product of the measure \( \mu \) with itself. A total subset of \( \mathcal{F} \) is given by the set of exponential vectors \( \epsilon(f) \in \mathcal{F} \) with \( f \in \mathcal{F} \),

\[
\epsilon(f) := (1, f, \ldots, (n!)^{-1/2} f^\otimes n, \ldots),
\]

with \( f^\otimes n(k_1, \ldots, k_n) := f(k_1)\ldots f(k_n), \mu \)-a.e. Let \( \mathcal{U}(\mathcal{K}) \) denote the set of unitary operators on some Hilbert space \( \mathcal{K} \) equipped with the topology associated with the strong convergence of bounded operators on \( \mathcal{K} \). Given \( f \in \mathcal{F} \) and \( U \in \mathcal{U}(\mathcal{F}) \), we let \( \mathcal{W}(f, U) \in \mathcal{U}(\mathcal{F}) \) denote the corresponding Weyl operator. We recall that it is determined by the prescription

\[
\mathcal{W}(f, U)\epsilon(g) := e^{-\|f\|^2/2 - \langle f, Ug \rangle} \epsilon(f + Ug), \quad g \in \mathcal{F},
\]

followed by linear and isometric extensions. The so obtained Weyl representation

\[
\mathcal{W} : \mathcal{F} \times \mathcal{U}(\mathcal{F}) \longrightarrow \mathcal{U}(\mathcal{F}), \quad (f, U) \longmapsto \mathcal{W}(f, U),
\]
is a strongly continuous projective representation of the semi-direct product of $\mathfrak{k}$ and $\mathscr{U}(\mathfrak{k})$. More precisely, we have the Weyl relations

$$\mathcal{W}(f_1, U_1)\mathcal{W}(f_2, U_2) = e^{-it\text{Im}(f_1U_1 f_2)}\mathcal{W}(f_1 + U_1 f_2, U_1 U_2),$$

for all $f_1, f_2 \in \mathfrak{k}$ and $U_1, U_2 \in \mathscr{U}(\mathfrak{k})$. As usual we abbreviate

$$\mathcal{W}(f) := \mathcal{W}(f, 1), \quad \Gamma(U) := \mathcal{W}(0, U), \quad f \in \mathfrak{k}, \ U \in \mathscr{U}(\mathfrak{k}). \quad (2.1)$$

Let $f \in \mathfrak{k}$. Then the above remarks imply that $\mathbb{R} \ni t \mapsto \mathcal{W}(-it f)$ is a strongly continuous unitary group on $\mathcal{F}$. Its selfadjoint generator is called the field operator associated with $f$. It is denoted by $\varphi(f)$, so that

$$\mathcal{W}(-it f) = e^{-it\varphi(f)}, \quad t \in \mathbb{R}.$$

In the whole article,

$$\omega : \mathcal{K} \to \mathbb{R}$$

is a measurable function that is $\mu$-a.e. strictly positive.

It has the physical interpretation of a boson dispersion relation. We shall use the same symbol $\omega$ to denote the associated selfadjoint multiplication operator in $\mathfrak{k}$. Then our remarks on the Weyl representation further imply that $\mathbb{R} \ni t \mapsto \Gamma(e^{-it\omega})$ is a strongly continuous unitary group on $\mathcal{F}$. Therefore, there exists a selfadjoint operator $d\Gamma(\omega)$ in $\mathcal{F}$ such that

$$\Gamma(e^{-it\omega}) = e^{-itd\Gamma(\omega)}, \quad t \in \mathbb{R}.$$

It is called the differential second quantization of $\omega$ and interpreted as the energy of the quantized radiation field.

Since the Nelson isometries $j_s : \mathfrak{k} \to \hat{\mathfrak{k}}$ introduced in Sect. 1.2 map into a Hilbert space different from $\mathfrak{k}$, the symbol $\Gamma(j_s)$ actually has to be understood in a sense generalizing (2.1). In fact, $\Gamma(j_s) : \mathcal{F} \to \hat{\mathcal{F}}$ is obtained by linear and isometric extension of the prescription $\Gamma(j_s)\epsilon(g) := \epsilon(j_s g) \in \hat{\mathcal{F}}, \ g \in \mathfrak{k}$, where $\hat{\mathcal{F}}$ is the bosonic Fock space modelled over $\hat{\mathfrak{k}}$.

We conclude this subsection by recalling the following standard relative bounds, where $\kappa : \mathcal{K} \to \mathbb{R}$ has the same properties as $\omega$ above,

$$\|\varphi(f)\psi\|_{\mathcal{F}} \leq 2^{1/2}\|f\|_{Q(\kappa^{-1})}\|\psi\|_{Q(d\Gamma(\kappa))},$$

$$\|\varphi(f)\varphi(g)\phi\|_{\mathcal{F}} \leq 8\|f\|_{Q(\kappa^{-1})}\|g\|_{Q(\kappa^{-1})}\|\phi\|_{\mathcal{D}(d\Gamma(\kappa))},$$

for all $f, g, \phi, \psi$ in the vectors spaces indicated by the respective subscripts; see, e.g., [27, Remark 2.10] for the second bound.

### 2.2. Vector-Valued Weak Derivatives

To deal with singular classical and quantized vector potentials it is most helpful to mimic the distributional techniques used in the study of magnetic Schrödinger operators in a vector-valued setting [27]. For the convenience of the reader we therefore recall the following fundamental definition:

Let $\mathcal{K}$ be a separable Hilbert space, $j \in \{1, \ldots, \nu\}$, and $\Psi \in L^1_{\text{loc}}(\Lambda, \mathcal{K})$. Then $\Psi$ is said to have a weak partial derivative with respect to $x_j$, iff there exists some (necessarily unique) vector $\partial_{x_j} \Psi \in L^1_{\text{loc}}(\Lambda, \mathcal{K'})$ such that

$$\int_{\Lambda} \langle \partial_{x_j} \eta(x), \Psi(x) \rangle_{\mathcal{K'}} \, dx = -\int_{\Lambda} \langle \eta(x), \partial_{x_j} \Psi(x) \rangle_{\mathcal{K'}} \, dx, \quad \eta \in \mathcal{D}(\Lambda, \mathcal{K'}).$$
2.3. Covariant Derivatives

Pick \( j \in \{1, \ldots, \nu\} \), \( A_j \in L^2_{\text{loc}}(\Lambda, \mathbb{R}) \) and let \( G_j : \Lambda \to \mathfrak{k} \) be in \( L^2_{\text{loc}}(\Lambda, \mathfrak{k}) \). For every \( \Psi \in L^2(\Lambda, Q(\text{d}\Gamma(1))) \), we define \( \varphi(G_j)\Psi \in L^1_{\text{loc}}(\Lambda, F) \) by

\[
(\varphi(G_j)\Psi)(x) := \varphi(G_j, x)\Psi(x), \quad \text{a.e. } x \in \Lambda.
\]

With this we define a symmetric operator \( v_{\Lambda,j} \) in \( L^2(\Lambda, F) \) by

\[
D(v_{\Lambda,j}) := \mathcal{D}(\Lambda, Q(\text{d}\Gamma(1))),
\]

\[
v_{\Lambda,j}\Psi := -i\partial_{x_j}\Psi - A_j\Psi - \varphi(G_j)\Psi, \quad \Psi \in D(v_{\Lambda,j}).
\]

Its adjoint \( v_{\Lambda,j}^* \) will play the role of a covariant derivative in the \( j \)-th coordinate direction in our Pauli–Fierz forms.

The approximation results proven in Sect. 5 depend crucially on the following theorem [27, Theorem 3.5] where, for any separable Hilbert space \( \mathcal{H} \) and any representative \( \Psi(\cdot) \) of \( \Psi \in L^1_{\text{loc}}(\Lambda, \mathcal{H}) \), we define

\[
S\Psi(x) := \begin{cases}
\|\Psi(x)\|_{\mathcal{H}}^{-1}\Psi(x), & \text{if } \Psi(\cdot) \neq 0, \\
0, & \text{if } \Psi(\cdot) = 0.
\end{cases}
\]  

(2.5)

**Theorem 2.1.** Let \( \Psi \in D(v_{\Lambda,j}^*) \). Then \( \|\Psi\|_{\mathcal{H}} \in L^2(\Lambda) \) has a weak partial derivative with respect to \( x_j \) which is given by

\[
\partial_{x_j}\|\Psi\|_{\mathcal{H}} = \text{Re}(S\Psi | iv_{\Lambda,j}^* \Psi) \in L^2(\Lambda).
\]  

(2.6)

2.4. Pauli–Fierz Forms and Dirichlet–Pauli–Fierz Operators

Assuming (1.5), (1.6), and (1.7) we first define a maximal Pauli–Fierz form,

\[
D(\mathfrak{h}_{\Lambda,N}) := L^2(\Lambda, Q(\text{d}\Gamma(\omega))) \cap Q(V 1_{\mathcal{F}}) \cap \bigcap_{j=1}^\nu D(v_{\Lambda,j}^*),
\]

\[
\mathfrak{h}_{\Lambda,N}[\Psi] := \frac{1}{2} \sum_{j=1}^\nu \|v_{\Lambda,j}^*\Psi\|^2 + \int_{\Lambda} V(x)\|\Psi(x)\|_{\mathcal{F}}^2 \, dx
\]

\[
+ \int_{\Lambda} \|d\Gamma(\omega)^{1/2}\Psi(x)\|_{\mathcal{F}}^2 \, dx, \quad \Psi \in D(\mathfrak{h}_{\Lambda,N}).
\]  

(2.7)

As a sum of nonnegative closed forms, \( \mathfrak{h}_{\Lambda,N} \) is itself closed and nonnegative. We further define a minimal closed forms, \( \mathfrak{h}_{\Lambda,D} \) is itself closed and nonnegative. We further define a minimal Pauli–Fierz form,

\[
\mathfrak{h}_{\Lambda,D} := \overline{\mathfrak{h}_{\Lambda,N} | \mathcal{D}(\Lambda, Q(\text{d}\Gamma(1\vee\omega)))} = \overline{\mathfrak{h}_{\Lambda}},
\]  

(2.8)

where in the second identity we used notation introduced in (1.3) and (1.4) of the introduction. In analogy to the Schrödinger case, the selfadjoint operator representing \( \mathfrak{h}_{\Lambda,D} \), we shall simply call it \( H_D \) dropping the subscript “D”, can be interpreted as the Dirichlet realization of the Pauli–Fierz operator on \( \Lambda \). (The subscript “N” is also borrowed from the Schrödinger theory where it stands for “Neumann”.)
3. Deriving Feynman–Kac Formulas for Dirichlet Realizations

In this section we explain how to derive Feynman–Kac formulas for proper open subsets $\Lambda \subset \mathbb{R}^\nu$ departing from known formulas in the case $\Lambda = \mathbb{R}^\nu$. This is done by a procedure which is standard for Schrödinger operators and originates from [35]; see also [1, Appendix B] for a helpful exposition treating Schrödinger operators with classical magnetic fields. All we do in this section is to carry through this procedure in a slightly abstracted setting covering the various nonrelativistic quantum field theoretic models we are interested in. The results of this section are, for instance, applied to the renormalized Nelson model in [15].

Let $\mathcal{H} \neq \{0\}$ be a separable Hilbert space. Suppose that $Q_{\mathbb{R}^\nu}$ and $Q_{\Lambda}$ are selfadjoint operators in $\mathcal{H} := L^2(\mathbb{R}^\nu, \mathcal{H})$ and its subspace $\mathcal{H}_\Lambda := 1_{\Lambda} L^2(\mathbb{R}^\nu, \mathcal{H})$, respectively, which are semi-bounded from below. Denote the corresponding quadratic forms by $q_{\mathbb{R}^\nu}$ and $q_{\Lambda}$, respectively. We assume that these two quadratic forms are related as follows:

We pick compact subsets $K_\ell$, $\ell \in \mathbb{N}$, of $\Lambda$ with $K_\ell \subset \bigcup_{\ell \in \mathbb{N}} K_\ell = \Lambda$. Furthermore, we pick $\vartheta_\ell \in C^\infty_0(\mathbb{R}^\nu)$ with $\vartheta_\ell = 1$ on $K_\ell$, $\vartheta_\ell = 0$ on $K_{\ell+1}$, and $0 \leq \vartheta_\ell \leq 1$, for all $\ell \in \mathbb{N}$. As we finally define a numerical function $Y_{\Lambda, \infty} : \mathbb{R}^\nu \to [0, \infty]$ by

$$Y_{\Lambda, \infty}(x) := \begin{cases} \text{dist}(x, \Lambda^c) - \sum_{\ell=1}^\infty |\nabla \vartheta_\ell|^2(x), & x \in \Lambda, \\ \infty, & x \in \Lambda^c; \end{cases}$$

for every fixed $x \in \Lambda$. This function defines a closed form in $\mathcal{H}$ with domain

$$\mathcal{Q}(Y_{\Lambda, \infty}) = \left\{ \Psi \in L^2(\mathbb{R}^\nu, \mathcal{H}) \mid \int_{\mathbb{R}^\nu} Y_{\Lambda, \infty}(x) \|\Psi(x)\|^2 \, dx < \infty \right\} \subset \mathcal{H},$$

which is not dense in general. We further set

$$\mathcal{D}(q_{\mathbb{R}^\nu, \Lambda}) := \mathcal{D}(q_{\mathbb{R}^\nu}) \cap \mathcal{Q}(Y_{\Lambda, \infty}) \subset \mathcal{H}.$$  

We now assume that the following:

**Hypothesis 3.1.** For at least one function $Y_{\Lambda, \infty}$ defined in the above fashion, Statements (a) and (b) hold, where:

(a) $\mathcal{D}(q_{\mathbb{R}^\nu, \Lambda})$ and the closure of $\mathcal{D}(q_{\mathbb{R}^\nu, \Lambda})$ with respect to the norm on $\mathcal{D}(q_{\Lambda})$ is equal to $\mathcal{D}(q_{\Lambda}).$

(b) $q_{\Lambda}[\Psi] = q_{\mathbb{R}^\nu} [\Psi]$, for all $\Psi \in \mathcal{D}(q_{\mathbb{R}^\nu, \Lambda}).$

The next remark explains the choice of the power $-3$ in (3.1). Any power strictly less than $-2$ would actually be sufficient for our applications in the later sections.

**Remark 3.2.** Let $t > 0$, let $I \subset \mathbb{R}$ be an interval containing $[0, t]$, and suppose that $\gamma : I \to \mathbb{R}^\nu$ is locally Hölder continuous of order $1/3$. Set

$$Y_{n, \Lambda} := n \wedge Y_{\Lambda, \infty}, \quad n \in \mathbb{N}.$$
Then we have the following equivalence, where the limit to the left always exists in $[0, \infty]$ by monotone convergence,

$$
\lim_{n \to \infty} \int_0^t Y_n^\Lambda(\gamma(s)) ds < \infty \iff \inf \{ s \geq 0 \mid s \in I, \gamma(s) \in \Lambda^c \} > t,
$$

with the common convention $\inf \emptyset = \infty$.

In fact, let $\tau \in [0, \infty]$ denote the infimum in (3.3). Assume first that $\tau > t$. Then $\gamma([0, t]) \subset \Lambda$. Thus $(Y_\infty^\Lambda \circ \gamma)_{|[0,t]}$ is a real-valued continuous function on the compact interval $[0, t]$. It is then clear that the limit as $n \to \infty$ of the integral to the left in (3.3) is finite. Next, assume that $\tau \leq t$. Since $\gamma$ is continuous and $\Lambda^c$ is closed, we then have $\gamma(\tau) \in \Lambda^c$. The local Hölder continuity of $\gamma$ thus implies

$$
Y_\infty^\Lambda(\gamma(s)) \geq |\gamma(s) - \gamma(\tau)|^{-3} \geq \frac{1}{C} |s - \tau|^{-1}, \quad s \in [0, t],
$$

for some $C > 0$. Consequently,

$$
\lim_{n \to \infty} \int_0^t Y_n^\Lambda(\gamma(s)) ds \geq \frac{1}{C} \int_0^t \frac{ds}{|s - \tau|} = \infty.
$$

### 3.1. Feynman–Kac Formulas for Dirichlet Realizations

Throughout this subsection we fix some $t > 0$. We work under the assumptions of the preceding subsection and the following hypothesis:

**Hypothesis 3.3.** There exist a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, for every $x \in \mathbb{R}^\nu$,

$\triangleright$ a strongly measurable map $M_t(x) : \Omega \to \mathcal{B}(\mathcal{H})$;

$\triangleright$ some pathwise continuous $\mathbb{R}^\nu$-valued stochastic process $X^x$ which $\mathbb{P}$-a.s. starts at 0 and whose paths are $\mathbb{P}$-a.s. locally Hölder continuous of order $1/3$;

such that the following holds:

$\triangleright$ For all $\Psi \in \mathcal{H}$ and $x \in \mathbb{R}^\nu$,

$$
\| M_t(x) \| \| \Psi(X^x_t) \| \in L^1(\mathbb{P}).
$$

(3.4)

$\triangleright$ For all bounded and continuous functions $v : \mathbb{R}^\nu \to \mathbb{R}$, the following Feynman–Kac type formula holds for all $\Psi \in \mathcal{H}$,

$$
(e^{-t(Q_{\mathbb{R}^\nu} + v)} \Psi)(x) = \mathbb{E}\left[ e^{-\int_0^t v(X^x_s) ds} M_t(x) \Psi(X^x_t) \right], \quad \text{a.e. } x \in \mathbb{R}^\nu.
$$

(3.5)

We further let $\tau_\Lambda(x) : \Omega \to [0, \infty]$ denote the first exit time of $X^x$ from $\Lambda$, i.e.,

$$
\tau_\Lambda(x) := \inf \{ s \geq 0 \mid X^x_s \in \Lambda^c \},
$$

with $\inf \emptyset = \infty$. Since $X^x$ is pathwise continuous and $\Lambda^c$ is closed, $\tau_\Lambda(x)$ is a stopping time with respect to the filtration generated by $X^x$. In particular,

$$
\{ \tau_\Lambda(x) > t \} \in \mathcal{F}.
$$

**Lemma 3.4.** In the situation described above, let $\Psi \in \mathcal{H}_\Lambda$. Then

$$
(e^{-tQ_\Lambda} \Psi)(x) = \mathbb{E}\left[ 1_{\{ \tau_\Lambda(x) > t \}} M_t(x) \Psi(X^x_t) \right], \quad \text{a.e. } x \in \mathbb{R}^\nu.
$$

(3.6)
Proof. Before we comment on the various steps of this proof we have to introduce some more notation:

For every \( \kappa > 0 \), we define \( \mathcal{D}(q_{R^\nu}^{\kappa,\infty}) := \mathcal{D}(q_{R^\nu}^{1,\infty}) \) (recall (3.2)) and

\[
q_{R^\nu}^{\kappa,\infty}[\Psi] := q_{R^\nu}[\Psi] + \kappa \int_{R^\nu} Y_n^\Lambda(x) \|\Psi(x)\|^2 dx, \quad \Psi \in \mathcal{D}(q_{R^\nu}^{\kappa,\infty}).
\]

Then \( q_{R^\nu}^{\kappa,\infty} \) is closed as a sum of closed semi-bounded forms. As remarked above, it is in general not densely defined as a form in \( \mathcal{H} \). By assumption (a) it is, however, a densely defined semi-bounded closed form on the sub-Hilbert space \( \mathcal{H}_\Lambda \). Therefore, there exists a unique selfadjoint operator in \( \mathcal{H}_\Lambda \) representing \( \mathcal{D}(q_{R^\nu}^{\kappa,\infty}) \) that we denote by \( Q_{R^\nu}^{\kappa,\infty} \). We further define the Hamiltonians

\[
Q_{R^\nu}^{\kappa,n} := Q_{R^\nu} + \kappa Y_n^\Lambda, \quad n \in \mathbb{N}, \kappa > 0,
\]

and denote the associated quadratic forms by \( q_{R^\nu}^{\kappa,n} \).

**Step 1.** Let \( \kappa > 0 \). We shall show that

\[
\left\| e^{-tQ_{R^\nu}^{\kappa,n}} \Psi - e^{-tQ_{R^\nu}^{\kappa,\infty}} \right\| \xrightarrow{n \to \infty} 0, \quad \Psi \in \mathcal{H}. \tag{3.7}
\]

We know that the form domain of \( Q_{R^\nu}^{\kappa,n} \) is \( \mathcal{D}(q_{R^\nu}^{\kappa,n}) \), which contains \( \mathcal{D}(q_{R^\nu}^{\kappa,\infty}) \). The monotone convergence theorem further shows that

\[
q_{R^\nu}^{\kappa,n}[\Psi] \uparrow q_{R^\nu}^{\kappa,\infty}[\Psi], \quad \Psi \in \mathcal{D}(q_{R^\nu}^{\kappa,\infty}) = \left\{ \Phi \in \mathcal{D}(q_{R^\nu}^{\kappa,n}) \right\| \sup_{n \in \mathbb{N}} q_{R^\nu}^{\kappa,n}[\Phi] < \infty \right\}.
\]

The convergence (3.7) now follows from a monotone convergence theorem for not necessarily densely defined quadratic forms [34, Theorem 4.1&Theorem 4.2].

**Step 2.** Let \( \kappa > 0 \) and \( \Psi \in \mathcal{H}_\Lambda \). We next show that

\[
(e^{-tQ_{R^\nu}^{\kappa,\infty}})(x) = \mathbb{E}\left[ e^{-\kappa \int_0^t Y_n^\Lambda(X_s^{x})ds} M_t(x)\Psi(X_t^{x}) \right], \tag{3.8}
\]

for a.e. \( x \in R^\nu \), where \( e^{-\infty} := 0 \). Owing to Step 1 we find natural numbers \( n_1 < n_2 < \ldots \) such that, a.e. on \( R^\nu \), the sequence \( (e^{-tQ_{R^\nu}^{\kappa,n}})_{t \in \mathbb{N}} \) converges to the vector \( e^{-tQ_{R^\nu}^{\kappa,\infty}} \). Furthermore, since the potentials \( \kappa Y_n^\Lambda \), \( n \in \mathbb{N}, \kappa > 0 \), are bounded and continuous, the Feynman–Kac type formula (3.5) applies to \( Q_{R^\nu}^{\kappa,n} \). We thus have

\[
(e^{-tQ_{R^\nu}^{\kappa,n}})(x) = \mathbb{E}\left[ e^{-\kappa \int_0^t Y_n^\Lambda(X_s^{x})ds} M_t(x)\Psi(X_t^{x}) \right], \quad \text{a.e. } x \in R^\nu, n \in \mathbb{N},
\]

as well as the domination

\[
\| e^{-\kappa \int_0^t Y_n^\Lambda(X_s^{x})ds} M_t(x)\Psi(X_t^{x}) \| \leq \| M_t(x)\| \|\Psi(X_t^{x})\| \leq L^1(\mathbb{P}), \quad n \in \mathbb{N}.
\]

Therefore, it remains to prove that, for every \( x \in R^\nu \),

\[
e^{-\kappa \int_0^t Y_n^\Lambda(X_s^{x})ds} \xrightarrow{n \to \infty} 1_{\{\tau_{\Lambda}(x) > t\}} e^{-\kappa \int_0^{\tau_{\Lambda}(x)} Y_n^\Lambda(X_s^{x})ds}, \quad \mathbb{P}\text{-a.s.}
\]

This follows, however, immediately from Remark 3.2 and the postulated \( \mathbb{P}\)-a.s. local 1/3-Hölder continuity of \( X^{x} \).

**Step 3.** We now claim that

\[
\left\| e^{-tQ_{R^\nu}^{\kappa,\infty}} \Psi - e^{-tQ_{R^\nu}^{\kappa,0}} \Psi \right\| \xrightarrow{\kappa \downarrow 0} 0, \quad \Psi \in \mathcal{H}_\Lambda. \tag{3.9}
\]
In fact, our assumption (a) ensures that $\mathcal{D}(q^{\kappa,\infty}_R) \subset \mathcal{D}(q_\Lambda)$, and using (b) we further observe that

$$q^{\kappa',\infty}_R[\Psi] \geq q^{\kappa,\infty}_R[\Psi] \geq q_R[\Psi] = q_\Lambda[\Psi], \quad \kappa' \geq \kappa > 0,$$

for all $\Psi \in \bigcup_{\kappa > 0} \mathcal{D}(q^{\kappa,\infty}_R) = \mathcal{D}(q^{1,\infty}_R)$. Thanks to the density requirement in (a), the convergence (3.9) now follows from a monotone convergence theorem for quadratic forms [19, Theorem VIII.3.11].

**Step 4.** Finally, let $\Psi \in \mathcal{H}_\Lambda$. We shall verify (3.6). By virtue of Step 3 we find $\kappa_n > 0$, $n \in \mathbb{N}$, with $\kappa_n \downarrow 0$, $n \to \infty$, such that, a.e. on $\mathbb{R}^\nu$, the sequence $(e^{-tQ^{\kappa_n,\infty}_R}\Psi)_{n \in \mathbb{N}}$ converges to the left hand side of (3.6). Thanks to Step 2 we further know that

$$e^{-tQ^{\kappa_n,\infty}_R\Psi}(x) = \mathbb{E}\left[1_{\{\tau_\Lambda(x) > t\}}e^{-\kappa_n \int_0^t Y^\Lambda_s(x^\nu_s)ds}M_t(x)\Psi(X^\nu_t)\right],$$

(3.10)

for a.e. $x \in \mathbb{R}^\nu$ and all $n \in \mathbb{N}$. Since we also have the domination

$$e^{-\int_0^t \kappa_n Y^\Lambda_s(x^\nu_s)ds}\|M_t(x)\Psi(B^\nu_t)\| \leq \|M_t(x)\|\|\Psi(X^\nu_t)\| \in L^1(\mathbb{P}), \quad n \in \mathbb{N},$$

the dominated convergence theorem implies that, for all $x \in \mathbb{R}^\nu$, the right hand side of (3.10) converges, as $n \to \infty$, to the right hand side of (3.6). \qed

### 3.2. Feynman–Kac Formulas for Semigroup Kernels of Dirichlet Realizations

Again we fix $t > 0$ and we assume:

**Hypothesis 3.5.** There exists a probability space $(\Omega, \mathcal{F}, \mathbb{P})$ and, for all $x, y \in \mathbb{R}^\nu$,

- an operator-valued map $M_t(x, y): \Omega \to \mathcal{B}(\mathcal{H})$;
- a continuous $\mathbb{R}^\nu$-valued stochastic process $(X^y_s)_{s \in [0, t]}$ that $\mathbb{P}$-a.s. starts at $y$ and whose paths are $\mathbb{P}$-a.s. locally Hölder continuous of order 1/3;

such that the following holds:

- For every $x \in \mathbb{R}^\nu$, the following map is measurable,

$$[0, t] \times \mathbb{R}^\nu \times \Omega \ni (s, y, \omega) \longmapsto X^y_s(\omega) \in \mathbb{R}^\nu.$$

- For every $x \in \mathbb{R}^\nu$, the following map is strongly measurable,

$$\mathbb{R}^\nu \times \Omega \ni (y, \omega) \longmapsto M_t(x, y)(\omega) \in \mathcal{B}(\mathcal{H}).$$

- For all $x \in \mathbb{R}^\nu$ and $\Psi \in \mathcal{H}$,

$$\int_{\mathbb{R}^\nu} \mathbb{E}[\|M_t(x, y)\|]\|\Psi(y)\|dy < \infty$$

(3.11)

- For all bounded and continuous functions $v: \mathbb{R}^\nu \to \mathbb{R}$, the relation

$$e^{-t(Q^v + v)}(x) = \int_{\mathbb{R}^\nu} \mathbb{E}[e^{-\int_0^t v(X^y_s)ds}M_t(x, y)\Psi(y)]dy,$$

(3.12)

a.e. $x \in \mathbb{R}^\nu$, holds for all $\Psi \in \mathcal{H}$. 


It might make sense to give the following remark, where \( \tau_\Lambda(y, x) : \Omega \to [0, \infty] \) denotes the first exit time of \( X^{y,x} \) from \( \Lambda \), i.e.,

\[
\tau_\Lambda(y, x) := \inf \{ s \in [0, t] \mid X^{y,x}_s \in \Lambda^c \}.
\]

**Remark 3.6.** Let \( x \in \mathbb{R}^\nu \). Then

\[
\{(y, \omega) \in \mathbb{R}^\nu \times \Omega \mid \tau_\Lambda(y, x)(\omega) = \infty\} \in \mathcal{B}(\mathbb{R}^\nu) \otimes \mathcal{F}.
\]

(3.13)

In fact, set \( Y_s(y, \omega) := X^{y,x}_s(\omega) \), for all \( s \in [0, t] \) and \( (y, \omega) \in \mathbb{R}^\nu \times \Omega \). Then \( (Y_s)_{s \in [0, t]} \) is a continuous stochastic process on \( (\mathbb{R}^\nu \times \Omega, \mathcal{B}(\mathbb{R}^\nu) \otimes \mathcal{F}, \beta \otimes \mathbb{P}) \), where \( \beta \) is an arbitrary Borel probability measure on \( \mathbb{R}^\nu \). Then its first exit time from \( \Lambda \), i.e., \( \tilde{\tau}_\Lambda := \inf\{s \in [0, t] \mid Y_s \in \Lambda^c \} \) is a stopping time with respect to the filtration generated by \( Y \). In particular, \( \mathcal{B}(\mathbb{R}^\nu) \otimes \mathcal{F} \ni \{\tilde{\tau}_\Lambda > t\} = \{\tilde{\tau}_\Lambda = \infty\} \) and by inspecting definitions we see that \( \{\tilde{\tau}_\Lambda = \infty\} \) is equal to the set in (3.13).

**Lemma 3.7.** In the situation described above, let \( \Psi \in \mathcal{H}_\Lambda \). Then

\[
(e^{-tQ_\Lambda}\Psi)(x) = \int_\Lambda \mathbb{E}\left[1_{\{\tau_\Lambda(y, x) = \infty\}} M_t(x, y)\Psi(y)\right] dy, \quad \text{a.e. } x \in \mathbb{R}^\nu
\]

(3.14)

**Proof.** The proof parallels the one of Lemma 3.4 and we shall again use some notation used there. Steps 1 and 3, dealing with the left hand sides of the Feynman–Kac formulas, are identical. Therefore, we only comment on the remaining two steps.

**Step 2.** We pick \( \kappa > 0 \) and \( \Psi \in \mathcal{H} \) and propose to show that, for a.e. \( x \in \mathbb{R}^\nu \),

\[
(e^{-tQ^{\kappa,\infty}}_n\Psi)(x) = \int_{\mathbb{R}^\nu} \mathbb{E}\left[1_{\{\tau_\Lambda(y, x) = \infty\}} e^{-\kappa \int_0^t Y^n_s(X^{y,x}_s)ds} M_t(x, y)\Psi(y)\right] dy.
\]

(3.15)

By assumption the following special cases of (3.12) hold, for a.e. \( x \in \mathbb{R}^\nu \),

\[
(e^{-tQ^{\kappa,\infty}}_n\Psi)(x) = \int_{\mathbb{R}^\nu} \mathbb{E}\left[e^{-\kappa \int_0^t Y^n_s(X^{y,x}_s)ds} M_t(x, y)\Psi(y)\right] dy, \quad n \in \mathbb{N}.
\]

(3.16)

Fix \( x \in \mathbb{R}^\nu \). Then \( \mathbb{E}[\|M_t(x, y)\|] \) is finite for a.e. \( y \in \mathbb{R}^\nu \) and, for every \( y \) for which this is the case, Remark 3.2 and the dominated convergence theorem imply that the expectation under the \( dy \)-integration in (3.16) converges to the expectation under the integral in (3.15), as \( n \to \infty \). Hence, (3.15) follows from Step 1 in the proof of Lemma 3.4, the bound (3.11), and another application of the dominated convergence theorem.

It is now obvious how to formulate the analogue of Step 4 in the proof of Lemma 3.4.

\[\square\]

**4. A Leibniz Rule for Vector-Valued Weak Derivatives**

Our goal in this section is to extend a version of the Leibniz rule for Sobolev functions we learned from [17, Lemma 2.3(i)] to the vector-valued case. This is done in Theorem 4.2 below. While most of the time Leibniz rules for
Sobolev functions are derived for a product of functions in $W^{1,p}$ and $W^{1,p'}$, respectively, with $p'$ denoting the conjugate exponent of $p$, the point about Theorem 4.2 is that it applies to two $W^{1,1}$ functions and merely the three products showing up in the Leibniz rule are assumed to be locally integrable. Similarly as in [17] we benefit from this generality in (5.8), (6.1), and (6.2) below.

The proof of Theorem 4.2 is slightly different from the one in [17], also in the case where all involved Hilbert spaces are one-dimensional.

First, however, we recall a standard mollifying procedure and prove a lemma: in the following paragraphs and the next lemma $\mathcal{K}$ is a separable Hilbert space. Let $p \in [1, \infty]$, $j \in \{1, \ldots, \nu\}$, and $\Psi \in L^p_{\text{loc}}(\Lambda, \mathcal{K})$ have a weak partial derivative with respect to $x_j$ such that $\partial_{x_j} \Psi \in L^p_{\text{loc}}(\Lambda, \mathcal{K})$. Pick a cutoff function $\rho$ such that

$$\rho \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R}), \quad \rho \geq 0, \quad \rho(x) = 0, \text{ for } |x| \geq 1, \quad \|\rho\|_1 = 1. \quad (4.1)$$

Furthermore, set

$$\Lambda_n := \left\{ y \in \Lambda \left| \text{dist}(y, \partial \Lambda) > \frac{1}{n} \right. \right\}, \quad \rho_n(x) := n^\nu \rho(nx), \quad x \in \mathbb{R}^\nu, \ n \in \mathbb{N}. \quad (4.2)$$

Finally, define the mollified functions

$$\Psi_n(x) := \int_\Lambda \rho_n(x - y) \Psi(y) \text{d}y, \quad x \in \Lambda_n, \ n \in \mathbb{N}. \quad (4.3)$$

Then $\Psi_n \in C^\infty(\Lambda_n, \mathcal{K})$, if $\Lambda_n \neq \emptyset$, and, for every compact subset $K \subset \Lambda$,

$$\|\Psi_n - \Psi\|_{L^p(K, \mathcal{K})} + \|\partial_{x_j} \Psi_n - \partial_{x_j} \Psi\|_{L^p(K, \mathcal{K})} \xrightarrow{n \to \infty} 0, \quad \text{if } p < \infty. \quad (4.4)$$

If $p = \infty$, then $\Psi_n \to \Psi$ and $\partial_{x_j} \Psi_n \to \partial_{x_j} \Psi$ a.e. on $\Lambda$. As remarked in [27, Remark 2.4] these assertions can be proved in virtually the same way as in the scalar case.

The next lemma will be used to compute weak derivatives of certain cutoff versions of vector-valued functions. In its statement and henceforth we abbreviate

$$Z_\delta(\Psi) := (\delta^2 + \|\Psi\|^2_{\mathcal{K}})^{1/2}, \quad \mathcal{G}_\delta, \Psi := Z_\delta(\Psi)^{-1} \Psi, \quad (4.5)$$

for all $\Psi \in L^1_{\text{loc}}(\Omega, \mathcal{K})$ and $\delta > 0$. We also use the notation $\mathcal{G}_\Psi$ introduced in (2.5).

**Lemma 4.1.** Let $j \in \{1, \ldots, \nu\}$, $p \in [1, \infty]$, and $\delta > 0$. Assume that $\Psi \in L^p_{\text{loc}}(\Lambda, \mathcal{K})$ has a weak partial derivative with respect to $x_j$ satisfying $\partial_{x_j} \Psi \in L^p_{\text{loc}}(\Lambda, \mathcal{K})$. Then $\|\Psi\|_{\mathcal{K}} \in L^p_{\text{loc}}(\Lambda)$ and $Z_\delta(\Psi) \in L^p_{\text{loc}}(\Lambda)$ have weak partial derivatives

$$\partial_{x_j} \|\Psi\|_{\mathcal{K}} = \text{Re} \langle \mathcal{G}_\Psi \partial_{x_j} \Psi, \mathcal{K} \rangle \in L^p_{\text{loc}}(\Lambda), \quad (4.6)$$

$$\partial_{x_j} Z_\delta(\Psi) = \text{Re} \langle \mathcal{G}_\delta, \Psi \partial_{x_j} \Psi, \mathcal{K} \rangle \in L^p_{\text{loc}}(\Lambda). \quad (4.7)$$

Furthermore, let $m \in \mathbb{N}$, $\varrho \in C^\infty(\mathbb{R}, \mathbb{R})$ such that $0 \leq \varrho \leq 1$, $\varrho = 1$ on $(-\infty, 1]$, and $\varrho = 0$ on $[2, \infty)$, and set $\varrho_m(t) := \varrho(m^{-1} \ln(t))$, $t > 0$, so that $|\varrho_m'(t)| \leq \|\varrho\|_{\infty}/mt$. Put $\beta_m := \varrho_m(Z_1(\Psi))$. Then $\beta_m \Psi \in L^\infty(\Lambda, \mathcal{K})$ has a
weak partial derivative with respect to $x_j$ satisfying $\partial_{x_j}(\beta_m \Psi) \in L^p_{\text{loc}}(\Lambda, \mathcal{H})$ and

$$\partial_{x_j}(\beta_m \Psi) = \delta_m(Z_1(\Psi)) \text{Re} \langle \mathcal{G}_{1, \Psi} | \partial_{x_j} \Psi \rangle, \mathcal{H} + \beta_m \partial_{x_j} \Psi.$$  \hspace{1cm} (4.8)

Proof. The relations (4.6) and (4.7) are derived in [27, Lemma 2.5], whence we only need to prove (4.8). With $\Psi_n$ as in (4.3) we define $\beta_{m,n} := \delta_m(Z_1(\Psi_n)) \in C^\infty(\Lambda_n)$, so that $\beta_{m,n} \Psi_n \in L^\infty(\Lambda_n, \mathcal{H}) \cap C^\infty(\Lambda_n, \mathcal{H})$, for all $n \in \mathbb{N}$. Then

$$\partial_{x_j} \beta_{m,n} = \delta_m(Z_1(\Psi_n)) \text{Re} \langle \mathcal{G}_{1, \Psi_n} | \partial_{x_j} \Psi_n \rangle, \mathcal{H} \text{ on } \Lambda_n, \ n \in \mathbb{N}.$$ \hspace{1cm} (4.9)

Let $\eta \in \mathcal{D}(\Lambda, \mathcal{H})$ and pick some compact $K \subset \Lambda$ with supp$(\eta) \subset \hat{K}$ as well as some $n_0 \in \mathbb{N}$ such that $K \subset \Lambda_{n_0}$. For all $n \geq n_0$, we then have

$$\int_K \langle \partial_{x_j} \eta, \beta_{m,n} \Psi_n \rangle, \mathcal{H} \text{ d}x = -\int_K \langle \eta, (\partial_{x_j} \beta_{m,n}) \Psi_n + \beta_{m,n} \partial_{x_j} \Psi_n \rangle, \mathcal{H} \text{ d}x.$$ \hspace{1cm} (4.10)

By virtue of the Riesz-Fischer theorem for $L^1(K, \mathcal{H})$ we find integers $n_0 \leq n_1 < n_2 < \ldots$ and dominating functions $R, R' \in L^1(K)$ such that $\Psi_{n_\ell} \to \Psi$ and $\partial_{x_j} \Psi_{n_\ell} \to \partial_{x_j} \Psi$, a.e. on $K$ as $\ell \to \infty$, and such that $\|\Psi_{n_\ell}\|_\mathcal{H} \leq R, \|\partial_{x_j} \Psi_{n_\ell}\|_\mathcal{H} \leq R'$, a.e. on $K$, for every $\ell \in \mathbb{N}$. On account of the bound $|\delta_m(t)| \leq \|\varrho\|_\infty/mt$, $t > 0$, and (4.9), $\|\partial_{x_j} \beta_{m,n_\ell} \Psi_{n_\ell}\|_\mathcal{H} \leq (\|\varrho\|_\infty/m)R'$, $\ell \in \mathbb{N}$. By dominated convergence, both sides of (4.10) thus converge, along the same subsequence, to the respective side of

$$\int_\Lambda \langle \partial_{x_j} \eta, \beta_m \Psi \rangle, \mathcal{H} \text{ d}x = -\int_\Lambda \langle \eta, (\delta_m(Z_1(\Psi)) \text{Re} \langle \mathcal{G}_{1, \Psi} | \partial_{x_j} \Psi \rangle, \mathcal{H} + \beta_m \partial_{x_j} \Psi \rangle, \mathcal{H} \text{ d}x.$$  

These remarks prove (4.8). $\square$

We are now in a position to prove the promised Leibniz rule:

**Theorem 4.2.** Let $\mathcal{H}_1, \mathcal{H}_2, \mathcal{H}_3$ be real or complex separable Hilbert spaces and

$$b: \mathcal{H}_1 \times \mathcal{H}_2 \longrightarrow \mathcal{H}_3$$

be real bilinear and continuous. Let $j \in \{1, \ldots, \nu\}$ and $\Psi_i \in L^1_{\text{loc}}(\Lambda, \mathcal{H}_i), \ i \in \{1, 2\}$, have weak partial derivatives $\partial_{x_j} \Psi_i \in L^1_{\text{loc}}(\Lambda, \mathcal{H}_i)$ such that

$$\|\Psi_1\|_{\mathcal{H}_1} \|\Psi_2\|_{\mathcal{H}_2} + \|\partial_{x_j} \Psi_1\|_{\mathcal{H}_1} \|\Psi_2\|_{\mathcal{H}_2} + \|\Psi_1\|_{\mathcal{H}_1} \|\partial_{x_j} \Psi_2\|_{\mathcal{H}_2} \in L^1_{\text{loc}}(\Lambda).$$

Then $b(\Psi_1, \Psi_2) \in L^1_{\text{loc}}(\Lambda, \mathcal{H}_3)$ has a weak partial derivative with respect to $x_j$ and

$$\partial_{x_j} b(\Psi_1, \Psi_2) = b(\partial_{x_j} \Psi_1, \Psi_2) + b(\Psi_1, \partial_{x_j} \Psi_2).$$ \hspace{1cm} (4.11)

Proof. Step 1. To start with we suppose in addition that $\Psi_i \in L^\infty(\Lambda, \mathcal{H}_i), \ i \in \{1, 2\}$. Putting $\Psi_i$ in place of $\Psi$ in (4.3) we construct mollified functions $\Psi_{i,n}, \ n \in \mathbb{N}, \ i \in \{1, 2\}$, such that $\Psi_{i,n} \to \Psi_i$ and $\partial_{x_j} \Psi_{i,n} \to \partial_{x_j} \Psi_i$ in $L^1(K, \mathcal{H}_i)$ for every compact $K \subset \Lambda$. Since $\rho_n$ in (4.3) is a probability density, we further have the dominations $\|\Psi_{i,n}\|_{\mathcal{H}_i} \leq \|\Psi_i\|_{\mathcal{H}_i} := \|\Psi_i\|_{L^\infty(\Lambda, \mathcal{H}_i)}$.

Now fix some compact $K \subset \Lambda$ and $n_0 \in \mathbb{N}$ with $K \subset \Lambda_{n_0}$. Employing the Riesz-Fischer theorem for $L^1(K, \mathcal{H}_i)$ we can find integers $n_0 \leq n_1 < n_2 < \ldots$
such that \( \Psi_{i,n} \to \Psi \) and \( \partial_{x_j} \Psi_{i,n} \to \partial_{x_j} \Psi \); a.e. on \( K \) as \( \ell \to \infty \), for \( i \in \{1, 2\} \). The Riesz-Fischer theorem further implies the existence of \( R_{i} \in L^{1}(K) \) such that \( \| \partial_{x_j} \Psi_{i,n} \|_{x_j} \leq R_{i} \), a.e. on \( K \), for all \( \ell \in \mathbb{N} \) and \( i \in \{1, 2\} \). Now the continuity of \( b \) implies

\[
\partial_{x_j} b(\Psi_{1,n}, \Psi_{2,n}) = b(\partial_{x_j} \Psi_{1,n}, \Psi_{2,n}) + b(\Psi_{1,n}, \partial_{x_j} \Psi_{2,n}) \quad \text{on } \Lambda_{n}, \ell \in \mathbb{N},
\]

where the right hand side converges a.e. on \( K \) to the right hand side of (4.11) and is dominated by \( \| b\| (R_{1,1} \| \Psi \|_{\infty} + \| \Psi \|_{\infty} R_{2}) \in L^{1}(K) \). Furthermore, the convergence \( b(\Psi_{1,n}, \Psi_{2,n}) \to b(\Psi_{1}, \Psi_{2}) \), \( \ell \to \infty \), and the bound \( \| b(\Psi_{1,n}, \Psi_{2,n}) \|_{x_{3}} \leq \| b\| \| \Psi \|_{\infty} \| \Psi \|_{\infty} \) hold a.e. on \( K \). Since \( K \subset \Lambda \) was an arbitrary compact subset, this proves (4.11) under the present extra assumptions.

**Step 2.** Next, we treat the general case with \( \Psi \) as in the statement. According to Step 1 and the last statement of Lemma 4.1 we may already apply (4.11) to \( \Phi_{i,n} := \beta_{i,n} \Psi \in L^{\infty}(\Lambda, \mathcal{H}) \), where \( \beta_{i,n} := \varrho_{n}(Z_{1}(\Psi_{i})) \), \( n \in \mathbb{N}, i \in \{1, 2\} \), and \( \varrho_{n} \) is defined as in the statement of Lemma 4.1. These remarks entail

\[
\partial_{x_j} b(\Phi_{1,n}, \Phi_{2,n}) = \beta_{1,n} \beta_{2,n} \left( b(\partial_{x_j} \Phi_{1,n}, \Phi_{2,n}) + b(\Phi_{1,n}, \partial_{x_j} \Phi_{2,n}) \right)
+ \beta_{2,n} \varrho'_{n}(Z_{1}(\Psi_{1})) \text{Re} \langle \mathcal{G}_{1} \Phi_{1} | \partial_{x_j} \Phi_{1} \rangle_{x} b(\Phi_{1}, \Psi_{2})
+ \beta_{1,n} \varrho'_{n}(Z_{1}(\Psi_{2})) \text{Re} \langle \mathcal{G}_{1} \Phi_{2} | \partial_{x_j} \Phi_{2} \rangle_{x} b(\Phi_{1}, \Psi_{2})
\quad \text{(4.12)}
\]

Since \( \beta_{i,n} \to 1 \), \( n \to \infty \), on \( \Lambda \) and \( |\varrho'_{n}(Z_{1}(\Psi_{i}))| ||\Psi_{i}||_{x_{j}} \leq ||\varrho'||_{\infty} / n \), the right hand side of (4.12) converges to the right hand side of (4.11) in \( L^{1}_{\text{loc}}(\Lambda, \mathcal{H}) \), as \( n \to \infty \), by the dominated convergence theorem, the boundedness of \( b \), and the assumptions \( ||\partial_{x_j} \Phi_{1}||_{x_{1}} \| \Phi_{2} \|_{x_{2}} \in L^{1}_{\text{loc}}(\Lambda) \) and \( ||\Phi_{1}||_{x_{1}}, ||\partial_{x_j} \Phi_{2}||_{x_{2}} \in L^{1}_{\text{loc}}(\Lambda) \). Since also \( b(\Phi_{1,n}, \Phi_{2,n}) \to b(\Phi_{1}, \Phi_{2}), n \to \infty \), in \( L^{1}_{\text{loc}}(\Lambda, \mathcal{H}) \) by dominated convergence, boundedness of \( b \), and our assumption \( ||\Phi_{1}||_{x_{1}}, ||\Phi_{2}||_{x_{2}} \in L^{1}_{\text{loc}}(\Lambda) \), this concludes the proof of (4.11) in full generality.

5. **Approximation with Respect to Pauli–Fierz Forms**

In this section we collect several fairly technical but crucial results on convergence and approximation with respect to the norm associated with the maximal Pauli–Fierz form \( \mathfrak{h}_{\Lambda,N} \) defined in (2.7). *In the whole section we will always assume* \( (1.5), (1.6), \) and \( (1.7) \). As prerequisites we shall need some more results of [27] which are collected in the first two of the following remarks:

**Remark 5.1.** Let \( j \in \{1, \ldots, \nu\} \) and \( \Psi \in \mathcal{D}(v^{*}_{\Lambda,j}) \). Consider the vectors

\[
\Psi_{\varepsilon} := N_{\varepsilon}^{-1/2} \Psi \in \mathcal{Q}(d\Gamma(1)), \quad \varepsilon > 0,
\quad \text{(5.1)}
\]

where

\[
N_{\varepsilon} := 1 + \varepsilon d\Gamma(1).
\quad \text{(5.2)}
\]

Introduce densely defined operators in \( \mathcal{F} \) by

\[
C_{\varepsilon}(G_{j,x}) \psi := N_{\varepsilon}^{-1/2} \varphi(G_{j,x}) \psi - \varphi(G_{j,x}) N_{\varepsilon}^{-1/2} \psi, \quad \psi \in \mathcal{Q}(d\Gamma(1)),
\]

where the boundedness of \( \varphi \) implies

\[
\varphi(G_{j,x}) N_{\varepsilon}^{-1/2} \psi \to \varphi(G_{j,x}) \psi, \quad \varepsilon \to 0,
\quad \text{(5.3)}
\]

and the dominated convergence theorem implies

\[
\varphi(G_{j,x}) N_{\varepsilon}^{-1/2} \psi \to \varphi(G_{j,x}) \psi, \quad \varepsilon \to 0,
\quad \text{(5.4)}
\]

The four remarks above provide a new proof of the convergence (1.5) and (1.6) for the general case.

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for all $x \in \Lambda$. Then
\begin{equation}
\|C_\varepsilon(G_{j,x})\| \leq (4/\pi)\varepsilon^{1/2}\|G_{j,x}\|_t, \quad x \in \Lambda.
\end{equation}
(5.3)
Moreover, $\Lambda \ni x \mapsto C_\varepsilon(G_{j,x})^{\ast} \in \mathcal{B}(\mathcal{F})$ is strongly measurable and the densely defined operators $C_\varepsilon(G_{j,x})^{\ast}N_\varepsilon^{1/2}$ are bounded with
\begin{equation}
\|C_\varepsilon(G_{j,x})^{\ast}N_\varepsilon^{1/2}\| \leq 2\varepsilon^{1/2}\|G_{j,x}\|_t, \quad x \in \Lambda.
\end{equation}
(5.4)
(To obtain (5.3) we choose the constant dispersion relation 1 in Lemma 2.9(1) of [27]. The bound (5.4) follows upon choosing $\varepsilon$ as dispersion relation in [27, Lemma 2.9(2)]. The asserted strong measurability is observed prior to Lemma 3.2 in [27].)

Now [27, Lemma 3.2] says that $\Psi_\varepsilon$ has a weak partial derivative with respect to $x_j$ given by
\begin{equation}
\partial_{x_j}\Psi_\varepsilon = iN_\varepsilon^{-1/2}v_{\Lambda,j}^{\ast}\Psi + iA_j\Psi_\varepsilon + i\varphi(G_j)\Psi_\varepsilon + iC_\varepsilon(G_j)^{\ast}\Psi \quad \text{in } L^1_{\text{loc}}(\Lambda, \mathcal{F}).
\end{equation}
(5.5)
(To see this we apply the quoted lemma with dispersion relation 1; notice that in fact $\varphi(G_j)\Psi_\varepsilon \in L^1_{\text{loc}}(\Lambda, \mathcal{F})$ by (2.2) with $\varkappa = 1$.)

**Remark 5.2.** Let $j \in \{1, \ldots, \nu\}$, $\Psi \in \mathcal{D}(v_{\Lambda,j}^{\ast})$, and let $\Psi_\varepsilon$ be given by (5.1) and (5.2). Under the additional condition that
\begin{equation}
\Lambda \ni x \mapsto \|G_{j,x}\|_t\|\Psi(x)\|_F \quad \text{is in } L^2(\Lambda),
\end{equation}
(5.6)
we observed in [27, Lemma 3.3] (here applied with dispersion relation 1) that $\Psi_\varepsilon \in \mathcal{D}(v_{\Lambda,j}^{\ast})$, for all $\varepsilon > 0$, and $\Psi_\varepsilon \rightarrow \Psi$, $\varepsilon \downarrow 0$, with respect to the graph norm of $v_{\Lambda,j}^{\ast}$.

In what follows, the symbol $L_0^\infty$ stands for essentially bounded functions of compact support.

**Remark 5.3.** Let $\Psi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N}) \cap L_0^\infty(\Lambda, \mathcal{F})$. Then the dominated convergence theorem implies that $V^{1/2}N_\varepsilon^{-1/2}\Psi \rightarrow V^{1/2}\Psi$ in $L^2(\Lambda, \mathcal{F})$ and $N_\varepsilon^{-1/2}\Psi \rightarrow \Psi$ in $L^2(\Lambda, Q(d\Gamma(\omega)))$, as $\varepsilon \downarrow 0$. Since $\Psi$ satisfies (5.6) for all $j \in \{1, \ldots, \nu\}$, we may thus infer from Remark 5.2 that $\Psi_\varepsilon \rightarrow \Psi$ with respect to the form norm of $\mathfrak{h}_{\Lambda,N}$. Of course, $\Psi_\varepsilon \in L_0^\infty(\Lambda, Q(d\Gamma(1)))$, for every $\varepsilon > 0$.

In particular, if $\{\Phi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N})|\Phi \in L_0^\infty(\Lambda, \mathcal{F})\}$ is a core for $\mathfrak{h}_{\Lambda,N}$, then the set $\{\Phi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N})|\Psi \in L_0^\infty(\Lambda, Q(d\Gamma(1)))\}$ is a core for $\mathfrak{h}_{\Lambda,N}$ as well.

In our first approximation lemma we treat cutoffs in the range of $\Psi$. Similar cutoffs have been used in [22, Lemma 2] and [17, Step 1 on p. 125] to study magnetic Schrödinger operators.

**Lemma 5.4.** Let $j \in \{1, \ldots, \nu\}$ and $\Psi \in \mathcal{D}(v_{\Lambda,j}^{\ast})$. Define the cutoff functions $\beta_n = \varphi_n(Z_1(\Psi))$, $n \in \mathbb{N}$, as in the statement of Lemma 4.1 so that $\beta_n\Psi \in L^\infty(\Lambda, \mathcal{F})$. Then $\beta_n\Psi \in \mathcal{D}(v_{\Lambda,j}^{\ast})$, for all $n \in \mathbb{N}$, and $\beta_n\Psi \rightarrow \Psi$, $n \rightarrow \infty$, with respect to the graph norm of $v_{\Lambda,j}^{\ast}$.
Proof. It is clear that $\beta_n\Psi \to \Psi, n \to \infty$, in $L^2(\Lambda, \mathcal{F})$. Let $\varepsilon > 0$ and consider the vector $\Psi_\varepsilon$ defined in (5.1). Combining (4.7) and (5.5) we obtain

$$
\partial_{x_j}Z_1(\Psi_\varepsilon) = \Re(\mathcal{G}_1, \Psi_\varepsilon | \partial_{x_j} \Psi_\varepsilon)_\mathcal{F}
= \Re(\mathcal{G}_1, \Psi_\varepsilon | iN_{\varepsilon}^{-1/2}v_{\Lambda,j}^* \Psi + iA_j \Psi_\varepsilon + i\varphi(G_j)\Psi_\varepsilon + iC_\varepsilon(G_j)^*\Psi)_\mathcal{F}
= \Re(\mathcal{G}_1, \Psi_\varepsilon | iN_{\varepsilon}^{-1/2}v_{\Lambda,j}^* \Psi + iC_\varepsilon(G_j)^*\Psi)_\mathcal{F} \in L^1_{\text{loc}}(\Lambda). (5.7)
$$

In the third step we used that $\Re(\Psi_\varepsilon | iA_j \Psi_\varepsilon)_\mathcal{F}$ and $\Re(\Psi_\varepsilon | i\varphi(G_j)\Psi_\varepsilon)_\mathcal{F}$ vanish a.e. on $\Lambda$ since $A_j$ is real and $\varphi(G_j, x)$ symmetric on $\mathcal{Q}(d\Gamma(1))$. Let also $n \in \mathbb{N}$. Applying the chain rule for distributional derivatives (see, e.g., [23, Theorem 6.16]) to compute the weak partial derivative of

$$
\beta_{n,\varepsilon} := \varrho_n(Z_1(\Psi_\varepsilon)),
$$

and combining the result with the Leibniz rule of Theorem 4.2, we further find

$$
\partial_{x_j}(\beta_{n,\varepsilon}\Psi_\varepsilon)
= \varrho'_n(Z_1(\Psi_\varepsilon))(\partial_{x_j}Z_1(\Psi_\varepsilon))\Psi_\varepsilon + \beta_{n,\varepsilon}\partial_{x_j} \Psi_\varepsilon \quad \text{in } L^1_{\text{loc}}(\Lambda, \mathcal{F}). (5.8)
$$

Here we took into account that, by the construction of $\varrho_n(t) = \varrho(n^{-1}\ln(t))$,

$$
|\varrho'_n(Z_1(\Psi_\varepsilon))||\Psi_\varepsilon|_\mathcal{F} \leq \|\varrho'\|_\infty \|\Psi_\varepsilon\|_\mathcal{F} \leq \|\varrho'\|_\infty \frac{1}{n}. (5.9)
$$

Together with (5.7) this shows that $|\partial_{x_j}\beta_{n,\varepsilon}|\Psi_\varepsilon|_\mathcal{F} \in L^1_{\text{loc}}(\Lambda)$, whence the Leibniz rule of Theorem 4.2 was indeed applicable.

Next, we subtract $i\beta_{n,\varepsilon}A_j \Psi_\varepsilon + i\beta_{n,\varepsilon}\varphi(G_j)\Psi_\varepsilon$ from both sides of (5.8). In view of (5.5) this results in

$$
\partial_{x_j}(\beta_{n,\varepsilon}\Psi_\varepsilon) - iA_j\beta_{n,\varepsilon}\Psi_\varepsilon - i\varphi(G_j)\beta_{n,\varepsilon}\Psi_\varepsilon
= \varrho'_n(Z_1(\Psi_\varepsilon))(\partial_{x_j}Z_1(\Psi_\varepsilon))\Psi_\varepsilon + \beta_{n,\varepsilon}(iN_{\varepsilon}^{-1/2}v_{\Lambda,j}^* \Psi + iC_\varepsilon(G_j)^*\Psi).
$$

(5.10)

In the next step we compute, a.e. on $\Lambda$, the $\mathcal{F}$-scalar product of both sides of (5.10) with $\eta \in \mathcal{D}(\Lambda, \mathcal{Q}(d\Gamma(1)))$, integrate the result with respect to $x \in \Lambda$, and pass to the limit $\varepsilon \downarrow 0$ afterwards. In doing so we observe that, as $\varepsilon \downarrow 0$,

(a) $\beta_{n,\varepsilon} \to \beta_n$ pointwise (recall $\beta_{n,\varepsilon} \leq 1$);

(b) $\beta_{n,\varepsilon}\Psi_\varepsilon \to \beta_n\Psi$ in $L^2(\Lambda, \mathcal{F})$;

(c) $N_{\varepsilon}^{-1/2}v_{\Lambda,j}^* \Psi \to v_{\Lambda,j}^* \Psi$ in $L^2(\Lambda, \mathcal{F})$;

(d) $C_\varepsilon(G_j)^*\Psi \to 0$ in $L^1_{\text{loc}}(\Lambda, \mathcal{F})$ by (5.3);

(e) $\partial_{x_j}Z_1(\Psi_\varepsilon) \to \Re(\mathcal{G}_1, \Psi | v_{\Lambda,j}^* \Psi)$ in $L^1_{\text{loc}}(\Lambda)$ by (5.7), (d), and (e);

(f) $\varrho'_n(Z_1(\Psi_\varepsilon))\Psi_\varepsilon \to \varrho'_n(Z_1(\Psi))\Psi$ pointwise with the $\varepsilon$-uniform bound (5.9).

We thus arrive at

$$
\langle v_{\Lambda,j}\eta | i\beta_n \Psi \rangle = \int_\Lambda \left( \eta(x) \Re(\mathcal{G}_1, \Psi(x) | i(v_{\Lambda,j}^* \Psi)(x)) \right)_\mathcal{F} \Psi(x) + i(\beta_n v_{\Lambda,j}^* \Psi)(x) dx.
$$
Next, we observe that the preceding integral is the scalar product of \( \eta \) with a vector in \( L^2(\Lambda, \mathcal{F}) \) since, in analogy to (5.9),
\[
|g'_n(Z_1(\Psi(x)))| \|\Psi\|_\mathcal{F} \leq \frac{|g'|_\infty}{n},
\]
and since \( \text{Re}(\mathcal{G}_1, \Psi|iv_{\Lambda,j}^* \Psi)_x \in L^2(\Lambda) \) and, of course, \( i\beta_n v_{\Lambda,j}^* \Psi \in L^2(\Lambda, \mathcal{F}) \).

By the definition of the adjoint operator \( v_{\Lambda,j}^* \), this reveals that \( \beta_n \Psi \in \mathcal{D}(v_{\Lambda,j}^*) \) with
\[
v_{\Lambda,j}^*(\beta_n \Psi) = -ig'_n(Z_1(\Psi))\text{Re}(\mathcal{G}_1, \Psi|iv_{\Lambda,j}^* \Psi)_x \Psi + \beta_n v_{\Lambda,j}^* \Psi.
\]

Taking into account (5.11), \( \text{Re}(\mathcal{G}_1, \Psi|iv_{\Lambda,j}^* \Psi)_x \in L^2(\Lambda) \), and \( \beta_n v_{\Lambda,j}^* \Psi \to v_{\Lambda,j}^* \Psi \) in \( L^2(\Lambda, \mathcal{F}) \), we further conclude that \( v_{\Lambda,j}^*(\beta_n \Psi) \to v_{\Lambda,j}^* \Psi \), as \( n \to \infty \).

\( \square \)

**Remark 5.5.** Let \( \Psi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N}) \) and consider again the cutoffs \( \beta_n \) appearing in Lemma 5.4. Then the dominated convergence theorem implies \( V^{1/2} \beta_n \Psi \to V^{1/2} \Psi \) in \( L^2(\Lambda, \mathcal{F}) \) and \( \beta_n \Psi \to \Psi \) in \( L^2(\Lambda, \mathcal{Q}(d\Gamma(\omega))) \). In conjunction with Lemma 5.4 this shows in particular that \( \{ \Phi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N}) | \Phi \in L^\infty(\Lambda, \mathcal{F}) \} \) is a core for \( \mathfrak{h}_{\Lambda,N} \).

We continue with a simple result on spatial cutoffs:

**Lemma 5.6.** Pick cutoff functions \( \vartheta_\ell \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R}), \ell \in \mathbb{N} \), satisfying
\[
0 \leq \vartheta_\ell \leq 1, \quad \vartheta_{\ell+1} = 1 \text{ on supp}(\vartheta_\ell), \quad \ell \in \mathbb{N}; \quad \Lambda \subset \bigcup_{\ell \in \mathbb{N}} \text{supp}(\vartheta_\ell).
\]

Let \( j \in \{1, \ldots, \nu\} \) and define \( \Theta_j : \mathbb{R}^\nu \to [0, \infty) \) by
\[
\Theta_j := \left( \sum_{\ell=1}^\infty |\partial_\ell \vartheta_\ell|^2 \right)^{1/2}.
\]

Finally, let \( \Psi \in \mathcal{D}(v_{\Lambda,j}^*) \) satisfy \( \Theta_j \Psi \in L^2(\Lambda, \mathcal{F}) \). Then \( \vartheta_n \Psi \to \Psi \), \( n \to \infty \), with respect to the graph norm of \( v_{\Lambda,j}^* \).

**Proof.** Of course, \( \vartheta_n \Psi \to \Psi \) in \( L^2(\Lambda, \mathcal{F}) \). Furthermore, it is straightforward to check that \( \vartheta_n \mathcal{D}(v_{\Lambda,j}^*) \subset \mathcal{D}(v_{\Lambda,j}^*) \) with \( v_{\Lambda,j}^*(\vartheta_n \Phi) = \vartheta_n v_{\Lambda,j}^* \Phi - i(\partial_\ell \vartheta_n) \Phi \), for all \( \Phi \in \mathcal{D}(v_{\Lambda,j}^*) \). The condition \( \Theta_j \Psi \in L^2(\Lambda, \mathcal{F}) \) and the dominated convergence theorem imply
\[
\| (\partial_\ell \vartheta_n) \Psi \| \leq \left( \sum_{\ell=n}^\infty |\partial_\ell \vartheta_\ell|^2 \right)^{1/2} \| \Psi \| \xrightarrow{n \to \infty} 0.
\]

Since also \( \vartheta_n v_{\Lambda,j}^* \Psi \to v_{\Lambda,j}^* \Psi \), these remarks show that \( v_{\Lambda,j}^*(\vartheta_n \Psi) \to v_{\Lambda,j}^* \Psi \), as \( n \to \infty \).

\( \square \)

**Lemma 5.7.** Assume that the cutoff functions in Lemma 5.6 are chosen such that \( \text{supp}(\vartheta_\ell) \subset \Lambda \), for all \( \ell \in \mathbb{N} \). Furthermore, assume that the functions \( \Theta_j \) defined in (5.12) satisfy
\[
\sum_{j=1}^\nu \Theta_j^2 \leq C + V, \quad \text{for some constant } C > 0.
\]

Then \( \{ \Phi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N}) | \Phi \in L^\infty_0(\Lambda, \mathcal{Q}(d\Gamma(1))) \} \) is a core for \( \mathfrak{h}_{\Lambda,N} \).
Proof. Let $\Psi \in D(\mathfrak{h}_{\Lambda,N})$. Then $V^{1/2}\vartheta_{\ell}\Psi \rightarrow V^{1/2}\Psi$ in $L^2(\Lambda,\mathcal{F})$ and $\vartheta_{\ell}\Psi \rightarrow \Psi$ in $L^2(\Lambda,Q(d\Gamma(\omega)))$, as $\ell \rightarrow \infty$, by dominated convergence. Since (5.13) entails $\Theta_{j}\Psi \in L^2(\Lambda,\mathcal{F})$, for all $j \in \{1,\ldots,\nu\}$, Lemma 5.6 implies that $\vartheta_{\ell}\Psi \rightarrow \Psi$ with respect to the graph norm of every $v_{\Lambda,j}^*$. Altogether this shows that $\vartheta_{\ell}\Psi \in D(\mathfrak{h}_{\Lambda,N})$, for all $\ell \in \mathbb{N}$, and $\vartheta_{\ell}\Psi \rightarrow \Psi$ with respect to the form norm on $D(\mathfrak{h}_{\Lambda,N})$. By virtue of Remark 5.5 we conclude that $\{\Phi \in D(\mathfrak{h}_{\Lambda,N})|\Phi \in L^\infty(\Lambda,\mathcal{F})\}$ is a core for $\mathfrak{h}_{\Lambda,N}$. Now the assertion follows directly from Remark 5.3.

In the next lemma we consider the choice $\Lambda = \mathbb{R}^\nu$:

Lemma 5.8. The set $\{\Phi \in D(\mathfrak{h}_{\mathbb{R}^\nu,N})|\Phi \in L^\infty(\mathbb{R}^\nu,Q(d\Gamma(1)))\}$ is a core for $\mathfrak{h}_{\mathbb{R}^\nu,N}$.

Proof. In the case $\Lambda = \mathbb{R}^\nu$, the functions $\vartheta_{\ell}$ appearing Lemma 5.6 can obviously be chosen such that $\Theta_1,\ldots,\Theta_\nu$ are bounded. Then (5.13) is satisfied, whence the assertion follows from Lemma 5.7.

Next, we study approximations by elements of

$\mathcal{C} \otimes \mathcal{E} := \text{span}_\mathbb{C}\{f\phi| f \in \mathcal{C}, \phi \in \mathcal{E}\},$

with suitable subspaces $\mathcal{C} \subset L^2(\Lambda)$ and $\mathcal{E} \subset \mathcal{F}$.

Lemma 5.9. Let $\Psi \in D(\mathfrak{h}_{\Lambda,N})$. Then the following holds:

(1) Assume in addition that $\Psi(x) \in Q(d\Gamma(1))$, for a.e. $x \in \Lambda$, and

$\Lambda \ni x \mapsto \|G_{j,x}\|_\|\Psi(x)\|_{Q(d\Gamma(1))}$ is in $L^2(\Lambda)$ for all $j \in \{1,\ldots,\nu\}$. (5.14)

Then there exist

$\Psi_n \in \{L^2(\Lambda) \otimes Q(d\Gamma(1))\} \cap D(\mathfrak{h}_{\Lambda,N}), \quad n \in \mathbb{N}, \quad (5.15)$

such that $\Psi_n \rightarrow \Psi$, $n \rightarrow \infty$, with respect to the form norm of $\mathfrak{h}_{\Lambda,N}$.

(2) Assume in addition that $\Psi \in L^\infty(\Lambda,Q(d\Gamma(1)))$. Then there exist

$\Psi_n \in \{L^\infty_0(\Lambda) \otimes Q(d\Gamma(1))\} \cap D(\mathfrak{h}_{\Lambda,N}), \quad n \in \mathbb{N}, \quad (5.16)$

such that $\Psi_n \rightarrow \Psi$, $n \rightarrow \infty$, with respect to the form norm of $\mathfrak{h}_{\Lambda,N}$.

Proof. We will always assume that $\Psi$ satisfies the additional condition imposed on it in Part (1) and we shall fix $j \in \{1,\ldots,\nu\}$ in the first four steps of this proof.

Step 1. We define a symmetric operator $w_{\Lambda,j}$ in $L^2(\Lambda,\mathcal{F})$ by setting $D(w_{\Lambda,j}) := \mathcal{D}(\Lambda,\mathcal{F})$ and

$w_{\Lambda,j}\Phi := -i\partial_{x,j}\Phi - A_j\Phi, \quad \Phi \in D(w_{\Lambda,j}).$

According to [27, Remark 3.1(1)] we then have $\Psi \in D(w_{\Lambda,j}^*)$ and

$v_{\Lambda,j}^*\Psi = w_{\Lambda,j}^*\Psi - \varphi(G_j)\Psi.$

With the help of (2.2) and (5.14), which together imply $\varphi(G_j)\Psi \in L^2(\Lambda,\mathcal{F})$, this is indeed straightforward to verify.

Let $Q \in \mathcal{B}(\mathcal{F})$ and write $(Q\Phi)(x) := Q\Phi(x)$, a.e. $x \in \Lambda$, for all $\Phi \in L^1_{\text{loc}}(\Lambda,\mathcal{F})$. Then it is clear that $Qw_{\Lambda,j}\Phi = w_{\Lambda,j}Q\Phi$, for every $\Phi \in D(w_{\Lambda,j})$, from which we infer that $Q\Psi \in D(w_{\Lambda,j}^*)$ with $w_{\Lambda,j}Q\Psi = Qw_{\Lambda,j}^*\Psi$. 
Suppose in addition that \( Q \Psi(x) \in Q(d\Gamma(1)) \), for \( x \in \Lambda \), with 
\[
\|Q \Psi\|_{Q(d\Gamma(1))} \leq C \Psi \|Q\|_{Q(d\Gamma(1))}
\] a.e. on \( \Lambda \), for some \( C > 0 \). Then it follows from (2.2) and (5.14) that \( \varphi(G_j)Q \Psi \in L^2(\Lambda, \mathcal{F}) \) and the definition of the adjoint operators \( v^*_{\Lambda,j} \) and \( w^*_{\Lambda,j} \) entails \( Q \Psi \in D(v^*_{\Lambda,j}) \) with
\[
v^*_{\Lambda,j}Q \Psi = w^*_{\Lambda,j}Q \Psi - \varphi(G_j)Q \Psi = Q w^*_{\Lambda,j} \Psi - \varphi(G_j)Q \Psi. \quad (5.17)
\]

**Step 2.** For every \( r \in \mathbb{N} \), we define \( Q_r \in \mathcal{B}(\mathcal{F}) \) by setting
\[
Q_r \psi = (\psi_0, \chi^{\otimes 1}_r \psi_1, \chi^{\otimes 2}_r \psi_2, \ldots, \chi^{\otimes r}_r \psi_r, 0, 0, \ldots), \quad (5.18)
\]
for every \( \psi = (\psi_n)_{n=0}^{\infty} \in \mathcal{F} \), where \( \chi^{\otimes m}_r \) denotes the characteristic function of the set
\[
\{(k_1, \ldots, k_m) \in \mathcal{K}^m \mid \omega(k_1) \leq r, \ldots, \omega(k_m) \leq r, m, r \in \mathbb{N} \}.
\]
Then \( Q_r \to 1, r \to \infty \), strongly in \( \mathcal{F} \) as well as in \( L^2(\Lambda, \mathcal{F}) \). By the remarks in Step 1 we know that \( Q_r \Psi \in D(v^*_{\Lambda,j}) \cap D(w^*_{\Lambda,j}) \), \( \varphi(G_j)Q_r \Psi \in L^2(\Lambda, \mathcal{F}) \), and (5.17) is satisfied with \( Q = Q_r \). Furthermore,
\[
\|\varphi(G_j)Q_r \Psi - \varphi(G_j)\|_{\mathcal{F}} \leq 2\|G_j\|_{\mathcal{F}}(Q - 1)\|Q_r\|_{Q(d\Gamma(1))} \overset{r \to \infty}{\longrightarrow} 0, \quad (5.19)
\]
pointwise a.e., since \( \Psi(x) \in Q(d\Gamma(1)) \) for a.e. \( x \in \Lambda \). Employing (5.14) and the dominated convergence theorem we deduce that \( \varphi(G_j)Q_r \Psi \to \varphi(G_j)\Psi \), \( r \to \infty \), in \( L^2(\Lambda, \mathcal{F}) \). Putting all these remarks together we conclude that
\[
v^*_{\Lambda,j}Q_r \Psi \to v^*_{\Lambda,j} \Psi. \quad \text{The dominated convergence theorem further implies that}
\]
\[
V^{1/2}Q_r \Psi \to V^{1/2} \Psi \quad \text{in } L^2(\Lambda, \mathcal{F}) \quad \text{and } Q_r \Psi \to \Psi \quad \text{in } L^2(\Lambda, Q(d\Gamma(1))).
\]

**Step 3.** We fix \( r \in \mathbb{N} \) in this and the next step. The definition of \( Q_r \) ensures that \( Q_r \Psi \in L^2(\Lambda, Q(d\Gamma(1 \lor \omega))) \). Let \( \{e_\ell : \ell \in \mathbb{N} \} \) be an orthonormal basis of \( Q(d\Gamma(1 \lor \omega)) \) and put
\[
P_n \phi := \sum_{\ell=1}^{n} \langle e_\ell | \phi \rangle Q(d\Gamma(1 \lor \omega)) e_\ell, \quad \phi \in Q(d\Gamma(1 \lor \omega)), n \in \mathbb{N}. \quad (5.20)
\]
Then \( P_n Q_r \Psi \to Q_r \Psi, n \to \infty, \) in \( L^2(\Lambda, Q(d\Gamma(1 \lor \omega))) \) by dominated convergence. As the canonical injections \( L^2(\Lambda, Q(d\Gamma(1 \lor \omega))) \subset L^2(\Lambda, Q(d\Gamma(\omega))) \subset L^2(\Lambda, \mathcal{F}) \) are continuous, we also have \( P_n Q_r \Psi \to Q_r \Psi, n \to \infty, \) in both \( L^2(\Lambda, \mathcal{F}) \) and \( L^2(\Lambda, Q(d\Gamma(\omega))) \). Likewise, \( V^{1/2}P_n Q_r \Psi = P_n V^{1/2}Q_r \Psi \to V^{1/2}Q_r \Psi \) in \( L^2(\Lambda, \mathcal{F}) \).

It remains to show that \( P_n Q_r \Psi \to Q_r \Psi, n \to \infty, \) with respect to the graph norm of \( v^*_{\Lambda,j} \), which is done in the next step.

**Step 4.** Since \( Q_r \) maps \( \mathcal{F} \) into \( Q(d\Gamma(1 \lor \omega)) \), we see that \( P_n Q_r \) defines a finite rank operator on \( \mathcal{F} \). Furthermore, we notice that \( P_n Q_r \Psi(x) \in Q(d\Gamma(1)) \), for a.e. \( x \in \Lambda \), with
\[
\|P_n Q_r \Psi\|_{Q(d\Gamma(1))} \leq \|P_n Q_r \Psi\|_{Q(d\Gamma(1 \lor \omega))} \leq \|Q_r \Psi\|_{Q(d\Gamma(1 \lor \omega))} \leq r^{1/2}\|Q_r \Psi\|_{Q(d\Gamma(1))} \leq r^{1/2}\|\Psi\|_{Q(d\Gamma(1))}, \quad (5.21)
\]
a.e. on \( \Lambda \). In the penultimate step we used that \( \chi^{\otimes m}(k_1, \ldots, k_m) \neq 0 \) entails
\[
1 + 1 \lor \omega(k_1) + \ldots + 1 \lor \omega(k_m) \leq 1 + rm \leq r(1 + m).
\]
Applying the remarks in Step 1 we conclude that
\[ P_n Q_r \Psi \in D(v_{\Lambda,j}^*) \cap D(w_{\Lambda,j}^*), \quad \varphi(G_j) P_n Q_r \Psi \in L^2(\Lambda, \mathcal{F}), \quad n \in \mathbb{N}, \]
and (5.17) holds with \( Q = P_n Q_r \). Since \( Q_r w_{\Lambda,j}^* \Psi \in L^2(\Lambda, \mathcal{Q}(\text{d}\Gamma(1 \lor \omega))) \), we further have \( P_n Q_r w_{\Lambda,j}^* \Psi \rightarrow Q_r w_{\Lambda,j}^* \Psi \), as \( n \rightarrow \infty \), in \( L^2(\Lambda, \mathcal{Q}(\text{d}\Gamma(1 \lor \omega))) \) and, hence, also in \( L^2(\Lambda, \mathcal{F}) \). Similarly to (5.19) we find
\[
\| \varphi(G_j) P_n Q_r \Psi - \varphi(G_j) Q_r \Psi \|_\mathcal{F} \leq 2 \| G_j \|_\mathcal{F} \| (P_n - 1) Q_r \Psi \|_{\mathcal{Q}(\text{d}\Gamma(1 \lor \omega))} \xrightarrow{n \to \infty} 0,
\]
a.e. on \( \Lambda \), because \( Q_r \Psi(\mathbf{x}) \in \mathcal{Q}(\text{d}\Gamma(1 \lor \omega)) \), a.e. \( \mathbf{x} \). On account of (5.21) we further have the uniform bounds
\[
\| \varphi(G_j) P_n Q_r \Psi - \varphi(G_j) Q_r \Psi \|_\mathcal{F} \leq 2 (r^{1/2} + 1) \| G_j \|_\mathcal{F} \| \Psi \|_{\mathcal{Q}(\text{d}\Gamma(1))} = L^2(\Lambda).
\]
Thus, \( \varphi(G_j) P_n Q_r \Psi \rightarrow \varphi(G_j) Q_r \Psi \), \( n \rightarrow \infty \), in \( L^2(\Lambda, \mathcal{F}) \) by dominated convergence. Altogether this shows that \( v_{\Lambda,j}^* P_n Q_r \Psi \rightarrow v_{\Lambda,j}^* Q_r \Psi \), as \( n \rightarrow \infty \).

**Step 5.** We can now conclude as follows: Let \( n \in \mathbb{N} \). According to Step 2 we then find some \( r_n \in \mathbb{N} \) such that \( \| Q_{r_n} \Psi - \Psi \|_{\mathcal{h}_{\Lambda,N}} < 1/2n \). After that Steps 3 and 4 permit to pick some \( m_n \in \mathbb{N} \) such that \( \| P_{m_n} Q_{r_n} \Psi - Q_{r_n} \Psi \|_{\mathcal{h}_{\Lambda,N}} < 1/2n \). This proves Part (1) with
\[
\Psi_n := P_{m_n} Q_{r_n} \Psi = \sum_{\ell=1}^{m_n} \langle e_\ell | Q_{r_n} \Psi \rangle \mathcal{Q}(\text{d}\Gamma(1 \lor \omega)) e_\ell, \quad n \in \mathbb{N}.
\]
Here \( \langle e_\ell | Q_{r_n} \Psi \rangle \mathcal{Q}(\text{d}\Gamma(1 \lor \omega)) \in L_0^\infty(\Lambda) \), whenever \( \Psi \in L_0^\infty(\Lambda, \mathcal{Q}(\text{d}\Gamma(1))) \). Since the latter condition on \( \Psi \) entails (5.14), this also proves Part (2).

Before we consider mollifications we note a simple observation that also is part of [27, Remark 3.1(2)];

**Remark 5.10.** Let \( j \in \{1, \ldots, \nu\}, \Psi \in D(v_{\Lambda,j}^*) \), and assume that
\[
\Lambda \ni \mathbf{x} \longmapsto \| G_{j,\mathbf{x}} \|_\mathcal{F} \| \Psi(\mathbf{x}) \|_{\mathcal{Q}(\text{d}\Gamma(1))} \text{ is in } L^1_{\text{loc}}(\Lambda). \tag{5.22}
\]
In view of (2.2) this entails \( \varphi(G_j) \Psi \in L^1_{\text{loc}}(\Lambda, \mathcal{F}) \) and it is clear that \( A_j \Psi \in L^1_{\text{loc}}(\Lambda, \mathcal{F}) \). By the definitions of the adjoint operator \( v_{\Lambda,j}^* \) and the weak partial derivatives, this implies that \( \partial_{x_j} \Psi \in L^1_{\text{loc}}(\Lambda, \mathcal{F}) \) exists and
\[
v_{\Lambda,j}^* \Psi = -i \partial_{x_j} \Psi - A_j \Psi - \varphi(G_j) \Psi \quad \text{(sum in } L^1_{\text{loc}}(\Lambda, \mathcal{F})). \tag{5.23}
\]

**Lemma 5.11.** Let \( \Psi \in \{ L_0^\infty(\Lambda) \otimes \mathcal{Q}(\text{d}\Gamma(1 \lor \omega)) \} \cap D(\mathcal{h}_{\Lambda,N}) \). Then there exist \( \Psi_n \in \mathcal{D}(\Lambda, \mathcal{Q}(\text{d}\Gamma(1 \lor \omega))) \), \( n \in \mathbb{N} \), such that \( \Psi_n \rightarrow \Psi \), \( n \rightarrow \infty \), with respect to the form norm of \( \mathcal{h}_{\Lambda,N} \).

**Proof.** Thanks to Remark 5.10 we know that the weak partial derivatives of \( \Psi \) with respect to every \( x_j \) exist and are given by the \( L^1_{\text{loc}}(\Lambda, \mathcal{F}) \)-sum
\[
\partial_{x_j} \Psi = iv_{\Lambda,j}^* \Psi + i A_j \Psi + i \varphi(G_j) \Psi.
\]
Together with the present assumptions on \( \Psi \), (2.2), (1.6), and (1.7) the latter formula reveals that actually \( \partial_{x_j} \Psi \in L^2(\Lambda, \mathcal{F}) \). Define \( \Psi_n \) as in (4.3), for all integers \( n \geq n_0 \) and some \( n_0 \in \mathbb{N} \) such that \( \text{dist}(\text{supp}(\Psi), \Lambda_{n_0}^c) \geq 1/n_0 \). Extending them by 0 outside \( \Lambda_n \), we obtain functions \( \Psi_n \in \mathcal{D}(\Lambda, \mathcal{Q}(\text{d}\Gamma(1 \lor \omega))) \), \( n \geq n_0 \), such that \( \Psi_n \rightarrow \Psi \) in \( L^2(\Lambda, \mathcal{Q}(\text{d}\Gamma(1 \lor \omega))) \) and \( \partial_{x_j} \Psi_n \rightarrow \partial_{x_j} \Psi \), \( j \in J \), in \( L^2(\Lambda, \mathcal{F}) \), as \( n \rightarrow \infty \).
∞. All $\Psi_n$ have their supports in a fixed compact subset of $\Lambda$. Recall the notation $N_1 := 1 + \text{di}'(1)$. Since $\Psi \in L^\infty_0(\Lambda) \otimes Q(\text{di}'(1 \lor \omega))$, we further have $\|\Psi_n\|_\infty \leq \|N_1^{1/2}\Psi_n\|_\infty \leq \|N_1^{1/2}\Psi\|_\infty < \infty$. (Here $\|\cdot\|_\infty$ stands for the essential supremum of $\|\cdot\|_F$-norms of Fock space-valued functions.) It is also clear that $N_1^{1/2}\Psi_n \to N_1^{1/2}\Psi$ in $L^2(\Lambda, \mathcal{F})$. Therefore, we find a subsequence of $\{\Psi_n\}_{n \geq n_0}$, call it $\{\Psi_n^\prime\}_{n \in \mathbb{N}}$, such that $N_1^{1/2}\Psi_n^\prime \to N_1^{1/2}\Psi$ a.e. on $\Lambda$. From these remarks and the dominated convergence theorem we infer that $A_j\Psi_n^\prime \to A_j\Psi$ in $L^2(\Lambda, \mathcal{F})$. In the same way we see that $V^{1/2}\Psi_n^\prime \to V^{1/2}\Psi$ in $L^2(\Lambda, \mathcal{F})$.

The above remarks, the dominated convergence theorem, and (2.2) further imply that $\varphi(G_j)\Psi_n^\prime \to \varphi(G_j)\Psi$ in $L^2(\Lambda, \mathcal{F})$. Moreover, it is clear that $\Psi_n^\prime \in \mathcal{D}(v_{\Lambda,j}) \subset \mathcal{D}(v_{\Lambda,j}^\ast)$ with $v_{\Lambda,j}^\ast\Psi_n^\prime = v_{\Lambda,j}\Psi_n^\prime = -i\partial_{x_j}\Psi_n - A_j\Psi_n - \varphi(G_j)\Psi_n^\prime$, and we conclude that $v_{\Lambda,j}^\ast\Psi_n \to v_{\Lambda,j}^\ast\Psi$ in $L^2(\Lambda, \mathcal{F})$, for all $j \in \{1, \ldots, \nu\}$.

The next lemma will be used to derive a diamagnetic inequality for resolvents of Dirichlet–Pauli–Fierz operators in Theorem 6.3 below.

**Lemma 5.12.** Let $\Psi \in \mathcal{D}(\mathfrak{h}_{\Lambda,N}) \cap L^\infty_0(\Lambda, \mathcal{F})$. Then $\Psi \in \mathcal{D}(\mathfrak{h}_{\Lambda,D})$.

**Proof.** We have to show that $\Psi$ can be approximated with respect to the form norm of $\mathfrak{h}_{\Lambda,N}$ by elements of $\mathcal{D}(\Lambda, Q(\text{di}'(1 \lor \omega)))$. But this follows upon combining Remark 5.3, Lemmas 5.9(2) and 5.11.

An example for the applicability of the above approximation results in the case $\Lambda = \mathbb{R}^\nu$ is the following analogue of a well-known result on Schrödinger forms [36]. The next theorem also generalizes [27, Corollary 4.7] by weakening the condition imposed on $G$ there.

**Theorem 5.13.** The maximal and minimal Pauli–Fierz forms on $\mathbb{R}^\nu$ agree, i.e.,

$$\mathfrak{h}_{\mathbb{R}^\nu,D} = \mathfrak{h}_{\mathbb{R}^\nu,N}. \quad (5.24)$$

**Proof.** Combine Lemmas 5.8, 5.9(2), and 5.11.

In view of the preceding theorem we abbreviate

$$\mathfrak{h}_{\mathbb{R}^\nu} := \mathfrak{h}_{\mathbb{R}^\nu,D} = \mathfrak{h}_{\mathbb{R}^\nu,N}, \quad (5.25)$$

and we shall refer to $\mathfrak{h}_{\mathbb{R}^\nu}$ simply as the Pauli–Fierz form on $\mathbb{R}^\nu$.

We conclude this section with a proposition providing a crucial technical ingredient needed to derive our Feynman–Kac formulas for $H_\Lambda$: We shall verify the conditions (a) and (b) of Hypothesis 3.1 when the forms $\mathfrak{h}_{\mathbb{R}^\nu}$ and $\mathfrak{h}_{\Lambda,D}'$ are put in place of $\mathfrak{q}_{\mathbb{R}^\nu}$ and $\mathfrak{q}_\Lambda$, respectively, where we use the following notational conventions:

For any function $\Phi : \Lambda \to \mathcal{F}$, we denote by $\Phi'$ its extension to $\mathbb{R}^\nu$ by 0. For a set $\mathcal{M}$ of functions from $\Lambda$ to $\mathcal{F}$, we put $\mathcal{M}' := \{\Phi' : \Phi \in \mathcal{M}\}$. Restrictions of functions on $\mathbb{R}^\nu$ to $\Lambda$ are denoted by a subscript $\Lambda$. Finally, we define

$$\mathfrak{h}_{\Lambda,D}[\Psi] := \mathfrak{h}_{\Lambda,D}[\Psi_\Lambda], \quad \Psi \in \mathcal{D}(\mathfrak{h}_{\Lambda,D}') := \mathcal{D}(\mathfrak{h}_{\Lambda,D})'. \quad (5.26)$$

In other words, $\mathfrak{h}_{\Lambda,D}'$ is $\mathfrak{h}_{\Lambda,D}$ considered as a form in $1_\Lambda L^2(\mathbb{R}^\nu, \mathcal{F})$ in the canonical way.
Proposition 5.14. Assume that $A \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu)$, $G \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu)$, and $0 \leq V \in L^1_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R})$. Let $Y^A_\infty$ be given by (3.1), the functions $\vartheta_\ell$ being chosen as in the paragraph preceding (3.1). Set
\[ \mathcal{D}(\mathfrak{h}_{RV}^{1,\infty}) := \left\{ \Psi \in \mathcal{D}(\mathfrak{h}_{RV}^\nu) \left| \int_{\mathbb{R}^\nu} Y^A_\infty(x) \| \Psi(x) \|^2 dx < \infty \right. \right\}. \]
Then $\mathcal{D}(\mathfrak{h}_{RV}^{1,\infty}) \subset \mathcal{D}(\mathfrak{h}_{AD}^V)$ and the closure of $\mathcal{D}(\mathfrak{h}_{RV}^{1,\infty})$ with respect to the form norm of $\mathfrak{h}_{AD}^V$ is $\mathcal{D}(\mathfrak{h}_{AD}^V)$. Furthermore, $\mathfrak{h}_{RV}^\nu [\Psi] = \mathfrak{h}_{AD}^V [\Psi]$, for every $\Psi \in \mathcal{D}(\mathfrak{h}_{RV}^{1,\infty})$.

Proof. Let $\Psi \in \mathcal{D}(\mathfrak{h}_{RV}^{1,\infty})$. Clearly, $\Psi = 0$ a.e. on $\Lambda^c$. If $\Phi \in \mathcal{D}(\Lambda, \mathfrak{Q}(d\Gamma(1)))$ and $j \in \{1, \ldots, \nu\}$, we find
\[ \langle v_{\Lambda,j}^\nu \Phi \mid \Psi_A \rangle_{L^2(\Lambda, \mathcal{F})} = \langle v_{\nu,j}^\nu \Phi' \mid \Psi \rangle_{L^2(\mathbb{R}^\nu, \mathcal{F})} = \langle \Phi' v_{\nu,j}^\nu \Psi \rangle_{L^2(\mathbb{R}^\nu, \mathcal{F})} = \langle \Phi' (v_{\nu,j}^\nu \Psi)_\Lambda \rangle_{L^2(\Lambda, \mathcal{F})}. \]

Thus, $\Psi_A \in \mathcal{D}(v_{\nu,j}^\nu)$ with
\[ v_{\nu,j}^\nu \Psi_A = (v_{\nu,j}^\nu \Psi)_\Lambda. \quad (5.27) \]
This shows that $\Psi_A \in \mathcal{D}(\mathfrak{h}_{AN}^Y) \subset \mathcal{D}(\mathfrak{h}_{AN})$, where the form $\mathfrak{h}_{AN}^Y$ is defined by putting the potential $(V + Y^A_\infty)_\Lambda$ in place of $V_\Lambda$ in the construction of $\mathfrak{h}_{AN}$. But $\{ \Phi \in \mathcal{D}(\mathfrak{h}_{AN}^Y) \mid \Phi \in L^\infty(\Lambda, \mathfrak{Q}(d\Gamma(1))) \}$ is a core for $\mathfrak{h}_{AN}^Y$ according to Lemma 5.7. Taking also Lemmas 5.9(2) and 5.11 into account we see that $\mathcal{D}(\Lambda, \mathfrak{Q}(d\Gamma(1 \lor \omega)))$ is a core for $\mathfrak{h}_{AN}^Y$ as well. Since $\mathfrak{h}_{AN} \subseteq \mathfrak{h}_{AN}^Y$, it is now clear that $\Psi_A$ can be approximated by elements of $\mathcal{D}(\Lambda, \mathfrak{Q}(d\Gamma(1 \lor \omega))) \subset \mathcal{D}(\mathfrak{h}_{AD}^{V'})$ with respect to the form norm of $\mathfrak{h}_{AN}^Y$, that is,
\[ \Psi_A \in \mathcal{D}(\mathfrak{h}_{AD}) \quad \text{and} \quad \mathfrak{h}_{AD}^{V'} [\Psi_A] = \mathfrak{h}_{AN}^Y [\Psi_A]. \quad (5.28) \]
In particular, $\Psi \in \mathcal{D}(\mathfrak{h}_{AD})$. Since $\Psi = 0$ a.e. on $\Lambda^c$, we further know that $v_{\nu,j}^\nu \Psi = 0$ a.e. on $\Lambda^c$, for every $j \in \{1, \ldots, \nu\}$; see [27, Lemma 3.4]. Employing (5.26), (5.27), and (5.28) we conclude that $\mathfrak{h}_{AD}^{V'} [\Psi] = \mathfrak{h}_{AN}^Y [\Psi_A] = \mathfrak{h}_{RV}^\nu [\Psi]$.

As $Y^A_\infty$ is locally bounded on $\Lambda$, it is also clear that $\mathcal{D}(\Lambda, \mathfrak{Q}(d\Gamma(1 \lor \omega)))' \subset \mathcal{D}(\mathfrak{h}_{RV}^{1,\infty})$. Moreover, by definition of $\mathfrak{h}_{AD}^{V'}$ and (5.26), $\mathcal{D}(\Lambda, \mathfrak{Q}(d\Gamma(1 \lor \omega)))'$ is a core for the form $\mathfrak{h}_{AD}^{V'}$. This reveals that the closure of $\mathcal{D}(\mathfrak{h}_{RV}^{1,\infty})$ with respect to the form norm of $\mathfrak{h}_{AD}^{V'}$ is $\mathcal{D}(\mathfrak{h}_{AD}^{V'})$. \hspace{1cm} \Box

6. A Diamagnetic Inequality for Resolvents

The purpose of this section is to derive a diamagnetic inequality comparing resolvents of Dirichlet–Pauli–Fierz operators and resolvents of Dirichlet–Schrödinger operators; see (6.5) in Theorem 6.3 below. This inequality will be used to discuss strong resolvent convergence of certain sequences of Dirichlet–Pauli–Fierz operators in the succeeding Sect. 7. Even for $\Lambda = \mathbb{R}^\nu$ and $A = 0$, Theorem 6.3 relaxes assumptions imposed on $G$ in earlier derivations [12,13,20] of the bound (6.5). The proofs in this section follow the lines of the corresponding ones in [17] but require additional arguments to deal with the quantized fields.
We start with a complement to Lemma 4.1. Recall that the symbols $Z_\delta(\Psi)$ and $\Theta_{\delta,\Psi}$ have been introduced in (4.5).

**Lemma 6.1.** Let $\mathcal{X}$ be a separable Hilbert space, $j \in \{1, \ldots, \nu\}$, $\delta > 0$, and let $\Psi \in L^p_{\text{loc}}(\Lambda, \mathcal{X})$ have a weak partial derivative with respect to $x_j$ satisfying $\partial_{x_j} \Psi \in L^p_{\text{loc}}(\Lambda, \mathcal{X})$. Then $\Theta_{\delta,\Psi} \in L^\infty(\Lambda, \mathcal{X})$ has a weak partial derivative with respect to $x_j$ which belongs to $L^p_{\text{loc}}(\Lambda, \mathcal{X})$ and is given by

$$\partial_{x_j} \Theta_{\delta,\Psi} = Z_\delta(\Psi)^{-1}(\partial_{x_j} \Psi - \text{Re}(\Theta_{\delta,\Psi}|\partial_{x_j} \Psi) \mathcal{X} \Theta_{\delta,\Psi}).$$

(6.1)

Furthermore, let $\chi \in W^{1,2}(\Lambda)$ satisfy $|\chi| \leq c Z_1(\Psi)$, for some $c > 0$. Then $\chi \Theta_{\delta,\Psi} \in L^2(\Lambda, \mathcal{X})$ has a weak partial derivative with respect to $x_j$ which is in $L^{2p}_{\text{loc}}(\Lambda, \mathcal{X})$ and given by

$$\partial_{x_j} (\chi \Theta_{\delta,\Psi}) = (\partial_{x_j} \chi) \Theta_{\delta,\Psi} + \frac{\chi}{Z_\delta(\Psi)} \left(\partial_{x_j} \Psi - \text{Re}(\Theta_{\delta,\Psi}|\partial_{x_j} \Psi) \mathcal{X} \Theta_{\delta,\Psi}\right).$$

(6.2)

**Proof.** Employing (4.7) and the usual chain rule for weak partial derivatives we compute

$$\partial_{x_j} Z_\delta(\Psi)^{-1} = -Z_\delta(\Psi)^{-2} \text{Re}(\Theta_{\delta,\Psi}|\partial_{x_j} \Psi) \mathcal{X} Z_\delta(\Psi) \in L^p_{\text{loc}}(\Lambda),$$

which in conjunction with Theorem 4.2 yields (6.1); notice that the product $(\partial_{x_j} Z_\delta(\Psi)^{-1})$ is indeed in $L^1_{\text{loc}}(\Lambda, \mathcal{X})$ so that Theorem 4.2 is applicable. We read off from (6.1) that $\partial_{x_j} \Theta_{\delta,\Psi} \in L^p_{\text{loc}}(\Lambda, \mathcal{X})$. Finally, (6.2) follows from (6.1) and Theorem 4.2: here we use that the product $\chi \partial_{x_j} \Theta_{\delta,\Psi}$ is in $L^1_{\text{loc}}(\Lambda, \mathcal{X})$ thanks to the postulated bound $|\chi| \leq c Z_1(\Psi)$. 

**Proposition 6.2.** Assume (1.6) and (1.7). Let $\delta > 0$, $j \in \{1, \ldots, \nu\}$, $\Psi \in \mathcal{D}(v_{\Lambda,j}^*)$, and let $\chi \in W^{1,2}(\Lambda)$ be nonnegative and satisfy $\chi \leq c Z_1(\Psi)$, for some $c > 0$. Then $\chi \Theta_{\delta,\Psi} \in \mathcal{D}(v_{\Lambda,j}^*)$ and, a.e. on $\Lambda$,

$$\text{Re}(v_{\Lambda,j}^*(\chi \Theta_{\delta,\Psi})|v_{\Lambda,j}^*) \mathcal{F} \geq \|\Theta_{\delta,\Psi}\|_{\mathcal{F}} \|\partial_{x_j} \chi\|_{\mathcal{F}} \|\Psi\|_{\mathcal{F}}.$$

(6.3)

**Proof.** We pick some $\varepsilon > 0$ and start by considering $\Psi_\varepsilon = N_\varepsilon^{-1/2} \Psi$ with $N_\varepsilon$ given by (5.2). According to Remark 5.1, $\Psi_\varepsilon$ has a weak partial derivative with respect to $x_j$ which is given by (5.5). Plugging $\Psi_\varepsilon$ and $\chi$ into (6.2), subtracting $i \chi A_j \Theta_{\delta,\Psi} + i \chi \varphi(G_j) \Theta_{\delta,b_\Psi}$ on both sides, and using the relations $\text{Re}(\Theta_{\delta,b_\Psi}|iA_j + \varphi(G_j))\psi_\varepsilon = 0$ and (5.5), we find

$$\partial_{x_j} (\chi \Theta_{\delta,\Psi_\varepsilon}) - iA_j (\chi \Theta_{\delta,\Psi_\varepsilon}) - i \varphi(G_j)(\chi \Theta_{\delta,\Psi_\varepsilon}) = (\partial_{x_j} \chi) \Theta_{\delta,\Psi_\varepsilon} + \frac{\chi}{Z_\delta(\Psi)} \left(iN_\varepsilon^{-1/2} v_{\Lambda,j}^* \Psi + iC_\varepsilon(G_j)^* \Psi - \text{Re}(\Theta_{\delta,b_\Psi}|iN_\varepsilon^{-1/2} v_{\Lambda,j}^* \Psi + iC_\varepsilon(G_j)^* \Psi) \mathcal{X} \Theta_{\delta,b_\Psi}\right)$$

in $L^1_{\text{loc}}(\Lambda, \mathcal{F})$. (6.4)

Next, we compute the $\mathcal{F}$-scalar product of $\eta \in \mathcal{D}(\Lambda, Q(d\Gamma(1)))$ with the vectors on both sides of (6.4) and integrate the result over $\Lambda$. After that we pass to the limit $\varepsilon \downarrow 0$ taking into account that

(a) $\delta \leq Z_\delta(\Psi_\varepsilon) \rightarrow Z_\delta(\Psi)$ pointwise;
(b) $N_\varepsilon^{-1/2} v_{\Lambda,j}^* \Psi \rightarrow v_{\Lambda,j}^* \Psi$ pointwise and in $L^2(\Lambda, \mathcal{F})$;
(c) by (a), (b), and $\chi \in L^2(\Lambda)$,

$$\frac{\chi}{Z_\delta(\Psi)} N_\delta^{-1/2} v_{\Lambda,j}^* \Psi \rightarrow \frac{\chi}{Z_\delta(\Psi)} v_{\Lambda,j}^* \Psi \quad \text{pointwise and in } L^1(\Lambda, \mathcal{F}),$$

with integrable majorant $|\chi||v_{\Lambda,j}^* \Psi| / \delta$;

(d) in view of $\chi \in L^2(\Lambda)$, $\|G_j\|_\ell \in L^2_{loc}(\Lambda)$, and (5.4),

$$\frac{|\chi|}{Z_\delta(\Psi)} \|C_\delta(G_j)^* \Psi\|_{\mathcal{F}} \leq 2 \varepsilon^{1/2} \|G_j\|_\ell |\chi| \frac{\|\Psi\|_{\mathcal{F}}}{Z_\delta(\Psi)} \rightarrow 0,$$

where the convergence is understood in $L^1_{loc}(\Lambda)$;

(e) $S_{\delta,\Psi} \rightarrow S_{\delta,\Psi}$ pointwise with $\|S_{\delta,\Psi}\|_{\mathcal{F}} \leq 1$.

In this way we arrive at the identity

$$i(v_{\Lambda,j}^* |\chi S_{\delta,\Psi})$$

$$= \int_{\Lambda} \langle \eta(x) | (\partial_{x_j} \chi)(x) S_{\delta,\Psi}(x) \rangle dx$$

$$+ \int_{\Lambda} \langle \eta(x) | \frac{\chi(x)}{Z_\delta(\Psi)} (iv_{\Lambda,j}^* \Psi - \Re(S_{\delta,\Psi}|iv_{\Lambda,j}^* \Psi) S_{\delta,\Psi})(x) \rangle dx.$$

Since we are assuming that $\partial_{x_j} \chi \in L^2(\Lambda)$ and $|\chi| \leq c Z_1(\Psi)$, the last two integrals can be read as scalar products of $\eta$ with vectors in $L^2(\Lambda, \mathcal{F})$. Thus, $\chi S_{\delta,\Psi} \in D(v_{\Lambda,j}^*)$ with

$$iv_{\Lambda,j}^* (\chi S_{\delta,\Psi}) = (\partial_{x_j} \chi) S_{\delta,\Psi} + \frac{\chi}{Z_\delta(\Psi)} (iv_{\Lambda,j}^* \Psi - \Re(S_{\delta,\Psi}|iv_{\Lambda,j}^* \Psi) S_{\delta,\Psi}).$$

From here on we can copy the proof of [17, Lemma 3.1]: Computing the $\mathcal{F}$-scalar product with $iv_{\Lambda,j}^* \Psi$ on both sides of the previous relation and taking real parts we arrive at

$$\Re(v_{\Lambda,j}^* (\chi S_{\delta,\Psi}) |v_{\Lambda,j}^* \Psi)_{\mathcal{F}}$$

$$= (\partial_{x_j} \chi) \Re(S_{\delta,\Psi}|iv_{\Lambda,j}^* \Psi)_{\mathcal{F}} + \frac{\chi}{Z_\delta(\Psi)} \left( \|v_{\Lambda,j}^* \Psi\|_{\mathcal{F}}^2 - \Re(S_{\delta,\Psi}|iv_{\Lambda,j}^* \Psi)_{\mathcal{F}}^2 \right)$$

$$\geq (\partial_{x_j} \chi) \Re(S_{\delta,\Psi}|iv_{\Lambda,j}^* \Psi)_{\mathcal{F}} = \|S_{\delta,\Psi}\|_{\mathcal{F}} (\partial_{x_j} \chi) \partial_{x_j} \|\Psi\|_{\mathcal{F}}.$$

Here we also used $\chi \geq 0$ in the penultimate step and (2.6) in the last one.

Now we are in a position to prove the promised diamagnetic inequality for resolvents. Recall that the Dirchlet–Pauli–Fierz operator $H_\Lambda$ has been defined in Sect. 2.4. By $S_\Lambda$ we denote the Dirichlet–Schrödinger operator with potential $V$ on $\Lambda$, i.e., the selfadjoint operator representing the nonnegative closed form

$$s_{\Lambda,D}[f] := \frac{1}{2} \|\nabla f\|^2 + \int_{\Lambda} V(x)|f(x)|^2 dx, \quad f \in D(s_{\Lambda,D}) := \dot{W}^{1,2}(\Lambda) \cap Q(V).$$

**Theorem 6.3.** Assume (1.5), (1.6), and (1.7). Let $\Phi \in L^2(\Lambda, \mathcal{F})$ and $E > 0$. Then, a.e. on $\Lambda$,

$$\|(H_\Lambda + E)^{-1}\Phi\|_{\mathcal{F}} \leq (s_\Lambda + E)^{-1}\|\Phi\|_{\mathcal{F}}. \quad (6.5)$$
Proof. We can adapt the proof of [17, Theorem 3.3]. Put \( \Phi := (H + E)^{-1} \Phi \in D(H) \subset D(h_{A,D}) \). Then [27, Corollary 4.1] implies \( \|\Psi\|_{\mathcal{F}} \in D(h_{A,D}) \). Pick some \( \delta > 0 \) and let \( \chi \in D(h_{A,D}) \subset W^{1,2}(\Lambda) \) be nonnegative, compactly supported, and bounded. Employing Proposition 6.2 we then infer that \( \chi \mathcal{G}_{\delta,\Psi} \in D(h_{A,N}) \). Since \( \chi \) is compactly supported and bounded, Lemma 5.12 now implies that actually \( \chi \mathcal{G}_{\delta,\Psi} \in D(h_{A,D}) \). Integrating (6.3), summing the result over \( j \in \{1,\ldots,\nu\} \), and observing \( \mathcal{G}_{\delta,\Psi}\|_{\mathcal{F}} = \|\mathcal{G}_{\delta,\Psi}\|_{\mathcal{F}} \) and

\[
\int_{\Lambda} |d\Gamma(\omega)|^{1/2} \chi(x) \mathcal{G}_{\delta,\Psi}(x)|d\Gamma(\omega)|^{1/2} \Psi(x) \rangle_{\mathcal{F}} dx \geq 0,
\]

we further find

\[
\frac{1}{2} \langle \|\mathcal{G}_{\delta,\Psi}\|_{\mathcal{F}} \nabla \|\Psi\|_{\mathcal{F}} \rangle_{L^2(\Lambda)} + \int_{\Lambda} |V(x) + E| \chi(x) \|\mathcal{G}_{\delta,\Psi}(x)\|_{\mathcal{F}} \|\Psi(x)\|_{\mathcal{F}} dx \leq |\langle h_{A,D} + E| \chi \mathcal{G}_{\delta,\Psi}, \Psi \rangle| \leq \langle \|\mathcal{G}_{\delta,\Psi}\|_{\mathcal{F}} \rangle_{L^2(\Lambda)}.
\]

Here we also used \( \chi \geq 0 \) and \( \|\mathcal{G}_{\delta,\Psi}\|_{\mathcal{F}} \leq 1 \) in the last step. (Furthermore, symbols like \( q[\phi,\psi] \) denote values of the sesquilinear form associated with a quadratic form \( q \).) By dominated convergence, we can pass to the limit \( \delta \downarrow 0 \) on the left hand side of the previous estimation. Since \( \nabla \|\Psi\|_{\mathcal{F}} = 0 \) a.e. on \( \{\Psi = 0\} \), we may drop the term \( \|\mathcal{G}_{\Psi}\|_{\mathcal{F}} \) found in this way whenever it is multiplied with \( \|\Psi\|_{\mathcal{F}} \) or \( \nabla \|\Psi\|_{\mathcal{F}} \). This yields

\[
\langle (h_{A,D} + E)\chi, \|\Psi\|_{\mathcal{F}} \rangle_{L^2(\Lambda)} \leq \langle \|\mathcal{G}_{\delta,\Psi}\|_{\mathcal{F}} \rangle_{L^2(\Lambda)}.
\]

The bound (6.6) is actually available for all nonnegative \( \chi \in D(h_{A,D}) \) since any such \( \chi \) can be approximated in the form norm of \( h_{A,D} \) by bounded and compactly supported nonnegative elements of \( W^{1,2}(\Lambda) \) (using [23, Corollary 6.18]). In particular, we may choose \( \chi := (S_{A} + E)^{-1}\eta \), for some \( \eta \in L^2(\Lambda) \) with \( \eta \geq 0 \), because \( D(S_{A}) \subset D(h_{A,D}) \) and the resolvent \( (S_{A,D}+E)^{-1} \) is positivity preserving. This yields (6.5) integrated with respect to the density \( \eta \). \( \square \)

7. Strong Resolvent Convergence

In the presence of singular electromagnetic fields, a Feynman–Kac formula is typically obtained in a chain of extension steps establishing the formula for ever more singular (vector) potentials. To ensure convergence of the functional analytic side of the Feynman–Kac formula, at least along suitable subsequences, when singular (vector) potentials are approximated by more regular ones, it is sufficient to prove strong resolvent convergence of the corresponding selfadjoint operators. For our model this is done in the present section. Since the approximation of electrostatic potentials is quite standard, we shall concentrate on the simultaneous approximation of classical and quantized vector potentials here.

Results for Schrödinger operators similar to Theorem 7.1 below appear in [18] for \( \Lambda = \mathbb{R}^{\nu} \) and in [24] for general open \( \Lambda \). Both the limiting vector potential and the ones approximating it are merely supposed to be in \( L_{loc}^{2} \) in [18]. In [24] results for even more general vector potentials can be found. We
shall restrict our attention to the situation we actually encounter later on as this admits a comparatively short proof.

In the next theorem and henceforth $C_b^\ell$ stands for bounded, $\ell$-times continuously differentiable maps with bounded derivatives of order $\ell \leq \ell$. Recalling (5.25) we further abbreviate $\mathfrak{h} := \mathfrak{h}_{\mathbb{R}^\nu}$ and refer to the selfadjoint operator $H := H_{\mathbb{R}^\nu}$ representing this form simply as the Pauli–Fierz operator on $\mathbb{R}^\nu$.

**Theorem 7.1.** Let $A \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu)$, $G \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu)$ and $A^n \in C_b^1(\mathbb{R}^\nu, \mathbb{R}^\nu)$, $G^n \in C_b^l(\mathbb{R}^\nu, \mathbb{R}^\nu)$, $n \in \mathbb{N}$, satisfy

$$
\int_K |A^n(x) - A(x)|^2 \, dx \xrightarrow{n \to \infty} 0, \tag{7.1}
$$

$$
\int_K \|G^n(x) - G(x)\|^2_{\mathfrak{h}} \, dx \xrightarrow{n \to \infty} 0, \tag{7.2}
$$

for all compact $K \subset \mathbb{R}^\nu$. Assume that $V \geq 0$ is measurable and bounded. Let $H$ be the Pauli–Fierz operator on $\mathbb{R}^\nu$ defined by means of $A$, $G$, and $V$. For every $n \in \mathbb{N}$, let $H^n$ be the Pauli–Fierz operator on $\mathbb{R}^\nu$ defined by means of $A^n$, $G^n$, and $V$. Then

$$
H^n \xrightarrow{n \to \infty} H \quad \text{in the strong resolvent sense.}
$$

**Proof.** Recall that, for each $z \in \mathbb{C} \setminus \mathbb{R}$, strong convergence of $(H^n - z)^{-1}$ to $(H - z)^{-1}$ is implied by weak convergence of $(H^n - z)^{-1}$ to $(H - z)^{-1}$, because

$$
\|(H^n - z)^{-1}\Psi\|^2 - \|(H - z)^{-1}\Psi\|^2 = \frac{1}{\text{Im}[z]} \text{Im} \langle \Psi \mid ((H^n - z)^{-1} - (H - z)^{-1})\Psi \rangle, \quad \Psi \in L^2(\mathbb{R}^\nu, \mathcal{F}),
$$

by the first resolvent equation, and because weak convergence of a sequence in a Hilbert space together with convergence of the norms of its elements implies norm convergence. In what follows we pick some $z \in \mathbb{C} \setminus \mathbb{R}$ with $\text{Re}[z] \leq -1$. Since $H_n \geq 0$, the resolvents $(H^n - z)^{-1}$ are uniformly bounded in $n \in \mathbb{N}$, whence it suffices to show that

$$
\langle \Xi \mid ((H^n - z)^{-1} - (H - z)^{-1})\Phi \rangle \xrightarrow{n \to \infty} 0,
$$

for all $\Xi$ and $\Phi$ in some dense subsets of $L^2(\mathbb{R}^\nu, \mathcal{F})$. We pick

$$
\Xi := (H - 1)\Xi', \quad \Xi' \in L^2(\mathbb{R}^\nu, \mathcal{F}) \cap L^\infty(\mathbb{R}^\nu, \mathcal{F}),
$$

noticing that $(H - 1)\Xi'$ is a bounded isomorphism on $L^2(\mathbb{R}^\nu, \mathcal{F})$ which in particular maps a dense subset onto another dense subset. In view of the diamagnetic inequality (6.5) this choice of $\Xi$ implies that

$$
\Upsilon := (H - 1)\Xi = (H - z)^{-1}\Xi' \quad \text{is bounded.}
$$

Furthermore, we know from [27, Theorem 5.5] that every $H^n$ with $n \in \mathbb{N}$ is essentially selfadjoint on $\mathcal{D}(\mathbb{R}^\nu, \mathcal{D}(d\Gamma(1 \vee \omega)))$. (Here we use that the Schrödinger operator $(1/2)(-i\nabla - A^n)^2 + V$ is essentially selfadjoint on $C_0^\infty(\mathbb{R}^\nu)$, exploiting that we work on the whole Euclidean space $\mathbb{R}^\nu$ and not on a proper open subset of it.) In particular, $(H^n - z)\mathcal{D}(\mathbb{R}^\nu, \mathcal{D}(d\Gamma(1 \vee \omega)))$
is a dense subspace of $L^2(\mathbb{R}^\nu, \mathcal{F})$, and we choose $\Phi := (H^n - z)\Psi$ for some $\Psi \in \mathcal{D}(\mathbb{R}^\nu, D(\mathrm{d}\Gamma(1 \lor \omega)))$. Then
\[
\langle \Xi | ((H^n - z)^{-1} - (H - z)^{-1}) \Phi \rangle = \langle \Xi | \Psi \rangle - \langle \Upsilon | (H^n - z) \Psi \rangle = h[\Upsilon, \Psi] - \langle \Upsilon | H^n \Psi \rangle.
\]
For all $j \in \{1, \ldots, \nu\}$ and $n \in \mathbb{N}$, we now abbreviate
\[
\tilde{A}_j^n := A_j^n - A_j, \quad \tilde{G}_j^n := G_j^n - G_j, \quad v_j^n := (-i\partial_{x_j} - A_j^n - \varphi(G_j^n)) [\mathcal{D}(\mathbb{R}^\nu, \mathcal{Q}(\mathrm{d}\Gamma(1))), \quad v_j := v_{\mathbb{R}^\nu \setminus \nu}.
\]
Furthermore, we pick $\Upsilon_m \in \mathcal{D}(\mathbb{R}^\nu, \mathcal{Q}(\mathrm{d}\Gamma(1 \lor \omega)))$, $m \in \mathbb{N}$, such that $\Upsilon_m \to \Upsilon$, $m \to \infty$, with respect to the form norm of $h$, which is possible because $\Upsilon \in \mathcal{D}(H) \subset \mathcal{D}(h)$. Then we obtain
\[
h[\Upsilon, \Psi] - \langle \Upsilon | H^n \Psi \rangle = \lim_{m \to \infty} h[\Upsilon_m, \Psi] - \lim_{m \to \infty} \langle \Upsilon_m | H^n \Psi \rangle
\]
\[
= \lim_{m \to \infty} \frac{1}{2} \sum_{j=1}^{\nu} \left\{ \langle v_j \Upsilon_m | v_j \Psi \rangle - \langle v_j^n \Upsilon_m | v_j^n \Psi \rangle \right\}
\]
\[
= \lim_{m \to \infty} \frac{1}{2} \sum_{j=1}^{\nu} \left\{ \langle v_j \Upsilon_m | (\tilde{A}_j^n + \varphi(\tilde{G}_j^n)) \Psi \rangle + \langle (\tilde{A}_j^n + \varphi(\tilde{G}_j^n)) \Upsilon_m | v_j^n \Psi \rangle \right\}.
\]
Next, we take into account that convergence of $\Upsilon_m$ with respect to the form norm of $h$ entails the convergences $v_j \Upsilon_m \to v_j \Upsilon$, for all $j \in \{1, \ldots, \nu\}$. On account of (2.2) and (2.3) we also know that $\tilde{A}_j^n v_j^n \Psi$ and $\varphi(\tilde{G}_j^n)v_j^n \Psi$ belong to $L^2(\mathbb{R}^\nu, \mathcal{F})$, for all $n \in \mathbb{N}$. We thus arrive at
\[
h[\Upsilon, \Psi] - \langle \Upsilon | H^n \Psi \rangle = \frac{1}{2} \sum_{j=1}^{\nu} \langle v_j^n \Upsilon | (\tilde{A}_j^n + \varphi(\tilde{G}_j^n)) \Psi \rangle
\]
\[
+ \frac{1}{2} \sum_{j=1}^{\nu} \int_{\mathbb{R}^\nu} \langle \Upsilon(x) | \tilde{A}_j^n(x)(v_j^n \Psi)(x) \rangle \mathcal{F} \, dx
\]
\[
+ \frac{1}{2} \sum_{j=1}^{\nu} \int_{\mathbb{R}^\nu} \langle \Upsilon(x) | \varphi(\tilde{G}_j^n(x))v_j^n \Psi(x) \rangle \mathcal{F} \, dx, \quad (7.4)
\]
for every $n \in \mathbb{N}$. Here $(\tilde{A}_j^n + \varphi(\tilde{G}_j^n)) \Psi \to 0$ in $L^2(\mathbb{R}^\nu, \mathcal{F})$ because of (2.2) and (7.2), since $\|\Psi\|_{\mathcal{F}}$ and $\|\Psi\|_{\mathcal{Q}(\mathrm{d}\Gamma(1)))}$ are compactly supported and bounded. Hence, the first term on the right hand side of (7.4) goes to zero as $n \to \infty$.
Next, we observe (using (2.2), (7.1), and (7.2)) that the vectors $v_j^n \Psi$ are supported in supp$(\Psi)$ and uniformly bounded in $L^2(\mathbb{R}^\nu, \mathcal{F})$. Together with (7.1) and (7.3) this shows that the term in the second line of (7.4) converges to zero as well. Furthermore, setting
\[
D_{j,x}^n \psi := N_1^{1/2} \varphi(G^n_{j,x})N_1^{-1/2} \psi - \varphi(G^n_{j,x}) \psi, \quad \psi \in \mathcal{Q}(\mathrm{d}\Gamma(1)),
\]
where $N_1 = 1 + \mathrm{d}\Gamma(1)$ as in (5.2), we obtain
\[
\|\varphi(G^n_{j,x})(v_j^n \Psi)(x)\|_{\mathcal{F}}
\]
Our objective in this section is to find Feynman–Kac formulas for the Pauli–Fierz operator on $\mathbb{R}^\nu$. Stochastic Analysis for Regular Coefficients

8. Stochastic Analysis for Regular Coefficients

Our objective in this section is to find Feynman–Kac formulas for the Pauli–Fierz operator on $\mathbb{R}^\nu$ with regular coefficients, more precisely, coefficients satisfying the hypotheses collected in Sect. 8.1. The main tools will be a stochastic differential equation (8.25 below) associated with the Pauli–Fierz model investigated in [8] and various results of the latter paper on the random functions $W_t(x)$ and $W_t(x,y)$ for $A = 0$. Before we can apply the findings of [8] and extend them to non-zero $A$, we have, however, to compare the formulas given in the introduction for $S_t(x)$, $K_t(x)$, $S_t(x,y)$, and $K_t(x,y)$ with more familiar expressions for Stratonovich type stochastic integrals. This is done in a discussion of the Feynman–Kac integrands in Sect. 8.3, after a more detailed explanation of the involved Brownian bridge processes and time reversed processes in Sect. 8.2. Finally, we verify in Sect. 8.4 that the probabilistic sides of the Feynman–Kac formula define a strongly continuous semigroup of bounded selfadjoint operators, whose generator is identified as the Pauli–Fierz operator on $\mathbb{R}^\nu$ in Sect. 8.5.

8.1. Assumptions on the Coefficients Used Throughout Sect. 8

In the entire Sect. 8 we assume

$$A \in C^1_b(\mathbb{R}^\nu, \mathbb{R}^\nu), \quad V \in C_b(\mathbb{R}^\nu, \mathbb{R}), \quad V \geq 0.$$  

Here $C_b := C^0_b(\mathbb{R}^\nu)$, and the notation $C^\ell_b$ has been explained in front of Theorem 7.1. Throughout this section we further assume $G$ to fulfill the following two hypotheses:

**Hypothesis 8.1.** It holds $G \in C^2(\mathbb{R}^\nu, \mathbb{R}^\nu)$, the components of $G_x$ and of $\partial_x_1 G_x, \ldots, \partial_x_\nu G_x$ are elements of $Q(\omega^{-1} + \omega^2)$, for every $x \in \mathbb{R}^\nu$, and the following map is continuous and bounded,

$$\mathbb{R}^\nu \ni x \longmapsto (G_x, \partial_{x_1} G_x, \ldots, \partial_{x_\nu} G_x) \in Q(\omega^{-1} + \omega^2)^{\nu(\nu+1)}.$$
Hypothesis 8.2. There exists a completely real subspace \( \mathfrak{F}_R \subset \mathfrak{F} \) such that

\[
G_x \in \mathfrak{F}^c_R, \quad e^{-t\omega} \mathfrak{F}_R \subset \mathfrak{F}_R,
\]

for all \( x \in \mathbb{R}^\nu \) and \( t > 0 \).

These two hypotheses have been imposed on \( G \) in [8]. The second one, Hypothesis 8.2, leads to some crucial cancellations in the analysis of Feynman–Kac integrands and their associated stochastic differential equations in [8]; it will not be used in a directly visible way in the present article.

8.2. Notation for Brownian Bridges and Time Reversed Processes

Recall that we fixed the filtered probability space \((\Omega, \mathfrak{F}, (\mathfrak{F}_t)_{t \geq 0}, \mathbb{P})\) satisfying the usual assumptions and the \((\mathfrak{F}_t)_{t \geq 0}\)-Brownian motion \( B \) starting in 0 in the introduction.

Let \( t > 0 \) in what follows. If \( x \in \mathbb{R}^\nu \) and \( q : \Omega \to \mathbb{R}^\nu \) is \( \mathfrak{F}_0 \)-measurable, then we let \( b_{t}^{x;q,x} \) denote a choice of the up to indistinguishability unique continuous semimartingale with respect to \((\mathfrak{F}_s)_{s \in [0,t]}\) which \( \mathbb{P}\)-a.s. solves the stochastic differential equation for a Brownian bridge in time \( t \) starting at \( q \) and ending at \( x \), i.e.,

\[
b_s = q + B_s + \int_0^s \frac{x - b_r}{t-r} \, dr, \quad s \in [0,t), \quad b_t = x. \tag{8.2}
\]

Next, we explain some notation for time reversals of Brownian motions and bridges; see [10,31] and [8, Appendix 4] for more details.

We denote by \((\hat{\mathfrak{F}}_s)_{s \geq 0}\) the standard extension of the filtration \((\mathfrak{F}_s)_{s \geq 0}\) where, for all \( s \in [0,t] \), \( \mathfrak{F}_s \) denotes the \( \sigma \)-algebra generated by \( B_{t-s} \) and all increments \( B_r - B_{t-r} \) with \( r \in [0,s] \), and where \( \mathfrak{F}_s = \mathfrak{F}_t \) for all \( s \geq t \). Let \( x \in \mathbb{R}^\nu \). Then the reversed process \( B_{t\mid x} \) defined in (1.10) is a semimartingale with respect to \((\hat{\mathfrak{F}}_s)_{s \in [0,t]}\). Furthermore, there exists a \((\hat{\mathfrak{F}}_s)_{s \in [0,t]}\)-Brownian motion \( \hat{B} \) such that \( B_{t\mid x} \) is \( \mathbb{P}\)-a.s. a solution to

\[
b_s = \hat{q} + \hat{B}_s + \int_0^s \frac{x - b_r}{t-r} \, dr, \quad s \in [0,t), \quad b_t = x, \tag{8.3}
\]

provided that we choose the \( \hat{\mathfrak{F}}_0 \)-measurable initial condition \( \hat{q} = B_t^x \).

We denote by \( \hat{b}_{t\mid y,x} \) the solution of (8.3) for the choice \( \hat{q} = y \). \tag{8.4}

We further denote by \((\tilde{\mathfrak{F}}_s)_{s \geq 0}\) the standard extension of the filtration \((\mathfrak{F}_s)_{s \geq 0}\) where, for all \( s \in [0,t] \), \( \tilde{\mathfrak{F}}_s \) denotes the \( \sigma \)-algebra generated by \( b_{t\mid y,x} \) and all increments \( B_r - B_{t-r} \) with \( r \in [0,s] \), and where \( \tilde{\mathfrak{F}}_s = \tilde{\mathfrak{F}}_t \) for all \( s \geq t \). Then the reversed process \( \hat{b}_{t\mid x\mid y} \) defined in (1.13) is a semimartingale with respect to \((\tilde{\mathfrak{F}}_s)_{s \in [0,t]}\), and there exists a \((\tilde{\mathfrak{F}}_s)_{s \in [0,t]}\)-Brownian motion \( \hat{B} \) such that \( \hat{b}_{t\mid x\mid y} \) is \( \mathbb{P}\)-a.s. a solution to

\[
b_s = x + \hat{B}_s + \int_0^s \frac{y - b_r}{t-r} \, dr, \quad s \in [0,t), \quad b_t = y.
\]
8.3. The Feynman–Kac Integrands for Regular Coefficients

To benefit from the results of [8], we first have to verify that the formulas (1.11), (1.12), (1.14), and (1.15) for the Stratonovich type integrals in our Feynman–Kac integrands generalize the ones used in the latter article:

Lemma 8.3. Let \( t > 0 \) and \( x, y \in \mathbb{R}^\nu \). Then the following identities hold \( \mathbb{P} \)-a.s.,

\[
S_t(x) = \int_0^t V(B^x_s) ds - i \int_0^t A(B^x_s) dB_s - \frac{i}{2} \int_0^t \text{div} A(B^x_s) ds, \quad (8.5)
\]

\[
K_t(x) = \int_0^t j_B G_{B^x_s} dB_s + \frac{1}{2} \int_0^t j_s \text{div} G_{B^x_s} ds, \quad (8.6)
\]

as well as

\[
S_t(x, y) = \int_0^t V(b^{t;y,x}_s) ds
\]

\[
- i \int_0^t A(b^{t;y,x}_s) db^{t;y,x}_s - \frac{i}{2} \int_0^t \text{div} A(b^{t;y,x}_s) ds, \quad (8.7)
\]

\[
K_t(x, y) = \int_0^t j_s G_{b^{t;y,x}_s} db^{t;y,x}_s + \frac{1}{2} \int_0^t j_s \text{div} G_{b^{t;y,x}_s} ds. \quad (8.8)
\]

Proof. Under the present conditions on \( G \), well-known results on Hilbert space-valued stochastic integrals reveal that

\[
\int_0^t j_s G_{B^x_s} dB^x_s = \lim_{n \to \infty} \sum_{\ell=1}^n j_{(\ell-1)t/n} G(B^x_{(\ell-1)t/n})(B_{\ell t/n} - B_{(\ell-1)t/n}),
\]

as well as

\[
\int_0^t j_{t-s} G_{B^x_s} dB^{t;x}_s = - \lim_{n \to \infty} \sum_{\ell=1}^n j_{\ell t/n} G(B^{x}_{\ell t/n})(B_{\ell t/n} - B_{(\ell-1)t/n}).
\]

Moreover, we verified in [8, Lemma 3.2] that the term on the right hand side of (8.6) equals

\[
\lim_{n \to \infty} \sum_{\ell=1}^n \frac{1}{2} (j_{\ell t/n} G_{B^{x}_{\ell t/n}} + j_{(\ell-1)t/n} G_{B^{x}_{(\ell-1)t/n}})(B_{\ell t/n} - B_{(\ell-1)t/n}).
\]

Altogether this proves (8.6). An analogous argument, again employing [8, Lemma 3.2], applies when \( b^{t;y,x} \) and \( \tilde{b}^{t;x,y} \) are put in place of \( B^x \) and \( B^{t;x} \), respectively. The relations (8.5) and (8.7) can be proved in the same fashion, using the more well-known (8.11) below. \( \square \)

In what follows we shall employ the following notation:

- \( \tilde{W}_t(x, y) \) is the random operator obtained upon replacing \( b^{t;y,x} \) by \( \tilde{b}^{t;y,x} \) in (8.7) and (8.8) and plugging the result into (1.17). Recall that \( \tilde{b}^{t;x,y} \) has been defined in (8.4).
- \( \hat{W}_t(x, y) \) is the random operator obtained upon replacing \( b^{t;y,x} \) by \( \hat{b}^{t;x,y} \) in (8.7) and (8.8) and plugging the result into (1.17); \( \hat{b}^{t;x,y} \) is defined in (1.13).
Theorem 8.4. Let \( t > 0 \) and \( \mathbf{x}, \mathbf{y} \in \mathbb{R}^\nu \). Then the following identities hold \( \mathbb{P}\text{-a.s.} \),

\[
W_t(\mathbf{x})^* = \hat{W}_t(\mathbf{x}, \mathbf{B}_t^x), \quad W_t(\mathbf{x}, \mathbf{y})^* = \hat{W}_t(\mathbf{y}, \mathbf{x}).
\]  

(8.9)

Furthermore, the random field \((\hat{W}_t(\mathbf{z}, \mathbf{z}))_{\mathbf{z} \in \mathbb{R}^\nu}\) can be modified such that the following map is continuous, for every \( \varpi \in \Omega \),

\[
\mathbb{R}^\nu \times \mathcal{F} \ni (\mathbf{z}, \psi) \longmapsto (\hat{W}_t(\mathbf{x}, \mathbf{z}))(\varpi)\psi \in \mathcal{F}.
\]  

(8.10)

Proof. For \( \mathbf{A} = 0 \), all assertions follow from [8, Theorem 9.2 and Lemma 10.2]. Assume without loss of generality that \( V = 0 \). Then

\[
S_t(\mathbf{x}) = -\lim_{n \to \infty} \sum_{\ell=1}^n \int_0^{\ell/n} A(\mathbf{B}_s^t\mathbf{x})d\mathbf{B}_s^t\mathbf{x} - \frac{i}{2} \int_0^{\ell/n} \text{div}A(\mathbf{B}_s^t\mathbf{x})ds,
\]  

(8.11)

Under the replacements \( \ell \to n - \ell + 1 \) we obviously obtain the complex conjugates of the approximating sums. Therefore,

\[
\overline{S}_t(\mathbf{x}) = -i \int_0^t A(\mathbf{B}_s^t\mathbf{x})d\mathbf{B}_s^t\mathbf{x} - \frac{i}{2} \int_0^t \text{div}A(\mathbf{B}_s^t\mathbf{x})ds,
\]  

(8.12)

where the stochastic integral on the right hand side is constructed with respect to the filtration \( (\hat{\mathcal{F}}_s)_{s \in [0,t]} \). Let \( \hat{S}_t(\mathbf{x}, \mathbf{y}) \) denote the random variable obtained upon putting \( \hat{b}^{t:y:x} \) in place of \( b^{t:y:x} \) on the right hand side of (8.7). Since \( \hat{B}_t^x \) solves (8.3) with the \( \hat{\mathcal{F}}_0 \)-measurable initial condition \( \hat{\mathbf{q}} := \hat{B}_0^x \) and since \( \mathbf{A} \in C_1^0(\mathbb{R}^\nu, \mathbb{R}^\nu) \), the random variable on the right hand side of (8.12) is \( \mathbb{P}\text{-a.s.} \) equal to \( \hat{S}_t(\mathbf{x}, \hat{\mathbf{q}}) \) (where the integrals are first computed along \( \hat{b}^{t:y:x} \), for each \( \mathbf{y} \in \mathbb{R}^\nu \), and \( \mathbf{y} = \hat{\mathbf{q}} \) is substituted afterwards). These remarks extend the first identity in (8.9) to non-vanishing \( \mathbf{A} \). The second identity in (8.9) can be proved, slightly more directly, in the same fashion. Finally, the last assertion extends to non-vanishing \( \mathbf{A} \in C_1^0(\mathbb{R}^\nu, \mathbb{R}^\nu) \) by standard properties of the stochastic integrals defining \( \hat{S}_t(\mathbf{x}, \mathbf{z}) \). \( \square \)

Next, we discuss a flow equation. For every \( r \geq 0 \), we set

\[
^r \mathbf{B}_t := \mathbf{B}_{t+r} - \mathbf{B}_r, \quad t \geq 0; \quad ^r \mathbf{B}^x := \mathbf{x} + ^r \mathbf{B}, \quad \mathbf{x} \in \mathbb{R}^\nu.
\]

so that \( ^r \mathbf{B} \) is a \( (\hat{\mathcal{F}}_{t+r})_{t \geq 0} \)-Brownian motion on the time-shifted filtered probability space \( (\Omega, \hat{\mathcal{F}}, (\hat{\mathcal{F}}_{t+r})_{t \geq 0}, \mathbb{P}) \). Denoting by \( (W_{r, r+t}(\mathbf{x}))_{t \geq 0} \) the process obtained upon putting \( ^r \mathbf{B} \) in place of \( \mathbf{B} \) in (8.5) and (8.6) and plugging the result into (1.16), we have the following result:

Theorem 8.5. By choosing a suitable version of the process \( (W_{r, r+t}(\mathbf{x}))_{t \geq 0} \), for each \( r \geq 0 \) and each \( \mathbf{x} \in \mathbb{R}^\nu \), we can achieve the following:

1. For all \( r \geq 0 \) and \( \varpi \in \Omega \), the following map is continuous,

\[
[r, \infty) \times \mathbb{R}^\nu \times \mathcal{F} \ni (t, \mathbf{x}, \psi) \longmapsto (W_{r, t}(\mathbf{x}))(\varpi)\psi \in \mathcal{F}.
\]

2. Fix \( r \geq 0 \) and \( \mathbf{x} \in \mathbb{R}^\nu \). Then \( W_{r, r}(\mathbf{x}) = \mathbb{1} \) and the following flow equations hold \( \mathbb{P}\text{-a.s.} \),

\[
W_{r, t}(\mathbf{x}) = W_{s, t}(^r \mathbf{B}^x_{s-r})W_{r, s}(\mathbf{x}), \quad t \geq s \geq r.
\]  

(8.13)
For all $t \geq r \geq 0$ and $x \in \mathbb{R}^\nu$, the random variable $W_{r,t}(x)$ is $\mathcal{F}_r$-independent.

**Proof.** For $A = 0$, all statements are contained in [8, Theorem 9.2]. By standard results on stochastic integrals they extend to non-vanishing $A$ in $C^1_b(\mathbb{R}^\nu, \mathbb{R}^\nu)$.

### 8.4. The Semigroup and Its Integral Kernel for Regular Coefficients

For all $\Psi \in L^2(\mathbb{R}^\nu, \mathcal{F})$, we abbreviate

$$
(T_t \Psi)(x) := \mathbb{E}[W_t(x)^* \Psi(\mathbf{B}^x_t)], \quad t \geq 0, \; x \in \mathbb{R}^\nu. \tag{8.14}
$$

In view of (1.20) this defines a bounded operator $T_t$ on $L^2(\mathbb{R}^\nu, \mathcal{F})$ satisfying

$$
\|T_t\| \leq 1, \quad t \geq 0. \tag{8.15}
$$

Recalling our notation (1.18) for the Euclidean heat kernel we further write

$$
T_t(x,y) := p_t(x,y)\mathbb{E}[W_t(x,y)], \quad t > 0, \; x,y \in \mathbb{R}^\nu; \tag{8.16}
$$

then recall Remark 1.3 concerning the existence of the $B(\mathcal{F})$-valued integral in (8.16).

**Proposition 8.6.** Let $t > 0$. Then

$$
(T_t \Psi)(x) = \int_{\mathbb{R}^\nu} T_t(x,y) \Psi(y)dy, \quad x \in \mathbb{R}^\nu, \tag{8.17}
$$

for all $\Psi \in L^2(\mathbb{R}^\nu, \mathcal{F})$, and

$$
T_t(x,y)^* = T_t(y,x), \quad x,y \in \mathbb{R}^\nu. \tag{8.18}
$$

In particular, $T_t$ is a bounded selfadjoint operator on $L^2(\mathbb{R}^\nu, \mathcal{F})$.

**Proof.** Let $t > 0$ and $x \in \mathbb{R}^\nu$. Combining (8.9) and (8.14) we find

$$
(T_t \Psi)(x) = \mathbb{E}[\hat{W}_t(x,B^x_t) \Psi(\mathbf{B}^x_t)] = \mathbb{E}[\mathbb{E}^{\mathbb{F}_0}[\hat{W}_t(x,B^x_t) \Psi(\mathbf{B}^x_t)]], \tag{8.19}
$$

where we also used the tower property of conditional expectations in the second equality. By definition of the reversed filtration $(\tilde{\mathbb{F}}_s)_{s \geq 0}$, the random functions $B^x_t$ and, hence, $\Psi(B^x_t)$ are $\tilde{\mathbb{F}}_0$-measurable. Furthermore, $\hat{W}_t(x,y)$ is $\tilde{\mathbb{F}}_0$-independent, as this is the case for the increments of solutions to (8.3) with a constant initial condition $\hat{q} = y$. In view of the continuity result stated in Theorem 8.4 we may thus apply the computation rule for conditional expectations of Example A.2 to the rightmost member in (8.19). This entails the first equality in

$$
(T_t \Psi)(x) = \mathbb{E}[\mathbb{E}[\hat{W}_t(x,y)]_{y=B^x_t} \Psi(B^x_t)] = \int_{\mathbb{R}^\nu} p_t(x,y)\mathbb{E}[\hat{W}_t(x,y)]\Psi(y)dy.
$$

In the second one we just used that the law of $B^x_t$ has density $p_t(x, \cdot)$. Since $\hat{W}_t(x,y)$ has the same law as $W_t(x,y)$, we arrive at (8.17).

The identity (8.18) follows from the second relation in (8.9) since the random functions $\hat{W}_t(y,x)$ and $W_t(y,x)$ have the same law.

In the next proposition we again use the notation introduced prior to Theorem 8.5:
Proposition 8.7. Let \( x \in \mathbb{R}^\nu \) and \( \Psi \in L^2(\mathbb{R}^\nu, \mathcal{F}) \). Then the following Markov property holds, for all \( t \geq s \geq r \geq 0 \),

\[
\mathbb{E}^{\tilde{\mathcal{F}}} [W_{r,t}(x)^* \Psi (rB^x_{t-r})] = W_{r,s}(x)^* (T_{t-s}\Psi)(rB^x_{s-r}), \quad \mathbb{P}\text{-a.s.} \tag{8.20}
\]

In particular, for all \( s, t \geq 0 \),

\[
(T_{s+t}\Psi)(x) = (T_s(T_t\Psi))(x). \tag{8.21}
\]

Proof. Since taking the adjoint is continuous on \( \mathcal{B}(\mathcal{F}) \), the map \( W_{u,v}(y)^*: \Omega \rightarrow \mathcal{B}(\mathcal{F}) \) is again measurable and separably valued, for all \( v \geq u \geq 0 \) and \( y \in \mathbb{R}^\nu \). Furthermore, \( W_{r,s}(x)^* \) is \( \tilde{\mathcal{F}}_s \)-measurable and \( W_{s,t}(y)^* \) is \( \tilde{\mathcal{F}}_s \)-independent, for all \( y \in \mathbb{R}^\nu \) by Theorem 8.5(3). The Markov property (8.20) thus follows from Parts (1) and (2) of Theorem 8.5 in conjunction with Example A.2. Taking the expectation of (8.20) with \( r = 0 \) we further obtain (8.21). \( \square \)

8.5. Feynman–Kac Formulas on \( \mathbb{R}^\nu \) for Regular Coefficients

In this subsection we shall often use the shorthand

\[ \theta := 1 + d\Gamma(\omega), \]

and abbreviate

\[ \hat{H}(x)\psi := \frac{1}{2} \varphi(G_x)^2 \psi - \frac{i}{2} \varphi(\text{div}G_x)\psi + d\Gamma(\omega)\psi, \quad \psi \in \mathcal{D}(d\Gamma(\omega)), x \in \mathbb{R}^\nu. \]

Lemma 8.8. Let \( x \in \mathbb{R}^\nu \), \( f \in C^2_b(\mathbb{R}^\nu, \mathbb{R}) \), and \( \psi \in \mathcal{D}(d\Gamma(\omega)) \). Then

\[ \mathcal{M}_*(x) := \int_0^\cdot \left( i(fA)(B^x_s) + f(B^x_s)i \varphi(G_{B^x_s}) + (\nabla f)(B^x_s) \right) W_s(x)\psi dB_s, \]

defines a continuous \( \mathcal{F} \)-valued \( L^2 \)-martingale \( \mathcal{M}(x) \) on \( [0, \infty) \) and, \( \mathbb{P}\)-a.s.,

\[
\begin{align*}
&f(B^x_t)W_t(x)\psi - f(x)\psi \\
&\quad = \int_0^t \left( \frac{1}{2}(\nabla + iA)^2 f - Vf \right) (B^x_s) - f(B^x_s)\hat{H}(B^x_s) \right) W_s(x)\psi ds \\
&\quad \quad + \int_0^t (i\nabla f - fA)(B^x_s) \cdot \varphi(G_{B^x_s}) W_s(x)\psi ds + \mathcal{M}_t(x), \quad t \geq 0.
\end{align*} \tag{8.23}
\]

Proof. According to [8, Lemma 7.6] there exists a monotone increasing function \( c: [0, \infty) \rightarrow (0, \infty) \) such that

\[
\sup_{z \in \mathbb{R}^\nu} \mathbb{E}\left[ \sup_{s \in [0,t]} \|\theta W_s(z)\psi\|^2 \right] \leq c(t)\|\theta \psi\|^2, \quad t \geq 0. \tag{8.24}
\]

In view of (1.20), (2.2), and (8.24) the integrand of the stochastic integral defining \( \mathcal{M}(x) \), call it \( (Y_s)_{s \geq 0} \), is a continuous adapted \( \mathcal{F} \)-valued stochastic process satisfying

\[
\mathbb{E}\|Y_s\|^2 \leq 3\{ \|A\infty\|_\infty^2 \|f\infty\|^2 \|\psi\|^2 + \|\nabla f\|^2 \|\psi\|^2 + c(s)\|\theta \psi\|^2 \},
\]
for all $s \geq 0$, where $c : [0, \infty) \to (0, \infty)$ is another monotone increasing function. Consequently, $\mathcal{M}(x)$ is a continuous $\mathcal{F}$-valued $L^2$-martingale.

Put $W_t^0(x) := \Gamma(j_1)^*e^{\psi(K_0(x))}\Gamma(j_0)$; compare this with (1.16). Thanks to [8, Theorem 5.3] we know that $(W_t^0(x)\psi)_{t \geq 0}$ is a $\mathcal{F}$-valued semimartingale whose paths $\mathbb{P}$-a.s. are continuous $\mathcal{D}(d\Gamma(\omega))$-valued functions and, $\mathbb{P}$-a.s.

$$W_t^0(x)\psi = \psi - \int_0^t \hat{H}(B^x_s)W_s^0(x)\psi ds + \int_0^t i\varphi(G_{B^x_s})W_s^0(x)\psi d\mathcal{B}_s,$$

(8.25)

for all $t \geq 0$. Thus, (8.23) follows from (8.25) and Itô’s formula. \hfill $\square$

Lemma 8.9. There exists $c > 0$ such that, for all $\psi \in \mathcal{F}$,

$$\sup_{x \in \mathbb{R}^\nu} \mathbb{E} \left[ \sup_{s \in [0,t]} \|\theta^{-1/2}(W_s(x) - 1)\psi\|^2 \right] \leq ct\|\psi\|^2, \quad t \geq 0. \quad (8.26)$$

Proof. Abbreviate $\psi_t := (W_t(x) - 1)\psi$, so that $\psi_0 = 0$. We may assume that $\psi \in \mathcal{D}(d\Gamma(\omega))$. (Otherwise approximate $\psi$ by the vectors $(1 + d\Gamma(\omega)/n)^{-1}\psi$, $n \in \mathbb{N}$, and take (1.20) into account.) We may also assume $\|\psi\| = 1$. In virtue of (8.23) with $f = 1$ and Itô’s formula, we $\mathbb{P}$-a.s. obtain, for all $t \geq 0$,

$$\|\theta^{-1/2}\psi_t\|^2 = -\int_0^t 2\text{Re}\langle\psi_s, \theta^{-1}(-\frac{1}{2}A^2 + \hat{H})(B^x_s)W_s(x)\psi\rangle ds$$

$$-\int_0^t 2\text{Re}\langle\psi_s, \theta^{-1}A(B^x_s) \cdot \varphi(G_{B^x_s})W_s(x)\psi\rangle ds$$

$$+ \int_0^t \|\theta^{-1/2}(A(B^x_s) + \varphi(G_{B^x_s}))W_s(x)\psi\|^2 ds$$

$$+ \int_0^t 2\text{Re}\langle\theta^{-1/2}\psi_s, i\theta^{-1/2}(A(B^x_s) + \varphi(G_{B^x_s}))W_s(x)\psi\rangle d\mathcal{B}_s.$$

(8.27)

On account of (2.2), (2.3), (8.1), and Hypothesis 8.1, the operators

$$\theta^{-1}(-\frac{1}{2}A^2 + \hat{H})(y), \quad \theta^{-1}A(y) \cdot \varphi(G_y), \quad \theta^{-1/2}(A(y) + \varphi(G_y)),$$

appearing here are well-defined on $\mathcal{D}(d\Gamma(\omega))$ and bounded uniformly in $y \in \mathbb{R}^\nu$. Furthermore, we have the pointwise bound $\|\psi_t\| \leq 2$, $t \geq 0$. From these remarks we infer in particular that the stochastic integral in the last line of (8.27), call it $\mathcal{M}$, is a martingale to which the Burkholder-Davis-Gundy inequality applies,

$$\mathbb{E} \left[ \sup_{s \leq t} |\mathcal{M}_s| \right] \leq c_0 \mathbb{E} \left[ \langle \mathcal{M} \rangle_t^{1/2} \right], \quad t \geq 0,$$

for some universal constant $c_0 > 0$. According to the above remarks the quadratic variation of $\mathcal{M}$ satisfies, however,

$$\langle \mathcal{M} \rangle_t = \int_0^t (2\text{Re}\langle\theta^{-1/2}\psi_s, i\theta^{-1/2}(A(B^x_s) + \varphi(G_{B^x_s}))W_s(x)\psi\rangle)^2 ds$$
Proposition 8.10. \((T_t)_{t \geq 0}\) is a strongly continuous semigroup of bounded self-adjoint operators on \(L^2(\mathbb{R}^\nu, \mathcal{F})\).

Proof. Boundedness and selfadjointness have already been observed in (8.15) and Proposition 8.6. In view of (8.15) it only remains to show that \(T_t \Psi \to \Psi\), as \(t \downarrow 0\), for all \(\Psi \in \mathcal{F}\) with \(||\theta^{1/2}\Psi\|_\mathcal{F} \in L^2(\mathbb{R}^\nu)\). (Vectors \(\Psi\) of the latter kind are dense in \(\mathcal{F}\).) For every such \(\Psi\), the convergence \(T_t \Psi \to \Psi\) follows, however, from an estimation which is virtually identical to the one in the proof of [8, Lemma 10.11]. Let us nevertheless repeat it here to demonstrate where and how Lemma 8.9 is used:

\[
\|\langle(T_t - I)\Psi\rangle\|^2 \\
= \int_{\mathbb{R}^\nu} \sup_{\phi \in \mathcal{F}:\|\phi\|=1} \left\| \phi \mathbb{E}\left[(W_t(x)^* - 1)\Psi(B_t^x)\right] \right\|^2 dx \\
= \int_{\mathbb{R}^\nu} \sup_{\phi \in \mathcal{F}:\|\phi\|=1} \left\{ \left\| \theta^{-1/2}(W_t(x) - 1)\phi \right\| \left\| \theta^{1/2}\Psi(B_t^x) \right\|^2 \right\} dx \\
\leq \sup_{y \in \mathbb{R}^\nu} \sup_{\phi \in \mathcal{F}:\|\phi\|=1} \mathbb{E}\left[ \left\| \theta^{-1/2}(W_s(y) - 1)\phi \right\|^2 \right] \int_{\mathbb{R}^\nu} \mathbb{E}\left[ \left\| \theta^{1/2}\Psi(B_t^x) \right\|^2 \right] dx,
\]

where the double supremum of the first expectation in the last line is \(\leq ct\) by Lemma 8.9 and the \(dx\)-integral in the same line is \(\leq \|\theta^{1/2}\Psi\|^2\). \(\square\)

Proposition 8.11. Let \(\Psi \in L^2(\mathbb{R}^\nu, \mathcal{F})\), \(t > 0\), and \(H := H_{\mathbb{R}^\nu}\). Then

\[
e^{-tH} \Psi = T_t \Psi.
\]  

(8.29)

Proof. We pick \(f \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R}), \psi \in \mathcal{D}(d\Gamma(\omega))\), scalar-multiply (8.23) with \(\phi \in \mathcal{D}(d\Gamma(\omega))\), and use the fact that \(\langle \phi, \mathcal{M}(x) \rangle\) is a martingale starting at zero to get

\[
\langle (T_t(f \phi))(x) | \psi \rangle - \langle f(x) \phi | \psi \rangle + t \langle (H(f \phi))(x) | \psi \rangle \\
= \int_0^t \mathbb{E}\left[ \left\langle (H(f \phi))(x) - W_s(x)^* (H(f \phi))(B_s^x) | \psi \right\rangle \right] ds =: I_\psi(t, x),
\]

for all \(t \geq 0\) and \(x \in \mathbb{R}^\nu\). Here

\[
\frac{1}{t^2} \int_{\mathbb{R}^\nu} \sup_{\psi \in \mathcal{D}(d\Gamma(\omega)):\|\psi\|=1} |I_\psi(t, x)|^2 dx \leq \frac{1}{t} \int_0^t \| (T_s - I) H(f \phi) \|^2 ds - t \to 0,
\]

as \(t \to 0\).
because \((T_s)_{s \geq 0}\) is strongly continuous. This shows that
\[
\frac{1}{t} (T_t(f) - f) \xrightarrow{t \to 0} H(f) \quad \text{in } L^2(\mathbb{R}', \mathcal{F}).
\]

Hence, \(\mathcal{D}(\mathbb{R}', D(d\Gamma(\omega)))\) is contained in the domain of the selfadjoint generator of \((T_t)_{t \geq 0}\) and the restriction of this generator to \(\mathcal{D}(\mathbb{R}', D(d\Gamma(\omega)))\) is equal to \(H|_{\mathcal{D}(\mathbb{R}', D(d\Gamma(\omega)))}\). Since \(H\) is essentially selfadjoint on \(\mathcal{D}(\mathbb{R}', D(d\Gamma(\omega)))\) (see, e.g., [27, Theorem 5.5]), this implies that \((T_t)_{t \geq 0}\) is generated by \(H\).

\(\square\)

### 9. Feynman–Kac Formulas for Singular Coefficients

In the first two subsections of this final section we give a precise meaning to all stochastic integrals appearing in the formulas for our Feynman–Kac integrands and observe a useful dominated convergence theorem for a particular class of stochastic integrals. After that we prove our main theorem for the special choice \(\Lambda = \mathbb{R}'\) and continuous, bounded \(V\) in Sect. 9.3. Ultimately, we obtain the theorem in full generality in Sect. 9.4, employing the results of Sect. 3 as well as an additional idea from [34]. Corollary 1.4 is proved in Sect. 9.4, too.

#### 9.1. Existence and Convergence of Path Integrals

Let \(\mathbb{K}\) be a separable real or complex Hilbert space and
\[
f \in L^2_{\text{loc}}(\mathbb{R}', \mathcal{H}').
\]

More precisely, we assume that a representative of \(f\) has been chosen so that \(f: \mathbb{R}' \to \mathbb{K}'\) is Borel measurable. Furthermore, we suppose that \(\mathbb{R} \ni s \mapsto J_s \in \mathcal{B}(\mathbb{K}', \hat{\mathbb{K}})\) is a strongly continuous family of isometries from \(\mathbb{K}\) into another separable Hilbert space \(\hat{\mathbb{K}}\). Relevant examples are \(j_s: \mathbb{K} \to \hat{\mathbb{K}}\) and \(\text{id}_{\mathbb{R}}: \mathbb{R} \to \mathbb{R}\). Finally, we fix \(t > 0\).

**Lemma 9.1.** There exist Borel zero sets \(N \subset \mathbb{R}'\) and \(N' \subset \mathbb{R}^{2}\) such that the two stochastic integral processes
\[
\left( \int_0^\tau J_s f(B_s^x) dB_s \right)_{\tau \in [0,t]}, \quad \left( \int_0^\tau J_{t-s} f(B_s^{t:x}) d\hat{B}_s^{t:x} \right)_{\tau \in [0,t]},
\]
are well-defined semimartingales, for all \(x \in \mathbb{R}' \setminus N\), and
\[
\left( \int_0^\tau J_s f(b_s^{t:y,x}) d\hat{b}_s^{t:y,x} \right)_{\tau \in [0,t]}, \quad \left( \int_0^\tau J_{t-s} f(\hat{b}_s^{t:y,x}) d\hat{b}_s^{t:y,x} \right)_{\tau \in [0,t]},
\]
are well-defined semimartingales for all \((x, y) \in \mathbb{R}^{2} \setminus N'\). The zero sets \(N\) and \(N'\) can be chosen independently of the choice of representative of \(f\). If this has been done, then, for all \(x \in \mathbb{R}' \setminus N\) and \((x, y) \in \mathbb{R}^{2} \setminus N'\), the semimartingales in (9.1) and (9.2), respectively, change only up to indistinguishability, if we pick another representative of \(f\).
Notice that the first integral processes in (9.1) and (9.2) are defined and semimartingales with respect to the filtration \((\mathcal{F}_s)_{s \in [0,t]}\), while the second one in (9.1) is constructed using \((\mathcal{H}_s)_{s \in [0,t]}\) and the second one in (9.2) by means of \((\mathcal{H}_s)_{s \in [0,t]}\).

**Proof.** As we neither specify \((J_s)_{s \in [0,t]}\), \((\mathcal{H}_s)_{s \in [0,t]}\), nor \(B\), we may ignore the second process in (9.2) in this proof.

Taking the strong continuity of \(s \mapsto J_s\) into account we first observe that all integrands in (9.1) and (9.2) are predictable with respect to the corresponding filtrations. In view of the stochastic differential equations solved by \(B^{t,x}\) and \(b^{t,y,x}\), we further have

\[
\int_0^\tau J_{t-s} f(B^{t,x}_s) dB^{t,x}_s = \int_0^\tau J_{t-s} f(B^{x}_{t-s}) dB_s - \int_0^\tau J_{t-s} f(B^{x}_{t-s}) \cdot \frac{B_{t-s}}{t-s} ds,
\]

and

\[
\int_0^\tau J_s f(b^{t,y,x}_s) db^{t,y,x}_s = \int_0^\tau J_s f(b^{y,x}_s) dB_s + \int_0^\tau J_s f(b^{y,x}_s) \cdot \frac{x - b^{y,x}_s}{t-s} ds,
\]

for all \(\tau \in [0,t]\). By the standard criterion for the existence of stochastic integrals along Brownian motions (see, e.g., [4, ß4.2]), the \(dB\)- and \(dB\)-integrals in the previous two formulas and the \(dB\)-integral to the left in (9.1) are well-defined, if

\[
P \left( \int_0^t \|J_s f(B_s^x)\|^2_{\mathcal{H}^\nu} ds < \infty \right) = P \left( \int_0^t \|J_{t-s} f(B^x_{t-s})\|^2_{\mathcal{H}^\nu} ds < \infty \right) = 1, \tag{9.3}
\]

\[
P \left( \int_0^t \|J_s f(b^{y,x}_s)\|^2_{\mathcal{H}^\nu} ds < \infty \right) = 1. \tag{9.4}
\]

Furthermore, the pathwise defined Bochner-Lebesgue integrals in the above two formulas exist and define processes having pathwise finite variation on \([0,t]\), \(P\)-a.s. at least, provided that

\[
P \left( \int_0^t \|J_s f(B_s^x)\|_{\mathcal{H}^\nu} \left| \frac{B_s}{s} \right| ds < \infty \right) = 1, \tag{9.5}
\]

\[
P \left( \int_0^t \|J_s f(b^{y,x}_s)\|_{\mathcal{H}^\nu} \left| \frac{x - b^{y,x}_s}{t-s} \right| ds < \infty \right) = 1. \tag{9.6}
\]

To verify (9.3) through (9.6), we may obviously ignore the isometries \(J_s\). Since \(\|f\|^2 := \|f\|^2_{\mathcal{H}^\nu}\) is locally integrable on \(\mathbb{R}^\nu\), it follows from [6, Lemma 2] that (9.3) is satisfied for a.e. \(x\). We shall, however, re-obtain this result in the following arguments which elaborate on the ones in [6].
Set $C_n := \{ |x| \leq n \}, \ n \in \mathbb{N}$. Then a weighted Cauchy-Schwarz inequality yields
\[
\mathbb{E} \left[ \int_0^t \| (1_{C_n} f)(B^x_s) \| \frac{|B^x_s|}{s} \, ds \right] 
\leq \mathbb{E} \left[ \int_0^t \frac{\| (1_{C_n} f)(B^x_s) \|^2}{s^{1/2}} \, ds \right]^{1/2} \mathbb{E} \left[ \int_0^t \| B^x_s \|^2 \, ds \right]^{1/2},
\]
where the rightmost expectation is a finite $(t, \nu)$-dependent constant and
\[
\int_{\mathbb{R}^\nu} \mathbb{E} \left[ \int_0^t \frac{\| (1_{C_n} f)(B^x_s) \|^2}{(s \wedge 1)^{1/2}} \, ds \right] \, dx 
\leq \int_0^t \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} p_s(x, y) \frac{\| (1_{C_n} f)(y) \|^2}{s^{1/2}} \, dx \, dy \, ds 
= 2t^{1/2} \| f \|_{L^2(C_n, \mathcal{X}^\nu)}^2.
\]
Therefore, we find Borel zero sets $N_n \subset \mathbb{R}^\nu$ such that
\[
\mathbb{E} \left[ \int_0^t \| (1_{C_n} f)(B^x_s) \|^2 \, ds \right] + \mathbb{E} \left[ \int_0^t \| (1_{C_n} f)(B^x_s) \| \frac{|B^x_s|}{s} \, ds \right] < \infty, \tag{9.8}
\]
for all $x \in \mathbb{R}^\nu \setminus N_n$. Since the expectation in the first line of (9.7) does not change when we pass to another representative of $f$, we can pick each $N_n$ independently of the choice of representative of $f$. We set $N := \bigcup_{n=1}^\infty N_n$. Since every path of the continuous process $(B^x_s)_{s \in [0,t]}$ must be contained in some $C_n$, it readily follows that (9.3) and (9.5) are satisfied for all $x \in \mathbb{R}^\nu \setminus N$.

Next, we define
\[
c_n := \sup_{x, y \in C_n} \frac{1}{p_t(x, y)} = \frac{1}{(2\pi t)^{\nu/2} e^{2n^2/t}}, \ n \in \mathbb{N},
\]
and recall that, for all $s \in (0, t)$, the law of $b^{t, u, x}_s$ is given by
\[
L_{s; y, x}(z) := \frac{p_s(y, z) p_{t-s}(z, x)}{p_t(x, y)}, \ z \in \mathbb{R}^\nu.
\]
Applying Fubini’s theorem we find
\[
\int_{C_n} \int_{C_n} \mathbb{E} \left[ \int_0^t ((t - s) \wedge 1)^{-1/2} \| (1_{C_n} f)(b^{t, u, x}_s) \| \, ds \right] \, dx \, dy 
\leq \int_0^t (t - s)^{-1/2} \int_{C_n} \int_{C_n} \int_{C_n} L_{s; y, x}(z) \| f(z) \|^2 \, dz \, dx \, dy \, ds 
\leq c_n \int_0^t (t - s)^{-1/2} \int_{C_n} \int_{\mathbb{R}^\nu} \int_{\mathbb{R}^\nu} p_s(y, z) p_{t-s}(z, x) \| f(z) \|^2_{\mathcal{X}^\nu} \, dx \, dy \, dz \, ds
= 2c_n t^{1/2} \| f \|_{L^2(C_n, \mathcal{X}^\nu)}^2 < \infty, \ n \in \mathbb{N}. \tag{9.9}
\]
Also employing the bound (see, e.g., [8, Lemma 15.2])
\[
\mathbb{E} \left[ \frac{|x - b^{t, u, x}_s|}{t - s} \right]^2 
\leq c_{\nu, t} \frac{1 + |x - y|}{(t - s) \wedge 1}, \ s \in (0, t),
\]
we thus find zero sets $N'_n \subset C_n \times C_n$ such that
\[
\mathbb{E}
\left[
\int_0^t \left\| (1_{C_n} f)(b_{s:y,x}^t) \right\| \frac{\|x - b_{s:y,x}^t\|}{t - s} \, ds
\right]
\leq \mathbb{E}
\left[
\int_0^t \left\| (1_{C_n} f)(b_{s:y,x}^t) \right\|^2 \, ds
\right]^{1/2}
\left(
\int_0^t (t - s)^{1/2} \mathbb{E}
\left[
\frac{\|x - b_{s:y,x}^t\|^2}{t - s}
\right] \, ds
\right)^{1/2}
< \infty,
\]
for all $(x, y) \in (C_n \times C_n) \setminus N'_n$ and $n \in \mathbb{N}$. Since $b_{s:y,x}^t$ is continuous, we conclude that (9.4) and (9.6) are satisfied for all $(x, y) \in \mathbb{R}^{2v} \setminus N'$ with $N' := \bigcup_{n=1}^{\infty} N'_n$. Again we can pick each $N'_n$ independently of the representative of $f$, since all representatives lead to the same integrand under the $(dx \, dy)$-integration in the first line of (9.9).

The last assertion is an easy consequence of Itô’s isometry for the $dB$- and $dB$-integrals, the continuity of stochastic integral processes, the isometry of $J$, and the fact that the laws of $B_s^x$ and $b_{s:y,x}^t$ with $s \in (0, t)$ are absolutely continuous with respect to the Lebesgue measure. □

We continue with a particular case of the dominated convergence theorem for stochastic integrals:

**Theorem 9.2.** Let $f^n \in L^2_{\text{loc}}(\mathbb{R}^v, \mathscr{X}^v)$, $n \in \mathbb{N} \cup \{\infty\}$, and $\alpha \in L^2_{\text{loc}}(\mathbb{R}^v)$. As a consequence of Lemma 9.1 we find Borel zero sets $N \subset \mathbb{R}^v$ and $N' \subset \mathbb{R}^{2v}$ such that all processes in (9.1) and (9.2) are well-defined, for $x \in \mathbb{R}^{2v} \setminus N$ and $(x, y) \in \mathbb{R}^{2v} \setminus N'$, respectively, when any pair $(J, f^n)$ with $n \in \mathbb{N} \cup \{\infty\}$ or $(\text{id}_R, \alpha)$ is put in place of $(J, f)$. Now, let $(I^n_{\tau})_{\tau \in [0, t]}$ be any of the processes in (9.1) or (9.2) defined by means of $(J, f^n)$ for some permitted value of $x$ (resp. $(x, y)$) on let $(I^\infty_{\tau})_{\tau \in [0, t]}$ denote the corresponding process defined by means of $(J, f^n)$. Assume that $\|f^n\|_{\mathscr{X}^v} \leq \alpha$ a.e. on $\mathbb{R}^v$, for each $n \in \mathbb{N}$, and $f^n \to f^\infty$ a.e. on $\mathbb{R}^v$, as $n \to \infty$. Then
\[
\sup_{\tau \in [0, t]} \|I^n_{\tau} - I^\infty_{\tau}\|_{\mathscr{X}^v} \xrightarrow{n \to \infty} 0 \quad \text{in probability.} \tag{9.10}
\]

**Proof.** By the last assertion in Lemma 9.1 we do not lose generality by assuming the bounds $\|J_s f^n\|_{\mathscr{X}^v} = \|f^n\|_{\mathscr{X}^v} \leq \alpha$, $n \in \mathbb{N}$, and the convergence $J_s f^n \to J_s f^\infty$ to hold everywhere on $\mathbb{R}^v$. If we do so, then (9.10) follows from the first assertion in Lemma 9.1 and the dominated convergence theorem for stochastic integrals; see, e.g., [29, Theorem 26.3] and the complementing remarks in the proof of [8, Theorem 2.13]. □

We shall apply the preceding theorem in conjunction with the following, presumably well-known observation, whose proof we include for the reader’s convenience:

**Lemma 9.3.** Let $f^n \in L^2_{\text{loc}}(\mathbb{R}^v, \mathscr{X}^v)$, $n \in \mathbb{N} \cup \{\infty\}$ and assume that $f^n \to f^\infty$ in $L^2_{\text{loc}}(\mathbb{R}^v, \mathscr{X}^v)$, as $n \to \infty$. Then there exist integers $1 \leq m_1 < m_2 < \ldots$ and some nonnegative $\alpha \in L^2_{\text{loc}}(\mathbb{R}^v)$ such that $\|f^{m_\ell}\|_{\mathscr{X}^v} \leq \alpha$ a.e. on $\mathbb{R}^v$, for each $\ell \in \mathbb{N}$, and $f^{m_\ell} \to f^\infty$ a.e. on $\mathbb{R}^v$, as $\ell \to \infty$. 

Proof. Let \( r \in \mathbb{N}_0 \) and abbreviate \( \mathcal{S}_r := \{ r < |x| \leq r + 1 \} \), if \( r \geq 1 \), and \( \mathcal{S}_0 := \{ |x| \leq 1 \} \). Then, given any subsequence of \( \{ f^n \}_{n \in \mathbb{N}} \), call it \( \{ f^{n_{r-1}, \ell} \}_{\ell \in \mathbb{N}} \), we can single out another subsequence, call it \( \{ f^{n_r, \ell} \}_{\ell \in \mathbb{N}} \), such that \( f^{n_{r}, \ell} \to f^\infty \) a.e. on \( \mathcal{S}_r \) as \( \ell \to \infty \). Furthermore, we find a dominating function \( \alpha_r \in L^2(\mathcal{S}_r) \) such that \( \| f^{n_r, \ell} \|_{\mathcal{S}_r} \leq \alpha_r \) a.e. on \( \mathcal{S}_r \), for each \( \ell \in \mathbb{N} \). (These assertions, including the existence of the dominating function, follow from the Riesz-Fischer theorem for \( L^2(\mathcal{S}_r, \mathcal{H}^\nu) \).) We employ this remark inductively with \( n_{0-1, \ell} := \ell \) and define \( \alpha := \sum_{r=0}^\infty \alpha_r \), where every \( \alpha_r \) is extended to a function on \( \mathbb{R}^\nu \) by setting it equal to 0 outside \( \mathcal{S}_r \). Then \( \alpha \in L^2_{\text{loc}}(\mathbb{R}^\nu) \) and the diagonal sequence \( \{ f^{m, \ell} \}_{\ell \in \mathbb{N}} := \{ f^{n_r, \ell} \}_{\ell \in \mathbb{N}} \) has all desired properties. \( \square \)

### 9.2. The Feynman–Kac Integrand for Singular Vector Potentials

Next, we explain how the observations of the preceding subsection can be used to make sense out of the stochastic integrals in (1.11), (1.12), (1.14), and (1.15), although \( A \) and \( G \) satisfying (1.6) and (1.9), respectively, might not have locally square-integrable extensions to the whole \( \mathbb{R}^\nu \).

Let \( \Lambda_n \subset \Lambda \) be open, proper subsets exhausting \( \Lambda \) in the sense that \( \overline{\Lambda_n} \subset \Lambda_{n+1} \) for all \( n \in \mathbb{N} \) and \( \bigcup_{n=1}^\infty \Lambda_n = \Lambda \). Then \( \Lambda_n A \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu) \) and \( \Lambda_n G \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu) \), after \( A \) and \( G \) have been extended to functions on \( \mathbb{R}^\nu \) by setting them equal to zero outside \( \Lambda \).

Let \( t > 0 \). According to the remarks in Sect. 9.1 we may pick zero sets \( N \subset \mathbb{R}^\nu \) and \( N' \subset \mathbb{R}^{2\nu} \) such that, for all \( x \in \mathbb{R}^\nu \setminus N \) and \( (x, y) \in \mathbb{R}^{2\nu} \), respectively, we obtain linear combinations of well-defined, \( \mathbb{P} \)-a.s. uniquely determined stochastic integrals,

\[
K^n_t(x) := \frac{1}{2} \int_0^t \hat{j}_s(1_{\Lambda_n} G) B^x_s \, dB_s^x - \frac{1}{2} \int_0^t j_{t-s}(1_{\Lambda_n} G) B^{t,x}_s \, dB^{t,x}_s ,
\]

\[
K^n_t(x, y) := \frac{1}{2} \int_0^t \hat{j}_s(1_{\Lambda_n} G) b^{t,x,y}_s \, dB^{t,x,y}_s - \frac{1}{2} \int_0^t j_{t-s}(1_{\Lambda_n} G) b^{t,x,y}_s \, dB^{t,x,y}_s ,
\]

for every \( n \in \mathbb{N} \). From the pathwise uniqueness property of stochastic integrals (see, e.g., Kor. 1 on page 188 of [9], whose proof extends to the Hilbert space-valued setting) we infer that, for all natural numbers \( m, n \) with \( m > n \),

\[
K^n_m(x) = K^m_t(x), \quad \mathbb{P} \text{-a.s. on } \{ \tau_{\Lambda_n}(x) > t \},
\]

as well as

\[
K^n(x, y) = K^m_t(x, y), \quad \mathbb{P} \text{-a.s. on } \{ \tau_{\Lambda_n}(t; y, x) = \infty \}.
\]

Modulo changes on \( \mathbb{P} \)-zero sets, we thus obtain well-defined random functions \( K_t(x) \) and \( K_t(x, y) \) defined on \( \{ \tau_{\Lambda}(x) > t \} = \bigcup_{n=1}^\infty \{ \tau_{\Lambda_n}(x) > t \} \) and \( \{ \tau_{\Lambda}(t; y, x) = \infty \} = \bigcup_{n=1}^\infty \{ \tau_{\Lambda_n}(t; y, x) = \infty \} \), respectively, by setting

\[
K_t(x) := K^n_t(x) \quad \text{on } \{ \tau_{\Lambda_n}(x) > t \}, \quad \text{(9.11)}
\]

\[
K_t(x, y) := K^n_t(x, y) \quad \text{on } \{ \tau_{\Lambda_n}(t; y, x) = \infty \}, \quad \text{(9.12)}
\]

for all \( n \in \mathbb{N} \). It is routine to check the independence of these definitions of the choice of the exhausting sequence of open proper subsets \( \{ \Lambda_n \}_{n \in \mathbb{N}} \).
This gives a precise meaning to the random functions in (1.12) and (1.15). Quite obviously, they are indeed differences of two stochastic integrals individually defined in the above fashion.

The stochastic integrals in (1.11) and (1.14) are defined in complete analogy; just replace \( t \) by \( \mathbb{R} \) and ignore the isometries \( j_n \) in the above construction. Furthermore, it is well-known (see [6, Lemma 2] and the estimations (9.7) and (9.9)) that the path integrals of \( V \) in (1.11) and (1.14) are well-defined for a.e. \( x \) and a.e. \( (x, y) \), respectively.

Altogether, this gives a clear, canonical meaning to all terms in the Feynman–Kac integrands in (1.16) and (1.17), which in the notation for the Weyl representation introduced in Sect. 2.1 read

\[
W_t(x)^* = e^{-S_t(x)} \Gamma(j_t^*) \mathcal{W}(-iK_t(x)) \Gamma(j_t), \tag{9.13}
\]

\[
W_t(x, y) = e^{-S_t(x, y)} \Gamma(j_t^*) \mathcal{W}(iK_t(x, y)) \Gamma(j_0). \tag{9.14}
\]

### 9.3. Feynman–Kac Formulas for Singular Vector Potentials and \( \Lambda = \mathbb{R}^\nu \)

In the next proof we shall work with the formulas (9.13) and (9.14), exploiting that

\[
\mathfrak{t} \ni f \mapsto \Gamma(j_t^*) \mathcal{W}(f) \Gamma(j_t) \text{ is strongly continuous,} \tag{9.15}
\]

\[
\| \Gamma(j_t^*) \mathcal{W}(f) \Gamma(j_t) \|_{\mathcal{B}(\mathcal{F})} \leq 1, \quad f \in \mathfrak{t}, \tag{9.16}
\]

for all \( s, t \geq 0 \). These two statements follow from the remarks in Sect. 2.1.

**Proposition 9.4.** Let \( \mathbf{A} \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu) \), \( \mathbf{G} \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu) \), and let \( V \geq 0 \) be in \( C_0(\mathbb{R}^\nu, \mathbb{R}) \). Pick some \( t > 0 \) and \( \Psi \in L^2(\mathbb{R}^\nu, \mathcal{F}) \). Then

\[
(e^{-tH}\Psi)(x) = \mathbb{E}[W_t(x)^*\Psi(B_t^x)], \quad \text{a.e. } x \in \mathbb{R}^\nu. \tag{9.17}
\]

Furthermore,

\[
(e^{-tH}\Psi)(x) = \int_{\mathbb{R}^\nu} p_t(x, y) \mathbb{E}[W_t(x, y)\Psi(y)] \, dy, \quad \text{a.e. } x \in \mathbb{R}^\nu. \tag{9.18}
\]

In (9.17) and (9.18) we again drop the subscript \( \mathbb{R}^\nu \) in the notation for Pauli–Fierz operators on \( \mathbb{R}^\nu \); recall the remarks preceding Theorem 7.1. The completely real subspace \( \mathfrak{t}_\mathbb{R} \subset \mathfrak{t} \) has the properties mentioned below (1.9).

**Proof. Step 1: Construction of approximating vector potentials.** Define the standard mollifier \( \rho_n \) as in (4.1) and (4.2). Pick some \( \chi \in C^\infty(\mathbb{R}, \mathbb{R}) \) with \( 0 \leq \chi \leq 1 \), \( \chi = 1 \) on \( (-\infty, 1] \) and \( \chi = 0 \) on \( [2, \infty) \). For every \( n \in \mathbb{N} \), define \( \chi_n(x) := \chi(|x|/n), \ x \in \mathbb{R}^\nu \), and

\[
\mathbf{A}^n := \rho_n * (\chi_n \mathbf{A}), \quad \mathbf{G}^n := \rho_n * (\chi_n 1_{[1/n, n]}(\omega) \mathbf{G}).
\]

Then \( \mathbf{A}^n \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu) \) and every \( \mathbf{G}^n \in C_0^\infty(\mathbb{R}^\nu, \mathbb{R}^\nu) \) with \( n \in \mathbb{N} \) fulfills Hypotheses 8.1 and 8.2. Defining \( W_t^n(x) \), \( W_t^n(x, y) \), and \( H^n \) by putting the pair \( (\mathbf{A}^n, \mathbf{G}^n) \) in place of \( (\mathbf{A}, \mathbf{G}) \) in the construction of \( W_t(x), W_t(x, y) \), and \( H \), respectively, we therefore have the following Feynman–Kac formulas for every \( n \in \mathbb{N} \),

\[
(e^{-tH^n}\Psi)(x) = \mathbb{E}[W_t^n(x)^*\Psi(B_t^x)], \quad \text{a.e. } x \in \mathbb{R}^\nu. \tag{9.19}
\]
as well as
\[(e^{-tH^n}\Psi)(x) = \int_{\mathbb{R}^\nu} p_t(x,y)\mathbb{E}[W^n_t(x,y)\Psi(y)]\,dy, \quad \text{a.e. } x \in \mathbb{R}^\nu. \] (9.20)
Furthermore, the following limit relations hold as \( n \to \infty \),
\[ A^n \to A \text{ in } L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu), \quad G^n \to G \text{ in } L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu). \] (9.21)
Here the first one is standard, while the second one follows from the following remarks:

Let \( C \subset \mathbb{R}^\nu \) be compact and choose \( n_0 \in \mathbb{N} \) so large that
\[ C_1 := \{ x \in \mathbb{R}^\nu \mid \text{dist}(x, C) \leq 1 \} \subset \{ \chi_{n_0} = 1 \}. \]
For every \( x \), we have \( \|(1 - 1_{[1/n,n]}(\omega))G_x\|_{\mathbb{R}^\nu} \to 0, \ n \to \infty \), by dominated convergence. Therefore, the generalized Minkowski inequality and the dominated convergence theorem further imply
\[
\left( \int_C \left( \int_{\mathbb{R}^\nu} \rho_n(x-y)\chi_n(y)(1 - 1_{[1/n,n]}(\omega))G_y\,dy \right)^2 \, dx \right)^{1/2} \\
\leq \left( \int_C \left( \int_{\mathbb{R}^\nu} \rho_n(z)(1 - 1_{[1/n,n]}(\omega))G_{x-z}\,dz \right)^2 \, dx \right)^{1/2} \\
\leq \int_{\mathbb{R}^\nu} \rho_n(z) \left( \int_{C-z} \|(1 - 1_{[1/n,n]}(\omega))G_x\|_{\mathbb{R}^\nu}^2 \, dx \right)^{1/2} \, dz \\
\leq \left( \int_{C_1} \|(1 - 1_{[1/n,n]}(\omega))G_x\|_{\mathbb{R}^\nu}^2 \, dx \right)^{1/2} \xrightarrow{n \to \infty} 0, \] (9.22)
Here we also used that every \( \rho_n \) is supported in the unit ball, which permitted to drop \( \chi_n \) for all \( n \geq n_0 \) in the first step and to replace \( C - z \) by the larger set \( C_1 \) in the last step. Likewise,
\[
\int_C \|\rho_n * (\chi_n G) - G\|_{\mathbb{R}^\nu}^2 \, dx = \int_C \|\rho_n * G - G\|_{\mathbb{R}^\nu}^2 \, dx \xrightarrow{n \to \infty} 0, \] (9.23)
where the equality holds for \( n \geq n_0 \) and the convergence is a special case of (4.4). Now the second relation in (9.21) follows from (9.22) and (9.23).

**Step 2. Convergence of the left hand side of the Feynman–Kac formulas.** Fix \( t > 0 \) in the rest of this proof. Theorem 7.1 shows that \( H^n \to H \), \( n \to \infty \), in strong resolvent sense, which implies the strong convergence \( e^{-tH^n} \to e^{-tH} \). Therefore, there exist integers \( 1 \leq n_1 < n_2 < \ldots \) such that
\[
(e^{-tH^{n_\ell}}\Psi)(x) \xrightarrow{\ell \to \infty} (e^{-tH}\Psi)(x), \quad \text{for a.e. } x \in \mathbb{R}^\nu. \] (9.24)

**Step 3. Application of the dominated convergence theorem.** Define \( K^n_t(x,y) \) and \( K^n_t(x,y) \) by putting \( G^n \) in place of \( G \) in the formulas for \( K_t(x) \) and \( K_t(x,y) \), respectively. Likewise, define \( S^n_t(x) \) and \( S^n_t(x,y) \) by substituting \( A^n \) for \( A \) in the expressions for \( S_t(x) \) and \( S_t(x,y) \), respectively. According to Lemma 9.1 we may in fact fix zero sets \( N \subset \mathbb{R}^\nu \) and \( N' \subset \mathbb{R}^{2\nu} \) in the rest of this proof such that these random functions are well-defined, for all \( x \in \mathbb{R}^\nu \setminus N \) and \( (x,y) \in \mathbb{R}^{2\nu} \setminus N' \), respectively. Combining (9.21), Theorem 9.2, and
Lemma 9.3 we now find a subsequence \( \{m_\ell\}_{\ell \in \mathbb{N}} \) of the index sequence \( \{n_\ell\}_{\ell \in \mathbb{N}} \) such that, as \( \ell \to \infty \),

\[
S_i^{m_\ell}(x) \to S_i(x) \quad \text{and} \quad K_i^{m_\ell}(x) \to K_i(x) \quad \text{in probability,}
\]

\[
(9.25)
\]

\[
S_i^{m_\ell}(x,y) \to S_i(x,y) \quad \text{and} \quad K_i^{m_\ell}(x,y) \to K_i(x,y) \quad \text{in probability,}
\]

\[
(9.26)
\]

for all \( x \in \mathbb{R}^v \setminus N \) in the first line and all \( (x,y) \in \mathbb{R}^{2v} \setminus N' \) in the second.

**Step 4. Convergence along a subsequence of the right hand side of (9.19).**

We fix \( x \in \mathbb{R}^v \setminus N \). Recall that convergence in probability implies \( \mathbb{P} \)-a.s. convergence along a subsequence. By virtue of (9.25) we therefore find a subsequence \( \{i_\ell\}_{\ell \in \mathbb{N}} \) of the index sequence \( \{m_\ell\}_{\ell \in \mathbb{N}} \) such that, \( \mathbb{P} \)-a.s.,

\[
K_i^{m_\ell}(x) \xrightarrow{\ell \to \infty} K_i(x) \quad \text{and} \quad S_i^{m_\ell}(x) \xrightarrow{\ell \to \infty} S_i(x) \quad \text{in} \ \mathbb{C}.
\]

Picking a representative \( \Psi(\cdot) \) of \( \Psi \in L^2(\mathbb{R}^v, \mathcal{F}) \) and taking (9.13), (9.15), and (9.16) into account, we deduce that \( W_i^{m_\ell}(x)^* \Psi(B_i^+ \overline{x}) \to W_i(x)^* \Psi(B_i^+ \overline{x}) \), as \( \ell \to \infty \), \( \mathbb{P} \)-a.s., with the pointwise domination \( \|W_i^{m_\ell}(x)^* \Psi(B_i^+ \overline{x})\|_{\mathcal{F}} \leq \|\Psi(B_i^+ \overline{x})\|_{\mathcal{F}} \leq 1 \), for every \( \ell \in \mathbb{N} \). Thus, by dominated convergence,

\[
\mathbb{E}[W_i^{m_\ell}(x)^* \Psi(B_i^+ \overline{x})] \xrightarrow{\ell \to \infty} \mathbb{E}[W_i(x)^* \Psi(B_i^+ \overline{x})] \quad \text{in} \ \mathcal{F},
\]

(9.27)

where \( x \in \mathbb{R}^v \setminus N \) was arbitrary. Combining (9.19), (9.24), and (9.27) we arrive at the Feynman–Kac formula (9.17).

**Step 5. Convergence along a subsequence of the right hand side of (9.20).**

In this step we cannot just mimic the argument of the preceding one because any choice of subsequence along which the convergences in (9.26) hold \( \mathbb{P} \)-a.s. would not only depend on \( x \) but also on \( y \).

Let us fix a representative \( \Psi(\cdot) : \mathbb{R}^v \to \mathcal{F} \) of \( \Psi \in L^2(\mathbb{R}^v, \mathcal{F}) \) in the rest of this proof. We also fix \( (x,y) \in \mathbb{R}^{2v} \setminus N' \) for the moment. Then the following map is continuous,

\[
\mathbb{C} \times \hat{\ell} \ni (z,f) \longmapsto F(z,f) := p_t(x,y) e^{-z \Gamma(j^i_0)} \mathbb{W}(if) \Gamma(j_0) \Psi(y) \in \mathcal{F}.
\]

Since \( W_n(x,y) \Psi(y) = F(S_n(x,y), K_n(x,y)) \), for every \( n \in \mathbb{N} \), and similarly for the limit processes, this permits to get

\[
W_i^{m_\ell}(x,y) \Psi(y) \xrightarrow{\ell \to \infty} W_i(x,y) \Psi(y) \quad \text{in} \ \mathcal{F} \quad \text{and in probability,}
\]

employing (9.26). We further have the uniform bounds

\[
\|W_i^{m_\ell}(x,y) \Psi(y)\|_{\mathcal{F}} \leq \|\Psi(y)\|_{\mathcal{F}}, \quad \ell \in \mathbb{N},
\]

showing in particular that the sequence \( \{W_i^{m_\ell}(x,y) \Psi(y)\}_{\ell \in \mathbb{N}} \) in \( L^1(\Omega, \mathcal{F}; \mathbb{P}) \) is uniformly integrable. Hence, by Vitali’s theorem in its vector-valued version,

\[
\mathbb{E}[W_i^{m_\ell}(x,y) \Psi(y)] \xrightarrow{\ell \to \infty} \mathbb{E}[W_i(x,y) \Psi(y)].
\]

(9.28)

Now, for a.e. \( x \), the cut \( N'_{x} := \{y \in \mathbb{R}^v \mid (x,y) \in N'\} \) has Lebesgue measure zero. Let us fix some \( x \in \mathbb{R}^v \) for which this is the case in the rest of the proof. Then (9.28) holds for a.e. \( y \) and we have the dominations

\[
p_t(x,y)\mathbb{E}[W_i^{m_\ell}(x,y) \Psi(y)]\|_{\mathcal{F}} \leq p_t(x,y)\|\Psi(y)\|_{\mathcal{F}}, \quad \text{a.e.} \ y \in \mathbb{R}^v, \ \ell \in \mathbb{N},
\]
where \( y \mapsto p_t(x, y)\|\Psi(y)\|_{\mathcal{F}} \) is in \( L^1(\mathbb{R}^\nu) \). The dominated convergence theorem, (9.20), and (9.24) now imply the desired formula (9.18).

\[\Box\]

### 9.4. Feynman–Kac Formulas for Singular Coefficients and General Open \( \Lambda \)

We are now in a position to prove our main theorem. We start by applying the results of Sect. 3, which is possible when \( A \) and \( G \) have locally square-integrable extension to the whole \( \mathbb{R}^\nu \).

**Proposition 9.5.** Let \( A \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu), \) \( G \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu), \) and \( V \in C_b(\mathbb{R}^\nu, \mathbb{R}) \). Pick some \( t > 0 \) and \( \Psi \in L^2(\mathbb{R}^\nu, \mathcal{F}) \). Then, for a.e. \( x \in \mathbb{R}^\nu, \)

\[
e^{-tH_\Lambda}\Psi(x) = \mathbb{E}\left[1_{\{\tau_\Lambda(x) > t\}} W_t(x)^* \Psi(B^x_t)\right] = \int_{\Lambda} p_t(x, y) \mathbb{E}\left[1_{\{\tau_\Lambda(t; y, x) = \infty\}} W_t(x, y)\Psi(y)\right] \, dy.
\]

(9.29)

**Proof.** It suffices to check the postulates in Sect. 3 when we set \( q_{\mathbb{R}^\nu} := h_{\mathbb{R}^\nu} \) and \( q_\Lambda := h_{\Lambda, D} \). That these two forms fulfill Hypothesis 3.1 has, however, already been observed in Proposition 5.14. The validity of Hypotheses 3.3 and 3.5 follows from Proposition 9.4. \(\Box\)

**Proof of Theorem 1.1.** First, we additionally assume that \( V \in C_b(\mathbb{R}^\nu, \mathbb{R}) \). To infer our main theorem from Proposition 9.5 in this case, we apply an idea from [34, B4]: Set \( \Lambda_n := \{x \in \Lambda | \text{dist}(x, \Lambda^c) > 1/n\} \) and \( A^n := 1_{\Lambda_n} A, \) \( G^n := 1_{\Lambda_n} G \), for all \( n \in \mathbb{N} \). Extend \( A^n \) and \( G^n \) to functions on \( \mathbb{R}^\nu \) by setting them equal to zero on \( \Lambda^c \). Then \( A^n \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu) \) and \( G \in L^2_{\text{loc}}(\mathbb{R}^\nu, \mathbb{R}^\nu) \). Let \( h_n \) denote the minimal Pauli–Fierz form on \( \Lambda_n \) defined by means of \( A^n \) and \( G^n \). Then it is clear that \( \mathcal{D}(h_n) \subset \mathcal{D}(h_m) \subset h_{\Lambda, D}, m > n, \) and \( h_{\Lambda, D}[\Phi] = \lim_{n < m \to \infty} h_m[\Phi], \) for all \( \Phi \in \mathcal{D}(h_n) \) and \( m \in \mathbb{N} \), where functions on \( \Lambda_n \) are tacitly extended by 0 to larger subsets of \( \Lambda \). Thus, [34, Theorem 4.1 and Theorem 4.2] imply that

\[e^{-tH_{\Lambda_n}}(\Psi |_{\Lambda_n}) \xrightarrow{n \to \infty} e^{-tH_\Lambda}\Psi \text{ in } L^2(\Lambda, \mathcal{F}),\]

(9.30)

where the \( e^{-tH_{\Lambda_n}}(\Psi |_{\Lambda_n}) \) are interpreted as functions on \( \Lambda \) that equal 0 on \( \Lambda \setminus \Lambda_n \). Along a suitable subsequence, the convergence in (9.30) also holds pointwise a.e. on \( \Lambda \). On the other hand, Proposition 9.5 in conjunction with (9.11), (9.12), and analogous relations for the complex actions \( S_t(x) \) and \( S_t(x, y) \) implies

\[
e^{-tH_{\Lambda_n}}(\Psi |_{\Lambda_n})(x) = \mathbb{E}\left[1_{\{\tau_{\Lambda_n}(x) > t\}} W_t(x)^* \Psi(B^x_t)\right] = \int_{\Lambda_n} p_t(x, y) \mathbb{E}\left[1_{\{\tau_{\Lambda_n}(t; x, y) = \infty\}} W_t(x, y)\Psi(y)\right] \, dy,
\]

(9.31)

for a.e. \( x \in \Lambda_n \) and all \( n \in \mathbb{N} \). Furthermore, \( 1_{\{\tau_{\Lambda_n}(x) > t\}} \to 1_{\{\tau_\Lambda(x) > t\}} \) and \( 1_{\{\tau_{\Lambda_n}(t; x, y) = \infty\}} \to 1_{\{\tau_\Lambda(t; x, y) = \infty\}} \) pointwise on \( \Omega \), as \( n \to \infty \), for all \( x, y \in \Lambda \). Hence, by dominated convergence, the expectation in the second line of (9.31) and the member in the third line of (9.31) converge to the corresponding terms in (9.29), for every \( x \in \Lambda \).

For merely measurable, bounded \( V \geq 0 \), all statements of Theorem 1.1 now follow from a standard mollifying procedure and, after that, they can be
extended to locally integrable $V \geq 0$ by approximation with $V \land n$, $n \in \mathbb{N}$; see, e.g., the proof of [8, Theorem 11.3] for more details.

Proof of Corollary 1.4. The first assertion in the corollary follows from the discussion in [27, ß4]. To prove the second one, we start by observing that Theorem 1.1 and Remark 1.3 extend trivially to locally integrable potentials that are bounded from below and in particular to every $V - U \land n$ with $n \in \mathbb{N}$. Furthermore, a monotone convergence theorem for quadratic forms [19, Theorem VIII.3.11] implies that $H^{U \land n}_\Lambda \to H^U_\Lambda$ in strong resolvent sense, as $n \to \infty$. Let $t > 0$ and $\Psi \in L^2(\Lambda, \mathcal{F})$. Then

$$e^{-tH^{U \land n}_\Lambda} \Psi \xrightarrow{\ell \to \infty} e^{-tH^U_\Lambda} \Psi \quad \text{a.e. on } \Lambda,$$

for a suitable subsequence $\{n_\ell\}_{\ell \in \mathbb{N}}$ of $\{n\}_{n \in \mathbb{N}}$. In view of (1.20) and the dominated convergence theorem it thus remains to verify the inequality in

$$\int_{\Lambda} p_t(x, y) \mathbb{E}\left[1_{\{\tau_\Lambda(y, x) = \infty\}} e^{\int_0^t U(B_{s}^{x,y}(x))ds} \right] \|\Psi(y)\|_{\mathcal{F}} dy$$

$$= \mathbb{E}\left[e^{\int_0^t (U(B_{s}^{x,y}(x))ds} \eta_{x}^{x,x} \right] < \infty,$$

for a.e. $x \in \Lambda$, where $\eta_{x}^{x,x} := 1_{\{\tau_\Lambda(x) > t\}} \|\Psi(B_{t}^{x})\|_{\mathcal{F}}$. (The equality in the previous relation is true for every $x \in \Lambda$ and follows upon substituting $U$ by $U \land n$ and applying the monotone convergence theorem.) We now argue similarly as in [37]: Denoting the Dirichlet-Laplacian on $\Lambda$ by $\Delta_\Lambda$, we know [19, Theorem VIII.3.11] that the operators $-\Delta_\Lambda/2 - U \land n$ have a limit in the strong resolvent sense. Denoting this limit by $L$, we find a subsequence $\{m_\ell\}_{\ell \in \mathbb{N}}$ of the index sequence $\{n_\ell\}_{\ell \in \mathbb{N}}$ such that, for a.e. $x \in \Lambda$,

$$(e^{-t(-\Delta_\Lambda/2 - U \land m_\ell)}\|\Psi\|_{\mathcal{F}})(x) \xrightarrow{\ell \to \infty} (e^{-tL}\|\Psi\|_{\mathcal{F}})(x) < \infty. \quad (9.32)$$

The monotone convergence theorem now implies that

$$\mathbb{E}\left[e^{\int_0^t U(B_{s}^{x,y}(x))ds} \eta_{x}^{x,x} \right] = \lim_{\ell \to \infty} \mathbb{E}\left[e^{\int_0^t (U\land m_\ell)(B_{s}^{x,y}(x))ds} \eta_{x}^{x,x} \right] < \infty, \quad (9.33)$$

for a.e. $x \in \Lambda$, since, again for a.e. $x \in \Lambda$, the expectations to the right in (9.33) are equal to the vectors to the left in (9.32).

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Appendix A. A useful rule for vector-valued conditional expectations

The following lemma should be well-known, also in the infinite dimensional setting, but we could not find an appropriate reference. Therefore, we prove it for the convenience of the reader.

\begin{align*}
\mathbb{E}\left[e^{\int_0^t U(B_{s}^{x,y}(x))ds} \eta_{x}^{x,x} \right] = \lim_{\ell \to \infty} \mathbb{E}\left[e^{\int_0^t (U\land m_\ell)(B_{s}^{x,y}(x))ds} \eta_{x}^{x,x} \right] < \infty,
\end{align*}
Lemma A.1. Let \((X, \mathcal{A}, P)\) be a probability space, \(\mathcal{C}\) be a sub-\(\sigma\)-algebra of \(\mathcal{A}\), and \(Y\) and \(Z\) separable Banach spaces equipped with their Borel \(\sigma\)-algebras \(\mathcal{B}(Y)\) and \(\mathcal{B}(Z)\), respectively. Let \(f : X \times Y \to Z\) be a function such that \(f(\cdot, y) : X \to Z\) is Bochner-Lebesgue integrable (in particular \(\mathcal{A} - \mathcal{B}(Z)\)-measurable) and \(\mathcal{C}\)-independent for every \(y \in Y\), and such that \(f(x, \cdot) : Y \to Z\) is continuous for every \(x \in X\). (This implies that \(f\) is \((\mathcal{A} \otimes \mathcal{B}(Y)) - \mathcal{B}(Z)\)-measurable.) Define
\[
\phi(y) := E[f(\cdot, y)] := \int_X f(x, y) dP(x), \quad y \in Y.
\]
(Then \(\phi : Y \to Z\) is in any case Borel measurable.) Finally, let \(g : X \to Y\) be \(\mathcal{C} - \mathcal{B}(Y)\)-measurable and assume that
\[
X \ni x \mapsto h(x) := f(x, g(x)) \in Z
\]
is Bochner-Lebesgue integrable. Then
\[
E^\mathcal{C}[h] = \phi(g), \quad P\text{-a.s.,}
\]
where \(E^\mathcal{C}\) denotes a version of the \(Z\)-valued conditional expectation with respect to \(P\) given the hypothesis \(\mathcal{C}\).

Proof. Let \(\chi \in C(\mathbb{R}, \mathbb{R})\) be such that \(0 \leq \chi \leq 1\) on \(\mathbb{R}\), \(\chi = 1\) on \((-\infty, 1]\), and \(\chi = 0\) on \([2, \infty)\). Put \(f_n := \chi(\|f\|_Z/n) f_n, \quad n \in \mathbb{N}\), so that each \(f_n\) enjoys all properties of \(f\) mentioned in the statement as well, and so that \(\|f_n\|_Z \leq 2n\) and \(f_n \to f, \quad n \to \infty\), pointwise on \(X \times Y\). Set \(h_n(x) := f_n(x, g(x)), \quad x \in X\), and \(\phi_n(y) := E[f_n(\cdot, y)], \quad y \in Y\). Then we have the dominations \(\|f_n(\cdot, y)\| \leq \|f(\cdot, y)\|\) and \(\|h_n\|_Z \leq \|h\|_Z\) on \(X\), for every \(n \in \mathbb{N}\). Hence, the dominated convergence theorem for the Bochner-Lebesgue integral implies that \(\phi_n(y) \to \phi(y), \quad n \to \infty\), for every \(y \in Y\), while the dominated convergence theorem for \(Z\)-valued conditional expectations implies that \(E^\mathcal{C}[h_n] \to E^\mathcal{C}[h], \quad n \to \infty\), \(P\)-a.s. Therefore, it only remains to show that \(E^\mathcal{C}[h_n] = \phi(g)\) holds \(P\)-a.s., for each fixed \(n \in \mathbb{N}\). Or, put differently, we may assume without loss of generality that \(f\) is bounded, which we shall do in the rest of this proof.

There exists a sequence of \(\mathcal{C} - \mathcal{B}(Y)\)-measurable functions \((g_n)_{n \in \mathbb{N}}\) such that the image \(g_n(X)\) is finite, for every \(n \in \mathbb{N}\), and such that \(g_n \to g, \quad n \to \infty\), pointwise on \(X\). Let \(n \in \mathbb{N}\). Then \(g_n\) has a standard representation \(g_n = \sum_{i=1}^{k_n} 1_{A^n_i} y^n_i\) for suitable \(k_n \in \mathbb{N}\), \(y^n_1, \ldots, y^n_{k_n} \in Y\), and suitable disjoint \(A^n_1, \ldots, A^n_{k_n} \in \mathcal{C}\) such that \(A^n_1 \cup \cdots \cup A^n_{k_n} = X\). Then
\[
\tilde{h}_n(x) := f(x, g_n(x)) = \sum_{i=1}^{k_n} 1_{A^n_i}(x) f(x, y^n_i), \quad x \in X.
\]
Since \(A^n_i \in \mathcal{C}\) and since \(f(\cdot, y^n_i) : X \to Z\) is \(\mathcal{C}\)-independent, well-known computation rules for the conditional expectation now imply
\[
E^\mathcal{C}[\tilde{h}_n] = \sum_{i=1}^{k_n} 1_{A^n_i} \phi(y^n_i) = \phi(g_n), \quad P\text{-a.s.,}
\]
where we again used that \(y^n_i = g_n\) on \(A^n_i\) in the second equality. Furthermore, by our present assumptions on \(f\), the functions \(\tilde{h}_n, \quad n \in \mathbb{N}\), are uniformly
bounded, and thanks to the continuity of \( y \mapsto f(x, y) \) for each \( x \), we know that \( h_n \to h, n \to \infty \), pointwise on \( X \). Hence, \( E[\hat{h}_n] \to E[h], n \to \infty \), \( P \)-a.s., by the dominated convergence theorem for \( Z \)-valued conditional expectations. Finally, we observe that \( \phi : Y \to Z \) is continuous by the boundedness of \( f \) and dominated convergence. Thus, \( \phi(g_n) \to \phi(g), n \to \infty \), pointwise on \( X \).

\[ \square \]

**Example A.2.** Let \( (X, \mathcal{A}, P) \) and \( C \) be as in Lemma A.1. Let \( Z \) be a separable Hilbert space, \( A(y) : X \to \mathcal{B}(Z) \) be measurable and separably valued, for every \( y \in \mathbb{R}^\nu \), such that \( \mathbb{R}^\nu \ni y \mapsto (A(y))(x) \) is strongly continuous for all \( x \in X \). Suppose that \( A(y) \) is \( C \)-independent and let \( g := (q, \Psi) : X \to \mathbb{R}^\nu \times Z \) be \( C \)-measurable with \( \int_X \| \Psi \|_Z dP < \infty \). Finally, assume there exists \( C > 0 \) such that \( \| A(y) \| \leq C \), \( P \)-a.s., for every \( y \in \mathbb{R}^\nu \). Then we can apply Lemma A.1 to the function \( f \) given by \( f(x, y, \psi) := A(y)\psi, (y, \psi) = \mathbb{R}^\nu \times Z \) with \( \phi(y, \psi) = E[f(\cdot, y, \psi)] = E[A(y)]\psi \). That is,

\[ E[\Psi] = E[A(y)] \mid_{y=q} \Psi. \]

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