Conformal invariants of manifolds of non-positive scalar curvature

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Abstract
Conformal invariants of manifolds of non-positive scalar curvature are studied in association with growth in volume and fundamental group.

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1. Introduction

Let \((M, g)\) be a complete Riemannian \(n\)-manifold. In this paper we consider the conformal invariant \(Q(M, g)\) defined by

\[
Q(M, g) = \inf \left\{ \frac{\int_M |\nabla u|^2 dv_g + \frac{n-2}{4(n-1)} \int_M S_g u^2 dv_g}{(\int_M |u|^{\frac{2n}{n-2}} dv_g)^{\frac{n-2}{n}}} \mid u \in C^\infty(M), u \not\equiv 0 \right\},
\]

where \(S_g\) is the scalar curvature of \((M, g)\). The conformal invariant \(Q(M, g)\) has been studied in association with the Yamabe problem [9]. The sign of \(Q(M, g)\) is an important conformal invariant. If \((M, g)\) is a compact Riemannian \(n\)-manifold with \(n \geq 3\), then \(Q(M, g)\) is negative (resp. zero, positive) if and only if \(g\) is conformal to a Riemannian metric of negative (resp. zero, positive) scalar curvature [13]. It is
shown in [13] that if there is a conformal map \( \Phi : (M,g) \rightarrow (S^n,g_o) \), then \( Q(M,g) = Q(S^n,g_o) > 0 \), where \( S^n \) is the unit sphere and \( g_o \) is the standard metric on \( S^n \). In particular, the conformal invariants for the Euclidean space \( \mathbb{R}^n \) and the hyperbolic space \( \mathbb{H}^n \) are both positive, even though in the Euclidean space, the scalar curvature is zero and in the hyperbolic space, the scalar curvature is a negative number. In this paper we show the following.

**Theorem 1.** Let \((M,g)\) be a complete non-compact Riemannian \(n\)-manifold with \(n \geq 3\) and \(S_g \leq 0\). Let \(x_o \in M\) and let \(B_R\) be the open ball on \(M\) with center at \(x_o\) and radius equal to \(R > 0\). If there exist positive constants \(C\) and \(\alpha \in (0,n/2)\) such that
\[
\text{Vol}(B_R) \leq CR^\alpha
\]
for all \(R > 0\), then \(Q(M,g) \leq 0\).

In particular, \(Q(M \times \mathbb{R}^m,g \times h_o) \leq 0\) if \(n > m\), where \((M,g)\) is a compact Riemannian \(n\)-manifold of non-positive scalar curvature and \(h_o\) is the Euclidean metric on \(\mathbb{R}^n\). If the scalar curvature of the manifold \((M,g)\) does not drop to zero too fast, and if \((M,g)\) has polynomial volume growth, then we can conclude that \(Q(M,g) < 0\). Next, we show the following.

**Theorem 2.** If \((M,g)\) is a compact conformally flat manifold with scalar curvature \(S_g \leq 0\), then the fundamental group of \(M\) has exponential growth unless \(S_g \equiv 0\).

By a result of Avez [1], a compact manifold of non-positive sectional curvature has exponentially growing fundamental group unless the manifold is flat. It is known that a compact manifold of non-positive sectional curvature is flat if and only if its fundamental group is almost solvable [4] (see also [2]). We show that if \((M,g)\) is a compact conformally flat manifold with scalar curvature \(S_g \leq 0\) and the fundamental group of \(M\) is almost solvable, then \((M,g)\) is flat.

2. Proofs

**Theorem 2.1.** Let \((M,g)\) be a complete non-compact Riemannian \(n\)-manifold with \(n \geq 2\) and \(S_g \leq 0\). Let \(x_o \in M\) and let \(B_R\) be the open ball on \(M\) with center at \(x_o\) and radius equal to \(R > 0\). If there exist positive constants \(C\) and \(\alpha \in (0,n/2)\) such
that
\[ \text{Vol}(B_R) \leq CR^\alpha \]
for all \( R > 0 \), then \( Q(M, g) \leq 0 \).

**Proof.** Assume that \( Q(M, g) > 0 \). Let \( \alpha' \in (0, n/2) \) be a positive constant such that \( \alpha < \alpha' \). There exist positive constants \( c \) and \( R_o \) such that
\[
\frac{2}{c^2} \leq \frac{Q(M, g)}{4} \quad \text{and} \quad \text{Vol}(B_R) \leq cR^{\alpha'}
\]
for all \( R \geq R_o \). For \( R > 0 \), let
\[
\lambda_R = \inf \left\{ \frac{\int_{B_R} |\nabla u|^2 \, dv_g}{(\int_{B_R} |u|^2 \, dv_g)} \mid u \in C^\infty_o(B_R), u \not\equiv 0 \right\}.
\]
Then
\[
\lambda_R \geq \inf \left\{ \frac{\int_{B_R} |\nabla u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_{B_R} S_g u^2 \, dv_g}{(\int_{B_R} |u|^2 \, dv_g)} \mid u \in C^\infty_o(B_R), u \not\equiv 0 \right\}
\]
\[
= \inf \left\{ \frac{(\int_{B_R} |u|^{\frac{2n}{n-2}} \, dv_g)^{\frac{n-2}{n}}}{\int_{B_R} |u|^2 \, dv_g} \times \left( \frac{\int_M |\nabla u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_M S_g u^2 \, dv_g}{(\int_M |u|^{\frac{2n}{n-2}} \, dv_g)^{\frac{n-2}{n}}} \right) \mid u \in C^\infty_o(B_R), u \not\equiv 0 \right\}.
\]
By Hölder’s inequality we have
\[
\int_{B_R} |u|^2 \, dv_g \leq \left( \int_{B_R} |u|^{\frac{2n}{n-2}} \, dv_g \right)^{\frac{n-2}{n}} \left( \int_{B_R} 1 \, dv_g \right)^{\frac{2}{n}}.
\]
Therefore
\[
\frac{(\int_{B_R} |u|^{\frac{2n}{n-2}} \, dv_g)^{\frac{n-2}{n}}}{\int_{B_R} |u|^2 \, dv_g} \geq \frac{1}{\left( \text{Vol}(B_R) \right)^{\frac{2}{n}}} \geq \frac{1}{c^2 R^{\frac{2\alpha'}{n}}}, \quad R \geq R_o.
\]
As \( Q(M, g) > 0 \), we have
\[
\frac{\int_M |\nabla u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_M S_g u^2 \, dv_g}{(\int_M |u|^{\frac{2n}{n-2}} \, dv_g)^{\frac{n-2}{n}}} > 0
\]
for all \( u \in C^\infty_o(M), u \not\equiv 0 \). Thus
\[
\lambda_R \geq \frac{1}{c^2 R^{\frac{2\alpha'}{n}}} \inf \left\{ \frac{\int_M |\nabla u|^2 \, dv_g + \frac{n-2}{4(n-1)} \int_M S_g u^2 \, dv_g}{(\int_M |u|^{\frac{2n}{n-2}} \, dv_g)^{\frac{n-2}{n}}} \mid u \in C^\infty_o(M), u \not\equiv 0 \right\}
\]
\[
\geq \frac{4}{R^{\frac{2\alpha'}{n}}}
\]
for $R \geq R_o$, as $c^2 \leq Q(M,g)/4$. Let $k > R_o$ be an integer and let $\varphi_k \in C^\infty(B_{k+1})$ be such that $\varphi_k(x) \geq 0$ for all $x \in B_{k+1}$, $\varphi_k(x) = 1$ for $x \in B_k$ and $|\nabla \varphi_k| \leq 2$ [cf. 11]. We have

$$\text{Vol} (B_{k+1}) - \text{Vol} (B_k) \geq \frac{1}{4} \int_{B_{k+1}} |\nabla \varphi_k|^2 d\nu_g \geq \frac{1}{4} \lambda_{k+1} \int_{B_{k+1}} |\varphi_k|^2 d\nu_g \geq \frac{\text{Vol} (B_k)}{(k+1)^{\frac{2n}{n}}}.$$ 

Therefore

$$\text{Vol} (B_{k+1}) \geq (1 + \frac{1}{(k+1)^{\frac{2n}{n}}}) \text{Vol} (B_k)$$

for all $k > R_o$. Let $\beta = 2\alpha'/n < 1$. Given an integer $m > n/2$, there exists an integer $k_o > R_o$ such that for any integer $k > k_o$, we have

$$\left(\frac{k+2}{k+1}\right)^m = (1 + \frac{1}{k+1})^m = 1 + \frac{c_1(m)}{k+1} + \frac{c_2(m)}{(k+1)^2} + \ldots + \frac{c_{m-1}(m)}{(k+1)^{m-1}} + \frac{1}{(k+1)^m} \leq 1 + \frac{C(m)}{k+1} \leq 1 + \frac{1}{(k+1)^\beta},$$

where $c_1(m), ..., c_{m-1}(m)$ are positive constants and $C(m)$ is a large positive constant.

We have

$$\text{Vol} (B_{k+1}) \geq (1 + \frac{1}{(k+1)^\beta})(1 + \frac{1}{k^\beta}) \cdots (1 + \frac{1}{(k_o+1)^\beta}) \text{Vol} (B_{k_o}) \geq \left(\frac{k+2}{k+1}\right)^m \times \left(\frac{k+1}{k}\right)^m \times \cdots \times \left(\frac{k_o+3}{k_o+2}\right)^m \times \left(\frac{k_o+2}{k_o+1}\right)^m \text{Vol} (B_{k_o}) = \left(\frac{k+2}{k_o+1}\right)^m \text{Vol} (B_{k_o})$$

for all $k > k_o$. This contradicts that $\text{Vol} (B_{k+1}) \leq c(k+1)^{\alpha'}$ with $\alpha' < n/2$, for all $k > k_o$. Q.E.D.

**Corollary 2.2.** Let $(M,g)$ be a complete Riemannian manifold of non-positive scalar curvature. If the volume of $(M,g)$ is finite, then $Q(M,g) \leq 0$.

**Corollary 2.3.** For $n \geq 3$, let $(M,g)$ be a simply connected, complete, non-compact, conformally flat Riemannian manifold of non-positive scalar curvature. Then for
some positive constant $C$ and for any $\alpha < n/(n + 2)$, $\text{Vol} B_R \geq CR^\alpha$ for all $R$ large.

**Proof.** As $(M, g)$ is a simply connected, complete, non-compact, conformally flat Riemannian manifold, we have [13]

$$Q(M, g) = Q(S^n, g_0) = \frac{n(n - 2)\omega_n^2}{4},$$

where $\omega_n$ denotes the volume of $(S^n, g_0)$. Let $C$ be a positive constant with $C\omega_n^2 \leq Q(M, g)/4$. If the statement that $\text{Vol} B_R \geq CR^\alpha$ for all $R$ large does not hold for a positive constant $\alpha < n/(n + 2)$, then there exists an increasing sequence of positive numbers $\{r_0, r_1, \ldots, r_k, \ldots\}$ such that

$$\lim_{k \to \infty} r_k = \infty \quad \text{and} \quad \text{Vol} B_{r_k} \leq Cr_k^\alpha$$

for $k = 1, 2, \ldots$. It follows from the proof of theorem 2.1 that for $k$ large,

$$\text{Vol} B_{r_k} \geq \prod (1 + \frac{1}{2d}) \text{Vol} B_{r_k-1} \geq \prod \frac{r_k}{2d} \text{Vol} B_1 \geq \frac{r_k}{2d} \text{Vol} B_1 \geq r_k^\alpha \text{Vol} B_1,$$

where $[r_k]$ is the integer part of $r_k$ and $\alpha'$ is a constant such that $\alpha < \alpha' < n/(n + 2)$. When $k$ is large enough, the last inequality contradicts that $\text{Vol} B_{r_k} \leq Cr_k^\alpha$ for all non-negative integer $k$. Q.E.D.

**Theorem 2.4.** Let $(M, g)$ be a complete non-compact Riemannian $n$-manifold with $n \geq 3$. Assume that $S_g \leq 0$ and there exist constants $C > 0$, $R_1 > 0$ and $\beta \in (0, 1)$ such that

$$S_g(x) \leq -Cd(x_o, x)^{-\beta}$$

for all $x \in M \setminus B_{R_1}$, where $x_o \in M$ is a fixed point and $d(x_o, x)$ is the distance between $x_o$ and $x$ in $(M, g)$. If there exist a positive constant $C'$ and a positive integer $m$ such
that
\[ \text{Vol}(B_R) \leq C'R^m \]
for all \( R > 0 \), then \( Q(M, g) < 0 \).

**Proof.** Given a positive number \( \delta \), let \( u_o \in C_0^\infty(M) \) be a smooth function such that
\[ \frac{4(n-1)}{n-2} \Delta u_o = \delta \quad \text{on} \quad B_{R_1} . \]
Assume that the support of \( u_o \) is inside \( B_{R_2} \) for some constant \( R_2 > R_1 \). We can find a positive constant \( c_o \) such that
\[ S_g(x)(c_o + u_o(x)) - \frac{4(n-1)}{n-2} \Delta u_o(x) \leq -\delta \]
for all \( x \in B_{R_2} \). This is because \( S_g \leq 0 \) and
\[ S_g(x) \leq -C R_2^{-\beta} \]
for all \( x \in B_{R_2} \setminus B_{R_1} \). Let \( u = c_o + u_o \). Then \( u \) is a smooth positive function on \( M \). Furthermore, \( u(x) = c_o \) for all \( x \in M \setminus B_{R_2} \). Let \( g' = u^{4/(n-2)}g \) and let \( S' \) be the scalar curvature of the metric \( g' \). Then [9]
\[ S'(x) = u^{-\frac{n-2}{4}}[S_g(x)u(x) - \frac{4(n-1)}{n-2} \Delta u_o(x)] \leq -\epsilon \]
for all \( x \in B_{R_2} \), where \( -\epsilon \) is a suitable constant. And
\[ S'(x) = c_o^{-\frac{n-2}{4}} S_g(x) \]
for all \( x \in M \setminus B_{R_2} \). There exist positive constants \( c_1 \) and \( C_1 \) such that \( c_1 \leq u \leq C_1 \) on \( M \). Therefore
\[ c_1^{-\frac{n-2}{4}} d'(x_o, x) \leq d(x_o, x) \leq C_1^{-\frac{n-2}{4}} d'(x_o, x) \]
for all \( x \in M \), where \( d'(x_o, x) \) is the distance between \( x_o \) and \( x \) with respect to the Riemannian metric \( g' \). Thus we can find a positive constant \( R'_1 \) such that
\[ S'(x) \leq -C'' d'(x_o, x)^{-\beta} \]
for all \( x \in M \) with \( d'(x_o, x) > R'_1 \), where \( C'' \) is a positive constant. And for \( d'(x_o, x) \leq R'_1 \), we can find a positive constant \( \epsilon' \) such that
\[ S'(x) \leq -\epsilon' . \]
Hence we can find a positive number \( R_o \) such that for all \( R > R_o \), we have
\[ S'(x) \leq -C_o R^{-\beta} \]
for all \( x \in M \) with \( d'(x_o, x) \leq R \), where \( C_o \) is a positive integer. As \( Q(M, g') = Q(M, g) \), by above, we may assume without loss of generality that for all \( R > R_o \),

\[
S_g(x) \leq -C_o R^{-\beta}
\]

for all \( x \in M \) with \( d(x_o, x) \leq R \).

Assume that \( Q(M, g) \geq 0 \). Take \( \beta' \in (0, 1) \) such that \( \beta' > \beta \). Then we can find a positive number \( R_3 > R_o \) such that for all \( R > R_3 \),

\[
S_g(x) \leq -\frac{4^2(n-1)}{n-2} \frac{1}{R^{\beta'}}
\]

for all \( x \in M \) with \( d(x_o, x) \leq R \). For \( R > R_3 \), we have

\[
\lambda_R \geq \inf \left\{ \frac{\int_{B_R} |\nabla u|^2 d\nu_g + \frac{n-2}{4(n-1)} \int_{B_R} S_g u^2 d\nu_g}{(\int_{B_R} |u|^2 d\nu_g)^{\frac{n-2}{n}}} \frac{\int_{B_R} S_g u^2 d\nu_g}{(\int_{B_R} |u|^2 d\nu_g)^{\frac{n-2}{n}}} \mid u \in C_o^\infty(B_R), u \not\equiv 0 \right\}
\]

\[
\geq \inf \left\{ \frac{\int_{B_R} |\nabla u|^2 d\nu_g + \frac{n-2}{4(n-1)} \int_{B_R} S_g u^2 d\nu_g}{(\int_{B_R} |u|^2 d\nu_g)^{\frac{n-2}{n}}} \mid u \in C_o^\infty(B_R), u \not\equiv 0 \right\} + 4 \frac{1}{R^{\beta'}}
\]

\[
\geq 4 \frac{1}{R^{\beta'}}
\]

as \( Q(M, g) \geq 0 \) implies that

\[
\frac{\int_{B_R} |\nabla u|^2 d\nu_g + \frac{n-2}{4(n-1)} \int_{B_R} S_g u^2 d\nu_g}{(\int_{B_R} |u|^2 d\nu_g)^{\frac{n-2}{n}}} \geq 0
\]

for all \( u \in C_o^\infty(B_R), u \not\equiv 0 \). It follows as in theorem 2.1 that for any positive integer \( m' > m \), there exists an integer \( k_o > R_3 \) such that for all \( k > k_o \), we have

\[
\text{Vol} (B_{k+1}) \geq (k + 2)^{m'} \frac{\text{Vol} (B_{k_o})}{(k_o + 1)^m}
\]

for all \( k > k_o \). This contradicts that \( \text{Vol} (B_{k+1}) \leq C' R^m \) with \( m < m' \), for all \( k > k_o \).

Q.E.D.

**Corollary 2.5.** Let \((M, g)\) be a simple connected, complete, non-compact conformally flat Riemannian \( n \)-manifold with \( n \geq 3 \). Assume that \( S_g \leq 0 \) and there exist constants \( C > 0 \), \( R_1 > 0 \) and \( \beta \in (0, 1) \) such that

\[
S_g(x) \leq -Cd(x_o, x)^{-\beta}
\]
for all $x \in M \setminus B_{R_1}$, where $x_o \in M$ is a fixed point. Then for any positive numbers $m$ and $C'$, there exists a positive constant $R_1$ such that

$$\text{Vol} \ (B_R) \geq C'R^m$$

for all $R > R_1$.

**Remark.** Corollary 2.5 shows that on the Euclidean space $\mathbb{R}^n$, $n \geq 3$, there does not exist a Riemannian metric $g'$ which is uniformly equivalent and conformal to the Euclidean metric and with non-positive scalar curvature $S'$ which satisfies

$$S'(x) \leq -C|x|^{-\beta}$$

for some constants $\beta \in (0, 1)$ and $C > 0$, for all $x$ outside a compact domain containing the origin. Such a metric exists if the scalar curvature is allowed to drop to zero faster. In fact, Ni [11] has constructed a metric $g'$ in $\mathbb{R}^n$ which is uniformly equivalent and conformal to the Euclidean metric, with $S' < 0$ and for all $x$ outside a compact domain containing the origin,

$$S'(x) \leq -C|x|^{-l}$$

for some constant $l > 2$ and $C > 0$.

**Theorem 2.6.** Let $(M, g)$ be a compact conformally flat $n$-manifold with $n \geq 3$ and scalar curvature $S_g \leq 0$. Then the fundamental group of $M$ has exponential growth unless $S_g \equiv 0$.

**Proof.** If $S_g \neq 0$, then $S_g$ is negative somewhere and hence the conformal invariant $Q(M, g)$ is negative. By a conformal change of the metric $g$, we may assume that $S_g \equiv -c^2$ with $c > 0$ being a constant [9]. Let $\tilde{M}$ be the universal covering of $M$, equipped with the pull back metric. Then

$$Q(\tilde{M}, g) = Q(S_n, g_o) = \frac{n(n-2)\omega_n^2}{4}.$$ 

Given a point $p \in \tilde{M}$ and $R > 0$, let $B_R$ be the ball in $\tilde{M}$ with center at $p$ and radius $R$. As in the proof of theorem 2.4, we have

$$\lambda_R \geq \frac{n-2}{4(n-1)}c^2$$

and hence there exist positive constants $C$ and $\delta$ such that (c.f. proposition 1.2 of [13])

$$\text{Vol} \ (B_R) \geq Ce^{\delta R}$$
for all $R$ large. Then an argument as in [10] shows that $\pi_1(M)$ has exponential growth. Q.E.D.

**Theorem 2.7.** For $n \geq 3$, let $(M, g)$ be a compact conformally flat $n$-manifold of non-positive scalar curvature. If $\pi_1(M)$ is almost solvable, then $(M, g)$ is a flat manifold.

**Proof.** Since $\pi_1(M)$ is almost solvable, the holonomy group of $(M, g)$ is also almost solvable. By a result in [6], $(M, g)$ is covered by a conformally flat manifold which is either conformally diffeomorphic to the $n$-sphere $S^n$, a flat $n$-torus, or a Hopf manifold $S^1 \times S^{n-1}$. If $M$ is covered by $S^1 \times S^{n-1}$, then $S^1 \times S^{n-1}$ admits a conformally flat metric of non-positive scalar curvature. Since $\pi_1(S^1 \times S^{n-1})$ does not have exponential growth, theorem 2.6 implies that the scalar curvature of $\bar{g}$ is identically equal to zero. Then by theorem 4.5 in [13], $(S^1 \times S^{n-1}, \bar{g})$ is conformally equivalent to $\Omega/\Gamma$, where $\Omega$ is the image of a developing map and $\Gamma$ is the corresponding holonomy group. Furthermore, $\Omega$ is the domain of discontinuity of $\Gamma$. Since $\Gamma \cong \mathbb{Z}$, by using a result in [6], we have $\Gamma$ is conjugate to a cyclic group of similarities. Hence $\Omega/\Gamma$ is conformally equivalent to $S^1 \times S^{n-1}$ with the standard product metric, denoted by $h_1$. If we denote

$$\psi : (S^1 \times S^{n-1}, \bar{g}) \to (S^1 \times S^{n-1}, h_1)$$

a conformal equivalence and $h^* = \psi^* h_1$, then there exists a positive function $u$ such that $\bar{g} = u^{4\frac{n}{n-2}} h^*$. Hence

$$\Delta \bar{g} u - c_n S_{\bar{g}} u = -c_n S_{h^*} u^{\frac{n+2}{n-2}},$$

where $S_{\bar{g}}$ and $S_{h^*}$ are the scalar curvature of $\bar{g}$ and $h^*$, respectively. Now $S_{\bar{g}} = 0$ and $S_{h^*} > 0$, the maximum principle implies that $u$ is a constant, which is impossible.

Suppose that $M$ is covered by $S^n$. By pulling back the metric $g$ to $S^n$, we obtain a metric $g'$. Then $g'$ is a conformally flat metric on $S^n$ with non-positive scalar curvature. Let $g_o$ be the standard metric on $S^n$. The developing map

$$\phi : (S^n, g') \to (S^n, g_o)$$

is a conformal equivalence. Let $g^* = \phi^* g_o$. Hence there exists a function $u > 0$ such that $g^* = u^2 g'$ and

$$\Delta g' u - c_n S_{g'} u = -c_n S_{g^*} u^{\frac{n+2}{n-2}},$$

where $S_{g'}$ and $S_{g^*}$ are the scalar curvature of $g'$ and $g^*$, respectively. In fact, $S_{g^*} = n(n-1)$ and $S_{g'} \leq 0$. The curvature conditions imply $\Delta g' u \leq 0$. Therefore $u$ is a
constant, which is impossible. So we conclude that $M$ is covered by a flat torus $T^n$. This implies that $\pi_1(M)$ is almost nilpotent and has polynomial growth [3]. Theorem 2.6 shows that the scalar curvature of $g$ is identically equal to zero. By pulling back the metric $g$ to $T^n$, we have a metric on $T^n$ which has zero scalar curvature. But any scalar flat metric on $T^n$ is flat [8]. So the pull back metric is flat, hence $(M, g)$ is flat. Q.E.D.

**Theorem 2.8.** For $n = 4, 6$, let $M$ be a compact, orientable $n$-manifold with zero Euler characteristic. If $\pi_1(M)$ does not have exponential growth, then any conformally flat metric on $M$ with non-positive scalar curvature is flat.

**Proof.** Let $g$ be a conformally flat metric on $M$ with non-positive scalar curvature. For $n = 4$, let $B$ be the Gauss-Bonnet integrand of $(M, g)$ and let $\mathcal{R}$ and $\text{Ric}$ be the curvature tensor and Ricci tensor of $g$, respectively. Then there exist universal constants [3] $\alpha, \beta$ and $\gamma$ such that

$$B = \alpha | \mathcal{R} |^2 + \beta | \text{Ric} |^2 + \gamma | S_g |^2.$$

Here $R$ is the scalar curvature of $g$. Since $(M, g)$ is conformally flat, we can write $| \mathcal{R} |^2$ in terms of the other two norms. That is, $B = a | \text{Ric} |^2 + b | S_g |^2$. Evaluate $a$ and $b$ on $S^1 \times S^3$ with the standard product metric and $S^4$, we obtain

$$a = -\frac{1}{6 \text{Vol} (S^4)},$$

and

$$b = \frac{1}{18 \text{Vol} (S^4)}.$$

Using the Gauss-Bonnet theorem, we obtain

$$0 = \chi(M) = \frac{-1}{6 \text{Vol} (S^4)} \int_M | \text{Ric} |^2 dv_g + \frac{1}{18 \text{Vol} (S^4)} \int_M | S_g |^2 dv_g.$$

Since $\pi_1(M)$ does not have exponential growth, theorem 2.6 implies that $S_g \equiv 0$. The above formula gives $\text{Ric} \equiv 0$. Since $g$ is conformally flat, the Weyl tensor is identically equal to zero. Therefore $g$ is flat.

For $n = 6$, we use equation (2.4) in [12] to obtain

$$384 \pi^3 \chi(M) = -\int_M | \nabla \mathcal{R} |^2 dv_g.$$
Since $\chi(M) = 0$, therefore $(M, g)$ is locally symmetric. Then the classification theorem of conformally flat symmetric spaces in dimension six [7] implies that the universal covering of $(M, g)$ is isometric to one of the following symmetric spaces:

$$\mathbb{R}^6, \quad S^6(c), \quad H^6(-c), \quad \mathbb{R} \times S^5(c), \quad \mathbb{R} \times H^5(-c), \quad S^2(c) \times H^4(-c),$$

$$S^4(c) \times H^2(c), \quad S^3(c) \times H^3(-c),$$

where $H^n(-c)$ is the $n$-dimensional simply connected complete manifold of constant sectional curvature equal to $-c$ and $S^n(c)$ is the sphere with sectional curvature equal to $c$. Since $(M, g)$ has non-positive scalar curvature, it cannot be covered by $S^6(c), \mathbb{R} \times S^5(c)$ or $S^4(c) \times H^2(c)$. If $M$ were covered by $H^6(-c), \mathbb{R} \times H^5(-c)$, or $S^3(c) \times H^3(-c)$, then $\pi_1(M)$ would have exponential growth, but this is not true. Hence $(M, g)$ can only be covered by $\mathbb{R}^6$ and so $g$ is flat. Q.E.D.
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