TAUT FOLIATIONS IN BRANCHED CYCLIC COVERS AND LEFT-ORDERABLE GROUPS

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Abstract. We study the left-orderability of the fundamental groups of cyclic branched covers of links which admit co-oriented taut foliations. In particular we do this for cyclic branched covers of fibered knots in integer homology 3-spheres and cyclic branched covers of closed braids. The latter allows us to complete the proof of the L-space conjecture for closed, connected, orientable, irreducible 3-manifolds containing a genus 1 fibered knot. We also prove that the universal abelian cover of a manifold obtained by generic Dehn surgery on a hyperbolic fibered knot in an integer homology 3-sphere admits a co-oriented taut foliation and has left-orderable fundamental group, even if the surgered manifold does not, and that the same holds for many branched covers of satellite knots with braided patterns.

1. Introduction

In this paper we study the left-orderability of the fundamental groups of rational homology 3-spheres $M$ which admit co-oriented taut foliations. Our primary motivation is the L-space conjecture:

Conjecture 1.1 (Conjecture 1 in [BGW], Conjecture 5 in [Ju]). Assume that $M$ is a closed, connected, irreducible, orientable 3-manifold. Then the following statements are equivalent.

(1) $M$ is not a Heegaard Floer L-space,

(2) $M$ admits a co-orientable taut foliation,

(3) $\pi_1(M)$ is left-orderable.

The conjecture is known to hold in a variety of situations, most notably when $M$ has positive first Betti number ([Ga1], [BRW]), or is a non-hyperbolic geometric 3-manifold ([BRW, LS, BGW]), or is a graph manifold ([BC, HRRW]). Condition (2) of the conjecture is known to imply condition (1) ([OS1, KR2, Bn]). Gordon and Lidman introduced the term excellent for manifolds...
satisfying conditions (2) and (3), and therefore (1), of the conjecture, and total L-space for manifolds satisfying neither (1) nor (3), and therefore neither (2). It is clear that Conjecture 1.1 holds for manifolds which are either excellent or total L-spaces and that the conjecture is equivalent to the statement that a closed, connected, irreducible, orientable 3-manifold is either excellent or a total L-space.

Given a closed, connected, irreducible, orientable 3-manifold \( M \), the available techniques for verifying that \( M \) satisfies conditions (1) and (2) of Conjecture 1.1 are far in advance of those available for verifying (3). An equivalent condition for (3) is the existence of a non-trivial homomorphism \( \pi_1(M) \to \text{Homeo}_+ (\mathbb{R}) \) ([BRW, Theorem 1.1]), but these are difficult to construct in general. One method for producing them is to consider a non-trivial representation \( \rho : \pi_1(M) \to PSL(2,\mathbb{R}) \) whose Euler class vanishes (cf. §5). Such a \( \rho \) lifts to a representation \( \pi_1(M) \to \widetilde{SL}_2 \leq \text{Homeo}_+ (\mathbb{R}) \), and so \( \pi_1(M) \) is left-orderable. A drawback of this approach is that it gives no insight into potential connections between condition (3) and conditions (1) and (2). To address this point, suppose that \( M \) satisfies (2) and let \( \rho : \pi_1(M) \to \text{Homeo}_+(S^1) \) be a non-trivial representation obtained through Thurston’s universal circle construction applied to a co-oriented taut foliation on \( M \) (cf. §6). As before, there is a characteristic class \( e(\rho) \in H^2(M) \) whose vanishing implies the left-orderability of \( \pi_1(M) \) (see §5). It is folklore that \( e(\rho) \) coincides with the Euler class of the foliation’s tangent bundle (see Proposition 7.1), and while the latter does not always vanish, one goal of this paper is to use contact geometry to show that it does in topologically interesting situations. In particular, we use this approach to investigate Conjecture 1.1 in the context of manifolds obtained as branched covers of knots and links in rational homology 3-spheres.

Gordon and Lidman initiated such a study for links in \( S^3 \) ([GLid1, GLid2]), focusing on torus links and certain families of satellite knots, including cables. Here we will be mainly concerned with cyclic branched covers of hyperbolic links. In this case, the cyclic covers are almost always hyperbolic ([BPH, Dun]).

Hyperbolic 2-bridge knots form one of the simplest families of hyperbolic knots and various aspects of Conjecture 1.1 have been studied for their branched covers. For instance, work of Dabkowski, Przytycki, and Togha [DPT] combines with that of Peters [Pe] to show that the branched covers of many genus one 2-bridge knots, including the figure eight knot, are total L-spaces. Hu showed that for large \( n \), the fundamental group of the \( n \)-fold branched cyclic cover of the \((p, q)\) 2-bridge knot is left-orderable if \( q \equiv 3 \mod 4 \) [Hu]. More generally, Gordon showed that the same conclusion holds for any 2-bridge knot with non-zero signature [Gor].

Before we state our results, we introduce some notation and terminology. See §2 for the details.

Given a 3-manifold \( V \) with a connected toroidal boundary, a slope on \( \partial V \) is a \( \partial V \)-isotopy class of essential simple closed curves contained in \( \partial V \). We identify slopes with \( \pm \)-classes of primitive elements of \( H_1(\partial V) \), in the usual way, and often represent them by primitive classes \( \alpha \in H_1(\partial V) \). The Dehn filling of \( V \) determined by a slope \( \alpha \) on \( \partial V \) will be denoted by \( V(\alpha) \).
Let $K$ be an oriented null-homologous knot in an oriented rational homology sphere $M$. We use $X(K)$ to denote the exterior of $K$ in $M$ and $\mu, \lambda \in H_1(\partial X(K))$ to denote, respectively, the longitudinal and meridional classes of $K$ (cf. §2.1). Since $K$ is null-homologous, $\{\mu, \lambda\}$ is a basis of $H_1(\partial X(K))$.

For each $n \geq 1$, $X_n(K) \to X(K)$ will be the canonical $n$-fold cyclic cover of $X(K)$ and $\Sigma_n(K) \to M$ the associated $n$-fold cyclic cover branched over $K$. There is a basis $\{\mu_n, \lambda_n\}$ of $H_1(\partial X_n(K))$ where the image of $\mu_n$ in $H_1(\partial X(K))$ is $n\mu$ and that of $\lambda_n$ is $\lambda$. By construction, $\Sigma_n(K) = X_n(K)(\mu_n)$ (§2.2).

Given a fibered knot $K$ in an irreducible rational homology sphere with monodromy $h$, one can define the fractional Dehn twist coefficient of its monodromy $h$, denoted by $c(h)$ (§4). When $K$ is hyperbolic and $c(h) \neq 0$, work of Roberts ([Rob]) can be used to show that if $n|c(h)| \geq 1$, the $n$-fold cyclic cover branched cover of such $K$ admits co-oriented taut foliations ([HKM2, Theorem 4.1]). We use the universal circle construction to show that under the same conditions, the branched covers have left-orderable fundamental groups (Theorem 1.2). Thus they are excellent.

**Theorem 1.2.** Let $K$ be an oriented hyperbolic fibered knot in an oriented integer homology 3-sphere $M$ with monodromy $h$.

1. $\Sigma_n(K)$ is excellent for $n|c(h)| \geq 1$. In particular, if the fractional Dehn twist coefficient $c(h) \neq 0$ and $g$ is the genus of $K$, then $\Sigma_n(K)$ is excellent for $n \geq 2(2g−1)$.

2. More generally, for $n \geq 1$, $X_n(K)(\mu_n + q\lambda_n)$ is excellent whenever $|nc(h) − q| \geq 1$.

For a fixed $n$, there are at most two values of $q$ for which $|nc(h) − q| < 1$ and if two, they are successive integers. Such exceptional values of $q$ are necessary as, for instance, $X_n(K)(\mu_n + q\lambda_n)$ could have a finite fundamental group. Compare Corollary 1.5.

It is known that the fractional Dehn twist coefficients of the monodromies of hyperbolic, fibred, strongly quasipositive knots are non-zero ([Hed, HKM1]). In particular, this true for $K$ an L-space knot as they are fibered and strongly quasipositive (cf. [Hed, Theorem 1.2], [Ni, Corollary 1.3] and the calculations of [OS2]).

**Corollary 1.3.** Suppose that $K$ is a hyperbolic, fibred, strongly quasipositive knot with monodromy $h$. Then $\Sigma_n(K)$ is excellent for $n \geq \frac{1}{|c(h)|}$. In particular, $\Sigma_n(K)$ is excellent if $n \geq 2(2g−1)$. □

Boileau, Boyer and Gordon have investigated the $n$-fold branched cyclic covers of strongly quasipositive knots [BBG] and have shown that in the fibered case they are not L-spaces for $n \geq 6$. Since $c(h)$ can be arbitrarily small for a hyperbolic fibered strongly quasipositive knot, the disparity between the sufficient condition $n \geq 6$ for condition (1) of the conjecture to hold and $n \geq \frac{1}{|c(h)|}$ for conditions (2) and (3) to hold is arbitrarily large. A major challenge is to develop techniques to bridge this gap.
Remarks 1.4.

(1) Theorem 1.2 and its corollaries (Corollary 1.5, Corollary 1.8) hold for hyperbolic fibered knots in oriented rational homology spheres under the assumption that the Euler class of the tangent plane bundle of the fibering of the exterior of the knot is zero (Proposition 9.1).

(2) In Theorem 1.2, the inequality $|nc(h) - q| \geq 1$ can be recast in terms of the distance $\Delta(\alpha, \beta)$ between slopes $\alpha, \beta$ on $\partial X(K)$. Thinking of $\alpha$ and $\beta$ as primitive classes in $H_1(\partial X(K))$ and using $\alpha \cdot \beta$ to denote their algebraic intersection number, $\Delta(\alpha, \beta)$ is defined to be $|\alpha \cdot \beta|$. If $c(h) = \frac{a}{b}$ where $a, b$ are coprime integers, then the degeneracy slope of $K$ is represented by the primitive class $\delta = b\mu + a\lambda$ ([Ga3, KR1]). Then $|nc(h) - q| < 1$ if and only if $\Delta(n\mu + q\lambda, \delta) = |na - qb| < |b| = \Delta(\lambda, \delta)$. Thus the theorem says that $X_n(K)(\mu_n + q\lambda_n)$ is excellent if $\Delta(n\mu + q\lambda, \delta) \geq \Delta(\lambda, \delta)$.

The universal abelian cover of a manifold $W$ is the regular cover $\tilde{W} \to W$ corresponding to the abelianisation homomorphism $\pi_1(W) \to H_1(W)$. It is simple to see that if $\gcd(n, q) = 1$, there is a universal abelian cover $X_n(K)(\mu_n + q\lambda_n) \to X(K)(n\mu + q\lambda)$.

**Corollary 1.5.** Let $K$ be a hyperbolic fibered knot in an integer homology 3-sphere with monodromy $h$. Given coprime integers $n \geq 1$ and $q$, then the universal abelian cover of $X(K)(n\mu + q\lambda)$ is excellent for $q \not\in \left\{ \begin{array}{l} \{nc(h)\} \quad \text{if } nc(h) \in \mathbb{Z} \\ \{\lfloor nc(h)\rfloor, \lceil nc(h)\rceil + 1\} \quad \text{if } nc(h) \not\in \mathbb{Z} \end{array} \right.$.

Corollary 1.5 is striking in that it says that the universal abelian cover of the generic Dehn surgery on a hyperbolic fibered knot in an integer homology 3-sphere is excellent even when the surgered manifold is not. Consider, for instance, a hyperbolic L-space knot $K \subset S^3$. Up to replacing $K$ by its mirror image, we can suppose that $n/q$-surgery of $K$ is an L-space if and only if $n/q \geq 2g(K) - 1$. The corollary implies that if $n/q \geq 2g(K) - 1$, then avoiding the specified values of $q$, $n/q$-surgery of $K$ is a non-excellent manifold whose universal abelian cover is excellent.

Assuming the truth of Conjecture 1.1, the corollary holds for all hyperbolic knots in the 3-sphere. For instance, if $K$ is a non-fibered hyperbolic knot in $S^3$, it admits no non-trivial surgeries which yield L-spaces ([Ni]). Conjecture 1.1 then implies that for $n$ and $q$ as in the corollary, the rational homology sphere $X(K)(n\mu + q\lambda)$ admits a co-orientable taut foliation. Hence the same is true for its universal abelian cover $X_n(K)(\mu_n + q\lambda_n)$. This cover also has a left-orderable fundamental group, and is therefore excellent, by Remark 6.3 and [BRW, Lemma 3.1].

**Conjecture 1.6.** Let $n, q$ be coprime integers with $nq \neq 0$ and let $K$ be a hyperbolic knot in $S^3$. If the universal abelian cover of $X(K)(n\mu + q\lambda)$ is not excellent, then $K$ is fibered and if $h$ is its monodromy, $q \in \left\{ \begin{array}{l} \{nc(h)\} \quad \text{if } nc(h) \in \mathbb{Z} \\ \{\lfloor nc(h)\rfloor, \lceil nc(h)\rceil + 1\} \quad \text{if } nc(h) \not\in \mathbb{Z} \end{array} \right.$.

**Problem 1.7.** Determine necessary and sufficient conditions for the universal abelian cover of an irreducible rational homology 3-sphere $M$ to be excellent.
For instance, is the existence of a representation $\pi_1(M) \to \Homeo_+(S^1)$ with non-abelian image necessary and sufficient for the universal abelian cover of an irreducible rational homology 3-sphere $M$ to be excellent?

Fix a knot $K$ in an integer homology 3-sphere $M$ and coprime integers $p > 0$ and $q$. Let $m$ be a positive integer and set $n = mp$. We can generalize Theorem 1.2(1) and Corollary 1.5 by considering the orbifold with underlying space $X(K)(p\mu + q\lambda)$ and singular set the core of the filling solid torus with isotropy groups $\mathbb{Z}/m$. Here $H_1(O) \cong \mathbb{Z}/n$ and the universal abelian cover of $O$ corresponds to an $n$-fold cyclic cover $X_n(K)(\mu_n + mq\lambda_n) \to X(K)(p\mu + q\lambda)$ branched over the core of the $(p\mu + q\lambda)$-filling torus with branching index $m$. When $p = 1$ and $q = 0$ this is the branched cover $\Sigma_n(K) \to M$.

**Corollary 1.8.** Let $K$ be a hyperbolic fibered knot in an integer homology 3-sphere $M$ with monodromy $h$ and consider coprime integers $p > 0$ and $q$ as well as a positive integer $m \geq 1$. The universal abelian cover of the orbifold with underlying space $X(K)(p\mu + q\lambda)$ and singular set the core of the filling solid torus with isotropy groups $\mathbb{Z}/m$ is an excellent 3-manifold if $m|pc(h) - q| \geq 1$.

We also have results on cyclic branched covers of non-fibered hyperbolic links in $S^3$. Here is a special case of Theorem 10.3.

**Theorem 1.9.** Let $b \in B_{2k+1}$ be an odd-strand braid whose closure $\hat{b}$ is an oriented hyperbolic link $L$ and let $c(b)$ be the fractional Dehn twist coefficient of $b$. Suppose that $|c(b)| \geq 2$. Then all even order cyclic branched covers of $\hat{b}$ are excellent.

Theorem 1.9 combines with results of Baldwin ([Bal]) and Li-Watson ([LW]) to prove that:

**Theorem 1.10.** Conjecture 1.1 holds for irreducible 3-manifolds which admit genus one open book decompositions with connected binding.

In its turn, Theorem 1.10 combines with Theorem 1.2 to determine precisely which branched covers of genus one fibered knots $K$ are excellent and which are total L-spaces. To describe this, let $T_1$ be the fibre of such a knot. It is known that the mapping class group $\text{Mod}(T_1)$ is generated by two right-handed Dehn twists $T_{c_1}$ and $T_{c_2}$ (cf. §11, especially Figure 7). Let

$$\delta = (T_{c_1}T_{c_2})^3$$

and note that $\delta^2$ is the right-handed Dehn twist along $\partial T_1$.

**Corollary 1.11.** Suppose that $K$ is a genus one fibered knot with monodromy $h$ in a closed, connected, orientable and irreducible 3-manifold $M$. Then either $\Sigma_n(K)$ is excellent for all $n \geq 2$ or it is a total L-space and one of the following conditions hold:

1. $h$ is pseudo-Anosov, $c(h) = 0$, and $n \geq 2$.  

Next we consider satellite links.

In [GLid1], Gordon and Lidman studied the cyclic branched covers of \((p,q)\)-cable knots in \(S^3\). These are satellite knots whose patterns are \((p,q)\)-torus knots embedded standardly as a \(q\)-braid in a solid torus. They showed that the \(n\)-fold cyclic branched covers of \((p,q)\)-cable knots are always excellent, except possibly for the case \(n = q = 2\) ([GLid1, Theorem 1.3]). In the latter case they showed that the 2-fold branched covers of a \((p,2)\)-cable knots are never L-spaces [GLid2, Theorem 1], and hence the truth of Conjecture 1.1 would imply that they are excellent.

**Conjecture 1.12.** (Gordon-Lidman) The \(n\)-fold cyclic branched cover \(\Sigma_n(K)\) of a prime, satellite knot \(K\) is excellent.

Satellite links whose patterns are closed braids and whose companions are fibered are a particularly interesting class to investigate as, for instance, all satellite L-space knots in \(S^3\) fall into this category ([BM, Theorem 7.3, Theorem 7.4]; also see [HRW, Theorem 35] and [Hom, Proposition 3.3]). Theorem 1.13 and Corollary 1.14 verify special cases of Conjecture 1.12.

**Theorem 1.13.** Assume that \(L\) is a satellite link in an integer homology 3-sphere \(M\) whose pattern is hyperbolic and contained in its solid torus as the closure of an \(m\)-strand braid \(b\), and whose companion is a fibered hyperbolic knot \(C\) in \(M\) with fibre \(S\) and monodromy \(h\).

1. If \(c(h) = 0\), then the \(n\)-fold cyclic branched cover of \(L\) is excellent whenever \(\gcd(m, n) = 1\).
2. If \(c(h) \neq 0\), then the \(n\)-fold cyclic branched cover of \(L\) is excellent when \(\gcd(m, n) = 1\) and \(n \geq \frac{2}{|c(h)|}\).

We remark that by Proposition 4.4, if \(c(h) \neq 0\), then \(c(h) \geq \frac{1}{2g(C) - 1}\) where \(g(C)\) is the genus of the companion knot \(C\). Hence the condition \(n \geq \frac{2}{|c(h)|}\) in Theorem 1.13(2) holds if \(n \geq 4(2g(C) - 1)\).

**Corollary 1.14.** Assume that \(L\) is a satellite link in an integer homology 3-sphere \(M\) whose pattern is a hyperbolic closed \(m\)-strand braid and whose companion is a fibered hyperbolic knot in \(M\). Then the \(n\)-fold cyclic branched cover of \(L\) is excellent when \(\gcd(m, n) = 1\) and \(n \gg 0\). □

Consider an L-space satellite knot \(K\). Baker and Motegi have shown that the pattern \(P\) is a closed braid [BM, §7]. Further, Hanselman, Rasmussen and Watson [HRW] have shown that the companion \(C\) is also an L-space knot. Hence the companion knot \(C\) is fibered and strongly quasipositive (cf. [Hed, Theorem 1.2], [Ni, Corollary 1.3] and the calculations of [OS2]), so its
fractional Dehn twist coefficient is non-zero (cf. [Hed, HKM2]). Up to replacing $K$ by its mirror image, we can suppose that the fractional Dehn twist of the monodromy of the companion knot $C$ is positive.

Boileau, Boyer and Gordon have shown that the cyclic branched covers of satellite L-space knots are never L-spaces [BBG, Corollary 6.4]. In the case that both pattern and companion are hyperbolic, and the fractional Dehn twist coefficient of the pattern braid is nonnegative (cf. [Hom, Question 1.8]), Theorem 1.15 below shows that $\Sigma_n(K)$ is excellent whenever $n$ is relatively prime to the braid index of the pattern.

**Theorem 1.15.** Assume that $L$ is a satellite link in an integer homology 3-sphere $M$ whose pattern $P$ is hyperbolic and contained in its solid torus as the closure of an $m$-strand braid $b$, and whose companion is a fibered hyperbolic knot $C$ in $M$ with fibre $S$ and monodromy $h$. Suppose that the fractional Dehn twist coefficients $c(b)$ and $c(h)$ are non-negative. Then for $n \geq 2$ relatively prime to $m$, the $n$-fold cyclic branched cover of $L$ is excellent.

Here is the plan of the paper. In §2 we introduce background material and notational conventions. Section 3 covers some basic concepts regarding mapping class groups and braids. In particular, we show that hyperbolic links are never the closure of a non-pseudo-Anosov braids (Proposition 3.1). Section 4 introduces fractional Dehn twist coefficients from two perspectives: isotopies (§4.1) and translation numbers (§4.2). The Euler classes of representations with values in $\text{Homeo}_+(S^1)$ and of oriented circle bundles are defined and related in §5. Section 6 is devoted to a description of the universal circle associated to a rational homology 3-sphere $M$ endowed with a co-oriented taut foliation. In §7 we prove a folklore result which identifies the Euler class of the universal circle representation with the Euler class of the associated foliation’s tangent bundle, and §8 uses this to deduce the left-orderability of $\pi_1(M)$ when this Euler class vanishes. The material of the previous sections is combined in §9 to study the left-orderability of 3-manifolds given by open books. In particular, Theorem 1.2, Corollary 1.5 and Corollary 1.8 are proved here. In §10 we prove Theorems 1.9 and 10.3, which are used in §11 to deduce Theorem 1.10 and Corollary 1.11. Finally in §12, we apply the results of §9 and §10 to study cyclic branched covers of satellite knots in order to prove Theorems 1.13 and 1.15.

2. **Some background results, terminology and notation**

We set some conventions in this section which will be used throughout the paper.

2.1. **Link exteriors in rational homology spheres.** Let $M$ be an oriented rational homology 3-sphere and $L$ an oriented null-homologous link in $M$. We use $N(L)$ to denote a closed tubular neighbourhood of $L$ and $X(L) = M \backslash N(L)$ to denote the exterior of $L$ in $M$.

If $L = \bigsqcup_i K_i$ is the decomposition of $L$ into its component knots, then $N(L) = \bigsqcup_i N(K_i)$ where $N(K_i)$ is a tubular neighbourhood of $K_i$.

A **meridional disk** of $K_i$ is any essential properly embedded disk in $N(K_i)$ which is oriented so that its intersection with the oriented knot $K_i$ is positive.
Lemma 2.1. Suppose that $i \in L_K$. In the case that $X$ is a knot, $\lambda_1$ represents the longitudinal class of $L = K_1$ in $H_1(\partial X(K_1))$.

A meridional class of $K_i$ in $H_1(X(L))$ is the image of $\mu_i$ under the inclusion-induced homomorphism $H_1(\partial N(K_i)) \to H_1(X(L))$.

The assumption that $L$ is null-homologous implies that there is a compact, connected, oriented surface $S$ properly embedded in $X(L)$ which intersects each component $\partial N(K_i)$ of $\partial X(L)$ in an oriented simple closed curve $\lambda_i$ isotopic in $N(K_i)$ to $K_i$. It is clear from the construction that

$$\mu_i \cdot \lambda_i = 1$$

for each $i$.

In the case that $L$ is a knot, $\lambda_1$ represents the longitudinal class of $L = K_1$ in $H_1(\partial X(K_1))$.

**Lemma 2.1.** Suppose that $K$ is a null-homologous knot in a rational homology 3-sphere $M$ with exterior $X_K$. Then $H_1(X_K) \cong H_1(M) \oplus \mathbb{Z}$ where the second factor is generated by a meridional class of $K$. Further, the inclusion-induced homomorphism $H^2(M) \to H^2(X_K)$ is an isomorphism.

**Proof.** Excision implies that

$$H_r(M, X_K) \cong H_r(N(K), \partial N(K)) \cong \begin{cases} \mathbb{Z} & \text{if } r = 2, 3 \\ 0 & \text{otherwise} \end{cases}$$

where $H_2(M, X_K) \cong \mathbb{Z}$ is generated by the class $\eta$ carried by a meridional disk of $N(K)$. Then the exact sequence of the pair $(M, X_K)$ yields a short exact sequence

$$(2.1.1) \quad 0 \to H_2(M, X_K) \xrightarrow{\partial} H_1(X_K) \to H_1(M) \to 0$$

where $\partial(\eta)$ is a meridional class $\mu$ of $K$. Since $K$ is null-homologous in $M$, there is a properly embedded, compact, connected, oriented surface $S$ in $X_K$ whose boundary represents the longitudinal class $\lambda$ of $K$ in $H_1(\partial X_K)$. Then $\partial([S]) \cdot \mu = \lambda \cdot \mu = \pm 1$, where $[S] \in H_2(X_K, \partial X_K)$ corresponds to the fundamental class of $S$. Hence the homomorphism $H_1(X_K) \to H_2(M, X_K), \alpha \mapsto (\alpha \cdot [S])\eta$, splits the sequence (2.1.1) up to sign, which proves the first assertion of the lemma.

For the second, consider the connecting map $H^1(X_K) \xrightarrow{\delta} H^2(M, X_K)$ from the cohomology exact sequence of the pair $(M, X_K)$. Excision shows that

$$H^2(M, X_K) \cong H^2(N(K), \partial N(K)) \cong \text{Hom}(H_2(N(K), \partial N(K)), \mathbb{Z}) \cong \mathbb{Z}$$

is generated by the homomorphism which takes the value 1 on the class $\eta \in H_2(M, X_K) = H_2(N(K), \partial N(K))$. On the other hand, since $H_1(X_K) \cong H_1(M) \oplus \mathbb{Z}$ where the $\mathbb{Z}$ factor is generated by $\partial \eta$, if $\nu \in \text{Hom}(H_1(X_K), \mathbb{Z}) \cong H^1(X_K)$ is the homomorphism which takes the value 1 on a meridian of $K$ and 0 on $H_1(M)$, then $\delta(\nu)$ is a generator of $H^2(N(K), \partial N(K))$. Thus $\delta$ is surjective. It then follows from the exact cohomology sequence of the pair $(M, X_K)$ that the homomorphism $H^2(M) \to H^2(X_K)$ is an isomorphism, which completes the proof. □
2.2. Cyclic branched covers of null-homologous links. Given a null-homologous oriented link \( L \) in an oriented rational homology sphere \( M \) and compact, connected, oriented surface \( S \) properly embedded in \( X(L) \) as above, let \([S]\in H_2(X(L),\partial X(L))\) correspond to the fundamental class of \( S \). For each \( n\geq 1 \), the epimorphism
\[
H_1(X(L)) \xrightarrow{\alpha \to \alpha \cdot [S]} \mathbb{Z} \xrightarrow{\text{mod } n \text{ reduction}} \mathbb{Z}/n,
\]
determines an \( n \)-fold cyclic cover
\[
X_n(L) \to X(L)
\]
and an \( n \)-fold cyclic cover
\[
(\Sigma_n(L),\tilde{L}) \xrightarrow{p} (M,L)
\]
branched over \( L \).

The link \( \tilde{L} \) decomposes into components \( \tilde{L} = \bigsqcup_i \tilde{K}_i \) where \( \tilde{K}_i = p^{-1}(K_i) \). Similarly, its closed tubular neighbourhood \( N(\tilde{L}) = \Sigma_n(L) \setminus X_n(L) \) splits into components \( N(\tilde{L}) = \bigsqcup_i N(\tilde{K}_i) \) where \( N(\tilde{K}_i) \) is a tubular neighbourhood of \( \tilde{K}_i \).

For each \( i \) there is a basis \( \{\tilde{\mu}_i,\tilde{\lambda}_i\} \) of \( H_1(\partial N(\tilde{K}_i)) \) determined by the property that
\[
\tilde{\mu}_i \xrightarrow{p_*} n\mu_i
\]
and
\[
\tilde{\lambda}_i \xrightarrow{p_*} \lambda_i.
\]
The surface \( S \) lifts to a properly embedded surface \( \tilde{S} \subset X_n(L) \) which intersects \( \partial \tilde{N}(K_i) \) in an oriented simple closed curve representing \( \tilde{\lambda}_i \).

By construction, \( \Sigma_n(L) \) is the \((\tilde{\mu}_1,\tilde{\mu}_2,\ldots,\tilde{\mu}_n)\)-Dehn filling of \( X_n(L) \).

2.3. Lifting contact structures to branched covers. Let \( M \) be an oriented rational homology 3-sphere and \( L \) an oriented null-homologous link in \( M \). Fix a compact, connected, oriented surface \( S \) properly embedded in \( X(L) \) and let \( p : (\Sigma_n(L),X_n(L),\tilde{L}) \xrightarrow{p} (M,X(L),L) \) be as above, where \( n \geq 1 \).

Let \( \xi = \ker(\alpha) \) be a positive contact structure on \( M \) determined by a smooth, nowhere zero 1-form \( \alpha \) and suppose that \( L \) is a positively transverse to \( \xi \). There is a lift of \( \xi \) to \( \Sigma_n(L) \), denoted by \( \tilde{\xi} \), which is the kernel of the pull-back form \( p^*(\alpha) \) on \( X_n(L) \) and is positively transverse to \( \tilde{L} \) (cf. [HKP, §2.5], [Gei, Theorem 7.5.4]). More precisely, \( \xi \) can be constructed as follows.

Recall that \( L = \bigsqcup_i K_i \) and \( \tilde{L} = \bigsqcup_i \tilde{K}_i \) where \( \tilde{K}_i = p^{-1}(K_i) \). For each \( i \), there is a suitable tubular neighborhoods \( N(K_i) \) and \( N(\tilde{K}_i) \), and cylindrical coordinates \((r,\theta,z)\) and \((\tilde{r},\tilde{\theta},\tilde{z})\) over the tubular neighborhoods \( N(K_i) \) and \( N(\tilde{K}_i) \) respectively. We may assume that the contact form \( \alpha \) restricted to \( N(K_i) \) is in the standard form \( \alpha|_{N(K_i)} = dz + r^2d\theta \) and the cyclic branched cover \( p \) restricts to \( N(\tilde{K}_i) \setminus \tilde{K}_i \) sends \((\tilde{r},\tilde{\theta},\tilde{z})\) to \((r,n\theta,z)\). The pull-back \( p^*(\alpha|_{N(K_i)}\setminus K_i) = d\tilde{z} + nr^2d\tilde{\theta} \) is a contact form over \( N(\tilde{K}_i) \setminus \tilde{K}_i \), which extends smoothly to \( N(\tilde{K}_i) \) by letting \( \tilde{\alpha}|_{\tilde{K}_i} = d\tilde{z} \). Extending \( p^*(\alpha|_{\Sigma_n(L)}\setminus \tilde{L}) \) in this way to the entire tubular neighborhood, we produce the desired contact form on \( \Sigma_n(L) \), denoted by \( \tilde{\alpha} \). Let \( \tilde{\xi} = \ker(\tilde{\alpha}) \).
2.4. **Fibered knots and open books.** An oriented knot $K$ in $M$ is called fibered with fibre $S$ if $S$ is a compact, connected, orientable surface properly embedded in $X(K)$ which has connected boundary and there is a locally-trivial fibre bundle $X(K) \to S^1$ with fibre $S$. Note that $S \cap \partial X(K)$ carries the longitudinal slope $\lambda$ of $K$.

A monodromy of $K$ is an orientation-preserving homeomorphism $h : S \to S$ such that $h|_{\partial S}$ is the identity, $X(K) \cong (S \times I)/((x, 1) \sim (h(x), 0))$, and if $x \in \partial S$, then the loop on $\partial X(K)$ determined by $\{x\} \times I$ carries the meridional slope $\mu$ of $K$. The monodromy $h$ is well-defined up to conjugation and an isotopy fixed on $\partial S$.

Conversely, given an orientation-preserving homeomorphism $h$ of a compact, connected, orientable surface $S$ with connected boundary which restricts to the identity on $\partial S$, there is a well-defined closed, connected, orientable 3-manifold $M$ obtained from the Dehn filling of $(S \times I)/((x, 1) \sim (h(x), 0))$ along the slope determined by the image of $\{x\} \times I$ for $x \in \partial S$. The core of the filling solid torus is a knot $K$ in $M$ which is fibered with fibre $S$ and monodromy $h$. The meridian of $K$ is carried by the image of $\{x\} \times I$. We call the pair $(S, h)$ an open book decomposition of $M$ with binding $K$.

### 3. Mapping class groups and closed braids

Throughout this section $S$ will denote an $m$-punctured ($m \geq 0$) smooth orientable compact surface with nonempty boundary. All diffeomorphisms of $S$ will be assumed to be orientation-preserving.

We use $\text{Mod}(S)$ to denote the mapping class group of isotopy classes of diffeomorphisms of $S$ which restrict to the identity on $\partial S$. Isotopies are assumed to be fixed on $\partial S$.

From time to time we will identify an element of $\text{Mod}(S)$ with one of its representative diffeomorphisms, though only when discussing properties held by all such representatives.

#### 3.1. The Nielsen-Thurston classification of mapping classes.** A homeomorphism $\varphi : S \to S$ is called pseudo-Anosov if it preserves a pair $(F_s^\varphi, \mu_s)$ and $(F_u^\varphi, \mu^u)$ of mutually transverse, measured, singular foliations on $S$, and there is a number $\lambda > 1$ such that $\varphi$ scales the transverse measure $\mu_s$ by $\lambda^{-1}$ and the transverse measure $\mu^u$ by $\lambda$. We call $(F_s^\varphi, \mu_s)$ the stable foliation of $\varphi$ and $(F_u^\varphi, \mu^u)$ the unstable foliation of $\varphi$.

By the Nilsen-Thurston classification, each element $f$ in $\text{Mod}(S)$ is freely isotopic to a map $\varphi : S \to S$ which is either

- a periodic diffeomorphism, i.e. $\varphi^n = 1$ for some $n > 0$, or
- a reducible diffeomorphism, i.e. there exists a nonempty collection $\mathcal{C} = \{c_1, \cdots, c_r\}$ of pairwise disjoint essential simple closed curves in $S$ such that $f(\mathcal{C}) = \mathcal{C}$, or
- a pseudo-Anosov homeomorphism.
We call $\varphi$ a Nielsen-Thurston representative of $f$ in $\text{Mod}(S)$ and $f$ is called periodic, reducible or pseudo-Anosov if its Nielsen-Thurston representative has the corresponding property.

It is known that a pseudo-Anosov mapping class is neither periodic nor reducible [Thu] (also see [FM]). A fundamental result of Thurston is that the interior of the mapping torus of $f \in \text{Mod}(S)$ is a finite volume hyperbolic manifold if and only if $f$ is pseudo-Anosov. It contains an essential torus if $f$ is reducible, and it is a Seifert fibre manifold if $f$ is periodic.

3.2. **The braid group** $B_m$ and $\text{Mod}(D_m)$. We use $B_m$ to denote the group of isotopy classes of smooth $m$-strand braids, where each strand of a braid is oriented upward. Let $\sigma_i$ be the standard $i^{th}$ Artin generator of the braid group $B_m, i = 1, \cdots, n - 1$ (Figure 1(A)).

![Figure 1](image-url)  
**Figure 1.** (A) $\sigma_i$ in $B_m$; (B) $\sigma_i$ in $\text{Mod}(D_m)$

It is well known that the braid group $B_m$ is isomorphic to the mapping class group $\text{Mod}(D_m)$, where $D_m$ denotes the $m$-punctured disk obtained by removing $m$ points on the real line from the interior of the unit disk $D^2$. We identify these two groups through the following correspondence. Given an $m$-strand braid the corresponding diffeomorphism of $D_m$ is obtained by sliding the $m$-punctured disk along the braid from the bottom to the top (see Figure 1). The product of two braids $b_1$ and $b_2$ is the braid obtained by placing $b_1$ on the top of $b_2$. When $b_1$ and $b_2$ are viewed as diffeomorphisms of the punctured disk $D_m$, we have $b_1 b_2(x) = b_1(b_2(x))$ for all $x \in D_m$.

A braid $b \in \text{Mod}(D_m)$ is called pseudo-Anosov, respectively periodic, respectively reducible, if it is freely isotopic to a homeomorphism of $D_m$ with the corresponding property.

3.3. **Hyperbolic links are closures of pseudo-Anosov braids.** The closure of a braid $b$, denoted $\hat{b}$, is an oriented link in $S^3$ obtained by closing the braid $b$ as illustrated in Figure 2. A classical theorem of Alexander [Al] asserts that for any oriented link $L$ in $S^3$, there is an $m \geq 1$ and a braid $b \in B_m$ such that $L$ is isotopic to $\hat{b}$.

Recall that a link $L$ in $S^3$ is hyperbolic if its exterior $S^3 \setminus L$ is hyperbolic, i.e., it admits a complete finite volume Riemannian metric of constant curvature $-1$. Ito has shown [Ito2, Theorem 1.3] that if the absolute value of the Dehornoy floor (Definition 10.4) of $b \in B_m$ is at least 2 and $\beta$ is a knot, then $\hat{b}$ is hyperbolic if and only if $b$ is pseudo-Anosov. While this may not hold in general, we do have the following result.
Proposition 3.1. Given a braid $b \in B_m$, if the closed braid $\hat{b}$ is a hyperbolic link, then $b$ is a pseudo-Anosov braid.

Proof. By the Nielsen-Thurston classification, it suffices to show that $b$ is neither periodic nor reducible.

Let $N_b \cong D_m \times [0,1]/(x,1) \sim (b(x),0)$ be the mapping torus of $b$. The boundary $T_0$ of $N_b$ is homeomorphic to a torus and if $\nu = \{p\} \times [0,1]/\sim$, a simple closed curve on $\partial N_b$, where $p$ is a point on $\partial D_m$ (see Figure 2), then the exterior of $\hat{b}$ is homeomorphic to the manifold $N_b(\nu)$, obtained by performing a Dehn filling $N_b$ along $\nu$.

If $b$ is periodic, then $N_b$ is Seifert fibered, and as each Dehn filling of a Seifert fibered manifold is a connected sum of Seifert fibered manifolds, our hypothesis implies that this case does not arise.

Suppose then that $b$ is reducible and fix a collection $C = \{c_1, \ldots, c_r\} \subset \text{int}(S)$ of pairwise disjoint essential simple closed curves in $D_m$ such that $b(C)$ is isotopic to $C$. After possibly replacing $f$ from within its mapping class we can assume that $b(C) = C$.

Each $c_i$ bounds a disk $E_i$ in the unit disk, and as $c_i$ is essential in $D_m$, some of the punctures in $D_m$ are contained in $E_i$ and some are not. By hypothesis, the union $\bigcup_i (c_i \times [0,1])/\sim$ gives rise to a collection of essential tori in $N_b$. Each of these tori is separating since $N_b$ is contained in $S^3$.

Let $T$ denote one of these tori. Then $N_b = M_1 \cup_T M_2$ where $T = M_1 \cap M_2 = \partial M_1 \cap \partial M_2$ and neither $M_1$ nor $M_2$ is the product of a torus and an interval. Without loss of generality we can assume that $T_0 \subset \partial M_1$.

The exterior of the closed braid $\hat{b}$ is obtained by performing a Dehn filling of $N_b$ along $\nu$. That is, it is given by $N_b(\nu) = M_1(\nu) \cup_T M_2$. If $T$ compresses in $N_b(\nu)$, it must compress in $M_1(\nu)$, and this occurs if and only if $M_1(\nu) \cong (S^1 \times D^2)\# W$ for some 3-manifold $W$, possibly $S^3$.

If $W \neq S^3$, then $N_b(\nu)$ is reducible, contrary to our assumption that it is hyperbolic. Thus $M_1(\nu) \cong S^1 \times D^2$, so in particular $M_1$ is compact. It follows that $M_2$ contains each of the ends of $N_b$. Fix integers $1 \leq i_1 < \ldots < i_k \leq r$ such that $T \cap D_m = c_{i_1} \cup \ldots \cup c_{i_k}$. Then each of the punctures of $D_m$ lie to the side of $c_{i_1}$ contained in $M_2$. But then $E_{i_1}$ contains each of
these punctures, contrary to construction. Thus $b$ cannot be reducible, which completes the proof.

4. Fractional Dehn twist coefficients

We assume that $S$ is an oriented hyperbolic surface with nonempty geodesic boundary. Given $h : S \to S$ representing an element of $\text{Mod}(S)$, let $H_t : S \to S$ denote a free isotopy between $H_0 = h$ and $H_1 = \varphi$, the Nielsen-Thurston representative of $h$.

We are interested in the fractional Dehn twist coefficient of $h$ with respect to a boundary component of $S$. Intuitively, the fractional Dehn twist coefficient of $h$ is a rational number representing the amount of twisting $\partial S$ undergoes during the isotopy $H_t$ from $h$ to $\varphi$. The concept was first defined in [HKM1] to study the tightness of the contact structure supported by an open book $(S,h)$. When $h$ is pseudo-Anosov, it is closely related to the degeneracy slope of a pseudo-Anosov homeomorphism [GO]. If, in addition, $\partial S$ is also connected, it gives a convenient criterion due to Honda, Kazez and Matic for the existence of co-oriented taut foliations in the open book $(S,h)$ (cf. Theorem 4.1) which provides a key element of the proofs of our main results.

Originally, the fractional Dehn twist coefficient was defined only for diffeomorphisms freely isotopic to pseudo-Anosov maps [HKM1], but it is clear that the concept of measuring the difference between $h$ and its Nielsen-Thurston representative over $\partial S$ makes sense for all types of diffeomorphisms.

We give two equivalent definitions of the fractional Dehn twist coefficient in Section 4.1 and Section 4.2 below, and will take advantage of both points of view. For simplicity, we also assume that $\partial S$ is connected and leave to the reader the simple task of extending the definition to the case that $\partial S$ is not connected (also see [HKM1]).

The following theorem summarizes results from [HKM2]. See Theorem 4.1, Theorem 4.3, and Lemma 4.4 of that paper for the details. (The results of [Bn, KR2] are needed for the proof of [HKM2, Theorem 4.3].)

**Theorem 4.1** ([HKM2]). Assume that $(S,h)$ is an open book decomposition of a closed oriented 3-manifold $M$, where the boundary $\partial S$ is connected and $h$ is freely isotopic to a pseudo-Anosov homeomorphism. If the fractional Dehn twist coefficient of $h$ satisfies $c(h) \geq 1$, then there exists a co-orientable taut foliation on $M$ which is transverse to the binding of $(S,h)$ and is homotopic to the contact structure supported by $(S,h)$.

4.1. Fractional Dehn twist coefficients via isotopies. In this section, we define the fractional Dehn twist coefficient following the idea of [HKM1] and extend the definition to all Nielsen-Thurston types of diffeomorphism $h$.

Let $C \cong S^1$ denote the boundary of $S$ and fix a periodic orbit $\{p_0, \ldots, p_{n-1}\} \subset C$ of the Nielsen-Thurston representative $\varphi$ of $h$. When $\varphi$ is pseudo-Anosov, we may choose $\{p_0, \ldots, p_{n-1}\}$ to be a subset of the singular points on $C$ of the stable singular foliation of $\varphi$. 
Assume that \( p_0, \ldots, p_{n-1} \) are indexed cyclically according to the orientation on \( C \) induced by that on \( S \). Since the set \( \{p_0, \ldots, p_{n-1}\} \) is preserved under \( \varphi \), there exists an integer \( k \in \{0, 1, \ldots, n-1\} \) such that \( \varphi(p_0) = p_k \). Then \( H_{t_0} : [0, 1] \rightarrow C \) defines a path on the boundary component \( C \) connecting \( H_0(p_0) = p_0 \) to \( H_1(p_0) = p_k \).

The orientation of \( C \) determines an oriented subarc \( \gamma_{p_0 p_k} \) of \( C \) from \( p_0 \) to \( p_k \). Let \( \tilde{\gamma}_{p_0 p_k} \) denote the same arc with the opposite orientation. Then \( H_t(p_0) \circ \tilde{\gamma}_{p_0 p_k} \), the concatenation of the paths \( H_t(p_0) \) and \( \tilde{\gamma}_{p_0 p_k} \), is a loop in \( C \) based at \( p_0 \). Hence there is a unique integer \( m \) such that

\[
[H_t] = [C]^m \in \pi_1(C, p_0)
\]

where \( [C] \) is the generator of \( \pi_1(C, p_0) \equiv \mathbb{Z} \) determined by the orientation of \( C \).

**Definition 4.2 ([HKM1]).** The fractional Dehn twist coefficient \( c(h) \) of the diffeomorphism \( h \) is defined to be

\[
c(h) = m + \frac{k}{n}.
\]

**Remark 4.3.** Since the connected components of \( \text{Homeo}(S) \) are contractible when \( S \) is hyperbolic (see Theorem 1.14 in [FM] and the references therein), any two paths \( H_t \) and \( H'_t \) between \( h \) and \( \varphi \) (as above) are homotopic rel \( \{h, \varphi\} \). Thus the paths \( H_t|_{p_0} \) and \( H'_t|_{p_0} \) are homotopic rel \( \{0, 1\} \), which shows that \( c(h) \) is independent of the choice of \( H_t \). A similar argument shows that \( c(h) \) depends only on the class of \( h \) in \( \text{Mod}(S) \), and so determines an invariant for each mapping class in \( \text{Mod}(S) \).

**Proposition 4.4.** (cf. [KR1, Theorem 4.4]) Let \( S \) be a compact orientable hyperbolic surface with connected boundary \( C \) and let \( h \) be a diffeomorphism of \( S \) which restricts to the identity on \( \partial S \). If \( h \) is pseudo-Anosov and \( c(h) \neq 0 \), then

\[
\left| c(h) \right| \geq \frac{1}{-2\chi(S)}.
\]

**Proof.** This is a straightforward consequence of the Euler-Poincaré formula [FM, Proposition 11.4].

Let \( \mathcal{F}^s \) be the stable singular foliation of the pseudo-Anosov homeomorphism \( \varphi \) that is freely isotopic to \( h \). By definition, \( c(h) \) can be written as a possibly unreduced fraction \( p/q \) with \( q > 0 \) being the number of the singular points of \( \mathcal{F}^s \) on \( C \).

Let \( \{x_i\} \) be the singular points of \( \mathcal{F}^s \) contained in the interior of \( S \). For each \( i \), \( n_i \geq 3 \) will denote the number of prongs of \( \mathcal{F}^s \) at \( x_i \). Then by the Euler-Poincaré formula, we have

\[
2(\chi(S) + 1) = (2 - q) + \sum_i (2 - n_i).
\]

Since \( n_i \geq 3 \), \( \sum x_i (2 - n_i) \leq 0 \) with equality only if \( \{x_i\} \) is empty. It follows that \(-2\chi(S) \geq q \).

By assumption, \( c(h) \neq 0 \). Therefore,

\[
\left| c(h) \right| = \frac{|p|}{q} \geq \frac{1}{q} \geq \frac{1}{-2\chi(S)}.
\]

\( \square \)
4.2. Fractional Dehn twist coefficients via translation numbers. Recall that the group Homeo$_+ (S^1)$ has the following central extension:

\[(4.2.1) \quad 1 \longrightarrow \mathbb{Z} \longrightarrow \tilde{\text{Homeo}}_+(S^1) \overset{\pi}{\longrightarrow} \text{Homeo}_+(S^1) \longrightarrow 1,\]

where $\tilde{\text{Homeo}}_+(S^1)$ is the universal covering group of $\text{Homeo}_+(S^1)$, consisting of the elements of $\text{Homeo}_+(\mathbb{R})$ which commute with translation by 1, which we denote by $\text{sh}(1)$. The kernel of the covering homomorphism $\pi$ is the group of integral translations of the real line.

Fix $x_0 \in \mathbb{R}$ and define the translation number of $\tilde{f} \in \tilde{\text{Homeo}}_+(S^1)$ to be the limit

\[\tau(\tilde{f}) = \lim_{n \to \infty} \frac{\tilde{f}^n(x_0) - x_0}{n}\]

This limit always exists and is independent of the choice of $x_0$.

Let $\tilde{S}$ be the universal cover of $S$ and $\tilde{C} \subset \tilde{S}$ be a lift of $\partial S = C$. By construction, we can take $\tilde{S}$ to be a closed subset of $\mathbb{H}^2$ with geodesic boundary. In particular, $\tilde{C}$ is geodesic. We use $\partial_\infty \tilde{S}$ to denote the intersection of the Euclidean closure of $\tilde{S}$ in $\mathbb{H}^2$ with $\partial \mathbb{H}^2$. Then $\partial \tilde{S} \cup \partial_\infty \tilde{S}$ is homeomorphic to a circle. The complement of the closure of $\tilde{C}$ in this circle is homeomorphic to $\mathbb{R}$ which we parameterise and orient so that the lift of the Dehn twist along $C$, denoted by $T_{\partial S}$, is the translation $\text{sh}(1)$.

Given any element $f \in \text{Mod}(S)$, let $\tilde{f} : \tilde{S} \to \tilde{S}$ denote the unique lift of $f$ satisfying $\tilde{f}|_{\tilde{C}} = \text{id}_{\tilde{C}}$. This correspondence defines an embedding of groups

\[\iota : \text{Mod}(S) \to \tilde{\text{Homeo}}_+(S^1),\]

with $\iota(T_{\partial S}) = \text{sh}(1)$. It was shown in [Mal] (see also [IK, Theorem 4.16]) that the fractional Dehn twist coefficient of $h$ in $\text{Mod}(S)$ satisfies

\[c(h) = \tau(\iota(h)).\]

Here are a few properties of fractional Dehn twist coefficients inherited from those of translation numbers (cf. [Ghy, §5]).

**Lemma 4.5.** The fractional Dehn twist coefficient map $c : \text{Mod}(S) \to \mathbb{Q}$ takes the value 1 on $T_{\partial S}$ and is invariant under conjugation. If $h_1, h_2 \in \text{Mod}(S)$ commute, then $c(h_1h_2) = c(h_1) + c(h_2)$. In particular, $c(h^n) = nc(h)$ for any $h \in \text{Mod}(S)$ and $n \in \mathbb{Z}$.

5. Euler classes of circle bundles and representations

In this section, we first review the definition of the the Euler class of an oriented $S^1$-bundle over a CW complex $X$ and how it relates to the problem of lifting a representation $\rho : \pi_1(X) \to \text{Homeo}_+(S^1)$ to a representation into $\text{Homeo}_+(S^1)$ (see (4.2.1) in §4.2).
5.1. Euler classes of circle bundles. Let \( \xi \) be an oriented circle bundle \( E \to X \) where \( X \) is an oriented CW complex. The Euler class \( e(\xi) \in H^2(X) \) is the obstruction to finding a section of \( \xi \) and its vanishing is equivalent to the triviality of \( \xi \) as a bundle. A representative cocycle for \( e(\xi) \) is constructed as follows.

Since \( S^1 \) is a \( K(\mathbb{Z}, 1) \) space, the only obstruction to the existence of a section of \( \xi \) arises when one tries to extend a section over the 1-skeleton \( X^{(1)} \) of \( X \) to the 2-skeleton \( X^{(2)} \). Fix a section \( \sigma : X^{(1)} \to E \) and define a cellular 2-cochain \( c_\sigma : C_2(X) \to \mathbb{Z} \) as follows. Let \( \varphi_\alpha : D^2 \to X^{(2)} \) be the characteristic map of a 2-cell \( e_\alpha \) and let \( \xi_{D^2} = (E_{D^2} \to D^2) \) denote the pull-back of \( \xi \) through \( \varphi_\alpha \). Then \( \sigma \) defines a section of \( \xi_{D^2} \) over \( \partial D^2 \). Since \( D^2 \) is contractible, \( \xi_{D^2} \) is trivial. By fixing a trivialization \( E_{D^2} \to D^2 \times S^1 \), one has the following composite map from \( S^1 \) to \( S^1 \)

\[
S^1 = \partial D^2 \to E_{D^2} \to D^2 \times S^1 \to S^1.
\]

The value of \( c_\sigma \) on \( e_\alpha \) is defined to be the degree of this map. (Here, the orientation on \( e_\alpha \) determines that of \( D^2 \).) This 2-cochain is actually a cocycle whose cohomology class \([c_\sigma]\) is independent of the choices made in its construction. Further, the class is equal to the Euler class \( e(\xi) \). See [Mor, Section 6.2], for instance.

Given an oriented 2-disk-bundle or an oriented \( \mathbb{R}^2 \)-bundle, there is an associated oriented circle bundle \( \xi \) over \( X \). The Euler class of the 2-disk-bundle or the \( \mathbb{R}^2 \)-bundle is defined to be \( e(\xi) \).

5.2. Euler classes and Thom classes. For later use, we record how to express the Euler class of an oriented \( S^1 \)-bundle \( \xi \) in terms of the Thom class of the associated disk bundle. For details, see [Spa, §5.7], where the Thom class is referred to as the orientation class and the Euler class is referred to as the characteristic class.

Consider the mapping cylinder \( D_\xi \to X \) of an oriented circle bundle \( E \to X \). This is an oriented 2-disk bundle and as such has a Thom class \( u_\xi \in H^2(D_\xi, E) \) uniquely characterised by the condition that for each disk fibre \( D \) of \( D_\xi \), the image of \( u_\xi \) under the restriction homomorphism \( H^2(D_\xi, E) \to H^2(D, \partial D) \) is the orientation generator. The Euler class \( e(\xi) \) of \( \xi \) is the image of \( u_\xi \) under the composition \( H^2(D_\xi, E) \to H^2(D_\xi) \xrightarrow{\partial} H^2(X) \).

5.3. Lifting representations with values in \( \text{Homeo}_+(S^1) \). Fix a representation \( \rho : \pi_1(X) \to \text{Homeo}_+(S^1) \). There is an associated oriented circle bundle \( E_\rho \to X \) whose total space is defined by

\[
E_\rho = \tilde{X} \times S^1 / (x, v) \sim (g \cdot x, \rho(g)v),
\]

where \( \tilde{X} \) is the universal cover of \( X \). The projection map \( \tilde{X} \times S^1 \to \tilde{X} \) descends to the bundle map \( E_\rho \to X \).

**Lemma 5.1** ([Mi1] Lemma 2). A representation \( \rho : \pi_1(X) \to \text{Homeo}_+(S^1) \) lifts to a representation \( \tilde{\rho} : \pi_1(X) \to \text{Homeo}_+(S^1) \) if and only if the Euler class of the circle bundle \( E_\rho \) vanishes. □

**Remark 5.2.** Each central extension of a group \( G \) by an abelian group \( A \) determines a class \( e \in H^2(G; A) \), called the characteristic class of the extension, and the correspondence is bijective.
(cf. [Ghy, §6.1]). Such an extension is isomorphic to the trivial extension \( 1 \to A \to G \times A \to G \to 1 \) if and only if its characteristic class is zero. It is known that the characteristic class of the extension (4.2.1), denoted by \( e_{S^1} \), generates \( H^2(\text{Homeo}_+(S^1)) \cong \mathbb{Z} \). See [MM, Example 2.12]. Hence given a representation \( \rho : G \to \text{Homeo}_+(S^1) \), \( \rho \) admits a lift to a representation with values in \( \text{Homeo}_+(S^1) \) if and only if \( e(\rho) = \rho^*(e_{S^1}) = 0 \). When \( G = \pi_1(M) \), the two obstruction classes described above coincide. More precisely, when \( M \) is irreducible, we have \( e(\rho) = \pm e(E_\rho) \in H^2(M) = H^2(\pi_1(M)) \). See the proof of [Mi1, Lemma 2].

5.4. **The vanishing of the Euler class of certain lifted contact structures.** Let \( M \) be an oriented rational homology 3-sphere and \( L \) an oriented null-homologous link in \( M \). Fix a compact, connected, oriented surface \( S \) properly embedded in \( X(L) \) and let

\[
(\Sigma_n(L), X_n(L), \tilde{L}) \xrightarrow{p} (M, X(L), L)
\]

be as in §2.2 where \( n \geq 1 \).

Let \( \xi = \ker(\alpha) \) be a positive contact structure on \( M \) determined by a smooth, nowhere zero 1-form \( \alpha \) and suppose that \( L \) is a positively transverse to \( \xi \). Recall the lift \( \tilde{\xi} \) of \( \xi \) to \( \Sigma_n(L) \) described in §2.3.

**Lemma 5.3.** If \( e(\tilde{\xi}) = 0 \), then \( e(\tilde{\xi}) = 0 \).

**Proof.** To simplify notation, we write \( N_i \) for \( N(K_i) \) and \( \tilde{N}_i \) for \( N(\tilde{K}_i) \). Let \( D_i \) be a meridian disk of \( N_i \) oriented coherently with \( K_i \) and \( M \). Then \( \tilde{D}_i = p^{-1}(D_i) \) is an oriented meridional disk in \( \tilde{N}_i \). We use \( \tilde{m}_i \) to denote the oriented boundary of \( \tilde{D}_i \). By construction, the \( n \)-fold cyclic branched cover \( \Sigma_n(L) \) is obtained from \( X_n(L) \) by attaching the meridian disk \( \tilde{D}_i \) to \( X_n(L) \) along \( \tilde{m}_i \) for each \( i \) and then plugging the boundary of the resultant 3-manifold with 3-cells.

Set \( \alpha_0 = \alpha \) where \( \alpha \) is the smooth, nowhere zero 1-form with \( \xi = \ker(\alpha) \) described above, and fix a one-parameter family of co-oriented 2-plane fields \( \xi_t = \ker(\alpha_t) \ (t \in [0, 1]) \) on \( \Sigma_n(L) \) such that \( \alpha_t|_{N_i} = dz_i + (1 - t)r_i^2d\theta_i \). Note that \( \xi_t|_{D_i} \) is the tangent bundle \( D_i \).

Let \( \tilde{\alpha}_t = p^*(\alpha_t) \) over \( \Sigma_n(L) \setminus \tilde{L} \) where \( \Sigma_n(L) \xrightarrow{p} M \) is the branched cover. According to our choice of tubular neighborhoods \( N_i, \tilde{N}_i \) and the coordinate systems on them (§2.3), we have

\[
\tilde{\alpha}_t|_{\tilde{N}_i \setminus \tilde{K}_i} = d\tilde{z}_i + n(1 - t)r_i^2d\tilde{\theta}_i.
\]

Then \( \tilde{\alpha}_t \) extends smoothly to \( \Sigma_n(L) \) with \( \tilde{\alpha}_t = d\tilde{z}_i \) over \( \tilde{K}_i \). Hence it defines a homotopy between co-oriented 2-plane fields \( \tilde{\xi} = \ker(\tilde{\alpha}_0) \) and \( \tilde{\xi}_1 = \ker(\tilde{\alpha}_1) \). To show \( e(\tilde{\xi}) = 0 \), we will show \( e(\tilde{\xi}_1) = 0 \).

By assumption \( e(\tilde{\xi}_1) = e(\tilde{\xi}) = 0 \) in \( H^2(M) \) so in particular, \( \xi_1 \) admits a nowhere zero section \( \sigma \). Since \( \xi_1|_{D_i} \) is the tangent bundle of \( D_i \), we may suppose that \( \sigma|_{D_i} = \partial x_i \), where \( x_i = r_i \cos(\theta_i) \) over \( D_i \). See Figure 3.

Let \( \tilde{\sigma} : X_n \to \tilde{\xi}_1 \) be a section of the restriction of \( \tilde{\xi}_1 \) to \( X_n \) obtained by lifting \( \sigma \). Then there is a 2-cocycle \( c_{\tilde{\sigma}} \) which vanishes on \( X_n \) determined by \( \tilde{\sigma} \) with \( [c_{\tilde{\sigma}}] = e(\tilde{\xi}_1) \) (cf. §5.1).
Let \( i : X_n(L) \to \Sigma_n(L) \) and \( j : (\Sigma_n(L), \emptyset) \to (\Sigma_n(L), X_n(L)) \) be the inclusions and consider the exact sequence
\[
\ldots \to H^1(\Sigma_n(L)) \to H^1(X_n(L)) \to H^2(\Sigma_n(L), X_n(L)) \to H^2(\Sigma_n(L)) \to H^2(X_n(L)) \to \ldots
\]
Since \( \tilde{\sigma} \) is defined over \( X_n(L) \), we have \( i^*(|c_{\tilde{\sigma}}|) = 0 \) and hence \( c_{\tilde{\sigma}} \) represents a cohomology class in \( H^2(\Sigma_n(L), X_n(L)) \), which we also denoted by \( [c_{\tilde{\sigma}}] \). We will show that this latter class lies in the image of \( \delta \) and therefore \( e(\xi_1) \), which equals \( j^*(|c_{\tilde{\sigma}}|) \), is 0.

To show that \( [c_{\tilde{\sigma}}] \in H^2(\Sigma_n(L), X_n(L)) \) lies in the image of \( \delta \), note that as \( H_1(\tilde{N}, \partial \tilde{N}) \cong H^2(\tilde{N}) = 0 \) we have
\[
H^2(\Sigma_n(L), X_n(L)) \cong H^2(\tilde{N}, \partial \tilde{N}) \cong \text{Hom}(H_2(\tilde{N}, \partial \tilde{N}), \mathbb{Z}) \cong \oplus_i \text{Hom}(H_2(\tilde{N}_i, \partial \tilde{N}_i), \mathbb{Z}),
\]
where \( \tilde{N} = \bigsqcup_i \tilde{N}_i \), as above. It follows that \( [c_{\tilde{\sigma}}] \in H^2(\Sigma_n(L), X_n(L)) \) is determined by the value of \( c_{\tilde{\sigma}} \) on the classes \([\tilde{D}_i]\) carried by the fundamental classes of the disks \( \tilde{D}_i \).

**Figure 3.** The normal vector field \( \partial_{\tilde{\sigma}} \) along \( \partial D \) lifts to the normal vector field \( \partial_{\sigma} \) along \( \partial \tilde{D} \). Also for any given point \( \tilde{x} \in \partial \tilde{D} \), the angle between \( \partial_{(\tilde{x})} \) and \( \tilde{\sigma}|_{\tilde{x}} \) is the same with the angle between \( \partial_{(\tilde{x})}|_{\tilde{A}_i} \) and \( \tilde{\sigma}|_{\tilde{A}_i} \). Based on these two observations, it is easy to draw \( \tilde{\sigma} \) along \( \partial \tilde{D} \). In this figure, we illustrate \( \tilde{\sigma} \) alone the boundary of a meridional disk \( \tilde{D} \) for 4-fold cyclic branched covers. It is also easy to see that from the point \( \tilde{A}_1 \) to \( \tilde{A}_2 \), the vector field \( \tilde{\sigma} \) rotates by an angle of \((2\pi - \frac{2\pi}{4})\) clockwise. Hence the total rotation of \( \tilde{\sigma} \) along \( \partial \tilde{D} \) is \(-\frac{2\pi - 2\pi}{4} \times 4 = -3 \times 2\pi \).

Therefore, by the construction of \( c_{\tilde{\sigma}} \), we have \( c_{\tilde{\sigma}}([\tilde{D}]) \equiv -3 \).

Figure 3 illustrates the calculation of the value \( c_{\tilde{\sigma}} \) on a meridional disk. In particular, by our choice of \( \tilde{\sigma} \), it follows that the value \( c_{\tilde{\sigma}}([\tilde{D}_i]) = 1 - n \) is independent on \( i \).

On the other hand, if we denote by \( u \in H^1(X_n(L)) \) the Poincaré dual of the element of \( H_2(X_n(L), \partial X_n(L)) \) carried by the fundamental class of the lift \( S \) of a Seifert surface \( S \), our assumptions on \( S \) and \( L \) imply that \( \delta(u) \in H^2(\Sigma_n(L), X_n(L)) \) evaluates to 1 on each \([\tilde{D}_i]\). It follows that \( [c_{\tilde{\sigma}}] \in H^2(\Sigma_n(L), X_n(L)) \) is \((1 - n)\delta(u) \). In particular, it lies in the image of \( \delta \), which completes the proof. \( \square \)

### 6. Universal circle actions

Throughout this section \( M \) will denote a closed connected oriented 3-manifold and \( \mathcal{F} \) a topological co-oriented taut foliation on \( M \). Such foliations are known to be isotopic to foliations whose
leaves are smoothly immersed and whose tangent planes vary continuously across $M$ ([Cal1]). We assume below that $\mathcal{F}$ satisfies this degree smoothness. Consequently, the tangent planes of $\mathcal{F}$ determine a 2-plane subbundle $T\mathcal{F}$ of $TM$. Orient $T\mathcal{F}$ so that its orientation together with the co-orientation of $\mathcal{F}$ determines the orientation of $M$.

We suppose that there is a Riemannian metric $g$ on $M$ whose restriction to the leaves of $\mathcal{F}$ has constant curvature $-1$. By this we mean that the restriction of $g$ to each plaque of $\mathcal{F}$ has constant curvature $-1$. Under this assumption, W. Thurston constructed a circle $S^1_{\text{univ}}$ associated to $\mathcal{F}$ and a non-trivial homomorphism $\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}_+(S^1_{\text{univ}})$. The key to the proof of our main results is determining sufficient conditions for the vanishing of $e(\rho_{\text{univ}})$. To that end, we prove Proposition 7.1, a folklore result which identifies the Euler class of the circle bundle $E_{\rho_{\text{univ}}}$ with the Euler class of the tangent bundle $T\mathcal{F}$ of $\mathcal{F}$.

In this section, we review the construction of $\rho_{\text{univ}}$ following the approach found in [CD] (see also [Cal2]). We prove Proposition 7.1 in §7.

6.1. Bundles from circles at infinity. Let $(M, g, \mathcal{F})$ be as above. As $\mathcal{F}$ is taut, the inclusion map induces an injection from the fundamental group of each of its leaves to $\pi_1(M)$ ([Nov]). Hence each leaf of the pull-back foliation $\tilde{\mathcal{F}}$ on the universal cover $\tilde{M}$ of $M$ is simply-connected.

The leaf space $\mathcal{L}$ of $\tilde{\mathcal{F}}$ is the quotient space of $\tilde{M}$ obtained by collapsing each leaf of $\tilde{\mathcal{F}}$ to a point. The simple-connectivity of $\tilde{M}$ implies that transversals to $\tilde{\mathcal{F}}$ map homeomorphically to their images in $\mathcal{L}$ and as such, the co-orientation on $\mathcal{F}$ determines an orientation on these images. Globally, $\mathcal{L}$ is an oriented, though not necessarily Hausdorff, 1-manifold (cf. [Cal2, Lemma 4.45]).

We use the Poincaré disk model for the hyperbolic plane $\mathbb{H}^2$. In particular, the underlying space of $\mathbb{H}^2$ is the open unit ball in $\mathbb{R}^2$ whose closure, denoted $\overline{\mathbb{H}}^2$, is the unit disk. The boundary of $\overline{\mathbb{H}}^2$ is the unit circle $S^1$ and is called the boundary of $\mathbb{H}^2$ at infinity. Given a point $p$ in $\mathbb{H}^2$ and a unit tangent vector $v \in UT_p\mathbb{H}^2$, there is a unique geodesic ray $\gamma_{p,v} : [0, \infty) \to \mathbb{H}^2$ for which $\gamma_{p,v}(0) = p$ and $\dot{\gamma}_{p,v}(0) = v$, and this geodesic ray limits to a unique point of $\partial \overline{\mathbb{H}}^2$. This correspondence determines a canonical homeomorphism between $UT_p\mathbb{H}^2$ and $\partial \overline{\mathbb{H}}^2$ for any $p \in \mathbb{H}^2$.

Since each leaf $\lambda$ of $\tilde{\mathcal{F}}$ is isometric to the hyperbolic plane $\mathbb{H}^2$ with respect to the pull-back $\tilde{g}$ of $g$ to $\tilde{M}$, each $\lambda$ gives rise to a circle at infinity which we denote by $\partial_\infty \lambda$. This association allows us to define two related $S^1$-bundles with fibres $\partial_\infty \lambda$. The first, denoted $\tilde{E}_\infty$, has base $\mathcal{L}$ and the second, denoted $E_\infty$, has base $\tilde{M}$. The topologies of these bundles are defined similarly.

Let $\{(U_\alpha, \varphi_\alpha)\}$ be a regular foliated atlas of $\tilde{\mathcal{F}}$ such that $\varphi_\alpha(U_\alpha) \cong R_\alpha \times B_\alpha$ where $R_\alpha$ is a rectangular region in $\mathbb{R}^2$ and $B_\alpha$ is an open interval in $\mathbb{R}$. We use $\mathcal{F}_\alpha$ to denote the foliation on $R_\alpha \times B_\alpha$ determined by the plaques $R_\alpha \times \{x\}$.

Recall that we have assumed that the transition maps $\varphi_\beta \circ \varphi_\alpha^{-1}$ are horizontally smooth. In particular, the differential $D(\varphi_\beta \circ \varphi_\alpha^{-1})$ is defined and varies continuously over $T\mathcal{F}_\alpha|_{\varphi_\alpha(U_\alpha \cap U_\beta)}$. Hence if $UT\tilde{\mathcal{F}}$ and $UT\mathcal{F}_\alpha$ denote the unit tangent bundles of $\tilde{\mathcal{F}}$ and $\mathcal{F}_\alpha$, the atlas $\{(U_\alpha, \varphi_\alpha)\}$
determines local trivialisations $\{\varphi_\alpha\}$ of $UT\bar{F}$:

$$UT\bar{F}|_{U_\alpha} \to (U TR_\alpha) \times B_\alpha \equiv (R_\alpha \times S^1) \times B_\alpha \equiv (R_\alpha \times B_\alpha) \times S^1 \cong U_\alpha \times S^1$$

whose transition functions $\varphi_\beta \circ \varphi_\alpha^{-1} : \varphi_\alpha(U_\alpha \cap U_\beta) \times S^1 \to \varphi_\alpha(U_\alpha \cap U_\beta) \times S^1$ are continuous.

Consider the fibre-preserving bijection

$$(6.1.1) \quad G : UT\bar{F} \to E_\infty, (\hat{p}, v) \mapsto \gamma(\hat{p}, v)$$

where $\gamma(\hat{p}, v)(t)$ is the geodesic ray on the leaf of $\bar{F}$ containing $\hat{p}$ which satisfies $\gamma(\hat{p}, v)(0) = \hat{p}$ and $\gamma(\hat{p}, v)(0) = v$. We use this bijection to topologise $E_\infty$ and endow it with the structure of a locally-trivial oriented $S^1$-bundle over $\bar{M}$ with transition functions $\{\varphi_\beta \circ \varphi_\alpha^{-1}\}$.

Defining the topology on $E_\infty$ is similar. Fix a transversal $\tau$ to $\bar{F}$. The simply-connectivity of $\bar{M}$ implies that $\tau$ embeds in $\mathcal{L}$ with image $l$, say. As above, there is a bijection

$$(6.1.2) \quad G_\tau : UT\bar{F}|_\tau \to \bar{E}_\infty|_l, (\hat{p}, v) \mapsto \gamma = \gamma(\hat{p}, v)$$

which we declare to be a homeomorphism. Distinct transversals with the same image in $\mathcal{L}$ determine the same topology on $\bar{E}_\infty|_l$ since the geometry of the leaves of $\bar{F}$ varies continuously over compact subsets of $\bar{M}$. See [CD, §2.8] for more details and discussion.

**Remark 6.1.** By construction, the deck transformations of the cover $\bar{M} \to M$ determine isometries between the leaves of $\bar{F}$ and as such, induce homeomorphisms between the fibres of $\bar{E}_\infty$ and $E_\infty$. The naturality of the topologies on $\bar{E}_\infty$ and $E_\infty$ is reflected in the fact these homeomorphisms determine actions of $\pi_1(M)$ on $\bar{E}_\infty$ and $E_\infty$ by bundle maps.

### 6.2. Circular orders and monotone maps.

A circular order on a set $O$ of cardinality 4 or more is a collection of linear orders $<_p$ on $O \setminus \{p\}$, one for each $p \in O$, such that for $p, q \in O$, the linear orders $<_p$ and $<_q$ differ by a cut on $O \setminus \{p, q\}$. (See [Cal2, Definition 2.34] for the details.) If $O = \{x, y, z\}$ has three elements, we add the condition that $y <_x z$ if and only if $z <_y x$. Subsets of cardinality 3 or more of circularly ordered sets inherit circular orders in the obvious way.

The archetypal example of a circularly ordered set is an oriented circle where the linear orders $<_p$ on $S^1 \setminus \{p\}$ are those determined by the orientation. More generally, any subset of cardinality 3 or more of an oriented circle inherits a circular order from the orientation on the circle.

Given a circularly ordered set $O$ of four or more elements, we define an ordered triple $(x, y, z) \in O^3$ to be *positively ordered* if there is a $p \in O \setminus \{x, y, z\}$ such that $x <_p y <_p z$. We call $(x, y, z)$ *negatively ordered* if there is a $p \in O \setminus \{x, y, z\}$ such that $y <_p x <_p z$. We leave it to the reader to verify that a positively ordered triple is never negatively ordered and vice versa. Further, a triple of distinct points $(x, y, z)$ is positively ordered, respectively negatively ordered, if and only if $(y, z, x)$ is positively ordered, respectively negatively ordered.

A totally ordered set $S$ admits a natural topology with basis consisting of the open interval $(x, y) = \{p \in S : x <_S p <_S y\}$. A map $f : S \to T$ between totally-ordered sets is called
monotone if it is surjective and if \( f^{-1}(t) \) is a closed interval \([x,y] = \{p \in S : x \leq_S p \leq_S y\}\) for each \( t \in T\). Monotone maps are continuous and satisfy \( f(s_1) \leq_T f(s_2) \) whenever \( s_1 <_S s_2 \).

We make analogous definitions for circularly ordered sets. Given distinct points \( x, y \) in a circularly ordered set \( O \), we define the open interval \((x, y)\) to be \( \{p \in O : (x, p, y) \) is positively ordered\}. Closed intervals are defined similarly. The complement of an open, respectively closed, interval is a closed, respectively open, interval.

The set of open intervals in \( O \) forms a basis of the order topology on \( O \). Closed intervals are closed in this topology. If \( O \) is a subset of an oriented circle with the induced circular order, then the order topology coincides with the subspace topology and the open intervals of \( O \) are intersections of \( O \) with open arcs of the circle.

A map \( f : O_1 \to O_2 \) between circularly ordered sets is called monotone if it is surjective and point inverses are closed intervals. Then for any \( p_2 \in O_2 \) and \( p_1 \in f^{-1}(p_2) \subset O_1 \), the restriction of \( f \) to \((O_1 \setminus f^{-1}(p_2), <_{p_1}) \to (O_2 \setminus \{p_2\}, <_{p_2})\) is a monotone map of totally ordered sets. Monotone maps are continuous.

6.3. Sections of \( \bar{E}_\infty \) and universal circles. The key to the proof of the existence of a universal circle \( S^1_{univ} \) of \( \mathcal{F} \) is the construction of a certain set of sections \( \mathcal{S} = \{ \sigma : \mathcal{L} \to \bar{E}_\infty \} \) of the circle bundle \( \bar{E}_\infty \) which is circularly orderable and is closed under the action of \( \pi_1(M) \) on \( \bar{E}_\infty \) ([CD, §6]). The set \( \mathcal{S} \) is separable with respect to the order topology and contains no pair of distinct elements \( \sigma_1, \sigma_2 \) such that \( (\sigma_1, \sigma_2) = \emptyset \) (i.e. \( \mathcal{S} \) has no gaps), so it can be embedded into an oriented circle as a dense ordered subspace. This circle turns out to be the universal circle \( S^1_{univ} \).

The action of \( \pi_1(M) \) on \( \mathcal{S} \) is order-preserving and continuous in the order topology, which implies that it extends to an orientation-preserving action on \( S^1_{univ} \), yielding a homomorphism \( \rho_{univ} : \pi_1(M) \to \text{Homeo}^+(S^1_{univ}) \).

For each leaf \( \lambda \) of \( \mathcal{L} \), the evaluation map \( e_\lambda : \mathcal{S} \to \partial_\infty \lambda \) is continuous and sends the closed interval \([\sigma_1, \sigma_2] \) into the interval \([\sigma_1(\lambda), \sigma_2(\lambda)] \). It extends to a monotone map \( \phi_\lambda : S^1_{univ} \to \partial_\infty \lambda \) which, in particular, is continuous of degree one.

It is shown in [CD] that these objects satisfy the conditions of the following definition.

**Definition 6.2.** ([CD, Definition 6.1]) Let \( \mathcal{F} \) be a co-oriented taut foliation of a closed oriented 3-manifold \( M \) and suppose that \( M \) admits a Riemannian metric whose restriction to each leaf of \( \mathcal{F} \) has constant curvature \(-1\). A universal circle for \( \mathcal{F} \) is a circle \( S^1_{univ} \) together with the following data:

1. A representation
   \[
   \rho_{univ} : \pi_1(M) \to \text{Homeo}^+(S^1_{univ})
   \]
2. For every leaf \( \lambda \) of the pull-back foliation \( \bar{\mathcal{F}} \), there is a monotone map
   \[
   \phi_\lambda : S^1_{univ} \to \partial_\infty \lambda
   \]
(3) For every leaf $\lambda$ of $\tilde{F}$ and every $\alpha \in \pi_1(M)$, the following diagram commutes:

$$
\begin{array}{ccc}
S^1_{\text{univ}} & \xrightarrow{\rho_{\text{univ}}(\alpha)} & S^1_{\text{univ}} \\
\phi_\lambda \downarrow & & \downarrow \phi_{\alpha \cdot \lambda} \\
\partial_\infty \lambda & \xrightarrow{\alpha} & \partial_\infty (\alpha \cdot \lambda)
\end{array}
$$

(4) If $\lambda$ and $\mu$ are incomparable leaves of $\tilde{F}$, then the core of $\phi_\lambda$ is contained in the closure of a single gap of $\phi_\mu$ and vice versa.

We refer the reader to [CD] for more details on condition (4), which will not play a role below.

**Remark 6.3.** Suppose that $F$ is a co-oriented taut foliation on $M$ which has a universal circle. We claim that the image of $\rho_{\text{univ}}$ is an infinite group and if $M$ is a rational homology 3-sphere, it is non-abelian. To see this, first note that $F$ must have a non-simply connected leaf. Otherwise each of its leaves is homeomorphic to $\mathbb{R}^2$, which implies that $M$ is the 3-torus ([Ros], [Ga2]). But this is impossible by [Pla, Lemma 7.2] and, for instance, [Cal2, Theorem 4.35]. Thus $F$ has a non-simply-connected leaf $\bar{\lambda}$. There is a leaf $\lambda$ of $\tilde{F}$ contained in the inverse image of $\bar{\lambda}$ which is invariant under the deck transformations corresponding to $\pi_1(\bar{\lambda}) \leq \pi_1(M)$. Since $\pi_1(\bar{\lambda})$ acts on the hyperbolic plane $\lambda$ by isometries, it induces a faithful action of $\pi_1(\bar{\lambda})$ on $\partial_\infty \lambda$. Hence as $\pi_1(\bar{\lambda})$ is non-trivial, it is infinite. So by (3) of the definition of a universal circle, the image of $\rho_{\text{univ}}$ is an infinite group. If $M$ is a rational homology 3-sphere, the image cannot be abelian as otherwise it would finite.

7. **The Euler class of the universal circle action**

Recall that $TF$ denotes the oriented 2-plane field over $M$ determined by $F$.

**Proposition 7.1.** Assume that $F$ is a co-oriented taut foliation on a closed oriented 3-manifold $M$ and that there is a Riemannian metric $g$ on $M$ which restricts to a metric of constant curvature $-1$ on each leaf $\lambda$ of $F$. Let $\rho_{\text{univ}} : \pi_1(M) \to \text{Homeo}_+(S^1_{\text{univ}})$ be a universal circle representation associated to $(M,g,F)$. Then the Euler class of $E_{\rho_{\text{univ}}}$ equals that of $TF$.

**Proof.** As above, $(\tilde{M},\tilde{g},\tilde{F})$ denotes the universal cover of $M$ equipped with the pull-back foliation $\tilde{F}$ and the pull-back metric $\tilde{g}$.

Let $\Phi : \tilde{M} \times S^1_{\text{univ}} \to E_\infty$ be the fibre-preserving map sending $(\tilde{p},\sigma)$ to $\phi_\lambda(\sigma)$, where $\lambda$ is the leaf of $F$ containing $\tilde{p}$ and $\phi_\lambda : S_{\text{univ}} \to \partial \lambda$ is the degree one monotone map of Definition 6.2(2). The continuity of $\Phi$ will be verified in Lemma 7.2 and we assume it for now.

Recall the bundle isomorphism $G : UT\bar{F} \to E_\infty$ with $(\tilde{p},v) \mapsto \gamma(\tilde{p},v)$ defined in §6.1. By composing $\Phi$ with $G^{-1}$, we obtain a fibre-preserving map

$$\bar{F} := G^{-1} \circ \Phi : \tilde{M} \times S^1_{\text{univ}} \to UT\tilde{F}$$
which restricts to a degree one monotone map between fibres.

We claim that \( \bar{F} \) descends to a fibre-preserving map \( F : E_{\text{univ}} \to \mathcal{U} \mathcal{T} \mathcal{F} \). To see this, note that \( \pi_1(M) \) acts on both \( \bar{M} \times S^1_{\text{univ}} \) and \( \mathcal{U} \mathcal{T} \bar{F} \) with quotient spaces \( E_{\text{univ}} \) and \( \mathcal{U} \mathcal{T} \mathcal{F} \) respectively. The existence of \( F \) will follow if we can show that \( h\bar{F} = \bar{F}h \) for any \( h \in \pi_1(M) \). But given \( \bar{p} \in \lambda \) on \( \bar{M} \) and \( \sigma \in S^1_{\text{univ}} \), we have

\[
\bar{F}(h \cdot (\bar{p}, \sigma)) = \bar{F}(h\bar{p}, \rho_{\text{univ}}(h)\sigma) = G^{-1} \circ \phi_{h\lambda}(\rho_{\text{univ}}(h)\sigma).
\]

By Definition 6.2(3), we have \( \phi_{h\lambda}(\rho_{\text{univ}}(h)\sigma) = h\phi_\lambda(\sigma) \). Hence

\[
\bar{F}(h \cdot (\bar{p}, \sigma)) = G^{-1} \circ h\phi_\lambda(\sigma)
\]

Since \( \pi_1(M) \) acts on \((\bar{M}, \bar{g})\) by isometries,

\[
\bar{F}(h \cdot (\bar{p}, \sigma)) = G^{-1} \circ h\phi_\lambda(\sigma) = hG^{-1} \circ \phi_\lambda = \bar{F}(\bar{p}, \sigma),
\]

which is what we needed to show.

Let \( D_{\text{univ}} \) and \( DT \mathcal{F} \) be the oriented disk bundles associated to \( E_{\text{univ}} \) and \( \mathcal{U} \mathcal{T} \mathcal{F} \) respectively (cf. \S 5), and let \( F_D : D_{\text{univ}} \to DT \mathcal{F} \) denote the map induced by \( F \). We have the following commutative diagram in which \( D_\infty \) denotes the fibre of \( D_{\text{univ}} \) at \( p \in M \):

\[
\begin{array}{ccc}
(D_{\infty}, S^1_{\text{univ}}) & \longrightarrow & (D_{\text{univ}}, E_{\text{univ}}) \\
\downarrow_{(F_D, F)|_p} & & \downarrow_{(F_D, F)} \\
(DT_{p, \mathcal{F}}, \mathcal{U}T_{p, \mathcal{F}}) & \longrightarrow & (DT \mathcal{F}, \mathcal{U} \mathcal{T} \mathcal{F})
\end{array}
\]

Since \( F \) restricts to a degree one map between fibres, \((F_D, F)|_p^* \) is an isomorphism which sends the orientation class in \( H^2(D_{\infty}, S^1_{\text{univ}}) \) to the orientation class in \( H^2(DT_{p, \mathcal{F}}, \mathcal{U}T_{p, \mathcal{F}}) \). Hence \((F_D, F)^* \) sends the Thom class of \( DT \mathcal{F} \) in \( H^2(DT \mathcal{F}, \mathcal{U} \mathcal{T} \mathcal{F}) \) to the Thom class of \( D_{\text{univ}} \) in \( H^2(D_{\text{univ}}, E_{\text{univ}}) \). Therefore, by definition, we have \( e(T \mathcal{F}) = e(\mathcal{U} \mathcal{T} \mathcal{F}) = e(E_{\text{univ}}) \) (see \S 5).

\[ \square \]

To complete the proof, it remains to prove that \( \Phi \) is continuous as we claimed.

**Lemma 7.2.** The map \( \Phi : \bar{M} \times S^1_{\text{univ}} \to E_\infty \) is continuous.

**Proof.** It suffices to show that if for any foliation chart \((U_\alpha, \varphi_\alpha)\) of \( \bar{F} \), the restriction \( \Phi|_{U_\alpha} : U_\alpha \times S^1_{\text{univ}} \to E_\infty|_{U_\alpha} \) is continuous. (See \S 6.1.)

Let \( l_\alpha \) be the open interval on \( \mathcal{L} \) corresponding to a transversal in \( U_\alpha \) and \( e_\alpha : \mathcal{S} \to S_{l_\alpha} \) the map which restricts a section in \( \mathcal{S} \) to \( l_\alpha \). Now define \( S_{l_\alpha} \) to be the image of \( e_\alpha \). That is,

\[
S_{l_\alpha} = \{ \sigma|_{l_\alpha} : \sigma \in \mathcal{S} \}
\]

The inverse image by \( e_\alpha \) of an element of \( S_{l_\alpha} \) is a closed interval in \( \mathcal{S} \), since sections in \( \mathcal{S} \) do not cross each other. Hence, the circular order on \( \mathcal{S} \) defines a circular order on \( S_{l_\alpha} \) and if we equip \( S_{l_\alpha} \) with the associated order topology, then \( e_\alpha \) is a monotone map between two circularly ordered sets. In particular, \( e_\alpha \) is continuous.
On the other hand, \( S_{l_\alpha} \) is a subset of the set of continuous functions \( C^0(l_\alpha, \bar{E}|_{l_\alpha}) \) from \( l_\alpha \) to \( \bar{E}|_{l_\alpha} \). One can check that the order topology on \( S_{l_\alpha} \) agrees with the subspace topology induced by the compact-open topology on \( C^0(l_\alpha, \bar{E}|_{l_\alpha}) \). We denote the closure of \( S_{l_\alpha} \) in \( C^0(l_\alpha, \bar{E}|_{l_\alpha}) \) by \( \bar{S}_{l_\alpha} \).

Note that for any leaf \( \lambda \in l_\alpha \), the evaluation map \( e_\lambda : S \to \partial_\infty \lambda \) factors through \( S_{l_\alpha} \). That is, the left-hand diagram immediately below commutes and its maps extend by continuity to yield the right-hand diagram.

Since the evaluation map from \( C^0(l_\alpha, \bar{E}|_{l_\alpha}) \times l_\alpha \to \bar{E}|_{l_\alpha} \) is continuous with respect to the compact open topology, it follows that \( \Phi_{l_\alpha} : U_\alpha \times \bar{S}_{l_\alpha} \to E_\infty|U_\alpha \) is continuous. Therefore, \( \Phi \) is continuous over \( U_\alpha \times S^1_{\text{univ}} \).

8. LEFT-ORDERABILITY OF 3-MANIFOLD GROUPS AND UNIVERSAL CIRCLES

A group \( G \) is said to be left-orderable if it is nontrivial and there exists a strict total order \( < \) on \( G \) such that \( a < b \) if and only if \( ca < cb \) for any \( a, b, c \) in \( G \).

The group \( \text{Homeo}_+(\mathbb{R}) \) is left-orderable (see, for instance, the proof of [Ghy, Theorem 6.8]) and serves as a universal host for countable left-orderable groups. Indeed, a countable group \( G \neq \{1\} \) is left-orderable if and only if it admits a faithful representation into \( \text{Homeo}_+(\mathbb{R}) \) (cf. [Ghy, Theorem 6.8]). If \( G \) is the fundamental group of an orientable irreducible 3-manifold, the condition that the representation be faithful can be removed.

**Theorem 8.1.** ([BRW, Theorem 1.1]) Assume that \( M \) is a compact, orientable, irreducible 3-manifold. Then \( \pi_1(M) \) is left-orderable if and only if it admits a homomorphism to \( \text{Homeo}_+(\mathbb{R}) \) with non-trivial image. Equivalently, if and only if it admits a left-orderable quotient.

Consequently,

**Corollary 8.2** ([HS, BRW]). Let \( M \) be a compact, orientable and prime 3-manifold and let \( b_1(M) \) denote its first Betti number. If \( b_1(M) > 0 \), then \( \pi_1(M) \) is left-orderable.

The following proposition states a known criterion for the left-orderability of the fundamental group of a rational homology 3-sphere.

**Theorem 8.3.** Let \( M \) be a rational homology 3-sphere which admits a co-orientable taut foliation whose tangent plane field has zero Euler class. Then \( \pi_1(M) \) is left-orderable.
Proof. First we observe that rational homology 3-spheres which admit co-orientable taut foliations are irreducible [Nov]. Hence \( \pi_1(M) \) will be left-orderable if it admits a homomorphism to \( \text{Homeo}_+(\mathbb{R}) \) with non-trivial image (Theorem 8.1).

Fix a Riemannian metric \( g \) on \( M \). Since \( \mathcal{F} \) is orientable, any leaf of \( \mathcal{F} \) which is not conformally negatively curved with respect to the induced metric gives rise to a nontrivial homology class in \( H_2(M;\mathbb{R}) \) ([Pla, Corollary 6.4]), contrary to the fact that \( M \) is a rational homology sphere. Consequently, each leaf of \( \mathcal{F} \) is conformally hyperbolic. Then by [Can, Theorem 4.1], \( g \) is conformal to a metric \( g' \) whose restriction to each leaf has constant curvature \(-1\). Hence there exists a universal circle action \( \rho: \pi_1(M) \to \text{Homeo}_+(S^1) \) which is non-trivial by Remark 6.3. By Proposition 7.1, we have \( e(E_{\rho}) = e(T\mathcal{F}) \), which is zero by hypothesis. Hence \( \rho \) lifts to a non-trivial action of \( \pi_1(M) \) on the real line (Lemma 5.1) and therefore by Theorem 8.1, the fundamental group \( \pi_1(M) \) is left-orderable.

\[ \square \]

9. THE LEFT-ORDERABILITY OF THE FUNDAMENTAL GROUPS OF CYCLIC BRANCHED COVERS OF FIBERED KNOTS

In this section we consider cyclic branched covers of hyperbolic fibered knots and prove Theorem 1.2, Corollary 1.5 and Corollary 1.8.

Proposition 9.1. Let \( M \) be an oriented rational homology sphere admitting an open book \((S,h)\), with the binding a knot \( K \) and the monodromy \( h \) pseudo-Anosov. Let \( \mathcal{F}_0 \) denote the foliation on the exterior of \( K \) given by locally-trivial fibre bundle structure. Suppose that \( e(T\mathcal{F}_0) = 0 \). If \( |c(h)| \geq 1 \), then \( M \) is excellent.

Proof. If \( c(h) \leq -1 \), we can consider the open book decomposition \((-S,h^{-1})\) of \( M \). By Lemma 4.5, \( c(h^{-1}) = -c(h) \geq 1 \). Hence, we may assume that \( c(h) \geq 1 \). By Theorem 4.1, \( M \) admits a co-oriented taut foliation \( \mathcal{F} \) whose tangent plane field is homotopic to the contact structure \( \xi \) supported by \((S,h)\). In particular, the restriction of \( \xi \) to the knot complement \( X(K) \) is homotopic to \( \mathcal{F}_0 \). It follows that the Euler class \( e(\xi) \) is sent to 0 under the inclusion-induced homomorphism \( H^2(M) \to H^2(X(K)) \). By Lemma 2.1, this homomorphism is an isomorphism and hence, \( e(T\mathcal{F}) = e(\xi) = 0 \). This implies that \( M \) has a left-orderable fundamental group by Theorem 8.3.

\[ \square \]

Proof of Theorem 1.2. Let \( K \) be a hyperbolic fibered knot in an oriented integer homology 3-sphere \( M \) with fibre \( S \) and monodromy \( h \). Then \( X_n(K)(\mu_n + q\lambda_n) \) has open book decomposition \((S,T_{\vartheta}^{-q}h^n)\) with binding the core of the filling solid torus, which we denote by \( \tilde{K} \). The exterior of \( K \) in \( X_n(K)(\mu_n + q\lambda_n) \) is \( X_n(K) \).

Since \( T_{\vartheta}^{-q}h^n \) is freely isotopic to \( h^n \), there is a fiber-preserving homeomorphism between the mapping tori of \( T_{\vartheta}^{-q}h^n \) and \( h^n \). Hence the foliation on \( X_n(K) \) determined by the open book \((S,h^n)\), denoted by \( \mathcal{F}_0 \), is isomorphic to the one determined by the open book \((S,T_{\vartheta}^{-q}h^n)\) and the same holds for their tangent plane fields. We show that \( e(T\mathcal{F}_0) = 0 \).
Since $M$ is an integer homology 3-sphere, $H^2(X(K)) \cong 0$, so that if $\mathcal{F}_0$ is the foliation of $X(K)$ determined by the open book $(S, h)$, then $e(T\mathcal{F}_0) = 0$. Since the Euler class $e(T\mathcal{F}_0)$ is the image of $e(\mathcal{F}_0)$ under the homomorphism $H^2(X(K)) \to H^2(X_n(K))$, it is also zero.

By Lemma 4.5, we have $|c(T_{g}^{a}h^{n})| = |nc(h) - q| \geq 1$. Hence by Proposition 9.1, if $X_n(K)(\mu_n + q\lambda_n)$ is a rational homology sphere, it is excellent. Otherwise, the first Betti number of $X_n(K)(\mu_n + q\lambda_n)$ is positive and it is also excellent ([BRW, Ga1]). This proves part (2) of Theorem 1.2. Part (1) is an immediate consequence of the part (2) and Proposition 4.4. □

Proof of Corollary 1.5. As before, the $n$-fold cyclic cover of $X(K)(\mu + q\lambda)$ is homeomorphic to the surgery manifold $X_n(K)(\mu_n + q\lambda_n)$, which is excellent if $|nc(h) - q| \geq 1$.

If $c(h) = 0$, then $nc(h) = 0$. Since $q \neq nc(h)$, in particular, $q \neq 0$. Hence $|nc(h) - q| = |q| \geq 1$.

If $c(h) \neq 0$, $|nc(h) - q| < 1$ only if either $q = nc(h) \in \mathbb{Z}$ or $nc(h) \notin \mathbb{Z}$ and $q \geq 1$ is either $[nc(h)]$ or $[nc(h)] + 1$.

This completes the proof. □

Proof of Corollary 1.8. Under the assumption, the $n$-fold cyclic branched cover of $X(K)(\mu + q\lambda)$ is homeomorphic to $X_n(K)(\mu_n + \lambda q_n)$, where $n = mp$ and $(p, q) = 1$. Then the claim follows from Theorem 1.2 Part (2). □

10. The left-orderability of the fundamental groups of cyclic branched covers of closed braids

In this section we examine Conjecture 1.1 for cyclic branched covers of hyperbolic closed braids.

10.1. Open book decomposition of cyclic branched covers of closed braids. Given a punctured surface $S$, we let $\hat{S}$ denote the compact surface obtained from $S$ by filling all punctures. If $f : S \to S'$ is a proper continuous map between two punctured surfaces, we use $\hat{f} : \hat{S} \to \hat{S}'$ to denote the continuous extension of $f$.

Given a $m$-braid $b : D_m \to D_m$, the pair $(\hat{D}_m, \hat{b})$ defines an open book decomposition of $S^{3}$ with pages diffeomorphic to the interior of the unit disk $\hat{D}_m$ and the monodromy $\hat{b} : \hat{D}_m \to \hat{D}_m$ is the extension $b$ to $\hat{D}_m$. This open book decomposition of $S^{3}$ lifts to an open book decomposition of the $n$-fold cyclic branched cover of $S^{3}$ along the closed braid $\hat{b}$, as we describe below.

Let $p : S_n \to D_m$ be the $n$-fold cyclic cover of the $m$-punctured disk associated with the epimorphism $\pi_1(D_m; p) \to \mathbb{Z}/n$ which maps each generator $x_i$ to the class of 1 in $\mathbb{Z}/n$. See Figure 4.

The group automorphism $b_\ast : \pi_1(D_m; p) \to \pi_1(D_m; p)$ induced by the diffeomorphism $b$, sends each generator $x_i$ to the conjugate of a generator $x_j$ for some $j$. Consequently, $b_\ast$ restricts to an automorphism of the kernel of $\pi_1(D_m; p) \to \mathbb{Z}/n$. Hence, $b : D_m \to D_m$ lifts to a diffeomorphism $\psi : S_n \to S_n$ such that $\psi|_{\partial S_n}$ is the identity on the boundary $\partial S_n$ and $b \circ p = p \circ \psi$. Hence
\[ \tilde{b} \circ \tilde{p} = \tilde{p} \circ \tilde{\psi} \] where \( \tilde{p} : \tilde{S}_n \to \tilde{D}_m \) is an \( n \)-fold cyclic branched cover of the disk \( \tilde{D}_m \) along \( m \) points.

It is routine to check that the pair \((\tilde{S}_n, \tilde{\psi})\) defines an open book decomposition of the \( n \)-fold cyclic branched cover \( \Sigma_n(\hat{b}) \) of \( S^3 \) along the closed braid \( \hat{b} \).

**Lemma 10.1.** If \( b : D_m \to D_m \) is a pseudo-Anosov braid, then \( \tilde{\psi} : \tilde{S}_n \to \tilde{S}_n \) is also pseudo-Anosov.

**Proof.** Let \( h_t : D_m \to D_m \) be a free isotopy from \( b \) to its pseudo-Anosov representative, denoted by \( \beta \), whose stable and unstable measured singular foliations are denoted by \((\mathcal{F}^s, \mu^s)\) and \((\mathcal{F}^u, \mu^u)\). The isotopy \( h_t \) lifts to an isotopy \( H_t : S_n \to S_n \) with \( H_0 = \psi \), where \( \psi : S_n \to S_n \) is the lift of the braid \( b \) as defined above.

By construction, \( H_t \) leaves each component of \( \partial S_n \) invariant.

Set \( \varphi := H_1 : S_n \to S_n \). We have \( \varphi \) is a pseudo-Anosov homeomorphism, whose stable and unstable singular foliations are lifts of singular foliations \((\mathcal{F}^s, \mu^s)\) and \((\mathcal{F}^u, \mu^u)\). We denote them by \((\tilde{\mathcal{F}}^s, \tilde{\mu}^s)\) and \((\tilde{\mathcal{F}}^u, \tilde{\mu}^u)\) respectively. Note that under this lift, any 1-pronged puncture on \( D_m \) is lifted to an \( n \)-pronged puncture on \( S_n \).

Hence \( \tilde{\mathcal{F}}^s \) and \( \tilde{\mathcal{F}}^u \) extend to a transverse pair of measured singular foliations on the branched cover \( \tilde{S}_n \) which are invariant under \( \tilde{\varphi} : \tilde{S}_n \to \tilde{S}_n \). Moreover, the extension \( \tilde{H}_t : \tilde{S}_n \to \tilde{S}_n \) of the isotopy \( H_t : S_n \to S_n \) defines a free isotopy between \( \tilde{\psi} \) and the pseudo-Anosov homeomorphism \( \tilde{\varphi} \). By definition, this shows that \( \tilde{\psi} \) is pseudo-Anosov. \( \square \)

It is easy to check that the boundary of \( S_n \) has \( (m,n) \) components. In particular, \( \partial S_n \) is not connected when \( m \) and \( n \) are not coprime. On the other hand, since the isotopy \( H_t \) in the proof of Lemma 10.1 is equivariant with respect to the deck transformations of \( \tilde{p} : S_n \to D_m \), the
fractional Dehn twist coefficient of $\psi$ with respect to any two boundary components of $S_n$ are equal. We denote this number by $c(\psi)$. Similarly, the fractional Dehn twist coefficients of $\bar{\psi}$ are equal with respect to all boundary components of the branched cover $\bar{S}_n$, which we denote by $c(\bar{\psi})$.

**Lemma 10.2.** $c(\bar{\psi}) = c(\psi) = \frac{(m,n)}{n} c(b)$.

**Proof.** We continue to use the notation developed in the proof of Lemma 10.1.

First of all, $c(\bar{\psi}) = c(\psi)$ since the two isotopies $H_t$ and $\bar{H}_t$ are identical over the collar neighborhoods of $\partial S_n$ and $\partial S_n$. It remains to show that $c(\psi) = \frac{(m,n)}{n} c(b)$.

Assume first that $(m,n) = 1$. In this case, $\partial S_n$ is connected, and we denote it by $C$, so the restriction $p|C : C \to \partial D_m$ is an $n$-fold cyclic cover. The proof that $c(\psi) = \frac{(m,n)}{n} c(b)$ is essentially contained in Figure 6. To write it down more precisely, we need some notation.

Let $\{p_0, \cdots, p_{N-1}\}$ be the set of singular points on $\partial D_m$ of the stable foliation $F^s$. Fix a preimage $q_0$ of $p_0$ on $C = \partial S_n$. For $k = 0, \cdots, n-1$, let $q^k_i \in C$ be the $k$th lift of the singular point $p_i$. That is, if $\gamma_{p_0 p_k}$ is the subarc of $\partial D_m$ with endpoints $p_0$ and $p_k$ (cf. §4.1), then $q^k_i$ is the end point of the unique lift of the path $C^k \cdot \gamma_{p_0 p_k}$ starting at $q_0$. To simplify notations, we denote $q^0_i$ by $q_i$. In particular, $q^0_0 = q_0$. Note that $\{q^k_i\}$ is the set of singular points on $\partial S_n = C$ of the stable foliation $\bar{F}^s$ of $\varphi$.

![Fig. 6](image)

**Figure 6.** In the figure, $\{q^0_0, \cdots, q^k_3\}_{k=0,1,2}$ and $\{p_0, \cdots, p_3\}$ are singular points of the stable foliations $\bar{F}^s$ and $F^s$ respectively. The path $H_t|q_0$ is the unique lift of the path $h_t|p_0$ starting at $q_0$. By Definition 4.2, we have $c(b) = 1 + \frac{1}{4} = \frac{5}{4}$ and $c(\psi) = \frac{5}{12} = \frac{1}{3} c(b)$.

Let $c(b) = l + \frac{k}{N}$. By Definition 4.2, this means that the path $h_t|p_0$ is homotopic to the path $[\partial D_m]^l \cdot \gamma_{p_0 p_k}$ on $\partial D_m$. 

We write \( l = ns + r \) where \( 0 \leq r < n \). Then by the uniqueness of path lifting, we have that \( H_t|_{q_0} \) is homotopic to the path \( C^s \cdot \gamma_{q_0q_0^r} \cdot \gamma_{q_0q_0^k} \), which is equal to \( C^s \cdot \gamma_{q_0q_0^k} \). Therefore,

\[
    c(\varphi) = s + \frac{rN + k}{Nn} = \frac{N(sn + r) + k}{Nn} = \frac{Nl + k}{Nn} = \frac{1}{n}c(b).
\]

This deals with the case that \((m, n) = 1\).

In case that \((m, n) \neq 1\), the degree of the covering map \( p : S_n \to D_m \) restricted to each boundary component of \( S_n \) is \( \frac{n}{(m, n)} \). Proceed as in the case that \((m, n) = 1\) to complete the proof. \( \square \)

10.2. The L-space conjecture and cyclic branched covers of closed braids. In this section we study the left-orderability of branched covers of closed braids. We begin with the proof of Theorem 1.9: Let \( b \in B_{2k+1} \) be an odd-strand braid whose closure \( \hat{b} \) is an oriented hyperbolic link \( L \) and let \( c(b) \in \mathbb{Q} \) be the fractional Dehn twist coefficient of \( b \). Suppose that \(|c(b)| \geq 2\). Then all even order cyclic branched covers of \( \hat{b} \) are excellent.

**Proof of Theorem 1.9.** First of all, the equivariant sphere theorem ([MSY]) and the positive solution of the Smith Conjecture ([MB]) imply that as \( \hat{b} \) is prime, all cyclic branched covers of \( \hat{b} \) are irreducible.

For each \( n \geq 1 \), there is an \( n \)-fold cyclic branched cover \( p_n : \Sigma_{2n}(\hat{b}) \to \Sigma_{2}(\hat{b}) \), branched over the lift \( \hat{L} \) of \( \hat{b} \) to \( \Sigma_{2}(\hat{b}) \), which the reader will verify is surjective on the level of fundamental groups. Hence if \( b_1(\Sigma_{2n}(\hat{b})) > 0 \), then \( b_1(\Sigma_{2n}(\hat{b})) > 0 \) for all \( n \). As such, Conjecture 1.1 holds for all even order cyclic branched covers of \( \hat{b} \).

Suppose then that \( \Sigma_{2}(\hat{b}) \) is a rational homology sphere. If \( c(b) \leq -2 \), then \( c(b^{-1}) = -c(b) \geq 2 \) by Lemma 4.5. Since \( \hat{b}^{-1} \) is the mirror image of \( \hat{b} \), their cyclic branched covers are diffeomorphic, so without loss of generality we may assume that \( c(b) \geq 2 \).

By Proposition 3.1, \( b \) is a pseudo-Anosov mapping class of \( D_{2k+1} \). Then Lemma 10.1 shows that the 2-fold cyclic branched cover of \( \hat{b} \) admits an open book decomposition \( (\hat{S}_2, \hat{\psi}) \), where \( \hat{S}_2 \) is the 2-fold cyclic branched cover of the unit disk branched over \( (2k + 1) \) points and the monodromy \( \hat{\psi} \) is pseudo-Anosov. By Lemma 10.2,

\[
    c(\hat{\psi}) = \frac{c(b)}{2} \geq 1.
\]

Since \( b \) is a braid on an odd number of strands, \( \partial \hat{S}_2 \) is connected. Then by Theorem 4.1, there exists a co-orientable taut foliation \( F \) on \( \Sigma_2(\hat{b}) \) and hence it cannot be an L-space [OS1, Bu, KR2]. Moreover, Theorem 4.1 says that the tangent plane field of the foliation \( F \) is homotopic to the contact structure supported by the open book \( (\hat{S}_2, \hat{\psi}) \). On the other hand, this contact structure is isotopic to the lift of the contact structure on \( S^3 \) that is supported by the open book \( (\hat{D}_{2k+1}, \hat{b}) \). (Here “lift” is used in the sense of §2.3.) Therefore, by Lemma 5.3 we have
e(TF) = 0. Applying Theorem 8.3, we conclude that \( \pi_1(\Sigma_2(\hat{b})) \) is left-orderable. This completes the proof for the 2-fold cyclic branched cover.

Now consider \( \Sigma_{2n}(\hat{b}) \) where \( n > 1 \) and recall that there is an \( n \)-fold branched cyclic cover \( p_n : \Sigma_{2n}(\hat{b}) \to \Sigma_2(\hat{b}) \), branched over \( \hat{L} \). We noted above that \( p_n \) is surjective on the level of \( \pi_1 \), so \( \pi_1(\Sigma_{2k}(\hat{b})) \) is left-orderable by Theorem 8.1. Finally, since \( \hat{L} \) intersects the foliation \( F \) on \( \Sigma_2(\hat{b}) \) transversely (cf. [HKM2, Lemma 4.4]), \( F \) lifts to a foliation \( F_n \) on \( \Sigma_{2n}(\hat{b}) \) which is easily seen to be co-oriented and taut. Consequently, \( \Sigma_{2n}(\hat{b}) \) is not an L-space for any \( n > 1 \), which completes the proof.

An identical argument yields the following more general statement. We omit the proof.

**Theorem 10.3.** Let \( b \in B_m \) whose closure is an oriented hyperbolic link. Assume that its fractional Dehn twist coefficient satisfies \( |c(b)| \geq N \). Then the \( nk \)-fold cyclic branched cover of the closed braid \( b \) admits a co-oriented taut foliation and has a left-orderable fundamental group for any \( n \) with \( 2 \leq n \leq N \), \( (m,n) = 1 \) and \( k \geq 1 \). □

### 10.3. Dehornoy’s braid ordering and cyclic branched covers of closed braids.

There is a special left order \( <_D \) on the braid group \( B_m \), due to Dehornoy, characterised by the condition that a braid \( b \) is positive if and only if there is a \( j \geq 1 \) such that \( b \) can be written (in the standard braid generators depicted in Figure 1) as a word containing \( \sigma_j \), but no \( \sigma_j^{-1} \), and not containing \( \sigma_i^\pm 1 \) for \( i < j \). Set \( \Delta_m = (\sigma_1 \sigma_2 \ldots \sigma_{m-1})(\sigma_1 \sigma_2 \ldots \sigma_{m-2}) \ldots (\sigma_1 \sigma_2)(\sigma_1) \) in \( B_m \). The centre of \( B_m \) is generated by \( \Delta_m^2 \) and for each \( b \in B_m \), there is an integer \( d > 0 \) such that

\[
\Delta_m^{-2d} <_D b <_D \Delta_m^{2d}
\]

In other words, the subgroup of \( B_m \) generated by \( \Delta_m \) is cofinal in \( B_m \) with respect to \( <_D \).

**Definition 10.4.** Given an element \( b \in B_m \), the Dehornoy floor \( [b]_D \) is the nonnegative integer defined to be

\[
[b]_D = \min\{k \in \mathbb{Z}_{\geq 0} \mid \Delta_m^{-2k-2} <_D b <_D \Delta_m^{2k+2}\}.
\]

Although the Dehornoy’s floor fails to be an topological invariant of the closed braid, it is proven a useful concept when studying links via closed braids [Ito1, Ito2].

Malyutin discovered a fundamental relationship between \( c(b) \) and \( [b]_D \). Though the Dehornoy’s floor defined by Malyutin [Mal, Definition 7.3] is slightly different from the one given above, it is easy to check that these two agree for \( b >_D 1 \) in \( B_m \).

**Proposition 10.5.** (cf. [Mal, Lemma 7.4]) For each \( b \in B_m \), \( [b]_D \leq |c(b)| \leq [b]_D + 1 \).

**Proof.** Let \( d = [b]_D \).

By Definition 10.4, this means either \( \Delta_m^{2d} \leq D b <_D \Delta_m^{2d+2} \) or \( \Delta_m^{-2d-2} <_D b \leq \Delta_m^{-2d} \). In the first case the claim follows directly from [Mal, Lemma 7.4]. In the second, the fact that \( \Delta_m^2 \) lies in the center of \( B_m \) implies that \( \Delta_m^{2d} \leq D b^{-1} <_D \Delta_m^{2d+2} \), so by the first case, \( [b]_D \leq D c(b^{-1}) \leq [b]_D + 1 \). Since \( c(b^{-1}) = -c(b) = |c(b)| \), this completes the proof. □
Remark 10.6. The proof of Proposition 10.5 shows that \( c(b) \geq 0 \) when \( b > D \) and \( c(b) \leq 0 \) when \( b < D \).

Given Proposition 10.5, Theorem 1.9 has the following consequence.

**Theorem 10.7.** Let \( b \in B_{2k+1} \) be an odd-strand braid whose closure \( \hat{b} \) is an oriented hyperbolic link in \( S^3 \). Suppose that \( |b|_D > 1 \). Then all even order cyclic branched covers of \( \hat{b} \) admit co-oriented taut foliations and have left-orderable fundamental groups.

11. **The L-space conjecture and genus one open books**

The goal of this section is to prove Theorem 1.10 and Theorem 1.11.

A simple Euler characteristic calculation shows that the 2-fold branched cover of a disk branched over three points is a genus 1 surface with one boundary component (see Figure 7). We denote this surface by \( T_1 \) and let \( \theta \) be its covering involution.

As in §10.1, every element in \( B_3 = \text{Mod}(D_3) \) admits a unique lift to \( \text{Mod}(T_1) \) which defines an embedding of groups \( B_3 \to \text{Mod}(T_1) \). The Artin generators \( \sigma_1 \) and \( \sigma_2 \) of \( B_3 \) lift to the right-handed Dehn twists \( T_{c_1} \) and \( T_{c_2} \) respectively, where \( c_i \) is the preimage of the segment connecting the \( i \)th and the \((i + 1)\)st punctures of the disk for \( i = 1, 2 \) as in Figure 7. Since Dehn twists \( T_{c_1} \) and \( T_{c_2} \) generates \( \text{Mod}(T_1) \), the embedding \( B_3 \to \text{Mod}(T_1) \) constructed above is indeed an isomorphism.

The identification between \( B_3 \) and the mapping class group \( \text{Mod}(T_1) \) can be used to show that any 3-manifold which admits an open book decomposition with \( T_1 \)-pages is the 2-fold cyclic branched cover of a closed 3-braid (§10.1). Hence, the following classification of 3-braids leads to a complete list of diffeomorphism classes of genus 1 open books with connected binding.

Set \( C = \Delta_3^2 = (\sigma_1\sigma_2)^3 \).

**Theorem 11.1.** ([Mur, Proposition 2.1], [Bal, Theorem 2.2]) Up to conjugation, any braid \( b \) in \( B_3 \) is equal to one of the following:

1. \( C^d \cdot \sigma_1\sigma_2^{-a_1} \cdots \sigma_1\sigma_2^{-a_n} \), where \( d \in \mathbb{Z} \), \( a_i \geq 0 \) and at least one of the \( a_i \) is nonzero;
2. \( C^d \cdot \sigma_2^m \) for some \( d \in \mathbb{Z} \) and \( m \in \mathbb{Z} \).
Remark 11.2. The conjugation classification of 3-braids detailed in Theorem 11.1 corresponds to Neilsen-Thurston classification of mapping classes in \( \text{Mod}(D_3) = B_3 \) (§3.1): braids of type (1) are pseudo-Anosov; those of type (2) are reducible; those of type (3) are periodic. This can be verified by considering their lifts in \( \text{Mod}(T_1) \), where the Nielsen-Thurston type of an element \( h \in \text{Mod}(T_1) \) is determined by the trace of the linear map \( h_* : H_1(T_1) \to H_1(T_1) \) (cf. [FM, §13.1])\(^1\).

Baldwin listed all closed 3-braids whose 2-fold branched covers are L-spaces.

Theorem 11.3. (Theorem 4.1 of [Bal]) The 2-fold cyclic branched cover \( \Sigma_2(\hat{b}) \) of a closed braid \( \hat{b} \) for \( b \in B_3 \) is an L-space if and only if \( b \) belongs to one of the following lists:

1. \( C^d \cdot \sigma_1^m \sigma_2^{-1} \), with \( d \in \mathbb{Z} \) and \( m \in \{-1, -2, -3\} \).
2. \( C^d \cdot \sigma_2^m \), for some \( m \in \mathbb{Z} \) and \( d = \pm 1 \).
3. \( C^d \cdot \sigma_1^m \sigma_2^{-1} \), with \( m \in \{-1, -2, -3\} \) and \( d \in \{-1, 0, 1, 2\} \).

Conjecture 1.1 holds for the manifolds listed in Theorem 11.3 since they admit no co-oriented taut foliations ([OS1, Bn, KR2]) and it was shown by Li and Watson ([LW]) that they have non-left-orderable fundamental groups.

Proof of Theorem 1.10. Let \( M \) denote an irreducible 3-manifold which admits a genus 1 open book decomposition with connected binding. Then \( M \) is diffeomorphic to the 2-fold cyclic branched cover of a closed 3-braid \( \hat{b} \), where \( b \in B_3 \) falls into one of the three families listed in Theorem 11.1.

Suppose first that \( b \) is in family (3). Then \( b = C^d \sigma_1^m \sigma_2^{-1} \) with \( m \in \{-1, -2, -3\} \). We noted above that this implies that \( b \) is periodic, so that using the notation from the proof of Proposition 3.1, \( N_b \) is Seifert fibered. The reader will verify that \( b \) has period 3, 4, or 6 depending on whether \( m \) is \(-3, -2, \) or \(-1 \). This implies that the fibre class on \( \partial N_b \) intersects \( \nu \) (see Figure 2) more than once algebraically. In particular, \( \nu \) is not the fibre class. Thus \( N_b(\nu) \), the exterior of \( \hat{b} \), is a Seifert manifold and therefore so is the irreducible manifold \( \Sigma_2(\hat{b}) \). Hence the theorem holds in this case by [BRW, BGW, LS].

If \( b \) is in family (2), then \( b = C^d \cdot \sigma_2^m \) with \( m \in \mathbb{Z} \). We can suppose that \( d \neq 0 \) as otherwise \( \hat{b} \) is a split link and therefore \( \Sigma_2(\hat{b}) \) is reducible, contrary to our hypotheses. Hence, given the paragraph immediately after Theorem 11.3, we can assume that \( |d| \geq 2 \). We can also suppose that \( m \neq 0 \) as otherwise \( \hat{b} \) is a \((3, 3d)\) torus link so that \( \Sigma_2(\hat{b}) \) is Seifert fibered and the theorem’s conclusion follows from [BRW, BGW, LS].

There is a circle \( O \) in \( \text{int}(D_3) \) which contains the second and third punctures in its interior, but not the first, and is invariant under \( b \). If \( T \) denotes the torus obtained from \( O \) in the mapping

\(^1\)The Nielsen-Thurston type of an element \( h \in \text{Mod}(T_1) \) is, by definition, the same as that of its projection in the mapping class group of the once-punctured torus, which is isomorphic to \( SL(2, \mathbb{Z}) \).
torus of $b$, the reader will verify that there is a genus 1 Heegaard splitting $V_1 \cup_T V_2$ of $S^3$ where $V_1 \cap \hat{b}$ is an $(m + 2d, 2)$ torus link standardly embedded in the interior of $V_1$ and $V_2 \cap \hat{b}$ is a $(d, 1)$ torus knot standardly embedded in the interior of $V_2$.

Then $\Sigma_2(\hat{b})$ is the union of a 2-fold branched cover $\Sigma_2(V_1, V_1 \cap \hat{b})$ of $V_1$ branched over $V_1 \cap \hat{b}$ and a 2-fold branched cover $\Sigma_2(V_2, V_2 \cap \hat{b})$ of $V_2$ branched over $V_2 \cap \hat{b}$. Since $V_1 \setminus \hat{b}$ fibres over the circle with fibre a twice-punctured disc, $\Sigma_2(V_1, V_1 \cap \hat{b})$ is homeomorphic to the product of a torus and an interval. On the other hand, $V_2$ admits a Seifert structure for which $V_2 \cap \hat{b}$ is a regular Seifert fibre, and therefore $\Sigma_2(V_2, V_2 \cap \hat{b})$ admits a Seifert structure. Further, as $|d| \geq 2$, $\Sigma_2(V_2, V_2 \cap \hat{b})$ is not a solid torus and so has incompressible boundary. Thus $\Sigma_2(\hat{b})$ is a graph manifold. In this case, Conjecture 1.1 has been confirmed in [BC, HRRW].

Finally suppose that $b$ is in family (1). Then $b = C^d \cdot \sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_n}$ is pseudo-Anosov, where $a_i \geq 0$ and $a_i \neq 0$ for some $i$. As some of the $a_i$ may be zero, we can write $b = C^d \cdot \sigma_1 b_1 \sigma_2^{-c_1} \cdots \sigma_1 b_k \sigma_2^{-c_k}$ where each $b_i$ and each $c_i$ is positive. Further, $k \geq 1$. We claim that the fractional Dehn twist coefficient $c(\sigma_1 b_1 \sigma_2^{-c_1} \cdots \sigma_1 b_k \sigma_2^{-c_k})$ is zero.

To see this, first observe that the conjugation by $\Delta_3 = \sigma_1 \sigma_2 \sigma_1$ in $B_3$ exchanges $\sigma_1$ and $\sigma_2$. Then by Lemma 4.5

\begin{equation}
(11.0.1) \quad c(\sigma_2^{b_1} \sigma_1^{-c_1} \cdots \sigma_2^{b_k} \sigma_1^{-c_k}) = c(\Delta_3(\sigma_2^{b_1} \sigma_1^{-c_1} \cdots \sigma_2^{b_k} \sigma_1^{-c_k}) \Delta_3^{-1}) = c(\sigma_2^{b_1} \sigma_1^{-c_1} \cdots \sigma_1 \sigma_2^{-c_k}).
\end{equation}

But by the definition of the Dehornoy’s order, $\sigma_2^{b_1 \sigma_1^{-c_1} \cdots \sigma_2^{b_k} \sigma_1^{-c_k}} <_D 1 <_D \sigma_1 \sigma_2^{-c_1} \cdots \sigma_1 \sigma_2^{-c_k}$.

Hence according to Remark 10.6, Equality (11.0.1) shows

\[0 \leq c(\sigma_2^{b_1} \sigma_2^{-c_1} \cdots \sigma_1 \sigma_2^{-c_k}) = c(\sigma_2^{b_1} \sigma_1^{-c_1} \cdots \sigma_2^{b_k} \sigma_1^{-c_k}) \leq 0.
\]

Thus $c(\sigma_2^{b_1} \sigma_2^{-c_1} \cdots \sigma_1 \sigma_2^{-c_k}) = 0$. But then as $C^d$ commutes with $\sigma_2^{b_1} \sigma_2^{-c_1} \cdots \sigma_1 \sigma_2^{-c_k}$,

\[c(b) = c(C^d) + c(\sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_m}) = d + c(\sigma_1 \sigma_2^{-a_1} \cdots \sigma_1 \sigma_2^{-a_m}) = d\]

by Lemma 4.5.

By the discussion immediately preceding the statement of the theorem we can suppose that $|c(b)| = |d| \geq 2$. The reader will verify that the argument in the proof of Theorem 1.9 goes through as long as $b$ is pseudo-Anosov and $\hat{b}$ is prime. The former holds by hypothesis while the latter holds by the assumption that $\Sigma_2(\hat{b})$ is irreducible. Thus $\Sigma_2(\hat{b})$ is not an L-space, admits a co-orientable taut foliation, and has left-orderable fundamental group.

Let $b(h) \in B_3$ be the image of $h$ under the isomorphism from $\text{Mod}(T_1)$ to $B_3$ described above. Note that $b(\delta) = C$, where $\delta = (T_{c_1} T_{c_2})^3$ as in Corollary 1.11.

\textbf{Proof of Corollary 1.11.} Suppose that $K$ is a genus one fibered knot in a closed, connected, orientable, irreducible 3-manifold $M$, with fiber $T_1$ and monodromy $h$. Each $\Sigma_n(K)$ is irreducible and contains a genus one fibered knot, Theorem 1.10 implies that $\Sigma_n(K)$ is excellent, respectively a total L-space, for some $n \geq 2$ if and only if it is not an L-space, respectively is an L-space.
Up to conjugation, we can suppose that the braid \( b(h) \in B_3 \) is one of the forms listed in Theorem 11.1.

First suppose that \( h \) is pseudo-Anosov. If \( c(h) \neq 0 \), Theorem 1.2(1) implies that \( \Sigma_n(K) \) is excellent for all \( n \geq 2 \). Suppose then that \( c(h) = 0 \) and let \( n \geq 2 \). Since \( h \) is pseudo-Anosov, \( b(h) = C^d \cdot \sigma_1^{-a_1} \cdots \sigma_2^{-a_n} \), where \( d \in \mathbb{Z} \), \( a_i \geq 0 \) and at least one of the \( a_i \) is nonzero. The proof of Theorem 1.10 and Lemma 10.2 show that

\[
(11.0.2) \quad 0 = 2c(h) = c(b(h)) = d.
\]

and therefore, \( b(h^n) = b(h)^n = (\sigma_1^{-a_1} \cdots \sigma_2^{-a_n})^n \). Theorem 11.3(1) now implies that \( \Sigma_n(K) \) is an L-space. This is case (1) of Corollary 11.1.

Next suppose that \( h \) is reducible. Then \( b(h) = C^d \cdot \sigma_2^m \) for some \( d \in \mathbb{Z} \) and \( m \in \mathbb{Z} \) (Theorem 11.1(2)). It follows that \( b(h^n) = b(h)^n = C^{nd} \cdot \sigma_2^{nm} \) and since \( nd \neq \pm 1 \) for \( n \geq 2 \), Theorem 11.3 shows that \( \Sigma_n(K) \) is not an L-space. Thus it is excellent.

Finally suppose that \( h \) is periodic. Then \( b(h) = C^d \cdot \sigma_1^{-m} \sigma_2^{-1} \) where \( d \in \mathbb{Z} \) and \( m \in \{-1, -2, -3\} \) (Theorem 11.1(3)). The reader will verify that if \( w_m = \sigma_1^{-m} \sigma_2^{-1} \), then

\[
w_m^r = \begin{cases} 
C^{-1} & \text{if } m = 1 \text{ and } r = 3 \\
C^{-1} & \text{if } m = 2 \text{ and } r = 2 \\
C^{-2} & \text{if } m = 3 \text{ and } r = 3
\end{cases}
\]

In particular,

\[
w_m^2 = \begin{cases} 
C^{-1}w_1^{-1} & \text{if } m = 1 \\
C^{-2}w_3^{-1} & \text{if } m = 3
\end{cases}
\]

We consider the cases \( m = 1, 2, 3 \) separately.

Suppose that \( m = 1 \) and let \( n = 3k + r \geq 2 \) where \( r \in \{0, 1, 2\} \). Then \( k \geq 1 \) if \( r = 0 \) or 1 and \( k \geq 0 \) otherwise. The identities above imply that

\[
b^n = (C^d w_1)^n = \begin{cases} 
C^{nd-k} & \text{if } r = 0 \\
C^{nd-k}w_1 & \text{if } r = 1 \\
(C^k+1-ndw_1)^{-1} & \text{if } r = 2
\end{cases}
\]

Theorem 11.1 then shows

\[
\Sigma_n(K) \text{ is an L-space if and only if } \begin{cases} 
d = k + 1 & \text{if } r = 0 \\
k - 1 \leq nd \leq k + 2 & \text{if } r = 1, 2
\end{cases}
\]

Thus,

\[
\Sigma_n(K) \text{ is an L-space if and only if } b = \begin{cases} 
w_1 \text{ and } n \leq 5 \\
w_1 \text{ and } n = 2
\end{cases}
\]

Next suppose that \( m = 2 \) and let \( n = 2k + r \geq 2 \) where \( r \in \{0, 1\} \). Then \( k \geq 1 \) and

\[
b^n = (C^d w_2)^n = \begin{cases} 
C^{nd-k} & \text{if } r = 0 \\
C^{nd-k}w_2 & \text{if } r = 1
\end{cases}
\]
Theorem 11.1 then shows

\[ \Sigma_n(K) \text{ is an L-space if and only if } \begin{cases} nd = k \pm 1 & \text{if } r = 0 \\ k - 1 \leq nd \leq k + 2 & \text{if } r = 1 \end{cases} \]

Thus,

\[ \Sigma_n(K) \text{ is an L-space if and only if } b = \begin{cases} w_2 \text{ and } n \leq 3 \\ Cw_2 \text{ and } n \leq 3 \end{cases} \]

Finally suppose that \( m = 3 \) and let \( n = 3k + r \geq 2 \) where \( r \in \{0, 1, 2\} \). Then \( k \geq 1 \) if \( r \) is 0 or 1 and \( k \geq 0 \) otherwise. We have,

\[ b^n = (C^d w_3)^n = \begin{cases} C^{nd - 2k} & \text{if } r = 0 \\ C^{nd - 2k} w_3 & \text{if } r = 1 \\ (C^{2(k+1)-nd} w_3)^{-1} & \text{if } r = 2 \end{cases} \]

Theorem 11.1 then shows

\[ \Sigma_n(K) \text{ is an L-space if and only if } \begin{cases} nd = 2k \pm 1 & \text{if } r = 0 \\ 2k - 1 \leq nd \leq 2k + 2 & \text{if } r = 1 \\ 2k \leq nd \leq 2k + 3 & \text{if } r = 2 \end{cases} \]

Thus,

\[ \Sigma_n(K) \text{ is an L-space if and only if } b = \begin{cases} w_3 \text{ and } n = 2 \\ Cw_3 \text{ and } n \leq 5 \end{cases} \]

This completes the proof. \( \square \)

12. The left-orderability of the fundamental groups of cyclic branched covers of satellite links

Let \( L \) be an oriented untwisted satellite link in an oriented integer homology 3-sphere \( M \) with pattern \( P \), a link in the solid torus \( N \), and companion \( C \), a knot in \( M \). The \( n \)-fold cyclic branched cover of \( L \), denoted \( \Sigma_n(L) \), is obtained in the usual way from the regular cover of the exterior of \( L \) determined by the homomorphism which sends the oriented meridians of the components of \( L \) to 1 (mod \( n \)). It can be obtained by gluing copies of the cyclic covers of the knot exterior \( X(C) \) to an \( n \)-fold cyclic cover of the solid torus \( N \) branched over \( P \).

In what follows we assume that \( L, P \) and \( C \) are as above and that \( P \) is the closure of an \( m \)-strand braid \( b \). The reader will verify that \( H_1(N \setminus P) \cong \mathbb{Z}^{|P|+1} \), where \( |P| \) is the number of components of \( P \), is freely generated by the meridian classes of the components of \( P \) and the class \( \nu \) carried by a longitudinal loop on \( \partial N = \partial X(C) \) (see Figure 2). We use \( N_n(P) \) to denote the \( n \)-fold cyclic branched cover of \( P \) in \( N \) determined by the homomorphism \( H_1(N \setminus P) \to \mathbb{Z}/n \) which sends the meridians of the components of \( L \) to 1 (mod \( n \)) and \( \nu \) to 0 (mod \( n \)).

We follow the notation developed in Section 10.1. Our discussion there shows that \( N_n(P) \) is homeomorphic to the mapping torus \( \tilde{S}_n \times [0,1]/(x,1) \sim (\tilde{\psi}(x),0) \), where \( \tilde{S}_n \) is the \( n \)-fold cyclic branched cover of the disk branched at \( m \) points and \( \tilde{\psi} \) is the unique lift of \( b \) satisfying \( \tilde{\psi}|_{\partial \tilde{S}_n} \) is the identity. (See §10.1 for the details.) The boundary of \( \tilde{S}_n \) is connected when \( \gcd(m,n) = 1 \),
so \( \partial N_n(P) \) is a torus. It is clear that each of the curves \( \partial \bar{S}_n \times t_0 \) carries the longitudinal slope of \( N_n(P) \). Let \( \mu \) is the class in \( H_1(\partial N_n(P)) \) carried by \( p_0 \times [0,1]/\sim\), \( p_0 \in \partial \bar{S}_n \).

**Proof of Theorem 1.15.** By assumption, \((S,h^n)\) is an open book decomposition of \( X_n(C) \). As in §9 we use \( \mu_n \) and \( \lambda_n \) to denote the meridional and longitudinal slopes on \( \partial X_n(C) \). Since \( \gcd(m,n) = 1 \), we have

\[
\Sigma_n(L) = N_n(P) \cup_\varphi X_n(C)
\]

where the homeomorphism \( \varphi : \partial N_n(P) \to \partial X_n(C) \) sends \( \mu \) to \( \lambda_n \) and \( \lambda \) to \( \mu_n \). We will show that \( \Sigma_n(L) \) admits a co-orientable taut foliation and has a left-orderable fundamental group (and is therefore excellent) by showing that the Dehn fillings \( N_n(P)(\mu - \lambda) \) and \( X_n(C)(\mu_n - \lambda_n) \) have the same properties.

Consider \( X_n(C) \) first. The Dehn filling \( X_n(C)(\mu_n - \lambda_n) \) has an open book decomposition \((S,T_\partial \circ h)\), where \( T_\partial \) denotes the right-handed Dehn twist along \( S \). The assumption that \( c(h) \geq 0 \) implies \( nc(h) + 1 \geq 1 \). By Theorem 1.2(1) we know that \( X_n(C)(\mu_n - \lambda_n) \) is excellent and hence has left-orderable fundamental group. However, to obtain a foliation on \( \Sigma_n(L) \), we need to use that fact that Theorem 4.1 guarantees a co-oriented taut foliation on \( X_n(C)(\mu_n - \lambda_n) \) which is transverse to the binding of the open book \((S,T_\partial \circ h)\). In our case, the binding is the core of the filling torus and consequently there is a co-oriented taut foliation on \( X_n(C) \) which restricts to a linear foliation on \( \partial X_n(C) \) of slope \( \mu_n - \lambda_n \).

Next consider \( N_n(P) \cong \bar{S}_n \times [0,1]/(x,1) \sim (\bar{\psi}(x),0) \). (See the discussion just prior to the proof of Theorem 1.15.) By Lemma 10.2, the assumption \( c(b) \geq 0 \) implies \( c(\bar{\psi}) = \frac{c(b)}{n} \geq 0 \). Since \( N_n(P)(\mu - \lambda) \) is homeomorphic to the open book \((\bar{S}_n,T_\partial \circ \bar{\psi})\), an argument analogous to that used in the previous paragraph shows that there is a co-orientable taut foliation \( \mathcal{F}_0 \) on \( N_n(P) \) which intersects \( \partial N_n(P) \) transversely in a linear foliation of slope \( \mu - \lambda \).

We need to show that \( \pi_1(N_n(P)(\mu - \lambda)) \) is left-orderable. Let \( \mathcal{F} \) denote the co-orientable taut foliation on \( N_n(P)(\mu - \lambda) \) induced by \( \mathcal{F}_0 \). The existence of \( \mathcal{F} \) implies that \( N_n(P)(\mu - \lambda) \) is irreducible ([Nov]). Hence, if \( b_1(N_n(P)(\mu - \lambda)) > 0 \), this is true by Corollary 8.2. Assume then that \( b_1(N_n(P)(\mu - \lambda)) = 0 \). In other words, \( N_n(P)(\mu - \lambda) \) is a rational homology 3-sphere. We show that \( e(T\mathcal{F}) = 0 \).

It is clear from our argument that the tangent plane field \( T\mathcal{F} \) is homotopic to the contact structure supported by the open book \((\bar{S}_n,T_\partial \circ \bar{\psi})\) (cf. Theorem 4.1). However, it is not immediately clear whether or not the Euler class of the foliation on \( N_n(P) \) given by the locally-trivial fibre bundle structure is zero. Hence Proposition 9.1 cannot be applied directly to show that \( e(T\mathcal{F}) = 0 \). Instead, we observe that the monodromy \( T_\partial \circ \bar{\psi} : \bar{S}_n \to \bar{S}_n \) is the lift of the braid \( \Delta_{m,n}^{2b} : D_m \to D_m \) (see §10.1). Hence by [HKP, Theorem 1.4], the contact structure supported by \((\bar{S}_n,T_\partial \circ \bar{\psi})\) has zero Euler class and therefore so does \( e(T\mathcal{F}) \). As a result, \( \pi_1(N_n(P)(\mu - \lambda)) \) is left-orderable by Theorem 8.3.

Finally, by piecing together the foliations on \( N_n(P) \) and \( X_n(C) \) constructed above we obtain a co-orientable taut foliation on \( \Sigma_n(L) \). Further, an application of [CLW, Theorem 2.7] implies that \( \pi_1(\Sigma(L)) \) is left-orderable. Thus \( \Sigma_n(L) \) is excellent. \( \square \)
Proof of Theorem 1.13. The proof of the theorem is similar to the one used to prove Theorem 1.15. As such, we only point out a few of the key points.

Note that when $c(h) \neq 0$ and $n \geq \frac{2}{|c(h)|}$, the fractional Dehn twist coefficient of the monodromy of the $n$-fold cyclic cover $X_n(C)$ satisfies $|c(h^n)| = n|c(h)| \geq 2$. Then the argument used in the proof of Theorem 1.15 to analyse $X_n(C)(\mu_n - \lambda_n)$ can be used to show that:

- $\pi_1(X_n(C)(\mu_n + \lambda_n))$ and $\pi_1(X_n(C)(\mu_n - \lambda_n))$ are left-orderable;
- there is a co-orientable taut foliation on $X_n(C)$ which induces a linear foliation on $\partial X_n(C)$ with leaves of slope $\mu_n + \lambda_n$;
- there is a co-orientable taut foliation on $X_n(C)$ which induces a linear foliation on $\partial X_n(C)$ with leaves of slope $\mu_n - \lambda_n$.

When $c(h) = 0$, it is simple to see that $X_n(C)$ possess the same properties.

Based on these observations, the rest of the proof follows exactly as in that of Theorem 1.15 when $c(b) \geq 0$. In the case that $c(b) < 0$, one can apply the arguments of Theorem 1.15 but replacing $\mu_n - \lambda_n$ by $\mu_n + \lambda_n$ and $\mu - \lambda$ by $\mu + \lambda$ to complete the proof.

□

References

[Al] J. W. Alexander, A Lemma on System of Knotted Curves, Proc. Nat. Acad. Sci. USA 9 (1923), 93–95.

[BM] K. Baker and K. Motegi, Seifert vs slice genera of knots in twist families and a characterization of braid axes, preprint (2017). arXiv:1705.10373.

[Bal] J. Baldwin, Heegaard Floer homology and genus one, one-boundary component open books, J. Topology 4 (2008), 963–992.

[BBG] M. Boileau, S. Boyer and C. McA. Gordon, Branched covers of quasipositive links and L-spaces, preprint 2017, arXiv:1710.07658.

[BC] S. Boyer and A. Clay, Foliations, orders, representations, L-spaces and graph manifolds, Adv. Math., 310 (2017), 159–234.

[BGW] S. Boyer, C. McA. Gordon and L. Watson, On L-spaces and left-orderable fundamental groups, Math. Ann. 356 (2013), 1213–1245.

[BPH] M. Boileau, J. Porti and M. Heusener, Geometrization of 3-orbifolds of cyclic type, Socit mathematique de France, 2001.

[BRW] S. Boyer, D. Rolfsen and B. Wiest, Orderable 3-manifold groups, Ann. Inst. Fourier 55 (2005), 243–288.

[Bn] J. Bowden, Approximating $C^0$-foliations by contact structures, Geo. Func. Anal. 26 (2005), 1255–1296.

[Cal1] D. Calegari, Leafwise smoothing laminations, Alg. & Geom. Topology 1 (2001), 579–585.

[Cal2] ———, Foliations and the geometry of 3-manifolds, Oxford Mathematical Monographs, Oxford University Press, Oxford, 2007.

[CD] D. Calegari and N. Dunfield Laminations and groups of homeomorphisms of the circle, Inv. Math. 152 (2003), 149–204.

[Cau] A. Candel, Uniformization of surface laminations, Ann. Sci. Ec. Norm. Sup. 26 (1993), 489–516.
[KR1] W. Kazez and R. Roberts, *Fractional Dehn twists in knot theory and contact topology*, Algebr. Geom. Topol., 13 (2013), 3603–3637.

[KR2] ———, *Approximating $C^1$-foliations*, in *Interactions between low-dimensional topology and mapping class groups*, 21–72, Geom. Topol. Monogr., 19, Geom. Topol. Publ., Coventry, 2015.

[LS] P. Lisca and A. Stipsicz, *Ozsváth-Szabó invariants and tight contact 3-manifolds, III*, J. Symplectic Geom. 5 (2007), 357–384.

[LW] Y. Li and L. Watson, *Genus one open books with non-left-orderable fundamental group*, Proc. Amer. Math. Soc. 142 (2014), 1425–1435.

[Mal] A. Malyutin, *Twist number of (closed) braids*, St. Petersburg Math. J. 16 (2005), 791–813.

[Mi1] J. Milnor, *On the existence of a connection with curvature zero*, Comm. Math. Helv. 32 (1958), 215–223.

[Mi2] ———, *Infinite cyclic coverings*, in *Conference on the Topology of Manifolds*, Prindle, Weber & Schmidt, Boston, Mass., 115–133

[MM] S. Matsumoto and S. Morita, *Bounded cohomology of certain groups of homeomorphisms*, Proc. Amer. Math. Soc. 94 (1985), 539–544.

[MSY] W. Meeks, L. Simon, and S. T. Yau, *Embedded minimal surfaces, exotic spheres, and manifolds with positive ricci curvature*, Ann. Math. 116 (1982), 621–659.

[MB] J. W. Morgan and H. Bass, *The Smith Conjecture*, Pure and Applied Math. 112, Academic Press, 1984.

[Mor] S. Morita, *Geometry of Differential Forms*, Translations of Mathematical Monographs, 201, American Mathematical Society, 2001.

[Mur] K. Murasugi, *On closed 3-braids*, Mem. Amer. Math. Soc. 151, American Mathematical Society, Providence, R.I., 1974.

[Ni] Y. Ni, *Knot Floer homology detects fibered knots*, Invent. Math. 170 (2007), 577–608.

[Nov] S. P. Novikov, *The topology of foliations*, Tru. Mosk. Mat. Obsc. 14 (1965), 248–278.

[OS1] P. Ozsváth and Z. Szabó, *Holomorphic disks and genus bounds*, Geom. & Top. 8 (2004), 311–334.

[OS2] ———, *On the Heegaard Floer homology of branched double-covers*, Adv. Math. 194 (2005), 1–33.

[Pc] T. Peters, *On L-spaces and non-left-orderable 3-manifold groups*, arXiv:0903.4495 (2009).

[Pla] J. Plante, *Foliations with measure preserving holonomy*, Ann. Math. 102 (1975), 327–361.

[Rob] R. Roberts, *Taut foliations in punctured surface bundles, II*, Proc. London Math. Soc. 83 (2001) 443–471.

[Ros] H. Rosenberg, *Foliations by planes*, Topology 7 (1968), 131–138.

[Spa] E. Spanier, *Algebraic topology*, Springer-Verlag 1995.

[Thu] W. Thurston, *On the geometry and dynamics of diffeomorphisms of surfaces*, Bull. Amer. Math. Soc. 19 (1988), 417–431.
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