Toric Bézier patches generalize the classical tensor-product triangular and rectangular Bézier surfaces, extensively used in CAGD. The construction of toric Bézier surfaces corresponding to multi-sided convex hulls for known boundary mass-points with integer coordinates (in particular for trapezoidal and hexagonal convex hulls) is given. For these toric Bézier surfaces, we find approximate minimal surfaces obtained by extremizing the quasi-harmonic energy functional. We call these approximate minimal surfaces as the quasi-harmonic toric Bézier surfaces. This is achieved by imposing the vanishing condition of gradient of the quasi-harmonic functional and obtaining a set of linear constraints on the unknown inner mass-points of the toric Bézier patch for the above mentioned convex hull domains, under which they are quasi-harmonic toric Bézier patches. This gives us the solution of the Plateau toric Bézier problem for these illustrative instances for known convex hull domains.

1. INTRODUCTION

The theory of minimal surfaces has its roots in the optimization problems of calculus of variations, based on the famous Euler- Lagrange equation which is a second order partial differential equation (pde). The solution of the Euler-Lagrange equation targets to find a function that extremizes a given functional and has many applications in the optimization theory. Many mathematicians have contributed to the subject of optimization theory and it has become a widely accepted discipline of Mathematics and Physics. A minimal surface is a surface which locally minimizes its area or equivalently a surface whose mean curvature vanishes everywhere on the surface. In the similar context, a problem known as the Plateau problem consists of finding the surface with least surface area bounded by a given boundary curve. It is named after Belgian physicist Joseph. A. Plateau who experimentally demonstrated in 1849 that minimal surfaces can be associated to the soap films spanned by wire frames of different shapes. In the meantime, many mathematicians developed their interest in finding a minimal surface spanned by a fixed boundary curve such as Schwarz (who studied the triply periodic surfaces namely the CLP (crossed layers of parallels), D (diamond), P (primitive), H (hexagonal) and T (tetragonal) surfaces, Weierstrass, Riemann and R. Garnier in the late 19th century. However, these were minimal surfaces for particular boundaries, until in 1931, American mathematician J. Douglas who studied the triply periodic surfaces namely the CLP (crossed layers of parallels), D (diamond), P (primitive), H (hexagonal) and T (tetragonal) surfaces, Weierstrass, Riemann and R. Garnier independently proved the existence of a minimal surface spanned by a closed curve by replacing the area functional by rather a simpler integral, now known as the Douglas-Dirichlet functional. The Douglas-Dirichlet functional does not have square root in its integrand as is the case with the area functional which makes it a suitable choice as an alternative to the area functional.

Exact mathematical solutions are known only for some specific boundaries. It is possible to find numerically the solution of a wide variety of problems giving rise to approximate minimal surfaces. Coppin and Greenspan used a computer model of molecular structure and forces to approximate a minimal surface. K. Koohestani also suggested the method involving non-linear force density to find minimal surfaces for membrane structures. Brakke used the finite element method to approximate parameterized minimal surfaces. Level set method was proposed by Chopp to cope with topological variations of a surface under linear convergence, whereas a variational approach to minimize the area of triply periodic surfaces was proposed by Jung et al. Ronquist and Träsk Dahl introduced an iterative scheme which involves parameterization of higher order polynomials to achieve a numerical approximation of a minimal surface with fixed boundaries. Similarly, Li et al. numerically approximated the minimal surfaces with geodesic constraints over boundary curves. Kassabov derived an equation of a canonical parameterized minimal surface and also pointed out its application. Xu et al. proposed a parametric form of polynomial minimal surface with varying degrees which possesses interesting properties helpful for geometric modeling in CAD.

Alternative energy functionals for minimization may be used to find an approximate minimal surface of a certain restricted class of surfaces. One of the widely used restriction is to find a minimal Bézier surface among all the Bézier...
surfaces

\[ x(u, v) = \sum_{i=0}^{n} \sum_{j=0}^{m} B_{i,j}^{n,m}(u, v) \mathbf{P}_{ij}, \]  

with

\[ B_{i,j}^{n,m}(u, v) = B_i^n(u) B_j^m(v), \]  

spanned by a given boundary in which \( \mathbf{P}_{ij} \) represents a two dimensional control net over the domain \( D = [0, 1] \times [0, 1] \) with \( u, v \) as the surface parameters, the bivariate functions \( \{B_{i,j}^{n,m}(u, v) : \mathbb{R}^2 \to \mathbb{R}\} \) are the blending functions to specify the shape of the surface and

\[ B_i^n(u) = \binom{n}{i} u^i (1-u)^{n-i} \]  

are the Bernstein polynomials of degree \( n \) with \( \binom{n}{i} = \frac{n!}{i!(n-i)!} \) as the binomial coefficients.

An extremal of discrete version of Dirichlet functional giving minimal Bézier surfaces can be seen in the Monterde work \[17\]. X. D. Chen, G. Xu, and Y. Wang. \[18\] found approximate minimal surfaces as the solution of Plateau-Bézier problem using extended Dirichlet functional and the extended bending energy functional, the surfaces depend on the parameters \( \lambda \) and \( \alpha \) (as they appear in eqs. (4) and (5) of the ref.\[18\]) for simple estimates of these parameters. Hao et al. \[19\] investigated the Plateau-quasi-Bézier problem, minimizing thereby the Dirichlet functional of surfaces for more generalized borders including the boundary curves like polynomial curves, catenaries and circular arcs. Another restriction could be to find a parametric polynomial minimal surface as has been proposed by Xu and Wang \[20\] to obtain a minimal surface for quintic parametric polynomial surface having the prescribed borders as polynomial curves. Ahmad and Masud \[21-23\] gave an algorithm to find a quasi-minimal surface, variationally improving the to obtain a minimal surface for quintic parametric polynomial surface having the prescribed borders as polynomial restriction could be to find a parametric polynomial minimal surface as has been proposed by Xu and Wang \[20\] et al. \[19\] investigated the Plateau-quasi-Bézier problem, minimizing thereby the Dirichlet functional of surfaces for \( \lambda \) parameters \[1, 25\]. This means that a positive definite metric in two dimensions

\[ ds^2 = E(x, y) dx^2 + 2F(x, y) dx dy + G(x, y) dy^2, \]  

defined in the neighbourhood of a surface \( x(x, y) \) in local coordinates \( (x, y) \) takes the form

\[ ds^2 = \lambda^2(x, y) \left( dx^2 + dy^2 \right), \]  

(\text{i.e. } E(x, y) = G(x, y) = \lambda^2(x, y), F(x, y) = 0) in the isothermal coordinates \( (x, y) \). If a surface is parameterized using the isothermal parameterization \[25\], then such a parameterization is minimal if the coordinate functions are harmonic. In other words, a surface with isothermal parameterization is a minimal surface if and only if it is a harmonic surface. This is also useful in finding a minimal surface associated to a class of surfaces namely the Bézier surfaces. Monterde and Ugal\[20\] indicated that harmonic Bézier surfaces can only be specified by opposite boundary control points and thus making it impracticable to generate a harmonic Bézier surface from the prescribed four boundary Bézier curves. In order to overcome this difficulty, Xu et al. \[21\] proposed the quasi-harmonic surfaces which serve as the solution surfaces for Plateau-Bézier problem. They also showed that in particular cases when the corners of Bézier surface are almost isothermal, quasi-harmonic surfaces are better approximations when compared to surfaces generated by Dirichlet method.

Polynomial functions and splines are widely used in many structural design program softwares. The fundamental units of modeling a surface geometrically are the classical Bézier triangles and rectangular tensor product patches \[27\] in computer aided geometric designing (CAGD), however some applications require a more generalized form of multi-sided \( C^\infty \) patches rather than the classical Bézier surfaces. J. Warren \[28\] realized the usage of real toric surfaces in CAGD. His notable contribution is construction of a hexagonal patch from a rational Bézier triangle with zero weights and the corresponding control points located appropriately. The multi-sided patches bear more flexibility and present interesting mathematical structures when dealt through Krassauskas’s toric Bézier patches \[29\]. Toric Bézier patches are the generalization of the classical Bézier patches that deal only with triangular or rectangular patches. In 2002, Krassauskas and Goldman \[30\] presented the construction of toric Bézier patches of depth \( d \) by using
the de Casteljau pyramid algorithm and blossoming algorithm for the associated patches. In recent work by Gang Xu, Tsz-Ho Kwok and Charlie C.L. Wang \[31\], a B-spline volumetric parameterization is constructed with semantic features for isogeometric analysis.

Further developments in toric Bézier surfaces include the work of García-Puente et al. \[32\], they illustrated the geometrical importance of the structural system of toric Bézier patches, Sun and Zhu \[33, 34\] discussed the $G^1$ continuity of toric Bézier surfaces and found approximate minimal toric Bézier surfaces by minimizing the Dirichlet functional.

In this paper, we construct quasi-harmonic toric Bézier patches defined over multi-sided convex hulls with prescribed boundary mass-points by extremizing the quasi-harmonic functional to generate a system of linear equations for the unknown inner mass-points. This enables us to write down the parametric form of the solution of the Plateau-toric Bézier problem. The paper is organized as follows: In section 2 we give the preliminary introduction to toric Bézier patch of depth $d$ in general and its construction consisting of indexing lattice polygon domains and the associated toric Bernstein polynomials. In the following sections 3 and 4, we utilize the quasi-harmonic energy functional as the objective functional to obtain the necessary and sufficient conditions for a toric Bézier patch to be a quasi-harmonic toric Bézier patch which serves as the solution to the Plateau-toric Bézier problem. Finally, in section 5 we construct quasi-harmonic toric Bézier patches defined over trapezoidal convex hulls and hexagonal convex hull as illustrative applications. Constraints on mass-points of the toric Bézier patches defined over the above mentioned multi-sided domains are obtained by solving the respective systems of linear equations for the inner unknown mass-points. For the prescribed boundary mass points, quasi-harmonic toric Bézier patches, as illustrative applications, have also been obtained and shown that the inner mass-points satisfy the computed constraints.

2. TORIC BÉZIER PATCHES AND RELATED TERMINOLOGY

In computer aided geometric designing (CAGD), three and four-sided patches namely the triangular and rectangular Bézier patches are commonly used for surface modeling but a multi-sided generalization of these Bézier schemes is required in order to fill $n$-sided holes. One of such schemes used to define multi-sided $C^\infty$ patches is the Krasauskas’s Toric Bézier patch as introduced in \[29\]. A scheme in section 4 is given to obtain quasi-harmonic toric Bézier surface by extremizing the quasi-harmonic functional introduced in the section 3. To comprehend the construction of these toric Bézier patches and then to extremize a given functional to find an approximate minimal surface, we give below the related terminology for the reader to get familiar with lattice polygons, Bernstein basis functions for these polygons, discrete convolution indexed by Minkowski sum and finally the construction of toric Bézier patches for given depth $d$.

**Definition 2.1. (Lattice Polygons)** The polygon formed by connecting the outer most sequence of points in the finite set $\sigma \in \mathbb{Z}^2$ in the plane is called the lattice polygon. The finite set $\sigma$ is used as the index set for control points $\{P_{\sigma_i}\}_{\sigma \in \sigma}$ to form a polygonal array of control points.

The lattice polygons for the classical tensor-product Bézier patch and triangular Bézier patch are lattice rectangle and lattice triangle respectively which form the array of their corresponding control points. Other examples of multi-sides lattice polygons are given in fig 1.

![FIG. 1: Multi-sided lattice polygons, a lattice pentagon (left) and a lattice hexagon (right) with inner lattice points (red dots).](image1)

**Definition 2.2. (Bernstein Polynomial Functions for Lattice Polygons)** Let $\sigma = \{\sigma_1, \sigma_2, ..., \sigma_m\} \in \mathbb{Z}^2$ be the set of finite integers in $uv$-plane. The lattice polygon $I_\sigma$ denotes the convex hull of $\sigma$ with corner points $v_1, v_2, ..., v_n$.
and \( T_k(u, v) = \alpha_k u + \beta_k v + \gamma_k \), \( k = 1, 2, \ldots, n \), the \( k\)th edge of the convex hull \( I_\sigma \). In addition, the direction of the normal vector \((\alpha_k, \beta_k)\) to the line \( L_k(u, v) \) is in the convex hull \( I_\sigma \) and \((\alpha_k, \beta_k)\) is the shortest normal vector with integer coordinates in that direction.

The Bernstein polynomials \( \beta_{\sigma_i}(u, v)_{\sigma_i \in \sigma} \) for \((u, v)\) in the convex hull \( I_\sigma \), for toric Bézier patch can be written as

\[
\beta_{\sigma_i}(u, v) = c_{\sigma_i} \left( L_1(u, v) \right) L_1(\sigma_i) \left( L_2(u, v) \right) L_2(\sigma_i) \cdots \left( L_n(u, v) \right) L_n(\sigma_i),
\]

where positive arbitrary normalizing constants \( c_{\sigma_i} \) are the coefficients of basis functions, chosen appropriately to get certain desired formulas. For toric Bézier patches, the Bernstein polynomials for lattice polygon \( \{ \beta_{\sigma_i}(u, v) \}_{(\sigma_i) \in I_\sigma} \) have the analogous properties as that of classical Bernstein polynomials \((1, 3)\) for which the classical bivariate functions \((eq. (2.1))\) indexed by the lattice polygon \( u, v \) are

\[
P^{m,n}_{i,j}(u, v) = \binom{n}{i} \binom{m}{j} u^i (1 - u)^{n-i} v^j (1 - v)^{m-j},
\]

(for \( i \in \{0, 1, \ldots, n\}, j \in \{0, 1, \ldots, m\} \) used to construct the triangular or rectangular Bézier patches. These Bernstein polynomials \( \{ \beta_{\sigma_i}(u, v) \}_{(\sigma_i) \in I_\sigma} \) indexed by the set \( \sigma \) with lattice polygon \( I_\sigma \) having corner points \( v_1, v_2, \ldots, v_n \) satisfy the following properties: 1) \( \beta_{\sigma_i}(u, v) > 0 \) inside the lattice polygon \( I_\sigma \), 2) \( \beta_{\sigma_i}(u, v) = 0 \) on the edge \( v_k v_{k+1} \), if and only if \( \sigma_i \notin v_k v_{k+1} \), 3) \( \beta_{\sigma_i}(u, v) = 1 \) if \( \sigma_i = v_k \) and 4) \( \{ \beta_{\sigma_i}(u, v) \} \) are polynomial functions.

**Definition 2.3. (Toric Bézier Patch)** A toric Bézier patch is a rational surface \( \mathcal{P}(u, v) \) in the real projective space \( \mathbb{R}^{d+4} \) of dimension 4 with control structure consisting of mass-points \( \{ (\omega_{\sigma_i} P_{\sigma_i}, \omega_{\sigma_i}) \} \) indexed by the lattice polygon \( I_\sigma \). The mass-points \( \{ (\omega_{\sigma_i} P_{\sigma_i}, \omega_{\sigma_i}) \} \) are four dimensional elements with \( \omega_{\sigma_i} \) as the scalar weights corresponding to control points \( P_{\sigma_i} \) in space. The Bernstein polynomials for lattice polygon \( \beta_{\sigma_i}(u, v) \) as given in eq. \((2.1)\) are the blending functions which serve as the basis functions for toric Bézier patches defined over the domain lattice polygon \( I_\sigma \) and they are chosen to obtain the desired shape of the surface. The toric Bézier surface \( \mathcal{P}(u, v) \) is defined by the expression

\[
\mathcal{P}(u, v) = \sum_{\sigma_i \in I_\sigma} \beta_{\sigma_i}(u, v) (\omega_{\sigma_i} P_{\sigma_i}, \omega_{\sigma_i}), \quad (u, v) \in I_\sigma,
\]

where Bernstein polynomials \( \{ \beta_{\sigma_i}(u, v) \}_{(\sigma_i) \in I_\sigma} \) are given in eq. \((2.1)\). A rational surface may be obtained by dividing the surface eq. \((2.3)\) by \( \sum_{\sigma_i \in I_\sigma} \beta_{\sigma_i}(u, v) \) provided that \( \sum_{\sigma_i \in I_\sigma} \beta_{\sigma_i}(u, v) \neq 0 \), throughout the domain. Krasauskas and Goldman \([30]\) introduced the concept of depth for toric Bézier patches which is the analogue of degree used to define the classical higher order Bézier surfaces. It is based on the depth of lattice polygons defined with the help of repeated Minkowski sums.

**Definition 2.4. (Minkowski sum)** Let \( A \) and \( B \) be any two sets of \( p\)-tuples. The Minkowski sum \( A \oplus B \) of these two sets is the set with the sum of all elements from \( A \) and all elements of \( B \) given by,

\[
A \oplus B = \{ a + b | a \in A, b \in B \}.
\]

**Definition 2.5. (Discrete convolution indexed by Minkowski sum)** Let \( P = \{ P_a | a \in A \} \) and \( Q = \{ Q_b | b \in B \} \) be two arrays. Then the discrete convolution \( P \otimes Q \) indexed by the Minkowski sum \( A \oplus B \) i.e., \( P \otimes Q = \{ (P \otimes Q)_c | c \in A \oplus B \} \) is defined as

\[
(P \otimes Q)_c = \sum_{a+b=c} P_a Q_b.
\]

The indexing of discrete convolution indexed by Minkowski sum may be used to define toric Bézier patches with depth \( d \), as given below. The depth \( d \) of toric Bézier patches as expressed by Krasauskas and Goldman \([30]\) is the analogue of degree used to define the classical higher order Bézier surfaces. It is based on the depth of lattice polygons defined with the help of repeated Minkowski sums as given above (definitions \([2.4, 2.5]\)).

**Definition 2.6. (Toric Bézier Patch with depth \( d \))** Let \( \sigma^d = \sigma \oplus \sigma \oplus \ldots \oplus \sigma \) be the \( d\)-fold Minkowski sum of \( \sigma \) and \( I^d \), the corresponding convex hull of \( \sigma^d \). Then the toric Bernstein basis functions \( \{ \beta_{\gamma}(u, v) \}_{\gamma \in \sigma^d} \) on \( I^d \) are given by convolution of the Bernstein basis function \( \{ \beta_{\sigma}(u, v) = \beta_{\sigma_i}(u, v) \}_{(\sigma_i) \in \sigma} \) indexed by \( \sigma^d \), i.e.,

\[
\{ \beta_{\gamma}(u, v) \}_{\gamma \in \sigma^d} = \beta_{\sigma}(u, v) \otimes \beta_{\sigma}(u, v) \otimes \ldots \otimes \beta_{\sigma}(u, v).
\]

\([2.4]\)
A toric Bézier patch defined on lattice polygon of depth $d$ and the corresponding convex hull $I^d$ of $\sigma^d$ in the projective space is a surface parameterized by the map $\mathcal{P} : I^d \rightarrow \mathbb{RP}^d$ for $(u, v) \in I^d$ is defined as,

$$\mathcal{P}(u, v) = \sum_{\gamma \in \sigma^d} \beta_\gamma^d(u, v) (\omega_\gamma p_\gamma, \omega_\gamma),$$

(2.5)

the control structure consists of the mass-points $\{(\omega_\gamma p_\gamma, \omega_\gamma)\}_{\gamma \in \sigma^d}$, where $\{p_\gamma\}_{\gamma \in \sigma^d}$ are the control points and $\{\omega_\gamma \geq 0\}_{\gamma \in \sigma^d}$ are the respective weights. $\mathcal{P}^d_\gamma(u, v)_{\gamma \in \sigma^d}$ are the blending functions, known as the toric Bernstein basis functions for $I^d$.

The toric Bézier patches are the rational surfaces lying in the affine or projective spaces. The derivative of a rational surface is not that straightforward in general but rather a little complicated. It is however advantageous to find the derivatives of the numerator and denominator parts of the rational surface first and then to apply the quotient rule of derivation to get the derivative of the quotient. Therefore, instead of derivative of the rational toric Bézier patch, the derivative of the corresponding toric Bézier surface in the space of mass-points is more useful. A detailed account of finding derivative of toric Bézier patch of depth $d$ w.r.t. the surface parameters $u$ and $v$ can be seen in [30] (pages 82-84). The partial derivative $w.r.t.$ $u$ of Bernstein polynomials $\beta^d_\gamma(u, v)$ for lattice polygons of depth $d$ is given by the following expression

$$\frac{\partial \beta^d_\gamma(u, v)}{\partial u} = d \sum_{\sigma_i \in \sigma} \frac{\partial \beta_{\sigma_i}(u, v)}{\partial u} \beta^d_{\gamma-\sigma_i}(u, v),$$

(2.6)

which leads to the first order partial differentiation $w.r.t.$ $u$ of the polynomial patch $\mathcal{P}(u, v)$ eq. (2.5) and is given by

$$\mathcal{P}_u(u, v) = d \sum_{\gamma \in \sigma^d} \left( \sum_{\sigma_i \in \sigma} \frac{\partial \beta_{\sigma_i}(u, v)}{\partial u} \beta^{d-1}_{\gamma-\sigma_i}(u, v) \right) (\omega_\gamma p_\gamma, \omega_\gamma).$$

(2.7)

The second order partial derivatives of toric Bézier patch $w.r.t.$ its parameters $u$ and $v$ (later to be used in next section) can be computed and they are

$$\mathcal{P}_{uu}(u, v) = d \sum_{\gamma \in \sigma^d} \left( \sum_{\sigma_i \in \sigma} \frac{\partial^2 \beta_{\sigma_i}(u, v)}{\partial u^2} \beta^{d-1}_{\gamma-\sigma_i}(u, v) \right) (\omega_\gamma p_\gamma, \omega_\gamma)$$

$$+ d(d-1) \sum_{\gamma \in \sigma^d} \left( \sum_{\sigma_i \in \sigma} \frac{\partial \beta_{\sigma_i}(u, v)}{\partial u} \left( \sum_{\delta_i \in \sigma, \delta_i \neq \sigma_i} \frac{\partial \beta_{\delta_i}(u, v)}{\partial u} \beta^{d-2}_{\gamma-\sigma_i-\delta_i}(u, v) \right) \right) (\omega_\gamma p_\gamma, \omega_\gamma),$$

(2.8)

$$\mathcal{P}_{vv}(u, v) = d \sum_{\gamma \in \sigma^d} \left( \sum_{\sigma_i \in \sigma} \frac{\partial^2 \beta_{\sigma_i}(u, v)}{\partial v^2} \beta^{d-1}_{\gamma-\sigma_i}(u, v) \right) (\omega_\gamma p_\gamma, \omega_\gamma)$$

$$+ d(d-1) \sum_{\gamma \in \sigma^d} \left( \sum_{\sigma_i \in \sigma} \frac{\partial \beta_{\sigma_i}(u, v)}{\partial v} \left( \sum_{\delta_i \in \sigma, \delta_i \neq \sigma_i} \frac{\partial \beta_{\delta_i}(u, v)}{\partial v} \beta^{d-2}_{\gamma-\sigma_i-\delta_i}(u, v) \right) \right) (\omega_\gamma p_\gamma, \omega_\gamma).$$

(2.9)

Above partial derivatives of $\mathcal{P}(u, v)$ are helpful in the extremization of the quasi-harmonic functional used as objective function to obtain quasi-harmonic toric Bézier patch as the solution of Plateau toric Bézier problem, the task accomplished in section 4. The next section gives a brief description of the energy functionals that can be used as objective functions for extremization purpose to obtain an approximate minimal surface.

3. QUASI-HARMONIC FUNCTIONAL

To find an approximate minimal surface, several energy functionals have been used instead of area functional itself which involves a square root in its integrand. These functionals may be extremized to obtain quasi-minimal surfaces with prescribed boundary in general. Following section gives a brief description of different energy functionals which may be used as objective functions to trigger the extremization process for different surfaces along with the quasi-harmonic functional that is used in our next section to obtain a quasi-harmonic Bézier patch as an approximate
solution to the Plateau-toric Bézier problem. In an optimization problem, one needs to minimize the area functional (eq. (3.1) for any surface \( x(u, v) \). The area functional of the toric Bézier surface \( P(u, v) \) is

\[
\mathcal{A}(P(u, v)) = \int_{I^d} |P(u, v)_u \times P(u, v)_v| dudv, \tag{3.1}
\]

where \( I^d \subset \mathbb{Z}^2 \) is the parametric domain over which the surface \( P(u, v) \) is defined as a map and \( P_u(u, v) \) and \( P_v(u, v) \) are the partial derivatives of \( P(u, v) \) with respect to parameters \( u \) and \( v \). However, the non-linearity of this functional makes it difficult to find the solution of Plateau problem in general. Douglas \[6\] replaced the area functional for a surface \( x(u, v) \) with respect to parameters \( u \) and \( v \) by extremizing the quasi-harmonic functional

\[
D(x(u, v)) = \frac{1}{2} \int_R \left( \|x_u\|^2 + \|x_v\|^2 \right) dudv. \tag{3.2}
\]

This functional was utilized by Monterde \[17\] to solve the Plateau-Bézier problem. Sun and Zhu \[34\] found the extremals of toric Bézier surfaces by minimizing the Dirichlet functional

\[
\mathcal{D}(x(u, v)) = \int_{I^d} \mathcal{L}(x(u, v)) dudv, \tag{3.3}
\]

with \( a, b, c, d \) and \( e \) being the real constants. By assigning different values to these constants, the functional could be reduced to other alternative functionals used for minimizing purposes such as Farin and Hansford functional \[36\], standard biharmonic functional introduced by Schneider and Kobbelt \[37\] or Bloor and Wilson’s modified biharmonic functional \[38\]. The solution of the area problem for Bézier patches by extremizing the quasi-harmonic functional

\[
\mathcal{H}(x(u, v)) = \int_R (x_{uu} + x_{vv})^2 dudv. \tag{3.4}
\]

for the surface \( x(u, v) \) is already known \[24\]. We choose this quasi-harmonic functional as an objective function to find the solution of Plateau’s toric Bézier problem, as mentioned earlier that the toric Bézier patches generalize the classical rational triangular and tensor-product Bézier surfaces defined over multi-sided domains. It gives \[24\] better approximation of surfaces with lesser area and smaller mean curvature values at arbitrary points when compared to the Dirichlet functional for Bézier surfaces. The quasi-harmonic functional, taken as an objective function, for the toric Bézier patch \( P(u, v) \) (eq. (2.3)) is given by

\[
\mathcal{H}(P(u, v)) = \int_{I^d} (P_{uu}(u, v) + P_{vv}(u, v))^2 dudv, \tag{3.5}
\]

where \( P_{uu}(u, v) \) and \( P_{vv}(u, v) \) are given by eqs. (2.8) and (2.9). In the following section, necessary and sufficient condition for a toric Bézier patch to be a quasi-harmonic toric Bézier is computed by extremizing the above mentioned quasi-harmonic functional eq. (3.5).

4. QUASIHARMONIC TORIC BÉZIER PATCHES FOR A GIVEN BOUNDARY

For the Plateau Toric Bézier problem, we minimize the quasi-harmonic functional to get a quasi-harmonic toric Bézier patch \( P(u, v) \). For this, we find the gradient of the \( \mathcal{H}(P(u, v)) \) with respect to the inner unknown mass-points \((\omega_p, \omega_\gamma)\) and equate it to zero to find the constraints as linear equations under which the \( P(u, v) \) is quasi-harmonic toric Bézier path.

**Theorem 4.1.** If the mass-points associated to the boundary lattice points of the convex hull \( I^d \) of the toric Bézier patch \( P(u, v) = \sum_{\gamma \in \sigma^d} \beta^d(u, v) (\omega_\gamma p_\gamma, \omega_\gamma) \) are given, the patch \( P(u, v) \) is quasi-harmonic toric Bézier surface if and only if the inner unknown mass-points \((\omega_p, \omega_\lambda)\) associated to the lattice points of the convex hull satisfy the following system of linear equations:

\[
\int_{I^d} \sum_{\gamma \in \sigma^d} \left( (\xi_{\lambda, u} u + (d-1)\eta_{\lambda, u}) + (\xi_{\lambda, v} v + (d-1)\eta_{\lambda, v}) \right) ((\xi_{\gamma, u} u + (d-1)\eta_{\gamma, u}) + (\xi_{\gamma, v} v + (d-1)\eta_{\gamma, v})) (\omega_\gamma p_\gamma, \omega_\gamma) dudv = 0, \tag{4.1}
\]
where the coefficients $\xi^{\gamma,u}$ and $\eta^{\gamma,u}$ are,

$$
ξ^{\gamma,u} = \sum_{σ_i ∈ σ} \frac{∂^2 β_{σ_i}(u,v)}{∂u^2} β_{d-1}^{\gamma-σ_i}(u,v),
$$

$$
η^{\gamma,u} = \sum_{σ_i ∈ σ} \frac{∂ β_{σ_i}(u,v)}{∂u} \left( \sum_{δ_i ∈ σ, δ_i ≠ σ_i} \frac{∂ β_{δ_i}(u,v)}{∂u} β_{d-2}^{\gamma-σ_i-δ_i}(u,v) \right).
$$

(4.2)

Other coefficients $ξ^{\gamma,v}$, $ξ^{λ,u}$, $ξ^{λ,v}$, $η^{γ,v}$, $η^{λ,u}$ and $η^{λ,v}$ are obtained by replacing $u$ by $v$ and $γ$ by $λ$ in above eq. (4.2).

**Proof.** The quasi-harmonic functional $\mathcal{H}(P(u,v))$ can be rewritten as

$$
\mathcal{H}(P(u,v)) = \int \mathcal{P}_{uu}(u,v) \mathcal{P}_{uu}(u,v) + \mathcal{P}_{vv}(u,v) \mathcal{P}_{vv}(u,v) + 2 \mathcal{P}_{uu}(u,v) \mathcal{P}_{vv}(u,v) \, du dv,
$$

(4.3)

where the operator $⟨,⟩$ denotes the inner product of the two functions. For an inner mass point $(ω_λ p_λ, ω_λ)$, $λ ∈ σ^d$ and $α ∈ \{1,2,3,4\}$, the gradient of the quasi-harmonic functional with respect to the coordinates of $(ω_λ p_λ, ω_λ)$ is given by

$$
\frac{∂ \mathcal{H}(P(u,v))}{∂ (ω_λ p_λ, ω_λ)^α} = 2 \int_{I^d} \left( \frac{∂ \mathcal{P}_{uu}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} \mathcal{P}_{uu}(u,v) + \frac{∂ \mathcal{P}_{vv}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} \mathcal{P}_{vv}(u,v) \right) + \left( \frac{∂ \mathcal{P}_{uu}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} \mathcal{P}_{vv}(u,v) + \frac{∂ \mathcal{P}_{vv}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} \mathcal{P}_{uu}(u,v) \right) \, du dv.
$$

(4.4)

Differentiating partially $\mathcal{P}_{uu}(u,v)$ and $\mathcal{P}_{vv}(u,v)$, the 2nd order partial derivatives (eqs. (2.8) and (2.9) respectively) of the toric Bezier patch $P(u,v)$ w.r.t. the inner mass-point coordinates $(ω_λ p_λ, ω_λ)$ gives us

$$
\frac{∂ \mathcal{P}_{uu}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} = d \left( \sum_{σ_i ∈ σ} \frac{∂^2 β_{σ_i}(u,v)}{∂u^2} β_{d-1}^{\gamma-σ_i}(u,v) \right) e^α + d(d-1) \left( \sum_{σ_i ∈ σ} \frac{∂ β_{σ_i}(u,v)}{∂u} \left( \sum_{δ_i ∈ σ, δ_i ≠ σ_i} \frac{∂ β_{δ_i}(u,v)}{∂u} β_{d-2}^{\gamma-σ_i-δ_i}(u,v) \right) \right) e^α,
$$

(4.5)

and

$$
\frac{∂ \mathcal{P}_{vv}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} = d \left( \sum_{σ_i ∈ σ} \frac{∂^2 β_{σ_i}(u,v)}{∂v^2} β_{d-1}^{\gamma-σ_i}(u,v) \right) e^α + d(d-1) \left( \sum_{σ_i ∈ σ} \frac{∂ β_{σ_i}(u,v)}{∂v} \left( \sum_{δ_i ∈ σ, δ_i ≠ σ_i} \frac{∂ β_{δ_i}(u,v)}{∂v} β_{d-2}^{\gamma-σ_i-δ_i}(u,v) \right) \right) e^α.
$$

(4.6)

It is to be noted that in above eqs. (4.5) and (4.6), the coefficients $β_{d-2}^{\gamma-σ_i-δ_i}(u,v) = 0$ if $λ - σ_i - δ_i \notin σ^{d-2}$, $e^α$ denote the $α$th vector of the standard basis, i.e. $e^1 = \{1,0,0,0\}$, $e^2 = \{0,1,0,0\}$, $e^3 = \{0,0,1,0\}$ and $e^4 = \{0,0,0,1\}$. Substituting the coefficients $ξ^{γ,u}, ξ^{λ,u}, ξ^{λ,v}$ and $η^{γ,v}, η^{λ,u}, η^{λ,v}$ (eqs. (4.2)) in eqs. (2.8)-(2.9), we get

$$
\mathcal{P}_{uu}(u,v) = d \sum_{γ ∈ σ^d} ξ^{γ,u}(ω_γ p_γ, ω_γ) + d(d-1) \sum_{γ ∈ σ^d} η^{γ,u}(ω_γ p_γ, ω_γ).
$$

(4.7)

$$
\mathcal{P}_{vv}(u,v) = d \sum_{γ ∈ σ^d} ξ^{γ,v}(ω_γ p_γ, ω_γ) + d(d-1) \sum_{γ ∈ σ^d} η^{γ,v}(ω_γ p_γ, ω_γ).
$$

(4.8)

so that the eqs. (4.5) and (4.6) reduce to

$$
\frac{∂ \mathcal{P}_{uu}(u,v)}{∂ (ω_λ p_λ, ω_λ)^α} = dξ^{λ,u} e^α + d(d-1)η^{λ,u} e^α,
$$

(4.9)
and
\[ \frac{\partial \mathcal{P}_{uv}(u,v)}{\partial (\omega_{\lambda p_1}, \omega_\lambda)^p} = d\xi^{\lambda^p} e^a + d(d - 1)\eta^{\lambda^p} e^a. \quad (4.10) \]

Now substitute eqs. (4.7) to (4.10) in eq. (4.4) to get
\[ \frac{\partial H(\mathcal{P}(u,v))}{\partial (\omega_{\lambda p_1}, \omega_\lambda)^p} = 2a^2 \int_{\mathbb{Z}^2} ((\xi^{\lambda^p} + (d - 1)\eta^{\lambda^p}) + (\xi^{\lambda^p} + (d - 1)\eta^{\lambda^p})) (\omega_{\gamma p_1}, \omega_\gamma) d\xi d\eta. \quad (4.11) \]

We can now obtain the set of linear system of equations as stated in eq. (4.1) for which the toric Bézier patch is quasi-harmonic surface by setting \( \frac{\partial H(\mathcal{P}(u,v))}{\partial (\omega_{\lambda p_1}, \omega_\lambda)^p} = 0. \)

5. QUASI-HARMONIC TORIC BÉZIER PATCHES OVER MULTI-SIDED CONVEX HULLS

In this section, we construct toric Bézier patches over two different convex hulls namely 1) the trapezoidal convex hull and 2) hexagonal convex hull. We use the linear set of equations given in eq. (4.1) to compute the inner unknown \( \omega \) for the trapezoidal convex hull is given as

\[ \sigma = \{(i, j) : 0 \leq j \leq n, \ 0 \leq i \leq m + pn + pj \} \quad (5.1) \]

be the collection of all the integers lattice points of the trapezoidal convex hull. The corresponding Bernstein polynomial for the trapezoidal convex hull is given as

\[ \beta_{ij}(u, v) = c_{ij} u^i (m + pn - pv - u)^{m + pn - p - \nu^j} v^j (n - v)^{n - j}. \quad (5.2) \]

Then the toric Bézier surface \( B(u, v) \) defined over a general trapezoidal hull is expressed as

\[ \mathcal{P}(u, v) = \sum_{(i, j) \in I} c_{ij} u^i (m + pn - pv - u)^{m + pn - p - \nu^j} v^j (n - v)^{n - j} (\omega_{ij} P_{ij}, \omega_{ij}), \quad (u, v) \in I_\sigma. \quad (5.3) \]

**Example 1.** In particular, for \( n = 2, m = p = 1 \), the eq. (5.1) gives us the following set of integer lattice-points

\[ \sigma = \{(0, 0), (1, 0), (2, 0), (0, 2), (0, 1), (2, 1), (1, 1)\}, \]

with only one inner unknown mass-point \( p_{10} \) associated to \( \sigma_i = (1, 1) \). The Bernstein polynomials

\[ \beta_{ij}(u, v) = c_{ij} u^i (2 - v)^{2 - j} v^j (3 - u - v)^{3 - i - j}, \quad (5.4) \]

for the respective lattice-points come out to be

\[ \beta_{00}(u, v) = \frac{1}{108} (2 - v)^2 (-u - v + 3)^3, \quad \beta_{10}(u, v) = \frac{1}{16} u(2 - v)^2 (-u - v + 3)^2, \]

\[ \beta_{20}(u, v) = \frac{1}{16} u^2 (2 - v)^2 (-u - v + 3), \quad \beta_{30}(u, v) = \frac{1}{108} u^3 (2 - v)^2, \]

\[ \beta_{01}(u, v) = \frac{1}{4} (2 - v) v (-u - v + 3)^2, \quad \beta_{02}(u, v) = \frac{1}{4} v^2 (-u - v + 3), \]

\[ \beta_{12}(u, v) = \frac{u v^2}{4}, \quad \beta_{21}(u, v) = \frac{1}{4} u^2 (2 - v)v, \]

\[ \beta_{11}(u, v) = u(2 - v)v (-u - v + 3), \]

\[ \beta_{00}(u, v) = \frac{1}{108} (2 - v)^2 (-u - v + 3)^3, \quad \beta_{10}(u, v) = \frac{1}{16} u(2 - v)^2 (-u - v + 3)^2, \]

\[ \beta_{20}(u, v) = \frac{1}{16} u^2 (2 - v)^2 (-u - v + 3), \quad \beta_{30}(u, v) = \frac{1}{108} u^3 (2 - v)^2, \]

\[ \beta_{01}(u, v) = \frac{1}{4} (2 - v) v (-u - v + 3)^2, \quad \beta_{02}(u, v) = \frac{1}{4} v^2 (-u - v + 3), \]

\[ \beta_{12}(u, v) = \frac{u v^2}{4}, \quad \beta_{21}(u, v) = \frac{1}{4} u^2 (2 - v)v, \]

\[ \beta_{11}(u, v) = u(2 - v)v (-u - v + 3), \]
FIG. 2: Trapezoidal domain with 1 inner lattice point, shown as the blue dot, indexing the corresponding unknown mass-point

in which \( c_{ij} \) have been chosen appropriately. The toric Bézier patch over the given trapezoidal convex hull \( I^\sigma \), as shown in fig. 2 with corresponding Bernstein polynomials defined over lattice points is expressed as

\[
P(u, v) = \sum_{(i,j) \in I} \beta_{ij}(u,v)(\omega_{ij} P_{ij}, \omega_{ij}),
\]

where \((u, v) \in I^\sigma\). We find the constraints for the toric Bézier patch with unknown inner mass-points to be quasi-harmonic by substituting the second order partial derivative and their gradient with respect to the inner unknown mass-point \( p_{11} \) in eq. (4.1). The toric Bézier patch is quasi-harmonic if and only if the mass-points of the patch satisfy the following constraint equation

\[
p_{11} = 0.0904p_{00} - 0.1973p_{01} + 0.01430p_{02} + 0.1970p_{10} + 0.1006p_{12} + 0.09269p_{20} - 0.1390p_{21} + 0.0438p_{30}.
\]

FIG. 3: A quasi-harmonic toric Bézier patch with 1 inner lattice point indexing the unknown mass-point which is computed by using the eq. (5.7)

A particular example of a toric Bézier patch over trapezoidal convex hull with 1 unknown inner mass-point is given in figure 3 by taking known mass-points on the boundary of the convex hull. The unknown inner mass-point \( p_{11} \) is computed by using the result as stated in eq. (5.7).

**Example 2.** For \( n = 2, m = 3, p = 1 \), the set of integer lattice points is given as

\[
\sigma = \{(0, 0), (1, 0), (2, 0), (3, 0), (4, 0), (5, 0), (0, 1), (0, 2), (1, 2), (2, 2), (3, 2), (2, 1), (3, 1), (4, 1), (1, 1)\}
\]

with 3 inner unknown mass-point \( p_{11}, p_{21} \) and \( p_{31} \). The Bernstein polynomials

\[
\beta_{ij}(u, v) = c_{ij}u^i (2-u)^{2-i} v^j (5-u-v)^{5-i-j},
\]

(5.8)
for the respective lattice points are
\[
\begin{align*}
\beta_{00}(u, v) &= \frac{1}{12500}(2 - v)^2(5 - u - v)^3, \\
\beta_{10}(u, v) &= \frac{1}{2500}u(2 - v)^2(5 - u - v)^4, \\
\beta_{20}(u, v) &= \frac{1}{1250}u^2(2 - v)^2(5 - u - v)^3, \\
\beta_{30}(u, v) &= \frac{1}{1250}u^3(2 - v)^2(5 - u - v)^2, \\
\beta_{40}(u, v) &= \frac{1}{1250}u^4(2 - v)(5 - u - v)^4v, \\
\beta_{50}(u, v) &= \frac{1}{108}(5 - u - v)^3v^2, \\
\beta_{01}(u, v) &= \frac{1}{512}(2 - v)(5 - u - v)^4v, \\
\beta_{11}(u, v) &= \frac{3}{256}u^2(2 - v)(5 - u - v)^2v, \\
\beta_{21}(u, v) &= \frac{1}{512}u^4(2 - v)v, \\
\beta_{31}(u, v) &= \frac{1}{36}u(5 - u - v) v^2, \\
\beta_{41}(u, v) &= \frac{1}{36}u^2(5 - u - v)v^2. \\
\beta_{32}(u, v) &= \frac{1}{108}(u^3v^2).
\end{align*}
\]

for an appropriate choice of \(c_{ij}\). The toric Bézier patch over the given convex hull \(I_\sigma\) is defined as
\[
P(u, v) = \sum_{(i,j) \in I} \beta_{ij}(u, v)(\omega_{ij}p_{ij}, \omega_{ij}).
\]

where \((u, v) \in I_\sigma\). We can find the constraints for the toric Bézier patch with unknown inner mass-points to be quasi-harmonic by substituting the second order partial derivatives and their gradient with respect to each unknown inner mass points, namely \(p_{11}, p_{21}\) and \(p_{31}\) in eq. (5.11). The toric Bézier patch over the given trapezoidal convex hull is quasi-harmonic if and only if the mass-points of the patch satisfy the following system of equations
\[
\begin{align*}
p_{11} &= 0.8157p_{00} - 1.027p_{01} + 0.3170p_{02} + 0.1815p_{10} + 0.2146p_{12} + 0.1106p_{20} - 0.0981p_{22} + 0.0652p_{30} \\
&\quad + 0.1122p_{32} + 0.0210p_{40} - 0.0708p_{41} + 0.0299p_{50}, \\
p_{21} &= -0.4561p_{00} + 0.545p_{01} - 0.1718p_{02} + 0.2450p_{10} + 0.0246p_{12} + 0.2527p_{20} + 0.7647p_{22} + 0.0338p_{30} \\
&\quad - 0.3465p_{32} + 0.02031p_{40} + 0.2384p_{41} - 0.09169p_{50}, \\
p_{31} &= 0.0871p_{00} - 0.0955p_{01} + 0.0341p_{02} - 0.0211p_{10} - 0.0221p_{12} + 0.1053p_{20} - 0.3099p_{22} + 0.3448p_{30} \\
&\quad + 0.6595p_{32} + 0.1629p_{40} - 0.4250p_{41} + 0.2278p_{50}.
\end{align*}
\]

5.2. Quasi-harmonic toric Bézier patches of depth 2

Consider a toric Bézier patch defined over hexagonal convex hull, shown by the dotted line in fig. 4 with lattice-points,
\[
\sigma = \{(0, 0), (0, 1), (1, 0), (1, 1), (1, 2), (2, 1), (2, 2)\},
\]

where the edges of the hexagonal convex hull \(I_\sigma\) are
\[
\begin{align*}
\mathcal{L}_1(u, v) &= v; \\
\mathcal{L}_2(u, v) &= -v + 2; \\
\mathcal{L}_3(u, v) &= -u + 2; \\
\mathcal{L}_4(u, v) &= u; \\
\mathcal{L}_5(u, v) &= v - u + 1; \\
\mathcal{L}_6(u, v) &= -v + u + 1.
\end{align*}
\]
The toric Bernstein polynomials for each lattice-point $\sigma_i \in \sigma$ can be defined using the following relation
\[
\beta_{\sigma_i} = c_{\sigma_i} L_1(u, v) L_2(u, v) L_3(u, v) L_4(u, v) L_5(u, v) L_6(u, v).
\]
(5.12)

Whereas, the Bernstein polynomials $\{\beta^d_{\gamma}\}_{\gamma \in \mathcal{I}}$ for the toric Bézier patch of depth $d = 2$ can be computed by convolving the Bernstein polynomials $\beta_{\sigma_i}(u, v)$ as stated above in eq. (5.12) indexed by the Minkowski sum $\sigma \oplus \sigma = \sigma^2$. The toric Bézier patch of depth 2 over the hexagonal convex hull (as shown as solid line in fig. 6) with corresponding Bernstein polynomials is defined as
\[
\mathcal{P}(u, v) = \sum_{\gamma \in \sigma^d} \beta^d_{\gamma}(u, v) (\omega_{\gamma} p_{\gamma}, \omega_{\gamma}),
\]
(5.13)

where $(u, v) \in \mathcal{I}$. Similarly, as we already have shown for the toric Bézier patches over trapezoidal convex hull, the constraints on the mass-points for this patch can also be computed by using eq. (4.1). The toric Bézier patch of depth 2 over the hexagonal convex hull with 7 unknown inner-mass points is quasi-harmonic if and only if these inner-mass points of the patch satisfy the following linear system of constraints

---

**FIG. 5:** A quasi-harmonic toric patch defined over trapezoidal convex hull with 3 inner lattice points indexing the unknown mass-points which are computed by using system of eqs. (5.11)

**FIG. 6:** A hexagonal convex hull (solid line) of depth $d = 2$ with 19 lattice points indexed by the set $\mathcal{I}_{\sigma^2}$ with 7 inner lattice points, marked as blue dots corresponding to the unknown mass-points. The dotted lines represent the hexagonal hull of $\mathcal{I}_{\sigma}$ for toric Bézier patch of depth $d = 1$.
prescribed entire or partial border. In particular, we find a solution to the Plateau-toric Bézier problem.

Toric Bézier patches defined over any polygonal convex hull of domains with prescribed boundary mass-points can be approximated to quasi-harmonic toric Bézier patch using the result stated in eq. (4.1).

6. CONCLUSION

In this paper, we considered the quasi-Plateau problem which consists of finding the quasi-minimal surface with prescribed entire or partial border. In particular, we find a solution to the Plateau-toric Bézier problem for toric Bézier surface, which is the generalization of classical rational triangular and tensor-product Bézier surfaces defined over multi-sided domains. Quasi harmonic functional is used as the objective functional which is extremized to obtain a toric Bézier patch among all the possible patches with prescribed boundary, which we termed as quasi-harmonic toric Bézier patch. This patch serves as the solution to quasi Plateau-toric Bézier problem. The vanishing condition for gradient of the quasi-harmonic functional yields the constraints on mass-points of the toric Bézier patch as system of linear equations under which it is a quasi-harmonic toric Bézier patch. This scheme is applied to toric Bézier patches for different prescribed borders defined over the multi-sided convex hulls to illustrate its effectiveness and flexibility.
[38] M.I.G. Bloor and M.J. Wilson. An analytic pseudo-spectral method to generate a regular 4-sided PDE surface patch. *Computer Aided Geometric Design, 22*(3):203 – 219, 2005.