INTEGRAL GEOMETRY OF PLANE CURVES
AND KNOT INVARIANTS

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Abstract. We study the integral expression of a knot invariant obtained as the second coefficient in the perturbative expansion of Witten’s Chern-Simons path integral associated with a knot. One of the integrals involved turns out to be a generalization of the classical Crofton integral on convex plane curves and it is related with invariants of generic plane curves defined by Arnold recently with deep motivations in symplectic and contact geometry. Quadratic bounds on these plane curve invariants are derived using their relationship with the knot invariant.

1. Introduction

The first and second order terms in the perturbative expansion of Witten’s Chern-Simons path integral associated with a knot in the 3-space was first analyzed by Guadagnini, Martellini and Mintchev [GMM] as well as Bar-Natan [BN] shortly after Witten’s seminal work. In an announcement appeared in 1992, Kontsevich perceived a construction of a vast family of knot invariants which, presumably, contains the same information as the family of coefficients in the perturbative expansion of the Chern-Simons path integral associated with a knot [K]. In a recent paper [BT], Bott and Taubes explored this construction in a much more detailed manner. At this stage, it seems that a rigorous foundation has been laid for studying the perturbative expansion of the Chern-Simons path integral associated with a knot. But, as it seems to us, a study of each individual knot invariant in this family as concrete and thorough as possible is what we are lacking of. The first term in the perturbative expansion turns out to be a classical quantity associated with a space curve with nowhere vanishing curvature, which was studied extensively under the name of Călugăreanu-Pohl-White self-linking formula. Although the second term as a knot invariant is also classical, we find that the approach suggested by perturbative theory of the Chern-Simons path integral provides a deep insight into some of its previously unknown

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geometric and topological contents. This will serve as the prototype of our further investigation.

The knot invariant we study here is, modulo a certain constant, the second coefficient of the Conway polynomial of a knot. This is a Vassiliev invariant of second order. So we denote it by $v_2$. Perturbative expansion of the Chern-Simons path integral leads to an expression of $v_2$ as the difference of two integrals, $I_X$ and $I_Y$, over the knot thought of as a space curve. Or rather, the functional $I_X - I_Y$ on simple closed space curves can be proved to be a knot invariant and identified with the second coefficient of the Conway polynomial, modulo a certain constant. When the knot approaches to its plane projection, the first integral $I_X$ will be concentrated at the crossings. On the other hand, the second integral $I_Y$ turns out to be well defined on the plane projection. Actually, it defines a functional on the space of generic plane curves. Using the fact that $I_X - I_Y$ is a knot invariant, we show that $I_Y$ is constant in each component of the space of generic plane curves. And we can go further to understand how $I_Y$ jumps when we pass through the discriminant (non-generic plane curves). This relates $I_Y$ and the invariants of plane curves constructed by Arnold [A1,2] with deep motivations in symplectic and contact geometry. We believe that this relationship is what Arnold expected in [A2].

We mentioned above that $v_2$ and the second coefficient of Conway polynomial can be identified only after modulo a certain constant. This constant is the value of $I_Y$ on a round plane curve. It was first calculated by Guadagnini, et al. [GMM]. But their computation is lengthy and not illuminating. As we couldn’t understand their computation, we started to look for our own. It seems that one should think of $I_Y$ as a 3-dimensional generalization of the Crofton formula (dated 1868) for convex plane curves in integral geometry. Our computation of $I_Y$ on a round circle is almost parallel to the classical proof of the Crofton formula. As the classical proof of the Crofton formula yields many consequences in integral geometry of plane curves (e.g., it implies that the measure of the set of lines intersecting a simple plane curve is equal to the length of that curve), we can’t stop seeking similar consequences of our generalized Crofton formula. Notice that Bott and Taubes [BT] have observed that the construction of these knot invariants looks rather similar to the construction in classical integral geometry. Our study here seems to make this observation more concrete.

In our investigation of the knot invariant $v_2$, we noticed a quadratic bound for the values of $v_2$ on knots with $n$ crossings. It is derived from a combinatorial formula for $v_2$. Such a quadratic bound agrees with the point of view that Vassiliev invariants should be thought of as polynomial functions on the set of knots. We conjecture that a similar bound exists in general. See Section 4. The combinatorial formula for $v_2$ also leads to quadratic bounds on Arnold’s invariants of plane curves via their relations with $I_Y$.

The paper is organized as follows. In Section 2, we will define the integral $I_X$
and \( I_Y \) and present some simple calculations and generalizations. In Section 3, we evaluate the integral \( I_Y \) on a round circle in the plane. This is done by imitating the classical proof of the Crofton formula. In Section 4, we study the combinatorics of the knot invariant \( v_2 \) by considering certain limiting behaviors of the integrals \( I_X \) and \( I_Y \). In Section 5, we relate the integral \( I_Y \) with invariants of so called *unicursal* plane curves defined by Arnold [A1].

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2. AN INTEGRAL KNOT INVARIANT

For \( x \in \mathbb{R}^3 \setminus \{0\} \), we denote by
\[
\omega(x) = \frac{1}{4\pi} \frac{[x, dx, dx]}{|x|^3}
\]
the unit area form of the unit 2-sphere \( S^2 \), where \([\cdot, \cdot, \cdot]\) is the mixed product in \( \mathbb{R}^3 \).

Let \( \gamma : S^1 \to \mathbb{R}^3 \) be a smooth imbedding where we identify \( S^1 \) with \( \mathbb{R}/\mathbb{Z} \). We denote
\[
\Delta_4 = \{(t_1, t_2, t_3, t_4) ; 0 < t_1 < t_2 < t_3 < t_4 < 1\}
\]
and
\[
\Delta_3(\gamma) = \{(t_1, t_2, t_3, z) ; 0 < t_1 < t_2 < t_3 < 1, z \in \mathbb{R}^3 \setminus \{\gamma(t_1), \gamma(t_2), \gamma(t_3)\}\}.
\]

Define
\begin{align}
I_X(\gamma) &= \int_{\Delta_4} \omega(\gamma(t_3) - \gamma(t_1)) \wedge \omega(\gamma(t_4) - \gamma(t_2)) \\
\end{align}
and
\begin{align}
I_Y(\gamma) &= \int_{\Delta_3(\gamma)} \omega(z - \gamma(t_1)) \wedge \omega(z - \gamma(t_2)) \wedge \omega(z - \gamma(t_3)).
\end{align}

We will call these integrals the *X-integral* and *Y-integral* respectively. They get their names from the diagrams they correspond to. See Figure 1.

First, we want to simplify the expressions of these two integrals a little bit.

**Lemma 2.1.** We have
\begin{align}
I_X(\gamma) &= -\frac{1}{(2\pi)^2} \int_{\Delta_4} \frac{[\gamma(t_3) - \gamma(t_1), \dot{\gamma}(t_3), \dot{\gamma}(t_1)]}{|\gamma(t_3) - \gamma(t)|^3} \\
&\quad \cdot \frac{[\gamma(t_4) - \gamma(t_2), \dot{\gamma}(t_4), \dot{\gamma}(t_2)]}{|\gamma(t_4) - \gamma(t_2)|^3} \, dt_1 \, dt_2 \, dt_3 \, dt_4.
\end{align}
Notice that \([\gamma(t) - \gamma(t'), \dot{\gamma}(t), \dot{\gamma}(t')]\) is the oriented volume of the parallelepiped spanned by \(\gamma(t) - \gamma(t'), \dot{\gamma}(t)\) and \(\dot{\gamma}(t').\)

**Lemma 2.2.** Let
\[
E(z, t) = \frac{(z - \gamma(t)) \times \dot{\gamma}(t)}{|z - \gamma(t)|^3}.
\]
Then,
\[
I_Y(\gamma) = -\frac{1}{(2\pi)^3} \int_{\Delta_3(\gamma)} [E(z, t_1), E(z, t_2), E(z, t_3)] \, d^3 z \, dt_1 \, dt_2 \, dt_3.
\]

Notice that \(E(z, t) \, dt = dB\) where \(B = B(z)\) is the magnetic field induced by the current \(\gamma(t)\).

Both of these two lemmas come from a straightforward computation.

**Theorem 2.3.** Let
\[
v_2(\gamma) = I_X(\gamma) - I_Y(\gamma).
\]
Then \(v_2\) is invariant under isotopy of \(\gamma\).

So \(v_2\) is a knot invariant. This was proved rigorously by Bar-Natan [BN] first in his Princeton thesis. See also [BT]. Furthermore, this knot invariant satisfies a crossing change formula which identifies itself with the second coefficient of the Conway polynomial modulo the constant \(v_2(\text{unknot})\). This also justifies the subscription of \(v_2\).

The first step in the proof of this theorem is to show that both integrals \(I_X(\gamma)\) and \(I_Y(\gamma)\) are finite. This is done in [BT] by compactifying the integral domains and showing that the integral forms extend as smooth forms to the compactification of integral domains. From this consideration, the following lemma should be quite obvious.

**Lemma 2.4.** For an immersion \(\gamma : S^1 \to \mathbb{R}^3\) with only finitely many singularities, the integral \(I_Y(\gamma)\) is finite.

**Proof:** When \(\gamma\) is an imbedding, \(\Delta_3(\gamma)\) is the total space of a fibration over \(\Delta_3 = \{(t_1, t_2, t_3) ; 0 < t_1 < t_2 < t_3 < 1\}\) whose fibre is \(\mathbb{R}^3\) with three distinct points deleted. In our case, we may define \(\Delta_3(\gamma)\) similarly and it will have degenerate fibres over a measure zero set of \(\Delta_3\). Similar to the case where \(\gamma\) is an imbedding, \(\Delta_3(\gamma)\) with degenerate fibres deleted can be compactified and the integral form of \(I_Y(\gamma)\) extends to the compactification. This implies that the integral \(I_Y(\gamma)\) is still finite when \(\gamma\) is an immersion with only finitely many singularities. \(\square\)
3. A generalized Crofton formula

Some remarkable integral formulae associated with convex sets in the plane are obtained by simple computations of the density \(dx\,dy\) in different coordinate systems. The classical Crofton formula is such an example [S].

Let \(D\) be a bounded convex set in the plane. Through each point \(P\) exterior to \(D\), there pass two supporting lines of \(D\). Let \(s\) and \(s'\) respectively be the lengths of the line segments from \(P\) to the corresponding supporting points \(H_1\) and \(H_2\), and let \(\alpha\) be the angle \(H_1PH_2\) between the supporting lines. See Figure 2. Then

\[
\int_{P \notin D} \frac{\sin \alpha}{s \cdot s'}\,dx\,dy = 2\pi^2
\]

(3.1)

Let \(A = A(P)\) be the area of the parallelogram spanned by \(PH_2\) and \(PH_1\), then (3.1) can be written as

\[
\int_{P \notin D} \frac{A}{s^2 \cdot s'^2}\,dx\,dy = 2\pi^2
\]

(3.2)

The Crofton formula (3.1) or (3.2) yields many consequences in integral geometry of plane curves. For example, it implied that the measure of the set of lines intersecting a simple plane curve is equal to the length of that curve.

It seems amazing that the Crofton integral (left side of (3.1) or (3.2)) is independent of the shape of \(C\). Assuming this for the moment, let us try to evaluate the Crofton integral when the boundary of \(D\) is the round circle \(\{(\cos 2\pi t, \sin 2\pi t) ; 0 \leq t \leq 1\}\) in the plane. It is done by a certain change of coordinates.

Let \(\phi : \mathbb{R}^2 \setminus D = \{(x, y) ; x^2 + y^2 > 1\} \rightarrow S^1 \times S^1\) be the map defined by sending each point \(P \notin D\) to the pair of angles \((\theta_1, \theta_2)\) with \(\theta_1 < \theta_2\) of the supporting points. It is easy to check:

- \(\phi\) is one-one;
- if \(P\) goes to infinity in the direction of angle \(\theta_0\), then \(\phi(P)\) goes to \((\theta_0 + \pi/2, \theta_0 + 3\pi/2)\);
- if \(P\) approaches the point \((\cos \theta_0, \sin \theta_0)\), then \(\phi(P)\) goes to \((\theta_0, \theta_0)\).

Therefore, the image of \(\phi\) covers exactly one half of the torus \(S^1 \times S^1\). By a direct computation, the Crofton integral is the same as the signed area covered by the image of \(\phi\). It follows that the Crofton integral is equal to \(1/2 \cdot (2\pi)^2 = 2\pi^2\).

In general, the same argument will go through since the angle of a supporting line of the convex set \(D\) is well defined once a center of \(D\) is chosen. See [S]

Let \(\gamma\) be a simple closed plane curve. The X-integral \(I_X\) of \(\gamma\) is 0 (see (2.3)). The fact that \(I_X - I_Y\) is a knot invariant implies that the Y-integral of \(\gamma\), \(I_Y(\gamma)\), is invariant under deformation of \(\gamma\) in the space of simple closed plane curves. The computation of the Y-integral for the round circle was first done by [GMM]. We
The setting of the generalized Crofton formula.

Figure 3. The setting of the generalized Crofton formula.

will provide a computation here which is similar to the proof of the Crofton formula described above.

Denote \( \mathbb{R}^3_+ = \{(z_1, z_2, z_3) \in \mathbb{R}^3 ; z_3 > 0\} \) and \( \mathbb{R}^2 = \{z_3 = 0\} \subset \mathbb{R}^3 \).

**Theorem 3.1. (Generalized Crofton formula)** Let \( \gamma = \gamma(t) : S^1 \rightarrow \mathbb{R}^2 \) be a simple closed plane curve in \( \mathbb{R}^2 \). Then

\[
\int_{\Delta_3} \int_{z \in \mathbb{R}^3_+} \frac{V}{\prod_{i=1}^3 |z - \gamma(t_i)|^3} d^3z dt_1 dt_2 dt_3 = \frac{\pi^3}{6}
\]

where \( V \) is the oriented volume of the parallelepiped spanned by \((z - \gamma(t_i)) \times \dot{\gamma}(t_i), i = 1, 2, 3\).

We will call the integral in (3.3) the **generalized Crofton integral**. See Figure 3.

**Proof:** Compare with (2.4), the integral in (3.3) is equal to a constant multiple of \( I_Y(\gamma) \). Since \( I_Y(\gamma) = -v_2(\gamma) \) in this case, it is invariant when \( \gamma \) is deformed by an isotopy of the plane. Therefore, it suffices to prove the theorem for the round circle \{\((\cos 2\pi t, \sin 2\pi t, 0) ; 0 \leq t \leq 1\} \in \mathbb{R}^2\).

Let \( \phi : \mathbb{R}^3_+ \times (S^1)^3 \rightarrow S^2_+ \times S^2_+ \times S^2_+ \) be the map defined by sending \((z, t_1, t_2, t_3)\) to

\[
\left( \frac{z - \gamma(t_1)}{|z - \gamma(t_1)|}, \frac{z - \gamma(t_2)}{|z - \gamma(t_2)|}, \frac{z - \gamma(t_3)}{|z - \gamma(t_3)|} \right).
\]

Then the generalized Crofton integral on the round circle is equal to the signed volume of the part of \((S^2_+)^3\) covered by the image of the map \( \phi \) multiplied by \((2\pi)^3/3!\).

**Claim:** There is a subset \( A \subset \text{Img}(\phi) \) of full measure in \((S^2_+)^3\) such that \( \phi|\phi^{-1}(A) \) is one-one.

It follows from the claim that the generalized Crofton integral on the round circle is \( \pm(2\pi)^3/3! \cdot (1/2)^3 = \pm \pi^3/6 \). By checking the orientations, we know the integral equals \( \pi^3/6 \).

**Proof of the claim:** Let \( v \) be a vector in the upper hemi-sphere \( S^2_+ \subset \mathbb{R}^3 \), and \( \phi_1 : \mathbb{R}^3_+ \times S^1 \rightarrow S^2_+ \) be the map defined by sending \((z, t)\) to

\[
\frac{z - \gamma(t)}{|z - \gamma(t)|}.
\]

Then \( \phi_1^{-1}(v) \) is the half infinite cylinder

\[
C_v = \{\gamma(t) + sv ; 0 \leq t \leq 1, s \geq 0\}.
\]

Let \((v_1, v_2, v_3)\) be a point in \((S^2_+)^3\). Then \( \phi^{-1}(v_1, v_2, v_3) \) is in one-one correspondence with the set of intersections of the three half infinite cylinders \( C_{v_1}, C_{v_2}, \) and \( C_{v_3} \). If \( v_1 \neq v_2 \), then \( C_{v_1}, C_{v_2} \) intersect in an arc lying on both \( C_{v_1} \) and \( C_{v_2} \) whose ends are a pair of antipode points of the round circle \( \gamma \). If \( v_1, v_2 \) and \( v_3 \) are pairwise distinct,
then $C_{v_1}$ and $C_{v_2}$ intersect in an arc $A_{12}$ on $C_{v_1}$, and similarly, $C_{v_1}$ and $C_{v_3}$ intersect in an arc $A_{13}$ on $C_{v_1}$. Let $p : \mathbb{R}^3 \to \mathbb{R}^2$ be the projection. If $p(v_2) - p(v_1)$ and $p(v_3) - p(v_1)$ are not colinear ($p(v_1)$, $p(v_2)$ and $p(v_3)$ are in general position), then the ends of $A_{12}$ and $A_{13}$ form two distinct pairs of antipode points on the round circle. This implies that $A_{12}$ and $A_{13}$ intersect at exactly one point in $\mathbb{R}^3_+$. So if $p(v_1)$, $p(v_2)$ and $p(v_3)$ are in general position, then $\phi^{-1}(v_1, v_2, v_3)$ is a single point in $\mathbb{R}^3_+ \times (S^1)^3$. The remnant in $(S^2)^3_+$ is of measure 0. This proves the claim and thus the theorem.

It seems very likely that the argument in the proof of the claim above also works if $\gamma$ is a convex curve in $\mathbb{R}^2$. If this is so, we will have a direct proof of the generalized Crofton formula for convex plane curves. Here is a very interesting intuitive interpretation of the claim in the proof of Theorem 3.1. Image that a fixed round circle in the plane starts to move in the plane with three non-colinear constant velocities respectively so that we will see three round circles of the same radius in a moment. Then there will be exactly one moment when these three circles have exactly one intersection. Intuitively, this should also be true if we start with a convex curve. We are not sure whether such a phenomenon has been discussed in the literatures.

**Corollary 3.2.** $v_2(\text{unknot}) = -1/24$.

**Proof:** It is easy to see that on a round circle, $I_Y$ is equal to the generalized Crofton integral on the round circle times $2 \cdot 1/(2\pi)^3 = 1/4\pi^3$. So $I_Y$ on a round circle is $1/24$. □

4. The combinatorics of the integral knot invariant $v_2$

As we mentioned before, the knot invariant $v_2$ can be identified, modulo the constant $v_2(\text{unknot}) = -1/24$, with the second coefficient of the Conway polynomial via a crossing change formula. From this identification, one can draw most of the conclusions about $v_2$ in this section. Therefore, the main interest of this section is probably to see how one can study $v_2$ by studying certain limiting behaviors of the integrals $I_X$ and $I_Y$. Such a consideration also leads to the discovery of the relationship between the knot invariant $v_2$ and invariants of plane curves discussed in the next section.

Let $\gamma : S^1 \to \mathbb{R}^3$ be an imbedding. Since it is very difficult to compute $v_2$ by evaluating both integrals $I_X(\gamma)$ and $I_Y(\gamma)$ directly, we study a limiting situation when the curve is pushed into a plane via a regular projection. It turns out that the limits of both integrals can be computed, and this gives us a new way of studying the knot invariant $v_2$. The same analysis applied to the Gauss linking formula give us the well-known combinatorial formula for the linking number.

When an imbedded curve $\gamma$ in $\mathbb{R}^3$ acquires a double point, the X-integral of $\gamma$ blows up. On the other hand, the Y-integral is still meaningful by Lemma 2.4.
Proposition 4.1. Let $\gamma : S^1 \rightarrow \mathbb{R}^3$ be an immersion with only transverse double points. Then

1. $I_Y(\gamma)$ does not depend on the parameterization or the orientation of $\gamma$.
2. Let $T : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ be an affine similarity. Then $I_Y(T \circ \gamma) = I_Y(\gamma)$.

The proof is immediate.

Let $K$ be the knot type of the imbedding $\gamma : S^1 \rightarrow \mathbb{R}^3$. Thought of as a knot diagram, $K$ can be drawn inside the plane except around each crossing. Assume that when we make an over-pass at a crossing of $K$, we go along a semi-circle of radius $\epsilon$ perpendicular to the plane. The other parts of $K$ lie completely in the plane. Denote such a diagram of $K$ by $K_\epsilon$. Also, we associate to each crossing a sign $\pm 1$ by the usual right-hand-rule.

When $\epsilon$ approaches 0, $K_\epsilon$ limits to a closed plane curve with only transverse double points. Denote this limiting plane curve by $K_0$. We associate to each double point the sign of the corresponding crossing, and we call $K_0$ with these signs the signed limiting plane curve. In general, let $C$ be an immersed circle in the plane with only transverse double points. If each double point is associated with a sign, then $C$ is called a signed immersed circle. For each closed plane curve with only transverse double points, we have a chord diagram defined as follows.

Definition 4.2. Let $C$ be an immersed circle with only transverse double points in the plane. The chord diagram (or Gauss diagram) of $C$ is the combinatorial pattern of a finite collection of chords with both ends sticking to the circle, connecting the points sent by the immersion to the same double point of the immersed circle. If each chord is associated with a sign, the chord diagram is called a signed chord diagram.

If $K_0$ is the limiting plane curve of a knot diagram $K_\epsilon$, then we associate to each chord of the chord diagram of $K_0$ the sign of the corresponding double point. For each pair of chords across to each other in the signed chord diagram of $K_0$, we assign to this pair of chords a sign equal to the product of the signs of the two chords.

Proposition 4.3. Let $K$ be a knot diagram with $n$ crossings. Then the limit

$$\lim_{\epsilon \to 0} I_X(K_\epsilon)$$

exists. Let $I_X(K_0)$ denote this limit. Then

$$I_X(K_0) = \frac{n}{16} + \frac{(c_+ - c_-)}{4}$$

where $c_+$ ($c_-$, respectively) is the number of pairs of chords in the signed chord diagram of $K_0$ with a positive (negative, respectively) sign.
Figure 4. The local picture of a knot projection at a crossing.

Proof: Our first observation is that when \( \dot{\gamma}(t) \) and \( \dot{\gamma}(t') \) are co-planar, then the Gaussian form \( \omega(\gamma(t) - \gamma(t')) \) is 0.

We may assume that the \( i \)-th crossing of \( K_\epsilon \) looks like the crossing depicted in Figure 4. Let \( C^i_\epsilon \) be the semi-circle of radius \( \epsilon \) at that crossing and \( A^i_a \) be a line segment \([-a, a]\) with a fixed small \( a > 0 \) on \( K_\epsilon \) running under \( C^i_\epsilon \). Furthermore, denote by \( C^i_\epsilon, a \) the union of \( C^i_\epsilon \) and line segments \([-a, -a + \epsilon]\) and \([a - \epsilon, a]\).

By the observation above, if both \( \gamma(t), \gamma(t') \) lie outside \( \bigcup_{i=1}^n C^i_\epsilon \), or they both lie inside some \( C^i_\epsilon, a \), then \( \omega(\gamma(t) - \gamma(t')) = 0 \). If \( \gamma(t) \in C^i_\epsilon \), and \( \gamma(t') \) lies outside \( C^i_\epsilon, a \cup A^i_a \), then \( |\gamma(t) - \gamma(t')| \) is bounded from below by a constant. It follows that when \( \epsilon \) approaches 0, the integral \( I_X \) over all those pairs goes to 0. Note that we need to fix a base point in order to evaluate the \( X \)-integral \( I_X \) for a curve. So we choose a base point on \( K_\epsilon \) which is not in any \( C^i_\epsilon, a \) or \( A^i_a \). Then all nonzero limits come from the following two cases:

(1) \( \gamma(t_1), \gamma(t_2) \in C^i_\epsilon, \) and \( \gamma(t_3), \gamma(t_4) \in A^i_a \);
(2) \( \gamma(t_1) \in C^i_\epsilon, \gamma(t_3) \in A^i_a \) or vice versa, and \( \gamma(t_2) \in C^j_\epsilon, \gamma(t_4) \in A^j_a \) or vice versa, with \( i \neq j \).

By a direct computation, the limit for case (1) is always 1/16. It is independent of the sign of the crossing.

Since \( t_1 < t_2 < t_3 < t_4 \), case (2) is possible if and only if the chords corresponding to the \( i \) and \( j \) cross each other. In this case, the limit is \( \epsilon_i \epsilon_j / 4 \), where \( \epsilon_i, \epsilon_j \) are the signs of the \( i \)-th and \( j \)-th crossings. This completes the proof. □

Let \( C \) be an immersed circle in the plane with only transverse double points. Then we can resolve \( C \) to knots by changing each double point to a crossing. There are \( 2^n \) resolutions if \( C \) has \( n \) double points.

Corollary 4.4. Let \( C \) be an immersed circle in the plane with only transverse double points.

(1) If \( C \) is resolved to two knots \( K^1 \) and \( K^2 \) of the same knot type, then \( I_X(K^1_\epsilon) \) and \( I_X(K^2_\epsilon) \) have the same limit.

(2) \( I_Y(C) \) is invariant when \( C \) is deformed in the plane without changing its chord diagram.

(3) If \( C \) is resolved to an unknot \( K^u \), then

\[
I_Y(C) = \frac{1}{24} + \frac{n}{16} + \frac{(c_+ - c_-)}{4}
\]

where \( c_\pm \) are computed using the signed chord diagram of \( K^u \).

Corollary 4.4 (2) implies thousands of integral formulae like the generalized Crofton formula.
Figure 5. An oriented surgery.

Proof: (1) This is because the limit of $I_Y(K^i_\epsilon)$ when $\epsilon$ approaches to 0 is $I_Y(C)$, for $i = 1, 2$, and $v_2(K^1) = v_2(K^2)$.

(2) Resolve $C$ to a knot $K$, since both $I_X(K_0)$ and $v_2(K)$ are invariant when $C$ is deformed in the plane without changing its chord diagram, so does $I_Y(C) = I_X(K_0) - v_2(K)$.

(3) This is a direct consequence of Corollary 3.2 and Lemma 4.3. □

Let $K$ be a knot diagram. At each crossing of $K$, the modification of the knot diagram depicted in Figure 5 changes $K$ into a link of two oriented components. Such a modification is called an oriented surgery at a crossing.

Corollary 4.5. Let $K$ be a knot diagram with $n$ crossings. And $K^u$ be an unknot obtained from $K$ by changing some crossings.

(1) Let $l_i$ be the linking number of the two component link obtained from the oriented surgery at the $i$-th crossing of $K$, and $l^u_i$ be the corresponding linking number from $K^u$. Then

$$v_2(K) = \frac{1}{2} \sum_{i=1}^{n} (l_i - l^u_i) - \frac{1}{24}; (4.2)$$

(2) $|v_2(K)| \leq n(n - 1)/4 + 1/24$;

(3) $v_2(K) + 1/24$ is an integer;

(4) $v_2(K)$ is independent of the orientation of $K$.

Proof: Let $K_0$ be the limiting plane curve of $K$. And $K^u_0$ be the limit plane curve of $K^u$. By Lemma 4.3 and Corollary 4.4 (3),

$$v_2(K) = \frac{c_+(K_0) - c_-(K_0)}{4} - \frac{c_+(K^u_0) - c_-(K^u_0)}{4} - \frac{1}{24}. (4.3)$$

The difference between the signed chord diagrams of $K_0$ and $K^u_0$ is that some signs of chords are changed.

(2) As the chord diagrams of $K_0$ and $K^u_0$ both have $n$-chords, there are at most $n(n - 1)/2$ intersections among chords. Thus, (4.3) implies $|v_2(K)| \leq n(n - 1)/4 + 1/24$.

(3) Let $d_i$ be the number of intersections of all chords with the $i$-th chord counted with signs. Then $d_i = 2l_i$, so $d_i$ is always an even integer.

Claim: Let $K^1_0$ be a signed plane curve obtained from $K_0$ by changing one sign of the double points. Then $I_X(K_0) - I_X(K^1_0)$ is an integer.

Proof of the claim: By (4.1), we need only count the changes of the intersections between the chord corresponding to the double point where the sign is changed and other chords, divided by 4. When the sign of the $i$-th chord is changed, $d_i$ is changed to $-d_i$. As $d_i$ is even, the total change $2d_i$ is divisible by 4.
Now we finish the proof of (3) as follows: choose a sequence of double points so that when we change the their signs one after another to get a sequence of signed plane curves \(K^1_0, \ldots, K^s_0\) with \(K^1_0 = K^1_0\) and \(K^s_0\) is the limit of an unknot. Then

\[
v_2(K) + \frac{1}{24} = I_X(K_0) - I_X(K^s_0) = \sum_{i=1}^{s-1} (I_X(K^i_0) - I_X(K^{i+1}_0)).
\]

Thus the proof of (3) is completed.

Finally, it is clear that (1) follows from the proof of (3). It is also clear that (4) can be derived in many ways and one way is via the formula (4.2) since the linking number will not change if one changes the orientations of both components of a link. \(\square\)

Corollary 4.5 (2) is of particular interest to us. Recall that a knot invariant is of finite type or a Vassiliev invariant if it vanishes on “higher order differences of knots”. One may think of such a knot invariant as a “polynomial function” on the set of knot types. See, for example, [B]. Since \(v_2\) is known to be a Vassiliev invariant of order 2, the bound for its values on knots with \(n\) crossings agrees with such a point of view. Notice that using the combinatorial formula for a Vassiliev invariant of order 3 given by Lannes [L], we can get a similar bound for values of Vassiliev invariants of order 3 on knots with \(n\) crossings. This leads us to the following conjecture.

**Conjecture 4.6.** For every Vassiliev invariant of order \(k\), say \(v_k\), there is a constant \(C\) such that if \(K\) is a knot with \(n\) crossings, then

\[
|v_k(K)| < Cn^k.
\]

Combinatorial formulae for Vassiliev invariants of lower orders were also discussed by Polyak and Viro.

### 5. Invariants of unicursal curves

Considering the space \(\mathcal{M}\) of all immersions of \(S^1\) into the plane. By a classical theorem of Whitney, components of this space can be indexed by \(\mathbb{Z}\) using the winding number. We will denote by \(\mathcal{M}_w\) the component of \(\mathcal{M}\) whose members all have winding number or index \(w\). If we want to look at \(\mathcal{M}\) more carefully, we will see generic immersions and non-generic ones. Generic immersions are those with only transverse double points. The set \(\Sigma\) of non-generic immersions or the discriminant of \(\mathcal{M}\) can be thought of as a stratified space whose top stratum have three components of particular interest to us. One component consists of immersions with exactly one transverse triple point, and the other two consist of, respectively, immersions with exactly one direct or inverse self-tangency point where two tangent branches of the curve concave in opposite direction. A self-tangency point is called **direct**, if two tangent vectors of the curve at the tangent point are in the same direction, or it is called **inverse** otherwise. It turns out that these three components of the top
stratum of $\Sigma$ corresponding respectively to immersions with exactly one transverse triple point, one direct self-tangency point or one inverse self-tangency point are all well "co-oriented". This means that we can talk about the positive or negative side of these three component of the top stratum of $\Sigma$ in $\mathcal{M}$. It is quite easy to see that a path in $\mathcal{M}$ can be perturbed so that it only crosses $\Sigma$ transversally through these three components at finitely many places. For a detailed study of the topology of $\Sigma \subset \mathcal{M}$, see [A1].

We will call a generic immersion of $S^1$ into the plane a unicusural curve. Two unicusural curves are equivalent if they belong to the same component of $\mathcal{M} \setminus \Sigma$. It is not hard to see that a path in $\mathcal{M} \setminus \Sigma$ is the same as a deformation of a unicusural curve without changing its chord diagram. An invariant of unicusural curves assigns values to every unicusural curve and equivalent unicusural curves should be assigned with the same value.

**Lemma 5.1.** $C \rightarrow I_y(C)$ is an invariant of unicusural curves.

This is simply a restatement of Corollary 4.4 (2).

In [A1], Arnold constructed three basic invariants of unicusural curve, $St$ and $J^{\pm}$. Up to an additive constant, they are completely determined by the way they jump when a deformation of unicusural curves crosses through a triple point, or a direct or inverse self-tangency point. To describe these invariants, we need to make some definition first.

**Definition 5.2.** (1) A transversal crossing of a self-tangency point is positive if the number of double points grows (by 2).

(2) A transversal crossing of a triple point is positive if the new-born vanishing triangle is positive.

Here for a given a unicusural curve $C$, a vanishing triangle of $C$ is a triangle formed by three branches of $C$ and no other branches of $C$ are allowed to run into such a triangle. At a transversal crossing of a triple point, one sees the death of one vanishing triangle and the birth of another one. The sign of a vanishing triangle is defined as follows. The orientation of the immersed circle defines a cyclic ordering of the sides of the vanishing triangle. Hence the sides of the triangle acquire orientations induced by the ordering. But each side has also its own direction which might coincide, or not, with the orientation defined by the ordering. For each vanishing triangle, let $q$ be the number of sides equally oriented by the ordering and their directions. Then the sign of a vanishing triangle is $(-1)^q$. It is easy to check that at a transversal crossing of a triple point, the dying vanishing triangle and the new-born vanishing triangle always have opposite signs.

**Theorem 5.3.** (Arnold) (1) There exists a unique (up to an additive constant) invariant of unicusural curves of fixed index whose value remains unchanged at a
transversal crossing of a self-tangency point, but increases by 1 at a positive transversal crossing of a triple point. This invariant is denoted by \( St \) with an appropriate normalization.

(2) There exists a unique (up to an additive constant) invariant of unicursal curves of fixed index whose value remains unchanged at a transversal crossing of a triple point or an inverse (respectively, direct) self-tangency point, but increases by 2 (respectively, \(-2\)) at a positive transversal crossing of a direct (respectively, inverse) self-tangency point. This invariant is denoted by \( J^+ \) (respectively, \( J^- \)) with an appropriate normalization. The invariants \( J^+ \) and \( J^- \) are related by \( J^+ - J^- = n \) on unicursal curves with \( n \) double points.

(3) These invariants are independent of the orientation of unicursal curves.

Here a normalization means to choose a unicursal curve \( C_w \) for each index \( w \in \mathbb{Z} \) and the value of the invariant in question on \( C_w \). One may think of these three invariants \( St \) and \( J^\pm \) of unicursal curves as dual to those three components of the top stratum of \( \Sigma \) corresponding to one triple point, one direct self-tangency point and one inverse self-tangency point respectively. Any invariant of unicursal curves which jumps by a constant at a transversal crossing of a triple point and a self-tangency point can be expressed uniquely as a linear combination of \( St \) and \( J^\pm \), modulo a constant depending on the index.

To simplify the terminology, we define several operations on unicursal curves analogous to the Reidermeister moves in knot theory. See Figure 6. A type I move on a unicursal curve kills one small kink on it. A type II\(^+\) (II\(^-\), respectively) move is a positive transversal crossing of a direct (inverse, respectively) self-tangency point. Finally, A type III move is a positive transversal crossing of a triple point.

**Definition 5.4.** Let \( C \) be a unicursal curve. Then we define

\[
\alpha(C) = I_Y(C) + \frac{n}{16} - \frac{1}{24}
\]

where \( n \) is the number of double points of \( C \).

As both the \( Y \)-integral and the number of double points are invariants of unicursal curves, so is \( \alpha \). If \( C \) is resolved to an unknot \( K^u \), then

\[
\alpha(C) = \frac{n}{8} + \frac{c_+ - c_-}{4}
\]

where \( c_\pm \) are computed using the signed chord diagram of \( K^u_0 \). From this formula, we see that each double point of \( C \) contributes \( 1/8 \) to \( \alpha \), and each pair of intersecting chords contributes \( \pm 1/4 \) to \( \alpha \).
Theorem 5.5. The invariant $\alpha$ of unicursal curves has the following properties:

1. $\alpha$ equals to 0 for every simple closed plane curve;
2. $\alpha$ is decreased by $1/8$ if a type $\text{I}^+$ move is performed;
3. $\alpha$ is unchanged if a type $\text{II}^+$ move is performed;
4. $\alpha$ is decreased by $1/4$ if a type $\text{III}$ move is performed.

As a consequence of (2), (3), and (4), $\alpha$ is increased by $1/4$ if a type $\text{II}^-$ move is performed. Furthermore, we have

5. $|\alpha(C)| \leq n^2/8$ where $n$ is the number of double points on $C$;
6. $\alpha(C)$ is independent of the orientation of $C$.

Proof: They are all consequences of (5.1).

1. This is obvious.
2. Note that the chord corresponding to the double point in a type I move is an isolated chord (it does not intersect any other chord). So this double point contributes only $1/8$ to $\alpha$.
3. We may assume that after a type $\text{II}^+$ move, the two new double points are resolved with opposite signs. They contribute $1/4$ to $\alpha$. But the two chords of the new double points intersect and every other chord either intersects them both or misses them both. So their contribution to $\alpha$ is $-1/4$. This implies (3)
4. Assume that a type $\text{III}$ move changes $C$ to $C'$ and the vanishing triangles on $C$ and $C'$ are resolved as depicted in Figure 7. This is done by choosing a base point and resolving double points according to the rule that the branch one walks through first is always above the branch one walks through second. The resulting knot is an unknot.

On the level of chord diagrams, there are two cases to study. See again Figure 7. In both cases, the chord diagrams for $C$ and $C'$ have the same number of chords. Let \{a, b, c\} and \{a', b', c'\} be the signed chords at the vertices of the vanishing triangles of $C$ and $C'$ respectively. The edges of the vanishing triangle of $C$ (respectively, $C'$) correspond to three disjoint arcs on $S^1$ and each chord in \{a, b, c\} (respectively, \{a', b', c'\}) connects end points of two distinct arcs. No other chords of $C$ (respectively, $C'$) with touch these arcs. Furthermore, when a type $\text{III}$ move changes $C$ to $C'$, the chord diagram of $C$ is changed to the chord diagram of $C'$ by switching every two end points of \{a, b, c\} paired as the end points of those arcs on $S^1$. The signs of chords will not be changed. So the contribution of \{a, b, c\} to $\alpha(C)$ will be $1/4$ more than the contribution of \{a', b', c'\} to $\alpha(C')$. Furthermore, if another chord intersects one chord in \{a, b, c\}, it will also intersects one in \{a', b', c'\} with the same sign. Therefore, $\alpha(C') = \alpha(C) - 1/4$.

5. This is a consequence of (5.1) and the bound $|c_+ - c_-| \leq n(n - 1)/2$.
6. This also follows easily from (5.1). □
Now by Theorem 5.4, it is very easy to compute \( \alpha \). Consequently, it is very easy to compute \( I_Y(C) \) for any unicursal curve \( C \). For example, \( \alpha(\infty) = 1/8 \). It follows that \( I_Y(\infty) = 5/48 \). Here the symbol \( \infty \) is used to denote a unicursal curve of the same shape.

**Corollary 5.6.** We have

\[
(5.2) \quad \alpha = -\frac{2St + J^-}{8}.
\]

**Proof:** The invariant in the right side of (5.2) changes in the same way as \( \alpha \) does under type II\( ^\pm \) and III moves. Checking the initial values of \( St \) and \( J^\pm \) given in [A1] verifies (5.2).

A result of F. Aicardi (see [A1]) says that \( 2St + J^+ = 0 \) holds if the chord diagram of \( C \) have no intersecting chords. Here is a generalization of this result.

**Corollary 5.7.** The identity \( 2St + J^+ = 0 \) holds for a unicursal curve \( C \) with \( n \) double points if and only if \( \alpha(C) = n/8 \), and if and only if a certain signed chord diagram coming from an unknot resolution of \( C \) have \( c_+ = c_- \).

**Proof:** For unicursal curves with \( n \) double points, we have \( J^+ - J^- = n \). So (5.2) gives us

\[
(5.3) \quad \alpha = -\frac{2St + J^+}{8} + \frac{n}{8}.
\]

on unicursal curves with \( n \) double points. This together with (5.1) proves the corollary.

Arnold’s triple \( St, J^+, J^- \) are related via some other index-type invariants. They are defined as follows.

Let \( C \) be a unicursal curve such that at each transverse double point, the two tangent vectors of \( C \) are orthogonal. At every double point, we may divide \( C \) into two branches \( C_1 \) and \( C_2 \). In fact, the preimage of that double point on \( S^1 \) cuts \( S^1 \) into two arcs and the images of these two arcs under the immersion are \( C_1 \) and \( C_2 \) respectively. These two branches \( C_1 \) and \( C_2 \) are ordered such that if the outgoing tangent vectors of \( C_1 \) and \( C_2 \) at the corresponding double point are \( v_1 \) and \( v_2 \), respectively, then the frame \( \{v_1, v_2\} \) has the same orientation as that of the plane.

**Definition 5.8.** The half-index \( i_1 \) (respectively, \( i_2 \)) of a double point is the angle of the rotation of the radius-vector connecting the double point to a point moving along \( C_1 \) (respectively, \( C_2 \)) from the double point to itself divided by \( \pi/2 \). The index of a double point is the difference \( i = i_1 - i_2 \).

The invariants \( I^\pm \) are defined to be

\[
I^\pm = \frac{\sum i \pm 2n}{4}
\]

(5.4)
where \( n \) is the number of double points, and the sum is over all double points.

Note that \( I^+ - I^- = n \). And we have

\[
(5.5) \quad J^\pm = I^\pm - 3St
\]

as shown in \([A1]\). So, among these three invariant \( St, J^+ \) and \( J^- \), there is only one which is essentially not of index-type. Corollary 5.10 below shows that \( \alpha \) is such an invariant.

The following theorem is essentially from \([A1]\).

**Theorem 5.9.** \( I^- \) is determined by the following properties:

1. \( I^- \) of a simple closed plane curve is 0;
2. \( I^- \) is decreased by \((i - 2)/4\) if a type I move is performed, where \( i \) is the index of this double point;
3. \( I^- \) is unchanged if a type \( \text{II}^+ \) move is performed, and \( I^- \) is decreased by 2 if a type \( \text{II}^- \) move is performed;
4. \( I^- \) is increased by 3 if a type \( \text{III} \) move is performed.

**Corollary 5.10.** We have the following identities:

1. \( St = I^- + 8\alpha \);
2. \( J^- = -2I^- - 24\alpha \);
3. \( J^+ = n - 2I^- - 24\alpha \).

**Proof:** The proof follows from (5.2), (5.5) and \( J^+ - J^- = n \). □

Let \( C \) be a unicursal curve, and \( x \) be a point which is not on \( C \). The winding number of \( C \) relative to \( x \) is the degree of the position map

\[
S^1 \to S^1 : t \mapsto \frac{C(t) - x}{|C(t) - x|}.
\]

The relative winding number remains unchanged if \( x \) moves in a connected component of \( \mathbb{R}^2 \setminus C \). We will use the relative winding number to estimate the index of double points.

Let us notice that the oriented surgery at a crossing on a knot diagram can be generalized to the oriented surgery at a double point on a unicursal curve. The oriented surgery at a double point \( x \) on a unicursal curve \( C \) will result two new unicursal curves \( C_1 \) and \( C_2 \) intersecting each other transversally. The component of \( \mathbb{R}^2 \setminus C_1 \cup C_2 \) where \( x \) lies is well defined.

**Lemma 5.11.** (1) Let \( w_1 \) and \( w_2 \) be the winding numbers of \( C_1 \) and \( C_2 \) relative to \( x \). Then the index of this double point \( x \) on \( C \) is \( 4w_1 - 4w_2 - 2 \).

(2) The index \( i \) of any double point on a unicursal curve with \( n \) double points satisfies the inequality \(|i| \leq 4n + 6 \).
Figure 8. Examples of unicursal curves with $\alpha \leq 0$.

Proof: (1) If
\[ i_1 = \frac{\theta_1}{\pi/2} \quad \text{and} \quad i_2 = \frac{\theta_2}{\pi/2}, \]
then
\[ w_1 = \frac{\theta_1 - \pi/2}{2\pi} \quad \text{and} \quad w_2 = \frac{\theta_2 + \pi/2}{2\pi}. \]
So we have $i = i_1 - i_2 = 4w_1 - 4w_2 - 2$.

(2) Let $C$ be a unicursal curve. Then $C$ divides the plane into many regions. Inside each region, pick a point. Then we assign to each region the winding number of $C$ relative to this point.

Claim: If $C$ has $n$ double points, then the maximum of the absolute value of the winding numbers for all regions is no greater than $n + 1$.

Proof of the claim: It is easy to check that if two regions are adjacent with a common edge, then their winding numbers differ by 1. If $C$ has $n$ double points, then $C$ has $2n$ edges, and the plane is divided into $n + 2$ regions (including the unbounded one). Note that the winding number for the unbounded region is always 0, and the claim follows.

Now the inequality in (2) follows from (1) and the claim. \( \square \)

Corollary 5.12. We have the following bounds on unicursal curves with $n$ double points:

1. $|I^-| \leq n^2 + 2n$;
2. $|St| \leq 2n^2 + 2n$;
3. $|J^-| \leq 5n^2 + 4n$.

These follow easily from Theorem 5.5 (5), Corollary 5.10 and Lemma 5.11.

It is an interesting question to study the extremal curves for each invariant. We can fix either the number of double points, or fix the index of the curve. For some conjectures about upper bounds of $St, J^\pm$, see [A1]. The extremal values of $\alpha$ are still unknown. The examples in Figure 8 show that there are curves with $\alpha \leq 0$. Actually, there is no bound from below for $\alpha$ if we do not fix the number of double points. Notice that both curves have only positive triangles. There are curves with only negative triangles, too.

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