GLOBAL STABILITY AND NON–VANISHING VACUUM STATES OF 3D
COMPRESSIBLE NAVIER–STOKES EQUATIONS

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Abstract. We investigate global stability and non–vanishing vacuum states of large solutions to the compressible Navier–Stokes equations on the torus $\mathbb{T}^3$, and the main purpose of this work is three-fold: First, under the assumption that the density $\rho(x, t)$ verifies $\sup_{t \geq 0} \|\rho(t)\|_{L^\infty} \leq M$, it is shown that the solutions converge to equilibrium state exponentially in $L^2$–norm. In contrast to previous related works where the density has uniform positive lower and upper bounds, this gives the first stability result for large strong solutions of the 3D compressible Navier–Stokes equations in the presence of vacuum. Second, by employing some new thoughts, we also show that the density converges to its equilibrium state exponentially in $L^\infty$–norm if additionally the initial density $\rho_0(x)$ satisfies $\inf_{x \in \mathbb{T}^3} \rho_0(x) \geq c_0 > 0$. Finally, we prove that the vacuum state will persist for any time provided that the initial density contains vacuum, which is different from the previous work of [H. L. Li et al., Commun. Math. Phys., 281 (2008), 401–444], where the authors showed that any vacuum state must vanish within finite time for the free boundary problem of the 1D compressible Navier–Stokes equations with density–dependent viscosity $\mu(\rho) = \rho^\alpha$ with $\alpha > 1/2$. This phenomenon implies the different behaviors for Navier–Stokes equations with different types of viscous effects, namely, degenerate or not.

1. Introduction

In this paper, we are concerned with the global stability and non–vanishing of vacuum states of large solutions to the compressible Navier–Stokes equations on the torus $\mathbb{T}^3$:

$$\begin{cases}
\rho_t + \text{div}(\rho u) = 0, \\
(\rho u)_t + \text{div}(\rho u \otimes u) + \nabla P(\rho) = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u.
\end{cases}$$

(1.1)

Here $\rho = \rho(x, t)$ and $u = (u^1(x, t), u^2(x, t), u^3(x, t))^T$ stand for the density and the velocity respectively, at position $x \in \mathbb{T}^3$ and time $t \geq 0$. The pressure $P(\rho) = \rho^\gamma$, where $\gamma > 1$ is the specific heat ratio. The constants $\mu$ and $\lambda$ are the shear viscosity and the bulk viscosity of the fluid satisfying the physical hypothesis:

$$\mu > 0 \quad \text{and} \quad 2\mu + 3\lambda \geq 0.$$ 

Finally, the system (1.1) is supplemented with the following initial condition:

$$(\rho, \rho u)|_{t=0} = (\rho_0, \vec{u}_0)(x), \quad x \in \mathbb{T}^3.$$ 

(1.2)

Without loss of generality, the mean value of total initial mass over $\mathbb{T}^3$ is taken to be one throughout this paper, i.e.,

$$\frac{1}{|\mathbb{T}^3|} \int_{\mathbb{T}^3} \rho_0(x) dx = 1.$$ 

(1.3)
1.1. History of the problem and main motivation. To put our results into context, let us highlight some progress on the topics of global well-posedness and stability for the multidimensional compressible Navier–Stokes equations. The global well-posedness of classical solutions in the whole space $\mathbb{R}^3$ was firstly established by Matsumura and Nishida [27] provided that the initial data are close to a non-vacuum equilibrium in $H^3$. With the help of the effective viscous flux, Hoff [15] [16] proved the global existence of weak solutions with discontinuous initial data, i.e., the initial density is close to a positive constant in $L^2$ and $L^\infty$, and the initial velocity is small in $L^2$ and bounded in $L^{2N}$, $N = 2, 3$ is the dimension of space. Under the framework of Besov space, Dančin [5] investigated existence and uniqueness of the global strong solutions under the hypothesis that the initial value are close to a non–vacuum equilibrium state, see also [2-3]. When the initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, Huang et al. [18] proved global existence and uniqueness of classical solutions with smooth initial data that are of small energy in whole space $\mathbb{R}^3$, see also [22]. For the existence of solutions with arbitrary initial data, the major breakthrough is due to Lions [23], where he used the renormalization skills introduced by DiPerna and Lions [8] to establish global weak solutions if $\gamma > 3N/(N + 2)$. Later, Feireisl et al. [11] improved Lions’s result to the case $\gamma > \frac{N}{2}$. When the initial data are assumed to have some spherically symmetric or axisymmetric properties, Jiang and Zhang [19] [20] proved the existence of global weak solutions for any $\gamma > 1$. Plotnikov and Weigant [31] obtained the global existence of weak solutions to the isothermal compressible Navier–Stokes equations in dimension two under some additional assumptions. Desjardins [7] studied the regularity of weak solutions for small time under periodic boundary conditions, and particularly showed that weak solutions in $\mathbb{T}^2$ turn out to be smooth as long as the density remains bounded in $L^\infty(\mathbb{T}^2)$. Due to the possible concentration of finite kinetic energy in very small domains, whether those results in [11] [23] still hold true for the case $\gamma \in [1, \frac{N}{2}]$ remains an outstanding open problem. Recently, Hu [17] considered the Hausdorff dimension of concentration for the compressible Navier–Stokes equations. If $\gamma \in [1, \frac{N}{2}]$, he proved that except for a space–time set with a Hausdorff dimension of less than or equal to $\Gamma(n)+1$ with

$$\Gamma(n) = \max \left\{ \gamma(n), n - \frac{n\gamma}{\gamma(n)+1} \right\} \quad \text{and} \quad \gamma(n) = \frac{n(n-1) - n\gamma}{n-\gamma},$$

no concentration phenomenon occurs.

In addition to the global–in–time existence, large time behavior of solutions is also an important topic in the mathematical theory of the physical world. Under the smallness assumption on the initial perturbation, the readers can refer to [6] [9] [13] [25] [26] [28] [32] and references therein for large time behavior of global smooth solutions to the compressible Navier–Stokes system. Recently, He et al. [14] investigated the global stability of large strong solutions to the 3D Cauchy problem. Under the hypothesis that the density $\rho(x, t)$ verifies $\inf_{x \in \mathbb{R}^3} \rho_0(x) \geq c_0 > 0$ and $\sup_{t \geq 0} \|\rho(t)\|_{C^\alpha} \leq M$ with arbitrarily small $\alpha$, they established a new approach for the convergence of the solutions to its associated equilibrium states with an explicit decay rate which is the same as that of the heat equation for the case $\mu > \frac{1}{2} \lambda$. The assumption $\sup_{t \geq 0} \|\rho(t)\|_{C^\alpha} \leq M$ played an essential role to derive the uniform positive lower bound of the density $\rho$ (See the proof of Proposition 2.3 in [14] for details). As concerned with long time behavior of large weak solutions, Feireisl and Petzeltová [12] first showed that any weak solution converges to a fixed stationary state as time goes to infinity via the weak convergence method. Under the assumptions that the density is essentially bounded and has uniform in time positive lower bound, Padula [29] proved that weak solutions decay exponentially to the equilibrium state in $L^2$-norm. With the help of the operator $B$ introduced by Bogovskii [1], Fang et al. [10] removed the restriction on the uniform positive lower bound of the density. Recently, Peng–Shi–Wu [30] improved those results of [10] [29] to the case that they didn’t need both upper and lower time–independent bounds of
density. Recently, Zhang et al. [35] showed that global regular solutions of the full compressible Navier–Stokes equations on the torus $\mathbb{T}^3$ converge to equilibrium with exponential rate provided that both the density $\rho$ and temperature $\theta$ possess uniform in time lower and upper bounds.

Motivated by [14] and [21], the main purpose of this paper is to investigate global stability and non–vanishing vacuum states of large strong solutions to the compressible Navier–Stokes equations on the torus $\mathbb{T}^3$. More precisely, we are concerned with the following three problems:

(i) Notice that the density has uniform positive lower and upper bounds in [14, 35]. Therefore, an important and interesting problem is: What about the stability of large strong solutions for the 3D compressible Navier–Stokes equations in presence of vacuum?

(ii) As mentioned before, assumptions that initial density $\rho_0(x)$ has uniform positive lower bound and $\sup_{t \geq 0} \|\rho(t)\|_{C^0} \leq M$ with arbitrarily small $\alpha$ in [14] played an essential role to derive the uniform positive lower bound of the density $\rho$. Therefore, the natural and interesting problem is: Can we show that the solutions converge to equilibrium state exponentially in $t$?

(iii) Li–Li–Xin [21] showed that any vacuum state will not exist within finite time for the free boundary problem of the 1D compressible Navier–Stokes equations with density–dependent viscosity $\mu(\rho) = \rho^\alpha$ with $\alpha > 1/2$. However, whether this result holds true for multidimensional case still remains an outstanding open problem. Therefore, a natural and important problem is: Provided that the initial density contains vacuum, whether the vacuum state persists or not for the 3D compressible Navier–Stokes equations?

The main purpose of this article is to give a clear answer to the above three problems.

1.2. Main results. Throughout this paper, we assume that the initial data satisfy

$$0 \leq \rho_0 \in W^{1, q}(\mathbb{T}^3), \quad \text{for some } q \in (3, 6) \text{ and } \mathbf{u}_0 \in H^2(\mathbb{T}^3).$$  \hspace{1cm} (1.4)

We use $C$ to denote a generic constant independent of time which may vary in different places. If $X$ is a Banach space, we will abbreviate the vector–valued space $X^3$ by $X$ for convenience. As in [11, 15, 16, 23], the effective viscous flux $F$ and vorticity $\mathbf{w}$ are defined by

$$F \overset{\text{def}}{=} (2\mu + \lambda) \text{div}\mathbf{u} - (P(\rho) - 1) \quad \text{and} \quad \mathbf{w} \overset{\text{def}}{=} \mathcal{P}\mathbf{u}, \hspace{1cm} (1.5)$$

where $\mathcal{P} = I + \nabla(-\Delta)^{-1}\text{div}$ denotes the projection on the space of divergence–free vector fields.

Now, we are ready to state our results. To begin with, we introduce the definition of strong solutions to the problem (1.1)–(1.2).

**Definition 1.1** (Strong solutions). For $T > 0$, a pair of function $(\rho, \mathbf{u})$ is said to be a strong solution of the problem (1.1)–(1.2) on $\mathbb{T}^3 \times [0, T]$, if for some $q \in (3, 6]$

\begin{align*}
0 &\leq \rho \in C([0, T]; W^{1, q}(\mathbb{T}^3)), \quad \rho_t \in L^q(\mathbb{T}^3), \\
\mathbf{u} &\in C([0, T]; L^2(\mathbb{T}^3) \cap L^2(0, T; W^{2, q}(\mathbb{T}^3))), \quad \mathbf{u}_t \in L^2(0, T; H^1(\mathbb{T}^3)), \quad \sqrt{\rho}\mathbf{u} \in L^\infty(0, T; L^2(\mathbb{T}^3)),
\end{align*}

and $(\rho, \mathbf{u})$ satisfies (1.1) a.e. on $\mathbb{T}^3 \times [0, T]$.

If the initial density contains vacuum, we have the following result on stability of a strong solution in $L^2$–norm to the problem (1.1)–(1.2).

**Theorem 1.1.** Assume that the initial data $(\rho_0, \mathbf{u}_0)$ satisfy (1.3)–(1.4), and $K := \|\rho_0 - 1\|_{L^2(\mathbb{T}^3)} + \|\sqrt{\rho_0}(\mathbf{u}_0 - \mathbf{m}_0)\|_{L^2(\mathbb{T}^3)} + \|\nabla \mathbf{u}_0\|_{L^2(\mathbb{T}^3)} < +\infty$ with $\mathbf{m}_0 = \int_{\mathbb{T}^3} \rho_0 \mathbf{u}_0(x) \, dx$. Let $(\rho, \mathbf{u})$ be a global strong solution to the problem (1.1)–(1.2) verifying that

$$\sup_{t \geq 0} \|\rho(t)\|_{L^\infty(\mathbb{T}^3)} \leq M, \hspace{1cm} (1.7)$$

for some positive constant $M$. Then, there exist two positive constants $C_1 > 0$ and $\eta_1 > 0$, which are dependent on $M$ and $K$, but independent of $t$, such that
\[
\|(\rho - 1)(\cdot, t)\|_{L^2(T^3)} + \|\sqrt{\rho(u - m_0)}(\cdot, t)\|_{L^2(T^3)} + \|\nabla u\|_{L^2(T^3)} \leq C_1 e^{-\eta_1 t},
\]
for any $t \geq 0$.

Remark 1.1. The question naturally arises whether the solution $(\rho, u)$ stated in Theorem 1.1 exists or not. When the initial density is allowed to vanish and the spatial measure of the set of vacuum can be arbitrarily large, Huang, Li and Xin [18] established the global existence and uniqueness of classical solutions in whole space $\mathbb{R}^3$ if the initial energy is small but the oscillations could be arbitrarily large, see also [22]. One of the key ingredients in [18, 22] is to derive a time–independent upper bound of the density. So, under the assumption that the initial energy is small, using the similar arguments as that in [18, 22], we can show that a strong solution $(\rho, u)$ satisfying (1.7) indeed exists.

Remark 1.2. Compared to [14, 35] where the density has uniform positive lower and upper bounds, this gives the first stability result for large strong solutions of the 3D compressible Navier–Stokes equations in the presence of vacuum.

Remark 1.3. It is interesting to make a comparison between Theorem 1.1 and those of Peng–Wu–Shi [30], where the authors give global exponential stability of finite energy weak solutions constructed by Lions and Feireisl etc. More precisely, for general large data, by both using the extra integrability of the density due to Lions and constructing a suitable Lyapunov functional, Peng–Wu–Shi [30] showed that
\[
\int_\Omega (|u|^2 + G(\rho, \rho_s)) \, dx \leq C \exp\{-Ct\},
\]
where $\rho_s = \frac{1}{|\Omega|} \int_\Omega \rho_0 dx$, and $G(\rho, \rho_s) := \rho \int_\rho^{\rho_s} \frac{h^\gamma - \rho_s^\gamma}{h^2} \, dh$. If the density $\rho$ has upper bound, it is easy to check that the above exponential decay estimate implies
\[
\|\rho - \rho_s\|_{L^2} + \|\sqrt{\rho} u\|_{L^2} \leq C \exp\{-Ct\}.
\]
Compared to the exponential decay estimate in [18], this gives no information for the large time behavior of $\|\nabla u\|_{L^2}$.

Remark 1.4. Our methods can be applied to investigate global stability of large strong solutions to full compressible Navier–Stokes equations on $\mathbb{T}^3$. When the initial density is allowed to vanish, i.e., $\rho_0(x) \geq 0$, we can prove global exponential stability of strong solutions provided that the density $\rho(x, t)$ verifies $\sup_{t \geq 0} \|\rho(\cdot, t)\|_{L^\infty(T^3)} \leq M$. This result will be reported in our forthcoming paper [34].

If the initial density possesses uniform positive lower bound, we have the following result on the stability of the density in $L^\infty$–norm to the problem (1.1)–(1.2).

Theorem 1.2. Assume that all conditions of Theorem 1.1 are in force. If additionally $\inf_{x \in \mathbb{T}^3} \rho_0(x) \geq c_0 > 0$, then there exist two positive constants $C_2 > 0$ and $\eta_2 > 0$, which are dependent on $c_0$, $M$ and $K$, but independent of $t$, such that
\[
\|(\rho - 1)(\cdot, t)\|_{L^\infty(T^3)} \leq C_2 e^{-\eta_2 t},
\]
for any $t \geq 0$. 

Remark 1.5. It is worth mentioning that to prove Theorem 1.2, we only assume that the density $\rho$ is bounded from above, while the theory on global stability of large solutions developed in [14] requires the additional assumption $\sup_{t \geq 0} \| \rho(t) \|_{C^\alpha} \leq M$ with arbitrarily small $\alpha$, which plays an essential role in deriving the uniform positive lower bound of $\rho$ in [14] (See the proof of Proposition 2.3 in [14] for details).

Remark 1.6. To prove Theorem 1.2, the key ingredient is to get a time–independent positive lower bound of the density $\rho$ (See Lemma 4.1). With the key time–independent positive upper and lower bounds of the density $\rho$ in hand, we can modify the methods of [35] to obtain the exponential decay rates of higher-order spatial derivatives of the solutions.

Provided that the vacuum states are present initially, we shall prove that the vacuum states will not vanish for any time.

Theorem 1.3. Assume that all conditions of Theorem 1.1 are in force. If additionally $\inf_{x \in \mathbb{T}^3} \rho_0(x) = 0$, then it holds that

$$\inf_{x \in \mathbb{T}^3} \rho(x, t) = 0,$$

for any $t \geq 0$.

Remark 1.7. Theorem 1.3 implies that the vacuum state will persist for any time provided that the initial density contains vacuum, which is different from the previous work of Li–Li–Xin [21]. Indeed, Li–Li–Xin [21] showed that for any global entropy weak solution, any (possibly existing) vacuum state must vanish within finite time for the free boundary problem of the 1D isentropic compressible Navier–Stokes equations with density–dependent viscosity:

$$\begin{cases}
\rho_t + (pu)_x = 0, \\
(pu)_t + \text{div}(p^2u) + \nabla \rho^\gamma - (\rho^\alpha v)_x = 0,
\end{cases}$$

where $\alpha > \frac{1}{2}$ and $\gamma \geq 1$. Therefore, there arises a natural question whether any vacuum state shall vanish within finite time or vacuum state is preserved for any time for the case $0 \leq \alpha \leq \frac{1}{2}$.

1.3. Outline of ideas. We make some comments on the main ideas of the proof and explain the main difficulties and techniques involved in the process.

The proof of Theorem 1.1 can be outlined as follows. Firstly, we prove the exponential decay estimate of $\| (\rho - 1, \sqrt{\rho} (u - m_0))(t) \|_{L^2(\mathbb{T}^3)}$. Set $v = u - m_0$ with $m_0 = \int_{\mathbb{T}^3} \rho_0 u_0(x) dx$. By making the basic energy estimate on the problem (1.1)–(1.2), one can derive an energy–dissipation inequality of the form

$$\frac{d}{dt} \mathcal{E}(t) + \mathcal{D}(t) \leq 0,$$

where energy $\mathcal{E}(t)$ is equivalent to $\| (\rho - 1, \sqrt{\rho} v) \|_{L^2}^2$, and dissipation $\mathcal{D}(t)$ is equivalent to $\| (\rho - 1, \nabla u) \|_{L^2}^2$. On the other hand, by making full use of momentum equation and Poincaré’s inequality, it is clear that $\| \nabla u \|_{L^2} \geq C \| \sqrt{\rho} v \|_{L^2}$, this particularly implies that $\mathcal{D}(t) \geq C \mathcal{E}(t)$. Consequently, the exponential decay estimate of $\| (\rho - 1, \sqrt{\rho} (u - m_0))(t) \|_{L^2(\mathbb{T}^3)}$ in Theorem 1.1 follows from (1.11) immediately (See also the Lyapunov–type energy inequality (3.13)). Secondly, we derive the exponential decay estimate of $\| \nabla u(t) \|_{L^2(\mathbb{T}^3)}$. To do this, we make full use of good properties of the effective viscous flux $F$ to get the energy estimate (3.24). To close the estimate (3.24), our main observation is that $\| \nabla (u - v) \|_{L^2}$ is sufficiently small for any large enough $t$. With this key observation in hand, we can take a linear combination of (3.16) and (3.24) to get the key
Lyapunov–type energy inequality (3.30). Then, (3.30) together with Gronwall’s inequality implies the exponential decay estimate of \( \| \nabla u(t) \|_{L^2(\mathbb{T}^3)} \) immediately.

To prove Theorem 1.2 and Theorem 1.3, the key ingredient is to establish the time–independent positive lower bound of \( \rho \). To achieve this goal, we will borrow some ideas from [7, 24] and make some key uniform estimate. To see this, we first rewrite the mass conservation equation (1.1) in terms of \( \log \rho \) (cf. (4.2)). Then, by defining \( H = (2\mu + \lambda) \log \rho + \Delta \mu \cdot \nabla \rho \), and fully using the momentum conservation equation (1.1) and Lagrangian coordinates, it is clear that along the particle trajectories \( H \) satisfies (4.4). Finally, we exploit some delicate energy estimates for (4.4) to get key time–independent negative lower bound of \( H \):

\[
H(t) \geq -C \quad \text{(for some constant } C > 0) 
\]

holds for any large enough \( t \). This together with (4.9) imply the time–independent positive lower bound of \( \rho \) immediately. The exponential decay estimate for \( \| (\rho - 1) \|_{L^\infty(\mathbb{T}^3)} \) is due to the damping mechanism of density. As a by–product, we finally show that the vacuum states will not vanish for any time provided that the vacuum states are present initially.

The rest of the paper is organized as follows. In Section 2, we recall some elementary facts and inequalities that will be used frequently in later analysis. Section 3 is devoted to proving Theorem 1.1. We prove Theorem 1.2 and Theorem 1.3 in Section 4.

2. Preliminaries

In this section, we list some elementary but useful facts and inequalities which will be used frequently in the sequel.

Set \( \varphi = \rho - 1 \) and define the potential energy density \( G \) by

\[
G(\rho) \triangleq \rho \int_1^\rho \frac{P(s) - 1}{s^2} \, ds. \tag{2.1}
\]

The following lemma is concerned with the estimates about \( P(\rho) - 1 \) and \( G(\rho) \), see [10].

**Lemma 2.1.** Let \( \gamma > 1 \) be arbitrary fixed constants. Then we have

\[
P(\rho) - 1 \sim \varphi \quad \text{and} \quad G(\rho) \sim \varphi^2
\]

if \( 0 \leq \rho \leq M \).

In virtue of (1.1), one has

\[
\Delta F = \text{div}(\rho \dot{u}) \quad \text{and} \quad \mu \Delta w = \mathcal{P}(\rho \dot{u}), \tag{2.2}
\]

where “\( \cdot \)” denotes the material derivative which is defined by

\[
\dot{f} \overset{\text{def}}{=} \partial_t f + u \cdot \nabla f.
\]

Applying the standard \( L^p \)–estimates of elliptic systems to (2.2), we have the following estimates.

**Lemma 2.2.** Let \( (\rho, u) \) be a strong solution to the problem (1.1)–(1.2). Then for any \( p \in (1, \infty) \), there exists a generic positive constant \( C \) which depends only on \( \mu, \lambda \) and \( p \) such that

\[
\| \nabla F \|_{L^p(\mathbb{T}^3)} + \| \nabla^2 w \|_{L^p(\mathbb{T}^3)} \leq C \| \rho \dot{u} \|_{L^p(\mathbb{T}^3)}, \tag{2.3}
\]

and

\[
\| \nabla u \|_{L^p(\mathbb{T}^3)} \leq C \left( \| F \|_{L^p(\mathbb{T}^3)} + \| \nabla w \|_{L^p(\mathbb{T}^3)} + \| (P(\rho) - 1) \|_{L^p(\mathbb{T}^3)} \right). \tag{2.4}
\]
Proof. Applying the standard $L^p$-estimate of elliptic systems to (2.2) and (2.3) follows immediately. Noticing that $-\Delta u = -\nabla \text{div} u + \nabla \times \nabla \times w$, one has
\[
\nabla u = \nabla \Delta^{-1} \nabla \text{div} u - \nabla \Delta^{-1} \nabla \times \nabla \times w,
\]
where $\Delta^{-1}$ denotes the inverse Laplacian with zero mean value on $T^3$. Thus, it follows the Marcinkiewicz multiplier theorem (see [33]) that
\[
\|\nabla u\|_{L^p(T^3)} \leq C(\|\nabla \text{div} u\|_{L^p(T^3)} + \|\nabla w\|_{L^p(T^3)})
\]
\[
\leq C(\|F\|_{L^p(T^3)} + \|\nabla w\|_{L^p(T^3)} + \|(P-1)\|_{L^p(T^3)})
\]
as claimed in (2.4). The proof the lemma is completed. $\square$

3. Proof of Theorem 1.1

In this section, we devote ourselves to proving Theorem 1.1. In order to deduce the a priori estimate, in what follows, we will give some energy estimates. Then, Theorem 1.1 is an easy consequence of Lemma 3.1 and Lemma 3.2.

The first lemma is concerned with the time–decay rate of $\|(\rho, \sqrt{\rho}(u - m_0))\|_{L^2(T^3)}$.

Lemma 3.1. Under the assumptions of Theorem 1.1, there exist two positive constants $C_3 > 0$ and $\eta_3 > 0$, which are dependent on $M$ and $K$, but independent of $t$, such that
\[
\|(\rho, t)\|_{L^2(T^3)} + \|(\sqrt{\rho}(u - m_0), t)\|_{L^2(T^3)} \leq C_3 e^{-\eta_3 t},
\]
for any $t \geq 0$.

Proof. We split the proof into three steps.

Step 1. $L^2$ estimate of $(\rho, v)$. Recalling $v = u - m_0$, multiplying the momentum conservation equation (1.1) by $u$, and then integrating the resultant equation over $T^3$, we have from integration by parts that
\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} |u|^2 dx + \int_{T^3} \nabla P(\rho) u dx + \int_{T^3} \mu |\nabla u|^2 + (\lambda + \mu) |\text{div} u|^2 dx = 0.
\]
It follows from mass conservation equation (1.1) and the definition of $G(\rho)$ in (2.1) that
\[
(G(\rho))_t + \text{div}(G(\rho) u) + (P(\rho) - 1) \text{div} u = 0.
\]
Integrating the above equation over $T^3$ and then adding the resulting equality to (3.2), one has
\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} |u|^2 + G(\rho) dx + \int_{T^3} \mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2 dx = 0.
\]
Noticing that
\[
\int_{T^3} \rho dx = 1, \text{ and } \int_{T^3} \rho u dx = m_0,
\]
we have
\[
\int_{T^3} \rho |v|^2 dx = \int_{T^3} \rho |(u - m_0)|^2 dx
\]
\[
= \int_{T^3} \rho |u|^2 dx - 2 \int_{T^3} \rho u \cdot m_0 dx + \int_{T^3} \rho |m_0|^2 dx
\]
\[
= \int_{T^3} \rho |u|^2 dx - |m_0|^2.
\]
Therefore, the equality (3.3) can be rewritten as follows
\[
\frac{d}{dt} \int_{T^3} \frac{1}{2} |\rho|^2 + G(\rho) dx + \int_{T^3} \mu |\nabla u|^2 + (\mu + \lambda) |\nabla v|^2 dx = 0. \tag{3.5}
\]

Step 2: Dissipation of $\rho$. From (1.11) and (1.12), we have
\[
(\rho v)_t + \text{div}(\rho v \otimes v) + \text{div}(\rho m_0 \otimes v) + \nabla(P(\rho) - 1) = \mu \Delta v + (\mu + \lambda) \nabla \text{div} v. \tag{3.6}
\]

Applying the operator $\Delta^{-1} \text{div}$ to (3.6), one has
\[
P(\rho) - 1 = -\partial_t \Delta^{-1} \text{div}(\rho v) + (2\mu + \lambda) \text{div} v - \mathcal{R}_i \mathcal{R}_j (\rho v^i v^j) - \mathcal{R}_i \mathcal{R}_j (\rho m_0^i v^j), \tag{3.7}
\]
where $\mathcal{R}_i = -(-\Delta)^{-1/2} \partial_{x_i}$ is the usual Riesz transform on $\mathbb{T}^3$. To achieve the dissipation on $\rho$, we take the $L^2$ inner product of the equation (3.7) with $\rho$ to get that
\[
\begin{align*}
\int_{T^3} (P(\rho) - 1) \rho dx &= - \int_{T^3} \partial_t \left[ \Delta^{-1} \text{div}(\rho v) \right] \rho dx + (2\mu + \lambda) \int_{T^3} \text{div} v \rho dx \\
&\quad - \int_{T^3} \mathcal{R}_i \mathcal{R}_j (\rho v^i v^j) \rho dx - \int_{T^3} \mathcal{R}_i \mathcal{R}_j (\rho m_0^i v^j) \rho dx \\
&\quad \triangleq I_{11} + I_{12} + I_{13} + I_{14}.
\end{align*}
\tag{3.8}
\]

For the term in the left–side of (3.8), it follows from Lemma 2.1 that there exists a positive constant $C_4$ such that
\[
\int_{T^3} (P(\rho) - 1) \rho dx \geq C_4 \|\rho\|_{L^2(T^3)}. \tag{3.9}
\]

We turn to estimate each term on the right–side of (3.8). For the term $I_{11}$, it follows from (1.11), (1.7), integration by parts, Parseval’s theorem, Marcinkiewicz multiplier theorem and Young’s inequality that
\[
I_{11} = \int_{T^3} \partial_t \left[ (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v) \right] (-\Delta)^{-\frac{1}{2}} \rho dx
\]
\[
= \frac{d}{dt} \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v)(-\Delta)^{-\frac{1}{2}} \rho dx - \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v)(-\Delta)^{-\frac{1}{2}} \partial_t \rho dx
\]
\[
= \frac{d}{dt} \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v)(-\Delta)^{-\frac{1}{2}} \rho dx + \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v)(-\Delta)^{-\frac{1}{2}} \text{div}(\rho u) dx
\]
\[
= \frac{d}{dt} \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v)(-\Delta)^{-\frac{1}{2}} \rho dx + \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v) dx
\]
\[
+ \int_{T^3} \left[ (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v) \right] \left[ (-\Delta)^{-\frac{1}{2}} \text{div}(\rho m_0) \right] dx
\]
\[
\leq \frac{d}{dt} \int_{T^3} (-\Delta)^{-\frac{1}{2}} \text{div}(\rho v)(-\Delta)^{-\frac{1}{2}} \rho dx + C \left( \|\rho v\|_{L^2(T^3)}^2 + \|\rho v\|_{L^2(T^3)} \|\rho\|_{L^2(T^3)} \right)
\]
\[
\leq - \frac{d}{dt} \int_{T^3} \Delta^{-1} \text{div}(\rho v) \rho dx + C \|v\|_{L^2(T^3)}^2 + \frac{C_4}{6} \|\rho\|_{L^2(T^3)}^2.
\tag{3.10}
\]

Using Young’s inequality, the term $I_{12}$ is controlled as
\[
I_{12} \leq C \|\nabla u\|_{L^2(T^3)}^2 + \frac{C_4}{6} \|\rho\|_{L^2(T^3)}^2. \tag{3.11}
\]
From (1.7), Marcinkiewicz multiplier theorem and Young’s inequality, the last two terms $I_{13}$ and $I_{14}$ can be bounded as

$$I_{13} + I_{14} \leq C \left( \| R_i R_j (p v_i v_j) \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)} + \| R_i R_j (p m_i^0 v_i) \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)} \right)$$

$$\leq C \left( \| p v^2 \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)} + \| p m_i^0 v_i \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)} \right)$$

$$\leq C \left( \| \tilde{\varphi} \|_{L^\infty(T^3)} \| v \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)} + \| \rho m_i \|_{L^\infty(T^3)} \| v \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)} \right)$$

$$\leq C \left( \| v \|_{L^2(T^3)}^2 + \| v \|_{L^2(T^3)}^2 + \| \nabla u \|_{L^2(T^3)}^2 \right).$$

Substituting (3.9)–(3.12) into (3.8), we obtain

$$\frac{d}{dt} \int_{T^3} \Delta^{-1} \text{div}(\rho v) \tilde{\varphi} \, dx + \frac{C_4}{2} \| \tilde{\varphi} \|_{L^2(T^3)}^2 \leq C \left( \| v \|_{L^2(T^3)}^2 + \| v \|_{L^2(T^3)}^2 + \| \nabla u \|_{L^2(T^3)}^2 \right).$$

### Step 3: Closing the estimates.

Choose a positive constant $D_1$ suitably large and define the temporal energy functional

$$\mathcal{M}_1(t) = D_1 \left( \int_{T^3} \frac{1}{2} \rho \| v \|^2 + G(\rho) \, dx \right) + \int_{T^3} \Delta^{-1} \text{div}(\rho v) \tilde{\varphi} \, dx,$$

for any $t \geq 0$. By virtue of (1.7), Lemma 2.1 Hölder’s inequality and Marcinkiewicz’s multiplier theorem, we have

$$\left\| \int_{T^3} \Delta^{-1} \text{div}(\rho v) \tilde{\varphi} \, dx \right\|$$

$$\leq \| \Delta^{-1} \text{div}(\rho v) \|_{L^6(T^3)} \| \tilde{\varphi} \|_{L^\infty(T^3)}$$

$$\leq C \| \rho v \|_{L^2(T^3)} \| \tilde{\varphi} \|_{L^2(T^3)}$$

$$\leq C \left( \int_{T^3} \frac{1}{2} \rho \| v \|^2 + G(\rho) \, dx \right).$$

Thus, $\mathcal{M}_1(t)$ is equivalent to $\| (\varphi, \sqrt{\rho} v) (t) \|_{L^2(T^3)}$ if we choose $D_1$ large enough.

From (1.7), Minkowski’s inequality, Hölder’s inequality and Poincaré’s inequality, we obtain

$$\| v \|_{L^r(T^3)} \leq \| u - \bar{u} \|_{L^r(T^3)} + \| \bar{u} - m_0 \|$$

$$\leq \| u - \bar{u} \|_{L^r(T^3)} + \left| \int_{T^3} (\rho \bar{u} - \rho u) \, dx \right|$$

$$\leq C \| u - \bar{u} \|_{L^r(T^3)},$$

(3.14)

Taking a linear combination of (3.13) and (3.14), and using (3.14), we obtain

$$\frac{d}{dt} \mathcal{M}_1(t) + \frac{\mathcal{M}_1(t)}{D_1} + \frac{\| \nabla u(t) \|_{L^2(T^3)}^2}{D_1} \leq 0,$$

(3.15)

for any $t \geq 0$. Integrating the above inequality with respect to $t$ over $[0, t]$, (3.15) follows immediately. The proof of lemma is completed.

In the following lemma, we derive the time–decay rate of $\| \nabla u \|_{L^2(T^3)}$. The main observation here is that $\| \nabla u(t) \|_{L^2(T^3)}$ is sufficiently small for any large enough $t$.

### Lemma 3.2.

Under the assumptions of Theorem 1.1, there exist two positive constants $C_5 > 0$ and $\eta_4 > 0$, which are dependent on $M$ and $K$, but independent of $t$, such that

$$\| \nabla u \|_{L^2(T^3)} \leq C_5 e^{-\eta_4 t},$$

(3.16)

for any $t \geq 0$. 

Proof. By the definition of material derivative, we can rewrite (3.1) as follows

\[ \rho \dot{u} + \nabla (P(\rho) - 1) = \mu \Delta u + (\mu + \lambda) \nabla \text{div} u. \] (3.17)

Multiplying (3.17) by $\dot{u}$ and then integrating the resultant equation over $\mathbb{T}^3$, one has

\[ \int_{\mathbb{T}^3} \rho \dot{u}^2 dx + \int_{\mathbb{T}^3} \nabla (P(\rho) - 1) \dot{u} dx = \int_{\mathbb{T}^3} (\mu \Delta u + (\mu + \lambda) \nabla \text{div} u) \dot{u} dx. \] (3.18)

Using (3.11) and integration by parts several times, the second term on the left–side of (3.18) can be rewritten as follows:

\[ \int_{\mathbb{T}^3} \nabla (P(\rho) - 1) \dot{u} dx \]

\[ = \int_{\mathbb{T}^3} \nabla (P(\rho) - 1) (u_t + u \cdot \nabla u) dx \]

\[ = - \frac{d}{dt} \int_{\mathbb{T}^3} (P(\rho) - 1) \text{div} u dx + \int_{\mathbb{T}^3} P'(\rho) \rho_t \text{div} u dx \]

\[ + \int_{\mathbb{T}^3} \left( P'(\rho) u \cdot \nabla \text{div} u + P(|\text{div} u|^2 - u_j^2 u_j^2) \right) dx \]

\[ = - \frac{d}{dt} \int_{\mathbb{T}^3} (P(\rho) - 1) \text{div} u dx + \int_{\mathbb{T}^3} \left( -\rho P'(\rho) |\text{div} u|^2 + P(|\text{div} u|^2 - u_j^2 u_j^2) \right) dx. \] (3.19)

Similarly, the term on the right–side of (3.18) can be rewritten as follows:

\[ \int_{\mathbb{T}^3} (\mu \Delta u + (\mu + \lambda) \text{div} \nabla u) \dot{u} dx \]

\[ = \int_{\mathbb{T}^3} (\mu \Delta u + (\mu + \lambda) \text{div} \nabla u) (u_t + u \cdot \nabla u) dx \]

\[ = - \frac{1}{2} \frac{d}{dt} \int_{\mathbb{T}^3} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2) dx - \mu \int_{\mathbb{T}^3} \left( u_j^2 u_j^2 - \frac{1}{2} |u_j^2|^2 \text{div} u \right) dx \]

\[ - (\mu + \lambda) \int_{\mathbb{T}^3} \left( u_j^2 \text{div} u - \frac{1}{2} (\text{div} u)^2 \right) dx. \] (3.20)

Substituting (3.19) and (3.20) into (3.18), one has

\[ \frac{d}{dt} \int_{\mathbb{T}^3} \left[ \frac{1}{2} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2) - (P(\rho) - 1) \text{div} u \right] dx + \int_{\mathbb{T}^3} \rho |\dot{u}|^2 dx \]

\[ = \int_{\mathbb{T}^3} \left( \rho P'(\rho) |\text{div} u|^2 - P(|\text{div} u|^2 - u_j^2 u_j^2) \right) dx - \mu \int_{\mathbb{T}^3} \left( u_j^2 u_j^2 - \frac{1}{2} |u_j^2|^2 \text{div} u \right) dx \]

\[ - (\mu + \lambda) \int_{\mathbb{T}^3} \left( u_j^2 \text{div} u - \frac{1}{2} (\text{div} u)^2 \right) dx \]

\[ = I_{21} + I_{22} + I_{23}. \] (3.21)

For the first term on the right–side of (3.21), it follows from (3.7) that

\[ |I_{21}| \leq C \|\nabla u\|_{L^2(\mathbb{T}^3)}^2. \] (3.22)
For the last two terms on the right–side of (3.21), by virtue of (1.7), Lemma 2.1, Lemma 2.2, Sobolev’s inequality and Young’s inequality, we have

\[ |I_{22}| + |I_{23}| \leq C \| \nabla u \|^3_{L^2(T^3)} \]

\[ \leq C \| \nabla u \|^3_{L^2(T^3)} \| \nabla \|_{L^6(T^3)}^2 \]

\[ \leq C \| \nabla u \|^3_{L^2(T^3)} \| (F, \nabla F, P(\rho) - 1) \|_{L^6(T^3)}^2 \]

\[ \leq C \| \nabla u \|^3_{L^2(T^3)} \left( \| (\nabla F, \nabla^2 w) \|_{L^2(T^3)}^2 + \| F \|_{L^2(T^3)}^2 + \| P(\rho) - 1 \|_{L^6(T^3)}^2 \right) \]

\[ \leq C \| \nabla u \|^3_{L^2(T^3)} \left( \| \sqrt{\rho} \|_{L^2(T^3)} + \| \rho \|_{L^6(T^3)} + \| \nabla u \|^2_{L^2(T^3)} + \| \rho \|^2_{L^6(T^3)} \right) \]

\[ \leq \frac{1}{2} \| \sqrt{\rho} \|_{L^2(T^3)}^2 + C \left( \| \nabla u \|^2_{L^2(T^3)} + \| \nabla u \|^3_{L^2(T^3)} + \| \nabla u \|^2_{L^2(T^3)} + \| \rho \|^2_{L^6(T^3)} \right). \]

(3.23)

Plugging (3.22)–(3.23) into (3.21), we have

\[
\frac{d}{dt} \int_{T^3} \left[ \frac{1}{2} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) - (P(\rho) - 1) \text{div}u \right] \rho \nabla \dot{u} \rho \nabla \dot{u} \, dx + \frac{1}{2} \int_{T^3} \rho |\nabla u|^2 \rho \nabla x \cdot dx 
\leq C \left( \| \nabla u \|^2_{L^2(T^3)} + \| \nabla u \|^3_{L^2(T^3)} + \| \nabla u \|^2_{L^2(T^3)} + \| \rho \|^2_{L^6(T^3)} \right). \]

(3.24)

This, combined with (1.6), (1.7), (3.3) and (3.1), yields

\[ \sqrt{\rho} \dot{u} \in L^2(0, \infty; L^2(T^3)), \]

and

\[ \int_{T^3} \left[ \frac{1}{2} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) - (P(\rho) - 1) \text{div}u \right] (t) dx \in C(0, \infty), \]

which together with (1.6) implies that

\[ \int_{T^3} \left[ \frac{1}{2} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) - (P(\rho) - 1) \text{div}u + D_2 |g|^2 \right] (t) dx \in C(0, \infty), \]

(3.25)

where \( D_2 \) is a suitably large positive constant. In light of (3.1) and (3.5), we obtain

\[ \int_0^\infty \int_{T^3} \left[ \frac{1}{2} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) - (P(\rho) - 1) \text{div}u + D_2 |g|^2 \right] \rho \nabla x \cdot dx dt < \infty. \]

(3.26)

Next, we choose a positive constant \( D_3 \) suitably large and define the temporal energy functional

\[ M_2(t) = D_3 M_1(t) + \int_{T^3} \left[ \frac{1}{2} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) - (P(\rho) - 1) \text{div}u + D_2 |g|^2 \right] (t) dx, \]

for any \( t \geq 0 \). Note that \( M_2(t) \) is equivalent to \( \| (\dot{\rho}, \sqrt{\rho} \nabla u, (\nabla u)) \|_{L^2(T^3)}^2 \) if we choose \( D_2 \) and \( D_3 \) large enough. Fix a positive constant \( \delta_1 \) that may be small. Then, it follows from (3.1) and (3.26) that there exists a positive constant \( T_1 > 0 \) such that

\[ M_2(T_1) < \delta_1. \]

(3.27)

Now, we claim that

\[ \int_{T^3} \left[ \frac{1}{2} \left( \mu |\nabla u|^2 + (\mu + \lambda) |\text{div}u|^2 \right) - (P(\rho) - 1) \text{div}u + D_2 |g|^2 \right] (t) dx < 2 \delta_1 \]

(3.28)
holds for any $t \geq T_1$. Assume this claim for the moment. Then, (3.28) implies that
\[
\int_{\mathbb{T}^3} \left[ (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2) \right](t) \, dx < 4\delta_1,
\] (3.29)
for any $t \geq T_1$. Let $\delta_1$ be small enough, then taking a linear combination of (3.19) and (3.24) yields
\[
\frac{d}{dt} M_2(t) + \frac{M_2(t)}{D_3} + \frac{\|\sqrt{\rho} u(t)\|_{L^2(\mathbb{T}^3)}^2}{D_3} \leq 0,
\] (3.30)
for any $t \geq T_1$. Integrating (3.30) with respect to $t$ over $[0, t]$ gives (3.16) immediately. Thus, to complete the proof of Lemma 3.2, it suffices to establish (3.28).

Next, we return to the proof of (3.28). If (3.28) is false, by (3.25), there exists a time $T_2 > T_1$ such that
\[
\int_{\mathbb{T}^3} \left[ \frac{1}{2} (\mu |\nabla u|^2 + (\mu + \lambda) |\text{div} u|^2) - (P(\rho) - 1) \text{div} u + D_2 |\sigma|^2 \right](T_2) \, dx = 2\delta_1.
\] (3.31)
Taking a minimal value of $T_2$ satisfying (3.31), then (3.28) holds for any $T_1 \leq t < T_2$. Integrating (3.30) from $T_1$ to $T_2$, one has
\[ M_2(T_2) \leq M_2(T_1) < \delta_1, \]
which contradicts (3.31). Hence (3.28) holds for any $t \geq T_1$. The proof of lemma is completed. □

4. PROOF OF THEOREM 1.2 AND THEOREM 1.3

We turn to prove Theorem 1.2 and Theorem 1.3 in this section. The following lemma is devoted to deriving uniform positive lower bound of $\rho$.

**Lemma 4.1.** Under the assumptions of Theorem 1.2, there exists a positive constant $c_1 > 0$, which is independent of $t$, such that
\[
\inf_{x \in \mathbb{T}^3} \rho(x, t) \geq c_1,
\] (4.1)
for any $t \geq 0$.

**Proof.** First, motivated by Desjardins [7], we rewrite mass conservation equation (1.1) as
\[
(\log \rho)_t + u \cdot \nabla \log \rho + \text{div} u = 0.
\] (4.2)
Defining $H \overset{\Delta}{=} (2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho v)$, and then combining (3.7) with (4.2), we have
\[
H_t + u \cdot \nabla H + (P(\rho) - 1) = [v^j, \mathcal{R}_i \mathcal{R}_j]([\rho v^i]) + [m_0^i, \mathcal{R}_i \mathcal{R}_j]([\rho v^i]),
\] (4.3)
where $[v^j, \mathcal{R}_i \mathcal{R}_j](v^i) = u^j \mathcal{R}_i \mathcal{R}_j(v^i) - \mathcal{R}_i \mathcal{R}_j(u^i v^j)$. Let $y \in \mathbb{T}^3$ and define the corresponding particle path $x(t, y)$ by
\[
\begin{aligned}
\frac{dx(t, y)}{dt} &= u(x(t, y), t), \\
x(t_0, y) &= y.
\end{aligned}
\]

Then, (4.3) can be reformulated as
\[
\frac{d}{dt} H(t) + (P(\rho) - 1) = [v^j, \mathcal{R}_i \mathcal{R}_j]([\rho v^i]) + [m_0^i, \mathcal{R}_i \mathcal{R}_j]([\rho v^i]).
\] (4.4)
In virtue of the results of Coifman, Lions, Meyer and Semmes [4], the following map
\[
W^{1, \alpha_1}(\mathbb{T}^N)^N \times L^{\alpha_2}(\mathbb{T}^N)^N \to W^{1, \alpha_3}(\mathbb{T}^N)^N,
\]
\[
(u, v) \to [u_j, \mathcal{R}_i \mathcal{R}_j]v_i,
\]

is continuous for any $N \geq 2$ as soon as $\frac{1}{2} + \frac{1}{2} = \frac{1}{2}$. Hence, using (4.7), Lemma 2.2 and 3.14, we can deduce that

$$
\left\| [v, R_i R_j] (\rho \nu) \right\|_{L^\infty(T^3)} + \left\| [m_0, R_i R_j] (\rho \nu) \right\|_{L^\infty(T^3)} \\
\leq C \left( \left\| [v, R_i R_j] (\rho \nu) \right\|_{W^{1,4}(T^3)} + \left\| [m_0, R_i R_j] (\rho \nu) \right\|_{W^{1,4}(T^3)} \right)
\leq C \left( \left\| v \right\|_{W^{1,4}(T^3)} + \left\| m_0 \right\|_{W^{1,4}(T^3)} \right) \left\| \rho \nu \right\|_{L^{12}(T^3)}
\leq C \left( \left\| \nabla u \right\|_{L^1(T^3)} + 1 \right) \left\| v \right\|_{L^{12}(T^3)}
\leq C \left( \left\| F \right\|_{L^6(T^3)} + \left\| \nabla w \right\|_{L^6(T^3)} + \left\| (P(\rho) - 1) \right\|_{L^6(T^3)} + 1 \right) \left\| \nabla u \right\|_{L^3(T^3)}
\leq C \left( \left\| \sqrt{\rho} u \right\|_{L^2(T^3)} + 1 \right) \left\| \Delta u \right\|_{L^2(T^3)}.
$$

(4.5)

On the other hand, it follows from (3.30) that

$$
\int_0^\infty \int_{T^3} \rho |\dot{u}|^2 dx dt < \infty,
$$

(4.6)

where we have used (3.10). Therefore, this together with Theorem 1.1 and (4.5) implies that

$$
\int_0^\infty \left\| [v, R_i R_j] (\rho \nu) \right\|_{L^\infty(T^3)} dt + \int_0^\infty \left\| [m_0, R_i R_j] (\rho \nu) \right\|_{L^\infty(T^3)} dt < \infty.
$$

(4.7)

In virtue (1.8), (4.4) and (4.7), it is clear that

$$
H(t) \in C[0, \infty),
$$

(4.8)

where we have abbreviated $H(x(t), t)$ by $H(t)$ for convenience. By virtue of (1.7) and (3.14), and Theorem 1.1 one has

$$
\left\| \Delta^{-1} \text{div}(\rho \nu) \right\|_{L^\infty(T^3)} \leq C \left\| \rho \nu \right\|_{L^4(T^3)} \leq C \left\| \nabla v \right\|_{L^4(T^3)} \leq C \left\| \nabla u \right\|_{L^2(T^3)} \leq C e^{-\eta t}.
$$

(4.9)

Fix a positive constant $\delta_2$ that may be small, in view of (4.7) and (4.9), there exists a positive constant $T_3 > 0$ such that

$$
\int_0^\infty \left\| \left( [v, R_i R_j] (\rho \nu), [m_0, R_i R_j] (\rho \nu) \right) \right\|_{L^\infty(T^3)} dt + \left\| \Delta^{-1} \text{div}(\rho \nu) (t) \right\|_{L^\infty(T^3)} \leq \delta_2,
$$

(4.10)

for any $t \geq T_3$. Combining (4.8) and (4.9), we see that $\| \log \rho(x, t) \|_{L^\infty(0, T_3; L^\infty(T^3))} \leq C(T_3)$. Assume that there exists a time $T_4 \geq T_3$ such that $0 < c_2 = \rho(T_4) \leq \frac{1}{2}$. Otherwise, we prove (1.1). Setting $\kappa = -(2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu)(T_4)$, then it is clear that $\kappa > 2\mu + \lambda$ if $\delta_2$ is small enough. Now, we claim that

$$
-(2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu))(t) < 2\kappa
$$

(4.11)

holds for any $t \geq T_4$. Assume this claim for the moment, then (1.1) follows immediately. Next, we return to the proof of (1.1). If (1.1) is false, by (4.8), there exists a time $T_6 > T_4$ such that

$$
-(2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu))(T_6) = \kappa.
$$

(4.12)

We take a minimal value of $T_6$ satisfying (4.12) and then choose a maximal value of $T_5 < T_6$ such that $-(2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu))(T_5) = \kappa$. Thus we have

$$
-(2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu))(t) \in [\kappa, 2\kappa].
$$

(4.13)

for any $t \in [T_5, T_6]$, which implies that $0 < \rho(t) < \frac{1}{2}$ for any $t \in [T_5, T_6]$. Using (4.10), and integrating (4.6) along particle trajectories from $T_5$ to $T_6$, we have

$$
-\kappa \geq -\int_{T_5}^{T_6} \left| P(\rho(t)) \right| dt - \int_{T_5}^{T_6} \left\| \left( [v, R_i R_j] (\rho \nu), [m_0, R_i R_j] (\rho \nu) \right) (t) \right\|_{L^\infty(T^3)} dt \geq -\delta_2,
$$

for any $t \in [T_5, T_6]$, which implies that $0 < \rho(t) < \frac{1}{2}$ for any $t \in [T_5, T_6]$. Using (4.10), and integrating (4.6) along particle trajectories from $T_5$ to $T_6$, we have
which is impossible if $\delta_2$ is small enough. We therefore conclude that there is no such time $T_6$, which is bigger than $T_4$, such that $-((2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu))(T_6) = 2\kappa$. Since $\nu \in \mathbb{T}^3$ is arbitrary, we have $((2\mu + \lambda) \log \rho + \Delta^{-1} \text{div}(\rho \nu))(t) > -2\kappa$ on $\mathbb{T}^3 \times [T_4, \infty)$, and (4.1) follows immediately. The proof of lemma is completed.

\[ \square \]

Now we are in a position to prove Theorem 1.2 and Theorem 1.3.

**Proof of Theorem 1.2.** Multiplying (4.4) by $H(t)$, we have

\[ \frac{1}{2} \frac{d}{dt} H^2(t) + \frac{P(\rho) - 1}{(2\mu + \lambda) \log \rho} H^2(t) = \left( [v^i, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i) + [m^i_0, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i) + \frac{(P(\rho) - 1) \Delta^{-1} \text{div}(\rho \nu)}{(2\mu + \lambda) \log \rho} \right) H(t). \]

In virtue of (1.7) and (4.1), we see that $\log \rho \sim P(\rho) - 1$. Hence, there exists a positive constant $\eta_5$ such that

\[
\frac{d}{dt} H^2(t) + \eta_5 H^2(t) \leq C \left( [v^i, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i), [m^i_0, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i), \Delta^{-1} \text{div}(\rho \nu) \right)(t) \left\| H(t) \right\|_{L^\infty(\mathbb{T}^3)},
\]

which implies that

\[
\frac{d}{dt} |H(t)| + \eta_5 |H(t)| \leq C \left( [v^i, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i), [m^i_0, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i), \Delta^{-1} \text{div}(\rho \nu) \right)(t) \left\| H(t) \right\|_{L^\infty(\mathbb{T}^3)}, \tag{4.14}
\]

Combining (1.3), (4.5) and (4.9) yields

\[
\left\| [v^i, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i), [m^i_0, \mathcal{R}_i \mathcal{R}_j](\rho \nu^i), \Delta^{-1} \text{div}(\rho \nu) \right)(t) \right\|_{L^\infty(\mathbb{T}^3)} \leq C \left( \| \sqrt{\rho} \hat{u} \|_{L^2(\mathbb{T}^3)} + 1 \right)^{\frac{3}{2}} e^{-\frac{2\kappa}{t}}.
\]

Substituting the above estimate into (4.14), we obtain

\[
\frac{d}{dt} |H(t)| + \eta_5 |H(t)| \leq C \left( \| \sqrt{\rho} \hat{u} \|_{L^2(\mathbb{T}^3)} + 1 \right)^{\frac{3}{2}} e^{-\frac{2\kappa}{t}},
\]

which implies

\[
\frac{d}{dt} \left( e^{\eta_5 t} |H(t)| \right) \leq C e^{\eta_5 t} \left( \| \sqrt{\rho} \hat{u} \|_{L^2(\mathbb{T}^3)} + 1 \right)^{\frac{3}{2}} e^{-\frac{2\kappa}{t}}. \tag{4.15}
\]
Integrating (4.15) along particle trajectories from 0 to \( t \), and using (4.6) and Hölder’s inequality, we obtain
\[
|H(t)| \leq e^{-\eta t} + C \int_0^t e^{-\eta(t-\tau)} (\| (\sqrt{\rho}) (\tau) \|_{L^2(\mathbb{T}^3)} + 1)^{\frac{3}{2}} e^{-\frac{3}{8} \eta \tau} d\tau
\]
\[
\leq e^{-\eta t} + C \int_0^t e^{-\eta(t-\tau)} (\| (\sqrt{\rho}) (\tau) \|_{L^2(\mathbb{T}^3)} + 1)^{\frac{3}{2}} e^{-\frac{3}{8} \eta \tau} d\tau
\]
\[
+ C \int_0^t e^{-\eta(t-\tau)} (\| (\sqrt{\rho}) (\tau) \|_{L^2(\mathbb{T}^3)} + 1)^{\frac{3}{2}} e^{-\frac{3}{8} \eta \tau} d\tau
\]
\[
\leq e^{-\eta t} + Ce^{-\frac{3}{8} \eta t} \left( \int_0^\frac{t}{2} (\| (\sqrt{\rho}) (\tau) \|_{L^2(\mathbb{T}^3)} d\tau) \right)^{\frac{1}{2}} \left( \int_0^t e^{-\frac{3}{8} \eta \tau} d\tau \right)^{\frac{1}{2}}
\]
\[
+ Ce^{-\frac{3}{8} \eta t} \int_0^\frac{t}{2} e^{-\frac{3}{8} \eta \tau} d\tau + Ce^{-\frac{3}{8} \eta t} \int_0^t e^{-\eta(t-\tau)} d\tau
\]
\[
+ Ce^{-\frac{3}{8} \eta t} \left( \int_0^\frac{t}{2} (\| (\sqrt{\rho}) (\tau) \|_{L^2(\mathbb{T}^3)} d\tau) \right)^{\frac{1}{2}} \left( \int_0^t e^{-\frac{3}{8} \eta \tau} d\tau \right)^{\frac{1}{2}}
\]
\[
\leq C \left( e^{-\frac{3}{8} \eta t} + e^{-\frac{3}{8} \eta t} \right)
\]
as claimed in (1.9). We complete the proof of Theorem 1.2. ∎

**Proof of Theorem 1.3** If the conclusion in Theorem 1.3 is false, then there exists a time \( T_7 \) such that \( \inf_{x \in \mathbb{T}^3} \rho(x, T_7) > 0 \). Due to Theorem 1.2, one deduces that
\[
\lim_{t \to \infty} \| (\rho - 1)(\cdot, t) \|_{L^\infty(\mathbb{T}^3)} = 0,
\]
which implies that there exists a time \( T_8(\geq T_7) \), such that for a.e. \( x \in \mathbb{T}^3 \),
\[
\frac{1}{2} \leq \rho(x, T_8) \leq \frac{3}{2}.
\]
(4.16)

By virtue of (1.6), we see that
\[
\rho(t) \in C[0, \infty),
\]
(4.17)
where we have abbreviated \( \rho(x(t), t) \) by \( \rho(t) \) for convenience. Due to (1.4) and \( \inf_{x \in \mathbb{T}^3} \rho_0(x) = 0 \), it is clear that for any \( \varepsilon > 0 \), there exists a positive constant \( T_\varepsilon < T_8 \) such that
\[
\inf_{x \in \mathbb{T}^3} \rho(x, T_\varepsilon) = \varepsilon \quad \text{and} \quad \inf_{t \in [T_\varepsilon, T_8]} \rho(t) \geq \varepsilon.
\]
(4.18)

Therefore, there exists a non–zero measurable \( A \) such that
\[
\varepsilon \leq \rho(y(T_\varepsilon), T_\varepsilon) \leq 2\varepsilon,
\]
(4.19)
for any \( y \in A \) if \( \varepsilon \) is sufficiently small. Integrating (4.4) along particle trajectories from \( T_\varepsilon \) to \( T_8 \), and using (1.7), (4.7), (4.9) and (4.19), we have
\[
\log \rho(x(T_8, y), T_8) \leq \log \rho(y(T_\varepsilon), T_\varepsilon) + C(M, T_8) \leq \log(2\varepsilon) + C(M, T_8),
\]
(4.20)
which contradicts (4.16) if \( \varepsilon \) is small enough. This completes the proof of Theorem 1.3. ∎
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