Local Improvement Gives Better Expanders

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Abstract. It has long been known that random regular graphs are with high probability good expanders. This was first established in the 1980s by Bollobás by directly calculating the probability that a set of vertices has small expansion and then applying the union bound.

In this paper we improve on this analysis by relying on a simple high-level observation: if a graph contains a set of vertices with small expansion then it must also contain such a set of vertices that is locally optimal, that is, a set whose expansion cannot be made smaller by exchanging a vertex from the set with one from the set’s complement. We show that the probability that a set of vertices satisfies this additional property is significantly smaller. Thus, after again applying the union bound, we obtain improved lower bounds on the expansion of random $\Delta$-regular graphs for $\Delta \geq 4$. In fact, the gains from this analysis increase as $\Delta$ grows, a fact we explain by extending our technique to general $\Delta$. Thus, in the end we obtain an improvement not only for some small special cases but on the general asymptotic bound on the expansion of $\Delta$-regular graphs given by Bollobás.

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1 Introduction

A graph is called an expander graph if for all partitions of its set of vertices into two sets the ratio of the number of edges with endpoints in both sets over the size of the smaller set is bounded from below by a constant independent of the graph size. Roughly speaking, an expander is a very well-connected graph and we are usually interested in expanders which are regular and sparse. Expander graphs have long been an object of intense study in discrete mathematics and theoretical computer science. Over time, they have been used in a wide number of applications, such as for example error-correcting codes [16], computational complexity [9,15], inapproximability for bounded-occurrence CSPs [14] and others too numerous to mention here (see [10]).

One of the most fundamental results in this area, established by Bollobás in 1988 [5], is that random $\Delta$-regular graphs are expanders with high probability. Thus, though the existence of some expander graph families had already been known [6,12], Bollobás managed to show that in fact almost all $\Delta$-regular graphs are expanders using surprisingly elementary methods (straightforward counting arguments, as opposed to the linear algebra techniques commonly used when dealing with expander graphs). Furthermore, Bollobás gave an asymptotic lower bound of $\frac{\Delta}{2} - \Theta(\sqrt{\Delta})$ on the expansion of random $\Delta$-regular graphs, and specific bounds for various small values of $\Delta$. Interestingly, the asymptotic bound was later found to be optimal up to the coefficient of $\sqrt{\Delta}$, as Alon showed that any $\Delta$-regular graph (not just random graphs) must have a bi-section with expansion at most $\frac{\Delta}{2} - \Theta(\sqrt{\Delta})$ [1].

The main objective of this paper is to revisit this classical result of Bollobás and to improve it through a more refined analysis. With this new analysis we will show higher lower bounds on the expansion of random $\Delta$-regular graphs.

Our results move in two main directions. First, we will obtain better lower bounds on the expansion of $\Delta$-regular graphs for various small values of $\Delta$. One of the reasons we are interested in this is because improvements in the currently known bounds could have applications in other areas, and in particular one example is the inapproximability of bounded occurrence CSPs. For instance, the fact that 6-regular graphs have expansion at least 1 (established in [5]) is invoked a number of times in this area (see [13,12]). It is still open whether $\Delta = 6$ is the smallest degree for which this is true, and resolving this question would be of immediate use in inapproximability reductions. Of course, beyond this, finding what is the expansion of random $\Delta$-regular graphs for small values of $\Delta$ is in itself a fundamental mathematical problem, intimately connected to a number of other basic graph-theoretic questions such as the size of the minimum bisection of random regular graphs. Unfortunately, little progress has been made since [5] and in most cases the bounds given by Bollobás are still the best known lower bounds, although some progress has been made in establishing corresponding upper bounds through results on the minimum bisection problem [13,7,18].

Second, we will use our refined analysis to also improve the asymptotic bound on the expansion as a function of $\Delta$, or more precisely, to show that the current bound is not tight. As mentioned, it is known that the correct value is $\frac{\Delta}{2} - \Theta(\sqrt{\Delta})$ but the coefficients of $\sqrt{\Delta}$ given by Bollobás in the lower bound ($\sqrt{\ln 2} \approx 0.83$) and Alon in the upper bound ($\frac{\sqrt{\Delta}}{16\sqrt{\ln 2}} \approx 0.13$) are relatively far apart. Again, this is a fundamental question given as an open problem in [1]. But furthermore, the main reason we would like to give such a general analysis is to show that the refinements we make in the analysis of small values of $\Delta$ are more than just an ad-hoc trick that improves some small special cases, but in fact our technique applies to all $\Delta$.

To explain the high-level idea of our method let us first recall the proof of the main result of [5]. The idea there is to calculate the probability that a certain fixed set of vertices of a random $\Delta$-regular graph has exactly $c$ edges connecting it to its complement. Clearly, as $c$ becomes smaller that probability also becomes smaller, so the trick here is to calculate the largest value of $c$ that still makes the probability $o(2^{-n})$. Afterwards, an application of the union bound ensures that with high probability no set has $c$ or fewer edges crossing the cut, giving a bound on the expansion. The best bound on $c$ is found by directly writing out the probability that exactly $c$ edges cross the cut and performing a tedious but straightforward calculation.

Since the probability that $c$ edges cross the cut is calculated exactly in [5] one might expect that the best place to look for an improvement may be in the application of the trivial union bound, which completely
ignores a very basic fact: The expansion ratios of two different sets of vertices are far from independent. Somewhat surprisingly, we give an argument that attempts to exploit this crucial fact while still relying on the union bound. The main idea is that if a set with small expansion exists, then there must exist a set with the same or smaller expansion which is locally optimal. A set is locally optimal in this context if exchanging a vertex from it with a vertex from its complement cannot decrease its expansion. Thus, to establish that a graph’s expansion is above a certain value it suffices to show that no locally optimal set exists with expansion below this value. Again, we want to find the maximum $c$ such that the probability that a fixed set of vertices is locally optimal and has $c$ edges leaving it is $o(2^{-n})$ and, as might be expected, this probability turns out to be significantly smaller than the probability that the set simply has $c$ edges leaving it, allowing us to end up with a larger $c$.

The main technical challenge arising here is to calculate this probability accurately. In the case of [5] it is a simple counting exercise to determine the number of configurations where exactly $c$ edges cross the cut obtaining a clean and relatively simple formula. In our case however, as we will argue, the local optimality condition imposes some constraints on the degrees of the vertices which make counting much more complex. Specifically, in a locally optimal set all vertices have the majority of their neighbors in their own set. Because of this complication a clean closed formula for the number of configurations that give a locally optimal set with $c$ edges coming out probably does not exist. Given that, our main technical effort is concentrated on determining the degree distribution of maximum probability. Using a (different kind of) local optimality argument we show that the number of neighbors each vertex has on the other side of the partition can be assumed to follow roughly a binomial distribution, with some undetermined parameters (Lemma 2).

Having arrived at this technical tool we then apply it to the problem at hand. First, as a warm-up we show that if one ignores the degree constraints it is not hard to calculate the parameters of the degree distribution. We thus, after some calculations, arrive at exactly the same bound given in [5], albeit through a rockier road and gaining some valuable insight in the kind of configurations that lead to bad cuts. Then we go on to exploit this insight combined with the local improvements idea to improve on the expansion lower bounds for specific values of $\Delta$. This leads to a set of optimization problems which we solve numerically to obtain our new results (note that the correctness of the bounds we give can however be verified by simple calculations). Our bounds improve on those given in [5] and for $\Delta \geq 4$ improve the state of the art (for cubic graphs the best result is still given in [11], but the complicated analysis there is restricted only to $\Delta = 3$). Finally, we show how our analysis can be applied for general $\Delta$ and that an improved asymptotic lower bound on the expansion of random $\Delta$-regular graphs can be obtained. The new coefficient of $\Delta$ we get offers only a small improvement over the one given in [5], so we do not expend much effort trying to calculate it exactly. Instead, for the sake of simplicity we simply show that a strictly smaller coefficient is achievable. Despite the lack of a large improvement in this constant, we believe there is still interest in this result, in that it indicates that the bound given by Bollobás is not the "right" bound, and also in that it shows that the local improvement method we employ here applies generally and not just for small values of $\Delta$.

2 Definitions and Notation

We use standard graph terminology and will be dealing with regular graphs only. We use $\Delta$ to denote the degree, $n$ to denote the number of vertices of a graph. When we talk about random regular graphs we mean a graph constructed via the following process. Take $\Delta n$ vertices (assume $\Delta n$ is even), numbered $0, \ldots, \Delta n - 1$. First, select a perfect matching on these vertices uniformly at random among all perfect matchings. Then, for all $k \in \{0, \ldots, n\}$ merge the vertices $k\Delta, k\Delta + 1, \ldots, (k + 1)\Delta - 1$ into a single vertex of degree $\Delta$. Observe that though this process may produce a multi-graph, the probability that it produces a simple graph is bounded away from 0 as $n \to \infty$ and all simple $\Delta$-regular graphs are equiprobable ([5]). The probability space therefore consists of $(\Delta n - 1)/(\Delta n - 3) \ldots$ matchings. We denote this product by $(\Delta n)!!$ (and generally we denote by $n!!$ the product of all odd positive integers which are less than or equal to $n$). We use $\log n$ to denote the binary logarithm of $n$ and $\ln n$ to denote the natural logarithm of $n$. 

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Let $S \subseteq V$ be a set of vertices with $|S| \leq n/2$. Let $c(S)$ denote the number of edges with exactly one endpoint in $S$. The edge expansion of $S$ is defined as $\frac{c(S)}{|S|}$. The edge expansion of $G$ is defined as $i(G) = \min_{|S| \leq n/2} \frac{c(S)}{|S|}$.

Given a partition of $V$ into $S, V \setminus S$ we define the out-degree of a vertex as the number of neighbors the vertex has on the other side of the partition. We will characterize cuts mainly by the number of vertices of each side of the partition. We will characterize cuts mainly by the number of vertices of each side of the partition. We will characterize cuts mainly by the number of vertices of each side of the partition. This gives the second factor. The third and fourth factors are similar for a specific set of vertices. Let $\Delta$ be the set of vertices in the original configuration where we take a random perfect matching. Therefore, if we re-prove the main theorem of [5], giving the additional insight that the distribution of out-degrees in the distribution of out-degrees in the main result gives a general form for the distribution of out-degrees of out-degrees occurs for

In this section we establish the main tools we will need in the analysis of the expansion of random regular graphs. The main result gives a general form for the distribution of out-degrees $s$ that has maximum probability, conditioned on the size of the cut (Lemma 2). In the end of this section we apply this tool to obtain improved results. However, for now we will work only with the basic case where there is no constraint on the out-degrees and $d = d' = \Delta$. As a result all sums given in this section can be read to range from $i = 0$ to $\Delta$.

We are given a random $\Delta$-regular graph $G(V, E)$. We want to examine the asymptotic (in $n$) behavior of the expansion of such $\Delta$-regular graphs. In other words, for each fixed $\Delta$ we would like to have a lower bound on a random graph’s expansion which holds with high probability for sufficiently large graphs. For concreteness, say we always assume that $n > \Delta^{20}$. To avoid inessential difficulties let us also assume that $n$ is even.

Fix a set of vertices $S$. Our first step is to write down exactly the probability that a certain configuration of out-degrees occurs for $S$ and $V \setminus S$ and that it produces a cut of size $c(S) = c$. More precisely, we will calculate the probability that, given $s = (s_0, s_1, \ldots, s_\Delta)$ and $s' = (s_0, s_1, \ldots, s_\Delta)$ such that $\sum_i s_i = |S|$, $\sum_i s'_i = |V \setminus S|$ and $\sum_i is_i = \sum_i is'_i = c$ the set $S$ has exactly $c$ edges coming out, and these are distributed in such a way that for all $i$ there are $s_i$ vertices in $S$ with $i$ neighbors in $V \setminus S$ and $s'_i$ vertices in $V \setminus S$ with $i$ neighbors in $S$. That probability is

$$P(s, s') = \frac{|S|!}{s_0! s_1! \ldots s_\Delta!} \prod_{i=0}^\Delta \left(\frac{\Delta}{i}\right)^{s_i} \frac{|V \setminus S|!}{s'_0! s'_1! \ldots s'_\Delta!} \prod_{i=0}^\Delta \left(\frac{\Delta}{i}\right)^{s'_i} c^\left(\Delta|S| - c\right)!! (\Delta|V \setminus S| - c)!! \left(\Delta n\right)!!$$

Let us explain this. The first factor comes from deciding how to partition the set $S$ into groups of vertices with the prescribed number of neighbors in $V \setminus S$. Recall that each vertex in our graph is derived from a group of $\Delta$ vertices in the original configuration where we take a random perfect matching. Therefore, if we decide that a vertex has $i$ edges to vertices in $V \setminus S$ we must decide which of the $\Delta$ vertices of the group are the endpoints of these edges. This gives the second factor. The third and fourth factors are similar for $V \setminus S$. The $c!$ comes from deciding how to match the vertices from each side that have neighbors on the other. Finally, we take the product of the total number of matchings for the rest of the vertices of $S$ and $V \setminus S$. This is divided by the total number of matchings overall. Recall now that we have written $P(s, s')$ for a specific set of vertices $S$. By union bound, the probability that some ”bad” set of vertices exists, such that its size is $u$, the cut has $c$ edges and the out-degree distributions of the set and the complement are described by $s, s'$ is at most $\binom{n}{u} P(s, s')$.

Let $U(u, c)$ be the set of all pairs $(s, s')$ such that $\sum_i s_i = u$, $\sum_i (s_i + s'_i) = n$ and $\sum_i is_i = \sum_i is'_i = c$. In other words, $U$ is the set of all possible vectors describing a configuration where $|S| = u$ and there are $c$ edges crossing the cut. Suppose we could show that
\[
\lim_{n \to \infty} \sum_{u=0}^{n/2} \sum_{c=0}^{\cdot} \sum_{(s, s') \in \mathcal{U}(u, c)} \binom{n}{u} P(s, s') = 0
\]  

for some parameter \( \phi \) depending on \( \Delta \). Then we would have that with high probability a random \( \Delta \)-regular graph has expansion at least \( \phi \) from a simple application of the union bound. Our goal is to determine the largest possible parameter \( \phi(\Delta) \) such that this holds, and recall that by the results of [3] and [4] we expect the correct answer to be \( \frac{\Delta}{2} - \Theta(\sqrt{\Delta}) \).

Observe that the above sum has at most \( \Delta n^{2\Delta+2} \) terms, since \( u \leq n, c \leq \Delta u \) and there are at most \( n^3 \) different vectors \( s \) such that \( |s| = u \) for all \( u \). Let \( s_M, s'_M \) be such that \( P(s_M, s'_M) \) is maximum among all \( (s, s') \in \mathcal{U}(u, c) \) for all \( u \leq n/2 \) and \( c \leq \phi u \). It is sufficient for us to have that \( \binom{n}{u} P(s_M, s'_M) = o(\Delta^{-1}n^{-2\Delta-2}) \). In fact, we will have the much stronger condition \( \binom{n}{u} P(s_M, s'_M) \leq \alpha^{-\Delta n} \) for some \( \alpha > 1 \), or in other words we will show that the probability that a bad set exists is exponentially small.

We have therefore reduced the question to that of determining a bad configuration with maximum probability and upper-bounding that probability. We would like to look for the vectors \( s, s' \), while avoiding some extreme cases. We can do that by using the fact that small perturbations in these vectors do not affect the probability too much. This is shown in the following lemma which essentially shows that changing some of the coordinates of a configuration vector \((s, s')\) has a small impact on the probability.

**Lemma 1.** If the configuration vectors \((s, s'), (p, p')\) satisfy \( \max_i |s_i - p_i| \leq n^{\frac{3}{2}} \) and \( \max_i |s'_i - p'_i| \leq n^{\frac{3}{2}} \) we have \( P(s, s') \leq 2^{\Theta(n)} P(p, p') \).

Lemma 1 allows us to neglect some annoying cases, such as for example sets with too small size \( o(n^{\frac{3}{2}}) \). We now only need to find the configuration of maximum probability among configurations where \( s_i > n^{\frac{3}{2}} \) for all \( i \) and make sure that the probability is \( 2^{-\Theta(\Delta n)} \). Then, by Lemma 1 we will know that looking for the maximum through the space of all configuration vectors would have made no difference.

Now, given fixed values of \( u, c \) let us try to find the configuration vector \( s \) that gives maximum probability. It is not hard to see that we are trying to maximize the following function:

\[
F(s) = \prod_{i=0}^{\Delta} \frac{(\Delta + s_i)}{s_i!} \frac{1}{s_U} s_1! \ldots s_\Delta!
\]

In other words, we want to determine the distribution of the out-degrees of \( u \) vertices so that \( \sum is_i = c \) and \( F(s) \) is maximum. Note that the problem for the vector \( s' \) is essentially identical.

**Lemma 2.** Let \( s^M \) be a configuration vector such that \( \sum is_i = c \), \( \sum is^M_i = c \), for all \( i, s^M_i > n^{\frac{3}{2}} \) and \( F(s^M) \) is maximum among all vectors that satisfy the previous conditions. Then \( |s^M_i - s^M_i (s^M_i) \frac{(\Delta)}{s_i!} (\Delta + s_i)\frac{1}{s_i!} | = o(n^{\frac{3}{2}}) \) for all \( i, 0 \leq i \leq \Delta \).

**Proof.** We define \( p^i \) for \( i \geq 2 \) to be the vector with \( p^i_0 = p^i_1 = 1, p^i_1 = p^i_{-1} = -1 \) and all other coordinates equal to 0. Intuitively, \( p^i \) is a small perturbation vector, which does not change either the size of \( S \) or the size of the cut. Since \( s^M \) maximizes the function \( F \), it must be the case that both \( s^M + p^i \) and \( s^M - p^i \) give smaller values. So we should have \( \frac{F(s^M)}{F(s^M + p^i)} \geq 1 \) and also \( \frac{F(s^M)}{F(s^M - p^i)} \geq 1 \).

The first inequality gives \( \frac{(s^M_i + 1)(s^M_{i+1})}{s^M_i s^M_{i-1}} \frac{\Delta(s^M_i)}{(s^M_i)!} \geq 1 \). The second inequality gives \( \frac{(s^M_i + 1)(s^M_{i-1})}{s^M_i s^M_{i+1}} \frac{\Delta(s^M_i)}{(s^M_i)!} \geq 1 \).

Observe that \( s^M_i s^M_{i+1} = O(\Delta^2) \). Also, \( s^M_i s^M_{i-1} = O(\frac{1}{\sqrt{n}}) \) because all coordinates of \( s^M \) are at least \( n^{\frac{3}{2}} \). Thus, the first inequality gives \( \frac{s^M_i s^M_{i+1}}{s^M_i s^M_{i-1}} \frac{\Delta(s^M_i)}{(s^M_i)!} \geq 1 - O(\frac{\Delta^3}{\sqrt{n}}) \). The second gives \( \frac{s^M_i s^M_{i-1}}{s^M_i s^M_{i+1}} \frac{\Delta(s^M_i)}{(s^M_i)!} \geq 1 - O(\frac{\Delta^3}{\sqrt{n}}) \). Together
these imply that \( \frac{s_i^M}{s_{i-1}^M} \frac{\Delta(s_i)}{(\frac{\Delta}{i})} - 1 = O(\frac{\Delta^2}{n}) \). Since the right-hand side is very small, intuitively what we will do is to assume it is essentially 0 and solve the recurrence relation.

More precisely, let \( p \) be the vector defined as follows: \( p_0 = s_0^M, p_1 = s_1^M \) and \( p_i = \frac{p_{i-1}}{p_0} \frac{\Delta_i}{\Delta(i-1)} \) for all \( i \geq 2 \). It is then not hard to see (or verify by induction) that \( p_i = s_0^M \left( \frac{\Delta_i}{\Delta_s} \right)^i \frac{\Delta_i}{i} \).

We have that
\[
\left( 1 - O\left( \frac{\Delta^2}{\sqrt{n}} \right) \right) \frac{s_i^M}{s_{i-1}^M} \leq p_i \leq \left( 1 + O\left( \frac{\Delta^2}{\sqrt{n}} \right) \right) \frac{s_i^M}{s_{i-1}^M} \frac{\Delta_i}{\Delta(i-1)}
\]

From this we get that \( (1 - O\left( \frac{\Delta^2}{\sqrt{n}} \right)) p_2 \leq s_2^M \leq (1 + O\left( \frac{\Delta^2}{\sqrt{n}} \right)) p_2 \). By induction we then have \( (1 - O\left( \frac{\Delta^2}{\sqrt{n}} \right))^i p_i \leq s_i^M \leq (1 + O\left( \frac{\Delta^2}{\sqrt{n}} \right))^i p_i \). Using the fact that \( \Delta \leq n^{1/3} \) we have \( ((1 + O\left( \frac{\Delta^2}{\sqrt{n}} \right)) \Delta = 1 + O\left( \frac{\Delta^2}{\sqrt{n}} \right) \).

Therefore, for all \( i \) we have \( |s_i^M - p_i| \leq p_i \cdot O\left( \frac{\Delta^2}{\sqrt{n}} \right) = O(\Delta^3 \sqrt{n}) = o(n^\frac{2}{3}). \)

Informally, as a result of Lemma 2 we can now assume that the vector describing the configuration of maximum probability has \( s_i = s_0 \left( \frac{\Delta_i}{\Delta_s} \right)^i \frac{\Delta_i}{i} \), since for sufficiently large \( n \) we know that the maximum has an edit distance from this vector smaller than the one allowed by Lemma 1. To ease notation slightly we introduce the parameters \( \beta, \gamma \) defined so that \( s_0 = \beta \sum_i s_i \) and \( s_1 = \gamma \Delta s_0 \). We will also use \( \beta', \gamma' \) similarly as parameters corresponding to the vector \( s' \).

Going back to equation (1) we would like to find the maximum probability that is achieved for bisections, that is, \( u = n/2 \). This is intuitively unsurprising, and it is stated by the following lemma.

**Lemma 3.** Let \( b, b' \) be two vectors such that \( \sum_i b_i = \sum_i b'_i = n/2, \sum_i i b_i = \sum_i i b'_i = c \) and these two vectors maximize the quantity \( \binom{n}{u} P(b, b') \) among all vectors that satisfy the previous two conditions. Then for all vectors \( s, s' \) with \( \sum_i s_i = u \leq n/2, \sum_i is_i = \sum_i is'_i = c \) we have \( \frac{n}{u} P(s, s') \leq 2^{o(n)} \binom{n}{u} P(b, b') \).

Thanks to Lemma 3 we are only interested in partitions of size exactly \( n/2 \), since these give the maximum probability of achieving a cut of size \( c \) (perhaps modulo a \( 2^{o(n)} \) factor which will prove insignificant). Therefore, for any expansion factor \( \phi \) the probability of finding a cut with expansion less than \( \phi \) is maximized when that set has size exactly \( n/2 \).

It is not hard to see that if the partition of the vertices is a bisection then the distribution of out-degrees that gives maximum probability is the same for the two parts. In other words, if \( s, s' \) are the two vectors that give maximum probability for a cut of size \( c \) and \( \sum_i s_i = \sum_i s'_i \) then it must be that \( s = s' \). Using this, the probability that a bisection which cuts \( c \) edges exists is upper bounded by
\[
P_B(s) \leq \binom{n}{n/2} \left( \frac{n/2}{n/2} \right)! \left( \frac{n/2}{s_0!s_1! \ldots s_n!} \right)^2 \prod_{i=0}^{\Delta} \frac{\Delta_i}{i} 2^{s_i} \frac{c! (\Delta_i - c)!}{(\Delta_n)!} \frac{2^{s_i} c! (\Delta_i - c)!}{(\Delta_n)!}
\]

where we have used the fact that \( (n!!)^2 < n! \). Now we would like to find the range of values of \( c \) for which the above quantity is very small, that is \( 2^{-\Theta(\Delta_n)} \). To ease notation, we write \( c = (1 - \eta) \frac{\Delta_n}{4} \), where \( \eta > 0 \), and look for the correct range of \( \eta \).

Using the facts that for large \( n \) we have \( \frac{n}{e} n < n! < \binom{n}{n/2} \) \( \log (n/2) \), \( \binom{n}{n/2} < 2^n \) and \( n!! > \left( \frac{n}{e} \right)^n \) we get that
\[
\log P_B(s) \leq n + n \log \frac{n}{2e} - 2 \sum_{i=0}^{\Delta} i \log \frac{s_i}{e(\frac{\Delta}{i})} + \frac{(1 - \eta) \Delta_n}{4} \log \frac{(1 - \eta) \Delta_n}{4} \frac{2n}{\Delta_n} \log \frac{\Delta_n}{e} + o(n)
\]
As mentioned, assuming that \( s_i = \beta \gamma^i (\frac{\Delta}{i+1}) \frac{n}{2} \) cannot affect the probability by more than \( 2^{o(n)} \), so with some calculations and using the fact that \( \sum s_i = \frac{n}{2} \) and \( \sum i s_i = (1 - \eta) \frac{\Delta n}{2} \), the upper bound becomes

\[
\log P_B \leq n - n \log \beta - (1 - \eta) \frac{\Delta n}{2} \log \gamma - \Delta n \\
+ \frac{(1 + \eta) \Delta n}{4} \log(1 + \eta) + \frac{(1 - \eta) \Delta n}{4} \log(1 - \eta) + o(n)
\]  

(2)

What remains now is to find the values of \( \beta, \gamma \) that give the maximum probability. Thankfully, because \( d = d' = \Delta \) we can use the binomial theorem to find a clean solution to this problem as follows. Recall that \( \sum_i s_i = \sum_i \beta \gamma^i (\frac{\Delta}{i+1}) \frac{n}{2} = \beta (1 + \gamma)^{\Delta} \frac{n}{2} = \frac{n}{2} \). Also, \( \sum_i i s_i = \sum_i i \beta \gamma^i (\frac{\Delta}{i+1}) \frac{n}{2} = \sum_i \Delta \beta \gamma^i (\frac{\Delta - 1}{i}) \frac{n}{2} = \beta \gamma (1 + \gamma)^{\Delta - 1} \frac{n}{2} = (1 - \eta) \frac{\Delta n}{4} \). Dividing these two equations gives \( \gamma = \frac{1 - \eta}{1 + \eta} \) and then \( \beta = (\frac{1 + \eta}{2})^\Delta \). Plugging these values into the upper bound for \( P_B \) we get

\[
\frac{4 \log P_B}{\Delta n} \leq \frac{4}{\Delta} - (1 - \eta) \log(1 - \eta) - (1 + \eta) \log(1 + \eta) + o(1/\Delta)
\]

The probability is therefore \( 2^{-\Omega(n)} \) whenever the right hand side is less than some arbitrarily small negative constant.

**Theorem 1.** For each \( \Delta \), if \( \eta > 0 \) satisfies \( (1 - \eta) \log(1 - \eta) + (1 + \eta) \log(1 + \eta) > \frac{4}{\Delta} \) then, for all sufficiently large \( n \), random \( n \)-vertex \( \Delta \)-regular graphs have expansion at least \( (1 - \eta) \frac{\Delta}{2} \) with high probability.

Thus, we have obtained again the main result of [5], albeit through a much more winding road. One bonus is that through Lemma 2 we now have a better understanding of the out-degree distribution of maximum probability. Informally, one way to interpret this is to observe that by Lemma 2 the fraction of vertices that have out-degree \( i \) is proportional to the probability that a binomial random variable with parameters \( (\Delta, \frac{\Delta}{1 + \Delta}) \) takes the value \( i \). This intuition will come in handy in the asymptotic analysis.

### 4 Local Improvements

In this section we introduce the idea of local improvements in the analysis. As mentioned, the main idea is to bound the probability that we find a partition that not only has small expansion, but also is locally optimal, in that flipping a pair of vertices cannot make the expansion smaller. The consequence of this property that we will rely on is that for a locally optimal set all vertices have out-degree at most (roughly) \( \frac{\Delta}{2} \). The analysis of the previous section will come in handy now, since the only difference is that \( d \) and \( d' \) are now smaller. Essentially all of it goes through unchanged, up to the point where we used the binomial theorem to find the values of \( \beta, \gamma \). Unfortunately, finding a clean formula for the values of \( \beta, \gamma \) in our case now becomes a much harder task. In this section we will be dealing with specific values of \( \Delta \), and we will therefore be able to calculate \( \beta, \gamma \) and the minimum \( \eta \) for which the graph is whp an \( (1 - \eta) \frac{\Delta}{2} \) expander numerically. In the next section we will discuss ways to bound the values of \( \beta, \gamma \) to obtain an improved asymptotic bound as \( \Delta \) tends to infinity.

We will say that a set \( S \subseteq V \), \( |S| \leq n/2 \) is locally improvable if there exist \( u \in S, v \in V \setminus S \) such that \( S \setminus \{u\} \cup \{v\} \) has a strictly smaller cut. If \( S \) is not locally improvable we will say that it is locally optimal. Recall that \( d \) denotes the maximum \( i \) such that \( s_i > 0 \) and \( d' \) the maximum \( i \) such that \( s'_i > 0 \). Observe that whenever there exists a set with expansion \( \phi \) there must also exist a locally optimal set with the same size and expansion \( \leq \phi \).

**Lemma 4.** If \( S \) is locally optimal then \( d + d' \leq \Delta + 1 \). Therefore, if \( S \) is locally optimal then \( \min\{d, d'\} \leq \lfloor \frac{\Delta}{2} \rfloor \).
Proof. Suppose for contradiction that $d + d' \geq \Delta + 2$. Let $u \in S$ have $d$ neighbors in $V \setminus S$ and $v \in V \setminus S$ have $d'$ neighbors in $S$. We swap $u$ and $v$. First, suppose that $u, v$ are not connected. The size of the new cut is $c - d - d' + \Delta - d + \Delta - d' = c + 2\Delta - 2(d + d') < c$ contradicting the local optimality of $S$. If $u, v$ are connected then the new cut has size $c - (d - 1) - (d' - 1) + \Delta - d + \Delta - d' < c$, again contradicting the local optimality of $S$. \hfill \Box

In fact, we can say slightly more than what Lemma 3 states. Notice that the thorny case is when the two vertices of maximum out-degrees happen to be connected. If there are $u \in S, v \in V \setminus S$ with out-degrees $d, d'$ respectively such that $u, v$ are not connected then with the same argument we can say that if $S$ is locally optimal then $d + d' \leq \Delta$. Now, if there are at least $\Delta + 1$ different vertices in $S$ with out-degree $d$, then we can always find such a pair $u, v$ and conclude that for a locally optimal set $d + d' \leq \Delta$. Furthermore, if there are at most $\Delta$ vertices of out-degree $d$ in $S$, then since $\Delta \leq n^{1/2} = o(n^{1/2})$ we can use Lemma 1 to say that the probability is essentially unchanged if we assume that there are no vertices of out-degree $d$, therefore the maximum out-degree in $S$ becomes $d - 1$. As a result, we will from now on assume that $d + d' \leq \Delta$ for locally optimal sets.

As mentioned, the analysis of the previous section goes through: even if vertices in $S$ have maximum out-degree $d$ and vertices in $V \setminus S$ have maximum out-degree $d'$ the probability of a certain configuration is still robust to small perturbations (Lemma 1) and the most likely degree distribution is still of the form $s_i = \beta \gamma_i(\Delta)/|S|$ (Lemma 2). Also, bisections are again the interesting case, so we may assume that $|S| = \frac{n}{2}$.

Revisiting the calculation that brought us to inequality (2) we now get:

$$
\frac{\log P_B}{n} \leq 1 - \frac{1}{2} \log \beta - \frac{1}{2} \log \beta' - (1 - \eta) \frac{\Delta}{4} \log \gamma - (1 - \eta) \frac{\Delta}{4} \log \gamma' - \Delta + \frac{(1 + \eta)\Delta}{4} \log(1 + \eta) + \frac{(1 - \eta)\Delta}{4} \log(1 - \eta) + o(1)
$$

(3)

Furthermore, we have

$$
\sum_{i=0}^{d} \beta \gamma_i(\Delta) = \sum_{i=0}^{d'} \beta' \gamma_i(\Delta) = 1
$$

(4)

$$
\sum_{i=1}^{d} \beta i \gamma_i(\Delta) = \sum_{i=1}^{d'} \beta' i \gamma_i(\Delta) = \frac{1 - \eta}{2} \Delta
$$

(5)

Recall that in the previous section where we had $d = d' = \Delta$ we were able to use the binomial theorem to simplify the above equations and solve for $\beta, \gamma, \beta', \gamma'$ as functions of $\eta, \Delta$. Plugging the result back into (2) and asking that the right hand side is negative led to Theorem 1. Here, it is not clear how to do something similar.

However, if $\Delta, d, d'$ have some known fixed values, we can do the following: pick a candidate value for $\eta$ and solve (numerically) equations (13) for $\beta, \gamma, \beta', \gamma'$. Plug the solutions into (3) and check if the right hand side is negative. In such a case we can conclude that whp a random $\Delta$-regular graph will not have a bisection with maximum out-degrees $d, d'$ and $(1 - \eta \Delta)$ edges crossing the cut. Given that in a locally optimal set $d + d' \leq \Delta$ and allowing the maximum out-degree to increase can only increase the probability, we are essentially looking for the minimum $\eta$ such that the right hand side of (3) is negative for all possible pairs $d, d'$ with $d + d' = \Delta$.

Performing this procedure for various small values of $\Delta$ we get the results listed in Table 1. Despite the fact that (as with the bound given in [3]) we need to rely on some numerical polynomial equation solver to find these numbers, their correctness can be verified relatively easily. Specifically, these bounds can be verified by checking that $\sum_{i=0}^{d} \beta \gamma_i(\Delta) = 1$, $\sum_{i=1}^{d} \beta i \gamma_i(\Delta) = \frac{1 - \eta}{2} \Delta$, the same conditions apply for $\beta', \gamma'$ and the logarithm of the probability (as given in (3)) evaluates to a negative number.
Table 1. Summary of numerically obtained lower bounds for various values of $\Delta$. The last column contains the numbers that follow by numerically solving the condition given in [5]. Due to space constraints we only list the final results for $\Delta \geq 12$.

Observe that the improvements we obtain, though quite modest (in the order of $10^{-2}$ for $\Delta \leq 10$ and $10^{-1}$ for somewhat larger $\Delta$) seem to grow with $\Delta$. This is a fact explained also by the results of the next section.

5 Asymptotic Bound

In this section we will extend the local improvement analysis to general $\Delta$ and show an asymptotic (in $\Delta$) lower bound on the expansion of random $\Delta$-regular graphs. As mentioned, this is interesting in part because it confirms the findings of the previous section that the gains from this analysis increase with $\Delta$.

Beyond this however, determining a tight asymptotic bound on this quantity is a fundamental mathematical question. Recall that the lower bound given by Bollobás is $\Delta / 2 - \sqrt{\Delta \ln 2}$. It is very natural to ask where the $\sqrt{\ln 2}$ comes from and whether it is the "right" constant here. One intuitive explanation might be the following: consider a bisection of the set of vertices. Each edge of the random graph has probability (essentially) $1/2$ of crossing the cut, therefore the expected number of edges crossing the cut is $\mu = \Delta n / 4$. If the edges were independent then the size of the cut would follow a binomial distribution with $\sigma^2 = \Delta n / 8$. What is the maximum size $c$ such that we can guarantee that a cut of this size has probability $o(2^{-n})$ of occurring? If we approximate the binomial distribution by a normal distribution we get that $\frac{1}{2\sqrt{\pi} \sigma} e^{-\frac{1}{2} \left( \frac{c - \mu}{\sigma} \right)^2}$ must be $o(2^{-n})$. Solving this gives $c = \frac{\mu}{2} - \sqrt{\Delta \ln 2}$. In other words, the analysis of [5] shows that the true probability that a fixed set of vertices has $c$ edges coming out can be approximated very well by assuming that edges are independent, an interesting and natural result.

Nevertheless, as mentioned the weakness in the analysis of [5] is the union bound, which would only be tight if the probability that two different sets of vertices are expanding were independent. In some way the local improvement idea plays on this non-independence and, though we make only very small progress...
towards bridging the gap between the known lower and upper bound, our results establish in an indirect way that $\sqrt{\ln 2}$ is not the right constant and that there is something significant lost by the union bound in the analysis of [5].

To simplify presentation, in this section we define a locally optimal partition as one where $\min\{d, d'\} \leq \lceil \frac{\Delta}{2} \rceil$. This is a slightly weaker definition than what we use in the previous section. Assume without loss of generality that $d \leq d'$. Following similar reasoning as before, if there exists a non-expanding set there must exist one with $d \leq \lceil \frac{\Delta}{2} \rceil$, and since allowing $d, d'$ to be larger can only increase the probability we can assume that $d = \frac{\Delta}{4}$ and $d' = \Delta$. Informally, we are applying the local improvement argument only to one side of the partition to simplify things. In particular, now we don’t need to check various combinations of $d, d'$, or to prove that the maximum probability is achieved when $d = d'$ (which would agree with the numerical results of the previous section but would likely be complicated to prove directly). Finally, as usual assume that we are dealing with bisections, since these are the most interesting case and we will assume that $\eta < \frac{2\sqrt{\ln 2}}{\sqrt{\Delta}}$, since otherwise we already know from [6] that the probability of such a set tends to 0.

Having set $d' = \Delta$ we can again use the binomial theorem and equations (4,5) to get $\gamma' = \frac{1-\eta}{1+\eta}$ and $\beta' = \left(\frac{1+\eta}{2}\right)^{\Delta}$. Inserting these into (6) gives

$$\log \frac{P_B}{n} \leq 1 - \frac{\Delta}{2} - \frac{1}{2} \log \beta - (1 - \eta) \frac{\Delta}{4} \log \gamma + o(1)$$

(6)

The question therefore now becomes to determine $\beta, \gamma$ and the smallest possible $\eta$ so that the right hand side of (6) is negative. Going back to the left hand side of equations (4,5) we observe that

$$\sum_{i=0}^{d} \beta \gamma^i \left(\frac{\Delta}{i}\right) = \beta(\gamma + 1)^\Delta \sum_{i=0}^{d} \left(\frac{\gamma}{\gamma + 1}\right)^i \left(\frac{1}{\gamma + 1}\right)^{\Delta-i} \left(\frac{\Delta}{i}\right)$$

We also have that

$$\sum_{i=1}^{d} \beta_i \gamma^i \left(\frac{\Delta}{i}\right) = \beta \Delta \sum_{i=1}^{d} \gamma^i \left(\frac{\Delta-1}{i-1}\right) = \beta \gamma \Delta (\gamma + 1)^{\Delta-1} \sum_{i=0}^{d-1} \left(\frac{\gamma}{\gamma + 1}\right)^i \left(\frac{1}{\gamma + 1}\right)^{\Delta-i-1} \left(\frac{\Delta-1}{i}\right)$$

where we used the fact that $\left(\frac{\Delta}{i}\right) = \Delta \binom{\Delta-1}{i-1}$.

The main intuition we are going to use now is that the two sums can be written more cleanly as probabilities using the binomial distribution. Specifically, let $P_1$ be defined as $P_1 = \Pr[B(\Delta, \frac{\gamma}{\gamma + 1}) \leq d]$, where by $B(n, p)$ we denote a random variable that follows the binomial distribution with parameters $n, p$. Also, let $P_2$ be defined as $P_2 = \Pr[B(\Delta - 1, \frac{\gamma}{\gamma + 1}) \leq d - 1]$. Informally, as $\Delta$ tends to infinity we expect $P_1$ and $P_2$ to become almost equal.

Now, with the help of the above equations (4,5) become

$$\beta(\gamma + 1)^\Delta P_1 = 1$$

$$\beta \gamma (\gamma + 1)^{\Delta-1} P_2 = \frac{1-\eta}{2}$$

(7)

(8)

Again, we will divide these two to get an equation for $\gamma$. To do this we would like to establish a relationship between $P_1$ and $P_2$. Let $P_3 = \Pr[B(\Delta, \frac{\gamma}{\gamma + 1}) = d]$.

Lemma 5. Let $P_1, P_2, P_3$ as defined above. Then $P_1 = P_2 + \frac{\Delta - d}{\Delta} P_3$.

To ease notation slightly let $\theta = \frac{\Delta - d}{\Delta} = \frac{P_3}{P_2}$. Dividing equations (7,8) we have $\frac{2+1}{1-\eta} = \frac{2}{\gamma}$. Solving this for $\gamma$ gives $\gamma = \frac{1-\eta}{1+\eta-2\theta}$. From this we get $\beta = \left(\frac{1+\eta-2\theta}{2}\right)^{\Delta} \frac{\Delta}{P_1}$. Plugging these into (6) we get
In this paper we gave improved lower bounds on the expansion of random \( \Delta \)-regular graphs. We also showed that the asymptotic lower bound given by Bollobás is not tight. Perhaps more importantly, we gave an alternative analysis of the result of [5] incorporating the distribution of out-degrees and showing that the bound for \( \Delta \) is strictly smaller than the one given in [5].

Conclusion

In this paper we gave improved lower bounds on the expansion of random \( \Delta \)-regular graphs for small values of \( \Delta \). We also showed that the asymptotic lower bound given by Bollobás is not tight. Perhaps more importantly, we gave an alternative analysis of the result of [5] incorporating the distribution of out-degrees and showing a general form. We believe that Lemma 2 could prove to be useful in establishing other properties of random regular graphs as well.

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A Omitted Proofs

A.1 Proof of Lemma 1

Proof. We want to upper bound $\frac{P(s, s')}{P(p, p')}$. Since $|s_i - p_i| < n^\frac{3}{2}$ we have $\frac{m_i}{s_i} < 2O(n^\frac{3}{2} \log n)$. There are $2\Delta$ factors of this form, so their total contribution is $2O(\Delta n^\frac{3}{2} \log n) = 2^o(n)$. There are also $2\Delta$ factors of the form $\binom{\Delta}{s_i-p_i}$, each of which contributes at most $2^{\Delta^2 n^\frac{3}{2}}$. Thus, their total contribution is at most $2\Delta^2 n^\frac{3}{2} = 2^o(n)$, using that $\Delta \leq n^\frac{3}{4}$. Similarly, the change in the remaining factors is also upper-bounded by $2\Delta^2 n^\frac{3}{2} = 2^o(n)$, since the number of edges crossing the cut cannot change by more than $\Delta^2 n^\frac{3}{2}$. □

A.2 Proof of Lemma 3

Proof. Consider a pair of vectors $s, s'$ as described above. We can assume without loss of generality that $s_i = \beta^\gamma_i \binom{\Delta}{i} u$ and similarly for $s_i' = \beta'^\gamma_i \binom{\Delta}{i}(n-u)$, since by Lemma 2 and Lemma 1 the actual maximum cannot differ from the maximum achieved by such vectors by a factor of more than $2^o(n)$.

First, assume that $\sum_i s_i = u < n/2 < \sum_i s_i'$. Therefore, we have

$$\sum_i \beta^\gamma_i \binom{\Delta}{i} u < \sum_i \beta'^\gamma_i \binom{\Delta}{i}(n-u)$$

$$\sum_i i\beta^\gamma_i \binom{\Delta}{i} u = \sum_i i\beta'^\gamma_i \binom{\Delta}{i}(n-u)$$

From these two we get that $\beta u < \beta'(n-u)$ or $s_0 < s_0'$. Consider now the configuration that we can obtain from $s, s'$ by decreasing $s_0'$ by one and increasing $s_0$ by one, thus increasing $u$. We claim that this new configuration achieves a value of $\sum_i \beta_i \binom{\Delta}{i} u$ that is at least as high. Indeed, because $\binom{n}{u}$ simplifies with the factors $|S|! = u!$ and $|V \setminus S|! = (n-u)!$ into $n!$, the only factors affected by this change are $\frac{1}{s_0 s_0'}$. But since $s_0 < s_0'$ this product does not decrease by the change and we have a configuration with larger $u$. Repeating this argument leads to $u = n/2$. □

A.3 Proof of Lemma 5

Proof. From the definitions of $P_1, P_2$ it is not hard to see that $P_1 = P_2 + \Pr[B(\Delta - 1, \frac{\Delta}{\gamma + 1}) = d - 1] \cdot \frac{1}{\gamma + 1}$. Intuitively, we will get at most $d$ successes in $\Delta$ trials if we either get at most $d-1$ in the first $\Delta - 1$, or we get exactly $d$ in the first $\Delta - 1$ and fail in the last.

But now $\Pr[B(\Delta - 1, \frac{\Delta}{\gamma + 1}) = d - 1] \cdot \frac{1}{\gamma + 1} = \binom{\Delta-1}{d} \frac{\Delta^d}{d} (\frac{\Delta}{\gamma + 1})^d = \frac{\Delta^d}{d} (\frac{\Delta}{\gamma + 1})^d$, where we have used the fact that $\binom{\Delta-1}{d} = \frac{\Delta^d}{d} (\frac{\Delta}{d})$. □