Ulam stability of an additive-quadratic functional equation in F-space and quasi-Banach spaces

Linlin Fu\textsuperscript{1}, Qi Liu\textsuperscript{1}\textsuperscript{*}, Yongjin Li\textsuperscript{1}

\textsuperscript{1} Department of Mathematics, Sun Yat-sen University, Guangzhou 510275, China

Abstract By adopting the direct method and fixed point method, we prove that the Hyers-Ulam stability of the following additive-quadratic functional equation

$$f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(x, w) = 0$$  \hspace{1cm} (1)

in \( \beta \)-homogeneous \( F \)-spaces and quasi-Banach spaces. There are some differences that we consider the target space with the \( \beta \)-homogeneous norm and quasi-norm. Overcoming the \( \beta \)-homogeneous norm and quasi-norm bottlenecks, we get some new results.

Keywords Hyers-Ulam stability, fixed point method, functional equation, \( F \)-space, quasi-Banach space

MSC 39B52, 47H10

1 Introduction and preliminaries

The stability problem of functional equations originated from a question of Ulam \cite{22} in 1940, concerning the stability of group homomorphisms.

Let \((G_1, \cdot)\) be a group and let \((G_2, \ast)\) be a metric group with the metric \(d(\cdot, \cdot)\). Given \(\delta > 0\), does there exist a \(\varepsilon > 0\), such that if a mapping \(h : G_1 \to G_2\) satisfies the inequality

$$d(h(x \ast y), h(x) \ast h(y)) \leq \delta$$

for all \(x, y \in G_1\), then there exists a homomorphism \(H : G_1 \to G_2\) with

$$d(h(x), H(x)) \leq \varepsilon$$

Corresponding author: Qi Liu, E-mail: liuq325@mail2.sysu.edu.cn
for all $x \in G_1$?

In 1941, Hyers [11] gave the first affirmative answer to the question of Ulam for Banach spaces. Let $f : E \to E'$ be a mapping between Banach spaces such that

$$\|f(x + y) - f(x) - f(y)\| \leq \delta$$

for all $x, y \in E$, and for some $\delta > 0$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \delta$$

for all $x \in E$. In 1978, Rassias [20] proved the following theorem.

**Theorem 1.** [20] Let $f : E \to E'$ be a mapping from a normed vector space $E$ into a Banach space $E'$ subject to the inequality

$$\|f(x + y) - f(x) - f(y)\| \leq \varepsilon (\|x\|^p + \|y\|^p)$$

(2)

for all $x, y \in E$, where $\varepsilon$ and $p$ are constants with $\varepsilon > 0$ and $p < 1$. Then there exists a unique additive mapping $T : E \to E'$ such that

$$\|f(x) - T(x)\| \leq \frac{2\varepsilon}{2 - 2p}\|x\|^p$$

(3)

for all $x \in E$. If $p < 0$ then inequality (2) holds for all $x, y \neq 0$, and (3) for $x \neq 0$. Also, if the function $t \mapsto f(tx)$ from $\mathbb{R}$ into $E'$ is continuous in $t \in \mathbb{R}$ for each fixed $x \in E$, then $T$ is $\mathbb{R}$-linear.

Although stability problems have been studied successfully in the framework of Banach spaces, there are not many relevant results in $F$-spaces. One of the most important reasons is that the nonlinear structure of infinite-dimensional $F$-spaces and the failure of triangle inequality bring us challenges and difficulties. Besides these, for $F$-spaces, several results can be consulted in [21] and the references therein. For more information about quasi-Banach spaces, the readers can refer to [23] and the references therein. Various more results for the stability of functional equations in quasi-Banach spaces can be seen in [7, 18].

Gilányi [10] showed that if $f$ satisfies the functional inequality

$$\|2f(x) + 2f(y) - f(x - y)\| \leq \|f(x + y)\|$$

(4)

then $f$ satisfies the Jordan-von Neumann functional equation

$$2f(x) + 2f(y) = f(x + y) + f(x - y).$$

Fechner [9] and Gilányi [10] proved the Hyers-Ulam stability of the functional inequality (4). The stability problems of functional equations and functional inequalities have been studied extensively by many authors (see [8, 16]).

Fixed point theory play an important role in functional analysis and other applied disciplines. Next, we recall a fundamental result in fixed point theory.
Theorem 2. [6] Let \((X, d)\) be a complete generalized metric space and let \(J : X \to X\) be a strictly contractive mapping with Lipschitz constant \(\alpha < 1\). Then for each given element \(x \in X\), either
\[d (J^n x, J^{n+1} x) = \infty\]
for all nonnegative integers \(n\) or there exists a positive integer \(n_0\) such that
(1) \(d (J^n x, J^{n+1} x) < \infty\), \(\forall n \geq n_0\);
(2) the sequence \(\{J^n x\}\) converges to a fixed point \(y^*\) of \(J\);
(3) \(y^*\) is the unique fixed point of \(J\) in the set \(Y = \{y \in X \mid d (J^n_0 x, y) < \infty\}\);
(4) \(d (y, y^*) \leq \frac{1}{1-\alpha} d(y, Jy)\) for all \(y \in Y\).

By using the new fixed point method, the stability problems of functional equations has been further studied extensively (see [4, 5, 8, 17, 19]).

Definition 1. Consider \(X\) be a linear space. A non-negative valued function \(\| \cdot \|\) achieves an \(F\)-norm if satisfies the following conditions:
(1) \(\| x \| = 0\) if and only if \(x = 0\);
(2) \(\| \lambda x \| = |\lambda| \| x \|\) for all \(\lambda, |\lambda| = 1\);
(3) \(\| x + y \| \leq \| x \| + \| y \|\) for all \(x, y \in X\);
(4) \(\| \lambda_n x \| \to 0\) provided \(\lambda_n \to 0\);
(5) \(\| \lambda x_n \| \to 0\) provided \(x_n \to 0\);
(6) \(\| \lambda_n x_n \| \to 0\) provided \(\lambda_n \to 0, x_n \to 0\).

Then \((X, \| \cdot \|)\) is called an \(F^*\)-space. An \(F\)-space is a complete \(F^*\)-space. An \(F\)-norm is called \(\beta\)-homogeneous \((\beta > 0)\) if \(\|tx\| = |t|^\beta \|x\|\) for all \(x \in X\) and all \(t \in \mathcal{C}\) (see [21, 24]).

If a quasi-norm is \(p\)-subadditive, then it is called \(p\)-norm \((0 < p < 1)\). In other words, if it satisfies
\[\|x + y\|^p \leq \|x\|^p + \|y\|^p, \ x, y \in X.\]

We note that the \(p\)-subadditive quasi-norm \(\| \cdot \|\) induces an \(F\)-norm. We refer the reader to [13] and [2] for background on it.

Definition 2. [3] A quasi-norm on \(\| \cdot \|\) on vector space \(X\) over a field \(K(\mathbb{R})\) is a map \(X \to [0, \infty)\) with the following properties:
(1) \(\| x \| = 0\) if and only if \(x = 0\)
(2) \(\| ax \| = |a| \| x \|\), \(a \in \mathbb{R}, x \in X\)
(3) \(\| x + y \| \leq C(\| x \| + \| y \|)\), \(x, y \in X\)
where \(C \geq 1\) is a constant independent of \(x, y \in X\). The smallest \(C\) for which (3) holds in the definition is called the quasi-norm constant of \((X, \| \cdot \|)\).

It is vital to emphasize the well-known theorem in nonlocally convex theory, that is, Aoki–Rolewicz theorem [21], which asserts that for some \(0 < p \leq 1\), every quasi-norm admits an equivalent \(p\)-norm.

In this paper, we study the stability of the additive-quadartic functional equation [11], which is closely related to the results by Inho Hwang and Choonkil
Park in 2020 [12]. There are some differences that we consider the target space with the $\beta$-homogeneous norm and quasi-norm. Overcoming the $\beta$-homogeneous norm and quasi-norm bottlenecks, we get some new results.

This paper is organized as follows: In Section 2, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1) in $F$-spaces and quasi-Banach spaces by using the direct method. In Section 3, we prove the Hyers-Ulam stability of the additive-quadratic functional equation (1) in $F$-spaces and quasi-Banach spaces by using the fixed point method.

2 Hyers-Ulam stability of the additive-quadratic functional equation (1): direct method

In this section, we study the additive-quadratic functional equation (1) in $F$-spaces. The following lemma plays a major role in our article.

Lemma 1. [12] Let $X, Y$ be vector spaces. If a mapping $f : X^2 \to Y$ satisfies $f(0, z) = f(x, 0) = 0$ and

$$f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(x, w) = 0$$

for all $x, y, z, w \in X$, then $f : X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

Note that if $f : X \to Y$ satisfies (5), then the mapping $f : X \to Y$ is called an additive-quadratic mapping.

Theorem 3. Let $X, Y$ be $\beta$-homogeneous $F$-spaces and $\varphi : X^2 \to [0, \infty)$ be a function satisfying

$$\Phi(x, y) := \sum_{j=1}^{\infty} 4^{(j-1)\beta} \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty$$

for all $x, y \in X$ and $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and

$$\|f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(x, w)\| \leq \varphi(x, y)\varphi(z, w)$$

for all $x, y, z, w \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x, z) - F(x, z)\| \leq \min \{\Psi(x, x)\varphi(z, 0), \varphi(x, 0)\Phi(z, z)\}$$

for all $x, z \in X$, where

$$\Psi(x, y) := \sum_{j=1}^{\infty} 2^{(j-1)\beta} \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right)$$

for all $x, y \in X$. 

Proof. Step 1 Setting \( w = 0 \) and \( y = x \) in (7), we can obtain
\[
\| f(2x, z) - 2f(x, z) \| \leq \varphi(x, x) \varphi(z, 0). \tag{8}
\]
This means that
\[
\| f(x, z) - 2f \left( \frac{x}{2}, z \right) \| \leq \varphi \left( \frac{x}{2}, \frac{x}{2} \right) \varphi(z, 0)
\]
for all \( x, z \in X \). Hence
\[
\left\| 2^l f \left( \frac{x}{2^l}, z \right) - 2^m f \left( \frac{x}{2^m}, z \right) \right\| \leq \sum_{j=l}^{m-1} 2^{j\beta} \left\| f \left( \frac{x}{2^j}, z \right) - 2f \left( \frac{x}{2^{j+1}}, z \right) \right\| \tag{9}
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x, z \in X \).

Applying (9), we can deduce that the sequence \( \{2^k f \left( \frac{x}{2^k}, z \right)\} \) is Cauchy for all \( x, z \in X \). Since \( Y \) is a \( F \)-space, the sequence \( \{2^k f \left( \frac{x}{2^k}, z \right)\} \) converges. Thus, we can define the mapping \( P : X^2 \rightarrow Y \) by
\[
P(x, z) := \lim_{k \to \infty} 2^k f \left( \frac{x}{2^k}, z \right)
\]
for all \( x, z \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (9), we get
\[
\| f(x, z) - P(x, z) \| \leq \Psi(x, x) \varphi(z, 0) \tag{10}
\]
for all \( x, z \in X \).

On the other hand, it follows from (6) and (7) that
\[
\left\| P(x + y, z + w) + P(x - y, z - w) - 2P(x, z) - 2P(x, w) \right\|
\]
\[
= \lim_{n \to \infty} \left\| 2^n \left( f \left( \frac{x + y}{2^n}, z + w \right) + f \left( \frac{x - y}{2^n}, z - w \right) - 2f \left( \frac{x}{2^n}, z \right) - 2f \left( \frac{x}{2^n}, w \right) \right) \right\|
\]
\[
\leq \lim_{n \to \infty} 2^{n\beta} \varphi \left( \frac{x}{2^n}, \frac{y}{2^n} \right) \varphi(z, w) = 0
\]
for all \( x, y, z, w \in X \).

Then we can deduce that
\[
P(x + y, z + w) + P(x - y, z - w) - 2P(x, z) - 2P(x, w) = 0
\]
for all \( x, y, z, w \in X \). By Lemma \( \| \) the mapping \( P : X^2 \rightarrow Y \) is additive in the first variable and quadratic in second variable.
Step 2 Now, let \(T : X^2 \to Y\) be another additive-quadratic mapping satisfying (10). Then we have
\[
\|P(x, z) - T(x, z)\| = \left\|2^q P\left(\frac{x}{2^q}, z\right) - 2^q T\left(\frac{x}{2^q}, z\right)\right\| \\
\leq \left\|2^q P\left(\frac{x}{2^q}, z\right) - 2^q f\left(\frac{x}{2^q}, z\right)\right\| + \left\|2^q T\left(\frac{x}{2^q}, z\right) - 2^q f\left(\frac{x}{2^q}, z\right)\right\| \\
\leq 2^{q\beta} + 1 \Psi\left(\frac{x}{2^q}, \frac{x}{2^q}\right) \varphi(z, 0) \\
= 2 \sum_{j=q}^{\infty} 2^{j\beta} \varphi(z, 0) \varphi\left(\frac{x}{2^{j+1}}, \frac{x}{2^{j+1}}\right),
\]
which tends to zero as \(q \to \infty\) for all \(x, z \in X\). Hence, \(P(x, z) = T(x, z)\) for all \(x, z \in X\).

Step 3 Letting \(y = 0\) and \(w = z\) in (7), we get
\[
\|f(x, 2z) - 4f(x, z)\| \leq \varphi(x, 0) \varphi(z, z) \tag{11}
\]
and so
\[
\|f(x, z) - 4f\left(x, \frac{z}{2}\right)\| \leq \varphi(x, 0) \varphi\left(\frac{z}{2}, \frac{z}{2}\right)
\]
for all \(x, z \in X\). Hence
\[
\left\|4^l f\left(x, \frac{z}{2^l}\right) - 4^m f\left(x, \frac{z}{2^m}\right)\right\| \leq \sum_{j=l}^{m-1} 4^{j\beta} \left\|f\left(x, \frac{z}{2^j}\right) - 4f\left(x, \frac{z}{2^{j+1}}\right)\right\| \tag{12}
\]
for all nonnegative integers \(m\) and \(l\) with \(m > l\) and all \(x, z \in X\). It follows from (12) that the sequence \(\{4^k f\left(x, \frac{z}{2^k}\right)\}\) is Cauchy for all \(x, z \in X\). Since \(Y\) is a \(F\)-space, the sequence \(\{4^k f\left(x, \frac{z}{2^k}\right)\}\) converges. Thus, we can define the mapping \(Q : X^2 \to Y\) by
\[
Q(x, z) := \lim_{k \to \infty} 4^k f\left(x, \frac{z}{2^k}\right)
\]
for all \(x, z \in X\). Moreover, letting \(l = 0\) and passing the limit \(m \to \infty\) in (12), we get
\[
\|f(x, z) - Q(x, z)\| \leq \varphi(x, 0) \Phi(z, z) \tag{13}
\]
for all \(x, z \in X\).

It follows from (6) and (7) that
\[
\left\|Q(x + y, z + w) + Q(x - y, z - w) - 2Q(x, z) - 2Q(x, w)\right\| \\
= \lim_{n \to \infty} \left\|4^n \left(f\left(x + y, \frac{z + w}{2^n}\right) + f\left(x - y, \frac{z - w}{2^n}\right) - 2f\left(x, \frac{z}{2^n}\right) - 2f\left(x, \frac{w}{2^n}\right)\right)\right\| \\
\leq \lim_{n \to \infty} 4^{n\beta} \varphi(x, y) \varphi\left(\frac{z}{2^n}, \frac{w}{2^n}\right) = 0
\]
for all $x, y, z, w \in X$. So
\[ Q(x + y, z + w) + Q(x - y, z - w) - 2Q(x, z) - 2Q(x, w) = 0 \]
for all $x, y, z, w \in X$. By Lemma \[\text{I}\] the mapping $Q : X^2 \to Y$ is additive in the first variable and quadratic in second variable.

**Step 4** Now, let $T : X^2 \to Y$ be another additive-quadratic mapping satisfying (13). Then we have
\[
\|Q(x, z) - T(x, z)\| = \left\|4^q Q\left(x, \frac{z}{2^q}\right) - 4^q T\left(x, \frac{z}{2^q}\right)\right\|
\leq 4^q \left\|Q\left(x, \frac{z}{2^q}\right) - 4^q f\left(x, \frac{z}{2^q}\right)\right\| + \left\|4^q T\left(x, \frac{z}{2^q}\right) - 4^q f\left(x, \frac{z}{2^q}\right)\right\|
\leq 2 \cdot \varphi(x, 0) \sum_{j = q}^{\infty} 4^q \varphi\left(\frac{x}{2^q}, \frac{z}{2^q}\right)
\]
which tends to zero as $q \to \infty$ for all $x, z \in X$. So we can conclude that $Q(x, z) = T(x, z)$ for all $x, z \in X$. This proves the uniqueness of $Q$.

**Step 5** It follows from (13) that
\[
2^n \beta \left\|f\left(\frac{x}{2^n}, z\right) - Q\left(\frac{x}{2^n}, z\right)\right\| \leq 2^n \beta \varphi\left(\frac{x}{2^n}, 0\right) \Phi(z, z)
\]
which tends to zero as $n \to \infty$ for all $x, z \in X$. Since $Q : X^2 \to Y$ is additive in the first variable, we get $\|P(x, z) - Q(x, z)\| = 0$. This means that $F(x, z) := P(x, z) = Q(x, z)$ for all $x, z \in X$. Thus there is an additive-quadratic mapping $F : X^2 \to Y$ such that
\[
\|f(x, z) - F(x, z)\| \leq \min \left\{\Psi(x, x)\varphi(z, 0), \varphi(x, 0)\Phi(z, z)\right\}
\]
for all $x, z \in X$, as desired.

\[\square\]

**Corollary 1.** Let $X, Y$ be quasi-Banach space and $\varphi : X^2 \to [0, \infty)$ be a function satisfying
\[
\Phi(x, y) := \sum_{j = 1}^{\infty} 4^{(j-1) \beta} \varphi\left(\frac{x}{2^j}, \frac{y}{2^j}\right) < \infty
\]
for all $x, y \in X$ and $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and
\[
\|f(x+y, z+w) + f(x-y, z-w) - 2f(x, z) - 2f(x, w)\| \leq \varphi^\frac{1}{\beta}(x, y)\varphi^\frac{1}{\beta}(z, w) \quad (14)
\]
for all $x, y, z, w \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that
\[
\|f(x, z) - F(x, z)\| \leq \min \left\{\Psi^\frac{1}{\beta}(x, x)\varphi^\frac{1}{\beta}(z, 0), \varphi^\frac{1}{\beta}(x, 0)\Phi^\frac{1}{\beta}(z, z)\right\}
\]
for all \( x, z \in X \), where
\[
\Psi(x, y) := \sum_{j=1}^{\infty} 2^{(j-1)p} \varphi\left( \frac{x_{2^j}}{2^j}, \frac{y_{2^j}}{2^j} \right)
\]
for all \( x, y \in X \).

**Proof.** Let \( \| \cdot \|_p = \| \cdot \|_p \), then it is obviously that \((Y, \| \cdot \|_p)\) is \( p \)-homogeneous \( F \)-space, so we can easily obtain the result from Theorem 3.

**Corollary 2.** Let \( X, Y \) be \( \beta \)-homogenous \( F \)-spaces, \( r > 2 \) and \( \theta \) be nonnegative real numbers and \( f : X^2 \to Y \) be a mapping satisfying \( f(x, 0) = f(0, z) = 0 \) and
\[
\| f(x + y, z + w) + f(x - y, z - w) - 2f(x, z) - 2f(x, w) \| \\
\leq \theta(\|x\|^r + \|y\|^r)(\|z\|^r + \|w\|^r)
\]
for all \( x, y, z, w \in X \). Then there exists a unique additive-quadratic mapping \( F : X^2 \to Y \) such that
\[
\| f(x, z) - F(x, z) \| \leq \frac{2\theta}{2^{\beta r} - 2^\beta\|x\|^r \|z\|^r}
\]
for all \( x, z \in X \).

**Proof.** The proof follows from Theorem 3 by taking \( \varphi(x, y) = \sqrt{\theta}(\|x\|^r + \|y\|^r) \) for all \( x, y \in X \), since
\[
\min \left\{ \frac{2\theta}{2^{\beta r} - 2^\beta\|x\|^r \|z\|^r}, \frac{2\theta}{2^{\beta r} - 2^\beta\|x\|^r \|z\|^r} \right\} = \frac{2\theta}{2^{\beta r} - 2^\beta\|x\|^r \|z\|^r}
\]
for all \( x, z \in X \).

**Theorem 4.** Let \( X, Y \) be \( \beta \)-homogenous \( F \)-spaces, \( \varphi : X^2 \to [0, \infty) \) be a function satisfying
\[
\Psi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^{2(j+1)\beta}} \varphi\left( 2^j x, 2^j y \right) < \infty
\]
for all \( x, y \in X \) and let \( f : X^2 \to Y \) be a mapping satisfying \( f(x, 0) = f(0, z) = 0 \) and \( \varphi(x, y) \) for all \( x, z \in X \). Then there exists a unique additive-quadratic mapping \( F : X^2 \to Y \) such that
\[
\| f(x, z) - F(x, z) \| \leq \min \{ \Psi(x, x) \varphi(z, 0), \varphi(x, 0) \Phi(z, z) \}
\]
for all \( x, z \in X \), where
\[
\Phi(x, y) := \sum_{j=0}^{\infty} \frac{1}{2^{2(j+1)\beta}} \varphi\left( 2^j x, 2^j y \right)
\]
for all \( x, y \in X \).
Proof. It follows from (8) that
\[ \|f(x, z) - \frac{1}{2} f(2x, z)\| \leq \frac{1}{2^\beta} \varphi(x, x) \varphi(z, 0) \]
for all \(x, z \in X\). Hence
\[
\left\| \frac{1}{2^l} f\left(2^l x, z\right) - \frac{1}{2^m} f\left(2^m x, z\right) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{2^j} f\left(2^j x, z\right) - \frac{1}{2^{j+1}} f\left(2^{j+1} x, z\right) \right\| \tag{18}
\]
for all nonnegative integers \(m\) and \(l\) with \(m > l\) and all \(x, z \in X\). It follows from (18) that the sequence \(\{\frac{1}{2^k} f\left(2^k x, z\right)\}\) is Cauchy for all \(x, z \in X\). Since \(Y\) is a \(\beta\)-homogeneous \(F\)-space, the sequence \(\{\frac{1}{2^k} f\left(2^k x, z\right)\}\) converges. So one can define the mapping \(P : X^2 \to Y\) by
\[ P(x, z) := \lim_{k \to \infty} \frac{1}{2^k} f\left(2^k x, z\right) \]
for all \(x, z \in X\). Moreover, letting \(l = 0\) and passing the limit \(m \to \infty\) in (18), we get
\[ \|f(x, z) - P(x, z)\| \leq \Psi(x, x) \varphi(z, 0) \tag{19} \]
for all \(x, z \in X\).

It follows from (7) and (16) that
\[
\|P(x + y, z + w) + P(x - y, z - w) - 2P(x, z) - 2P(x, w)\|
\leq \lim_{n \to \infty} \left\| \frac{1}{2^n} \left[f\left(2^n (x + y), z + w\right) + f\left(2^n (x - y), z - w\right) - 2f\left(2^n x, z\right) - 2f\left(2^n x, w\right)\right] \right\|
\leq \lim_{n \to \infty} \frac{1}{2^n} \varphi\left(2^n x, 2^n y\right) \varphi(z, w) = 0
\]
for all \(x, y, z, w \in X\). So
\[ P(x + y, z + w) + P(x - y, z - w) - 2P(x, z) - 2P(x, w) = 0 \]
for all \(x, y, z, w \in X\). By Lemma II the mapping \(P : X^2 \to Y\) is additive in the first variable and quadratic in second variable.

Now, let \(T : X^2 \to Y\) be another additive-quadratic mapping satisfying (19). Then we have
\[
\|P(x, z) - T(x, z)\| = \left\| \frac{1}{2^q} P\left(2^q x, z\right) - \frac{1}{2^q} T\left(2^q x, z\right) \right\|
\leq \left\| \frac{1}{2^q} P\left(2^q x, z\right) - \frac{1}{2^q} f\left(2^q x, z\right) \right\| + \left\| \frac{1}{2^q} T\left(2^q x, z\right) - \frac{1}{2^q} f\left(2^q x, z\right) \right\|
\leq \frac{2}{2^{q\beta}} \Psi\left(2^q x, 2^q x\right) \varphi(z, 0)
\]
which tends to zero as \( q \to \infty \) for all \( x, z \in X \). So we can conclude that \( P(x, z) = T(x, z) \) for all \( x, z \in X \). This proves the uniqueness of \( P \).

It follows from (20) that
\[
\| f(x, 2^l z) - 4f(x, z) \| \leq \varphi(x, 0)\varphi(z, z) \tag{20}
\]
and so
\[
\| f(x, z) - \frac{1}{4}f(x, 2^l z) \| \leq \frac{1}{4^l} \varphi(x, 0)\varphi(2^l z, 2^l z)
\]
for all \( x, z \in X \). Hence
\[
\left\| \frac{1}{4^j} f \left( x, 2^j z \right) - \frac{1}{4^m} f \left( x, 2^m z \right) \right\| \leq \sum_{j=l}^{m-1} \left\| \frac{1}{4^j} f \left( x, 2^j z \right) - \frac{1}{4^{j+1}} f \left( x, 2^{j+1} z \right) \right\| \tag{21}
\]
\[
\leq \sum_{j=l}^{m-1} \frac{1}{4^{(j+1)b}} \varphi(x, 0)\varphi \left( 2^j z, 2^j z \right)
\]
for all nonnegative integers \( m \) and \( l \) with \( m > l \) and all \( x, z \in X \). It follows from (21) that the sequence \( \left\{ \frac{1}{4^k} f \left( x, 2^k z \right) \right\} \) is Cauchy for all \( x, z \in X \). Since \( Y \) is a \( F \)-space, the sequence \( \left\{ \frac{1}{4^k} f \left( x, 2^k z \right) \right\} \) converges. So one can define the mapping \( \mathcal{Q} : X^2 \to Y \) by
\[
\mathcal{Q}(x, z) := \lim_{k \to \infty} \frac{1}{4^k} f \left( x, 2^k z \right)
\]
for all \( x, z \in X \). Moreover, letting \( l = 0 \) and passing the limit \( m \to \infty \) in (21), we get
\[
\| f(x, z) - \mathcal{Q}(x, z) \| \leq \varphi(x, 0)\Phi(z, z) \tag{22}
\]
for all \( x, z \in X \).

It follows from (7) and (16) that
\[
\| Q(x + y, z + w) + Q(x - y, z - w) - 2Q(x, z) - 2Q(x, w) \|
\]
\[
= \lim_{n \to \infty} \left\| \frac{1}{4^n} \left( f(x + y, 2^n(z + w)) + f(x - y, 2^n(z - w)) - 2f(x, 2^n z) - 2f(x, 2^n w) \right) \right\|
\]
\[
\leq \lim_{n \to \infty} \frac{1}{4^n} \varphi(x, y)\varphi \left( 2^n z, 2^n w \right) = 0
\]
for all \( x, y, z, w \in X \). So
\[
\mathcal{Q}(x + y, z + w) + \mathcal{Q}(x - y, z - w) - 2\mathcal{Q}(x, z) - 2\mathcal{Q}(x, w) = 0
\]
for all \( x, y, z, w \in X \). By Lemma, the mapping \( \mathcal{Q} : X^2 \to Y \) is additive in the first variable and quadratic in second variable.
Now, let $T : X^2 \to Y$ be another additive-quadratic mapping satisfying (22). Then we have
\[
\|Q(x, z) - T(x, z)\| = \left\| \frac{1}{4^q} Q(x, 2^q z) - \frac{1}{4^q} T(x, 2^q z) \right\|
\leq \left\| \frac{1}{4^q} Q(x, 2^q z) - \frac{1}{4^q} f(x, 2^q z) \right\| + \left\| \frac{1}{4^q} T(x, 2^q z) - \frac{1}{4^q} f(x, 2^q z) \right\|
\leq \frac{2}{4^{q\beta}} \varphi(x, 0) \Phi(2^q z, 2^q z),
\]
which tends to zero as $q \to \infty$ for all $x, z \in X$. So we can conclude that $Q(x, z) = T(x, z)$ for all $x, z \in X$. This proves the uniqueness of $Q$.

It follows from (22) that
\[
\frac{1}{2^n} \left\| f(2^n x, z) - Q(2^n z, z) \right\| \leq \frac{1}{2^n} \varphi(2^n x, 0) \Phi(z, z)
\]
which tends to zero as $n \to \infty$ for all $x, z \in X$. Since $Q : X^2 \to Y$ is additive in the first variable, we get $\|P(x, z) - Q(x, z)\| = 0$, i.e., $F(x, z) := P(x, z) = Q(x, z)$ for all $x, z \in X$. Thus there is an additive-quadratic mapping $F : X^2 \to Y$ such that
\[
\|f(x, z) - F(x, z)\| \leq \min \{ \Psi(x, x) \varphi(z, 0), \varphi(x, 0) \Phi(z, z) \}
\]
for all $x, z \in X$.

Corollary 3. Let $X$ be a quasi-Banach space, $\varphi : X^2 \to [0, \infty)$ be a function satisfying
\[
\Psi(x, y) := \sum_{j=1}^{\infty} \frac{1}{2^{(j-1)p}} \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right) < \infty
\]
for all $x, y \in X$ and $f : X^2 \to Y$ be a mapping satisfying $f(x, 0) = f(0, z) = 0$ and (14) for all $x, y, z, w \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that
\[
\|f(x, z) - F(x, z)\| \leq \min \left\{ \Psi^p(x, x) \varphi^p(z, 0), \varphi^p(x, 0) \Phi^p(z, z) \right\}
\]
for all $x, z \in X$, where
\[
\Phi(x, y) := \sum_{j=1}^{\infty} 4^{(j-1)p} \varphi \left( \frac{x}{2^j}, \frac{y}{2^j} \right)
\]
for all $x, y \in X$.

Proof. Let $\| \cdot \|_p = \| \cdot \|^p$, then it is obviously that $(Y, \| \cdot \|_p)$ is $p$-homogeneous $F$-space, so we can easily obtain the result from Theorem 4.

\[\square\]
Corollary 4. Let $X, Y$ be $\beta$-homogeneous $F$-spaces, $r < 1$ and $\theta$ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying (13) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x, z) - F(x, z)\| \leq \frac{2\theta}{(2^\beta - 2^r\beta)} \|x\|^r \|z\|^r$$

for all $x, z \in X$.

Proof. The proof follows from Theorem 1 by taking $\varphi(x, y) = \sqrt{\theta(\|x\|^r + \|y\|^r)}$ for all $x, y \in X$, since $\min \left\{ \frac{2\theta}{(2^\beta - 2^r\beta)} \|x\|^r \|z\|^r, \frac{2\theta}{(2^\beta - 2^r\beta)} \|x\|^r \|z\|^r \right\} = \frac{2\theta}{(2^\beta - 2^r\beta)} \|x\|^r \|z\|^r$ for all $x, z \in X$. \qed

3 Hyers-Ulam stability of the additive-quadratic functional equation (1): fixed point method

Using the fixed point method, we prove the Hyers-Ulam stability of the additive quadratic functional equation (1) in complex $F$-spaces.

Theorem 5. Let $X, Y$ be $\beta$-homogeneous $F$-spaces, $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with

$$\varphi\left(\frac{x}{2}, \frac{y}{2}\right) \leq \frac{L}{2^\beta} \varphi(x, y) \leq \frac{L}{2^\beta} \varphi(x, y)$$

(23)

for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying (7) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that

$$\|f(x, z) - F(x, z)\| \leq \min \left\{ \frac{L}{2^\beta(1 - L)} \varphi(x, x)\varphi(z, 0), \frac{L}{4^\beta(1 - L)} \varphi(x, 0)\varphi(z, z) \right\}$$

(24)

for all $x, z \in X$.

Proof. Letting $w = 0$ and $y = x$ in (7), we get

$$\|f(2x, z) - 2f(x, z)\| \leq \varphi(x, x)\varphi(z, 0)$$

(25)

for all $x, z \in X$. Consider the set

$$S := \{ h : X^2 \to Y, \ h(x, 0) = h(0, z) = 0 \ \forall x, z \in X \}$$

and introduce the generalized metric on $S$:

$$d(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \|g(x, z) - h(x, z)\| \leq \mu \varphi(x, x)\varphi(z, 0), \ \forall x, z \in X \}$$
where, as usual, \( \inf \emptyset = +\infty \). It is easy to show that \((S, d)\) is complete.

Now we consider the linear mapping \( J : S \to S \) such that

\[
Jg(x, z) := 2g \left( \frac{x}{2}, z \right)
\]

for all \( x, z \in X \). Let \( g, h \in S \) be given such that \( d(g, h) = \varepsilon \). Then

\[
\|g(x, z) - h(x, z)\| \leq \varepsilon \varphi(x, x) \varphi(z, 0)
\]

for all \( x, z \in X \). Hence

\[
\|Jg(x, z) - Jh(x, z)\| = \left\| 2g \left( \frac{x}{2}, z \right) - 2h \left( \frac{x}{2}, z \right) \right\|
\]

\[
\leq 2^{\beta} \varepsilon \varphi \left( \frac{x}{2}, \frac{x}{2} \right) \varphi(z, 0)
\]

\[
\leq 2^{\beta} \varepsilon \frac{L}{2^\beta} \varphi(x, x) \varphi(z, 0)
\]

\[
= L \varepsilon \varphi(x, x) \varphi(z, 0)
\]

for all \( x, z \in X \). So \( d(g, h) = \varepsilon \) implies that \( d(Jg, Jh) \leq L \varepsilon \). This means that

\[
d(Jg, Jh) \leq Ld(g, h)
\]

for all \( g, h \in S \).

It follows from (25) that

\[
\left\| f(x, z) - 2f \left( \frac{x}{2}, z \right) \right\| \leq \varphi \left( \frac{x}{2}, \frac{x}{2} \right) \varphi(z, 0)
\]

\[
\leq \frac{L}{2^\beta} \varphi(x, x) \varphi(z, 0)
\]

for all \( x, z \in X \). So \( d(f, Jf) \leq \frac{L}{2^\beta} < \infty \).

By Theorem 2 there exists a mapping \( P : X^2 \to Y \) satisfying the following:

(i) \( P \) is a fixed point of \( J \), i.e.,

\[
P(x, z) = 2P \left( \frac{x}{2}, z \right)
\]

(26)

for all \( x, z \in X \). The mapping \( P \) is a unique fixed point of \( J \). This implies that \( P \) is a unique mapping satisfying (26) such that there exists a \( \mu \in (0, \infty) \) satisfying

\[
\|f(x, z) - P(x, z)\| \leq \mu \varphi(x, x) \varphi(z, 0)
\]

for all \( x, z \in X \).

(ii) \( d(J^l f, P) \to 0 \) as \( l \to \infty \). This implies the equality

\[
\lim_{l \to \infty} 2^l f \left( \frac{x}{2^l}, z \right) = P(x, z)
\]
for all \( x, z \in X \).

(iii) \( d(f, P) \leq \frac{1}{2L}d(f, Jf) \), which implies

\[
\| f(x, z) - P(x, z) \| \leq \frac{L}{2^{\beta} (1 - L)} \varphi(x, x) \varphi(z, 0)
\]

for all \( x, z \in X \). By the same reasoning as in the proof of Theorem 4 one can show that the mapping \( P : X^2 \to Y \) is additive in the first variable and quadratic in the second variable.

Letting \( z = w \) and \( y = 0 \) in (7), we get

\[
\| f(x, 2z) - 4f(x, z) \| \leq \varphi(x, 0) \varphi(z, z)
\]

for all \( x, z \in X \). Consider the set

\[
S := \{ h : X^2 \to Y, \quad h(x, 0) = h(0, z) = 0 \quad \forall x, z \in X \}
\]

and introduce the generalized metric on \( S \):

\[
d'(g, h) = \inf \{ \mu \in \mathbb{R}_+ : \| g(x, z) - h(x, z) \| \leq \mu \varphi(x, 0) \varphi(z, z), \forall x, z \in X \}
\]

where, as usual, \( \inf \phi = +\infty \). It is easy to show that \((S, d')\) is complete. Now we consider the linear mapping \( J' : S \to S \) such that

\[
J'g(x, z) := 4g\left(\frac{x}{2}, z\right)
\]

for all \( x, z \in X \). Let \( g, h \in S \) be given such that \( d'(g, h) = \varepsilon \). Then

\[
\| g(x, z) - h(x, z) \| \leq \varepsilon \varphi(x, 0) \varphi(z, z)
\]

for all \( x, z \in X \). Hence

\[
\| J'g(x, z) - J'h(x, z) \| = \left\| 4g\left(\frac{x}{2}, \frac{z}{2}\right) - 4h\left(\frac{x}{2}, \frac{z}{2}\right) \right\| \leq 4\varepsilon \varphi(x, 0) \varphi\left(\frac{z}{2}, \frac{z}{2}\right)
\]

\[
\leq 4\varepsilon \frac{L}{4^{\beta}} \varphi(x, 0) \varphi(z, z) = L \varepsilon \varphi(x, 0) \varphi(z, z)
\]

for all \( x, z \in X \). So \( d'(g, h) = \varepsilon \) implies that \( d'(J'g, J'h) \leq L \varepsilon \). This means that

\[
d'(J'g, J'h) \leq Ld'(g, h)
\]

for all \( g, h \in S \).

It follows from (27) that

\[
\| f(x, z) - 4f\left(\frac{x}{2}, \frac{z}{2}\right) \| \leq \varphi(x, 0) \varphi\left(\frac{z}{2}, \frac{z}{2}\right) \leq \frac{L}{4^{\beta}} \varphi(x, 0) \varphi(z, z)
\]

for all \( x, z \in X \). So \( d'(f, J'f) \leq \frac{L}{4^{\beta}} < \infty \).

By Theorem 4 there exists a mapping \( Q : X^2 \to Y \) satisfying the following:
(i) \( Q \) is a fixed point of \( J' \), i.e.,
\[
Q(x, z) = 4Q\left( x, \frac{z}{2} \right)
\]
(28)
for all \( x, z \in X \). The mapping \( Q \) is a unique fixed point of \( J' \). This implies that \( Q \) is a unique mapping satisfying (28) such that there exists a \( \mu \in (0, \infty) \) satisfying
\[
\| f(x, z) - Q(x, z) \| \leq \mu \varphi(x, 0)\varphi(z, z)
\]
for all \( x, z \in X \).

(ii) \( d(J^l f, Q) \to 0 \) as \( l \to \infty \). This implies the equality
\[
\lim_{l \to \infty} 4^lf\left( x, \frac{z}{2^l} \right) = Q(x, z)
\]
for all \( x, z \in X \).

(iii) \( d(f, Q) \leq \frac{1}{1-L} d(f, J' f) \), which implies
\[
\| f(x, z) - Q(x, z) \| \leq \frac{L}{4^\beta(1-L)} \varphi(x, 0)\varphi(z, z)
\]
for all \( x, z \in X \). By the same reasoning as in the proof of Theorem 3 one can show that the mapping \( Q : X^2 \to Y \) is additive in the first variable and quadratic in the second variable.

By the same reasoning as in the proof of Theorem 3 we get
\[
\| P(x, z) - Q(x, z) \| = 0 , \text{ i.e., } F(x, z) := P(x, z) = Q(x, z) \text{ for all } x, z \in X.
\]
Theorem 4.
Then there exists a unique additive-quadratic mapping \( F : X^2 \to Y \) such that
\[
\| f(x, z) - F(x, z) \| \leq \min \left\{ \frac{L}{2^\beta(1-L)} \varphi(x, x)\varphi(z, 0), \frac{L}{4^\beta(1-L)} \varphi(x, 0)\varphi(z, z) \right\}
\]
for all \( x, z \in X \).

**Corollary 5.** Let \( X, Y \) be \( \beta \)-homogeneous \( F \)-spaces, \( r > 1 \) and \( \theta \) be nonnegative real numbers and \( f : X^2 \to Y \) be a mapping satisfying (15) and \( f(x, 0) = f(0, z) = 0 \) for all \( x, z \in X \). Then there exists a unique additive-quadratic mapping \( F : X^2 \to Y \) such that
\[
\| f(x, z) - F(x, z) \| \leq \frac{2\theta}{2r^\beta - 2^\beta} ||x||^r ||z||^r
\]
for all \( x, z \in X \).

**Proof.** The proof follows from Theorem 5 by taking \( L = \frac{2r^\beta - 2^\beta}{2r^\beta - 2^\beta + 1} \) and \( \varphi(x, y) = \sqrt{\theta}(||x||^r + ||y||^r) \) for all \( x, y \in X \), since
\[
\min \left\{ \frac{2\theta}{2(r-1)^\beta - 1} ||x||^r ||z||^r, \frac{2\theta}{2r^\beta - 2^\beta} ||x||^r ||z||^r \right\} = \frac{2\theta}{2r^\beta - 2^\beta} ||x||^r ||z||^r
\]
for all \( x, z \in X \).
Theorem 6. Let $X, Y$ be $\beta$-homogeneous $F$-spaces and $\varphi : X^2 \to [0, \infty)$ be a function such that there exists an $L < 1$ with
\[
\varphi(x, y) \leq 2^\beta L \varphi \left( \frac{x}{2}, \frac{y}{2} \right) \leq 4^\beta L \varphi \left( \frac{x}{2}, \frac{y}{2} \right) \quad (29)
\]
for all $x, y \in X$. Let $f : X^2 \to Y$ be a mapping satisfying $f(x,0) = f(0,z) = 0$ for all $x, z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that
\[
\| f(x, z) - F(x, z) \| \leq \min \left\{ \frac{1}{2^\beta (1-L)} \varphi(x, x) \varphi(z, 0), \frac{1}{4^\beta (1-L)} \varphi(x, 0) \varphi(z, z) \right\}
\]
for all $x, z \in X$.

Proof. Consider the complete metric spaces $(S, d)$ and $(S, d')$ given in the proof of Theorem 5.

Now we consider the linear mapping $J : S \to S$ such that
\[
Jg(x, z) := \frac{1}{2} g(2x, z)
\]
for all $x, z \in X$. It follows from (25) that
\[
\left\| f(x, z) - \frac{1}{2} f(2x, z) \right\| \leq \frac{1}{2^\beta} \varphi(x, x) \varphi(z, 0)
\]
for all $x, z \in X$. So $d(f, Jf) \leq \frac{1}{2^\beta}$.

By the same reasoning as in the proof of Theorem 3, one can show that there exists a unique additive-quadratic mapping $P : X^2 \to Y$ such that
\[
\| f(x, z) - P(x, z) \| \leq \frac{1}{2^\beta (1-L)} \varphi(x, x) \varphi(z, 0)
\]
for all $x, z \in X$.

By the same reasoning as in the proof of Theorem 3, one can show that the mapping $P : X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

Now we consider the linear mapping $J' : S \to S$ such that
\[
J'g(x, z) := 4g \left( \frac{x}{2}, z \right)
\]
for all $x, z \in X$. It follows from (27) that
\[
\left\| f(x, z) - \frac{1}{4} f(x, 2z) \right\| \leq \frac{1}{4^\beta} \varphi(x, 0) \varphi(z, z)
\]
for all $x, z \in X$. So $d'(f, J'f) \leq \frac{1}{4^\beta}$. 
By the same reasoning as in the proof of Theorem 3, one can show that there exists a unique additive-quadratic mapping $Q : X^2 \to Y$ such that

$$\| f(x, z) - Q(x, z) \| \leq \frac{1}{4^\beta (1 - L)} \varphi(x, 0) \varphi(z, z)$$

for all $x, z \in X$.

By the same reasoning as in the proof of Theorem 3, one can show that the mapping $Q : X^2 \to Y$ is additive in the first variable and quadratic in the second variable.

By the same reasoning as in the proof of Theorem 3, we get $\| P(x, z) - Q(x, z) \| = 0$, i.e., $F(x, z) := P(x, z) = Q(x, z)$ for all $x, z \in X$. Thus there is an additive-quadratic mapping $F : X^2 \to Y$ such that

$$\| f(x, z) - F(x, z) \| \leq \min\left\{ \frac{1}{2^\beta (1 - L)} \varphi(x, x) \varphi(z, 0), \frac{1}{4^\beta (1 - L)} \varphi(x, 0) \varphi(z, z) \right\}$$

for all $x, z \in X$.

Corollary 6. Let $X, Y$ be $\beta$-homogeneous $F$-spaces, $r < 1$ and $\theta$ be nonnegative real numbers and $f : X^2 \to Y$ be a mapping satisfying (15) and $f(x, 0) = f(0, z) = 0$ for all $x, z \in X$. Then there exists a unique additive-quadratic mapping $F : X^2 \to Y$ such that

$$\| f(x, z) - F(x, z) \| \leq \frac{2\theta}{4^\beta - 2^r \beta} \| x \|^r \| z \|^r$$

for all $x, z \in X$.

**Proof.** The proof follows from Theorem 3 by taking $L = 2^{\beta(r - 2)}$ and $\varphi(x, y) = \sqrt{\theta} (\| x \|^r + \| y \|^r)$ for all $x, y \in X$, since

$$\min\left\{ \frac{2\theta}{4^\beta - 2^r \beta} \| x \|^r \| z \|^r, \frac{2^{\beta + 1} \theta}{4^\beta - 2^{r+1} \beta} \| x \|^r \| z \|^r \right\} = \frac{2\theta}{4^\beta - 2^{r+1} \beta} \| x \|^r \| z \|^r$$

for all $x, z \in X$. \hfill \Box

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