Radiative Transfer and Limb Darkening of Accretion Disks

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Abstract

The transfer equation in a geometrically thin accretion disk is reexamined under the plane-parallel approximation with a finite optical depth. The emergent intensity is analytically obtained in cases with or without internal heating. For large or infinite optical depth, the emergent intensity exhibits the usual limb-darkening effect, where the intensity changes linearly as a function of the direction cosine. For a small optical depth, on the other hand, the angle-dependence of the emergent intensity changes drastically. In the case without heating, but with uniform incident radiation at the disk equator, the emergent intensity becomes isotropic for a small optical depth. In the case with uniform internal heating, limb brightening takes place for a small optical depth. We also emphasize and discuss the limb-darkening effect in an accretion disk for several cases.

Key words: accretion, accretion disks — black holes — galaxies: active — radiative transfer — relativity

1. Introduction

Accretion disks are now widely believed to be energy sources in various active phenomena in the universe: in protoplanetary nebulae and around young stellar objects, in cataclysmic variables and supersoft X-ray sources, in galactic X-ray binaries and microquasars, and in active galaxies and quasars. Accretion-disk models have been extensively studied during the past three decades (see Kato et al. 1998 for a review).

Radiative transfer in an accretion disk has been investigated in relation to the structure of a static disk atmosphere and the spectral energy distribution from the disk surface (e.g., Meyer, Meyer-Hofmeister 1982; Cannizzo, Wheeler 1984; Kříž, Hubeny 1986; Shaviv, Wehrse 1986; Adam et al. 1988; Hubeny 1990; Ross et al. 1992; Artemova et al. 1996; Hubeny, Hubeny 1997, 1998; Hubeny et al. 2000, 2001; Davis et al. 2005; Hui et al. 2005). In many cases the diffusion approximation, or Eddington one, was employed; it provides a satisfactory description at a large optical depth, although the emergent radiation field originates at an optical depth on the order of unity. Furthermore, gray and non-gray models of accretion disks were constructed under numerical treatments (Kříž, Hubeny 1986; Shaviv, Wehrse 1986; Adam et al. 1988; Ross et al. 1992; Shimura, Takahara 1993; Hubeny, Hubeny 1997, 1998; Hubeny et al. 2000, 2001; Davis et al. 2005; Hui et al. 2005) and under analytical ones (Hubeny 1990; Artemova et al. 1996).

In these studies, however, the vertical movement and the mass loss were not considered. Hence, recently, in relation to the radiative disk wind, radiative transfer in a moving disk atmosphere was also investigated (e.g., Fukue 2005a, b, 2006a, b). In contrast to a static atmosphere, in a moving atmosphere the boundary condition at the surface of zero optical depth should be modified (Fukue 2005a, b). Moreover, the usual Eddington approximation is violated in a highly relativistic flow (Fukue 2005b; see also Turolla, Nobili 1988; Turolla et al. 1995; Dullemond 1999), and thus a velocity-dependent variable Eddington factor was proposed (Fukue 2006b).

In the usual studies of radiative transfer in a disk, the emergent intensity has not been fully obtained, since attention has usually been focused on the disk internal structure, such as the temperature distribution. In addition, the effect of limb darkening has not been well examined, except for a few cases related to cataclysmic variables (e.g., Diaz et al. 1996; Wade, Hubeny 1998; see also Fukue 2000; Hui et al. 2005 for high energy cases).

In this paper, we thus reexamine radiative transfer in an accretion disk with a finite optical depth under the plane-parallel approximation, and analytically obtain an emergent intensity for the cases with or without internal heating. Besides cataclysmic variables, we also emphasize and discuss the limb-darkening effect in an accretion disk for various cases.

In the next section we describe the basic equations. In section 3, we show analytical solutions. In section 4, we discuss several cases of accretion-disk models, and emphasize the importance of the limb-darkening effect. The final section is devoted to concluding remarks.

2. Basic Equations

We here assume the followings: (i) The disk is steady and axisymmetric. (ii) It is also geometrically thin and plane parallel. (iii) As a closure relation, we use the Eddington approximation. (iv) The gray approximation, where the opacity does not depend on the frequency, is adopted. (v) The viscous heating rate is concentrated at the equator, or uniform in the vertical direction.

The radiative transfer equations are given in several articles (Chandrasekhar 1960; Mihalas 1970; Rybicki, Lightman 1979; Mihalas, Mihalas 1984; Shu 1991; Kato et al. 1998). For the plane-parallel geometry in the vertical direction, the frequency-integrated transfer equation, the zeroth-moment equation, and the first-moment equation become, respectively,
\[
\cos \theta \frac{dI}{dZ} = \rho \left[ \frac{j}{4\pi} - (\kappa_{\text{abs}} + \kappa_{\text{sca}}) I + \kappa_{\text{sca}} \frac{c}{4\pi} E \right],
\]
(1)

\[
\frac{dF}{dz} = \rho(j - c\kappa_{\text{abs}} E),
\]
(2)

\[
\frac{dP}{dz} = -\frac{\rho(\kappa_{\text{abs}} + \kappa_{\text{sca}})}{c} F,
\]
(3)

where \(\theta\) is the polar angle, \(I\) the frequency-integrated specific intensity, \(E\) the radiation energy density, \(F\) the vertical component of the radiative flux, \(P\) the \(zz\)-component of the radiation stress tensor, \(\rho\) the gas density, and \(c\) the speed of light. The mass emissivity, \(j\), and the opacity, \(\kappa_{\text{abs}}\) and \(\kappa_{\text{sca}}\), are assumed to be independent of the frequency (gray approximation).

For matter, the vertical momentum balance and the energy equation are, respectively,

\[
0 = -\frac{d\psi}{dz} - \frac{1}{\rho} \frac{dp}{dz} + \frac{\kappa_{\text{abs}} + \kappa_{\text{sca}}}{c} F,
\]
(4)

\[
0 = q_{\text{vis}}^+ - \rho(j - c\kappa_{\text{abs}} E),
\]
(5)

where \(\psi\) the gravitational potential, \(p\) the gas pressure, and \(q_{\text{vis}}\) the viscous-heating rate. In this paper, we do not solve the hydrostatic equilibrium (4). Generally speaking, when the contribution of the radiative flux is small, compared with the pressure gradient term, the gas pressure dominates in the atmosphere, and the density distribution will not be constant. When the radiative flux is strong, on the other hand, the radiation pressure dominates, and the density may be approximately constant throughout much of the disk. In any case, we suppose that the density distribution should be adjusted so as to hold the hydrostatic equilibrium (4) through the main part of the disk atmosphere, under the radiative flux obtained later.

Using this energy equation (5) and introducing the optical depth, defined by

\[
d\tau = -\rho(\kappa_{\text{abs}} + \kappa_{\text{sca}})dz,
\]
(6)

we rewrite the radiative transfer equations:

\[
\mu \frac{dI}{d\tau} = I - \frac{c}{4\pi} E - \frac{1}{4\pi} \frac{1}{\kappa_{\text{abs}} + \kappa_{\text{sca}}} q_{\text{vis}}^+ \rho,
\]
(7)

\[
\frac{dF}{d\tau} = -\frac{1}{\kappa_{\text{abs}} + \kappa_{\text{sca}}} q_{\text{vis}}^+ \rho,
\]
(8)

\[
\frac{dP}{d\tau} = F,
\]
(9)

\[
cP = \frac{1}{3} cE,
\]
(10)

where \(\mu \equiv \cos \theta\). The final equation is the usual Eddington approximation.

As for the boundary condition at the disk surface of \(\tau = 0\), we impose the usual condition:

\[
3cP_s = cE_s = 2F_s \quad \text{at} \quad \tau = 0,
\]
(11)

where the subscript \(s\) denotes the values at the disk surface.

For internal heating, we consider two extreme cases: (i) No heating \((q_{\text{vis}}^+ = 0)\), where the viscous heating is concentrated at the disk equator and there is no heating source in the atmosphere. (ii) Uniform heating in the sense that \(q_{\text{vis}}^+/(\kappa_{\text{abs}} + \kappa_{\text{sca}})\rho = \text{constant}\). The latter case means that the kinematic viscosity, \(v\), is constant in the vertical direction, since \(q_{\text{vis}}^+/\rho = v(r d\Omega/dr)^2\), as long as the opacities are constant.

Finally, the disk total optical depth becomes

\[
\tau_0 = -\int_H^0 \rho(\kappa_{\text{abs}} + \kappa_{\text{sca}})dz,
\]
(12)

where \(H\) is the disk half-thickness.

3. Analytical Solutions

Except for the emergent intensity, \(I\), several analytical expressions for the moments as well as the temperature distributions were obtained by several researchers (e.g., Laor, Netzer 1989; Hubeny et al. 2005; Artemova et al. 1996). For completeness, we recalculate them as well as the intensity, \(I\).

3.1. No Heating Case

We first consider the case without heating in the disk atmosphere, \(q_{\text{vis}}^+ = 0\), but with a uniform incident intensity, \(I_0\), from the disk equator, where the viscous heating is assumed to be concentrated.

In this case, the analytical solutions of moment equations are easily given as

\[
F = F_s = \pi I_0,
\]
(13)

\[
3cP = cE = 3F_s \left( \frac{2}{3} + \tau \right).
\]
(14)

This is a familiar solution under the Milne–Eddington approximation for a plane-parallel geometry. It should be noted that the vertical radiative flux, \(F\), is conserved, and equals to \(\pi I_0\) at the disk equator.

Since we obtain the radiation energy density, \(E\), in an explicit form, we can now integrate the radiative transfer equation (7). After several partial integrations, we obtain both an outward intensity, \(I(\tau, \mu)\) \((\mu > 0)\), and an inward intensity, \(I(\tau, -\mu)\) as

\[
I(\tau, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau + \mu - \left( \frac{2}{3} + \tau_0 + \mu \right) e^{(\tau - \tau_0)/\mu} \right] + I(\tau_0, \mu) e^{(\tau - \tau_0)/\mu},
\]
(15)

\[
I(\tau, -\mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau - \mu - \left( \frac{2}{3} - \mu \right) e^{-\tau/\mu} \right],
\]
(16)

where \(I(\tau_0, \mu)\) is the boundary value at the midplane of the disk.

In a geometrically thin disk with a finite optical depth, \(\tau_0\), and a uniform incident intensity, \(I_0\), from the disk equator, the boundary value \(I(\tau_0, \mu)\) of the outward intensity, \(I\), consists of two parts:

\[
I(\tau_0, \mu) = I_0 + I(\tau_0, -\mu),
\]
(17)

where \(I_0\) is the uniform incident intensity, and \(I(\tau_0, -\mu)\) is the inward intensity from the backside of the disk beyond the midplane. Determining \(I(\tau_0, -\mu)\) from equation (16),
we finally obtain the outward intensity as
\[ I(\tau, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau + \mu - 2\mu e^{(\tau-\tau_0)/\mu} \right. \]
\[ \left. - \left( \frac{2}{3} - \mu \right) e^{(\tau-2\tau_0)/\mu} \right] + I_0 e^{(\tau-\tau_0)/\mu} \]
\[ = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau + \mu + \left( \frac{4}{3} - 2\mu \right) e^{(\tau-\tau_0)/\mu} \right. \]
\[ \left. - \left( \frac{2}{3} - \mu \right) e^{(\tau-2\tau_0)/\mu} \right], \quad (18) \]

where we have used \( F_s = \pi I_0 \).

For a sufficiently large optical depth, \( \tau_0 \), this equation (18) reduces to the usual Milne–Eddington solution,
\[ I = \frac{3F_s}{4\pi} \left( \frac{2}{3} + \tau + \mu \right). \quad (19) \]

Finally, the emergent intensity, \( I(0, \mu) \), emitted from the disk surface for a finite optical depth becomes
\[ I(0, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \mu + \left( \frac{4}{3} - 2\mu \right) e^{-\tau_0/\mu} \right. \]
\[ \left. - \left( \frac{2}{3} - \mu \right) e^{-2\tau_0/\mu} \right]. \quad (20) \]

In figure 1, the emergent intensity, \( I(0, \mu) \), normalized by the isotropic value, \( T (= F_s/\pi) \), is shown for several values of \( \tau_0 \) as a function of \( \mu \).

As can be easily seen in figure 1, for a large optical depth (\( \tau_0 > 10 \)) the angle-dependence of the emergent intensity is very close to the case for a usual plane-parallel case with an infinite optical depth. Therefore, the usual limb-darkening effect is seen. Namely, in the case of a semi-infinite disk with a large optical depth, the energy density increases linearly with the optical depth in the atmosphere, and the temperature increases accordingly. As a result, an observer at a pole-on position of \( \mu = 1 \) will see deeper into the disk, where the temperature (and, therefore, the source function) is larger than that observed by an observer at an edge-on position of \( \mu = 0 \). Thus, the observed intensity will be higher at \( \mu = 1 \). This is just the usual limb-darkening.

For a small optical depth, however, the angle-dependence is drastically changed. When the optical depth is \( \sim 2–3 \), the vertical intensity (\( \mu \sim 1 \)) decreases due to the finiteness of the optical depth. That is, we cannot see the ‘deeper’ position in the atmosphere, compared with the case of a semi-infinite disk. Furthermore, when the optical depth is less than unity, the intensity in the direction of small \( \mu \) increases, and the emergent intensity becomes isotropic with a uniform value of \( I_0 \) at the disk equator; the limb-darkening effect disappears. Indeed, for the limiting case of \( \tau_0 \to 0 \), \( I(0, \mu) \to F_s/\pi \). That is, in this case of a very small optical depth, the source function is dominated by the isotropic source at the midplane.

### 3.2. Uniform Heating Case

Now, we consider the case with uniform heating:
\[ q_{\text{vis}}/\left( \kappa_{\text{abs}} + \kappa_{\text{sca}} \right) \rho = \text{constant}. \]

Integrating equation (8) under the boundary conditions
\[ F = 0 \quad \text{at} \quad \tau = \tau_0, \]
\[ F = F_s \quad \text{at} \quad \tau = 0, \]
we obtain
\[ F = F_s \left( 1 - \frac{\tau}{\tau_0} \right). \quad (22) \]

The radiative flux, \( F \), linearly increases from 0 to the surface value, \( F_s \).

Substituting equation (22) into equation (9), and integrating the resultant equation under boundary condition (11), we obtain
\[ 3cP = cE = 3F_s \left( \frac{2}{3} + \tau - \frac{\tau^2}{2\tau_0} \right). \quad (23) \]

This expression for a finite optical depth can be seen in, e.g., Laor and Netzer (1989). A similar, but more general, expression was obtained by Hubeny (1990). In any case, this expression reduces to the Milne–Eddington solution for a sufficiently large optical depth. In the case of a finite optical depth, the radiation energy density and pressure decrease from the midplane to the surface in a quadratic form. It should be noted that at the midplane of the disk of \( \tau = \tau_0 \),
\[ 3cP = cE = 3F_s \left( \frac{2}{3} + \frac{\tau_0}{2} \right). \quad (24) \]

As already mentioned by Hubeny (1990), the energy density at the disk midplane is half of the corresponding stellar atmospheric one. This is explained by the fact that radiation from the disk midplane may escape equally to both sides of the disk.

Since we obtained the radiation energy density, \( E \), in an explicit form (23), we can now integrate the radiative transfer equation (7). After several partial integrations, we obtain both
an outward intensity, $I(\tau, \mu)$ ($\mu > 0$), and an inward intensity, $I(\tau, -\mu)$, as

$$I(\tau, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau + \mu + \frac{1}{\tau_0} \left( \frac{1}{3} - \frac{\tau^2}{2} - \mu \tau - \mu^2 \right) \right]$$

$$- \left( \frac{2}{3} - \mu + \frac{1}{3\tau_0} - \frac{\mu^2}{\tau_0} \right) e^{(-\tau_0)/\mu} + I(\tau_0, \mu) e^{(\tau-\tau_0)/\mu},$$

and we finally obtain the outward intensity as

$$I(\tau, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau + \mu + \frac{1}{\tau_0} \left( \frac{1}{3} - \frac{\tau^2}{2} - \mu \tau - \mu^2 \right) \right]$$

$$- \left( \frac{2}{3} - \mu + \frac{1}{3\tau_0} - \frac{\mu^2}{\tau_0} \right) e^{(-\tau_0)/\mu}. \quad \text{(26)}$$

where $I(\tau_0, \mu)$ is the boundary value at the midplane of the disk.

In the case with uniform heating and without incident intensity, the boundary value $I(\tau_0, \mu)$ of the outward intensity, $I$, is

$$I(\tau_0, \mu) = I(\tau_0, -\mu), \quad \text{(27)}$$

and we finally obtain the outward intensity as

$$I(\tau, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \tau + \mu + \frac{1}{\tau_0} \left( \frac{1}{3} - \frac{\tau^2}{2} - \mu \tau - \mu^2 \right) \right]$$

$$- \left( \frac{2}{3} - \mu + \frac{1}{3\tau_0} - \frac{\mu^2}{\tau_0} \right) e^{(-\tau_0)/\mu}. \quad \text{(28)}$$

For a sufficiently large optical depth, $\tau_0$, this equation also reduces to the usual Milne–Eddington solution (19).

Finally, the emergent intensity, $I(0, \mu)$, emitted from the disk surface for the finite optical depth becomes

$$I(0, \mu) = \frac{3F_s}{4\pi} \left[ \frac{2}{3} + \mu + \frac{1}{\tau_0} \left( \frac{1}{3} - \mu^2 \right) \right]$$

$$- \left( \frac{2}{3} - \mu + \frac{1}{3\tau_0} - \frac{\mu^2}{\tau_0} \right) e^{-2\tau_0/\mu}. \quad \text{(29)}$$

In figure 2, the emergent intensity, $I(0, \mu)$, normalized by the isotropic value, $T (= F_s/\pi)$, is shown for several values of $\tau_0$ as a function of $\mu$.

As can be easily seen in figure 2, for a large optical depth ($\tau_0 > 10$) the angle-dependence of the emergent intensity is very close to the case with the usual plane-parallel case with an infinite optical depth. Therefore, the usual limb-darkening effect is seen, as already stated at the end of subsection 3.1.

For a small optical depth, however, the angle-dependence is drastically changed in a way similar to the case without heating. In the vertical direction of $\mu \sim 1$, the emergent intensity decreases as the optical depth decreases. This is due to the finiteness of the optical depth. That is, we cannot see the ‘deeper’ position in the atmosphere, compared with the case of a semi-infinite disk. In the inclined direction of small $\mu$, on the other hand, the emergent intensity becomes larger than that in the case of an infinite optical depth. Moreover, when the optical depth is less than unity, the emergent intensity for a small $\mu$ is greater than unity; limb brightening takes place. Indeed, in the limiting case of $\tau_0 \sim 0$, $I(0, \mu) \sim (F_s/\pi)/(2\mu)$. This is because that the path length is longer for such a case of a small $\mu$. That is, in this case for a low optical depth, the source function is very uniform. This, coupled with the absence of an isotropic source at the midplane, is why the geometric effect (longer path length) is dominant and one finds limb ‘brightening’.

3.3. Validity of the Eddington Approximation

In this subsection, we briefly discuss the validity of the closure relation in the present treatment. In this paper, we have adopted the usual Eddington approximation, where the ratio of the radiation pressure to the energy density is fixed at 1/3, to close the moment equations. As is well known, this approximation is correct in the limit of an isotropic radiation field. Hence, in the problem of limb-darkening, where a deviation from isotropy is essential, this approximation is only approximately correct, although it is used in the usual Milne–Eddington approximation.

For example, let us suppose the case of a semi-infinite disk with an infinite optical depth of $\tau_0$. In this case, we can easily calculate the energy density as well as the radiation pressure from the derived intensity (19) of the Milne–Eddington solution. At a deeper position in the atmosphere, where the integration is done in all directions, the re-calculated variables satisfy the condition $P/E = 1/3$. However, at the surface of the disk, where integration should be done in a semi-sphere, this is not true. At the surface of $\tau = 0$, we integrate the emergent intensity to yield:

$$cE = \frac{3F_s}{2} \int_0^1 \left( \frac{2}{3} + \mu \right) d\mu = \frac{3}{8} F_s \frac{7}{12}. \quad \text{(30)}$$

$$cP = \frac{3F_s}{2} \int_0^1 \left( \frac{2}{3} + \mu \right) \mu^2 d\mu = \frac{1}{3} F_s \frac{17}{24}. \quad \text{(31)}$$

or $P/E = (1/3)(17/14)$. Hence, in the case under the usual
limb-darkening effect, the ratio of the radiation pressure to the energy density is slightly larger than 1/3 at the surface. This is just the peaking effect originated from the anisotropic radiation field near to the disk surface.

On the other hand, in the limb-brightening case of a small optical depth with uniform heating, the situation is reversed. In this case, the ratio of the radiation pressure to the energy density becomes smaller than 1/3, and the Eddington approximation holds approximately. This also originates from the anisotropy, and may be called an anti-peaking effect. As can be seen in figure 2, the limb-brightening becomes stronger and stronger for a small optical depth. Hence, for such a case of a very small optical depth, the Eddington approximation would not be good, although the qualitative properties would not be changed.

In order to obtain the intensity distribution more precisely, we, for example, introduce a variable Eddington factor, which is beyond the scope of the present paper.

4. Discussion

As derived in the previous section, the emergent intensity, $I$, of the accretion disk depends on the disk total optical depth, $\tau_0$, as well as the direction cosine, $\mu$. The limb-darkening effect is considerably modified for a small $\tau_0$, compared with the usual case for an infinite optical depth. Even for the case with a sufficiently large optical depth, limb darkening in the luminous accretion disk must be important, and should be examined more carefully.

In this section, we discuss several cases in turn, and call attention to the importance of the limb-darkening effect.

4.1. Standard Disk

The optical depth at the midplane in the inner region of a geometrically thin standard disk (Shakura, Sunyaev 1973; see also Kato et al. 1998) is expressed as

$$\tau_0 = \frac{1}{\kappa \Sigma} = \frac{20 \alpha^{-1} m^{-1} \dot{m}^{3/2}}{\left(1 - \sqrt{\frac{3}{\dot{r}}} \right)^{-1}}. \quad (32)$$

where $\kappa$ is the electron-scattering optical depth, $\alpha$ the viscous parameter, $m$ the mass-accretion rate normalized by the critical rate, $\dot{M} (= L_E/c^2)$, $L_E$ being the Eddington luminosity of the central object, and $\dot{r}$ the radius normalized by the Schwarzschild radius $r_s (= 2GM/c^2)$.

This optical depth becomes small for a large $\dot{m}$ and/or a small $\dot{r}$. For a slightly large accretion rate, inside some critical radius

$$r_{ct} = 2 \dot{m}, \quad (33)$$

does not shift to a supercritical regime, whereas the disk is a standard regime outside $r_{ct}$ (Fukue 2004). At this critical radius, the optical depth becomes

$$\tau_{ct} \sim 57 \alpha^{-1} m^{1/2}. \quad (34)$$

Hence, in the usual situation the optical depth of the inner region of the standard disk is greater than several tens of value.

In such a situation, however, due to the usual limb-darkening effect for a semi-infinite medium, the emergent radiation toward the pole-on direction is enchanced by 20 percent, while the emergent radiation seen from the edge-on direction is diminished by 50 percent. Thus, in calculating the flux and spectrum of the standard disk, we carefully consider the limb-darkening effect.

In the region inside the inner edge at $3r_s$, the disk gas freely falls toward the central black hole, and the surface density (i.e., the disk optical depth) quickly drops. Hence, the emergent spectrum from the innermost region inside the inner edge would be greatly modified from the optically thick case and the optically thin one.

4.2. Supercritical Disk

In a supercritical accretion disk, where the mass-accretion rate exceeds the critical rate, the expression for the disk optical depth is changed. For example, in the self-similar model without mass loss (Fukue 2000), the optical depth at the midplane of the disk is

$$\tau_0 = \frac{\kappa}{4\pi c^2 \alpha} \frac{1}{\sqrt{GMr}} = \frac{\dot{m}}{\sqrt{2c_1 \alpha}}. \quad (35)$$

where $c_1$ is a coefficient on the order of unity. At the critical radius, the disk optical depth is

$$\tau_{ct} \sim \frac{1}{2c_1 \alpha} \dot{m}^{1/2}. \quad (36)$$

In a critical accretion disk (Fukue 2004), where the mass accretion rate exceeds the critical one, but the excess mass is expelled by the wind mass loss, the optical depth at the midplane of the disk is

$$\tau_0 = \frac{16\sqrt{6}}{\alpha} \dot{m}^{1/2} = 39.2 \alpha^{-1} \dot{m}^{1/2}. \quad (37)$$

At the critical radius, the disk optical depth is

$$\tau_{ct} \sim 55 \alpha^{-1} \dot{m}^{1/2}. \quad (38)$$

Hence, in the usual situation the optical depth of the supercritical/critical disk is also greater than several tens of value.

In such a situation, however, the usual limb-darkening effect for a semi-infinite medium is also important (see Fukue 2000). In addition to the limb-darkening effect, geometrical effects, such as a projection effect and a self-occultation, should be considered when calculating the flux and spectrum of the supercritical disk (e.g., Fukue 2000; Watarai et al. 2005; Kawata et al. 2006).

4.3. Disk Corona and ADAF

If a luminous disk is sandwiched by a disk corona, the situation is similar to the case without heating, but with incident intensity from the midplane. Hence, when the optical depth of the corona is sufficiently smaller than unity, the emergent intensity is just that of the disk, except for a very small direction cosine. However, when the optical depth of the corona is $\sim 2$–3, limb darkening takes place, and the emergent intensity toward the edge-on direction is remarkably reduced.

On the other hand, if the accretion rate is quite small, and the inner region of the disk becomes an optically-thin advection dominated state (ADAF), the situation is similar to the case with internal heating, although, rigorously speaking, the plane-parallel approximation may be invalid. When the optical
The depth of an ADAF region is sufficiently smaller than unity, limb brightening would take place.

In both cases with hot gas, however, the effect of Compton scattering may be important. Hence, the angle dependence of the intensity would be modified by the angle dependence of the Compton scattering, and the transfer problem should be treated more carefully.

In addition, in the latter case of ADAF, the accretion flow is supposed to be conical or spherical. Hence, when the opening angle is small, limb brightening would qualitatively take place. When the opening angle is large, on the other hand, the extension of the emitting region is large, and the angle dependence of the emergent intensity would become much more complicated than in the present simple case.

4.4. Relativistic Disk

In the case of the relativistic standard disk (e.g., Novikov, Thorne 1973; Page, Thorne 1974), the situation is similar to the non-relativistic case. However, the direction cosine of the local emergent intensity in the disk to the observer is changed for two additional reasons: (i) the light trajectory is bent by the space-time curvature, and (ii) the emission from the gas rotating around a black hole suffers from a special relativistic aberration. The limb-darkening effect in a comoving frame for two additional reasons: (i) the light trajectory is bent by the space-time curvature, and (ii) the emission from the gas rotating around a black hole suffers from a special relativistic aberration. The limb-darkening effect in a comoving frame.

In calculating the spectra from, e.g., cataclysmic variables, the effect of limb darkening was considered (e.g., Diaz et al. 1996; Wade, Hubeny 1998). In the cases of high energy and relativistic regimes, we should also consider limb darkening in some detail in their non-LTE calculation of the accretion-disk spectra around intermediate-mass black holes.

5. Concluding Remarks

In this paper we analytically solve the radiative transfer problem of a geometrically thin accretion disk with finite optical depth, and obtain analytical expressions for emergent intensity from the surface of the disk. For a small optical depth, the angle-dependence of the emergent intensity drastically changes from the case with a large optical depth. In the case of a black hole, however, the usual limb darkening was not considered.

Examples of silhouettes of a dressed black hole are shown in figures 3 and 4. In figure 3 an edge-on view with an inclination angle of 80° is shown, while a pole-on view is expressed in figure 4. In both figures, the left panels are for the case without limb darkening, whereas the right panels are for the case with limb darkening.

In the edge-on view (figure 3), the image of a limb-darkening disk darkens as expected. This is due mainly to the usual limb-darkening effect for a small direction cosine. Surprisingly, on the other hand, in the pole-on view (figure 4), the image of a limb-darkening disk also darkens! This is because the local direction cosine becomes small due to light aberration associated with disk rotation.

Thus, in calculating the spectra and the observed fluxed of relativistic disks as well as in taking black hole silhouettes, we should carefully consider limb darkening. For example, besides Fu, Taam (1990) and Gierliński et al. (2001), Hui et al. (2005) considered limb darkening in some detail in their non-LTE calculation of the accretion-disk spectra around intermediate-mass black holes.
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References

Adam, J., Störzer, H., Shaviv, G., & Wehrse, R. 1988, A&A, 193, L1
Artemova, I. V., Bisnovatyi-Kogan, G. S., Igumenshchev, I. V., & Novikov, I. D. 2001, ApJ, 549, 1050
Cannizzo, J. K., & Wheeler, J. C. 1984, ApJS, 55, 367
Chandrasekhar, S. 1960, Radiative Transfer (New York: Dover Publishing, Inc.)
Davis, S. W., Blaes, O. M., Hubeny, I., & Turner, N. J. 2005, ApJ, 621, 372
Diaz, M. P., Wade, R. A., & Hubeny, I. 1996, ApJ, 459, 236
Dullemond, C. P. 1999, A&A, 343, 1030
Fanton, C., Calvani, M., de Felice, F., & Čadež, A. 1997, PASJ, 49, 159
Fu, A., & Taam, R. E. 1990, ApJ, 349, 553
Fukue, J. 2000, PASJ, 52, 829
Fukue, J. 2003, PASJ, 55, 155
Fukue, J. 2004, PASJ, 56, 569
Fukue, J. 2005a, PASJ, 57, 841
Fukue, J. 2005b, PASJ, 57, 1023
Fukue, J. 2006a, PASJ, 58, 187
Fukue, J. 2006b, PASJ, 58, 461
Fukue, J., & Yokoyama, T. 1988, PASJ, 40, 15
Gierliński, M., Maciolek-Niedźwiecki, A., & Elbisawa, K. 2001, MNRAS, 325, 1253
Hubeny, I. 1990, ApJ, 351, 632
Hubeny, I., Agol, E., Blaes, O., & Krolik, J. H. 2000, ApJ, 533, 710
Hubeny, I., Blaes, O., Krolik, J. H., & Agol, E. 2001, ApJ, 559, 680
Hubeny, I., & Hubeny, V. 1997, ApJ, 484, L37
Hubeny, I., & Hubeny, V. 1998, ApJ, 505, 558
Hui, Y., Krolik, J. H., & Hubeny, I. 2005, 625, 913
Jaroszyński, M., Wambsganss, J., & Paczyński, B. 1992, ApJ, 396, L65
Karas, V., Vokrouhlický, D., & Polnarev, A. G. 1992, MNRAS, 259, 569
Kato, S., Fukue, J., & Mineshige, S. 1998, Black-Hole Accretion Disks (Kyoto: Kyoto University Press)
Kawata, A., Watarai, K., & Fukue, J. 2006, PASJ, 58, 477
Kříž, S., & Hubeny, I. 1986, BAIC, 37, 129
Laor, A., & Netzer, H. 1989, MNRAS, 238, 897
Luminet, J.-P. 1979, A&A, 75, 228
Meyer, F., & Meyer-Hofmeister, E. 1982, A&A, 106, 34
Mihalas, D. 1970, Stellar Atmospheres (San Francisco: W. H. Freeman & Co.)
Mihalas, D., & Mihalas, B. W. 1984, Foundations of Radiation Hydrodynamics (Oxford: Oxford University Press)
Novikov, I. D., & Thorne, K. S. 1973, in Black Holes, ed. C. DeWitt & B. S. DeWitt (New York: Gordon and Breach)
Page, D. N., & Thorne, K. S. 1974, ApJ, 191, 499
Ross, R. R., Fabian, A. C., & Mineshige, S. 1992, MNRAS, 258, 189
Rybicki, G. B., & Lightman, A. P. 1979, Radiative Processes in Astrophysics (New York: John Wiley & Sons)
Shakura, N. I., & Sunyaev, R. A. 1973, A&A, 24, 337
Shaviv, G., & Wehrse, R. 1986, A&A, 159, L5
Shimura, T., & Takahara, F. 1993, ApJ, 419, 78
Shu, F. H. 1991, The Physics of Astrophysics Vol. 1: Radiation (Mill Valley: University Science Books)
Stübs, D. W. N. 1971, ApJ, 168, 155
Takahashi, R. 2004, ApJ, 611, 996
Takahashi, R. 2005, PASJ, 57, 273
Turolla, R., & Nobili, L. 1988, MNRAS, 235, 1273
Turolla, R., Zampieri, L., & Nobili, L. 1995, MNRAS, 272, 625
Wade, R. A., & Hubeny, I. 1998, ApJ, 509, 350
Watarai, K., Ohsuga, K., Takahashi, R., & Fukue, J. 2005, PASJ, 57, 513