Quenched dynamics of two-dimensional solitary waves and vortices in the Gross–Pitaevskii equation

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Abstract
We consider a two-dimensional (2D) counterpart of the experiment that led to the creation of quasi-1D bright solitons in Bose–Einstein condensates (BECs) (2002 Nature 417 150–3). We start by identifying the fundamental state of the 2D Gross–Pitaevskii equation for repulsive interactions, with a harmonic-oscillator (HO) trap, and with or without an optical lattice (OL). Subsequently, we switch the sign of the interaction to induce interatomic attraction and monitor the ensuing dynamics. Regions of stable self-trapping and a catastrophic collapse of 2D fundamental states are identified in the parameter plane of the OL strength and BEC norm. The increase of the OL strength expands the persistence domain for the solitary waves to larger norms. For single-charged solitary vortices, in addition to the survival and collapse regimes, an intermediate one is identified, where the vortex resists the collapse but loses its structure, transforming into a single-hump state. The same setting may also be implemented in the context of optical solitons and vortices, using photonic-crystal fibers.

Keywords: quenched dynamics, Gross–Pitaevskii equation, collapse, solitary waves, vortices

(Some figures may appear in colour only in the online journal)

1. Introduction

The past decade has brought about a very substantial amount of research efforts in the physics of atomic Bose–Einstein condensates (BECs) [42, 43]. These studies have revealed a wide array of interesting phenomena, not only thanks to the precise control over experimental settings and the use of accurate and relatively simple theoretical models, which is a particularly appealing feature of this area [12, 29], but also due to direct connections to other areas of physics, including superfluidity, superconductivity, quantum and nonlinear optics, and nonlinear wave theory.

One of the main topics for which these connections have been pursued is the study of the nonlinear dynamics of matter waves in BECs. Diverse experimental techniques have been used to produce a broad array of matter-wave excitations. In particular, phase engineering has been used to create vortices [36, 51] and dark solitons [4, 8, 13, 15, 16]. Stirring of the BECs has led to the formation of vortices [24, 32] and vortex-lattices [2, 3, 17]. The switch of the scattering length, from positive (repulsive) to negative (attractive), via Feshbach resonances, has been used to produce bright matter-wave solitons and soliton trains [11, 30, 46, 47]. These modes have been studied extensively, to the extent that numerous reviews (and even books [29]) are dedicated to bright solitons [1, 7], dark solitons [20] and vortices [18, 19, 26].

In this work, we aim to revisit a fundamental aspect associated with some of the principal experiments used to produce bright solitons, especially those carried out by the Rice group [46, 47]. (Note that a similar method was used for the creation of solitons in optical fibers in the pioneering works in that field [22, 35].) Precisely, we study
the modulational instability (MI) [28] of the solitary wave and vortex after the quench, i.e., after switching the BEC system from repulsive to attractive interactions. While this mechanism was extremely efficient in the experimentally elaborated cigar-shaped setting, to the best of our knowledge it has not yet been systematically explored in quasi-two-dimensional (2D) pancake-shaped BECs. This is the subject of the present work, with emphasis on the formation of single-hump states and solitary vortices. More specifically, we examine the results of the sign switching of the nonlinearity from repulsive to attractive in the 2D BEC, trapped via a combination of a harmonic-oscillator (HO), i.e., magnetic, and periodic optical lattice (OL) potentials [7, 40]. Our purpose is to cast into the following dimensionless form:

\( iu_t + \frac{1}{2}(u_{xx} + u_{yy}) + g|u|^2 u - V_x u = 0, \)  

with potential

\[ V_x = \frac{1}{2} \Omega^2 r^2 - \epsilon [\cos 2x + \cos 2y]. \]  

In this normalized GPE, the wavefunction is rescaled as \( \psi \rightarrow \sqrt{|g|}\psi \) \( \exp[i(V_0/E_{\text{OL}})t], \) and the sign parameter is \( g = \pm 1, \) with \( g = -1 \) and \( g = +1 \) corresponding respectively, to repulsive and attractive interatomic interactions. Further, the lattice depth in equation (4) is measured in units of \( 4E_{\text{rec}} \), while the normalized HO strength is \( \Omega = \alpha_{\text{HO}}/a_{\perp} \equiv \omega_x/\omega_y, \) where \( a_{\perp} = \sqrt{\hbar/m_0} \) is the transverse trapping length.

We are interested in the dynamics of solitary waves and vortices when the nonlinearity is switched from defocusing \((g < 0)\) to focusing \((g > 0)\). More precisely, we first fix \( g = -1 \) and solve the imaginary-time GPE version [10] to find the respective steady state for a given configuration, and then solve the GPE with \( g = 1, \) using that steady state as the initial condition. All the parameters stay the same except that \( g \) changes sign. In all the presented examples, \( \Omega = 0.1 \) is assumed in equation (4), yet our findings should be relevant to a wide range of \( \Omega \)'s within the pancake setting.

Another way (a faster one) of finding such a steady state that we employed is to plug the ansatz \( u = e^{-i\mu t}v(x, y) \) into equation (3) to derive a nonlinear eigenvalue problem, which is then solved using Newton’s method. Solutions produced by these two approaches are found to be virtually identical, with the maximum pointwise difference being between \( 10^{-6} \) and \( 10^{-10} \) for most configurations.

Unless specified otherwise, Newton’s solution is used as the initial condition in all the simulations. Further, the fourth-order split-step Fourier method [6, 39] is used to solve the GPE in time. The corresponding domain size is \([-8\pi, 8\pi] \times [-8\pi, 8\pi], \) with 256 Fourier modes in each direction and time step \( \Delta t = 0.001. \) The simulations were run up to \( T = 2000, \) which is large enough to observe the stability of the final states (if they are stable). Figure 1 shows some examples of the initial conditions, i.e., the steady states obtained by Newton’s method.

Control parameters will be the OL strength \( \epsilon \) and the norm of the initial condition (i.e., the normalized number of atoms)

\[ N = \iint |u_0(x, y)|^2 \, dx \, dy. \]

The following main features of the solution will be computed: amplitude \((|u|)_{\text{max}}\), and the total angular momentum,

\[ M = (-i) \iint \left( \frac{\partial u^*}{\partial \theta} - \frac{\partial u}{\partial \theta} u^* \right) \, dx \, dy. \]
where $\theta$ is the angular coordinate. Note that $M$ is conserved in the isotropic system that does not include the OL, and is not conserved with the presence of the OL. In particular, for isotropic solutions with integer vorticity $S$,\[ u = e^{-i\mu t + iS\theta} U(r), \]the relation between the angular momentum and the norm is $M = 2SN$. We consider both the solitary wave with $S = 0$ and solitary vortex with topological charge $S = 1$, with or without the OL.

### 3. Fundamental-state quench dynamics

We start with the fundamental state with no topological charge. We perform the nonlinearity-sign switch (quench) for a wide range of OL strengths and initial norms. The former is used as a representative parameter associated with the potential, while the latter is employed for scanning through the set of initial data. The resulting two-dimensional map of the stability region of the fundamental state is plotted in figure 2. There are two regions with one denoted by open squares, wherein the fundamental state persists in the attractive regime in the form of a stable single-hump solution (see, e.g., [9, 23] for detailed discussions of the stability of such states), and the other denoted by filled circles, which corresponds to the catastrophic wave collapse, a well-known phenomenon for equations of the nonlinear-Schrödinger type [5, 48]. Clearly, the OL plays a critical role towards the stabilization of the fundamental state, since the critical norm increases as the OL gets stronger. When the OL is absent ($\epsilon = 0$), the single-hump state exhibits breathing dynamics when $N \leq 5.81$, and will collapse at $N > 5.91$. Note that the collapse threshold corresponding to the Townes soliton is $N_{cr} = 5.85$ [5].

In the presence of the OL, the dynamics is more complex, and, in particular, the angular momentum will be generated when $\epsilon < 0$. In such cases, the solution’s center initially coincides with a local maximum of the OL potential, hence it will slide down from this position. In fact, in the beginning of the simulation, the solution center moves back and forth between its initial position and the centers of other OL cells that are located along a straight line. (It is relevant to note that a 2D soliton can travel more than one cell, especially when the lattice is not very strong [45].) When the solution center starts to deviate from this straight line at a certain time, the generation of the angular momentum commences. The subsequent motion does not follow any simple pattern. The
Figure 2. The stability diagram for the single-hump state resulting from the quench of the fundamental state of the repulsive BEC. The squares denote stable configurations that support breathing dynamics, while the dots denote unstable configurations that lead to the collapse.

trajectory of the solution’s center for one such case is shown in figure 3.

4. The variational approximation

To develop an understanding of the breathing regime exhibited by the simulations, it is reasonable to apply the VA [33]. The starting point is the nonstationary GPE (3), but with a time-dependent nonlinearity coefficient \(g(t)\). This equation can be derived from the Lagrangian, \(L = \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy \mathcal{L}(u)\), with density

\[
\mathcal{L} = \frac{i}{2} \left( \frac{\partial u}{\partial t} + u^* \frac{\partial}{\partial x} - u \frac{\partial}{\partial x} \right) - \frac{1}{2} \left( \frac{\partial u}{\partial x} \right)^2 + g(t) |u|^4 + \left\{ -\frac{1}{2} \Omega^2 r^2 + \epsilon [\cos(2x) + \cos(2y)] \right\} |u|^2.
\]

(8)

The variational ansatz for the fundamental state is based on the isotropic Gaussian,

\[
u(r, t) = A(t) \exp \left( -\frac{r^2}{2W^2(t)} + \frac{1}{2} b(t) r^2 + i\phi(t) \right). \tag{9}\]

where \(A, W, b\) and \(\phi\) are, respectively, the amplitude, width, chirp and overall phase, which are assumed to be real functions of time. Following the standard procedure, we insert the ansatz into density (8) and calculate the corresponding effective Lagrangian,

\[
L_{\text{eff}} = 2\pi \int_0^\infty \mathcal{L} dr. \tag{10}\]

The result of the calculation is

\[
L_{\text{eff}} = -N \frac{d\phi}{dt} - \frac{1}{2} NW^2 \frac{db}{dt} - \frac{N}{2W^2} - \frac{1}{2} NW^2 b^2 + \frac{N^2 g(t)}{4\pi W^2} - \frac{1}{2} \Omega^2 NW^2 + 2\epsilon Ne^{-w^2}. \tag{11}\]

where the overdot stands for the time derivative, and the norm of the ansatz is, cf equation (5):

\[
N \equiv \int_{-\infty}^{+\infty} dx \int_{-\infty}^{+\infty} dy |u(x, y)|^2 = \pi A^2 W^2. \tag{12}\]

The variational equation \(\delta L_{\text{eff}}/\delta \phi = 0\) reproduces the conservation of the norm, \(dN/dt = 0\), hence \(N\) may be treated as a constant. Then, equation \(\delta L_{\text{eff}}/\delta b = 0\) yields an expression for the chirp:

\[
b = \frac{1}{W} \frac{dW}{dt}. \tag{13}\]

and the final equation, \(\delta L_{\text{eff}}/\delta (W^2) = 0\), leads to a closed-form evolution equation for the width, in which \(b\) is eliminated by means of equation (13):

\[
\frac{d^2W}{dt^2} = \frac{2\pi - Ng(t)}{2\pi W^3} - \Omega^2 W - 4\epsilon Ne^{-w^2}. \tag{14}\]

For constant \(g > 0\), equation (14) yields the well-known variational approximation for the critical norm, \(N_{\text{cr}}^{(\text{var})} = 2\pi/g\) [14].

Figure 3. The trajectory of the center of the single-hump state. The OL strength is \(\epsilon = -0.5\), while the solution’s parameters are \(\mu = 0.06, N = 6.0294\). The left and right panels are shown from different angles to better represent the rather complex motion.
Aiming to compare results of the VA to those of the full simulations, we have to note that, as the initial profiles in the direct simulations were taken as per the steady state in the case with repulsive interactions, they are much wider than the OL cell. For this reason, the model including the OL cannot be adequately tackled by means of the (radially symmetric) ansatz (9), which does not include the density modulation induced by the OL. Therefore, the comparison is only carried out for the fundamental state in the absence of the OL.

A number of examples of the breathing regime of the resulting fundamental states are displayed in figure 4. These are obtained by solving equation (14), for which two initial conditions are needed. The first initial condition \( W(t=0) \) is simply the width of the initial single-hump state, while the second initial condition \( \frac{dW}{dt}(t=0) \) is approximated by a first-order finite difference based on the widths of the solution in direct simulations at \( t = 0 \) and \( t = 0.001 \).

In figure 4, one observes that, for sufficiently small \( N \), the resulting amplitude of the solution closely follows the VA prediction. However, as \( N \) increases, there arises a beating effect in the full GPE dynamics that is presumably not accounted for by the simplified VA dynamics of equation (14). Nevertheless, the VA still captures the principal features of the oscillatory dynamics of the width of the fundamental state.

5. The quench-induced dynamics of vortices

With the introduction of initial vorticity, the most interesting observation is the rather delicate character of the stability of trapped solitary vortices under the self-attractive nonlinearity (see [38] and references therein). For the vortical initial inputs, we have performed an extensive numerical analysis similar to that reported in section 3 for the simpler case of the fundamental state. The respective two-parameter (the OL strength and initial norm) stability region is shown in figure 5. Here, we categorize both coherent and less coherent (see the examples below) states generated by the vortical inputs as stable when the collapse does not occur by \( T = 2000 \) (our reporting horizon). When \( \varepsilon = 0 \) (i.e., the OL is absent), the critical norm for the quench process is found to be

\[
N^{(S=1)}_{\text{cr}} \approx 11.81,
\tag{15}
\]

which is essentially larger than the known value, \( N^{(S=1)}_{\text{max}} = 7.79 \), i.e., the boundary of the existence of (numerically) exact stable trapped vortices with topological charge \( S = 1 \) [38]. Thus, the effective stability range of dynamical (breathing) vortices may be essentially broader than that of their static counterparts. For the simulation time exceeding \( T = 2000 \), the critical norm may be found to vary slightly, as the stable solutions found at \( N \) very close to \( N^{(S=1)}_{\text{cr}} \) are still in the process of splittings and recombinations at \( T = 2000 \). These
The dynamics becomes much more complicated when the optical lattice is present. For these cases, the main conclusions are as follows.

(i) The stability region becomes smaller for a weak OL ($\varepsilon = \pm 0.1$). Yet, the stable solutions still behave like a coherent vortex ring, provided that norm $N$ is not too large. With an increase of $N$, the solutions mimic the splitting-recombination scenario observed in the absence of the OL (see above). Two typical solutions are displayed in figure 6.

(ii) In the case of the OL with a moderate strength ($\varepsilon = \pm 0.5$), there are two regions in which the solution stays stable for up to $T = 2000$. In particular, it is stable when its initial norm lies in two intervals,

$$0 < N \leq 8.04, \quad (16)$$
$$17.59 \leq N \leq 25.17. \quad (17)$$

In the stability region (16), there are two kinds of solutions observed. When the norm is small enough, all the solutions are similar to the one shown in the top panel of figure 7. Basically, it is built of eight peaks forming two groups. The first group includes the peaks at the 3, 6, 9 and 12 o’clock positions, and this set is always present. The second group consists of the remaining peaks set along the diagonals, which breathe as a function of time (thus becoming more or less noticeable). The overall topological charge is still preserved$^3$. The second kind of dynamics eventually leads to the loss of the vorticity, and transformation of the vortex into a single-hump state, as shown in the bottom panel of figure 7. Its location may vary, depending on the initial norm of the solution. The largest observed norm contained in such a stable peak is $N = 5.43$. When $N$ is increased to be out of the stability region (16), the solution eventually collapses. In the stability region (17), the solutions for all the considered configurations feature four peaks. The largest total norm is $N = 21.53$, with 5.38 in each individual peak. A solution of this type is displayed at figure 8. However, only some of the stable solutions are true vortices (marked by circles in figure 5), bearing the topological charge of $S = 1$. For other solutions, the loss of the global coherence among the peaks occurs in the course of the evolution, at the same moment of time when

$^3$ A video illustrating this case can be found at: [www.math.umass.edu/~qchen/S1_OLE05_N1.zip](http://www.math.umass.edu/~qchen/S1_OLE05_N1.zip).
Figure 7. Example solutions with topological charge $S = 1$, for the OL strength $\varepsilon = 0.5$. Top: $N = 1.3383$, the final solution still being a vortex. Bottom: $N = 3.9573$, the solution evolving towards a single-hump state.

Figure 8. A vortex solution from the stability region (17). Here the lattice strength is $\varepsilon = 0.5$.

the symmetry-breaking occurs (induced by the numerical noise), i.e., the peaks start to have different height. Two examples are presented in figure 9, with one preserving the vorticity (similar to the vortices on discrete lattice, although there is complete lack of the isotropy in the system [25, 34]), and one becoming incoherent and thus shedding the vorticity off.

(iii) For large strength of the OL (e.g., $\varepsilon = \pm 0.9$), the first stability region, corresponding to equation (16), expands, similar to what is the case for the fundamental states in section 3. On the other hand, the second stability region, which corresponds to equation (17), practically disappears. The final stable solutions (at $T = 2000$) exhibit breathing behavior. Yet aside from their
Figure 9. Examples of the loss of symmetry for the quasi-discrete vortices from the stability region (17). The OL strength is ε = 0.5. Left panels: the largest difference among the four peak amplitudes. Right panels: phase shifts (in units of π) between the peaks. Top: N = 17.92; bottom: N = 25.17.

6. Conclusions and future challenges

In this work, we have examined the quenched dynamics of both the fundamental states and vortices with topological charge S = 1 in BEC. The quench consists of the sudden reversal of the nonlinearity sign from repulsive to attractive. The resulting states were investigated by systematic simulations, addressing both the impact of parameters, such as the strength of the OL (optical lattice), and the effect of initial conditions, by considering a range of values of the initial norm, N (which is proportional to the number of atoms in the BEC). A principal result is that the OL expands the range of initial norms which do not lead to the collapse of both the fundamental and vortex states. In fact, this expansion occurs well over the interval of norms for which static vortices were previously identified as stable states via the linear stability analysis. For the fundamental states, the application of the VA (variational approximation) is more efficient for the lower norm of the initial state. Additional beating effects, not captured by the VA, were found close to the collapse threshold. On the other hand, a particular finding, in the case of the vortex, is that, in addition to the regimes where a vortex survives in the breathing form or collapses, there is an intermediate regime where the vortex (in the presence of the OL) loses its topological character, yet remains immune to the collapse. Furthermore, a second stability region was identified for the vortex, in the presence of the OL with an intermediate strength, where the collapse was avoided due to the formation of a robust quasi-discrete vortex.

The same system may also be implemented in nonlinear optics, where the combination of the HO trap and OL potential corresponds to photonic-crystal fibers, while the switch of the nonlinearity corresponds to a junction of two waveguides made of self-defocusing and self-focusing materials [49].

We believe that these results provide a potential for further studies on this theme, both at the theoretical and at the experimental level. At the theoretical level, it may be useful to develop 3D generalizations of the analysis developed in this paper, either for the full case of spherically symmetric BECs or for strongly anisotropic traps. The effect of anisotropy in 2D would be interesting to examine too, as
it departs from the configuration of the isotropic cylindrical trap. On the other hand, at the experimental level, recent advances in producing [41, 50] and monitoring [21, 37] the dynamics of vortices and vortex clusters, in conjunction with the well-established control of the BEC dynamics by means of the Feshbach resonance [44], render particularly appealing and accessible examination of such quenches (or the corresponding adiabatic transitions) between the repulsive and attractive regimes.

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