THE GEOMETRY OF POSITIVELY CURVED KÄHLER METRICS ON TUBE DOMAINS

GABRIEL KHAN, JUN ZHANG, AND FANGYANG ZHENG

Abstract. In this article, we study a class of Kähler manifolds defined on tube domains in $\mathbb{C}^n$, and in particular those which have $O(n) \times \mathbb{R}^n$ symmetry. For these, we prove a uniqueness result showing that any such manifold which is complete and has non-negative orthogonal bisectional curvature ($n \geq 3$) or non-negative bisectional curvature ($n \geq 2$) is biholomorphically isometric to $\mathbb{C}^n$. We also consider another curvature tensor called the “orthogonal anti-bisectional” curvature and find necessary and sufficient conditions for a complete $O(n)$-symmetric tube domain to have non-negative orthogonal anti-bisectional curvature. We provide several examples of complete metrics which satisfy this condition. These examples are also of interest to optimal transport, as they can be used to generate new examples of cost functions which only depend on the Euclidean distance between points and satisfy the weak MTW condition. Finally, we discuss how the interplay between optimal transport and complex geometry can be used to define a “synthetic” version of curvature bounds for Kähler manifolds whose associated potential is merely $C^3$.

1. INTRODUCTION

In this paper, we study Kähler metrics defined on tube domains, which are domains of the form

$$T\Omega = \{ z \in \mathbb{C}^n \mid z = x + iy, x \in \Omega \}$$

where $\Omega$ is a convex domain in $\mathbb{R}^n$. In particular, we focus on metrics $\omega_{\Psi}$ whose potential $\Psi$ (in the $z$-coordinates) is independent of $y$, and so admit a natural translation symmetry. Our primary focus in this paper are Kähler metrics which are complete and whose curvature is non-negative (in several different senses). In general, positive curvature on Kähler manifolds is a very strong assumption which greatly restricts the geometry and topology. On the other hand, positively curved metrics have very interesting geometric properties and in many geometric or analytic applications, an assumption of positive curvature is necessary to establish results. Our results fall in line with both these expectations, showing that positive curvature is quite restrictive but also has interesting applications.

Before discussing our results, it is worth noting that the tube domain $T\Omega$ has a natural interpretation as the tangent bundle of the domain $\Omega$. From this perspective, the $\mathbb{R}^n$-symmetric Kähler metric on $T\Omega$ coincides with the so-called Sasaki metric$^1$. To distinguish this class of spaces from more general Kähler manifolds, we will refer to translation-symmetric Kähler metrics on tube domains as Kähler Sasaki metrics.

$^1$More generally, the Sasaki metric is an almost-Hermitian metric defined on the tangent bundle of a Riemannian manifold $(M, g)$ with an affine connection $\nabla$. This metric need not be Kähler, but will be for all of the spaces of interest in this paper.
1.1. Positively curved $O(n)$-symmetric metrics. In the first several sections of this paper, we focus on Kähler Sasaki metrics which are complete and $O(n)$-symmetric. In other words, we take $\Omega$ to be the ball $B_a \subseteq \mathbb{R}^n$ of radius $a$ (with $a$ possibly infinite) and suppose that rotations of $\Omega$ preserve the Kähler metric on $T\Omega$. In Section 3 we study the bisectional and orthogonal bisectional curvatures of these spaces. We find the following uniqueness theorem.

Proposition. Let $\Psi$ be a strictly convex function on the ball $B_a \subseteq \mathbb{R}^n$ of radius $0 < a \leq \infty$. Assume that $\Psi(x)$ depends solely on the norm of $x$ (i.e. the metric is rotationally symmetric) and that the associated Kähler metric $h_\Psi$ on $TB_a$ is complete. If either $n = 2$ and the bisectional curvature is everywhere nonnegative, or if $n \geq 3$ and the orthogonal bisectional curvature is everywhere nonnegative, then $a = \infty$ and the metric $h_\Psi$ is the flat Euclidean metric on $\mathbb{C}^n$.

Put more simply, this shows that the only Kähler Sasaki metrics which is complete and rotationally symmetric, and whose bisectional curvature is non-negative is the standard flat metric on $\mathbb{C}^n$.

Since this result rules out any (non-trivial) examples with positive bisectional curvature, we turn our attention to other notions of curvature. In Section 4 we consider the anti-bisectional curvature, defined as follows:

$$\mathfrak{A}(U, V) = 4R^h(U, \nabla U, V, \nabla V)$$

for two holomorphic vector fields $U$ and $V$. For Kähler Sasaki metrics, there is a notion of “non-negative anti-bisectional curvature” (abbreviated (NAB)) and “non-negative orthogonal anti-bisectional curvature” (abbreviated (NOAB)), which is defined precisely in Section 4. For $O(n)$-symmetric tube domains, we find several integral-differential inequalities which are equivalent to (NOAB) (see Proposition 6 for a precise statement). Using this, we find several new examples of such metrics. As a result, (NOAB) does not imply the same type of uniqueness properties as with the bisectional curvature.

Following the theme that positively curved metrics have desirable analytic properties, recent work by the first two authors [14] found a connection between orthogonal anti-bisectional curvature and optimal transport. More precisely, given a (NOAB) metric on a tube domain, it is possible to construct a cost function $c : X \times Y \to \mathbb{R}$ (with $X, Y \subset \mathbb{R}^n$) satisfying the (MTW) condition, which plays a crucial role in the regularity theory of optimal transport (see Section 5 for more details).

As such, we can use our examples of (NOAB) Kähler Sasaki metrics to find costs whose optimal transport has good regularity properties. For instance, we provide the following example.

Example. The cost function $c : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ given by $c(x, y) = \|x - y\| - C \log(\|x - y\| + C)$ for $C > 0$ satisfies (MTW).

For Kähler Sasaki metrics which are $O(n)$-symmetric, the induced cost function depends only on the Euclidean distance between $x$ and $y$. As such, by finding $O(n)$-symmetric metrics with (NOAB), we can find cost functions which have a natural geometric interpretation.

\footnote{Or whose orthogonal bisectional curvature is non-negative when $n \geq 3$.}

\footnote{As a note of caution, (NAB) (respectively (NOAB)) are weaker assumptions than assuming $\Re \geq 0$ for all (respectively all orthogonal) type $(1,0)$-vectors.}
1.2. Synthetic curvature of Kähler Sasaki metrics. Having seen how the study of tube domains has applications to optimal transport, we then use ideas from optimal transport to better understand the geometry of Kähler Sasaki metrics. In particular, we study synthetic curvature bounds when the Kähler potential is not $C^4$, (and thus the usual curvature expressions are not well defined). By analogy, these results are similar to the CAT(κ)-inequality, which provides a synthetic version of lower bounds on the sectional curvature.

As a preliminary example in Kähler geometry, we provide a synthetic formulation for Ricci bounds on tube domains. Although simple, this is an instructive example and serves as a proof of concept. We then use ideas from optimal transport to provide a synthetic characterization of non-negative orthogonal anti-bisectional curvature and non-negative anti-bisectional curvature. Our work is based off a paper by Guillen and Kitagawa, which introduced a condition known as quantitive quasi-convexity (abbreviated (QQConv)) [13] and showed that it provides a synthetic version of (MTW). Using this, we prove the following.

**Theorem.** Suppose $\Psi : \Omega \to \mathbb{R}$ is a $C^4$ strongly convex function. The associated Kähler Sasaki metric on $T\Omega$ has (NOAB) if and only if the cost function $c(x, y) = \Psi(x - y)$ satisfies (QQConv) for all pairs of sufficiently small Euclidean balls $B(p, \varepsilon), B(q, \varepsilon) \subset \Omega$.

Furthermore, we use a result of Figalli, Kim and McCann [7] to provide a synthetic version of (NAB).

1.3. Organization of the paper. Section 2 provides background information on Hessian manifolds and Kahler metrics on tube domains. In Section 3, we discuss the bisectional and orthogonal bisectional curvatures of $O(n)$-symmetric tube domains. In Section 4, we study the anti-bisectional curvature of rotational symmetric tube domains. Section 5 discusses the relationship between orthogonal anti-bisectional curvature and optimal transport. Finally, Section 6 studies curvature bounds on tube domains when the Kähler potential is not $C^4$.

There are also two appendices at the end of the paper. The first verifies that a particular family of metrics satisfies (NOAB). The second proves a lemma needed in Section 6 which also addresses a question first asked on MathOverflow [20].

2. Background on Hessian metrics and Kähler metrics on tube domains

In this section, we provide some background on Hessian manifolds and the Kähler metrics on their tube domains. A Riemannian manifold $(\Omega, g)$ is said to be a Hessian manifold if

1. $\Omega$ admits a flat connection $D$,
2. such that around every point $x \in \Omega$, there is an open neighborhood $U_x \subset \Omega$,
3. and a function $\Psi : U_x \to \mathbb{R}$, such that $g = D^2 \Psi$.

For convex domains in Euclidean space, we can construct Hessian metrics at will, simply by choosing a convex function and using the connection induced by differentiation in coordinates.

More precisely, we consider a convex domain $\Omega \subseteq \mathbb{R}^n$ with standard coordinates $x = (x_1, \ldots, x_n)$ and a strongly convex function $\Psi : \Omega \to \mathbb{R}$. For now, we will assume that

---

4Since Hessian manifolds must be affine (i.e. admit a flat connection), it is non-trivial to find compact examples which are not tori.
Clearly, the length of $\gamma$ is of infinite length as $\gamma \to \infty$. We have $|\gamma(s)|_h \geq |x'(s)|_g$, so if the curve $x(s)$ goes to the boundary of $\Omega$, then $x(s)$, hence $\gamma(s)$, is of infinite length as $g$ is complete. If, on the other hand, the curve $x(s)$ is contained in a compact subset $K$ of $\Omega$, then $y(s)$ must tend to infinity. Take $\varepsilon > 0$ such that $g \geq \varepsilon g_0$ on $K$, where $g_0$ is the Euclidean metric of $\mathbb{R}^n$, we have $|\gamma(s)|_h \geq |\gamma(s)|_{g_0} \geq \varepsilon |\gamma(s)|_{g_0}$. Hence the length of $\gamma(s)$ is again infinite. This completes the proof of the lemma. \hfill $\Box$

We now turn our attention to the curvature of Kähler Sasaki metrics. For brevity, we will not derive the full expressions for the curvature, which can be found in full detail in Satoh \cite{Satoh}. Following the convention of \cite{Harvey}, we denote $\Psi_{ij} = \frac{\partial^2 \Psi}{\partial x_i \partial x_j}$, etc., and use $\Psi^{i\ell}$ to denote the elements of the matrix inverse to $\Psi_{ij}$. For the Kähler manifold $(T\Omega, h)$, under the natural frame of the holomorphic coordinate $z$, we have $\frac{\partial}{\partial z_i} = \frac{1}{2}(\frac{\partial}{\partial x_i} - \frac{\partial}{\partial y_i})$, so we have

$$h_{\overline{i}j} = \Psi_{ij}, \quad h_{\overline{i}j\overline{k}} = \frac{1}{2}\Psi_{ijk}, \quad h_{\overline{i}j\overline{k}\ell} = \frac{1}{4}\Psi_{ijk\ell}, \quad h^{\overline{j}i} = \Psi^{ij}.$$
The components of the curvature tensor \( R^{(b)}_{ijkl} \) of \( h \) are given by

\[
R^{(b)}_{ijkl} = -\frac{1}{4} \Psi_{ijkl} + \sum_{p,q} \frac{1}{4} \Psi_{ipk} \Psi_{jql} \Psi^{pq}.
\]  

(2.1)

In particular, for tangent vectors \( u = \sum u_i \frac{\partial}{\partial x_i} \) and \( v = \sum v_i \frac{\partial}{\partial x_i} \), the bisectional curvature of \( h \) is given by

\[
R_{\varphi\varphi}^{(b)} = \sum_{p,q} \frac{1}{4} \Psi_{\varphi p\varphi q} + \frac{1}{4} \Psi_{\varphi \varphi p} \Psi_{\varphi \varphi q} \Psi^{pq}.
\]  

(2.2)

Here we denoted by \( \Psi_{\varphi p\varphi q} = \sum_{i,j} u_i u_j \Psi_{ijp\varphi q} \), etc. for the sake of simplicity.

2.2. \( O(n) \)-symmetric Hessian metrics. We now specialize our attention to the case when \( \Psi \) is rotationally symmetric. That is to say, we take \( \Omega \) to be the ball \( B_a \subseteq \mathbb{R}^n \) of radius \( a \) \((0 < a \leq \infty)\) centered at the origin. Furthermore, we suppose that \( \Psi(x) = \phi(r) \), where \( r = |x| = \sqrt{x_1^2 + \cdots + x_n^2} \) and \( \phi \) is a convex function.

We will write \( f(r) = \frac{1}{r} \phi'(r) \), so \( f \in C^\infty(0, a) \) and

\[
\Psi_{ij} = f \delta_{ij} + \frac{f'}{r} x_i x_j.
\]

For convenience, we write \( h = f + rf' \). It is straightforward to see that \( g_{\psi} \) is a metric if and only if both \( f \) and \( h \) are positive on \([0, a]\), as the eigenvalues of the matrix \((\Psi_{ij})\) are \( h \) and \((n - 1)\) copies of \( f \).

For such metrics, we provide an alternate characterization of completeness.

**Lemma 2.** A rotationally symmetric Hessian metric \( g_{\psi} \) is complete if and only if

\[
\int_0^a \sqrt{h} \, dr = \infty.
\]

**Proof.** Assume that \( g_{\psi} \) is complete. Consider \( \gamma(r) = (r, 0, \ldots, 0), \ 0 \leq r < a \). We have \( \gamma'(r) = \frac{\partial}{\partial x_1} \), so \( |\gamma'(r)|_g^2 = \Psi_{11} = f + rf' = h \). Therefore the length of \( \gamma \) is equal to \( \int_0^a \sqrt{h} \, dr \), which must be finite for the metric to be complete.

Conversely, assume that this integral is infinite. Let \( \gamma(s) = (x_1(s), \ldots, x_n(s)), 0 \leq s < b \) (where \( 0 < b \leq \infty \)), be a smooth curve in \( B_a \) approaching the boundary. We may assume that \( s \) is the arc-length parameter in the Euclidean metric \( g_0 \), namely, \( x_1^2 + \cdots + x_n^2 = 1 \).

We have

\[
|\gamma'(s)|_g^2 = \sum_{i,j=1}^n x_i x_j' \Psi_{ij} = \sum_{i=1}^2 x_i^2 + \frac{f'}{r} \left( \sum_{i=1}^n x_i x_i' \right)^2 = f + \frac{f'}{r} (rr')^2.
\]

By the Cauchy-Schwartz inequality, we have

\[
r^2 r'^2 = \left( \sum x_i x_i' \right)^2 \leq \sum x_i^2 \cdot \sum x_i'^2 = r^2 \cdot 1,
\]

so \( r'^2 \leq 1 \), and

\[
|\gamma'|_g^2 = f + f'rr'^2 \geq f r'^2 + f'rr'^2 = hr'^2,
\]

thus \( |\gamma'|_g \geq \sqrt{h} r' \), and we have

\[
\int_0^b |\gamma'(s)|_g \, ds \geq \int_0^b \sqrt{h} r' \, ds = \int_0^a \sqrt{h} \, dr = \infty.
\]

This proves that \( g_{\psi} \) is complete. \qed
3. **O(n)**-symmetric Kähler Sasaki metrics with non-negative bisectional curvature

In this section, we consider **O(n)**-symmetric Kähler metrics with non-negative bisectional curvature. For these metrics, we can simplify the curvature formulas by computing several further derivatives of the rotationally symmetric potential function \( \Psi(x) = \phi(r) \). In particular, we find that

\[ \Psi_i = \frac{P'}{r} x_i = f x_i, \quad \Psi_{ij} = f \delta_{ij} + \frac{f'}{r} x_i x_j, \quad \Psi^{ij} = \frac{1}{f} \left( \delta_{ij} - \frac{f'}{r h} x_i x_j \right). \tag{3.1} \]

For the third and fourth derivatives, we have

\[ \Psi_{ijk} = \dot{f} (\delta_{ij} x_k + \delta_{jk} x_i + \delta_{ik} x_j) + \ddot{f} x_i x_j x_k \tag{3.2} \]

\[ \Psi_{ijkl} = \dot{f} (\delta_{ij} \delta_{kl} + \delta_{jk} \delta_{il} + \delta_{ik} \delta_{jl}) + \dddot{f} x_i x_j x_k x_l + 2 \ddot{f} (\delta_{ij} x_k x_l + \delta_{jk} x_i x_l + \delta_{ik} x_j x_l + \delta_{jl} x_i x_k + \delta_{il} x_j x_k + \delta_{kl} x_i x_j) \tag{3.3} \]

where we have used the notation

\[ \dot{f} = \frac{1}{r} f', \quad \ddot{f} = \frac{1}{r} \left( \frac{1}{r} f' \right)', \quad \dddot{f} = \frac{1}{r} \left( \frac{1}{r} \left( \frac{1}{r} f' \right)' \right)'. \]

Now suppose that \( u = (u_1, \ldots, u_n) \) and \( v = (v_1, \ldots, v_n) \) are unit vectors in \( \mathbb{C}^n \) (in Euclidean norm). We denote

\[ \alpha_u = \langle u, x \rangle = \sum_i x_i u_i, \quad \alpha_v = \langle v, x \rangle, \quad \beta = \langle u, v \rangle, \quad \lambda = \langle u, v \rangle. \]

Using this notation, we find that

\[ \Psi_{uvp} = \dot{f} (\beta x_p + r \alpha_u v_p + r \alpha_v u_p) + r^2 \ddot{f} \alpha_u \alpha_v x_p \tag{3.4} \]

\[ \Psi_{uex} = \sum_{p=1}^n \Psi_{uvp} x_p = r^2 \dot{f} (\beta + 2 \alpha_u \alpha_v) + r^4 \ddot{f} \alpha_u \alpha_v \tag{3.5} \]

\[ \Psi_{uex} = \dot{f} \left( 1 + |\beta|^2 + |\lambda|^2 \right) + r^2 \dot{f} \left( |\alpha_u|^2 + |\alpha_v|^2 + 2 \text{Re} (\beta \overline{\alpha_u} \alpha_v + \lambda \overline{\alpha_u} \alpha_v) \right) \]

\[ + r^4 \dddot{f} |\alpha_u|^2 |\alpha_v|^2 \tag{3.6} \]

Furthermore, it follows that

\[ \sum_p |\Psi_{uvp}|^2 = r^2 \dot{f}^2 (|\beta|^2 + |\alpha_u|^2 + |\alpha_v|^2 + 4B + 2C) \]

\[ + r^6 \ddot{f}^2 |\alpha_u \alpha_v|^2 + r^4 \dddot{f}^2 (4|\alpha_u \alpha_v|^2 + 2B) \]

and

\[ |\Psi_{uex}|^2 = r^4 \dot{f}^2 (|\beta|^2 + 4|\alpha_u \alpha_v|^2 + 4B) \]

\[ + r^8 \ddot{f}^2 |\alpha_u \alpha_v|^2 + r^6 \dddot{f}^2 (4|\alpha_u \alpha_v|^2 + 2B) \]
where $\mathcal{B} = \text{Re}(\overline{\alpha_u} \alpha_v)$, and $\mathcal{C} = \text{Re}(\lambda \overline{\alpha_u} \alpha_v)$. Using the fact that $h - r^2 \hat{f} = f$, we get

$$
\sum_{p,q} \Psi_{uvp} \overline{\Psi_{uvq}} \Psi^{pq} = \frac{1}{f} \sum_p |\Psi_{uwp}|^2 - \frac{\hat{f}}{fh} |\Psi_{uvx}|^2 = \frac{r^2 \hat{f}^2}{f} \left( |\alpha_u|^2 + |\alpha_v|^2 + 2\mathcal{C} \right) + \frac{r^2 \hat{f}^2}{h} |\beta|^2 + |\alpha_u \alpha_v|^2 \left( \frac{4r^4 \hat{f} \hat{\bar{f}} + r^6 \hat{\bar{f}}^2}{fh} - \frac{4r^4 \hat{f}^3}{fh} \right) + \mathcal{B} \frac{2r^4 \hat{f} \hat{\bar{f}} + 4r^2 \hat{f}^2}{h}.
$$

Combining (3.6) with (3.7), we obtain

$$
4R_{\alpha \pi \beta \tau}^{(b)} = -\Psi_{u \pi \tau} \overline{\Psi_{uvq}} \Psi^{pq} = -\hat{f} \left( 1 + |\lambda|^2 \right) - \frac{\hat{f}}{h} |\beta|^2 + \left( \frac{r^2 \hat{f}^2}{f} - r^2 \hat{f} \right) \left( |\alpha_u|^2 + |\alpha_v|^2 + 2\mathcal{C} \right) + |\alpha_u \alpha_v|^2 \left( \frac{4r^4 \hat{f} \hat{\bar{f}} + r^6 \hat{\bar{f}}^2}{fh} - \frac{4r^4 \hat{f}^3}{fh} - r^4 \hat{f} \right) + \mathcal{B} \frac{4r^2 \hat{f}^2 - 2r^2 \hat{f}^2}{h}.
$$

In the special case when $u = \frac{\pi}{2}$ and $v \perp u$, we have $\beta = \lambda = \alpha_v = 0$, $\alpha_u = 1$, and $\mathcal{C} = \mathcal{B} = 0$, so in this case the curvature becomes

$$
-\hat{f} + \frac{r^2 \hat{f}^2}{f} - r^2 \hat{f} = -f (\log f)''.
$$

Note that in this case we have $\Psi_{\alpha \pi} = 0$, so $-f (\log f)''$ is a value of the orthogonal bisectional curvature of the Kähler Sasaki metric $h_\Psi$.

**Proposition 3.** Let $\Psi$ be a strongly convex rotationally symmetry function on the ball $B_a \subseteq \mathbb{R}^n$ of radius $0 < a \leq \infty$. Assume that the Kähler Sasaki metric $h_\Psi$ is complete. If either $n = 2$ and the bisectional curvature is everywhere nonnegative, or if $n \geq 3$ and the orthogonal bisectional curvature is everywhere nonnegative, then $a = \infty$ and the metric $h_\Psi$ is the flat Euclidean metric on $\mathbb{C}^n$.

**Proof.** Suppose that $h_\Psi$ were to have everywhere nonnegative orthogonal bisectional curvature ($n \geq 3$) or nonnegative bisectional curvature ($n = 2$). If $\alpha_u = \alpha_v = 0$, then $\mathcal{C} = \mathcal{B} = 0$, and

$$
4R_{\alpha \pi \beta \tau}^{(b)} = -\hat{f} \left( 1 + |\lambda|^2 + \frac{\hat{f}}{h} |\beta|^2 \right).
$$

Note that when $n \geq 3$, we can choose such $u$ and $v$ so that $\Psi_{\alpha \pi} = 0$. So under the assumptions on the curvature, we always have $\hat{f} \leq 0$ and $(\log f)'' \leq 0$. For convenience, denote $F = \log \hat{f}$, which is a smooth function on $[0, a)$. We have $F' \leq 0$, $F'' \leq 0$, and

$$
h = (rf)' = (1 + rF')e^F > 0.
$$

If $a < \infty$, then since $h \leq e^F \leq e^F(0)$ as $F$ is non-increasing, we see that $\int_0^a \sqrt{h} \, dr < \infty$, contradicting with the completeness of the metric. So we may assume that $a = \infty$. Since $h > 0$, we find that

$$
0 \geq F' > -\frac{1}{r}, \quad 0 < r < \infty.
$$
This forces $\lim_{r \to \infty} F(r) = 0$. But $F'$ is nonpositive and non-increasing, so must be constantly zero. This implies that $f$ is a positive constant, hence $h_\Psi$ the ($f$-multiple of the standard) flat complex Euclidean metric on $\mathbb{C}^n$. □

It is worth noting that Yau’s Uniformization Conjecture (partially proven by Liu [17]) states that any complete Kähler metric with non-negative bisectional curvature is biholomorphic to $\mathbb{C}^n$. We have made strong assumptions on the structure of our manifold (in particular $O(n) \times \mathbb{R}^n$-symmetry), which is why it is possible to conclude that the metric is actually flat, and not simply that it is biholomorphic to $\mathbb{C}^n$.

4. The Anti-Bisectional Curvature

For Kähler Sasaki metrics, there is also a notion of “anti-bisectional curvature,” which is defined as follows:

$$A(u, v) = \sum_{i,j,k,\ell} (-\Psi_{ijk\ell} + \sum_{p,q} \Psi_{ijp} \Psi_{k\ell q} \Psi_{pq}) u_i u_j v_k v_\ell,$$

(4.1)

where $u = (u_1, \ldots, u_n)$, $v = (v_1, \ldots, v_n)$ are vectors in $\mathbb{R}^n$.

**Definition 4** (Non-negative anti-bisectional curvature). The Kähler metric $h_\Psi$ has non-negative orthogonal anti-bisectional curvature (abbreviated (NAB)), if $A(u, v) \geq 0$ for any $u, v \in \mathbb{R}^n$.

**Definition 5** (Non-negative orthogonal anti-bisectional curvature). The Kähler metric $h_\Psi$ has non-negative orthogonal anti-bisectional curvature (abbreviated (NOAB)), if $A(u, v) \geq 0$ for any $u, v \in \mathbb{R}^n$ such that $\Psi_{uv} = \sum_{i,j} u_i v_j \Psi_{ij} = 0$.

Note that if we write $U = \sum_i u_i \frac{\partial}{\partial z_i}$ and $V = \sum_i v_i \frac{\partial}{\partial z_i}$. Then we have

$$A(u, v) = 4R^{(b)}(U, \nabla U, V, \nabla V).$$

(4.2)

So (NOAB) and (NAB) are positivity conditions on the curvature of $h$, but are different from requiring $h$ to have nonnegative orthogonal bisectional curvature or non-negative bisectional curvature. If one requires a Kähler metric $\omega$ to satisfy the curvature condition $R(X, \nabla X, Y, \nabla Y) \geq 0$ for any type (1,0) vectors $X$ and $Y$, then it is necessary for $\omega$ to have constant holomorphic sectional curvature and so be a complex space form. As such, the above (NOAB) and (NAB) conditions are specialized to Kähler Sasaki metrics and do not generalize naively to arbitrary Kähler metrics. We leave it as an open question to find meaningful generalization of these conditions for more general Kähler metrics.

4.1. The orthogonal anti-bisectional curvature of $O(n)$-symmetric metric. We now return to the case where the tube domain is rotationally symmetric. We want understand when $TB_\alpha$ will have (NOAB), which means

$$A(u, v) = -\Psi_{uv} + \sum_{p,q} \Psi_{up} \Psi_{vq} \Psi_{pq} \geq 0$$

(4.3)

for any $u, v \in \mathbb{R}^n$ such that $\Psi_{uv} = 0$. To find conditions which ensure (NOAB), we will rewrite the above expression in terms of some auxiliary functions.
Without loss of generality, we may assume that $|u| = |v| = 1$ under the Euclidean norm. Let us write $\beta = \langle u, v \rangle = \sum_i u_i v_i$, $\alpha_u = \langle u, \tilde{u} \rangle$, and $\alpha_v$ similarly. Then since

$$0 = \Psi_{uv} = \Psi_{ij} u_i v_j = f \beta + r^2 f \alpha_u \alpha_v,$$

we find

$$\beta = -\frac{r^2 f}{f} \alpha_u \alpha_v = \left(1 - \frac{h}{f}\right) \alpha_u \alpha_v. \quad (4.4)$$

From the formulas for $\Psi_{ij}$, $\Psi_{ijk}$, and $\Psi_{ijkl}$, we get

$$\Psi_{uk} = \frac{\partial}{\partial x_k} (r^2 f \alpha_u u_k) + r^2 f \alpha_u^2 x_k \quad (4.5)$$

$$\Psi_{ux} = -r^2 f (1 + 2 \alpha_u^2) + r^2 \tilde{f} \alpha_u \quad (4.6)$$

$$\Psi_{uuv} = f(1 + 2 \beta^2) + r^2 f (\alpha_u^2 + \alpha_v^2 + 4 \beta \alpha_u \alpha_v) + r^4 f \tilde{f} \alpha_u^2 \quad (4.7)$$

Combining Equations (4.5) and (4.6), we find

$$\Psi_{uuk} \Psi_{vkk} = r^2 f^2 (1 + 2 \alpha_u^2 + 2 \alpha_v^2 + 4 \beta \alpha_u \alpha_v) + r^4 f \tilde{f} (\alpha_u^2 + \alpha_v^2 + 4 \beta \alpha_u \alpha_v) + r^6 f \tilde{f} \alpha_u^2 \quad (4.8)$$

$$\Psi_{uxx} \Psi_{vxx} = r^4 f^2 (1 + 2 \alpha_u^2 + 2 \alpha_v^2 + 4 \beta \alpha_u \alpha_v) + r^6 f \tilde{f} (\alpha_u^2 + \alpha_v^2 + 4 \beta \alpha_u \alpha_v) + r^8 f \tilde{f} \alpha_u^2 \quad (4.9)$$

where $k$ is summed in the first line. Since $\beta = (1 - \frac{h}{f}) \alpha_u \alpha_v$ and

$$\frac{1}{f} - \frac{\tilde{f}}{fh} = \frac{1}{f} - \frac{h - f}{fh} = \frac{1}{h},$$

we obtain from Equations (4.8) and (4.9) that

$$\sum_{p,q} \Psi_{up} \Psi_{vq} \Psi_{pq} = \frac{1}{f} \sum_h \Psi_{uuk} \Psi_{vkk} - \frac{\tilde{f}}{fh} \Psi_{uxx} \Psi_{vxx}$$

$$= \frac{1}{f} \tilde{f}^2 \left(-4 \frac{h}{f}\right) \alpha_u^2 \alpha_v^2 + \frac{1}{h} r^2 f^2 (1 + 2 \alpha_u^2 + 2 \alpha_v^2 + 4 \beta \alpha_u \alpha_v) + r^4 f \tilde{f} \alpha_u^2 + \frac{1}{h} r^6 f \tilde{f} \alpha_u^2 \alpha_v^2.$$

Using the above identity and Equation (4.7), we find that

$$\Delta(u, v) = A + B (\alpha_u^2 + \alpha_v^2) + C \alpha_u \alpha_v^2, \quad (4.10)$$

where

$$A = \frac{1}{h} r^2 \tilde{f}^2 - \tilde{f},$$

$$B = \frac{2}{h} r^2 \tilde{f}^2 + \frac{1}{h} r^4 \tilde{f} \tilde{f} - r^2 \tilde{f},$$

$$C = \frac{4h}{f^2} r^2 \tilde{f}^2 + \frac{4}{h} r^2 \tilde{f}^2 + \frac{4}{h} r^4 \tilde{f} \tilde{f} + \frac{1}{h} r^6 \tilde{f}^2 - 2(1 - \frac{h}{f}) \tilde{f} - 4(1 - \frac{h}{f}) r^2 \tilde{f} - r^4 \tilde{f}.$$

Using the relation $r^2 \tilde{f} = h - f$ to simplify $A, B$ and $C$, we find

$$A = \frac{-\tilde{f}}{h}, \quad (4.11)$$

$$B = \frac{r^2}{h} (2 \tilde{f}^2 - \tilde{f}), \quad (4.12)$$

$$C = \left(-4 \frac{f}{h} + 2 + 8 \frac{h}{f} - 6 \frac{h^2}{f^2}\right) \tilde{f} + 4 \left(\frac{h}{f} - \frac{\tilde{f}}{h}\right) r^2 \tilde{f} + \frac{r^6}{h} \tilde{f}^2 - r^4 \tilde{f}. \quad (4.13)$$
Now let us try to understand the (NOAB) condition for these metrics. Recall that this says that \( \Psi(u, v) \geq 0 \) for any unit vectors \( u, v \in \mathbb{R}^n \) such that \( \Psi_{uv} = 0 \) (or equivalently, vectors for which \( \beta = (1 - \frac{h}{f})\alpha_u\alpha_v \)). Fixing the unit vector \( \hat{e} \) and denoting its orthogonal complement in \( \mathbb{R}^n \) by \( W \), we write

\[
\begin{align*}
u = \alpha_u \frac{r}{u} + u', \quad v = \alpha_v \frac{r}{v} + v',
\end{align*}
\]

where \( u', v' \in W \). Since \( u, v \) are unit vectors, we have \( |u'|^2 = 1 - \alpha_u^2 \) and \( |v'|^2 = 1 - \alpha_v^2 \). Also,

\[
\beta = \langle u, v \rangle = \alpha_u\alpha_v + \langle u', v' \rangle,
\]

so we get

\[
-h \int u_\alpha v = \langle u', v' \rangle = |u'|\ |v'| \cos \theta
\]

where \( \theta \) is the angle between \( u' \) and \( v' \). When \( n = 2 \), \( W \) is one dimensional, hence we get

\[
\frac{h^2}{f^2} \alpha_u^2 \alpha_v^2 \leq (1 - \alpha_u^2)(1 - \alpha_v^2)
\]

(4.14)

When \( n \geq 3 \), we find instead that

\[
\frac{h^2}{f^2} \alpha_u^2 \alpha_v^2 \leq (1 - \alpha_u^2)(1 - \alpha_v^2)
\]

(4.15)

Writing \( s = \alpha_u^2 \) and \( t = \alpha_v^2 \), we know that (NOAB) simply means that the function \( F(s, t) = A + B(s + t) + Cst \) is nonnegative for all \( s, t \in [0, 1] \) such that \( \frac{h^2}{f^2} st = (1 - s)(1 - t) \) when \( n = 2 \) or \( \frac{h^2}{f^2} st \leq (1 - s)(1 - t) \) when \( n \geq 3 \). Let \( \gamma \) be the curve segment:

\[
\frac{h^2}{f^2} st = (1 - s)(1 - t), \quad 0 \leq s, t \leq 1
\]

in the \( st \)-plane. If \( h = f \), then \( \gamma \) is the line segment \( s + t = 1 \). If \( h \neq f \), then \( \gamma \) is the segment of the hyperbola contained in the unit square, and it goes from \((1, 0)\) to \((0, 1)\). The intersection of \( \gamma \) with the diagonal line is \((s_0, s_0)\), where \( s_0 = \left(1 + \frac{h}{f}\right)^{-1} \). Let us define

\[
D = \left(1 + \frac{h}{f}\right)^2 A + 2 \left(1 + \frac{h}{f}\right) B + C,
\]

(4.16)

where \( A, B, \) and \( C \) are given by Equations 4.11, 4.12, and 4.13. We have the following.

**Proposition 6.** For a rotationally symmetric convex function \( \Psi \) on the ball \( B_n \subseteq \mathbb{R}^n \), \((TB_n, h_{\Psi})\) has (NOAB) if and only if

1. when \( n = 2 \): \( A + B, D \) are everywhere non-negative
2. when \( n \geq 3 \): \( A, A + B, D \) are everywhere non-negative.

Here, \( A, B \) and \( D \) are given by 4.11, 4.12, and 4.16, respectively.

**Proof.** As noted above, (NOAB) means that \( F(s, t) = A + B(s + t) + Cst \geq 0 \) for any \( 0 \leq s, t \leq 1 \) satisfying the above constraint condition, which means that \( s, t \) lies on the curve segment \( \gamma \) if \( n = 2 \) and \( s, t \) lies in the sub-region \( \Omega \) in the unit square bounded by the coordinate axes and \( \gamma \).

First let us assume that \( n = 2 \). Since \( F(1, 0) = A + B \) and \( F(s_0, s_0) = s_0^2 D \), we know that both \( A + B \) and \( D \) are non-negative when (NOAB) is satisfied. Conversely, if both \( A + B \)
and $D$ are non-negative, then it is easy to see that $F$ is non-negative along the entire curve segment $\gamma$, hence (NOAB) is satisfied.

Now assume that $n \geq 3$. If (NOAB) is satisfied, then $A = F(0,0)$, $A + B = F(1,0)$, $D = F(s_0, s_0)$ are all non-negative. Conversely, suppose that $A$, $A + B$, $D$ are all non-negative. Then it is not hard to see that $F$ is non-negative along the boundary of the domain $\Omega$. We claim that $F$ is also non-negative in the interior of $\Omega$. To see this, suppose that $(s_1, t_1)$ is a critical point of $F$ in the interior of $\Omega$. Then we have $B + Cs_1 = B + Ct_1 = 0$ and $s_1, t_1 \in (0,1)$. So $B$ and $C$ are non-zero and with opposite sign.

We consider two separate cases.

1. If $C > 0$, then $s_1 = t_1 = -\frac{B}{C}$ so $0 < -B < C$. On the other hand, since $A + B \geq 0$, we have $-B \leq A$. So $B^2 \leq AC$, hence
   \[ F(s_1, t_1) = A - \frac{B^2}{C} = \frac{1}{C}(AC - B^2) \geq 0. \]

2. On the other hand, if $C < 0$, then $AC - B^2 \leq AC \leq 0$, hence $F(s_1, t_1) = \frac{1}{C}(AC - B^2) \geq 0$.

In either case, $F(s_1, t_1) \geq 0$, so the orthogonal anti-bisectional curvature is non-negative. This completes the proof of the proposition. \( \square \)

The above proposition says that to find $O(n)$-symmetric metrics with (NOAB), we need only to focus on the derived functions $A$, $A + B$, and $D$. As such, we will analyze these terms more carefully. First let us try to understand the second order derivative term $A + B$.

To make the equation more tractable, it is helpful to define further auxiliary functions. To this end, we define
\[
\ell = \frac{1}{f}, \quad \lambda = \ell - \ell' \tag{4.17}
\]
Recall that in terms of the original convex function $\phi : \mathbb{R}_{\geq 0} \to \mathbb{R}$ used to define a radially symmetric potential, $\ell = \frac{\phi}{f}$. It follows immediately that $\ell$ is a positive function. From the formula $h = (rf)' = \frac{\ell - \ell'}{\ell} = \frac{\lambda}{\ell}$, we know that $\lambda$ is also a positive function. By a straightforward computation, we find
\[
A = \frac{\ell'}{r\lambda\ell}, \quad A + B = \frac{\ell''}{\ell \lambda} \tag{4.18}
\]
\[
D = \left( -\frac{1}{r\lambda\ell} + \frac{1}{r\ell^2} - \frac{3\lambda}{r\ell^3} + \frac{6\lambda^2}{r\ell^4} \right) \ell' + \left( \frac{7}{\lambda\ell^2} + \frac{6}{\ell^3} + \frac{8\lambda}{\ell^4} \right) \ell''
+ \left( \frac{4r}{\lambda\ell^3} + \frac{6r}{\ell^4} \right) \ell'^3 + \left( \frac{4r^2}{\lambda\ell^4} + \frac{6r}{\ell^5} - \frac{4\lambda}{\ell^6} \right) \ell'^4
- \left( \frac{2r}{\lambda\ell^4} + \frac{6r}{\ell^5} \right) \ell' \ell'' - \left( \frac{4r^2}{\lambda\ell^5} + \frac{r^2}{\ell^6} \right) \ell'^2 \ell'' + \frac{r^2}{\ell^6} \ell'''. \tag{4.20}
\]
From this, we get the following:

**Proposition 7.** Let $\Psi$ be a strictly convex rotationally symmetric function on the ball $B_a \subseteq \mathbb{R}^n$ (where $0 < a \leq \infty$) so that the Kähler Sasaki metric $g_\Psi$ is complete and satisfies (NOAB). If $n \geq 3$, then $a = \infty$. 
Proof. By the previous proposition, we have $A, A + B, D$ all non-negative. Thus $\ell' \geq 0, \ell'' \geq 0$. So $h = \frac{1}{\sqrt{\ell}}$ is non-increasing since $\lambda' = -r\ell'' \leq 0$. The completeness of $g_\Psi$ means that the integral $\int_0^a \sqrt{h} dr = \infty$. So $a$ must be $\infty$. \hfill \square

In particular, any such metric must be biholomorphic to $\mathbb{C}^n$ with its standard complex structure. This analysis also allows us to find several examples of such metrics, two of which we provide here.

Example 8. Consider the function $\ell = c + r$ on $[0, \infty)$, where $c > 0$ is a constant. We have $\ell' = 1, \ell'' = 0$, and $\lambda = c$. Hence

$$r\ell^4 D = \frac{1}{c} \left( 4r^3 + 4r^2\ell - 7r\ell^2 - \ell^3 \right) + \left( 6r^2 + 6r\ell + \ell^2 \right) + c \left( 8r - 3\ell \right) + 6c^2$$

$$= \frac{1}{c} \left( 13c^2 + 10c^2 r + c^3 \right) + \left( 13r^2 + 8cr + c^2 \right) + c \left( 5r - 3c \right) + 6c^2$$

$$= 3cr + 3c^2 = 3c \ell.$$

So the metric has (NOAB). Also, since $h = \frac{1}{\sqrt{\ell}}$, we have $\int_0^\infty \sqrt{h} dr = \infty$, hence $g_\Psi$ is complete.

We provide a second example, although verifying it is more involved and we postpone a proof that $D \geq 0$ to Appendix A.

Example 9. Consider the function $\ell = r + \frac{1}{L^2}, \quad L = \log(c + r)$

for $c \log(c)^3 \geq 2$. In this case, if we write $R = c + r$, then we have that

$$\lambda = \frac{1}{L^2} + \frac{2r}{RL^3} > 0, \quad \ell' = 1 - \frac{2}{RL^3} > 0, \quad \ell'' = \frac{2}{R^2L^3} + \frac{6}{R^2L^3} > 0,$$

and $\sqrt{h} = \frac{\sqrt{c}}{R} \sim \frac{\sqrt{c}}{R}$, so $\int_0^\infty \sqrt{h} dr = \infty$ and the metric is complete. Furthermore, $D \geq 0$ so this is another examples of an $O(n)$-symmetric complete metric with (NOAB).

5. The MTW tensor and the Regularity of Optimal Transport

Apart from complex geometry, the primary motivation for considering anti-bisectional curvature arises from optimal transport. To explain this, we first discuss some preliminary background on optimal transport. For a more complete reference on this topic, we refer the reader to the survey paper of DePhilippis and Figalli [5] or the book by Villani [25].

The original transport problem was considered by Monge in 1781 [22]. In his work, he sought to find the most cost-efficient way to transport rubble ($d^\prime eblais$) into a desired configuration to build a fortification ($remblais$). In the modern setting, this problem is formalized in terms of the Kantorovich formulation.

Given probability spaces $(X, \mu)$ and $(Y, \nu)$ and a lower semi-continuous cost function $c(x, y) : X \times Y \to \mathbb{R}$, the Kantorovich problem seeks to find a coupling $\pi$ of $\mu$ and $\nu$ which achieves

$$\inf_{\pi \in \Pi(\mu, \nu)} \int_{X \times Y} c(x, y) d\pi.$$  \hfill (5.1)

\footnote{In Monge’s work, the cost of transporting a unit of mass from $x$ to $y$ was taken to be $|x - y|$.}
Here, \( \Pi(\mu, \nu) \) is the set of all couplings of \((X, \mu)\) and \((Y, \nu)\) (i.e. probability measures on \(X \times Y\) whose marginals are \(\mu\) and \(\nu\), respectively). Under mild regularity assumptions on \(\mu, \nu\) and \(c\), such a coupling exists, which intuitively describes how mass from \(X\) is transported to \(Y\). In general, the optimal coupling may split mass at a single point and distribution it throughout \(Y\). However, when \(X\) and \(Y\) are domains in Euclidean space (or domains in a smooth manifold) and certain technical conditions hold, Gangbo and McCann \[9\] showed that the optimal coupling is induced by a map \(T: X \to Y\). More precisely, they showed the following result, which is based off an earlier work of Brenier \[2\] for the cost function \(c(x, y) = |x - y|^2\).

**Theorem.** Let \(X\) and \(Y\) be two open domains of \(\mathbb{R}^n\) and consider a cost function \(c: X \times Y \to \mathbb{R}\). Suppose that \(d\mu\) is a smooth probability density supported on \(X\) and that \(d\nu\) is a smooth probability density supported on \(Y\). Suppose that the following conditions hold:

1. The cost function \(c\) is of class \(C^4\) with \(\|c\|_{C^4(X \times Y)} < \infty\).
2. The following two conditions hold, which are collectively called (Twist):
   a. For any \(x \in X\), the map \(Y \ni y \mapsto D_x c(x, y) \in \mathbb{R}^n\) is injective.
   b. For any \(y \in Y\), the map \(X \ni x \mapsto D_y c(x, y) \in \mathbb{R}^n\) is injective.
3. The mixed Hessian matrix \(c_{i,j} = \frac{\partial^2}{\partial x^i \partial y^j} c(x, y)\) is invertible for all \((x, y) \in X \times Y\). In other words, \(\det(c_{i,j}(x, y)) \neq 0\). Through the rest of the paper, we will denote this condition as (NonDeg).

Then:

1. There exists a unique solution to the Kantorovich problem \((5.1)\).
2. This solution is induced by a measurable map \(T: X \to Y\) satisfying \(T_\sharp \mu = \nu\), which is injective \(d\mu\)-a.e.
3. There exists a function \(u: X \to \mathbb{R}\) such that \(T_u(x) := c \cdot \exp_x(\nabla u(x))\), where \(c \cdot \exp\) is the so-called \(c\)-exponential map.
4. The potential \(u\) satisfies the following Monge-Ampère type equation
   \[
   |\det(\nabla T_u(x))| = \frac{d\mu(x)}{d\nu(T_u(x))}\quad \mu - \text{a.e.} \tag{5.2}
   \]

In other words, the optimal transport is solved by a transport map, which sends each point in \(X\) to a unique point in \(Y\). Furthermore, we can solve for this map by solving a fully non-linear equation of Monge-Ampère type. In this case, it is of interest to determine the continuity properties of \(T\), which is to ask whether nearby points in \(X\) are sent to nearby points in \(Y\). Even for smooth costs and measures, a priori this potential is merely Lipschitz\(^6\) and so the transport map may be discontinuous. Determining the regularity of \(T\) is known as the regularity problem of optimal transport, and is an active area of research.

In the years after Brenier’s initial work, much of the focus for the regularity problem was for the cost function \(c(x, y) = |x - y|^2\), in which case the Monge-Ampère equation \((5.2)\) takes a simple form. For this cost function, Caffarelli and others established a priori interior \(C^2\)-estimates for weak solutions to \((5.2)\) when

1. \(X\) and \(Y\) are strictly convex domains, and
2. the associated densities \(d\mu\) and \(d\nu\) are bounded away from 0 and \(\infty\).

\(^6\)A deep theorem of De Phillipis and Figalli \[4\] shows that for smooth costs and measures, the transport is smooth away from a singular set of measure zero.
These estimates imply a $C^1$-estimate for $T$ (and also higher-order estimates using elliptic bootstrapping for the linearized operator). However, this work did not address the regularity for more general cost functions.

In 2005, breakthrough work of Ma, Trudinger and Wang found two structural conditions, one on the cost function and another on the domains $X$ and $Y$ that are sufficient to prove regularity for the optimal transport. We will discuss the condition on $X$ and $Y$ in Section 6, but for now, we will focus on the condition on the cost function, which is that the so-called MTW tensor is non-negative. In order to define the MTW tensor, we first introduce some notation. In the following, we use $c_{I,J}$ to denote $\partial |I| \partial_x I \partial |J| \partial_y J$ for multi-indices $I$ and $J$ and $c^{i,j}$ to denote the matrix inverse of the mixed Hessian $c_{i,j}$. Using this notation, for a $C^4$ cost function which satisfies (NonDeg), the MTW tensor (denoted $\mathcal{S}$) is defined as follows:

$$\mathcal{S}(\xi, \eta) = \sum_{i,j,k,l,p,q,r,s} (c_{ij,p} c_{p,q} c_{q,rs} - c_{ij,rs}) c^{r,k} c^{s,l} \xi^i \xi^j \eta^k \eta^l \quad (5.3)$$

In this formula, $\eta$ is a vector and $\xi$ is a covector. The MTW tensor is a fourth-order quantity which scales quadratically in $\xi$ and $\eta$. Although it is not immediately obvious, this expression transforms tensorially under change of coordinates.

**Definition.** A $C^4$ cost function with invertible mixed Hessian satisfies:

1. (MTW) If $\mathcal{S}(\xi, \eta) \geq 0$ for all vector-covector pairs satisfying $\eta(\xi) = 0$. Such cost functions are also said to be weakly-regular.
2. (MTW($\kappa$)) If $\mathcal{S}(\xi, \eta) \geq \kappa |\xi|^2 |\eta|^2$ for all vector-covector pairs satisfying $\eta(\xi) = 0$. This is also known as strong MTW non-negativity.
3. (NNCC) If $\mathcal{S}(\xi, \eta) \geq 0$ for all vector-covector pairs, not necessarily orthogonal. This condition is also known as non-negative cost curvature.

As mentioned previously, to prove regularity for optimal transport, it is necessary to prove a priori estimates for equations of the form (5.2). A full overview of this line of research would take us too far from the main focus of this paper. However, to motivate our considerations, we present one example of such a regularity result, proven by Figalli, Kim, and McCann [8].

**Theorem (8, Theorem 2.1).** Let $X$ and $Y$ be two domains in $\mathbb{R}^n$ and let $c$ be a cost function $c : X \times Y \to \mathbb{R}$. Consider two probability densities $f(x)$ and $g(y)$ supported on $X$ and $Y$ and suppose that the following conditions hold:

1. The cost function $c$ is of class $C^4$ with $\|c\|_{C^4(X \times Y)} < \infty$
2. The cost function satisfies (Twist) and (NonDeg)
3. The density $f$ is bounded from above on $X$ and the density $g$ bounded away from both zero and infinity on $Y$.
4. The domains $X$ and $Y$ are uniformly relatively $c$-convex. (See Definition [13])
5. The cost function satisfies (MTW).

Then the optimal transport from $f$ to $g$ is induced by a map $T \in C^\alpha(\overline{X}) \ X' \subset X$ is an open set with $f$ bounded uniformly away from zero.

5.1. **The Anti-Bisectional Curvature and the MTW tensor.** To relate the MTW tensor to the curvature of tube domains, we specialize our attention to cost functions of the form $c(x, y) = \Psi(x - y)$ for some strongly convex function $\Psi : \Omega \to \mathbb{R}$ (henceforth $\Psi$-costs).
Such cost functions were first studied by Gangbo and McCann [9], although they did not use this terminology.

For a $C^4$ $\Psi$-cost, the MTW tensor is proportional to the orthogonal anti-bisectional curvature of the associated Kähler Sasaki metric (Theorem 6 of [14]). As a result of this, a $\Psi$-cost satisfies (MTW) iff the associated Kähler Sasaki metric satisfies (NOAB). Furthermore, the cost-curvature is proportional to the anti-bisectional curvature, so (NNCC) for a $\Psi$-cost corresponds to (NAB) for the associated Kähler Sasaki metric.

From this observation, we can use our examples to generate $\Psi$-costs with (MTW). Furthermore, these costs will satisfy a growth condition at infinity, which corresponds to completeness for the Kähler Sasaki metric (or equivalently completeness of the underlying Hessian manifold). As a shorthand for this, we say that a $\Psi$-cost is complete if the associated Hessian manifold is complete as a Riemannian manifold.

Example 10. The cost function $c(x, y) = \|x - y\| - c \log(\|x - y\| + c)$ is a complete cost function which satisfies (MTW).

This is the cost function corresponding to Example 8 (after integrating out to solve for $\phi$). Unfortunately, it is not possible to write out a closed form cost associated with Example 9, as $r \cdot f$ does not have elementary anti-derivative.

6. Synthetic Notions of Curvature in Complex Geometry

In the previous section, we used analysis from complex geometry to find cost functions of interest to optimal transport. Here, we do the converse, and use optimal transport to better understand the complex geometry. More precisely, we show how optimal transport can be used to define synthetic curvature bounds in Kähler geometry. We use the term “synthetic curvature” in the sense of defining curvature for low-regularity metric spaces, which may not be smooth enough for the Riemann curvature tensor to be defined. This notion has also been called “coarse curvature” (see, e.g., [1]), but we will not use this terminology. Although the concept is perhaps best understood by analogy, we will use the following definition for synthetic curvature bounds.

**Definition** (Synthetic curvature bounds). A condition $Q_\kappa$ is a synthetic lower bound for a curvature tensor $S$ if the following two conditions hold:

1. On a smooth manifold $M$ where $S$ is defined, $S \geq \kappa \iff Q_\kappa$.
2. The condition $Q_\kappa$ is well-defined for spaces with low regularity (where $S$ is not well-defined).

One can define synthetic curvature upper bounds analogously. Note that we have purposely left the condition $Q_\kappa$ and the curvature tensor $S$ ambiguous, so as to make this definition as general as possible. Depending on the context, $S$ might be the sectional curvature, Ricci curvature, scalar curvature, or any other sort of curvature. The main goal of this section is to define synthetic versions of (NOAB) and (NAB) on tube domains. However, it is instructive to first consider several examples of synthetic curvature.

To motivate the definition of a synthetic curvature bound, it is worth considering the $\text{CAT}(\kappa)$-inequality, which is the prototypical example.
Theorem (CAT(\(\kappa\))-inequality). Suppose \(M\) is a Riemannian manifold with sectional curvature \(S\) satisfying \(S \geq \kappa\). Denote the distance function on \(M\) by \(d\). Let \(\triangle pqr\) be a geodesic triangle in \(M\) (i.e. a triangle whose sides are geodesics) such that

1. the sides \(\overline{pq}\), \(\overline{pr}\) and \(\overline{qr}\) are minimal, and
2. if \(\kappa > 0\), all of the sides have length at most \(\frac{\pi}{\sqrt{\kappa}}\).

For comparison, let \(M_\kappa\) be a simply connected space of constant curvature \(\kappa\) and consider \(\triangle p'q'r'\) a geodesic triangle in \(M_\kappa\) with

1. length\((\overline{pq'})\) = length\((\overline{p'q'})\),
2. length\((\overline{pr'})\) = length\((\overline{p'r'})\), and
3. length\((\overline{qr'})\) = length\((\overline{q'r'})\).

For any pair of points \((x, y) \in \overline{pq} \times \overline{pr} \subset M \times N\), consider the pair \((x', y') \in \overline{p'q'} \times \overline{p'r'} \subset M_\kappa \times M_\kappa\) satisfying \(d(p, x) = d'(p', x')\) and \(d(p, y) = d'(p', y')\).

Then the following inequality holds.

\[
d(x, y) \leq d'(x', y').
\] (6.1)

In fact, this result characterizes sectional curvature bounds, in that whenever a Riemannian manifold has some sectional curvature smaller than \(\kappa\), it is possible to find a small geodesic triangle where inequality (6.1) fails. Furthermore, if we use Inequality 6.1 as the definition for sectional curvature bounds, this has the additional advantage in that it is well-defined on spaces which are not smooth manifolds. In this vein, a complete geodesic space which satisfies the inequality 6.1 is said to be a CAT(\(\kappa\))-spaces [11] and play an important role in metric geometry and geometric group theory.

6.1. Synthetic Ricci bounds on Kähler manifolds. For a simple though instructive example of this idea in complex geometry, we now discuss a synthetic formulation for Ricci bounds. On a smooth Kähler manifold, the Ricci form is given by the formula

\[
\rho = -\sqrt{-1} \partial \bar{\partial} \log \det \partial \bar{\partial} \Psi,
\] (6.2)

where \(\Psi\) is the Kähler potential (i.e. The Kähler form \(\omega\) satisfies \(\omega = \partial \bar{\partial} \Psi\)). The Ricci curvature is bounded below (respectively above) by a constant \(\kappa\) if

\[
\rho \geq \kappa \omega \quad \text{(respectively \(\leq \kappa \omega\)).}
\] (6.3)

The above inequality should be interpreted in the sense of \((1,1)\)-forms. That is to say, given a holomorphic vector \(X\), the above inequality implies that \(\rho(X, \overline{X}) \geq \kappa \omega(X, \overline{X})\). To rephrase this in synthetic terms, we consider the function \(Q_\kappa = \log \det \partial \bar{\partial} \Psi + \kappa \Psi\) and say that the Ricci curvature is bounded above (or below) by \(\kappa\) whenever \(Q_\kappa\) is plurisubharmonic (plurisuperharmonic)\(^8\). When \(\Psi\) is \(C^4\), this is equivalent to Ricci bounds in the normal sense. However, this definition does not require \(Q_\kappa\) to be \(C^2\), so we are able to define Ricci curvature bounds when the potential is only \(C^3\) (which is the natural regularity so that the Kähler condition \(d\omega = 0\) is well-defined).

\(^8\)This is a slight abuse of notation from our definition of synthetic curvature bounds, where \(Q_\kappa\) was a condition, instead of a function, but this is not important.
For Kähler Sasaki metrics on tube domains, we can simplify this further. For these metrics, the Ricci form simplifies to
\[ \rho_{ij} = \frac{\partial^2}{\partial x^i \partial x^j} \log \det \left[ \frac{\partial^2 \Psi}{\partial x^k \partial x^l} \right]. \]

Therefore, if we define
\[ Q_\kappa = \log \det \left[ \frac{\partial^2 \Psi}{\partial x^k \partial x^l} \right] + \kappa \Psi \tag{6.4} \]
we can see that the Ricci curvature of a Sasaki metric is bounded below by \( \kappa \) if and only if \( Q_\kappa \) is convex (on \( \Omega \)). This immediately implies the following proposition.

**Proposition 11.** A \( C^4 \) Kähler Sasaki metric has Ricci curvature bounded below by \( \kappa \) iff the function \( Q_\kappa \) satisfies
\[ \lambda Q_\kappa(p_1) + (1 - \lambda) Q_\kappa(p_2) \geq Q_\kappa(\lambda p_1 + (1 - \lambda) p_2) \]
for all \( p_1, p_2 \in \Omega \) and \( 0 < \lambda < 1 \).

As such, convexity (or concavity) of \( Q_\kappa \) gives a way to define bounds on the Ricci curvature. For \( C^4 \) potentials, this is equivalent to Ricci curvature bounds in the usual sense, but has the advantage of being defined for less smooth potentials. It is worth noting that there are many other ways of defining synthetic bounds for Ricci curvature, several of which can be defined for less regular metric-measure spaces. We refer to the work of Villani [26] and Ache and Warren [1] for some references on this topic.

### 6.2. A synthetic formulation of (NOAB).

We now come to the main focus of this section, which is to provide a synthetic version of non-negative orthogonal bisectional curvature. In order to do so, it is first necessary to provide some additional background on Hessian manifolds and optimal transport.

#### 6.2.1. Dual coordinates and \( c \)-segments.

Given a Hessian manifold on a domain \( \Omega \) with metric
\[ g_{ij} = \frac{\partial^2 \Psi}{\partial x^i \partial x^j}, \]
there are a set of dual coordinates \( \theta^i \in \Omega^* \) which are given by
\[ \theta^i = \frac{\partial \Psi}{\partial u^i}. \tag{6.5} \]

As implied by the name, these functions form coordinates for the Hessian manifold. Furthermore, in these coordinates, the metric is also given by the second derivative of a convex function. The potential function in the dual coordinates is the Legendre dual \( \Psi^* \), which satisfies
\[ \Psi^*(\theta) = \sup_{x \in \Omega} \langle \theta, x \rangle - \Psi(x). \]

These dual coordinates play an essential role in the study of Hessian manifolds, and also in the optimal transport of \( \Psi \)-costs. To explain this, we first provide a definition of \( c \)-segments and relative \( c \)-convexity.
Definition 12 (c-segment). For a $C^1$ cost function $c : X \times Y \rightarrow \mathbb{R}$, a c-segment in $X$ with respect to a point $y$ is a solution set $\{x\}$ to $D_y c(x, y) \in \ell$ for $\ell$ a line segment in $\mathbb{R}^n$. A c∗-segment in $Y$ with respect to a point $x$ is a solution set $\{y\}$ to $D_x c(x, y) \in \ell$ where $\ell$ is a line segment in $\mathbb{R}^n$.

Definition 13 (c-convexity). A set $X$ is c-convex relative to a set $Y$ if for any two points $x_0, x_1 \in X$ and any $y \in Y$, the c-segment relative to $y$ connecting $x_0$ and $x_1$ lies in $X$. Similarly we say $Y$ is c∗-convex relative to $X$ if for any two points $y_0, y_1 \in Y$ and any $x \in X$, the c∗-segment relative to $x$ connecting $y_0$ and $y_1$ lies in $Y$.

For a cost function of the form $\Psi(x \cdot y)$, c-segments correspond to line segments in the $\theta$-coordinates. As such, for costs of this form, we have the following result.

Proposition (14, Proposition 8). For a $\Psi$-cost, a set $Y$ is c-convex relative to $X$ if and only if, for all $x \in X$, the set $x - Y \subseteq \Omega$ is convex in terms of the dual coordinates $\theta$.

6.2.2. A Synthetic Version of (MTW). With the notions of dual coordinates and c-segments in hand, we now provide a synthetic version of MTW non-negativity. To this end, we introduce the condition known as quantitative quasiconvexity, which was introduced by Guillen and Kitagawa [13].

Definition (Quantitative Quasiconvexity). A cost function is said to be quantitatively quasiconvex (denoted (QQConv)) if there is a universal constant $M \geq 1$ such that for any points $x, x_0, x_1 \in X$ and $y, y_0, y_1 \in Y$

\[
-c(x, y(t)) + c(x, y_0) - (-c(x_0, y(t)) + c(x_0, y_0)) \\
\leq Mt(-c(x, y_1) + c(x, y_0) - (-c(x_0, y_1) + c(x_0, y_0))), \quad \forall t \in [0, 1] \\
-c(x(s), y) + c(x_0, y) - (-c(x(s), y_0) + c(x_0, y_0)) \\
\leq Ms(-c(x_1, y) + c(x_0, y) - (-c(x_1, y_0) + c(x_0, y_0))), \quad \forall s \in [0, 1].
\]

Here, $y(t)$ is the c-segment with respect to $x_0$ to $y_1$, and $x(s)$ is the c-segment with respect to $y_0$ from $x_0$ to $x_1$. If this inequality holds with $M = 1$, then we say that the function is quantitatively convex, which we denote (QuantConv).

Quantitative quasiconvexity only requires $C^1$ smoothness for the cost function (so that c-segments are well defined), and so plays an important role in low-regularity optimal transport.

Lemma (13, Lemma 2.23). Suppose $c$ is a $C^4$ cost function\footnote{The actual regularity required of $c$ is slightly less, but $C^4$-regularity is sufficient for the MTW tensor to be defined.} and that $c, X,$ and $Y$ satisfy the following conditions.

1. $c$ satisfies (Twist).
2. $c$ satisfies (Nondeg) (i.e. det $(D_x D_y c(x, y)) \neq 0$).
3. The domains $X$ and $Y$ are uniformly relatively c-convex. (i.e. $X$ and $Y$ satisfy (DomConv)).

Then the cost $c$ satisfies (MTW) iff it satisfies (QQConv).

We refer to [13] for a proof of the lemma. For our purposes, this immediately implies the following corollary.
Corollary 14. Suppose $\Psi: \Omega \to \mathbb{R}$ is a $C^4$ convex function and that $X$ and $Y$ are subsets of $\mathbb{R}^n$ with $S = X - Y \subset \Omega$. Suppose that for all $x_0 \in X$, $y_0 \in Y$, the sets $x_0 - Y$ and $X - y_0$ are convex when viewed in the dual coordinates $\theta$. Then $\Psi$ induces a Kähler Sasaki metric with (NOAB) on $T_\Omega$ if and only if the associated $\Psi$-cost satisfies (QQConv).

Proof. To prove this, it suffices to show that the (Twist) and (Nondeg) assumptions in the previous lemma are satisfied. For a $\Psi$-cost, the map $y \mapsto -D_{x_0} c(x_0, y)$ simplifies to $y \mapsto -D_{x_0} \Psi(x_0 - y)$, which is simply the point $x_0 - y$ expressed in terms of the dual $\theta$ coordinates. For a strongly convex function, the transition maps from primal to dual coordinates are invertible, which is equivalent to (Twist).

Furthermore, when $\Psi$ is strongly convex, $\text{Hess}(\Psi) > 0$. This immediately implies that the cost is non-degenerate as well. Finally, for a $\Psi$-cost, (DomConv) is equivalent to convexity of $x - Y$ and $X - y$ within the dual coordinates, which is exactly the second assumption in this corollary. From this, we can apply the result of Guillen and Kitagawa to show that (QQConv) is equivalent to (MTW) for the cost $\Psi(x - y)$, which furthermore implies that $\Psi$ induces a Kähler Sasaki metric with (NOAB).

6.2.3. The role of (DomConv). In Corollary 14, the assumption of (DomConv) is somewhat awkward. It plays a crucial role in the regularity of optimal transport, but is a bit out of place from the perspective of curvature bounds. When the potential is strongly convex and bounded in $C^4$, it is possible to choose $X$ and $Y$ so that (DomConv) is always satisfied. To do this, we use the following lemma, proven in the appendix (Section B).

Lemma 15. Suppose $\Omega$, $g$ is a Hessian manifold with local potential $\Psi$ that is strongly convex and bounded in $C^4$. Then there exists an $c_0 > 0$ so that all balls $B(p, \epsilon)$ ($p \in \Omega$, $\epsilon < c_0$) in the primal coordinates $x$ are convex when expressed in terms of the dual coordinates $\theta$.

Using this result, for a point $p \in \Omega$, we choose $x_0$ and $y_0$ so that $p = x_0 - y_0$. We then set $X$ and $Y$ so that $X - y_0$ and $x_0 - Y$ are small balls of radius $\epsilon$ in the primal coordinates. For sufficiently small balls, the previous lemma shows that $x - Y$ and $X - y$ are convex sets in dual coordinates for every $x \in X$ and $y \in Y$, which implies $X$ and $Y$ are relatively $c$-convex. Combining these results, we obtain the following theorem.

Theorem 16. Suppose $\Psi: \Omega \to \mathbb{R}$ is a $C^4$ strongly convex function. The associated Kähler Sasaki metric on $T_\Omega$ satisfies (NOAB) if and only if there exists an $c > 0$ so that the $\Psi$-cost $c(x, y) = \Psi(x - y)$ satisfies (QQConv) for all pairs of Euclidean balls $B(x_1, \epsilon)$, $B(x_2, \epsilon)$ for $x_1, x_2 \in \Omega$.

Strictly speaking, this is not a synthetic version of (NOAB), as Lemma 15 requires $C^4$-regularity of $\Psi$ to guarantee pairs of small Euclidean balls are relatively $c$-convex. However, it may be possible to refine this result to give a version of (NOAB) when $\Psi$ is less regular than $C^4$. At least $C^3$-regularity is necessary for the Kähler form to satisfy $d\omega = 0$. Furthermore, Guillen and Kitagawa were able to establish continuity for the solution to the Monge problem when the cost function is $C^3$ and satisfies (QQConv). This suggests that $C^3$ smoothness of $\Psi$ is a critical threshold for optimal transport.

We can also provide a synthetic version of non-negative anti-bisectional curvature. This is based on the equivalence between (NNCC) and (QuantConv) for $C^4$ cost functions, which was proven by Figalli, Kim and McCann (see Lemma 6.1 of [7]). We thank Professor Kitagawa for informing us of this result. Using this observation, we have the following.
Corollary 17. Suppose $\Psi : \Omega \rightarrow \mathbb{R}$ is a $C^4$ strongly convex function. Then $\Psi$ induces a Kähler Sasaki metric with non-negative bisectional curvature on $T\Omega$ if and only if there exists an $\epsilon > 0$ so that the $\Psi$-cost $c(x, y) = \Psi(x - y)$ satisfies (QuantConv) for all pairs of Euclidean balls $B(x_1, \epsilon), B(x_2, \epsilon)$ for $x_1, x_2 \in \Omega$.

We leave as a future question to determine a synthetic version of strong MTW non-negativity (i.e. $(MTW(\kappa))$), and in turn strong lower bounds for the orthogonal anti-bisectional curvature. It would also be of interest to modify (QuantConv) to find a synthetic version of non-negative holomorphic sectional curvature (possibly by restricting the choices of $x$ and $x_0$).

As a final remark, we note that there is also a natural pseudo-Riemannian framework of optimal transport [15] [16]. In this framework, the MTW tensor is the sectional curvature of light-like planes. As such, we expect that there are versions of Theorem 16 and Corollary 17 which provide synthetic curvature bounds for pseudo-Riemannian metrics. In this paper, we have focused on the complex framework, so will not address the pseudo-Riemannian framework further.

7. Acknowledgements

The first author would like to thank Jun Kitagawa for his helpful comments and discussions.

References

[1] Ache, A. G., & Warren, M. W. (2017). Coarse Ricci Curvature as a Function on $M \times M$. Results in Mathematics, 72(4), 1823-1837.
[2] Brenier, Y. Decomposition polaire et réarrangement monotone des champs de vecteurs. C. R. Acad. Sci. Paris Ser. I Math. 305, 19 (1987)
[3] Chen, X. X. (2007). On Kähler manifolds with positive orthogonal bisectional curvature. Advances in Mathematics, 215(2), 427-445.
[4] De Philippis, G., & Figalli, A. (2015). Partial regularity for optimal transport maps. Publications mathématiques de l'IHÉS, 121(1), 81-112.
[5] De Philippis, G., & Figalli, A. (2014). The Monge-Ampère equation and its link to optimal transportation. Bulletin of the American Mathematical Society, 51(4), 527-580.
[6] Dombrowski, P. (1962). On the Geometry of the Tangent Bundle. Journal für Mathematik. Bd, 210(1/2), 10.
[7] Figalli, A., Kim, Y. H., & McCann, R. J. (2011). When is multidimensional screening a convex program?. Journal of Economic Theory, 146(2), 454-478.
[8] Figalli, A., Kim, Y. H., & McCann, R. J. (2013). Hölder continuity and injectivity of optimal maps. Archive for Rational Mechanics and Analysis, 209(3), 747-795.
[9] Gangbo, W., & McCann, R. J. (1995). Optimal maps in Monge’s mass transport problem. Comptes Rendus de l’Académie des Sciences-Série I-Mathematique, 321(12), 1653.
[10] Gangbo, W., & McCann, R. J. (1996). The geometry of optimal transportation. Acta Mathematica, 177(2), 113-161.
[11] Gromov, M. (1987). Hyperbolic groups. In Essays in group theory (pp. 75-263). Springer, New York, NY.
[12] H.L. Gu & Z.H. Zhang, An Extension of Moks Theorem on the Generalized Frankel Conjecture, Sci. China Math. 53 (2010), no. 5, 12531264, MR 2653275, Zbl 1204.53058
[13] Guillen, N., & Kitagawa, J. (2015). On the local geometry of maps with $c$-convex potentials. Calculus of Variations and Partial Differential Equations, 52(1-2), 345-387.
[14] G. Khan & J. Zhang, On the Kähler geometry of certain optimal transport problems, preprint.
[15] Kim, Y. H., & McCann, R. J. (2010). Continuity, curvature, and the general covariance of optimal transportation. Journal of the European Mathematical Society, 12(4), 1009-1040.
[16] Kim, Y. H., McCann, R. J., & Warren, M. (2010). Pseudo-Riemannian geometry calibrates optimal transportation. Mathematical Research Letters, 17(5), 1183-1197.
[17] Liu, G. (2019). On Yau’s uniformization conjecture. Cambridge Journal of Mathematics, 7(1/2), 33-70.
[18] Loeper, G. (2009). On the regularity of solutions of optimal transportation problems. Acta mathematica, 202(2), 241-283.
[19] Ma, X. N., Trudinger, N. S., & Wang, X. J. (2005). Regularity of potential functions of the optimal transportation problem. Archive for rational mechanics and analysis, 177(2), 151-183.
[20] Macbeth, H. Question 322535 of February 9, 2019. “Convexity in co-ordinate charts of geodesic balls.”
[21] Molitor, M. (2014). Gaussian distributions, Jacobi group, and Siegel-Jacobi space. Journal of Mathematical Physics, 55(12), 122102.
[22] Monge, G. (1781). Mémoire sur la théorie des déblais et des remblais. Histoire de l’Académie Royale des Sciences de Paris.
[23] Satoh, H. (2007). Almost Hermitian structures on tangent bundles. In Proceedings of The Eleventh International Workshop on Differential Geometry, Kyungpook Nat. Univ., Taegu (Vol. 11, pp. 105-118).
[24] Trudinger, N. S., & Wang, X. J. (2009). On the second boundary value problem for Monge-Ampere type equations and optimal transportation. Annali della Scuola Normale Superiore di Pisa-Classe di Scienze-Serie IV, 8(1), 143.
[25] Villani, C. (2008). Optimal transport: old and new (Vol. 338). Springer Science & Business Media.
[26] Villani, C. (2016). Synthetic theory of Ricci curvature bounds. Japanese Journal of Mathematics, 11(2), 219-263.
In this section, we prove that Example 9 on page 12 has non-negative anti-bisectional curvature.

Recall that in this example, we set

$$\ell = r + \frac{1}{\ell^2}, \quad L = \log(c + r) \text{ for } c \log(c)^3 \geq 2.$$  

Here, $\ell = \frac{r}{\phi'(r)}$ where $\phi$ is the convex function satisfying $\Psi(x) = \phi(|x|)$. A tedious but straightforward computation shows the following.

$$A = \frac{\log(c + r)^2(-2 + (c + r)\log(c + r)^3)}{r(2r + (c + r)\log(c + r))(1 + r \log(c + r)^2)}$$

$$B = \frac{\log(c + r)(6r + 2(c + 2r)\log(c + r) - (c + r)^2\log(c + r)^4)}{r(c + r)(2r + (c + r)\log(c + r))(1 + r \log(c + r)^2)}$$

$$C = \frac{-60r^3 - 4r^2(19c + 26r)\log(c + r) - 2r(17c^2 + 49cr + 2r^2(17 + 3r))\log(c + r)^2}{r(c + r)^3\log(c + r)(2r + (c + r)\log(c + r))(1 + r \log(c + r)^2)^3}$$

$$D = \frac{-60r^3 - 4r^2(15c + 22r)\log(c + r) - 2r(9c^2 + 29cr + 2r^2(11 + 3r))\log(c + r)^2}{r(c + r)^3\log(c + r)(2r + (c + r)\log(c + r))(1 + r \log(c + r)^2)^3}$$

Whenever $c \log(c)^3 \geq 2$, we have that $A$ and $A + B$ are greater than 0, so what remains to show is that $D \geq 0$.

We start by consider the denominator of $D$

$$Denom[D] = r(c + r)^3\log(c + r)(2r + (c + r)\log(c + r))(1 + r \log(c + r)^2)^3.$$  

This is positive for $r \geq 0$ whenever $c \geq 1$, so the denominator is positive. As such, what remains to show is that the numerator is also positive.

$$Numer[D] = \frac{-60r^3 - 4r^2(15c + 22r)\log(c + r) - 2r(9c^2 + 29cr + 2r^2(11 + 3r))\log(c + r)^2}{r(c + r)^3\log(c + r)(2r + (c + r)\log(c + r))(1 + r \log(c + r)^2)^3}$$

Momentarily treating powers of $\log(c + r)$ as if they were constants, then this appears to be a quartic polynomial in $r$. Arranging the terms in this fashion, we find

$$Numer[D] = A_0 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4,$$
We will now show that each of these terms are non-negative, which completes the proof that \( D \geq 0 \).

\( \mathcal{A}_0 \geq 0 \). Note that \( c^3 \log[c + r]^3 > 0 \) and we have assumed that \( c \log[c]^3 \geq 2 \), so both factors are non-negative. As such, \( \mathcal{A}_0 \geq 0 \).

\( \mathcal{A}_1 > 0 \). Divide \( \mathcal{A}_1 \) by \( c^2 \log[c + r]^2 > 0 \) to obtain

\[
\mathcal{A}_1' = 2(-9 - 8 \log[c + r] + 6c \log[c + r]^3 + 6c \log[c + r]^4).
\]

By the assumption that \( c \log[c]^3 \geq 2 \), we have that

\[
\mathcal{A}_1' \geq 2(-9 - 8 \log[c + r] + 12 + 12 \log[c + r]) = 3 + 4 \log[c + r] > 0.
\]

As such, we have that \( \mathcal{A}_1 > 0 \).

\( \mathcal{A}_2 > 0 \). Simplifying \( \mathcal{A}_2 \) by dividing out \( c \log[c + r] \), we obtain

\[
\mathcal{A}_2 \frac{c \log[c + r]}{c \log[c + r]} = 2 \left( -30 - 29 \log[c + r] - 9 \log[c + r]^2 + 15c \log[c + r]^3 \right)
+ 21c \log[c + r]^4 + 9c \log[c + r]^5 \right).
\]

Once again using our assumption on \( c \), we have that

\[
\frac{\mathcal{A}_2}{c \log[c + r]} \geq 2 \left( -29 \log[c + r] - 9 \log[c + r]^2 + 42 \log[c + r] + 18 \log[c + r]^2 \right)
= 2(13 \log[c + r] + 9 \log[c + r]^2) > 0
\]

As such, \( \mathcal{A}_2 > 0 \).

\( \mathcal{A}_3 > 0 \). To show that this term is positive, we cannot simply bound each term by below, as we did for the previous terms. Instead, consider \( \mathcal{A}_3(r, c) \) as a function of \( r \) and \( c \). We first show that when \( c \) satisfies \( c \log[c]^3 \geq 2 \), \( \mathcal{A}_3(0, c) > 0 \). To see this, observe that

\[
\mathcal{A}_3(0, c) = -60 - 88 \log[c] - 44 \log[c]^2 - 8 \log[c]^3 + 12c \log[c]^3
+ 54c \log[c]^4 + 44c \log[c]^5 + 12c \log[c]^6.
\]

We estimate this term from below as follows.

\[
\mathcal{A}_3(0, c) \geq -60 - 88 \log[c] - 44 \log[c]^2 - 8 \log[c]^3
+ 24 + 108 \log[c] + 88 \log[c]^2 + 24 \log[c]^3
= -36 + 20 \log[c] + 44 \log[c]^2 + 16 \log[c]^3
\]

From the fact that \( c \log[c]^3 \geq 2 \), we have that \( \log[c] > .87^3 \) as a lower bound for \( \log[c] \) in the previous inequality, we have that

\[
\mathcal{A}_3(0, c) > -36 + 20 \cdot .87 + 44 \cdot (.87)^2 + 16 \cdot (.87)^3 = 25.239648 > 0.
\]

\(^{10}\)Note that \( .87^3 \exp[.87] < .87^3 \cdot .87 = 1.975509 < 2 \), which is how we obtain this lower bound.
Finally, we show that \((r + c) \frac{\partial}{\partial r} A_3(r, c) > 0\).

\[
(r + c) \frac{\partial}{\partial r} A_3(r, c) = 4 \left( \frac{-22 - 22 \log(c + r) + (-6 + 9c) \log(c + r)^2}{+54c \log(c + r)^3 + 55c \log(c + r)^4 + 18c \log(c + r)^3} \right) \\
\geq 4 \left( \frac{-22 - 22 \log(c + r) + (-6 + 9c) \log(c + r)^2}{+108 + 110 \log(c + r) + 36 \log(c + r)^2} \right) \\
= 4(86 + 88 \log(c + r) + (27 + 9c) \log(c + r)^2) > 0.
\]

This implies that \(\frac{\partial}{\partial r} A_3(r, c) > 0\), which further implies that \(A_3(r, c) > A_3(0, c)\) for all \(r > 0\). Since \(A_3(0, c) > 0\), this implies that \(A_3 > 0\).

\(A_4 > 0\). Simplifying \(A_4\) by dividing out \(\log(c + r)^2\), we obtain

\[A'_4 = -12 + 20 \log(c + r)^2 + 14 \log(c + r)^3 + 3 \log(c + r)^4.\]

Now, using the fact that \(\log(c + r) > .87\) (as shown above), we have that

\[
A'_4 > -12 + 20(.87)^2 + 14(.87)^3 + 3(.87)^4 = 14.07573483 > 0.
\]

Since all these terms are non-negative, this implies that

\[\text{Numer} \frac{1}{D} = A_0 + A_1 r + A_2 r^2 + A_3 r^3 + A_4 r^4 \geq 0.\]

Since the denominator of \(D\) is also positive, this implies that \(D > 0\), and so this metric has non-negative anti-bisectional curvature.

**Appendix B. Convexity of geodesic balls in scale-controlled coordinates**

In this section, we prove two lemmas which address relatively \(c\)-convex subsets. In short, we show that for a \(\Psi\)-cost with strongly convex and \(C^3\) potential, sufficiently small balls in the primal coordinates are relatively \(c\)-convex. This shows that the (DomConv) assumption can be avoided by simply considering sufficiently small local neighborhoods.

To begin, we first prove the following more general lemma about convexity of geodesic balls in scale controlled coordinates.

**Lemma 18.** Suppose that \(g_{ij}\) is a Riemannian metric tensor on an open subset \(\Omega \subset \mathbb{R}^n\). Suppose further that \(g_{ij}\) is scale-\(C^2\)-controlled in the following way:

1. \(Q^{-1} \delta_{ij} \leq g_{ij} \leq Q \delta_{ij}\) (in the sense of positive definite matrices),
2. \(r \frac{\partial g_{ij}}{\partial r} \leq Q - 1\),
3. \(r^2 \frac{\partial^2 g_{ij}}{\partial x^2} \leq Q - 1\)

Then there exists an \(\epsilon_0(r, Q)\) so that whenever \(\epsilon < \epsilon_0\), the geodesic ball (with respect to the metric \(g\)) \(B_g(p, \epsilon)\) is convex in \(\Omega\) as a subset of Euclidean space.

**Proof.** We start by fixing a point \(p\), which will be the center of the ball throughout. For this proof, it is necessary to consider two different Riemannian metrics on \(\Omega\). The first, (denoted \(g\)) is the Riemannian metric of interest. The second, which we denote \(g_0\), is the flat metric satisfying \((g_0)_{ij}(x) = g_{ij}(p)\) for all \(x \in \Omega\) (i.e. we consider the metric at \(p\) and do not change the components throughout \(\Omega\)). We then consider two separate distance functions.
The first, which we denote $d$, is the distance from $p$ in the $g$ metric. The second, which we denote $\delta_0$, is the distance from $p$ in the $g_0$ metric.

As a broad overview, the goal is to show that $d$ and $\delta_0$ are $C^2$-close. Then, using the implicit function theorem, we show that their level sets are also $C^2$-close. However, since the level sets of $\delta_0$ are ellipsoids with bounded eccentricity, its level sets are uniformly strongly convex, which implies that the level sets of $d$ must also be convex.

We first show that in a small ball, the functions $d$ and $\delta_0$ are close in the $C^0$-sense. To see this, consider a second point $q$. We consider two separate paths from $p$ to $q$. One, which we denote $\gamma$, is the geodesic with respect to $g$. The other, denoted $\gamma_0$, is the geodesic with respect to $g_0$ (and is simply a straight line in the coordinates). Assume that both are parametrized so that $\gamma(0) = \gamma_0(0) = p$ and $\gamma(1) = \gamma_0(1) = q$.

Using the bounds on the metric, we have that
\[ d(q)^2 = \int_0^1 g(\dot{\gamma}, \dot{\gamma}) \, ds \geq Q^{-1} \int_0^1 g_0(\dot{\gamma}, \dot{\gamma}) \, ds. \]
\[ \geq Q^{-1} \inf_{\tilde{\gamma}} \{ \tilde{\gamma} \mid \tilde{\gamma}(0) = p, \tilde{\gamma}(1) = q \} \int_0^1 g_0(\dot{\gamma}, \dot{\gamma}) \, ds \]
\[ = Q^{-1} \delta_0(q)^2. \]

Similarly, by considering the curve $\gamma_0$ and repeating the same argument, we can show that $d(q)^2 \leq Q \delta_0(q)^2$. This shows that for $q$ close to $p$, the two distance functions are close in $C^0$-sense.

We now show that the functions are close in $C^1$-sense as well. To do so, we bound the acceleration of the $g$-geodesics in the $x$-coordinates. This then implies that for $p$ and $q$ close, the gradients of $d(q)$ and $\delta_0(q)$ are very similar. Consider a point $q$ which is close to $p$.

On the other hand, the unit speed geodesic for the metric $g$ from $p$ to $q$ satisfy the equations
\[ \frac{d^2 \gamma^i}{ds^2} + \Gamma^i_{jk} \frac{d\gamma^j}{ds} \frac{d\gamma^k}{ds} = 0 \] (B.1)
where
\[ \Gamma^i_{jk} = \frac{1}{2} g^{il} \left( \frac{\partial g_{jl}}{\partial x^k} + \frac{\partial g_{kl}}{\partial x^j} - \frac{\partial g_{jk}}{\partial x^l} \right). \] (B.2)

However, from the scale-$C^1$-control, we can estimate that
\[ \left| \frac{d^2 \gamma^i}{ds^2} \right| \leq \sum_{j,k} \frac{Q}{2} \frac{3(Q - 1)}{r} \left| \frac{d\gamma^j}{ds} \right| \left| \frac{d\gamma^k}{ds} \right| \]
along the entire geodesic. In order to make the estimates readable, we will absorb any constants involving $n, r$ and $Q$ using the notation $f_1 \lesssim f_2$ whenever $f_1 \leq C f_2$ for some

\footnote{We can assume that $p$ and $q$ are close in terms of $d$ or $\delta_0$, since we have shown that these distances control each other.}
constant $C$ depending only on $n, Q$, and $r$. We will also use the notation
\[ \left\| \frac{d^k \gamma}{ds^k} \right\|_{L^1} \bigg|_{s=\tau} = \sum_{i=1}^{n} \left| \frac{d^k \gamma^i(\tau)}{ds^k} \right| \]
and the same corresponding for the $L^2$ norm as well.

In this notation, we find that
\[ \left| \frac{d^2 \gamma^i}{ds^2} \right| \lesssim \left| \frac{d \gamma^j}{ds} \right| \left| \frac{d \gamma^k}{ds} \right| . \]

Summing over the $i$ index, this implies that
\[
\left\| \frac{d^2 \gamma}{ds^2} \right\|_{L^1} \bigg|_{s=\tau} \lesssim \left\| \frac{d \gamma}{ds} \right\|_{L^2} \bigg|_{s=\tau} \lesssim \left\| 1 + \int_0^\tau \left\| \frac{d^2 \gamma}{ds^2} \right\|_{L^1} dt \right\|_{L^2}^2 \leq 2 + \left( \int_0^\tau \left\| \frac{d^2 \gamma}{ds^2} \right\|_{L^1} dt \right)^2 .
\]

We now define $F(\tau) = \int_0^\tau \left\| \frac{d^2 \gamma}{ds^2} \right\|_{L^1} dt$. When written in terms of $F$, the above estimate shows that
\[ \frac{dF}{d\tau} \lesssim 1 + F^2 . \]

Dividing both sides by $1 + F^2$ and integrating, we find that
\[ \arctan F(\tau) \lesssim \tau . \]

For $\tau$ small, this provides an upper bound for $F$. However, since $\gamma$ is a unit speed geodesic, this shows that for small $d$ (equivalently $\delta_0$), the acceleration (in coordinates) of $\gamma$ is very small. As a result, $\gamma$ is $C^1$-close to a line segment from $p$ to $q$ (in coordinates). Therefore, the gradient of $d$ at $q$ is close to the gradient of $\delta_0$ at $q$, which implies that the functions are $C^1$-close as well. Since the choice of $q$ was arbitrary, this shows that $\|d - \delta_0\|_{C^1(B_{\delta_0}(p, \epsilon) \setminus \{p\})}$ can be made arbitrarily small by taking $\epsilon$ small.

We can use this same idea to show that $d$ and $\delta_0$ are $C^2$-close as well. Since the argument is essentially identical to the $C^1$ estimate, we will provide a sketch rather than a fully detailed argument. When it is unambiguous, we will also drop the subscript indicating which particular value of $s$ to consider.

1. First differentiate Equations B.1 with respect to $s$. Differentiating the Christoffel symbols (Equations B.2) and using the scale-$C^2$-control, we obtain an estimate of the form
   \[ \left\| \frac{d^3 \gamma}{ds^3} \right\|_{L^1} \lesssim \left\| \frac{d^2 \gamma}{ds^2} \right\|_{L^1} \left\| \frac{d \gamma}{ds} \right\|_{L^1}^2 + \left\| \frac{d \gamma}{ds} \right\|_{L^1}^3 . \]

2. Using Young’s inequality, this shows that
   \[ \left\| \frac{d^3 \gamma}{ds^3} \right\|_{L^1} \lesssim \left\| \frac{d^2 \gamma}{ds^2} \right\|_{L^1}^2 + \left\| \frac{d \gamma}{ds} \right\|_{L^1}^4 + \left\| \frac{d \gamma}{ds} \right\|_{L^1}^3 . \]
(3) Using the bound on $F$ from the $C^1$ estimate, for small time we can control $\left\| \frac{d\gamma}{ds} \right\|$ by a constant. This then implies that

$$\left\| \frac{d^2\gamma}{ds^2} \right\|_{L^1} \lesssim \left\| \frac{d^2\gamma}{ds^2} \right\|_{L^1}^2 + 1.$$ 

(4) Using the fact that

$$\left\| \frac{d^2\gamma}{ds^2} \right\|_{s=\tau} \leq \int_0^\tau \left\| \frac{d^3\gamma}{ds^3} \right\|_{s=t} \, dt,$$

we find that

$$\left\| \frac{d^3\gamma}{ds^3} \right\|_{s=\tau} \lesssim \left( \int_0^\tau \left\| \frac{d^3\gamma}{ds^3} \right\|_{s=t} \, dt \right)^2 + 1.$$ 

(5) From this estimate, we can integrate out the differential inequality and bound

$$\int_0^\tau \left\| \frac{d^3\gamma}{ds^3} \right\|_{s=t} \, dt \lesssim \tan(\tau).$$

For sufficiently small times $\tau$, this provides a small bound on the total jerk of $\gamma$.

(6) As a result, $\gamma$ is $C^2$-close to a line segment from $p$ to $q$. Therefore, the gradient of $d$ is $C^1$-close to the gradient of $\delta_0$ at $q$, which implies that the two distance functions are $C^2$-close.

With these estimates, we can show that $B_{q_0}(p, \varepsilon)$ is convex in the $x$-coordinates. Consider a point $q$ with $d(p, q)$ small and the hyperplane (in $x$-coordinates) $V$ through $q$ which is perpendicular to the line segment from $p$ to $q$. Near $q$, both the functions $d$ and $\delta_0$ have gradients which are transverse to this hyperplane. As such, by the implicit function theorem we can locally find two functions $\ell_1, \ell_2 : V \to \mathbb{R}$ so that $d(v, \ell_1(v)) = d(q)$ and $d(v, \ell_2(v)) = \delta_0(q)$. In other words, we use the implicit function theorem to express the level sets of $d$ and $\delta_0$ as graphs in a small neighborhood of $q$. Furthermore, we can write the derivatives of $\ell_1$ and $\ell_2$ in terms of the derivatives of $d(q)$ and $\delta_0(q)$, respectively.

Since $d(q)$ and $\delta_0(q)$ are $C^2$-close, it follows that $\ell_1$ and $\ell_2$ are also $C^2$-close. However, the level sets of $\delta_0$ are ellipsoids with bounded eccentricity, and thus $\ell_2$ is a uniformly strongly convex function. For small $d$, this implies that the level sets of $d$ are also convex, since $\ell_1$ is $C^2$-close to a strongly convex function (and so must have non-negative definite Hessian). This then implies that the ball $B_q(p, d)$ is convex as a subset of Euclidean space.

This lemma addresses a question posed by Macbeth [20] on Mathoverflow, so may be of independent interest. Originally, she asked for explicit estimates on the radius of geodesic balls whose image in coordinates is convex. In theory, it is possible to find closed form estimates by carefully keeping track of all of the constants involving $Q$, $n$ and $r$ and making the estimates from the implicit function theorem explicit. However, we have not done this here.

For our purposes, the previous lemma shows that for a Hessian manifold whose potential is strongly convex and bounded in $C^4$, small balls in the primal coordinates are convex in the dual coordinates.

**Lemma 19.** Suppose $\Omega$ is a Hessian manifold with local potential $\Psi$ that is strongly convex and bounded in $C^4$. Then there exists an $\varepsilon > 0$ so that all balls $B(p, r)$ $(p \in \Omega$, $r < \varepsilon$) in the primal coordinates $x$ are convex in the dual coordinates $\theta$. 
Proof. Consider the primal $x$-coordinates with domain $\Omega$. We induce $\Omega$ with its natural inner product as an open domain of Euclidean space, which we denote by $\tilde{g}$. More precisely,
\[
\tilde{g} \left( \frac{\partial}{\partial x^i}, \frac{\partial}{\partial x^j} \right) = \delta_{ij}.
\]
Note that this is not the metric on $\Omega$ as a Hessian manifold, which is why we have included the tilde. We now show that the dual coordinates $\theta$ are $C^2$-scale-controlled with respect to $\tilde{g}$. To see this, note that we can express the primal coordinates in terms of the dual coordinates as
\[
x^i = \frac{\partial \Psi^*}{\partial \theta^i},
\]
where $\Psi^*$ is the Legendre dual of $\Psi$. As such, we have the following expression for $\tilde{g}$ in the $\theta$ coordinates.
\[
\tilde{g} \left( \frac{\partial}{\partial \theta^i}, \frac{\partial}{\partial \theta^j} \right) = \frac{\partial^2 \Psi^*}{\partial \theta^i \partial \theta^k} \frac{\partial^2 \Psi^*}{\partial \theta^j \partial \theta^k} = \delta_{ij}.
\]
Using the symmetry of mixed partials, this shows that as a matrix, $\tilde{g}$ satisfies
\[
\tilde{g} = \left[ \text{Hess}_\theta \Psi^* \right]^\top \left[ \text{Hess}_\theta \Psi^* \right]
\]
where $[\text{Hess}_\theta \Psi^*]_{ij} = \frac{\partial^2 \Psi^*}{\partial \theta^i \partial \theta^j}$. As such, to show that $\tilde{g}$ is $C^2$-scale-controlled in the $\theta$-coordinates, we need a lower bound on the singular values of $h$ and a $C^4$-estimate on $\Psi^*$.

To bound the singular values, we return our attention to the primal coordinates $x^i$ and note the following. Since $\Psi$ is strongly convex and $C^4$, there exists a $Q > 1$ so that
\[
Q^{-1} \delta_{ij} \leq [\text{Hess}_x \Psi] \leq Q \delta_{ij},
\]
where $\text{Hess}_x \Psi = \frac{\partial^2 \Psi}{\partial x^i \partial x^j}$. Here, the first inequality follows from strong convexity, and the second inequality follows from the $C^4$-estimate (which implies a $C^2$-estimate).

Now, in the $\theta$ coordinates, it is a well known fact about Legendre duality that $[\text{Hess}_\theta \Psi^*]$ is given by the inverse of $[\text{Hess}_x \Psi]$. From this, we immediately obtain that
\[
Q^{-1} \delta_{ij} \leq [\text{Hess}_\theta \Psi^*] \leq Q \delta_{ij},
\]
which implies that
\[
Q^{-2} \delta_{ij} \leq \tilde{g} \leq Q^2 \delta_{ij}.
\]

Furthermore, we can express the third and fourth derivatives of $\Psi^*$ in terms of the first four derivatives of $\Psi$ and the inverse of the Hessian of $\Psi$. Using the $C^4$-estimate on $\Psi$ and the bound from strong convexity, this provides a $C^4$-estimate on $\Psi^*$. As such, $\tilde{g}$ is scale controlled in the $\theta$-coordinates, which implies from the previous lemma that small balls (in the sense of $\tilde{g}$) are convex in the $\theta$-coordinates.

□
