QCD in the Infrared with Exact Angular Integrations

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Abstract

In a previous paper we have shown that in quantum chromodynamics the gluon propagator vanishes in the infrared limit, while the ghost propagator is more singular than a simple pole. These results were obtained after angular averaging, but in the current paper we go beyond this approximation and perform an exact calculation of the angular integrals. The powers of the infrared behaviour of the propagators are changed substantially. We find the very intriguing result that the gluon propagator vanishes in the infrared exactly like $p^2$, whilst the ghost propagator is exactly as singular as $1/p^4$. We also find that the value of the infrared fixed point of the QCD coupling is much decreased from the $y$-max estimate: it is now equal to $4\pi/3$.

Following a recent study by von Smekal et al.[1], we analyzed in Ref.[2] the coupled Dyson-Schwinger equations for the gluon and ghost form factors $F$ and $G$. The approximations were two-fold: firstly the vertices were taken bare, and secondly angular averaging was introduced (the so-called $y$-max approximation). Deferring to later work an improvement of the vertices, in this paper we seek to remove the deficiency of the $y$-max approximation. On the one hand the results might be regarded simply as quantitative adjustments to the $y$-max calculations; but on the other hand they are far from negligible. The numerical value of the infrared fixed point is reduced by a factor of almost three; and the finding that the gluon propagator has a simple zero, while the ghost propagator has a double pole, might perhaps be deemed a qualitatively new result.

As an improvement on the $y$-max approximation used in Ref.[2], we now solve the coupled integral equations for the gluon and ghost propagators with an exact treatment of the angular

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integrals. Although the y-max approximation is good in the ultraviolet region, where the form factors run logarithmically, we will see that it is a crude approximation in the infrared region, where the form factors exhibit power behaviour. The y-max approximation apparently leads to a difficulty in the ghost equation, as the asymmetry in the treatment of the gluon and ghost momenta in the loop gives an ambiguous result. In our previous study we used the ghost momentum for the radial integration and the gluon momentum for the angular integration. However, if we exchange these momenta we no longer find a consistent solution to the integral equation. We will show later why the first choice is a better approximation than the latter, thus motivating the choice made in Ref.\[2\]. Furthermore we will show that both choices are equivalent if we treat the angular integrals exactly, although this equivalence is by no means trivial.

With the radial integration over the ghost momentum, the ghost form factor satisfies

$$
\frac{1}{G(x)} = \tilde{Z}_3 - \frac{6\lambda}{\pi} \int_0^{\Lambda^2} dy yG(y) \int_0^\pi d\theta \sin^4 \theta \frac{F(z)}{y^2},
$$

whereas with the radial integration over the gluon momentum, we have instead

$$
\frac{1}{G(x)} = \tilde{Z}_3 - \frac{6\lambda}{\pi} \int_0^{\Lambda^2} dy F(y) \int_0^\pi d\theta \sin^4 \theta \frac{G(z)}{y},
$$

where $\lambda = g^2/16\pi^2$, $g$ being the strong coupling constant and where $z = x + y - 2\sqrt{xy}\cos \theta$. Here we have used the fact that $\tilde{Z}_1$, the ghost-gluon vertex renormalization constant, is equal to unity in the Landau gauge.

We will show that

$$
F(x) = Ax^{2\kappa} \quad G(x) = B x^{-\kappa}
$$

are solutions of Eq. (1) and Eq. (2), yielding identical conditions on $\lambda AB^2$, but that this equivalence does not hold in the y-max approximation.

Substituting Eq. (3) in Eq. (1) and evaluating the angular integral in terms of the hypergeometric function, $\,_{2}F_{1}$, we obtain

$$
\frac{x^\kappa}{B} = \tilde{Z}_3 - \frac{9\lambda AB}{4} \int_0^{\Lambda^2} dy y^{-\kappa+1}y_{>2\kappa-2}^{x} \,_{2}F_{1}\left(-2\kappa + 2,-2\kappa;3;\frac{y_{<}}{y_{>}}\right)
$$

$$
= \tilde{Z}_3 - \frac{9\lambda AB}{4} \left\{ \int_0^{x} dy y^{-\kappa+1} \,_{2}F_{1}\left(-2\kappa + 2,-2\kappa;3;\frac{y}{x}\right) \right\} + \int_0^{\Lambda^2} dy y^{\kappa-1} \,_{2}F_{1}\left(-2\kappa + 2,-2\kappa;3;\frac{x}{y}\right) \right\}.
$$
Set \( t = y/x \) in the infrared integration and \( t = x/y \) in the ultraviolet integration and take \( \Lambda^2 \to \infty \):
\[
x^\kappa \frac{B}{B} = -\frac{9\lambda AB}{4} x^\kappa \int_0^1 dt \left( t^{-\kappa+1} + t^{-\kappa-1} \right) {\text{I}}_1 (-2\kappa + 2, -2\kappa; 3; t) .
\] (4)

Note that we have dropped \( \tilde{Z}_3 \). This is in accordance with standard regularization procedure: the integral in Eq. (1) is convergent if \( \text{Re} \, \kappa < 0 \), whereas a subtraction is necessary if \( 0 \leq \kappa < 1 \).

By identifying \( \tilde{Z}_3 \) with this subtraction constant, we ensure that \( G(x) \) is defined by analytic continuation in \( \kappa \) beyond \( \text{Re} \, \kappa = 0 \). This continuation is made explicit in terms of the generalized hypergeometric function \( \text{F}_2 \) (see Eq. (17)). After matching the coefficients of \( x^\kappa \) we obtain
\[
\frac{1}{\lambda AB^2} = -\frac{9}{4} \left[ \frac{1}{2 - \kappa} \text{F}_2 (-2\kappa + 2, -2\kappa, 2 - \kappa; 3, 3 - \kappa; 1) \right. \\
- \left. \frac{1}{\kappa} \text{F}_2 (-2\kappa + 2, -2\kappa, -\kappa; 3, 1 - \kappa; 1) \right]
\] (5)

The y-max approximation that we made in [2] amounts to replacing \( \kappa \) by zero in the hypergeometric function in Eq. (4). The result was
\[
\frac{1}{\lambda AB^2} = -\frac{9}{4} \left[ \frac{1}{2 - \kappa} - \frac{1}{\kappa} \right].
\] (6)

Next we make a similar analysis of Eq. (2):
\[
x^\kappa \frac{B}{B} = \tilde{Z}_3 - \frac{9\lambda AB}{4} \int_0^{\Lambda^2} dy y^{2\kappa} y^{-\kappa-1} \text{F}_1 (\kappa + 1, \kappa - 1; 3; y < y >). 
\] (7)

After defining \( t = y/x \) in the infrared integration and \( t = x/y \) in the ultraviolet integration, taking \( \Lambda^2 \to \infty \) and eliminating \( \tilde{Z}_3 \) as before, we obtain
\[
x^\kappa \frac{B}{B} = -\frac{9\lambda AB}{4} x^\kappa \int_0^1 dt \left( t^{2\kappa} + t^{-\kappa-1} \right) \text{F}_1 (\kappa + 1, \kappa - 1; 3; t) .
\] (8)

Using Eq. (17) in Eq. (8) and equating the coefficients of \( x^\kappa \) we find
\[
\frac{1}{\lambda AB^2} = -\frac{9}{4} \left[ \frac{1}{2\kappa + 1} \text{F}_2 (\kappa + 1, \kappa - 1; 2\kappa + 1, 3, 2\kappa + 2; 1) \\
- \frac{1}{\kappa} \text{F}_2 (\kappa + 1, \kappa - 1, -\kappa; 3, 1 - \kappa; 1) \right].
\] (9)

The y-max approximation of this expression is once more obtained by replacing \( \kappa \) by zero in the hypergeometric function in Eq. (8), yielding
\[
\frac{1}{\lambda AB^2} = -\frac{9}{4} \left[ \frac{1}{2\kappa + 1} - \frac{1}{3(2\kappa + 2)} - \frac{1}{\kappa} + \frac{1}{3(\kappa - 1)} \right].
\] (10)
This approximation is quite different from that of Eq. (10), illustrating the precariousness of the y-max approximation.

However, the exact forms Eq. (5) and Eq. (9) are identical, as we verified by using Mathematica to check the numerical equivalence over a domain of $\kappa$ values.

We now consider the gluon equation, and will show that it too is solved by a pure power behaviour if we keep only the ghost loop. Even in the presence of the gluon loop, when the power behaviour is no longer an exact solution of the equation, it will still represent the correct leading order infrared asymptotic behaviour. Omitting then the gluon loop, we write the equation for the gluon form factor as follows:

$$\frac{1}{F(x)} = Z_3 + \frac{2\lambda}{\pi} \int_0^{\Lambda^2} \frac{dy}{x} G(y) \int_0^{\pi} d\theta \sin^2 \theta M(x, y, z) G(z),$$

where

$$M(x, y, z) = \left(\frac{x + y}{2} - \frac{y^2}{x}\right) \frac{1}{z} + \frac{1}{2} + \frac{2y}{x} - \frac{z}{x}.$$ 

We now substitute the solution (3) into Eq. (11), obtaining

$$\frac{x^{-2\kappa}}{A} = Z_3 + \frac{2\lambda B^2}{\pi} \int_0^{\Lambda^2} \frac{dy}{x} y^{-\kappa} \int_0^{\pi} d\theta \sin^2 \theta \left[\left(\frac{x + y}{2} - \frac{y^2}{x}\right) z^{-\kappa-1} + \left(\frac{1}{2} + \frac{2y}{x}\right) z^{-\kappa - \frac{1}{x}}\right].$$

We evaluate the angular integral with the help of Eq. (16):

$$\frac{x^{-2\kappa}}{A} = Z_3 + \lambda B^2 \int_0^{\Lambda^2} \frac{dy}{x} y^{-\kappa} \left[\left(\frac{x + y}{2} - \frac{y^2}{x}\right) y_<^{-\kappa-1} \left(\frac{1}{2} + \frac{2y}{x}\right) y_\sim^{\kappa-1} - \left(\frac{1}{2} + \frac{2y}{x}\right) y_\sim^{-\kappa-1} \right] F_1 \left(\kappa + 1, \kappa; 2; \frac{y_<}{y_>}\right) - F_1 \left(\kappa + 1, \kappa; 2; \frac{y_<}{y_>}\right) - F_1 \left(\kappa + 1, \kappa; 2; \frac{y_<}{y_>}\right).$$

We split the integration region and define $t = y/x$ in the infrared integration and $t = x/y$ in the ultraviolet integration, as before, and finally implicitly implement the analytic regularization, taking $\Lambda^2 \to \infty$. After some rearrangement we obtain

$$\frac{x^{-2\kappa}}{A} = \lambda B^2 x^{-2\kappa} \int_0^1 dt \left\{ \frac{1}{2} \left( t^{-\kappa} + t^{-\kappa+1} + t^{2\kappa-1} + 2t^{2\kappa-2} \right) - \left( t^{-\kappa+2} + t^{2\kappa-3} \right) \right\} F_1 \left(\kappa + 1, \kappa; 2; t\right) + \left[ \frac{1}{2} \left( t^{-\kappa} + t^{2\kappa-2} \right) + 2 \left( t^{-\kappa+1} + t^{2\kappa-3} \right) \right] F_1 \left(\kappa + 1, \kappa; 2; t\right) - \left[ t^{-\kappa} + t^{2\kappa-3} \right] F_1 \left(\kappa + 1, \kappa; 2; t\right).$$
Using Eq. (17) in Eq. (12) and equating coefficients of $x^{\kappa}$, we find

\[
\frac{1}{\lambda AB^2} = \frac{1}{2(1 - \kappa)} \binom{3}{2} \binom{2}{1 - \kappa} + \frac{1}{2(2 - \kappa)} \binom{3}{2} \binom{2}{2 - \kappa} + \frac{1}{4\kappa} \binom{3}{2} \binom{2}{2\kappa + 1} + \frac{1}{2(2\kappa - 1)} \binom{3}{2} \binom{2}{2\kappa} - \frac{1}{2(1 - \kappa)} \binom{3}{2} \binom{2}{2 - \kappa} - \frac{1}{2(2\kappa - 1)} \binom{3}{2} \binom{2}{2\kappa}.
\]

Some of these generalized hypergeometric functions are in fact infinite for certain values of $\kappa$; they should be understood by replacing the final argument '1' in each generalized hypergeometric function by $\tau$, and then by taking the limit $\tau \to 1$. Since this amounts to replacing the upper integration limit in Eq. (12) by $\tau$, the procedure is clearly legitimate.

Finally, to demonstrate that Eq. (3) really is a solution of the coupled integral equations, for a particular value of $\kappa$, we equate the right-hand side of Eq. (5) or Eq. (9) with that of Eq. (13) and solve for $\kappa$. We used the routine FindRoot of Mathematica and discovered that, in the limit $\kappa \to 1$,

\[
\kappa \to 1,
\]

both expressions become equal. This limit has been checked analytically in App. [4]. We find

\[
\lambda AB^2 = \frac{1}{3}
\]

which means that the infrared fixed point for the gauge invariant running coupling, $\alpha(p^2) = 4\pi \lambda Z_1^2 F(p^2) G^2(p^2)$, has the value

\[
\alpha(0) = \frac{4\pi}{3} \approx 4.19
\]

which is quite different from the value we found using the y-max approximation. There we had $\alpha(0) \approx 11.47$, nearly three times the exact result, corresponding to $\kappa \approx 0.77$.

Our findings for the propagators in the infrared region can be summarized by the following formulas:
gluon propagator in Landau gauge

\[ D_{\mu\nu}^{ab}(p) \sim -\delta^{ab} \left[ g_{\mu\nu} - \frac{p_\mu p_\nu}{p^2} \right] p^2, \]  

(14)

ghost propagator in Landau gauge

\[ G^{ab}(p) \sim -\delta^{ab} \frac{1}{p^4}. \]  

(15)

A Angular and Radial Integrations

Throughout we used the angular integration formula

\[ \int_0^\pi d\theta \sin^2 \theta \ z^n = B \left( r + \frac{1}{2}, \frac{1}{2} \right) y^n_\leq 2F1 \left( -n, -n - r; r + 1; \frac{y_\leq}{y_\geq} \right). \]  

(16)

The radial integrations of hypergeometric functions are evaluated as follows:

\[ \int_0^1 dt \ t^\nu \ 2F1 (a, b; c; t) = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n \ n!} \int_0^1 dt \ t^{\nu+n} = \sum_{n=0}^\infty \frac{(a)_n (b)_n}{(c)_n \ n!} \frac{1}{\nu+n+1} = \frac{1}{\nu+1} 3F2 (a, b, \nu + 1; c, \nu + 2; 1). \]  

(17)

B Limit of \( \lambda A B^2 \) for \( \kappa \to 1 \)

We check analytically that the limit of \( \lambda A B^2 \) for \( \kappa \to 1 \) is identical for Eqs. (13, 14, 15). Starting from Eq. (13) we find

\[ \frac{1}{\lambda A B^2} = -\frac{9}{4} \left\{ 3F2 (0, -2, 1; 3, 2; 1) - \lim \limits_{\kappa \to 1} 3F2 (2 - 2\kappa, -2, -1; 3, 1 - \kappa; 1) \right\} = -\frac{9}{4} \left[ 1 - \left( 1 + \frac{4}{3} \right) \right] = 3 \]

and from Eq. (14)

\[ \frac{1}{\lambda A B^2} = -\frac{9}{4} \left\{ \frac{1}{3} 3F2 (2, 0, 3; 3, 4; 1) - \lim \limits_{\kappa \to 1} 3F2 (2, \kappa - 1, -1; 3, 1 - \kappa; 1) \right\} = -\frac{9}{4} \left[ \frac{1}{3} - \left( 1 + \frac{2}{3} \right) \right] = 3. \]
We make an analogous check for the gluon equation. To circumvent convergence problems, we replace the argument of the generalized hypergeometric functions by $\tau$, and take the limit $\tau \to 1$ at the end. From Eq. (13),

$$\frac{1}{\lambda AB^2} =
\lim_{\tau \to 1} \lim_{\kappa \to 1} \left\{ \frac{1}{2(1 - \kappa)} F_2 (2, 1, 1 - \kappa; 2, 1; \tau) + \frac{1}{2(1 - \kappa)} F_2 (1, \kappa - 1, 1 - \kappa; 2, 1; \tau) \right\}
+ \frac{5}{2} F_2 (1, 0, 1; 2; \tau)
- \frac{5}{2} F_2 (\kappa - 1, -1, 1 - \kappa; 2, 1; \tau)
- \frac{1}{2(1 - \kappa)} F_2 (\kappa - 1, -1, 2\kappa - 2; 2, 1; \tau)
$$

$$= \lim_{\tau \to 1} \lim_{\kappa \to 1} \left\{ \frac{1}{2(1 - \kappa)} \sum_{n=1}^{\infty} \frac{(1 - \kappa)_n + (2\kappa - 2)_n}{n!} \tau^n + \frac{1}{4} F_1 (1, 1; 2; \tau) - \frac{1}{4} F_1 (1, 2; 3; \tau) \right\} + \frac{5}{2}
$$

$$= \lim_{\tau \to 1} \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{2n} + \frac{1}{n + 1} - \frac{1}{2(2 + n)} \right) \tau^n \right\} + \frac{13}{4}
$$

$$= 3 + \lim_{\tau \to 1} \left\{ \sum_{n=1}^{\infty} \left( \frac{1}{2n} + \frac{1}{n + 1} - \frac{1}{2n} \right) \tau^n \right\}
$$

$$= 3$$.

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