AN APPLICATION OF SCATTERING THEORY TO THE SPECTRUM OF THE LAPLACE-BELTRAMI OPERATOR

FRANCESCA ANTOCI

Abstract. Applying a theorem due to Belopol’ski and Birman, we show that the Laplace-Beltrami operator on 1-forms on $\mathbb{R}^n$ endowed with an asymptotically Euclidean metric has absolutely continuous spectrum equal to $[0, +\infty)$.

1. Introduction

The relationships between the geometric properties of a complete non-compact Riemannian manifold and the spectrum of the Laplace-Beltrami operator have been intensively investigated by many authors.

Among them, H. Donnelly since the late seventies studied the spectra of the Laplacian and of the Laplace-Beltrami operators on particular manifolds, such as the hyperbolic space ([1]), manifolds with negative sectional curvature ([3]), asymptotically euclidean manifolds ([2]). However, to our knowledge, the case of the Laplace-Beltrami operator acting on $p$-forms on asymptotically euclidean manifolds has been left aside up to now.

The purpose of this paper is to contribute to the investigation of this case. We study the absolutely continuous spectrum of the Laplace-Beltrami operator on $\mathbb{R}^n$ endowed with an asymptotically euclidean metric, that is with a Riemannian metric satisfying conditions (2.5), (2.6) and (2.7). The tool employed is classical scattering theory in the wave operators approach. In this case, the problem is reduced to a problem of scattering for vector-valued operators.

An inherent restriction of the proof, however, that makes it difficult an extension to more general cases, is the use of the Fourier transform, which has a crucial role in our considerations, particularly in connection with Lemma 3.2. The lack of this tool in the more general case of manifolds with an asymptotically controlled Riemannian metric is a major obstacle to the extension of the theorem.

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2. Preliminaries

Let \((\mathbb{R}^n, e)\) be the euclidean \(n\)-dimensional space, and \((\mathbb{R}^n, g)\) be the same space, endowed with a complete Riemannian metric \(g\).

We will denote by \(\Lambda^1_c(\mathbb{R}^n)\) the vector space of all smooth, compactly supported 1-forms on \(\mathbb{R}^n\), and by \(L^2_1(\mathbb{R}^n, e)\) the completion of \(\Lambda^1_c(\mathbb{R}^n)\) with respect to the norm
\[
\|\omega\|^2_{L^2_1(\mathbb{R}^n, e)} = \int_{\mathbb{R}^n} \omega(x)(\omega(x))^e dx,
\]
where \(dx\) denotes the Euclidean volume element and
\[
<\omega(x), \omega(x)^e>^e = \sum_i \omega_i^2(x)
\]
is the fiber norm for 1-forms induced by the Euclidean metric.

\(L^2_1(\mathbb{R}^n, g)\) will stand for the completion of \(\Lambda^1_c(\mathbb{R}^n)\) with respect to the norm:
\[
\|\omega\|^2_{L^2_1(\mathbb{R}^n, g)} = \int_{\mathbb{R}^n} \omega(x)(\omega(x))^g \sqrt{g} dx,
\]
where \(\sqrt{g} dx\), as usual, denotes the volume element induced by the Riemannian metric \(g\) and
\[
<\omega(x), \omega(x)^g>^g = g^{ij}(x)\omega_i(x)\omega_j(x)
\]
is the fiber norm for 1-forms induced by \(g\). (Here, as everywhere throughout the paper, the repeated indices convention is adopted.)

The action of the Laplace-Beltrami operator \(\Delta_e\) on 1-forms \(\omega \in \Lambda^1_c(\mathbb{R}^n)\) acts componentwise as:
\[
(\Delta_e \omega)_k = -\sum_j \frac{\partial^2 \omega_k}{\partial x_j^2}.
\]
It is well-known that \(\Delta_e\) is essentially selfadjoint on \(\Lambda^1_c(\mathbb{R}^n)\), and its closure \(H_0\) has purely absolutely continuous spectrum equal to
\[
\sigma(H_0) = [0, +\infty),
\]
with constant multiplicity.

We will denote by \(h_0[\omega]\) the quadratic form associated to \(\Delta_e\) on \(\Lambda^1_c(\mathbb{R}^n)\):
\[
(2.2) \quad h_0[\omega] = \int_{\mathbb{R}^n} \omega(x)(\omega(x))^e dx = \int_{\mathbb{R}^n} \sum_{i,j=1}^n \left(\frac{\partial \omega_i}{\partial x^j}\right)^2 dx.
\]

The action of the Laplace-Beltrami operator \(\Delta_g\) on 1-forms \(\omega \in \Lambda^1_c(\mathbb{R}^n)\) is given in local coordinates by the Weitzenböck formula:
\[
(\Delta_g \omega)_k = -(g^{ij} \nabla_i \nabla_j \omega)_k + R^i_k \omega_i,
\]
where \(\nabla_i\) is the covariant derivative with respect to the connection induced by the metric \(g\), and \(R^i_k\) is the Ricci tensor.
Since the Riemannian metric $g$ is complete, $\Delta_g$ is essentially selfadjoint on $\Lambda^1_c(\mathbb{R}^n)$. We will denote its closure by $H_1$.

Moreover, we will denote by $h_1[\omega]$ the quadratic form associated to $\Delta_g$ on $\Lambda^1_c(\mathbb{R}^n)$

\begin{equation}
(2.4) \quad h_1[\omega] = \int_{\mathbb{R}^n} <\Delta_g \omega, \omega>_g \sqrt{g} \, dx = \\
= \int_{\mathbb{R}^n} |\nabla \omega|_g^2 \sqrt{g} \, dx + \int_{\mathbb{R}^n} <R \omega, \omega>_g \sqrt{g} \, dx,
\end{equation}

where

$$|\nabla \omega|_g^2 = g^{ij} g^{\alpha\beta} \nabla_i \omega_\alpha \nabla_j \omega_\beta$$

and

$$<R \omega, \omega>_g = g^{\alpha\beta} R^i_\alpha \omega_i \omega_\beta.$$

In the next section we will show how it is possible, under suitable hypothesis on the asymptotic behaviour of $g$, to get information about the spectrum of $H_1$ from the knowledge of the spectrum of $H_0$, proving the following

**Theorem 2.1.** Let $\mathbb{R}^n$ be endowed with a Riemannian metric $g$ such that

(1) for every $i, j$,

\begin{equation}
(2.5) \quad |g^{ij}(x) - \delta^{ij}| < \frac{C}{|x|^k}
\end{equation}

for some $k > n$;

(2) for every $i, j, k, l$

\begin{equation}
(2.6) \quad \left| \frac{\partial g_{il}}{\partial x_j} \right| < \frac{C}{|x|^k}
\end{equation}

(2.7) \quad \left| \frac{\partial^2 g_{il}}{\partial x_j \partial x_k} \right| < \frac{C}{|x|^k}

for some $k > n$.

Then the Laplace-Beltrami operator $\Delta_g$ acting on 1-forms has absolutely continuous spectrum equal to $[0, +\infty)$:

$$[0, +\infty) = \sigma_{ac}(H_1) = \sigma(H_1).$$

In particular, it has no discrete spectrum. (There might be singularly continuous spectrum or embedded eigenvalues.)

The main tool for the proof is Belopol’ski-Birman theorem (see [5], [6]), which provides a sufficient condition so that two selfadjoint operators have the same absolutely continuous spectrum.

We recall it briefly:
Theorem 2.2. Let $H_0$, $H_1$ be selfadjoint operators acting respectively on Hilbert spaces $\mathcal{H}_0$, $\mathcal{H}_1$, and let $E_\Omega(H_0)$, $E_\Omega(H_1)$, for $\Omega \subset \mathbb{R}$, be the associated spectral measures.

If $J \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1)$ satisfies the conditions:

1. $J$ has a bounded two-sided inverse;
2. for every bounded interval $I \subset \mathbb{R}$, (2.8) $E_I(H_1)(J - JH_0)E_J(H_0) \in \mathcal{I}_1(\mathcal{H}_0, \mathcal{H}_1)$, where $\mathcal{I}_1(\mathcal{H}_0, \mathcal{H}_1) = \{ A \in \mathcal{L}(\mathcal{H}_0, \mathcal{H}_1) \mid (A^*A)^{1/2} \in \mathcal{I}_1(\mathcal{H}_0) \}$ and $\mathcal{I}_1(\mathcal{H}_0)$ denotes, as usual, the set of trace-class operators on $\mathcal{H}_0$;
3. for every bounded interval $I \subset \mathbb{R}$, $(J^*J - I)E_I(H_0)$ is compact;
4. $JQ(H_0) = Q(H_1)$, where $Q(H_i)$ is the form domain of the operator $H_i$, for $i = 0, 1$,

then the wave operators $W^\pm(H_1, H_0; J)$ exist, are complete, and are partial isometries with initial space $P_{ac}(H_0)$ and final space $P_{ac}(H_1)$, where $P_{ac}(H_i)$ denotes, as usual, the absolutely continuous space of $H_i$, for $i = 0, 1$.

As a consequence, the absolutely continuous spectra of $H_0$ and $H_1$ do coincide.

Remark 2.3. We recall (see [4]) that if $H$ is a densely defined, essentially selfadjoint, positive operator on a Hilbert space $\mathcal{H}$ and $h$ is the associated quadratic form, the form domain $Q(H)$ of the selfadjoint operator $\overline{H}$ is the domain of the closure $\overline{h}$ of the form $h$, that is to say: $Q(H)$ is the set of those $u \in \mathcal{H}$ such that there exists a sequence $\{u_n\} \subset D(H)$ converging to $u$ in $\mathcal{H}$ such that

$$h[u_n - u_m] \to 0$$

as $n, m \to +\infty$.

3. Proof of Theorem 2.1

We will prove that, for a suitable $J : L^2(\mathbb{R}^n, e) \to L^2(\mathbb{R}^n, g)$, $H_1 = \overline{\Delta g}$, $H_0 = \overline{\Delta e}$ and $J$ satisfy the conditions of Theorem 2.2 for $\mathcal{H}_0 = L^2(\mathbb{R}^n, e)$ and $\mathcal{H}_1 = L^2(\mathbb{R}^n, g)$.

We begin with the following

Lemma 3.1. Let $g$ be as in Theorem 2.1. Then there exist $C, C_1 > 0$, $D, D_1 > 0$ such that

1. for every $x \in \mathbb{R}^n$

   $$(3.1) \quad C \leq \sqrt{g(x)} \leq C_1;$$

2. for every $x \in \mathbb{R}^n$, $v$ in the cotangent space at $x$, $T^*_x(\mathbb{R}^n)$

   $$(3.2) \quad D \sum_i v_i^2 \leq g^{ij}(x)v_i v_j \leq D_1 \sum_i v_i^2.$$
Proof. (3.1) follows immediately observing that, for every \(x\), \(\sqrt{g(x)}\) is strictly positive and \(\sqrt{g(x)} \to 1\) as \(|x| \to +\infty\).

As for (3.2), since the matrix \(g^{ij}(x)\), which expresses the Riemannian metric \(g\) in contravariant form, is a continuous function of \(x\) and is positive, its eigenvalues \(\lambda_1(x), \ldots, \lambda_n(x)\) depend continuously on \(x\) and are strictly positive. Hence, the functions \(f\) and \(h\) defined by

\[ f(x) := \inf_i \lambda_i(x) \]

and

\[ h(x) := \sup_i \lambda_i(x), \]

are continuous and strictly positive. Moreover, since the metric \(g\) is asymptotically euclidean, \(f(x) \to 1\) and \(h(x) \to 1\) as \(|x| \to +\infty\). As a consequence, there exist \(D, D_1 > 0\) such that, for every \(x \in \mathbb{R}^n\),

\[ D \leq f(x) \leq h(x) \leq D_1, \]

which yields (3.2). \(\square\)

Lemma 3.1 implies that there is a natural identification between \(L^2_1(\mathbb{R}^n, g)\) and \(L^2_1(\mathbb{R}^n, e)\), and, moreover, (2.1) and (2.3) are equivalent norms. As a consequence, the identity map on \(\Lambda^1_c(\mathbb{R}^n)\) extends to a bounded linear operator \(J : L^2_1(\mathbb{R}^n, e) \to L^2_1(\mathbb{R}^n, g)\), with bounded two-sided inverse, and condition 1 of Theorem 2.2 is satisfied.

In order to prove (2.8), we need two Lemmas:

**Lemma 3.2.** Let \(A : \xi \mapsto A_\xi\) be a \(n \times n\)-matrix-valued function on \(\mathbb{R}^n\), and let \(A\) be the linear operator

\[ A : D(A) \subset L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n) \to L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n) \]

of the form

\[ f(x)A(-i\nabla_x), \]

where \(f(x)\) is a function on \(\mathbb{R}^n\) and \(A(-i\nabla_x)\) is the operator

\[ A(-i\nabla_x) = \mathcal{F} \circ \hat{A}_\xi \circ \mathcal{F}^{-1}, \]

\(\mathcal{F}\) being the Fourier transform and \(\hat{A}_\xi\) the multiplication operator

\[ v \mapsto \hat{A}_\xi v \quad (\hat{A}_\xi v)(\xi) = A(\xi)v(\xi). \]

Let \(L^2_\alpha(\mathbb{R}^n)\) be the space of functions \(h\) such that

\[ \|h\|_{L^2_\alpha}^2 = \|(1 + |x|^2)^{\frac{\alpha}{2}}h(x)\|_{L^2} < \infty. \]

If, for some \(\delta > \frac{n}{2}\), \(f(x) \in L^2_\delta(\mathbb{R}^n)\) and, for every pair of indices \((\alpha, \beta)\), \(A_{\alpha\beta}(\xi) \in L^2_\delta(\mathbb{R}^n)\), then \(A\) is a trace-class operator.
Proof of Lemma 3.2. It suffices to show that, for every fixed \((\alpha, \beta)\), the operator 
\[ A_\alpha^\beta : D(A_\alpha^\beta) \subset L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n) \]

\[ \omega = (\omega_1, \ldots, \omega_n) \longmapsto \begin{pmatrix} 0, \ldots, 0, f(x)A_\alpha^\beta(-i\nabla_x)\omega_\beta, 0, \ldots, 0 \end{pmatrix} \]

is trace-class. But this latter operator coincides with the composition
\[ I_\alpha \circ \left( f(x)A_\alpha^\beta(-i\nabla_x) \right) \circ P_\beta, \]

where \(P_\beta\) is the projection
\[ P_\beta : L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \]

\[ \omega = (\omega_1, \ldots, \omega_\beta, \ldots, \omega_n) \longmapsto \omega_\beta, \]

\(I_\alpha\) is the immersion
\[ I_\alpha : L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n) \]

\[ \omega \longmapsto (0, \ldots, 0, \omega_\beta, \ldots, 0), \]

and \(A_\alpha^\beta(-i\nabla_x)\) is the operator
\[ D(A_\alpha^\beta(-i\nabla_x)) \subset L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \]

\[ A_\alpha^\beta(-i\nabla_x) = \mathcal{F} \circ (\hat{A}_\xi)^\beta_\alpha \circ \mathcal{F}^{-1}, \]

where \(\mathcal{F}\) is the Fourier transform and \((\hat{A}_\xi)^\beta_\alpha\) is the multiplication operator associated to the scalar function \((A_\xi)^\beta_\alpha\).

The conclusion follows from the fact that \(P_\beta\) and \(I_\alpha\) are bounded operators and (see [5], Theorem XI.21) any operator
\[ L^2(\mathbb{R}^n) \longrightarrow L^2(\mathbb{R}^n) \]

of the form \(f(x)h(-i\nabla_x)\) is trace-class if \(f(x)\) and \(h(\xi)\) belong to \(L^2_\delta(\mathbb{R}^n)\).

\[ \square \]

Lemma 3.3. If \(f : \mathbb{R}^n \longrightarrow \mathbb{R}\) is continuous and such that for some \(k > n\)
\[ |f(x)| < \frac{C}{|x|^k} \]

when \(|x| > > 0\), then \(f \in L^2_\delta(\mathbb{R}^n)\) for some \(\delta > \frac{n}{2}\).

Proof. Choosing \(\epsilon > 0\) such that \(k > n + \epsilon\), then
\[ |f(x)| < \frac{C}{|x|^k} \]
for $|x| >> 0$; as a consequence, a straightforward computation in polar coordinates shows that, for $\delta = \frac{\pi}{2} + \epsilon$,

$$\int_{\mathbb{R}^n} |f(x)|^2 (1 + |x|^2)^\delta \, dx < +\infty.$$ 

□

Now, to prove (2.8), it suffices to see that for every bounded interval $I \subset \mathbb{R}$,

$$(H_1 - H_0)E_I(H_0) \in \mathcal{L}_1(L^2(\mathbb{R}^n) \oplus \ldots \oplus L^2(\mathbb{R}^n)).$$

Let $\Gamma^\alpha_{ik}$ be the Christoffel symbols of the Riemannian connection induced by $g$; then the difference $H_1 - H_0$ is given by

$$(3.4) \quad ((H_1 - H_0)\omega)_k = (-g^{ij} + \delta^{ij}) \frac{\partial^2 \omega_k}{\partial x^i \partial x^j} + g^{ij} \Gamma^\alpha_{jk} \frac{\partial \omega_k}{\partial x^i} + g^{ij} \Gamma^\alpha_{ij} \frac{\partial \omega_k}{\partial x^i}$$

$$+ g^{ij} \Gamma^\alpha_{ik} \frac{\partial \omega_j}{\partial x^i} + g^{ij} \delta \Gamma^\alpha_{ik} \omega_i - g^{ij} \Gamma^\alpha_{ij} \Gamma^\beta_{ik} \omega_j - g^{ij} \Gamma^\alpha_{ik} \Gamma^\beta_{jk} \omega_i + R^i_k \omega_i.$$ 

A direct computation shows that conditions (2.5), (2.6), (2.7), and hypothesis 3 in Theorem 2.1 imply that $|g^{ij} \Gamma^\alpha_{jk}|, |g^{ij} \Gamma^\alpha_{ij}|, |g^{ij} \Gamma^\alpha_{ik}|, |g^{ij} \frac{\partial \Gamma^\alpha_{ij}}{\partial x^i}|, |g^{ij} \Gamma^\alpha_{ij} \Gamma^\beta_{ik}|, |g^{ij} \Gamma^\alpha_{ik} \Gamma^\beta_{ij}|, |R^i_k|$ are all bounded from above by $\frac{C}{|x|^k}$ for some constant $C > 0$ and some $k > n$.

$$(H_1 - H_0)(E_I(H_0))$$

is a sum of operators of type $f(x)A(-i\nabla_x)$, with $f(x) \in L^2_\delta(\mathbb{R}^n)$ in view of Lemma 3.3, and $A(\xi)$ smooth and compactly supported.

Thus, thanks to Lemma 3.2, $(H_1 - H_0)(E_I(H_0))$ is trace-class and condition 2 is fulfilled.

As for condition 3, first of all we observe that the adjoint of $J$

$$J^* : L^2_1(\mathbb{R}^n, g) \longrightarrow L^2_1(\mathbb{R}^n, e)$$

satisfies the equation

$$\int_{\mathbb{R}^n} g^{ij} \omega_i \phi_j \sqrt{g} \, dx = \int_{\mathbb{R}^n} \delta^{ij} \omega_i (J^* \phi)_j \, dx,$$

and therefore

$$(J^* \phi)_k = \delta_{ik} g^{ij} \phi_j \sqrt{g}.$$ 

As a consequence, in local coordinates

$$((J^* J - I)\phi)_k = (\sqrt{g} g^{jk} - \delta^{jk}) \phi_j;$$

now,

$$\left| \sqrt{g} g^{jk} - \delta^{jk} \right| \leq \left| \sqrt{g} \right| \left| g^{jk} - \delta^{jk} \right| + |(\sqrt{g} - 1)| \delta^{ij}.$$ 

By (2.5), there exists $C > 0$ such that

$$\left| g^{jk} - \delta^{jk} \right| \leq \frac{C}{|x|^k}.$$
for some \( k > n \) for \( |x| > |x| \); moreover,

\[
|((\sqrt{g} - 1)| = \frac{1}{2}|1 - g| + o\left(\frac{1}{|x|^k}\right) \leq K\frac{1}{|x|^k}
\]

for some \( K > 0 \) and some \( k > n \) as \( |x| \to +\infty \).

Thus, \( \sqrt{g}g^k - \delta^k \) belongs to \( L_2^k(\mathbb{R}^n) \). Hence \((J^\ast J - I)E_I(H_0)\) is an operator of type \( f(x)A(-i\nabla_x) \), with \( f(x) \in L_2^k(\mathbb{R}^2) \) and \( A(\xi) \) smooth and compactly supported; by Lemma 3.2 it is trace-class, and therefore it is compact.

As for condition 4, thanks to Remark 2.3, \( Q(H_0) \) and \( Q(H_1) \) can be characterized as follows:

**Lemma 3.4.** \( Q(H_0) \) is the set of those \( \omega \in L_2^2(\mathbb{R}^n, e) \) for which there exists a sequence \( \{\omega^{(n)}\} \subset \Lambda_1^1(\mathbb{R}^n) \) such that

\[
\omega^{(n)} \to \omega \quad \text{in} \quad L_2^2(\mathbb{R}^n, e)
\]

and

\[
h_0[\omega^{(n)} - \omega^{(m)}] \to 0
\]

as \( n, m \to +\infty \).

Analogously, \( Q(H_1) \) is the set of those \( \omega \in L_2^2(\mathbb{R}^n, g) \) such that there exists \( \{\psi^{(n)}\} \subset \Lambda_1^1(\mathbb{R}^n) \) for which

\[
\psi^{(n)} \to \omega \quad \text{in} \quad L_2^2(\mathbb{R}^n, g)
\]

and

\[
h_1[\psi^{(n)} - \psi^{(m)}] \to 0
\]

as \( n, m \to +\infty \).

We prove now that

\[
Q(H_0) \subseteq Q(H_1).
\]

For \( \omega \in Q(H_0) \), there exists a sequence \( \{\omega^{(n)}\} \subset \Lambda_1^1(\mathbb{R}^n) \) satisfying (3.5) and (3.6). Due to the equivalence of the norms (2.1) and (2.3),

\[
\omega^{(n)} \to \omega \quad \text{in} \quad L_2^2(\mathbb{R}^n, g);
\]

hence, in order to see that \( \omega \in Q(H_1) \) it suffices to prove that

\[
h_1[\omega^{(n)} - \omega^{(m)}] \to 0
\]

as \( m, n \to +\infty \).

To establish this fact, we consider first the curvature part of \( h_1[\omega^{(n)} - \omega^{(m)}] \),

\[
\int_{\mathbb{R}^n} R(\omega^{(n)} - \omega^{(m)}), (\omega^{(n)} - \omega^{(m)}) \geq \sqrt{g} \, dx.
\]

The following Lemma holds:
Lemma 3.5. There exists $C > 0$ such that

$$\int_{\mathbb{R}^n} < R\omega, \omega >_g \sqrt{g} \, dx \leq C \|\omega\|^2_{L^2(T^*\mathbb{R}^n,e)}$$

for every $\omega \in L^2_1(\mathbb{R}^n,e)$.

Proof. Consider for every $x \in \mathbb{R}^n$ the quadratic form on $T^*_x(\mathbb{R}^n)$

$$\omega \mapsto g^{\alpha\beta}(x)R^i_\alpha(x)\omega_i \omega_\beta = R^{i\beta}(x)\omega_i \omega_\beta.$$ 

Since the matrix $R^{i\beta}(x)$ depends continuously on $x$, its eigenvalues $\lambda_1(x),\ldots,\lambda_n(x)$ are continuous functions of $x$. Hence the function

$$f(x) := \sup_i \lambda_i(x),$$

is continuous. Moreover, since the metric $g$ is asymptotically euclidean, $f(x) \to 0$ as $|x| \to +\infty$. As a consequence, there exists $C > 0$ such that $|f(x)| \leq C$ for every $x \in \mathbb{R}^n$. This in turn implies

$$|\lambda^{i\beta}(x)\omega_i \omega_\beta| \leq C \||\omega||^2_{T^*_x(\mathbb{R}^n)}$$

for every $x \in \mathbb{R}^n$ and for every $\omega \in T^*_x(\mathbb{R}^n)$, which yields (3.11). \qed

Since $\{\omega^{(n)}\}$ is a Cauchy sequence, the preceding Lemma implies that

$$\int_{\mathbb{R}^n} < R(\omega^{(n)} - \omega^{(m)}), (\omega^{(n)} - \omega^{(m)}) >_g \sqrt{g} \, dx \to 0$$

as $n, m \to +\infty$.

As for the gradient part of $h_1[\omega^{(n)} - \omega^{(m)}]$,

$$\int_{\mathbb{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|^2_g \sqrt{g} \, dx,$$

we begin by proving

Lemma 3.6. There exist $C, D > 0$ such that

$$\int_{\mathbb{R}^n} |\eta|^2_e \, dx \leq \int_{\mathbb{R}^n} |\eta|^2_g \sqrt{g} \, dx \leq D \int_{\mathbb{R}^n} |\eta|^2_e \, dx$$

for every smooth, compactly supported tensor $\eta = \eta_{ij}$ of rank 2, where

$$|\eta|^2_e = \sum_{i,j=1}^n \eta_{ij}^2$$

and

$$|\eta|^2_g = g^{ij}(x)g^{kl}(x)\eta_{ik}\eta_{jl}.$$
Proof. Consider, for every \( x \in \mathbb{R}^n \), the quadratic form
\[
\mathbb{R}^{n^2} \times \mathbb{R}^{n^2} \rightarrow \mathbb{R}
\]
defined by
\[
\eta = \eta_{ij} \mapsto a(x)[\eta] = g^{ij}(x)g^{kl}(x)\eta_{ik}\eta_{jl}.
\]
Since this quadratic form is positive and depends continuously on \( x \), its eigenvalues \( \lambda_k(x) \), for \( k = 1, ..., n^2 \), are continuous, positive functions of \( x \). Moreover,
\[
a(x)[\eta] \rightarrow \sum_{i,j=1}^{n} \eta_{ij}^2
\]
as \( |x| \rightarrow +\infty \), implying that \( \lambda_k(x) \rightarrow 1 \), for every \( k = 1, ..., n^2 \), as \( |x| \rightarrow +\infty \).

As a consequence, there exist \( C, D > 0 \) such that
\[
C|\eta|^2 \leq a(x)[\eta] \leq D|\eta|^2
\]
for every \( x \in \mathbb{R}^n, \eta \in \mathbb{R}^{n^2} \), which implies (3.14). \( \square \)

Setting \( \eta_{ik} = \nabla_i(\omega_k^{(n)} - \omega_k^{(m)}) \), (3.14) yields
\[
\int_{\mathbb{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|^2 e \, dx \leq \frac{1}{C} \int_{\mathbb{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|^2 \sqrt{g} \, dx
\]
and
\[
\int_{\mathbb{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|^2 \sqrt{g} \, dx \leq D \int_{\mathbb{R}^n} |\nabla(\omega^{(n)} - \omega^{(m)})|^2 e \, dx.
\]

Now,
\[
\nabla_i \omega_k = \frac{\partial \omega_k}{\partial x_i} - \Gamma^\alpha_{ik} \omega_\alpha,
\]
whence an easy computation shows that for every \( i, k = 1, ..., n \)
\[
\left\| \nabla_i \left( \omega_k^{(n)} - \omega_k^{(m)} \right) \right\|_{L^2(\mathbb{R}^n, e)} \leq \left\| \frac{\partial (\omega_k^{(n)} - \omega_k^{(m)})}{\partial x_i} \right\|_{L^2(\mathbb{R}^n, e)} + K \left\| \omega^{(n)} - \omega^{(m)} \right\|_{L^2(\mathbb{R}^n, e)}.
\]

As a consequence,
\[
h_1[\omega^{(n)} - \omega^{(m)}] \rightarrow 0
\]
as \( n, m \rightarrow 0 \). Thus, \( Q(H_0) \subseteq Q(H_1) \).

We complete the proof of Theorem 2.1 showing that \( Q(H_1) \subseteq Q(H_0) \).

For any \( \omega \in Q(H_1) \) there exists a sequence \( \{ \psi^{(n)} \} \subset \Lambda^1_c(\mathbb{R}^n) \) such that (3.7) and (3.8) hold.
Thanks to the equivalence of the norms (2.1) and (2.3),

$$\psi(n) \rightarrow \omega \text{ in } L^2_1(\mathbb{R}^n, e).$$

Thus, in order to see that \(\omega \in Q(H_0)\) it suffices to prove that

$$h_0[\psi(n) - \psi(m)] \rightarrow 0$$

as \(m, n \rightarrow +\infty\).

Now, (3.7) and (3.8), together with (3.10), imply that

$$\int_{\mathbb{R}^n} |\nabla(\psi(n) - \psi(m))|_g^2 \sqrt{g} \, dx \rightarrow 0$$

as \(n, m \rightarrow +\infty\).

For every \(i, k = 1, ..., n\)

$$\left| \frac{\partial (\psi_k(n) - \psi_k(m))}{\partial x_i} \right|_{L^2(\mathbb{R}^n, e)} \leq$$

$$\leq \left\| \nabla_i (\psi_k(n) - \psi_k(m)) \right\|_{L^2(\mathbb{R}^n, e)} + \left\| \Gamma_{ik}^\alpha (\psi_\alpha(n) - \psi_\alpha(m)) \right\|_{L^2(\mathbb{R}^n, e)} \leq$$

$$\leq \left\| \nabla_i (\psi_k(n) - \psi_k(m)) \right\|_{L^2(\mathbb{R}^n, e)} + C \left\| \psi(n) - \psi(m) \right\|_{L^2_1(\mathbb{R}^n, e)},$$

Then, in view of (3.15),

$$h_0[\psi(n) - \psi(m)] \rightarrow 0$$

as \(n, m \rightarrow +\infty\), and \(Q(H_1) \subseteq Q(H_0)\).

Therefore,

$$J(Q(H_0)) = Q(H_1).$$

Remark 3.7. Theorem 2.1 holds, more in general, for \(p\)-forms, with \(p = 1, ..., n\), with arguments following the same patterns of the ones developed for \(p = 1\). Indeed, estimates like (3.1), (3.2), (3.4) hold for \(p\)-forms, showing that the identification \(J = I : L^2_p(\mathbb{R}^n, g) \rightarrow L^2_p(\mathbb{R}^n, e)\) is continuous with two-sided bounded inverse. To establish the validity of conditions 2.,3. of Belopol’ski-Birman theorem requires replacing \(\Delta_g\) with the Laplace-Beltrami operator on \(p\)-forms, given by

$$(\Delta_g(p)\omega)_{i_1...i_p} = - \sum_{\alpha, \beta} g^{\alpha \beta} \nabla_\alpha \nabla_\beta \omega_{i_1...i_p} + \sum_{j, \alpha} R^\alpha_{ij} \omega_{i_1...i_j...i_p} +$$

$$- \sum_{j \neq i, \alpha, \beta} R^\beta_{i,j} \omega_{\alpha i_1...\beta i_j...i_p},$$

where \(R^\beta_{i,j}\) is the Riemann curvature tensor, which satisfies the condition

$$|R^\beta_{i,j}| \leq \frac{C}{|x|}$$

for \(|x| >> 0\).
The quadratic forms $h_0$ and $h_1$ have now to be replaced by

$$h_0(p) = \int_{\mathbb{R}^n} \sum_{i_1, \ldots, i_p, j=1}^n \left( \frac{\partial \omega_{i_1 \ldots i_p}}{\partial x_j} \right)^2 dx$$

and by $h_1(p)$ expressed by

$$h_1[p] = \int_{\mathbb{R}^n} |\nabla \omega|_g^2 \sqrt{g} dx + \int_{\mathbb{R}^n} <\tilde{R}\omega, \omega > g \sqrt{g} dx,$$

where

$$|\nabla \omega|_g^2 = g^{\alpha \beta} g^{i_1 j_1} \ldots g^{i_p j_p} \nabla_\alpha \omega_{i_1 \ldots i_p} \nabla_\beta \omega_{j_1 \ldots j_p}$$

and

$$<\tilde{R}\omega, \omega > g = g^{i_1 j_1} \ldots g^{i_p j_p} R^\alpha_{i_1 j_1 \ldots i_p} \omega_{i_1 \ldots i_p} \omega_{j_1 \ldots j_p} +$$

$$+ g^{i_1 j_1} \ldots g^{i_p j_p} R^\alpha_{i_1 j_1 \ldots i_p} \omega_{i_1 \ldots \beta \ldots i_p} \omega_{j_1 \ldots j_p}.$$
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Dipartimento di Matematica, Politecnico di Torino

E-mail address: antoci@dimat.unipv.it