Lepton Mixing from $\Delta(3n^2)$ and $\Delta(6n^2)$ and CP

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Abstract

We perform a detailed study of lepton mixing patterns arising from a scenario with three Majorana neutrinos in which a discrete flavor group $G_f = \Delta(3n^2)$ or $G_f = \Delta(6n^2)$ and a CP symmetry are broken to residual symmetries $G_e = Z_3$ and $G_\nu = Z_2 \times CP$ in the charged lepton and neutrino sectors, respectively. While we consider all possible $Z_3$ and $Z_2$ generating elements, we focus on a certain set of CP transformations. The resulting lepton mixing depends on group theoretical indices and one continuous parameter. In order to study the mixing patterns comprehensively for all admitted $G_e$ and $G_\nu$, it is sufficient to discuss only three types of combinations. One of them requires as flavor group $\Delta(6n^2)$. Two types of combinations lead to mixing patterns with a trimaximal column, while the third one allows for a much richer structure. For the first type of combinations the Dirac as well as one Majorana phase are trivial, whereas the other two ones predict in general all CP phases to be non-trivial and also non-maximal. Already for small values of the index $n$ of the group, $n \leq 11$, experimental data on lepton mixing can be accommodated well for particular choices of the parameters of the theory. We also comment on the relation of the used CP transformations to the automorphisms of $\Delta(3n^2)$ and $\Delta(6n^2)$.
1 Introduction

Lepton mixing is encoded in the Pontecorvo-Maki-Nakagawa-Sakata (PMNS) mixing matrix $U_{PMNS}$ that contains three mixing angles and up to three CP phases, one Dirac phase $\delta$ and two Majorana phases $\alpha$ and $\beta$. By now, all three lepton mixing angles have been measured in neutrino oscillation experiments [1] (for other global fits reaching similar results see [2,3])

$$\sin^2 \theta_{13} = 0.0219^{+0.0010}_{-0.0011}, \quad \sin^2 \theta_{12} = 0.304^{+0.012}_{-0.012}, \quad \sin^2 \theta_{23} = \begin{cases} 0.451^{+0.06}_{-0.03}, \\ 0.577^{+0.027}_{-0.033}, \end{cases}$$

while there is only a weak indication for a preferred value of the Dirac phase $\delta$ [4]

$$\delta = 4.38^{+1.17}_{-1.03}$$

and no measurement of the Majorana phases $\alpha$ and $\beta$. An interesting approach is based on the idea that a flavor symmetry $G_f$ might be responsible for the peculiar mixing pattern, observed among leptons [4,5]. This symmetry is usually chosen to be discrete, non-abelian and finite and is assumed to be broken to residual groups $G_e$ and $G_\nu$ in the charged lepton and neutrino sectors, respectively [6,10]. All mixing angles and the Dirac phase $\delta$ are then determined by $G_f$ and its breaking, if the three generations of left-handed (LH) leptons form an irreducible three-dimensional representation 3 of $G_f$. The residual symmetry $G_e$ is taken as a (product of) cyclic group(s) with $G_e = Z_3$ being the simplest choice, while the group $G_\nu$ is fixed to be (a subgroup of) a Klein group $Z_2 \times Z_2$ for Majorana neutrinos [6]. A drawback of this approach is that Majorana phases cannot be constrained. In addition, surveys of mixing patterns which can be derived from flavor symmetries $G_f$ being subgroups of $SU(3)$ have shown that the form of these mixing patterns is rather restricted [13,18], e.g. frequently one of the columns of the PMNS mixing matrix turns out to be trimaximal [19] and/or the Dirac phase is trivial.

For this reason we follow here the approach [20] (see also [21,22]) and consider a theory with a flavor and a CP symmetry which are broken to residual symmetries $G_e$ and $G_\nu$ in the charged lepton and neutrino sectors, respectively. The CP symmetry is represented by the CP transformation X that acts on flavor space. Combining the latter consistently with $G_f$ requires certain conditions to be fulfilled and thus constrains the choice of X [20,22]. The residual group $G_e$ is, like in the approach without a CP symmetry, taken to be an abelian subgroup of $G_f$ that allows the three generations of charged leptons to be distinguished. In contrast, the symmetry $G_\nu$ is assumed to be the direct product of a $Z_2$ group contained in $G_f$ and the CP symmetry. All mixing angles and CP phases are then fixed in terms of a single free continuous parameter $\theta$, up to the possible permutations of rows and columns of $U_{PMNS}$. These are admitted, since fermion masses are not constrained in this approach. All observables are thus strongly correlated and, in particular, predictions for Majorana phases are obtained.

In this paper we focus on the groups $\Delta(3n^2)$ and $\Delta(6n^2)$ as flavor symmetries $G_f$. Throughout our analysis we consider groups whose index $n$ is not divisible by three and, if necessary, even. We choose a class of CP transformations X which fulfill all requirements in order to be consistently combined with $G_f$. As $G_e$ we consider the minimal possible symmetry, namely a $Z_3$ group, while for the residual $Z_2$ symmetry in the neutrino sector we study all possible choices. We then find that the mixing arising from all such combinations can be comprehensively studied by considering only three types of combinations.

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1If the residual symmetry $G_\nu$ is only a subgroup of a Klein group, i.e. $G_\nu = Z_3$, then lepton mixing is not only determined by the symmetry breaking pattern of $G_f$, but a free parameter $\theta$ is present [11]. For neutrinos being Dirac particles $G_\nu$ can be any abelian subgroup of $G_f$ which allows the three generations of neutrinos to be distinguished like in the case of charged leptons, see e.g. [12,13].
For the first type of combination, called case 1), the mixing angles only depend on the continuous parameter \( \theta \) and are thus the same for all groups \( \Delta(3n^2) \) and \( \Delta(6n^2) \). In addition, the Dirac phase as well as one of the Majorana phases are trivial, while the other Majorana phase depends on the chosen CP transformation \( X \). The mixing angles, the Dirac phase \( \delta \) and the Majorana phase \( \beta \) obtained for the second type of combination, called case 2), depend in general not only on the continuous parameter \( \theta \), but also on an integer one whose value is determined by the choice of the CP transformation \( X \). The other Majorana phase \( \alpha \) instead is dependent not only on these two parameters, but also on a third one that is again related to the choice of the CP transformation \( X \). One characteristic feature of the PMNS mixing matrix resulting from both these types of combinations is that its second column is trimaximal. This originates from the choice of the generator of the residual \( Z_2 \) symmetry in the neutrino sector. The mixing arising from the third type of combination has a richer structure. In particular, we can classify the mixing in two different cases, called case 3 a) and case 3 b.1). The reactor and the atmospheric mixing angles depend in case 3 a) only on the integer characterizing the residual \( Z_2 \) symmetry in the neutrino sector and the index \( n \) of the flavor group. The expressions of the solar mixing angle and the CP phases instead depend on \( \theta \), on the residual \( Z_2 \) group as well as on the CP transformation \( X \). In case 3 b.1) all mixing angles and CP phases depend on these three parameters. Nevertheless, the requirement to accommodate the experimental data on the mixing angles well selects the residual \( Z_2 \) symmetry in the neutrino sector as well as requires particular values of the parameter \( \theta \). Furthermore, it is interesting to note that in case 3 b.1) a particular choice of the residual \( Z_2 \) symmetry allows the PMNS mixing matrix to have a first column whose elements have the same absolute values as those of the first column of the tribimaximal (TB) mixing matrix \( [23] \). We also perform a numerical analysis in each of these cases and tabulate our results for the smallest (even and odd) values of the index \( n \) that admit a reasonably good fit to the experimental data on the mixing angles. We show that in most cases it is sufficient to consider groups with an index \( n \leq 11 \).

Some particular cases of groups \( \Delta(3n^2) \) and \( \Delta(6n^2) \) combined with a CP symmetry have already been discussed in the literature: the groups with the smallest index \( n = 2 \), \( A_4 \) and \( S_4 \) \( [20] \), as well as the groups with \( n = 4 \), \( \Delta(48) \) \( [24] \) and \( \Delta(96) \) \( [25] \). In \( [27] \) the groups \( \Delta(6n^2) \) for an arbitrary index \( n \) are combined with a CP symmetry. The fundamental difference between our approach and the one discussed there lies in the fact that the latter requires the residual symmetry in the neutrino sector to be a Klein group \( Z_2 \times Z_2 \) and a CP symmetry (that do not necessarily form a direct product), while we only require one \( Z_2 \) and a CP symmetry to be preserved. An immediate consequence is that the authors in \( [27] \) only discuss groups \( \Delta(6n^2) \) with an even index \( n \), whereas we also admit groups with an odd index, see case 3 a) and case 3 b.1). The residual symmetry in the charged lepton sector, on the other hand, is in both approaches chosen as a \( Z_3 \) group. Since the symmetry preserved in the neutrino sector is larger in \( [27] \) than in our approach, their results are more constrained, in particular all mixing angles are fixed, up to the possible permutations of rows and columns of the PMNS mixing matrix, one column of \( U_{PMNS} \) is always trimaximal, the Dirac phase is trivial as well as one of the Majorana phases, while the other one depends on the chosen CP transformation. We can reproduce these results from ours for particular choices of the continuous parameter \( \theta \), as we show explicitly in the discussion of case 1).

The paper is organized as follows: in section \( [2] \) we recapitulate the essential ingredients of the approach with a flavor and a CP symmetry and how lepton mixing is derived. Furthermore, we detail the relevant properties of the groups \( \Delta(3n^2) \) and \( \Delta(6n^2) \) that we employ as flavor symmetries. In section \( [3] \) we list all possible elements of \( \Delta(3n^2) \) and \( \Delta(6n^2) \) that generate a \( Z_3 \)

\footnote{For a study of the group \( \Delta(27) \) combined with a CP symmetry see \( [26] \).}
or a $Z_2$ group and thus can be used as generators of residual symmetries in the charged lepton
and neutrino sectors, respectively. As regards the CP transformations $X$, we focus on a certain
set and show that these can be consistently combined with the flavor groups under discussion
and with the residual $Z_2$ group in the neutrino sector. We also comment on the relation of these
CP transformations to the automorphisms of $\Delta(3 \, n^2)$ and $\Delta(6 \, n^2)$ and study their properties,
especially, the question whether they can be ‘class-inverting’ or not [28]. The possibility to
have accidental CP symmetries in the theory is mentioned as well. Three types of different
combinations of $Z_3$ and $Z_2$ generators and CP transformations $X$ turn out to be representative
for all possible ones and for these lepton mixing is discussed in detail in section 4; we present
analytic formulae for mixing angles and CP invariants/phases, study constraints put on the
parameters of the theory by the experimental data, discuss the possible presence of accidental
CP symmetries, and analyze each mixing pattern numerically. In doing so, we first study the
general dependences of mixing angles and CP phases on the parameters of each combination
and then perform a $\chi^2$ analysis in order to find the smallest values of the index $n$ that admit a
good agreement with experimental data. Our results are shown in various tables, see tables 3-5,
7-12. In section 5 we summarize our main results and conclude. Our conventions for mixing
angles, CP invariants and phases are found in appendix A together with a summary of the
global fit results [1] and details of the $\chi^2$ analysis. Appendix B contains details about how to
reduce the number of combinations of residual $Z_3$ and $Z_2$ symmetries and CP transformations
$X$ to only three types that lead to distinct mixing patterns.

2 Approach

In this section we recapitulate the conditions which have to be fulfilled in order to consistently
combine a flavor and a CP symmetry, represented by the CP transformation $X$, and repeat
the derivation of lepton mixing in a such a theory. Furthermore, we briefly summarize some
relevant properties of the groups $\Delta(3 \, n^2)$ and $\Delta(6 \, n^2)$.

2.1 Combination of flavor and CP symmetry and derivation of lepton mixing

We consider in the following a theory that is invariant under a discrete, non-abelian and finite
flavor symmetry $G_f$ and a CP symmetry which in general also acts in a non-trivial way on the
flavor space. Since we are interested in the description of lepton mixing and motivated by the
existence of three generations, we focus on irreducible three-dimensional representations $3$ of
$G_f$ to which we will assign the three generations of LH leptons. The elements of the group
$G_f$ can be represented by unitary three-by-three matrices $g$ in $3$ and also the CP symmetry is
represented by a three-by-three matrix $X$. This matrix has to be unitary and symmetric [20]

$$XX^\dagger = XX^* = 1.$$  \hspace{1cm} (3)

The latter constraint arises, because we only consider CP transformations that correspond to
automorphisms of order two (involutions). As has been shown in [20], constraints on the choice
of $X$ arise from the requirement that the subsequent application of the CP transformation,
the flavor symmetry and the CP transformation can still be represented by an element of the
flavour group, i.e.

$$(X^{-1} g X)^* = g'$$  \hspace{1cm} (4)

with $g$ and $g'$ representing two elements of the flavor group $G_f$ which are in general not equal.
In order to show that $X$ fulfills this condition it is sufficient to check that it holds for a set
of generators $g$ of $G_f$. The fact that the residual symmetry of the neutrino sector should be
a direct product of a $Z_2$ symmetry contained in $G_f$ and the CP symmetry imposes a further constraint

$$XZ^* - ZX = 0 \text{ with } Z^2 = 1$$

(5)

with $Z$ being the generator of this $Z_2$ symmetry in the representation 3. The lepton mixing is derived in this theory from the requirement that residual symmetries $G_e$ and $G_\nu = Z_2 \times \text{CP}$ are present in the charged lepton and neutrino sectors, respectively. For $Q$ being the realization of the generator of $G_e$ in 3 we know that the combination $m_\ell^\dagger m_\ell$ (where $m_\ell$ is written in a basis with right-handed (RH) charged leptons on the left-hand side and LH leptons on the right-hand side) fulfills

$$Q^\dagger m_\ell^\dagger m_\ell Q = m_\ell^\dagger m_\ell .$$

(6)

For non-degenerate eigenvalues of $Q$ (i.e. we have the possibility to distinguish the three generations with the help of this symmetry) the unitary matrix $U_e$ which diagonalizes $Q$ is determined, up to permutations of its columns and overall phases of each column, by the requirement

$$U_e^\dagger Q U_e$$

(7)

is diagonal. Given (6) the matrix $U_e$ also diagonalizes $m_\ell^\dagger m_\ell$, i.e. also

$$U_e^\dagger m_\ell^\dagger m_\ell U_e$$

(8)

is diagonal. The fact that lepton masses are not constrained in this approach is reflected by the possible permutations of the columns of $U_e$. Analogously, the neutrino sector and thus the light neutrino mass matrix $m_\nu$ (for three Majorana neutrinos) is invariant under the residual symmetry $G_\nu = Z_2 \times \text{CP}$. Concretely, the matrix $m_\nu$ is constrained by the conditions

$$Z^T m_\nu Z = m_\nu \text{ and } X m_\nu X = m_\nu^* .$$

(9)

Applying the basis transformation induced by the unitary matrix $\Omega$ that fulfills

$$X = \Omega \Omega^T \text{ and } \Omega^T Z \Omega = \begin{pmatrix}
(-1)^{z_1} & 0 & 0 \\
0 & (-1)^{z_2} & 0 \\
0 & 0 & (-1)^{z_3}
\end{pmatrix}$$

(10)

with $z_i = 0, 1$ and two $z_i$ being equal, we see that the combination $\Omega^T m_\nu \Omega$ is constrained to be block-diagonal and real. Thus, this matrix diagonalized by a rotation $R_{ij}(\theta)$ through an angle $\theta$, $0 \leq \theta < \pi$, in the $(ij)$-plane. This plane is determined by the $(ij)$-subspace of the matrix $\Omega^T Z \Omega$ which has degenerate eigenvalues. In addition, a diagonal matrix $K_\nu$ with elements equal to $\pm 1$ and $\pm i$ is necessary for making neutrino masses positive. This matrix can be parametrized without loss of generality as

$$K_\nu = \begin{pmatrix}
1 & 0 & 0 \\
0 & i^{k_1} & 0 \\
0 & 0 & i^{k_2}
\end{pmatrix},$$

(11)

3Throughout the analysis we only discuss the case in which the group $G_e$ can be generated by a single generator. However, the generalization to the case in which $G_e$ is a (direct) product of cyclic symmetries is straightforward, see [20].

4 We define the three different matrices $R_{ij}(\theta)$ as

$$R_{12}(\theta) = \begin{pmatrix}
\cos \theta & \sin \theta & 0 \\
-\sin \theta & \cos \theta & 0 \\
0 & 0 & 1
\end{pmatrix}, \quad R_{13}(\theta) = \begin{pmatrix}
\cos \theta & 0 & \sin \theta \\
0 & 1 & 0 \\
-\sin \theta & 0 & \cos \theta
\end{pmatrix}, \quad R_{23}(\theta) = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos \theta & \sin \theta \\
0 & -\sin \theta & \cos \theta
\end{pmatrix}.$$
with \( k_{1,2} = 0, 1, 2, 3 \). So, the original matrix \( m_{\nu} \) can be brought into diagonal form with positive entries on its diagonal via the unitary matrix

\[
U_{\nu} = \Omega R_{ij}(\theta) K_{\nu}.
\]

Also the masses of the light neutrinos are not fixed and thus permutations of the columns of the matrix \( U_{\nu} \) are admitted. Altogether, we find that the lepton mixing matrix \( U_{PMNS} \) is of the form

\[
U_{PMNS} = U_{\nu}^\dagger U_{\nu} = U_{\nu}^\dagger \Omega R_{ij}(\theta) K_{\nu},
\]

up to possible unphysical phases and permutations of rows and columns. Thus, in our analysis of the groups \( \Delta(3n^2) \) and \( \Delta(6n^2) \) we always consider 36 possible permutations of rows and columns for a given combination \((Q, Z, X)\).

Before concluding the discussion about the general approach let us mention that the formulae in (3-5) are co-variant under the basis transformation with a unitary matrix \( \tilde{\Omega} \), i.e.

\[
\tilde{Z} = \tilde{\Omega}^\dagger Z \tilde{\Omega} \quad \text{and} \quad \tilde{X} = \tilde{\Omega}^\dagger X \tilde{\Omega}^* \quad \text{(14)}
\]

do also fulfill the conditions in (3-5). If we also transform the generator \( Q \) of \( G_e \) in this way

\[
\tilde{Q} = \tilde{\Omega}^\dagger Q \tilde{\Omega},
\]

we see that the PMNS mixing matrix in (13) does not change, since its result does not depend on the transformation \( \tilde{\Omega} \). Thus, both combinations \((Q, Z, X)\) and \((\tilde{Q}, \tilde{Z}, \tilde{X})\) related by \( \tilde{\Omega} \) lead to the same results for lepton mixing.

### 2.2 Group theory of \( \Delta(3n^2) \) and \( \Delta(6n^2) \)

The groups \( \Delta(3n^2) \) are isomorphic to the semi-direct product \((Z_n \times Z_n) \rtimes \mathbb{Z}_3\) with the index \( n \) being in general an integer. Here we always assume that \( n \) is not divisible by three, i.e. \( 3 \nmid n \). These groups can be defined with the help of three generators \( \tilde{a}, \tilde{c} \) and \( \tilde{d} \) that fulfill the relations

\[
\tilde{a}^3 = e, \quad \tilde{c}^n = e, \quad \tilde{d}^n = e, \\
\tilde{c} \tilde{d} = \tilde{d} \tilde{c}, \quad \tilde{a} \tilde{c} \tilde{a}^{-1} = \tilde{c}^{-1} \tilde{d}^{-1}, \quad \tilde{a} \tilde{d} \tilde{a}^{-1} = \tilde{c} \quad \text{(16)}
\]

with \( e \) denoting the neutral element of the group \( \Delta(3n^2) \). The explicit form of these generators in the irreducible three-dimensional representations can be chosen as

\[
\tilde{a} = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 0 & 0 \end{pmatrix}, \quad \tilde{c} = \begin{pmatrix} 0 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-k-l} \end{pmatrix}, \quad \tilde{d} = \begin{pmatrix} \eta^{-k-l} & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^k \end{pmatrix} \quad \text{(17)}
\]

with \( k, l = 0, 1, ..., n - 1 \) and \( \eta = e^{2\pi i/n} \), i.e. \( \eta^n = 1 \). The indices \( k \) and \( l \) label the three-dimensional representations (excluding the case \( k = l = 0 \)). Since this labeling leads to an over-counting of representations, we find in general that this type of group has \( \frac{n^2 - 1}{3} \) inequivalent three-dimensional irreducible representations. In the following we choose \( k = n - 1 \) and \( l = 1 \), i.e.

\[
\tilde{c} = \begin{pmatrix} \eta & 0 & 0 \\ 0 & \eta^{-1} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{d} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta & 0 \\ 0 & 0 & \eta^{-1} \end{pmatrix}, \quad \text{(18)}
\]

\(^5\)Some useful relations that can be easily derived are: \( \tilde{c}^{-1} \tilde{a}^2 \tilde{a} \tilde{c} \frac{\tilde{c}^{-1}}{\tilde{d}^{-1}} \tilde{a}^2 \tilde{a} \tilde{c}^{-1} \) and \( \tilde{a} \tilde{c} \tilde{d} = \tilde{d}^{-1} \tilde{a} \).
that always give rise to a faithful representation of $\Delta(3n^2)$. Note that, for notational simplicity, we do not distinguish between the elements of the abstract group $\Delta(3n^2)$ and the representatives of these elements in the representation 3 which we employ in our discussion of lepton mixing patterns. This we can do, especially, since the representation 3 is faithful and thus each element of the abstract group is represented by a different matrix representative. It is convenient to change to a basis in which $\tilde{a}$ becomes diagonal

$$ a = U_a^\dagger \tilde{a} U_a = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \quad \text{with} \quad \omega = e^{2\pi i/3} \quad (19) $$

and the unitary matrix $U_a$ reads

$$ U_a = \frac{1}{\sqrt{3}} \begin{pmatrix} 1 & \omega & \omega^2 \\ 1 & \omega^2 & \omega \\ 1 & 1 & 1 \end{pmatrix}. \quad (20) $$

The generator $\tilde{c}$ reads in this basis

$$ c = U_a^\dagger \tilde{c} U_a = \frac{1}{3} \begin{pmatrix} 1 + 2 \cos \phi_n & 1 - \cos \phi_n - \sqrt{3} \sin \phi_n & 1 - \cos \phi_n + \sqrt{3} \sin \phi_n \\ 1 - \cos \phi_n + \sqrt{3} \sin \phi_n & 1 + 2 \cos \phi_n & 1 - \cos \phi_n + \sqrt{3} \sin \phi_n \\ 1 - \cos \phi_n - \sqrt{3} \sin \phi_n & 1 - \cos \phi_n + \sqrt{3} \sin \phi_n & 1 + 2 \cos \phi_n \end{pmatrix} \quad (21) $$

where we introduced the abbreviation

$$ \phi_n = \frac{2\pi}{n}. \quad (22) $$

The form of the remaining generator $\tilde{d}$ can also be easily computed in the new basis, e.g. by using the relation $d = a^2 c a$ in (16). It is important to note that all elements of the group can be written in the form

$$ g = a^\alpha c^\gamma d^\delta \quad \text{with} \quad \alpha = 0, 1, 2, \quad 0 \leq \gamma, \delta \leq n - 1. \quad (23) $$

In order to generate the groups $\Delta(6n^2)$ that are isomorphic to $(Z_n \times Z_n) \rtimes S_3$ four generators $\tilde{a}, \tilde{b}, \tilde{c}$ and $\tilde{d}$ are necessary that fulfill the relations in (16) and also

$$ \tilde{b}^2 = e, \quad (\tilde{a} \tilde{b})^2 = e, \quad \tilde{b} \tilde{c} \tilde{b}^{-1} = \tilde{d}^{-1}, \quad \tilde{b} \tilde{d} \tilde{b}^{-1} = \tilde{c}^{-1}. \quad (24) $$

Following [30] we define $\tilde{a}$ in the irreducible three-dimensional representations as in (17), while $\tilde{c}$ and $\tilde{d}$ now depend on a single index $l$, $l = 1, ..., n - 1$,

$$ \tilde{c} = \begin{pmatrix} \eta^l & 0 & 0 \\ 0 & \eta^{-l} & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad \tilde{d} = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \eta^l & 0 \\ 0 & 0 & \eta^{-l} \end{pmatrix}, \quad (25) $$

and the additional generator $\tilde{b}$ is chosen to be of the form

$$ \tilde{b} = \pm \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}. \quad (26) $$

One can show that the usage of a faithful three-dimensional representation different from 3 does not give rise to any new results for lepton mixing, see also [14].
As can be checked, 2 \((n - 1)\) inequivalent irreducible three-dimensional representations are obtained. If we want to match \(\tilde{c}\) and \(\tilde{d}\) in (25) to the ones already chosen for \(\Delta(3n^2)\) in (18) we have to take \(l = 1\). We also apply the change of basis induced by \(U_a\) in (20) to \(\tilde{b}\)

\[
b = U_a^\dagger \tilde{b} U_a = \pm \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & \omega^2 \\ 0 & \omega & 0 \end{pmatrix}.
\]

(27)

Without loss of generality we can choose “+” in (27). Similar to (23) all elements of the group \(\Delta(6n^2)\) can be uniquely written in the form

\[
g = a^\alpha b^\beta c^\gamma d^\delta \quad \text{with} \quad \alpha = 0, 1, 2, \beta = 0, 1, \quad 0 \leq \gamma, \delta \leq n - 1.
\]

(28)

For reasons which we discuss below we not only assume that \(n\) is not divisible by three, but for \(\Delta(3n^2)\), also always that \(n\) is even. For \(\Delta(6n^2)\) the latter assumption is only made for case 1) and case 2).

3 Possible choices of \(Q, Z\) and CP transformation \(X\)

In this section we detail our choices of the generator \(Q\) of the residual symmetry \(G_e\), the possible choices for \(Z\), the generator of the \(Z_2\) symmetry present in the neutrino sector, as well as our choice of the CP transformation \(X\). We also comment on the properties of the automorphisms corresponding to the presented \(X\) as well as discuss the possible existence of accidental CP symmetries for certain choices of combinations \((Q, Z, X)\).

3.1 Discussion of choices of \(Q\)

As regards the groups \(\Delta(3n^2)\), it is clear that the generator \(Q\) of \(G_e\) has to be of the form

\[
Q = a c^\gamma d^\delta \quad \text{or} \quad Q = a^2 c^\gamma d^\delta \quad \text{with} \quad 0 \leq \gamma, \delta \leq n - 1,
\]

(29)

since the remaining form \(c^\gamma d^\delta\), see (23), would lead to a generator \(Q\) which commutes with all the possible choices of \(Z_2\) symmetry generating elements, see (31). The admissible choices of \(Q\) for \(\Delta(3n^2)\) thus all generate a \(Z_3\) symmetry \(((ac^\gamma d^\delta)^3 = e\) and \((a^2 c^\gamma d^\delta)^3 = e\) with \(0 \leq \gamma, \delta \leq n - 1\)). Indeed, these are also all elements of the groups \(\Delta(3n^2)\) that can give rise to a \(Z_3\) symmetry for an index \(n\) that is not divisible by three\footnote{If the latter constraint did not hold, also elements of the form \(c^\gamma d^\delta\) can give rise to a \(Z_3\) symmetry, see also [4].}. Thus, the choice of \(Q\) in (29) is the most general one for the groups \(\Delta(3n^2)\). In the case of the groups \(\Delta(6n^2)\) we still stick to the same choice for \(Q\) and thus discuss in this case comprehensively only the case \(G_e = Z_3\) (again, additional \(Z_3\) generating elements exist, if \(n\) is divisible by three). As we show in Appendix B it is sufficient to consider the choice

\[
Q = a
\]

(30)

in order to comprehensively study all cases \(G_e = Z_3\).

3.2 Discussion of choices of \(Z\)

In the case of \(\Delta(3n^2)\) the index \(n\) has to be even in order for the group to have \(Z_2\) generating elements. These are

\[
Z = c^{n/2}, \quad Z = d^{n/2} \quad \text{and} \quad Z = (cd)^{n/2}.
\]

(31)
The number of \( \mathbb{Z}_2 \) generating elements considerably increases, if we choose \( \Delta(6n^2) \), since also elements of the form
\[
Z = b c^m d^n, \quad Z = a b c^n \quad \text{and} \quad Z = a^2 b d^n \quad \text{with} \quad 0 \leq m \leq n - 1
\]
give rise to a \( \mathbb{Z}_2 \) symmetry. Depending on whether \( n \) is odd or even, we thus have \( 3n \) or \( 3(n+1) \) elements at our disposal as generator of the residual \( \mathbb{Z}_2 \) symmetry in the neutrino sector for \( \Delta(6n^2) \), see also [14].

3.3 Discussion of choices of CP transformation \( X \)

We do not attempt to perform a comprehensive study of all possible admissible CP transformations \( X \). Rather we would like to focus on a particular set. One representative of this set is the CP transformation \( X_0 \)
\[
X_0 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix} = P_{23} .
\]

A viable choice of \( \Omega \) fulfilling \( X_0 = \Omega \Omega^T \) is
\[
\Omega = P_{123} U_a \quad \text{with} \quad P_{123} = \begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix} .
\]

As one can check \( X \) fulfills (4) for the generators \( a, c \) and \( d \) as given in (19) and (21)
\[
X_0 a^* X_0^* = a , \quad X_0 c^* X_0^* = c^{-1} , \quad X_0 d^* X_0^* = d^{-1} .
\]

As shown in [22,28], CP transformations correspond to automorphisms of the flavor symmetry \( G_f \), here \( \Delta(3n^2) \). The action of the automorphism corresponding to \( X_0 \) on the generators of the group is as follows
\[
a \to a , \quad c \to c^{-1} \quad \text{and} \quad d \to d^{-1} .
\]

Since this transformation exchanges classes of \( \Delta(3n^2) \), e.g. the class \( \{ c, (cd)^{-1}, d \} \) is mapped into \( \{ c^{-1}, cd, d^{-1} \} \) that is different, if \( n \neq 2 \), we conclude that this automorphism is an outer one. We also see that since \( a \) is mapped into \( a \) and \( a \) is not in the same class as \( a^{-1} = a^2 \) that this automorphism cannot be ‘class-inverting’.

If we choose the CP transformation \( X_0 \) for \( G_f = \Delta(6n^2) \) we can additionally check that
\[
X_0 b^* X_0^* = b
\]
and thus the automorphism corresponding to this CP transformation maps
\[
b \to b .
\]

We also note that in the case of \( G_f = \Delta(6n^2) \) \( X_0 \) can be written in terms of the matrices \( a \) and \( b \)
\[
P_{23} = a b .
\]

---

8\(^8\)Notice that \( X_0 \) in the “un-rotated” basis is given by \( \tilde{X}_0 = U_a X_0 U_a^T = \mathbb{I} \).

9\(^9\)Notice that \( c^{-1} \) has the same form as \( c \) in [21] with \( \phi_n \to -\phi_n \). This is also the form of the matrix \( c^T \) in this basis.

10\(^{10}\)For a discussion of the necessity of using CP transformations that correspond to ‘class-inverting’ automorphisms see [28].
By studying only the classes containing the generators of the group one could be tempted to claim this to be an inner automorphism (mainly because \(c, d \text{ and } c^{-1}, d^{-1} \) are now in the same class). However, for example the class \(\{c^{d^{-1}}, c^{c^{-2}d^{-1}}, c^{d^2}\} \) \((\rho = 1)\) is mapped into \(\{c^{-1}d, c^2d, c^{-1}d^{-2}\} \) \((\rho = n - 1)\) that is different (the general form of this type of classes is \(\{c^{\rho d}, c^{c^{-2}\rho d}, c^{c^{-2}\rho d^\rho}\}\) with \(\rho = 1, \ldots, n - 1\)) \([30]\). Thus, again some classes are exchanged and the automorphism must be outer, unless we choose \(n = 2\) (the flavor symmetry is then \(S_4\)), since in this particular case the automorphism coincides with the identity mapping, simply because \(c^2 = e \text{ and } d^2 = e\). Considering the class structure of \(\Delta(6n^2)\) we see that the elements of the form \(ac^2dy\) and \(a^{2}c^{-y}dz\), \(y, z = 0, \ldots, n - 1\), belong to the same class, i.e. especially \(a\) is now similar to \(a^2 = a^{-1}\). Furthermore, \(b = b^{-1}\) because it has order two. Thus, we might guess that the automorphism is ‘class-inverting’ with respect to the group \(\Delta(6n^2)\). This guess is confirmed by an explicit computation at the end of this subsection.

As is known, if \(X_0\) is an admissible CP transformation also CP transformations of the form

\[
X = gX_0 = \alpha^a c^\beta d^\gamma P_{23} \quad \text{and} \quad X = gX_0 = \alpha^a b^\beta c^\gamma d^\delta P_{23} \quad (40)
\]

with \(\alpha = 0, 1, 2, \beta = 0, 1 \text{ and } 0 \leq \gamma, \delta \leq n - 1\) are admissible for \(G_f = \Delta(3n^2)\) and \(\Delta(6n^2)\), respectively, since they are conjugated with an element of the group, as long as they lead to symmetric matrices in the representation \(3\). Applying this constraint we find four types of CP transformations \(X\)

\[
X = c^s d^t P_{23}, \quad X = b c^sd^{n-s} P_{23}, \quad X = a b c^sd^{2s} P_{23} \quad \text{and} \quad X = a^2b c^{2t}d^t P_{23} \quad (41)
\]

with \(0 \leq s, t \leq n - 1\). The exponents of the last three choices are constrained by the requirement that \(X\) is symmetric. In particular, we cannot find any \(X\) of the form \(a c^s P_{23}\) or \(a^2 c^s d^t P_{23}\) that corresponds to a symmetric matrix. If we just count the number of admissible choices of \(X\) that arise from \(X_0\) and its conjugation with an element of the flavor group, we arrive at \(n^2\) such choices for \(\Delta(3n^2)\) and \(n(n+3)\) possibilities for \(G_f = \Delta(6n^2)\). This, however, does not imply that a CP transformation of the form as one of the last three in \([41]\) is in general not admitted, if \(G_f = \Delta(3n^2)\) is selected. It just implies that such a CP transformation is not related to the automorphism corresponding to \(X_0\) and we have to carefully check the properties of this new automorphism. This indeed happens in the first case in \([52]\).

Comment on ‘class-inverting’ automorphisms

A simple test to see whether an automorphism can be ‘class-inverting’, i.e. can have the property that each element is mapped into an element that belongs to the same class as the inverse of the former, is related to the following observation: for an automorphism \(\iota\) that is an involution and ‘class-inverting’ the twisted Frobenius-Schur indicator \(\epsilon_\iota(r)\) equals \(\pm 1\) for all irreducible representations \(r\). If all \(\epsilon_\iota(r) = 1\), the automorphism \(\iota\) is a Bickerstaff-Damhus automorphism \([28]\). The definition of \(\epsilon_\iota(r)\) is

\[
\epsilon_\iota(r) = \frac{1}{|G|} \sum_{g \in G} \chi_r(g^{-1}g) \quad (42)
\]

for a group \(G\), here \(G_f, |G|\) being the number of elements of \(G\), \(\chi_r(h)\) the character of the element \(h\) and \(g^{-1}g\) being the image of the element \(g\) under the automorphism \(\iota\). According to \([31]\) for an automorphism \(\iota\) being an involution and a finite group \(G\) the following holds

\[
\sum_r \chi_r(h) \epsilon_\iota(r) = |\{g \in G \mid g^{-1}g = h\}| \quad (43)
\]
for any element \( h \) of \( G \) and summing over all irreducible representations \( r \) on the left-hand side. In particular, it is true for \( h \) being the neutral element of the group that \( \epsilon_i(r) = 1 \) for every irreducible representation \( r \) of \( G \) if and only if
\[
\sum_r \chi_r(e) = \left| \left\{ g \in G \mid g = g^{-1} \right\} \right| \tag{44}
\]
with \( \chi_r(e) \) being the character of the neutral element in the representation \( r \), i.e., we sum over the dimensions of all irreducible representations of \( G \) on the left-hand side of (44).

So, we can check for all CP transformations \( X \) mentioned in (41) whether the equality in (44) is fulfilled. If so, the automorphism must be ‘class-inverting’. The explicit computation shows that the right-hand side of (44) turns out to be equal to \( n(n+3) \) for all \( X \) in the case of \( G_f = \Delta(6n^2) \), whereas it can be maximally \( n^2 \) for \( X \) in (41), if \( G_f = \Delta(3n^2) \). We can compare this result to the sum of the dimensions of the irreducible representations, i.e., the left-hand side of (44), and see that for \( \Delta(3n^2) \) it is always equal to \( 3 + n^2 - 1 = n^2 + 2 \) for \( 3 \nmid n \), and \( n^2 + 6 \), if we considered \( 3 \mid n \), while for \( \Delta(6n^2) \) we always get \( 2 + 2 + 6(n-1) + (n-1)(n-2) = n(n+3) \) for \( 3 \nmid n \) and \( n(n+3) + 4 \) for \( 3 \mid n \). Thus, we find equality of left- and right-hand side of (44) for \( \Delta(6n^2) \), \( 3 \nmid n \), whereas in the other cases the value of the right-hand side is smaller than the one of the left-hand side. So, we know that the CP transformations \( X \) for \( \Delta(6n^2) \), \( 3 \nmid n \), correspond to ‘class-inverting’ automorphisms that are of Bickerstaff-Damhus type. For \( \Delta(3n^2) \) instead this cannot be deduced and, indeed, the arguments given above showed that the CP transformation \( X_0 \) is not ‘class-inverting’. As a consequence [28], we have to expect that the representation content of a theory with \( \Delta(3n^2) \) and CP as symmetries is changed compared to a theory with only \( \Delta(3n^2) \) as symmetry. However, since the CP transformations \( X \) corresponding to these automorphisms still fulfill all requirements in (3-5), this conclusion does not affect our considerations for lepton mixing derived from such a theory.

### 3.4 Accidental CP symmetries

Before summarizing all possible choices of combinations \((Q,Z,X)\) that we will study in the subsequent section we pay attention to the possibility that an accidental CP symmetry is present, different from the one corresponding to the CP transformation \( X \) that we impose in our theory. To remind the reader: an (accidental) CP symmetry corresponding to a CP transformation \( Y \) exists, if \( Y \) fulfills the conditions [20]
\[
Y^* m_1^\dagger m_1 Y = (m_1^\dagger m_1)^* \quad \text{and} \quad Y m_\nu Y^* = m_\nu^* . \tag{45}
\]
Clearly, then all CP phases \( \delta \), \( \alpha \) and \( \beta \) have to be trivial
\[
\sin \delta = 0 \ , \ \sin \alpha = 0 \quad \text{and} \quad \sin \beta = 0 . \tag{46}
\]
If \( Y \) and \( m_\nu \) only fulfill
\[
Y^* m_\nu^\dagger m_\nu Y = (m_\nu^\dagger m_\nu)^* , \tag{47}
\]
then Majorana phases are in general non-trivial, while the Dirac phase \( \delta \) has to be 0 or \( \pi \). As has been shown in [20], the first equality in (45) is fulfilled, if
\[
Q Y - YQ^T = 0 , \tag{48}
\]
while the fulfillment of the second equality implies
\[
Y Z^* - Z Y = 0 \quad \text{and} \quad X Y^* - Y X^* = 0 . \tag{49}
\]
Table 1: Different types of $Z_2$ generators $Z$, contained in $G_f$, that can be combined with a CP transformation $X$ of the form $a^\gamma b^\delta c^\alpha d^\beta P_{23}$, when requesting the fulfillment of $[5]$. If not stated differently, $m$, $s$ and $t$ take integer values between 0 and $n-1$. Obviously, for the first three choices of $Z$ the index $n$ of the flavor symmetry has to be even. Note furthermore that the last three types of $Z_2$ generators are only admitted for the groups $\Delta(6n^2)$.

| $Z = e^{\pi i/2}$ | $X = e^\gamma d P_{23}$, $X = a b c^\alpha d^\beta P_{23}$ |
| $Z = d^{\pi i/2}$ | $X = e^\gamma d P_{23}$, $X = a^2 b c^\alpha d^\beta P_{23}$ |
| $Z = (c d)^{\pi i/2}$ | $X = e^\gamma d P_{23}$, $X = b c^\alpha d^\beta P_{23}$ |
| $Z = b c^m d^m$ | $X = e^\gamma d P_{23}$ with $t = n - 2m - s$, $X = b c^\delta d^\gamma P_{23}$ |
| $Z = a b c^n$ | $X = e^\gamma d P_{23}$ with $t = 2(m + s)$, $X = a b c^\delta d^\gamma P_{23}$ |
| $Z = a^2 b d^m$ | $X = e^\gamma d P_{23}$ with $s = 2(m + t)$, $X = a^2 b c^\delta d^\gamma P_{23}$ |

as well as that the CP transformation $Y$ is diagonal and real in the neutrino mass basis, i.e.

$$\tilde{Y} = U_\nu^\dagger Y U_\nu^*$$

has to be diagonal and real. If only the equality in $[47]$ should be fulfilled, it is sufficient that the first equation in $[49]$ is satisfied together with the condition that $\tilde{Y}$ has to be diagonal. In this case the (in general non-trivial) Majorana phases are determined by the differences of the phases of the diagonal elements of $\tilde{Y}$, see $[20]$.

In particular, we see that the most general form of $Y$ compatible with a charged lepton mass matrix invariant under the residual symmetry $G_e$ generated by $Q = a$ ($Q = a^2$) is

$$Y = \begin{pmatrix} e^{iy_1} & 0 & 0 \\ 0 & e^{iy_2} & 0 \\ 0 & 0 & e^{iy_3} \end{pmatrix}$$

with $0 \leq y_i \leq 2\pi$.\(^{11}\) As we will see in the following section such a CP transformation $Y$ can also be, for certain values of $y_i$, accidentally present in the neutrino sector, if the latter is required to be invariant under $G_\nu = Z_2 \times CP$.

An accidental CP symmetry that is always present in the neutrino sector for given transformations $Z$ and $X$ is the one represented by the CP transformation $Y = ZX$ that fulfills the constraints in $[31]$ and $[32]$.

### 3.5 Summary of choices ($Q$, $Z$, $X$)

We take the residual symmetry $G_e$ in the charged lepton sector to be a $Z_3$ symmetry that is generated by $Q = a c^\gamma d^\delta$ or $Q = a^2 c^\gamma d^\delta$, $0 \leq \gamma, \delta \leq n - 1$. As generators of the $Z_2$ symmetry, we can use the ones mentioned in $[31]$ and $[32]$ and our possible choices of CP transformations $X$ are given in $[41]$. Since we require $G_\nu$ to be a direct product of the $Z_2$ symmetry generated by

\[^{11}\text{If we had chosen } Q' = a c^\gamma d^\delta \text{ or } Q' = a^2 c^\gamma d^\delta, 0 \leq \gamma, \delta \leq n - 1, \text{ as generator of } G_e, \text{ the accidental CP transformation } Y' \text{ fulfilling } Q' Y' - Y' Q'^T = 0 \text{ would be of the form } Y' = g^T Y g \text{ with } g = c^\alpha d^\beta \text{ being the similarity transformation relating } Q' = a c^\gamma d^\delta \text{ (} Q' = a^2 c^\gamma d^\delta \text{) to } Q = a (Q = a^2) \text{ via } Q' = g^T Q g \text{ for certain values of } x \text{ and } y.\]

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Z and the CP symmetry corresponding to X, we additionally have to ensure that the relation in (51) is fulfilled. In doing so, we see that the six different types of \( Z_2 \) generating elements can be each combined with two types of CP transformations X, that we list in table 1. Thus, we should consider any generator \( Q \) giving rise to a \( Z_3 \) symmetry with any of the twelve possible combinations of \( Z \) and \( X \). Instead of doing so, we can show, see appendix B, that it is sufficient to only consider the following three types of choices of \((Q, Z, X)\):

\[
(Q = a, Z = c^{n/2}, X = a b c^s d^{2s} P_{23}) ,
\]

\[
(Q = a, Z = c^{n/2}, X = c^s d^l P_{23}) ,
\]

\[
(Q = a, Z = b c^m d^m, X = b c^s d^{n-s} P_{23}) ,
\]

in order to comprehensively study the lepton mixing patterns.

4 Mixing patterns derived from \((Q, Z, X)\)

In the following we discuss the mixing patterns arising from the choices of \((Q, Z, X)\) shown in (52). We first present (one possible) form of the matrix \( \Omega \) and the PMNS mixing matrix. We then discuss the patterns originating from the 36 possible permutations of rows and columns of the latter matrix and detail analytical formulae for mixing angles and CP invariants \( J_{CP} \), \( I_1 \) and \( I_2 \) for the permutations that allow the mixing angles to be in accordance with the experimental data for particular values of the indices related to the choice of the flavor symmetry and the residual symmetries in the neutrino sector as well as of the continuous parameter \( \theta \). Furthermore, we explain why and under which conditions (some) CP phases are trivial. A numerical study shows the dependence of the mixing parameters on the quantities of the theory. The smallest values of the group index \( n \) that admit a reasonably good fit to the experimental data are found via a \( \chi^2 \) analysis and are displayed in various tables.

4.1 Case 1 \((Q = a, Z = c^{n/2}, X = a b c^s d^{2s} P_{23})\)

The first case for which we analyze the lepton mixing in detail can be represented by the following choice of the generator \( Z \) of the \( Z_2 \) symmetry in the neutrino sector and of the CP transformation \( X \)

\[
Z = c^{n/2} \quad \text{and} \quad X = a b c^s d^{2s} P_{23}
\]

(53)

with \( 0 \leq s \leq n-1 \). Since the CP transformation \( X \) is a combination of the element \( a b c^s d^{2s} \) and the CP transformation \( X_0 \), this case assumes as underlying flavor symmetry \( \Delta(6 n^2) \). Nevertheless, it can also be realized for \( G_f = \Delta(3 n^2) \). However, in the latter case the automorphism corresponding to the CP transformation \( X \) is different from the one related (via an inner automorphism) to \( X_0 \). The form of \( Z \) is independent of \( n \)

\[
Z = c^{n/2} = \frac{1}{3} \begin{pmatrix} -1 & 2 & 2 \\ 2 & -1 & 2 \\ 2 & 2 & -1 \end{pmatrix} .
\]

12 Using (39) we can rewrite \( X \) as \( X = c^s d^{2s} \) with \( X_0 \) now being \( \mathbb{1} \) corresponding to the automorphism that maps \( a \rightarrow a^2 \), \( c \rightarrow c \) and \( d \rightarrow c^{-1} d^{-1} \). Since both elements do not belong to the same class in \( \Delta(3 n^2) \) in general (only for \( n = 2 \), see below (36), this automorphism cannot be ‘class-inverting’. For \( G_f = \Delta(6 n^2) \) the CP transformation \( X_0 = 1 \) also maps \( b \rightarrow a^2 b \) and, as discussed, the corresponding automorphism is ‘class-inverting’ in these groups, see below (44).
The non-degenerate eigenvalue of $Z$ is $+1$ and its corresponding eigenvector reads

$$Z v_{+1} = +v_{+1} \text{ with } v_{+1} \propto \begin{pmatrix} 1 \\ 1 \\ 1 \end{pmatrix}. \tag{55}$$

Thus, one of the columns of the resulting PMNS mixing matrix has to be trimaximal (up to phases). In order to achieve compatibility with the experimental data on lepton mixing angles this column must be identified with the second one of $U_{PMNS}$. As is well-known $[19]$, this implies a lower bound on the solar mixing angle

$$\sin^2 \theta_{12} \gtrsim \frac{1}{3}. \tag{56}$$

A choice of $\Omega$ which fulfills the conditions in $[10]$ for $X$ and $Z$ in $[53]$ is

$$\Omega_1 = e^{i \phi_s} U_{TB} \begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{-3i \phi_s} & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{57}$$

with $U_{TB}$ being the TB mixing matrix

$$U_{TB} = \begin{pmatrix} \sqrt{\frac{2}{3}} & \frac{1}{\sqrt{3}} & 0 \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{2}} \\ -\frac{1}{\sqrt{6}} & \frac{1}{\sqrt{3}} & -\frac{1}{\sqrt{2}} \end{pmatrix} \tag{58}$$

and

$$\phi_s = \frac{\pi s}{n}. \tag{59}$$

In particular, $Z$ reads after the basis transformation $\Omega_1$

$$\Omega_1^\dagger Z \Omega_1 = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \tag{60}$$

and thus the rotation $R_{ij}(\theta)$ has to be applied in the $(13)$-plane. So, the contribution to lepton mixing from the neutrino sector is of the form

$$U_{\nu,1} = \Omega_1 R_{13}(\theta) K_{\nu}, \tag{61}$$

up to permutations of columns, with $K_\nu$ defined as in $[11]$. Given that the generator $Q = a$ of the residual symmetry in the charged lepton sector is diagonal in our chosen basis, it results $U_e = 1$, up to permutations of columns, and thus, the PMNS mixing matrix is, up to possible permutations of rows and columns, of the form

$$U_{PMNS,1} = \Omega_1 R_{13}(\theta) K_{\nu}. \tag{62}$$

### 4.1.1 Analytical results

Out of the 36 possible permutations of rows and columns only twelve lead to a pattern compatible with data. As mentioned above, these are the ones with the second column being trimaximal (the others either give rise to $\sin^2 \theta_{13} = 1/3$ or to a relation between solar and reactor mixing...
angle which does not allow both to be fitted well simultaneously). Six of these twelve permutations lead to the same mixing pattern, if a possible shift in the continuous parameter $\theta$ and a possible re-labeling of $k_1$ and $k_2$ (including their sum or difference) are taken into account. Using the actual form of the PMNS mixing matrix as quoted in (62), we find

$$\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1}{2} \left( 1 + \frac{\sqrt{3} \sin 2\theta}{2 + \cos 2\theta} \right)$$

(63)

and for the CP invariants we get

$$J_{CP} = 0, \quad I_1 = \frac{2}{9} (-1)^{k_1+1} \cos^2 \theta \sin 6 \phi_s, \quad I_2 = 0.$$  

(64)

The remaining six permutations lead to very similar results with the only difference that the atmospheric mixing angle reads

$$\sin^2 \theta_{23} = \frac{1}{2} \left( 1 - \frac{\sqrt{3} \sin 2\theta}{2 + \cos 2\theta} \right),$$

(65)

i.e. the relative sign among the two terms in the expression of $\sin^2 \theta_{23}$ in (63) changes. This pattern, for example, originates from the PMNS mixing matrix in (62) with second and third rows exchanged. It is noteworthy that the mixing angles only depend on the continuous parameter $\theta$ and so all groups $\Delta(3n^2)$ and $\Delta(6n^2)$ lead to the same results. Thus, it is sufficient to consider the smallest such group, i.e. the case $n = 2$. Indeed, this case has already been studied in the literature and our results coincide with those, see case II in [20].

The size of the parameter $\theta$ is mainly determined by the requirement to fit the reactor mixing angle well, i.e. we expect $\theta$ to be either small ($0.17 \lesssim \theta \lesssim 0.2$) or close to $\pi$ ($2.94 \lesssim \theta \lesssim 2.97$).

Since the mixing angles only depend on $\theta$, they fulfill certain (approximate) sum rules

$$\sin^2 \theta_{12} = \frac{1}{3 \cos^2 \theta_{13}} \approx \frac{1}{3} \left( 1 + \sin^2 \theta_{13} \right) \quad \text{and} \quad \sin^2 \theta_{23} \approx \frac{1}{2} \left( 1 \pm \sqrt{2} \sin \theta_{13} \right)$$

(66)

with “+” for $\theta < \pi/2$ and “−” for $\theta > \pi/2$. These have also been found in [24]. For $\sin^2 \theta_{13} = 0.0219$ ($\theta \approx 0.18$ or $\theta \approx 2.96$) which is the best fit value from the latest global fit [1] we find

$$\sin^2 \theta_{12} \approx 0.341 \quad \text{and} \quad \sin^2 \theta_{23} \approx \left\{ \begin{array}{l} 0.605 \\ 0.395 \end{array} \right.$$

(67)

As we see, the Dirac phase is trivial as well as one of the two Majorana phases, since $J_{CP}$ and $I_2$ both vanish. The vanishing of the former indicates an accidental CP symmetry common to the charged lepton sector and to the combination $m^1_\nu m^\nu$ of the neutrino mass matrix (see section 2.4 of [20]) that we explicitly confirm, see (70). The Majorana invariant $I_1$ is in general non-vanishing and can take different values. We can easily extract the value of the Majorana phase $\alpha$ from $I_1$

$$\sin \alpha = (-1)^{k_1+1} \sin 6 \phi_s.$$  

(68)

For the particular case $n = 2$ which has been studied in the literature (see case II in [20]) $I_1$ vanishes, as the only allowed values of $s$ are $s = 0$ and $s = 1$ ($\phi_s = 0$ or $\phi_s = \pi/2$). For $n = 4$ which has been presented in [24, 25] instead also non-vanishing $I_1$ can be achieved – by the choice $s = 1$ or $s = 3$ (corresponding to $\phi_s = \pi/4$ or $\phi_s = 3\pi/4$) – which both
lead to a maximal Majorana phase $\alpha$. The behavior of $\sin \alpha$ for general values of $n$ and $s$ can be read off from the figure in the bottom-left panel of figure 4 that belongs to case 2), $(Q = a, Z = c^{n/2}, X = c^s d^t P_{23})$, if we identify $6 s/n$ with $v/n$ (setting $k_1$ to zero in (68)).

We can understand the vanishing of the CP invariant $J_{CP}$ by recognizing that the accidental CP symmetry $Y$ of the charged lepton sector, see [51], fulfills the following conditions: the one involving $Z$ in [5], if
\[ Y = e^{iy} \mathbb{1} \quad \text{with} \quad 0 \leq y < 2\pi , \]
and it takes a diagonal form in the neutrino mass basis
\[ \tilde{Y} = U_{\nu,1}^\dagger Y U_{\nu,1}^* = e^{i(y-2\phi_s)} \begin{pmatrix} 1 & 0 & 0 \\ 0 & (-1)^{k_1} e^{i\phi_s} & 0 \\ 0 & 0 & (-1)^{k_2} \end{pmatrix} . \quad (70) \]

As discussed in [20], the fulfillment of these conditions tells us that the CP symmetry $Y$ of the charged lepton sector is in this case also a CP symmetry of the combination $m_1^\dagger m_{\nu}$ of the neutrino mass matrix. Furthermore, the values of the Majorana phases $\alpha$ and $\beta$ can be read off from $\tilde{Y}$ in (70)
\[ |\sin \alpha| = |\sin 6 \phi_s| \quad \text{and} \quad \sin \beta = 0 \quad (71) \]
that are consistent with our results for the CP invariants $I_1$ and $I_2$, see (64) and (68). Only if $\tilde{Y}$ is also real [20], all CP violation vanishes, i.e. if $e^{6i\phi_s} = \pm 1$ which is equivalent to $\sin 6 \phi_s = 0$. This holds for $s = 0$ and $s = \frac{n}{2}$. In these cases and for $y = 0$ or $y = \pi$ the two CP symmetries $X$ and $Y$ also commute, see second equality in (40). The two values given for $s$ are the only admissible ones, since $0 \leq s \leq n - 1$ and three does not divide $n$.

At the end of this subsection, we would like to comment on the relations of the presented results to those found in the literature. Our case 1) leads to results very similar those obtained in [27] where an additional $Z_2$ symmetry is present in the neutrino sector. The CP invariants $J_{CP}$ and $I_2$ vanish in general like in [27]. Furthermore, the second column of the PMNS mixing matrix is also trimaximal. If we identify the continuous parameter $\theta$ with $-\frac{\pi}{2} \gamma$ (\(\gamma\) is related to one of the $Z_2$ symmetries, while $n$ is the index of the group $\Delta(6 n^2)$), we can achieve the same results for the mixing angles as found in [27]. In order to reproduce their result for the non-trivial Majorana phase, $6 \phi_s$ in (64) should be identified with $-(\varphi_1 - \varphi_2)$ of [27], since both parameter combinations depend on the choice of the CP transformation (Here we implicitly have set $k_1 = 0$.) Thus, $s = \gamma + x$ (see equation (3.40) in [27]).

\[ \textbf{4.2 Case 2} (Q = a, Z = c^{n/2}, X = c^s d^t P_{23}) \]

Also the choice $(Q = a, Z = c^{n/2}, X = c^s d^t P_{23})$ requires $n$ to be even. The results of this choice have certain similarities with those found in the first case, but have a richer structure, since now the mixing angles not only depend on the continuous parameter $\theta$, but also on the chosen CP transformation $X$, i.e. on a certain combination of the exponents $s$ and $t$, see [22]. In addition, all CP violating phases are in general non-trivial and depend on the continuous $\theta$ as well as on $s$ and $t$ that characterize the CP transformation $X$.

Since also in this case $Z = c^{n/2}$, we know that the resulting PMNS mixing matrix will have a second column which is trimaximal. Consequently, the value of $\sin^2 \theta_{12}$ is bounded from below, $\sin^2 \theta_{12} \gtrsim 1/3$, as is also confirmed in the numerical analysis, see tables 3,4.

It is useful to define the two parameters $u$ and $v$
\[ u = 2s - t \quad \text{and} \quad v = 3t \quad (72) \]
that take integer values in the intervals
\[-(n - 1) \leq u \leq 2(n - 1) \quad \text{and} \quad 0 \leq v \leq 3(n - 1), \quad (73)\]
since \(s\) and \(t\) are constrained to be \(0 \leq s, t \leq n - 1\). Furthermore, we also define, analogously to \(\phi_s\) in (59) for the first case,
\[\phi_u = \frac{\pi u}{n} \quad \text{and} \quad \phi_v = \frac{\pi v}{n}. \quad (74)\]
Then, the form of \(\Omega_2\) which fulfills the conditions in (10) for \(Z = c^{n/2}\) and \(X = e^{s^d}p_{23}\) can be chosen as
\[\Omega_2 = e^{i \frac{\phi_u}{6}} U_{TB} R_{13} \left( -\frac{\phi_u}{2} \right) \left( \begin{array}{ccc} 1 & 0 & 0 \\ 0 & e^{-i \frac{\phi_v}{2}} & 0 \\ 0 & 0 & -i \end{array} \right). \quad (75)\]
The \(Z_2\) generator \(Z\) is given as
\[\bar{Z} = \Omega_2^\dagger Z \Omega_2 = \left( \begin{array}{ccc} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{array} \right) \quad (76)\]
in the basis transformed with \(\Omega_2\) and thus also here the appropriate rotation \(R_{ij}(\theta)\) is in the (13)-plane. The mixing matrix \(U_{\nu,2}\) in the neutrino sector hence reads, up to permutations of its columns,
\[U_{\nu,2} = \Omega_2 R_{13}(\theta) K_\nu \quad (77)\]
and consequently, the PMNS mixing matrix, called \(U_{PMNS,2}\) in the following, is of the same form, up to permutations of rows and columns.

### 4.2.1 Analytical results

As in the first case, also in this case only twelve out of the 36 possible permutations can lead to a mixing pattern compatible with experimental data, namely those whose second column is trimaximal. Similarly to the above, also here all twelve permutations lead to the same type of results for the mixing angles and CP invariants. Taking into account possible shifts in the parameter \(\theta\) and a possible re-labeling of \(k_1\) and \(k_2\) two out of the twelve permutations of the PMNS mixing matrix in (77) lead to
\[
\begin{align*}
\sin^2 \theta_{13} &= \frac{1}{3} \left( 1 - \cos \phi_u \cos 2\theta \right), \\
\sin^2 \theta_{12} &= \frac{1}{2 + \cos \phi_u \cos 2\theta}, \\
\sin^2 \theta_{23} &= \frac{1}{2} \left( 1 + \sqrt{3} \sin \phi_u \cos 2\theta \right)
\end{align*}
\quad (78)\]
and
\[
\begin{align*}
J_{CP} &= -\frac{\sin 2\theta}{\sqrt{3}}, \\
I_2 &= \frac{1}{9} (-1)^{k_2} \sin 2\phi_u \sin 2\theta, \\
I_1 &= \frac{1}{5} (-1)^{k_1+1} \left[ \cos \phi_u + \cos 2\theta \right] \sin \phi_v - \sin \phi_u \cos \phi_v \sin 2\theta.
\end{align*}
\quad (79)\]
We easily see that the mixing angles fulfill the following sum rules
\[
\begin{align*}
\sin^2 \theta_{12} &= \frac{1}{3} \cos^2 \theta_{13} \approx \frac{1}{3} \left( 1 + \sin^2 \theta_{13} \right) \quad (80) \\
6 \sin^2 \theta_{23} \cos^2 \theta_{13} &= 3 + \sqrt{3} \tan \phi_u - 3 \left( 1 + \sqrt{3} \tan \phi_u \right) \sin^2 \theta_{13}.
\end{align*}
\]
\[
\begin{array}{|c|c|c|}
\hline
u \rightarrow u + n & \frac{\pi}{2} - \theta & \sin^2 \theta_{ij}, J_{CP}, I_2 \text{ are invariant} \\
(\phi_u \rightarrow \phi_u + \pi) & & I_1 \text{ changes sign} \\
\hline
u \rightarrow n - u & \theta \rightarrow \theta + \frac{\pi}{2} & \sin^2 \theta_{13}, \sin^2 \theta_{12}, I_2 \text{ are invariant} \\
(\phi_u \rightarrow \pi - \phi_u) & & \sin^2 \theta_{23} \text{ becomes } 1 - \sin^2 \theta_{23}; J_{CP} \text{ and } I_1 \text{ change sign} \\
\hline
u \rightarrow 2n - u & \pi - \theta & \sin^2 \theta_{13}, \sin^2 \theta_{12}, I_1 \text{ and } I_2 \text{ are invariant} \\
(\phi_u \rightarrow 2\pi - \phi_u) & & \sin^2 \theta_{23} \text{ becomes } 1 - \sin^2 \theta_{23}; J_{CP} \text{ changes sign} \\
\hline
\end{array}
\]

Table 2: Case 2). Symmetry transformations of the formulae for mixing angles and CP invariants in (78) and (79).

Obviously, the first sum rule coincides with the one found in the first case. Another two of the twelve permutations give rise to the same formulae, but \(\sin^2 \theta_{23}\) becomes \(1 - \sin^2 \theta_{23}\), i.e.

\[
\sin^2 \theta_{23} = \frac{1}{2} \left( 1 - \sqrt{3} \sin \phi_u \cos 2\theta + \cos \phi_u \cos 2\theta \right) \tag{81}
\]

and the corresponding sum rule reads

\[
6 \sin^2 \theta_{23} \cos^2 \theta_{13} = 3 - \sqrt{3} \tan \phi_u - 3 \left( 1 - \sqrt{3} \tan \phi_u \right) \sin^2 \theta_{13} \tag{82}
\]

The results for the mixing angles and CP invariants calculated from the other eight permutations of the PMNS mixing matrix can be cast into the form of these formulae, if not only a shift in the continuous parameter \(\theta\) and a re-labeling of \(k_1\) and \(k_2\) is taken into account, but also a shift of \(\pm \frac{n}{3}\) of the integer parameter \(u\) (which means \(\phi_u\) is shifted into \(\phi_u \pm \frac{\pi}{3}\)). It is important to mention that the latter shift does in general lead to physically different results, since we consider a shift of an integer parameter through a non-integer number \(\frac{n}{3}\) (remember three does not divide \(n\)). For this reason we discuss the numerical results of mixing angles and CP invariants for \(u\) and \(u\) shifted into \(u \pm \frac{\pi}{3}\) separately. In particular, if we consider the PMNS mixing matrix in (77) and multiply it from the left with the matrix

\[
P_1 = \begin{pmatrix}
0 & 1 & 0 \\
0 & 0 & 1 \\
1 & 0 & 0
\end{pmatrix},
\tag{83}
\]

i.e. cyclicly permute the rows of this matrix, we can obtain the corresponding mixing angles and CP invariants from the formulae in (78) and (79) by simply shifting \(u\), \(\theta\) and by re-defining \(k_1\)

\[
u \rightarrow u - \frac{n}{3}, \quad \theta \rightarrow \frac{\pi}{2} - \theta \quad \text{and} \quad k_1 \rightarrow k_1 + 1, \tag{84}
\]

while for a PMNS mixing matrix that is multiplied from the left by the matrix

\[
P_2 = \begin{pmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{pmatrix} = P_1^2 = P_1^T,
\tag{85}
\]
we get the formulae for mixing angles and CP invariants from (78) and (79), if we perform as transformations

\[ u \to u + \frac{n}{3}, \quad \theta \to \frac{\pi}{2} - \theta \quad \text{and} \quad k_1 \to k_1 + 1. \quad (86) \]

From (78) and (79) we see that \( J_{CP} \) only depends on \( \theta \), while the mixing angles and \( I_2 \) depend on \( \theta \) as well as on \( u \). \( I_1 \) eventually is the only quantity which also depends on \( v \). The formulae in (78) and (79) have several symmetry properties which help us to understand the numerical results and which we summarize in table 2. The first of these symmetries is also valid, if we consider the formulae after applying one of the transformations in (84) or (86). The other two instead relate results for \( u - \frac{n}{3} \) with those for \( u + \frac{n}{3} \), i.e. if the operations \( u \to n - u \) and \( \theta \to \theta + \frac{\pi}{2} \) are applied to the formulae (78) and (79) that are transformed with (84), we recover expressions that result from performing the transformation in (86) on mixing angles and CP invariants in (78) and (79). Since the third symmetry in table 2 is obtained from applying the other two ones subsequently (the ordering of the two transformations is irrelevant), also in this case we relate results for \( u - \frac{n}{3} \) with those for \( u + \frac{n}{3} \). Furthermore, we note that the formulae in (78) in the original version as well as if one of the transformations in (84) or (86) is applied, remain invariant, if we replace \( \theta \) with \( \pi - \theta \). Thus, we expect to find in our numerical analysis for each value \( \theta = \theta_{bf} \) that allows to accommodate the experimental data of the mixing angles well the same good fit for \( \theta = \pi - \theta_{bf} \). The CP invariants, on the other hand, do not remain invariant, if \( \theta \) is replaced by \( \pi - \theta \), but instead \( J_{CP} \) and \( I_2 \) change their sign, while \( I_1 \) does not transform in a definite way, since it contains terms that are even functions in \( \theta \), but also one that is odd in \( \theta \).

Note that for \( u = 0 \) (\( \phi_u = 0 \)) the same results of the mixing angles are obtained for all \( n \), i.e.

\[ \sin^2 \theta_{13} = \frac{2}{3} \sin^2 \theta, \quad \sin^2 \theta_{12} = \frac{1}{2 + \cos 2\theta}, \quad \sin^2 \theta_{23} = \frac{1}{2}. \quad (87) \]

The formulae of reactor and solar mixing angles are the same as in case 1), see (63), while atmospheric mixing turns out to be maximal. Since the value of \( \theta \) that gives rise to the best fit of the experimental data is (mainly) determined by \( \theta_{13} \), also in this case the preferred values of \( \theta \) are \( \theta \approx 0.18 \) and \( \theta \approx 2.96 \). Furthermore, the Dirac phase extracted from \( J_{CP} \) in (79) and (87) is maximal, \( |\sin \delta| = 1 \) (for \( \theta \neq 0, \pi \)), while the Majorana phase \( \beta \) is trivial (\( I_2 \) vanishes independently of \( \theta \)) and from

\[ I_1 = \frac{2}{9} \left(-1\right)^{k_1} \cos^2 \theta \sin \phi_v. \quad (88) \]

we can derive \( |\sin \alpha| = |\sin \phi_v| \). Note that this formula for \( I_1 \) coincides with the one in case 1), see (64), if we identify \( \phi_v \) with 6 \( \phi_s \). As one can check also for \( u = n \) (\( \phi_u = \pi \)) the mixing angles becomes independent of \( n \); a case that is clearly related to \( u = 0 \) via a symmetry found in table 2.

If \( u, v \) and \( n \) are divisible by the same factor, \( \phi_u \) and \( \phi_v \) do not change their values, e.g. a case with \( u \) and \( v \) even can always be reduced to a case with smaller \( u' = \frac{u}{2} \) and \( v' = \frac{v}{2} \), as long as \( n \) is divisible by four, since also \( n' \) has to be even. Thus, it frequently happens that the same results of mixing angles and CP invariants are achieved with different groups \( \Delta(3n^2) \) (and \( \Delta(6n^2) \)). In the case in which only \( u \) and \( n \) (but not \( v \)) have a common divisor \( \rho \) larger than one, the mixing angles, the Dirac and the Majorana phase \( \beta \) are the same, if computed for \( u \) and \( n \) as well as for \( u/\rho \) and \( n/\rho \), however, different values of the Majorana phase \( \alpha \) can be obtained for the “original” and the “reduced” pair of \( n \) and \( u \). In the numerical analysis, in particular in tables 3-5, we only mention the smallest value of \( n \) and \( u \) that lead to a certain result for the mixing parameters.
The results for mixing angles and CP phases obtained in the present case reduce to the ones found in case 1), if
\[ \theta = 0 \quad , \quad \phi_u = 2 \theta_1 \quad \text{and} \quad v = 6 \, s_1 \]  
(89)
with \( \theta_1 \) and \( s_1 \) being \( \theta \) and \( s \) as defined in case 1), see (63) and (64). Here we assume that \( n \) is the same in both cases. Since we have to identify the discrete parameter \( \phi_u \) with the continuous one \( \theta_1 \), it is clear that in general the results obtained are slightly different, see results for \( n = 8 \) and \( u = \mp 1 \) in table [3]. Using the symmetry transformations displayed in table [2] we see that a very similar identification can be made for \( \theta = \pi/2, i.e. \)
\[ \theta = \frac{\pi}{2} \quad , \quad \phi_u = 2 \theta_1 + \pi \quad , \quad v = 6 \, s_1 \quad \text{and} \quad k_1 = k_1^1 + 1 \]  
(90)
with \( k_1^1 \) denoting the parameter \( k_1 \) in case 1). Indeed, such cases are also found in the numerical analysis, see table [5]. In these cases \( J_{CP} \) and \( I_2 \) vanish, while the Majorana phase \( \alpha \) fulfills
\[ |\sin \alpha| = |\sin \phi_u| \]
Coming back to general formulae in (78) and (79), the smallness of \( \theta_{13} \) requires that
\[ \cos \phi_u \cos 2 \theta \approx 1 \]  
(91)
which is fulfilled for the following combinations of \((\phi_u, \theta)\)
\[ (\phi_u, \theta) \approx (0, 0), (0, \pi), (2 \pi, 0), (2 \pi, \pi) \quad \text{or} \quad (\phi_u, \theta) \approx (\pm \pi, \pi/2) \].
Taking into account the symmetries of the formulae in \((u, \theta)\) we see that it is sufficient to focus on the case \( u \approx 0 \) (\( \phi_u \approx 0 \)) and \( \cos 2 \theta \approx 1 \). Requiring \( \theta_{13} \) to be within the experimentally preferred \( 3\sigma \) interval we find an upper bound on \( \phi_u \)
\[ |\phi_u| \lesssim 0.39 \quad \text{corresponding to} \quad |u/n| \lesssim 0.12 \].  
(92)
Thus, for \( n = 20 \) the maximum value of \( u \) which can give rise to a good fit of the experimental data is \( |u| \lesssim 0.12 \, n \approx 2.4 \). This is confirmed in our numerical analysis and, indeed, one finds \( u = \mp 1 \) for \( n = 20 \) in table [3] as well as the case \( u = \mp 2 \) and \( n = 20 \) that can be “reduced” to \( u = \mp 1 \) and \( n = 10 \) – a case that is also mentioned in table [3]. Obviously, applying the symmetries listed in table [2] further choices of \( u \) can be found that lead to the same good accordance with experimental data. However, since these values are easily obtained by the table we refrain from listing them explicitly in the following.
If we consider instead a pattern with \( u \) shifted into \( u \pm \frac{n}{3} \), we see that not \( \phi_u \), but \( \phi_u \pm \frac{\pi}{3} \) is constrained to be smaller or larger than 0.39 or \(-0.39\) in order to fit the reactor mixing angle well. Thus, the allowed range for \( u/n \) we can derive is
\[ 0.66 \lesssim |\phi_u| \lesssim 1.44 \quad \text{corresponding to} \quad 0.21 \lesssim |u/n| \lesssim 0.46 \]  
(93)
For \( n = 20 \) we shall expect a good fit to the experimental data for
\[ 4.2 \lesssim |u| \lesssim 9.2 \]  
(94)
Also this result can be compared with the findings of the numerical analysis and, indeed, the values \( u = 7 \) and \( u = 9 \) are mentioned in table [5] for \( n = 20 \) in the case of a PMNS mixing matrix that leads to (78) and (79) with replacements as in (84). The other three values that should lead to a good fit, \( u = 5, u = 6 \) and \( u = 8 \), only appear implicitly, namely in table [4] since their common divisor with \( n = 20 \) is larger than one \((u = 5, n = 20 \) is reduced to \( u = 1, n = 4 \) and \( u = 6, u = 8 \) and \( n = 20 \) to \( u = 3, u = 4 \) and \( n = 10 \)).
In this case in general all CP violating phases are non-trivial. However, also here for particular choices of the parameters \( \theta, u \) and \( v \) some or all of these phases can vanish. When this happens usually an accidental CP symmetry common to the charged lepton and the neutrino sector is present, see (51). Consider again the CP symmetry \( Y \) accidentally present in the charged lepton sector. Since \( Z \) is the same as in case 1), also here \( Y \) is constrained to be of the form as in (69). Its form is block-diagonal in the neutrino mass basis

\[
\bar{Y} = U_{\nu,2}^\dagger Y U_{\nu,2} = e^{i \left( y - \frac{\pi}{6} \right)} \begin{pmatrix}
\cos 2\theta & 0 & (-i)^{k_2} \sin 2\theta \\
0 & (-1)^{k_1} e^{i\phi_v} & 0 \\
(-i)^{k_2} \sin 2\theta & 0 & (-1)^{k_2+1} \cos 2\theta
\end{pmatrix}.
\] (95)

As one sees, \( \sin 2\theta = 0 \) leads to a diagonal form of \( \bar{Y} \) and thus must imply the vanishing of \( J_{CP} \). This is consistent with the findings that \( J_{CP} \) is proportional to \( \sin 2\theta \). The Majorana phases can then be read off as

\[
|\sin \alpha| = |\sin \phi_v| \quad \text{and} \quad \sin \beta = 0.
\] (96)

This is again consistent with the form of the CP invariants \( I_1 \) and \( I_2 \) shown above. Note that for \( v = 0 \) all CP violation vanishes. This is only fulfilled if \( t = 0 \), since we only consider groups with an index \( n \) that is not divisible by three. The CP transformation \( \bar{Y} \) in (95) becomes then real (and diagonal) for the choice \( y = k \pi, k = 0, 1, \) and fulfills the second equation in (49).

We note that expressions corresponding to those in (78) and (79) have been obtained in \[24\]-\[25\] for the particular choice \( n = 4 \), i.e. for \( \Delta(48) \) \[24\] and \( \Delta(96) \) \[25\]. In particular, the sum rules in (80) and (82) with \( \phi_u = \pi/12 \) were also found in \[25\].

### 4.2.2 Numerical results

Here we study numerically mixing angles and CP phases that are obtained for the choice \( Q = a, Z = c^{n/2}, X = c^d P_{23} \). Since the mixing angles depend on \( u/n (\phi_u) \) and the continuous parameter \( \theta \) we display the contour regions for the 3\( \sigma \) experimental bounds of \( \sin^2 \theta_{ij} \) as well as their experimental best fit values in the plane \( \theta \) versus \( u/n \) in figure 1, using the data from the global fit analysis given in \[1\] and summarized in appendix A.2. In this figure we can restrict the discussion to the interval \(-1 < u/n \leq 1 (-\pi < \phi_u \leq \pi) \), since the mixing angles depend on \( \cos \phi_u \) and \( \sin \phi_u \), see (78). As one can clearly see, the tightest constraint on the parameters \( u/n \) and \( \theta \) arises from the requirement to accommodate the reactor mixing angle within the experimentally preferred 3\( \sigma \) interval (red ring-shaped areas in figure 1). If this is the case, also the value of the solar mixing angle is within its 3\( \sigma \) range (green disk). Furthermore, we note that for this mixing angle only the upper 3\( \sigma \) bound, \( \sin^2 \theta_{12} = 0.344 \), is visible in the figure, since the trimaximal column of the PMNS mixing matrix \( U_{PMNS} \) constrains \( \theta_{12} \) to fulfill \( \sin^2 \theta_{12} \gtrsim 1/3 \). As discussed in the preceding section, there are three mixing patterns that can be distinguished for \( Q = a, Z = c^{n/2}, X = c^d P_{23} \) corresponding to three different permutations of \( U_{PMNS} \). If no permutation is applied, the formulae in (78) are obtained for the mixing angles and these are used in the left panel of figure 1. In this case we confirm the analytical estimates that \( |u/n| \) has to be small and \( \theta \) close to 0 or \( \pi \), see (91) and (92), or \( u/n \) close to \( \pm 1 \) and \( \theta \approx \pi/2 \), if the symmetries in table 2 are taken into account. If we instead apply the permutations \( P_1 \) and \( P_2 \) to \( U_{PMNS} \), respectively, as described above, and thus obtain the formulae (78) with (84) and (86), the corresponding figure of \( \sin^2 \theta_{ij} \) in the plane \( \theta \) versus \( u/n \) is the one in the right panel of figure 1. The shifts in the parameters \( u/n \) and \( \theta \) are clearly visible from figure 1 and also the analytical estimates of \( |u/n| \) that leads to a viable fit of the experimental data, see (93), are confirmed. The other regions in the \( u/n - \theta \) plane that are indicated to accommodate the data well are, as expected, related to the former region through
Figure 1: Case 2). Contour plots of $\sin^2 \theta_{ij}$ in the plane $\theta$ versus $u/n$. The blue, green and red contour lines are associated with the atmospheric, solar and reactor mixing angles, respectively. The thick (thin) plain lines represent the upper (lower) $3\sigma$ bounds of the lepton mixing angles, while the dashed lines refer to the corresponding best fit values. The $3\sigma$ colored regions in the left panel are computed from (78). In the right panel, regions in foreground and background follow from a different permutation of the PMNS mixing matrix $U_{PMNS}$ that leads to mixing angles given by the formulae in (78) with replacements (84) and (86), respectively. The plain black lines in both panels indicate maximal atmospheric mixing $\theta_{23} = \pi/4$.

We would like to emphasize that the figures in the left and the right panel of figure 1 do lead in general to different results for mixing angles, simply because the shift $\pm \frac{u}{3}$ in the parameter $u$ that is necessary to relate these two figures is not an integer for $3 \nmid n$ so that for a particular choice of $n$ and $u$ the application of this shift will lead to a parameter $u$ that is not an integer and thus not admitted in our scenario. For this reason we present separate tables with values of $n$ and $u$ for which the experimental data of the mixing angles are accommodated well.

With a $\chi^2$ analysis that includes the three mixing angles, uses the global fit results found in [1], see also appendix A.2 and that is described in detail in appendix A.3, we evaluate for even $n \leq 20$, $3 \nmid n$, and all corresponding values of $u$ whether the continuous parameter $\theta$ can take values such that a good fit to the experimental data ($\chi^2_{\text{tot}} \lesssim 27$ and all mixing angles within their $3\sigma$ intervals) can be achieved. Our results from such an analysis using the formulae in (78) for the mixing angles are summarized in table 3, where we list for each case the values of $n$, $u$, the resulting $\chi^2_{\text{tot}}$ obtained for the “best fitting” value $\theta = \theta_{\text{bf}}$, the results for the mixing angles $\sin^2 \theta_{ij}$ for this set of parameters as well as the values of the CP invariants $J_{CP}$ and $I_2$ and the corresponding CP phases $\sin \delta$ and $\sin \beta$. The results for $I_1$ and thus $\sin \alpha$ are not reported in this table, since these quantities depend on an additional parameter $\phi_v (v/n)$, and are discussed in detail below, see (97) and the figure in the bottom-left panel in figure 4. A second best fitting value for $\theta$ that leads to exactly the same results for the mixing angles is found at $\theta = \pi - \theta_{\text{bf}}$, since the formulae of the mixing angles in (78) remain invariant, if $\theta$ is replaced by $\pi - \theta$.

As can be seen from table 3, $n$ has to be at least 8 or $u$ has to be chosen as $u = 0 (n)$, see
Table 3: Case 2). Results for fixed values of \( n \) and \( u \). Expressions for \( \sin^2 \theta_{12}, J_{CP} \) and \( I_2 \) are taken from (78) and (79) with \( k_1 = k_2 = 0 \). For all cases presented \( \chi^2_{\text{tot}} \lesssim 27 \) and the mixing angles lie in their experimentally preferred 3\( \sigma \) intervals. A second solution with the same \( \chi^2_{\text{tot}} \) is obtained in each case for \( \theta = \pi - \theta_{bf} \), however \( J_{CP} \) and \( I_2 \) change sign. Furthermore, different values of \( u \) are obtained from the symmetry transformations in Table 2. Notice that we do not display \( I_1 \) and the Majorana phase \( \alpha \) since they depend also on \( \phi_v (v/u) \), see (79), and thus several different values of \( \sin \alpha \) can be achieved for a particular value of \( n \) and \( u \), see (97) for example and the figure in the bottom-left panel in figure 4. The trivial Dirac phase \( \delta \) for \( n = 8 \) and \( u = \pm 1 \) is related to an accidental CP symmetry \( \tilde{Y} \), see (95), that arises in this case since \( \theta_{bf} \) is zero. Additionally, \( \sin \beta \) vanishes in this case, see (96). However, the other Majorana phase \( \alpha \) is in general non-zero, see (97). Here and in the following tables lower signs, if present, refer to the values given in parentheses.

discussion around (87) and below for the latter case. For the smallest value of \( n \), \( n = 8 \), the requirement to accommodate the mixing angles well \( \chi^2_{\text{tot}} \lesssim 27 \) leads to \( \theta_{bf} = 0 \) that implies the presence of an accidental CP symmetry in the theory, see (95), such that the Dirac phase is trivial. Then also one of the Majorana phases \( \beta \) becomes trivial, see (96). Nevertheless, the remaining phase \( \alpha \) is in general non-trivial and for \( n = 8 \) and \( u = \mp 1 \) it can take several values, since it also depends on \( \phi_v (v/u) \), see (79). Using the definition of \( u \) and \( v \), see (72), and the information that \( 0 \leq s \leq t \leq n-1 \) (thus smaller/equal 7 in this case) we find that \( v \) can take the values 3, 9, 15 and 21 and the corresponding value of \( \sin \alpha \) reads (for \( k_1 = 0 \))

\[
\sin \alpha = -\sin (3\pi/8) = -\sin (21\pi/8) \approx -0.924 , \sin \alpha = -\sin (9\pi/8) = -\sin (15\pi/8) \approx 0.383 . \quad (97)
\]

As already mentioned above, cases in which \( \theta \) vanishes reveal the same mixing pattern as the one obtained in case 1), if the identifications in (89) are made. The case \( n = 8 \) and \( u = \mp 1 \) shows a characteristic feature common to the other cases, namely \( u = -1 \) entails \( \theta_{23} \) smaller than \( \pi/4 \), while values larger than \( \pi/4 \) are achieved for \( u = 1 \), see table 2. If smaller values of the index \( n \) of the flavor group (and thus smaller groups) are desired, a possibility that, independently of the value of \( n \), always admits a reasonable fit to the experimental data is to choose the parameter \( u = 0 \) (\( \phi_u = 0 \)), i.e. require a certain form of the CP transformation \( X_{CP} \)

\[
X = e^{i \phi_u} P_{23} ,
\]

or, related by the symmetries in Table 2 \( u = n \) (\( \phi_u = 1 \)). As shown in (87), this case always entails maximal atmospheric mixing, contributing \( \chi^2_{23} \approx 0.69 \) to \( \chi^2_{\text{tot}} \), while the other two mixing angles can be accommodated equally well as in the other cases with \( u \neq 0 \), see table 3. Furthermore, the Dirac phase is fixed to its maximal value; for the value of \( \theta_{bf} \) displayed in Table 3 \( \sin \delta \) is positive, while \( \sin \delta = -1 \) is obtained for the choice
\( \theta = \pi - \theta_{bd} \approx 0.18 (1.39) \). One of the Majorana phases is trivial, \( \sin \beta = 0 \), while the other one is determined by the parameter \( v/n \): \( \sin \alpha = -\sin \phi_v = -\sin(\pi v/n) \) for \( k_1 = 0 \), see above. In particular, for the smallest value of the index \( n \), \( n = 2 \), also this phase is trivial, since the only possible value of \( v \) is \( v = 0 \). This feature has already been observed in [20]. For the next smallest choice \( n = 4 \), also already known in the literature [24, 25], the Majorana phase \( \alpha \) is either trivial (for \( v = 0 \)) or maximal (\( \sin \alpha = -1 \) for \( v = 6 \); taking \( k_1 = 1, 3 \) also \( \sin \alpha = 1 \) can be achieved). Clearly, for larger values of \( n \) also other values of \( \alpha \) can be achieved that all lie on the curve displayed in the bottom-left panel in figure 4. Larger values of \( n \), \( n > 8 \), all allow for \( u = \pm 1 \) a reasonable fit to the experimental data of the lepton mixing angles and, at the same time, in general predict non-trivial CP phases. Results corresponding to different choices of the parameter \( u \) can be derived by applying the symmetry transformations reported in table 3. These all lead to the same results for the mixing parameters that are shown in table 3.

The results presented in table 3 together with the results for \( n = 100 \) are displayed in the plane \( \theta \) versus \( u/n \) in figure 2 restricting the range of \( u/n \) to \( |u/n| \lesssim 0.12 \), as estimated in [92]. These are superimposed with the red ring-shaped area indicating the 3\( \sigma \) interval of the reactor mixing angle. The results for \( n = 100 \) are shown in figure 2 (empty circles) in order to improve the figure and to indicate the limit of large \( n \). The results in the left panels of figures 3 and 4 for the Dirac phase \( \sin \delta \) and the Majorana phase \( \sin \beta \) are shown as functions of \( u/n \) for \(-1 < u/n \leq 1 \). They are computed for all the pairs \((u/n, \theta_{bd})\) shown in figure 2. The possible

\[ \text{Figure 2: Case 2.} \text{ Pairs } (u/n, \theta_{bd}) \text{ that predict the lepton mixing angles in agreement with the experimental data, resulting from our } \chi^2 \text{ analysis. The two plots correspond to two independent permutations of the PMNS mixing matrix. The red ring-shaped region and the dashed contour lines in the left and right panels are extracted from figure 1. The discrete points in the plane of the left (right) panel can be found in table 3 (tables 4 and 5) for } n \leq 20 \text{, taking into account the second solution with } \theta = \pi - \theta_{bd} \text{ in the various cases. Notice that the red ring-shaped region is deformed in the figure on the left due to the scales chosen for the axes.} \]
values of the other Majorana phase \( \sin \alpha \) are shown in the lower panel of figure 4. We show the predictions of \( \sin \alpha \) obtained for the cases reported in table 3, i.e., the value of \( u \) is chosen as \( \mp 1 \) and \( \theta = \theta_{\text{bf}} \), as well as for \( n = 100 \) for \( u = \mp 1 \) and the corresponding \( \theta_{\text{bf}} \) so that \( v/n \) remains as variable, see (79). As its fundamental interval we consider \( 0 \leq v/n \leq 2 \), since \( I_1 \) is a periodic function in \( \phi_u = \pi v/n \) with periodicity \( 2 \pi \). However, notice that for each \( n \) some of the allowed values of \( \sin \alpha \) are actually obtained for values of \( v \) in the interval \( 2 < v/n < 3 \), as it happens for example for \( n = 8 \), \( u = \mp 1 \) in (97).

In exactly the same manner we can discuss the results for the mixing originating from the permutation of the PMNS mixing matrix \( U_{\text{PMNS2}} \) that leads to the \( 3 \sigma \) allowed regions as displayed in the foreground in the right panel of figure 1, namely the mixing angles and CP invariants obtained in (78) and (79) with replacements (84), i.e., with \( u \) shifted into \( u - \frac{a}{3} \). The results of our analysis for \( n \leq 20 \) are collected in tables 4 and 5. The numbers mentioned in square brackets are obtained, if the second and third rows of the PMNS mixing matrix are exchanged, and represent a solution with the atmospheric mixing angle in the other octant (and \( J_{\text{CP}} \) changes its sign). Additionally, as in the case above, we find a further best fitting value \( \theta \) at \( \theta = \pi - \theta_{\text{bf}} \) in each case that leads to the same mixing angles. The smallest value of \( n \) that allows a reasonable fit to the experimental data is for this type of mixing pattern \( n = 4 \) and \( u = 1 \). As one can see in this case, the phases \( \delta \) and \( \beta \) are non-trivial and also the Majorana phase \( \alpha \), depending on \( v \) is non-trivial for the two choices of the parameter \( t \), see (72), admitted by the constraint \( u = 1 \)

\[
\sin \alpha \approx 0.731 \quad (v = 3, \ t = 1) \quad \text{and} \quad \sin \alpha \approx 0.683 \quad (v = 9, \ t = 3). 
\]  

These results are in agreement with those found in 24, 25. Let us focus on the two particular cases \( n = 14, \ u = 3 \) and \( n = 20, \ u = 9 \) in table 5 that both lead to predictions \( \sin \delta = 0 \) and \( \sin \beta = 0 \). This result is obtained, since the best fitting value \( \theta_{\text{bf}} \) is in both cases \( \pi/2 \). For this value, as discussed above, the accidental CP symmetry of the charged lepton sector is also a
CP symmetry of the neutrino mass matrix combination $m^\dagger_\nu m_\nu$. This explains $\sin \delta = 0$. The fact that also the Majorana phase $\beta$ is trivial is due to the special form of CP transformation $\tilde{Y}$ in the neutrino mass basis, see \ref{95} and \ref{96}. Instead the Majorana phase $\alpha$ takes in both cases only non-trivial values $|\sin \alpha| = |\sin \phi_\alpha| = |\sin \pi v/n|$. In particular, for $n = 14$ and $u = 3$ we find (for $k_1 = 0$)

$$\sin \alpha \approx 0.623 \quad \sin \alpha \approx 0.901 \quad \sin \alpha \approx -0.223 \quad \text{and} \quad \sin \alpha = -1$$ \hspace{1cm} (99)

valid for $v = 3, 39, v = 9, 33, v = 15, 27$ and the maximal value of the Majorana phase $\alpha$ is attained for $v = 21$. Notice that $\sin \alpha = 1$ cannot be achieved, simply because the parameter $v$ is always constrained to be divisible by three, see its definition in \ref{72}. Likewise, we find for $n = 20$ and $u = 9$ also always a non-vanishing value for the CP phase $\alpha$. Again, we set $k_1 = 0$ and achieve

$$\sin \alpha \approx 0.454 \quad \sin \alpha \approx 0.988 \quad \sin \alpha \approx 0.707 \quad \sin \alpha \approx -0.156 \quad \text{and} \quad \sin \alpha \approx -0.891$$ \hspace{1cm} (100)

valid for $v = 3, 57, v = 9, 51, v = 15, 45, v = 21, 39$ and $v = 27, 33$. Numerical values of $\sin \alpha$ for all choices of $n$ and $u$ reported in tables \ref{4} and \ref{5} are displayed in figure \ref{4} (bottom-right panel). Similarly to the mixing pattern derived from $U_{PMNS2}$ in \ref{77} also in this case additional values of $u$ that lead to the same results for the mixing angles are found, if the symmetry transformations in table \ref{2} are applied. However, note that the second and third transformations now relate the pattern with $u$ shifted into $u - \frac{n}{3}$ to the one with $u$ shifted into $u + \frac{n}{3}$. Again, we show in figure \ref{2} all pairs $(u/n, \theta_{ud})$, this time in the right panel of the figure, that reproduce the experimental data on the lepton mixing angles well, for $8 \leq n \leq 20$ and $n = 100$. As in the left panel of figure \ref{2} we restrict the interval of $u/n$ to the one estimated in \ref{93}, $0.21 \leq u/n \leq 0.46$, since this embraces all solutions $u/n \approx 1/3$ that allow a reasonable fit to the experimental data. The (upper) figure on the right in figure \ref{3} and \ref{4} is obtained in
the analogous way as those on the left for the other permutation of the PMNS mixing matrix. Instead for the figure of \( \sin \alpha \) on the bottom-right in figure 4 the values of \( u \) used for \( n = 100 \) are now three, namely \( u = 23, 29 \) and 31, each of them leading to a set of fifty different values of \( \sin \alpha \). Notice that some of them lie in the interval \( 2 < v/n < 3 \) that we do not report in the figure.

We end the discussion of case 2) by deriving approximate expressions for the sines of the CP phases that help to understand the distribution of the points in figures 3 and 4. In fact, we can always express \( \theta \) as a function of \( \phi_u \left( u/n \right) \) and the best fit value of \( \sin^2 \theta_{13} \) determined from the global fit analysis, namely \( (\sin^2 \theta_{13})_{\text{bf}} = 0.0219 \) \( [1] \), using (78) (either for \( u \) or for \( u - n/3 \)). Thus, we can write \( \sin \delta \) and \( \sin \beta \), see (79), in terms of \( u/n \) only and can expand in the parameter \( \phi_u \) (always setting \( k_1 = k_2 = 0 \)) around \( \bar{\phi} \)

\[
\sin \delta \approx \pm 1 \mp 3.3 \left( \phi_u - \bar{\phi} \right)^2, \\
\sin \beta \approx \mp 5.6 \left( \phi_u - \bar{\phi} \right) \pm 23 \left( \phi_u - \bar{\phi} \right)^3,
\]

with \( \bar{\phi} = 0, \pm \pi \left( -2\pi/3, \pi/3 \right) \). The different values of \( \bar{\phi} \) correspond to the left (right) panel in figures 3 and 4. This approximation is reasonably good for \( |\phi_u - \bar{\phi}| \lesssim 0.3 \) in both cases. The different signs in (101) refer to the different possible values of \( \theta \), namely the upper ones are valid for \( \theta > \pi/2 \) and the lower ones for \( \theta < \pi/2 \). These different solutions are represented with dashed and continuous lines, respectively, in figures 3 and 4. The approximation of \( \sin \delta \) nicely shows that a large CP phase can be achieved for small values of \( u/n \) and for \( u = 0 \) it becomes maximal, see also table 3. Likewise, we obtain for the sine of the Majorana phase \( \alpha \) and at leading order in \( |\phi_u - \bar{\phi}| \)

\[
\sin \alpha \approx -\sin \phi_v \quad \text{for} \quad \bar{\phi} = 0 \quad \text{and} \quad \sin \alpha \approx \sin \phi_v \quad \text{for} \quad \bar{\phi} = \pi/3. 
\]

This approximation is shown as continuous line in the lower plots in figure 4 and, indeed, agrees very well with the numerical solutions in the whole range of \( v/n \). The next term in the expansion contributing to \( \sin \alpha \) reads \( \pm 0.18 \cos \phi_v (\phi_u - \bar{\phi}) \) for \( \theta < \pi/2 \), with the upper (lower) sign valid for \( \bar{\phi} = 0 \left( \pi/3 \right) \). In the case \( \theta > \pi/2 \) the sign should be reversed. The form of the approximation in (102) coincides with the exact result for the Majorana phase \( \alpha \) derived in case 1), see (68). The approximation in (102) and figure 4 show that large values of the Majorana phase \( \alpha \) are achieved for \( v/n \approx 1/2 \) and \( v/n \approx 3/2 \), while the choice \( v/n \approx 0, 1, \ldots \) leads to small values of \( \sin \alpha \).
Figure 4: Case 2). Predictions for the Majorana phases $\beta$ and $\alpha$ for different choices of $n$. The plots in the left and right panels are related to two independent permutations of the PMNS mixing matrix. In the upper panels $\sin \beta$ is displayed against $u/n$ for the pairs $(u/n, \theta_{bd})$ shown in figure 2 that result from our $\chi^2$ analysis. Similarly, in the lower panels $\sin \alpha$ against $v/n$ is presented with $\sin \alpha$ computed for the values of $\theta_{bd}$ and $u$ reported in table 3 (left lower panel) and in tables 4 and 5 (right lower panel). In the case $n = 100$, we set $u = \pm 1$ ($u = 23, 29, 31$) for the plot in the left (right) lower panel. The analytic approximations represented by the dashed and continuous lines in $\sin \beta$ are given in (101) and are valid for $\theta_{bd} \gtrless \pi/2$. For $\sin \alpha$ an excellent analytic approximation is found, see (102), that is indicated by the continuous line in $\sin \alpha$. 
Table 5: **Case 2**. Results for \( n = 14, 16 \) and 20 using the same permutation of the PMNS mixing matrix as in Table 4. The values in the square brackets as well as the opposite sign of \( J_{\text{CP}} \) are related to the mixing pattern resulting from an additional permutation of the second and third rows of the PMNS mixing matrix. Due to the properties of the formulae (78) and (79) the same good fit, i.e. the same \( \chi^2_{\text{tot}} \), is obtained for \( \theta = \pi - \theta_{\text{bf}} \), while \( J_{\text{CP}} \) and \( I_2 \) change sign. Additional values of \( u \) giving rise to the same results for the mixing angles are found applying the symmetry transformations mentioned in Table 2 and by taking into account the comments made above. For \( n = 14 \) and \( u = 3 \), the Dirac phase \( \delta \) is trivial, since the value \( \theta_{\text{bf}} = \pi/2 \) leads to an accidental CP symmetry, see (95), in the neutrino sector. Additionally, the Majorana phase \( \beta \) is trivial. However, the other Majorana phase is in general non-vanishing, see (99). Similarly, an additional CP symmetry guarantees that \( \sin \delta = 0 \) for \( n = 20 \) and \( u = 9 \) and additionally \( \sin \beta = 0 \). Also in this case, the remaining CP phase \( \alpha \) is in general non-trivial, see (100). Numerical values of \( \sin \alpha \) for the choices of \( n \) and \( u \) in the table are displayed in Figure 4 (bottom-right panel).

| \( n \) | 3 | 4 | 5 | 6 | 7 | 7 | 9 |
|---|---|---|---|---|---|---|---|
| \( \chi^2_{\text{tot}} \) | 14.5 [12.2] | 9.41 [10.4] | 12.0 [9.62] | 9.48 [10.7] | 9.67 [12.2] | 9.77 [11.5] | 12.6 [9.74] | 10.6 [12.9] |
| \( \theta_{\text{bf}} \) | \( \pi/2 \) | 1.40 | 1.75 | 1.47 | 1.39 | 1.49 | 1.39 | \( \pi/2 \) |
| \( \sin^2 \theta_{12} \) | 0.341 | 0.341 | 0.341 | 0.341 | 0.341 | 0.341 | 0.341 | 0.341 |
| \( \sin^2 \theta_{13} \) | 0.0230 | 0.0218 | 0.0218 | 0.0218 | 0.0218 | 0.0218 | 0.0218 | 0.0221 |
| \( \sin^2 \theta_{23} \) | 0.392 [0.608] | 0.458 [0.542] | 0.521 [0.479] | 0.585 [0.415] | 0.482 [0.518] | 0.594 [0.406] | 0.514 [0.486] | 0.606 [0.394] |
| \( J_{\text{CP}} \) | 0 | \( \mp 0.0314 \) | \( \pm 0.0336 \) | \( \mp 0.0200 \) | \( \mp 0.0337 \) | \( \pm 0.0155 \) | \( \mp 0.0339 \) | 0 |
| \( \sin \delta \) | 0 | \( \mp 0.921 \) | \( \pm 0.981 \) | \( \mp 0.594 \) | \( \mp 0.986 \) | \( \pm 0.460 \) | \( \mp 0.991 \) | 0 |
| \( I_2 \) | 0 | \( -0.0107 \) | \( -0.0058 \) | 0.0130 | \( -0.0051 \) | 0.0109 | 0.0041 | 0 |
| \( \sin \beta \) | 0 | \( -0.761 \) | \( -0.411 \) | 0.928 | \( -0.362 \) | 0.774 | 0.291 | 0 |
4.3 Case 3 \((Q = a, Z = b c^m d^m, X = b c^s d^{m-s} P_{23})\)

The last case can be represented by the choice

\[
Z = b c^m d^m, \quad X = b c^s d^{m-s} P_{23}
\]  

(103)

with \(0 \leq m, s \leq n - 1\). Since the \(Z_2\) generator contains the element \(b\), this case can only be realized, if the flavor symmetry is \(\Delta(6n^2)\). First, we note that in the case \(Z = b c^m d^m\) the eigenvector belonging to the non-degenerate eigenvalue is proportional to

\[
\frac{1}{\sqrt{6}} \begin{pmatrix} -1 + e^{2i\phi_m} \\ -\omega^2 + e^{2i\phi_m} \\ -\omega + e^{2i\phi_m} \end{pmatrix}
\]

(104)

with

\[
\phi_m = \frac{\pi m}{n}.
\]

If this eigenvector is identified with the third column of the PMNS mixing matrix, the reactor as well as the atmospheric mixing angle are only determined by the ratio \(m/n\). This case we call case 3 a) in the following and twelve possible permutations of the PMNS mixing matrix represent this situation. As we see, the smallness of the reactor mixing angle can be explained by small \(m/n\) (or small \(1 - m/n\)). However, the particular choice \(m = 0\) is excluded. If we consider instead \(m = \frac{n}{2}\), the eigenvector in (104) takes the special form

\[
\frac{1}{\sqrt{6}} \begin{pmatrix} -2 \\ \omega \\ \omega^2 \end{pmatrix}
\]

(106)

whose components have the same absolute value as the ones of the first column of the TB mixing matrix, see (58). Thus, such a vector can be identified with the first column of the PMNS mixing matrix. This is a particular choice in our case 3 b.1). As regards the mixing, we know that in this case the solar mixing angle is bounded from above \(\sin^2 \theta_{12} \lesssim \frac{1}{3}\). A possible choice of the matrix \(\Omega\) that satisfies both equalities in (10) for \(Z\) and \(X\) in (103) is

\[
\Omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix} \Omega_1 R_{13}(\phi_m)
\]

(108)

with \(\Omega_1\) as in (57). Note that \(\Omega_1\) contains as parameter \(s\) (\(\phi_s\)) that is constrained to be \(0 \leq s \leq n - 1\). Applying \(\Omega_3\) to \(Z\) we find

\[
\Omega_3^\dagger Z \Omega_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.
\]

(109)

Thus, the appropriate rotation \(R_{ij}(\theta)\) is in the (12)-plane. The neutrino mixing matrix takes the form

\[
U_{\nu,3} = \Omega_3 R_{12}(\theta) K_{\nu},
\]

(110)

as usual, up to permutations of its columns. Again, since \(U_e = 1\), up to permutations of columns, the PMNS mixing matrix \(U_{\nu,3}\) is of the same form as \(U_{\nu,3}\), up to permutations of rows and columns. In this case, none of the 36 possible permutations can be obviously excluded and thus we study all of them in the following. We can distinguish two types of mixing
a) twelve permutations that lead to $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ depending on $n$ and $m$, but not on $s$ and $\theta$; for these permutations the third column of the PMNS mixing matrix is identified with the eigenvector of $Z$ mentioned in (104).

b.1) twelve permutations that lead to $\sin^2 \theta_{13}$ depending on $n$, $m$, $s$ as well as $\theta$; here the first column of $U_{PMNS,3}$ is identified with the eigenvector in (104).

b.2) eventually, twelve permutations with the second column of the PMNS mixing matrix corresponding to (104).

As we will show below, case 3 b.2) cannot accommodate the experimental data on the mixing angles well.

Before discussing the lepton mixing patterns analytically and also numerically we first comment on the possible presence of the accidental CP symmetry $Y$ of the charged lepton sector in the neutrino sector: in general, there are two possible structures of $Y$ that fulfill the first equality in (49) for $Z$ in (103) either

$$Y_1 = e^{iy} \begin{pmatrix} \omega^2 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & 1 \end{pmatrix} \quad \text{with} \quad 0 \leq y \leq 2\pi \quad (111)$$

or, if we take $m = 0$,

$$Y_2 = \begin{pmatrix} e^{iy_1} & 0 & 0 \\ 0 & e^{iy_2} & 0 \\ 0 & 0 & e^{iy_2} \end{pmatrix} \quad \text{with} \quad 0 \leq y_i \leq 2\pi. \quad (112)$$

However, the latter case is only of theoretical use, since $m = 0$ cannot be chosen, if we want to accommodate the experimental data of all mixing angles well.

4.3.1 Case 3 a)

**Analytical results**

We first discuss the case in which the rows and columns of the PMNS mixing matrix in (110) are not permuted. The mixing angles are found to be of the form

$$\sin^2 \theta_{13} = \frac{2}{3} \sin^2 \phi_m, \quad \sin^2 \theta_{23} = \frac{1}{2} \left(1 + \frac{\sqrt{3} \sin 2 \phi_m}{2 + \cos 2 \phi_m}\right), \quad (113)$$

$$\sin^2 \theta_{12} = \frac{1 + \cos 2 \phi_m \sin^2 \theta + \sqrt{2} \cos \phi_m \cos 3 \phi_s \sin 2 \theta}{2 + \cos 2 \phi_m}$$

and for the CP invariants we find

$$J_{CP} = -\frac{1}{6\sqrt{6}} \sin 3 \phi_m \sin 3 \phi_s \sin 2 \theta, \quad (114)$$

$$I_1 = \frac{1}{9} (-1)^{k_1+1} \cos \phi_m \sin 3 \phi_s \left(4 \cos \phi_m \cos 3 \phi_s \cos 2 \theta + \sqrt{2} \cos 2 \phi_m \sin 2 \theta \right),$$

$$I_2 = \frac{4}{9} (-1)^{k_2} \sin^2 \phi_m \sin 3 \phi_s \sin \theta \left(\cos 3 \phi_s \sin \theta - \sqrt{2} \cos \phi_m \cos \theta \right).$$

All other permutations of the PMNS mixing matrix that leave the third column untouched also give rise to this mixing pattern, if shifts in the continuous parameter $\theta$, re-labeling of $k_1$ and $k_2$...
as well as shifts in the integer parameter \( m \) are taken into account. In particular, \( m \) has to be shifted into \( n - m \) and/or \( m \pm \frac{n}{3} \). Since \( n \) is not divisible by three, the latter type of shifts in \( m \) does lead in general to different results for the mixing angles and is thus treated separately in our numerical analysis.\(^{15}\) The shift of \( m \) into \( n - m \) also embraces the case in which the relative sign in the bracket of \( \sin^2 \theta_{23} \) is changed.\(^{16}\)

\[
\sin^2 \theta_{23} = \frac{1}{2} \left( 1 - \frac{\sqrt{3} \sin 2 \phi_m}{2 + \cos 2 \phi_m} \right) . \tag{115}
\]

It is interesting to note that the formulae in (113) and (114) reveal certain symmetries that are collected in table 6. Furthermore, we note that the solar mixing angle and CP invariants become even or odd functions in \( \theta \) for \( s = \frac{n}{2} \) (assuming \( n \) is even), since terms with \( \cos 3 \phi_s \) vanish.

| \( m \rightarrow n - m \) | \( \theta \rightarrow \pi - \theta \) | \( \sin^2 \theta_{13}, \sin^2 \theta_{12}, \ I_1, \ I_2 \) are invariant \( \sin^2 \theta_{23} \) becomes \( 1 - \sin^2 \theta_{23} \); \( J_{CP} \) changes sign |
|---|---|---|
| \( \phi_m \rightarrow \pi - \phi_m \) | \( \phi_s \rightarrow \pi - \phi_s \) | \( \sin^2 \theta_{13}, \sin^2 \theta_{12}, \ J_{CP} \) are invariant \( \sin^2 \theta_{23} \) becomes \( 1 - \sin^2 \theta_{23} \); \( I_1 \) and \( I_2 \) change sign |

Table 6: \textit{Case 3}). Symmetry transformations of the formulae for mixing angles and CP invariants in (113) and (114).

Since \( \sin^2 \theta_{13} \) only depends on \( m/n \), the latter is fixed by the experimental value of \( \theta_{13} \) and has to be small or close to one. Thus, we already know that for a certain group \( \Delta(6 n^2) \) only very few choices of the \( Z_2 \) symmetry generator \( Z \) are admissible, since the value of the parameter \( m \) characterizes this symmetry, see (103). For \( 0.0188 \lesssim \sin^2 \theta_{13} \lesssim 0.0251 \)\(^{14}\) we find as allowed range of \( m/n \)

\[
0.054 \lesssim m/n \lesssim 0.062 \quad \text{or} \quad 0.938 \lesssim m/n \lesssim 0.946 , \tag{116}
\]

since \( 0 \leq m \leq n - 1 \). Also \( \sin^2 \theta_{23} \) only depends on \( m/n \) and thus we can express it in terms of

\(^{15}\) Examples of permutations of the PMNS mixing matrix in (110) which are related to the original pattern by such shifts in \( m \) are: the PMNS mixing matrix that is multiplied from the left with the matrix \( P_1 \) in (113) leads to mixing angles and CP invariants as in (113) and (114), if we replace in these formulae \( m \) with \( m + \frac{n}{3} \) and \( \theta \) with \( \pi - \theta \); while the PMNS mixing matrix in (110) multiplied from the left with \( P_2 \) in (113) gives rise to mixing angles and CP invariants whose dependence from the parameters \( m, n, s \) and \( \theta \) is obtained, if \( m \) is replaced by \( m - \frac{s}{2} \) and \( \theta \) by \( \pi - \theta \) in (113) and (114).

\(^{16}\) This mixing pattern can be easily achieved by exchanging the second and third rows of the PMNS mixing matrix in (110) and all mixing angles and CP invariants can be obtained from (113) and (114) by replacing \( m \) with \( n - m \) and \( \theta \) with \( \pi - \theta \) in these formulæ.
\[
\sin^2 \theta_{13} = \frac{1}{2} \left( 1 \pm \sin \theta_{13} \frac{\sqrt{2 - 3 \sin^2 \theta_{13}}}{1 - \sin^2 \theta_{13}} \right) \approx \frac{1}{2} \left( 1 \pm \sqrt{2} \sin \theta_{13} \right) \tag{117}
\]

with “+” being valid for \(m/n\) small and “-” for \(1 - m/n\) small and it varies in the interval
\[
0.387 \lesssim \sin^2 \theta_{23} \lesssim 0.403 \text{ for } 1 - m/n \text{ small and } 0.597 \lesssim \sin^2 \theta_{23} \lesssim 0.613 \text{ for } m/n \text{ small}.
\]

We can also approximate the result of the solar mixing angle for \(\cos \phi_m \approx \pm 1\) (and thus \(\cos 2\phi_m \approx 1\)) by
\[
\sin^2 \theta_{12} \approx \frac{1}{3} \left( 1 + \sin^2 \theta \pm 2 \sqrt{2} \sin 3 \phi_s \cos 2\theta \right) \tag{118}
\]
which is close to 1/3, if
\[
\sin \theta \left( \sin \theta \pm 2 \sqrt{2} \sin 3 \phi_s \cos \theta \right) \approx 0 \tag{119}
\]

This allows two types of solutions
\[
\theta \approx 0, \pi \text{ or } \tan \theta \approx \mp 2 \sqrt{2} \cos 3\phi_s \tag{120}
\]
with “-” holding for \(m/n\) small and “+” being relevant for \(1 - m/n\) small. These solutions are also found in the numerical analysis, see figure 5 (in particular, the black lines represent the second type of solution for small \(m/n\)). We note that for \(s = n/2\) only the solution \(\theta \approx 0, \pi\) remains and the solar mixing angle is bounded from below \(\sin^2 \theta_{12} \gtrsim 1/3\), since
\[
\sin^2 \theta_{12} \approx \frac{1}{3} \left( 1 + \sin^2 \theta \right) \tag{121}
\]
Thus, the experimental best fit value of the solar mixing angle can be accommodated best for \(\theta_{1d} = 0\). This entails together with \(s = n/2\) that all CP phases vanish, see table 7.

Using the fact that \(m/n\) or \(1 - m/n\) is small and \(\theta\) is constrained to fulfill (120) we can also derive approximations for the sines of the CP phases from the expressions in (114). For the Dirac phase \(\delta\) we get
\[
\sin \delta (\theta \approx 0, \pi) \approx 0 \text{ and } | \sin \delta \left( \tan \theta \approx \mp 2 \sqrt{2} \cos 3\phi_s \right) | \approx \left| \frac{3 \sin 6 \phi_s}{5 + 4 \cos 6 \phi_s} \right| \tag{122}
\]
showing that we can achieve a maximal Dirac phase in the latter case e.g. for \(s/n \approx 0.13\) and \(s/n \approx 0.2\). For the Majorana phase \(\alpha\) we find analogously
\[
| \sin \alpha | \approx | \sin 6 \phi_s | \tag{123}
\]
for all possible values of \(\theta\) in (120). The second Majorana phase \(\beta\) instead behaves similar to the Dirac phase, i.e.
\[
\sin \beta (\theta \approx 0, \pi) \approx 0 \text{ and } | \sin \beta \left( \tan \theta \approx \mp 2 \sqrt{2} \cos 3\phi_s \right) | \approx 2 | \sin 6 \phi_s | \left| \frac{2 + \cos 6 \phi_s}{5 + 4 \cos 6 \phi_s} \right| \tag{124}
\]
We see that this phase cannot be maximal and its maximally achieved value is \(| \sin \beta | = \sqrt{3}/2 \approx 0.866\) for e.g. \(s/n \approx 0.11\) and \(s/n \approx 0.22\). For \(\theta \neq 0, \pi\) it becomes very small for \(s/n\) close to \(k/6, k = 0, \ldots, 5\). All statements made are consistent with our numerical results, see figure 6.
The CP transformation $Y_1$ in (111) reads in the neutrino mass basis as follows

$$
\tilde{Y}_1 = U_{\nu3}^\dagger Y_1 U_{\nu3}^* = e^{i(y-2\phi_s)} \begin{pmatrix}
\cos^2 \theta + e^{6i\phi_s} \sin^2 \theta & -(i)^{k_1}e^{3i\phi_s} \sin 3\phi_s \sin 2\theta & 0 \\
-(i)^{k_1}e^{3i\phi_s} \sin 3\phi_s \sin 2\theta & ((i)^{k_1}e^{6i\phi_s} \cos^2 \theta + \sin^2 \theta) & 0 \\
0 & 0 & (\omega)^{k_2}
\end{pmatrix}. 
$$

(125)

This matrix becomes diagonal for $\sin 2\theta = 0$ or $s = 0$ (which is the only solution, since $0 \leq s \leq n - 1$ and three does not divide $n$). In the first case, $\sin 2\theta = 0$, we find for the remaining diagonal matrix

$$
\tilde{Y}_1 (\theta = 0) = \omega^2 e^{i(y-2\phi_s)} \begin{pmatrix}
1 & 0 & 0 \\
0 & (-1)^{k_1} e^{6i\phi_s} & 0 \\
0 & 0 & (-1)^{k_2}
\end{pmatrix}
$$

(126)

and

$$
\tilde{Y}_1 (\theta = \pi/2) = \omega^2 e^{i(y-2\phi_s)} \begin{pmatrix}
1 & 0 & 0 \\
0 & 0 & 0 \\
0 & (-1)^{k_1} & 0 \\
0 & 0 & (-1)^{k_2}
\end{pmatrix}
$$

(127)

and thus for the Majorana phases

$$
|\sin \alpha| = |\sin 6\phi_s| \quad \text{and} \quad \sin \beta = 0
$$

(128)

and

$$
|\sin \alpha| = |\sin \beta| = |\sin 6\phi_s|
$$

(129)

respectively, while for $s = 0$ we find

$$
\tilde{Y}_1 (s = 0) = \omega^2 e^{iy} \begin{pmatrix}
1 & 0 & 0 \\
0 & (-1)^{k_1} & 0 \\
0 & 0 & (-1)^{k_2}
\end{pmatrix}
$$

(130)

and thus all CP phases are trivial. This observation is consistent with the fact that $J_{CP}$, $I_1$ and $I_2$ are all proportional to $\sin 3\phi_s$ and is, indeed, confirmed by the numerical analysis, see tables (7), (9). The CP transformation $\tilde{Y}_1(s = 0)$ in (130) fulfills the second equation in (49), if we choose $y = \frac{2\pi}{3} + k\pi$, $k = 0, 1$.

**Numerical results**

As in the second case, we first discuss our numerical results for the mixing pattern derived from $U_{PMNS,3}$ in (110) and found in (113) and (114). Since in this case groups $\Delta(6n^2)$ with an even as well as those with an odd index $n$ are admitted, we present results for both types of choices. The ratio $m/n$ ($1 - m/n$) is practically fixed by the requirement to accommodate the reactor and the atmospheric mixing angles well, see (116), and we find as smallest indices $n$ and $m$ that allow for a good fit $n = 16$ and $m = 1$ ($m = 15$) and $n = 17$ and $m = 1$ ($m = 16$). For these cases we study the dependence of the solar mixing angle as well as on the CP phases on the continuous parameter $\theta$ as well as on $s$ that characterizes the chosen CP transformation $X$, see (103).

For $n = 16$ and $m = 1$ ($m = 15$) the values of the reactor and the atmospheric mixing angle read $\sin^2 \theta_{13} \approx 0.0254$ and $\sin^2 \theta_{23} \approx 0.613$ ($\sin^2 \theta_{23} \approx 0.387$). Note that $\theta_{23}$ is in agreement with the $3\sigma$ range given in the current global fit analysis (1), whereas $\theta_{13}$ is marginally too large. The contribution to $\chi^2_{tot}$ from each of these quantities is $\chi^2_{13} \approx 12.1$ and $\chi^2_{23} \approx 1.81$ ($\chi^2_{23} \approx 4.31$), respectively. In figure (5) we display the $3\sigma$ contour region for $\sin^2 \theta_{12}$ in the plane $\theta$ versus $s/n$, see (113), for the choice $m/n = 1/16$. The thick and thin plain lines in green correspond to
Figure 5: Case 3 a). Contour region of $\sin^2 \theta_{12}$ defined in (113), in the plane $\theta$ versus $s/n$ for $m/n = 1/16$. The colored region in the plane is realized by taking the $3\sigma$ limits of $\sin^2 \theta_{12}$ (continuous green lines) and thus all values in this area lead to $\chi^2_{12} \lesssim 9$. The dashed lines indicate the best fit value, $(\sin^2 \theta_{12})^{bf} = 0.304$. The black plain curve is an analytic approximation, assuming $\sin^2 \theta_{12} \approx 1/3$ and is given in the second equation in (120) with “-”, since $m/n \ll 1$. For this choice of $m/n$ the other two mixing angles are fixed to $\sin^2 \theta_{13} \approx 0.0254$ and $\sin^2 \theta_{23} \approx 0.613$. In order to obtain the corresponding figure for $m/n = 15/16$ we have to reflect the $3\sigma$ contour region of $\sin^2 \theta_{12}$ in the line defined by $\theta = \pi/2$, as can be seen using the first transformation in table 6. In this case the atmospheric mixing angle is given by $\sin^2 \theta_{23} \approx 0.387$.

the experimental upper and lower bounds of $\sin^2 \theta_{12}$, as reported in appendix A.2, while the dashed curve represents its best fit value, $(\sin^2 \theta_{12})^{bf} = 0.304$. As discussed above, there are two possible ways to accommodate the solar mixing angle within its $3\sigma$ interval: either $\theta \approx 0, \pi$ for $s/n$ arbitrary, then $\sin^2 \theta_{12} \approx 1/3$, see (113), or the value of $\theta$ depends on $s/n$ and in general values of $\sin^2 \theta_{12}$ smaller than $1/3$ can be achieved. An analytic approximation of the latter is given in (120) (with the choice “-”, since $m/n$ is small) and is indicated by the thick black lines in figure 5. For the choice $m/n = 15/16$ the corresponding figure of $\sin^2 \theta_{12}$ in the $\theta$-$s/n$ plane is obtained from the one in figure 5 by performing a reflection of the $3\sigma$ regions in the line defined by $\theta = \pi/2$, compare to first symmetry transformation in table 6. Even though we have chosen the value $m/n = 1/16$ in figure 5, the latter will practically be the same for other choices of $m/n$, provided these fall into the range given in (116). In particular, the figure obtained for $m/n = 1/17$ will be very similar and is thus not separately reported here.

We continue with the discussion of the CP phases and show results for them in form of contour plots in the $s/n$-$\theta$ plane in figure 6. We compute the CP phases from (114) for $k_1 = k_2 = 0$. Again, we choose $m/n = 1/16$. The black areas in the figures represent the $3\sigma$ allowed regions of $\sin^2 \theta_{12}$ in the same plane, also displayed in figure 5. As can be seen, the CP phases $\delta$ and $\alpha$ can assume maximal values in these areas. This observation has also been made using the analytic approximations in (122) and (123). For example, we can read off from figure 6

$\sin \delta \gtrsim 0.9$ (\lesssim -0.9) for $0.11 \lesssim s/n \lesssim 0.14$ ($0.19 \lesssim s/n \lesssim 0.22$) and $2.2 \lesssim \theta \lesssim 2.6$ ($0.5 \lesssim \theta \lesssim 0.9$)

and

$\sin \alpha \gtrsim 0.9$ (\lesssim -0.9) for $0.055 \lesssim s/n \lesssim 0.11$ ($0.23 \lesssim s/n \lesssim 0.28$) and $2 \lesssim \theta \lesssim 2.2$ ($1 \lesssim \theta \lesssim 1.2$).
Predictions for the Majorana and Dirac phases obtained from (113) and (114) for \( m/n = 1/16 \). The shaded areas in the three figures correspond to the 3\( \sigma \) regions of \( \sin^2 \theta_{12} \), see figure 5 for details. The parameters \( k_1 \) and \( k_2 \) set to zero. For the choice \( m/n = 15/16 \) the corresponding contour plots of the CP phases are obtained by performing a transformation \( \theta \rightarrow \pi/2 - \theta \) and \( \theta \rightarrow \pi - \theta \) for \( \sin \delta \) and \( \sin \alpha, \sin \beta \), respectively, see (131). For numerical values obtained from a \( \chi^2 \) analysis see table 7.

On the other hand, as also remarked in the analytical study, the absolute value of the Majorana phase \( \beta \) has a non-trivial upper limit \( |\sin \beta| \lesssim 0.87 \) and large values are obtained for e.g.

\[
0.6 \lesssim \sin \beta \lesssim 0.87 \quad \text{for} \quad 0.05 \lesssim s/n \lesssim 0.15 \quad \text{and} \quad 1.9 \lesssim \theta \lesssim 2.7
\]

and

\[
-0.87 \lesssim \sin \beta \lesssim -0.6 \quad \text{for} \quad 0.18 \lesssim s/n \lesssim 0.28 \quad \text{and} \quad 0.44 \lesssim \theta \lesssim 1.2.
\]

The various points in the plots in figure 6 in which all contour lines converge correspond to points at which the CP phase(s) are not physical, because some of the mixing angles vanish or become \( \pi/2 \). In particular, the points found in the figures of all three CP phases indicate \( \cos \theta_{12} = 0 \) (\( \theta_{12} = \pi/2 \)), while those present only in the figures of \( \sin \delta \) and \( \sin \alpha \) correspond to points with \( \sin \theta_{12} = 0 \). Since these points are far away from the regions in which the
solar mixing angle is accommodated well, these have no impact on our results. In the case
$m/n = 15/16$ the contour lines for the CP phases can be obtained from those shown in the
figures in figure 6 by applying the following identities

\begin{align*}
\sin\delta(n - m, \theta) &= \sin\delta(m, \pi/2 - \theta), \\
\sin\alpha(n - m, \theta) &= \sin\alpha(m, \pi - \theta), \quad \sin\beta(n - m, \theta) = \sin\beta(m, \pi - \theta).
\end{align*}

Note that the appearance of $\pi/2 - \theta$ as argument on the right-hand side of the first equality in
\cite{131} takes into account that $\mathcal{J}_{CP}$ changes sign, if the first transformation in table 6 is applied.

In table 7 the results of our $\chi^2$ analysis for $n = 16$ and $m = 1$ are shown (again, always
setting $k_1 = k_2 = 0$). Since the value of the reactor and the atmospheric mixing angles are fixed
by the choice of $n$ and $m$, we display in the table only the solar mixing angle for each value of the
parameter $s$ for which a value of the parameter $\theta$ is found that permits a reasonably good fit to the experimental data. As can be seen from the table and also from figure 5, this, indeed, happens for all value of $s$. We only show those that fulfill $s \leq n/2 = 8$, since the results for the others are easily obtained by exploiting the symmetry transformations found in table 6.

As already mentioned when discussing figures 5 and 6 also the results for the choice $n = 16$ and $m = 15$ can be derived by making use of the symmetries shown in table 6. For this choice, obviously, the value of $\chi^2_{\text{tot}}$ is slightly different, since the atmospheric mixing angle then reads $\sin^2\theta_{23} \approx 0.387$, see also caption of table 7. As regards the solar mixing angle, it is interesting to note that in most cases two values of “best fitting” $\theta$ are obtained and $\sin^2\theta_{12}$ at these points coincides with the experimentally preferred best fit value $(\sin^2\theta_{12})_{\text{bf}} = 0.304$. If only one value of $\theta_{12}$ appears in the table, the solar mixing angle is not accommodated so well (still within its $3\sigma$ range). We comment on this observation in more detail at the end of this subsection.

Furthermore, also notice that the CP invariant $I_1$ and the Majorana phase $\phi_3$ evaluated at the
two different best fitting points $\theta_{12}$ have opposite signs and vanish, if only one value $\theta_{12}$ exists,
while the other two CP phases take in general different values at the different $\theta_{12}$ and are still
non-vanishing in the cases with only one $\theta_{12}$, see $s = 3$. Also this behavior can be understood,
as is shown at the end of this subsection. The cases $s = 0$ and $s = 8$ are peculiar, since in both
cases all CP phases are trivial. Thus, an accidental CP symmetry must be present in the
theory. This, indeed, happens, since $s = 0$ always entails an accidental CP symmetry, see \cite{130},
while for $s = 8$ no CP violation is observed, because the best fitting value of $\theta$ is $\theta_{12} = 0$, see
\cite{126}, and, in addition, $\sin 6\phi_3$ vanishes, see \cite{128}.

As expected, the results for $n = 17$ and $m = 1$, corresponding to the smallest value of an
odd index $n$ for which the reactor and the atmospheric mixing angle can be accommodated well,
are pretty similar to those obtained for $n = 16$ and $m = 1$. Due to the slightly smaller value of
$m/n$ both mixing angles agree slightly better with the data in this case: $\sin^2\theta_{13} \approx 0.0225$ and $\sin^2\theta_{23} \approx 0.607$ leading to contributions to $\chi^2_{\text{tot}}$ of $\chi^2_{13} \approx 0.371$ and $\chi^2_{23} \approx 1.21$, respectively.
An atmospheric mixing angle in the first octant is obtained for the choice $n = 17$ and $m = 16$, $\sin^2\theta_{23} \approx 0.393$, as expected. Its contribution to $\chi^2_{\text{tot}}$ is $\chi^2_{23} \approx 3.46$. Detailed numerical results for this case are found in table 8. As already mentioned for $n = 16$, $m = 1$, the solar mixing angle at $\theta_{12}$ is for most values of the parameter $s$ equal to the experimentally preferred best fit value. If so, also here two different values of $\theta_{12}$ are found. If not, see $s = 3$ in table 8 only one value of $\theta_{12}$ is found. All statements made above concerning the CP phases are also valid in this case.

In particular, the statements referring to the values of $I_1$ and, consequently, $\sin\alpha$ as well as the observation that for $s = 0$ an accidental CP symmetry is present are true. For values $s > n/2 = 17/2$ and for $n = 17$ and $m = 16$ numerical results are easily deduced from table 8 simply by applying the symmetry transformations in table 6.

Up to now, we have focussed on the mixing angles and CP invariants, shown in \cite{113} and
\cite{114}, that are derived from $U_{PMNS,3}$ in \cite{110}. However, it is also interesting to consider the
case in which this matrix is multiplied by $P_2$ from the left such that we have to replace $m$ and $\theta$ in the formulae in (113) and (114) by $m - \frac{n}{3}$ and $\pi - \theta$. Interestingly enough, in this case a smaller (odd) value of $n$ is sufficient for achieving a good fit to the reactor and the atmospheric mixing angle. For $n = 11$ and $m = 3$ we obtain $\sin^2 \theta_{13} \approx 0.0239$ and $\sin^2 \theta_{23} \approx 0.390$ leading to contributions to $\chi^2_{\text{tot}}$ of $\chi^2_{13} \approx 3.91$ and $\chi^2_{23} \approx 3.86$. As one can see from table 9 that is subject to the same conventions as tables 7 and 8, also here the solar mixing angle can be fitted for all shown choices of $s$, but one ($s = 2$), to $(\sin^2 \theta_{12})^{bf} = 0.304$. The behavior of the CP invariants and the corresponding CP phases can be described in the same way as for $n = 16$, $m = 1$ and $n = 17$, $m = 1$. A value of the atmospheric mixing belonging to the second octant is in this case easily achieved by considering the PMNS mixing matrix with an additional exchange of $s$ and $\alpha$ as CP invariant $I$ is satisfied, assuming $\cos 2\theta_{\text{min}}$. Lastly, we comment on the fact that the solar mixing angle is either accommodated to its best fit value $(\sin^2 \theta_{12})^{bf} = 0.304$, if two different “best fitting” points $\theta_{bf}$ exist, or its value is larger and then only one value for $\theta_{bf}$ is given, as displayed in tables 7 and 8. Similarly, the application of the other two symmetries in this table allows to recover numerical results for a PMNS mixing matrix $U_{PMNS}\mathcal{P}$ that is the product of $P_1$ and the matrix displayed in (110), i.e. the mixing parameters are given by (113) and (114), replacing $m$ and $\theta$ by $m + \frac{n}{3}$ and $\pi - \theta$. If the index $n$ shall be even instead of odd, the smallest possible choice of $n$ that admits a reasonable fit to the experimental data is $n = 22$ and $m = 6$. This is clear, since the ratio $m/n$ is the same as in the case $n = 11$ and $m = 3$ that we have just discussed. Thus, also the numerical results found in table 9 apply in this case. In addition, there are results originating from odd values of $s$ for $n = 22$ and $m = 6$ that cannot be obtained for $n = 11$ and $m = 3$.

Lastly, we comment on the fact that the solar mixing angle is either accommodated to its best fit value $(\sin^2 \theta_{12})^{bf} = 0.304$, if two different “best fitting” points $\theta_{bf}$ exist, or its value is larger and then only one value for $\theta_{bf}$ is given, as displayed in tables 7 and 8. Since the reactor and the atmospheric mixing angles, and hence their contributions to the $\chi^2$ function, are fixed by the choice of $m/n$, effectively only the contribution $\chi^2_{12}$ depends on the variation of $\theta$ for a given value of the parameter $s$. In the case in which two different values of $\theta_{bf}$ are mentioned, the minimum of the solar mixing angle as function of $\theta$ is smaller than the experimentally preferred best fit value, $\sin^2 \theta_{12} (\theta_{\text{min}}) < 0.304$ and thus it is possible to obtain $\sin^2 \theta_{12} = (\sin^2 \theta_{12})^{bf}$ for some value of $\theta$ (and consequently $\chi^2_{12} = 0$). Since $\sin^2 \theta_{12}$ is a symmetric function with respect to $\theta_{\text{min}}$ in its vicinity, we indeed find two such values $\theta_{bf}$, $\theta_{bf,1} < \theta_{bf,2}$ that fulfill $\theta_{bf,2} - \theta_{bf,1} = \theta_{\text{min}} - \theta_{bf,1}$. If the minimum value of $\sin^2 \theta_{12}$ attained turns out to be larger than $(\sin^2 \theta_{12})^{bf}$, the choice of $\theta_{bf} = \theta_{\text{min}}$ minimizes the $\chi^2$ function (however, $\chi^2_{12} > 0$). At $\theta_{\text{min}}$ the relation

$$\tan 2\theta_{\text{min}} = -2 \sqrt{2} \frac{\cos \phi_m}{\cos 2 \phi_m} \cos 3 \phi_s,$$

is satisfied, assuming $\cos 2\theta_{\text{min}} \neq 0$. If we plug (132) into the expression for $I_1$ in (114), we find that $I_1$ vanishes at $\theta_{\text{min}}$, independently of the choice of $s$, $m$ or $n$ and, hence, the Majorana phase $\alpha$ is trivial. If $\theta_{bf} = \theta_{\text{min}}$ we thus find vanishing $I_1$ and $\sin \alpha$, see $s = 3$ in tables 7 and 8 and $s = 2$ in table 9. If we instead find two different values of $\theta$, $\theta_{bf,1}$ and $\theta_{bf,2}$, we see that the CP invariant $I_1$ fulfills $I_1 (\theta_{bf,1}) = -I_1 (\theta_{bf,2})$, since the expression of $I_1$ in (114) can be written as

$$I_1 = \frac{\sqrt{2}}{9} (-1)^{k+1} \frac{\cos \phi_m}{\cos 2 \phi_m} \frac{2 \phi_m}{\cos 2 \phi_{\text{min}}} \sin 3 \phi_s \sin 2 (\theta - \theta_{\text{min}}),$$

and thus also the Majorana phase $\alpha$ fulfills $\sin \alpha (\theta_{bf,1}) = -\sin \alpha (\theta_{bf,2})$. Concerning the other CP phases $\delta$ and $\beta$ no such statement can be made, since they are neither even nor odd functions with respect to $\theta = \theta_{\text{min}}$. 

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| $s$ | 0 | 1 | 2 | 3 | 4 | 5 | 6 | 7 | 8 |
|-----|---|---|---|---|---|---|---|---|---|
| $\chi^2_{\text{tot}}$ | 13.9 | 13.9 | 13.9 | 15.0 | 13.9 | 13.9 | 13.9 | 13.9 | 23.9 |
| $\theta_{bf}$ | 1.93 (3.10) | 2.00 (3.09) | 2.40 (3.02) | 0.265 (1.07) | 0.0584 (1.20) | 0.0415 (1.18) | 0.0441 (0.955) | 0.0758 (0.955) | 0 |
| $\chi^2_{12}$ | 0 | 0 | 0 | 1.16 | 0 | 0 | 0 | 0 | 10.0 |
| $\sin^2 \theta_{12}$ | 0.304 | 0.304 | 0.304 | 0.317 | 0.304 | 0.304 | 0.304 | 0.304 | 0.342 |
| $J_{\text{CP}}$ | 0 | 0.0159 (0.0021) | 0.0348 (0.0082) | 0.0348 | 0.0348 | 0.0348 | 0.0348 | 0.0348 | 0.0348 |
| $\sin \delta$ | 0 | 0.458 (0.0594) | 0.9995 (0.234) | 0.458 | 0.458 | 0.458 | 0.458 | 0.458 | 0.458 |
| $I_1$ | 0 | ±0.189 | ±0.116 | 0 | ±0.201 | ±0.0792 | ±0.146 | ±0.177 | 0 |
| $\sin \alpha$ | 0 | ±0.939 | ±0.579 | 0 | ±0.998 | ±0.394 | ±0.725 | ±0.882 | 0 |
| $I_2$ | 0 | 0.0114 (0.00066) | 0.0135 (0.0026) | 0.0114 | 0.0114 | 0.0114 | 0.0114 | 0.0114 | 0 |
| $\sin \beta$ | 0 | 0.662 (0.0383) | 0.784 (0.152) | 0.662 | 0.662 | 0.662 | 0.662 | 0.662 | 0.662 |

Table 7: Case 3 a). Results of the $\chi^2$ analysis for $n = 16$ and $m = 1$ obtained for the mixing angles and CP invariants given in (113) and (114). The choice $n = 16$ is the smallest even $n$ that provides $\chi^2_{\text{tot}} \lesssim 27$. The values of $\sin^2 \theta_{13}$ and $\sin^2 \theta_{23}$ only depend on the ratio $m/n$ and read for $m/n = 1/16$: $\sin^2 \theta_{13} \approx 0.0254$ and $\sin^2 \theta_{23} \approx 0.613$. Their contributions to $\chi^2_{\text{tot}}$ are $\chi^2_{13} \approx 12.1$ and $\chi^2_{23} \approx 1.81$. The one resulting from the fit of $\sin^2 \theta_{12}$ depends on the parameters $s$ and $\theta_{bf}$ and is displayed in table 7. As one can see, for most $s$ the experimentally preferred value of $\sin^2 \theta_{12}$ can be achieved. The CP invariants $I_{1,2}$ and the sines of the Majorana phases are computed for $k_1 = k_2 = 0$. For $s = 0$ and $s = 8$ (that results in $\theta_{bf} = 0$) an accidental CP symmetry is present in the charged lepton and neutrino sectors and thus all CP phases are trivial, see (125)-(130). The corresponding results for $s > 8$ are achieved by exploiting the symmetry transformations reported in table 6. Furthermore, the choice $n = 16$ and $m = 15$ leads to the same reactor mixing angle, but the atmospheric mixing angle is smaller than $\pi/4$, i.e. $\sin^2 \theta_{23} \approx 0.387$, contributing $\chi^2_{23} \approx 4.31$ to the value of $\chi^2_{\text{tot}}$. Again, the results for $m = 15$ and all possible $s$ can be obtained from those shown here using the symmetry transformations in table 6.
### Table 8: Case 3 a). Results for the smallest odd value of the index $n$, $n = 17$ together with $m = 1$, leading to $\chi^2_{\text{tot}} \lesssim 27$ for the PMNS mixing matrix $U_{\text{PMNS,3}}$ in (110) that is also considered in table 7. The ratio $m/n = 1/17$ fixes the reactor and the atmospheric mixing angles to $\sin^2 \theta_{13} \approx 0.0225$ and $\sin^2 \theta_{23} \approx 0.607$, contributing $\chi^2_{13} \approx 0.371$ and $\chi^2_{23} \approx 1.21$ to the value of $\chi^2_{\text{tot}}$, respectively. The parameters $k_1$ and $k_2$ are set to zero. As expected, the choice $s = 0$ leads to an accidental CP symmetry that results in trivial CP phases, see (130). If we consider $n = 17$ and $m = 16$, the atmospheric mixing angle is fixed to $\sin^2 \theta_{23} \approx 0.393$ that contributes $\chi^2_{23} \approx 3.46$ to $\chi^2_{\text{tot}}$. Results for this case as well as for $n = 17$, $m = 1$ and $s > 8$ can be obtained by applying the symmetry transformations given in table 6 to the results presented here.
The mixing angles read $\pm 0.197$ for $I_1$ and $(0.0132)$ for $I_2$, respectively. Again, the CP invariants $I_{1,2}$ and Majorana phases are shown for $k_1 = k_2 = 0$. Like for the other choices of $n$ and $m$, for $s = 0$ all CP phases are trivial indicating the presence of the accidental CP symmetry, see (130). Results for $s > 5$ are obtained from those given here by using the symmetry transformations in table 6. A permutation of the second and third rows of the PMNS mixing matrix leads to an atmospheric mixing angle $\sin^2 \theta_{23} \approx 0.610$ giving rise to $\chi^2_{tot} \approx 1.49$. If we consider the mixing pattern resulting in (113) and (114) with the replacements $m \to m + \frac{3}{2}$ and $\theta \to \pi - \theta$, we find $m = 8$ for $n = 11$ and the results of the $\chi^2$ analysis of this case can be deduced from those presented here by exploiting the symmetry transformations in table 6. If the index $n$ shall be even, the smallest value is $n = 22$ (and $m = 6$) whose results (for even $s$) coincide with those given here.

### 4.3.2 Case 3 b.1)

#### Analytical results

This mixing pattern is obtained by using the matrix $U_{PMNS,3}$ with the columns permuted by the matrix $P_1$, defined in (83), i.e. we apply this matrix from the right to the PMNS mixing matrix in (110). The mixing angles read

\begin{align}
\sin^2 \theta_{13} &= \frac{1}{3} \left( 1 + \cos 2 \phi_m \sin^2 \theta + \sqrt{2} \cos \phi_m \cos 3 \phi_s \sin 2\theta \right), \\
\sin^2 \theta_{23} &= \frac{1}{2} \left( 1 + \frac{2\sqrt{3} \sin \phi_m \sin \theta [\sqrt{2} \cos 3 \phi_s \cos \theta - \cos \phi_m \sin \theta]}{2 - \cos 2 \phi_m \sin^2 \theta - \sqrt{2} \cos \phi_m \cos 3 \phi_s \sin 2\theta} \right), \\
\sin^2 \theta_{12} &= 1 - \frac{2 \sin^2 \phi_m}{2 - \cos 2 \phi_m \sin^2 \theta - \sqrt{2} \cos \phi_m \cos 3 \phi_s \sin 2\theta},
\end{align}

Table 9: Case 3 a). Results of the $\chi^2$ analysis for $n = 11$ and $m = 3$ that is the smallest value of $n$ leading to $\chi^2_{tot} \leq 27$ for mixing angles and CP invariants as given in (113) and (114) with the replacements $m \to m - \frac{3}{2}$ and $\theta \to \pi - \theta$. The reactor and the atmospheric mixing angles read $\sin^2 \theta_{13} \approx 0.0239$ and $\sin^2 \theta_{23} \approx 0.390$, contributing $\chi^2_{13} \approx 3.91$ and $\chi^2_{23} \approx 3.86$ to $\chi^2_{tot}$, respectively. Again, the CP invariants $I_{1,2}$ and Majorana phases are shown for $k_1 = k_2 = 0$. Like for the other choices of $n$ and $m$, for $s = 0$ all CP phases are trivial indicating the presence of the accidental CP symmetry, see (130). Results for $s > 5$ are obtained from those given here by using the symmetry transformations in table 6. A permutation of the second and third rows of the PMNS mixing matrix leads to an atmospheric mixing angle $\sin^2 \theta_{23} \approx 0.610$ giving rise to $\chi^2_{tot} \approx 1.49$. If we consider the mixing pattern resulting in (113) and (114) with the replacements $m \to m + \frac{3}{2}$ and $\theta \to \pi - \theta$, we find $m = 8$ for $n = 11$ and the results of the $\chi^2$ analysis of this case can be deduced from those presented here by exploiting the symmetry transformations in table 6. If the index $n$ shall be even, the smallest value is $n = 22$ (and $m = 6$) whose results (for even $s$) coincide with those given here.
and the CP invariants are given by

\[ J_{CP} = -\frac{1}{6\sqrt{6}} \sin 3\phi_m \sin 3\phi_s \sin 2\theta, \]

\[ I_1 = \frac{4}{9} (-1)^{k_2+1} \sin^2 \phi_m \sin 3\phi_s \sin \theta \left( \cos 3\phi_s \sin \theta - \sqrt{2} \cos \phi_m \cos \theta \right), \]

\[ I_2 = \frac{4}{9} (-1)^{k_1+k_2+1} \sin^2 \phi_m \sin 3\phi_s \cos \theta \left( \cos 3\phi_s \cos \theta + \sqrt{2} \cos \phi_m \sin \theta \right), \]

(135)

Again, twelve permutations lead to this mixing, if possible shifts in \( \theta \), but also in \( m \), like in case 3 a), are taken into account. Indeed, the permutations that allow us to generate mixing patterns with \( m \) being replaced by \( n - m \), \( m - \frac{n}{3} \) or \( m + \frac{n}{3} \) are the same as in case 3 a). Furthermore, the symmetries found in table 6 are also symmetries of the formulae in (134) and (135) and, if \( m \) is replaced by \( m - \frac{n}{3} \) or \( m + \frac{n}{3} \), we find the same modifications to these symmetries as in case 3 a). Eventually, also the formulae in (134) and (135) exhibit for \( s = \frac{n}{2} \) (for \( n \) even) [and independent of the value of \( m \)] a well-defined transformation behavior, if \( \theta \) is replaced by \( \pi - \theta \), i.e. the expressions of the mixing angles are even functions in \( \theta \), whereas the CP invariants are odd functions and thus change sign, if \( \theta \) is changed into \( \pi - \theta \).

In order to study this case analytically we define the following quantities

\[ p = \cos 2\phi_m \sin^2 \theta + \sqrt{2} \cos \phi_m \cos 3\phi_s \sin 2\theta, \]

\[ q = 2 \sin \phi_m \sin \theta (\cos \phi_m \sin \theta - \sqrt{2} \cos 3\phi_s \cos \theta). \]

(136)

(137)

and see that we can write the formulae for the mixing angles as

\[ \sin^2 \theta_{13} = \frac{1}{3} \left( 1 + p \right), \quad \sin^2 \theta_{12} = 1 - \frac{2 \sin^2 \phi_m}{2 - p}, \quad \sin^2 \theta_{23} = \frac{1}{2} \left( 1 - \frac{\sqrt{3}q}{2 - p} \right). \]

(138)

We can express \( \sin^2 \theta_{12} \) and \( \sin^2 \theta_{23} \) in terms of \( \sin^2 \theta_{13} \)

\[ \sin^2 \theta_{12} = 1 - \frac{2 \sin^2 \phi_m}{3 \left( 1 - \sin^2 \theta_{13} \right)} \quad \text{and} \quad \sin^2 \theta_{23} = \frac{1}{2} \left( 1 - \frac{\sqrt{3}q}{\sqrt{3} \left( 1 - \sin^2 \theta_{13} \right)} \right). \]

(139)

Since the solar mixing angle is to good approximation \( \sin^2 \theta_{12} \approx 1/3 \), the ratio \( m/n \) is constrained to fulfill

\[ \sin^2 \phi_m \approx 1, \]

i.e.

\[ m \approx \frac{n}{2} \quad \text{for} \quad n \text{ even \ and} \quad m \approx \frac{n \pm 1}{2} \quad \text{for} \quad n \text{ odd}. \]

(140)

(141)

Then

\[ \cos \phi_m \approx 0, \quad \cos 2\phi_m \approx -1, \quad \sin 2\phi_m \approx 0, \]

(142)

and consequently

\[ -\sin^2 \theta \approx p = 3 \sin^2 \theta_{13} - 1. \]

(143)

This relation determines the value of \( \theta \) to be

\[ \theta_0 \approx 1.31 \quad \text{or} \quad \theta_0 \approx 1.83, \]

(144)

if \( \sin^2 \theta_{13} \) is set to its experimental best fit value, \( (\sin^2 \theta_{13})^{bf} = 0.0219. \) These two values of \( \theta_0 \) are related by the transformation \( \theta \rightarrow \pi - \theta \), see also the first symmetry transformation in table 6. Indeed, the reactor and the solar mixing angles only depend on the continuous
parameter $\theta$ and not on $s$ (the choice of the CP transformation $X$) for $m = n/2$, see left panel in figure 7 and fulfill the sum rule

$$\sin^2 \theta_{12} = \frac{1 - 3 \sin^2 \theta_{13}}{3(1 - \sin^2 \theta_{13})}$$

(145)

that has also been found in [33]. For $\theta_0$ in (144) the solar mixing angle takes the value $\sin^2 \theta_{12} \approx 0.318$, see also table 10 in the numerical analysis. This value is well within the experimentally preferred $3\sigma$ range [1]. Note that if we had neglected non-zero $\theta_{13}$ in (143), the solution would have been $\theta_0 = \frac{\pi}{4}$. Indeed, for $m = n/2$ ($\phi_m = \pi/2$) and $\theta = \pi/2$ mixing is TB. Using (140) we find

$$\sin^2 \theta_{23} \approx \frac{1}{2} \left(1 + \sqrt{\frac{2}{3} \cos 3 \phi_s \sin 2 \theta_0} \right)$$

(146)

that tells us that the allowed values of $\phi_s$ ($s/n$) are constrained by the request to accommodate the atmospheric mixing angle well, e.g. for $\theta_0 \approx 1.31$ we find as allowed intervals

$$0.09 \lesssim s/n \lesssim 0.23, \quad 0.44 \lesssim s/n \lesssim 0.58 \quad \text{and} \quad 0.75 \lesssim s/n \lesssim 0.90.$$  \hspace{1cm} (147)

The constraints derived with $\theta_0 \approx 1.83$ are very similar. Thus, the mixing angle $\theta_{23}$ can be accommodated well for a large range of $s/n$. If we refine our analysis and consider $m \neq n/2$, i.e. $m = n(1/2 + \kappa/\pi)$, $\kappa \ll 1$, we can derive a relation between the deviation of $\theta$ from $\theta_0$, $\theta = \theta_0 + \epsilon$ and $\kappa$

$$\epsilon \approx -\sqrt{2} \cos 3 \phi_s \kappa.$$  \hspace{1cm} (148)

For $n = 20$ and $m = 11$, $\kappa \approx 0.16$ we find $\epsilon \approx -0.22 \cos 3 \phi_s$. This approximation is displayed as dotted line in the right panel of figure 7 and fits the exact result reasonably well.

If $m = n/2$, the CP invariants in (135) read

$$J_{CP} = \frac{1}{6 \, \sqrt{6}} \sin 3 \phi_s \sin 2 \theta, \quad I_1 = \frac{2}{9} (-1)^{k_2+1} \sin 6 \phi_s \sin^2 \theta, \quad I_2 = \frac{2}{9} (-1)^{k_1+k_2+1} \sin 6 \phi_s \cos^2 \theta.$$  \hspace{1cm} (149)

For $\theta_0$ and $s/n$ as chosen in (144) and (147) the Dirac phase attains a lower value of

$$|\sin \delta| \gtrsim 0.71$$  \hspace{1cm} (150)

and a maximal value can be obtained, if $s/n = 1/6$, $s/n = 1/2$ or $s/n = 5/6$. The first and the third possibilities are excluded, since we do not consider the case $3 \mid n$. However, the case $s = n/2$ is allowed for all even $n$. Interestingly enough, the two Majorana phases $\alpha$ and $\beta$ depend for $m = n/2$ only on the parameter $s/n$, i.e. the absolute value of both reads

$$|\sin \alpha| = |\sin \beta| = |\sin 6 \phi_s|.$$  \hspace{1cm} (151)

We can use the results obtained for $\tilde{Y}_1$ in the case 3 a), see (125)-(130), applying the permutation $P_1$ to the rows and columns of $\tilde{Y}_1$, $P_1^T \tilde{Y}_1 P_1$. Thus, the conclusions regarding the CP phases are very similar to those above. The only difference is that now the Majorana phases for $\tilde{Y}_1$ ($\theta = 0$) read

$$\sin \alpha = 0 \quad \text{and} \quad |\sin \beta| = |\sin 6 \phi_s|,$$  \hspace{1cm} (152)

while for $\tilde{Y}_1$ ($\theta = \frac{\pi}{2}$) they read

$$|\sin \alpha| = |\sin 6 \phi_s| \quad \text{and} \quad \sin \beta = 0.$$  \hspace{1cm} (153)

As in case 3 a), all CP phases are trivial for $s = 0$, see (130).
Numerical results

We proceed with the presentation of our numerical results for this case. As has been noted, in this case the index $n$ of the group $\Delta(6n^2)$ can be even as well as odd. Furthermore, the parameter $m$ is constrained by the condition in (141). In figures 7-9 we display the results obtained for mixing angles and CP invariants using the formulae in (134) and (135) for the choice $n = 20$. We do so, since for this value of $n$ not only the choice $m = n/2 = 10$, but also $m = 11$ ($m = 9$ as well) allow a reasonably good fit to the experimental data with $\chi^2_{tot} \lesssim 27$ for certain values of $s$ and the continuous parameter $\theta$. This can be clearly seen from figure 7 where we show the 3 $\sigma$ contour regions of $\sin^2 \theta_{ij}$ in the $s/n$-$\theta$ plane for the case $m = n/2 = 10$ in the left panel and for $m = 11$ in the right one (the color coding is the same as in figure 1). Since the solar mixing angle fulfills $\sin^2 \theta_{12} \lesssim 1/3$ for $m/n = 1/2$, see above, no contour line associated with the 3 $\sigma$ upper limit of $\sin^2 \theta_{12}$ is present in the left panel of figure 7. As has been estimated in (144) for $m/n = 1/2$ and is obvious from figure 7 (as well as confirmed by the results found in table 10), the parameter $\theta$ is practically fixed by the requirement to accommodate the reactor mixing angle well. For $m = n/2$ it takes the values in (144) independent of $s$, while for the choice $n = 20$ and $m = 11$ the “best fitting” $\theta_{bf}$ reveals a certain dependence on the parameter $s$ which can be approximated by the expression in (148). This approximation is presented as dotted curves in the right panel of figure 7 and agrees with the exact result to a certain extent. In the case $m = n/2$ both best fit values of the atmospheric mixing angle, $(\sin^2 \theta_{23})_{bf} = 0.451$ and $(\sin^2 \theta_{23})_{bf} = 0.577$, overlap with the red areas, whereas for $n = 20$ and $m = 11$ only the value $(\sin^2 \theta_{23})_{bf} = 0.577$ has a non-vanishing overlap. The figure corresponding to the choice $m = 9$ and $n = 20$ can be obtained from the one shown in the right panel of figure 7 by applying the first symmetry transformation in table 6, i.e. by reflecting the 3 $\sigma$ contour regions.
for $m = 11$ and $n = 20$ in the line defined by $\theta = \pi/2$ where now the blue region indicates $0.385 \leq \cos^2 \theta_{23} \leq 0.644$, since $\sin^2 \theta_{23}$ is replaced by $\cos^2 \theta_{23}$. Consequently, values for $\theta_{23}$ smaller than $\pi/4$ (and thus close to $(\sin^2 \theta_{23})^{bf} = 0.451$) are accommodated well.

Turning to the CP phases we first discuss them for $n = 20$ and $m = n/2 = 10$. As has been shown in (151), the Majorana phases $\alpha$ and $\beta$ do not depend on the parameter $\theta$ in this case. Thus, we only plot $\sin \delta$ in the $s/n-\theta$ plane in the left panel of figure 8. The black areas indicate the regions in which all three lepton mixing angles are within their experimentally preferred $3\sigma$ ranges. As estimated in (150), the absolute value of the Dirac phase has a non-trivial lower limit in this case and can also attain a maximal value. We note some peculiarities of the plot in the left panel of figure 8: for $\theta = 0, \pi/2$ and $\pi$ the Dirac phase is not physical, since for these values either the reactor or the solar mixing angle vanishes, as can be seen from

$$\sin^2 \theta_{13} = \frac{1}{3} \cos^2 \theta \quad \text{and} \quad \sin^2 \theta_{12} = \frac{\sin^2 \theta}{2 + \sin^2 \theta}.$$  

We can expand $\sin \delta$ around these particular values of $\theta$, $\theta = \bar{\theta} + \varepsilon$ with $\bar{\theta} = 0, \pi/2, \pi$ and $|\varepsilon| \ll 1$, and find at leading order in $\varepsilon$

$$\sin \delta = (-1)^k \sgn(\varepsilon) \sin 3 \phi_s$$

with $k = 0$ for $\bar{\theta} = 0, \pi$ and $k = 1$ for $\bar{\theta} = \pi/2$, respectively. Also the points in the left panel of figure 8 in which all contour lines converge correspond to unphysical values of the Dirac phase, since in these points the atmospheric mixing angle either vanishes or becomes $\pi/2$.

If we consider $n = 20$ and $m = 11$ the results for the Dirac phase are different, as can be seen in the right panel of figure 8. In particular, this phase cannot attain maximal values anymore in the regions in which all three mixing angles are within their experimentally preferred $3\sigma$ range (black areas in the figure). Instead, its maximal value is $|\sin \delta| \approx 0.75$. Also here the points
Figure 9: Case 3 b.1. Predictions for the Majorana phases $\sin \alpha$ and $\sin \beta$ for $n = 20$ and $m = 11$ obtained from the CP invariants $I_{1,2}$ in (135) for $k_1 = k_2 = 0$. Again, in the black regions all three lepton mixing angles are within their experimentally preferred $3\sigma$ intervals, compare figure 7.

In which all the contour lines converge indicate unphysical values of the Dirac phase, since either the reactor, solar or atmospheric mixing angle vanishes or $\theta_{23} = \pi/2$ holds. As regards the predictions for the Majorana phases $\alpha$ and $\beta$ for $n = 20$ and $m = 11$, these are displayed in the $s/n$-$\theta$ plane in figure 9. Again, the black areas indicate the regions in which all three lepton mixing angles are within their $3\sigma$ intervals. Note that we have set $k_1 = k_2 = 0$, when computing $\sin \alpha$ and $\sin \beta$ from (135). Also in these figures points in which all contour lines converge correspond to unphysical values of the CP phases, because either the solar (relevant for $\sin \alpha$) or the reactor mixing angle (relevant for $\sin \beta$) vanishes. The figures of the CP phases for the choice $n = 20$ and $m = 9$ can be easily deduced from those for $n = 20$ and $m = 11$, if we apply the first symmetry transformation in table 6, i.e. for $\sin \alpha$ and $\sin \beta$ the figures are the same as in figure 9, replacing only $\theta$ with $\pi - \theta$, whereas for $\sin \delta$ we have to not only replace $\theta$ by $\pi - \theta$ in the right panel of figure 8, but to also change the sign of $\sin \delta$.

In tables 10 and 11 we present the results of our $\chi^2$ analysis for the smallest even and odd values of the index $n$ that allow for $\chi^2_{\text{tot}} \lesssim 27$ and all lepton mixing angles within their experimentally preferred $3\sigma$ ranges, using the formulae in (134) and (135) with $k_1 = k_2 = 0$. As can be read off from table 10 the smallest even value of $n$ is $n = 2$ with $m = 1$ and $s = 1$ (last column of the table). This case has already been studied in the literature [20] and leads to a maximal Dirac phase and vanishing Majorana phases. The smallest even $n$ that also permits non-trivial Majorana phases is $n = 4$ with $m = 2$ and $s = 1$. This case is implicitly contained in table 10 since the result for $n = 8$, $m = 4$ and $s = 2$ can be “reduced” to the former set of parameters $n$, $m$ and $s$ by dividing out the common factor two of all parameters $n$, $m$ and $s$. This is very similar to what has been described in case 2) (there for the parameters $n$, $u$ and $v$), see discussion in the paragraph below (88). Indeed, in this case both Majorana phases are maximal, while the Dirac phase is large. This is consistent with the findings in [25]. However, in this case the value of the atmospheric mixing angle is very close to the upper $3\sigma$ limit [1]. As shown for $n = 20$ and $m = n/2 = 10$ in the left panel in figure 7 there are (mostly) two “best fitting” values.
Table 10: Case 3 b.1). Results of the $\chi^2$ analysis for the smallest even values of $n$ that allow $\chi^2_{\text{tot}} \lesssim 27$. These results are obtained using the formulae in (134) and (135). The integers $k_{1,2}$ are set to zero. The fit of the reactor and solar mixing angles contribute $\chi^2_{12} \lesssim 0.08$ and $\chi^2_{23} \lesssim 1.5$ to $\chi^2_{\text{tot}}$, respectively. The (absolute) values of $\sin \alpha$ and $\sin \beta$ are always equal for $m = n/2$, see (151), and for $n = 8$, $m = 4$ and $s = 2$ we find maximal Majorana phases. Note that this case can be “reduced” to $n = 4$, $m = 2$ and $s = 1$. For choices of the parameter $s > n/2$ results can be obtained by applying the symmetry transformations in table 6 to those presented here. The fact that in most cases two different values of $\theta_{bf}$ lead to a reasonable fit with $\theta_{23} \gtrapprox \pi/4$ (and opposite sign for $J_{CP}$) is also observed in the left panel in figure 7. The choice of parameters mentioned in the last column always allows for a good fit of the experimental data, if $n$ is even. Thus, the smallest value of the index $n$ for that this choice can be realized is $n = 2$.

$\theta_{bf}$ one leading to $\theta_{23}$ smaller than $\pi/4$ and one larger than $\pi/4$. If we consider the particular choice $m = n/2$ and $s = n/2$, the atmospheric mixing angle is maximal, see (134). The value of the reactor mixing angle instead is accommodated very close to $(\sin^2 \theta_{13})_{\text{bf}} = 0.0219$ in all cases. This value entails, as explained in the analytical study, $\sin^2 \theta_{12} \approx 0.318$. Since the two best fitting values $\theta_{bf}$ are related by $\theta_{bf,2} = \pi - \theta_{bf,1}$, compare to the first symmetry in table 6 the CP invariant $J_{CP}$ has opposite sign for the two values, while the Majorana invariants $I_1$ and $I_2$ are the same. The estimated lower bound for $|\sin \delta|$ mentioned in (150) is clearly fulfilled in the cases in table 10. As already observed in (151), the sines of the Majorana phases $\alpha$ and $\beta$ have the same absolute value in these cases (the sign depends on $k_1$ and $k_2$). In order to obtain numerical results for values of $s$ that are not shown in table 10 we can make use of the third symmetry transformation in table 6.

The smallest value of an odd index $n$ that allows for $\chi^2_{\text{tot}} \lesssim 27$ is $n = 11$ and we display results for $m = 5$ (remember $m/n \approx 1/2$ is required) and $n = 11$ in table 11. In this case for all admitted values of the parameter $s$, $0 \leq s \leq 10$, (at least) one value of $\theta$ can be found for which all lepton mixing angles are fitted reasonably well. In table 11 only values $s < n/2 = 11/2$ are presented, since the results for those larger than $s = 5$ can be obtained from table 11 by exploiting the symmetry transformations in table 6. Only in the case $s = 0$ all CP phases vanish, since in this case an accidental CP symmetry is present in the charged lepton and neutrino sectors, see (130). As observed for $n = 20$ and $m = 11$, see right panel in figure 8 also here the value of $\delta$ cannot be maximal, $|\sin \delta| \lesssim 0.7$. For $n = 11$ not only $m = 5$, but also the choice $m = 6$ leads to a good agreement with the experimental data on lepton mixing angles.
Table 11: Case 3 b.1. Results for the smallest odd value of $n$ that allows for $\chi^2_{\text{tot}} \lesssim 27$, namely $n = 11$. Also here the mixing angles and CP invariants are computed using the expressions in (134) and (135) with $k_1 = k_2 = 0$. Two values of the parameter $m$, $m = 5$ and $m = 6$, are admitted by the fit. Here we only display $m = 5$, since results for $m = 6$ are obtained via the symmetry transformations in table 6. The most notable difference lies in the fact that $m = 6$ usually leads to $\sin^2 \theta_{23} > 1/2$ at $\theta_{bf}$ and that $\chi^2_{\text{tot}}$ is slightly larger than for $m = 5$, i.e. $\chi^2_{\text{tot}} \gtrsim 6$. With the help of table 6 also the results for $s > n/2 = 11/2$ can be obtained, showing that for all values of the parameter $s$, $0 \leq s \leq 10$, reasonable fits are possible. The fit of reactor and solar mixing angles contributes $\chi^2_{\text{bf}} \lesssim 0.02$ and $\chi^2_{\text{tot}} \lesssim 5.5$ to $\chi^2_{\text{tot}}$, respectively. As explained, the choice $s = 0$ implies the presence of an accidental CP symmetry entailing trivial CP phases.

Results for this case can be obtained, as before, by applying the symmetry transformations in table 6. For the choice $m = 6$ in general the value of the atmospheric mixing angle is larger than $\pi/4$ at the best fitting point(s) $\theta_{bf}$. The values of $\chi^2_{\text{bf}}$ obtained in these cases fulfill $\chi^2_{\text{bf}} \gtrsim 0.6$.

If we consider instead the formulae in (134) and (135) with the replacements $m \to m - \frac{3}{2}$ and $\theta \to \pi - \theta$, i.e. we want to study the results for a different permutation of the PMNS mixing matrix in (110) than before, the smallest odd and even values of $n$ are $n = 5$ and $n = 8$, respectively. In particular, the case $n = 5$ that requires the choice $m = 4$ is interesting, since the associated flavor group $\Delta(150)$ is quite small. If the results are shown in table 12. For all values of the parameter $s$, $0 \leq s \leq 4$, a reasonably good fit to the experimental data can be achieved and we choose as representatives $s \leq 2$, since the results for the other two values $s = 3$ and $s = 4$ can be straightforwardly deduced from table 12 using table 6. Similarly, the choice $n = 8$ and $m = 7$ allows to accommodate the mixing angles well for all possible choices of the parameter $s$ for a certain value $\theta_{bf}$. The results for the values $s > 4$ are not displayed in table 12 but can be obtained from the latter with the help of the symmetry transformations in table 6. Note that for $n = 8$, $m = 7$ and $s = 4$ due to the choice $s = n/2$ two values of $\theta_{bf}$ lead to the same reasonable fit to the experimental data. These two values $\theta_{bf,1}$ and $\theta_{bf,2}$ are related by $\theta_{bf,2} = \pi - \theta_{bf,1}$, see third symmetry transformation in table 6. Furthermore, we see that in this case all CP invariants for $\theta_{bf,1}$ have opposite sign as those for $\theta_{bf,2}$. In addition, the choice $s = n/2 = 4$ tells us that the CP invariants $I_1$ and $I_2$ have to have the same absolute values (their signs, obviously, depend on the choice of $k_1$ and $k_2$, see (135)). This observation is independent from the other parameters $n$, $m$ and $\theta$. As expected in both cases the choice $s = 0$ leads to an accidental CP symmetry that enforces trivial CP phases, see (130).

---

| $s$ | $\chi^2_{\text{tot}}$ | $\theta_{bf}$ | $\sin^2 \theta_{23}$ | $\sin^2 \theta_{12}$ | $\sin^2 \theta_{13}$ | $\delta_C$ | $\sin \delta$ | $I_1$ | $\sin \alpha$ | $I_2$ | $\sin \beta$ |
|-----|-----------------|--------------|------------------|-----------------|-----------------|----------|-----------|------|----------|------|-----------|
| 0   | 5.55            | 1.52         | 0.0438           | 0.332           | 0.0220          | 0        | 0.0071    | -0.209 | -0.986   | 0    | -0.436    |
| 1   | 5.53            | 1.50         | 0.0157           | 0.332           | 0.0220          | 0        | -0.436   | 0.209  | 0.228    | 0    | -0.0063   |
| 2   | 5.80 (8.49)     | 1.71 (1.38)  | 0.290 (2.98)     | 0.332           | 0.0220          | 0        | 0.0071   | -0.209 | 0.0485   | 0    | -0.0012   |
| 3   | 5.54            | 1.64         | 0.0321           | 0.332           | 0.0220          | 0        | -0.436   | 0.209  | 0.228    | 0    | -0.0012   |
| 4   | 5.55            | 1.63         | 0.0411           | 0.332           | 0.0220          | 0        | -0.436   | 0.209  | 0.228    | 0    | -0.0012   |
| 5   | 5.51            | 1.67         | 0.000010         | 0.332           | 0.0220          | 0        | -0.436   | 0.209  | 0.228    | 0    | -0.0012   |

---

\footnote{This group has also been discussed in [17,34].}
\[ n \]
\[ m \]
\[ s \]
\[ 0 \]
\[ 1 \]
\[ 2 \]
\[ 3 \]
\[ 4 \]
\[ \chi^2_{\text{tot}} \]
\[ 5.06 \]
\[ 3.61 \]
\[ 5.76 \]
\[ 7.68 \]
\[ 6.02 \]
\[ 7.10 \]
\[ 7.56 \]
\[ 4.69 \]
\[ \theta_{\text{bf}} \]
\[ 1.68 \]
\[ 1.40 \]
\[ 1.45 \]
\[ 1.50 \]
\[ 1.45 \]
\[ 1.66 \]
\[ 1.65 \]
\[ 1.39 \]
\[ \chi^2_{23} \]
\[ 1.72 \]
\[ 0.308 \]
\[ 2.42 \]
\[ 2.93 \]
\[ 1.33 \]
\[ 2.38 \]
\[ 2.82 \]
\[ 0.0080 \]
\[ \sin^2 \theta_{23} \]
\[ 0.531 \]
\[ 0.484 \]
\[ 0.523 \]
\[ 0.517 \]
\[ 0.537 \]
\[ 0.523 \]
\[ 0.518 \]
\[ 0.574 \]
\[ \sin^2 \theta_{12} \]
\[ 0.326 \]
\[ 0.326 \]
\[ 0.326 \]
\[ 0.330 \]
\[ 0.330 \]
\[ 0.330 \]
\[ 0.330 \]
\[ 0.330 \]
\[ \sin^2 \theta_{13} \]
\[ 0.0222 \]
\[ 0.0220 \]
\[ 0.0218 \]
\[ 0.0219 \]
\[ 0.0218 \]
\[ 0.0218 \]
\[ 0.0218 \]
\[ 0.0220 \]
\[ J_{\text{CP}} \]
\[ 0 \]
\[ -0.0208 \]
\[ 0.0094 \]
\[ 0 \]
\[ -0.0141 \]
\[ 0.0080 \]
\[ -0.0036 \]
\[ \pm 0.0225 \]
\[ \sin \delta \]
\[ 0 \]
\[ -0.612 \]
\[ 0.276 \]
\[ 0 \]
\[ -0.416 \]
\[ 0.236 \]
\[ -0.107 \]
\[ \pm 0.667 \]
\[ I_1 \]
\[ 0 \]
\[ 0.115 \]
\[ -0.201 \]
\[ 0 \]
\[ -0.143 \]
\[ 0.212 \]
\[ -0.151 \]
\[ \mp 0.0144 \]
\[ \sin \alpha \]
\[ 0 \]
\[ 0.547 \]
\[ -0.958 \]
\[ 0 \]
\[ -0.676 \]
\[ 0.9997 \]
\[ -0.715 \]
\[ \mp 0.0683 \]
\[ I_2 \]
\[ 0 \]
\[ 0.0142 \]
\[ -0.0080 \]
\[ 0 \]
\[ -0.0114 \]
\[ 0.0069 \]
\[ -0.0032 \]
\[ \pm 0.0144 \]
\[ \sin \beta \]
\[ 0 \]
\[ 0.981 \]
\[ -0.544 \]
\[ 0 \]
\[ -0.793 \]
\[ 0.484 \]
\[ -0.225 \]
\[ \pm 1 \]

Table 12: Case 3 b.1). Results of the $\chi^2$ analysis for $n = 5$ ($m = 4$) and $n = 8$ ($m = 7$) that are the smallest odd and even values of the index $n$ for which $\chi^2_{\text{tot}} \lesssim 27$, if we consider the formulae in (134) and (135) and $m$ and $\theta$ are replaced by $m - \frac{\pi}{2}$ and $\pi - \theta$, respectively. The parameters $k_{1,2}$ in (136) are taken to be zero. The contributions to $\chi^2_{\text{tot}}$ arising from the fit of the reactor and the solar mixing angles are $\chi^2_{13} \lesssim 0.08$ (0.02) and $\chi^2_{12} \lesssim 3.3$ (4.7) for $n = 5$ (8), respectively. For $s = 0$ all CP phases are trivial due to the presence of an accidental CP symmetry, see (130). Taking into account the third symmetry transformation in table 6 we see that all admitted values of $s$, $0 \leq s \leq n - 1$, allow for a good fit. The other symmetries in that table show that for $n = 5$, $m = 1$ and $n = 8$, $m = 1$ also reasonable fits are obtained, belonging to a mixing pattern given by the formulae in (134) and (135) with $m$ and $\theta$ replaced by $m + \frac{\pi}{2}$ and $\pi - \theta$, respectively. In the case $n = 8$, $m = 7$ and $s = 4$ the first symmetry in table 6 explains the presence of two different values for $\theta_{bf}$ (given by $\theta$ and $\pi - \theta$) leading to the same best fitted values of the mixing angles and opposite signs for the three CP invariants. In addition, $s = n/2$ explains why the (absolute) values of $I_1$ and $I_2$ are identical.

we notice that exploiting the relation between results for $m$ and $n - m$, see table 6, we find that also $n = 5$ and $m = 1$ as well as $n = 8$ and $m = 1$ allow us to accommodate well the experimental data. (Note that these solutions correspond to a different permutation of the PMNS mixing matrix in (110), i.e. the one that leads to formulae for mixing angles and CP invariants in (134) and (135) with $m$ and $\theta$ replaced by $m + \frac{\pi}{2}$ and $\pi - \theta$.) While the results for the reactor and the solar mixing angles are the same as for the displayed cases, the atmospheric mixing angle takes values in the opposite octant, according to the fact that $\sin^2 \theta_{23}$ becomes
replaced by \( \cos^2 \theta_{23} \). In this case, the value of \( \chi^2_{\text{tot}} \) is for almost all values of \( s \) smaller than the one reported in table [12] except for \( s = 1 \).

4.3.3 Case 3 b.2)

The last type of mixing pattern can be obtained from the PMNS mixing matrix \( U_{\text{PMNS},3} \) in [110] by exchanging its second and third columns. Since this corresponds to taking the PMNS mixing matrix of case 3 b.1) and exchanging its first and second columns, we find the same results for \( \sin^2 \theta_{13} \) and \( \sin^2 \theta_{23} \) as in [134], while \( \sin^2 \theta_{12} \) becomes \( \cos^2 \theta_{12} \) in this case. Furthermore, the signs of \( J_{\text{CP}} \) and of \( I_1 \) are changed with respect to those in [135], whereas \( I_2 \) has now a different dependence on the parameters

\[
I_2 = \frac{1}{9} (-1)^{k_1+1} \cos \phi_m \sin 3 \phi_s \left( 4 \cos \phi_m \cos 3 \phi_s \cos 2\theta + \sqrt{2} \cos 2 \phi_m \sin 2\theta \right).
\]

(156)

Again, the results for mixing angles and CP invariants obtained for the remaining eleven permutations are related through shifts in \( \theta \) and/or in the parameter \( m \) to those presented here. The crucial difference between this case and case 3 b.1) is the change of \( \sin^2 \theta_{12} \) into \( \cos^2 \theta_{12} \), i.e. \( \sin^2 \theta_{12} \) can now be written as

\[
\sin^2 \theta_{12} = \frac{2 \sin^2 \phi_m}{3 (1 - \sin^2 \theta_{13})}.
\]

(157)

Again, its value should be close to \( 1/3 \) meaning that

\[
\sin^2 \phi_m \approx \frac{1}{2}
\]

(158)

has to be fulfilled that requires in turn

\[
m \approx \frac{n}{4} \text{ or } m \approx \frac{3n}{4}.
\]

(159)

Furthermore, we see

\[
\sin \phi_m \approx \frac{1}{\sqrt{2}}, \quad \cos \phi_m \approx \pm \frac{1}{\sqrt{2}}, \quad \cos 2 \phi_m \approx 0, \quad \sin 2 \phi_m \approx \pm 1,
\]

(160)

with “+”, if \( m/n \approx 1/4 \), and “−” for \( m/n \approx 3/4 \). The parameter \( p \) in [136] is required to fulfill \( p \approx -1 \), as can be derived from [138] when neglecting \( \theta_{13} \), and it implies here

\[
\cos 3 \phi_s \sin 2\theta \approx \mp 1
\]

(161)

with “−” for \( m/n \approx 1/4 \) and “+” for \( m/n \approx 3/4 \). At the same time, this condition tells us that \( \sin 2\theta \approx \pm 1 \) and hence also \( \sin \theta \approx \mp \frac{1}{\sqrt{2}} \) for \( 0 \leq \theta < \pi \) and consequently we find that \( q \) in [137] is determined

\[
q \approx \mp \frac{3}{2} \quad \text{with “+” for } m/n \approx 1/4 \quad \text{and “−” for } m/n \approx 3/4,
\]

(162)

such that the atmospheric mixing angle in [138] results to be

\[
\sin^2 \theta_{23} \approx \frac{1}{2} \left( 1 \mp \frac{\sqrt{3}}{2} \right) \approx \begin{cases}
0.067 & \text{for } m/n \approx 1/4 \\
0.933 & \text{for } m/n \approx 3/4
\end{cases},
\]

(163)

\[18\text{The index } n \text{ should be divisible by four or the formulae have to be modified in such a way that } n \text{ on the right-hand side of the equations is replaced by } n - k \text{ with } k \text{ chosen so that } n - k \text{ is divisible by four.}\]
Figure 10: Case 3 b.2). Similar to figure we plot the $3\sigma$ contour regions of $\sin^2 \theta_{ij}$ in the $s/n-\theta$ plane for $m/n = 1/4$. As one can clearly see, it is impossible to accommodate simultaneously all three lepton mixing angles well in this case.

i.e. this mixing angle cannot be in accordance with experimentally measured values, if $\theta_{13}$ and $\theta_{12}$ are accommodated well. A refined numerical analysis, e.g. taking into account the non-zero value of $\theta_{13}$, confirms this result as can be clearly seen from figure 10 in which we display the $3\sigma$ contour regions for $\sin^2 \theta_{ij}$ in the $s/n-\theta$ plane, using the expressions of the lepton mixing angles in case 3 b.2) for $m/n = 1/4$. As can be checked, also for the choice $m/n = 3/4$ the experimentally preferred $3\sigma$ ranges of the three different mixing angles do not overlap.

5 Summary and conclusions

We have analyzed in detail lepton mixing patterns that arise from a theory in which a flavor symmetry $G_f = \Delta(3n^2)$ or $G_f = \Delta(6n^2)$ (with an index $n$ not divisible by three) and a CP symmetry are broken to residual symmetries $G_e = Z_3$ and $G_\nu = Z_2 \times CP$ in the charged lepton and neutrino sectors, respectively. All mixing angles and CP phases are determined by group theoretical indices (characterizing the flavor group and the generators of the residual symmetries as well as the CP transformation $X$, representing the CP symmetry) and by one continuous free parameter $\theta$, that can take values between 0 and $\pi$. We have studied all possible $Z_3$ and $Z_2$ subgroups of $\Delta(3n^2)$ and $\Delta(6n^2)$ that can function as residual symmetries. As regards the CP symmetry, we have focussed on a set of CP transformations that can be consistently combined with $G_f$ as well as with the residual $Z_2$ group in the neutrino sector. Furthermore, we have dealt with the question whether these CP transformations can correspond to ‘class-inverting’ automorphisms for $\Delta(3n^2)$ and $\Delta(6n^2)$.

We have shown that it is sufficient to consider only three types of combinations of residual symmetries, represented by case 1), case 2), case 3 a) and case 3 b.1), in the charged lepton and neutrino sectors in order to comprehensively discuss lepton mixing. Especially, the generator of the residual symmetry $G_e$ can always be fixed to the generator $a$ of $\Delta(3n^2)$ and $\Delta(6n^2)$. Due to the choice of the $Z_2$ symmetry the first two types of combinations, case 1) and case 2), can be realized for $G_f = \Delta(3n^2)$ as well as $G_f = \Delta(6n^2)$, whereas the third type of combination,
case 3 a) and case 3 b.1), is only admitted for $\Delta(6^2)$. Furthermore, the choice of the $Z_2$ group constrains the index $n$ of the flavor group to be even for the first two types. Interestingly enough, this choice is also responsible for the fact that the second column of the PMNS mixing matrix has to be trimaximal in case 1) and case 2).

The mixing angles derived from the first type of combination only depend on the continuous parameter $\theta$ and their experimentally preferred values can be accommodated well for certain choices of $\theta$. The Dirac phase and one of the Majorana phases vanish, while the value of the other Majorana phase depends on the chosen CP transformation $X$. The second type of combination instead leads to mixing angles and two CP phases, $\delta$ and $\beta$, that depend on two parameters, $\theta$ as well as on an integer related to the choice of the CP transformation, whereas the Majorana phase $\alpha$ is fixed not only by these parameters, but in addition by a third one that also characterizes the CP transformation $X$. As a consequence, for each set of parameters that leads to mixing angles in good agreement with the experimental data, see tables 3-5, we can obtain a variety of different values of the Majorana phase $\alpha$, see figure 4. We find that for small and moderate values of the index $n$, $2 \leq n \leq 20$, the data on lepton mixing angles can be accommodated very well for certain choices of the CP transformation $X$ and the parameter $\theta$.

The third type of combination allows for a richer structure of mixing patterns and, indeed, we can divide the resulting mixing patterns in two categories, case 3 a) and case 3 b.1): for case 3 a) the reactor and the atmospheric mixing angles are determined by the choice of the residual $Z_2$ symmetry in the neutrino sector (characterized by the parameter $m$) and the index $n$ of the flavor group, while the solar mixing angle as well as the CP invariants depend, in general, also on the continuous parameter $\theta$ and the choice of the CP transformation $X$. We find that for a good agreement with the experimental data the index $n$ has to be at least $n = 11$. The solar mixing angle can be fitted to its best fit value in most cases, see tables 7-9. The CP phases are in general all non-trivial (unless a certain CP transformation $X$ is employed). The Dirac as well as the Majorana phase $\alpha$ can obtain (close to) maximal values, while the absolute value of the other Majorana phase has a non-trivial bound, $|\sin \beta| \lesssim 0.87$, see figure 6. The mixing pattern belonging to the second category, case 3 b.1), reveals the most complex structure, since all mixing angles and CP invariants depend on the parameter $\theta$, the choice of the residual $Z_2$ symmetry as well as on the choice of the CP transformation $X$. However, the condition to accommodate the mixing angles well strongly constrains the choice of the $Z_2$ group, i.e. the parameter $m$, as well as the value of $\theta$, see (141), (144) and figure 7. For the various choices of $X$ different predictions for the CP phases, as shown in figures 8 and 9, are obtained. In particular, for the choice $m = n/2$ the sines of the Majorana phases turn out to be equal up to a sign and to depend only on the choice of CP transformation, while the Dirac phase has in general a non-trivial lower bound, $|\sin \delta| \gtrsim 0.7$. If $m$ is not chosen as $n/2$, also smaller values are obtained for $|\sin \delta|$ and the Majorana phases mildly depend on the continuous parameter $\theta$.

As shown in tables 10-12 reasonably good agreement with the experimental data is achieved for small values of the index $n$, corresponding to a moderately sized flavor group $\Delta(6^2)$. In particular, we find that $n = 5$, i.e. $\Delta(150)$, admits a very good fit to the mixing angles together with non-vanishing and also non-maximal values of all three CP phases, see table 12.

Given the promising results obtained here it is worth to extend our study. For example, we could consider other choices for the residual symmetry in the charged lepton sector for $G_f = \Delta(6^2)$ or we could employ a different set of CP transformations. It would also be interesting to exploit the presented results in studies of phenomena that involve CP phases, such as neutrinoless double beta decay and leptogenesis. Furthermore, the construction of concrete models in which the breaking pattern of the flavor and CP symmetry is achieved dynamically, see e.g. [24,25,35], is another interesting direction, since in such models also constraints on the lepton mass spectrum can be achieved.
A Conventions for mixing angles and CP invariants, global fit results and $\chi^2$ analysis

In this appendix we fix our conventions of mixing angles and of the CP invariants $J_{CP}$, $I_1$ and $I_2$, list the latest global fit results \cite{1} and describe our $\chi^2$ analysis.

A.1 Conventions for mixing angles and CP invariants

As parametrization of the PMNS mixing matrix we use

$$U_{PMNS} = \tilde{U} \text{ diag}(1,e^{i\alpha/2},e^{i(\beta/2+\delta)})$$ \hspace{1cm} (164)

with $\tilde{U}$ being of the form of the Cabibbo-Kobayashi-Maskawa (CKM) matrix $V_{CKM}$ \cite{36}

$$\tilde{U} = \begin{pmatrix} c_{12}c_{13} & s_{12}c_{13} & s_{13}e^{-i\delta} \\ -s_{12}c_{23} - c_{12}s_{23}s_{13}e^{i\delta} & c_{12}c_{23} - s_{12}s_{23}s_{13}e^{i\delta} & s_{23}c_{13} \\ s_{12}s_{23} - c_{12}c_{23}s_{13}e^{i\delta} & -c_{12}s_{23} - s_{12}c_{23}s_{13}e^{i\delta} & c_{23}c_{13} \end{pmatrix}$$ \hspace{1cm} (165)

and $s_{ij} = \sin \theta_{ij}$ and $c_{ij} = \cos \theta_{ij}$. The mixing angles $\theta_{ij}$ range from 0 to $\pi/2$, while the Majorana phases $\alpha, \beta$ as well as the Dirac phase $\delta$ take values between 0 and $2\pi$. The Jarlskog invariant $J_{CP}$ reads \cite{37}

$$J_{CP} = \text{Im}[U_{PMNS,11}U_{PMNS,13}^*U_{PMNS,31}^*U_{PMNS,33}]$$

$$= \frac{1}{8} \sin 2\theta_{12} \sin 2\theta_{23} \sin 2\theta_{13} \cos \theta_{13} \sin \delta$$ \hspace{1cm} (166)

Similar invariants, called $I_1$ and $I_2$, can be defined which depend on the Majorana phases $\alpha$ and $\beta$ \cite{38} (see also \cite{39,41})

$$I_1 = \text{Im}[U_{PMNS,12}^*(U_{PMNS,11})^2] = s_{12}^2 c_{12}^2 c_{13}^2 \sin \alpha$$ \hspace{1cm} (167)

$$I_2 = \text{Im}[U_{PMNS,13}^*(U_{PMNS,11})^2] = s_{13}^2 c_{12}^2 c_{13}^2 \sin \beta$$ \hspace{1cm} (168)

Notice that the Dirac phase has a physical meaning only if all mixing angles are different from 0 and $\pi/2$, as indicated by the data. Analogously, the vanishing of the invariants $I_{1,2}$ only implies $\sin \alpha = 0, \sin \beta = 0$, if solutions with $\sin 2\theta_{12} = 0, \cos \theta_{13} = 0$ or $\sin 2\theta_{13} = 0, \cos \theta_{12} = 0$ are discarded. Furthermore, notice that one of the Majorana phases becomes unphysical, if the lightest neutrino mass vanishes.
A.2 Global fit results

We use in our numerical analysis the results of mixing angles and the CP phase $\delta$ taken from [1] as given in the left table of table 1, i.e. the results obtained by including the short baseline reactor data (called RSBL in [1]) and leaving reactor fluxes free in the fit (see free fluxes in [1]). The best fit values of $\sin^2 \theta_{ij}$, the 1 $\sigma$ errors as well as 3 $\sigma$ ranges are

$$\sin^2 \theta_{13} = 0.0219^{+0.0010}_{-0.0011} \quad \text{and} \quad 0.0188 \leq \sin^2 \theta_{13} \leq 0.0251$$

$$\sin^2 \theta_{12} = 0.304^{+0.012}_{-0.012} \quad \text{and} \quad 0.270 \leq \sin^2 \theta_{12} \leq 0.344$$

$$\sin^2 \theta_{23} = \begin{cases} 
[0.451^{+0.031}_{-0.030}] \\
0.577^{+0.027}_{-0.035}
\end{cases} \quad \text{and} \quad 0.385 \leq \sin^2 \theta_{23} \leq 0.644$$

(169)

where the value $\sin^2 \theta_{23} < 0.5$ is a local minimum. The 1 $\sigma$ errors of this best fit value refer to itself and not to the global minimum, as done in [1]. We have read these errors off the figure given in [1]. The CP phase $\delta$, here given in radian, is constrained at the 1 $\sigma$ level

$$\delta = 4.38^{+1.17}_{-1.03} \quad \text{and} \quad 0 \leq \delta \leq 2\pi \quad \text{at} \quad 3\sigma .$$

(170)

A.3 $\chi^2$ analysis

In our numerical analysis we use a $\chi^2$ function in order to evaluate which mixing patterns agree well with the experimental data on the mixing angles. This function is defined in the usual way

$$\chi^2_{\text{tot}} = \chi^2_{12} + \chi^2_{13} + \chi^2_{23}$$

(171)

$$\chi^2_{ij} = \left( \frac{\sin^2 \theta_{ij} - (\sin^2 \theta_{ij})_{bf}}{\sigma_{ij}} \right)^2 \quad \text{for} \quad ij = 12, 13, 23$$

(172)

with $\sin^2 \theta_{ij}$ being the mixing angles derived in the different cases, e.g. $[63, 78, 113, 134]$, that depend in general on several discrete parameters $n, u, v, s, m$ as well as on the continuous parameter $\theta$, $0 \leq \theta < \pi$, $(\sin^2 \theta_{ij})_{bf}$ the best fit values and $\sigma_{ij}$ the 1 $\sigma$ errors given in (169).

Note that these errors also depend on whether $\sin^2 \theta_{ij}$ is larger or smaller than the best fit value. Since the atmospheric mixing angle has a global minimum at $(\sin^2 \theta_{23})_{bf} = 0.577$ as well as a local one at $(\sin^2 \theta_{23})_{bf} = 0.451$, we compute $\chi^2_{23}$ using $(\sin^2 \theta_{23})_{bf} = 0.451$, if $\sin^2 \theta_{23}$ for $n, u, v, s, m$ and $\theta$ is smaller or equal 0.5, and use $(\sin^2 \theta_{23})_{bf} = 0.577$ otherwise. A mixing pattern is considered to agree reasonably well with the experimental data, if $\chi^2_{\text{tot}} \lesssim 27$ and all mixing angles $\sin^2 \theta_{ij}$ are within the 3 $\sigma$ intervals in (169). For the different mixing patterns that allow for such a situation we present in tables 3–5 and 7–12 values of $n, u, v, s, m$ and $\theta = \theta_{bf}$ for which the $\chi^2$ function is minimized. Since the indication of a preferred value of the Dirac phase $\delta$ coming from global fit analyses, see (170), is rather weak, i.e. below the 3 $\sigma$ significance, we do not include any information on $\delta$ in the $\chi^2$ function in (171).

B Relations among the different choices $(Q, Z, X)$

First, we show that we can reduce all possible combinations of $(Q, Z, X)$ to $(\tilde{Q} = a, \tilde{Z}, \tilde{X})$ with $\tilde{Z}$ and $\tilde{X}$ being of the same type as $Z$ and $X$, respectively, if the residual symmetry $G_e$ of the charged lepton sector is a $\mathbb{Z}_3$ group, i.e. it is generated by $Q = a c^\gamma d^\delta$ or by $Q = a^2 c^\gamma d^\delta$ with $0 \leq \gamma, \delta \leq n - 1$. Then, we prove that the twelve types of combinations $(Q = a, Z, X)$,
given by the twelve possible combinations of $Z$ and $X$, collected in table 1, can be reduced to three distinct types $(Q = a, Z, X)$, either by applying the similarity transformations $\tilde{\Omega} = a$ and $\tilde{\Omega} = a^2$ or by exploiting the fact that also $Y = ZX$ is an admissible CP transformation (in the neutrino sector), if $X$ is such a transformation and $Z$ is the generator of a $Z_2$ symmetry fulfilling the condition in [3], see end of subsection [3.4].

B.1 Relations among different $(Q = a c^\gamma d^\delta, Z, X)$ and $(Q = a^2 c^\gamma d^\delta, Z, X)$

Here we argue that choosing $G_e = Z_3$ always allows us to reduce all combinations of $(Q, Z, X)$ to the triple $(\tilde{Q} = a, \tilde{Z}, \tilde{X})$ or $(\tilde{Q} = a^2, \tilde{Z}, \tilde{X})$ that leads to the same $Z_3$ symmetry in the charged lepton sector. As noted in subsection [3.1], all $Z_3$ subgroups of $\Delta(3n^2)$ and $\Delta(6n^2)$ with $3 \nmid n$, are generated by elements of the form

$$Q = a c^\gamma d^\delta \quad \text{or} \quad Q = a^2 c^\gamma d^\delta \quad \text{with} \quad 0 \leq \gamma, \delta \leq n - 1.$$ 

Since all elements of the form $a c^\gamma d^\delta$ belong to the same class in $\Delta(3n^2)$ and $\Delta(6n^2)$, we know that a similarity transformation $\tilde{\Omega}$ must exist which relates $Q = ac^\gamma d^\delta$ to $\tilde{Q} = a$. As can be checked such a transformation is of the form $\tilde{\Omega} = c^\delta d^h$ with $0 \leq f, h \leq n - 1$. One can compute $f$ and $h$ which should lead to the correct transformation from the two conditions

$$\gamma + f + h = 0 \pmod{n} \quad \text{and} \quad \delta - f + 2h = 0 \pmod{n}.$$ 

These equations can be solved for any combination of $\gamma$ and $\delta$. The same type of transformation $\tilde{\Omega}$ also relates $Q = a^2 c^\gamma d^\delta$ to $\tilde{Q} = a^2$ (these elements also always belong to the same class of the groups $\Delta(3n^2)$ and $\Delta(6n^2)$). The conditions that determine $f$ and $h$ in this case are

$$\gamma + 2f - h = 0 \pmod{n} \quad \text{and} \quad \delta + f + h = 0 \pmod{n}.$$ 

In the next step we apply $\tilde{\Omega} = c^\delta d^h$ with arbitrary $f$ and $h$ to all the twelve pairs $(Z, X)$ that we have collected in table 1 in subsection [3.5]. Clearly, the form of $Z$ does not change when $\tilde{\Omega}$ is applied, if it is an element containing only $c$ and $d$, i.e. $\tilde{Z} = Z$ for $Z = c^{s/2}$, $Z = d^{n/2}$ and $Z = (cd)^{n/2}$. We compute for $X = c^{\epsilon d^f} P_{23}$ that

$$\tilde{X} = \tilde{\Omega} X \Omega^* = c^{\epsilon' d^f} P_{23}, \quad \text{with} \quad s' = s - 2f, \quad t' = t - 2h,$$

and, thus, the form of $X$ remains the same. Similarly, we see that $X = ab c^s d^{2s} P_{23}$ does not change its form, since

$$\tilde{X} = ab c^{s'} d^{2s'} P_{23}, \quad \text{with} \quad s' = s - h.$$ 

Also $X = a^2 b c^{2t} d^f P_{23}$ which gets transformed via $\tilde{\Omega}$ into

$$\tilde{X} = a^2 b c^{2t'} d^f P_{23}, \quad \text{with} \quad t' = t - f$$

has the same form as the original $X$. Lastly, the CP transformation $X = b c^{s} d^a P_{23}$ reads, after applying $\tilde{\Omega} = c^\delta d^h$,

$$\tilde{X} = b c^{s'} d^{a - s'} P_{23}, \quad \text{with} \quad s' = s + h - f.$$ 

Thus, we have shown that all pairs $(Z, X)$ that are mentioned in the first three lines in table 1 still have the same structure in the transformed basis.

Proceeding in the same way in the case of the combination $(Z = b c^{m} d^{m}, X = c^{\epsilon d^f} P_{23})$ with the condition $t = n - 2m - s$ we find that this pair is transformed into

$$\tilde{Z} = b c^{m'} d^{m'}, \quad \tilde{X} = c^{s'} d^{t'} P_{23}, \quad \text{with} \quad m' = m + f + h, \quad s' = s - 2f, \quad t' = t - 2h.$$ 

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so that the form of \((Z, X)\) as well as of the condition are maintained, i.e. it also holds \(t' = n - 2m' - s'\). Next we consider the combination \((Z = abc^m, X = c^dP_{23})\) with the condition \(t = 2(m + s)\): \(Z\) is transformed into

\[ \tilde{Z} = abc^{m'} \quad \text{with} \quad m' = m + 2f - h \]

via the similarity transformation \(\tilde{\Omega}\). Since \(\tilde{X} = c^{s'}d^{n'}P_{23}\) with \(s' = s - 2f, t' = t - 2h\), we also recover the form of the condition, namely \(t' = 2(m' + s')\). The combination \((Z = a^2bd^m, X = c^dP_{23})\) together with the condition \(s = 2(m + t)\) is transformed into

\[ \tilde{Z} = a^2bd^{m'} \quad \text{with} \quad m' = m + 2f - h \quad \text{and} \quad X = c^{s'}d^{n'}P_{23} \quad \text{with} \quad s' = s - 2f, t' = t - 2h \]

fulfilling the constraint \(s' = 2(m' + t')\). Eventually, using these results it is immediate to see that also the three remaining pairs \((Z = bcd^m, X = bcd^{n-s}P_{23}), (Z = abcd^m, X = abc^dP_{23})\) and \((Z = a^2bd^m, X = a^2bc^dP_{23})\) keep their structure when the transformation \(\tilde{\Omega}\) is applied.

In summary, we have shown that all combinations \((Q = ac^d, Z, X)\) and \((Q = a^2c^d, Z, X)\) can be related via a similarity transformation to \((\tilde{Q} = a, \tilde{Z}, \tilde{X})\) and \((\tilde{Q} = a^2, \tilde{Z}, \tilde{X})\) where \(\tilde{Z}\) and \(\tilde{X}\) have the same structure as \(Z\) and \(X\), respectively. Thus, it is sufficient to consider only cases with \(Q = a\) in the case of the groups \(\Delta(3n^2)\) as well as \(\Delta(6n^2), 3 \nmid n\), in order to perform a comprehensive analysis of the cases in which \(G_e\) is a \(Z_3\) symmetry. In the next subsection we show that also the number of pairs \((Z, X)\) that needs to be discussed can be reduced.

### B.2 Relations among the different choices \((Q = a, Z, X)\)

As we will see, it is sufficient to consider the similarity transformations \(\tilde{\Omega} = a\) and \(\tilde{\Omega} = a^2\) as well as the possibility that also \(Y = ZX\) is a viable CP transformation that leads to the same results for the mixing in order to reduce the twelve different types of \((Q = a, Z, X)\) to only three. We start with

\[ Q = a , \quad Z = c^{n/2} \quad \text{and} \quad X = c^dP_{23}. \]

Taking \(\tilde{\Omega} = a\) we find

\[ \tilde{Z} = d^{n/2} \quad \text{and} \quad \tilde{X} = c^{s'}d^{t'}P_{23} \quad \text{with} \quad s' = n - t , \quad t' = s - t \]

with \(s'\) and \(t'\) taking all possible values between \(0\) and \(n - 1\). If we use instead \(\tilde{\Omega} = a^2\), we see that

\[ \tilde{Z} = (cd)^{n/2} \quad \text{and} \quad \tilde{X} = c^{s'}d^{t'}P_{23} \quad \text{with} \quad s' = -s + t , \quad t' = n - s \]

with again \(s'\) and \(t'\) taking all possible values between \(0\) and \(n - 1\). Obviously, \(\tilde{Q} = a\) in these two (and in the following) cases. Next we study

\[ Q = a , \quad Z = c^{n/2} \quad \text{and} \quad X = abcd^{2s}P_{23}. \]

Again, we first take \(\tilde{\Omega} = a\) and find

\[ \tilde{Z} = d^{n/2} \quad \text{and} \quad \tilde{X} = a^2bcd^{t'}P_{23} \quad \text{with} \quad s' = 2(n - s) , \quad t' = n - s \]

so that \(s' = 2t'\) as required. For \(\tilde{\Omega} = a^2\), on the other hand, we get

\[ \tilde{Z} = (cd)^{n/2} \quad \text{and} \quad \tilde{X} = bcd^{s}d^{t'}P_{23} \quad \text{with} \quad s' = s , \quad t' = n - s \]

and thus \(t' = n - s'\) as needed. We, similarly, find that starting with

\[ Q = a , \quad Z = bcd^{m}d^{n} \quad \text{and} \quad X = bc^dP_{23}. \]
the application of $\tilde{\Omega} = e$ leads to
\[
\tilde{Z} = abc^{m'} \quad \text{and} \quad \tilde{X} = abc^{s'}d^tP_{23} \quad \text{with} \quad m' = n - m, \quad s' = s, \quad t' = 2s
\]
and thus $t' = 2s'$. Applying $\tilde{\Omega} = a^2$ instead gives rise to
\[
\tilde{Z} = a^2bd^{m'} \quad \text{and} \quad \tilde{X} = a^2bc^{s'}d^tP_{23} \quad \text{with} \quad m' = n - m, \quad s' = 2(n - s), \quad t' = n - s
\]
and so that $s' = 2t'$. Furthermore, we can relate
\[
Q = a, \quad Z = b^m\quad \text{and} \quad X = c^s d^t P_{23} \quad \text{with} \quad t = n - 2m - s
\]
via the transformation $\tilde{\Omega} = e$ to
\[
\tilde{Q} = a, \quad \tilde{Z} = abc^{m'} \quad \text{and} \quad \tilde{X} = c^{s'}d^tP_{23} \quad \text{with} \quad m' = n - m, \quad s' = n - t, \quad t' = s - t
\]
with $t' = 2(m' + s')$ being fulfilled as well as via the transformation $\tilde{\Omega} = a^2$ to
\[
\tilde{Q} = a, \quad \tilde{Z} = a^2bd^{m'} \quad \text{and} \quad \tilde{X} = c^{s'}d^tP_{23} \quad \text{with} \quad m' = n - m, \quad s' = t - s, \quad t' = n - s
\]
so that $s' = 2(m' + t')$. Eventually, we notice that the case
\[
Q = a, \quad Z = b^m d^m \quad \text{and} \quad X = c^s d^t P_{23}
\]
can be related to another case by exploiting that $Y = ZX$ can function as CP transformation
\[
Y = ZX = c^{s'}d^tP_{23} \quad \text{with} \quad s' = s - m, \quad t' = n - s - m
\]
so that $t' = n - 2m - s'$. Obviously, $Q = a$ and $Z = b^m d^m$ remain untouched. This allows us to recover the combination
\[
Z = b^m d^m \quad \text{and} \quad X = c^s d^t P_{23} \quad \text{with} \quad t = n - 2m - s
\]
So, starting with twelve different allowed combinations ($Q = a, Z, X$) we end up with three, namely
\[
Q = a, \quad Z = c^{n/2} \quad \text{and} \quad X = c^s d^t P_{23}
\]
\[
Q = a, \quad Z = c^{n/2} \quad \text{and} \quad X = abcd^{2s} P_{23}
\]
\[
Q = a, \quad Z = bc^m d^m \quad \text{and} \quad X = bc^d^{n-s} P_{23}
\]
for which we study the lepton mixing.
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