LARGE-TIME BEHAVIOR OF SOLUTIONS TO THE BIPOLAR QUANTUM EULER-POISSON SYSTEM WITH CRITICAL TIME-DEPENDENT OVER-DAMPING

Qiwei Wu*

Department of Mathematics, Shanghai University
Shanghai 200444, China

(Communicated by Renjun Duan)

Abstract. We shall investigate the large-time behavior of solutions to the Cauchy problem for the one-dimensional bipolar quantum Euler-Poisson system with critical time-dependent over-damping. By means of the time-weighted energy method, we prove that the smooth solutions to the Cauchy problem exist uniquely and globally, and time-asymptotically converge to the nonlinear diffusion waves when the initial perturbation around the nonlinear diffusion waves are small enough. Particularly, we show the optimal decay rates of solutions toward the nonlinear diffusion waves.

1. Introduction. In this paper, we are interested in the following one-dimensional bipolar quantum Euler-Poisson system with time-dependent damping

\[
\begin{align*}
    n_{1t} + J_{1x} &= 0, \\
    J_{1t} + \left( \frac{j_2}{n_1} + p(n_1) \right) - n_1 \left( \frac{\sqrt{n_1} \lambda}{\sqrt{n_1}} \right)_x &= n_1 E - \frac{\mu}{(1+t)^\lambda} J_1, \\
    n_{2t} + J_{2x} &= 0, \\
    J_{2t} + \left( \frac{j_2}{n_2} + p(n_2) \right) - n_2 \left( \frac{\sqrt{n_2} \lambda}{\sqrt{n_2}} \right)_x &= -n_2 E - \frac{\mu}{(1+t)^\lambda} J_2, \\
    E_x &= n_1 - n_2.
\end{align*}
\]

The unknown functions \( n_i = n_i(x,t) > 0, J_i = J_i(x,t)(i = 1,2) \) represent the particles densities, the current densities and the electric field, respectively. \( p(n_i)(i = 1,2) \) are the pressure-density functions. The terms \( \frac{\mu}{(1+t)^\lambda} J_i(i = 1,2) \) with physical coefficients \( \lambda \in \mathbb{R}, \mu > 0 \) are called the time-dependent damping effects, which play the key role for the regularity of the solutions as well as the large-time behavior. For \( \lambda > 0 \), the damping effects in (1) become time-gradually-degenerate, and is asymptotically vanishing as \( t \to \infty \). This case is called the under-damping. For \( \lambda < 0 \), the damping effects in (1) become time-gradually-enhancing, and is asymptotically enhancing to infinity as \( t \to \infty \). This case is called the over-damping.

2020 Mathematics Subject Classification. Primary: 35Q35, 35B40; Secondary: 76W05.

Key words and phrases. Bipolar quantum Euler-Poisson system, critical over-damping, large-time behavior, smooth solutions, nonlinear diffusion waves.

The author is supported in part by Science and Technology Commission of Shanghai Municipality (Grant No. 20JC1413600).

*Corresponding author: Qiwei Wu.
For the case that $\lambda = 0$, system (1) is the bipolar quantum hydrodynamic model [6, 8] which is used to model and simulate the charged particles transport in ultra-small sub-micron semiconductor devices, for instance, resonant tunneling diodes, where the quantum effects arise [2, 5, 16] and cannot be simulated by classical hydrodynamic models. In recent years, many progress have been made on the bipolar quantum Euler-Poisson system. Unterreiter [30] first studied the multi-dimensional stationary model in bounded domain. Liang and Zhang [24] studied the existence and asymptotic limits of the steady-state solutions in a bounded domain of $\mathbb{R}(1 \leq d \leq 3)$ . Zhang and Zhang [35] first proved the existence and uniqueness of thermal equilibrium solution to the multi-dimensional model in the whole space, and then studied the semi-classical limit and a combined Plank-Debye length limit. Zhang, et al [34] studied the local and global existence of strong solution to the Cauchy problem and analyzed the semiclassical and relaxation limits in $\mathbb{R}^3$, later they investigated the optimal decay rates of the strong solutions in [20]. Li [21, 22] studied the Global existence and asymptotic behavior of the smooth solutions to the 1-D bipolar quantum hydrodynamic model in the whole line and half line, respectively, and proved that the smooth solutions converge to the corresponding nonlinear diffusion waves in both cases. Hu, et al [10] considered the initial-value problem to a 1-D bipolar quantum hydrodynamic model, first established the existence and uniqueness of the stationary solution to the corresponding stationary model, then obtained the exponentially asymptotic stability of the stationary solution and the semi-classical limit. Hu, et al [11] studied the stationary solutions to the 1-D bipolar quantum hydrodynamic model with general doping profile. For classical bipolar hydrodynamic model, we refer the interesting readers, however not limit, to [3, 7, 13, 15, 14, 23, 17, 26] and the reference therein.

When $\lambda \neq 0$, the over-damping and under-damping make the structure of solutions become more complex and challenging, and the results are very limit. For the bipolar classical Euler-Poisson equations with time-dependent damping, Li, et al [18] studied the Cauchy problem for $-1 < \lambda < 1, \mu > 0$ when the pressure functions are identical. Later, Wu, et al studied the case that two pressure functions are different and the doping profile is non-zero in [33]. Luan, et al [25] studied the Cauchy problem for critical under-damping case $\lambda = 1, \mu > 2$. Wu and Li [32] investigated the initial boundary value problem for $-1 < \lambda < 1, \mu > 0$. For the bipolar quantum Euler-Poisson system with time-dependent damping, Wu and Li [31] proved the global existence of smooth solutions to the Cauchy problem and showed the optimal converge rates of the smooth solutions toward the nonlinear diffusion waves.

As a subsequent study of [31], we will investigate the critical over-damping case $\lambda = -1$ in this article, which is more interesting and challenging. Without loss of generality, we set $\mu = 1$, then system (1) can be reduced to

$$\begin{cases}
    n_{1t} + J_{1x} = 0,
    J_{1t} + \left( \frac{J_{1}^2}{n_{1}} + p(n_{1}) \right)_x - n_{1} \left( \frac{(\sqrt{n_{1}})_x}{\sqrt{n_{1}}} \right)_x = n_{1}E - (1 + t)J_{1}, \\
    n_{2t} + J_{2x} = 0,
    J_{2t} + \left( \frac{J_{2}^2}{n_{2}} + p(n_{2}) \right)_x - n_{2} \left( \frac{(\sqrt{n_{2}})_x}{\sqrt{n_{2}}} \right)_x = -n_{2}E - (1 + t)J_{2}, \\
    E_x = n_{1} - n_{2},
\end{cases}$$

(2)

where $t_0$ is a given positive constant. We prescribe the initial data and far field conditions as follows
\[
\begin{cases}
(n_1, n_2, J_1, J_2)(x, t_0) = (n_{10}, n_{20}, J_{10}, J_{20})(x) \to (n_\pm, n_\pm, J_\pm, J_\pm) \text{ as } x \to \pm \infty, \\
E(-\infty, t) = E^-(t) = 0.
\end{cases}
\]

Here \(n_i(x) > 0\) \((i = 1, 2)\) are assumed to satisfy
\[
\int_\mathbb{R} (n_{10}(x) - n_{20}(x))dx = 0,
\]
which is a technical one, and we will discuss the more general case in the future. \(n_\pm > 0\) and \(J_\pm\) are given constants.

As in [18, 19], the asymptotic profiles the solutions to IVP (2)–(3) are expected to be the solutions to the following system
\[
\begin{cases}
\dot{n}_t + \dot{J}_x = 0, \\
p(n)x = -(1 + t)\dot{J}, \quad (x, t) \in \mathbb{R} \times (t_0, +\infty), \\
(n, J) \to (n_\pm, 0) \quad \text{as} \quad x \to \pm \infty.
\end{cases}
\]

As shown in [4], when \(p'(s) > 0\) for \(s > 0\), this system possesses a unique self-similar solution in the form of
\[
(n, J)(x, t) = (n, J)(\frac{x}{\sqrt{\ln(1 + t)}}),
\]
which is the so called the nonlinear diffusion wave.

Before stating our main results, we first give some notations which will be used throughout this paper. The symbol \(C > 0\) denotes a generic constant, which is independent of time, and \(C_i > 0\) \((i = 1, 2, \cdots)\) stands for some specific constant. \(L^p(\mathbb{R})\) \((1 \leq p < +\infty)\) represents the space of measurable functions whose \(p\)-powers are integrable on \(\mathbb{R}\), with the norm \(\| \cdot \|_{L^p} = (\int_{\mathbb{R}} |pdx|)^{\frac{1}{p}}\). For the special case that \(p = 2\), we simply denote \(\| \cdot \|_{L^2} = \| \cdot \|\). And \(L^\infty(\mathbb{R})\) is the space of essentially bounded measurable functions on \(\mathbb{R}\), with the norm \(\| \cdot \|_{L^\infty} = \text{ess sup}_{x \in \mathbb{R}} |\cdot|\). Moreover, for a nonnegative integer \(m\), \(H^m(\mathbb{R})\) denotes the Hilbert space with the norm \(\| \cdot \|_m\), especially \(\| \cdot \|_0 = \| \cdot \|\). We also denote \((f_1, f_2, \cdots, f_k)^{m} := \|f_1\|_{m}^{2} + \|f_2\|_{m}^{2} + \cdots + \|f_k\|_{m}^{2}\). Furthermore, let \(T > t_0\), we denote by \(C([t_0, T]; H^m(\mathbb{R}))\) (resp. \(L^2([t_0, T]; H^m(\mathbb{R}))\)) the space of continuous (resp. square integrable) functions on \([t_0, T]\) with values taken in \(H^m(\mathbb{R})\).

The main purpose in this paper is to study the asymptotic behavior of solutions to the IVP (2)–(3), and the main results are stated as follows.

**Theorem 1.1.** Let \(p(s)\) be a smooth function with \(p'(s) > 0\) for \(s > 0\). Assume that \((\varphi_{10}, \eta_{10}) \in H^5(\mathbb{R}) \times H^3(\mathbb{R})\) \((i = 1, 2)\). Set \(\Phi_0 := \| (\varphi_{10}, \varphi_{20}) \|_5 + \| (\eta_{10}, \eta_{20}) \|_3\) and \(\delta_0 := |n_+ - n_-| + |J_+| + |J_-|\). Then, there exists a sufficiently small constant \(\varepsilon_0 > 0\) such that if \(\Phi_0 + \delta_0 \leq \varepsilon_0\), then IVP (2)–(3) has a unique global smooth solution \((n_1, n_2, J_1, J_2, E)(x, t)\), which satisfies
\[
\sum_{k=0}^{2} \ln^{k+1} (1 + t) \| \partial^k_x (n_1 - \bar{n}) n_2 - \bar{n} - \bar{n}) (t) \|^2 \\
+ \ln^3 (1 + t) \sum_{k=3}^{4} \| \partial^k_x (n_1 - \bar{n}) n_2 - \bar{n} - \bar{n}) (t) \|^2
\]
electric field. But in the critical over-damping case of the Klein-Gordon equation, and the authors drive the exponential decay rate of the electric field shown in [31] as well as [18], the electric field is optimal when the initial perturbation around the nonlinear diffusion waves are small enough, one can see that through analyze the corresponding linearized equations. Remarks 1.2.

The details can be found in [19]. Under the assumptions of Theorem 1.1, it holds
\[ \|n_1 - \hat{n}, n_2 - \hat{n}\|(t) \leq C(\Phi_0^2 + \delta_0)^{\frac{3}{2}} \ln^{-\frac{3}{2}}(1 + t), \]
\[ \|J_1 - \hat{J}, J_2 - \hat{J}\|(t) \leq C(\Phi_0^2 + \delta_0)^{\frac{1}{2}} (1 + t)^{-\frac{1}{2}} \ln^{-\frac{1}{2}}(1 + t), \]
and
\[ \|E(t)\| \leq C(\Phi_0^2 + \delta_0)^{\frac{3}{2}} (1 + t)^{-1}. \]

**Corollary 1.** Under the assumptions of Theorem 1.1, it holds
\[ \|n_1 - \hat{n}, n_2 - \hat{n}\|(t) \leq C(\Phi_0^2 + \delta_0)^{\frac{3}{2}} \ln^{-\frac{3}{2}}(1 + t), \]
\[ \|J_1 - \hat{J}, J_2 - \hat{J}\|(t) \leq C(\Phi_0^2 + \delta_0)^{\frac{1}{2}} (1 + t)^{-\frac{1}{2}} \ln^{-\frac{1}{2}}(1 + t), \]
\[ \|E(t)\| \leq C(\Phi_0^2 + \delta_0)^{\frac{3}{2}} (1 + t)^{-1}. \]

**Remarks 1.2.**

1. The decay rates of the solutions shown in Theorem 1.1 and Corollary 1 are optimal when the initial perturbation around the nonlinear diffusion waves are small enough, one can see that through analyze the corresponding linearized equations. The details can be found in [19].

2. The main difficulties in the proof of Theorem 1.1 are as follows. The first is the coupling and interaction between the two carriers. For the case that $-1 < \lambda < 1$ shown in [31] as well as [18], the electric field $E$ satisfies the time-dependent damped “Klein-Gordon” equation, and the authors drive the exponential decay rate of the electric field. But in the critical over-damping case $\lambda = -1$, the electric field $E$ is no longer the time-exponentially decaying and is just time-algebraically decay, which leads us cannot ignore the effect of the electric field. Next, due to the decay rates of the nonlinear diffusion waves are in logarithmic form, the method that constructing the polynomial time weights used in [18, 32, 31] is no longer applicable in this case. To overcome this difficulty, inspired by [19], we technically construct logarithmic time weights when establishing the a-priori estimates. The third is the additional dispersion terms. More precisely, the third order nonlinear terms $f_{14x}$ and $f_{24x}$ shown in (23) can not be controlled by the left-hand side terms of (23) when applying the energy method to construct the a-priori estimates. To overcome this difficulty, we need to introduce new variables (25) and (29), and reformulated the original problem to (30)–(31), then $f_{14x}$ and $f_{24x}$ can be controlled by the left-hand side terms of (30).
3. We need to mention that for the critical under-damping case $\lambda = 1$, the system (1) exhibit different mathematical structures, and the asymptotic behavior of the solutions may different from what we study here. For time-dependent damped Euler and classical bipolar Euler-Poisson equations in critical under-damping case that $\lambda = 1$, one can refer to \[1, 9, 25, 28, 29\]. Next, we also should mention that for the corresponding general case that two pressure functions are different and the doping profile is non-zero, the asymptotic profiles of the solutions are expected to be the corresponding stationary waves rather than the nonlinear diffusion waves, which is more physical and interesting, but more challenging. These are expected to be studied in the forthcoming papers.

The rest of this paper is organized as follows. In Section 2, we first investigate the behavior of solutions at far fields $x = \pm \infty$, then we construct some correction functions to delete the gaps between the original solutions and the nonlinear diffusion waves. We reformulate the original problem in terms of the perturbation variables in Section 3. Section 4 is devoted to establish the a-priori estimates by means of the energy method, which is the key part of this article.

2. Preliminaries. We first investigate the behavior of solutions at far fields $x = \pm \infty$, then we construct some correction functions to delete the gaps between the original solutions and the nonlinear diffusion waves. Set

$$f^{\pm}(t) := f(\pm \infty, t), \quad f \in \{n_1, n_2, J_1, J_2, E\}.$$ 

On the one hand, integrating (2)$_5$ with respect to $x$ over $\mathbb{R}$, then taking $t = t_0$, we have

$$E^+(t_0) = \int_\mathbb{R} (n_{10}(x) - n_{20}(x)) dx = 0. \quad (5)$$

On the other hand, differentiating (2)$_5$ in $t$ and integrating the resultant equation respect to $x$ over $\mathbb{R}$ gives

$$\frac{d}{dt} E^+(t) = -(J^+_1(t) - J^+_2(t)) + (J^-_1(t) - J^-_2(t)), \quad \frac{d}{dt} E^+(t_0) = 0, \quad (6)$$

which together with (5) leads to

$$E^+(t) = \int_{t_0}^t \left[ (J^-_1(s) - J^-_2(s) - (J^+_1(s) - J^+_2(s))) \right] ds. \quad (7)$$

Then, as $x \to \pm \infty$, (2)-(3) can be reduced to

$$\begin{cases}
\frac{d}{dt} n^+_i(t) = 0, \text{ i.e., } n^+_i(t) = n_{i\pm}, \\
\frac{d}{dt} J^+_i(t) = (-1)^{i+1} n_+ E^+(t) - (1 + t) J^+_i(t), \\
\frac{d}{dt} J^-_i(t) = -(1 + t) J^-_i(t), \text{ i.e., } J^-_i(t) = J^-_i e^{\frac{1}{2} \left[ (1 - (1 + t)^2) \right]} , \\
E^+(t) = \int_{t_0}^t \left[ (J^+_1(\tau) - J^+_2(\tau)) + (J^-_1(\tau) - J^-_2(\tau)) \right] d\tau, \\
(n^\pm_i, J^\pm_i, E^+)(0) = (n_{i\pm}, J_{i\pm}, E_+), \quad (8)
\end{cases}$$

for $i = 1, 2$. By (8)$_2$, we have

$$\frac{d}{dt} (J^+_1(t) - J^+_2(t)) = 2n_+ E^+(t) - (1 + t) \left( J^+_1(t) - J^+_2(t) \right). \quad (9)$$
It follows from (7), (8) and (9) that
\[
\begin{aligned}
\frac{d}{dt} E^+(t) + (1 + t) \frac{d}{dt} E^+(t) + 2n_+ E^+(t) &= 0, \\
E^+(t_0) &= 0, \\
\frac{d}{dt} E^+(t_0) &= 0.
\end{aligned}
\]  
(10)

It is clear that (10) is well-posed, and it has a unique solution
\[
E^+(t) = 0, 
\]
(11)

which together with (8) yields
\[
J^+_1(t) = J^+_2(t) = J_+ e^{\frac{1}{2} (1 + t)^2}.
\]  
(12)

To delete the gaps between the original solutions and the nonlinear diffusion waves at far fields, we need to construct some correction functions. For this, we define
\[
\hat{n}(x, t) := (J_+ - J_-)m_0(x) \int_t^\infty e^{\frac{1}{2} (1 + s)^2} ds,
\]  
(13)

\[
\hat{J}(x, t) := J_0 e^{\frac{1}{2} (1 + t)^2} + (J_+ - J_-) e^{\frac{1}{2} (1 + t)^2} \int_{-\infty}^x m_0(y) dy.
\]  
(14)

Here \(m_0(x)\) is chosen as
\[
m_0(x) \geq 0, \quad m_0(x) \in C^\infty_0(\mathbb{R}), \quad \text{supp} m_0(x) \subset [-L_0, L_0], \quad \int_{-\infty}^{+\infty} m_0(x) dx = 1,
\]
(15)

with some constant \(L_0 > 0\). One can verify the \((\hat{n}, \hat{J})(x, t)\) satisfies
\[
\begin{aligned}
\hat{n}_t + \hat{J}_x &= 0, \\
\hat{J}_t &= -(1 + t) \hat{J}, \\
\hat{n}(\pm \infty, t) &= 0, \quad \hat{J}(\pm \infty, t) = J^\pm(t).
\end{aligned}
\]  
(16)

3. The reformulated problems. Since it is convenient to regard the solution \((n_1, n_2, J_1, J_2, E)(x, t)\) of the IVP (2)–(3) as the perturbation of \((\bar{n}, \bar{n}, \bar{J}, \bar{J}, 0)\), we are going to reformulate the original problem in terms of the perturbation variables in this section. To begin with, from (2), (4) and (16), we see
\[
\begin{aligned}
(n_1 - \bar{n} - \hat{n})_t + (J_1 - \bar{J} - \hat{J})_x &= 0, \\
(J_1 - \bar{J} - \hat{J})_t + \left(\frac{f_1^2}{n_1} + p(n_1) - p(\bar{n})\right)_x - n_1 \left(\frac{\sqrt{n_1}}{\sqrt{n_1}}\right)_x \\
&= n_1 E - (1 + t) (J_1 - \bar{J} - \hat{J}) + f_0, \\
(n_2 - \bar{n} - \hat{n})_t + (J_2 - \bar{J} - \hat{J})_x &= 0, \\
(J_2 - \bar{J} - \hat{J})_t + \left(\frac{f_2^2}{n_2} + p(n_2) - p(\bar{n})\right)_x - n_2 \left(\frac{\sqrt{n_2}}{\sqrt{n_2}}\right)_x \\
&= -n_2 E - (1 + t) (J_2 - \bar{J} - \hat{J}) + f_0, \\
E_x &= (n_1 - \bar{n} - \hat{n}) - (n_2 - \bar{n} - \hat{n}).
\end{aligned}
\]  
(17)

Here \(f_0 := (1 + t)^{-1} p(\bar{n})_x - (1 + t)^{-2} p(\bar{n})_x\), and \((\bar{n}, \bar{J}) = (\bar{n}, \bar{J})(x + x_0, t)\) is the shifted diffusion wave, where \(x_0 \in \mathbb{R}\) is a constant which will be determined as
follows. Due to
\[
\frac{d}{dt} \int_R (n_i(x,t) - \bar{n}(x + x_0, t) - \hat{n}(x, t)) dx
\]
\[
= \int_R (-J_{ix}(x, t) + \bar{J}_2(x + x_0, t) + \hat{J}_x(x, t)) dx
\]
\[
= 0,
\]
then it is trivial that
\[
\int_R (n_i(x, t) - \bar{n}(x + x_0, t) - \hat{n}(x, t)) dx = \int_R (n_{i0}(x) - \bar{n}(x + x_0, t_0) - \hat{n}(x, t_0)) dx.
\]
Noting that \(\int_R (n_{10}(x) - n_{20}(x)) dx = 0\), then we take
\[
h(y) = \int_R (n_{10}(x) - \bar{n}(x + y, t_0) - \hat{n}(x, t_0)) dx,
\]
and \(x_0\) is selected such that \(h(x_0) = 0\). Since \(h'(y) = -\int_R \bar{n}_2(x + y, t_0) dx = -(n_+ - n_-)\), then
\[
h(x_0) = h(0) + \int_0^{x_0} h'(y) dy = h(0) - (n_+ - n_-)x_0,
\]
which gives, with \(h(x_0) = 0\), that
\[
x_0 = \frac{1}{n_+ - n_-} \int_R (n_{10}(x) - \bar{n}(x, t_0) - \hat{n}(x, t_0)) dx,
\]
thus
\[
\int_R (n_1(x, t) - \bar{n}(x + x_0, t) - \hat{n}_1(x, t)) dx = \int_R (n_2(x, t) - \bar{n}(x + x_0, t) - \hat{n}_2(x, t)) dx = 0.
\]
Now we take the perturbation variables \((\varphi_1, \varphi_2, \eta_1, \eta_2)(x, t)\) as
\[
(\varphi_i, \eta_i) := \left( \int_{-\infty}^x (n_i(y, t) - \bar{n}(y + x_0, t) - \hat{n}(y, t)) dy, J_i(x, t) - \bar{J}(x + x_0, t) - \hat{J}(x, t)) \right),
\]
for \(i = 1, 2\). Next, using the fact that
\[
n_i \left( \frac{(\sqrt{m_i})^2}{\sqrt{m_i}} \right)_x = \frac{1}{2} n_{ixxx} - \frac{1}{2} \left( \frac{n_x^2}{n_i} \right)_x, \quad i = 1, 2,
\]
we get the following equations for perturbation variables
\[
\begin{cases}
\varphi_{1t} + \eta_1 = 0, \\
\eta_{1t} + \left( \frac{n_1 + J_x}{\varphi_1x + n \bar{n}} \right)_x + p(\varphi_{1x} + \bar{n} + \hat{n}) - p(\bar{n}) \left( \frac{n_x^2}{\varphi_1x + n \bar{n}} \right)_x - \frac{1}{2} (\varphi_{1xxx} + \bar{n}_{xxx} + \hat{n}_{xxx})
\end{cases}
\]
\[
+ \frac{1}{2} \left( \frac{(\varphi_{1xx} + \bar{n}_{xx} + \hat{n}_{xx})^2}{\varphi_{1x} + n \bar{n}} \right)_x = (\varphi_{1x} + \bar{n} + \hat{n}) E - (1 + t) \eta_1 + f_0,
\]
\[
\varphi_{2t} + \eta_2 = 0,
\]
\[
\eta_{2t} + \left( \frac{n_2 + J_x}{\varphi_2x + n \bar{n}} \right)_x + p(\varphi_{2x} + \bar{n} + \hat{n}) - p(\bar{n}) \left( \frac{n_x^2}{\varphi_2x + n \bar{n}} \right)_x - \frac{1}{2} (\varphi_{2xxx} + \bar{n}_{xxx} + \hat{n}_{xxx})
\]
\[
+ \frac{1}{2} \left( \frac{(\varphi_{2xx} + \bar{n}_{xx} + \hat{n}_{xx})^2}{\varphi_{2x} + n \bar{n}} \right)_x = -(\varphi_{2x} + \bar{n} + \hat{n}) E - (1 + t) \eta_2 + f_0,
\]
\[
E = \varphi_1 - \varphi_2,
\]
with the initial data
\[
(\varphi_1, \varphi_2, \eta_1, \eta_2)(x, 0) = (\varphi_{10}, \varphi_{20}, \eta_{10}, \eta_{20}),
\]
where
\[
\varphi_{10}(x) := \int_0^x (n_{10}(y) - \bar{n}(y + x_0, 0) - \hat{n}(y, 0)) \, dy,
\]
\[
\eta_{10}(x) := J_{10}(x) - \tilde{f}(x + x_0, 0) - \tilde{J}(x, 0)
\]
for \(i = 1, 2\). Further, substituting \((21)_1\) into \((21)_2\), and \((21)_3\) into \((21)_4\), respectively, we obtain the following IVP for \(\varphi_1\) and \(\varphi_2\) that
\[
\begin{cases}
\varphi_{1tt} + (1 + t)\varphi_{1t} + \frac{1}{2}\phi_{1xxx} - (p'((\bar{n})\varphi_{1x})_x + \bar{n}E = -\varphi_{1x} + \hat{n})E - f_1 + f_{12x} + f_{13x} + f_{14x}, \\
\varphi_{2tt} + (1 + t)\varphi_{2t} + \frac{1}{2}\phi_{2xxx} - (p'((\bar{n})\varphi_{2x})_x - \bar{n}E = (\varphi_{2x} + \hat{n})E - f_1 + f_{22x} + f_{23x} + f_{24x}, \\
E = \varphi_1 - \varphi_2,
\end{cases}
\]
with
\[
(\varphi_1, \varphi_2)(x, 0) = (\varphi_{10}, \varphi_{20})(x), \quad (\varphi_{1t}, \varphi_{2t})(x, 0) = (-\eta_{10}, -\eta_{20})(x).
\]
Here
\[
f_1 = (1 + t)^{-1}p((\bar{n})_{xt} - (1 + t)^{-2}p((\bar{n})_x + \frac{1}{2}(\bar{n}_{xxx} + \hat{n}_{xxx}),
\]
and for \(i = 1, 2\)
\[
f_{i2} := \frac{(-\varphi_{it} + \tilde{J} + \tilde{f})^2}{\varphi_{ix} + \bar{n} + \hat{n}}, \quad f_{i3} := p(\varphi_{ix} + \bar{n} + \hat{n}) - p((\bar{n}) - p'((\bar{n})\varphi_{ix}),
\]
\[
f_{i4} := \frac{1}{2} (\varphi_{ixx} + \bar{n}_{xx} + \hat{n}_{xx})^2.
\]
Furthermore, due to the additional dispersion term, it is difficult to estimate \((\varphi_1, \varphi_2, E)\) from \((23)-(24)\) directly. More precisely, the third order nonlinear terms \(f_{14x}\) and \(f_{24x}\) in \((23)\) can not be controlled by the left hand side terms. To overcome this difficulty, we introduce the new variables
\[
w_1 = \sqrt{n_1}, \quad w_2 = \sqrt{n_2}.
\]
Then, from \((2)-(3)\), we have
\[
\begin{cases}
2w_1w_{1tt} + J_{1t} = 0, \\
J_{1t} + \left(\frac{J_{1x}^2}{w_1^2} + p(w_1^2)\right)_x - w_1^2 \left(\frac{w_{1xx}}{w_1^2}\right)_x = w_1^2 E - (1 + t)J_1, \\
2w_2w_{2tt} + J_{2t} = 0, \\
J_{2t} + \left(\frac{J_{2x}^2}{w_2^2} + p(w_2^2)\right)_x - w_2^2 \left(\frac{w_{2xx}}{w_2^2}\right)_x = -w_2^2 E - (1 + t)J_2, \\
E_x = w_1^2 - w_2^2,
\end{cases}
\]
with the initial data
\[
(w_1, w_2, J_1, J_2) = (\sqrt{n_{10}}, \sqrt{n_{20}}, J_{10}, J_{20})(x).
\]
Further, substituting (26) into (26), and (26) into (26), respectively, we have

\[
\begin{align*}
\psi_{1tt} + (1 + t)\psi_{1t} + \frac{1}{2}\psi_{1xxx} &= -\frac{w_1^2}{w_1} - \frac{1}{2w_1}(w_1^2 E)_x \\
+ \frac{1}{2w_1} \left( \frac{\rho^2}{w_1^2} + p(\phi_1^2) \right)_{xx} + \frac{w_1^2}{2w_1}, \\
\psi_{2tt} + (1 + t)\psi_{2t} + \frac{1}{2}\psi_{2xxx} &= -\frac{w_2^2}{w_2} + \frac{1}{2w_2}(w_2^2 E)_x \\
+ \frac{1}{2w_2} \left( \frac{\rho^2}{w_2^2} + p(\phi_2^2) \right)_{xx} + \frac{w_2^2}{2w_2}.
\end{align*}
\]

Moreover, denoting \( \bar{n} = \sqrt{n} \), and introducing

\[
\psi_1(x, t) = w_1(x, t) - \bar{n}(x + x_0, t), \quad \psi_2(x, t) = w_2(x, t) - \bar{n}(x + x_0, t),
\]

which together with (28) leads to the following IVP for \( \psi_1 \) and \( \psi_2 \)

\[
\begin{align*}
\psi_{1tt} + (1 + t)\psi_{1t} + \frac{1}{2}\psi_{1xxx} + \frac{1}{2}\bar{n}E_x &= -\bar{n}_{tt} - \frac{1}{2}\bar{n}_{xxxx} - g_{11} - g_{12} + g_{13} + g_{14}, \\
\psi_{2tt} + (1 + t)\psi_{2t} + \frac{1}{2}\psi_{2xxx} - \frac{1}{2}\bar{n}E_x &= -\bar{n}_{tt} - \frac{1}{2}\bar{n}_{xxxx} - g_{21} + g_{22} + g_{23} + g_{24}, \\
E_x &= (\psi_1 + \psi_2 + 2\bar{n})(\psi_1 - \psi_2),
\end{align*}
\]

with the initial data

\[
\begin{align*}
\psi_1(x, 0) &= \psi_{10}(x) := \sqrt{n_{10}(x)} - \sqrt{n(x + x_0, 0)}, \\
\psi_2(x, 0) &= \psi_{20}(x) := \sqrt{n_{20}(x)} - \sqrt{n(x + x_0, 0)}, \\
\psi_{1t}(x, 0) &= \psi_{11}(x) := -\frac{J_{10}(x)}{2\sqrt{n_{10}(x)}} + \frac{J_{1}(x + x_0, 0)}{2\sqrt{n(x + x_0, 0)}}, \\
\psi_{2t}(x, 0) &= \psi_{21}(x) := -\frac{J_{20}(x)}{2\sqrt{n_{20}(x)}} + \frac{J_{1}(x + x_0, 0)}{2\sqrt{n(x + x_0, 0)}},
\end{align*}
\]

where for \( i = 1, 2 \)

\[
\begin{align*}
g_{11} &:= \frac{(\psi_{tt} + \bar{n})^2}{\psi + \bar{n}}, \quad g_{22} := (\psi_{tt} + \bar{n})E + \frac{1}{2}\psi E, \\
g_{33} &:= \frac{p((\psi + \bar{n})^2)_{xx}}{2(\psi + \bar{n})} - \frac{p(\bar{n})_{xx}}{2\bar{n}} + \frac{1}{2(\psi + \bar{n})} \left( \frac{(-\varphi_{tt} + \tilde{J} + \tilde{J})^2}{(\psi + \bar{n})^2} \right)_{xx}, \\
g_{44} &:= \frac{(\psi_{xx} + \bar{n})^2}{2(\psi + \bar{n})}.
\end{align*}
\]

Here we have used the fact that

\[-(1 + t)\bar{n}_t = -(1 + t)\frac{\bar{n}_t}{2\bar{n}}, \quad -(1 + t)\frac{(1 + t)^{-1}p(\bar{n})_{xx}}{2\bar{n}} = -\frac{p(\bar{n})_{xx}}{2\bar{n}}.\]

**Remark 1.** The definitions of \( \psi_1, \psi_2 \) imply that

\[
\psi_i = \frac{\varphi + \bar{n}_i}{\sqrt{n_i} + \sqrt{n}}, \quad i = 1, 2.
\]

Now we are going to state our main results on new variables.
Theorem 3.1. Under the assumptions of Theorem 1.1, IVP (23)–(24) and (30)–(31) has a unique global smooth solution \((\varphi_1, \varphi_2, \psi_1, \psi_2, E)(x, t)\), which satisfies

\[
\sum_{k=0}^{3} \ln^k (1 + t) \| \partial_x^k (\varphi_1, \varphi_2, \psi_1, \psi_2)(t) \|^2 + \ln^3 (1 + t) \| \partial_x^3 (\varphi_1, \varphi_2, \psi_1, \psi_2)(t) \|^2 + \sum_{k=0}^{1} (1 + t)^2 \ln^{k+2} (1 + t) \| \partial_x^k (\varphi_{1t}, \varphi_{2t})(t) \|^2 + \sum_{k=0}^{1} (1 + t)^2 \ln^{k+1} (1 + t) \| \partial_x^k (\psi_{1t}, \psi_{2t})(t) \|^2 + (1 + t)^2 \ln^2 (1 + t) \| (\varphi_{1tt}, \varphi_{2tt}, \psi_{1tt}, \psi_{2tt})(t) \|^2 \leq C(\Phi_0^2 + \delta_0),
\]

and

\[
(1 + t) \| E(t) \|_2^2 + (1 + t) \| E_t(t) \|_1^2 + (1 + t)^2 \ln^2 (1 + t) \| (\partial_x^2 E, E_{tt})(t) \|_1^2 \leq C(\Phi_0^2 + \delta_0).
\]

Proof of Theorem 1.1. Once Theorem 3.1 is proved, then by the relations (20), (21)_1, (21)_3, and (32), Theorem 1.1 is immediately obtained.

To prove Theorem 3.1, we shall employ the standard continuation argument based on the local existence and the \textit{a-priori} estimates. The local existence of the smooth solution \((n_1, n_2, J_1, J_2, E)(x, t)\) to IVP (2)–(2) can be obtained by Galerkin method together with iterative argument (refer to [12, 27, 34] for details). Then from the relations (20), (21)_1, (21)_3, (25), and (29), we know that IVP (23)–(24) and (30)–(31) has a unique local smooth solution \((\varphi_1, \varphi_2, \psi_1, \psi_2, E)(x, t)\). Thus, the main effort in the rest of this article is to establish the \textit{a-priori} estimates for the solution \((\varphi_1, \varphi_2, \psi_1, \psi_2, E)(x, t)\). For this, we define the solution space for \(T > t_0\) as follows

\[
X(T) := \{ (\varphi_1, \varphi_2, \psi_1, \psi_2, E)(x, t) \mid \partial_t^i \varphi_i \in C([t_0, T]; H^{5-2j}(\mathbb{R})), \partial_t^i \psi_i \in C([t_0, T]; H^{4-2j}(\mathbb{R})), \partial_t^i E \in C([t_0, T]; H^{5-2j}(\mathbb{R})), i = 1, 2, j = 0, 1, 2, \}
\]

and introduce

\[
N(T)^2 := \sup_{t_0 \leq t \leq T} \left\{ \sum_{i=1}^{2} \left[ \sum_{k=0}^{3} \ln^k (1 + t) \| \partial_x^k (\varphi_i, \psi_i)(t) \|^2 + \ln^3 (1 + t) \| \partial_x^3 (\varphi_i, \psi_i)(t) \|^2 + \sum_{k=0}^{1} (1 + t)^2 \ln^{k+2} (1 + t) \| \partial_x^k \varphi_{it}(t) \|^2 + \sum_{k=0}^{1} (1 + t)^2 \ln^{k+1} (1 + t) \| \partial_x^k \psi_{it}(t) \|^2 + (1 + t)^2 \ln^2 (1 + t) \| (\varphi_{1tt}, \psi_{1tt})(t) \|^2 \right] \right\},
\]

Then, to prove Theorem 3.1, it is sufficiently to prove the following \textit{a-priori} estimates for the solutions to IVP (23)–(24) and (30)–(31)
Proposition 1. For $T > t_0$, let $(\varphi_1, \varphi_2, \psi_1, \psi_2, E)(x,t)$ be the solution to the IVP (23)–(24) and (30)–(31) in the time interval $[t_0, T]$, then there exists a sufficiently small constant $\varepsilon_1 > 0$ such that if $N(T) + \delta \leq \varepsilon_1$, then

$$
\sum_{k=0}^{3} \ln^k(1 + t)\|\partial_x^k(\varphi_1, \varphi_2, \psi_1, \psi_2)(t)\|^2 \\
+ \ln^3(1 + t)\|\partial_x^3(\varphi_1, \varphi_2, \psi_1, \psi_2)(t)\|^2 \\
+ \sum_{k=0}^{1} (1 + t)^2\ln^{k+2}(1 + t)\|\partial_x^k(\varphi_{1t}, \varphi_{2t})(t)\|^2 \\
+ \sum_{k=0}^{1} (1 + t)^2\ln^{k+1}(1 + t)\|\partial_x^k(\psi_{1t}, \psi_{2t})(t)\|^2 \\
+ (1 + t)^2\ln^2(1 + t)\|\partial_x^2(\varphi_{1t}, \varphi_{2t}, \psi_{1t}, \psi_{2t})(t)\|^2 \\
+ (1 + t)^2\ln^2(1 + t)\|\varphi_{1t}, \varphi_{2t}, \psi_{1t}, \psi_{2t})(t)\|^2 \leq C(\Phi_0^2 + \delta_0),
$$

and

$$(1 + t)\|E(t)\|^2 + (1 + t)^3\|E_0(t)\|^2 + (1 + t)^2\ln^2(1 + t)\|\partial_x^2E_0, E_t(t)\|^2 \leq C(\Phi_0^2 + \delta_0).$$

4. Proof of Proposition 3.3. In this section, we are going to prove Proposition 3.3. To begin with, we list the decay estimates of the nonlinear diffusion wave $(\bar{n}, \bar{J})$ and $\tilde{w}$ as follows (see [19]).

Lemma 4.1. It holds that

$$
\begin{cases}
\min\{n_+ + n_-, n_+\} \leq \bar{n}(x,t) \leq \max\{n_+ + n_-, n_+\}, \\
\min\{\sqrt{n_+}, \sqrt{n_-}\} \leq \tilde{w}(x,t) \leq \max\{\sqrt{n_+}, \sqrt{n_-}\}, \\
\|\partial_x^n(\bar{n}, \bar{w})(t)\|^2 \leq C|n_+ + n_-|^2\ln^{-\frac{2k+1}{2}}(1 + t), k \geq 1, \\
\|\partial_x^n(\tilde{n}, \tilde{w})(t)\|^2 \leq C|n_+ + n_-|^2(1 + t)^{-2}\ln^{-\frac{2k+2}{2}}(1 + t), k \geq 0, l \geq 1, \\
\|\partial_x^n(\tilde{J})(t)\|^2 \leq C|n_+ + n_-|^2(1 + t)^{-2l}\ln^{-\frac{2k+1}{2}}(1 + t), k, l \geq 0.
\end{cases}
$$

Furthermore,

$$
\begin{cases}
\|\partial_x^n(\bar{n}, \bar{w})(t)\|_{L^\infty} \leq C|n_+ + n_-|\ln^{-\frac{k}{2}}(1 + t), k \geq 1, \\
\|\partial_x^n(\tilde{n}, \tilde{w})(t)\|_{L^\infty} \leq C|n_+ + n_-|(1 + t)^{-l}\ln^{-\frac{k+2}{2}}(1 + t), k \geq 0, l \geq 1, \\
\|\partial_x^n(\tilde{J})(t)\|_{L^\infty} \leq C|n_+ + n_-|^2(1 + t)^{-l}\ln^{-\frac{k+1}{2}}(1 + t), k, l \geq 0.
\end{cases}
$$

It is easy to see that owing to the Sobolev inequality $\|f\|_{L^\infty} \leq C\|f\|^\frac{1}{2}\|f_x\|^\frac{1}{2}$, if $N(T) + \delta_0 \ll 1$, there then exists some constants $C_1, C_2 > 0$ such that

$$
0 < \frac{1}{C_1} \leq \varphi_{ix} + \bar{n} + \bar{n} \leq C_1, 0 < \frac{1}{C_2} \leq \psi + \tilde{w} \leq C_2, i = 1, 2.
$$

Moreover, from the a-priori assumptions (33) and the Sobolev inequality as well as the relation (32), we have for $i = 1, 2$
\[
\sum_{k=0}^{2} \ln^{\frac{k+1}{2}} \left( 1 + t \right) \| \partial_x^k \varphi(t) \|_{L^\infty} + \ln^{\frac{3}{2}} \left( 1 + t \right) \| (\partial_x^2 \varphi, \partial_x^2 \varphi_1)(t) \|_{L^\infty} \\
+ \left( 1 + t \right) \ln^{\frac{1}{2}} \left( 1 + t \right) \| (\varphi_{it}, \varphi_{iext})(t) \|_{L^\infty} + (1 + t) \ln \left( 1 + t \right) \| (\partial_x^2 \varphi_{it}, \varphi_{iext})(t) \|_{L^\infty} \\
+ \sum_{k=1}^{2} \ln^{\frac{k+1}{2}} \left( 1 + t \right) \| \partial_x^k \psi(t) \|_{L^\infty} + \ln^{\frac{3}{2}} \left( 1 + t \right) \| (\partial_x^2 \psi, \partial_x^2 \psi_1)(t) \|_{L^\infty} \\
+ \left( 1 + t \right) \ln^{\frac{1}{2}} \left( 1 + t \right) \| \psi_{it}(t) \|_{L^\infty} + (1 + t) \ln \left( 1 + t \right) \| \partial_x \psi_{it}(t) \|_{L^\infty} \\
\leq C(N(T) + \delta_0). \\
\tag{34}
\]

and
\[
\sum_{k=0}^{3} \ln^{\frac{k+1}{2}} \left( 1 + t \right) \| \partial_x^k (n_x, w_1)(t) \|_{L^\infty} + (1 + t) \ln^{\frac{1}{2}} \left( 1 + t \right) \| J_i(t) \|_{L^\infty} \\
+ \left( 1 + t \right) \ln \left( 1 + t \right) \| (n_x, w_x, J_x, J_{xx})(t) \|_{L^\infty} \leq C(N(T) + \delta_0). \tag{35}
\]

Now we are going to drive the estimates for the electric field \(E\). For this, subtracting \((23)_2\) from \((23)_1\) and using \((23)_3\), we get the following IVP for \(E\)

\[
E_{tt} + (1 + t)E_t + \frac{1}{2} E_{xxxx} - (p'(\bar{n})E_x)_x + 2\bar{n}E = F_1, \\
\tag{36}
\]

with

\[
E(x, 0) = (\varphi_{10} - \varphi_{20})(x), \ E_t(x, 0) = (-\eta_{10} + \eta_{20})(x). \\
\tag{37}
\]

Here

\[
F_1 := -(\varphi_{1x} + \varphi_{2x} + 2\bar{n})E + (f_{12} - f_{22})_x + (f_{13} - f_{23})_x + (f_{14} - f_{24})_x.
\]

Due to the additional dispersion term, it is difficult to estimate \(E\) from \((36)-(37)\) directly. For this, we set

\[
\chi = \psi_1 - \psi_2, \\
\tag{38}
\]

then subtracting \((30)_2\) from \((30)_1\), and using the relation

\[
\chi = \frac{E_x}{w_1 + w_2}, \\
\tag{39}
\]

we get

\[
\chi_{tt} + (1 + t)\chi_t + \frac{1}{2} \chi_{xxxx} + \bar{w}(w_1 + w_2)\chi = G_1, \\
\tag{40}
\]

with

\[
\chi(x, 0) = (\psi_{10} - \psi_{20})(x), \ \chi_t(x, 0) = (\psi_{11} - \psi_{21})(x). \\
\tag{41}
\]

Here

\[
G_1 := -(g_{11} - g_{21}) - (g_{12} + g_{22}) + (g_{13} - g_{23}) + (g_{14} - g_{24}).
\]

**Lemma 4.2.** There exists a sufficiently small constant \(\varepsilon_2 > 0\) such that if \(N(T) + \delta \leq \varepsilon_2\), then it holds

\[
(1 + t)\|E(t)\|_3^3 + (1 + t)^3\|E_t(t)\|_3^2 + (1 + t)\| (\partial_x^2 E_t, E_{tt})(t) \|_3^2 \leq C(\Phi_0 + \delta_0).
\]

**Proof.** We divide the proof into three steps.
Step1. Multiplying (36) by $2E_t + (1 + t)^{-1} E$, then integrating it with respect to $x$ over $\mathbb{R}$, we have

\[
\frac{d}{dt} \int_{\mathbb{R}} \left( E_t^2 + \frac{1}{2} E_{xx}^2 + p'(\bar{n})E_x^2 + (1 + t)^{-1} E E_t + 2\bar{n}E^2 \right) dx \\
+ \frac{1}{2} (1 + t)^{-2} E^2 + \frac{1}{2} E^2 dx \\
+ \int_{\mathbb{R}} \left[ (1 + t) E_t^2 + \frac{1}{2} (1 + t)^{-1} E_{xx}^2 + (1 + t)^{-1} p'(\bar{n})E_x^2 
+ (1 + t)^{-3} E^2 + 2\bar{n}(1 + t)^{-1} E^2 \right] dx = 
\int_{\mathbb{R}} (p''(\bar{n})\bar{n}_t E_x^2 + 2\bar{n}_t E^2) dx \\
+ \int_{\mathbb{R}} (2E_t + (1 + t)^{-1} E) \left[ - (\varphi_{1x} + \varphi_{2x} + 2\bar{n}) E - (f_{12} - f_{22})_x \right. \\
+ (f_{13} - f_{23})_x + (f_{14} - f_{24})_x \right] dx. \tag{42}
\]

The right-hand side of (42) can be estimated as follows. First, employing Cauchy-Schwartz’s inequality and using Lemma 4.1, (13) and (34), we have

\[
\int_{\mathbb{R}} (p''(\bar{n})\bar{n}_t E_x^2 + 2\bar{n}_t E^2) dx + \int_{\mathbb{R}} (2E_t + (1 + t)^{-1} E) \left[ - (\varphi_{1x} + \varphi_{2x} + 2\bar{n}) E \right] dx \\
\leq C(\|\bar{n}_t(t)\|_{L^\infty}) \int_{\mathbb{R}} (E^2 + E_x^2) dx \\
+ C(\|\varphi_{1x}(t)\|_{L^\infty} + \|\varphi_{2x}(t)\|_{L^\infty} + \|\bar{n}_t(t)\|_{L^\infty}) \int_{\mathbb{R}} (|E| E_t + (1 + t)^{-1} |E|^2) dx \\
\leq C(N(T) + \delta_0) \int_{\mathbb{R}} (1 + t)^{-1} (E^2 + E_x^2 + (1 + t) E_t^2) dx. \tag{43}
\]

Second, taking integration by parts and using (34), (13) and (14), we have

\[
\int_{\mathbb{R}} (2E_t + (1 + t)^{-1} E) (f_{12} - f_{22}) dx \\
\leq \int_{\mathbb{R}} (2E_t + (1 + t)^{-1} E) \left[ - \frac{J_2^2}{n_1^2} E_{xx} - 2J_1 E_{xt} \right] dx \\
+ C(N(T) + \delta_0) (1 + t)^{-1} \int_{\mathbb{R}} \left( |E_t| + (1 + t)^{-1} |E| \right) \left( |\bar{n}_x| + |J_x| \right) + |E_t| + |E_x| dx \\
\leq \frac{d}{dt} \int_{\mathbb{R}} \frac{J_2^2}{n_1^2} E_x^2 dx + C(N(T) + \delta_0) \int_{\mathbb{R}} (1 + t)^{-1} (E^2 + E_x^2) + (1 + t) E_t^2 dx \\
+ C\delta_0 e^{-\tau}. \tag{44}
\]

Next, taking integration by parts and using the mean value theorem, we have

\[
\int_{\mathbb{R}} (2E_t + (1 + t)^{-1} E) (f_{13} - f_{23}) dx \\
= \int_{\mathbb{R}} (2E_t + (1 + t)^{-1} E) \left( (p'(n_1) - p'(\bar{n})) E_{xx} + (p'(n_1) - p'(n_2))(\varphi_{2xx} + \bar{n}_x) \right) dx \\
\leq - \frac{d}{dt} \int_{\mathbb{R}} (p'(n_1) - p'(\bar{n})) E_x^2 dx + C(N(T) + \delta_0) \int_{\mathbb{R}} (1 + t)^{-1} (E^2 + E_x^2) \\
+ (1 + t) E_t^2 dx - \int_{\mathbb{R}} (p'(n_1) - p'(\bar{n}))(1 + t)^{-1} E_x^2 dx. \tag{45}
\]
Finally, using the fact that
\[(f_{14} - f_{24})_x = 4w_1w_{1xx} - 4w_2w_{2xx} = \frac{2n_1x}{w_1} \chi_{xx} + \left(\frac{2n_2xx}{w_2} - \frac{n^2_2}{w^2_2}\right) \chi_x,\]
then by Cauchy-Schwartz’s inequality and (34), we see
\[
\int_{\mathbb{R}} (2E_t + (1 + t)^{-1}E)(f_{14} - f_{24})_x \, dx
\leq C(N(T) + \delta_0) \int_{\mathbb{R}} ((1 + t)^{-1}(E^2 + \chi_x^2 + \chi_{xx}^2) + (1 + t)E_t^2) \, dx. \tag{46}
\]
Substituting (43)–(45) into (42), we obtain
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( E_t^2 + \frac{1}{2} E_{xx}^2 + \left(p'(n_1) - \frac{j_1^2}{n_1^2}\right) E_x^2 + (1 + t)^{-1} E_t E_x + 2\tilde{n} E^2\right) \, dx
+ \frac{1}{2} (1 + t)^{-2} E_t^2 + \frac{1}{2} E_x^2 \right) \, dx + \int_{\mathbb{R}} \left[ (1 + t) E_t^2 + \frac{1}{2} (1 + t)^{-1} E_{xx}^2 \right) \, dx
+ (1 + t)^{-1} E \cdot \left( \chi \right) E_x \right) \, dx
\leq C(N(T) + \delta_0) \int_{\mathbb{R}} ((1 + t)^{-1}(E^2 + E_x^2 + \chi_x^2 + \chi_{xx}^2) + (1 + t)E_t^2) \, dx
+ C \delta_0 e^{-Ct^2}. \tag{47}
\]
Next, multiplying (40) by $2 \chi_t + (1 + t)^{-1} \chi$, then integrating the resultant equation with respect to $x$ over $\mathbb{R}$ yields
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \chi_t^2 + \frac{1}{2} \chi_{xx}^2 + (1 + t)^{-1} \chi \chi_t + \frac{1}{2} \bar{\omega}(w_1 + w_2) \chi^2 \right) \, dx
+ \frac{1}{2} (1 + t)^{-2} \chi_t^2 + \frac{1}{2} \chi_{xx}^2 \right) \, dx + \int_{\mathbb{R}} \left[ (1 + t) \chi_t^2 + \frac{1}{2} (1 + t)^{-1} \chi_{xx}^2 \right) \, dx
+ (1 + t)^{-3} \chi^2 + (1 + t)^{-1} \bar{\omega}(w_1 + w_2) \chi^2 \right) \, dx
= \frac{1}{2} \int_{\mathbb{R}} \left[ \bar{\omega}_t(w_1 + w_2) + \bar{\omega}(w_{1t} + w_{2t}) \right] \chi^2 \, dx
\leq \int_{\mathbb{R}} \left( 2 \chi_t + (1 + t)^{-1} \chi \right) \left[ - (g_{11} - g_{21}) - (g_{12} + g_{22}) \right) \, dx
+ (g_{13} - g_{23}) + (g_{14} - g_{24}) \right) \, dx. \tag{48}
\]
The right-hand side of (48) can be estimated as follows. First, as in (43), we have
\[
\frac{1}{2} \int_{\mathbb{R}} \left( \bar{\omega}_t(w_1 + w_2) + \bar{\omega}(w_{1t} + w_{2t}) \right) \chi^2 \, dx
\leq C(N(T) + \delta_0) \int_{\mathbb{R}} ((1 + t)^{-1}(E^2 + E_x^2 + \chi^2) + (1 + t)\chi_t^2) \, dx. \tag{49}
\]
Next, noting that
which together with (47) leads to

\[
g_{13} - g_{23} = \left( \frac{J_1^2}{w_1 n_1} - \frac{J_2^2}{w_2 n_2} \right) + \left( \frac{J_1 J_{1xx}}{w_1 n_1} - \frac{J_2 J_{2xx}}{w_2 n_2} \right)
- \left( \frac{2 J_1 J_{1xx} n_{1xx}}{w_1 n_1^2} - \frac{J_2 J_{2xx} n_{2xx}}{w_2 n_2^2} \right) - \left( \frac{p''(n_1) n_{1xx}^2}{2 w_1} - \frac{p''(n_2) n_{2xx}^2}{2 w_2} \right) + \left( \frac{p'(n_1) n_{1xx}}{2 w_1} - \frac{p'(n_2) n_{2xx}}{2 w_2} \right),
\]

then by a tedious but in a similar way as (44) and (45), we can show

\[
\frac{1}{2} \int_R (2 \chi_t + (1 + t)^{-1} \chi)(g_{13} - g_{23}) dx
\leq - \frac{d}{dt} \int_R \left( \chi^2 + \frac{1}{2} \chi_x^2 + (p' - n_1) \chi_x^2 + (1 + t)^{-1} \chi \chi_t + \frac{1}{2} \hat{\omega}(w_1 + w_2) \chi^2 \right)
+ \frac{1}{2} (1 + t)^{-2} \chi^2 + \frac{1}{2} \chi_x^2 \right) dx + \int_R \left( (1 + t) \chi_t^2 + \frac{1}{2} (1 + t)^{-1} \chi_x^2 \right) dx
+ (1 + t)^{-1} p'(n_1) \chi_x^2 + (1 + t)^{-3} \chi^2 + (1 + t)^{-1} \hat{\omega}(w_1 + w_2) \chi^2 \right] dx
\leq C(N(T) + \delta_0) \int_R ((1 + t)^{-1} (E_t^2 + \chi_t^2) + (1 + t) E_t^2 + \chi_t^2 + \chi_x^2 + \chi_{xx}^2) dx,
\]

Finally, as in (44), we can prove

\[
\frac{1}{2} \int_R (2 \chi_t + (1 + t)^{-1} \chi)(g_{14} - g_{24}) dx
\leq C(N(T) + \delta_0) \int_R ((1 + t)^{-1} (\chi_x^2 + \chi_{xx}^2) + (1 + t) \chi_t^2) dx.
\]

Substituting (49)–(51) into (48), we have

\[
\frac{d}{dt} \int_R \left( \chi_t^2 + \chi_x^2 + \chi_{xx}^2 + (p' - n_1) \chi_x^2 + (1 + t)^{-1} \chi \chi_t + \frac{1}{2} \hat{\omega}(w_1 + w_2) \chi^2 \right)
+ \frac{1}{2} (1 + t)^{-2} \chi^2 + \frac{1}{2} \chi_x^2 \right) dx + \int_R \left( (1 + t) \chi_t^2 + \frac{1}{2} (1 + t)^{-1} \chi_x^2 \right) dx
+ (1 + t)^{-1} p'(n_1) \chi_x^2 + (1 + t)^{-3} \chi^2 + (1 + t)^{-1} \hat{\omega}(w_1 + w_2) \chi^2 \right] dx
\leq C(N(T) + \delta_0) \int_R ((1 + t)^{-1} (E_t^2 + \chi_t^2) + (1 + t) E_t^2 + \chi_t^2 + \chi_x^2 + \chi_{xx}^2) dx,
\]

which together with (47) leads to

\[
\frac{d}{dt} \int_R \left( (E_t^2 + \chi_t^2) + \frac{1}{2} (E_{xx}^2 + \chi_{xx}^2) + (p'(n_1) - \frac{J_1^2}{n_1^2}) (E_{xx}^2 + \chi_{xx}^2) \right)
+ (1 + t)^{-1} (E_t^2 + \chi_t^2) + 2 \hat{\omega}(w_1 + w_2) \chi^2 + (1 + t)^{-2} \chi^2 \right) dx + \int_R \left( (1 + t)^{-1} (E_{xx}^2 + \chi_{xx}^2) \right) dx
+ (1 + t)^{-1} p'(n_1) (E_{xx}^2 + \chi_{xx}^2) + (1 + t)^{-3} (E_t^2 + \chi_t^2) + (1 + t)^{-1} (2 \hat{\omega} E_t)
+ \hat{\omega}(w_1 + w_2) \chi^2 \right] dx \leq C(N(T) + \delta_0) \int_R ((1 + t)^{-1} (E_t^2 + \chi_t^2 + \chi_{xx}^2) dx + (1 + t) E_t^2 dx + C \delta_0 e^{-Ct^2}.
\]
Lemma 4.3. There exists a sufficiently small constant $0 < \varepsilon_3 < \varepsilon_2$ such that if $N(T) + \delta \leq \varepsilon_3$, then it holds for $i = 1, 2$

\[
\left\| (\varphi_i, \psi_i)(t) \right\|^2 + \ln(1 + t) \left\| (\varphi_{ix}, \varphi_{it}, \varphi_{ixx}, \psi_{ix}, \psi_{it}, \psi_{ixx})(t) \right\|^2 \\
+ \int_{t_0}^{t} (1 + s)^{-3} \left\| (\varphi_i, \psi_i)(s) \right\|^2 ds + \int_{t_0}^{t} (1 + s)^{-1} \left\| (\varphi_{ix}, \varphi_{ixx}, \psi_{ix}, \psi_{ixx})(s) \right\|^2 ds \\
+ \int_{t_0}^{t} (1 + s) \ln(1 + s) \left\| (\varphi_{it}, \psi_{it})(s) \right\|^2 ds \leq C(\Phi_0^2 + \delta_0).
\]

Proof. We divide the proof into two steps.
Step 1. Multiplying (23) by 2$\varphi_{1t} + (1 + t)^{-1}\varphi_1$, then integrating the resultant equations respect to $x$ over $\mathbb{R}$ gives
\[
\frac{d}{dt} \int_\mathbb{R} \left( \varphi_{1t}^2 + \frac{1}{2} \varphi_{xx}^2 + p'(\bar{n})\varphi_{1x}^2 + (1 + t)^{-1}\varphi_1\varphi_{1t} + \frac{1}{2} \varphi_1^2 + \frac{1}{2}(1 + t)^{-2}\varphi_1^2 \right) dx
\]
\[
+ \int_\mathbb{R} \left[ (1 + t)^{-3}\varphi_1^2 + (1 + t)^{-1}p'(\bar{n})\varphi_{1x}^2 + (1 + t)\varphi_1^2 + \frac{1}{2}(1 + t)^{-1}\varphi_{1xx}^2 \right] dx
\]
\[
+ \int_\mathbb{R} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)nE dx = \int_\mathbb{R} p''(\bar{n})\bar{n}_1\varphi_{1x}^2 dx
\]
\[
+ \int_\mathbb{R} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1) \left[ - (\varphi_1 + \hat{n})E - f_1 + f_{12x} + f_{13x} + f_{14x} \right] dx. \tag{56}
\]

First, from Lemma 4.1, (13) and (34), we have
\[
\int_\mathbb{R} p''(\bar{n})\bar{n}_1\varphi_{1x}^2 dx + \int_\mathbb{R} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)[- (\varphi_1 + \hat{n})E] dx
\]
\[
\leq C\|\bar{n}_t(t)\|_{L^\infty} \int_\mathbb{R} \varphi_{1x}^2 dx
\]
\[
+ C(\|\varphi_{1x}(t)\|_{L^\infty} + \|\bar{n}(t)\|_{L^\infty}) \int_\mathbb{R} (|\varphi_{1t}E| + (1 + t)^{-1}|\varphi_1E|) dx
\]
\[
\leq C(N(T) + \delta_0) \int_\mathbb{R} ((1 + t)^{-1}(\varphi_{1x}^2 + E^2) + (1 + t)^{-3}\varphi_1^2 + (1 + t)\varphi_{1t}^2) dx, \tag{57}
\]

and
\[
\int_\mathbb{R} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)(-f_1) dx
\]
\[
= - \int_\mathbb{R} 2\varphi_{1t} \left[ (1 + t)^{-1}p''(\bar{n})\bar{n}_x\bar{n}_t + (1 + t)^{-1}p(\bar{n})\bar{n}_{xt} - (1 + t)^{-2}p'(\bar{n})\bar{n}_x \right.
\]
\[
+ \frac{1}{2}(\bar{n}_{xxx} + \bar{n}_{xxxx}) \right] dx + \int_\mathbb{R} \left[ (1 + t)^{-2}\varphi_{1x}p'(\bar{n})\bar{n}_t + (1 + t)^{-3}p'(\bar{n})\bar{n}_x\varphi_1 \right.
\]
\[
+ \frac{1}{2}(1 + t)^{-1}\varphi_{1x}(\bar{n}_{xx} + \bar{n}_{xxxx}) \right] dx
\]
\[
\leq C\delta_0 \int_\mathbb{R} ((1 + t)\varphi_{1t}^2 + (1 + t)^{-1}\varphi_{1x}^2 + (1 + t)^{-3}\varphi_1^2) dx
\]
\[
+ C\delta_0 \int_\mathbb{R} \left[ (1 + t)^{-3}(|\bar{n}_x\bar{n}_t|^2 + |\bar{n}_{xt}|^2 + |\bar{n}_t|^2 + |\bar{n}_x|^2) + (1 + t)^{-5}|\bar{n}_x|^2 \right.
\]
\[
+ (1 + t)^{-1}(|\bar{n}_{xxx}|^2 + |\bar{n}_{xxxx}|^2 + |\bar{n}_{xx}|^2 + |\bar{n}_{xxxx}|^2) \right] dx
\]
\[
\leq C\delta_0 \int_\mathbb{R} ((1 + t)^2\varphi_{1t}^2 + (1 + t)^{-1}\varphi_{1x}^2 + (1 + t)^{-3}\varphi_1^2) dx
\]
\[
+ C\delta_0 (1 + t)^{-1}ln^{-\frac{1}{2}}(1 + t). \tag{58}
\]

Next, taking integrating by parts and using (35), we see
\[
\int_\mathbb{R} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)f_{12x} dx
\]
\[
= \int_\mathbb{R} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1) \left[ \frac{2J_t}{n_1}(-\varphi_{1xt} + \hat{J}_x + \hat{J}_x) \right.
\]
\[
\leq C\delta_0 \int_\mathbb{R} ((1 + t)^{-1}\varphi_{1t}^2 + (1 + t)^{-3}\varphi_{1x}^2 + (1 + t)^{-1}\varphi_1^2) dx
\]
\[
+ C\delta_0 (1 + t)^{-1}ln^{-\frac{1}{2}}(1 + t). \tag{59}
\]
Moreover, employing Taylor’s formula, we obtain

\[
\int_R (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)f_{13x}dx
\]

\[
= \int_R 2\varphi_{1t} \left[ (p'(n_1) - p'(\bar{n}))\varphi_{1xx} + (p'(n_1) - p'(\bar{n}))\bar{n}_x + p'(n_1)\bar{n}_x - p''(\bar{n})\bar{n}_x\varphi_{1x} \right]dx - \int_R (1 + t)^{-1}\varphi_{1x}(p(n_1) - p(\bar{n}) - p'(\bar{n})\varphi_{1x})dx
\]

\[
\leq -\frac{d}{dt} \int_R (p'(n_1) - p'(\bar{n}))\varphi_{1x}^2dx + C(N(T) + \delta_0) \int_R (1 + t)^{-1}\varphi_{1x}^2 + (1 + t)\varphi_{1t}^2)dx + C\delta_0e^{-Ct^2}. \quad (60)
\]

Finally, using the fact that \( f_{14x} = 4w_{1x}w_{1xx} \), then

\[
\int_R (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)f_{14x}dx
\]

\[
\leq C(N(T) + \delta_0) \int_R ((1 + t)^{-1}(\varphi_{1x}^2 + \varphi_{1xx}^2 + \psi_{1xx}^2) + (1 + t)\varphi_{1t}^2)dx
\]

\[
+C\delta_0(1 + t)^{-1}\ln^{-\frac{3}{2}}(1 + t). \quad (61)
\]

Substituting (57)–(61) into (56), we obtain

\[
\frac{d}{dt} \int_R \left( \varphi_{1t}^2 + \frac{1}{2}\varphi_{1xx}^2 + (p'(n_1) - \frac{J_2}{n_1^2})\varphi_{1x}^2 + (1 + t)^{-1}\varphi_{1t}\varphi_{1x} \right)
\]

\[
+ \left( \frac{1}{2}\varphi_{1t}^2 + \frac{1}{2}(1 + t)^{-2}\varphi_{1x}^2 \right) dx + \int_R \left[ (1 + t)^{-3}\varphi_{1x}^2 + (1 + t)^{-1}p'(\bar{n})\varphi_{1x}^2 \right]
\]

\[
+ (1 + t)\varphi_{1t}^2 + \frac{1}{2}(1 + t)^{-1}\varphi_{1xx}^2 \right] dx + \int_R (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)\bar{n}E dx
\]

\[
\leq C(N(T) + \delta_0) \int_R \left( (1 + t)^{-3}\varphi_{1x}^2 + (1 + t)\varphi_{1t}^2 \right)
\]

\[
+ (1 + t)^{-1}((\varphi_{1x}^2 + \varphi_{1xx}^2 + \psi_{1xx}^2 + E^2)) dx + C\delta_0(1 + t)^{-1}\ln^{-\frac{3}{2}}(1 + t). \quad (62)
\]

Next, multiplying (23) by \( 2\varphi_{2t} + (1 + t)^{-1}\varphi_2 \), then analysis similar to above, we have

\[
\frac{d}{dt} \int_R \left( \varphi_{2t}^2 + \frac{1}{2}\varphi_{2xx}^2 + (p'(n_2) - \frac{J_2}{n_2^2})\varphi_{2x}^2 + (1 + t)^{-1}\varphi_{2t}\varphi_{2x} + \frac{1}{2}\varphi_{2t}^2 \right)
\]

\[
+ \left( \frac{1}{2}(1 + t)^{-2}\varphi_{2x}^2 \right) dx + \int_R \left[ (1 + t)^{-3}\varphi_{2x}^2 + (1 + t)^{-1}p'(\bar{n})\varphi_{2x}^2 + (1 + t)\varphi_{2t}^2 \right]
\]

\[
+ \frac{1}{2}(1 + t)^{-1}\varphi_{2xx}^2 \right] dx - \int_R (2\varphi_{2t} + (1 + t)^{-1}\varphi_2)\bar{n}E dx
\]

\[
\leq C(N(T) + \delta_0) \int_R ((1 + t)^{-3}\varphi_{2x}^2 + (1 + t)\varphi_{2t}^2
\]
\[\int_{\mathbb{R}} (2\varphi_{1t} + (1 + t)^{-1}\varphi_1)\bar{n}Edx - \int_{\mathbb{R}} (2\varphi_{2t} + (1 + t)^{-1}\varphi_2)\bar{n}Edx\]
\[= \frac{d}{dt} \int_{\mathbb{R}} \bar{n}E^2 dx + \int_{\mathbb{R}} (1 + t)^{-1}\bar{n}E^2 dx - \int_{\mathbb{R}} \bar{n}_t E^2 dx\]
\[\geq \frac{d}{dt} \int_{\mathbb{R}} \bar{n}E^2 dx + \int_{\mathbb{R}} (1 + t)^{-1}\bar{n}E^2 dx - C\delta_0 \int_{\mathbb{R}} (1 + t)^{-1}E^2 dx. \quad (64)\]

Now adding (63) to (62), and using (64), then integrating the resultant equation over \((t_0, t)\) and using the smallness of \(N(T) + \delta_0\), we get for \(i = 1, 2\)
\[\|(\varphi_i, \varphi_{ix}, \varphi_{it}, \varphi_{ixx}, E)(t)\|^2 + \int_{t_0}^{t} (1 + s)^{-3}||\varphi_i(s)||^2 ds + \int_{t_0}^{t} (1 + s)||\varphi_{it}(s)||^2 ds\]
\[+ \int_{t_0}^{t} (1 + s)^{-1}||\varphi_{ixx}(s)||^2 ds\]
\[\leq C(N(T) + \delta_0) \int_{t_0}^{t} (1 + s)^{-1}||\psi_{ixx}(s)||^2 ds + C(\Phi_0^2 + \delta_0). \quad (65)\]

On the other hand, multiplying (30)\(_1\) by \(2\psi_{1t} + (1 + t)^{-1}\psi_1\), then integrating the resultant equation with respect to \(x\) over \(\mathbb{R}\), we have
\[\frac{d}{dt} \int_{\mathbb{R}} \left[ \psi_{1t}^2 + \frac{1}{2} \psi_{1xx}^2 + (1 + t)^{-1}\psi_{1x}^2 + \frac{1}{2} \psi_{1t}^2 + (1 + t)^{-2} \psi_{1t}^2 \right] dx\]
\[+ \int_{\mathbb{R}} \left[ (1 + t)^{-3} \psi_{1t}^2 + (1 + t)^{-1} \psi_{1x}^2 + \frac{1}{2} (1 + t)^{-1} \psi_{1t}^2 \right] dx\]
\[+ \int_{\mathbb{R}} \bar{w}E_x (2\varphi_{1t} + (1 + t)^{-1}\varphi_1) dx\]
\[= \int_{\mathbb{R}} (2\psi_{1t} + (1 + t)^{-1}\psi_1) \left( - \bar{w}_{tt} - \frac{1}{2} \bar{w}_{xxxx} - g_{11} - g_{12} + g_{13} + g_{14} \right) dx. \quad (66)\]

First, as in (57) and (59), we see
\[\int_{\mathbb{R}} (2\psi_{1t} + (1 + t)^{-1}\psi_1) \left( - \bar{w}_{tt} - \frac{1}{2} \bar{w}_{xxxx} - g_{11} - g_{12} + g_{14} \right) dx\]
\[\leq C(N(T) + \delta_0) \int_{\mathbb{R}} ((1 + t)^{-3} \psi_{1t}^2 + (1 + t) \psi_{1t}^2)\]
\[+ (1 + t)^{-1} (\psi_{1x}^2 + \psi_{1xx}^2 + E^2 + E_x^2) dx + C\delta_0 (1 + t)^{-1} \ln^{-\frac{3}{2}} (1 + t). \quad (67)\]

Next, taking integration by parts, we have
\[\int_{\mathbb{R}} 2\psi_{1t} g_{13} dx\]
\[= - \int_{\mathbb{R}} (2p'(w_t^2)w_{1x} - 2p'(\bar{w}^2)\bar{w}_x)\psi_{1xt} dx\]
\[+ \int_{\mathbb{R}} \left[ \frac{w_{1x} n_{1x}}{n_1} p'(n_1) - \frac{\bar{w}_x \bar{n}_x}{\bar{n}} p'(\bar{n}) \bar{n}'(\bar{n}) \right] \psi_{1t} dx\]
\[- \int_{\mathbb{R}} \frac{1}{w_1} \left( \frac{2J_1 J_{1x}}{w_1^2} - \frac{2J_1^2 w_{1x}}{w_1^3} \right) \psi_{1xt} dx + \int_{\mathbb{R}} \frac{w_{1x}}{w_1^2} \left( \frac{2J_1 J_{1x}}{w_1^2} - \frac{2J_1^2 w_{1x}}{w_1^3} \right) \psi_{1t} dx\]
First, by a tedious computation, utilizing Young’s inequality and mean value theorem, and using (34)–(35), we have

\[ I_1 = -\int_R (2p'(w^2)w_{1x} - 2p'(-\bar{w})w_{1x})\psi_{1x}t\,dx \]

\[ = -\int_R 2p'(n_1)\psi_{1x}t\,dx - \int_R 2(p'(n_1) - p'(\bar{n}))\bar{w}_x\psi_{1x}t\,dx \]

\[ = -\frac{d}{dt} \int_R p'(n_1)\psi_{1x}t\,dx + \int_R p'(n_1)n_{1t}\psi_{1x}^2\,dx + \int_R 2(p'(n_1) - p'(\bar{n}))\bar{w}_{xx}\psi_{1t}\,dx \]

\[ + \int_R 2(p''(n_1)n_{1x} - p''(\bar{n})\bar{n}_x)\bar{w}_x\psi_{1t}\,dx \]

\[ \leq -\frac{d}{dt} \int_R p'(n_1)\psi_{1x}^2\,dx + C(N(t) + \delta_0) \int_R (1 + t)^{-1}(\psi_{1x}^2 + \varphi_{1x}^2 + \varphi_{1xx}^2)\,dx \]

\[ + C\delta_0 e^{-ct^2}, \quad (68) \]

and

\[ I_2 = \int_R \left( \frac{w_{1x}}{n_1} p'(n_1) - \frac{w_x\bar{n}_x}{\bar{n}} p'(\bar{n}) \right) \psi_{1t}\,dx \]

\[ = \int_R \psi_{1t} n_{1x} p'(n_1) + w_x p'(n_1)(\varphi_{1xx} + \bar{n}_x) + \bar{w}_x n_x (p'(n_1) - p'(\bar{n})) \psi_{1t}\,dx \]

\[ - \int_R \frac{w_x \bar{n}_x p'(\bar{n})}{n_1 \bar{n}} (\varphi_{1x} + \bar{n}) \psi_{1t}\,dx \]

\[ \leq C(N(t) + \delta_0) \int_R ((1 + t)^2 + (1 + t)^{-1}(\psi_{1x}^2 + \varphi_{1x}^2 + \varphi_{1xx}^2))\,dx \]

\[ + C\delta_0 e^{-ct^2}. \quad (69) \]

Next, using the fact that \( J_{1x} = -2w_1w_{1t} \), and utilizing (34)–(35), one gets

\[ I_3 = -\int_R \frac{1}{w_1^2} \left( \frac{2J_1J_{1x}}{w_1^2} - \frac{2J_{1x}^2}{w_1^2} \right) \psi_{1x}t\,dx \]

\[ = \int_R \frac{2J_1 J_{1x}}{w_1^2} (\psi_{1t} + \bar{w}_t) \psi_{1x}t\,dx + \int_R \frac{2J_{1x}^2}{w_1^2} (\psi_{1x} + \bar{w}_x) \psi_{1x}t\,dx \]

\[ \leq \frac{d}{dt} \int_R \frac{J_{1x}^2}{w_1^2} \psi_{1x}^2\,dx + C(N(t) + \delta_0) \int_R ((1 + t)^2 + (1 + t)^{-1}\psi_{1x}^2)\,dx \]

\[ + C\delta_0 (1 + t)^{-5} \ln^{-\frac{3}{2}}(1 + t), \quad (71) \]

and

\[ I_4 = \int_R \frac{w_{1x}}{w_1^2} \left( \frac{2J_1 J_{1x}}{w_1^2} - \frac{2J_{1x}^2}{w_1^2} \right) \psi_{1t}t\,dx \]

\[ = -\int_R \frac{4J_1 w_{1x}}{w_1^2} (\psi_{1t} + \bar{w}_t)\,dx - \int_R \frac{2J_{1x}^2}{w_1^2} (\psi_{1x} + \bar{w}_x) \psi_{1t}\,dx \]

\[ \leq C(N(t) + \delta_0) \int_R ((1 + t)^2 + (1 + t)^{-1}\psi_{1x}^2)\,dx \]

\[ + C\delta_0 (1 + t)^{-5} \ln^{-\frac{3}{2}}(1 + t). \quad (72) \]
Combining (68)-(72), we get
\[
\int_{\mathbb{R}} 2\psi_1 g_{13} dx \leq -\frac{d}{dt} \int_{\mathbb{R}} p'(n_1)\psi_{1x}^2 dx + C(N(t) + \delta_0) \int_{\mathbb{R}} ((1 + t)\psi_{1t}^2 + (1 + t)^{-1}(\psi_{1x}^2 + \varphi_{1xx}^2)) dx + C\delta_0(1 + t)^{-5}\ln^{-\frac{3}{2}}(1 + t). \tag{73}
\]
Next, in a similar way, however a little easier, we can prove
\[
\int_{\mathbb{R}} (1 + t)^{-1}\psi_1 g_{13} dx \leq -\frac{1}{2} \int_{\mathbb{R}} (1 + t)^{-1}p'(n_1)\psi_{1x}^2 dx \\
+ C(N(t) + \delta_0) \int_{\mathbb{R}} ((1 + t)\psi_{1t}^2 + (1 + t)^{-1}(\psi_{1x}^2 + \varphi_{1xx}^2)) dx \\
+ C\delta_0(1 + t)^{-2}\ln^{-\frac{3}{2}}(1 + t). \tag{74}
\]
Substituting (67) and (73)-(74) into (66), we have
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \psi_{1t}^2 + \frac{1}{2} \psi_{1xx}^2 + (p'(n_1) - \frac{J_1^2}{n_1^2})\psi_{1x}^2 + (1 + t)^{-1}\psi_1 \psi_{1t} + \frac{1}{2} \psi_1^2 \right) dx \\
+ \frac{1}{2} (1 + t)^{-2} \psi_1^2 dx + \int_{\mathbb{R}} \left[ (1 + t)^{-3} \psi_1^2 + (1 + t)\psi_{1t}^2 + \frac{1}{2} (1 + t)^{-1}(p'(n_1)\psi_{1x}^2 \\
+ \psi_{1xx}^2) \right] dx + \int_{\mathbb{R}} \frac{1}{2} \bar{w} E_x(2\varphi_{1t} + (1 + t)^{-1}\varphi_1) dx \\
\leq C(N(t) + \delta_0) \int_{\mathbb{R}} ((1 + t)^{-3} \psi_1^2 + (1 + t)\psi_{1t}^2 + (1 + t)^{-1}(\psi_{1x}^2 + \varphi_{1xx}^2) \\
+ \varphi_{1x}^2 + \varphi_{1xx}^2) dx + C\delta_0(1 + t)^{-1}\ln^{-\frac{3}{2}}(1 + t). \tag{75}
\]
Next, multiplying (30) by (1 + t)^{-1}\psi_2, and in a completely similar way, we can show
\[
\frac{d}{dt} \int_{\mathbb{R}} \left( \psi_{2t}^2 + \frac{1}{2} \psi_{2xx}^2 + (p'(n_2) - \frac{J_2^2}{n_2^2})\psi_{2x}^2 + (1 + t)^{-1}\psi_2 \psi_{2t} + \frac{1}{2} \psi_2^2 \right) dx \\
+ \frac{1}{2} (1 + t)^{-2} \psi_2^2 dx + \int_{\mathbb{R}} \left[ (1 + t)^{-3} \psi_2^2 + (1 + t)\psi_{2t}^2 + \frac{1}{2} (1 + t)^{-1}(p'(n_2)\psi_{2x}^2 \\
+ \psi_{2xx}^2) \right] dx - \int_{\mathbb{R}} \frac{1}{2} \bar{w} E_x(2\varphi_{2t} + (1 + t)^{-1}\varphi_2) dx \\
\leq C(N(t) + \delta_0) \int_{\mathbb{R}} ((1 + t)^{-3} \psi_2^2 + (1 + t)\psi_{2t}^2 + (1 + t)^{-1}(\psi_{2x}^2 + \varphi_{2xx}^2) \\
+ \varphi_{2x}^2 + \varphi_{2xx}^2) dx + C\delta_0(1 + t)^{-1}\ln^{-\frac{3}{2}}(1 + t). \tag{76}
\]
Moreover, the terms related to the electric field on the left-hand side of (75) and (76) can be estimated as follows.
\[
\frac{1}{2} \bar{w} E_x(2\psi_{1t} + (1 + t)^{-1}\psi_1) dx - \frac{1}{2} \bar{w} E_x(2\psi_{2t} + (1 + t)^{-1}\psi_2) dx \\
= \frac{d}{dt} \int_{\mathbb{R}} \frac{\bar{w}}{2(w_1 + w_2)} E_x^2 dx + \int_{\mathbb{R}} \left( (1 + t)^{-1} \frac{\bar{w}}{2(w_1 + w_2)} E_x^2 dx \\
- \int_{\mathbb{R}} \frac{\bar{w}}{2(w_1 + w_2)} \frac{\bar{w}(\psi_{1t} + \psi_{2t} + 2\psi_{1t})}{2(w_1 + w_2)^2} E_x^2 dx \right) \\
\geq \frac{d}{dt} \int_{\mathbb{R}} \frac{\bar{w}}{2(w_1 + w_2)} E_x^2 dx + \int_{\mathbb{R}} \left( (1 + t)^{-1} \frac{\bar{w}}{2(w_1 + w_2)} E_x^2 dx \right)
\]
Then adding (76) to (75), and using (77), and integrating the resultant equation over \((t_0, t)\) and using the smallness of \(N(T) + \delta\), we get for \(i = 1, 2\)

\[
\|((\psi_i, \psi_{ix}, \psi_{ixx}, E_x)(t))\|^2 + \int_{t_0}^{t} (1 + s)^{-3}\|\psi_i(s)\|^2 ds \\
+ \int_{t_0}^{t} (1 + s)^{-1}\|((\psi_{ix}, \psi_{ixx}, E_x)(s))\|^2 ds \\
\leq C(N(T) + \delta_0) \int_{t_0}^{t} (1 + s)^{-1}((\varphi_{ixx}, \varphi_{ixxx})(s))\|^2 ds + C(\Phi_0^2 + \delta_0),
\]

which together with (65) leads to

\[
\|((\varphi_i, \varphi_{ix}, \varphi_{ixx}, \psi_i, \psi_{ix}, \psi_{ixx}, E, E_x)(t))\|^2 \\
+ \int_{t_0}^{t} (1 + s)^{-3}\|((\varphi_i, \psi_i)(s))\|^2 ds + \int_{t_0}^{t} (1 + s)^{-1}\|((\varphi_{ix}, \psi_{ix})(s))\|^2 ds \\
+ \int_{t_0}^{t} (1 + s)^{-1}\|((\varphi_{ixx}, \varphi_{ixxx}, \psi_{ixx}, \psi_{ixxxx}, E, E_x)(s))\|^2 ds \\
\leq C(\Phi_0^2 + \delta_0), \quad i = 1, 2.
\]

**Step 2.** Using Lemma 4.2, we can obtain the following estimates

\[
\mathcal{J}_{12} = \int_\mathbb{R} (22\ln(1 + t)\varphi_{ix}(\varphi_{ix} + \bar{n} + \hat{n})E + \ln(1 + t)\psi_{ix}nE_x) dx \\
\leq \frac{1}{4} \int_\mathbb{R} (1 + t)(\varphi_{ix}^2 + \psi_{ix}^2) dx + C \int_\mathbb{R} (1 + t)^{-1}\ln(1 + t)(E^2 + E_x^2) dx \\
\leq \frac{1}{4} \int_\mathbb{R} (1 + t)(\varphi_{ix}^2 + \psi_{ix}^2) dx + C\delta_0(1 + t)^{-2}\ln(1 + t).
\]

Next, by performing \(\int_\mathbb{R} ((23) \times 2\ln(1 + t)\varphi_{ix} + (30) \times 2\ln(1 + t)\psi_{ix}) dx\) for \(i = 1, 2\), then analysis similar to Step 1 and using (80) and (79), we obtain the desired estimates. Therefore the proof is complete.

Next, for \(i = 1, 2\), differentiating (23)i, (30)i in \(x\), respectively, we get

\[
\varphi_{ixxt} + (1 + t)\varphi_{ix} + \frac{1}{2}\varphi_{ixxx} - (\varphi_{ix}^2)_{xx} = (-1)^i((\varphi_{ix} + \hat{n})E_x - f_{ix} \\
+ f_{ii}xx + f_{i3}xx + f_{i4}xx),
\]

and

\[
\psi_{ixtt} + (1 + t)\psi_{ix} + \frac{1}{2}\psi_{ixxxx} = (-1)^i(\bar{w}E_x)_x - \bar{w}_{xxt} - \frac{1}{2}\bar{w}_{xxxx} \\
- g_{ix}x + (-1)^i g_{ixx} + g_{i3}x + g_{i4}x.
\]

By performing \(\int_\mathbb{R} ((81) \times (2\ln(\alpha + t)\varphi_{ix}) + (82) \times (2\ln(\alpha + t)\psi_{ix}) dx, \int_\mathbb{R} ((81) \times 2\ln(1 + t)\varphi_{ix} + (82) \times 2\ln(1 + t)\psi_{ix}) dx\) and \(\int_\mathbb{R} (\partial_{(81)} \times (2\ln(\alpha + t)\varphi_{ix}) + (1 + t)^{-1}\ln(\alpha + t)\varphi_{ix}) \partial_x(82) \times (2\ln(\alpha + t)\psi_{ix}) dx\) for some large number \(\alpha > 0\), then analysis similar to Lemma 4.3, we have the following two Lemmas.
Lemma 4.4. There exists a sufficiently small constant $0 < \varepsilon_4 < \varepsilon_3$ such that if $N(T) + \delta \leq \varepsilon_4$, then it holds for $i = 1, 2$
\[ \ln^2(1 + t)\|\partial_x(\varphi_{ix}, \varphi_{it}, \varphi_{iixx}, \psi_{ix}, \psi_{iit})(t)\|^2 \]
\[ + \int_{t_0}^t (1 + s)\ln^2(1 + t)\|\partial_x(\varphi_{it})(s)\|^2 ds \]
\[ + \int_{t_0}^t (1 + s)^{-1}\ln(1 + t)\|\partial_x^2(\varphi_{i}, \varphi_{ix}, \psi_{i}, \psi_{ix})(s)\|^2 ds \leq C(\Phi_0^2 + \delta_0). \]

Lemma 4.5. There exists a sufficiently small constant $0 < \varepsilon_5 < \varepsilon_4$ such that if $N(T) + \delta \leq \varepsilon_5$, then it holds for $i = 1, 2$
\[ \ln^3(1 + t)\|\partial_x^2(\varphi_{ix}, \varphi_{it}, \varphi_{iixx}, \psi_{ix}, \psi_{iit})(t)\|^2 \]
\[ + \int_{t_0}^t (1 + s)\ln^3(1 + t)\|\partial_x^2(\varphi_{it})(s)\|^2 ds \]
\[ + \int_{t_0}^t (1 + s)^{-1}\ln^2(1 + t)\|\partial_x^2(\varphi_{i}, \varphi_{ix}, \psi_{i}, \psi_{ix})(s)\|^2 ds \leq C(\Phi_0^2 + \delta_0). \]

Lemma 4.6. There exists a sufficiently small constant $0 < \varepsilon_6 < \varepsilon_5$ such that if $N(T) + \delta \leq \varepsilon_6$, then it holds for $i = 1, 2$
\[ (1 + t)^2\ln^2(1 + t)\|\varphi_{it}(t)\|^2 + (1 + t)^2\ln^3(1 + t)\|\varphi_{iit}(t)\|^2 \]
\[ + (1 + t)^2\ln^2(1 + t)\|\psi_{iit}(t)\|^2 + (1 + t)^2\ln^2(1 + t)\|\psi_{iixt}(t)\|^2 \]
\[ + \int_{t_0}^t (1 + s)^3\ln^2(1 + t)\|\varphi_{iit}(s)\|^2 ds \leq C(\Phi_0^2 + \delta_0). \]

Proof. For $i = 1, 2$, taking
\[ \int_{\mathbb{R}} \left( \partial_t(\varphi_{itt}) \times (2(\beta + t)^2\ln(\alpha + t)\varphi_{iit} + (\beta + t)\ln(\alpha + t)\varphi_{it}) \right) dx \]
and
\[ \int_{\mathbb{R}} \left( \partial_t(\varphi_{itt}) \times (2(\beta + t)^2\ln(\alpha + t)\psi_{iit} + (\beta + t)\ln(\alpha + t)\psi_{it}) \right) dx, \]
for some large number $\alpha, \beta > 0$, and analysis similar to Lemma 4.3, we can prove
\[ (1 + t)^2\ln^2(1 + t)\|\varphi_{it}(t)\|^2 \]
\[ + (1 + t)^2\ln^2(1 + t)\|\varphi_{iixx}(t)\| + \|\varphi_{iixx}(t)\|^2 + \|\varphi_{iixx}(t)\|^2 \]
\[ + \int_{t_0}^t (1 + s)^3\ln^2(1 + t)\|\varphi_{iit}(s)\|^2 ds \leq C(\Phi_0^2 + \delta_0), \]
we omit the details here. Finally, from (23) and (82), we have for $i = 1, 2$
\[ (1 + t)^2\|\varphi_{it}(t)\|^2 \leq C(\|\varphi_{iit}(t)\|^2 + \|\varphi_{iixx}(t)\|^2 + \|\varphi_{iixx}(t)\|^2) \]
\[ + \|((\varphi_{ix} + \bar{n} + \bar{n})E(t))\|^2 + \|f_1(t)\|^2 + \|f_{i2x}(t)\|^2 + \|f_{i3x}(t)\|^2 + \|f_{i4x}(t)\|^2, \]
and
\[ (1 + t)^2\|\varphi_{ixx}(t)\|^2 \leq C(\|\varphi_{ixx}(t)\|^2 + \|\varphi_{ixx}(t)\|^2 + \|\varphi_{ixx}(t)\|^2) \]
\[ + \|((\varphi_{ix} + \bar{n} + \bar{n})E_x(t))\|^2 + \|f_{i1x}(t)\|^2 + \|f_{i2xx}(t)\|^2 + \|f_{i3xx}(t)\|^2 + \|f_{i4xx}(t)\|^2, \]}
Then employing Lemma 4.1–4.5 and (83), we get

\[(1 + t)^2 \ln (1 + t) \left\| \varphi_{it} (t) \right\|^2 + (1 + t)^2 \ln (1 + t) \left\| \varphi_{ixt} (t) \right\|^2 \leq C(\Phi_0^2 + \delta_0). \quad (84)\]

Combining (83) with (84), we arrive the desired estimates. We complete the proof.

Acknowledgments. The author is grateful to the two referees’ valuable comments and suggestions, which led a significant improvement of the original manuscript. The author would like to express his gratitude to Prof. Peicheng Zhu for his helpful suggestions. The author is supported in part by Science and Technology Commission of Shanghai Municipality (Grant No. 20JC1413600).

REFERENCES

[1] S. Chen, H. Li, M. Mei and K. Zhang, Global and blow-up solutions to compressible Euler equations with time-dependent damping, *J. Differential Equations*, 268 (2020), 5035–5077.
[2] P. Degond and C. Ringhofer, Quantum moment hydrodynamics and the entropy principle, *J. Stat. Phys.*, 112 (2003), 587–628.
[3] D. Donatelli, M. Mei, B. Rubino and R. Sampalmieri, Asymptotic behavior of solutions to Euler-Poisson equations for bipolar hydrodynamic model of semiconductors, *J. Differential Equations*, 255 (2013), 3150–3184.
[4] C. J. van Duyn and L. A. Peletier, A class of similiary solutions of the nonlinear diffusion equations, *Nonlinear Anal.*, 1 (1977), 223–233.
[5] D. K. Ferry and J.-R. Zhou, Form of the quantum potential for use in hydrodynamic equations for semiconductor device modeling, *Phys. Rev. B*, 48 (1993), 7944–7950.
[6] C. L. Gardner, The quantum hydrodynamic model for semiconductors devices, *SIAM J. Appl. Math.*, 54 (1994), 409–427.
[7] I. Gasser, L. Hsiao and H. Li, Large time behavior of solutions of the bipolar hydrodynamical model for semiconductors, *J. Differential Equations*, 192 (2003), 326–359.
[8] I. Gasser and P. A. Markowich, Quantum hydrodynamics, Wigner transforms and the classical limit, *Asymptot. Anal.*, 14 (1997), 97–116.
[9] S. Geng, Y. Lin and M. Mei, Asymptotic behavior of solutions to Euler equations with time-dependent damping in critical case, *SIAM J. Math. Anal.*, 52 (2020), 1463–1488.
[10] H. Hu, M. Mei and K. Zhang, Asymptotic stability and semi-classical limit for bipolar quantum hydrodynamic model, *Commun. Math. Sci.*, 14 (2016), 2331–2371.
[11] J. Hu, Y. Li and J. Liao, The stationary solution of a one-dimensional bipolar quantum hydrodynamic model, *J. Math. Anal. Appl.*, 493 (2021), 124537.
[12] F. Huang, H.-L. Li and A. Matsumura, Existence and stability of steady-state of one-dimensional quantum Euler-Poisson system for semiconductors, *J. Differential Equations*, 225 (2006), 1–25.
[13] F. Huang and Y. Li, Large time behavior and quasineutral limit of solutions to a bipolar hydrodynamic model with large data and vacuum, *Dis. Contin. Dyn. Sys., Ser. A*, 24 (2009), 455–470.
[14] F. Huang, M. Mei and Y. Wang, Large time behavior of solutions to n-dimensional bipolar hydrodynamic models for semiconductors, *SIAM J. Math. Anal.*, 43 (2011), 1595–1630.
[15] F. Huang, M. Mei, Y. Wang and T. Yang, Long-time behavior of solution to the bipolar hydrodynamic model of semiconductors with boundary effect, *SIAM J. Math. Anal.*, 44 (2012), 1134–1164.
[16] N. C. Klusdahl, A. M. Kriman, D. K. Ferry and C. Ringhofer, Self-consistent study of the resonant-tunneling diode, *Phys. Rev. B*, 39 (1989), 7720–7735.
[17] C. Lattanzio, On the 3-D bipolar isentropic Euler-Poisson model for semiconductors and the drift-diffusion limit, *Math. Models Methods Appl. Sci.*, 10 (2000), 351–360.
[18] H. Li, J. Li, M. Mei and K. Zhang, Asymptotic behavior of solutions to bipolar Euler-Poisson equations with time-dependent damping, *J. Math. Anal. Appl.*, 437 (2019), 1081–1121.
[19] H. Li, J. Li, M. Mei and K. Zhang, Optimal convergence rate to nonlinear diffusion waves for Euler equations with critical overdamping, *Appl. Math. Lett.*, 113 (2021), 106882.
BIPOLAR QUANTUM EULER-POISSON SYSTEM

[20] H.-L. Li, G. Zhang and K. Zhang, Algebraic time-decay for the bipolar quantum hydrodynamic model, *Math. Models Methods Appl. Sci.*, 18 (2008), 859–881.

[21] Y. Li, Long-time self-similarity of classical solutions to the bipolar quantum hydrodynamic models, *Nonlinear Anal.*, 74 (2011), 1501–1512.

[22] Y. Li, Global existence and large time behavior of solutions for the bipolar quantum hydrodynamic models in the quarter plane, *Math. Meth. Appl. Sci.*, 36 (2013), 1409–1422.

[23] Y. Li and X. Yang, Global existence and asymptotic behavior of the solutions to the three dimensional bipolar Euler-Poisson systems, *J. Differential Equations*, 252 (2012), 768–791.

[24] B. Liang and K. Zhang, Steady-state solutions and asymptotic limits on the multi-dimensional semiconductor quantum hydrodynamic model, *Math. Models Methods Appl. Sci.*, 17 (2007), 253–275.

[25] L. Luan, M. Mei, B. Rubino and P. Zhu, Large-time behavior of solutions to Cauchy problem for bipolar Euler-Poisson system with time-dependent damping in critical case, *Commun. Math. Sci.*, 19 (2021), 1207–1231.

[26] M. Mei, B. Rubino and R. Sampalmieri, Asymptotic behavior of solutions to the bipolar hydrodynamic model of semiconductors in bounded domain, *Kinet. Relat. Models*, 5 (2012), 537–550.

[27] S. Nishibata and M. Suzuki, Initial boundary value problems for a quantum hydrodynamic model of semiconductors: Asymptotic behaviors and classical limits, *J. Differential Equations*, 244 (2008), 836–874.

[28] X. Pan, Global existence of solutions to 1-d Euler equations with time-dependent damping, *Nonlinear Anal.*, 132 (2016), 327–336.

[29] X. Pan, Blow up of solutions to 1-d Euler equations with time-dependent damping, *J. Math. Anal. Appl.*, 442 (2016), 435–445.

[30] A. Unterreiter, The thermal equilibrium solution of a generic bipolar quantum hydrodynamic model, *Commun. Math. Phys.*, 188 (1997), 69–88.

[31] Q.-W. Wu and Y.-P. Li, Asymptotic behavior of solutions to the bipolar quantum Euler-Poisson system with time-dependent damping, preprint, 2021.

[32] Q. Wu, Y. Li and R. Xu, Large-time behavior of solutions to bipolar Euler-Poisson equations with time-dependent damping in the half space, *J. Math. Anal. Appl.*, 508 (2022), 125899.

[33] Q. Wu, J. Zheng and L. Luan, Large-time behavior of solutions to the time-dependent damper bipolar Euler-Poisson system, *Appl. Anal.*, (2021), in press.

[34] G. Zhang, H.-L. Li and K. Zhang, Semiclassical and relaxation limits of bipolar quantum hydrodynamic model for semiconductors, *J. Differential Equations*, 245 (2008), 1433–1453.

[35] G. Zhang and K. Zhang, On the bipolar quantum Euler-Poisson system: The thermal equilibrium model solution and semiclassical limit, *Nonlinear Anal.*, 66 (2007), 2218–2229.

Received for publication September 2021; early access January 2022.

E-mail address: wuqivei_shu@163.com