The plurigenera and birational models of nondegenerate toric hypersurfaces

Julius Giesler
University of Tübingen

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Abstract

In this article we present a formula for the plurigenera of minimal models of nondegenerate toric hypersurfaces, which is valid in any dimension and which expresses these invariants through lattice points on the Fine interior. Besides we consider other birational models of toric hypersurfaces and study their singularities from the point of view of the minimal model program. We show that the first irregularity \( q(Y) \) of a minimal model of a toric hypersurface is always 0. Restricting to surfaces in toric 3-folds we compute the difference in the rank of the Picard group when switching between birational models.

1 Introduction

In the article (Bat20) it was shown how to construct minimal models and other birational models with mild singularities of nondegenerate toric hypersurfaces. That is we start with a nondegenerate Laurent polynomial \( f \) with an \( n \)-dimensional Newton polytope \( \Delta \subset M_{\mathbb{R}} \), consider the zero set \( Z_f := \{ f = 0 \} \subset (\mathbb{C}^*)^n \) and ask for a projective variety \( Z_\Sigma \) birational to \( Z_f \) with at most terminal singularities and with \( K_{Z_\Sigma} \) nef, which we also call a minimal model of \( Z_f \).

The idea is to consider some modifications of the Newton polytope \( \Delta \) of \( f \), namely the Fine interior \( F(\Delta) \), the canonical closure \( C(\Delta) \), their Minkowski sum \n
\[ \tilde{\Delta} := C(\Delta) + F(\Delta) \]
and another fan $\Sigma$ refining the normal fan of $\tilde{\Delta}$.

Let $\mathbb{P}_{F(\Delta)}$, $\mathbb{P}_{C(\Delta)}$ and $\mathbb{P}_{\tilde{\Delta}}$ denote the projective toric varieties to the normal fans of these polytopes and $\mathbb{P}_{\Sigma}$ the toric variety to the fan $\Sigma$. $\mathbb{P}_{\Sigma}$ is projective as well, but not that easy to describe via a polytope of which $\Sigma$ is the normal fan. Then take the closure of $Z_f$ in these toric varieties and denote these closures of $Z_f$ by $Z_{F(\Delta)}$, $Z_{C(\Delta)}$, $Z_{\tilde{\Delta}}$ and $Z_{\Sigma}$.

In ([Bat20 Thm.5.4]) it is shown that $Z_{\Sigma}$ gets a minimal model of $Z_f$ and since the normal fan of $\tilde{\Delta}$ refines the normal fans both of $C(\Delta)$ and of $F(\Delta)$ we obtain a diagram of toric morphisms and morphisms between the hypersurface closures

\[
\begin{array}{cccc}
\mathbb{P}_{\Sigma} & \xrightarrow{\pi} & \text{terminal sing.} & \mathbb{P}_{F(\Delta)} \\
\downarrow{\rho} & & & \downarrow{\theta} \\
\mathbb{P}_{\tilde{\Delta}} & \xrightarrow{\pi} & \text{canonical sing.} & \mathbb{P}_{C(\Delta)} \\
\mathbb{P}_{\Delta} & \xleftarrow{\pi} & \mathbb{P}_{F(\Delta)} & \xrightarrow{\theta} \mathbb{P}_{C(\Delta)} \\
\end{array}
\]

where $\rho$ and $\pi$ are birational and $\theta$ is birational if $\dim \Delta = \dim F(\Delta)$. The toric variety $\mathbb{P}_{\Sigma}$ ($\mathbb{P}_{\tilde{\Delta}}$) as well as the hypersurface $Z_{\Sigma}$ ($Z_{\tilde{\Delta}}$) have at most terminal (canonical) singularities.

In this article we continue this investigation by computing the plurigenera of $Z_{\Sigma}$ and the irregularity $q(Z_{\Sigma})$, and by dealing also with the singularities of $Z_{C(\Delta)}$ and $Z_{F(\Delta)}$, at least in some special cases. For the closure $Z_{\Delta}$ of $Z_f$ in the toric variety $\mathbb{P}_{\Delta}$ to the Newton polytope $\Delta$ the first plurigenus, which is also called the geometric genus, and the irregularity are well known (compare [DK86]), at least if $\mathbb{P}_{\Delta}$ is $\mathbb{Q}$-factorial. We generalize these results by computing all plurigenera and the irregularity and adjust them to the setting, that we take the closures of $Z_f$ in other toric varieties than $\mathbb{P}_{\Delta}$.

After giving some necessary background on toric geometry and nondegenerate toric hypersurfaces in section 2 we shortly present the definitions of $F(\Delta)$, $C(\Delta)$, $\tilde{\Delta}$ and $\Sigma$ and the corresponding known results in section 3. In section 4 we compute the plurigenera of the minimal model $Z_{\Sigma}$ of $Z_f$:

**Theorem 1.1.** Let $\Delta$ be an $n$-dimensional lattice polytope, where $n \geq 2$, with $k := \dim F(\Delta) \geq 0$. Let $Y := Z_{\Sigma}$ be a minimal model of $Z_f$. Then for
\(m \geq 1\) we get for the plurigenera \(P_m(Y) := h^0(Y, mK_Y)\)

\[
P_m(Y) = \begin{cases} 
  l(m \cdot F(\Delta)) - l^*(m - 1) \cdot F(\Delta), & k = n \\
  l(m \cdot F(\Delta)) + l^*((m - 1) \cdot F(\Delta)), & k = n - 1 \\
  l(m \cdot F(\Delta)), & k < n - 1,
\end{cases}
\]

As an illustration of this formula we deal with smooth surfaces in \(\mathbb{P}^3\), where the Newton polytope is just a multiple \(d \cdot \Delta_3\) of the standard simplex \(\Delta_3\) and \(F(d \cdot \Delta_3) = (d - 4)\Delta_3\).

Having understood the singularities of \(\mathbb{P}_\Sigma, Z_\Sigma, \mathbb{P}_\Delta\) and \(Z_\Delta\), the problem with \(\mathbb{P}_{F(\Delta)}\) and \(Z_{F(\Delta)}\) is that the Fine interior \(F(\Delta)\) might be of lower dimension than \(\Delta\), in which case \(Z_{F(\Delta)}\) is not birational to \(Z_f\). We show in Theorem 6.2 that at least if

\[\dim \Delta = \dim F(\Delta) = 3\]

and \(C(\Delta)\) is again a lattice polytope, in which case by ([Bat20, Thm.6.2]) \(Z_\Sigma\) is a surface of general type, then \(Z_{F(\Delta)}\) gets a canonical model of \(Z_\Sigma\) and the morphism \(\theta \circ \pi : Z_\Sigma \to Z_{F(\Delta)}\) contracts exactly the \((-2)\)-curves on \(Z_\Sigma\). On the other hand side concerning \(C(\Delta)\) this polytope does always contain \(\Delta\) and in fact often we have \(C(\Delta) = \Delta\). Thus we put more emphasis on the morphisms \(\rho\) and \(\pi\).

In Lemma 3.10, Remark 3.8 and Proposition 5.1 we deal with the following basic questions: Is the closure \(Z_{C(\Delta)}\) always a normal variety? Does for example the pullback \(\rho^*(Z_{C(\Delta)})\) coincide with \(Z_\Delta\) and what can we say about the exceptional sets of the maps \(\rho\) and \(\theta\) in fact we will see that the questions on normality and pullback have a positive answer and that \(\rho\)-exceptional and for \(\dim F(\Delta) = \dim \Delta\) also the \(\theta\)-exceptional curves on \(\mathbb{P}_\Delta\) have some intersection properties with respect to the canonical class \(K_{\mathbb{P}_\Delta}\). The latter result was already mentioned in ([Bat20]).

In Proposition 5.6 we show that the closure \(Z_\Delta\) of \(Z_f\) in \(\mathbb{P}_\Delta\) always has at most rational singularities and the same holds for \(Z_{C(\Delta)}\), at least under the additional assumption that \(C(\Delta)\) is a lattice polytope again (see Corollary 5.7). Just as the condition \(F(\Delta) \neq \emptyset\) guarantees the existence of a minimal model the condition that \(C(\Delta)\) is a lattice polytope ensures that \(Z_{C(\Delta)} \subset \mathbb{P}_{C(\Delta)}\) is a Cartier divisor.

Since the closure \(Z_\Delta\) of \(Z_f\) in \(\mathbb{P}_\Delta\) is always an ample divisor, by a Theorem of Lefschetz ([DK86, Prop.3.4]) we have \(h^1(Z_\Delta, \mathbb{C}) = 0\). The divisor \(Z_\Sigma\) is in general not ample any longer, thus we are not able to apply this Theorem of Lefschetz to deduce \(h^1(Z_\Sigma, \mathbb{C}) = 0\). But from the result that \(Z_\Delta\) has at most
rational singularities and \( h^1(Z_\Delta, \mathbb{C}) = 0 \) it follows \( h^1(Z_\Delta, \mathcal{O}_{Z_\Delta}) = 0 \) and in this way we were able to deduce the following result

**Proposition 1.2.** Let \( \Delta \) be a \( n \)-dimensional lattice polytope with \( F(\Delta) \neq \emptyset \). Then we get for \( Y := Z_\Sigma \) or \( Y := Z_\Delta \) and if \( \mathbb{P}_{C(\Delta)} \) is \( \mathbb{Q} \)-Gorenstein, also for \( Y := Z_{C(\Delta)} \) that

\[
q(Y) := h^0(Y, \Omega^1_Y) = 0.
\]

In section 6 we restrict ourselves to surfaces in toric 3-folds, although some of the results might extend to higher dimensions. In Proposition 6.2 we show that if \( \Delta \) is a 3-dimensional lattice polytope with at least one interior lattice point, such that \( C(\Delta) \) is again a lattice polytope, \( Y := Z_\Sigma \), then the canonical divisor \( K_Y \) as well as all its multiples are basepointfree. This result might be useful in studying examples of minimal surfaces \( Y := Z_\Sigma \) in \( \mathbb{P}_\Sigma \), especially if \( Y \) is of general type, since it implies that then the pluricanonical maps are always morphisms.

Further we consider the Picard numbers \( \rho(Z_{\Sigma,f}) \) and \( \rho(Z_{C(\Delta),f}) \) of \( Z_{\Sigma,f} \) and \( Z_{C(\Delta),f} \) in the case of surfaces in toric 3-folds, where we keep \( f \) in the notation since the Picard number might depend on the particular Laurent polynomial \( f \). We then show in Theorem 6.7 that independently of the nondegenerate Laurent polynomial \( f \) the relative Picard number

\[
\rho(Z_{\Sigma,f}/Z_{C(\Delta),f}) := \rho(Z_{\Sigma,f}) - \rho(Z_{C(\Delta),f})
\]

is given by the number of \( (\rho \circ \pi) \)-exceptional curves between \( Z_\Sigma \) and \( Z_{C(\Delta)} \).

This article should provide the basic results for two further upcoming articles of the author on surfaces in toric 3-folds and their Hodge theory.

### 2 Background on toric varieties

In this article \( M \) always denotes an \( n \)-dimensional lattice \( \mathbb{Z}^n \) with dual lattice \( N \). We write \( M_\mathbb{R} \) for \( M \otimes \mathbb{R} \). Let \( T := N \otimes \mathbb{C}^* \cong (\mathbb{C}^*)^n \) be the \( n \)-dimensional torus. We denote by \( \Delta \) always a lattice polytope in \( M_\mathbb{R} \) and think of it as the Newton polytope of some Laurent polynomial \( f \), that is

\[
f = \sum_{m \in M \cap \Delta} a_m x^m, \quad a_m \in \mathbb{C}, \quad (1)
\]
where $a_m \neq 0$ for $m$ a vertex of $\Delta$. Here $x^m$ is an abbreviation for $x_1^{m_1} \cdot \ldots \cdot x_n^{m_n}$.

By a rational polytope $F \subset M_{\mathbb{R}}$ we mean a polytope, whose vertices have coordinates in $\mathbb{Q}$. We may represent a rational polytope $F$ as

$$F = \{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geq -a_i, i = 1, \ldots, r \},$$

where $\nu_i \in N$ are primitive and $a_i \in \mathbb{Q}$ and we may even have $a_i \in \mathbb{Z}$ but $F$ just rational. Then we associate to $F$ its normal fan $\Sigma_F$ which has as rays $\Sigma_F[1] = \{\nu_1, \ldots, \nu_r\}$. For $\Delta$ an $n$-dimensional lattice polytope or more generally a rational polytope let

$$\text{ord}_\Delta(\nu) := \min_{m \in \Delta \cap M} \langle m, \nu \rangle$$

for $\nu \in N$. Then the facet presentation of $\Delta$ becomes

$$\{x \in M_{\mathbb{R}} | \langle x, \nu_i \rangle \geq \text{ord}_\Delta(\nu_i), \nu_i \in \Sigma_\Delta[1] \}.$$

With a complete fan $\Sigma$ we associate a complete toric variety, which we denote by $P_\Sigma$. If $\Sigma = \Sigma_F$ is the normal fan to $F$, the toric variety $P_{\Sigma_F}$ is projective and we denote it by $P_F$. If $\nu_i \in \Sigma[1]$ is a ray of a complete fan $\Sigma$, then to $\nu_i$ is associated a toric divisor on $P_\Sigma$ which we denote by $D_i$.

### 2.1 Nondegenerate hypersurfaces in toric varieties

Given a Laurent polynomial $f$ with Newton polytope $\Delta$, we denote the zero set in the torus $T$ by $Z_f$. We may take the closure of $Z_f$ in any $n$-dimensional toric variety, but there is a particular natural choice for the toric variety, namely the toric variety $P_\Delta$ to the Newton polytope $\Delta$ of $f$. We denote this closure by $Z_\Delta$ or $Z_{\Delta,f}$ if we want to stress the dependence of the polynomial $f$.

For an $n$-dimensional lattice polytope $\Delta$ we denote by $L(\Delta)$ the vector space over $\mathbb{C}$ with basis the lattice points of $\Delta$ and by $L^*(\Delta)$ the vector space over $\mathbb{C}$ with basis the interior lattice points of $\Delta$, where for the latter $\Delta$ is considered as a subset of an affine space of the same dimension as $\Delta$. The dimensions of $L(\Delta)$ and $L^*(\Delta)$ are denoted by $l(\Delta)$ and $l^*(\Delta)$. Given a Laurent polynomial $f$ with Newton polytope $\Delta$, by identifying monomials with lattice points, we also write $f \in L(\Delta)$.

**Remark 2.1.** We may embed $\mathbb{P}_\Delta$ into $\mathbb{P}^{l(\Delta)-1}$ and under this embedding $Z_\Delta$ gets a hyperplane section. Thus $Z_\Delta \subset \mathbb{P}_\Delta$ is a very ample Cartier divisor.
Similarly if $F \subset M_\mathbb{R}$ is a rational polytope the hypersurface $Z_F \subset \mathbb{P}_F$ gets an ample $\mathbb{Q}$-Cartier divisor, that is some positive multiple of $Z_F$ is ample and Cartier.

**Definition 2.2.** Given a Laurent polynomial $f$ with Newton polytope $\Delta$ we call $f$ nondegenerate with respect to $\Delta$ (or $\Delta$-regular) if $Z_f$ is smooth and $Z_\Delta$ intersects the toric strata of $\mathbb{P}_\Delta$ transversally. In this situation we sometimes also call $Z_\Delta$ nondegenerate. We denote the set of $\Delta$-regular Laurent polynomials by $U_{\text{reg}}(\Delta)$.

**Remark 2.3.** This condition may also be expressed by saying that $Z_f$ is smooth and for every $k$-dimensional face $\Gamma$ of $\Delta$

$$f|_\Gamma, \frac{\partial f|_\Gamma}{\partial x_1}, \ldots, \frac{\partial f|_\Gamma}{\partial x_n}$$

have no common zero in the torus orbit $(\mathbb{C}^*)^k$ corresponding to $\Gamma$, where

$$f = \sum_{m \in M \cap \Delta} a_m x^m, \quad f|_\Gamma := \sum_{m \in M \cap \Gamma} a_m x^m.$$  

The singular locus $(\mathbb{P}_\Delta)_{\text{sing}}$ of $\mathbb{P}_\Delta$ is closed and the union of the closures of some torus orbits. If $p \in (\mathbb{P}_\Delta)_{\text{sing}}$ and the nondegenerate hypersurface $Z_\Delta$ passes through $p$, then $p$ is also a singular point on $Z_\Delta$. This follows from the jacobian criterion for smoothness and the above characterization of nondegeneracy. Conversely if $p \in \mathbb{P}_\Delta \setminus (\mathbb{P}_\Delta)_{\text{sing}}$ and $p \in Z_\Delta$ the by the nondegeneracy $p$ is also a nonsingular point of $Z_\Delta$.

**Remark 2.4.** By Bertini’s Theorem the condition for $f \in L(\Delta)$ to be $\Delta$-regular is a Zariski open condition on the coefficients $(a_m)_{m \in M \cap \Delta}$. Often properties of the hypersurface closure $Z_{\Delta,f}$ only depend on the Newton polytope $\Delta$ and not on the particular Laurent polynomial $f$, at least as long as $f$ is $\Delta$-regular. This justifies the notation $Z_\Delta$ instead of $Z_{\Delta,f}$.

We also want to take the closures of $Z_f$ in other toric varieties: For this if $\Sigma$ is an $n$-dimensional fan we write $Z_\Sigma$ or $Z_{\Sigma,f}$ for the closure of $Z_f$ in $\mathbb{P}_\Sigma$. For $F \subset M_\mathbb{R}$ a rational polytope we also write $Z_F$ or $Z_{F,f}$ for the closure $Z_{\Sigma_F,f}$.

**Construction 2.5.** By ([Bat20, Prop.5.1]) for $f \in L(\Delta)$ the closure $Z_\Sigma$ in the toric variety $\mathbb{P}_\Sigma$ to an $n$-dimensional complete fan $\Sigma$ is as a Weil divisor
linear equivalent to

\[ Z_\Sigma \sim_{\text{lin}} - \sum_{\nu_i \in \Sigma[1]} \text{ord}_\Delta(\nu_i) D_i \]

(2)

\( Z_\Sigma \) is a Cartier divisor if and only if \( \text{ord}_\Delta \) is a support function, that is

\[ \text{ord}_\Delta : N_\mathbb{R} \to \mathbb{R} \]

is linear on each cone of \( \Sigma \) and \( \text{ord}_\Delta(N) \subset \mathbb{Z} \). Similarly \( Z_\Sigma \) is \( \mathbb{Q} \)-Cartier if we just have \( \text{ord}_\Delta(N) \subset \mathbb{Q} \).

**Remark 2.6.** In this article if \( D \) and \( D' \) are integral divisors we write \( D \sim_{\text{lin}} D' \) or sometimes even just \( D = D' \), to mean the linear equivalence of Weil divisors, which coincides with the linear equivalence of Cartier divisors, if \( D \) and \( D' \) are Cartier. To a Cartier (Weil) divisor \( D \) on an algebraic variety \( Y \) is associated a locally free sheaf (rank 1 reflexive sheaf) \( \mathcal{O}_Y(D) \), such that \( \mathcal{O}_Y(D) \) is isomorphic to \( \mathcal{O}_Y(D') \) if and only if \( D \) is linear equivalent to \( D' \) (see [Reid79, App. to §1]). We denote by

\[ |D| := \mathbb{P}H^0(Y, \mathcal{O}_Y(D)) \]

the complete linear system associated to \( D \).

**Proposition 2.7.** A projective toric variety \( \mathbb{P}_\Sigma \) is \( \mathbb{Q} \)-factorial if and only if each cone \( \sigma \in \Sigma \) is simplicial, that is the generators \( \nu_i \in \sigma[1] \) are linearly independent over \( \mathbb{R} \).

**Remark 2.8.** For \( D \) a Cartier on an projective variety \( Y \) and \( \sigma : Y' \to Y \) a proper surjective morphism we have by the projection formula ([Hart77, Ch.3, Ex.8.3])

\[ \sigma_*\sigma^*(D) = D. \]

If \( D \) is \( \mathbb{Q} \)-Cartier and \( mD \) is Cartier we define the pullback \( \sigma^*(D) \) as

\[ \sigma^*(D) := \frac{1}{m}\sigma^*(mD). \]

This might be just a \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisor, that is some multiple of it is integral and Cartier. \( \mathbb{Q} \)-Cartier \( \mathbb{Q} \)-divisors \( D \) and \( D' \) are linearly equivalent if there is an \( m \in \mathbb{N}_{\geq 1} \) with \( mD \) and \( mD' \) integral and Cartier and \( mD \sim_{\text{lin}} mD' \). We will mainly use that in this situation \( D \) and \( D' \) are numerically equivalent,
that is they have the same intersection numbers with all curves. We denote
this equivalence by \( D \equiv D' \). If \( \sigma \) is birational and \( D \) a \( \mathbb{Q} \)-Cartier divisor, we
will use that by the projection formula for intersection numbers
\[
\sigma_* \sigma^*(D) \equiv D.
\]

**Proposition 2.9.** Given two fans \( \Sigma \) and \( \Sigma' \) such that \( \Sigma[1] \) and \( \Sigma'[1] \) belong to
the same lattice \( N \) and \( \Sigma' \) refines \( \Sigma \) there is an induced birational morphism
\( p : \mathbb{P}_{\Sigma'} \to \mathbb{P}_{\Sigma} \).

**Remark 2.10.** If \( Z_\Sigma \) is a \( \mathbb{Q} \)-Cartier divisor with support function \( \text{ord} \Delta \), then
by ([CLS11, Prop.6.2.7]) the pullback \( p^*(Z_\Sigma) \) under a birational morphism
\( p : \mathbb{P}_{\Sigma'} \to \mathbb{P}_{\Sigma} \) still has support function \( \text{ord}_\Delta \), that is
\[
p^*(Z_\Sigma) \sim_{\text{lin}} \sum_{\nu_i \in \Sigma'[1]} \text{ord}_\Delta(\nu_i)D_i.
\]
By formula 2 in this case the pullback \( p^*(Z_\Sigma) \) equals the closure \( Z_{\Sigma'} \) of \( Z_f \):
\[
Z_{\Sigma'} \sim_{\text{lin}} p^*(Z_\Sigma).
\]
Note that for \( \Delta \) a lattice polytope and if \( Z_\Sigma \) is a \( \mathbb{Q} \)-Cartier divisor, the
pullback \( p^*(Z_\Sigma) \) is again an integral divisor by the definition of \( \text{ord}_\Delta \).

### 2.2 Polytopes associated to toric divisors

We recall here some results on ample and basepointfree divisors and their
associated polytopes. To every Weil divisor
\[
D = \sum_{i=1}^{r} a_i D_i, \quad a_i \in \mathbb{Z}
\]
on a toric variety \( \mathbb{P}_{\Sigma} \) to a complete fan \( \Sigma \) is associated a polytope
\[
P_D := \{ x \in M_\mathbb{R} | \langle x, \nu_i \rangle \geq -a_i, \quad \nu_i \in \Sigma[1] \}, \quad \text{(3)}
\]
which is a rational polytope, and which computes the global sections of \( D \),
that is (compare [CLS11, Prop.4.3.3])
\[
H^0(\mathbb{P}_{\Sigma}, O_{\mathbb{P}_{\Sigma}}(D)) \cong L(P_D).
\]
Note that for \( k \geq 1 \) the polytope \( P_{kD} \) associated to \( k \cdot D \) equals \( k \cdot P_D \).
Construction 2.11. For \( D \) a basepointfree Cartier divisor on \( \mathbb{P}_\Sigma \), the polytope \( P_D \) is a lattice polytope ([CLS11 Thm.6.1.10]) and for \( D \) ample and Cartier the operation of assigning a polytope to a divisor is invertible: To a lattice polytope \( P_D \subset M_\mathbb{R} \) with presentation (3) is associated the ample divisor

\[
D_{P_D} := \sum_{i : \nu_i \in \Sigma_{[1]}} a_iD_i
\]  

(4)
on \( \mathbb{P}_{P_D} \) and \( D_{P_D} = D \) ([CLS11 Cor.6.1.15]). In this sense the Newton polytope \( \Delta \) is associated to the divisor \( Z_\Delta \). Similarly if \( P_D \) as in (3) is just a rational polytope, the divisor \( D_{P_D} \) as defined in (4) is \( \mathbb{Q} \)-Cartier and ample. On a complete toric variety a Cartier divisor is basepointfree if and only if it is nef ([CLS11 Theorem 6.3.12]), such that an ample Cartier divisor is always basepointfree. Now if \( D \) is a basepointfree Cartier divisor on \( \mathbb{P}_\Sigma \), \( D_{P_D} \) is an ample Cartier divisor on \( \mathbb{P}_{P_D} \), and we may still relate these two divisors (compare [CLS11 Thm.6.2.8]): There is a proper toric morphism

\[
\theta : \mathbb{P}_\Sigma \to \mathbb{P}_{P_D} \text{ with } \theta^*(D_{P_D}) = D,
\]

that is \( D \) is the pullback of an ample divisor on the toric variety to the polytope \( P_D \).

3 Minimal models of toric hypersurfaces

3.1 The Fine interior \( F(\Delta) \)

In the article ([Bat20]) it is shown how to construct minimal models of non-degenerate toric hypersurfaces. Since we will heavily exploit these constructions, we recall them here:

Definition 3.1. Given an \( n \)-dimensional lattice polytope \( \Delta \subset M_\mathbb{R} \) with presentation

\[
\{ x \in M_\mathbb{R} | \langle x, \nu_i \rangle \geq \text{ord}_\Delta(\nu_i), \nu_i \in \Sigma_\Delta[1] \},
\]

we define the Fine interior \( F(\Delta) \) of \( \Delta \) as

\[
F(\Delta) := \{ x \in M_\mathbb{R} | \langle x, \nu \rangle \geq \text{ord}_\Delta(\nu) + 1, \nu \in N \setminus \{0\} \},
\]
Remark 3.2. The Fine interior was introduced by J. Fine in [Fine83]. In general it is only a rational polytope although if dim $\Delta = 2$ it is always a lattice polytope, namely it equals the convex span of the interior lattice points of $\Delta$ ([Bat17, Prop.2.9]).

Further we define the support $S_F(\Delta)$ as follows:

**Definition 3.3.** The set of lattice points $\nu \in \mathbb{N} \setminus \{0\}$ with

$$\text{ord}_{F(\Delta)}(\nu) = \text{ord}_{\Delta}(\nu) + 1$$

is called the support $S_F(\Delta)$ of $F(\Delta)$ to $\Delta$.

### 3.2 A diagram of toric morphisms

A simplicial fan $\Sigma$ with rays $\Sigma[1] = S_F(\Delta)$ defines a $\mathbb{Q}$-factorial toric variety $\mathbb{P}_\Sigma$ and chosen appropriately $\mathbb{P}_\Sigma$ will become the surrounding toric variety for a minimal model of a toric hypersurface (Theorem 3.10).

Unfortunately we need not have $\Sigma[1] \subset S_F(\Delta)$ although such a property would be desirable, for then we would get a birational morphism $\mathbb{P}_\Sigma \to \mathbb{P}_\Delta$. But this problem can be solved by passing to another polytope $C(\Delta)$, which might be slightly larger than $\Delta$:

**Definition 3.4.** The polytope

$$C(\Delta) := \{ x \in M_\mathbb{R} \mid \langle x, \nu \rangle \geq \text{ord}_{\Delta}(\nu) \quad \forall \nu \in S_F(\Delta) \}$$

is called the canonical closure of $\Delta$. We call $\Delta$ canonically closed if $C(\Delta) = \Delta$.  

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Lemma 3.5. ([Bat20, Prop.1.17a], Prop.2.3]
An $n$-dimensional lattice polytope $\Delta$ with $F(\Delta) \neq \emptyset$ is canonically closed if and only if $\Sigma[1]_{\Delta} \subset S_F(\Delta)$. $C(\Delta)$ is always canonically closed and thus $\Sigma[1]_{C(\Delta)} \subset S_F(\Delta)$. Further for $\nu \in S_F(\Delta)$ we have
\[ \text{ord}_{C(\Delta)}(\nu) = \text{ord}_\Delta(\nu). \]

Remark 3.6. The canonical closure $C(\Delta)$ is in general again just a rational polytope. In dimension 2 it is always a lattice polytope ([Bat20, Prop.2.4]) and it dimension 3 it seems to be unknown whether for any lattice polytope $\Delta$ with $F(\Delta) \neq \emptyset$ the canonical closure $C(\Delta)$ is a lattice polytope again.

Definition 3.7. Let $\Delta \subset M_\mathbb{R}$ be an $n$-dimensional lattice polytope with Fine interior $F(\Delta) \neq \emptyset$. Then we define $\tilde{\Delta}$ as Minkwoski sum
\[ \tilde{\Delta} := C(\Delta) + F(\Delta). \]

In this way the normal fan $\Sigma_{\tilde{\Delta}}$ gets the coarsest refinement of $\Sigma_{C(\Delta)}$ and $\Sigma_{F(\Delta)}$ (see [CLS11, Prop.6.2.13]) and thus we get toric morphisms $\rho : \mathbb{P}_{\tilde{\Delta}} \to \mathbb{P}_{C(\Delta)}$ and $\theta : \mathbb{P}_{\tilde{\Delta}} \to \mathbb{P}_{F(\Delta)}$. By ([Bat20, Thm.4.3]) we still have
\[ \Sigma[1]_{\tilde{\Delta}} \subset S_F(\Delta). \]

and thus we may define a simplicial fan $\Sigma$ with $\Sigma[1] = S_F(\Delta)$ and which refines $\Sigma_{\tilde{\Delta}}$. We arrive at a diagram of toric morphisms and morphisms between hypersurfaces (by abuse of notation we use the same letters for the maps and its restrictions)

\[ \begin{array}{ccc}
\mathbb{P}_\Sigma & \xrightarrow{\pi} & \mathbb{P}_{\tilde{\Delta}} \\
\downarrow & & \downarrow \\
\mathbb{P}_{C(\Delta)} & \xrightarrow{\rho} & \mathbb{P}_{F(\Delta)} \\
\mathbb{P}_\Sigma & \xleftarrow{\theta} & \mathbb{P}_F(\Delta) \\
\downarrow & & \downarrow \\
\mathbb{P}_{\tilde{\Delta}} & \xrightarrow{\rho} & \mathbb{P}_{F(\Delta)} \\
\downarrow & & \downarrow \\
\mathbb{P}_{\Sigma} & \xrightarrow{\theta} & \mathbb{P}_{F(\Delta)} \\
\end{array} \]

where $\pi$ and $\rho$ are birational and $\theta$ is birational if $\dim F(\Delta) = \dim \Delta$. 

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Remark 3.8. $Z_{\Delta} \subset \mathbb{P}_\Delta$ is an ample Cartier divisor and by Lemma 3.5 and Construction 2.5

$$Z_{C(\Delta)} \sim_{\text{lin}} - \sum_{\nu_i \in \Sigma_{C(\Delta)}[1]} \text{ord}_{C(\Delta)}(\nu_i) D_i$$

is also an ample divisor, Cartier if $C(\Delta)$ is again a lattice polytope, since then $Z_{C(\Delta)}$ has $\text{ord}_{C(\Delta)}$ has its support function. We have to be careful: Although

$$\text{ord}_{\Delta}(\nu_i) = \text{ord}_{C(\Delta)}(\nu_i) \quad \nu_i \in \Sigma_{C(\Delta)}[1]$$

$\text{ord}_{\Delta}$ is not a support function for $Z_{C(\Delta)}$ since it need not be linear on the cones of $\Sigma_{C(\Delta)}$. But at least for $\nu \in \Sigma[1] = S_F(\Delta)$ we have

$$\text{ord}_{\Delta}(\nu) = \text{ord}_{C(\Delta)}(\nu)$$

by Lemma 3.5 and thus we get with Construction 2.5

$$\rho^*(Z_{C(\Delta)}) = Z_{\Delta}, \quad \pi^*Z_{\Delta} = Z_\Sigma.$$  

Thus if $C(\Delta)$ is a lattice polytope, then $Z_{C(\Delta)}$, $Z_{\Delta}$ and $Z_\Sigma$ are all Cartier divisors.

3.3 Birational models with terminal and canonical singularities

Definition 3.9. ([KM98, Def.2.34])

A normal algebraic variety $Y$ (over $\mathbb{C}$) is said to have at most terminal (canonical) singularities if $K_Y$ is $\mathbb{Q}$-Cartier and if $m$ is the smallest natural number such that $mK_Y$ is Cartier, then for every resolution of singularities $\sigma: Y' \to Y$ we have

$$mK_{Y'} = \sigma^*(mK_Y) + \sum_{i=1}^r a_i E_i,$$

(8)

where $a_i$ are integers with $a_i > 0$ ($a_i \geq 0$). Here $E_1, ..., E_r$ are the exceptional divisors of $\sigma$.

In fact by ([KM98, Cor.2.32(1)]) it is enough to check the condition (8) for one resolution of singularities. The following two Theorems and the Corollary to it are contained in the statements of ([Bat20, Cor.4.4, Cor.4.5, Thm.5.4])
Theorem 3.10. Let $\Delta \subset \mathbb{R}^n$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then $\mathbb{P}_\Sigma$ has at most terminal singularities and the adjoint divisor $K_{\mathbb{P}_\Sigma} + Z_\Sigma$ is $\mathbb{Q}$-Cartier and nef.

Theorem 3.11. In the same situation as in Theorem 3.10 $\mathbb{P}_\tilde{\Delta}$ has at most canonical singularities and the morphism $\pi : \mathbb{P}_\Sigma \rightarrow \mathbb{P}_\tilde{\Delta}$ is crepant, that is $\pi^*(K_{\mathbb{P}_\tilde{\Delta}}) = K_{\mathbb{P}_\Sigma}$.

Corollary 3.12. The closure $Z_\Sigma$ gets a minimal model of $Z_\Delta$, i.e. $Z_\Sigma$ is birational to $Z_\Delta$, has at most terminal singularities and $K_{Z_\Sigma}$ is nef. $\mathbb{P}_\tilde{\Delta}$ has at most canonical singularities and $\pi : Z_\Sigma \rightarrow Z_\Delta$ is crepant, that is $\pi^*(K_{Z_\Delta}) \sim_{\text{lin}} K_{Z_\Sigma}$ as integral Weil divisors.

Remark 3.13. By Construction 2.5

$$Z_\Sigma \sim_{\text{lin}} - \sum_{\nu_i \in S_F(\Delta)} \text{ord}_\Delta(\nu_i) D_i, \quad K_{\mathbb{P}_\Sigma} = - \sum_{\nu_i \in S_F(\Delta)} D_i.$$ 

Thus to the divisor $Z_\Sigma + K_{\mathbb{P}_\Sigma}$ is associated the polytope

$$\{ x \in \mathbb{R}^n \mid \langle x, \nu_i \rangle \geq \text{ord}_\Delta(\nu_i) + 1 \quad \nu_i \in S_F(\Delta) \},$$

which is exactly $F(\Delta)$. In particular the lattice points of $F(\Delta)$ count the global sections of $Z_\Sigma + K_{\mathbb{P}_\Sigma}$. Let

$$m := \min\{ n \in \mathbb{N} \geq 1 \mid n \cdot F(\Delta) \text{ is a lattice polytope} \},$$

then

$$m \cdot \text{ord}_{F(\Delta)}(n) = \min_{w \in F(\Delta) \cap \mathbb{Z}} \langle m \cdot w, n \rangle \in \mathbb{Z}, \quad \text{for } n \in \mathbb{N}$$

induces a support function for $m(Z_\Sigma + K_{\mathbb{P}_\Sigma})$ such that this divisor is Cartier. By Remark 2.11 we have an equality

$$m(Z_\Sigma + K_{\mathbb{P}_\Sigma}) = (\theta \circ \pi)^*(mD_{F(\Delta)})$$

of Cartier divisors and thus

$$Z_\Sigma + K_{\mathbb{P}_\Sigma} \equiv (\theta \circ \pi)^*(D_{F(\Delta)}).$$

Note also that if $Z_\Sigma$ is Cartier, then by the adjunction formula ([KM98, Prop.5.73]) we have

$$(Z_\Sigma + K_{\mathbb{P}_\Sigma})_{Z_\Sigma} = K_{Z_\Sigma}.$$
Corollary 3.14. Let $\Delta$ be canonically closed with $F(\Delta) \neq \emptyset$ and assume that $\mathbb{P}_\Delta$ has at most canonical (terminal) singularities, then $\rho$ ($\rho$ and $\pi$) are isomorphisms.

Proof. This could be seen via the characterization of canonical and terminal singularities of toric varieties in ([Reid83 (1.11), (1.12)]).

Remark 3.15. Note that by ([CLS11, Prop.11.4.24(b)]) as a normal projective toric $\mathbb{P}_{C(\Delta)}$ has at most log-terminal singularities, that is $a_i > -1$ in equation (8), if it is $\mathbb{Q}$-Gorenstein, that is $K_{\mathbb{P}_{C(\Delta)}}$ is $\mathbb{Q}$-Cartier. To check this condition we may restrict again to one resolution if we choose an snc-resolution $\sigma$ (see ([KM98 Cor.2.32(2)])). By this we mean that the exceptional set of $\sigma$ is an snc-divisor, that is a reduced divisor with smooth irreducible components meeting transversally.

Lemma 3.16. Let $\Delta$ be the Newton polytope of a nondegenerate Laurent polynomial $f$. Then the closure $Z_{C(\Delta)}$ is a normal variety.

Proof. For $f \in U_{\text{reg}}(\Delta)$ the closure $Z_\Delta \subset \mathbb{P}_\Delta$ intersects the toric strata transversally and by this is a normal variety. By ([Tre10 Prop.5.1.3]) the closure $Z_{\Delta+C(\Delta)}$ then also intersects the toric strata of $\mathbb{P}_{\Delta+C(\Delta)}$ transversally. It follows from this that the hypersurface $Z_{C(\Delta)} \subset \mathbb{P}_{C(\Delta)}$ still intersects any toric divisor $D_i \subset \mathbb{P}_{C(\Delta)}$ transversally outside of the intersection of $D_i$ with some other $D_j \subset \mathbb{P}_{C(\Delta)}$. Thus since $\mathbb{P}_{C(\Delta)}$ is normal $Z_{C(\Delta)}$ has a singular locus of codimension $\geq 2$. Since by ([KM98 Def.5.1, Prop.5.3(1)]) $Z_{C(\Delta)}$ is Cohen-Macaulay it follows by Serre’s criterion ([Rei87 (3.18)]) that $Z_{C(\Delta)}$ is normal.

Remark 3.17. By ([CLS11 Proof of Thm.11.2.2]) given a normal projective toric variety $\mathbb{P}$ and a resolution of singularities $\mathbb{P}' \to \mathbb{P}$, there exists an snc-resolution $\mathbb{P}'' \to \mathbb{P}$ of singularities, which factors as $\mathbb{P}'' \to \mathbb{P}' \to \mathbb{P}$. This resolution could be even chosen as an isomorphism over the smooth locus of $\mathbb{P}$ ([CLS11 Def.11.1.1]).

It follows from the pullback formulas in Remark 3.8 Remark 3.15 and the adjunction formula that if $C(\Delta)$ is a lattice polytope and $\mathbb{P}_{C(\Delta)}$ is $\mathbb{Q}$-Gorenstein, then also $Z_{C(\Delta)}$ has at most log-terminal singularities.

4 The plurigenera of minimal models

First the Kodaira dimension of $Z_\Sigma$, which by definition is the Kodaira dimension of a resolution of singularities of $Z_\Sigma$, has already been computed:
Theorem 4.1. ([Bat20, Thm.6.2])
Let $\Delta \subset M_\mathbb{R}$ be a lattice polytope with $F(\Delta) \neq \emptyset$. Let $k := \dim F(\Delta)$. Then $Y := Z_\Sigma$ has Kodaira dimension

$$\kappa(Y) = \begin{cases} 
  n - 1 & k = n \\
  n - 1 & k = n - 1 \\
  k & k < n - 1,
\end{cases}$$

In the following Theorem we also compute the plurigenera of $Z_\Sigma$:

Theorem 4.2. Let $\Delta$ be an $n$-dimensional lattice polytope, where $n \geq 2$, with $k := \dim F(\Delta) \geq 0$. Let $Y := Z_\Sigma$ be a minimal model. Then for $m \geq 1$ we get for the plurigenera $P_m(Y) := h^0(Y, mK_Y)$

$$P_m(Y) = \begin{cases} 
  l(m \cdot F(\Delta)) - l^*( (m - 1) \cdot F(\Delta) ) & k = n \\
  l(m \cdot F(\Delta)) + l^*( (m - 1) \cdot F(\Delta) ) & k = n - 1 \\
  l(m \cdot F(\Delta)) & k < n - 1,
\end{cases}$$

Proof. If we take cohomology groups of a divisor $D$ we always mean the cohomology groups of the sheaf $\mathcal{O}(D)$ associated to $D$. By Remark 3.13 we have

$$H^0(\mathbb{P}_\Sigma, m(K_{\mathbb{P}_\Sigma} + Y)) \cong L(m \cdot F(\Delta)) \quad m \in \mathbb{Z}_{>0}$$

Then we use an ideal sheaf sequence of $Y$

$$0 \to H^0(\mathbb{P}_\Sigma, (m - 1)(K_{\mathbb{P}_\Sigma} + Y) + K_{\mathbb{P}_\Sigma}) \to H^0(\mathbb{P}_\Sigma, m(K_{\mathbb{P}_\Sigma} + Y))$$
$$\to H^0(Y, mK_Y) \to H^1(\mathbb{P}_\Sigma, (m - 1)(K_{\mathbb{P}_\Sigma} + Y) + K_{\mathbb{P}_\Sigma}) \to 0$$

where

$$H^1(\mathbb{P}_\Sigma, m(K_{\mathbb{P}_\Sigma} + Y)) = 0$$

by Demazure’s vanishing ([CLS11, Thm.9.2.3]) for the divisor $K_{\mathbb{P}_\Sigma} + Y$, which is $\mathbb{Q}$-Cartier and nef by Theorem 3.10. By ([Perl09, Prop.4.22]) we may apply Serre-duality to the $\mathbb{Q}$-Cartier divisor $K_{\mathbb{P}_\Sigma}$. Thus for $m = 1$ we have $h^0(\mathbb{P}_\Sigma, K_{\mathbb{P}_\Sigma}) = 0$ and

$$h^1(\mathbb{P}_\Sigma, K_{\mathbb{P}_\Sigma}) = h^{n-1}(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}) = 0$$

by Demazure’s vanishing Theorem again. Thus $P_1(Y)$ is given by

$$P_1(Y) = l(F(\Delta)) = l^*(\Delta).$$

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For \( m \geq 2 \) we apply Serre duality to the \( \mathbb{Q} \)-Cartier divisor \( (m - 1)(K_{\mathbb{P}_\Sigma} + Y) \) and use the Batyrev-Borisov vanishing Theorem ([CLS11, Thm.9.2.7]), which states the following:

For a \( \mathbb{Q} \)-Cartier nef divisor \( D \) on a complete toric variety \( \mathbb{P}_\Sigma \), the cohomology groups \( H^p(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(-D)) \) are all zero, except for possibly one \( p \). More precisely if \( P_D \) denotes the polytope associated to \( D \) then

\[
H^p(\mathbb{P}_\Sigma, \mathcal{O}_{\mathbb{P}_\Sigma}(-D)) = \begin{cases} 0 & p \neq \dim P_D \\ L^*(P_D) & p = \dim P_D \end{cases}
\]

In our case \( D = (m - 1)(K_{\mathbb{P}_\Sigma} + Y) \) and \( P_D = (m - 1) \cdot F(\Delta) \), thus

\[
H^0(\mathbb{P}_\Sigma, (m - 1)(K_{\mathbb{P}_\Sigma} + Y) + K_{\mathbb{P}_\Sigma})^* \cong H^n(\mathbb{P}_\Sigma, (1 - m)(K_{\mathbb{P}_\Sigma} + Y))
\]

\[
\cong \begin{cases} 0, & \dim F(\Delta) \leq n - 1 \\ L^*((m - 1)F(\Delta)), & \dim F(\Delta) = n \end{cases}
\]

and

\[
H^1(\mathbb{P}_\Sigma, (m - 1)(K_{\mathbb{P}_\Sigma} + Y) + K_{\mathbb{P}_\Sigma})^* \cong H^3(\mathbb{P}_\Sigma, (1 - m)(K_{\mathbb{P}_\Sigma} + Y))
\]

\[
\cong \begin{cases} L^*((m - 1)F(\Delta)), & \dim F(\Delta) = n - 1 \\ 0, & \dim F(\Delta) \neq n - 1 \end{cases}
\]

The result follows by adding the dimensions in the exact sequence.

**Remark 4.3.** Note that if \( Y \) is a smooth complex projective variety and \( D \) a nef \( \mathbb{Q} \)-Cartier divisor, which is additionally big, then for \( p \neq \dim Y \)

\[
H^p(Y, \mathcal{O}_Y(-D)) = 0.
\]

This vanishing result is due to Kawamata and Viehweg ([Laz00, Thm.4.3.1]). The vanishing result of Batyrev-Borisov assumes a complete normal toric variety \( \mathbb{P}_\Sigma \) and a nef \( \mathbb{Q} \)-Cartier divisor \( D \) on it. By ([CLS11, Lemma 9.3.9]) \( D \), or equivalently \( mD \) for \( mD \) Cartier, is big if and only if \( \dim(P_D) = \dim Y \). Thus this result contains the result of Kawamata-Viehweg applied to toric varieties as a special case.

**Example 4.4.** Let \( \Delta_3 \) be the standard 3-simplex and \( d \in \mathbb{N} \geq 1 \). The Fine interior \( F(d \cdot \Delta_3) \) is nonempty if and only if \( d \geq 4 \). We claim that for \( d \geq 4 \)

\[
F(d \cdot \Delta_3) = (d - 4)\Delta_3.
\]
To see this note that the convex span of the interior lattice points of $d \cdot \Delta_3$ equals $(d - 4)\Delta_3$ and is contained in $F(d \cdot \Delta_3)$. Further if we translate just the facets of $\Delta_3$ one step into the interior, we already get $(d - 4)\Delta_3$.

In particular the inner facet normal vectors belong to the support $S_F(\Delta)$ and thus by Lemma 3.5 we have $\Delta = C(\Delta)$. Now since the toric variety $\mathbb{P}^3$ to $d \cdot \Delta_3$ is smooth it has in particular at most terminal singularities. Thus by Corollary 3.14 we get $S_F(\Delta) = \Sigma \Delta[1]$ and $\mathbb{P}_\Sigma = \mathbb{P}^3$. With the formula

$$l(d \cdot \Delta_3) = \frac{1}{6} \cdot (d + 1)(d + 2)(d + 3),$$

which is easily checked, the plurigenera of smooth surfaces in $\mathbb{P}^3$ could be computed more explicitly. The same method applies to smooth hypersurfaces in $\mathbb{P}^n$.

**Remark 4.5.** For $Y$ a minimal surface of general type the plurigenera of $Y$ are given by

$$P_m(Y) = \chi(Y, \mathcal{O}_Y) + \frac{m(m - 1)}{2}K_Y^2.$$  

Similarly the plurigenera of surfaces $Y$ with $\kappa(Y) = 1$, that is properly elliptic surfaces, could be deduced from the canonical bundle formula (see [FrMo94, Ch.1, Prop.3.22]). But in general it is difficult to compute the plurigenera already of a 3-fold, say with at most terminal singularities (compare [Rei87, Cor.(10.3)])

**Remark 4.6.** We have to be careful with any conclusions from the formula in Theorem 4.2: If $p_g(Y) = 0$ it might well be the case that the plurigenera are not monotone increasing in $m$, since $F(\Delta)$ is just a rational polytope and in the formula we just count the lattice points (see [Bat20, Example 1.6] for an example).

## 5 The singular birational models of hypersurfaces

### 5.1 The exceptional sets of $\rho$ and $\theta$

**Proposition 5.1.** Let $\Delta$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$ and let $K := K_{F_\Delta}$, $Z := Z_\Delta$. Then the birational morphism $\rho : \mathbb{P}_\Delta \to \mathbb{P}_{C(\Delta)}$
is $K$-positive, whereas if $\dim F(\Delta) = \dim \Delta$ the morphism $\theta : \mathbb{P}_F(\Delta) \to \mathbb{P}_{F(\Delta)}$ is $K$-negative.

By this we mean that if $R \subset \mathbb{P}_F(\Delta)$ is a compact curve, contained in the exceptional set of $\rho$, then

$$R.K > 0$$

and $R.K < 0$ for a compact curve contracted by $\theta$.

**Proof.** On $\mathbb{P}_{C(\Delta)}$ we have the ample $\mathbb{Q}$-Cartier divisor $Z_{C(\Delta)}$ and on $\mathbb{P}_{F(\Delta)}$ the ample $\mathbb{Q}$-Cartier divisor $D_{F(\Delta)}$ associated to the polytope $F(\Delta)$. Their pullbacks $\rho^* (Z_{C(\Delta)})$ and $\theta^* (D_{F(\Delta)})$ are big and nef and have intersection number 0 exactly with the $\rho$ or $\pi$ exceptional curves.

Since the normal fan of $\Delta$ is the coarsest refinement of the normal fans of $C(\Delta)$ and $F(\Delta)$ there are no common exceptional curves of $\rho$ and $\pi$ and the divisor

$$D := \rho^* (Z_{C(\Delta)}) + \theta^* (D_{F(\Delta)})$$

is ample by the Nakai-Moishezon criterion. By Remark 3.13 and since $\pi$ is crepant we have

$$\theta^* (D_{F(\Delta)}) \equiv K + Z,$$

whereas by formula (7) we have $\rho^* (Z_{C(\Delta)}) = Z$. Thus we get

$$D \equiv 2 \cdot Z + K.$$

and for a curve $R \subset \mathbb{P}$

$$0 < D.R = 2 \cdot Z.R + K.R$$

(9)

If $R$ is $\rho$-exceptional, then $Z.R = \rho^* (Z_{C(\Delta)}).R = 0$, thus $K.R > 0$, whereas if $R$ is $\theta$-exceptional, then

$$0 = \theta^* (D_{F(\Delta)}).R = K.R + Z.R$$

and together with the inequality (9) we get $Z.R > 0$ and thus $K.R < 0$. 

**Corollary 5.2.** In the same situation as in Proposition 5.1 if $C(\Delta)$ is a lattice polytope the birational morphism $\rho : Z_{\Delta} \to Z_{C(\Delta)}$ is $K_{Z_{\Delta}}$-positive.

**Proof.** This follows from Proposition 5.1 and the adjunction formula applied to the Cartier divisor $Z_{\Delta}$. 

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Corollary 5.3. If the exceptional set of \( \rho \) is of codimension \( \geq 2 \) in \( \mathbb{P}_\Delta \), then \( K_{P_{C(\Delta)}} \) is not \( \mathbb{Q} \)-Cartier, in particular \( P_{C(\Delta)} \) is not \( \mathbb{Q} \)-factorial.

Proof. The proof relies on a standard argument ([KM98, 2.6 Case 3]): First by Proposition 5.1 we have with \( K := K_{P_\Delta} \)

\[
R.K > 0, \quad i = 1, ..., k
\]

for \( R \) contained in the exceptional locus. If \( K_{P_{C(\Delta)}} \) were \( \mathbb{Q} \)-Cartier, then \( \rho^*(K_{P_{C(\Delta)}}) \) would be linear equivalent to \( K \). But

\[
\rho^*K_{P_{C(\Delta)}}.R = 0, \quad K.R > 0,
\]

a contradiction. \( \square \)

5.2 Rational singularities

Definition 5.4. ([KM98, Def.5.8, Thm.5.10])
A variety \( Z \) has rational singularities if for one (or equivalently every) resolution of singularities \( \eta : Z' \to Z \) we have

1) \( \eta_*\mathcal{O}_{Z'} = \mathcal{O}_Z \),
2) \( R^i\eta_*\mathcal{O}_{Z'} = 0 \quad i > 0 \).

Remark 5.5. By ([Hart77, Ch.3, Cor.11.4]) the condition 1) is always true if \( Z \) is normal and moreover by ([CLS11, Thm.11.4.2]) normal toric varieties have at most rational singularities.

Proposition 5.6. Let \( \Delta \) be an \( n \)-dimensional lattice polytope with toric variety \( \mathbb{P}_\Delta \). Then \( Z_\Delta \) has at most rational singularities.

Proof. Let \( \eta : \mathbb{P}_{\Sigma'} \to \mathbb{P}_\Delta \) be a resolution of singularities. Then \( \eta \) induces a resolution of singularities \( Z_{\Sigma'} \to Z_\Delta \) and to show that the singularities of \( Z_\Delta \) are at most rational it is enough to argue that

\[
R^i\eta_*\mathcal{O}_{Z_{\Sigma'}} = 0 \quad i > 0, \quad (10)
\]

since \( \eta_*\mathcal{O}_{Z_{\Sigma'}} = \mathcal{O}_{Z_\Delta} \) holds by normality of \( Z_\Delta \). To show (10) we start with the exact sequence

\[
0 \to \mathcal{O}_{\mathbb{P}_{\Sigma'}}(-Z_{\Sigma'}) \to \mathcal{O}_{\mathbb{P}_{\Sigma'}} \to \mathcal{O}_{Z_{\Sigma'}} \to 0. \quad (11)
\]
Since the divisor $Z_\Delta$ is Cartier and $\eta^*(Z_\Delta) = Z_\Sigma'$ we may apply the projection formula to deduce

$$R^i \eta_* \mathcal{O}_{\mathbb{P}_\Sigma'}(-Z_\Sigma') \cong \mathcal{O}_{\mathbb{P}_\Delta}(-Z_\Delta) \otimes R^i \eta_* \mathcal{O}_{\mathbb{P}_\Sigma'} = 0, \quad i > 0$$

since the normal toric variety $\mathbb{P}_\Delta$ has at most rational singularities, that is

$$R^i \eta_* \mathcal{O}_{\mathbb{P}_\Sigma'} = 0 \quad i > 0.$$ 

We conclude by taking the derived sequence of (11) (compare [Hart77, Ch.3.1])

$$\ldots \rightarrow R^1 \eta_* \mathcal{O}_{\mathbb{P}_\Sigma'}(-Z_\Sigma') \rightarrow R^1 \eta_* \mathcal{O}_{\mathbb{P}_\Sigma'} \rightarrow R^1 \eta_* \mathcal{O}_{Z_\Sigma'} \rightarrow R^2 \eta_* \mathcal{O}_{\mathbb{P}_\Sigma'}(-Z_\Sigma') \rightarrow \ldots$$

\[\Box\]

**Corollary 5.7.** Let $\Delta$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$ and with $C(\Delta)$ again a lattice polytope, then $Z_{C(\Delta)}$ also has at most rational singularities.

**Proof.** It is enough to show that

$$R^i (\rho \circ \pi)_* \mathcal{O}_{Z_\Sigma} = 0, \quad i > 0. \quad (12)$$

For $Z_\Sigma$ has at most terminal and thus by ([KM98, Thm.5.22]) rational singularities. Choose a resolution of singularities $\sigma : Z' \rightarrow Z_\Sigma$ then by ([KM98, Prop.2.69]) $R^i (\rho \circ \pi \circ \sigma)_* \mathcal{O}_{Z'} = 0$ is equivalent to

$$H^i(Z', (\rho \circ \pi \circ \sigma)^*(H)) = 0$$

for $H$ a sufficiently ample Cartier divisor. The same holds for $\rho \circ \pi$ and if (12) is true then

$$H^i(Z_\Sigma, (\rho \circ \pi)^*(H)) = 0.$$

But since $Z_\Sigma$ has at most rational singularities we get by ([Hart77, Ch.3, Ex.8.1]) and the projection formula

$$H^i(Z_\Sigma, (\rho \circ \pi)^*(H)) \cong H^i(Z', (\rho \circ \pi \circ \sigma)^*(H))$$

and we are done. To show (12) use that $Z_{C(\Delta)}$ is Cartier, $(\rho \circ \pi)^*(Z_{C(\Delta)}) = Z_\Sigma$ by (7) and argue just as in the proof of the Proposition above. \[\Box\]
5.3 The first irregularity

**Definition 5.8.** For \( Z \) an \( n \)-dimensional normal algebraic variety and \( 1 \leq p \leq n \) we define \( \Omega^p_Z := i_* \Omega^p_U \), where \( i : U \to Z \) denotes the inclusion of the smooth locus of \( Z \). Then let

\[
q(Z) := h^0(Z, \Omega^1_Z)
\]

be the first irregularity of \( Z \).

**Remark 5.9.** If \( Y \) is an \( n \)-dimensional projective variety with at most canonical singularities and \( K_Y \) is Cartier, then for \( \sigma : Y' \to Y \) a resolution of singularities we have

\[
\sigma_*(K_{Y'}) = K_Y.
\]

In ([GKKP11, Thm.1.4]) this is extended to log-terminal singularities and even more general types of singularities of pairs, further to the case that \( K_Y \) is just \( \mathbb{Q} \)-Cartier and to differential \( p \)-forms for \( p \leq n \). Then we still have

\[
\sigma_*(\Omega^p_{Y'}) = \Omega^p_Y \quad 1 \leq p \leq n
\]

at least if \( \sigma \) is an snc-resolution. This applies for example to resolutions of singularities of \( Z_{\Sigma} \), \( Z_{\Delta} \) or \( Z_{C(\Delta)} \) if \( \mathbb{P}_{C(\Delta)} \) is \( \mathbb{Q} \)-Gorenstein. It also applies to \( \mathbb{P}_{\Sigma} \), \( \mathbb{P}_{\Delta} \) or \( \mathbb{P}_{C(\Delta)} \) if \( \mathbb{P}_{C(\Delta)} \) is \( \mathbb{Q} \)-Gorenstein, and we may take \( \sigma \) to be a toric morphism by Remark 3.15.

**Remark 5.10.** Let \( \Delta \) be a \( n \)-dimensional lattice polytope. By a Theorem of Lefschetz ([DK86, Prop.3.4]) applied to the ample divisor \( Z_{\Delta} \subset \mathbb{P}_{\Delta} \) we get

\[
H^1(Z_{\Delta}, \mathbb{C}) = 0.
\]

If \( Z_{\Delta} \) is smooth then \( H^1(Z_{\Delta}, \mathbb{C}) \) decomposes as

\[
H^1(Z_{\Delta}, \mathbb{C}) \cong H^0(Z_{\Delta}, \Omega^1_{Z_{\Delta}}) \oplus H^1(Z_{\Delta}, \mathcal{O}_{Z_{\Delta}}),
\]

where \( h^0(Z_{\Delta}, \Omega^1_{Z_{\Delta}}) = h^1(Z_{\Delta}, \mathcal{O}_{Z_{\Delta}}) \) and it follows \( q(Z_{\Delta}) = 0 \). In general since \( Z_{\Delta} \) has at most rational singularities the natural sheaf map \( \mathbb{C} \to \mathcal{O}_{Z_{\Delta}} \) still induces a surjection (compare [Kol95, Ch.12])

\[
H^1(Z_{\Delta}, \mathbb{C}) \to H^1(Z_{\Delta}, \mathcal{O}_{Z_{\Delta}}),
\]

such that \( H^1(Z_{\Delta}, \mathcal{O}_{Z_{\Delta}}) = 0 \).
Proposition 5.11. Let $\Delta$ be an $n$-dimensional lattice polytope with $F(\Delta) \neq \emptyset$. Then we get for $Y := Z_\Sigma$ or $Y := Z_{\Delta}$, and if $\mathbb{P}_{C(\Delta)}$ is $\mathbb{Q}$-Gorenstein also for $Y := Z_{C(\Delta)}$, that

$$q(Y) = 0.$$  

Proof. We choose a common toric resolution of singularities

$$
\begin{array}{c}
\mathbb{P}_\Sigma' \\
\sigma \\
\mathbb{P}_\Sigma \\
\eta \\
\mathbb{P}_\Delta
\end{array}
$$

(13)

where $\sigma$ is assumed to be an snc-resolution. Then $\sigma$ and $\eta$ induce birational maps between the hypersurfaces, which resolve the singularities of $Z_\Sigma$ and $Z_\Delta$. Since $Z_\Delta$ has rational singularities we get

$$0 = h^1(Z_\Delta, \mathcal{O}_{Z_\Delta}) = h^1(Z_{\Sigma'}, \mathcal{O}_{Z_{\Sigma'}}).$$

By the Hodge decomposition we get $q(Z_{\Sigma'}) = 0$ and the results follow by Remark 5.9.

6 Further properties of the birational models of hypersurfaces for $\dim \Delta = 3$

In dimension 2 the minimal model $Z_\Sigma$ is smooth, whereas $Z_\Delta$ has at most rational double points ([KM98, Thm.4.5]) Further in dimension 2 there are no small contractions, that is the birational morphisms $\rho$ and $\pi$ contract some curves or are isomorphisms. The singularities of $Z_\Delta$, $Z_{C(\Delta)}$ and $Z_{\Delta}$ are isolated and if $E$ is an exceptional curve for $\pi$ or $\rho$, then $E^2 < 0$ by ([BHPV04 Ch.3, Thm.2.1]).

Theorem 6.1. Let $\Delta$ be a 3-dimensional lattice polytope such that $\dim F(\Delta) = 3$ and $C(\Delta)$ is again a lattice polytope. Then $Z_{F(\Delta)}$ gets a canonical model of a surface of general type and the morphism $Z_\Sigma \to Z_{F(\Delta)}$ is the minimal resolution of singularities.

Proof. Since $C(\Delta)$ is a lattice polytope $Z_\Sigma$ is a Cartier divisor and the adjunction formula could be applied. Since $D_{F(\Delta)}$ is ample we get for a curve $C \subset Z_\Sigma$ with Remark 3.13

$$(\theta \circ \pi)(C) = \text{pt.} \iff 0 = (\theta \circ \pi)^*(D_{F(\Delta)}).C = (K_{F(\Delta)} + Z_\Sigma).C = K_{Z_\Sigma}.C,$$
that is \((\theta \circ \pi)\) contracts exactly the \((-2)\)-curves of \(Z_\Sigma\) and the result follows.

\[ \square \]

**Proposition 6.2.** Let \(\Delta\) be a 3-dimensional lattice polytope with \(l^*(\Delta) > 0\) and such that \(C(\Delta)\) is again a lattice polytope. Then for \(Y := Z_\Sigma\) the canonical divisor \(K_Y\) and all its multiples \(mK_Y\) for \(m \geq 2\) are basepointfree.

**Remark 6.3.** Set \(P := P_\Sigma\) then the adjoint divisor \(K_P + Y\) is \(\mathbb{Q}\)-Cartier and nef. If it is Cartier then as mentioned in Construction 2.11 it is basepointfree and so is its restriction \(K_Y = (K_P + Y)|_Y\). In general the singular locus \(P_{\text{sing}}\) of \(P\) consists of finitely many points (compare [KM98, Cor.5.18]). In the proof below we work with a resolution of singularities of \(P\), which is an isomorphism over \(P \setminus P_{\text{sing}}\) and use that \(Y\) does not pass through \(P_{\text{sing}}\) since it is smooth. In this way we show that at least the restriction

\[ K_Y = (K_P + Y)|_Y \]

is basepointfree. Then since \(H^0(Y, K_Y) \subset H^0(Y, mK_Y)\) for \(m \geq 2\) we are done.

**Proof.** Let \(\mathbb{P} = P_\Sigma\) and note that \(Y = (\rho \circ \pi)^*(Z_{C(\Delta)})\) is Cartier under the assumption on \(C(\Delta)\). Choose an snc-resolution of singularities \(\sigma : \mathbb{P}' \to \mathbb{P}\) which is an isomorphism over \(\mathbb{P} \setminus \mathbb{P}_{\text{sing}}\). By Remark 5.9 we have \(\sigma_*K_{\mathbb{P}'} = K_P\). Since \(\sigma_*(Y') = \sigma_*\sigma^*(Y) = Y\) by the projection formula, where \(Y'\) is the isomorphic preimage of \(Y\) in \(\mathbb{P}'\), we get

\[ H^0(\mathbb{P}', K_{\mathbb{P}'} + Y') \cong H^0(\mathbb{P}, K_P + Y). \]  

(14)

Let \(m \in \mathbb{N}_{\geq 1}\) be an integer such that \(m(K_P + Y)\) is Cartier. Then by Construction 2.11 \(m(K_P + Y)\) is basepointfree and thus so is \(\sigma^*(m(K_P + Y))\). Since

\[ m(K_{\mathbb{P}'} + Y') \sim_{\text{lin}} \sigma^*(m(K_P + Y)) + \sum_{i=1}^{s} a_i E_i, \quad a_i \in \mathbb{N}_{\geq 1} \]

also \(m(K_{\mathbb{P}'} + Y')\) is basepointfree. Thus \(K_{\mathbb{P}'} + Y'\) is nef, but \(K_{\mathbb{P}'} + Y'\) is an integral Cartier divisor and thus it is basepointfree. Thus by (14) \(K_P + Y\) is basepointfree outside of \(P_{\text{sing}}\), in particular on \(Y\) and we are done by the adjunction formula.

\[ \square \]
6.1 The relative Picard number

Definition 6.4. Let $Z$ be a normal algebraic surface and let

$$NS(Z) := \text{Im}(c_1 : \text{Pic}(Z) \to H^2(Z, \mathbb{Z}))$$

be the Néron-Severi group of $Z$, where $c_1$ maps a line bundle to its first chern class. Then the rank of $NS(Z)$ is called the Picard number of $Z$ and is denoted by $\rho(Z)$. If $Z' \to Z$ is a birational morphism between normal surfaces we let

$$\rho(Z'/Z) := \rho(Z') - \rho(Z)$$

be the relative Picard number of $Z'$ over $Z$.

Remark 6.5. If $Z$ is smooth and $q(Z) = 0$ like in the case of Proposition 5.11 then by the exponential exact sequence $c_1 : \text{Pic}(Z) \to NS(Z)$ is in fact an isomorphism. In general if $L \in \text{Pic}(Z) \cong H^1(Z, \mathcal{O}_Z^*)$ with $c_1(L) = 0$, then by ([Laz00, Rem.1.1.20]) and ([Hart10, Prop.2.6]) $L \in H^1(Z, \mathcal{O}_Z)$.

Remark 6.6. Let $Y$ be a normal irreducible algebraic surface and $D \subset Y$ a Cartier divisor which is nef and big. Then $D$ behaves similarly to an ample divisor: For example by ([Laz00, Cor.2.2.10]) for $m \gg 0$

$$H^0(Y, \mathcal{O}_Y(mD)) \neq 0$$

and by ([Laz00, Remark after Def.2.2.1]) the rational map associated to $|mD|$ is birational for $m \gg 0$.

Theorem 6.7. Let $\Delta$ be a 3-dimensional Newton polytope with $l^*(\Delta) > 0$ and such that $C(\Delta)$ is again a lattice polytope. Let $f \in U_{\text{reg}}(\Delta)$,

$$\sigma := \rho \circ \pi : Z_{\Sigma,f} \to Z_{C(\Delta),f}$$

and let $E_1, ..., E_s$ be the $\sigma-$exceptional curves. Then

$$\rho(Z_{\Sigma,f}/Z_{C(\Delta),f}) = \rho(Z_{\Sigma,f}) - \rho(Z_{C(\Delta),f}) = s. \quad (15)$$

Remark 6.8. The proof works very similarly as in ([KM98, Rem. 3.16 Step 7]), but makes no use of a difficult basepoint-freeness theorem. Since the proof works independently of $f \in U_{\text{reg}}(\Delta)$ we skip the Laurent polynomial $f$ in the notations.
Proof. By Remark 6.8 the Néron-Severi group of $Z_\Sigma$ equals the Picard group $\text{Pic}(Z_\Sigma)$. By Corollary 5.7 $Z_{C(\Delta)}$ has at most rational singularities. Since $Z_{C(\Delta)}$ is normal $\sigma_*\mathcal{O}_{Z_\Sigma} = \mathcal{O}_{Z_{C(\Delta)}}$ and thus

$$h^1(Z_{C(\Delta)}, \mathcal{O}_{Z_{C(\Delta)}}) = h^1(Z_\Sigma, \mathcal{O}_{Z_\Sigma}) = 0$$

and by Remark 6.8 we may work with the Picard group $\text{Pic}(Z_{C(\Delta)})$ instead of $\text{NS}(Z_{C(\Delta)})$. In particular we may use the projection formula for Cartier divisors. For $\pi : Z_\Sigma \to Z_\Delta$ the theorem is well known since $Z_\Delta$ has at most rational double points.

Thus we set $P := P_\Delta, Y := Z_\Delta$ and assume that $\sigma := \rho : P \to P_{C(\Delta)}$ contracts a single toric divisor $G$ to a point or a curve. Then

$$G|_Y = E_1 + \ldots + E_s,$$

where the $E_i$ contract to different points, and we prove that the following sequence is exact

$$0 \to \text{Pic}(Z_{C(\Delta)}) \xrightarrow{\sigma^*} \text{Pic}(Y) \xrightarrow{D \mapsto (D, E_i)} \mathbb{Z}^s.$$

Obviously we have an injection $\sigma^* : \text{Pic}(Z_{C(\Delta)}) \to \text{Pic}(Y)$, essentially due to the projection formula and since $\sigma_* \mathcal{O}_Y = \mathcal{O}_{Z_{C(\Delta)}}$. The exceptional divisors $E_1, \ldots, E_s$ are linearly independent over $\mathbb{Q}$: For we have

$$E_i.\sigma^*(D) = 0, \quad E_i.E_j = 0, \quad D \in \text{Pic}(Z_{C(\Delta)})$$

since the $E_i$ contract to different points and

$$E_i^2 < 0 \quad i = 1, \ldots, s.$$

Thus if there is a relation

$$\sum_{i=1}^s \lambda_i E_i \sim_{\text{lin}} 0,$$

we may intersect with $E_i$ to get $\lambda_i = 0$. We are left to show that if $D \in \text{Pic}(Y)$ with $D.E_i = 0$ for $i = 1, \ldots, s$ then $D \in \sigma^*(\text{Pic}(Z_{C(\Delta)}))$.

Choose an ample Cartier divisor $A'$ on $P_{C(\Delta)}$ and let $A := A'|_{Z_{C(\Delta)}}$. The aim is to show that for $a, m, k \gg 0$ the divisor

$$a(mD + k \cdot \sigma^*(A))$$

is in the image of $\sigma^*$.
is basepointfree and induces the birational morphism $\sigma$. The remaining part of the proof then follows easily.

The pullback $\sigma^*(A)$ is nef and big since $\sigma$ is generically finite. Further for $C \subset Y$ an irreducible curve we have $\sigma^*(A).C = 0$ if and only if $C = E_i$ for some $i$. Since $D.E_i = 0$ we may arrange it by choosing $A$ sufficiently ample, that $D + \sigma^*(A)$ remains nef and big. But then by Remark 6.6

$$H^0(Y, \mathcal{O}_Y(m(D + \sigma^*A))) \neq 0 \quad \text{for } m \gg 0, \quad (16)$$

that is $m(D + \sigma^*(A))$ is linear equivalent to an effective divisor. Letting $K := K_{\mathbb{P}_\Delta}$ we have $K.E_i > 0$ by Proposition 5.1 and thus for $A$ sufficiently ample the divisor

$$\sigma^*(A) + K|_Y$$

is ample. Then a sufficiently high multiple $k \cdot (\sigma^*(A) + K|_Y)$ is very ample and in particular basepointfree. Thus if $H^0(Y, -K|_Y) \neq 0$ it follows that $k \cdot \sigma^*(A)$ is basepointfree as well for $k \gg 0$.

To see that $H^0(Y, -K|_Y) \neq 0$ take the exact sequence

$$0 \to H^0(\mathbb{P}, \mathcal{O}(-K - Y)) \to H^0(\mathbb{P}, \mathcal{O}(-K)) \to H^0(Y, \mathcal{O}(-K|_Y)).$$

Obviously $H^0(\mathbb{P}, \mathcal{O}(-K)) \neq 0$ since $-K$ is effective. Under the assumption $l^*(\Delta) > 0$ the divisor $K + Y$ is either effective or 0 by the formula in Construction 2.5. If it is effective then $H^0(\mathbb{P}, -K - Y) = 0$ whereas if $Y \sim_{\text{lin}} -K$ then $F(\Delta)$ is just a point and $\sigma = \rho$ the identity morphism.

Thus $k \cdot \sigma^*(A)$ is basepointfree for $k \gg 0$ and the remaining part follows as in ([KM98]): By (16)

$$m \cdot (D + \sigma^*(A)) + k \cdot \sigma^*(A) = mD + (m + k)\sigma^*(A) \quad m, k \gg 0$$

is basepointfree. The morphism associated to the complete linear system of some multiple of this divisor is birational by Remark 6.6 and it contracts exactly the curves $E_1, ..., E_s$, thus we obtain the morphism $\sigma : Y \to Z_{C(\Delta)}$. But then

$$D_a := amD + a(m + k)\sigma^*(A) \in \sigma^*(\text{Pic}(Z_{C(\Delta)})) \quad a, m, k \gg 0$$

and

$$mD + (m + k)\sigma^*(A) = D_{a+1} - D_a \in \sigma^*(\text{Pic}(Z_{C(\Delta)})) \quad k, m \gg 0$$

and finally $D = (m + 1)D + (m + 1 + k - 1)\sigma^*(A) - mD - (m + k)\sigma^*(A) \in \sigma^*(\text{Pic}(Z_{C(\Delta)}))$. 

$\square$
Remark 6.9. On $Z_{\Sigma,f}$ for $f$ a very general $\Delta$-regular Laurent polynomial, that is $f$ outside a countable union of proper subvarieties of $U_{reg}(\Delta)$, there are Picard classes that come from the surrounding toric variety, but there might be also some additional Picard classes. By a Theorem of Lefschetz ([Vo02, Thm.11.30]) we have

$$\text{Pic}(Z_{\Sigma,f}) \cong H^{1,1}(Z_{\Sigma,f}) \cap H^2(Z_{\Sigma,f}, \mathbb{Z}).$$

We guess that for $f$ very general the rank of this group could be calculated via Hodge theory. The Theorem above then implies that these additional Picard classes are the same on $Z_{\Sigma,f}$ and on $Z_{C(\Delta),f}$.

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