Abstract. Let $\Gamma$ be a lattice of a semisimple Lie group $L$. Suppose that one parameter Ad-diagonalizable subgroup $\{g_t\}$ of $L$ acts ergodically on $L/\Gamma$ with respect to the probability Haar measure $\mu$. For certain proper subgroup $U$ of the unstable horospherical subgroup of $\{g_t\}$ we show that given $x \in L/\Gamma$ for almost every $u \in U$ the trajectory $\{g_tux : 0 \leq t \leq T\}$ is uniformly distributed with respect to $\mu$ as $T \to \infty$.

1. Introduction

Let $(X, B, \mu, T)$ be a probability measure preserving system, i.e. $\mu$ is a probability measure on the measurable space $(X, B)$ and the measurable map $T : X \to X$ preserves $\mu$. The Birkhoff ergodic theorem says that if $T$ is ergodic then for every $f \in L^1_\mu(X)$

\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} f(T^nx) = \int_X f \, d\mu
\end{equation}

for almost every $x \in X$.

Suppose that $X$ is a locally compact second countable Hausdorff topological space and $B$ is the Borel sigma algebra of $X$. Given $x \in X$ the condition that (1.1) holds for every $f$ belonging to the set $C_c(X)$ of continuous functions with compact support is equivalent to

\begin{equation}
\lim_{N \to \infty} \frac{1}{N} \sum_{n=0}^{N-1} \delta_{T^nx} = \mu
\end{equation}

in the space of finite measures on $X$ under the weak* topology. Here $\delta_y$ denotes the Dirac measure supported on $y \in X$. A Radon measure
ν on X is said to be (T, µ) generic if (1.2) holds for ν almost every x ∈ X. A natural question is whether a measure ν (usually singular to µ) is (T, µ) generic.

This question is studied by several authors for natural dynamical systems on X = \mathbb{R}/\mathbb{Z}. Let m, n be coprime positive integers greater than or equal to 2. Suppose that µ_X is the Lebesgue measure on X and T_n = x_n modulo Z. Host [12] shows that any T_m invariant and ergodic probability measure ν on X with positive entropy is (T_n, µ_X) generic.

This result is strengthened by Hochman and Shmerkin [11] where they prove that for any C^2 diffeomorphism \varphi : \mathbb{R} → \mathbb{R}, the push forward of ν modulo Z is (T_n, µ_X) generic. The reader can find detailed references of related results in [11].

The aim of this paper is to address this question for one parameter flows in homogeneous space. Let Γ be a lattice of a Lie group L. Every subgroup H of L acts on L/Γ by left translations and this action preserves the probability Haar measure µ_{L/Γ}. We use (H, L/Γ) to denote this measure preserving system. There are two basic types of one parameter subgroup t → g_t ∈ L in terms of its image under the adjoint representation Ad : L → GL(l) where l is the Lie algebra of L. If Ad(g_t) is unipotent, then according to Ratner’s uniform distribution theorem [20] the Dirac measure δ_x of any point x ∈ L/Γ is generic with respect to some \{g_t : t ∈ \mathbb{R}\} ergodic homogeneous probability measure. If the one parameter subgroup is Ad-diagonalizable, i.e. Ad(g_t) is diagonalizable over \mathbb{R}, the unstable horospherical subgroup of \{g_t : t ∈ \mathbb{R}\} is defined by

U_L^+ = \{h ∈ L : g_t^{-1} h g_t → e \text{ as } t → ∞\}.

Here and throughout the paper we use the bold faced letter e to denote neutral element of group. A variant of Birkhoff ergodic theorem says that if \{g_t : t ∈ \mathbb{R}\}, L/Γ is ergodic then given any x ∈ L/Γ and any f ∈ C_c(L/Γ)

(1.3) \lim_{T → \infty} \frac{1}{T} \int_0^T f(g_t u x) \, dt = \int_{L/Γ} f \, dμ_{L/Γ}

holds for almost every u ∈ U_L^+ with respect to the Haar measure of U_L^+. Suppose that μ is a \{g_t : t ∈ \mathbb{R}\} invariant probability measure on L/Γ. We say that a Radon measure ν on U_L^+ or more generally on L is (g_t, µ) generic at x ∈ L/Γ if for any f ∈ C_c(L/Γ) and ν almost every u we have (1.3) holds. We remark here that the property of being (g_t, µ) generic only depends on the equivalence class of the measure ν.

Unlike one parameter Ad-unipotent subgroup few results are known for Ad-diagonalizable subgroup when ν on U_L^+ is singular to the Haar
measure. We do know many examples of probability measure $\nu$ whose pushforward image under $g_t$ as $t \to \infty$ or trajectory under $\{g_t : 0 \leq t \leq T\}$ as $T \to \infty$ is equidistributed with respect to some probability homogeneous measure. The reader can find precise description of these measures for asymptotic results in Shah \[21\][22][23][24][25], Shah and Weiss \[26\]; and for average results by author in \[27\][28\].

We investigate pointwise equidistribution for measures studied in \[21\] and \[26\] above. Let $G \leq L$ be a connected semisimple Lie group without compact factors. Ratner’s theorem \[20\] implies that for $x \in L/\Gamma$ the orbit closure $Gx$ is a finite volume homogeneous space, i.e. $Gx = Hx$ where $H = \{g \in L : gGx = Gx\}$ and there is a unique $H$ invariant probability measure (denoted by $\mu_{Gx}$) supported on $Gx$.

The main result of this paper is:

**Theorem 1.1.** Let $\{g_t : t \in \mathbb{R}\}$ be an Ad-diagonalizable one parameter subgroup of a connected semisimple Lie group $G$ without compact factors. Suppose that the projection of $g_t$ to each simple factor of $G$ is nontrivial. Let $\Gamma$ be a lattice of a Lie group $L$ which contains $G$. Then for every $x \in L/\Gamma$ the Haar measure of $U^+_G$ is $(g_t, \mu_{Gx})$ generic at $x$.

Our result is new in the following simple case: \(G = \begin{pmatrix} SL_2(\mathbb{R}) & 0 \\ 0 & 1 \end{pmatrix}\), $g_t = \text{diag}(e^t, e^{-t}, 1)$, $L = SL_3(\mathbb{R})$, $\Gamma = SL_3(\mathbb{Z})$. The key property we use for the group $U^+_G$ is the $g_1$ expanding property which we describe now. Let $\{g_t : t \in \mathbb{R}\}$ and $G$ be as in Theorem 1.1. Every representation $\rho$ of $G$ on a finite dimensional real vector space $V$ splits into a direct sum $V^+ \oplus V^0 \oplus V^-$ of $\rho(g_1)$ invariant subspaces so that restrictions of $\rho(g_1)$ to the spaces $V^+, V^0, V^-$ have eigenvalues $> = < 1$ respectively. Let $\pi_+$ be the projection from $V$ to $V^+$. A connected subgroup $U$ of $G$ normalized by $g_t$ is said to be $g_1$ expanding if for every nontrivial irreducible representation $\rho$ of $G$ on $V$ and every nonzero vector $v \in V$ one has that the map

$$U \to V \quad \text{given by} \quad u \to \pi_+(\rho(u)v)$$

is not identically zero. It can be proved that $U$ is $g_1$ expanding if and only if $U \cap U^+_G$ is $g_1$ expanding.

One family of $g_1$ expanding subgroups comes from epimorphic subgroups of algebraic groups introduced by Bien and Borel \[6\]. Suppose that $G$ is the connected component of real points of some semisimple linear algebraic group defined over $\mathbb{R}$. Let $S \leq G$ be a one dimensional $\mathbb{R}$ split algebraic torus and let $U$ be a unipotent algebraic subgroup of $G$ normalized by $S$. Let $H$ be the subgroup generated by $S$ and $U$. The group $H$ is epimorphic in $G$ if any $H$ fixed vector of an algebraic
representation of $G$ is also fixed by $G$. It is proved in [26] Proposition 2.2 that if $H$ is an epimorphic subgroup of $G$ then $U$ is $g_1$ expanding for some choice of the parameterization of the connected component of $S$.

Under an additional abelian assumption for $g_1$ expanding subgroup $U$ we prove the following

**Theorem 1.2.** Let \( \{g_t : t \in \mathbb{R}\} \) be an Ad-diagonalizable one parameter subgroup of a connected semisimple Lie group $G$ without compact factors. Let $\Gamma$ be a lattice of a Lie group $L$ which contains $G$. Suppose that $U \leq U^+_G$ is a connected $g_1$ expanding abelian subgroup of $G$. Then for every $x \in L/\Gamma$ the Haar measure of $U$ is \((g_t, \mu \mid_{Gx})\) generic at $x$.

Here we give some concrete examples that Theorem 1.2 applies. Let $m, n$ be two positive integers and let \( v = (a_1, \ldots, a_m, -b_1, \ldots, -b_n) \) where $a_i, b_j > 0$ and $a_1 + \cdots + a_m = b_1 + \cdots + b_n$. For every $\xi \in M_{mn}$ where $M_{mn}$ is the set of $m \times n$ matrices, we let $u(\xi) = \begin{pmatrix} I_m & \xi \\ 0 & I_n \end{pmatrix}$ where $I_m$ and $I_n$ are identity matrices of order $m$ and $n$ respectively. Let diagonal matrix $g_v = \text{diag}(e^{a_1 t}, \ldots, e^{a_m t}, e^{-b_1 t}, \ldots, e^{-b_n t})$. It follows from Kleinbock and Weiss [13] Proposition 2.4 that the group $U = \{u(\xi) : \xi \in M_{mn}\}$ is $g_v$ expanding. Therefore as a special case of Theorem 1.2 we have

**Corollary 1.3.** Let $\Gamma$ be a lattice of $G = \text{SL}(m + n, \mathbb{R})$ and let $\mu$ be the probability Haar measure on $G/\Gamma$. Then for every $x \in G/\Gamma$ the additive Haar measure of $U = \{u(\xi) : \xi \in M_{mn}\}$ is \((g_v, \mu)\) generic at $x$.

The abelian assumption of Theorem 1.2 for the group $U$ might be superfluous. The only place where we essentially need it is the shadowing Lemma 4.6 and its variant Lemma 5.2 which are links between random walks and flows. We do not know how to get shadowing lemma and simultaneously the contraction property Lemma 3.4 even in the case where $U$ is the two step Heisenberg group. This is also the main obstruction that we cannot apply our method to the case of volume measures of curves studied by [22][23][24][25], e.g. nonplanar analytic curves in $U^+_G$ where $G = \text{SL}(n, \mathbb{R})$ and $g_t = \text{diag}(e^{(n-1)t}, e^{-t}, \ldots, e^{-t})$. Theorem 1.1 is deduced from Theorem 1.2 and the asymptotic equidistribution of measures proved in [26]. This type of deduction might be able to prove pointwise equidistribution in some other cases where $U$ is not abelian.

The proof of Theorem 1.2 is based on quantitative estimate of the \( \{g_t : 0 \leq t \leq T\} \) trajectory of measures. The method is inspired
by Chaika and Eskin [7] where they prove Birkhoff type ergodic theorem for Teichmüller geodesic flows on moduli spaces and by Benoist and Quint [4] where they prove almost everywhere equidistribution of Random walks on homogeneous space.

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2. Outline of the proof

We first outline the proof of Theorem 1.2 and leave details of the proof of Proposition 2.2, 2.3 and 2.5 for later sections. Let $G, g_t, U$ be as in Theorem 1.2. In particular $U \leq U^+_G$ is a connected abelian $g_1$ expanding subgroup of $G$. It follows from Ratner [19] Proposition 1.3 that $U$ is simply connected. We fix an isomorphism of Lie groups

$$u : \mathbb{R}^m \rightarrow U$$

so that there are positive real numbers $b_1, \ldots, b_m$ such that for standard basis $\{e_i\}_{1 \leq i \leq m}$ of $\mathbb{R}^m$ one has

$$g_t u(e_i)g_{-t} = u(e^{tb_i} e_i).$$

It is not hard to see that Theorem 1.2 follows from

Theorem 2.1. Let $\{g_t : t \in \mathbb{R}\}$ be an Ad-diagonalizable one parameter subgroup of a connected semisimple Lie group $G$ without compact factors. Let $\Gamma$ be a lattice of a Lie group $L$ which contains $G$. Suppose that $U \leq U^+_G$ is a connected $g_1$ expanding abelian subgroup of $G$. Let $x \in L/\Gamma$, let the interval $I = [-1, 1]$ and let $u$ be a fixed isomorphism as in (2.1) so that (2.2) holds. Then for almost every $w \in I^m$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g_t u(w)\delta_x dt = \mu_{\Gamma x}.$$  

Here $g_t u(w)\delta_x$ is the pushforward of the Dirac measure $\delta_x$ by $g_t u(w)$ and it is equal to $\delta_{g_t u(w)x}$. In the rest of this section we assume the notation and assumption in Theorem 2.1. To prove it we first establish unipotent invariance:

Proposition 2.2. For almost every $w \in I^m$, if $\nu_w$ is any weak* limit point of $\frac{1}{T} \int_0^T g_t u(w)\delta_x dt$ as $T \to \infty$, then $\nu_w$ is invariant under $U$.

Next we prove nonescape of mass (Corollary 2.4) using quantitative nonescape of mass for $\{g_t : 0 \leq t \leq T\}$ trajectory of the measure associated to the Lebesgue measure on $I^m$. For every measurable subset
$K$ of $X$, positive real number $T$ and $w \in I^m$ we use $A^T_K(w)$ to denote the proportion of the trajectory $\{g_t u(w)x : 0 \leq t \leq T\}$ in $K$. More precisely,
\begin{equation}
(2.4) \quad A^T_K(w) := \frac{1}{T} \int_0^T 1_K(g_t u(w)x) \, dt
\end{equation}
where $1_K$ is the characteristic function of $K$. For every measurable subset $J$ of $\mathbb{R}^m$ we let $|J|$ to denote the Lebesgue measure of $J$.

**Proposition 2.3.** For every $0 < \epsilon < 1$, there is a compact subset $K$ of $L/\Gamma$ and a positive real number $c < 1$ such that
\begin{equation}
(2.5) \quad \left| \left\{ w \in I^m : A^T_K(w) \leq 1 - \epsilon \right\} \right| \leq c^T
\end{equation}
for every $T \geq 0$.

**Corollary 2.4.** For almost every $w \in I^m$, any weak$^*$ limit point of $\frac{1}{T} \int_0^T g_t u(w)\delta_x \, dt$ as $T \to \infty$ is a probability measure.

**Proof.** Given $0 < \epsilon < 1$, according to Proposition 2.3 there exists a compact subset $K$ of $L/\Gamma$ and a positive number $c < 1$ so that (2.5) holds as $T$ runs through all the positive integers. So Borel-Cantelli Lemma implies that
\begin{equation}
(2.6) \quad \liminf_{n \to \infty, n \in \mathbb{N}} A^n_K(w) \geq 1 - \epsilon
\end{equation}
for almost every $w \in I^m$. It follows from (2.6) that for almost every $w \in I^m$
\begin{equation}
\liminf_{T \to \infty} A^T_K(w) \geq 1 - \epsilon.
\end{equation}
Since we can take $\epsilon$ arbitrarily close to zero, the conclusion follows. \qed

Let $H$ be the group generated by $\{g_t : t \in \mathbb{R}\}$ and $U$. It follows from Mozes \cite{17} Theorem 1 that any $H$ invariant probability measure on $L/\Gamma$ is $G$ invariant. A closed subset $Y$ of $L/\Gamma$ is said to be a finite volume homogeneous subspace if a closed subgroup $L'$ of $L$ acts transitively on $Y$ and $L'$ preserves a probability measure $\mu_Y$ on $Y$. We say $Y$ is $G$ ergodic if $G$ acts ergodically on $(Y, \mu_Y)$. Let $C_L(G)$ be the group of centralizers of $G$ in $L$. It follows from \cite{14} Proposition 2.1 that $G$ ergodic probability measures on $L/\Gamma$ is at most a countable union of the set
\begin{equation}
C_L(G)\mu_Y := \{g\mu_Y : g \in C_L(G)\}
\end{equation}
where $Y$ is a $G$ ergodic finite volume homogeneous subspace. Without loss of generality we may assume that $\overline{Gx} = L/\Gamma$. We show that for almost every $w \in I^m$ any weak$^*$ limit $\nu_w$ of $\frac{1}{T} \int_0^T g_t u(w)\delta_x \, dt$ as $T \to \infty$ does not put any mass on $C_L(G)Y$ for any proper $G$ ergodic
finite volume homogeneous subspace $Y$. This is proved by a similar quantitative result for $\{g_t : 0 \leq t \leq T\}$ trajectory of the measure associated to the Lebesgue measure of $I^m$.

**Proposition 2.5.** Suppose that $Gx$ is dense in $L/\Gamma$. Let $Y$ be a proper $G$ ergodic finite volume homogeneous subspace. For any compact subset $F$ of $C_L(G)$ and any $\epsilon_1 > 0$, there exists a compact subset $K_1$ of $L/\Gamma$ with $K_1 \cap FY = \emptyset$ and a positive number $c_1 < 1$ such that
\[
\left| \{w \in I^m : \mathcal{A}^T_{K_1}(w) \leq 1 - \epsilon_1\} \right| \leq c_1^T
\]
for every $T \geq 0$.

**Corollary 2.6.** Suppose that $Gx$ is dense in $L/\Gamma$. Let $Y$ be a proper $G$ ergodic finite volume homogeneous subspace. Then for almost every $w \in I^m$ one has $\nu_w(C_L(G)Y) = 0$ for any weak* limit $\nu_w$ of $\frac{1}{T} \int_0^T g_t u(w) \delta_x dt$ as $T \to \infty$.

The proof uses Proposition 2.5 and is the same as that of Corollary 2.4 so we omit the details here.

**Proof of Theorem 2.7.** It follows from Ratner’s orbit closure theorem [20] that $Gx$ is dense in a $G$ ergodic finite volume homogenous subspace of $L/\Gamma$. So we can without loss of generality assume that $Gx$ is dense in $L/\Gamma$.

It follows from Proposition 2.2, Corollary 2.4 and Corollary 2.6 that there exists a subset $J$ of $I^m$ with full measure such that for any $w \in J$, any weak* limit $\nu_w$ of $\frac{1}{T} \int_0^T g_t u(w) \delta_x dt$ as $T \to \infty$ has the following properties:

- $\nu_w$ is invariant under $U$ and hence invariant under $G$;
- $\nu_w$ is a probability measure;
- $\nu_w(C_L(G)Y) = 0$ for any proper $G$ ergodic finite volume homogeneous subspace $Y$.

Therefore for any $w \in J$ we have (2.3) holds. This completes the proof. \[\square\]

Here we describe a general strategy of using Theorem 1.2 to prove pointwise equidistribution for other $g_1$ expanding subgroups not necessarily abelian. In particular we derive Theorem 1.1 from it. We need to use the following

**Theorem 2.7 (21, 26).** Let $U'$ be a connected $\text{Ad}$-unipotent $g_1$ expanding subgroup of $G$. Suppose that $\mu$ is a probability measure on $U'$ absolutely continuous with respect to the Haar measure. Let $\mu_x$ be the push forward of $\mu$ to $L/\Gamma$ with respect to the map $u \in U' \to ux$. Then
\[
g_t \mu_x \to \mu_{Gx} \quad \text{as } t \to \infty.
\]
This result is not explicitly stated in both of the papers but it is a simple consequence. Let $H$ be the subgroup of $G$ generated by $\{g_t : t \in \mathbb{R}\}$ and $U'$. It is easy to see that $g_1$ expanding property implies that the Zariski closure of $\rho(H)$ is an epimorphic subgroup of $\rho(G)$ for any finite dimensional real representation $\rho$ of $G$. Furthermore the ray $\{g_t : t > 0\}$ is contained in the cone of $[26]$ Lemma 2.1 for any nontrivial irreducible representation $\rho$. Therefore Theorem 2.7 follows from $[26]$ Theorem 1.4.

In case $U'$ is abelian and $g_1$ expanding we obtain in Theorem 1.2 that the Haar measure of $U'$ is generic at $x \in L/\Gamma$. This seems to be true when $U'$ is not assumed to be abelian. In view of Theorem 2.7 it suffices to show that for almost every $u \in U'$

$$\lim_{T \to \infty} \frac{1}{T} \int_0^T g_t u \delta_x dt$$

exists in the space of probability measures on $L/\Gamma$. Theorem 1.2 will give this almost everywhere existence if there is an abelian subgroup subgroup $U_a$ of $U'$ normalized by $g_t$ and a connected semisimple subgroup $G_1$ of $G$ without compact factors so that the following holds:

$$(*) \{g_t : t \in \mathbb{R}\} \text{ is a subgroup of } G_1 \text{ and } U_a \text{ is a } g_1 \text{ expanding subgroup of } G_1.$$

For example, if $G$ is the real rank one group $SU(2,1)$ then the unstable horospherical subgroup $U_G^+$ is not abelian. But we can find a subgroup with Lie algebra $sl_2(\mathbb{R})$ containing the group $\{g_t : t \in \mathbb{R}\}$. More generally we have

**Lemma 2.8.** Under the assumption of Theorem 1.1 there is a connected semisimple subgroup $G_1 \subset G$ without compact factors and an abelian subgroup $U_a$ of $U_G^+$ such that property $(*)$ holds.

The proof of this lemma uses strongly orthogonal system of simple root systems and will be given in the appendix. Lemma 2.8 together with Theorem 1.2 and Theorem 2.7 proves Theorem 1.1.

3. Some auxiliary results

3.1. Large deviation. In this section we prove a large deviation result. Our argument is inspired by [3] and [1].

Let $(W, B, \mu)$ be a standard Borel space with probability measure $\mu$. The conditional expectation of a nonnegative Random variable $\tau$ (i.e. a measurable map $\tau : W \to [0, \infty]$) with respect to a sub sigma algebra $A$ of $B$ is an $A$ measurable function $E(\tau|A)$ such that for any $A \in A$ one has $\int_A \tau(w) d\mu(w) = \int_A E(\tau|A)(w) d\mu(w)$. The conditional
probability of $A \in \mathcal{B}$ is the function $\mu(A|\mathcal{A}) := E(\mathbb{1}_A|\mathcal{A})$ where $\mathbb{1}_A$ is the characteristic function of $A$. For a nonnegative random variable $\tau$ and $a \in \mathbb{R}$ we will follow the the convention of probability theory to write $\mu(\tau \geq a)$ for $\mu(\{w \in W : \tau(w) \geq a\})$ and $E(\tau)$ for $\int_W \tau(w) \, dw$

In this paper the set of natural numbers is $\mathbb{N} = \{0, 1, 2, \ldots \}$. A measurable map $\tau : W \to \mathbb{N} \cup \{\infty\}$ with $\tau(w) < \infty$ almost surely is called $\mathbb{N}$ valued random variable. A sequence of random variables $(\tau_i)_{i \in \mathbb{N}}$ is said to be increasing if $\tau_i(w) \geq \tau_{i-1}(w)$ for any $i \geq 1$ and almost every $w \in W$. A sequence of sub sigma algebras $(\mathcal{A}_i)_{i \in \mathbb{N}}$ of $\mathcal{B}$ is said to be a filtration if $\mathcal{A}_{i-1} \subseteq \mathcal{A}_i$. In the rest of this section the relations $= \text{ and } \leq$ for functions on $W$ are meant to hold almost surely.

Lemma 3.1. Let $(\tau_i)_{i \in \mathbb{N}}$ be an increasing sequence of $\mathbb{N}$ valued Random variables on $W$. Let $(\mathcal{A}_i)_{i \in \mathbb{N}}$ be a sequence of filtrations of sub sigma algebras of $\mathcal{B}$ such that $\tau_i$ is $\mathcal{A}_i$ measurable. Suppose that there exits $\vartheta_0 > 0$ and $Q_0 > 0$ such that

$$
(3.1) \quad \mu(\tau_i - \tau_{i-1} \geq q|\mathcal{A}_{i-1}) \leq e^{-\vartheta_0 q}
$$

for every $q \geq Q_0$ and $i \geq 1$. Then for every $\epsilon > 0$ there exists $\vartheta > 0$ such that for all sufficiently large $Q$ and any positive integer $n$ we have

$$
(3.2) \quad \mu \left( \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_Q(\tau_i(w) - \tau_{i-1}(w)) \geq \epsilon \right) \leq e^{-\vartheta n}
$$

where $\mathbb{1}_Q : \mathbb{N} \to \mathbb{N}$ is defined by

$$
(3.3) \quad \mathbb{1}_Q(q) = \begin{cases} 
q & \text{if } q \geq Q \\
0 & \text{otherwise.}
\end{cases}
$$

Remark: It can be seen from the proof below that $Q$ and $\vartheta$ only depend on $Q_0, \epsilon$ and $\vartheta_0$ but not on the probability space, the the sequence of random variables or the filtration of sigma algebras.

Proof. We will show that if $\vartheta = \epsilon \vartheta_0 / 4$ and $Q \geq Q_1$ where

$$
(3.4) \quad Q_1 = \max \left\{ \frac{2 \log[(\epsilon^{\vartheta_0/4} - 1)(1 - e^{-\vartheta_0/2})]}{-\vartheta_0}, Q_0 \right\}
$$

then (3.2) holds.

For every positive integer $n$ we define a function $f_n$ on $W$ by

$$
f_n(w) = \exp \left( \frac{\vartheta_0}{2} \sum_{i=1}^{n} \mathbb{1}_Q(\tau_i(w) - \tau_{i-1}(w)) \right).
$$
By monotone convergence theorem for conditional expectations of non-negative random variables we have

\[ E(f_n | A_{n-1}) \leq f_{n-1} \left( 1 + \sum_{q \geq Q} e^{\theta_0 q/2} \mu(\tau_n - \tau_{n-1} = q | A_{n-1}) \right) \]

\[ \leq f_{n-1} \frac{1 - e^{-\theta_0/2} + e^{-Q\theta_0/2}}{1 - e^{-\theta_0/2}}. \]

By induction on \( n \) one gets

\[ E(f_n) \leq \left( 1 - e^{-\theta_0/2} + e^{-Q\theta_0/2} \right)^n. \]

On the other hand by Chebyshev inequality

\[ E(f_n) \geq e^{\epsilon n \theta_0/2} \mu \left( \frac{1}{n} \sum_{i=1}^n 1_Q(\tau_n - \tau_{n-1}) \geq \epsilon \right). \]

Therefore

\[ \mu \left( \frac{1}{n} \sum_{i=1}^n 1_Q(\tau_n - \tau_{n-1}) \geq \epsilon \right) \leq \left( \frac{1 - e^{-\theta_0/2} + e^{-Q\theta_0/2}}{e^{\epsilon \theta_0/2} (1 - e^{-\theta_0/2})} \right)^n. \]

In view of the first lower bound of \( Q_1 \) in (3.4) the conclusion follows.

3.2. unipotent invariance. The aim of this section is to prove Proposition 2.2. Our argument is modeled on [7] §3. Let \( L, \Gamma, G, g_t, U, x \) be as in Theorem 2.1.

We observe that there exists a countable dense subset of \( C_c(L/\Gamma) \) consisting of smooth functions. Also if \( s_1, s_2 \) are linearly independent over \( Q \), then the closure of the group \( \langle u(s_1 e_j), u(s_2 e_j) : 1 \leq j \leq m \rangle \) is \( U \). Therefore Proposition 2.2 will follow if we can show that for every \( \psi \in C_c^\infty(L/\Gamma) \), \( s > 0 \) and \( 1 \leq i \leq m \) we have for almost every \( w \in \Gamma \)

\[ \frac{1}{T} \int_0^T \psi_t(w) \, dw \to 0 \quad \text{as} \ T \to \infty \]

where

\[ \psi_t(w) = \psi[gu(w)x] - \psi[u(se_i)gu(w)x]. \]

We will prove (3.6) using law of large numbers. The key is the following estimate:

**Lemma 3.2.** There exists \( \vartheta > 0 \) and \( C > 0 \) such that for any \( t, l > 0 \)

\[ \int_{\Gamma^m} \psi_t(w) \psi_l(w) \, dw \leq Ce^{-\vartheta |l-t|}. \]

Lemma 3.2 allows us to use the following lemma to complete the proof of (3.6) and hence Proposition 2.2.
**Lemma 3.3** ([17] Lemma 3.4). Suppose that $\psi_t : I^m \to \mathbb{R}$ are bounded functions satisfying (3.7) (for some $C > 0$ and $\vartheta > 0$). Additionally, assume that $\psi_t(w)$ are Lipschitz functions of $t$ for each $w \in I^m$. Then (3.6) holds for almost every $w \in I^m$.

**Proof of Lemma 3.3** We fix a right invariant Riemannian metric on $L$ and let $d(\cdot, \cdot)$ be the induced distance function. We note that the function $\psi$ is Lipschitz, i.e. $|\psi(gy) - \psi(hy)| \ll d(g, h)$ for any $g, h \in L$ and $y \in L/\Gamma$.

Without loss of generality we assume that $l > t$ and $i = 1$. Let $b = b_1 > 0$ which is defined in the beginning of [12] i.e. $g_1 u(e_1) g_1^{-1} = u(e^b e_1)$. Then

$$
\psi_t(w) = \psi[g_t u(w)x] - \psi[g_t u(w + se^{-bl} e_1)x].
$$

We will show that for $\vartheta = b/2$ there exists $C > 0$ so that (3.7) holds.

We divide $[-1, 1]$ consecutively into intervals of the form

$$
I(r) = [r - e^{-(l+t)b/2}, r + e^{-(l+t)b/2}]
$$

except for the last part which will not affect the validity of (3.7) since it has length less than $2e^{-(l+t)b/2}$. For every $s_1 \in \mathbb{R}$ with $|s_1| \leq e^{-(l+t)b/2}$ we have

$$
d(g_t u(s_1 e_1), g_t) = d(u(e^{bt} s_1 e_1), e) \ll e^{-(l-t)b/2}.
$$

As noted above that the function $\psi$ is Lipschitz, so for every $s_1 \in I(r)$ and $w \in \{0\} \times I^{m-1}$ one has

$$
|\psi_t(s_1 e_1 + w) - \psi_t(r e_1 + w)| \ll e^{-(l-t)b/2}.
$$

Therefore for any $w \in \{0\} \times I^{m-1}$

$$
\frac{1}{|I(r)|} \int_{I(r)} \psi_t(s_1 e_1 + w) \psi_t(s_1 e_1 + w)\, ds_1
= \frac{\psi_t(r e_1 + w)}{|I(r)|} \int_{I(r)} \psi_t(s_1 e_1 + w)\, ds_1 + O(e^{-(l-t)b/2}).
$$

Since the interval $I(r)$ and $I(r) + se^{-bl}$ have overlaps except for ends whose length are $se^{-bl}$, we have

$$
\frac{1}{|I(r)|} \int_{I(r)} \psi_t(s_1 e_1 + w)\, ds_1 \ll 2se^{-(l-t)b/2}.
$$

We sum up the integral of $\psi_t \psi_t$ over a covering of $[-1, 1]$ by consecutive intervals of the form $I(r)$, then (3.8), (3.9) and Fubini theorem with respect to $I \times I^{m-1}$ give (3.7).
3.3. Linear representations. Let $G, g_t, U$ be as in Theorem 2.1 and $H$ be the subgroup of $G$ generated by $U$ and $\{g_t : t \in \mathbb{R}\}$. The main result of this section is

**Lemma 3.4.** Let normed real vector spaced $V$ be a finite dimensional representation of $G$ without nonzero $G$ invariant vectors. Then there exists $\vartheta_0 > 0$ so that the following holds: for every $0 < \vartheta < \vartheta_0$ and every $a > 0$ there exists $T_0 > 0$ such that if $\tau : I^m \to \mathbb{R}_{\geq 0}$ is any measurable function with $\inf_{w \in I^m} \tau(w) \geq T_0$ then

$$\sup_{\|v\|=1} \int_{I^m} \frac{dw}{\|g_\tau(w)u(w)v\|} \leq a$$

where $\| \cdot \|$ is the norm on $V$.

We decompose $V$ as $V^+ \oplus V^0 \oplus V^-$ according to the eigenvalues of $g_1$, i.e. $V^+$ is the sum of eigen spaces of $g_1$ whose eigenvalues are greater than one, etc. Let $\pi_+$ be the projection from $V$ to $V^+$. For every $v \in V, r > 0$ we set

$$D^+(v, r) = \{w \in I^m : \|\pi_+(u(w)v)\| \leq r\}.$$

**Lemma 3.5.** Let $V$ be as in Lemma 3.4. Then there exists $0 < \vartheta_0 \leq 1$ such that

$$\sup_{\|v\|=1, r > 0} \left| D^+(v, r) \right| < \infty.$$  \hspace{1cm} (3.11)

*Proof.* Recall that $U$ is assumed to be $g_1$ expanding in $G$. When $v$ varies in the unit sphere of $V$ the family of maps which send $w \in I^m \to \pi_+(u(w)v)$ are polynomials in $w$ with degree uniformly bounded from above and maximum of coefficients in absolute value uniformly bounded from below by some positive constant. So the lemma follows from the $(C, \alpha)$-good property of polynomial functions proved in [5].

Proof of Lemma 3.4. Our proof basically follows that of [9] Lemma 5.1. We take $\vartheta_0 > 0$ so that (3.11) holds. For fixed $a > 0$ and $0 < \vartheta < \vartheta_0$ we need to find $T_0$ so that (3.10) holds for any $\tau$ whose value on $I^m$ is bounded from below by $T_0$.

First we need some preparation. As $V$ is finite dimensional there exists $C_1 > 1$ such that for every vector $v_1 \in V$ one has $\|\pi_+(v_1)\| \leq C_1 \|v_1\|$. Let $b > 0$ so that $e^b$ is the smallest eigenvalue of $g_1$ in $V^+$. Let $C$ be the constant in (3.11) and let

$$r = \sup_{\|v\|=1, w \in I^m} \|\pi_+(u(w)v)\|.$$
We will show that for $T_0 > 0$ which satisfies
\begin{equation}
\frac{2CC_1 r^{\vartheta_0 - \vartheta}}{1 - 2^{\vartheta - \vartheta_0}} e^{-bT_0 \vartheta} = a
\end{equation}
the conclusion holds.

We fix a unit vector $v \in V$, function $\tau$ on $I^m$ with $\inf \tau \geq T_0$ and estimate the integral of
\[
f_{\tau, v}(w) := \|g_{\tau(w)}u(w)v\|^{-\vartheta}.
\]
Since $\|g_{\tau(w)}u(w)v\| \geq C_1^{-1} e^{bT_0} \|\pi_+ [u(w)v]\|$ one has
\begin{equation}
f_{\tau, v}(w) \leq C_1 e^{-bT_0 \vartheta} \|\pi_+ [u(w)v]\|^{-\vartheta}
\end{equation}
for every $w \in I^m$. For every nonnegative integer $n$, (3.11) and (3.13) imply that
\begin{equation}
\int_{D^+(v, r2^{-n}) \setminus D^+(v, r2^{-n-1})} f_{\tau, v}(w) \, dw \leq e^{-bT_0 \vartheta} 2CC_1 r^{\vartheta_0 - \vartheta} 2^{-n(\vartheta_0 - \vartheta)}.
\end{equation}
We write
\[I^m = D^+(v, 0) \bigcup_{n \geq 0} [D^+(v, 2^{-n}r) \setminus D^+(v, 2^{-n-1}r)].\]
Since $|D^+(v, 0)| = 0$, we have
\[
\int_{I^m} f_{\tau, v}(w) \, dw = \sum_{n=0}^{\infty} \int_{D^+(v, r2^{-n}) \setminus D^+(v, r2^{-n-1})} f_{\tau, v}(w) \, dw
\]
by (3.14) \leq \frac{2CC_1 r^{\vartheta_0 - \vartheta}}{1 - 2^{\vartheta - \vartheta_0}} e^{-bT_0 \vartheta}
by (3.12) = a.

4. NONESCAPE OF MASS

The aim of this section is to prove Proposition 2.3. Let $L, \Gamma, G, g_t, U, x$ be as in Theorem 2.1 and let $X = L/\Gamma$. The main tool is the contraction property of a function $\alpha$ (we call it height function) on $X$ which measures whether points in $X$ are close to $\infty$. The height function with the contraction property on homogeneous space is introduced by Eskin, Margulis and Mozes [9]. A significant improvement is given by Benoist and Quint [2] which will be used in this paper.
4.1. Existence of height function.

Lemma 4.1. Given a compact subset $Z$ of $X$ and a positive number $a < 1$, for $t$ sufficiently large (depending on $a$) there exists a lower semicontinuous function $\alpha : X \to [0, \infty]$ and $b > 0$ with the following properties:

1. For every $y \in X$

\[
\int_{I_m} \alpha(g_t u(w)y) \, dw \leq a\alpha(y) + b
\]

where $dw$ is the usual Lebesgue measure;

2. $\alpha$ is finite on $GZ$;

3. $\alpha$ is Lipschitz, i.e. for every compact subset $F_0$ of $G$ there exists $C > 0$ such that $\alpha(gy) \leq C\alpha(y)$ for every $y \in X$ and $g \in F_0$;

4. $\alpha$ is proper, i.e. if $\alpha(Z_0)$ is bounded for some subset $Z_0$ of $X$ then $Z_0$ is relatively compact.

Remark: Here lower semicontinuity implies that for every positive number $M$ the subset $\alpha^{-1}([0, M])$ is closed and hence compact by (4).

We first deal with the case where $\Gamma$ is arithmetic. For the moment we assume that $L = SL_d(\mathbb{R})$ and $\Gamma = SL_d(\mathbb{Z})$ where $d \geq 2$. It is well known that the space $X = SL_d(\mathbb{R})/SL_d(\mathbb{Z})$ can be identified with the set of unimodular lattices in $\mathbb{R}^d$. For every $y \in X$, let $\Lambda_y$ be the lattice in $\mathbb{R}^d$ corresponding to it, i.e. $\Lambda_y = gZ^d$ if $y = gSL_d(\mathbb{Z})$. A vector $v \in \wedge^* \mathbb{R}^d := \bigoplus_{0 \leq i \leq d} \wedge^i \mathbb{R}^d$ is monomial if $v = v_1 \wedge \cdots \wedge v_i$ where $v_1, \ldots, v_i \in \mathbb{R}^d$. We say $v$ is $y$-integral monomial if we can take $v_i \in \Lambda_y$.

We review the height function defined in [2]. Since $G$ is a connected semisimple Lie group contained in $L = SL_d(\mathbb{R})$, it is the connected component of real points of some real algebraic group. We fix a maximal connected diagonalizable subgroup $A$ of $G$ containing $\{g_t : t \in \mathbb{R}\}$. Let $\Phi(G, A)$ be the relative root system, i.e. the set of nonzero weights of $A$ appeared in the adjoint representation. We fix a positive system $\Phi(G, A)^+$ such that $\lambda(g_t) \geq 1$ for every $\lambda \in \Phi(G, A)^+$. We endow a partial order on the set $P$ of algebraic characters of $A$ by $\lambda \leq \mu$ if and only if $\mu - \lambda$ is nonnegative linear combination of $\Phi(G, A)^+$. For any irreducible finite dimensional real representation of $G$, the set of weights of $A$ in this representation has a unique maximal element called highest weight of the representation. Let $P^+$ be the set of all the highest weights appearing in $\wedge^* \mathbb{R}^d$.

For each $\lambda \in P^+$, let $q_\lambda$ be the projection from $\wedge^* \mathbb{R}^d$ to the subspace consisting of all the irreducible sub representations with highest weight.
λ. Let \( \| \cdot \| \) be the usual Euclidean norm on \( \wedge^* \mathbb{R}^d \). One of the key ingredients of \cite{2} is the following Mother Inequality:

**Lemma 4.2** (\cite{2} Proposition 3.1). There exists \( C_1 > 0 \) such that for any monomials \( u, v, w \) in \( \wedge^* \mathbb{R}^d \) one has the inequality

\[
\| q_\lambda(u) \| \cdot \| q_\mu(u \wedge v \wedge w) \| \leq C_1 \max_{\nu, \rho \in P^+} \| q_\nu(u \wedge v) \| \cdot \| q_\rho(u \wedge w) \|.
\]

We fix the following index:

\[
\delta_i = (d - i)i \quad \text{and} \quad \delta_\lambda = \log(\lambda(g_1))
\]

where \( 0 \leq i \leq d \) and \( \lambda \in P^+ \setminus 0 \) where 0 is the trivial character of \( A \). Recall that \( U \) is \( g_1 \) expanding, so for \( \lambda \in P^+ \setminus 0 \) we have \( \delta_\lambda > 0 \). Also we take

\[
\kappa = \left( \min_{\lambda \in P^+ \setminus 0} \delta_\lambda \right)^{-1} \quad \text{and} \quad \kappa_1 = \left( \max_{\lambda \in P^+ \setminus 0} \delta_\lambda \right)^{-1}.
\]

Let \( \varepsilon > 0 \) and \( 0 < i < d \). Following \cite{2} for every \( v \in \wedge^i \mathbb{R}^d \) we let

\[
\varphi_\varepsilon(v) = \begin{cases} 
\min_{\lambda \in P^+ \setminus 0} \delta_\lambda \| q_\lambda(v) \|^{-1} & \text{if} \quad \| q_0(v) \| < \varepsilon \delta_i \\
0 & \text{otherwise}
\end{cases}
\]

We remark here that \( \varphi_\varepsilon(v) = \infty \) if \( v = q_0(v) \) and \( \| v \| < \varepsilon \delta_i \).

**Lemma 4.3.** There exists \( \vartheta_1 > 0 \) such that for every \( \vartheta \) with \( 0 < \vartheta < \vartheta_1 \) and \( 0 < \alpha < 1 \) the following holds: for \( t \) sufficiently large and for every \( v \in \wedge^i \mathbb{R}^d \) with \( 0 < i < d \) one has

\[
\int_{t^m} \varphi_\varepsilon^\vartheta(g_t u(w)v) \, dw \leq a \varphi_\varepsilon^\vartheta(v) \quad \text{for any} \ 0 < \varepsilon < 1.
\]

**Proof.** Let \( V \) be the subspace of \( \wedge^* \mathbb{R}^d \) complementary to the subspace of \( G \) invariant vectors. For the representation \( G \) on \( V \) we fix \( \vartheta_0 \) given by the conclusion of Lemma \ref{lemma:3.3} and take \( \vartheta_1 = \vartheta_0 / \kappa \).

There are two trivial cases: if either \( \| q_0(v) \| \geq \varepsilon \delta_i \) or \( q_0(v) = v \) and \( \| v \| < \varepsilon \delta_i \), then both sides of (4.3) are either 0 or \( \infty \) respectively. In general if \( v \neq q_0(v) \) and \( \| q_0(v) \| < \varepsilon \delta_i \), then the conclusion follows from Lemma \ref{lemma:3.4} and the fact that the integral of the minimum of finite functions is less than or equal to the minimum of integrals. \( \square \)

Following \cite{2} we define \( \alpha_\varepsilon : X \to [0, \infty] \) by

\[
\alpha_\varepsilon(y) = \max \varphi_\varepsilon(v)
\]

where the maximum is taken over all the non-zero \( y \)-integral monomials \( v \in \wedge^i \mathbb{R}^d \) with \( 0 < i < d \).
Lemma 4.4. Given $\vartheta > 0$ sufficiently small and $0 < a < 1$, for every $t$ sufficiently large (depending on $\vartheta$ and $a$) and $\varepsilon > 0$ sufficiently small (depending on $t$) there exists $b > 0$ such that

\begin{equation}
\int_{I^m} \alpha_\varepsilon^\vartheta (g_t u)(w) y \, dw \leq a \alpha_\varepsilon^\vartheta (y) + b
\end{equation}

for every $y \in X$.

Proof. We fix $t > 0$ sufficiently large so that according to Lemma 4.3 one has

\[ \int_{I^m} \varphi_\varepsilon^\vartheta (g_t u)(w) v \, dw \leq \frac{a}{2d} \varphi_\varepsilon^\vartheta (v) \]

for every $0 < \varepsilon < 1$ and $v \in \wedge^i \mathbb{R}^d$ with $0 < i < d$.

Let $C_0 = \sup \{ \| g_t u(w) \| + \| (g_t u(w))^{-1} \| : w \in I^m \} \geq 1$ where $\| \cdot \|$ is the operator norm for elements of $G$ acting on $\wedge^* \mathbb{R}^d$. We take $\varepsilon$ small enough so that

\[ C_0^{2\kappa}(C_1\varepsilon)^{\kappa_1/2} < 1 \]

where $C_1$ is the constant given in Lemma 4.2 and $\kappa, \kappa_1$ are defined in (4.2). Let

\[ b_1 = \sup \varphi_\varepsilon(v) < \infty \]

where the supremum is taken over all the monomials $v \in \wedge^i \mathbb{R}^d$ with $\| v \| \geq 1$. We will show that for

\[ b = 2m(C_0^\kappa \max\{ b_1, C_0^{2\kappa} \})^\vartheta \]

(4.4) holds.

It follows from the definition of $C_0$ that for every monomial $v \in \wedge^i \mathbb{R}^d$ with $0 < i < d$ one has

\[ C_0^{-\kappa} \varphi_\varepsilon(v) \leq \varphi_\varepsilon(g_t u)(w) v \leq C_0^{\kappa} \varphi_\varepsilon(v). \]

If $\alpha_\varepsilon(y) \leq \max\{ b_1, C_0^{2\kappa} \}$, then

\[ \int_{I^m} \alpha_\varepsilon^\vartheta (g_t u)(w) y \, dw \leq b. \]

Let $\Psi$ be the finite set of primitive $y$-integral and monomial elements $v$ of $\wedge^* \mathbb{R}^d$ with degree in $(0, d)$ such that

\[ \varphi_\varepsilon(v) \geq C_0^{-2\kappa} \alpha_\varepsilon(y). \]

It follows from claim (5.9) in the proof of [2] Proposition 5.9 that if $\alpha_\varepsilon(y) > \max\{ b_1, C_0^{2\kappa} \}$ then $\Psi$ contains at most one element up to sign change in each degree $i$. Therefore in this case one has

\[ \int_{I^m} \alpha_\varepsilon^\vartheta (g_t u)(w) y \, dw \leq \sum_{v \in \Psi} \int_{I^m} \varphi_\varepsilon^\vartheta (g_t u)(w) v \, dw \leq \frac{a}{2d} \sum_{v \in \Psi} \varphi_\varepsilon^\vartheta (v) \leq a \alpha_\varepsilon^\vartheta (y). \]

\[ \square \]
We fix $\vartheta$ and $\varepsilon$ sufficiently small so that $\alpha_{\vartheta}^\varepsilon$ is finite on $Z$ and Lemma 4.4 holds. It is easy to see that $\alpha = \alpha_{\vartheta}^\varepsilon$ satisfies properties (1)-(4) of Lemma 4.1. Therefore we have proved Lemma 4.1 in the case where $L = SL_d(\mathbb{R})$ and $\Gamma = SL_d(\mathbb{Z})$. The general case will be reduced to this case and the real rank one case. We need the following lemma which is straightforward to check so we omit the details of proof.

Lemma 4.5. Let $\Gamma$ be a lattice of a connected Lie group $L_1$. Let $\varphi : L \to L_1$ be a surjective homomorphism of Lie groups so that $\varphi(G)$ is nontrivial. Suppose that $\varphi(\Gamma) \subset \Gamma$ and the induced map $X = L/\Gamma \to L_1/\Gamma_1$ is proper. If Lemma 4.1 holds for $L_1/\Gamma_1, \varphi(g_t), \varphi(U)$ or it holds for $L/\Gamma, g_t, U$ where $\Gamma'$ is a finite index subgroup of $\Gamma$, then it holds for $X, g_t, U$.

Proof of Lemma 4.1. Let $\mathfrak{r}$ be the largest amenable ideal of the Lie algebra $\mathfrak{l}$ of $L$, $\mathfrak{s} := \mathfrak{l}/\mathfrak{r}$, $S := \text{Aut}(\mathfrak{s})$. Let $R$ be the kernel of the adjoint representation $Ad_s : L \to S$. It follows from [2] Lemma 6.1 that $\Gamma \cap R$ is a cocompact lattice in $R$ and the image group $\Gamma_S := Ad_s(\Gamma)$ is a lattice in $S$. Therefore the map $L/\Gamma \to S/\Gamma_S$ is proper. So according to Lemma 4.5 it suffices to prove the case where $L$ is a connected semisimple center free Lie group without compact factors.

Under this assumption we can write $L = \prod_{i=1}^q L_i$ as a direct product of connected semisimple Lie groups such that $L_i \cap \Gamma$ is an irreducible lattice in $L_i$. We can assume that $\Gamma = \prod_{i=1}^q L_i \cap \Gamma$ since the latter has finite index in $\Gamma$. Let $\pi_i : L \to L_i$ be the natural quotient map. If $\pi_i(G)$ is nontrivial then $\pi_i(g_t)$ is a nontrivial $\text{Ad}$-diagonalizable one parameter subgroup of $\pi_i(G)$ and $\pi_i(U)$ is $\pi_i(g_t)$ expanding. Suppose that Lemma 4.1 holds for every $L_i/\Gamma_i$ with $\pi_i(G)$ nontrivial. Let $\alpha_i : L_i/\Gamma_i \to [0, \infty]$ be a lower semicontinuous function associated to the compact subset $\pi_i(Z) \subset L_i/\Gamma_i$, $0 < a < 1$ and $t > 0$. If $\pi_i(G)$ is trivial, we set $\alpha_i = (1 - 1_{\pi_i(Z)}) \cdot \infty$. Then the function $\alpha$ on $X$ defined by

$$\alpha(y_1, \ldots, y_q) = \alpha_1(y_1) + \cdots + \alpha_q(y_q) \quad \text{where} \quad y_i \in L_i/\Gamma_i$$

satisfy properties (1)-(4) of Lemma 4.1 with respect to $Z, a$ and $t$. Therefore it suffices to prove the case where $L$ is a connected center free semisimple Lie group without compact factors and $\Gamma$ is an irreducible lattice.

If the real rank of $L$ is bigger than or equal to two, then Margulis arithmeticity theorem (see e.g. [29] Theorem 6.1.2) implies that there is an injective map

$$\varphi : L \to SL_d(\mathbb{R})$$
such that \( \varphi(\Gamma) \) is commensurable with \( \varphi(L) \cap SL_d(\mathbb{Z}) \). So Lemma 4.1 follows from Lemma 4.5 and the case where \( L = SL_d(\mathbb{R}) \) and \( \Gamma = SL_d(\mathbb{Z}) \).

Otherwise \( L \) has real rank one. If \( X = L/\Gamma \) is compact, then we take \( \alpha(y) = 1 \) for any \( y \in X \). Suppose that \( X \) is noncompact. It follows from [10] (cf. [14] Proposition 3.1 and [2] page 54) and the proof of [8 Proposition 2.7] that there exists a finite dimensional real representation \( V \) of \( G \) with norm \( \| \cdot \| \) and finite nonzero vectors \( v_1, \ldots, v_r \) of \( V \) with the following properties:

(a) \( \Gamma v_i \) is closed and hence discrete in \( V \) for \( 1 \leq i \leq r \);
(b) For any \( F \subset L \), the set \( F\Gamma \subset L/\Gamma \) is relatively compact if and only if there exists \( c > 0 \) such that \( \| g\gamma v_i \| > c \) for any \( \gamma \in \Gamma, g \in F \) and \( 1 \leq i \leq r \);
(c) There exists \( c_0 > 0 \) such that for any \( g \in L \) there exists at most one \( v \in \bigcup_{1 \leq i \leq r} \Gamma v_i \) such that \( \| gv \| < c_0 \);
(d) There exists \( C' > 0 \) such that for every \( 1 \leq i \leq r \) and every \( g \in L \) one has \( \| \pi(gv_i) \| \geq C'\| gv_i \| \) where \( \pi \) is the projection to the subspace complementary to the subspace of \( G \) invariant vectors.

Let

\[
\hat{\alpha}_\vartheta(g\Gamma) = \max_{1 \leq i \leq r} \max_{\gamma \in \Gamma} \| g\gamma v_i \|^{-\vartheta}.
\]

In this case Lemma 4.1 follows from properties (a)-(d) listed above and Lemma 3.4 by taking \( \alpha = \hat{\alpha}_\vartheta \) for some \( \vartheta \) sufficiently small. \( \square \)

4.2. **Exponential recurrence to cusp.** For \( Z = \{ x \} \) and \( a = \frac{1}{4} \) we choose \( t > 0, b > 0 \) and \( \alpha : X \to [0, \infty] \) so that Lemma 4.1 holds. We first use inequality (4.1) to study discrete trajectory

\[
\{ g_{nt}u(w)x : n \in \mathbb{N} \}
\]

where \( \mathbb{N} = \{ 0, 1, 2, \ldots \} \). Recall that \( \{ e_i : 1 \leq i \leq m \} \) is the standard basis of \( \mathbb{R}^m \) and \( b_i > 0 \) satisfies (2.2). Let

\[
w = \sum_{i=1}^{m} a_i e_i \quad \text{and} \quad w' = \sum_{i=1}^{m} a'_i e_i.
\]

If \( |a_i - a'_i| \leq 2e^{-ntb_i} \), then two points \( g_{nt}u(w)x \) and \( g_{nt}u(w')x \) can always be translated to each other by elements in a fix compact subset of \( G \). In view of property (3) of \( \alpha \) we consider them as at the same height. The following lemma plays a key role to link random walks with respect to \( g_t u(I^m) \) and trajectory (4.5).
**Lemma 4.6 (Shadowing Lemma).** For $1 \leq i \leq m$ let $J_i \subset [-1, 1]$ be an interval with length $|J_i| \geq e^{-ntb_i}$. Then for any nonnegative measurable function $\psi$ on $X$ and $J = \prod_{i=1}^{m} J_i$ one has

\[
\int_{J} \psi(g_{(n+1)}u(w)x) \, dw \leq \int_{J} \int_{j} \psi(g_{it}u(w_1)g_{nt}u(w)x) \, dw_1 \, dw.
\]

The proof is an elementary exercise of calculus using change of variables

\[(w_1, w) = ((s'_i), (s_i)) \rightarrow ((s'_i), (s_i + s_i' e^{-ntb_i})).\]

Here abelian assumption is essential to us. If we drop the abelian assumption, then we need to change the domain of the integral for $w_1$ to something that depends on $J$. In that case it is not clear to the author how to get (3.10) uniformly in terms of $T_0$ for various domains determined by $J$ and hence the contraction property (4.1).

For every positive integer $n$ we need to divide the interval $[-1, 1]$ into intervals of size $e^{-ntb_i}$ for each component of $I^m$ to form a box so that the above shadowing lemma holds and $g_{nt}u(w)x$ are bounded for $w$ in each box. We can do this consecutively except for the last interval which we allow to have length bigger than $e^{-ntb_i}$ but no more than $2e^{-ntb_i}$. We want the partition for $n+1$ to be a refinement of that for $n$ so we do this by induction on $n$. The first step we divide $I^m$ into boxes of the form

\[
\prod_{1 \leq i \leq m} [-1 + je^{-tb_i}, -1 + (j + 1)e^{-tb_i})
\]

with slight modifications for the end intervals. For every $w \in I^m$ we use $I(w)$ to denote the box containing $w$. In the second step we divide each box above into smaller boxes of the form

\[
\prod_{1 \leq i \leq m} [-1 + je^{-tb_i} + ke^{-2tb_i}, -1 + je^{-tb_i} + (k + 1)e^{-2tb_i})
\]

and we use $I_2(w)$ to denote the one containing $w$. By the same construction we do it for all $n$ and define $I_n(w)$ accordingly. We also take $I_0(w) = I^m$ and $I_\infty(w) = \{w\}$ for every $w \in I^m$. We fix a compact subset $F_0$ of $G$ so that for any $n \in \mathbb{N}$, $w \in I^m$ and $w' \in I_n(w)$ one has

\[
g_{nt}u(w')x = hg_{nt}u(w)x \quad \text{for some} \quad h \in F_0.
\]

For every $n \in \mathbb{N}$ let $\mathcal{B}_n$ be the smallest sigma algebra of $I^m$ such that $I_j(w) \in \mathcal{B}_n$ for every $0 \leq j \leq n$ and $w \in I^m$. It is not hard to see that the atom of $w$ in $\mathcal{B}_n$ is $I_n(w)$ and the sequence $\langle \mathcal{B}_n \rangle_{n \in \mathbb{N}}$ is a filtration of sigma algebras.
Lemma 4.7. For every $J \in B_n$ where $n \in \bN$ one has
\[
\int_J \alpha(g_{(n+1)}tu(w)x) \, dw \leq \frac{1}{4} \int_J \alpha(g_{nt}u(w)x) \, dw + b|J|.
\]

Proof. It follows from shadowing Lemma 4.6 and the linear inequality (4.1).

Let us fix a positive real number $l_0$ with the following properties:

(i) $b/l_0 < 1/4$;

(ii) $x \in X_{l_0}$ where $X_{l_0} = \{y \in X : \alpha(y) \leq l_0\};$

We define a sequence of measurable functions $\sigma_i : \mathbb{N} \to \mathbb{N} \cup \{\infty\}$ which represents $i$th return time to the compact subset $X_{l_0}$. To begin with we set $\sigma_0(w) = 0$. To apply shadowing lemma we want \{$w \in \mathbb{N} : \sigma_i(w) = n$\} to be $B_n$ measurable. The formal definition is
\[
\sigma_i(w) = \inf\{n > \sigma_{i-1}(w) : g_{nt}u(w_1)x \in X_{l_0} \text{ for some } w_1 \in I_n(w)\}.
\]

If $\sigma_i(w) = \infty$ for some $i$ then we set $\sigma_j(w) = \infty$ for every $j > i$. It follows from the definition that \{$w \in \mathbb{N} : \sigma_i(w) > n$\} is $B_n$ measurable. To simplify notation we set
\[
I(\sigma_n, w) := I_{\sigma_n(w)}(w).
\]

Lemma 4.8. There exists $Q_0 > 0$ and $\vartheta_0 > 0$ such that for any integer $q \geq Q_0, n \in \bN$ and $w_0 \in \mathbb{N}$ with $\sigma_n(w_0) < \infty$ the measure of the set
\[
J_{n,q}(w_0) = \{w \in I(\sigma_n, w_0) : \sigma_{n+1}(w) - \sigma_n(w) \geq q\}
\]

is less than or equal to $e^{-\vartheta_0 q}I(\sigma_n, w_0)$.

Remark: It follows from Lemma 4.8 that $\sigma_n(w) < \infty$ almost surely for every $n \in \bN$.

Proof. We fix $w_0$, $n$ and write $\sigma_n = \sigma_n(w_0), J_q = J_{n,q}(w_0)$ for simplicity. Let
\[
s_q := \int_{J_{q+1}} \alpha(g_{(\sigma_n+q)}tu(w)x) \, dw \leq \int_{J_q} \alpha(g_{(\sigma_n+q)}tu(w)x) \, dw.
\]

Since $J_q$ is $B_{\sigma_n+q-1}$ measurable, Lemma 4.7 implies
\[
s_q \leq \frac{1}{4}s_{q-1} + b|J_q| \leq \left(\frac{1}{4} + \frac{b}{l_0}\right)s_{q-1} \leq \frac{1}{2} s_{q-1}.
\]

A simple induction on $q$ implies that $s_q \leq s_0 2^{-q}$. Chebyshev inequality and property (3) of $\alpha$ in Lemma 4.7 implies that there exists $C_1 > 0$ not depending on $w_0$ or $n$ such that $|J_{q+1}| \leq 2^{-q}C_1 I(\sigma_n, w_0)$. Hence the existence of $Q_0$ and $\vartheta_0$ follows.
Recall that the proportion of the trajectory \( \{g_t u(w)x : 0 \leq t \leq T\} \) in a subset \( K \) of \( X \) is defined in (2.4). Similarly, a discrete version of this function is defined as
\[
D_n^K(w) := \frac{1}{n} \sum_{i=0}^{n-1} \mathbb{1}_K(g_i u(w)x)
\]
where \( n \) is a positive integer and \( \mathbb{1}_K \) is the characteristic function of \( K \).

Lemma 4.9. For every \( 0 < \epsilon_0 < 1 \) there exists a compact subset \( K_0 \) of \( X \) and \( 0 < c_0 < 1 \) so that
\[
\left| \left\{ w \in I^m : D_{K_0}^n(w) \leq 1 - \epsilon_0 \right\} \right| \leq c_0^n
\]
for every positive integer \( n \).

Proof. We choose \( l_0 > 0 \) so that properties (i) and (ii) listed after Lemma 4.7 hold. It follows from Lemma 4.8 that there exists a positive integer \( Q_0 \) such that for every \( q \geq Q_0 \) we get the exponential decay for the measure of the set \( J_{n,q}(w) \). It follows from Lemma 3.1 that there exists \( Q > 0, 0 < c_0 < 1 \) and integer \( N_0 \geq 1 \) such that for \( n \geq N_0 \) the measure of the set
\[
J_n = \left\{ w \in I^m : \frac{1}{n} \sum_{i=1}^{n} \mathbb{1}_Q(\sigma_i(w) - \sigma_{i-1}(w)) \geq \epsilon_0 \right\}
\]
is less than or equal to \( c_0^n \). Here \( \mathbb{1}_Q \) is the truncation of the identity function defined in (3.3).

We claim that the lemma holds for
\[
K_0 = \bigcup_{0 \leq s \leq (Q+N_0)T} g_s F_0 X_{l_0}.
\]
To see this we note that \( x \in X_{l_0} \). Therefore if \( g_j u(w)x \notin K_0 \) for some \( j \in \mathbb{N} \), then \( \sigma_i(w) - \sigma_{i-1}(w) \geq Q \) for some \( i \leq j \). So if \( w \in I^m \) satisfies \( D_{K_0}^n(w) \leq 1 - \epsilon_0 \) then \( w \in J_n \). It is easy to see that for \( n < N_0 \) we have \( D_{X_{l_0}}^n(w) = 1 \) for any \( w \in I^m \). Therefore the conclusion follows from the exponential decay of the measure \( |J_n| \). \qed

The following lemma allows us to deduce the continuous version of exponential decay from the discrete version in Lemma 4.9.

Lemma 4.10. Let \( 0 < \epsilon_0, c_0 < 1 \) and let \( K_0 \) be a compact subset of \( X \). Suppose that (4.11) holds for every positive integer \( n \). Then there exists a positive number \( c < 1 \) and a compact subset \( K \) of \( X \) such that
\[
\left| \left\{ w \in I^m : A_{K}^T(w) \leq 1 - 2\epsilon_0 \right\} \right| \leq c^T.
\]
Proof. We fix $T_0 > 0$ so that for $T \geq T_0$ we have

\[ (1 - \frac{t}{T}) (1 - \epsilon_0) \geq 1 - 2\epsilon_0 \quad \text{and} \quad T \geq 2t. \]

Let $K'$ be a compact subset of $X$ so that for any $w \in I^m$ and $0 \leq s \leq T_0$ we have $g_s u(w)x \in K'$. We claim that the compact subset $K = \left( \bigcup_{s \in [0, t]} g_s K_0 \right) \cup K'$ and $c = \frac{c_0}{2}$ satisfies (4.12).

To prove the claim it suffices to consider the case $T \geq T_0$. Given $i \in \mathbb{N}$, if $g_i u(w)x \in K_0$ then $g_s u(w)x \in K$ for $s \in [it, (i+1)t]$. Therefore in view of the first inequality of (4.13) we have if $A^T_K(w) \leq 1 - 2\epsilon_0$ then $D^\lfloor T/t \rfloor K_0(w) \leq 1 - \epsilon_0$ where $\lfloor T/t \rfloor$ is the biggest integer less than or equal to $T/t$. A simple calculation using second inequality of (4.13) gives (4.12). □

Proof of Proposition 2.3. It follows from Lemma 4.9 and Lemma 4.10. □

5. Exponential recurrence to singular subspace

The aim of this section is to prove Proposition 2.5. Let $L, \Gamma, G, g_t, U, x$ be as in Theorem 2.1, let $Y, K_1, C_L(G), F, \epsilon_1$ be as in Proposition 2.5 and let $X = L/\Gamma$. Let $S = \{ g \in L : gY = Y \}$ and let $s, c, l, g$ be the Lie algebras of $S, C_L(G), L$ and $G$, respectively. Let $t$ be a $G$ invariant subspace of $l$ complementary to $s + c$ with respect to the adjoint action.

We first set up some constants and review some results from previous section. For every $w \in I^m$ and $n \in \mathbb{N}$ we let $I_n(w)$ to be the box defined in (4.2). Let $F_0$ be a compact subset of $G$ so that (4.7) holds. For $a = 1/4, Z = \{ x \}$ we fix $t > 0$ so that there exists $\alpha : X \to [0, \infty]$ and $b > 0$ satisfying Lemma 4.11. We fix $l_0 > 0$ so that properties (i) and (ii) listed after Lemma 4.7 hold. We let $\sigma_i : I^m \to \mathbb{N} \cup \{ \infty \}$ be the $i$th return time to $X_{l_0}$ defined in (4.8). By Lemma 4.8 there exists $Q_0$ and $\vartheta_0 > 0$ such that for $q \geq Q_0$ the measure of $J_{n,q}(w)$ defined in (4.9) is less than or equal to $e^{-\vartheta_0 q} |I(\sigma_n, w)|$.

We fix a norm $\| \cdot \|$ on $g$ and use $\| g \|$ to denote the operator norm of $g \in G$ with respect to the adjoint representation. There exits $\vartheta' > 0$ such that

\[ \max (\| g_n u(w) \|, |(g_n u(w))^{-1} \|) \leq e^\vartheta' \]

for every $w \in I^m$ and positive integer $n$. Let $0 < \vartheta < 1$ be sufficiently small so that $\frac{\vartheta}{2} - \vartheta' \vartheta > 0$. According to Lemma 3.4 by making $\vartheta$ smaller we can find a positive integer $p$ such that for any measurable
map $\tau : I^m \to \mathbb{N} \cup \{\infty\}$ with $\inf_{w \in I^m} \tau(w) \geq pt$ and $\tau(w) < \infty$ almost surely

\begin{equation}
\sup_{\|v\|=1} \int_{I^m} \frac{d \omega}{\|g_{\tau(w)}u(w)v\|} < \frac{1}{4}.
\end{equation}

To use contraction property (5.2) we need to modify $i$th return function $\sigma_i$ and define inductively

$$
\kappa_0(w) = 0 \quad \text{and} \quad \kappa_i(w) = \min\{\sigma_n(w) : \sigma_n(w) \geq \kappa_{i-1}(w) + p\}.
$$

To simplify notation we set

$$
I(\kappa_n, w) := I_{\kappa_n(w)}(w).
$$

It follows from Lemma 4.8 that for any $n \in \mathbb{N}$, $w_0 \in I^m$ and $q \geq Q_0 + 2p$ we have

\begin{equation}
|\{w \in I(\kappa_n, w_0) : \kappa_{n+1}(w) - \kappa_n(w) \geq q\}| \leq e^{-q/2q_0}|I(\kappa_n, w_0)|.
\end{equation}

Following [3] §6.8 we will define a height function $\beta : X \to [0, \infty]$ which roughly speaking measures whether elements of a fixed compact subset are close to $FY$. Let $N \supset F_0$ be a relatively compact open neighborhood of identity in $G$. We choose a positive number $\varepsilon \leq 1$, an open neighborhood $O$ of identity in $C_L(G)$ and finite number of elements $f_1, \ldots, f_k \in F$ with $F \subset O f_1 \cup \cdots \cup O f_k$ so that the following holds: for any $y \in NX_{t_0} := \bigcup_{h \in N} hX_{t_0}$ and any $f_i$ there exists at most one $v \in t$ with $\|v\| \leq \varepsilon$ and $y \in \exp(v)O f_i Y$. By shrinking $O$ we also assume that $x \not\in \overline{FY}$ where $\overline{O}$ is the closure of $O$. For any $y \in NX_{t_0}$ and $1 \leq i \leq k$, we set

$$
\beta_i(y) = \begin{cases}
\|v\|^{\vartheta} & \text{if } y \in \exp(v)O f_i Y \text{ with } v \in t \text{ and } \|v\| \leq \varepsilon \\
\varepsilon^{\vartheta} & \text{otherwise}
\end{cases}
$$

and $\beta(y) = \beta_1(y) + \cdots + \beta_k(y)$. We also set $\beta(y) = 0$ if $y \not\in NX_{t_0}$. It is easy to see that $\beta$ satisfies the following properties:

(I) $\beta$ is lower semicontinous;
(II) $\beta$ is Lipschitz on $NX_{t_0}$, i.e. for every compact subset $F_2$ of $G$, there exists $C_2 > 1$ such that $\beta(y) \leq C_2 \beta(y)$ for any $y \in NX_{t_0};$
(III) $\beta(y) = \infty$ if and only if $y \in NX_{t_0} \cap (\cup O f_i) Y$.

Our strategy is in principle the same as that of previous section. The key ingredient is Lemma 5.2 which is a variant of Lemma 4.7. The following lemma is a preparation for the proof of Lemma 5.2.

**Lemma 5.1.** For every $y \in NX_{t_0}$ and bounded measurable function $r : I^m \to \mathbb{Z}_{\geq p}$, there exists $b' > 0$ depending on the upper bound of the
function \( r \) such that
\[
\int_{I_m} \beta(g_r(w)t u(w) y) dw \leq \frac{1}{4} \beta(y) + \beta'.
\]

Proof. Since the function \( r \) is bounded, there exists \( C \geq 1 \) such that \( \max\{\|g_r(w) t u(w)\|, \|g_r(w) t u(w)^{-1}\|\} \leq C \) for every \( w \in I_m \). Let
\[
J_i = \{w \in I_m : \beta_i(g_r(w)t u(w)y) \geq (C\varepsilon^{-1})^{\theta} \}
\]
and \( J'_i = I_m \setminus J_i \). If \( w \in J_i \), then \( y = \exp(v_i) \) \( O f_i Y \) with \( \|v_i\| = \beta_i(y) \) and \( \beta_i(g_r(w)t u(w)y) = \|g_r(w)t u(w)v_i\| \). Therefore according to (5.2)
\[
\int_{J_i} \beta_i(g_r(w)t u(w)y) dw \leq \frac{1}{4} \beta_i(y).
\]
The lemma follows by taking \( b' = 2^m k(C\varepsilon^{-1})^\theta \).

Lemma 5.2. There exists \( b'' > 0 \) such that for any \( n \in \mathbb{N} \), \( w_0 \in I_m \) with \( \kappa_n(w_0) < \infty \) and \( J = I(\kappa_n, w_0) \) one has
\[
(5.4) \quad \int_J \beta(g_{\kappa_n+1(w)} u(w)x) dw \leq \frac{1}{3} \int_J \beta(g_{\kappa_n(w)} u(w)x) dw + b''|J|.
\]

Remark: Let \( C_n \) be the smallest sigma algebra of \( I_m \) generated by \( I(\kappa_i, w) \) for \( 0 \leq i \leq n \) and \( w \in I_m \) with \( \kappa_i(w) < \infty \). Since modulo null sets every element of \( C_n \) is countable or finite disjoint union of sets of the form \( I(\kappa_n, w) \), the lemma also holds for \( J \in C_n \).

Proof. Since the function \( \kappa_n(w) \) is fixed on \( J \) we simply write \( \kappa_n \) for \( \kappa_n(w) \). Here \( \kappa_{n+1}(w) - \kappa_n \) varies for different \( w \) and might be unbounded, so we can not use the idea of shadowing Lemma 4.6 directly. To overcome this difficulty we fix a positive integer \( Q \geq p \) which will be specified afterwards and define a truncation for the function \( \kappa_{n+1}(w) - \kappa_n \) by
\[
r(w) = \begin{cases} 
\kappa_{n+1}(w) - \kappa_n & \text{if } w \in J \text{ and } \kappa_{n+1}(w) - \kappa_n < Q \\
Q & \text{otherwise.}
\end{cases}
\]

It follows from Lipschitz property of \( \beta \) that there exists \( C_2 \geq 1 \) such that
\[
(5.5) \quad \beta(g_{\kappa_n t u(w_0)x})/C_2 \leq \beta(g_{\kappa_n t u(w)x}) \leq C_2 \beta(g_{\kappa_n t u(w_0)x})
\]
for any \( w \in J \). We take \( Q \) to be the smallest integer greater than or equal to
\[
(5.6) \quad \max \left\{ Q_0 + 2p, \frac{\log(12C_2^2) - \log(1 - e^{\theta_0/2 - \theta'} \vartheta)}{\vartheta_0/2 - \theta'} \right\}.
\]
Let $b' > 0$ be the constant given by Lemma 5.1 with respect to the truncated function $r$. We will show that (5.4) holds for $b'' = b'$. Note that $C_2$ and hence $b'$ does not depend on $n$ or $w_0$.

We divide $J$ into two sets:

$$J_1 = \{ w \in J : r(w) < Q \} \quad \text{and} \quad J_2 = J \setminus J_1 = \{ w \in J : r(w) = Q \}.$$

Let $w_n' = (e^{-\kappa t b_1}, \ldots, e^{-\kappa t b_m})$ where $b_i > 0$ satisfies (2.2) and let $w_1 \cdot w_n'$ be the usual inner product on $\mathbb{R}^m$. We have

$$\int_{J_1} \beta(g_{\kappa+1}(w)u(w)x) \, dw \leq \int_J \beta(g_{r(w)+\kappa t}u(w)x) \, dw$$

$$\leq \int_J \int_{J_1} \beta(g_{r(w)+w_1'w_n'}u(w_1)g_{\kappa t}u(w)x) \, dw_1 \, dw$$

by Lemma 5.1

$$\leq \frac{1}{4} \int_J \beta(g_{\kappa t}u(w)x) \, dw + b'|J|.$$

Let $B_q = \{ w \in J : \kappa_{n+1}(w) - \kappa_n = q \}$. In view of (5.1) we have

$$\int_{J_2} \beta(g_{\kappa_{n+1}(w)}u(w)x) \, dw \leq \sum_{q \geq Q} \int_{B_q} e^{q^{\varphi q}} \beta(g_{\kappa_{n+1}(w)}u(w)x) \, dw$$

by (5.5)

$$\leq \sum_{q \geq Q} \int_{B_q} e^{q^{\varphi q}} C_2 \beta(g_{\kappa_{n+1}(w_0)}u(x)) \, dw$$

by (5.3)

$$\leq \sum_{q \geq Q} e^{-q(\varphi_{0/2} - \varphi q)} C_2 \int_J \beta(g_{\kappa_{n+1}(w_0)}u(x)) \, dw$$

by (5.5)

$$\leq \frac{e^{-Q(\varphi_{0/2} - \varphi q)}}{1 - e^{\varphi_{0/2} - \varphi q}} C_2^2 \int_J \beta(g_{\kappa_{n+1}(w)}u(x)) \, dw.$$

The second lower bound for $Q$ in (5.6) implies that

$$\frac{e^{-Q(\varphi_{0/2} - \varphi q)}}{1 - e^{\varphi_{0/2} - \varphi q}} C_2^2 \leq \frac{1}{12}$$

which completes the proof. \qed

We fix a positive number $l$ with $\beta(x) < l$ and $b''/l < 1/12$. Let

$$(5.7) \quad X_I^Y = \{ y \in \nabla X_{l_0} : \beta(y) \leq l \}.$$ 

The $i$th return time to $X_I^Y$ is the function $\tau_i : I^m \to \mathbb{N} \cup \{ \infty \}$ defined inductively as follows: $\tau_0(w) = 0$ and

$$\tau_i(w) = \inf\{ n : n > \tau_{i-1}(w), \kappa_n(w) < \infty \text{ and } g_{\kappa_n(w)}u(w_1)x \in X_I^Y \text{ for some } w_1 \in I(\kappa_n(w)) \}.$$
We make it convention that \( \tau_n(w) = \infty \) if the set where we take infimum is empty. It will also be convenient to set \( \kappa_\infty(w) = \infty \). We set
\[
\kappa_{\tau_n}(w) := \kappa_{\tau_n}(w) \quad \text{and} \quad I(\kappa_{\tau_n}, w) := I(\kappa_{\tau_n}(w), w)
\]
to simplify the notation. The following two lemmas are preparations for the proof of Lemma 5.5 which is similar to Lemma 4.8.

**Lemma 5.3.** There exists \( Q_1 > 0 \) and \( \vartheta_1 > 0 \) such that for any \( q \geq Q_1, n \in \mathbb{N} \) and \( w_0 \in I^m \) with \( \tau_n(w_0) < \infty \) the measure of the set
\[
B_{n,q}(w_0) := \{ w \in I(\kappa_{\tau_n}, w_0) : \tau_{n+1}(w) - \tau_n(w) \geq q \}
\]
is less than or equal to \( e^{-\vartheta_1 q}|I(\kappa_{\tau_n}, w_0)| \).

Remark: It follows from Lemma 5.3 that for almost every \( w \in I^m \) we have \( \tau_n(w) < \infty \) for any \( n \in \mathbb{N} \).

**Proof.** We fix \( n, w_0 \) and set \( B_q = B_{n,q}(w_0), i = \tau_n(w_0) \). It is easy to see that
\[
s_q := \int_{B_{q+1}} \beta(g_{\kappa_{i+q}}(w)u(w)x) \, dw \leq \int_{B_q} \beta(g_{\kappa_{i+q}}(w)u(w)x) \, dw.
\]
Note that \( B_q \in C_{i+q-1} \) where \( C_{i+q-1} \) is defined in the remark of Lemma 5.2. So Lemma 5.2 implies that
\[
s_q \leq \frac{1}{3} \int_{B_q} \beta(g_{\kappa_{i+q-1}}(w)u(w)x) \, dw + b''|B_q| \leq \frac{1}{2}s_{q-1}.
\]
The rest of proof is the same as that of Lemma 4.8 \( \square \)

**Lemma 5.4.** There exists \( Q_2 > 0 \) and \( \vartheta_2 > 0 \) such that for any integers \( i \geq 0, j > 0 \) and any \( w_0 \in I^m \) with \( \kappa_i(w_0) < \infty \) the measure of the set
\[
C_{i,j}(w_0) := \{ w \in I(\kappa_i, w_0) : \kappa_{i+j}(w) - \kappa_i(w) \geq Q_2j \}
\]
is less than or equal to \( e^{-\vartheta_2 j}|I(\kappa_i, w_0)| \).

**Proof.** It follows from (5.3), Lemma 3.1 and its remark that there exists \( \vartheta > 0 \) and \( Q \geq Q_0 + 2p \) such that the measure of the set
\[
C'_{i,j}(w_0) := \left\{ w \in I(\kappa_i, w_0) : \sum_{s=1}^{j} 1_Q(\kappa_{i+s}(w) - \kappa_{i+s-1}(w)) \geq j \right\}
\]
is less than or equal to \( e^{-\vartheta j}|I(\kappa_i, w_0)| \). Suppose that \( \kappa_{i+j}(w) - \kappa_i(w) \geq 2Qj \) for some \( w \in I(\kappa_i, w_0) \), then \( w \in C'_{i,j}(w_0) \). Therefore the Lemma follows by taking \( Q_2 = 2Q \) and \( \vartheta_2 = \vartheta \). \( \square \)
Lemma 5.5. There exists $Q_3 > 0$, $\vartheta_3 > 0$ such that for any $n \in \mathbb{N}$, $q \geq Q_3$ and $w_0 \in I^m$ with $\tau_n(w_0) < \infty$ the measure of the set

$$A_{n,q}(w_0) := \{w \in I(\kappa_{\tau_n}, w_0) : \kappa_{\tau_{n+1}(w)} - \kappa_{\tau_n(w)} \geq q\}$$

is less than or equal to $e^{-\vartheta_3 q |I(\kappa_{\tau_n}, w_0)|}$.

Proof. We fix positive numbers $Q_1, \vartheta_1$ and $Q_2, \vartheta_2$ so that Lemma 5.3 and Lemma 5.4 hold respectively. We show that for $\vartheta_3 = 1/2$ and $Q_3 = \max\{Q_1 + 2, \log \vartheta_3\}$ the conclusion holds.

We fix $n, w_0$ and $q \geq Q_3$. Let $\lfloor q/Q_2 \rfloor$ is the biggest integer less than or equal to $q/Q_2$ and let

$$A' = \{w \in A_{n,q}(w_0) : \tau_{n+1}(w) - \tau_n(w) \geq \lfloor q/Q_2 \rfloor\}.$$

It follows from Lemma 5.3 that

$$|A'| \leq e^{-\vartheta_1 q |I(\kappa_{\tau_n}, w_0)|}.$$

On the other hand it is easy to see that $A_{n,q}(w_0) \setminus A' \subset C_{\tau_n(w_0),\lfloor q/Q_2 \rfloor}$ where the latter is defined in (5.9). So Lemma 5.4 implies that

$$|A_{n,q}(w_0) \setminus A'| \leq e^{-\vartheta_2 q |I(\kappa_{\tau_n}, w_0)|}.$$

A simple calculation shows that (5.10) holds. \qed

Proof of Proposition 2.3. It follows from Lemma 5.5 and Lemma 3.1 that there exists positive number $c_0 < 1$, positive integers $Q$ and $M_0$ such that for every integer $n \geq M_0$ the measure of the set

$$J_n = \left\{w \in I^m : \frac{1}{n} \sum_{i=1}^{n} : 1_Q(\kappa_{\tau_i}(w) - \kappa_{\tau_{i-1}(w)}) \leq 1 - \frac{\epsilon_1}{2}\right\}$$

is less than or equal to $c_0^n$. Therefore by taking

$$K_0 = \bigcup_{0 \leq s \leq (Q+M_0)\, \mu} g_s \mathbb{N}X^Y_l,$$

the same proof as that of Lemma 4.9 shows that

$$\left|\left\{w \in I^m : D^n_{K_0}(w) \leq 1 - \epsilon_1/2\right\}\right| \leq c_0^n$$

for any positive integer $n$. The conclusion follows from (5.11) and Lemma 4.10. \qed
APPENDIX A.

The aim of this section is to prove Lemma 2.8. Let $G, g_t, U^+_G$ be as in Theorem 1.1. We first give a characterization of $g_1$ expanding subgroup.

**Lemma A.1.** Let $U$ be a connected $Ad$-unipotent subgroup of $G$ normalized by $\{g_t : t \in \mathbb{R}\}$. Suppose that for any nonzero $U$ stabilized vector $v$ in a nontrivial irreducible finite dimensional representation of $G$ the projection $\pi_+(v)$ to the expanding subspace of $g_1$ is nonzero, then $U$ is a $g_1$ expanding subgroup of $G$.

Remark: It is easy to see that the converse of Lemma A.1 also holds.

**Proof.** We fix a nonzero vector $v$ in a nontrivial finite dimensional irreducible representation $V$ of $G$. Let $W = \{v_1 \in V : uv_1 = v_1 \text{ for any } u \in U\}$. It is easy to see that $W$ is invariant under the one parameter subgroup $\{g_t : t \in \mathbb{R}\}$. According to the assumption we have that $W \subset V^+$. Let $W'$ be a $\{g_t : t \in \mathbb{R}\}$ invariant complement of $W$ in $V$ and let $\pi_W : V \rightarrow W$ be the projection to $W$ with respect to $W'$. It follows from [21] Lemma 5.1 that $\pi_W(Uv)$ is not identically zero. Therefore $\pi_+(Uv)$ is not identically zero. So $U$ is a $g_1$ expanding subgroup of $G$ according to the definition. □

The key ingredient of the proof of Lemma 2.8 is the following result about abstract root systems.

**Lemma A.2.** Let $\Delta$ be an irreducible abstract root system and let $E = \text{span}_\mathbb{R}\Delta$. Suppose that $E$ has dimension $n$ and $(\cdot, \cdot)$ is the inner product of $E$ invariant under the Weyl group of $\Delta$. Let $\Delta^+ \subset \Delta$ be a positive system dominated by some $\alpha \in E$, i.e. $(\alpha, \beta) \geq 0$ for any $\beta \in \Delta^+$. Then there exists a basis $\beta_1, \ldots, \beta_n \in \Delta^+$ of $E$ such that

\[(A.1) \quad \alpha = c_1\beta_1 + \cdots + c_n\beta_n\]

where $c_i \geq 0$ and $\beta_i + \beta_j \not\in \Delta^+$ for any $i, j$.

**Proof.** According to the classification of irreducible abstract root systems, see e.g. [15] Chapter 2, it suffices to consider the case where $\Delta$ is reduced. Let $\Pi = \{\alpha_1, \ldots, \alpha_n\}$ be simple roots determined by $\Delta^+$ and let $A$ be the associated Cartan matrix.

Recall that two roots $\beta$ and $\gamma$ are said to be strongly orthogonal if $(\beta, \gamma) = 0$ and a subset $O$ of $\Delta^+$ is called strongly orthogonal system if elements of $O$ are pairwise strongly orthogonal. It follows from Oh [18] that if $\Delta$ is of type $B, C, E_7, E_8, F_4, G_2, D_n (n$ is even) then there is a strongly orthogonal system $O$ consisting of $n$ elements. In these cases
\( \alpha \) is a linear combination of elements in \( \mathcal{O} \) satisfying the conclusion of the lemma.

We will prove the rest cases one by one. Let \( \| \cdot \| \) be the induced norm on \( E \). We assume without loss of generality that \( \| \alpha_i \| = 1 \) if \( \Delta \) is of type \( A_n, D_n \) or \( E_6 \). It follows from Lusztig and Tits [16] that \( A^{-1} \) has positive rational entries. So we have

\[
\alpha = a_1 \alpha_1 + \cdots + a_n \alpha_n
\]

where \( a_i \in \mathbb{R}_{>0} \).

**Case \( A_n \).**

\[
\begin{array}{cccccc}
  & - & - & - & - & - & \circ & - & - & - & - & - & \circ \\
\alpha_1 & \alpha_2 & \alpha_{k-1} & \alpha_k & \\
\end{array}
\]

We assume that our simple roots are ordered so that \( a_1 \geq a_2 \) and the corresponding Dynkin diagram is as above. Since \( \alpha \) is dominated we have

\[
(\alpha, \alpha_2) = a_2 - \frac{1}{2}a_1 - \frac{1}{2}a_3 \geq 0
\]

which implies \( a_2 \geq a_3 \). Hence a simple induction implies that \( a_i \geq a_{i+1} \) for \( i \leq k - 1 \). Therefore \( \Pi \) can be rearranged so that \( a_i \geq a_{i+1} \) and \( \alpha_{i+1} \) is connected with one of \( \{\alpha_1, \ldots, \alpha_i\} \) in the Dynkin diagram. So if we take \( \beta_i = \alpha_1 + \cdots + \alpha_i \in \Delta^+ \) the conclusion of the lemma follows.

**Case \( D_n \) where \( n \) is odd.** In this case the strongly orthogonal \( \mathcal{O} \) constructed in [18] contains \( n - 1 \) elements and the highest root is not in \( \mathcal{O} \). If we take there elements as \( \beta_1, \ldots, \beta_n \), then they satisfy \( \beta_i + \beta_j \notin \Delta \) but it is not clear to the author how to prove \( c_i \geq 0 \).

\[
\begin{array}{cccccc}
  & \circ & \circ & - & - & - & - & - & - & - & - & - & - & \circ \\
\alpha_1 & \alpha_2 & \alpha_{n-3} & \alpha_{n-2} & \alpha_{n-1} & \circ & \circ & \\
\end{array}
\]

We assume that \( \Pi \) is ordered so that its Dynkin diagram is as above. There is a complete list of \( \Delta^+ \) in terms of \( \Pi \) with \( E = \mathbb{R}^n \) given in Knapp [15] Appendix C as follows: \( \alpha_i = e_i - e_{i+1} \) for \( i < n \) and \( \alpha_n = e_{n-1} + e_n \); \( \Delta^+ = \{e_i \pm e_j : i < j\} \) where \( \{e_1, \ldots, e_n\} \) is the standard basis of \( \mathbb{R}^n \).

Since \( \alpha \) is dominated we have

\[
2(\alpha, \alpha_1) = 2a_1 - a_2 \geq 0
\]

Assume that \( (i + 1)a_i - ia_{i+1} \geq 0 \) for \( i \leq n - 4 \). Then

\[
2(i + 1)(\alpha, \alpha_{i+1}) + (i + 1)a_{i} - ia_{i+1} = (i + 2)a_{i+1} - (i + 1)a_{i+2} \geq 0
\]
Therefore we have

\[(A.2) \quad (i + 1)a_i - ia_{i+1} \geq 0 \quad \text{for } 1 \leq i \leq n - 3.\]

By calculating inner products of \(\alpha\) with \(\alpha_{n-1}\) and \(\alpha_n\) we have

\[(A.3) \quad \begin{cases} 2a_n \geq a_{n-2} \\ 2a_{n-1} \geq a_{n-2}. \end{cases}\]

It follows from (A.3) and \((\alpha, \alpha_{n-2}) \geq 0\) that

\[a_{n-3} \leq a_{n-2}.\]

Using \((\alpha, \alpha_{n-2-i}) \geq 0\) for \(1 \leq i \leq n - 4\) it can be shown inductively that

\[(A.4) \quad a_{n-2-i} \geq a_{n-3-i}.\]

For \(1 \leq i \leq (n-3)/2\) we take

\[
\begin{align*}
\beta_{2i} &= \alpha_{2i-1} \\
\beta_{2i} &= \alpha_{2i-1} + 2(\alpha_{2i} + \cdots + \alpha_{n-2}) + \alpha_{n-1} + \alpha_n.
\end{align*}
\]

It follows from (A.2), (A.3) and (A.4) that there are nonnegative integers \(c_1, \ldots, c_{n-3}, b_{n-2}, b_{n-1}, b_n\) such that

\[\alpha - c_1\alpha_1 - \cdots - c_{n-3}\alpha_{n-3} = b_{n-2}\alpha_{n-2} + b_{n-1}\alpha_{n-1} + b_n\alpha_n.\]

If \(b_{n-2} \geq b_{n-1} \geq b_n\) we take \(\beta_{n-2} = \alpha_{n-2} + \alpha_{n-1} + \alpha_n, \beta_{n-1} = \alpha_{n-2} + \alpha_{n-1}\) and \(\beta_n = \alpha_{n-2}\). It is easy to see the existence of nonnegative numbers \(c_{n-2}, c_{n-1}, c_n\). The fact that \(\beta_i + \beta_j \notin \Delta^+\) follows from the list of \(\Delta^+\) and

\[
\begin{align*}
\beta_{2i-1} &= e_{2i-1} - e_{2i} \quad \text{for } 1 \leq i \leq (n-3)/2 \\
\beta_{2i} &= e_{2i-1} + e_{2i} \quad \text{for } 1 \leq i \leq (n-3)/2 \\
\beta_{n-2} &= e_{n-2} + e_{n-1} \\
\beta_{n-1} &= e_{n-2} - e_n \\
\beta_n &= e_{n-2} - e_{n-1}.
\end{align*}
\]

The rest cases can be proved similarly by taking \(\beta_{n-2} = \alpha_{n-2} + \alpha_{n-1} + \alpha_n\), so we only list the choices of \(\beta_{n-1}\) and \(\beta_n\). If \(b_{n-2} \geq b_n > b_{n-1}\) we take \(\beta_{n-1} = \alpha_{n-2} + \alpha_n\) and \(\beta_n = \alpha_{n-2}\); if \(b_{n-1} > b_{n-2} \geq b_n\) we take \(\beta_{n-1} = \alpha_{n-2} + \alpha_{n-1}\) and \(\beta_n = \alpha_{n-1}\); if \(b_n > b_{n-2} \geq b_{n-1}\) we take \(\beta_{n-1} = \alpha_{n-2} + \alpha_n\) and \(\beta_n = \alpha_n\); if \(b_n > b_{n-2} > b_{n-1}\) we take \(\beta_{n-1} = \alpha_{n-1}\) and \(\beta_n = \alpha_n\).

Case \(E_6\).
We assume that $\Pi$ is ordered so that its Dynkin diagram is as above. A simple calculation using $(\alpha, \alpha_i) \geq 0$ implies

$$
\begin{align*}
2a_1 &\geq a_3 \\
2a_6 &\geq a_5 \\
3a_3 &\geq 2a_4 \\
3a_5 &\geq 2a_4 \\
2a_4 &\geq 3a_2.
\end{align*}
$$

Let $\beta_1 = \alpha_1 + 2\alpha_2 + 2\alpha_3 + 3\alpha_4 + 2\alpha_5 + \alpha_6$ to be the highest root. Then it follows from the above calculation that there exist nonnegative numbers $b_1, b_3, b_4, b_5, b_6$ such that

$$
\alpha - \frac{a_2}{2} \beta_1 = b_1 \alpha_1 + b_3 \alpha_3 + \cdots + b_6 \alpha_6.
$$

We take $\beta_i$ to be the linear combination of $\alpha_1, \alpha_3, \alpha_4, \alpha_5, \alpha_6$ with coefficients 0 or 1 depending on the decreasing order of $b_i$. For example if $b_1 \geq b_3 \geq b_5 \geq b_6 \geq b_4$, then we take $\beta_2 = \alpha_1 + \alpha_3 + \alpha_4 + \alpha_5 + \alpha_6$, $\beta_3 = \alpha_5 + \alpha_6$, $\beta_4 = \alpha_1 + \alpha_3$, $\beta_5 = \alpha_5$ and $\beta_6 = \alpha_1$. The existence of nonnegative coefficients $c_i$ follows easily. We can check the condition that $\beta_i + \beta_j \not\in \Delta$ by simply list all the roots.

Proof of Lemma 2.8 Let $a$ be a maximal $\mathbb{R}$ split Cartan subalgebra of the Lie algebra $g$ of $G$ so that for some $v \in a$ we have $g_t = \exp tv$. Let $B$ be the Killing form of $g$ and let $\theta$ be the Cartan involution of $g$ with $a$ belonging to the eigenspace of $-1$. The inner product of $w, w' \in a$ is defined by $(w, w') = -B(w, \theta w')$. For every $w \in g$ we associate $\alpha_w \in a^*$ where $\alpha_w(w') = (w, w')$. This defines an isomorphism of real vector spaces $a \rightarrow a^*$. The inner product on $a$ can be transferred to an inner product on $a^*$ via this isomorphism.

Let $\Phi(g, a)^+$ be a positive system dominated by $v$ in the relative root system $\Phi(g, a) \subset a^*$. Suppose that $a$ has dimension $n$. It follows from Lemma A.2 that there exist nonnegative real numbers $c_1, \ldots, c_n$ and $\beta_1, \ldots, \beta_n \in \Phi(g, a)^+$ such that

$$
\alpha_v = c_1 \beta_1 + \cdots + c_n \beta_n \quad \text{and} \quad \beta_i + \beta_j \not\in \Phi(g, a).
$$

For each $i$ with $c_i > 0$ we choose some $w_i$ in the root space of $\beta_i$ and fix a natural $\mathfrak{sl}_2$ triple $(v_i, w_i, w_i^-)$ where $w_i^-$ belongs to the root...
space of $-\beta_i$ and $v_i \in \mathfrak{a}$ satisfies $\alpha_{v_i} = b_i\beta_i$ for some $b_i > 0$, cf. Proposition 6.52. Let $G_1 \subset G$ be the connected Lie group whose Lie algebra $\mathfrak{g}_1$ is generated by these $\mathfrak{sl}_2$ triples. It is straightforward to check using Lemma A.1 that the connected subgroup $U_a$ whose lie algebra is generated by $\{w_i : c_i \neq 0\}$ satisfies property ($\ast$). □

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