Proof of a conjecture on unimodality

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Abstract

Let \( P(x) \) be a polynomial of degree \( m \), with nonnegative and non-decreasing coefficients. We settle the conjecture that for any positive real number \( d \), the coefficients of \( P(x + d) \) form a unimodal sequence, of which the special case \( d \) being a positive integer has already been asserted in a previous work. Further, we explore the location of modes of \( P(x + d) \) and present some sufficient conditions on \( m \) and \( d \) for which \( P(x + d) \) has the unique mode \( \left\lceil \frac{m-d}{d+1} \right\rceil \).

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1 Introduction

Let \( a_0, \ldots, a_m \) be a sequence of nonnegative real numbers. We say that the sequence is unimodal if there exists an index \( 0 \leq t \leq m \) such that \( a_0 \leq \cdots \leq a_{t-1} \leq a_t \geq a_{t+1} \geq \cdots \geq a_m \). Such an index \( t \) is called a mode of the sequence. A property closely related to unimodality is log-concavity. We say that the sequence is log-concave if \( a_{i-1}a_{i+1} \leq a_i^2 \) for all \( 1 \leq i \leq m-1 \). The sequence is said to have no internal zeros if there are not three indices \( i < j < k \) such that \( a_i, a_k \neq 0 \) and \( a_j = 0 \). It is well known that a log-concave sequence with no internal zeros is unimodal (see [3, Proposition 2.5.1] for instance). Unimodal and log-concave sequences occur naturally in many branches of mathematics. See the survey articles [8] and [4] for various techniques, problems, and results about unimodality and log-concavity.

Let \( P(x) = \sum_{i=0}^{m} a_i x^i \) be a polynomial with nonnegative coefficients. We say that \( P(x) \) is unimodal (respectively, log-concave, non-decreasing, etc.) if the sequence of coefficients \( a_0, a_1, \ldots, a_m \) of \( P(x) \) enjoys the corresponding property. A mode of \( a_0, \ldots, a_m \) is also called a mode of \( P(x) \).

It is well known that if \( P(x) \) is log-concave with no internal zeros, then \( P(x + 1) \) is log-concave (see [4, Corollary 8.4] or [7, Theorem 2]). Actually, it may also be shown that
$P(x + d)$ is log-concave for any positive number $d$ by using [3, Theorem 2.5.3]. In the present paper, we consider the analogue problem concerning unimodality. Let $P(x)$ be nonnegative and non-decreasing. It is shown that $P(x + 1)$ is unimodal in [2] and more generally, that $P(x + n)$ is unimodal when $n$ is a positive integer in [1]. Further, the following is conjectured.

**Conjecture 1.1** ([1]). Let $P(x)$ be a polynomial of degree $m$ and with nonnegative coefficients. Suppose that $P(x)$ is non-decreasing and that $d$ is a positive real number. Then $P(x + d)$ is unimodal.

In this paper we settle the above conjecture. Moreover, we will explore the number and location of modes of the polynomial $P(x + d)$. Let $M_s(P, d)$ and $M^*(P, d)$ be the smallest and the greatest mode of $P(x + d)$ respectively. Denote $\overline{m}(d) = \lceil \frac{m - d}{d + 1} \rceil$ and $\underline{m}(d) = \lfloor \frac{m}{d + 1} \rfloor$ where $[x]$ and $\lceil x \rceil$ denote the least integer $\geq x$ and the greatest integer $\leq x$ respectively. It is not difficult to see that $\overline{m}(d)$ and $\underline{m}(d)$ coincide when $d$ is a positive integer. In [1], it is shown that $\underline{m}(d)$ is a mode of $P(x + d)$ when $d$ is a positive integer. The statement is not true when $d$ is only a positive number. Generally speaking, the number and location of modes of $P(x + d)$ are related not only to $m$ and $d$, but also to coefficients of the polynomial $P(x)$. The matter is somewhat different when $d \geq 1$. In this case, we can show that $P(x + d)$ has at most two modes $\overline{m}(d)$ and $\overline{m}(d) + 1$ if $P(x) = ax^m$, or $\overline{m}(d) - 1$ and $\overline{m}(d)$ otherwise. We will also present certain sufficient conditions on $m$ and $d$ that $P(x + d)$ has the unique mode $\overline{m}(d)$, including the case when $d$ is a positive integer larger than 1.

Throughout this paper, let $m$ be a positive integer and $d$ a positive real number. We denote by $P_m^+$ the set of monic polynomials of degree $m$, with nonnegative and non-decreasing coefficients. When there is no danger of confusion, we simply write $\overline{m}$ to mean $\overline{m}(d)$. By definition, it follows immediately that

$$m - d \leq (d + 1)\overline{m} < m + 1,$$

which will be used repeatedly in the sequel.

## 2 Proof of Conjecture 1.1

To prove Conjecture 1.1 we need the following two lemmas.

**Lemma 2.1.** Suppose that the polynomial $f(x)$ is unimodal with the smallest mode $t$ and that $d > 0$. Then $(x + d)f(x)$ is unimodal with the smallest mode $t$ or $t + 1$.

**Proof.** Let $f(x) = \sum_{i=0}^{n} c_i x^i$ where $c_0 \leq \cdots \leq c_{t-1} < c_t \geq c_{t+1} \geq \cdots \geq c_n$. Then

$$(x + d)f(x) = c_0 d + (c_0 + c_1 d)x + \cdots + (c_{t-2} + c_{t-1} d)x^{t-1} + (c_{t-1} + c_t d)x^t + (c_t + c_{t+1} d)x^{t+1} + \cdots + (c_{n-1} + c_n d)x^{n-1} + c_n x^n.$$ 

Clearly, $c_0 \leq c_0 + c_1 d \leq \cdots \leq c_{t-2} + c_{t-1} d < c_{t-1} + c_t d$ and $c_t + c_{t+1} d \geq \cdots \geq c_{n-1} + c_n d \geq c_n$. So the statement follows. \qed
Lemma 2.2. Let $P(x) = \sum_{i=0}^{m} a_ix^i$ be a polynomial of degree $m$, with nonnegative coefficients and $d > 0$. Suppose that $P(x + d) = \sum_{j=0}^{m} b_jx^j$. Then $b_{\bar{m}} \geq b_{\bar{m}+1} \geq \cdots \geq b_{m}$.

Furthermore, if $d \geq (m - 1)/2$, then $P(x + d)$ is unimodal and has the mode 0 or 1. In particular, if $d \geq m$ then $P(x + d)$ is non-increasing.

Proof. We have $b_j = P^{(j)}(d)/j! = \sum_{i=j}^{m} a_id^{i-j} {i \choose j}$, which yields that

$$(j + 1)d^{j+1}(b_{j+1} - b_j) = \sum_{i=j}^{m} a_id^j [i(i - 1) - (d + 1)(j + 1)].$$  \hfill (2)

Now let $j \geq \bar{m}$. Then $(d+1)(j+1) \geq (d+1)(\bar{m}+1) \geq m+1$ by (1). Every term in the sum (2) is therefore non-positive, and thus $b_{j+1} \leq b_j$. Finally, note that $(m - 1)/2 \leq d < m$ implies $\bar{m} \leq 1$, and that $d \geq m$ implies $\bar{m} = 0$. So the statement follows.

Proof of Conjecture 1.1. Let $P(x) = \sum_{i=0}^{m} a_ix^i$ and $P(x + d) = \sum_{j=0}^{m} b_jx^j$. We need to show that $b_0, \ldots, b_m$ is unimodal. We do this by induction on $m$. If $m = 1$, the result is obvious, so we proceed to the inductive step. By Lemma 2.2, it suffices to consider the case $m > 2d + 1$.

Let $P(x) = a_0 + xf(x)$ where $f(x) = \sum_{i=0}^{m-1} a_{i+1}x^i$. Then

$$P(x + d) = a_0 + (x + d)f(x + d).$$

By the induction hypothesis, $f(x + d)$ is unimodal, so is $(x + d)f(x + d)$ by Lemma 2.1. Thus $b_1, b_2, \ldots, b_m$ is unimodal.

Let $r = \lfloor d \rfloor$. Then $r < d + 1 < m$. By (2) we have

$$b_1 - b_0 = \sum_{i=0}^{m} a_i d^{i-1} (i - d)$$

$$= \sum_{i=r+1}^{m} a_i d^{i-1} (i - d) - \sum_{i=0}^{r} a_i d^{i-1} (d - i)$$

$$\geq a_r \sum_{i=r+1}^{m} d^{i-1} (i - d) - a_r \sum_{i=0}^{r} d^{i-1} (d - i)$$

$$= a_r [d + 2d^2 + \cdots + (m - 1)d^{m-1} - d^m]$$

$$\geq a_r [(m - 1)d^{m-1} - d^m]$$

$$= a_r (m - d - 1)d^{m-1}$$

$$\geq 0.$$

Thus $b_0, b_1, \ldots, b_m$ is still unimodal. This completes the proof.

Corollary 2.1. Let $P(x) \in \mathbb{P}_m^+$ and $d > 0$. Suppose that $P(x) \neq x^m$. Then

$$M^*(P, d) \leq \bar{m}.$$
Proof. Let $P(x) = \sum_{i=0}^{m} a_i x^i$ and $P(x+d) = \sum_{j=0}^{m} b_j x^j$. We have by (2)

$$\sum_{i=m}^{m+1} (b_{m+1} - b_m) = \sum_{i=m}^{m+1} a_i d^i \left[ \frac{(i+1) - (d+1)(m+1)}{m+1} \right].$$

By (1), $(i+1) - (d+1)(m+1) \leq (m+1) - (d+1)(m+1) \leq 0$ for each $i \leq m$. In particular, $m - (d+1)(m+1) \leq -1 < 0$. On the other hand, $a_m \neq 0$ since $P(x) \neq x^m$. Hence $b_{m+1} < b_m$. This implies that the unimodal sequence $\{b_j\}$ has no mode larger than $m$, and the proof is therefore complete.

3 Modes of $(x+d)^m$ and $\sum_{i=0}^{m} (x+d)^i$

This section is devoted to studying modes of $P(x+d)$ for two basic polynomials $P(x) = x^m$ and $P(x) = \sum_{i=0}^{m} x^i$ respectively, which will play a key role in investigating modes of $P(x+d)$ for generic polynomials $P(x) \in \mathbb{P}_m^m$.

Proposition 3.1. Let $d > 0$. If $\frac{m+1}{d+1} \in \mathbb{Z}^+$, then $(x+d)^m$ has two modes $m$ and $m+1$; otherwise $(x+d)^m$ has the unique mode $m$.

Proof. Let $(x+d)^m = \sum_{i=0}^{m} c_i x^i$ where $c_i = \binom{m}{i} d^{m-i}$. Denote $f(x) = \frac{m-x+1}{dx}$. Then $\frac{c_i}{c_{i-1}} = f(i)$. Clearly, $f(x)$ is strictly decreasing and $f(\frac{m+1}{d+1}) = 1$. Now $i \leq m$ implies $i < \frac{m+1}{d+1}$, and $i \geq m+1$ implies $i \geq \frac{m+1}{d+1}$. So the statement follows.

Let $Q_m(x) = \sum_{i=0}^{m} x^i$ and $Q_m(x+d) = \sum_{j=0}^{m} d_j x^j$ where

$$d_j = \sum_{i=j}^{m} d^{i-j} \binom{i}{j}, \quad j = 0, 1, \ldots, m. \tag{3}$$

Then the sequence $\{d_j\}$ is log-concave with no internal zeros (see Brenti[3, Theorem 2.5.3] for instance). Actually, we have the following stronger result.

Proposition 3.2. The sequence $\{d_j\}$ is strictly log-concave, i.e., $d_{j-1}d_{j+1} < d_j^2$ for all $0 < j < m$, and is therefore unimodal with at most two modes.

Proof. Note that

$$d_{j-1} = \sum_{i=j-1}^{m} d^{i-j+1} \binom{i}{j-1} = \sum_{i=j-1}^{m} d^{i-j+1} \left[ \binom{i+1}{j} - \binom{i}{j} \right] = (1-d)d_j + d^{m-j+1} \binom{m+1}{j}.$$
Thus we have
\[
d_j^2 - d_{j-1}d_{j+1} = d_j^2 - \left[(1 - d)d_j + d^{m-j+1}\binom{m+1}{j}\right]d_{j+1}
\]
\[
= [d_j - (1 - d)d_{j+1}]d_j - d^{m-j+1}\binom{m+1}{j}d_{j+1}
\]
\[
= d^{m-j}\binom{m+1}{j+1}d_j - d^{m-j+1}\binom{m+1}{j}d_{j+1}
\]
\[
= \sum_{i=j}^m \left[\binom{m+1}{j+1} - \binom{m+1}{i}d^{m-i-2j}\right]
\]
\[
= \sum_{i=j}^m \frac{m-i+1}{j+1}\binom{m+1}{i}d^{m+i-2j}
\]
\[
> 0,
\]
the desired inequality.

In what follows we explore the location of modes of the sequence \(\{d_j\}\). We first consider the case \(d \geq 1\). The matter is rather simple when \(d = 1\).

**Proposition 3.3.** If \(m\) is even then \(Q_m(x + 1)\) has two modes \(\frac{m}{2} - 1\) and \(\frac{m}{2}\); otherwise \(Q_m(x + 1)\) has the unique mode \(\frac{m}{2} - 1\).

**Proof.** Since \(Q_m(x) = \frac{1}{x - 1}(x^{m+1} - 1)\), we have
\[
Q_m(x + 1) = \frac{1}{x}[(x + 1)^{m+1} - 1].
\]

By Proposition 3.1, \((x + 1)^{m+1}\) has two modes \(m + 1\) and \(m + 1 + 1\) for \(m\) even, or only one mode \(m + 1\) otherwise, so does \((x + 1)^{m+1} - 1\). Thus the statement follows.

**Proposition 3.4.** Let \(d \geq 1\). Then \(Q_m(x + d)\) has at most two modes \(\overline{m} - 1\) and \(\overline{m}\). In particular, if \(m + 1 = \overline{m} + 1\), then \(Q_m(x + d)\) has the unique mode \(\overline{m}\).

**Proof.** By Lemma 2.2, it suffices to consider the case \(1 \leq d < m\). We have
\[
(x + d - 1)Q_m(x + d) = (x + d)^{m+1} - 1.
\]

By Proposition 3.1, \((x + d)^{m+1}\) has the smallest mode \(m + 1\), so does \((x + d)^{m+1} - 1\). Thus \(M_s(Q_m, d) \geq m + 1 - 1\) by Lemma 2.1. On the other hand, we have \(M_s(Q_m, d) \leq \overline{m}\) by Corollary 2.1. Note that \(\overline{m} = \overline{\overline{m}}\) or \(\overline{m} + 1\) since
\[
\frac{m - d}{d + 1} < \frac{m + 1 - d}{d + 1} < \frac{m - d}{d + 1} + 1.
\]

Hence \(Q_m(x + d)\) has at most two modes \(\overline{m} - 1\) and \(\overline{m}\), and in particular, only one mode \(\overline{m}\) provided \(m + 1 = \overline{m} + 1\). This completes the proof.

**Corollary 3.1.** If \(d \geq 1\) and \(\frac{d+1}{d+1} \in \mathbb{Z}^+\), then \(Q_m(x + d)\) has the unique mode \(\overline{m}\).
Proof. If $\frac{m+1}{d+1} \in \mathbb{Z}^+$, then $\frac{m-d}{d+1} \in \mathbb{Z}^+$, and so $\overline{m} = \frac{m-d}{d+1}$. On the other hand,
\[
\overline{m} + 1 = \left\lfloor \frac{m + 1 - d}{d + 1} \right\rfloor = \left\lfloor \frac{m + 1}{d + 1} - \frac{d}{d + 1} \right\rfloor = \frac{m + 1}{d + 1}.
\]
Thus $\overline{m} + 1 = \overline{m} + 1$. So the statement follows from Proposition 3.4.

Proposition 3.5. If $d > 1$ and $dm \in \mathbb{Z}^+$, then $Q_m(x + d)$ has the unique mode $\overline{m}$.

Proof. By Proposition 3.4, it suffices to prove $d^m m - d^{m-1}$. By (2), we have
\[
\overline{m} d^m (d_m - d_{m-1}) = \sum_{i=m-1}^{m} d^i \left( \frac{i}{m - 1} \right) [(i + 1) - (d + 1)\overline{m}].
\]
The sum contains terms of both signs. Let $r = [(d + 1)\overline{m}] - 1$. Denote
\[
S_1 = \sum_{i=r}^{m} d^i \left( \frac{i}{m - 1} \right) [(i + 1) - (d + 1)\overline{m}]
\]
and
\[
S_2 = \sum_{i=m-1}^{r-1} d^i \left( \frac{i}{m - 1} \right) [(d + 1)\overline{m} - (i + 1)].
\]
Then $\overline{m} d^m (d_m - d_{m-1}) = S_1 - S_2$. Thus we need to prove $S_1 > S_2$.

Since $(d + 1)\overline{m} < m + 1$ by (1) and the left is an integer by the assumption, we have $r \leq m - 1$. So
\[
S_1 \geq d^{r+1} \left( \frac{r + 1}{m - 1} \right) [(r + 2) - (d + 1)\overline{m}] = d^{r+1} \left( \frac{r + 1}{m - 1} \right).
\]
On the other hand,
\[
S_2 \leq \sum_{i=m-1}^{r-1} d^{r-1} \left( \frac{i}{m - 1} \right) [(r + 1) - (i + 1)]
\]
\[
\leq d^{r-1} \left[ (r + 1) \sum_{i=m-1}^{r-1} \left( \frac{i}{m - 1} \right) - \overline{m} \sum_{i=m-1}^{r-1} \left( \frac{i + 1}{m - 1} \right) \right]
\]
\[
= d^{r-1} \left[ (r + 1) \left( \frac{r}{m} \right) - \overline{m} \left( \frac{r + 1}{m + 1} \right) \right]
\]
\[
= d^{r-1} \left( \frac{r + 1}{m + 1} \right).
\]
Thus we have
\[
\frac{S_1}{S_2} \geq \frac{d^{r+1} \left( \frac{r + 1}{m - 1} \right)}{d^{r-1} \left( \frac{r + 1}{m + 1} \right)} = \frac{d^2 \overline{m} (m + 1)}{(r - \overline{m} + 1)(r - \overline{m} + 2)} = \frac{d(m + 1)}{d\overline{m} + 1} > 1,
\]
the desired inequality.
Corollary 3.2. If \( d > 1 \) and \( d \in \mathbb{Z}^+ \), then \( Q_m(x + d) \) has the unique mode \( \overline{m} \).

Corollary 3.3. If \( d > 1 \) and \( \frac{m}{d+1} \in \mathbb{Z}^+ \), then \( Q_m(x + d) \) has the unique mode \( \overline{m} \).

Proof. If \( \frac{m}{d+1} \in \mathbb{Z}^+ \), then

\[
\overline{m} = \left\lceil \frac{m - d}{d+1} \right\rceil = \left\lceil \frac{m}{d+1} - \frac{d}{d+1} \right\rceil = \frac{m}{d+1}.
\]

Thus \( d\overline{m} = m - \overline{m} \in \mathbb{Z}^+ \), and the statement follows from Proposition 3.5. \[\square\]

We next consider the case \( 0 < d < 1 \), which is more complicated. For example, modes of \( Q_m(x + d) \) may be neither \( \overline{m} - 1 \) nor \( m \) (see Remark 3.1). The following is some rough estimate for location of modes of \( Q_m(x + d) \).

Proposition 3.6. Let \( 0 < d < 1 \). Then

(i) \( \left\lceil \frac{m}{2} \right\rceil \leq M_0(Q_m, d) \leq M^*(Q_m, d) \leq \min\{m - 1, \overline{m}\} \).

(ii) If \( 0 < d < 1/\left(\frac{m}{2}\right) \), then \( Q_m(x + d) \) has the unique mode \( m - 1 \). The converse is also true.

(iii) If \( 0 < 1 - d \leq 1/m \), then \( Q_m(x + d) \) has at most two modes \( \overline{m} - 1 \) and \( \overline{m} \). In particular, if \( \frac{m+1}{d+1} \in \mathbb{Z}^+ \), then \( Q_m(x + d) \) has the unique mode \( \overline{m} \).

(iv) There exists a positive number \( \varepsilon \) such that for \( 0 < 1 - d < \varepsilon \), \( Q_m(x + d) \) has the unique mode \( \left\lceil \frac{m}{2} \right\rceil \).

Proof. By the definition, \( M^*(Q_m, d) \) is the greatest integer \( j \) no larger than \( m \) such that

\[
d_j - d_{j-1} = \sum_{i=j}^{m} \binom{i}{j} d^{i-j} - \sum_{i=j-1}^{m} \binom{i}{j-1} d^{i-j+1}
= \sum_{i=j-1}^{m-1} \binom{i+1}{j} d^{i-j+1} - \sum_{i=j-1}^{m} \binom{i}{j-1} d^{i-j+1}
= \sum_{i=j}^{m-1} \binom{i}{j} d^{i-j+1} - \binom{m}{j-1} d^{m-j+1}.
\]

Hence

\[
M^*(Q_m, d) = \max \left\{ 1 \leq j \leq \overline{m} : \sum_{i=j}^{m-1} \binom{i}{j} d^{i-j+1} - \binom{m}{j-1} d^{m-j+1} > 0 \right\}.
\]

When \( 0 < d < 1 \), we have

\[
\sum_{i=j}^{m-1} \binom{i}{j} d^{i-j+1} \geq d^{m-j} \sum_{i=j}^{m-1} \binom{i}{j} = \binom{m}{j+1} d^{m-j}.
\]

It is not difficult to see that

\[
\binom{m}{j+1} d^{m-j-1} - \binom{m}{j-1} d^{m-j} > 0
\]
is equivalent to
\[(m - j)(m - j + 1) - dj(j + 1) > 0.\]

Now let \(h(x) = (m - x)(m - x + 1) - dx(x + 1)\). Then \(h(x)\) is a decreasing function in the interval \(0 \leq x \leq m\) since \(h'(x) < 0\). Thus \(h(x_0) > 0\) for some \(x_0 \in (0, m)\) implies that \(M_s(Q_m, d) \geq \lfloor x_0 \rfloor\).

(i) Since \(h \left( \frac{m}{2} \right) = \frac{m}{2} \left( \frac{m}{2} + 1 \right) (1 - d) > 0\), we have \(M_s(Q_m, d) \geq \lfloor \frac{m}{2} \rfloor\).

It remains to show that \(M^*(Q_m, d) \leq m - 1\). It suffices to prove \(d_{m-1} > d_m\), which is obvious since \(d_m = 1\) and \(d_{m-1} = 1 + md\).

(ii) By (i), \(m - 1\) is the unique mode of \(Q_m(x + d)\) if and only if \(d_{m-1} > d_{m-2}\). Note that \(d_{m-1} = 1 + md\) and \(d_{m-2} = 1 + (m - 1)d + \left(\frac{m}{2}\right) d^2\). Hence \(Q_m(x + d)\) has the unique mode \(m - 1\) if and only if \(0 < d < 1/(\frac{m}{2})\).

(iii) If \(0 < 1 - d \leq 1/m\), then
\[
\frac{m(1 - x)}{d + 1} = \frac{d(m + 1)}{(d + 1)^2}[3d + 1 - (1 - d)m] > 0,
\]
which implies that \(M_s(Q_m, d) \geq \lceil \frac{m - d}{d + 1} \rceil\). On the other hand, \(M^*(Q_m, d) \leq \overline{m} = \left\lceil \frac{m - d}{d + 1} \right\rceil\) by Corollary 1.4. Note that
\[
\lfloor x \rfloor = \begin{cases} 
\lceil x \rceil, & \text{if } x \in \mathbb{Z}; \\
\lceil x \rceil - 1, & \text{otherwise}.
\end{cases}
\]
Hence \(Q_m(x + d)\) has at most two modes \(\overline{m}\) and \(\overline{m} - 1\), and in particular, only one mode \(\overline{m}\) if \(\frac{m - d}{d + 1}\) is an integer.

(iv) Denote \(t = \left\lceil \frac{m}{2} \right\rceil\). Then \(M_s(Q_m, d) \geq t\) by (i). On the other hand, we have by \(\text{(1)}\)
\[
d_{t+1} - d_t = \sum_{i=t+1}^{m-1} \binom{i}{t+1} d^{i-t} - \binom{m}{t} d^{m-t}
\]
\[
\quad \rightarrow \sum_{i=t+1}^{m-1} \binom{i}{t+1} - \binom{m}{t}
\]
\[
= \binom{m}{t+2} - \binom{m}{t}
\]
when \(d\) tends to 1. Note that \(\binom{m}{t+2} - \binom{m}{t} < 0\). Hence \(d_{t+1} - d_t < 0\) if \(d\) is sufficiently close to 1, which implies that \(Q_m(x + d)\) has the unique mode \(t\). \(\Box\)

**Remark 3.1.** It is worth pointing out that modes of \(Q_m(x + d)\) may be neither \(\overline{m} - 1\) nor \(\overline{m}\) when \(0 < d < 1\). For example, let \(1/(\frac{m}{2}) < d < 1/m\). Then \(\overline{m} = m\). However, each mode of \(Q_m(x + d)\) is smaller than \(m - 1\) since \(d_{m-2} > d_{m-1}\).

### 4 Modes in General Case

The following theorem shows the importance of two basic polynomials considered in the last section.
**Theorem 4.1.** Let \( P(x) \in \mathbb{P}_m^+ \) and \( d > 0 \). Then

\[
M_*(Q_m, d) \leq M_*(P, d) \leq M^*(P, d) \leq M^*(x^m, d).
\]

Moreover, if \( Q_m(x + d) \) has the mode \( \overline{m} \), then so does \( P(x + d) \). In particular, if \( Q_m(x + d) \) has the unique mode \( \overline{m} \), then so does \( P(x + d) \) unless \( P(x) = x^m \) and \((m+1)/(d+1) \in \mathbb{Z}^+\).

**Proof.** The inequality \( M^*(P, d) \leq M^*(x^m, d) \) follows from Corollary 2.1 and Proposition 3.1 so it suffices to prove the inequality \( M_*(Q_m, d) \leq M_*(P, d) \).

Let \( P(x) = \sum_{j=0}^m a_j x^j \) and \( P(x + d) = \sum_{j=0}^m b_j x^j \). For \( 1 \leq t \leq \overline{m} \), let \( r = \lceil (d+1)t \rceil - 1 \). Then \( t \leq r \leq m \). By (2), we have

\[
td^i(b_r - b_{r-1}) = \sum_{i=t-1}^{m} a_i d^{i-t} \binom{i}{t-1} [(i+1) - (d+1)t] \\
= \sum_{i=r}^{m} a_i d^i \binom{i}{t-1} [(i+1) - (d+1)t] \\
- \sum_{i=t-1}^{r-1} a_i d^i \binom{i}{t-1} [(d+1)t - (i+1)] \\
\geq a_r \sum_{i=r}^{m} d^i \binom{i}{t-1} [(i+1) - (d+1)t] \\
- a_r \sum_{i=t-1}^{r-1} d^i \binom{i}{t-1} [(d+1)t - (i+1)] \\
= a_r \sum_{i=r}^{m} d^i \binom{i}{t-1} [(i+1) - (d+1)t] \\
= a_r td^i(d_r - d_{r-1}),
\]

and the equality holds if and only if all \( a_i \)'s are equal, i.e., \( P \) coincides with \( Q_m \).

Take \( t = M_*(Q_m, d) \). Then \( d_t > d_{t-1} \) by the definition. Thus \( b_t > b_{t-1} \), which implies that \( M_*(P, d) \geq t \), the desired inequality.

Assume now that \( \overline{m} \) is a mode of \( Q_m(x + d) \). Then \( d_0 \leq d_1 \leq \cdots \leq d_{\overline{m}} \). Thus \( b_0 \leq b_1 \leq \cdots \leq b_{\overline{m}} \). However, \( b_{\overline{m}} \geq b_{\overline{m}+1} \geq \cdots \geq b_m \) by Corollary 2.1. Hence \( \overline{m} \) is a mode of \( P(x + d) \).

In particular, if \( \overline{m} \) is the unique mode of \( Q_m(x + d) \), then \( M_*(P, d) \geq \overline{m} \). Thus \( \overline{m} \) is the unique mode of \( P(x + d) \) if and only if \( b_{\overline{m}} > b_{\overline{m}+1} \), which holds if and only if \( P(x) = x^m \) and \((m+1)/(d+1) \in \mathbb{Z}^+\) by Corollary 2.1 and Proposition 3.1. This completes the proof of the theorem.

Combining Theorem 4.1, Corollary 2.1, and the results of the last section we conclude that

**Corollary 4.1.** Let \( P \in \mathbb{P}_m^+ \) and \( d \geq 1 \). Then \( P(x + d) \) has at most two modes \( \overline{m} \) and \( \overline{m} + 1 \) if \( P(x) = x^m \), or \( \overline{m} - 1 \) and \( \overline{m} \) otherwise.
Corollary 4.2. Let $P \in \mathbb{P}^m$. Then $P(x + 1)$ has the mode $\left\lfloor \frac{m-1}{2} \right\rfloor$. In particular, if $P(x)$ is neither $x^m$ nor $\sum_{i=0}^{m} x^i$, then $\left\lfloor \frac{m-1}{2} \right\rfloor$ is the unique mode of $P(x + 1)$.

Corollary 4.3. Let $d > 1$ and $P \in \mathbb{P}^m$ be such that $P(x) \neq x^m$. Suppose that one of the following conditions holds:

(i) $m + 1 = m + 1$;
(ii) $\frac{m+1}{d+1} \in \mathbb{Z}^+$;
(iii) $\frac{d}{m} \in \mathbb{Z}^+$;
(iv) $d \in \mathbb{Z}^+$;
(v) $\frac{m}{d+1} \in \mathbb{Z}^+$.

Then $P(x + d)$ has the unique mode of $\overline{m}$.

Corollary 4.2 and Corollary 4.3(iv) strengthen the main results of [2] and [11], respectively.

In the case $0 < d < 1$, the number and location of modes of $P(x + d)$ depend heavily on coefficients of $P(x)$. Since we are mainly concerned with those properties of modes satisfied by generic polynomials in $\mathbb{P}^m$, we will not dwell on this case $0 < d < 1$ any further but give one useful consequence of Proposition 3.6 and Theorem 4.1, as follows.

Theorem 4.2. Let $0 < d < 1$ and $P \in \mathbb{P}^m$. Suppose that $P(x) \neq x^m$. Then

$$\left\lfloor \frac{m}{2} \right\rfloor \leq M_*(P,d) \leq M^*(P,d) \leq \overline{m}.$$

5 Remarks and Open Problems

Our results can be restated in terms of sequences instead of polynomials. For example, the statement of Conjecture [11] is equivalent to the following.

Theorem 5.1. Suppose that $0 \leq a_0 \leq a_1 \leq \cdots \leq a_m$ and that $d > 0$. Then the sequence

$$b_j = \sum_{i=j}^{m} a_id^{i-j} \binom{i}{j}, \quad j = 0, 1, \ldots, m$$

is unimodal.

It often occurs that unimodality of a sequence is known, but to find out the exact number and location of modes of the sequence is a much more difficult task. For example, it is well known that, for each positive integer $n$, the Stirling number of the second kind $S(n, k)$ is unimodal in $k$ with at most two modes $K_n, K_n + 1$, and that $K_n \sim n/\ln n$. However it is very difficult to determine whether the mode of $S(n, k)$ is unique or not. See [5, 6] for the related results.

We end our paper by proposing the following.

Conjecture 5.1. Suppose that $P \in \mathbb{P}^m$ and that $0 < d_1 < d_2$. Then $M_*(P,d_1) \geq M_*(P,d_2)$ and $M^*(P,d_1) \geq M^*(P,d_2)$.
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References

[1] J. Alvarez, M. Amadis, G. Boros, D. Karp, V. H. Moll and L. Rosales, An extension of a criterion for unimodality, Electron. J. Combin. 8(2001) #R30.

[2] G. Boros and V. H. Moll, A criterion for unimodality, Electron. J. Combin. 6(1999) #R10.

[3] F. Brenti, Unimodal, log-concave, and Pólya frequency sequences in combinatorics, Mem. Amer. Math. Soc. 81(1989) no. 413.

[4] F. Brenti, Log-concave and unimodal sequences in algebra, combinatorics, and geometry: an update, Contemp. Math. 178(1994) 71-89.

[5] E. R. Canfield, On the location of the maximum Stirling number(s) of the second kind, Studies in Appl. Math. 59(1978) 83-93.

[6] L. H. Harper, Stirling behavior is asymptotically normal, Ann. Math. Stat. 31(1967) 410-414.

[7] S. G. Hoggar, Chromatic polynomials and logarithmic concavity, J. Combin. Theory Ser. B 16(1974) 248-254.

[8] R. P. Stanley, Log-concave and unimodal sequences in algebra, combinatorics, and geometry, Ann. New York Acad. Sci. 576(1989) 500-534.