Universal Scaling of Strong-Field Localization in an Integer Quantum Hall Liquid

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Abstract

We study the Landau level localization and scaling properties of a disordered two-dimensional electron gas in the presence of a strong external magnetic field. The impurities are treated as random distributed scattering centers with parameterized potentials. Using a transfer matrix for a finite-width strip geometry, we calculate the localization length as a function of system size and electron energy. The finite-size localization length is determined by calculating the Lyapunov exponents of the transfer matrix. A detailed finite-size scaling analysis is used to study the critical behavior near the center of the Landau bands. The influence of varying the impurity concentration, the scattering potential range and its nature, and the Landau level index on the scaling behavior and on the critical exponent is systematically investigated. Particular emphasis is put on studying the effects of finite range of the disorder potential and Landau level coupling on the quantum localization behavior. Our numerical results, which are carried out on systems much larger than those studied before, indicate that pure \( \delta \)-function disorder in the absence of any Landau level coupling gives rise to non-universal localization properties with the critical exponents in the lowest two Landau
levels being substantially different. Inclusion of a finite potential range and/or Landau
level mixing may be essential in producing universality in the localization.

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I. Introduction

The quantum Hall effect [1] phenomenon is now well accepted to be closely related to the strong-field Landau level localization problem in a disordered two-dimensional (2D) electron system. In a disordered 2D system, without any external magnetic fields, it is well-known that all electronic states are localized [2] and strictly there is no metal-insulator transition. In the presence of a strong external magnetic field the energy spectrum of a 2D system is a set of impurity-broadened Landau bands. While the electronic states in the tails of the Landau bands remain localized, the states at Landau band centers delocalize and become extended with a mobility edge $E_c$ near the center of each Landau band. When the Fermi energy is in the localized regime, the Hall resistance ($\rho_{xy}$) is quantized with vanishing dissipative or longitudinal resistance ($\rho_{xx} = 0$) at $T = 0$. When the Fermi energy moves to the delocalized regime, there is a metal-insulator transition at $E_c$, and the system behaves as a metal with non-zero $\rho_{xx}$ and non-quantized $\rho_{xy}$ which takes us from one quantized Hall plateau to the next. This transition regime, where $\rho_{xx}$ is non-zero and $\rho_{xy}$ is jumping from one plateau to the next, plays an important fundamental role in localization and quantum Hall studies. Experimentally [3-9], this transition regime becomes sharper and narrower as $T$ decreases with the $T = 0$ limit thought to be infinitely sharp with only the state at $E = E_c$ being extended and all other states localized. Existence of these extended states (a set of measure zero at $T = 0$) near the Landau level centers is of crucial importance to the quantum Hall effect phenomenon and, conversely, the quantum Hall effect demonstrates the existence of extended states (and, of metal-insulator transitions) in the strong-field 2D system. In this paper, we provide a detailed numerical study of the localization properties of the two lowest Landau levels in the strong-field 2D system using a finite-size scaling analysis. There exist several recent studies of this type in the literature [10-13]. Utilizing the advantages provided by massively parallel computing machines, we have been able to go to system sizes substantially larger (by a factor of four) than those hitherto existing in the literature. In addition, our emphasis, in contrast to the earlier numerical studies [10,11] which mostly concentrate on the lowest Landau level in the presence of zero-range disorder potential, is
on the effects of finite range disorder potentials and Landau level coupling effects on the localization properties of the two lowest Landau levels. We are particularly interested in the issue of universality (i.e. whether the critical localization exponents are always the same independent of Landau level index, type of disorder potential, etc.) in strong-field Landau level localization, a topic which has created some controversy both in the experimental and in the theoretical literature on the subject [3-14].

The critical localization behavior of the quantum Hall effect can be summarized as follows. The quantum Hall resistance $\rho_{xy}$ and the longitudinal dissipative resistance $\rho_{xx}$ of a strong-field 2D disordered system are measured at different temperatures ($T$) in the transition regime by varying the magnetic field ($B$). In the transition regime it is found [3,4] that the maxima of $d^n \rho_{xy}/dB^n$ diverge as $T^{-n\kappa}$ ($n = 1, 2, 3$) and the width of $\rho_{xx}$ peak, $\Delta B$, is proportional to $T^\kappa$. Here, $\Delta B$ is the width of the transition regime where $\rho_{xx}$ is non-zero and $\rho_{xy}$ is unquantized. The exponent $\kappa$ of the temperature dependence arises from a competition between two microscopic length scales at a finite $T$ [14]. One is the inelastic scattering or the phase breaking length $L_{in}$ which scales with temperature with an exponent $p$ ($L_{in}(T) \propto T^{-p/2}$). The other is the localization length $\xi$ for the electronic state at the Fermi energy, which scales with energy with an exponent $\nu$. The localization length diverges as the Fermi level approaches the mobility edge critical energy $E_c$ in the middle of each Landau band i.e. $\xi(E) \propto |E - E_c|^{-\nu}$. When $E_F$ is at the tails of Landau bands, $\xi \ll L_{in}$, the system is insulating and localization plays the major role in producing quantum Hall plateaus. When $E_F$ is near the critical energy, $\xi \gg L_{in}$, inelastic scattering length acts as a cut-off making the system behave metallic. Using scaling arguments [14] it is easy to combine these two exponents to show that the temperature dependence of $\rho_{xx}$ and $\rho_{xy}$ in the transition regime (i.e. in the metallic phase with $E_F \sim E_c$) would be controlled by a composite exponent $\kappa = p/2\nu$, which is directly measured experimentally by studying the $\rho_{xx}$ peak and the associated $d\rho_{xy}/dB$. (We have, in fact, recently shown [15] that the scaling analysis can be extended even to the $\rho_{xx}$ minima, i.e. where $\rho_{xy}$ is quantized, provided the conduction mechanism is activated transport and not variable range hopping.) A key question in this
problem is whether the exponent $\kappa$ is universal or not. Earlier experimental investigation [3,4] involving studies of the temperature dependence of $\rho_{xx}$ peak and the associated $d\rho_{xy}/dB$ was controversial with $\kappa$ being independent of Landau level index in one study and dependent (by a factor of two) on the Landau level index in the other. There are also some reports [5] of the dependence of the localization exponent on the impurity scattering strength. In our opinion, the most thorough and reliable temperature dependent experiments tend to indicate a universal (Landau level independent) $\kappa$. The problem with experimental measurements of $\kappa$ is that it is a composite scaling exponent, $\kappa = p/2\nu$, being dependent on the localization exponent $\nu$ and the phase breaking exponent $p$. While calculations, such as the one carried out in this paper, can provide a reasonably accurate numerical value of $\nu$ (for non-interacting electrons), the value of the inelastic scattering exponent $p$ in the strong-field situations is simply not known. Comparison between theory and temperature-dependent localization experiments has been carried out mainly under the assumption that $p \approx 2$ which is the corresponding zero-field clean limit result for electron-electron scattering in two dimensional Fermi liquids [10]. (We mention that under the assumption $p = 2$, one gets $\kappa \approx 0.4$ using the numerically calculated $\nu_0 \approx 2.3$, obtaining excellent agreement between theory and experiment.) There are, however, some difficulties with this “standard” theoretical analysis of the localization data. First, we do not see any justification for applying the zero-filed theoretical result for $p$ to the strong-field situation. Second, there is no justification for using the clean limit result ($p \approx 2$) rather than the disordered 2D result ($p \approx 1$) except perhaps the (fortuitous?) “agreement” obtained between theory and experiment using $p = 2$. Third, even if $\nu$ is universal (i.e. Landau level independent, etc.), it is very hard to see how $p$ could be universal – for example, in the zero-field situation $p = 1(2)$ depending on whether the 2D system is dirty (clean). Because universality of $\kappa$ requires universality of both $p$ and $\nu$, we believe that a complete understanding of the experimental results awaits a theory for inelastic scattering in the strong-field 2D system. In this respect, we note that temperature dependent experiments measuring the width $\Delta B$ of $\rho_{xx}$ peaks and of the associated $d\rho_{xy}/dB$ show that $\kappa$ in GaAs samples[9] achieve the “universal” value ($\sim 0.4$) only for $T < 200 mK$ whereas earlier
similar experiments[3] on InP heterostructures produce the universal $\kappa \sim 0.4$ for $T \sim 4K$. Thus, the size of the critical scaling regime (which is determined by an interplay between localization and inelastic scattering) is clearly sample-dependent. There are also systematic studies of Si-MOSFETs[4,6,7] and GaAs heterostructures[5] in the literature which find that the measured value of $\kappa$ is non-universal, with the reported dependence of $\kappa$ on both the Landau level index and sample quality. A recent size-dependent experimental localization study[8] of narrow GaAs heterostructures, which claims separate experimental measurements of $\nu$ and $p$, concludes that while $\nu(\approx 2.3)$ is universal (independent of both Landau level index and the sample quality), $p$ varies from 2.7 to 3.4, depending on the Landau level index and the sample quality. Thus, $\kappa$ itself ($\kappa = p/2\nu$) may not be universal. We emphasize that our work is solely on an accurate numerical calculation of the localization exponent $\nu$ and the universality being discussed in this paper refers entirely to the localization aspect of the problem (i.e., we are concerned with whether or not $\nu$ is independent of Landau level index, the nature of the impurity scattering potential, etc.). We have nothing to say about the exponent $p$ which is often needed in understanding the experimental quantum Hall localization data.

The interesting recent experimental attempt [8], alluded to above, circumvent the problem of having a composite exponent $\kappa$ by direct and independent measurement of $\nu$ (and $p$) using experimental 2D strip systems of various widths warrants a discussion. The competing length scales in this type of experimental “finite-size scaling” studies are the localization length and the system size (with the inelastic scattering length, which is not playing an active role, assumed to be a constant, larger than the system size, because the temperature is held fixed in these experiments). Such direct measurements of $\nu$ show that it is indeed universal with a value $\nu \approx 2.3$ in excellent agreement with the existing (and our own) theoretical calculations. While the basic premise of these experiments is novel and this procedure certainly eliminates the difficulty associated with $\kappa(= p/2\nu)$ being a composite exponent, we feel that there are some inherent problems in the interpretation of these experimental results as well. Suppose the temperature is such that $L_s > L_{in}$, where $L_s$ is the finite size of the experimental
system. In that situation, obviously $L_s$ is irrelevant for the localization behavior of the system and the competition between the localization length $\xi$ and the inelastic scattering length $L_{in}$ controls the scaling behavior. Thus, for the finite-size scaling experiments to be meaningful, one must have $L_{in} \gg L_s$, which is precisely the mesoscopic conductance fluctuation regime [16] where the system is manifestly non-self-averaging and one should see quantum interference induced conductance fluctuations because the system size is smaller than the phase coherence length. Such strong mesoscopic fluctuations are not present in the experimental results and, therefore, it is unclear to what extent the necessary condition $L_{in} \gg L_s$ is satisfied. We emphasize that, while the weak-field conductance fluctuation phenomenon is currently well understood, there has been very little work on strong-field conductance fluctuation phenomenon, and, in principle, it is possible that there is a very good (but presently unknown) reason why conductance fluctuations are absent in the strong-field 2D systems even in the mesoscopic $L_{in} \gg L_s$ regime. Without such a theory explaining the absence of strong-field mesoscopic conductance fluctuations, we feel that a complete understanding of the novel finite-size scaling experiments remains elusive.

In this paper, we study localization and scaling properties of a disordered two-dimensional system of finite width in the presence of a strong magnetic field. The impurities are treated as random distributed scattering potential centers. By choosing an appropriate cut-off in the scattering range, we can divide our system into cells with only the nearest-neighbor intercell scattering being appreciable. This allows us to set up an intercell transfer matrix for evaluating the exact single particle non-interacting electron wavefunction in the presence of disorder. The finite-size localization length of the electron is determined by calculating the Lyapunov exponents of the transfer matrix on a massive parallel processing machine. Finite-size scaling method is applied to study the critical behavior near the center of the Landau bands. The influence of varying the impurity concentration, the scattering potential and the Landau level index on the critical exponent is systematically investigated. A short report of our work has earlier appeared in the literature[17].

The rest of the paper is organized in the following way: in Sec. II we give the details of
our theoretical method with the relevant formulas and equations; in Sec. III we present our numerical results; we provide a discussion with the conclusion in Sec. IV.

II. Theoretical Method

For a non-interacting 2D electron gas in the presence of a strong external magnetic field $\mathbf{B} \equiv (0,0,B)$ perpendicular to the 2D $x-y$ plane, the single electron energy spectrum is given by a set of discrete Landau levels (denoted by the index $N = 0, 1, 2, ...$):

$$E_N = (N + \frac{1}{2})\hbar \omega_c,$$

where $\omega_c = eB/mc$ is the cyclotron frequency, and $m$ the electron effective mass. Each Landau level is highly degenerate with a macroscopic degeneracy given by $(2\pi l_c^2)^{-1}$ per unit area where $l_c = (\hbar c/eB)^{1/2}$ is the magnetic length. By choosing the Landau gauge for the vector potential $\mathbf{A} = (0,Bx,0)$, and periodic boundary conditions in the $y$ direction, the electron wavefunction becomes a simple harmonic oscillator eigenfunction in the $x$ direction, and a free-electron plane wave in the $y$ direction. The one electron Landau wavefunction of the pure system can then be written as

$$\phi_{Nk} = \frac{1}{\sqrt{M}} e^{-iky} H_n\left(\frac{x - kl_c^2}{l_c}\right) e^{-\frac{1}{2}\left(\frac{x - kl_c^2}{l_c}\right)^2}.$$

Here $M$ is the width of the 2D system in $y$ direction, $H_n$ is the Hermite polynomial, and $k = \frac{2\pi n}{M}$ where $n$ is an integer. Note that the simple harmonic oscillator has a displaced center with the oscillation center at $x \equiv X = kl_c^2$. We choose these one electron Landau eigenstates as a complete basis in the Hilbert space for our localization study of the disordered 2D system.

We consider a two-dimensional disordered system of length $L$ (along the $x$ direction) and width $M$ in the strip geometry with periodic boundary conditions applied in the $y$ direction. We use $l_o = \sqrt{2\pi l_c}$ as the length unit in this problem. The single electron Hamiltonian in the presence of disorder can be written as

$$H = \sum_{NX} |NX>(N + \frac{1}{2})\hbar \omega_c <NX| + \sum_{NX} \sum_{N'X'} |NX><NX|V|N'X'><N'X'|.$$
where \( |NX> \) is the Landau state for the \( N \)th Landau level with the center of oscillation \( X \) taking discrete values with spacing \( 1/M \) due to the periodic boundary conditions. The disorder potential \( V \), which is treated as the one electron potential arising from a random distribution of impurity scattering centers, is given by

\[
V(r) \equiv \sum_i V_i(r - r_i) \tag{4}
\]

where \( r_i \) is the location of the \( i \)th impurity scattering center and \( V_i \) depends on the type of scattering potential. We consider two models for the random disorder potential, namely, the short-range potential , \( V_i = V_o \delta(r - r_i) \), and, the finite range Gaussian potential, \( V_i = \frac{V_o \sqrt{\pi d}}{d} e^{-|r - r_i|^2/d^2} \). The potentials are chosen to be all attractive, or all repulsive, or an equal mixture of both. The concentration of impurities, which along with \( V_o \) determines the scattering strength, is denoted by \( c_i \). An important numerical quantity in our calculation is \( l_{\text{cutoff}} \) which determines the length scale above which the impurity scattering potential does not connect the different Landau states, \( i.e. \) we only need to consider scattering between states \( X, X' \) satisfying \( |X - X'| \leq l_{\text{cutoff}} \). Obviously \( l_{\text{cutoff}} \) is determined by the characteristics of the Landau wavefunctions and the impurity potential and concentration. Our criterion to determine \( l_{\text{cutoff}} \) is that the matrix element \( < NX | V | N'X' > \) for \( |X - X'| \geq l_{\text{cutoff}} \) should be much smaller (less than 1%) than that for \( |X - X'| \leq l_{\text{cutoff}} \). In our calculations, we choose \( l_{\text{cutoff}} = 1 \) for the \( N = 0 \) Landau level, and \( l_{\text{cutoff}} = 2 \) for higher Landau levels\((N = 1)\). Our choice of \( l_{\text{cutoff}} \) is consistent with the others existing in the literature \([10,11]\). In addition to the inter-Landau-level energy separation \( \hbar \omega_c \), there is another energy scale in this problem which is the Landau level broadening \( \Gamma_N \) determined by the strength of the impurity-scattering potential. Within the self-consistent-Born-approximation(SCBA), and in the strong field limit \( (i.e. \) no Landau level coupling), the density of states(DOS) for the short-range impurity potential is given by \([18]\):

\[
D(E) = 2 \sum_{N=0} (\pi \Gamma)^{-1} \left[ 1 - \left( \frac{E - E_N}{\Gamma} \right)^2 \right]^{1/2}, \tag{5}
\]

where \( \Gamma = 2V_o \sqrt{c_i} \ll \hbar \omega_c \). Defining \( \gamma = \Gamma/\hbar \omega_c \), we note that \( \gamma \ll 1 \) defines the weak disorder strong-field limit when scattering to other Landau levels may be ignored. In some
situations we would consider the strong disorder case, $\gamma \sim 1$, where we have to take into account Landau level coupling in our calculations. We choose $\hbar \omega_c$ as the unit of energy in our calculations (which is the most obvious natural choice).

Our localization calculation proceeds in two steps: First, we obtain the localization length for a 2D strip of a finite width $M$ and an essentially infinite length of $L = 10^5 (>> M = 4 - 256)$ by calculating the system Lyapunov exponent on a massively parallel processing machine; then, we carry out a finite size scaling analysis to obtain the localization length in the $M \to \infty$ limit. In order to calculate the localization length for a 2D strip of length $L$ and width $M$, we start by dividing the system into cells with length $l_{cutoff}$ and width $M$ such that the next-nearest-neighbor intercell interaction can be neglected. The number of states in each cell for each Landau level is $n = l_{cutoff} M$. Expanding the wavefunction $|\Psi>$ of the disordered system for energy $E$ in the complete basis of Landau wavefunctions, we get

$$|\Psi> = \sum_{N_i} a_{N_i} |NX_i>$$

with

$$H|\Psi> = E|\Psi>.$$  

Combining Eqs.(1),(3),(6) and (7) we get

$$(N + \frac{1}{2})\hbar \omega_c a_{N,i+n} + \sum_{N'} \sum_{k=0}^{2n} <NX_{i+n}|V|NX_{i+k}> a_{N',i+k} = Ea_{N,i+n}.\tag{8}$$

Defining column vectors

$$A_i = \begin{pmatrix} a_{0i} \\ a_{1i} \\ \vdots \\ a_{N_L-1,i} \end{pmatrix}, \tag{9}$$

and

$$\psi^{(i)} = \begin{pmatrix} A_{i+2n-1} \\ \vdots \\ A_i \end{pmatrix}, \tag{10}$$

$$9$$
where $N_L$ is the number of Landau levels being included in our numerical calculation, Eq. (8) can be transformed to the following set of linear equations:

$$
\sum_{k=0}^{2n} H_{i+k} A_{i+k} = E A_{i+n},
$$

(11)

where

$$
H_{i+k} = \begin{pmatrix}
\frac{1}{2} \hbar \omega_c & 0 & \ldots & 0 \\
0 & \frac{3}{2} \hbar \omega_c & \ldots & 0 \\
\vdots & \vdots & \ddots & \vdots \\
0 & 0 & \ldots & (N_L - \frac{1}{2}) \hbar \omega_c
\end{pmatrix} \delta_{kn}
$$

(12)

Equation (11) shows that $A_{i+2n}$ is determined by $A_i, A_{i+1}, \ldots, A_{i+2n-1}$ i.e. we can set up the transfer matrix $T^{(i)}$ such that

$$
\psi^{(i+1)} = T^{(i)} \psi^{(i)}.
$$

(13)

Here the transfer matrix $T^{(i)}$ is a $(2n \times N_L) \times (2n \times N_L)$ dimensional matrix:

$$
T^{(i)} =
\begin{pmatrix}
-H_{i+2n-1}^{-1} H_{i+2n-1} & \ldots & -H_{i+2n-1}^{-1} (EI - H_{i+2n}) & \ldots & -H_{i+2n-1}^{-1} H_{i+1} & -H_{i+2n-1}^{-1} H_i \\
I & \ldots & 0 & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & I & \ldots & 0 & 0 \\
\vdots & \ddots & \vdots & \ddots & \vdots & \vdots \\
0 & \ldots & 0 & \ldots & I & 0
\end{pmatrix}
$$

(14)

where $I$ is $N_L \times N_L$ unit matrix. Note that if Landau level coupling is neglected in the calculations (as has been done in all the existing calculations), $N_L = 1$. We have done calculations for single Landau levels ($N = 0, 1$) with $N_L = 1$ and for coupling between the lowest two Landau levels with $N_L = 2$. $T^{(i)}$ is calculated by summing up the contribution from each impurity scatterer, and, therefore, depends on the details of the local distribution of...
the random impurities. For example, for the short-range impurity potential $V_i = V_0 \delta(r - r_i)$, we obtain the following non-zero matrix elements:

$$<0X_{n_1} | V_i | 0X_{n_2}> = \frac{\sqrt{2}V_0}{M} e^{i\pi (n_1-n_2)y_i/M} e^{-\pi [(x_i-n_1/M)^2 + (x_i-n_2/M)^2]},$$

(15)

$$<0X_{n_1} | V_i | 1X_{n_2}> = \frac{\sqrt{2}V_0}{M} 2\sqrt{\pi} (x_i - n_2/M) e^{i\pi (n_1-n_2)y_i/M}$$

$$\times e^{-\pi [(x_i-n_1/M)^2 + (x_i-n_2/M)^2]},$$

(16)

$$<1X_{n_1} | V_i | 0X_{n_2}> = \frac{\sqrt{2}V_0}{M} 4\pi (x_i - n_1/M)(x_i - n_2/M) e^{i\pi (n_1-n_2)y_i/M}$$

$$\times e^{-\pi [(x_i-n_1/M)^2 + (x_i-n_2/M)^2]},$$

(17)

$$<0X_{n_1} | V_i | 0X_{n_2}> = \frac{\sqrt{2}V_0}{M} 4\pi (x_i - n_1/M)(x_i - n_2/M) e^{i\pi (n_1-n_2)y_i/M}$$

$$\times e^{-\pi [(x_i-n_1/M)^2 + (x_i-n_2/M)^2]}.$$  

(18)

We do not show here the explicit matrix elements for the Gaussian impurity potential which can also be calculated analytically.

As mentioned before, once the transfer matrix $T^{(i)}$ is formed, we calculate the localization length by evaluating the Lyapunov exponent for the transfer matrix. Calculation of the Lyapunov exponent for maps such as the one defined in Eq.(13) is standard in non-linear dynamics, and has been successfully used in localization calculations in other contexts [19]. For the transformation $T$ which transforms a vector $\psi$ according to Eq.(13), the Lyapunov exponents are given by

$$L_y = \lim_{k \to \infty} \frac{1}{k} \ln \frac{|\psi^{(k+1)}|}{|\psi^{(1)}|} = \lim_{k \to \infty} \frac{1}{k} \ln \frac{1}{\psi^{(1)}} \prod_{i=1}^{k} T^{(i)} \psi^{(1)}. \tag{19}$$

For our $(L, M)$ strip geometry, with $L$ very large and $M$ small, $|\psi^{(1)}| \sim |\psi^{(k+1)}| \sim |\psi(L)|$, $k = \frac{L}{2\pi} = LM$. For localized states, the localization length $\lambda_M$ is defined to be the characteristic length controlling exponential decay of the wavefunction, i.e. $|\psi(L)| \sim e^{-L/\lambda_M}$, so that

$$\lambda_M = -\frac{1}{ML_y}. \tag{20}$$

Mathematically there exist $2n \times N_L$ Lyapunov exponents for the transfer matrix $T$, but the relevant one for the localization length is the smallest negative Lyapunov exponent.
We employ the following technique to calculate the smallest negative Lyapunov exponent. Denote $S$ as an upper triangular matrix, i.e. $S_{i>j} = 0$, $Q$ as a lower triangular matrix, i.e. $Q_{i<j} = 0$ with $\det Q = 1$. $T^{(1)}$ can be expressed as $T^{(1)} = Q^{(1)} S^{(1)}$, then $T^{(2)} Q^{(1)} = Q^{(2)} S^{(2)}$ and so on. Finally,

$$\prod_{i=1}^{k} T^{(i)} = Q^{(k)} \prod_{i=1}^{k} S^{(i)}$$

(21)

The relevant Lyapunov exponent we need is

$$L_y = \lim_{k \to \infty} \frac{1}{k} \ln | \prod_{i=1}^{k} S^{(i)} |_n^{N_L+1,n \times N_L+1}$$

(22)

The problem of computing the Lyapunov exponent now reduces to a simple matrix multiplication problem. The problem is, therefore, ideally suited for the parallel processing capability of massive parallel processing computers. We have used both CM2 and CM5 parallel processing computers of the Thinking Machine Co. to carry out our calculations. Note that, since scattering from various impurities is independent of each other, we can use a parallel processing algorithm to calculate their contribution to $T$ as well.

Once the localization length, $\lambda_M(E)$, at a particular energy $E$ and for a specific finite value of $M$ is obtained from the Lyapunov exponent calculation, we carry out a finite-size scaling analysis to go to the 2D limit (i.e. the $M \to \infty$ limit). Finite-size scaling analysis has been extensively used in localization calculations and for details we refer to the literature [10,11]. The idea is that one should see one-parameter scaling for large enough sample width $M \geq M_{sc}$ and for energies in the critical regime $| E - E_c | \leq E_{sc}$ (note that the system length $L$ is very large).

$$\frac{\lambda_M(E)}{M} = f \left( \frac{M}{\xi(E)} \right).$$

(23)

Here $f(x)$ is the universal scaling function for the localization problem, with $\xi(E)$ the localization length for the infinite system. This means that in the critical regime a change in energy is equivalent to a change in the system size. As we go to large system sizes, the localized states far away from the critical region have $\xi(E) \ll M$, and $\lambda_M \sim \xi$, so the scaling function has the asymptotic behavior of $f(x) \sim 1/x (x \gg 1)$. On the other hand, for the states very close to the critical energy $E \sim E_c$, $\lambda_M \sim M$, and $\xi \gg M$, so that $f(x) \sim$ constant
for $x \ll 1$. By plotting the numerically calculated $\lambda_M(E)/M$ against $M$ on a log-log plot, we can calculate $\xi(E)$ by using a least-square optimization fit to make all the data points collapse on a single “smooth” scaling curve for $\lambda_M(E)/M$ against $M/\xi(E)$ obeying Eq.(23) and the correct asymptotic properties. This finite-size scaling analysis immediately gives us $\xi(E)$, the localization length for the infinite system. Using log-log plot of the calculated $\xi(E)$ against $E$, we can obtain the critical exponent $\nu$, i.e.

$$\xi(E) \propto |E - E_c|^{-\nu}.$$  

(24)

Obeying of both Eq.(23) and (24) together puts rather stringent requirements on the finite-size scaling analysis ensuring the correct evaluation of the critical exponent $\nu$ from the optimization procedure. Note that the long strip-geometry topology of our model ensures self-averaging for the Lyapunov exponent calculation of the transfer matrix and no additional averaging is required.

### III. Numerical Results

In this section we present detailed numerical results obtained by applying the theoretical method described in II. We have carried out calculations on system sizes up to $M = 256$, which is larger than those studied before [10,11]. The sample length $L(\sim 10^5)$ is chosen such that the numerical error in the calculation of Lyapunov exponent is less than 1%. We consider a wide range of impurity concentrations from $c_i = 2$ to $c_i = 32$. We consider the short-range $\delta$-function scattering potential and Gaussian long-range potentials with the potential range $d/l_c = 0.5$ and 1. For the weak disorder case, we carry out calculations for single (i.e. $N_L = 1$) Landau levels $N = 0, 1$, and for the strong disorder case, we take into account Landau level coupling ($N_L = 2$) between $N = 0$ and 1 Landau levels.

In Figs.1 and 2 we present our numerical results for Landau levels $N = 0$ and 1 respectively in the weak disorder (i.e. neglecting Landau level coupling), symmetric (i.e. with equal number of repulsive and attractive $\delta$-function scatterers) case. The weak disorder results for different impurity concentrations with symmetric $\delta$-function scatterers are summarized
in Table 1. The calculated localization critical exponent $\nu$ depends strongly on the Landau level index, but varies little with the impurity concentration. This finding of a strong variation of $\nu$ with the Landau level index $N$ is consistent with earlier findings in the literature [10,11](using substantially smaller system sizes). While our numerical results show reasonably good one-parameter scaling (Fig.1(b) and Fig.2(b)) properties for both the Landau levels, the asymptotic value of the scaling function seems to depend on the Landau level index. In our scaling plots, the saturation values for $M/\xi \ll 1$ are quite different for the different Landau levels (for $N = 0$, $\lambda_M/M \rightarrow 1$; for $N = 1$, $\lambda_M/M \sim 2$). The insensitivity of the scaling function to a variation in the impurity concentration $c_i$ is, however, quite robust.

For random distributed weak-disorder repulsive $\delta$ impurities, the calculated critical exponents are presented in Table 2 and a typical set of scaling data is shown in Fig.3. In this asymmetric situation with all repulsive scatterers, the critical energy $E_c$ is shifted away from the exact center of the Landau band, but as the impurities get denser, the critical energy approaches the Landau band center. In the dilute impurity limit, the critical exponents are asymmetric, being somewhat different for the lower and the higher energy branches of the Landau band. It is known that in the low impurity concentration case, the density of states (DOS) is asymmetric with the maxima of DOS shifting away from the center of the Landau band (for repulsive scatterers, the maxima shift to lower energies) [20]. We believe the asymmetry in our calculated critical exponents arises from this finite impurity concentration effect, and should disappear if Landau level coupling is included in the calculation.

Our results for the values of $\nu$ in the presence of Gaussian impurity scattering potential (without any Landau level coupling) is summarized in Table 3, and some typical scaling plots are shown in Fig.4 (for $N = 0$) and Fig.5 (for $N = 1$). Basic trends are similar to those found in the $\delta$-function case with one important difference. While for the $N = 0$ Landau level, the critical exponent remains essentially a constant ($\nu \simeq 2.2$) as the potential range is varied, the exponent for the $N = 1$ Landau level clearly decreases with increasing the range of the impurity potential. In particular, the value of $\nu$ decreases from around 5.5 for zero-range potential to around 2.5 for finite range potentials (the value of $\nu_0$, however, remains
constant around 2.2). Based on this rather limited numerical data, one may speculate that universality is restored by finite-range scatterers and that $\delta$-function scattering is quite a special case.

In the presence of strong disorder, the Landau level broadening $\Gamma$ is comparable to inter-Landau-level energy difference $\hbar\omega_c$, and, therefore, we must include Landau level coupling in the calculation. We consider the simplest situation taking into account the coupling between $N = 0$ and $N = 1$ Landau levels. The strength of the Landau level coupling is parameterized by $\gamma = \Gamma/\hbar\omega_c$. Our calculated numerical results for the critical exponents (using the $\delta$-function potential model) are summarized in Table 4. In Fig.6 we show one set of representative scaling data for the finite $\gamma$ Landau level-coupled case. We observe that the calculated critical exponents for the lower and the higher energy branches of the same Landau band are different due to the strong asymmetry introduced by Landau level coupling effect ($E_c \neq 0$). This asymmetry is introduced by our model which considers coupling only between the lowest two Landau levels. The coupling obviously plays a much stronger role in the higher(lower) energy branch of the $N = 0(1)$ Landau band because the coupling is the strongest among those states. The most important result of our Landau level coupled calculations is that $\nu_0$ and $\nu_1$ come closer together in the presence of Landau level coupling, allowing us to speculate that universality may be restored even for the $\delta$-function scattering potential once Landau level coupling is included in the theory.

We have also carried out calculations for two coupled Landau levels in the presence of finite range impurity potentials. These calculations, which are necessarily on somewhat smaller system sizes, bear out our basic conclusions that finite potential range and Landau level coupling bring $\nu_0$ and $\nu_1$ close together numerically. We emphasize that by necessity our numerical results for the finite range disorder potential and for the Landau level coupling case are much less quantitatively accurate than the simple zero-range uncoupled calculations. The qualitative trends, however, are reliable and can be trusted. For quantitatively accurate results in the presence of finite range disorder potential and Landau level coupling effects, one needs to consider substantially larger values of system width which are not currently
computationally accessible. We refrain from showing error bars for our finite range disorder and Landau level coupling calculations except to mention here that the error bars in Tables 2-4 are at least an order of magnitude larger than those in Table 1.

IV. Discussions and Conclusion

In this paper, we study localization and scaling properties of a disordered two-dimensional electron system in the presence of a strong external magnetic field, with emphasis on the question of universality of the localization transition in various Landau levels. The impurities are treated as random distributed scattering centers with short range δ-function or finite range Gaussian potentials. Our localization calculation employs a strip geometry for which we obtain the Lyapunov exponent by setting up a suitable transfer matrix. This is done rather efficiently in a parallel algorithm. The Lyapunov exponents of the transfer matrix are calculated to determine the finite-size localization length. The critical behavior of the disordered two-dimensional system near the center of the Landau bands is then studied using the standard one-parameter finite-size scaling analysis. We systematically investigate the influence of varying the impurity concentration, the scattering potential, and the Landau level index on the critical exponent. We also study the effect of Landau level coupling on the localization exponent.

In the weak disorder limit, we consider scattering only within a single Landau level, neglecting any Landau level coupling coupling effect. We find that the critical exponent and the scaling property both depend on the Landau level index, particularly for the δ-function disorder. Within a single Landau level, especially the lowest Landau level, we observe universal scaling behavior independent of impurity concentration, scattering potential type (repulsive or attractive or both, short range or long range). The calculated critical exponent varies a little for different impurity concentrations which we attribute to finite-size or finite-concentration effect. We believe that the dependence of the critical exponent on the Landau level index comes from the different symmetry properties of the Landau wavefunction in different Landau levels. For short-range scattering potentials with the range smaller than
or comparable to the size of the Landau wavefunction, the detailed nature of the Landau wavefunction may play a role in the critical behavior, leading to different values of $\nu_N$ for $N = 0$ and 1. In the presence of only repulsive scattering potential, we observe an expected asymmetry in the critical exponent arising from the asymmetry of the density of states which is due to the finite impurity concentration effect. For finite range scattering potentials, we see the clear trend of decreasing critical exponent with increasing potential range for the higher Landau level $N = 1$(without much effect on $\nu_0$). We speculate that for real long range scattering potentials with range larger than the size of Landau wavefunctions, the dependence of the critical behavior on the details of the symmetry of the Landau wavefunction will disappear, producing true universality in the critical localization behavior. This percolation limit, beyond the scope of our work, has actually been studied both analytically and numerically [21-23].

In the strong disorder limit, the Landau level broadening due to the presence of disorder is comparable to the inter-Landau-level energy difference, so the coupling among Landau levels should be taken into account. It is, in fact, not justified to ignore lower Landau levels even in the single-Landau-level approximation when one is considering the localization properties of the excited levels$(e.g. N = 1)$. In our numerical calculation, we just consider coupling between the lowest two Landau levels. This is, of course, a very crude approximation for Landau level coupling which we are forced to make because of our limited numerical capability. We find that Landau level coupling, even in this perturbative sense, has some qualitative effect. For the $\delta$-function potential we find that in the strong disorder Landau level coupling regime, the numerical values of the critical exponents for different Landau levels get closer as we increase the impurity concentration while the scaling properties remain the same as in the non-coupling case. The critical energies shift from the center of each Landau band because of the asymmetry introduced by Landau level coupling. This new critical behavior arises from the mixing of symmetry of different Landau wavefunctions. A similar trend of decreasing critical exponent in the higher$(N = 1)$ Landau level is seen for finite range Gaussian impurities as well. We believe that both finite range disorder and Landau level
coupling reduce the role of the detailed symmetry of the Landau wavefunction in the critical localization problem and restore universality in the Landau level localization properties. Our limited numerical results only indicate a trend and much larger simulations are needed to definitely establish this claim.

There have been several earlier numerical investigations [10-13] of strong-field localization in disordered two-dimensional electron systems. Our parameterization of the random disorder potential (as arising from randomly distributed point impurity scattering centers characterized by short-range $\delta$-function or finite-range Gaussian potentials) is similar to that used by Ando and Aoki [11] who concentrated mainly on the large impurity concentration limit($c_i = 40$), neglecting Landau level coupling, and used rather small system sizes (up to $M = 16$ only). Ando and Aoki [11] concluded that $\nu_0 \simeq 2$ and $\nu_1 \simeq 4$ for $\delta$-function scatterers compared with our results $\nu_0 = 2.2 \pm 0.1$ and $\nu_1 \simeq 5.5 \pm 0.5$. For finite-range scatterers, they found $\nu_0 \simeq 2$ for $d = 2$, but did not calculate $\nu_1$. We also find that $\nu_0(\simeq 2.2)$ remains unaffected by the finite range of the scattering potential whereas $\nu_1$ is reduced considerably($\nu_1 \leq 3$), bringing it much closer to $\nu_0$ and thus restoring universality. The work of Ando and Aoki was criticized (for having used system sizes far too small to see conclusive evidence of scaling) by Huckestein and Kramer [10] who went to larger system sizes($M = 64$) and concluded that $\nu_0 = 2.34 \pm 0.04$ for short-range white-noise random potential. These latter authors [10] did not systematically investigate the exponents in the higher Landau levels or the effects of having a finite disorder potential range. They also neglected Landau level coupling effects. Our $\delta$-function potential numerical results for the lowest Landau level are quantitatively and qualitatively consistent with the findings of Huckestein and Kramer. It should be mentioned that Huckestein and Kramer [10] model the disorder potential somewhat differently from us – instead of working with random distributed impurity centers, they use a random white-noise distribution of disorder potential matrix elements with a vanishing correlation length. We point out that the largest system sizes($M = 256$) used in our work are much larger than those used by Huckestein and Kramer. (It is, in some sense, essential that we use larger systems because we are interested in investigating
higher Landau levels, finite range random potentials, and Landau level coupling effects.) For slowly varying random disorder potentials, the strong-field quantum localization problem is equivalent to a classical percolation problem, which has been analytically solved [21] to produce $\nu = 4/3$. If quantum tunneling is included in the percolation calculation[23], one gets $\nu = 7/3 \approx 2.33$ which is very close to our numerically calculated $\nu_0$. It is worthwhile to mention that in the percolation limit, the Landau levels are necessarily coupled and the disorder is necessarily long-range – it is, therefore, reassuring to know that our finite-range disorder and Landau level coupled calculations produce $\nu_0 \approx \nu_1$ which are consistent with the percolation results. There has also been a numerical calculation [22] near the percolation threshold, including effects of quantum interference and tunneling, leading to $\nu = 2.5 \pm 0.5$.

Existing numerical and analytical calculations [10-13,21-23] clearly establish that the localization critical exponent is around 2.3 independent of the Landau level index for finite-range potentials and/or in the presence of Landau level coupling, and for the $\delta$-function potential at least in the lowest Landau level. We emphasize that the percolation limit necessarily includes Landau level coupling and long-range disorder, and is thus complementary to the $\delta$-function short-range disorder in the absence of Landau level coupling. The fact that both the percolation calculations and the lowest Landau level finite size scaling analysis for $\delta$-potentials gives the same critical exponent $\nu \approx 2.3$ provides evidence in favor of universality in the Landau level localization. As mentioned before, our finite range and/or Landau level coupled numerical calculations are semi-quantitatively consistent with the percolation results. On the other hand, there does seem to be a problem with the $\delta$-function random potentials in the higher($N = 1$) Landau levels(in the absence of any Landau level coupling) where at least four independent studies [10,11,12,17](including our own, which employs the largest system sizes) find the critical exponent $\nu_1$ to be substantially different from that in the lowest Landau level($\nu_1 \approx 5, \nu_0 \approx 2.3$). It is, of course, possible (but highly unlikely in our opinion) that this lack of universality for the short-range $\delta$-function disorder is purely a finite-size phenomenon and future calculations involving larger system sizes will restore universality (i.e. $\nu_1 = \nu_0$) even for $\delta$-function disorder. This scenario is unlikely in our view.
for two reasons: (1) Early calculations involving small system sizes ($M = 16$) all the way to our large system ($M = 256$), we have some limited results for $M = 512$) calculations consistently show $\nu_1 \gtrsim 2\nu_0$ for $\delta$-function disorder (in the absence of Landau level coupling) – the quality of scaling for $N = 1$ Landau level in our calculations is quite comparable to that for $N = 0$ level, and, therefore, it is unclear how increasing system sizes further would reduce $\nu_1$ by more than a factor of two to bring it in agreement with $\nu_0$; (2) for $\delta$-function disorder, we see no obvious intrinsic length scale in the problem which demands that we go to system sizes substantially larger than this characteristic length scale to obtain the true (rather than effective) exponents – thus it is very hard to understand the nature of any crossover behavior which may be dominating the finite-size scaling analysis for $N = 1$ producing $\nu_1 \approx 5$ (we do expect that one has to use somewhat larger sizes for analyzing the critical behavior of the $N = 1$ level compare to the $N = 0$ behavior because the free electron Landau wavefunction for $N = 1$ is more spread out than that for the $N = 0$ level). Based on these considerations we feel that (in the absence of Landau level coupling) the range of the disorder potential may actually be a relevant perturbation, and the localization transition in higher Landau levels may be non-universal for short-range disorder as our numerical calculations indicate. We believe that this lack of universality arises from the fact (Fig.7) that the matrix element of the disorder potential projected onto a particular Landau level shows qualitatively different spatial behavior for the higher Landau levels compared with the lowest Landau level as a function of the range of disorder. This is explicitly shown in Fig.7. In particular, the spatial behavior of the lowest Landau level matrix element remains qualitatively the same (namely, it is a monotonically decreasing function of distance) independent of the range of the disorder potential (because the wavefunction for the lowest Landau level is nodeless). On the other hand, the higher Landau levels being excited states have nodes and, therefore, the spatial behavior of disorder matrix elements for the higher Landau levels has a qualitative dependence on the range of the disorder, showing spatial oscillations for short-range disorder and becoming qualitatively similar to the lowest Landau level situation only when the disorder range exceeds the spatial extent of the excited wavefunction. Without a real theory for the
localization transition, it is difficult to quantify these qualitative considerations. We believe, however, that disorder invariably mixes Landau levels and the independent Landau level approximation breaks down, making the localization transition universal even for short-range disorder as demonstrated by our numerical results in the presence of Landau level coupling effects.

Before concluding we discuss a number of experimental strong-field localization issues which are not understood currently. While experiment seems to be in “agreement” with theory that the localization exponent $\nu \simeq 2.3$, it is hard to understand why a non-interacting theoretical calculation of the localization exponent should agree with the measured experimental value because, in general, electron-electron interaction is known\cite{24} to be a relevant perturbation which changes the universality class. In this respect, it is reassuring that the measured localization exponent ($\kappa \approx 0.4$) for the fractional quantum Hall transition\cite{25} is the same as the integer quantum Hall case. It is important to have some theoretical idea about why interaction does not change the universality class in the strong-field situation (or, if it does, why the measured value of $\kappa$ is the same for integer and fractional situations in apparent agreement with the non-interacting calculation of the localization exponent $\nu$). The experimentally found dependence\cite{26} of the measured $\kappa$ on the spin degeneracy of the Landau level is also not understood at the present time. In particular, $\kappa$ for the spin-uncpolarized case is found\cite{26} to be half ($\approx 0.21$) the value of the spin-polarized situation ($\approx 0.43$). In a non-interacting localization model, it is unclear how the electron spin can be a relevant perturbation. Inclusion of interaction effects, however, leads to a spin dependence of fermion localization properties in the zero-field case\cite{27}. Whether the observed spin dependence of the exponent $\kappa$ is somehow related to interaction effects or not is totally unknown at this stage. We suggest a simple scenario for the spin dependence of $\kappa$ which can be experimentally tested. Suppose that there is a small (but unresolved) spin-splitting $\Delta E$ in the spin degenerate unpolarized case. In that case, it is possible that, if $\Delta E << k_B T$, the experimental measurement of $\kappa$ measures only an effective $\kappa$ which is smaller than the real $\kappa$ because there are two unresolved critical energies separated by a small (but finite)
$\Delta E$ and one is observing the combined effects of both the localization transitions without resolving them. (The argument fails if $\Delta E = 0$, but the spin-splitting is unlikely to be exactly zero.) A simple analysis shows that this scenario leads to a temperature dependent effective $\kappa$ which asymptotically approaches the real $\kappa(\approx 0.4)$ in the high temperature limit but is lower than the real $\kappa$ in the experimentally feasible regime, approaching, in fact, a saturation ($i.e. \kappa = 0$) logarithmically in the low temperature limit. Thus, a measurement of this effective $\kappa$ in an intermediate temperature range could produce $\kappa \approx 0.2$ as observed experimentally. This proposed scenario should be tested experimentally through detailed temperature dependent measurements in the spin degenerate situation. Finally, we mention that very recent microwave frequency-dependent measurements[28] show clear evidence for finite-frequency ($f$) dynamic scaling of the strong-field localization transition in the integer quantum Hall regime. The experimental finding is that for $f \gtrsim 1 GHz$, $\sigma_{xx}$ peaks broaden (at fixed temperature) with increasing frequency, roughly as $\Delta B \sim f^\gamma$ where $\gamma \approx 0.4(0.2)$ for spin-split (spin-degenerate) peaks. Since one expects $\gamma = 1/\nu z$ where $z$ is the dynamical exponent, one concludes that $z = 1$ and $\nu = 2.3(4.6)$ for spin-split (spin-degenerate) situations. The spin-dependence of the microwave experiments is, of course, consistent with that of the temperature dependent measurements and remains unexplained (unless our “trivial” explanation applies!). The interesting finding of $z = 1$ is consistent with the recent speculation[29] for dirty boson systems in the context of superconductor-insulator transition in thin metal films. Statistical transmutation properties of two dimensional quantum systems allow mapping between fermion and boson systems, and it is, therefore, tempting to speculate[29] that $z = 1$ is a universal feature of all two dimensional quantum phase transitions[30]. One problem in this context is that theory[29] predicts a universal value of the critical resistance at the metal-insulator transition whereas experimentally[31] the value of the $\sigma_{xx}$ peak seems to be non-universal and substantially below the universal theoretical value of $e^2/h$. One should, however, bear in mind that while the theory applies strictly at $T = 0$, the experimental measurements are necessarily at finite temperature, and a simple extrapolation to $T = 0$ to obtain the critical conductance may not work. This issue also requires further
Our conclusion is that the strong-field two-dimensional Landau level localization is indeed universal (i.e. \( \nu_0 = \nu_1 \), etc.) except for the short-range random disorder potential which, in the absence of any Landau level coupling, gives rise to non-universal localization with different Landau levels having different localization exponents. Our work indicates, however, that inclusion of a finite potential range and/or coupling between Landau levels restores localization universality making \( \nu_0 = \nu_1 \).

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**Table and Figure Caption**

**Table 1.** Critical exponents for different Landau levels without Landau level coupling for weak $\delta$-function impurities.

**Table 2.** Critical exponents for weak repulsive $\delta$-function impurities without Landau level coupling.

**Table 3.** Critical exponents for weak finite range Gaussian impurity-potentials without Landau level coupling.

**Table 4.** Critical exponents for strong $\delta$-function impurity-potentials considering coupling between $N = 0$ and $N = 1$ Landau levels.

**Figure 1.** $N = 0$ Landau level, for weak $\delta$-function impurities ($c_i = 8$): (a) The renormalized finite system localization length $\lambda_M/M$ as a function of system size $M$, different symbols represent different energies; (b) One-parameter scaling function, with inset: Localization length $\xi(E)$, $\nu = 2.32$

**Figure 2.** $N = 1$ Landau level, for weak $\delta$-function impurities ($c_i = 4$): (a) $\lambda_M/M$ vs. $M$, different symbols represent different energies; (b) One-parameter scaling function, with inset: Localization length $\xi(E)$, $\nu = 5.95$

**Figure 3.** In the presence of weak repulsive $\delta$-function impurities, $N = 0, c_i = 4$, the scaling function for the lower(a) and higher(b) energy branches of the Landau band. Corresponding localization length in the critical regime are shown in the insets for lower energy branch, $E_c = -0.15, \nu = 3.08$, and for higher energy branch, $E_c = -0.156, \nu = 2.05$.

**Figure 4.** For weak finite range Gaussian impurity-potentials, $N = 0$ Landau band, the scaling function with localization length in the insets are presented for (a) $d/l_c = 0.5, c_i = 8, \nu = 2.21$; (b) $d/l_c = 1.0, c_i = 4, \nu = 2.3$. 

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Figure 5. For weak finite range Gaussian impurity-potentials, $N = 1$ Landau band, the scaling function with localization length in the insets are presented for (a) $d/l_c = 0.5, c_i = 2, \nu = 3.6$; (b) $d/l_c = 1.0, c_i = 4, \nu = 2.51$.

Figure 6. For strong $\delta$-function impurities considering coupling between $N = 0$ and $N = 1$ Landau levels, the scaling function are presented for $c_i = 8$ (a) $N = 0$ lower energy branch; (b) $N = 0$ higher energy branch; (c) $N = 1$ lower energy branch. Corresponding localization length in the critical regime are presented in the insets of (a) $E_c = 0.029, \nu = 1.55$; (b) $E_c = 0.100, \nu = 2.24$; (c) $E_c = 0.872, \nu = 2.85$.

Figure 7. Shows the spatial behavior of the calculated matrix elements of the impurity potential for $N = 0$(solid line), $N = 1$(dotted line), and $N = 2$(dashed line) Landau levels, and, for four different values of the potential range $d/l_c = 0(a),1(b),2(c),$ and $4(d)$. The impurity is located at $x_i$, and $l_o = \sqrt{2\pi l_c}$ where $l_c$ is the magnetic length.

Table 1.

| $c_i$ | $\nu_0$ | $E_c$ | $\nu_1$ | $E_c$ |
|------|--------|------|--------|------|
| 4    | 2.21 ± 0.02 | 0    | 5.9 ± 0.1 | 0    |
| 8    | 2.32 ± 0.02 | 0    | 5.4 ± 0.2 | 0    |
| 16   | 2.22 ± 0.04 | 0    | 5.5 ± 0.1 | 0    |
| 32   | 2.22 ± 0.02 | 0    |        |      |
| Ando et al | ≤ 2 | 0 | ≤ 4 | 0 |
| Kramer et al | 2.34 ± 0.04 | 0 | ~ 4 | 0 |
Table 2.

| $c_i$ | $\nu_{0l}$ | $E_c$ | $\nu_{0h}$ | $E_c$ |
|-------|------------|-------|------------|-------|
| 4     | 3.08       | -0.150| 2.05       | -0.156|
| 8     |            | 2.07  | -0.109     |       |
| 16    |            | 2.22  | -0.090     |       |
| 32    |            | 2.29  | -0.075     |       |

Table 3.

| $d/l_c$ | $c_i$ | $\nu_0$ | $E_c$ | $\nu_1$ | $E_c$ |
|---------|-------|---------|-------|---------|-------|
| 0.5     | 2     | 1.86    | 0     | 3.60    | 0     |
|         | 4     | 2.21    | 0     | 3.56    | 0     |
|         | 8     | 2.21    | 0     | 4.00    | 0     |
| 1.0     | 2     | 2.13    | 0     | 2.8     | 0     |
|         | 4     | 2.30    | 0     | 2.51    | 0     |
|         | 8     | 2.14    | 0     | 2.8     | 0     |

Table 4.

| $\gamma = \Gamma/\hbar\omega_c$ | $c_i$ | $\nu_{0l}$ | $E_c$ | $\nu_{0h}$ | $E_c$ | $\nu_{1l}$ | $E_c$ |
|----------------------------------|-------|------------|-------|------------|-------|------------|-------|
| 0.5                              | 2     | 0.86       | 0.009 | 2.02       | 0.059 | 3.65       | 0.975 |
|                                  | 4     | 1.50       | 0     | 2.30       | 0.082 | 3.04       | 0.886 |
|                                  | 8     | 1.55       | 0.029 | 2.24       | 0.100 | 2.85       | 0.872 |
|                                  | 16    | 1.39       | 0.003 | 2.06       | 0.105 | 2.77       | 0.841 |