Global Uniform Boundedness of Solutions to viscous 3D Primitive Equations with Physical Boundary Conditions

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Abstract

Global uniform boundedness of solutions to 3D viscous Primitive Equations in a bounded cylindrical domain with physical boundary condition is proved in space $H^m$ for any $m \geq 2$. A bounded absorbing set for the solutions in $H^m$ is obtained. These results seem quite difficult to be proved using the methods recently developed in [8] and [10]. A completely different approach based on hydrostatic helmholtz decomposition is presented, which is also applicable to the cases with other boundary conditions. Several important results about hydrostatic Leray projector are obtained and utilized. These results are expected to be of general interest and will be helpful for solving some other problems for 3D viscous Primitive Equations which appeared hard previously for the cases with non-periodic boundary conditions (see e.g. [9]).

Keywords: Primitive Equations, physical boundary condition, global uniform estimates, hydrostatic helmholtz decomposition, hydrostatic Leray projector.

MSC: 35B40, 35B30, 35Q35, 35Q86.

1 Introduction

Let $D$ be a bounded open subset of $\mathbb{R}^2$ with smooth boundary $\partial D$ and

$$\Omega = D \times (-h, 0) \subset \mathbb{R}^3,$$

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where \( h \) is a positive constant. Consider in the cylinder \( \Omega \) the system of viscous Primitive Equations (PEs) of Geophysical Fluid Dynamics:

**Conservation of horizontal momentum:**

\[
v_t + (v \cdot \nabla)v + wv_z + \nabla p + fv^\perp + L_1 v = 0;
\]

**Hydrostatic balance:**

\[ p_z + \theta = 0; \]

**Continuity equation:**

\[ \nabla \cdot v + w_z = 0; \]

**Heat conduction:**

\[ \theta_t + v \cdot \nabla \theta + w\theta_z + L_2 \theta = Q. \]

The unknowns in the above system of 3D viscous PEs are the fluid velocity field \((v, w) = (v_1, v_2, w) \in \mathbb{R}^3\) with \( v = (v_1, v_2) \) and \( v^\perp = (-v_2, v_1) \) being horizontal, the temperature \( \theta \) and the pressure \( p \). The Coriolis rotation frequency \( f = f_0(\beta + y) \) in the \( \beta \)-plane approximation and the heat source \( Q \) are given. The differential operators \( L_1 \) and \( L_2 \) are defined respectively as:

\[
L_i := -\nu_i \Delta - \mu_i \partial_z^2,
\]

with positive constants \( \nu_i, \mu_i \) for \( i = 1, 2 \). In the above equations, \( \nabla \) and \( \Delta \) denote horizontal gradient and Laplacian:

\[
\nabla := (\partial_x, \partial_y) \equiv (\partial_1, \partial_2), \quad \Delta := \partial_x^2 + \partial_y^2 \equiv \sum_{i=1}^{2} \partial_i^2.
\]

In the sequel, we also denote

\[
\nabla_3 := (\nabla, \partial_z) = (\partial_x, \partial_y, \partial_z).
\]

The boundary of \( \Omega \) is partitioned into three parts: \( \partial\Omega = \Gamma_t \cup \Gamma_b \cup \Gamma_l \), where

\[
\Gamma_t := \{(x, y, z) \in \overline{\Omega} : z = 0\}, \\
\Gamma_b := \{(x, y, z) \in \overline{\Omega} : z = -h\}, \\
\Gamma_l := \{(x, y, z) \in \overline{\Omega} : (x, y) \in \partial D\}.
\]
The following set of physical boundary conditions will be used:

- on $\Gamma_t$: $v_z + \alpha_1 v = 0$, $w = 0$, $\theta_z + \alpha_2 \theta = 0$,
- on $\Gamma_b$: $v = 0$, $w = 0$, $\theta_z = 0$,
- on $\Gamma_l$: $v = 0$, $\partial_n \theta = 0$,

where $\alpha_1, \alpha_2$ are non-negative constants and $n$ is the normal vector of $\Gamma_l$.

Let $p_s(x, y, t)$ be the pressure on the top $\Gamma_t$. Then, the above system of PEs can be re-written as

$$ v_t + L_1 v + (v \cdot \nabla) v + w v_z + \nabla p_s + \int_0^z \nabla \theta(x, y, \xi, t) d\xi + f v^l = 0. \quad (1.1) $$

$$ \theta_t + L_2 \theta + v \cdot \nabla \theta + w \theta_z = Q; \quad (1.2) $$

$$ w(x, y, z, t) = \int_0^z \nabla \cdot v(x, y, \xi, t) d\xi, \quad (1.3) $$

$$ (v_z + \alpha_1 v)|_{z=0} = v|_{z=-h} = v|_{(x,y) \in \partial D} = 0, \quad (1.4) $$

$$ w|_{z=-h} = 0, \quad (1.5) $$

$$ (\theta_z + \alpha_2 \theta)|_{z=0} = \theta_z|_{z=-h} = \partial_n \theta|_{(x,y) \in \partial D} = 0. \quad (1.6) $$

The above system of PEs will be solved with suitable initial conditions:

$$ v(x, y, z, 0) = v_0(x, y, z), \quad \theta(x, y, z, 0) = \theta_0(x, y, z). \quad (1.7) $$

Assume $Q$ is independent of time for simplicity of discussion, since results to be presented for autonomous case can be extended to non-autonomous case with proper modifications.

The notions of weak and strong solutions were introduced in [16], where existence of weak solutions was proved, though uniqueness of weak solutions is still not resolved yet. Local (in time) existence and uniqueness of strong solutions were obtained in [4] and [20]. Global (in time) existence of strong solutions was proved in [2] for the case when $v$ satisfies a set of Neumann type boundary conditions. See [11] for a different approach. For the case when $v$ satisfies the boundary conditions (1.4), see [12] for a proof of global regularity of strong solutions. For the case when $v$ satisfies a set of related boundary conditions, see [5] for a different approach. Uniform boundedness
of strong solutions was proved in [6] for the case with Neumann type boundary conditions and in [13] for the case with physical boundary conditions. For many more other results and studies of 3D Primitive Equations, refer [14],[18] and [20].

An initial motivation of this paper is to study uniform boundedness of $H^2$ solutions to the 3D primitive equations for the case with physical boundary conditions (1.4). Global existence of the $H^2$ solutions for the case with periodic boundary condition was proved in [17]. The method of [17] uses extensively integrations by parts. For non-periodic cases, the boundary terms arising from those integrations by parts would cause trouble for a priori estimates. The first success overcoming this difficulty was achieved in [10], which proved global uniform boundedness of $H^2$ solutions for the 3D Primitive Equations with $v$ satisfying a set of Neumann type boundary conditions. An interesting aspect of this approach is that not only it works for both the periodic and the non-periodic case, it also requires less demanding condition on $Q$ than [17]. This approach was further improved in [8] which eliminated all technical restrictions completely. Roughly speaking, the main idea of [8] and [10] is obtaining uniform boundedness of $\|(v_t, \theta_t)\|_{L^2}$ and $\|v_z\|_{H^1}$ first, then using elliptic regularity to get uniform boundedness of $\|(v, \theta)\|_{H^2}$, since boundary conditions of $v_t, \theta_t$ and $v_z$ can be used in this approach. However, it seems the approach of [8] and [10] can not help for the case with the physical boundary condition (1.4). The reason is that uniform boundedness of $\|v_z\|_{H^1}$ is difficult for this case; while the nonlinear term $wv_z$ in (1.1) is not controllable with just uniform boundedness of $\|(v_t, \theta_t)\|_{L^2}$ alone. Another drawback of the method of [8] and [10] is that it seems not helpful for uniform estimate of higher regularity, even for the case with Neumann boundary conditions.

In this paper, a completely different approach is presented. The new approach can prove uniform boundedness of $H^m$ norm of $(v, \theta)$ for any $m \geq 2$, if $\partial D$ and $Q$ are sufficiently smooth. Moreover, this approach works not only for the case with boundary condition (1.4), but also for other typical boundary conditions. The main new idea is the use of hydrostatic Leray projector to be defined in Section 3. It allows treatment of pressure and boundary
conditions more convenient. To realize the goal, several fundamental results for hydrostatic Leray projector and a new anisotropic multi-linear estimate in Sobolev space are established. With the help of these results, the proofs of the new approach for uniform boundedness appear direct and simplified. The results on hydrostatic Leray projector are expected to be of general interest and can be helpful for solving some other problems about the system of 3D Primitive Equations which appeared rather difficult previously, due to non-periodic boundary conditions. For example, analyticity in Gevery class for the solutions of 3D Primitive Equations with non-periodic boundary conditions can be established now with the help of hydrostatic Leray projector ([9]).

The rest of this article is organized as follows:

In Section 2, we give the notations, briefly review the background results and recall some important facts which are useful for later analysis. We will also state and prove Lemma 2.2, which gives a useful new anisotropic multi-linear estimate in Sobolev space and will be used a few times in Sections 4 and 5.

In Section 3, we present and prove Theorems 3.1, 3.2, 3.3 and 3.4. These fundamental results on hydrostatic Helmholtz decomposition and hydrostatic Leray projector will be used in several key steps of the analyses of Section 4 and Section 5.

In Section 4, the result of global uniform boundedness of $H^2$ solutions is stated as Theorem 4.1 and is proved. Theorem 4.2 on global uniform boundedness of $\|(v_t, \theta_t)\|$ will also be proved.

In Section 5, global uniform boundedness for higher regularity will be presented as Theorem 5.1 and be proved briefly.

2 Preliminaries

Notations to be used are basically standard. $C$ and $c$ denote generic positive constants, the values of which may vary from one occurrence to another. The
relation operators ≼ and ≈ are used such that, for real numbers $A$ and $B$,

$$A \preceq B \quad \text{if and only if} \quad A \leq C \cdot B,$$

$$A \simeq B \quad \text{if and only if} \quad c \cdot A \leq B \leq C \cdot A,$$

for some positive constants $C$ and $c$ independent of $A$ and $B$.

$L^p(\Omega)$ and $L^p(D)$ ($1 \leq p \leq \infty$) are the classic Lebesgue $L^p$ spaces with the norm denoted as

$$\| \cdot \|_p := \| \cdot \|_{L^p}, \quad \| \cdot \| := \| \cdot \|_2.$$

$H^m(\Omega)$ and $H^m(D)$ ($m \geq 1$) (with norm $\| \cdot \|_{H^m}$) denote the classic Sobolev spaces for $L^2$ functions with weak derivatives up to order $m$ also in $L^2$. Notations for vector and scalar function spaces may not be distinguished when they are self-evident from the context.

Define $\overline{\varphi}$ as the vertical average of $\varphi$ on $\Omega$:

$$\overline{\varphi}(x, y) = \frac{1}{h} \int_{-h}^{0} \varphi(x, y, z) \, dz.$$

By the Hölder inequality, it is easy to see that, for $\varphi \in L^p(\Omega)$,

$$\| \overline{\varphi} \|_{L^p(\Omega)} = h^{\frac{1}{p}} \| \varphi \|_{L^p(D)} \leq \| \varphi \|_p, \quad \forall p \in [1, +\infty]. \quad (2.1)$$

Define the bilinear forms:

$$a_1(v, u) = \int_{\Omega} \left( \left( \sum_{i=1}^{2} \nabla v_i \cdot \nabla u_i + \mu_1 v_z \cdot u_z \right) \cdot d\Omega + \alpha_1 \mu_1 \int_{\Gamma_t} v \cdot u \, dD,$$

$$a_2(\theta, \eta) = \int_{\Omega} \left( \nabla \theta \cdot \nabla \eta + \mu_2 \theta_z \eta_z \right) \cdot d\Omega + \alpha_2 \mu_2 \int_{\Gamma_t} \theta \eta \, dD,$$

We use $d\Omega$ and $dD$ to denote $dxdydz$ and $dxdy$ in integrals in $\Omega$ and $D$ respectively, or we may simply omit them.
and the linear operators $A_i : V_i \mapsto V'_i, \ i = 1, 2$, such that

$$\langle A_1 v, u \rangle = a_1(v, u), \ \forall v, u \in V_1; \ \langle A_2 \theta, \eta \rangle = a_2(\theta, \eta), \ \forall \theta, \eta \in V_2,$$

where $V'_i (i = 1, 2)$ is the dual space of $V_i$ and $\langle \cdot, \cdot \rangle$ is the corresponding scalar product between $V'_i$ and $V_i$. We also use $\langle \cdot, \cdot \rangle$ to denote the inner products in $H_1$ and $H_2$.

Define:

$$D(A_i) = \{ \phi \in V_i, A_i \phi \in H_i \}, \ i = 1, 2.$$

Since $A_i^{-1}$ is a self-adjoint compact operator in $H_i$, by the classic spectral theory, the power $A_i^s$ can be defined for any $s \in \mathbb{R}$. Then

$$D(A_i)' = D(A_i^{-1})$$

is the dual space of $D(A_i)$ and, with the denotation $\Lambda_i = A_i^{\frac{1}{2}}$,

$$V_i = D(A_i^{\frac{1}{2}}) = D(\Lambda_i), \ V'_i = D(A_i^{-\frac{1}{2}}) = D(\Lambda_i^{-1}).$$

Moreover,

$$D(A_i) \subset V_i \subset H_i \subset V'_i \subset D(A_i)'$$

where the embeddings above are all compact and each space above is dense in the one following it.

Define the norm $\| \cdot \|_{V_i}$ by:

$$\| \cdot \|_{V_i}^2 = a_i(\cdot, \cdot) = \langle A_i^{\frac{1}{2}} \cdot, A_i^{\frac{1}{2}} \cdot \rangle = \langle \Lambda_i \cdot, \Lambda_i \cdot \rangle, \ i = 1, 2.$$

Then, for any $\phi = (\phi_1, \phi_2) \in V_1$ and $\psi \in V_2$

$$\| \phi \| \lesssim \| \phi \|_{V_1}, \ \| \psi \| \lesssim \| \psi \|_{V_2}.$$

Therefore, for any $\phi = (\phi_1, \phi_2) \in V_1$ and $\psi \in V_2$,

$$\| \phi \|_{V_1} \approx \| \phi \|_{H^1}, \ \| \psi \|_{V_2} \approx \| \psi \|_{H^1}. \ \ (2.2)$$

Recall the following definitions of weak and strong solutions:
**Definition 2.1** Suppose $Q \in L^2(\Omega)$, $(v_0, \theta_0) \in H$ and $T > 0$.

The pair $(v, \theta)$ is called a weak solution of the 3D viscous PEs (1.1)-(1.7) on the $[0,T]$ if it satisfies (1.1)-(1.7) in weak sense (see e.g. [16] for details) and that

$$(v, \theta) \in L^\infty([0,T]; H) \cap L^2(0,T; V).$$

Moreover, if $(v_0, \theta_0) \in V$, a weak solution $(v, \theta)$ is called a strong solution of (1.1)-(1.7) on the time interval $[0,T]$ if, in addition, it satisfies

$$(v, \theta) \in L^\infty([0,T]; V) \cap L^2(0,T; D(A_1) \times D(A_2)).$$

Recall the following lemma which will be used in the *a priori* estimates in Sections 4 and 5. See [1] and [6] for the proof.

**Lemma 2.1** Suppose that $\nabla v, \varphi \in H^1(\Omega)$ and $\psi \in L^2(\Omega)$. Then,

$$\left| \left( \int \nabla \cdot v(x, y, \xi) d\xi \right) \varphi, \psi \right| \leq \|\nabla v\|^\frac{1}{2}\|\varphi\|^\frac{1}{2}(\|\phi\| + \|\phi_z\|)^\frac{1}{2}(\|\psi\| + \|\nabla \psi\|)\frac{1}{2}$$

Finally, we prove the following anisotrophic estimate in Sobolev spaces. It will be used as well in Sections 4 and 5.

**Lemma 2.2** Suppose $\phi, \psi \in H^1(\Omega)$ and $\varphi \in L^2(\Omega)$. Then,

$$\int_{\Omega} |\phi \psi \varphi| \leq \int_{-h}^0 \|\phi\|_{L^4(D)}\|\psi\|_{L^4(D)}\|\varphi\|_{L^2(D)} \, dz$$

$$\leq \int_{-h}^0 \|\phi\|^\frac{1}{2}\|\nabla \phi\|^\frac{1}{2}\|\psi\|^\frac{1}{2}\|\nabla \psi\|^\frac{1}{2}\|\varphi\|^\frac{1}{2}\|\nabla \varphi\|^\frac{1}{2} \, dz$$

**Proof:**

$$\int_{\Omega} |\phi \psi \varphi| \leq \int_{-h}^0 \|\phi\|_{L^4(L^2)}\|\psi\|_{L^4(L^2)}\|\varphi\|_{L^4(L^2)} \, dz$$

$$\approx \int_{-h}^0 \|\phi\|^\frac{1}{2}\|\nabla \phi\|^\frac{1}{2}\|\psi\|^\frac{1}{2}\|\nabla \psi\|^\frac{1}{2}\|\varphi\|^\frac{1}{2}\|\nabla \varphi\|^\frac{1}{2} \, dz$$

$$\leq \|\phi\|_{L^2(L^2)}\|\psi\|_{L^2(L^2)}\|\varphi\|_{L^2(L^2)}$$

$$\leq \|\phi\|^\frac{1}{2}\|\nabla \phi\|^\frac{1}{2}\|\psi\|^\frac{1}{2}\|\nabla \psi\|^\frac{1}{2} \, dz$$

$$\leq \|\phi\|^\frac{1}{2}\|\nabla \phi\|^\frac{1}{2}\|\psi\|^\frac{1}{2}\|\nabla \psi\|^\frac{1}{2} \, dz$$
Notice that
\[ \parallel \phi \parallel_{L^2(D, L^\infty)}^2 = \int_D \parallel \phi \parallel_{L^\infty}^2 \, dD \]
\[ \lesssim \int_D \parallel \phi \parallel_{L^2}^2 (\parallel \phi \parallel_{L^2} + \parallel \phi_z \parallel_{L^2}) \, dD \lesssim \parallel \phi \parallel_2 (\parallel \phi \parallel_2 + \parallel \phi_z \parallel_2). \]
This finishes the proof of Lemma 2.2. \(\square\)

3 Hydrostatic Decomposition

First, we prove the following Helmholtz type hydrostatic decomposition. Recall that the space \( H_1 \) was defined in Section 2.

**Theorem 3.1**
\[ (L^2(\Omega))^2 = H_1 \oplus G, \]

where
\[ G := \{ u \in (L^2(\Omega))^2 \mid u_z = 0, \ u = \nabla q, \ q \in H^1(D) \}. \]

**Proof:** Let \((H_1)^\perp\) be the orthogonal complement of \( H_1 \) with respect to the inner product in \((L^2(\Omega))^2\), i.e.
\[ (L^2(\Omega))^2 = H_1 \oplus (H_1)^\perp. \]

We need only to show that \( G = (H_1)^\perp \).

**Claim 1:** \( G \subset (H_1)^\perp. \)

**Proof of Claim 1:** Let \( u \in G \). Thus, \( u = \nabla q \) for some \( q \in H^1(D) \) and \( u_z = 0 \). Let \( v \in V_1 \), where
\[ V_1 := \{ v \in (C^\infty(\Omega))^2 \mid v \text{ vanishes in a neighborhood of } \Gamma_b \cup \Gamma_t, \ \nabla \cdot \bar{v} = 0 \}. \]

Then,
\[ \langle u, v \rangle = h \int_D (\nabla q) \cdot \bar{v} \, dD = -h \int_D q \nabla \cdot \bar{v} = 0. \]
Recall that \( V_1 \) is dense in \( V_1 \) (see [15]). Hence, \( V_1 \) is dense in \( H_1 \), since \( V_1 \) is dense in \( H_1 \). Therefore, \( u \in (H_1)^\perp \). This proves Claim 1.
Claim 2: \((H_1)^{\perp} \subset G\).

Proof of Claim 2: Let \(u = (u_1, u_2) \in (H_1)^{\perp}\). Then, \(u \in (L^2(\Omega))^2\) and
\[
\langle u, v \rangle = 0, \quad \forall v \in H_1.
\]

Step 1. Choose the special \(v = (\varphi_z, 0)\) with \(\varphi \in C_0^\infty(\Omega)\). Then, \(v \in H_1\), since \(\overline{v} = (0, 0)\). Thus,
\[
\langle u_1, \varphi_z \rangle = \langle u, v \rangle = 0, \quad \forall \varphi \in C_0^\infty(\Omega).
\]

Therefore, \(\partial_z u_1 = 0\) as the weak derivative of \(u_1\). Similarly, \(\partial_z u_2 = 0\). Thus,
\[
u_z = 0, \quad \text{and } u \in (L^2(D))^2.
\]

Step 2. Choose \(v \in (L^2(\Omega))^2\) such that \(v_z = 0\) and
\[
\nabla \cdot v = 0, \quad n \cdot v = 0\big|_{\partial D} = 0.
\]

Since \(v \in H_1\), we have \(\langle u, v \rangle = 0\) for all such \(v\)’s. Thus, there exists a \(q \in H^1(D)\), such that
\[
u = \nabla q.
\]

Therefore, \(u \in G\). This proves Claim 2. \(\square\)

An immediate application of Theorem 3.1 is the definition of the hydrostatic Leray projector \(\mathbb{P}\) as the orthogonal projection of \((L^2(\Omega))^2\) onto \(H_1\) with respect to the inner product of \((L^2(\Omega))^2\).

Next, we give the following refined decomposition result for \((L^2(\Omega))^2\):

**Theorem 3.2** Let \(D\) be an open bounded set of class \(C^2\). Then,
\[(L^2(\Omega))^2 = H_1 \oplus G_1 \oplus G_2,\] (3.1)

where \(H_1, G_1\) and \(G_2\) are mutually orthogonal spaces and
\[
G_1 := \{ u \in (L^2(\Omega))^2 \mid u_z = 0, \ u = \nabla q, \ q \in H^1(D), \ \Delta q = 0 \},
\]
\[
G_2 := \{ u \in (L^2(\Omega))^2 \mid u_z = 0, \ u = \nabla q, \ q \in H^1_0(D) \}.
\]

Moreover, the following decomposition is valid:
\[
u = \mathbb{P}u + \nabla (q_1 + q_2), \quad \forall u \in (L^2(\Omega)),\] (3.2)
where

\[ \Delta q_1 = 0, \quad \partial_n q_1 |_{\partial D} = n \cdot (\bar{u} - \nabla q_2), \quad (3.3) \]
\[ \Delta q_2 = \nabla \cdot \bar{u} \in H^{-1}(D), \quad q_2 \in H^1_0(D). \quad (3.4) \]

Proof: In the following, we prove (3.2), from which (3.1) follows as well. For any given \( u \in (L^2(\Omega))^2 \), the Dirichlet problem (3.4) has a unique solution \( q_2 \). Therefore, the Neumann problem (3.3) is well defined. Moreover, by the Stokes formula and (3.4), we have

\[ \int_{\partial D} n \cdot (\bar{u} - \nabla q_2) \, ds = \int_D \nabla \cdot (\bar{u} - \nabla q_2) \, dD = \int_D (\nabla \bar{u} - \Delta q_2) \, dD = 0. \]

Then, the Neumann problem (3.3) has a solution \( q_1 \in H^1(D) \), which is unique up to an additive constant. Obviously,

\[ \nabla q_i \in Q_i, \quad i = 1, 2. \]

It is easy to see that \( \nabla q_1 \) and \( \nabla q_2 \) are orthogonal in \( (L^2(\Omega))^2 \), since

\[ \int_{\Omega} \nabla q_1 \cdot \nabla q_2 \, d\Omega = -h \int_D \nabla q_1 \cdot \nabla q_2 \, dD = h \int_{\partial D} (\partial_n q_1) q_2 ds - h \int_D (\Delta q_1) q_2 \, dD = 0, \]

where the Stokes formula and the definitions of \( q_1 \) and \( q_2 \) given by (3.3) and (3.4) are used. Finally, we show that

\[ u_* := u - \nabla (q_1 + q_2) \in H_1. \]

Notice that, by (3.3) and (3.4),

\[ \nabla \cdot \bar{u}_* = \nabla \cdot \bar{u} - \Delta q_1 - \Delta q_2 = \nabla \cdot \bar{u} - \Delta q_2 = 0, \]

and

\[ \bar{u}_* \cdot n |_{\partial D} = (n \cdot \bar{u} + \partial_n q_1 + \partial_n q_2) |_{\partial D} = 0. \]

Thus, \( u_* \in H_1 \). This proves (3.2) and hence (3.1) as well. \( \square \)

The importance of (3.2) is that it gives more concrete information about the decomposition (3.1). Especially, it provides detailed relation between \( P u \) and \( u \), via (3.3) and (3.4). An immediate and very important consequence of (3.2) is the following result.
Theorem 3.3 For any $u \in (L^2(\Omega))^2$, the following statements are valid:

(a) Suppose $|u_z| \in L^1_{\text{loc}}(\Omega)$. Then, in the sense of distribution,
$$ (Pu)_z = u_z. $$

(b) Suppose $|u|, |\nabla u| \in L^r(\Omega)$, $r \in (1, \infty)$ and $\partial D \in C^2$. Then, there exists a constant $c = c(r, D) > 0$ such that:
$$ \|\nabla (Pu)\|_{L^r(\Omega)} \leq c(r, D)(\|u\|_{L^r(\Omega)}^2 + \|\nabla u\|_{L^r(\Omega)}). $$

(c) Suppose $u \in (W^{1,r}(\Omega))^2$, $r \in (1, \infty)$ and $\partial D \in C^2$. Then, there exists a constant $c = c(r, D) > 0$ such that:
$$ \|\nabla (\nabla \cdot \partial_z Pu)\|_{L^r(\Omega)} \leq c(r, D)\|u\|_{(W^{1,r}(\Omega))^2}. $$

(d) Suppose $u \in (W^{m,r}(\Omega))^2$, $r \in (1, \infty)$ and $\partial D \in C^{m+1}$. Then, there exists a constant $c = c(r, D) > 0$ such that:
$$ \|\nabla^m (Pu)\|_{L^r(\Omega)} \leq c(r, D)\|u\|_{(W^{m,r}(\Omega))^2}. $$

Proof: Theorem 3.3 (a) follows from (3.2) immediately. By (3.2), we also have
$$ \|\nabla (Pu)\|_{L^r(\Omega)} \leq \|\nabla u\|_{(L^r(\Omega))^2} + \|\nabla u\|_{(L^r(\Omega))^2} + \|\nabla^2 q_2\|_{L^r(\Omega)}. $$

From (3.3), it follows that
$$ \|\nabla^2 q_1\|_{L^r(D)} \leq \left|\nabla - \nabla q_2\right|_{W^{1,1/r, r}(\partial D)} \leq \left|\nabla - \nabla q_2\right|_{W^{1,1/r, r}(D)} + \|\nabla q_2\|_{W^{1,1/r, r}(D)}^2 \leq h^{-\frac{1}{2}}(\|u\|_{L^r(\Omega)} + \|\nabla u\|_{L^r(\Omega)}) + \|\nabla q_2\|_{W^{1,1/r, r}(D)}^2 $$

From (3.4), it follows that
$$ \|\nabla q_2\|_{L^r(D)} \leq \|\nabla \cdot \bar{u}\|_{L^r(D)} \leq h^{-\frac{1}{2}}\|\nabla u\|_{L^r(\Omega)}. $$

In the above estimates, (2.1) has been used. Therefore,
$$ \|\nabla (Pu)\|_{(L^r(\Omega))^2} \leq \|u\|_{L^r(\Omega)} + \|\nabla u\|_{L^r(\Omega)}. $$
This proves Theorem 3.3 (b). Theorem 3.3 (c) is an immediate consequence of (a) and (b). Proof of Theorem 3.3 (d) is similar to that of (c).

The last important result of the section is about the relation between the anisotropic Laplace operator $L_1$ and the hydrostatic Stokes operator $A_1$ through the hydrostatic Leray projector $\mathbb{P}$. Recall that the operator $A_1$ was already defined in Section 2.

**Theorem 3.4** As an isomorphism from $V_1$ onto $V_1'$, $A_1$ satisfies:

$$A_1 u = \mathbb{P} L_1 u, \quad \forall u \in V_1.$$

**Proof:** It suffices to prove $A_1 = \mathbb{P} L_1$ only. Suppose $u \in D(A_1)$ and $\varphi \in V_1$. Then, $L_1 u \in (L^2(\Omega))^2$. By Theorem 3.1, there exists a $q \in G$ such that

$$\langle \mathbb{P} L_1 u, \varphi \rangle = \langle L_1 u - \nabla q, \varphi \rangle = \langle L_1 u, \varphi \rangle = -\int_{\Omega} [\nu_1 \Delta u \cdot \varphi + \mu_1 u_{zz} \cdot \varphi] d\Omega.$$

Since $u \in D(A_1)$ and $\varphi \in V_1$, we have

$$\langle \Delta u, \varphi \rangle = \int_{-h}^{0} \left( \int_D \Delta u \cdot \varphi dD \right) dz = \int_{-h}^{0} \left( \int_D \sum_{i=1}^{2} \Delta u_i \varphi_i dD \right) dz$$

$$= \int_{-h}^{0} \left( \int_{\partial D} \sum_{i=1}^{2} \partial_{n_i} u_i \cdot \varphi_i ds - \int_D \sum_{i=1}^{2} \nabla u_i \cdot \nabla \varphi_i dD \right) dz$$

$$= -\int_{\Omega} \sum_{i=1}^{2} \nabla u_i \cdot \nabla \varphi_i d\Omega,$$

$$\langle u_{zz}, \varphi \rangle = \int_D \left( \int_{-h}^{0} u_{zz} \cdot \varphi dz \right) dD = \int_D \left( \int_{-h}^{0} u_z \cdot \varphi dz \right) dD$$

$$= -\alpha_1 \int_{\Gamma_1} u \cdot \varphi dD - \int_{\Omega} u_z \cdot \nabla \varphi z d\Omega.$$

Thus,

$$\langle \mathbb{P} L_1 u, \varphi \rangle = \langle A_1 u, \varphi \rangle, \quad \forall \varphi \in V_1,$$

that is

$$\mathbb{P} L_1 u = A_1 u, \quad \forall u \in D(A_1).$$
Notice that \( D(A_1) \) is dense in \( V_1 \). Therefore, considered as an element of \( (V_1)' \),
\[
\mathbb{P}L_1 u = A_1 u, \quad \forall \ u \in V_1.
\]

\[\Box\]

4 Uniform Boundedness of \( \| (A_1 v, A_2 \theta) \| \)

In this section, we prove uniform boundedness of \( \| (A_1 v, A_2 \theta) \| \) and existence of a bounded absorbing set of it in \( \mathbb{R}_+ \). More precisely, we prove the following theorem.

**Theorem 4.1** Suppose \( Q \in L^2(\Omega) \) and \( (v_0, \theta_0) \in D(A_1) \times D(A_2) \). Let \( (v, \theta) \) be the unique strong solution of problem (1.1)-(1.7). Then,
\[
(A_1 v, A_2 \theta) \in L^\infty(0, \infty; H), \quad (A_1 v, \theta_t) \in L^2(0, T; V), \quad \forall \ T \in (0, \infty).
\]

Moreover, there exists a bounded absorbing set of \( \| (A_1 v, A_2 \theta) \| \) in \( \mathbb{R}_+ \).

**Proof:** Step 1. Estimate of \( \| A_1 v \| \).

First, apply the hydrostatic Leray projector \( \mathbb{P} \) to (1.1) and use Theorem 3.4 to get
\[
v_t + A_1 v = -\mathbb{P} \left[ (v \cdot \nabla)v + wv_z + \int_{\mathbb{R}} \nabla \theta \, d\xi + f v^\perp \right]. \quad (4.1)
\]

Next, apply \( A_1 \) to (4.1) and then take inner product with \( A_1 v \) to arrive at
\[
\frac{1}{2} \frac{d}{dt} \| A_1 v \|^2 + \| A_1^2 v \|^2
\leq - \left\langle A_1 \mathbb{P} \left[ (v \cdot \nabla)v + wv_z + \int_{\mathbb{R}} \nabla \theta \, d\xi + f v^\perp \right], A_1 v \right\rangle
\leq \left\| A_1^2 \mathbb{P} \left[ (v \cdot \nabla)v + wv_z + \int_{\mathbb{R}} \nabla \theta \, d\xi + f v^\perp \right] \right\| \| A_1^2 v \|.
\]

Notice that \( \mathbb{P} \) does not commute with \( A_1 \) due to no-slip boundary condition. Therefore, Theorem 3.3 is crucial here in dealing with the right-hand side.
of the above inequality, from which it then follows that
\[
\frac{1}{2} \frac{d}{dt} \| A_1 v \|^2 + \| A^4_1 v \|^2 \\
\leq \left\| P \left[ (v \cdot \nabla) v + w v_z + \int_z^0 \nabla \theta \, d\xi + f v^{rac{1}{2}} \right] \right\|_{H^1} \| A^4_1 v \| \\
\leq \left\| (v \cdot \nabla) v + w v_z + \int_z^0 \nabla \theta \, d\xi + f v^{rac{1}{2}} \right\|_{H^1} \| A^4_1 v \|,
\]
where we have also used norm equivalence (2.2). Noticing that \( v|_{\Gamma_b} = 0 \), we have
\[
\| (v \cdot \nabla) v + w v_z + \int_z^0 \nabla \theta \, d\xi + f v^{rac{1}{2}} \|_{H^1} + \| v \|_{V_1} \\
\leq \| \nabla_3[(v \cdot \nabla) v + w v_z] \| + \| v \|_{V_1} + \| A_2 \theta \|.
\]
Therefore,
\[
\frac{d}{dt} \| A_1 v \|^2 + \| A^4_1 v \|^2 \leq \| \nabla_3[(v \cdot \nabla) v + w v_z] \|^2 + \| v \|_{V_1}^2 + \| A_2 \theta \|^2. \tag{4.2}
\]
Now, we estimate the first term on the right side of (4.2). By Lemma 2.2, we have
\[
\| \nabla [(v \cdot \nabla) v] \| \leq \| \nabla |\nabla^2 v| \| + \| \nabla v \|^2 \\
\leq \| v \|_{V_1} \frac{1}{2} \| \nabla^2 v \| + \| \nabla v \|^2 \\
\leq \| v \|_{V_1} \frac{1}{2} \| \nabla v \|_3 \| A_1 v \| + \| v \|_{V_1} \| A_1 v \| \frac{3}{2} \| A_1 v \| \frac{3}{2},
\]
where
\[
\| \partial_z [(v \cdot \nabla) v] \| \leq \| (v \cdot \nabla) v_z \| + \| (v_z \cdot \nabla) v \| \\
\leq \| v \|_{V_1} \frac{1}{2} \| \nabla v \|_3 \| \nabla v \|_3 + \| v \|_{V_1} \| \nabla v \|_3 \| \nabla v \|_3 \\
\leq \| v \|_{V_1} \| \nabla v \|_3 \| A_1 v \| + \| v \|_{V_1} \| A_1 v \| \frac{3}{2} \| A_1 v \| \frac{3}{2}.
\]
Therefore,
\[
\| \nabla_3[(v \cdot \nabla) v] \|^2 \leq \| v \|_{V_1} \| \nabla v \|_3 \| A_1 v \| + \| v \|_{V_1} \| A_1 v \| \frac{3}{2} \| A_1 v \| \frac{3}{2} \\
\leq C \| v \|_{V_1} ^{10} \| A_1 v \|^2 + \| v \|_{V_1} \| A_1 v \|^3 + \| v \|_{V_1} \| A_1 v \|^3 + \| v \|_{V_1} \| A_1 v \|^3 + \| v \|_{V_1} \| A_1 v \|^3 \tag{4.3}
\]
Next, we use (1.3) and Lemma 2.1 and to get
\[
\| \nabla (wv_z) \| \leq \| w(\nabla v_z) \| + \| (\nabla w) \otimes v_z \|
\]
\[
\approx \| \nabla v \|^{\frac{1}{2}} \| \nabla v_z \|^{\frac{1}{2}} \| \nabla v_z \|^{\frac{1}{2}}
\]
\[
+ \| \nabla v \|^{\frac{1}{2}} \| \nabla^2 v \|^{\frac{1}{2}} \| v_z \|^{\frac{1}{2}} \| v_z \|^{\frac{1}{2}}
\]
\[
\approx \| v \|^{\frac{1}{2}} \| A_1 v \| \| A_1^{\frac{3}{2}} v \|^{\frac{1}{2}}.
\]

Similar to the above estimates, we use Lemma 2.1 and Lemma 2.2 to get
\[
\| \partial_z (wv_z) \| \leq \| wv_{zz} \| + \| (\nabla \cdot v) v_z \|
\]
\[
\approx \| \nabla v \|^{\frac{1}{2}} \| \nabla v_z \|^{\frac{1}{2}} \| v_z \|^{\frac{1}{2}} \| v_z \|^{\frac{1}{2}}
\]
\[
+ \| \nabla v \|^{\frac{1}{2}} \| \nabla v \|^{\frac{1}{2}} \| v_z \|^{\frac{1}{2}} \| v_z \|^{\frac{1}{2}}
\]
\[
\approx \| v \|^{\frac{1}{2}} \| A_1 v \| \| A_1^{\frac{3}{2}} v \|^{\frac{1}{2}} + \| v \|^{\frac{1}{2}} \| A_1 v \|^{\frac{1}{2}}.
\]

Therefore,
\[
\| \nabla_3 (wv_z) \| \leq \| v \| \| A_1 v \|^{\frac{3}{2}} \| A_1^{\frac{3}{2}} v \|^{\frac{1}{2}} + \| v \| \| A_1 v \|^{\frac{3}{2}}
\]
\[
\approx C \| v \| \| A_1 v \|^{4} + \| v \| \| A_1 v \|^{3} + \varepsilon \| A_1^{\frac{3}{2}} v \|^{2}.
\]

Combining (4.2), (4.3) and (4.4) with \( \varepsilon > 0 \) sufficiently small, we have
\[
\frac{d}{dt} \| A_1 v \|^{2} + \| A_1^{\frac{3}{2}} v \|^{2}
\]
\[
\approx \| v \|^{2} \| v \|^{2} \| A_1 v \|^{2} + \| v \| \| A_1 v \|^{3}
\]
\[
+ \| v \|^{4} \| A_1 v \|^{4} + \| v \|^{2} \| A_2 \theta \|^{2}
\]
\[
\approx \| v \|^{2} \| v \|^{2} \| A_1 v \|^{2} + \| v \| \| A_1 v \|^{3}
\]
\[
+ \| v \|^{4} \| A_1 v \|^{4} + \| v \|^{2} \| A_2 \theta \|^{2}.
\]

Now, an application of Gronwall lemma to (4.5) immediately yields the following boundedness:
\[
A_1 v \in L^\infty (0, T; H_1) \cap L^2 (0, T; V_1), \ \forall \ T \in (0, \infty).
\]

Let
\[
y(t) := \| A_1 v(t) \|^{2}, \ \ h(t) := \| v \|_{V_1}^{2} + \| A_2 \theta \|^{2}.
\]
\[ g_1(t) := \|v\|^2 \|v\|_{V_1}^6 + \|v\|_{V_1} \|A_1 v\| + \|v\|_{V_1}^4 \|A_1 v\|^2. \]

The following integrals with \( t \geq 0 \)
\[
\int_t^{t+1} y(s) ds, \int_t^{t+1} h(s) ds, \int_t^{t+1} g_1(s) ds
\]
are all uniformly bounded for strong solutions. Then, an application of the uniform Gronwall lemma (see [3] and [19]) to (4.5) yields the following uniform boundedness:

\[ A_1 v \in L^\infty(0, \infty; H_1) \cap L^2(0, T; V_1), \ \forall \ T \in (0, \infty). \]

Moreover, there is a bounded absorbing set for \( A_1 v \) in \( H_1 \). This finishes proof of uniform boundedness of \( \|A_1 v\| \).

**Step 2. Estimate of \( \|A_2 \theta\| \).**

We will estimate \( \|A_2 \theta\| \) in a way different from that of Step 1. This is because the method of Step 1 will not work for \( Q \in L^2(\Omega) \).

Take inner product of (1.2) with \( A_2 \theta \), we get
\[
\|A_2^{\frac{3}{2}} \theta_t\|^2 + \frac{1}{2} \frac{d}{dt} \|A_2 \theta\|^2 = \langle Q, A_2 \theta_t \rangle + \left\langle A_2^{\frac{1}{2}} [v \cdot \nabla \theta + w \theta_z], A_2^{\frac{3}{2}} \theta_t \right\rangle \]
\[
= \frac{d}{dt} \langle Q, A_2 \theta \rangle + \left\langle A_2^{\frac{1}{2}} [v \cdot \nabla \theta + w \theta_z], A_2^{\frac{3}{2}} \theta_t \right\rangle \tag{4.6}
\]

Therefore,
\[
\|A_2^{\frac{3}{2}} \theta_t\|^2 + \frac{1}{2} \frac{d}{dt} \|A_2 \theta\|^2 \leq 2 \frac{d}{dt} \langle Q, A_2 \theta \rangle + \|A_2^{\frac{1}{2}} [v \cdot \nabla \theta + w \theta_z]\|^2.
\]

Noticing that \( v|_{\Gamma_{\text{in}} \cup \Gamma_{\text{out}}} = w|_{\Gamma_{\text{in}} \cup \Gamma_{\text{out}}} = 0 \), we have
\[
\|A_2^{\frac{3}{2}} \theta_t\|^2 + \frac{d}{dt} [\|A_2 \theta\|^2 - 2 \langle Q, A_2 \theta \rangle] \leq \|\nabla_3 [v \cdot \nabla \theta + w \theta_z]\|^2. \tag{4.7}
\]

Now, we estimate the right-hand side of (4.7). We have
\[
\|\nabla_3 (v \cdot \nabla \theta)\| \leq \|v \cdot \nabla \nabla_3 \theta\| + \|\nabla_3 v\| \|\nabla \theta\| \leq \|A_1 v\| \|A_2 \theta\| + \|\nabla_3 v\| \|\nabla \theta\|_6,
\]
\[
\|\nabla (w \theta_z)\| \leq \|w \nabla \theta_z\| + \|\theta_z \nabla w\| \leq \|w\|_{\infty} \|A_2 \theta\| + \|\nabla^2 v\| \|\theta_z\|_6,
\]
\[
\|\partial_z (w \theta_z)\| \leq \|w \theta_{zz}\| + \|\theta_z \nabla v\| \leq \|w\|_{\infty} \|A_2 \theta\| + \|\nabla v\| \|\theta_z\|_6.
\]
By interpolation inequality and Sobolev embedding, we have
\[\|\nabla^3 v\|_3 \leq \|v\|_{V_1}^{\frac{1}{2}} \|A_1 v\|^{\frac{1}{2}}, \quad \|\nabla^2 v\|_3 \leq \|A_1 v\|^{\frac{3}{2}} \|A_1^2 v\|^{\frac{1}{2}}, \quad \|\nabla \theta\|_6, \|\theta_z\|_6 \leq \|A_2 \theta\|.
\]

Using (1.3), we have
\[|w(x, y, z)| = \left| \int_{-h}^{z} \nabla \cdot v(x, y, \xi) \, d\xi \right| \leq \int_{-h}^{0} |\nabla v(x, y, \xi)| \, d\xi.
\]

By Agmon’s inequality,
\[|w(x, y, z)| \leq \int_{-h}^{0} \|\nabla v(\cdot, \cdot, \xi)\|_{(L^2(D))^2} \|\nabla v(\cdot, \cdot, \xi)\|_{(H^2(D))^2} \, d\xi \leq h^\frac{1}{2} \|v\|_{V_1} \|A_1^2 v\|^{\frac{1}{2}}.
\]

Thus,
\[\|w\|_\infty \leq h^\frac{1}{2} \|v\|_{V_1} \|A_1^2 v\|^{\frac{1}{2}}.
\]

Collecting all the above estimates after (4.7), we obtain
\[\|\nabla^3 (v \cdot \nabla \theta + w \theta_z)\|^2 \leq g_2(t) \|A_2 \theta\|^2,
\]

where
\[g_2(t) := \|v\|_{V_1} \|A_1 v\| + \|A_1 v\|^2 + h \|v\|_{V_1} \|A_1^2 v\| + \|A_1 v\| \|A_1^3 v\|.
\]

Then, (4.7) implies
\[\|A_2^2 \theta_t\|^2 + \frac{d}{dt} \|A_2 \theta - Q\|^2 \leq g_2(t) \|A_2 \theta - Q\|^2 + \|Q\|^2. \quad (4.8)
\]

By (4.5) and uniform boundedness of \(\|A_1 v\|\), we have
\[g_2 \in L^2(0, T), \quad \forall \ T \in (0, \infty).
\]

Thus, we conclude immediately from (4.8) by Gronwall lemma that
\[A_2 \theta \in L^\infty(0, T; H_2), \quad \theta_t \in L^2(0, T; V_2), \quad \forall \ T \in (0, \infty).
\]

Moreover, by (4.5), we see that \(\int_t^{t+1} g_2(s) \, ds\) is uniformly bounded for all \(t \geq 0\) and it is also obvious that
\[\int_t^{t+1} \|A_2 \theta(s) - Q\|^2 \, ds \leq \|Q\|^2 + \int_t^{t+1} \|A_2 \theta(s)\|^2 \, ds
\]
is uniformly bounded for all $t \geq 0$, since $(v, \theta)$ is a strong solution. Thus, an application of uniform Gronwall lemma to (4.8) yields an absorbing set for $\|A_2\theta\|$ in $\mathbb{R}_+$ and the uniform boundedness:

$$A_2\theta \in L^\infty(0, \infty; H_2).$$

$\square$

**Remark:** If $Q \in H^1(\Omega)$, then, it is easy to see that one can use the method in *Step 1* of the proof of Theorem 4.1 to estimate $\|A_2\theta\|$ and obtain the following additional regularity for $\theta$:

$$A_2\theta \in L^2(0, T; V_2), \ \forall \ T \in (0, \infty).$$

$\square$

With Theorem 4.1, we obtain the following Theorem 4.2. Notice that only $(u_0, \theta_0) \in D(A)$ is assumed here. Especially, $(u_t(0), \theta_t(0)) \in H$ is *not* assumed, nor is it assumed that the system of 3D primitive equations is valid at $t = 0$, as the later implies $(u_t(0), \theta_t(0)) \in H$ when $(u_0, \theta_0) \in D(A)$.

**Theorem 4.2** Suppose $Q \in L^2(\Omega)$ and $(v_0, \theta_0) \in D(A_1) \times D(A_2)$. Let $(v, \theta)$ be the unique strong solution of the problem (1.1)-(1.7). Then,

$$(v_t, \theta_t) \in L^\infty(0, \infty; H) \cap L^2(0, T; V), \ \forall \ T \in (0, \infty).$$

Moreover, there exists a bounded absorbing set of $\|(v_t, \theta_t)\|$ in $\mathbb{R}_+$.

**Proof:**

By (4.1) and Lemma 2.1, we have

$$\|v_t\| \leq \|A_1 v\| + \|(v \cdot \nabla)v\| + \|wv_z\| + \|
abla \theta\| + \|v\|$$

$$\leq \|A_1 v\| + \|v\|_6 \|\nabla v\|_3 + \|\nabla v\|_3 \|\nabla v\|_V^\frac{1}{2} \|v_z\|_V^\frac{3}{2} + \|\nabla \theta\| + \|v\|$$

$$\leq (1 + \|v\|_V) \|A_1 v\| + \|v\|_V^\frac{3}{2} \|A_1 v\|_V^\frac{1}{2} + \|\nabla \theta\| + \|v\|.$$  \hfill (4.9)

By (1.2), we have

$$\|\theta_t\| \leq \|A_2\theta\| + \|(v \cdot \nabla)\theta\| + \|w\theta_z\|$$

$$\leq \|A_2\theta\| + \|A_1 v\| \|\nabla \theta\| + \|A_1 v\| \|A_2\theta\|.$$  \hfill (4.10)
Thus, uniform boundedness of $\|(v_t, \theta_t)\|$ and existence of an absorbing ball of it in $\mathbb{R}_+$ follow from (4.9), (4.10) and Theorem 4.1.

Denote

$$u := v_t = \partial_t v, \quad \zeta := \theta_t = \partial_t \theta.$$  

By (1.1) and (1.2), we have

$$u_t + L_1 u + (u \cdot \nabla) v + (v \cdot \nabla) u + w_t v_z + w u_z$$

$$+ \nabla (p_s)_t + \int_0^z \nabla \zeta(x, y, \xi, t) d\xi + f u^\perp = 0,$$

(4.11)

$$\zeta_t + L_2 \zeta + u \cdot \nabla \theta + v \cdot \nabla \zeta + w_t \theta_z + w \zeta_z = 0.$$  

(4.12)

Taking the inner product of (4.9) with $u$ and using the boundary conditions (1.4) and (1.6), we obtain

$$\frac{1}{2} \frac{d}{dt} \|u\|^2 + \|u\|^2_{V_1}$$

$$= - \langle (u \cdot \nabla) v, u \rangle - \langle (v \cdot \nabla) u, u \rangle - \langle w_t v_z, u \rangle - \langle w u_z, u \rangle$$

$$- \langle \nabla (p_s)_t, u \rangle - \left\langle \int_z^0 \nabla \zeta, u \right\rangle - \left\langle f u^\perp, u \right\rangle$$

(4.13)

$$= - \langle (u \cdot \nabla) v, u \rangle - \langle w_t v_z, u \rangle + \left\langle \int_z^0 \nabla \zeta, u \right\rangle$$

where we have used the fact that

$$\langle \nabla (p_s)_t, u \rangle = \left\langle f u^\perp, u \right\rangle = \langle (v \cdot \nabla) u, u \rangle + \langle w u_z, u \rangle = 0.$$

Taking the inner product of (4.10) with $\zeta$ and using (1.6), we obtain

$$\frac{1}{2} \frac{d}{dt} \|\zeta\|^2 + \|\zeta\|^2_{V_2} + \mu \alpha \|\zeta(z = 0)\|^2$$

$$= - \langle u \cdot \nabla \theta, \zeta \rangle - \langle v \cdot \nabla \zeta, \zeta \rangle - \langle w_t \theta_z, \zeta \rangle - \langle w \zeta_z, \zeta \rangle$$

(4.14)

$$= - \langle u \cdot \nabla \theta, \zeta \rangle - \langle w_t \theta_z, \zeta \rangle$$

where we have used the fact that

$$\langle v \cdot \nabla \zeta, \zeta \rangle + \langle w \zeta_z, \zeta \rangle = 0.$$

Following the a priori estimates of [7] using (4.13) and (4.14), we obtain

$$\frac{d}{dt} \left( \|u\|^2 + \|\zeta\|^2 \right) + \|u\|^2_{V_1} + \|\zeta\|^2_{V_2} \leq g(t) \left( \|u\|^2 + \|\zeta\|^2 \right),$$

(4.15)
where
\[ g(t) := 1 + \|\nabla v\|^2 + \|\nabla \theta\|^4 + \|v_z\|^2\|\nabla v_z\|^2 + \|\theta_z\|^2\|\nabla \theta_z\|^2. \]

Thus, from (4.15), we obtain
\[ (v_t, \theta_t) \in L^2(0, T; V), \quad \forall \ T > 0. \]

This finishes proof of Theorem 4.2. \(\square\)

## 5 Uniform Boundedness for Higher Regularity

**Theorem 5.1** Suppose the integer \(m \geq 3\), \(Q \in H^{m-1}(\Omega)\) and \((v_0, \theta_0) \in V \cap (H^m(\Omega))^3\). Let \((v, \theta)\) be the unique strong solution of problem (1.1)-(1.7). Then,
\[ (v, \theta) \in L^\infty(0, \infty; (H^m(\Omega))^3), \quad (v, \theta) \in L^2(0, T; (H^{m+1}(\Omega))^3), \quad \forall \ T \in (0, \infty). \]

Moreover, there exists a bounded absorbing set of \(\|(v, \theta)\|_{(H^m(\Omega))^3}\) in \(R^+\).

**Proof:** The idea of the proof is to use induction and the method of the proof of Theorem 4.1. We give an outline of the proof here. To simplify the presentation without loss of generality, let us consider the following simplified equation:
\[ v_t + A_1 v = -\mathbb{P} [(v \cdot \nabla)v + vw_z]. \tag{5.1} \]

Applying \(A_1^m\) to (5.1) and taking inner product with \(v\), we get
\[ \frac{1}{2} \frac{d}{dt} \|\Lambda_1^m v\|^2 + \|A_1^{m+1} v\|^2 = - \langle \Lambda_1^{m-1} \mathbb{P} [(v \cdot \nabla)v + vw_z], A_1^{m+1} v \rangle \]
\[ \leq \frac{1}{2} \|\Lambda_1^{m-1} \mathbb{P} [(v \cdot \nabla)v + vw_z]\|^2 + \frac{1}{2} \|A_1^{m+1} v\|^2, \]

that is
\[ \frac{d}{dt} \|\Lambda_1^m v\|^2 + \|A_1^{m+1} v\|^2 \leq \|\Lambda_1^{m-1} \mathbb{P} [(v \cdot \nabla)v + vw_z]\|^2 \tag{5.2} \]

Now, we can use induction to prove the expected uniform boundedness of \(\|v\|_{(H^m(\Omega))^2}\) for \(m \geq 3\). We already have the result valid for \(m = 0, 1, 2\).
Assume it is already valid for \( \|v\|_{(H^k(\Omega))^2} \), with \( 0 \leq k \leq m - 1 \). Next, we show uniform boundedness of \( \|v\|_{(H^m(\Omega))^2} \). Using Theorem 3.3 as in the proof of Theorem 4.1, we have

\[
\|A_4^{m-1} P [(v \cdot \nabla)v + wv_z] \|^2 \lesssim \|[(v \cdot \nabla)v + wv_z] \|_{(H^{m-1}(\Omega))^2}^2.
\]

To estimate the right-hand side of the above inequality, we just need to estimate \( L^2 \) norm of

\[
\partial_z^{m-1} [(v \cdot \nabla)v + wv_z],
\]

since if these terms which contain the highest order derivatives can be well controlled, then those terms involving lower order derivatives can be treated exactly in the same way. To simplify our presentation, we just need to estimate \( \|\partial_z^{m-1} [(v \cdot \nabla)v]\| \), and \( \|\nabla^{m-1}(wv_z)\| \), since all other terms appearing in (5.3) can be treated similarly and behave no worse. By Leibniz rule, we have

\[
\|\partial_z^{m-1} [(v \cdot \nabla)v]\| \leq \sum_{k=0}^{m-1} \|\partial_z^k \partial_z \partial_z^{m-1-k} v\|.
\]

By Lemma 2.2,

\[
\|\partial_z^k \partial_z \partial_z^{m-1-k} v\| \lesssim \|v\|_{H^k} \|v\|_{H^{k+1}} \|v\|_{H^m-k} \|v\|_{H^m-k+1}.
\]

Hence, by induction hypothesis for uniform boundedness of \( \|v\|_{(H^{m-1})^2} \), we have

\[
\|\partial_z^{m-1} [(v \cdot \nabla)v]\| \lesssim \sum_{k=0}^{m-1} \|v\|_{H^k} \|v\|_{H^{k+1}} \|v\|_{H^m-k} \|v\|_{H^m-k+1}
\]

\[
\lesssim \left( \max_{0 \leq k \leq m-1} \|v\|_{H^k} \right) \sum_{k=0}^{m-1} \|v\|_{H^{k+1}} \|v\|_{H^m-k} \|v\|_{H^m-k+1}
\]

\[
\lesssim \sum_{k=0}^{m-1} \|v\|_{H^{k+1}} \|v\|_{H^m-k} \|v\|_{H^m-k+1}
\]

\[
\lesssim \|v\|_{H^m} \|v\|_{H^{m+1}} + \|v\|_{H^m}^2.
\]
Therefore,
\[
\|\partial_t^{m-1} [(v \cdot \nabla)v]\| \lesssim \|v\|_{H^m} \|v\|_{H^{m+1}} + \|v\|_{H^m}^2
\lesssim 1 + C\varepsilon \|v\|_{H^m}^2 + \varepsilon \|v\|_{H^{m+1}}^2.
\]

(5.4)

Similarly, we have
\[
\|\nabla^{m-1}(wv_z)\| \lesssim \sum_{k=0}^{m-1} \| (\nabla^k w) \otimes (\nabla^{m-1-k}v_z) \|,
\]

where we use \(\otimes\) to include all the products occurring here. By (1.3) and Lemma 2.1, we get
\[
\|\nabla^{m-1}(wv_z)\|^2 \lesssim \sum_{k=0}^{m-1} \|v\|_{H^{k+1}} \|v\|_{H^{k+2}} \|v\|_{H^{m-k}} \|v\|_{H^{m+1-k}}
\lesssim \|v\|_{H^m} \|v\|_{H^{m+1}} \leq C\varepsilon \|v\|_{H^m}^2 + \varepsilon \|v\|_{H^{m+1}}^2.
\]

(5.5)

Therefore, from (5.2) with consideration of (5.4) and (5.5), we have for \(m \geq 3\),
\[
\frac{d}{dt} \|\Lambda_1^m v\|^2 + \|\Lambda_1^{m+1} v\|^2 \lesssim 1 + \|v\|_{H^m}^2.
\]

(5.6)

Applying Gronwall lemma first and then uniform Gronwall lemma to (5.6) proves uniform boundedness of \(\|v\|_{H^m}\) for (5.1) and existence of a bounded absorbing set for \(\|v\|_{H^m}\) in \(\mathbb{R}_+\). The proof of the theorem for problem (1.1)-(1.7) is similar and is thus omitted.

\[\square\]

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