Cooperative loss and decoherence in quantum computation and communication

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Abstract

Cooperative effects in the loss (the amplitude damping) and decoherence (the phase damping) of the qubits (two-state quantum systems) due to the inevitable coupling to the same environment are investigated. It is found that the qubits undergo the dissipation coherently in this case. In particular, for a special kind of input states (called the coherence-preserving states), whose form depends on the type of the coupling, loss and decoherence in quantum memory are much reduced. Based on this phenomenon, a scheme by encoding the general input states of the qubits into the corresponding coherence-preserving states is proposed for reducing the cooperative loss and decoherence in quantum computation or communication.

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In recent years, quantum computation and communication have undergone a dramatic evolution [1]. New algorithms such as factoring [2,3] were developed and some individual quantum logic gates had been implemented in experiments [4,5]. Quantum computers act as sophisticated, nonlinear interferometers. The coherent interference pattern between the multitude of superpositions is essential for taking advantage of quantum parallelism. Unfortunately, decoherence of the qubits caused by the interaction with the environment will collapse the state of the quantum computer and make the result of the computation no longer correct [6]. To overcome this difficulty, Shor [7] has shown it is possible to restore a desired state using only partial knowledge of the state of the quantum computer. This scheme is called quantum-error correction, which operates in a subtle way, essentially by embedding the quantum information to be protected in a subspace so oriented in a larger state space as to leak no or little information to the environment. Many kinds of quantum-error correcting codes have since been discovered which correct for specific interactions [8-15].

In the existing quantum-error correction codes, it is generally assumed that the qubits decohere independently, which implies that the different qubits couple to separate environments. A natural question is, what will occur if the qubits couple cooperatively to the same environment? This situation may be more practical. For example, the intrinsic decoherence in the ion trap quantum computers just results from the cooperative coupling of the ions to the environment [16]. If one only consider the phase damping, it has been shown [17] that the cooperative dissipation results in the coherent decoherence. And more interestingly, for a special kind of input states, i.e., the coherence-preserving states, the qubits undergo no decoherence at all even in the noisy memory. In this letter, we consider
the general dissipation, including the amplitude damping (loss) and the phase damping (decoherence). The decoherence time is obtained for the qubits. Cooperative effects in the loss and decoherence are examined. Interestingly, there still exist the coherence-preserving states, whose form depends on the type of the coupling between the qubits and the environment. For these states, decoherence of the qubits is much reduced. Based on this phenomenon, we furthermore propose a scheme for reducing the cooperative loss and decoherence in quantum computation or communication. The scheme operates by encoding the general input states of the qubits into the corresponding coherence-preserving states in a slightly larger Hilbert space. The cost of encoding $L$ qubits varies from $2L$ qubits to a function that approaches $L$ qubits asymptotically as $L$ grows. So this scheme is very efficient.

The qubits in the memory, which may be spin-$\frac{1}{2}$ electrons or two-level atoms, can be described by Pauli’s operators $\vec{\sigma}_l$ ($l$ marks different qubits). The environment is modeled by a bath of oscillators with infinite degrees of freedom. The general dissipation of the qubits, including the phase damping and the amplitude damping, is described by the following coupling Hamiltonian

$$H = \hbar \left\{ \omega_0 \sum_{l=1}^{L} \sigma^z_l + \sum_{l=1}^{L} \sum_{\omega} \left[ \left( \lambda^{(1)}_{ol} \sigma^+_{l} \sigma^-_{l} + \lambda^{(2)}_{ol} \sigma^y_{l} \sigma^y_{l} + \lambda^{(3)}_{ol} \sigma^z_{l} \right) \left( a^+_{\omega} + a_{\omega} \right) \right] + \sum_{\omega} \omega a^+_{\omega} a_{\omega} \right\},$$

where $L$ is the number of qubits. $a_{\omega}$ indicates the bath operator. The coupling constants $\lambda^{(1)}, \lambda^{(2)}, \lambda^{(3)}$ may be independent of $\omega$ and $l$. This coupling system is nonlinear and very complicated, which makes it impossible to find its exact solutions. However, in quantum computation or communication, we mainly take interest in the decoherence time. This time gives a mark after which the state of the qubits is obviously collapsed. To obtain the decoherence time, we only need study the short-time behavior of the coupling system. Very recently, Kim, etc.,
[18] propose a short-time perturbative scheme for studying coherence loss. Here we follow this method. The reduced density of the qubits is indicated by \( \rho_1(t) \). A simple and direct measure of the degree of decoherence of the qubits is thus provided by the "idempotency defect" of the density \( \rho_1(t) \) [19], which is written in the equation below, where it is furthermore subjected to a short-time power series expansion:

\[
\delta(t) = \text{tr} [\rho_1(t) - \rho_1^2(t)]
\]

(2)

where \( \delta_0, \tau_1, \tau_2 \) can be expressed by \( \rho(0) \) (the initial density of the whole system), \( H \) (the Hamiltonian) and their commutators. Since the qubits and the environment are not entangled at the beginning and usually the input states of the qubits are pure, \( \rho(0) \) can be factorized as \( \rho(0) = |\Psi_1(0)\rangle \langle \Psi_1(0)| \otimes \rho_2(0) \), where \( \rho_2(0) \) indicates the reduced density of the environment, which is generally in a mixed state. Under this condition, \( \delta_0 = \frac{1}{\tau_1} = 0 \) and \( \tau_2 \), which marks the decoherence time, is expressed as

\[
\frac{\hbar^2}{2\tau_2^2} = \langle H^2 \rangle_{1,2} + \langle H \rangle_{1,2}^2 - \langle \langle H \rangle_{1,2}^2 \rangle_1 - \langle \langle H \rangle_{1,2} \rangle_2
\]

(3)

where \( \langle H \rangle_{1(2)} \) stands for the average of the Hamiltonian over the subsystem 1(2), i.e., \( \langle H \rangle_1 = \langle \Psi_1(0) | H | \Psi_1(0) \rangle \), \( \langle H \rangle_2 = \text{tr} (\rho_2(0) H) \), \( \langle H \rangle_{1,2} = \langle \Psi_1(0) | \langle H \rangle_2 | \Psi_1(0) \rangle \).

Only with pure input states for the qubits, the decoherence time can be simplified to Eq.(3).

Now, consider the Hamiltonian (1), which can be rewritten as

\[
H = H_1 + H_2 + \sum_{l=1}^{L} \sum_{\mu=1}^{3} H_1^{(\mu)} H_2^{(\mu)}
\]

(4)

where \( H_1 = \hbar \omega_0 \sum_{l=1}^{L} \sigma_l^x \), \( H_2 = \sum_{\omega} \hbar \omega a_\omega^+ a_\omega \), \( H_1^{(1,2,3)} = \sigma_l^{(x,y,z)} \), \( H_2^{(\mu)} = \sum_{\omega} \hbar \lambda_{\omega l}^{(\mu)} (a_\omega^+ + a_\omega) \).
The environment is supposed initially in thermal equilibrium, i.e.,

$$\rho_2(0) = \prod_\omega \int d^2\alpha_\omega \frac{1}{\pi \langle N_\omega \rangle} \exp \left( -\frac{|\alpha_\omega|^2}{\pi \langle N_\omega \rangle} \right) |\alpha_\omega\rangle \langle \alpha_\omega|$$

(5)

with the mean photon number

$$\langle N_\omega \rangle = \frac{1}{\exp \left( \frac{\hbar \omega}{k_B T} \right) - 1}.$$  

(6)

With the density (5) of the environment, we obviously have $$\langle H(\mu) \rangle^2_i = \langle H_2(\mu) \rangle^2_i = 0.$$ Under this condition, Eq. (3) gives

$$\frac{\hbar^2}{2\tau_2^2} = \sum_{1 \leq i,j \leq L} \sum_{1 \leq \mu,\nu \leq 3} \langle H_{2i}^{(\mu)} H_{2j}^{(\nu)} \rangle \left( \langle H_{1i}^{(\mu)} H_{1j}^{(\nu)} \rangle - \langle H_{1i}^{(\mu)} \rangle \langle H_{1j}^{(\nu)} \rangle \right).$$

(7)

In practice, the coupling constants $$\lambda_{\omega l}^{(\mu)} (\mu = 1, 2, 3)$$ often factor as $$\lambda_{\omega l}^{(\mu)} = \lambda_{l}^{(\mu)} \kappa(\omega).$$ Then Eq. (7) can be further simplified. To show this, let

$$\Omega^2 = 2 \left\langle \left[ \sum_\omega \kappa(\omega) \left( a_\omega^+ + a_\omega \right) \right]^2 \right\rangle = 4 \int d\omega \kappa^2(\omega) \left( \langle N_\omega \rangle + \frac{1}{2} \right)$$

(8)

and define

$$A = \sum_{l,\mu} \lambda_l^{(\mu)} H_{1l}^{(\mu)} = \sum_{l=1}^L \left( \lambda_l^{(1)} \sigma_l^x + \lambda_l^{(2)} \sigma_l^y + \lambda_l^{(3)} \sigma_l^z \right),$$

(9)

the decoherence time $$\tau_2$$ thus becomes

$$\frac{1}{\tau_2^2} = \Omega^2 \langle (\Delta A)^2 \rangle.$$ 

(10)

Eq.(10) suggests that the qubits decohere coherently when they couple cooperatively to the same environment. This fact can be more clearly seen by comparison with the decoherence in the case that the qubits interact independently with separate environments. Very simple calculation yields the decoherence time in the latter situation

$$\frac{1}{\tau_2^2} = \Omega^2 \sum_{l=1}^L \langle (\Delta A_l)^2 \rangle,$$

(11)
where $A_l = \lambda^{(1)}_l \sigma_x^{l} + \lambda^{(2)}_l \sigma_y^{l} + \lambda^{(3)}_l \sigma_z^{l}$. The decoherence rate $\frac{1}{\tau'_2}$ increases with $L$ monotonically. Its typical behavior is $\frac{1}{\tau'_2} \propto L$. This decoherence is insensitive to the input states of the qubits. In contrast, for the cooperative dissipation of the qubits, the decoherence rate $\frac{1}{\tau'_2}$ depends greatly on the type of the input states. For some input states, $\frac{1}{\tau'_2}$ increases with $L$ very rapidly, whereas for some other input states, the decoherence rate does not increase with $L$ at all. In particular, for the eigenstates of the operator $A$, $\frac{1}{\tau'_2} = 0$, which suggests, for this kind of input states the decoherence is much reduced.

We briefly discuss the eigenstates of the operator $A$, which may be called the coherence-preserving states. The case that the coupling constants $\lambda^{(\mu)}_l$ are equal for different qubits $l$ is of special interest. Then $\lambda^{(\mu)}_l$ is just indicated by $\lambda^{(\mu)}$. The Hermitian operators $A_l$ satisfy $tr(A_l) = 0$, so their eigenvalues are $\pm a$, where $a$ is a real number. Without loss of generality, their corresponding eigenstates can be indicated by $|\pm 1\rangle_l$. For example, if there is only the phase damping, $\lambda^{(1)} = \lambda^{(2)} = 0$, so $|\pm 1\rangle_l = |\pm \rangle_l$, where $|\pm \rangle_l$ stand for the eigenvectors of $\sigma_z^l$. On the other hand, if there is only the amplitude damping, i.e., $\lambda^{(2)} = \lambda^{(3)} = 0$, $|\pm 1\rangle_l = \frac{1}{\sqrt{2}} (|+\rangle_l \pm |-\rangle_l)$, which are the eigenvectors of $\sigma_x^l$. The eigenstates of the operator $A$ can be easily constructed from the states $|\pm 1\rangle_l$. They are

$$|\Psi_{2L}\rangle_{m} = \sum_{\{i_1,i_2,\ldots,i_{2L}\} \atop {\sum_{i=1}^{2L} i = m}} c_{\{i\}} |\{i\}\rangle,$$

(12)

where the constants $m$ stand for the eigenvalues of the operator $A$. The state $|\{i\}\rangle$ indicates $|i_1\rangle \otimes |i_2\rangle \otimes \cdots \otimes |i_{2L}\rangle$ and $i_l = \pm 1$. Here we have supposed there are $2L$ qubits. With $m = 0, \pm 2, \pm 4, \cdots, \pm 2L$, the dimensions of the state spaces expanded by the vectors $|\Psi_{2L}\rangle_{0}, |\Psi_{2L}\rangle_{\pm 2}, \cdots, |\Psi_{2L}\rangle_{\pm 2L}$ are respectively
The sum of these dimensions satisfies
\[
\binom{2L}{L}, 2 \binom{2L}{L-1}, \cdots, 2 \binom{2L}{0}.
\]
The sum of these dimensions satisfies
\[
\binom{2L}{L} + 2 \binom{2L}{L-1} + \cdots + 2 \binom{2L}{0} = 2^{2L}.
\]
So all the eigenstates of the operator \( A \) make a complete basis for the Hilbert space of \( 2L \) qubits.

Now the question is how to exploit these coherence-preserving states to reduce the cooperative decoherence in quantum memory. In the following we show this may be achieved by encoding arbitrary input states of the qubits into the corresponding coherence-preserving states in a larger state space. The input states of \( L \) qubits can be generally expressed as
\[
|\Psi_L\rangle = \sum_{\{i_l\}} c_{\{i_l\}} |\{i_l\}\rangle, \quad (13)
\]
where \(|\{i_l\}\rangle\) is just the abbreviation of \(|i_1, i_2, \cdots, i_L\rangle\). If only one qubit with the state \( c_1 |1\rangle + c_{-1} |-1\rangle \) is input, we encode the input state into the state \( c_1 |1, -1\rangle + c_{-1} |-1, 1\rangle \) of two qubits. Obviously, the latter is a coherence-preserving state. Similarly, there is one-to-one correspondence between the input states of \( L \) qubits and the following coherence-preserving states in the Hilbert space of \( 2L \) qubits
\[
|\Psi_{2L}\rangle_{coh} = \sum_{\{i_l\}} c_{\{i_l\}} |\{i_l, -i_l\}\rangle, \quad (14)
\]
where \(|\{i_l, -i_l\}\rangle\) represents \(|i_1, -i_1, i_2, -i_2, \cdots, i_L, -i_L\rangle\). We encode the input states (13) into the corresponding states in the form of Eq. (14) before storing them into the memory. The encoded states undergo reduced decoherence in the noisy memory and then they can be decoded into the original states. By this scheme the cooperative decoherence is much reduced, especially when the storing time is short.

The encoding and decoding in the above scheme can be easily realized in quantum computers by the elementary logic gates, the quantum controlled-NOT.
Some applications of this sort of logic gates have been commented in Ref. [20] with stress on the appearance of a conditional quantum dynamics. The quantum controlled-NOT gate is defined as that which effects the unitary operation on two qubits, which in a chosen orthonormal basis \{\ket{-}, \ket{1}\} reproduces the classical controlled-NOT operation

\[
\ket{\varepsilon_1}_1 \ket{\varepsilon_2}_2 \xrightarrow{C_{12}} \ket{-\varepsilon_1}_1 \ket{-\varepsilon_1 \cdot \varepsilon_2}_2,
\]  

(15)

Here and in the following the first subscript of \( C_{ij} \) refers to the control bit and the second to the target bit. To encode \( L \) qubits in the state (13), \( L \) ancillary qubits \( 1', 2', \ldots, L' \) need be prearranged in the state \( |\Psi_{1'2'\cdots L'}\rangle = |1\rangle_{1'} \otimes |1\rangle_{2'} \otimes \cdots \otimes |1\rangle_{L'} \). From the definition (15), the state (13) is transformed into the coherence-preserving state (14) of the \( 2L \) qubits \( 1, 1', 2, 2', \ldots, L, L' \) by \( L \) times controlled-NOT operations

\[
|\Psi_{12\cdots L}\rangle \otimes |\Psi_{1'2'\cdots L'}\rangle \xrightarrow{\bigotimes_{l=1}^{L} C_{11}' C_{22}' \cdots C_{LL}'} \sum_{\{i_l\}} c_{\{i_l\}} \ket{\{i_l, -i_l\}} \rangle = |\Psi_{11'22'\cdots LL'}\rangle_{coh}. 
\]  

(16)

This transformation can be reversed by applying the same controlled-NOT operations again. So the decoding is fulfilled by

\[
|\Psi_{11'22'\cdots LL'}\rangle_{coh} \xrightarrow{\bigotimes_{l=1}^{L} C_{11}' C_{22}' \cdots C_{LL}'} |\Psi_{12\cdots L}\rangle \otimes |\Psi_{1'2'\cdots L'}\rangle. 
\]  

(17)

The above encoding scheme is very simple but not efficient, since only the coherence-preserving states in the form of Eq. (14) are used. Suppose there are \( 2L \) qubits. The maximum dimension of the eigenspace of the operator \( A = \sum_{l=1}^{2L} A_l \) is \( \binom{2L}{L} \), with the eigenvalue \( m = 0 \). If all the coherence-preserving states in this eigenspace are fully used, the efficiency \( \eta \) of the encoding attains the maximum, which is

\[
\eta_M = \log_2 \left( \frac{2L}{L} \right) \approx 1 - \frac{1}{4L} \log_2 (\pi L). 
\]  

(18)
In Eq. (18) the approximation $L >> 1$ is introduced and the stirling formula $L! \approx \sqrt{2\pi L^{L+\frac{1}{2}}} e^{-L}$ is used. As $L$ grows, the efficiency $\eta_M$ tends to 1. So in the perfect encoding scheme, the input states can be transformed into the coherence-preserving states almost without expanding of the number of qubits. Of course, to raise the efficiency, the encoding scheme will correspondingly become complicated and involve much more logic gates.

It is interesting to compare this strategy with the quantum-error correction schemes. In the error-correction schemes, the decoherence time for a qubit is not increased. What one does is to repeatedly restore the original state of the qubits from the decohered encoded state by unitary transformations and measurements on some ancillary qubits. The quantum state should be restored at time intervals much less than the decoherence time, and to make the restoration possible, one needs at least $5L$ qubits to encode $L$ qubits [12,15]. On the other hand, in the present scheme what we do is to increase the decoherence time. We deal with the case that the qubits decohere cooperatively. The cooperative dissipation of the qubits results in the coherence-preserving states. By encoding the input states into the corresponding coherence-preserving states, the decoherence time for a qubit is increased. The cost of encoding $L$ qubits varies from $2L$ qubits to $L + \frac{1}{2} \log_2 \left( \frac{\pi}{2} L \right)$ qubits (from Eq. (18)). So by this scheme the decoherence is reduced at little cost.

In this letter, the decoherence time is obtained by short-time expansion. If the coherence-preserving states are input, the second-order decoherence rate equals zero. What about the higher order contributions? It has been shown that if there is only the phase damping, all the higher order contributions disappear at the same time [17]. So in this case the cooperative decoherence can be eliminated by encoding the input states into the coherence-preserving states. But in
general cases, especially when the amplitude damping is dominant, the higher order contributions do not equal zero. Therefore, by this scheme the decoherence is reduced but can not be eliminated. This is the main disadvantage of the scheme. However, this shortcoming can be overcome by combining the scheme with the quantum-error correction. In quantum-error correction, if the unitary transformations and the measurements are perfect, the error rate can be made arbitrarily small by repeatedly restoring the quantum state. But in practice, each time one gets rid of the decoherence, he introduces some extra error. So there is also a small amount of error which is hard to be eliminated by the error correction schemes. However, if we combine these two schemes together, it is possible to further reduce the error rate, and also, the efficiency of the encoding may be raised. So it is of interest to find a scheme for correcting quantum-error caused by the cooperative decoherence. This question needs further investigation, since the existing quantum-error correction schemes are devoted to reducing the independent decoherence.

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