Optical probing in a bilayer dark-bright condensate system

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We consider a bilayer system of two-dimensional (2D) Bose-Einstein-condensed dipolar dark excitons (upper layer) and bright ones (bottom layer). We demonstrate that the interlayer interaction leads to a mixing between excitations from different layers. This mixing leads to the appearance of a second spectral branch in the spectrum of bright condensate. The excitation spectrum of the condensate of dark dipolar excitons then becomes optically accessible during luminescence spectra measurements of the bright condensate, which allow probing its kinetic properties. This approach is relevant for experimental setups, where detection via conventional techniques remains challenging, in particular, the suggested method is useful for studying dark dipolar excitons in transition metal dichalcogenide (TMDC) monolayers.

I. INTRODUCTION

Achieving full control over quantum many-body systems is of significant importance both in fundamental science and possible applications [1]. Remarkable progress in experiments with atomic and molecular gases in degenerate regimes makes them a perfect playground for revealing novel phases and many-body states of ultracold matter [2–4] and also realizing quantum technologies [4–6]. Another promising platforms in this context are many-body systems of quasiparticles, such as excitons in solid-state materials [7–10]. Creating excitons with spatially separated electrons and holes increases their lifetime significantly [9–14], which is quite favorable for investigating their collective properties at sufficiently high temperatures [15–20]. Coupled quantum wells separated by a barrier [16, 17, 21] and single quantum wells in electric fields [22–24] with long-lived two-dimensional (2D) excitons have been intensively studied experimentally (see Ref. [14]). Recently, significant attention has been paid to excitons in graphene monolayers separated by an insulating barrier [25–28], thin films of topological insulators [29–34], and TMDC monolayers [35–40].

In the systems mentioned above, excitons can be either optically bright or dark. Properties of bright-exciton condensates, such as condensate density and coherence time, can be directly obtained from emission pattern (luminescence) measurements [16], whereas the detection of dark excitons with existing optical techniques remains challenging. Understanding properties of dark excitons is of fundamental importance for a wide variety of processes in semiconductors. This is also crucial for potential applications such as light harvesting [41–43] and quantum-state engineering [44]. Recent proposals for the detection of dark excitons include using near-field coupling to surface plasmon polaritons [45] and monitoring their interactions with a polariton mode [46].

In this paper, we study a system of 2D Bose-Einstein-condensed dipolar dark (upper layer) and bright (bottom layer) excitons in a bilayer geometry where the excitons are oriented perpendicularly to the layers (see Fig. 1). We demonstrate that this setup offers a possibility of direct probing kinetic properties of the dark-exciton condensate via luminescence spectra measurements of the bright condensate. This is feasible since the interlayer interaction leads to the appearance of a second spectral branch in the spectrum of bright condensate. This allows measuring excitation spectrum and kinetic properties of the dark condensate with the use of conventional luminescence spectra measurements. Recent experiments on studying excitons in TMDC monolayers [35–40, 45] could be accomplished by this method. We also note that bilayer systems of dipolar particles have been widely studied in the context of ultracold quantum gases [47–54], including interesting findings on interlayer superfluidity [47, 54].

The paper is organized as follows. In Sec. II, we describe the system of bilayer dark-bright condensates. In
Sec. III, we reveal the impact of the interlayer interaction between excitons and calculate the excitation spectrum. In Sec. IV, we discuss experimental conditions for optical probing the excitation spectrum of the condensate of dark dipolar excitons. We give our conclusion in Sec. V.

II. BILAYER DARK-BRIGHT CONDENSATE SYSTEM

We study 2D system of Bose-Einstein-condensed dipolar bright and dark excitons in a bilayer geometry with all dipoles oriented in the same direction. The Hamiltonian, $\hat{H} = \hat{H}_b + \hat{H}_d + \hat{H}_{\text{int}}$, includes terms of the following form:

$$\hat{H}_{b(d)} = \int \hat{\psi}^+_b(r) \left( -\frac{\hbar^2}{2m_{b(d)}} \Delta - \mu_{b(d)} \right) \hat{\psi}_b(r) dr + \int \hat{\psi}^+_b(r) \hat{\psi}^+_d(s) U_{b(d)}(r-r') \psi_b(s) \psi_d(r) \frac{dr dr'}{2},$$

Here $\hat{\psi}(r)$ is the exciton bosonic field operator, $m$ is the exciton mass, the subindex ‘$b$’ indicates on the bright-exciton condensate and ‘$d$’ on the dark-exciton condensate, $r = \{x, y\}$ is the 2D position vector in the layer. We consider excitons as bosons since the overlap integral of exciton wavefunctions is exponentially small (see Ref. [55]). We also note that chemical potentials $\mu_{b(d)}$ may not be equal to each other since we assume no interlayer hopping. For bright layers (GaAs) we use Coulomb-like interaction potential. The calculation of the in-layer interaction potential for MoS$_2$ layers is different from the standard treatment due to a specific response in mono-layers (see Appendix A and Appendix B).

In the dilute regime, in the first Born approximation for both bright and dark layers the in-layer interaction is as follows:

$$U_{b(d)}(k) - g_{b(d)} = U^0_{b(d)}(k) - U^0_{b(d)}(0), \quad (2)$$

where $g_{b(d)}$ is the coupling constant of excitons in the bright (dark) layer, and $k$ is the 2D momentum. However, we cannot use $g_{b(d)} = \int U^0_{b(d)}(r) dr$ explicitly because of the divergency of the interaction potential of rigid dipoles at $r \to 0$. Therefore, for the weakly correlated system, we dress the bare interaction by the ladder diagrams [56] (see also Fig. 2a). The quantitative approach for calculating $g_{b(d)}$ is the same as in Ref. [57]. To distinguish dressed and bare interactions we use “0” superscript below.

Assuming no interlayer hopping, the interaction part of the Hamiltonian is as follows:

$$\hat{H}_{\text{int}} = \int \hat{\psi}^+_b(r_b) \hat{\psi}_b(r_b)V(r_b-r_d)\hat{\psi}^+_d(r_d)\hat{\psi}_d(r_d) dr_b dr_d. \quad (3)$$

The interlayer interaction may be still taken in the Coulomb form since the interlayer separation $L$ is much larger than the effective screening length in MoS$_2$ monolayer:

$$V(R) = \frac{\epsilon^2}{\varepsilon} \left( -\frac{1}{\sqrt{R^2 + L^2}} - \frac{1}{\sqrt{R^2 + (L + D_b + D_d)^2}} + \frac{1}{\sqrt{R^2 + (L + D_b)^2}} + \frac{1}{\sqrt{R^2 + (L + D_d)^2}} \right), \quad (4)$$

where $L$ is the interlayer spacing, $D_{b(d)}$ is the effective electron-hole separation in bright(dark) layer, $e$ is the electron charge, and $\epsilon$ is the interlayer dielectric constant. In experimentally relevant situations, it is assumed to be equal to the dielectric constant in a bright layer (GaAs). This realizes since the dark layer (TMDC) is very thin and lies on a thicker structure of GaAs quantum well.

In the first Born approximation the interaction potential (4) satisfies the relation $h = \int V(R) dR = 0$, which complicates the ordinary method of finding a bound state in 2D potentials finite at the origin [58, 59]. If the bound state energy is significant then one can expect the formation of interlayer exciton biexcitons (dimers). How-
ever, in the considered experimental setup, we expect that the bound state energy is small so the biexciton (dimer) physics is still not important. We also note that the effects of the formation of bound states in bilayer dipolar systems have been studied [11, 47, 49, 60].

III. EXCITATION SPECTRA

The key of our work is to reveal the impact of the interlayer interaction in the luminescence spectrum of the optically accessible bright condensate. Assuming the system to be homogeneous, it can be expressed in the following form (see, e.g., Ref. [61]):

\[ I_b(\varphi, \theta, \omega_{\text{phot}}) \propto \frac{E_g}{2\pi \tau} \hat{G}_k^+(\omega). \] (5)

In the dilute regime for a BEC system normal \( G_k(\omega) \) and anomalous \( F_k(\omega) \) Green’s functions can be obtained from the standard Belyaev system [62], which can be presented as follows:

\[
\begin{align*}
G_k(\omega) &= G_k^0(\omega) + G_k^0(\omega)\Sigma_{k}^{11}(\omega)G_k(\omega) + G_k^0(\omega)\Sigma_{k}^{00}(\omega)F_k(\omega), \\
F_k(\omega) &= G_k^0(-\omega)\Sigma_{k}^{11}(\omega)F_k(\omega) + G_k^0(-\omega)\Sigma_{k}^{00}(\omega)G_k(\omega).
\end{align*}
\] (7)

Using the explicit expression for Green’s function of free boson \( G_k^{(0)}(\omega) = \hbar / (\hbar \omega - T_k + \mu + i\delta) \), where \( T_k = \hbar^2 \kappa^2 / (2m) \), we express the formal solution of the Belyaev system (7) in the following form:

\[
\begin{align*}
G_k(\omega) &= \frac{\hbar \omega + T_k + S_k(\omega) - A_k(\omega) - \mu}{(\hbar \omega - A_k(\omega))^2 - (T_k + S_k(\omega) - \mu)^2 + \Sigma_{k}^{00}(\omega)^2}, \\
F_k(\omega) &= -\frac{\hbar (\hbar \omega - A_k(\omega))^2 - (T_k + S_k(\omega) - \mu)^2 + \Sigma_{k}^{00}(\omega)\Sigma_{k}^{02}(\omega)}{\Sigma_{k}^{00}(\omega)^2},
\end{align*}
\] (8)

where \( A_k(\omega) = \Sigma_{k}^{11}(\omega) - \Sigma_{k}^{11}(-\omega)/2 \) and \( S_k(\omega) = (\Sigma_{k}^{11}(\omega) + \Sigma_{k}^{11}(-\omega))/2 \). Corresponding diagrams are presented in Fig. 3a. By comparing diagrams in Fig. 3a and Eq. (7), we find explicit graphical expressions for self-energy terms, which are presented in the diagrammatic form in Fig. 3b.

For deriving analytical expressions for self-energy terms one need to obtain the dynamical structural factor (see Fig 2d), which can be expressed using Green’s functions \( \tilde{G}_k(\omega) \) and \( \tilde{F}_k(\omega) \) (see Fig 2e and Fig 2f). Calculating normal \( G_k(\omega) \) and anomalous \( F_k(\omega) \) Green’s functions for the dark layer also requires the use of the
Here the following notations are introduced:

\[ \tilde{G}_k(\omega) = \frac{\tilde{G}_k^{(0)}(\omega)}{\omega - \epsilon_k^0 + i\delta} + \frac{\tilde{G}_k^{(0)}(\omega)}{\omega + \epsilon_k^0 - i\delta}, \]

\[ \tilde{F}_k(\omega) = \frac{\tilde{F}_k^{(0)}(\omega)}{\omega - \epsilon_k^0 + i\delta} + \frac{\tilde{F}_k^{(0)}(\omega)}{\omega + \epsilon_k^0 - i\delta}, \]

where we use the following notations: \( U_k = n_0 U(k), \tilde{U}_k = \tilde{n}_0 U(k), V_k = n_0 V(k), \tilde{V}_k = \tilde{n}_0 V(k), \) where \( V(k) = \int V(r) \exp(-ikr) \text{d}r, \) \( n_0 \) is condensate density and \( \tilde{\mu} = g\tilde{n}_0 + h\tilde{n}_0 \) is the chemical potential of the dark layer.

By solving these equations and substituting \( \tilde{\alpha}_k \) into the expression for the structural factor (Fig 2d) we obtain the solution of the Belyaev system (7) in the following form:

\[ G_k(\omega) = \frac{\tilde{G}_k^{(0)}(\omega)}{\omega - \epsilon_k^0 + i\delta} + \frac{\tilde{G}_k^{(0)}(\omega)}{\omega + \epsilon_k^0 - i\delta}, \]

\[ F_k(\omega) = \frac{\tilde{F}_k^{(0)}(\omega)}{\omega - \epsilon_k^0 + i\delta} + \frac{\tilde{F}_k^{(0)}(\omega)}{\omega + \epsilon_k^0 - i\delta}, \]

with the following notation for the self-energy:

\[ \Sigma_k(\omega) = U_k + \frac{2V_k \tilde{V}_k^2}{\omega^2 - (\tilde{T}_k + \Sigma_k(\omega))^2 + \Sigma_k(\omega)^2}, \]

By expanding these expressions in the form of simple fractions, we obtain:

\[ F_k(\omega) = -\frac{\hbar^2}{\omega - \epsilon_k^0 + i\delta} + \frac{\hbar^2}{\omega - \epsilon_k^0 - i\delta}, \]

\[ G_k(\omega) = \frac{\hbar^2}{\omega - \epsilon_k^0 + i\delta} - \frac{\hbar^2}{\omega - \epsilon_k^0 - i\delta}, \]

Here the following notations are introduced:

\[ u_k^{\pm} = (U_k + T_k + \epsilon_k^0)((\epsilon_k^0)^2 - \epsilon_k^0) - 2\tilde{T}_k \tilde{V}_k V_k \]

\[ v_k^{\pm} = (U_k + T_k - \epsilon_k^0)((\epsilon_k^0)^2 - \epsilon_k^0) - 2\tilde{T}_k \tilde{V}_k V_k, \]

\[ u_k^{\pm}v_k^{\pm} = \frac{U_k((\epsilon_k^0)^2 - \epsilon_k^0) - 2\tilde{T}_k \tilde{V}_k V_k}{2(\epsilon_k^0)^2} \]

where \( \epsilon_k^0 = \frac{(\epsilon_k^0)^2 - \epsilon_k^0}{2} \), 0, and \( \epsilon_k^0 > 0 \) (for details, see Appendix D).

The obtained Green’s functions given by Eqs. (12)-(13) correspond exactly to the the originally considered bilayer system with the following diagonalized Hamiltonian:

\[ \hat{\mathcal{H}} = \sum_{k \neq 0} \tilde{\alpha}_k \hat{\alpha}_k + \epsilon_k^0 \tilde{\beta}_k \hat{\beta}_k. \]

Here the new annihilation operators may be expressed in terms of exciton annihilation operators \( \hat{a}_k \) and \( \hat{b}_k \) for bright and light layer correspondingly, from the following equations:

\[ \hat{a}_k = \frac{\epsilon_k^0 - \epsilon_k^0}{\epsilon_k^0 + \epsilon_k^0} - \frac{\epsilon_k^0 - \epsilon_k^0}{\epsilon_k^0 + \epsilon_k^0}, \]

\[ \hat{b}_k = \frac{\epsilon_k^0 - \epsilon_k^0}{\epsilon_k^0 + \epsilon_k^0} - \frac{\epsilon_k^0 - \epsilon_k^0}{\epsilon_k^0 + \epsilon_k^0}. \]

To obtain expressions for \( \tilde{\alpha}_k, \tilde{\beta}_k \), we use Eqs. (14)-(15), swapping all tilde expressions with the ones without tilde. By substituting these operators in the expression for the Keldysh correlation function (6), we obtain the following result:

\[ iG_k(\omega) = 2\pi \hbar (\epsilon_k^0 + \epsilon_k^0) + \epsilon_k^0 - \epsilon_k^0, \]

where \( n_k = u_k^{\pm} + v_k^{\pm} \).

By analyzing expressions (15), (17), and (21), we conclude that in the case of the absence of the interlayer interaction one has two spectral branches, but only one branch has non-zero population, i.e. it is optically accessible. Taking into account interlayer interaction leads to the significantly large occupation of second branch in certain momentum regions. Therefore, by inclusion of the interlayer interaction the excitation spectrum of the dark condensate becomes optically accessible during luminescence spectra measurements of the bright condensate, which allows probing its kinetic properties. Fig. 4 clearly demonstrates two areas in which the effect of mixing excitations leads to a noticeable population of both branches near the certain momentum region. The latter would indicate the existence of a large-scale coherence in both layers. We note that due to the in-plane momentum conservation the value \( k \) is bounded by \( hE_0/c_0 \) (the radiation zone in free space).
We rely that the suggested approach for optical probing in a bilayer dark-bright condensate system is relevant for studying excitons in TMDC monolayers. In particular, we consider a bilayer structure with bright excitons in a GaAs/AlGaAs/GaAs layer and dark excitons in a MoS$_2$/hBN/MoS$_2$ structure. For numerical estimation we use parameters from Table I and assume interlayer separation $L = 15$ nm.

$$n_0, 10^{10} \text{cm}^{-2} \quad m/m_0 \quad m_r/m_0 \quad D, \text{nm} \quad \epsilon \quad \alpha_{2D}, A$$

|            | TMDC     | GaAs    |
|------------|----------|---------|
| $n_0$, $10^{10} \text{cm}^{-2}$ | 15       | 1       |
| $m/m_0$    | 0.25     | 0.22    |
| $m_r/m_0$  | 1.333    | 0.0467  |
| $D, \text{nm}$ | 8.7      | 12      |
| $\epsilon$ | 6.60     | 12.5    |
| $\alpha_{2D}$ | 6.60     | –       |

Table I. Here $n_0$ is the condensate density, $m/m_0$ and $m_r/m_0$ are ratios of exciton mass and reduced mass to free electron mass, $D$ is the effective electron-hole separation within the single layer, $\epsilon$ is the dielectric constant of surrounding medium (see Appendix B). Interlayer dielectric constant is assumed equal to the one in GaAs (see text). $\alpha_{2D}$ is 2D polarizability of MoS$_2$ monolayer [64].

We would like to note that there is a huge inhomogeneous broadening of the exciton resonance in TDMC structures. The advantage of our method is that it is sensitive to the inhomogeneous broadening in a bright layer in the GaAs structure, which is very small, but not to the inhomogeneous broadening in the dark layer in TDMC structures. We also note that if the interlayer hopping may be neglected, effects of excitation mixing and splitting spectral branches take place if both layers are in the BEC phase [65]. If there is no condensate in the dark layer, then there is no occupation of the second spectral branch [66]. We expect that the suggested approach is relevant for experimental setups, where detection via conventional techniques remains challenging, in particular, the considered method is useful for studying excitons in TMDC monolayers.

To take into account finite coherence lengths $\xi_b$ ($\xi_d$) and coherence times $\tau_b$ ($\tau_d$), the excitation spectrum should be treated with considering uncertainties in momentum and energy of order $\max(\hbar/\xi_b, \hbar/\xi_d)$ and $\max(\hbar/\tau_b, \hbar/\tau_d)$, correspondingly.

**V. CONCLUSION**

In the present work, we have considered the system of bilayer dark-bright condensates. We have revealed the impact of the interlayer interaction between excitons and calculated the excitation spectrum. We have demonstrated that the excitation spectrum of the condensate of dark dipolar excitons becomes accessible for optical probing at the realistic experimental conditions.

We would like to emphasize, that our approach is still applicable in the finite temperature case. Neglecting the loop diagrams is still valid up to sufficiently high temperatures of the same order of magnitude as BKT crossover temperature (see Ref. [57] Fig. 3a). However, a quantitative analysis is beyond the scope of the current paper.

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**APPENDIX A: KELDYSH-RYTOVA POTENTIAL FOR A BILAYER SYSTEM**

Here we consider a bilayer system separated by a distance $D$ with 2D polarizabilities of single monolayers $\alpha_1$ and $\alpha_2$, which are embedded into a dielectric medium with the dielectric constant $\epsilon$ and a point charge $e$, located in the first layer at the origin. In order to find the electrostatic potential in the whole space, we start from the Poisson’s equation of the following form (in the way it is done in Ref. [67]):

$$\Delta \varphi = - 4\pi n(r_{3D}),$$

(A1)

where $n$ is the total charge density, which is a sum $n = n_{\text{ext}} + n_{\text{ind}}$ of external (point charge) and induced densities. Then $n_{\text{ind}}$ has three contributions of the following form:

$$n_{\text{ind}} = - \text{div} P = \frac{\epsilon - 1}{4\pi} \Delta \varphi(r, z) +$$

$$+ \delta(z) \alpha_1 \Delta \varphi(r, z = 0) +$$

$$+ \delta(z - D) \alpha_2 \Delta \varphi(r, z = D).$$

(A2)
The first one corresponds to the polarization of the dielectric environment, whereas the remaining ones are charges confined in monolayers. Here we use the following notation: \( r_{3D} = r e_z + ze_z \), where \( r \) is the in-plane coordinate and \( z \) is the normal coordinate. We then arrive to the following equation:

\[
\epsilon \Delta \varphi = -4\pi \varepsilon [\delta(\mathbf{r}_{3D}) + \delta(z)\alpha_1 \Delta_r \varphi(r, z = 0) + \\
+ \delta(z - D)\alpha_2 \Delta_r \varphi(r, z = D)].
\]

One may see that the impact of the dielectric environment may be replaced by applying \( \varepsilon, \alpha_1 \) and \( \alpha_2 \). Thus it is possible to find the potential for the vacuum case first, and then reduce these values in \( \varepsilon \) times.

Applying the Fourier transform to Eq. (A3), we obtain:

\[
(k_z^2 + k^2)\varphi(k_z, k) + 2\alpha_2 k^2 e^{-ikD}\int dk_z' \varphi(k_z', k)e^{ik'D} = 4\pi e - 2\alpha_1 k^2 \int dk_z' \varphi(k_z', k),
\]

(A4)

where \( k \) and \( k_z \) are in-plane and normal components of the wave vector, respectively. Here and further we use the following expressions for the screening lengths: \( \rho_1 = 2\pi \alpha_1 \) and \( \rho_2 = 2\pi \alpha_2 \).

From Eq. (A4) one can conclude that the potential has the following form:

\[
\varphi(k_z, |k|) = \frac{c_1(|k|)e^{-ikD} + c_2(|k|)}{k_z^2 + k^2}.
\]

(A5)

Substituting Eq. (A5) into Eq. (A4) and then integrating over \( k_z' \), we obtain the following equations for \( c_1 \) and \( c_2 \) (\( k = |k| \)):

\[
\begin{aligned}
&c_1 + k\rho_2[c_1 + c_2e^{-kD}] = 0, \\
&c_2 + k\rho_1[c_1e^{-kD} + c_2] = 4\pi e.
\end{aligned}
\]

(A6)

By solving these equation, we arrive to the following expression for the electrostatic potential:

\[
\varphi(k_z, k, D) = \frac{c_1(k)e^{-ikD} + c_2(k)}{k_z^2 + k^2} = \frac{4\pi e}{(k_z^2 + k^2)(1 + k\rho_1)(1 + k\rho_2) - k^2\rho_1\rho_2 e^{-2kD}}.
\]

(A7)

In order to calculate the interexciton interaction potential, we need only special cases of \( z = 0 \) and \( z = D \).

Taking the inverse Fourier transform with fixed \( z \) we obtain:

\[
\varphi_2(k, D) = \frac{2\pi e e^{-kD}}{k((1 + k\rho_1)(1 + k\rho_2) - k^2\rho_1\rho_2 e^{-2kD})}.
\]

(A8)

and

\[
\varphi_2(k, 0) = \frac{2\pi e (1 + k\rho_2 (1 - e^{-2kD}))}{k((1 + k\rho_1)(1 + k\rho_2) - k^2\rho_1\rho_2 e^{-2kD})}.
\]

(A9)

We note that Eq. (A9) is not the same as the Keldysh-Rytova potential \([68, 69]\) for the monolayer case since though \( z = 0 \), here it is calculated in the presence of the second layer. However, by setting \( D = 0 \) we obtain the Keldysh-Rytova potential with the effective screening length \( \rho_{\text{eff}} = \rho_1 + \rho_2 \).

**APPENDIX B: IN-LAYER INTERACTION POTENTIALS**

In the case of the bright (GaAs) layer, we use the standard Coulomb potential as follows:

\[
U_{b(d)}^0(k) = \frac{4\pi e^2 D_{b(d)}}{\epsilon_{b(d)}} \left( 1 - e^{-kD_{b(d)}} \right). \tag{B1}
\]

In order to describe the interaction in the dark (MoS2) layer, we use Eq. (A8) and Eq. (A9) for the system considered on Fig. (1) with the use of several simplifications. First, we consider the bilayer system of MoS2 embedded in the dielectric medium of with the dielectric constant \( \varepsilon_d = (\varepsilon_{\text{GaAs}} + \varepsilon_{\text{TiO}_2})/2 \), which includes neglecting the thickness of the 3hBN/ MoS2/2hBN/MoS2/3hBN system. Second, we neglect the thickness of MoS2 monolayers and treat them as two absolutely thin sheets separated by the distance \( D_g = 4/3 \) nm.

Reducing charges and screening lengths in \( \varepsilon_d \) times and using the fact that both layers are the same (\( \rho_1 = \rho_2 = \rho \)), we obtain the following expression for the in-layer interexciton interaction:

\[
U_{\text{a}(\text{b})}^0(k) = \frac{4\pi e^2}{\epsilon_d} \left( k(1 + k\rho/\epsilon_d)^2 - (k\rho/\epsilon_d)^2 e^{-2kD_{\text{a}(\text{b})}} \right) \times \\
\times \left( 1 - e^{-kD_{\text{a}(\text{b})}} \right) \left( k\rho/\epsilon_d \left( 1 - e^{-2kD_{\text{a}(\text{b})}} \right) \right). \tag{B2}
\]

Although interaction in bright and dark layers has different functional form, they have the same divergent behavior at \( r \to 0 \) limit. In order to check it, we integrate the angular part of Eq. (B1) and Eq. (B2), and arrive at the following expression:

\[
U_{\text{b(d)}}(r) = \int_0^\infty f_{\text{b(d)}}(r,k) J_0(kr) dk. \tag{B3}
\]

If \( r \to 0 \), the main contribution arises from \( k \approx [0; 1/r] \) region due to oscillatory behaviour of Bessel function at \( kr \gg 1 \). So, small \( r \) behaviour of interaction potential is governed by \( U(k) \) at small momenta. Expanding (B1) and (B2), we get the same asymptotic form:

\[
U_{\text{b(d)}}^0(k) = \frac{4\pi e^2 D_{\text{b(d)}}^2}{\epsilon_{\text{b(d)}}} \left( \frac{2\pi e D_{\text{b(d)}}^2}{\epsilon_{\text{b(d)}}} k \right). \tag{B4}
\]

We note that the first term is the bare coupling constant, which has to be dressed.
Figure D1. $C_k(|k|)$ is shown as a function of the momenta. This value does not cross $k$-axis, which confirms stability of the homogeneous system under consideration.

APPENDIX C: SPIN EFFECTS

In the boson limit the spin relaxation of excitons is suppressed [70]. Therefore, both condensate and non-condensate occupies the lowest spin branch. This justifies the consideration of only one spin component for excitons at $T = 0$ in the weakly interacting regime. However, this is not the case for sufficiently low finite temperatures. To prevent the significant thermal occupation of higher spin branches, one may take into account the exchange splitting or use the Zeeman effect. Neglecting the loop diagrams is still valid up to sufficiently high temperatures, i.e. temperatures of the same order of magnitude as the BKT crossover temperature. As we may see in Ref. [57] (Fig. 3), the BKT crossover temperature dependence on a magnetic field is quite weak. However, a quantitative analysis is beyond the scope of the present research.

APPENDIX D: HOMOGENEOUS SYSTEM STABILITY CONDITIONS

Stability conditions in both layers without the inter-layer interaction correspond to the following expressions:

$$\langle \varepsilon_k^0 \rangle^2 > 0, \quad \langle \varepsilon_k^0 \rangle^2 > 0. \quad (D1)$$

The additional condition for the bilayer case has the following form:

$$\langle \varepsilon_k^0 \rangle^2 > 0. \quad (D2)$$

From Eq. (18) one can see that $\Delta_k^2 > 0$, since $V_k < 0, V_k < 0$. This provides $\langle \varepsilon_k^0 \rangle^2 > 0$. For $\langle \varepsilon_k^0 \rangle^2$ to be non-negative, we have:

$$\frac{\langle \varepsilon_k^0 \rangle^2 + \langle \varepsilon_k^0 \rangle^2}{4} > \Delta_k^2. \quad (D3)$$

Expanding Eq. (D3) we obtain the following expression:

$$C_k^2 = T_k \tilde{T}_k + 2T_k U_k + 2T_k ^2 U_k + 4V_k \tilde{V}_k - 4V_k \tilde{V}_k > 0. \quad (D4)$$

For the bilayer system under consideration this inequality holds, which can be shown by calculation (see Fig. D1).
