Birkhoff Theorem and Matter

Rituparno Goswami and George F R Ellis

AGCG and Department of Mathematics and Applied Mathematics, University of Cape Town, Rondebosch, 7701, South Africa

(Dated: February 2, 2012)

Birkhoff’s theorem for spherically symmetric vacuum spacetimes is a key theorem in studying local systems in general relativity theory. However, realistic local systems are only approximately spherically symmetric and only approximately vacuum. In a previous paper, we showed that the theorem remains approximately true in an approximately spherically symmetric vacuum space-time. In this paper we prove the converse case: the theorem remains approximately true in a spherically symmetric, approximately vacuum space time.

PACS numbers:

I. BIRKHOFF’S THEOREM

Birkhoff’s theorem (see e.g. [11]) is a key theorem in general relativity theory. It underlies the way local astronomical systems decouple from the expansion of the universe. It states that if a spacetime domain is locally (a) spherically symmetric and (b) empty, then it necessarily has an extra symmetry: it is either static or spatially homogeneous. That is, either the spacetime is locally flat, or it is locally part of a Schwarzschild solution: either the exterior part of a Schwarzschild solution outside the event horizon (as in the solar system) or the part of the solution inside the event horizon (as in collapse of a star to a singularity).

The theorem actually applies to a somewhat wider class of solutions than spherically symmetric spacetimes: it applies to all vacuum locally rotationally symmetric (LRS) class II solutions, that is vorticity-free solutions with a preferred spatial axis that are invariant under rotations about that axis [10, 11]. We emphasize here that this is a local result: it does not depend on boundary conditions at infinity.

However real astronomical systems are neither exactly spherically symmetric, nor exactly empty. While it remains valid for the case of an electrovac solution (3), section 18.1), Birkhoff’s theorem is not true in general when matter is present, as is shown for example by the Lemaitre-Tolman-Bondi solutions [4, 5]. It remains true when matter is present, as is shown for example by the Solar System, which is neither exactly spherically symmetric, nor exactly empty. While it is true in general. These results do not include crucial boundary conditions at infinity.

In a previous paper [14] we showed that the result is stable to small geometric perturbations: it remains true if spacetime is not exactly spherically symmetric. Here we show that the result is stable to small matter perturbations: it remains true if spacetime is not exactly vacuum, as for example in the case of the solar system.

II. BIRKHOFF THEOREM IN LRS-II SPACETIMES

We prove the result by using the 1+1+2 covariant formalism [9]. First we give a brief outline of the proof of the exact result [14] and then the approximate result is a straightforward generalization of the exact result using the 1+1+2 covariant perturbation theory.

A. 1+1+2 Covariant formalism

In 1+3 covariant approach [6–8], first we define a timelike congruence by a timelike unit vector \( u^a \) \( (u^a u_a = -1) \). Then the spacetime is locally split in the form \( R \otimes V \) where \( R \) denotes the timeline along \( u^a \) and \( V \) is the tangent 3-space perpendicular to \( u^a \). Then any vector \( X^a \) can be projected on the 3-space by the projection tensor \( h^a_b = g^a_b + u^a u_b \). The vector \( u^a \) is used to define the covariant time derivative (denoted by a dot) for any tensor \( T^{a,b}_{\ldots \cdot d} \) along the observers’ worldlines defined by

\[
\dot{T}^{a,b}_{\ldots \cdot d} = u^c \nabla_c T^{a,b}_{\ldots \cdot d},
\]

and the tensor \( h_{ab} \) is used to define the fully orthogonally projected covariant derivative \( D \) for any tensor \( T^{a,b}_{\ldots \cdot d} \)

\[
D_{\ldots \cdot d} T^{a,b}_{\ldots \cdot d} = h^c_j h^p_c \ldots h^q_a h^r_e \nabla_r T^{f,g}_{\ldots \cdot p,q},
\]

with total projection on all the free indices.

In the (1+1+2) approach we further split the 3-space \( V \), by introducing a spacelike unit vector \( e^a \) orthogonal to \( u^a \) so that

\[
e_a u^a = 0, \quad e_a e^a = 1.
\]

Then the projection tensor

\[
h^a_b \equiv \delta_a^b - e_a e^b = g^a_b + u_a u^b - e_a e^b, \quad N^a_a = 2.
\]
projects vectors onto the tangent 2-surfaces orthogonal to \( e^a \) and \( u^a \), which, following \cite{12}, we will refer to as ‘sheets’. Hence it is obvious that \( e^a N_{ab} = 0 = u^a N_{ab} \). In \((1+3)\) approach any second rank symmetric 4-tensor can be split into a scalar along \( u^a \), a 3-vector, a scalar part on the 3-space orthogonal to \( u^a \), and a projected symmetric trace free (PSTF) 3-tensor. In \((1+1+2)\) slicing, we can this split further by splitting the 3-vector and PSTF 3-tensor with respect to \( e^a \). For example, in the \(1+3\) splitting, the Energy Momentum Tensor \( T_{ab} \) can be written as

\[
T_{ab} = \mu u_a u_b + q_a u_b + u_a q_b + p h_{ab} + \pi_{ab}
\]

Where the scalars \( \mu = T_{ab} u^a u^b \) and \( p = (1/3) T_{ab} h^{ab} \) are the energy density and isotropic pressure respectively. The 3-vector, \( q^a = T_{ab} u^b h^{ca} \), is the heat flux and the PSTF 3-tensor, \( \pi_{ab} = T_{ac} h^c_{ba} h^{ab} \), defines the anisotropic stress. In \(1+1+2\) splitting, we further split the fluid variables \( q^a \) and \( \pi_{ab} \) as

\[
q^a = Qe^a + Q^a,
\]

\[
\pi_{ab} = \Pi \left[ e_a e_b - \frac{1}{2} N_{ab} \right] + 2 \Pi_\mu (e_b) + \Pi_{ab}.
\]

The sheet carries a natural 2-volume element, the alternating Levi-Civita 2-tensor:

\[
\varepsilon_{ab} \equiv \varepsilon_{abcd} e^c e^d = \eta_{dabc} e^c u^d,
\]

where \( \varepsilon_{abc} \) is the 3-space permutation symbol the volume element of the 3-space and \( \eta_{abcd} \) is the space-time permutation or the 4-volume.

Now apart from the ‘time’ (dot) derivative, of an object (scalar, vector or tensor) which is the derivative along the timelike congruence \( u^a \), we now introduce two new derivatives, which \( e^a \) defines, for any object \( \psi_{a...b...c...d}^\text{...} \):

\[
\dot{\psi}_{a...b...c...d} \equiv e^f D_f \psi_{a...b...c...d},
\]

\[
\delta_f \psi_{a...b...c...d} \equiv N_{a...f} N_{b...g} N_{c...h} N_{d...j} D_f \psi_{g...h...i...j}.
\]

The hat-derivative is the derivative along the \( e^a \) vector-field in the surfaces orthogonal to \( u^a \). The \( \delta \) -derivative is the projected derivative onto the sheet, with the projection on every free index.

We can now decompose the covariant derivative of \( e^a \) in the direction orthogonal to \( u^a \) into its irreducible parts giving

\[
D_a e_b = e_a e_b + \frac{1}{2} \phi N_{ab} + \xi e_a + \zeta_{ab},
\]

where

\[
a_a \equiv e^c D_c e_a = \dot{e}_a,
\]

\[
\phi \equiv \delta_a e^a,
\]

\[
\xi \equiv \frac{1}{2} e^{ab} \delta_a e_b,
\]

\[
\zeta_{ab} \equiv \delta_{(a} e_{b)}.
\]

We see that along the spatial direction \( e_a \), \( \phi \) represents the expansion of the sheet, \( \xi_{ab} \) is the shear of \( e^a \) (i.e. the distortion of the sheet) and \( \alpha^a \) its acceleration. We can also interpret \( \xi \) as the vorticity associated with \( e^a \) so that it is a representation of the “twisting” or rotation of the sheet. The other derivative of \( e^a \) is its change along \( u^a \),

\[
\dot{e}_a = Au_a + \alpha_a,
\]

where we have \( A = e^a u_a \) and \( \alpha_a = N_{ac} \dot{e}_c \). Also we can write the \((1+3)\) kinematical variables and Weyl tensor as follows

\[
\Theta = h^b_b \nabla_b u^a
\]

\[
\dot{u}^a = A e^a + A^a,
\]

\[
\omega^a = \Omega e^a + \Omega^a,
\]

\[
\sigma_{ab} = \Sigma (e_a e_b - \frac{1}{2} N_{ab}) + 2 \Sigma (e_b + \Sigma_{ab},
\]

\[
E_{ab} = \mathcal{E} (e_a e_b - \frac{1}{2} N_{ab}) + 2 \mathcal{E} (e_b + \mathcal{E}_{ab},
\]

\[
H_{ab} = \mathcal{H} (e_a e_b - \frac{1}{2} N_{ab}) + 2 \mathcal{H} (e_b + \mathcal{H}_{ab} + \mathcal{H}_{ab} \),
\]

where \( E_{ab} \) and \( H_{ab} \) are the electric and magnetic part of the Weyl tensor respectively. Therefore the key variables of the \((1+1+2)\) formalism are

\[
[\Theta, A, \Omega, \Sigma, \mathcal{E}, \mathcal{H}, \phi, \xi, \mu, p, \Pi, Q, A^a, \Omega^a, Q^a, \Pi^a, \Sigma^a, \alpha^a, \mathcal{E}^a, \mathcal{H}^a, \Sigma_{ab}, \mathcal{E}_{ab}, \mathcal{H}_{ab}, \mathcal{H}_{ab}, \mathcal{H}_{ab} \].
\]

These variables (scalars, 2-vectors and PSTF 2-tensors) form an irreducible set and completely describe a spacetime locally.

Using the above described \((1+1+2)\) variables, the full covariant derivatives of \( e^a \) and \( u^a \) are

\[
\nabla_a e_b = -Au_a u_b - u_a a_b + \left( \Sigma + \frac{1}{3} \Theta \right) e_a u_b + (\Sigma_a - \varepsilon_{ac} \Omega^c) u_b + e_a a_b + \frac{1}{2} \phi N_{ab} + \xi e_a + \zeta_{ab},
\]

\[
\nabla_a u_b = -u_a (A e_b + A b) + e_a e_b \left( \frac{1}{3} \Theta + \Sigma \right) + e_a (\Sigma_b + \varepsilon_{ac} \mathcal{E}^c) + (\Sigma_a - \varepsilon_{ac} \mathcal{E}^c) e_b + N_{ab} \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right) + \Omega e_a + \Omega_{ab}.
\]

### B. Field equations for LRS-II spacetimes

We know that for LRS-II spacetimes \cite{10} (which are rotation free Locally Rotationally Symmetric spacetimes)
the 1+1+2 covariant variables \( \{ A, \Theta, \phi, \Sigma, \varepsilon, \mu, p, \Pi, Q \} \) fully characterize the kinematics. The propagation, evolutions and constraint equations for these variables in such spaces are:

\[
\dot{\phi} = -\frac{1}{2} \phi^2 + \left( \frac{1}{3} \Theta + \Sigma \right) \left( \frac{2}{3} \Theta - \Sigma \right)
\]

\[
\dot{\Sigma} - \frac{2}{3} \dot{\Theta} = - \frac{3}{2} \phi \Sigma - Q ,
\]

\[
\dot{E} - \frac{1}{3} \dot{\mu} + \frac{1}{2} \dot{\Pi} = - \frac{3}{2} \phi \left( E + \frac{1}{2} \Pi \right)
\]  

\[ + \left( \frac{1}{2} \Sigma - \frac{1}{3} \Theta \right) Q . \]  

This gives the 3-Ricci-scalar as

\[
3R = -2 \left[ \phi + \frac{3}{4} \phi^2 - K \right] 
\]

where \( K \) is the Gaussian curvature of the sheet, \( \frac{2}{3} R_{ab} = K N_{ab} \). From this equation and (20) an expression for \( K \) is obtained in the form [13]

\[
K = \frac{1}{3} \mu - E - \frac{1}{2} \Pi + \frac{1}{4} \phi^2 - \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right)^2
\]

From (26-31), the evolution and propagation equations of \( K \) can be determined as

\[
\dot{K} = - \left( \frac{2}{3} \Theta - \Sigma \right) K ,
\]

\[
\dot{K} = - \phi K .
\]

From equation (39), it follows that whenever the Gaussian curvature of the sheet is non-zero and constant in time, then the shear is always proportional to the expansion as \( \Sigma = \frac{2}{3} \Theta \).

### C. Vacuum LRS-II spacetimes and Birkhoff Theorem

To covariantly investigate the geometry of the vacuum LRS-II spacetime, we write the Killing equation for a vector of the form

\[
\xi_a = \Psi u_a + \Phi e_a ,
\]

where \( \Psi \) and \( \Phi \) are scalars. The Killing equation gives

\[
\nabla_a (\Psi u_b + \Phi e_b) + \nabla_b (\Psi u_a + \Phi e_a) = 0 .
\]

which in this case becomes the following differential equations and constraints:

\[
\dot{\Psi} + 4 \Phi = 0 ,
\]

\[
\dot{\Psi} - \Phi - \Psi A + \Phi ( \Sigma + \frac{1}{3} \Theta ) = 0 ,
\]

\[
\dot{\Phi} + \Psi ( \frac{1}{3} \Theta + \Sigma ) = 0 ,
\]

\[
\Psi ( \frac{2}{3} \Theta - \Sigma ) + \Phi \phi = 0 .
\]

Now we know \( \xi_a \xi^a = - \Psi^2 + \Phi^2 \). If \( \xi^a \) is timelike (that is \( \xi_a \xi^a < 0 \)), then because of the arbitrariness in choosing the vector \( u^a \), we can always make \( \Phi = 0 \). On the other hand, if \( \xi^a \) is spacelike (that is \( \xi_a \xi^a > 0 \)), we can make \( \Psi = 0 \).

Let us assume that \( \xi^a \) is timelike and set \( \Phi = 0 \). In that case Killings equations (43-46) become

\[
\dot{\Psi} = 0 ,
\]

\[
\dot{\Psi} - \Psi A = 0 ,
\]

\[
\Psi ( \frac{1}{3} \Theta + \Sigma ) = 0 ,
\]

\[
\Psi ( \frac{2}{3} \Theta - \Sigma ) = 0 .
\]
We know that the solution of equations (47) and (48) always exists while the constraints (49) and (50) together imply that in general, (for a non trivial \( \Psi \)), \( \Theta = \Sigma = 0 \). When these are plugged into the field equations (51-3), we see that the “dot” derivative of all the quantities vanish and the remaining field equations are as follows:

\[
\begin{align*}
\dot{\phi} &= - \frac{1}{2} \phi^2 - \mathcal{E}, \\
\dot{\mathcal{E}} &= - \frac{3}{2} \phi \mathcal{E} \\
\mathcal{E} &= -A \dot{\phi}, \\
\dot{A} &= - (A + \phi) A.
\end{align*}
\]

Also the local Gaussian curvature of the 2-sheets are given as

\[
K = -\mathcal{E} + \frac{1}{4} \dot{\phi}^2
\]

From [14] we know that the resultant set of equations has a unique solution (for \( K > 0 \)), which gives the Schwarzschild metric. Similarly if the Killing vector is spacelike we have \( A = \phi = 0 \). In that case the spacetime is spatially homogeneous as the ‘hat’ derivative of all the quantities vanish and the resultant solution (for \( K > 0 \)) is the Schwarzschild interior.

Hence the Birkhoff Theorem for LRS-II spacetime says that there always exists a Killing vector in the local \([u,e]\) plane for a vacuum LRS-II spacetime. If the Killing vector is timelike then the spacetime is locally static, and if the Killing vector is spacelike the spacetime is locally spatially homogeneous. For \( K > 0 \), we get the known result, any \( C^2 \) solution of Einstein’s equations in empty space which is spherically symmetric in an open set \( S \) is locally equivalent to part of maximally extended Schwarzschild solution in \( S \).

From [14], we know that for vacuum LRS-II spacetime

\[
\mathcal{E} = CK^{3/2}.
\]

That is, the \( 1+1+2 \) scalar of the electric part of the Weyl tensor is always proportional to a power of the Gaussian curvature of the 2-sheet. The proportionality constant \( C \) sets up a scale in the problem. We can immediately see that for Minkowski spacetime \( C = 0 \). Also it is interesting to note that the modulus of the proportionality constant in equation (57) is exactly equal to the Schwarzschild radius:

\[
C = RS = 2M
\]

where \( M \) is the mass of the star in the unit of \( 8\pi G = c = 1 \).

### III. ALMOST VACUUM LRS-II SPACETIMES

The result obtained in the previous section is not true if spacetime is not a vacuum (empty) spacetime, for the degrees of freedom available through a matter source generically invalidate the result, as is shown for example by the family of Lemaître-Tolman-Bondi (LTB) models [3]. However we would like to ask the question, that how much matter can be present if the above theorem is to remain approximately true. In other words, we would like to perturb a vacuum LRS-II spacetime by introducing a small amount of general matter in the spacetime. In this section we only deal with the static exterior background as that is astrophysically more interesting.

#### A. Matter

We know from the covariant linear perturbation theory, any quantity which is zero in the background is considered as the first order quantity and is automatically gauge-invariant by virtue of the Stewart and Walker lemma [15]. Hence the set \( \{ \Theta, \Sigma, \mu, p, \Pi, Q \} \), describes the first order quantities. As we have already seen that the vacuum spacetime has an covariant scale given by the the Schwarzschild radius which sets up the scale for perturbation. Let us locally introduce general matter on a static Schwarzschild background such that

\[
\frac{\mu}{K^{(3/2)}}, \frac{|p|}{K^{(3/2)}}, \frac{\Pi}{K^{(3/2)}}, \frac{|Q|}{K^{(3/2)}} << C,
\]

and

\[
\frac{|\mu|}{K^{(3/2)}}, \frac{|\dot{p}|}{K^{(3/2)}}, \frac{|\Pi|}{K^{(3/2)}}, \frac{|\dot{Q}|}{K^{(3/2)}} << \phi C
\]

where \( C \) is the proportionality constant of (57), which is also the Schwarzschild radius.

#### B. Domains

Now we need to make clear in what domain these equations will hold. The application will be to the spherically symmetric exterior domain of a star of mass \( M \) and Schwarzschild radius \( R_S = 2M \), in the units of \( 8\pi G = c = 1 \). We will define Finite Infinity \( F \) as a 2-sphere of radius \( R_F \gg R_M \) surrounding the star: this is infinity for all practical purposes [16, 17]. We assume the relations (59, 60) hold in the domain \( D_F \) defined by \( r_S < r < R_F \) where \( r_S > r_M \) is the radius of the surface of the star. This is the local domain where our results will apply. In the case of the solar system, \( R_F \) can be taken to be about a light year (we return to this issue in Section 4).

It is important to make this restriction, else eventually we will reach a radius \( r \) where these inequalities may no
longer hold; but this will be unphysical, as in the real universe asymptotically flat regions are always of finite size, being replaced at larger scales by galactic and cosmological conditions. The result we wish to prove is a local result, applicable to the locally restricted nature of real physical systems.

C. Equations

Now subtracting the background equations (61)-(64), from the field equations (20)-(24), and neglecting the higher order quantities, we get the following linearised equations for the first order quantities

\[ \dot{\Sigma} - \frac{2}{3} \dot{\phi} = \frac{3}{2} \phi = Q, \quad (61) \]

\[ \dot{\phi} = -\frac{1}{2} (\mu + 3p), \quad (62) \]

\[ \dot{\Sigma} - \frac{2}{3} \dot{\phi} = \frac{1}{3} (\mu + 3p) + \frac{1}{2} \Pi, \quad (63) \]

\[ \dot{\phi} = \left( \Sigma - \frac{2}{3} \phi \right) (A - \frac{1}{2} \phi) + Q, \quad (64) \]

\[ \frac{1}{3} \dot{\mu} - \frac{1}{2} \Pi = \frac{3}{4} \phi \Pi, \quad (65) \]

\[ \dot{\phi} = \left( \frac{3}{2} \Sigma - \phi \right) \xi + \frac{1}{2} \phi Q, \quad (66) \]

\[ \dot{\mu} + \dot{\Pi} = - (\phi + 2A) Q, \quad (67) \]

\[ \dot{Q} + \dot{\rho} + \dot{\Pi} = - \left( \frac{3}{2} \phi + A \right) \Pi - (\mu + p) A, \quad (68) \]

Equations (65)-(68) are linearised matter conservation equations. From these equations we can see that if (59) and (60) are locally satisfied at any epoch, within the domain \( D_F \), then the time variation of the matter variables are of same order of smallness as themselves. Hence there exists an open set \( S \) within where the amount of matter remains “small”, if the amount is small at any epoch in \( S \) and only small amounts of matter enter \( D_F \) across \( F \). One could attempt to determine the same kinds of inequality as those above for matter crossing \( F \), but one can resolve this issue in another way: we have not yet specified the time evolution of \( F \). We now do so in the following manner: choose it in a suitable manner in some initial surface \( t = t_0 \), and then propagate it to the future by dragging it along world lines that are integral curves of the timelike eigenvector of the Ricci tensor \( R_{ab} \) (this will be unique for any realistic non-zero matter). As these are then timelike eigenvectors of the stress tensor \( T_{ab} \) (because of the field equations), equal amount of energy density will convect in and out across \( F \) due to random motions of matter [7]; the total amount of matter inside \( F \) will be conserved, and if the inequalities (59, 60) are satisfied at some initial time they will be satisfied at later times, unless major masses enter the \( F \) locally in some region. If this is so, we do not have an isolated system and the extended Birkhoff’s theorem need not apply.

Hence we will define the time evolution of \( F \) in the way just indicated, and suppose that (59, 60) are then satisfied at later times; if this is not the case the local system considered is not isolated and our result is not applicable.

D. Almost symmetries

Now from equations (61)-(64), it is evident that if the matter variables remains “small” as defined by (59), then the spatial and temporal variance of the expansion \( \Theta \) and the shear \( \Sigma \) are of the same order of smallness as the matter. In that case we see that a timelike vector will not exactly solve the Killing equations (43)-(46) in general, although it may do so approximately. To see this explicitly, let us set \( \Phi = 0 \) in (41) and consider the following symmetric tensor

\[ K_{ab} := \nabla_a (\Psi u_b) + \nabla_b (\Psi u_a). \quad (69) \]

This tensor vanishes if \( \Psi u^a \) is a Killing vector. This is the case of an exact symmetry when the spacetime is exactly static. However, in the perturbed scenario, to see how close the vector \( \xi_a = \Psi u_a \) is to a Killing vector, let us consider the scalars constructed by contracting the above tensor by the vectors \( u^a \), \( e^a \) and the projection tensor \( N_{ab} \). If the conditions

\[ \left| \frac{|K_{ab} u^a u^b|}{K^{3/2}} \right|, \left| \frac{|K_{ab} u^a e^b|}{K^{3/2}} \right|, \left| \frac{|K_{ab} e^a e^b|}{K^{3/2}} \right| < C \]

are satisfied, then we can say that \( \xi_a = \Psi u_a \) is close to a Killing vector and the spacetime is approximately static.

E. The Main Result

From equations (24) and (25), we see that there always exists a non-trivial solution of the scalar \( \Psi \) for which \( |K_{ab} u^a u^b| \) and \( |K_{ab} u^a e^b| \) vanishes; we choose \( \Psi \) accordingly. However for a general matter perturbation, as \( \Theta \) and \( \Sigma \) are non-zero, from (24) and (25) it is evident that \( |K_{ab} e^a e^b|^2 \) and \( |K_{ab} N_{ab}|^2 \) are non-zero. However, subtracting the background equation (60) from (35), we get

\[ \left( \frac{1}{3} \Theta - \frac{1}{2} \Sigma \right)^2 \approx \frac{1}{3} \mu - \frac{1}{2} \Pi. \quad (71) \]
Similarly subtracting (51) from (26) we get
\[
\left(\frac{1}{3} \Theta + \Sigma\right) \left(\frac{2}{3} \Theta - \Sigma\right) \approx \frac{2}{3} \mu + \frac{1}{2} \mathbf{H}.
\] (72)

Using the above equations (71) and (72), we immediately see that if the amount of matter is "small", that is the condition (59) is satisfied, then the following conditions are satisfied
\[
|K_{ab}e^a e^b|^2 = \Psi^2 \left(\frac{1}{3} \Theta + \Sigma\right)^2 \ll CK^{-3/2},
\] (73)
\[
|K_{ab}N^{ab}|^2 = \Psi^2 \left(\frac{2}{3} \Theta - \Sigma\right)^2 \ll CK^{-3/2}.
\] (74)

Therefore we can say that there always exists a timelike vector that satisfies (70). This vector then almost solves the Killing equations in $S$ and hence the spacetime is almost static in $S$. Also the resultant field equations are the zeroth order equations (51)-(54) with $O(\epsilon)$ terms added to it. Hence for $K > 0$, the local spacetime is described by an almost Schwarzschild metric.

The above conditions, (59) and (60), can also be written in another way.

\[
\left[\frac{|R|}{K^{(3/2)}}, \frac{|R_{ab} u^a u^b|}{K^{(3/2)}}, \frac{|R_{<ab>} e^a e^b|}{K^{(3/2)}}, \frac{|R_{<ab>} e^a e^b|}{K^{(3/2)}}\right] \ll C
\] (75)

and

\[
\left[\frac{|\hat{R}|}{K^{(3/2)}}, \frac{|\hat{R}_{ab} u^a u^b|}{K^{(3/2)}}, \frac{|\hat{R}_{<ab>} u^a e^b|}{K^{(3/2)}}, \frac{|\hat{R}_{<ab>} e^a e^b|}{K^{(3/2)}}\right] \ll \phi C
\] (76)

In other words the ratio of the scalars constructed from the Ricci tensor using the vectors $u^a$ and $e^a$ (and their spatial variations) to the $(3/2)$th power of the local Gaussian curvature of the 2-sphere should be much smaller than the Schwarzschild radius if the Birkhoff theorem is to remain approximately true. Equations (75) and (76) are easier to use, in case of presence of multifluids in the spacetime.

IV. COMMENTS ON THE SOLAR SYSTEM

In case of the solar system [18], we know that the interplanetary medium includes interplanetary dust, cosmic rays and hot plasma from the solar wind. Its density is very low at about 5 particles per cubic centimeter in the vicinity of the Earth; it decreases with increasing distance from the sun, in inverse proportion to the square of the distance. In this section, to compare our result with the observed astronomical data, we will use SI units for clarity.

The density of interplanetary medium is variable, and may be affected by magnetic fields and events such as coronal mass ejections. It may rise to as high as 100 particles/cm$^3$. These particles are mostly Hydrogen nuclei, and hence the maximum density per cubic meter will be approximately of the order of $10^{-19}$ Kilograms, and the local Gaussian curvature of the heliocentric celestial sphere in the vicinity of the earth is of the order of $10^{-22}$ m$^{-2}$. Hence the ratio of the maximum interplanetary density to the $(3/2)$th power of the local Gaussian curvature is of the order of $10^{14}$ Kilograms, which is much smaller then the solar mass ($10^{30}$ Kilograms). Also the large amplitude waves in the medium are comparable to the energy density of the unperturbed medium, which makes the spatial variation of energy density to be of the same order of smallness as itself. This satisfies (59) and (60) and hence in the solar system the Birkhoff theorem remains almost true.

We can relate the discussion to the Finite Infinity concept for the solar system. We know that the outer edge of the solar system is the boundary between the flow of the solar wind and the diffused interstellar medium. This boundary, which is known as the Heliosphere, is at a radius of approximately $10^{13}$ meters. The interplanetary medium thus fills the roughly spherical volume contained within the heliopause. As the density of the interplanetary medium decreases in inverse proportion to the square of the distance, the density near the heliopause is of the order of $10^{-23}$ Kilograms per cubic meter. Hence the ratio of the density to the $(3/2)$th power of the local Gaussian curvature is of the order of $10^{16}$ Kilograms and still remains much smaller than the solar mass. Also the amount of matter crossing the heliopause to the diffused interstellar medium is of the same order. Hence we can easily define the heliopause as the boundary of our domain $D_x$. As the conditions (59) and (60) are true at the boundary of the domain, they should be true everywhere inside the domain, unless the matter outside the star is highly clustered locally. But we are considering the case of a low density diffuse gas where this is not the case. the conditions (75) and (76) will be satisfied in this domain.

For the massive planets inside the solar system (e.g. Jupiter or Saturn), these conditions may be violated in their very close vicinity, but in that case the local spacetime no longer remains spherically symmetric. However as the vast fraction of the solar system’s mass (more than 99%) is in the sun, on average these massive planets have a very tiny effect on the system as a whole and the approximate theorem remains true. Hence the local spacetime within the solar system is “almost” described by a Schwarzschild metric.

V. CONCLUSION

Our previous paper showed an “Almost Birkhoff theorem” holds if a vacuum spacetime is almost spherically
symmetric. This paper shows such a result also holds for an almost vacuum spherically symmetric spacetime.

It seems clear that the generic result – needed for the real universe application – will be true: an “Almost Birkhoff theorem” will hold for an almost–vacuum almost–spherically symmetric spacetime. We leave that proof, combining the results of this paper and the previous one, for another investigation.

[1] S W Hawking and G F R Ellis (1973) *The large scale structure of spacetime* (Cambridge: Cambridge University Press), Appendix 2.
[2] S Capozziello and V Faraoni (2011) *Beyond Einstein Gravity: A Survey of Gravitational Theories for Cosmology* (Dordrecht: Springer).
[3] R D’Inverno (1992) *Introducing Einstein’s Relativity* (Oxford: Clarendon Press).
[4] H Bondi (1947) “Spherically symmetric models in general relativity”. *Mon. Not. Roy. Astr. Soc.* **107**, 410.
[5] A Krasinski (1997) *Inhomogeneous Cosmological Models* (Cambridge: Cambridge University Press)
[6] J. Ehlers (1961) “Contributions to the Relativistic Mechanics of Continuous Media” *Abh. Mainz Akad. Wiss. u. Litt. (Math. Nat. kl)* 11. Reprinted as a Golden Oldie: Gen. Rel. Grav. 25, no 12, 1225-66 (1993).
[7] G. F. R. Ellis (1971), “Relativistic cosmology”. In *General Relativity and Cosmology, Proceedings of XLVII Enrico Fermi Summer School*, ed. R. K. Sachs (New York Academic Press)
[8] G. F. R. Ellis and H. van Elst (1999) “Cosmological models” In *Theoretical and Observational Cosmology*, (Cargese Summer School 1998) edited by M. Lachi`eze-Rey, p. 1 (Kluwer, Dordrecht).
[9] C. Clarkson (2007) “A covariant approach for perturbations of rotationally symmetric spacetimes” Phys. Rev. D **76**, 104034 [arXiv:0708.1398[gr-qc]].
[10] G F R Ellis (1967) “The dynamics of pressure-free matter in general relativity”. *Journ Math Phys* **8**, 1171 – 1194.
[11] H. van Elst and G. F. R. Ellis (1996) “The Covariant Approach to LRS Perfect Fluid Spacetime Geometries” *Class. Quantum Grav.* **13**, 1099, [gr-qc/9510044].
[12] C. A. Clarkson and R. K. Barrett (2003) “Covariant Perturbations of Schwarzschild Black Holes”, *Class. Quant. Grav.* **20**, 3855 [gr-qc/0209051].
[13] G. Betschart and C. A. Clarkson (2005) “Scalar field and electromagnetic perturbations on Locally Rotationally Symmetric spacetimes”, *Class. Quantum Grav.* **21** 5587 [arXiv:gr-qc/0404116].
[14] R. Goswami, G F R Ellis (2011), “Almost Birkhoff theorem in general relativity ” *GRG* **43**: 2157-2170 [arXiv:1101.4520].
[15] J. M. Stewart (1990) “Perturbations of Friedmann-Robertson-Walker cosmological models”, *Class. Quantum Grav.* **7**, 1169.
[16] G F R Ellis (1984): “Relativistic cosmology: its nature, aims and problems”. In *General Relativity and Gravitation*, Ed B Bertotti et al (Reidel), 215-288.
[17] G F R Ellis and W R Stoeger (2009): ”The Evolution of Our Local Cosmic Domain: Effective Causal Limits:” MNRAS Volume 398, Number 3, September 2009 , pp. 1527-1536(10) [arXiv:1001.4572].
[18] Kenneth R. Lang (2000), “The sun from space”, Vol:1, (Springer).