Explicit heat kernels of a model of distorted Brownian motion on spaces with varying dimension

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Abstract. In this paper, we study a particular model of distorted Brownian motion (dBM) on state spaces with varying dimension. Roughly speaking, the state space of such a process consists of two components: a 3-dimensional component and a 1-dimensional component. These two parts are joined together at the origin. The restriction of dBM on the 3- or 1-dimensional component receives a strong “push” towards the origin. On each component, the “magnitude” of the “push” can be parametrized by a constant $\gamma > 0$. In this article, using probabilistic method, we get the exact expressions for the transition density functions of dBM with varying dimension for any $0 < t < \infty$.

AMS 2010 Mathematics Subject Classification: Primary 60J60, 60J35; Secondary 60J45, 60J65.

Keywords: Distorted Brownian motions, Dirichlet forms, varying dimension, transition density

1 Introduction

The concept of 3-dimensional distorted Brownian motion arises in statistical physics. To give a brief description to 3-dimensional dBM, we consider the the standard 3-dimensional Brownian motion on the path space denoted by $(\Omega, \mathcal{F}, \mathbb{P}_x)_{x \in \mathbb{R}^3}$, $\omega(t)$, $t \geq 0$). For the Hamiltonian we select $H(\omega) = \int_0^t 1_{\{\|x\| \leq \epsilon\}}(\omega(s))ds$. For $A(\epsilon) = \frac{\pi^2}{8\epsilon^2} + \frac{\gamma}{\epsilon}$ where $\gamma$ is a positive parameter, we define the Gibbs measure $\mathbb{P}_{x,\beta,t}^\epsilon$ by setting

$$\frac{d\mathbb{P}_{x,\beta,t}^\epsilon}{d\mathbb{P}_x} = \frac{\exp\left\{A(\epsilon) \int_0^t 1_{\{\|x\| \leq \epsilon\}}(\omega(s))ds\right\}}{Z_{\beta,t}(x)},$$

where

$$Z_{\beta,t}(x) = E_x\left[\exp\left\{A(\epsilon) \int_0^t 1_{\{\|x\| \leq \epsilon\}}(\omega(s))ds\right\}\right]$$

is the normalizing constant making $\mathbb{P}_{x,\beta,t}^\epsilon$ a probability measure. This model arises from the discrete homopolymer model: The latter is similar to the model described above, with the only changes being that 3-dimensional Brownian path $\omega(t)$ is replaced with 3-dimensional continuous time simple random walk on $\mathbb{Z}^3$, $1_{\{\|x\| \leq \epsilon\}}$ is replaced with $\delta_0$, and that $A(\epsilon)$ is replaced with $\gamma$.

For the continuous model on $\mathbb{R}^3$ we introduce above, as $\epsilon \to 0$, there is a limiting process associated with it. The rigorous meaning of the “limit” can be found in [3]. Roughly speaking, as $\epsilon \to 0$, the resolvents converge to another family of resolvents which has a Markov process associated with it. We call such a limiting process a 3-dimensional distorted Brownian motion with parameter $\gamma$. 
Many interesting properties of 3-dimensional dBM have been investigated in [3], [4], and [5], including its explicit transition densities and behaviors near the origin. Later in [6], Fitzsimmons and Li give a very thorough description to this process by means of its associated Dirichlet form.

Unlike a 3-dimensional standard Brownian motion which does not hit any singleton, a 3-dimensional dBM is subject to a strong push towards the origin. Therefore, it is recurrent and has positive capacity at the origin, which allows us to study such a process on a state space with varying dimension. Such a state space with varying dimension consists of two components: a 3-dimensional component and a 1-dimensional component. These two parts are joined together at the origin. The study of Markov processes with varying dimension was originated in [2], where the model is constructed by joining together a 2-dimensional Brownian motion and a 1-dimensional Brownian motion on half line. Since 2-dimensional Brownian motion does not hit any singleton, the construction of such a process with varying dimension utilizes the method of “darning”, i.e., setting the resistance on a 2-dimensional disc equal to zero. The disc is centered at the intersection of the plane and the pole. The model studied in [2] is a toy model of Markov processes with varying dimension, but many important properties as well as techniques of analyzing such process have been developed in that article.

In this paper, we first give a more precise description to dBM with varying dimension. The state space of such a process is embedded in $\mathbb{R}^4$. We let $\mathbb{R}^4 \supset E_1 := \{(x, 0_1) : x \in \mathbb{R}^3\} \cong \mathbb{R}^3$ and $\mathbb{R}^4 \supset E_2 := \{(0_3, x) : x \in \mathbb{R}_+\} \cong \mathbb{R}_+$. Set $E := E_1 \cup E_2$.

Clearly, $E_1 \cap E_2 = (0_3, 0_1) =: 0 \in \mathbb{R}^4$. The restriction of dBM with varying dimension on $E_1$ and $E_2$ behaves like a 3-dimensional and an 1-dimensional distorted Brownian motion, respectively.

The main result of this paper is obtaining the explicit expression for the transition density function of distorted Brownian motion with varying dimension for all $t > 0$, for the case that the “parameter of distortion” $\gamma > 0$ is the same on both the 3-dimensional and 1-dimensional components. The key observation is that for this case, the signed radial process of this process is symmetric about 0. Therefore the “absolute” radial process is actually a Brownian motion reflected at zero with a constant drift pushing towards the origin. From here, realizing that the distribution of the signed radial process can be “decomposed” into a Brownian motion with drift reflected at the origin and a Brownian motion with drift killed at the origin, both 1-dimensional, we derive the explicit global transition density of the process for all $t > 0$.

Before we state the main results, we introduce the underlying measure and the metric equipped on the state space. Fix a parameter $\gamma > 0$. The measure $m_\gamma$ on $E$ is given as

$$m_\gamma(dx) := \begin{cases} 
\frac{\gamma}{2\pi} e^{-\gamma|\underline{x}|} |\underline{x}|^2 d\underline{x}, & \text{on } E_1, \\
2\gamma e^{-\gamma|\underline{x}|} |\underline{x}| dx, & \text{on } E_2.
\end{cases}$$

The above $dx$ or $d\underline{x}$ means the 1-dimensional or 3-dimensional Lebesgue measure. $m_\gamma$ is well-defined because 0 is of zero-Lebesgue-measure for both 1-dimensional and 3-dimensional spaces.

Throughout this paper, we denote by $|x - y|$ the Euclidean distance between $x$ and $y$ if either $x, y \in E_1$ or $x, y \in E_2$. This can either be viewed as Euclidean distance on $\mathbb{R}^4$, or its projection onto $\mathbb{R}_+$ or $\mathbb{R}^3$. By slightly abusing the notation, we let

$$|x - y| := |x - 0_1| + |y - 0_1|, \quad \text{if } x \in E_1, y \in E_2.$$
Heat kernels for dBM with varying dimension

In this paper, we denote the main process of interest, the distorted Brownian motion with varying dimension by \( M \), whose rigorous definition is given in Definition 2.1. For any connected \( C^{1,1} \) open subset \( D \) of \( E \), we let \( M^D \) be the part process of dBM with varying dimension killed upon exiting \( D \), and denote by \( p_D(t, x, y) \) its transition density function.

The main result of this paper is the following explicit transition density function for dBM with varying dimension.

**Theorem 1.** Fix \( \gamma > 0 \). With respect to the measure \( m_\gamma \) given in (1.1), for all \( t > 0 \), the transition density of \( M \), denoted by \( p(t, x, y) \), has the following expression:

\[
\begin{align*}
(i) & \quad p(t, x, y) = q(t, x, y) + \frac{1}{2} \left( p^\Y(t, |x|, |y|) - p^Y_{\mathbb{R}^+}(t, |x|, |y|) \right), \quad x, y \in E_1; \\
(ii) & \quad p(t, x, y) = \frac{1}{2} \left( p_{E_2}(t, x, y) + p^\Y(t, |x|, |y|) \right), \quad x, y \in E_2; \\
(iii) & \quad p(t, x, y) = \frac{1}{2} \left( p^\Y(t, |x|, |y|) - p^Y_{\mathbb{R}^+}(t, |x|, |y|) \right), \quad x \in E_1, y \in E_2,
\end{align*}
\]

where the explicit expressions of \( q(t, x, y) \), \( p^\Y(t, x, y) \), \( p^Y_{\mathbb{R}^+}(t, |x|, |y|) \), and \( p_{E_2}(t, x, y) \) are given in (4.3), (4.5), (4.8), and (4.6) respectively.

**Remark 1.** (i) \( q(t, x, y) \) denotes the transition density of a 3-dimensional distorted Brownian motion on \( E_1 \) killed upon hitting 0, with respect to \( m_\gamma \).

(ii) \( Y \) is the signed radial process of \( M \), defined at the beginning of Section 3.2. \( p^\Y(t, x, y) \) is the density of \( Y \) with respect to the measure \( \tilde{m} \) characterized in (3.3). In fact, \( \tilde{m} \) is the symmetrizing measure for \( Y \).

(iii) \( \tilde{Y} := |M| = |Y| \). With respect to the measure \( \tilde{m} \), \( p^\Y(t, x, y) \) is the density of \( \tilde{Y} \).

(iv) \( p_{E_2}(t, x, y) \) is the density of the part process of \( M \) on \( E_2 \). It is with respect to \( m_\gamma|_{E_2} \).

(v) Since \( p(t, x, y) \) is symmetric in \( (x, y) \) with respect to \( m_\gamma \), the three cases (i)-(iii) essentially cover all the cases for \( x, y \in E \).

2 Preliminary

In this section, we give an introductory overview on distorted Brownian motion and dBM with varying dimension. Most the results in this section can be found in [6].

2.1 3-dimensional distorted BM

This is a summary of [6]. Fix

\[
\psi_\gamma(x) := \sqrt{\frac{\gamma}{2\pi}} \cdot \frac{e^{-|x|^2}}{|x|}, \quad x \in \mathbb{R}^3. \tag{2.1}
\]

Note that \( \int \psi_\gamma(x)^2 dx = 1 \). Set \( m(dx) := \psi_\gamma(x)^2 dx \) on \( \mathbb{R}^3 \) and define an energy form on \( L^2(\mathbb{R}^3, m) \) as follows:

\[
\mathcal{E}(f, g) := \frac{1}{2} \int_{\mathbb{R}^3} \nabla f(x) \cdot \nabla g(x) m(dx), \quad f, g \in \mathcal{F}.
\]
The next theorem includes some facts about \((\mathcal{E}, \mathcal{F})\). The second one (0 is of positive capacity) is critical for us to construct dBM with varying dimension.

**Theorem 2 (Cf. [6]).** The following statements hold:

(i) \((\mathcal{E}, \mathcal{F})\) is a regular Dirichlet form on \(L^2(\mathbb{R}^3, m)\) with \(C_c^\infty(\mathbb{R}^3)\) being its special standard core. Denote its associated Markov process by \(X = \{X_t\}_{t \geq 0}\).

(ii) Any singleton \(x \neq 0\) is \(\mathcal{E}\)-polar, but 0 \(\in \mathbb{R}^3\) is of positive capacity.

(iii) \((\mathcal{E}, \mathcal{F})\) or \(X\) is recurrent, conservative and irreducible. Particularly, \(m\) is an invariant measure of \(X\).

(iv) \(X\) is not a semi-martingale.

The diffusion \(X\) is called a 3-dimensional distorted Brownian motion. In the following we provide more detailed description to it. The third assertion of Theorem 2 states that \(X\) is irreducible recurrent. This implies (by [8, Theorem 4.7.1])

\[
\mathbb{P}_x(\sigma_0 < \infty) = 1, \quad \text{for q.e. } x \in \mathbb{R}^3.
\]

Particularly, \(\mathbb{P}_0(\sigma_0 < \infty) = 1\). In fact, we have \(\mathbb{P}_0(\sigma_0 = 0) = 1\). This means 0 is a regular point. Heuristically speaking, \(X\) behaves like a one-dimensional Brownian motion near 0.

Next we give some remarks on the rotational invariance of \(X\), some of which will be used later in this paper. These results can be found in [6, Section 3].

(i) \(X\) is isotropic in the sense that if \(T : \mathbb{R}^3 \rightarrow \mathbb{R}^3\) is isotropic (i.e. \(T : x = (r, \theta, \varphi) \mapsto (r, \theta + \theta_0, \varphi + \varphi_0)\) for some given \(\theta_0\) and \(\varphi_0\), then \(X\) and \(T(X)\) are equivalent (i.e. they share the same Dirichlet form).

(ii) It holds

\[
X_t = (r_t, \vartheta_{A_t}), \quad t \geq 0,
\]

where \(r_t\) is a diffusion on \([0, \infty)\), \(\vartheta\) is a BM on \(S^2\) and \(A_t\) is a PCAF of \((r_t)\). The Revuz measure of \(A\) is \(\mu_A(du) = \frac{l(du)}{u^2} (l\) is given in (2.2)).

(iii) The radial part \(r_t\) of \(X\) is a diffusion reflected at 0 with the scale function

\[
s(u) = \frac{1}{4\gamma^2} e^{2\gamma u}, \quad u \in [0, \infty)
\]

and speed measure

\[
l(du) = 2\gamma e^{-2\gamma u} du.
\]

(iv) \(r_t\) satisfies

\[
r_t - r_0 = \beta_t - \gamma t + \pi \gamma \cdot l_t^0,
\]

where \(\beta_t\) is a one-dimensional BM and \(l_t^0\) is the local time of \(r\) in the sense of Revuz measure at 0.

### 2.2 Distorted Brownian motion on spaces with varying dimension

In this subsection, we rigorously give the definition for distorted Brownian motion with varying dimension \(M\) on the state space \(E\). Recall that we have mentioned in Section 1 that \(E = E_1 \cup E_2\). The restriction of dBM with varying dimension \(M\) on \(E_1\) is induced by the 3-dimensional distorted Brownian motion \(X\) in Section 2.1. Set \(m_1 := m\) on \(\mathbb{R}^3\). Denote by \((\mathcal{E}^1, \mathcal{F}^1) := (\mathcal{E}, \mathcal{F})\)
the Dirichlet form of the 3-dimensional process $X$. Here we use the superscript “1” to emphasize that this corresponds to $E_1$ defined in Section 1, the 3-dimensional component of $E$. Set

$$\iota_1 : \mathbb{R}^3 \to E_1, \quad x \mapsto (x, 0).$$

Then $M^1 := \iota_1(X)$ is a distorted Brownian motion on $E_1$ associated with the Dirichlet form on $L^2(E_1, m_1)$ ($m_1 := m_1 \circ \iota_1^{-1}$)

\begin{align*}
\mathcal{F}^1 &:= \{ f : f \circ \iota_1 \in \mathcal{F}^1 \}, \\
\mathcal{E}^1(f, g) &:= \mathcal{E}^1(f \circ \iota_1, g \circ \iota_1), \quad f, g \in \mathcal{F}^1.
\end{align*}

To introduce the second part, for $\gamma > 0$ we first define

$$\phi_\gamma(u) = \sqrt{2\gamma e^{-\gamma u}}, \quad \text{for } u \in \mathbb{R}_+.$$  \hspace{1cm} (2.3)

We now consider the following Dirichlet form on $L^2(\mathbb{R}_+, m_2)$:

\begin{align*}
\mathcal{F}^2 &:= \{ f \in L^2(\mathbb{R}_+, m_2) : \nabla f \in L^2(\mathbb{R}_+, m_2) \}, \\
\mathcal{E}^2(f, g) &:= \frac{1}{2} \int_{\mathbb{R}_+} \nabla f(u) \nabla g(u) m_2(du), \quad f, g \in \mathcal{F}^2,
\end{align*}

where

$$m_2(du) := \phi_\gamma(u)^2 du = 2\gamma e^{-2\gamma u} du.$$

Write

$$\iota_2 : \mathbb{R}_+ \to E_2, \quad u \mapsto (0, u).$$

Denote its associated diffusion by $X^2$. Then $M^2 := \iota_2(X^2)$ is a diffusion on $E_2$ associated with (on $L^2(E_2, m_2) := L^2(E_2, m_2 \circ \iota_2^{-1})$)

\begin{align*}
\mathcal{F}^2 &:= \{ f : f \circ \iota_2 \in \mathcal{F}^2 \}, \\
\mathcal{E}^2(f, g) &:= \mathcal{E}^2(f \circ \iota_2, g \circ \iota_2), \quad f, g \in \mathcal{F}^2.
\end{align*}

Now we are ready to introduce the definition of dBM with varying dimension, as well as its associated Dirichlet form.

**Proposition 1.** Let

\begin{align*}
\mathcal{F} &:= \left\{ f \in L^2(E, m) : f|_{E_1} \in \mathcal{F}^1, f|_{E_2} \in \mathcal{F}^2, \tilde{f}|_{E_1}(0) = \tilde{f}|_{E_2}(0) \right\}, \\
\mathcal{E}(f, g) &:= \mathcal{E}^1(f|_{E_1}, g|_{E_1}) + \mathcal{E}^2(f|_{E_2}, g|_{E_2}), \quad f, g \in \mathcal{F}.
\end{align*}

(2.4)  \hspace{1cm} (2.5)

Then $(\mathcal{E}, \mathcal{F})$ is a strongly local regular Dirichlet form on $L^2(E, m_\gamma)$. Therefore there exists a unique diffusion process associated with it.

**Definition 1** (Distorted Brownian motion with varying dimension). Let $\gamma > 0$ be fixed. The diffusion process associated with $(\mathcal{F}, \mathcal{E})$ defined in (2.4) and (2.5) is called a distorted Brownian motion with varying dimension and is denoted by $M$. 
3 Basic properties of $M$ and its associated signed radial process

In this Section, we quickly remark on some of the basic properties of $M$ that are mostly reflected through its Dirichlet form expression. We give the rigorous statement that 0 is of positive capacity with respect to $M$, therefore can be hit with probability one starting from quasi-everywhere. Then we present the isotropic properties as well as the rotational invariance of $M$. We give the SDE characterization for the radial process of $M$ which is needed in Section 4. Finally, the existence of the transition density function is established.

3.1 Basic properties of $M$

We denote by Cap the capacity of $(\mathcal{E}, \mathcal{F})$.

Proposition 2. For any $u \in E_2$, Cap($\{u\}$) > 0. However, for any $x \in E_1 \setminus \{0\}$, Cap($\{x\}$) = 0. Particularly, Cap($\{0\}$) > 0. Furthermore, $\mathbb{P}_x(\sigma_0 < \infty) = 1$ for q.e. $x \in E$.

Proof. This is obvious. See, for example, [1, Theorem 7.5.4].

To introduce the isotropic property and rotational invariance of $M$, we define

- $M^0$: part process of $M$ on $E \setminus \{0\}$;
- $M^{1,0}$: part process of $M^1$ on $E_1 \setminus \{0\}$;
- $M^{2,+}$: part process of $M^2$ on $E_2^+$, where $E_2^+ := \{(0,3, x) : x > 0\}$;
- $M^{2,-}$: part process of $M^2$ on $E_2^-$, where $E_2^- := \{(0,3, x) : x < 0\}$.

The following proposition is similar to the same properties of 3-dimensional dBMs. The proofs are straightforward and thus omitted.

Proposition 3.

$|M^0| \overset{d}{=} |M^{1,0}| \overset{d}{=} M^{2,+} \overset{d}{=} -M^{2,-}$.

We state the following proposition which establishes the existence of transition density.

Proposition 4. Let $(P_t(x,\cdot))_{t \geq 0}$ be the semigroup of $M$. Then for any $x \in E, t > 0$, we have $P_t(x,\cdot) \ll m_x$. Thus there exists a density function $\{p(t,x,y) : t > 0, x, y \in E\}$ such that $P_t(x,dy) = p(t,x,y)m_x(dy)$.

Proof. By [8, Theorem 4.2.4], it suffices to show that any $m$-polar set is polar. This is obviously true. See, for example, [1, Theorem 3.1.3].

3.2 Signed radial process $Y$

To introduce the signed radial process of $M$, we define

$$u(\mathbf{r}) := \begin{cases} |\mathbf{r}|, & \mathbf{r} = (x,0) \in E_1, \\ -x, & \mathbf{r} = (0,3,x) \in E_2 \end{cases} \quad (3.1)$$

and let $Y_t := u(M_t)$ for any $t \geq 0$. 

Proposition 5. $Y = (Y_t)_{t \geq 0}$ defined above is a diffusion process on $\mathbb{R}$ characterized by the following SDE:

$$Y_t - Y_0 = B_t + \gamma \int_0^t 1_{(-\infty,0)}(Y_s) ds - \gamma \int_0^t 1_{(0,\infty)}(Y_s) ds, \quad t \geq 0, \quad (3.2)$$

where $(B_t)_{t \geq 0}$ is a 1-dimensional standard Brownian motion.

Proof. Set $\tilde{m} := m \circ u^{-1}$, which is a fully supported Radon measure on $\mathbb{R}$. In practice, one can easily obtain

$$\tilde{m}(dx) = 2\gamma e^{-2\gamma|x|} dx|_{(0,\infty)} + 2\alpha e^{-2\alpha|x|} dx|_{(-\infty,0)}. \quad (3.3)$$

Standard argument (can be provided upon request) shows that $Y$ is a symmetric Markov process associated with the Dirichlet form on $L^2(\mathbb{R}, \tilde{m})$:

$$\mathcal{F}^Y = \{ f : f \circ u \in \mathcal{F} \}, \quad \mathcal{E}^Y(f, f) = \mathcal{E}(f \circ u, f \circ u), \quad f \in \mathcal{F}^Y. \quad (3.4)$$

Next we take $f(x) := x \in \mathcal{F}^Y_{\text{loc}}$ and consider the Fukushima’s decomposition:

$$f(Y_t) - f(Y_0) = M^f_t + N^f_t. \quad \text{The martingale part } M^f \text{ is determined by its energy measure } \mu_{\langle f \rangle} \text{ and for any } g \in C_0^\infty(\mathbb{R}),$$

$$\int g d\mu_{\langle f \rangle} = 2\mathcal{E}^Y(fg, f) - \mathcal{E}^Y(f^2, g) = \int g d\tilde{m}. \quad \text{It follows that } \mu_{\langle f \rangle} = \tilde{m} \text{ and hence } M^f \text{ has the same distribution as one-dimensional Brownian motion. For the zero-energy part } N^u, \text{ we note}$$

$$-\mathcal{E}^Y(f, g) = -\frac{1}{2} \int_\mathbb{R} \tilde{m}(dx) = \gamma \int_0^\infty g(x)\tilde{m}(dx) - \gamma \int_0^\infty g(x)\tilde{m}(dx). \quad \text{Thus } N^u \text{ is of bounded variation, and}$$

$$\mu_{N^u} = \gamma \cdot \tilde{m}|_{(-\infty,0)} - \gamma \cdot \tilde{m}|_{(0,\infty)}. \quad \text{Eventually, it follows from } \mathbb{R} \text{ Theorem 5.5.5) that}$$

$$Y_t - Y_0 = B_t + \gamma \int_0^t 1_{(-\infty,0)}(Y_s) ds - \gamma \int_0^t 1_{(0,\infty)}(Y_s) ds, \quad t \geq 0,$$

where $(B_t)_{t \geq 0}$ is a standard Brownian motion. \qed
4 Heat kernel of $M$: Proof to Theorem \[1\]

Throughout the rest of this paper, for $M$, we use $\tilde{p}(t,x,y)$ to denote the transition density with respect to the measure on $E$ induced by 3- or 1-dimensional Lebesgue measure, and we let $p(t,x,y)$ denote the transition density with respect to $m_\gamma$. Thus

$$p(t,x,y) = \frac{1}{h_\gamma(y)^2} \tilde{p}(t,x,y), \quad (4.1)$$

where

$$h_\gamma := \begin{cases} \psi_\gamma, & \text{on } E_1, \\ \phi_\gamma, & \text{on } E_2, \end{cases}$$

where $\phi_\gamma$ is defined in \[23\]. We denote by $p^Y(t,x,y)$ and $\tilde{p}^Y(t,x,y)$ the densities of $Y$ and $\tilde{Y}$ respectively, both with respect to $\tilde{m}$ characterized in \[33\]. We denote by $\tilde{p}^Y(t,x,y)$ and $\hat{p}^Y(t,x,y)$ the densities of $Y$ and $\tilde{Y}$ with respect to the 1-dimensional Lebesgue measure.

The first key ingredient of the proof is that we establish the explicit transition density for 3-dimensional dBM killed upon hitting 0. The second key ingredient is to find the explicit density function for part dBM with varying dimension restricted on $E_2$. The global density function for $M$ can be obtained by combining these two key ingredients, as well as the exact density function for 1-dimensional Brownian motion with constant drift (pushing towards 0) reflected at 0, which was established in \[10\].

Recall that we denote the 3-dimensional distorted Brownian motion by $X$. We let $q(t,x,y)$ denote the transition density function of the part process of $X$ on $E_1 \cong \mathbb{R}^3$ killed upon hitting 0, i.e., for any non-negative function $f \geq 0$ on $E_1$,

$$\int_{E_1} q(t,x,y) f(y) m_1(dy) = \mathbb{E}_x [f(X_t); t < \sigma_0]. \quad (4.2)$$

The following proposition gives the explicit transition density function of a 3-dimensional distorted Brownian motion killed up hitting 0, for all $t > 0$.

**Proposition 6.**

$$q(t,x,y) = \frac{1}{(2\pi t)^{3/2}} e^{-\gamma^2 t/2 - |x-y|^2/(2t)} \frac{1}{\psi_\gamma(x) \psi_\gamma(y)}, \quad x,y \in E_1, \ t > 0. \quad (4.3)$$

Let $X^0$ be the 3-dimensional distorted Brownian motion killed upon hitting 0. It is associated with the Dirichlet form $(\mathcal{E}^{1,0}, \mathcal{F}^{1,0})$ on $L^2(E_1, m_1)$ where

$$\mathcal{F}^{1,0} = \{ f \in \mathcal{F}^1 : \tilde{f}(0) = 0 \},$$

$$\mathcal{E}^{1,0}(f,g) = E_1(f,g), \quad f,g \in \mathcal{F}^{1,0}. \quad (4.4)$$

Note that $\mathcal{C}_0 := C_c^\infty(E_1 \setminus \{0\})$ is a special standard core of $(\mathcal{E}^{1,0}, \mathcal{F}^{1,0})$. Set

$$\mathcal{G} := \{ u \in L^2(E_1, dx) : u/\psi_\gamma \in \mathcal{F}^{1,0} \},$$

$$\mathcal{A}(u,v) := \mathcal{E}^{1,0}(u/\psi_\gamma, v/\psi_\gamma), \quad u,v \in \mathcal{G}. \quad (4.4)$$
It is easy to verify that \((\mathcal{A}, \mathcal{G})\) is a closed form on \(L^2(E_1, dx)\) and \(\mathcal{G}_0 \cdot \psi_\gamma := \{ f \cdot \psi_\gamma : f \in \mathcal{G}_0 \}\) is \(\mathcal{A}_1\)-dense in \(\mathcal{G}\). Since \(\psi_\gamma\) is smooth on \(E_1 \setminus \{0\}\), it follows that \(\mathcal{G}_0 \cdot \psi_\gamma = \mathcal{G}_0\). Hence \(\mathcal{G}_0\) is \(\mathcal{A}_1\)-dense in \(\mathcal{G}\). Take \(u, v \in \mathcal{G}_0\). Mimicking the proof of \([9\text{, Theorem 2.1]}\), one can obtain
\[
\mathcal{A}(u, v) = \mathcal{A}^1,0(u/\psi_\gamma, v/\psi_\gamma) = \frac{1}{2} \int_{E_1} \nabla u(x) \cdot \nabla v(x) dx + \frac{\gamma^2}{2} \int_{E_1} u(x) v(x) dx.
\]
As a result, \(\mathcal{G} = H^1(E_1)\) and \((\mathcal{A}, \mathcal{G})\) is a regular Dirichlet form on \(L^2(E_1)\) associated with the killed Brownian motion with the ratio \(\gamma^2/2\). From \([4, 6]\), we can eventually conclude \([13]\), which completes the proof.

Denote by \(\hat{Y} := |Y|\). The following proposition says that \(\hat{Y}\) can be viewed as a reflected Brownian motion with a constant drift.

**Proposition 7.**
\[
d\hat{Y}_t = dB_t - \gamma dt + dL^0_t, \quad t \geq 0,
\]
where \(L^0\) is semimartingale local time with respect to \(\hat{Y}\).

**Proof.** This is an immediate consequence of applying Tanaka’s formula to \(3.2\).

The following transition density (with respect to Lebesgue measure) of reflected Brownian motion with constant drift was established by Linetsky in \([10\text{, Section 4.2]}\):
\[
\hat{p}(t, x, y) = 2\gamma e^{-2\gamma y} + \frac{2}{\pi} e^{\gamma(x-y)-\gamma^2 t/2} \times \int_0^\infty \frac{e^{-s^2 t/2}}{s^2 + \gamma^2} [s \cos(sx) - \gamma \sin(sx)] [s \cos(sy) - \gamma \sin(sy)] ds, \quad x, y \in [0, +\infty).
\]
By a simple change of measure, we get
\[
\hat{p}(t, x, y) = \hat{p}(t, x, y) \frac{1}{\phi_\gamma(y)^2}
\]
\[
= 1 + \frac{1}{\pi \gamma} e^{\gamma(x+y)-\gamma^2 t/2} \int_0^\infty \frac{e^{-s^2 t/2}}{s^2 + \gamma^2} [s \cos(sx) - \gamma \sin(sx)] [s \cos(sy) - \gamma \sin(sy)] ds, \quad x, y \in [0, +\infty).
\]
(4.5)

Letting \(p_{E_2}(t, x, y)\) denote the transition density of the part process of \(M\) restricted on \(E_2\) (i.e., isomorphic to an 1-dimensional distorted Brownian motion killed upon hitting \(0\)), we first record the following lemma regarding \(p_{E_2}(t, x, y)\).

**Lemma 1.**
\[
p_{E_2}(t, x, y) = \frac{1}{\gamma \sqrt{2\pi t}} e^{-\gamma^2 t/2+\gamma(|x|+|y|)} \left( e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)} \right), \quad x, y \in E_2, \quad t > 0.
\]
(4.6)

**Proof.** The idea of this proof is very similar to that of Proposition \([3]\).

Let \(X^0\) be an 1-dimensional distorted Brownian motion killed upon hitting \(0\). It is associated with the Dirichlet form \((\mathcal{E}^{2,0}, \mathcal{F}^{2,0})\) on \(L^2(E_2, \mathcal{m}_2)\) where
\[
\mathcal{F}^{2,0} = \{ f \in \mathcal{F}^2 : f(0) = 0 \}, \quad \mathcal{E}^{2,0}(f, g) = \mathcal{E}^2(f, g), \quad f, g \in \mathcal{F}^{2,0}.
\]
\((\mathcal{E}^2, \mathcal{F}^2)\) is defined in Section 2.2) Note that \(\mathcal{C}_0 := C^\infty_c(E_2 \setminus \{0\})\) is a special standard core of \((\mathcal{E}^2, \mathcal{F}^2)\). Set
\[
\mathcal{G} := \{ u \in L^2(E_2, dx) : u/\phi_\gamma \in \mathcal{F}^2 \},
\]
\[
\mathcal{A}(u, v) := \mathcal{E}^2(u/\phi_\gamma, v/\phi_\gamma), \quad u, v \in \mathcal{G}.
\]

(4.7)

It is easy to see that \((\mathcal{E}^2, \mathcal{F}^2)\) is an \(h\)-transform of \((\mathcal{A}, \mathcal{G})\), where \(h = \phi_\gamma\). We need to show \(\mathcal{G} = H^1(E_2)\) and \((\mathcal{A}, \mathcal{G})\) is a regular Dirichlet form on \(L^2(E_2)\) associated with the 1-dimensional part Brownian motion on \(\mathbb{R}_+\) killed at a ratio \(\gamma^2/2\). The approach is similar to Proposition 6.2. Below we spell out the details. It is easy to verify that \((\mathcal{A}, \mathcal{G})\) is closed form on \(L^2(E_2, dx)\) and \(\mathcal{C}_0 \cdot \phi_\gamma := \{ f \cdot \phi_\gamma : f \in \mathcal{C}_0 \}\) is \(\mathcal{A}^\ast\)-dense in \(\mathcal{G}\). Since \(\phi_\gamma\) is smooth on \(E_2 \setminus \{0\}\), it follows that \(\mathcal{C}_0 \cdot \phi_\gamma = \mathcal{C}_0\). Hence \(\mathcal{C}_0\) is \(\mathcal{A}^\ast\)-dense in \(\mathcal{G}\). Taking \(u, v \in \mathcal{C}_0\), we have
\[
\mathcal{E}^2(u, v) = \frac{1}{2} \int_{\mathbb{R}_+} u'(x)v'(x) 2\gamma e^{-2\gamma|x|} dx.
\]

Therefore,
\[
\mathcal{A}(u, v) = \mathcal{E}^2(u/\phi_\gamma, v/\phi_\gamma)
\]
\[
= \left( -\frac{1}{2} \left( \frac{v(x)}{\sqrt{2\gamma}} e^{\gamma|x|} \right)^\prime \right) + \gamma \left( \frac{v(x)}{\sqrt{2\gamma}} e^{\gamma|x|} \right)^\prime, u(x)\phi(x)^{-1} \right) 2\gamma e^{-2\gamma|x|} dx.
\]

Consequently, \(\mathcal{G} = H^1(E_2)\) and \((\mathcal{A}, \mathcal{G})\) is a regular Dirichlet form on \(L^2(E_2)\) associated with the part Brownian motion on \(\mathbb{R}_+\) killed at the ratio \(\gamma^2/2\). In view of (4.7), on account of the transition density for 1-dimension part Brownian motion killed upon reaching 0 which is explicitly known,
we can eventually conclude
\[
p_{E_2}(t,x,y) = \frac{1}{\sqrt{2\pi t}} e^{-\gamma^2 t/2} \left( e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)} \right) \frac{1}{\phi_{\gamma}(x)\phi_{\gamma}(y)}, \quad x, y \in E_2, \ t > 0,
\]
where \( \phi_{\gamma}(x) = \sqrt{2\gamma} e^{-\gamma|x|} \), i.e.,
\[
p_{E_2}(t,x,y) = \frac{1}{\gamma\sqrt{8\pi t}} e^{-\gamma^2 t/2+\gamma(|x|+|y|)} \left( e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)} \right), \quad x, y \in E_2, \ t > 0.
\]
\[
\square
\]

**Remark 2.** We notice that \( E_2 = \{ (0, x) : x \in \mathbb{R}_+ \} \cong \mathbb{R}_+ \), and the radial process \( M \) restricted on \( E_2 \) has the same distribution as \( Y \) restricted on \( \mathbb{R}_- \) by switching the sign. Also \( Y \) is symmetric about zero, i.e.,
\[
p_{E_2}(t, -x, y) = p_{E_2}(t, x, y), \quad t > 0, \ x, y \in (0, +\infty).
\]
Namely,
\[
p_{E_2}^Y(t, |x|, |y|) = p_{E_2}^Y(t, |x|, -|y|) = p_{E_2}^M(t, x, y)
\]
\[
= \frac{1}{\gamma\sqrt{8\pi t}} e^{-\gamma^2 t/2+\gamma(|x|+|y|)} \left( e^{-(x-y)^2/(2t)} - e^{-(x+y)^2/(2t)} \right), \quad x, y \in E_2, \ t > 0. \quad (4.8)
\]
Before presenting the proof of finding the global transition density for \( M \), we first record the following simple proposition which will be used repeatedly.

**Proposition 8.** It holds for all \( x, y \in E \) and \( t > 0 \) that

(i) \[
\hat{p}^Y(t, |x|, |y|; \sigma_{\{0\}} < t) + \hat{p}^Y(t, |x|, -|y|) + \hat{p}^Y(t, |x|, |y|) = \hat{p}^Y(t, |x|, |y|).
\]

(ii) \[
\hat{p}^Y(t, |x|, |y|; \sigma_{\{0\}} < t) = \hat{p}^Y(t, |x|, -|y|) = \frac{1}{2} \left( \hat{p}^Y(t, |x|, |y|) - \hat{p}^Y(t, |x|, |y|) \right).
\]

**Proof.** Since \( \hat{Y} = |Y| \), for any \( x, y \in E \), for any \( \delta > 0 \) such that \([|y| - \delta, |y| + \delta] \subset (0, +\infty)\), it holds
\[
\mathbb{P}_{|x|} (Y_t \in [|y| - \delta, |y| + \delta]) + \mathbb{P}_{|x|} (Y_t \in [-|y| - \delta, -|y| + \delta]) = \mathbb{P}_{|x|} \left( \hat{Y}_t \in [|y| - \delta, |y| + \delta] \right).
\]
Therefore,
\[
\mathbb{P}_{|x|} (Y_t \in [|y| - \delta, |y| + \delta]; \sigma_{\{0\}} > t) + \mathbb{P}_{|x|} (Y_t \in [|y| - \delta, |y| + \delta]; \sigma_{\{0\}} < t)
\]
\[
+ \mathbb{P}_{|x|} (Y_t \in [-|y| - \delta, -|y| + \delta]) = \mathbb{P}_{|x|} \left( \hat{Y}_t \in [|y| - \delta, |y| + \delta] \right).
\]
This justifies (4.9). To justify (4.10), observing that \( Y \) is symmetric about 0, we have
\[
\hat{p}^Y(t, |x|, |y|; \sigma_{\{0\}} < t) = \int_0^t \mathbb{P}_{|x|} \left( \sigma_{\{0\}}^Y \in ds \right) \hat{p}^Y(t - s, 0, |y|)
\]
\[
= \int_0^t \mathbb{P}_{|x|} \left( \sigma_{\{0\}}^Y \in ds \right) \hat{p}^Y(t - s, 0, -|y|) = \hat{p}^Y(t, |x|, -|y|). \quad (4.11)
\]
Now (4.10) readily follows from (4.9).
In the following, we divide our discussion into three cases depending on the locations of \( x, y \):

- **Case (i):** \( x, y \in E_1 \);
- **Case (ii):** \( x \in E_1, y \in E_2 \);
- **Case (iii):** \( x, y \in E_2 \).

### 4.1 Case (i): both \( x, y \in E_1 \)

For this case, we recall that the density of 3-dimensional distorted Brownian motion killed upon hitting 0, \( q(t, x, y) \), has been computed in Proposition 6. We first notice that for \( y \in E_1 \), \( p^M(t, 0, y) \) is rotationally invariant in \( y \). Therefore, for any \( y \in E_1 \), there exists

\[
\hat{p}^M(t, 0, r) := \hat{p}^M(t, 0, y), \quad \text{for } r = |y|, \tag{4.12}
\]

Using this notation and polar coordinates, we have for any pair of \( a > b > 0 \),

\[
\int_a^b \hat{p}^Y(t, 0, r)dr = \mathbb{P}^Y_0(a \leq Y_t \leq b) = \mathbb{P}^M_0(M_t \in E_1 \text{ with } a \leq |M_t| \leq b)
\]

\[
= \int_{\{y \in E_1: a \leq |y| \leq b\}} \hat{p}^M(t, 0, y)dy
\]

\[
= \int_a^b 4\pi r^2 \hat{p}^M(t, 0, r)dr.
\]

This implies that

\[
\hat{p}^Y(t, 0, r) = 4\pi r^2 \hat{p}^M(t, 0, r) = 4\pi |y|^2 \hat{p}^M(t, 0, y), \quad \text{for all } y \in E_1 \text{ and } r = |y|.
\]

We now have

\[
p(t, x, y) = q(t, x, y) + \int_0^t \mathbb{P}_x(\sigma^{M(0)} \in ds) p^M(t - s, 0, y)
\]

\[
= q(t, x, y) + \int_0^t \mathbb{P}_x(\sigma^{M(0)} \in ds) \hat{p}^M(t - s, 0, y) \cdot \frac{2\pi |y|^2}{\gamma} e^{2\gamma |y|}
\]

\[
= q(t, x, y) + \int_0^t \mathbb{P}_x(\sigma^{Y(0)} \in ds) \hat{p}^Y(t - s, 0, |y|) \cdot \frac{1}{4\pi |y|^2} \frac{2\pi |y|^2}{\gamma} e^{2\gamma |y|}
\]

\[
= q(t, x, y) + \int_0^t \mathbb{P}_x(\sigma^{Y(0)} \in ds) \hat{p}^Y(t - s, 0, |y|) \cdot \frac{1}{2\gamma} e^{2\gamma |y|}
\]

\[
= q(t, x, y) + \frac{1}{2\gamma} e^{2\gamma |y|} \hat{p}^Y(t, |x|, |y|; \sigma^{Y(0)} < t).
\]

Applying (4.11) to the right hand side of (4.14), we have

\[
p(t, x, y) = q(t, x, y) + \frac{1}{4\gamma} e^{2\gamma |y|} \left( \hat{p}^Y(t, |x|, |y|) - \hat{p}^Y_\sigma(t, |x|, |y|) \right).
\]

Note that

\[
\hat{p}^Y(t, x, y) = \hat{p}^Y(t, x, y) \frac{1}{\phi_\gamma(y)^2} = \hat{p}^Y(t, x, y) \frac{1}{2\gamma} e^{2\gamma |y|}, \quad y \in E_2.
\]

\[
\hat{p}^Y(t, x, y) = \hat{p}^Y(t, x, y) \frac{1}{\phi_\gamma(y)^2} = \hat{p}^Y(t, x, y) \frac{1}{2\gamma} e^{2\gamma |y|}, \quad y \in E_2.
\]
Applying (4.20) to the right hand side of (4.19) yields
\[ y = \frac{1}{\phi_s(y)^2} \hat{p}^y_{\mathbb{R}^+}(t, x, y) = \hat{p}^y_{\mathbb{R}^+}(t, x, y) \frac{1}{2\gamma} e^{2\gamma|y|}, \quad y \in E_2, \] (4.17)
respectively. Replacing the second term on the right hand side of (4.13) with (4.16) and (4.17) yields
\[ p(t, x, y) = q(t, x, y) + \frac{1}{2} \left( p^Y(t, |x|, |y|) - p^Y_{\mathbb{R}^+}(t, |x|, |y|) \right), \]
where \( p^Y(t, x, y) \) and \( p^Y_{\mathbb{R}^+}(t, x, y) \) are given in (4.5), (4.8), respectively.

4.2 Case (ii): \( x \in E_1, \ y \in E_2 \)
For this case, we notice that any path of \( M \) has to pass 0 in order to travel from \( x \) to \( y \). It therefore follows (note that \( y \in E_2 \))
\[ \hat{p}(t, x, y) = \int_0^t \mathbb{P}_x \left( \sigma^M_{\{0\}} \in ds \right) \hat{p}^M(t - s, 0, y) = \int_0^t \mathbb{P}_{|x|} \left( \sigma^Y_{\{0\}} \in ds \right) \hat{p}^Y(t - s, 0, -|y|) = \hat{p}^Y(t, |x|, -|y|) = \frac{1}{2} \left( \hat{p}^Y(t, |x|, |y|) - \hat{p}^Y_{\mathbb{R}^+}(t, |x|, |y|) \right). \]
By (4.16) and (4.17), it immediately follows
\[ p(t, x, y) = \frac{1}{2} \left( p^Y(t, |x|, |y|) - p^Y_{\mathbb{R}^+}(t, |x|, |y|) \right). \] (4.18)

4.3 Case (iii): both \( x, y \in E_2 \)
Finally, the remaining case is that both \( x \) and \( y \) are in the 1-dimensional component \( E_2 \). We note that similar to the previous two cases, it holds
\[ p(t, x, y) = p_{E_2}(t, x, y) + p^M \left( t, x, y; \sigma^M_{\{0\}} < t \right) = p_{E_2}(t, x, y) + p^Y \left( t, -|x|, -|y|; \sigma^Y_{\{0\}} < t \right) = p_{E_2}(t, x, y) + \frac{1}{2\gamma} e^{2\gamma|y|} \hat{p}^Y(t, -|x|, -|y|; \sigma^Y_{\{0\}} < t). \] (4.19)
Again we notice that \( Y \) is symmetric about 0, so
\[ \hat{p}^Y(t, -|x|, -|y|; \sigma^Y_{\{0\}} < t) = \hat{p}^Y(t, |x|, |y|; \sigma^Y_{\{0\}} < t) = \frac{1}{2} \left( \hat{p}^Y(t, |x|, |y|) - \hat{p}^Y_{\mathbb{R}^+}(t, |x|, |y|) \right). \] (4.20)
Applying (4.20) to the right hand side of (4.19) yields
\[ p(t, x, y) = p_{E_2}(t, x, y) + \frac{1}{4\gamma} e^{2\gamma|y|} \left( \hat{p}^Y(t, |x|, |y|) - \hat{p}^Y_{\mathbb{R}^+}(t, |x|, |y|) \right) = p_{E_2}(t, x, y) + \frac{1}{2} \left( \hat{p}^Y(t, |x|, |y|) - \hat{p}^Y_{\mathbb{R}^+}(t, |x|, |y|) \right) \frac{1}{\phi(y)^2} = p_{E_2}(t, x, y) + \frac{1}{2} \left( p^Y(t, |x|, |y|) - p^Y_{\mathbb{R}^+}(t, |x|, |y|) \right) = \frac{1}{2} \left( p^Y(t, |x|, |y|) + p_{E_2}(t, x, y) \right), \]
where the last “=” is due to (4.8). Combining all the three cases above, we have completed the proof to Theorem 1.
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