Classical/quantum correspondence for pseudo-hermitian systems

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In this work, a classical/quantum correspondence for a pseudo-hermitian system with finite energy levels is proposed and analyzed. We show that the presence of a complex external field can be described by a pseudo-hermitian Hamiltonian if there is a suitable canonical transformation that links it to a real field. We construct a covariant quantization scheme which maps canonically related pseudoclassical theories to unitarily equivalent quantum realizations, such that there is a unique metric-inducing isometry between the distinct Hilbert spaces. In this setting, the pseudo-hermiticity condition for the operators induces an involution which guarantees the reality of the corresponding symbols, even for the complex field case. We assign a physical meaning for the dynamics in the presence of a complex field by constructing a classical correspondence. As an application of our theoretical framework, we propose a damped version of the Rabi problem and determine the configuration of the parameters of the setup for which damping is completely suppressed.

Keywords: canonical quantization, pseudo-hermitian operators, pseudoclassical theory, damped Rabi problem

I. INTRODUCTION

The simplest system with non-trivial dynamics that we can build in quantum mechanics is the two-level system. But despite its simplicity, one cannot underestimate the power of this setup. For instance, two-level models are the best understood quantum systems and adequately describe several physically relevant scenarios. Moreover, they play an important role in the understanding of more intricate arrangements. In general, one can treat a quantum two-level system as a spin-1/2 particle interacting with an external magnetic field if the spatial dynamics is not taken into account. Thus, a two-level system is governed by the Pauli equation in (0 + 1) dimension,

\[
i \frac{\partial v}{\partial t} = \hat{H} v, \text{ with } \hat{H} = \frac{\sigma}{2} \cdot \mathbf{F} \text{ and } v = \left( \begin{array}{c} v_1(t) \\ v_2(t) \end{array} \right).
\]

In Eq. (1), \( v \) is a two-component spinor, \( \sigma = (\sigma_1, \sigma_2, \sigma_3) \) are the Pauli matrices and \( \mathbf{F} = (F_1(t), F_2(t), F_3(t)) \) represents an external field. Therefore, solving a two-level system is equivalent to solving (1), to which will be referred as the spin equation (SE).

Among the exact solutions of the SE, we highlight the Rabi problem [1, 2], which has applications in a wide variety of fields, such as quantum optics, condensed matter, molecular, atomic and particle physics and quantum computing. Two-level systems can also be used as a model for open systems, those which interact with the environment in which they are embedded. Although the interaction problem is well-formulated in classical physics, it is not yet fully comprehended at the quantum level. One of the reasons for the lack of a proper quantum description of the interacting process is that open systems are often described by non-hermitian Hamiltonians [3], and consequently, by non-unitary theories. Due to the probabilistic interpretation of quantum mechanics, the notion of a non-unitary theory raises important questions. Despite that, non-unitary theories have drawn some attention in the physics community through the study of a certain class of non-hermitian operators called pseudo-hermitian operators (PHOs). PHOs define the so-called pseudo-hermitian quantum mechanics (PHQM). In PHQM, the freedom in defining an inner-product in the physical Hilbert spaces is explored to recover unitarity. Therefore, one may think that the notion of non-unitarity arises because one is using the “wrong” inner product.

The freedom in choosing the inner product has already been studied [4, 5]. These early developments attempted to recover unitarity from systems using what they called indefinite-metrics quantum theories (the terminology “indefinite-metrics” stands for non-positive-definite inner products). More recently, non-hermitian Hamiltonians with real eigenvalues were considered (see for example [10]). Later on, a series of papers [11–15] exploring whether a Hamiltonian

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5 We are setting \( \gamma = -1 \), where \( \gamma = gq/2m \) with \( q \), \( m \) and \( g \) being, respectively, the charge, mass and the \( g \)-factor of the spin-1/2 particle. Also, in this description, \( \mathbf{F} \) has dimension of energy.
must be hermitian or not were proposed. The authors argued that a weaker and physically transparent condition for the reality of the spectrum of $\hat{H}$ is the presence of $\mathcal{PT}$ symmetry, where $\mathcal{P}$ stands for the parity operator and $\mathcal{T}$ stands for the time-reversal operator. Also, it was shown that if $\hat{H}$ has an unbroken $\mathcal{PT}$ symmetry, there is an operator $\mathcal{C}$, commuting with $\hat{H}$, that allows one to define a positive-definite inner product, with a metric operator given by $\eta = \mathcal{CPT}$.

The issue of what are the necessary and sufficient conditions for the reality of the spectrum of a linear operator were explored in [16–20]. It turns out that the answer to this problem propelled the research in PHQM. It was shown that, albeit relevant, the role played by the $\mathcal{PT}$ symmetry and the $\mathcal{C}$ operator is not a fundamental one. Indeed, it can be seen from PHQM that $\eta = \mathcal{CPT}$ is just an example of a positive-definite metric operator [21]. In fact, the existence of a preferred metric, and its physical meaning, is an open issue in the PHQM. There are several contexts where pseudo-hermitian operators appear [21]. In special, recent treatments of topological aspects of non-hermitian systems use the framework of PHQM [22–30].

A subtle point regarding quantization in general, and quantization in the PHQM framework in particular, is that canonical transformations, which are transformation on the level of the algebra of operators, do not necessarily translate as isometries or unitary transformations between the Hilbert spaces upon which these operators act [31]. When one is faced with non-unitary canonical transformations, for instance, in the infinite-dimensional case, a physical meaning for these transformations can be established by looking at the classical limit of the theory [19, 21, 32]. This procedure is called $\eta$-pseudo-hermitian canonical quantization.

For the present work, the important observation is that there is no usual classical analog for a system with fermionic degrees of freedom. Nevertheless, quantization schemes can still be defined in the context of pseudoclassical mechanics [33–35], in which Grassmann variables are used as phase-space coordinates. In this picture, the Grassmannian degrees of freedom should be quantized with anti-commutation relations, rather than with commutation relations. The latter is of course a well-known scheme for quantization of fermionic degrees of freedom, such as spin.

In this paper, the pseudo-hermitian treatment will be extended to the pseudoclassical framework. Despite the existing treatments concerning pseudoclassical mechanics, its relation with pseudo-hermitian theories was not yet fully analyzed. The aim of this work is to exploit the latter at the level of canonical transformations, considering both the pseudo-hermitian quantum theory and its pseudoclassical limit. For this purpose, complex external fields, associated to non-unitary systems, will be considered. We then study the classical correspondence in order to assign a physical meaning for the complex fields. We construct a covariant quantization scheme which maps canonically related pseudoclassical theories with real and complex external fields to unitarily equivalent quantum realizations, such that there is a unique metric-inducing isometry between the distinct Hilbert spaces. In this setting, the pseudo-hermiticity condition for the operators induces an involution which guarantees the reality of the corresponding symbols, even in the presence of complex external fields. We apply these developments to propose a damped version of the Rabi problem, which could have important implications in related areas. Furthermore, possible experimental tests for the theory are proposed.

This work is organized as follows. In section II, the basic theoretical setup is established, with a revision of the notation used in the present development. In section III, a pseudo-hermitian/pseudoclassical correspondence is proposed and explored. A physical realization of the proposed theoretical framework is constructed in section IV, where the Rabi problem is extended and its generalization analyzed. In section V, final remarks and future perspectives are presented. Units where $\hbar = 1$ are used in this work, except where otherwise indicated.

## II. PSEUDO-HERMITIAN AND PSEUDOCCLASSICAL FRAMEWORKS

### A. Pseudo-hermitian theories

Simply put, pseudo-hermitian operators are operators which are not hermitian or symmetric with respect to the canonical or natural inner product, but which are hermitian with respect to some (positive-definite) inner product. The treatment of pseudo-hermitian operators starts with the observation that non-hermitian matrices (that is, matrices that are not equal to their own conjugate transpose) can have real eigenvalues. It follows that the spectra of the related operators can be associated with physical observables in the quantum description of a system. Taking a pseudo-hermitian operator as the Hamiltonian of the system, an evolution operator can be constructed in such way that the time evolution is unitary [21]. This formalism is the base of the pseudo-hermitian quantum mechanics.

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2 $(\langle x, \mathcal{P} \psi (t) \rangle = \psi (-x, t)$ and $\langle x, \mathcal{T} \psi (t) \rangle = \bar{\psi} (x, -t)$, where the bar denotes complex conjugation.
3 We note that there is a broader definition of pseudo-hermiticity where the product is not necessarily positive definite [30]. This characterization takes into account operators whose eigenvalues appear as complex-conjugate pairs.
Pseudo-hermitian operators in general won’t have orthogonal eigenvectors corresponding to distinct eigenvalues, as do hermitian and normal operators. Despite of this problem, the familiar probabilistic interpretation of quantum mechanics can be recovered with a convenient choice of inner product.

Let us consider the pseudo-hermitian formalism associated to the problem at hand. Let $\mathcal{H}$ be a finite-dimensional Hilbert space isomorphic to $\mathbb{C}^n$ with the canonical inner product $\langle \cdot, \cdot \rangle$, $\mathcal{H} \cong (\mathbb{C}^n, \langle \cdot, \cdot \rangle)$. We denote the adjoint of an operator $T$ with respect to the canonical inner product to be $T^\dagger$.

Now let $\eta : \mathcal{H} \to \mathcal{H}$ and define
\[
\langle x, y \rangle_\eta \equiv \langle x, \eta y \rangle, \quad \forall x, y \in \mathbb{C}^n .
\]
(2)
The sesquilinear form $\langle \cdot, \cdot \rangle_\eta$ is an inner product in $\mathbb{C}^n$ if and only if $\eta = P^\dagger P$ for some invertible $P$. Let us denote this new Hilbert space as $\mathcal{H}_\eta \cong (\mathbb{C}^n, \langle \cdot, \cdot \rangle_\eta)$. We denote $\eta$ as the metric operator.

In this case, an operator $T : \mathcal{H} \to \mathcal{H}$ is pseudo-hermitian or $\eta$-hermitian if and only if it is symmetric with respect to the inner product (2). In other words, $T : \mathcal{H} \to \mathcal{H}$ is pseudo-hermitian if and only if it is hermitian as an operator on $\mathcal{H}_\eta$. It follows that an $\eta$-hermitian operator $T$ satisfies
\[
T = \eta^{-1}T^\dagger \eta .
\]
(4)

It should be noticed that the metric operator $\eta$ is not unique. In fact, if $A$ is any invertible operator which commutes with the $\eta$-hermitian operator $T$, then $T$ is hermitian with respect to the inner product $\langle \cdot, \cdot \rangle_\tilde{\eta} = A^\dagger \eta A$.

For the specific case of the generic two-level system SE in Eq. (1), defined in terms of the Hamiltonian operator
\[
\hat{H} = \frac{1}{2} \begin{pmatrix}
B_3 & B_1 - iB_2 \\
B_1 + iB_2 & -B_3
\end{pmatrix},
\]
(5)
with eigenvalues
\[
E_{\pm} = \pm \frac{1}{2} \sqrt{B_1^2 + B_2^2 + B_3^2} , \quad \eta
\]
(6)
one sees that the operator $\hat{H}$ is pseudo-hermitian if and only if
\[
\det(\hat{H}) = -\frac{1}{4} (B_1^2 + B_2^2 + B_3^2) \in \mathbb{R}_- ,
\]
(7)since this corresponds to real eigenvalues \[36\].

As we will show in section III A, a choice of metric $\eta$ induces an isometry $\mathcal{M}$ between the Hilbert spaces $\mathcal{H}$ and $\mathcal{H}_\eta$, such that the hermitian operators on $\mathcal{H}$ are mapped to hermitian operators on $\mathcal{H}_\eta$. On the other hand, these operators can be seen as images of quantization maps on pseudoclassical phase spaces which are themselves related by canonical transformations. The symbols of these operators, according to each quantization map, are real functions in the respective pseudoclassical phase space.

B. Pseudoclassical theories

In this section we give a brief presentation of a simple non-relativistic model for a spinning particle in the context of pseudoclassical mechanics. Following \[35\], one considers a phase-space formulation where dynamical variables are functions on a Grassmann algebra, such that, upon quantization, their Poisson brackets provide the correct commutation relations.

The Grassmann algebra $G_n(\xi)$ is an algebra over the complex field $\mathbb{C}$ whose generators $\xi_i$, $i = 1, \ldots, n$ satisfy the relations
\[
\xi_i \xi_j + \xi_j \xi_i = 0 .
\]
(8)

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4 The canonical inner product $\langle \cdot, \cdot \rangle$ is defined as $\langle z, w \rangle = \bar{z}w_1 + \cdots \bar{z}_n w_n$, where $z, w \in \mathbb{C}^n$. 

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Functions $f(\xi)$ on $G_n(\xi)$ are polynomials in the generators $\xi_i$. Hence, one can define a derivative operator acting on monomials and, by extension, on functions, as the right-derivatives

$$\frac{\partial}{\partial \xi_i} \xi_{i_1} \cdots \xi_{i_k} = \sum_{j=1}^{k} (-1)^{k-j} \delta_{i_j} \xi_{i_1} \cdots \xi_{i_{j-1}} \cdots \xi_{i_{j+1}} \cdots \xi_{i_k}.$$  \hfill (9)

For our purposes, it is enough to consider Grassmann algebras with three generators, that is, $n = 3$. Thus, a general function $f(\xi)$ on the Grassmann algebra $G_3(\xi)$ is given by

$$f(\xi) = f_0 + f_i \xi_i + f_{ij} \xi_i \xi_j + \frac{i}{3!} f_{ijk} \xi_i \xi_j \xi_k,$$ \hfill (10)

where $f_0, f_i, f_{ij}, f_{ijk} \in \mathbb{C}$ and $f_{ij} = -f_{ji}$. Odd-parity functions $f$ are sums of homogeneous terms with odd numbers of the Grassmann generators, and we write $P_f = 1$. Even-parity functions $f$ are those containing even number of generators, and we write $P_f = 0$.

A relevant non-relativistic pseudoclassical model is given by

$$S = \int_{t_i}^{t_f} \left( L(\xi, \dot{\xi}) \right) dt, \quad L = \frac{i}{2} \xi^i \dot{\xi}_i - H(\xi),$$

where $H(\xi)$ is some even function of the $\{\xi^i\}$, $P_H = 0$. One can proceed as in usual mechanics, and define the conjugate momenta

$$\pi_i = \frac{\partial L}{\partial \dot{\xi}_i} = \frac{i}{2} \xi_i,$$ \hfill (12)

with the derivatives always taken from the right, as defined in Eq. (9). As a result, one finds the canonical Hamiltonian

$$H_c(\xi, \pi) = H(\xi) + (\pi_i - \frac{i}{2} \xi_i) \xi_i.$$ \hfill (13)

There is a natural Poisson bracket in the coordinates $(\xi, \pi)$. Let $f$ and $g$ be functions of the Grassmann variables of definite parity. Then, the Poisson bracket between them is defined as

$$\{f, g\} = \frac{\partial f}{\partial \xi_i} \frac{\partial g}{\partial \pi_i} - (-1)^{P_f P_g} \frac{\partial g}{\partial \xi_i} \frac{\partial f}{\partial \pi_i},$$ \hfill (14)

where derivatives are taken from the right as usual. Thus, the Poisson brackets between the canonical pairs are

$$\{\xi^i, \pi_j\} = \{\pi_j, \xi^i\} = \delta^i_j.$$ \hfill (15)

It should be noticed that the equations (12) are constraints, which we denote as

$$\phi_i = \pi_i - \frac{i}{2} \xi_i.$$ \hfill (16)

Their conservation in time fixes the velocities $\{\dot{\xi}^i\}$:

$$\{\phi_i, H_c\} = 0 \implies \dot{\xi}_i = \frac{i}{\partial H}{\partial H_c{\partial \xi_i}}.$$ \hfill (17)

Therefore, according to Dirac’s terminology for constrained systems, the model (11) is a second-class theory, that is, there are no first-class constraints and the dynamics is completely determined on the constraint surface $\phi = 0$. Following Dirac’s quantization scheme for second-class theories (17), we first define the Dirac brackets over the set of second-class constraints $\phi$ as

$$\{f, g\}_{D(\phi)} = \{f, g\} - \{f, \phi_i\} C^{ij} \{\phi_j, g\},$$ \hfill (18)

where $C^{ij}$ denotes the inverse matrix to $C_{ij} = \{\phi_i, \phi_j\} = -i \delta_{ij}$, and again $f$ and $g$ are parity-definite functions of the Grassmann variables. Thus the non-vanishing Dirac brackets between canonical variables are

$$\{\xi_i, \xi_j\}_{D(\phi)} = -i \delta_{ij}, \{\pi_i, \pi_j\}_{D(\phi)} = \frac{i}{4} \delta_{ij}, \{\xi_i, \pi_j\}_{D(\phi)} = \frac{1}{2} \delta_{ij}.$$ \hfill (19)
One can use the constraints $\phi$ to eliminate the momenta from the description, so that one is only left with coordinates $\xi_i$. Also, by requiring that $\xi_i$ transform as a vector under $O(3)$, it is natural to consider a rotational- and parity-invariant theory. In this case, $H$ must be of the form

$$H = H_B = -\frac{1}{2}\varepsilon_{ijk}\xi_i B_k,$$

(20)

where $B_k$ transforms as a pseudo-vector (for instance, like the magnetic field). Thus, the equation of motion for $\xi_i$ becomes

$$\dot{\xi}_i = \{\xi_i, H\}_{D(\phi)} = -\varepsilon_{ijk}\xi_j B_k,$$

(21)

which is recognized as the classical precession equation, like a magnetic moment immersed in a magnetic field $B = (B_1, B_2, B_3)$.

Of particular interest for the present work is the role of involution and canonical transformations in the pseudo-classical formalism. For a general function $f(\xi)$ as in Eq. (11), we define an involution $*: G_3(\xi) \rightarrow G_3(\xi)$ such that its action on the generators $\xi_i$ is given by

$$\xi_i^* = \xi_i, \quad i = 1, 2, 3.$$  

(22)

Therefore, elements of the real subalgebra (those for which $f^* = f$) are given by Eq. (11) with $f_0, f_i, k_j \in \mathbb{R}$ and $f_{ij} = f_{ji}$. In particular, the $*$-involution as defined above yields $H_B(\xi)$ to be real when $B \in \mathbb{R}^3$. That is,

$$B \in \mathbb{R}^3 \iff H_B(\xi) = H_B^*(\xi).$$

(23)

Suppose we consider a linear canonical transformation on the pseudo-mechanical phase space, defined as a map $(\xi, \pi) \rightarrow (\zeta, \varpi)$, which preserves the symplectic structure in that the only non-vanishing Poisson brackets between the new coordinates are $\{\zeta_i(\xi, \pi), \varpi_j(\xi, \pi)\} = \delta_{ij}$. Due to the constraints $\phi$ in Eq. (11), we observe that $\pi$ is proportional to $\xi$, so we write the linear canonical transformation simply as

$$\zeta_i = R_{ik} \xi_k \text{ and } \varpi_j = R_{jl} \pi_l.$$  

(24)

Then, demanding that this transformation is canonical implies $RR^T = I$, that is, $R$ is an orthogonal matrix. In principle, $R$ can have complex entries, so $R \in O(3, \mathbb{C})$. Furthermore, under this transformation the Hamiltonian function (20) becomes

$$H_F(\zeta) = -\frac{1}{2}\varepsilon_{ijk}\zeta_i F_k,$$

(25)

where

$$F_k = (\text{det} R)R_{kl}B_l.$$  

(26)

Relation (26) implies that

$$F^2 = F_k F_i = (\text{det} R)^2 R_{ij}R_{ik}B_j B_k = \delta_{jk} B_j B_k = B^2.$$  

(27)

Thus, if $B$ is a real field, then from the previous relation it follows that $F^2$ is a positive real number for an arbitrary complex field $F$.

Indeed, considering a complex field $F$, one can define an involution such that (25) is real with respect to the new involution. Initially, let us look at functions on the Grassmann algebra $G_3(\zeta)$ with generators $\{\zeta_i\}_{i=1}^3$, which are given by

$$g = g^0 + g^1_i \zeta_i + g^2_i \zeta_i \zeta_j + ik_g \frac{1}{3!}\varepsilon_{ijk}\zeta_i \zeta_j \zeta_k.$$  

(28)

Then an involution $+: G_3(\zeta) \rightarrow G_3(\zeta)$ can be defined, whose action on generators is given by

$$\zeta^+ = \zeta^*,$$  

(29)

where the $*$-involution is presented in Eq. (22) and the $\zeta$-terms above are taken as function of $\xi$. As a result, the even subalgebra of $G_3(\zeta)$ is given by the functions (28) with $g^0 \in \mathbb{R}$, $g^1 = RR^T g^1$, $R^T g^2 R = (R^T g^2 R)^T$, and $k_g \in \mathbb{R}$. One can also show that the even subalgebras of $G_3(\xi)$ and $G_3(\zeta)$ are isomorphic, since $f = f^* \iff g = g^*$ where $f(\xi) = g(\zeta(\xi))$. It follows that the Hamiltonian function $H_F(\zeta)$ in Eq. (25) is real with respect to the $+$-involution, that is,

$$B \in \mathbb{R}^3 \iff H_F(\zeta) = H^+_F(\zeta).$$  

(30)
III. PSEUDO-HERMITIAN/PSEUDOCLASSICAL CORRESPONDENCE

A. Quantization and hermiticity

Exceptionally in this subsection we restore $\hbar$. Let us define a quantization map $Q : G_3(\xi) \to L(H)$, where $G_3(\xi)$ is the Grassmann algebra with generators $\{\xi_i\}_{i=1}^3$ and $L(H)$ is the set of bounded linear operators on the Hilbert space $H = (C^2, \langle \cdot, \cdot \rangle)$. It suffices to define the map on monomials, following the anti-symmetrization rule

$$Q(\xi_1, \xi_2, \ldots, \xi_n) = \frac{1}{n!} \sum_{\text{perm}} (-1)^{\sigma(\text{perm})} Q(\xi_i) Q(\xi_i) \cdots Q(\xi_i),$$

and extend it linearly to all functions. Furthermore, the quantization map $Q$ is required to map the unit to the identity in $H$, $Q(1) = 1$. In our case, the above requirements imply the following for general classical functions:

$$Q(f) = f_0 1 + f_i Q(\xi_i) + f_{ij} Q(\xi_i) Q(\xi_j) + \frac{1}{3!} k_f \varepsilon_{ijk} Q(\xi_i) Q(\xi_j) Q(\xi_k).$$

It should be noticed that the quantization map $Q$ satisfies

$$f = f^* \implies \langle x, Q(f) y \rangle = \langle Q(f^*) x, y \rangle.$$  

That is, for real functions $f$, $Q(f)$ is symmetric, $Q^*(f) = Q(f)$. The map $Q$ is also required to satisfy the correspondence principle

$$\{f, h\}_D(\phi) = \lim_{\hbar \to 0} \frac{1}{\hbar} [Q(f), Q(h)],$$

where $[\cdot, \cdot]$ is a $Z_2$-graded commutator:

$$[Q(f), Q(h)] = Q(f) Q(h) - (-1)^{P^i P^j} Q(h) Q(f),$$

for all homogeneous functions $f$ and $h$. Thus, one has for the dynamical variables $\{\xi_i\}_{i=1}^3$ the basic anti-commutation relations

$$[Q(\xi_i), Q(\xi_j)] = i \hbar \delta_{ij}.$$

As in the classical case [24], let us consider the canonical transformation $\zeta_i = R_{ij} \xi_j$ with $R \in O(3, \mathbb{C})$. A quantization map $Q' : G_3(\zeta) \to L(H_\eta)$ can be defined in an analogous manner. A natural question then is what the relation between $Q(f)$ and $Q'(\eta)$, where $g(\zeta) = f(\xi(\zeta))$. To address this issue, let us take $P = M^{-1}$ in the expression [3], $\eta = P^i P$, so

$$\eta = (\mathcal{M} M^i)^{-1}.$$  

Then, from Eq. [24], we see that $\mathcal{M} : H \to H_\eta$ is the isometry

$$\langle \phi, \psi \rangle = \langle \mathcal{M} \phi, \mathcal{M} \psi \rangle_\eta$$

for all $\phi, \psi \in C^2$. Thus, for $\phi' = M \phi$ and $\psi' = M \psi$, one has

$$\langle \phi', Q'(g) \psi' \rangle_\eta = \langle \phi, \mathcal{M}^{-1} Q'(g) \mathcal{M} \psi \rangle.$$  

Since $f$ and $g$ represent the same classical state (i.e., are related by a canonical transformation), one has the familiar relation between the operators of the corresponding functions:

$$Q'(g) = \mathcal{M} Q(f) \mathcal{M}^{-1}.$$  

Moreover, let $Q'^+(g)$ denote the adjoint of $Q'(g)$ in the inner product $\langle \cdot, \cdot \rangle_\eta$ in Eq. [24]. It follows from the definition [24] that $Q'^+(g) = \eta^{-1} Q'^*(g) \eta$. Using the results [35] and [40] it is obtained that $Q'^+(g) = \mathcal{M} Q^*(f) \mathcal{M}^{-1}$. Thus, for real $g$ (with respect to the $+$-involution presented in Eq. [24]), the corresponding operator is symmetric, $Q'^+(g) = Q'(g)$, since real $g \ (g^+ = g)$ implies real $f \ (f = f^*)$, and $Q^*(f) = Q(f)$. The similarity relation [40] preserves the canonical relation [35], and can be regarded as a quantum canonical transformation induced by the classical canonical transformation [24].
By means of the relation \( \eta = (\mathcal{M} \mathcal{M}^\dagger)^{-1} \), we see that \( \eta \rightarrow \eta \) if \( \mathcal{M} \rightarrow \mathcal{M} U \), for unitary \( U, U^\dagger = U^{-1} \). Let us call \( Q'_U \) the quantization map with isometry \( \mathcal{M} U \). Then, the relation between \( Q' \) in Eq. (10) and \( Q'_U \) is \( Q'_U = S^+ Q'S \) where \( S = \mathcal{M}(\mathcal{M}U)^{-1} \). That is \( Q'_U \) is \(+\)-unitarily equivalent\(^5\) to \( Q' \):

\[
\langle \phi, Q'_U \psi \rangle_\eta = \langle S \phi, Q' S \psi \rangle_\eta ,
\]

and

\[
\langle S \phi, S \psi \rangle_\eta = \langle \phi, \psi \rangle_\eta .
\]

A unitary representation of the Clifford algebra \((35)\) on \( \mathbb{C}^2 \) is given by the Pauli matrices \( \sigma_i \) as

\[
Q(\xi) = \sqrt{\frac{\hbar}{2}} \sigma_i .
\]

Then, following the quantization rule \((31)\), the Hamiltonian operator \( \hat{H}_B \equiv Q(\hat{H}_B) \) (image of \((20)\) by the quantization map \( Q \)) is

\[
\hat{H}_B = \frac{\hbar}{2} \sigma \cdot B .
\]

We recognize \( \hat{H}_B \) as the Hamiltonian for the spin equation \((1)\). Given a realization of the algebra \((35)\), it is immediate to write a realization for the operators \( Q(g(\zeta)) \) using relation \((40)\). For the particular case of the Hamiltonian function \( H_F \) in Eq. \((25)\), one has

\[
Q' (H_F) = \hat{H}_F = \mathcal{M} \hat{H}_B \mathcal{M}^{-1} .
\]

Since the above relation is a similarity transformation, both operators \( \hat{H}_F \) and \( \hat{H}_B \) have the same eigenvalues, so from this point of view \( \mathcal{M} \) is a mere change of basis in \( \mathbb{C}^2 \).

There is a unique realization of the \( Q'(\zeta) \) algebra, up to the sign of \( \det R \), such that the Hamiltonian operator in both quantizations have the same form, and that realization is

\[
Q' (\zeta_k) = \det R \sqrt{\frac{\hbar}{2}} \sigma_k .
\]

In other words, up to a sign, if the \( Q' \) quantization is realized in the usual representation by Pauli matrices, \( \hat{H}_F \) is given by the operator

\[
\hat{H}_F = \frac{\hbar}{2} \sigma \cdot F .
\]

Thus, starting from this requirement, one fixes the isometry \( \mathcal{M} \) that will give \((17)\) from \((14)\), and because of the result \((37)\), the \( \eta \)-inner product is also fixed. As a result, the \( Q' \)-quantization of \( \hat{H}_F \) will give the operator \((17)\). Furthermore, one sees from this procedure that the isometry \( \mathcal{M} \) is unique. In section \((III.5)\) we will provide a systematic way of constructing the isometry.

In conclusion, \( \hat{H}_B \) describes a quantum theory of a spin system interacting with a real field \( B \), such that \( \hat{H}_B = \hat{H}_B^1 \). At the same time, \( \hat{H}_F \) describes a quantum theory of a spin system interacting with a complex field \( F \) (with \( \text{Im}(F) \neq 0 \)), such that \( \hat{H}_F = \hat{H}_F^1 \). In this sense, what we have achieved so far is to connect the description of a non-relativistic spinning particle under a real field \( B \) with another one with a complex field \( F \), such that the respective Hamiltonians are real under their classic involutions, while the corresponding operators are symmetric (or hermitian) with respect to the inner products of the Hilbert spaces whereupon they act. Both fields are connected by the complex canonical transformation \( R \) by Eq. \((26)\) which implies the important algebraic relation \((27)\).

An important remark following from Eq. \((24)\) should be stressed here. The condition \( F^2 \in \mathbb{R}_+ \) is exactly the condition \((7)\) that the quantum Hamiltonian needs to fulfill so that it is pseudo-hermitian. In other words, the existence of a real field \( B \), canonically related to a field \( F \) with \( \text{Im}(F) \neq 0 \), ensures the reality of the spectrum of \( \hat{H}_F \), according to Eq. \((3)\). This result implies in the existence of a metric operator \( \eta \) that renders \( \hat{H}_F \) hermitian. Furthermore, the same canonical transformation connects the two pseudoclassical models whose Hamiltonians are real with respect to the corresponding involutions.

\(^5\) \( S^+ = \eta^{-1} S^\dagger \eta \) is the adjoint with respect to the \( \eta \)-inner product.
B. Canonical limit and classical correspondence

Unlike the usual description of pseudo-hermitian theories, where the metric operator is not unique, we have seen in section [114] that the metric derived from the isometry [35] is actually unique. We present in this section a schematic way to construct this metric operator, which we call “the canonical limit.”

Besides giving the explicit form of the metric, the canonical limit also furnishes a physical interpretation to our pseudo-hermitian setup. Vectors related by the isometry \( \mathcal{M} \) describe the same physical system. In other words, the Hilbert spaces \( \mathcal{H} \simeq (\mathbb{C}^2, \langle \cdot, \cdot \rangle) \) and \( \mathcal{H}_\eta \simeq (\mathbb{C}^2, \langle \cdot, \cdot \rangle_\eta) \) represent two physically equivalent quantum descriptions (quantizations) of the same classical model, with two classical description that differ by a canonical transformation. Therefore, in order to give correct measurable results, the states must be handled with the appropriate metric.

Let us consider an orthonormal basis \( \{ \phi_\pm \} \) in \( \mathcal{H} \). So, the states \( \phi_\pm \in \mathcal{H} \) were prepared (or measured) by the observer associated to the canonical metric in his quantum description. While the states

\[
\phi'_\pm = \mathcal{M} \phi_\pm
\]

were prepared by an observer associated to the \( \eta \) metric. The states \( \{ \phi'_\pm \} \) form a orthonormal basis of \( \mathcal{H}_\eta \). One observer does not agree about the orthogonality of the states prepared by the other. Thus these observers are using different measurement apparatus to construct the quantum description (of the same system). The use of the canonical metric on the state \( \phi'_\pm \) (or the metric \( \eta \) on \( \phi_\pm \)) is physically meaningless. In the present work, the states whose probabilities must be calculated with the \( \eta \) metric are denoted by primes. The physical description by the observer associated to the metric \( \eta \) is compatible with the presence of an (effective) complex field \( \mathcal{F} \) and the observer associated to the canonical metric measures a real field \( \mathcal{B} \). In other words, we distinguish the observables \( \hat{H}_F : \mathcal{H}_\eta \rightarrow \mathcal{H}_\eta \) and \( \hat{H}_B : \mathcal{H} \rightarrow \mathcal{H} \). For every operator \( A \) acting on \( \mathcal{H} \) there is an equivalent operator \( A' = \mathcal{M} \mathcal{A} \mathcal{M}^{-1} \) acting on \( \mathcal{H}_\eta \).

The classical and quantum descriptions of both observers, especially their notion of orthogonality, must coincide when \( \text{Im}(\mathcal{F}) \rightarrow 0 \). In order to achieve this requirement it is necessary to choose \( \phi'_\pm \) and \( \phi_\pm \) in Eq. (48) to be, respectively, the eigenvectors of \( \hat{H}_F = \hat{H}_F^+ \) and \( \hat{H}_B = \hat{H}_B^+ \).

The imaginary part of \( \mathcal{F} \) can be written as \( \text{Im}(\mathcal{F}_i) = \alpha_i V_i \), where \( \{ \alpha_i \} \) are three dimensionless parameters measuring how far the Hamiltonian \( \hat{H}_F \) is from being canonically hermitian. Thus, we are interested in systems where the canonical hermiticity of \( \hat{H}_F \) is broken continuously, namely, with a well-defined limit \( \alpha_i \rightarrow 0 \). In this limit, \( \hat{H}_F \) becomes hermitian with respect to the canonical inner product, and both theories (defined by \( \hat{H}_B \) and \( \hat{H}_F \)) will differ at most by a unitary transformation. In order to implement this requirement, for a given \( \mathcal{F} \), we choose the real field \( \mathcal{B} \) such that

\[
\lim_{\alpha_i \rightarrow 0} \mathcal{F} = \lim_{\alpha_i \rightarrow 0} \mathcal{B} \in \mathbb{R}^3.
\]

(49)

As we will see in a future example, Eq. (48) gives us a prescription such that, when \( \alpha_i \rightarrow 0 \),

\[
\phi'_\pm \rightarrow \phi_\pm \Rightarrow \mathcal{M} \rightarrow \mathbb{I} \Rightarrow \eta \rightarrow \mathbb{I}.
\]

(50)

In summary, the canonical limit is defined to be the prescription (48), together with the unique isometry which defines \( \eta \) and relates the eigenvectors of \( \hat{H}_F \) and \( \hat{H}_B \).

We turn now our attention to the classical correspondence of two quantum theories: one with a non-hermitian Hamiltonian, and another with a hermitian Hamiltonian. From now on we will assume that non-hermitian operators are those for which there is no inner product with respect to which they are hermitian.

We construct the classical correspondence by taking mean values of operators. The dynamical variables are real numbers that we expect to be related with the measurable behavior of the system. As will see, for non-hermitian Hamiltonians this averaging procedure does not recover the classical equations of motion. On the other hand, the classical equations of motion are recovered for pseudo-hermitian Hamiltonians.

In order to show the above statement, let us first consider the following non-hermitian Hamiltonian \( \hat{H} \),

\[
\hat{H} = \frac{1}{2} \sigma \cdot [\text{Re}(\mathcal{F}) + i \text{Im}(\mathcal{F})],
\]

(51)

which is non-hermitian by construction since its eigenvalues are not real. We generally define the classical correspondence as the normalized mean value (with the appropriate inner product) of the spin operators \( \{ \sigma_i \} \), that is,

\[
n_i \equiv \frac{\langle \psi, \sigma_i \psi \rangle}{\langle \psi, \psi \rangle}, \quad n^2 = 1.
\]

(52)
In the present case, because the Hamiltonian is non-hermitian and there is no suitable inner product, we used the canonical inner product. For \( \psi \) a solution of the time-dependent Schrödinger equation, we have

\[
\dot{n}_i = \frac{1}{\langle \psi, \psi \rangle} \left[ i \langle \psi, (\hat{H}^\dagger \sigma_i - \sigma_i \hat{H}) \psi \rangle - n_i \frac{d}{dt} \langle \psi, \psi \rangle \right],
\]

or, in a vector notation,

\[
\dot{n} = -n \times \text{Re} (F) - n \times [n \times \text{Im} (F)],
\]

It follows that, when the external field is real, that is, when \( \text{Im} (F) = 0 \), relation (54) coincides with Feynman’s results in [38]. Also, in this case the result (54) reproduces the precession equation (21) of the pseudoclassical theory. However, for \( \text{Im} (F) \neq 0 \), Eq. (54) has an additional term that leads to damping of the dynamics of \( n \). The damping term cannot be obtained classically from Eq. (11) by taking the external field to have imaginary entries from the very start, since real and complex fields provide the same equation of motion (21).

At this point, we should mention that, when dealing with a real field \( B \), one has the usual physical interpretation for the spin equation (1), that is, of a charged particle interacting with an external magnetic field. However, when dealing with a complex field, this notion does not hold. Therefore, in order to give a physical meaning for a complex field, we can look at Eq. (54) as

\[
\dot{n} = -n \times F_{\text{eff}}, \text{ with } F_{\text{eff}} = \text{Re} (F) + n \times \text{Im} (F).
\]

In (55), \( F_{\text{eff}} \) plays the role of an effective field in the precession equation. Therefore, when there is damping, the system interacts with the environment in such manner that all the resulting combinations of external and internal fields produce an effective field, which can be represented as a complex external field. In the following section we will give a concrete example.

Consider now the case where \( \hat{H} \) is pseudo-hermitian and therefore \( F^2 \in \mathbb{R}_+ \). We will show that in this case the theory is unitary, and there are no damping terms in the equations of motion. Let \( \langle \cdot, \cdot \rangle_\eta \) be the inner product with respect to which \( \hat{H} \) is hermitian. Then the classical correspondence gives

\[
n_i(t) = \frac{\langle \psi, \sigma_i \psi \rangle_\eta}{\langle \psi, \psi \rangle_\eta} = \langle \psi, \sigma_i \psi \rangle_\eta,
\]

rather than Eq. (52). In this case \( \hat{H}^+ = \hat{H} \) and we have

\[
\dot{n}_i (t) = i \langle \psi, [\hat{H}, \sigma_i] \psi \rangle_\eta = -\varepsilon_{ijk} n_j (t) F_k,
\]

or, in a vector notation,

\[
\dot{n} = -n \times F.
\]

The previous equation corresponds to the pseudoclassical equations of motion (21) even when the external field has an imaginary part. The pseudoclassical equations of motion are recovered from the classical correspondence with the identification \( n \to \zeta \).

We conclude that a non-hermitian Hamiltonian does indeed describe damping. On the other hand, when the Hamiltonian is pseudo-hermitian, the external field fulfills the condition (27) and the system does not presents a damping behavior. In particular, starting with a non-hermitian Hamiltonian, we can change the parameters of the effective field (55) such that the condition \( F^2 \in \mathbb{R}_+ \) (with \( \text{Im} (F) \neq 0 \)) is satisfied. In this case, there is a configuration of \( F \) such that the damping is completely suppressed. In the following we use this property to propose a possible measurable effect. For \( F^2 \in \mathbb{R}_+ \), we can summarize the results in the commutative diagram presented in Figure [1].

We emphasize that the classical-correspondence map in the diagram means that we are able to formally obtain the pseudoclassical equations of motion after the identification of \( n \) with corresponding Grassmann variable, either \( \xi \) or \( \zeta \).

In order to consolidate the physical meaning to this correspondence, as well as the physical interpretation of a complex field, let us introduce a concrete scenario in the next section.
From Eq. (61) we see that
\[ R = \frac{1}{B_1^2 + B_2^2} \begin{pmatrix} F_1 B_1 - B_3 F_3 & 0 & F_1 B_3 + B_1 F_3 \\ 0 & -B_1^2 - B_3^2 & 0 \\ F_1 B_3 + B_1 F_3 & 0 & -(F_1 B_1 - B_3 F_3) \end{pmatrix}. \] (60)

As one can explicitly check, \( R \in SO(3, \mathbb{C}) \) for arbitrary complex vectors \( F \) and \( B \) is an explicit solution to the equation \( F_k = R_{kl} B_l \). In other words, \( \det(R) = 1 \) and \( R \) preserves the symplectic structure (19). In the particular case (60), one has additionally \( R = R^{-1} \). Moreover, one can show that equation (27) under the restriction (59),
\[ F_1^2 + F_3^2 = B_1^2 + B_3^2, \] (61)
is a sufficient condition for the existence of \( R \). As shown in subsection [11] for \( B \in \mathbb{R}^3 \), the Hamiltonians
\[ H_B(\xi) = -i(B_1 \xi_2 \xi_3 + B_3 \xi_1 \xi_3) \quad \text{and} \quad H_F(\zeta) = -i(F_1 \xi_2 \zeta_3 + F_3 \zeta_1 \zeta_3) \] (62)
are real in the sense of the involutions,
\[ H_B(\xi) = H_B^+(\xi) \quad \text{and} \quad H_F(\zeta) = H_F^+(\zeta). \] (63)

Following our prescription for the canonical limit, we now use the eigenvectors of \( \hat{H}_B \) and \( \hat{H}_F \) in order to construct the metric operator \( \eta \). Maintaining the convention of using primes to indicate the states whose probabilities must be calculated with the \( \eta \) metric, we write the eigenvectors \( \phi_{\pm}' \) of \( \hat{H}_F \), with eigenvalues \( E_{F\pm} \), as
\[ \phi_{\pm}' = \frac{1}{F_1} \begin{pmatrix} F_3 \pm E_F \\ F_1 \end{pmatrix}, \quad E_{F\pm} = \pm \frac{E_F}{2} = \pm \frac{1}{2} \sqrt{F_1^2 + F_3^2}, \] (64)
and the eigenvector \( \phi_{\pm} \) of \( \hat{H}_B \), with eigenvalues \( E_{B\pm} \), as
\[ \phi_{\pm} = \frac{1}{B_1} \begin{pmatrix} B_3 \pm E_B \\ B_1 \end{pmatrix}, \quad E_{B\pm} = \pm \frac{E_B}{2} = \pm \frac{1}{2} \sqrt{B_1^2 + B_3^2}. \] (65)

From Eq. (61) we see that \( E_F = E_B \equiv E \). The isometry can be read off from relation (15) for the eigenvector \( \phi_{\pm} \) and \( \phi_{\pm}' \),
\[ M = \frac{1}{F_1} \begin{pmatrix} B_1 & F_3 - B_3 \\ 0 & F_1 \end{pmatrix}, \] (66)
and the metric operator \( \psi \) in \( \mathcal{H}_\eta \) will be given by
\[
\eta = \frac{1}{B^2} \left( \frac{|F_1|^2}{F_1 (B_3 - F_3)} \frac{\bar{F}_1 (B_3 - F_3)}{B_1^2 + |B_3 - F_3|^2} \right).
\]
(67)

As expected from the general theory, one has the hermiticity conditions \( \hat{H}_B = \hat{H}_B^\dagger \) and \( \hat{H}_F = \hat{H}_F^\dagger \). Besides, by the canonical limit, if \( B = F \) we have \( \mathcal{M} = \mathbb{I}, \eta = \mathbb{I}, \hat{H}_B = \hat{H}_F. \)

Assuming that the operator \( \hat{H}_F \) is time-independent, the dynamics is simply obtained by exponentiation of \( \hat{H}_F. \) For instance, if one wishes to evaluate a transition amplitude between the eigenvectors of \( \alpha \) can be read from Eq. (69). In some descriptions, the value \( \sqrt{E} \) agrees with the one in [39]. However, unlike in [39], here the evolution is unitary and states do not lose their normalization condition under time evolution. In general, there is a critical value \( \alpha_c \) of \( \alpha \) for which \( \text{Im}(E = \sqrt{\omega^2 - \alpha^2}) \neq 0 \) if \( \alpha > \alpha_c \). In the illustrated example presented in this subsection, this critical value \( \alpha_c = V \) can be read from Eq. (69). In some descriptions, the value \( \alpha_c \) can be associated with symmetry breaking and a consequent phase transition [40]. In this article, conditions (61) and \( E_F = E_B \in \mathbb{R} \) are assumed.

**B. Rabi problem and the Gilbert damping term**

Let us now consider the more elaborate Rabi problem [1, 2]. This is a two-level system, consisting of a single electron fixed in the space, in interaction with an external magnetic field given by
\[
B_R = (B \cos \omega t, B \sin \omega t, B_z),
\]
(73)
with \( B, B_z \) and \( \omega \) real constants. We can eliminate the second component of the \( B \) field by changing to a rotating reference frame with the help of the rotation
\[
R_z(\omega t) = \exp \left( \frac{i \omega \sigma_3 t}{2} \right).
\]
(74)
In this rotating reference frame we have
\[ B_1 = B,\ B_2 = 0,\ B_3 = \delta,\ \delta = B_z - \omega, \tag{75} \]
and time-independent Hamiltonian
\[ \hat{H}_R = \frac{1}{2} (\delta \sigma_3 + B \sigma_1). \tag{76} \]
The transition amplitude between spin-up and spin-down states \((\sigma_3 \psi_+ = \pm \psi_+)\) is given by the Rabi oscillations
\[ \langle \psi_+, \exp \left( -i \hat{H}_R t \right) \psi_- \rangle = -i \frac{B}{\Omega_R} \sin \left( \frac{\Omega_R t}{2} \right), \quad \Omega_R^2 = B^2 + \delta^2. \tag{77} \]
The \(\delta\) factor is called detuning, while \(\Omega_R\) and \(\omega = B_z\) denote the Rabi frequency and resonance frequency respectively.

As we have seen in section III B, a damped precession is characteristic of a non-unitary evolution. Indeed, we can see that the damping term in Eq. (54) arises exactly from the imaginary part of field, which is what breaks the hermiticity of the Hamiltonian. Therefore, one can consider a damped version of the Rabi problem by introducing an imaginary term in the field (75). For this reason, we choose the external field to be
\[ F_1 = \frac{1 + i \alpha}{1 + \alpha^2} B,\ F_2 = 0,\ F_3 = \frac{1 + i \alpha}{1 + \alpha^2} B_z - \omega, \quad \alpha \in \mathbb{R}. \tag{78} \]
In the limit \(\alpha \to 0\), this field configuration reduces to the original Rabi problem characterized by (75) in the rotating frame. For arbitrary values of the parameters \(B,\ B_z,\ \omega,\ \alpha\), the Hamiltonian \(\hat{H}_F\) is non-hermitian, resulting in a damped behavior.

The time-dependent field configuration for the damped Rabi setup in the non-rotating frame is
\[ \mathbf{F}_R = (F_1 \cos (\omega t), F_1 \sin (\omega t), F_3) = \frac{1 + i \alpha}{1 + \alpha^2} \mathbf{B}_R, \tag{79} \]
which reduces to the original Rabi problem described by \(\mathbf{B}_R\) in (73) when \(\alpha\) is set to zero. We can now obtain the classical correspondence, which we interpret as the behavior of the damped system as actually measured. Substituting the field configuration (79) in Eq. (54), we have
\[ \dot{\mathbf{n}} = -\frac{1}{1 + \alpha^2} \mathbf{n} \times \mathbf{B}_R - \frac{\alpha}{1 + \alpha^2} \mathbf{n} \times (\mathbf{n} \times \mathbf{B}_R). \tag{80} \]
A physical interpretation can now be provided for the parameter \(\alpha\). The above equation describes a damped precession of the magnetic moment. As is well known, this phenomenon can be adequately described by the Landau-Lifshitz-Gilbert (LLG) equation [41], which consists of introducing an \textit{ad hoc} term in the undamped equation of motion. The LLG equation, for the unit magnetization \(\mathbf{n}\), subject to a magnetic field \(\mathbf{B}\), has the form [42]
\[ \frac{d\mathbf{n}}{dt} = -\frac{1}{1 + \alpha^2} \mathbf{n} \times \mathbf{B} - \frac{\alpha}{1 + \alpha^2} \mathbf{n} \times (\mathbf{n} \times \mathbf{B}), \tag{81} \]
where \(\alpha\) is the Gilbert damping parameter. By comparing the LLG equation (81) with the relation (80) obtained via classical correspondence, we see that the \(\alpha\) parameter introduced in Eq. (78) can be identified with the Gilbert damping parameter.

Even though we have just addressed the damped Rabi problem, the identification of \(\alpha\) with the Gilbert damping term is valid for a general effective field \(\mathbf{F}\) in the form
\[ \mathbf{F} = \frac{1 + i \alpha}{1 + \alpha^2} \mathbf{B}, \tag{82} \]
for any \(\mathbf{B} \in \mathbb{R}^3\). This follows from the fact that the classic correspondence equation (51) is exactly the LLG equation for the field configuration (52).
C. The pseudo-hermitian version of the Rabi problem

In this section we choose the parameters $B, B_z, \omega$ and $\alpha$ such that the restriction (61) is satisfied, so that $\hat{H}_F$ is (pseudo) hermitian, $\hat{H}_F^+ = \hat{H}_F$. We introduce the notation

$$F \equiv F_1 = \frac{1 + i \alpha}{1 + \alpha^2} B \quad \text{and} \quad \Delta \equiv F_3 = \frac{1 + i \alpha}{1 + \alpha^2} B_z - \omega,$$

(83)

to label the field components satisfying the condition (61). Now the classical Hamiltonian

$$H_F = -i (F \zeta_2 \zeta_3 + \Delta \zeta_1 \zeta_2),$$

(84)

is real ($H_F^+ = H_F$). The specific choice of parameters can be found from Eq. (7), i.e., from $\text{Im}(F^2 + \Delta^2) = 0$,

$$B^2 + \delta^2 - \alpha^2 \omega^2 + \delta \omega (1 - \alpha^2) = 0,$$

(85)

It follows that

$$F^2 + \Delta^2 = -\delta \omega = B_1^2 + B_2^2.$$  

(86)

Even though $B$ has not yet been determined, the eigenvalues (65) of $\hat{H}_B$ are known, because of relation (61). The Hamiltonian $\hat{H}_F$ has the eigenvectors $\phi_\pm$ and eigenvalues $E_\pm$:

$$\phi_\pm = \frac{1}{B} \left( \begin{array}{c} \Delta \pm \Omega \\ F \end{array} \right), \quad E_\pm = \pm \frac{\Omega}{2}, \quad \Omega^2 = F^2 + \Delta^2.$$  

(87)

From Eq. (86), for $\delta \omega > 0$ the eigenvalues are purely imaginary, however we only consider the case where $\delta \omega < 0$, that is, the case of real eigenvalues. Considering that the limit $\alpha \to 0$ implies

$$\Delta \to \delta, \quad F \to B, \quad \Omega \to \Omega_R,$$

(88)

we use the canonical limit to construct the eigenvectors $\phi_\pm$ of $\hat{H}_B$, which has the same eigenvalues $E_\pm$:

$$\phi_\pm = \frac{1}{B} \left( \begin{array}{c} \delta \pm \Omega_R \\ B \end{array} \right).$$  

(89)

After calculating the eigenvectors $\phi_\pm$ in (87) and (89), one can determine the isometry $M$,

$$M = \frac{1}{F \Omega_R} \left( \begin{array}{cc} B \Omega & \Delta \Omega_R - \delta \Omega \\ 0 & F \Omega_R \end{array} \right),$$

(90)

and the metric operator $\eta$,

$$\eta = \frac{1}{B^2 \Omega^2} \left( \begin{array}{cc} |F|^2 \Omega_R^2 & \tilde{F} \Omega_R (\delta \Omega - \Delta \Omega_R) \\ F \Omega_R (\delta \Omega - \Delta \Omega_R) & B^2 \Omega^2 + |\delta \Omega - \Delta \Omega_R|^2 \end{array} \right).$$

(91)

The expression for $\eta$ in (91) satisfies the canonical limit $\eta \to \mathbb{I}$ when $\alpha \to 0$. The explicit form of $\hat{H}_B$ can be obtained from result (40), i.e., $\hat{H}_B = M^{-1} \hat{H}_F M$. Moreover, one can determine the $B$ field,

$$B = \frac{\Omega}{\Omega_R} (B, 0, \delta),$$

(92)

and the canonical transformation $R$ from Eq. (60).

In order to obtain the pseudo-hermitian version of the damped Rabi problem in the original (non-rotating) frame, one must rotate back the reference frame with the rotation $R_z' = M R_z M^{-1}$, where $R_z (-\omega)$ is given in Eq. (74), that is,

$$\hat{H}_F' = i \frac{\partial R_z'}{\partial t} (R_z')^{-1} + R_z' \hat{H}_F (R_z')^{-1} = \mathcal{M} \hat{H}_B \mathcal{M}^{-1},$$

(93)
with
\[
\hat{H}_B' = \frac{1}{2\Omega_R} \begin{pmatrix} \delta \Omega + \omega \Omega_R & B\Omega \exp(-i\omega t) \\ B\Omega \exp(i\omega t) & -\delta \Omega + \omega \Omega_R \end{pmatrix}.
\] (94)

As expected, \(\hat{H}_B'\) is the field obtained from (92) by the usual rotation (74). The Hamiltonian \(\hat{H}_F'\) keeps its pseudo-hermiticity. In the canonical limit, not only do we verify that \(\hat{H}_F' \rightarrow \hat{H}_B'\), but we also recover the Hamiltonian associated to the Rabi problem in the non-rotating frame (73).

Let us consider the dynamics of this model. Using Eq. (81) we can determine the transition amplitude (88) between spin-up and spin-down states,
\[
\langle \psi_+', \exp(-i\hat{H}_F t) \psi_- \rangle = -\frac{B}{\Omega_R} \sin \left( \frac{\Omega t}{2} \right).
\] (95)

The frequency \(\omega = B_z\) (\(\delta = 0 \rightleftharpoons \Omega = 0\)) represents a critical point, which can be associated with symmetry breaking. From relations (80) and (85) we have
\[
\Omega^2 = \begin{cases} \Omega_R^2 + \frac{\alpha^2}{1+\alpha^2} (\Omega_R^2 - \omega^2) & \text{for } \alpha \neq \pm 1 \\ |\delta \Omega_R| & \text{for } \alpha = \pm 1 \end{cases}.
\] (96)

In summary, when condition (61) holds, the theory is unitary, and there is no damping term in the equations of motion.

Previous results can furnish possible measurable effects. The main point is that \(\hat{H}_F\) is non-hermitian, and thus there would be a damping term in the equations of motion for any value of the external field, except if (85) is valid. When condition (85) is satisfied, the damping effect disappears and the evolution of the system becomes unitary. From Eq. (80) we see that, when \(\omega > 0\), the pseudo-hermitian regime can only be reached for \(\delta < 0\). This means that it is not possible to suppress damping with a frequency below the resonance frequency of the usual Rabi problem. In addition, we can use Eq. (85) to determine, for example, \(B\) as a function of the other parameters:
\[
B^2 = B_z \left[ \omega (1 + \alpha^2) - B_z \right] \text{ for } B, B_z, \alpha \neq 0.
\] (97)

We interpret the condition (97) as the configuration of the \(B\) field which injects energy in the system at the same rate the system dissipates energy. In this case, the damping effect is completely suppressed and the classical limit is again a precession movement described by (88), and not by the LLG equation (81).

V. FINAL REMARKS

In this work, a classical/quantum correspondence for a pseudo-hermitian system with finite energy levels is proposed and analyzed. A dictionary connects particles subjected to real and complex fields (\(B\) and \(F\), related by a canonical transformation. The quantization map ensures hermiticity of operators, whose symbols are real functions in the respective pseudoclassical phase space. The commutativity of the quantization map relates canonical transformations between symbols to unitary transformations between the corresponding operators. In particular, the Hamiltonians associated to \(B\) and \(F\) are real under their classic involutions, and the corresponding operators are symmetric (or hermitian) with respect to the inner products of the Hilbert spaces whereupon they act. An important point in our development is the notion that there isn’t a fundamental distinction between hermitian and pseudo-hermitian (Hamiltonian) operators, or even between ordinary quantum mechanics and pseudo-hermitian quantum mechanics for that matter, as long as the relations summarized by Figure 1 are satisfied. That is, as long as the possibility of a metric redefinition which reestablishes hermiticity in quantum theory can be seen as a consequence of a proper choice of coordinates in the pseudoclassical theory. The only non-trivial physical statement is that non-hermitian operators can become pseudo-hermitian under certain regimes.

Furthermore, we show that there is a unique isometry between the Hilbert spaces \((\mathbb{C}^n, \langle \cdot, \cdot \rangle)\) and \((\mathbb{C}^n, \langle \cdot, \cdot \rangle_{\eta})\) that preserves the representation of the Clifford algebra chosen in both settings (real and complex), implying a unique metric. A systematic way of constructing this metric is provided. In addition, we apply the classical correspondence to the two-level quantum system coupled to a complex field. For non-hermitian Hamiltonians, this correspondence describes damping and does not recover the classical equations of motion. When the Hamiltonian is pseudo-hermitian, this correspondence does not imply damping and the classical equations of motion are recovered.

As an application, we propose a damped version of the Rabi setup, considering a complex field associated to a non-hermitian Hamiltonian. We identify the parameter that controls the intensity of the imaginary part as the Gilbert
damping parameter of the Landau-Lifshitz-Gilbert equation. In this setup, we find a specific configuration of the parameters where the damping is completely suppressed. In this case, the classical correspondence describes again a precession movement for the spinning particle. We interpret this arrangement as the configuration where the applied field completely compensates the damping effect. It may be identified with the so-called steady-state precession [43], where an external field cancels the spin damping, generating a constant angle precession. The steady-state regime could be observed with measurements involving ferromagnetic resonance methods [44]. We believe that the presented developments could be verified in laboratory tests.

In addition, the classical/quantum correspondence for pseudo-hermitian systems may have practical applications. The precise manipulation of the spin has several technological consequences and a description of damping process is essential in this manipulation. For example, in the emerging technologies of spintronic devices. In nowadays applications, the dynamics of the magnetization in the digital storage process is described by the LLG equation and any deviation from this description should have practical implications. The possibility of suppressing the damping behavior could lead to a faster and more energy-efficient spin manipulation. Phenomena in the steady-state precession regime have also consequences in processes involving magnetic resonance [45].

Recent developments of the pseudo-hermitian setup suggest interesting perspectives for the theoretical framework presented here. Effects involving non-hermiticity enhances the dynamics of the topological-phase transitions, bringing up new effects considering the scenarios involving the usual hermitian framework [46]. Topological properties of the theory can be explored, by evaluating quantities such as the Berry phase. A second-quantization approach of the semiclassical damped Rabi problem proposed in the present work can be investigated following a treatment in the same lines as the one presented in [47]. Finally, the developed formalism might be extendable to lattice systems [48], and in this case topological phase transitions in the exceptional points could be investigated, as done for optical lattices [49].

ACKNOWLEDGMENTS

K. R. acknowledges the support of Coordenação de Aperfeiçoamento de Pessoal de Nível Superior (CAPES), Finance Code 001, Brazil; and the National Council for Scientific and Technological Development (CNPq), Brazil, with grant #141264/2020-9. R. F. acknowledges the support of São Paulo Research Foundation (FAPESP), Brazil, with grant #2016/03319-6. C. M. acknowledges the support of National Council for Scientific and Technological Development (CNPq), Brazil, with grant #420878/2016-5.

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