Unruh Effect for General Trajectories

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Abstract

We consider two-level detectors coupled to a scalar field and moving on arbitrary trajectories in Minkowski space-time. We first derive a generic expression for the response function using a (novel) regularization procedure based on the Feynmann prescription that is explicitly causal, and we compare it to other expressions used in the literature. We then use this expression to study, analytically and numerically, the time dependence of the response function in various non-stationarity situations. We show that, generically, the response function decreases like a power in the detector’s level spacing, \( E \), for high \( E \). It is only for stationary world-lines that the response function decays faster than any power-law, in keeping with the known exponential behavior for some stationary cases. Under some conditions the (time dependent) response function for a non-stationary world-line is well approximated by the value of the response function for a stationary world-line having the same instantaneous acceleration, torsion, and hyper-torsion. While we cannot offer general conditions for this to apply, we discuss special cases; in particular, the low energy limit for linear space trajectories.
1 Introduction

The physics of accelerated quantum systems in Minkowski space-times share some interesting features with quantum gravity phenomena that are much less tractable; e.g. with the Hawking evaporation of black holes\[^{[1]}\] and quantum aspects of expanding universes and inflation\[^{[2]}\]. Beside being of great interest on their own right, such Minkowskian phenomena may shed light on their full fledged curved space time cousins.

A moving two-level detector (so called a Unruh or de Witt detector) that couples to the modes of a field of interest is a simple and very useful paradigm for studying quantum effects of accelerated systems. The most renowned of the phenomena studied with such a device is the Unruh effect\[^{[3]}\], whereby a linear, uniformly accelerated detector in flat space-time perceives the vacuum as a thermal bath of the modes to which it couples. (Whenever we refer in the paper to space characteristics of a world-line, such as its being linear, circular, etc, we mean it to be so in some Lorentz frame.) One starts by defining a vacuum—the ground state of the various fields of interest—based on a set of creation-annihilation operators associated with inertial observers. This means that modes of any free field are eigenmodes with respect to the proper-time of such inertial observers.

The vacuum effects associated with the detector are encapsulated in the response function, which is the object we study in this paper. The correct way to regularize the divergent expression for the two-point Wightman function, which appears in the definition of the response function, is still a moot issue. For example, following work by Takagi\[^{[4]}\], Schlicht\[^{[5]}\] proposed to start with a detector of finite spatial width in the proper frame. In this case, the Wightman function is regular, and the point detector response function is gotten by taking the width to zero. This gives well-defined expressions, which have been generalized by others\[^{[6, 7]}\]. We discuss the merits of this approach, and propose an alternative that is, in our view, superior in some regards.

The response function is an explicit function of \(\tau\), the proper-time along the world-line, and of \(E\); it is also a functional of the world-line. Since the response function is Poincaré invariant, it is reduced, in \(3 + 1\) dimensions, to a functional of the three world-line invariants: the acceleration \(a(\tau)\), the torsion \(T(\tau)\), and the hyper-torsion \(H(\tau)\). Beside the trivial, inertial case, only two world-lines have so far lent themselves to analytic evaluation of the response function: the original, Unruh case (constant \(a, T = H = 0\) ), and planar cusp motion (constant \(a = T, H = 0\) ), which is also the infinite-Lorentz-factor limit of uniform, circular motion response function, as we show below. The high-\(\gamma\), circular Unruh effect has been discussed extensively (e.g., by Bell and Leinaas\[^{[8]}\], and by Unruh\[^{[9]}\]), who obtained analytic results for the infinite Lorentz factor case, in the hope that it can be detected in electrons moving in storage rings. Letaw and Pfautsch\[^{[10]}\] considered, numerically, circular motion with finite velocities. Experimental manifestations of the linear Unruh effect have been discussed in \[^{[11, 12]}\]. The above are all examples of stationary world-lines, for which the response function is constant. Our main goal is to get some insight into properties of the response function for more general, non-stationary world-lines.

All along we assume that the interaction of the detector with the relevant field is ever present and constant with time. Otherwise the response function will be affected. For example even inertial detectors will be characterized by a non-zero response function if the interaction strength is variable.

The metric is \((+,-,-,-)\) and, except when expressed explicitly, the value of the constants \((c, \hbar, G, k_B)\) is taken as unity.

Section 2 is dedicated to the discussion of some general properties of the response function including our proposed regularization. This is then applied in section 3 to derive various numerical and analytical results for non-stationary trajectories. For example, we give integral representations of the transition rate as function of the ratios \(E/a\) and \(E/T\) in \(1 + 1\) and \(2 + 1\)
dimensions. We also constrain the asymptotic behavior of the response function in the high $E$ regime. Section 4 summarizes our conclusions.

2 The response function

2.1 Preliminaries

We start with some well documented preliminaries. One considers a massless, chargeless scalar field $\Phi(t, \vec{x})$. In a flat Minkowski space-time one can use particle states associated with inertial observers and decompose the field operator into these states according to the standard definition:

$$\Phi(t, \vec{x}) = \int \int \int_{-\infty}^{+\infty} \frac{d\vec{k}}{2(2\pi)^3||\vec{k}||} \left( a_\vec{k} e^{-ik_\mu x^\mu} + a^{+}_\vec{k} e^{+ik_\mu x^\mu} \right),$$

where the annihilation and creation operators satisfy the usual commutation rules $[a_\vec{k}, a^{+}_\vec{k}'] = \delta(\vec{k} - \vec{k}')$. We couple this field to a point-like moving detector constrained to follow the time-like world-line $x^\mu(\tau) = [t(\tau), \vec{x}(\tau)]$, where $\tau$ is the proper-time along the trajectory.

Following the two-level-detector paradigm, the detector’s intrinsic structure is described by two levels: the ground level $|\downarrow\rangle$ and an excited one $|\uparrow\rangle$, separated by an energy gap $E$ in the proper frame. It is standard to take the interaction Hamiltonian for the model, in the interaction picture, as

$$H_{int}(\tau) = g \left( e^{iE\tau} |\uparrow\rangle \langle \downarrow| + e^{-iE\tau} |\downarrow\rangle \langle \uparrow| \right) \Phi[x^\mu(\tau)],$$

where $g$ is the coupling constant. By this expression, $H_{int}$ can be interpreted as a sort of boundary condition imposed on the field along the world-line $x^\mu(\tau)$. The state of the system is the product of the field’s particle content and of the detector’s state. Denoting the field vacuum by $|0\rangle$, the ground level of the entire system is $|0, \downarrow\rangle$. In the absence of interaction this is an eigenstate of the free Hamiltonian and so remain stable. When the interaction is on, the field vacuum becomes unstable and gets populated by quanta while the detector makes transitions. Starting from the system vacuum at $\tau_0$, one gets, at a later time $\tau$, a system state

$$|\Psi(\tau)\rangle = \hat{T} e^{-i \int_{\tau_0}^{\tau} d\tau' H_{int}(\tau')} |0, \downarrow\rangle,$$

where $\hat{T}$ stands for time ordering. The probability $P_+(\tau)$ to find the detector in the upper level at $\tau$ (independently of the field particle content) is given by the expectation value of the projector $\Pi_+ = |\uparrow\rangle\langle \uparrow|$ in the above mentioned state. Another useful quantity—the subject of our discussion—is the transition rate (response function) defined as $R_+(\tau) \equiv \frac{dP_+}{d\tau}$. Retaining the lowest (second) order contribution in the interaction strength $g$, we have for $\tau > \tau_0$

$$P_+(\tau) \equiv \langle \Psi(\tau)|\uparrow\rangle \langle \uparrow|\Psi(\tau) \rangle \quad (4)$$

$$= g^2 \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau} d\tau_2 e^{-iE(\tau_1 - \tau_2)} W(\tau_1, \tau_2)$$

$$= 2g^2 \text{Re} \int_{\tau_0}^{\tau} d\tau_1 \int_{\tau_0}^{\tau} d\tau_2 e^{-iE\tau_2} W(\tau_1, \tau_1 - \tau_2) \quad (5)$$

$$R_+(\tau) = 2g^2 \text{Re} \int_{0}^{\tau - \tau_0} d\tau' e^{-iE\tau'} W(\tau, \tau - \tau'),$$
where

\[ W(\tau_1, \tau_2) \equiv \langle 0 \Phi[x^\mu(\tau_1)] \Phi[x^\mu(\tau_2)]0 \rangle \]  

(7)

is the Wightman two-point (vacuum) function evaluated along the trajectory, and use was made of its property: \( W(\tau_1, \tau_2)^* = W(\tau_2, \tau_1) \). The Wightman function can be written explicitly in terms of the trajectory:

\[
W(\tau_1, \tau_2) = \frac{1}{2(2\pi)^3} \iiint_{-\infty}^{+\infty} \frac{\tilde{k}}{||\tilde{k}||} e^{-ik_{\mu}[x^\mu(\tau_1)-x^\mu(\tau_2)]} .
\]  

(8)

The probability \( P_- \) and the corresponding response function \( R_- \) related to the de-excitation process are obtained by replacing \( E \) by \(-E\) in the expressions above. If a steady-state is reached, the ratio of level populations is \( n_+/n_- = R_+/R_- \).

Equations(5) and (6) exhibit causality explicitly: to get these quantities at time \( \tau \) we only have to know the trajectory prior to this time.

The quantities we deal with here enjoy an obvious and useful scaling relation stemming from invariance under change in the units of length: If two world-lines, \( x^\mu(\tau) \) and \( \hat{x}^\mu(\hat{\tau}) \) are related by \( \hat{x}^\mu(\hat{\tau}) = \lambda x^\mu(\tau/\lambda) \) then their respective response functions are related by

\[ \hat{R}_+(\hat{\tau}, E) = \lambda^{-1} R_+(\hat{\tau}/\lambda, \lambda E) . \]  

(9)

The intrinsic, Poincaré invariant, quantities that describe a world-line \( x^\mu(\tau) \) in 3 + 1 space-time can be chosen as the acceleration \( a(\tau) \), the torsion \( T(\tau) \), and the hyper-torsion \( H(\tau) \), defined as

\[
a(\tau) = [-a^{\mu} a_{\mu}]^{1/2}, \quad T(\tau) = a^{-1} [\dot{a}^2 - a^{\mu} \ddot{a}_{\mu}]^{1/2}, \quad H(\tau) = a^{-3} T^{-2} \epsilon^{\alpha\beta\gamma\delta} \ddot{x}_{\alpha} \dot{a}_{\beta} \dot{a}_{\gamma} \ddot{a}_{\delta} ,
\]

(10-12)

where \( \dot{\tau} \equiv \frac{d}{d\tau} \), \( a^{\mu} = \ddot{x}^\mu(\tau) \), and \( \epsilon^{\alpha\beta\gamma\delta} \) is the Levi-Civita anti-symmetric tensor (compare Letaw [13]). A world-line for which the three invariants are \( \tau \) independent is called stationary. For such world-lines all points are equivalent and, clearly, \( R_+(\tau) \) is constant.

The response function, being itself Poincaré invariant, can be written as a functional of these [instead of one of the frame specific \( x^\mu(\tau) \)]. The above scaling relation implies for \( f(\tau) = a, H, T \)

\[ \hat{f}(\hat{\tau}) = \lambda^{-1} f(\tau/\lambda) , \]

(13)

so that

\[ R_+ [\tau, E; a(\tau), T(\tau), H(\tau)] = \lambda^{-1} R_+ [\lambda^{-1} \tau, \lambda E; \lambda a(\lambda \tau), \lambda T(\lambda \tau), \lambda H(\lambda \tau)] . \]  

(14)

(We use the convention whereby a dependent variable is a functional of the function variables appearing after the semicolon.) This can be used to reduce the number of variables. For instance if we take \( \lambda = E^{-1} \) we get

\[ R_+ [\tau, E; a(\tau), T(\tau), H(\tau)] = E \hat{R}_+ \left[ E \tau; \frac{a(E \tau)}{E}, \frac{T(E \tau)}{E}, \frac{H(E \tau)}{E} \right] , \]

(15)

where \( \tau_E = \tau/E \).
2.2 Regularization of the Wightman function

As expressed in Eq. (5), the Wightman function is not well defined since the integrand behaves as $k$ times an oscillating term in the UV limit. Various regularization procedures can be found in the literature. For instance, one can add a small imaginary part $-i\epsilon$ term to $t(\tau_1) - t(\tau_2)$, as in Birrell and Davies\cite{14}. This deformation multiplies the integrand in Eq. (5) by $e^{-k\epsilon}$, which ensures convergence. This gives a Wightman function of the form

$$W^{BD}(\tau_1, \tau_2) = -\frac{1}{4\pi^2} \frac{1}{t(\tau_1) - t(\tau_2) - i\epsilon} \frac{1}{\|\vec{x}(\tau_1) - \vec{x}(\tau_2)\|^2}.$$ \hspace{1cm} (16)

This procedure has been criticized by Takagi\cite{4} and by Schlicht\cite{5} who showed that it leads to non-physical results even for some of the most simple examples. Takagi proposed a regularization based on endowing the detector with a small, finite extent, which is then described with the help of Fermi-Walker coordinates. Schlicht\cite{5} improved Takagi’s procedure, retaining causality explicitly in each step. This regularization is implemented by replacing

$$e^{-ik_\mu x^\mu(\tau)} \Rightarrow e^{-ik_\mu x^\mu(\tau)} e^{-\epsilon k_\mu \dot{x}^\mu(\tau)},$$ \hspace{1cm} (17)

where $\epsilon$ is now a measure of the proper spatial extension of the detector. Since $k^\mu$ is light-like and $\dot{x}^\mu$ is future oriented and time-like, $k_\mu \dot{x}^\mu(\tau) > 0$ ensures the integrability for large $|\vec{k}|$. The resulting expression for the Wightman function is

$$W^S(\tau_1, \tau_2) = -\frac{1}{4\pi^2} \frac{1}{\{x^\mu(\tau_1) - x^\mu(\tau_2) - i\epsilon[\dot{x}^\mu(\tau_1) + \dot{x}^\mu(\tau_2)]\}^2}.$$ \hspace{1cm} (18)

This procedure has been extended to various space-time topologies\cite{6} as well as for more general kinds of spatial extents\cite{7}. Without the regulator (i.e., putting $\epsilon = 0$) these two formulations of the Wightman function are inversely proportional to the square of geodesic distance

$$d^2(\tau_1, \tau_2) = \eta_{\mu\nu}[x^\mu(\tau_1) - x^\mu(\tau_2)][x^\nu(\tau_1) - x^\nu(\tau_2)].$$ \hspace{1cm} (19)

2.2.1 Proper-time regularization

Motivated by several requirements that we deem desirable in the Wightman function, we would like to propose another regularization. The Wightman function always appears as a weight function in integrations that result in observable quantities such as the transition rate, the flux emitted\cite{15}, the conditional energy\cite{16}, the flux correlation function\cite{17}, etc. Hence, $W(\tau, \tau - \tau')$ should be viewed in Eq. (6) as a distribution of the running variable $\tau'$. In this respect, the analytic properties of $W$ is of key importance. More specifically, one needs to know the poles distribution of the Wightman function and their relative weights in order to express analytically quantities such as the transition rate.

As we previously noticed, apart from the coincidence point $\tau' = 0$ where it diverges, the Wightman function is inversely proportional to the square of geodesic distance. This means that the analytic properties of $W$ (its poles) come directly from the zeros of the geodesic distance in the complex plane. We see then that the Wightman function formally possesses only one real pole at $\tau' = 0$ and (possibly) an infinity of poles symmetrically distributed above and below the real axis, since the geodesic distance is real. The regularization is only needed to cure the divergence at $\tau' = 0$. More exactly, the regularization’s role is to isolate this pole such that it does not contribute to $R_+$, only to $R_-$. As a consequence, one finds that the difference between the excitation and de-excitation response functions, for every trajectory, is precisely the contribution of this pole (see next section).
From the last remarks, one deduces that it is desirable in a regularization that it displace the \( \tau' = 0 \) pole above the real axis such it does not contribute to \( R_+ \), and preserve the distribution of other poles in. The above two procedures satisfy the first requirement but not the second. A way to shift the zero pole while preserving the characteristics of the world-line in a consistent way is to introduce the regulator by replacing \( x^\mu(\tau) \) with \( x^\mu(\tau \pm i\epsilon) \). The wave-function in the integrand of Eq.\( \{27\} \) becomes

\[
e^{-i\epsilon\mu x^\mu(\tau)} \Rightarrow e^{-i\epsilon\mu x^\mu(\tau - i\epsilon)},
\]

which gives the regularized Wightman function

\[
W^{pt}(\tau_1, \tau_2) = -\frac{1}{4\pi^2} \frac{1}{[x^\mu(\tau_1 - i\epsilon) - x^\mu(\tau_2 + i\epsilon)]^2}.
\]

Schlicht’s Wightman function is the first-order expansion of our Eq.\( \{21\} \) in \( \epsilon \). This means that the two regularizations have the same \( \epsilon \to 0 \) limit and should give the same results for observables in this limit. The superiority of one choice over the other may, however, show up when we want to (or have to) calculate with a finite value of \( \epsilon \), e.g. for computational convenience. The non-vanishing of \( \epsilon \) matters only near the coincidence point \( \tau' = 0 \) where it appears in the denominator in Eqs.\( \{13\} \) and \( \{21\} \) respectively as \(-4\epsilon^2 \) and \(-4\epsilon^2 + 4\epsilon^2a^2(\tau)/3 + \mathcal{O}(\epsilon^6) \). Therefore, Schlicht’s Taylor expansion agrees with ours (barring the yet higher order terms) as long as \( \epsilon \ll 1/|a(\tau)| \).

Consider, for example the canonical linear motion with constant acceleration \( a_0 \), for which all the expressions can be calculated exactly for finite \( \epsilon \). The motion is hyperbolic, with

\[
x^\mu(\tau) = \left[ \frac{\sinh(a_0\tau)}{a_0}, \frac{\cosh(a_0\tau)}{a_0}, 0, 0 \right].
\]

Schlicht’s regularization gives for finite \( \epsilon \)

\[
W^{S}_{acc}(\tau, \tau - \tau') = -\frac{a_0^2}{16\pi^2} \left[ \sinh\left(a_0\tau'/2\right) - ia_0\epsilon \cosh\left(a_0\tau'/2\right) \right]^{-2},
\]

whereas our proposed proper-time regularization Eq.\( \{21\} \), gives

\[
W^{pt}_{acc}(\tau, \tau - \tau') = -\frac{a_0^2}{16\pi^2} \sinh^{-2}(a_0(\tau' - 2i\epsilon)/2),
\]

The corresponding transition rates are

\[
R^S_+ (\epsilon) = R^U_+ \times \frac{\frac{2E}{a_0} \arctan(ea_0)}{1 + \epsilon^2 a_0^2},
\]

\[
R^{pt}_+ (\epsilon) = R^U_+ \times \frac{\frac{2E}{a_0} \left( \frac{ea_0}{\pi} \right)^{3/2}}{1 + \epsilon^2 a_0^2}.
\]

where \( R^U_+ \) is the Unruh \( \{3\} \) expression, which is the \( \epsilon = 0 \) limit of the two regularization schemes \( ([x] \) is the integer part of \( x \)). To obtain our result, we have used the pole decomposition \( 1/\sinh^2(x) = \sum_{n=\infty}^{\infty} 1/(x + in\pi)^2 \) and integrated by the method of residues. As expected, expressions \( \{25\} \) and \( \{26\} \) agree to first order in \( a_0 \epsilon \) for which they both differ from the Unruh expression by a factor \( e^{2E\epsilon} \). So, both schemes still require \( \epsilon E \ll 1 \). However, for large values of \( ea_0 \) Schlicht’s formulation leads to a vanishing transition rate whereas ours always differs from
the correct expression by a finite factor between 1 and \( e^{2\pi E/a_0} \), which is near one for \( E \ll a_0 \) for arbitrary values of \( \epsilon \).

For world-lines with bounded values of the acceleration we can choose a value of \( \epsilon \) that is much smaller than \( 1/a \) at all times, and then the two regularizations should give very similar results. The situation is, however, different for world-lines with unbounded acceleration. Consider as an example the tractable case of a linear space trajectory with \( a(\tau) = -\alpha/\tau, \ -\infty < \tau < 0 \). For \( \alpha \neq 1 \) this is gotten, for example, for the world-line

\[
x^\mu(\tau) = \left[ \frac{1}{2} \left( \frac{(-\tau)^{1+\alpha}}{1+\alpha} - \frac{(-\tau)^{1-\alpha}}{1-\alpha} \right), \frac{1}{2} \left( \frac{(-\tau)^{1+\alpha}}{1+\alpha} - \frac{(-\tau)^{1-\alpha}}{1-\alpha} \right), 0, 0 \right];
\]

(28)

for \( \alpha = 1 \) a representative world-line is

\[
x^\mu(\tau) = \left[ \frac{-\log(-\tau \kappa)}{2\kappa} - \frac{\tau^2 \kappa}{4}, \frac{-\log(-\tau \kappa)}{2\kappa} - \frac{\tau^2 \kappa}{4}, 0, 0 \right].
\]

(29)

\( R_+(\tau) \) does not depend on \( \kappa \), as different values of \( \kappa \) are related by a Poincaré transformation. This trajectory is particularly interesting because when followed by a moving mirror it generates a constant thermal flux \( \propto \kappa^2 \) with temperature \( \propto \kappa \).)

For these world-lines, the scaling law, for a finite value of \( \epsilon \), becomes (in both regularization schemes) \( R_+(\epsilon/\lambda, \tau/\lambda, \alpha, E/\lambda)/\lambda = R_+(\epsilon, \tau, \alpha, E) \). Choosing \( \lambda = 1/a \) we can write \( R_+(\epsilon, \tau, \alpha, E) = a\tilde{R}_+(a\epsilon, \alpha, E/a) \). The question we ask at this junction is: “how well does the expression for finite \( \epsilon \) approximate the correct (\( \epsilon = 0 \)) value?”. We present in Fig. (1) the departure factor \( R_+(\epsilon, \tau, \alpha, E)/R_+(0, \tau, \alpha, E) = \tilde{R}_+(a\epsilon, \alpha, E/a)/\tilde{R}_+(0, \alpha, E/a) \) as a function of \( \epsilon a \), for several values of the pair \( \alpha, E/a \) using Schlicht’s regularization, as well as ours. We see that, in line with what we saw in the constant acceleration case, at least for low energies the proper-time regularization works well for values of \( \epsilon a \) even an order larger than for Schlicht’s regularization.

![Figure 1](image_url)

**Figure 1:** The ratio \( R_+(\epsilon, \tau, \alpha, E)/R_+(0, \tau, \alpha, E) = \tilde{R}_+(a\epsilon, \alpha, E/a)/\tilde{R}_+(0, \alpha, E/a) \) as a function of \( a\epsilon \) for \( 2\pi E/a = 10^{-2} \) (left panel) and \( 2\pi E/a = 1 \) (right panel) and for two values of \( \alpha = 5, 100 \), as marked. Continuous lines correspond to the proper-time regularization, dashed curves to Schlicht’s regularization.

### 2.2.2 The subtracted Wightman function

The troublesome singularity in the Wightman function that we wish to regularize appears in the same way (same pole position and residuum) in all trajectories including the inertial ones...
for which the transition rate vanishes. Being time-like, all trajectories have for small \( \tau \) lapses:
\[
d^2(\tau, \tau - \tau') = \tau'^2 + O(\tau'^4),
\]
(The \( \tau'^2 \) term vanishes identically for all world-lines.) So there is some \( f \) such that
\[
d^2(\tau, \tau - \tau') = \tau'^2 + \tau'^4 f(\tau, \tau - \tau'),
\]
where \( f \) is regular at \( \tau' = 0 \); in fact \( f(\tau, \tau) = a(\tau)^2/12 \). We can then write using Eq. (21)
\[
R_+(\tau) = -\frac{g^2}{2\pi^2} \Re \int_0^\infty d\tau' \frac{e^{-iE\tau'}}{(\tau' - 2i\epsilon)^2} + \frac{g^2}{2\pi^2} \int_0^\infty d\tau' \frac{\cos(E\tau')f(\tau, \tau - \tau')}{1 + \tau'^2 f(\tau, \tau - \tau')},
\]
where we put \( \epsilon = 0 \) in the second term, which is now without the pole at \( \tau' = 0 \). The first integral vanishes for positive \( E \) and \( \epsilon \), as we close the integration contour around the lower complex plane. (In fact, this term is just the response function for an inertial world-line.) For the de-excitation rate we have
\[
R_-(\tau) = \frac{g^2 E}{2\pi} + R_+(\tau).
\]
The same pole free expression was used in [10, 13] for stationary orbits, and was generalized to non-stationary world-lines, though with some restrictions, in [12], using Schlicht’s regularization. These authors extended the applicability of the pole-free expression to finite time interaction [Eq.(6.1) in [7]]. We note that this finite interaction time expression follows straightforwardly in the proper-time regularization scheme[Eqs.(30)(31)] applied to the general response function Eq.(6). It turns out that the pole-free expression is not necessarily more useful, in numerical determinations of the response function, than the finite \( \epsilon \) regularizations. The reason is that the function \( f \) in the integrand is defined as the difference between two diverging expressions and its numerical evaluation requires, in the end, some sort of regularization.

### 3 The transition rate–some derived properties

In this section we discuss two issues regarding the response function of general world-lines. The first is its behavior for high detector energy gaps for stationary and non-stationary trajectories. The second issue concerns the applicability of the “quasi-stationary” approximation, whereby the response function is represented by that of a stationary world-line with the instantaneous values of acceleration, torsion, and hyper-torsion.

#### 3.1 High energy behavior

To study the high energy behavior of the response function we use the subtracted Wightman function Eq.(31). One can then write:
\[
R_+(\tau) = \int_0^\infty d\tau' \cos(E\tau')F_\tau(\tau'),
\]
where
\[
F_\tau(\tau') = \frac{g^2}{2\pi^2} \frac{f(\tau, \tau - \tau')}{1 + \tau'^2 f(\tau, \tau - \tau')},
\]
Following an idea in [7] we integrate by parts \( 2N \) times to get for any \( N \)
\[ R_+ (\tau) = \sum_{n=1}^{N} \left( \frac{-1}{E^2} \right)^n F_{\tau}^{(2n-1)} (0) + \left( \frac{-1}{E^2} \right)^N \int_0^{\infty} d\tau' \cos (E \tau') F_{\tau}^{(2N)} (\tau'), \tag{35} \]

where we used the fact that \( F_{\tau}^{(n)} (\tau' \to \infty) = 0 \) since \( F_{\tau} (\tau') \) decreases at least as rapidly as \( \tau'^{-2} \). Using the Riemann-Lebesgue lemma\(^{[19]}\), the second term is a \( \mathcal{O}(x^{-2N}) \). So, Eq.\(^{35}\) can be used to pin down the asymptotic behavior of the response function for \( E \to \infty \). This we proceed to do below, first for stationary trajectories, then for non-stationary ones.

The first three odd derivatives of \( F_{\tau} \) are useful and are given in terms of the world-line derivatives as:

\[
\left( \frac{g^2}{2\pi^2} \right)^{-1} F_{\tau}' (0) = -\frac{a\dot{a}}{12}, \tag{36}
\]

\[
\left( \frac{g^2}{2\pi^2} \right)^{-1} F_{\tau}^{(3)} (0) = -\frac{a\ddot{a}}{12} + \frac{a^3\dot{a}}{20} - \frac{a a^{(3)}}{30} + \frac{a\dot{a}T^2}{60} + \frac{a^2 T \dot{T}}{60}, \tag{37}
\]

\[
\left( \frac{g^2}{2\pi^2} \right)^{-1} F_{\tau}^{(5)} (0) = -\frac{a^5 \dot{a}}{24} - \frac{a^3 \dot{a} T^2}{9} - \frac{13 a a^3}{63} - \frac{a^4 T \ddot{T}}{18} + \frac{5 a^2 T \dot{T}}{42} + \frac{5 a \ddot{a} T^2}{42} + \frac{53 a^2 \dot{a} \ddot{a}}{126} + \frac{5 \ddot{a} a T^2}{84} + \frac{5 a \ddot{a} T \dot{T}}{42} + \frac{5 a^2 \ddot{T} \dot{T}}{42} + \frac{5 a a^{(3)}}{14} + \frac{5 a (3)^2 T^2}{252} - \frac{5 a a^{(3)}}{252} + \frac{5 a^2 T \dot{T}^{(3)}}{252} - \frac{5 a \ddot{a} a^{(4)}}{72} - \frac{a a^{(5)}}{56} + \frac{x_{\mu}(4) x_{\mu}(5)}{168}. \tag{38}
\]

### 3.1.1 Stationary trajectories

For stationary world-lines the geodesic distance, Eq.(19), is a function of the lapse \( \tau' = \tau_1 - \tau_2 \); so, by Eq.(6), the transition rate is constant (provided the interaction strength is constant): We can write with the proper-time regularization, using \( W(\tau, \tau - \tau') \Rightarrow W^{\text{stat}} (\tau' - i\epsilon) \), and the general property \( W^{\text{stat}} (\tau')^* = W^{\text{stat}} (-\tau') \)

\[
R^{\text{stat}}_+ = g^2 \int_{-\infty}^{+\infty} d\tau' e^{-iE\tau'} W^{\text{stat}} (\tau' - i\epsilon), \tag{39}
\]

These world-lines were classified and analyzed for example in \([13]\). They can be grouped into six classes, equivalent to the orbits of time-like Killing vector fields. As said above, they are all characterized by constant values of \( a, T, \) and \( H \). For \( a = T = H = 0 \) we get inertial world-lines; linear, uniformly accelerated ones have \( T = H = 0, a \neq 0 \); and circular, constant speed ones have \( (a < T, H = 0) \) (\( H = 0 \) for planar trajectories).

The general property \( d^2(\tau_1, \tau_2) = d^2(\tau_2, \tau_1) \) leads, for stationary trajectories, to \( F_{\tau} (-\tau') = F_{\tau} (\tau') \). Thus, all the odd derivatives of \( F_{\tau} \) at \( \tau' = 0 \) vanish, and with them the first sum in Eq.(35), for any \( N \). We conclude then that for all stationary trajectories \( R_+ \) vanishes faster than any power of \( E \) in the high energy limit. This is clearly in line with the known thermality of the response function for uniformly accelerated trajectories (the original Unruh case Eq.(24)). We discuss other examples below.
Note first that from the general scaling law \[15\], we can reduce by one the number of independent variables in the stationary case. For example we have in this case:

\[ R_+(E, a, T, H) = E \tilde{R}_+(a/E, T/E, H/E) . \]  

(40)

For general planar trajectories (up to a Poincaré transformation) the hyper-torsion vanishes, and the stationary trajectories are characterized by the acceleration and the torsion. Circular trajectories, which are defined by \( T > a \), can also be defined by some orbital radius \( R \) and a frequency \( \Omega \) in the frame in which the center of the orbit is at rest, such that \( a = \gamma^2 \Omega^2 R \) and \( T = \gamma a / \sqrt{\gamma^2 - 1} \), where \( \gamma = (1 - R^2 \Omega^2)^{-1/2} \). \( T < a \) defines hyperbolically unbound trajectories, and \( T = a \), which is the the \( \gamma \to \infty \) limit of the circular case, corresponds to the so called “cusp motion”. It is realized, for example, by the world-line

\[ x^\mu_{\text{cusp}}(\tau) = \left[ \tau + \frac{a^2 \tau^3}{6}, \frac{a \tau^2}{2}, \frac{a^2 \tau^3}{6}, 0 \right] \]  

(41)

A circular world-line is realized, for example, by:

\[ x^\mu_{\text{circ}}(\tau) = \left[ \frac{T \tau}{\omega}, \frac{a}{\omega^2} \cos \left( \frac{\omega \tau}{2} \right), \frac{a}{\omega^2} \sin \left( \frac{\omega \tau}{2} \right), 0 \right] , \]  

(42)

where \( \omega \equiv (T^2 - a^2)^{1/2} \) (for circular orbits \( \omega = \gamma \Omega \)). The geodesic distances for these three types of world-lines is given by

\[ d^2(\tau, \tau - \tau') = \frac{T^2}{\omega^4} \left[ \omega^2 \tau^2 - \frac{4a^2}{T^2} \sin^2 \left( \frac{\omega \tau}{2} / 2 \right) \right] , \]  

(43)

and is applicable also for imaginary (hyperbolically unbound case), and vanishing (cusp case) \( \omega \).

The response function for the three stationary cases is of the form \( E \) times a function of \( \nu \equiv a/E \) and \( \mu \equiv T/E \). Defining \( y = E \tau / 2 \) and \( \kappa = \omega / E = (\mu^2 - \nu^2)^{1/2} \) we obtain with the proper-time regularization:

\[ R_+ = \frac{g^4 E}{4\pi^2} \int_{-\infty}^{+\infty} \frac{dy}{\kappa^2 \mu^2 \nu^2 (y - i\epsilon)^2 - \nu^2 \sin^2 \left( \kappa (y - i\epsilon) \right)} e^{-2iyy} . \]  

(44)

An analytic expression is known for the cusp motion[13]:

\[ R^\text{cusp}_+ = \frac{g^2 a e^{-2\sqrt{3}E/a}}{8\pi \sqrt{3}} , \]  

(45)

which, again, decreases with \( E \) faster than any power, as we found for all stationary trajectories. This is also the response function for the circular case in the limit of infinite \( \gamma \), which was derived before by Bell and Leinaas[8] and by Unruh[9]. (They proceeded by reducing the circular geodesic distance Eq.(13) to the contribution of its first three zeros, integrating the Wightman function by the method of residues. It can easily be understood that this gives the same result as using the cusp geodesic distance Eq.(13) for \( \omega = 0 \).) This infinite \( \gamma \) “circular Unruh effect” is quasi thermal in the sense that it can be described by a “temperature” that varies slowly with \( E \), monotonically from \( T^< = \frac{\gamma}{4\sqrt{3}} \) at low energies to \( T^> = \frac{\gamma}{2\sqrt{3}} \) at high energies. Letaw and Pfautsch[10] used the subtracted Wightman function to numerically calculate the behavior of the transition rate for finite \( a \) and \( T \). It is of interest to have an analytic expression for \( R^\text{circ}_+ \) for finite values of \( \gamma \). We derive the first order correction in \( \gamma^{-2} \). For the geodesic distances one finds

\[ d^2_{\text{circ}} = d^2_{\text{cusp}} [1 - \frac{a^4 \tau^4}{300 + 30a^2 \tau^2} \frac{1}{\gamma^2} + u(a \tau) \mathcal{O}(\gamma^{-4})] , \]  

(46)
where \( d^2_{\text{cusp}} \) is the expression for the cusp world-line [Eq. (43) with \( \omega = 0 \)]; From this one deduces for fixed \( E/a \) in the limit of large \( \gamma \)

\[
R_+^{\text{irc}}(E, a, \gamma) = R_+^{\text{cusp}}(E, a) \left[ 1 + \frac{1}{5\gamma^2} \left( \frac{E}{a/2\sqrt{3}} - 1 \right) + s(E/a)O(\gamma^{-4}) \right].
\] (47)

We verified this with numerical results for \( R_+ \). Using this and the expression for \( R_- \) we see that the effective temperature in the limit \( E \to 0 \) is modified to

\[
\mathbf{T}_\gamma^c \approx \mathbf{T}^c(1 - 1/5\gamma^2).
\]

We do not have an analytic expression for fixed \( \gamma \) and high \( E \).

### 3.1.2 Non-stationary trajectories

We start again with the subtracted Wightman function as in Eq. (35). We noted that for a stationary trajectory all the odd derivatives vanish because the expression for the geodesic distance from \( \tau \) to \( \tau - \tau' \) is symmetric in \( \tau' \) at \( \tau' = 0 \). The opposite is also true: if there is such a symmetry for all values of \( \tau \) the trajectory is stationary. Thus, if the trajectory is not stationary there will be \( \tau \) values (generically, all values of \( \tau \)) where at least some of the odd derivatives do not vanish. For such \( \tau \) the response function decays only as a power-law at high energies, and thus, in particular, the spectrum is not thermal. For example, if \( \dot{a} \neq 0 \) the dominant high energy behavior, from Eq. (36), is

\[
R_+(\tau) \approx R_+^0(\tau) = \frac{g^2}{24\pi^2} \frac{a\dot{a}}{E^2}.
\] (48)

If the acceleration is constant on a world-line but the torsion is not, the dominant contribution is, generically, proportional to the derivative of the torsion \( T(\tau) \). From Eq. (37) we get

\[
R_+(\tau) \approx \frac{g^2}{120\pi^2} \frac{a^2TT}{E^4}.
\] (49)

(Note that if \( a \) vanishes at an isolated point, the first term in Eq. (37) gives a finite contribution to the response function at that time.) If \( T \) is also constant the next order will dominate and it contains the derivative of the hyper-torsion through \( x^{\mu(1)}x^{\mu(5)} \propto HH \). If this too vanishes the trajectory is a stationary one. Conversely, if the world-line is non-stationary, so either \( a \), \( T \), or \( H \) do not vanish identically, at least one of the first three orders doesn’t vanish identically. We conclude that generically the high-energy behavior of the transition rate for non-stationary trajectories is given by

\[
R_+(\tau) \approx g^2\alpha(\tau)E^{-2n},
\]

where \( \alpha(\tau) \) contains the acceleration, the torsion or the hyper-torsion and their derivatives and where \( n = 1, 2 \) or 3.

We checked numerically the validity of the first order approximation for the linear trajectory Eqs. (25) (29), having \( a(\tau) = -\alpha/\tau \). In Fig. (2) we plot the ratio \( \xi \equiv R_+/R_+^0 \), calculated numerically with the proper-time regularization. Using the scaling law we can write

\[
R_+(\tau, \alpha, E) = E\tilde{R}_+(E/a, \alpha).
\]

Since \( R_+^0 \) is obtained from a Taylor expansion of \( R_+ \), it also satisfies this scaling. Thus, the ratio \( \xi \equiv R_+/R_+^0 \) is a function of \( \alpha \) and \( E/a \). Fig. (2) shows this ratio as a function of \( E/a \) for different values of \( \alpha \). We see that indeed Eq. (48) is a very good approximation for the response function at large values of \( E/a \). For \( \alpha \) larger than about 5 the power-law behavior starts at the same value of \( E/a \) regardless of the value of \( \alpha \). In the large-\( \alpha \)-low-\( E/a \) regime \( \xi \) can, in fact, be approximated analytically very well using the quasi-stationary approximation—see section 3.2 and Fig. (5), so the behavior displayed in Fig. (2) for this regime can be reproduced analytically.

In the above example, once \( E/a \gg 1 \), the dominant power in \( E/a \) is a good description of the response function for all values of \( \alpha > 1 \). This can be traced to the fact that in this case

\[
E^{-2}F_{\tau}^{(3)}(0)/F_{\tau}^0(0) = O((a/E)^2) \quad (T \equiv 0 \text{ for the present, linear case}).
\]

In general, the contribution
of the higher orders hinges also on other characteristics of the world-line as reflected, e.g., in the additional terms in $F_\tau^{(3)}(0)$. To better demonstrate this we considered another example: Take a linear trajectory with acceleration law $a(\tau) = 2 + \tanh \tau$ in some arbitrary units ($\tau$ in the inverse of the same unit); $a$ is bound between 1 in the far past and 3 in the far future. Such an acceleration law is gotten for example for the linear world-line:

$$x^\mu(\tau) = \left[-e^{-\tau} + \frac{e^\tau}{4} + \frac{e^{3\tau}}{12} + \arctan(e^{-\tau}), e^{-\tau} + \frac{e^\tau}{4} + \frac{e^{3\tau}}{12} - \arctan(e^{-\tau}), 0, 0\right].$$

At large negative times the detector has a nearly constant acceleration with the characteristic time for acceleration change $t_a = a/\dot{a} \simeq (3/4)e^{-2\tau} \gg t_a = 1/a \simeq 1$. The first term in the expression for $F_\tau^{(3)}(0)$ dominates again and then we expect that for $E \gg 1$ the first order power-law will be a good description. As $\tau$ approaches zero the additional terms in the expression for $F_\tau^{(3)}(0)$ become important. We first show in Fig. 3 the computed transition rate itself for a single value of the energy $E = 20$ (in the same units). It follows $R_+^0 \propto \dot{a}(\tau)a(\tau) =$
Figure 4: The ratio $\xi \equiv R_+/R_0^+$ for the linear trajectory Eq. (50) with acceleration $a(\tau) = 2 + \tanh \tau$ and for different values of the energy.

$(2 + \tanh \tau)/\cosh^2 \tau$ rather closely (as expected since $E \gg a(\tau)$ for all $\tau$). The latter is maximal at $\tau_0 = \ln \left[ (2 + \sqrt{7})/3 \right]/2$ at which it takes the value $2(10 + 7\sqrt{7})/27 \approx 2.11$.

In Fig. 4 we plot the ratio $\xi = R_+/R_0^+$ for different values of $E$. For $E$ larger than about 5 all the curves cross a value of 1 at about the same value of $\tau \simeq -0.906313$, which is a zero of $F_\tau^{(3)}(0)$.

This is a reminder that all our results are valid only to the lowest order in $g^2$. So, for example, if we speak of asymptotic behavior it is to be understood in the following sense: Given a high enough energy, $E$, we assume that $g$ is small enough for the lowest order to dominate at $E$. We have no reason to think that our results on the asymptotic behavior are invalidated by higher order corrections. However, even if they are, our claim is even strengthened. Take the non-stationary case. If higher order terms decay even faster with $E$, then obviously they do not affect our results. If, in some case, higher order terms decay even slower than the first order behavior we found, then this even strengthen our conclusion regarding the non-thermal behavior and the power-law decay. In the stationary case where we found the first order response function to decay faster than any power, it might in principle happen that higher order terms decay slower. However, this is known not to be the case at least for the Unruh case where $R^+$ is known to all orders.

### 3.2 The quasi-stationary approximation

One might think that if the world-line is characterized by $a$, $T$, and $H$ that vary slowly enough, the response function is well approximated by the expression

$$R_+[\tau, E; a(\tau), T(\tau), H(\tau)] \approx R_0^+[E, a(\tau), T(\tau), H(\tau)] = E \frac{a(\tau)}{E} \frac{T(\tau)}{E} \frac{H(\tau)}{E}. \quad (51)$$

Here, the value of $R_+$ at $\tau$, which is a functional of $a(\tau)$, $T(\tau), H(\tau)$, is approximated by its value for a stationary world-line with the instantaneous values of these parameters. For example, Letaw [13] remarks on linear trajectories: “if a detector was moving with a very slowly increasing linear acceleration one would expect the spectrum to be Planckian, though with a slowly increasing temperature”. Our analysis of the high energy behavior of $R_+$ shows that this cannot be correct across the spectrum. For any given non-stationary world-line, no matter how slow the variations in $a, H, T$ are, the quasi-stationary approximation has to break down.
at high enough energies as the spectrum becomes power-law as opposed to that of stationary world-lines. However, at low enough energies, slowness of variation of the three invariants might suffice to validate the quasi-stationary approximation. It is difficult to give general criteria for this so we treat numerically several linear motions. In this case we write

\[ R_+ (\tau, E) = R_+^q (\tau, E) + D(\tau, E), \]

\[ R_+^q (\tau, E) \equiv \frac{g^2 E}{2\pi} \frac{1}{e^{2\pi E/a(\tau)} - 1}, \]

where \( R_+^q \) is the stationary (Unruh) expression for the response function. We wish to know under what conditions \( |D(\tau, E)| \ll R_+^q (\tau, E) \). As we said above, one necessary condition is that \( E \) is small enough, and it is convenient to express this smallness in terms of \( E/a \) (provided \( a \neq 0 \) locally). Another condition it that the acceleration varies slowly enough on the world-line: the time scale of variation of \( a \): \( t_{tr} \equiv a/\dot{a} \), has to be long compared with \( t_a \equiv 1/a \). In other words \( \dot{a}/a^2 \ll 1 \). In general, \( t_{tr} \) is not the only measure of the orbital time variations, but in the examples that we consider below it is a good single measure. We considered, numerically again, the two families of world-lines discussed before. In the first instance we looked at the world-line with \( a(\tau) = -\alpha/\tau \). In this case, the time ratio \( a^2/\dot{a} = \alpha \) is constant along the world-line, and it also represents, up to some numerical factor the time ratio for all the possible time scales defined by the trajectory. We show in Fig.5 the ratio \( \zeta \equiv R_+ / R_+^q \) as a function of \( E/a \) for several values of the parameter \( \alpha \). Because of the scaling properties mentioned above, \( \zeta \) is a function of only these two variables, and is given by:

\[ \zeta = \frac{e^{2\pi E/a}}{\pi a E/a} - 1 \int_0^\infty d\tau' \cos (\alpha \tau' E/a) \left\{ \frac{\alpha^2 - 1}{[1 - (1 + \tau')^{1-\alpha}] [1 - (1 + \tau')^{1+\alpha}]} + \frac{1}{\tau'^2} \right\}. \]

**Figure 5:** The ratio \( \zeta \equiv R_+ / R_+^q \) as a function of \( E/a \) for different values of \( \alpha \) (as marked) for the linear trajectory with acceleration \( a(\tau) = -\alpha/\tau \).

We see that indeed the quasi-stationary approximation is good (\( \zeta \approx 1 \)) when \( E/a \) is small and the orbital variations are slow (high \( \alpha \)), but it breaks down for all values of \( \alpha \) plotted for large \( E/a \), and for all values of \( E \) if \( \alpha \) is small. The parameter \( \zeta \) has a well defined low energy limit, which for high \( \alpha \) behaves, according to the numerical results, as \( \zeta(E = 0, \alpha) \approx 1 - \alpha^{-1} \).

Next, we reconsidered the linear trajectory Eq.50 with acceleration \( a(\tau) = 2+\tanh \tau \). Since the acceleration is bounded between 1 and 3, the condition \( E \ll a \) is tantamount to \( E \ll 1 \). The ratio of time scales \( \dot{a}/a^2 \) is small for large, positive and negative, values of \( \tau \) where it decreases...
expansively, but is of order one near \( \tau = 0 \). In Fig. (6) we plot \( \zeta \) as a function of \( \tau \) for different values of \( E \). For \( E = 2 \) which corresponds to \( E/a \approx 2 \) for large negative \( \tau \), and \( E/a \approx 2/3 \) for high positive \( \tau \)—we see that \( \zeta \) departs from one appreciably even when the acceleration varies very slowly. Also for very small values of \( E \) the quasi-stationary approximation is valid only when the acceleration varies on time scales longer than \( 1/a \) (i.e., away from \( \tau = 0 \)).

Finally, we consider a linear trajectory with an acceleration law \( a(\tau) = \tau - 2/3 \) \((-\tau) - 1/3 \) \((\tau < 0)\). It is realized, e.g., by the world-line
\[
x^\mu(\tau) = \left[ \tau_0 \sqrt{\frac{2}{3}} \left( \int_0^A dx \sinh x^2 - A \sinh A^2 \right), -\tau_0 \sqrt{\frac{2}{3}} \left( \int_0^A dx \cosh x^2 - A \cosh A^2 \right), 0, 0 \right],
\]
(55)
where \( A = \sqrt{\frac{3}{2}} \left( \frac{-\tau}{\tau_0} \right)^{1/3} \). This acceleration law might be of interest because it reproduces the time dependence of the surface acceleration of an evaporating black hole (in which case \( \tau \) is the proper-time of a static observer at infinity). Accelerating detectors and black hole are related (e.g., [3]): An analogy can be drawn between the Unruh thermal bath for a uniform acceleration \( a \), and Hawking radiation from a constant mass black hole with surface gravity \( \kappa = 1/M \), as the temperatures for the two cases are given respectively by
\[
k_B T_U = \frac{h a}{2 \pi c}, \quad k_B T_H = \frac{h \kappa}{2 \pi c}.
\]
(56)
If we apply the semiclassical treatment down to complete evaporation through a massless scalar field, and take the time \( \tau = 0 \) at this event, one finds that \( \kappa = T_P^{-2/3} (-\tau)^{-1/3} \), where \( T_P = t_P/\sqrt{80\pi} \) and \( t_P \) is the Planck time (the numerical factor relating \( T_P \) and \( t_P \) varies when considering other fields according to their mass, spin, etc).

In discussing black hole evaporation one usually assumes the analogue of our quasi-stationary approximation; i.e., that at any given time a black hole radiates as a black body corresponding to its momentary mass. We set to check to what extent such an approximation is justified in our mock-evaporation case\(^1\). Of course, our chosen acceleration law is based on the quasi-stationary

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\(^1\)The process of Hawking radiation has been proven to keep its thermal character until the mass of the black-hole reaches \( \mathcal{O}(M_{\text{Pl}}) \) (see [20] for a heuristic picture, [21] for a direct numerical treatment and [22] for a semiclassical back-reaction analysis). Our analogy is an alternative, heuristic check.
approximation in deducing the time dependence of the mass. It is also clear that the analogy becomes meaningless beyond about $\tau = -T_P$. But we still consider the analogy at all times as it may be of interest for its own sake: in general our parameter $\tau_0$ need not be related to the Planck time. Again, we consider the parameter $\zeta = R_+/R_+^0$ as a measure of the departure from quasi-stationarity. The scaling law for $R_+$ tells us that this quantity can be written as a function of two parameters; for example $\zeta = \zeta(\tau/\tau_0, E/a)$, or alternatively $\zeta = \zeta(\tau/\tau_0, E\tau_0)$. The first helps to follow the time dependence of $R_+$ at a fixed value of $E/a$, which tells us where on the black body spectrum we are; the second representation follows the time evolution for a fixed value of the energy. The time variable $\tau/\tau_0 = (3\dot{a}/a^2)^{-3/2}$ is also the parameter we use to measures the fastness of the variation of $a$.

We plot the above two representations of $\zeta$ (computed with the proper-time regularization) in Figs.(7) and (8) respectively. In the first $\zeta$ is plotted vs. $\tau/\tau_0$ for different values of $E/a$ and in the second for different values of $E\tau_0$. (For values of these parameters below the minimum ones plotted the curves remain the same.) We see that, in line with our previous results,
for high enough energies, or rapid enough variations in the acceleration, the quasi-stationary approximation breaks down appreciably. For $-\tau/\tau_0 > 100$, where $\dot{a}/a^2$ is about 0.015, the quasi-stationary approximation is valid for all energies around and below the black body peak at $E/a = 1/2\pi$. In the black hole analogy this value of $\tau/\tau_0$ would correspond to a black hole mass of $\approx 0.07M_{Pl}$ ($M_{Pl}$ is the Planck mass). So our results would indicate that the quasi-stationary approximation for black holes holds in the whole classical regime.

One can show that, in the limit $\tau \to 0$, $R_+$ tends to an energy-dependent finite value $R^0_+(E)$ (unlike the $a(\tau) = -\alpha/\tau$ case, where $R_+$ diverges in this limit). We found numerically that $R^0_+(E) \overset{E \to 0}{\sim} (2/15\pi)(g^2/\tau_0)$.

We see that at least for the above three examples $\dot{a}/a^2 \ll 1$ is indeed a sufficient condition for the validity of the quasi-stationary approximation, provided $E/a \lesssim 1$.

4 Conclusion

We considered several issues concerning the behavior of the transition rate for a moving two-level detector in the vacuum. We introduced the Feynmann prescription for regularizing the Wightman function in order to secure causality for the response function of a two-level detector. We then use this to derive certain properties of the response function. We showed that for stationary world-lines the response function decreases faster than any power-law in the high energy limit, in concordance with the known exponential decrease for a couple of known special cases. In contrast, non-stationary motions are characterized by a power-law decline of the response function at high detector energies: Generically, the response function at any given time vanishes as $E^{-2}$ at high energies; in special cases the decline is faster: either as $E^{-4}$ or as $E^{-6}$. We also considered some aspects of the applicability of the quasi-stationary approximation, whereby the response function is approximated by it value for a stationary trajectory with the instantaneous values of the invariant parameters $a$, $T$, $H$. Our result on the high energy behavior precludes this approximation for high energies (generically $1 \lesssim E/a$), but in the low energy regime we find for several examples studied numerically that the approximation is good when $\dot{a}/a^2 \ll 1$.

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