Quasimap counts and Bethe eigenfunctions

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Abstract

We associate an explicit equivalent descendent insertion to any relative insertion in quantum K-theory of Nakajima varieties.

This also serves as an explicit formula for off-shell Bethe eigenfunctions for general quantum loop algebras associated to quivers and gives the general integral solution to the corresponding quantum Knizhnik-Zamolodchikov and dynamical $q$-difference equations.

1 Introduction

1.1 Overview

1.1.1

The problem solved in this paper has a representation-theoretic side and a geometric side.

In representation theory of quantum affine algebras, and its applications to exactly solvable models of mathematical physics, a very important role is played by certain $q$-difference equations. These are the quantum Knizhnik-Zamolodchikov equations ($q$KZ), see [7,14] and the corresponding commuting dynamical equations [8,9,11,12,52,55]. A lot of research has been focused on solving these equations by integrals of Mellin-Barnes type, see e.g. [7,28,45,53,54,56,57]. Such integrals, in particular, give explicit formulas for Bethe eigenvectors in the stationary phase $q \to 1$ limit. Here we give a general integral solution for tensor products of evaluation representations of quantum affine Lie algebras associated to quivers as in [27]. These include, in particular, double loop algebras of the form $\mathfrak{g}_h(\hat{\mathfrak{gl}}_n)$, which are known under many different names and play a very important role in many branches of modern mathematical physics, see [32] for a detailed introduction and further references.

For us, these representation-theoretic problems are reflections of certain geometric questions about enumerative K-theory of quasimaps to Nakajima quiver varieties (see [39] for an introduction). In mathematical physics, Nakajima varieties appear in supersymmetric gauge theories as Higgs branches of moduli of vacua, and K-theoretic quasimaps counts may be interpreted as Higgs
branch computations of 3-dimensional supersymmetric indices. Nekrasov and Shatashvili [37, 38] were the first to make the connection between these indices and Bethe equations, see also [34]. The actual problem solved here is to associate an explicit equivalent descendent insertion to any relative insertion in enumerative K-theory of quasimaps to Nakajima varieties, see below and [39] for an explanation of these terms.

Our results are complementary to the recent important work of Smirnov [51] who associates an equivalent relative insertion to any descendent insertion in terms of a certain graphical calculus and canonical tensors associated to the quantum group. Here we allow a wider supply of descendent insertions, and get a simple formula (with an arguably simpler proof) for a map going in the opposite direction.

For quivers of affine ADE type, quasimap counts compute the K-theoretic Donaldson-Thomas invariants of threefolds fibered in ADE surfaces. Finding an equivalence between relative and descendent insertions in Donaldson-Thomas theories of threefolds is a well-known problem of crucial technical importance for the developments of the theory, see [26] for an early discussion and [42, 43] for major further progress in cohomology. Our formulas are both more explicit and work in K-theory.

1.1.2

Let \( g \) be a Lie algebra associated to a quiver with a vertex set \( I \) as in [27]. For example, modulo center, \( g \) is the corresponding simple Lie algebra for quivers of finite ADE type and \( g = \hat{gl}_\ell \) for the cyclic quiver \( \hat{A}_{\ell-1} \) with \( \ell \) vertices.

Extending the work of Nakajima [31], tensor products of fundamental evaluation representations \( F_i(a), i \in I \), of the the corresponding quantum loop algebra \( \mathcal{H}(\hat{g}) \) may be realized geometrically using equivariant K-groups of Nakajima quiver varieties [27, 41].

Let \( X = \mathcal{M}(v,w) \) be a Nakajima variety indexed by dimension vectors \( v, w \in \mathbb{N}^I \) and let \( T \) be a torus of automorphisms of \( X \). It scales the canonical symplectic form \( \omega \) on \( X \) and

\[
\mathbf{h} = \text{weight of } \omega \in K_T(\text{pt})
\]

is the deformation parameter in \( \mathcal{H}(\hat{g}) \). We set \( A = \text{Ker } \mathbf{h} \) and assume that \( A \) contains the torus

\[
A \supset \left\{ \begin{pmatrix} a_{i1} & \cdots & a_{i\ell} \\ \vdots & \ddots & \vdots \\ a_{i1} & \cdots & a_{i\ell} \end{pmatrix} \right\} \subset \prod GL(W_i) \subset \text{Aut}(X) \tag{1}
\]

while the Mellin-Barnes integrals may be interpreted as the equivalent Coulomb branch computations, see e.g. [41] for further discussion.

Those include local curves, that is, threefolds fibered in \( A_0 = \mathbb{C}^2 \).

Equivariant K-theory is similarly the natural setting of Smirnov’s formulas [51].
acting on the framing spaces $W_i$ of the quiver.

A certain integral form of $U_h(\hat{g})$ acts by correspondences between equivariant K-theories of Nakajima varieties so that

$$K_T(X) \otimes_{K_T(pt)} \text{field} \cong \left( \bigotimes_{i \in I} \bigotimes_{j=1}^{w_i} F_i(a_{i,j}) \right)_{\text{weight}=v}$$

where the weight is with respect to the Cartan subalgebra $\mathfrak{h} \subset \mathfrak{g}$ acting by linear function of $w$ and $v$.

1.1.3

Quantum Knizhnik-Zamolodchikov equations of I. Frenkel and N. Reshetikhin [7, 14] are certain canonical $q$-difference equations for a function of the variables $a_{ij}$ in (11) with values in the vector space (2). The shift $q \in \mathbb{C}^\times$ here is a free parameter related to the loop-rotation automorphism of $U_h(\hat{g})$. In the original setup of [14], qKZ equations appeared as difference equation for conformal blocks of $U_h(\hat{g})$ at a fixed level and there was a relation between $q$, the deformation parameter $h$, and the level. The geometric meaning of $q$ will be explained below.

As a parameter, qKZ equations take

$$z \in \mathbb{Z} = \text{group-like elements of } U_h(\hat{g}) / \text{center}$$

or, equivalently, of the torus corresponding to the $v$-part in $\mathfrak{h}$. The monomials $z^v$ are the characters $\mathbb{Z}$.

Compatible systems of $q$-difference equations in $z$ were studied in detail by Etingof, Tarasov, Varchenko, and others in the case of finite-dimensional algebras $\mathfrak{g}$, see [52] and also for example [8, 9, 11, 12, 53–57]. In particular, for finite-dimensional $\mathfrak{g}$, the commuting equations were understood in terms of the lattice part in the dynamical quantum affine Weyl group of $U_h(\hat{g})$ in [9]. These dynamical difference equations are intrinsic to $U_h(\hat{g})$ and make sense in an arbitrary weight space even in the absence of tensor product structure and associated qKZ equations.

For general $\mathfrak{g}$, the dynamical difference equations were constructed in [11].

1.1.4

First Chern classes of tautological bundles give a natural map

$$\mathbb{Z}^I \to H^2(X, \mathbb{Z})$$

which is known to be surjective [29]. The dual map sends the group algebra of $H_2(X, \mathbb{Z})$ to $\mathbb{C}[\mathbb{Z}]$ and makes the monomials $z^v$ degree labels for curve counts in $X$. The variables $z$ are known as the Kähler variables for $X$ in the parlance
of enumerative geometry. The so-called Kähler moduli space is, in the case of Nakajima varieties, a certain toric compactification $\overline{Z} \supset Z$.

With the identification $[2]$, the qKZ and dynamical equations become the quantum difference equations in enumerative K-theory of quasimaps to $X$ $[39, 41]$. These $q$-difference equations shift the equivariant variables $a$ and the Kähler variables $z$ by the fundamental weight $q$ of the group

$$C_q^X = Aut(\mathbb{P}^1, 0, \infty)$$

that acts on the moduli spaces of quasimaps

$$QM(X) = \{ f : \mathbb{P}^1 \to X \} / \cong$$

by automorphisms of the domain, see $[39]$ for an introduction.

1.1.5

While the natural evaluation map

$$QM(X) \ni f \mapsto (f(0), f(\infty)) \in \mathfrak{X} \times \mathfrak{X}$$

only goes to the stack quotient

$$\mathfrak{X} = \left[ \frac{\text{prequotient}}{G} \right] \supset \frac{\text{stable locus}}{G} = X, \tag{3}$$

one can impose constraints on $f$ or modify the moduli spaces to turn enumerative counts into correspondences on $X$, or correspondences between $\mathfrak{X}$ and $X$. Conditions imposed at $0, \infty \in \mathbb{P}^1$ are customary called insertions, just like insertions in functional integrals.

K-theoretic counts of quasimaps with different insertions at $0, \infty \in \mathbb{P}^1$ give objects of different nature as functions of $a$, $z$, and other parameters. For certain insertions, we get a fundamental solutions of the quantum difference equations, while for other insertions we get integrals of Mellin-Barnes type.

1.1.6

By an integral of Mellin-Barnes type we mean an integral of the form

$$I_{\alpha \beta}(z, \ldots) = \int_{\gamma \subset T_G / W_G} f_\alpha(x) g_\beta(x) e(x, z) \prod \frac{\phi(x^{\lambda_i} b_i)}{\phi(x^{\lambda_i} c_i)} \prod \frac{dx_k}{2\pi i x_k} \tag{4}$$

up to multiplicative shift$^4$ in $z$, where

$^4$The exact form of this multiplicative shift, which is of no importance here, is discussed in the Appendix.
the integration is over a middle-dimensional cycle in the quotient of a torus $T_G$ by a finite group $W_G$. Concretely, $T_G \subset G$ is a maximal torus of the group $G$ in \cite{3}, with Weyl group $W_G$. Geometrically, the coordinates on $T_G/W_G$ are the characteristic classes of the universal bundles on $X$. Since these are known to span the $K$-theory of $X$ \cite{22}, we have a natural embedding

\[
\text{Spec } K_T(X) \xrightarrow{\iota} T \times T_G/W_G
\]

finite over the torus $T$ of equivariant parameters. The variables in $T$ including $\hbar$ and $a$ are parameters in \cite{4} and the integral should be viewed as an integral in the fibers of the projection $\pi_T$.

- the cycle $\gamma$ extracts the residues of the integrand at $q$-translates of the pole at the image of $\iota$ in \cite{5}.
- the function $\phi(y) = \prod_{n=0}^{\infty} (1 - q^n y)$ solves the simplest $q$-difference equation and replaces the reciprocal of the $\Gamma$-function in the $q$-world. Ratios of the form $\frac{\phi(x^{a b})}{\phi(x^{a c})}$ generalize complex powers of linear forms ubiquitous in hypergeometric integrals. Instead of hyperplanes, we have translates of codimension 1 subtori in $T \times T_G$.

- the weights $\lambda_i$ and the shifts $b_i, c_i$ involve the roots of $G$ and the weights of $T \times G$ action on the prequotient in \cite{3}. For Nakajima varieties, \cite{3} is an algebraic symplectic reduction of a cotangent bundle, and the self-duality of this setup implies

\[
\{b_i, c_i\} = \{q t^{\nu_i}, \hbar t^{\nu_i}\}
\]

for a certain weight $t^{\nu_i}$ of $T$ on the prequotient in \cite{3}.

- the function

\[
e(x, z) = \exp \left( (\ln q)^{-1} \sum_{i,k} \ln x_{i,k} \ln z_i \right)
\]

where the coordinate $x_{i,k}$ are grouped according to $G = \prod_{i \in I} GL(v_i)$ solves monomial $q$-difference equations in $x$ and $z$ and makes the integral \cite{1} a $q$-difference analog of Fourier or Mellin transform.

- the function $g_\beta(x)$ is an elliptic function on $x$ (that is, a constant, from the viewpoint of $q$-difference equations) regular at the location of $\gamma$. It is convenient to use a suitable basis of such functions as a mechanism to generate a basis in the $\text{rk} K(X)$-dimensional space of solutions of the quantum difference equations.
From the perspective of \cite{1,2}, see in particular Section 6.2 in \cite{2} and Section 5.4 in \cite{1} for detailed examples, it is natural to use elliptic stable envelopes to build functions \( g_\beta(x) \). Our focus in this paper, however, is on the functions \( f_\alpha(x) \), and their relations to \( K \)-theoretic stable envelopes.

- the Bethe subscheme

\[
\mathcal{B} = \left\{ \frac{\partial}{\partial x} \mathcal{W} = 0 \right\} \subset \mathcal{Z} \times T \times T_{G/W_G}
\]

where

\[
\mathcal{W} = \lim_{q \to 1} \ln(q) \ln \left( e(x, z) \prod \frac{\phi(x^{\lambda_i}b_i)}{\phi(x^{\lambda_i}c_i)} \right)
\]

appears as the critical points of the integral in the \( q \to 1 \) limit. It is the joint spectrum of the corresponding commuting operators on \( K_T(X) \) and the map

\[ K_T(X) \ni \alpha \mapsto f_\alpha \mapsto \mathbb{C}[\mathcal{B}] \]

gives the Jordan normal form of the \( \mathbb{C}[\mathcal{B}] \)-action on \( K_T(X) \). The fiber of \( \mathcal{B} \) over \( 0 \in \mathcal{Z} \) is the spectrum of K-theory of \( X \) in \cite{5}. The concrete form of Bethe equations is recalled in the Appendix.

The connection between Bethe equations and quiver gauge theories whose Higgs branch is \( X \) is one of the main points of a very influential sequence of papers by Nekrasov and Shatashvili, see \cite{37,38}.

- finally, the function \( f_\alpha(x) \) is a rational function of \( x \) that depends linearly on \( \alpha \in K_T(X) \) and restricts to \( \alpha \) on the image of \( \iota \) in \cite{5}. It is known under various names including “off-shell Bethe eigenfunction” and “weight function”. This function \( f_\alpha(x) \) will be the most important player in this paper.

Partition functions of supersymmetric gauge theories can be often expressed as integrals of the general form (4), see e.g. \cite{30,33} for prominent examples of such computation. The group \( G \) in this case is the complexification of the gauge group and integration corresponds, via Weyl integration formula, to extracting invariants of constant gauge transformations.\footnote{The function \( \mathcal{W} \) is known as the Yang-Yang function.} See e.g. \cite{1} for and introductory mathematical discussion and an explanation of how integrals of the form (4) appear in enumerative theory of quasimaps to \( X \) with descendent insertions. See also e.g. \cite{41} for a detailed discussion of the Nekrasov-Shatashvili connection between Bethe equation and enumerative theory of quasimaps that does not make an explicit use of Mellin-Barnes integrals.

\footnote{Alternatively, the quotient \( T_{G/W_G} \) is closely related to the Coulomb branch of vacua of the theory and the integral (4) may be interpreted as an equivalent direct computation on the Coulomb branch.}
1.1.7

The space of possible descendent insertions at $0 \in \mathbb{P}^1$

$$\left\{ \text{descendent insertions} \right\} = K_{T \times G}(pt) = \mathbb{Z}[T \times T_G/W_G]$$

corresponds to all possible Laurent polynomials $f_\alpha(x)$ in (4). A choice of $g_\beta$ corresponds to a nonsingular insertion at $\infty \in \mathbb{P}^1$. There is a third flavor of insertions, called relative and they take a class $\alpha \in K_T(X)$ as an input. This is explained in Section 1.2 and, in more details, in [39].

By a geometric argument, K-theoretic count of quasimaps with a relative insertion at 0 and a nonsingular insertion at $\infty$ gives a fundamental solution of the quantum difference equations, see Section 8 in [39] for details.

1.1.8

In this paper, we will describe a linear map

$$\left\{ \text{relative insertions} \right\} = K_T(X) \ni \alpha \mapsto f_\alpha \in \mathbb{Q}(T \times T_G/W_G) = \left\{ \text{localized descendent insertions} \right\}$$

that preserves K-theoretic counts, and therefore makes the Mellin-Barnes integral (4) a solution of the quantum difference equations. Among all quasimaps, there are degree zero, that is, constant quasimaps, which means

$$f_\alpha|_{K_T(X)} = \alpha$$

in the diagram (5).

In (8), we allow only very specific denominators

$$f_\alpha = \frac{s_\alpha}{\Delta_h}, \quad s_\alpha \in \mathbb{Z}[T \times T_G/W_G]$$

where $\Delta_h$ is the Koszul complex for the moment map equations for $X$, that is,

$$\Delta_h = \sum_k (-h)^k \Lambda^k \text{Lie}(G)$$

$$= \prod_i \prod_{k,l} (1 - h x_{i,k}/x_{i,l})$$

with the coordinates $x_{i,k}$ grouped as in (6).

The numerator $s_\alpha$ of $f_\alpha$ is such that the counts are still defined in integral, that is, nonlocalized K-theory. This integrality is crucial and the geometric mechanism responsible for it will be explained in Section 2.3. In particular, we will make precise the mechanism of restriction (9) of a rational function to a locus that may be contained in the divisor of poles.
1.1.9

The denominator in the correspondence (8) is what differentiates our approach from other results in the literature, notably from a very general result of Smirnov [51] who gives a map

\[ \{ \text{descendent insertions} \} \rightarrow K_T(X) \otimes \mathbb{Q}(z, q) = \{ \text{relative insertions} \} \otimes \mathbb{Q}(z, q) \tag{12} \]

which preserves K-theoretic counts. Restricted to \( z = 0 \), the map (12) is the pullback \( \iota^* \) in (5) and hence any set of tautological classes that forms a basis of \( K_T(X) \) can be used to write integral formulas for solutions of quantum difference equations.

1.1.10

Our main result, Theorem 1 in Section 2.4.1, is an equivalence between a relative insertion \( \alpha \) and the corresponding insertion \( f_\alpha \) in enumerative theory of quasimaps to \( X \).

For \( f_\alpha \), we give a simple formula in terms of K-theoretic stable envelopes, see Definition 1 in Section 2.1.8. A representation-theoretic translation of this formula is given in (18) and (19) in Section 1.3, see also Section 3.1. An introduction to K-theoretic stable envelopes may be found in [39].

An interesting feature of our formula for \( f_\alpha \) is that it does not depend on variables \( z \) or \( q \), in marked contrast to (12).

As an special case, we give explicit formulas for \( f_\alpha \) for cyclic quivers \( \widehat{A}_\ell \), that is, for the quantum double loop algebras \( \mathbb{U}_\hbar(\widehat{gl}_\ell) \), see Section 3.2. These formulas can be seen as an instance of an abelianization formula for stable envelopes in the style of [48, 50]. We make the formulas particularly explicit in the important case of the Hilbert scheme of points in \( \mathbb{C}^2 \) in Section 3.2.6.

1.1.11

For \( g = gl_n \), our formulas specialize, with a very different proof, to integrals studied by Tarasov and Varchenko [52, 57]. A connection between what they call the weight function and stable envelopes was observed, in this instance, in the papers [46, 47]. These papers were an important source of inspiration for the work presented here.

For \( g = \widehat{gl}_1 \), Bethe eigenvectors are obtained in [10] in the shuffle algebra realization. Presumably, these formulas may be extended to \( g = \widehat{gl}_\ell \) using e.g. the shuffle algebra techniques developed in [32].

Here we don’t use any specific features of \( gl_\ell \) and solve a more general problem, namely the \( q \)-difference equations that generalize the eigenvalue problem solved in [10].
1.2 Insertions in quantum K-theory

1.2.1

In enumerative geometry of regular maps $f: C \to X$, it is natural and important to be able to constrain the values $f(c)$ of $f$ at specific points $c \in C$. For example, the quantum product in $H^\bullet(X)$ is defined using counts of 3-pointed rational curves

$$f: (C, c_1, c_2, c_3) \to X$$

such that the points

$$(f(c_1), f(c_2), f(c_3)) \in X^3$$

meet 3 given cycles in $X$.

Unlike regular maps, quasimaps may be singular at a finite set of points of $C$, whence the difficulties with using the rational map

$$ev_c : QM(X) \ni f \dashrightarrow f(c) \in X$$

in enumerative K-theory of the moduli space $QM(X)$ of stable quasimaps to $X$. There are at least 3 ways around this difficulty, namely:

— one can restrict to the open set $QM(X)_{\text{nonsing}}$ of quasimaps nonsingular at $c$. While the evaluation map is not proper on this subset, the equivariant counts are well defined if $c \in \{0, \infty\} \subset \mathbb{P}^1 \cong C$ and one works equivariantly with respect to $\mathbb{C}_q^\times = \text{Aut}(\mathbb{P}^1, 0, \infty)$.

— one can use a resolution of the map (13)

$QM_{\text{relative}} c \rightarrow QM_{\text{nonsing}} c \rightarrow X$

provided by the moduli space of quasimaps relative the point $c \in C$. The domain of a relative quasimap is allowed to sprout off a chain of rational curves joining the new evaluation point $c$ to its old location on $C$.

— tautological bundles $\mathcal{V}_i$ on $C$ are part of the quasimap data and one can use Schur functors of their fibers at $c$ to impose constraints on $f(c)$. These are known as descendent insertions in the parlance. Recall that Nakajima varieties are constructed as quotients by $G = \prod GL(V_i)$ and the natural map (sometimes called the K-theoretic analog of the Kirwan map)

$$K_G(\text{pt}) \to K(X)$$

is known to be surjective [29]. Precisely because of the singularities, the bundles $\mathcal{V}_i$ are not pulled back from $X$ by $f$ and, therefore, descendent insertions do not factor through the Kirwan map [15].
1.2.2

Since the options listed above express in different precise languages the same intuitive idea of constraining the value \( f(c) \), one expects to have a translation between e.g. relative and descendent insertions at \( c \).

This turns out to be a highly nontrivial problem with important geometric applications, for instance, in Donaldson-Thomas theory. Early discussion of it may be found in [26] and, in cohomology, a very important progress on this problem was achieved by Pandharipande and Pixton in [42, 43]. Geometric representation theory provides a different and perhaps more powerful approach to these problems, as demonstrated by A. Smirnov in [51].

1.2.3

In a fully equivariant theory, with the action of \( \mathbb{C}_q^\times \) included, it is possible to mix and match the type of insertions at the \( \mathbb{C}_q^\times \)-fixed points \( \{0, \infty\} \) of the domain \( C \). It is natural to interpret 2-pointed counts as correspondences acting on \( K(X) \) or as correspondences between \( X \) and the stack \( X \).

More precisely, one has to localize \( K(X) \) in the presence of nonsingular insertions and work in formal power series in the variables

\[
z^{\deg f} \in \text{semigroup algebra of } H_2(X, \mathbb{Z})_{\text{effective}}
\]

that keep track of the degree of a quasimap. These are usually called the Kähler variables, as opposed to the the equivariant variables which include \( q \) and the coordinates on a maximal torus

\[
T = A \times \mathbb{C}_h^\times \subset \text{Aut}(X) ,
\]

where \( h \) is the \( T \)-weight of the symplectic form on \( X \) and \( A = \text{Ker} \, h \).

1.2.4

The geometric, representation-theoretic, and functional nature of the resulting operators strongly depends on the type of insertions chosen, as illustrated by the following list. In this list we indicate the type of insertion at \( 0 \) followed by the type of insertion at \( \infty \). Obviously, the roles of \( 0 \) and \( \infty \) may be switched by the automorphism of \( \mathbb{P}^1 \) that permutes them and sends \( q \) to \( q^{-1} \).

relative/relative, also known as the glue operator \( G \), is a generalization of the longest element in the quantum dynamical Weyl group of the nonaffine subalgebra

\[
\mathcal{U}_h(\mathfrak{g}) \subset \mathcal{U}_h(\check{\mathfrak{g}}) ,
\]

see [41] and also [40]. It does not depend on \( q \) and is a rational function of the Kähler variables \( z \). It also does not depend the variables \( a \) in \( A \) in certain special bases of \( K_T(X) \), see Section 10.3 in [39].
relative/nonsingular, also known as the capping operator $J$, gives a fundamental solution to $q$-difference equations in both Kähler and equivariant variables. Difference equations with respect to $z$ may be interpreted as the action of the lattice inside the quantum dynamical affine Weyl group of $\mathcal{U}_h(\hat{g})$. Difference equations with respect to $a \in A$ are the quantum Knizhnik-Zamolodchikov equations.

descendent/nonsingular is also known as the vertex with descendents\footnote{The vertex without descendents refers to having no insertions at 0.}, or the so-called big-I function in the more conventional nomenclature that goes back to Givental. Its computation by $\mathbb{C}_q^\times$-localization may be converted into a Mellin-Barnes type integral over a certain middle-dimensional cycle $\gamma$ in a maximal torus of $G$. Such integrals are a standard practice in SUSY gauge theory literature, and can be also explained mathematically, see e.g. the Appendix in \cite{1}. Descendent insertions become functions $f_\alpha$ in (1).

descendent/relative, also known as the capped vertex, is the essential piece in the correspondence between descendent and relative insertions. As shown in \cite{39}, quantum correction to the capped vertex vanish for any fixed insertions and sufficiently large framing. This property is called large framing vanishing. Smirnov shows in \cite{51} how to use it to obtain an explicit representation-theoretic formula for the capped vertex, which is manifestly a rational function in all variables.

1.2.5

The technical crux of the paper is the analysis of the capped vertex with our specific insertions $f_\alpha$. This is done in Section 2.4.4.

Just like the proof of large framing vanishing, this is fundamentally a rigidity result in the classical spirit of Atiyah, Hirzebruch, Krichever, and others \cite{3, 22, 23}. The main ingredients in this analysis are the integrality established in Section 2.3 and bounds on equivariant weights from Section 2.2.

1.3 Algebraic Bethe Ansatz reformulation

1.3.1

In the study of vertex models of statistical physics, from which quantum groups originated, one associates a representation $F$ of $\mathcal{U}_h(\hat{g})$ to lines in a 4-valent oriented planar graph and an interaction tensor

$$R_{F,F'} : F \otimes F' \to F \otimes F'$$

to the vertices of the graph, as in Figure 1. This tensor is the R-matrix for $\mathcal{U}_h(\hat{g})$ and the Yang-Baxter equation satisfied by it is central to integrability of such models. See e.g. \cite{4, 16, 25, 39} for an introduction.
In the approach of [27], one first constructs geometrically a tensor structure on the K-theory of Nakajima varieties, which then yields R-matrices and the quantum group itself, see [39, 40] for an introduction.

1.3.2

Tensor structure is realized geometrically using certain correspondences called stable envelopes, and the R-matrix is computed as composition of one stable envelope with the inverse on another. A certain triangularity inherent in stable envelopes implies that matrix elements of the form

\[ F \otimes \text{vac} \to \text{vac} \otimes F' \]

where vac ∈ F and vac′ ∈ F′ are the vacuum, that is, lowest weight vectors, satisfy

\[ R(\alpha \otimes \text{vac}')_{\text{vac} \otimes F'} = \Pi^{-1} \text{Stab}(\alpha \otimes \text{vac}')_{\text{vac} \otimes F'} \]  \hspace{1cm} (17)

for a certain invertible operator Π on F′. This operator Π belongs to a very specific commutative algebra

\[ \mathcal{B}_0 \subset \text{End}(F') \]

which may be identified with

— the image of the quantum loop algebra \( \mathcal{H}(\mathfrak{h}) \) for the Cartan subalgebra \( \mathfrak{h} \subset \mathfrak{g} \).

— the algebra of multiplication operators in the geometric realization of \( F' \) as a K-theory of a certain algebraic variety. Such realization makes \( F' \) a commutative ring and, in fact, a quotient of a ring of \( W_G \)-invariant Laurent polynomials. It is in this language that \( \Pi \) is presented in (20) below.

— \( \mathcal{B}_0 \) is the limit of Baxter’s algebra \( \mathcal{B}_z \) of commuting transfer matrices \( \text{(65)} \) as the parameter \( z \) goes to 0.

This is reviewed in Section 3.1.4.
1.3.3

Our formula for $s_\alpha$ is of the form

$$s_\alpha = \text{Stab}(\alpha \otimes \text{vac}')|_{\text{vac}} \otimes \star$$  \hspace{1cm} (18)

where $\star$ is a specific point in the geometric realization of $F'$. Its structure sheaf $\mathcal{O}_\star$ is the unique, up to multiple, eigenvector of $\mathcal{B}_0$ with a certain eigenvalue computed in Section 3.1, where further details may be found.

This gives

$$f_\alpha = \frac{\Pi}{\Delta_h} \cdot \text{specific partition function}$$  \hspace{1cm} (19)

where

$$\Pi = \prod_{i \in I} \prod_{k=1}^{v_i} \prod_{l=1}^{w_i} (1-hx_{i,k}/a_{i,l})$$  \hspace{1cm} (20)

and the boundary conditions for the partition function in (19) are explained in Figure 2.

![Diagram](image)

*Figure 2: The partition function that computes the function $f_\alpha$. The $\emptyset$-signs denote the vacuum vectors. The boundary conditions indicated by stars form an eigenvector of the algebra $\mathcal{B}_0$.*

In Figure 2, we make the fundamental representations $F_i, i \in I$, evaluated at points $x_{i,k}$ where $k = 1, \ldots, v_i$ run along the NE-SW lines. Along the NW-SE line runs the representation in which $K(X)$ is a weight subspace. We draw this line as a multiple line in reference to a tensor structure that this module typically possesses. As the boundary condition at SW corner, we chose a certain specific eigenvector of $\mathcal{B}_0$.

The eigenvector property of the boundary conditions means the following identity

$$f_{\alpha \otimes \text{vac}_{\delta w}} = \frac{\Pi'}{\Pi_{w=\delta w}} f_\alpha$$  \hspace{1cm} (21)

where $\text{vac}_{\delta w}$ is the vacuum vector of weight $\delta w$ and

$$\Pi' = \prod_{i \in I} \prod_{k=1}^{v_i} \prod_{l=1}^{w_i} h^{1/2} (1-x_{i,k}/a_{i,l}) .$$
Pictorially, the eigenvalue property (21) may be represented as follows:

\[ \varnothing = \Pi' \mid_{w=\delta w} \varnothing \]

Explicit formulas for the eigenvector \( \varnothing \) may, in turn, be given in terms of stable envelopes. This is a reflection of the basic fact that the dual of the stable envelope is again a stable envelope with opposite parameters, see [39].

1.3.4

Let

\[ p(x_{i,k}) \in \mathbb{Z}[T_G/W_G] \]

by a symmetric polynomial of \( x_{i,k} \), that is, a characteristic class \( p(\{V_i\}) \) of the tautological bundles \( V_i \) on \( \mathcal{M} \). Formula (9) means

\[ (\alpha, p)_{K_T(X)} = \chi(\alpha \otimes p(\{V_i\})) = \int_{\gamma_0} f_\alpha p(x_{i,k}) \ldots , \quad (22) \]

where \( \gamma_0 \) is the part of \( \gamma \) that encircles the image of \( \iota \) in (5) and the integration measure omitted in (22) is, among other things, the specialization of the integration measure in (4) to quasimaps of degree 0.

Using (22), we can read the operators in Figure 2 backwards and interpret that picture as an operator formula for the off-shell Bethe eigenfunction. In the familiar context of the spin 1/2 XXZ spin chain, this becomes the classic formula

\[ \text{off-shell Bethe eigenfunction} = B(x_1) \ldots B(x_v) \text{ vac} \]

of the algebraic Bethe Ansatz, further generalized in [24] and countless papers since.

1.4 Acknowledgments

1.4.1

Our interactions with Pavel Etingof, Boris Feigin, Edward Frenkel, Davesh Maulik, Nikita Nekrasov, Nikolai Reshetikhin, and Andrei Smirnov played a very important role in the development of the ideas presented here.

1.4.2

As already explained, the connection between Bethe equations and supersymmetric gauge theories (specifically, enumerative theory of quasimaps) goes back to the pioneering work of Nekrasov and Shatashvili [37, 38]. As a next step, Bethe eigenvectors found a gauge theoretic interpretation in Nekrasov’s study of orbifold defects in gauge theories, see [34, 36].
1.4.3

This paper looks from a somewhat different angle on the problem which Smirnov essentially already solved in [51] building on the large framing vanishing of [39]. Smirnov’s result is used in [1] to solve qKZ by Mellin-Barnes integrals and the present work was very much motivated by the desire to bring the formulas of [1] closer to those of Tarasov and Varchenko. In this, we were guided by the papers [46,47] of Rimányi, Tarasov, and Varchenko and also by the older papers of Matsuo [28] and Reshetikhin [45].

1.4.4

In this paper, we present complete integral solutions to the dynamical and qKZ equations for tensor products of evaluation representations of quantum affine algebras associated to quivers. As a special case, this includes diagonalization of Baxter-Bethe commuting operators acting in these spaces. That problem goes back to a 1931 paper of Hans Bethe and is the subject of an immense body of literature both in mathematics and physics.

It is unrealistic to analyze how the great many different threads present in that literature enter implicitly or explicitly in what we do here. We cannot attempt to survey the literature and only include those references that influenced our work. Of the many different approaches to Bethe Ansatz, we suspect the one based on the so-called universal weight function [6, 13, 20, 21] may be the closest. Stable envelopes which we use here give a geometric Gauss factorization of the $R$-matrices in the style of Khoroshkin and Tolstoy and this is closely related to universal weights functions.

1.4.5

Another paper which is particularly close to direction of this work is [10], where the authors prove a formula for Bethe eigenvectors for $\mathcal{U}_h(\widehat{\mathfrak{g}}_1)$ which is a close relative of our formula (19), see Section 4 in [10].

Instead of taking the eigenvector boundary condition in Figure 2, the authors of [10] take the $(\emptyset, \emptyset)$-matrix element of the $R$-matrix as a universal map $K_T(X) \to \mathcal{U}_h(\widehat{\mathfrak{g}})$, which they further compose with a shuffle algebra realization of $\mathcal{U}_h(\widehat{\mathfrak{g}})$ to get to functions of $x_{i,k}$. Our formulation bypasses the need to work with shuffles, and also solves a more general problem — the $q$-difference equations. For eigenvalue problems, overall factors, such as our denominators $\Delta_h$, are not relevant, which explains the discrepancy with [10], where the square $\Delta_h|_{h=1}$ of the Vandermonde determinant appears in the denominators.

1.4.6

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2 Main result

2.1 Descendent insertions from stable envelopes

2.1.1

For a given oriented framed quiver like the one in Figure 3, let \( \text{Rep}(v, w) \) denote the linear space of quiver representation with dimension vectors \( v \) and \( w \), where

\[
v_i = \dim V_i, \quad w_i = \dim W_i.
\]

Let

\[
\mu : T^* \text{Rep}(v, w) \to \text{Lie}(G)^*, \quad G = \prod GL(V_i),
\]

be the algebraic moment map and let

\[
\mathcal{Z}(v, w) = \mu^{-1}(0)
\]

denote its zero locus.

By definition, a Nakajima variety \( X \) is an algebraic symplectic reduction

\[
X = \mathcal{M}(v, w) = \mathcal{Z}(v, w) \sslash G
\]

where a certain choice of a GIT stability condition is understood, see e.g. \[15\] for an introduction. The stability choices are parametrized by vectors

\[
\theta \in \mathbb{R}^I = \text{characters}(G) \otimes \mathbb{Z} \mathbb{R},
\]

which must avoid a finite number of rational hyperplanes, up to a positive proportionality. We also consider quotient stacks

\[
X \subset \mathcal{X} = \left[ \frac{\mathcal{Z}(v, w)}{G} \right] \subset \mathcal{R} = \left[ \frac{T^* \text{Rep}(v, w)}{G} \right]
\]
obtained by forgetting the stability condition and the moment map equations, respectively.

2.1.2

Our goal in this section is to construct a certain $K_T(pt)$-linear map

$$K_T(X) \ni \alpha \mapsto s_\alpha \in K_T(\mathcal{R}) \quad (25)$$

such that $s_\alpha$ is supported on $X \subset \mathcal{R}$ and

$$s_\alpha \big|_{\text{neighborhood of } X} = \iota_X \ast \alpha, \quad (26)$$

where $\iota_X : X \hookrightarrow \mathcal{R}$ is the inclusion. One can thus view (25) as an extension of $\iota_X \ast \alpha$ to a K-theory class on $\mathcal{R}$.

This extension is canonical once certain further choices are made. Its construction involves stable envelopes on a larger Nakajima variety $\mathcal{M}(v, w + v)$.

2.1.3

The dependence of what follows on the stability condition (23) may be summarized as follows. Let $i \in I$ be a vertex of the quiver and let $\delta_i \in \mathbb{N}^I$ be the delta-function at $i$. Consider

$$T^* \text{Rep}(\delta_i, \delta_i) = T^* \text{Hom}(W_i, V_i) \oplus T^* \text{Hom}(V_i, V_i)^g$$

where $\dim W_i = \dim V_i = 1$ and

$$g = \text{number of loops at } i.$$

The moment map equations take the form

$$ab = 0, \quad a \in \text{Hom}(W_i, V_i), b \in \text{Hom}(V_i, W_i),$$

and

$$\mathcal{Z}(\delta_i, \delta_i)_{\text{stable}} = \begin{cases} a \neq 0, \\ b \neq 0, \end{cases} \quad \text{depending on } \theta_i \gtrless 0. \quad (27)$$

For either choice of stability, this gives

$$\mathcal{M}(\delta_i, \delta_i) \cong \mathbb{C}^{2g} \quad (28)$$

equivariantly with respect to $Sp(2g) \subset \text{Aut} \mathcal{M}(v, w)$.

Since Nakajima varieties are unchanged under flips of edge orientation, we may assume that the direction of the invertible map in (27) coincides with the orientation. To simplify the exposition we will assume that the framing edges are oriented in the direction of $\text{Hom}(W_i, V_i)$ in Figure 3. It will be clear how to modify this in the general case.
2.1.4

By convention, the group $\mathbb{C}_h^\times$ scales the cotangent direction of $T^* \text{Rep}(v,w)$ by $h^{-1}$ and thus scales the canonical symplectic form $\omega$ on $X$ with weight $h$. Note that this splitting of the exact sequence

$$1 \to \text{Aut}(X, \omega) \to \text{Aut}(X) \to GL(\mathbb{C} \omega) \to 1$$

depends on the choice of the orientation.

2.1.5

Let $V'_i$ be collection of vector spaces of dim $V'_i = v_i$ and denote

$$G' = \prod GL(V'_i) \cong G .$$

Define

$$Y = \mathcal{Z}(v, w + v)_{\text{iso}} / G'$$

where the framing spaces are of the form $W_i \oplus V'_i$ and the subscript refers to the locus of points where the framing maps

$$V'_i \to V_i, \quad \text{respectively} \quad V_i \to V'_i,$$

are isomorphism, according to the orientation explained in Section 2.1.4. In what follows, we will assume that $V'_i \to V_i$. Clearly,

$$\mathcal{Z}(v, w + v)_{\text{iso}} \subset \mathcal{Z}(v, w + v)_{G\text{-stable}} .$$

2.1.6

There is a $G$-equivariant map

$$\iota : T^* \text{Rep}(v, w) \hookrightarrow Y$$

which supplements quiver maps by

$$(\phi, -\phi^{-1} \circ \mu_{GL(V_i)}) \in \text{Hom}(V'_i, V_i) \oplus \text{Hom}(V_i, V'_i)$$

for a framing isomorphism

$$V'_i \overset{\phi}{\to} V_i .$$

The dependence on $\phi$ is precisely taken out by the quotient by $G'$. We denote the induced map

$$\iota : \mathfrak{M} \hookrightarrow [Y/G] = [\mathcal{M}(v, v + w)_{\text{iso}} / G']$$

by the same symbol. Formula (30) implies

$$\mu_G' = -\phi^{-1} \circ \mu_G \circ \phi$$
and thus $X \subset \mathcal{R}$ is cut out but pullback via $\iota$ of the moment map equations for $G'$. In other words, we have a pull-back diagram

$$
\begin{array}{ccc}
X & \xrightarrow{\mu_{G'} \circ \iota} & \mathcal{R} \\
\downarrow & & \downarrow \\
[0/G'] & \xrightarrow{\cdot} & [\text{Lie}(G')^*/G']
\end{array}
$$

(32)

2.1.7

Let

$$U \cong \mathbb{C}^\times \subset \text{center}(G')$$

be the group acting with its defining weight $u$ on each $V'_i$. We have

$$X \cup \mathcal{M}(v, v) \subset \mathcal{M}(v, w + v)^U$$

(33)

and we can choose attracting directions for $U$ so that $\mathcal{M}(v, v)$ lies in the full attracting set of $X$.

We apply the general machinery of stable envelopes, an introduction to which may be found in \cite{39}, to this action of $U$. Since $U$ commutes with $T \times G'$, stable envelopes give a $K_{T \times G'}(X)$-linear map

$$\text{Stab} : K_{T \times G'}(X) \rightarrow K_{T \times G'}(\mathcal{M}(v, w + v))$$

(34)

that depends on two pieces of additional data, namely:

— a fractional line bundle $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{R}$, called the \textit{slope}. The slope $\mathcal{L}$ should be away from the walls of a certain periodic locally finite rational hyperplane arrangement in $\text{Pic}(X) \otimes \mathbb{R}$ and stable envelopes depend only on the alcove of that arrangement that contains $\mathcal{L}$. We fix the slope to be

$$\mathcal{L} = \varepsilon \cdot \text{ample bundle}, \quad 0 < \varepsilon \ll 1.$$  

(35)

This choice is not material for showing (26), but will be crucial for what comes later.

— a polarization $T^{1/2}$ which is a solution of the equation

$$T^{1/2} + h^{-1} \left( T^{1/2} \right)^\vee = \text{tangent bundle}$$

in equivariant K-theory. Polarization is an auxiliary piece of data in that stable envelopes corresponding to different polarizations differ by a shift of the slope. Polarization is also required to set up quasimap counts, see Section 6.1 in \cite{39}, and so we assume that a polarization of $X$ has been chosen and set

$$T^{1/2}_{\mathcal{M}(v, w + v)} = T^{1/2} X + \sum_i h^{-1} \text{Hom}(V_i, V'_i),$$

(36)

that is, we select the directions \textit{opposite} to the framing maps $V'_i \rightarrow V$ that are assumed to be invertible.
2.1.8

Because \( \text{Stab}(\alpha) \) is \( G' \)-equivariant, it descends to a class on \( Y/G \). We make the following

**Definition 1.** We set

\[
s_{\alpha} = \iota^* \text{Stab}(\alpha) \in K_T(\mathcal{R})
\]

where the slope of the stable envelope is chosen as in (35) and the polarization is as in (36).

**Proposition 1.** The class (37) is supported on \( X \subset \mathcal{R} \) and satisfies (26).

*Proof.* The moment map \( \mu_{G'} \) for the group \( G' \) is an \( U \)-invariant map to an affine variety. Since \( X \) in (33) lies in the zero fiber of this map, the full attracting set of \( X \) does, too. From (32), we conclude that

\[
\text{supp } s_{\alpha} \subset X.
\]

Now let \( \text{Attr}(X) \) denote the attracting manifold of \( X \) in (33). It fits into the diagram

\[
\begin{array}{ccc}
\text{Attr}(X) & \overset{\pi_{\text{Attr}}}{\longrightarrow} & \mathcal{M}(v, w + v) \\
\downarrow{\iota_{\text{Attr}}} & & \\
X & & \\
\end{array}
\]

in which

- the map \( \pi_{\text{Attr}} \) forgets the maps \( V'_i \to V_i \),
- the map \( \iota_{\text{Attr}} \) sets to zero the maps \( V_i \to V'_i \).

Our choice of the polarization (36) and the conventions for the normalization of the stable envelope explained in Section 9.1 of [39] imply

\[
\text{Stab}(\alpha) \big|_{\text{neighborhood of } X} = \iota_{\text{Attr}}^* \pi_{\text{Attr}}^* \alpha,
\]

whence the conclusion. \( \square \)

### 2.2 Restriction to the origin

#### 2.2.1

As a polynomial in universal bundles, the insertion \( s_{\alpha} \) is determined by its restriction to the origin \( 0 \in \mathcal{R} \).

Our next goal is to bound the \( G \)-weights that appear in this restriction. The origin is a fixed point of \( G \) and under the inclusion \( \iota \) it corresponds to the point

\[
* = \{ V'_i \sim \to V_i, \text{ all other maps } = 0 \}
\]

—that is, an equivariant map to a variety with a trivial \( U \)-action.

---

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which can be viewed as a point in either \( Y^G \) or \( \mathcal{M}(v, w+v)^{G'} \), the isomorphism in (39) giving an identification of \( G \) and \( G' \).

To bound the \( G \)-weight in \( \mathfrak{s}_\alpha\big|_0 \) is thus same as to bound the \( G' \) weights of \( \text{Stab}(\alpha)\big|_\star \). This is equivalent to bounding the \( A' \)-weights, where \( A' \subset G' \) is a maximal torus.

### 2.2.2

The torus \( A' \) contains \( U \). Since \( X \subset \mathcal{M}(v, w+v) \) is fixed by the whole torus \( A' \), the triangle lemma for stable envelopes implies

\[
\text{Stab}_{A'}(\alpha) = \text{Stab}_U(\alpha)
\]

for the same slope, polarization, and a small perturbation of the 1-parameter subgroup. See Section 9.2 in [39] for a discussion of the triangle lemma.

### 2.2.3

By definition of stable envelopes, the torus weights in their restriction to fixed points are bounded in terms of the polarization, after a shift by the slope

\[
\mathcal{L} \in \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} = \text{characters}(G) \otimes_{\mathbb{Z}} \mathbb{R}.
\]

The identification in (40) sees \( \det V_i \) as a line bundle on \( X \) and as a character of \( G \).

While we made a specific choice of \( \mathcal{L} \) in (35), the following proposition is true for an arbitrary slope.

**Proposition 2.** The \( G \)-weights of the restriction of \( \mathfrak{s}_\alpha \) to the origin \( 0 \in \mathbb{R} \) are contained in

\[
\mathcal{L} + \text{convex hull} \left( \text{weights of } \Lambda^* \left( T_{1/2}^{1/2} \mathcal{M}(v, w+v) \right) \right) \subset \text{weights} \otimes_{\mathbb{Z}} \mathbb{R}.
\]

Here vee and star denote the dual representation and the restriction to (39), that is, to \( V' = V \), respectively.

**Proof.** Follows directly from the definition of stable envelopes. \( \square \)

### 2.2.4

In general, a polarization of a Nakajima variety is a virtual bundle on the prequotient in which either the tangent bundle to \( G \)-orbits or the target of the moment map equations enters with the minus sign. Note, however, that this term is precisely added back in (36) after the specialization to \( V' = V \). This means that \( T_{1/2}^{1/2} \mathcal{M}(v, w+v)\big|_\star \) is an actual representation of \( G \) modulo balanced classes, and thus the exterior algebra in (41) is an actual \( G \)-module.

Recall from [39] that a virtual representation of \( G \) is called **balanced** if it is of the form \( V - V^\vee \), for some \( V \in \mathcal{K}_G(\text{pt}) \). For balanced classes, one defines

\[
\Lambda^*(V - V^\vee) = (-1)^{\dim V} \det V.
\]
2.2.5

The bound in Proposition 2 means that $s_\alpha$ may be seen as a stable envelope extension of the class $\iota_X\cdot\alpha$ to the stack $\mathcal{R}$ in the sense of D. Halpern-Leistner and his collaborators, see [17–19].

2.3 Integrality of $f_\alpha$-insertions

2.3.1

Let $\text{QM}(\mathcal{R})$ denote the moduli space of stable quasimaps

$$f : C \rightarrow \mathcal{R},$$

as defined in [5], see also e.g. [39] for an informal introduction.

By definition, a point of $\text{QM}(\mathcal{R})$ is a collection of vector bundles $\mathcal{V}_i$ on $C$ of rank $v_i$ together with a section of the associated bundles like $\text{Hom}(\mathcal{V}_i, \mathcal{V}_j)$ or $\text{Hom}(\mathcal{V}_i, \mathcal{W}_j)$ per every arrow in the doubled quiver, where $\mathcal{W}_j$ is a trivial bundle of rank $w_j$. A quasimap is stable if it evaluates to a stable point of $\mathcal{R}$ at the generic point of $C$. We set

$$\deg f = (\ldots, \deg \mathcal{V}_i, \ldots) \in \mathbb{Z}^I$$

by definition.

The image of the natural inclusion

$$\iota_{\text{QM}(X)} : \text{QM}(X) \hookrightarrow \text{QM}(\mathcal{R})$$

is cut out by the moment map equations imposed pointwise.

2.3.2

Consider the pull-back

$$\text{ev}_0^* s_\alpha \in K_{T \times C_q^*}(\text{QM}(\mathcal{R}))$$

of the class $s_\alpha$ under the evaluation map

$$\text{ev}_0 : \text{QM}(\mathcal{R}) \ni f \mapsto f(0) \in \mathcal{R}.$$  

By Proposition 11 every quasimap in the support of this class satisfies $f(0) \in X$. Therefore, the obstruction theory for $\text{QM}(X)$, restricted to the support of $\text{ev}_0^* s_\alpha$ has a trivial factor

$$\text{Obs}_{\text{QM}(X)} \big|_{\text{supp} \text{ev}_0^* s_\alpha} \rightarrow h^{-1} \bigoplus \text{Hom}(\mathcal{V}_i|_0, \mathcal{V}_i|_0) \rightarrow 0, \quad (42)$$

corresponding to the moment map equations at $0 \in C$. We can take the kernel of (42) as a new reduced obstruction theory for $\text{QM}(X)$ to produce a reduced virtual fundamental class

$$\mathcal{O}_{\text{QM}(X), \text{reduced}}^\text{vir} \in K_{T \times C_q^*}(\text{QM}(\mathcal{R})).$$
2.3.3

The difference between the virtual fundamental class $\mathcal{O}^{\text{vir}}_{\text{QM}(X)}$ and its reduced version is a factor of

$$\Delta_\hbar = \text{Koszul complex of } \left( \bigoplus h^{-1} \text{Hom}(V_i, V_i) \right).$$

We make the following

**Definition 2.** We set

$$f_\alpha = \Delta_\hbar^{-1} s_\alpha$$

and we define the product

$$\text{ev}_0^*(f_\alpha) \otimes \mathcal{O}^{\text{vir}}_{\text{QM}(X)} \in K_{T \times C^*_0}(\text{QM}(X))$$

by the equality of K-classes

$$\iota_{\text{QM}(X),*} \left( \text{ev}_0^*(f_\alpha) \otimes \mathcal{O}^{\text{vir}}_{\text{QM}(X)} \right) = \text{ev}_0^*(s_\alpha) \otimes \mathcal{O}^{\text{vir}}_{\text{QM}(X),\text{reduced}}$$

on $\text{QM}(\mathcal{R})$.

In actual quasimap counting, one uses the so-called symmetrized virtual structure sheaves $\hat{\mathcal{O}}^{\text{vir}}$, see Section 6.1 in [39] and (46) below. Those differ from $\mathcal{O}^{\text{vir}}$ by a twist by a line bundle, which is the same line bundle on both sides in (45).

2.3.4

The following is clear from construction

**Proposition 3.** The class (44) is an integral K-theory class which equals $\alpha$ for $\text{QM}(X)_{\text{degree}=0} \cong X$.

Integral formulas for descendent insertions generalize verbatim to (44) with the insertion of the rational function (43).

2.4 Equivalence of descendent and relative insertions

2.4.1

Our next goal is to prove the following

**Theorem 1.** A relative insertion of $\alpha \in K_T(X)$ at $0 \in C$ equals the descendent insertion of $f_\alpha$ at the same point in equivariant quasimap counts with arbitrary insertions at points away from $0 \in C$. 

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In Theorem 1 a certain alignment between the polarization used to define $f_\alpha$ and a polarization required in settings up the quasimap counts is understood. Recall that a polarization of $T^{1/2}$ of $X$ induces a virtual bundle $\mathcal{T}^{1/2}$ on the domain of the quasimap and one defines the symmetrized virtual structure sheaf by

$$\hat{O}_{\text{vir}} = O_{\text{vir}} \otimes \left( \frac{\det \mathcal{T}_{\text{vir}}^{1/2}}{\det \mathcal{T}_{0}^{1/2}} \right)^{1/2},$$

where the subscripts denote the fibers of $\mathcal{T}$ at $0, \infty \in C$. The quasimap counts from [39] are defined using (46). Note that they depend on the polarization only via its determinant.

In Theorem 1 we assume that the determinants of the two polarizations are inverse of each other, up to equivariant constants. In [39], equivariant correspondences are interpreted as operators from the fiber at $\infty$ to the fiber at $0$, which is why it is natural to use dual bases for the fiber at $0$. Stable envelopes, in particular, change both the slope and polarization to opposite

$$(\mathcal{L}, T^{1/2}) \mapsto (-\mathcal{L}, h^{-1} (T^{1/2})^\vee)$$

under duality. It is easier to implement the flipping of the polarization in the statement of Theorem 1 than to work with the opposite polarization throughout the paper.

With this change, the localization contributions at 0 take the form

$$\hat{O}_{\text{vir}} = \frac{1}{\Lambda_-} \left( \frac{\det \mathcal{T}_{0}^{1/2}}{\det \mathcal{T}_{\text{vir}}^{1/2}} \right)^{\vee} \otimes \ldots$$

(47)

where the dots stand for terms with a finite limit as $q^{\pm 1} \to \infty$ and

$$\Lambda_- = \sum_k (-1)^k \Lambda^k.$$  

(48)

See Section 7.3 of [39] for details on the localization formula (47).

2.4.2

The proof of Theorem 1 proceeds in several steps.

As a first step, we can equivariantly degenerate $C$ to a union

$$C \leadsto C_1 \cup_{\text{node}} C_2$$

so that $0 \in C_1 \setminus \{\text{node}\}$ and all other insertions lie in $C_2 \setminus \{\text{node}\}$. By the degeneration formula, it is therefore enough to show that the counts in Theorem 1 coincide when we impose a relative insertion $\beta \in K_T (X)$ at $\infty \in C \cong \mathbb{P}^1$. 

24
2.4.3

Since the count of quasimaps relative $0, \infty \in \mathbb{P}^1$ is the glue matrix $G$, the Theorem is equivalent to showing that the operator

$$
\alpha \mapsto \tilde{ev}_{\infty,*} \left( ev_0^*(f_{\alpha}) \otimes \hat{\partial}_{\text{vir}} z^{\text{deg}} \right) \in K_{T}(X)[q^{\pm 1}][[z]]
$$

(49)
equals $G$, where $\tilde{ev}$ is the relative evaluation map as in (14). Here we get polynomials in $q$ because the map $\tilde{ev}_{\infty}$ is proper and $\mathbb{C}_{q}^{\times}$-invariant.

Recall that the glue matrix does not depend on $q$, which can be explicitly seen by its analysis as $q^{\pm 1} \to 0$ as in Section 7.1 of [39]. This analysis is based on $\mathbb{C}_{q}^{\times}$-equivariant localization and we can apply the same reasoning to (49).

The $\mathbb{C}_{q}^{\times}$-fixed quasimaps are constant on $\mathbb{P}^1 \setminus \{0, \infty\}$ and the contributions from $0$ and $\infty$ essentially decouple. The contributions from $\infty$ are literally the same as for the glue matrix. They are computed using the push-pull in the following diagram

$$
K \left( \text{QM}(X)^{\mathbb{C}_{q}^{\times}}_{\text{nonsing at } 0, \text{ relative } \infty} \right) \xrightarrow{ev_0^*} K(X) \xrightarrow{\tilde{ev}_{\infty,*}} K(X),
$$

(50)

where we tensor with $\hat{\partial}_{\text{vir}} z^{\text{deg}}$ on the middle stage. The analysis in Section 7.1 of [39] shows

$$
\text{operator from } (50) \to \begin{cases} G, & q \to 0, \\ 1, & q \to \infty. \end{cases}
$$

(51)

The contributions from $0 \in C$ in the localization formula for are computed using a parallel push-pull diagram

$$
K \left( \text{QM}(X)^{\mathbb{C}_{q}^{\times}}_{\text{nonsing at } \infty} \right) \xrightarrow{ev_0^*} K(X) \xrightarrow{ev_{\infty,*}} K(X),
$$

(52)

which computes the so-called vertex with descendents, see Section 7.2 in [39]. The $\mathbb{C}_{q}^{\times}$-fixed locus in (52) has a concrete description as a certain space of flags of quiver representations, see Section 7.2 of [39] and also [1].

Since (49) is a product of the two operators, Theorem I follows from the following

**Proposition 4.** The vertex with descendent $f_{\alpha}$ remains bounded in the $q \to \infty$ limit and goes to $\alpha$ in the limit $q \to 0$.

Note that a vertex with descendents is a power series in $z$ and Proposition I implies all terms of nonzero degree in $z$ in that series vanish in the $q \to 0$ limit.
2.4.4

Proof of Proposition 4. On the fixed locus, the bundles $\mathcal{V}_i$ can be written in the form

$$\mathcal{V}_i = \oplus_j O_C(d_{i,j}[0])$$

with their natural linearization. This means that their fiber $\mathcal{V}_i|_\infty$ at infinity is a trivial $\mathbb{C}_q$-module while the $\mathbb{C}_q$-weights in the fiber $\mathcal{V}_i|_0$ at zero are $\{q^{d_{i,j}}\}$. This means that the insertion $ev_0^* s_\alpha$ is a Laurent polynomial in $\{q^{d_{i,j}}\}$ with coefficients in K-theory of the fixed locus.

This polynomial does not depend on the quiver maps and therefore we may assume that all quiver maps are zero. The Newton polygon of $ev_0^* s_\alpha$ is thus bounded by the formula in Proposition 2. We find

$$\mathbb{C}_q$$-weights of $ev_0^* s_\alpha \subset (d, \mathcal{L}) + \text{conv} \left( \text{weights of } \Lambda^* \left( \mathcal{T}_{\mathcal{M}(v,w+v)}^{1/2} \right)_0 \right)$$

and this inclusion is strict if $(d, \mathcal{L}) \neq 0$ because it is true for an open set of $\mathcal{L}$.

Here $\mathcal{T}_{\mathcal{M}(v,w+v)}^{1/2}$ is the virtual bundle on $C$ obtained by plugging the bundles $\mathcal{V}_i$ and $\mathcal{W}_j$ into the formula (36) and subscript refers to its fiber at $0 \in C$. As observed in Section 2.2.4, the exterior algebra here is a well-defined $\mathbb{C}_q$-module.

Also in (53) we have the natural pairing of the degree of the quasimap

$$d = (d_i) \in \mathbb{Z}^I = H_2(X, \mathbb{Z}), \quad d_i = \sum_j d_{i,j}$$

with a fractional bundle $\mathcal{L} \in \text{Pic}(X) \otimes \mathbb{R}$. The moduli spaces of quasimaps of degree $d$ are empty unless $d$ is effective, see Section 7.2 in [39], so we assume that $d$ is effective in what follows. Since $\mathcal{L}$ was assumed to be an ample bundle, we have

$$(d, \mathcal{L}) = 0 \iff d = 0.$$  

From (53), we have

$$\Lambda_* \left( \mathcal{T}_{\mathcal{M}(v,w+v)}^{1/2} \right)_0^\vee = \frac{1}{\Delta_h} \Lambda_* \left( \mathcal{T}_{\mathcal{M}(v,w+v)}^{1/2} \right)_0^\vee$$

and therefore from (47) we conclude

$$q^{-(d, \mathcal{L})} ev_0^* f_\alpha \otimes \widehat{\theta}_{\text{vir}} \to 0, \quad q \to 0, \infty, \quad d \neq 0.$$  

Since $\mathcal{L}$ was assumed to be a very small ample bundle, we have

$$0 < (d, \mathcal{L}) \ll 1, \quad d \neq 0.$$  

Therefore for $d \neq 0$ we have

$$ev_0^* f_\alpha \otimes \widehat{\theta}_{\text{vir}} \to \begin{cases} 0, & q \to 0, \\ \text{bounded}, & q \to \infty, \end{cases}$$

as was to be shown. □
3 Reformulations and examples

3.1 R-matrices and Bethe eigenfunctions

3.1.1 In the setup of Section 2.1.7 consider the R-matrix for the action of \( U^{-1} \circ Stab \in \text{End}_{K^T \times G'}(M(v, w + v)^U) \) localized

where the map Unstab is defined as in (34) with the same choice of slope and polarization, but the opposite choice of the 1-parameter subgroup.

Our next goal is to express \( s_\alpha \) in terms of the restriction of \( R(\alpha) \) to \( M(v, v) \) in (33) and more concretely in term of its restriction to the \( G' \)-fixed point \( \ast \in M(v, v) \) as in (39). Recall that \( s_\alpha \) is completely determined by its restriction to the point \( \ast \).

3.1.2 By our choice of the 1-parameter subgroup, \( M(v, v) \) was at the bottom of the attracting order among components of the \( U \) fixed locus. Since this order is reversed for Unstab, we have

\[
\text{Unstab}(\beta)|_{M(v, v)} = \beta \otimes \Lambda^-(N^\vee_{\text{repell}}) \otimes \ldots ,
\]

for any \( \beta \in K_{T \times G'}(M(v, v)) \), where dots stand for a certain line bundle and \( N_{\text{repell}} \) is the repelling part of the of the normal bundle \( N \) to \( M(v, v) \). We have

\[
N = \sum_i \text{Hom}(W_i, V_i) + h^{-1} \sum_i \text{Hom}(V_i, W_i).
\]

Fixing the line bundle in (56) requires fixing a polarization of \( X \). For simplicity, we assume that the polarization of the framing maps for \( X \) is the same as in the new framing terms in (36), that is

\[
T^{1/2}X = h^{-1} \sum_i \text{Hom}(V_i, W_i) + \text{non-framing terms}.
\]

Recall from Section (2.1.3) that such choice of orientation on framing edges was dependent on the stability parameter \( \theta \), and that both orientation and polarization should be flipped if the entries of \( \theta \) change sign.

3.1.3 With the assumption (58), the repelling directions in (57) coincide with the normal directions chosen by polarization and hence the dots in (56) are trivial. In other words

\[
\text{Unstab}(\beta)|_{M(v, v)} = \beta \otimes \Pi.
\]
where

$$\Pi = \Lambda^* \left( \hbar \sum_i \text{Hom}(W_i, V_i) \right) = \prod_{i \in I} \prod_{k=1}^{v_i} \prod_{l=1}^{w_i} (1 - \hbar x_{i,k}/a_{i,l}) .$$  \hspace{1cm} (60)

The variables $x_{i,k}$ and $a_{i,l}$ in (60) are the Chern roots of $V_i$ and $W_i$, respectively, as in $\Pi$. We deduce the following

**Proposition 5.** We have

$$s_\alpha \big|_0 = \Pi \ R(\alpha) \big|_\star .$$  \hspace{1cm} (61)

### 3.1.4

It remains to characterize the fiber at $\star$, which is, abstractly, a linear form

$$K_{G'}(\mathcal{M}(v,v)) \ni \mathcal{F} \mapsto \mathcal{F} \big|_\star = \chi(\mathcal{F} \otimes \Theta_\star) \in K_{G'}(pt) ,$$  \hspace{1cm} (62)

in representation-theoretic terms.

The structure sheaf

$$\Theta_\star \in K_{G'}(\mathcal{M}(v,v))$$

of the $G'$-fixed point (39) is an eigenvector of operators of multiplication in $K_{G'}(\mathcal{M}(v,v))$, namely

$$\mathcal{F} \otimes \Theta_\star = \mathcal{F} \big|_\star \cdot \Theta_\star$$  \hspace{1cm} (63)

for any $\mathcal{F} \in K_{G'}(\mathcal{M}(v,v))$.

Following [27], we recall how express generators of the commutative algebra of operators (63) in terms of the vacuum matrix elements of R-matrices. These are operators in $K_{G'}(\mathcal{M}(v,v))$ defined by

$$R_{w,\varnothing,\varnothing}(\beta) = R(\beta)\big|_{\mathcal{M}(v,v)} ,$$  \hspace{1cm} (64)

where $R$ is our current R-matrix defined in (55). Its dependence on the dimension vector $w$ is made explicit in (64). Obviously

$$R_{w,\varnothing,\varnothing} = \lim_{z \to 0} \text{tr}_{1st \ factor} (z^w \otimes 1) R$$  \hspace{1cm} (65)

and so the operators (64) are the limit of Baxter’s commuting transfer matrices as $z \to 0$.

In the description (57) of the normal bundle, the repelling direction for Stab are the attracting directions for Unstab and they are precisely opposite to the polarization. Therefore

$$\text{Stab}(\beta) = \Pi' \otimes \beta , \quad \beta \in K_{G'}(\mathcal{M}(v,v)) ,$$

where

$$\Pi' = \hbar^{1/2} \text{rk} N \Lambda^* \left( \sum_i \text{Hom}(W_i, V_i) \right)$$

$$= \prod_{i \in I} \prod_{k=1}^{v_i} \prod_{l=1}^{w_i} \hbar^{1/2} (1 - x_{i,k}/a_{i,l}) .$$  \hspace{1cm} (66)
From this and (59) it follows that
\[
R_{w,\varnothing,\varnothing} = \prod'_{i} \otimes \in \text{End } K_{G'}(\mathcal{M}(v, v)) \otimes \mathbb{Q}(A').
\] (67)

3.1.5

Recall that \(A' \subset G'\) denote the maximal torus. Extending the analysis of Section 2.1.3 it is easy to see that
\[
\mathcal{M}(v, v)^{A'_{\text{component of } \star}} = \prod_{i} \mathcal{M}(\delta_{i}, \delta_{i})^{v_{i}}.
\]

This is a vector space with origin \(\star\). The Weyl group of \(G'\) acts on it by permutations of factors. Since the K-theory of this fixed component is trivial, we have the following

**Proposition 6.** The structure sheaf \(\mathcal{O}_{\star}\) is the unique, up to multiple, eigenvector of the operators \(R_{w,\varnothing,\varnothing}\) with eigenvalue
\[
R_{w,\varnothing,\varnothing}(\mathcal{O}_{\star}) = \prod'_{i, k} a_{i,k}' \mathcal{O}_{\star}
\] (68)

and (62) is the unique, up to multiple, linear form in the dual of this eigenspace. The normalization may be fixed by e.g. (9).

To connect with the notations of Section 1.3 of the Introduction, it suffices to make the inverse substitution \(a_{i,k}' = x_{i,k}\).

3.2 Example: \(\mathcal{U}_{h}(\widehat{\mathfrak{gl}_{\ell}})\)

3.2.1

Our goal here is to produce an explicit basis of the functions \(f_{\alpha}\) for quivers of cyclic type \(\widehat{A}_{\ell-1}\) with \(\ell\) vertices. The corresponding Nakajima varieties are moduli spaces of framed sheaves, including Hilbert schemes of points, on the \(A_{\ell-1}\)-surfaces, that is, minimal resolutions of
\[xy = z^{\ell},\]
starting with the affine plane \(A_{0} = \mathbb{C}^{2}\) for \(\ell = 1\). In particular, K-theoretic counts of quasimaps to these Nakajima varieties are directly related to K-theoretic Donaldson-Thomas theory of threefold fibered in \(A_{\ell-1}\)-surfaces.

The Lie algebra \(\mathfrak{g}\) corresponding to the cyclic quiver is the affine Lie algebra \(\widehat{\mathfrak{gl}_{\ell}}\), hence the action of a double affine algebra \(\mathcal{U}_{h}(\widehat{\mathfrak{gl}_{\ell}})\) on the K-theories of these Nakajima varieties. Its direct link to important questions in enumerative geometry and mathematical physics makes \(\mathcal{U}_{h}(\widehat{\mathfrak{gl}_{\ell}})\) a very interesting object of study. See in particular [32] for a detailed discussion and many references.
As a special case, cyclic quiver varieties include quiver varieties for the linear quiver $A_\ell$, for which we recover the action of $\mathfrak{gl}_\ell$ and the formulas of Tarasov and Varchenko. The connection between those formulas and stable envelopes has already been observed in [46, 47].

### 3.2.2

For explicit formulas, it is convenient to choose a particularly symmetric polarization of $X$. We start with a polarization

$$T^{1/2} = \bigoplus_{\sigma \to \sigma'} \text{Hom}(\sigma, \sigma') - \bigoplus_i \text{Hom}(V_i, V_i)$$  \hspace{1cm} (69)

obtained from an orientation of the framed quiver in Figure 3. The first sum in (69) is over all oriented edges. The weights in the corresponding stable envelopes are then bounded by the weights in $\Lambda^{-}_- (T^{1/2})^\vee$, which is a product of expressions like

$$\Lambda^{-}_- \text{Hom}(V, V')^\vee = \prod (1 - x_i/x'_j).$$  \hspace{1cm} (70)

Here $\Lambda^{-}_-$ is the alternating sum of exterior powers as in (48) and $\{x_i\}, \{x'_j\}$ are the Chern roots of $V$ and $V'$, respectively.

We define

$$\Lambda^o \text{Hom}(V, V') = \Lambda^{-}_- \text{Hom}(V, V') \otimes (\text{det } V)^{rk V'}$$

$$= \hat{a} (\text{Hom}(V', V)) \otimes (\text{det } V)^{1/2} \otimes (\text{det } V')^{1/2} \otimes (\text{det } V)^{rk V}$$

$$= \prod (x_i - x'_j)$$  \hspace{1cm} (71)

which is, up to a sign, symmetric in $V$ and $V'$. Since (70) and (71) differ by a sign and a line bundle, we have

$$\Lambda^o T^{1/2} = \pm \Lambda^{-}_- \left(T_o^{1/2}\right)^\vee$$  \hspace{1cm} (72)

for a certain polarization $T_o^{1/2}$. In what follows, we consider stable envelopes with this polarization; their weights are bounded by (72).

It is convenient to extend the definition (71) by linearity in the second factor

$$\Lambda^o \left(\text{Hom}(V, V') \otimes M\right) = \Lambda^o \text{Hom}(V, V' \otimes M) = \prod_{i,j,k} (x_i - m_k x'_j)$$  \hspace{1cm} (73)

where $M$ is a multiplicity bundle and $\{m_k\}$ are its Chern roots. Recall that for Nakajima varieties may have nontrivial automorphisms acting on edge multiplicity spaces. The rank of the group of such automorphisms is the 1st Betti number of the quiver. A review of these basis facts may be found e.g. in the introductory material in [27].
3.2.3

Let $A \subset T$ denote the subtorus preserving the symplectic form $\omega$. The torus $A$ includes a maximal torus of the framing group $GL(W)$ and an additional $C_\text{loop}$ for the loop in the quiver. In the moduli of sheaves interpretation, this $C_\text{loop}$ acts by symplectic automorphisms of the surface.

We have

$$X^A = \bigcup_{\sum \tilde{v}^{(ij)} = v} \prod_{i \in I} \prod_{j=1}^{w_i} M_{\text{linear}}(\tilde{v}^{(ij)}, \delta_i)$$

(74)

where the $M_{\text{linear}}$ denotes the Nakajima variety corresponding to the infinite linear quiver $A_\infty$ and the equality $\sum \tilde{v}^{(ij)} = v$ in (74) involves summing over the fibers of the map

$$A_\infty \xrightarrow{\text{universal cover}} \hat{A}_\ell.$$  

(75)

The fixed locus (74) may be interpreted as a Nakajima variety associated to a (disconnected) fixed-point quiver

$$Q^A = |w|\text{-many copies of } A_\infty$$

(76)

with dimension vector $\tilde{v} = (\tilde{v}_k^{(ij)})$, where $|w| = \sum w_i$.

3.2.4

Note that

$$M_{\text{linear}}(v, \delta_i) = \begin{cases} \text{pt}, & v \text{ corresponds to a partition } \lambda, \\ \emptyset, & \text{otherwise}, \end{cases}$$

(77)

where the first case means that

$$v_j = \# \text{ of squares in } \lambda \text{ of content } j - i,$$

with

$$\text{content}(\square) = \text{column}(\square) - \text{row}(\square).$$

Indeed, the nonempty moduli spaces in (77) form a basis of a level one Fock module for $\hat{gl}_\infty$, also known as a fundamental representation of this Lie algebra. Those are labelled by an integer $i$ and this is the index $i$ in the formulas above.

3.2.5

Let $F$ be a component of the fixed locus (74). It corresponds to a homomorphism

$$\phi_F : A \to G$$

which makes all spaces $V_i$ and the Hom-spaces between them $A$-graded. In particular, the fixed locus $F$ itself parameterizes $A$-invariant quiver maps, modulo the action of the centralizer $G^A \subset G$. 

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We choose a generic 1-parameter subgroup in $A$ to partition all nonzero weights into attracting and repelling. In particular, the polarization $T^{1/2}$ decomposes
\[ T^{1/2} \big|_F = \left( T^{1/2} \right)_{\text{attracting}} \oplus \left( T^{1/2} \right)_{A-\text{fixed}} \oplus \left( T^{1/2} \right)_{\text{repelling}} \] (78)
according to the $A$ weights. We define
\[ f_F = \sum_{w \in W_G/W_{G^A}} w \cdot \Lambda^\circ \left( \left( T^{1/2} \right)_{\text{repelling}} \oplus \hbar \left( T^{1/2} \right)_{\text{attracting}} \right) , \] (79)
where the Weyl group acts by permuting the Chern roots of the bundles. Since the decomposition (78) is $G^A$-equivariant, the group the Weyl group $W_{G^A}$ of $G^A$ acts trivially and the summation in (79) is over the cosets of $W_{G^A}$.

Let $\mathcal{L}$ be line bundle of the form
\[ \mathcal{L}_0 = \bigotimes (\det V_i)^{\epsilon_i} , \quad 0 < \epsilon_i \ll 1 . \] (80)
The following proposition may be seen as an instance of an abelianization formula for stable envelopes, see e.g. [2, 48, 50]. Closely related constructions also appear in [18, 19].

**Proposition 7.** The functions $f_F$ for all components $F$ of the fixed locus (74) form a $\mathbb{Q}(T)$-basis of the space of functions $f_\alpha$ for cyclic quiver varieties for polarization (72) and slope (80).

Note that if the rest of the terms and the cycle of integration in (4) are symmetric then there is no need to symmetrize under the integral sign.

### 3.2.6

For example, let
\[ X = \text{Hilb}(\mathbb{C}^2, n) \]
be the Hilbert scheme of $n$ points in the plane $\mathbb{C}^2$, which corresponds to
\[ \ell = 1 , \quad w = 1 , \quad v = n . \]
The tori
\[ T = \left\{ \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \right\} \supset A = \left\{ \begin{pmatrix} t_1 \\ t_1^{-1} \end{pmatrix} \right\} \]
acts naturally on $\mathbb{C}^2$ and $\text{Hilb}(\mathbb{C}^2, n)$ and
\[ h = \frac{1}{t_1 t_2} . \]
The fixed points of $T$ and $A$ are indexed by partitions $\lambda$ of $n$ and
\[ V|_\lambda = \sum_{\Box = (i,j) \in \lambda} t_1^{-i} t_2^{-i} . \]
as a \( T \)-module. In particular, the \( A \)-weights in \( V \) are given by minus contents of the boxes. As a polarization, we may take
\[
T^{1/2} = V + (t_1 - 1) \text{Hom}(V, V)
\]
\[
= \sum x_i + (t_1 - 1) \sum_{i,j} x_i/x_j
\]
where \( \{x_i\} \) are the Chern roots of \( V \). A fixed point is specified by the assignment of \( x_i \) to the boxes of \( \lambda \), up to permutation.

If we take \( t_1 \) to be a repelling weight for \( A \) then
\[
T_{\geq}^{1/2} = \sum_{c(i) \geq 0} x_i + t_1 \sum_{c(i) \geq c(j)+1} x_i/x_j - \sum_{c(i) \leq c(j)} x_i/x_j
\]
where
\[
T_>^{1/2} = T_{\text{attracting}}^{1/2}, \quad T_<^{1/2} = T_{\text{repelling}}^{1/2},
\]
and \( c(i) \) is the content of the box in \( \lambda \) assigned to \( x_i \). Therefore, up to an \( h \) multiple, we have
\[
f_\lambda = \text{symmetrization of } \frac{\Pi_1 \Pi_2}{\Pi_3}
\]
where
\[
\Pi_1 = \prod_{c(i)<0} (1-x_i) \prod_{c(i)>0} (t_1 t_2 - x_i)
\]
and
\[
\Pi_2 = \prod_{c(i)<c(j)+1} (x_j - t_1 x_i) \prod_{c(i)>c(j)+1} (t_2 x_j - x_i)
\]
\[
\Pi_3 = \prod_{c(i)<c(j)} (x_j - x_i) \prod_{c(i)>c(j)} (t_1 t_2 x_j - x_i).
\]
These are formulas for K-theoretic stable envelopes for \( \text{Hilb}(\mathbb{C}^2, n) \) with the polarization and slope as in Proposition 7. They are a direct K-theoretic generalization of the formulas from [45, 50].

Note that in all cases treated by the formula (79) the slope is near an integral line bundle. Much more interesting functions appear at fractional slopes, but they seem to be not required in the context of Bethe Ansatz.

### 3.2.7

The proof of Proposition 7 takes several steps. As a first step, we clarify the geometric meaning of the formula (79).

We separate the numerator and denominator in (79) by writing
\[
\left(T^{1/2}\right)_{\text{repelling}} \oplus h \left(T^{1/2}\right)_{\text{attracting}} = \rho_+ - \rho_-
\]
as a difference of two $A$-modules. Then $\Lambda^\rho_+$ is the numerator in (79), while $\Lambda^\rho_-$ is the denominator. We note that
\[
\Lambda^\rho_+ = \pm \mathcal{O}_{\text{Attr}(F)} \otimes \cdots \in K_{T \times P}(T^* \text{Rep})
\] (81)
where dots stand for a character. Here
\[
\text{Attr}(F) \subset T^* \text{Rep}
\]
is the $A$-attracting manifold and $P \subset G$ is the the parabolic subgroup with
\[
p = \text{Lie} \, P = g_{\text{attracting}}.
\]
It acts on the character in (81) via the homomorphism
\[
1 \to \text{unipotent radical } N \to P \to G^A \to 1
\] (82)
to its Levi subgroup $G^A = P^A$.

3.2.8

Formula (81) illustrates two general facts. First, this is an instance of stable envelopes for abelian quotients and abelian stacks. In general, in the abelian case, stable envelopes are structure sheaves of the attracting locus, up to line bundles.

The second general principle apparent in (81) is summarized in the following, in which $Y$ is an abstract variety or stack for which stable envelopes are defined.

**Lemma 1.** Let $P$ in
\[
A \subset P \subset \text{Aut}(Y)
\]
be an algebraic group such that the $A$-weights in $p$ are attracting. Stable envelopes define a map
\[
K_P(Y^A) \to K_P(Y)
\]
where $P$ acts on $Y^A$ via the projection to $P^A$.

**Proof.** Our assumption on $P$ implies that it preserves attracting manifolds. We then argue inductively using the attracting order on the components $F_i$ of $Y^A$. For the very bottom component, the stable envelope is the push-pull in the $P$-equivariant diagram
\[
\begin{array}{ccc}
\text{Attr}(F_{\text{bottom}}) & \xleftarrow{\text{projection}} & F_{\text{bottom}} \\
& \searrow & \downarrow \text{inclusion} \\
& & Y
\end{array}
\] (83)
up to a line bundle pulled back from $F_{\text{bottom}}$. For all other components $F$, stable envelopes are uniquely determined by having the same structure near $F$ and being orthogonal to all lower stable envelopes in the sense of [17], whence the conclusion.

\[\Box\]
3.2.9

By construction
\[ \rho_- = \mathfrak{g}/\mathfrak{p} \oplus \hbar \mathfrak{n}, \]
where \( \mathfrak{n} = \text{Lie } N \) is the nilradical of \( \mathfrak{p} \). The second term here has the following interpretation.

Since the moment map is a \( A \)-equivariant map, we have
\[ \mu : \text{Attr}(F) \to \mathfrak{g}^\vee \text{attracting} = n^\perp. \]
Therefore there is no need to impose the moment map in \( n^\vee \). Equivalently, if we planning to multiply by the Koszul complex \( \Delta_h \) of \( h^{-1}\mathfrak{g}^\vee \) to get a class supported on \( X \), we may divide by
\[ \text{Koszul complex of } h^{-1}n^\vee = \pm \Lambda^\circ(\hbar \mathfrak{n}) \otimes \ldots, \]
where dots stand for an unspecified character, as before.

3.2.10

The meaning of the first term in (84) is the following. Given a \( P \)-equivariant sheaf on a \( G \)-variety \( Y \), we can induce it to a \( G \)-equivariant by first, making a \( G \)-equivariant sheaf on \( G/P \times Y \) and then pushing it forward to \( Y \).

In the case at hand, up to a line bundle, the denominators \( \Lambda^\circ(\mathfrak{g}/\mathfrak{p}) \) and the summation over \( W_G/W_{GA} \) in (79) quite precisely come from an equivariant localization on \( G/P \). We conclude the following

**Proposition 8.** The symmetric polynomial \( \Delta_h f_F \) represents a class in \( KT(\mathfrak{g}) \) supported on the full \( A \)-attracting set of \( F \) in \( X \).

3.2.11

*Proof of Proposition [7]* Note that the formula (79) is universal for all dimension vectors. By the logic of our Definition [1] to prove (79) for a specific \( X = \mathcal{M}(v,w) \) we need to check something for a larger Nakajima variety \( \mathcal{M}(v,w+v) \). Namely, together with \( X \), the fixed locus \( F \) embeds in \( \mathcal{M}(v,w+v) \) and we need to bound \( U \)-weights in the corresponding function \( f_{F,\mathcal{M}(v,w+v)} \).

From this angle, there is nothing special about the framing dimension being increased by exactly \( v \), and we can more generally assume that an action of \( U \cong \mathbb{C}^* \) is defined by a decomposition of the framing spaces
\[ W = W' + u W'', \]
in which \( u \) is the defining weight of \( U \) and \( W',W'' \) are trivial \( U \)-modules. We have
\[ X^U = \bigcup_{v'+v''=v} \mathcal{M}(v',w') \times \mathcal{M}(v'',w'') \]
and we choose the attracting directions so that components with larger \(v''\) are attracted to those with smaller \(v'''\). For the bundle \(\mathcal{L}_o\) from (80) we have

\[
\text{weight } \mathcal{L}_o \big|_{X^U} = \varepsilon \cdot v''.
\]

In the context of our Definition 1,
— we assume that \(F\) lies in the \(w'' = 0\) component of the fixed locus \(X^U\),
— also assume that the attracting direction for \(A\) agree with those for \(U \subset A\),
— and we need to prove that

\[
\left. u^{-\varepsilon v''} \frac{f_F}{\Lambda^o \, \frac{T^{1/2}}{A\text{-fixed}}} \right|_{X^U} = O(1), \quad u^{\pm 1} \to \infty,
\]

for \(0 < \varepsilon_i \ll 1\).

In (79), we select the attracting and repelling directions in the decomposition (78). Since in (85) this is compared with the whole polarization, the bound (85) follows from

\[
\left. u^{-\varepsilon v''} \frac{1}{\Lambda^o \, \frac{T^{1/2}}{A\text{-fixed}}} \right|_{X^U} = O(1), \quad u^{\pm 1} \to \infty,
\]

which will now be established.

In \(f_F\), the Chern classes \(x_{ij}\) of the universal bundles are partitioned into various groups according to their \(A\)-grading. The sizes of these groups are given by the dimension vector \(\tilde{v}\) of the quiver (76). Restricted to \(X^U\), this dimension vector further splits

\[
\tilde{v} = \tilde{v}' + \tilde{v}''
\]

into components of weight 0 or 1 with respect to \(U\).

For computations of degree in \(u\), it is natural to use the quadratic form associated to the quiver (76). In general, for any quiver \(Q\) with dimension vector \(v\), one defines

\[
(v, v)_Q = \sum_{i \to j} v_i v_j
\]

where \(i \to j\) means that \(i\) and \(j\) are connected by an edge of \(Q\). Together with the corresponding dot product

\[
v \cdot Q \, v' = \sum_{i \in \text{vertices}(Q)} v_i v'_i
\]

the form (87) enters the dimension formula for Nakajima varieties

\[
\frac{1}{2} \dim \mathcal{M}_Q(v, w) = (v, v)_Q + v \cdot Q (w - v).
\]
To prove (86), we consider the cases $u \to 0$ and $u \to \infty$ limits separately. In the $u \to 0$ limit, we have

$$\frac{1}{\Lambda^o T_{A\text{-fixed}}} \bigg|_{X^u} = O(u^{e_0}), \quad u \to 0,$$

where

$$e_0 = -\bar{v}'' \cdot Q^A \bar{v}'' + (\bar{v}'', \bar{v}'')_{Q^A}$$

because $w'' = 0$ by construction. Since $Q^A$ is a union of quivers of type $A_\infty$ the quadratic form in (90), which is proportional to the Cartan-Killing form for the corresponding Lie algebra, is negatively defined. Therefore

$$e_0 < 0 \quad \text{for} \quad \bar{v}'' \neq 0$$

and the $u \to 0$ case of (86) is established.

In the opposite limit we have

$$\frac{1}{\Lambda^o T_{A\text{-fixed}}} \bigg|_{X^u} = O(u^{e_\infty}), \quad u \to \infty,$$

where

$$e_\infty = \frac{1}{2} \dim M_{Q^A}(\bar{v}, w) - \frac{1}{2} \dim M_{Q^A}(\bar{v}', w).$$

From (77), we conclude

$$e_\infty = 0$$

and the proof of (86) is complete.

**Appendix: Bethe equations**

For completeness, we recall the Bethe equations first derived in the current context by Nekrasov and Shatashvili [37, 38]. Here we derive them formally as equations for the critical points of the integrand in (4). See e.g. [44] for a discussion which does not explicitly involve integral representation.

Let

$$TX = T (T^* \text{Rep}(v, w)) - \sum_i (1 + \hbar^{-1}) \text{End}(V_i)$$

be the tangent bundle of $X$ viewed as an element of $K_{T \times G}(\mathfrak{R})$. This is a Laurent polynomial in $x_{i,k}$ and the characters of $T$. The negative terms in it reflect the moment map equations and the quotient by $G$.

Let the transformation $\tilde{a}$ be defined by

$$\tilde{a} \left( \sum n_i \chi_i \right) = \prod \left( \chi_i^{1/2} - \chi_i^{-1/2} \right)^{n_i}, \quad n_i \in \mathbb{Z},$$

where $\chi_i$ are weights of $T \times G$. This is a homomorphism from the group algebra of the weight lattice to rational functions on a double cover of the maximal torus.

The following is a restatement of a result of Nekrasov and Shatashvili [37, 38].
Proposition 9. The critical points in the \( q \to 1 \) asymptotics of the integral (4) satisfy the following Bethe equations

\[
\hat{a} \left( x_{i,k} \frac{\partial}{\partial x_{i,k}} TX \right) = z_i
\]  

(A.2)

for all \( i \in I \) and \( k = 1, \ldots, v_i \).

The exact form of the right-hand side in (A.2) depends on the shift of variable \( z \), which was mentioned but not made explicit in the discussion of (4). In \cite{39} it is explained why it is natural to use

\[
z\# = z \left( -\hbar^{1/2} \right) - \det T^{1/2}
\]

in place of \( z \) in (4), see also \cite{1}. It is directly related to the shift by the canonical theta-characteristic in \cite{27}. With this shift, the equation (A.2) take the stated form.

Note that

\[
\det T^{1/2} = \prod_{\chi \in T^{1/2}X} \chi
\]

is a line bundle on \( X \) and hence a cocharacter of the Kähler torus \( Z \). It therefore makes sense to shift the variables \( z \) by the value of this cocharacter at \(-\hbar^{1/2}\). Concretely, the coordinates of (A.3) in the lattice of cocharacters are the exponents of \( x_{i,k} \) in (A.3). Note that these exponents do not depend on \( k \).

Proof of Proposition 9. Let \( \Phi \) denote the term with \( \phi \)-functions in (4). We recall from \cite{1} that

\[
\Phi = \prod_{\chi \in T^{1/2}X} \frac{\phi(q \chi)}{\phi(h \chi)}
\]

(A.4)

where the product is over the weight \( \chi \) in a polarization \( T^{1/2}X \) of (A.1). By definition of a polarization, we have

\[
TX = \sum_{\chi \in T^{1/2}X} \left( \chi + \frac{1}{h \chi} \right) .
\]

(A.5)

Approximating a sum by a Riemann integral gives

\[
\ln \frac{\phi(q \chi)}{\phi(h \chi)} \sim \frac{1}{\ln q} \int_1^h \ln (1 - s \chi) \frac{ds}{s}, \quad q \to 1.
\]

Elementary manipulations give

\[
x \frac{\partial}{\partial x} \int_1^h \ln (1 - s \chi) \frac{ds}{s} = -x \frac{\partial}{\partial x} \chi \ln \frac{1 - \chi}{1 - h \chi} =
\]

\[= -\ln \hat{a} \left( x \frac{\partial}{\partial x} \left( \chi + \frac{1}{h \chi} \right) \right) + \ln(-h^{1/2}) x \frac{\partial}{\partial x} \ln \chi .
\]

(A.6)
Note that summed over $\chi$ the first term on the second line of (A.6) gives

\[
\ln \hat{a} \left( x \frac{\partial}{\partial x} T X \right).
\]

The other exponentially large term in (4) is $e(x, z_{\#})$, where $z_{\#}$ denotes the Kähler variables $z$ shifted by $(-\hbar^{1/2})^{-\det T^{1/2}}$, as above. By definition, this means

\[
x_{i,k} \frac{\partial}{\partial x_{i,k}} \ln e(x, z_{\#}) = \frac{1}{\ln q} \left( \ln z_i - \ln(-\hbar^{1/2}) \sum_{\chi} x_{i,k} \frac{\partial}{\partial x_{i,k}} \ln \chi \right). \tag{A.7}
\]

Summing (A.6) and (A.7) gives

\[
\ln \hat{a} \left( x_{i,k} \frac{\partial}{\partial x_{i,k}} T X \right) = \ln z_i
\]

as equations for the critical point of the function $W$ in (7), as claimed.

\[\square\]

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