Markovian Persuasion with Stochastic Revelations

Ehud Lehrer* and Dimitry Shaiderman†

May 6, 2022

Abstract

In the classical Bayesian persuasion model an informed player and an uninformed one engage in a static interaction. The informed player, the sender, knows the state of nature, while the uninformed one, the receiver, does not. The informed player partially shares his private information with the receiver and the latter then, based on her belief about the state, takes an action. This action, together with the state of nature, determines the utility of both players. This paper analyzes a dynamic Bayesian persuasion model where the state of nature evolves according to a Markovian law. Here, the sender always knows the realized state, while the receiver randomly gets to know it. We discuss the value of the sender when he becomes more and more patient and its relation to the revelation rate, namely the probability at which the true state is revealed to the receiver at any stage.

Keywords: Markovian persuasion, Dynamic Bayesian persuasion, Markov chain, Asymptotic value, Revelation rate.

JEL Codes:...
1 Introduction

The vast majority of the literature on Bayesian persuasion is devoted to static situations. In such models a state of nature is realized, an informed agent, the sender, knows it and he then discloses some information about it to an uninformed agent, the receiver, who takes an action.Kamenica and Gentzkow [23] present a case where a prosecutor, who is fully informed about the state of nature, i.e., whether the suspect is guilty or innocent, wishes to persuade a judge, e.g., to convict the suspect of a crime. This is a static scenario: upon the prosecutor’s disclosure, the judge takes a decision and the game is over.

Here we study a dynamic model where the interaction between the sender, who has a commitment power, and the receiver, evolves over time. As in the static model, the sender is informed at each period about the realized state of nature and it is at his discretion to decide how much information to disclose to the receiver.

The sender publicly announces an information provision policy to which he is committed throughout. The receiver knows the details of this policy and thus, when she receives a signal from the sender, she updates her belief about the true state accordingly. She then takes action based solely on her posterior belief. This action, together with the realized state, determines not only her own payoff, but also that of the sender.

The new ingredient of the model we currently address is the following. In any stage, and after the receiver took an action, the true state and particularly, her payoff, are revealed to her with a positive probability, called the revelation rate. When the state is revealed to the receiver it enables her to further update her belief and consequently to improve her action. This might well affect the sender’s payoff as well.

We assume that the evolution of states follows a Markov chain. The dynamic of the receiver’s belief is therefore governed by both the disclosure policy of the sender and the Markov transition law. In case of no state revelation, based solely on the signal sent by the sender, the receiver updates her belief according to Bayes’ law. In case of state revelation, the receiver knows the true state and proceeds with Bayesian updating any subsequent time period at which she is informed of a new piece of information. This Bayesian updating process induces a dynamic on the set of the receiver’s belief, that is, on the set of probability distributions on the state space. Since all terms are known to both players, the nature of the shift from one
belief to another is also known to both.

When committing to an information provision policy, the sender takes into account the resulting posterior belief of the receiver, which has two effects. The first is on the action taken by the receiver and directly on his own stage payoff. In a dynamic setting, as discussed here, there is another effect: on the belief over future states, and consequently on future payoffs. The posterior resulting from a Bayesian updating in one period is then shifted by the Markov transition matrix and becomes the initial belief in the next one. Balancing between these two, potentially contradicting effects, is the dynamic programming problem faced by the sender.

In this paper we study the long-run optimal values that the sender may achieve in the case of an irreducible Markov chain. While in a static model this function is the payoff function of the sender, here, this function takes into account the underlying dynamics and combines present and future payoffs.

Two of the main results (Theorems 1 and 3) deal with patient players. In this case the stationary distribution of the Markov chain plays a central role. The reason is that in the long run the frequency of visiting each state converges to its probability with respect to the invariant distribution. We show in Theorem 1 that as the sender becomes increasingly patient, his expected payoff in the dynamic game converges to a weighted average of the discounted values (in the game without revelations) with the priors being the rows of the Markov matrix. In other words, the asymptotic value of the game with revelations is characterized by the discounted values of the game with no revelation.

The second result applies Theorem 1 to highlight a particular monotonicity aspect of the discounted values of the game with no revelations. As far as we know, there is no result in Markovian models that refers to the monotonicity of the discounted values of the game as the discount factor varies. Here, however, the introduction of the revelation factor enables us to obtain a positive result. Theorem 2 states that a certain average of discounted values of games without revelations is monotonically non-decreasing as a function of the discount factor. The last result of this paper, stated in Theorem 3, applies this observation to zero-sum Markovian chain games studies initially by Renault [35].

The closest papers to the present one are those studying Markov persuasion problems with no revelations. Renault et. al. [36] deals with a specific type of Markov chains, homothetic ones, and with a utility function which gives a fixed amount to the sender in a certain region and zero in its comple-
ment. Renault et al. [36] show that in this model the optimal information provision policy is a greedy one (starting from a certain random time period on). Namely, the sender’s greedy policy instructs him to maximize his current stage payoff at any time period, ignoring future effects of this policy. Ely [10] and Farhadi and Teneketzis [12], study a case where there are two states, whose evolution is described by a Markov chain with one absorbing state. In these two works, the receiver is interested in detecting the jump to the absorbing state, whereas the sender seeks to prolong the duration of time until detection. Lehrer and Shaiderman [25] study the asymptotic value of general irreducible and aperiodic Markov persuasion problems.

In a broader context, this paper belongs to a growing body of literature concerned with information design problems, and more specifically with the analysis of situations where a sender is required to statically or sequentially convey signals to a receiver. Without elaborating on the exact details of their models, whose frameworks employ diverse methodologies, we refer the reader to Mailath and Samuelson [29], Mathevet et al. [31], Honryo [18], Orlov et al. [32], Phelan [34], Lingenbrink and Iyer [28], Wiseman [38], Kolotilin et al. [24], Athey and Bagwell [3], Arieli and Babichenko [1], Pei [33], Best and Quigley [7], Arieli et al. [2], Au [4], Dziuda and Gradwohl [9], Margaria and Smolin [30],Escobar and Toikka [11], Guo and Shmaya [17], Hörner et al. [21], Ganglmair and Tarantino [15], Guo and Shmaya [16], and Augenblick and Bodoh-Creed [5] for an acquaintance with the literature.

The paper is organized as follows. Section 2 presents the model. Section 3 provides a reduction of the model to a Markov decision problem (MDP). The main results appear in Section 4. Section 5 introduces an application to zero-sum Markov chain games with a lack of information on one player. The proofs are found in Sections 6,7, and in the Appendixes.

2 The Model

We consider a two-player repeated game between a sender and a receiver, called Markovian persuasion game with stochastic revelations. The roles of the players remain fixed throughout the different stages of the game.

In order to formally define the game, we need the following notations.

- $K = \{1, ..., k\}$ is a finite set of states.
• \((X_n)_{n \geq 1}\) is an irreducible Markov chain over \(K\) with prior probability \(p \in \Delta(K)\) and a transition rule given by the stochastic matrix \(M\). We assume that \((X_n)_{n \geq 1}\) are defined on the probability space \((\Omega, \mathcal{F}, \mathbb{P})\).

• \((Z_n)_{n \geq 1}\) is a sequence of i.i.d. Bernoulli trials, defined on \((\Omega, \mathcal{F}, \mathbb{P})\). The success probability of each \(Z_n\) is \(x \in (0, 1]\). The parameter \(x\) is called the revelation rate. We assume that for every \(n \geq 1\), \((Z_n)\) is independent of \((X_n)_{n \geq 1}\).

During the game the state evolves according to the Markov chain \((X_n)_{n \geq 1}\) while the receiver obtains information from the sender and in addition, gets to know the realized state when the respective (i.e., \(Z_n\) at time \(n\)) Bernoulli trial turns out to be a success.

Formally, each stage \(n \geq 1\) of the game consists of three parts, evolving sequentially according to the following timeline.

**Sender’s role.** The sender is informed of the realization \(x_n\) of \(X_n\). He then sends to the receiver a signal \(s_n\), from a finite set of signals \(S\) whose cardinality is at least \(k\).

**Receiver’s role.** The receiver takes an action \(b_n\) from a set \(B\), which is assumed to be a compact metric space. This decision may take into account the information she received so far, namely, the \(n\) signals \(s_1, \ldots, s_n\) sent by the sender, and the realized states up to time \(n - 1\) at those periods when the Bernoulli trials \(Z_1, \ldots, Z_{n-1}\) were successes. The decision of the receiver, \(b_n\), together with the current state \(x_n \in K\) determine the payoffs of the sender and the receiver, given by the utilities \(v(x_n, b_n)\) and \(w(x_n, b_n)\), respectively.

**Random revelations.** After completing their roles in the \(n^{th}\) stage, both players observe the outcome of the Bernoulli trial \(Z_n\). In case this trial is a success, the state \(x_n\) is publicly announced, becoming common knowledge between the two players. In particular, if \(Z_n\) is a success, the receiver learns his payoff at the \(n^{th}\) stage. When \(Z_n\) is a success, we say that a revelation occurs.

A signaling strategy \(\sigma\) of the sender in \(\Gamma_3^\lambda(p)\) is described by a sequence of stage strategies \((\sigma_n)\), where each \(\sigma_n\) is a mapping \(\sigma_n : (K \times S \times \{0, 1\})^{n-1} \times K \rightarrow \Delta(S)\). That is, the signal \(s_n\) sent by the sender at time \(n\) is distributed by the lottery \(\sigma_n\), which may depend on all past states \(x_1, \ldots, x_{n-1}\), past signals \(s_1, \ldots, s_{n-1}\), and past outcomes of \(Z_1, \ldots, Z_{n-1}\) (where we decode a

\[1\] The purpose of this assumption is to make sure that the sender can disclose \((x_n)\).
success (i.e., revelation) by 1 and a failure by 0) together with the current state $x_n$. Let $\Sigma$ be the space of all signaling strategies.

The game $\Gamma_x^\lambda(p)$ can be thought of as a generalization of the Markovian persuasion game $\Gamma_\lambda(p)$, studied in [25]. The game $\Gamma_\lambda(p)$ coincides with the description of $\Gamma_x^\lambda(p)$, except for the fact that in former there are no revelations. That is, in $\Gamma_\lambda(p)$ the receiver knows only what the sender had sent her.

The space $\Sigma$ is richer than the space of signaling strategies, $\Xi$, in the Markovian persuasion game $\Gamma_\lambda(p)$. Indeed, in the latter there is no exogenous information given to the receiver, and each $\xi \in \Xi$ is a sequence of mappings $\xi_n : (K \times S)^{n-1} \times K \to \Delta(S)$.

In both games, $\Gamma_x^\lambda(p)$ and $\Gamma_\lambda(p)$, we make the standard assumption of commitment by the sender. That is, we assume that the sender commits to a signaling strategy at the start of the game, and makes it known to the receiver. The commitment assumption enables the receiver to update her beliefs about the distribution of states $(x_n)$ based on the signals she obtains during the game. Formally, by Kolmogorov’s Extension Theorem, each signaling strategy $\sigma \in \Sigma$ together with $(X_n)_{n \geq 1}$ and $(Z_n)_{n \geq 1}$ induces a unique probability measure $P_{p,x,\sigma}$ on the space $Y = (K \times S \times \{0, 1\})^\mathbb{N}$. This measure is determined by the following law:

$$P_{p,x,\sigma}(x_1, s_1, z_1, ..., x_n, s_n, z_n) = p(x_1) \prod_{i=1}^{n-1} M_{x_i,x_{i+1}} \times \left( \prod_{i=1}^{n} \sigma_i(x_1, s_1, z_1, ..., x_{i-1}, s_{i-1}, z_{i-1}, x_{i}, s_{i}) \right) (s_i) \times \left( \prod_{i=1}^{n} x^z(1-x)^{1-z} \right).$$

Only when $Z_n = 1$ the receiver knows the realized state $x_n$. In order to define the posterior probabilities $p_n := (p_n^\ell)_{\ell \in K}$ that the receiver assigns to the events $\{X_n = \ell\}$, $\ell \in K$, given the signals $s_1, ..., s_n$, the first $n-1$ outcomes $z_1, ..., z_{n-1}$ of $Z_1, ..., Z_{n-1}$, and the strategy $\sigma$, we need the following notation. Let $r_n = x_n$ if $Z_n = 1$ and $r_n = \emptyset$, representing a null signal, otherwise.

The formula for the posterior $p_n$ is

$$p_n^\ell = P_{p,x,\sigma}(X_n = \ell | s_1, r_1, ..., s_{n-1}, r_{n-1}, s_n).$$

Similarly, in the Markovian persuasion game $\Gamma_\lambda(p)$, each signaling strategy $\xi \in \Xi$ together with $(X_n)_{n \geq 1}$ induces a unique probability measure $P_{p,\xi}$.
on the space $\mathcal{O} = (K \times S)^\mathbb{N}$. The corresponding sequence of posteriors $(\eta_n)$, where $\eta_n = (\eta^\ell_n)_{\ell \in K}$, is defined by $\eta^\ell_n = \mathbb{P}_{p,\xi}(X_n = \ell | s_1, ..., s_n)$ for every $\ell \in K$.

A second key assumption of our model is that the receiver’s decision at any time period $n$ in both $\Gamma^{\pi}_x(p)$ and $\Gamma^{\lambda}_x(p)$ depends only on her current posterior, $p_n$. Such an assumption is realistic in cases where the receiver maximizes her expected payoff based on her current belief. It is a natural assumption in several scenarios. The first is when a sequence of transitory short-lived receivers (e.g., Jackson and Kalai [22]) are involved in a recurring game, where a stage game is played sequentially by different players having social memory of the information provided (as is often the case in online markets). The second scenario is when at any stage a particular receiver is chosen from a large population of anonymous receivers, and the sender is a political party or a media outlet trying to persuade the general public.

Denote by $\theta : \Delta(K) \to B$ the strategy of the receiver, that is, the mapping that associates an action to any belief. As in previous related models, we assume that the decision policy of the receiver is known to the sender. The last assumption of our model is that the function $u : \Delta(K) \to \mathbb{R}$ defined by $u(q) = \sum_{\ell \in K} q^\ell v(\ell, \theta(q))$ is continuous and non-negative. To summarize, our assumptions imply that the signaling strategies of the sender determine his payoff at any time period $n$. Thus, the total expected payoff to the sender in $\Gamma^{\pi}_x(p)$ under the signaling strategy $\sigma \in \Sigma$ can be written as

$$\gamma^{\pi}_x(p, x, \sigma) := \mathbb{E}_{p,x,\sigma} \left[ (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} u(p_n) \right],$$

where $\mathbb{E}_{p,x,\sigma}$ is the expectation w.r.t. $\mathbb{P}_{p,x,\sigma}$, whereas the total expected payoff to the sender in $\Gamma^{\lambda}_x(p)$ under the signaling strategy $\xi \in \Xi$ takes the form

$$\gamma^{\lambda}_x(p, \xi) := \mathbb{E}_{p,\xi} \left[ (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} u(\eta_n) \right],$$

where $\mathbb{E}_{p,\xi}$ is the expectation w.r.t. $\mathbb{P}_{p,\xi}$. The value of $\Gamma^{\pi}_x(p)$ is $v^{\pi}_x(p) := \sup_{\sigma \in \Sigma} \gamma^{\pi}_x(p, \sigma)$, and $v^{\lambda}_x(p) := \sup_{\xi \in \Xi} \gamma^{\lambda}_x(p, \xi)$ is the value of the game $\Gamma^{\lambda}_x(p)$.

3 The Reduced MDP

As first noted in Renault et al. [36] in regards to $\Gamma^{\lambda}_x(p)$, one may describe the latter as well as the game $\Gamma^{\pi}_x(p)$ in terms of MDP’s over the belief space.
\(\Delta(K)\), which will be denoted by \(\mathcal{G}\) and \(\mathcal{G}_x\), respectively. In both \(\mathcal{G}\) and \(\mathcal{G}_x\) the set of actions available to the sender at each belief \(q \in \Delta(K)\) consists of the possible set of splits of \(q\) to \(|S|\) beliefs in \(\Delta(K)\), denoted \(\mathcal{S}_q\), and given by

\[
\mathcal{S}_q := \left\{ \left\{ (q_i, \alpha_i) \right\}_{i=1}^{|S|} : q_i \in \Delta(K) \ \forall i, \ (\alpha_i)_{i=1}^{|S|} \in \Delta(S), \ \text{s.t.} \ \sum_{i=1}^{|S|} \alpha_i q_i = q \right\}.
\]

On the one hand, when the elements of \(K\) are distributed according to \(q\), any lottery over the signal set \(S\) that depends on the realized states clearly defines an element of \(\mathcal{S}_q\). On the other hand, by the Splitting Lemma (e.g., Blackwell [8], Aumann and Maschler [6]), any split in \(\mathcal{S}_q\) may be achieved by conducting a lottery on \(|S|\) which is correlated with the state. The MDP’s \(\mathcal{G}_x\) and \(\mathcal{G}\) differ in their transition rules, each is determined by the current belief \(q\) and action (split) at \(q\). To describe their respective transition rules, we need to consider the (random) sequences of beliefs \((y_n)_{n\geq1}\) and \((\zeta_n)_{n\geq1}\) describing the beliefs at the start of each stage \(n \geq 1\) in \(\Gamma^\alpha(x)\) and \(\Gamma^\alpha(p)\), respectively. Clearly, \(y_1 = \zeta_1 = p\). Consider \((\zeta_n)_{n\geq1}\) first. Given \(\zeta_n\), upon taking the split \(\{(q_i, \alpha_i)\} \in \mathcal{S}_{\zeta_n}\), the posterior \(\eta_n\) equals \(q_i\) with probability \(\alpha_i\). Therefore, as \((X_n)_{n\geq1}\) evolves according to \(M\), given \(\eta_n = q_i\), the belief at the start of the \((n + 1)\)'st stage of \(\Gamma^\alpha(p)\), \(\zeta_{n+1}\), is equal to \(q_i M\). We therefore define the transition rule \(\rho\) of the MDP \(\mathcal{G}\) as

\[
\rho(q, \{(q_i, \alpha_i)\}_{i=1}^{|S|}) = q_i M, \ \text{with prob.} \ \alpha_i, \ i = 1, \ldots, |S|.
\]

The situation in \(\Gamma^\alpha(x)\) is different. Indeed, given \(y_n\), assume that the sender decides to take the split \(\{(q_i, \alpha_i)\} \in \mathcal{S}_{y_n}\) at the \(n\)'th stage of \(\Gamma^\alpha(x)\). Then, when the receiver is told (with probability \(x\)) the state \(X_n\), by the Markov property of \((X_n)_{n\geq1}\), \(y_{n+1}\) is equal to \(\delta x_n M\). Moreover, the time line of \(\Gamma^\alpha(x)\) dictates that revelation might occur only after the sender and the receiver have taken their actions. In case the split \(\{(q_i, \alpha_i)\}\) employed by the sender results in \(q_i\) (which occurs with probability \(\alpha_i\)) and revelation occurs, the observed value of \(X_n\) will be interpreted by the receiver as the outcome of a lottery over \(K\) with distribution \(q_i\).

As \(Z_n\) is independent of \((X_n)_{n\geq1}\) and as a result of the sender’s strategy, in the case \(Z_n = 1\), we have \(y_{n+1} = \delta x_n M = \delta i M\) with probability \(\sum_{i=1}^{|S|} \alpha_i q_i^\ell = y_n^\ell\) for any \(\ell \in K\). In view of this, upon revelation, the transition rule from \(y_n\) to \(y_{n+1}\) is independent of the split used: it depends only on \(y_n\). In the other
case, namely when the receiver is not told (with probability $1 - x$) the state $X_n$, $y_{n+1}$ is equal to $q_iM$ with probability $\alpha_i$. To summarize, the transition rule of the MDP $G_x$, denoted $\rho_x$, is given by

$$
\rho_x(q, \{(q_i, \alpha_i)\}_{i=1}^{|S|}) = \begin{cases} 
q_iM, \text{ with prob. } (1 - x)\alpha_i, & i = 1, ..., |S| 
\delta\ell M, \text{ with prob. } xq_\ell, & \ell \in K.
\end{cases}
$$

Lastly, we need to specify the payoff functions in $G_x$ and $G$. First, keeping the notation described above, we have that the posteriors $p_n$ and $\eta_n$ are determined by $y_n$ and $\zeta_n$, respectively, and the relevant split employed. Therefore, in both $\Gamma_x(p)$ and $\Gamma_\lambda(p)$ the $n$'th stage payoff is determined by the distribution of the $n$'th posterior, $p_n$ and $\eta_n$, respectively. For that reason, we set the payoff in $G_x$ and $G$, denoted by $r$, to equal

$$
r(q, \{(q_i, \alpha_i)\}_{i=1}^{|S|}) := \sum_{i=1}^{|S|} \alpha_i u(q_i), \quad \{(q_i, \alpha_i)\}_{i=1}^{|S|} \in \mathcal{S}_q.
$$

### 3.1 A Recursive Formula

By the dynamic programming principle for the MDP $G_x$ and the discussion above, we get the following recursive formula for $v_x^\lambda(p)$:

$$
v_x^\lambda(p) = \max_{\{(q_i, \alpha_i)\}_{i=1}^{|S|} \in \mathcal{S}_p} \{(1 - \lambda) \sum_{i=1}^{|S|} \alpha_i u(q_i) + \lambda(1 - x) \sum_{i=1}^{|S|} \alpha_i v_x^\lambda(q_i) + \lambda x \sum_{\ell \in K} p_\ell v_x^\lambda(\delta\ell M)\}.
$$

Consider the operator $\phi : \Delta(K) \to \Delta(K)$ defined by $\phi(q) = qM$. Since $|S| \geq k$, Carathéodory’s Theorem (see, e.g., Corollary 17.1.5 in [37]) implies that Eq. (3) can be rewritten as follows:

$$
v_x^\lambda(p) = (\text{Cav} \{(1 - \lambda)u + \lambda(1 - x) (v_\lambda \circ \phi)\})(p) + \lambda x \langle p, \xi_x^\lambda \rangle,
$$

where $\xi_x^\lambda = (v_x^\lambda(\delta_1 M), ..., v_x^\lambda(\delta_k M))$, and $\langle \cdot, \cdot \rangle$ denotes the scalar product in $\mathbb{R}^k$. As Carathéodory’s Theorem states that the Cav of any scalar function $f$ with domain $\Delta(K)$ can be achieved by a split to exactly $k$ points, we have the following.
Observation 1. The value \( v^x_\lambda(p) \) is the same for all signal sets \( S \) of cardinality at least \( k \).

Lemma 1. Assume that \( p \in \Delta(K) \) satisfies \( u(p) = (\text{Cav } u)(p) \). Then, for any \( \lambda \in [0, 1] \) and any \( x \in (0, 1] \), there exists an optimal signaling strategy \( \sigma_\lambda \) in \( \Gamma^x_\lambda(p) \) that would instruct the sender to reveal no information at \( p \) (i.e., take the split \( \{(q_i, \alpha_i) \in S_p \} \) with \( q_i = p \) for all \( i \)).

Proof of Lemma 1. On the one hand, we have by Eq. (4) that \( v^x_\lambda : \Delta(K) \to \mathbb{R} \) is concave for every \( \lambda \in [0, 1] \) and \( x \in (0, 1] \). As \( \phi \) is linear, \( v_\lambda \circ \phi \) is also concave. Then, by the definition of Cav, we infer from Eq. (4) the inequality

\[
v_\lambda(p) \leq (1 - \lambda)(\text{Cav } u)(p) + \lambda(1 - x)(v_\lambda \circ \phi)(p) + \lambda x (p, \xi^x_\lambda). \tag{5}
\]

On the other hand, since the sender can always decide to not reveal any information at \( p \), and thereafter play optimally in the game \( \Gamma^x(pM) \), we also have that \( v_\lambda(p) \geq (1 - \lambda)u(p) + \lambda(v_\lambda \circ \phi)(p) \). As \( u(p) = (\text{Cav } u)(p) \), the latter combined with Eq. (5) completes the proof. \( \square \)

4 The Main Results

The first main theorem takes an asymptotic approach. It provides an approximation to the value of the game \( \Gamma^x_\lambda(p) \) when the sender is sufficiently patient. This approximation uses two ingredients of the model: the revelation rate \( x \) and the Markov matrix \( M \).

It turns out that the asymptotic values, as the sender grows infinitely patient, becomes arbitrarily close to a number \( \mathcal{V}(x) \) expressed in terms of (i) the invariant distribution of \( M \), denoted \( \pi_M \); and (ii) the values of the Markovian persuasion games with \( 1 - x \) as the discount factor and \( m_\ell, \ell \in K \), as the prior distributions, where \( m_\ell \) is the \( \ell \)th row of \( M \).

There are two interesting features of the asymptotic value. First, it is characterized in terms of the values associated with a fixed discount factor (i.e., \( 1 - x \)) related to some priors: the rows of \( M \). Second, it does not depend on the initial distribution, \( p \).

Theorem 1. For every revelation rate \( x \in (0, 1] \) there exists a number \( \mathcal{V}(x) \), such that for every \( \varepsilon > 0 \) there exists \( 0 < \delta < 1 \) such that

\[
|v^x_\lambda(p) - \mathcal{V}(x)| < \varepsilon, \quad \forall \lambda > \delta, \quad \forall p \in \Delta(K).
\]
Moreover, we have the following formula for $V(x)$:

$$V(x) = \pi_1^M v_{1-x}(m_1) + \cdots + \pi_k^M v_{1-x}(m_k), \quad \forall x \in (0, 1].$$

The proof of Theorem 1 hinges on the following observations and heuristic arguments.

**The evolution of information.** By the Markov property, upon a revelation of the current state of $(X_n)_{n \geq 1}$ the information obtained by the receiver from all previous signals perishes. For instance, if the state at the first time a revelation occurred was $\ell \in K$, the belief about the following state is $m_\ell$ regardless of past signals observed. In view of this, one can think of the revelations as reboots: after each reboot there is a renewal of information. For this reason, our analysis concentrates on games within the game $\Gamma^x_x(p)$ that consist of the stages between any two successive revelations. These games can be interpreted as random-duration Markovian persuasion games (without stochastic revelations). They are defined and studied in Subsection 6.1.

**Uniform Tauberian theorem for MDP’s.** Probabilistic tools such as the central limit theorem and the ergodic theorem will be required for our analysis. As such tools are adapted to $N$-Cesáro valuations rather then $\lambda$-discounted valuations, we would like to reduce our analysis to the former. This will be made available by a uniform Tauberian theorem for dynamic programming of Lehrer and Sorin [26] applied for the MDP $G_x$. This useful result asserts that the statement of Theorem 1 holds if and only if $v^x_N(p)$ converges uniformly (over $\Delta(K)$) to $V(x)$, where $v^x_N(p)$ is the value

$$v^x_N(p) := \sup_{\sigma \in \Sigma} \left\{ \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{n=1}^{N} u(p_n) \right) \right\}.$$  

Using this reduction, we state in Subsection 6.4 two propositions, Propositions 1 and 2, from whom Theorem 1 follows.

**Probabilistic interpretation of discounted valuations.** The $\lambda$-discounted valuation admits a well-known probabilistic interpretation. On each day, extinction occurs with probability $1 - \lambda$. Therefore, the probability to survive for $n$ days equals $\lambda^{n-1}(1 - \lambda)$. The $\lambda$-discounted valuation thus awards the sender with his payoff of the last day, $u(p_n)$, for surviving $n$ days in $\Gamma^x_\lambda(p)$. A
similar interpretation applies to the number of stages between any two successive revelations, since it follows a geometric distribution with parameter \( x \). However, in the latter, the payoff obtained when the game lasts for \( n \) days is the expected sum of payoffs on these days (and thus the convenience of working with Cesáro valuations), and not just the expected payoff of the last day. Under such a payoff, the value of the random-duration Markovian persuasion game between any two successive revelations is shown to be equal (see Claim 1) to \( v_{1-x}(\delta_t M)/x = v_{1-x}(m_t)/x \) whenever the state at the time of the first revelation of the two was \( \ell \in K \).

**The two-dimensional Markov chain** \((X_n, Z_n)_{n \geq 1}\). In view of the former arguments, it seems natural to leap to the conclusion that Theorem 1 holds, as by the ergodic theorem for the two-dimensional Markov chain \((X_n, Z_n)_{n \geq 1}\), the asymptotic frequency (limit of the \( N \)-Cesáro valuations) of stages in which the revelation occurs when the state \( \ell \in K \) is \( x \cdot \pi_{M}^L P_{p,x,\sigma} \)-a.s., regardless of the signaling strategy \( \sigma \) of the sender. A further argument, based on properties of negative binomial distributions (see Subsection 6.2) and the standard normal distribution arising from the central limit theorem (see Subsection 6.3), is required for a probabilistic analysis of the on-average behavior of the revelation times.

Discounted or finite dynamic games with lack of information on one side (such as those analyzed by Aumann and Maschler [6], Renault [35] and Lehrer and Shaiderman [25]) are characterized by the discount factor or by the length of the game. Here, we introduce a new parameter: the revelation rate, at which the uninformed player obtains extra information about the realized state. It turns out that adding this new parameter enables us to learn more about games with no revelations. In other words, expanding the scope by allowing positive revelation rates enables one to learn more about traditional games in which revelations are absent.

In the Aumann and Maschler [6] zero-sum games when the state is selected once and for all, the value of the discounted game decreases with the discount rate. The intuition is clear: a greater discount factor enables the uninformed player (the minimizer) to learn more to her advantage.

When the state is dynamic, however, the value is no longer monotone with respect to the discount factor. It turns out that the introduction of positive revelation rates enables us to provide some monotonicity results about the values of Markovian persuasion games: this time not about a specific prior,
but rather about a combination of values corresponding to a few priors. We think that the observations provided here and the proof techniques have a potential to be useful also in other models (see also Section 5).

The next theorem states that a convex combination of the values \( v_\lambda(m_\ell), \ell \in K \), is monotonically non-decreasing. It implies, in particular, that there is no range of discount factors in which all these values are simultaneously decreasing.

**Theorem 2.** The mapping \( \Psi : [0, 1) \rightarrow \mathbb{R}_+ \) defined by

\[
\Psi(\lambda) = \pi_M^1 v_\lambda(m_1) + \cdots + \pi_M^k v_\lambda(m_k)
\]

is non-decreasing.

Here we change perspective and look at \( v_\lambda(x) \) as a function of \( x \in (0, 1] \) for fixed \( \lambda \in [0, 1) \) and \( p \in \Delta(K) \). By Theorem 1, if \( x \mapsto v_\lambda(x) \) is non-increasing for every \( \lambda \) and \( p \), then so is \( \mathcal{V}(x) \). As \( \Psi(\lambda) = \mathcal{V}(1-\lambda) \), we obtain that \( \Psi(\lambda) \) is non-decreasing, as desired.

The main argument leading to the monotonicity of \( x \mapsto v_\lambda(x) \) goes as follows. Fix \( 0 < x < y \leq 1 \). For each \( n \geq 2 \), if \( Z_{n-1} = 0 \), the sender can reveal the state \( x_{n-1} \) with probability \( (y-x)/(1-x) \). By doing so, the state \( x_{n-1} \) will be revealed to the receiver (prior to taking his \( n \)'th action \( b_n \)) with probability \( y \), independently across all stages \( n \). Based on this observation, we will argue formally in Section 7 that the sender can guarantee \( v_\lambda(p) \) in \( \Gamma_\lambda(p) \).

5 **An Application to Markov Games with Lack of Information on One Side**

Markov chain games with lack of information on one side were first introduced and studied by Renault [35]. Those are zero-sum repeated games generalizing the classic model of zero-sum repeated games with incomplete information on one side of Aumann and Maschler [6]. The parameters of a Markov chain game with lack of information on one side, denoted \( \Gamma^*(p) \), consist of (i) a finite state space \( K = \{1, ..., k\} \); (ii) finite action sets \( I, J \) for Players 1,2, respectively; (iii) a payoff function \( g : K \times I \times J \rightarrow \mathbb{R}_+ \); (iv) a stochastic matrix \( M \); and (v) an initial probability \( p \in \Delta(K) \).
The play of $\Gamma^*(p)$ evolves in stages. At the start of the first stage, nature chooses a state $k_1 \in K$ according to $p$, and informs Player 1 of its choice. Player 2 knows only $p$. Players 1 and 2 then take simultaneously actions $i_1 \in I$ and $j_1 \in J$, respectively, which are then publicly announced. The payoff for the first stage equals $g(k_1, i_1, j_1)$, and thus is known to Player 1 only. At each subsequent stage $n \geq 2$, Player 1 is informed of the current state $k_n \in K$, which evolves according to the Markov chain with parameters $(p, M)$. Upon obtaining this private information, Players 1, 2 take actions $i_n \in I$ and $j_n \in J$, which are publicly announced. The payoff for that stage equals $g(k_n, i_n, j_n)$, and therefore is again known to Player 1 only.

As pointed out in Renault [35], the game $\Gamma^*(p)$ is equivalent to a zero-sum stochastic game over the compact set $\Delta(K)$, where the compact action sets of Players 1, 2 are $\Delta(I)$ and $\Delta(J)$, respectively. By abuse of notation we will denote this stochastic game by $\Gamma^*(p)$ as well. The transition rule depends only on the state and the action $\alpha = (\alpha^\ell)_{\ell \in K} \in \Delta(I)^K$ of Player 1. By similar arguments to those given for the MDP $G$ in Section 3, given the state $q \in \Delta(K)$ and the action $\alpha = (\alpha^\ell)_{\ell \in K} \in \Delta(I)^K$ the new state becomes $\hat{q}(\alpha, i)M$ with probability $\alpha(q)(i)$, where

$$\alpha(q)(i) := \sum_{\ell \in K} q^\ell \alpha^\ell(i),$$

$$\hat{q}(\alpha, i) := \left(\frac{q^\ell \alpha^\ell(i)}{\alpha(q)(i)}\right) \in \Delta(K).$$

The payoff function assigns to the state $q \in \Delta(K)$ and actions $\alpha \in \Delta(I)^K$ and $\beta \in \Delta(J)$ the quantity $\sum_{\ell \in K} q^\ell g(\ell, \alpha^\ell, \beta)$, where we extend $g$ linearly to $K \times \Delta(I) \times \Delta(J)$. Denote by $V_N(p)$ and $V_\lambda(p)$ the values of $\Gamma^*(p)$ under the $N$-Cesáro and $\lambda$-discounted payoff valuations, respectively.

Analogously to the Markov persuasion game with stochastic revelations, we consider for any $x \in (0, 1]$, the generalized Markov chain game with lack of information on one side, denoted $\Gamma^{*,x}(p)$. In the current setup, this extension means that at any stage $n \geq 1$, Player 2 learns the current state $k_n$ with probability $x$ (independently of all other events) by observing $Z_n$, right after he and his adversary take their actions $\beta_n \in \Delta(J)$ and $\alpha_n \in \Delta(I)^K$, respectively. The game $\Gamma^{*,x}(p)$ is a stochastic game as well, sharing the same parameters as the stochastic game $\Gamma^*(p)$, except for the transition rule. Analogously to the transition rule in the MDP $G_x$ discussed in Section 3, the transition rule in $\Gamma^{*,x}(p)$, given the state $q \in \Delta(K)$ and action $\alpha \in \Delta(I)^K$,
chooses the new state $q'(\alpha, i)M$ with probability $x(q'(i)) \cdot (1-\alpha)$ for every $i \in I$, and $\delta \epsilon M$ with probability $q^t \cdot x$. Denote by $V^x_N(p)$ and $V^x_\lambda(p)$ the values of $\Gamma^{x,r}(p)$ under the $N$-Cesàro and $\lambda$-discounted payoff valuations, respectively.

Define the mapping $\Phi : [0, 1) \rightarrow \mathbb{R}_+$ by

$$\Phi(\lambda) := \pi^1_M V^1_\lambda(m_1) + \cdots + \pi^k_M V^k_\lambda(m_k).$$

**Theorem 3.** Assume that $M$ is irreducible. We have

(i) $V^x_\lambda$ converges uniformly to $\Phi(1-x)$ as $\lambda \rightarrow 1$.

(ii) $\Phi$ is non-decreasing on $[0, 1)$.

The proof of Theorem 3 builds on similar techniques and arguments showcased in the proofs of Theorems 1 and 2. The main additional tool required is a uniform Tauberian theorem for zero-sum stochastic games of Ziliotto [39]. The exact explanations where Theorem 3 follows from can be found in Appendix B.

6 Proof of Theorem 1

6.1 The Random-Duration Game

Let $W$ be a discrete random variable defined on $\mathcal{O}$, which has a geometric distribution with parameter $x$, and which is independent of the Markov chain $(X_n)$ and any signaling strategy $\xi \in \Xi$ of the sender in the Markovian persuasion game. We define the random-duration Markovian persuasion game, denoted $\hat{\Gamma}(p, x)$, as follows. The signaling strategy space of the sender is $\Xi$, i.e., the same as in the Markovian persuasion game. The difference is in the expected payoff valuation, which associates to $\xi \in \Xi$ the payoff

$$\hat{\gamma}(p, x, \xi) = E_{p, \xi} \left( \sum_{n=1}^{\infty} u(\eta_n) \mathbb{1}\{n \leq W\} \right).$$

Let $\hat{v}(p, x) := \sup_{\xi \in \Xi} \hat{\gamma}(p, x, \xi)$ be the value of $\hat{\Gamma}(p, x)$. The interpretation of $\hat{\Gamma}(p, x)$ is that the sender and the receiver engage in a Markovian persuasion game which terminates at random after $W$ stages. The payoff of this game is the sum of the stage payoffs until termination has occurred.

We have the following relationship between $\hat{v}(p, x)$ and $v_\lambda(p)$.

---

$^2$Recall that $m_\ell$ is the $\ell$th row of $M$. 

14
Claim 1. For every \( p \in \Delta(K) \) and \( x \in (0, 1] \) it holds that
\[
\hat{v}(p, x) = \frac{v_{1-x}(p)}{x}.
\] (7)

Proof. Since \( W \) is independent of \((X_n)\) and \((p_n)\) we may disintegrate it from (6), and then change the order of summation to obtain that, for every \( \xi \in \Xi \),
\[
\hat{\gamma}(p, x, \xi) = \sum_{N=1}^{\infty} (1-x)^{N-1} x \mathbb{E}_{p, \xi} \left( \sum_{n=1}^{N} u(\eta_n) \right)
= \sum_{n=1}^{\infty} \left( \sum_{N=n}^{\infty} (1-x)^{N-1} x \right) \mathbb{E}_{p, \xi}(u(\eta_n))
= \frac{1}{x} \sum_{n=1}^{\infty} x(1-x)^{n-1} \mathbb{E}_{p, \xi}(u(\eta_n)) = \frac{1}{x} \gamma_{1-x}(p, \xi).
\] (8)
Optimizing over \( \xi \in \Xi \) yields the claim. \( \square \)

6.2 The Negative Binomial Distribution
Recall that a discrete random variable \( Y \) is said to have a negative binomial distribution with parameters \( (r, p) \in \mathbb{N} \times [0, 1] \), denoted \( Y \sim NB(r, p) \), if its density obeys the law
\[
f_Y(y) = \binom{y+r}{y} p^r (1-p)^y, \quad \forall y \geq 0.
\]
In words, one can think of \( Y \) as counting the number of failures in a sequence of i.i.d. Bernoulli trials having success (i.e., an outcome of 1) probability \( p \), until the first time \( r \) successes were monitored. The mean of \( Y \), denoted \( \mathbb{E}Y \), is known to equal \( n(1 - x)/x \).

The negative binomial distribution arises in our setup naturally. Indeed, let us define by iteration the sequence of random variables \( (\kappa_n)_{n \geq 1} \) by,
\[
\kappa_1 := \inf \{ n \geq 1 : Z_n = 1 \},
\kappa_n := \inf \{ n \geq 1 : Z_{\kappa_1+\cdots+\kappa_{n-1}+n} = 1 \}, \quad n \geq 2.
\]
By the strong Markov property of \((Z_n)\) we have that \((\kappa_n)\) are i.i.d. random variables having geometric distribution with parameter \( x \). In words, the sequence \((\kappa_n)\) counts the times between successive revelations.
Claim 2. For every $n \geq 1$, $\kappa_1 + \cdots + \kappa_n - n$ has negative binomial distribution with parameters $(n, x)$.

Proof of Claim 2: We observe that $\kappa_1 + \cdots + \kappa_n - n$ counts the number of non-revelations (‘failures’) until the first $n$ revelations (‘successes’) were monitored. The claim now follows from the fact that $Z_n$ are i.i.d. Bernoulli with parameter $x$.

The proof of Theorem 1 will require us to estimate the mean of a lower truncated negative Binomial distribution. To do so, we will use the following formula (e.g., Eq. (16) in Geyer [13]):

Fact 1. For any $n \in \mathbb{N}$, if $Y \sim NB(r, p)$ then

$$E(Y | Y > n) = E(Y) + \frac{n + 1}{p(1 + \beta_{n+1}(r, p))},$$

where $\beta_{n+1}(r, p) := \mathbb{P}(Y > n + 1)/ \mathbb{P}(Y = n + 1)$.

6.3 The Central Limit Theorem

A second probabilistic tool which will be of significance in the proof of Theorem 1 is the Central Limit Theorem (abbreviated to CLT from here on forward). Let us now illustrate how it comes into play. Define for any $N \in \mathbb{N}$, the random integer $T_N$ by,

$$T_N := Z_1 + \cdots + Z_N.$$

In words, $T_N$ stands for the number of revelations at the end of the $N$’th stage of $\Gamma(p, x)$. One can also describe $T_N$ in terms of the stopping times $(\kappa_n)$ as follows:

$$T_N = \max \{ n \in \{0, 1, \cdots, N\} : \kappa_1 + \cdots + \kappa_n \leq N \}.$$ (10)

The CLT states that for any $z \in \mathbb{R}$ it holds that

$$\lim_{N \to \infty} \mathbb{P} \left( \sqrt{N} \left( \frac{T_N}{N} - x \right) \leq z \right) = \Phi \left( \frac{z}{\sqrt{x(1-x)}} \right)$$

where $\Phi$ is the cumulative distribution function of a standard normal random variable. Let us now fix $0 < \varepsilon < 1/2$ till the end of the proof of Theorem
1. Choose $z_\varepsilon \in \mathbb{R}_+$ so that $z_\varepsilon / \sqrt{x(1-x)}$ is the $(1 - \varepsilon/2)$-quantile of the distribution of a standard normal random variable. That is,

$$\int_{-\infty}^{z_\varepsilon / \sqrt{x(1-x)}} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{x^2}{2} \right) = 1 - \frac{\varepsilon}{2}.$$ 

By the symmetry of a standard normal random variable, we have that $-z_\varepsilon / \sqrt{x(1-x)}$ is the $\varepsilon/2$-quantile of the latter’s distribution. Therefore, if we define the sequences $(a_N)$ and $(b_N)$ by $a_N := Nx - z_\varepsilon \sqrt{N}$ and $b_N := N x + z_\varepsilon \sqrt{N}$, (11) ensures the existence of $N_\varepsilon \in \mathbb{N}$ for which the following three properties hold:

(i) $a_N \geq 2$ for every $N \geq N_\varepsilon$.

(ii) $\mathbb{P}(a_N \leq T_N) \geq 1 - \varepsilon$ for every $N \geq N_\varepsilon$.

(iii) $\mathbb{P}(T_N \leq b_N) \geq 1 - \varepsilon$ for every $N \geq N_\varepsilon$.

The following simple claim will be of importance to the proofs of Propositions 1 and 2.

**Claim 3.** $z_\varepsilon \varepsilon \leq \sqrt{\frac{2}{x(1-x)}} \varepsilon$.

**Proof.** We have

$$1 - \Phi\left( \sqrt{\frac{2}{\varepsilon}} \right) \leq \sqrt{\frac{\varepsilon}{2}} \frac{1}{\sqrt{2\pi}} \exp\left( -\frac{1}{\varepsilon} \right) \leq \frac{1}{2\sqrt{2\pi}} \exp\left( -\frac{1}{\varepsilon} \right) < \varepsilon/2, \quad (12)$$

where the first inequality is follows from the famous relation $1 - \Phi(x) \leq (1/(\sqrt{2\pi} x)) \exp(-x^2/2)$ due to Gordon (1941) (see Eq. (7) in [14]), whereas the second and third inequalities follow from the fact that $\varepsilon < 1/2$. The choice of $z_\varepsilon$ coupled with Eq. (12) implies that

$$z_\varepsilon \sqrt{x(1-x)} \leq \sqrt{\frac{2}{\varepsilon}},$$

thus proving the claim. \qed
6.4 A Reduction using a Uniform Tauberian Theorem for MDP’s.

By a uniform Tauberian theorem for Markov decision problems over Borel state spaces (e.g., Theorem 1 and discussion in Section 6 in Lehrer and Sorin [26]) in order to prove Theorem 1 it suffices to show that \( v_N^x : \Delta(K) \rightarrow \mathbb{R} \) converges uniformly to \( V(x) \) as \( N \rightarrow \infty \), where \( v_N^x := \sup_{\sigma \in \Sigma} \gamma_N^x(p, \sigma) \) and \( \gamma_N^x : \Delta(K) \times \Sigma \rightarrow \mathbb{R}_+ \) is defined by

\[
\gamma_N^x(p, \sigma) := \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{n=1}^{N} u(p_n) \right).
\]

In other words, \( v_N^x \) is the value of the MDP \( G_x \) where the payoffs at every stage are evaluated according to the \( N \)’th Cesàro average.

For each \( N \in \mathbb{N} \) Let us set \( \ell_N := \kappa_0 + \kappa_1 + \cdots + \kappa_T \), where \( \kappa_0 := 0 \). The random variable \( \ell_N \) describes the total number of stages from the start of the game up to the final revelation till the end of stage \( N \). In particular \( \ell_N \leq N \). For any \( N \geq 1 \) and \( \sigma \in \Sigma \) we have

\[
\gamma_N^x(p, \sigma) = \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{n=1}^{\kappa_1 \wedge N} u(p_n) \right) := A_N(p, x, \sigma)
\]

\[
+ \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{n=\kappa_1 \wedge N+1}^{\ell_N} u(p_n) \right),
\]

\[
+ \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{n=\ell_N+1}^{N} u(p_n) \right), := C_N(p, x, \sigma)
\]

where we use the convention that \( \sum_{n=L+1}^{L} u(p_n) = 0 \) for any \( L \in \mathbb{N} \). Let us now analyze \( A_N(p, x, \sigma) \) and \( C_N(p, x, \sigma) \) separately. We begin with the former. Since \( u \) is non-negative and \( \kappa_1 \) has mean \( 1/x \), we have for all \( \sigma \in \Sigma \) that

\[
0 \leq A_N(p, x, \sigma) \leq \mathbb{E}_{p,x,\sigma} \left( \frac{(\kappa_1 \wedge N)}{N} \|u\|_\infty \right)
\]

\[
\leq \frac{\|u\|_\infty}{N} \mathbb{E}_{p,x,\sigma}(\kappa_1) = \frac{\|u\|_\infty}{Nx}
\]

18
Next, as it pertains to $C_N(p, x, \sigma)$, we use the non-negativity of $u$ to deduce that for any $\sigma \in \Sigma$ it holds,

$$0 \leq C_N(p, x, \sigma) \leq \mathbb{E}_{p, x, \sigma} \left( \frac{(N - \ell_N)}{N} \|u\|_\infty \right).$$

(15)

We need to estimate $\mathbb{E}_{p, x, \sigma} (N - \ell_N)$. Let us claim that for every $1 \leq n \leq N$ we have

$$\{N - \ell_N = n\} = \{Z_{N-n} = 1, Z_t = 0, \forall t \in \{N - n + 1, ..., N\}\}.$$

Indeed, as $\ell_N$ is the time of the last revelation before the end of the $N$'th stage of $\Gamma(p, x)$, the event $\{N - \ell_N = n\}$ states that the last revelation till time $N$ took place at time $N - n$. Therefore, we have for every $\sigma \in \Sigma$ that,

$$\mathbb{E}_{p, x, \sigma} (N - \ell_N) = \sum_{n=0}^{N} n \mathbb{P}_{p, x, \sigma} (\{N - \ell_N = n\})$$

$$= \sum_{n=1}^{N} n \mathbb{P}_{p, x, \sigma} (\{Z_{N-n} = 1, Z_t = 0, \forall t \in \{N - n + 1, ..., N\}\})$$

$$= \sum_{n=1}^{N} n x(1-x)^n \leq (1-x) \sum_{n=1}^{\infty} n(1-x)^{n-1}x = \frac{1-x}{x}.$$

(16)

Hence, Eq. (16) in conjunction with Eq. (15) implies that for every $\sigma \in \Sigma$, and every $N \in \mathbb{N}$ we have

$$0 \leq C_N(p, x, \sigma) \leq \frac{1-x}{x} \cdot \frac{\|u\|_\infty}{N}.$$

(17)

Next, consider the event $\{\kappa_1 + \kappa_2 \leq N\}$. In particular, on this event, we have that $\kappa_1 \wedge N = \kappa_1$. Using the latter, together with Eqs. (13), (14), (17), and the reversed triangle inequality we obtain that for any $\sigma \in \Sigma$ and $N \in \mathbb{N}$

$$|\gamma_N^\tau(p, \sigma) - B_N(p, x, \sigma)| \leq$$

$$\left( \left( \frac{1}{x} + \frac{1-x}{x} \right) \frac{\|u\|_\infty}{N} + 2 \mathbb{P}_{p, x, \sigma} (\{\kappa_1 + \kappa_2 > N\}) \|u\|_\infty \right) / \mathbb{P}_{p, x, \sigma} (\{\kappa_1 + \kappa_2 \leq N\}),$$

(18)
where we set
\[
B_N(p, x, \sigma) := \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{n=\kappa_1+1}^{t_N} u(p_n) \Big| \{\kappa_1 + \kappa_2 \leq N\} \right)
\]
\[
= \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=2}^{T_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\cdots+\kappa_{j-1}+i}) \right) \Big| \{\kappa_1 + \kappa_2 \leq N\} \right).
\]

As \(\mathbb{P}_{p,x,\sigma}(\{\kappa_1 + \kappa_2 \leq N\})\) is independent of \(\sigma\) and tends to 1 as \(N\) tends to infinity, we obtain from (18) and Claim 3 that in order to show that \(v^*_N\) converges uniformly to \(V(x)\) as \(N \to \infty\) it suffices to prove the following two propositions:

**Proposition 1.** There exist \(\sigma^* \in \Sigma\), a positive integer \(L^*\), and positive constants \(C^*, D^*\) such that
\[
B_N(p, x, \sigma^*) \geq V(x) - C^* \varepsilon - D^* z_\varepsilon \varepsilon, \quad \forall p \in \Delta(K), \quad \forall N \geq L^*.
\]

**Proposition 2.** There exist a positive integer \(L\) and positive constants \(C, D\) such that for every \(\sigma \in \Sigma\) it holds
\[
B_N(p, x, \sigma) \leq V(x) + C \varepsilon + D z_\varepsilon \varepsilon, \quad \forall p \in \Delta(K), \quad \forall N \geq L.
\]

In the next subsection we will prove Proposition 1. The proof of Proposition 2 is of similar nature, and is therefore postponed to Appendix A.

### 6.5 Proof of Proposition 1

We begin with the description of the strategy \(\sigma^*\).

**The strategy \(\sigma^*\):** Play the same arbitrary action until stage \(\kappa_1\). Then, for each \(n \geq 1\), upon observing \(X_{\kappa_1+\cdots+\kappa_n}\), play an optimal strategy in \(\tilde{\Gamma}(\delta_{X_{\kappa_1+\cdots+\kappa_n}} M, x)\) starting from the \(\kappa_1 + \cdots + \kappa_n + 1\)’st stage until stage \(\kappa_1 + \cdots + \kappa_{n+1}\) (included).

In words, the strategy \(\sigma^*\) prescribes the sender to play optimally in the random duration games between any two consecutive revelations. The initial probability in the random duration game is determined by the state of \((X_n)\) at the time of the revelation.

For each \(N \in \mathbb{N}\) consider the event \(A_N := \{\kappa_1 + \kappa_2 \leq N\} \cap \{a_N \leq T_N\}\).

By the choice of \(N_\varepsilon\) and Eq. (10) we have that \(A_N = \{a_N \leq T_N\}\) for every
\( N \geq N_\varepsilon \). The latter, coupled with the fact that \( u \) is non-negative, implies that for every \( N \geq N_\varepsilon \) it holds

\[
B_N(p, x, \sigma^*) \geq P_{p, x, \sigma^*}(A_N) \mathbb{E}_{p, x, \sigma^*}
\left( \frac{1}{N} \sum_{j=2}^{N} \left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right) \right)
\left| A_N \right|
\]

\[
\geq P_{p, x, \sigma^*}(A_N) \mathbb{E}_{p, x, \sigma^*}
\left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right) \right)
\left| A_N \right|
\]

\[
= \mathbb{E}_{p, x, \sigma^*}
\left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right) \right)
\]

\[ := I_N^1 \]

\[ - P_{p, x, \sigma^*}(A_N^c) \mathbb{E}_{p, x, \sigma^*}
\left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right) \right)
\left| A_N^c \right|
\]

\[ = \mathbb{E}_{p, x, \sigma^*}
\left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right) \right)
\]

\[ := I_N^2 \]

\[ \text{(19)} \]

Using the linearity of the expectation operator, the fact that the \( \kappa_j \)'s are i.i.d. geometric with parameter \( x \), and the definition of \( \sigma^* \), we obtain

\[
I_N^1 = \frac{1}{N} \sum_{j=2}^{a_N} \mathbb{E}_{p, x, \sigma^*}
\left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right)
\]

\[ = \frac{1}{N} \sum_{j=2}^{a_N} \mathbb{E}_{p, x, \sigma^*}
\left( \mathbb{E}_{p, x, \sigma^*}
\left( \sum_{i=1}^{\kappa_j} u(p_{k_1+\cdots+k_{j-1}+i}) \right| \kappa_1 + \cdots + \kappa_{j-1}, X_{k_1+\cdots+k_{j-1}} \right) \right)
\]

\[ = \frac{1}{N} \sum_{j=2}^{a_N} \mathbb{E}_{p, x, \sigma^*}
\left( \hat{v}(\delta_{X_{k_1+\cdots+k_{j-1}} M}) \right). \]

\[ \text{(20)} \]

The latter, together with the fact that by Claim 1 \( \| \hat{v} \|_\infty \leq \| u \|_\infty / x \), and the definitions of \( (a_N) \) and \( (b_N) \) implies

\[
I_N^1 \geq \frac{1}{N} \sum_{j=2}^{b_N} \mathbb{E}_{p, x, \sigma^*}
\left( \hat{v}(\delta_{X_{k_1+\cdots+k_{j-1}} M}) \right) - \frac{2\varepsilon \| u \|_\infty}{x \sqrt{N}}
\]

\[ = \sum_{\ell \in K} \mathbb{E}_{p, x, \sigma^*}
\left( \frac{1}{N} \sum_{j=2}^{b_N} \mathbb{1} \{ X_{k_1+\cdots+k_{j-1}} = \ell \} \right) \hat{v}(\delta_{\ell M}) - \frac{2\varepsilon \| u \|_\infty}{x \sqrt{N}}
\]

\[ \text{(21)} \]
Next, by the choice of $N_\varepsilon$ and the choice of $(b_N)$ we obtain that for every $\ell \in K$ and every $N \geq N_\varepsilon$ it holds that

$$E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{b_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \right) \geq$$

$$E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{b_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \mathbb{1}\{T_N \leq b_N\} \right) \geq E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{T_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \right) \geq E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{T_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \mathbb{1}\{T_N \leq b_N\} \right)$$

$$\geq E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{T_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \right) - \mathbb{P}_{p,x,\sigma^*}(\{T_N > b_N\}) E_{p,x,\sigma^*} \left( \frac{T_N}{N} \right) \{T_N > b_N\} \geq E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{T_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \right) - \varepsilon, \quad (22)$$

where in the last inequality we used the fact that $T_N \leq N$. We continue our analysis, observing that the Ergodic Theorem for irreducible Markov Chains (e.g., Theorem C.1 in [27]) implies that for any $\ell \in K$

$$\frac{1}{N} \sum_{j=2}^{T_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} = \#\{n \leq N : (X_n, Z_n) = (\ell, 1)\} - 1 \rightarrow \pi_\ell \cdot x \quad \mathbb{P}_{p,x,\sigma^*}-\text{a.s.} \quad (23)$$

as $N \rightarrow \infty$. Therefore, since $K$ is finite, the Bounded Convergence Theorem ensures the existence of a positive integer $N_\varepsilon^1 \geq N_\varepsilon$ for which the inequality

$$E_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{T_N} \mathbb{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \right) \geq \pi_\ell \cdot x - \varepsilon \quad (24)$$

holds across all $\ell \in K$ and every $N \geq N_\varepsilon^1$. It thus follows from Eqs. (21),
the positivity of $\hat{v}$, and Claim 1 that for every $N \geq N^1_\varepsilon$ we have

\[
I^1_N \geq \sum_{\ell \in K} (\pi^\ell_M \cdot x - \varepsilon - \varepsilon) \hat{v}(\delta_{\ell}M) - \frac{2z_\varepsilon \|u\|_\infty}{x \sqrt{N}} - \frac{2k \|u\|_\infty}{x \sqrt{N}} + \frac{2z_\varepsilon \|u\|_\infty}{x \sqrt{N}} - \frac{2k \|u\|_\infty}{x \sqrt{N}} = \mathcal{V} - \frac{2k \|u\|_\infty}{x \sqrt{N}} - \frac{2z_\varepsilon \|u\|_\infty}{x \sqrt{N}}.
\]

Upon obtaining a lower bound for $I^1_N$ which will suffice for our future purposes, we move on to an analysis of the expression

\[
I^2_N := \mathbb{E}_{p,x,\sigma^*} \left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1 + \cdots + \kappa_{j-1} + i}) \right) - \mathcal{A}_N \right),
\]

which by Eq. (19) is imperative to obtaining a lower bound on $B_N(p, x, \sigma^*)$ for large values of $N$. Our analysis begins by noticing that since $A_N := \{\kappa_1 + \kappa_2 \leq N\} \cap \{a_N \leq T_N\} = \{a_N \leq T_N\}$ for every $N \geq N^1_\varepsilon$ (as $N^1_\varepsilon \geq N_\varepsilon$), it follows that

\[
A^c_N = \{T_N < a_N\} = \{\kappa_1 + \cdots + \kappa_{a_N} > N\} = \{\kappa_1 + \cdots + \kappa_{a_N} - a_N > N-a_N\},
\]

where the second inequality follows from the definition of $T_N$ - the number of revelations up to time $N$, and the fact that $\kappa_1 + \cdots + \kappa_{a_N}$ is the number of stages until the $a_N$’th revelation. Moreover, As $\kappa_1 + \cdots + \kappa_{a_N} - a_N$ has negative binomial distribution with parameters $(a_N, x)$ by Claim 2, we may
asses their lower truncated mean by means of Fact 1, to obtain:

\[
I_N^2 = \mathbb{E}_{p, x, \sigma^*} \left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1, \ldots, \kappa_{j-1}+1}) \right) \right) \left\{ \kappa_1 + \cdots + \kappa_{a_N} - a_N > N - a_N \right\}
\]

\[
< \frac{\|u\|_\infty}{N} \mathbb{E}_{p, x, \sigma^*} (\kappa_1 + \cdots + \kappa_{a_N}) \left\{ \kappa_1 + \cdots + \kappa_{a_N} - a_N > N - a_N \right\}
\]

\[
\leq \frac{\|u\|_\infty}{N} \mathbb{E}_{p, x, \sigma^*} (\kappa_1 + \cdots + \kappa_{a_N} - a_N) \left\{ \kappa_1 + \cdots + \kappa_{a_N} - a_N > N - a_N \right\}
\]

\[
+ \frac{\|u\|_\infty a_N}{N}
\]

\[
\leq \frac{\|u\|_\infty}{N} \left( \frac{a_N(1-x)}{x} + \frac{N-a_N+1}{x} \right) + x\|u\|_\infty
\]

\[
\leq \frac{\|u\|_\infty}{N} \left( N + \frac{2N}{x} \right) + x\|u\|_\infty = \|u\|_\infty (1 + \frac{2}{x} + x),
\]

(27)

where in the third and fourth inequality we used the fact that \(a_N \leq N_x\).

Plugging the bounds given in Eqs. (25) and (27) back to Eq. (19), we obtain that for every \(N \geq N^1_x\) we have:

\[
B_N(p, x, \sigma^*) \geq \mathcal{V}(x) - \frac{2k\|u\|_\infty}{x} \frac{\varepsilon - \frac{2z\|u\|_\infty}{x\sqrt{N}}}{x\sqrt{N}} - \mathbb{P}_{p, x, \sigma^*}(A_N^c)\|u\|_\infty (1 + \frac{2}{x} + x)
\]

\[
\geq \mathcal{V}(x) - \frac{2k\|u\|_\infty}{x} \frac{\varepsilon - \frac{2z\|u\|_\infty}{x\sqrt{N}}}{x\sqrt{N}} - \|u\|_\infty (1 + \frac{2}{x} + x)\varepsilon.
\]

(28)

Hence by choosing \(L^* \geq N^1_x\) such that \(1/\sqrt{L^*} \leq \varepsilon\), we may conclude the proof of Proposition 1 by setting \(C^* := (\frac{2k}{x} + 1 + \frac{2}{x} + x)\|u\|_\infty\) and \(D^* := \frac{2}{x}\|u\|_\infty\).

### 7 Proof of Theorem 2

The following proposition is the key step towards the proof of Theorem 2.

**Proposition 3.** For every \(p \in \Delta(K)\) and \(\lambda \in [0, 1]\) the function \(x \mapsto v^\lambda_x(p)\) is non-increasing.
Proof of Proposition 3. Fix \( x < y \). By the dynamic principle formula for \( v^\lambda(p) \) (e.g., Eq. (3)) there exists a stationary optimal strategy \( \hat{\sigma}^2 \) in \( \Gamma_\lambda(p, y) \). In other words, \( \hat{\sigma}^2 \) can be viewed as a mapping from \( \Delta(K) \) to \( \bigcup_{p \in \Delta(K)} S_p \), assigning to each state (belief) of the MDP \( G_p \) the same action (split). Let \((\hat{p}_n)_{n \geq 1}\) be the sequence of posteriors induced by \( \hat{\sigma}^2 \) in \( \Gamma_\lambda(p, y) \).

By the definition of the payoff given in Eq. (1), to prove the proposition, it suffices to describe a behavioral strategy \( \sigma \in \Sigma \), such that the sequence of posteriors \((p_n)_{n \geq 1}\) it induces in \( \Gamma_\lambda(p, x) \) has the same distribution as \((\hat{p}_n)_{n \geq 1}\). By the same distribution we mean that \( p_n \) and \( \hat{p}_n \) have the same law for each \( n \geq 1 \).

To describe such a \( \sigma \), we first may utilize Observation 1 to our advantage, and assume that the signal set available to the sender in \( \Gamma_\lambda(p, x) \) is of the form \( \{K \cup \{0\}\} \times S \), where \( S \) is the signal set of the sender in \( \Gamma_\lambda(p, y) \). Also, let us introduce the new notation \((Z^n)\) specifying the revelation rate \( r \in [0, 1] \) of the revelation process \((Z_n)\), and denote by \( z^n_0 \) its realized values\(^3\). We now define \( \sigma = (\sigma_n) \in \Sigma \) by the following rules; for \( n \geq 1 \), \( \sigma_1(x_1) := \hat{\sigma}^2_1(x_1) \) and for each \( n \geq 2 \),

\[
\sigma_n(x_1, (t_1, s_1), z^x_1, ..., x_{n-1}, (t_{n-1}, s_{n-1}), 1, x_n) := \delta_{\emptyset} \otimes \hat{\sigma}^2_n(x_1, s_1, z^x_1, ..., x_{n-1}, s_{n-1}, 1, x_n),
\]

and

\[
\sigma_n(x_1, (t_1, s_1), z^x_1, ..., x_{n-1}, (t_{n-1}, s_{n-1}), 0, x_n) := \delta_{x_{n-1}} \otimes \hat{\sigma}^2_n(x_1, s_1, z^x_1, ..., x_{n-1}, s_{n-1}, 1, x_n),\quad \text{with prob. } \frac{y-x}{1-x},
\]

\[
\delta_{\emptyset} \otimes \hat{\sigma}^2_n(x_1, s_1, z^x_1, ..., x_{n-1}, s_{n-1}, 0, x_n),\quad \text{with prob. } 1 - \frac{y-x}{1-x}.
\]

for every \( n \geq 1 \), where \( \otimes \) is denotes the product measure on \( \{K \cup \{0\}\} \times S \), with respect to two given measures on each coordinate. Intuitively speaking, \( \sigma_n \) completes the missing information of the receiver at the start of the \( n \)’th stage in \( \Gamma_\lambda(p, x) \) compared to \( \Gamma_\lambda(p, y) \), by revealing the state \( x_{n-1} \) (with probability \( \frac{y-x}{1-x} \)) whenever \( Z^x_{n-1} \) failed to do so (an event of probability \( 1-x \)). Upon ‘transmitting’ the information, \( \sigma_n \) copycats \( \hat{\sigma}^2_n \). Let us now prove by induction that \( p_n \) and \( \hat{p}_n \) have the same law for each \( n \geq 1 \).

For \( n = 1 \) this follows immediately, as the distribution of both \( p_1 \) and \( \hat{p}_1 \) depends only on \( p, X_1 \) and \( \hat{\sigma}^2_1 \). Let us now show the induction step \( n \rightarrow n + 1 \).

\(^3\)This is required as we now deal with different values of \( t \).
We have by the induction assumption that
\[
P_{p,x,\sigma}(\{p_{n+1} = h\}) = \sum_{\zeta \in \text{supp}(p_n)} P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta\}) P_{p,x,\sigma}(\{p_n = \zeta\})
\]
\[
= \sum_{\zeta \in \text{supp}(\hat{p}_n)} P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta\}) P_{p,y,\hat{\sigma}^2}(\{\hat{p}_n = \zeta\}).
\]
(29)

Take some \(\zeta \in \text{supp}(\hat{p}_n)\). Since \(Z_{x_n}^x\) is independent of \(p_n\) we have
\[
P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 1\}) = x P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 1\}) + (1 - x) P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 0\}).
\]
(30)

Let us focus on the first summand in the right-side of Eq. (30). By the definition of \(\sigma\) we have
\[
x P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 1\}) = x \sum_{\ell \in K} \zeta^\ell \hat{\sigma}^2(\delta_\ell M)[h],
\]
(31)

where we recall that \(\hat{\sigma}^2\) is stationary, and hence we may denote by \(\hat{\sigma}^2(\delta_\ell M)[h]\) the convex weight (distribution mass) assigned to the belief \(h\), by the optimal split prescribed by \(\hat{\sigma}^2\) at the belief \(\delta_\ell M\). We shift our focus to the second summand in the right-side of Eq. (30). By the definition of \(\sigma\) we have
\[
(1 - x) P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 0\}) =
\]
\[
(1 - x) \left( \frac{y - x}{1 - x} \right) P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 0, t_{n+1} \neq \{\emptyset\} \}) +
\]
\[
(1 - x) \left( 1 - \frac{y - x}{1 - x} \right) P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 0, t_{n+1} = \{\emptyset\} \}).
\]
(32)

As by the definition of \(\sigma\) we have
\[
P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 0, t_{n+1} \neq \{\emptyset\} \}) = \sum_{\ell \in K} \zeta^\ell \hat{\sigma}^2(\delta_\ell M)[h],
\]
and
\[
P_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta, Z_{x_n}^x = 0, t_{n+1} = \{\emptyset\} \}) = \hat{\sigma}^2(\zeta M)[h].
\]
Next Eqs. (30), (31), and (32) imply that

$$\mathbb{P}_{p,x,\sigma}(\{p_{n+1} = h \mid p_n = \zeta\}) = y \sum_{\ell \in K} \zeta^\ell \hat{\sigma}^2(\delta_{\ell} M)[h] + (1 - y)\hat{\sigma}^2(\zeta M)[h]$$

$$= \mathbb{P}_{p,y,\hat{\sigma}^2}(\{\hat{p}_{n+1} = h \mid \hat{p}_n = \zeta\}),$$

where the second equality follows by a use of the complete expectation formula with the events \(\{Z_n^y = 1\}\) and \(\{Z_n^y = 0\}\), and the fact that \(\hat{\sigma}^2\) is stationary. Now Eq. (33) in conjunction with Eq. (29) implies that

$$\mathbb{P}_{p,x,\sigma}(\{p_{n+1} = h\}) = \sum_{\zeta \in \text{supp}(\hat{p}_n)} \mathbb{P}_{p,y,\hat{\sigma}^2}(\{\hat{p}_{n+1} = h \mid \hat{p}_n = \zeta\}) \mathbb{P}_{p,y,\hat{\sigma}^2}(\{\hat{p}_n = \zeta\})$$

$$= \mathbb{P}_{p,y,\hat{\sigma}^2}(\{\hat{p}_{n+1} = h\}),$$

thus proving the induction step. \(\square\)

Finally, we provide the proof of Theorem 2.

**Proof of Theorem 2.** Define the sequence of functions \((f_n) : (0,1] \rightarrow \mathbb{R}_+\) by \(f_n(x) = v_{1/n}^p(p)\) for some \(p \in \Delta(K)\). By Theorem 1, \((f_n)\) converges (uniformly) to \(V\). Since by Proposition 3 each \(f_n\) is non-increasing, \(V\) must also be non-increasing. Finally, as \(\Psi(\lambda) = V(1 - \lambda)\) for every \(\lambda \in [0,1)\), we deduce that \(\Psi\) is non-decreasing. \(\square\)

**References**

[1] Arieli, I. and Babichenko, Y. (2019), Private Bayesian Persuasion. *Journal of Economic Theory*, 182, 185–217.

[2] Arieli, I., Babichenko, Y., Smorodinsky, R. and Yamashita, T. (2020), Optimal Persuasion via Bi-Pooling. In: *The Twenty-First ACM Conference on Economics and Computation*.

[3] Athey, S. and Bagwell, K. (2008), Collusion with Persistent Cost Shocks. *Econometrica*, 76(3), 493-540.

[4] Au, P.H. (2015), Dynamic Information Disclosure. *The RAND Journal of Economics*, 46(4): 791–823.
[5] Augenblick, N. and Bodoh-Creed, A. (2018), To Reveal or not to Reveal: Privacy Preferences and Economic Frictions. *Games and Economic Behavior*, 110, 318–329.

[6] Aumann, R.J. and Maschler, M. (1995), *Repeated Games with Incomplete Information*. With the collaboration of R. Stearns, MIT Press, Cambridge, MA.

[7] Best, J. and Quigley, D. (2017), Persuasion for the Long Run. *Working Paper*.

[8] Blackwell, D. (1951), Comparison of Experiments. In: *Proceedings of the Second Berkeley Symposium on Mathematical Statistics and Probability, 1950*, pp. 93–102. University of California Press, Berkeley and Los Angeles, CA.

[9] Dziuda, W. and Gradwohl, R. (2015), Achieving Cooperation Under Privacy Concerns. *Am. Econ. J. Microecon.*, 7, 142–173.

[10] Ely, J.C. (2017), Beeps. *American Economic Review*, 107(1), 31–53.

[11] Escobar, J.F. and Toikka, J. (2013), Efficiency in Games with Markovian Private Information. *Econometrica*, 81, 1887–1934.

[12] Farhadi, F. and Teneketzis, D. (2021), Dynamic Information Design: A Simple Problem on Optimal Sequential Information Disclosure. *Dyn Games Appl*.

[13] Geyer, C. J. (2021), Lower-Truncated Poisson and Negative Binomial Distributions. University of Minnesota, MN.

[14] Gordon R.D. (1941), Values of Mills’ Ratio of Area to Bounding Ordinate of the Normal Probability Integral for Large Values of the Argument. *Annals of Mathematical Statistics*, 12, 364–366.

[15] Ganglmair, B. and Tarantino, E. (2014), Conversation With Secrets. *Rand J. Econ.*, 45, 273–302.

[16] Guo, Y. and Shmaya, E. (2018), The Value of Multistage Persuasion. *Working paper*. 
[17] Guo, Y. and Shmaya, E. (2019), The Interval Structure of Optimal Disclosure. *Econometrica* 87 (2), 653–675.

[18] Honryo, T. (2018), Dynamic Persuasion. *J. Econom. Theory*, 178, 36–58.

[19] Horn, R.A. and Johnson, C.R. (2013), *Matrix analysis. Second edition*. Cambridge University Press, Cambridge.

[20] Hörner, J., Rosenberg, D., Solan, E., and Vieille, N. (2010), On a Markov Game with One-Sided Information. *Oper. Res.* 58 (4), part 2, 1107–1115.

[21] Hörner, J., Takahashi, S., and Vieille, N. (2015), Truthful Equilibria in Dynamic Bayesian Games. *Econometrica*, 83(5), 1795–1848.

[22] Jackson, M. O., and Kalai, E. (1997), Social Learning in Recurring Games, *Games Econ. Behav.* 21(1-2), 102–134.

[23] Kamenica, E. and Gentzkow, M. (2011), Bayesian Persuasion. *American Economic Review*, 101 (6), 2590–2615.

[24] Kolotilin, A., Mylovanov, T., Zapechelnyuk, A., and Li, M. (2017), Persuasion of a Privately Informed Receiver. *Econometrica*, 85(6), 1949–1964.

[25] Lehrer, E. and Shaiderman, D. (2021), Markovian persuasion. *arXiv preprint arXiv:2111.14365*.

[26] Lehrer, E. and Sorin, S. (1992), A Uniform Tauberian Theorem in Dynamic Programming. *Math. Oper. Res.* 17, no. 2, 303–307.

[27] Levin, D. A. and Peres, Y. (2017), *Markov Chains and Mixing Times*, second edition. With contributions by Elizabeth L. Wilmer. With a chapter on “Coupling from the past” by James G. Propp and David B. Wilson. American Mathematical Society, Providence, RI.

[28] Lingenbrink, D. and Iyer, K. (2019) Optimal Signaling Mechanisms in Unobservable Queues with Strategic Customers. *Oper. Res.* 67, no. 5, 1397-1416.

[29] Mailath, G. and Samuelson, L. (2001), Who Wants a Good Reputation? *Rev. Econom. Stud.*, 68, no. 2, 415–441.
[30] Margaria, C. and Smolin, A. (2018), Dynamic Communication with Biased Senders. *Games Econ. Behav.*, 110, 330–339.

[31] Mathevet, L., Pearce, D., and Stacchetti, E. (2022), Reputation for a Degree of Honesty. *Working Paper.*

[32] Orlov, D., Skrzypacz, A., and Zryumov, P. (2020), Persuading the Principal to Wait. *Journal of Political Economy*, 128(7), 2542–2578.

[33] Pei, H. (2021), Trust and Betrayals: Reputational Payoffs and Behaviors Without Commitment. *Theor. Econ.* 16, no. 2, 449–475.

[34] Phelan, C. (2006), Public Trust and Government Betrayal. *J. Econom. Theory*, 127(1), 27–43.

[35] Renault, J. (2006), The value of Markov chain games with lack of information on one side. *Math. Oper. Res.* 31, no. 3, 433–648.

[36] Renault, J., Solan, E., and Vieille, N. (2017), Optimal Dynamic Information Provision. *Games Econ. Behav.* 104, 329–349.

[37] Rockafellar, R. T. (1970), *Convex Analysis*. Princeton University Press.

[38] Wiseman, T. (2008), Reputation and Impermanent Types. *Games Econ. Behav.*, 62, 190–210.

[39] Ziliotto, B. (2016), A Tauberian Theorem for Nonexpansive Operators and Applications to Zero-Sum Stochastic Games. *Math. Oper. Res.* 41, no. 4, 1522–1534.

A Proof of Proposition 2

We begin our analysis by recalling that \( \ell_N := \kappa_0 + \kappa_1 + \cdots + \kappa_{T_N} \leq N \), which together with the non-negativity of \( u \) yields the following string of
inequalities, which holds for every $\sigma \in \Sigma$ and every $N \in \mathbb{N}$:

$$B_N(p, x, \sigma) = \mathbb{E}_{p, x, \sigma} \left( \frac{1}{N} \sum_{\kappa_{1+1}}^{\ell_N} u(p_n) \mathbb{1}\{T_N \leq b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right)$$

$$+ \mathbb{E}_{p, x, \sigma} \left( \frac{1}{N} \sum_{\kappa_{1+1}}^{\ell_N} u(p_n) \mathbb{1}\{T_N > b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right)$$

$$\leq \mathbb{E}_{p, x, \sigma} \left( \frac{1}{N} \sum_{\kappa_{1+1}}^{\ell_N} u(p_n) \mathbb{1}\{T_N \leq b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right)$$

$$+ \mathbb{E}_{p, x, \sigma} \left( \frac{\ell_N}{N} \|u\|_\infty \mathbb{1}\{T_N > b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right)$$

$$\leq \mathbb{E}_{p, x, \sigma} \left( \frac{1}{N} \sum_{\kappa_{1+1}}^{\ell_N} u(p_n) \mathbb{1}\{T_N \leq b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right)$$

$$+ \|u\|_\infty \mathbb{P}_{p, x, \sigma} \left( \{T_N > b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right).$$

By the choice of $N_\varepsilon$ we have for every $N \geq N_\varepsilon$ and every $\sigma \in \Sigma$ that

$$\mathbb{P}_{p, x, \sigma} \left( \{T_N > b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right) = \frac{\mathbb{P}_{p, x, \sigma}(\{T_N > b_N\})}{\mathbb{P}_{p, x, \sigma}(\{\kappa_1 + \kappa_2 \leq N\})}$$

$$\leq \frac{\mathbb{P}_{p, x, \sigma}(\{T_N > b_N\})}{\mathbb{P}_{p, x, \sigma}(\{a_n \leq T_N\})} \leq \frac{\varepsilon}{1 - \varepsilon} \leq 2\varepsilon,$$

where the last inequality follows from the fact that $\varepsilon \leq 1/2$. Hence, we may weaken the bound in Eq. (35) for every $N \geq N_\varepsilon$ and every $\sigma \in \Sigma$ as follows:

$$B_N(p, x, \sigma) \leq \mathbb{E}_{p, x, \sigma} \left( \frac{1}{N} \sum_{\kappa_{1+1}}^{\ell_N} u(p_n) \mathbb{1}\{T_N \leq b_N\} \mid \{\kappa_1 + \kappa_2 \leq N\} \right)$$

$$+ \|u\|_\infty 2\varepsilon.$$
Since $u$ is non-negative we have for every $N \geq N_\varepsilon$ and every $\sigma \in \Sigma$ that

$$J_N(\sigma) \leq \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=2}^{b_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\ldots+\kappa_{j-1}+i}) \right) \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right)$$

$$= \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\ldots+\kappa_{j-1}+i}) \right) \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right)$$

$$+ \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=a_N+1}^{b_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\ldots+\kappa_{j-1}+i}) \right) \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right).$$

(38)

Since $a_N \geq 2$ for every $N \geq N_\varepsilon$, and the $(\kappa_j)$’s are i.i.d. geometrically distributed with parameter $x$, we get that for every $N \geq N_\varepsilon$ and every $\sigma \in \Sigma$,

$$\mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=a_N+1}^{b_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\ldots+\kappa_{j-1}+i}) \right) \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right) \leq \|u\|_{\infty} \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=a_N+1}^{b_N} \kappa_j \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right) = \|u\|_{\infty} \frac{b_N - a_N}{N} \cdot \frac{1}{x}. \quad (39)$$

The definition of $(a_N)$ and $(b_N)$, coupled with Eqs. (38) and (39) implies that

$$J_N(\sigma) \leq \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\ldots+\kappa_{j-1}+i}) \right) \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right) + \frac{2\|u\|_{\infty} \varepsilon}{x\sqrt{N}} \quad (40)$$

for every $\sigma \in \Sigma$ and every $N \geq N_\varepsilon$. Thus, in view of Eqs. (37) and (40) it remains to analyze the quantity $J^1_N(\sigma)$, defined for every $N \geq N_\varepsilon$ and every $\sigma \in \Sigma$ as

$$J^1_N(\sigma) := \mathbb{E}_{p,x,\sigma} \left( \frac{1}{N} \sum_{j=2}^{a_N} \left( \sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\ldots+\kappa_{j-1}+i}) \right) \left| \{\kappa_1 + \kappa_2 \leq N\} \right. \right). \quad (41)$$
Utilizing once again the fact that $u$ is non-negative, we obtain

$$J_N^1(\sigma) \leq \frac{1}{\mathbb{P}_{p,x,\sigma}\{a_N \leq T_N\}} \mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{a_N} \left(\sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\cdots+\kappa_{j-1}+i})\right)\right),$$

(42)

where the last inequality follows from the fact that $\{\kappa_1 + \kappa_2 \leq N\} = \{2 \leq T_N\} \supseteq \{a_N \leq T_N\}$ for every $N \geq N_\varepsilon$. Since the $(\kappa_j)$’s are i.i.d. geometrically distributed random variables with parameter $x$, we have for every $2 \leq j \leq a_N$,

$$\mathbb{E}_{p,x,\sigma}\left(\sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\cdots+\kappa_{j-1}+i})\right) =$$

$$\mathbb{E}_{p,x,\sigma}\left(\mathbb{E}_{p,x,\sigma}\left(\sum_{i=1}^{\kappa_j} u(p_{\kappa_1+\cdots+\kappa_{j-1}+i}) | \kappa_1 + \cdots + \kappa_{j-1}, X_{\kappa_1+\cdots+\kappa_{j-1}}\right)\right) \leq$$

$$\mathbb{E}_{p,x,\sigma}\left(\tilde{v}(\delta_{X_{\kappa_1+\cdots+\kappa_{j-1}}M})\right),$$

(43)

where the last inequality follows from the definition of the random duration game. Using Eq. (43) we can further weaken the upper bound given in Eq. (42) across all $\sigma \in \Sigma$ and $N \geq N_\varepsilon$ as follows:

$$J_N^1(\sigma) \leq \frac{1}{\mathbb{P}_{p,x,\sigma}\{a_N \leq T_N\}} \mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{a_N} \tilde{v}(\delta_{X_{\kappa_1+\cdots+\kappa_{j-1}}M})\right) =$$

$$\frac{1}{\mathbb{P}_{p,x,\sigma}\{a_N \leq T_N\}} \sum_{\ell \in K} \mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{a_N} \mathbbm{1}\{X_{\kappa_1+\cdots+\kappa_{j-1}} = \ell\}\right) \tilde{v}(\delta_{\ell M}).$$

(44)

Since $T_N \leq N$ we have for every $\ell \in K$,

$$\mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{a_N} \mathbbm{1}\{X_{\kappa_1+\cdots+\kappa_{j-1}} = \ell\}\right) \leq \mathbb{E}_{p,x,\sigma}\left(\frac{a_N}{N} \mathbbm{1}\{a_N > T_N\}\right)$$

$$+ \mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{T_N} \mathbbm{1}\{X_{\kappa_1+\cdots+\kappa_{j-1}} = \ell\} \mathbbm{1}\{a_N \leq T_N\}\right).$$

(45)
Let us now focus on the second summand on the right-hand side of (45) and observe that for every $\ell \in K$ we have

$$\mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{T_N} \mathbbm{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\} \mathbbm{1}\{a_N \leq T_N\}\right)$$

$$\leq \frac{1}{\mathbb{P}_{p,x,\sigma}(\{a_N \leq T_N\})} \mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{T_N} \mathbbm{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\}\right). \quad (46)$$

As in the proof of Proposition 1, the ergodic theorem for irreducible Markov chains (e.g., Theorem C.1 in [27]) shows that for any $\ell \in K$,

$$\frac{1}{N} \sum_{j=2}^{T_N} \mathbbm{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\}$$

$$= \frac{\#\{n \leq N : (X_n, Z_n) = (\ell, 1)\}}{N} \Rightarrow \pi_{\ell}^{\ell} \cdot x \quad \mathbb{P}_{p,x,\sigma}\text{-a.s.} \quad (47)$$

for any $\sigma \in \Sigma$ as $N \to \infty$. Moreover, as the rate of convergence depends solely on $(X_n, Z_n)$, and is independent of $\sigma \in \Sigma$, we can use the bounded convergence theorem to choose a positive integer $N_2 \geq N_\varepsilon$, independent of $\sigma \in \Sigma$, such that

$$\mathbb{E}_{p,x,\sigma}\left(\frac{1}{N} \sum_{j=2}^{T_N} \mathbbm{1}\{X_{\kappa_1+\ldots+\kappa_{j-1}} = \ell\}\right) \leq \pi_{\ell}^{\ell} \cdot x + \varepsilon \quad (48)$$

for every $\ell \in K$, $\sigma \in \Sigma$ and every $N \geq N_2^2$. Now, Eqs. (44), (45), (46), and (48) imply that for every $\sigma \in \Sigma$, and every $N \geq N_2$ it holds that:

$$J_{N_1}(\sigma) \leq \frac{1}{\mathbb{P}_{p,x,\sigma}(\{a_N \leq T_N\})} \sum_{\ell \in K} \left(\mathbb{E}_{p,x,\sigma}\left(\frac{a_N}{N} \mathbbm{1}\{a_N > T_N\}\right)\right) \hat{v}(\delta_{\ell} M)$$

$$+ \frac{1}{\mathbb{P}_{p,x,\sigma}(\{a_N \leq T_N\})^2} \sum_{\ell \in K} \left(\pi_{\ell}^{\ell} \cdot x + \varepsilon\right) \hat{v}(\delta_{\ell} M). \quad (49)$$

As $\mathbb{P}_{p,x,\sigma}(\{a_N \leq T_N\}) \geq 1 - \varepsilon$ for every $N \geq N_\varepsilon^2$, $\|\hat{v}\|_{\infty} \leq \|u\|_{\infty}/x$ (by Claim 1), and $a_N \leq Nx \leq N$, we may further weaken the inequality in Eq. (49)
and obtain that, for every $\sigma \in \Sigma$, and every $N \geq N_\varepsilon^2$,

$$J_N^1(\sigma) \leq \frac{k\|u\|_\infty\cdot x}{1-\varepsilon} + \frac{1}{(1-\varepsilon)^2} \left( \sum_{\ell \in K} \pi_M^\ell v_{1-x}(\delta_\ell M) + \frac{k\|u\|_\infty\cdot x}{1-\varepsilon} \right)$$

$$< \frac{k\|u\|_\infty\cdot x}{1-\varepsilon} + \frac{6k\|u\|_\infty\cdot x}{1-\varepsilon} \varepsilon + 3\|u\|_\infty\varepsilon.$$

Here we used the inequalities $\varepsilon/(1-\varepsilon) < 2\varepsilon$, and $1/(1-\varepsilon)^2 < 1 + 6\varepsilon < 4$, which are both valid when $\varepsilon < 1/2$, and the fact that $\mathcal{V}(x) \leq \|u\|_\infty$ for every $x \in (0,1]$. The latter together with Eqs. (37) and (40) implies that, for every $\sigma \in \Sigma$ and every $N \geq N_\varepsilon^2$,

$$B_N(p, x, \sigma) \leq \mathcal{V}(x) + \frac{6k\|u\|_\infty\cdot x}{1-\varepsilon} \varepsilon + 3\|u\|_\infty\varepsilon + 2\|u\|_\infty\varepsilon.$$

To complete the proof of Proposition 2 we choose $L \geq N_\varepsilon^2$ so that $1/\sqrt{L} \leq \varepsilon$, and set $C = \frac{6k\|u\|_\infty\cdot x}{1-\varepsilon} + 5\|u\|_\infty$ and $D = \frac{2\|u\|_\infty}{x\sqrt{N}}$.

### B. Proof of Theorem 3

We begin by specifying the behavioral strategy spaces of Players 1, 2 in the games $\Gamma^*(p)$ and $\Gamma^{*,x}(p)$. In the former the behavioral strategy spaces of Player 1 and Player 2, denoted $\Sigma^*$ and $\mathcal{T}^*$, are defined by

$$\Sigma^* = \{ \sigma = (\sigma_n)_{n \geq 1} : \sigma_n : (K \times I \times J)^{n-1} \times K \to \Delta(I) \},$$

and

$$\mathcal{T}^* = \{ \tau = (\tau_n)_{n \geq 1} : \tau_n : (I \times J)^{n-1} \to \Delta(J) \}.$$

Indeed, at each stage $n$, Player 1 may take a mixed action that depends on past states $k_1, ..., k_{n-1}$ and past observed actions $i_1, j_1, ..., i_{n-1}, j_{n-1}$, together with the current state $k_n$. On the opposite, at each stage $n$, Player 2 can choose a mixed action based on his information, i.e., past observed actions $i_1, j_1, ..., i_{n-1}, j_{n-1}$. Using Kolmogorov’s extension theorem, we may associate with each triplet $p \in \Delta(K)$, $\sigma \in \Sigma^*$, $\tau \in \mathcal{T}^*$ a probability measure $\mathbb{P}^*_p, \sigma, \tau$ on the product space $(K \times I \times J)^N$ which is consistent with finite
histories. Analogously to the random-duration Markovian persuasion game defined in Subsection 6.1, we now define the random-duration Markov chain game $\hat{\Gamma}^*(p, x)$, in which the strategy space are $\Sigma^*$ for Player 1 and $\mathcal{T}^*$ for Player 2, so that the payoff associated with each pair $(\sigma, \tau) \in \Sigma^* \times \mathcal{T}^*$, prior distribution $p \in \Delta(K)$ and $x \in (0, 1]$ is given by

$$\gamma^*(p, x, \sigma, \tau) := \mathbb{E}_{p, \sigma, \tau}^* \left( \sum_{n=1}^{\infty} g(k_n, i_n, j_n) \mathbb{I}\{n \leq W\} \right),$$

where $\mathbb{E}_{p, \sigma, \tau}^*$ is the expectation operator w.r.t. $\mathbb{P}_{p, \sigma, \tau}^*$, and $W$ is a geometric random variable with parameter $x$ which is independent of $(X_n)_{n \geq 1}$ and the actions of the players. If we denote by $\hat{V}(p, x)$ the value of $\hat{\Gamma}^*(p, x)$, then the same arguments used in the proof of Claim 1 imply that

$$\hat{V}(p, x) = V_{1-x}(p)/x.$$

We now turn to the description of $\Sigma^{*, \mathcal{R}}$ and $\mathcal{T}^{*, \mathcal{R}}$, the behavioral strategy spaces of players 1 and 2, respectively, in $\Gamma^{*, \mathcal{R}}(p)$. Those are given by

$$\begin{align*}
\Sigma^{*, \mathcal{R}} &= \{\sigma = (\sigma_n)_{n \geq 1} : \sigma_n : (K \times I \times J \times \{0, 1\})^{n-1} \times K \to \Delta(I)\}, \\
\mathcal{T}^{*, \mathcal{R}} &= \{\tau = (\tau_n)_{n \geq 1} : \tau_n : (I \times J \times \{0, 1\})^{n-1} \to \Delta(J)\}.
\end{align*}$$

The interpretation is the following. In $\Gamma^{*, \mathcal{R}}(p)$, Player 1 may base his mixed action in stage $n$, $\sigma_n$, on the past history of states $k_1, ..., k_{n-1}$, actions $i_1, j_1, ..., i_{n-1}, j_{n-1}$, and revelations $Z_1, ..., Z_{n-1}$ together with the current state $k_n$. Player 2 may condition his mixed action at the $n$'th stage of $\Gamma^{*, \mathcal{R}}(p)$, $\tau_n$, only on past actions $i_1, j_1, ..., i_{n-1}, j_{n-1}$, and past revelations $Z_1, ..., Z_{n-1}$. As before, by Kolmogorov’s extension theorem, we may associate with each triplet $p \in \Delta(K)$, $x \in (0, 1]$, $(\sigma, \tau) \in \Sigma^{*, \mathcal{R}} \times \mathcal{T}^{*, \mathcal{R}}$ a probability measure $\mathbb{P}_{p, x, \sigma, \tau}^*$ on the product space $(K \times I \times J \times \{0, 1\})^{\mathbb{N}}$ which is consistent with finite histories. By the von Neumann minimax theorem, we have that, for every $\lambda \in [0, 1)$,

$$V_x^\lambda(p) = \max_{\sigma \in \Sigma^{*, \mathcal{R}}} \min_{\tau \in \mathcal{T}^{*, \mathcal{R}}} \mathbb{E}_{p, x, \sigma, \tau}^* \left( (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} g(k_n, i_n, j_n) \right),$$

$$= \min_{\sigma \in \Sigma^{*, \mathcal{R}}} \max_{\tau \in \mathcal{T}^{*, \mathcal{R}}} \mathbb{E}_{p, x, \sigma, \tau}^* \left( (1 - \lambda) \sum_{n=1}^{\infty} \lambda^{n-1} g(k_n, i_n, j_n) \right).$$
and

\[ V^x_N(p) = \max_{\sigma \in \Sigma^*, \tau \in \mathcal{T}^*, R} \min_{p, x, \sigma, \tau} \left[ \frac{1}{N} \sum_{n=1}^{N} g(k_n, i_n, j_n) \right] \]

\[ = \min_{\sigma \in \Sigma^*, \tau \in \mathcal{T}^*, R} \max_{p, x, \sigma, \tau} \left[ \frac{1}{N} \sum_{n=1}^{N} g(k_n, i_n, j_n) \right]. \]

By a uniform Tauberian theorem for non-expansive operators of Ziliotto [39], it suffices to show that \( V^x_N \) converges uniformly to \( \Phi(1 - x) \) as \( N \to \infty \) in order to deduce item (i) of Theorem 3. The fact that Ziliotto’s theorem is applicable to the stochastic game \( \Gamma^{*, x}(p) \) follows from the argument given in Subsection 3.3 of [39].

By performing the same probabilistic analysis as in Subsection 6.4, we reduce the proof to showing the two following propositions.

**Proposition 4.** There exist \( \sigma^* \in \Sigma^*\mathcal{R} \), a positive integer \( L^* \), and positive constants \( C^*, D^* \) such that for every \( \tau \in \mathcal{T}^*\mathcal{R} \) it holds that

\[ B^*_{N}(p, x, \sigma^*, \tau) \geq \Phi(1 - x) - C^* \varepsilon - D^* z \varepsilon, \quad \forall p \in \Delta(K), \quad \forall N \geq L^*. \]

For each \( (\sigma, \tau) \in \Sigma^*\mathcal{R} \times \mathcal{T}^*\mathcal{R} \) define \( B^*_{N}(p, x, \sigma, \tau) \) analogously to \( B_{N}(p, x, \sigma) \) (whose definition is given in Proposition 1 above) by

\[ B_{N}(p, x, \sigma, \tau) = \mathbb{E}_{p, x, \sigma, \tau}^* \left( \frac{1}{N} \sum_{n=N_{\kappa_1+1}}^{\kappa_2} g(k_n, i_n, j_n) \{ \kappa_1 + \kappa_2 \leq N \} \right). \]

**Proposition 5.** There exist \( \tau^* \in \mathcal{T}^*\mathcal{R} \), a positive integer \( L \) and positive constants \( C, D \) such that for every \( \sigma \in \Sigma^*\mathcal{R} \),

\[ B^*_{N}(p, x, \sigma, \tau^*) \leq \Phi(x) + C \varepsilon + D z \varepsilon, \quad \forall p \in \Delta(K), \quad \forall N \geq L. \]

The proof of Proposition 4 is carried out using the same probabilistic analysis as in the proof of Proposition 1, with one adjustment: \( \sigma^* \in \Sigma^*\mathcal{R} \) is now defined as follows. Play the same arbitrary action until stage \( \kappa_1 \). Then, for each \( n \geq 1 \), upon observing \( X_{\kappa_1 + \cdots + \kappa_n} \), play an optimal strategy in \( \hat{\Gamma}^*(\delta_{X_{\kappa_1 + \cdots + \kappa_n}}, M, x) \) starting from stage \( \kappa_1 + \cdots + \kappa_n + 1 \) until stage \( \kappa_1 + \cdots + \kappa_n + 1 \), included.

The proof of Proposition 5 is carried out analogously to the proof of Proposition 2. This time \( \tau^* \in \mathcal{T}^*\mathcal{R} \) is defined as follows. Play the same
arbitrary action until stage $\kappa_1$. Then, for each $n \geq 1$, upon observing $X_{\kappa_1+\cdots+\kappa_n}$, play an optimal strategy in $\hat{\Gamma}^*(\delta_{X_{\kappa_1+\cdots+\kappa_n}}M, x)$ starting from stage $\kappa_1 + \cdots + \kappa_n + 1$ until stage $\kappa_1 + \cdots + \kappa_{n+1}$, included.

We can therefore deduce that item (i) of Theorem 3 holds as well. As for item (ii) of Theorem 3, we use the same idea of ‘monotonicity through information’ for $V^x_\lambda$. Namely, we consider a generalization of $\Gamma^*_\lambda(x)$ in which Player 2 is told at each stage $n$ the state $k_{n-1}$ at the previous stage with probability $(y - x)(1 - x)$ each time that $Z_{n-1} = 0$. Clearly, in this new setup Player 1 cannot guarantee more than $V^x_\lambda(p)$, as we deal with a zero-sum game. However, he can guarantee in this setup $V^y_\lambda(p)$ because this generalized game is equivalent from a probabilistic standpoint to the game $\Gamma^*_y(p)$. Together with item (i) of Theorem 3 this suffices to deduce item (ii).