Spontaneous symmetry breaking in three-Higgs-doublet $S_3$-symmetric models

D Emmanuel-Costa$^{1,5}$, O M Ogreid$^2$, P Osland$^3$ and M N Rebelo$^{1,4}$

1 Centro de Física Teórica de Partículas – CFTP and Dept de Física Instituto Superior Técnico – IST, Universidade de Lisboa (UL), Av. Rovisco Pais, P-1049-001 Lisboa, Portugal
2 Western Norway University of Applied Sciences, Postboks 7030, N-5020 Bergen, Norway
3 Department of Physics and Technology, University of Bergen, Postboks 7803, N-5020 Bergen, Norway
4 Theoretical Physics Department, CERN, CH-1211, Geneva 23, Switzerland

E-mail: omo@hvl.no, Per.Osland@uib.no and rebelo@tecnico.ulisboa.pt

Abstract. The talk summarises work done by the authors consisting of a detailed study of the possible vacua in models with three Higgs doublets with $S_3$ symmetry and without explicit CP violation. Different vacua require special regions of the parameter space which were analysed in our work. We establish the possibility of spontaneous CP violation in this framework and we also show which complex vacua conserve CP. In our work we discussed constraints from vacuum stability. The results presented here are relevant for model building.

1. Introduction

In the Standard Model of Particle Physics (SM) there is one Higgs doublet responsible for spontaneous electroweak symmetry breaking and for the mechanism that gives mass to fermions and to electroweak gauge bosons. The model predicts the existence of one Higgs boson. In 2012 a scalar boson was discovered at the LHC [1, 2] with properties consistent with those predicted by the SM. However there are good motivations to consider models with more than one Higgs doublet such as the possibility of having CP symmetry broken spontaneously [3] or new sources of CP violation. Supersymmetric models require two Higgs doublets. Furthermore, models with two Higgs doublets have a rich phenomenology with many interesting possible manifestations of physics beyond the SM [4, 5]. Extensions of the SM with more than one Higgs doublet are good candidates to explain some of the present flavour anomalies and to solve some of the puzzles left unanswered by the SM.

Models with more than one Higgs doublet can give substantial contributions to flavour changing neutral currents (FCNC). Current experimental bounds require these to be strongly suppressed. One possibility is to completely forbid Higgs mediated FCNC at tree level via a symmetry, as is the case in models with natural flavour conservation (NFC) [6, 7] where only one Higgs doublet is allowed to couple to each charge quark sector. In the case of two Higgs doublets this is achieved by means of a $Z_2$ symmetry and as a result neither spontaneous nor hard CP violation can occur in the Higgs sector. It is possible to have CP violation in the scalar sector,

5 Affiliation valid till the end of 2016
with two Higgs doublets and NFC, with the $Z_2$ symmetry softly broken in the Higgs potential [8]. Three Higgs doublets and NFC with exact $Z_2$ symmetries allow for CP to be violated either explicitly [9] or spontaneously [10]. FCNC can also be eliminated at tree level assuming alignment of Yukawa couplings in flavour space [11], in this case complex Yukawa couplings give rise to new sources of CP violation. There have been several attempts at obtaining Yukawa alignment in several extensions of the SM [12, 13, 14, 15]. Another very interesting possibility to avoid the problem of having too large FCNC with two Higgs doublets, is to build models allowing for Higgs mediated FCNC which are under control by means of a symmetry that leads to couplings suppressed by small off-diagonal elements of the quark mixing matrix $V_{CKM}$. The first models of this type, based on a symmetry, were proposed by Branco, Grimus and Lavoura (BGL) [16]. Implications of BGL-type models have been extensively analysed recently, in the light of the LHC results, for several different implementations [17, 18, 19, 20, 21, 22]. BGL-type models can have FCNC either in the up sector or in the down sector but not in both sectors at the same time. Recently, a generalisation of BGL-type models allowing for FCNC in both sectors was built [23]. BGL models models can also be extended to the case of three doublets [17].

Three-Higgs-doublet models may provide good dark matter candidates [24, 25, 26, 27, 28, 29]. One may also speculate that nature is such that three generations of fermions come with three Higgs doublets, which is, of course, an issue to be settled by experiment. As the number of doublets increases so does the complexity of the scalar potential and the number of free parameters in the theory [30]. Discrete symmetries play an important role in reducing this number and lead at the same time to testable predictions. Symmetries also play an important role in stabilising dark matter [31, 32, 33].

As mentioned above, CP can be spontaneously violated in models with three Higgs doublets with $Z_2$ symmetries [10]. Such is also the case for an $S_3$ symmetry [34, 35, 36].

This talk is based on the work done in Ref. [35] where the possible vacuum solutions were analysed and the possibility of having spontaneous CP violation in the context of three Higgs doublets, with an $S_3$ symmetry, was studied. In this work we concentrated our attention on the scalar potential. Several authors have considered implications of three-Higgs-doublet models with an $S_3$ symmetry for flavour physics (see, for example: [37, 38, 39, 40, 41, 42, 43, 44]).

2. Vacua of $S_3$-symmetric three-Higgs-doublet potential

The $S_3$ group is the permutation group involving three Higgs doublets $\phi_1$, $\phi_2$ and $\phi_3$, identified as the defining representation and constituting a reducible triplet of Higgs doublets. The study of the $S_3$-symmetric three-Higgs-doublet potential can be done in two different frameworks, either in terms of the defining representation [45, 46], or in terms of the irreducible representations [47]. In what follows, we classify the vacua in terms of constraints on the potential. Vacua can also be classified in terms of their residual symmetries [48].

2.1. The scalar potential

The $S_3$ symmetric potential has a quadratic and a quartic part, which in terms of the defining representation can be written [45]:

$$V = V_2 + V_4$$

(1)
\[ V_2 = -\lambda \sum_i \phi_i^\dagger \phi_i + \frac{1}{2} \gamma \sum_{i<j} [\phi_i^\dagger \phi_j + \text{h.c.}], \quad (2a) \]

\[ V_4 = A \sum_i (\phi_i^\dagger \phi_i)^2 + \sum_{i<j} \{ C(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \overline{C}(\phi_i^\dagger \phi_j)(\phi_j^\dagger \phi_i) + \frac{1}{2} D[(\phi_i^\dagger \phi_j)^2 + \text{h.c.}] \}
+ \frac{1}{2} E_1 \sum_{i \neq j} [(\phi_i^\dagger \phi_i)(\phi_j^\dagger \phi_j) + \text{h.c.}] + \sum_{i \neq j \neq k \neq l, i < k} \left( \frac{1}{2} E_2 [(\phi_i^\dagger \phi_j)(\phi_k^\dagger \phi_l) + \text{h.c.}] \right.
+ \frac{1}{2} E_3 [(\phi_i^\dagger \phi_k)(\phi_j^\dagger \phi_l) + \text{h.c.}] + \frac{1}{2} E_4 [(\phi_i^\dagger \phi_l)(\phi_j^\dagger \phi_k) + \text{h.c.}]. \quad (2b) \]

Here all fields appear on an equal footing. This representation is not irreducible, it splits into two irreducible representations consisting of a singlet, \( h_S \) and a doublet of \( S_3 \) with components \( h_1 \) and \( h_2 \). The decomposition into these two irreducible representations is given by:

\[
\begin{pmatrix}
  h_1 \\
  h_2 \\
  h_S
\end{pmatrix}
= \begin{pmatrix}
  \frac{1}{\sqrt{2}} & -\frac{1}{\sqrt{2}} & 0 \\
  \frac{1}{\sqrt{6}} & \frac{1}{\sqrt{6}} & -\frac{2}{\sqrt{6}} \\
  \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}} & \frac{1}{\sqrt{3}}
\end{pmatrix}
\begin{pmatrix}
  \phi_1 \\
  \phi_2 \\
  \phi_3
\end{pmatrix}
\]

(3)

This matrix has a striking similarity with the tribimaximal mixing matrix [49] which is very close to the observed leptonic mixing.

The scalar potential written in term of fields from irreducible representations has the form [47, 41, 50]

\[
V_2 = \mu_0^2 h_S^\dagger h_S + \mu_1^2 (h_1^\dagger h_1 + h_2^\dagger h_2),
\]

\[
V_4 = \lambda_1 (h_1^\dagger h_1 + h_2^\dagger h_2)^2 + \lambda_2 (h_1^\dagger h_2 - h_2^\dagger h_1)^2 + \lambda_3 [(h_1^\dagger h_1 - h_2^\dagger h_2)^2 + (h_1^\dagger h_2 + h_2^\dagger h_1)^2]
+ \lambda_4 [(h_1^\dagger h_1)(h_2^\dagger h_2)(h_1^\dagger h_1 - h_2^\dagger h_2) + \text{h.c.}] + \lambda_5 (h_S^\dagger h_S)(h_1^\dagger h_1 + h_2^\dagger h_2)
+ \lambda_6 [(h_S^\dagger h_1)(h_2^\dagger h_S)(h_S^\dagger h_2) + \lambda_7 [(h_S^\dagger h_1)(h_2^\dagger h_S)(h_S^\dagger h_2) + \text{h.c.}]
+ \lambda_8 (h_S^\dagger h_S)^2.
\]

(4b)

In this form the potential has no symmetry for the interchange of \( h_1 \) and \( h_2 \) but there is a \( Z_2 \) symmetry of the form \( h_1 \to -h_1 \). There is an equivalent doublet representation which has also been used in the literature [40]:

\[
\begin{pmatrix}
  \tilde{\chi}_1 \\
  \tilde{\chi}_2
\end{pmatrix}
= \frac{1}{\sqrt{2}} \begin{pmatrix}
  i & 1 \\
  -i & 1
\end{pmatrix}
\begin{pmatrix}
  h_1 \\
  h_2
\end{pmatrix},
\]

(5)

where the symmetry appears as a symmetry for the interchange of the fields \( \tilde{\chi}_1 \) and \( \tilde{\chi}_2 \). Both expressions for the potential describe the same physics and the coefficients in the different frameworks are related through linear equations.

With the special choice of \( \lambda_4 = 0 \) the potential acquires a continuous \( SO(2) \) symmetry defined by:

\[
\begin{pmatrix}
  h_1' \\
  h_2'
\end{pmatrix}
= \begin{pmatrix}
  \cos \theta & -\sin \theta \\
  \sin \theta & \cos \theta
\end{pmatrix}
\begin{pmatrix}
  h_1 \\
  h_2
\end{pmatrix},
\]

(6)

Spontaneous breaking of this symmetry leads to a massless scalar which of course must be avoided.

\( S_3 \) has three irreducible representations, a doublet, a singlet and a pseudosinglet, \( h_A \). The latter has no direct translation into the initial fields \( \phi_1, \phi_2 \) and \( \phi_3 \) and transforms under \( S_3 \) into
(-h_A). From a group theoretical point of view we can choose to write an S_3 symmetric potential in terms of the doublet and h_A. The new potential becomes:

\[ V_2 = \mu_0^2 h_A^4 + \mu_1^2 (h_1^4 h_1 + h_2^4 h_2), \]

\[ V_4 = \lambda_1 (h_1^4 h_1 + h_2^4 h_2) + \lambda_2 (h_1^4 h_2 - h_2^4 h_1)^2 + \lambda_3 [(h_1^4 h_1 - h_2^4 h_2)^2 + (h_1^4 h_2 + h_2^4 h_1)^2] + \lambda_4 (h_A^4 h_2)(h_1^4 h_2 + h_2^4 h_1) - (h_A^4 h_1)(h_1^4 h_1 - h_2^4 h_2) + \text{h.c.} + \lambda_5 (h_A^4 h_A)(h_1^4 h_1 + h_2^4 h_2) + \lambda_6 (h_A^4 h_1)(h_2^4 h_A) + \lambda_7 [(h_A^4 h_1)(h_1^4 h_1) + (h_A^4 h_2)(h_1^4 h_2) + \text{h.c.}] + \lambda_8 (h_A^4 h_A)^2, \]

which reduces to the same potential we had before with h_1 and h_2 interchanged. At this stage there is no new physics from this choice of representations. However this may change depending on how the couplings to the fermions are introduced.

In order to study the possibility of having spontaneous CP violation we start with a potential with real coefficients. This choice guarantees, without loss of generality, that the potential conserves CP. In this case we are left with ten independent parameters irrespective of the choice of representations. This potential does not fall into a CP conserving potential with irremovable complex parameters [51].

We use the following field notations for the decomposition of the SU(2) Higgs doublets:

\[ \phi_i = \left( \begin{array}{c} \varphi_i^+ \\ (\rho_i + \eta_i + i\chi_i)/\sqrt{2} \end{array} \right), \quad i = 1, 2, 3, \]

\[ h_i = \left( \begin{array}{c} h_i^+ \\ (w_i + \eta_i + i\chi_i)/\sqrt{2} \end{array} \right), \quad i = 1, 2, \quad h_S = \left( \begin{array}{c} h_S^+ \\ (w_S + \eta_S + i\chi_S)/\sqrt{2} \end{array} \right). \]

### 2.2. Real vacuum solutions

Real vacuum solutions do not violate CP spontaneously. It is interesting to understand what are the possible real solutions for the vacuum. In this case one has to solve three minimisation conditions corresponding to the vanishing of the three relevant derivatives of the potential. In the irreducible framework these conditions can be solved in terms of \( \mu_0^2 \) and \( \mu_1^2 \) leading to \([50]\):

\[ \mu_0^2 = \frac{1}{2w_S^2} \left[ \lambda_1 (w_2^2 - 3w_1^2)w_2 - (\lambda_5 + \lambda_6 + 2\lambda_7)(w_1^2 + w_2^2)w_S - 2\lambda_8 w_3^2 \right], \]

\[ \mu_1^2 = -\frac{1}{2} \left[ 2(\lambda_1 + \lambda_3)(w_1^2 + w_2^2) + 6\lambda_4 w_2 w_S + (\lambda_5 + \lambda_6 + 2\lambda_7)w_S^2 \right], \]

\[ \mu_2^2 = -\frac{1}{2} \left[ 2(\lambda_1 + \lambda_3)(w_1^2 + w_2^2) - 3\lambda_4 (w_2^2 - w_1^2) w_S^2 / w_2 + (\lambda_5 + \lambda_6 + 2\lambda_7)w_S^2 \right]. \]

The first equation comes from the derivative of the potential with respect to \( w_2 \) and the second and third from the derivatives with respect to \( w_1 \) and \( w_2 \). Eqs. \((10b)\) and \((10c)\) were obtained dividing by \( w_1 \) and \( w_2 \) respectively. Clearly, these two equations are not automatically consistent. There are several possible consistency cases:

- for \( w_1 = 0 \) the corresponding derivative is zero and there is no clash with the determination of \( \mu_1^2 \) from Eq. \((10c)\).
- otherwise, \( \lambda_4 (3w_2^2 - w_1^2)w_S = 0 \) is required. This can be achieved in three different ways: \( \lambda_4 = 0 \) or \( w_1 = \pm \sqrt{3}w_2 \) or \( w_S = 0 \).
- for \( w_S = 0 \) a special condition arises from Eq. \((10a)\): \( \lambda_4 w_2 (3w_2^2 - w_1^2) = 0 \) so that in addition we must have \( \lambda_4 = 0 \) or \( w_2 = \pm \sqrt{3}w_1 \), or \( w_2 = 0 \).
Table 1. Possible real vacua (partly after Derman and Tsao [46]). This classification uses the notation $R$-$X$-$y$, where $R$ refers to “real”. The roman numeral $X$ gives the number of constraints on the parameters of the potential that arise from solving the stationary-point equations. The letter $y$ is used to distinguish different vev’s that have the same $X$, and $\lambda_a$ is defined in Eq. (11).

| Vacuum | $\rho_1, \rho_2, \rho_3$ | $w_1, w_2, w_S$ | Comment |
|--------|----------------|----------------|---------|
| R-0    | 0, 0, 0         | 0, 0, 0        | Not interesting |
| R-I-1  | $x, x, x$       | 0, 0, $w_S$    | $\mu_0^2 = -\lambda_8 w_S^2$ |
| R-I-2a | $x, -x, 0$      | $w, 0, 0$      | $\mu_1^2 = - (\lambda_1 + \lambda_3) w_1^2$ |
| R-I-2b | $x, 0, -x$      | $w, \sqrt{3}w, 0$ | $\mu_1^2 = -\frac{4}{3} (\lambda_1 + \lambda_3) w_1^2$ |
| R-I-2c | 0, $x, -x$      | $w, -\sqrt{3}w, 0$ | $\mu_1^2 = -\frac{4}{3} (\lambda_1 + \lambda_3) w_2^2$ |
| R-II-1a| $x, x, y$       | 0, 0, $w_S$    | $\mu_0^2 = \frac{2}{3} \lambda_4 w_4^2 - \frac{1}{2} \lambda_a w_2^2 - \lambda_8 w_S^2$, $\mu_1^2 = -(\lambda_1 + \lambda_3) w_2^2 + \frac{2}{3} \lambda_4 w_4 w_S - \frac{1}{2} \lambda_a w_2^2$ |
| R-II-1b| $y, x, x$       | $w, -w/\sqrt{3}, w_S$ | $\mu_0^2 = -4 \lambda_4 w_4^2 - 2 \lambda_a w_2^2 - \lambda_8 w_S^2$, $\mu_1^2 = -4 (\lambda_1 + \lambda_3) w_2^2 - 3 \lambda_4 w_4 w_S - \frac{1}{2} \lambda_a w_2^2$ |
| R-II-1c| $y, x, x$       | $w, w/\sqrt{3}, w_S$ | $\mu_0^2 = -4 \lambda_4 w_4^2 - 2 \lambda_a w_2^2 - \lambda_8 w_S^2$, $\mu_1^2 = -4 (\lambda_1 + \lambda_3) w_2^2 - 3 \lambda_4 w_4 w_S - \frac{1}{2} \lambda_a w_2^2$ |
| R-II-2 | $x, x, -2x$     | 0, 0, 0        | $\mu_1^2 = -(\lambda_1 + \lambda_3) w_1^2$, $\lambda_4 = 0$ |
| R-II-3 | $x, y, -x - y$  | $w_1, w_2, 0$  | $\mu_1^2 = -(\lambda_1 + \lambda_3) (w_1^2 + w_2^2)$, $\lambda_4 = 0$ |
| R-III  | $\rho_1, \rho_2, \rho_3$ | $w_1, w_2, w_S$ | $\mu_0^2 = -\frac{1}{2} \lambda_a (w_1^2 + w_2^2) - \lambda_8 w_S^2$, $\mu_1^2 = -(\lambda_1 + \lambda_3) (w_1^2 + w_2^2) - \frac{1}{2} \lambda_a w_2^2$, $\lambda_4 = 0$ |

Derman and Tsao [46] analysed spontaneous symmetry breaking with real vacua taking also into account the residual symmetries. Their work was done in the reducible framework where the condition $\lambda_4 = 0$ corresponds to $4A - 2(C + \bar{C} + D) - E_1 + E_2 + E_3 + E_4 = 0$. This condition was obtained before by Derman [45] who considered it very unnatural, since in his context it was not clear that it was associated to an additional symmetry. With $\lambda_4 \neq 0$ there were only three possible real solutions [46]:

- $(x, x, x)$ leaving $S_3$ unbroken and translating into the doublet-singlet notation as $(0, 0, w_S)$; consistency condition: $w_1 = 0$ (also verifies $w_1 = \pm \sqrt{3}w_2$).
- $(x, x, y)$ leaving a residual $S_2$ symmetry. In terms of the reducible representation any ordering of the vev’s is equivalent, however, in the definition of the doublet of $S_3$ a special direction is chosen. As a result, different orderings correspond to different translations: $(x, x, y)$ translates into $(0, w_2, w_S)$; consistency condition: $w_1 = 0$.
- $(x, x, y)$ translates into $(w_1, -\frac{1}{\sqrt{3}}w_1, w_S)$; consistency condition: $w_1 = -\sqrt{3}w_2$.
- $(y, x, x)$ translates into $(w_1, \frac{1}{\sqrt{3}}w_1, w_S)$; consistency condition: $w_1 = \sqrt{3}w_2$.
- $(x, y, z)$ $(x, -x, 0)$ leaving a residual $S_2$ symmetry. This is the only possible real solution with all three vevs different from each other, unless one imposes $4A - 2(C + \bar{C} + D) - E_1 + E_2 + E_3 + E_4 = 0$ ($\lambda_4 = 0$). The translation into the irreducible representation is now:
- $(x, -x, 0)$ translates into $(w_1 = \sqrt{2}x, 0, 0)$; consistency conditions: $w_S = 0$ together with $w_2 = 0$.
- $(x, 0, -x)$ translates into $(w_1 = \frac{1}{\sqrt{2}}x, w_2 = \frac{\sqrt{3}}{\sqrt{2}}x, 0)$; consistency conditions: $w_S = 0$ together with $w_2 = \sqrt{3}w_1$.
- $(0, x, -x)$ translates into $(w_1 = -\frac{1}{\sqrt{2}}x, w_2 = \frac{\sqrt{3}}{\sqrt{2}}x, 0)$; consistency conditions: $w_S = 0$
Table 2. Complex vacua. Notation: \( \epsilon = 1 \) and \(-1\) for C-III-d and C-III-e, respectively; \( \xi = \sqrt{-3 \sin 2\rho_1/\sin 2\rho_2} \), \( \psi = \sqrt{[3 + 3 \cos(\rho_2 - 2\rho_1)]/(2 \cos \rho_2)} \). Due to the constraints the vacua labelled with an asterisk (*) are in fact real.

| IRF (Irreducible Rep.) | RRF (Reducible Rep.) |
|------------------------|----------------------|
| \( w_1, w_2, w_S \)   | \( \rho_1, \rho_2, \rho_3 \) |
| C-I-a                  | \( \hat{w}_1, \pm i\hat{w}_1, 0 \) |
| C-II-a                 | \( 0, \hat{w}_2e^{i\sigma_2}, \hat{w}_S \) |
| C-II-b                 | \( \pm \hat{w}_1, 0, \hat{w}_S \) |
| C-II-c                 | \( \hat{w}_1e^{i\sigma_1}, \hat{w}_2e^{i\sigma_2}, 0 \) |
| C-II-d                 | \( \pm i\hat{w}_1, \hat{w}_2, \hat{w}_S \) |
| C-II-e                 | \( \hat{w}_1e^{i\sigma_1}, \hat{w}_2, \hat{w}_S \) |
| C-III-a                | \( \sqrt{3\hat{w}_2e^{i\sigma_2}}, \pm \hat{w}_2e^{i\sigma_2}, \hat{w}_S \) |
| C-III-b                | \( \hat{w}_1e^{i\sigma_1}, 0, \hat{w}_S \) |
| C-III-c                | \( \sqrt{1 + 2\cos^2\sigma_2} \hat{w}_2, \hat{w}_S \) |
| C-III-d                | \( \hat{w}_1e^{i\sigma_1}, \pm \hat{w}_2e^{i\sigma_2}, \hat{w}_S \) |
| C-III-e                | \( \sqrt{2 + \frac{\cos(\sigma_1 - 2\sigma_2)}{\cos \sigma_1}} \hat{w}_2e^{i\sigma_1}, \hat{w}_S \) |
| C-V*                   | \( \hat{w}_1e^{i\sigma_1}, \hat{w}_2e^{i\sigma_2}, \hat{w}_S \) |

2.3. Complex vacuum solutions

In the discussion of possible complex vacua we now adopt a convention where \( w_S \) is real and non-negative and take

\[
    w_1 = \hat{w}_1 e^{i\sigma_1}, \quad w_2 = \hat{w}_2 e^{i\sigma_2},
\]

with the \( \hat{w}_i \) also real and non-negative. With this convention \( w_S \) is also denoted by \( \hat{w}_S \). A systematic analysis of possible solutions was performed in [35]. The results are summarised in Table 2. The list of the constraints on the potential that are consistent with each solution is not given here, it can be found in Ref. [35].

Several solutions require \( \lambda_4 = 0 \). This is not a new feature, it also happened in the context of real solutions. For \( \lambda_4 = 0 \) the potential acquires a continuous SO(2) symmetry which can be
broken spontaneously by the vacuum solutions, therefore, leading to a massless scalar. Massless scalars are ruled out by experiment. It is possible to avoid this problem by introducing soft breaking terms. The most general form for the $V_2$ part of the potential with soft breaking terms would be:

$$V_2 = \mu_0^2 h_1^\dagger h_S + \mu_1^2 (h_1^\dagger h_1 + h_2^\dagger h_2) + \mu_2^2 (h_1^\dagger h_2 - h_2^\dagger h_1) + \frac{1}{2} \nu^2 (h_2^\dagger h_2 - h_1^\dagger h_1). \quad (13)$$

However, soft breaking terms involving $h_S$ and one $h_i$ are not consistent with $\lambda_4 = 0$.

In Table 3 we collect all possible complex vacuum solutions indicating whether or not they require $\lambda_4$ equal to zero and whether or not they allow for spontaneous CP violation. One important conclusion from our analysis is that there are cases where CP can be violated spontaneously, however, no solution requiring $\lambda_4 = 0$ can lead to spontaneous CP violation.

In order to confirm that CP could indeed be violated spontaneously we used a powerful tool based on CP-odd Higgs-basis-invariant conditions, verifying that there were indeed conditions that were violated. There are several such conditions which were especially built for the analysis of the Higgs potential [52, 53, 54, 55, 56]. In the next subsection we discuss spontaneous CP violation using a few illustrative examples.

### Table 3. Spontaneous CP violation

| Vacuum | $\lambda_4$ | SCPV | Vacuum | $\lambda_4$ | SCPV | Vacuum | $\lambda_4$ | SCPV |
|--------|-------------|------|--------|-------------|------|--------|-------------|------|
| C-I-a  | X           | no   | C-III-f,g | 0           | no   | C-IV-c | X           | yes  |
| C-III-a| X           | yes  | C-III-h | X           | yes  | C-IV-d | 0           | no   |
| C-III-b| 0           | no   | C-III-i | X           | no   | C-IV-e | 0           | no   |
| C-III-c| 0           | no   | C-IV-a | 0           | no   | C-IV-f | X           | yes  |
| C-III-d,e| X           | no   | C-IV-b | 0           | no   | C-V    | 0           | no   |

#### 2.4. Spontaneous CP violation

Spontaneous CP violation can only occur if the Lagrangian conserves CP but the vacuum does not. This can only happen when there is no transformation that can be identified with a CP transformation leaving both the Lagrangian and the vacuum invariant. Under a CP transformation a single Higgs doublet $\Phi$ transforms into its complex conjugate. In models with several Higgs doublets the most general CP transformation is given by:

$$\Phi_i \xrightarrow{\text{CP}} U_{ij} \Phi_j^*.$$  \quad (14)

Here, $U$ is a unitary matrix mixing different Higgs doublets and corresponds to a Higgs basis transformation. Higgs basis transformations do not change the physics.

If all the coefficients of the potential are real the potential conserves CP explicitly and the above equation is verified for $U$ the identity matrix. Checking for explicit CP invariance of a multi-Higgs potential may be a non-trivial task since Higgs basis transformations, in general, can transform couplings that are real in one Higgs basis into couplings that are complex in another basis. For this purpose CP-odd Higgs basis invariants are of great help [53, 54]. Once

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This transformation is often referred to as a “generalized” CP transformation, thus suggesting that there is also a “non-generalized” CP transformation. This is, of course, misleading.
it is known that a Lagrangian conserves CP it remains to check whether or not CP is violated spontaneously. It has been shown [57] that in order for the vacuum to conserve CP the following relation has to be obeyed:
\[ U_{ij} \langle 0 | \Phi_j | 0 \rangle^* = \langle 0 | \Phi_i | 0 \rangle \]  
(15)
with \( U \) now a unitary matrix corresponding to a symmetry of the Lagrangian. This relation is very powerful and allows to show that vacua that would at first sight violate CP are indeed CP conserving. This can be illustrated with a few examples taken from Table 3. For a full discussion see Ref. [35].

- Let us consider the vacuum identified as C-I-a, given by \( \left( x, xe^{\pm \frac{2\pi i}{3}}, xe^{\mp \frac{2\pi i}{3}} \right) \) in the reducible representation. It is not possible to rephase the three Higgs doublets in such a way that the three vevs become real keeping at the same time the potential real. This is a vacuum solution with calculable non-trivial phases, fixed by the symmetry of the potential with no explicit dependence on the parameters of the potential. Such phases are called geometrical phases [57]. It was shown in Ref. [57] that this vacuum does not violate CP since Eq. (15) can be verified for \( U \) given by:
\[ U = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}. \]  
(16)
This matrix makes use of the symmetry of the potential for the interchange of \( \phi_2 \) and \( \phi_3 \).

- Another interesting example is the C-III-c vacuum which is of the form \( \left( \hat{w}_1 e^{i\sigma}, \hat{w}_2 e^{i\sigma}, 0 \right) \) in the irreducible representation framework. It can also be written, without loss of generality, through an overall phase rotation, in the form \( \left( \hat{w}_1 e^{i\sigma}, \hat{w}_2, 0 \right) \). At first sight this vacuum looks like a CP violating vacuum, especially taking into consideration the fact that the moduli of \( \hat{w}_1 \) and \( \hat{w}_2 \) are different from each other. However, once again we can use Eq. (15) to show that this vacuum conserves CP. Notice that this solution requires an overall phase rotation of the three Higgs doublets by \( \exp[-i(\delta_1 + \delta_2)/2] \), so that now the first two vevs acquire symmetric phases: \( \left( ae^{i\delta}, ae^{-i\delta}, 0 \right) \), \( \left( 0, 0, 0 \right) \).

\[ e^{i(\delta_1 + \delta_2)} \begin{pmatrix} \cos \theta & \sin \theta & 0 \\ -\sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \cos \theta & -\sin \theta & 0 \\ \sin \theta & \cos \theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 e^{i\sigma} \\ \hat{w}_2 \\ 0 \end{pmatrix} \]  
(17)

or
\[ e^{i(\delta_1 + \delta_2)} \begin{pmatrix} \sin 2\theta & \cos 2\theta & 0 \\ \cos 2\theta & -\sin 2\theta & 0 \\ 0 & 0 & 1 \end{pmatrix} \begin{pmatrix} \hat{w}_1 e^{i\sigma} \\ \hat{w}_2 \\ 0 \end{pmatrix} = \begin{pmatrix} \hat{w}_1 e^{i\sigma} \\ \hat{w}_2 \\ 0 \end{pmatrix}. \]  
(18)

In this example the matrix \( U \) has several components:
- an \( SO(2) \) rotation of \( h_1 \) and \( h_2 \) by an angle \( \theta \), which should be chosen as:
\[ \tan 2\theta = \frac{\hat{w}_1^2 - \hat{w}_2^2}{2\hat{w}_1 \hat{w}_2 \cos \sigma}. \]  
(19)
With this choice, the vevs of the new \( S_3 \) doublet fields acquire the same modulus and the new vacuum acquires the form \( \left( ae^{i\delta_1}, ae^{i\delta_2}, 0 \right) \),
- an overall phase rotation of the three Higgs doublets by \( \exp[-i(\delta_1 + \delta_2)/2] \), so that now the first two vevs acquire symmetric phases: \( \left( ae^{i\delta}, ae^{-i\delta}, 0 \right) \),
- finally we just need to use the symmetry for the interchange \( h'_1 \leftrightarrow h'_2 \) in the \( S_3 \) doublet representation.
The last example illustrates how powerful the condition given by Eq. (15) can be, but at the same time it shows that, as complexity grows, it may be non-trivial, in cases where such a matrix exists, to build the necessary matrix $U$. In fact this may require special insight and there is the danger of missing it, in a CP conserving case. In Ref. [58] we propose an alternative simple method, which is very useful in such cases, and allows to detect or eliminate the possibility of having spontaneous CP violation in multi-Higgs models. The three tools, consisting of the use of CP-odd invariant conditions, the relation given by Eq. (15) and the simple method proposed in Ref. [58], combined together, provide a reliable procedure to determine whether or not a given Higgs potential violates CP spontaneously.

3. Conclusions
We have presented here a summary of the work done in Ref. [35]. We have focused on some important features of three-Higgs-doublet models with an $S_3$ symmetry with emphasis on the discussion of spontaneous CP violation. Some aspects which were dealt with in the paper were not included in this short presentation. We refer the reader to the original work for a more detailed discussion of these aspects and for other topics such as ideas about constraining the potential by the vevs, relations among complex and real vacua, and a discussion on positivity beyond the necessary conditions given by Das and Dey [50] following the approach of Refs. [59, 24] (see also [60]). Models with multi-Higgs doublets such as those discussed in our work are very interesting and can in principle provide answers for several open questions. In particular they can provide viable dark matter candidates. These and other questions such as ways of generating realistic fermion masses and mixing in this context or looking for viable models with spontaneous CP violation are still challenging despite the fact that a lot of work has been already done along these lines. These questions are very timely due to the potential for being tested at the LHC.

Acknowledgements
MNR thanks the local organising committee of the Symposium Discrete 2016 for the stimulating scientific atmosphere and the warm hospitality in Warsaw, and acknowledges financial support from the National Science Centre, Poland, the HARMONIA project under contract UMO-2015/18/M/ST2/00518 (2016-2019) to participate in the Symposium and in the 1st Harmonia Meeting that took place immediately after the Symposium. Special thanks go to Maria Krawczyk. The work of MNR is partially supported by Fundação para a Ciência e a Tecnologia (FCT, Portugal) through the projects CERN/FIS-NUC/0010/2015, CFTP-FCT Unit 777 (UID/FIS/00777/2013) which are partially funded through POCTI (FEDER), COMPETE, QREN and EU.

References
[1] Aad G et al. [ATLAS Collaboration], 2012 “Observation of a new particle in the search for the Standard Model Higgs boson with the ATLAS detector at the LHC,” Phys. Lett. B 716 1 [arXiv:1207.7214 [hep-ex]].
[2] Chatrchyan S et al. [CMS Collaboration], 2012 “Observation of a new boson at a mass of 125 GeV with the CMS experiment at the LHC,” Phys. Lett. B 716 30 [arXiv:1207.7235 [hep-ex]].
[3] Lee T D, 1973 “A Theory of Spontaneous T Violation,” Phys. Rev. D 8 1226.
[4] Gunion J F, Haber H E, Kane G L and Dawson S, 2000 “The Higgs Hunter’s Guide,” Front. Phys. 80 1.
[5] Branco G C, Ferreira P M, Lavoura L, Rebelo M N, Sher M and Silva J P, 2012 “Theory and phenomenology of two-Higgs-doublet models,” Phys. Rept. 516 1 [arXiv:1106.0934 [hep-ph]].
[6] Glashow S L and Weinberg S, 1977 “Natural Conservation Laws for Neutral Currents,” Phys. Rev. D 15 1958.
[7] Paschos E A, 1977 “Diagonal Neutral Currents,” Phys. Rev. D 15 1966.
[8] Branco G C and Rebelo M N, 1985 “The Higgs Mass in a Model With Two Scalar Doublets and Spontaneous CP Violation,” Phys. Lett. 160B 117.
[9] Weinberg S, 1976 “Gauge Theory of CP Violation,” Phys. Rev. Lett. 37 657.
[10] Branco G C, 1980 “Spontaneous CP Nonconservation and Natural Flavor Conservation: A Minimal Model,” Phys. Rev. D 22 2901.
[11] Pich A and Tuzon P, 2009 “Yukawa Alignment in the Two-Higgs-Doublet Model,” Phys. Rev. D 80 091702 [arXiv:0908.1554 [hep-ph]].
[12] Serodio H, 2011 “Yukawa Alignment in a Multi Higgs Doublet Model: An effective approach,” Phys. Lett. B 700 133 [arXiv:1104.2545 [hep-ph]].
[13] De Medeiros Varzielas I, 2011 “Family symmetries and alignment in multi-Higgs doublet models,” Phys. Lett. B 701 597 [arXiv:1104.2601 [hep-ph]].
[14] Celis A, Fuentes-Martín J and Serodio H, 2014 “Effective Aligned 2HDM with a DFSZ-like invisible axion,” Phys. Lett. B 737 185 [arXiv:1407.0971 [hep-ph]].
[15] Botella F J, Branco G C, Coutinho A M, Rebelo M N and Silva-Marcos J I, 2015 “Natural Quasi-Alignment with two Higgs Doublets and RGE Stability,” Eur. Phys. J. C 75 286 [arXiv:1501.07435 [hep-ph]].
[16] Branco G C, Grimus W and Lavoura L, 1996 “Relating the scalar flavor changing neutral couplings to the CKM matrix,” Phys. Lett. B 380 119 [hep-ph/9601383].
[17] Botella F J, Branco G C and Rebelo M N, 2010 “Minimal Flavour Violation and Multi-Higgs Models,” Phys. Lett. B 687 194 [arXiv:1011.1753 [hep-ph]].
[18] Botella F J, Branco G C, Nebot M and Rebelo M N, 2011 “Two-Higgs Leptonic Minimal Flavour Violation,” JHEP 1110 037 [arXiv:1102.0520 [hep-ph]].
[19] Bhattacharyya G, Das D, Pal P B and Rebelo M N, 2013 “Scalar sector properties of two-Higgs-doublet models with a global U(1) symmetry,” JHEP 1310 081 [arXiv:1308.4297 [hep-ph]].
[20] Botella F J, Branco G C, Carmona A, Nebot M, Pedro L and Rebelo M N, 2014 “Physical Constraints on a Class of Two-Doublet Higgs Models with FCNC at tree level,” JHEP 1407 078 [arXiv:1401.6147 [hep-ph]].
[21] Bhattacharyya G, Das D and Kundu A, 2014 “Feasibility of light scalars in a class of two-Higgs-doublet models and their decay signatures,” Phys. Rev. D 89 095029 [arXiv:1402.0364 [hep-ph]].
[22] Botella F J, Branco G C, Nebot M and Rebelo M N, 2016 “Flavour Changing Higgs Couplings in a Class of Two Higgs Doublet Models,” Eur. Phys. J. C 76 no.3, 161 [arXiv:1508.05101 [hep-ph]].
[23] Alves J M, Botella F J, Branco G C, Cornet-Gomez F and Nebot M, 2017 “Controlled Flavour Changing Neutral Couplings in Two Higgs Doublet Models,” arXiv:1703.03796 [hep-ph].
[24] Grzadkowski B, Ogreid O M and Olsand P, 2009 “Natural Multi-Higgs Model with Dark Matter and CP Violation,” Phys. Rev. D 80 055013 [arXiv:0904.2173 [hep-ph]].
[25] Grzadkowski B, Ogreid O M, Olsand P, Pukhov A and Purmohammadi M, 2011 “Exploring the CP-Violating Inert-Doublet Model,” JHEP 1106 003 [arXiv:1012.4680 [hep-ph]].
[26] Keus V, King S F, Moretti S and Sokolowska S, 2014 “Dark Matter with Two Inert Doublets plus One Higgs Doublet,” JHEP 1411 016 [arXiv:1407.7850 [hep-ph]].
[27] Machado A C B and Pleitez V, 2016 “A model with two inert scalar doublets,” Annals Phys. 364 53 [arXiv:1205.0995 [hep-ph]].
[28] Forcetas E C F S, Machado A C B, Montaño J and Pleitez V, 2015 “Scalar dark matter candidates in a two inert Higgs doublet model,” J. Phys. G 42 105003 [arXiv:1407.4749 [hep-ph]].
[29] Forcetas E C F S, Machado A C B, Montaño J and Pleitez V, 2015 “Prediction of $h \rightarrow \gamma Z$ from $h \rightarrow \gamma \gamma$ at LHC for the IMDS$_3$ Model,” J. Phys. G 42 115001 [arXiv:1408.0780 [hep-ph]].
[30] Olausen K, Olsand P and Solberg M A, 2011 “Symmetry and Mass Degeneration in Multi-Higgs-Doublet Models,” JHEP 1107 070 [arXiv:1107.1424 [hep-ph]].
[31] Ma E, 2006 “Verifiable radiative seesaw mechanism of neutrino mass and dark matter,” Phys. Rev. D 73 077301 [hep-ph/0601225].
[32] Barbieri R, Hall L J and Rychkov V S, 2006 “Improved naturalness with a heavy Higgs: An Alternative road to LHC physics,” Phys. Rev. D 74 015007 [hep-ph/0603188].
[33] Lopez Honorez L, Nezri E, Oliver J F and Tytgat M H G, 2007 “The Inert Doublet Model: An Archetype for Dark Matter,” JCAP 0702 028 [hep-ph/0612275].
[34] Barradas-Guevara E, Félix-Beltrán O and Rodríguez-Jáuregui E, 2015 “CP breaking in S(3) flavoured Higgs model,” arXiv:1507.05180 [hep-ph].
[35] Emmanuel-Costa D, Ogreid O M, Olsand P and Rebelo M N, 2016 “Spontaneous symmetry breaking in the $S_3$-symmetric scalar sector,” JHEP 1602 154 Erratum: [JHEP 1608 169] [arXiv:1601.04654 [hep-ph]].
[36] Barradas-Guevara E, Félix-Beltrán O and Rodríguez-Jáuregui E, 2016 “Higgs sector with spontaneous CP violation in S(3) Standard Model,” JNPMSRA, Vol. 4 [arXiv:1606.07773 [hep-ph]].
[37] Pakvasa S and Sugawara H, 1978 “Discrete symmetry and Cabibbo Angle,” Phys. Lett. 73B 61.
[38] Ma E, 2000 “Permutation symmetry for neutrino and charged lepton mass matrices,” Phys. Rev. D 61 033012 [hep-ph/9909249].
[39] Araki T, Kubo K and Paschos E A, 2006 “S(3) flavor symmetry and leptogenesis,” Eur. Phys. J. C 45 465 [hep-ph/0502164].
[40] Bhattacharyya G, Leser P and Pas H, 2011 “Exotic Higgs boson decay modes as a harbinger of $S_3$ flavor symmetry,” Phys. Rev. D 83 011701 doi:10.1103/PhysRevD.83.011701 [arXiv:1006.5597 [hep-ph]].

[41] Teshima T, 2012 “Higgs potential in $S_3$ invariant model for quark/lepton mass and mixing,” Phys. Rev. D 85 105013 [arXiv:1202.4528 [hep-ph]].

[42] Canales F G, Mondragón A, Mondragón M, Saldaña Salazar U J and Velasco-Sevilla L, 2013 “Fermion mixing in an $S_3$ model with three Higgs doublets,” J. Phys. Conf. Ser. 447 012053.

[43] Das D, Dey U K and Pal P B, 2016 “$S_3$ symmetry and the quark mixing matrix,” Phys. Lett. B 753 315 [arXiv:1507.06509 [hep-ph]].

[44] Cruz A A and Mondragón M, “Neutrino masses, mixing, and leptogenesis in an $S_3$ model,” arXiv:1701.07929 [hep-ph].

[45] Derman E, 1979 “Flavor Unification, $\tau$ Decay and $b$ Decay Within the Six Quark Six Lepton Weinberg-Salam Model,” Phys. Rev. D 19 317.

[46] Derman E and Tsao H S, 1979 “SU(2) X U(1) X $S(n)$ Flavor Dynamics and a Bound on the Number of Flavors,” Phys. Rev. D 20 1207.

[47] Kubo J, Okada H and Sakamaki F, 2004 “Higgs potential in minimal $S(3)$ invariant extension of the standard model,” Phys. Rev. D 70 036007 [hep-ph/0402089].

[48] Ivanov I P and Nishi C C, 2015 “Symmetry breaking patterns in 3HDM,” JHEP 1501 021 [arXiv:1410.6139 [hep-ph]].

[49] Harrison P F, Perkins D H and Scott W G, 2002 “Tri-bimaximal mixing and the neutrino oscillation data,” Phys. Lett. B 530 167 [hep-ph/0202074].

[50] Das D and Dey U K, 2014 “Analysis of an extended scalar sector with $S_3$ symmetry,” Phys. Rev. D 89 095002 Erratum: [Phys. Rev. D 91 (2015) no.3, 039905] [arXiv:1404.2491 [hep-ph]].

[51] Ivanov I P and Silva J P, 2016 “CP-conserving multi-Higgs model with irremovable complex coefficients,” Phys. Lett. B 753 187 [arXiv:1512.09276 [hep-ph]].

[52] Lavoura L and Silva J P, 1994 “Fundamental CP-violating quantities in a SU(2) x U(1) model with many Higgs doublets,” Phys. Rev. D 50 4619 [hep-ph/9404276].

[53] Branco G C, Rebelo M N and Silva-Marcos J I, 2005 “CP-odd invariants in models with several Higgs doublets,” Phys. Lett. B 614 187 [hep-ph/0502118].

[54] Gunion J F and Haber H E, 2005 “Conditions for CP-violation in the general two-Higgs-doublet model,” Phys. Rev. D 72 095002 [hep-ph/0506227].

[55] de Medeiros Varzielas I, King S F, Luhn C and Neder T, 2016 “CP-odd invariants for multi-Higgs models: applications with discrete symmetry,” Phys. Rev. D 94 056007 [arXiv:1603.06942 [hep-ph]].

[56] Branco G C, Lavoura L and Silva J P, 1999 CP Violation, Int. Ser. Monogr. Phys. 103 1.

[57] Branco G C, Gerard J M and Grimus W, 1984 “Geometrical T Violation,” Phys. Lett. B 136 383.

[58] Ogreid O M, Oslund P and Rebelo M N, 2017 “A Simple Method to detect spontaneous CP Violation in multi-Higgs models,” arXiv:1701.04768 [hep-ph].

[59] El Kaffas A W, Khater W, Ogreid O M and Oslund P, 2007 “Consistency of the two Higgs doublet model and CP violation in top production at the LHC,” Nucl. Phys. B 775 45 [hep-ph/0605142].

[60] Emmanuel-Costa D, Félix-Beltran O, Mondragon M and Rodriguez-Jauregui E, 2007 “Stability of the tree-level vacuum in a minimal S(3) extension of the standard model,” AIP Conf. Proc. 917 390.