Propagating torsion from first principles

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Abstract

A propagating torsion model is derived from the requirement of compatibility between minimal action principle and minimal coupling procedure in Riemann-Cartan spacetimes. In the proposed model, the trace of the torsion tensor is derived from a scalar potential that determines the volume element of the spacetime. The equations of the model are written down for the vacuum and for various types of matter fields. Some of their properties are discussed. In particular, we show that gauge fields can interact minimally with the torsion without the braking of gauge symmetry.
I. INTRODUCTION

Many “connection-dynamic” theories of gravity with propagating torsion have been proposed in the last decades. Contrary to the usual Einstein-Cartan (EC) gravity [1], in such theories one could in principle have long-range torsion mediated interactions. In the same period, we have also witnessed a spectacular progress in the experimental description of the solar system [2]. Many important tests using the parameterized post-Newtonian (PPN) formalism have been performed. Tight limits for the PPN parameters have been establishing and several alternatives theories to General Relativity (GR) have been ruled out. Indeed, such solar system experiments and also observations of the binary pulsar 1913 + 16 offer strong evidence that the metric tensor must not deviate too far from the predictions of GR [2]. Unfortunately, the situation with respect to the torsion tensor is much more obscure. The interest in experimental consequences of propagating torsion models has been revived recently [3,4]. Carroll and Field [3] have examined the observational consequences of propagating torsion in a wide class of models involving scalar fields. They conclude that for reasonable models the torsion must decay quickly outside matter distribution, leading to no long-range interaction which could be detected experimentally. Nevertheless, as also stressed by them, this does not mean that torsion has not relevance in Gravitational Physics.

Typically, in propagating torsion models the Einstein-Hilbert action is modified in order to induce a differential equation for the torsion tensor, allowing for non-vanishing torsion configurations to the vacuum. In almost all cases a dynamical scalar field is involved, usually related to the torsion trace or pseudo-trace. Such modifications are introduced in a rather arbitrary way; terms are added to the Lagrangian in order to produce previously desired differential equations for the torsion tensor.

The goal of this paper is to present a propagating torsion model obtained from first principles of EC theory. By exploring some basic features of the Einstein-Hilbert action in spacetimes with torsion we get a model with a new and a rather intriguing type of propagating torsion involving a non-minimally coupled scalar field. We write and discuss the metric
and torsion equations for the vacuum and in the presence of different matter fields. Our
model does not belong to the large class of models studied in [3]. The work is organized as
follows. Section II is a brief revision of Riemann-Cartan (RC) geometry, with special empha-
sis to the concept of parallel volume element. In the Section III, we show how a propagating
torsion model arises from elementary considerations on the compatibility between minimal
action principle and minimal coupling procedure. The Section IV is devoted to study of the
proposed model in the vacuum and in presence of various type of matter. Section V is left
to some concluding remarks.

II. RC MANIFOLDS AND PARALLEL VOLUME ELEMENTS

A RC spacetime is a differentiable four dimensional manifold endowed with a metric
tensor \( g_{\alpha\beta}(x) \) and with a metric-compatible connection \( \Gamma^\mu_{\alpha\beta} \), which is non-symmetrical in its
lower indices. We adopt in this work \( \text{sign}(g_{\mu\nu}) = (+, -, -, -) \). The anti-symmetric part of
the connection defines a new tensor, the torsion tensor,

\[
S^\gamma_{\alpha\beta} = \frac{1}{2} \left( \Gamma^\gamma_{\alpha\beta} - \Gamma^\gamma_{\beta\alpha} \right),
\]

(2.1)
The metric-compatible connection can be written as

\[
\Gamma^\gamma_{\alpha\beta} = \left\{^\gamma_{\alpha\beta} \right\} - K^\gamma_{\alpha\beta},
\]

(2.2)
where \( \left\{^\gamma_{\alpha\beta} \right\} \) are the usual Christoffel symbols and \( K^\gamma_{\alpha\beta} \) is the contorsion tensor, which is
given in terms of the torsion tensor by

\[
K^\gamma_{\alpha\beta} = -S^\gamma_{\alpha\beta} + S^\gamma_{\beta\alpha} - S^\gamma_{\alpha\beta},
\]

(2.3)
The connection (2.2) is used to define the covariant derivative of vectors,

\[
D_\nu A^\mu = \partial_\nu A^\mu + \Gamma^\mu_{\nu\rho} A^\rho,
\]

(2.4)
and it is also important to our purposes to introduce the covariant derivative of a density
\( f(x) \),
\[ D_\mu f(x) = \partial_\mu f(x) - \Gamma^\rho_{\mu\nu} f(x). \quad (2.5) \]

The contorsion tensor \( \text{[2.3]} \) can be covariantly split in a traceless part and in a trace,

\[ K_{\alpha\beta\gamma} = \tilde{K}_{\alpha\beta\gamma} - \frac{2}{3} (g_{\alpha\gamma} S_\beta - g_{\alpha\beta} S_\gamma), \quad (2.6) \]

where \( \tilde{K}_{\alpha\beta\gamma} \) is the traceless part and \( S_\beta \) is the trace of the torsion tensor, \( S_\beta = S^\alpha_{\alpha\beta} \). In four dimensions the traceless part \( \tilde{K}_{\alpha\beta\gamma} \) can be also decomposed in a pseudo-trace and a part with vanishing pseudo-trace, but for our purposes \( \text{[2.6]} \) is sufficient. The curvature tensor is given by:

\[ R^\beta_{\alpha\nu\mu} = \partial_\alpha \Gamma^\beta_{\nu\mu} - \partial_\nu \Gamma^\beta_{\alpha\mu} + \Gamma^\beta_{\alpha\rho} \Gamma^\rho_{\nu\mu} - \Gamma^\beta_{\nu\rho} \Gamma^\rho_{\alpha\mu}. \quad (2.7) \]

After some algebraic manipulations we get the following expression for the scalar of curvature \( R \), obtained from suitable contractions of \( \text{[2.7]} \),

\[ R \left( g_{\mu\nu}, \Gamma^\gamma_{\alpha\beta} \right) = g^{\mu\nu} R_{\alpha\mu\nu}^\gamma = \mathcal{R} - 4 D_\mu S^\mu + \frac{16}{3} S_\mu S^\mu - \tilde{K}_{\nu\rho\alpha} \tilde{K}^{\alpha\nu\rho}, \quad (2.8) \]

where \( \mathcal{R} \left( g_{\mu\nu}, \{\gamma_{\alpha\beta}\} \right) \) is the Riemannian scalar of curvature, calculated from the Christoffel symbols.

In order to define a general covariant volume element in a manifold, it is necessary to introduce a density quantity \( f(x) \) which will compensate the Jacobian that arises from the transformation law of the usual volume element \( d^4 x \) under a coordinate transformation,

\[ d^4 x \rightarrow f(x) d^4 x = d\text{vol}. \quad (2.9) \]

Usually, the density \( f(x) = \sqrt{-g} \) is took to this purpose. However, there are natural properties that a volume element shall exhibit. In a Riemannian manifold, the usual covariant volume element

\[ d\text{vol} = \sqrt{-g} d^4 x, \quad (2.10) \]

is parallel, in the sense that the scalar density \( \sqrt{-g} \) obeys

\[ D_\mu \sqrt{-g} = 0, \quad (2.11) \]
where $D_\mu$ is the covariant derivative defined using the Christoffel symbols. One can infer that the volume element (2.10) is not parallel when the spacetime is not torsionless, since

$$D_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \Gamma^\rho_{\mu\nu} \sqrt{-g} = -2S_\mu \sqrt{-g},$$

as it can be checked using Christoffel symbols properties. This is the main point that we wish to stress, it will be the basic argument to our claim that the usual volume element (2.10) is not the most appropriate one in the presence of torsion, as it will be discussed in the next section.

The question that arises now is if it is possible to define a parallel volume element in RC manifolds. In order to do it, one needs to find a density $f(x)$ such that $D_\mu f(x) = 0$. Such density exists only if the trace of the torsion tensor, $S_\mu$, can be obtained from a scalar potential

$$S_\beta(x) = \partial_\beta \Theta(x),$$

and in this case we have $f(x) = e^{2\Theta} \sqrt{-g}$, and

$$d\text{vol} = e^{2\Theta} \sqrt{-g} \ d^4 x,$$

that is the parallel RC volume element, or in another words, the volume element (2.14) is compatible with the connection in RC manifolds obeying (2.13). It is not usual to find in the literature applications where volume elements different from the canonical one are used. Non-standard volume elements have been used in the characterization of half-flats solutions of Einstein equations, in the description of field theory on Riemann-Cartan spacetimes and of dilatonic gravity, and in the study of some aspects of BRST symmetry. In our case the new volume element appears naturally; in the same way that we require compatibility conditions between the metric tensor and the linear connection we can do it for the connection and volume element.

With the volume element (2.14), we have the following generalized Gauss’ formula

$$\int d\text{vol} \ D_\mu V^\mu = \int d^4 x \partial_\mu e^{2\Theta} \sqrt{-g} V^\mu = \text{surface term},$$

(2.15)
where we used that
\[
\Gamma^\rho_{\mu} = \partial_\mu \ln e^{2\theta} \sqrt{-g}
\] (2.16)
under the hypothesis (2.13). It is easy to see that one cannot have a generalized Gauss’
formula of the type (2.15) if the torsion does not obey (2.13). We will return to discuss the
actual role of the condition (2.13) in the last section.

III. MINIMAL COUPLING PROCEDURE AND MINIMAL ACTION PRINCIPLE

As it was already said, our model arises from elementary considerations on the minimal
coupling procedure and minimal action principle. Minimal coupling procedure (MCP) pro-
vides us with an useful rule to get the equations for any physical field on non-Minkowskian
manifolds starting from their versions of Special Relativity (SR). When studying classical
fields on a non-Minkowskian manifold \(X\) we usually require that the equations of motion
for such fields have an appropriate SR limit. There are, of course, infinitely many covariant
equations on \(X\) with the same SR limit, and MCP solves this arbitrariness by saying that
the relevant equations should be the “simplest” ones. MCP can be heuristically formulated
as follows. Considering the equations of motion for a classical field in the SR, one can get
their version for a non-Minkowskian spacetime \(X\) by changing the partial derivatives by the
\(X\) covariant ones and the Minkowski metric tensor by the \(X\) one. MCP is also used for the
classical and quantum analysis of gauge fields, where the gauge field is to be interpreted as
a connection, and it is in spectacular agreement with experience for QED an QCD.

Suppose now that the SR equations of motion for a classical field follow from an action
functional via minimal action principle (MAP). It is natural to expect that the equations
obtained by using MCP to the SR equations coincide with the Euler-Lagrange equations of
the action obtained via MCP of the SR one. This can be better visualized with the help of
where $E(\mathcal{L})$ stands to the Euler-Lagrange equations for the Lagrangian $\mathcal{L}$, and $C_L$ is the equivalence class of Lagrangians, $\mathcal{L}'$ being equivalent to $\mathcal{L}$ if $E(\mathcal{L}') = E(\mathcal{L})$. We restrict ourselves to the case of non-singular Lagrangians. The diagram (3.1) is verified in GR. We say that MCP is compatible with MAP if (3.1) holds. We stress that if (3.1) does not hold we have another arbitrariness to solve, one needs to choose one between two equations, as we will shown with a simple example.

It is not difficulty to check that in general MCP is not compatible with MAP when spacetime is assumed to be non-Riemannian. Let us examine for simplicity the case of a massless scalar field $\varphi$ in the frame of Einstein-Cartan gravity [5]. The equation for $\varphi$ in SR is

$$\partial_\mu \partial^\mu \varphi = 0,$$

which follows from the extremals of the action

$$S_{SR} = \int d\text{vol} \eta^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi.$$  

Using MCP to (3.3) one gets

$$S_X = \int d\text{vol} g^{\mu\nu} \partial_\mu \varphi \partial_\nu \varphi,$$

and using the Riemannian volume element for $X$, $d\text{vol} = \sqrt{g} dx$, we get the following equation from the extremals of (3.4)

$$\frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu \varphi = 0.$$  

It is clear that (3.3) does not coincide in general with the equation obtained via MCP of (3.2)
\[
\partial_\mu \partial^\mu \varphi + \Gamma^\mu_{\mu\alpha} \partial^\alpha \varphi = \frac{1}{\sqrt{g}} \partial_\mu \sqrt{g} \partial^\mu \varphi + 2 \Gamma^\mu_{[\mu\alpha]} \partial^\alpha \varphi = 0. \tag{3.6}
\]

We have here an ambiguity, the equations (3.5) and (3.6) are in principle equally acceptable ones, to choose one of them corresponds to choose as more fundamental the equations of motion or the action formulation from MCP point of view. As it was already said, we do not have such ambiguity when spacetime is assumed to be a Riemannian manifold. This is not a feature of massless scalar fields, all matter fields have the same behaviour in the frame of Einstein-Cartan gravity.

An accurate analysis of the diagram (3.1) reveals that the source of the problems of compatibility between MCP and MAP is the volume element of \( \mathcal{X} \). The necessary and sufficient condition to the validity of (3.1) is that the equivalence class of Lagrangians \( \mathcal{C}_\mathcal{L} \) be preserved under MCP. With our definition of equivalence we have that

\[
\mathcal{C}^\mathcal{L}_{\mathcal{SR}} \equiv \{ \mathcal{L}'_{\mathcal{SR}} | \mathcal{L}'_{\mathcal{SR}} - \mathcal{L}_{\mathcal{SR}} = \partial_\mu V^\mu \}, \tag{3.7}
\]

where \( V^\mu \) is a vector field. The application of MCP to the divergence \( \partial_\mu V^\mu \) in (3.7) gives \( D_\mu V^\mu \), and in order to the set

\[
\{ \mathcal{L}'_{\mathcal{X}} | \mathcal{L}'_{\mathcal{X}} - \mathcal{L}_{\mathcal{X}} = D_\mu V^\mu \} \tag{3.8}
\]

be an equivalence class one needs to have a Gauss-like law like (2.15) associated to the divergence \( D_\mu V^\mu \). As it was already said in Section II, the necessary and sufficient condition to have such a Gauss law is that the trace of the torsion tensor obeys (2.13).

With the use of the parallel volume element in the action formulation for EC gravity we can have qualitatively different predictions. The scalar of curvature (2.8) involves terms quadratic in the torsion. Due to (2.13) such quadratic terms will provide a differential equation for \( \Theta \), what will allow for non-vanishing torsion solutions for the vacuum. As to the matter fields, the use of the parallel volume element, besides of guarantee that the diagram (3.1) holds, brings also qualitative changes. For example, it is possible to have a minimal interaction between Maxwell fields and torsion preserving gauge symmetry. The
next section is devoted to the study of EC equations obtained by using the parallel volume element (2.14).

**IV. THE MODEL**

Now, EC gravity will be reconstructed by using the results of the previous sections. Spacetime will be assumed to be a Riemann-Cartan manifold with the parallel volume element (2.14), and of course, it is implicit the restriction that the trace of the torsion tensor is derived from a scalar potential, condition (2.13). With this hypothesis, EC theory of gravity will predict new effects, and they will be pointed out in the following subsections.

**A. Vacuum equations**

According to our hypothesis, in order to get the EC gravity equations we will assume that they can be obtained from an Einstein-Hilbert action using the scalar of curvature (2.8), the condition (2.13), and the volume element (2.14),

\[
S_{\text{grav}} = - \int d^4x e^{2\Theta} \sqrt{-g} R
\]

\[= - \int d^4x e^{2\Theta} \sqrt{-g} \left( R + \frac{16}{3} \partial_\mu \Theta \partial^\mu \Theta - \tilde{K}_{\nu\rho\alpha} \tilde{K}^{\nu\rho\alpha} \right) + \text{surf. terms}, \tag{4.1}
\]

where the generalized Gauss’ formula (2.15) was used.

The equations for the \(g^{\mu\nu}\), \(\Theta\), and \(\tilde{K}_{\nu\rho\alpha}\) fields follow from the extremals of the action (4.1). The variations of \(g^{\mu\nu}\) and \(S_{\mu\nu}^{\rho}\) are assumed to vanish in the boundary. The equation \(\frac{\delta S_{\text{grav}}}{\delta \tilde{K}_{\nu\rho\alpha}} = 0\) implies that \(\tilde{K}_{\nu\rho\alpha} = 0\), \(\frac{\delta S_{\text{grav}}}{\delta \tilde{K}_{\nu\rho\alpha}}\) standing for the Euler-Lagrange equations for \(\delta \tilde{K}_{\nu\rho\alpha}\).

For the other equations we have

\[- e^{-2\Theta} \frac{\delta}{\sqrt{-g}} \frac{\delta g^{\mu\nu}}{\delta g_{\mu\nu}} S_{\text{grav}} \bigg|_{\tilde{K}=0} = R_{\mu\nu} - 2D_\mu \partial_\nu \Theta \]

\[- \frac{1}{2} g_{\mu\nu} \left( R + \frac{8}{3} \partial_\rho \Theta \partial^\rho \Theta - 4 \Box \Theta \right) = 0, \tag{4.2}
\]

\[- e^{-2\Theta} \frac{\delta}{2\sqrt{-g}} \frac{\delta S_{\text{grav}}}{\delta \Theta} \bigg|_{\tilde{K}=0} = R + \frac{16}{3} (\partial_\mu \Theta \partial^\mu \Theta - \Box \Theta) = 0, \]
where $\mathcal{R}_{\mu\nu} \left( g_{\mu\nu}, \{^{\gamma}_{\alpha\beta} \} \right)$ is the usual Ricci tensor, calculated using the Christoffel symbols, and $\Box = D_\mu D^\mu$.

Taking the trace of the first equation of (4.2),

$$\mathcal{R} + \frac{16}{3} \partial_\mu \Theta \partial^\mu \Theta = 6 \Box \Theta,$$

(4.3)

and using it, one finally obtains the equations for the vacuum,

$$\mathcal{R}_{\mu\nu} = 2 D_\mu \partial_\nu \Theta - \frac{4}{3} g_{\mu\nu} \partial^\rho \Theta \partial^\rho \Theta = 2 D_\mu S_\nu - \frac{4}{3} g_{\mu\nu} S_\rho S_\rho,$$

$$\Box \Theta = \frac{e^{-2\Theta}}{\sqrt{-g}} \partial_\mu e^{2\Theta} \sqrt{-g} \partial^\mu \Theta = D_\mu S^\mu = 0,$$  

(4.4)

$$\tilde{K}_{\alpha\beta\gamma} = 0.$$ 

The vacuum equations (4.4) point out new features of our model. It is clear that torsion, described by the last two equations, propagates. The torsion mediated interactions are not of contact type anymore. The traceless tensor $\tilde{K}_{\alpha\beta\gamma}$ is zero for the vacuum, and only the trace $S_\mu$ can be non-vanishing outside matter distributions. As it is expected, the gravity field configuration for the vacuum is determined only by boundary conditions, and if due to such conditions we have that $S_\mu = 0$, our equations reduce to the usual vacuum equations, $S_{\alpha\gamma\beta} = 0$, and $\mathcal{R}_{\alpha\beta} = 0$. Note that this is the case if one considers particle-like solutions (solutions that go to zero asymptotically). Equations (4.4) are valid only to the exterior region of the sources. For a discussion to the case with sources see [11].

The first term in the right-handed side of the first equation of (4.4) appears to be non-symmetrical under the change ($\mu \leftrightarrow \nu$), but in fact it is symmetrical as one can see using (2.13) and the last equation of (4.4). Of course that if $\tilde{K}_{\alpha\beta\gamma} \neq 0$ such term will be non-symmetrical, and this is the case when fermionic fields are present, as we will see.

It is not difficult to generate solutions for (4.4) starting from the well-known solutions of the minimally coupled scalar-tensor gravity [12].
B. Scalar fields

The first step to introduce matter fields in our discussion will be the description of scalar fields on RC manifolds. In order to do it, we will use MCP according to Section II. For a massless scalar field one gets

$$S = S_{grav} + S_{scal} = - \int d^4x e^{2\Theta} \sqrt{-g} \left( R - \frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi \right)$$

(4.5)

$$= - \int d^4x e^{2\Theta} \sqrt{-g} \left( \mathcal{R} + \frac{16}{3} \partial_\mu \Theta \partial^\mu \Theta - \tilde{K}_{\nu\rho} \tilde{K}^{\nu\rho} - \frac{g^{\mu\nu}}{2} \partial_\mu \phi \partial_\nu \phi \right),$$

where surface terms were discarded. The equations for this case are obtained by varying (4.5) with respect to $\phi$, $g^{\mu\nu}$, $\Theta$, and $\tilde{K}_{\alpha\beta\gamma}$. As in the vacuum case, the equation $\frac{\delta S}{\delta \tilde{K}} = 0$ implies $\tilde{K} = 0$. Taking it into account we have

$$- e^{-2\Theta} \frac{\delta S}{\delta \phi} \bigg|_{K=0} = \frac{e^{-2\Theta}}{\sqrt{-g}} \partial_\mu e^{2\Theta} \sqrt{-g} \partial^\mu \phi = \square \phi = 0,$$

$$- e^{-2\Theta} \frac{\delta S}{\delta g^{\mu\nu}} \bigg|_{K=0} = \mathcal{R}_{\mu\nu} - 2D_\mu S_\nu - \frac{1}{2} g_{\mu\nu} \left( \mathcal{R} + \frac{8}{3} S_\rho S^\rho - 4D_\rho S^\rho \right)$$

$$- \frac{1}{2} \partial_\mu \phi \partial_\nu \phi + \frac{1}{4} g_{\mu\nu} \partial_\rho \phi \partial^\rho \phi = 0,$$

(4.6)

$$- e^{-2\Theta} \frac{\delta S}{\delta \Theta} \bigg|_{K=0} = \mathcal{R} + \frac{16}{3} \left( S_\mu S^\mu - D_\mu S^\mu \right) - \frac{1}{2} \partial_\mu \phi \partial^\mu \phi = 0.$$

Taking the trace of the second equation of (4.6),

$$\mathcal{R} + \frac{16}{3} S_\mu S^\mu = 6D_\mu S^\mu + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi,$$

(4.7)

and using it, we get the following set of equations for the massless scalar case

$$\square \phi = 0,$$

$$\mathcal{R}_{\mu\nu} = 2D_\mu S_\nu - \frac{4}{3} g_{\mu\nu} S_\rho S^\rho + \frac{1}{2} \partial_\mu \phi \partial_\nu \phi,$$

(4.8)

$$D_\mu S^\mu = 0,$$

$$\tilde{K}_{\alpha\beta\gamma} = 0.$$

As one can see, the torsion equations have the same form than the ones of the vacuum case (4.4). Any contribution to the torsion will be due to boundary conditions, and not due
to the scalar field itself. It means that if such boundary conditions imply that $S_\mu = 0$, the equations for the fields $\varphi$ and $g_{\mu\nu}$ will be the same ones of the GR. One can interpret this by saying that, even feeling the torsion (see the second equation of (4.8)), massless scalar fields do not produce it. Such behavior is compatible with the idea that torsion must be governed by spin distributions.

However, considering massive scalar fields,

$$S_{\text{scal}} = \int d^4 x e^{2\Theta} \sqrt{-g} \left( \frac{g^{\mu\nu}}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{m^2}{2} \varphi^2 \right), \quad (4.9)$$

we have the following set of equations instead of (4.8)

$$(\Box + m^2) \varphi = 0,$$

$$\mathcal{R}_{\mu\nu} = 2D_\mu S_\nu - \frac{4}{3} g_{\mu\nu} S_\rho S^\rho + \frac{1}{2} \partial_\mu \varphi \partial_\nu \varphi - \frac{1}{2} g_{\mu\nu} m^2 \varphi^2, \quad (4.10)$$

$$D_\mu S^\mu = \frac{3}{4} m^2 \varphi^2,$$

$$\tilde{K}_{\alpha\beta\gamma} = 0.$$  

The equation for the trace of the torsion tensor is different than the one of the vacuum case, we have that massive scalar field couples to torsion in a different way than the massless one. In contrast to the massless case, the equations (4.10) do not admit as solution $S_\mu = 0$ for non-vanishing $\varphi$ (Again for particle-like solutions we have $\phi = 0$ and $S_\mu = 0$). This is in disagreement with the traditional belief that torsion must be governed by spin distributions. We will return to this point in the last section.

C. Gauge fields

We need to be careful with the use of MCP to gauge fields. We will restrict ourselves to the abelian case in this work, non-abelian gauge fields will bring some technical difficulties that will not contribute to the understanding of the basic problems of gauge fields on Riemann-Cartan spacetimes.

Maxwell field can be described by the differential 2-form
\[ F = dA = d(A_\alpha dx^\alpha) = \frac{1}{2} F_{\alpha\beta} dx^\alpha \wedge dx^\beta, \quad (4.11) \]

where \( A \) is the (local) potential 1-form, and \( F_{\alpha\beta} = \partial_\alpha A_\beta - \partial_\beta A_\alpha \) is the usual electromagnetic tensor. It is important to stress that the forms \( F \) and \( A \) are covariant objects in any differentiable manifolds. Maxwell equations can be written in Minkowski spacetime in terms of exterior calculus as

\[ dF = 0, \quad (4.12) \]
\[ d^* F = 4\pi^* J, \]

where \( ^* \) stands for the Hodge star operator and \( J \) is the current 1-form, \( J = J_\alpha dx^\alpha \). The first equation in (4.12) is a consequence of the definition (4.11) and of Poincaré’s lemma. In terms of components, one has the familiar homogeneous and non-homogeneous Maxwell’s equations,

\[ \partial_{[\gamma} F_{\alpha\beta]} = 0, \quad (4.13) \]
\[ \partial_\mu F^{\mu\nu} = 4\pi J^\nu, \]

where \( [ \quad ] \) means antisymmetrization. We know also that the non-homogenous equation follows from the extremals of the following action

\[ S = - \int \left( 4\pi J^\nu A_\alpha - \frac{1}{2} F_{\alpha\beta} F^{\alpha\beta} \right) dx^4. \quad (4.14) \]

If one tries to cast (4.14) in a covariant way by using MCP in the tensorial quantities, we have that Maxwell tensor will be given by

\[ F_{\alpha\beta} \rightarrow \tilde{F}_{\alpha\beta} = F_{\alpha\beta} - 2S_{\alpha\beta}^\rho A_\rho, \quad (4.15) \]

which explicitly breaks gauge invariance. With this analysis, one usually arises the conclusion that gauge fields cannot interact minimally with Einstein-Cartan gravity. We would stress another undesired consequence, also related to the breaking of gauge symmetry, of the use of MCP in the tensorial quantities. The homogeneous Maxwell equation, the first
of (4.13), does not come from a Lagrangian, and of course, if we choose to use MCP in the tensorial quantities we need also apply MCP to it. We get

$$\partial_{\alpha} \tilde{F}_{\beta\gamma} + 2 S_{[\alpha}{}^{\rho} \tilde{F}_{\beta\gamma]}{}_{\rho} = 0,$$

(4.16)

where $\tilde{F}_{\alpha\beta}$ is given by (4.13). One can see that (4.16) has no general solution for arbitrary $S_{\alpha\beta}{}^{\rho}$. Besides the breaking of gauge symmetry, the use of MCP in the tensorial quantities also leads to a non-consistent homogeneous equation.

However, MCP can be successfully applied for general gauge fields (abelian or not) in the differential form quantities [7]. As consequence, one has that the homogeneous equation is already in a covariant form in any differentiable manifold, and that the covariant non-homogeneous equations can be gotten from a Lagrangian obtained only by changing the metric tensor and by introducing the parallel volume element in the Minkowskian action (4.14). Considering the case where $J^\mu = 0$, we have the following action to describe the interaction of Maxwell fields and Einstein-Cartan gravity

$$S = S_{\text{grav}} + S_{\text{Maxw}} = -\int d^4x e^{2\Theta} \sqrt{-g} \left( R + \frac{1}{4} F_{\mu\nu} F^{\mu\nu} \right).$$

(4.17)

As in the previous cases, the equation $\tilde{K}_{\alpha\beta\gamma} = 0$ follows from the extremals of (4.17). The other equations will be

$$e^{-2\Theta} \sqrt{-g} \partial_{\mu} e^{2\Theta} \sqrt{-g} F^{\mu\nu} = 0,$$

$$R_{\mu\nu} = 2 D_{\mu} S_{\nu} - \frac{4}{3} g_{\mu\nu} S_\rho S^\rho - \frac{1}{2} \left( F_{\mu\alpha} F_{\nu}{}^{\alpha} + \frac{1}{2} g_{\mu\nu} F_{\omega\rho} F^{\omega\rho} \right),$$

$$D_{\mu} S^\mu = -\frac{3}{8} F_{\mu\nu} F^{\mu\nu}.$$

(4.18)

One can see that the equations (4.18) are invariant under the usual $U(1)$ gauge transformations. It is also clear from the equations (4.18) that Maxwell fields can interact with the non-Riemannian structure of spacetime. Also, as in the massive scalar case, the equations do not admit as solution $S_\mu = 0$ for arbitrary $F_{\alpha\beta}$, Maxwell fields are also sources to the spacetime torsion. Similar results can be obtained also for non-abelian gauge fields [4].
D. Fermion fields

The Lagrangian for a (Dirac) fermion field with mass \( m \) in the Minkowski spacetime is given by

\[
\mathcal{L}_F = \frac{i}{2} \left( \overline{\psi} \gamma^a \partial_a \psi - \left( \partial_a \overline{\psi} \right) \gamma^a \psi \right) - m \overline{\psi} \psi, \tag{4.19}
\]

where \( \gamma^a \) are the Dirac matrices and \( \overline{\psi} = \psi^\dagger \gamma^0 \). Greek indices denote spacetime coordinates (holonomic), and roman ones locally flat coordinates (non-holonomic). It is well known [1] that in order to cast (4.19) in a covariant way, one needs to introduce the vierbein field, \( e^\mu_a(x) \), and to generalize the Dirac matrices, \( \gamma^\mu(x) = e^\mu_a(x) \gamma^a \). The partial derivatives also must be generalized with the introduction of the spinorial connection \( \omega_\mu \),

\[
\partial_\mu \psi \to \nabla_\mu \psi = \partial_\mu \psi + \omega_\mu \psi,
\]

\[
\partial_\mu \overline{\psi} \to \nabla_\mu \overline{\psi} = \partial_\mu \overline{\psi} - \overline{\psi} \omega_\mu, \tag{4.20}
\]

where the spinorial connection is given by

\[
\omega_\mu = \frac{1}{8} [\gamma^a, \gamma^b] e_\nu^a \left( \partial_\mu e_{\nu b} - \Gamma^\rho_{\mu \nu} e^b_\rho \right).
\]

where

\[
\omega_\mu = \frac{1}{8} \left( \gamma^\nu \partial_\mu \gamma_\nu - (\partial_\mu \gamma_\nu) \gamma^\nu - [\gamma^\nu, \gamma_\rho] \Gamma^\rho_{\mu \nu} \right).
\]

The last step, according to our hypothesis, shall be the introduction of the parallel volume element, and after that one gets the following action for fermion fields on RC manifolds

\[
S_F = \int d^4xe^{2\Theta} \sqrt{-g} \left\{ \frac{i}{2} \left( \overline{\psi} \gamma^\mu(x) \nabla_\mu \psi - \left( \nabla_\mu \overline{\psi} \right) \gamma^\mu(x) \psi \right) - m \overline{\psi} \psi \right\}. \tag{4.22}
\]

Varying the action (4.22) with respect to \( \overline{\psi} \) one obtains:

\[
\frac{e^{-2\Theta} \delta S_F}{\sqrt{-g} \delta \overline{\psi}} = \frac{i}{2} (\gamma^\mu \nabla_\mu \psi + \omega_\mu \gamma^\mu \psi) - m \psi + \frac{i}{2} e^{-2\Theta} \partial_\mu e^{2\Theta} \sqrt{-g} \gamma^\mu \psi = 0. \tag{4.23}
\]

Using the result

\[
[\omega_\mu, \gamma^\nu] \psi = - \left( \frac{e^{-2\Theta}}{\sqrt{-g}} \partial_\mu e^{2\Theta} \sqrt{-g} \gamma^\mu \right) \psi, \tag{4.24}
\]
that can be check using (4.21), (2.16), and properties of ordinary Dirac matrices and of the vierbein field, we get the following equation for $\psi$ on a RC spacetime:

$$i\gamma^\mu(x)\nabla_\mu \psi - m\psi = 0. \quad (4.25)$$

The equation for $\overline{\psi}$ can be obtained in a similar way,

$$i\left(\nabla_\mu \overline{\psi}\right)\gamma^\mu(x) + m\overline{\psi} = 0. \quad (4.26)$$

We can see that the equations (4.25) and (4.26) are the same ones that arise from MCP used in the minkowskian equations of motion. In the usual EC theory, the equations obtained from the action principle do not coincide with the equations gotten by generalizing the minkowskian ones. This is another new feature of the proposed model.

The Lagrangian that describes the interaction of fermion fields with the Einstein-Cartan gravity is

$$S = S_{\text{grav}} + S_F \quad (4.27)$$

$$= -\int d^4x e^{2\Theta} \sqrt{-g} \left\{ R - \frac{i}{2} \left( \overline{\psi} \gamma^\mu \partial_\mu \psi - \left( \partial_\mu \overline{\psi} \right) \gamma^\mu \psi \right\}$$

$$\quad + \overline{\psi} \left[ \gamma^\mu, \omega_\mu \right] \psi + m\overline{\psi} \psi \}$$

$$= -\int d^4x e^{2\Theta} \sqrt{-g} \left\{ R - \frac{i}{2} \left( \overline{\psi} \gamma^\mu \partial_\mu \psi - \left( \partial_\mu \overline{\psi} \right) \gamma^\mu \psi \right\}$$

$$\quad + \overline{\psi} \left[ \gamma^\mu, \tilde{\omega}_\mu \right] \psi - \frac{i}{8} \overline{\psi} \tilde{K}_{\mu\nu\omega} \gamma^{[\mu, \gamma^\nu, \gamma^\omega]} \psi + m\overline{\psi} \psi \right\},$$

where it was used that $\gamma^a \left[ \gamma^b, \gamma^c \right] + \left[ \gamma^b, \gamma^c \right] \gamma^a = 2\gamma^{[a} \gamma^b \gamma^{c]}$, and that

$$\omega_\mu = \tilde{\omega}_\mu + \frac{1}{8} \tilde{K}_{\mu\nu\rho} \left[ \gamma^\nu, \gamma^\rho \right], \quad (4.28)$$

where $\tilde{\omega}_\mu$ is the Riemannian spinorial connection, calculated by using the Christoffel symbols instead of the full connection in (4.21).

The peculiarity of fermion fields is that one has a non-trivial equation for $\tilde{K}$ from (4.27). The Euler-Lagrange equations for $\tilde{K}$ is given by

$$\frac{e^{-2\Theta}}{\sqrt{-g}} \delta S \delta \tilde{K} = \tilde{K}^{\mu\nu\omega} + \frac{i}{8} \overline{\psi} \left[ \gamma^\mu, \gamma^\nu, \gamma^\omega \right] \psi = 0. \quad (4.29)$$
Differently from the previous cases, we have that the traceless part of the contorsion tensor, \( \tilde{K}_{\alpha\beta\gamma} \), is proportional to the spin distribution. It is still zero outside matter distribution, since its equation is an algebraic one, it does not allow propagation. The other equations follow from the extremals of (4.27). The main difference between these equations and the usual ones obtained from standard EC gravity, is that in the present case one has non-trivial solution for the trace of the torsion tensor, that is derived from \( \Theta \). In the standard EC gravity, the torsion tensor is a totally anti-symmetrical tensor and thus it has a vanishing trace.

**V. FINAL REMARKS**

In this section, we are going to discuss the role of the condition (2.13) and the source for torsion in the proposed model. The condition (2.13) is the necessary condition in order to be possible the definition of a parallel volume element on a manifold. Therefore, we have that our approach is restrict to spacetimes which admits such volume elements. We automatic have this restriction if we wish to use MAP in the sense discussed in Section II. Although it is not clear how to get EC gravity equations without using a minimal action principle, we can speculate about matter fields on spacetimes not obeying (2.13). Since it is not equivalent to use MCP in the equations of motion or in the action formulation, we can forget the last and to cast the equations of motion for matter fields in a covariant way directly. It can be done easily, as example, for scalar fields [3]. We get the equation (3.6), which is, apparently, a consistent equation. However, we need to define a inner product for the space of the solutions of (3.6) [13], and we are able to do it only if (2.13) holds. We have that the dynamics of matter fields requires some restrictions to the non-riemannian structure of spacetime, namely, the condition (2.13). This is more evident for gauge fields, where (2.13) arises directly as an integrability condition for the equations of motion [7]. It seems that condition (2.13) cannot be avoided.

We could realize from the matter fields studied that the trace of the torsion tensor is not
directly related to spin distributions. This is a new feature of the proposed model, and we naturally arrive to the following question: What is the source of torsion? The situation for the traceless part of the torsion tensor is the same that one has in the standard EC theory, only fermion fields can be sources to it. As to the trace part, it is quite different. Take for example $\tilde{K}_{\alpha \beta \gamma} = 0$, that corresponds to scalar and gauge fields. In this case, using the definition of the energy-momentum tensor

$$e^{-2\Theta} \frac{\delta S_{\text{mat}}}{\sqrt{-g}} \delta g^{\mu \nu} = -\frac{1}{2} T_{\mu \nu}, \quad (5.1)$$

and that for scalar and gauge fields we have

$$e^{-2\Theta} \frac{\delta S_{\text{mat}}}{\sqrt{-g}} \delta \Theta = 2 \mathcal{L}_{\text{mat}}, \quad (5.2)$$

one gets

$$D_{\mu} S^\mu = \frac{3}{2} \left( \mathcal{L}_{\text{mat}} - \frac{1}{2} T \right), \quad (5.3)$$

where $T$ is the trace of the energy-momentum tensor. The quantity between parenthesis, in general, has nothing to do with spin, and it is the source for a part of the torsion, confirming that in our model part of torsion is not determined by spin distributions. See also [11] for a discussion on possible source terms to the torsion.

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