Willmore Function on Curvatures of The Curve-Surface Pair Under Mobius Transformation

Filiz Ertem Kaya

ABSTRACT: We find a geometric invariant of the curve-surface pairs on Willmore functions with the mean and Gauss curvatures. Similar to the work in [5,19], in this work, we define Willmore functions on curve-surface pair and give new characterizations about Willmore functions with necessary and sufficient condition with strip theory in Euclidean 3-space for the first time. In this paper Willmore function on curvatures of the curve-surface pair under Möbius transformation is provided invariant.

Key Words: Curve-surface pair, mobius transformation, curvature, Willmore function.

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1. Introduction

Möbius differential geometry is a classical subject that was extensively developed in the nineteenth and early twentieth centuries, culminating with the publication of Blascke’s Vorlesungen über Differentialgeometrie III:Differentialgeometrie der Kreise und Kugeln [3] in 1929.

In 3-dimensional Euclidean Space, a regular curve is described by its curvatures $k_1$ and $k_2$ and also a curve-surface pair is described by its curvatures $k_n$, $k_g$ and $t_r$. The relations between the curvatures of a curve-surface pair and the curvatures of the curve can be seen in many differential books and papers.

Möbius transformations are the automorphisms of the extended complex plane $\mathbb{C}_\infty : \mathbb{C} \cup \{\infty\}$, that is the metamorphic bijections [5,9].

Several authors including Fubini [21], Thomsen [22] and White [23] have proven that the two form $H^2 - K dA$ is Möbius invariant. It is called Willmore functional [5,19].

In this paper we provide that Willmore function on curve-surface of the curve-surface pair under Möbius transformation is invariant.

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2. The Curve-Surface pairs

In this section, we give some basic definitions from differential geometry and curve-surface pairs.

**Definition 2.1.** Let $M$ and $\alpha$ be a surface in $E^3$ and a curve in $M \subset E^3$. We define a surface element of $M$ is the part of a tangent plane at the neighbour of the point. The locus of these surface element along the curve $\alpha$ is called a curve-surface pair and is shown as $(\alpha, M)$.

**Definition 2.2.** Let \( \{\vec{t}, \vec{n}, \vec{b}\} \) and \( \{\vec{\xi}, \vec{\eta}, \vec{\zeta}\} \) be the curve and curve-surface pair’s vector fields. The curve-surface pair’s tangent vector field, normal vector field and binormal vector field is given by

\[
\vec{t} = \vec{\xi}, \quad \vec{\eta} = \vec{\zeta} - \vec{\xi} \times \vec{\eta} \quad \text{[7, 10 – 18]}
\]

2.0.1. Curvatures of the curve-surface pair and Curvatures of the Curve. Let $k_n = -b$, $k_g = c$, $t_r = a$ be the normal curvature, the geodesic curvature, the geodesic torsion of the strip. Let \( \{\vec{\xi}, \vec{\eta}, \vec{\zeta}\} \) be the curve-surface pair’s vector fields on $\alpha$. Then we have

\[
\begin{align*}
\vec{\xi}' &= c\vec{\eta} - b\vec{\zeta} \\
\vec{\eta}' &= -c\vec{\xi} + a\vec{\zeta} \\
\vec{\zeta}' &= b\vec{\xi} - a\vec{\eta}
\end{align*}
\]

(3)

We know that a curve $\alpha$ has two curvatures $\kappa$ and $\tau$. A curve has a strip and a strip has three curvatures $k_n$, $k_g$ and $t_r$. Let $k_n$, $k_g$ and $t_r$ be the -b, c, a [4, 6]. From (3) we have $\vec{\xi} = c\vec{\eta} - b\vec{\zeta}$. If we substitute $\vec{\zeta} = \vec{t}$ in last equation, we obtain

\[
\vec{\xi}' = \kappa\vec{n}
\]

and

\[
\begin{align*}
b &= -\kappa \sin \varphi \\
c &= \kappa \cos \varphi
\end{align*}
\]

[7, 8, 10 – 18]. From last two equations we obtain,

\[
\kappa^2 = b^2 + c^2.
\]

This equation is a relation between the curvature $\kappa$ of a curve $\alpha$ and normal curvature and geodesic curvature of a curve-surface pair [4, 5, 7, 10 – 18].

By using similar operations, we obtain a new equation as follows

\[
\tau = a + \frac{bc - b\dot{c}}{b^2 + c^2}
\]

([4, 5, 7, 10 – 18]). This equation is a relation between $\tau$ (torsion or second curvature of $\alpha$) and $a, b, c$ curvatures of a curve-surface pair that belongs to the curve $\alpha$.

And also we can write

\[
a = \dot{\varphi} + \tau.
\]

**The special case:** if $\varphi$ =constant, then $\dot{\varphi} = 0$. So the equation is $a = \tau$. That is, if the angle is constant, then torsion of the curve-surface pair is equal to torsion of the curve.

**Definition 2.3.** Let $\alpha$ be a curve in $M \subset E^3$. If the geodesic curvature (torsion) of the curve $\alpha$ is equal to zero, then the curve-surface pair $(\alpha, M)$ is called a curvature curve-surface pair [4, 5, 7, 10 – 18].
2.1. Willmore Function on Curvatures of the Curve-Surface Pair Under Möbius

The most outstanding problem in Möbius differential geometry is the Willmore Conjecture [5,19]. This conjecture is most naturally formulated in terms of surfaces in $R^3$ rather than $S^3$. Let $f : M^2 \to R^3$ be a compact surface immersed in $R^3$[5,19]. Let $\kappa$ and $\tau$ denote principal curvatures of $f$, $H = (\kappa + \tau)/2$ and $K = \kappa\tau$ denote the mean and Gauss curvatures of $f$, respectively [5,19]. In 1965 Willmore [5,19] proposed the study of the functional. So it can be written $\tau(f, M^2)$ on the curve surface pair

$$\tau(f, M^2) = \int_{M^2} \frac{\sqrt{b^2 + c^2} + \left( a + \frac{bc - bc'}{b^2 + c^2} \right)}{2} \, dA$$

where $dA$ is the area form on $(f, M^2)$ induced by the immersion $f$. Several authors includinf Fubini [21], Thomsen[22] and White [23] have proven that the two form $H^2 - KdA$ is Möbius invariant. It so-called Willmore functional. Now it is:

$$W(f, M^2) = \int_{M^2} \left\{ \frac{\sqrt{b^2 + c^2} + \left( a + \frac{bc - bc'}{b^2 + c^2} \right)}{2} \right\} - \sqrt{b^2 + c^2} \left( a + \frac{bc - bc'}{b^2 + c^2} \right) \, dA$$

is Möbius invariant on curve-surface pair. Thus the Gauss-Bonnet Theorem states that

$$\int_{M^2} \sqrt{b^2 + c^2} \left( a + \frac{bc - bc'}{b^2 + c^2} \right) \, dA = 2\pi\chi(f, M^2)$$

, where $\chi(f, M^2)$ is the Euler characteristic of $(f, M^2)$, we have

$$W(f, M^2) = \int_{M^2} \left\{ \frac{\sqrt{b^2 + c^2} + \left( a + \frac{bc - bc'}{b^2 + c^2} \right)}{2} \right\} - \sqrt{b^2 + c^2} \left( a + \frac{bc - bc'}{b^2 + c^2} \right) \, dA = \tau(f, M^2) - 2\pi\chi(f, M^2)$$

and then $\tau(f, M^2) = W(f, M^2) + 2\pi\chi(f, M^2)$ is also Möbius invariant. Note that

$$\frac{\sqrt{b^2 + c^2} + \left( a + \frac{bc - bc'}{b^2 + c^2} \right)}{2} - \sqrt{b^2 + c^2} \left( a + \frac{bc - bc'}{b^2 + c^2} \right) = \frac{1}{4} \left[ \sqrt{b^2 + c^2} - a + \frac{bc - bc'}{b^2 + c^2} \right]^2$$

so the Willmore functional on curve-surface pair has the property that its integrand is non-negative, it vanishes at umbilic point where $\sqrt{b^2 + c^2} = a + \frac{bc - bc'}{b^2 + c^2}$.

Conflict of Interests

The authors declare that there is no conflict of interests regarding the publication of this paper.

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Filiz Ertem Kaya,
Department of Mathematics,
Faculty of Science and Arts,
University of Nigde Omer halisdemir Campus, Nigde, TURKEY.
E-mail address: fertem@ohu.edu.tr