On the Correspondence Between the Strongly Coupled 2-Flavor Lattice Schwinger Model and the Heisenberg Antiferromagnetic Chain

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Abstract

We study the strong coupling limit of the 2-flavor massless Schwinger model on a lattice using staggered fermions and the Hamiltonian approach to lattice gauge theories. Using the correspondence between the low-lying states of the 2-flavor strongly coupled lattice Schwinger model and the antiferromagnetic Heisenberg chain established in a previous paper, we explicitly compute the mass spectrum of this lattice gauge model: we identify the low-lying excitations of the Schwinger model with those of the Heisenberg model and compute the mass gaps of other excitations in terms of vacuum expectation values (v.e.v.’s) of powers of the Heisenberg Hamiltonian and spin-spin correlation functions. We find a satisfactory agreement with the results of the continuum theory already at the second order in the strong coupling expansion. We show that the pattern of symmetry breaking of the continuum theory is well reproduced by the lattice theory; we see indeed that in the lattice theory the isoscalar \(\langle \bar{\psi}\psi \rangle\) and isovector \(\langle \bar{\psi}\sigma^a\psi \rangle\) chiral condensates are zero to every order in the strong coupling expansion. In addition, we find that the chiral condensate \(\langle \bar{\psi}_L^{(2)} \psi_L^{(1)} \psi_R^{(1)} \psi_R^{(2)} \rangle\) is non zero also on the lattice; this is the relic in this lattice model of the axial anomaly in the continuum theory.

We compute the v.e.v.’s of the spin-spin correlators of the Heisenberg model which are pertinent to the calculation of the mass spectrum and obtain an explicit construction of the lowest lying states for finite size Heisenberg antiferromagnetic chains.
1 Introduction

One of the analytical approaches to the study of gauge theories with confining spectra is the strong coupling expansion. In the strong coupling limit, confinement is explicit, the confining string is a stable object, and some other qualitative features of the spectrum are easily obtained. Formulation of the strong coupling expansion requires a gauge invariant ultraviolet cutoff, which is most conveniently implemented using a lattice regularization.

One of the most difficult problems of the strong coupling approach to lattice gauge theory is its extrapolation to the continuum limit, which usually occurs at weak coupling. One symptom of this problem is that many choices of strong coupling theory produce identical continuum physics. In spite of this difficulty, there are strong coupling computations which claim some degree of success. It is well known that between strongly coupled lattice gauge theories and quantum spin systems there is an intimate relationship which is most useful to analyze chiral symmetry breaking in a variety of lattice models.

A useful test of strong coupling expansions in lattice gauge theories is the study of models whose solution in the continuum is known even in the strong coupling regime. For example, the 1-flavor Schwinger model has been studied using the Hamiltonian lattice field theory within the strong coupling expansion; several relevant parameters, analytically computed on the lattice, are shown to be in good agreement with those of the exact continuum solution. Furthermore, issues involving the realization of chiral symmetry on the lattice can be studied systematically. The 1-flavor Schwinger model has also been used as an example of the fact that quantum link models may reproduce the physics of conventional Hamiltonian lattice gauge theories. The physically more interesting case of the 2-flavor Schwinger model has been studied in detail; in addition to the issues of spectrum and chiral symmetry breaking, one sees that the strong coupling limit of this model is mapped on a relevant quantum spin model—the one-dimensional spin-1/2 quantum Heisenberg antiferromagnet. This correspondence is very useful since the ground state of the antiferromagnetic chain is known and its energy has been computed using the algebraic Bethe ansatz.

There are by now many hints at a correspondence between quantized gauge theories and quantum spin models, aimed at analyzing new phases relevant for condensed matter systems. Recently, Laughlin has argued that there is an analogy between the spectral data of gauge theories and strongly correlated electron systems. Moreover, certain spin ladders have been related to the 2-flavor Schwinger model in Ref. where a relation between the physical parameters of the spin and the gauge systems was also found.

In this context, the mapping between the strongly coupled 2-flavor Schwinger model and the quantum Heisenberg antiferromagnetic chain provides a concrete computational scheme in which the issue of the correspondence between quantized gauge theories and quantum spin models may be investigated. Because of the dimensionality of the coupling constant in (1+1)-dimensions, the infrared behavior is governed by the strong coupling limit, and it is tempting to conjecture the existence of an exact correspondence between the infrared limits of the Heisenberg and 2-flavor Schwinger models.

In this paper we revisit the strong coupling limit of the 2-flavor lattice Schwinger model in the Hamiltonian formalism with staggered fermions. Using the results of Ref. we analyze in detail the spectrum of the model: the gauge theory vacuum is the ground state of the Heisenberg antiferromagnetic chain and we find that the low-lying excitations of the gauge model have the same quantum numbers of the states of the spin chain. In addition we compute the masses of the excitations to the second order in the strong coupling expansion; as pointed out in Ref. the computation needs the knowledge of some spin-spin correlators of the quantum Heisenberg antiferromagnetic chain, which we explicitly compute. Our analysis hints to the existence of two other massive isotriplet states in addition to the expected pseudoscalar massive isosinglet.

We analyze then the pattern of symmetry breaking in the lattice theory. Even though the continuum axial symmetry is broken explicitly by the staggered fermions, a discrete axial symmetry
remains, it corresponds to a chiral rotation of $\pi/2$ and appears in the lattice theory as a translation by one lattice site. Since the ground state of the Heisenberg chain is unique and translationally invariant - at variance with the one flavor Schwinger model - in the 2-flavor model the discrete axial symmetry is unbroken. The chiral anomaly in this model is realized on the lattice via the explicit breaking of the $U_A(1)$-symmetry induced by the staggered fermions. Therefore, the pattern of symmetry breaking on the lattice reproduces faithfully the one of the continuum theory.

We also compute the chiral condensates in the strongly coupled 2-flavor lattice Schwinger model showing that the results of the lattice are in agreement with those of the continuum. In the continuum there is neither an isoscalar $\langle \bar{\psi}\psi \rangle$ nor an isovector $\langle \bar{\psi}\sigma^a\psi \rangle$ chiral condensate, since this is forbidden by the Coleman theorem [19]. We show that on the lattice these fermion condensates are also zero to all the orders in the strong coupling expansion. We find that just as in the continuum theory - the non-vanishing chiral condensate is given by the v.e.v. $\langle \bar{\psi}_L^{(2)}\psi_L^{(1)}\psi_R^{(1)}\psi_R^{(2)} \rangle$, which we compute up to the second order in the strong coupling expansion.

We finally use both the mass spectrum and the non-vanishing chiral condensate to compute, by means of suitable “Padé approximants”, the physical parameters of this lattice model and then compare their numerical values with the exact results of the continuum theory.

In section 2 we review known results [20, 21] of the continuum 2-flavor Schwinger model and define its Hamiltonian lattice version. We also identify the lattice counterparts of the relevant symmetries used in [20] for the description of the spectrum of the continuum theory.

In section 3 we expand the results of Ref. [6] and set up the formalism needed for the strong coupling analysis of the lattice 2-flavor Schwinger model. We show that the ground and excited states of the antiferromagnetic Heisenberg model have the spectrum and quantum numbers which are expected for the ground state and massless excitations of the 2-flavor Schwinger model.

In section 4 we construct the operators that create the massive excitations when acting on the ground state. We set up the strong coupling expansion and compute the corrections to the energies of the ground state and of the low-lying massive bosonic excitations. Subtracting the energy of the ground state from those of the excitations defines the lattice meson masses.

In section 5 we show that both the isoscalar and the isovector chiral condensates are zero to every perturbative order in the strong coupling expansion due to the translational invariance of the strong coupling ground state. We also compute, up to the second order in the strong coupling expansion, the chiral condensate $\langle \bar{\psi}_L^{(2)}\psi_L^{(1)}\psi_R^{(1)}\psi_R^{(2)} \rangle$, which is the order parameter for the breaking of the $U_A(1)$ symmetry.

Section 6 is devoted to the comparison of the lattice results with the continuum theory; there we shall see that the lattice theory, properly extrapolated to the continuum via the use of Padé approximants, well reproduces the parameters of the continuum theory already at the second order of the strong coupling expansion.

Section 7 is devoted to some concluding remarks.

In the appendix A, after reviewing known results about the Bethe Ansatz solution [14] of the antiferromagnetic Heisenberg chain, we study explicitly the complete spectrum of finite size Heisenberg antiferromagnetic chains of 4 and 6 sites in order to compare it to the Bethe Ansatz solution obtained in the thermodynamic limit. We write down explicitly the ground states of 4, 6 and 8 site chains and we find that these systems, even for these very small systems, the results exhibited in [14] are well reproduced. This provides an useful intuitive picture of the strong coupling ground state of the 2-flavor lattice Schwinger model.

In appendix B we comment on the computation of the spin-spin correlator of the Heisenberg chain and provide a link between the results given in [17] and the method used in [18].
2 Two flavor Schwinger model in the continuum and on the lattice

The action of the 1+1-dimensional electrodynamics with two charged Dirac spinor fields is

\[
S = \int d^2 x \left[ 2 \sum_{\alpha=1}^2 \overline{\psi}_\alpha (i\gamma_\mu \partial^\mu + \mu \gamma_\mu) \psi_\alpha - \frac{1}{4e^2} F_{\mu \nu} F^{\mu \nu} \right] \tag{2.1}
\]

The theory has an internal \(SU_L(2) \otimes SU_R(2)\)-flavor isospin symmetry; the Dirac fields are an isodoublet whereas the electromagnetic field is an isosinglet. It is well known that in 1+1 dimensions there is no spontaneous breakdown of continuous internal symmetries, unless there are anomalies or the Higgs phenomenon occurs. Neither mechanism is possible in the 2-flavor Schwinger model for the \(SU_L(2) \otimes SU_R(2)\)-symmetry: isovector currents do not develop anomalies and there are no gauge fields coupled to the isospin currents. The particles belong then to isospin multiplets. For what concerns the \(U(1)\) gauge symmetry there is an Higgs phenomenon [23].

The action is invariant under the symmetry \(SU_L(2) \otimes SU_R(2) \otimes U_V(1) \otimes U_A(1)\)

\[
SU_L(2) \otimes SU_R(2) \otimes U_V(1) \otimes U_A(1) \tag{2.2}
\]

The group generators act on the fermion isodoublet to give

\[
SU_L(2) : \quad \psi_a (x) \rightarrow (e^{i\theta_a} \sigma^a P_L)_{ab} \psi_b (x) \quad \overline{\psi}_a (x) \rightarrow \overline{\psi}_b (x) (e^{-i\theta_a} \frac{\sigma^a P_R}{2})_{ba} \tag{2.3}
\]

\[
SU_R(2) : \quad \psi_a (x) \rightarrow (e^{i\theta_a} \frac{\sigma^a P_R}{2})_{ab} \psi_b (x) \quad \overline{\psi}_a (x) \rightarrow \overline{\psi}_b (x) (e^{-i\theta_a} \sigma^a P_L)_{ba} \tag{2.4}
\]

\[
U_V(1) : \quad \psi_a (x) \rightarrow (e^{i(\theta(x)1)_{ab}}) \psi_b (x) \quad \overline{\psi}_a (x) \rightarrow \overline{\psi}_b (x) (e^{-i(\theta(x)1)_{ba}}) \tag{2.5}
\]

\[
U_A(1) : \quad \psi_a (x) \rightarrow (e^{i\alpha \gamma_5 1_{ab}}) \psi_b (x) \quad \overline{\psi}_a (x) \rightarrow \overline{\psi}_b (x) (e^{-i\alpha \gamma_5 1_{ba}}) \tag{2.6}
\]

where \(\sigma^a\) are the Pauli matrices, \(\theta_a\), \(\theta(x)\) and \(\alpha\) are real coefficients and

\[
P_L = \frac{1}{2}(1 - \gamma_5) \quad , \quad P_R = \frac{1}{2}(1 + \gamma_5) \quad . \tag{2.7}
\]

At the classical level the symmetries (2.3–2.6) lead to conservation laws for the isovector, vector and axial currents

\[
j_\mu^a (x)_R = \overline{\psi}_a (x) \gamma^\mu P_R (\frac{\sigma^a}{2})_{ab} \psi_b (x) \tag{2.8}
\]

\[
j_\mu^a (x)_L = \overline{\psi}_a (x) \gamma^\mu P_L (\frac{\sigma^a}{2})_{ab} \psi_b (x) \tag{2.9}
\]

\[
j_\mu (x) = \overline{\psi}_a (x) \gamma^\mu 1_{ab} \psi_b (x) \tag{2.10}
\]

\[
j_5^a (x) = \overline{\psi}_a (x) \gamma^5 1_{ab} \psi_b (x) \tag{2.11}
\]

It is well known that at the quantum level the vector and axial currents cannot be simultaneously conserved, due to the anomaly phenomenon [23]. If the regularization is gauge invariant, so that the vector current is conserved, then the axial current acquires the anomaly which breaks the \(U_A(1)\)-symmetry

\[
\partial_\mu j_5^a (x) = 2 \frac{e^2}{2\pi} \epsilon_{\mu \nu} F^{\mu \nu} (x) \tag{2.12}
\]

The isoscalar and isovector chiral condensates are zero due to the Coleman theorem [19]: in fact, they would break not only the \(U_A(1)\) symmetry of the action, but also the continuum internal symmetry \(SU_L(2) \otimes SU_R(2)\) down to \(SU_V(2)\). There is, however, a \(SU_L(2) \otimes SU_R(2)\) invariant operator, which is non-invariant under the \(U_A(1)\)-symmetry; it can acquire a non-vanishing v.e.v without violating Coleman’s theorem and consequently may be regarded as a good order parameter for the \(U_A(1)\)-breaking. Its expectation value is given by [24, 25]

\[
< F > = -< \psi_L^{(2)} \psi_L^{(1)} \psi_R^{(1)} \psi_R^{(2)} > = \left( \frac{e^2}{4\pi} \right)^2 \frac{2}{\pi} e^2 . \tag{2.13}
\]
It describes a process in which two right movers are annihilated and two left movers are created.

Note that \( F \), being quadrilinear in the fields, is actually invariant under chiral rotations of \( \pi/2 \), namely under the discrete axial symmetry

\[
\psi_a(x) \to \gamma^5 \psi_a(x) \quad \bar{\psi}_a(x) \to -\bar{\psi}_a(x) \gamma_5 .
\]  

As a consequence, this part of the chiral symmetry group is not broken by the non-vanishing v.e.v. of \( F \).  

As we shall see in section 6, the lattice theory faithfully reproduces the pattern of symmetry breaking of the continuum theory; this happens even if on the lattice the \( SU(2) \)-flavor symmetry is not protected by the Coleman theorem. The isoscalar and isovector chiral condensates are zero also on the lattice, whereas the operator \( F \) acquires a non-vanishing v.e.v. due to the coupling of left and right movers induced by the gauge field. The continuous axial symmetry is broken explicitly by the staggered fermion, but the discrete axial symmetry \( (2.14) \) remains.

The action \( (2.1) \) may be presented in usual abelian bosonized form \( [20] \). Setting

\[
\psi^a \gamma^\mu \psi_a := \frac{1}{\sqrt{\pi}} \epsilon^{\mu\nu} \partial_\nu \Phi^a , \quad a = 1, 2
\]  

the electric charge density and the action read

\[
j_0 =: \psi_1^\dagger \psi_1 + \psi_2^\dagger \psi_2 := \frac{1}{\sqrt{\pi}} \partial_x (\Phi_1 + \Phi_2)
\]  

\[
S = \int d^2x \left[ \frac{1}{2} \partial_\mu \Phi_1 \partial^\mu \Phi_1 + \frac{1}{2} \partial_\mu \Phi_2 \partial^\mu \Phi_2 - \frac{e_c^2}{2\pi} (\Phi_1 + \Phi_2)^2 \right] .
\]

By changing the variables to

\[
\Phi_+ = \frac{1}{\sqrt{2}} (\Phi_1 + \Phi_2)
\]  

\[
\Phi_- = \frac{1}{\sqrt{2}} (\Phi_1 - \Phi_2)
\]

one has

\[
S = \int d^2x \left( \frac{1}{2} \partial_\mu \Phi_+ \partial^\mu \Phi_+ + \frac{1}{2} \partial_\mu \Phi_- \partial^\mu \Phi_- - \frac{e_c^2}{\pi} \Phi_+^2 \right) .
\]

The theory describes two scalar fields, one massive and one massless. \( \Phi_+ \) is an isosinglet as evidenced from Eq.(2.16); its mass \( m_S = \sqrt{\frac{2}{\pi}} e_c \) comes from the anomaly Eq.(2.12) \[22\]. Local electric charge conservation is spontaneously broken, but no Goldstone boson appears because the Goldstone mode may be gauged away. \( \Phi_- \) represents an isotriplet; it has rather involved nonlinear transformation properties under a general isospin transformation. All three isospin currents can be written in terms of \( \Phi_- \) but only the third component has a simple representation in terms of \( \Phi_- \); namely

\[
j^3_\mu(x) =: \psi_1(x) \gamma_\mu (\sigma_3^2)_{ab} \psi_2(x) :=
\]

\[
= \frac{1}{2} : \psi_1(x) \gamma_\mu \psi_1(x) - \psi_2(x) \gamma_\mu \psi_2(x) := (2\pi)^3 e^{i\mu\nu} \partial_\nu \Phi_- .
\]

The other two isospin currents \( j^1_\mu(x) \) and \( j^2_\mu(x) \) are complicated nonlinear and nonlocal functions of \( \Phi_- \); a more symmetrical treatment of the bosonized form of the isotriplet currents is available within the framework of nonabelian bosonization [25]. For the multiflavor Schwinger model this approach has been carried out in [27], providing results in agreement with [20].

The excitations are most conveniently classified in terms of the quantum numbers of \( P \)-parity and \( G \)-parity; \( G \)-parity is related to the charge conjugation \( C \) by

\[
G = e^{i\pi \frac{\Phi}{2}} C .
\]
\( \Phi_- \) is a \( G \)-even pseudoscalar, while \( \Phi_+ \) is a \( G \)-odd pseudoscalar

\[
\Phi_- : \quad I^{PG} = 1^{-+} \quad \text{(2.23)}
\]
\[
\Phi_+ : \quad I^{PG} = 0^{-+} \quad \text{(2.24)}
\]

The massive meson \( \Phi_+ \) is stable by \( G \) conservation since the action (2.20) is invariant under \( \Phi_+ \rightarrow -\Phi_+ \).

In the massive \( SU(2) \) Schwinger model – when the mass of the fermion \( m \) is small compared to \( e^2 \) (strong coupling) – Coleman [20] showed that – in addition to the triplet \( \Phi_- (I^{PG} = 1^{-+}) \) – the low-energy spectrum exhibits a singlet \( I^{PG} = 0^{++} \) lying on top of the triplet \( \Phi_- \). In this limit the gauge theory is mapped to a sine-Gordon model and the low-lying excitations are soliton-antisoliton states. When \( m \rightarrow 0 \), these soliton-antisoliton states become massless [21]; in this limit, the analysis of the many body wave functions, carried out in Ref. [21], hints to the existence of a whole class of massless states with positive \( G \)-parity; \( P \)-parity however cannot be determined with the procedure developed in [21]. These are not the only excitations of the model: way up in mass there is the isosinglet \( I^{PG} = 0^{-=} \) (2.23), already discussed in Ref. [20]. The model exhibits also triplets, whose mass – of order \( m_S \) or greater – stays finite [21]; among the triplets there is a \( G \)-even state \( I^{PG} = 0^{-=} \).

The Hamiltonian, gauge constraint and non-vanishing (anti-)commutators of the continuum 2-flavor Schwinger model are

\[
H = \int dx \left[ \frac{e^2}{2} E^2(x) + \sum_{a=1}^{2} \psi_a^\dagger(x) \alpha \left(i \partial_x + e A(x)\right) \psi_a(x) \right] \quad \text{(2.25)}
\]

\[
\partial_x E(x) + \sum_{a=1}^{2} \psi_a^\dagger(x) \psi_a(x) \sim 0 \quad \text{(2.26)}
\]

\[
[A(x), E(y)] = i \delta(x-y), \left\{ \psi_a(x), \psi_b^\dagger(y) \right\} = \delta_{ab} \delta(x-y) \quad \text{(2.27)}
\]

A lattice Hamiltonian, constraint and (anti-) commutators reducing to (2.25,2.26,2.27) in the naive continuum limit are

\[
H_S = \frac{e^2 a}{2} \sum_{x=1}^{N} E_x^2 - \frac{i t}{2a} \sum_{x=1}^{N} \sum_{a=1}^{2} \left( \psi_{a,x+1}^\dagger e^{i A_x} \psi_{a,x} - \psi_{a,x}^\dagger e^{-i A_x} \psi_{a,x+1} \right) \quad \text{(2.28)}
\]

\[
E_x - E_{x-1} + \psi_{1,x}^\dagger \psi_{1,x} + \psi_{2,x}^\dagger \psi_{2,x} - 1 \sim 0 \quad \text{(2.28)}
\]

\[\{ A_x, E_y \} = i \delta_{xy}, \left\{ \psi_{a,x}, \psi_{b,y}^\dagger \right\} = \delta_{ab} \delta_{xy}.\]

The fermion fields are defined on the sites, \( x = 1, \ldots, N \), the gauge and electric fields, \( A_x \) and \( E_x \), on the links \( [x; x+1] \), \( N \) is an even integer and, when \( N \) is finite it is convenient to impose periodic boundary conditions. When \( N \) is finite, the continuum limit is the 2-flavor Schwinger model on a circle [23]. The coefficient \( t \) of the hopping term in (2.28) plays the role of the lattice light speed. In the naive continuum limit, \( e_L = e_c \) and \( t = 1 \).

The Hamiltonian and gauge constraint exhibit the discrete symmetries

- Parity \( P \):

\[
A_x \rightarrow -A_{-x-1}, \quad E_x \rightarrow -E_{-x-1}, \quad \psi_{a,x} \rightarrow (-1)^x \psi_{a,-x}^\dagger, \quad \psi_{a,x}^\dagger \rightarrow (-1)^x \psi_{a,-x} \quad \text{(2.29)}
\]

- Discrete axial symmetry \( \Gamma \):

\[
A_x \rightarrow A_{x+1}, \quad E_x \rightarrow E_{x+1}, \quad \psi_{a,x} \rightarrow \psi_{a,x+1}, \quad \psi_{a,x}^\dagger \rightarrow \psi_{a,x+1}^\dagger \quad \text{(2.30)}
\]

- Charge conjugation \( C \):

\[
A_x \rightarrow -A_{x+1}, \quad E_x \rightarrow -E_{x+1}, \quad \psi_{a,x} \rightarrow \psi_{a,x}^\dagger, \quad \psi_{a,x}^\dagger \rightarrow \psi_{a,x} \quad \text{(2.31)}
\]

\(^1\text{K. Harada private communication.}\)
G-parity:

\[
\begin{align*}
A_x &\rightarrow -A_{x+1}, \quad E_x \rightarrow -E_{x+1} \\
\psi_{1,x} &\rightarrow \psi_{1,x+1}^\dagger, \quad \psi_{2,x} \rightarrow \psi_{2,x+1} \\
\psi_{2,x} &\rightarrow -\psi_{1,x+1}^\dagger, \quad \psi_{1,x} \rightarrow -\psi_{1,x+1}^\dagger.
\end{align*}
\] 

(2.32)

The lattice 2-flavor Schwinger model is equivalent to a one dimensional quantum Coulomb gas on the lattice with two kinds of particles. To see this one can fix the gauge, \( A_x = A \) (Coulomb gauge). Eliminating the non-constant electric field and using the gauge constraint, one obtains the effective Hamiltonian

\[
H_S = H_u + H_p = \left[ \frac{e^2}{2N} E^2 + \frac{e^2 a}{2} \sum_{x,y} \rho(x)V(x-y)\rho(y) \right] + \\
+ \left[ -\frac{it}{2a} \sum_{x,a=1}^2 \left( \psi_{a,x+1}^\dagger e^{-iA}\psi_{a,x} - \psi_{a,x}^\dagger e^{-iA}\psi_{a,x+1} \right) \right],
\] 

(2.33)

where the charge density is

\[
\rho(x) = \psi_{1,x}^\dagger \psi_{1,x} + \psi_{2,x}^\dagger \psi_{2,x} - 1,
\]

(2.34)

and the potential

\[
V(x-y) = \frac{1}{N} \sum_{n=1}^{N-1} e^{i2\pi n(x-y)/N} \frac{1}{4 \sin^2 \frac{\pi n}{N}}
\]

is the Fourier transform of the inverse laplacian on the lattice for non zero momentum. The constant modes of the gauge field decouple in the thermodynamic limit \( N \rightarrow \infty \).

3 The strong coupling limit and the antiferromagnetic Heisenberg Hamiltonian

In a previous paper [6] we showed that the low-lying spectrum of the 2-flavor lattice Schwinger model in the strong coupling limit is equivalent to the spectrum of the antiferromagnetic Heisenberg model. There we also showed that the mass of the massive excitations can be computed in terms of correlators of the Heisenberg model. It is our purpose in this section to further explore this equivalence and to set up the formalism needed in what follows.

In the thermodynamic limit the Schwinger Hamiltonian (2.33), rescaled by the factor \( e^2 a/2 \), reads

\[
H = H_0 + \epsilon H_h
\]

(3.36)

with

\[
H_0 = \sum_{x>y} \left[ \frac{(x-y)^2}{N} - (x-y) \right] \rho(x)\rho(y)
\]

(3.37)

\[
H_h = -i(R-L)
\]

(3.38)

and \( \epsilon = t/e^2 a^2 \). In Eq. (3.38) the right \( R \) and left \( L \) hopping operators are defined \( (L = R^\dagger) \) as

\[
R = \sum_{x=1}^N R_x = \sum_{x=1}^N \sum_{a=1}^2 R^{(a)}_x = \sum_{x=1}^N \sum_{a=1}^2 \psi_{a,x+1}^\dagger e^{iA}\psi_{a,x}
\]

(3.39)

On a periodic chain the commutation relation

\[
[R, L] = 0
\]

is satisfied.
We shall consider the strong coupling perturbative expansion where the Coulomb Hamiltonian (3.37) is the unperturbed Hamiltonian and the hopping Hamiltonian (3.38) the perturbation. Due to Eq.(2.34) every configuration with one particle per site has zero energy, so that the ground state of the Coulomb Hamiltonian (3.37) is \(2^N\) times degenerate. The degeneracy of the ground state can be removed only at the second perturbative order since the first order is trivially zero.

At the second order the lattice gauge theory is effectively described by the antiferromagnetic Heisenberg Hamiltonian. The vacuum energy – at order \(\epsilon^2\) – reads
\[
E^{(2)}_0 = \text{<} H^\dagger_h \Pi \frac{H_h}{E^{(0)}_0 - H_0} H_h \text{>}
\]
where the expectation values are defined on the degenerate subspace and \(\Pi\) is the operator projecting on a set orthogonal to the states with one particle per site. Due to the vanishing of the charge density on the ground states of \(H_0\), the commutator
\[
[H_0, H_h] = H_h
\]
holds on any linear combination of the degenerate ground states. Consequently, from Eq.(3.41) one finds
\[
E^{(2)}_0 = -2 \text{<} RL \text{>}
\]
On the ground state the combination \(RL\) can be written in terms of the Heisenberg Hamiltonian. By introducing the Schwinger spin operators
\[
\vec{S}_x = \psi_{a,x}^\dagger \sigma_{ab} \psi_{b,x}
\]
the Heisenberg Hamiltonian \(H_J\) reads
\[
H_J = \sum_{x=1}^{N} \left( \vec{S}_x \cdot \vec{S}_{x+1} - \frac{1}{4} \right) = \sum_{x=1}^{N} \left( -\frac{1}{2} L_x R_x - \frac{1}{4} \rho(x) \rho(x+1) \right)
\]
and, on the degenerate subspace, one has
\[
\text{<} H_J \text{>} = \left\langle \sum_{x=1}^{N} \left( \vec{S}_x \cdot \vec{S}_{x+1} - \frac{1}{4} \right) \right\rangle = \left\langle \sum_{x=1}^{N} \left( -\frac{1}{2} L_x R_x \right) \right\rangle.
\]
Taking into account that products of \(L_x\) and \(R_y\) at different points have vanishing expectation values on the ground states, and using Eq.(3.46), Eq.(3.43) reads
\[
E^{(2)}_0 = 4 \text{<} H_J \text{>}
\]
The ground state of \(H_J\) singles out the correct vacuum, on which to perform the perturbative expansion. In one dimension \(H_J\) is exactly diagonalizable \[12, 29\]. In the spin model a flavor 1 particle on a site can be represented by a spin up, a flavor 2 particle by a spin down. The spectrum of \(H_J\) exhibits \(2^N\) eigenstates; among these, the spin singlet with lowest energy is the non degenerate ground state \(|\text{g.s.}>\).

We shall construct the strong coupling perturbation theory of the 2-flavor Schwinger model using \(|\text{g.s.}>\) as the unperturbed ground state. \(|\text{g.s.}>\) is invariant under translations by one lattice site, which amounts to invariance under discrete chiral transformations. As a consequence, at variance with the 1-flavor model \[4\], chiral symmetry cannot be spontaneously broken even in the infinite coupling limit.

\(|\text{g.s.}>\) has zero charge density on each site and zero electric flux on each link
\[
\rho(x)|\text{g.s.}> = 0, \quad E_x|\text{g.s.}> = 0 \quad (x = 1, ..., N).
\]
$|\text{g.s.}\rangle$ is a linear combination of all the possible states with $N/2$ spins up and $N/2$ spins down. The coefficients are not explicitly known for general $N$. In the appendix A we shall exhibit $|\text{g.s.}\rangle$ explicitly for finite size systems of 4, 6 and 8 sites. The Heisenberg energy of $|\text{g.s.}\rangle$ is known exactly and, in the thermodynamic limit, is \[ H_J|\text{g.s.}\rangle = (-N \ln 2)|\text{g.s.}\rangle. \tag{3.49} \]

Eq. (3.49) provides the second order correction Eq. (3.47) to the vacuum energy, $E_{\text{g.s.}}^{(2)} = -4N \ln 2$.

There exist two kinds of excitations created from $|\text{g.s.}\rangle$: one kind involves only spin flipping and has lower energy since no electric flux is created, the other involves fermion transport besides spin flipping and thus has a higher energy. For the latter excitations the energy is proportional to the coupling times the length of the electric flux: the lowest energy is achieved when the fermion is transported by one lattice spacing. Of course only the excitations of the first kind can be mapped into states of the Heisenberg model.

In \[4\] the antiferromagnetic Heisenberg model excitations have been classified. There it was shown that any excitation may be regarded as the scattering state of quasiparticles of spin-$1/2$: every physical state contains an even number of quasiparticles and the spectrum exhibits only integer spin states. The two simplest excitations of lowest energy in the thermodynamic limit are a triplet and a singlet \[4\]: they have a dispersion relation depending on the momenta of the two quasiparticles. For vanishing total momentum (relative to the ground state momentum $P_{\text{g.s.}} = 0$ for $N$ even, $P_{\text{g.s.}} = \pi$ for $\frac{N}{2}$ odd) in the thermodynamic limit they are degenerate with the ground state.

In the appendix A we show that even for finite size systems, the excited states can be grouped in families corresponding to the classification given in \[4\]. We explicitly exhibit all the energy eigenstates for $N = 4$ and $N = 6$. The lowest lying are a triplet and a singlet, respectively; they have a well defined relative (to the ground state) $P$-parity and $G$-parity $-1^-$ for the triplet and $0^{++}$ for the singlet. Since they share the same quantum numbers these states can be identified, in the limit of vanishing fermion mass, with the soliton-antisoliton excitations found by Coleman in his analysis of the 2-flavor Schwinger model. A related analysis about the parity of the lowest lying states in finite size Heisenberg chains, has been given in \[31\].

Moreover in \[4\] a whole class $-\mathcal{M}_{AF}$ of gapless excitations at zero momentum was singled out in the thermodynamic limit; these states are eigenstates of the total momentum and consequently have positive $G$-parity at zero momentum. The low-lying states of the Schwinger model also contain \[21\] many massless excitations with positive $G$-parity; they are identified \[1\] with the excitations belonging to $\mathcal{M}_{AF}$. The mass of these states in the Schwinger model can be obtained from the differences between the excitation energies at zero momentum and the ground state energy. The energies of the states $|\text{ex.}\rangle$ belonging to the class $\mathcal{M}_{AF}$ have the same perturbative expansion of the ground state. Consequently, the states $|\text{ex.}\rangle$ at zero momentum up to the second order in the strong coupling expansion have the same energy of the ground state \(3.41\), $E_{\text{ex.}}^{(2)} = -4N \ln 2$. To this order the mass gap is zero. Higher order corrections may give a mass gap.

## 4 The meson masses

In this section we determine the masses for the states obtained by fermion transport of one site on the Heisenberg model ground state. Our analysis shows that besides the $G$-odd pseudoscalar isosinglet $0^{--}$ with mass $m_S = e\sqrt{2/\pi}$, there are also a $G$-even pseudoscalar isosinglet $0^{++}$ and isotriplet $1^{-+}$ and a $G$-odd scalar isotriplet $1^{+-}$ with masses of the order of $m_S$ or greater. The quantum numbers are relative to those of the ground state $I_{\text{g.s.}}^{PG} = 0^{++}$ for $N/2$ even $I_{\text{g.s.}}^{PG} = 0^{--}$ for $N/2$ odd.

Two states can be created using the spatial component of the vector $j_1^i(x)$ Eq. (2.10) and isovector $j_2^i(x)$ Eqs. (2.8,2.9) Schwinger model currents. They are the $G$-odd pseudoscalar isosinglet $I^{PG} = 0^{--}$ and the $G$-even pseudoscalar isotriplet $I^{PG} = 1^{-+}$. The lattice operators with the correct
quantum numbers creating these states at zero momentum, when acting on \( |g.s.> \), read

\[
S = R + L = \sum_{x=1}^{N} j^1(x) \tag{4.50}
\]

\[
T_+ = (T_-)^\dagger = R^{(12)} + L^{(12)} = \sum_{x=1}^{N} j^2_+(x) \tag{4.51}
\]

\[
T_0 = \frac{1}{\sqrt{2}}(R^{(11)} + L^{(11)} - R^{(22)} - L^{(22)}) = \sum_{x=1}^{N} j^3_+(x) \tag{4.52}
\]

\( R^{(ab)} \) and \( L^{(ab)} \) in (4.51, 4.52) are the right and left flavor changing hopping operators \( (L^{(ab)} = (R^{(ab)})^\dagger) \)

\[
R^{(ab)} = \sum_{x=1}^{N} \psi_{a,x+1} e^{iA} \psi_{b,x} .
\]

The states are given by

\[
|S> = |0^{-}>= S|g.s.> \quad (4.53)
\]

\[
|T_{\pm}> = |1^{\pm}, \pm 1> = T_{\pm}|g.s.> \quad (4.54)
\]

\[
|T_{0}> = |1^{\pm}, 0> = T_{0}|g.s.> . \quad (4.55)
\]

They are normalized as

\[
<S|S> = <g.s.|S^\dagger S|g.s.> = -4 <g.s.|H_J|g.s.> = 4N \ln 2 \quad (4.56)
\]

\[
<T_{\pm}|T_{\pm}> = \frac{2}{3}(N+ <g.s.|H_J|g.s.>) = \frac{2}{3}N(1 - \ln 2) \quad (4.57)
\]

and

\[
<T_{0}|T_{0}> = <T_{-}|T_{-}> = <T_{+}|T_{+} > . \quad (4.58)
\]

In Eqs.(4.56, 4.57, 4.58) \( <g.s.|g.s.> = 1 \).

The isosinglet energy, up to the second order in the strong coupling expansion, is

\[
E_S = E^{(0)}_S + \epsilon^2 E^{(2)}_S \tag{4.59}
\]

with

\[
E^{(0)}_S = \frac{<S|H_0|S>}{<S|S>} = 1 
\]

\[
E^{(2)}_S = \frac{<S|H^\dagger_S \Lambda_S H_S|S>}{<S|S>} \tag{4.60}
\]

\( \Lambda_S = \frac{\Pi_S}{E^{(0)}_S - H_0} \) and \( 1 - \Pi_S \) a projection operator onto \( |S> \). On \( |g.s.> \)

\[
[H_0, (\Pi_S H_S)^n S] = (n + 1)(\Pi_S H_S)^n S, \quad (n = 0, 1, \ldots) \tag{4.61}
\]

holds; Eq.(4.60) may then be written in terms of spin correlators as

\[
E^{(2)}_S = E^{(2)}_{g.s.} + 4 - \sum_{x=1}^{N} \frac{<g.s.|\vec{S}_x \cdot \vec{S}_{x+2} - \frac{1}{2} |g.s.>}{<g.s.|H_J|g.s.>} . \tag{4.62}
\]

One immediately recognizes that the excitation spectrum is determined once \( <g.s.|\vec{S}_x \cdot \vec{S}_{x+2}|g.s.> \) is known. Equations similar to Eq.(4.62) may be established also at a generic order of the strong coupling expansion.

At the zeroth perturbative order the pseudoscalar triplet is degenerate with the isosinglet \( E^{(0)}_T = E^{(0)}_S = 1 \). Following the same procedure as before one may compute the energy of the states (4.54)
They are normalized as
\[ N+ < g.s.|H_j|g.s. > \]

The corresponding lattice operators at zero momentum are
\[ N \]

in analogy with the one flavor Schwinger model where, as shown in Ref. [3], the chiral current
\[ \langle j^5(x) \rangle \]

where in terms of the vector operator \( \vec{V} = \sum_{x=1}^{N} \bar{S}_x \wedge \bar{S}_{x+1} \), one can write \( \Delta_{DS}(T) \) as
\[ \Delta_{DS}(T_{\pm}) = 12 \frac{< g.s. | (V_1)^2 | g.s. > + < g.s. | (V_2)^2 | g.s. >}{N+ < g.s.|H_j|g.s. >} \]

The v.e.v. of each squared component of \( \vec{V} \) on the rotationally invariant singlet \(|g.s.>\) give the same contribution i.e. \( \Delta_{DS}(T_{\pm}) = \Delta_{DS}(T_0) \); the triplet states (as in the continuum theory) have a degenerate mass gap. This is easily verified by direct computation on finite size systems; when the size of the system is finite one may also show that \( \Delta_{DS} \) is of zeroth order in \( N \).

The excitation masses are given by \( m_S = \frac{\sqrt{2}}{2}(E_S - E_{g.s.}) \) and \( m_T = \frac{\sqrt{2}}{2}(E_T - E_{g.s.}) \). Consequently, the \( (N\text{-dependent}) \) ground state energy terms appearing in \( E_S^{(2)} \) and \( E_T^{(2)} \) cancel and what is left are only \( N \) independent terms. This is a good check of our computation, being the mass an intensive quantity.

In principle one should expect also excitations created acting on \(|g.s.>\) with the chiral currents, in analogy with the one flavor Schwinger model where, as shown in Ref. [3], the chiral current creates a two-meson bound state. The chiral currents operators for the two flavor Schwinger model are given by
\[ j^5(x) = \bar{\psi}(x)\gamma^5\psi(x) \]
\[ j^5_0(x) = \bar{\psi}_a(x)\gamma^5(x)\gamma^a\gamma^b\psi_b(x) \]

The corresponding lattice operators at zero momentum are
\[ S_5 = R - L = \sum_{x=1}^{N} j^5(x) \]
\[ T_{5+} = (T_5^2)^1 = R(12) - L(12) = \sum_{x=1}^{N} j_{5+}^i(x) \]
\[ T_{50} = \frac{1}{\sqrt{2}}(R(11) - L(11) - R(22) + L(22)) = \sum_{x=1}^{N} j_{50}^i(x) \]

The states created by \( S_5, T_{5+}, T_{50} \) when acting on \(|g.s.>\), are
\[ |S^5> = |0^{++}> = S_5|g.s.> \]
\[ |T_{5+}^0> = |1^{+-}, \pm 1 > = T_{5+}^0|g.s.> \]
\[ |T_{50}^-> = |1^{+-}, 0 > = T_{50}^-|g.s.> \]

They are normalized as
\[ <S^5|S^5> = <g.s.|S^5|S^5|g.s.> = -4 <g.s.|H_j|g.s.> = 4N \log 2 \]
\[ <T_{5+}^0|T_{5+}^0> = \frac{2}{3}(N+ <g.s.|H_j|g.s.>) \]
\[ <T_{50}^-|T_{50}^-> = \frac{2}{3}N(1 - \log 2) \]

and
\[ <T_{5+}^0|T_{50}^-> = <T_{5+}^0|T_{50}^-> = <T_{5+}^0|T_{50}^-> \]

Following the computational scheme used to study \(|S>\) and \(|T>\), one finds for the state \(|S^5>\)
\[ E_{S_5}^{(0)} = 1 \]
\[ E_{S_5}^{(2)} = E_{g.s.}^{(2)} + 12 - 3 \sum_{x=1}^{N} <g.s.|\bar{S}_x \cdot \bar{S}_{x+2} - \frac{1}{4}|g.s.> \]
\[ <g.s.|H_j|g.s.> \]
For the triplet \(|T^5\rangle\) one gets

\[
E^{(0)}_{T^5} = 1
\]

\[
E^{(2)}_{T^5} = E^{(2)}_{g.s.} + \sum_{x=1}^{N < g.s. | \vec{S}_x \cdot \vec{S}_{x+2} - \frac{1}{4} | g.s. > - 4 < g.s. | H_f | g.s. > \over N+ < g.s. | H_f | g.s. > .
\]

(4.80)

(4.81)

5 The correlators \(\sum_x < g.s. | \vec{S}_x \cdot \vec{S}_{x+2} | g.s. >\) and \(< g.s. | \vec{V}^2 | g.s. >\)

In this section we compute the spin-spin correlators needed to evaluate the second order energies of the isosinglet and isotriplets derived in section 4. The explicit computation of spin-spin correlation functions is far from being trivial since \(G(r) \equiv < g.s. | \vec{S}_0 \cdot \vec{S}_r | g.s. >\) is not known for arbitrary lattice separations \(r\). For \(r = 2\) it was computed by M. Takahashi \([17]\) in his perturbative analysis of the half filled Hubbard model in one dimension. For \(r > 2\) no exact numerical values of \(G(r)\) are known. In \([18]\) were given two representations of \(G(r)\), while in \([21, 22]\) the exact asymptotic \((r \to \infty)\) expression of \(G(r)\) was derived.

In order to explicitly compute the second order energies Eq.(4.62) and Eq.(4.63) one has to evaluate the correlation function

\[
G(2) = \frac{1}{N} \sum_{x=1}^{N} < g.s. | \vec{S}_x \cdot \vec{S}_{x+2} | g.s. >
\]

(5.82)

which has been exactly computed in \([17]\) and is given by

\[
G(2) = \frac{1}{4} (1 - 16 \ln 2 + 9 \zeta(3)) = 0.1820 .
\]

(5.83)

In the appendix B we shall show how the knowledge of this correlator allows one to compute explicitly the first three “emptiness formation probabilities”, used in Ref.\([18]\) in the study of the Heisenberg chain correlators, \(G(r)\).

In order to compute the mass of the isotriplet one has to evaluate also the correlation functions appearing in Eq.(4.63); namely \(< g.s. | \vec{V} \cdot \vec{V} | g.s. >\), where \(\vec{V} = \sum_x S_x \wedge S_{x+1} \). The explicit expression of this function is not known. For its evaluation it is most useful to rearrange this correlator as

\[
< g.s. | \vec{V} \cdot \vec{V} | g.s. > = \sum_{x,y=1}^{N} ( < g.s. | (\vec{S}_x \cdot \vec{S}_y) (\vec{S}_{x+1} \cdot \vec{S}_{y+1}) | g.s. > 
\]

\[- < g.s. | (\vec{S}_x \cdot \vec{S}_{y+1}) (\vec{S}_{x+1} \cdot \vec{S}_y) | g.s. > + \sum_{x=1}^{N} < g.s. | \vec{S}_x \cdot \vec{S}_{x+1} | g.s. > .
\]

(5.84)

It is possible to extract a numerical value from Eq.(5.84) only within the random phase approximation \([17, 23]\). For this purpose it is first convenient to rewrite the unconstrained sum over the sites \(x\) and \(y\) as a sum where all the four spins involved in the v.e.v.’s lie on different sites,

\[
< g.s. | \vec{V} \cdot \vec{V} | g.s. > = \sum_{y \neq x \pm 1} ( < g.s. | (\vec{S}_x \cdot \vec{S}_y) (\vec{S}_{x+1} \cdot \vec{S}_{y+1}) | g.s. > 
\]

\[- < g.s. | (\vec{S}_x \cdot \vec{S}_{y+1}) (\vec{S}_{x+1} \cdot \vec{S}_y) | g.s. > + \frac{3}{8} N 
\]

\[- \frac{1}{2} \sum_{x=1}^{N} < g.s. | \vec{S}_x \cdot \vec{S}_{x+1} | g.s. > + \sum_{x=1}^{N} < g.s. | \vec{S}_x \cdot \vec{S}_{x+2} | g.s. >
\]

(5.85)

and then factorize the four spin operators in Eq.(5.85) as

\[
< g.s. | \vec{V} \cdot \vec{V} | g.s. > \simeq N \sum_{r=2}^{\infty} ( < g.s. | \vec{S}_0 \cdot \vec{S}_r | g.s. >^2 - < g.s. | \vec{S}_0 \cdot \vec{S}_{r+1} | g.s. > < g.s. | \vec{S}_1 \cdot \vec{S}_r | g.s. > )
\]
\[ \sum_{x=1}^{N} \langle \text{g.s.} | \vec{S}_x \cdot \vec{S}_{x+1} | \text{g.s.} \rangle - \sum_{x=1}^{N} \langle \text{g.s.} | \vec{S}_x \cdot \vec{S}_{x+2} | \text{g.s.} \rangle \]  
\hspace{1cm} (5.86)

Of course, Eq.(5.86) provides an answer larger than the exact result; terms such as \( \langle \ldots | \ldots \rangle \) contain negative contributions which are eliminated once one factorizes them in the form \( \langle \ldots \rangle \langle \ldots \rangle \). This is easily checked also by direct computation on finite size systems.

The spin-spin correlation functions \( G(r) \) are exactly known for \( r = 1, 2 \). For the spin-spin correlation functions \( G(r) \) up to a distance of \( r = 30 \) the results are reported in table 1 \([31, 34]\).

| \( r \) | \( G(r) \) | \( r \) | \( G(r) \) |
|-----|-----|-----|-----|
| 1   | -0.4431 | 16  | 0.0305 |
| 2   | 0.1821  | 17  | -0.0296 |
| 3   | -0.1510 | 18  | 0.02274 |
| 4   | 0.1038  | 19  | -0.0267 |
| 5   | -0.0925 | 20  | 0.0249 |
| 6   | 0.0731  | 21  | -0.0242 |
| 7   | -0.0671 | 22  | 0.0228 |
| 8   | 0.0567  | 23  | -0.0233 |
| 9   | -0.0532 | 24  | 0.0211 |
| 10  | 0.0465  | 25  | -0.0206 |
| 11  | -0.0442 | 26  | 0.0196 |
| 12  | 0.0395  | 27  | -0.0193 |
| 13  | -0.0379 | 28  | 0.0183 |
| 14  | 0.0344  | 29  | -0.0181 |
| 15  | -0.0332 | 30  | 0.0172 |

For \( r > 30, \) one may write \([31]\)

\[ G(r) = \frac{3}{4} \sqrt{\frac{2}{\pi \alpha}} \frac{1}{r \sqrt{g(r)}} [1 - \frac{3}{16} g(r)^2 + \frac{156\zeta(3)}{384} g(r)^3 + O(g(r)^4) - \frac{0.4}{2r} ((-1)^r + 1 + O(g(r)) + O(\frac{1}{r^2})] \]  
\hspace{1cm} (5.87)

with \( g(r) \) satisfying

\[ g(r) = \frac{1}{C(r)} (1 + \frac{1}{2} g(r) \ln(g(r))) \]  
\hspace{1cm} (5.88)

and

\[ C(r) = \ln(2\sqrt{2\pi e^{\gamma+1}} r) \]  
\hspace{1cm} (5.89)

Eq.(5.88) may be solved by iteration. To the lowest order in \( \frac{1}{r} \) one finds

\[ g(r) \approx \frac{1}{C(r)} - \frac{1}{C(r)^2} \ln C(r) \]  
\hspace{1cm} (5.90)

Inserting (5.90) in (5.87) leads to

\[ G(r) \approx \sqrt{2\pi} \frac{1}{r} \sqrt{C(r)} [1 + \frac{1}{4C(r)} \ln C(r)] + O(\frac{1}{C(r)^2}) \]  
\hspace{1cm} (5.91)
Putting (5.91) in (5.86), one finally gets
\[
<\text{g.s.}|\vec{V} \cdot \vec{V}|\text{g.s.}> = 0.3816N . \tag{5.92}
\]

Using Eq.(5.83), the isosinglet mass reads as
\[
\frac{m_S}{e^2a} = \frac{1}{2} + 1.9509 \; \epsilon^2 . \tag{5.93}
\]

For what concerns the isotriplet mass, since the double sum in Eq.(4.6 3) is given by
\[
\Delta_{DS}(T) = 8 \frac{<\text{g.s.}|\vec{V} \cdot \vec{V}|\text{g.s.}>}{N + <\text{g.s.}|H_J|\text{g.s.}>} , \tag{5.94}
\]

using Eq.(5.92), one gets
\[
\frac{m_T}{e^2a} = \frac{1}{2} + 0.0972 \; \epsilon^2 . \tag{5.95}
\]

The existence of massive isotriplets was already noticed in [21], and their mass in the continuum theory was numerically computed for various values of the fermion mass. In particular there is a $G$-parity even isotriplet with mass approximately equal to the mass of the isosinglet $0^{--}$.

The mass of the $|S_5\rangle$ isosinglet and the $|T_5\rangle$ isotriplet is
\[
\frac{m_{S5}}{e^2a} = \frac{1}{2} + 5.85 \epsilon^2 \tag{5.96}
\]
\[
\frac{m_{T5}}{e^2a} = \frac{1}{2} + 4.4069 \epsilon^2 . \tag{5.97}
\]

Equations (5.93), (5.95), (5.96) and (5.97) provide the values of $m_S$, $m_T$ and $m_{T5}$ for small values of $z = \epsilon^2 = \frac{e^2}{16\pi^2}$ up to the second order in the strong coupling expansion. Whereas (5.93) is only approximate (5.96) and (5.97) are exact at the second order in the $\epsilon$ expansion. In section 6 we shall extrapolate these masses to the continuum limit using the standard technique of the Padé approximants.

### 6 The chiral condensate

In the following we shall first prove that also on the lattice either the isoscalar $\langle \bar{\psi}\psi \rangle$ or the isovector $\langle \bar{\psi}\sigma^a\psi \rangle$ chiral condensates are zero to every order of perturbation theory. This should be verified by explicit computation since on the lattice the symmetry $SU_L(2) \otimes SU_R(2)$ is already broken by introducing staggered fermions; thus, there is no symmetry to prevent the formation of such chiral condensates. In the continuum theory, instead, the breaking of the $SU_L(2) \otimes SU_R(2)$ down to $SU_V(2)$ is prevented by the Coleman theorem [19].

In the staggered fermion formalism the isoscalar condensate is given by
\[
\sum_{a=1}^{2} \bar{\psi}_a(x)\psi_a(x) \rightarrow \frac{(-1)^x}{2a}(\psi_{1,x}^\dagger\psi_{1,x} + \psi_{2,x}^\dagger\psi_{2,x} - 1) ; \tag{6.98}
\]

it is obtained by considering the mass operator
\[
M = \frac{1}{2Na} \sum_{x=1}^{N} (-1)^x(\psi_{1,x}^\dagger\psi_{1,x} + \psi_{2,x}^\dagger\psi_{2,x}) \tag{6.99}
\]

and evaluating its expectation value on the perturbed states $|p_{g.s.}\rangle$ generated by applying $H_h$ to $|g.s.\rangle$. To the second order in the strong coupling expansion, one has
\[
|p_{g.s.}\rangle = |g.s.\rangle + \epsilon |p_{g.s.}^{1}\rangle + \epsilon^2 |p_{g.s.}^{2}\rangle + \ldots \tag{6.100}
\]
where
\[ |p_{g.s.}^1 > = -H_{g.s.} > \]
\[ |p_{g.s.}^2 > = \frac{\Pi_{g.s.}}{2} H_{h} |g.s. > . \]  
(6.101)

(6.102)

To the fourth order, (6.99) is given by
\[ \chi_{isos.} = \frac{< p_{g.s.} | M | p_{g.s.} >}{< p_{g.s.} | p_{g.s.} >} = \frac{< g.s. | M | g.s. > + \epsilon^2 < p_{g.s.}^1 | M | p_{g.s.}^1 > + \epsilon^4 < p_{g.s.}^2 | M | p_{g.s.}^2 > + \ldots}{< g.s. | g.s. >} \]
(6.103)

It is very easy to see that \( \chi_{isos.} \) is zero to all orders in the strong coupling expansion. Let us introduce the translation operator
\[ \hat{T} = e^{ip_{t}} \]  
(6.104)

using
\[ \hat{T} M \hat{T}^{-1} = -M \]
(6.105)

\[ \hat{T} H_{h} \hat{T}^{-1} = H_{h} \]
(6.106)

and
\[ \hat{T} |g.s. > = \pm |g.s. > \]
(6.107)

one gets order by order in the strong coupling expansion in Eq.(6.103)
\[ \chi_{isos.} = -\chi_{isos.} \]  
(6.108)

In Eq.(6.107) the + appears when \( N/2 \) is even and the - when \( N/2 \) is odd.

The isovector chiral condensate is given by the expectation value of the operator
\[ \Sigma = \frac{1}{2} \sum_{x} \sum_{a} (-1)^{x} \bar{\psi}_a \sigma \psi_b \]
(6.109)

on the perturbed states \( |p_{g.s.} > \). Taking into account that
\[ \hat{T} \Sigma \hat{T}^{-1} = -\Sigma \]  
(6.110)

one gets
\[ \chi_{isov.} = -\chi_{isov.} \]  
(6.111)

also the isovector chiral condensate is identically zero.

In the continuum there is, as evidenced in section 2, only a non-vanishing chiral condensate associated to the anomalous breaking of the \( U_A(1) \) symmetry. Since
\[ \bar{\psi}^a L(x) \psi^a_R(x) = \frac{1}{2} \gamma_5 (e^\gamma_5 2 \pi e - (e^\gamma_5 2 \pi e)^2 m_S) \]  
(6.112)

the pertinent operator is \( F = \bar{\psi}^2 L(1) \psi^1 R(1) - \bar{\psi}^1 L(2) \psi^2 R(2) \) and is given by
\[ \langle \bar{\psi}^2 L(1) \psi^1 R(1) \psi^2 R(2) \rangle = \frac{(e^\gamma_5 2 \pi e)^2 m_S^2}{4 \pi} \]  
(6.113)

On the lattice one has
\[ \bar{\psi}^a L(x) \psi^a_R(x) \rightarrow \frac{1}{2a} \sum_{x} \left( \psi_{a,x}^T \psi_{a,x} - \psi_{a,x+1}^T \psi_{a,x+1} + L_x^{(a)} - R_x^{(a)} \right) \]  
(6.114)

The factor \( 1/2a \) is due to the doubling of the lattice spacing in the antiferromagnetic bipartite lattice. Upon introducing the occupation number operators \( n_{x}^{(a)} = \psi_{a,x}^T \psi_{a,x} \), the umklapp operator \( F \) is represented on the lattice by
\[ F = -\frac{1}{16a^2N} \sum_{x} \left( (n_{x}^{(1)} - n_{x+1}^{(1)})(n_{x}^{(2)} - n_{x+1}^{(2)}) + (L_x^{(1)} - R_x^{(1)})(L_x^{(2)} - R_x^{(2)}) \right) \]  
(6.115)

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The strong coupling expansion carried up to the second order in $\epsilon = \frac{t^2}{e_L^4a^4}$, yields

$$
<F > = \frac{<p_{g.s.}|F|p_{g.s.}>}{<p_{g.s.}|p_{g.s.}>} = \frac{<g.s.|F|g.s.> + \epsilon^2 <p_{g.s.}|F|p_{g.s.}>}{<g.s.|g.s.> + \epsilon^2 <p_{g.s.}|p_{g.s.}>}
$$

(6.116)

for the lattice chiral condensate. Since

$$
<g.s.|g.s.> = 1
$$

(6.117)

and

$$
<p_{g.s.}|p_{g.s.}> = -4 <g.s.\cdot H|g.s.>
$$

(6.118)

and taking into account that

$$
<p_{g.s.}|F|g.s.> = \frac{1}{8a^2N} <g.s.|H|g.s.>
$$

(6.119)

$$
<p_{g.s.}|p_{g.s.}> = \frac{1}{4a^2N}(-2 <g.s.|(H_J)^2|g.s.> - \frac{5}{3} <g.s.|H_J|g.s.> + \frac{5}{12} N)
$$

$$
- \frac{2}{3} \sum_{x=1}^{N} <g.s.|S_x \cdot S_{x+2} - \frac{1}{4}(g.s.)>
$$

(6.120)

from Eqs.(3.49) and (8.223), one gets

$$
<F > = \frac{1}{a^2} (0.0866 - 0.4043\epsilon^2)
$$

(6.121)

A nonvanishing value of the lattice chiral condensate is due to the coupling — induced by the lattice gauge field — between the right and left fermions. This is the relic in the lattice of the $U_A(1)$ anomaly in the continuum theory.

7 Lattice versus continuum

We now want to compare our lattice results with the exact results of the continuum model; to do this, one should extrapolate the strong-coupling expansion derived under the assumption that the parameter $z = \epsilon^2 = \frac{t^2}{e_L^4a^4} \ll 1$ to the region in which $z \gg 1$; this corresponds to take the continuum limit since $e_L^4a^4 \rightarrow 0$ when $z \rightarrow \infty$. To make the extrapolation possible, it is customary to make use of Padé approximants, which allow to extrapolate a series expansion beyond the convergence radius. Strong-coupling perturbation theory improved by Padé approximants should then provide results consistent with the continuum theory. As we shall see the strong-coupling expansion derived in this paper provides accurate estimates of the meson masses, already at the first order in powers of $z$.

Let us now evaluate $m_S$ and the lattice light velocity $t$. We first compute the ratio between the continuum value of the meson mass $m_S = \sqrt{\frac{2}{\pi}e_c}$ and the lattice coupling constant $e_L$ by equating the lattice chiral condensate, Eq.(6.111), to its continuum counterpart Eq.(5.113)

$$
\frac{1}{a^2} (0.0866 - 0.4043z) = \left(\frac{e_L^4}{4\pi}\right)^2 m_S^2
$$

(7.122)

Eq.(7.122) is true only when Padé approximants are used since, as it stands, the left hand side holds only for $z \ll 1$, while the right-hand side provides the value of the chiral condensate to be obtained when $z = \infty$. Using

$$
a = \frac{t^2}{e_Lsz^\frac{3}{2}}
$$

(7.123)

one gets from Eq.(7.122)

$$
\left(\frac{m_S}{e_L}\right)^2 = \left(\frac{4\pi}{e_L^2}\right)^2 \frac{z^\frac{1}{2}}{t} (0.0866 - 0.4043z)
$$

(7.124)
As in Refs. [3, 4], due to the factor \( z^2 \), the second power of Eq. (7.124) should be considered in order to construct a non diagonal Padé approximant. Since the strong coupling expansion has been carried out up to second order in \( z \), one is allowed to construct only the \([0, 1]\) Padé approximant for the polynomial written in Eq. (7.124). One gets

\[
\left( \frac{m_S}{e_L} \right)^4 = \left( \frac{4\pi}{e^7} \right)^4 \frac{1}{t^2} \frac{0.0074z}{1 + 9.3371z},
\]

and, taking the continuum limit \( z \to \infty \), one finds

\[
\left( \frac{m_S}{e_L} \right)^4 = \left( \frac{4\pi}{e^7} \right)^4 \frac{0.0008}{t^2}.
\]

Next we compute the same mass ratio by equating the singlet mass gap given in Eq. (5.93) to its continuum counterpart \( m_S \)

\[
e^2 a \left( \frac{1}{2} + 1.9509z \right) = m_S.
\]

Again, Eq. (7.127) is true only when Padé approximants are used. Dividing both sides of Eq. (7.127) by \( e_L \) and taking into account that

\[
e_L a = \frac{t^{\frac{1}{2}}}{z^4}
\]

one gets

\[
\frac{m_S}{e_L} = \frac{t^{\frac{1}{2}}}{z^4} \left( \frac{1}{2} + 1.9509z \right).
\]

Taking the fourth power and constructing the \([1, 0]\) Padé approximant for the right hand side of Eq. (7.129) one has

\[
\left( \frac{m_S}{e_L} \right)^4 = t^2 \frac{0.9754}{z}.
\]

when \( z \to \infty \), Eq. (7.130) gives

\[
\left( \frac{m_S}{e_L} \right)^4 = t^2 0.9754.
\]

The numerical value of the hopping parameter \( t \), determined if one equates Eq. (7.131) and Eq. (7.126), is

\[
t = \frac{4\pi}{e^7} 0.1692 = 1.1940
\]

and lies 19% above the exact value. Putting this value of \( t \) in Eq. (7.126) or Eq. (7.131) one gets

\[
\frac{m_S}{e_L} = 1.0969
\]

which lies 37% above the exact value \( \sqrt{2} \). It is comforting to see that the lattice reproduces in a sensible way the continuum results even if we use just first order (in \( z \)) results of the strong coupling perturbation theory.

Using the value of \( t \) given in Eq. (7.132) one gets for the isotriplet mass Eq. (7.139)

\[
\frac{m_T}{e_L} = 0.5143.
\]

By direct computation on an 8 sites chain one gets

\[
\frac{m_T}{e_L} = 1.3524.
\]

The discrepancy between Eq. (7.134) and Eq. (7.135) is mainly due to the approximation involved in the computation of \( <\text{g.s.}|V^2|\text{g.s.}> \). However, it is safe to believe that our lattice computation implies the existence of a massive isotriplet \( 1^+ \) with a mass between the lower bound (7.134) and
the upper bound \((7.135)\). This is in agreement with the results provided for the continuum theory in \([21]\).

Using a similar procedure one may also compute the masses of the singlet \(0^{++}\) and the triplet \(1^{+-}\). From Eqs.\((5.96,5.97)\) one gets

\[
\frac{m_S}{e_L} = 1.4290 \quad (7.136)
\]

\[
\frac{m_T}{e_L} = 1.3347 \quad (7.137)
\]

The triplet \(|T^5\rangle\), being \(G\)-odd, is a scattering state of a \(0^{--}\) singlet with a \(1^{-+}\) triplet, which are the fundamental excitations of the system. The mass of this \(1^{-+}\) triplet should be larger than the mass of the massive \(1^{-+}\) triplet, which should be a scattering state of massless \(1^{-+}\) triplets.

Putting \(t = c = 1\), \(i.e.\ e_La = \frac{1}{2}\), the lattice mass spectrum gets closer to its continuum counterpart; for the isosinglet mass, one gets

\[
\frac{m_S}{e} = 0.9938 \quad (7.138)
\]

while for the isotriplet one gets

\[
\frac{m_T}{e_L} = 0.4695 \quad (7.139)
\]

Eq.\((7.138)\) provides a value of the isosinglet mass lying 24\% above the exact answer. Again the triplet mass is reproduced with lesser accuracy due to the random phase approximation used in the computation of the pertinent correlator; a better answer is given however by a direct computation on the 8 sites chain yielding the value 1.2346 for \(m_T/e_L\).

8 Concluding remarks

In this paper we used the correspondence between the 2-flavor strongly coupled lattice Schwinger model and the antiferromagnetic Heisenberg Hamiltonian established in \([3]\) to investigate the spectrum of the gauge model. Using the analysis of the excitations of the finite size chains given in the appendix, we showed the equality of the quantum numbers of the states of the Heisenberg model and the low-lying excitations of the 2-flavor Schwinger model. We provided also the spectrum of the massive excitations of the gauge model; in order to extract numerical values for the masses, we explicitly computed the pertinent spin-spin correlators of the Heisenberg chain. Although the spectrum is determined only up to the second order in the strong coupling expansion the agreement with the continuum theory is satisfactory.

The massless and the massive excitations of the gauge model are created from the spin chain ground state with two very different mechanisms: massless excitations involve only spin flipping while massive excitations are created by fermion transport besides spin flipping and do not belong to the spin chain spectrum. As in the continuum theory, due to the Coleman theorem \([20]\), the massless excitations are not Goldstone bosons, but may be regarded as the gapless quantum excitations of the spin-\(\frac{1}{2}\) antiferromagnetic Heisenberg chain \([35]\).

In computing the chiral condensate we showed that, also in the lattice theory, the expectation value of the umklapp operator \(F\) is different from zero, while both \(\langle \bar{\psi} \psi \rangle\) and \(\langle \bar{\psi} \sigma^a \psi \rangle\) are zero to every order in the strong coupling expansion. This implies that both on the lattice and the continuum the \(SU(2)\) flavor symmetry is preserved whereas the \(U_A(1)\) axial symmetry is broken. The umklapp operator \(F\) is the order parameter for this symmetry, but being quadri-linear in the fermi fields, is invariant, in the continuum, under chiral rotation of \(\pi/2\) and on the lattice under the corresponding discrete axial symmetry \((2.30)\) (translation by one lattice site). This shows that the discrete axial symmetry is not broken in both cases. Our lattice computation enhance this result since the ground state of the strongly coupled 2-flavor Schwinger model is translationally invariant.

The pattern of symmetry breaking of the continuum is exactly reproduced even if the Coleman theorem does not apply on the lattice and the anomalous symmetry breaking is impossible due to
the Nielsen-Ninomiya theorem. At variance with the strongly coupled 1-flavor lattice Schwinger model, the anomaly is not realized in the lattice theory via the spontaneous breaking of a residual chiral symmetry, but, rather, by explicit breaking of the chiral symmetry due to staggered fermions. The non-vanishing of \(< F >\) may be regarded as the only relic, in the strongly coupled lattice theory, of the anomaly of the continuum 2-flavor Schwinger model. It is due to the coupling induced by the gauge field, between the right and left-movers on the lattice.

It is quite straightforward to generalize our analysis to an \(SU(N)\)-flavor group. For this, one should observe that the results are very different depending on if \(N\) is even or odd. When \(N\) is odd, the ground state energy in the strong coupling limit is proportional to \(e^2\) but, when \(N\) is even, the ground state energy is of order 1. This difference arises since, on the lattice, the charge density operator is odd under charge conjugation; therefore, the constant \(\frac{N}{2}\) should be subtracted from the charge density operator. As a consequence, when \(N\) is odd, the ground state supports electric fluxes while this becomes impossible when \(N\) is even.

When the fermion mass \(m\) is different from zero, some further difference arises between \(N\) odd and \(N\) even. When \(N\) is odd, the mass term induces a translational non invariant ground state, generating a spontaneous chiral symmetry breaking. When \(N\) is even, the ground state remains translationally invariant in the strong coupling limit, \textit{i.e.} \(e^2 \gg m\). In the weak coupling limit, \(m \gg e^2\), the discrete chiral symmetry is spontaneously broken for every \(N\). For \(N = 2\), the soliton-antisoliton excitations acquire a mass.

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Appendix A:
Exact diagonalization of finite size antiferromagnetic Heisenberg chains

In the following we shall provide the exact diagonalization of the Heisenberg antiferromagnetic model for finite size chains of \(N = 4, 6\) and 8 sites and compare the results with the Bethe Ansatz solution provided in [14]. In order to make our arguments self-contained, we shall outline the steps involved in deriving the exact solution in the thermodynamic limit. We shall show that already very small finite size chains exhibit spectra that match very well with the thermodynamic limit solution. Furthermore the analysis of finite size chains is very useful since it allows the comparison between the quantum numbers of the Schwinger and the Heisenberg model excitations. The interested reader may also look up references [14, 36, 29].

The one dimensional isotropic Heisenberg model describes a system of \(N\) interacting spin-\(\frac{1}{2}\) particles. The Hamiltonian of the model is

\[
H_J = J \sum_{x=1}^{N} (\vec{S}_x \cdot \vec{S}_{x+1} - \frac{1}{4}) .
\]

(8.140)

where \(J > 0\) (\(J < 0\) would describe a ferromagnet) and the spin operators have the following form

\[
\vec{S}_x = 1_1 \otimes 1_2 \otimes \ldots \otimes \frac{\vec{\sigma}_x}{2} \otimes \ldots \otimes 1_N .
\]

(8.141)
They act nontrivially only on the Hilbert space of the $x^{th}$ site. Periodic boundary conditions are assumed.

The Hamiltonian is invariant under global rotations in the spin space, generated by

$$\vec{S} = \sum_{x=1}^{N} \vec{S}_x$$  \hspace{1cm} (8.142)

Due to the periodic boundary conditions, under translations generated by the operator $\hat{T}$,

$$\hat{T} : \vec{S}_x \rightarrow \vec{S}_{x-1}$$  \hspace{1cm} (8.143)

the Hamiltonian is invariant and $[\vec{S}, \hat{T}] = 0$.

The energy and the momentum of a given state with $M$ spins down can be written as

$$E_M = \sum_{\alpha=1}^{M} \epsilon_\alpha = -\frac{J}{2} \sum_{\alpha=1}^{M} \frac{1}{\lambda_\alpha^2 + \frac{i}{2}}$$  \hspace{1cm} (8.144)

$$P_M = i \ln T = \sum_{\alpha=1}^{M} p_\alpha = i \sum_{\alpha=1}^{M} \ln \left(\frac{\lambda_\alpha - i}{\lambda_\alpha + i}\right)$$  \hspace{1cm} (8.145)

Energy and momentum are thus additive as if there were $M$ independent particles and the parameters $\lambda_\alpha$ must satisfy the Bethe Ansatz equations

$$\left(\frac{\lambda_\alpha - i}{\lambda_\alpha + i}\right)^N = -\prod_{\beta=1}^{M} \frac{\lambda_\alpha - \lambda_\beta - i}{\lambda_\alpha - \lambda_\beta + i}$$  \hspace{1cm} (8.146)

in order for $E_M$ and $P_M$ to be eigenvalues of the Hamiltonian and momentum operator. See for a derivation of Eqs.(8.146).

The solution of the antiferromagnetic Heisenberg chain is reduced to the solution of the system of the $M$ algebraic equations (8.146). This, in general, is not an easy task. It can be shown, however, that, in the thermodynamic limit $N \rightarrow \infty$, the complex parameters $\lambda_\alpha$ have the form

$$\lambda_\alpha = \lambda_{j,L} + il \hspace{1cm} l = -L, -L + 1, \ldots, L - 1, L;$$  \hspace{1cm} (8.147)

where $L$ is a non-negative integer or half-integer, $\lambda_{i,L}$ is the real part of the solution of (8.146) and we shall define shortly the set of allowed values for the integer index $j$. The $\lambda$’s that, for a given $\lambda_{j,L}$, are obtained varying $l$ between $[-L, L]$ by integer steps, form a string of length $2L + 1$, see fig.(1). This arrangement of $\lambda$’s in the complex plane is called the “string hypothesis” [14]. In the following we shall verify that, even on finite size systems, the “string hypothesis” is very well fulfilled.

In a generic Bethe state with $M$ spins down, there are $M$ solutions to (8.146), which can be grouped according to the length of their strings. Let us denote by $\nu_L$ the number of strings of length $2L + 1$, $L = 0, \frac{1}{2}, \ldots$; strings of the same length are obtained by changing the real parts, $\lambda_{j,L}$, of the $\lambda$’s in (8.147); as a consequence $j = 1, \ldots, \nu_L$. If one denotes the total number of strings by $q$ one has

$$q = \sum_{L} \nu_L \hspace{1cm} M = \sum_{L} (2L + 1) \nu_L$$  \hspace{1cm} (8.148)

The set of integers $(M, q, \{\nu_L\})$ constrained by (8.148), characterizes Bethe states up to the fixing of the $q$ numbers $\lambda_{j,L}$; this set is called the “configuration”. Varying $M$, $q$ and $\nu_L$, one is able to construct all the $2^N$ eigenstates of an Heisenberg antiferromagnetic chain of $N$ sites [14]. The energy and momentum of the Bethe’s state, corresponding to a given configuration – within exponential accuracy as $N \rightarrow \infty$ – consist of $q$ summands representing the energy and momentum.
of separate strings. For the parameters $\lambda_{j,L}$ of the given configuration, taking the logarithm of (8.146) the following system of equations is obtained in the thermodynamic limit

$$2N \arctan \frac{\lambda_{j,L_1}}{L_1 + \frac{1}{2}} = 2\pi Q_{j,L_1} + \sum_{L_2} \sum_{k=1}^{\nu_{L_2}} \Phi_{L_1,L_2}(\lambda_{j,L_1} - \lambda_{k,L_2}) ,$$

where

$$\Phi_{L_1,L_2}(\lambda) = 2 \sum_{L=|L_1-L_2|\neq0} (\arctan \frac{\lambda}{L} + \arctan \frac{\lambda}{L+1}) .$$

Integer and half integer numbers $Q_{j,L}$ parametrize the branches of the arcotangents and, consequently, the possible solutions of the system of Eqs.(8.149). In ref.\[14\] it was shown that the $Q_{j,L}$ are limited as

$$-Q_{L_1}^{\text{max}} \leq Q_{1,L} < Q_{2,L} < \ldots < Q_{\nu_{L_2},L} \leq Q_{L_2}^{\text{max}}$$

with $Q_{L_2}^{\text{max}}$ given by

$$Q_{L_2}^{\text{max}} = \frac{N}{2} - \sum_{L'} J(L,L') \nu_{L'}, - \frac{1}{2}$$

and

$$J(L_1,L_2) = \begin{cases} 2\text{min}(L_1,L_2) + 1 & \text{if } L_1 \neq L_2 \\ 2L_1 + \frac{1}{2} & \text{if } L_1 = L_2 \end{cases} .$$

The admissible values for the numbers $Q_{j,L}$ are called the “vacancies” and their number for every $L$ is denoted by $P_L$

$$P_L = 2Q_{L_2}^{\text{max}} + 1 .$$

The main hypothesis formulated in \[14\] is that to every admissible collection of $Q_{j,L}$ there corresponds a unique solution of the system of equations (8.149). The solution always provides, in a multiplet, the state with the highest value of the third spin component $S^3$.

Let us now consider some simple example. The simplest configuration has only strings of length 1, i.e. all the $\lambda$‘s are real. The singlet associated to this configuration

$$M = q = \nu_0 = \frac{N}{2} \quad , \quad \nu_L = 0 \quad , \quad L > 0 \quad ,$$

Figure 1: Strings for $L = 0, \frac{1}{2}, 1, \frac{3}{2}$
is the ground state. The vacancies of the strings of length 1, i.e. the admissible values of \( Q_{j,0} \), due to eqs. (8.151, 8.152, 8.153), belong to the segment
\[
-\frac{N}{4} + \frac{1}{2} \leq Q_{j,0} \leq \frac{N}{4} - \frac{1}{2}.
\]
Therefore they are \( N/2 \). All these vacancies must then be used to find the \( N/2 \) strings of length 1. As a consequence this state is uniquely specified and no degeneracy is possible.

Next we consider the configuration that provides a singlet with 1 string of length 2 and all the others of length 1:

\[
M = \frac{N}{2}, \quad q = \frac{N}{2} - 1, \quad \nu_0 = \frac{N}{2} - 2, \quad \nu_2 = 1, \quad \nu_L = 0, \quad L > \frac{1}{2}.
\]

For the strings of length 1 the number of vacancies is again \( N/2 \); for the string of length 2 there is one vacancy and the only admissible \( Q_{j,1} \) equals 0. Thus, since the number of strings of length 1 is \( \nu_0 = \frac{N}{2} - 2 \), there are two vacancies for which Eqs. (8.149) have no solution; they are called “holes” and are denoted \( Q_1^{(h)} \) and \( Q_2^{(h)} \). This configuration is determined by two parameters: the positions of two “holes” which vary independently in the interval (8.156).

There is another state with only 2 holes: the triplet corresponding to the configuration

\[
M = q = \nu_0 = \frac{N}{2} - 1, \quad \nu_L = 0, \quad L > 0
\]

The number of vacancies for the strings of length 1 equals \( \frac{N}{2} + 1 \), while \( \nu_0 = \frac{N}{2} - 1 \).

The excitations determined by the configurations (8.157, 8.158) belong to the configuration class called in [14] \( \mathcal{M}_{AF} \). The class \( \mathcal{M}_{AF} \) is characterized as follows: the number of strings of length 1 in each configuration belonging to this class differs by a finite quantity from \( \frac{N}{2} \), \( \nu_0 = \frac{N}{2} - k_0 \) where \( k_0 \) is a positive finite constant, so that the number of strings of length greater than 1 is finite. From (8.154) we then have

\[
P_0 = \frac{N}{2} + k_0 - 2 \sum_{L > 0} \nu_L
\]

\[
P_L = 2k_0 - 2 \sum_{L' > 0} J(L, L') \nu_{L'}, \quad L > 0
\]

so that

\[
P_0 \geq \frac{N}{2}, \quad P_L < 2k_0, \quad L > 0.
\]

From (8.159) follows that the number of holes for the strings of length 1 is always even and equals 2 only for the singlet and the triplet excitations discussed above. One can imagine the class \( \mathcal{M}_{AF} \) as a “sea” of strings of length 1 with a finite number of strings of length greater than 1 immersed into it. It was proven in [14] that, in the thermodynamic limit, the states belonging to \( \mathcal{M}_{AF} \) have finite energy and momentum with respect to the antiferromagnetic vacuum, whereas each of the states which corresponds to a configuration not included in the class \( \mathcal{M}_{AF} \) has an infinite energy relative to the antiferromagnetic vacuum.

Let us now sketch the computation of the thermodynamic limit ground state energy. Eqs. (8.149) for the ground state have the form

\[
\arctan 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N} \sum_{k=1}^{N/2} \arctan(\lambda_j - \lambda_k).
\]

Taking the thermodynamic limit \( N \to \infty \), one has

\[
\frac{Q_j}{N} \to x, \quad -\frac{1}{4} \leq x \leq \frac{1}{4}, \quad \lambda_j \to \lambda(x),
\]

(8.163)
and Eqs. (8.162) can be rewritten in the form

\[
\arctan 2\lambda(x) = \pi x + \int_{-\frac{1}{4}}^{\frac{1}{4}} \arctan(\lambda(x) - \lambda(y))dy .
\]

(8.164)

Upon introducing the density of the numbers \(\lambda(x)\) in the interval \(d\lambda\)

\[
\rho(\lambda) = \frac{1}{d\lambda(x)|_{x=x(\lambda)}}
\]

(8.165)

and differentiating Eqs. (8.164), one gets

\[
\rho(\lambda) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{e^{-|\xi|}}{1 + e^{-|\xi|}} e^{-i\lambda|\xi|} d\xi = \frac{1}{2} \cosh \pi \lambda .
\]

(8.166)

The density \(\rho(\lambda)\) introduced in this way is normalized to \(1/2\). It is now easy to compute the energy and the momentum of the ground state

\[
E_{g.s.} = \sum_{\alpha=1}^{N/2} \epsilon_{\alpha} = N \int_{-\infty}^{\infty} \epsilon(\lambda)\rho(\lambda)d\lambda = -\frac{JN}{4} \int_{-\infty}^{\infty} d\lambda \left( \frac{\lambda^2}{\cosh \pi \lambda} \right) = -JN \ln 2
\]

(8.167)

\[
P_{g.s.} = \sum_{\alpha=1}^{N/2} p_{\alpha} = N \int_{-\infty}^{\infty} \rho(\lambda)d\lambda = \frac{\pi}{2} \int_{-\infty}^{\infty} d\lambda \frac{\pi}{\cosh \pi \lambda} = \frac{N}{2} \pi \pmod{2\pi}
\]

(8.168)

According to Eq. (8.168), \(P_{g.s.} = 0 \pmod{2\pi}\) for \(N\) even, and \(P_{g.s.} = \pi \pmod{2\pi}\) for \(N\) odd. The ground state, as expected, is a singlet, in fact the spin \(S\) is given by

\[
S = \frac{N}{2} - \sum_{\alpha=1}^{N/2} 1 = \frac{N}{2} - N \int_{-\infty}^{\infty} \rho(\lambda)d\lambda = 0
\]

(8.169)

Let us analyze the triplet described by Eq. (8.158); Eqs. (8.149) take the form

\[
\arctan 2\lambda_j = \pi Q_j N + \frac{1}{N} \sum_{k=1}^{N-1} \arctan(\lambda_j - \lambda_k)
\]

(8.170)

where now the numbers \(Q_j\) lie in the segment \([-\frac{N}{4}, \frac{N}{4}]\) and have two holes, \(Q_1^{(h)}\) and \(Q_2^{(h)}\) with \(Q_1^{(h)} < Q_2^{(h)}\). Taking the thermodynamic limit one gets

\[
\frac{Q_1^{(h)}}{N} \rightarrow x_1 , \quad \frac{Q_2^{(h)}}{N} \rightarrow x_2 , \quad \frac{Q_j}{N} \rightarrow x + \frac{1}{N}(\theta(x - x_1) + \theta(x - x_2))
\]

(8.171)

where \(\theta(x)\) is the Heaviside function. Eqs. (8.170) become

\[
\arctan 2\lambda(x) = \pi x + \frac{\pi}{N}(\theta(x - x_1) + \theta(x - x_2)) + \int_{-\frac{1}{4}}^{\frac{1}{4}} \arctan(\lambda(x) - \lambda(y))dy .
\]

(8.172)

Eq. (8.172) gives, for this triplet, the density of \(\lambda\), \(\rho(\lambda) = \frac{d\lambda}{dx}\)

\[
\rho(x) = \rho(\lambda) + \frac{1}{N}(\sigma(\lambda - \lambda_1) - \sigma(\lambda - \lambda_2))
\]

(8.173)

where \(\rho(\lambda)\) is given in Eq. (8.166) and

\[
\sigma(\lambda) = -\frac{1}{2\pi} \int_{-\infty}^{\infty} \frac{1}{1 + e^{-i\lambda\xi}} e^{-i\lambda\xi} d\xi
\]

(8.174)

22
\( \lambda_1 \) and \( \lambda_2 \) are the parameters of the holes, \( \lambda_i = \lambda(x_i), \, i = 1, 2 \). The energy and the momentum of this state measured from the ground state are now easily computed

\[
\epsilon_T(\lambda_1, \lambda_2) = N \int_{-\infty}^{\infty} \epsilon(\lambda)(\rho(\lambda) - \rho(\lambda))d\lambda = \epsilon(\lambda_1) + \epsilon(\lambda_2) \tag{8.175}
\]

\[
p_T(\lambda_1, \lambda_2) = N \int_{-\infty}^{\infty} p(\lambda)(\rho(\lambda) - \rho(\lambda))d\lambda = p(\lambda_1) + p(\lambda_2) \pmod{2\pi} \tag{8.176}
\]

where

\[
\epsilon(\lambda) = \int_{-\infty}^{\infty} \epsilon(\mu)\sigma(\lambda - \mu)d\mu = J_\pi \frac{\pi}{2\cosh \pi \lambda} \tag{8.177}
\]

\[
p(\lambda) = \int_{-\infty}^{\infty} p(\mu)\sigma(\lambda - \mu)d\mu = \arctan \sinh \pi \lambda - \frac{\pi}{2}, \quad -\pi \leq p(\lambda) \leq 0 . \tag{8.178}
\]

From Eqs. \((8.177,8.178)\) one gets

\[
\epsilon = - \frac{J\pi}{2} \sin p . \tag{8.179}
\]

The momentum \( p_T(\lambda_1, \lambda_2) \) varies over the interval \([0, 2\pi]\), when \( \lambda_1 \) and \( \lambda_2 \) run independently over the whole real axis. The spin of this state can be computed by the formula

\[
S = - \int_{-\infty}^{\infty} (\sigma(\lambda - \lambda_1) + \sigma(\lambda - \lambda_2))d\lambda = 1 . \tag{8.180}
\]

Let us finally analyze the singlet excitation characterized by the configuration \((8.157)\). Denoting by \( \lambda_S \) the only number among the \( \lambda_j, 1/2 \) which characterizes the string of length 2 and by \( \lambda_j \) the numbers \( \lambda_j, 0 \) for the strings of length 1, Eqs. \((8.149)\) read

\[
\arctan 2\lambda_j = \frac{\pi Q_j}{N} + \frac{1}{N} \Phi(\lambda_j - \lambda_S) + \frac{1}{N} \sum_{k=1}^{N-2} \arctan(\lambda_j - \lambda_k) \tag{8.181}
\]

\[
\arctan \lambda_S = \frac{1}{N} \sum_{j=1}^{N-2} \Phi(\lambda_S - \lambda_j) \tag{8.182}
\]

with

\[
\Phi(\lambda) = \arctan 2\lambda + \arctan \frac{2}{3} \lambda . \tag{8.183}
\]

The \( \frac{N}{2} - 2 \) numbers \( Q_j \) vary in the segment \([-\frac{N}{4} + \frac{1}{2}, \frac{N}{4} - \frac{1}{2}]\); among them there are the two holes \( Q_1^{(h)} \) and \( Q_2^{(h)} \). Taking the thermodynamic limit one finds the density of \( \lambda \)'s for the singlet

\[
\rho(\lambda_S) = \rho(\lambda) + \frac{1}{N}(\sigma(\lambda - \lambda_1) + \sigma(\lambda - \lambda_2) + \omega(\lambda - \lambda_S)) \tag{8.184}
\]

where \( \rho \) and \( \sigma \) were given in Eqs. \((8.166, 8.174)\) and where

\[
\omega(\lambda) = - \frac{1}{2\pi} \int_{-\infty}^{\infty} e^{-\frac{1}{2}(|\xi| - i\lambda \xi)}d\xi = - \frac{2}{\pi(1 + 4\lambda^2)} . \tag{8.185}
\]

In \([14]\) it was demonstrated that the string parameter \( \lambda_S \) is uniquely determined by the \( \lambda \)'s parametrizing the two holes

\[
\lambda_S = \frac{\lambda_1^{(h)} + \lambda_2^{(h)}}{2} . \tag{8.186}
\]

In \([14]\) it was also proved the remarkable fact that the string of length 2 does not contribute to the energy and momentum of the excitation, so that the singlet and the triplet have the same dispersion relations

\[
\epsilon_S(\lambda_1, \lambda_2) = \epsilon_T(\lambda_1, \lambda_2) = \epsilon(\lambda_1) + \epsilon(\lambda_2) \tag{8.187}
\]

\[
p_S(\lambda_1, \lambda_2) = p_T(\lambda_1, \lambda_2) = p(\lambda_1) + p(\lambda_2) \pmod{2\pi} . \tag{8.188}
\]
The spin of this excitation is, of course, zero

\[ S = -2 - \int_{-\infty}^{\infty} (2\sigma(\lambda) + \omega(\lambda))d\lambda = 0 \quad (8.189) \]

The only difference between the state whose configuration is given in Eq. (8.158) and the state of Eq. (8.157) is the spin.

To summarize, the finite energy excitations of the antiferromagnetic Heisenberg chain are only those belonging to the class \( \mathcal{M}_{AF} \) and are described by scattering states of an even number of quasiparticles or kinks. The momentum \( p \) of these kinks runs over half the Brillouin zone \( -\pi \leq p \leq 0 \), the dispersion relation is \( \epsilon(p) = \frac{2\pi}{\sqrt{P}} \sin p \), Eq. (8.179), and the spin of a kink is \( 1/2 \). The singlet and the triplet excitations described above are the only states composed of two kinks, the spins of the kinks being parallel for the triplet and antiparallel for the singlet. For vanishing total momentum all the states belonging to \( \mathcal{M}_{AF} \) have the same energy of the ground state so that they are gapless excitations. Since the eigenstates of \( H \) always contain an even number of kinks, the dispersion relation is determined by a a set of two-parameters: the momenta of the even number of kinks whose scattering provides the excitation. There are no bound states of kinks.

Let us now turn to the computation of the spectrum of finite size quantum antiferromagnetic chains by exact diagonalization. We shall see that already for very small chains, the spectrum is well described by the Bethe ansatz solution in the thermodynamic limit. Furthermore, an intuitive picture of the ground state and of the lowest lying excitations of the strongly coupled 2-flavor lattice Schwinger model emerges.

The states of an antiferromagnetic chain are classified according to the quantum numbers of spin, third spin component, energy and momentum \( |S, S^3, E, p > \). For a 4 site chain the momenta allowed for the states are: \( 0, \frac{\pi}{2}, \frac{3\pi}{2} \) mod \( 2\pi \). The ground state is

\[ |g.s. > = |0, 0, -3J, 0 > = \frac{1}{\sqrt{12}} (2|\uparrow\uparrow\downarrow\downarrow> + |\uparrow\downarrow\uparrow\downarrow> - |\uparrow\uparrow\downarrow\downarrow> - |\uparrow\downarrow\uparrow\downarrow> - |\downarrow\uparrow\uparrow\downarrow> - |\downarrow\uparrow\downarrow\downarrow>) \quad (8.190) \]

This state is \( P \)-parity even. In fact, by the definition of \( P \)-parity given in Eq. (2.23), the \( P \)-parity inverted state Eq. (8.190) is obtained by reverting the order of the spins in each vector \( |\ldots > \) appearing in Eq. (8.190), e.g. \( |\downarrow\downarrow\uparrow\uparrow> \rightarrow |\uparrow\uparrow\downarrow\downarrow> \).

The \( \lambda \)'s associated to the ground state (solution of the Bethe ansatz equations (8.146)) are \( \lambda_1 = -\frac{1}{\sqrt{2}} \) and \( \lambda_2 = \frac{1}{\sqrt{2}} \). There is also an excited singlet

\[ |0, 0, -J, \pi > = \frac{1}{\sqrt{4}} (|\downarrow\uparrow\uparrow\uparrow> - |\downarrow\uparrow\downarrow\downarrow> - |\uparrow\downarrow\uparrow\downarrow> + |\uparrow\downarrow\downarrow\downarrow>) \quad (8.191) \]

It is \( P \)-even, so that it is a \( S^P = 0^+ \) excitation, with the same quantum numbers (the isospin is replaced by the spin) of the lowest lying singlet excitation of the strongly coupled Schwinger model discussed by Coleman [20]. The state Eq. (8.191) also coincides with the excited singlet described by the configuration (8.157). It has only two complex \( \lambda \)'s which arrange themselves in a string approximately of length 2, \( \lambda_1 = -\lambda_2 = \frac{i}{\sqrt{2}} \) and there are two holes with \( Q_1^{(h)} = -\frac{1}{2} \) and \( Q_2^{(h)} = \frac{1}{2} \).

There are also three excited triplets, whose highest weight states are

\[ |1, 1, -J, \pi > = \frac{1}{\sqrt{4}} (|\uparrow\uparrow\uparrow\uparrow> + i|\uparrow\uparrow\downarrow\downarrow> - |\uparrow\downarrow\uparrow\downarrow> - i|\uparrow\downarrow\downarrow\downarrow>) \quad (8.192) \]
\[ |1, 1, -2J, \pi > = \frac{1}{\sqrt{4}} (|\downarrow\uparrow\uparrow\uparrow> - |\uparrow\uparrow\downarrow\downarrow> + |\uparrow\downarrow\uparrow\downarrow> - |\uparrow\downarrow\downarrow\downarrow>) \quad (8.193) \]
\[ |1, 1, -J, 3\pi > = \frac{1}{\sqrt{4}} (|\downarrow\uparrow\uparrow\uparrow> - i|\uparrow\uparrow\downarrow\downarrow> - |\uparrow\downarrow\uparrow\downarrow> + i|\uparrow\downarrow\downarrow\downarrow>) \quad (8.194) \]

Among these, only the non-degenerate state with the lowest energy has a well defined \( P \)-parity (8.193). It is a \( S^P = 1^- \) like the lowest lying triplet of the 2-flavor strongly coupled Schwinger model. The degenerate states can be always combined to form a \( P \)-odd state.
We thus see that within the states in a given configuration there is always a representative state with well defined parity, the others are degenerate and can be used to construct state of well defined energy and parity. Moreover the parity of the representative states (with respect to the parity of the ground state) is the same of the one of the lowest-lying Schwinger model excitations in strong coupling.

All the triplets in (8.194) have one real $\lambda$ and two holes; they can be associated with the family of triplets (8.158). In table (2) we summarize the triplet $\lambda$'s and $Q^{(h)}$'s. The spectrum exhibits also

Table 2: Triplet internal quantum numbers

| TRIPLET       | $\lambda$ | $Q_1^{(h)}$ | $Q_2^{(h)}$ |
|---------------|-----------|-------------|-------------|
| $|1, 1, -J, \frac{\pi}{2}>$ | $\frac{1}{2}$ | $-1$ | $0$ |
| $|1, 1, -2J, \pi>$ | $0$ | $-1$ | $1$ |
| $|1, 1, -J, \frac{3\pi}{2}>$ | $-\frac{1}{2}$ | $0$ | $1$ |

a quintet, whose highest weight state is

$$|2, 2, 0, 0> = |\uparrow\uparrow\uparrow\uparrow>$$

(8.195)

We report in fig.(2) the spectrum of the 4 sites chain.

Let us analyze the spectrum of the 6 site antiferromagnetic chain. The momenta allowed for the states are now $0, \frac{\pi}{3}, \frac{2\pi}{3}, \pi, \frac{4\pi}{3}, \frac{5\pi}{3}$ mod $2\pi$. The ground state is

$$|g.s.> = |0, 0, -J \frac{2(5 + \sqrt{13})}{\sqrt{26 - 6\sqrt{13}}}, \pi> = \frac{1}{\sqrt{26 - 6\sqrt{13}}}(|\downarrow\uparrow\downarrow\uparrow\uparrow\downarrow> - |\uparrow\downarrow\uparrow\uparrow\downarrow\uparrow>)$$

Figure 2: Four sites chain spectrum
\[
+ \frac{1 - \sqrt{13}}{6}(| \uparrow\uparrow\uparrow\downarrow\downarrow > - | \uparrow\uparrow\downarrow\uparrow\downarrow > + | \downarrow\uparrow\downarrow\uparrow\uparrow > - | \downarrow\uparrow\downarrow\uparrow\downarrow > + | \uparrow\uparrow\uparrow\downarrow\downarrow > - | \downarrow\uparrow\uparrow\downarrow\downarrow > \\
- | \downarrow\downarrow\uparrow\uparrow\uparrow > + | \uparrow\downarrow\uparrow\uparrow\uparrow > - | \uparrow\uparrow\downarrow\uparrow\uparrow > - | \uparrow\downarrow\uparrow\uparrow\uparrow > + | \down\uparrow\uparrow\uparrow\uparrow > + | \up\uparrow\uparrow\uparrow\uparrow > \\
+ \frac{4 - \sqrt{13}}{3}(| \up\up\up\up\up > - | \up\up\up\down\down > + | \up\up\down\up\down > - | \down\up\up\up\down > + | \down\down\up\up\up > - | \down\down\down\up\up > )}
\tag{8.196}
\]

This state is odd under \( P \)-parity. The spectrum of the six sites chain is reported in fig.(3). There are 9 triplets in the spectrum. In \([37]\) it was already pointed out that the number of lowest lying triplets for a finite system with \( N \) sites is \( N(N + 2)/8 \), so for \( N = 6 \) there are 6 lowest lying triplet states. In order to identify these 6 states among the 9 that are exhibited by the spectrum of fig.(3), it is necessary to compute their \( \lambda \)'s and their \( Q \)'s. In this way in fact, we can find out which are the triplets characterized by two holes and thus belonging to the triplet of type \((8.158)\). In table (3) we report the internal quantum numbers of the lowest lying triplets. The \( Q^{(h)} \)'s vary in the segment \([-\frac{3}{2}, \frac{3}{2}]\). The highest weight state of the triplet of zero momentum and energy \(-(J/2)(5 + \sqrt{5})\) reads

\[
|0, 0, -\frac{J}{2}(5 + \sqrt{5}), 0 > = \frac{1}{\sqrt{45 - 15\sqrt{5}}}(\frac{-3 + \sqrt{5}}{2}( | \down\down\up\up\up > + | \up\up\down\up\up > + | \up\up\up\down\down > + | \down\up\up\up\down > + | \down\down\down\up\up > + | \down\down\down\down\up > + | \up\down\up\up\up > + | \down\up\down\up\up > + | \up\up\down\up\up > ) \\
+ (1 - \sqrt{5})( | \up\up\down\up\up > + | \up\up\up\down\down > + | \down\up\up\up\down > + | \down\up\down\up\up > )}
\tag{8.197}
\]

One can get the triplet of energy \(-(J/2)(5 - \sqrt{5})\) from \([8.197]\) by changing \( \sqrt{5} \rightarrow -\sqrt{5} \). As can be explicitly checked from \([8.197]\), the two non-degenerate triplets of zero momentum are then \( P \)-parity even, namely they have opposite parity with respect to that of the ground state, as it happens for the lowest lying triplet excitations of the 2-flavor Schwinger model. For what concerns the degenerate triplets of momenta \( \pi/3 \) and \( 5\pi/3 \) (or \( 2\pi/3 \) and \( 4\pi/3 \)) they do not have definite \( P \)-parity, but it is always possible to take a linear combination of them with parity opposite to the ground state.

| Table 3: Triplet internal quantum numbers |
|---|---|---|---|
| TRIPLET | \( \lambda_1 \) | \( \lambda_2 \) | \( Q_1^{(h)} \) | \( Q_2^{(h)} \) |
| \(|1, 1, -\frac{5+\sqrt{5}}{2}J, 0 >\) | \(-\sqrt{\frac{5-2\sqrt{5}}{20}}\) | \(\sqrt{\frac{5-2\sqrt{5}}{20}}\) | \(-\frac{1}{2}\) | \(\frac{1}{2}\) |
| \(|1, 1, -\frac{5-\sqrt{5}}{2}J, 0 >\) | \(-\sqrt{\frac{5+2\sqrt{5}}{20}}\) | \(\sqrt{\frac{5+2\sqrt{5}}{20}}\) | \(-\frac{3}{2}\) | \(\frac{3}{2}\) |
| \(|1, 1, -\frac{3J}{2}, \frac{\pi}{3} >\) | \(-\sqrt{\frac{3+\sqrt{5}}{8}}\) | \(-\sqrt{\frac{3-\sqrt{5}}{8}}\) | \(-\frac{3}{2}\) | \(\frac{1}{2}\) |
| \(|1, 1, -\frac{7+\sqrt{17}}{4}J, \frac{2\pi}{3} >\) | \(-\sqrt{\frac{3-\sqrt{5}}{8}}\) | \(\sqrt{\frac{5+\sqrt{5}}{2}}\) | \(-\frac{3}{2}\) | \(\frac{1}{2}\) |
| \(|1, 1, -\frac{7+\sqrt{17}}{4}J, \frac{4\pi}{3} >\) | \(\sqrt{\frac{3-\sqrt{5}}{8}}\) | \(\sqrt{\frac{5+\sqrt{5}}{2}}\) | \(\frac{3}{2}\) | \(\frac{3}{2}\) |

The remaining three triplets in fig.(3) have no real \( \lambda \)'s and are characterized by a string of length 2 and four holes for \( Q = -\frac{3}{2}, -\frac{1}{2}, \frac{1}{2}, \frac{3}{2} \), \( i.e. \) do not belong to the type \((8.158)\). More precisely, two triplets have a string approximately of length 2, due to the finite size of the system, while the triplet
of momentum \( \pi \) has a string exactly of length 2. In table (4) we summarize the quantum numbers of these triplets.

**Table 4: Four holes triplet internal quantum numbers**

| TRIPLET | \( \lambda_1 \) | \( \lambda_2 \) |
|---------|----------------|----------------|
| \(|1, 1, -\frac{7-\sqrt{17}}{4} J, \frac{2\pi}{3}\) | \( \frac{2\sqrt{3}-i\sqrt{2+2\sqrt{17}}}{2+2\sqrt{17}} \) | \( \frac{2\sqrt{3}+i\sqrt{2+2\sqrt{17}}}{2+2\sqrt{17}} \) |
| \(|1, 1, -J, \pi\) | \(-\frac{3}{2}\) | \(\frac{1}{2}\) |
| \(|1, 1, -\frac{7-\sqrt{17}}{4} J, \frac{4\pi}{3}\) | \(\frac{2\sqrt{3}+i\sqrt{2+2\sqrt{17}}}{2-2\sqrt{17}}\) | \(\frac{2\sqrt{3}-i\sqrt{2+2\sqrt{17}}}{2-2\sqrt{17}}\) |

In fig. (3) it is shown that the spectrum exhibits five singlet states. The lowest lying state at momentum \( \pi \) is the ground state. Then there are three excited singlets characterized by the configuration with two holes \((8.157)\), i.e., they have one real \( \lambda \) and a string of length almost 2. In table (5) we summarize their quantum numbers. Among these singlets, those which are not degenerate, have \( P \)-parity equal to that of the ground state (odd) as it happens in the 2-flavor Schwinger model. The non-degenerate singlet in fact reads

\[
|0, 0, -3J, 0\rangle = \frac{1}{\sqrt{12}} \{ |\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\downarrow\downarrow\uparrow\rangle + |\downarrow\uparrow\downarrow\uparrow\rangle + |\downarrow\uparrow\uparrow\downarrow\rangle + |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle - |\uparrow\downarrow\downarrow\uparrow\rangle - |\downarrow\uparrow\downarrow\uparrow\rangle - |\uparrow\downarrow\uparrow\downarrow\rangle - |\downarrow\uparrow\uparrow\downarrow\rangle - |\uparrow\uparrow\downarrow\uparrow\rangle \} \quad (8.198)
\]

The degenerate singlets are again not eigenstates of the \( P \)-parity, but it is always possible to take a linear combination of them with the \( P \)-parity that coincides with that of the representative state \((8.198)\) of the configuration.

**Table 5: Singlet internal quantum numbers**

| SINGLET | \( \lambda \) | \( \lambda_s \) | \( Q_1^{(h)} \) | \( Q_2^{(h)} \) |
|---------|-----------|-----------|-------------|-------------|
| \(|0, 0, -3J, 0\rangle\) | 0 | 0 | -1 | 1 |
| \(|0, 0, -2J, \frac{\pi}{3}\rangle\) | \(-\frac{\sqrt{3}+2\sqrt{6}}{14}\) | \(\frac{-2+3\sqrt{6}}{\sqrt{3}(4+\sqrt{2})}\) | 0 | 1 |
| \(|0, 0, -2J, \frac{5\pi}{4}\rangle\) | \(\frac{\sqrt{3}+2\sqrt{6}}{14}\) | \(\frac{2-3\sqrt{6}}{\sqrt{3}(4+\sqrt{2})}\) | -1 | 0 |

The remaining singlet \(|0, 0, -\frac{5-\sqrt{13}}{2} J, \pi\rangle\) is not of the type \((8.157)\). It is characterized by a string approximately of length 3 with \( \lambda_{1,1} = i\sqrt{\frac{2+2\sqrt{13}}{12}} \), \( \lambda_{2,1} = 0 \) and \( \lambda_{3,1} = -i\sqrt{\frac{5+2\sqrt{13}}{12}} \).

Even in finite systems very small like the 4 and 6 sites chains, the “string hypothesis” is a very good approximation and it allows us to classify and distinguish among states with the same spin.

The ground state of the antiferromagnetic Heisenberg chain with \( N \) sites is a linear combination of all the \( \binom{N}{N/2} \) states with \( \frac{N}{2} \) spins up and \( \frac{N}{2} \) spins down. These states group themselves into
sets with the same coefficient in the linear combination according to the fact that the ground state
is translationally invariant (with momentum 0 (\(\pi\)) for \(N\) even (odd)), it is an eigenstate of \(P\)-parity
and it is invariant under the exchange of up with down spins. The states belonging to the same set
have the same number of domain walls, which ranges from \(N\), for the two Néel states, to 2 for the
states with \(N\) adjacent spins up and \(N\) adjacent spins down.

The ground state of the 8 sites chain is

\[
|g.s.\rangle = \frac{1}{\sqrt{N}}(|\psi_8\rangle + \alpha|\psi_6^{(1)}\rangle + \beta|\psi_6^{(2)}\rangle + \gamma|\psi_4^{(1)}\rangle + \delta|\psi_4^{(2)}\rangle + \epsilon|\psi_4^{(3)}\rangle + \zeta|\psi_2\rangle) \tag{8.199}
\]

where

\[
|\psi_8\rangle = |\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\downarrow\rangle + |\downarrow\downarrow\uparrow\downarrow\uparrow\downarrow\uparrow\uparrow\rangle \tag{8.200}
\]

\[
|\psi_6^{(1)}\rangle = |\uparrow\uparrow\uparrow\downarrow\downarrow\downarrow\rangle + |\uparrow\uparrow\uparrow\downarrow\uparrow\uparrow\rangle + \text{translated states} \tag{8.201}
\]

\[
|\psi_6^{(2)}\rangle = |\uparrow\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\uparrow\downarrow\uparrow\uparrow\uparrow\rangle + \text{translated states} \tag{8.202}
\]

\[
|\psi_4^{(1)}\rangle = |\uparrow\uparrow\downarrow\downarrow\downarrow\rangle + \text{translated states} \tag{8.203}
\]

\[
|\psi_4^{(2)}\rangle = |\uparrow\uparrow\uparrow\downarrow\downarrow\rangle + |\uparrow\uparrow\downarrow\uparrow\uparrow\rangle + \text{translated states} \tag{8.204}
\]

\[
|\psi_4^{(3)}\rangle = |\uparrow\uparrow\uparrow\uparrow\downarrow\rangle + |\uparrow\uparrow\downarrow\uparrow\uparrow\rangle + \text{translated states} \tag{8.205}
\]

\[
|\psi_2\rangle = |\uparrow\uparrow\uparrow\uparrow\uparrow\rangle + \text{translated states} \tag{8.206}
\]

By direct diagonalization one gets

\[
\alpha = -0.412773 \tag{8.207}
\]
\[
\begin{align*}
\beta &= 0.344301 \quad (8.208) \\
\gamma &= 0.226109 \quad (8.209) \\
\delta &= -0.087227 \quad (8.210) \\
\epsilon &= 0.136945 \quad (8.211) \\
\zeta &= 0.018754 \quad (8.212)
\end{align*}
\]

\[N = 2 + 16\alpha^2 + 8\beta^2 + 4\gamma^2 + 16\delta^2 + 16\epsilon^2 + 8\zeta^2 = 6.30356 \quad (8.213)\]

The energy of the ground state is

\[E_{\text{g.s.}} = -5.65109J \quad (8.214)\]

Eq. (8.214) differs only by 1.8% from the thermodynamic limit expression \(E_{\text{g.s.}} = -8\ln 2 = -5.54518\). Moreover also the correlation function of distance 2 Eq. (5.83) computed for the 8 sites chain is \(G(2) = 0.1957N\), value which is 7% higher than the exact answer Eq. (5.83).

In the analysis of finite size systems we were able to find the coefficient \(\beta\) of the first set of states containing \(N-2\) domain walls in the ground state. These states are obtained interchanging two adjacent spins in the Néel states. The \(\beta\) is for a generic chain of \(N\)-sites

\[
\beta = \frac{N + 2E_{\text{g.s.}}}{N} = 1 - 2\ln 2 \quad (8.215)
\]

Appendix B: The correlator \(G(2)\) in terms of spin configuration probabilities

In this appendix we shall establish a relation between the Heisenberg model correlator \(G(2)\) and the probabilities of finding, in the antiferromagnetic vacuum, certain groups of spin in a given position.

The isotropy of the Heisenberg model implies that

\[
\sum_{x=1}^{N} <g.s.|\hat{S}_x \cdot \hat{S}_{x+2}|g.s.> = 3 \sum_{x=1}^{N} <g.s.|\hat{S}_x^3 \cdot \hat{S}_{x+2}^3|g.s.> \quad (8.216)
\]

Let us introduce the probability \(P_3\) for finding three adjacent spins in a given position in the Heisenberg antiferromagnetic vacuum. Taking advantage of the isotropy of the Heisenberg model ground state and of its translational invariance, it is easy to see that the correlator can be written in terms of the \(P_3\)'s as

\[
\sum_{x=1}^{N} <g.s.|\hat{S}_x^3 \cdot \hat{S}_{x+2}^3|g.s.> = N \frac{1}{4} \left( 2 P_3(\uparrow\uparrow\uparrow) + P_3(\uparrow\downarrow\uparrow) - P_3(\uparrow\uparrow\downarrow) - P_3(\downarrow\uparrow\uparrow) \right) \quad (8.217)
\]

The factor 2 appears in (8.217) due again to the isotropy of the Heisenberg model: the probability of a configuration and of the configuration rotated by \(\pi\) around the chain axis, are the same.

In \[18\] the so called “emptiness formation probability” \(P(x)\) was introduced.

\[
P(x) = <g.s.\prod_{j=1}^{x} P_j|g.s.> \quad (8.218)
\]

where

\[
P_j = \frac{1}{2}(\sigma_j^3 + 1) \quad (8.219)
\]

and \(\sigma_j^3\) is the Pauli matrix. \(P(x)\) determines the probability of finding \(x\) adjacent spins up in the antiferromagnetic vacuum. One gets

\[
\begin{align*}
P(\uparrow\uparrow\uparrow) &= P(3) \quad (8.220) \\
P(\uparrow\downarrow\uparrow) &= P(1) - 2P(2) + P(3) \quad (8.221) \\
P(\downarrow\uparrow\uparrow) &= P(\uparrow\uparrow\downarrow) = P(2) - P(3) \quad (8.222)
\end{align*}
\]
so that Eq. (5.82) reads
\[
\frac{G(2)}{3} = 2P(3) - 2P(2) + \frac{1}{2} P(1) .
\] (8.223)

Using the exact value the correlator \(G(2)\) computed in \[17\] from (8.223) and from the known values of \(P(2)\) and \(P(1)\) given in \[18\]

\[
P(1) = \frac{1}{2} \quad (8.224)
\]
\[
P(2) = \frac{1}{3}(1 - \ln 2) \quad (8.225)
\]
\[
(8.226)
\]
we get

\[
P(3) = \frac{1}{4} - \ln 2 + \frac{3}{8} \zeta(3) . \quad (8.227)
\]

For the general emptiness formation probability \(P(x)\) of the antiferromagnetic Heisenberg chain, an integral representation was given in \[18\], but, to our knowledge, the exact value of \(P(3)\) \((8.227)\) was not known.

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