The Atick-Witten free energy, closed tachyon condensation and deformed Poincaré symmetry

Michele Maggiore

Département de Physique Théorique, Université de Genève,
24 quai Ernest-Ansermet, CH-1211 Genève 4
E-mail: michele.maggiore@physics.unige.ch

ABSTRACT: The dependence of the free energy of string theory on the temperature at $T \gg T_{Hag}$ was found long ago by Atick and Witten and is $F(T) \sim \Lambda T^2$, where $\Lambda$ diverges because of a tachyonic instability. We show that this result can be understood assuming that, above the Hagedorn transition, Poincaré symmetry is deformed into a quantum algebra. Physically this quantum algebra describes a non-commutative spatial geometry and a discrete euclidean time. We then show that in string theory this deformed Poincaré symmetry indeed emerges above the Hagedorn temperature from the condensation of vortices on the world-sheet. This result indicates that the endpoint of the condensation of closed string tachyons with non-zero winding is an infinite stack of spacelike branes with a given non-commutative world-volume geometry. On a more technical side, we also point out that $T$-duality along a circle with antiperiodic boundary conditions for spacetime fermions is broken by world-sheet vortices, and the would-be $T$-dual variable becomes non-compact.

KEYWORDS: Strings at finite temperature, Tachyon condensation, Quantum groups
1. Introduction

One of the long-standing problems of string theory is to understand what happens above the Hagedorn temperature $T_{\text{Hag}}$. The existence of many analogies with the deconfinement transition in QCD suggests that at $T > T_{\text{Hag}}$ it could be possible to discover a more fundamental level of description and more fundamental degrees of freedom.

One of the best hints for understanding the physics above $T_{\text{Hag}}$ comes from the Atick-Witten computation of the closed strings free energy \[ F(T) = V\Lambda T^2 \] (1.1). In the limit $T \gg T_{\text{Hag}}$ the result, both for the bosonic string and for superstrings, is

\[ F(T) \rightarrow V\Lambda T^2, \quad (T \rightarrow \infty) \]

where $V$ is the spatial volume and $\Lambda$ a divergent cosmological constant. This should be compared with the behavior of the free energy of field theory in $D$ space-time dimensions, $F(T) \sim VT^D$, which would have rather suggested $F(T) \sim VT^{26}$ for the bosonic string and $F(T) \sim VT^{10}$ for superstrings. Eq. (1.1) seems therefore to indicate a vast reduction in the number of degrees of freedom.

The divergence of $\Lambda$ is related to the fact that at $T_{\text{Hag}}$ a tachyonic instability develops. Understanding the physics behind eq. (1.1) means therefore understanding what is the
endpoint of the condensation of this closed string tachyon. If one is able to identify the correct vacuum, it becomes possible to compute around it, and to find a finite value for $\Lambda$.

In this paper we will show that eq. (1.1) expresses the fact that above $T_{Hag}$ there is a phase transition and Poincaré symmetry is deformed, i.e. it becomes a quantum algebra with a deformation parameter $a$ with dimensions of length. Physically, this quantum algebra describes a system in which the fictitious euclidean time used to study finite temperature becomes discrete, and $a$ is the corresponding lattice spacing.

The physics of this deformation of Poincaré symmetry will be discussed in sect. 2. The quantization of a system described by this deformed algebra is particularly interesting, and we will find that it automatically implies a non-commutativity between the spatial coordinates, as well as a generalized uncertainty principle between coordinates and momenta. In fact, we will just reobtain the generalized uncertainty principle that we found in ref. [2]. Following a recent suggestion [3], we will then see that in a system with deformed Poincaré symmetry, in the limit where $T$ is much larger than any scale and in particular than the deformation scale, $aT \gg 1$, the free energy comes out automatically of the form (1.1), in any number of dimensions; $\Lambda$ is finite, and is related to the deformation parameter $a$, $\Lambda \sim (1/a)^{D-2}$.

In sect. 3 we apply these ideas to the Hagedorn transition in string theory. We will first recall in sect. 3.1 some basic facts about the Atick-Witten result, emphasizing some aspects (as the lack of thermal duality for superstrings) that will play an important role in our analysis. Our main goal in sects. 3.2–3.4 will be to show that the deformation of Poincaré symmetry discussed in sect. 2 indeed emerges in string theory from the world-sheet dynamics. A crucial role will be played by world-sheet vortices. It is well known [4, 5] that the discretizations of the action of a compact string coordinate

$$S = \frac{1}{4\pi\alpha'} \int d^2\xi \left( \partial_\alpha X \right)^2, \quad X \sim X + 2\pi R \tag{1.2}$$

fall into two different universality classes, depending on whether they admit or not vortices. The issue is irrelevant at large $R$, since vortices are anyway dynamically suppressed, but becomes crucial below a critical value $R_c$, where world-sheet vortices, if included, dominate the partition function. We must therefore ask what is the correct definition, in the application to string theory, of the formal continuum action (1.2). In sect. 3.2 we will show that in superstring theory the correct prescription is that vortices should not be included when, compactifying along $X$, we impose periodic boundary conditions both on spacetime bosons and fermions, while we must include them if we instead impose antiperiodic boundary conditions on fermions, i.e. if we compactify on $S^1/(−1)^F$ with $F$ the spacetime fermion number. The difference in the prescription between $S^1$ and $S^1/(−1)^F$ has its roots in the fact that in these two cases there are also two different prescriptions for the GSO projection [1, 6]. In particular, vortices must be retained when we compactify $X^0$ to study finite temperature, since in this case we impose the $(−1)^F$ twist.

Retaining the vortices in the compactification of $X^0$ (or of any variable compactified with a $(−1)^F$ twist) has crucial effects on the world-sheet dynamics below a critical radius $R_c$. At $R = R_c$ there is in fact on the world-sheet a Kosterlitz-Thouless (KT) phase
transition driven by vortex condensation. When $X = X^0$ is the euclidean time direction, $1/(2\pi R_c)$ coincides with the Hagedorn temperature \cite{7, 8, 3, 4}, and therefore this phase transition corresponds, on the world-sheet, to the Hagedorn phase transition in spacetime.

Using standard results from the renormalization group analysis of the KT phase transition, we will see that below $R_c$ vortices have two crucial effects. First, T-duality is broken, and the variable $\tilde{X}^0 = X^0_L - X^0_R$ that usually describes the T-dual physics and lives on a circle with dual radius $R' = \alpha'/R$ becomes uncompactified, and lives now on the whole real line. Second, vortices generate an effective potential for $\tilde{X}^0$

$$V(\tilde{X}^0) = -\mu \cos \left( \frac{R}{\alpha'} \tilde{X}^0 \right), \quad (1.3)$$

with $\mu$ going to infinity under renormalization group transformations on the world-sheet. This potential breaks the translation invariance of $\tilde{X}^0$ and localizes it on the minima of the cosine, and therefore gives rise to a lattice spacing $a = 2\pi \alpha'/R$. This is in full agreement with results already obtained some time ago by Gross and Klebanov \cite{4, 5} using matrix model techniques. We will confirm this result also from a $\sigma$-model analysis of tachyon condensation in sect. 3.4.

In turn this will allow us to put forward a definite proposal for the condensation of closed string tachyons; we will claim that the endpoint of the condensation of closed string tachyons with non-zero winding is a stack of spacelike branes, with a non-commutative world-volume geometry fixed by the deformation of the Poincaré algebra. It would be very interesting to understand whether these branes have a CFT description in terms of strings with Dirichlet boundary conditions in the temporal direction, along the lines discussed recently in refs. \cite{10, 11}. We will also compare this picture with recent results on the condensation of type 0 tachyons \cite{12, 13, 14, 15}, and we will find elements which support our proposal.

2. The physics of deformed Poincaré symmetry

2.1 Deformed algebras and discretized spacetime

Let us begin by considering, for simplicity, a 1+1 dimensional system with continuous time and discrete space, with lattice spacing $a$. In this setting a Klein-Gordon equation reads

$$(-\partial_t^2 + \Delta_x^2 - m^2) \phi = 0 \quad (2.1)$$

with $\Delta_x \phi(t, x) = [\phi(t, x + a) - \phi(t, x - a)]/2a$. The dispersion relation that follows is

$$E^2 = \frac{\sin^2 \alpha p}{a^2} + m^2. \quad (2.2)$$

The physics of eq. (2.2) is clear. Momentum is periodically identified, and the lattice can sustain travelling waves only up to a maximum energy $E_{\text{max}} = (a^{-2} + m^2)^{1/2}$. If we replace $c = 1$ with a speed $v < 1$, this equation just describes phonons in 1+1 dimensions. From a Lie algebra point of view, the symmetry of the system is described by the generator
$H$ of continuous time translations and the generator $P$ of discrete spatial translations, satisfying the Lie algebra $[H, P] = 0$, supplemented by the identification $P \sim P + 2\pi/a$. The symmetry under rotations in the $(t, x)$ plane, i.e. boosts, is broken, and no generator is associated to it.

There is however an alternative description of the symmetry of this system, based on a deformed algebra \cite{16}. One introduces also the boost generator $K$, and considers the algebra

$$[P, H] = 0, \quad [K, P] = iH, \quad [K, H] = i \frac{\sin(2aP)}{2a}. \quad (2.3)$$

In the limit $a \to 0$ this reduces to the standard Poincaré algebra of a 1+1 continuous relativistic system. The structure \eqref{2.3} is however well defined even at finite $a$, because it is easy to see that the commutators indeed obey the Jacobi identities\footnote{In fact, we can equate the commutator $[K, H]$ to an arbitrary function of $P$, and still the Jacobi identities are trivially satisfied. In this way we can obtain an algebra corresponding to an arbitrary discretization of the spatial derivative.}. Eq. \eqref{2.3} is an example of a quantum algebra, or deformed algebra; $a$ is the deformation parameter.

The physical relevance of this construction emerges from the observation that this quantum algebra has a quadratic Casimir $C_2$ given by

$$C_2 = H^2 - \frac{\sin^2(aP)}{a^2}, \quad (2.4)$$

as well as the realization, in position space

$$P_{\mu} = -i \partial_{\mu}, \quad K = ix \partial_t - t \frac{\sin(-2ia\partial_x)}{2a}, \quad (2.5)$$

where $P^\mu = (H, P)$ and we use $\eta_{\mu\nu} = (-, +)$. Therefore the discrete KG equation \eqref{2.1}, or equivalently the dispersion relation \eqref{2.2}, is simply the condition $C_2 = m^2$, and in this sense this deformed Poincaré algebra can be considered as the symmetry of a relativistic system living in discrete one-dimensional space and continuous time.

Comparing the Lie algebra and the deformed algebra descriptions of this system we see that in the Lie algebra approach, when $a \neq 0$, there are only two generators, $H$ and $P$; $a = 0$ is a point of enhanced symmetry, where a new generator $K$ suddenly pops out. In the deformed algebra description instead we always have the three generators $H, P, K$ even for finite $a$, so in a sense we always have the information about the existence of a symmetry group with three generators, but we pay this with a non-linear structure. The point $a = 0$ is the point where the algebraic structure linearizes.

At this classical level, however, the Lie algebra and the deformed algebra decscriptions of this system contains basically the same amount of informations.\footnote{Both when we consider a single particle system and composite systems. In the case of composite systems there is some confusion in the literature, and we clarify the issue in the appendix \ref{appendix_2}.} The dynamics of the classical system is completely specified by its equation of motion. For the deformed algebra, this comes out from the quadratic Casimir, while from the Lie algebra point of view eq. \eqref{2.1} reflects the covariance under continuous time translation and discrete spatial translations.
Our real interest, however, is in systems with discrete time and continuous space, and in this case we will find in sect. 2.2 that, after quantization, the description based on the deformed algebra leads to very substantial differences from the description based on the Lie algebra. Consider therefore a system, again for the moment 1+1 dimensional and with minkoskian signature, in which time is discrete and space is continuous. The KG equation reads \((-\Delta^2 + \partial^2_x - m^2)\phi = 0\) and the dispersion relation is

\[
\frac{\sin^2 aE}{a^2} = p^2 + m^2. \tag{2.6}
\]

Even if our starting point, a KG with a finite time difference, seemed reasonably simple, the physics of this dispersion relation is quite peculiar, and there is a maximum allowed momentum, and even a maximum mass, \(p^2 + m^2 \leq 1/a^2\). It is easy to find a quantum algebra description of this system, simply exchanging the role of \(H\) and \(P\) in eq. (2.3),

\[
[P,H] = 0, \quad [K,H] = iP, \quad [K,P] = \frac{i \sin(2aH)}{2a}. \tag{2.7}
\]

and again the KG equation is reproduced by the Casimir, which now is

\[
C_2 = \frac{\sin^2(aH)}{a^2} - P^2. \tag{2.8}
\]

A third deformation of the Poincaré algebra, which will turn out to be the most relevant for our purposes, is obtained discretizing euclidean time and then rotating back into Minkowski. In this case we start from an euclidean KG equation \((\Delta^2 + \partial^2_x - m^2)\phi = 0\), leading to a dispersion relation \(-\frac{\sin^2 aE}{a^2} = p^2 + m^2\). When we rotate back into Minkowski space, \(E \rightarrow -iE\), the dispersion relation becomes

\[
\frac{\sinh^2 aE}{a^2} = p^2 + m^2. \tag{2.9}
\]

Of course both eq. (2.9) and eq. (2.6) reduce to the standard Minkoskian dispersion relation \(E^2 = p^2 + m^2\) in the limit \(a \rightarrow 0\). However, at finite \(a\) they are different and they describe very different physics. For instance, in eq. (2.3) there is maximum momentum, which is not the case for eq. (2.9). Therefore physically a system with a discrete minkoskian time, eq. (2.6), has nothing to do with the system obtained discretizing first time in euclidean space and then rotating back into Minkowski space. However formally eqs. (2.6) and (2.9) are related by \(a \rightarrow ia\). Substituting \(a \rightarrow ia\) into eq. (2.7) we therefore find a deformation of the Poincaré algebra whose Casimir reproduces the dispersion relation (2.9),

\[
[P,H] = 0, \quad [K,H] = iP, \quad [K,P] = \frac{i \sinh(2aH)}{2a}. \tag{2.10}
\]

Eqs. (2.7) and (2.10) are special cases of deformations of the Poincaré algebra that can be written in any number of dimensions [17, 18, 19]. The deformation relevant for a system with \(d\) spatial dimensions, in Minkowski space, with discrete Minkowski time, (i.e. the generalization of (2.7)) is as follows: all commutators involving the angular momentum \(J_{ij}\) are the same as in the undeformed Poincaré algebra. Hence, the group of spatial
rotations is not deformed. Similarly for spacetime translations still holds \([P_\mu, P_\nu] = 0\). The commutators involving the boosts \(K_i = J_{i0}\) are instead

\[
[K_i, H] = iP_i, \quad [K_i, P_j] = i\delta_{ij} \frac{\sin(2aH)}{2a}, \tag{2.11}
\]

\[
[K_i, K_j] = -iJ_{ij} \cos(2aH) - i\alpha^2 P^k (P_i J_{jk} + P_j J_{ki} + P_k J_{ij}). \tag{2.12}
\]

The quadratic Casimir is

\[
C_2 = \frac{\sin^2(aH)}{a^2} - \mathbf{P}^2. \tag{2.13}
\]

The deformation relevant instead for a system with \(d\) spatial dimensions, obtained discretizing first euclidean time and then rotating back to Minkowski is obtained with \(a \to ia\) and is therefore \([18, 19]\)

\[
[K_i, H] = iP_i, \quad [K_i, P_j] = i\delta_{ij} \frac{\sinh(2aH)}{2a}, \tag{2.14}
\]

\[
[K_i, K_j] = -iJ_{ij} \cosh(2aH) + i\alpha^2 P^k (P_i J_{jk} + P_j J_{ki} + P_k J_{ij}). \tag{2.15}
\]

with all other commutators undeformed, and with

\[
C_2 = \frac{\sinh^2(aH)}{a^2} - \mathbf{P}^2. \tag{2.16}
\]

2.2 Quantization: non-commutative geometry and generalized uncertainty principle

The real difference between the Lie algebra and the deformed algebra description of a system with discrete time appears when we quantize the system. Compare in fact what happens in the two cases (2.2) (discrete space) and (2.6) (discrete Minkowski time) when we quantize a particle imposing \([x_i, p_j] = i\delta_{ij}\). In momentum space the operator \(x_i\) is represented as \(i\frac{\partial}{\partial p_i}\), and the velocity of the particle in the Heisenberg representation is given by

\[
\dot{x}_i = i[H, x_i] = \frac{\partial E}{\partial p_i}. \tag{2.17}
\]

In the familiar case of a phonon this of course gives the standard expression for the group velocity: using eq. (2.2) (and setting for simplicity \(m = 0\)), one gets

\[
\frac{\partial E}{\partial p} = \cos(ap), \tag{2.18}
\]

i.e. the standard group velocity of a massless particle on a regular spatial lattice. Instead, for a lattice in Minkowski time, using eq. (2.6) we get

\[
\frac{\partial E}{\partial p_i} = \left( \frac{p_i}{\sqrt{p^2 + m^2}} \right) \frac{1}{\cos(aE)}. \tag{2.19}
\]

The term in parenthesis is just the standard expression for the velocity in terms of momentum. However, the cosine at the denominator makes no sense, and if we would take
eq. (2.19) as an expression for the velocity we would find \( v > 1 \), and even \( v \to \infty \), when \( aE \) approaches \( \pi/2 \), i.e. near the maximum momentum.

Clearly something has gone wrong, and we cannot quantize in this way a particle on a space with discrete Minkowski time. The simplest attitude would be to say that, when time is discrete, time derivatives make no sense and should be replaced by finite differences, so we must give up equations like \( v_i = i[H, x_i] \). This is the approach that would be taken within the Lie algebra description of the symmetries of the system, and it is completely analogous to the fact that in the Lie algebra approach the boost generator \( K \) only appears at \( a = 0 \).

The description of the symmetries in terms of a deformed algebra suggests however a different route. Namely, we do not give up the possibility of having boosts, and therefore velocities, even at finite \( a \), but we accept to pay this with the introduction of a non-linear structure.

In particular, we can retain the equation \( v_i = i[H, x_i] \), but we modify the definition of the position operator requiring that the relation between velocity and momentum is not deformed,\(^3\) \( v_i = p_i/\sqrt{p^2 + m^2} \). This is easily done defining, in momentum space,

\[
x_i = i \cos(aE) \frac{\partial}{\partial p_i} = i \sqrt{1 - a^2(p^2 + m^2)} \frac{\partial}{\partial p_i},
\]

where in the second equality we used the dispersion relation. By construction we now have

\[
v_i = i[H, x_i] = \cos(aE) \frac{\partial E}{\partial p_i} = \frac{p_i}{\sqrt{p^2 + m^2}},
\]

and \( x_i \) obviously has the correct limit for \( a \to 0 \). However, using eq. (2.20), we can compute explicitly the \([x_i, x_j]\) and \([x_i, p_j]\) commutators, and we find

\[
[x_i, x_j] = ia^2 J_{ij},
\]
\[
[x_i, p_j] = i \delta_{ij} \sqrt{1 - a^2(p^2 + m^2)},
\]

where we have defined

\[
J_{ij} = -i \left( p_i \frac{\partial}{\partial p_j} - p_j \frac{\partial}{\partial p_i} \right).
\]

Note that \( J_{ij} \) satisfies the usual commutation relations of angular momentum, so that the group of spatial rotations is not deformed.

This result is quite surprising; first of all, space is now non-commutative, and eq. (2.22) is just of the type first discussed, long ago, by Snyder \(^4\). Second, the uncertainty principle is modified. Amazingly, at the maximum allowed value of the momentum the \([x, p]\) commutator vanishes.

These commutation relations are quite interesting by themselves, but they are not yet the setting that will be used to reproduce the Atick-Witten free energy. A partition

---

\(^3\)This choice for \( v_i \) is physically very reasonable, and the deeper reason for it is explained in app. A.

\(^4\)More precisely, in ref. [20] there was also a non-commutativity between time and spatial coordinates, whose commutator closed on the boosts, which is not the case for us.
function at finite temperature $T$ is formally equivalent to the path integral for a system with imaginary “time” ranging from zero to $1/T$, and it is this fictitious euclidean time that we want to discretize. As we discussed in sect. 2.1, this is obtained from the case with discrete Minkowski time simply with the replacement $a \rightarrow ia$. Then the dispersion relation is eq. (2.9), the position operator becomes

$$x_i = i \cosh(aE) \frac{\partial}{\partial p_i}$$

(2.25)

and the commutation relations become

$$[x_i, x_j] = -ia^2 J_{ij},$$

(2.26)

$$[x_i, p_j] = i \delta_{ij} \sqrt{1 + a^2 (p^2 + m^2)}.$$  

(2.27)

Eq. (2.27) shows that in this case at large energies the volume of the cells of the phase space increases. Expanding at first order in $a^2$ one finds [2] that eq. (2.27) implies a generalized uncertainty principle of the form

$$\Delta x \geq \frac{1}{\Delta p} + a^2 \Delta p$$

(2.28)

and therefore a minimum observable length of order $a$. A generalized uncertainty principle of this form has been obtained in string theory, studying planckian scattering in the eikonal limit [22, 23, 24, 25], and also in quantum gravity from gedanken black-hole experiments [26]. We will see that eq. (2.28), or better the full expression (2.27), is also relevant in the high temperature regime of string theory.

The commutators (2.22, 2.23) and (2.26, 2.27) where obtained some time ago [2] using the following argument, that illustrates their uniqueness. Suppose that we look for the most general deformation of the Heisenberg algebra, $[x_i, x_j] = 0, [x_i, p_j] = i\delta_{ij}$, in $d$ spatial dimensions, which depends on a deformation parameter $a$ with dimensions of length, and such that for $a \rightarrow 0$ we recover the standard Heisenberg algebra. Impose further the conditions that the group of spatial rotations is not deformed, so that the $J_{ij}$’s still close the standard algebra of $SO(d)$, and also that the group of spatial translations is not deformed, so that $[p_i, p_j] = 0$. We then look for the most general deformed algebra that can be constructed using only $x_i, p_i$ and $J_{ij}$. The commutator $[x_i, x_j]$ can only be proportional to $J_{ij}$, since it is the only available antisymmetric tensor with the same transformation properties under rotation (and parity). The proportionality factor $a^2$ is dictated by dimensional considerations and a factor of $i$ by hermiticity, so the most general form is

$$[x_i, x_j] = ia^2 g(p^2) J_{ij},$$

(2.29)

with $g$ an arbitrary real function. Invariance under rotations and translations requires that $g$ can depend only on $p^2$ (in particular, a dependence on $x^2$ or $x \cdot p$ is forbidden

---

5A more precise analysis uses the deformation of the Newton-Wigner position operator, which is hermitean with respect to the scalar product invariant under deformed Poincaré symmetry. This analysis is explained in detail in ref. [21], but the final result for the $[x, x]$ and $[x, p]$ commutators is the same.
by translation invariance). Similarly, invariance under spatial rotations and translations
requires a general form
\[ [x_i, p_j] = if(p^2) \delta_{ij}. \] (2.30)

The remarkable fact is that the Jacobi identities fix uniquely the functions \( f(p^2), g(p^2) \).
Consider first the identity with three \( x \)'s,
\[ 0 = [x_i, [x_j, x_k]] + \text{cyclic} = [x_i, g(p^2)J_{jk}] + [x_j, g(p^2)J_{ki}] + [x_k, g(p^2)J_{ij}] =
\]
\[ = g(p^2) ((x_i, J_{jk}) + [x_j, J_{ki}] + [x_k, J_{ij}]) + ([x_i, g]J_{jk} + [x_j, g]J_{ki} + [x_k, g]J_{ij}) \]. (2.31)

The first parenthesis in the last line vanishes automatically, using the fact that the rotation
group is undeformed, so that it still holds \([J_{ij}, V_k] = i(\delta_{ik}V_j - \delta_{jk}V_i)\), for any vector \( V_i \).
To compute \([x_i, g]\) observe that eq. (2.30) implies that in momentum space \( x_i \) can be
represented as
\[ x_i = if(p^2) \frac{\partial}{\partial p_i}. \] (2.32)
Then we get
\[ 0 = 2if \frac{\partial g}{\partial p^2} (p_iJ_{jk} + p_jJ_{ki} + p_kJ_{ij}) \] (2.33)
The term \((p_iJ_{jk} + p_jJ_{ki} + p_kJ_{ij})\) vanishes automatically if \( J_{ij}\) is the orbital angular momentum (2.24), i.e. for spin zero particles. However, it is non zero on a generic representation
of the rotation group. Since the Jacobi identities must hold independently of the representation,
we must have either \( f = 0 \) or \( g = \text{const.} \) The choice \( f = 0 \) of course does not reproduce the standard commutators in the limit \( a \to 0 \), so we conclude that \( g \) is a constant. With a redefinition of \( a \), we can set it to \( \pm 1 \). We therefore conclude that the
most general result for the \([x, x]\) commutator is
\[ [x_i, x_j] = \pm ia^2 J_{ij}. \] (2.34)

The other Jacobi identities now fix the function \( f \). The Jacobi identity with three \( p \)'s and
that with one \( x \) and two \( p \)'s are trivially satisfied as a consequence of \([p_i, p_j] = 0\). The last
Jacobi identity is
\[ 0 = [x_i, [x_j, p_k]] + [x_j, [p_k, x_i]] + [p_k, [x_i, x_j]] = \delta_{jk}[x_i, f(p^2)] - \delta_{ik}[x_j, f(p^2)] \pm a^2[p_k, J_{ij}] \]. (2.35)

Using again eq. (2.32), eq. (2.35) becomes
\[ 2f \frac{\partial f}{\partial p^2} (\delta_{jk}p_i - \delta_{ik}p_j) = \mp a^2(\delta_{jk}p_i - \delta_{ik}p_j), \] (2.36)
or
\[ \frac{\partial f^2}{\partial p^2} = \mp a^2. \] (2.37)
The solution (imposing further that \( f \) is actually a function of the combination \( p^2 + m^2 \)
rather than only of \( p^2 \), and that \( f = 1 \) when \( a = 0 \), is
\[ f(p^2) = \left[ 1 \mp a^2(p^2 + m^2) \right]^{1/2}, \] (2.38)
which reproduces eqs. (2.23) and (2.27).

In general, it is not obvious that a deformation of a given Lie algebra exists, since the Jacobi identities provide very stringent requirements on non-linear structures. Here we see, first of all, that it is possible to deform the Heisenberg algebra with a dimensionful parameter and, second, that this deformation is unique, modulo the replacement \( a \leftrightarrow ia \), and within the rather general assumptions that we have discussed. Furthermore, the functions \( f(p^2) \) and \( g(p^2) \) are the same in any number of spatial dimensions.

Finally we observe that, with a non-local redefinition of coordinates, 
\[ x_i \rightarrow x_i / f(p^2), \]
we can always reduce the \([x, p]\) commutator to the standard form \([x_i, p_j] = i\delta_{ij}\). However, we would pay this at the level of the action, which would become non-local if we started from a local expression, or equivalently at the level of the dispersion relation. Thus, once we say that our starting point is, e.g., a Klein-Gordon equation \((-\Delta^2_t + \partial^2_i - m^2) \phi = 0\), the physical definition of coordinates and momenta has been fixed and we find that there is a non-trivial deformation of the Heisenberg algebra, and this deformation is unique, with the assumptions made.

2.3 Statistical mechanics: \( F(T) = \Lambda V T^2 \)

A deformed quantization procedure implies a deformation of the standard rules of statistical mechanics. The following important observation has been made recently by Kalyana Rama [3]. Consider a system in \( d \) spatial dimensions \((d = D - 1)\), described by the deformed Poincaré algebra with dispersion relation (2.9) and therefore quantized according to the deformed commutation relations (2.24, 2.27). The volume of the cells of the phase space is not anymore \( (2\pi \hbar)^d \), but rather \( (2\pi \hbar f(E))^d \), where
\[ f(E) \equiv \sqrt{1 + a^2 (p^2 + m^2)} = \cosh(aE). \] (2.39)

As a consequence, all averages over the phase space should be performed using this new measure,
\[ \int \frac{d^d q d^d p}{(2\pi \hbar)^d} \rightarrow \int \frac{d^d q d^d p}{2\pi f(E) \hbar^d} \] (2.40)

Let us first use for illustration the Maxwell-Boltzmann statistics; the free energy at temperature \( T \) of a such a system is given by
\[ -\frac{F(T)}{T} = \int \frac{d^d q d^d p}{2\pi f(E)^d} e^{-E/T} \frac{\Omega_d}{(2\pi)^d} \int \frac{p^{d-1} dp}{\cosh^d(aE)} e^{-E/T}, \] (2.41)

where we have set again \( \hbar = 1 \), and \( \Omega_d \) is the solid angle.

In string theory at \( T \ll T_{Hag} \) the free energy is dominated by the massless string modes and reproduces the standard field theoretic behavior \( F(T) \sim V T^D \), plus exponentially small corrections from the massive modes. As \( T \) approaches \( T_{Hag} \), the cumulative effect of the massive modes becomes important, because of the exponentially raising density of states and, above the transition, it takes over. The result \( F(T) \sim V T^2 \) therefore can be seen as a consequence of the cumulative effects of all string massive modes. To understand physically this result means to be able to explain it in terms of the effect of a few new light
modes, which would then be the more appropriate degrees of freedom for describing the new phase.

Since we aim at explaining the result for the free energy in terms of light modes, we can neglect the mass term and write the dispersion relation simply as
\[ p = \frac{1}{a} \sinh aE, \] (2.42)
and \[ dp = \cosh(aE) dE. \]

Then eq. (2.41) becomes
\[ -\frac{F(T)}{T} = V \frac{\Omega_d}{(2\pi)^da^{d-1}} \int_0^\infty dE (\tanh aE)^{d-1} e^{-E/T} = \]
\[ = VT \frac{\Omega_d}{(2\pi)^da^{d-1}} \int_0^\infty dx [\tanh(aTx)]^{d-1} e^{-x}. \] (2.43)

In the limit \( aT \to \infty \) at fixed \( x \) we have \( \tanh(aTx) \to 1 \) and since the integrand is regular near \( x = 0 \) we can take the limit inside the integral,
\[ \lim_{aT \to \infty} \int_0^\infty dx [\tanh(aTx)]^{d-1} e^{-x} = \int_0^\infty dx e^{-x} = 1. \] (2.44)

Correspondingly, one finds at large \( T \) \[ F(T) \approx -\left( \frac{\Omega_d}{(2\pi)^da^{d-1}} \right) VT^2. \] (2.45)

Note that the \( T^2 \) dependence holds in any number of spatial dimensions since, apart from the overall constant, \( d \) appears only in \( [\tanh(aTx)]^{d-1} \), which saturates to 1. The physical mechanism behind this result is that the growth of the volume of the cells of the phase space, combined with the modified dispersion relation, compensates exactly the phase space factor \( p^{d-1} dp \).

The behavior \( F(T) \sim VT^2/a^{d-1} \) can be obtained also using Bose-Einstein or Fermi-Dirac statistics. For Bose-Einstein the result follows from
\[ \lim_{aT \to \infty} \int_0^\infty dx [\tanh(aTx)]^{d-1} (-1) \log(1 - e^{-x}) = -\int_0^\infty dx \log(1 - e^{-x}) = \frac{\pi^2}{6} \] (2.46)
and for Fermi-Dirac
\[ \lim_{aT \to \infty} \int_0^\infty dx [\tanh(aTx)]^{d-1} \log(1 + e^{-x}) = \int_0^\infty dx \log(1 + e^{-x}) = \frac{\pi^2}{12}. \] (2.47)

Therefore the result for the free energy in the large \( T \) limit is
\[ F(T) \to \Lambda VT^2, \quad (aT \to \infty) \] (2.48)
with
\[ \Lambda = -\left( (2n_B + n_F) \frac{\pi^2}{12} \frac{\Omega_d}{(2\pi)^d} \right) \frac{1}{a^{d-1}}, \] (2.49)
where \( n_B \) is the number of light boson species and \( n_F \) the number of light fermion species that constitute the relevant degrees of freedom in the phase with deformed Poincaré symmetry.
If we remove the deformation parameter, i.e. $a \to 0$, while keeping $T$ fixed, we cannot use eqs. (2.46) and (2.47), since they have been obtained in the limit $aT \to \infty$. Instead, if $a \to 0$ at fixed $T$ we obviously reobtain the undeformed result $F(T) \sim VT^D$. However, if we first take the large $T$ limit at fixed $a$ and then we take the limit $a \to 0$ in eq. (2.49), then we see that $\Lambda$ diverges. As we will recall in sect. 3.1, in string theory, taking first the large $T$ limit, one finds a result just of the form (2.48), with $\Lambda$ divergent, because of a tachyonic instability. This divergence must disappear if one is able to identify the endpoint of tachyon condensation and compute around the correct vacuum. We will indeed claim that the true vacuum above the Hagedorn temperature has a deformed Poincaré symmetry stemming from a discrete lattice structure in euclidean time, and that the divergence of $\Lambda$ in the string computation is a consequence of having neglected this discreteness.

2.4 An unexpected duality and time (de)construction

A surprising result emerges looking in more detail into the structure of the deformed algebra which corresponds to discrete euclidean time. In this case the spacetime consists of a stack of spacelike surfaces, separated by a spacing $a$ in euclidean time, see fig. [1], and on each spacelike surface are defined the angular momentum operators $J_{ij}$, the position operators $x_i$ and the momenta $p_i$. As we have seen, the $J_{ij}$ satisfy the undeformed commutation relation of $SO(d)$,

$$[J_{ij}, J_{kl}] = -i(\delta_{il} J_{jk} + \delta_{jk} J_{il} - \delta_{ik} J_{jl} - \delta_{jl} J_{ik}).$$ (2.50)

The $x_i, p_i$ have the standard commutation relations of a vector with $J_{ij}$, and furthermore we have found

$$[x_i, x_j] = -ia^2 J_{ij},$$ (2.51)

$$[x_i, p_j] = i \delta_{ij} \sqrt{1 + a^2(p^2 + m^2)},$$ (2.52)

while the momenta commute,

$$[p_i, p_j] = 0.$$ (2.53)

Now observe that, if we introduce the notation

$$J_{0i} \equiv \frac{x_i}{a},$$ (2.54)

eq. (2.51) becomes

$$[J_{0i}, J_{0j}] = -iJ_{ij}$$ (2.55)

which is the commutation relation between boosts in a (undeformed) Lorentz group.

Furthermore, since the $x_i$ are vectors, so are the $J_{0i}$, and therefore also the boosts-angular momenta commutators $[J_{0i}, J_{0j}]$ are the same as in the Lorentz group. Therefore eqs. (2.50) and (2.51) reconstruct a Lorentz group $SO(d,1)$. We can go even further observing that, in terms of the variables $J_{0i}$, eq. (2.52) reads

$$[J_{0i}, p_j] = i \delta_{ij} \frac{1}{a} \sqrt{1 + a^2(p^2 + m^2)} = i \delta_{ij} \sqrt{p^2 + m^2 + \frac{1}{a^2}}.$$ (2.56)
However, this is nothing but the boost-momentum commutator of a (undeformed) Poincaré group with Hamiltonian
\[ H = \left[ p^2 + m^2 + \frac{1}{a^2} \right]^{1/2}. \] (2.57)

This is just the Hamiltonian of a particle of mass \( m^2 + (1/a^2) \) and therefore has automatically also the correct commutation relations with momenta and angular momenta, \([p_i, H] = [J_{ij}, H] = 0\), as well as with boosts, \([J_{0i}, H] = ip_i\). Therefore, together with eq. (2.53), we have reconstructed a full undeformed Poincaré symmetry in \( d + 1 \) spacetime dimensions! This is a truly surprising result, since our starting point was a spacelike \( d \)-dimensional surface with no time direction, with a non-commutative spatial geometry on it, and we now discover that this can be interpreted as a \( d + 1 \) dimensional commutative spacetime. We see that the non-commutative spatial geometry has generated a timelike dimension, somewhat similarly to the (de)construction of a dimension discussed in ref. [27] (and generalized to a time dimension in ref. [28]; see also ref. [29] for a related approach based on non-commutative geometry).

It is clear that the physical interpretations of \( x_i \), either as a position operator or as \( a \) times a boost, are dual to each other. In the limit of \( a \) small compared to all other scales (\( a|p| \ll 1, am \ll 1 \)) the non-commutativity of the \( x_i \) is small, and it is appropriate to interpret them as coordinates on a manifold. At the same time the lattice spacing between the spacelike sheets in fig. 1 goes to zero and a continuous time is recovered. Instead when \( a \to 0 \) eq. (2.57) gives \( H \simeq 1/a \to \infty \), so that the extra time dimension with respect to which \( x_i/a \) is a boost becomes unaccessible.

The opposite situation takes place in the limit \( a \to \infty \). In this case the separation between the spacelike sheets in fig. 1 goes to infinity and we are apparently left with a single spatial surface, infinitely separated in euclidean time by all others, so the original time has disappeared. However, in this limit the non-commutativity of the \( x_i \) becomes infinitely strong and it makes no sense to interpret them as coordinates of a manifold. The correct description is now in terms of boosts operators, and we recover a full (undeformed) Poincaré symmetry, and therefore a new minkowskian time variable has emerged. Furthermore, in the limit \( a \to \infty \) the Hamiltonian in eq. (2.57) becomes our original starting point before deformation, \( H = \sqrt{p^2 + m^2} \). Therefore at \( a = \infty \) we recover, in an unexpected way, the symmetries of the undeformed theory with \( a = 0 \).

3. Application to string theory above the Hagedorn temperature

3.1 The Atick-Witten free energy

We find useful to recall in this section some known facts about the closed string free energy at finite temperature, and in particular its properties under T-duality. In the closed bosonic
string the contribution to the torus partition function of a single string coordinate $X$ with a periodic identification $X \sim X + 2\pi R$ is (see e.g. ref. [30], sect. 8.2)

$$Z_X(\tau, R) = |\eta(\tau)|^{-2} \sum_{n,w=-\infty}^{\infty} \exp \left[ -\pi \tau_2 \left( \frac{\alpha'n^2}{R^2} + \frac{w^2R^2}{\alpha'} \right) + 2\pi i \tau_1 nw \right],$$  \hspace{1cm} (3.1)

with $\tau$ the modular parameter of the torus and $\eta$ the Dedekind eta function. With a Poisson resummation this can be written as

$$Z_X(\tau, R) = 2\pi R Z_X(\tau) \sum_{m,w=-\infty}^{\infty} \exp \left( -\frac{\pi R^2|m - w\tau|^2}{\alpha'\tau_2} \right),$$  \hspace{1cm} (3.2)

where $Z_X(\tau)$ is the partition function of the non compact theory. Eq. (3.1) shows explicitly the T-duality symmetry $R \rightarrow \alpha'/R, n \leftrightarrow w$, while eq. (3.2) shows explicitly the modular invariance: $\tau \rightarrow \tau + 1$ is compensated by a change of variable $m \rightarrow m + w$, and $\tau \rightarrow -1/\tau$ by $m \rightarrow -w, w \rightarrow m$.

Eqs. (3.1, 3.2) hold also when the periodic field is $X^0$, in which case we are studying finite temperature with $T = 1/(2\pi R)$ and, after including the contribution of all the spatial $X$'s and of ghosts and integrating over the fundamental domain of the torus, one gets $-F_1(T)/(VT)$, where $F_1(T)$ is the one-loop free energy and $V$ is the spatial volume. Taking explicitly the large $T$ limit, Atick and Witten find the result

$$F_1(T) \rightarrow \Lambda_1 VT^2, \hspace{1cm} (T \rightarrow \infty)$$  \hspace{1cm} (3.3)

where $\Lambda_1$ is the (divergent) one-loop cosmological constant of the bosonic string. To get eq. (3.3) it is necessary to take the large $T$ limit inside the integral over the moduli space, and then one can replace the summation over windings with an integral over continuous variables. Of course, strictly speaking these manipulations cannot be justified, since they are performed on a divergent integral. However, comparison with field theory shows [1] that this interchange is dangerous only near the corners of moduli space which correspond to UV regions, like $\tau \rightarrow 0$ for the torus. Since these regions are absent in string theory, there are good reasons to believe that eq. (3.3) catches the correct $T$ dependence in the large $T$ limit.\(^6\)

Another way to get the same result is to use the invariance under T-duality of eq. (3.1). Then, in units of the self-dual temperature $T_{\text{self,dual}} = (2\pi \sqrt{\alpha'})^{-1}$, we have $F(T)/T = TF(1/T)$ so that, at large $T$, $F(T) \simeq T^2 F(0)$ [1, 30]. However, while $Z_X(\tau, R)$ at fixed $\tau$ is certainly T-dual, as we see from the explicit and well-defined expression (3.1), the T-duality of $F(T)/T$ is rather formal, since $F(T)/T$ is obtained performing the integral over the fundamental domain of the torus of a T-dual integrand, but this integration diverges at $\tau_2 \rightarrow \infty$ because of the tachyon. We will come back to this point later.

\(^6\)The conclusion of ref. [1] was indeed criticized in ref. [31], where it was remarked that the result is in contradiction with the $T$-duality of the heterotic string, and the blame was put on this exchange between the large $T$ limit and the integration. We will see however that at finite temperature $T$-duality along $X^0$ is broken by world-sheet vortices above the Hagedorn temperature.
In the bosonic string many finite temperature issues are obscured by the presence of the zero temperature tachyon. It is therefore more instructive to consider type II strings. The great difference in superstring theory comes from the GSO projection. The important point here is that when a coordinate X is compact and we use periodic boundary conditions both on spacetime bosons and fermions (so that spacetime supersymmetry is preserved) the GSO projection is the same as in the non-compact space. However, when we impose periodic boundary conditions on spacetime bosons and antiperiodic on spacetime fermions (which is the case for instance when \(X = X^0\) and we study finite temperature) the GSO projection is different \(^1\) \(^2\), and there are additional minus signs which depend on the windings. The one-loop free energy with this GSO projection is \(^1\)

\[
\frac{F_1}{V T} = \frac{2\pi R}{16} \left(\frac{1}{4\pi^2 \alpha'}\right)^5 \int_\tau \frac{d^2 \tau}{\tau_0^2} |\eta(\tau)|^{-24} \sum_{m, w = -\infty}^\infty Z_f(\tau; m, w) \exp \left(-\frac{\pi R^2 |m - w\tau|^2}{\alpha' \tau_2}\right),
\]

(3.4)

where \(Z_f(\tau; m, w)\) comes from the contribution of the world-sheet fermions and is a combination of theta functions (see eq. (5.20) of ref. \(^1\)). A crucial point is that \(Z_f(\tau; m, w)\) depends on \(m, w\) because of the modified GSO projection. Modular invariance is respected, and in fact the GSO projection has been fixed just requiring it. To examine T-duality, we rewrite eq. (5.20) of ref. \(^1\) using the Poisson resummation \(\sum_m \exp \left[-\pi (m - b)^2/a\right] = a^{1/2} \sum_n \exp \left(-\pi an^2 + 2\pi ib n\right)\) and we get

\[
\frac{F_1}{V T} = \frac{\pi \sqrt{\alpha'}}{16(4\pi^2 \alpha')^{5/2}} \int_\tau^{11/2} \frac{d^2 \tau}{\tau_2^{12}} |\eta(\tau)|^{-24} \sum_{n, w = -\infty}^\infty \left\{e^{-\pi \tau_2 \left(\frac{\alpha' n^2}{R^2} + \frac{R^2 \theta_2^2}{\alpha'}\right) + 2\pi i \tau_1 w n} \left[|\vartheta_2|^8 + |\vartheta_3|^8 + |\vartheta_4|^8 - e^{i\pi w} (\vartheta_3^2 \vartheta_4^2 + \vartheta_3^4 \vartheta_4^4)\right] + e^{-\pi \tau_2 \left(\frac{(\alpha'(n - 1)^2)}{R^2} + \frac{R^2 \theta_2^2}{\alpha'}\right) + 2\pi i \tau_1 w(n - 1/2)} \left[e^{i\pi w} (\vartheta_2^2 \vartheta_4^4 + \vartheta_3^4 \vartheta_4^4) - (\vartheta_3^2 \vartheta_2^4 + \vartheta_3^4 \vartheta_2^4)\right]\right\}.
\]

(3.5)

where \(\vartheta_i(\tau) = \vartheta_i(0|\tau)\) are the Jacobi theta functions. We see that the dependence of \(Z_f\) on \(m, w\) has generated a more complicated dependence on \(n, w\), and the above expression is not symmetric under \(R \rightarrow \alpha'/R\), \(n \leftrightarrow w\). As a result, \(T\)-duality in the temporal direction is broken, and the origin of this breaking is in the modified GSO projection \(^1\). The fact that the symmetry between the momentum and winding modes is broken by the GSO projection can also be seen directly on the spectrum. Consider for instance the winding and momentum modes of the tachyon, \(|0; n, w\rangle\). In the sector \(w = 0\) all momentum modes \(|0; n, w = 0\rangle\) are eliminated by the GSO projection, for all \(n\) (including of course the zero temperature tachyon \(n = 0\)). Instead, in the sector \(n = 0\), the winding modes \(|0; n = 0, w\rangle\) with \(w\) odd survive. The mass formula for the momentum and winding modes of the tachyon in type II strings is

\[
\alpha' m^2 = -2 + \frac{\alpha'}{R^2} n^2 + \frac{R^2}{\alpha'} w^2,
\]

(3.6)
so we see that at low temperature the lowest lying momentum modes of the tachyon with \( w = 0 \) would have been themselves tachyonic, so it is very welcome that they are all eliminated by the GSO projection.\(^7\)

The effect of the GSO projection can also be checked expanding the integrand in eq. (3.5) for \( \tau^2 \to \infty \). Recall that the spectrum of a string theory in D spacetime dimensions can be read off the asymptotics of the torus partition function at \( \tau^2 \to \infty \), which has the general form \(^{30}\)

\[
\sim \int_\infty \tau_2^{-(D/2)} \sum_i \exp \left( -\pi \tau_2 \alpha' m_i^2 \right),
\]

(3.7)

where \( m_i \) are the masses of the states in the theory. Expanding the integrand of eq. (3.5) and retaining the terms corresponding to the momentum and winding modes of the tachyon, one finds that the asymptotic behavior at large \( \tau_2 \) is

\[
\sim \int_\infty \tau_2^{-11/2} \sum_n \sum_{w=-\infty}^{\infty} \left[ 2 + 480 e^{-2\pi \tau_2} - (-1)^w (2 - 32 e^{-2\pi \tau_2}) \right] e^{-\pi \tau_2 \left( -2 + \frac{\alpha'}{\alpha''} n^2 + \frac{R^2}{\alpha'} w^2 \right)}.
\]

(3.8)

We see that, because of the factor \((-1)^w\), the winding modes of the tachyon with \( w \) even are eliminated from the spectrum while those with \( w \) odd survive\(^8\) (the prefactor corresponds to \( D = 9 \) because we have separated the effect of the compact direction to obtain an effective mass in the remaining 9 dimensions). The winding modes at large \( R \) are heavy, so at low temperatures the spectrum is free of tachyonic instabilities. As we decrease \( R \) below a critical value \( R_{\text{Hag}} \), however, the two states \( |0; n = 0, w = \pm 1 \rangle \) become tachyonic. From eq. (3.4), this happens at

\[
R_{\text{Hag}} = \sqrt{2\alpha'} \quad \Rightarrow \quad T_{\text{Hag}} = \frac{1}{2\pi \sqrt{2\alpha'}} \quad \text{(type II)}
\]

(3.9)

which is indeed the Hagedorn temperature of type II strings. The Hagedorn transition is therefore signalled by the fact that, in an otherwise tachyon free theory, a winding mode which is not removed by the GSO projection becomes tachyonic above \( T_{\text{Hag}} \).\(^{7, 8, 9, 1}\)

For the bosonic string again the states \( w = \pm 1 \) become tachyonic exactly at its Hagedorn temperature,

\[
R_{\text{Hag}} = 2\sqrt{\alpha'} \quad \Rightarrow \quad T_{\text{Hag}} = \frac{1}{4\pi \sqrt{\alpha'}} \quad \text{(bosonic)}.
\]

(3.10)

However in the bosonic string the meaning of this finite temperature tachyon is obscured by the fact that the theory is already tachyonic at zero temperature, and that there are also

---

\(^7\)Of course, eq. (3.6) gives a mass in the remaining nine-dimensional euclidean space, since we have separated the effect of the momentum and winding in the compact direction. To really interpret it as a mass of a state in Minkowski spacetime we should rotate back the theory to Minkowski along one of these nine directions rather than along \( X^0 \), and keep \( X^0 \) as a compact spatial coordinate. However, this is an issue of interpretation which is quite irrelevant. The real point is that, when \( m^2 \) in eq. (3.6) is negative, there are divergent contributions to the partition function, exactly as if there were a tachyon in Minkowski space.

\(^8\)I thank Maria Alice Gasparini for performing this check.
all momentum modes of the tachyon, which instead switch from tachyonic to non-tachyonic as we decrease $R$.

Using eq. (3.4) and performing explicitly the large $T$ limit of the one loop free energy, Atick and Witten find also for type II strings the result $F_1(T) = \Lambda_1 V T^2$. Thus, the $T^2$ dependence of the free energy still appears, but it is not anymore a consequence of T-duality, which now is explicitly broken. Indeed, taking the small $R$ limit of type IIB theory we do not find type IIA at large $R$, as would be the case for the compactification in the absence of the $(-1)^F$ twist, but rather type 0 theory [1]. The spacetime supersymmetry of type II strings is broken by the $(-1)^F$ twist in the boundary conditions; fermions have half-integral momenta while boson have integral momenta. In the limit $R \to 0$ all fermions are therefore removed from the spectrum, and we are left with the bosonic spectrum of the type 0A theory at $R \to 0$. The tachyon that develops at the Hagedorn transition becomes, in the limit $R \to 0$, the tachyon of the type 0 theory.

In the large $T$ limit the quantity $\Lambda_1$, computed from string theory around this tachyonic vacuum, is given by the type 0 partition function [1],

$$\Lambda_1 = \frac{1}{16} \left( \frac{1}{4\pi^2\alpha'} \right)^4 \int \frac{d^2 \tau}{\tau_2^6} |\eta(\tau)|^{-24} \left[ |\vartheta_2|^8 + |\vartheta_3|^8 + |\vartheta_4|^8 \right].$$

(3.11)

The integration over the fundamental region of the torus moduli space is divergent at $\tau_2 \to \infty$ because of the type 0 tachyon.

Finally, the behavior that we have discussed so far is the free energy at one-loop. The contributions at $k$-loops have the form [1]

$$F_k(T) \sim V T^2 (g^2 T^2 4\pi^2 \alpha')^{k-1}$$

(3.12)

The factor in parenthesis is just $g^2 \alpha'/R^2$. Therefore in the region

$$g\sqrt{\alpha'} \ll R \ll \sqrt{2\alpha'}$$

(3.13)

the leading contribution to the free energy is the one-loop term and $F(T) \sim V T^2$. At $R = g\sqrt{\alpha'}$ all higher loop contributions become comparable and a change of regime takes place. Of course, we know nowadays that what happens at this scale is that D-branes become important and, in type IIA theory, we see the opening up of the 11th dimension. With hindsight, it is interesting to observe that the importance of the mass scale $1/(g\sqrt{\alpha'})$ in string theory could have been inferred already from the higher loop behavior of the free energy.

### 3.2 World-sheet vortices and T-duality

The partition function of the closed bosonic string at finite temperature is computed working with the euclidean field $X^0(\sigma_1, \sigma_2)$ periodically identified, $X^0 \sim X^0 + 2\pi R$. The dynamics of $X^0$ is governed by the action

$$S = \frac{1}{4\pi \alpha'} \int d^2\xi \left( \partial_\alpha X^0 \right)^2 \equiv \beta \int d^2\xi \frac{1}{2} \left( \partial_\alpha \theta \right)^2$$

(3.14)
where $X^0 \equiv R\theta$ so that $\theta \sim \theta + 2\pi$, and
\[
\beta \equiv \frac{R^2}{2\pi \alpha'}.
\]

There is however a crucial subtle point. Because of the identification $\theta \sim \theta + 2\pi$, this is a non trivial theory and the continuum definition (3.14) is still somewhat formal. In fact, its possible discretizations fall into two different universality classes, depending on whether they admit or not vortices. To elucidate this point, let us recall that in the continuum limit a vortex is a classical configuration singular at one point, such that as we encircle the singular point once, the field $\theta$ does not come back to itself but rather to $\theta + 2\pi v$, with $v$ integer. The form of this configuration is $\theta(\sigma_1, \sigma_2) = v \arctan(\sigma_2/\sigma_1)$ which, substituted into eq. (3.14), gives the vortex action
\[
S_{\text{vortex}} = \frac{\beta v^2}{2} \int \frac{d^2 \xi}{|\xi|^2} = \pi \beta v^2 \log(L/\epsilon),
\]
where we used a lattice discretization to regularize the integral in the UV and a finite volume of the world-sheet to regularize it in the infrared.\footnote{Actually, on a lattice we cannot follow $\theta$ continuously as we encircle a point, and the definition of vortices must be modified. We will come back later to the correct lattice definition of vortices; however for $\epsilon \rightarrow 0$ these lattice vortices will reduce to the continuum definition and for a first estimate we can use eq. (3.16).} $S_{\text{vortex}}$ is therefore divergent as $\epsilon \rightarrow 0$ and vortices might seem to be irrelevant in the continuum limit. However, they have a collective coordinate which is the position of the center, so their multiplicity is equal to the number of lattice sites and the contribution to the partition function from the vortices with $v = \pm 1$ is
\[
Z_{v=\pm 1} \sim \left( \frac{L}{\epsilon} \right)^2 e^{-\pi \beta \log(L/\epsilon)} = e^{(2-\pi \beta) \log(L/\epsilon)}.
\]

We see that for $\beta > 2/\pi$ vortices are indeed irrelevant, but for $\beta < 2/\pi$ vortices with $v = \pm 1$ dominate; this is the famous Kosterlitz-Thouless (KT) phase transition \cite{32, 33, 34}; $\beta_c = 2/\pi$ corresponds to $R = 2\sqrt{\alpha'}$ and therefore to $T = 1/(4\pi \sqrt{\alpha'})$, which coincides with the Hagedorn temperature of the bosonic string \cite{33, 34}. The Hagedorn phase transition in spacetime is therefore signalled by a KT transition on the world-sheet.

The same analysis can be repeated for the supersymmetric KT transition, and one finds now a critical value $\beta_c = 1/\pi$ \cite{35, 36}, that corresponds to $T = 1/(2\pi \sqrt{2\alpha'})$, so that also for type II strings (and for the heterotic string \cite{37}) we recover the correct value of the Hagedorn temperature.

It is however also possible to find different discrete formulations in which vortices are explicitly suppressed, and in this case one finds that the KT transition is absent \cite{36}. The question is which of the two types of discretizations should be taken as fundamental, that is, which of the two is the correct definition for the formal expression (3.14), in the application to string theory. A clue to this issue is the fact that vortices and the KT transition break T-duality. This is clear from the fact that in the large $\beta$ phase the theory has an infinite correlation length while in the low $\beta$ phase a standard lattice strong coupling expansion
shows that a finite correlation length is generated. If however one explicitly suppresses the vortices, one finds that the KT transition is eliminated and T-duality is restored.

The answer that we propose is therefore the following: when the compactification of a superstring is done along a circle with periodic boundary conditions on bosons and fermions, we know that T-duality is respected, and therefore the definition that suppresses the vortices is the correct one. But when we compactify $X^0$ to study finite temperature, or more in general when we compactify a coordinate on a circle with a $(-1)^F$ twist, the situation is different. In sect. 2 we have seen that in type II theory thermal duality $T/T_{\text{self-dual}} \to T_{\text{self-dual}}/T$ is broken. Therefore when we compactify with a $(-1)^F$ twist we must chose a discretization which retains the vortices. The fact that we have two different prescriptions for the definition of eq. (3.14) on $S^1$ and on $S^1/(-1)^F$ goes back to the fact that in these two cases we have also two different prescriptions for the GSO projection.

In the bosonic string instead the question of whether thermal duality $F(T)/T = T F(1/T)$ really holds is not well posed, since $F(T)$ diverges at all temperatures because of the tachyon. In the bosonic string one has the tendency to ask questions forgetting about the tachyon, since this will be cured by the superstring. However, the mechanism that in the superstring eliminates the zero temperature tachyon and, at finite temperature, all its tachyonic momentum modes, is the GSO projection, which is the same mechanism that breaks $T$-duality in the temporal direction. Therefore the two issues cannot be separated. However, as long as we wish to use the bosonic string as a simplified model to learn something about superstrings, we must also use the same prescription. In the following analysis we will use for simplicity the bosonic string, retaining the vortices when compactifying $X^0$.

Similarly, using this prescription on the field $X^0$ of the heterotic string, one finds again a KT phase transition on the world-sheet at a value of $R$ corresponding to the Hagedorn temperature, and above $T_{\text{Hag}}$ T-duality is broken. The need for a mechanism that breaks thermal duality in the heterotic string was discussed in ref. [1].

### 3.3 Decompactification and dynamical localization of $\tilde{X}^0$

Let us therefore recall in more detail what is the effect of vortices, following the review [36]. Consider a lattice discretization of eq. (3.14), so that the partition function reads

$$Z = \int \left[ \prod_r d\theta(r) \right] \exp \left\{ -\beta \sum_{r,\alpha} \frac{1}{2} (\Delta_{\alpha} \theta)^2 \right\},$$

where $\Delta_{\alpha} \theta(\xi) = (\theta(\xi + \epsilon \hat{\alpha}) - \theta(\xi - \epsilon \hat{\alpha}))/2$, $r$ is an index labelling the lattice sites and $\epsilon$ is the world-sheet lattice spacing, to be eventually sent to zero.$^{10}$

With standard manipulations (see e.g. ref. [36], eqs. (7.19)-(7.28), or ref. [37]) it can be rewritten as

$$Z = \int \left[ \prod_r d\phi(r) \right] \sum_{\{m(r)\} = -\infty}^{\infty} \exp \left\{ -\frac{1}{\beta} \sum_{r,\alpha} \frac{1}{2} (\Delta_{\alpha} \phi)^2 + 2\pi i \sum_r m(r) \phi(r) \right\}. \quad (3.19)$$

$^{10}$As $\epsilon \to 0$, $\Delta_{\alpha} \to \epsilon \partial_{\alpha}$ to conform with the notations of ref. [36].
Here \( \phi \) is a real scalar field that, contrary to \( \theta \), is not subject to any periodic identification, i.e. \( \phi \) lives on \( R \) rather than on \( S^1 \), and \( m(r) \) is an integer valued field.\footnote{This is a particular case of a more general construction \cite{38}: if \( N \) is a non-simply connected manifold, \( M \) its universal covering space and \( N = M/G \) with \( G \) a group freely acting on \( M \), then a field on \( N \) can be similarly replaced by a field on \( M \) plus a field in \( G \); here \( S^1 = R/Z \), so \( \phi \in R \) and \( m \in Z \). When \( G \) is a continuous group the field in \( G \) is a gauge field.} At small \( \beta \), when \( \theta \) is strongly coupled, \( \phi \) is instead weakly coupled and vice versa. Thus \( \phi \) is the most convenient variable in the region \( \beta < 2/\pi \), i.e. \( T > T_{\text{Hag}} \), where it describes an ordinary massless scalar field, weakly coupled and unconstrained. The integer valued field \( m(r) \) describes instead the vortices. This can be shown integrating out the \( \phi \) field; then one finds the vortices partition function

\[
Z_{\text{vortices}} \sim \sum_{\{m(r)\}=-\infty}^{\infty} \exp \left\{ -2\pi^2 \beta \sum_{r,r'} m(r)G(r-r')m(r') \right\}. \tag{3.20}
\]

\( G(r-r') \) is the lattice propagator, \( \Delta^2 G(r) = \delta_{r,0} \), and at \( r = r' \) it diverges logarithmically, \( G(0) \simeq (1/2\pi) \log(L/\epsilon) \). Therefore the term with \( r = r' \) in the sum in eq. \( 3.20 \) gives a contribution to the action

\[
S(\{m(r)\}) = \pi \beta \left( \sum_r m(r) \right)^2 \log(L/\epsilon). \tag{3.21}
\]

Comparison with eq. \( 3.16 \) shows that the configuration with \( m(r) = v \delta_{r,r_0} \), with \( v \) integer, can be identified with a vortex centered at \( r_0 \) and with winding \( v \), and this provides the lattice definition of the vortex. The contribution of \( G(r-r') \) at \( r \neq r' \) provides instead an interaction term between the vortices, which grows logarithmically with \( r-r' \).

For our purposes it is more useful to integrate out the vortices and remain with an effective field theory for \( \phi \). We follow again refs. \cite{36, 37}. Because of the long-range interaction between vortices, this is a non-trivial many body problem that can be addresses using the renormalization group (RG). A RG transformation of the partition function \( 3.19 \) generates also a term \( \sim \sum_r m^2(r) \) in the action, so it is convenient to start directly with

\[
Z = \int [\prod_r d\phi(r)] \sum_{\{m(r)\}=-\infty}^{\infty} \exp \left\{ -\frac{1}{\beta} \sum_{r,\alpha} \frac{1}{2} (\Delta_\alpha \phi)^2 + (\log y) \sum_r m^2(r) + 2\pi i \sum_r m(r)\phi(r) \right\}. \tag{3.22}
\]

The initial value for the RG transformation is \( y = 1 \). Defining \( x = \pi \beta - 2 \), the RG flow in the \((x,y)\) plane turns out to have a line of fixed points at \( y = 0, x \geq 0 \). At \( y \) close to zero eq. \( 3.22 \) simplifies because we can retain only the states with \( m = 0, \pm 1 \). Using

\[
\sum_{m(r)=0,\pm 1} e^{((\log y)m^2(r)+2\pi im(r)\phi(r))} = 1 + 2y \cos [2\pi \phi(r)] \simeq \exp \{ 2y \cos [2\pi \phi(r)] \} \tag{3.23}
\]

and rescaling \( \phi \to \phi \sqrt{\beta} \), eq. \( 3.22 \) becomes

\[
Z \simeq \int [\prod_r d\phi(r)] \exp \left\{ -\frac{1}{2} \sum_{r,\alpha} (\Delta_\alpha \phi)^2 + 2y \sum_r \cos \left[ 2\pi \sqrt{\beta} \phi(r) \right] \right\}, \tag{3.24}
\]
and we recognize a sine-Gordon theory.

The RG flow in the \((x, y)\) plane is given by the well-known diagram reproduced in fig. 2 [33, 34, 36] (and the general KT picture has been confirmed by many Monte Carlo simulations and lattice strong coupling expansions, see e.g. refs. 39–12 and references therein, while the existence of the phase transition was rigorously proved in ref. 13). From fig. 2 we see that there is a critical value \(\beta_c\) and therefore a critical temperature \(T_c\) (in a first approximation \(T_c = T_{Hag}\), but we will come back to this point below), such that for \(T < T_c\) the RG trajectories flow toward \(y = 0, x > 0\) and then stop, i.e. we have a line of fixed points. At \(y = 0\) vortices are completely suppressed, as we read from eq. (3.22), and because of this the issue of whether to include or not vortices in the regularized theory is irrelevant in the low temperature phase. The critical properties are the same as those of a free scalar field, and the correlation length is infinite.

Above \(T_c\), however, \(y\) flows toward large values and vortices are important. In this regime, we have seen that the weakly coupled variable is \(\phi\) rather than \(\theta\). We therefore describe the string at \(T > T_c\) by \(\tilde{X}^0\) and the \(X^i\), where

\[
\tilde{X}^0 \equiv \sqrt{2\pi\alpha'} \phi.
\]

(3.25)

The normalization of \(\tilde{X}^0\) has been chosen so that its kinetic term has the standard string normalization (we will discuss in sect. 3.4 the relation of \(\tilde{X}^0\) to \(X^0_L - X^0_R\)). Observe that \(\phi\) is not subject to any periodic identification, and therefore we have no periodic identification on \(\tilde{X}^0\) either: the domain of definition of \(\tilde{X}^0\) is the whole real line, rather than a circle. Since the effect of vortices has already been taken into account, we can use without ambiguity a continuum notation, and the action for \(\tilde{X}^0\) reads

\[
S = \int d^2\sigma \left[ \frac{1}{4\pi\alpha'} (\partial_{\sigma} \tilde{X}^0)^2 - \mu \cos \left( \frac{R}{\alpha'} \tilde{X}^0 \right) \right],
\]

(3.26)

with \(\mu = 2y/\epsilon^2\). Therefore below \(\beta_c\) we have an effective potential for \(\tilde{X}^0\)

\[
V(\tilde{X}^0) = -\mu \cos \left( \frac{R}{\alpha'} \tilde{X}^0 \right),
\]

(3.27)

with \(\mu \to \infty\). This potential breaks the continuous translation symmetry of \(\tilde{X}^0\) to discrete translations, and localizes \(\tilde{X}^0\) on the minima of the cosine, i.e. on an infinite lattice with spacing \(a = 2\pi\alpha'/R\).

The same results on the decompactification and dynamical localization of the compact coordinate below \(R_c\) that we have read off this well-known RG analysis of the KT transition have also been obtained some time ago by Gross and Klebanov [4, 5] with matrix model techniques, and their result [5] is indeed that below \(R_c\) the model defined by a single
compact string coordinate coincides with an infinite set of decoupled $c = 0$ one matrix models (see also refs. \[14, 15\]).\(^{12}\)

At the critical temperature the lattice spacing jumps from zero to the finite value

$$a_c = \frac{2\pi \alpha'}{R_c} \simeq \pi \sqrt{\alpha'}.$$  \hspace{1cm} (3.28)

Therefore $a$ is the order parameter of a first order phase transition in spacetime. The fact that the transition is first order is in agreement with the result of ref. \[1\].

The value of the critical temperature $T_c$ is not exactly equal to $T_{Hag}$, i.e. $\beta_c$ is not exactly equal to $2/\pi$, as was suggested by the result (3.17). The RG analysis \[33, 34, 36\] shows that the critical value of $\beta$ is determined by

$$\pi \beta_c - 2 = (2c) \exp \left\{ -\frac{\pi^2}{2} \beta_c \right\},$$  \hspace{1cm} (3.29)

with $c \simeq 1.3\pi$ a positive constant. Physically, the correction term comes from the interaction between vortices, while eq. (3.17) was obtained in the dilute vortex approximation, and the positivity of $c$ reflects the formation of an effective dielectric constant greater than one \[33\]. This gives a critical temperature $T_c$ slightly smaller than $T_{Hag} = 1/(4\pi \sqrt{\alpha'})$,

$$\left( \frac{T_{Hag}}{T_c} \right)^2 = 1 + c \exp \left\{ -\pi \left( \frac{T_{Hag}}{T_c} \right)^2 \right\},$$  \hspace{1cm} (3.30)

which, solved numerically, gives $T_c \simeq 0.94T_{Hag}$. Eq. (3.30) is compatible with a first order phase transition in spacetime, and in particular with the fact that a first order transition proceeds via tunneling before $T_{Hag}$ is reached; denoting by $t$ and $t^*$ the winding modes of the tachyon with winding $w = 1$ and $w = -1$, respectively, the spacetime dynamics is governed by an effective potential of the form \[1\]

$$V(t^*t) = m^2(T) t^* t + u(T)(t^* t)^2 + \ldots$$  \hspace{1cm} (3.31)

where

$$m^2(T) = \frac{4}{\alpha'} \left( \frac{T_{Hag}^2}{T^2} - 1 \right)$$  \hspace{1cm} (3.32)

is the mass squared of $t$, $t^*$, and becomes zero at $T_{Hag}$; if $u(T_{Hag})$ were positive, the transition could be second order (depending also on the sign of the higher order terms) in which case it would take place when $m(T)$ vanishes, i.e. at $T = T_{Hag}$. However $u(T_{Hag})$ turns out to be negative because of the tachyon-dilaton coupling \[1\], and the transition is then first order and proceeds via tunnelling when $m^2(T)$ is still positive, i.e. at $T_c < T_{Hag}$.

The fact that the transition is first order means that the Hagedorn temperature is not limiting neither for closed nor for open strings, since the transition takes place via tunneling before $T_{Hag}$ is reached, independently of whether an infinite energy would be needed to reach $T_{Hag}$ (which appears to be the case for open strings \[15\]).

\(^{12}\)It is also interesting to compare with the case of a boundary sine-Gordon theory, corresponding to the insertion of the vertex operator of open string tachyons. In this case the model is exactly solvable by Bethe ansatz techniques \[41\] and one finds that the boundary value of $X$ is pinned to the minima of the potential, so that the IR fixed point is a stack of D-branes \[47\].
3.4 Closed tachyon condensation

In the last few years there has been formidable progress in understanding the condensation of open string tachyons \[49\]. For open strings, the vertex operators of the tachyon live on the boundary of the world-sheet and, at small string coupling, their effect is just to modify the boundary conditions of the world-sheet theory. Because of this, the condensation of open string tachyons does not have a dramatic influence on the structure of spacetime itself. The typical process that they describe is the annihilation of brane-antibrane systems, whose endpoint turns out to be simply the flat vacuum. It is instead believed that the condensation of closed string tachyon is a much harder problem, because it involves a bulk perturbation of the world-sheet. However, the results of the previous section give an answer to what happens when a closed string tachyon with non-zero winding condense. First of all, it is instructive to recover the same results with a $\sigma$-model analysis of tachyon condensation.

The vertex operator for the winding modes $w = \pm 1$ of the bosonic string tachyon is

\[
V(0, n = 0, w = \pm 1) = g_c \int d^2 z : \exp\{-i(k_L^0 X^0_L + k_R^0 X^0_R) + ik^i X^i_1\} : = g_c \int d^2 z : \exp\{\mp i\frac{R}{\alpha'}(X^0_L - X^0_R) + ik^i X^i_1\} : ,
\]

with $k_L = n/R + wR/\alpha' = \pm R/\alpha'$, $k_R = n/R - wR/\alpha' = \mp R/\alpha'$, and the mass shell condition is $k^2 = (4/\alpha')(1 - T_{Hag}^2/T^2)$, so that $k_i = 0$ just at $T = T_{Hag}$.

The $\sigma$-model relevant for the condensation of tachyons with $w = \pm 1$ is then

\[
S = \int d^2 z \frac{1}{4\pi\alpha'} (\partial_\alpha X^0)^2 + t(X)e^{i\frac{R}{\alpha'}(X^0_L - X^0_R)} + t^*(X)e^{-i\frac{R}{\alpha'}(X^0_L - X^0_R)} ,
\]

where $t(X)$ is the complex tachyon field that describes the two real tachyons with $w = \pm 1$. Observe that the kinetic term of $X^0 = X^0_L + X^0_R$ is equal to minus the kinetic term of $X^0_L - X^0_R$, since $\partial_\alpha X^0)^2 = 2\partial_\alpha X^0_L \partial_\alpha X^0_R$; therefore, the action (3.34) can be written as a functional of the combination $X^0_L - X^0_R$ only, and this is a peculiarity of a tachyon which is a winding mode of the zero temperature tachyon, because it is just in this case that its vertex operator depends only on $X^0_L - X^0_R$. Writing $t(X) = |t| e^{i\vartheta}$, eq. (3.34) becomes

\[
S = - \int d^2 z \frac{1}{4\pi\alpha'} |\partial_\alpha (X^0_L - X^0_R)|^2 - 2|t(X)| \cos \left[\frac{R}{\alpha'}(X^0_L - X^0_R) + \vartheta\right] .
\]

Comparison with eq. (3.26) shows that the two actions are the same\(^\text{13}\) if we take the tachyon field constant, we identify $2|t|$ with $\mu$, and we set

\[
\tilde{X}^0 = (X^0_L - X^0_R) + \frac{\alpha'}{R} \vartheta .
\]

\(^\text{13}\)Neglecting an overall minus sign of the action. This has no influence on the Noether charges of the theory and therefore on spacetime quantities, since in any case the sign of the Noether charges is unrelated to the sign of the action, and is fixed by physical requirements.
that, when a string coordinate $X = X_L + X_R$ is periodically identified as $X \sim X + 2\pi R$, then $X_L - X_R$ is its T-dual variable and is also periodically identified, with a periodicity $2\pi \alpha'/R$. Since by definition also $\vartheta \sim \vartheta + 2\pi$, this seems to suggest that the right-hand side of eq. (3.36) lives on a circle of radius $2\pi \alpha'/R$.

However, when $X = X^0$ (or in general when we compactify on $S^1/(-1)^F$) this conclusion is incorrect. The reason why one usually says that $X_L - X_R$ is periodic is the following. If $X = X_L + X_R$ is taken by definition to live on a circle of radius $R$, then the expansion of $X_L, X_R$ on the complex plane is

$$X_L = x_L - i\frac{\alpha'}{2}p_L \log z + \text{oscillators}$$

$$X_R = x_R - i\frac{\alpha'}{2}p_R \log \bar{z} + \text{oscillators}$$

with $p_L = (n/R) + (wR/\alpha'), p_R = (n/R) - (wR/\alpha')$. Under a $2\pi$ rotation in the complex plane $X_L \rightarrow X_L + \alpha' \pi p_L$, $X_R \rightarrow X_R - \alpha' \pi p_R$ and therefore $X_L - X_R \rightarrow X_L - X_R + n2\pi R'$, with $R' = \alpha'/R$. Now, if $X_L - X_R$ is a single valued functions, we conclude that $X_L - X_R$ and $X_L - X_R + 2\pi R'$ must be identified, and therefore also $X_L - X_R$ lives on a circle.

However, we have seen that when we compactify the time direction we must allow for vortex configurations, and these are not single valued. Below $T_{\text{Hag}}$ they are anyway dynamically irrelevant, so all configurations of $X_L, X_R$ that contribute to the path integral are single valued, and $X_L^0 - X_R^0$ indeed lives on the dual circle. Instead, above $T_{\text{Hag}}$ the path integral is dominated just by the non single-valued configurations, and therefore the above argument does not go through and $X_L^0 - X_R^0$ has no constraint and lives on the real line. The identification (3.36) is therefore possible, and the sigma-model approach gives the same answer as the Kosterlitz-Thouless analysis of the previous section.

We have therefore understood that, as we reach the Hagedorn temperature from the low $T$ side, tachyon condensation leads us to a state that can be described as a stack of spacelike surfaces, separated by a spacing $a_c \simeq \pi \sqrt{\alpha'}$ in the euclidean time direction, as in fig. 1. Similarly, if we ask what is the endpoint of tachyon condensation of a theory compactified at a radius $R < R_c$, we find a stack of spacelike surfaces with $a = 2\pi \alpha'/R$. The situation is depicted in fig. 3.

The light degrees of freedom in the new phase are naturally identified with the fluctuation modes of these spacelike surfaces. The peculiarity of these modes is that they evolve in a space with a discretized euclidean time and, as we have seen in sect. 2.2, in this setting it is quite natural to quantize them imposing the deformed commutation relations (2.26, 2.27). Therefore, if after reaching $T_{\text{Hag}}$ we inject further energy into the system, the free energy of these modes, at large $T$, will have the Atick-Witten form $F(T) = V A T^2$ with $\Lambda$ finite and given by eq. (2.49).

We conclude this section with a few comments on the literature. First, the fact that the condensation of closed string tachyons can produce a discrete spacetime was nicely shown, in a different setting, in ref. [50]. These authors considered string theory in D=2 (at zero
temperature). In this case the would-be zero temperature tachyon is actually massless, but they put it slightly off-shell, \( k^2 = 2 - \epsilon \), so that the tachyon operator becomes slightly relevant and it suffices to compute the beta function at order \( \epsilon \) to find the endpoint of the condensation of this tachyon. Guided by the analogy with Landau theory of solidification, they find that the final spacetime is a two-dimensional lattice. This fits very nicely with our results.

Recently there have been a number of investigations on the condensation of type 0 tachyons \([12, 13, 14, 15]\), and signals of the formation of a lattice structure in spacetime have also been found in some of these works. In particular, Adams, Polchinski and Silverstein \([14]\) consider type II theory in ten dimensions, and replace a plane, say \((89)\), with an orbifold \( C/Z_n \), i.e. identify \( z \sim z \exp\{2\pi i/n\} \), where \( z \) is the complex coordinate of the \((89)\) plane. If \( z = re^{i\theta} \), at fixed \( r \neq 0 \) the variable \( r\theta \) parametrizes a circle of length \( 2\pi r/n \), so for given fixed \( r \), in the large \( n \) limit we have a small circle. Furthermore in the large \( n \) limit the orbifold projection is such that going around this circle the fermions pick a minus sign, i.e. we have a \((-1)^F\) twist. Taking the T-dual of this circle one has type 0 theory in the bulk, and the authors find that translation symmetry in the T-dual variable is broken, and that there is a set of equally spaced branes (defined broadly as defects which break translation invariance) in the T-dual variable. This is in agreement with the result that we have found. However, in our case we have also found that the would-be T-dual variable becomes non-compact, and lives on the whole real line rather than on the dual circle. The difference might be due to the fact that the tachyon studied in ref. \([14]\) resembles a winding mode only far from the tip of the cone.

Another hint in this direction has been found in ref. \([15]\). The authors consider type II theories compactified on twisted circles, that interpolate between type II on an ordinary circle and type 0 theories, and in a supergravity analysis they find a tachyon carrying a non-vanishing momentum in the compact direction, therefore breaking again the translation invariance and producing a regular lattice.

It is also interesting to see what our results suggest for the condensation of type 0 tachyons. Type 0 theory is obtained in the limit \( R \to 0 \), and in this limit the lattice spacing \( a \to \infty \). Taking \( a \to \infty \), we are actually isolating a single spacelike surface. However, as discussed in sect. 2.4, in this limit the description in terms of a spacelike surface is not really adequate: the \( x_i \)'s lose their interpretation as coordinates, and become boost operators; the deformed Poincaré symmetry on the single \( d \)-dimensional spacelike surface at fixed time becomes equivalent to a full undeformed Poincaré symmetry in a \( d + 1 \) dimensional spacetime. However, this is exactly the symmetry of the ground state of a (non-tachyonic) string theory. This result is therefore consistent with the recent suggestion that type 0 theories decay to type II theories \([12, 13, 14, 15]\).
4. Conclusions

The behavior of strings above the Hagedorn temperature is, since a long time, one of the crucial mysteries of string theory. Discussing the result $F(T) \sim T^2$ that they found, Atick and Witten, back in 1988, commented “A new version of Heisenberg’s principle - some non-commutativity where it does not usually arise - may be the key to the thinning of the degrees of freedom that is needed to describe string theory correctly” [1]. The results that we have presented fully vindicate this intuition, and show that the non-commutativity emerges in a quite subtle way, with a phase transition in which the Poincaré algebra is deformed to a quantum algebra.

The deformations of the Poincaré algebra that we have discussed are also quite interesting structures by themselves. Research in this direction had somehow got stuck since many years into a (false) problem on the description of composite systems, see app. A. Once one interprete these deformed symmetries properly, basically as symmetries of the one-particles states only, they offer a viable and interesting possibility. The results that we have found in sect. 2 of this paper show that, at the quantum level, they are the natural symmetry of system with a discrete time, and that they are remarkably rich structures embodying, in a single conceptually well motivated scheme, a non-commutative geometry on the spatial coordinates, a generalized uncertainty principle and a minimal length, deformed dispersion relations and, in the limit of large deformation parameter, a surprising form of time (de)construction. The results of sect. 3 indicate that these ideas can find their place in string theory.

Finally, in a more general perspective, the concept of a phase transition where a symmetry group, rather than being broken to a subgroup, is deformed into a non-linear structure is possibly interesting in itself and might have broader applications.

Acknowledgments

I am grateful to Maura Brunetti, Maria Alice Gasparini, Gabriele Veneziano and Konstantin Zarembo for useful discussions.

A. Composite systems and deformed Poincaré symmetry

In this appendix we clarify a problem on the treatement of multiparticle systems within the deformed Poincaré algebra. Consider a system made of two particles, with generators $H_1, P_1$ and $H_2, P_2$, respectively. Since deformed algebras are non-linear structures, if $H_1, P_1$ and $H_2, P_2$ separately satisfy a deformed algebra, then $H_1 + H_2$ and $P_1 + P_2$ do not satisfy the deformed algebra. It has been observed that there is a non-linear combination of energy and momenta, known as the coproduct, that satisfies the algebra. For instance, for the algebra (2.3) this combination is given by $E_{12} = \exp\{iaP_2\}E_1 + \exp\{-iaP_1\}E_2$ and $P_{12} = P_1 + P_2$. Using the experience from other types of quantum groups, in particular deformations of the angular momentum algebra, there have been attempts to interprete these expressions as the energy and momentum of the composite system. However, these
attempts have immediately failed. First of all, $E_{12}$ is not even real, nor symmetric under the exchange of the two particles, so that $E_{21}$ defines a different composition law. The situation does not improve considering any of the other deformed algebras discussed above. In the algebra (2.7) now energies add simply, $E_{12} = E_1 + E_2$, but momenta have a non-trivial, and again physically unacceptable coproduct, $P_{12} = \exp\{iaE_2\}P_1 + \exp\{-iaE_1\}P_2$. The algebra (2.10) instead has again $E_{12} = E_1 + E_2$ and $P_{12} = \exp\{aE_2\}P_1 + \exp\{-aE_1\}P_2$. This latter composition law is, at least, real, but it is still physically meaningless, since it states that the total momentum of two infinitely separated particles is a combination (non even symmetric) of the momenta of one-particle and the exponential of the energy of the other. The point of view that $a$ is very small and therefore the effects are not observable is also untenable. Combining a sufficiently large number of particles with this coproduct rule we would find that even macroscopic objects obey these unphysical rules for the composition of energy and momenta.

If however we do not lose sight of the physics, the situation is quite clear. For instance, the algebra (2.3) is a useful description of a system of 1+1 phonons. The usefulness comes from the fact that its Casimir reproduces the wave equation, but certainly phonons do not satisfy these strange composition laws (while, even in 1+1 dimensions, the coproduct would give a strange result for two antiparallel phonons). So this is an explicit counterexample showing that the deformed Poincaré algebra can be a useful tool in the description of a system with a standard composition law for energy and momenta, $E_{\text{tot}} = E_1 + E_2, P_{\text{tot}} = P_1 + P_2$. In other words, $E_{\text{tot}}$ and $P_{\text{tot}}$ do not have to satisfy the deformed algebra.

In general, the physical reason why the generators of a composite system are expected to satisfy the same algebra as the constituents is that we can imagine to put two elementary constituents into a black box, and there is no way to distinguish this composite system from an elementary object. However, two particles with energy and momenta $E_1, P_1$ and $E_2, P_2$ cannot be put in the same finite size black box, unless they are exactly collinear and with the same speed, even more so in relativistic quantum theory, where particles are only defined as asymptotic states. Therefore the argument which says that the total energy and momentum of the composite system must satisfy the same algebra as the constituents does not go through. The system of two free particles moving in two different directions is in no way the same as a single localized system. Of course in the undeformed case the total energy and momenta still satisfy the Poincaré algebra; however this comes out for free because the algebra is linear. In the deformed case instead this does not come out automatically, and there is no argument that requires it, and indeed phonons provide an explicit counterexample. The deformed Poincaré algebras discussed in sect. 2 are a useful description of discretized systems, and the action of the generators $P_1^\mu, P_2^\mu$ on a composite system $|1\rangle|2\rangle$ of two particles not exactly collinear is simply given by

$$\left(e^{-ia_\mu P_1^\mu}|1\rangle\right)\left(e^{-ia_\mu P_2^\mu}|2\rangle\right)$$

(A.1)

rather than by $\exp\{-ia_\mu P_{12}^\mu\}|1\rangle|2\rangle$, with $P_{12}^\mu$ given by the coproduct. As asymptotic states, $|1\rangle$ and $|2\rangle$ are infinitely separated and the generators $P_1^\mu, P_2^\mu$ act separately on their respective single particle states.
There is however a point to be checked. In the (somewhat idealized) case when the two particles are exactly collinear and with exactly the same speed, the “black box argument” can be applied, and therefore the coproduct must give a meaningful answer. To understand what is a meaningful answer in this context observe that, when the dispersion relation is deformed, we cannot have at the same time the standard relation between momentum and velocity, \( p = \gamma m v \), with \( \gamma = (1 - v^2)^{-1/2} \), and the standard relation between momentum and energy, \( p = E v \), since one follows from the other upon use of \( E^2 = p^2 + m^2 \). Suppose that \( p = E v \) is modified into \( p = u(E)v \), with \( u(E) \) a given function. The non-trivial coproduct must simply express the obvious fact that, when \( u(E) \) is not linear in \( E \), the momentum of a system with energy \( E_1 + E_2 \) is not just the sum of the momentum of a system with energy \( E_1 \) and of that of a system with energy \( E_2 \). Rather, one has \( P_{12} = u(E_{12})v \).

To verify this, we use for instance the algebra \((2.14, 2.15)\), which is the more interesting for application to string theory. The coproduct in this case is \( E_{12} = E_1 + E_2 \) and \( P_{12} = e^{aE_2}P_1 + e^{-aE_1}P_2 = (P_1 \cosh aE_2 + P_2 \cosh aE_1) + (P_1 \sinh aE_2 - P_2 \sinh aE_1) \).

Thus, we want to find a function \( u(E) \) such that, if we take \( P_1 \) and \( P_2 \) to be collinear, and with modulus \( p_1 = u(E_1)v \) and \( p_2 = u(E_2)v \) with the same \( v \), then eq. \((A.2)\) becomes \( P_{12} = u(E_1 + E_2)v \). It is easy to see that the only function that does the job (and reduces to \( u(E) = E \) if \( a \to 0 \)) is \( u(E) = \sinh(aE)/a \), i.e.

\[
p = \frac{\sinh aE}{a} v,
\]

since then eq. \((A.2)\) becomes

\[
P_{12} = \frac{\sinh a(E_1 + E_2)}{a} v. \tag{A.4}
\]

As a bonus, we have understood that eq. \((A.3)\) is the correct definition of the velocity when Poincaré symmetry is deformed. Eliminating \( \sinh aE \) with the help of the dispersion relation, we find that this relation simply means that the standard relation between momentum and velocity, \( p = \gamma m v \), with \( \gamma = (1 - v^2)^{-1/2} \), is not deformed \([21]\). The velocity therefore still ranges between zero and one. The same procedure can be applied to any of the other deformations of the Poincaré algebra that we have discussed. For the algebra representing discrete Minkowski time it suffices to exchange \( a \to ia \), i.e. to define the velocity from \( ap = \sin(aE)v \). Again, using the dispersion relation, we see that \( p = \gamma mv \) is not modified. Then the coproduct for two collinear particles with the same speed turns out to be real, symmetric under the exchange of the two particles, and reproduces correctly the non-linearity of the dispersion relation. In all other kinematical configurations, the coproduct of the deformed Poincaré algebra has no physical interest.

References

[1] J. J. Atick and E. Witten, “The Hagedorn transition and the number of degrees of freedom of string theory,” Nucl. Phys. B 310 (1988) 291.
[2] M. Maggiore, “The algebraic structure of the generalized uncertainty principle,” Phys. Lett. B 319 (1993) 83 [arXiv:hep-th/9309034].

[3] S. Kalyana Rama, “Some consequences of the generalised uncertainty principle: Statistical mechanical, cosmological, and varying speed of light,” Phys. Lett. B 519 (2001) 103 [arXiv:hep-th/0107255].

[4] D. J. Gross and I. R. Klebanov, “One-dimensional string theory on a circle,” Nucl. Phys. B 344 (1990) 475.

[5] D. J. Gross and I. R. Klebanov, “Vortices and the nonsinglet sector of the c = 1 Matrix Model,” Nucl. Phys. B 354 (1991) 459.

[6] R. Rohm, “Spontaneous supersymmetry breaking in supersymmetric string theories,” Nucl. Phys. B 237 (1984) 553.

[7] B. Sathiapalan, “Vortices on the string world sheet and constraints on toral compactification,” Phys. Rev. D 35 (1987) 3277.

[8] Y. I. Kogan, “Vortices on the world sheet and string’s critical dynamics,” JETP Lett. 45 (1987) 709 [Pisma Zh. Eksp. Teor. Fiz. 45 (1987) 556].

[9] K. H. O’Brien and C. I. Tan, “Modular invariance of thermopartition function and global phase structure of heterotic string,” Phys. Rev. D 36 (1987) 1184.

[10] C. M. Hull, “Timelike T-duality, de Sitter space, large N gauge theories and topological field theory,” JHEP 9807 (1998) 021 [arXiv:hep-th/9806146].

[11] M. Gutperle and A. Strominger, “Spacelike branes,” [arXiv:hep-th/0202210].

[12] M. S. Costa and M. Gutperle, “The Kaluza-Klein Melvin solution in M-theory,” JHEP 0103 (2001) 027 [arXiv:hep-th/0012072].

[13] M. Gutperle and A. Strominger, “Fluxbranes in string theory,” JHEP 0106 (2001) 035 [arXiv:hep-th/0104136].

[14] A. Adams, J. Polchinski and E. Silverstein, “Don’t panic! Closed string tachyons in ALE space-times,” JHEP 0110 (2001) 029 [arXiv:hep-th/0108075].

[15] J. R. David, M. Gutperle, M. Headrick and S. Minwalla, “Closed string tachyon condensation on twisted circles,” JHEP 0202 (2002) 041 [arXiv:hep-th/0111212].

[16] F. Bonechi, E. Celeghini, R. Giachetti, E. Sorace and M. Tarlini, Phys. Rev. Lett. 68 (1992) 3718 [arXiv:hep-th/9201002].

[17] J. Lukierski, H. Ruegg, A. Nowicki and V. N. Tolstoi, “Q deformation of Poincaré algebra,” Phys. Lett. B 264 (1991) 331.

[18] J. Lukierski, A. Nowicki and H. Ruegg, “New quantum Poincaré algebra and k deformed field theory,” Phys. Lett. B 293 (1992) 344.

[19] J. Lukierski and H. Ruegg, “Quantum Kappa Poincaré in any Dimension,” Phys. Lett. B 329 (1994) 189 [arXiv:hep-th/9310117].

[20] H. S. Snyder, “Quantized Space-Time”, Phys. Rev. 71 (1947) 38.

[21] M. Maggiore, “Quantum groups, gravity and the generalized uncertainty principle,” Phys. Rev. D 49 (1994) 5182 [arXiv:hep-th/9305163].

[22] G. Veneziano, “A Stringy Nature Needs Just Two Constants,” Europhys. Lett. 2 (1986) 199.
[23] D. Amati, M. Ciafaloni and G. Veneziano, “Superstring Collisions at Planckian Energies,” Phys. Lett. B 197 (1987) 81; “Classical and Quantum Gravity Effects from Planckian Energy Superstring Collisions,” Int. J. Mod. Phys. A 3 (1988) 1615; “Higher Order Gravitational Deflection and Soft Bremsstrahlung in Planckian Energy Superstring Collisions,” Nucl. Phys. B 347 (1990) 550.

[24] D. J. Gross and P. F. Mende, “The High-Energy Behavior of String Scattering Amplitudes,” Phys. Lett. B 197 (1987) 129; “String Theory beyond the Planck Scale,” Nucl. Phys. B 303 (1988) 407.

[25] K. Konishi, G. Paffuti and P. Provero, “Minimum Physical Length and the Generalized Uncertainty Principle in String Theory,” Phys. Lett. B 234 (1990) 276; “On the Short Distance Behavior of String Theories,” Mod. Phys. Lett. A 6 (1991) 1487.

[26] M. Maggiore, “A Generalized Uncertainty Principle in Quantum Gravity,” Phys. Lett. B 304 (1993) 65 [arXiv:hep-th/9301067].

[27] N. Arkani-Hamed, A. G. Cohen and H. Georgi, “(De)constructing dimensions,” Phys. Rev. Lett. 86 (2001) 4757 [arXiv:hep-th/0104003].

[28] Z. Berezhiani, A. Gorsky and I. I. Kogan, “On the deconstruction of time,” arXiv:hep-th/0203016.

[29] M. Alishahiha, “(De)constructing dimensions and non-commutative geometry,” Phys. Lett. B 517 (2001) 406 [arXiv:hep-th/0105153].

[30] J. Polchinski, “String Theory. Vol. I and II” Cambridge, UK: Univ. Pr. (1998).

[31] I. Antoniadis and C. Kounnas, “Superstring phase transition at high temperature,” Phys. Lett. B 261 (1991) 369.

[32] V. L. Berezinsky, “Destruction of Long Range Order in One-Dimensional and Two-Dimensional Systems having a Continuous Symmetry Group. 1. Classical Systems,” Sov. Phys. JETP 32 (1971) 493.

[33] J. M. Kosterlitz and D. J. Thouless, “Ordering, Metastability and Phase Transitions in Two-Dimensional Systems,” J. Phys. CC 6 (1973) 1181.

[34] J. M. Kosterlitz, “The Critical Properties of the Two-Dimensional XY Model,” J. Phys. C 7 (1974) 1046.

[35] Y. Y. Goldschmidt, “A Kosterlitz-Thouless phase transition associated with supersymmetric sine-Gordon theory,” Nucl. Phys. B 270 (1986) 29.

[36] J. B. Kogut, “An Introduction to Lattice Gauge Theory and Spin Systems,” Rev. Mod. Phys. 51 (1979) 659.

[37] J. V. Jose, L. P. Kadanoff, S. Kirkpatrick and D. R. Nelson, “Renormalization, vortices, and symmetry-breaking perturbations in the two-dimensional planar model,” Phys. Rev. B 16 (1977) 1217.

[38] A. A. Abrikosov and Y. I. Kogan, “Vortices on the string and superstring world sheets,” Int. J. Mod. Phys. A 6 (1991) 1501 [Sov. Phys. JETP 69 (1989) 235].

[39] R. Gupta, J. DeLapp, G. G. Batrouni, G. C. Fox, C. F. Baillie and J. Apostolakis, “The Phase Transition in the 2-D XY Model,” Phys. Rev. Lett. 61 (1988) 1996.
[40] M. Hasenbusch, M. Marcu and K. Pinn, “High precision renormalization group study of the roughening transition,” Physica A 208 (1994) 124 [arXiv:hep-lat/9404016].

[41] R. Kenna and A. C. Irving, “The Kosterlitz-Thouless universality class,” Nucl. Phys. B 485 (1997) 583 [arXiv:hep-lat/9601029].

[42] M. Campostrini, A. Pelissetto, P. Rossi and E. Vicari, “A strong-coupling analysis of two-dimensional O(N) sigma models with $N \leq 2$ on square, triangular and honeycomb lattices,” Phys. Rev. B 54 (1996) 7301 [arXiv:hep-lat/9603002].

[43] J. Frohlich and T. Spencer, “The Kosterlitz-Thouless transition in two-dimensional abelian spin systems and the Coulomb gas,” Commun. Math. Phys. 81 (1981) 527.

[44] D. Kutasov, “Irreversibility of the renormalization group flow in two-dimensional quantum gravity,” Mod. Phys. Lett. A 7 (1992) 2943 [arXiv:hep-th/9207064].

[45] E. Hsu and D. Kutasov, “The gravitational sine-Gordon model,” Nucl. Phys. B 396 (1993) 693 [arXiv:hep-th/9212023].

[46] P. Fendley, H. Saleur and N. P. Warner, “Exact solution of a massless scalar field with a relevant boundary interaction,” Nucl. Phys. B 430 (1994) 577 [arXiv:hep-th/9406125].

[47] J. A. Harvey, D. Kutasov and E. J. Martinec, “On the relevance of tachyons,” [arXiv:hep-th/0003101].

[48] K. R. Dienes, E. Dudas, T. Gherghetta and A. Riotto, “Cosmological phase transitions and radius stabilization in higher dimensions,” Nucl. Phys. B 543 (1999) 387 [arXiv:hep-ph/9809406].

[49] A. Sen, “Stable non-BPS states in string theory,” JHEP 9806 (1998) 007 [arXiv:hep-th/9803194]; “Stable non-BPS bound states of BPS D-branes,” JHEP 9808 (1998) 010 [arXiv:hep-th/9805013]; “Tachyon condensation on the brane antibrane system,” JHEP 9808 (1998) 012 [arXiv:hep-th/9805170].

[50] S. Elitzur, A. Forge and E. Rabinovici, “Some global aspects of string compactifications,” Nucl. Phys. B 359 (1991) 581.