BULK ASYMPOTOTICS FOR POLYANALYTIC CORRELATION KERNELS

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ABSTRACT. For a weight function $Q : \mathbb{C} \to \mathbb{R}$ and a positive scaling parameter $m$, we study reproducing kernels $K_{q,mQ,n}$ of the polynomial spaces $A_{q,mQ,n} := \text{span}_\mathbb{C}\{z^r\bar{z}^j : 0 \leq r \leq q - 1, 0 \leq j \leq n - 1\}$ equipped with the inner product from the space $L^2(e^{-mQ(z)}dA(z))$. Here $dA$ denotes a suitably normalized area measure on $\mathbb{C}$. For a point $z_0$ belonging to the interior of certain compact set $S$ and satisfying $\Delta Q(z_0) > 0$, we define the rescaled coordinates $z = z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, \quad w = z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}}.$ The following universality result is proved in the case $q = 2$:

$$\frac{1}{m\Delta Q(z_0)}|K_{q,mQ,n}(z,w)|e^{-\frac{1}{2}mQ(z) - \frac{1}{2}mQ(w)} \to |L_{q-1}^1(\xi - \lambda)|^2 e^{-\frac{1}{2}(|\xi - \lambda|^2)}$$

as $m, n \to \infty$ while $n \geq m - M$ for any fixed $M > 0$, uniformly for $(\xi, \lambda)$ in compact subsets of $\mathbb{C}^2$. The notation $L_{q-1}^1$ stands for the associated Laguerre polynomial with parameter 1 and degree $q - 1$. This generalizes a result of Ameur, Hedenmalm and Makarov concerning analytic polynomials to bianalytic polynomials. We also discuss how to generalize the result to $q > 2$. Our methods include a simplification of a Bergman kernel expansion algorithm of Berman, Berndtsson and Sjöstrand in the one complex variable setting, and extension to the context of polyanalytic functions. We also study off-diagonal behaviour of the kernels $K_{q,mQ,n}$.

1. INTRODUCTION

1.1. Notation. We will write $\partial X$ and $\text{int}(X)$ for the boundary and the interior of a subset $X$ of the complex plane $\mathbb{C}$. By $1_X$ we mean the characteristic function of the set $X$. We let $dA(z) = \pi^{-1}dxdy$, where $z = x + iy \in \mathbb{C}$, be the normalized area measure in $\mathbb{C}$, and use the standard Wirtinger derivatives

$$\partial_z := \frac{1}{2}(\partial_x - i\partial_y), \quad \bar{\partial}_z := \frac{1}{2}(\partial_x + i\partial_y).$$

Will will often omit the subscripts if there is no risk of confusion. We write $\Delta = \partial\bar{\partial}$, and it can be observed that this equals to one quarter of the usual Laplacian. We write $\mathbb{D}$ for the open unit disk, and more generally $\mathbb{D}(z, r)$ for the disk with center $z$ and radius $r$. Given a Lebesgue measurable function $w : \mathbb{C} \to \mathbb{R}$, we denote by $L^2(w)$ the space of measurable functions $\mathbb{C} \to \mathbb{C}$ which are square-integrable with respect to the measure $w(z)dA(z)$.

1.2. Spaces of polyanalytic polynomials. Let $Q : \mathbb{C} \to \mathbb{R}$ be a continuous function satisfying

$$Q(z) \geq (1 + \epsilon)|z|^2, \quad |z| \geq C$$

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for two positive numbers $\epsilon$ and $C$. This function will be referred to as the weight. We set

$$\text{Pol}_{q,n} := \text{span}_\mathbb{C}\{z^r z^j \mid 0 \leq r \leq q - 1, 0 \leq j \leq n - 1\},$$

and

$$A_{q,mQ,n} := \text{Pol}_{q,n} \cap L^2(e^{-mQ(z)}).$$

The space $A_{q,mQ,n}$ is a finite dimensional, and thus closed, subspace of $L^2(e^{-mQ(z)})$. We see that when $m \geq n + q - 1$, the growth condition on $Q$ implies that $A_{q,mQ,n}$ contains the whole $\text{Pol}_{q,n}$.

Notice that $A_{1,mQ,n}$ consists of analytic polynomials of degree at most $n - 1$. For a more general $q \geq 1$, functions in the spaces $A_{q,mQ,n}$ will be called $q$-analytic polynomials.

By a $q$-analytic function, we mean a continuous function that satisfies the equation $\partial^q f = 0$ in the sense of distribution theory. A function which is $q$-analytic for some $q \geq 1$ is called polyanalytic. Obviously, $q$-analytic polynomials form a subclass of $q$-analytic functions. It is also easy to see that a $q$-analytic function $f$ can be written as

\begin{equation}
\tag{1.2}
f(z) = \sum_{j=0}^{q-1} z^j f_j(z)
\end{equation}

for some analytic functions $f_j$. The decomposition provides a correspondence between $q$-analytic functions and vector-valued analytic functions; this connection is explained in [22] in more detail.

The space $A_{q,mQ,n}$ possesses the reproducing kernel

$$K_{q,mQ,n}(z, w) = \sum_j e_j(z) \overline{e_j(w)},$$

where $\{e_j\}$ is any orthonormal basis for $A_{q,mQ,n}$. It is well-known that $K_{q,mQ,n}$ does not depend on the choice of the basis and that the reproducing property

$$p(z) = \int_{\mathbb{C}} p(w) K_{q,mQ,n}(z, w) e^{-mQ(w)} dA(w),$$

holds for all $p \in A_{q,mQ,n}$.

1.3. Determinantal point processes. Assuming $m \geq n + q - 1$, which implies that $A_{q,mQ,n}$ is $nq$-dimensional, we use $K_{q,mQ,n}$ to define the following probability distribution on $C^{nq}$.

\begin{equation}
\tag{1.3}
dP_{q,mQ,n}(z_1, \ldots, z_{nq}) := \Lambda_{q,mQ,n}(z_1, \ldots, z_{nq}) dA(z_1) \ldots dA(z_{nq}),
\end{equation}

where

\begin{equation}
\tag{1.4}
\Lambda_{q,mQ,n}(z_1, \ldots, z_{nq}) := \frac{1}{(nq)!} \det \left[ K_{q,mQ,n}(z_i, z_j) \right]_{i,j=1}^{nq} e^{-m \sum_{j=1}^{nq} Q(z_j)}.
\end{equation}

This is a particular instance of a so called determinantal point process. That $dP_{q,mQ,n}$ is a probability measure follows from standard arguments (see any book on random matrices, e.g., [17], [29]); this depends only on the fact that $K_{q,mQ,n}$ is a kernel of a projection to a $nq$-dimensional subspace of $L^2(e^{-mQ})$. It is customary to identify all the $nq$ copies of the complex plane and permutations of the points $z_1, \ldots, z_{nq}$, and think the process as a random configuration of $nq$ unlabelled points in $\mathbb{C}$.

For $1 \leq k \leq nq$, let us define the $k$-point intensity functions $\Gamma^k_{q,mQ,n}$ by

$$\Gamma^k_{q,mQ,n}(z_1, \ldots, z_k) := \frac{(nq)!}{(nq-k)!} \int_{C^{nq-k}} \Lambda(z_1, \ldots, z_{nq}) dA(z_{k+1}, \ldots, z_{nq}).$$

It is a well-known fact about determinantal point processes that all the intensity functions are easily expressed with the kernel $K_{q,mQ,n}$:

$$\Gamma^k_{q,mQ,n}(z_1, \ldots, z_k) = \det \left[ K_{q,mQ,n}(z_i, z_j) \right]_{i,j=1}^{k} e^{-m \sum_{j=1}^{k} Q(z_j)}.
\end{equation}

The one-point intensity $\Gamma^1_{q,mQ,n}(z) = K_{q,mQ,n}(z, z) e^{-mQ(z)}$ is particularly important, since integrating it over a set $A$ gives the expected number of points in $A$. 

The intensity functions are often called correlation functions in the literature, and the weighted kernel \( K_{q,mQ,n}(z, w)e^{-\frac{1}{2}mQ(z)-\frac{1}{2}mQ(w)} \) is referred to as the correlation kernel of the process.

1.4. Weighted potential theory. To discuss previous results on analytic polynomial kernels, we need to recall some facts from weighted potential theory. Let us assume that the weight \( Q \) is \( C^1 \)-regular in \( \mathbb{C} \) and satisfies the growth condition \((1.1)\). We define \( \mathcal{N}_+ \) and \( \mathcal{N}_{+,0} \) to be the sets

\[
\mathcal{N}_+ := \{ w \in \mathbb{C} : \Delta Q(w) > 0 \}, \quad \mathcal{N}_{+,0} := \{ w \in \mathbb{C} : \Delta Q(w) \geq 0 \}.
\]

The equilibrium weight \( \hat{Q} \) is defined as the largest subharmonic function in \( \mathbb{C} \) which is \( \leq Q \) everywhere and has the growth bound

\[
\hat{Q}(z) = \log |z|^2 + O(1), \quad \text{as } |z| \to +\infty.
\]

The function \( \hat{Q} \) is then subharmonic and it follows from general theory for obstacle problems that it is \( C^{1,1} \) regular (see [19]). There is also an elementary proof of this fact due to Berman [7]. The coincidence set of the obstacle problem is

\[
\mathcal{S} := \{ Q(z) = \hat{Q}(z) \}.
\]

This is a compact set and because \( \hat{Q} \) is subharmonic, we have \( \mathcal{S} \subset \mathcal{N}_{+,0} \). It is known that \( \hat{Q} \) is harmonic on \( \mathbb{C} \setminus \mathcal{S} \). The set \( \mathcal{S} \) is a central object in weighted potential theory as it arises as the support of the unique solution to the energy minimization problem

\[
\min_{\sigma} I(\sigma),
\]

where

\[
I(\sigma) := \frac{1}{2} \int_{\mathbb{C}^2} \log \frac{1}{|z-w|^2} d\sigma(z)d\sigma(w) + \int_{\mathbb{C}} Q(w)d\sigma(w).
\]

The infimum is taken over all compactly supported Borel probability measures. Existence and uniqueness in this problem is due to Frostman. For more details, see [33]. Note that the functional \((1.6)\) coincides with the standard logarithmic energy in the special case \( Q = 0 \). The solution to energy minimization problem can be written as (see [19] or the earlier preprint [18])

\[
d\hat{\sigma} = \Delta \hat{Q}dA = \Delta Q1_{\mathcal{S}}dA.
\]

The measure \( \hat{\sigma} \) will be referred to as the equilibrium measure.

1.5. Bergman kernels for analytic polynomials. Kernels \( K_{1,mQ,n} \) and associated probability densities \( \Lambda_{1,mQ,n} \) were studied by Ameur, Hedenmalm and Makarov in [3], [4] and [5]. There are three possible interpretations for these point processes: in terms of Coulomb gas, free fermions or eigenvalues of random normal matrices. For more details, we refer the reader to [37].

As \( m, n \to +\infty \) while \( n = m + o(n) \), Hedenmalm and Makarov ([18], [19]) showed, building on the work of Johansson [24], that

\[
\frac{(n-k)!}{n!} \Gamma_{1,mQ,n}^{k} dA^{\otimes k} \to d\hat{\sigma}^{\otimes k}
\]

in the weak-* sense of measures for all fixed \( k \). Here, the notation \( \mu^{\otimes k} \) stands for the \( k \)’th tensor power of the measure \( \mu \). As the authors showed in [19], this result can be used to show that for any bounded continuous function \( g \), we have the convergence

\[
\frac{1}{n} \sum_{j=1}^{n} g(\lambda_j) \to \int_{\mathcal{S}} g(w)d\hat{\sigma}(w), \quad n, m \to \infty, n = m + o(n).
\]

in distribution. Recalling \((1.7)\), the intuitive interpretation of this is that the points from the determinantal process defined via \( K_{1,mQ,n} \) tend to accumulate on \( \mathcal{S} \) with density \( \Delta Q \) as \( m, n \to +\infty \) while \( n = m + o(n) \). The set \( \mathcal{S} \) was called droplet for this reason.

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1 This result is just a special case of their theorem, which holds for Coulomb gas models in arbitrary temperatures.
Let us write bulk for the interior of the set $\mathcal{S} \cap \mathcal{N}_\omega$. The results of [4] show that for a bulk point $z$ and a $n$-tuple $(z_1, \ldots, z_n)$ picked from the density
\[
\Lambda_{1,mQ,n}(z_1, \ldots, z_n) dA(z_1) \ldots dA(z_n),
\]
the local blow-up process at $z$ with coordinates
\[
\xi_j := m^{1/2} |\Delta Q(z)|^{1/2} (z_j - z),
\]
converges to the Ginibre($\infty$) process, as $m,n \to +\infty$ while $n = m + o(1)$. If we write $\overline{\Gamma}^k_{1,mQ}$ for the $k$-point intensity of the blow-up process, this means that
\[
\overline{\Gamma}^k_{1,mQ}(\xi_1, \ldots, \xi_k) \to \det \left[e^{\xi_i \xi_j} \right]_{i,j=1}^k e^{-\sum_{i=1}^k |\xi_i|^2}
\]
as $n, m \to \infty$ while $n = m + o(1)$, for all $k$ and $(\xi_1, \ldots, \xi_k) \in \mathbb{C}^k$. Notice that the kernel $e^{\xi_\bar{\xi}}$ appearing in the determinant on the right hand side is the reproducing kernel of the Bargmann-Fock space.

This fact could be interpreted as a universality result in the spirit of random matrix theory and related fields. In general, universality means that there exists a scaling limit which does not depend on particularities of the model (see [14] for more discussion). In our setting, this is reflected by the fact that the limiting process is the same for all weights $Q$.

One can also formulate a universality result more directly in terms of the kernel $K_{1,mQ,n}$. To do this, let us define the Berezin density centered at $z$ as
\[
\Gamma^{(z)}_{1,mQ,n}(w) := \frac{|K_{1,mQ,n}(w,z)|^2}{K_{1,mQ,n}(z,z)} e^{-mQ(w)}.
\]
The main theorem in [3] was as follows:

**Theorem 1.1** (Ameur, Hedenmalm, Makarov). Fix $z \in \text{int} \mathcal{S} \cap \mathcal{N}_\omega$ and suppose that $Q$ is real-analytic in some neighborhood of $z$. Then
\[
\frac{1}{m|\Delta Q(z)|} \Gamma^{(z)}_{1,mQ,n}(z + \frac{\xi}{\sqrt{|\Delta Q(z)|} m}) \to e^{-|\xi|^2}, \quad n, m \to \infty, m = n + o(1),
\]
where the convergence holds in $L^1(\mathbb{C})$.

It should be mentioned that Berman [8] proved a similar result independently, also in a higher-dimensional setting.

Our main result will be a generalization of a slight reformulation of theorem 1.1 to the context of more general polyanalytic polynomial kernels $K_{q,mQ,n}$.

### 1.6. Polyanalytic Ginibre ensembles

In a joint paper with Hedenmalm [21], we studied kernels $K_{q,mQ,n}$ with the weight $Q(z) = |z|^2$ and called the associated determinantal point processes *Polyanalytic Ginibre ensembles*. As we explained in this paper, these point processes describe systems of free (i.e. non-interacting) electrons in $\mathbb{C}$ in a constant magnetic field of strength $m$ perpendicular to the plane, so that each of the first $q$ Landau levels contains $n$ particles. This model has also been studied in physics literature, see Dunne [15].

The analysis was in terms of the Berezin density, which was defined as in the case $q = 1$:
\[
\Gamma^{(z)}_{q,mQ,n}(w) := \frac{|K_{q,mQ,n}(w,z)|^2}{K_{q,mQ,n}(z,z)} e^{-mQ(w)}.
\]
In the macroscopic length scales, we showed that,
\[
\Gamma^{(z)}_{q,mQ,n}(w) dA(w) \to \delta_z, \quad |z| < 1
\]
\[
\Gamma^{(z)}_{q,mQ,n}(w) dA(w) \to \omega_z, \quad |z| > 1,
\]
as $n, m \to \infty$ while $|n - m| = O(1)$. Here $\delta_z$ stands for the Dirac point mass at $z$ and $\omega_z$ for the harmonic measure at $z$ with respect to the domain $\mathbb{C} \setminus \overline{B}$. Notice that both limits are clearly independent of $q$.

For microscopic length scales in the bulk $\{|z| < 1\}$, we obtained
\[
\Gamma^{(z)}_{q,mQ,n}(z + \frac{\xi}{\sqrt{m}}) \to \frac{1}{q} L^1_{q-1}(|\xi|^2) e^{-|\xi|^2}, \quad m, n \to \infty, m = n + O(1)
\]
Figure 1.1. The limiting Berezin density for polyanalytic Ginibre process with \( q = 3 \) exhibiting a Fresnel-type ring pattern. Here, white is high and black is low Berezin density.

where \( L_{q-1}^1 \) is the associated Laguerre polynomial with parameter 1 and degree \( q - 1 \).

It might be interesting to recall that the Laguerre polynomial \( L_{q-1}^1 \) has \( q - 1 \) zeros on the positive real axis. In terms of the points process, this means that around each point \( z \) from the process, there are \( q - 1 \) rings around \( z \), the radii of which are of the order of magnitude \( m^{-1/2} \), so that there is no repulsion between \( z \) and points on those rings in the limit as \( m, n \to \infty \). So, one would expect the electrons to accumulate on those circles around any given electron.

It should be emphasized that this phenomenon is not present in the analytic case \( q = 1 \).

We want to mention here that in [21], we also studied the Berezin kernels near the edge \( \{|z| = 1\} \). We will not discuss such issues in this paper, and will only refer the interested reader to the original article.

1.7. Main results. Our aim is to generalize the results from polyanalytic Ginibre ensembles to more general weights \( Q \). For simplicity, we mostly work with \( q = 2 \), but the proof methods should work for any \( q \). We start by analyzing the one point function \( \Lambda_{q, mQ,n}^{2, mQ,n}(z) \). In section 3, it is shown in that this expression tends to zero for \( z \in \mathbb{C} \setminus S \) in an exponential rate as \( m, n \to \infty \). This means that the points tend to accumulate on \( S \) as in the analytic case \( q = 1 \).

Within \( S \), the following theorem presents more detailed information. It states that the bulk scaling limits obtained for polyanalytic Ginibre ensembles are universal.

**Theorem 1.2.** Set \( q = 2 \). Fix \( z_0 \in \text{int} S \cap N_+ \) and \( M > 0 \). Assume that \( Q \) is \( C^2 \)-smooth, satisfies the growth condition (1.1) and is real-analytic in a neighborhood of \( z_0 \). Then, there exists a number \( m_0 \) such that for all \( m \geq m_0 \), we have

\[
(1.12) \quad \left| \frac{1}{m\Delta Q(z_0)} K_{q, mQ,n}(z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}}) \right| \\
	imes e^{-\frac{1}{2}mQ(z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}})} e^{-\frac{1}{2}mQ(z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}})} = |L_{q-1}^1(|\xi - \lambda|^2)|e^{-\frac{1}{2}|\xi - \lambda|^2} + O(m^{-1/2}),
\]

as \( m \to \infty \) and \( n \geq m - M \). The convergence is uniform on compact sets of \( \mathbb{C}^2 \).

One can check that with the weight \( Q(z) = |z|^2 \), the theorem is a slight reformulation of (1.11).
Basic structure of our argument will be the same as in [3]. There, the authors relied on two main techniques: algorithm of Berman-Berndtsson-Sjöstrand [9] to compute asymptotic expansions for Bergman kernels, and Hörmander’s $\partial$-estimates. First, we will simplify the method of Berman-Berndtsson-Sjöstrand (in the one complex variable context only) and then extend it to polyanalytic functions. In a joint paper with Hedenmalm [22], we already showed how to obtain asymptotic expansions in the polyanalytic setting, but the approach we will take here will be more elementary and also computationally simpler.

Whereas in [3] certain estimates for the $\partial$-operator are used, we need similar results for the operators $\partial^q$ with $q > 1$. As a consequence, we also obtain an off-diagonal decay estimate for bianalytic Bergman kernels, which, informally speaking, says that correlations are short range in $S$. Comparing with the similar result in [3], one sees that the decay is essentially as strong as in the case $q = 1$. Again, we present a proof for the case $q = 2$, but the method should generalize to any $q \geq 2$.

**Theorem 1.3.** Suppose that $Q$ is $C^2$-smooth. Fix a compact set $K$ in the interior of $S \cap N_+$ and constant $M > 0$. Set

$$r_{0,K} := \frac{1}{4} \text{dist}(K, \mathbb{C} \backslash (S \cap N_+)).$$

Then, there exist positive constants $C, \epsilon$ and $m_0$ such that for any $z_0 \in K$ and $z_1 \in S$, it holds

$$|K_{2,mQ,n}(z_0, z_1)|^2 e^{-mQ(z_0)} - mQ(z_1) \leq C m^{2} e^{-\epsilon \sqrt{m} \min\{r_{0,K}, |z_0 - z_1|\}}$$

where we assume $m \geq \max\{m_0, M - 1\}$ and $n \geq m - M + 1$. The constants $C, \epsilon$ and $m_0$ only depend on $Q, K$ and $M$.

One can of course ask what happens if the point $z_1$ is allowed to be outside $S$. The answer will be provided in section 3 where we show that even stronger decay holds as $m \to \infty$.

It is possible that this off-diagonal decay estimate, or a variant of it, could also be used in other contexts to extend known results about analytic functions to polyanalytic functions. We should mention at least the work of Ortega-Cerdá and Ameur [4] concerning Fekete points as well as that of Ortega-Cerdá and Seip [22] on description of sampling and interpolation sets. The latter topic in spaces of polyanalytic functions is related to time frequency analysis (see Abreu [1]).

It would also be natural to study asymptotics of $K_{q,mQ,n}$ near the edge of the droplet but this question remains open even in the case $q = 1$.

1.8. **Further questions: letting $q$ tend to infinity.** It is also possible to let all the parameters $q, m$ and $n$ tend to infinity in our model. We explained already in [21] that the rescaled Berezin transform

$$\frac{1}{mq} B_{q,m|z|^2,n}^{(z_0)}(z_0 + \frac{\xi}{\sqrt{mq}}) \quad |z_0| < 1$$

converges to the limit $J^1(2|\xi|)/(2|\xi|)$ if we first let $n, m \to \infty$ while $n = m + O(1)$ and then let $q \to \infty$ afterwards. Here, $J^1$ is the standard Bessel function. Because of theorem 1.2, a similar result also holds when the weight is more general. Interestingly, this Bessel kernel could be viewed as a two-dimensional analogue of the sine kernel from Hermitian random matrix theory: the latter is the Fourier transform of a characteristic function of an interval while the former is the Fourier transform of a characteristic function of a disk.

It would interesting to study the asymptotics for $K_{q,mQ,n}$ as $q$ and $n$ go tend to infinity simultaneously. To get a very symmetric model, one could set $n = m$, require that $n + q \leq N$ and then let $N$ tend to infinity. It seems likely that the above Bessel kernel would also arise here in the limit.

1.9. **Further questions: fluctuations.** In [4] and [5], fluctuation field of the random normal matrix model was shown to converge to Gaussian free field, in the first paper with the restriction that the test function is supported in the interior of $S$. The methods of the first paper should apply to the polyanalytic setting, given the technology we develop in the present paper. The argument of [5], where the case of more general test functions was treated using so called Ward identities, seems harder to generalize.
2. Construction of local polyanalytic Bergman kernels

In this section, we present an algorithm to compute asymptotic expansions for polyanalytic Bergman kernels near the diagonal. For analytic functions, this is a well-studied topic in several complex variables literature see e.g. [11, 38, 34, 31]. The algorithm we present here is based on the work of Berman, Berndtsson and Sjöstrand [9], whose method relies on a certain technique from microlocal analysis (for an detailed exposition in the one complex variables setting, see [3]). Here we will show that at least in the one-dimensional case, this technique can be dispensed with. In particular, we get an alternative way to obtain results of Ameur, Hedenmalm and Makarov in the case \( q = 1 \). We then show how to extend this modified algorithm to polyanalytic functions. This provides a simplification of the method of [22], which was based on the original microlocal analysis technique.

We take an arbitrary \( z_0 \in N \) and assume that \( Q \) is real-analytic in a neighborhood of \( z_0 \). We will also pick \( r > 0 \) such that the following conditions are satisfied:

1. \( Q \) is real-analytic in \( D(z_0, r) \) and \( \Delta Q(z) > \epsilon > 0 \) on \( D(z_0, r) \).
2. There exists a local polarization of \( Q \in D(z_0, r) \), i.e. a function \( Q : D(z_0, r) \times D(z_0, r) \rightarrow \mathbb{C} \) which is analytic in the first and anti-analytic in the second variable, and satisfies \( Q(z, z) = Q(z) \).

For \( z, w \in D(z_0, r) \), we have \( \partial_{\bar{z}} Q(z, w) \neq 0 \) and \( \partial_{\bar{z}} Q(z, w) \neq 0 \); these conditions are made possible by condition (1). Here \( \theta \) is the phase function which is defined below.

(4) Taylor expansion of \( Q(z, w) \) gives

\[
2 \text{Re} Q(z, w) - Q(w) - Q(z) = -\Delta Q(z)|w - z|^2 + O(|z - w|^3).
\]

For details, see p. 1555 in [3]. We require that for \( z, w \in D(z_0, r) \),

\[
2 \text{Re} Q(z, w) - Q(w) - Q(z) \leq -\frac{1}{2} \Delta Q(z_0)|w - z|^2.
\]

We define the phase function \( \theta : D(z_0, r) \times D(z_0, r) \) as

\[
\theta(z, w) = \frac{Q(w) - Q(z, w)}{w - z}.
\]

Notice that \( \theta \) is analytic in the first variable and real-analytic in the second. It can be analytically continued to the diagonal, and we have \( \theta(z, z) = \partial_{\bar{z}} Q(z) \). More generally,

\[
\theta(z, w) = \frac{Q(w, w) - Q(z, w)}{w - z} = \sum_{j=0}^{\infty} \frac{1}{(j + 1)!}(w - z)^j \partial_{\bar{z}}^{j+1} Q(z, w).
\]

This leads to the following Taylor expansion, which we will need repeatedly:

\[
\partial_{\bar{z}} \theta(z, w) = b(z, w) + \frac{1}{2}(w - z)\partial_{\bar{z}}b(z, w) + \frac{1}{6}(w - z)^2 \partial_{\bar{z}}^2 b(z, w) + \ldots
\]

where

\[
b(z, w) := \partial_{\bar{z}} \partial_{\bar{z}} Q(z, w).
\]

We fix \( \chi_0 : \mathbb{R} \rightarrow \mathbb{R} \) to be a smooth and non-negative cut-off function which equals 1 on the interval \( (-\frac{2}{3}, \frac{2}{3}) \) and is supported on \( (-1, 1) \). We then define \( \chi(z) = \chi_0\left(\frac{2 - \text{Re} z}{r}\right) \); this function will then be supported on \( D(z_0, r) \) and equal 1 on \( D(z_0, \frac{2}{3} r) \).

We will write \( A^2_{q, mQ} \) for the subspace of \( q \)-analytic functions in \( L^2(e^{-mQ}) \):

\[
A^2_{q, mQ} := \{ f : \mathbb{C} \rightarrow \mathbb{C} \mid \bar{\partial} f = 0, ||f||^2_m := \int_{\mathbb{C}} |f|^2 e^{-mQ} dA < \infty \}.
\]

Clearly, the spaces \( A^2_{q, mQ, n} \) are closed subspaces of \( A^2_{q, mQ} \). We will use the notation \( \| \cdot \|_m \) for the norm in the spaces \( L^2(e^{-mQ(z)}) \), \( A^2_{q, mQ} \) and \( A^2_{q, mQ, n} \).

We start by proving a lemma that will be frequently used in the sequel.

**Lemma 2.1.** Let \( m \geq 1 \) and \( A_m : D(z_0, r) \times D(z_0, r) \setminus \{(z, w) \in \mathbb{C}^2 : z = w\} \rightarrow \mathbb{C} \) be \( q \)-analytic in the first variable and real-analytic in the second. We also assume that there exists \( K > 0 \) and a positive integer \( N \) such that

\[
|\bar{\partial}^k_w A_m(z, w)| \leq m^N K, \quad 0 \leq k \leq q - 1,
\]
for all \( z \in \mathbb{D}(z_0, \frac{1}{3} r) \) and \( w \in \mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r) \). Then, given integers \( k \) and \( l \) satisfying \( 0 \leq k \leq q-1 \) and \( l \geq 1 \), there exists \( \delta > 0 \) such that

\[
(2.6) \quad \left| \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} \partial_{\bar{w}}^k u(w) \partial^l_w \chi(w) A(z, w) e^{m(z-w)\theta(z,w)} dA(w) \right| = O(\|u\|_m e^{\frac{1}{2}mQ(z)} e^{-\delta m})
\]

for any \( u \in A^2_{q,mQ} \) and \( z \in \mathbb{D}(z_0, \frac{1}{3} r) \). The number \( \delta \) and the constant of the error term are independent of \( u, m \) and \( z \).

**Proof.** Let us start with \( k = 0 \). By (2.2), we have

\[
(2.7) \quad \left| \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} u(w) \partial^l_w \chi(w) A(z, w) e^{m(z-w)\theta(z,w)} dA(w) \right|
\]

\[
\leq m^N Ke^\frac{1}{2}mQ(z) \left[ \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} \left| u(w) \partial^l_w \chi(w) \right| e^{-\frac{1}{2}mQ(w)} e^{-\frac{1}{2}m\Delta Q(z_0)|w-z|^2} dA(w) \right]^\frac{1}{2}
\]

\[
\leq m^N Ke^\frac{1}{2}mQ(z) \|u\|_m \left[ \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} \left| \partial^l_w \chi(w) \right|^2 e^{-\frac{1}{2}m\Delta Q(z_0)|w-z|^2} dA(w) \right]^\frac{1}{2}
\]

\[
\leq Ce^\frac{1}{2}mQ(z) \|u\|_m e^{-\frac{1}{2}m\Delta Q(z_0)z^2} \left[ \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} \left| \partial^l_w \chi(w) \right|^2 dA(w) \right]^{\frac{1}{2}},
\]

for some positive constants \( C \) and \( \delta \). This shows the desired statement for the case \( k = 0 \).

For \( k > 0 \), we integrate by parts:

\[
(2.8) \quad \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} \partial_{\bar{w}}^k u(w) \partial^l_w \chi(w) A(z, w) e^{m(z-w)\theta(z,w)} dA(w)
\]

\[
= (-1)^k \int_{\mathbb{D}(z_0, r) \setminus \mathbb{D}(z_0, \frac{2}{3} r)} u(w) \partial^l_w \left[ \partial_{\bar{w}}^k \chi(w) A(z, w) e^{m(z-w)\theta(z,w)} \right] dA(w).
\]

The statement follows after carrying out the differentiation and analyzing each term in the resulting sum as in the case \( k = 0 \). \( \square \)

Next, we will prove an approximate reproducing identity for polyanalytic functions. For the case \( q = 2 \), this was already done in [22]. Here we present the argument for general \( q \geq 1 \).

**Proposition 2.2.** There exists \( \delta > 0 \), independent of \( m \), such that for all \( z \in \mathbb{D}(z_0, \frac{1}{3} r) \) and \( u \in A^2_{q,mQ} \), we have

\[
(2.9) \quad u(z) = \int_{\mathbb{D}(z_0, r)} u(w) \chi(w) R_{q,m}(z, w) e^{m(z-w)\theta(z,w)} dA(w) + O(\|u\|_m e^{mQ(z)/2 - \delta m}),
\]

where

\[
(2.10) \quad R_{q,m}(z, w) = m \sum_{s=k}^{q-1} \frac{q!(-1)^k}{(q - 1 - k)!k!(k + 1)!} (\bar{z} - \bar{w})^k \partial_{\bar{w}}^k (\partial_w \theta e^{m(z-w)\theta}) e^{-m(z-w)\theta}.
\]

The constant of the error term in (2.9) is independent of \( u, z \) and \( m \).

**Proof.** We will use the fundamental solution \( \frac{1}{(q-1)!} \frac{\bar{w}^{q-1}}{w} \) of the operator \( \partial^q_{\bar{w}} \) (recall that our reference measure \( dA \) is the usual area measure divided by \( \pi \), so there is no need for that
normalization here). Because of lemma 2.1 and $q$-analyticity of $u$, we have

\begin{equation}
\frac{1}{(q-1)!} \int \frac{u(w)\chi(w)\partial_q^k}{(q-1)!} \left( \frac{(\bar{w} - z)^{q-1}}{w - z} e^{m(z-w)\theta} \right) dA(w)
\end{equation}

\begin{equation}
= \frac{(-1)^q}{(q-1)!} \int \frac{\partial_q^k}{(q-1)!} \left( \frac{(\bar{w} - z)^{q-1}}{w - z} e^{m(z-w)\theta} \right) dA(w)
\end{equation}

\begin{equation}
= \frac{(-1)^q}{(q-1)!} \sum_{k=0}^{q-1} \frac{q!(-1)^k}{(q-1)!k!(k+1)!} (\bar{z} - \bar{w})^k \partial_w^k \left( \frac{(\bar{w} - z)^{q-1}}{w - z} e^{m(z-w)\theta} \right) dA(w)
\end{equation}

\begin{equation}
= O(\|u\|_m e^{mQ(z)/2 - \delta m})
\end{equation}

for some $\delta > 0$.

Denoting by $\delta_z(w)dA(w)$ the Dirac point mass at $z$, we get

\begin{equation}
u(z) = \int \frac{u(w)\chi(w)\delta_z}{(q-1)!} \left( \frac{(\bar{w} - z)^{q-1}}{w - z} e^{m(z-w)\theta} \right) dA(w) + O(\|u\|_m e^{\frac{1}{2} mQ(z) - \delta m})
\end{equation}

where

\begin{equation}
R_{q,m}(z,w) e^{m(z-w)\theta} dA(w) = \left[ \delta_z(w) - \delta_z^{mQ(z) - \delta m} \right] dA(w).
\end{equation}

Applying Leibniz rule in the sense of distribution theory shows that

\begin{equation}
R_{q,m}(z,w) e^{m(z-w)\theta} = m \sum_{k=0}^{q-1} \frac{q!(-1)^k}{(q-1)!k!(k+1)!} (\bar{z} - \bar{w})^k \partial_w^k \left( \frac{(\bar{w} - z)^{q-1}}{w - z} e^{m(z-w)\theta} \right).
\end{equation}

So actually, the singularity in (2.13) cancels and $R_{q,m}$ is $q$-analytic in $z$ and real-analytic in $w$.

A function $L_{q,m} : \mathbb{D}(z_0, r) \times \mathbb{D}(z_0, r) \to \mathbb{C}$ which is $q$-analytic in the first variable and real-analytic in the second will be called a local $q$-analytic reproducing kernel mod$(e^{-\delta m})$ if for any $u \in A^2_{q,m\theta}$ and $z \in \mathbb{D}(z_0, \frac{1}{3} r)$, we have

\begin{equation}
u(z) = \int_{\mathbb{D}(z_0, r)} u(w) \chi(w) L_{q,m}(z, w) e^{-mQ(w)} dA(w) + O(\|u\|_m e^{mQ(z)/2 - \delta m}),
\end{equation}

where the constant of the error term can depend on $Q$, $z_0$ and $r$ but not on $u$, $z$ and $m$. Clearly, proposition 2.2 shows that $R_{q,m}(z, w) e^{mQ(z,w)}$ satisfies this condition. We define a local reproducing kernel mod$(m^{-k})$ similarly, just by replacing the factor $e^{-\delta m}$ in the error term by $m^{-k}$. If a local reproducing kernel $L_{q,m}$ with any error term is $q$-analytic in $\bar{w}$ (i.e. satisfies $\partial_{\bar{w}} L_{q,m}(z,w) = 0$), it will be called a local $q$-analytic Bergman kernel.

In [21], we presented an algorithm producing local $q$-analytic Bergman kernels mod$(m^{-k})$ for arbitrary $k$, based on a microlocal analysis technique of Berman-Bernardsson-Sjöstrand [9] in the analytic case $q = 1$. We will next show that when $q = 1$, Taylor expansion and partial integration is enough. After this we show in the case $q = 2$, how this approach can be extended to more general polyanalytic functions. Later, in section 5, we will show that when $z_0 \in \mathbb{S} \cap \mathbb{N}_+$, local bianalytic Bergman kernels actually provide a near-diagonal approximation of the kernel $K_{2,m\theta,n}$ as $m, n \to \infty$.

2.1. Computation of local analytic Bergman kernels. The aim is to show how to compute local analytic Bergman kernels mod$(m^{-k-\frac{1}{2}})$ in the form

\begin{equation}
\left( m\delta_{1,0}(z,w) + \delta_{1,1}(z,w) + m^{-1}\delta_{1,2}(z,w) + \cdots + m^{-k}\delta_{2,k+1}(z,w) \right) e^{mQ(z,w)}
\end{equation}
where all the coefficient functions $g_{1,j}$ are analytic in $z$ and $\bar{w}$. We denote by $X_j(z, w)$ a function on $D(z_0, r) \times D(z_0, r)$ which is analytic in $z$ and real-analytic in $w$ but whose exact form is not of interest to us. The number $\delta$ will denote a positive number that can change at each step.

Let $u \in A^2_1, mQ$ and $z \in D(z_0, \frac{1}{3} r)$. We check from (2.10) that

\[(2.17) \quad R_{1,m}(z, w) = m\tilde{d}_w \theta(z, w).\]

Then, recalling proposition 2.2 and the Taylor expansion 2.4, we compute

\[(2.18) \quad u(z) = m \int_{D(z_0, r)} u(w)\chi(w) \left[ b(z, w) + (w - z)\frac{1}{2} \partial_z b(z, w) + (w - z)^2 X_1(z, w) \right] e^{m(z-w)\theta} dA(w) + O(||u||_m e^{\frac{b}{2} mQ(z) - \delta m})
\]

= \[m \int_{D(z_0, r)} u(w)\chi(w) b(z, w) e^{m(z-w)\theta} dA(w)
\]

- \[\int_{D(z_0, r)} u(w)\chi(w) \left( \frac{1}{2} \partial_{w} b(z, w) + (w - z) X_1(z, w) \right) \tilde{d}_w e^{m(z-w)\theta} dA(w)
\]

= \[m \int_{D(z_0, r)} u(w)\chi(w) b(z, w) e^{m(z-w)\theta} dA(w)
\]

\[+ \int_{D(z_0, r)} u(w)\chi(w) \tilde{d}_w \left( \frac{1}{2} \partial_{w} b(z, w) + (w - z) X_1(z, w) \right) e^{m(z-w)\theta} dA(w)
\]

+ \[O(||u||_m e^{\frac{b}{2} mQ(z) - \delta m}),
\]

where for the last equality, we needed an application of lemma 2.1. Here and later in computations of this nature, the choice of $\delta$ and the error constant is independent of $u$, $m$ and $z$.

We Taylor expand \(\frac{1}{\partial_{w} \theta} = \frac{1}{b} + (w - z) X_2\) using (2.4), and continue the analysis:

\[(2.19) \quad u(z) = \int_C u(w)\chi(w) \left( \frac{1}{b} + \frac{1}{2} \partial_{w} \frac{\partial_z b}{b} \right) e^{m(z-w)\theta} dA(w)
\]

+ \[\int_{D(z_0, r)} u(w)\chi(w)(w - z) X_3 e^{m(z-w)\theta} dA(w) + O(||u||_m e^{\frac{b}{2} mQ(z) - \delta m})
\]

= \[\int_{D(z_0, r)} u(w)\chi(w) \left( \frac{1}{b} + \frac{1}{2} \partial_{w} \frac{\partial_z b}{b} \right) e^{m(z-w)\theta} dA(w)
\]

- \[\frac{1}{m} \int_{D(z_0, r)} u(w)\chi(w) X_3 \frac{1}{\partial_{w} \theta} \partial_{w} e^{m(z-w)\theta} dA(w) + O(||u||_m e^{\frac{b}{2} mQ(z) - \delta m})
\]

= \[\int_{D(z_0, r)} u(w)\chi(w) \left( \frac{1}{b} + \frac{1}{2} \partial_{w} \frac{\partial_z b}{b} \right) e^{m(z-w)\theta} dA(w)
\]

+ \[\frac{1}{m} \int_{D(z_0, r)} u(w)\chi(w) \frac{1}{\partial_{w} \theta} X_3 e^{m(z-w)\theta} dA(w) + O(||u||_m e^{\frac{b}{2} mQ(z) - \delta m})
\]

= \[\int_{D(z_0, r)} u(w)\chi(w) \left( \frac{1}{b} + \frac{1}{2} \partial_{w} \frac{\partial_z b}{b} \right) e^{m(z-w)\theta} dA(w)
\]

+ \[O(||u||_m e^{\frac{b}{2} mQ(z) m^{-\frac{3}{2}}})].
\]

For the third equality, lemma 2.1 was again used. The fact that we get a factor $m^{-3/2}$ in last error term is a consequence of a small computation which we include here for the convenience of the reader. Let $Y : D(z_0, r) \times D(z_0, r) \to \mathbb{C}$ be $C^2$-smooth and assume

\[\max_{z, w \in D(z_0, r)} |Y(z, w)| \leq C\]
for a constant $C > 0$. Then, using (2.20) and Cauchy-Schwarz inequality,

\begin{equation}
\frac{1}{m} \left| \int_{D(z_0, r)} u(w) \chi(w) Y(z, w)e^{m(z-w)\theta} dA(w) \right| \\
\leq \frac{C}{m} \int_{D(z_0, r)} |u(w)|e^{mReQ(z,w) - mQ(w)} dA(w) \\
\leq \frac{C}{m} \|u\|_m e^{\frac{1}{2}mQ(z)} \left[ \int_{D(z_0, r)} e^{-\frac{1}{2}m\Delta Q(z_0)|w-z|^2} dA(w) \right]^{\frac{1}{2}} \\
= \frac{C}{m^{3/2} \sqrt{\Delta Q(z_0)}} \|u\|_m e^{\frac{1}{2}mQ(z)} \left[ \int_{D(z_0, r)} e^{-\frac{1}{2}|w-z|^2} dA(w) \right]^{\frac{1}{2}} \\
= O(\|u\|_m e^{\frac{1}{2}mQ(z)} m^{-\frac{3}{2}}).
\end{equation}

The conclusion is that

\[ m b(z, w) + \frac{1}{2} \partial_w b(z, w) \] \[ e^{mQ(z,w)} \]

is a local analytic Bergman kernel \( \mod (m^{-\frac{3}{2}}) \). It is possible to continue in the same way and compute local Bergman kernels \( \mod (m^{-k-\frac{3}{2}}) \) for any positive integer $k$; in order to do this, one just has to use higher order Taylor expansions of $\partial_w \theta$. Notice that the local Bergman kernels provided by this process are indeed conjugate analytic in $w$; this follows from the fact that the coefficient functions in the Taylor expansion (2.4) have this property.

**Remark 2.3.** In the computation of local Bergman kernels, we do not necessarily need to require that $u \in A^2_{1, mQ}$: it is enough to assume that $u$ is analytic in $D(z_0, r)$ and then replace $\|u\|_m^2$ by $\int_{D(z_0, r)} |u(w)|^2 e^{-mQ(w)} dA(w)$ in the error terms.

### 2.2. Local bianalytic Bergman kernels

We will now explain how to extend the above method to a more general polyanalytic setting. We focus on the case $q = 2$. The functions satisfying $\partial^2 u = 0$ will be called bianalytic. The intention is to show how to compute local bianalytic Bergman kernels \( \mod (m^{-k-\frac{3}{2}}) \) in the form

\begin{equation}
\left( m^2 \partial_{2,0}(z, w) + m \partial_{2,1}(z, w) + \cdots + m^{-k} \partial_{2,2+k}(z, w) \right) e^{mQ(z,w)},
\end{equation}

where $z, w \in D(z_0, r)$ and all the coefficient functions are bianalytic in $z$ and $\bar{w}$.

Proceeding as in the case $q = 1$, we will expand the kernel $R_{2,m}$ in powers of $w - z$ so that the coefficients are bianalytic in $\bar{w}$. The following proposition will replace the partial integration that was performed in the analytic setting.

**Proposition 2.4.** Let $A : D(z_0, r) \times D(z_0, r) \to \mathbb{C}$ be a $C^2$-smooth function. Then, there exists $\delta > 0$ such that for any $u \in A^2_{q, mQ}$ and $z \in D(z_0, \frac{1}{2}r)$, we have

\begin{equation}
\int u(w) \chi(w) m^2 (z - w)^2 A(z, w) e^{m(z-w)\theta} dA(w) \\
= \int u(w) \chi(w) \left[ -\partial_w A(z, w) \frac{1}{(\partial_w \theta)^2} + m(z - w) \left( -2 \frac{\partial_w A(z, w)}{\partial_w \theta} + 3 \frac{A(z, w) \partial_w^2 \theta}{(\partial_w \theta)^2} \right) \right] e^{m(z-w)\theta} dA(w) \\
+ O(\|u\|_m e^{\frac{1}{2}mQ(z)-\delta m}).
\end{equation}

The constant of the error term is independent of $u$, $m$ and $z$. 
Proof. The proof involves only partial integration.

(2.23) \[ \int_{\mathbb{D}(z_0, r)} u(w) \chi(w) m^2(z - w)^2 A(z, w) e^{m(z - w)\theta} dA(w) \]

\[
= \int_{\mathbb{D}(z_0, r)} u(w) \chi(w) m(z - w) \frac{A(z, w)}{\partial_w \theta} \partial_w e^{m(z - w)\theta} dA(w) \\
= - \int_{\mathbb{D}(z_0, r)} \partial_w u(w) \chi(w) m(z - w) \frac{A}{\partial_w \theta} e^{m(z - w)\theta} dA(w) \\
- \int_{\mathbb{D}(z_0, r)} \partial_w u(w) \chi(w) m(z - w) \frac{A}{\partial_w \theta} e^{m(z - w)\theta} dA(w) \\
- \int_{\mathbb{D}(z_0, r)} \partial_w u(w) \chi(w) m(z - w) \frac{A}{\partial_w \theta} e^{m(z - w)\theta} dA(w) \\
- \int_{\mathbb{D}(z_0, r)} \partial_w u(w) \chi(w) m(z - w) \frac{A}{\partial_w \theta} e^{m(z - w)\theta} dA(w) + O(\|u\| m e^{\frac{1}{2}mQ(z) - \delta m}),
\]

where we used lemma 2.1 to get the last equality.

We leave the first integral in the last expression of (2.23) as such and continue with the analysis of the second.

(2.24) \[ - \int \partial_w u(w) \chi(w) m(z - w) \frac{A}{\partial_w \theta} e^{m(z - w)\theta} dA(w) \]

\[
= - \int \partial_w u(w) \chi(w) \frac{A}{(\partial_w \theta)^2} \partial_w e^{m(z - w)\theta} dA(w) \\
= \int \partial_w u(w) \chi(w) \partial_w \left( \frac{A}{(\partial_w \theta)^2} \right) e^{m(z - w)\theta} dA(w) \\
+ \int \partial_w u(w) \partial_w \chi(w) \left( \frac{A}{(\partial_w \theta)^2} \right) e^{m(z - w)\theta} dA(w) \\
= \int \partial_w u(w) \chi(w) \partial_w \left( \frac{A}{(\partial_w \theta)^2} \right) e^{m(z - w)\theta} dA(w) + O(\|u\| m e^{\frac{1}{2}mQ(z) - \delta m}) \\
= - \int u(w) \partial_w \left[ \chi(w) \partial_w \left( \frac{A}{(\partial_w \theta)^2} \right) e^{m(z - w)\theta} \right] dA(w) + O(\|u\| m e^{\frac{1}{2}mQ(z) - \delta m}) \\
= - \int u(w) \chi(w) \partial_w \left[ \partial_w \left( \frac{A}{(\partial_w \theta)^2} \right) e^{m(z - w)\theta} \right] dA(w) + O(\|u\| m e^{\frac{1}{2}mQ(z) - \delta m}),
\]

where the third and the fifth equality depended on lemma 2.1. After carrying out the differentiations in the last integrand, the assertion follows from combining (2.23) with (2.24).

We will now illustrate how the result can be used by computing two first terms of the expansion (2.21). We let \( X_j \) stand for a function defined on \( \mathbb{D}(z_0, r) \times \mathbb{D}(z_0, r) \) which is bianalytic in the first variable and real-analytic in the second.

We see from (2.10) that

(2.25) \[ R_{2,m}(z, w) = 2m \partial_w \theta(z, w) - m(z - w) \partial_w^2 \theta(z, w) - m^2 |z - w|^2 [\partial_w \theta(z, w)]^2. \]
By Taylor expanding this expression with (2.4), we get

\[(2.26) \quad u(z) = \int u(w) \chi(\tilde{w}) \left[ 2m(b + \frac{1}{2}(w - z)\partial_z b) - m(\bar{z} - \bar{w})(\partial_w b + \frac{1}{2}(w - z)\partial_w \partial_z b) \right.
\]
\[\left. - m^2|z - w|^2b^2 + m^2(z - w)^2(\bar{z} - \bar{w})b\partial_z b \right] + m(z - w)^2X_1(z, w) + m^2(z - w)^3X_2(z, w) \int e^{m(z-w)\theta} dA(w) \]
\[+ O(\|u\|_m e^{\frac{1}{2}mQ(z)} e^{-\delta m}). \]

We require the coefficients in (2.21) to be bianalytic in \(z\) and \(\bar{w}\), which means in particular that they may not contain the factor \((z - w)^2\) to any degree higher than 1. Only the last three terms in (2.26) contain \((z - w)^2\), so they are the only ones which require further analysis. For the term with \(X_1\), proposition 2.4 shows that

\[(2.27) \quad \int u(w) \chi(w) m(z - w)^2X_1(z, w) e^{m(z-w)\theta} dA(w) \]
\[= \int u(w) \chi(w) \left( \frac{1}{m}X_3(z, w) + (z - w)X_4(z, w) \right) e^{m(z-w)\theta} dA(w) \]
\[+ O(\|u\|_m e^{\frac{1}{2}mQ(z) - \delta m}) = O(\|u\|_m e^{\frac{1}{2}mQ(z)m^{-1/2}}). \]

As we are only interested in the coefficients for \(m^2\) and \(m\), we conclude that this term is negligible for our purposes. Note that the extra factor \(m^{-1/2}\) in the error term comes from the computation similar to (2.20). For the term with \(X_2\), two applications of proposition 2.4 are needed to show this:

\[(2.28) \quad \int u(w) \chi(w) m^2(z - w)^3X_2(z, w) e^{m(z-w)\theta} dA(w) \]
\[= \int u(w) \chi(w) \left( (z - w)X_5(z, w) + m(z - w)^2X_6(z, w) \right) e^{m(z-w)\theta} dA(w) \]
\[+ O(\|u\|_m e^{\frac{1}{2}mQ(z) - \delta m}) = O(\|u\|_m e^{\frac{1}{2}mQ(z)m^{-1/2}}). \]

We now set

\[A_1(z, w) = (\bar{z} - \bar{w})b\partial_z b, \]

and analyze the corresponding term in (2.26) with the proposition 2.4

\[(2.29) \quad \int u(w)m^2(z - w)^2A_1(z, w) e^{m(z-w)\theta} dA(w) \]
\[= \int u(w) \left[ - \partial^2_w A_1(z, w) + m(z - w)A_2(z, w) \right] e^{m(z-w)\theta} dA(w) \]
\[+ O(\|u\|_m e^{\frac{1}{2}mQ(z) - \delta m}), \]

where

\[(2.30) \quad A_2(z, w) = \left( - 2 \frac{\partial_w A(z, w)}{\partial \theta} + 3 \frac{A(z, w) \partial^2_w \theta}{(\partial_w \theta)^2} \right). \]

We expand \(A_2\) in powers in \(z - w\):

\[(2.31) \quad A_2(z, w) = -2 \frac{\partial_w A_1(z, w)}{\partial \theta} + 3 \frac{A_1(z, w) \partial^2_w \theta}{(\partial_w \theta)^2} = -2 \frac{\partial_w A_1(z, w) \partial^2_w \theta}{b} + 3 \frac{A_1(z, w) \partial_w b}{b^2} + (z - w)X_9
\]
\[= 2\partial_z b + (\bar{z} - \bar{w}) \left( -2 \partial_w \partial_z b + \frac{\partial_z b \cdot \partial_w b}{b} \right) + (z - w)X_9. \]
We implement this into (2.29) and see by the same argument as in (2.27) that the term with $X_0$ is negligible. We now put everything together within (2.26):

\begin{equation}
(2.32) \quad u(z) = \int_{\mathbb{C}} u(w) \chi(w) \left[ m^2 \bar{\sigma}_{2,0} + m \bar{\sigma}_{2,1} \right] e^{m(z-w)B} dA(w) + O\left( \|u\|_m - \frac{1}{2} e^{\frac{1}{4} m Q(z)} \right),
\end{equation}

where

\begin{equation}
(2.33) \quad \bar{\sigma}_{2,0} = -|z - w|^2 b^2
\end{equation}

and

\begin{equation}
(2.34) \quad \bar{\sigma}_{2,1} = 2b + (z - w) \partial_z b - (\bar{z} - \bar{w}) \bar{\partial}_w b + |z - w|^2 \left( - \frac{3}{2} \partial_w \partial_z b + \frac{\partial_z b \cdot \bar{\partial}_w b}{b} \right).
\end{equation}

We conclude that

\begin{equation}
(2.35) \quad K_{2,m}^{(2)}(z, w) := \left\{ - m^2 |z - w|^2 b^2 + m \left[ 2b + (z - w) \partial_z b - (\bar{z} - \bar{w}) \bar{\partial}_w b + |z - w|^2 \left( - \frac{3}{2} \partial_w \partial_z b + \frac{\partial_z b \cdot \bar{\partial}_w b}{b} \right) \right] \right\} e^{m Q(z, w)}
\end{equation}

is a local bianalytic Bergman kernel \quad mod \quad (m^{-1/2}).

We have also computed the third term:

\begin{equation}
(2.36) \quad \bar{\sigma}_{2,2} = 2 \partial_w \partial_z \log b + (\bar{w} - \bar{z}) \partial_z \partial_w b + (z - w) \bar{\partial}_w \partial_z \log b + |z - w|^2 M(z, w),
\end{equation}

where

\begin{equation}
(2.37) \quad M = \frac{3}{2} \frac{\partial_w \partial_z b \cdot \partial_z b}{b^2} - \frac{13}{2} \frac{\partial_w b \cdot \bar{\partial}_w \partial_z b \cdot \bar{\partial}_w b}{b^3} + \frac{3}{2} \frac{(\bar{\partial}_w b \partial_z b)^2}{b^2} - \frac{(\partial_z b)^2 (\partial_w b)^2}{b^3} + \frac{17}{4} \frac{(\partial_z b)^2 \cdot (\bar{\partial}_w b)^2}{b^4} - \frac{2}{3} \frac{\partial_w^2 \partial_z^2 b}{b} + \frac{3}{2} \frac{\partial_w^2 \bar{\partial}_w b \cdot \bar{\partial}_w b}{b^2} - \frac{(\partial_z b^2) (\bar{\partial}_w b)^2}{b^3} + \frac{1}{3} \frac{\partial_z b \cdot \partial_z b}{b^2}.
\end{equation}

The rather long computations are presented in the appendix.

### 3. Some estimates for the one-point function

In this section, we provide an a priori bound for the one point function

\begin{equation}
\Gamma_{2,mQ,n}^{(1)}(z) := K_{2,mQ,n}(z, z) e^{-m Q(z)}
\end{equation}

for $z \in S$ and show that for $z \in \mathbb{C} \setminus \mathcal{S}$ we have convergence to zero with a rate that is exponential in $m$.

In [3], analogous results were shown for kernels $K_{1,m,n}$ relying heavily on the fact that $\log |f|$ is subharmonic whenever $f$ is an analytic function. This is no longer true when $f$ is more general polynomial analytic function, so we must use a different strategy.

Let us recall the following result, which is just proposition 8.1 of [22] in slightly altered form. Later, in lemma 4.2, we will show how a more general version of this result follows from proposition 4.1 of that paper.

**Lemma 3.1.** For any bianalytic function $u$ and $z \in \mathbb{C}$, we have

\begin{equation}
(3.1) \quad |u(z)|^2 e^{-m Q(z)} \leq m (8 + 48 A^2) e^A \int_{B(z,m^{-1/2})} |u(w)|^2 e^{-m Q(w)} dA(w)
\end{equation}

and

\begin{equation}
(3.2) \quad |\partial u(z)|^2 e^{-m Q(z)} \leq 3 m^2 e^A \int_{D(z,m^{-1/2})} |u(w)|^2 e^{-m Q(w)} dA(w),
\end{equation}

where

\begin{equation}
A := \sup_{w \in D(z,m^{-1/2})} |\Delta Q(w)|
\end{equation}
As an implication, we get two useful estimates for the kernel $K_{2,mQ,n}$. Namely, taking $u(w) := K_{2,mQ,n}(w,z)$ for some fixed $z$ and using
\[
\int_{\mathbb{C}} |K_{2,mQ,n}(w,z)|^2 e^{-mQ(w)} dA(w) = K_{2,mQ,n}(z,z),
\]
we get the following estimate for the one-point function on $\mathcal{S}$:
\[
(3.3) \quad K_{2,mQ,n}(z,z) e^{-mQ(z)} \leq m(8 + 48A_S^2)e^{As}, \quad z \in \mathcal{S}, \quad m \geq 1,
\]
where
\[
A_S := \sup_{\text{dist}(w,\mathcal{S}) \leq 1} |\Delta Q(w)|.
\]
Combining (3.3) with Cauchy-Schwarz inequality gives
\[
(3.4) \quad |K_{2,mQ,n}(z,w)|^2 e^{-mQ(z) - mQ(w)} \leq K_{2,mQ,n}(z,z) e^{-mQ(z)} K_{2,mQ,n}(w,w) e^{-mQ(w)} \leq m^2(8 + 48A_S^2)^2 e^{2As}, \quad z, w \in \mathcal{S}.
\]

We now apply the lemma 3.1 to provide an analogue of lemma 3.5 in [3]. We will need a simple weighted maximum principle for analytic polynomials provided by lemma 3.4 of the same paper (see also theorem III.2.1 in [33]). The proof is based on using the definition of $\hat{Q}$ and an application of maximum principle.

**Lemma 3.2.** Suppose $m \geq 1$ and $n \leq m + 1$. Let $u$ be an analytic polynomial of degree $\leq n - 1$ satisfying
\[
|u(z)|^2 e^{-mQ(z)} \leq 1, \quad z \in \mathcal{S}.
\]
Then
\[
|u(z)|^2 e^{-m\hat{Q}(z)} \leq 1, \quad z \in \mathbb{C}.
\]

**Proposition 3.3.** Suppose $m \geq 1$ and $n \leq m$. Then, there exists a constant depending only on $Q$ such that for all $u \in A_{2,mQ,n}^2$ and $z \in \mathbb{C}$, it holds
\[
|u(z)|^2 \leq Cm^2\|u\|_{mQ}^2 e^{mQ(z)}, \quad m \geq 1.
\]

**Proof.** We take $z_0 \in \mathbb{C}$ which minimizes the quantity
\[
\max_{z \in \mathcal{S}} |z - z_0|.
\]
Let $c$ be the value of this expression corresponding to the optimal choice of $z_0$. We write $u(z) = p(z) + (z - z_0)q(z)$ for analytic $p$ and $q$. According to (3.2),
\[
|(z - z_0)q(z)|^2 e^{-mQ(z)} \leq 3m^2e^{As} e^2\|u\|^2_m, \quad z \in \mathcal{S},
\]
and
\[
(3.5) \quad |p(z)|^2 e^{-mQ(z)} \leq 2[|u(z)|^2 + |z - z_0|^2|q(z)|^2] e^{-mQ(z)} \leq 2[(8 + 48A_S^2)e^{As}m + 3e^{As}m^2]\|u\|^2_m, \quad z \in \mathcal{S},
\]
where
\[
A_S := \sup_{\text{dist}(z,\mathcal{S}) \leq 1} |\Delta Q(z)|.
\]
We can now apply lemma 3.2 to get
\[
(3.6) \quad |(z - z_0)q(z)|^2 e^{-m\hat{Q}(z)} \leq me^{As} e^2\|u\|^2_m, \quad z \in \mathbb{C}
\]
and
\[
(3.7) \quad |p(z)|^2 e^{-m\hat{Q}(z)} \leq 2[(8 + 48A_S^2)e^{As}m + 3e^{As}m^2]\|u\|^2_m, \quad z \in \mathbb{C}.
\]
The statement of the proposition follows from putting (3.6) and (3.7) together:
\[
(3.8) \quad |u(z)|^2 e^{-m\hat{Q}(z)} \leq 2|p(z)|^2 e^{-m\hat{Q}(z)} + 2|(z - z_0)|^2|q(z)|^2 e^{-m\hat{Q}(z)} \leq Cm^2\|u\|^2_m,
\]
where the constant $C$ only depends on $A_S$ and $c$. □
From this result, we deduce

\[(3.9) \quad K_{2,mQ,n}(z,z)^2 e^{-mQ(z)} \leq C m^2 \|K_{2,mQ,n}(\cdot,z)\|_m^2 e^{-m(Q(z)-\hat{Q}(z))}, \quad n \leq m\]

and because

\[\|K_{2,mQ,n}(\cdot,z)\|_m = K_{2,mQ,n}(z,z),\]

we get

\[(3.10) \quad K_{2,mQ,n}(z,z) e^{-mQ(z)} \leq C m^2 e^{-m(Q(z)-\hat{Q}(z))}, \quad n \leq m.\]

for all \(z \in \mathbb{C}\). Notice that this estimate is only interesting for \(z \in \mathbb{C} \setminus \mathcal{S}\), because for \(z \in \mathcal{S}\), we already have a better estimate in (3.3). We conclude that the one-point function \(K_{2,mQ,n}(z,z) e^{-mQ(z)}\) decays exponentially to zero for a fixed \(z \in \mathbb{C} \setminus \mathcal{S}\) as \(m \to \infty\). Moreover, the growth conditions (1.1) and (1.5) imply that for any neighborhood \(\mathcal{D}\) of \(\mathcal{S}\), we have

\[\int_{\mathcal{D}} K_{q,mQ,n}(z,z) e^{-mQ(z)} \, dA(z) \to 0, \quad m \to \infty, n \leq m.\]

Finally, we want to record the following off-diagonal analogue of (3.10):

\[(3.11) \quad |K_{2,mQ,n}(w,z)|^2 e^{-mQ(w)-mQ(z)} \leq C m^2 K_{2,mQ,n}(z,z) e^{-mQ(z)} e^{-m(Q(w)-\hat{Q}(w))}\]

\[\leq C^2 m^4 e^{-m(Q(z)-\hat{Q}(z))} e^{-m(Q(w)-\hat{Q}(w))}, \quad n \leq m.\]

In particular, for fixed \(z\) and \(w\), we have again exponential decay to 0 whenever one of the two points is in \(\mathbb{C} \setminus \mathcal{S}\). This should be compared with the off-diagonal damping theorem (1.3) which deals with the case when both \(z\) and \(w\) belong to \(\mathcal{S}\).

4. HöRMANDER-TYPE ESTIMATES FOR \(\bar{\partial}^2\)

The purpose of this section is to prove theorem (4.5) which is an estimate for a solution of the equation \(\bar{\partial}^2 u = f\), where a certain growth condition is imposed on the solution near infinity. This result will be used in section (8) to prove an off-diagonal decay estimate for bianalytic Bergman kernels.

We will assume that \(\hat{Q} \geq 1\). We can always arrange this by adding a sufficiently big constant to \(Q\). This just means that the corresponding reproducing kernel will be multiplied by a constant and so the problem remains essentially unchanged.

We start by fixing \(z_0 \in \text{int} \mathcal{S} \cap \mathcal{N}_+\) and setting

\[r_0 = \frac{1}{4} \text{dist}(z_0, \mathbb{C} \setminus (\mathcal{S} \cap \mathcal{N}_+)), \quad \alpha = \inf_{z \in \mathbb{D}(z_0, 2r_0)} \Delta Q(z), \]

\[A = \sup_{z \in \mathcal{S} \cap \mathcal{N}_+} |\Delta Q(z)|, \quad \beta = \sup_{z \in \mathcal{S}} Q(z),\]

\[l = \inf_{z \in \mathcal{S}} \frac{1}{1 + |z|^2} = \inf_{z \in \mathcal{S}} \Delta \log(1 + |z|^2).\]

We have \(\alpha > 0\); this is because \(Q\) is strictly subharmonic in the interior of \(\mathcal{S} \cap \mathcal{N}_+\). We choose two positive numbers \(M_0\) and \(M_1\) so that

\[(4.1) \quad M_1 \log(1 + |z|^2) \leq M_0 \hat{Q}(z), \quad z \in \mathbb{C};\]

this is possible because of the assumption \(\hat{Q} \geq 1\) and the growth condition

\[\hat{Q}(z) = \log |z|^2 + O(1), \quad z \to \infty.\]

Notice that this implies immediately \(M_1 \leq M_0\). We will also need to assume that \(M_1 > 2\).

We define

\[\phi_m = mQ, \quad \hat{\phi}_m(z) = (m - M_0)\hat{Q} + M_1 \log(1 + |z|^2).\]

The definitions are exactly as in [3]. Intuitively, we would like to think \(\hat{\phi}_m\) to be equal to \(m\hat{Q}\); it however turns out that two correction terms involving the constants \(M_0\) and \(M_1\) are needed.
The purpose of the logarithmic correction term $M_1 \log(1 + |z|^2)$ is that now the weight $\hat{\phi}_m$ becomes strictly subharmonic in the whole plane:

$$\Delta \hat{\phi}_m(z) = (m - M_0)\Delta \hat{Q}(z) + M_1(1 + |z|^2)^{-2} \geq M_1(1 + |z|^2)^{-2},$$

for a.e. $z \in \mathbb{C}$ and $m \geq M_0$. The effect of substracting the term $M_0\hat{Q}$ is

$$\hat{\phi}_m \leq \phi_m, \quad z \in \mathbb{C}.$$  

We also see that

$$\phi_m \leq \hat{\phi}_m + M_0\beta.$$  

We fix $r_1 < r_0$ and suppose that $2m^{-1/2} \leq r_1$. We will work with a function $\rho_m$ which is $C^{1,1}$-smooth on $\mathbb{C}$, zero on $D(z_0, m^{-1/2})$, constant on $\mathbb{C} \setminus D(z_0, r_1 - m^{-1/2})$ and satisfies the properties

$$-\frac{m\alpha}{2} + M_0\alpha \leq \Delta \rho_m \leq mA$$  

$$|\partial \rho_m|^2 \leq \frac{\alpha}{27 \cdot 16 \cdot e^{2A+1+M_0\beta}} \left(\frac{1}{2}m\alpha + M_1 l\right)$$  

$$|\partial^2 \rho_m + (\partial \rho_m)^2|^2 \leq \left(\frac{1}{4}m\alpha + M_1 l\right)\left(\frac{1}{4}m\alpha + M_1 l\right) \cdot 9e^{M_0\beta}$$

for $m \geq 1$. The role of these conditions will become clear in the proof of theorem 4.5. A consequence of the estimate (4.5) is that

$$\Delta(\hat{\phi}_m + \rho_m) \geq \frac{m\alpha}{2} + M_1 l$$

for a.e. $z \in D(z_0, 2r_0)$. We also have

$$\Delta(\hat{\phi}_m + \rho_m) \geq M_1 \frac{1}{(1 + |z|^2)^2}$$

for a.e. $z \in \mathbb{C}$.

Our argument is based on iterating the elementary one-dimensional version of Hörmander’s $\bar{\partial}$-estimate. This result states that for $f \in L^2_{\text{loc}}(\mathbb{C})$ and $\psi \in C^{1,1}(\mathbb{C})$ satisfying $\Delta \psi > 0$ a.e., there exists a solution to the equation

$$\bar{\partial}u = f$$

satisfying

$$\int_{\mathbb{C}} |u|^2 e^{-\psi} \leq \int_{\mathbb{C}} |f|^2 e^{-\psi} \Delta \psi$$

provided that the right hand side is finite. It is also known that there exists a unique norm-minimal solution $u_0$ to (4.10). Namely, given any solution $u_1 \in L^2(e^{-\psi})$ to the equation, $u_0$ can be written as

$$u_0 = u_1 - P_{\psi}[u_1],$$

where $P_{\psi}$ denotes the projection to the subspace of analytic functions within $L^2(e^{-\psi})$.

**Theorem 4.1.** Let $f \in L^\infty(\mathbb{C})$ be supported on $D(z_0, r_0)$ and let $\rho_m$ satisfy the conditions above. Then, there exists a solution $u_2$ to the equation

$$\bar{\partial}^2 u = f$$

satisfying

$$\int_{\mathbb{C}} |u_2|^2 e^{-\langle \hat{\phi}_m + \rho_m \rangle} dA \leq \frac{1}{(\frac{1}{4}m\alpha + M_1 l)(\frac{1}{4}m\alpha + M_1 l)} \int_{D(z_0, r_0)} |f|^2 e^{-\langle \hat{\phi}_m + \rho_m \rangle} dA$$

for all $m \geq m_0$, where $m_0$ is a positive constant depending only on the parameters $M_1, l, \alpha$ and $r_0$. 


Proof. Let us consider the equation
$\bar{\partial} u = u_1,$
where $u_1 \in L^2_{\text{loc}}(\mathbb{C})$ is some function to be specified in a moment.

According to Hörmander’s result and (4.9), there exists a solution $u_2$ to the equation with the norm control
\begin{equation}
\int_{\mathbb{C}} |u_2|^2 e^{-(\phi_m + \rho_m)} \, dA \leq \int_{\mathbb{C}} |u_1|^2 \frac{e^{-(\phi_m + \rho_m)}}{\Delta(\phi_m + \rho_m)} \, dA \leq \int_{\mathbb{C}} |u_1|^2 e^{-(\phi_m + \rho_m) + \log \Delta(\phi_m + \rho_m)} \, dA.
\end{equation}

We would like to proceed by using Hörmander’s estimate again with the weight $(\phi_m + \rho_m) + \log \Delta(\phi_m + \rho_m)$. But this is not possible, because the function $\log \Delta(\phi_m + \rho_m)$ is not $C^{1,1}$-smooth. We proceed by replacing it with another function which is smooth enough. Building on (4.8) and (4.9), we estimate
$$
\log \Delta(\hat{\phi}_m + \rho_m) \geq \theta_m(|z - z_0|),
$$
where $\theta_m$ is any $C^{1,1}$-smooth function satisfying
$$
\theta_m(x) \leq \log \left( \frac{1}{2} \alpha m + M_1 l \right), \quad 0 < x \leq 2r_0,
$$
and
$$
\theta_m(x) \leq \log M_1 - 2 \log (1 + |z|^2)
$$
for $x \geq 2r_0$. It will be convenient to make the change of variables $x = e^t$, so that we can rewrite
$$
\Delta \theta_m(|z - z_0|) = \frac{1}{4|z - z_0|^2} \frac{d^2}{dt^2} \theta_m(e^t)|_{t = \log |z - z_0|}.
$$
We define another function by $\sigma_m(t) := \theta_m(e^t)$, and set
$$
\sigma_m(t) = \log \left( \frac{1}{2} m \alpha + M_1 l \right), \quad -\infty < t \leq \log r_0,
$$
and
$$
\sigma_m(t) = \log M_1 - 2 \log (1 + e^{2t}), \quad t \geq \log 2 + \log r_0.
$$
It remains to define $\sigma_m$ on the interval $[\log r_0, \log r_0 + \log 2]$. We set
$$
\sigma_m(t) := \log \left( \frac{1}{2} m \alpha + M_1 l \right) - \frac{m \alpha_0^2}{2} (t - \log r_0)^2, \quad \log r_0 \leq t \leq t_m,
$$
where
$$
t_m := \log r_0 + \frac{\sqrt{2}}{\sqrt{m \alpha_0}} \sqrt{\log \left( \frac{m \alpha}{2M_1} + l \right)}.
$$
The motivation for the definition of $t_m$ is that it solves the equation $\sigma(t_m) = \log M_1$. We are here assuming that $m \geq m_0$, where $m_0$ is so large that
$$
\log \left( \frac{m \alpha}{2M_1} + l \right) > 0
$$
and $t_m \leq \log r_0 + \frac{1}{2} \log 2$ for all such $m$. Notice that this definition of $\sigma_m$ implies that on the annulus $\mathbb{D}(z_0, t_m) \setminus \mathbb{D}(z_0, r_0)$ we have
$$
\Delta \theta_m(|z - z_0|) = -\frac{m \alpha_0^2}{4|z - z_0|^2} \geq -\frac{m \alpha}{4}.
$$

For the remaining interval $[t_m, \log 2 + \log r_0]$, the choice of $\sigma_m$ is rather insignificant. We can namely choose any decreasing function so that the resulting function defined on the whole real line becomes $C^{1,1}$ and whose second derivative is not smaller than $-\alpha \sigma_0^2 m$.

With this construction of $\sigma_m$, we now see from (4.8) and (4.9) that
$$
\Delta (\hat{\phi}_m(z) + \rho_m(z) + \theta_m(|z - z_0|)) \geq \frac{1}{4} m \alpha + M_1 l,
$$
for \( z \in \mathbb{D}(z_0, 2r_0) \) and
\[
\Delta (\tilde{\phi}_m(z) + \rho_m(z) + \theta_m(\|z - z_0\|)) \geq M_1 (1 + |z|^2)^{-2} - 2(1 + |z|^2)^{-2}
\]
for \( z \in \mathbb{C}\setminus\mathbb{D}(z_0, 2r_0) \). In any case, \( \tilde{\phi}_m + \rho_m + \theta_m(\|z - z_0\|) \) is strictly subharmonic a.e. in the whole complex plane (we use here the assumption \( M_1 > 2 \) from the beginning of this section).

We can now apply Hörmander’s estimate the second time. We take \( u_1 \) to be the solution of
\[
\tilde{\partial} u = f
\]
which is norm-minimal in \( L^2(e^{-\tilde{\phi}_m + \rho_m + \theta_m(\|z - z_0\|)}) \). We can now continue from (4.12):
\[
(4.13) \quad \int_{\mathbb{C}} |u_1(z)|^2 e^{-\tilde{\phi}_m + \rho_m + \log \Delta (\tilde{\phi}_m + \rho_m))} dA(z) \leq \int_{\mathbb{C}} |u_1(z)|^2 e^{-\tilde{\phi}_m + \rho_m + \theta_m(\|z - z_0\|)} dA(z)
\]
\[
\leq \int_{\mathbb{D}(z_0, r_0)} |f(z)|^2 \frac{e^{-\tilde{\phi}_m(z) + \rho_m(z) + \theta_m(\|z - z_0\|)}}{\Delta (\tilde{\phi}_m(z) + \rho_m(z) + \theta_m(\|z - z_0\|))} dA(z)
\]
\[
\leq \left( \frac{1}{2} m \alpha + M_1 l \right) \left( \frac{1}{2} m \alpha + M_1 l \right) \int_{\mathbb{D}(z_0, r_0)} |f(z)|^2 e^{-\tilde{\phi}_m(z) + \rho_m(z)} dA(z),
\]
and the proof is complete. \( \square \)

We will now prove a generalization of lemma 3.1.

**Lemma 4.2.** Take \( m > 0, z \in \mathbb{C} \) and let \( \psi_m \) be a \( C^{1,1} \)-smooth real-valued function on \( \mathbb{C} \). Then, for any bianalytic function \( u \), we have
\[
(4.14) \quad |u(z)|^2 e^{-\psi_m(z)} \leq m(8 + 48 A^2) e^{A_m} \int_{\mathbb{D}(z, m^{-1/2})} |u(w)|^2 e^{-\psi_m(w)} dA(w)
\]
and
\[
(4.15) \quad |\tilde{\partial} u(z)|^2 e^{-\psi_m(z)} \leq 3 m^2 e^{A_m} \int_{\mathbb{D}(z, m^{-1/2})} |u(w)|^2 e^{-\psi_m(w)} dA(w),
\]
where
\[
A_m := \frac{1}{m} \text{ess sup}_{w \in \mathbb{D}(z, m^{-1/2})} |\Delta \psi_m(w)|
\]

**Proof.** We assume \( z = 0 \) without loss of generality. In [22], it was proved that for a bianalytic function \( v \) on \( \mathbb{D} \) and a subharmonic function \( \Psi \in C^{1,1}(\mathbb{D}) \) satisfying
\[
\int_{\mathbb{D}} (1 - |w|^2) \Delta \Psi(w) dA(w),
\]
it holds
\[
(4.16) \quad |v(0)|^2 e^{\Psi(0)} \leq \left[ 8 + 12 |G[\Delta \Psi](0)|^2 \right] \int_{\mathbb{D}} |v(w)|^2 e^{\Psi(w)} dA(w),
\]
where \( G[\mu] \) denotes the Green’s potential of the measure \( \mu \):
\[
G[\mu](z) := \int_{\mathbb{D}} \log \frac{|z - w|^2}{1 - z \overline{w}} d\mu(w).
\]
We will set \( u_m(\xi) = u(m^{-1/2} \xi) \) and \( \Psi_m(\xi) = A_m |\xi|^2 - \psi_m(m^{-1/2} \xi) \) for \( \xi \in \mathbb{D} \). We have \( \Delta \Psi_m \geq 0 \) and
\[
|G[\Delta \Psi_m](0)| \leq 2 A_m.
\]
An application of (4.16) gives
\[
(4.17) \quad |u(0)|^2 e^{-\psi_m(0)} = |u_m(0)|^2 e^{\Psi_m(0)} \leq \left[ 8 + 12 |G[\Delta \Psi_m](0)|^2 \right] \int_{\mathbb{D}} |u_m|^2 e^{\Psi_m} dA
\]
\[
\leq (8 + 48 A^2) e^{A_m} \int_{\mathbb{D}} |u_m(\xi)|^2 e^{-\psi_m(m^{-1/2} \xi)} dA(\xi)
\]
\[
= m(8 + 48 A^2) e^{A_m} \int_{\mathbb{D}(0, m^{-1/2})} |u(w)|^2 e^{-\psi_m(w)} dA(w),
\]
which proves the first inequality. The second inequality follows the same way from
\begin{equation}
(4.18)
|\partial v(0)|^2 e^{\phi(0)} \leq 3 \int_{\Delta} |v(w)|^2 e^{\phi(w)} dA(w),
\end{equation}
which also was proved in [22].

In the following lemma and later, we will use the notation \(|x|\) for the largest integer smaller than or equal to a real number \(x\).

**Lemma 4.3.** Let \(f\) and \(g\) be analytic functions, and suppose that
\[ u(z) := f(z) + zg(z) \in L^2(e^{-\phi_m}), \quad m \geq M_0. \]
Then, \(f\) and \(g\) are polynomials of degree \(\leq |m - M_0 + M_1 + 1|\).

**Proof.** From lemma 4.2 we get
\[ |g(z)|^2 e^{-\phi_m(z)} \leq 3m^2 e^{A_{z,m}} \int_{\Delta_2(z,m^{-1/2})} |u(w)| e^{-\phi_m(w)} dA(w), \]
where
\[ A_{z,m} := \frac{1}{m} \text{ess sup}_{w \in \Delta_2(z,m^{-1/2})} |\Delta \phi_m(w)|. \]
Using (4.2), we see that
\begin{equation}
(4.19)
A_{z,m} \leq K + 1, \quad z \in \mathbb{C}, \quad m \geq M_0,
\end{equation}
where
\[ K := \text{ess sup}_{z \in \mathcal{S}} \Delta \hat{Q}(z) = \text{ess sup}_{z \in \mathbb{C}} \Delta \hat{Q}(z). \]
The last equality followed from harmonicity of \(\hat{Q}\) in \(\mathbb{C}\setminus\mathcal{S}\). By (4.19), we have
\[ |g(z)|^2 \leq 3m^2 e^{K+1} ||u||^2_{L^2(e^{-\phi_m})} e^{\phi_m(z)}. \]
for all \(z \in \mathbb{C}\). This, together with the growth condition
\begin{equation}
(4.20)
\hat{\phi}_m(z) = (m - M_0 + M_1) \log |z|^2 + O(1)
\end{equation}
shows that \(g\) is a polynomial of degree \(\leq |m - M_0 + M_1|\). For \(f\), we first estimate
\[ |f(z)|^2 \leq 2(|u(z)|^2 + |zg(z)|^2). \]
We use (1.14) with \(\psi_m = \hat{\phi}_m\) for the first term:
\[ |u(z)|^2 \leq m[8 + 48(K + 1)^2] e^{K+1} ||u||^2_{L^2(e^{-\phi_m})} e^{\phi_m(z)}. \]
We get
\begin{equation}
(4.21)
|f(z)|^2 \leq 2(|u(z)|^2 + |zg(z)|^2)
\end{equation}
\[ \leq 2m[8 + 48(K + 1)^2] e^{K+1} ||u||^2_{L^2(e^{-\phi_m})} e^{\phi_m(z)} + 2|zg(z)|^2. \]
The desired assertion for \(f\) now follows from the growth condition (4.20) and that \(zg(z)\) is a polynomial of degree \(\leq |m - M_0 + M_1 + 1|\)

We now turn to a definition of certain subspaces of \(L^2(e^{-\psi})\), where \(\psi\) is some real-valued \(C^{1,1}\) smooth weight function on \(\mathbb{C}\). We assume that \(\int_{\mathbb{C}} e^{-\psi} dA < \infty\) so that all constant functions belong to the space. Let us first fix some notation:
\[ A^2_{2,\psi} := \{ f \mid f \in L^2(e^{-\psi}), \bar{\partial}^2 f = 0 \} \]
and
\[ A^2_{2,\psi,n} := L^2(e^{-\psi}) \cap \text{Pol}_2, n. \]
The spaces in question are
\begin{equation}
(4.22)
L^2_{2,\psi,n} := \{ f \in L^2(e^{-\psi}) \mid f \in C^1(\mathbb{C}\setminus K) \text{ for some compact set } K, \]
\[ |f(z) - z\bar{\partial}f(z)| = O(|z|^{n-1}), \bar{\partial}f(z) = O(|z|^{n-1}) \text{ as } z \to \infty \}. \]
We see immediately that these spaces are not in general closed in $L^2(e^{-\psi})$. An important fact however is that

\[(4.23) \quad A^2_{2,\psi,n} = L^2_{2,\psi,n} \cap A^2_{2,\psi},\]

and $A^2_{2,\psi,n}$ is always closed in $L^2(e^{-\psi})$. This makes it possible to speak about $L^2_{2,\psi,n}$-minimal solutions to $\bar{\partial}^2$-equations.

We will make use of the fundamental solution $\frac{1}{z}$ of the operator $\bar{\partial}^2$ and denote the associated solution operator by $C_2$:

\[
C_2[f](z) = \int_{\mathbb{C}} f(w) \frac{z-w}{z-w} dA(w)
\]

for a compactly supported $L^\infty$-function $f$. We also write $P_{2,\psi,n}$ for the orthogonal projection from $L^2(e^{-\psi})$ to $A^2_{2,\psi,n}$.

Because $C_2[f] \in L^2_{2,\psi,n}$, we have by (4.23) that

\[
u_n := C_2[f] - P_{2,\psi,n}[C_2[f]]
\]

is the unique $L^2_{2,\psi,n}$-minimal solution to the equation

\[
\bar{\partial}^2 u = f.
\]

This solution is characterized among the solutions in the class $L^2_{2,\psi,n}$ by the condition

\[
\int_{\mathbb{C}} u_n(z) \overline{h(z)} e^{-\psi} dA(z) = 0
\]

which should hold for all $h \in A^2_{2,\psi,n}$.

The following lemma is an adaption of lemma 4.2 in [3] to our bianalytic context.

**Lemma 4.4.** For any compactly supported function $f \in L^\infty(\mathbb{C})$, the equation

\[
\bar{\partial}^2 u = f
\]

has a solution in $L^2(e^{-(\hat{\phi}_m + \rho_m)})$. The $L^2(e^{-(\hat{\phi}_m + \rho_m)})$-minimal solution is of class $L^2_{2,\phi_m,n}$ when $n \geq m - M_0 + M_1 + 1$.

**Proof.** The function $C_2[f]$ solves the equation and satisfies $C_2[f] = O(1)$ as $z \to \infty$. Therefore, it belongs to the class $L^2(e^{-(\hat{\phi}_m + \rho_m)})$. The $L^2(e^{-(\hat{\phi}_m + \rho_m)})$-minimal solution $v$ can be written as

\[
v = C_2[f] + g
\]

for some bianalytic function $g \in A^2_{2,\phi_m + \rho_m}$. Because we assume that $\rho_m$ is $C^{1,1}$-smooth and constant in $\mathbb{D}(z_0, r_1 - m^{-1/2})$, the spaces $A^2_{2,\phi_m + \rho_m}$ and $A^2_{2,\phi_m}$ are equal as sets, and therefore, by (4.3) and lemma 4.3 we get the inclusion

\[
A^2_{2,\phi_m + \rho_m} \subset A^2_{2,\phi_m,n}.
\]

Hence, $g \in A^2_{2,\phi_m,n}$, and also $g \in L^2_{2,\phi_m,n}$. Because $C_2[f]$ is bounded and $\bar{\partial}C_2[f] = O(|z|^{-1})$ as $z \to \infty$, we see that $C_2[f] \in L^2_{2,\phi_m,n}$. The assertion is now proved. \(\square\)

**Theorem 4.5.** Fix a number $r_1$ such that $r_1 \leq r_0$ and let $f \in L^\infty(\mathbb{C})$ be supported on $\mathbb{D}(z_0, r_1)$ and equal to 0 on $\mathbb{D}(z_0, r_1)$. Assume

\[
m \geq \max\{m_0, M_0\}, \quad 2m^{-1/2} < r_1, \quad n \geq |m - M_0 + M_1 + 1|,
\]

where $m_0$ is as in proposition 4.1. Let $\rho_m$ be a $C^{1,1}$-smooth function which is supported on $\mathbb{D}(z_0, r_1 - m^{-1/2})$, zero on $\mathbb{D}(z_0, m^{-1/2})$ and satisfies the conditions (4.3), (4.6) and (4.7). Then, the $L^2_{2,mQ,n}$-minimal solution $u_n$ to the equation $\bar{\partial}^2 u = f$ satisfies

\[(4.24) \quad \int_{\mathbb{C}} |u_n|^2 e^{\rho_m - mQ} \leq \frac{9e^{M_0 \beta}}{(\frac{1}{4} m \alpha + M_1 l)(\frac{1}{2} m \alpha + M_1 l)} \int_{\mathbb{D}(z_0, r_n)} |f|^2 e^{\rho_m - mQ}.
\]

In the special case $\rho_m = 0$, the conditions $2m^{-1/2} < r_1$ and $f = 0$ on $\mathbb{D}(z_0, r_1)$ are not needed.
Proof. The norm-minimality condition for $u_n$ can be rephrased as
\[ (4.25) \quad \int_C u_ne^{\rho_m} \hat{h}e^{-\phi_m} dA = 0 \quad \text{for all } h \in A^2_{2,\phi_m+\rho_m,n}. \]
This is because $A^2_{2,\phi_m,n}$ and $A^2_{2,\phi_m+\rho_m,n}$ are equal as sets. Because $\rho_m$ is supported on a compact set, we have $u_n e^{\rho_m} \in L^2_{2,\phi_m+\rho_m,n}$. Therefore, $u_n e^{\rho}$ is the $L^2_{2,\phi_m+\rho_m,n}$-minimal solution to the equation
\[ (4.26) \quad \tilde{\partial} u = \tilde{\partial}^2 (u_n e^{\rho_m}) = \left[ u_n (\tilde{\partial}^2 \rho_m + (\tilde{\partial} \rho_m)^2) + 2\tilde{\partial} u_n \tilde{\partial} \rho_m + f \right] e^{\rho_m}. \]
Let $u_2$ be the $L^2_{2,\phi_m+\rho_m,n}$-minimal solution of the problem (4.26), the existence of which is guaranteed by lemma 4.4. We deduce from the same lemma that $u_2 \in L^2_{2,\phi_m,n}$, and consequently $u_2 \in L^2_{2,\phi_m+\rho_m,n}$. Hence
\[ (4.27) \quad \int_C |u_n|^2 e^{\rho_m - \phi_m} dA = \int_C |u_n e^{\rho_m}|^2 e^{-(\phi_m + \rho_m)} dA \leq \int_C |u_2|^2 e^{-(\phi_m + \rho_m)} dA \]
\[ (4.28) \quad \leq \int_C |u_2|^2 e^{-(\phi_m + \rho_m)} dA. \]
For the last inequality here, we applied (4.3). The proposition 4.1 gives
\[ (4.29) \quad \int_C |u_2|^2 e^{-(\phi_m + \rho_m)} dA \leq \frac{1}{(\frac{1}{2} m \alpha + M_1)} \int_C |\tilde{\partial}^2 (u_n e^{\rho_m})|^2 e^{-(\phi + \rho_m)} dA \]
\[ \leq \frac{1}{(\frac{1}{2} m \alpha + M_1)} \left[ \int_{\mathbb{D}(z_0, r_1 - m^{-1/2})} |u_n|^2 |\tilde{\partial}^2 \rho_m + (\tilde{\partial} \rho_m)^2|^2 e^{\rho_m - \phi} dA + 12 \int_{\mathbb{D}(z_0, r_0)} |\tilde{\partial} u_n|^2 |\tilde{\partial} \rho_m|^2 e^{\rho_m - \phi} dA + 3 \int_{\mathbb{D}(z_0, r_0)} |f|^2 e^{\rho_m - \phi} dA \right]. \]
We proceed to estimate the three terms in this expression. For the first term, we use (4.7):
\[ (4.30) \quad \frac{3}{(\frac{1}{2} m \alpha + M_1)} \int_{\mathbb{D}(z_0, r_1 - m^{-1/2})} |u_n|^2 |\tilde{\partial}^2 \rho_m + (\tilde{\partial} \rho_m)^2|^2 e^{\rho_m - \phi} dA \]
\[ \leq \frac{1}{(\frac{1}{2} m \alpha + M_1)} \int_{\mathbb{D}(z_0, r_1 - m^{-1/2})} |u_n|^2 e^{\rho_m - \phi} dA \]
\[ \leq \frac{1}{3} \int_{\mathbb{D}(z_0, r_1 - m^{-1/2})} |u_n|^2 e^{\rho_m - \phi} dA. \]
Let us turn to the second term in (4.29). The condition $f = 0$ on $\mathbb{D}(z_0, r_1)$ means that $u_n$ is bianalytic in this disk. We will employ the estimate (4.15) for this function, with the weight $\psi_m = \phi_m - \rho_m$. We first check that
\[ (4.31) \quad \sup_{w \in \mathbb{D}(z_0, r_1)} |\Delta (\phi_m(w) - \rho_m(w))| \leq \frac{1}{m} [(m - M_0) A + M_1 + m A] \leq 2 A, \]
where we used $|\Delta \rho_m| \leq m A$, which is a direct consequence of (4.5). We get with (4.6)
\[ (4.32) \quad \frac{12}{(\frac{1}{2} m \alpha + M_1)} \int_{\mathbb{D}(z_0, r_1 - m^{-1/2})} |\tilde{\partial} u_n|^2 |\tilde{\partial} \rho_m|^2 e^{\rho_m - \phi} dA \]
\[ \leq \frac{12 \alpha m^2}{(\frac{1}{2} m \alpha + M_1)} e^{M_0 S} \int_{\mathbb{D}(z_0, r_1 - m^{-1/2})} |u_n(w)|^2 e^{\rho_m(w) - \phi_m(w)} dA(w) dA(z) \]
\[ \leq \frac{12 \alpha m^2}{(\frac{1}{2} m \alpha + M_1)} e^{M_0 S} \int_{\mathbb{D}(z_0, r_1)} |u_n(w)|^2 e^{\rho_m(w) - \phi_m(w)} dA(w) \]
\[ = \frac{1}{3} \int_{\mathbb{D}(z_0, r_1)} |u_n(w)|^2 e^{\rho_m(w) - \phi_m(w)} dA(w). \]
The third term, we only want to replace $\tilde{\phi}_m$ by $\phi_m$:

\begin{equation}
(4.32) \quad \frac{3}{(\frac{1}{2}ma + M1)} \int_{D(z_0,r_0)} |f|^2 e^{\rho_m - \tilde{\phi}_m} dA \leq \frac{3e^{M_0b}}{(\frac{1}{2}ma + M1)} \int_{D(z_0,r_0)} |f|^2 e^{\rho_m - \phi_m} dA.
\end{equation}

Now, combining (4.27), (4.29), (4.30), (4.31) and (4.32) gives,

\begin{equation}
(4.33) \quad \int_{\mathbb{C}} |u_n|^2 e^{\rho_m - \phi_m} dA \leq \frac{2}{3} \int_{\mathbb{C}} |u_n|^2 e^{\rho_m - \phi_m} dA + \frac{3e^{M_0b}}{(\frac{1}{2}ma + M1)} \int_{D(z_0,r_1)} |f|^2 e^{\rho_m - \phi_m} dA,
\end{equation}

from which the assertion of the proposition follows immediately.

Notice that the condition $f = 0$ on $D(z_0,r_1)$ was only used in (4.31). If $\rho_m = 0$, this assumption is clearly not needed. For the same reason, it is unnecessary to require that $2m^{-1/2} \leq r_1$.

We will make use of this theorem twice, in the proofs of both main theorems 1.2 and 1.3. For the first one, it is enough to use the special case $\rho_m = 0$.

5. LOCAL BLOW-UP FOR BIANALYTIC BERGMAN KERNELS IN THE BULK

We are now ready to present the proof the theorem 1.2. The main ingredients of the argument are the construction of local bianalytic Bergman kernels in section 2.2 and the estimate for $\partial^2$-operator in theorem 1.5. The structure of the proof is the same as in 22; the main difference is that there it was enough to apply a simpler $\partial^2$-estimate.

**Proof of theorem 1.2.** We pick a point $z_0 \in S \cap N_+ \cap S \cap N_+$. We require additionally that

\[ r \leq r_0 = \frac{1}{4} \text{dist}(z_0, \mathcal{C} \setminus S \cap N_+). \]

Recall that we constructed a local bianalytic Bergman kernel mod($m^{-\frac{1}{2}}$) in the form

\begin{equation}
(5.1) \quad K_{2,mQ}^{(2)}(z,w) := \mathfrak{g}_{2,mQ}^{(2)}(z,w)e^{mQ(z,w)}
\end{equation}

where $\mathfrak{g}_{2,0}$ and $\mathfrak{g}_{2,1}$ are as in (2.33), (2.34).

Let us define the associated integral operator as

\[ I_{2,mQ}^{(2)}[f](\zeta) = \int_{\Omega} f(\xi)K_{2,mQ}^{(2)}(\zeta,\xi)\chi(\xi)e^{-mQ(\xi)} dA(\xi), \quad \zeta \in D(z_0,r). \]

We write $K_{2,mQ,n;\zeta}(w) := K_{2,mQ,n}(w,z)$, with a similar notation for kernels $K_{2,mQ}^{(2)}$. It is easily verified that

\[ P_{2,mQ,n}K_{2,mQ,n;\zeta}(w) = I_{2,mQ}^{(2)}[K_{2,mQ,n;\zeta}](w), \quad z, w \in D(z_0,r), \]

where $P_{2,mQ,n}$ denotes the projection within $L^2(e^{-mQ})$ to the subspace $A_{2,mQ,n}$.

Our first step is to make the splitting

\begin{equation}
(5.2) \quad \left| K_{2,mQ,n}(z,w) - K_{2,mQ}^{(2)}(z,w) \right| \leq \left| K_{2,mQ,n}(w,z) - I_{q,m}^{(2)}[K_{2,mQ,n}](w) \right| + \left| P_{2,mQ,n}K_{2,mQ,n;\zeta}(w) - K_{2,mQ,n;\zeta}(w) \right|, \quad z, w \in D(z_0, \frac{1}{3}r).
\end{equation}
For the first term on the right hand side, we use that \( K_{2,mQ}^{(2)} \) is a local bianalytic Bergman kernel mod \( (m^{-\frac{1}{2}}) \). This fact can be rephrased as

\[
(5.3) \quad \left| K_{2,mQ}(w,z) - I_{2,mQ}^{(2)}[K_{2,mQ};z](w) \right| \leq C_1 m^{-\frac{1}{2}} e^{\frac{4}{3}mQ(z)} \| K_{2,mQ};z \|_m
\]

\[
= C_1 m^{-\frac{1}{2}} e^{\frac{4}{3}mQ(z)} K_{2,mQ}(z,z)^{1/2} \leq C_2 e^{\frac{1}{3}m(Q(z)+Q(z))}, \quad z,w \in \mathbb{D}(z_0, \frac{1}{3} r)
\]

where the last inequality is a consequence of \((5.3)\). Here, and later in the proof, \( C_j \) with \( j \geq 1 \) refers to a constant that can only depend on \( z_0,Q \) and \( r \).

We turn to the second term in \((5.2)\). By \((4.23)\), the function \( u_0(z) := K_{2,mQ,w}^{(2)}(z) - P_{2,mQ,n}[K_{2,mQ,n,w}^{(2)}](z) \) is the \( L^2_{2,mQ,n} \)-minimal solution to the equation

\[
(5.4) \quad \bar{\partial}^2 u = \bar{\partial}^2 [K_{2,mQ,w}^{(2)}(\xi) - K_{2,mQ,w}^{(2)}(\xi)] = 2\bar{\partial}^2 K_{2,mQ,w}^{(2)} \bar{\partial} \chi(z) + K_{2,mQ,w}^{(2)} \bar{\partial}^2 \chi(z).
\]

The support of this function is contained in \( \mathbb{D}(z_0,r) \), so we can apply theorem \((4.3)\) in the special case \( \rho_m = 0 \). First, we record the estimate

\[
|2\bar{\partial}K_{2,mQ,w}^{(2)} \bar{\partial} \chi(z) + K_{2,mQ,w}^{(2)} \bar{\partial}^2 \chi(z)|^2 \leq C_3 m^6 e^{2mRe(Q(z),w)} \quad , \quad z,w \in \mathbb{D}(z_0,r), m \geq 1,
\]

which follows directly from the definition of \( K_{2,mQ}^{(2)} \). Here \( C_3 \) is a constant depending only on \( z_0,Q \) and \( r \).

Assuming that \( m \geq \max\{m_0, M_0\} \) and \( n \geq |m - M_0 + M_1 + 1| \), we obtain

\[
(5.5) \quad \| u_0 \|_m^2 \leq \frac{72 e^{M_0}}{m^2 \alpha^2} \int_{\frac{1}{3}r \leq |z-z_0| \leq r} \left| 2\bar{\partial}K_{2,mQ,w}^{(2)} \bar{\partial} \chi(z) + K_{2,mQ,w}^{(2)} \bar{\partial}^2 \chi(z) \right|^2 e^{-mQ(z)} dA(z)
\]

\[
\leq C_4 m^4 e^{mQ(w)} e^{-\delta m}.
\]

We now make the assumption that \( m^{-1/2} \leq \frac{1}{3} r \), and apply \((3.1)\):

\[
(5.6) \quad |u_0(z)|^2 \leq C_5 m \| u_0 \|_m^2 e^{mQ(z)}, \quad z \in \mathbb{D}(z_0, \frac{1}{3} r).
\]

Notice that the use of \((3.1)\) was justified given our extra assumption for \( m \), because \( u_0 \) is bianalytic for all \( z \in \mathbb{D}(z_0, \frac{1}{3} r) \).

We have by \((5.5)\) and \((5.6)\) that

\[
(5.7) \quad |u_0(z)|^2 \leq C_6 m^5 e^{mQ(w)+mQ(z)} e^{-\delta m} \leq C_7 e^{mQ(w)+mQ(z)}, \quad z,w \in \mathbb{D}(z_0, \frac{1}{3} r).
\]

Putting \((5.3)\) and \((5.7)\) together, we get

\[
(5.8) \quad \left| K_{2,mQ,n}(z,w) - K_{2,mQ}^{(2)}(z,w) \right| \leq C_8 e^{\frac{2}{3}(Q(z)+Q(w))}.
\]

This implies that

\[
(5.9) \quad \left| K_{2,mQ,n}(z,w) e^{-\frac{1}{3}(Q(z)+Q(w))} - K_{2,mQ}^{(2)}(z,w) e^{-\frac{2}{3}(Q(z)+Q(w))} + O(1) \right|
\]

Now, we take

\[
z = z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, \quad w = z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}},
\]

where \( (\xi, \lambda) \) is restricted to be in some compact subset of \( \mathbb{C}^2 \) and \( m \) is assumed to be so big that \( z,w \in \mathbb{D}(z_0, \frac{1}{3} r) \).

Using the explicit form for \( \tilde{e}_{2,mQ}^{(2)}(z,w) \), we compute

\[
(5.10) \quad \tilde{e}_{2,mQ}^{(2)}(z,w) = \left[ -m b(z,w) \Delta Q(z_0) \left| \xi - \lambda \right|^2 + m^2 b(z,w) + O(1) \right]
\]

\[
= m\Delta Q(z_0)(-|\xi - \lambda|^2 + 2) + O(m^{1/2}),
\]
Remark 5.1. The formula \((5.8)\) provides justification for the claim that local bianalytic kernels approximate the kernel \(K_{2,mQ,n}\) in the bulk (in the presence of appropriate weight factors, of course). By using a local kernel with more terms, one would obtain a better approximation.

6. An off-diagonal estimate for bianalytic Bergman kernels

In this section we use the \(\overline{\partial}^2\)-estimate of \([4.5]\) to prove theorem \([1.3]\).

Proof of theorem \([1.3]\). We fix \(z_0 \in \text{int} \mathcal{S} \cap \mathcal{N}_+\) and use the same definitions for the parameters \(r_0, \alpha, A, \beta, l, M_0, M_1\) as in section \([4]\).

We also take arbitrary \(z_1 \in \mathcal{S}\), and require

\[9m^{-1/2} \leq \min\{r_0, |z_1 - z_0|\}.\]

We then set

\[r_1 := \frac{1}{2} \min\{r_0, |z_1 - z_0| - m^{-1/2}\}\]

and see that

\[4m^{-1/2} \leq r_1 \quad \text{and} \quad 2r_1 \leq r_0.\]

The two last conditions clearly imply the assumptions \(2m^{-1/2} < r_1\) and \(r_1 < r_0\) that were made in the section \([4]\). Furthermore, we will also require that \(m \geq m_0\) and

\[n \geq |m_0 - M_0 + M_1 + 1|,\]

where \(M_1 > 2\) and \(m_0\) is so big that the theorem \([4.5]\) holds. By making \(M_0\) bigger if needed, we see that the assumptions \(n \geq m - M\) and \(m \geq m_0\) (which were made in the statement of the theorem) actually imply \((6.2)\).

We choose a smooth cut-off function \(\chi\) that satisfies \(\chi = 0\) on \(\mathbb{D}(z_0, r_1)\), \(\chi = 1\) on \(\mathbb{C} \setminus \mathbb{D}(z_0, 2r_1)\) and \(0 < \chi(w) < 1\) in the annulus \(\mathbb{D}(z_0, 2r_1) \setminus \mathbb{D}(z_0, r_1)\). We can arrange it so that

\[\int_{\mathbb{C}} |\overline{\partial} \chi(w)|^2 dA(w) \leq K_1\]

and

\[\int_{\mathbb{C}} |\overline{\partial}^2 \chi(w)|^2 dA(w) \leq \frac{K_2}{r_1^2},\]

for some absolute constants \(K_1\) and \(K_2\).

We consider the equation

\[\overline{\partial}^2 u = \overline{\partial}^2 [K_{2,mQ,n}(\cdot, z_0)\chi(\cdot)].\]

The \(L^2_{2,mQ,n}\)-minimal solution \(u_{2,mQ,n}\) can be written as

\[u_{2,mQ,n} = K_{2,mQ,n}(w, z_0)\chi(w) - P_{2,mQ,n}[K_{2,mQ,n}(\cdot, z_0)\chi(\cdot)](w).\]

Because \(\chi(z_0) = 0\), we have

\[\int_{\mathbb{C}} |K_{2,mQ,n}(w, z_0)|^2 \chi(w)e^{-mQ(w)} dA(w) \geq e^{-mQ(z_0)}\]

where for the last equality, the Taylor expansion

\[b(z, w) = \Delta Q(z_0) + O(m^{-1/2}).\]

was applied. Furthermore, by \([2.1]\),

\[e^{m\text{Re}Q(z,w)-\frac{1}{2}m(Q(z)+Q(w))} = e^{-\frac{1}{2}(|\xi-\lambda|^2 + O(m^{-\frac{1}{2}}))}.\]

Finally, recall that \(L_1^1(x) = -x + 2\). The theorem now follows from \([5.9]\), \([5.10]\) and \([5.11]\).
Because $\bar{\partial}^2[K_{2,mQ,n}(w,0)\chi(w)] = 0$ in $\mathbb{D}(z_0, m^{-1/2})$, we can apply the inequality (3.1) to get

\begin{equation}
(6.6) \quad |u_{2,mQ,n}(z_0)|^2e^{-mQ(z_0)}
\leq (8 + 48A^2)e^A m \int_{\mathbb{D}(z_0, m^{-1/2})} |u_{2,mQ,n}(w)|^2e^{-mQ(w)}dA(w)
\leq (8 + 48A^2)e^A m \int_{\mathbb{C}} |u_{2,mQ,n}(w)|^2e^{\rho_m(w)-mQ(w)}dA(w).
\end{equation}

Here and later, $\rho_m$ is any function which satisfies the hypotheses required by theorem 4.5. In particular, for the last inequality we use that $\rho_m = 0$ in $\mathbb{D}(z_0, m^{-1/2})$.

We are now ready to apply theorem 4.5

\begin{equation}
(6.7) \quad (8 + 48A^2)e^A m \int_{\mathbb{C}} |u_{2,mQ,n}(w)|^2e^{\rho_m(w)-mQ(w)}dA(w)
\leq \frac{C_1}{m} \int_{\mathbb{C}} |\bar{\partial}^2[K_{2,mQ,n}(w,0)\chi(w)]|^2e^{\rho_m(w)-mQ(w)}dA(w)
\leq \frac{8C_1}{m} \int_{r_1 \leq |w-z_0| \leq 2r_1} |\bar{\partial}_w K_{2,mQ,n}(w,z_0)|^2 |\bar{\partial}_w \chi(w)|^2e^{\rho_m(w)-mQ(w)}dA(w)
+ \frac{2C_1}{m} \int_{r_1 \leq |w-z_0| \leq 2r_1} |K_{2,mQ,n}(w,z_0)|^2 |\bar{\partial}_w \chi(w)|^2e^{\rho_m(w)-mQ(w)}dA(w),
\end{equation}

where the constant $C_1 > 0$ only depends on the parameters $A, \alpha, M_0, M_1, l$ and $\beta$. We analyze the two terms separately. For the first one, we estimate

\begin{equation}
(6.8) \quad \int_{r_1 \leq |w-z_0| \leq 2r_1} |\bar{\partial}_w K_{2,mQ,n}(w,z_0)|^2 |\bar{\partial}_w \chi(w)|^2e^{\rho_m(w)-mQ(w)}
\leq \sup_{r_1 \leq |w-z_0| \leq 2r_1} \left\{ |\bar{\partial}_w K_{2,mQ,n}(w,z_0)|^2 e^{\rho_m(w)-mQ(w)} \right\} \int_{r_1 \leq |w-z_0| \leq 2r_1} |\bar{\partial}_w \chi(w)|^2 dA(w)
\end{equation}

From (3.2), we get

\begin{equation}
(6.9) \quad |\bar{\partial}_w K_{2,mQ,n}(w,z_0)|^2 e^{-mQ(w)}
\leq 3e^A m^2 \int_{|\bar{w}-w| \leq m^{-1/2}} |K_{2,mQ,n}(\bar{w},z_0)|^2 e^{-mQ(\bar{w})} dA(\bar{w})
\leq 3e^A m^2 K_{2,mQ,n}(z_0,0).
\end{equation}

for any $w \in \mathbb{D}(z_0, 2r_1)$. Furthermore, by (3.3),

\begin{equation}
3e^A m^2 K_{2,mQ,n}(z_0,0) \leq 3e^{2A}(48 + 12A^2)m^3 e^{mQ(z_0)}.
\end{equation}

After applying (6.3) to the second factor of (6.8), we can conclude

\begin{equation}
(6.10) \quad \int_{r_1 \leq |w-z_0| \leq 2r_1} |\bar{\partial}_w K_{2,mQ,n}(w,z_0)|^2 |\bar{\partial}_w \chi(w)|^2e^{\rho_m(w)-mQ(w)}
\leq C_2 m^3 \sup_{r_1 \leq |w-z_0| \leq 2r_1} e^{\rho_m(w)-mQ(z_0)}
\end{equation}

for a constant $C_2$ depending only on $\alpha, A, \beta$ and $M_0$.

We now move to the second term of the last expression in (6.8).

\begin{equation}
(6.11) \quad \int_{r_1 \leq |w-z_0| \leq 2r_1} |K_{2,mQ,n}(w,z_0)|^2 |\bar{\partial}^2 \chi(w)|^2e^{\rho_m(w)-mQ(w)}dA(w)
\leq \sup_{r_1 \leq |w-z_0| \leq 2r_1} \left\{ e^{\rho_m(w)} |K_{2,mQ,n}(w,z_0)|^2 e^{-mQ(w)} \right\} \int_{r_1 \leq |w-z_0| \leq 2r_1} |\bar{\partial}^2 \chi(w)|^2 dA(w)
\end{equation}

For the second factor, we use that

\begin{equation}
\int_{\mathbb{C}} |\bar{\partial}^2 \chi(w)|^2 dA(w) \leq \frac{K_2}{r_1^2} \leq \frac{K_2 \cdot m}{16},
\end{equation}

where $K_2$ is a constant depending only on $A, \alpha, M_0, M_1, l$ and $\beta$. Therefore, we get

\begin{equation}
(6.12) \quad \int_{r_1 \leq |w-z_0| \leq 2r_1} |K_{2,mQ,n}(w,z_0)|^2 |\bar{\partial}^2 \chi(w)|^2e^{\rho_m(w)-mQ(w)}dA(w)
\end{equation}

\begin{equation}
\leq C_3 m^3 \sup_{r_1 \leq |w-z_0| \leq 2r_1} e^{\rho_m(w)-mQ(z_0)}
\end{equation}

where $C_3$ is a constant depending only on $\alpha, A, \beta$ and $M_0$.
where we used (6.1). The first factor in (6.11) is analyzed with (3.1) and (3.3):

\[
|K_{2,mQ,n}(w, z_0)|^2 e^{-mQ(w)} \
\leq m(8 + 48A^2) e^A \int_{\mathbb{D}(z_0, m^{-1/3})} |K_{2,mQ,n}(\bar{w}, z_0)|^2 e^{-mQ(\bar{w})} dA(w) \
\leq m(8 + 48A^2) e^A K_{2,mQ,n}(z_0, z_0) \leq m^2(8 + 48A^2)^2 e^{2A} e^{mQ(z_0)}
\]

for any \( w \in \mathbb{D}(z_0, 2r_1) \).

Summarizing the discussion so far, we have

\[
\int_{\mathbb{C}} |K_{2,mQ,n}(w, z_0)|^2 \chi(w) e^{-mQ(w)} dA(w) \leq C_3 m^2 \sup_{r_1 \leq |w - z_0| \leq 2r_1} \{ e^{\frac{1}{2} \rho_m(w)} \},
\]

for a constant \( C_3 > 0 \) depending only on \( \alpha, A, \beta, l, M_0 \) and \( M_1 \). This leads to

\[
\int_{\mathbb{C}} |K_{2,mQ,n}(w, z_0)|^2 \chi(w) e^{-mQ(w)} dA(w) e^{-mQ(z_0)} \leq \sqrt{C_3 m} \sup_{r_1 \leq |w - z_0| \leq 2r_1} \{ e^{\frac{1}{2} \rho_m} \}.
\]

We can use the estimate (6.1) to deduce

\[
|K_{2,mQ,n}(z_0, z_1)|^2 e^{-mQ(z_0) - mQ(z_1)} \leq C_4 m^2 \sup_{r_1 \leq |w - z_0| \leq 2r_1} \{ e^{\frac{1}{2} \rho_m} \},
\]

with \( C_4 \) depending only on \( \alpha, A, \beta, l, M_0 \) and \( M_1 \). To get a good off-diagonal estimate for \( K_{2,mQ,n} \), it remains to choose a function \( \rho_m \) such that the conditions of the section 4 are satisfied and that \( \sup_{r_1 \leq |z - z_0| \leq 2r_1} \rho_m(z) \) is small. We present one possible way to choose \( \rho_m \) in the end of this section. Here, we only need to know that this choice of \( \rho_m(z) \) only depends on the distance \( |z - z_0| \), and we have

\[
\sup_{r_1 \leq |z - z_0| \leq 2r_1} \rho_m(z) = \rho_m(w), \quad \rho_m(w) = -k(\sqrt{m}r_1 - 3)
\]

whenever \( |w - z_0| = r_1 \). Here, \( k \) is a constant which depends only on \( \alpha, A, M_0, M_1, \beta \) and \( l \). By a simple computation, we get

\[
\rho_m(w) \leq -\frac{1}{2}k\sqrt{m} \min\{r_0, |z_1 - z_0|\} + \frac{7}{2}k.
\]

We have now obtained the estimate

\[
|K_{2,mQ,n}(z_0, z_1)|^2 e^{-mQ(z_0) - mQ(z_1)} \leq C e^{-\frac{1}{2}k\sqrt{m}\min\{r_0, |z_1 - z_0|\}},
\]

for a constant \( C \) depending only on the parameters \( \alpha, A, M_0, M_1, \beta \) and \( l \). Recall that this was proven under the assumption

\[
9m^{-1/2} \leq \min\{r_0, |z_1 - z_0|\}.
\]

We now investigate what happens when this condition does not hold. Let us recall the estimate (3.4):

\[
|K_{2,mQ,n}(z_0, z_1)|^2 e^{-mQ(z) - mQ(w)} \leq m^2(8 + 48A^2)^2 e^{2A}.
\]

This and the negation of (6.16) imply that

\[
|K_{2,mQ,n}(z_0, z_1)|^2 e^{-mQ(z) - mQ(z_1)} \leq Cm^2 e^{-\sqrt{m}\min\{r_0, |z_1 - z_0|\}}
\]

holds with

\[
C = (8 + 48A^2)^2 e^{2A+9}.
\]

We have now showed the conclusion of the theorem 1.3 for a fixed \( z_0 \) in the interior of \( S \cap \mathcal{N}_+ \). The constants \( C \) and \( \epsilon \) depend only on \( M_0, M_1, \alpha, A, \beta \) and \( l \). The number \( m_0 \) may additionally depend on \( r_0 \).
Finally, it is clear from the proof that we can find $C$ and $\epsilon$ which work uniformly over any $z_0$ in a given compact set $K$ in the interior of $\mathcal{S} \cap N_+$. In fact, $r_0$ and $\alpha$ are the only parameters that depend on $z_0$, and they can be replaced by

$$r_{0,K} := \frac{1}{4} \text{dist}(K, \mathbb{C}\backslash \mathcal{S} \cap N_+).$$

and

$$\alpha_K := \inf_{w \in K} \Delta Q(w).$$

where

$$\hat{K} := \bigcup_{z_0 \in K} \mathbb{D}(z_0, 2r_0).$$

The theorem is now proved.

We now give one possible construction for the function $\rho_m$ that was used in the theorem. This will be done under the assumption $4m^{-1/2} \leq r_1$, which was also made in the proof.

We set

$$\rho_m(z) = -k\sigma_m(|z - z_0|),$$

where $k$ is a constant to be specified later, and

$$\sigma_m(x) = \begin{cases} 0, & 0 < x \leq m^{-1/2} \\ \frac{1}{2}m(x - m^{-1/2})^2, & m^{-1/2} < x \leq 2m^{-1/2} \\ \frac{1}{2}m(x - 2m^{-1/2}) + \frac{1}{2}, & 2m^{-1/2} < x \leq r_1 - 2m^{-1/2} \\ \frac{1}{2}m(x - r_1 + m^{-1/2})^2 + \sqrt{mr_1} - 3, & r_1 - 2m^{-1/2} < x \leq r_1 - m^{-1/2} \\ \sqrt{mr_1} - 3, & r_1 - m^{-1/2} < x.
\end{cases}$$

(6.17) $\sigma_m(x) = \begin{cases} 0, & 0 < x \leq m^{-1/2} \\ \frac{1}{2}m(x - m^{-1/2})^2, & m^{-1/2} < x \leq 2m^{-1/2} \\ \frac{1}{2}m(x - 2m^{-1/2}) + \frac{1}{2}, & 2m^{-1/2} < x \leq r_1 - 2m^{-1/2} \\ \frac{1}{2}m(x - r_1 + m^{-1/2})^2 + \sqrt{mr_1} - 3, & r_1 - 2m^{-1/2} < x \leq r_1 - m^{-1/2} \\ \sqrt{mr_1} - 3, & r_1 - m^{-1/2} < x.
\end{cases}$

One can quickly verify that this function is $C^{1,1}$-smooth and the derivatives satisfy $\sigma_m'(x) \leq \sqrt{m}$ for all $x > 0$ and $|\sigma_m''(x)| \leq m$ for all almost every $x > 0$. Now, we compute the derivatives of $\rho_m$ which are relevant from the point of view of the conditions (4.5), (4.6) and (4.7):

$$\bar{\partial}\rho_m(z) = -k \frac{(z - z_0)}{2|z - z_0|} \sigma_m'(|z - z_0|),$$

(6.18) $\bar{\partial}\rho_m(z) = -k \frac{(z - z_0)}{2|z - z_0|} \sigma_m'(|z - z_0|)$

$$\bar{\partial}^2 \rho_m(z) = k \left[ \frac{(z - z_0)^2}{4|z - z_0|^2} \sigma_m'(|z - z_0|) - \frac{(z - z_0)^2}{4|z - z_0|^2} \sigma_m''(|z - z_0|) \right],$$

(6.19) $\bar{\partial}^2 \rho_m(z) = k \left[ \frac{(z - z_0)^2}{4|z - z_0|^2} \sigma_m'(|z - z_0|) - \frac{(z - z_0)^2}{4|z - z_0|^2} \sigma_m''(|z - z_0|) \right]$

$$\Delta \rho_m(z) = -k \frac{1}{4} \left[ \frac{1}{|z - z_0|} \sigma_m'(|z - z_0|) + \sigma_m''(|z - z_0|) \right].$$

(6.20) $\Delta \rho_m(z) = -k \frac{1}{4} \left[ \frac{1}{|z - z_0|} \sigma_m'(|z - z_0|) + \sigma_m''(|z - z_0|) \right].$

We can assume $|z - z_0| \geq m^{-1/2}$, because $\rho_m = 0$ otherwise. It is then easily checked that absolute value of the first of these derivatives is uniformly bounded by $k\sqrt{m}/2$ for all $z$. The two others are bounded by $km/2$ almost everywhere. Now, allowing $k$ to depend on the parameters $\alpha, A, M_0, M_1, \beta$ and $l$, we can fix it to be so small that the conditions (4.5), (4.6) and (4.7) will be satisfied.

7. Remarks on the case $q > 2$

In this section, we aim to explain how one could prove theorem 1.2 for more general $q > 2$. The main issue here is to identify those terms in the local $q$-analytic Bergman kernel mod ($m^{-\frac{q}{2}}$) which are the most dominant after the blow-up

$$z = z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, \quad w = z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}}$$

around $z_0 \in \mathcal{S} \cap N_+$.

We take $z_0$, $r$ and $\chi$ as in section 2. We also recall the kernel $R_{q,m}$ from proposition 2.2

$$R_{q,m}(z, w) = m \sum_{k=0}^{q^{-1}} \frac{q!(-1)^k}{(q - 1 - k)!(k + 1)!k!} (z - w)^k \bar{\partial}^k \left( \bar{\partial}_w \theta e^{m(z-w)\theta} \right) e^{-m(z-w)\theta}. \quad (7.1)$$

$$R_{q,m}(z, w) = m \sum_{k=0}^{q^{-1}} \frac{q!(-1)^k}{(q - 1 - k)!(k + 1)!k!} (z - w)^k \bar{\partial}^k \left( \bar{\partial}_w \theta e^{m(z-w)\theta} \right) e^{-m(z-w)\theta}. \quad (7.1)$$
Let us define functions \( A_{k,j}(z,w) \) by
\[
\overline{\partial}_w^k (\overline{\partial}_w \theta e^{m(z-w)\theta}) = \left[ \sum_{j=0}^k m^j (z-w)^j A_{k,j}(z,w) \right] e^{m(z-w)\theta}.
\]

The functions \( A_{k,j} \) are analytic in \( z \) and \( \bar{w} \), and \( A_{k,k} = (\overline{\partial}_w \theta)^{k+1} \). We can now rewrite (7.1) as
\[
(7.2) \quad R_{q,m}(z,w) = m \sum_{k=0}^{q-1} \sum_{j=0}^k \frac{q!(-1)^k}{(q-1-k)!(k+1)!} (\bar{z} - \bar{w})^k m^j (z-w)^j A_{k,j}(z,w)
\]

We say that a term \( A(z,w) \) is of order \( l - \frac{1}{2}k - \frac{1}{2}j \), if we can write
\[
(7.3) \quad A(z,w) = m^l (\bar{z} - \bar{w})^k (z-w)^j X_{l,k,j}(z,w),
\]
where \( X_{l,k,j}(z,w) \) is analytic in \( z \), real-analytic in \( w \) and cannot be divided by \( \bar{z} - \bar{w} \) or \( z - w \) so that the divided function would also satisfy these two properties. We also use the notation
\[
\text{ord}[A] = l - \frac{1}{2}k - \frac{1}{2}j.
\]

We define the order of a finite sum of such terms to be the maximum of the orders of the individual terms.

The formula (7.2) shows that all terms in \( R_{q,m} \) are of order \( \leq 1 \), and the order 1 terms are those for which \( j = k \). The order 1 terms can be written more compactly as
\[
m \bar{\partial}_w \theta L_{q-1}^1 (m \bar{\partial}_w \theta |z-w|^2),
\]
where \( L_{q-1}^1 \) is the associated Laguerre polynomial with parameter 1 and degree \( q - 1 \). The remaining terms are of order \( \leq \frac{1}{2} \).

We will need a generalization of the proposition 2.4.

**Proposition 7.1.** For any \( C^q \)-smooth function \( A : \mathbb{D}(z_0,r) \times \mathbb{D}(z_0,r) \to \mathbb{C} \), there exists \( \delta > 0 \) such that for any \( u \in A^{2}_{q,mQ} \) and \( z \in \mathbb{D}(z_0, \frac{1}{3}r) \) we have
\[
(7.4) \quad \int u(w) \chi(w) m^q(z-w)^q A(z,w) e^{m(z-w)\theta} dA(w)
= \int u(w) \chi(w) \left[ \sum_{j=0}^{q-1} m^j (z-w)^j \sum_{k=0}^{q-j} X_{k,j}(z,w) \bar{\partial}_w^k A(z,w) \right] \times e^{m(z-w)\theta} dA(w)
\leq O(\|u\|_m e^{\frac{1}{2}mQ(z)-\delta m})
\]
for certain functions \( X_{k,j}(z,w) \) which are analytic in \( z \) and \( \bar{w} \) and can be written as a rational functions of \( \bar{\partial}_w \)-derivatives of \( \bar{\partial}_w \theta \). The number \( \delta \) can depend on \( Q, z_0, r \) and \( A \).
Proof. Set \( j = 0 \). We compute using lemma 2.1

\[
(7.5) \quad \int \tilde{\partial}_w^j u(w) \chi(w) m^{q-j} (z - w)^{q-j} \frac{A(z, w)}{(\partial_w \theta)^{q-j+1}} e^{m(z-w)\theta} \, dA(w)
\]

\[
= \int \tilde{\partial}_w^{j+1} u(w) \chi(w) m^{q-j} (z - w)^{q-j} \frac{A(z, w)}{(\partial_w \theta)^{j+1}} \tilde{\partial}_w (e^{m(z-w)\theta}) \, dA(w)
\]

\[
- \int \tilde{\partial}_w^j u(w) \chi(w) m^{q-j} (z - w)^{q-j} \tilde{\partial}_w \left[ \frac{A(z, w)}{(\partial_w \theta)^{j+1}} \right] e^{m(z-w)\theta} \, dA(w)
\]

\[
+ O(\|u\|_m e^{\frac{1}{4}mQ(z) - \delta m})
\]

\[
= - \int \tilde{\partial}_w^{j+1} u(w) \chi(w) m^{q-j} (z - w)^{q-j} \frac{A(z, w)}{(\partial_w \theta)^{j+1}} e^{m(z-w)\theta} \, dA(w)
\]

\[
+ (-1)^{j+1} \int u(w) \chi(w) m^{q-j} (z - w)^{q-j} \tilde{\partial}_w \left[ \frac{A(z, w)}{(\partial_w \theta)^{j+1}} \right] e^{m(z-w)\theta} \, dA(w)
\]

\[
+ O(\|u\|_m e^{\frac{1}{4}mQ(z) - \delta m})
\]

The second integral will then be left as such and the first integral will be analyzed again in the same way after setting \( j \rightarrow j + 1 \). The process stops when \( j = q \).

The algorithm to compute local \( q \)-analytic Bergman kernels involves Taylor expanding \( R_{q,m} \) in powers of \( (z-w) \) and using this proposition. The proposition will only be applied to functions \( A(z, w) \) which can be written in the form \((7.3)\). We then have

\[
\text{ord} \left[ \tilde{\partial}_w^k A(z, w) \right] \leq \text{ord} A + \frac{k}{2},
\]

and therefore

\[
\text{ord} \left[ \sum_{j=0}^{q-1} m^j (z - w)^j \sum_{k=0}^{q-j} X_{k,j}(z, w) \tilde{\partial}_w^k A(z, w) \right] \leq \text{ord} \left[ m^q(z-w)^q A \right].
\]

In other words, applying the proposition \((7.1)\) to a term of the form \( m^q(z-w)^q A(z, w) \) cannot increase the order.

Suppose that we are after a local \( q \)-analytic Bergman kernel \( \mod (m^{-\frac{1}{2}}) \), which will be in the form

\[
(7.6) \quad K^{(q)}_{q,m}(z, w) = G^{(q)}_{q,m}(z, w) e^{mQ(z-w)}
\]

\[
= (m^q G_{q,0}(z, w) + m^{q-1} G_{q,1}(z, w) + \cdots + m G_{q,q-1}(z, w)) e^{mQ(z-w)}.
\]

Proceeding as in the case \( q = 2 \), the first step is to Taylor expand \( R_{q,m}(z, w) \) in powers of \( (z-w) \) with \((2.4)\). One gets the presentation

\[
(7.7) \quad R_{q,m}(z, w) = \sum_{l=1}^{q-1} \sum_{k=1}^{q-l-1} m^l (z - w)^k (z - w)^l D_{l,k,j}(z, w)
\]

\[
+ \sum_{l=1}^{q-1} \sum_{k=1}^{q-l} m^l (z - w)^k (z - w)^{q+l} E_{l,k}(z, w),
\]

where the functions \( D_{l,k,j} \) are \( q \)-analytic in \( (z, \bar{w}) \) and \( E_{l,k} \) are \( q \)-analytic in \( z \) and real-analytic in \( w \). Applying the previous proposition \( l \) times shows that the terms containing functions \( E_{l,k} \) contribute \( \mathcal{O}(m^{-1/2} \|u\|_m e^{\frac{1}{4}mQ(z)}) \) and can therefore be neglected for the purpose at hand.

Taylor expansion of the order 1 terms of \( R_{q,m} \) yields

\[
(7.9) \quad m \tilde{\partial}_w \theta L^1_{q-1} (m \tilde{\partial}_w \theta |z-w|^2) = mb(z, w) L^1_{q-1} (mb(z, w)|z-w|^2)
\]

\[
+ \text{terms of order 1/2 or less + negligible terms.}
\]
On the other hand, the terms which remain from the Taylor expansion of order \( \leq \frac{1}{2} \) terms of \( R_{q,m} \) also are of order \( \leq \frac{1}{2} \). We can therefore write

\[
R_{q,m}(z,w) = mb(z,w)L_{q-1}^1(mb(z,w)|z-w|^2)
\]

+ terms of order 1/2 or less + negligible terms.

The order 1 terms on the right hand side are \( q \) analytic in \( z \) and \( \bar{w} \), so they will be left intact in the subsequent steps of the algorithm. The process continues by applying proposition 7.1 to those terms in (7.7) that contain \( D_{i,k,j} \) for \( j \geq q \) (the other ones are either negligible or \( q \)-analytic in \( \bar{w} \)) and then Taylor expanding the result in powers of \( (z-w) \) again. Then one ends up with a presentation of the form (7.7) again, but now the highest possible power for the parameter \( m \) is \( q - 1 \).

A crucial point is that further steps of the algorithm cannot produce any new order 1 terms. This follows from two facts. The first one is that if one Taylor expands a term \( m^l(\bar{z} - \bar{w})^k(z - w)^jX(z,w) \), where \( X \) is some function which is \( q \)-analytic in \( z \) and real-analytic in \( w \), in powers of \( (z-w) \), then each of the resulting terms has order which is less or equal to the order of \( m^l(\bar{z} - \bar{w})^k(z - w)^jX(z,w) \). The second fact is that the application of proposition 7.1 cannot increase the order. Therefore, when the algorithm is finished (i.e. when all terms that are not \( q \)-analytic in \( \bar{w} \) are of the form \( m^l(\bar{z} - \bar{w})^k(z - w)^jX(z,w) \) for \( l \leq 0 \)), we have produced a local poly-Bergman kernel \( K^{(q)}_{q,m}(z,w) = \bar{g}^{(q)}_{q,m}(z,w)e^{mQ(z,w)} \), so that

\[
\bar{g}^{(q)}_{q,m}(z,w) = mb(z,w)L_{q-1}^1(mb(z,w)|z-w|^2) + \text{terms of order 1/2 or less.}
\]

Let us now put

\[
z = z_0 + \frac{\xi}{\sqrt{m\Delta Q(z_0)}}, \quad w = z_0 + \frac{\lambda}{\sqrt{m\Delta Q(z_0)}}
\]

and apply the Taylor expansions

\[
b(z,w) = \Delta Q(z_0) + O(m^{-1/2})
\]

and

\[
e^{m\text{Re}Q(z,w) - \frac{1}{2}m(\xi + \lambda)} = e^{-\frac{1}{2}|\xi - \lambda|^2} + O(m^{-\frac{1}{2}})
\]

which hold uniformly for \( (\xi, \lambda) \) in any given compact subset of \( \mathbb{C}^2 \) when \( m \) is so big that \( z, w \in \mathcal{D}(z_0, \frac{1}{3}r) \). We get

\[
\frac{1}{m\Delta Q(z_0)}|K^{(q)}_{q,m}(z,w)e^{mQ(z,w)}| = |L_{q-1}^1((\xi - \lambda)^2)|e^{-\frac{1}{2}|\xi - \lambda|^2} + O(m^{-\frac{1}{2}}),
\]

where the result holds uniformly for all \( (\xi, \lambda) \) in any given compact subset of \( \mathbb{C}^2 \).

To show that the same result holds when the local polyanalytic Bergman kernel \( K^{(q)}_{q,m} \) is replaced by \( K_{q,mQ,n} \), one should proceed as in section 5. One would also need to generalize the proposition 4.5 to solutions of \( \bar{\partial}^m \)-equations for general \( q \) (the special case \( \rho_m = 0 \) would suffice here). We expect this to be a straightforward but rather laborious task.

The reader may have noticed that apart from the construction of local polyanalytic Bergman kernels, the lemma 3.1 has been a crucial ingredient in our arguments. The proof method presented in [22] should work also for more general \( q \)-analytic functions, but argument would become rather cumbersome. Therefore, it might be of value to quickly sketch another way of proving similar statements which easily works for any \( q \). We fix a point \( z \) and consider the Riesz decomposition

\[
Q(w) = h(w) + G[\Delta Q](w), \quad w \in \mathcal{D}(z, m^{-1/2}),
\]

where \( h \) is a harmonic function and \( G[\Delta Q](w) \) is the Green’s potential of \( \Delta Q \) in the disk \( \mathcal{D}(z, m^{-1/2}) \). One can then find an analytic function \( H_m \) such that \( |H_m|^2 = e^{-mh} \), and we get

\[
|H_m(w)|^2 \leq Ce^{-mQ(w)} \quad |z-w| \leq m^{-1/2},
\]

for some constant \( C \). Using this fact, one can easily make a reduction to the case \( Q = 0 \). The desired inequalities in this unweighted case could be worked out with the work of Koshelev [25].
who identified explicit reproducing kernels for polyanalytic Bergman spaces on the unit disk with constant weight.

8. Appendix: The third term of the bianalytic Bergman kernel

We will here compute the third term of the local bianalytic Bergman kernel. The Taylor expansion Lemma 2.1 and Proposition 2.4 will be casually used without further notice. We start as before by Taylor expanding (2.12).

\begin{equation}
\begin{aligned}
(8.1) \quad u(z) &= \int_{\mathcal{C}} u(w) \chi(w) \left[ 2m \left( b + \frac{1}{2} (w - z) \partial_{\bar{z}} b \right) - m(z - \bar{w}) (\partial_w b + \frac{1}{2} (w - z) \partial_w \partial_{\bar{z}} b) 
\right. \\
&\quad - m^2 |z - w|^2 b^2 + m^2 (z - w)^2 A_1(z, w) + m(z - w)^2 B_1(z, w) \\
&\left. + m^2 (z - w)^3 C_1(z, w) \right] e^{m(z-w)\theta} dA(w) + O\left( \|u\|_m e^{\frac{1}{4} mQ(z)} m^{-\frac{3}{2}} \right),
\end{aligned}
\end{equation}

where

\begin{equation}
\begin{aligned}
(8.2) \quad A_1(z, w) &= (\bar{z} - \bar{w}) b \partial_w b \\
B_1(z, w) &= \frac{1}{3} \partial_{\bar{z}} b - \frac{1}{6} (\bar{z} - \bar{w}) \partial_w \partial_{\bar{z}} b, \\
C_1(z, w) &= (\bar{w} - \bar{z}) \left( \frac{1}{3} b \partial_{\bar{z}} b + \frac{1}{4} (\partial_{\bar{z}} b)^2 \right).
\end{aligned}
\end{equation}

Only these last three terms require further analysis. We analyze \( B_1 \) first:

\begin{equation}
\begin{aligned}
(8.3) \quad m \int_{\mathcal{C}} u(w) \chi(w) (z - w)^2 B_1(z, w) e^{m(z-w)\theta} dA(w) \\
&= - \int_{\mathcal{C}} u(w) \chi(w) \left[ \frac{1}{m} \partial_{\bar{w}} \partial_{\bar{z}} B_1 \right. \\
&\quad + (z - w) \left( 2 \partial_w B_1 - \frac{3 B_1 \partial_w \partial_{\bar{z}} b}{\partial_{\bar{w}} \partial_{\bar{z}} b} \right) \right] e^{m(z-w)\theta} dA(w) \\
&\quad + O\left( \|u\|_m e^{\frac{1}{4} mQ(z)} e^{-\delta m} \right) \\
&= \int_{\mathcal{C}} u(w) \chi(w) (z - w) \left[ - \frac{2 \partial_w \partial_{\bar{z}} b}{3 b} - \frac{1 \partial_w \partial_{\bar{z}} b}{b^2} + \frac{\partial_{\bar{z}} b \cdot \partial_{\bar{w}} b}{b^2} \right] e^{m(z-w)\theta} dA(w) + O\left( \|u\|_m e^{\frac{1}{4} mQ(z)} m^{-\frac{3}{2}} \right).
\end{aligned}
\end{equation}

We turn to the term with \( C_1 \).

\begin{equation}
\begin{aligned}
(8.4) \quad \int_{\mathcal{C}} u(w) \chi(w) m^2 (z - w)^3 C_1 e^{m(z-w)\theta} dA(w) \\
&= \int_{\mathcal{C}} u(w) \chi(w) \left[ (z - w) \partial_{\bar{w}} \frac{C_1}{\partial_{\bar{w}} \partial_{\bar{z}} \theta} + m(z - w)^2 \left( 2 \partial_w C_1 - \frac{3 C_1 \partial_w \partial_{\bar{z}} \theta}{\partial_{\bar{w}} \partial_{\bar{z}} \theta} \right) \right] e^{m(z-w)\theta} dA(w) \\
&\quad + O\left( \|u\|_m e^{\frac{1}{4} mQ(z)} e^{-\delta m} \right) \\
&= \int_{\mathcal{C}} u(w) \chi(w) \left[ (z - w) \partial_{\bar{w}} \frac{C_1}{b^2} + m(z - w)^2 C_2(z, w) \right] e^{m(z-w)\theta} dA(w),
\end{aligned}
\end{equation}
where

\begin{equation}
(8.5) \quad -\partial^2_w \frac{C_1}{b^2} = -\partial_w \left[ (\bar{w} - \bar{z}) \left( \frac{1}{3} \frac{\partial^2 b}{b} + \frac{1}{4} \frac{(\partial_z b)^2}{b^2} \right) \right] \\
= -\frac{2}{3} \frac{\partial_w \partial^2 b}{b^2} - \frac{\partial_z b \cdot \partial_w \partial_z b}{b^2} + \frac{(\partial_z b)^2 \cdot \partial_w b}{b^2} + \frac{2}{3} \frac{\partial^2 b \cdot \partial_w b}{b^2} \\
+ (\bar{z} - \bar{w}) \left[ \frac{1}{3} \frac{\partial^2 b}{b} - \frac{2}{3} \frac{\partial_w \partial^2 b \cdot \partial_z b}{b^2} - \frac{1}{3} \frac{\partial^2 b \cdot \partial_w^2 b}{b^2} \right] \\
+ \frac{2}{3} \frac{\partial^2 b \cdot (\partial_w b)^2}{b^2} + \frac{1}{2} \left( \frac{\partial_w \partial_z b}{b^2} + \frac{\partial_z b \cdot \partial_w \partial_z b}{b^2} - \frac{1}{2} \frac{(\partial_z b)^2 \partial_w^2 b}{b^2} \right) - \frac{1}{2} \frac{(\partial_z b)^2 \partial_w^2 b}{b^2} + \frac{3}{2} \frac{(\partial_z b) \cdot (\partial_w b)^2}{b^2},
\end{equation}

and

\begin{equation}
(8.6) \quad C_2 := 2 \partial_w C_1 - 3 \frac{C_1 \partial^2 \theta}{(\partial \theta)^2} \\
= \frac{2}{3} \partial^2 b + \frac{1}{2} \left( \frac{\partial_z b}{b} \right)^2 + (\bar{w} - \bar{z}) \left[ \frac{2}{3} \frac{\partial_w b \cdot \partial^2 b}{b} + \frac{2}{3} \frac{\partial_w \partial^2 b}{b} + \frac{\partial_z b \cdot \partial_w \partial_z b}{b} \right] \\
- (\bar{w} - \bar{z}) \left( \frac{\partial^2 b}{b} + \frac{3}{4} \frac{(\partial_z b)^2}{b^2} \right) \partial_w b.
\end{equation}

We continue by analyzing (8.4) by with proposition 2.4

\begin{equation}
(8.7) \quad -\int \mathcal{C} u(w) \chi(w) m(z - w)^2 C_2 e^{m(z - w)^2} dA(w) \\
= \int \mathcal{C} u(w) \chi(w) \left[ \frac{1}{m} \partial^2_w \frac{C_2}{(\partial \theta)^2} + (z - w) \left( 2 \frac{\partial_w C_2}{b} - 3 \frac{C_2 \partial^2 \theta}{b} \right) \right] e^{m(z - w)^2} dA(w) \\
\quad + O\left( \|u\|_m e^{\frac{1}{4} m Q(z)} e^{-\delta m} \right) \\
= \int \mathcal{C} u(w) \chi(w) \left[ \frac{1}{m} \partial^2_w \frac{C_2}{(\partial \theta)^2} + (z - w) \left( 2 \frac{\partial_w C_2}{b} - 3 \frac{C_2 \partial^2 \theta}{b^2} \right) \right] e^{m(z - w)^2} dA(w) \\
\quad + O\left( \|u\|_m e^{\frac{1}{4} m Q(z)} m^{-\frac{1}{2}} \right).
\end{equation}

The first term is negligible for our purposes. The second term equals

\begin{equation}
(8.8) \quad \frac{2}{b} \partial_w \frac{C_2}{b} - 3 \frac{C_2 \partial^2 \theta}{b^2} \\
= \frac{8}{3} \frac{\partial_w}{b} + \frac{4}{3} \frac{\partial_z b \cdot \partial_w \partial_z b}{b^2} - \frac{8}{3} \frac{\partial_w b \cdot \partial^2 b}{b^2} - \frac{4}{3} \frac{(\partial_z b)^2 \cdot \partial_w b}{b^3} \\
+ (\bar{w} - \bar{z}) \left[ - \frac{2}{3} \frac{\partial^2 b}{b^2} - \frac{5}{3} \frac{(\partial_w b)^2 \cdot \partial^2 b}{b^2} + \frac{4}{3} \frac{\partial^2 b^2}{b^2} + \frac{2}{3} \frac{(\partial_w b)^2}{b} \right] \\
+ 2 \frac{\partial_z b \cdot \partial_w \partial_z b}{b^2} - \frac{3}{2} \frac{(\partial_w b)^2 \partial^2 b}{b^2} + \frac{21}{4} \frac{(\partial_z b)^2 \partial_w b}{b^2}.
\end{equation}
We now turn to the term \( A_1 \) in (8.1).

\[
(8.9) \quad \int_C u(w) \chi(w) m^2 (z - w)^2 A_1 \, dA
= \int_C u(w) \chi(w) \left[ -\frac{\partial_{w}^{2} A_1}{(\partial_{w} \theta)^2} + m(z - w) \left( -2 \frac{\partial_{w} A_1}{\partial_{w} \theta} + 3 \frac{\partial_{w}^{2} \theta}{(\partial_{w} \theta)^2} \right) \right] e^{m(z-w)\theta} + O\left(\|u\|_m e^{z m Q(z)} e^{-\delta m}\right)
= \int_C u(w) \chi(w) \left\{ A_2 + m(z - w) \left[ -2 \frac{1}{b} \partial_w \left( \bar{z} \partial_z b \cdot b \right) + 3(\bar{z} - w) \frac{\partial_z b \cdot \partial_w b}{b} \right]
+ m(z-w)^2 A_3 \right\} e^{m(z-w)\theta} dA(w) + O\left(\|u\|_m e^{z m Q(z)} m^{-\frac{2}{3}}\right),
\]

where

\[
(8.10) \quad A_2 := -\partial_{w}^{2} \left[ (\bar{z} - \bar{w}) \partial_z b \cdot b \left( \frac{1}{b^2} - (w - z) \frac{\partial_z b}{b^3} \right) \right]
= 2 \frac{\partial_w \partial_z b}{b} - 2 \frac{\partial_z b \cdot \partial_w b}{b^2}
+ (\bar{z} - w) \left\{ -\frac{\partial_{w}^{2} \partial_z b}{b} + 2 \frac{\partial_w \partial_z b \cdot \partial_w b}{b^2} + \frac{\partial_z b \cdot \partial_{w}^{2} b}{b^2} - 2 \frac{\partial_z b \cdot (\partial_{w} \theta)^2}{b^3} \right\}
\]

and

\[
(8.11) \quad A_3 := -\partial_{w} \partial_z \left[ (\bar{z} - \bar{w}) \partial_z b \cdot b \right] + (\bar{z} - w) \frac{\partial_z b \cdot \partial_{w}^{2} b}{b^2} + 3(\bar{z} - \bar{w}) \frac{\partial_z b \cdot \partial_{w} \partial_z b}{b}
= \frac{(\partial_z b)^2}{b} + (\bar{z} - w) \left[ -\frac{5}{2} \frac{\partial_z b \cdot \partial_{w} \partial_z b}{b} + 2 \frac{(\partial_z b)^2 \partial_{w} b}{b^2} \right].
\]

We continue with the term \( A_3 \).

\[
(8.12) \quad \int_C u(w) \chi(w) m(z - w)^2 A_3 e^{m(z-w)\theta} dA(w)
= \int_C u(w) \chi(w) \left\{ -\frac{1}{m} \frac{\partial_{w}^{2} A_3}{\partial_{w} \theta} + (z - w) \left[ -2 \frac{\partial_w \partial_{w} A_3}{\partial_{w} \theta} + 3 \frac{\partial_{w}^{2} \theta}{(\partial_{w} \theta)^2} \right] \right\} e^{m(z-w)\theta} dA(w) + O\left(\|u\|_m e^{z m Q(z)} e^{-\delta m}\right)
= \int_C u(w) \chi(w) \left\{ -\frac{1}{m} \frac{\partial_{w}^{2} A_3}{\partial_{w} \theta} + (z - w) \left[ -2 \frac{\partial_w \partial_{w} A_3}{b} + 3 \frac{A_3 \partial_{w}^{2} \theta}{b} \right] \right\} e^{m(z-w)\theta} dA(w)
+ O\left(\|u\|_m e^{z m Q(z)} m^{-\frac{2}{3}}\right).
\]

The first term is negligible, and the second term equals

\[
(8.13) \quad -2 \frac{\partial_w A_3}{b^2} + 3 \frac{A_3 \partial_{w}^{2} \theta}{b} = -9 \frac{\partial_z b \cdot \partial_{w} \partial_z b}{b^2} + 9 \frac{(\partial_z b)^2 \partial_{w} b}{b^3}
+ (\bar{z} - w) \left[ -5 \frac{(\partial_w \partial_{w} b)^2}{b^2} - 5 \frac{\partial_z b \cdot \partial_{w}^{2} \partial_z b}{b^2} - \frac{41}{2} \frac{\partial_z b \cdot \partial_{w} \partial_z b \cdot \partial_{w} b}{b^3}
\right.
\]

\[
- \frac{4}{b^3} \frac{(\partial_z b)^2 \partial_{w}^{2} b}{b^3} + 14 \frac{(\partial_z b)^2 \cdot (\partial_{w} b)^2}{b^4}\right].
\]
This concludes the analysis of $A_1$ from (8.1). We combine everything together within (8.1) to get the third term (i.e. constant order contribution) for the bianalytic Bergman kernel. We omit the laborious summation.

\[
\mathfrak{A}_{2,2} = 2\tilde{\partial}_w \partial_z \log b + (\bar{w} - z)\tilde{\partial}_w \partial_z \log b + (z - w)\partial_w \partial_z^2 \log b + |z - w|^2 M(z, w),
\]

where

\[
\begin{align*}
M &= + \frac{3}{2} \frac{\partial^2 \tilde{\partial}_w \partial_z b \cdot \partial_z b}{b^2} - \frac{13}{2} \frac{\partial_z b \cdot \tilde{\partial}_w \partial_z b \cdot \partial_z b}{b^3} + \frac{3}{2} \frac{(\tilde{\partial}_w \partial_z b)^2}{b^2} \\
&\quad - \frac{(\partial_z b)^2 (\tilde{\partial}_w b)^2}{b^3} + \frac{17}{4} \frac{(\partial_z b)^2 \cdot (\tilde{\partial}_w b)^2}{b^3} - \frac{2}{3} \frac{\partial^2 \tilde{\partial}_w b \cdot \partial_z b}{b^2} + \frac{3}{2} \frac{\partial^2 \tilde{\partial}_w b \cdot \partial_z b}{b^2} \\
&\quad - \frac{(\partial_z b)^2 (\tilde{\partial}_w b)^2}{b^3} + \frac{1}{3} \frac{\partial^2 \tilde{\partial}_w b \cdot \partial^2 b}{b^2}.
\end{align*}
\]

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