Gravitational Lensing in the metric theory proposed by Sobouti

Tula Bernal \(^1\) & Sergio Mendoza \(^2\)

Instituto de Astronomía, Universidad Nacional Autónoma de México, Ciudad Universitaria, México.

Abstract

Recently, Y. Sobouti (2007) has provided a metric theory \(f(R)\) that can account for certain dynamical anomalies observed in spiral galaxies. Mendoza & Rosas-Guevara (2007) have shown that in this theory there is an extra-bending as compared to standard general relativity. In the present work we have developed in more specific detail this additional lensing effect and we have made evaluations of the \(\alpha\) parameter used in the model adjusting the theory to observations in X-rays of 13 clusters of galaxies with gravitational lensing (Hoekstra (2007)).

1 Introduction

Some modified theories of gravity are an actual alternative to the dark matter paradigm, which in spite of its success to explain astrophysical problems like the flat rotation curves of spiral galaxies, the gravitational lensing by clusters of galaxies and the formation of structure in the early Universe, is not the ultimate word of the story.

In the different alternatives we find the modifications to the Hilbert-Einstein action, which introduce an arbitrary function of the Ricci scalar \(R\) in the action. This modification is relevant, because there’s no exist an \textit{a priori} reason to restrict the gravitational action to \(R\) and has the advantage to be a metric theory of gravitation that carry out with several physical principles and at the same time accounts to the energy-momentum conservation, as well as the equations of the theory are invariants.

In this sense, Sobouti (2007) has developed a metric theory \(f(R)\) that reproduces flat rotation curves in different spiral galaxies (with \(f(R) = R^{1-\alpha/2}\) and \(\alpha \ll 1\)), reproduces the Tully-Fisher relation, converges to a version of MOND in the weak field approximation and the resulting differential field equations are of the second order.

\(^1\)tbernal@astrosca.unam.mx
\(^2\)sergio@mendozza.org
2 Modified field equations

The model introduces the following modification to the Hilbert-Einstein action (Capozziello (2002); Capozziello et al. (2003)):

\[ S_g = - \int \frac{1}{2} f(R) \sqrt{-g} \, d\Omega. \]  \hspace{1cm} (1)

Variation of the action \( S = S_g + S_m \) with respect to the metric \( g_{\mu\nu} \) gives the following field equations (Capozziello et al. (2003)):

\[ f'(R) R_{\mu\nu} - \frac{1}{2} g_{\mu\nu} = T_{\mu\nu} - (f'(R))_{;\mu\nu} - (f'(R))_{;\lambda} g_{\mu\nu}. \]  \hspace{1cm} (2)

For an application of this theory to different galactic systems, the metric is chosen like a Schwarzschild-like one given by (Cognola et al. (2005); Sobouti (2007)):

\[ ds^2 = B(r) dt^2 - A(r) dr^2 - r^2 (d\theta^2 + \sin^2 \theta d\phi^2). \]  \hspace{1cm} (3)

If the function \( f \) differs not too much from general relativity at first order of approximation, the exterior solution (\( \rho = 0 = p \)) to the field equations is obtained with the assumption that \( v = df / dR \) is a parametric function \( v(r, \alpha) \) with \( \alpha \) a real number such that \( \alpha \ll 1 \) (for \( \alpha = 0 \) the solution converges to G.R.). Proposing a solution of the form \( v \propto r^\beta, \beta \in \mathbb{R} \), results in \( v \simeq r^\alpha \) and the solutions for the metric are (Sobouti (2007)):

\[ 1/A(r) = \frac{1}{1 - \alpha} \left[ 1 - \left( \frac{s}{r} \right)^{1-\alpha/2} \right]; B(r) = \left( \frac{r}{s} \right)^{\alpha} \frac{1}{A(r)}; \]  \hspace{1cm} (4)

where \( s := 2GM \) is the Schwarzschild radius. For \( \alpha \ll 1 \), \( f(R) \) is given by:

\[ f(R) = (3\alpha)^{\alpha/2} s^{-\alpha} R^{1-\alpha/2} \simeq R \left[ 1 - \frac{\alpha}{2} \ln (s^2 R) + \frac{\alpha}{2} \ln(3\alpha) \right]. \]  \hspace{1cm} (5)

3 The deflection of light and gravitational lensing

Mendoza & Rosas-Guevara (2007) have shown that the deflection angle \( \beta \) for a spherically symmetric mass distribution in this metric theory (cf. eq.3), is given by:

\[ \beta = \pi \left[ \frac{2\sqrt{1-\alpha}}{2-\alpha} - 1 \right] + 2\sqrt{1-\alpha} \left( \frac{s}{r_m} \right)^{1-\alpha/2}, \]  \hspace{1cm} (6)
where \( r_m \) is the impact parameter (the closest approach to the lens). When \( \alpha = 0 \), the deflection angle equals the Einstein’s one: \( \beta_E = 2s/r_m \).

For an application in gravitational lensing for a symmetric object (e.g. a cluster of galaxies) with mass \( M \), the lens equation is given by (Mendoza & Rosas-Guevara (2007)):

\[
\Theta^2 - \Theta(\Phi + C_1) - C_2 \Theta^{\alpha/2} = 0,
\]

where:\n\[
\begin{align*}
C_1 &:= \frac{\pi D_{LS}}{\theta_E D_S} \left[ \frac{2\sqrt{1-\alpha}}{2-\alpha} - 1 \right]; \\
C_2 &:= \theta_E^{-\alpha/2} \left( \frac{1}{2} \frac{D_S}{D_{LS}} \right)^{-\alpha/2} \sqrt{1-\alpha}; \\
\Theta &:= \theta/\theta_E; \\
\Phi &:= \gamma/\theta_E; \\
\gamma & is the angular position of the source, \\
\theta & is the angular position of the image, \\
D_S, D_L, and D_{LS} & are the observer-source, observer-lens and lens-source distances, respectively.
\]

The solution to the lens equation at first order, with \( \Theta_1 \) a small linear perturbation to the solution of general relativity, \( \Theta_0 = \frac{1}{2} \left( \Phi \pm \sqrt{\Phi^2 + 4} \right) \), is given by (Mendoza & Rosas-Guevara (2007)):

\[
\Theta_1 = \frac{C_1 \Theta_0 + C_2 \Theta_0^{\alpha/2} - 1}{2 \Theta_0 - \Phi - C_1 - \frac{\alpha}{2} C_2 \Theta_0^{\alpha/2 - 1}}.
\]

With this perturbation the magnification factor is (Mendoza & Rosas-Guevara (2007)):

\[
\mu = \det [J(\theta)]^{-1} = \left| \frac{\Theta d\Theta}{\Phi d\Phi} \right| \approx \frac{\Theta_0 + \Theta_1 d\Theta_0}{\Phi d\Phi},
\]

where \( J(\theta) \) is the jacobian of the transformation \( \theta \to \gamma \) (from the lens plane to the source plane).

For \( \det [J(\theta)] = 0 \) (when the magnification factor \( \mu \) diverges), the critical curve for \( \Phi = 0 \) and for \( \theta^+_0(0) = \theta_E \), is given by:

\[
\theta^+_C = \frac{1 + C_2 \left( 1 - \frac{\alpha}{2} \right)}{2 - C_1 - \frac{\alpha}{2} C_2} \theta_E,
\]

and for \( \theta^-_0(0) = 2\pi - \theta_E \) by:

\[
\theta^-_C = \frac{2(\theta^-_0)^2 - 1 + C_2 (\Theta^-_0)^{\alpha/2} \left( 1 - \frac{\alpha}{2} \right)}{2(\theta^-_0 - C_1 - \frac{\alpha}{2} C_2 (\Theta^-_0))^{\alpha/2 - 1}} \theta_E.
\]

From the symmetry of the problem we have \( (2\pi - \theta^-_C) = \theta^+_C := \theta_C \), and we can take \( \theta_C \) as the equivalent to the Einstein angle \( \theta_E \), i.e. an "Einstein ring" in this theory have an aperture of \( \theta_C \), and \( \theta_C = \theta_E \) for \( \alpha = 0 \).
4 Approximation method to $\alpha$

Since these results are derived from a metric with spherical symmetry, we use observations in X-rays of gravitational lensing by 13 clusters of galaxies that approximate well to this symmetry (see Hoekstra (2007) and references therein). In order to estimate the parameter $\alpha$, we assume that the estimated values of the Einstein angle from the observations approximates $\theta_C$. We use this as a first way to measure the order of magnitude to $\alpha$.

The cosmological model used is: $H_0 = 70 \text{km}\text{s}^{-1}\text{Mpc}^{-1}$, $\Omega_m = 0.3$ and $\Omega_\Lambda = 0.7$. To calculate the Einstein angle we use the virial mass of the system with the mean velocity dispersion observed in a particular system (Borgani et al. (1999); Girardi & Mezzetti (2001)).

Using this method we can solve numerically the equation (10) to obtain $\alpha$.

5 Results

Sobouti’s $f(R)$ theory is a first approximation to a modified metric theory of gravity in order to account for phenomena usually ascribed to dark matter. However, there is a caveat with Sobouti’s approach. The mass dependency of $\alpha$ destroys one of the most important facts of an ordinary action: it should be invariant under the changes of sources. Also, this mass dependency means that the function $f(R)$ varies from source to source. However, as mentioned by Sobouti (2007), perhaps we need to re-think whether this postulate has to be accepted when modifications to standard general relativity are done.

The way Sobouti (2007) calculated the value for the parameter $\alpha$ was using a spherically symmetric metric, applied to spiral galaxies. This is of course wrong since these galaxies are far from spherical symmetry. This is the reason as to why we decided to calibrate the $\alpha$ parameter using spherical symmetric astrophysical systems: cluster of galaxies. From the many cluster of galaxies known we have chosen a set of very well spherically symmetric objects.

To summarize, our main results are as follows:

- The caustic in the source plane is the point colinear with the lens and the observer. This is an expected result since the metric is a Schwarzschild-like one for a spherically symmetric matter distribution.
\[
\alpha = (3.5 \pm 0.3) \times 10^{-9} (M/M_\odot)^{1/2}
\]

Figure 1: Plot of \(\alpha\) vs. \(M/10^{14} M_\odot\)^{1/2}. The best linear fit is shown by the line.

- The \(\alpha\)-mass relation from Sobouti (2007) is given by \(\alpha = \alpha_0 \left( \frac{M}{M_\odot} \right)^{1/2}\).

  Our best linear fit gives a value of \(\alpha_0 = (3.5 \pm 0.3) \times 10^{-9}\) (see Fig. 1), that differs in three orders of magnitude from the value obtained by Sobouti (2007), \(\alpha_0 = (3.07 \pm 0.18) \times 10^{-12}\), for 31 spiral galaxies with \(\alpha = 2v_\infty^2/c^2\) (\(v_\infty\) is the asymptotic constant tangential speed of the galaxy).

- The log-log plot of \(\alpha\) vs. \(M/M_\odot\), leads the power law fit: \(\alpha = [(1.47 \pm 0.51) \times 10^{-5}] \left( \frac{M}{M_\odot} \right)^{0.25}\), i.e. a relation of the form:

\[
\alpha(M) = \alpha_1 \left( \frac{M}{M_\odot} \right)^{1/4}.
\]

- Although the calculations are only a first approximation to the order of magnitude to the \(\alpha\) parameter, this is the correct way for evaluating it, since the distribution of barionic matter approximates to a spherical symmetric one in clusters of galaxies.

- One could be tempted to use the results also for quite “spherical” elliptical galaxies. However, this approximation fails because the Einstein angle in these galaxies is very small and so the variables \(C_1\) and \(C_2\) (cf. eq.7) diverge.
6 \alpha = (1.47 \pm 0.51) \times 10^{-5} (M/M_\odot)^{0.25}

Figure 2: Plot of \log(\alpha) vs. \log(M/M_\odot). The best linear fit is shown by the line.

• **Future work**: In section 4 we approximated the value \alpha by using the estimated Einstein angles and equate them to \theta_C. This is a first approximation to the real value of \alpha and the only way to correct this is to develop a complete theory of gravitational lensing for the metric theory proposed by Sobouti. In this way, we intend to determine with much precision the exact value for the \alpha parameter, and check whether this theory can account for different astrophysical problems.

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