DISCORRELATION BETWEEN PRIMES IN SHORT INTERVALS AND POLYNOMIAL PHASES

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Abstract. Let $H = N^\theta$. We obtain estimates for the exponential sum over primes in short intervals:

$$\sum_{N < n \leq N + H} \Lambda(n)e(\alpha n^k),$$

which are valid for all $k \geq 1$ and $\theta > 2/3$. As a consequence of this, we deduce a short interval version of the Waring-Goldbach problem. We also investigate more general sums of the form

$$\sum_{N < n \leq N + H} \Lambda(n)e(g(n)),$$

where $g$ is a polynomial of degree $k$.

1. Introduction

Let $N \geq 2$ be a positive integer, and let $H = N^\theta$ for some $0 < \theta \leq 1$. The purpose of this paper is to obtain estimates for the sum

$$\sum_{N < n \leq N + H} \Lambda(n)\psi(n),$$

(1.1)

where $\Lambda$ is the von-Mangoldt function, and $\psi$ is a polynomial phase of the form $\psi(n) = e(g(n))$ (with the notation $e(x) = \exp(2\pi i x)$) for some polynomial $g$. We would like to obtain results with $\theta$ as small as possible.

In the case of summing over a long interval (i.e. $\theta = 1$), the task of estimating (1.1) is well understood. When $\deg g = 0$, asymptotic formula for (1.1) is given by the Prime Number Theorem. When $\deg g = 1$, estimates for the exponential sum

$$\sum_{N < n \leq 2N} \Lambda(n)e(\alpha n)$$

for $\alpha \in \mathbb{R}$ were obtained and used by Vinogradov to solve the ternary Goldbach problem. More generally, for $\psi$ a fixed nilsequence (which includes polynomial phases as special examples), Green and Tao [4] showed the discorrelation estimate

$$\sum_{N < n \leq 2N} \mu(n)\psi(n) \ll A \frac{N}{(\log N)^A},$$

for any $A \geq 2$. This leads to a discorrelation estimate for (1.1), when $\psi$ is in “minor arc” or when $\Lambda$ is “W-tricked” (see [3, Proposition 10.2]).

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In the case of summing over a short interval, the case deg $g = 0$ corresponds to the classical problem of counting primes in short intervals. Huxley’s zero density estimate [8] implies an asymptotic formula for primes in short intervals when $\theta > 7/12$ (see the discussion in [9, Chapter 10]). When $\text{deg} \, g = 1$, (1.1) becomes the exponential sum estimate

$$\sum_{N < n \leq N + H} \Lambda(n)e(\alpha n).$$

This has been studied quite extensively due to its implication on Vinogradov’s theorem with almost equal summands. The best threshold for $\theta$ in this problem is $\theta > 5/8$ due to Zhan [13]. In the more general case when $g(n) = \alpha n^k$ is a monomial of degree $k$, Huang [7, Theorem 2] obtained estimates for (1.1) when $\theta > 19/24$. When dealing with $\mu$ instead of $\Lambda$, Huang [6] obtained the estimate

$$\sum_{N < n \leq N + H} \mu(n)e(\alpha n^k) \ll A H (\log N)^A$$

for any $A \geq 2$, in the wider region $\theta > 3/4$. In this paper we establish estimates for (1.1) when $\theta > 2/3$ and $g(n) = \alpha n^k$.

**Theorem 1.1.** Let $H = N^\theta$ for some fixed $\theta > 2/3$. Let $\alpha \in \mathbb{R}$ and let $k$ be a positive integer. Suppose that

$$\left| \sum_{N < n \leq N + H} \Lambda(n)e(\alpha n^k) \right| \geq \frac{H}{(\log N)^A}$$

for some $A \geq 2$. Then there exists a positive integer $q \leq (\log N)^{O_k(A)}$ such that

$$\| q\alpha \| \leq \frac{(\log N)^{O_k(A)}}{N^{k-1} H}.$$

Note that if $q \approx 1$ and $\| q\alpha \| \approx 1/(N^{k-1} H)$, then the phase $\alpha n^k$ is almost constant on $(N, N + H)$ after dividing it into residue classes modulo $q$. This major arc case will thus correspond to the classical prime number theorem in short intervals (in residue classes modulo $q$).

Via the circle method, Theorem 1.1 leads to a short interval version of the Waring-Goldbach problem. For a prime $p$ and a positive integer $k$, let $\tau = \tau(k, p)$ be the largest integer such that $p^\tau \mid k$. Define

$$\gamma(k, p) = \begin{cases} \tau + 2 & \text{if } p = 2 \text{ and } \tau > 0, \\ \tau + 1 & \text{otherwise.} \end{cases}$$

Define $R(k) = \prod p^{\gamma(k, p)}$, where the product is taken over all primes $p$ with $(p - 1) \mid k$.

**Theorem 1.2.** Fix $k \geq 2$, $s \geq k(k + 1) + 3$, and $\theta > 2/3$. Then every sufficiently large positive integer $N \equiv s \pmod{R(k)}$ can be written as

$$N = p_1^k + \ldots + p_s^k,$$

where $p_1, \ldots, p_s$ are primes satisfying $|p_i - (N/s)^{1/k}| \leq N^\theta$. 

This was proved for $\theta > 19/24$ and $s \geq \max(7, 2k(k-1)+1)$ by Huang [6]. We refer the reader to [6] for the historical development of this problem. The improvement on the threshold for $\theta$ comes from Theorem 1.1, whereas the improvement on the number of variables $s$ is due to the recent resolution of the main conjecture in Vinogradov’s mean value theorem [1] (which was unavailable to previous authors). Indeed, given Vinogradov’s mean value conjecture, Huang’s result would require $\theta > 19/24$ and $s \geq k(k+1) + 1$. Unfortunately Theorem 1.2 is worse in the $s$ respect.

Our original motivation for studying (1.1) is, in fact, to obtain the short interval version of the aforementioned Green-Tao theorem on discorrelation between primes and nilsequences. We are unable to get any results for general nilsequences $\psi$, but the following theorem provides a much weaker version of Theorem 1.1 in the case $\psi(n) = e(g(n))$ for a general polynomial $g$.

**Theorem 1.3.** Let $k$ be a positive integer. Let $H = N^\theta$ for some fixed $\theta > \theta_k$, where

$$\theta_k = \begin{cases} 2/3 & k = 2, \\ 23/32 & k = 3, \\ (2k + 2)/(2k + 5) & k \geq 4. \end{cases}$$

Let $g$ be a polynomial of degree $k$ of the form

$$g(n) = \sum_{j=1}^{k} \alpha_j (n - N)^j$$

for some $\alpha_1, \ldots, \alpha_k \in \mathbb{R}$. Suppose that

$$\left| \sum_{N < n \leq N + H} \Lambda(n)e(g(n)) \right| \geq \frac{H}{(\log N)^A}$$

for some $A \geq 2$. Then there exists a positive integer $q \leq (\log N)^{O_k(A)}$ such that

$$\|q\alpha_j\| \leq \frac{(\log N)^{O_k(A)}}{H^3}$$

for all $1 \leq j \leq k$.

Again note that if $q \approx 1$ and $\|q\alpha_j\| \approx H^{-j}$ for all $j$, then $g(n)$ is almost constant on $(N, N + H]$ after dividing it into residue classes modulo $q$. This is once again a major arc case corresponding to the classical prime number theorem in short intervals.

We end the introduction by mentioning a few related results. In this paper we focus on a fixed short interval, but one can also ask the same question for almost all short intervals. For example, Huxley’s zero density estimate implies that one can count primes in almost all short intervals of length $H = N^\theta$ with $\theta > 1/6$. In this direction, Matomäki and Radziwiłł [10] made the breakthrough showing that

$$\sum_{n_0 < n \leq n_0 + H} \mu(n) = o(H)$$
for almost all \(n_0 \sim N\), provided that \(H = H(N) \to \infty\). For the degree 1 case involving exponential sums, Matomäki, Radziwiłł, and Tao \[11\] showed that
\[
\sup_{\alpha \in \mathbb{R}} \left| \sum_{n_0<n \leq n_0+H} \mu(n)e(\alpha n) \right| = o(H)
\]
for almost all \(n_0 \sim N\), provided that \(H = N^\theta\) for any fixed \(\theta > 0\). Unfortunately these results do not apply to \(\Lambda\).

The rest of this paper is organized as follows. In Section 2 we outline the general structure of our argument, which naturally splits into a “minor arc” case and a “wide major arc” case. We also explain the reasons behind having worse threshold for \(\theta\) in the case of general \(g\), and the lack of results for general nilsequences \(\psi\). In Section 3 we show how the minor arc case and the wide major arc case are combined to yield Theorems 1.1 and 1.3. In Sections 4 and 5, we deal with the wide major arc case and the minor arc case, respectively. Finally in Section 6, we deduce Theorem 1.2 via the circle method.

2. Overview of proof

From now on we will always write \(H = N^\theta\) and all implied constants are allowed to depend on the degree \(k\). The argument of proving Theorems 1.1 and 1.3 is divided into two cases, usually referred to as the minor arc case and the wide major arc case in the literature.

**Proposition 2.1** (Minor arc). Fix \(\theta > 2/3\). Let \(\delta \in (0,1/2)\), with \(\delta \geq N^{-c(\theta - 2/3)}\) for some sufficiently small constant \(c = c(k) > 0\). Let
\[
g(n) = \sum_{j=1}^{k} \alpha_j (n - N)^j
\]
be a polynomial of degree \(k\). If
\[
\sum_{N<n \leq N+H} \Lambda(n)e(g(n)) \geq \delta H (\log N)^4,
\]
then there exists a positive integer \(q \leq \delta^{-O_h(1)}\) such that
\[
\|q(j\alpha_j + (j+1)N\alpha_{j+1})\| \leq \delta^{-O_h(1)} \frac{N}{H^{j+1}}
\]
for all \(1 \leq j \leq k\), with the convention that \(\alpha_{k+1} = 0\).

This will be proved in Section 5 using Vaughan’s identity. The Diophantine information in the conclusion of Proposition 2.1 is perhaps unexpected, but from the argument in Section 5 one can see that if
\[
\|j\alpha_j + (j+1)N\alpha_{j+1}\| \approx \frac{N}{H^{j+1}}
\]
for all \(j\), then the type-II sum could be large:
\[
\sum_{N<\ell m \leq N+H} a_\ell b_m e(g(\ell m)) \approx H
\]
for certain coefficients \(\{a_\ell\}\) and \(\{b_m\}\). In other words, the conclusion in Proposition 2.1 is the best one can extract out of using Vaughan’s identity, unless one uses more precise information about the coefficients \(\{a_\ell\}\) and \(\{b_m\}\).

Specializing Proposition 2.1 to the case \(g(n) = \alpha n^k\), we will obtain in Section 5 the following corollary, which should be compared with [7, Theorem 1] where \(\theta > 3/4\) is required.

**Corollary 2.2** (Minor arc). Fix \(\theta > 2/3\). Let \(\delta \in (0,1/2)\), with \(\delta \geq N^{-c(\theta-2/3)}\) for some sufficiently small constant \(c = c(k) > 0\). Let \(\alpha \in \mathbb{R}\). If

\[
\left| \sum_{N < n \leq N + H} \Lambda(n)e(\alpha n^k) \right| \geq \delta H (\log N)^4,
\]

then there exists a positive integer \(q \leq \delta^{-O_k(1)}\) such that

\[
\|q\alpha\| \leq \delta^{-O_k(1)} \frac{1}{N^{k-2}H^2}.
\]

Now we turn to the wide major arcs.

**Proposition 2.3** (Wide major arcs). Fix \(\theta > \theta_k\), where

\[
\theta_k = \begin{cases} 
5/8 & k \leq 2, \\
23/32 & k = 3, \\
(2k+2)/(2k+5) & k \geq 4. 
\end{cases}
\]

Fix \(A \geq 2\). Let \(\chi (\text{mod } q)\) be a Dirichlet character with \(q \leq (\log N)^A\). Then for any \(|t| \leq (N/H)^{k+1}(\log N)^A\), we have

\[
\left| \sum_{N < n \leq N + H} \Lambda(n)\chi(n)n^it \right| \ll_A \frac{H}{(\log N)^A},
\]

unless \(|t| \leq N(\log N)^A/H\) and \(\chi = \chi_0\) is the principal character.

This will be proved in Section 4 using zero density estimates. Let us remark here how this is related to exponential sums. By Taylor expansion, one can show that if \(|t| \approx (N/H)^{k+1}\), then \(n^it \approx e(g(n))\) on \((N, N + H)\) for a certain polynomial \(g\) of degree \(k\). Moreover, the coefficients of \(g\) satisfy the conclusion of Proposition 2.1 (for details see Section 4). This means that, in order to estimate (1.1), it is necessary to estimate sums over primes twisted by \(n^it\) for \(t\) up to roughly \((N/H)^{k+1}\). Moreover, the cases handled by Propositions 2.1 and 2.3 are exactly complementary to each other, and hence they can be combined to yield Theorems 1.1 and 1.3.

Proposition 2.3 becomes weaker as \(k\) gets larger, as we will need control on the zeros of \(L(s, \chi)\) in a vertical strip of the from \(|\text{Im } s - T^{k+1}| \leq T\) where \(T \approx N/H\). In the special case \(g(n) = \alpha n^k\), it turns out that \(\alpha n^k\) can only resemble \(n^it\) on \((N, N + H)\) when \(|t|\) is up to roughly \((N/H)^2\). Thus this special case only requires the \(k = 1\) case of Proposition 2.3.

We end this section by speculating on what happens with general nilsequences \(\psi\). We expect that the following rough statement can be proved by our minor arc argument, using the quantitative Leibman theorem due to
Green and Tao [5, Theorem 2.9] in place of Weyl’s inequality. See [5] for the precise definitions of the terms below.

Let \( G/\Gamma \) be a nilmanifold, \( g \) be a polynomial sequence on \( G \), and \( \varphi \) be a smooth function on \( G/\Gamma \). Let \( \psi(n) = \varphi(g(n)\Gamma) \). If

\[
\left| \sum_{N < n \leq N + H} \Lambda(n) \psi(n) \right| \approx H,
\]

then there is a nontrivial horizontal character \( \chi \) on \( G \) (with bounded modulus), such that the coefficients of the polynomial

\[
\chi \circ g(n) = \sum_{j=1}^{k} \alpha_j (n - N)^j
\]
satisfies

\[
\| j \alpha_j + (j + 1)N \alpha_{j+1} \| \leq \frac{N}{H^{j+1}}
\]

for all \( 1 \leq j \leq k \). This is the same as saying that the polynomial \( \chi \circ g(n) \) is roughly the same as \( n^k \) on \((N, N+H]\), but we do not know how to use this information to say something about the nilsequence \( \psi \).

3. Combining the wide major arc and the minor arc estimates

**Proof of Theorem 1.3 assuming Propositions 2.1 and 2.3.** We start by applying Proposition 2.1 with \( \delta = (\log N)^{-A-4} \) to find a positive integer \( q \leq (\log N)^{O(A)} \) such that

\[
\| q(j \alpha_j + (j + 1)N \alpha_{j+1}) \| \leq (\log N)^{O(A)} \frac{N}{H^{j+1}}
\]

for all \( 1 \leq j \leq k \), with the convention that \( \alpha_{k+1} = 0 \). Let \( B = CA \) for some sufficiently large constant \( C = C(k) \), and let \( H_0 = H(\log N)^{-B} \). We divide \((N, N+H]\) into arithmetic progressions of the form

\[
P = \{ n_0 < n \leq n_0 + H_0 : n \equiv a \pmod{k!q} \},
\]

where \( n_0 \in [N, N+H] \) and \( (a, k!q) = 1 \). Our hypothesis implies that for at least one such progression \( P \), we have

\[
\left| \sum_{n \in P} \Lambda(n)e(g(n)) \right| \gg \frac{|P|}{(\log N)^A}.
\]

For the remainder of the proof we fix such a progression \( P \). We claim that there exists some \( \eta \) with \( |\eta| = 1 \) and some \( t \) with \( |t| \leq (N/H)^{k+1}(\log N)^{O(A)} \), such that

\[
e(g(n)) = \eta n^t (1 + O((\log N)^{-2A}))
\]

for all \( n \in P \). To see this, first write

\[
g(n) = \sum_{j=1}^{k} \alpha_j (n - N)^j = \sum_{j=1}^{k} \beta_j (n - n_0)^j,
\]

so that

\[
\beta_j = \sum_{i=j}^{k} \binom{i}{j} (n_0 - N)^{i-j} \alpha_i.
\]
After some algebra one derives that

\[ j\beta_j + (j + 1)n_0\beta_{j+1} = \sum_{i=j}^{k} \binom{i}{j} (n_0 - N)^{i-j} (i\alpha_i + (i + 1)N\alpha_{i+1}) \]

for all \( 1 \leq j \leq k \), with the convention that \( \beta_{k+1} = 0 \). Hence

\[ \|q(j\beta_j + (j + 1)n_0\beta_{j+1})\| \ll \sum_{i=j}^{k} H^{i-j} (\log N)^{O(A)} \frac{N}{H^{i+1}} \leq (\log N)^{O(A)} \frac{N}{H^{j+1}}. \]

Now shift each \( \beta_j \) by \( (qj)^{-1}a_j \) for an appropriate \( a_j \in \mathbb{Z} \) to get \( \beta'_j \), so that

\[ |q(j\beta'_j + (j + 1)n_0\beta'_{j+1})| \leq (\log N)^{O(A)} \frac{N}{H^{j+1}} \] (3.2)

for all \( 1 \leq j \leq k \). Let

\[ g'(n) = \sum_{j=1}^{k} \beta'_j (n - n_0)^j. \]

Note that for \( n \in P \) we have

\[ e(g(n)) = e(g'(n)) e \left( \sum_{j=1}^{k} \frac{a_j}{q_j} (n - n_0)^j \right) = \eta e(g'(n)), \]

for some \( \eta \) (independent of \( n \)) with \( |\eta| = 1 \), since all \( n \in P \) lie in the same residue class modulo \( q_j \) for each \( j \). By induction one can deduce from (3.2) that

\[ |\beta'_j - \frac{(-1)^{j-1}}{j!n_0^{j-1}} \beta'_1| \leq (\log N)^{O(A)} \frac{1}{H^j} \] (3.3)

for all \( 1 \leq j \leq k + 1 \). In particular when \( j = k + 1 \) this gives

\[ |\beta'_1| \leq (\log N)^{O(A)} \frac{N^k}{H^{k+1}}. \]

Set \( t = 2\pi n_0 \beta'_1 \), so that

\[ |t| \leq (\log N)^{O(A)} \left( \frac{N}{H} \right)^{k+1}. \]

For \( n \in P \) we have

\[ n^it = n_0^it \exp \left( it \log \left( 1 + \frac{n - n_0}{n_0} \right) \right). \]

Using the Taylor expansion

\[ \log \left( 1 + \frac{n - n_0}{n_0} \right) = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \left( \frac{n - n_0}{n_0} \right)^j + O \left( \left( \frac{|n - n_0|}{N} \right)^{k+1} \right), \]

we get

\[ n^it = n_0^it e(g(n)) \left( 1 + O \left( |t| \left( \frac{H_0}{N} \right)^{k+1} \right) \right), \]
where
\[ \tilde{g}(n) = \frac{t}{2\pi} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \left( \frac{n-n_0}{n_0} \right)^j = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j^{n_0-1}} \beta'_1 (n-n_0)^j. \]

The error term above can be made \( O((\log N)^{-2A}) \) by choosing \( B \) in the definition of \( H_0 \) large enough. Hence
\[ n^{it} = n_0^{it} e(\tilde{g}(n))(1 + O((\log N)^{-2A})). \]

Note that for \( n \in P \) we have
\[ |g'(n) - \tilde{g}(n)| \leq \sum_{j=1}^{k} \left| \beta'_j - \frac{(-1)^{j-1}}{j^{n_0-1}} \beta'_1 \right| (n-n_0)^j \leq \sum_{j=1}^{k} (\log N)^{O(A)} \frac{H_0}{H^j}, \]
which can again be made \( \leq (\log N)^{-2A} \) by choosing \( B \) large enough. It follows that
\[ n^{it} = n_0^{it} e(g'(n))(1 + O((\log N)^{-2A})) \]
for \( n \in P \), and hence
\[ e(g(n)) = (\eta n_0^{it}) n^{it} (1 + O((\log N)^{-2A})). \]
This establishes the claim (3.1). It then follows that
\[ \left| \sum_{n \in P} \Lambda(n)n^{it} \right| \gg \frac{|P|}{(\log N)^A}. \]

Decomposing this sum using Dirichlet characters mod \( k!q \), we get
\[ \left| \sum_{n_0 < n \leq n_0 + H_0} \Lambda(n) \chi(n)n^{it} \right| \gg \frac{H_0}{q(\log N)^A} \]
for some \( \chi \pmod{k!q} \). We can ensure that the modulus \( k!q \leq (\log N)^B \), and moreover the lower bound above is \( \gg H_0(\log N)^{-B+1} \). Now Proposition 2.3 is applicable with \( n_0 \) in place of \( N \), \( H_0 \) in place of \( H \), and \( B \) in place of \( A \), since \( \theta \) exceeds the required threshold and
\[ |t| \leq (\log N)^{O(A)} \left( \frac{N}{H^j} \right)^{k+1} \leq (\log N)^B \left( \frac{n_0}{H_0} \right)^{k+1}. \]
Hence we can conclude from Proposition 2.3 that
\[ |t| \leq \frac{n_0}{H_0}(\log N)^B \leq \frac{N}{H}(\log N)^{4B}. \]

By the definition of \( t \), it follows that
\[ |\beta'_1| \leq \frac{(\log N)^{4B}}{H}, \]
and then by (3.3) we get
\[ |\beta'_j| \leq \frac{(\log N)^{4B}}{N^{j-1}H^j} + \frac{(\log N)^{O(A)}}{H^j} \leq \frac{(\log N)^{O(A)}}{H^j}. \]
Hence
\[ \| (k!q)\beta_j \| = \|(k!q)\beta'_j \| \leq \frac{(\log N)^{O(A)}}{H^j}. \]
Finally, using the relation
\[ \alpha_j = \sum_{i=j}^{k} \binom{i}{j} (N - n_0)^{i-j} \beta_i, \]
one arrives at the desired inequality
\[ \| (k!q)\alpha_j \| \leq \sum_{i=j}^{k} \binom{i}{j} H^{i-j} \frac{(\log N)^{O(A)}}{H^i} \leq \frac{(\log N)^{O(A)}}{H^j}. \]

\[ \square \]

**Proof of Theorem 1.1 assuming Corollary 2.2 and Proposition 2.3.** The deduction is very similar as the argument above. By Corollary 2.2 applied with \( \delta = (\log N)^{-A-4} \), there exists a positive integer \( q \leq (\log N)^{O(A)} \) such that
\[ \| q\alpha \| \leq (\log N)^{O(A)} \frac{1}{N^{k-2}H^2}. \]

Let \( B = CA \) for some sufficiently large constant \( C = C(s) \), and let \( H_0 = H(\log N)^{-B} \). Divide \( (N, N+H] \) into arithmetic progressions of the form \( P = \{ n_0 < n \leq n_0 + H_0 : n \equiv a \pmod{q} \} \).

Fix one such progression with the property that
\[ \left| \sum_{n \in P} \Lambda(n)e(\alpha n^k) \right| \gg \frac{|P|}{(\log N)^A}. \]

Shift \( \alpha \) by an appropriate integral multiple of \( 1/q \) to get \( \alpha' \) so that
\[ \| q\alpha' \| \leq (\log N)^{O(A)} \frac{1}{N^{k-2}H^2}. \]

Note that for \( n \in P \) we have
\[ e(\alpha n^k) = \eta e(\alpha' n^k) \]
for some \( \eta \) (independent of \( n \)) with \( |\eta| = 1 \). Set \( t = 2\pi kn_0 \alpha' \), so that
\[ |t| \leq (\log N)^{O(A)} \frac{N^2}{H^2}. \]

Using Taylor expansion as before, we have for \( n \in P \) that
\[ n^{it} = n_0^{it}e(\tilde{g}(n))(1 + O((\log N)^{-2A})), \]
where
\[ \tilde{g}(n) = \frac{t}{2\pi} \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} \left( \frac{n - n_0}{n_0} \right)^j = \sum_{j=1}^{k} \frac{(-1)^{j-1}}{j} kn_0^{k-j} \alpha'(n - n_0)^j. \]

Comparing this with the identity
\[ \alpha' n^k = \sum_{j=1}^{k} \binom{k}{j} n_0^{k-j} \alpha'(n - n_0)^j, \]
we deduce that
\[ |\tilde{g}(n) - \alpha' n^k| \ll \sum_{j=2}^{k} n_0^{k-j} (n - n_0)^j |\alpha'| \ll N^{k-2}H_0^2 |\alpha'| \leq (\log N)^{-2A}. \]
for \( n \in P \). Combining everything together we obtain
\[
e(\alpha n^k) = \eta n^t (1 + O((\log N)^{-2A}))
\]
for all \( n \in P \), where \( \eta \) is a constant independent of \( n \) with \( |\eta| = 1 \). Hence
\[
\left| \sum_{n_0 < n \leq n_0 + H_0} \Lambda(n) \chi(n) n^t \right| \geq \frac{H_0}{q(\log N)^A}
\]
for some Dirichlet character \( \chi \) (mod \( q \)). We now apply Proposition 2.3 as before, but note crucially that our bound on \( |t| \) means that we can apply the \( k = 1 \) case of Proposition 2.3, so that only \( \theta > 5/8 \) is required. It follows that
\[
|t| \leq \frac{N}{H} (\log N)^{4B},
\]
which implies by our definition of \( t \) that
\[
|\alpha'| \leq \frac{(\log N)^{4B}}{N^k - 1 H}.
\]
The conclusion follows since \( \| q\alpha \| = \| q\alpha' \|. \)

4. The wide major arcs: Proof of Proposition 2.3

The method of proof of Proposition 2.3 is rather routine. We use Perron’s formula to reduce matters to obtaining a certain zero density estimate for \( L(s, \chi) \). Because of the twist \( n^t \), we need to control the number of zeros of \( L(s, \chi) \) in a short vertical strip of the form \( |\text{Im } s - t| \leq T \), where \( t \) could be up to \( T^k + 1 \). A good bound for this was obtained by Zhan [14].

We start with the following twisted version of the explicit formula, whose proof is standard.

**Lemma 4.1.** Let \( 2 \leq T \leq x \), \( |t_0| \leq x^{O(1)} \), and \( \sigma_0 \in (0, 1) \). Let \( \chi \) (mod \( q \)) be a Dirichlet character with \( q \leq x \). Let \( \delta(\chi, t_0, T) = 1 \) if \( \chi \) is the principal character and \( |t_0| \leq T \), and let \( \delta(\chi, t_0, T) = 0 \) otherwise. Then
\[
\sum_{n \leq x} \Lambda(n) \chi(n) n^{it_0} = \delta(\chi, t_0, T) \frac{x^{1+it_0}}{1+it_0} - \sum_{\rho} \frac{x^\rho}{\rho} + O \left( \left( \frac{x}{T} + x^{\sigma_0} \right) \log^2 x \right),
\]
where the summation is over those \( \rho \) satisfying \( \sigma_0 \leq \text{Re}(\rho) \leq 1 \), \( |\text{Im}(\rho)| \leq T \), \( L(\rho - it_0, \chi) = 0 \).

**Proof.** By Perron’s formula, we have
\[
\sum_{n \leq x} \Lambda(n) \chi(n) n^{it_0} = \frac{1}{2\pi i} \int_{c-iT}^{c+iT} \left( -\frac{L'(s - it_0, \chi)}{L(s - it_0, \chi)} \right) \frac{x^s}{s} ds + O \left( \frac{x}{T} \log^2 x \right),
\]
where \( c = 1 + 1/\log x \). Consider the contour integral along the rectangle with vertices \( \sigma_0 \pm iT, c \pm iT \). By the residue theorem, the integral along this contour is
\[
\delta(\chi, t_0, T) \frac{x^{1+it_0}}{1+it_0} - \sum_{\rho} \frac{x^\rho}{\rho}.
\]
Using standard bounds for \( L(s, \chi) \), one can show that the contribution from the integrals along the top, bottom, and the left side of this rectangle are
all acceptable, after possibly perturbing $\sigma_0$ and $T$ a little bit to avoid zeros on the boundary. Moreover, the extra and missing terms in the summation over $\rho$ due to this perturbation can be shown to make a contribution of $O(x^{\sigma_0} \log^2 x)$.

\[ \square \]

To prove Proposition 2.3, we apply Lemma 4.1 with $T = (N/H)(\log N)^{A+2}$ and $\sigma_0 < \theta$ to get

\[
\sum_{N < n \leq N + H} \Lambda(n) \chi(n)n^{it} = \delta(\chi, t, T) \frac{(N + H)^{1+it} - N^{1+it}}{1 + it} - \sum_{\rho} \frac{(N + H)^\rho - N^\rho}{\rho} + O(H(\log N)^{-A}),
\]

(4.1)

where the summation is over those $\rho$ satisfying $\sigma_0 \leq \Re \rho \leq 1$, $|\Im \rho| \leq T$, $L(s, \chi) = 0$.

The first term is 0 unless $\chi = \chi_0$, and if $|t| > N(\log N)^A/H$ then the first term is $O(H(\log N)^{-A})$ since the numerator is $O(N)$. Hence it suffices to show that the sum over $\rho$ is $O(H(\log N)^{-A})$. By the fundamental theorem of calculus, we have

\[
\frac{(N + H)^\rho - N^\rho}{\rho} \ll H N^{\Re \rho - 1}
\]

for any $\rho \in \mathbb{C} \setminus \{0\}$. Denote by $N(\sigma, -t, T, \chi)$ the number of zeros of $L(s, \chi)$ in the box

\[
\sigma \leq \Re s \leq 1, \quad -t - T \leq \Im s \leq -t + T.
\]

Then the sum over $\rho$ in (4.1) can be bounded by

\[
H \sum_{\rho} N^{\Re(\rho) - 1} \ll H(\log N) \int_{\sigma_0}^1 N(\sigma, -t, T, \chi) N^{\sigma - 1} \, d\sigma.
\]

From the zero-free region of $L(s, \chi)$, we only need to integrate up to $\sigma_1 = 1 - 1/(\log N)^{0.9}$ (say). Assuming for the moment that for some $c \geq 2$ the estimate

\[
N(\sigma, -t, T, \chi) \ll (qT)^{c(1-\sigma)}(\log qT)^{O(1)}
\]

holds for all $\sigma_0 \leq \sigma \leq 1$, and moreover that

\[
c(1 - \theta) < 1.
\]

Then it follows that the sum over $\rho$ in (4.1) is

\[
\ll H(\log N)^{O(1)} q^c \int_{\sigma_0}^{\sigma_1} \left( \frac{T}{N} \right)^{1-\sigma} \, d\sigma \ll H(\log N)^{O(A)} \int_{\sigma_0}^{\sigma_1} N^{[c(1-\theta)-1](1-\sigma)} \, d\sigma.
\]

The integral above is

\[
\ll N^{[c(1-\theta)-1](1-\sigma_1)} \ll_B (\log N)^{-B}
\]

for any $B \geq 2$. Hence the sum over $\rho$ term in (4.1) is $O(H(\log N)^{-A})$. 

It remains to determine the value of \( c \) we can take. Zhan [14] showed that if \( T \geq |t|^{1/3} \) then we can take
\[
c = \begin{cases} 
8/3 & \text{if } \sigma_0 \leq 3/4, \\
2/\sigma_0 & \text{if } \sigma_0 > 3/4.
\end{cases}
\]
From our assumption on \( |t| \) and our choice of \( T \), we have \( |t| \leq T^{k+1} \).

**Case \( k \leq 2 \).** In this case we have \( |t| \leq T^3 \) and \( \theta > 5/8 \). Thus Zhan’s result directly implies that we can take \( c = 8/3 \) in (4.2), and \( c(1 - \theta) < 1 \) is satisfied.

**Case \( k = 3 \).** Zhan’s result applied with \( T \) replaced by \( T' = \max(T, |t|^{1/3}) \) implies that
\[
N(\sigma, -t, T, \chi) \ll (qT')(8/3)(1-\sigma) (\log qT)^{O(1)}.
\]
Since \( |t| \leq T^4 \), we have \( T' \leq T^{4/3} \), and thus we can take \( c = (4/3) \cdot (8/3) = 32/9 \) in (4.2). The inequality \( c(1 - \theta) < 1 \) is satisfied since \( \theta > 23/32 \).

**Case \( k \geq 4 \).** Since \( \theta > 3/4 \) in this case, we can ensure that \( \sigma_0 > 3/4 \). As before, Zhan’s result applied with \( T \) replaced by \( T' = \max(T, |t|^{1/3}) \) implies that
\[
N(\sigma, -t, T, \chi) \ll (qT')(2/\sigma_0)(1-\sigma) (\log qT)^{O(1)}.
\]
Since \( T' \leq T^{(s+1)/3} \), we can take
\[
c = \frac{s + 1}{3} \cdot \frac{2}{\sigma_0}
\]
in (4.2). The inequality \( c(1 - \theta) < 1 \) is satisfied by taking \( \sigma_0 \) sufficiently close to \( \theta \).

5. **The minor arcs: Proof of Proposition 2.1 and Corollary 2.2**

We need the following reformulation of Weyl’s inequality. This is a direct consequence of [5, Proposition 4.3], and also a special case of a more general quantitative equidistribution result on nilsequences [5, Theorem 2.9].

**Lemma 5.1.** Let \( N \geq 2 \). Let \( g(n) = \alpha_1 n + \cdots + \alpha_k n^k \) be a polynomial of degree \( k \). If
\[
\left| \sum_{n \in I} e(g(n)) \right| \geq \delta N
\]
for some interval \( I \subset \{1, 2, \cdots, N\} \) and some \( \delta \in (0, 1/2) \), then there exists a positive integer \( q \leq \delta^{-O_k(1)} \) such that
\[
\| q\alpha_j \| \leq \delta^{-O_k(1)} \frac{1}{N^j}
\]
for all \( 1 \leq j \leq k \).

We use Vaughan’s identity to reduce proving Proposition 2.1 to estimating a type-I sum and a type-II sum. Here and in the sequel, we use \( m \sim M \) to denote the dyadic range \( M < m \leq 2M \).
Proposition 5.2 (Type-I estimate). Let $M \leq N^{1/3}$, let $|b_m| \leq 1$, and let $\delta \in (0, 1/2)$. Let

$$g(n) = \sum_{j=1}^{k} \alpha_j (n - N)^j$$

be a polynomial of degree $k$. Assume that $H \geq \delta^{-C} M$ for some sufficiently large constant $C = C(k) > 0$. If

$$\sum_{\ell, m : m \sim M, N < \ell m \leq N + H} \psi(\ell) b_m e(g(\ell m)) \geq \delta H \log N$$

for either $\psi(\ell) = 1$ or $\psi(\ell) = \log \ell$, then there exists a positive integer $q \leq \delta^{-O_k(1)}$ such that

$$\|q \alpha_j\| \leq \delta^{-O_k(1)} \frac{1}{H^j}$$

for all $1 \leq j \leq k$.

Proof. From the hypothesis we have

$$\delta H \log N \leq \sum_{m \sim M} \left| \sum_{N/m < \ell \leq (N + H)/m} \psi(\ell) e(g(\ell m)) \right|.$$ 

Denote by $\mathcal{M}$ the set of $m \sim M$ such that

$$\left| \sum_{N/m < \ell \leq (N + H)/m} \psi(\ell) e(g(\ell m)) \right| \gg \frac{\delta H}{M} \log N.$$ 

It follows that $|\mathcal{M}| \gg \delta M$. For $m \in \mathcal{M}$, let $\ell_0 = \lfloor N/m \rfloor$ be the starting point of the range of summation over $\ell$. We can conclude that

$$\left| \sum_{\ell_0 < \ell \leq \ell_1} e(g(\ell m)) \right| \gg \frac{\delta H}{M}$$

for some $\ell_1 \leq (N + H)/m$. Indeed, when $\psi(\ell) = 1$ this is trivial, and when $\psi(\ell) = \log \ell$ this follows from partial summation. We will apply Lemma 5.1 to the shifted sequence $\ell \mapsto g((\ell_0 + \ell)m)$. Note that

$$g((\ell_0 + \ell)m) = \sum_{i=1}^{k} \alpha_i (\ell m + b)^i,$$

where $b = \ell_0 m - N$. The only property we will use about $b$ is the bound $|b| \leq H$. The coefficient of $\ell^j$ in this polynomial is given by

$$\beta_j := \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} m^j b^{i-j}.$$ 

By Lemma 5.1, there exists a positive integer $q \leq \delta^{-O(1)}$ such that

$$\|q \beta_j\| \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^j$$

for all $1 \leq j \leq k$ and $m \in \mathcal{M}$. Below we will allow ourselves to enlarge $q$ by multiplying it with a positive integer at most $\delta^{-O(1)}$, and this process will
be done $O(1)$ times, so the bound $q \leq \delta^{-O(1)}$ will remain to hold in the end. 
We will show by induction the desired Diophantine information on $\alpha_j$:

$$\|qa_j\| \leq \delta^{-O(1)} \frac{1}{H}$$

for all $1 \leq j \leq k$.

The base case $j = k$. Since $\beta_k = \alpha_k m^k$, we have

$$\|qa_k m^k\| \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^k.$$

This holds for $\gg \delta M$ values of $m \sim M$. Hence by [5, Lemma 4.5] (which is applicable by our assumption that $M/H \leq \delta^C$), we conclude that

$$\|qa_k\| \leq \delta^{-O(1)} \frac{1}{H^k}$$

as desired.

The induction step. Now let $1 \leq j < k$, and assume that the claim has already been proved for larger values of $j$. Then

$$\|q(\beta_j - \alpha_j m^j)\| = \left\| q \sum_{j<i \leq k} \alpha_i \binom{i}{j} m^i b^{i-j} \right\|$$

$$\ll M^j \sum_{j<i \leq k} |b|^{i-j} \|qa_i\| \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^j.$$

It follows that

$$\|qa_j m^j\| \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^j.$$

This holds for $\gg \delta M$ values of $m \sim M$. Hence by [5, Lemma 4.5] (which is again applicable by our assumption that $M/H \leq \delta^C$), we conclude that

$$\|qa_j\| \leq \delta^{-O(1)} \frac{1}{H^j}.$$

This completes the proof. $\square$

**Proposition 5.3** (Type-II estimate). Let $M \in [N^{1/3}, N^{2/3}]$ and $L = N/M$, let $|a_\ell| \leq \tau(\ell)$, $|b_m| \leq 1$, and let $\delta \in (0, 1/2)$. Let

$$g(n) = \sum_{j=1}^k \alpha_j (n - N)^j$$

be a polynomial of degree $k$. Assume that $H \geq \delta^{-C} \max(L, M)$ for some sufficiently large constant $C = C(k) > 0$. If

$$\sum_{\ell, m \sim M \atop N < \ell m \leq N + H} a_\ell b_m e(g(\ell m)) \geq \delta H (\log N)^2,$$

then there exists a positive integer $q \leq \delta^{-O_k(1)}$ such that

$$\|q(j \alpha_j + (j+1)N \alpha_{j+1})\| \leq \delta^{-O_k(1)} \frac{N}{H^{j+1}}.$$
for all \(1 \leq j \leq k\), with the convention that \(\alpha_{k+1} = 0\).

**Proof.** From the hypothesis we have

\[
\delta H (\log N)^2 \leq \sum_{\substack{m, m' \sim M \atop |m - m'| \leq 2H/L}} \tau(\ell) \left| \sum_{\substack{m \sim M \atop N < \ell m \leq N + H}} b_m e(g(\ell m)) \right|.
\]

By the Cauchy-Schwarz inequality, we have

\[
\delta^2 H^2 \ll L \sum_{\substack{m, m' \sim M \atop |m - m'| \leq 2H/L}} \left| \sum_{\substack{m \sim M \atop N < \ell m \leq N + H}} b_m e(g(\ell m)) \right|^2.
\]

Expanding the square and changing the order of summation, we obtain

\[
\sum_{\substack{m, m' \sim M \atop |m - m'| \leq 2H/L}} \left| \sum_{\substack{m \sim M \atop N < \ell m, \ell m' \leq N + H}} e(g(\ell m) - g(\ell m')) \right| \gg \delta^2 H^2.
\]

We will restrict \(m, m'\) to intervals of length \(2H/L\). Let \(J = [m_0, m_0 + 2H/L]\) for some \(m_0 \sim M\). Then the inequality

\[
\sum_{m, m' \sim J} \left| \sum_{\substack{m \sim M \atop N < \ell m, \ell m' \leq N + H}} e(g(\ell m) - g(\ell m')) \right| \gg \delta^2 \left( \frac{H}{L} \right)^2 \frac{H}{M}
\]

holds for \(\gg \delta^2 M\) choices of \(m_0 \sim M\). For the moment we fix one such choice of \(m_0\) and \(J\), but towards the end of the argument we will allow \(m_0\) to vary. Denote by \(\mathcal{M}\) the set of all pairs \((m, m') \in J \times J\) such that

\[
\sum_{\substack{m, m' \sim J \atop |m - m'| \leq 2H/L \atop N < \ell m, \ell m' \leq N + H}} e(g(\ell m) - g(\ell m')) \gg \delta^2 \frac{H}{M}.
\]

It follows that \(|\mathcal{M}| \gg \delta^2 |J|^2\). For \((m, m') \in \mathcal{M}\), let \(I_{m, m'}\) be the range of summation for \(\ell\):

\[
I_{m, m'} = \{ L/2 \leq \ell \leq 2L : N < \ell m, \ell m' \leq N + H \}.
\]

Note that all of these \(I_{m, m'}\) are contained in a common interval \(I = [\ell_0 + 1, \ell_0 + |I|]\) of length \(|I| = O(H/M)\) for some \(\ell_0 = N/m_0 + O(H/M)\), which depends on \(m_0\) but not on \(m, m'\). We are now in a position to apply Lemma 5.1 to the shifted sequence

\[
\ell \mapsto g((\ell_0 + \ell)m) - g((\ell_0 + \ell)m').
\]

Note that

\[
g((\ell_0 + \ell)m) - g((\ell_0 + \ell)m') = \sum_{i=1}^k \alpha_i ((\ell m + b)^i - (\ell m' + b')^i),
\]
where $b = \ell_0 m - N$, $b' = \ell_0 m' - N$ (bear in mind the dependence of $b, b'$ on $m, m'$). The coefficient of $\ell^j$ in this polynomial is given by

$$\beta_j(m, m') := \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} (m^j b^{i-j} - m'^j b'^{i-j}).$$

By Lemma 5.1, there exists a positive integer $q \leq \delta^{-O(1)}$ such that

$$\|q\beta_j(m, m')\| \leq \delta^{-O(1)} \frac{1}{|J|} \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^j$$

for all $1 \leq j \leq k$ and $(m, m') \in M$. In the rest of the arguments we will always allow ourselves to enlarge $q$ by multiplying it with a positive integer at most $\delta^{-O(1)}$, and this process will be done $O(1)$ times so that the bound $q \leq \delta^{-O(1)}$ will remain to hold in the end. Let

$$\gamma_j(m) = m^j \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} (\ell_0 m - N)^{i-j},$$

so that $\beta_j(m, m') = \gamma_j(m) - \gamma_j(m')$. The Diophantine information on $\beta_j(m, m')$ implies that $\|q\gamma_j(m)\|$ lies in an arc of length $\delta^{-O(1)} (M/H)^j$ for $\gg \delta^2 |J|$ values of $m \in J$, and thus $\|q\gamma_j(m_0 + m)\|$ lies in an arc of length $\delta^{-O(1)} (M/H)^j$ for $\gg \delta^2 H/L$ values of $1 \leq m \leq 2H/L$. Using this we will obtain the desired Diophantine information:

$$\|q(j \alpha_j + (j + 1) N \alpha_{j+1})\| \leq \delta^{-O(1)} \frac{N}{H^{j+1}}$$

by induction on $j$.

The base case $j = k$. Note that

$$\gamma_k(m_0 + m) = \alpha_k (m_0 + m)^k.$$

As a polynomial in $m$, its linear coefficient is $km_0^{k-1} \alpha_k$. By [5, Lemma 4.5] (which is applicable by our assumption that $M/H \leq \delta^C$ for some sufficiently large $C$), we deduce that

$$\|q km_0^{k-1} \alpha_k\| \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^k \frac{L}{H}.$$

Recall that this holds for $\gg \delta^2 M$ values of $m_0 \sim M$, so by [5, Lemma 4.5] again (which is again applicable by our assumption that $M/H, L/H \leq \delta^C$), we conclude that

$$\|q k \alpha_k\| \leq \delta^{-O(1)} \frac{ML}{H^{k+1}} = \delta^{-O(1)} \frac{N}{H^{k+1}},$$

as desired.

The induction step. Now let $1 \leq j < k$, and assume that the claim has already been proved for larger values of $j$. Note that

$$\gamma_j(m_0 + m) = (m_0 + m)^j \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} (\ell_0 m + h_0)^{i-j},$$

Recall that this holds for $\gg \delta^2 M$ values of $m_0 \sim M$, so by [5, Lemma 4.5] again (which is again applicable by our assumption that $M/H, L/H \leq \delta^C$), we conclude that

$$\|q k \alpha_k\| \leq \delta^{-O(1)} \frac{ML}{H^{k+1}} = \delta^{-O(1)} \frac{N}{H^{k+1}},$$

as desired.
where \( h_0 = \ell_0 m_0 - N \) satisfies \(|h_0| = O(H)\). As a polynomial in \( m \), its linear coefficient is
\[
\lambda = \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} (i - j) \ell_0^{i-j} h_0^{i-j-1} + j m_0^{i-1} \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} h_0^{i-j},
\]
and by [5, Lemma 4.5] we have
\[
\|q\lambda\| \leq \delta^{-O(1)} \left( \frac{M}{H} \right)^j \left( \frac{L}{H} \right).
\]

The expression for \( \lambda \) can be rewritten as
\[
\lambda = \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} ((i - j) \ell_0 m_0 h_0^{i-j-1} + j h_0^{i-j}).
\]
Since \( \ell_0 m_0 = h_0 + N \), we have
\[
\lambda = \sum_{j \leq i \leq k} \alpha_i \binom{i}{j} ((i - j) N h_0^{i-j-1} + i h_0^{i-j}).
\]
By regrouping terms according to the exponent of \( h_0 \) we obtain
\[
\lambda = \sum_{j \leq i \leq k} h_0^{i-j} \left( i \alpha_i \binom{i}{j} + (i + 1 - j) N \alpha_{i+1} \binom{i+1}{j} \right) = \sum_{j \leq i \leq k} h_0^{i-j} \binom{i}{j} (i \alpha_i + (i + 1) N \alpha_{i+1}).
\]
By induction hypothesis, we know that all summands above with \( i > j \) are bounded by
\[
\ll H^{i-j} \delta^{-O(1)} \frac{N}{H^{j+1}} \leq \delta^{-O(1)} \frac{N}{H^{j+1}},
\]
and thus
\[
\|q\lambda\| = \|q m_0^{j-1} (j \alpha_j + (j + 1) N \alpha_{j+1})\| + O \left( \delta^{-O(1)} \frac{M^{j-1} N}{H^{j+1}} \right).
\]
Hence the bound on \( \|q\lambda\| \) implies that
\[
\|q m_0^{j-1} (j \alpha_j + (j + 1) N \alpha_{j+1})\| \leq \delta^{-O(1)} \frac{M^{j-1} N}{H^{j+1}}.
\]
Since this is true for \( \gg \delta^2 M \) values of \( m_0 \sim M \), another application of [5, Lemma 4.5] leads to
\[
\|q (j \alpha_j + (j + 1) N \alpha_{j+1})\| \leq \delta^{-O(1)} \frac{N}{H^{j+1}},
\]
as desired. This completes the proof. \( \square \)

**Proof of Proposition 2.1.** Suppose that
\[
\left| \sum_{N < n \leq N + H} \Lambda(n) e(g(n)) \right| \geq \delta H (\log N)^4.
\]
By Vaughan’s identity (see [9, Proposition 13.4]), at least one of the following three cases must occur:
(1) (Type-I) For some $M \leq N^{1/3}$ and $|b_m| \leq 1$, we have
\[ \sum_{\ell,m} b_m e(g(\ell m)) \gg \delta \log N. \]

(2) (Type-I) For some $M \leq N^{1/3}$ and $|b_m| \leq 1$, we have
\[ \sum_{\ell,m} e(\log \ell) b_m e(g(\ell m)) \gg \delta \log N. \]

(3) (Type-II) For some $M \in \left[ N^{1/3}, N^{2/3} \right]$, $|a_{\ell}| \leq \tau(\ell)$, $|b_m| \leq 1$, we have
\[ \sum_{\ell,m} a_{\ell} b_m e(g(\ell m)) \gg \delta (\log N)^2. \]

Type-I case. In either case (1) or (2), Proposition 5.2 implies that there exists a positive integer $q \leq \delta - O(1)$ such that
\[ \| q \alpha_j \| \leq \delta - O(1) \frac{1}{H^j} \]
for all $1 \leq j \leq k$. This immediately implies that
\[ \| q(j \alpha_j + (j + 1) N \alpha_{j+1}) \| \leq \delta - O(1) \frac{N}{H^{j+1}}. \]

Type-II case. In case (3), the desired conclusion follows from Proposition 5.3, which is applicable because
\[ \delta - C \leq N^{\theta - 2/3} \leq \min(H/L, H/M). \]

This completes the proof. \(\square\)

Proof of Corollary 2.2. If we write
\[ g(n) = \alpha n^k = \sum_{j=1}^{k} \alpha_j (n - N)^j, \]
then $\alpha_j = \binom{k}{j} N^{k-j} \alpha$, and thus
\[ j \alpha_j + (j + 1) N \alpha_{j+1} = \binom{k}{j} N^{k-j} \alpha. \]

Hence Proposition 2.1 implies that there exists a positive integer $q \leq \delta - O(1)$ such that
\[ \left\| q k \binom{k}{j} N^{k-j} \alpha \right\| \leq \delta - O(1) \frac{N}{H^{j+1}} \] (5.1)
for all $1 \leq j \leq k$. Let $q'$ be the least common multiple of $q k \binom{k}{j}$ ($1 \leq j \leq k$), so that $q' \leq \delta - O(1)$. We will show by induction on $j$ that
\[ \| q' \alpha \| \leq \delta - O(1) \frac{1}{N^{k-1-j} H^{j+1}} \]
for all $1 \leq j \leq k$, and the conclusion follows from the $j = 1$ case of this.
When $j = k$, the claim follows from (5.1) with $j = k$. Now let $1 \leq j < k$,
and assume that the claim has already been proven for \( j + 1 \). The induction hypothesis implies that
\[
N^{k-j} \|q' \alpha\| \leq 2 \delta^{-O(1)} \frac{N^{k-j}}{H^{j+2}} \leq 2 \delta^{-O(1)} \frac{N^2}{H^{j+2}} < \frac{1}{2}
\]
by the assumption on \( \delta \). Combining this with
\[
\|N^{k-j} q' \alpha\| \leq 2 \delta^{-O(1)} N^{-1}
\]
from (5.1) leads to
\[
N^{k-j} \|q' \alpha\| \leq 2 \delta^{-O(1)} \frac{N}{H^{j+1}},
\]
which completes the induction step. ⧫

6. Application to the Waring-Goldbach problem

Now that we are equipped with the exponential sum estimate Theorem 1.1, we can deduce Theorem 1.2 via the circle method. In this section we sketch this standard deduction. Let \( X = \left( \frac{N}{s} \right)^{1/k} \), \( H = X^\theta \), and let
\[
f(\alpha) = \sum_{|n-X| \leq H} \Lambda(n)e(\alpha n^k).
\]
Then the (weighted) number of ways to write
\[
N = p_1^k + \cdots + p_s^k
\]
with \( p_1, \ldots, p_s \) primes satisfying \( |p_i - X| \leq X^\theta \) is
\[
\rho(N) = \int_0^1 f(\alpha)^s e(-N\alpha) d\alpha.
\]
Set \( Q = (\log N)^A \) for a sufficiently large constant \( A \). For \( 1 \leq a \leq q \leq Q \) with \( (a, q) = 1 \), define
\[
\mathcal{M}(q, a) = \left\{ \alpha \in [0, 1]: |q \alpha - a| \leq \frac{Q}{X^{k-1}H} \right\}.
\]
Let \( \mathcal{M} \) be the union of all such \( \mathcal{M}(q, a) \), and let \( \mathcal{M} \) be the complement \( [1/(X^{k-1}H), 1 + 1/(X^{k-1}H)] \setminus \mathcal{M} \). We caution that our definition of \( \mathcal{M} \) here consists only of the genuine major arcs, while the definition of \( \mathcal{M} \) in [12, Section 2] consists also of the wide major arcs. We have \( \rho(N) = \rho(N; \mathcal{M}) + \rho(N; \mathcal{M}) \), where
\[
\rho(N; \mathcal{M}) = \int_{\mathcal{M}} f(\alpha)^s e(-N\alpha) d\alpha, \quad \rho(N; \mathcal{M}) = \int_{\mathcal{M}} f(\alpha)^s e(-N\alpha) d\alpha.
\]
Theorem 1.2 follows once we show that
\[
\rho(N; \mathcal{M}) \gg \frac{H^{s-1}}{X^{k-1}}, \quad \rho(N; \mathcal{M}) = o \left( \frac{H^{s-1}}{X^{k-1}} \right).
\]
Analysis of $\rho(N; \mathfrak{M})$. The width of our major arc is chosen so that if $\alpha \in \mathfrak{M}(q, a)$, then $f(\alpha)$ can be estimated by counting primes in short intervals in residue classes modulo $q$. Since $\theta > 7/12$, we may use Huxley’s result on primes in short intervals \([8]\) to get

$$f\left(\frac{a}{q} + \beta\right) = \varphi(q)^{-1} S(q, a) v(\beta) + O\left(\frac{H}{(\log X)^{10}}\right)$$

for $|\beta| \leq Q/(X^{k-1}H)$, where

$$S(q, a) = \sum_{1 \leq b \leq q \atop (b, q) = 1} e\left(\frac{ab^k}{q}\right),$$

and

$$v(\beta) = k^{-1} \sum_{(X-H)^s \leq m \leq (X+H)^s} m^{-1+1/k} e(\beta m).$$

From this point on, the standard theory of the major arc contributions in the Waring-Goldbach problem can be applied to yield the estimate

$$\rho(N; \mathfrak{M}) = \mathfrak{G}(N) \mathfrak{J}(N) + O\left(\frac{H^{s-1}}{X^{k-1}(\log X)^{10}}\right),$$

where $\mathfrak{G}(N)$ is the singular series

$$\mathfrak{G}(N) = \sum_{q=1}^{\infty} \varphi(q)^{-s} \sum_{1 \leq a \leq q \atop (a, q) = 1} S(q, a)^s e(-aN/q),$$

and $\mathfrak{J}(N)$ is the singular integral

$$\mathfrak{J}(N) = \int_{0}^{1} v(\beta)^s e(-\beta N) d\beta.$$

See \([12, \text{Section 2}]\) and the references therein. Moreover, under the assumption on $s$ and the congruence condition on $N$, it can be shown that

$$\mathfrak{G}(N) \asymp 1, \quad \mathfrak{J}(N) \asymp \frac{H^{s-1}}{X^{k-1}}.$$

Hence $\rho(N; \mathfrak{M}) \gg H^{s-1}/X^{k-1}$ as desired.

Analysis of $\rho(N; \mathfrak{m})$. Let $t = k(k + 1)/2 + 1$ and choose $B > 2t/(s - 2t)$. Since $A$ can be chosen sufficiently large in terms of $B$, Theorem 1.1 implies that $|f(\alpha)| \leq H(\log X)^{-B}$ for $\alpha \in \mathfrak{m}$. Thus

$$\rho(N; \mathfrak{m}) \ll \left(\frac{H}{(\log X)^B}\right)^{s-2t} \int_{0}^{1} |f(\alpha)|^{2t} d\alpha.$$

It suffices to establish the following mean value estimate:

$$\int_{0}^{1} |f(\alpha)|^{2t} \ll \frac{H^{2t-1}}{X^{k-1}} (\log X)^{2t}.$$

This is basically \([12, \text{Proposition 2.2}]\); we just need to apply the Vinogradov mean value theorem without the $X^\varepsilon$ loss. Let

$$F(\alpha) = \sum_{|n-X| \leq H} e(\alpha n^k).$$
By considering the underlying Diophantine equations, we get
\[ \int_0^1 |f(\alpha)|^{2t} d\alpha \ll (\log X)^{2t} \int_0^1 |F(\alpha)|^{2t} d\alpha. \]

An argument of Daemen [2] (see [12, Lemma 3.1]) shows that
\[ \int_0^1 |F(\alpha)|^{2t} d\alpha \ll \frac{H^{k(k+1)/2-1}}{X^{k-1}} J_{t,k}(H), \]
where \( J_{t,k}(H) \) is the number of integral solutions to the system of Diophantine equations
\[ x_1^j + \cdots + x_t^j = y_1^j + \cdots + y_t^j, \quad 1 \leq j \leq k, \]
with \( 1 \leq x_1, \cdots, x_t, y_1, \cdots, y_t \leq H \). The Vinogradov mean value conjecture (see [1, Section 5]) gives that
\[ J_{t,k}(H) \ll H^{2t-k(k+1)/2} \]
for \( t > k(k+1)/2 \). Combining the inequalities above together, we get
\[ \int_0^1 |f(\alpha)|^{2t} d\alpha \ll (\log X)^{2t} \frac{H^{2t-1}}{X^{k-1}}. \]
Hence \( \rho(N;m) = o(H^{s-1}/X^{k-1}) \) by our choice of \( B \).

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