The solar system mimics a hydrogen atom.

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Abstract

The solar system and the hydrogen atom are two well known systems on different scales and look unrelated: The former is a classical system on the scale of about billions of kilometers and the latter a quantum system of about tens of picometers. Here we show a connection between them. Specifically, we find that the orbital radii of the planets mimic the mean radii of the energy levels of a quantum system under the Coulomb-like potential. This connection might be explained by very light dark matter which manifests quantum behavior in the solar system, thereby hinting at a dark matter mass around $8 \times 10^{-14}$ electron-volts.

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FIG. 1: The solar system mimics a hydrogen atom. The orbital radii of the planets are shown in the middle row. The gray levels illustrate the mean radii of thirty \( s \)-state energy levels of a hydrogen atom which has been enlarged about \( 8 \times 10^{19} \) times. The eight numbers in the top row are the quantum numbers of the energy levels which the planets conform to. The ticks and the numbers at the bottom denote the distance from the center (the tip on the left) in unit of AU; the distance below 1AU is on a linear scale and that beyond 1AU on a log scale. (1AU \( \simeq 1.5 \times 10^{11} \) meters.)

For a quantum system under the Coulomb-like potential \( V(r) \propto -1/r \) such as a hydrogen atom, the mean radius for an energy eigenstate with zero azimuthal quantum number \( (s \text{-state}) \) is proportional to \( n^2 \), where the principle quantum number \( n = 1, 2, \ldots \) (e.g., see [1]). Here we examine whether the solar system [2] has the same feature, regarding the orbital radii of the planets. Specifically, we attempt to find smaller positive integers \( n_i \) \((i = 1, 2, \ldots, 8)\) such that \( n_i^2 r_0 \) are close to the orbital radii of the planets, where the proportionality constant \( r_0 \) is the ground-state radius of the quantum system.

| Planet          | Mercury | Venus | Earth | Mars | Jupiter | Saturn | Uranus | Neptune |
|-----------------|---------|-------|-------|------|---------|--------|--------|---------|
| Mass (10^{24}kg) | 0.330   | 4.87  | 5.97  | 0.642| 1898    | 568    | 86.8   | 102     |
| Orbital radius (10^6km)* | 57.9 | 108.2 | 149.6 | 227.9| 778.6  | 1433.5 | 2872.5 | 4495.1  |
| Radius \( n_i^2 r_0 \) (10^6km)† | 56.8 | 101.0 | 157.9 | 227.3| 764.1  | 1420.9 | 2784.9 | 4603.7  |
| Quantum number \( n_i \) | 3      | 4     | 5     | 6    | 11     | 15     | 21     | 27      |
| Ground-state radius | \( r_0 = 6.315 \times 10^6 \text{km} \) |
| Fractional error (%) | -1.84 | -6.62 | 5.53  | -0.245| -1.86  | -0.880 | -3.05  | 2.41    |
| Root-mean-square of the eight fractional errors: 3.49% |

* The semi-major axis, i.e., the average distance from a planet to the sun.
† The expectation value of the radius for an \( s \)-state energy level.
The conformity of the solar system with a hydrogenlike quantum system is depicted in Fig. 1. The corresponding quantum numbers $n_i$ of the planets are presented in both Fig. 1 and Table I. As shown in the table, the orbital radii of the planets are close to the mean radii $n_i^2 r_0$ for the $s$-state energy levels. The fractional errors between them are several percent or smaller; the root-mean-square (rms) of the eight fractional errors is merely 3.49%. (Henceforth the rms of the fractional errors will simply be termed the error.)

The conformity with the lower energy levels is special and might be a sign of some fundamental physics behind. Note that although the excellent conformity with very large $n_i$ can easily occur—the error can go to zero when $n_i$ go to infinity—, however, the conformity we found with smaller $n_i$ is unusual. We will show its rareness compared to the cases of randomly generated orbital radii. We generate 10,000 sets of eight random radii; for each set we find the best-fit of the positive integers $n_i$ under some low-energy-level condition (to be specified later) and obtain the error. With the distribution of the 10,000 errors we show the smallness of the error in the real case compared to those in the random cases, e.g., via its deviation from the median of the 10,000 errors.

Two kinds of random radii are considered here. First, we attempt to see how the goodness of conformity changes when the planets are relocated to other orbits slightly different from...
FIG. 3: The histogram of the errors in the third case of the upper panel in Fig. 2. The median (8.65%) of the errors is denoted by the white dot, the real-case error (3.49%) by the white triangle, and the 1σ–3σ confidence intervals for the median by the gray regions.

For this case, Fig. 3 plots the distribution of the 10,000 errors together with the confidence intervals (gray regions), the median (white dot) and the real-case error (white triangle). These results show that the conformity of the solar system with a hydrogenlike quantum system is special: A change of the orbital radii of the planets will more likely reduce the

| Changes of orbital radii | Deviation from the median |
|-------------------------|--------------------------|
| < 10%                   | 1.17σ (75.7%CL)          |
| < 20%                   | 1.81σ (92.9%CL)          |
| < 30%                   | 1.92σ (94.5%CL)          |

Fitting condition: \( n_1 \leq 3 \).
FIG. 4: The histogram of the errors in the third case of the lower panel in Fig. 2. The median (24.3%) of the errors is denoted by the white dot, the real-case error (4.42%) by the white triangle, and the $1\sigma$–$3\sigma$ confidence intervals for the median by the gray regions.

Second, we concentrate on the inner planets that are in conformity with the energy levels $n_{1,2,3,4} = 3, 4, 5, 6$. Note that having the conformity of the inner planets with the lower levels is more difficult, therefore more significative, than that of the outer planets with the higher levels. Moreover, as to be shown, the conformity with four levels in a row is truly exceptional. We generate 10,000 sets of four positive random numbers with a uniform probability.\(^1\) For each set, we obtain the best-fit of $n_i$ under the conditions that $n_{1,2,3,4}$ are four positive integers in a row and that $n_1 \leq 10, 5, \text{or } 3$. The results are shown in the lower panel of Fig. 2. The smallness of the real-case error (4.42%) compared to the random ones is even more significant than the previous cases, with the deviation from the median as follows:

\[
\begin{align*}
  n_1 & \leq 10 : & 1.76\sigma & (92.2\%\text{CL}) \\
  n_1 & \leq 5 : & 2.17\sigma & (97.0\%\text{CL}) \\
  n_1 & \leq 3 : & 2.79\sigma & (99.5\%\text{CL})
\end{align*}
\]

As expected, the deviation gets larger when the condition on $n_i$ gets tighter. Even in the case with a rather loose condition, $n_1 \leq 10$, the real-case error has already sat outside the 90% confidence interval. It further goes beyond the 99% confidence interval in the case.

\(^1\) An upper bound to the random numbers is required. However, its value is irrelevant here because it does not change the error but simply changes the proportionality constant $r_0$. 


where \( n_1 \leq 3 \). For this case, Fig. 4 plots the distribution of the 10,000 errors together with the confidence intervals, the median and the real-case error. These results show that it is very rare to have the conformity with four energy levels in a row as good as that of the inner planets of the solar system.

The conformity might be explained by very light dark matter. In cosmology it is widely believed that dark matter helps the formation of the cosmic structures such as galaxies, galaxy clusters etc: The dark matter structures formed beforehand and baryons followed later; i.e., after the recombination of electrons and protons, baryons fell into the gravitational potentials provided by the dark structures. Here we speculate the possible role of dark matter in the formation of the solar system and give a sketch of the scenario. Specifically, we consider the dark matter which is so light that its de Broglie wavelength is on the scale of the solar system and therefore its mass distribution in the solar system manifests quantum behavior.\(^2\)

The mass distribution of the dark matter structure formed beforehand may respect the wave functions of the energy eigenstates. Since the radial probability distribution of an energy eigenstate is peaked roughly around its mean radius, with a smaller width for a lower energy level, the mass density of dark matter is larger around these mean radii of the energy levels. Later, when the embryos of the planets formed via the accretion of dust grains, the denser regions of dark matter may provide nuclei for the accretion; thereby the planetary orbits can be related to the quantum energy levels of dark matter.

The formation of the planets guided by different energy levels may have different fates:

1. The first few levels are too close to the sun to form planets.
2. Each of the next several lower levels helps to form an inner planet.
3. Several of the higher levels together help to form an outer, more massive planet, encircled by a ring system and orbited by many moons.

These three situations are exhibited in Table I: no planet at \( n = 1, 2 \), one planet at each of \( n = 3, 4, 5, 6 \), and one planet every 4–6 higher levels. Note that Fate 3 for the higher levels is possibly due to the wide spread of the probability (therefore the mass) distribution as well as the small energy difference between the nearby levels that makes the transition between them easy to occur.

\(^2\) For even lighter dark matter with its quantum nature manifest at galactic scales or beyond, see, e.g., 3.
To estimate the mass of dark matter, we consider dark matter under the central potential $V(r) = -GM⊙m/r$, where $M⊙$ is the mass of the sun and $m$ the dark matter mass. Under this potential the mean radius for an $s$-state energy level is $(3/2)n^2a_0$, where $a_0 \equiv (GM⊙m^2)^{-1}$ (analogous to the Bohr radius), and accordingly the ground-state radius $r_0 = (2GM⊙m^2/3)^{-1}$. Considering $n_i$ and $r_0$ in Table I as the best-fit with the $s$-state energy levels, we obtain the dark matter mass $m \simeq 8 \times 10^{-14}$ electron-volts.

The conformity of the solar system with a quantum system indicates the quantum nature of the solar system, which is possibly carried by very light dark matter. It suggests the possibility that dark matter or some other quantum source plays an important role in the formation of the planets. It invites the study of the formation and the evolution of the solar system with dark matter taken into consideration. This may give a different story of the solar system. It is also worth investigating the exoplanet systems, e.g., examining the conformity of their orbits with the energy levels of a quantum system and, if explained by dark matter, evaluating its mass. If many planet systems exhibit this conformity and suggest similar mass of dark matter, they will give a strong support to this scenario of the planet formation and to very light dark matter, and can serve as a new probe of dark matter.

[1] R. Eisberg and R. Resnick, “Quantum Physics of Atoms, Molecules, Solids, Nuclei, and Particles,” John Wiley & Sons (1985).

[2] NASA, “Planetary Fact Sheet,” [http://nssdc.gsfc.nasa.gov/planetary/factsheet/](http://nssdc.gsfc.nasa.gov/planetary/factsheet/)

[3] W. H. Press, B. S. Ryden and D. N. Spergel, “Single Mechanism for Generating Large Scale Structure and Providing Dark Missing Matter,” Phys. Rev. Lett. 64, 1084 (1990);
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