Numerical investigations of gravitational collapse

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Abstract. Some properties of a new framework for simulating generic 4-dimensional spherically symmetric gravitating systems are discussed. The framework can be used to investigate spacetimes that undergo complete gravitational collapse. The analytic setup is chosen to ensure that our numerical method is capable to follow the time evolution everywhere, including the black hole region.

1. Introduction
All those people who has far reaching memory about the history of ideas in general relativity may recall the time when cosmic censor hypotheses were formulated by Penrose. The strong cosmic censor hypothesis expresses our expectation that generic spacetimes are globally hyperbolic (Penrose 1979). This, in particular, rules out the formation of Cauchy horizons and, in turn, it is expected to justify that there is no limitation on the predictive power of general relativity. One of the famous drawings by Penrose (see Fig. 1) depicts the diagram of a cosmological situation where spacetime is expanding steadily but black holes are also formed. According to results of Geroch (Geroch 1967, 1970) the base manifold $M$ of such a globally hyperbolic spacetime—i.e.,
the Cauchy development of some three-dimensional achronal hypersurface $\Sigma$—has to possess the product space structure $\Sigma \times \mathbb{R}$. Then the entire spacetime may be foliated by suitably chosen Cauchy surfaces that are diffeomorphic to each other. Whenever black holes are formed the Cauchy slices should bend around them—as it is suggested by the drawing of Penrose—in order to get them close to and simultaneously keep them away from the developing singularity.

* Dedicated to the memory of Péter Csizmadia who was missing—along with three other young Hungarian climbers—at the end of October 2009 while taking part in the first ascent attempt of Ren Zhong Feng in China.
It is worth to recall now that in numerical relativity there is a huge number of simulations where black holes play significant role. However, in most of these numerical investigations, one of the aims was to avoid entering to the trapped regions. This was achieved by making use of techniques of trapped region avoiding slicing and black hole excision (Thornburg 1987, Seidel 1992). The unsatisfactory aspect of this trouble avoiding attitude is that—opposed to the strong cosmic censor hypothesis—we cannot investigate evolution everywhere. This limitation is what we intended to overcome and what was successfully done for the first time in (Csizmadia 2010). In doing so we applied two tricks. The first one provided us the chance to have stable evolution in the trapped region while the second one was applied to get the largest possible part of the spacetime be covered by the time slices. Although the concept of “maximal Cauchy development” was introduced in (Choquet-Bruhat 1969). However, in proving the existence of the largest possible Cauchy development Zorn’s Lemma has to be used. Therefore, it is of crucial importance to have a practical method providing us the largest possible Cauchy development.

2. The dynamical system

For simplicity we consider only the evolution of a real self-interacting scalar field with Lagrangian $\mathcal{L} = \frac{1}{2} g^{\mu \nu} \nabla_{\mu} \psi \nabla_{\nu} \psi - V$ in Einstein’s theory of gravity. Accordingly the energy-momentum tensor is $T_{\alpha \beta} = \nabla_{\alpha} \psi \nabla_{\beta} \psi - g_{\alpha \beta} \left[ \frac{1}{2} g^{\mu \nu} \nabla_{\mu} \psi \nabla_{\nu} \psi - V \right]$ and the field equations are

\begin{equation}
E_{\alpha \beta} = R_{\alpha \beta} - \frac{1}{2} g_{\alpha \beta} R - 8 \pi T_{\alpha \beta} = 0, \quad (1)
\end{equation}

\begin{equation}
\nabla_{\alpha} \nabla^{\alpha} \psi + \frac{\partial V}{\partial \psi} = 0. \quad (2)
\end{equation}

The metric of the 4-dimensional spherically symmetric spacetime $(M, g_{\alpha \beta})$ was chosen to be

\begin{equation}
d s^2 = \alpha \beta^2 d \tau^2 - \alpha d \rho^2 - r^2 \left( d \theta^2 + \sin^2 \theta \ d \phi^2 \right), \quad (3)
\end{equation}

where the coordinates $\tau$ and $\rho$ label the points of the two-dimensional timelike surfaces transverse to the transitivity surfaces of the rotation group, and $\alpha$, $\beta$, $r$ are smooth functions of $(\tau, \rho)$.

Before providing the basic equations satisfied by the dynamical variables $\alpha$, $r$ and $\psi$ let us recall that in spherically symmetric spacetimes any $(1,1)$-type tensor field $E^{\mu \nu}$ must have the structure

\begin{equation}
E^{\mu \nu} = \begin{pmatrix}
E^{\tau \tau} & E^{\tau \rho} & 0 & 0 \\
E^{\rho \tau} & E^{\rho \rho} & 0 & 0 \\
0 & 0 & E^{\theta \theta} & 0 \\
0 & 0 & 0 & E^{\phi \phi}
\end{pmatrix}, \quad (4)
\end{equation}

where $E^{\nu \phi} = E^{\theta \phi}$. If in addition $E_{\mu \nu}$ is symmetric and the metric is given by the line element (3) then we also have $E^{\rho \tau} = -\beta^2 E^{\tau \rho}$. Whenever $E_{\alpha \beta}$ is defined by the r.h.s. of (1), the evolution equations for $\alpha, r$ can be seen to be given by the combinations

\begin{equation}
E^{\phi \phi} + E^{\theta \theta} - E^{\rho \rho} - E^{\tau \tau} = 0, \quad E^{\rho \rho} + E^{\tau \tau} = 0, \quad (5)
\end{equation}

while the constraint equations read as

\begin{equation}
E_{\rho}^{\rho} = 0, \quad E_{\tau}^{\tau} = 0. \quad (6)
\end{equation}

Thus (5) and (6) are equivalent to Einstein’s equations, i.e., requiring the four non-identically zero components of $E^{\mu \nu}$ to vanish. It can also be shown that, if the evolution equations hold,

\begin{equation}
\partial_{\tau} E_{\rho}^{\tau} - \partial_{\rho} E_{\tau}^{\tau} + \frac{2}{r} \left[ r_{\tau} E_{\rho}^{\tau} - r_{\rho} E_{\tau}^{\tau} \right] + \frac{(\alpha_{\tau} \beta + \alpha_{\beta} \tau) E_{\rho}^{\tau} - (2 \alpha_{\rho} \tau + \alpha_{\beta} \rho) E_{\tau}^{\tau}}{\alpha \beta} = 0, \quad (7)
\end{equation}

\begin{equation}
\partial_{\tau} E_{\tau}^{\tau} - \beta^2 \partial_{\rho} E_{\rho}^{\tau} + \frac{2}{r} \left[ r_{\tau} E_{\tau}^{\tau} - \beta^2 r_{\rho} E_{\rho}^{\tau} \right] + \frac{\alpha_{\tau} E_{\tau}^{\tau} - \beta (\alpha_{\rho} \beta + 3 \alpha_{\beta} \rho) E_{\rho}^{\tau}}{\alpha} = 0 \quad (8)
\end{equation}
are satisfied. Since these equations are linear and homogeneous in the variables $E_\tau^\alpha$ and $E_\rho^\tau$, they have identically zero solution for vanishing initial data, i.e., the constraints propagate whence it is completely satisfactory to impose them only on the initial data hypersurface.

2.1. The field equations

The simpleninded use of the above setup leaded to unstable evolutions because close to an origin the expression $\frac{\alpha^2 + \beta^2 r^2}{r^2}$—that appears in (5), with $n = 1$ and $n = 2$, respectively—are of 0/0 type. To cure these instabilities it was crucial to use the Misner-Sharp mass $m$

$$m = \frac{r}{2} \left(1 + g^{ab} \partial_a r \partial_b r\right) = \frac{r}{2} \left(\frac{\alpha \beta^2 + r^2_c - \beta^2 r^2_0}{\alpha \beta^2}\right).$$ (9)

By applying $m$ as an additional new variable the basic field equations could be given as

$$\partial_\tau \alpha_r = \beta^2 \partial_\rho \alpha_\rho + \frac{4m \beta^2 \alpha^2}{r^3} + 8\pi \alpha^2 \beta^2 \left(T_\varphi^\tau + T_\varphi^\rho - T_\tau^\tau - T_\rho^\rho\right) + \frac{\alpha^2}{\alpha} - \frac{\beta^2 \alpha^2}{\beta} + \frac{\beta_\tau \alpha_\tau + \beta^2 \beta_\rho \alpha_\rho}{\beta} + 2\alpha \partial_\rho \beta_\rho,$$ (10)

$$\partial_\tau r_r = \beta^2 \partial_\rho r_\rho + 4\pi \alpha \beta^2 \left(T_\tau^\tau + T_\rho^\rho\right) - \frac{2m \beta^2 \alpha}{r^2} + \frac{\beta_\tau r_\tau + \beta^2 \beta_\rho r_\rho}{\beta},$$ (11)

$$\partial_\tau \rho_\tau = 4\pi r^2 \left(r_\tau T_\rho^\rho - r_\rho T_\tau^\tau\right),$$ (12)

$$\partial_\tau \psi_\tau = \beta^2 \partial_\rho \psi_\rho - \frac{2 \psi_\tau r_\tau - \beta^2 \psi_\rho r_\rho}{\rho} + \frac{\beta_\tau \psi_\tau + \beta^2 \beta_\rho \psi_\rho}{\beta} - \frac{\alpha \beta^2 \partial V}{\partial \psi},$$ (13)

where the abbreviating notation $f_\sigma = \partial_\sigma f$ is applied (for $f = \alpha$, $\beta$, $r$ and $\psi$). Notice that there is no evolution equation for $\beta$.

The full set of evolution equations can be shown (Csizmadia 2010) to possess the form of a first order strongly hyperbolic system $\partial_\tau \mathbf{u} = \mathbf{A} \partial_\rho \mathbf{u} + \mathbf{B}$ for the 10-dimensional vector variable $\mathbf{u} = (m, \alpha, \alpha_r, \alpha_\rho, r, r_\tau, r_\rho, \psi, \psi_\tau, \psi_\rho)^T$ and with a suitable source term $\mathbf{B}$, the nontrivial components of which can be determined by making use of (10), (11), (12) and (13). Note that to these type of evolution equations the existence and uniqueness of solutions is guaranteed.

3. Numerical results

The first order strongly hyperbolic system of evolution equations were solved numerically with our finite difference code called GridRipper AMR, within which the time integration is done by making use of the method of lines based on a fourth order Runge-Kutta scheme. GridRipper does also incorporate the techniques of adaptive mesh refinement [Csizmadia 2007].

3.1. The tricks

Before presenting the our first trick that enabled us to enter to the trapped region let us recall that—as it was shown in (Csizmadia 2010)—a 2-sphere $S$ of radius $r$, that is invariant under the action of the $SO(3)$ symmetry group, is untrapped or trapped (future or past) if and only if $1 - 2m/r$ is positive or negative, respectively. This characterization follows from the fact that the principal null expansions $\theta_\pm$—with respect to the principal null directions $n^\pm_\pm = \left(\frac{\partial}{\partial \tau}\right)^a \pm \beta \left(\frac{\partial}{\partial \rho}\right)^a$—can be related to the Misner-Sharp mass as

$$1 - \frac{2m}{r} = \frac{\beta^2 \rho^2 - r^2_0}{\alpha \beta^2} = - \frac{r^2_0}{4 \alpha \beta^2} \cdot \theta_+ \theta_-.$$ (14)
Our first "trick" was nothing more than keeping the evolution equation (10) for $\alpha$ instead of determining it by the relation $\alpha = (1 - \frac{2m}{r})^{-1} \frac{\beta^2 r^2 - r^2}{\beta^2}$ as in the latter case we had fast growing numerical errors at the marginally trapped surfaces.

The second trick enabling us to get the largest possible Cauchy development was the introduction of a dynamical lapse function. As $\beta$ is freely specifiable we chose it to evolve according to the equation

$$\beta = -p \left[ \frac{m \tau r - 3 m}{r^4} \right] \left( \frac{r \tau r}{r} \right)^3 \beta,$$

where $p$ is a positive real parameter, starting with the initial value $\beta|_{\Sigma_0} = 1$. This way we were able to slow down the evolution at those parts of the time level surfaces which got too close to the singularity providing thereby apparently the largest possible spacetime domain associated with the chosen initial time slices.

3.2. The dynamics of time slices

In representing the yielded spacetime the use of the conformal time $\tau_c = \int_0^\tau \beta d\tau$ turned to be advantageous since the radial null geodesics are well approximated by $\tau_c \pm \rho = \text{const}$ lines. Its definition comes from the fact that along radial null geodesics $\beta d\tau \pm d\rho = 0$ holds. This, along with $d\tau_c = \beta d\tau + \left[ \int_0^\tau \beta d\tau \right] d\rho$ and that the $\tau$-integral of $\beta \rho$ is negligible, implies that in the $\rho - \tau_c$ coordinate plane for the radial null geodesics $d\tau_c/d\rho = \mp 1 + \left[ \int_0^\tau \beta \rho d\tau \right] \approx \mp 1$ holds.

The spacetime diagrams below depict the time evolution relevant for initial data surfaces with topology $\Sigma_0 = \mathbb{R}^3$, $S^3$ and $S^1 \times S^2$. The applied initial data in each of these cases is exactly the same as it was in the corresponding simulations of (Csizmadia P 2010). The main point here is the way the $\tau = \text{const}$ time level surfaces—indicated by the thin dotted lines—bend as the evolution slows down close to the $r = 0$ singularity—represented by the thick dashed lines—while the evolution remains regular everywhere else. The future and past apparent horizons are also indicated by the thick and thin continuous lines.

Acknowledgments

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