A class of exactly solvable rationally extended non-central potentials in Two and Three Dimensions

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Abstract

We start from a seven parameters (six continuous and one discrete) family of non-central exactly solvable potential in three dimensions and construct a wide class of ten parameters (six continuous and four discrete) family of rationally extended exactly solvable non-central real as well as \textit{PT} symmetric complex potentials. The energy eigenvalues and the eigenfunctions of these extended non-central potentials are obtained explicitly and it is shown that the wave eigenfunctions of these potentials are either associated with the exceptional orthogonal polynomials (EOPs) or some type of new polynomials which can be further re-expressed in terms of the corresponding classical orthogonal polynomials. Similarly, we also construct a wide class of rationally extended exactly solvable non-central real as well as complex \textit{PT}-invariant potentials in two dimensions.

1 Introduction

In non-relativistic quantum mechanics the exactly solvable (ES) problems play an important role in the understanding of different quantum mechanical systems associated with any branch of theoretical physics. For many of the quantum mechanical systems, whose exact solutions are unknown, these ES potentials are generally considered as a starting
potential to get their approximate eigenspectrum. Most of the ES potentials are either one dimensional or are central potentials which are essentially one dimensional on the half line.

There are only few examples of exactly solvable non-central potentials such as anisotropic harmonic oscillator whose solutions are well known, see for example \cite{1}. In a comprehensive study, Khare and Bhaduri \cite{2} have discussed a number of exactly solvable non-central potentials by considering the Schrödinger equation in three dimensional spherical polar and two dimensional polar co-ordinates. The classical orthogonal polynomials (such as Hermite, Laguerre and Jacobi polynomial etc) are playing a fundamental role in the construction of the bound state solutions to these ES potentials. The solution of most of the ES non-central potentials is connected with the above well known orthogonal polynomials. It has been observed that the solution of all the non-central potentials obtained in Ref. \cite{2} is associated either with the classical Laguerre or Jacobi orthogonal polynomials.

After the recent discovery of the two new families of orthogonal polynomials namely the exceptional $X_m$ Laguerre and $X_m$ Jacobi orthogonal polynomials \cite{3, 4}, a number of new exactly solvable potentials have been discovered \cite{5, 6, 7, 8, 9, 10, 11, 12, 13, 14}. In most of these cases, these new potentials are the rational extension of the corresponding conventional potentials \cite{15, 16}. Various properties of these new extended potentials have also been studied by different groups \cite{17, 18, 19, 20, 21, 22, 23, 24, 25, 26}. It is then natural to consider the rational extension of the conventional non-central potentials discussed in Ref. \cite{2} and discover new exactly solvable non-central potentials whose solutions are in terms of the rational extension of the Jacobi or Laguerre polynomials. The purpose of this paper is precisely to address this issue.

In this present manuscript, our aim is to obtain the rational extension of all the non-central conventional potentials discussed in Ref. \cite{2}. Further, one of the major development after the work of \cite{2} has been the discovery of $PT$-invariant complex potentials with real energy eigenvalues. The concept of $PT$ (combined parity ($P$) and time reversal ($T$)) symmetric quantum mechanics \cite{27} also plays a crucial role in the understanding of the complex quantum mechanical systems. Bender et.al., \cite{27} have showed that the eigenspectrum of such non-hermitian $PT$ symmetric complex systems are real provided the $PT$-symmetry is not spontaneously broken. The second purpose of this paper is to consider $PT$-symmetric, noncentral, exactly solvable, potentials and obtain their rational extensions. It turns out that the bound state eigenfunctions of some of these potentials are not in the exact form of EOPs rather they are written in the form of some new types of polynomials which can be further written in terms of the corresponding classical orthogonal polynomials.

The paper is organized as follows:

In section 2, we start from the non-central potentials with seven parameters (six continuous and one discrete) in spherical polar co-ordinates and explain how one can obtain its rationally extended solution in terms of ten parameters (six continuous and four discrete) solutions. Various forms of $r$, $\theta$ and $\phi$ dependent potential terms are also mentioned. For an illustration, we discuss in detail in Sec. 2.1 and 2.2 respectively two
examples (one real and one complex and PT symmetric) of ten parameters rationally extended non-central potentials. In particular, we show that the corresponding eigenfunctions are product of the exceptional Jacobi, exceptional Laguerre and/or some type of new orthogonal polynomials. A list of possible forms of $\theta$ and $\phi$ dependent terms with the corresponding eigenfunctions as well as the other parametric relations are given in Tables I and II respectively. Some examples of RE non-central potentials in two dimensional polar co-ordinates are also mentioned in brief in Sec. 3. In particular, we start from the five parameter (four continuous and one discrete) families of non-central potentials in polar coordinates and obtain the corresponding rationally extended non-central potentials with seven parameters (four continuos and three discrete). Finally, we summarize our results in section 4.

2 Non central potential in 3-dimensional spherical polar co-ordinates

In spherical polar coordinates $(r, \theta, \phi)$, consider a non central potential \[ V(r, \theta, \phi) = \tilde{U}(r) + \frac{V(\theta)}{r^2} + \frac{U(\phi)}{r^2 \sin^2(\theta)}. \] (1)

The Schrödinger equation corresponding to this potential i.e., \[ \left[ -\left( \frac{\partial^2 \Psi}{\partial r^2} + \frac{2}{r} \frac{\partial \Psi}{\partial r} \right) - \frac{1}{r^2} \left( \frac{\partial^2 \Psi}{\partial \theta^2} + \cot \theta \frac{\partial \Psi}{\partial \theta} \right) - \frac{1}{r^2 \sin^2 \theta} \frac{\partial^2 \Psi}{\partial \phi^2} \right] = (E - V(r, \theta, \phi)) \Psi, \] (2)

has been solved exactly \cite{2} by using the fact that the eigenfunction can be written in the product form \[ \Psi(r, \theta, \phi) = \frac{R(r)}{r} \frac{\Theta(\theta)}{\sin(\theta)^{1/2}} \Phi(\phi). \] (3)

Using Eq. (3) in Eq. (2), we obtain three exactly solvable uncoupled equations given by

- $- \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + U(\phi) \Phi(\phi) = m^2 \Phi(\phi)$, \[ - \frac{\partial^2 \Phi(\phi)}{\partial \phi^2} + U(\phi) \Phi(\phi) = m^2 \Phi(\phi), \] (4)

- $- \frac{\partial^2 \Theta(\theta)}{\partial \theta^2} + \left[ V(\theta) + \left( m^2 - \frac{1}{4} \right) \csc^2 \theta \right] \Theta(\theta) = \ell^2 \Theta(\theta)$ \[ - \frac{\partial^2 \Theta(\theta)}{\partial \theta^2} + \left[ \tilde{U}(r) + \left( \frac{\ell^2 - 1/4}{r^2} \right) \right] \Theta(\theta) = \ell^2 \Theta(\theta) \] (5)

and

- $- \frac{\partial^2 R(r)}{\partial r^2} + \left[ \tilde{U}(r) + \left( \frac{\ell^2 - 1/4}{r^2} \right) \right] R(r) = ER(r)$ \[ - \frac{\partial^2 R(r)}{\partial r^2} + \left[ \tilde{U}(r) + \left( \frac{\ell^2 - 1/4}{r^2} \right) \right] R(r) = ER(r). \] (6)

By considering different forms of a seven parameter family of potentials, the above three Eqs. (4) - (6) have been solved exactly \cite{2} and in this way one has constructed several non-central potentials $V(r, \theta, \phi)$ by considering different forms of $\tilde{U}(r), U(\theta)$ and $U(\phi)$. 
with the corresponding eigenvalues being $E, \ell^2$ and $m^2$ respectively. The eigenfunctions corresponding to these three equations (4) - (6) are either in terms of classical Laguerre or Jacobi orthogonal polynomials. The complete eigenfunctions are obtained by using Eq. (3). The forms of the potential $V(r, \theta, \phi)$ with their solutions can be found in detail in Ref. [2]. As an illustration, one of the seven parameter family of potential considered in [2] is given by

$$V(r, \theta, \phi) = \frac{\omega^2 r^2}{4} + \frac{\delta}{r^2} + \frac{C}{r^2 \sin^2 \theta} + \frac{D}{r^2 \cos^2 \theta} + \frac{G}{r^2 \sin^2 \theta \sin^2 p\phi} + \frac{F}{r^2 \sin^2 \theta \cos^2 p\phi},$$  \hspace{1cm} (7)

In this work, if we change the potential $V(r, \theta, \phi) \Rightarrow V_{m_1, m_2, m_3}(r, \theta, \phi)$ by redefining the extended form $\tilde{U}(r) \Rightarrow \tilde{U}_{m_1, \text{ext}}(r)$, $V(\theta) \Rightarrow V_{m_2, \text{ext}}(\theta)$, $U(\phi) \Rightarrow U_{m_3, \text{ext}}(\phi)$ i.e.,

$$V(r, \theta, \phi) \Rightarrow V_{m_1, m_2, m_3}(r, \theta, \phi) = \tilde{U}_{m_1, \text{ext}}(r) + \frac{1}{r^2} V_{m_2, \text{ext}}(\theta) + \frac{1}{r^2 \sin^2 \theta} U_{m_3, \text{ext}}(\phi),$$  \hspace{1cm} (8)

and the eigenfunction

$$\Psi(r, \theta, \phi) \Rightarrow \Psi_{m_1, m_2, m_3}(r, \theta, \phi) = \frac{R_{m_1}(r)}{r} \frac{\Theta_{m_2}(\theta)}{(\sin \theta)^{\frac{1}{2}}} \Phi_{m_3}(\phi),$$  \hspace{1cm} (9)

then Eq. (2) becomes

$$\left[ - \frac{1}{R_{m_1}} \frac{\partial^2 R_{m_1}}{\partial r^2} + \tilde{U}_{m_1, \text{ext}}(r) - \frac{1}{4r^2} \right] + \frac{1}{r^2} \left[ - \frac{1}{\Theta_{m_2}^{(h)}} \frac{\partial^2 \Theta_{m_2}^{(h)}}{\partial \theta^2} + V_{m_2, \text{ext}}^{(h)}(\theta) - \frac{1}{4} \text{cosec}^2 \theta \right]$$

$$+ \frac{1}{r^2 \sin^2 \theta} \left[ - \frac{1}{\Phi_{m_3}^{(h)}} \frac{\partial^2 \Phi_{m_3}^{(h)}}{\partial \phi^2} + U_{m_3, \text{ext}}^{(h)}(\phi) \right] = E.$$  \hspace{1cm} (10)

Here $h = I, II$ correspond to real potential while $h = (PT)_1, (PT)_2$ correspond to the complex and $PT$ symmetric potential in $\theta$ and/or $\phi$.

Similar to Eqs. (4)-(6), the above Eq. (10) can also be easily uncoupled into three new exactly solvable equations given by

$$- \frac{\partial^2 \Phi_{m_3}^{(h)}(\phi)}{\partial \phi^2} + U_{m_3, \text{ext}}^{(h)}(\phi) \Phi_{m_3}^{(h)}(\phi) = m^2 \Phi_{m_3}^{(h)}(\phi),$$  \hspace{1cm} (11)

$$- \frac{\partial^2 \Theta_{m_2}^{(h)}(\theta)}{\partial \theta^2} + \left[ V_{m_2, \text{ext}}^{(h)}(\theta) \right] + \left( m^2 - \frac{1}{4} \text{cosec}^2 \theta \right) \Theta_{m_2}^{(h)}(\theta) = \ell^2 \Theta_{m_2}^{(h)}(\theta)$$  \hspace{1cm} (12)
and

\[- \frac{\partial^2 R_m(r)}{\partial r^2} + \left[ \tilde{U}_{m,ext}(r) + \left( \frac{\ell^2 - 1}{4} \right) r^2 \right] R_m(r) = E R_m(r). \quad (13)\]

By knowing the solutions of the above Eqs. (11)-(13), a complete solution of the extended non-central potential \( V_{m_1,m_2,m_3}(r, \theta, \phi) \) can be obtained by using Eq. (9) with the energy eigenvalues \( E \).

We show that there is one choice of \( \tilde{U}_{m,ext}(r) \), four choices of \( V_{m_2,ext}^{(h)}(\theta) \) (two real and two PT symmetric) and three choices of \( U_{m_3,ext}^{(h)}(\phi) \) (two real and one PT symmetric) for which one can obtain exact solutions of the non-central potential. These choices are:

(a) Form of \( \tilde{U}_{m,ext}(r) \):

\[ \tilde{U}_{m,ext}(r) = \frac{\omega^2 r^2}{4} + \frac{\delta}{r^2} + \tilde{U}_{m,rat}(r), \quad (14) \]

where

\[
\tilde{U}_{m,rat}(r) = -2m_1 \omega - \omega^2 r^2 \frac{L_{m_1-2}^{(\delta+1)}(-\omega r^2/2)}{L_{m_1}^{(\delta-1)}(-\omega r^2/2)} \\
+ \omega(\omega r^2 + 2(\delta - 1)) \frac{L_{m_1-1}^{(\delta)}(-\omega r^2/2)}{L_{m_1}^{(\delta-1)}(-\omega r^2/2)} \\
+ 2\omega^2 r^2 \left( \frac{L_{m_1-1}^{(\delta)}(-\omega r^2/2)}{L_{m_1}^{(\delta-1)}(-\omega r^2/2)} \right)^2 .
\]

Here \( L_{m_1}^{(\delta)}(-\omega r^2/2) \) is a classical Laguerre polynomial.

(b) Forms of \( V_{m_2,ext}^{(h)}(\theta) \):

(i) For \( h = I \)

\[ V_{m_2,ext}^{(I)}(\theta) = \frac{C}{\sin^2 \theta} + \frac{D}{\cos^2 \theta} + V_{m_2,rat}^{(I)}(\theta), \quad (15) \]

where the rational part \( V_{m_2,rat}^{(I)}(\theta) \) is given by

\[
V_{m_2,rat}^{(I)}(\theta) = 4 \left[ -2m_2(\alpha - \beta - m_2 + 1) - (\alpha - \beta - m_2 + 1)(\alpha + \beta + (\alpha - \beta + 1) \cos(2\theta)) \right] \times \frac{P_{m_2-1}^{(-\alpha,\beta)}(\cos(2\theta))}{P_{m_2}^{(-\alpha-1,\beta-1)}(\cos(2\theta))} + \frac{(\alpha - \beta - m_2 + 1)^2 \sin^2(2\theta)}{2} \left( \frac{P_{m_2-1}^{(-\alpha,\beta)}(\cos(2\theta))}{P_{m_2}^{(-\alpha-1,\beta-1)}(\cos(2\theta))} \right)^2 .
\]

(ii) For \( h = II \)

\[ V_{m_2,ext}^{(II)}(\theta) = \frac{C}{\sin^2 \theta} + \frac{D}{\sin \theta \tan \theta} + V_{m_2,rat}^{(II)}(\theta), \quad (16) \]
with the rational part

\[
V_{m_2,\text{rat}}^{(II)}(\theta) = \left[ -2m_2(\alpha - \beta - m_2 + 1) - (\alpha - \beta - m_2 + 1)(\alpha + \beta + (\alpha - \beta + 1)\cos \theta) \right. \\
\left. \times \left( \frac{P_{m_2-1}^{(-\alpha,\beta)}(\cos \theta)}{P_{m_2}^{(-\alpha-1,\beta-1)}(\cos \theta)} \right) + \frac{(\alpha - \beta - m_2 + 1)^2 \sin^2 \theta}{2} \left( \frac{P_{m_2-1}^{(-\alpha,\beta)}(\cos \theta)}{P_{m_2}^{(-\alpha-1,\beta-1)}(\cos \theta)} \right)^2 \right].
\]

(iii) For \( h = (PT)_1 \)

\[
V_{m_2,\text{ext}}^{(PT)_1}(\theta) = \frac{C}{\sin^2 \theta} + \frac{iD}{\tan \theta} + V_{m_2,\text{rat}}^{(PT)_1}(\theta),
\]

(17)

is a complex and \( PT \) symmetric form of \( V_{m_2,\text{ext}}^{(h)}(\theta) \) with the corresponding complex and \( PT \) symmetric rational term

\[
V_{m_2,\text{rat}}^{(PT)_1}(\theta) = -2\csc^2 \theta \left[ 2i \cot \theta \frac{q_{m_2}^{(A,B)}(z)}{d_{m_2}^{(A,B)}(z)} - \csc^2 \theta \right. \\
\left. \times \left( \frac{d_{m_2}^{(A,B)}(z)}{q_{m_2}^{(A,B)}(z)} - \left( \frac{d_{m_2}^{(A,B)}(z)}{q_{m_2}^{(A,B)}(z)} \right)^2 \right) - m_2 \right].
\]

(iv) For \( h = (PT)_2 \)

The potential given in case (ii) can also be made complex and \( PT \) symmetric by multiplying the potential parameter \( D \) by imaginary number \( i \) and get

\[
V_{m_2,\text{ext}}^{(PT)_2}(\theta) = \frac{C}{\sin^2 \theta} + \frac{iD}{\tan \theta} + V_{m_2,\text{rat}}^{(PT)_2}(\theta),
\]

(18)

with the complex rational part

\[
V_{m_2,\text{rat}}^{(PT)_2}(\theta) = \left[ -2m_2(\alpha - \beta - m_2 + 1) - (\alpha - \beta - m_2 + 1)(\alpha + \beta + (\alpha - \beta + 1)\cos \theta) \right. \\
\left. \times \left( \frac{P_{m_2-1}^{(-\alpha,\beta)}(\cos \theta)}{P_{m_2}^{(-\alpha-1,\beta-1)}(\cos \theta)} \right) + \frac{(\alpha - \beta - m_2 + 1)^2 \sin^2 \theta}{2} \left( \frac{P_{m_2-1}^{(-\alpha,\beta)}(\cos \theta)}{P_{m_2}^{(-\alpha-1,\beta-1)}(\cos \theta)} \right)^2 \right].
\]

Here the potential parameters \( \alpha \) and \( \beta \) are complex.

(c) Forms of \( V_{m_3,\text{ext}}^{(h)}(\phi) \):

(i) For \( h = I \)

\[
V_{m_3,\text{ext}}^{(I)}(\phi) = \frac{G}{\sin^2(p\phi)} + \frac{F}{\cos^2(p\phi)} + V_{m_3,\text{rat}}^{(I)}(\phi),
\]

(19)

where

\[
U_{m_3,\text{rat}}^{(I)}(\phi) = 4p^2 \left[ -2m_3(\bar{\alpha} - \bar{\beta} - m_3 + 1) - (\bar{\alpha} - \bar{\beta} - m_3 + 1)(\bar{\alpha} + \bar{\beta} + (\bar{\alpha} - \bar{\beta} + 1)\cos(2p\phi)) \right. \\
\left. \times \left( \frac{P_{m_2-1}^{(-\bar{\alpha},\bar{\beta})}(\cos(2p\phi))}{P_{m_2}^{(-\bar{\alpha}-1,\bar{\beta}-1)}(\cos(2p\phi))} + \frac{(\bar{\alpha} - \bar{\beta} - m_2 + 1)^2 \sin^2(2p\phi)}{2} \right) \left( \frac{P_{m_2-1}^{(-\bar{\alpha},\bar{\beta})}(\cos(2p\phi))}{P_{m_2}^{(-\bar{\alpha}-1,\bar{\beta}-1)}(\cos(2p\phi))} \right)^2 \right].
\]
Note that here \( p \) is any positive integer.

\[ V_{m,ext}^{(II)}(\phi) = \frac{G}{\sin^2(p\phi)} + \frac{F}{\sin(p\phi)\tan(p\phi)} + V_{m,\text{rat}}^{(II)}(\phi), \quad (20) \]

where

\[ U_{m,\text{rat}}^{(II)}(\phi) = p^2 \left[ -2m_3(\tilde{\alpha} - \tilde{\beta} - m_3 + 1) - (\tilde{\alpha} - \tilde{\beta} - m_3 + 1)(\tilde{\alpha} + \tilde{\beta} + (\tilde{\alpha} - \tilde{\beta} + 1)\cos(p\phi) \right. \]
\[ \times \left. \frac{P_{m_3-1}^{(-\tilde{\alpha},\tilde{\beta})}(\cos(p\phi))}{P_{m_3}^{(-\tilde{\alpha}-1,\tilde{\beta}-1)}(\cos(p\phi))} \right] + \frac{(\tilde{\alpha} - \tilde{\beta} - m_3 + 1)^2\sin^2(p\phi)}{2} \]
\[ \times \left. \frac{P_{m_3-1}^{(-\tilde{\alpha},\tilde{\beta})}(\cos(p\phi))}{P_{m_3}^{(-\tilde{\alpha}-1,\tilde{\beta}-1)}(\cos(p\phi))} \right]^2. \]

Here \( p \) is any odd positive integer.

\[ U_{m,\text{rat}}^{(PT)}(\phi) = p^2 \left[ -2m_3(\tilde{\alpha} - \tilde{\beta} - m_3 + 1) - (\tilde{\alpha} - \tilde{\beta} - m_3 + 1)(\tilde{\alpha} + \tilde{\beta} + (\tilde{\alpha} - \tilde{\beta} + 1)\cos(p\phi) \right. \]
\[ \times \left. \frac{P_{m_3-1}^{(-\tilde{\alpha},\tilde{\beta})}(\cos(p\phi))}{P_{m_3}^{(-\tilde{\alpha}-1,\tilde{\beta}-1)}(\cos(p\phi))} \right] + \frac{(\tilde{\alpha} - \tilde{\beta} - m_3 + 1)^2\sin^2(p\phi)}{2} \]
\[ \times \left. \frac{P_{m_3-1}^{(-\tilde{\alpha},\tilde{\beta})}(\cos(p\phi))}{P_{m_3}^{(-\tilde{\alpha}-1,\tilde{\beta}-1)}(\cos(p\phi))} \right]^2. \]

Note that here \( p \) is any odd positive integer.

Here \( P_{m_3}^{(\tilde{\alpha},\tilde{\beta})}(z) \) and \( P_{m_3}^{(\tilde{\alpha},\tilde{\beta})}(z) \) are classical Jacobi polynomials.

By taking various combinations of these allowed choices, we then have twelve different, rational, exactly solvable non-central potentials in three dimensions, each with ten parameters. These choices of potentials, the corresponding eigenvalues and eigenfunctions are given in Tables I and II. In particular, in Table I we give the four forms of possible \( V_{m,\text{ext}}^{(h)}(\theta) \) (two real and two complex but \( PT \)-invariant) with their corresponding eigenvalues and eigenfunctions. In Table II, we similarly give two real and one complex and \( PT \) symmetric forms of \( U_{m,\text{ext}}^{(h)}(\phi) \) and the corresponding eigenvalues and eigenfunctions.

As an illustration, we discuss two examples in detail, one real and one \( PT \)-invariant complex case in Secs. 2.1 and 2.2 respectively.
2.1 Example of Rationally Extended (RE) non-central real potential

In this section, we discuss an example of the ten parameters (six continuous and four discrete) RE non-central real potential and its bound state solutions explicitly. We consider the potential of the form

\[ V_{m_1,m_2,m_3}(r, \theta, \phi) = \omega^2 r^2 + \frac{\delta}{r^2} + \hat{U}_{m_1, \text{rat}}(r) + \frac{C}{r^2 \sin^2 \theta} + \frac{D}{r^2 \cos^2 \theta} + \frac{1}{r^2} V_{m_2, \text{rat}}^{(l)}(\theta) + \frac{1}{r^2 \sin^2 \theta \sin^2 p\phi} + \frac{G}{r^2 \sin^2 \theta \cos^2 p\phi} + \frac{F}{r^2 \sin^2 \theta} U_{m_3, \text{rat}}^{(l)}(\phi), \]  

(22)

where the six parameters \( \omega, \delta, C, D, F \) and \( G \) are continuous parameters while the rest four i.e., \( p, m_1, m_2 \) and \( m_3 \) are discrete parameters. In particular, each of them can take any integral value. The rational terms \( \hat{U}_{m_1, \text{rat}}(r), V_{m_2, \text{rat}}^{(l)}(\theta) \) and \( U_{m_3, \text{rat}}^{(l)}(\phi) \) are given by Eqs. (14), (15) and (19) respectively. It is easy to show that the eigenvalues of this extended non-central potential are the same as that of the conventional case given by Eq. (2) but the eigenfunctions are different which are obtained in terms of EOPs. The complete eigenfunction is given by Eq. (9) which is ultimately a product of these EOPs.

To solve the above extended non-central potential, first we consider a simple case of \( m_1 = m_2 = m_3 = 1 \) and then we generalize it to any arbitrary positive integer values of \( m_1, m_2 \) and \( m_3 \).

Case (i): For \( m_1 = m_2 = m_3 = 1 \)

In this case, the ten parameters RE non-central potential is reduced to a seven parameters RE non-central potential

\[ V_{1,1,1}(r, \theta, \phi) = \frac{\omega^2 r^2}{4} + \frac{\delta}{r^2} + \bar{U}_{1, \text{rat}}(r) + \frac{C}{r^2 \sin^2 \theta} + \frac{D}{r^2 \cos^2 \theta} + \frac{1}{r^2} V_{1, \text{rat}}^{(l)}(\theta) + \frac{1}{r^2 \sin^2 \theta \sin^2 p\phi} + \frac{G}{r^2 \sin^2 \theta \cos^2 p\phi} + \frac{F}{r^2 \sin^2 \theta} U_{1, \text{rat}}^{(l)}(\phi), \]  

(23)

here \( p \) is any positive integer. To get the exact solution of the above Eq. (23), we define the rational terms \( \bar{U}_{1, \text{rat}}(r), V_{1, \text{rat}}^{(l)}(\theta) \) and \( U_{1, \text{rat}}^{(l)}(\phi) \) as (by putting \( m_1 = m_2 = m_3 = 1 \)) in the rational parts of Eqs. (14), (15) and (19)

\[ \bar{U}_{1, \text{rat}}(r) = \frac{4 \omega}{(\omega r^2 + 2 \delta)} - \frac{16 \omega \delta}{(\omega r^2 + 2 \delta)^2}, \]  

(24)

\[ V_{1, \text{rat}}^{(l)}(\theta) = \frac{8(\alpha + \beta)}{((\alpha + \beta) - (\beta - \alpha) \cos(2\theta))} - \frac{8((\alpha + \beta) - (\beta - \alpha) \cos(2\theta))^2}{((\alpha + \beta) - (\beta - \alpha) \cos(2\theta))^2}, \]  

(25)

and

\[ U_{1, \text{rat}}^{(l)}(\phi) = 4p^2 \left[ \frac{2(\bar{\alpha} + \bar{\beta})}{((\bar{\alpha} + \bar{\beta}) - (\bar{\beta} - \bar{\alpha}) \cos(2p\phi))} - \frac{2((\bar{\alpha} + \bar{\beta}) - (\bar{\beta} - \bar{\alpha}) \cos(2p\phi))^2}{((\bar{\alpha} + \bar{\beta}) - (\bar{\beta} - \bar{\alpha}) \cos(2p\phi))^2} \right]. \]  

(26)
On comparing Eq. (23) with Eq. (8) (for $m_1 = m_2 = m_3 = 1$ and $h = I$), we get the rationally extended trigonometric Pöschl-Teller potential [5, 7]

$$U_{1, ext}^{(I)}(\phi) = U_{con}^{(I)}(\phi) + U_{1, rat}^{(I)}(\phi),$$

(27)

where

$$U_{con}^{(I)}(\phi) = G\cosec^2(p\phi) + F\sec^2(p\phi)$$

(28)

is the corresponding conventional potential. The unnormalized $\phi$ dependent wave function of Eq. (11) (for $m_3 = 1$) with the extended potential (27) in terms of $X_1$ exceptional orthogonal polynomials $\hat{P}^{(\alpha, \beta)}_{n_3+1}(z)$ is well known and given by [5, 7]

$$\Phi_{1,n_3}^{(I)}(\phi) \propto (1 - z)\frac{\hat{\alpha}^2 + \frac{1}{4} + z}{((\alpha + \beta) - (\beta - \alpha)\cos(2p\phi))} \hat{P}^{(\alpha, \beta)}_{n_3+1}(z); \quad 0 \leq p\phi \leq \pi/2,$$

(29)

where $n_3 = 0, 1, 2, 3..., \quad z = \cos(2p\phi)$ and the positive constant parameters

$$\hat{\alpha} = \frac{1}{2}\sqrt{1 + \frac{4G}{p^2}}; \quad \hat{\beta} = \frac{1}{2}\sqrt{1 + \frac{4F}{p^2}}.$$

(30)

The eigenvalue spectrum of this extended potential is same (i.e. isospectral) as that of the conventional potential $U_{con}(\phi)$ which is given by

$$m^2 = p^2(2n_3 + \hat{\alpha} + \hat{\beta} + 1)^2.$$  

(31)

Again from Eqs. (23) and (8), the rationally extended $\theta$ dependent potential is given by

$$V_{1, ext}^{(I)}(\theta) = V_{con}^{(I)}(\theta) + V_{1, rat}^{(I)}(\theta),$$

(32)

where the conventional potential

$$V_{con}^{(I)}(\theta) = C\cosec^2(\theta) + D\sec^2(\theta).$$

(33)

The Schrödinger equation (12) (for $m_2 = 1$ and $h = I$) becomes

$$-\frac{\partial^2 \Theta_1^{(I)}(\theta)}{\partial \theta^2} + \left[\left(C + m^2 - \frac{1}{4}\right)\cosec^2\theta + D\sec^2\theta + V_{1, rat}^{(I)}(\theta)\right] \Theta_1^{(I)}(\theta) = \ell^2 \Theta_1^{(I)}(\theta).$$

(34)

Using the rational term $V_{1, rat}^{(I)}(\theta)$ from Eq. (25), the wave function and the eigenspectrum of the above Eq. (34) are thus given by

$$\Theta_{1,n_2}^{(I)}(\theta) \propto (1 - z)\frac{\hat{\alpha}^2 + \frac{1}{4} + z}{((\alpha + \beta) - (\beta - \alpha)\cos(2\theta))} \hat{P}^{(\alpha, \beta)}_{n_2+1}(z); \quad 0 \leq \theta \leq \pi/2,$$

(35)

and

$$\ell^2 = (2n_2 + \alpha + \beta + 1)^2; \quad n_2 = 0, 1, 2,...,$$

(36)
where \( z = \cos(2\theta) \) and the parameters
\[
\alpha = \sqrt{C + m^2}, \\
\beta = \frac{1}{2}\sqrt{1 + 4D}.
\] (37)

Note that the eigenvalue spectrum is unchanged while the eigenfunctions are different from those of the nonrational case.

Similar to the above cases, from Eqs. (23) and (8), the radial component of the extended potential is given by
\[
\tilde{U}_{1,\text{ext}}(r) = \tilde{U}_{\text{con}}(r) + \tilde{U}_{1,\text{rat}}(r),
\] (38)
where the conventional radial oscillator potential
\[
\tilde{U}_{\text{con}}(r) = \frac{\omega r^2}{4} + \frac{\delta}{r^2}.
\] (39)

From Eq. (13) (for \( m_1 = 1 \)), finally, we get the exactly solvable Schrödinger equation
\[
-\frac{\partial^2 R_1(r)}{\partial r^2} + \left[ \frac{\omega^2 r^2}{4} + \left( \frac{\delta + \ell^2 - 1/4}{r^2} \right) + \tilde{U}_{1,\text{rat}}(r) \right] R_1(r) = E R_1(r),
\] (40)
with the solution in term of \( X_1 \) Laguerre EOPs \( \hat{L}_{n_1+1}^{(\delta)}(\frac{\omega r^2}{2}) \) given by \([5,7]\)
\[
R_{1,n_1}(r) \propto \frac{r^{(\delta+1/2)}}{(\omega r^2 + 2\delta)} \hat{L}_{n_1+1}^{(\delta)}(\frac{\omega r^2}{2}); \quad 0 < r < \infty.
\] (41)

The energy eigenvalue \( E \) which depends on \( n_1, n_2 \) and \( n_3 \) is given by
\[
E_{n_1,n_2,n_3} = \omega(2n_1 + 1 + \tilde{\delta}),
\] (42)
where \( \tilde{\delta} = \sqrt{\delta + \ell^2} \) and
\[
\ell^2 = \left[ (2n_2+1) + \sqrt{D + 1/4} + \left\{ C + \left( \sqrt{F + p^2/4} + \sqrt{G + p^2/4} + p(2n_3+1) \right) \right\}^{1/2} \right]^2. \] (43)

Again note that the energy eigenvalues are unchanged while the corresponding eigenfunctions are different from the nonrational case.

**Case (ii): For any positive integer values of \( m_1, m_2 \) and \( m_3 \)**

In this case, we consider a more general form of the ten parameters potential given in Eq. (22). Similar to the \( X_1 \) case, on comparing Eqs. (22) and (8), we obtain the rationally extended trigonometric Pöschl-Teller equation
\[
U_{m_3,\text{ext}}^{(l)}(\phi) = U_{\text{con}}^{(l)}(\phi) + U_{m_3,\text{rat}}^{(l)}(\phi),
\] (44)
where $U^{(l)}_{\text{con}}(\phi)$ and $U^{(l)}_{\text{m3, rat}}(\phi)$ are given by Eqs. (28) and (20). The unnormalized wavefunction of Eq. (11) with the extended potential (44) in terms of the $X_{m3}$ exceptional Jacobi polynomial $\hat{P}_{n3+m3}^{(\tilde{\alpha}, \tilde{\beta})}(z)$ is given by [7]

$$\Phi_{m3,n3}^{(l)}(\phi) \propto \frac{(1-z)^{\frac{3}{2}+\frac{1}{2}}(1+z)^{\frac{3}{2}+\frac{1}{2}}}{P_{m3}^{(-\tilde{\alpha}-1, \tilde{\beta}-1)}(z)} \hat{P}_{n3+m3}^{(\tilde{\alpha}, \tilde{\beta})}(z); \quad 0 \leq p\phi \leq \pi/2;$$

(45)

where $n_3 = 0, 1, 2, \ldots$; $m_3 = 1, 2, 3, \ldots$; $z = \cos(2p\phi)$ and the parameters $\tilde{\alpha}$ and $\tilde{\beta}$ will be same as obtained in Eq. (30). The eigenvalue of this extended potential is same (i.e. isospectral) as that of the conventional potential $U^{(l)}_{\text{con}}(\phi)$ given by Eq. (31).

Similar to the above case from Eqs. (22) and (8), the $\theta$ and $r$ dependent extended potentials which depend on parameters $m_2$ and $m_3$ respectively are given as

$$V_{m2, ext}(\theta) = V^{(l)}_{\text{con}}(\theta) + V_{m2, rat}^{(l)}(\theta),$$

(46)

and

$$\tilde{U}_{m1, ext}(r) = \tilde{U}_{\text{con}}(r) + \tilde{U}_{m1, rat}(r).$$

(47)

The corresponding conventional potential terms $V^{(l)}_{\text{con}}(\theta)$ and $\tilde{U}_{\text{con}}(r)$ and the rational terms $V_{m2, rat}^{(l)}(\theta)$ and $\tilde{U}_{m1, rat}(r)$ are given in Eqs. (32), (38) and (16), (15). Following the same procedure as in the $X_1$ case, the solutions of the the Eqs. (12) and (13) with the corresponding potentials $V_{m2, ext}^{(l)}(\theta)$ and $\tilde{U}_{m1, ext}(r)$ can be obtained in a simple way. Thus the bound state wavefunctions corresponding to these potentials are given by

$$\Theta_{m2,n2}^{(l)}(\theta) \propto \frac{(1-z)^{\frac{3}{2}+\frac{1}{2}}(1+z)^{\frac{3}{2}+\frac{1}{2}}}{P_{m2}^{(-\alpha-1, \beta-1)}(z)} \hat{P}_{n2+m2}^{(\alpha, \beta)}(z); \quad 0 \leq \theta \leq \pi/2;$$

(48)

and

$$R_{m1,n1}(r) \propto \frac{r^{(\sqrt{\beta+r^2}+1/2)} \exp \left( -\frac{\sqrt{r^2}}{4} \right)}{L_{m1}^{(\sqrt{\beta+r^2})} \left( -\frac{\sqrt{r^2}}{2} \right)} \hat{L}_{n1+m1}^{(\sqrt{\beta+r^2})} \left( \frac{\omega r^2}{2} \right); \quad 0 < r < \infty;$$

(49)

where $\hat{P}_{n2+m2}^{(\alpha, \beta)}(z)$ and $\hat{L}_{n1+m1}^{(\sqrt{\beta+r^2})} \left( \frac{\omega r^2}{2} \right)$ are $X_{m2}$ exceptional Jacobi and $X_{m1}$ exceptional Laguerre polynomials respectively. The energy eigenspectrum and the other parametric relations will be same as that of the $X_1$ case.

### 2.2 Example of Rationally Extended $PT$ symmetric complex non-central potential

Similar to the above example of real case, here we consider an example of RE non-central potential which is complex but symmetric under the combined operation of the parity ($P$) ($P: r \to r, \theta \to \pi - \theta, \phi \to \phi + \pi$) and the time reversal ($T$) ($T: t \to -t, i \to -i$) operators and given by
\[ V^{(PT)}_{m_1,m_2,m_3}(r, \theta, \phi) = \frac{\omega^2 r^2}{4} + \frac{\delta}{r^2} + \bar{U}_{m_1, \text{rat}}(r) + \frac{C}{r^2 \sin^2 \theta} + \frac{i D}{r^2 \tan \theta} + \frac{1}{r^2} V^{(PT)}_{m_2, \text{rat}}(\theta) + \frac{G}{r^2 \sin^2 \theta \sin^2 p \phi} + \frac{F}{r^2 \sin^2 \theta \cos^2 p \phi} + \frac{1}{r^2} U^{(I)}_{m_3, \text{rat}}(\phi), \]  

where \( V^{(PT)}_{m_2, \text{rat}}(\theta) \) is given by Eq. (11). In this case, we obtain a complete solution of this potential by considering the same form of the \( \phi \) and \( r \) dependent terms (as defined in the first example) with a new form of \( \theta \) dependent term which is now complex but PT-invariant.

**Case (i) For \( m_1 = m_2 = m_3 = 1 \)**

For this particular case, on comparing the above Eq. (50) with Eq. (8) (by defining \( h = PT \) for \( \theta \) dependent term and \( h = I \) for \( \phi \) dependent term), we get the PT symmetric extended potential

\[ V^{(PT)}_{1, \text{ext}}(\theta) = V^{PT}_{\text{con}}(\theta) + V^{(PT)}_{1, \text{rat}}(\theta), \]  

where

\[ V^{PT}_{\text{con}}(\theta) = C \cosec^2 \theta + i D \cot \theta, \quad (52) \]

is the conventional PT symmetric trigonometric Eckart potential\(^1\) and the associated rational term

\[ V^{(PT)}_{1, \text{rat}}(\theta) = \frac{1}{A^2(A-1)^2} \left[ \frac{-4iB[A^2(A-1)^2 - B^2]}{(iB + A(A-1) \cot \theta)} \right] + \frac{2[A^2(A-1)^2 - B^2]^2}{(iB + A(A-1) \cot \theta)^2}. \]  

(53)

The form of the \( \phi \) and the \( r \) dependent extended terms will be same as defined by Eqs. (27) and (38). The solution of Eq. (12) (for \( m_2 = 1 \) and \( h = PT \)) with the above potential (51) is not in the exact form of EOPs rather they are written in the form of some types of new polynomials (discussed in detail in Ref. [11]) given as

\[ \Theta^{PT}_{1,n_2}(\theta) \propto \frac{(z - 1)^{\alpha_{n_2}}(z + 1)^{\beta_{n_2}}}{(iB + A(A-1) \cot \theta)} y_{n_2}^{(A,B)}(z), \]

(54)

with \( z = i \cot \theta \). Here the polynomial function \( y_{n_2}^{(A,B)}(z) \) can be expressed in terms of the classical Jacobi polynomials \( P_{n_2}^{(\alpha_{n_2}, \beta_{n_2})}(z) \) as

\[ y_{n_2}^{(A,B)}(z) = \frac{2(n_2 + \alpha_{n_2})(n_2 + \beta_{n_2})}{(2n_2 + \alpha_{n_2} + \beta_{n_2})} q_{1}^{(A,B)}(z) P_{n_2-1}^{(\alpha_{n_2}, \beta_{n_2})}(z) - \frac{2(1 + \alpha_{1})(1 + \beta_{1})}{(2 + \alpha_{1} + \beta_{1})} P_{n_2}^{(\alpha_{n_2}, \beta_{n_2})}(z). \]  

(55)

\(^1\)Which is easily obtained by complex co-ordinate transformation \( x \rightarrow ix \) of the rationally extended hyperbolic Eckart potential given in [11].
Here \( q_{1}^{(A,B)}(z) = P_{1}^{(\alpha_{1},\beta_{1})}(z) \) (Classical Jacobi polynomial \( P_{n_{2}}^{(\alpha_{1},\beta_{1})}(z) \) for \( n_{2} = 1 \)). The parameters \( \alpha_{n_{2}} \) and \( \beta_{n_{2}} \) in terms of \( A \) and \( B \) are given by

\[
\alpha_{n_{2}} = -(A - 1 + n_{2}) + \frac{B}{(A - 1 + n_{2})}; \quad \beta_{n_{2}} = -(A - 1 + n_{2}) - \frac{B}{(A - 1 + n_{2})}.
\]

(56)

with

\[
A = \frac{1}{2} + \sqrt{C + m^{2}}, \quad \text{and} \quad B = \frac{D}{2}.
\]

(57)

The other two parameters \( \alpha_{1} \) and \( \beta_{1} \) are simply obtained by putting \( n_{2} = 1 \) in \( \alpha_{n_{2}} \) and \( \beta_{n_{2}} \). The energy eigenvalues are given by

\[
\ell^{2} = (A - 1 + n_{2})^{2} + \frac{B^{2}}{(A - 1 + n_{2})^{2}}; \quad n_{2} = 0, 1, 2, ...
\]

(58)

Thus the complete wavefunction associated with the extended \( PT \) symmetric complex non-central potential Eq. 43 (for \( m_{1} = m_{2} = m_{3} = 1 \)) is obtained by using Eq. 44, which is a product of the \( X_{1} \) Jacobi polynomial (as given by Eq. 23), \( X_{1} \) Laguerre polynomial (as given by Eq. 44) times a new polynomial given in Eq. 44.

**Case (ii) For any positive integer values of \( m_{1}, m_{2} \) and \( m_{3} \)**

Again by considering the same form of the \( \phi \) and \( r \) dependent terms for any arbitrary values of \( m_{3} \) and \( m_{1} \), the above complex potential can be generalized easily for any non-zero positive integer values of \( m_{2} \) by defining

\[
V_{(P T)}^{(m_{2},\alpha_{n_{2}},\beta_{n_{2}})}(\theta) = V_{(P T)}^{(m_{2},\alpha_{n_{2}},\beta_{n_{2}})}(\theta) + V_{(P T)}^{(m_{2},\alpha_{n_{2}},\beta_{n_{2}})}(\theta),
\]

(59)

where \( V_{(P T)}^{(m_{2},\alpha_{n_{2}},\beta_{n_{2}})}(\theta) \) and \( V_{(P T)}^{(m_{2},\alpha_{n_{2}},\beta_{n_{2}})}(\theta) \) are given by Eqs. 52 and 17 respectively. The wavefunction associated with this potential corresponding to the Eq. 12 is given by

\[
\Theta_{m_{2},\alpha_{n_{2}},\beta_{n_{2}}}(\theta) \propto \frac{(z - 1)^{\alpha_{n_{2}}}(z + 1)^{\beta_{n_{2}}}}{q_{m_{2}}^{(A,B)}(z)} y_{\nu,m_{2}}^{(A,B)}(z); \quad \nu = n_{2} + m_{2} - 1,
\]

(60)

where \( q_{m_{2}}^{(A,B)}(z) = P_{m_{2}}^{(\alpha_{m_{2}},\beta_{m_{2}})}(z) \) and the polynomial function \( y_{\nu,m_{2}}^{(A,B)}(z) \) is

\[
y_{\nu,m_{2}}^{(A,B)}(z) = \frac{2(n_{2} + \alpha_{n_{2}})(n_{2} + \beta_{n_{2}})}{(2n_{2} + \alpha_{n_{2}} + \beta_{n_{2}})} q_{m_{2}}^{(A,B)}(z) P_{n_{2}}^{(\alpha_{n_{2}},\beta_{n_{2}})}(z) \]

- \( \frac{2(m_{2} + \alpha_{m_{2}})(m_{2} + \beta_{m_{2}})}{(2m_{2} + \alpha_{m_{2}} + \beta_{m_{2}})} q_{m_{2} - 1}^{(A+1,B)}(z) P_{n_{2}}^{(\alpha_{n_{2}},\beta_{n_{2}})}(z),
\]

(61)

with the parameters

\[
\alpha_{m_{2}} = -(A - 1 + m_{2}) + \frac{B}{(A - 1 + m_{2})}; \quad \beta_{m_{2}} = -(A - 1 + m_{2}) - \frac{B}{(A - 1 + m_{2})}.
\]

(62)
The energy eigenvalues will be same as given by Eq. (58). Thus the complete wavefunction and the eigenvalues of this complex non-central extended potential are obtained by using Eqs. (9) and (42).

Table I. In this table all the four forms of $V_{m_2,\text{ext}}(\theta)$ (for $h = I, II, (PT)_1$ and $(PT)_2$) with their corresponding energy eigenvalues ($l^2$) and the eigenfunctions ($\Theta_{m_2,n_2}(\theta)$) are given. Cases (i) and (iii) are discussed in detail in the text.

| $V_{m_2,\text{ext}}(\theta)$ | $l^2$ | $\Theta_{m_2,n_2}(\theta)$ |
|-----------------------------|------|--------------------------|
| (i) $V_{m_2,\text{ext}}^{(I)}(\theta)$ | $(2n_2 + \alpha + \beta + 1)^2$ | | |
| $n_2 = 0,1,2,\ldots$ | | $(1-z)^{\frac{a}{2}+\frac{1}{4}(1+z)} \frac{n_2}{\beta_{n_2+m_2}(\theta)}(z)$; | |
| $\alpha = \sqrt{C + m^2}$ | | $z = \cos 2\theta$ | |
| $\beta = \frac{1}{2}\sqrt{1+4D}$ | | $m_2 = 1,2\ldots$ | |
| (ii) $V_{m_2,\text{ext}}^{(II)}(\theta)$ | $(n_2 + \frac{\alpha + \beta + 1}{2})^2$ | | |
| $\alpha = \sqrt{C + m^2 - D}$ | | $(1-z)^{\frac{a}{2}+\frac{1}{4}(1+z)} \frac{n_2}{\beta_{n_2+m_2}(\theta)}(z)$; | |
| $\beta = \sqrt{C + m^2 + D}$ | | $z = \cos \theta$ | |
| (iii) $V_{m_2,\text{ext}}^{(PT)_1}(\theta)$ | $(A-1 + n_2)^2 + \frac{B^2}{(A-1+n_2)^2}$ | | |
| $A = \frac{1}{2} + \sqrt{C + m^2}; B = \frac{D}{2}$ | | $(z-1)^{\frac{a}{2}+\frac{1}{4}(z+1)} \frac{n_2}{\beta_{n_2+m_2}(\theta)}(z)$; | |
| $\alpha_{n_2} = -(A-1 + n_2) + \frac{B}{(A-1+n_2)}$ | | $z = i \cot \theta$ | |
| $\beta_{n_2} = -(A-1 + n_2) - \frac{B}{(A-1+n_2)}$ | | $\nu = n_2 + m_2 - 1$, | |
| (iv) $V_{m_2,\text{ext}}^{(PT)_2}(\theta)$ | $(n_2 + \frac{\alpha + \beta + 1}{2})^2 = (n_2 + A)^2$ | | |
| $\alpha = \sqrt{C + m^2 - iD}$ | | $(1-z)^{\frac{a}{2}+\frac{1}{4}(1+z)} \frac{n_2}{\beta_{n_2+m_2}(\theta)}(z)$; | |
| $\beta = \sqrt{C + m^2 + iD}$ | | $z = \cos \theta$ | |
| $C + m^2 = -B^2 + (A-\frac{1}{2})^2$ | | | |
| $D = 2AB$ | | | |
Table II. The three different forms of \( \phi \) dependent terms \( V^{(h)}_{m3,ext}(\phi) \) (for \( h = I, II, (PT)_1 \)) with their corresponding energy eigenvalues \( (m^2) \) and the eigenfunctions \( (\Phi^{(h)}_{m3,n3}(\phi)) \) are given. Out of these, case (i) is already considered in detail in the text.

| \( U^{(h)}_{m3,ext}(\phi) \) | \( m^2 \) | \( \Phi^{(h)}_{m3,n3}(\phi) \) |
|-------------------------------|----------------|-----------------------------------|
| (i) \( U^{(I)}_{m3,ext}(\phi) \) | \( p^2(2n_3 + \bar{\alpha} + \bar{\beta} + 1)^2 \) | \( (1-z)^{\bar{\alpha} + \frac{1}{2} (1+z) \frac{1}{2} + \frac{1}{2}} \frac{p^{(\bar{\alpha},\bar{\beta})}}{n_{3+m3} (z)} \); |
| \( \bar{\alpha} = \frac{1}{2} \sqrt{1 + 4G^2} \) | \( z = \cos(2p\phi) \); | \( p = 1, 2, 3, ... \); |
| \( \bar{\beta} = \frac{1}{2} \sqrt{1 + 4G^2} \) | \( m_3 = 1, 2, 3, ... \); | |
| (ii) \( U^{(II)}_{m3,ext}(\phi) \) | \( p^2(n_3 + \frac{\bar{\alpha} + \bar{\beta} + 1}{2})^2 \) | \( (1-z)^{\bar{\alpha} + \frac{1}{2} (1+z) \frac{1}{2} + \frac{1}{2}} \frac{p^{(\bar{\alpha},\bar{\beta})}}{n_{3+m3} (z)} \); |
| \( \bar{\alpha} = \frac{1}{2} \sqrt{1 + 4G^2 - \frac{4F}{p^2}} \) | \( z = \cos(p\phi) \); | \( p = 1, 3, 5, ... \); |
| \( \bar{\beta} = \frac{1}{2} \sqrt{1 + 4G^2 + \frac{4F}{p^2}} \) | | |
| (iii) \( U^{(PT)_1}_{m3,ext}(\phi) \) | \( p^2(n_3 + \frac{\bar{\alpha} + \bar{\beta} + 1}{2})^2 = (n_3p + A)^2 \) | \( (1-z)^{\bar{\alpha} + \frac{1}{2} (1+z) \frac{1}{2} + \frac{1}{2}} \frac{p^{(\bar{\alpha},\bar{\beta})}}{n_{3+m3} (z)} \); |
| \( \bar{\alpha} = \frac{1}{2} \sqrt{1 + 4G^2 - \frac{4F}{p^2}} \) | \( z = \cos(p\phi) \); | \( p = 1, 3, 5, ... \); |
| \( \bar{\beta} = \frac{1}{2} \sqrt{1 + 4G^2 + \frac{4F}{p^2}} \) | | |
| \( G = A^2 - B^2 - Ap \) | | |
| \( F = B(2A - p) \) | | |

3 RE non-central potentials in 2-Dimensions

In two dimensional polar co-ordinates \((r, \phi)\), the Schrödinger equation corresponding to the non-central potential \( V_{m1,m3}(r, \phi) \) is given by \( (\hbar = 2m = 1) \)

\[
\left[ -\frac{d^2}{dr^2} \frac{1}{r} \frac{d}{dr} - \frac{1}{r^2} \frac{d^2}{d\phi^2} \right] \Psi(r, \phi) + V_{m1,m3}(r, \phi) \Psi(r, \phi) = E \Psi(r, \phi). \quad (63)
\]

The forms of the non-central potential in this co-ordinate system is given by

\[
V^{(h)}_{m1,m3}(r, \phi) = U_{m1,ext}(r) + \frac{1}{r^2} U^{(h)}_{m3,ext}(\phi), \quad (64)
\]

with \( h = 1, 2, (PT)_1 \) and \( (PT)_2 \). The above Schrödinger equation is exactly solvable, if we define the wave function in the form

\[
\Psi(r, \phi) = \frac{R_{m1}(r)}{r^{1/2}} \Phi^{(h)}_{m3}(\phi). \quad (65)
\]

Using \( \Psi(r, \phi) \) in Eq. \( (63) \), the angular component of the wave function satisfies the equation

\[
\left[ -\frac{d^2}{dr^2} + U_{m3,ext}(\phi) \right] \Phi^{(h)}_{m3}(\phi) = m^2 \Phi^{(h)}_{m3}(\phi), \quad (66)
\]
and the radial component satisfies

\[
- \frac{d^2}{dr^2} + \tilde{U}_{m_1,\text{ext}}(r) + \left( \frac{m^2 - \frac{1}{4}}{r^2} \right) R_{m_1}(r) = E R_{m_1}(r),
\]

(67)

where \( m^2 \) is the eigenvalue of the angular equation.

These two equations (66) and (67) are identical to the corresponding equations (11) and (13) obtained in the three dimensional case except the parameter \( \ell \) in the three dimensional case is now replaced by \( m \) in the radial component of the Eq. (67). In this case, we have one choice of \( \tilde{U}_{m_1,\text{ext}}(r) \) and four choices of \( V_{m_3,\text{ext}}(\phi) \) (two real and two PT symmetric). The form of \( \tilde{U}_{m_1,\text{ext}}(r) \) will be same as in the case of three dimensions with the parameter \( \tilde{\delta} = \sqrt{\delta + m^2} \). Out of these four choices of \( V_{m_3,\text{ext}}(\phi) \), three (two real and one PT symmetric, \( h = I, II, (PT)_1 \)) are already discussed in Table II of the previous section while the fourth form of the potential (\( h = (PT)_2 \)) is special to the two dimensions.

The form of this complex potential is given by

\[
V_{m_3,\text{ext}}^{(PT)_2}(\phi) = \frac{G}{\sin^2(p\phi)} + \frac{iF}{\tan(p\phi)} + V_{m_2,\text{rat}}^{(PT)_2}(\phi),
\]

(68)

where

\[
V_{m_3,\text{rat}}^{(PT)_2}(\phi) = -2p^2 \csc^2(p\phi) \left[ 2i \cot(p\phi) \frac{q_{m_3}(A/p,B/p)(z)}{q_{m_3}(A/p,B/p)(z)} - \csc^2(p\phi) \right] \times \left[ \left( \frac{q_{m_3}(A/p,B/p)(z)}{q_{m_3}(A/p,B/p)(z)} \right)^2 - m_3 \right]; \quad m_3 = 1, 2, 3, ...
\]

with \( z = \cos(p\phi); \quad 0 \leq p\phi \leq \pi \) and

\[
q_{m_3}(A/p,B/p)(z) = P_{m_3}^{(\tilde{\alpha}_{m_3}, \tilde{\beta}_{m_3})}(z)
\]

(69)

is a classical Jacobi polynomials with the parameters

\[
\tilde{\alpha}_{m_3} = -(A/p - 1 + m_3) + \frac{B/p}{(A/p - 1 + m_3)} \quad \tilde{\beta}_{m_3} = -(A/p - 1 + m_3) - \frac{B/p}{(A/p - 1 + m_3)}.
\]

(70)

Here \( p \) is restricted to the positive odd integers only and a dot on \( q_{m_3}^{(A/p,B/p)}(z) \) indicates single derivative with \( z \).

An explanation is in order as to why the potential as given by Eq. (68) is a PT-symmetric complex potential in two dimensions but only a complex but not PT-symmetric in three dimensions. The point is that unlike three space dimensions, parity in two space
dimensions correspond to say \( x \to -x, y \to +y \), i.e. it corresponds to \((P : r \to r, \phi \to \pi - \phi)\). The time reversal corresponds to \((T : t \to -t, i \to -i)\) symmetry, the above potential \((68)\) is \(PT\) symmetric in 2-dimensions but not in three space dimensions since in three dimensions \( \phi \to \pi + \phi \).

Of course one can also consider the potential as given by Eq. \((68)\) in three dimensions and there it is merely complex but non \(PT\)-symmetric potential. However, the spectrum is still real thereby confirming the well known fact that \(PT\)-symmetry is sufficient but not necessary for the spectrum to be real. Note that we also have considered this type of \(\theta\)-dependent potential in case \((iii)\) of Table I (detail solution is also given in Section 2.2), since such a potential is indeed complex and \(PT\)-invariant term. This is because, in three dimensions, under parity, unlike \(\phi, \theta \to \pi - \theta\). Following the same procedure as in the three dimensional case, the solution of this 2-dimensional \(PT\) symmetric non-central potential

\[
V_{m_1,m_3}^{(PT)}(r, \phi) = \tilde{U}_{m_1,ext}(r) + \frac{1}{r^2}U_{m_3,ext}^{(PT)}(\phi),
\]

(71)
can also be obtained in a straightforward way. In particular, the solution for the special case of \(m_1 = m_3 = 1\) is straightforward one, therefore we only consider the general case of any arbitrary positive integers \(m_1\) and \(m_3\).

Using Eq. \((68)\) in the angular equation \((66)\) we get the solution of the form

\[
\Phi_{m_3,n_3}^{(PT)}(\phi) \propto \frac{(z - 1)\hat{\alpha}_{n_3} + (z + 1)\hat{\beta}_{n_3}}{\tilde{q}_{m_3}(A/p,B/p)} y_{\nu,m_3}(z),
\]

(72)

where

\[
\hat{\alpha}_{n_3} = -(A/p - 1 + n_3) - \frac{B/p}{(A/p - 1 + n_3)},
\]

(73)

\[
\hat{\beta}_{n_3} = -(A/p - 1 + n_3) + \frac{B/p}{(A/p - 1 + n_3)},
\]

and the form of \(y_{\nu,m_3}(z)\) will be same as given by Eq. \((61)\) (where we replace \(n_2 \to n_3, m_2 \to m_3, A \to A/p, B \to B/p \) and \(z \to \cos(p\phi)\)). The energy eigenvalue \(m^2\) is given by

\[
m^2 = p^2 \left[ (A/p - 1 + n_3)^2 + \frac{B^2/p^2}{(A/p - 1 + n_3)^2} \right]; \quad n_3 = 0, 1, 2, 3...
\]

(74)

The parameters \(A\) and \(B\) in terms of \(F\) and \(G\) are related as

\[
\frac{A}{p} = \frac{1}{2} + \frac{1}{2} \sqrt{1 + 4G/p^2}; \quad 2B = F.
\]

(75)

Now using equation \((14)\) in the radial part of the Schrödinger equation \((67)\) we get

\[
R_{m_1,n_1}(r) \propto \frac{r^{(\sqrt{\phi + m^2} + 1/2)} \exp \left( -\frac{\omega r^2}{2} \right) L_{n_1 + m_1}^{(\sqrt{\phi + m^2})} \left( \frac{\omega r^2}{2} \right)}{L_{m_1}^{(\sqrt{\phi + m^2} - 1)} \left( -\frac{\omega r^2}{2} \right)}; \quad 0 < r < \infty,
\]

(76)
with the energy eigenvalues

\[ E_{n_1,n_3} = \omega(2n_1 + 1 + \tilde{\delta}), \]  

(77)

where

\[ \tilde{\delta} = \sqrt{\delta} + m^2. \]  

(78)

Thus the complete wavefunction and the eigenspectrum for the above seven parameter (four continuous and three discrete) family of potential (71) are given by Eqs. (65) and (77). In the particular case of \( m_1 = m_3 = 0 \), these potentials are reduced to their corresponding conventional potentials whose solutions are associated with the classical orthogonal polynomials.

4 Summary

In this work, we have constructed twelve rationally extended non-central real and \( PT \) symmetric complex potentials in three dimensional spherical polar co-ordinates. The solutions of these potentials are obtained by using the recently discovered rationally extended potentials whose solutions are in terms of \( X_{m_1}, X_{m_2} \) or \( X_{m_3} \) exceptional Laguerre and (or) Jacobi orthogonal polynomials. The eigenfunctions and the energy eigenvalues of these twelve extended non-central potentials are obtained explicitly and shown that the eigenfunctions of these extended non-central potentials are the product of Laguerre and Jacobi EOPs. It is found that the three dimensional Schrödinger equation is exactly solvable for the one possible choice for \( \tilde{U}_{m_1,\text{ext}}(r) \), four possible choices for \( V_{m_2,\text{ext}}^{(h)}(\theta) \) and three choices for \( U_{m_3,\text{ext}}^{(h)}(\phi) \). All possible choices of \( \theta \) and \( \phi \) dependent potentials and the corresponding solutions are listed in Tables I and II. The various combinations of \( \tilde{U}_{m_1,\text{ext}}(r), V_{m_2,\text{ext}}^{(h)}(\theta) \) and \( U_{m_3,\text{ext}}^{(h)}(\phi) \) lead to the total twelve different forms (four real and eight \( PT \) symmetric complex) of the RE non-central potentials. In the examples of \( PT \) symmetric cases, some of the solutions corresponding to the \( \theta \) dependent term is not in the exact form of EOPs, they are written in the forms of some types of new orthogonal polynomials \( (y_{a,b;\nu}^{(A,B)}(z)) \) which are simplified further in the terms of classical Jacobi polynomials. In this works we have only consider one choice of \( r \) dependent extended potential as a RE radial oscillator case. One can also replace the RE radial oscillator part with the conventional coulomb \( U_{\text{con}}(r) = -\frac{e^2}{r} + \frac{\delta}{r^2} \) (as shown in Ref. [2]) then one will have nine parameters RE non-central potential and the spectrum can also be obtained easily. Few attempts at rational extension of Coulomb have been done [13], but they are not very general, so we are not mentioning them.

In a particular case of \( m_1 = m_2 = m_3 = 0 \), these potentials are reduced to their conventional counterparts (which are non-rational) with seven parameters (six continuous and one discrete) whose solutions are in terms of classical orthogonal polynomials. Out of these twelve conventional cases, the eight \( PT \) symmetric complex non-central seven parameters conventional potentials are also new and not discussed earlier.
We have also considered the Schrödinger equation in two dimensional polar co-ordinates and constructed four possible forms (two real and two $PT$ symmetric complex) of the seven parameters (four continuous and three discrete) RE non-central potentials. The solutions of these potentials are also obtained in terms of EOPs.

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