Divisors on overlapped intervals and multiplicative functions

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Abstract
Consider the real numbers
\[ \ell_{n,k} = \ln \left( \frac{3}{2} k + \sqrt{\left( \frac{3}{2} k \right)^2 + 3n} \right) \]
and the intervals \( L_{n,k} = [\ell_{n,k} - \ln 3, \ell_{n,k}] \). For all \( n \geq 1 \), define
\[ \frac{L_n(q)}{q^{n-1}} = \sum_{d|n} \sum_{k \in \mathbb{Z}} 1_{L_{n,k}} (\ln d) q^k, \]
where \( 1_A(x) \) is the characteristic function of the set \( A \). Let \( \sigma(n) \) be sum of divisors of \( n \). We will prove that \( A002324(n) = 4 \sigma(n) - 3 L_n(1) \) and \( A096936(n) = L_n(-1) \), which are well-known multiplicative functions related to the number of representations of \( n \) by a given quadratic form.

1 Introduction
For a given integer \( n \geq 1 \), consider the two-sided sequence
\[ p_{n,k} = \ln \left( k + \sqrt{k^2 + 2n} \right), \]
where \( k \in \mathbb{Z} \) and define the intervals
\[ P_{n,k} = [p_{n,k} - \ln 2, p_{n,k}] \].
Kassel and Reutenauer [1] introduced the polynomials¹,

\[ \frac{P_n(q)}{q^{n-1}} = \sum_{d|n} \sum_{k \in \mathbb{Z}} \sum_{P_{n,k}} (\ln d) q^k, \]

where \(1_A(x)\) is the characteristic function of the set \(A\), i.e. \(1_A(x) = 1\) if \(x \in A\), otherwise \(1_A(x) = 0\). Each polynomial \(P_n(q)\) is monic of degree \(2n - 2\), its coefficients are non-negative integers and it is self-reciprocal (see [2]). The evaluations of \(P_n(q)\) at some complex roots of 1 have number-theoretical interpretations (see [2]), e.g.

\[ \sigma(n) = P_n(1), \]
\[ \frac{r_{1,0,1}(n)}{4} = P_n(-1), \]
\[ \frac{r_{1,0,2}(n)}{2} = |P_n(\sqrt{-1})|, \]
\[ \frac{r_{1,1,1}(n)}{6} = \text{Re } P_n \left( \frac{-1 + \sqrt{-3}}{2} \right), \]

where \(\sigma(n)\), \(\frac{r_{1,0,1}(n)}{4}\), \(\frac{r_{1,0,2}(n)}{2}\) and \(\frac{r_{1,1,1}(n)}{6}\) are multiplicative functions² given by

\[ \sigma(n) = \sum_{d|n} d, \]
\[ r_{a,b,c}(n) = \# \{(x, y) \in \mathbb{Z}^2 : ax^2 + bxy + cy^2 = n\}. \]

Furthermore, for \(q = \frac{1 + \sqrt{-3}}{2}\), the same sequence \(n \mapsto P_n(q)\) is related to \(r_{1,0,1}(n)\) in three ways (see [3]), depending on the congruence class of \(n\) in \(\mathbb{Z}/3\mathbb{Z}\),

\[ \left| P_n \left( \frac{1 + \sqrt{-3}}{2} \right) \right| = \begin{cases} 
\frac{r_{1,0,1}(n)}{4} & \text{if } n \equiv 0 \pmod{3}, \\
\frac{r_{1,0,1}(n)}{4} & \text{if } n \equiv 1 \pmod{3}, \\
\frac{r_{1,0,1}(n)}{4} & \text{if } n \equiv 2 \pmod{3}.
\end{cases} \]

For any integer \(n \geq 1\), consider the two-sided sequence

\[ \ell_{n,k} = \ln \left( \frac{3}{2} k + \sqrt{\left( \frac{3}{2} k \right)^2 + 3n} \right) \]

and the intervals

\[ \mathcal{L}_{n,k} = [\ell_{n,k} - \ln 3, \ell_{n,k}], \]

where \(k\) runs over the integers. Define the following variation of the polynomials \(P_n(q)\),

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¹The original definition of \(P_n(q)\), which we refer to as Kassel-Reutenauer polynomials, given in [1] is rather different, but equivalent, to the one presented here. We preferred to take the logarithm of the divisors in place of the divisors themselves in order to work with intervals \(P_{n,k}\) of constant length.

²The proofs that the functions \(\frac{r_{1,0,1}(n)}{4}\), \(\frac{r_{1,0,2}(n)}{2}\) and \(\frac{r_{1,1,1}(n)}{6}\) are multiplicative can be found in [4], pages 413, 417 and 431 respectively.
For example, in order to compute $L_6(q)$ from the definition, we need to consider the intervals $[\ell_{6,k} - \ln 3, \ell_{6,k}]$ on the real line and to count the number of values of $\ln d$ inside each interval, where $d$ runs over the divisors of $n$. This data is shown in Fig 1, where the numbers $\ell_{6,k}$ are plotted on the line below (the corresponding values of $k$ are labelled) whereas the numbers $\ln d$ are plotted on the line above (the corresponding values of $d$ are labelled). Counting the number of intersections between the horizontal and the vertical lines, we obtain that the coefficients of $L_6(q)$ are as follows,

$$L_6(q) = \sum_{d|n} \sum_{k \in \mathbb{Z}} 1_{\ell_{n,k}} (\ln d) \ q^k.$$  

Like $P_n(q)$, the polynomial $L_n(q)$ is monic of degree $2n - 2$, self-reciprocal and its coefficients are non-negative integers. The aim of this paper is to express the multiplicative functions $r_{1,1,1}(n)$ and $r_{1,0,3}(n)$ in terms of the evaluations of $L_n(q)$ at roots of the unity. More precisely, we will prove the following result.

**Theorem 1.** For each $n \geq 1$,

$$A002324(n) \overset{\text{def}}{=} \frac{r_{1,1,1}(n)}{6} = 4 \sigma(n) - 3 \ L_n(1),$$ \hspace{1cm} (1)

$$A096936(n) \overset{\text{def}}{=} \frac{r_{1,0,3}(n)}{2} = L_n(-1).$$ \hspace{1cm} (2)

### 2 Auxiliary results for the identity (1)

For any $n \geq 1$, we will use the following notation,

$$d_{a,m}(n) := \# \{ d | n : \ d \equiv a \pmod{m} \}.$$  

3The proof that the function $\frac{r_{1,0,3}(n)}{2}$ is multiplicative can be found in [4], page 421.
Lemma 2. For all integers \( n \geq 1 \),
\[
\frac{r_{1,1}(n)}{6} = d_{1,3}(n) - d_{2,3}(n).
\]

Proof. This result can be found as equation (3) in [5].

Lemma 3. For any integer \( n \geq 1 \),
\[
3 \left\lfloor 3^{-1} n \right\rfloor - n = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{3}, \\
2 & \text{if } n \equiv 1 \pmod{3}, \\
1 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]
\[
n - 3 \left\lfloor 3^{-1} n \right\rfloor = \begin{cases} 
0 & \text{if } n \equiv 0 \pmod{3}, \\
1 & \text{if } n \equiv 1 \pmod{3}, \\
2 & \text{if } n \equiv 2 \pmod{3}.
\end{cases}
\]

Proof. It is enough to evaluate \( 3 \left\lfloor 3^{-1} n \right\rfloor - n \) and \( 3 \left\lfloor 3^{-1} n \right\rfloor - n \) at \( n = 3k + r \), for \( k \in \mathbb{Z} \) and \( r \in \{0, 1, 2\} \).

Lemma 4. For any pair of integers \( n \geq 1 \) and \( k \), the inequalities
\[
\ell_{n,k} - \ln 3 < \ln d \leq \ell_{n,k}
\]
hold if and only if the inequalities
\[
3^{-1} d - \frac{n}{d} \leq k < d - 3^{-1} \frac{n}{d}
\]
hold.

Proof. The inequalities
\[
\ell_{n,k} - \ln 3 < \ln d \leq \ell_{n,k}
\]
are equivalent to
\[
\ln d \leq \ell_{n,k} < \ln d + \ln 3.
\]

Applying the strictly increasing function \( x \mapsto e^x - n e^{-x} \) to the last inequalities we obtain the following equivalent inequalities
\[
3^{-1} d - \frac{n}{d} \leq k < d - 3^{-1} \frac{n}{d}
\]
Indeed,
\[
\frac{e^{\ln d} n e^{-\ln d}}{3} = 3^{-1} d - \frac{n}{d},
\]
\[
\frac{e^{\ln d + \ln 3} n e^{-(\ln d + \ln 3)}}{3} = d - 3^{-1} \frac{n}{d}
\]
and
\[
\frac{e^{\ell_{n,k}}}{3} - n e^{-\ell_{n,k}} = \frac{\frac{3}{2} k + \sqrt{\left(\frac{3}{2} k\right)^2 + 3 n}}{3} - \frac{n}{\frac{3}{2} k + \sqrt{\left(\frac{3}{2} k\right)^2 + 3 n}}
\]
\[
= \frac{3}{2} k + \frac{\sqrt{\left(\frac{3}{2} k\right)^2 + 3 n}}{3} + \frac{\frac{3}{2} k - \sqrt{\left(\frac{3}{2} k\right)^2 + 3 n}}{3}
\]
\[
= k.
\]
So, the lemma is proved.
Lemma 5. Let $n \geq 1$ be an integer. For all $d \mid n$,

$$\sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_{n,k}}(\ln d) = \left[d - 3^{-1} \frac{n}{d}\right] - \left[3^{-1} d - \frac{n}{d}\right].$$

Proof. For all integers $n \geq 1$ and $k$,

$$\sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_{n,k}}(\ln d) = \# \left\{ k \in \mathbb{Z} : \ell_{n,k} - \ln 3 < \ln d \leq \ell_{n,k} \right\}$$

$$= \# \left\{ k \in \mathbb{Z} : 3^{-1} d - \frac{n}{d} \leq k < d - 3^{-1} \frac{n}{d} \right\} \quad \text{(Lemma 4)}$$

$$= \# \left\{ k \in \mathbb{Z} : \left[3^{-1} d - \frac{n}{d}\right] \leq k < \left[d - 3^{-1} \frac{n}{d}\right] \right\},$$

$$= \left[d - 3^{-1} \frac{n}{d}\right] - \left[3^{-1} d - \frac{n}{d}\right].$$

So, the lemma is proved. \qed

3 Auxiliary results for the identity (2)

Lemma 6. The function $\frac{r_{1,0,3}(n)}{2}$ is multiplicative.

Proof. See page 421 in [4]. \qed

Lemma 7. For all integers $n \geq 1$,

$$\frac{r_{1,0,3}(n)}{2} = d_{1,3}(n) - d_{2,3}(n) + 2 (d_{4,12}(n) - d_{8,12}(n)).$$

Proof. This result can be found as equation (1) in [5]. \qed

Recall that the nonprincipal Dirichlet character mod 3 is the 3-periodic arithmetic function $\chi_3(n)$ given by $\chi_3(0) = 0$, $\chi_3(1) = 1$ and $\chi_3(2) = -1$.

Lemma 8. For all $n \geq 1$,

$$\frac{(-1)^{\left\lfloor 3^{-1} n \right\rfloor} - (-1)^{\left\lceil 3^{-1} n \right\rceil}}{2} = (-1)^{n-1} \chi_3(n).$$

Proof. It is enough to substitute $n = 3k + r$, with $k \in \mathbb{Z}$ and $r \in \{0, 1, 2\}$, in both sides in order to check that they are equal. \qed

Lemma 9. For all $n \geq 1$,

$$\sum_{d \mid n} (-1)^{\delta(n)} (-1)^{d-1} \chi_3(d) = (-1)^{n-1} \frac{r_{1,0,3}(n)}{2}.$$
Proof. By Lemma 6, the function $\frac{r_{1,0,3}(n)}{2}$ is multiplicative. Also, it is easy to check that the functions $(-1)^{n-1}$ and $\chi_3(n)$ are multiplicative. So, the functions $f(n) = (-1)^{n-1} \frac{r_{1,0,3}(n)}{2}$ and $(-1)^{n-1} \chi_3(n)$ are multiplicative, because the multiplicative property is preserved by ordinary product. The function $g(n) = \sum_{d|n} (-1)^{\frac{n}{d}-1} (-1)^{d-1} \chi_3(d)$ is multiplicative, because Dirichlet convolution preserves the multiplicative property. So, it is enough to prove that $f(p^k) = g(p^k)$ for each prime power $p^k$.

Consider the case $p = 2$. The following elementary equivalences hold for any integer $m \geq 0$,

\[
2^m \equiv 1 \pmod{3} \iff m \equiv 0 \pmod{2},
2^m \equiv 2 \pmod{3} \iff m \equiv 1 \pmod{2},
2^m \equiv 4 \pmod{12} \iff m \equiv 0 \pmod{2} \text{ and } m \neq 0,
2^m \equiv 8 \pmod{12} \iff m \equiv 1 \pmod{2} \text{ and } m \neq 1.
\]

So, for each integer $k \geq 1$,

\[
\begin{align*}
d_{1,3}(2^k) &= \# [0, k] \cap 2\mathbb{Z} = \left\lfloor \frac{k}{2} \right\rfloor + 1, \\
d_{2,3}(2^k) &= \# [1, k] \cap (2\mathbb{Z} + 1) = \left\lceil \frac{k}{2} \right\rceil, \\
d_{4,12}(2^k) &= \# [2, k] \cap 2\mathbb{Z} = \left\lfloor \frac{k}{2} \right\rfloor, \\
d_{8,12}(2^k) &= \# [3, k] \cap (2\mathbb{Z} + 1) = \left\lceil \frac{k}{2} \right\rceil - 1.
\end{align*}
\]

For any $k \geq 1$, it follows that
\[ g(2^k) = \sum_{j=0}^{k} (-1)^{2^{k-j}-1} (-1)^{2^{j-1}} \chi_3(2^j) \]
\[ = \sum_{j=0}^{k} (-1)^{2^{k-j}-1} (-1)^{2^{j-1}} (-1)^j \]
\[ = -1 - (-1)^k + \sum_{j=1}^{k-1} (-1)^j \]
\[ = -1 - (-1)^k + \frac{-1 - (-1)^k}{2} \]
\[ = -3 \frac{1 + (-1)^k}{2} \]
\[ = -3 \left( 1 + \left\lceil \frac{k}{2} \right\rceil - \left\lfloor \frac{k}{2} \right\rfloor \right) \]
\[ = - \left( \left( \left\lceil \frac{k}{2} \right\rceil + 1 \right) - \left\lfloor \frac{k}{2} \right\rfloor + 2 \left( \left\lceil \frac{k}{2} \right\rceil - \left( \left\lfloor \frac{k}{2} \right\rfloor - 1 \right) \right) \right) \]
\[ = (-1)^{2^{k-1}} \left( d_{1,3}(2^k) - d_{2,3}(2^k) + 2 \left( d_{4,12}(2^k) - d_{8,12}(2^k) \right) \right) \]
\[ = f(2^k) \quad \text{(Lemma 7)}. \]

Let \( p \) and \( k \geq 1 \) be an odd prime and an integer respectively. Notice that \((-1)^{p^j-1} = 1\) for all \( 0 \leq j \leq k \). Also, \( d_{4,12}(p^k) = d_{8,12}(p^k) = 0 \), because \( p^k \) has no even divisor. So, for any \( k \geq 1 \),

\[ g(p^k) = \sum_{j=0}^{k} (-1)^{p^k-j-1} (-1)^{p^j-1} \chi_3(p^j) \]
\[ = \sum_{j=0}^{k} \chi_3(p^j) \]
\[ = d_{1,3}(p^k) - d_{2,3}(p^k) \]
\[ = (-1)^{p^k-1} \left( d_{1,3}(p^k) - d_{2,3}(p^k) + 2 \left( d_{4,12}(p^k) - d_{8,12}(p^k) \right) \right) \]
\[ = f(p^k) \quad \text{(Lemma 7)}. \]

Therefore, \( f(n) = g(n) \) for all \( n \geq 1 \).

\[ \text{Lemma 10. For each } d|n, \]
\[ \sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_{n,k}}(\ln d) (-1)^k = \frac{1}{2} \left( (-1)^{\lceil \frac{3^{-1} \cdot d - \frac{n}{d}}{2} \rceil} - (-1)^{\lceil \frac{d - 3^{-1} \cdot \frac{n}{d}}{2} \rceil} \right). \]
Proof. For any integer \( n \geq 1 \) and any \( d \mid n \),

\[
\sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_n,k} ( \ln d ) (-1)^k = \sum_{3^{-1} d^{-\frac{n}{d}} \leq k < d - 3^{-1} \frac{n}{d}} (-1)^k \quad \text{(Lemma 4)}
\]

\[
= \sum_{[3^{-1} d^{-\frac{n}{d}}] \leq k < [d - 3^{-1} \frac{n}{d}]} (-1)^k.
\]

Substituting \( a = \lceil 3^{-1} d - \frac{n}{d} \rceil \), \( b = \lceil d - 3^{-1} \frac{n}{d} \rceil \) and \( q = -1 \) in the geometric sum

\[
\sum_{a \leq k < b} q^k = \frac{q^a - q^b}{1 - q}
\]

we obtain

\[
\sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_n,k} ( \ln d ) (-1)^k = \frac{(-1)^{\lceil 3^{-1} d - \frac{n}{d} \rceil} - (-1)^{\lceil d - 3^{-1} \frac{n}{d} \rceil}}{1 - (-1)}
\]

\[
= \frac{1}{2} \left( (-1)^{\lceil 3^{-1} d - \frac{n}{d} \rceil} - (-1)^{\lceil d - 3^{-1} \frac{n}{d} \rceil} \right).
\]

So, the lemma is proved. \( \square \)

4 Proof of the main result

We proceed now with the proof of the main result of this paper.

Proof of Theorem 1. Identity (1) follows from the following transformations,
\[ L_n(1) = \sum_{d|n} \sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_{n,k}}(\ln d) \]
\[ = \sum_{d|n} \left( \left\lfloor d - 3^{-1} \frac{n}{d} \right\rfloor - \left\lfloor \frac{n}{d} - 3^{-1} d \right\rfloor \right) \quad \text{(Lemma 5)} \]
\[ = \sum_{d|n} \left( d + \frac{n}{d} \right) + \sum_{d|n} \left\lfloor -3^{-1} \frac{n}{d} \right\rfloor - \sum_{d|n} \left\lfloor -3^{-1} d \right\rfloor \]
\[ = \sum_{d|n} \left( d + \frac{n}{d} \right) - \sum_{d|n} \left\lfloor 3^{-1} \frac{n}{d} \right\rfloor - \sum_{d|n} \left\lfloor 3^{-1} d \right\rfloor \]
\[ = \frac{2}{3} \sum_{d|n} \left( d + \frac{n}{d} \right) + \frac{1}{3} \sum_{d|n} \left( \frac{n}{d} - 3 \left\lfloor 3^{-1} \frac{n}{d} \right\rfloor \right) - \frac{1}{3} \sum_{d|n} \left( 3 \left\lfloor 3^{-1} d \right\rfloor - d \right) \]
\[ = \frac{4\sigma(n)}{3} + \frac{d_{1,3}(n) + 2d_{2,3}(n)}{3} - \frac{2d_{1,3}(n) + d_{2,3}(n)}{3} \quad \text{(Lemma 3)} \]
\[ = \frac{4\sigma(n)}{3} - \frac{d_{1,3}(n) - d_{2,3}(n)}{3} \]
\[ = \frac{4}{3} \sigma(n) - \frac{1}{3} \frac{r_{1,1,1}(n)}{6} \quad \text{(Lemma 2)}. \]

Identity (2) follows from the following transformations,

\[ \frac{L_n(-1)}{(-1)^{n-1}} = \sum_{d|n} \sum_{k \in \mathbb{Z}} 1_{\mathcal{L}_{n,k}}(\ln d) (-1)^k \]
\[ = \sum_{d|n} \frac{1}{2} \left( (-1)^{\left\lfloor -3^{-1} d - \frac{n}{d} \right\rfloor} - (-1)^{\left\lfloor \frac{n}{d} - 3^{-1} d \right\rfloor} \right) \quad \text{(Lemma 10)} \]
\[ = \sum_{d|n} \frac{1}{2} \left( (-1)^{\left\lfloor 3^{-1} d - \frac{n}{d} \right\rfloor} - (-1)^{\frac{n}{d} - \left\lfloor -3^{-1} d \right\rfloor} \right) \]
\[ = \sum_{d|n} (-1)^{\frac{n}{d} - 1} \frac{(-1)^{\left\lfloor 3^{-1} d \right\rfloor} - (-1)^{\left\lfloor -3^{-1} d \right\rfloor}}{2} \]
\[ = \sum_{d|n} (-1)^{\frac{n}{d} - 1} (-1)^{d-1} \chi_3(d) \quad \text{(Lemma 8)} \]
\[ = (-1)^{n-1} \frac{r_{1,0,3}(n)}{2} \quad \text{(Lemma 9)}. \]

So, the theorem is proved. \[ \square \]

5 Final remarks

1. Let \( k \) be a field and \( \mathcal{R} \) be a \( k \)-algebra. The codimension of an ideal \( I \) of \( \mathcal{R} \) is the dimension of the quotient \( \mathcal{R}/I \) as a vector space over \( k \).
Consider the free abelian group of rank 2, denoted $\mathbb{Z} \oplus \mathbb{Z}$. Let $k = \mathbb{F}_q$ be the finite field with $q$ elements and $R = \mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ its group algebra. Kassel and Reutenauer [1] proved that, for any prime power $q$, the number of ideals of codimension $n \geq 1$ of $\mathbb{F}_q[\mathbb{Z} \oplus \mathbb{Z}]$ is $(q - 1)^2 P_n(q)$. So, it is natural to look for connections between the values of $L_n(q)$, when $q$ is a prime power, and the algebraic structures related to $\mathbb{F}_q$.

2. The polynomials $P_n(q)$ are generated by the product (see [2])

$$
\prod_{m \geq 1} \frac{(1 - t^m)^2}{(1 - qt^m)(1 - q^{-1}t^m)} = 1 + (q + q^{-1} - 2) \sum_{n=1}^{\infty} \frac{P_n(q)}{q^{n-1}} t^n.
$$

It would be interesting to find a similar generating function for $L_n(q)$.

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