Charged-particle multiplicity moments as described by shifted Gompertz distribution in $e^+e^-$, $p\bar{p}$ and $pp$ collisions at high energies

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Abstract

In continuation of our earlier work, in which we analysed the charged particle multiplicities in leptonic and hadronic interactions at different center-of-mass energies in full phase space as well as in restricted phase space using the shifted Gompertz distribution, a detailed analysis of the normalized moments and normalized factorial moments is reported here. A two-component model in which a probability distribution function is obtained from the superposition of two shifted Gompertz distributions, as introduced in our earlier work, has also been used for the analysis. This is the first analysis of the moments with the shifted Gompertz distribution. Analysis has also been performed to predict the moments of multiplicity distribution for the $e^+e^-$ collisions at $\sqrt{s} = 500$ GeV at a future collider.

Keywords: Charged multiplicities, Probability Distribution Functions, Factorial moments

1. Introduction

In one of our recent papers, we introduced a statistical distribution, the shifted Gompertz distribution to investigate the multiplicity distributions of charged particles produced in $e^+e^-$ collisions at the LEP, $p\bar{p}$ interactions at
the SPS and \( pp \) collisions at the LHC at different center of mass energies in full phase space as well as in restricted phase space [1]. A distribution of the largest of two independent random variables, the shifted Gompertz distribution was introduced by Bemmaor [2] as a model of adoption of innovations. One of the parameters has an exponential distribution and the other has a Gumbel distribution, also known as log-Weibull distribution. The non-negative fit parameters define the scale and shape of the distribution. Subsequently, the shifted Gompertz distribution has been widely studied in various contexts [3, 4, 5]. In our earlier work [1] by studying the charged particle multiplicities, we showed that this distribution can be successfully used to study the statistical phenomena in high energy \( e^+ e^- \), \( p\bar{p} \) and \( pp \) collisions at the LEP, SPS and LHC colliders, respectively.

A multiplicity distribution is represented by the probabilities of \( n \)-particle events as well as by its moments or its generating function. The aim of the present work is to extend the analysis by calculating the higher moments of a multiplicity distribution. Because the moments are calculated as derivatives of the generating function, the moment analysis is a powerful tool which helps to unfold the characteristics of multiplicity distribution. The multi-particle correlations can be studied through the normalized moments and normalized factorial moments of the distribution. The dependence of moments on energy can also reveal the KNO (Koba, Nielsen and Olesen) scaling [6, 7, 8] conservation or violation. Several analyses of moments have been done at different energies, using different probability distribution functions and different types of particles [9, 10, 11, 12]. The higher moments also can identify the correlations amongst produced particles.

In section 2, formulae for the Probability Distribution Function (PDF) of the shifted Gompertz distribution, normalized moments and the normalized factorial moments used for the analysis are given. A two-component model has been used and modification of distributions carried out, in terms of these two components; one from soft events and another from semi-hard events. Superposition of distributions from these two components, by using appropriate weights is done
to build the full multiplicity distribution. When multiplicity distribution is fitted with the weighted superposition of two shifted Gompertz distributions, we find that the agreement between data and the model improves considerably. The details of these fits are published in [1]. The distributions have been fitted both in full phase space as well as in restricted rapidity windows for $p\bar{p}$ and $pp$ data and in five rapidity windows and in full phase space only for $e^+e^-$, in terms of soft and semi-hard components.

Section 3 presents the evaluations of moments from experimental data, the fitted shifted Gompertz distributions and the fitted modified shifted Gompertz distributions. Section 4 details the method of estimating uncertainties on the moments. Discussion and conclusion are presented in Section 5.

2. Shifted Gompertz distribution and Moments

The particle production dynamics can be understood by analysing the charged particle multiplicity distribution as its measurements can provide relevant constraints for particle-production models. Charged particle multiplicity is defined as the average number of charged particles $n$, produced in a collision at a given energy in the center of mass system.

In addition, the analysis of moments of the distribution is often used to study the patterns and correlations in the multi-particle final state of high-energy collisions in the presence of statistical fluctuations. The fractal structures present in the multiplicity distributions have often been studied to search for the embedded constraints on the underlying particle production mechanism [13, 14]. The observation of fractal structures is of great interest because it imposes strong constraints on the underlying particle-production mechanism. We define different kinds of moments as follows.

Let $X$ be any non-negative random variable having the shifted Gompertz distribution with parameters $b$ and $\beta$, where $b > 0$ is a scale parameter and $\beta > 0$ is a shape parameter. The probability distribution function (PDF) of $X$...
is given by

\[ P_X(x; b, \beta) = be^{-(bx+\beta e^{-bx})}(1 + \beta(1 - e^{-bx})) \text{, where } x > 0 \]  

(1)

The raw moments \((c_n)\) and factorial moments \((f_n)\) are defined as:

\[ c_n = E[X^n] \text{ and } f_n = E[(X)(X-1)(X-2)...(X-(n-1))] \]  

(2)

Whereas, the normalized moments \((C_n)\) and normalized factorial moments \((F_n)\) are defined as following:

\[ C_n = \frac{E[X^n]}{(E[X])^n} \text{ and } F_n = \frac{E[(X)(X-1)(X-2)...(X-(n-1))]}{(E[X])^n} \]  

(3)

\(n\) as a natural number ranging from 1 to \(\infty\). The Mean value \((E[X])\) of Shifted Gompertz distribution is given by

\[ E[X] = \frac{1}{b}(\gamma + \ln \beta + \frac{1 - e^{-\beta}}{\beta} + \Gamma[0, \beta]) \]  

(4)

and the \(c_2\) moment is given by

\[ c_2 = E[X^2] = \frac{2}{b^2\beta}\left(\gamma + \Gamma[0, \beta] + \beta^2 3F_3\{\{1, 1, 1\}, \{2, 2, 2\}, -\beta + \ln \beta\}\right) \]  

(5)

The higher order raw moments \((c_n)\) can be found by the Moment Generating Function of the Shifted Gompertz distribution\([b, \beta, t]\)

\[ e^{-\beta} - (1 + \frac{t}{b\beta})\beta^t(\Gamma[1 - \frac{t}{b}] - \Gamma[1 - \frac{t}{b}, \beta]) \]  

(6)

Also \(f_2\) Moment is given by

\[ f_2 = E[(X)(X-1)] = \frac{2}{b^2\beta}\left(\gamma + \Gamma[0, \beta] + \beta^2 3F_3\{\{1, 1, 1\}, \{2, 2, 2\}, -\beta + \ln \beta\}\right) \]  

\[ - \frac{(1 - e^{-\beta} + \beta(\gamma + \Gamma[0, \beta] + \ln \beta))}{b\beta} \]  

(7)

The higher factorial moments \((f_n)\) can be found by the Generating Function of Shifted Gompertz Distribution\([b, \beta, t]\)

\[ e^{-\beta} - \beta \ln \frac{t}{b}(\Gamma[1 - \frac{\ln t}{b}] - \Gamma[1 - \frac{\ln t}{b}, \beta])(1 + \frac{\ln t}{b\beta}) \]  

(8)
where

(i) \( \gamma \approx 0.5772156 \) stands for the Euler constant (also referred to as Euler-Mascheroni constant).
(ii) \( \Gamma[s] \) the Euler Gamma function and \( \Gamma[s, x] \) the incomplete Gamma function defined below;

\[
\Gamma[s] = \int_{0}^{\infty} t^{s-1} e^{-t} dt \quad \Gamma[s, x] = \int_{x}^{\infty} t^{s-1} e^{-t} dt
\] (9)

(iv) \( _3F_3 \) is a Generalized Hypergeometric function.

\[
_3F_3\left[ \{1, 1, 1\}, \{2, 2, 2\}, -\beta \right] = \sum_{k=1}^{\infty} \frac{(-1)^{k+1} \beta^{k+1}}{k!k^2}
\] (10)

2.1. Modified Shifted Gompertz Distribution

It is well established that at high energies, charged particle multiplicity distribution in full phase space becomes broader than a Poisson distribution. The most widely adopted, Negative Binomial distribution \[15\] to describe the multiplicity spectra, fails to explain the experimental data. As a corrective measure to explain the failure, a two-component approach, was introduced by A. Giovannini et al \[15\]. The details are included in our earlier publication \[1\] on the Shifted Gompertz distribution.

To better explain the data at high energies, a superposition of two shifted Gompertz distributions, which are interpreted as soft and hard components, is used. The multiplicity distribution is produced by adding weighted superposition of multiplicity in soft events and multiplicity distribution in semi-hard events. This approach combines merely two classes of events and not two different particle-production mechanisms. Therefore, no interference terms need to be introduced. The final distribution is the superposition of the two independent distributions. We call it 'modified shifted Gompertz distribution'.

\[
P(n) = \alpha P_{soft}^{Gomp}(n) + (1 - \alpha) P_{semi-hard}^{Gomp}(n)
\] (11)
Adopting this approach for the multiplicity distributions in $e^+e^-$, $pp$ and $p\bar{p}$ collisions at high energies, the data at different energies are fitted with the distribution which involves five parameters as given below:

$$P_n(\alpha : b_1, \beta_1; b_2, \beta_2) = \alpha P_n(soft) + (1 - \alpha)P_n(semi-hard)$$

where $\alpha$ is the fraction of soft events, $(b_1, \beta_1)$ and $(b_2, \beta_2)$ are respectively the scale and shape parameters of the two distributions.

Values of $\alpha$ evaluated for different interactions, were published in the tables in our previous paper [1]. In figure 2, we show the variation of $\alpha$ as a function of collision energy and pseudorapidity for $pp$ interactions at LHC energies. The values show that the alpha decreases with collision energy as well as with increasing pseudo-rapidity window.

3. Analysis and Results

Calculations of the normalized moments and the normalized factorial moments are presented by using the data from different experiments and following three collision types;

i) $e^+e^-$ annihilations at different collision energies, from 91 GeV up to the highest energy of 206 GeV at LEP2, from two experiments L3 [16] and OPAL [17, 18, 19, 20] are analysed.

ii) $pp$ collisions at LHC energies from 900 GeV, 2360 GeV and 7000 GeV [21] are analysed for five intervals of increasing extent in pseudorapidity $|\eta| < 0.5$ up to $|\eta| < 2.4$.

iii) $p\bar{p}$ collisions at energies from 200 GeV, 540 GeV and 900 GeV [7, 8] are analysed in full phase space as well as in pseudorapidity intervals from $|\eta| < 0.5$ up to $|\eta| < 5.0$, where $\eta$ is defined as $-\ln[tan(\theta/2)]$, and $\theta$ is the polar angle of the particle with respect to the counter-clockwise beam direction.

The PDF defined by equation (1) is used to fit the experimental data on charged particle multiplicity distributions, for the shifted Gompertz function and the modified (two-component) function. Results from these fits to the above mentioned data were published in our earlier work [1]. As an example, results for L3
data are shown in figure 1. To avoid repetition, the details of other figures are not given here. It was shown that the data are very well explained by the modified shifted Gompertz distribution and the $\chi^2$ values for the fits in almost all cases reduce substantially. In the present analysis we calculate the normalized moments and the normalized factorial moments defined in equations (2-8).

Figures 3-6 show the normalized moments ($C_q$) and the normalized factorial moments ($F_q$) calculated from the data of two experiments at LEP, at different energies for $e^+e^-$ collisions and also from the fitted shifted Gompertz and modified shifted Gompertz distributions. The values of the moments are documented in table 1.

Figures 7-10 show the normalized moments ($C_q$) and the normalized factorial moments ($F_q$) calculated from the data and from the modified shifted Gompertz distribution for the $p\bar{p}$ data at energies from 200, 540 and 900 GeV, in four rapidity bins and full phase space. The values of the moments are documented in table 2.

Figures 11-14 show the normalized moments ($C_q$) and the normalized factorial moments ($F_q$) calculated from the data and from the modified shifted Gompertz distribution for the $pp$ data at energies from 900, 2360 and 7000 GeV in five rapidity bins. The values of the moments are documented in table 3.

It is observed that the moments decrease with the increase in rapidity window, at all energies. It is also interesting to compare the results from $pp$ and $p\bar{p}$ collisions at the same c.m. energy. From the comparison at 900 GeV, we find that the moments have higher values in case of $p\bar{p}$ than $pp$. For example, we include figure 15 to show the dependence for $|\eta| < 0.5$. However values of multiplicity for $pp$ collisions at 900 GeV in full phase space are not available. In addition, for $pp$ and for $p\bar{p}$ the rapidity intervals are also different. So a direct comparison is not feasible.

Figures 5-14, described above, are shown only for the modified shifted Gompertz distributions. To avoid repetition and cluttering of figures, the figures for shifted Gompertz distributions are only included for $e^+e^-$ collisions and are not included for $pp$ and $p\bar{p}$ data. However the values of the moments are given in
The predictions for normalized moments and normalized factorial moments are also made for $e^+e^-$ collisions at 500 GeV at a future collider. By using the shifted Gompertz distribution, the prediction for probability distribution is made, as shown in figure 16. Using this predicted distribution, $1\sigma$ confidence interval band, moments have been calculated, as given in table 4.

It is observed that in case of $pp$ and $pp$ interactions, $C_2$, $C_3$ and $F_2$, $F_3$ remain roughly constant with energy while higher moments $C_4$, $C_5$, $F_4$, $F_5$ show an increase with increasing energy. The increase becomes more evident for larger rapidity windows. This leads to the depiction of violation of KNO scaling at high energies. Same conclusions have been reported in the reference [21] for $pp$ collisions at the LHC. However for the $e^+e^-$ collisions, the moments are roughly independent of energy. This is expected as the collision energy is low, nearly at the onset of energy range, which marks the start of KNO scaling violation.

4. Uncertainties on Moments

Given a distribution $P(n)$ which is normalized to unity with an uncertainty $\epsilon_n$, and assuming that the errors on the individual bins are uncorrelated, the moment errors can be calculated by using the method described in [22], using the partial derivatives:

$$\frac{\partial C_q}{\partial P_n} = \frac{n^n\langle n \rangle - \langle n^q \rangle qn}{\langle n \rangle^{q+1}}$$

$$\frac{\partial F_q}{\partial P_n} = \frac{n(n-1)\ldots(n-q+1)\langle n \rangle - \langle n(n-1)\ldots(n-q+1)q n \rangle}{\langle n \rangle^{q+1}}$$

The total error is then

$$E^2_q = \sum_n \left( \frac{\partial X_q}{\partial P_n} \epsilon_n \right)^2$$

where $X_q$ is $C_q$ or $F_q$. 
In the published data, multiplicities are given either as (value + statistical error + systematic error) or as (value + error). The error on the multiplicities have been taken as the total error, by adding the statistical and systematic errors in quadrature.

5. Conclusion

An analysis of moments of multiplicity distributions described within a newly proposed statistical distribution, the shifted Gompertz distribution and its modified form has been done. We had proposed and shown that the use of this statistical distribution for studying the multiplicity distributions in high energy collisions reproduces the results in $e^+e^−$, $p\bar{p}$ and $pp$ collisions very well.

A good agreement between the normalized moments as well as normalized factorial moments obtained from the shifted Gompertz distribution and its modified form with the experimental values, serves as a good test of the validity of the proposed distribution. The results have reproduced the violation of KNO scaling as observed for higher moments in the measured data. At higher energies, particularly at LHC energies, the moments strongly are dependent on the energy. The information dissemination from such an analysis is often used to study the patterns and correlations in the multi-particle final state of high-energy collisions in the presence of statistical fluctuations. In this connection we find that the factorial moments are large indicating correlations amongst the produced particles. The predictions for normalized moments and normalized factorial moments are also made for $e^+e^−$ collisions at 500 GeV at a future collider.

6. Data Availability

All the data used in the paper can be obtained from the references quoted or from the authors.
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Figure 1: Charged multiplicity distributions from L3 experiment. Solid lines represent the shifted Gompertz distribution and the modified shifted Gompertz distributions.
Figure 2: Dependence of $\alpha$ on c.m. energy and pseudrapidity for pp interaction
Figure 3: Normalized moments of shifted Gompertz distribution for the $e^+e^-$ collisions recorded by the L3 and OPAL experiments at different energies.
Figure 4: Normalized factorial moments of shifted Gompertz distribution for the $e^+e^-$ collisions recorded by the L3 and OPAL experiments at different energies.
Figure 5: Normalized moments of modified shifted Gompertz distribution for the $e^+e^-$ collisions recorded by the L3 and OPAL experiments at different energies.
Figure 6: Normalized factorial moments of modified shifted Gompertz distribution for the $e^+e^-$ collisions recorded by the L3 and OPAL experiments at different energies.
Figure 7: Normalized factorial moments of modified shifted Gompertz distribution for the $p\bar{p}$ collisions recorded by the UA5 experiment at different energies and in different rapidity bins.
Figure 8: Normalized factorial moments of modified shifted Gompertz distribution for the $pp$ collisions recorded by the UA5 experiment at different energies and in different rapidity bins.
Figure 9: Normalized factorial moments of modified shifted Gompertz distribution for the $p\bar{p}$ collisions recorded by the $UA5$ experiment at different energies and in different rapidity bins.
Figure 10: Normalized factorial moments of modified shifted Gompertz distribution for the $pp$ collisions recorded by the UA5 experiment at different energies and in different rapidity bins.
Figure 11: Normalized moments of modified shifted Gompertz distribution for the pp collisions recorded by the CMS experiment at different energies and in different rapidity bins.
Figure 12: Normalized moments of modified shifted Gompertz distribution for the pp collisions recorded by the CMS experiment at different energies and in different rapidity bins.
Figure 13: Normalized factorial moments of modified shifted Gompertz distribution for the $pp$ collisions recorded by the CMS experiment at different energies and in different rapidity bins.
Figure 14: Normalized factorial moments of modified shifted Gompertz distribution for the \(pp\) collisions recorded by the CMS experiment at different energies and in different rapidity bins.
Figure 15: Comparison of normalised factorial moments for $pp$ and $p\bar{p}$ interactions at the same c.m. energy in the same rapidity window.

Figure 16: Probability distribution predicted from shifted Gompertz distribution, in $1\sigma$ confidence interval band, for the $e^+e^-$ collisions at 500 GeV.
| Energy (GeV) | Normalized moments (Experiment) | Normalized factorial moments (Experiment) |
|-------------|---------------------------------|------------------------------------------|
|             | C_0                             | C_1                                      |
|             | C_2                             | C_3                                      |
|             | C_4                             | C_5                                      |
|             | C_6                             | C_7                                      |
|             | F_2                             | F_3                                      |
|             | F_4                             | F_5                                      |
|             | F_6                             | F_7                                      |
|             | F_8                             | F_9                                      |
| L3          | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
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|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |
|             | 26.404 ± 0.075                   | 26.404 ± 0.075                           |

Table 1: Moments of Experimental, shifted Gompertz and Modified shifted Gompertz distributions for $e^+e^-$ collisions
| Energy (GeV) | < n > | Normalized moments (Experiment) | Normalized factorial moments (Experiment) |
|-------------|-------|---------------------------------|-------------------------------------|
|              |       | C2 | C3 | C4 | C5 | F2 | F3 | F4 | F5 | F6 |
| 200         |       |    |    |    |    |    |    |    |    |    |
| 0.01        | 1.644 ± 0.070 | 1.914 ± 0.010 | 1.895 ± 0.026 | 1.714 ± 0.070 | 0.734 ± 0.040 | 1.51 ± 0.053 | 1.64 ± 0.070 | 1.89 ± 0.026 | 1.71 ± 0.070 | 0.73 ± 0.040 |
| 3/4         | 1.709 ± 0.068 | 1.864 ± 0.040 | 1.789 ± 0.062 | 1.512 ± 0.040 | 0.674 ± 0.030 | 1.41 ± 0.035 | 1.51 ± 0.040 | 1.79 ± 0.062 | 1.51 ± 0.040 | 0.67 ± 0.030 |
| 1           | 1.548 ± 0.202 | 1.828 ± 0.102 | 1.608 ± 0.202 | 1.368 ± 0.102 | 0.704 ± 0.080 | 1.29 ± 0.070 | 1.61 ± 0.102 | 1.61 ± 0.202 | 1.37 ± 0.102 | 0.70 ± 0.080 |
| 3           | 2.085 ± 0.479 | 1.79 ± 0.163 | 1.608 ± 0.263 | 1.373 ± 0.213 | 0.642 ± 0.083 | 1.29 ± 0.070 | 1.61 ± 0.163 | 1.61 ± 0.263 | 1.37 ± 0.213 | 0.64 ± 0.083 |
| 5           | 2.737 ± 0.234 | 1.48 ± 0.193 | 1.302 ± 0.302 | 1.148 ± 0.193 | 0.514 ± 0.051 | 1.29 ± 0.070 | 1.61 ± 0.193 | 1.61 ± 0.302 | 1.37 ± 0.193 | 0.64 ± 0.051 |
| 10          | 3.584 ± 0.185 | 1.215 ± 0.185 | 1.057 ± 0.185 | 0.884 ± 0.185 | 0.384 ± 0.038 | 1.29 ± 0.070 | 1.61 ± 0.185 | 1.61 ± 0.185 | 1.37 ± 0.185 | 0.64 ± 0.038 |
| 20          | 4.571 ± 0.130 | 1.18 ± 0.180 | 0.993 ± 0.180 | 0.835 ± 0.180 | 0.374 ± 0.037 | 1.29 ± 0.070 | 1.61 ± 0.180 | 1.61 ± 0.180 | 1.37 ± 0.180 | 0.64 ± 0.037 |
| 50          | 6.210 ± 0.125 | 1.171 ± 0.125 | 1.075 ± 0.125 | 0.915 ± 0.125 | 0.394 ± 0.039 | 1.29 ± 0.070 | 1.61 ± 0.125 | 1.61 ± 0.125 | 1.37 ± 0.125 | 0.64 ± 0.039 |
| 100         | 8.094 ± 0.137 | 1.163 ± 0.137 | 1.063 ± 0.137 | 0.903 ± 0.137 | 0.384 ± 0.038 | 1.29 ± 0.070 | 1.61 ± 0.137 | 1.61 ± 0.137 | 1.37 ± 0.137 | 0.64 ± 0.038 |
| 300         | 11.301 ± 0.227 | 1.154 ± 0.227 | 1.056 ± 0.227 | 0.901 ± 0.227 | 0.384 ± 0.038 | 1.29 ± 0.070 | 1.61 ± 0.227 | 1.61 ± 0.227 | 1.37 ± 0.227 | 0.64 ± 0.038 |
| 500         | 16.068 ± 0.312 | 1.147 ± 0.312 | 1.047 ± 0.312 | 0.901 ± 0.312 | 0.384 ± 0.038 | 1.29 ± 0.070 | 1.61 ± 0.312 | 1.61 ± 0.312 | 1.37 ± 0.312 | 0.64 ± 0.038 |

Table 2: Moments of Experimental, shifted Gompertz and modified shifted Gompertz distributions for $\bar{pF}$ collisions
| Energy (GeV) | h | C1 | C2 | C3 | C4 | C5 | F1 | F2 | F3 | F4 | F5 |
|-------------|---|----|----|----|----|----|----|----|----|----|----|
| 900         | 0.5 | 4.426 ± 0.064 | 5.584 ± 0.056 | 3.389 ± 0.052 | 9.981 ± 0.054 | 27.013 ± 0.387 | 1.956 ± 0.032 | 2.410 ± 0.013 | 1.122 ± 0.056 | 12.119 ± 0.115 |
|             | 0.55 | 4.597 ± 0.051 | 5.707 ± 0.043 | 3.272 ± 0.047 | 9.821 ± 0.043 | 22.142 ± 1.148 | 1.945 ± 0.027 | 2.387 ± 0.009 | 1.109 ± 0.034 | 13.058 ± 0.143 |
|             | 1.20 | 5.144 ± 0.041 | 6.756 ± 0.034 | 4.092 ± 0.036 | 7.751 ± 0.031 | 19.015 ± 0.999 | 2.215 ± 0.036 | 2.718 ± 0.013 | 1.262 ± 0.020 | 14.354 ± 0.314 |
|             | 1.50 | 5.207 ± 0.041 | 6.914 ± 0.032 | 4.216 ± 0.035 | 8.088 ± 0.030 | 18.093 ± 0.951 | 2.260 ± 0.036 | 2.693 ± 0.017 | 1.289 ± 0.020 | 14.265 ± 0.314 |
|             | 2.00 | 6.442 ± 0.024 | 8.174 ± 0.020 | 5.179 ± 0.026 | 9.378 ± 0.025 | 16.424 ± 0.897 | 1.683 ± 0.020 | 2.562 ± 0.007 | 1.780 ± 0.015 | 12.749 ± 0.217 |
| 2000        | 0.5 | 5.269 ± 0.059 | 6.565 ± 0.051 | 3.774 ± 0.050 | 11.900 ± 0.163 | 24.685 ± 1.178 | 1.751 ± 0.030 | 2.723 ± 0.012 | 1.019 ± 0.030 | 11.487 ± 0.475 |
|             | 1.20 | 5.832 ± 0.041 | 7.568 ± 0.034 | 4.798 ± 0.032 | 7.990 ± 0.030 | 19.913 ± 0.912 | 2.344 ± 0.035 | 2.606 ± 0.012 | 1.304 ± 0.023 | 11.798 ± 0.325 |
|             | 1.50 | 5.892 ± 0.041 | 7.660 ± 0.035 | 4.909 ± 0.034 | 8.071 ± 0.030 | 18.851 ± 0.906 | 2.314 ± 0.035 | 2.578 ± 0.012 | 1.286 ± 0.020 | 11.760 ± 0.324 |
|             | 2.00 | 7.234 ± 0.023 | 8.900 ± 0.020 | 6.276 ± 0.025 | 9.675 ± 0.026 | 15.547 ± 0.851 | 1.735 ± 0.026 | 2.569 ± 0.008 | 1.798 ± 0.015 | 11.507 ± 0.217 |
| 3000        | 0.5 | 4.004 ± 0.057 | 5.172 ± 0.043 | 2.175 ± 0.043 | 7.745 ± 0.043 | 16.128 ± 0.910 | 1.724 ± 0.030 | 2.741 ± 0.012 | 1.008 ± 0.029 | 11.306 ± 0.443 |
|             | 1.20 | 5.145 ± 0.041 | 6.570 ± 0.034 | 3.867 ± 0.036 | 10.949 ± 0.151 | 23.731 ± 1.380 | 1.395 ± 0.030 | 2.772 ± 0.012 | 1.209 ± 0.027 | 11.747 ± 0.321 |
|             | 1.50 | 5.248 ± 0.041 | 6.770 ± 0.035 | 4.198 ± 0.035 | 11.250 ± 0.156 | 22.410 ± 1.370 | 1.435 ± 0.030 | 2.727 ± 0.012 | 1.180 ± 0.026 | 11.610 ± 0.314 |
|             | 2.00 | 5.762 ± 0.041 | 7.256 ± 0.036 | 4.930 ± 0.037 | 11.971 ± 0.163 | 19.906 ± 1.318 | 1.427 ± 0.030 | 2.696 ± 0.012 | 1.262 ± 0.020 | 11.742 ± 0.317 |
| 4000        | 0.5 | 3.775 ± 0.040 | 4.269 ± 0.033 | 1.941 ± 0.033 | 6.902 ± 0.033 | 15.635 ± 0.963 | 1.235 ± 0.029 | 2.487 ± 0.012 | 0.878 ± 0.027 | 10.310 ± 0.341 |
|             | 1.20 | 4.658 ± 0.039 | 5.750 ± 0.032 | 3.486 ± 0.032 | 9.800 ± 0.032 | 19.149 ± 0.951 | 1.668 ± 0.029 | 2.413 ± 0.012 | 1.130 ± 0.023 | 10.812 ± 0.334 |
|             | 1.50 | 4.782 ± 0.039 | 5.952 ± 0.033 | 3.673 ± 0.033 | 10.150 ± 0.032 | 18.304 ± 0.943 | 1.617 ± 0.029 | 2.390 ± 0.012 | 1.106 ± 0.020 | 10.722 ± 0.333 |
|             | 2.00 | 5.275 ± 0.037 | 6.503 ± 0.032 | 4.273 ± 0.032 | 11.750 ± 0.032 | 17.240 ± 0.935 | 1.617 ± 0.029 | 2.390 ± 0.012 | 1.106 ± 0.020 | 10.722 ± 0.333 |

Table 3: Moments of Experimental, shifted Gompertz and Modified shifted Gompertz distributions for pp collisions
| Energy (GeV) \( (e^+e^-) \) | \(< n >\) | Normalized moments (Predicted) | Normalized factorial moments (Predicted) |
|-----------------|---------|-------------------------------|---------------------------------------|
|                 | C₂      | C₃       | C₄       | C₅       | F₂       | F₃       | F₄       | F₅       |
| 500             | 37.041 ± 0.007 | 1.145 ± 0.003 | 1.407 ± 0.008 | 2.187 ± 0.004 | 1.496 ± 0.008 | 1.138 ± 0.003 | 1.396 ± 0.008 | 1.936 ± 0.008 | 2.948 ± 0.002 |

Table 4: Predicted moments at \( \sqrt{s} = 500 \text{ GeV} \) from shifted Gompertz distribution for \( e^+e^- \) collisions