A NEW RELATIONSHIP BETWEEN THE DILATATION OF
PSEUDO-ANOSOV BRAIDS AND FIXED POINT THEORY

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Abstract. A relation between the dilatation of pseudo-Anosov braids
and fixed point theory was studied by Ivanov. In this paper we reveal a
new relationship between the above two subjects by showing a formula
for the dilatation of pseudo-Anosov braids by means of the representa-
tions of braid groups due to B. Jiang and H. Zheng.

1. Introduction

The purpose of this paper is to reveal a new relationship between the
dilatation of pseudo-Anosov braids and fixed point theory. For this purpose
we obtain a new formula to determine the dilatation of pseudo-Anosov braids
from the representation $\zeta_{n,m}$ due to Jiang and Zheng [14].

Let us recall the notion of pseudo-Anosov braids. Let $\Sigma_g$ be a closed
surface of genus $g$ and $P_n$ be an $n$-point subset of $\Sigma_g$. We denote by $\Sigma_{g,n}$
the subset of $\Sigma_g$ deleting $P_n$. We consider the case when $\Sigma_{g,n}$ has negative
Euler characteristic. Let $f$ be a homeomorphism of $\Sigma_g$ fixing $P_n$ setwise.
We recall that $f$ is periodic if $f^k$ equals identity for some $k > 0$, and it is reducible if there exists an $f$-invariant closed 1-manifold $J \subset \Sigma_{g,n}$ whose
complementary components in $\Sigma_{g,n}$ have negative Euler characteristic or else
are Möbius bands. We refer to $J$ as a reduction of $f$. Finally, $f$ is pseudo-
Anosov if there exists a number $\lambda > 1$ and a pair $F^s, F^u$ of transverse
measured foliations with singularities modelled on $k$-prongs, $k = 1, 2, \ldots$ in
Figure 1 such that the equalities $f(F^s) = (1/\lambda)F^s$ and $f(F^u) = \lambda F^u$ hold.
Furthermore, the one-prong singularities of these foliations are allowed to
occur only at the punctures. For an isotopy class $\varphi$ of homeomorphisms
of $\Sigma_g$, $\varphi$ is periodic if there exists a periodic element in $\varphi$. Similarly, $\varphi$ is reducible if there exists a reducible element in $\varphi$ and $\varphi$ is pseudo-Anosov if there exists a pseudo-Anosov element in $\varphi$.

In [21], Thurston classified the isotopy classes of homeomorphisms on
$\Sigma_g$ fixing $P_n$ into periodic, reducible and pseudo-Anosov. Since we can
regard the braid group $B_n$ on $n$ strands as the mapping class group of
disk with $n$ punctures, every element of $B_n$ is also classified into periodic,
reducible and pseudo-Anosov types. In [3], Bestvina and Handel obtained an
algorithm which gave the classification for surface homeomorphisms. Using
this algorithm, they established a method to calculate the dilatation of a
pseudo-Anosov mapping class $\varphi$.

Dilatations themselves are related to many fields and have been inten-
sively studied by many authors. For example, it is known that the logarithm
of the dilatation of pseudo-Anosov maps is the same as the topological en-
tropy of pseudo-Anosov maps, which is an important subject in ergodic
theory. Also in [10], Ivanov showed that the logarithm of the asymptotic Nielsen number, which appeared in fixed point theory, coincides with the entropy. In this paper, we obtain a new formula to determine the dilatation of pseudo-Anosov braids from the representation $\zeta_{n,m}$ due to B. Jiang and H. Zheng [14].

The growth rate of a sequence $\{a_n\}$ of complex numbers is defined by
\[
\text{Growth}_{n \to \infty} a_n = \max \left\{ 1, \limsup_{n \to \infty} |a_n|^{1/n} \right\}.
\]
Let us notice that the above growth rate could be infinity. When the inequality $\text{Growth}_{n \to \infty} a_n > 1$ holds, we say that the sequence grows exponentially.

For any set $S$, $\mathbb{Z}S$ denotes the free abelian group with the specified basis $S$. If $x = \sum_{s \in S} k_s s$ is a finite sum, we define the norm of $x$ in $\mathbb{Z}S$ by
\[
\|x\| = \sum_{s \in S} |k_s|.
\]
For any matrix $A = (a_{ij})$ with coefficients in $\mathbb{Z}S$, the norm of $A$ is the matrix defined by $\|A\| = (\|a_{ij}\|)$ when $a_{ij}$ is a finite sum for all $i$ and $j$.

Let $P_n$ be a finite subset of $\text{int} \, D^2$ of $n \geq 0$ points and we set $D_n = D^2 \setminus P_n$. For integers $n, m \geq 0$, we consider three types of configuration spaces as follows: The space of $m$-tuples of distinct points in $D_n$ denoted by $F_{n,m}(D^2) = \{(z_1, \ldots, z_m) \in (D_n)^m \mid z_i \neq z_j \text{ for all } i \neq j\}$, the space of subsets of distinct $m$ elements in $D_n$ denoted by $\mathcal{C}_{n,m}(D^2) = F_{n,m}(D^2)/S_m$ and the space $\text{IT}_{n,m}(D^2)$ of pairs of disjoint subsets of $n$ distinct elements and $m$ distinct elements in $D^2$ denoted by $\text{IT}_{n,m}(D^2) = F_{0,n+m}(D^2)/S_n \times S_m$, where the symmetric group $S_m$ acts on $F_{n,m}(D^2)$ by permuting components of an $m$-tuple and similarly, the subgroup $S_n \times S_m$ of $S_{n+m}$ acts on $\mathcal{C}_{n,m}(D^2)$ and $\text{IT}_{n,m}(D^2)$. The subgroup $S_n \times S_m$ of $S_{n+m}$ acts on $\mathcal{C}_{n,m}(D^2)$ and $\text{IT}_{n,m}(D^2)$.

Figure 1. local chart around the singularities

1-prong singularity

3-prong singularity
The elements of $C_n,m(D^2)$ and the intertwining by this action.

These two actions, $B$ is given in Section 4.1. The braid group $\pi_1$ projection. We suppose $Z$ long to phism of the free Abelian group generated by $\Gamma$.

For any pseudo-Anosov braid $\beta$ dilatation of $B$.

In $\cite{4}$, $\cite{18}$ and $\cite{19}$, Bigelow and Krammer show ed the faithfulness of the Lawrence-Krammer-Bigelow representation independently.

We set $E_{n,m}(D^2) = \pi_1(I\Gamma_n,m(D^2), b)$. We note that, under the basis $E_{n,m}$, all matrix elements of $\zeta_{n,m}(\beta)$ belong to $Z\Gamma_{\beta,m}$, where $\Gamma_{\beta,m}$ is the subgroup of $B_{n+m}$ generated by $\beta$ and $B_{n,m}(D^2)$. Therefore, $\zeta_{n,m}(\beta)$ can be naturally regarded as an endomorphism of the free $Z\Gamma_{\beta,m}$-module generated by $E_{n,m}$.

Our main result is stated as follows.

**Theorem 1.1.** For any pseudo-Anosov braid $\beta \in B_n$, we denote by $\lambda$ the dilatation of $\beta$. Then we obtain

$$\frac{\mathrm{Growth}_{k \to \infty} \| \text{tr}_{\beta^{k},m} \zeta_{n,m}(\beta^k) \|}{\text{Growth}_{m \to \infty} \| \text{tr}_{\beta^{k},m} \zeta_{n,m}(\beta^k) \|} = \lambda^m$$

The representations $\zeta_{n,m}$ are related to homological representations of braid groups in the following way. For $m = 1$, there exists a homomorphism $\rho_B : E_{n,1}(D^2) \to \mathbb{Z}$ such that the representation induced by $\rho_B$ is equivalent to the reduced Burau representation. Similarly for $m \geq 2$, there exists a homomorphism $\rho_{LKB} : E_{n,m}(D^2) \to \mathbb{Z} \oplus \mathbb{Z}$ such that the representation induced by $\rho_{LKB}$ is equivalent to Lawrence-Krammer-Bigelow representation. The Lawrence-Krammer-Bigelow representations of the braid groups were studied by Lawrence $\cite{20}$ in relation with Hecke algebra representations of the braid groups. In $\cite{4}$, $\cite{18}$ and $\cite{19}$, Bigelow and Krammer showed the faithfulness of the Lawrence-Krammer-Bigelow representation independently.
In [8], Fried proved that the entropy of pseudo-Anosov braids is bounded below by the logarithm of the spectral radius of the Burau matrix $B(t)$ of pseudo-Anosov braids after substituting a complex number of modulus 1 in place of $t$. In [17], Kolev proved the same estimation directly with different methods. The estimate will be called the Burau estimate. In [2], Band and Boyland showed that the spectral radius of the Burau matrix $B(t)$ of pseudo-Anosov braids after substituting the root of unity in place of $t$ is the dilatation itself of pseudo-Anosov braids only if $t = -1$. Furthermore, Band and Boyland showed that the spectral radius of $B(-1)$ is the dilatation of pseudo-Anosov braids if and only if the invariant foliations for pseudo-Anosov maps in the classes of pseudo-Anosov braids have odd order singularities at all punctures and all interior singularities are even order.

In [16], Koberda proved that the square of the dilatation of pseudo-Anosov braids is bounded below by the spectral radius of Lawrence-Krammer-Bigelow representation $LKB(q,t)$ of pseudo-Anosov braids after substituting complex numbers of modulus 1 in place of $q$ and $t$. In this paper we recover the following result of [8], [17] and [16].

**Theorem 1.2.** (Fried [8], Kolev [17] and Koberda [16]) For a pseudo-Anosov braid $\beta$, the dilatation of $\beta$ is equal to or greater than the spectral radius of the Burau matrix $B(t)$ of $\beta$ after substituting a complex number of modulus 1 in place of $t$ and the $m$-th power of the dilatation of $\beta$ is equal to or greater than the spectral radius of the Lawrence-Krammer-Bigelow matrix $LKB_m(q,t)$ of $\beta$ after substituting complex numbers of modulus 1 in place of $q$ and $t$.

This paper is organized as follows. In Section 2 we recall the definition of the topological entropy due to Adler, Konheim and McAndrew [1]. In Section 3 we review asymptotic fixed point theory. We recall asymptotic fixed point theory for compact spaces due to Jiang [13] and a version of relative Nielsen theory due to Jiang, Zhao and Zheng [15] and Jiang and Zheng [14]. In Section 4 we construct the representation $\zeta_{n,m}$ due to Jiang and Zheng [13] and state the relation between the trace of $\zeta_{n,m}$ and the number of essential fixed points of some good self map. In Section 5 we prove the main theorem using the relation among dilatation, entropy and fixed point theory. In Section 6 we recover from our main theorem the estimation of the dilatation of pseudo-Anosov braids in [8], [17] and [16] by means of the homological representation.

2. Preliminaries

2.1. **Topological entropy.** The most widely used measure for the complexity of a dynamical system is the topological entropy. We refer the readers to [22] for an introductory treatment. We recall basic notions of the topological entropy due to Adler, Konheim and McAndrew [1]. Originally the topological entropy is defined in [1]. We recall [1] for the definition of the topological entropy. For any open cover $\alpha$ of $X$, let $N(\alpha)$ denote the number of sets in a subcover of minimal cardinality. For open covers $\alpha$ and $\beta$ of $X$, their join is the open cover consisting of all sets of the form $A \cap B$ with $A \in \alpha$ and $B \in \beta$. Similarly, we can define the join $\bigvee_{i=1}^n \alpha_i$ of any finite collection $\{\alpha_i\}$
of open covers of $X$. For a continuous self map $T$ of $X$, $T^{-1} \alpha$ denotes the open cover consisting of all sets $T^{-1}A$ with $A \in \alpha$. The entropy $h(T, \alpha)$ of a map $T$ with respect to a cover $\alpha$ is defined as $\lim_{n \to \infty} \frac{1}{n} \log N \left( \bigvee_{i=0}^{n-1} T^{-i} \alpha \right)$. The topological entropy $h(T)$ of a map $T$ is defined as $\sup h(T, \alpha)$, where the supremum is taken over all open covers $\alpha$.

For a compact surface $X$ with negative Euler characteristic and a pseudo-Anosov homeomorphism $f$ of $X$ with the dilatation $\lambda > 1$, $h(f) = \log \lambda$ is the minimal entropy in the homotopy class of $f$ ([7, p. 194]).

3. Asymptotic Nielsen theory for stratified maps

In [13], Jiang studied fixed point theory using mapping torus. In [15], Jiang, Zhao and Zheng studied fixed point theory for some good noncompact spaces. In [14], Jiang and Zheng studied fixed point theory for configuration spaces using the method in [15]. In this section we will review some of the relevant materials from [13], [14] and [15] about fixed point theory.

3.1. Mapping torus. Subsections 3.1 and 3.2 are devoted to recall basic notions of fixed point theory due to [13]. In [13], Jiang studied fixed points by using mapping torus. Let $X$ be a topological space and $f : X \to X$ be a continuous self map. The mapping torus $T_f$ of $f$ is the space obtained from $X \times \mathbb{R}_+$ by identifying $(x, s + 1)$ with $(f(x), s)$ for any element $x \in X$ and $s \in \mathbb{R}_+$, where $\mathbb{R}_+$ stands for the real interval $[0, \infty)$. On $T_f$ there exists the natural semi-flow

$$\varphi : T_f \times \mathbb{R}_+ \to T_f, \quad \varphi((x,s),t) = (x, s + t) \text{ for all } t \geq 0.$$

A point $x$ of $X$ and a positive number $\tau > 0$ determine the time-$\tau$ orbit curve $\varphi_{(x,\tau)} = \{\varphi_t(x,0)\}_{0\leq t \leq \tau}$ in $T_f$. We may identify $X$ with the cross-section $X \times \{0\} \subset T_f$, then the map $f : X \to X$ is just the return map of the semi-flow $\varphi$.

We take the base point $v$ of $X$ as the base point of $T_f$. We define $\Gamma$ to be the fundamental group $\pi_1(T_f, v)$ of $T_f$ and let $\Gamma_c$ be the set of conjugacy classes of $\Gamma$. Then $\Gamma_c$ is independent of the base point of $T_f$ and can be regarded as the set of free homotopy classes of closed curves in $T_f$. By the van Kampen Theorem, $\Gamma$ is obtained from $G$ by adding a new generator $z$ represented by the loop $\varphi_{(v,1)}w^{-1}$, and the relations $z^{-1}gz = f_G(g)$ for all $g \in G$:

$$\Gamma = \langle G, z \mid zg = zf_G(g) \text{ for all } g \in G \rangle.$$

We note that $x$ is a fixed point of $f$ if and only if its time-1 orbit curve is closed on the mapping torus $T_f$. For fixed points $x$ and $y$ of $f$, we define $x$ and $y$ to be in the same fixed point class if and only if their time-1 orbit curves are freely homotopic in $T_f$. Therefore every fixed point class $F$ gives rise to a conjugacy class $\text{cd}(F)$ in $\Gamma_c$, called the coordinate of $F$. A fixed point class $F$ is called essential if its index $\text{ind}(f, F)$ is nonzero.

Remark 3.1. We take an arbitrary path $c$ from $v$ to a fixed point $x$. In the light of the continuous map $H : I \times I \to T_f$ defined by $H(s,t) = (c(t), s)$, $\varphi_{(x,1)}$ is homotopic to the loop $c^{-1}\varphi_{(v,1)}f(c) = c^{-1}zf(c)$ and we obtain

$$\text{cd}(x) = [zwf(c)c^{-1}],$$
where $[\gamma]$ is a free homotopy class obtained by $\gamma$.

Given a nontrivial $n$-strand braid $\beta$, there exists a connecting isotopy $\{h_t : D^2 \to D^2\}_{0 \leq t \leq 1}$ from $\text{id}$ such that the curves $\{h_t(P_n)\}_{0 \leq t \leq 1}$ represent the braid $\beta$. We set $f_{\beta} = h_1$. The map $f_{\beta}$ induces a map $\tilde{f}_{\beta} : \mathcal{C}_{n,m}(D^2) \to \mathcal{C}_{n,m}(D^2)$ given by

$$\tilde{f}_{\beta}(\{x_1, \ldots, x_m\}) = \{f_{\beta}(x_1), \ldots, f_{\beta}(x_m)\}.$$ 

In [14], Jiang and Zheng showed that the fundamental group $\Gamma_{\beta, m}$ of $T_{\tilde{f}_{\beta}}$ is isomorphic to the subgroup in $\mathcal{B}_{n+m}$ generated by $\beta$ and $\mathcal{B}_{n,m}(D^2)$.

### 3.2. Periodic orbit classes.

In [13], Jiang studied the periodic orbit of $f$, i.e. the fixed points of the iterates of $f$.

The **periodic point set** of $f$ is the set of points $(x, n)$ in $X \times \mathbb{N}$ satisfying $x = f^n(x)$ and is denoted by $\text{PP}(f)$. An **$n$-point** of $f$ is a fixed point $x$ of $f^n$. For an $n$-point $x$ of $f$, an **$n$-orbit** of $f$ at $x$ is the $f$-orbit $\{x, \ldots, f^{n-1}(x)\}$ in $X$. An $n$-orbit of $f$ at $x$ is a **primary $n$-orbit** if $n$ is the least period of the periodic point $x$.

An **$n$-point class** of $f$ is a fixed point class $P^n$ of $f^n$. Two points $x$ and $x'$ in $\text{Fix}(f^n)$ are said to be in the same **$n$-orbit class** of $f$ if and only if there exist natural numbers $i$ and $j$ such that $f^i(x)$ and $f^j(x')$ are in the same $n$-point class of $f$. The set $\text{Fix}(f^n)$ splits into a disjoint union of $n$-orbit classes. On the mapping torus $T_f$, we observe that $(x, n)$ is in the periodic point set of $f$ if and only if the time-$n$ orbit curve $\varphi_{(x,n)}$ is closed. The free homotopy class $[\varphi_{(x,n)}] \in \Gamma_c$ of the closed curve $\varphi_{(x,n)}$ is called the **$\Gamma$-coordinate** of $(x, n)$ and is denoted by $\text{cd}_\Gamma(x, n)$. Every $n$-orbit class $O^n$ gives rise to a conjugacy class $\text{cd}_\Gamma(O^n)$ in $\Gamma_c$, called the **$\Gamma$-coordinate** of $O^n$.

An important notion in the Nielsen theory for periodic orbits is the notion of reducibility. Suppose $m$ is a divisor of $n$ and $n$ is less than $n$. An $n$-orbit class $O^n$ is **reducible to period $m$** if $\text{cd}_\Gamma(O^n)$ has an $(n/m)$-th root and is **irreducible** if $\text{cd}_\Gamma(O^n)$ has no nontrivial root.

An $n$-orbit class $O^n$ is called **essential** if its index $\text{ind}(O^n, f^n)$ is nonzero. For each natural number $n$, the generalized Lefschetz number with respect to $\Gamma$ is defined as

$$L_\Gamma(f^n) = \sum_{O^n} \text{ind}(O^n, f^n) \cdot \text{cd}_\Gamma(O^n) \in \mathbb{Z}\Gamma_c,$$

where the summation is taken over all essential $n$-orbit classes $O^n$ of $f$. The **Nielsen number of $n$-orbits** $N_\Gamma(f^n)$ is the number of nonzero terms in $L_\Gamma(f^n)$ and the indices of the essential fixed point classes appear as the coefficients in $L_\Gamma(f^n)$. Clearly it is a lower bound for the number of $n$-orbits of $f$. The **Nielsen number of irreducible $n$-orbits** $N_\Gamma(f^n)$ is the number of nonzero primary terms in $L_\Gamma(f^n)$. It is the number of irreducible essential $n$-orbit classes. It is a lower bound for the number of primary $n$-orbits of $f$. These are homotopy invariants.

### 3.3. Asymptotic Nielsen theory.

In [13] Jiang defines the **asymptotic Nielsen number** of $f$ to be the growth rate of the Nielsen numbers

$$N^\infty(f) = \lim_{n \to \infty} N_\Gamma(f^n),$$
the asymptotic irreducible Nielsen number of $f$ to be the growth rate of the Nielsen numbers of irreducible orbits

$$NI_{\infty}(f) = \text{Growth}_{n \to \infty} N_{\Gamma}(f^n)$$

and the asymptotic absolute Lefschetz number of $f$ to be the growth rate of the norm of generalized Lefschetz numbers

$$L_{\infty}(f) = \text{Growth}_{n \to \infty} \| L_{\Gamma}(f^n) \|.$$

In [13] all these asymptotic numbers are shown to enjoy the homotopy invariance.

**Remark 3.2.** Since the inequality $NI_{\Gamma}(f) \leq N_{\Gamma}(f) \leq \| L_{\Gamma}(f) \|$ holds, we obtain $NI_{\infty}(f) \leq N_{\infty}(f) \leq L_{\infty}(f)$. In [13], Jiang showed that a sufficient condition for the equality $NI_{\infty}(f) = N_{\infty}(f)$ is that $f$ satisfies the following Property of Essential Irreducibility: The number $E_n$ of essentially irreducible $n$-point classes that are reducible is uniformly bounded in $n$. Also in [13], Jiang showed that a sufficient condition for the equality $N_{\infty}(f) = L_{\infty}(f)$ is that $f$ satisfies the following Property of Bounded Index: The maximum absolute value $B_n$ of the indices of $n$-point classes $F_n^m$ is uniformly bounded in $n$. These conditions are not strong. For example, every homeomorphism of $D_n$ satisfies the Property of Essential Irreducibility and the Property of Bounded Index.

In [10], Ivanov showed that the logarithm of the asymptotic Nielsen number $N_{\infty}(f)$ of a self map $f$ coincides with the entropy of a self map $f$.

**Theorem 3.3.** (Ivanov [10]) Let $X$ be a compact surface with negative Euler characteristic and $f$ be a self map of $X$. Then the entropy of $f$ coincides with $\log N_{\infty}(f)$.

For a compact surface $X$ with negative Euler characteristic, we take a pseudo-Anosov homeomorphism $f$ of $X$ with the dilatation $\lambda > 1$. Then we obtain that

$$h(f) = \log \lambda = \log N_{\infty}(f)$$

is the minimal entropy in the homotopy class of $f$.

4. The representation $\zeta_{n,m}$ and fixed points

4.1. The definition of $\zeta_{n,m}$. In [6], Bigelow defined the triangle corresponding to the embedded edge for $m = 2$. Triangles are elements of the relative homology of some abelian covering of the configuration space $C_{n,m}(D^2)$. In this subsection we define $\zeta_{n,m}$ due to Jiang and Zheng by using the lifts of triangles to the universal covering. Let $R_B$ denote the group ring $\mathbb{Z}[B_{n,m}(D^2)]$ and $R$ denote the group ring $\mathbb{Z}[E_{n,m}(D^2)]$.

We introduce some relative homology of the universal covering of the configuration space $C_{n,m}(D^2)$. Let $p : \tilde{C}_{n,m}(D^2) \to C_{n,m}(D^2)$ be the universal covering of $C_{n,m}(D^2)$ and fix $\tilde{c} \in p^{-1}(c)$ as a base point of $\tilde{C}_{n,m}(D^2)$. For $\varepsilon > 0$, we define $V_{\varepsilon}$ to be the set of points $\{x_1, \ldots, x_m\}$ in $\tilde{C}_{n,m}(D^2)$ such that at least one of the pair $(x_i, x_j)$ is within distance $\varepsilon$ of each other. We
define \( \tilde{V}_\varepsilon \) to be the preimage of \( V_\varepsilon \) in \( \tilde{C}_{n,m}(D^2) \). The relative homology \( H_m(\tilde{C}_{n,m}(D^2), \partial \tilde{C}_{n,m}(D^2) \cup \tilde{V}_\varepsilon) \) is nested by inclusion.

\( f_\beta \) has a unique lift \( \tilde{f}_\beta : (\tilde{C}_{n,m}(D^2), \tilde{c}) \to (\tilde{C}_{n,m}(D^2), \tilde{c}) \) and induces an automorphism of the left \( R_B \) module

\[
\lim_{\varepsilon \to 0} H_m(\tilde{C}_{n,m}(D^2), \partial \tilde{C}_{n,m}(D^2) \cup \tilde{V}_\varepsilon).
\]

The induced automorphism is independent of the choice of the representative and denoted by \( \beta_* \).

The intertwining \((n, m)\)-braid group \( E_{n,m}(D^2) \) is isomorphic to the subgroup \( E_{n,m} \) of \( B_{n+m} \) generated by

\[
\sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2, \sigma_{n+1}, \ldots, \sigma_{n+m-1}
\]

and \( B_{n,m}(D^2) \) is isomorphic to the subgroup \( B_{n,m} \) of \( B_{n+m} \) generated by

\[
A_{1,n+1}, \ldots, A_{n,n+1}, \sigma_{n+1}, \ldots, \sigma_{n+m-1},
\]

where \( A_{ij} \) is defined by

\[
A_{ij} = \sigma_{j-1} \ldots \sigma_{i+1} \sigma_i^2 \sigma_{i+1} \ldots \sigma_{j-1}.
\]

Therefore \( B_n \) acts on \( E_{n,m}(D^2) \) by the right multiplication and so there exists an induced action of \( \beta \) on the \( R \). Moreover, since \( B_{n,m}(D^2) \) is included in \( E_{n,m}(D^2) \), \( R \) is a right \( R_B \) module. Using the \( \mathbb{Z} \) module automorphism \( \tilde{\beta}_* \) and the action on \( E_{n,m}(D^2) \) by \( B_n \), we construct an automorphism \( \beta \otimes \tilde{\beta}_* \) on the left \( R \) module

\[
R \otimes_{R_B} \lim_{\varepsilon \to 0} H_m(\tilde{C}_{n,m}(D^2), \partial \tilde{C}_{n,m}(D^2) \cup \tilde{V}_\varepsilon)
\]

by

\[
(\beta \otimes \tilde{\beta}_*)(h \otimes c) = h\beta \otimes \tilde{\beta}_*(c).
\]

Clearly \( \beta \otimes \tilde{\beta}_* \) is a \( R \)-homomorphism.

From now on, we define a representation \( \zeta_{n,m} \) of \( B_n \) over the free left \( R \) module generated by \( \mathcal{E}_{n,m} \). The cardinality \( d_{n,m} \) of the basis \( \mathcal{E}_{n,m} \) is

\[
\begin{pmatrix}
 n + m - 2 \\
 m
\end{pmatrix}.
\]

We now introduce some other relative homology and an intersection pairing. Henceforth every path is a continuous map from \( I = [0, 1] \). For \( \varepsilon > 0 \), we define \( U_\varepsilon \) to be the set of points \( \{x_1, \ldots, x_m\} \in C_{n,m}(D^2) \) such that at least one of them is within distance \( \varepsilon \) of some puncture point. We define \( \tilde{U}_\varepsilon \) to be the preimage of \( p \) in \( \tilde{C}_{n,m}(D^2) \). The relative homology \( H_m(\tilde{C}_{n,m}(D^2), \tilde{U}_\varepsilon) \) is nested by inclusion.

We set

\[
\begin{align*}
p_i &= \left( \frac{i}{2n}, 0 \right), & P_n &= \{p_1, \ldots, p_n\}, \\
d_j &= \left( \cos \frac{j}{3n} \pi, \sin \frac{j}{3n} \pi \right), & c &= \{d_1, \ldots, d_m\}, \\
N_i &= \left\{ x = \frac{2i + 1}{4n} \right\} \cap D^2, & \alpha_i &= \left\{ (x, 0) \mid \frac{i}{2n} < x < \frac{i + 1}{2n} \right\}, \\
z_i' &= \left( \frac{2i + 1}{4n}, \sin \frac{j}{3n} \pi \right)
\end{align*}
\]
and let $\alpha_i^j$ be a polygonal line connecting $p_i$, $z_i^j$ and $p_i+1$. We call $\alpha_i^j$ fork. For $\mu \in \mathcal{E}_{n,m}$, we set

$$F_\mu = \{ \{x_1, \ldots, x_m\} \in \mathcal{C}_{n,m}(D^2) \mid \#(\{x_1, \ldots, x_m\} \cap N_i) = \mu_i \}$$

and

$$S_\mu = \prod_{i=1}^{n-1} \prod_{j=u_i+1}^{u_i+1} \text{int} \alpha_i^j,$$

where $u_i = \sum_{j=1}^{i-1} \mu_j$. We take line segments $\theta_j$ on $D_n$ from $c_j$ to $z_i^j$, where $u_i < j \leq u_i+1$. We notice that they are disjoint. Let $z_\mu$ be the endpoint of $\Theta_\mu = \{\theta_1, \ldots, \theta_m\}$. We take a lift $\tilde{z}_\mu$ of $z_\mu$ so that the lift $\tilde{\Theta}_\mu$ of $\Theta_\mu$ is starting at $\tilde{c}$ and ending at $\tilde{z}_\mu$. We take lifts $\tilde{F}_\mu$ and $\tilde{S}_\mu$ of $F_\mu$ and $S_\mu$ containing $\tilde{z}_\mu$ respectively. Let $[X]$ denote the element of certain relative homology corresponding to the $m$-dimensional subspace $X$ of $\tilde{\mathcal{C}}_{n,m}(D^2)$. We set

$$\mathcal{H}_F = \bigoplus_{\mu \in \mathcal{E}_{n,m}} R_B \left[ \tilde{F}_\mu \right] \subset \lim_{\varepsilon \to 0} H_m(\tilde{\mathcal{C}}_{n,m}(D^2), \partial \tilde{\mathcal{C}}_{n,m}(D^2) \cup \tilde{V}_\varepsilon)$$
and

\[ \mathcal{H}_S = \bigoplus_{\mu \in \mathcal{E}_{n,m}} R_B \left[ \mathcal{S}_\mu \right] \subset \lim_{\varepsilon \to 0} H_m(\mathcal{C}_{n,m}(D^2), \mathcal{U}_\varepsilon). \]

For \( x \in \mathcal{H}_S \) and \( y \in \mathcal{H}_F \), let \((x \cdot y) \in \mathbb{Z}\) denote the standard intersection number. In [6] for \( m = 2 \) and [3], Bigelow defined an intersection pairing. Similarly, we define an intersection pairing

\[ \langle \cdot, \cdot \rangle : \mathcal{H}_S \times \mathcal{H}_F \to R_B \text{ by } \langle x, y \rangle = \sum_{\beta \in R_B} (x \cdot \tilde{\beta}_s(y)) \beta. \]

We notice that \( \left( \left[ \mathcal{S}_\mu \right], \left[ \mathcal{F}_\nu \right] \right) \) equals 1 when \( \mu = \nu \) and 0 otherwise. Therefore \( \left\{ \left[ \mathcal{F}_\nu \right] \right\}_{\nu \in \mathcal{E}_{n,m}} \) is linearly independent. We define elements \( d_{\mu \nu}^{(\beta)} \) of \( R_B \) so that \( \{ d_{\mu \nu}^{(\beta)} \}_{\mu, \nu \in \mathcal{E}_{n,m}} \) satisfies the relations \( \sum_{\nu} d_{\mu \nu}^{(\beta)} \left[ \mathcal{F}_\nu \right] = \tilde{\beta}_s \left( \left[ \mathcal{F}_\mu \right] \right) \). Using the intersection pairing, we obtain

\[ d_{\mu \nu}^{(\beta)} = \tau \left( \left( \left[ \mathcal{S}_\mu \right], \tilde{\beta}_s \left( \left[ \mathcal{F}_\mu \right] \right) \right) \right), \]

where \( \tau \) is an automorphism of \( R_B \) with \( \tau(\beta) = \beta^{-1} \). There exists a homomorphism

\[ \zeta'_{n,m} : B_n \to \text{Aut}_R (R \otimes_{R_B} \mathcal{H}_F) \]

defined by \( \zeta'_{n,m}(\beta) = (\beta \otimes \tilde{\beta}_s)|_{\mathcal{H}_F} \). We notice that \( R \otimes_{R_B} \mathcal{H}_F \cong \bigoplus_{\mu \in \mathcal{E}_{n,m}} R \left[ \mathcal{F}_\mu \right] \) and this gives the representation \( \zeta_{n,m} \) to the matrix group \( \text{GL}(d_{n,m}, R) \). We set \( \zeta_{n,m}(\beta) = (c_{\mu \nu}^{(\beta)}) \) and notice that \( c_{\mu \nu}^{(\beta)} = \beta d_{\mu \nu}^{(\beta)} \) in \( R \). It is straightforward that the map \( \zeta_{n,m} \) is a group homomorphism.

We recall the definition of trace. Let \( \Gamma \) be a group, \( \mathbb{Z}\Gamma \) its group ring, \( \Gamma_c \) the set of conjugacy classes, \( \mathbb{Z}\Gamma_c \) the free Abelian group generated by \( \Gamma_c \), and \( \pi_\Gamma : \mathbb{Z}\Gamma \to \mathbb{Z}\Gamma_c \) the natural projection. Let \( \zeta \) be an endomorphism of a free \( \mathbb{Z}\Gamma \)-module satisfying \( \zeta(v_i) = \sum_{j=1}^{k} a_{ij} \cdot v_j \) for a basis \( \{v_1, \ldots, v_k\} \). The trace of \( \zeta \) is defined as

\[ \text{tr}_\Gamma \zeta = \pi_\Gamma \left( \sum_{i=1}^{k} a_{ii} \right) \in \mathbb{Z}\Gamma_c. \]

The definition is independent of the choice of the basis and for two endomorphism \( \zeta \) and \( \xi \), we have \( \text{tr}_\Gamma \zeta \circ \xi = \text{tr}_\Gamma \xi \circ \zeta \).

We note that, under the basis \( \mathcal{E}_{n,m} \), all matrix elements of \( \zeta_{n,m}(\beta) \) belong to \( \mathbb{Z}\Gamma_{\beta, m} \). Therefore \( \zeta_{n,m}(\beta) \) can naturally be regarded as an endomorphism of the free \( \mathbb{Z}\Gamma_{\beta, m} \)-module generated by \( \mathcal{E}_{n,m} \). In this way, the notations \( \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta) \) and \( \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta^k) \) in the main theorem are well-defined.

**Theorem 4.1.** For any pseudo-Anosov braid \( \beta \in B_n \), we denote by \( \lambda \) the dilatation of \( \beta \). Then we obtain

\[
\begin{align*}
\text{Growth}_{k \to \infty} \| \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta^k) \| &= \text{Growth}_{k \to \infty} \| \zeta_{n,m}(\beta^k) \| = \lambda^m, \\
\text{Growth}_{m \to \infty} \| \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta) \| &= \lambda.
\end{align*}
\]
4.2. The work of Jiang and Zheng. The representation $\zeta_{n,m}$ is the same as the representation due to Jiang and Zheng [14]. We compactify $D_n$ to a 2-disk with $n$ holes and denote it by $Y_n$, and assume further that there exists a homeomorphism $f_\beta : Y_n \to Y_n$ such that $f_\beta$ is the map restricting $\overline{f_\beta}$ on $\text{int} Y_n$. We identify $\text{int} Y_n \cup \partial D^2$ with $D_n$. We decompose the surface $Y_n$ into an annulus and $n - 1$ foliated rectangles, as shown in Figure 3.

We define $U = U_1 \cup \cdots \cup U_{n-1}$ to be the union of the $n - 1$ foliated open rectangles. We define a partial ordering on $U$ such that $x_1 \prec x_2$ if either $x_1$ lies in a rectangle to the right of $x_2$ or $x_1$ lies in a strictly lower leaf of the same rectangle as $x_2$. For example, the order of the three points in Figure 3 is $x_1 \prec x_2 \prec x_3$.

We set $V = \{\{x_1, \ldots, x_m\} \in C_{m,0}(Y_n) \mid x_i \in U, \ x_{\eta(1)} \prec \cdots \prec x_{\eta(m)} \forall \eta \in S_m \text{ s.t.} \ x_i \in \text{int} Y_n \cup \partial D^2 \}$.

Then we have $V = \bigcup_{\mu \in E_{n,m}} V_\mu$, where

$$V_\mu = \{\{x_1, \ldots, x_m\} \in V \mid \#\{x_1, \ldots, x_m\} \cap U_i = \mu_i\}.$$  

Each $V_\mu$ is connected; thus the elements of $E_{n,m}$ are in one-to-one correspondence to the components of $V$.

Illustrated in Figure 4 and Figure 5 are two embeddings $\phi_i$ and $\overline{\phi}_i$, which can be understood as the action of the elementary mapping $\sigma_i$ and $\sigma_i^{-1}$ on $Y_n$ respectively. Both push the annulus outward, irrationally rotate the outmost boundary, keep the foliations of $(\phi_i)^{-1}(U)$ and $(\overline{\phi}_i)^{-1}(U)$, uniformly contract along the leaves of the foliations, and uniformly expand along the transversal direction.

For every $\phi \in \{\phi_1, \ldots, \phi_{n-1}, \overline{\phi}_1, \ldots, \overline{\phi}_{n-1}\}$, we have

$$V_\mu \cap \phi^{-1}(V_\nu) = \bigcup_{\eta \in S_m} W^{(\phi)}_{\mu \nu \eta}.$$

---

**Figure 3.** Decomposition of $Y_n$
where

\[ W_{\mu \nu \eta}^{(\phi)} = \left\{ x \in V_{\mu} \cap \phi^{-1}(V_{\nu}) \mid \begin{array}{l}
\text{there exist } x_1, \ldots, x_m \text{ s.t. } \\
x = \{x_1, \ldots, x_m\}, \\
x_\eta(1) \prec \cdots \prec x_\eta(m), \\
\phi(x_1) \prec \cdots \prec \phi(x_m), \\
x_\mu \cap \phi^{-1}(V_{\nu})
\end{array} \right\}. \]

Each \( W_{\mu \nu \eta}^{(\phi)} \) is connected; thus the elements of the set \( \{ \eta \in \mathcal{S}_m \mid W_{\mu \nu \eta}^{(\phi)} \neq \emptyset \} \) are in one-to-one correspondence to the components of \( V_{\mu} \cap \phi^{-1}(V_{\nu}) \).

We choose a base point \( b = \{b_1, \ldots, b_m\} \) in \( \text{int } A \). For every element \( x = \{x_1, \ldots, x_m\} \) in \( V \) with \( x_1 \prec \cdots \prec x_m \), the disjoint “descending” paths connecting \( b_k \) to \( x_k \) in \( Y_n \) give rise to a path \( \gamma_x \) in \( C_{n,m}(Y_n) \). Similarly, the disjoint “ascending” paths connecting \( b_k \) to \( \phi(b_k) \) give rise to a path \( \gamma_{\phi(b)} \) in \( C_{n,m}(Y_n) \). For every nonempty \( W_{\mu \nu \eta}^{(\phi)} \), we choose a point \( x \in W_{\mu \nu \eta}^{(\phi)} \) and \( \alpha_{\mu \nu \eta}^{(\phi)} \) denotes the element of \( \pi_1(C_{n,m}(Y_n), b) \) represented by the loop \( \gamma_{\phi(b)} \cdot \phi(\gamma_x) \cdot \gamma_{\phi(x)}^{-1} \). We note that \( \alpha_{\mu \nu \eta}^{(\phi)} \) is independent of the choices of \( x, \gamma_x, \gamma_{\phi(b)} \) and \( \gamma_{\phi(x)}^{-1} \).

In [14], Jiang and Zheng showed that the equations

\[
\begin{align*}
\mu \cdot \zeta_{n,m}(\sigma_i) &= \sum_{\nu \in \mathcal{E}_{n,m}} c_{\mu \nu}^{(i)} \cdot \nu, \\
\mu \cdot \zeta_{n,m}(\sigma_i^{-1}) &= \sum_{\nu \in \mathcal{E}_{n,m}} d_{\mu \nu}^{(i)} \cdot \nu,
\end{align*}
\]

In [14], Jiang and Zheng showed that the equations

\[
\begin{align*}
\mu \cdot \zeta_{n,m}(\sigma_i) &= \sum_{\nu \in \mathcal{E}_{n,m}} c_{\mu \nu}^{(i)} \cdot \nu, \\
\mu \cdot \zeta_{n,m}(\sigma_i^{-1}) &= \sum_{\nu \in \mathcal{E}_{n,m}} d_{\mu \nu}^{(i)} \cdot \nu,
\end{align*}
\]
where
\[ c^{(i)}_{\mu \nu} = (-1)^{\nu_i} \cdot \sigma_i \cdot \sum_{\eta; W^{(\phi_i)}_{\mu \nu \eta} \neq \emptyset} \text{sgn} \cdot \alpha^{(\phi_i)}_{\mu \nu \eta}, \]
\[ d^{(i)}_{\mu \nu} = (-1)^{\nu_i} \cdot \sigma_i^{-1} \cdot \sum_{\eta; W^{(\phi_i)}_{\mu \nu \eta} \neq \emptyset} \text{sgn} \cdot \alpha^{(\phi_i)}_{\mu \nu \eta}, \]
give rise to a group representation of \( B_n \) over the free \( \mathbb{Z}B_{n+m} \) module generated by \( \mathcal{E}_{n,m} \).

We take the base point \( b \) in \( \Theta_\mu \cap A \). We can take the base point \( b \) independent of \( \mu \) because of the definition of \( \Theta_\mu \) and \( A \). Let \( \Theta_b \) be a path from \( b \) to \( \Theta_\mu(1) \) along \( \Theta_\mu \) and \( \Theta'_b \) be a path from \( b \) to \( \Theta_\mu(0) \) along \( \Theta_\mu \). We identify \( \pi_1(\mathcal{C}_{n,m}(D^2),c) \) with \( \pi_1(\mathcal{C}_{0,m}(Y_n),b) \) by the map induced by \( \Theta_b \).

**Proposition 4.2.** The representation defined above and the representation \( \zeta_{n,m} \) give the same matrix for any braid under the above identification.

**Proof.** We consider the case \( \beta = \sigma_i \) and the case \( \beta = \sigma_i^{-1} \) is similar. We notice that \( F_\mu \) is given by shrinking \( V_\mu \) along the leaves of foliations and then \( \hat{\phi}(W^{(\phi_i)}_{\mu \nu \eta}) \) is homotopy equivalent to \( F_\nu \). Therefore the nonzero terms of \( \tilde{\sigma}_i \Phi \left( F_\mu \right) \) are in one-to-one correspondence to the components of \( V_\mu \cap \phi^{-1}(V_\nu) \), which are in one-to-one correspondence to the elements of the set \( \{ \eta \in \mathcal{S}_m \mid W^{(\phi_i)}_{\mu \nu \eta} \neq \emptyset \} \).

There exists a homotopy \( \{ H : D_n \times I \to D_n \} \) with \( H(x,0) = \phi_i(x) \) and \( H(x,1) = f_\beta(x) \) such that a map \( H(\cdot,t) \) defined by \( H(\cdot,t)(x) = H(x,t) \) is injective for any \( t \). Let \( \tilde{H} : \mathcal{C}_{n,m}(D^2) \times I \to \mathcal{C}_{n,m}(D^2) \) be the map defined by \( \tilde{H}(\{ x_1, \ldots, x_m \},t) = \{ H(x_1,t), \ldots, H(x_m,t) \} \) and \( \tilde{H}(x,\cdot) \) be the path defined by \( \tilde{H}(x,\cdot)(t) = \tilde{H}(x,t) \).

For nonempty \( W^{(\phi_i)}_{\mu \nu \eta} \), we take an element \( x \in W^{(\phi_i)}_{\mu \nu \eta} \cap F_\mu \). We take \( \gamma_x \) the composition of two paths \( \Theta_b \) and the path from \( z_\mu \) to \( x \) in \( F_\mu \). Since \( \gamma_{\phi_i}(b) \) is homotopic to the composition of two paths \( \Theta'_b \) and \( \hat{\phi}(\Theta'_b) \) relative to the endpoints, the loop \( \tilde{f}_\beta(\gamma_x) \gamma_x^{-1} \) is identified with \( \alpha^{(\phi_i)}_{\mu \nu \eta} \) by the above identification. Therefore \( \alpha^{(\phi_i)}_{\mu \nu \eta} \) is the term of \( \tilde{\sigma}_i \Phi \left( F_\mu \right) \) corresponding to \( W^{(\phi_i)}_{\mu \nu \eta} \) and the signature is \( (-1)^{\nu_i} \text{sgn} \eta \). Finally, left multiplication of \( \sigma_i \) and tensoring \( \sigma_i \) from left induce the same action on \( R \). Therefore \( \zeta_{n,m} \) and the representation due to Jiang and Zheng \cite{14} give the same matrix for all \( \beta \in B_n \). \( \square \)

### 4.3. Trace of \( \zeta_{n,m} \) and fixed points

In this subsection, we prove the key lemma of the proof of main theorem. We define \( e\text{Fix} \) to be the set of essential fixed points of \( f \). We choose a word \( \beta = \tau_1 \ldots \tau_N \), where \( \tau_i \) is an element of \( \{ \sigma_1^{\pm 1}, \ldots, \sigma_{n-1}^{\pm 1} \} \). We put \( \varphi_i = \phi_{j_i} \) if there exists a number \( j_i \) satisfying \( \tau_i = \sigma_{j_i} \) and \( \varphi_i = \hat{\phi}_{j_i} \) if there exists a number \( j_i \) satisfying \( \tau_i = \sigma_{j_i}^{-1} \). Then the embedding \( g = \varphi_N \ldots \varphi_1 : Y_n \to Y_n \) induces a map \( \hat{g} : B_{n,m}(Y_n) \to B_{n,m}(Y_n) \). It is immediate from the definition of \( \phi_i \) and \( \hat{\phi}_i \) that \( \text{Fix} \hat{g} \) is a subset of \( V \).

We prove the next lemma whose proof is similar to that of \cite{14} Proposition 4.3.] by Jiang and Zheng.
Lemma 4.3. There exists a positive number $B$ such that we have the inequality
\[ \# \text{eFix}(\hat{g}^k) \leq \left\| \text{tr}_{\beta,m} \zeta_{n,m}(\beta^k) \right\| \leq B \# \text{eFix}(\hat{g}^k). \]

Proof. Without loss of generality, we only have to prove the case $k = 1$. We note that each of the components $W^1_{\mu}$ of $\bigcup_{\mu \in \mathcal{E}_{n,m}} V_{\mu} \cap (\hat{g})^{-1}(V_{\mu})$ is homeomorphic to $\mathbb{R}^{2m}$. Since $\hat{g}$ is a hyperbolic map on $W^1_{\mu}$, there exists precisely one fixed point of $\hat{g}$ on $W^1_{\mu}$. Let $x_j \in W^1_{\mu}$ be the fixed point of $\hat{g}$ on $W^1_{\mu}$. We notice that the fixed point class containing $x_j$ consists of one element $x$. We set
\[ \alpha^g(x_j) = \gamma_{\hat{g}(c)}(\hat{g})(\gamma_{x_j}) \cdot \gamma_{x_j}^{-1}. \]

We obtain
\[ \text{cd}(x_j) = [z\gamma_{\hat{g}(c)}(\hat{g})(\gamma_{x_j}) \cdot \gamma_{x_j}^{-1}] = \beta[[\alpha^g(x_j)]] \in (\Gamma_{\beta,m})_c \]
by Remark 3.1 and recall that
\[ \text{ind}(\hat{g}, x_j) = \langle \text{diag}(C_{n,m}(D^2)), \text{graph}(\hat{g}) \rangle |_{x_j} \]
is the definition of $\text{ind}(\hat{g}, x_j)$.

On the other hand, we take a lift $\tilde{x}$ of $x$ so that the lift $\tilde{\gamma}_x$ of $\gamma_x$ is starting at $\tilde{c}$ and ending at $\tilde{x}$. Then we obtain $\hat{g}(\tilde{x}_j) = \alpha^g(x_j)\tilde{x}_j$. Computing the fixed point index $\text{ind}(\hat{g}, x_j)$ of $\hat{g}$ at $x_j$, we obtain
\[ \text{ind}(\hat{g}, x_j) = (-1)^m \left( \alpha^g(x_j)\tilde{S}_\mu \cdot (\hat{g})_*(\tilde{F}_\mu) \right). \]
Therefore we obtain
\[ (-1)^m[[\epsilon^{(\beta)}]] = \sum_j \text{ind}(\hat{g}, x_j)\text{cd}(x_j), \]
where $[[\epsilon]]$ is the element of the free abelian group $\mathbb{Z}(\Gamma_{\beta,m})_c$ projecting $c$, and
\[ (-1)^m \text{tr}_{\beta,m} \zeta_{n,m}(\beta) = \sum_{x \in \text{Fix} \hat{g}} \text{ind}(\hat{g}, x) \cdot \text{cd}(x). \]
In the above equality, the number of nonzero terms in the right hand side is $\text{eFix}(\hat{g})$. By Remark 3.2 there exists a positive number $B$ such that the inequality
\[ \# \text{eFix}(\hat{g}) \leq \left\| \text{tr}_{\beta,m} \zeta_{n,m}(\beta) \right\| \leq B \# \text{eFix}(\hat{g}) \]
holds. 

We count the number of essential fixed points of $\hat{g}^k$. Let $\{x_1, \ldots, x_m\}$ be a fixed point of $\text{Fix}(\hat{g}^k)$. Then there exists an $m$-tuple $(n_1, \ldots, n_m)$ of natural numbers with $\sum_{i=1}^{m} i n_i = m$ such that there exist $n_i$ periodic orbits of $g^k$ of period $i$ in $\{x_1, \ldots, x_m\}$ for all $1 \leq i \leq m$. Let $A_m$ be the set of such $m$-tuples and $D^k_i$ be the number of essential periodic points of $g^k$ of period $i$. Then there exist $D^k_i/i$ periodic orbits of $g^k$ of period $i$ and we obtain
\[ \# \text{eFix}(\hat{g}^k) = \sum_{(n_1, \ldots, n_m) \in A_m} \prod_{i=1}^{m} \binom{D^k_i}{n_i}. \]
Remark 4.4. When we consider the period of periodic points of $g^k$ of period $i$ as periodic points of $g$, we notice that $D^k_i = D^1_{g(k,i)i}$, where $g(k,i)$ is the greatest common divisor of $k$ and $i$. Moreover, if $i$ a divisor of $k$ then periodic orbits of $g$ of period $i$ is contained in some periodic orbits of $g$ of period $k$ and $D^1_i/i$ is equal to or greater than $D^1_k/k$. Therefore we have

$$D^k_i/i = D^1_{g(k,i)i}/i \geq D^1_{ki}/l(ki),$$

where $l(k,i)$ is the least common multiplier, and we obtain

$$\# \text{eFix}(\hat{g}^k) \geq \sum_{(n_1, \ldots, n_m) \in A_m} \prod_{i=1}^m \left( \frac{D^1_{ki}/l(k,i)}{n_i} \right).$$

5. Proof of the main theorem

In this section we conclude the proof of main theorem. We denote by $\lambda$ the dilatation of a pseudo-Anosov braid $\beta$.

**Proposition 5.1.** For any pseudo-Anosov braid $\beta \in B_n$, the inequalities

$$\text{Growth}_{k \to \infty} \left\| \text{tr}_{\Gamma_{\beta, km}} \zeta_{n,m}(\beta^k) \right\| \geq \lambda^m$$

$$\text{Growth}_{m \to \infty} \left\| \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta) \right\| \geq \lambda$$

hold.

**Proof.** We recall that $NI_{\Gamma_{\beta, l}}((g^k)^i)$ defined in Section 3.2 is a lower bound for the number of primary $i$-orbits of $g^k$. In other words, we have the inequality $D^k_i/i \geq NI_{\Gamma_{\beta, l}}((g^k)^i)$. When we use this inequality and Remark 4.3 and consider the case $(n_1, \ldots, n_m) = (0, \ldots, 0, 1)$, we obtain the inequality

$$\left\| \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta^k) \right\| \geq \# \text{eFix}(\hat{g}^k) = \sum_{(n_1, \ldots, n_m) \in A_m} \prod_{i=1}^m \left( \frac{D^1_{ki}/l(k,i)}{n_i} \right) \geq \frac{D^k_m}{m} = \frac{D^1_{km}}{l(k,m)} \geq g(k,m)NI_{\Gamma_{\beta, l}}((g^{km}).$$

Since $g$ is homotopic to $f_{\beta}$, we obtain

$$\text{Growth}_{k \to \infty} \left\| \text{tr}_{\Gamma_{\beta, km}} \zeta_{n,m}(\beta^k) \right\| \geq \text{Growth}_{k \to \infty} g(k,m)NI_{\Gamma_{\beta, l}}((g^{km}) = \lambda^m,$$

$$\text{Growth}_{m \to \infty} \left\| \text{tr}_{\Gamma_{\beta, m}} \zeta_{n,m}(\beta) \right\| \geq \text{Growth}_{m \to \infty} NI_{\Gamma_{\beta, l}}((g^{m}) = \lambda.$$

□

**Proposition 5.2.** For any pseudo-Anosov braid $\beta \in B_n$, the inequality

$$\text{Growth}_{k \to \infty} \left\| \zeta_{n,m}(\beta^k) \right\| \leq \lambda^m$$

holds.
Figure 6. The case when $\mu_1 = 6, K^k_{11} = 4, K^k_{12} = 2, K^k_{13} = 3, \rho_{11} = 1, \rho_{12} = 4, \rho_{13} = 1$

Proof. By (4.1), the $(\mu, \nu)$-entry of $\zeta_{m,n}(\beta^k)$ is $\langle \tilde{S}_\nu, \tilde{\beta}^k \left( \tilde{F}_{\mu} \right) \rangle$. We notice that $\langle \tilde{S}_\nu, \tilde{\beta}^k \left( \tilde{F}_{\mu} \right) \rangle$ is equal to or less than the number of intersections of $S_\nu$ and $\hat{g}^k(\hat{F}_\mu)$. We define $K^k_{ij}$ to be the number of intersections of $\alpha^k_i$ and $g^k(N_j)$ and set $A^k = \sum_{ij} K^k_{ij}$. We set

$$M(n, \mu, \nu) = \left\{ \rho \in M(n-1, \mathbb{N}) \mid \sum_{i=1}^{n-1} \rho_{ij} = \nu_j, \sum_{j=1}^{n-1} \rho_{ij} = \mu_i \right\}.$$ 

For every $i, j$ and $\rho \in M(n, \mu, \nu)$, we can choose $\rho_{ij}$ paths from $\mu_i$ forks and choose one intersection from $K^k_{ij}$ intersections for each forks; see Figure 6. Therefore we obtain

$$\langle \tilde{S}_\nu, \tilde{\beta}^k \left( \tilde{F}_{\mu} \right) \rangle \leq \sum_{\rho \in M(n, \mu, \nu)} \prod_{i=1}^{n-1} \mu_i \prod_{j=1}^{n-1} \frac{1}{\rho_{ij}!} (K^k_{ij})^{\rho_{ij}} \leq \left( \prod_{i=1}^{n-1} \mu_i \right) \sum_{\rho \in M(n, \mu, \nu)} \prod_{i=1}^{n-1} \frac{1}{\rho_{ij}!} (A^k)^{\rho_{ij}} = \left( \prod_{i=1}^{n-1} \mu_i \right) (A^k)^m \sum_{\rho \in M(n, \mu, \nu)} \prod_{i=1}^{n-1} \frac{1}{\rho_{ij}!}. $$
and
\[
\text{Growth}_{k \to \infty} \left\| \left[ S_\nu \right], \beta_k \left( \left[ \tilde{F}_\mu \right] \right) \right\| \leq \left( \text{Growth}_{k \to \infty} A^k \right)^m.
\]

It suffices to show \( \text{Growth}_{k \to \infty} A^k \leq \lambda \). We set
\[
U_i \cap g^{-1}(U_j) = \prod_{l=1}^{K_{ij}^l} V_{ijkl}
\]
and take an open cover \( \alpha = \{ V_{ijk} \mid 1 \leq i, j \leq n-1, 1 \leq k \leq K_{ij}^1 \} \cup A' \) of the compact set \( Y_n \), where \( A' \) does not contain any intersections of \( g^{-1}(\alpha_i) \) and \( N_j \).

**Lemma 5.3.** Each element of \( \bigvee_{p=0}^{k-1} g^{-p}(\alpha) \) contains at most one intersection of \( g^{-k}(\alpha_j) \) and \( N_i \).

**Proof.** Every nonempty element of \( \bigvee_{p=0}^{k-1} g^{-p}(\alpha) \) can be written as
\[
B = V_{i_0i_1i_2} \cap \cdots \cap g^{-k+1}(V_{i_{k-1}i_ki_k})
\]
with \( i_0 = i \) and \( i_k = j \). By the definition of \( \phi \) and \( \tilde{\phi}, g^k|_B : B \to U_j \) is bijective. Therefore \( (g^k|_B)^{-1}(\alpha_j) \) is one leaf of \( U_i \) and there exists only one intersection of \( g^{-k}(\alpha_j) \) and \( N_i \). \( \square \)

It follows from Lemma 5.3 that
\[
A^\ell = \sum_{i,j} K_{ij}^\ell \leq N \left( \bigvee_{i=0}^{\ell-1} g^{-i}(\alpha) \right)
\]
and by (3.1), the growth rate of \( N \left( \bigvee_{i=0}^{\ell-1} g^{-i}(\alpha) \right) \) is equal to or less than the dilatation of \( \beta \). Therefore the proposition follows. \( \square \)

**Proposition 5.4.** For any pseudo-Anosov braid \( \beta \in B_n \), the inequality
\[
\text{Growth}_{m \to \infty} \left\| \text{tr}_{\Gamma, \beta, m} \zeta_{n, m}(\beta) \right\| \leq \lambda
\]
holds.

**Proof.** By Lemma 4.3 \( \left\| \text{tr}_{\Gamma, \beta, m} \zeta_{n, m}(\beta^k) \right\| \) is equal to or greater than the number of essential fixed points of \( \tilde{g}^k \). For \( m = 1 \), we notice that \( \tilde{g}^k \) is \( g^k \). Therefore \( \left\| \text{tr}_{\Gamma, \beta, 1} \zeta_{n, 1}(\beta^k) \right\| \) is equal to or greater than the number of essential periodic points of \( g \) whose period is a divisor of \( k \). In particular, we obtain \( \left\| \text{tr}_{\Gamma, \beta, 1} \zeta_{n, 1}(\beta^k) \right\| \geq D_k^1/k \). Therefore we obtain
\[
\left\| \text{tr}_{\Gamma, \beta, m} \zeta_{n, m}(\beta) \right\| \leq B \# \text{eFix} \tilde{g} = B \sum_{(n_1, \ldots, n_m) \in A_m} \prod_{i=1}^{m} \left( \frac{D_i^1/i}{n_i} \right)
\]
\[
\leq B \sum_{(n_1, \ldots, n_m) \in A_m} \prod_{i=1}^{m} \left( \left\| \text{tr}_{\Gamma, \beta, 1} \zeta_{n, 1}(\beta^i) \right\| \right).
\]
By Proposition 5.2, there exists a monotonically increasing sequence \{a_i\} of real numbers such that
\[ \| \text{tr}_{\beta, 1} \zeta_{n, 1}(\beta^i) \| \leq (a_i \lambda)^i \text{ and } \limsup_{i \to \infty} a_i = 1 \]
holds. Therefore we obtain
\[ \| \text{tr}_{\beta, m} \zeta_{n, m}(\beta) \| \leq B \sum_{(n_1, \ldots, n_m) \in A_m} \prod_{i=1}^{m} (a_i \lambda)^{m_i} \leq B (a_m \lambda)^m S_m, \]
where \( S_m \) is the number of elements of \( A_m \).

**Lemma 5.5.** The equality \( \lim_{m \to \infty} S_m^{1/m} = 1 \) holds.

**Proof.** We suppose that \( m - c_m > c_m d_m \), where
\[ c_m = 4(\lfloor \sqrt[4]{m} \rfloor + 1)^2, \quad d_m = 4(\lfloor \sqrt[4]{m} \rfloor + 2) \]
and \( \lfloor x \rfloor \) is the floor function. Let \( C_m \) be the subset of \( A_m \) satisfying the following condition
\[ \sum_{i=1}^{c_m} n_i = d_m \text{ and } n_m - \sum_{i=1}^{c_m} m_i = 1. \]
Then \( C_m \) is in one-to-one correspondence with the \( d_m \)-combinations with repetition from \( c_m \) elements. Therefore we obtain the inequality
\[ S_m \geq \left( \frac{c_m + d_m - 1}{d_m} \right) = \left( \frac{4(\lfloor \sqrt[4]{m} \rfloor + 2)(\lfloor \sqrt[4]{m} \rfloor + 1)}{4(\lfloor \sqrt[4]{m} \rfloor + 2)} \right) \geq \left( \frac{\lfloor \sqrt[4]{m} \rfloor + 1}{\lfloor \sqrt[4]{m} \rfloor + 2} \right)^2 = m^{\lfloor \sqrt[4]{m} \rfloor + 2}. \]
We set
\[ A_{m,k} = \{ (n_1, \ldots, n_m) \in A_m \mid \max\{i \mid n_i \neq 0\} = k \}, \]
and let \( S_{m,k} \) be the number of the elements of \( A_{m,k} \). Then clearly
\[ S_m = \sum_{k=1}^{m} S_{m,k} \]
holds and the recursion formula
\[ (5.1) \quad S_{m+1,k+1} = S_{m,k} + S_{m-k,k+1} \]
follows from the equality \( A_{m,k} = \bigsqcup_{j=1}^{k} A_{m-k,j} \). Moreover, \( S_{m,k} \) is less than the number of how to put \( m \) balls in distinct \( k \) boxes, which is \( m^k \).

We assume that \( \max k S_{m,k} = S_{m,k_0} \). Since \( S_m \leq m S_{m,k_0} \) holds, we obtain
\[ m^{k_0} \geq S_{m,k_0} \geq \frac{1}{m} S_m \geq m^{\sqrt[4]{m}} \]
and \( k_0 \geq \sqrt[4]{m} \). From (5.1), we obtain
\[ S_{m,k_0} \leq S_{2(m-k_0),m-k_0} = S_{m-k_0}. \]
Since \( S_m \) is monotonically increasing for \( m \), we obtain
\[ S_m \leq m S_{m,k_0} \leq m S_{m-k_0} \leq m S_{m-\sqrt[4]{m}}. \]
There exists a natural number \( N \) such that the assumption holds for all \( m \geq N \). We set \( f(m) = m - \sqrt{m} \) and \( n_N(m) = \min\{i \mid f^i(m) \leq N\} \). Then we obtain \( S_m \leq m^{n_1(m)}S_N \). We notice that if \( x \) is larger than \((\sqrt{m} - 1)^4\), then \( x - f(x) = \sqrt{m}x \) is larger than \( \sqrt{m} - 1 \). Therefore we obtain

\[
f[\sqrt{m^2 - 2\sqrt{m} + 2}] + 1 \leq m - (\sqrt{m}(\sqrt{m} - 1)(\sqrt{m}^2 - 2\sqrt{m} + 2)) = (\sqrt{m} - 1)^4.
\]

Therefore we obtain

\[
n_N(m) \leq \sum_{k=1}^{m} [4k^2 - 2k + 2] + 1 \leq \sqrt{m}(4\sqrt{m^2 - 2\sqrt{m} + 3}) \leq 4m^{3/4}
\]

and

\[
1 < \sqrt[4]{S_m} \leq (m^{n_1(m)}S_{f^1(m)}(m))^{1/m} \leq \sqrt[4]{S_N m^{4m - 4}}.
\]

Since the limit \( \lim_{m \to \infty} \sqrt[4]{S_N m^{4m - 4}} \) equals 1, squeeze theorem leads to the conclusion \( \lim_{m \to \infty} S_m^{1/m} = 1 \). By this lemma, we obtain

\[
\limsup_{m \to \infty} \|\text{tr}_{\Gamma_{\beta,m}} \zeta_{n,m}(\beta)\|^{1/m} \leq \limsup_{m \to \infty} (BS_m)^{1/m} a_m \lambda = \lambda.
\]

**Proof of Theorem 6.1.** Since we have the inequality \( \text{tr}(A) \geq \|A\| \) for any matrix \( A \) with coefficients in Laurent polynomial ring, we obtain

\[
\lambda^m \leq \text{Growth} \|\text{tr}_{\beta^k,m} \zeta_{n,m}(\beta^k)\| \leq \text{Growth} \|\zeta_{n,m}(\beta^k)\| \leq \lambda^m
\]

by Proposition 5.1 and Proposition 6.2. Therefore we have

\[
\text{Growth} \|\text{tr}_{\beta^k,m} \zeta_{n,m}(\beta^k)\| = \text{Growth} \|\zeta_{n,m}(\beta^k)\| = \lambda^m.
\]

We have

\[
\lambda \leq \text{Growth} \|\text{tr}_{\beta,m} \zeta_{n,m}(\beta)\| \leq \lambda
\]

by Proposition 5.1 and Proposition 5.4 and we have \( \text{Growth} \|\text{tr}_{\beta,m} \zeta_{n,m}(\beta)\| = \lambda \). □

6. Homological representation of braid groups

6.1. Homological representation of braid groups. In [20], Lawrence construct a monodromy representation of braid groups. We review the representation. We take a homomorphism

\[
\rho_B : B_{n,1}(D^2) \cong \langle \sigma_1, \ldots, \sigma_{n-1}, \sigma_n^2 \rangle \to \mathbb{Z}
\]

defined by \( \rho_B(\sigma_i) = 0 \) for all \( 1 \leq i < n \) and \( \rho_B(\sigma_n^2) = 1 \). Let \( p_B : \tilde{D}_n \to D_n \) be the covering corresponding to \( \text{Ker} \rho_B \) and fix \( \tilde{d} \in p_B^{-1}(d_1) \). For an \( n \)-braid \( \beta \), we take a representative \( f \). Let

\[
\tilde{f}_B : (\tilde{D}_n, \tilde{d}) \to (\tilde{D}_n, \tilde{d})
\]

be the lift of \( f \). Then \( \tilde{f}_B \) acts on \( H_1(\tilde{D}_n, \partial \tilde{D}_n) \) as \( \mathbb{Z}[\mathbb{Z}] \)-homomorphism. The linear representation \( B \) defined by \( B(\beta) = \tilde{f}_B^* \) is called the reduced Burau representation. Let \( t \) denote the generator of covering transformation.
of \( D_n^B \) corresponding to \( 1 \in \mathbb{Z} \). Then the ring \( \mathbb{Z}[\mathbb{Z}] \) is isomorphic to the Laurent polynomial ring \( \mathbb{Z}[t^{\pm 1}] \) and \( B(\beta) \) can be regarded as a matrix with coefficients in the Laurent polynomial ring \( \mathbb{Z}[t^{\pm 1}] \). Similarly for \( m \geq 2 \), we take a homomorphism

\[
\rho_{LKB} : B_{n,m}(D^2) \cong (\sigma_1, \ldots, \sigma_{n-1}, \sigma_{n+1}, \ldots, \sigma_{n+m-1}) \to \mathbb{Z} \oplus \mathbb{Z}
\]
defined by \( \rho_{LKB}(\sigma_i) = 0 \oplus 0 \) for all \( 1 \leq i < n \), \( \rho_{LKB}(\sigma_i^2) = 1 \oplus 0 \) and \( \rho_{LKB}(\sigma_{n+j}) = 0 \oplus 1 \) for all \( 1 \leq j < m \). Let \( p_{LKB} : \mathcal{C}_{n,m}^{LKB}(D^2) \to C_{n,m}(D^2) \) be the covering corresponding to \( \text{Ker} \rho_{LKB} \) and fix \( \mathcal{C}_{n,m}^{LKB} \in p_{LKB}^{-1}(c) \). For \( \beta \in B_n \), we take a representative \( f \). Let

\[
\tilde{f}^{LKB} : (\mathcal{C}_{n,m}^{LKB}(D^2), \mathcal{C}_{n,m}^{LKB}) \to (\mathcal{C}_{n,m}^{LKB}(D^2), \mathcal{C}_{n,m}^{LKB})
\]
be the lift of \( \tilde{f} \). Then \( \tilde{f}^{LKB} \) acts on \( H_2(\tilde{B}_{n,m}^{LKB}(D^2)) \) as an \( \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \)-homomorphism.

The linear representation \( LKB_m \) defined by \( LKB_m(\beta) = \tilde{f}_*^{LKB} \) is called the Lawrence-Krammer-Bigelow representations. Let \( q \) and \( t \) denote the generator of covering transformation of \( \mathcal{C}_{n,m}^{LKB}(D^2) \) corresponding to \( 1 \oplus 0 \in \mathbb{Z} \oplus \mathbb{Z} \) and \( 0 \oplus 1 \in \mathbb{Z} \oplus \mathbb{Z} \) respectively. Then the ring \( \mathbb{Z}[\mathbb{Z} \oplus \mathbb{Z}] \) is isomorphic to the Laurent polynomial ring \( \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \) and \( LKB_m(\beta) \) can be regarded as a matrix with coefficients in the 2-variable Laurent polynomial ring \( \mathbb{Z}[q^{\pm 1}, t^{\pm 1}] \).

The homological representation of braid groups has been also intensively studied. The Lawrence-Krammer-Bigelow representations of the braid groups were studied by Lawrence [20] in relation with Hecke algebra representations of the braid groups. In [3], [18] and [19], Bigelow and Krammer showed the faithfulness of the Lawrence-Krammer-Bigelow representation for \( m = 2 \) independently.

In [8], Fried showed how to estimate the entropy of a pseudo-Anosov braid by using the Burau matrix \( B(t) \) of a pseudo-Anosov braid. In [17], Kolev proved the same estimation directly with different methods. The following theorem is the estimate and this estimate is called the Burau estimate.

**Theorem 6.1.** (Fried [8], Kolev [17]) Let \( f \) be a homeomorphism of \( D^2 \) fixing \( P_n \) setwise and \( \beta \) be an \( n \)-braid represented by \( f \). Then the topological entropy of \( f \) is equal to or greater than the logarithm of the spectral radius of the Burau matrix \( B(t) \) of \( \beta \) after substituting a complex number of modulus 1 in place of \( t \).

If the inequality is an equality for \( \eta = \eta_0 \), then the Burau estimate is said to be sharp at \( \eta_0 \). In [2], Band and Boyland determined a necessary and sufficient condition when the Burau estimate is sharp at the root of unity.

**Theorem 6.2.** (Band and Boyland [2]) For a pseudo-Anosov braid \( \beta \), the Burau estimate is sharp at the root of unity \( \eta_0 \) only if \( \eta_0 = -1 \). Furthermore, the Burau estimate is sharp at \(-1\) if and only if the invariant foliations for a pseudo-Anosov map in the class represented by \( \beta \) have odd order singularities at all punctures and all interior singularities are even order.

In [16], Koberda shows the similar estimate by using Lawrence-Krammer-Bigelow representation.

**Theorem 6.3.** (Koberda [16]) For a pseudo-Anosov braid \( \beta \), the \( m \)-th power of the dilatation of \( \beta \) is equal to or greater than the spectral radius of the
Lawrence-Krammer-Bigelow matrix $LKB_m(q,t)$ of $\beta$ after substituting complex numbers of modulus 1 in place of $q$ and $t$.

6.2. **Homological estimation and Theorem 1.1.** In this section, we recover the estimation in [8], [17] and [16] using Theorem 1.1. If we have a homomorphism $\rho$ from $E_{n,m}(D^2)$ to some group $G$, we have another representation $\rho_*(\zeta_{n,m})$ on the free $\mathbb{Z}[G]$-module defined by $\rho_*(\zeta_{n,m}) = (\rho_*(c^{(i)}_{\mu\nu}))$. Moreover, if $G$ is a finitely generated free abelian group, $\mathbb{Z}[G]$ can be embedded in $\mathbb{C}$ and in this way, $\rho_*(\zeta_{n,m})$ gives rise to a linear representation $\rho'_*(\zeta_{n,m})$ over $\mathbb{C}$.

When $m = 1$, Let $\rho'_B : E_{n,1}(D^2) \to \mathbb{Z}$ be a the homomorphism defined by $\rho'_B(\sigma_i) = 0$ for all $1 \leq i < n$ and $\rho'_B(\sigma_n^2) = 1$. When $m \geq 2$, let $\rho'_{LKB} : E_{n,m}(D^2) \to \mathbb{Z} \oplus \mathbb{Z}$ be a homomorphism defined by $\rho'_{LKB}(\sigma_i) = 0 \oplus 0$ for all $1 \leq i < n$, $\rho'_{LKB}(\sigma_n^2) = 1 \oplus 0$ and $\rho'_{LKB}(\sigma_{n+j}) = 0 \oplus 1$. We consider the homomorphism from $\text{Aut}_R(R \otimes_{R_B} H_F)$ induced by $\rho'_{LKB}$. Since $\rho'_{LKB}(\sigma_i)$ is $0 \oplus 0$ for all $1 \leq i < n$, the action as the right multiplication becomes trivial and $(\rho'_{LKB})_* (\zeta_{n,m})$ is equivalent to the Lawrence-Krammer-Bigelow representations for all $m \geq 2$. Similarly, $(\rho'_B)_*(\zeta_{n,m})$ is equivalent to the reduced Burau representation.

For any matrix $A$ with coefficients in $n$-variable Laurent polynomial ring and complex numbers $x_1, \ldots, x_n$, we denote by $A(x_1, \ldots, x_n)$ the matrix with coefficients in $\mathbb{C}$ substituting $x_i$ for $i$-th variable. For any matrix $A$ with coefficients in $\mathbb{C}$, we denote by $\text{sr } A$ the spectral radius of $A$. We state the main result of this section.

**Proposition 6.4.** For any matrix $A$ with coefficients in the Laurent polynomial ring $\mathbb{Z}[x_1, \ldots, x_n]$, we have

$$\text{Growth}_{k \to \infty} \left\| \text{tr } A^k \right\| = \sup_{x_i \in S^1} \text{sr } A(x_1, \ldots, x_n).$$

Let $I = (i_1, \ldots, i_n)$ be a multi index and $x^I = \prod_{k=0}^{n} x_k^{i_k}$.

**Lemma 6.5.** We suppose $f(x_1, \ldots, x_n) = \sum_{i=0}^{M} \cdots \sum_{i=0}^{M} a_I x^I$ is an $n$-variable polynomial of degree $M$. Then we have the inequality

$$\sum_{I} |a_I| \leq (M + 1)^n \sup_{x_i \in S^1} |f(x_1, \ldots, x_n)|$$

**Proof.** First of all, we prove the case $n = 1$. Then $f(x)$ is a polynomial $\sum_{i=0}^{M} a_i x^i$ of degree $M$. We consider the Vandermonde matrix

$$V = V_{M+1}(x_0, \ldots, x_M) = \begin{pmatrix} 1 & x_0 & \cdots & x_0^M \\ 1 & x_1 & \cdots & x_1^M \\ \vdots & \vdots & \ddots & \vdots \\ 1 & x_M & \cdots & x_M^M \end{pmatrix}.$$ 

Then we have $V a = A$, where

$$a = \begin{pmatrix} a_0 \\ \vdots \\ a_M \end{pmatrix} \quad \text{and} \quad A = \begin{pmatrix} f(x_0) \\ \vdots \\ f(x_M) \end{pmatrix}.$$
We denote by $\sigma_m$ the $m$-th elementary symmetric function in the $(M + 1)$ variables $x_0, \ldots, x_M$. In other words, we have

$$\sigma_m = \sigma_m(x_0, \ldots, x_M) = \sum_{\nu \in S_m} x_{\nu(1)} \cdots x_{\nu(m)}$$

for all $1 \leq m \leq M + 1$ and $\sigma_0 = 1$. We use the notation $\sigma_m^i$ to denote the $m$-th elementary symmetric function in the $M$ variables $x_k$ with $x_i$ missing. In other words, we have

$$\sigma_m^i = \sigma_m(x_0, \ldots, x_{i-1}, x_{i+1}, \ldots, x_M).$$

We set $V^{-1} = (v_{ij})_{0 \leq i, j \leq M}$. It is well known (see [9]) that we have

$$v_{ij} = (-1)^i \frac{\sigma_{M-i}^j}{\prod_{k \neq j}(x_k - x_j)}$$

We put $\theta = \pi/M + 1$ and $x_k = \exp(2\sqrt{-1}k\theta)$. Since $x_i$’s are all the roots of $z^{M+1} - 1 = 0$, we obtain $\sigma_m(x_0, \ldots, x_M) = 0$ for all $1 \leq m \leq M$. Since the recursion formula $\sigma_{m+1}^i = \sigma_m - x_i\sigma_m^i$ holds, we obtain $\sigma_{m+1}^i = -x_i\sigma_m^i$ and $\sigma_m^i = (-x_i)^m$. Then we obtain

$$|v_{ij}| = \left| (-1)^{i-1} \frac{\sigma_{M-i}^j}{\prod_{k \neq j}(x_k - x_j)} \right| = \frac{1}{\prod_{k=1}^{M}(2\sin k\theta)}.$$  

Since we have $a = V^{-1}A$, we have the inequality

$$|a_i| \leq \frac{M + 1}{\prod_{k=1}^{M}(2\sin k\theta)} \max_k |f(x_k)| \leq \frac{M + 1}{\prod_{k=1}^{M}(2\sin k\theta)} \sup_{x \in S^1} |f(x)|.$$  

**Lemma 6.6.** The equality $\prod_{k=1}^{M}(2\sin k\theta) = M + 1$ holds.

**Proof.** We set

$$\cos(2n - 1)\theta = \cos \theta f_n(\cos \theta), \quad \sin 2n\theta = 2\theta g_n(\cos \theta)$$

for $n \geq 1$. Since

$$\begin{cases} 
\cos(2n + 3)\theta + \cos(2n - 1)\theta = 2\cos 2\theta \cos(2n + 1)\theta \\
\sin 2(n + 2)\theta + \sin 2n\theta = 2\cos 2\theta \sin 2(n + 1)\theta,
\end{cases}$$

hold, we obtain recursion formulae $f_{n+2}(x) = 2(2x^2 - 1)f_{n+1}(x) - f_n(x)$ and $g_{n+2}(x) = 2(2x^2 - 1)g_{n+1}(x) - g_n(x)$. Moreover, because of the initial conditions $f_1(x) = 1$, $f_2(x) = 4x^2 - 3$, $g_1(x) = 1$ and $g_2(x) = 4x^2 - 2$, $f_n(x)$ and $g_n(x)$ are polynomials of degree $2(n - 1)$. Solving the recursion formulae of leading coefficient and constant term, we find that the leading coefficients of $f_n(x)$ and $g_n(x)$ is $4^n$, the constant term of $f_n(x)$ is $(2n - 1)(-1)^{n-1}$ and the constant term of $g_n(x)$ is $n(-1)^{n-1}$.

There exist distinct $2(n - 1)$ solutions

$$\pm \sin(k\pi/(2n - 1)) = \cos(\pi/2 \pm k\pi/(2n - 1)) \quad k = 1, \ldots, n - 1$$

of $f_n(x) = 0$ and distinct $2(n - 1)$ solutions

$$\pm \sin(k\pi/2n) = \cos(\pi/2 \pm k\pi/2n) \quad k = 1, \ldots, n - 1$$

of $g_n(x) = 0$. Vieta’s formula implies $\prod_{k=1}^{M}(2\sin k\theta) = M + 1$.  

□
Lemma 6.6 implies $\sum_{i=0}^{M} |a_i| \leq (M+1) \sup_{x \in S^1} |f(x)|$.

Now we consider the general case. For any $n$-variable polynomial

$$f(x_1, \ldots, x_n) = \sum_{i_1=0}^{M} \cdots \sum_{i_n=0}^{M} a_{i_1}x_1^{i_1}$$

of degree $M$, we set

$$f(x_1, \ldots, x_n) = \sum_{i_n=0}^{M} f_i(x_1, \ldots, x_{n-1})x_n^{i_n}.$$  

Then we obtain

$$\sup_{x_1, \ldots, x_{n-1} \in S^1} \sum_{i} |f_i(x_1, \ldots, x_{n-1})| \leq (M+1) \sup_{x_1, \ldots, x_n \in S^1} |f(x_1, \ldots, x_n)|.$$  

Repeating this $n$ times shows the inequality

$$\sum_{i} |a_i| \leq (M+1)^n \sup_{x_1, \ldots, x_n \in S^1} |f(x_1, \ldots, x_n)|.$$  

\[ \square \]

**Proof of Proposition 6.4.** We notice that

$$\sup_{x_i \in S^1} \left| \sum_{i_1=m}^{M} \cdots \sum_{i_n=m}^{M} a_{i_1}x_1^{i_1} \right| = \sup_{x_i \in S^1} \left| \sum_{i_1=0}^{M-m} \cdots \sum_{i_n=0}^{M-m} a_{i_1}x_1^{i_1} \right|$$

holds. We denote by $A$ a matrix with coefficients in $n$-variable Laurent polynomial ring. Let $M$ and $m$ be the maximum and minimum degree of all entries of $A$. Then the maximum degree of all entries of $A^k$ is equal to or less than $kM$ and the minimum degree of all entries of $A^k$ is equal to or greater than $km$. Using Lemma 6.5 we obtain

$$\sup_{x_i \in S^1} |\text{tr} A^k(x_1, \ldots, x_n)| \leq \left\| \text{tr} A^k \right\| \leq (k(M-m)+1)^n \sup_{x_i \in S^1} |\text{tr} A^k(x_1, \ldots, x_n)|.$$  

Therefore we obtain

$$\text{Growth} \left\| \text{tr} A^k \right\| = \text{Growth} \sup_{x_i \in S^1} |\text{tr} A^k(x_1, \ldots, x_n)|.$$  

Cayley-Hamilton theorem shows

$$\text{tr} A^k(x_1, \ldots, x_n) = \lambda_1^k + \cdots + \lambda_N^k,$$  

where $\lambda_1, \ldots, \lambda_N$ are the eigenvalues of $A(x_1, \ldots, x_n)$. Therefore we obtain

$$\text{Growth} \sup_{x_i \in S^1} |\text{tr} A^k(x_1, \ldots, x_n)| = \sup_{x_i \in S^1} \text{sr} A(x_1, \ldots, x_n).$$  

\[ \square \]

Using Proposition 6.4 we recover the estimation in [8], [17] and [16].

**Corollary 6.7.** For a pseudo-Anosov braid $\beta$, the dilatation of $\beta$ is equal to or greater than the spectral radius of the Burau matrix $B(t)$ of $\beta$ after substituting a complex number of modulus 1 in place of $t$ and the $m$-th power of the dilatation of $\beta$ is equal to or greater than the spectral radius of the Lawrence-Krammer-Bigelow matrix $LKB_m(q,t)$ of $\beta$ after substituting complex numbers of modulus 1 in place of $q$ and $t$.  

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Proof. Since $\|\text{tr}(\rho)_{*}(\zeta_{n,m})(\beta^{k})\|$ is equal to or less than $\|\text{tr}_{\beta^{k},m}\zeta_{n,m}(\beta^{k})\|$, we obtain

$$\text{Growth}_{k \to \infty} \left\| \text{tr}(\rho'_{B})_{*}(\zeta_{n,1})(\beta^{k}) \right\| \leq \lambda$$

and

$$\text{Growth}_{k \to \infty} \left\| \text{tr}(\rho'_{LKB})_{*}(\zeta_{n,m})(\beta^{k}) \right\| \leq \lambda^{m}.$$  

From Proposition 6.4 we obtain

$$\text{Growth}_{k \to \infty} \left\| \text{tr}(\rho'_{B})_{*}(\zeta_{n,1})(\beta^{k}) \right\| = \sup_{t \in S^{1}} B(t)$$

and

$$\text{Growth}_{k \to \infty} \left\| \text{tr}(\rho'_{LKB})_{*}(\zeta_{n,m})(\beta^{k}) \right\| = \sup_{q,t \in S^{1}} LKB_{m}(q,t).$$

Therefore we obtain

$$\sup_{t \in S^{1}} B(t) \leq \lambda \quad \text{and} \quad \sup_{q,t \in S^{1}} LKB_{m}(q,t) \leq \lambda^{m}.$$  

□

On the other hand, it is not known whether $\text{Growth}_{m \to \infty} \left\| \text{tr}(\rho_{LKB})_{*}(\zeta_{n,m})(\beta) \right\|$ is $\lambda$ or not. If $\text{Growth}_{m \to \infty} \left\| \text{tr}(\rho_{LKB})_{*}(\zeta_{n,m})(\beta) \right\|$ is not necessarily $\lambda$, there exists some sufficient condition for $\text{Growth}_{m \to \infty} \left\| \text{tr}(\rho_{LKB})_{*}(\zeta_{n,m})(\beta) \right\| = \lambda$. Clearly the condition in Theorem 6.2 is a sufficient condition for the above equality. We want to reveal whether this sufficient condition is the best condition or not.

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