On uniform convergence of the inverse Fourier transform for differential equations and Hamiltonian systems with degenerating weight

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Abstract
We study pseudospectral and spectral functions for Hamiltonian system \( Jy' - B(t)y = \lambda \Delta(t)y \) and differential equation \( l[y] = \lambda \Delta(t)y \) with matrix-valued coefficients defined on an interval \( I = [a, b] \) with the regular endpoint \( a \). It is not assumed that the matrix weight \( \Delta(t) \geq 0 \) is invertible a.e. on \( I \). In this case a pseudospectral function always exists, but the set of spectral functions may be empty. We obtain a parametrization \( \sigma = \sigma_\tau \) of all pseudospectral and spectral functions \( \sigma \) by means of a Nevanlinna parameter \( \tau \) and single out in terms of \( \tau \) and boundary conditions the class of functions \( y \) for which the inverse Fourier transform \( y(t) = \int \varphi(t, s) d\sigma(s) \hat{y}(s) \) converges uniformly. We also show that for scalar equation \( l[y] = \lambda \Delta(t)y \) the set of spectral functions is not empty. This enables us to extend the Kats–Krein and Atkinson results for scalar Sturm–Liouville equation \( -(p(t)y')' + q(t)y = \lambda \Delta(t)y \) to such equations with arbitrary coefficients \( p(t) \) and \( q(t) \) and arbitrary non trivial weight \( \Delta(t) \geq 0 \).

KEYWORDS
generalized Fourier transform, pseudospectral function, spectral function, uniform convergence

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1 | INTRODUCTION

We consider the differential equation of an even order \( 2r \)

\[
l[y] = \sum_{k=0}^{r} (-1)^k (p_{r-k}(t)y^{(k)})^{(k)} = \lambda \Delta(t)y, \quad t \in I = [a, b), \quad -\infty < a < b \leq \infty, \tag{1.1}
\]

and its natural generalization – the Hamiltonian differential system

\[
Jy' - B(t)y = \lambda \Delta(t)y, \quad t \in I = [a, b), \quad -\infty < a < b \leq \infty, \tag{1.2}
\]

on an interval \( I = [a, b] \) with the regular endpoint \( a \) and arbitrary (regular or singular) endpoint \( b \). It is assumed that the coefficients \( p_j \) and the weight \( \Delta \) in (1.1) are functions on \( I \) with values in the set \( B(\mathbb{C}^m) \) of all linear operators in \( \mathbb{C}^m \).
(or equivalently $m \times m$-matrices) such that $p_j = p_j^*$, $\Delta \geq 0$ (a.e. on $I$) and $p_0^{-1}$, $p_1, \ldots, p_r, \Delta$ are locally integrable. As to system (1.2), we assume that $J \in B(\mathbb{C}^n)$ ($n = 2p$) is given by

$$J = \begin{pmatrix} 0 & -I_p \\ I_p & 0 \end{pmatrix}: \mathbb{C}^p \oplus \mathbb{C}^p \rightarrow \mathbb{C}^p \oplus \mathbb{C}^p$$  \hspace{1cm} (1.3)$$

and $B$ and $\Delta$ are locally integrable $B(\mathbb{C}^n)$-valued functions on $I$ such that $B = B^*$ and $\Delta \geq 0$ a.e. on $I$. Equation (1.1) (system (1.2)) is called regular if $b < \infty$ and $p_0^{-1}$, $p_1, \ldots, p_r, \Delta$ (resp. $B$, $\Delta$) are integrable on $I$; otherwise it is called singular.

Following to [4] we call the weight $\Delta$ definite if it is invertible a.e. on $I$ and semi-definite in the opposite case. Moreover, the weight $\Delta$ in the scalar equation (1.1) is called nontrivial if the equality $\Delta(t) = 0$ (a.e. on $I$) does not hold. Clearly, non triviality is the weakest restriction on $\Delta$, which saves the interest to studying of (1.1).

As is known a spectral function is a fundamental concept in the spectral theory of differential equations [11, 32, 34, 35] and Hamiltonian systems [1, 21, 33]. Let $\varphi(\cdot, \lambda)\in B(\mathbb{C}^p, \mathbb{C}^p \oplus \mathbb{C}^p))$ be an operator solution of (1.2) such that $\varphi(a, \lambda) = (-\sin A, \cos A)^T$ with some $A = A^* \in B(\mathbb{C}^p)$. Then a spectral function of the system (1.2) is defined as an operator-valued (or, equivalently, matrix-valued) distribution function $\sigma(s)\in B(\mathbb{C}^p)$ such that the generalized Fourier transform

$$L^2_\Delta(I) \ni f(t) \rightarrow \hat{f}(s) = \int_I \varphi^*(t, s)\Delta(t)f(t) \, dt$$  \hspace{1cm} (1.4)$$

induces an isometry $V_\sigma$ from the Hilbert space $L^2_\Delta(I)$ of all vector-functions $f(t)\in \mathbb{C}^n$ such that $\int_I (\Delta(t)f(t), f(t)) \, dt < \infty$ to the Hilbert space $L^2(\sigma; \mathbb{C}^p)$. Similarly one defines a spectral function $\sigma(s)\in B((\mathbb{C}^m)^r)$ of Equation (1.1). If $\sigma(\cdot)$ is a spectral function of (1.1) or (1.2), then for each $y \in L^2_\Delta(I)$ the inverse Fourier transform is

$$y(t) = \int_\mathbb{R} \varphi(t, s) \, d\sigma(s) \hat{y}(s),$$  \hspace{1cm} (1.5)$$

where the integral converges in $L^2_\Delta(I)$. Recall also that a spectral function $\sigma(\cdot)$ is called orthogonal if $V_\sigma$ is a unitary operator.

Existence of a spectral function for Equation (1.1) and system (1.2) with the definite weight is a classical result (see, e.g., [35]). This result was extended by I. S. Kats [19, 20] to the scalar Sturm–Liouville equation

$$l[y] = -(p(t)y')' + q(t)y = \lambda \Delta(t)y, \quad t \in I = [a, b), \quad \lambda \in \mathbb{C},$$  \hspace{1cm} (1.6)$$

with $p(t) \equiv 1$ and the semi-definite weight $\Delta$. Moreover, I. S. Kats and M. G. Krein parameterized in [22, §14] all spectral functions of such an equation under the following additional conditions:

(A1) there is no interval $(a, b') \subset I$ such that $\Delta(t) = 0$ a.e. on $(a, b')$ and there is no interval $(a', b) \subset I$ such that $\Delta(t) = 0$ a.e. on $(a', b)$;

(A2) if $\Delta(t) = 0$ a.e. on an interval $(a', b') \subset I$, then $q(t) = 0$ a.e. on $(a', b')$.

The Kats–Krein parametrization can be formulated as the following theorem.

**Theorem 1.1.** Consider scalar regular equation (1.6) such that $p(t) \equiv 1$ and (A1) and (A2) are satisfied. Let $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ be solutions of (1.6) with

$$\varphi(a, \lambda) = -\sin \alpha, \quad \varphi'(a, \lambda) = \cos \alpha, \quad \psi(a, \lambda) = -\cos \alpha, \quad \psi'(a, \lambda) = -\sin \alpha$$  \hspace{1cm} (1.7)$$

and let $\tilde{R} = R \cup \{\tau(\lambda) \equiv \infty\}$, where $R$ is the class of all complex-valued Nevanlinna functions $\tau(\lambda)$ (see Section 2.1). Then the equalities

$$m(\lambda) = \frac{\psi(b, \lambda)\tau(\lambda) - \varphi'(b, \lambda)}{\varphi(b, \lambda)\tau(\lambda) - \varphi'(b, \lambda)}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},$$  \hspace{1cm} (1.8)$$
establish a bijective correspondence $\sigma(\cdot) = \sigma_\tau(\cdot)$ between all functions $\tau \in \tilde{R}[\mathbb{C}]$ and all (real valued) spectral functions $\sigma(\cdot)$ of (1.6) (with respect to the Fourier transform (1.4)). Moreover, $\sigma_\tau(\cdot)$ is orthogonal if and only if $\tau(\lambda) \equiv \delta(= \delta)$ or $\tau(\lambda) \equiv \infty$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$.

As is known each orthogonal spectral function $\sigma(\cdot)$ of Equation (1.1) with definite weight is associated with a certain self-adjoint operator $\tilde{S}$ in $L^2_\Delta(I)$. Moreover, a classical result claims that for each function $y$ from the domain of $\tilde{S}$ the $L^2_\Delta(I)$-convergence in (1.5) can be improved to uniform convergence on each compact interval $[a, c] \subset I$ (see, e.g., [11, Theorem XIII.5.16]. In the case of the Sturm–Liouville equation this result yields the following theorem (see, e.g., [6]).

**Theorem 1.2.** Consider the eigenvalue problem for scalar regular Sturm–Liouville equation (1.6) with the definite weight $\Delta$

subject to self-adjoint boundary conditions

$$\cos \alpha \cdot y(a) + \sin \alpha \cdot (py')(a) = 0, \quad \cos \beta \cdot y(b) + \sin \beta \cdot (py')(b) = 0.$$  

(1.10)

Then each function $y \in AC(I)$ such that $py' \in AC(I)$, $\Delta^{-1}[y] \in L^2_\Delta(I)$ and (1.10) is satisfied admits the eigenfunction expansion

$$y(t) = \sum_{k=1}^{\infty} (y, v_k)_{\Delta} v_k(t), \quad t \in I,$$

(1.11)

which converges absolutely and uniformly on $I$. In (1.11) $\{v_k\}_{1}^{\infty}$ are orthonormal eigenfunctions of the problem (1.6), (1.10).

F. Atkinson in [2, Theorem 8.9.1] extended Theorem 1.2 to the scalar regular equations (1.6) with semi-definite weight $\Delta$ satisfying the condition $0 \leq p(t) \leq \infty$, $t \in I$, and assumptions (A1) and (A2) before Theorem 1.1. Moreover, Theorem 1.2 was extended to eigenvalue problems for regular scalar equations (1.6) [13, 15] and (1.1) [3] with definite weight subject to boundary conditions linearly dependent on the eigenparameter $\lambda$. It is worth to note that these papers deal in fact with a special class of nonorthogonal spectral functions. Observe also that various properties (existence and behavior of eigenvalues, oscillation of eigenfunctions etc.) of eigenvalue problems for Sturm–Liouville equations with semi-definite weight was studied in [4].

It turns out that a spectral function of the system (1.2) and Equation (1.1) with semi-definite weight may not exist and hence definition of a spectral function requires a certain modification. To this end one defines a pseudospectral function of the system (1.2) as an operator-valued distribution function $\sigma(s)(\in B(\mathbb{C}^p))$ such that the generalized Fourier transform (1.4) induces a partial isometry $V_{\sigma} : L^2_\Delta(I) \to L^2(\sigma; \mathbb{C}^p)$ with the minimally possible kernel $\ker V_{\sigma}$ (see [1, 21, 33] for regular systems and [29] for singular ones). If $\sigma(\cdot)$ is a pseudospectral function, then the inverse Fourier transform (1.5) holds only for functions $y \in L^2_\Delta(I) \ominus \ker V_{\sigma}$. It turns out that a pseudospectral function exists for any system (1.2); moreover, either the set of spectral functions of a given system is empty or it coincides with the set of pseudospectral ones. The Kats–Krein parametrization of spectral functions was extended in [1, 29, 33] to Hamiltonian systems (1.2). In these papers a parametrization $\sigma(\cdot) = \sigma_\tau(\cdot)$ of pseudospectral functions $\sigma(\cdot)$ is given in terms of the parameter $\tau = \tau(\lambda)$, which takes on values in the set of all relation-valued Nevanlinna functions (for more details see Theorem 3.14).

In the present paper we extend the above results concerning the uniform convergence of the inverse Fourier transform (1.5) to arbitrary (possibly nonorthogonal) pseudospectral and spectral functions of differential equation (1.1) and Hamiltonian system (1.2) with matrix-valued coefficients and semi-definite weight $\Delta$. This enables us to extend Theorems 1.1 and 1.2 to scalar regular Sturm–Liouville equation (1.6) with arbitrary coefficients $p$ and $q$ and semi-definite nontrivial weight $\Delta$.

First we consider Hamiltonian system (1.2). Assume for simplicity that the set of spectral functions of this system is not empty. Let $\tau = \tau(\lambda)$ be a Nevanlinna parameter and let $\sigma(\cdot) = \sigma_\tau(\cdot)$ be the corresponding spectral function of the system. We prove the following statement:

$$\sigma_\tau(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{\delta} \text{Im} m_\tau(u + i\varepsilon) \, du$$  

(1.9)
(S) If \( y \in L^2_\Delta(I) \) is an absolutely continuous vector-function such that the equality \( Jy' - By = \Delta f_y \) holds with some \( f_y \in L^2_\Delta(I) \) and the boundary conditions

\[
\begin{align*}
\cos A, \sin A \ y(a) &= 0, \\
\Gamma_b y &\in \eta_r
\end{align*}
\]

are satisfied, then the inverse Fourier transform (1.5) converges absolutely and uniformly on each compact interval \([a, c] \subset I\). In (1.12) \( A = A^* \in B(C) \), \( \Gamma_b y \) is a singular boundary value of \( y \) at the endpoint \( b \) (in the case of the regular system one can put \( \Gamma_b y = y(b) \)) and \( \eta_r \) is a linear relation defined in terms of the asymptotic behavior of the parameter \( \tau(\lambda) \) at infinity.

If \( \tau(\lambda) \equiv \theta \) is a self-adjoint parameter, then the spectral function \( \sigma(\cdot) = \sigma(\cdot) \) is orthogonal, \( \eta_r = \theta \) and (1.12) turn into self-adjoint boundary conditions, which defines a self-adjoint operator \( \tilde{T} \) in \( L^2_\Delta(I) \). So in this case under the additional assumption of definiteness of \( \Delta \) statement (S) gives rise to known results on the uniform convergence [11]. Note also that in fact we prove statement (S) for pseudospectral functions (see Theorem 4.4).

As is known [23] Equation (1.1) is equivalent to a certain special system (1.2). Therefore the concept of a pseudospectral function and relative results can be readily transformed to Equation (1.1) with matrix-valued coefficients and semi-definite weight (see Theorems 5.7 and 5.8). Nevertheless it turns out that scalar equation (1.1) with semi-definite nontrivial weight possesses an essential peculiarity. Namely, we show (see Theorem 5.13) that the set of spectral functions of such an equation is not empty. Moreover, we parameterize all these spectral functions by means of a Nevanlinna parameter \( \tau \) and single out in terms of \( \tau \) and boundary conditions the class of functions \( y \in L^2_\Delta(I) \) for which the inverse Fourier transform (1.5) converges uniformly on each compact interval \([a, c] \subset I\) (see Theorems 5.13, 5.14 and 5.15). In the case of the Sturm–Liouville equation these results can be formulated in the form of the following theorem.

**Theorem 1.3.** Consider scalar regular equation (1.6) on \( I = [a, b] \) with real-valued coefficients \( p, q \) and semi-definite non-trivial weight \( \Delta(t) \geq 0, (p^{-1}, q, \Delta \in L^1(I)) \). Denote by \( dom l \) the set of all functions \( y \in AC(I) \) such that \( y^{[1]} := py' \in AC(I) \) and let \( \lim_{y \to \infty} \tau(y) \not\equiv 0 \), then \( \cos \alpha \cdot y(a) + \sin \alpha \cdot y^{[1]}(a) = 0 \) and \( y(b) = 0 \);

\[
\text{(bc1) if } \lim_{y \to \infty} \frac{\tau(y)}{iy} \not\equiv 0, \text{ then } \cos \alpha \cdot y(a) + \sin \alpha \cdot y^{[1]}(a) = 0 \text{ and } y(b) = 0;
\]

\[
\text{(bc2) if } \lim_{y \to \infty} \frac{\tau(y)}{iy} = 0 \text{ and } \lim_{y \to \infty} y \text{ Im } \tau(iy) < \infty,
\]

\[
\text{then } \cos \alpha \cdot y(a) + \sin \alpha \cdot y^{[1]}(a) = 0 \text{ and } y^{[1]}(b) = D(\tau)y(b) \text{ (here } D(\tau) = \lim_{y \to \infty} \text{ Im } \tau(iy));
\]

\[
\text{(bc3) if } \lim_{y \to \infty} \frac{\tau(y)}{iy} = 0 \text{ and } \lim_{y \to \infty} y \text{ Im } \tau(iy) = \infty, \text{ then }
\]

\[
\cos \alpha \cdot y(a) + \sin \alpha \cdot y^{[1]}(a) = 0, \ y(b) = 0 \text{ and } y^{[1]}(b) = 0.
\]

Then for each function \( y \in F \)

\[
y(t) = \int \varphi(t, s) \hat{y}(s) d\sigma(s),
\]

where the integral converges absolutely and uniformly on \( I \).

Note that statement (i) of Theorem 1.3 extends the Katse existence theorem [19, 20] and Kats–Krein parametrization of spectral functions to Sturm–Liouville equations (1.6) with \( p(t) \not\equiv 1 \) and semi-definite nontrivial weight \( \Delta \) (cf. Theorem 1.1). Moreover, by using Theorem 1.3 we extend to such equations Theorem 1.2 (see Corollary 5.16). In other words, we show that in the case \( p(t) < \infty \) Theorem 1.2 remains valid without Atkinson’s assumptions.
In conclusion note that our investigations are based on the results of [31] (see also [10]), where compression $C := P_{\mathfrak{F}} \tilde{A} \uparrow \mathfrak{K}$ of an exit space extension $\tilde{A} = \tilde{A}^*$ of an operator $A \subset A^*$ in the Hilbert space $\mathfrak{K}$ are characterized in terms of abstract boundary conditions. We show that in the case of a nonorthogonal spectral function $\sigma(\cdot)$ the integral in (1.5) converges uniformly for any $y$ from the domain of the respective compression $C$ (for Equation (1.1) with definite weight this fact was proved in [5, 7]). Next we apply to $C$ the results of [31].

2 PRELIMINARIES

2.1 Notations

The following notations will be used throughout the paper: $\tilde{\mathfrak{K}}, \mathfrak{H}$ denote separable Hilbert spaces; $\mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ is the set of all bounded linear operators defined on $\mathfrak{H}_1$ with values in $\mathfrak{H}_2$; $A \uparrow \mathcal{L}$ is a restriction of the operator $A \in \mathcal{B}(\mathfrak{H}_1, \mathfrak{H}_2)$ to the linear manifold $\mathcal{L} \subset \mathfrak{H}_1$; $P_{\mathcal{L}}$ is the orthoprojection in $\mathfrak{K}$ onto the subspace $\mathcal{L} \subset \tilde{\mathfrak{K}}$; $C_+ (C_-)$ is the open upper (lower) half-plane of the complex plane; $\mathcal{A}$ is the $\sigma$-algebra of Borel sets in $\mathbb{R}$ and $\mu$ is the Borel measure on $\mathcal{A}$. For a set $B \subset \mathbb{R}$ we denote by $\chi_B(\cdot)$ the indicator of $B$, i.e., the real-valued function on $\mathbb{R}$ given by $\chi_B(t) = 1$ for $t \in B$ and $\chi_B(t) = 0$ for $t \in \mathbb{R} \setminus B$.

Recall that a linear manifold $T$ in the Hilbert space $\mathfrak{H}_0 \oplus \mathfrak{H}_1 (\mathfrak{H} \oplus \mathfrak{H})$ is called a linear relation from $\mathfrak{H}_0$ to $\mathfrak{H}_1$ (resp. in $\mathfrak{H}$). The set of all closed linear relations from $\mathfrak{H}_0$ to $\mathfrak{H}_1$ (in $\mathfrak{H}$) will be denoted by $\tilde{C}(\mathfrak{H}_0, \mathfrak{H}_1)$ (resp. $\tilde{C}(\mathfrak{H})$). Clearly for each linear operator $T : \text{dom} \, T \to \mathfrak{H}_1$, $\text{dom} \, T \subset \mathfrak{H}_0$, its graph $\text{gr} \, T = \{ \{ f, T f \} : f \in \text{dom} \, T \}$ is a linear relation from $\mathfrak{H}_0$ to $\mathfrak{H}_1$. This fact enables one to consider an operator $T$ as a linear relation. In the following we denote by $C(\mathfrak{H}_0, \mathfrak{H}_1)$ the set of all closed linear operators $T : \text{dom} \, T \to \mathfrak{H}_1$, $\text{dom} \, T \subset \mathfrak{H}_0$. Moreover, we let $C(\mathfrak{H}) = C(\mathfrak{H}_0, \mathfrak{H}_1)$.

For a linear relation $T$ from $\mathfrak{H}_0$ to $\mathfrak{H}_1$ we denote by $\text{dom} \, T$, $\text{ker} \, T$, $\text{ran} \, T$ and $\text{mul} \, T : = \{ h_1 \in \mathfrak{H}_1 : \{ 0, h_1 \} \in T \}$ the domain, kernel, range and multivalued part of $T$ respectively. Denote also by $T^{-1}$ and $T^*$ the inverse and adjoint linear relations of $T$ respectively. Clearly, $T$ is an operator if and only if $\text{mul} \, T = \{ 0 \}$.

We will use the following notations:

(i) $R[H]$ is the set of all Nevanlinna $\mathcal{B}(\mathfrak{H})$-valued functions, i.e., the set of all holomorphic operator functions $M(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \mathcal{B}(\mathfrak{H})$ such that $\text{Im} \lambda \cdot M(\lambda) \geq 0$ and $M^*(\lambda) = M(\bar{\lambda})$, $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(ii) $R_0[H]$ is the set of all functions $M(\cdot) \in R[H]$ such that $(\text{Im} \, M(\lambda))^{-1} \in \mathcal{B}(\mathfrak{H})$ for all $\lambda \in \mathbb{C} \setminus \mathbb{R}$;
(iii) $\tilde{R}(\mathfrak{H})$ is the set of all Nevanlinna relation-valued functions (see, e.g., [8]), which in the case $\mathfrak{H} = \mathbb{C}^m$ can be defined as the set of all functions $\tau(\cdot) : \mathbb{C} \setminus \mathbb{R} \to \tilde{C}(\mathbb{C}^m)$ such that $\text{mul} \tau(\lambda) : = \mathcal{K}$ does not depend on $\lambda \in \mathbb{C} \setminus \mathbb{R}$ and the decompositions

$$ C^m = \mathfrak{H}_0 \oplus \mathcal{K}, \quad \tau(\lambda) = \text{gr} \, \tau_0(\lambda) \oplus \tilde{\mathcal{K}}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, $$

(2.1)

hold with $\tilde{\mathcal{K}} = \{ 0 \} \oplus \mathcal{K}$ and $\tau_0(\cdot) \in R[\mathfrak{H}_0]$ (the operator function $\tau_0(\cdot)$ is called the operator part of $\tau(\cdot)$).

It is clear that $R[H] \subset \tilde{R}(\mathfrak{H})$.

2.2 Boundary triplets and compressions of exit space extensions

Recall that a linear relation $T$ in $\mathfrak{K}$ is called symmetric (self-adjoint) if $T \subset T^*$ (resp. $T = T^*$). In the following we denote by $A$ a closed symmetric linear relation in a Hilbert space $\mathfrak{K}$. Let $\mathfrak{N}_A(A) = \text{ker} \, (A^* - \lambda) (\lambda \in \mathbb{C} \setminus \mathbb{R})$ be a defect subspace of $A$ and let $n_\pm(A) : = \dim \mathfrak{N}_A(A)$, $\lambda \in C_\pm$, be deficiency indices of $A$. Denote by $\text{exf}(A)$ the set of all closed proper extensions of $A$ (i.e., the set of all relations $\tilde{A} \subset \tilde{C}(\mathfrak{K})$ such that $A \subset \tilde{A} \subset A^*$).

It is easy to see that $A$ is a densely defined operator if and only if $\text{mul} \, A^* = \{ 0 \}$.

As is known a linear relation $\tilde{A} = \tilde{A}^*$ in a Hilbert space $\tilde{\mathfrak{K}} \supset \mathfrak{K}$ is called an exit space extension of $A$ if $A \subset \tilde{A}$ and the minimality condition $\text{span} \{ \mathfrak{K}, (\tilde{A} - \lambda)^{-1} \mathfrak{K} : \lambda \in \mathbb{C} \setminus \mathbb{R} \} = \tilde{\mathfrak{K}}$ is satisfied.
Definition 2.1 [14]. A collection \( \Pi = \{ H, \Gamma_0, \Gamma_1 \} \) consisting of a Hilbert space \( H \) and linear mappings \( \Gamma_j : A^* \to H, \ j \in \{ 0, 1 \} \), is called a boundary triplet for \( A^* \), if the mapping \( \Gamma = (\Gamma_0, \Gamma_1) \) from \( A^* \) into \( H \oplus H \) is surjective and the following abstract Green’s identity holds:

\[
(f', g) - (f, g') = (\Gamma_1 \hat{f}, \Gamma_0 \hat{g}) - (\Gamma_0 \hat{f}, \Gamma_1 \hat{g}), \quad \hat{f} = (f, f'), \ \hat{g} = (g, g') \in A^*.
\]

Theorem 2.2 [9, 26]. Let \( \Pi = \{ H, \Gamma_0, \Gamma_1 \} \) be a boundary triplet for \( A^* \). Then:

(i) The mapping

\[
\theta \to A_\theta := \{ f \in A^* : \{ \Gamma_0 \hat{f}, \Gamma_1 \hat{f} \} \in \theta \}
\]

establishes a bijective correspondence \( A = A_\theta \) between all linear relations \( \theta \in \overline{C}(H) \) and all extensions \( \overline{A} = \overline{\text{ext}}(A) \). Moreover \( A_\theta \) is symmetric (self-adjoint) if and only if \( \theta \) is symmetric (resp. self-adjoint).

(ii) The equality \( P_\theta (\overline{A}_\tau - \lambda)^{-1} \| \phi = (A_{-\tau(\lambda)} - \lambda)^{-1}, \ \lambda \in \mathbb{C} \setminus \mathbb{R}, \) gives a bijective correspondence \( \overline{A} = \overline{A}_\tau \) between all functions \( \tau = \tau(\cdot) \in \overline{C}(H) \) and all exit space extensions \( \overline{A} = \overline{A}_\tau \) of \( A \). Moreover, if \( \tau(\lambda) \equiv \theta(\tau^*), \ \lambda \in \mathbb{C} \setminus \mathbb{R}, \) then \( \overline{A}_\tau = A_{-\theta} \) (see (2.2)).

Note that the same parametrization \( \overline{A} = \overline{A}_\tau \) of exit space extensions \( \overline{A} \) of \( A \) can be also given by means of the Krein formula for generalized resolvents (see, e.g., [9, 24, 26]).

Definition 2.3. The linear relation \( C(\overline{A}) \) in \( H \) defined by

\[
C(\overline{A}) := P_\theta \overline{A} \| H = \{ \{ f, f' \} : \{ \Gamma_0 \hat{f}, \Gamma_1 \hat{f} \} \in \overline{A}, \ f \in H \}
\]

is called the compression of the exit space \( \overline{A} = \overline{A}_\tau \) of \( A \).

Clearly, \( C(\overline{A}) \) is a symmetric extension of \( A \). Note also that the equality

\[
\Phi(\overline{A}) := \{ \{ P_\theta f, P_\theta f' \} : \{ f, f' \} \in \overline{A}, \ f \in H \}
\]

defines a linear relation \( \Phi(\overline{A}) \subset A^* \) (see, e.g. [8]).

A characterization of the compression \( C(\overline{A}_\tau) \) in terms of the parameter \( \tau \) is given by the following theorem obtained in our paper [31].

Theorem 2.4. Assume that \( \Pi = \{ \mathbb{C}^m, \Gamma_0, \Gamma_1 \} \) is a boundary triplet for \( A^* \) (in this case \( n_+(A) = n_-(A) = m \)). Let \( \tau \in \overline{C}(\mathbb{C}^m) \), let \( \overline{A}_\tau = \overline{A}_\tau^+ \) be the corresponding exit space extension of \( A \) and let \( C(\overline{A}_\tau) \) be the compression of \( \overline{A}_\tau \) of \( A \). Assume also that \( \tau_0 \in \overline{R}(H_0) \) and \( \mathcal{K} \) are the operator and multivalued parts of \( \tau \) respectively (see (2.1)). Then:

(i) the equalities \( B_{\tau_0} = \lim_{y \to \infty} \frac{1}{iy} \tau_0(iy) \) and

\[
\text{dom} \ D_{\tau_0} = \left\{ h \in H_0 : \lim_{y \to \infty} y \text{Im}(\tau_0(iy)h, h) < \infty \right\}, \ D_{\tau_0} h = \lim_{y \to \infty} \tau_0(iy)h, \ h \in \text{dom} \mathcal{N}_{\tau_0},
\]

correctly define the nonnegative operator \( B_{\tau_0} \in B(H_0) \) and the operator \( D_{\tau_0} : \text{dom} \ D_{\tau_0} \to H_0 \) (\( \text{dom} \ D_{\tau_0} \subset H_0 \));

(ii) \( C(\overline{A}_\tau) = A_{\eta_\tau} \) with the symmetric linear relation \( \eta_\tau \in \overline{C}(\mathbb{C}^m) \) given by

\[
\eta_\tau = \{ \{ h, -D_{\tau_0} h + B_{\tau_0} h' + k \} : h \in \text{dom} \ D_{\tau_0}, h' \in H_0, k \in \mathcal{K} \}.
\]

2.3 The spaces \( L^2(\sigma; \mathbb{C}^m) \) and \( L^2(\sigma; \mathbb{C}^m) \)

Recall that a non-decreasing operator function \( \sigma(\cdot) : \mathbb{R} \to B(\mathbb{C}^m) \) is called a distribution function if it is left continuous and satisfies \( \sigma(0) = 0 \).
The following theorem was originally proved by I. S. Kac in [18]. Other proofs and further development can be found in [11, 27].

Theorem 2.5. Let \( \sigma(\cdot) : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{C}^m) \) be a distribution function. Then:

1. There exist a scalar measure \( \nu \) on \( A \) and a function \( \Psi : \mathbb{R} \times \mathcal{B}(\mathbb{C}^m) \) (uniquely defined by \( \nu \) up to \( \nu \)-a.e.) such that \( \Psi(s) \geq 0 \) \( \nu \)-a.e. on \( \mathbb{R} \), \( \nu([\alpha, \beta)) < \infty \) and \( \sigma(\beta) - \sigma(\alpha) = \int_{[\alpha, \beta)} \Psi(s) \, d\nu \) for any finite interval \( [\alpha, \beta) \subset \mathbb{R} \).

2. The set \( L^2(\sigma; \mathbb{C}^m) \) of all Borel-measurable functions \( f = f(\cdot) : \mathbb{R} \rightarrow \mathbb{C}^m \) satisfies

\[
\|f\|_{L^2(\sigma; \mathbb{C}^m)}^2 = \int_{\mathbb{R}} (d\sigma(s)f(s), f(s)) = \int_{\mathbb{R}} (\Psi(s)f(s), f(s))_{\mathbb{C}^m} \, d\nu < \infty
\]

is a semi-Hilbert space with the semi-scalar product

\[
(f, g)_{L^2(\sigma; \mathbb{C}^m)} = \int_{\mathbb{R}} (d\sigma(s)f(s), g(s)) = \int_{\mathbb{R}} (\Psi(s)f(s), g(s))_{\mathbb{C}^m} \, d\nu, \quad f, g \in L^2(\sigma; \mathbb{C}^m).
\]

Definition 2.6 [11]. The Hilbert space \( L^2(\sigma; \mathbb{C}^m) \) is a Hilbert space of all equivalence classes in \( L^2(\sigma; \mathbb{C}^m) \) with respect to the seminorm \( \| \cdot \|_{L^2(\sigma; \mathbb{C}^m)} \).

In the following we denote by \( \pi_\sigma \) the quotient map from \( L^2(\sigma; \mathbb{C}^m) \) onto \( L^2(\sigma; \mathbb{C}^m) \). Two functions \( f_1, f_2 \in L^2(\sigma; \mathbb{C}^m) \) are said to be \( \sigma \)-equivalent if \( \pi_\sigma f_1 = \pi_\sigma f_2 \), i.e., if \( \Psi(s)f_1(s) = \Psi(s)f_2(s) \) \( \nu \)-a.e. on \( \mathbb{R} \).

With a distribution function \( \sigma(\cdot) : \mathbb{R} \rightarrow \mathcal{B}(\mathbb{C}^m) \) one associates the \( \mathcal{B}(\mathbb{C}^m) \)-valued measure \( \mu_\sigma \) on \( A \) given by

\[
\mu_\sigma(B) = \int_B \Psi(s) \, d\nu, \quad B \in A.
\]

This measure is a continuation of the measure \( \mu_{\sigma|_{[\alpha, \beta)}} \) on finite intervals \( [\alpha, \beta) \subset \mathbb{R} \) defined by \( \mu_{\sigma|_{[\alpha, \beta)}}([\alpha, \beta)) = \sigma(\beta) - \sigma(\alpha) \).

Let \( \sigma(s) \in \mathcal{B}(\mathbb{C}^m) \) be a distribution function. For Borel measurable functions \( Y(s) \in \mathcal{B}(\mathbb{C}^m, \mathbb{C}^k) \) and \( g(s) \in \mathbb{C}^m \) on \( \mathbb{R} \) we let

\[
\int_{\mathbb{R}} Y(s) \, d\sigma(s)g(s) = \int_{\mathbb{R}} Y(s)\Psi(s)g(s) \, d\nu \quad (\in \mathbb{C}^k)
\]

(2.6)

where \( \nu \) and \( \Psi(\cdot) \) are defined in Theorem 2.5, (1).

3 | PSEUDOSPECTRAL AND SPECTRAL FUNCTIONS OF HAMILTONIAN SYSTEMS

3.1 | Notations

Let \( I = (a, b) \) be an interval of the real line (the endpoint \( b < \infty \) might be either included to \( I \) or not). Denote by \( AC(I; \mathbb{C}^n) \) the set of functions \( f(\cdot) : I \rightarrow \mathbb{C}^n \) which are absolutely continuous on each segment \( [a, b'] \subset I \).

An operator-function \( Y(\cdot) : I \rightarrow \mathcal{B}(\mathbb{C}^n) \) is called locally integrable if \( \int_{[a, b']} \|Y(t)\| \, dt < \infty \) for each \( b' \in I \). Assume that \( \Delta(\cdot) : I \rightarrow \mathcal{B}(\mathbb{C}^n) \) is a locally integrable function such that \( \Delta(t) \geq 0 \) a.e. on \( I \). Denote by \( L^2_{\Delta}(I; \mathbb{C}^n) \) the semi-Hilbert space of Borel measurable functions \( f(\cdot) : I \rightarrow \mathbb{C}^n \) satisfying \( \|f(\cdot)\|_{L^2_{\Delta}}^2 = \int_I (\Delta(t)f(t), f(t)) \, dt < \infty \) (see e.g. [11, Chapter 13.5]). The semi-definite inner product \( \langle \cdot, \cdot \rangle_{L^2_{\Delta}} \) is defined by \( \langle f(\cdot), g(\cdot) \rangle_{L^2_{\Delta}} = \int_I (\Delta(t)f(t), g(t)) \, dt \). Moreover, let \( L^2_{\Delta}(I; \mathbb{C}^n) \) be the Hilbert space of the equivalence classes in \( L^2_{\Delta}(I; \mathbb{C}^n) \) with respect to the semi-norm \( \| \cdot \|_{L^2_{\Delta}} \). Denote also by \( \pi_{\Delta} \) the quotient map from \( L^2_{\Delta}(I; \mathbb{C}^n) \) onto \( L^2_{\Delta}(I; \mathbb{C}^n) \) and let \( \bar{\pi}_{\Delta}(f(\cdot), g(\cdot)) := \{ \pi_{\Delta}f(\cdot), \pi_{\Delta}g(\cdot) \} \), \{ f(\cdot), g(\cdot) \} \in (L^2_{\Delta}(I; \mathbb{C}^n))^2 \). Clearly, \( \ker \pi_{\Delta} \) coincides with the set of all Borel measurable functions \( f(\cdot) : I \rightarrow \mathbb{C}^n \) such that \( \Delta(t)f(t) = 0 \) a.e. on \( I \).
3.2 Hamiltonian systems

Let as above \( I = [a, b) \) \((−\infty < a < b ≤ \infty)\) be an interval in \( \mathbb{R} \), let \( p \in \mathbb{N} \) and let \( n = 2p \). Recall that a Hamiltonian system of the dimension \( n \) on an interval \( I \) (with the regular endpoint \( a \)) is a system of differential equations

\[
Jy' - B(t)y = \lambda \Delta(t)y, \quad t \in I, \quad \lambda \in \mathbb{C}, \tag{3.1}
\]

where \( B(\cdot) \) and \( \Delta(\cdot) \) are locally integrable \( B(\mathbb{C}^n) \)-valued functions on \( I \) satisfying \( B(t) = B^*(t) \) and \( \Delta(t) \geq 0 \) a.e. on \( I \) and \( J \in B(\mathbb{C}^n) \) is the operator given by (1.3). Together with system (3.1) we consider the inhomogeneous system

\[
Jy' - B(t)y = \Delta(t)f(t), \quad t \in I, \tag{3.2}
\]

where \( f(\cdot) \in L^2_{\Delta}(I; \mathbb{C}^n) \). A function \( y(\cdot) \in AC(I, \mathbb{C}^n) \) is a solution of (3.1) (resp. (3.2)) a.e. on \( I \). A function \( Y(\cdot, \lambda) : I \to B(\mathbb{C}^k, \mathbb{C}^n) \) is an operator solution of (3.1) if \( y(t) = Y(t, \lambda)h \) is a (vector) solution of (3.1) for every \( h \in \mathbb{C}^k \). In the sequel we denote by \( Y_0(\cdot) \) the \( B(\mathbb{C}^n) \)-valued operator solution of the system

\[
Jy' - B(t)y = 0 \tag{3.3}
\]
such that \( Y_0(0) = I_n \). As is known, \( Y_0(t) \) satisfies the identities

\[
Y_0^*(t)JY_0(t) = J, \quad Y_0(t)JY_0^*(t) = J. \tag{3.4}
\]

By using the second identity in (3.4) one can easily verify that each solution \( y(\cdot) \) of (3.2) admits the representation

\[
y(t) = z(t) - Y_0(t)J \int_{[a,t]} Y_0^*(u)\Delta(u)f(u) du, \tag{3.5}
\]

where \( z(\cdot) \in AC(I, \mathbb{C}^n) \) is the solution of (3.3) with \( z(a) = y(a) \).

As it is known (see, e.g., [21, 25]) system (3.1) gives rise to the maximal linear relations \( T_{\max} \) and \( T_{\max} \) in \( L^2_{\Delta}(I; \mathbb{C}^n) \) and \( L^2_{\Delta}(I; \mathbb{C}^n) \) respectively. Namely, \( T_{\max} \) is the set of all pairs \( \{y(\cdot), f(\cdot)\} \in (L^2_{\Delta}(I; \mathbb{C}^n))^2 \) such that \( y(\cdot) \in AC(I, \mathbb{C}^n) \) and (3.2) holds a.e. on \( I \), while \( T_{\max} = \pi_{\Delta}T_{\max} \). Moreover for any \( y(\cdot), z(\cdot) \in \text{dom} \ T_{\max} \) there exists the limit

\[
[y, z]_b := \lim_{t \uparrow b}(Jy(t), z(t)).
\]

Next, define the linear relation \( T_a \) in \( L^2_{\Delta}(I; \mathbb{C}^n) \) and the minimal linear relation \( T_{\min} \) in \( L^2_{\Delta}(I; \mathbb{C}^n) \) by setting

\[
T_a = \{\{y(\cdot), f(\cdot)\} \in T_{\max} : y(a) = 0 \quad \text{and} \quad [y, z]_b = 0 \quad \text{for every} \quad z \in \text{dom} \ T_{\max}\}
\]

and \( T_{\min} = \pi_{\Delta}T_a \). Then \( T_{\min} \) is a closed symmetric linear relation in \( L^2_{\Delta}(I; \mathbb{C}^n) \) and \( T_{\min} = T_{\max} \) [21, 25, 28].

The null manifold \( \mathcal{N} \) of the system (3.1) is defined as a linear space of all solutions \( y(\cdot) \) of (3.3) such that \( \Delta(t)y(t) = 0 \) (a.e. on \( I \)).

In the sequel we denote by \( \mathcal{N}_\lambda, \lambda \in \mathbb{C} \), the linear space of solutions of the system (3.1) belonging to \( L^2_{\Delta}(I; \mathbb{C}^n) \). The numbers \( N_+ = \dim \mathcal{N}_+ \) and \( N_- = \dim \mathcal{N}_- \) are called the formal deficiency indices of the system (3.1). It was shown in [23, 25] that \( N_+ = \dim \mathcal{N}_+ \), \( \lambda \in \mathbb{C}_+ \) (i.e., \( \dim \mathcal{N}_+ \) does not depend on \( \lambda \) in either \( \mathbb{C}_+ \) or \( \mathbb{C}_- \)) and \( p \leq N_+ \leq n \). Moreover, deficiency indices of \( T_{\min} \) are \( n_\pm(T_{\min}) = N_\pm - \dim \mathcal{N} \).

Recall that system (3.1) is called definite if \( \mathcal{N} = \{0\} \).

**Definition 3.1.** Let \( U \in B(\mathbb{C}^n, \mathbb{C}^p) \) be an operator such that

\[
UJU^* = 0 \quad \text{and} \quad \text{ran} \ U = \mathbb{C}^p. \tag{3.6}
\]

System (3.1) is called \( U \)-definite if for each \( y \in \mathcal{N} \) the equality \( Uy(a) = 0 \) yields \( y = 0 \). System (3.1) is called \( U \)-definite on an interval \( I' \subset I \) if its restriction on \( I' \) is \( U \)-definite.

Clearly each definite system is \( U \)-definite for any \( U \).
It was proved in [23] that for each definite system there is a compact interval \([a, b'] \subset I\) such that the system is definite on \([a, b']\). In the same way one proves the following proposition.

**Proposition 3.2.** If system (3.1) is \(U\)-definite, then there is a compact interval \([a, b'] \subset I\) such that the system is \(U\)-definite on \([a, b']\).

### 3.3 Pseudospectral and spectral functions

Below we suppose that \(U \in B(\mathbb{C}^n, \mathbb{C}^p)\) is an operator satisfying (3.6). Then the following assertion holds (see [30, Lemma 3.3]).

**Assertion 3.3.** The equality

\[
T = \left\{ \bar{\pi}_\Delta(y, f) : \{y, f\} \in \mathcal{T}_{max}, \; Uy(a) = 0 \text{ and } [y, z]_b = 0, \; z \in \text{dom } \mathcal{T}_{max} \right\}
\]  

(3.7)

defines a (closed) symmetric extension \(T\) of \(T_{min}\). Moreover, \(T^* = \bar{\pi}_\Delta \mathcal{T}_s\), where \(\mathcal{T}_s\) is the linear relation in \(L^2_\Delta(I; \mathbb{C}^n)\) given by

\[
\mathcal{T}_s = \left\{ \{y(\cdot), f(\cdot)\} \in \mathcal{T}_{max} : Uy(a) = 0 \right\}.
\]  

(3.8)

Clearly the domain of \(\mathcal{T}_s\) is

\[
\text{dom } \mathcal{T}_s = \{y(\cdot) \in AC(I, \mathbb{C}^n) \cap L^2_\Delta(I; \mathbb{C}^n) : Jy'(t) - B(t)y(t) = \Delta(t)f_y(t) \text{ (a.e. on } I) \text{ with some } f_y(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) \text{ and } Uy(a) = 0\}.
\]  

(3.9)

Note that \(f_y(\cdot)\) in (3.9) is defined by \(y(\cdot)\) uniquely up to the equivalence with respect to the seminorm \(|| \cdot ||_\Delta\).

In what follows we put \(\mathcal{H} := L^2_\Delta(I; \mathbb{C}^n)\) and \(\mathcal{H}_0 := \mathcal{H} \ominus \text{mul } T\). Since \(T\) is a symmetric relation in \(\mathcal{H}\), the decompositions

\[
\mathcal{H} = \mathcal{H}_0 \oplus \text{mul } T, \quad T = \text{gr } T_0 \oplus \hat{\text{mul }} T
\]  

(3.10)

hold with \(\hat{\text{mul }} T = \{0\} \oplus \text{mul } T\) and a (not necessarily densely defined) symmetric operator \(T_0\) in \(\mathcal{H}_0\) (this operator is called the operator part of \(T\)).

Below we denote by \(L'\), \(L_0\) and \(D\) the linear manifolds in \(L^2_\Delta(I; \mathbb{C}^n)\) defined by

\[
L' = \left\{ f(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) : \text{ there exists a solution } y(\cdot) \text{ of (3.2) such that } \Delta(t)y(t) = 0 \text{ (a.e. on } I), \; Uy(a) = 0 \text{ and } [y, z]_b = 0, \; z \in \text{dom } \mathcal{T}_{max} \right\},
\]  

(3.11)

\[
L_0 = \left\{ f(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) : \langle f(\cdot), g(\cdot) \rangle_\Delta = 0 \text{ for any } g(\cdot) \in L' \right\},
\]  

(3.12)

\[
D = \left\{ y(\cdot) \in \text{dom } \mathcal{T}_s : f_y(\cdot) \in L_0 \right\}.
\]  

(3.13)

Clearly, \(\text{mul } T = \pi_\Delta L'\) and \(\mathcal{H}_0 = \pi_\Delta L_0\).

Let \(\varphi_U(\cdot, \lambda) \in B(\mathbb{C}^p, \mathbb{C}^n)\), \(\lambda \in \mathbb{C}\), be the operator solution of (3.1) with the initial value \(\varphi_U(a, \lambda) = -JU^*\). One can easily prove that for each function \(f(\cdot) \in L^2_\Delta(I; \mathbb{C}^n)\) and each point \(c \in I\) the equality

\[
\hat{f}_c(s) = \int_I \varphi_U^+(t, s)\Delta(t)\chi_{[a,c]}(t)f(t) \, dt
\]  

(3.14)

defines a continuous function \(\hat{f}_c(\cdot) : \mathbb{R} \to \mathbb{C}^p\) (the integral in (3.14) is understood as the Lebesgue integral).
Definition 3.4 [29]. A distribution function \( \sigma(\cdot) : \mathbb{R} \to \mathcal{B}(\mathbb{C}^p) \) is called a pseudospectral function of the system (3.1) if:

(i) for each function \( f(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) \) and each \( c \in I \) one has \( \hat{f}_c(\cdot) \in L^2(\sigma; \mathbb{C}^p) \) and there exists a function \( \hat{f}(\cdot) \in L^2(\sigma; \mathbb{C}^p) \) such that

\[
\lim_{c \uparrow b} ||| \hat{f} - \hat{f}_c |||_{L^2(\sigma; \mathbb{C}^p)} = 0;
\]

(3.15)

(ii) \( ||| \hat{f} |||_{L^2(\sigma; \mathbb{C}^p)} = 0 \) for \( f(\cdot) \in L' \) and the Parseval equality \( ||| \hat{f} |||_{L^2(\sigma; \mathbb{C}^p)} = ||| f |||_{L^2_\Delta(I; \mathbb{C}^n)} \) holds for all \( f(\cdot) \in L_0 \).

Clearly, the function \( \hat{f}(\cdot) \) in Definition 3.4 is defined by \( f(\cdot) \) uniquely up to the \( \sigma \)-equivalence. This function is called the (generalized) Fourier transform of a function \( f(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) \).

Definition (3.15) of \( \hat{f}(\cdot) \) can be written as

\[
\hat{f}(s) = \int_I \varphi^*(t,s)\Delta(t)f(t)dt,
\]

(3.16)

where the integral converges in the seminorm of \( L^2(\sigma; \mathbb{C}^p) \).

Definition 3.5. A distribution function \( \sigma(\cdot) : \mathbb{R} \to \mathcal{B}(\mathbb{C}^p) \) is called a spectral function of the system (3.1) if for each function \( f(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) \) with compact support the corresponding Fourier transform (3.16) (with the Lebesgue integral in the right hand side) satisfies the Parseval equality \( ||| \hat{f} |||_{L^2(\sigma; \mathbb{C}^p)} = ||| f |||_{L^2_\Delta(I; \mathbb{C}^n)} \).

Clearly, for a spectral function \( \sigma(\cdot) \) the Fourier transform (3.16) (with the integral convergent in \( L^2(\sigma; \mathbb{C}^p) \)) satisfies the Parseval equality \( ||| \hat{f} |||_{L^2(\sigma; \mathbb{C}^p)} = ||| f |||_{L^2_\Delta(I; \mathbb{C}^n)} \) for every \( f(\cdot) \in L^2_\Delta(I; \mathbb{C}^p) \).

Remark 3.6. If \( \sigma(\cdot) \) is a pseudospectral function, then the equality

\[
\mathcal{V}_\sigma \tilde{f} = \pi_\Delta \hat{f}(\cdot), \quad \tilde{f} \in \mathcal{S},
\]

(3.17)

where \( \hat{f}(\cdot) \) is the Fourier transform of a function \( f(\cdot) \in \tilde{f} \), defines a partial isometry \( \mathcal{V}_\sigma \in \mathcal{B}(\mathcal{S}, L^2(\sigma; \mathbb{C}^p)) \) such that \( \ker \mathcal{V}_\sigma = \text{mul} T \) and \( ||| \mathcal{V}_\sigma \tilde{f} ||| = ||| \tilde{f} ||| \), \( \tilde{f} \in \mathcal{S}_0 \) (see (3.10)). Clearly, \( \mathcal{V}_\sigma \) is an isometry if and only if \( \sigma(\cdot) \) is a spectral function.

Definition 3.7. A pseudospectral (spectral) function \( \sigma(\cdot) \) of (3.1) is called orthogonal if \( \ker \mathcal{V}_\sigma = L^2(\sigma; \mathbb{C}^p) \).

Proposition 3.8 [29]. Let \( \sigma(\cdot) \) be a pseudospectral function of the system (3.1). Then for each function \( g(\cdot) \in L^2(\sigma; \mathbb{C}^p) \) the following holds:

(i) for each bounded Borel set \( B \subset \mathbb{R} \) the equality

\[
\bar{g}_B(t) = \int_{\mathbb{R}} \varphi_U(t,s)d\sigma(s)\chi_B(s)g(s), \quad t \in I,
\]

(3.18)

defines a function \( \bar{g}_B(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) \) (the integral in (3.18) exists as the Lebesgue integral, see (2.6))

(ii) there exists a function \( \bar{g}(\cdot) \in L^2_\Delta(I; \mathbb{C}^n) \) such that for each sequence \( \{B_n\}_{n=1}^\infty \) of bounded Borel sets \( B_n \subset \mathbb{R} \) satisfying \( B_n \subset B_{n+1} \) and \( \mu_T(R \setminus \bigcup_{n \in \mathbb{N}} B_n) = 0 \) the following equality holds:

\[
\lim_{n \to \infty} ||| \bar{g}(\cdot) - \bar{g}_{B_n}(\cdot) |||_{L^2_\Delta(I; \mathbb{C}^n)} = 0.
\]

(3.19)

Equality (3.19) is written as \( \bar{g}(t) = \int_{\mathbb{R}} \varphi_U(t,s)d\sigma(s)g(s) \), where the integral converges in the seminorm of \( L^2_\Delta(I; \mathbb{C}^n) \).

Moreover, for each \( \bar{g} \in L^2(\sigma; \mathbb{C}^p) \) one has

\[
\mathcal{V}_\sigma \bar{g} = \pi_\Delta \bar{g}(\cdot) = \pi_\Delta \left( \int_{\mathbb{R}} \varphi_U(\cdot,s)d\sigma(s)g(s) \right), \quad g(\cdot) \in \bar{g}.
\]

(3.20)
Corollary 3.9. Let $\sigma(\cdot)$ be a pseudospectral function of the system (3.1), let $f(\cdot) \in L_0$ and let $\hat{f}(\cdot)$ be the Fourier transform of $f(\cdot)$. Then

$$ f(t) = \int_{\mathbb{R}} \varphi_U(t, s) \, d\sigma(s) \hat{f}(s), $$

(3.21)

where the integral converges in the seminorm of $L^2_A(I; \mathbb{C}^n)$.

Remark 3.10. The equality (3.21) is called the inverse Fourier transform of a function $f(\cdot)$. Clearly, (3.21) is valid for each $f(\cdot) \in L^2_A(I; \mathbb{C}^n)$ if and only if $\sigma(\cdot)$ is a spectral function.

Remark 3.11. According to [29] a distribution function $\sigma(\cdot) : \mathbb{R} \to B(\mathbb{C}^p)$ is called a $q$-pseudospectral function of the system (3.1) if the condition (i) of Definition 3.4 is satisfied and the Fourier transform $V_\sigma$ of the form (3.17) is a partial isometry from $\mathfrak{H}$ to $L^2(\sigma; \mathbb{C}^p)$. According to [29, Proposition 3.8] for each $q$-pseudospectral function $\sigma(\cdot)$ one has $\operatorname{mul} T \subset \ker V_\sigma$. This implies that for a pseudospectral function $\sigma(\cdot)$ the Fourier transform $V_\sigma$ has the minimally possible kernel $\ker V_\sigma$ among all $q$-pseudospectral functions and hence the inverse Fourier transform (3.21) is valid for functions $f(\cdot)$ from the maximally possible set (namely, from the set $L_0$). This facts justify our interest to pseudospectral functions.

Proposition 3.12 [29]. Assume that:

(A1) system (3.1) has equal formal deficiency indices $N_+ = N_- = d$;

(A2) $U \in B(\mathbb{C}^n, \mathbb{C}^p)$ is an operator satisfying (3.6) and system (3.1) is $U$-definite;

(A3) $\Gamma_b = (\Gamma_{0b}, \Gamma_{1b})^T : \text{dom } T_{\text{max}} \to \mathbb{C}^{d-p} \oplus \mathbb{C}^{d-p}$ is a surjective operator satisfying

$$ [\gamma, z]_b = (\Gamma_{0b}\gamma, \Gamma_{1b}z) - (\Gamma_{1b}\gamma, \Gamma_{0b}z), \quad \gamma, z \in \text{dom } T_{\text{max}} $$

(such an operator exists in view of [29, Lemma 4.1]).

Moreover, let $T$ be the symmetric extension (3.7) of $T_{\min}$. Then:

(i) for each pair $\{\tilde{\gamma}, \tilde{f}\} \in T^*$ there exists a unique function $y(\cdot) \in \text{dom } T_\sigma$ such that $\pi_\Delta y(\cdot) = \tilde{\gamma}$ and $\pi_\Delta f(y(\cdot)) = \tilde{f}$;

(ii) the collection $\Pi = \{\mathbb{C}^{d-p}, \Gamma_0, \Gamma_1\}$ with operators $\Gamma_j : T^* \to \mathbb{C}^{d-p}$ given by

$$ \Gamma_0\{\tilde{\gamma}, \tilde{f}\} = \Gamma_{0b}\tilde{\gamma}, \quad \Gamma_1\{\tilde{\gamma}, \tilde{f}\} = -\Gamma_{1b}\tilde{\gamma}, \quad \{\tilde{\gamma}, \tilde{f}\} \in T^*, $$

(3.22)

is a boundary triplet for $T^*$ (in (3.22) $y(\cdot) \in \tilde{\gamma}$ is a function from statement (i)).

Remark 3.13. In the case of the system (3.1) on a compact interval $I = [a, b]$ one has $d = 2p$. In this case one can put $\Gamma_b y = y(b)$, $y \in \text{dom } T_{\text{max}}$.

Theorem 3.14 [29]. Let the assumptions (A1) and (A2) in Proposition 3.12 be satisfied. Then the set of pseudospectral functions of the system (3.1) is not empty and there exists a Nevanlinna operator function

$$ M(\lambda) = \begin{pmatrix} m_0(\lambda) & M_2(\lambda) \\ M_3(\lambda) & M_4(\lambda) \end{pmatrix} : \mathbb{C}^p \oplus \mathbb{C}^{d-p} \to \mathbb{C}^p \oplus \mathbb{C}^{d-p}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, $$

(3.23)

such that $M_4(\cdot) \in R_u[\mathbb{C}^{d-p}]$ and the equalities

$$ m_\tau(\lambda) = m_0(\lambda) - M_2(\lambda)(\tau(\lambda) + M_4(\lambda))^{-1}M_3(\lambda), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}, $$

(3.24)

$$ \sigma_\tau(s) = \lim_{\delta \to +0} \lim_{\varepsilon \to +0} \frac{1}{\pi} \int_{-\delta}^{\varepsilon - \delta} \text{Im } m_\tau(u + i\varepsilon) \, du $$

(3.25)
establish a bijective correspondence \( \sigma(\cdot) = \sigma(\cdot) \) between all functions \( \tau(\cdot) \in \tilde{R}(C^{d-p}) \) satisfying the admissibility condition

\[
\lim_{y \to \infty} \frac{1}{i y} (\tau(iy) + M_4(iy))^{-1} = \lim_{y \to \infty} \frac{1}{i y} (\tau^{-1}(iy) + M_4^{-1}(iy))^{-1} = 0 \tag{3.26}
\]

and all pseudospectral functions \( \sigma(\cdot) \). Moreover, the following statements hold: (i) all functions \( \tau(\cdot) \in \tilde{R}(C^{d-p}) \) satisfying (3.26) if and only if \( \text{mul} T = \text{mul} T^* \); (ii) a pseudospectral function \( \sigma(\cdot) \) is orthogonal if and only if \( \tau(\lambda) \equiv \bar{\tau}(\lambda^*) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \).

Note that the matrix \( M(\lambda) \) in (3.23) is defined in terms of the boundary values of certain operator solutions of (3.1) at the endpoints \( a \) and \( b \) (see [29, Proposition 4.9]).

**Definition 3.15.** A function \( \tau \in \tilde{R}(C^{d-p}) \) satisfying (3.26) will be called an admissible boundary parameter.

Clearly, Theorem 3.14 gives a parametrisation of all pseudospectral functions of the system (3.1) in terms of the admissible boundary parameter \( \tau \).

**Remark 3.16.** The operator function \( m_4(\cdot) \) in (3.24) coincides with the \( m \)-function of the system (3.1) corresponding to the admissible boundary parameter \( \tau \) (see [29]). Note that \( m_4(\cdot) \in R[C^d] \) and (3.25) is the Perron–Stieltjes formula for \( m_4 \). In the case of the constant-valued admissible boundary parameter \( \tau(\lambda) \equiv \pi(\tau) \), \( \lambda \in \mathbb{C} \setminus \mathbb{R} \), the function \( m_4(\cdot) \) turns into the \( m \)-function (Titchmarsh–Weyl function) of the system in the sense of [16, 17].

**Proposition 3.17.**

(i) For system (3.1) the following equivalences are valid:

\[
\mathcal{L}' \subset \ker \pi_\Delta \iff \mathcal{L}_0 = \mathcal{L}_0^2(I; C^n) \iff \text{mul} T = \{0\} \iff D = \text{dom} \tau. \tag{3.27}
\]

If \( \Delta(t) \) is invertible a.e on \( I \), then all the relations in (3.27) hold.

(ii) Let for system (3.1) the assumptions (A1) and (A2) be satisfied. Then the set of spectral functions of the system (3.1) is not empty if and only if at least one (and hence all) of the equivalent conditions in (3.27) are satisfied. Moreover, in this case the sets of spectral and pseudospectral functions coincide and hence Theorem 3.14 holds for spectral functions.

**Proof.** (i) The first and second equivalences in (3.27) are obvious. Next, by (3.10) one has \( T^* = T_0^* \oplus \text{mul} T \), where \( T_0^* \in \mathcal{L}(\mathcal{S}_0) \). This yields the third equivalence in (3.27).

Statement (ii) directly follows from [29, Theorem 5.12]. \[ \square \]

## 4 UNIFORM CONVERGENCE OF THE INVERSE FOURIER TRANSFORM FOR HAMILTONIAN SYSTEMS

**Lemma 4.1.** Suppose that system (3.1) is given on a compact interval \( I = [a, b] \) and satisfies the assumption (A2) in Proposition 3.12. Let \( \mathcal{N}_0 \) be a linear space of all solutions \( y(\cdot) \) of the system (3.3) satisfying \( Uy(a) = 0 \) (clearly, \( \mathcal{N}_0 \subset \mathcal{L}_0^2(I; C^n) \)), let \( y(\cdot) \in \mathcal{N}_0 \) and let \( \{y_n(\cdot)\}_{n=1}^{\infty} \) be a sequence of functions \( y_n(\cdot) \in \mathcal{N}_0 \) such that \( ||y_n(\cdot) - y(\cdot)||_\Delta \to 0 \). Then

\[
\lim_{n \to \infty} \sup_{t \in I} ||y(t) - y_n(t)|| = 0. \tag{4.1}
\]

**Proof.** Let \( y(\cdot) \in \mathcal{N}_0 \) and \( (y(\cdot), y(\cdot))_\Delta = 0 \). Then \( \Delta(t)y(t) = 0 \) and hence \( y(\cdot) \in \mathcal{N} \). Since \( Uy(a) = 0 \) and the system is \( U \)-definite, the equality \( y = 0 \) holds. Thus \( \mathcal{N}_0 \) is a finite dimensional Hilbert space with the inner product \( (\cdot, \cdot)_\Delta \). Clearly, the relation \( \mathcal{N}_0 \ni y(\cdot) \to y(0) \in \ker U \) defines a linear isomorphism of \( \mathcal{N}_0 \) onto \( \ker U \). Therefore the condition \( ||y_n(\cdot) - y(\cdot)||_\Delta \to 0 \) yields \( y_n(0) \to y(0) \), which implies (4.1). \[ \square \]

**Proposition 4.2.** Let system (3.1) satisfy the assumption (A2). Assume also that \( \{y(\cdot), f(\cdot)\} \in \mathcal{T}_\pi \) and let \( \{y_n(\cdot)\}_{n=1}^{\infty} \) and \( \{f_n(\cdot)\}_{n=1}^{\infty} \) be sequences of functions \( y_n(\cdot), f_n(\cdot) \in \mathcal{L}_0^2(I; C^n) \) such that \( \{y_n(\cdot), f_n(\cdot)\} \in \mathcal{T}_\pi \) and
\[ ||y_n(\cdot) - y(\cdot)||_\Delta \to 0, \ ||f_n(\cdot) - f(\cdot)||_\Delta \to 0. \] Then for each compact interval \([a, c] \subset I\) one has

\[ \lim_{n \to \infty} \sup_{t \in [a, c]} ||y(t) - y_n(t)|| = 0. \] (4.2)

**Proof.**

(i) First suppose that system (3.1) is given on a compact interval \(I = [a, b]\). Since \(y(\cdot)\) and \(y_n(\cdot)\) are solutions of (3.2) with \(f(\cdot)\) and \(f_n(\cdot)\) respectively, it follows from (3.5) that

\[ y(t) = z(t) + g(t), \quad y_n(t) = z_n(t) + g_n(t), \quad t \in I, \] (4.3)

where \(z(\cdot)\) and \(z_n(\cdot)\) are solutions of (3.3) with \(z(a) = y(a)\) and \(z_n(a) = y_n(a)\) and

\[ g(t) = -Y_0(t)J \int_a^t Y^*_0(u)\Delta(u)f(u)du, \quad g_n(t) = -Y_0(t)J \int_a^t Y^*_0(u)\Delta(u)f_n(u)du. \] (4.4)

Let

\[ r(t) = \int_a^t Y^*_0(u)\Delta(u)f(u)du, \quad r_n(t) = \int_a^t Y^*_0(u)\Delta(u)f_n(u)du. \]

Then for any \(t \in I\) and \(h \in \mathbb{C}^n\) one has

\[ |(r(t) - r_n(t), h)| = \left| \int_a^t (Y^*_0(u)\Delta(u)(f(u) - f_n(u)), h)du \right| \]

\[ = \left| \int_a^t (\Delta(u)(f(u) - f_n(u)), Y_0(u)h)du \right| \]

\[ = \left| (f(\cdot) - f_n(\cdot), Y_0(\cdot)h)_{L^2([a,t])} \right| \]

\[ \leq ||f(\cdot) - f_n(\cdot)||_{L^2([a,t])} \cdot ||Y_0(\cdot)h||_{L^2([a,t])} \]

\[ \leq ||f(\cdot) - f_n(\cdot)||_\Delta \cdot ||Y_0(\cdot)h||_\Delta. \]

This implies that

\[ \lim_{n \to \infty} \sup_{t \in I} |(r(t) - r_n(t), h)| = 0, \quad h \in \mathbb{C}^n, \]

and hence

\[ \lim_{n \to \infty} \sup_{t \in I} ||r(t) - r_n(t)|| = 0. \] (4.5)

Since by (4.4) \(g(t) - g_n(t) = -Y_0(t)J(r(t) - r_n(t))\) and the operator function \(Y_0(t)\) is bounded in \(I\), it follows from (4.5) that

\[ \lim_{n \to \infty} \sup_{t \in I} ||g(t) - g_n(t)|| = 0. \] (4.6)

Therefore \(||g(\cdot) - g_n(\cdot)||_\Delta \to 0\) and by (4.3) \(||z(\cdot) - z_n(\cdot)||_\Delta \to 0\). Since \(Uz(a) = Uy(a)\) and \(Uz_n(a) = Uy_n(a)\), it follows from (3.8) that \(Uz(a) = Uz_n(a) = 0\). Therefore \(z(\cdot), z_n(\cdot) \in \mathcal{N}_0'\) (for \(\mathcal{N}_0'\) see Lemma 4.1) and by Lemma 4.1

\[ \lim_{n \to \infty} \sup_{t \in I} ||z(t) - z_n(t)|| = 0. \] (4.7)
Now combining (4.3) with (4.6) and (4.7) we arrive at the equality
\[
\lim_{n \to \infty} \sup_{t \in I} ||y(t) - y_n(t)|| = 0.
\]

(ii) Now consider system (3.1) on an interval \( I = [a, b) \), \( b \leq \infty \). According to Proposition 3.2 there is a segment \( I_0 = [a, c_0) \subset I) \) such that the system is \( U \)-definite on \( I_0 \). Let \( I_1 = [a, c] \) be a segment in \( I \) and let \( I' = [a, c') \subset I \) be a segment such that \( c_0 < c' \) and \( I_1 \subset I' \). Then the system is \( U \)-definite on \( I' \). Let \( T_{\alpha} \) be the linear relation \( (3.8) \) corresponding to the restriction of the system (3.1) on \( I' \) and let \( \hat{y}(\cdot), \hat{f}(\cdot), \hat{f}_n(\cdot) \) be restrictions of the functions \( y(\cdot), y_n(\cdot), f(\cdot), f_n(\cdot) \) on \( I' \) respectively. Clearly, \( \{\hat{y}(\cdot), \hat{f}(\cdot)\} \subset T', \{\hat{y}_n(\cdot), \hat{f}_n(\cdot)\} \subset T_{\alpha} \) and 
\[
||\hat{y}_n(\cdot) - \hat{y}(\cdot)||_{L^2(I', \mathbb{C}^p)} \to 0, ||\hat{f}_n(\cdot) - \hat{f}(\cdot)||_{L^2(I', \mathbb{C}^p)} \to 0.
\]
Therefore by statement (i)
\[
\lim_{n \to \infty} \sup_{t \in I'} ||y(t) - y_n(t)|| = 0 \tag{4.8}
\]
and the inclusion \( I_1 \subset I' \) means that (4.8) holds with \( I_1 = [a, c) \) instead of \( I' \).

Let \( T \) be a symmetric relation (3.7) and let \( \Pi = \{C^{d-p}, \Gamma_0, \Gamma_1\} \) be a boundary triplet (3.22) for \( T^* \). Moreover, let \( \tau \in \overline{R}(C^{d-p}) \) be an admissible boundary parameter and let \( \overline{T}_\tau = T_{\alpha}^\tau \) be the corresponding exit space extension of \( T \) (see Theorem 2.2, (ii)). Assume that \( \overline{T}_\tau \) is a linear relation in a Hilbert space \( \mathfrak{H} \supset \mathfrak{H}_0 \). Then according to [29, Proposition 5.3] \( \text{mul}\overline{T}_\tau = \text{mul} T \) and the equalities (3.10) for \( \overline{T}_\tau \) take the form
\[
\mathfrak{H} = \mathfrak{H}_0 \oplus \text{mul} T, \quad \overline{T}_\tau = \text{gr} \overline{T}_{0\tau} \oplus \text{mul} T, \tag{4.9}
\]
where \( \mathfrak{H}_0 = \mathfrak{H} \cap \text{mul} T \) and \( \overline{T}_{0\tau} \) is a self-adjoint operator in \( \mathfrak{H}_0 \).

Combining (4.9) with (3.10) one obtains that \( \mathfrak{H}_0 \subset \mathfrak{H} \) and \( \overline{T}_{0\tau} \) is an exit space extension of \( T_0 \).

**Proposition 4.3.** Let for system (3.1) the assumptions (A1) and (A2) in Proposition 3.12 be satisfied. Moreover, let \( \tau \in \overline{R}(C^{d-p}) \) be an admissible boundary parameter, let \( \sigma(\cdot) = \sigma(-\cdot) \) be a pseudospectral function of the system (3.1), let \( \overline{T}_\tau = T_{\alpha}^\tau \) be an exit space extension of \( T \) in the Hilbert space \( \mathfrak{H} \supset \mathfrak{H}_0 \), let \( \overline{T}_{0\tau} \) be the operator part of \( \overline{T}_\tau \) (see decompositions (4.9)) and let \( E(\cdot) \) be the orthogonal spectral measure of \( \overline{T}_{0\tau} \). Next assume that \( \{B_n\} \subset \mathbb{R} \) is a sequence of bounded Borel sets \( B_n \subset \mathbb{R} \) such that \( \bigcup_{n \in \mathbb{N}} B_n = \bigcup_{n \in \mathbb{N}} B_n \) and let \( B \) be a pair of functions such that \( \hat{y} = \pi_{\Delta} y(\cdot) \in \text{dom} \overline{T}_{0\tau} \cap \mathfrak{H}_0 \) and \( \overline{f} = \pi_{\Delta} f(\cdot) = P_{\mathfrak{H}} \overline{T}_{0\tau} \hat{y} \) (for \( \mathfrak{H}_0 \) see (3.10)) and let \( \overline{y}_B = P_{\mathfrak{H}_0} E(B) \hat{y}, \overline{f}_B = P_{\mathfrak{H}_0} E(B) \overline{T}_{0\tau} \hat{y} \). Then:

(i) \( \{\overline{y}_B, \overline{f}_B\} \subset T^* \) and hence there exists a pair of functions \( y_B(\cdot), f_B(\cdot) \subset T^* \) such that \( \pi_{\Delta} y_B(\cdot) = \overline{y}_B \) and \( \pi_{\Delta} f_B(\cdot) = \overline{f}_B \);
(ii) if \( \hat{y}(\cdot) \) is the Fourier transform of \( y(\cdot) \), then for each compact interval \( [a, c] \subset I \) one has
\[
\lim_{n \to \infty} \sup_{t \in [a, c]} ||y(t) - \int_{\mathbb{R}} \varphi_u(t, s) \sigma(s) \chi_{B_n}(s) \hat{y}(s)|| = 0. \tag{4.10}
\]
(iii) if in addition \( \mu_{\Delta}(\mathbb{R} \setminus B) = 0 \), then for each \( [a, c] \subset I \) one has
\[
\lim_{n \to \infty} \sup_{t \in [a, c]} ||y(t) - \int_{\mathbb{R}} \varphi_u(t, s) \sigma(s) \chi_{B_n}(s) \hat{y}(s)|| = 0. \tag{4.11}
\]

**Proof.**

(i) Since \( \{E(B) \overline{y}, E(B) \overline{f}\} \in \text{gr} \overline{T}_{0\tau} \), it follows from (2.3) that \( \{\overline{y}_B, \overline{f}_B\} \subset \Phi(\overline{T}_{0\tau}) \) and hence \( \overline{y}_B, \overline{f}_B \subset T_{\alpha}^\tau \). Since obviously \( T_{\alpha}^\tau \subset T^* \), this implies that \( \overline{y}_B, \overline{f}_B \subset T^* \).

(ii) Let \( \mathcal{K} := \mathcal{V}_\sigma \mathfrak{H}_0 (\subset L^2(\sigma; \mathbb{C}^p)) \) and let \( \mathcal{V}_0 \in \mathcal{B}(\mathfrak{H}_0, \mathcal{K}) \) be a unitary operator given by \( \mathcal{V}_0 \overline{f} = \mathcal{V}_\sigma \overline{f} \), \( \overline{f} \in \mathfrak{H}_0 \). Denote by \( \Lambda_{\sigma} \) the multiplication operator in \( L^2(\sigma; \mathbb{C}^p) \) defined by
\[
\text{dom} \Lambda_{\sigma} = \{g \in L^2(\sigma; \mathbb{C}^p) : \text{sg}(s) \in L^2(\sigma; \mathbb{C}^p) \text{ for some (and hence for all) } g(\cdot) \in \mathfrak{g}\}, \quad \Lambda_{\sigma} \mathfrak{g} = \pi_{\sigma} (\text{sg}(s)), \quad \mathfrak{g} \in \text{dom} \Lambda_{\sigma}, \quad g(\cdot) \in \mathfrak{g}.
\]
As is known, $\Lambda^* = \Lambda$ and the orthogonal spectral measure $E_\sigma(\cdot)$ of $\Lambda$ is

$$E_\sigma(B)g = \pi_\sigma(\chi_B(\cdot)g(\cdot)), \quad B \in A, \quad g \in L^2(\sigma;C^n), \quad g(\cdot) \in \overline{g}.$$  (4.12)

According to [29, Proposition 5.6] there exists a unitary operator $\tilde{V} \in B(\mathfrak{S}_0, L^2(\sigma;C^p))$ such that $\tilde{V} |_{\mathfrak{S}_0} = V_\sigma |_{\mathfrak{S}_0}$ and the operators $\tilde{T}_0\tau$ and $\Lambda_\sigma$ are unitarily equivalent by means of $\tilde{V}$. This implies that

$$P_{\mathfrak{S}_0} E(B_n) |_{\mathfrak{S}_0} = \tilde{V}_0^* P_{K_\sigma} E_\sigma(B_n) \tilde{V}_0,$$  (4.13)

$$P_{\mathfrak{S}_0} E(B_n) \tilde{T}_0\tau | (\text{dom } \tilde{T}_0\tau \cap \mathfrak{S}_0) = \tilde{V}_0^* P_{K_\sigma} E_\sigma(B_n)\Lambda_\sigma V_\sigma | (\text{dom } \tilde{T}_0\tau \cap \mathfrak{S}_0).$$  (4.14)

Since $\tilde{V}_0^* P_{K_\sigma} \overline{g} = V_\sigma^* \overline{g}$, $\overline{g} \in L^2(\sigma;C^p)$, the equalities (4.13) and (4.14) can be written as

$$P_{\mathfrak{S}_0} E(B_n) |_{\mathfrak{S}_0} = V_\sigma^* E_\sigma(B_n) V_\sigma,$$  (4.15)

$$P_{\mathfrak{S}_0} E(B_n) \tilde{T}_0\tau | (\text{dom } \tilde{T}_0\tau \cap \mathfrak{S}_0) = V_\sigma^* E_\sigma(B_n)\Lambda_\sigma V_\sigma | (\text{dom } \tilde{T}_0\tau \cap \mathfrak{S}_0).$$  (4.16)

Let $\tilde{y}_n := P_{\mathfrak{S}_0} E(B_n)\tilde{y}$ and $\tilde{f}_n := P_{\mathfrak{S}_0} E(B_n)\tilde{T}_0\tau \tilde{y}$. Then by (4.15) and (4.16) one has

$$\tilde{y}_n = V_\sigma^* E_\sigma(B_n)\tilde{y}, \quad \tilde{f}_n = V_\sigma^* E_\sigma(B_n)\Lambda_\sigma \tilde{y}.$$  (4.17)

Combining (4.17) with (3.17), (4.12) and (3.20) one gets $\tilde{y}_n = \pi_\Delta y_n(\cdot)$ and $\tilde{f}_n = \pi_\Delta f_n(\cdot)$, where $y_n(\cdot)$ and $f_n(\cdot)$ are functions from $L^2(\tau;C^n)$ given by

$$y_n(t) = \int_{\mathbb{R}} \varphi_U(t,s) d\sigma(s) \chi_{B_n}(s) \tilde{y}(s), \quad f_n(t) = \int_{\mathbb{R}} \varphi_U(t,s) d\sigma(s) \chi_{B_n}(s) \tilde{y}(s).$$  (4.18)

It was shown in the proof of Proposition 5.5 in [29] that $\{y_n(\cdot), f_n(\cdot)\} \in \mathfrak{S}_*$ and, moreover, since $||\tilde{y}_n - \tilde{y}_B||_{\mathfrak{S}_0} \to 0$ and $||\tilde{f}_n - \tilde{f}_B||_{\mathfrak{S}_0} \to 0$, it follows that $||y_n(\cdot) - y_B(\cdot)||_{\Delta} \to 0$ and $||f_n(\cdot) - f_B(\cdot)||_{\Delta} \to 0$. Therefore by Proposition 4.2 for each segment $[a, c] \subset I$ the equality (4.10) is valid.

(iii) Assume that $\mu_\sigma(\mathbb{R} \setminus B) = 0$. Since the operators $E_\sigma(B)$ and $E(B)$ are unitarily equivalent, this implies that $E(\mathbb{R} \setminus B) = 0$ and hence $E(B) = I_{\mathfrak{S}_0}$. Therefore $\tilde{y}_B = \tilde{y}$, $\tilde{f}_B = \tilde{f}$ and, consequently, $\pi_\Delta y(\cdot) = \pi_\Delta y_B(\cdot)$, $\pi_\Delta f(\cdot) = \pi_\Delta f_B(\cdot).$ Thus by Proposition 3.12, (i) $y(\cdot) = y_B(\cdot)$ and (4.10) yields (4.11).

□

The main results of this section are given in the following two theorems.

**Theorem 4.4.** Let for system (3.1) the assumptions (A1)–(A3) in Proposition 3.12 be satisfied and let $D \subset \subset \text{dom } \tau$ be the linear manifold (3.13). Assume also that $\tau \in \widetilde{\mathcal{R}}(C^{d-p})$ is an admissible boundary parameter, $\sigma(\cdot) = \sigma_\tau(\cdot)$ is a pseudospectral function of the system and $\eta_\tau \in \widetilde{\mathcal{L}}(C^{d-p})$ is the linear relation defined in Theorem 2.4. Then for each function $y(\cdot) \in D$ satisfying the boundary condition $\{\Gamma_0 b y(\cdot), -\Gamma_1 b y(\cdot)\} \in \eta_\tau$ the following statements hold:

(i) If $\tilde{y}(\cdot)$ is the Fourier transform of $y(\cdot)$, then $\int_{\mathbb{R}} ||\varphi_U(t,s)\Psi(s)\tilde{y}(s)|| d\nu < \infty$, $t \in I$, and the inverse transform for $y(\cdot)$ is

$$y(t) = \int_{\mathbb{R}} \varphi_U(t,s) d\sigma(s) \chi_{B_n}(s) \tilde{y}(s), \quad t \in I.$$  (4.18)

Here $\Psi(\cdot)$ and $\nu$ are the operator function and Borel measure for $\sigma(\cdot)$ defined in Theorem 2.5 (the integral in (4.18) exists as the Lebesgue integral).

(ii) The integral in (4.18) converges uniformly on each compact interval $[a, c] \subset I$, that is for each sequence $\{B_n\}_{n=1}^\infty$ of bounded Borel sets $B_n \subset \mathbb{R}$ satisfying $B_n \subset B_{n+1}$ and $\mu_\sigma(\mathbb{R} \setminus \bigcup_{n \in \mathbb{N}} B_n) = 0$ the equality (4.10) holds. This implies that

$$\lim_{n \to \infty} \sup_{t \in [a,c]} \left| y(t) - \int_{\mathbb{R}} \varphi_U(t,s) d\sigma(s) \chi_{[a,b]}(s) \tilde{y}(s) \right| = 0.$$  (4.19)
Proof. Assume that \( y(\cdot) \in D \) and \( \{ \Gamma_{0b} y(\cdot), -\Gamma_{1b} y(\cdot) \} \in \eta \). Then according to (3.9) \( \{ y(\cdot), f_y(\cdot) \} \in \mathcal{T}^* \) with some \( f_y(\cdot) \) and hence the pair \( \{ \tilde{y}, \tilde{f} \} = \tilde{\pi}(y, f_y) \) belongs to \( T^* \). Let \( \Pi = \{ C^{-p}, \Gamma_0, \Gamma_1 \} \) be the boundary triplet (3.22) for \( T^* \). Then \( \{ \Gamma_0 \{ \tilde{y}, \tilde{f} \}, -\Gamma_1 \{ \tilde{y}, \tilde{f} \} \} \in \eta \) and by Theorem 2.4 \( \{ \tilde{y}, \tilde{f} \} \in \mathcal{C}(\tilde{T}_e) \), where \( \mathcal{C}(\tilde{T}_e) \) is the compression of the exit space extension \( \tilde{T}_e = \tilde{T}_0^+ \) of \( T \) with \( \text{mul} \tilde{T}_e = \text{mul} T \). One can easily verify that

\[
\mathcal{C}(\tilde{T}_e) = \text{gr} \mathcal{C}(\tilde{T}_0) \oplus \text{mul} T
\]

(4.20)

where \( \mathcal{C}(\tilde{T}_0) = P_{\mathcal{D}_0} \tilde{T}_0 \upharpoonright \mathcal{D}_0 \cap \text{dom} \tilde{T}_0 \) is the compression of the operator part \( \tilde{T}_0 \) of \( \tilde{T} \) (see (4.9)). Since \( f_y(\cdot) \in \mathcal{L}_0 \), it follows that \( \tilde{f} \in \mathcal{D}_0 \) and by (4.20) \( \{ \tilde{y}, \tilde{f} \} \in \mathcal{C}(\tilde{T}_0) \), that is \( \tilde{y} \in \text{dom} \tilde{T}_0 \cap \mathcal{D}_0 \) and \( \tilde{f} = P_{\mathcal{D}_0} \tilde{T}_0 \tilde{y} \). Therefore by Proposition 4.3 for any \( t \in I \) and for any sequence \( \{ B_n \}_{n=1}^\infty \) of bounded Borel sets \( B_n \subset \mathbb{R} \) satisfying \( B_n \subset B_{n+1} \) there exists \( C > 0 \) such that

\[
\left\| \int_{\mathbb{R}} |\varphi_u(t, s)\Psi(s)\hat{y}(s)| \, d\nu(s) \right\| \leq C
\]

and Proposition 4.3, (iii) yields (4.18) and statement (ii).

□

Theorem 4.5. Let the assumptions be the same as in Theorem 4.4. Moreover, let at least one (and hence all) of the equivalent conditions in (3.27) be satisfied (in particular this assumption is fulfilled if \( \Delta(t) \) is invertible a.e. on \( I \)). Then \( \sigma(\cdot) = \sigma_\tau(\cdot) \) is a spectral function and statements (i) and (ii) of Theorem 4.4 hold for any function \( y(\cdot) \in \mathcal{AC}(I, \mathbb{C}^n) \cap \mathcal{L}^2(\Delta; \mathbb{C}^n) \) such that:

(a) the equality \( Jy'(t) - B(t)y(t) = \Delta(t)f_y(t) \) (a.e. on \( I \)) holds with some \( f_y(\cdot) \in \mathcal{L}^2_\Delta(I; \mathbb{C}^n) \);

(b) the boundary conditions \( U_y(a) = 0, \{ \Gamma_{0b} y(\cdot), -\Gamma_{1b} y(\cdot) \} \in \eta \) are satisfied.

Proof. It follows from Proposition 3.17, (ii) that \( \sigma(\cdot) \) is a spectral function. Next, assume that \( y(\cdot) \) satisfies the conditions of the theorem. Then \( y(\cdot) \in \text{dom} \mathcal{T}^* \) and the last condition in (3.27) yields \( y(\cdot) \in D \). Moreover, \( \{ \Gamma_{0b} y(\cdot), -\Gamma_{1b} y(\cdot) \} \in \eta \) and by Theorem 4.4 statements (i) and (ii) of this theorem hold.

□

5 | UNIFORM CONVERGENCE OF THE INVERSE FOURIER TRANSFORM FOR DIFFERENTIAL EQUATIONS

5.1 | Preliminary results

In this section we apply the above results to ordinary differential operators of an even order on an interval \( I = (a, b) \) \((-\infty < a < b \leq \infty)\) with the regular endpoint \( a \).

Assume that

\[
l[y] = \sum_{k=1}^{r} (-1)^k \left( p_{r-k}(t)y^{(k)} \right)^{(k)} + p_r(t)y
\]

(5.1)

is a symmetric differential expression of an even order \( n = 2r \) with operator valued coefficients \( p_j(\cdot) : I \to B(\mathbb{C}^m) \) satisfying \( p_0^{-1}(t) \in B(\mathbb{C}^m) \) and \( p_j(t) = p^*_j(t), \ t \in I \). Moreover, it is assumed that the operator-functions \( p_0^{-1}(t) \) and \( p_j(t), \ j \in \{1, ..., r\} \), are locally integrable.

The quasi-derivatives \( y^{[j]}(\cdot), \ j \in \{0, ..., 2r\} \), of a function \( y(\cdot) : I \to \mathbb{C}^m \) are defined as follows [23, 35]:

\[
y^{[j]} = y^{(j)}, \ j \in \{0,1,...,r-1\}, \quad y^{[r]} = p_0y^{(r)}
\]

(5.2)

\[
y^{[r+j]} = -\left( y^{[r+j-1]} \right)' + p_jy^{(r-j)}, \ j \in \{1,...,r\}.
\]

(5.3)

The quasi-derivatives \( Y^{[j]}(\cdot) \) of an operator-valued function \( Y(\cdot) : I \to B(\mathbb{C}^m, \mathbb{C}^m) \) are defined by (5.2)–(5.3) with \( Y \) instead of \( y \).

Denote by \( \text{dom} l \) the set of all functions \( y(\cdot) : I \to \mathbb{C}^m \) such that \( y^{[j]}(\cdot) \in \mathcal{AC}(I, \mathbb{C}^m) \) for all \( j \in \{0,1,...,2r-1\} \) and let \( l[y] = y^{[2r]}, \ y \in \text{dom} l \).
Next assume that \( \Delta(\cdot) : I \to B(C^m) \) is a locally integrable operator function satisfying \( \Delta(t) \geq 0 \) a.e. on \( I \). We consider the differential equation
\[
l[y] = \lambda \Delta(t)y, \quad t \in I, \quad \lambda \in \mathbb{C},
\]
and the corresponding inhomogeneous equation
\[
l[y] = \Delta(t)f(t), \quad t \in I,
\]
where \( f(\cdot) \in L^2(I; C^m) \).

A function \( y(\cdot) \in \text{dom} \ l \) is a solution of (5.4) (resp. (5.5)) a.e. on \( I \). An operator function \( Y(\cdot) : I \to B(C^\nu, C^m) \) is a solution of (5.4), if the quasi-derivatives \( Y^{[j]}(\cdot), \ j \in \{0, 1, \ldots, 2r - 1\} \), are absolutely continuous on each segment \([a, c] \subset I\) and the equality \( Y^{[2r]}(t) = \lambda \Delta(t)Y(t) \) holdsa.e. on \( I \).

**Definition 5.1.** Differential equation (5.4) is called regular if it is given on a compact interval \( I = [a, b] \) (this implies that \( \int_I \|p_0(t)\|^{-1} \|dt < \infty \), \( \int_I \|p_k(t)\| \|dt < \infty, \ k \in \{1, 2, \ldots, r\} \), and \( \int_I \|\Delta(t)\| \|dt < \infty \)).

With a function \( y(\cdot) \in \text{dom} \ l \) one associates a function \( y(\cdot) : I \to (C^m)^{2r} \), given by
\[
y(t) = y(t) \oplus y^{[1]}(t) \oplus \cdots \oplus y^{[r-1]}(t) \oplus y^{[2r-1]}(t) \oplus \cdots \oplus y^{[r]}(t).
\]
(5.6)

With an operator solution \( Y(\cdot) : I \to B(C^\nu, C^m) \) of (5.4) one associates the operator function \( Y(\cdot) : I \to B(C^\nu, (C^m)^{2r}) \) given by
\[
Y(t) = \begin{bmatrix} Y(t), \ldots, Y^{[r-1]}(t), Y^{[2r-1]}(t), \ldots, Y^{[r]}(t) \end{bmatrix}^T \in B(C^\nu, (C^m)^{2r}).
\]
(5.7)

Equation (5.4) gives rise to the maximal linear relations \( S_{\text{max}} \) in \( L^2_\Delta(I; C^n) \) and \( S_{\text{max}} \) in \( L^2_\Delta(I; C^n) \) defined as follows:
\( S_{\text{max}} \) is the set of all pairs \( \{y(\cdot), f(\cdot)\} \in (L^2_\Delta(I; C^n))^2 \) such that \( y(\cdot) \in \text{dom} \ l \) and (5.5) holds a.e. on \( I \), while \( S_{\text{max}} = \pi_\Delta S_{\text{max}} \).

It turns out that Equation (5.4) is equivalent in fact to a certain Hamiltonian system. More precisely, the following proposition is implied by the results of [23].

**Proposition 5.2.** Let \( l[y] \) be the expression (5.1) and let
\[
J = \begin{pmatrix} 0 & -I_{mr} \\ I_{mr} & 0 \end{pmatrix} : (C^m)^r \oplus (C^m)^r \to (C^m)^r \oplus (C^m)^r,
\]
(5.8)

\[\Delta(t) = \begin{pmatrix} \Delta(t) & 0 & \cdots & 0 \\ 0 & \ddots & \cdots & 0 \\ \vdots & \ddots & \ddots & \vdots \\ 0 & 0 & \cdots & 0 \end{pmatrix} : C^m \oplus C^m \oplus \cdots \oplus C^m \to C^m \oplus C^m \oplus \cdots \oplus C^m \]
(2r times)
(5.9)

where \( \Delta(t) \) is taken from (5.4). Then there exists a locally integrable operator function \( B(t) = B^r(t) \in B((C^m)^{2r}) \), \( t \in I \), (defined in terms of \( p_j \)) such that the Hamiltonian system
\[
Jy' - B(t)y = \lambda \Delta(t)y, \quad t \in I, \quad \lambda \in \mathbb{C},
\]
(5.10)

and the corresponding inhomogeneous system
\[
Jy' - B(t)y = \Delta(t)f(t), \quad t \in I,
\]
(5.11)

possesses the following properties:

(i) The relation \( Y(\cdot, \lambda) \to Y(\cdot, \lambda) \), where \( Y(\cdot, \lambda) \) is given by (5.7), gives a bijective correspondence between all \( B(C^\nu, C^m) \)-valued operator solutions \( Y(\cdot, \lambda) \) of (5.4) and all \( B(C^\nu, (C^m)^{2r}) \)-valued operator solutions \( Y(\cdot, \lambda) \) of (5.10).
(ii) Let $T_{\text{max}}$ be the maximal linear relation in $L^2_{\Delta}(I; (\mathbb{C}^m)^{2r})$ induced by system (5.10). Then the equality $U_1\{y(\cdot), f(\cdot)\} = \{y(\cdot), f(\cdot)\} \in S_{\text{max}}$, where

$$f(t) = f(t) \oplus 0 \oplus \ldots \oplus 0 \ (\in (\mathbb{C}^m)^{2r}), \quad (5.12)$$

defines a bijective linear operator $U_1$ from $S_{\text{max}}$ onto $T_{\text{max}}$.

(iii) Let $T_{\text{max}}$ be the maximal relation in $L^2_{\Delta}(I; (\mathbb{C}^m)^{2r})$ induced by system (5.10). Then the equality

$$U_2 \tilde{f} = \pi_{\Delta}f(\cdot), \quad \tilde{f} \in L^2_{\Delta}(I; C^m), \quad f(\cdot) \in \tilde{f},$$

defines a unitary operator $U_2$ from $L^2_{\Delta}(I; C^m)$ onto $L^2_{\Delta}(I; (\mathbb{C}^m)^{2r})$ such that

$$(U_2 \oplus U_2)S_{\text{max}} = T_{\text{max}}. \quad (5.13)$$

Let $T_{\text{max}}, T_{\text{max}}$ and $T_{\text{min}}, T_{\text{min}}$ be maximal and minimal relations for system (5.10) corresponding to Equation (5.4) (see Proposition 5.2). It follows from Proposition 5.2, (ii) that there exists the limit

$$[y, z]_b := \lim_{t \uparrow b} (\mathbf{y}(t), z(t)), \quad y, z \in \text{dom} S_{\text{max}}.$$ 

This fact enables one to define the linear relation $S_a$ in $L^2_{\Delta}(I; C^m)$ and the minimal linear relation $S_{\text{min}}$ in $L^2_{\Delta}(I; C^m)$ for Equation (5.4) by setting

$$S_a = \{\{y(\cdot), f(\cdot)\} \in S_{\text{max}} : y(a) = 0 \text{ and } [y, z]_b = 0 \text{ for every } z \in \text{dom} S_{\text{max}}\}$$

and $S_{\text{min}} = \overline{\pi}_{\Delta}S_a$. It follows from Proposition 5.2 that

$$(U_2 \oplus U_2)S_{\text{min}} = T_{\text{min}}, \quad (5.14)$$

where $U_2$ is a unitary operator defined in Proposition 5.2, (iii). This and (5.13) imply that $S_{\text{min}}$ is a closed symmetric linear relation in $L^2_{\Delta}(I; C^m)$ and $S^*_{\text{min}} = S_{\text{max}}$.

For $\lambda \in \mathbb{C}$ denote by $\mathcal{N}_{\Delta}$ the linear space of all solutions $y(\cdot)$ of (5.4) belonging to $L^2_{\Delta}(I; C^m)$. The numbers $N_+ = \dim \mathcal{N}_{\lambda}$ and $N_- = \dim \mathcal{N}_{-\lambda}$ will be called the formal deficiency indices of Equation (5.4). It follows from Proposition 5.2, (i) that $N_{\pm}$ are formal deficiency indices of the system (5.10). Therefore $N_+ = \dim \mathcal{N}_{\lambda}, \lambda \in \mathbb{C}_{\pm}$, and $mr \leq N_{\pm} \leq 2mr$.

### 5.2 Differential equations with matrix-valued coefficients

Similarly to Hamiltonian systems Equation (5.4) is called definite if there is only a trivial solution $y = 0$ of the equation $l[y] = 0$ satisfying $\Delta(t)y(t) = 0$ (a.e. on $I$).

Let $J$ be the operator (5.8). Below we suppose that $U \in B((\mathbb{C}^m)^{2r}, (\mathbb{C}^m)^{r})$ is an operator satisfying

$$UJU^* = 0 \quad \text{and} \quad \text{ran} U = (\mathbb{C}^m)^{r}. \quad (5.15)$$

**Definition 5.3.** Equation (5.4) will be called $U$-definite if there exists only the trivial solution $y = 0$ of the equation $l[y] = 0$ such that $Uy(a) = 0$ and $\Delta(t)y(t) = 0$ (a.e. on $I$).

It follows from Assertion 3.3 and Proposition 5.2 that the equality

$$S = \{\overline{\pi}_{\Delta}y, f\} : \{y, f\} \in S_{\text{max}}, \quad Uy(a) = 0 \quad \text{and} \quad [y, z]_b = 0, \quad z \in \text{dom} S_{\text{max}}\} \quad (5.16)$$

defines a symmetric extension $S$ of $S_{\text{min}}$ and $S^* = \overline{\pi}_{\Delta}S_s$, where $S_s$ is a linear relation in $L^2_{\Delta}(I; C^m)$ given by $S_s = \{\{y, f\} \in S_{\text{max}} : Uy(a) = 0\}$. Clearly, the domain of $S_s$ is

$$\text{dom} S_s = \{y \in \text{dom} l \cap L^2_{\Delta}(I; C^m) : l[y] = \Delta(t)f_Y(t) \quad (\text{a.e. on } I) \text{ with some } f_Y(\cdot) \in L^2_{\Delta}(I; C^m) \quad \text{and} \quad Uy(a) = 0\}. \quad (5.17)$$
In the following we put $\mathcal{S}' := L^2_\Delta(I; \mathbb{C}^m)$ and $\mathcal{S}'_0 := \mathcal{S}' \ominus \text{mul} \mathcal{S}$. We will also denote by $\mathcal{K}'$, $\mathcal{K}_0$ and $\mathcal{E}$ the linear manifolds in $L^2_\Delta(I; \mathbb{C}^m)$ defined by

$$\mathcal{K}' = \{ f(\cdot) \in L^2_\Delta(I; \mathbb{C}^m) : \text{there exists a solution } y(\cdot) \in \text{dom } I \text{ of (5.4) such that } \Delta(t)y(t) = 0 \text{ (a.e. on } I), Uy(a) = 0 \text{ and } [y, z]_0 = 0, z \in \text{dom } S_{\max} \},$$

(5.18)

$$\mathcal{K}_0 = \{ f(\cdot) \in L^2_\Delta(I; \mathbb{C}^m) : (f(\cdot), g(\cdot))_\Delta = 0 \text{ for any } g(\cdot) \in \mathcal{K}' \},$$

(5.19)

$$\mathcal{E} = \{ y(\cdot) \in \text{dom } S_\ast : f y(\cdot) \in \mathcal{K}_0 \}.$$

(5.20)

Clearly, $\text{mul} \mathcal{S} = \pi_\Delta \mathcal{K}'$ and $\mathcal{S}'_0 = \pi_\Delta \mathcal{K}_0$.

Let $\varphi_U(\cdot, \lambda) \in B((\mathbb{C}^m)^r, \mathbb{C}^m)$ be the operator solution of (5.4) such that the corresponding operator-function $\varphi_U(t, \lambda) : I \to B((\mathbb{C}^m)^r, (\mathbb{C}^m)^{2r})$ given by

$$\varphi_U(t, \lambda) = \left( \varphi_U(t, \lambda), \ldots, \varphi_U^{[r-1]}(t, \lambda), \varphi_U^{[2r-1]}(t, \lambda), \ldots, \varphi_U^{[r]}(t, \lambda) \right)^T$$

(5.21)

satisfies $\varphi_U(a, \lambda) = -J U^*$. 

**Definition 5.4.** A distribution function $\sigma(\cdot) : \mathbb{R} \to B((\mathbb{C}^m)^r)$ is called a pseudospectral function of Equation (5.4) if:

(i) for each function $f(\cdot) \in L^2_\Delta(I; \mathbb{C}^m)$ there exists a function $\hat{f}(\cdot) \in L^2(\sigma; (\mathbb{C}^m)^r)$ such that

$$\hat{f}(s) = \int_I \varphi_U^+(t, s) \Delta(t)f(t) \, dt$$

(5.22)

(the integral in (5.22) converges in $L^2(\sigma; (\mathbb{C}^m)^r)$, c.f. Definition 3.4, (i));

(ii) $\pi_\sigma \hat{f}(\cdot) = 0$, $f(\cdot) \in \mathcal{K}'$, and $|||\hat{f}(\cdot)|||_{L^2(\sigma; (\mathbb{C}^m)^r)} = |||f(\cdot)|||_{L^2_\Delta(I; \mathbb{C}^m)}$, $f(\cdot) \in \mathcal{K}_0$.

The operator-function $\hat{f}(\cdot) \in L^2(\sigma; (\mathbb{C}^m)^r)$ defined by (5.22) is called the (generalized) Fourier transform of a function $f(\cdot) \in L^2_\Delta(I; \mathbb{C}^m)$. Clearly, the function $\hat{f}(\cdot)$ is defined by $f(\cdot)$ uniquely up to the $\sigma$-equivalence.

**Definition 5.5.** A distribution function $\sigma(\cdot) : \mathbb{R} \to B((\mathbb{C}^m)^r)$ is called a spectral function of Equation (5.4) if for each function $f(\cdot) \in L^2_\Delta(I; \mathbb{C}^m)$ with compact support the Parseval equality $|||\hat{f}(\cdot)|||_{L^2(\sigma; (\mathbb{C}^m)^r)} = |||f(\cdot)|||_{L^2_\Delta(I; \mathbb{C}^m)}$ holds.

Note that Remark 3.6 and Definition 3.7 of an orthogonal pseudospectral (spectral) function remain valid, with the obvious modifications, for Equation (5.4).

By using Proposition 5.2 one can easily prove the following assertion.

**Assertion 5.6.** A distribution function $\sigma(\cdot) : \mathbb{R} \to B((\mathbb{C}^m)^r)$ is a pseudospectral (spectral) function of the system (5.10) with respect to the Fourier transform

$$\hat{f}(s) = \int_I \varphi_U^+(t, s) \overline{\Delta(t)f(t)} \, dt, \quad f(\cdot) \in L^2_\Delta(I; (\mathbb{C}^m)^{2r})$$

(5.23)

if and only if it is a pseudospectral (resp. spectral) function of Equation (5.4) with respect to the Fourier transform (5.22); moreover, $\hat{y}(s) = \hat{y}(s)$, $y(\cdot) \in \text{dom } S_{\max}$.

Applying Theorem 3.14, Proposition 3.17 and Theorems 4.4, 4.5 to system (5.10) and taking Assertion 5.6 into account we arrive at the following theorems.

**Theorem 5.7.** Assume that:

(A1') Equation (5.4) has equal formal deficiency indices $N_+ = N_- = \, \vdots \, d$;

(A2') $U \in B((\mathbb{C}^m)^{2r}, (\mathbb{C}^m)^r)$ is an operator satisfying (5.15) and Equation (5.4) is $U$-definite.
Then: (i) there exists a Nevanlinna operator function $M(\cdot)$ of the form (3.23) (with $p = mr$) such that the equalities (3.24) and (3.25) establish a bijective correspondence $\sigma(\cdot) = \sigma(\tau(\cdot))$ between all functions $\tau = \tau(\cdot) \in \tilde{R}(C^{d-mr})$ satisfying the condition (3.26) (i.e., all admissible boundary parameters) and all pseudospectral functions $\sigma(\cdot)$ of Equation (5.4). Moreover, all functions $\tau(\cdot) \in \tilde{R}(C^{d-mr})$ satisfy (3.26) if and only if $mul S = mul S^*$. 

(ii) The set of spectral functions of Equation (5.4) is not empty if and only if $\mathcal{K}^\prime \subset ker \pi \Delta$ (or, equivalently, $mul S = 0$). Moreover, in this case the sets of spectral and pseudospectral functions coincide and hence statement (i) holds for spectral functions.

**Theorem 5.8.** Let for the differential equation (5.4) the assumptions (A1') and (A2') in Theorem 5.7 and the following assumption (A3') be satisfied:

(A3') $(G_{0b}, G_{1b})^T : \text{dom } S_{\text{max}} \to C^{d-mr} \oplus C^{d-mr}$ is a surjective linear operator satisfying

$$\langle y, z \rangle_b = (G_{0b}y, G_{1b}z) - (G_{1b}y, G_{0b}z), \quad y, z \in \text{dom } S_{\text{max}}.$$ 

Assume also that $\mathcal{E} \subset \text{dom } S_s$ is linear manifold (5.20) and let $\tau(\cdot) \in \tilde{R}(C^{d-mr})$ be a relation-valued function satisfying (3.26), let $\sigma(\cdot) = \sigma(\tau(\cdot))$ be the corresponding pseudospectral function of the equation and let $\eta(\cdot) \in \tilde{C}(C^{d-mr})$ be the linear relation defined in Theorem 2.4. Then for each function $y(\cdot) \in \mathcal{E}$ satisfying the boundary condition $\{G_{0b}y(\cdot), -G_{1b}y(\cdot)\} \in \eta$, the following statements hold:

(i) If $\hat{y}(\cdot)$ is the Fourier transform (5.22) of $y(\cdot)$, then for each $t \in I$

$$y^{[k]}(t) = \int_{\mathbb{R}} \varphi_U^{[k]}(t, s) d\sigma(s)\hat{y}(s), \quad k \in \{0, 1, \ldots, 2r - 1\},$$

(5.24)

where the integral exists as the Lebesgue integral (in the same sense as the integral in (4.18)).

(ii) The integral in (5.24) converges uniformly on each compact interval $[a, c] \subset I$ in the same sense as the integral in (4.18) (see Theorem 4.4, (ii)).

If in addition $\mathcal{K}^\prime \subset ker \pi \Delta$ (or, equivalently, $mul S = 0$), then $\sigma(\cdot) = \sigma(\cdot)$ is a spectral function and statements (i) and (ii) hold for any function $y(\cdot) \in \text{dom } S_s$ satisfying the boundary condition $\{G_{0b}y(\cdot), -G_{1b}y(\cdot)\} \in \eta$.

**Remark 5.9.**

(i) In the case of the regular equation (5.4) one has $d = 2mr$. In this case for $y(\cdot) \in \text{dom } S_{\text{max}}$ one can put

$$G_{0b}y = y(b) \oplus y^{[1]}(b) \oplus \cdots \oplus y^{[r-1]}(b), \quad G_{1b}y = y^{[2r-1]}(b) \oplus y^{[2r-2]}(b) \oplus \cdots \oplus y^{[r]}(b).$$

(ii) If the weight $\Delta(t)$ is invertible a.e. on $I$, then the condition $\mathcal{K}^\prime \subset ker \pi \Delta$ in the last statement of Theorem 5.8 is obviously satisfied.

### 5.3 Scalar differential equations

In the case $m = 1$ the differential expression $l[y]$ of the form (5.1) and Equation (5.4) will be called a scalar expression and scalar equation respectively. Clearly, in this case the coefficients $p_j(\cdot)$ and the weight $\Delta(\cdot)$ are real-valued functions.

It is easy to see that for the scalar equation (5.4) the assumption (A1') in Theorem 5.7 is automatically satisfied.

**Lemma 5.10.** Let $l[y]$ be a scalar expression (5.1) on an interval $I = [a, b]$, let $B \subset I$ be a Borel set and let $y(\cdot) \in \text{dom } l$ be a function such that $y(t) = 0$ (a.e. on $B$). Then $y^{[k]}(t) = 0$ (a.e. on $B$), $k \in \{0, 1, \ldots, 2r\}$, that is there is a Borel set $B_0 \subset B$ such that $\mu(B \setminus B_0) = 0$, $y^{[2r]}(t)$ exists for each $t \in B_0$ and $y^{[k]}(t) = 0$, $t \in B_0$, $k \in \{0, 1, \ldots, 2r\}$. 
Proof. Clearly, it is sufficient to prove the lemma for the case of a compact interval \( I = [a, b] \). Moreover, we may assume without loss of generality that \( y(t) = 0, \ t \in B \).

Since \( y(\cdot) \) is absolutely continuous, there exists a Borel set \( B' \subset I \) such that \( \mu(I \setminus B') = 0 \), the derivative \( y'(t) \) exists for each \( t \in B' \) and \( y'(\cdot) \) is a Borel measurable function on \( B' \). Let \( B_1 := B' \cap B \). Then \( B_1 \subset B \), \( B_1 \subset A, \mu(B \setminus B_1) = 0 \) and \( y' \mid B_1 \) is a Borel measurable function. Hence for the set \( B'_0 := \{ t \in B_1 : y'(t) = 0 \} \) one has \( B'_0 \subset B_1 \subset B, B'_0 \subset A \) and \( y'(t) = 0, \ t \in B'_0 \). Next we show that \( \mu(B \setminus B'_0) = 0 \).

Denote by \( B_2 \) the set of all limit points of \( B_1 \) belonging to \( B_1 \). Assume that \( t \subset B_2 \). Then there exists a sequence \( \{ t_n \}_{n=1}^{\infty} \) such that \( t_n \in B_1, t_n \neq t \) and \( t_n \to t \). Moreover, \( t_n, t \in B \) and, consequently, \( y(t_n) = y(t) = 0 \). Note also that \( t \in B_1 \) and hence there exists the derivative

\[
y'(t) = \lim_{n \to \infty} \frac{y(t_n) - y(t)}{t_n - t} = 0.
\]

Thus \( B_2 \subset B'_0 \subset B_1 \) and, consequently, \( (B_1 \setminus B'_0) \subset (B_1 \setminus B_2) \). Recall that the lower Lebesgue measure \( \mu_*(B) \) of the set \( B \subset I \) is defined by

\[
\mu_*(B) = \sup \{ \mu(F) : F \subset B \text{ and } F \text{ is closed} \}
\]

and \( \mu(B) = \mu_*(B) \) for \( B \subset A \). Since \( B_1 \setminus B_2 \) is the set of all isolated points of \( B_1 \), it follows that all points of a closed set \( F \subset (B_1 \setminus B_2) \) are isolated. Since \( F \) is bounded, this implies that \( F \) is finite and hence \( \mu(F) = 0 \). Therefore \( \mu_*(B_1 \setminus B_2) = 0 \) and the relations

\[
0 \leq \mu(B_1 \setminus B'_0) = \mu_*(B_1 \setminus B'_0) \leq \mu_*(B_1 \setminus B_2) = 0
\]

show that \( \mu(B_1 \setminus B'_0) = 0 \). Moreover, \( B \setminus B'_0 = (B_1 \setminus B'_0) \cup (B \setminus B_1) \), which yields the required equality \( \mu(B \setminus B'_0) = 0 \).

Since \( y^{[1]}(t) = y'(t) \) (a.e. on \( I ) \), this implies that there is a Borel set \( B_{00} \subset B \) such that \( \mu(B \setminus B_{00}) = 0 \) and \( y^{[1]}(t) = 0, \ t \in B_{00} \). Now by using the above method one proves step by step the existence of Borel sets \( B_{0k} \subset B \) such that \( \mu(B \setminus B_{0k}) = 0 \) and \( y^{[k]}(t) = 0, \ t \in B_{0k}, k \in \{0, 1, \ldots, 2r\} \). Finally, letting \( B_0 = \bigcap_{k=0}^{2r} B_{0k} \) we obtain the set \( B_0 \) with the required properties. □

As usual we denote by \( \mu(\Delta > 0) \) the Borel measure of the set

\[
B_+ := \{ t \in I : \Delta(t) > 0 \}.
\]

Proposition 5.11. For the scalar equation (5.4) the following statements are equivalent:

(i) The weight function \( \Delta(\cdot) \) is nontrivial, that is

\[
\mu(\Delta > 0) \neq 0.
\]

(ii) Equation (5.4) is definite.

(iii) Equation (5.4) is \( U \)-definite for any operator \( U \in \mathcal{B}(\mathbb{C}^{2r}, \mathbb{C}^r) \) satisfying

\[
UJU^* = 0 \quad \text{and} \quad \text{ran} \ U = \mathbb{C}^r.
\]

(iv) There exists an operator \( U \in \mathcal{B}(\mathbb{C}^{2r}, \mathbb{C}^r) \) such that (5.26) holds and Equation (5.4) is \( U \)-definite.

Proof. (i) ⇒ (ii). Assume that a function \( y(\cdot) \in \text{dom} \ l \) satisfies \( l[y] = 0 \) and \( \Delta(t)y(t) = 0 \) (a.e. on \( I ) \). Then \( y(t) = 0, \ t \in B_+ \), and by Lemma 5.10 there is a Borel set \( B_0 \subset B_+ \) such that \( \mu(B_+ \setminus B_0) = 0 \) and \( y^{[k]}(t) = 0, \ t \in B_{0k}, k \in \{0, 1, \ldots, 2r-1\} \). Since \( \mu(B_+) > 0 \), it follows that \( B_0 \neq 0 \) and hence \( y(t) = 0, \ t \in I \). Thus Equation (5.4) is definite.

The implications (ii) ⇒ (iii) and (iii) ⇒ (iv) are obvious.

(iv)⇒(i). If \( \mu(\Delta > 0) = 0 \), then \( \Delta(t)y(t) = 0 \) (a.e. on \( I ) \) for each solution \( y(\cdot) \) of the equation \( l[y] = 0 \) satisfying \( Uy(a) = 0 \) and hence Equation (5.4) is not \( U \)-definite. This implies that \( \mu(\Delta > 0) \neq 0 \). □
Theorem 5.12. In the case of a scalar differential equation (5.4) the corresponding minimal relation \( S_{\text{min}} \) is a densely defined operator in \( S' \).

Proof. Let for scalar equation (5.4) \( B_0' := I \setminus B_+ = \{ t \in I : \Delta(t) = 0 \} \). Assume that \( y(\cdot) \in \text{dom} \ l \) and \( \Delta(t)y(t) = 0 \) (a.e. on \( I \)). Then obviously \( y(t) = 0 \) (a.e. on \( B_+ \)) and by Lemma 5.10 the following statement is valid:

(S) If \( y(\cdot) \in \text{dom} \ l \) and \( \Delta(t)y(t) = 0 \) (a.e. on \( I \)), then \( l[y] = 0 \) (a.e. on \( B_+ \)).

Let \( \mathcal{L}'' \) be the set of all functions \( f(\cdot) \in L^2_\Delta(I; \mathbb{C}) \) such that there exists a solution \( y(\cdot) \in \text{dom} \ l \) of (5.5) satisfying \( \Delta(t)y(t) = 0 \) (a.e. on \( I \)). In view of statement (S) for each \( f(\cdot) \in \mathcal{L}'' \) one has \( \Delta(t)f(t) = 0 \) (a.e. on \( I \)) and hence

\[ \pi_\Delta f(\cdot) = 0, \quad f(\cdot) \in \mathcal{L}'' \]  

(5.27)

Since obviously \( \text{mul} S_{\text{max}} = \pi_\Delta \mathcal{L}' \), it follows from (5.27) that \( \text{mul} S_{\text{max}} = \{ 0 \} \). This yields the required statement.

\[ \square \]

Theorem 5.13. Let for the scalar differential equation (5.4) the weight function \( \Delta(t) \) satisfies (5.25) and let \( U \in B(C^{2r}, C^r) \) be an operator satisfying (5.26). Then the set of spectral functions \( \sigma(\cdot)(\in B(C^r)) \) of this equation (with respect to the Fourier transform (5.22)) is not empty and there exists a Nevanlinna operator-function (3.23) (with \( p = r \)) such that the equalities (3.24) and (3.25) give a bijective correspondence \( \sigma(\cdot) = \sigma_\tau(\cdot) \) between all (arbitrary) functions \( \tau(\cdot) \in \tilde{R}(C^{d-r}) \) and \( \tilde{R}(C^{d-r}) \) and all spectral functions \( \sigma(\cdot) \) of (5.4). Moreover, a spectral function \( \sigma_\tau(\cdot) \) is orthogonal if and only if \( \tau(\lambda) \equiv \delta(= \delta^*), \lambda \in C \setminus \mathbb{R} \).

Proof. First observe that by Proposition 5.11 Equation (5.4) is \( U \)-definite and hence the assumptions \((A1')\) and \((A2')\) in Theorem 5.7 are satisfied. Next, the relation \( S \) (see (5.16)) is a symmetric extension of \( S_{\text{min}} \) and by Theorem 5.12 \( S_{\text{min}} \) is a densely defined operator. Therefore \( S \) is a densely defined operator as well and hence

\[ \text{mul} S = \text{mul} S^* = \{ 0 \}. \]  

(5.28)

Now the required statement follows from Theorem 5.7. \[ \square \]

In the following theorem we provide sufficient conditions for the uniform convergence of integrals in (5.24) with a spectral function \( \sigma(\cdot) \) of the scalar equation.

Theorem 5.14. Let for the scalar differential equation (5.4) the assumptions of Theorem 5.13 be satisfied and let the assumption \((A3')\) in Theorem 5.8 be fulfilled. Moreover, let \( \tau(\cdot) \in \tilde{R}(C^{d-r}) \), let \( \sigma(\cdot) = \sigma_\tau(\cdot) \) be the corresponding spectral function of (5.4) (see Theorem 5.13) and let \( \eta_\tau \in \tilde{R}(C^{d-r}) \) be the linear relation defined in Theorem 2.4. Denote by \( P \) the set of all functions \( y(\cdot) \in \text{dom} \ l \cap L^2_\Delta(I; \mathbb{C}) \) satisfying the equality \( l[y] = \Delta(t)f_y(t) \) (with some \( f_y(\cdot) \in L^2_\Delta(I; \mathbb{C}) \)) and the boundary conditions

\[ U y(a) = 0, \quad \{ G_{0b}y(\cdot), -G_{1b}y(\cdot) \} \in \eta_\tau. \]  

(5.29)

Then for each function \( y(\cdot) \in P \) statements (i) and (ii) of Theorem 5.8 hold.

Proof. First observe that by Proposition 5.11 Equation (5.4) is \( U \)-definite and hence the assumptions \((A1')-(A3')\) in Theorems 5.7 and 5.8 are satisfied.

Assume that \( y(\cdot) \in P \). Then \( y(\cdot) \in \text{dom} S \) (see (5.17)) and \( \{ G_{0b}y(\cdot), -G_{1b}y(\cdot) \} \in \eta_\tau \). Moreover, by (5.28) \( \text{mul} S = \{ 0 \} \). This and the last statement in Theorem 5.8 yield the required statement. \[ \square \]

Next consider the scalar regular equation (5.4) on an interval \( I = [a, b] \) (see Definition 5.1). Clearly for such equation one has \( d(= N_\perp) = 2r \).

Let \( U \in B(C^{2r}, C^r) \) be an operator satisfying (5.26). Then there exists an operator \( U' \in B(C^{2r}, C^r) \) such that the operator \( U = (U', U) \in B(C^{2r}, C^r) \) satisfies \( U^* J U = J \). Let as before \( \varphi_U(\cdot, \lambda)(\in B(C^r, C^r)) \) be an operator solution of (5.4) satisfying \( \varphi_U(a, \lambda) = -JU^* \) and let \( \psi(\cdot, \lambda) \) be similar solution with \( \psi(\cdot, \lambda) = J(U')^* \). Clearly, \( \varphi_U(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) are components of the solution \( Y(t, \lambda) = (\varphi_U(t, \lambda), \psi(t, \lambda))(\in B(C^r \oplus C^r, C^r)) \) of (5.4) satisfying \( U Y(a, \lambda) = I_{2r} \).
Below with a function \( \tau(\cdot) \in \tilde{R}(\mathbb{C}^r) \) represented in the “canonical” form (2.1) we associate a pair of operator functions \( C_j(\cdot) : \mathbb{C} \setminus \mathbb{R} \rightarrow \mathcal{B}(\mathbb{C}^r), \ j \in \{0,1\} \), given by

\[
C_0(\lambda) = \text{diag}(-\tau_0(\lambda), I_{k}), \quad C_1(\lambda) = \text{diag}(I_{H_0}, 0), \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\] (5.30)

It is easy to see that

\[
\tau(\lambda) = \{\{h, h'\} \in \mathbb{C}^r \oplus \mathbb{C}^r : C_0(\lambda)h + C_1(\lambda)h' = 0\}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R}.
\]

In the case of a regular equation (5.4) Theorem 5.13 can be reformulated in the form of the following theorem.

**Theorem 5.15.** Let for the regular scalar equation (5.4) the assumptions of Theorem 5.13 be satisfied and let \( w_j(\lambda)(\in \mathcal{B}(\mathbb{C}^r)) \) be the operator functions given by

\[
w_1(\lambda) = \left( \varphi_U(t, \lambda), \varphi_U^{[1]}(t, \lambda), \ldots, \varphi_U^{[r-1]}(t, \lambda) \right)^T,
\]

(5.31)

\[
w_2(\lambda) = \left( \psi(t, \lambda), \psi^{[1]}(t, \lambda), \ldots, \psi^{[r-1]}(t, \lambda) \right)^T,
\]

(5.32)

\[
w_3(\lambda) = \left( \varphi_U^{[2r-1]}(t, \lambda), \varphi_U^{[2r-2]}(t, \lambda), \ldots, \varphi_U^{[r]}(t, \lambda) \right)^T,
\]

(5.33)

\[
w_4(\lambda) = \left( \psi^{[2r-1]}(t, \lambda), \psi^{[2r-2]}(t, \lambda), \ldots, \psi^{[r]}(t, \lambda) \right)^T.
\]

(5.34)

Then the equality

\[
m_\tau(\lambda) = \left( C_0(\lambda)w_1(\lambda) + C_1(\lambda)w_3(\lambda) \right)^{-1}\left( C_0(\lambda)w_2(\lambda) + C_1(\lambda)w_4(\lambda) \right)
\]

(5.35)

together with (3.25) gives a bijective correspondence \( \sigma(\cdot) = \sigma_\tau(\cdot) \) between all functions \( \tau = \tau(\cdot) \in \tilde{R}(\mathbb{C}^r) \) and all spectral functions \( \sigma(\cdot) \) of (5.4) (with respect to the Fourier transform (5.22)).

**Proof.** Consider the Hamiltonian system (5.10) corresponding to Equation (5.4) (see Proposition 5.2). Let \( T \) be symmetric relation (3.7) for system (5.10) and let \( S \) be symmetric relation (5.16) for Equation (5.4). Then by (5.28) and Proposition 5.2 \( \text{mul} T = \{0\} \) and by Theorem 3.14 and Proposition 3.17 the equalities (3.24) and (3.25) give a parametrization of all spectral functions \( \sigma(\cdot) \) of (5.10) in terms of functions \( \tau(\cdot) \in \tilde{R}(\mathbb{C}^r) \).

Let \( \varphi_U(t, \lambda) = \left( \varphi_{0U}(t, \lambda), \varphi_{1U}(t, \lambda) \right)^T \) and \( \psi(t, \lambda) = \left( \psi_0(t, \lambda), \psi_1(t, \lambda) \right)^T \) be \( \mathcal{B}(\mathbb{C}^r, \mathbb{C}^r \oplus \mathbb{C}^r) \)-valued operator solutions of (5.10) with the initial values \( \varphi_U(a, \lambda) = -JU^* \) and \( \psi(a, \lambda) = J(U')^* \). Then according to [29, 33] the equality (3.24) can be written in the form (5.35) with \( w_1(\lambda) = \varphi_{0U}(b, \lambda), w_2(\lambda) = \psi_0(b, \lambda), w_3(\lambda) = \varphi_{1U}(b, \lambda) \) and \( w_4(\lambda) = \psi_1(b, \lambda) \). Moreover, by Proposition 5.2, (i) \( w_j(\lambda) \) admit the representation (5.31)–(5.34) and Assertion 5.6 yields the required statement.

5.4 | Scalar Sturm–Liouville equations

The results of this section take an especially simple form in the case \( m = 1 \) and \( r = 1 \), i.e., in the case of the scalar Sturm–Liouville equation (1.6). Below we give the proof of Theorem 1.3 concerning this equation.

**Proof.**

(i) It is clear that the operators \( U = (-\cos \alpha, -\sin \alpha) \) and \( U' = (-\sin \alpha, \cos \alpha) \) satisfy the assumptions before Theorem 5.15 and the corresponding solutions \( \varphi_u(\cdot, \lambda) = \varphi(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) of (1.6) are defined by initial values specified in the theorem. This and Theorem 5.15 give statement (i).

(ii) In view of (2.4) the linear relation \( \eta_\tau \) in \( \mathbb{C} \) is defined as follows:

1. If \( \lim_{y \to \infty} \frac{r(y)}{i} \neq 0 \), then \( \eta_\tau = \{0\} \oplus \mathbb{C} \);
2. If (1.13) holds, then \( \eta_\tau = h \oplus (-D_\tau h), \ h \in \mathbb{C}, \) with \( D_\tau = \lim_{y \to \infty} r(y) \);
3. If \( \lim_{y \to \infty} \frac{r(y)}{i} = 0 \) and \( \lim_{y \to \infty} y \Im r(iy) = \infty \), then \( \eta_\tau = \{0\} \).
Note also that according to Remark 5.9 one can put in (5.29) $G_{0y}y = y(b)$ and $G_{1b}y = y^{[1]}(b)$. Now statement (ii) follows from Theorem 5.14.

For given $\alpha, \beta \in \mathbb{R}$ consider the eigenvalue problem (1.6), (1.10) (cf. Theorem 1.2). We assume that $p, q$ and $\Delta$ in (1.6) are real-valued functions on a compact interval $I = [a, b]$ such that $\frac{1}{p}, q$ and $\Delta$ are integrable on $I$ and $\Delta(t) \geq 0$, $t \in I$ (we do not assume that $\Delta(t) > 0$, $t \in I$). A function $y \in \text{dom } l$ is called a solution of the problem (1.6), (1.10) if $l[y] = \lambda \Delta(t)y$ (a.e. on $I$) and (1.10) is satisfied. The set of all solutions of this problem will be denoted by $L_\lambda$ (it is clear that $L_\lambda$ is a finite-dimensional subspace in $L^2(I; \mathbb{C})$). Denote also by $EV$ the set of all eigenvalues of the problem (1.6), (1.10), i.e., the set of all $\lambda \in \mathbb{C}$ such that $\lambda \notin \{0\}$. For each $\lambda \in EV$ the subspace $L_\lambda \subset L^2(I; \mathbb{C})$ is called an eigenspace and a function $y \in L_\lambda$ is called an eigenfunction.

**Corollary 5.16.** Let the weight function $\Delta(\cdot)$ in (1.6) satisfies $\mu(\Delta > 0) \neq 0$. Then:

(i) $EV$ is an infinite countable subset in $\mathbb{R}$ without finite limit points and $\dim L_\lambda = 1$, $\lambda \in EV$.

(ii) If in addition $p(t) \geq 0$, $t \in I$, then the set $EV$ has properties from statement (i) and, moreover, it is bounded from below (the latter means that there exists $\lambda_0 \in EV$ such that $\lambda_0 \leq \lambda$, $\lambda \in EV$).

(iii) Let $\lambda_k$ be a sequence of all eigenvalues $\lambda_k \in EV$ and let $E_k \in L_{\lambda_k}$ be an eigenfunction with $\|E_k\|_{L^2(I; \mathbb{C})} = 1$, $k \in \mathbb{N}$.

Denote by $P'$ the set of all functions $y \in \text{dom } l$ such that $l[y] = \Delta f_y$ (a.e. on $I$) with some $f_y \in L^2(I; \mathbb{C})$ and the boundary conditions (1.10) are satisfied. Then each function $y \in P'$ admits an eigenfunction expansion (1.11), which converges absolutely and uniformly on $I$.

**Proof.** First we give the proof for the case $\sin \beta \neq 0$. In this case (1.10) is equivalent to

$$
\cos \alpha \cdot y(a) + \sin \alpha \cdot y^{[1]}(a) = 0, \quad y^{[1]}(b) = \partial y(b),
$$

where $y^{[1]}(t)$ is the same as in Theorem 1.3 and $\partial = -\cot \beta$.

(i) Let $U = (-\cos \alpha, -\sin \alpha)$, let $\varphi(\cdot, \lambda)$ and $\psi(\cdot, \lambda)$ be solutions of (1.6) from Theorem 1.3 and let $\tau \in \mathbb{R}[\mathbb{C}]$ be given by $\tau(\lambda) \equiv \bar{\theta}(= \bar{\theta})$, $\lambda \in \mathbb{C}$. Then $\varphi(\cdot, \lambda) = \varphi_U(\cdot, \lambda)$ and by Theorem 1.3, (i) the equality (1.8) with $\tau(\lambda) \equiv \theta$ defines a function $m(\cdot) = m_U(\cdot) \in \mathbb{R}[\mathbb{C}]$ such that formula (1.9) gives a spectral function $\sigma(\cdot) = \sigma_U(\cdot)$ of Equation (1.6). Since the function $m(\cdot)$ is a quotient of two entire functions, it follows that $m(\cdot)$ is a meromorphic function with the finite or countable set $P = \{\lambda_k\}_{n=1}^\infty$ of poles, which lies in $\mathbb{R}$ and has no finite limit points. Hence $\sigma(\cdot)$ is a jump function with jumps $\sigma_k > 0$ at points $\lambda_k \in P$.

Next assume that $S$ is a symmetric relation (5.16). Then by (5.28) $S$ is a densely defined operator in $L^2(I; \mathbb{C})$. Put

$$
L_* = \left\{ y \in \text{dom } l : \cos \alpha \cdot y(a) + \sin \alpha \cdot y^{[1]}(a) = 0 \right\}
$$

and $l[y] = \Delta f_y$ (a.e. on $I$) with some $f_y \in L^2(I; \mathbb{C})$.

Then the adjoint $S^*$ of $S$ is given by

$$
\text{dom } S^* = \{ \pi \Delta y : y \in L_* \}, \quad S^*(\pi \Delta y) = \pi \Delta f_y, \quad y \in L_*.
$$

It follows from Proposition 5.11 that Equation (1.6) is $U$-definite. Therefore

$$
\ker (\pi \Delta | L_*) = \{ 0 \}
$$

and combining of Proposition 5.2 with Proposition 3.12 and Remark 5.9 implies that the equalities $\Gamma_0(\pi \Delta y) = y(b)$, $\Gamma_1(\pi \Delta y) = -y^{[1]}(b)$, $y \in L_*$, define a boundary triplet $\Pi = \{ \mathbb{C}, \Gamma_0, \Gamma_1 \}$ for $S^*$. Let $\tilde{S}_r$ be a self-adjoint extension of $S$ corresponding to $\tau(\lambda) \equiv \bar{\theta}$ (in the triplet $\Pi$) and let

$$
L_r = \{ y \in L_* : y^{[1]}(b) = \partial y(b) \}.
$$
Then by Theorem 2.2, (ii) \( \tilde{S}_r \) is an operator in \( L^2_\Delta (I; \mathbb{C}) \) given by
\[
\text{dom} \tilde{S}_r = \{ \pi_\Delta y : y \in \mathcal{L}_r \}, \quad \tilde{S}_r (\pi_\Delta y) = \pi_\Delta f_y, \quad y \in \mathcal{L}_r.
\] (5.39)

In the following we denote by \( \Sigma(\tilde{S}_r) \) spectrum of \( \tilde{S}_r \).

According to [29] the Fourier transform (5.22) defines a unitary operator \( V_\sigma(\pi_\Delta y) = \hat{y}, \quad y \in \mathcal{L}_2(\mathbf{s}; \mathbb{C}) \), acting from \( L^2_\Delta (\mathbf{s}; \mathbb{C}) \) onto \( L^2(\sigma; \mathbb{C}) \); moreover,
\[
V_\sigma^* g = \pi_\Delta \left( \int_{\mathbb{R}} \varphi(\cdot, s) g(s) d\sigma(s) \right), \quad g \in L^2(\sigma; \mathbb{C}),
\] (5.40)

and the operator \( \tilde{S}_r \) is unitarily equivalent to the multiplication operator \( \Lambda_\sigma \) in \( L^2_2(\sigma; \mathbb{C}) \) by means of \( V_\sigma \). Therefore \( \Sigma(\tilde{S}_r) = P = \{ \lambda_k \}_{k=1}^n \), \( n \leq \infty \), which implies that \( \Sigma(\tilde{S}_r) \) coincides with the set of all eigenvalues \( \lambda_k \) of \( \tilde{S}_r \) and dim ker \( (\tilde{S}_r - \lambda_k) = 1 \), \( \lambda_k \in \Sigma(\tilde{S}_r) \). Also, it follows from (5.25) that dim \( L^2_\Delta (I; \mathbb{C}) = \infty \) and hence the set \( \Sigma(\tilde{S}_r) \) is infinite (that is \( n = \infty \)). Next, in view of (5.39) and (5.38) ker \( (\tilde{S}_r - \lambda) = \pi_\Delta L_\lambda, \lambda \in \mathbb{C} \), and (5.37) implies that ker \( (\pi_\Delta \mid L_\lambda) = \{ 0 \} \). Hence \( EV = \Sigma(\tilde{S}_r) \) and dim \( L_\lambda = \text{dim ker} (\tilde{S}_r - \lambda = 1, \lambda \in EV \). This proves statement (i).

Statement (ii) can be proved in the same way as Theorem 5 in [32, §19].

(iii) Let \( y \in \mathbf{F}^l \), so that (5.36) is satisfied with \( \theta = \bar{\theta} \). Let \( \tau(\cdot) \in \mathbb{R}[\mathbb{C}] \) be given by \( \tau(\lambda) \equiv \theta \). Then (1.13) is satisfied, \( D_r = \theta \) and hence \( y \) satisfies boundary conditions (bc2) in Theorem 1.3, (ii). Let \( \mathcal{V}_k(t) = \pi(\lambda_k) \tau^2 \psi(\lambda_k, t, \psi(0, \lambda_k) = 0, \psi'(0, \lambda_k) = 1 \). Hence \( \mathcal{V}_k(t) = y \) with some \( y \in \mathcal{L}_r(\mathbf{s}; \mathbb{C}) \). On the other hand \( \mathcal{V}_k \in \mathcal{L}_r \) and (5.37) implies that \( \mathcal{V}_k \in L^2_\lambda \). In the case \( \sin \beta = 0 \) one proves the required statements in the same way by setting \( \tau(\lambda) \equiv \{ 0 \} \oplus \mathbb{C}, \lambda \in \mathbb{C} \).

Remark 5.17. Statement (ii) of Corollary 5.16 was proved by other methods in [12].

5.5 | Example

Consider the scalar regular Sturm–Liouville equation
\[
-\ddot{y} = \lambda y, \quad t \in [0, 1], \quad \lambda \in \mathbb{C},
\] (5.41)
on an interval \( I = [0, 1] \). Let
\[
\varphi(t, \lambda) = \cos \left( \sqrt{\lambda} t \right), \quad \psi(t, \lambda) = \frac{1}{\sqrt{\lambda}} \sin \left( \sqrt{\lambda} t \right).
\]
The immediate checking shows that \( \varphi(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) are solutions of (5.41) with \( \varphi(0, \lambda) = 1, \varphi'(0, \lambda) = 0 \) and \( \psi(0, \lambda) = 0, \psi'(0, \lambda) = 1 \). Hence \( \varphi(\cdot, \lambda) \) and \( \psi(\cdot, \lambda) \) satisfy (1.7) with \( \alpha = -\frac{\pi}{2} \) and
\[
\varphi(1, \lambda) = \cos \sqrt{\lambda}, \quad \varphi'(1, \lambda) = -\sqrt{\lambda} \sin \sqrt{\lambda}, \quad \psi(1, \lambda) = \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}}, \quad \psi'(1, \lambda) = \cos \sqrt{\lambda}.
\]
Therefore by Theorem 1.3, (i) the equality
\[
m_{r}(\lambda) = \left( \frac{\sin \sqrt{\lambda}}{\sqrt{\lambda}} \cdot \tau(\lambda) - \cos \sqrt{\lambda} \right) \left( \cos \sqrt{\lambda} \cdot \tau(\lambda) + \sqrt{\lambda} \sin \sqrt{\lambda} \right)^{-1}
\] (5.42)
together with (1.9) describes in terms of the parameter \( \tau \in \mathbb{R}[\mathbb{C}] \) all spectral functions of Equation (5.41) with respect to the Fourier transform

\[
\hat{y}(s) = \int_{[0,1]} \cos(\sqrt{s} t) y(t) dt, \quad y(\cdot) \in L^2[0,1], \quad s \in \mathbb{R}.
\] (5.43)

Let \( \tau = \tau(\lambda) = \sqrt{\lambda} \) and let \( \sigma(\cdot) = \sigma_\tau(\cdot) \) be the corresponding spectral function of (5.41). Then by (5.42)

\[
m_\tau(\lambda) = \sin \sqrt{\lambda} - \cos \sqrt{\lambda}, \quad \lambda \in \mathbb{C} \setminus \mathbb{R},
\] (5.44)

and (1.9) implies that \( \sigma(\cdot) \in AC((−\infty,0);\mathbb{R}) \) and

\[
\sigma'(s) = \frac{1}{\pi} \Im m_\tau(s) = \frac{2}{\pi \sqrt{-s}(e^{2\sqrt{-s}} + e^{-2\sqrt{-s}})}, \quad s \in (−\infty,0).
\] (5.45)

Moreover, \( m_\tau(\cdot) \) is meromorphic on \( \mathbb{C} \setminus (−\infty,0) \) with poles \( a_k \in (0,\infty) \) given by

\[
a_k = \pi^2 \left( k - \frac{1}{4} \right)^2, \quad k \in \mathbb{N}.
\] (5.46)

Hence \( \sigma(s) \) is constant on intervals \((0,a_1)\) and \((a_k,a_{k+1})\), \( k \in \mathbb{N} \), with jumps \( \sigma_k \) in \( a_k \) given by

\[
\sigma_k = -\frac{(\sin \sqrt{s} - \cos \sqrt{s})_{s=a_k}}{(\sqrt{s}(\cos \sqrt{s} + \sin \sqrt{s}))'_{s=a_k}} = -\frac{\sin \sqrt{a_k} - \cos \sqrt{a_k}}{\frac{1}{2}(\cos \sqrt{a_k} - \sin \sqrt{a_k})} = 2.
\] (5.47)

Note also that by (5.43)

\[
\hat{y}(s) = \frac{1}{2} \int_{[0,1]} \left( e^{\sqrt{-s} t} + e^{-\sqrt{-s} t} \right) y(t) dt, \quad s \in (−\infty,0),
\] (5.48)

\[
\hat{y}(a_k) = \int_{[0,1]} \cos \left( \pi \left( k - \frac{1}{4} \right) t \right) y(t) dt, \quad k \in \mathbb{N}.
\] (5.49)

Now we are ready to prove the following assertion.

**Assertion 5.18.** Let \( y \) be a complex-valued function on \( I = [0,1] \) such that \( y' \) is absolutely continuous on \( I \), \( y'' \in L^2(I) \) and \( y'(0) = 0, \ y(1) = y'(1) = 0 \). Then the function \( y \) admits the representation

\[
y(t) = \frac{2}{\pi} \int_{(0,\infty)} \frac{e^{ut} + e^{-ut}}{e^{2u} + e^{-2u}} \tilde{y}(u) du + 2 \sum_{k=1}^{\infty} \alpha_k \cos \left( \pi \left( k - \frac{1}{4} \right) t \right),
\] (5.50)

where

\[
\alpha_k = \int_{[0,1]} y(t) \cos \left( \pi \left( k - \frac{1}{4} \right) t \right) dt, \quad k \in \mathbb{N},
\] (5.51)

\[
\tilde{y}(u) = \frac{1}{2} \int_{[0,1]} \left( e^{ut} + e^{-ut} \right) y(t) dt, \quad u \in (0,\infty).
\]

The integral and series in (5.50) converge absolutely and uniformly on \( I \).
Proof. Let a function $y(\cdot)$ satisfy the assumption of the assertion. Since $\lim_{y \to +\infty} \frac{\tau(iy)}{iy} = 0$ and $\lim_{y \to +\infty} y \cdot \Im \tau(iy) = \infty$, it follows that $y$ belongs to the set $P$ from Theorem 1.3. Moreover, the equality (1.14) takes the form

$$y(t) = \int_{(-\infty,0)} \varphi_U(t,s)\sigma'(s)\hat{y}(s)\,ds + \sum_{k=1}^{\infty} \varphi_U(t,a_k)\sigma_k\alpha_k,$$

(5.52)

where $\hat{y}(s)$ and $\alpha_k = \hat{y}(a_k)$ are given by (5.48) and (5.51), $\sigma'(s)$ is given by (5.45),

$$\varphi_U(t,s) = \cos \left( i \sqrt{-s} t \right) = \frac{1}{2} \left( e^{\sqrt{-s} t} + e^{-\sqrt{-s} t} \right), \quad s \in (-\infty,0), \quad t \in [0,1],$$

and in view of (5.47) $\sigma_k = 2$. It follows from Theorem 1.3 that

$$y(t) = \frac{1}{\pi} \int_{(-\infty,0)} \frac{e^{\sqrt{-s} t} + e^{-\sqrt{-s} t}}{\sqrt{-s} \left( e^{\sqrt{-s} t} + e^{-2\sqrt{-s} t} \right)} \hat{y}(s)\,ds + 2 \sum_{k=1}^{\infty} \alpha_k \cos \left( \pi \left( k - \frac{1}{4} \right) t \right)$$

(5.53)

where $\hat{y}(s)$ is given by (5.48), $\alpha_k$ is defined by (5.51) and the integral and series converge uniformly on $I$. Now the substitution $s = -u^2$ in (5.48) and (5.53) yields the result. □

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