A MINIMAL TRIANGULATION OF COMPLEX PROJECTIVE PLANE
ADMITTING A CHESS COLOURING OF FOUR-DIMENSIONAL SIMPLICES

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ABSTRACT. In this paper we construct and study a new 15-vertex triangulation \( X \) of the complex projective plane \( \mathbb{C}P^2 \). The automorphism group of \( X \) is isomorphic to \( S_4 \times S_3 \). We prove that the triangulation \( X \) is the minimal by the number of vertices triangulation of \( \mathbb{C}P^2 \) admitting a chess colouring of four-dimensional simplices. We provide explicit parametrizations for simplices of \( X \) and show that the automorphism group of \( X \) can be realized as a group of isometries of the Fubini–Study metric. We provide a 33-vertex subdivision \( \tilde{X} \) of the triangulation \( X \) such that the classical moment mapping \( \mu: \mathbb{C}P^2 \to \Delta^2 \) is a simplicial mapping of the triangulation \( X \) onto the barycentric subdivision of the triangle \( \Delta^2 \). We study the relationship of the triangulation \( X \) with complex crystallographic groups.

INTRODUCTION

The complex projective plane \( \mathbb{C}P^2 \) has a 9-vertex triangulation \( K \) constructed by W. Kühnel in 1980 (see [1], [2]). The simplicial complex \( K \) is a remarkable combinatorial object and has many interesting properties.

- The triangulation \( K \) is the minimal by the number of vertices triangulation of \( \mathbb{C}P^2 \). Moreover, \( K \) is minimal among all four-dimensional triangulated manifolds not homeomorphic to the sphere.
- The triangulation \( K \) is 3-neighbourly, that is, any set of 3 vertices of \( K \) spans a two-dimensional simplex.
- The automorphism group of \( K \) has order 54. Besides, there is a homeomorphism \( |K| \approx \mathbb{C}P^2 \) such that the automorphism group of \( K \) acts by isometries of the Fubini-Study metric.
- The simplicial complex \( K \) has an interesting interpretation in terms of the affine plane over the 3-element field [3].
- The triangulation \( K \) has deep relationship with theory of complex crystallographic groups. In particular, there exist a rectilinear triangulation \( \tilde{K} \) of \( \mathbb{C}^2 \) and a complex crystallographic group \( \Gamma \) acting on \( \mathbb{C}^2 \) such that \( \mathbb{C}^2/\Gamma \approx \mathbb{C}P^2 \) and \( \tilde{K}/\Gamma = K \) (see [4], [5]; see also Section 5 of the present paper).

In the present paper we construct and study a new 15-vertex triangulation \( X \) of \( \mathbb{C}P^2 \), which possesses several interesting properties. The vertices of \( X \) admit a regular colouring in 5 colours. (The colouring is said to be regular if any four-dimensional simplex contains exactly one vertex of each colour.) The four-dimensional simplices of \( X \) admit a chess colouring, that is, a colouring in two colours, black and white, such that any two simplices possessing a common facet are of different colours. Moreover, \( X \) is the minimal by the number of vertices triangulation of \( \mathbb{C}P^2 \) admitting a regular colouring of vertices.
and a chess colouring of four-simplices. (It is easy to prove that a triangulation of a simply-connected manifold admits a regular colouring of vertices if and only if it admits a chess colouring of maximal-dimensional simplices.) The study of triangulations admitting either regular colourings of vertices or chess colourings of maximal-dimensional simplices is motivated by the works of I. A. Dynnikov and S. P. Novikov [6]–[8]. In these papers such triangulations are very important for the discretization of differential geometric connections and complex analysis (for details, see section 2).

The automorphism group of $X$, which we denote by $\text{Sym}(X)$, has order 144 and is isomorphic to the direct product $S_4 \times S_3$, where $S_n$ is the permutation group of an $n$-element set. In section 3 we shall give an explicit formula for the homeomorphism $f : |X| \to \mathbb{CP}^2$ such that the group $\text{Sym}(X)$ is realized by a group of isometries of the Fubini-Study metric.

Let us divide the complex projective plane $\mathbb{CP}^2$ into three four-dimensional disks

$$B_1 = \{(z_1 : z_2 : z_3) \mid |z_1| \geq |z_2| \text{ and } |z_1| \geq |z_3|\};$$

$$B_2 = \{(z_1 : z_2 : z_3) \mid |z_2| \geq |z_1| \text{ and } |z_2| \geq |z_3|\};$$

$$B_3 = \{(z_1 : z_2 : z_3) \mid |z_3| \geq |z_1| \text{ and } |z_3| \geq |z_2|\}.$$

The intersections $\Pi_1 = B_2 \cap B_3$, $\Pi_2 = B_3 \cap B_1$, and $\Pi_3 = B_1 \cap B_2$ are solid tori. The intersection $T = B_1 \cap B_2 \cap B_3$ is the two-dimensional torus given by the equation $|z_1| = |z_2| = |z_3|$. The boundary of each disk $B_j$ is divided by the torus $T$ into two solid tori $\Pi_k$ and $\Pi_l$, where $\{j, k, l\} = \{1, 2, 3\}$. In the paper [9] T. F. Banchoff and W. Kühnel introduced the notion of an equilibrium triangulation of $\mathbb{CP}^2$. According to their definition, a triangulation of the complex projective plane is said to be equilibrium if the disks $B_j$, the solid tori $\Pi_j$, and the torus $T$ are the subcomplexes of this triangulation. T. F. Banchoff and W. Kühnel constructed a series of equilibrium triangulations of $\mathbb{CP}^2$ with interesting automorphism groups. The simplest of these triangulations has 10 vertex and the automorphism group of order 42. The triangulation $X$ is equilibrium too, though it does not belong to the series constructed by T. F. Banchoff and W. Kühnel.

For each two-dimensional simplex of $X$ the number of four-dimensional simplices containing it is either 4 or 6. It is interesting to compare this fact with a result of N. Brady, J. McCammond, and J. Meier on triangulations of three-dimensional manifolds. In [10] they proved that every closed three-dimensional manifold has a triangulation in which each edge is contained in exactly 4, 5, or 6 tetrahedra. Besides it is known that every three-dimensional manifold admitting a triangulation in which each edge is contained in not more than 5 tetrahedra can be covered by the three-dimensional sphere (see [11], [12]). Hence it is interesting to ask which four-dimensional manifolds possess triangulations such that each two-dimensional simplex is contained in exactly 4, 5, or 6 four-dimensional simplices. It is also interesting for which four-dimensional manifolds one can dispense with one of the three numbers 4, 5, and 6.

Another interesting problem closely related with the problem of constructing triangulations of manifolds is the problem of constructing triangulations of mappings of manifolds. Under a triangulation of a mapping $g : M \to N$ we mean a pair of triangulations of the manifolds $M$ and $N$ with respect to which the mapping $g$ is simplicial. (Recall that a mapping of simplicial complexes is called simplicial if it takes vertices of the first complex to vertices of the second complex and maps linearly every simplex of the first complex onto some simplex of the second complex.) For example, K. V. Madahar and K. S. Sarkaria [13] constructed a minimal by the number of vertices triangulation of the Hopf mapping $S^3 \to S^2$. We shall consider the classical moment mapping $\mu : \mathbb{CP}^2 \to \Delta^2$
given by

$$
\mu(z_1 : z_2 : z_3) = \frac{(|z_1|^2, |z_2|^2, |z_3|^2)}{|z_1|^2 + |z_2|^2 + |z_3|^2},
$$

(1)

where $\Delta^2 \subset \mathbb{R}^3$ is the equilateral triangle with vertices $(1,0,0)$, $(0,1,0)$, and $(0,0,1)$. In section 4 we shall use the triangulation $X$ to construct a triangulation of the moment mapping $\mu$. For a triangulation of the complex projective plane we shall take a 33-vertex $\text{Sym}(X)$-invariant subdivision $\overline{X}$ of $X$, for a triangulation of the triangle $\Delta^2$ we shall take its barycentric subdivision.

Like Kühnel’s triangulation $K$, the triangulation $X$ is related with complex crystallographic groups. This relationship is obtained in section 5.

1. Simplicial complex $X$

An abstract simplicial complex on a vertex set $V$ is a set $K$ of finite subsets of $V$ such that $\emptyset \in K$ and for any subsets $\tau \subset \sigma \subset V$ if $\sigma$ belongs to $K$ and $\tau \subset \sigma$, then $\tau$ belongs to $K$ too. The geometric realization of a simplicial complex $K$ is denoted by $|K|$. A full subcomplex of $K$ spanned by a vertex set $W \subset V$ is a simplicial complex consisting of all simplices $\sigma \in K$ such that $\sigma \subset W$. The link of a simplex $\sigma \in K$ is the subcomplex of $K$ consisting of all simplices $\tau$ such that $\tau \cap \sigma = \emptyset$ and $\tau \cup \sigma \in K$. A simplicial complex is said to be an $n$-dimensional combinatorial sphere if its geometric realization is piecewise linearly homeomorphic to the boundary of an $(n + 1)$-dimensional simplex. A simplicial complex is said to be an $n$-dimensional combinatorial manifold if the links of all its vertices are $(n - 1)$-dimensional combinatorial spheres. A simplicial mapping of a simplicial complex $K_1$ on vertex set $V_1$ to a simplicial complex $K_2$ on vertex set $V_2$ is a mapping $f : V_1 \to V_2$ such that $f(\sigma) \in K_2$ whenever $\sigma \in K_1$. An isomorphism of simplicial complexes is a simplicial mapping with a simplicial inverse.

Construction 1.1. We construct a 15-vertex abstract simplicial complex $X$ in the following way. For the vertex set of $X$ we take the 15-element set

$$
V = (V_4 \setminus \{e\}) \cup \{(1,2,3,4) \times \{1,2,3\}\},
$$

where $V_4 \subset S_4$ is the Klein four group. Thus the vertices of $X$ are the permutations $(12)(34)$, $(13)(24)$, and $(14)(23)$ and the pairs of integers $(a,b)$ such that $1 \leq a \leq 4$ and $1 \leq b \leq 3$. The four-dimensional simplices of $X$ are spanned by the sets

$$
\nu, (1,b_1), (2,b_2), (3,b_3), (4,b_4), \quad \nu \in V_4 \setminus \{e\}, 1 \leq b_a \leq 3, a = 1, 2, 3, 4,
$$

such that $b_\nu(a) \neq b_a$ for $a = 1, 2, 3, 4$. The simplices of dimensions less than 4 are spanned by all subsets of the above sets.

The vertices of $X$ admit a regular colouring in 5 colours. To obtain such colouring one should paint the vertices $\nu \in V_4 \setminus \{e\}$ in colour 0 and the vertices $(a,b)$ in colour $a$, $a = 1, 2, 3, 4$. It is easy to compute that the $f$-vector of the triangulation $X$ is equal to $(15, 90, 240, 270, 108)$. Later we shall prove that $X$ is a combinatorial manifold. In sections 3 and 5 we shall prove in two different ways that the geometric realization $|X|$ is piecewise linearly homeomorphic to $\mathbb{C}P^2$.

Now we compute the automorphism group of $X$. We define actions of the groups $S_4$ and $S_3$ on the set $V$ by

$$
\theta \cdot \nu = \theta \nu \theta^{-1}, \quad \theta \cdot (a,b) = (\theta(a), b), \quad \theta \in S_4;
$$

$$
\kappa \cdot \nu = \nu, \quad \kappa \cdot (a,b) = (a, \kappa(b)), \quad \kappa \in S_3.
$$
Figure 1. The link of the vertex \( (12)(34) \)

Obviously, these actions take simplices of \( X \) to simplices of \( X \). Besides, these actions commute. Hence the group \( S_4 \times S_3 \) acts on \( X \) by automorphisms. Obviously, the kernel of this action is trivial. Therefore the automorphism group \( \text{Sym}(X) \) of \( X \) contains the subgroup isomorphic to \( S_4 \times S_3 \). Indeed, this subgroup coincides with the whole group \( \text{Sym}(X) \).

**Proposition 1.2.** The automorphism group \( \text{Sym}(X) \) of the simplicial complex \( X \) is isomorphic to \( S_4 \times S_3 \).

The vertices of \( X \) are divided into two \( \text{Sym}(X) \)-orbits. The first orbit consists of the 3 vertices \( (12)(34), (13)(24), \) and \( (14)(23) \). The second orbit consists of 12 vertices \( (a,b) \).

**Proposition 1.3.** The simplicial complex \( X \) is a four-dimensional combinatorial manifold.

**Proof.** We need to prove that the links of all vertices of \( X \) are three-dimensional combinatorial spheres. Obviously, the links of all vertices in the same \( \text{Sym}(X) \)-orbit are pairwise isomorphic. Thus we suffice to prove that the links of the vertices \( (12)(34) \) and \( (1,1) \) are three-dimensional combinatorial spheres.

The link of the vertex \( (12)(34) \) consists of the simplices

\[(1, b_1), (2, b_2), (3, b_3), (4, b_4)\]

such that \( b_1 \neq b_2 \) \( b_3 \neq b_4 \). Hence it is isomorphic to the join of two closed 6-vertex circles shown in Fig. 1.

The link of the vertex \( (1,1) \) is more complicated. We denote it by \( \mathcal{N} \). First, let us consider the full subcomplex \( J \) of \( \mathcal{N} \) spanned by the set of all vertices of \( \mathcal{N} \) except for the vertices \( (2,1), (2,2), \) and \( (2,3) \). This complex can be realized in a three-dimensional sphere as it is shown in Fig. 2. (Here the three-dimensional sphere is identified with the one-point compactification of \( \mathbb{R}^3 \).) We imply that the vertex \( (12)(34) \) is placed at the infinitely remote point, the vertices \( (3,2) \) and \( (4,2) \) are somewhat “lifted” above the plane of the figure and the vertices \( (3,3) \) and \( (4,3) \) are somewhat “lowered” below the plane of the figure. The complex \( J \) is the union of three two-dimensional disks. The first disk consists of the triangles

\[
\begin{align*}
(12)(34), (3,1), (4,2); \\
(12)(34), (3,1), (4,3); \\
(12)(34), (3,2), (4,3); \\
(12)(34), (3,2), (4,1); \\
(12)(34), (3,3), (4,1);
\end{align*}
\]

\[
\begin{align*}
(12)(34), (3,3), (4,2); \\
(13)(24), (3,2), (4,1); \\
(13)(24), (3,3), (4,1); \\
(14)(23), (3,1), (4,2); \\
(14)(23), (3,1), (4,3).
\end{align*}
\]
The second disk consists of the triangles

\[(13)(24), (3, 3), (4, 2); \quad (14)(23), (3, 2), (4, 2);\]
\[(13)(24), (3, 2), (4, 2); \quad (14)(23), (3, 2), (4, 3).\]

The third disk consists of the triangles

\[(13)(24), (3, 2), (4, 3); \quad (14)(23), (3, 3), (4, 3);\]
\[(13)(24), (3, 3), (4, 3); \quad (14)(23), (3, 3), (4, 2).\]

The complement of $J$ in the three-dimensional sphere is the union of three open three-dimensional disks. The first disk $D_1$ is the interior of the octahedron with vertices $(13)(24)$, $(14)(23)$, $(3, 2)$, $(3, 3)$, $(4, 2)$, $(4, 3)$, the second disk $D_2$ is situated “below the plane of the figure”, and the third disk $D_3$ is situated “above the plane of the figure”. We place the vertices $(2, 1)$, $(2, 2)$, and $(2, 3)$ in the interiors of the disks $D_1$, $D_2$, and $D_3$ respectively and triangulate the disks $D_1$, $D_2$, and $D_3$ as cones over their boundaries with vertices $(2, 1)$, $(2, 2)$, and $(2, 3)$ respectively. The obtained triangulation of the three-dimensional sphere is isomorphic to the simplicial complex $\mathcal{N}$. □

Let us now describe the links of edges and two-dimensional simplices of $X$. The edges of $X$ are divided into three $\text{Sym}(X)$-orbits whose representatives are the edge with vertices $(1, 1)$ and $(2, 1)$, the edge with vertices $(12)(34)$ and $(1, 1)$, and the edge with vertices $(1, 1)$ and $(2, 2)$ respectively. The first orbit consists of 18 edges, either of the second and the third orbits consists of 36 edges. The link of an edge in the first orbit is isomorphic to the boundary of an octahedron. The link of an edge in the second orbit is isomorphic to the suspension over the boundary of a hexagon. The link of an edge in the third orbit is isomorphic to the triangulation of a two-dimensional sphere that can be obtained by gluing together two examples of the triangulation shown in Fig. 3 along their boundaries.

The link of a codimension 2 simplex of a combinatorial manifold is always isomorphic to the boundary of a polygon and is completely characterized by the number of vertices of this polygon, which is equal to the number of maximal-dimensional simplices containing this codimension 2 simplex. The two-dimensional simplices of $X$ are divided
to the boundary of a quadrangle. Thus every two-dimensional simplex of
the first orbit and in the fifth orbit are isomorphic to the boundary of a hexagon, the links
either in 4 or in 6 four-dimensional simplices.

\[ \text{Let us now describe a 33-vertex Sym}(X)\)-invariant subdivision \( \overline{X} \) of \( X \), which will be
needed in section 4 to construct a triangulation of the moment mapping. As it has been
mentioned before the set of edges of \( X \) is divided into three Sym\((X)\)-orbits. Consider
the orbit consisting of 18 edges \((a_1, b), (a_2, b), a_1 \neq a_2\). At the midpoint of every such
dge we introduce a new vertex denoted by \((\widehat{a}_1\widehat{a}_2, b)\). (Notations \((\widehat{a}_1\widehat{a}_2, b)\)
and \((\widehat{a}_2\widehat{a}_1, b)\) correspond to the same vertex.) The set of four-dimensional simplices of \( X \) is divided
into two Sym\((X)\)-orbits. The first orbit consists of 72 simplices
\[ (a_1a_2)(a_3a_4), (a_1, b_1), (a_2, b_2), (a_3, b_1), (a_4, b_3), \] (2)
where the numbers \( a_1, a_2, a_3, \) and \( a_4 \) are pairwise distinct and the numbers \( b_1, b_2, \) and
\( b_3 \) are pairwise distinct. The second orbit consists of 36 simplices
\[ (a_1a_2)(a_3a_4), (a_1, b_1), (a_2, b_2), (a_3, b_1), (a_4, b_2), \] (3)
where the numbers \( a_1, a_2, a_3, \) and \( a_4 \) are pairwise distinct and \( b_1 \neq b_2 \). We divide every
four-dimensional simplex (2) into 2 four-dimensional simplices
\[ (a_1a_2)(a_3a_4), (\widehat{a}_1a_3, b_1), (a_1, b_1), (a_2, b_2), (a_4, b_3), \]
\[ (a_1a_2)(a_3a_4), (\widehat{a}_1a_3, b_1), (a_1, b_1), (a_2, b_2), (a_4, b_2), \] (4)
and divide every four-dimensional simplex (3) into 4 four-dimensional simplices
\[ (a_1a_2)(a_3a_4), (\widehat{a}_1a_3, b_1), (\widehat{a}_2a_4, b_2)(a_1, b_1), (a_2, b_2), \]
\[ (a_1a_2)(a_3a_4), (\widehat{a}_1a_3, b_1), (\widehat{a}_2a_4, b_2)(a_1, b_1), (a_4, b_2), \]
\[ (a_1a_2)(a_3a_4), (\widehat{a}_1a_3, b_1), (\widehat{a}_2a_4, b_2)(a_3, b_1), (a_2, b_2), \]
\[ (a_1a_2)(a_3a_4), (\widehat{a}_1a_3, b_1), (\widehat{a}_2a_4, b_2)(a_3, b_1), (a_4, b_2). \] (5)

As a result we obtain a 33-vertex Sym\((X)\)-invariant subdivision \( \overline{X} \) of \( X \) with 288 four-
dimensional simplices. It can be easily checked that the four-dimensional simplices of \( X \)
are divided into two Sym\((X)\)-orbits. The first orbit consists of 144 simplices (4) and the
second orbit consists of 144 simplices (5).
Let $M$ be a closed $n$-dimensional manifold.

**Definition 2.1.** A *triangulation* of a manifold $M$ is a simplicial complex $K$ together with a homeomorphism $\varphi : |K| \to M$. A triangulation is said to be *combinatorial* or *piecewise linear* if $K$ is a combinatorial manifold.

In this paper we shall deal only with manifolds of dimension not greater than 4. It is well known that in this case every triangulation is combinatorial. We shall often not care about a concrete homeomorphism $\varphi$ and say that a simplicial complex $K$ is a triangulation of $M$ if $|K| \approx M$.

By $T(M)$ we denote the set of all triangulations of $M$ up to an isomorphism. The set $T(M)$ contains the following interesting subsets.

1. $T_{\text{even}}(M) \subset T(M)$ is the subset consisting of all triangulations $K$ such that every $(n-2)$-dimensional simplex of $K$ is contained in even number of $n$-dimensional simplices.
2. $T_{\text{bw}}(M) \subset T(M)$ is the subset consisting of all triangulations admitting a chess colouring of $n$-dimensional simplices, that is, a colouring in black and white colours such that any two simplices possessing a common facet have distinct colours.
3. $T_{\text{colour}}(M) \subset T(M)$ is the subset consisting of all triangulations $K$ admitting a regular colouring of vertices, that is, a colouring of vertices in colours of some $(n+1)$-element set such that every $n$-dimensional simplex of $K$ contains exactly one vertex of each colour. (Usually we denote the colours by the numbers $0, 1, \ldots, n$.)

The classes of triangulations $T_{\text{even}}(M), T_{\text{bw}}(M),$ and $T_{\text{colour}}(M)$ are very important in theory of discretization of differential geometric connections and complex analysis developed in works of I. A. Dynnikov and S. P. Novikov [6]–[8]. In these papers an *operator of discrete connection* on a triangulation $K$ of a manifold $M$ is defined to be an operator that takes a function on the vertex set of $K$ to a function on the set of $n$-dimensional simplices of $K$ by

$$ (Q\psi)_{\sigma} = \sum_{v \in \sigma} b_{\sigma,v} \psi_v, \quad (6) $$

where $b_{\sigma,v}$ is a fixed set of coefficients. A discrete connection is called *canonical* if all coefficients $b_{\sigma,v}$ are equal to 1. We shall always deal with canonical discrete connections only. A *fat path* in a triangulation $K$ is a sequence of $n$-dimensional simplices such that any two consecutive simplices have a common facet. Solving equation (6) along the fat path corresponding to the circuit around an $(n-2)$-dimensional simplex $\tau$, we obtain the *local holonomy* or the *curvature* of a discrete connection at the simplex $\tau$. For the canonical discrete connection the local holonomy at simplex $\tau$ is trivial if and only if the simplex $\tau$ is contained in even number of $n$-dimensional simplices. Hence $T_{\text{even}}(M)$ is exactly the class of all triangulations whose canonical discrete connections have zero curvature. If the local holonomy is trivial, then the *global holonomy* homomorphism $\rho : \pi_1(M) \to S_{n+1}$ is well defined. It is easy to see that $T_{\text{colour}}(M) \subset T_{\text{even}}(M)$ is exactly the subclass consisting of all triangulations whose canonical discrete connections have trivial global holonomy.

Following [8] we denote by $\rho_1$ the composite homomorphism

$$ \pi_1 \xrightarrow{\rho} S_{n+1} \xrightarrow{\text{sgn}} \mathbb{Z}_2. $$

Two other homomorphisms considered in [8] are the orientation homomorphism $\rho_2 : \pi_1(M) \to \mathbb{Z}_2$ and the homomorphism $\rho_3 : \pi_1(M) \to \mathbb{Z}_2$ that takes each homotopy class to the parity of the number of $n$-dimensional simplices in a fat path.
representing this homotopy class. (The homomorphism $\rho_3$ is well defined if the triangulation belongs to $T_{even}(M)$.) Obviously, the subclass $T_{bw}(M) \subset T_{even}(M)$ consists of all triangulations for which the homomorphism $\rho_3$ is trivial. As it was mentioned in [8], the homomorphisms $\rho_1, \rho_2, \rho_3$ satisfy the relation $\rho_1 \rho_2 = \rho_3$, where we use the multiplicative notation for the group $\mathbb{Z}_2$. It easily follows from this relation that $T_{colour}(M) \subset T_{bw}(M)$ if the manifold $M$ is orientable and $T_{colour}(M) \cap T_{bw}(M) = \emptyset$ if the manifold $M$ is non-orientable.

Notice that the holonomy group of the canonical discrete connection coincides with the *projectivity group* of the triangulation, which was introduced by M. Joswig [15], [16]. Hence in M. Joswig’s terminology the class $T_{colour}(M)$ is exactly the class of all triangulations with trivial projectivity groups.

It follows from the above reasoning that $T_{colour}(M) = T_{bw}(M) = T_{even}(M)$ if $M$ is simply-connected. Moreover, $T_{bw}(M) = T_{even}(M)$ if $\pi_1(M)$ has no non-trivial homomorphisms to $\mathbb{Z}_2$ and $T_{colour}(M) = T_{even}(M)$ if $\pi_1(M)$ has no non-trivial homomorphisms to $S^{n+1}$.

Triangulations with regular colourings of vertices appear as well in many other problems. For example, in the paper [14] by M. W. Davis and T. Januszkiewicz such triangulations lead to an important class of manifolds called *small coverings induced from a linear model over simple polytopes* and in the author’s paper [17] such triangulations are important for a construction of combinatorial realization of cycles.

Two-dimensional triangulations with chess colourings of triangles are very important for the discretization of complex analysis [8].

A *minimal* triangulation of a manifold $M$ is a triangulation of $M$ with the smallest possible number of vertices. The problem of finding minimal triangulations of manifolds is a very interesting and hard problem. It is solved only for a few manifolds. A good survey of results on minimal triangulations is the paper [18] by F. Lutz. For the complex projective plane a minimal triangulation has 9 vertex and is unique up to an isomorphism. It was constructed by W. Kühnel in 1980 (see [1], [2]). In the present paper we are interested in the problems of finding minimal triangulations of manifolds in the classes $T_{even}$, $T_{bw}$, and $T_{colour}$. For simply-connected manifolds these three problems coincide.

**Example 2.2.** Let $e_0, e_1, \ldots, e_n$ be the standard basis in $\mathbb{R}^{n+1}$. The convex hull of the points $\pm e_0, \pm e_1, \ldots, \pm e_n$ is called a *cross-polytope*. It is the regular polytope dual to the $(n+1)$-dimensional cube. The boundary of the cross-polytope is a triangulation of the $n$-dimensional sphere. This triangulation admits a regular colouring of vertices. To obtain such colouring one should paint the vertices $\pm e_j$ in colour $j$. Obviously, this triangulation is minimal in the class $T_{even}(S^n) = T_{bw}(S^n) = T_{colour}(S^n)$. Indeed, if a triangulation of an $n$-dimensional manifold admits a regular colouring of vertices, then this triangulation should contain at least 2 vertices of each colour and, hence, at least $2n + 2$ vertices.

We have the following simple proposition.

**Proposition 2.3.** If a combinatorial triangulation of an $n$-dimensional manifold admits a regular colouring of vertices in $n + 1$ colours and contains less than $3n + 3$ vertices, then this manifold is piecewise linearly homeomorphic to an $n$-dimensional sphere.

**Proof.** If there is a colour with only two vertices $v_1$ and $v_2$ coloured by it, then the triangulation is the suspension with vertices $v_1$ and $v_2$ over the full subcomplex spanned by all other vertices of the triangulation. A combinatorial manifold that is a suspension is piecewise linearly homeomorphic to a sphere. \qed
Example 2.4. It is well known that a minimal triangulation of the two-dimensional torus is unique up to an isomorphism and has 7 vertices. The most symmetric realization of this triangulation is shown in Fig. 4a. (We imply the identification of the opposite sides of the hexagon.) One can easily check that this triangulation belongs to the classes $T_{\text{even}}(T^2)$ and $T_{\text{bw}}(T^2)$ and does not belong to the class $T_{\text{colour}}(T^2)$. Proposition 2.3 implies that a minimal triangulation in the class $T_{\text{colour}}(T^2)$ has at least 9 vertices. Actually such triangulation has 9 vertices and is unique up to an isomorphism. This triangulation is shown in Fig. 4b.

Example 2.5. A minimal triangulation of the real projective plane $\mathbb{RP}^2$ is unique, has 6 vertices, and can be obtained from the boundary of a regular icosahedron by identifying every pair of antipodal points. This triangulation belongs to none of the classes $T_{\text{even}}(\mathbb{RP}^2)$, $T_{\text{bw}}(\mathbb{RP}^2)$, and $T_{\text{colour}}(\mathbb{RP}^2)$. A minimal triangulation in the classes $T_{\text{even}}(\mathbb{RP}^2)$ and $T_{\text{bw}}(\mathbb{RP}^2)$ is unique and has 7 vertices (see Fig. 5a). A minimal triangulation in the class $T_{\text{colour}}(\mathbb{RP}^2)$ is also unique and has 9 vertices (see Fig. 5b).

Example 2.6. The complex projective plane is simply-connected. Therefore, $T_{\text{colour}}(\mathbb{CP}^2) = T_{\text{bw}}(\mathbb{CP}^2) = T_{\text{even}}(\mathbb{CP}^2)$. By proposition 2.3 a minimal triangulation in the class $T_{\text{colour}}(\mathbb{CP}^2)$ has at least 15 vertices. On the other hand, the combinatorial
manifold $X$ constructed in section 1 has 15 vertices and admits a regular colouring of vertices in 5 colours. In section 3 we shall prove that $X$ is a piecewise linear triangulation of $\mathbb{C}P^2$. Hence $X$ is a minimal triangulation in the class $\mathcal{T}_{\text{colour}}(\mathbb{C}P^2)$. The author does not know whether a minimal triangulation in this class is unique.

3. Explicit realization of $X$ as a triangulation of $\mathbb{C}P^2$

First, let us construct a representation of the group $\text{Sym}(X) = S_4 \times S_3$ to the isometry group of the Fubini–Study metric on $\mathbb{C}P^2$.

Let $c : \mathbb{C}^4 \to \mathbb{C}^3$ be the operator of coordinatewise complex conjugation, $c(z_1, z_2, z_3) = (\bar{z}_1, \bar{z}_2, \bar{z}_3)$. By $\hat{U}(3)$ we denote the group of $\mathbb{R}$-linear automorphisms of $\mathbb{C}^3$ generated by the unitary group $U(3)$ and the operator $c$. Any element of $\hat{U}(3)$ is either $\mathbb{C}$-linear or $\mathbb{C}$-antilinear. The unitary group $U(3) \subset \hat{U}(3)$ is a subgroup of index 2 and coincides with the intersection $\hat{U}(3) \cap GL(3, \mathbb{C})$. By $D$ we denote the group of diagonal unitary matrices, $D = \{\lambda E | |\lambda| = 1\}$. Then $D \subset \hat{U}(3)$ is a normal subgroup. (Notice that the subgroup $D$ is not central.) We put $P\hat{U}(3) = \hat{U}(3)/D$. By $[g]$ we denote the image of an element $g \in \hat{U}(3)$ in $P\hat{U}(3)$. The projective unitary group $PU(3)$ is a subgroup of index 2 of $P\hat{U}(3)$. The group $PU(3)$ acts on $\mathbb{C}P^2$ by isometries of the Fubini–Study metric. The operator $c$ also induces the isometry $(z_1 : z_2 : z_3) \mapsto (\bar{z}_1 : \bar{z}_2 : \bar{z}_3)$. Therefore the group $P\hat{U}(3)$ is the subgroup of the isometry group of the Fubini–Study metric. Indeed, $P\hat{U}(3)$ coincides with the group of all isometries of $\mathbb{C}P^2$.

Now we construct a projective representation $R : S_4 \times S_3 \to P\hat{U}(3)$ in the following way. We start with the standard representation $\rho : S_4 \to O(3, \mathbb{R}) \subset U(3)$ that realizes the group $S_4$ as the symmetry group of a regular tetrahedron $T \subset \mathbb{R}^3$ with vertices numbered by 1, 2, 3, 4. It is convenient to us to place the tetrahedron $T$ in $\mathbb{R}^3$ so that the vectors $e_1, e_2, e_3$ of the standard orthonormal basis of $\mathbb{R}^3$ coincide with the vectors from the center of the tetrahedron $T$ to the midpoints of the edges 14, 24, and 34 respectively. Then the representation $\rho$ is given on the standard generators of $S_4$ by

$$\rho((12)) = \begin{pmatrix} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}; \quad \rho((23)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}; \quad \rho((34)) = \begin{pmatrix} 0 & -1 & 0 \\ -1 & 0 & 0 \\ 0 & 0 & 1 \end{pmatrix}. $$

Now we define a representation $\hat{\rho} : S_4 \to \hat{U}(3)$ by putting $\hat{\rho}(\theta) = \rho(\theta)$ if $\theta$ is an even permutation and $\hat{\rho}(\theta) = \rho(\theta)c$ if $\theta$ is an odd permutation. The representation $\hat{\rho}$ is well defined, since real matrices commute with $c$ and $c$ has order 2.

Now we define a representation $\eta : S_3 \to \hat{U}(3)$ on generators by

$$\eta((12)) = c; \quad \eta((123)) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \omega & 0 \\ 0 & 0 & \omega^2 \end{pmatrix},$$

where $\omega = e^{\frac{2\pi i}{3}}$ is the cubic root of unity. It can be immediately checked that the representation $\eta$ is well defined and the following proposition holds.

**Proposition 3.1.** If $\theta \in S_4$ and $\tau \in S_3$, then the commutator

$$(\hat{\rho}(\theta), \eta(\tau)) = \hat{\rho}(\theta)\eta(\tau)\hat{\rho}(\theta)^{-1}\eta(\tau)^{-1}$$

lies in $D$.

This proposition implies that the formula

$$R((\theta, \tau)) = [\hat{\rho}(\theta)\eta(\tau)]$$
yields a well-defined projective representation \( R : S_4 \times S_3 \to \mathbb{P}\mathbb{U}(3) \). In the sequel we always regard the group \( \mathbb{P}\mathbb{U}(3) \) as an isometry group of the Fubini–Study metric and identify the group \( S_4 \times S_3 \) with a subgroup of \( \mathbb{P}\mathbb{U}(3) \) by the representation \( \eta \).

Recall that \( \mathbb{C}\mathbb{P}^2 \) can be decomposed into three four-dimensional disks \( B_1, B_2, \) and \( B_3 \) whose pairwise intersections are solid tori \( \Pi_1, \Pi_2, \) and \( \Pi_3 \) and whose triple intersection is a two-dimensional torus \( T \). A triangulation of \( \mathbb{C}\mathbb{P}^2 \) is said to be \textit{equilibrium}, if the disks \( B_j \), the solid tori \( \Pi_j \), and the torus \( T \) are subcomplexes of this triangulation. T. F. Banchoff and W. Kühnel constructed a series of equilibrium triangulations of \( \mathbb{C}\mathbb{P}^2 \) depending on two coprime natural numbers \( p \) and \( q \). The triangulation corresponding to a pair \( (p, q) \) has \( p^2 + pq + q^2 + 3 \) vertices. It is based on a \( (p^2 + pq + q^2) \)-vertex triangulation of the two-dimensional torus \( T \). This triangulation of the torus \( T \) is the \textit{regular map} \( \{3, 6\}_{p, q} \) on the torus (see [19]). The triangulation \( \{3, 6\}_{p, q} \) can be obtained in the following way. Let us consider the standard triangulation of plane by rectilinear triangles with vertices in the points of the lattice generated by two unit vectors \( e_1 \) and \( e_2 \) with angle \( \frac{2\pi}{3} \) between them. We factorize this triangulation by the sublattice generated by the vectors with coordinates \((p - q, 2p + q)\) and \((2p + q, p + 2q)\) in the basis \((e_1, e_2)\). If \( p + q \geq 3 \), then we obtain a well-defined \((p^2 + pq + q^2)\)-vertex triangulation of the two-dimensional torus. For coprime \( p \) and \( q \), T. F. Banchoff and W. Kühnel found a way to extend the triangulation \( \{3, 6\}_{p, q} \) to a triangulation of a solid torus without adding new vertices. Since the triangulation \( \{3, 6\}_{p, q} \) is invariant under rotation by \( \frac{2\pi}{3} \), such extension automatically yields three different extensions, which are taken to each other by rotations. Realize these three extensions on the solid tori \( \Pi_1, \Pi_2, \) and \( \Pi_3 \). Now we introduce three new vertices in the interiors of the disks \( B_1, B_2, \) and \( B_3 \) and triangulate every of these disks as a cone over the constructed triangulation of its boundary. The simplest of the triangulations obtained corresponds to the pair \((2, 1)\) and has 10 vertices. The corresponding triangulation \( \{3, 6\}_{2, 1} \) of the torus is the minimal triangulation of the torus (see Fig. [19]).

Let us now realize the combinatorial manifold \( X \) as an equilibrium triangulation of \( \mathbb{C}\mathbb{P}^2 \). Our construction is based on the 12-vertex triangulation \( \{3, 6\}_{2, 2} \) of the two-dimensional torus. We shall follow Banchoff–Kühnel’s method everywhere except for the way of constructing triangulations of solid tori \( \Pi_j \) because Banchoff–Kühnel’s method does not work if the numbers \( p \) and \( q \) are not coprime.

Let us consider the subcomplex \( T \subset X \) consisting of all simplices \( \sigma \) such that \( \sigma \cup \{\nu\} \) is a simplex of \( X \) for any vertex \( \nu \in V_4 \setminus \{e\} \). It is easy to check that \( T \) is a triangulation the two-dimensional torus shown in Fig. [8]. This triangulation is isomorphic to the triangulation \( \{3, 6\}_{2, 2} \). Now let us consider the subcomplex \( P_1 \subset X \) consisting of all simplices \( \sigma \) such that \( \sigma \cup \{(12)(34)\} \) and \( \sigma \cup \{(13)(24)\} \) are simplices of \( X \). (The subcomplexes \( P_2 \) and \( P_3 \) are defined in a similar way.) The one-dimensional skeleton of \( P_1 \) is obtained from the one-dimensional skeleton of \( T \) by adding the 6 edges \((1, b), (4, b)\) and \((2, b), (3, b)\), \( b = 1, 2, 3 \). (These edges are drawn by dotted arcs in Fig. [6].) The complex \( P_1 \) consists of the three-dimensional simplices

\[
(1, b_1), (4, b_1), (2, b_2), (3, b_3), \quad (2, b_1), (3, b_1), (1, b_2), (4, b_3), \quad (1, b_1), (4, b_1), (2, b_2), (3, b_2),
\]

where \( b_1, b_2, b_3 \) are pairwise distinct, and faces of these three-dimensional simplices. It is easy to see that \( P_1 \) is a 12-vertex triangulation of a solid torus with boundary \( T \). The triangulation \( P_1 \) is shown in Fig. [6]. Here we imply that the left face and the right face of the parallelepiped shown in this figure are identified after rotation by \( \pi \). Similarly, \( P_2 \) and \( P_3 \) are also triangulated solid tori with the same vertex sets. Besides, it is easy to show that the complex \( X \) is obtained from the complex \( P_1 \cup P_2 \cup P_3 \) by adding cones over the
subcomplexes $\mathcal{P}_1 \cup \mathcal{P}_2$, $\mathcal{P}_1 \cup \mathcal{P}_3$, and $\mathcal{P}_2 \cup \mathcal{P}_3$ with vertices $(12)(34)$, $(13)(24)$, and $(14)(23)$ respectively. Thus the complex $X$ can be realized as an equilibrium triangulation of $\mathbb{C}P^2$ in the following way. Firstly, we realize the subcomplex $\mathcal{T}$ as a triangulation of the torus $T$. Secondly, we realize the subcomplexes $\mathcal{P}_j$ as triangulations of solid tori $\Pi_j$, $j = 1, 2, 3$. Finally, we put vertices $(14)(23)$, $(13)(24)$, and $(12)(34)$ inside the four-dimensional disks $B_1$, $B_2$, and $B_3$ respectively and add to the triangulation the cones with these vertices over the triangulations $\mathcal{P}_2 \cup \mathcal{P}_3$, $\mathcal{P}_1 \cup \mathcal{P}_3$, and $\mathcal{P}_1 \cup \mathcal{P}_2$ respectively.

Now we shall construct these realizations explicitly so that the action of the automorphism group $\text{Sym}(X)$ on $X$ will be identified with the constructed action of the group $S_4 \times S_3$ by isometries of $\mathbb{C}P^2$. First, to vertices $s$ of $X$ we assign the following points $v_s \in \mathbb{C}P^2$,

$$
\begin{align*}
  v_{(14)(23)} &= (1 : 0 : 0), & v_{(13)(24)} &= (0 : 1 : 0), & v_{(12)(34)} &= (0 : 0 : 1), \\
  v_{(1,b)} &= (-1 : \omega^b : \omega^{2b}), & v_{(2,b)} &= (1 : -\omega^b : \omega^{2b}), & v_{(3,b)} &= (1 : \omega^b : -\omega^{2b}), \\
  v_{(4,b)} &= (1 : \omega^b : \omega^{2b}), & b &= 1, 2, 3.
\end{align*}
$$

It can be immediately checked that this set of points is $(S_4 \times S_3)$-invariant and $R(h)v_s = v_{h \cdot s}$ for any element $h \in S_4 \times S_3$ and any vertex $s$ of $X$. 

---

Figure 6. The 12-vertex triangulation $\mathcal{T}$ of the two-dimensional torus

Figure 7. The 12-vertex triangulation $\mathcal{P}_1$ of the solid torus
The points \( v_{(a,b)} \) lie in the torus \( T \). The torus \( T \) is flat in the Fubini–Study metric and the 12 points \( (1 : \pm \omega^b : \pm \omega^{2b}) \) lie in it exactly at vertices of the triangulation \( \{3,6\}_{2,2} \) consisting of rectilinear triangles. Thus the triangulation \( \mathcal{T} \) is realized as a flat triangulation of the torus \( T \) with vertices at the points \( v_{(a,b)} \). The edges of \( \mathcal{T} \) are realized by the shortest geodesic segments. For example, the edge with endpoints \( (1,1) \) and \( (2,2) \) is realized by the geodesic segment \( (e^{-\frac{i\pi}{6}(1+t)} : e^{-\frac{i\pi}{3}(2-t)} : 1), \ t \in [0,1] \). Other edges can be obtained from this one by the action of the group \( S_4 \times S_3 \).

Let us now construct an explicit triangulation of the solid torus \( \Pi_1 \). We realize the edge \( (1,1), (4,1) \) by the geodesic segment \( (-t : \omega : \omega^2), \ t \in [-1,1] \). Other edges \( (a_1,b), (a_2,b) \) can be obtained from this one by the action of the group \( S_4 \times S_3 \). To realize three-dimensional simplices it is convenient to endow the solid torus \( \Pi_1 \) by a flat metric coinciding with the Fubini–Study metric on the torus \( T \). This flat metric is defined in the following way. We parametrize the torus \( T \) by

\[
t(\varphi, \psi) = (e^{i\varphi} : e^{i\psi} : e^{-i\psi}), \quad \varphi, \psi \in \mathbb{R}/(2\pi\mathbb{Z}).
\]

This parametrization is two-to-one since \( t(\varphi + \pi, \psi + \pi) = t(\varphi, \psi) \). The restriction of the Fubini–Study metric to the torus \( T \) is equal to \( \frac{2}{5}d\varphi^2 + \frac{2}{3}d\psi^2 \). The length of the circle \( \psi = c \), which is homological to zero in the solid torus \( \Pi_1 \), is equal to \( \frac{2\sqrt{5\pi}}{3} \). Now we parametrize (two-to-one) the solid torus \( \Pi_1 \) by

\[
p(x, y, h) = \left( 2(|x| + |y|) \left( \sin \frac{\pi x}{2(|x| + |y|)} + i \sin \frac{\pi y}{2(|x| + |y|)} \right) : \right.
\]

\[
= \left( \frac{-i\sqrt{2}h}{2} - |x| - |y| \right) e^{i\sqrt{2}h} : \left( \frac{-i\sqrt{2}h}{2} - |x| - |y| \right) e^{-i\sqrt{2}h}, \quad |x| + |y| \leq \frac{\pi}{6}, \ h \in \mathbb{R}/\left( 2\sqrt{\frac{2}{3}\pi}\mathbb{Z} \right)
\]

and endow it with the flat metric \( dx^2 + dy^2 + dh^2 \). It is easy to check that the restriction of this metric to the torus \( T \) coincides with the Fubini–Study metric. Indeed, every section \( h = c \) of the solid torus is the square with vertices \( (\pm \frac{\pi}{6}, 0), (0, \pm \frac{\pi}{6}) \) and perimeter \( \frac{2\sqrt{5\pi}}{3} \) in the Euclidean coordinates \( (x, y) \).

The coordinates \( (x, y, h) \) yield an isometric embedding of the universal covering of the solid torus \( \Pi_1 \) into \( \mathbb{R}^3 \). The image of this embedding is an infinite cylinder \( C \) over a square with side length \( \frac{\sqrt{2\pi}}{6} \). The cylinder \( C \) is given by the inequality \( |x| + |y| \leq \frac{\pi}{6} \). The solid torus \( \Pi_1 \) is the quotient of the cylinder \( C \) by the action of the infinite cyclic group \( \langle S \rangle \) generated by the isometry \( S : (x, y, h) \mapsto (-x, -y, h + \sqrt{\frac{2}{3}\pi}) \). Let \( \widetilde{P}_1 \) be the universal covering of \( P_1 \). Under the constructed embedding the vertices of \( \widetilde{P}_1 \) go to the points \( \left( \pm \frac{\pi}{6}, 0, \sqrt{\frac{\pi}{3}k} \right), \left( 0, \pm \frac{\pi}{6}, \sqrt{\frac{\pi}{3}(\frac{\pi}{6} + \frac{\pi}{3})} \right), \ k \in \mathbb{Z} \), and the edges of \( \widetilde{P}_1 \) go to rectilinear segments. Let us decompose the cylinder \( C \) into convex tetrahedra as it is shown on Fig. [7]. This decomposition consists of three series of tetrahedra

1) \( \left( \pm \frac{\pi}{6}, 0, \sqrt{\frac{\pi}{3}k} \right), \left( 0, \pm \frac{\pi}{6}, \sqrt{\frac{\pi}{3}(\frac{\pi}{6} + \frac{\pi}{3})} \right), \left( \pm \frac{\pi}{6}, 0, \sqrt{\frac{\pi}{3}(\frac{\pi}{6} + \frac{\pi}{3})} \right), \ k \in \mathbb{Z}, \ \varepsilon = \pm 1; \)

2) \( \left( 0, \pm \frac{\pi}{6}, \sqrt{\frac{\pi}{3}k} \right), \left( \pm \frac{\pi}{6}, 0, \sqrt{\frac{\pi}{3}(\frac{\pi}{6} + \frac{\pi}{3})} \right), \left( 0, \pm \frac{\pi}{6}, \sqrt{\frac{\pi}{3}(\frac{\pi}{6} + \frac{\pi}{3})} \right), \ k \in \mathbb{Z}, \ \varepsilon = \pm 1; \)

3) \( \left( \pm \frac{\pi}{6}, 0, \sqrt{\frac{\pi}{3}k} \right), \left( 0, \pm \frac{\pi}{6}, \sqrt{\frac{\pi}{3}(\frac{\pi}{6} + \frac{\pi}{3})} \right), \ k \in \mathbb{Z}. \)

The triangulation constructed is invariant under the isometry \( S \) and, hence, yields a triangulation of the solid torus \( \Pi_1 \). The latter triangulation is the required realization of
the simplicial complex $\mathcal{P}_1$. The triangulations of the solid tori $\Pi_2$ and $\Pi_3$ are constructed similarly.

Now we need to construct explicitly triangulations of the disks $B_j$. We consider the coordinates $Z_2 = \frac{z_2}{z_1(|z_1|+|z_2|+|z_3|)}$ and $Z_3 = \frac{z_3}{z_1(|z_1|+|z_2|+|z_3|)}$. In these coordinates the set $B_1$ is given by the inequalities $2|Z_1| + |Z_2| \leq 1$, $2|Z_2| + |Z_1| \leq 1$. The boundary of this convex set is the union of the solid tori $\Pi_2$ and $\Pi_3$ with triangulations constructed above. We triangulate the disk $B_1$ as the affine cone over the constructed triangulation of the union $\Pi_2 \cup \Pi_3$ with vertex $v_{(14)(23)} = (0, 0)$. Similarly, we triangulate the disks $B_2$ and $B_3$.

The above construction allows us to obtain an explicit formula for the ($S_4 \times S_3$)-equivariant piecewise smooth homeomorphism $f : |X| \to \mathbb{C}P^2$. We shall give explicit formulae for the restrictions of the homeomorphism $f$ to the four-dimensional simplices

$$\Delta_1 = \{(12)(34), (1, 3), (2, 1), (3, 2), (4, 3)\},$$
$$\Delta_2 = \{(12)(34), (1, 3), (2, 1), (3, 1), (4, 3)\}. \quad (7)$$

The restrictions of $f$ to other four-dimensional simplices of $X$ can be obtained from the restrictions to the simplices $\Delta_1$ and $\Delta_2$ by the action of the group $S_4 \times S_3$. The restrictions of $f$ to the simplices $\Delta_1$ and $\Delta_2$ are given by

$$(\xi_0, \xi_1, \ldots, \xi_4) \mapsto \left( \begin{array}{c} |\xi_4 - \xi_1| + \xi_2 + \xi_3 \\ 3 \\ \sin \frac{\pi(\xi_4 - \xi_1)}{2(|\xi_4 - \xi_1| + \xi_2 + \xi_3)} + \\ + i \sin \frac{\pi(\xi_2 + \xi_3)}{2(|\xi_4 - \xi_1| + \xi_2 + \xi_3)} \end{array} \right) \left( \begin{array}{c} 1 - \xi_0 \\ 2 \\ \left(1 - \xi_0\right) - \frac{|\xi_4 - \xi_1| + \xi_2 + \xi_3}{6} \\ e^{\frac{i\pi(\xi_2 - \xi_3)}{6(1 - \xi_0)}} \end{array} \right), \quad (8)$$

$$(\zeta_0, \zeta_1, \ldots, \zeta_4) \mapsto \left( \begin{array}{c} |\zeta_4 - \zeta_1| + |\zeta_2 - \zeta_3| \\ 3 \\ \sin \frac{\pi(\zeta_4 - \zeta_1)}{2(|\zeta_4 - \zeta_1| + |\zeta_2 - \zeta_3|)} + \\ + i \sin \frac{\pi(\zeta_2 - \zeta_3)}{2(|\zeta_4 - \zeta_1| + |\zeta_2 - \zeta_3|)} \end{array} \right) \left( \begin{array}{c} 1 - \zeta_0 \\ 2 \\ \left(1 - \zeta_0\right) - \frac{|\zeta_4 - \zeta_1| + |\zeta_2 - \zeta_3|}{6} \\ e^{\frac{i\pi(\zeta_2 + \zeta_3)}{6(1 - \zeta_0)}} \end{array} \right), \quad (9)$$

where $(\xi_0, \xi_1, \ldots, \xi_4)$ and $(\zeta_0, \zeta_1, \ldots, \zeta_4)$, $\sum_{j=0}^4 \xi_j = \sum_{j=0}^4 \zeta_j = 1$, are the barycentric coordinates in the simplices $\Delta_1$ and $\Delta_2$ respectively. (The barycentric coordinates are numbered corresponding to those orderings of vertices of simplices $\Delta_1$ and $\Delta_2$ which are given by (7).)

In section 1 we defined the subdivision $\overline{X}$ of the simplicial complex $X$. It is convenient to rewrite formulae (8) and (9) for the homeomorphism $f$ in the barycentric coordinates for simplices of $\overline{X}$. The vertices $(\overline{a_1a_2}, b)$ of $\overline{X}$ are the midpoints of those edges of $X$ that are contained in the subcomplex $\mathcal{P}_1 \cup \mathcal{P}_2 \cup \mathcal{P}_3$ but are not contained in the subcomplex $T$. Hence under the homeomorphism $f$ these vertices go to the points

$$v_{(12,b)} = (1 : -\omega^b : 0), \quad v_{(13,b)} = (-\omega^b : 0 : 1), \quad v_{(23,b)} = (0 : 1 : -\omega^b),$$
$$v_{(34,b)} = (1 : \omega^b : 0), \quad v_{(24,b)} = (\omega^b : 0 : 1), \quad v_{(41,b)} = (0 : 1 : \omega^b).$$
Recall that four-dimensional simplices of $\overline{X}$ are divided into two $(S_4 \times S_3)$-orbits with representatives

\[
\sigma_1 = \{(12)(34), (14, 3), (2, 1), (3, 2), (4, 3)\}, \\
\sigma_2 = \{(12)(34), (14, 3), (23, 1), (2, 1), (4, 3)\}.
\]

Let $(p_0, p_1, \ldots, p_4)$ be the barycentric coordinates in the simplex $\sigma_1$. (The vertices of $\sigma_1$ are ordered as they are listed in (10).) The simplex $\sigma_1$ is contained in the simplex $\Delta_1$ and we have $\xi_0 = p_0$, $\xi_1 = \frac{p_1}{2}$, $\xi_2 = p_2$, $\xi_3 = p_3$, $\xi_4 = p_4 + \frac{p_1}{2}$. Thus formula (8) implies that the restriction of the homeomorphism $f$ to the simplex $\sigma_1$ is given by

\[
(p_0, p_1, \ldots, p_4) \mapsto \left(\frac{p_2 + p_3 + p_4}{3} e^{\frac{i \pi (p_2 + p_3)}{6(1 - p_0)}}, \frac{p_1}{2} + \frac{p_2 + p_3 + p_4}{3} e^{\frac{i \pi (p_2 - p_3)}{6(1 - p_0)}} \right) : \\
\left(\frac{p_1}{2} + \frac{p_2 + p_3 + p_4}{3} e^{\frac{i \pi (p_2 - p_3)}{6(1 - p_0)}}\right).
\]

Similarly, let $(q_0, q_1, \ldots, q_4)$ be the barycentric coordinates in the simplex $\sigma_2$. (The vertices of $\sigma_2$ are ordered as they are listed in (10).) Then the restriction of the homeomorphism $f$ to the simplex $\sigma_2$ is given by

\[
(q_0, q_1, \ldots, q_4) \mapsto \left(\frac{q_3 + q_4}{3} e^{\frac{i \pi (q_3 + q_4)}{6(1 - q_0)}}, \frac{q_1 + q_2}{2} + \frac{q_3 + q_4}{3} e^{\frac{i \pi (q_1 + q_3)}{6(1 - q_0)}} \right) : \\
\left(\frac{q_1 + q_2}{2} + \frac{q_3 + q_4}{3} e^{\frac{i \pi (q_1 + q_3)}{6(1 - q_0)}}\right).
\]

The formulae for the restrictions of $f$ to other four-dimensional simplices of $\overline{X}$ can be obtained from formulae (11) and (12) by the action of the group $S_4 \times S_3$.

### 4. Triangulation of the Moment Mapping

In this section we shall construct a triangulation of the classical moment mapping $\mu : \mathbb{C}P^2 \to \Delta^2$ given by (1). This means that we shall construct a triangulation of $\mathbb{C}P^2$ and a triangulation of the triangle $\Delta^2$ such that the mapping $\mu$ is simplicial with respect to this pair of triangulations. For a triangulation of $\mathbb{C}P^2$ we take the triangulation $\overline{X}$. (Recall that the explicit homomorphism $f : [\overline{X}] \to \mathbb{C}P^2$ was constructed in section 3.) For a triangulation of the triangle $\Delta^2$ we take its barycentric subdivision $(\Delta^2)'$.

We introduce an action of the group $S_4 \times S_3$ on the triangle $\Delta^2$ such that the multiplier $S_3$ acts trivially and the multiplier $S_4$ acts by linear mappings permuting the vertices of the triangle $\Delta^2$ according to the homomorphism $S_4 \to S_4/V_4 \cong S_3$. Then the action is given by

\[
(12) \cdot (t_1, t_2, t_3) = (34) \cdot (t_1, t_2, t_3) = (t_2, t_1, t_3); \\
(23) \cdot (t_1, t_2, t_3) = (t_1, t_3, t_2)
\]

on the generators of $S_4$. Recall that the group $S_4 \times S_3$ acts on $\mathbb{C}P^2$ by isometries and the action is given by the projective representation $R$ (see section 3). It is easy to check that the mapping $\mu$ is equivariant with respect to the above pair of actions.
We define a simplicial mapping \( m : \overline{X} \to (\Delta^2)' \) on the vertices of \( \overline{X} \) by
\[
\begin{align*}
m((12)(34)) &= (0, 0, 1); & m((13)(24)) &= (0, 1, 0); & m((14)(23)) &= (1, 0, 0); \\
m((\hat{1}2, b)) &= m((\hat{3}4, b)) = \left( \frac{1}{2}, \frac{1}{2}, 0 \right); & m((\hat{1}3, b)) &= m((\hat{2}4, b)) = \left( \frac{1}{2}, 0, \frac{1}{2} \right); \end{align*}
\]
\[
m((\hat{2}3, b)) = m((\hat{1}4, b)) = \left( 0, \frac{1}{2}, \frac{1}{2} \right); & m((a, b)) = \left( \frac{1}{3}, \frac{1}{3}, \frac{1}{3} \right).
\]

It can be immediately checked that the simplicial mapping \( m \) is well defined and \((S_4 \times S_3)\)-equivariant. The mapping of the geometric realizations of the complexes \( \overline{X} \) and \((\Delta^2)' \) induced by the mapping \( m \) will also be denoted by \( m \).

Ahead with the mapping \( \mu \), it is convenient to consider the mapping \( \tilde{\mu} : \mathbb{C}P^2 \to \Delta^2 \) given by
\[
\tilde{\mu}(z_1 : z_2 : z_3) = \frac{|z_1|, |z_2|, |z_3|}{|z_1| + |z_2| + |z_3|}.
\]

We have \( \tilde{\mu} = \mu \circ g \), where \( g : \mathbb{C}P^2 \to \mathbb{C}P^2 \) is the homeomorphism given by
\[
g(z_1 : z_2 : z_3) = \left( \frac{z_1}{\sqrt{|z_1|}} : \frac{z_2}{\sqrt{|z_2|}} : \frac{z_3}{\sqrt{|z_3|}} \right).
\]

Here one should replace \( \frac{z_j}{\sqrt{|z_j|}} \) by 0 if \( z_j = 0 \). Obviously, the mappings \( \tilde{\mu} \) and \( g \) are equivariant.

**Proposition 4.1.** The simplicial mapping \( m \) triangulates the mapping \( \tilde{\mu} \), that is, there is a commutative diagram
\[
\begin{array}{ccc}
|\overline{X}| & \xrightarrow{f} & \mathbb{C}P^2 \\
\downarrow m & & \downarrow \tilde{\mu} \\
|\Delta^2'| & \longrightarrow & \Delta^2
\end{array}
\]

**Proof.** Both mappings \( m \) and \( \tilde{\mu} \circ f \) are \((S_4 \times S_3)\)-equivariant. Hence we suffices to check that they coincide on two four-dimensional simplices of \( \overline{X} \) representing different \((S_4 \times S_3)\)-orbits, for example, on the simplices \( \sigma_1 \) and \( \sigma_2 \) (see (10)). The coincidence of the mappings \( m \) and \( \tilde{\mu} \circ f \) on the simplices \( \sigma_1 \) and \( \sigma_2 \) follows easily from formulae (11) and (12).

Now to obtain a triangulation of the moment mapping \( \mu \) one should replace the homeomorphism \( f : |\overline{X}| \to \mathbb{C}P^2 \) by the homeomorphism \( g \circ f \).

The preimage of the barycenter of the triangle \( \Delta^2 \) under the mapping \( m \) is the subcomplex \( T \subset \overline{X} \) isomorphic to the 12-vertex triangulation of the two-dimensional torus shown in Fig. [3]. The preimage of the midpoint of every edge of the triangle \( \Delta^2 \) is a subcomplex of \( \overline{X} \) isomorphic to the boundary of the hexagon. The preimage of every vertex of \( \Delta^2 \) is a vertex of \( \overline{X} \).

5. **Relationship with complex crystallographic groups**

In this section we shall conveniently regard the group \( S_3 \) as the group of permutations of the set \( \mathbb{Z}_3 = \{0, 1, 2\} \) rather than the set \( \{1, 2, 3\} \).

Recall that a **crystallographic group** is a cocompact discrete group of isometries of a finite-dimensional Euclidean space, that is, a cocompact subgroup of the semidirect product \( \mathbb{R}^n \rtimes O(n) \). Similarly, a **complex crystallographic group** is a cocompact discrete subgroup of the group \( \mathbb{C}^n \rtimes U(n) \), which is the group of those transformations of a
finite-dimensional Hermitian space that are compositions of unitary transformations and translations.

The relationship of Kühnel’s 9-vertex triangulation of $\mathbb{CP}^2$ with complex crystallographic groups was discovered by B. Morin and M. Yoshida \[4\] (see also \[5\]). In this section we recall some results of the papers \[4], \[5\] and explain the relationship between the constructed 15-vertex triangulation $X$ with complex crystallographic groups.

Let $\tau$ be a complex number with positive imaginary part. By $L = L(\tau)$ we denote the lattice in $\mathbb{C}$ with basis $(1, \tau)$. We consider the complex torus $T^2 = \mathbb{C}/L$, the 6-dimensional torus $T^6 = T^2 \times T^2 \times T^2$, and the subtorus

$$T^4 = \{(z_1, z_2, z_3) \in T^2 \times T^2 \times T^2 \mid z_1 + z_2 + z_3 = 0\}$$

of the torus $T^6$. The group $S_3$ acts on $T^6$ by permutations of the multipliers $T^2$. The torus $T^4$ is invariant under this action. The following proposition is well known.

**Proposition 5.1.** The quotient $T^4/S_3$ is homeomorphic to $\mathbb{CP}^2$.

**Proof.** The complex torus $T^2$ can be realized as an elliptic curve $E \subset \mathbb{CP}^2$. Three points $z_1, z_2, z_3 \in T^2$ satisfy the equality $z_1 + z_2 + z_3 = 0$ if and only if the corresponding three points of $E$ lie in a complex line. Thus we obtain a homeomorphism between the space $T^4/S_3$, which is the space of unordered triples of points $z_1, z_2, z_3 \in T^2$ such that $z_1 + z_2 + z_3 = 0$, and the space of complex lines in $\mathbb{CP}^2$. The latter space is obviously homeomorphic to $\mathbb{CP}^2$. \[\square\]

This proposition immediately implies that $\mathbb{CP}^2$ is the quotient of $\mathbb{C}^2$ by a complex crystallographic group. Let us describe this group explicitly. Consider the two-dimensional subspace

$$W = \{(z_1, z_2, z_3) \in \mathbb{C}^3 \mid z_1 + z_2 + z_3 = 0\} \subset \mathbb{C}^3$$

and the vectors $h_0 = (1, -1, 0)$, $h_1 = (0, 1, -1)$, and $h_2 = (-1, 0, 1)$ belonging to it. Obviously, $h_0 + h_1 + h_2 = 0$. Let us identify the group $S_3$ with the group of transformations of $\mathbb{C}^3$ that permutes the coordinates. The subspace $W$ is invariant under this group. Consider the lattice $\Lambda = Lh_0 + Lh_1 + Lh_2 \subset W$ and the complex crystallographic group $\Gamma = \Lambda \rtimes S_3$. Then $W/\Gamma = T^4/S_3 \approx \mathbb{CP}^2$.

Suppose $f_0 = \frac{h_2 - h_1}{3}$, $f_1 = \frac{h_0 - h_2}{3}$, and $f_2 = \frac{h_1 - h_0}{3}$; then $h_0 = f_1 - f_2$, $h_1 = f_2 - f_0$, and $h_2 = f_0 - f_1$. The lattice $\Lambda \mathbb{R} = \sum_{j=0}^2 \mathbb{Z} h_j$ is the subgroup of index 3 in the lattice $\overline{\Lambda}_\mathbb{R} = \sum_{j=0}^2 \mathbb{Z} f_j$ and the lattice $\Lambda$ is the subgroup of index 9 in the lattice $\overline{\Lambda} = \sum_{j=0}^2 L f_j$. Decompose the space $W$ into the direct sum $W = W_\mathbb{R} \oplus \tau W_\mathbb{R}$, where $W_\mathbb{R}$ is the two-dimensional real subspace spanned by the vectors $h_0, h_1, h_2$. Similarly, we have the decompositions $\Lambda = \Lambda \mathbb{R} \oplus \tau \Lambda \mathbb{R}$, $\overline{\Lambda} = \overline{\Lambda} \mathbb{R} \oplus \tau \overline{\Lambda} \mathbb{R}$. The lattice $\overline{\Lambda} \mathbb{R}$ is hexagonal. By $\hat{Z}$ we denote the corresponding triangulation of the plane $W_\mathbb{R}$ by rectilinear triangles. The direct (though not orthogonal) product of the triangulations $\hat{Z}$ and $\tau \hat{Z}$ is a decomposition of the space $W$ into convex prisms $\Delta^2 \times \Delta^2$. This decomposition, which will be denoted by $\hat{Q}$, is invariant under the action of $\Gamma$ and $Q = \hat{Q}/\Gamma$ is a decomposition of $\mathbb{CP}^2$ into prisms $\Delta^2 \times \Delta^2$. (Later we shall see that two prisms in the latter decomposition can possess several common facets.) B. Morin and M. Yoshida \[4\] constructed a $\Gamma$-invariant rectilinear triangulation $\hat{K}$ of the space $W \approx \mathbb{C}^2$ such that $K = \hat{K}/\Gamma$ is a well-defined triangulation of $\mathbb{CP}^2$ isomorphic to Kühnel’s 9-vertex triangulation. P. Arnoux and A. Marin \[6\] noticed that the triangulation $\hat{K}$ is a subdivision of the decomposition $\hat{Q}$ such that every prism $\Delta^2 \times \Delta^2$ is divided into 6 four-dimensional simplices without adding new vertices.
The case $\tau = \omega = e^{\frac{2\pi i}{3}}$ is of a special interest since for $\tau = \omega$ the triangulation $\tilde{K}$ becomes invariant under a bigger crystallographic group $\overline{\Gamma} \supset \Gamma$ (see [4]). The group $\overline{\Gamma}$ is the semidirect product $\overline{\Gamma} = G_{18}$, where $G_{18} \subset U(2)$ is the group of order 18 generated by the subgroup $S_3 \subset U(2)$ and the operator of multiplication by $\omega$. The quotient group $\overline{\Gamma}/\Gamma$ has order 27 and is a subgroup of index 2 of the automorphism group of $K$.

Let us now describe how the triangulation $X$ appears in the context of complex crystallographic groups. The multiplication by $\omega$ will not be important in this construction. Hence we can again suppose that $\tau$ is an arbitrary complex number with positive imaginary part.

First, let us consider the decomposition $\tilde{Q} = \tilde{Q}/\Lambda$ of the torus $T^4$. This decomposition is the direct product of the two identical decompositions $\tilde{Z} = \tilde{Z}/\Lambda_{\mathbb{R}}$ of the two-dimensional torus $W_{\mathbb{R}}/\Lambda_{\mathbb{R}}$ into triangles. The decomposition $\tilde{Z}$ is shown in Fig. 8. (We imply the identification of opposite sides of the hexagon.) This decomposition is not a triangulation since all 6 triangles of this decomposition have the same 3 vertices. The group $S_3$ acts on the torus $W_{\mathbb{R}}/\Lambda_{\mathbb{R}}$ so that the transpositions $\sigma_0 = (12)$, $\sigma_1 = (20)$, and $\sigma_2 = (01)$ act by symmetries in the lines $\ell_0$, $\ell_1$, and $\ell_2$ respectively. The group $S_3$ acts transitively on the set of triangles of the decomposition $\tilde{Z}$. We mark the triangles of $\tilde{Z}$ by elements of $S_3$ in such a way that the action of $S_3$ on $\tilde{Z}$ coincides with the action of $S_3$ on itself by left shifts. Then four-dimensional cells of the decomposition $\tilde{Q} = \tilde{Z} \times \tilde{Z}$ are marked by ordered pairs of elements of $S_3$ and the action of $S_3$ is diagonal. Therefore the decomposition $Q = \tilde{Q}/S_3$ contains 6 four-dimensional cells which can be marked by permutations $\kappa \in S_3$ so that under a factorization by the action of $S_3$ the cell with mark $(\kappa_1, \kappa_2)$ goes to the cell with mark $\kappa_1^{-1}\kappa_2$. The four-dimensional cell of $Q$ with mark $\kappa$ will be denoted by $P_\kappa$.

We mark vertices of the decomposition $\tilde{Z}$ by elements $0, 1, 2 \in \mathbb{Z}_3$ as it is shown in Fig. 8. Then vertices of the decomposition $\tilde{Q}$ are marked by pairs $(a_1, a_2) \in \mathbb{Z}_3 \times \mathbb{Z}_3$. The vertex corresponding to a pair $(a_1, a_2)$ will be denoted by $u(a_1, a_2)$. The vertices $u(a_1, a_2)$ are fixed by the action of $S_3$. Hence these vertices are exactly the vertices of the decomposition $Q$. Thus $Q$ is a 9-vertex decomposition of $\mathbb{C}P^2$ into 6 four-dimensional cells $P_\kappa$. Every cell $P_\kappa$ is a prism $\Delta^2 \times \Delta^2$ and the vertices of each multiplier $\Delta^2$ are marked by pairwise distinct elements 0, 1, 2. Faces of the prism $\Delta^2 \times \Delta^2$ are in one-to-one correspondence with pairs of subsets $A_1, A_2 \subset \mathbb{Z}_3$. To a pair $(A_1, A_2)$ we assign the face spanned by all vertices $u(a_1, a_2)$ such that $a_1 \in A_1$ and $a_2 \in A_2$. The face corresponding to a pair $(A_1, A_2)$ is called a face of type $(A_1, A_2)$.

\begin{figure}[h]
\centering
\includegraphics[width=0.5\textwidth]{figure8.png}
\caption{The decomposition $\tilde{Z}$ of the torus $W_{\mathbb{R}}/\Lambda_{\mathbb{R}}$ into 6 triangles}
\end{figure}
Now, for two distinct four-dimensional cells $P_{\kappa_1}$ and $P_{\kappa_2}$ of $Q$ we compute the number of common facets of the cells $P_{\kappa_1}$ and $P_{\kappa_2}$. If one of the permutations $\kappa_1$ and $\kappa_2$ is even and the other is odd, then the cells $P_{\kappa_1}$ and $P_{\kappa_2}$ have two common facets of types $(\{a_1+1, a_1+2\}, \{1, 2, 3\})$ and $(\{1, 2, 3\}, \{a_2+1, a_2+2\})$, where $\kappa_2 = \sigma_a \kappa_1 = \kappa_1 \sigma_a$.

If the permutations $\kappa_1$ and $\kappa_2$ are either both even or both odd, then the cells $P_{\kappa_1}$ and $P_{\kappa_2}$ have no common facets. Thus the decomposition $Q$ contains 9 quadrangular two-dimensional cells each of which is contained in exactly two three-dimensional cells and exactly two four-dimensional cells. Perform the following operation with the decomposition $Q$. Delete from this decomposition those 9 quadrangular two-dimensional faces and unite the pair of common facets of each two prisms $P_{\kappa_1}$ and $P_{\kappa_2}$ so as to obtain one three-dimensional cell with combinatorial type of the polytope shown in Fig. 9. Four-dimensional cells of the obtained decomposition are just four-dimensional cells of $Q$. However the cell decompositions of their boundaries are different. Every four-dimensional cell of the obtained decomposition is a four-dimensional disk with boundary decomposed into 3 three-dimensional cells combinatorially equivalent to the polytope shown in Fig. 9. The obtained cell decomposition of $\mathbb{CP}^2$ will be denoted by $Q^{(1)}$ and its four-dimensional cells will be denoted by $P_{\kappa}^{(1)}$.

Now we consider the two-dimensional skeleton of $Q^{(1)}$. It consists of triangular and quadrangular cells. Every quadrangular cell has vertices $u(a, b)$, $u(a+1, b)$, $u(a+1, b+1)$, and $u(a, b+1)$ for some $a, b \in \mathbb{Z}$. Decompose every such quadrangular cell into two triangular cells by the diagonal with vertices $u(a, b)$ and $u(a+1, b+1)$. We denote the obtained cell decomposition by $Q^{(2)}$ and denote its four-dimensional cells by $P_{\kappa}^{(2)}$. Certainly, the combinatorial types of three-dimensional and four-dimensional cells have changed under this operation. In particular, all three-dimensional cells have become isomorphic to the suspension over a hexagon (see Fig. 9, b). Notice that every two distinct four-dimensional cells $P_{\kappa_1}^{(2)}$ and $P_{\kappa_2}^{(2)}$ of $Q^{(2)}$ have no common facets if the permutations $\kappa_1$ and $\kappa_2$ are either both even or both odd and have a unique common facet if one of the permutations $\kappa_1$ and $\kappa_2$ is even and the other is odd. (The same assertion holds for $Q^{(1)}$.)

The two-dimensional skeleton of the decomposition $Q^{(2)}$ is a decomposition into triangles. It can be immediately checked that the two-dimensional skeleton of $Q^{(2)}$ is a simplicial complex, that is, it does not contain multiple edges and triangles with coinciding boundary. (This fact will be very important for us.) Indeed, the construction described above provides an explicit description of two-dimensional faces of $Q^{(2)}$. Every two-dimensional face of $Q^{(2)}$ is a triangle spanned by a set of vertices of one of the following types.
1) \(u(a, 0), u(a, 1), u(a, 2)\).
2) \(u(0, a), u(1, a), u(2, a)\).
3) \(u(a + 1, b + 1), u(a + 2, b + 1), u(a + 2, b + 2)\).
4) \(u(a + 1, b + 1), u(a + 1, b + 2), u(a + 2, b + 2)\).

Suppose that \(\kappa_1, \kappa_2 \in S_3\) are permutations one of which is even and the other is odd. Then the triangle of type 1) is contained in \(P^{(2)}_{\kappa_1} \cap P^{(2)}_{\kappa_2}\) if and only if \(\kappa_2 \neq \sigma_a \kappa_1\), the triangle of type 2) is contained in \(P^{(2)}_{\kappa_1} \cap P^{(2)}_{\kappa_2}\) if and only if \(\kappa_2 \neq \kappa_1 \sigma_a\), and the triangle of type 3) or of type 4) is contained in \(P^{(2)}_{\kappa_1} \cap P^{(2)}_{\kappa_2}\) if and only if exactly one of the two equalities \(\kappa_2 = \sigma_a \kappa_1\) and \(\kappa_2 = \kappa_1 \sigma_a\) holds.

Now let us construct a 15-vertex triangulation \(Y\) of \(\mathbb{C}P^2\) in the following way. Introduce a new vertex \(u(\kappa)\) in the interior of every four-dimensional cell \(P^{(2)}_{\kappa}\). Decompose every cell \(P^{(2)}_{\kappa}\) into the cones over its facets with vertex \(u(\kappa)\). Every three-dimensional face \(F\) of \(Q^{(2)}\) is contained in two four-dimensional cells \(P^{(2)}_{\kappa_1}\) and \(P^{(2)}_{\kappa_2}\). Uniting the cones over \(F\) with vertices \(u(\kappa_1)\) and \(u(\kappa_2)\) we obtain the suspension over \(F\). Triangulate this suspension as the join of the segment with endpoints \(u(\kappa_1)\) and \(u(\kappa_2)\) and the triangulation of \(\partial F\). (Recall that the two-dimensional skeleton of \(Q^{(2)}\) is a simplicial complex.) Performing the described operation for all three-dimensional faces of \(Q^{(2)}\), we obtain a triangulation, which we denote by \(Y\). The triangulation \(Y\) has 15 vertices among which there are 9 vertices \(u(a, b)\) and 6 vertices \(u(\kappa)\). Four-dimensional simplices are spanned by the following sets of vertices.

1) \(u(\kappa), u(\sigma_a \kappa), u(0, a), u(1, a), u(2, a), a \neq b;\)
2) \(u(\kappa), u(\sigma_a \kappa), u(0, a), u(1, a), u(2, a), a \neq b;\)
3) \(u(\kappa), u(\sigma_a \kappa_a), u(a + 1, b + 1), u(a + 2, b + 1), u(a + 2, b + 2), a \neq \kappa(b);\)
4) \(u(\kappa), u(\sigma_a \kappa_a), u(a + 1, b + 1), u(a + 1, b + 2), u(a + 2, b + 2), a \neq \kappa(b);\)
5) \(u(\kappa), u(\sigma_a \kappa_a), u(a + 1, b + 1), u(a + 2, b + 1), u(a + 2, b + 2), a \neq \kappa(b);\)
6) \(u(\kappa), u(\sigma_a \kappa_a), u(a + 1, b + 1), u(a + 1, b + 2), u(a + 2, b + 2), a \neq \kappa(b).\)

The number of simplices of each type is equal to 18.

**Proposition 5.2.** The triangulation \(Y\) is isomorphic to the simplicial complex \(X\).

**Proof.** Arrange a one-to-one correspondence between the vertices of \(X\) and the vertices of \(Y\) in the following way.

\[
(12)(34) \mapsto u(\sigma_0); \quad (13)(24) \mapsto u(\sigma_1); \quad (14)(23) \mapsto u(\sigma_2);
\]

\[
(4, b) \mapsto u((012)^b), \quad b = 1, 2, 3;
\]

\[
(a, b) \mapsto u(-a - b, -a + b), \quad a, b = 1, 2, 3,
\]

where the sums in the last formula are taken modulo 3. Notice that a set of pairwise distinct vertices of \(X\) spans a simplex if and only if it contains no pair of vertices of the form \((a, b_1), (a, b_2)\) or of the form \(\nu_1, \nu_2\) and no triple of vertices of the form \((a_1, b), (a_2, b), (a_3, b)\) or of the form \(\nu, (a, b), (\nu(a), b)\). A set of pairwise distinct vertices of \(Y\) spans a simplex if and only if it contains no pair of vertices of the form \(u(a_1, b_1), u(a_2, b_2)\) with \(a_1 + b_1 = a_2 + b_2\) or of the form \(u(\kappa_1)\), \(u(\kappa_2)\) with \(\kappa_1\) and \(\kappa_2\) either both even or both odd and no triple of vertices of one of the forms

\[
u(a_0, a_0), u(a_1, a_1 + 1), u(a_2, a_2 + 2);\]
\[
u(\kappa), u(a + 1, \kappa(a) + 1), u(a + 2, \kappa(a) + 2);\]
\[
u(u), u(\sigma_a \kappa_a), u(\kappa(b), b).\]

To prove that the complexes \(X\) and \(Y\) are isomorphic one suffices to notice the constructed one-to-one correspondence between their vertices takes the “prohibited” pairs and triples
of vertices for the complex $X$ to the “prohibited” pairs and triples of vertices for the complex $Y$. The latter assertion can be checked immediately.

**Remark 5.3.** Recall that the obtained result on the relationship of the triangulation $X$ with complex crystallographic groups is weaker than a similar result of B. Morin and M. Yoshida for Kühnel’s triangulation $K$. First, the author does not know whether the triangulation $X$ can be obtained as the quotient of a rectilinear triangulation of $\mathbb{C}^2$ by a crystallographic group. Second, the above construction of the triangulation $Y$ does not explain why the automorphism group of the triangulation $Y$ is so big.

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