A REMARK ON THE COMPONENT GROUP OF THE SATO–TATE GROUP

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Abstract. In this paper we give a complete characterization of the component group of the Sato–Tate group of an abelian variety $A$ of arbitrary dimension, defined over a number field $K$, in terms of the connectedness of the Lefschetz group associated to $A$.

1. Introduction

In the past six decades, the Sato–Tate conjecture has been largely studied since it has a strong connection with the generalized Riemann hypothesis. The Sato–Tate conjecture for an elliptic curve without complex multiplication defined over a number field, predicts the equidistribution of traces of Frobenius automorphisms with respect to the Haar measure of the corresponding Sato–Tate group. On the other hand the generalized Riemann hypothesis predicts where are located the zeros of the $L$-function, associated with the elliptic curve.

Despite all the efforts, only a few cases are known so far [Tay08, ACC+18]. It is known, that when considering elliptic curves over number fields, the Sato–Tate group can only be one of the following three groups $SU(2)$, $U(1)$ and $N(U(1))$. Nonetheless, when considering higher-dimensional abelian varieties, the classification of the possible Sato–Tate groups become more difficult to study [FKRS12 and FKS21]. It is of utmost importance to determine Sato–Tate groups. The first approach is to study their component groups rather than the groups themselves in some particular settings. The goal of this paper is to give an explicit determination of the component group of the Sato–Tate group.

1.1. The main result. The algebraic Sato–Tate conjecture, introduced by Serre and developed later on by Banaszak and Kedlaya [BK15, BK16], is a key input in the direction of determining the Sato–Tate group. Notice that once we have a better understanding of the Sato–Tate group we could potentially determine new instances of the generalized Sato–Tate conjecture for higher-dimensional abelian varieties. This latter conjecture predicts the equidistribution of the normalized factors of the $L$-function associated to an abelian variety. One of the main goals of this paper is to give a characterization of the component group of the Sato–Tate group when enough information is provided. The initial motivation of this paper

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was a previous result of Banaszak and Kedlaya [BK15, Thm. 6.1]. In this theorem, the authors gave very specific conditions for the algebraic Sato–Tate conjecture to hold. More precisely, the algebraic Sato–Tate conjecture is known to be true when: 1) the abelian variety is fully of Lefschetz type (i.e. the Mumford–Tate conjecture is satisfied and the Hodge group coincides with the Lefschetz group), and 2) the twisted Lefschetz group is connected. Notice that the second condition does not always hold once \( g > 3 \). Hence, we wonder what can be said when the second condition is not anymore satisfied. The spirit of our main result lies in the fact that in the case of simple abelian varieties of type I\( II \), in the sense of Albert’s classification, the twisted Lefschetz group is not connected. Let us recall that this type in Albert’s classification only appears when the dimension of the abelian variety is at least 4. Our main result can be stated as follows:

**Theorem 1.1.** Let \( A \) be an abelian variety defined over a number field \( K \). Consider its twisted Lefschetz group \( DL_K(A) \) and the smallest finite field extension \( K_e/K \) where all endomorphisms of \( A \) are defined over. Under some hypotheses\(^1\) we obtain the following direct product decomposition:

\[
\pi_0(DL_K(A)) = \pi_0(DL_{id}^K(A)) \times \text{Gal}(K_e/K).
\]

Notice that, when assuming that the abelian variety is fully of Lefschetz type\(^2\), we have the following isomorphisms:

\[
\pi_0(G_{\text{alg}}^\ell,K,1_\mathbb{Q}) \cong \pi_0(\text{AST}_K(A)) \cong \pi_0(\text{ST}_K(A)) \cong \pi_0(DL_K(A)).
\]

Hence, we are able to concretely determine, as a corollary, the component group of the Sato–Tate group in terms of the connectedness of the Lefschetz group.

**Corollary 1.2.** Under the hypotheses of Theorem 1.1 and the assumption that \( A \) is fully of Lefschetz type, we have the following isomorphism:

\[
\pi_0(\text{ST}_K(A)) \cong \pi_0(DL_K(A)) \cong \pi_0(DL_{id}^K(A)) \times \text{Gal}(K_e/K).
\]

In particular if \( DL_{id}^K(A) \) is connected then:

\[
\pi_0(\text{ST}_K(A)) \cong \pi_0(DL_K(A)) \cong \text{Gal}(K_e/K).
\]

In other words, we provide a way to determine the component group of the Sato–Tate group in terms of endomorphisms of \( A \) when enough information is known. We refer the reader to Section 2 for the notation introduced above. In Section 3 we present a proof of our main result.

2. Preliminaries

Throughout this paper \( A \) will denote an abelian variety defined over a number field \( K \) with the rational endomorphism algebra \( D := \text{End}_{\mathbb{Q}}(A) \otimes \mathbb{Q} \). Let \( V \) denote the first homology group \( V(A) := H_1(A(\mathbb{C}), \mathbb{Q}) \) and let \( V_\ell \) denote the \( \mathbb{Q}_\ell \)-vector space \( V_\ell(A) := V(A) \otimes \mathbb{Q}_\ell \) for every prime number \( \ell \). Let \( G_K := \text{Gal}(\overline{K}/K) \) be the absolute Galois group and

\[
\rho_\ell : G_K \to \text{GL}(V_\ell)
\]

\(^1\)We refer the reader to Section 3, Theorem 3.6

\(^2\)See Definition 2.4
the ℓ-adic representation attached to \( A \). Consider as well the \( \mathbb{Q} \)-bilinear non degenerate alternating form \( \psi : V \times V \to \mathbb{Q} \) coming from the polarization of \( A \). Equivalently, for every prime number \( \ell \), we have \( \psi_\ell : V_\ell \times V_\ell \to \mathbb{Q}_\ell \).

Recall the definition of general symplectic groups associated with forms \( \psi \) and \( \psi_\ell \):

\[
\text{GSp}_{(V, \psi)} := \{ g \in \text{GL}(V) ; \, \psi(gv, gw) = \chi(g)\psi(v, w), \, \forall v, w \in V \},
\]

where \( \chi \) and \( \chi_\ell \) are the following associated characters:

\[
\chi : \text{GSp}_{(V, \psi)} \to \mathbb{G}_{m, \mathbb{Q}}
\]

\[
\chi_\ell : \text{GSp}_{(V, \psi_\ell)} \to \mathbb{G}_{m, \mathbb{Q}_\ell}.
\]

The special symplectic groups are given by

\[
\text{Sp}_{(V, \psi)} := \{ g \in \text{GL}(V) ; \, \psi(gv, gw) = \psi(v, w), \, \forall v, w \in V \},
\]

\[
\text{Sp}_{(V, \psi_\ell)} := \{ g \in \text{GL}(V_\ell) ; \, \psi_\ell(gv, gw) = \psi_\ell(v, w), \, \forall v, w \in V_\ell \}.
\]

### 2.1. The endomorphism field extension

When studying abelian varieties, defined over a number field \( K \), we will examine a particular finite field extension of \( K \) which is related to the endomorphisms of the abelian variety. Consider the following continuous representation \[ \text{BK15, BK16} \]

\[
\rho_e : G_K \to \text{Aut}_\mathbb{Q}(D)
\]

and denote by \( K_e \) the fixed field of the kernel of \( \rho_e \), that is \( K_e := \overline{K}^{\ker \rho_e} \).

**Notation 2.1.** The finite field extension \( K_e \) will be called the endomorphism field extension.

### 2.2. The Lefschetz group and the twisted Lefschetz group

**Definition 2.2.** The Lefschetz group of an abelian variety \( A \) is defined as follows:

\[
\mathcal{L}(A) := C_D(\text{Sp}_{(V, \psi)})^\circ.
\]

For each \( \tau \in \text{Gal}(K_e/K) \) we have a closed subscheme of \( \text{Sp}_{(V, \psi)} \):

\[
\text{DL}_K^\tau(A) := \{ g \in \text{Sp}_{(V, \psi)} ; \, g\beta g^{-1} = \rho_e(\tau)(\beta), \, \forall \beta \in D \}.
\]

Taking disjoint union, we obtain the twisted Lefschetz group \[ \text{BK15 Def. 5.2} \]:

\[
\text{DL}_K(A) := \bigsqcup_{\tau \in \text{Gal}(K_e/K)} \text{DL}_K^\tau(A).
\]

**Remark 2.3.** We have the following equalities (see \[ \text{BK15 \S 5}, \text{BK16 \S 3} \]):

1. \( \text{DL}_K^\text{id}(A) = \{ g \in \text{Sp}_{(V, \psi)} ; \, g\beta g^{-1} = \beta, \, \forall \beta \in D \} = C_D(\text{Sp}_{(V, \psi)}) \),
2. \( \text{DL}_K^\text{id}(A)^\circ = C_D(\text{Sp}_{(V, \psi)})^\circ = \mathcal{L}(A) \),
3. \( \text{DL}_K(A)^\circ = \text{DL}_K^\text{id}(A)^\circ \),
4. \( \text{DL}_K(A) = \text{DL}_K^\text{id}(A) = \text{DL}_K^\circ(A) \).
2.3. The \(\ell\)-adic monodromy groups and their twists. For the representation (2.1) define the \(\ell\)-adic monodromy group \(G_{\ell,K}^{\text{alg}}(V_\ell, \psi_\ell)\) as the Zariski closure of the image of the \(\ell\)-adic representation in \(\text{GSp}_{(V_\ell, \psi_\ell)}\). When no ambiguities exist regarding the \(\mathbb{Q}_\ell\)-vector space, we will denote the \(\ell\)-adic monodromy group by \(G_{\ell,K}^{\text{alg}}\) rather than \(G_{\ell,K}^{\text{alg}}(V_\ell, \psi_\ell)\). Let us define the following algebraic group defined over \(\mathbb{Q}_\ell\):

\[
G_{\ell,K,1}^{\text{alg}} = G_{\ell,K,1}^{\text{alg}}(V_\ell, \psi_\ell) := G_{\ell,K}^{\text{alg}} \cap \text{SP}(V_\ell, \psi_\ell).
\]

For all \(\tau \in \text{Gal}(K_\ell/K)\) consider the following twists of the monodromy group.

\[
(G_{\ell,K}^{\text{alg}})^\tau := \{ g \in G_{\ell,K}^{\text{alg}} : g\beta g^{-1} = \rho_\ell(\tau)(\beta) \quad \forall \beta \in D \},
\]

\[
(G_{\ell,K,1}^{\text{alg}})^\tau := (G_{\ell,K}^{\text{alg}})^\tau \cap G_{\ell,K,1}^{\text{alg}}.
\]

Notice that we have

\[
(\mathcal{G}_{\ell,K,1}^{\text{alg}})^\tau \subseteq \text{DL}_K^\tau(A)_{\mathbb{Q}_\ell}.
\]

Let \(\tilde{\tau} \in G_K\) be a lift of \(\tau \in \text{Gal}(K_\ell/K)\) and remark that the Zariski closure of \(\rho_\ell(\tilde{\tau} G_K) = \rho_\ell(\tilde{\tau}) \rho_\ell(G_K)\) in \(\text{GSp}_{(V_\ell, \psi_\ell)}\) is \(\rho_\ell(\tilde{\tau}) G_{\ell,K}^{\text{alg}}\), where \(G_K\) is the absolute Galois group of the endomorphism field \(K_\ell\). Moreover we have the following properties \([\text{Bek}16\text{a}]\) \text{Rmk. 5.13]}:

\[
G_{\ell,K}^{\text{alg}} = \bigcup_{\tau \in \text{Gal}(K_\ell/K)} (G_{\ell,K}^{\text{alg}})^\tau,
\]

\[
G_{\ell,K,1}^{\text{alg}} = \bigcup_{\tau \in \text{Gal}(K_\ell/K)} (G_{\ell,K,1}^{\text{alg}})^\tau.
\]

By the equations (2.11) and (2.13) we get:

\[
(G_{\ell,K,1}^{\text{alg}})^\tau \subseteq \text{DL}_K^\tau(A)_{\mathbb{Q}_\ell}.
\]

In addition, \(\rho_\ell(\tilde{\tau}) G_{\ell,K_\ell}^{\text{alg}} = (G_{\ell,K}^{\text{alg}})^\tau\) for all \(\tau\). Hence, we obtain the following equality

\[
(G_{\ell,K,1}^{\text{alg}})^{\text{id}} = G_{\ell,K,1}^{\text{alg}}
\]

and the following natural isomorphism:

\[
G_{\ell,K}^{\text{alg}} / (G_{\ell,K}^{\text{alg}})^{\text{id}} \cong \text{Gal}(K_\ell/K).
\]

Since \(\text{DL}_K^{\text{id}}(A) = \text{DL}_{K_\ell}(A)\) (from Remark 2.3 (4)), we obtain the inclusion

\[
G_{\ell,K,1}^{\text{alg}} \subseteq \text{DL}_{K_\ell}(A)_{\mathbb{Q}_\ell},
\]

and the following natural isomorphisms:

\[
G_{\ell,K,1}^{\text{alg}} / (G_{\ell,K,1}^{\text{alg}})^{\text{id}} \cong \text{DL}_K(A) / \text{DL}_K^{\text{id}}(A) \cong \text{Gal}(K_\ell/K).
\]
2.4. The maps $d$ and $d^{id}$. From the isomorphisms (2.16) and (2.18) we obtain the following exact sequences:

\[(2.19)\quad 1 \to \pi_0((G_{\ell,K}^{\text{alg}})^{id}) \to \pi_0(G_{\ell,K}^{\text{alg}}) \to \text{Gal}(K_\ell/K) \to 1\]

\[(2.20)\quad 1 \to \pi_0((G_{\ell,K,1}^{\text{alg}})^{id}) \to \pi_0(G_{\ell,K,1}^{\text{alg}}) \to \text{Gal}(K_\ell/K) \to 1\]

A natural question that arises is to know under which conditions exact sequences (2.19) and (2.20) split as direct products or as semi-direct products. Furthermore, it will be interesting to explore the consequences of having such exact sequences.

Consider the natural homomorphisms:

\[(2.21)\quad d : \pi_0(G_{\ell,K,1}^{\text{alg}}) \to \pi_0(DL_K(A)),\]

\[(2.22)\quad d^{id} : \pi_0((G_{\ell,K,1}^{\text{alg}})^{id}) \to \pi_0(DL_K^{id}(A)).\]

If the homomorphism (2.21) is an epimorphism then by the inclusion (2.11) the homomorphism (2.22) is an epimorphism. Serre proved that there is natural isomorphism (cf. [BK15, Thm. 3.3], [BK16, Thm. 4.8])

\[(2.23)\quad \pi_0(G_{\ell,K,1}^{\text{alg}}) \cong \pi_0(G_{\ell,K}^{\text{alg}}).\]

Hence the homomorphism (2.21) gives a natural homomorphism

\[(2.24)\quad \pi_0(G_{\ell,K}^{\text{alg}}) \to \pi_0(DL_K(A)).\]

Under the inclusion (2.11) we have the following commutative diagram:

\[
\begin{array}{ccccccccc}
1 & \to & \pi_0((G_{\ell,K,1}^{\text{alg}})^{id}) & \overset{j^e}{\to} & \pi_0(G_{\ell,K,1}^{\text{alg}}) & \overset{\pi_\ell}{\to} & \text{Gal}(K_\ell/K) & \to & 1 \\
\downarrow d^{id} & & \downarrow d & & \downarrow \pi & & \downarrow 1 \\
1 & \to & \pi_0(DL_K^{id}(A)) & \overset{j}{\to} & \pi_0(DL_K(A)) & \overset{\pi}{\to} & \text{Gal}(K_\ell/K) & \to & 1 \\
\end{array}
\]

Diagram 1.

2.5. Mumford–Tate conjecture and abelian varieties fully of Lefschetz type. Consider a complex abelian variety $A$, then we can attach to it a reductive algebraic group defined over $\mathbb{Q}$ - the Mumford–Tate group associated to $A$. To define it, we need to introduce the following morphism:

\[(2.25)\quad h : \text{Res}_{\mathbb{C}/\mathbb{R}} G_{m,\mathbb{C}} \to \text{GL}(V)_\mathbb{R} \quad z \mapsto h(z),\]

where $V = H_1(A, \mathbb{Q})$ is the first homology group. We can determine the Hodge structure decomposition of $V_\mathbb{C}$, which in turn, is equivalent to the data of the linear map $h(z)$. Indeed, we know that

\[(2.26)\quad V_\mathbb{C} = H_1(A, \mathbb{C}) = V^{-1,0} \oplus V^{0,-1},\]
and
\[
\begin{align*}
\quad h(z) & : V^{-1,0} \oplus V^{0,-1} \rightarrow V^{-1,0} \oplus V^{0,-1}, \\
\quad v & \mapsto zv \quad \text{for } v \in V^{-1,0}, \\
\quad v & \mapsto \bar{z}v \quad \text{for } v \in V^{0,-1}.
\end{align*}
\] (2.27)

The Mumford–Tate group $MT(A)$ is then defined as the smallest algebraic group, defined over $\mathbb{Q}$, such that $h$ factors through $MT(A) \mathbb{R}$. Notice that the Mumford–Tate group is connected. There is smaller algebraic group called the Hodge group associated to $A$ and defined by
\[
(2.28) \quad Hg(A) = (MT(A) \cap SL(V))^\circ.
\]

Both groups are related via the following equality:
\[
(2.29) \quad MT(A) = \mathbb{G}_m \cdot Hg(A).
\]

Moreover, the Hodge group $Hg(A)$ is contained in $C_D(\text{Sp}_h(V))$. Because $Hg(A)$ is connected we have the following inclusion:
\[
(2.30) \quad Hg(A) \subset L(A).
\]

**Definition 2.4.** The abelian variety $A$ is fully of Lefschetz type if and only if
1. $Hg(A) = L(A)$,
2. the Mumford–Tate Conjecture hold for $A$.

**Remark 2.5.** By work of Deligne, Piatetski-Šapiro and Borovoǐ we know that
\[
(2.31) \quad G_{\ell,K,1}^{\text{alg}} \subseteq Hg(A)_{\mathbb{Q}_\ell} \subseteq L(A)_{\mathbb{Q}_\ell} = DL_K(A)_{\mathbb{Q}_\ell}.
\]

Therefore $A$ is fully of Lefschetz type if and only if the following isomorphisms:
\[
(2.32) \quad G_{\ell,K,1}^{\text{alg}} \circ \cong Hg(A)_{\mathbb{Q}_\ell} \cong L(A)_{\mathbb{Q}_\ell},
\]

hold for every prime number $\ell$. The first isomorphisms in (2.32) is the Mumford–Tate conjecture. We refer the readers to the following survey paper [Moo17] for further details about the Mumford–Tate conjecture.

3. **Proof of the main result and further applications**

3.1. **Preliminaries.** The goal of this section is to determine $\pi_0(DL_K(A))$ in terms of the endomorphism field extension $K_e$. In order to do so, we need to consider two possibilities: either $DL_K^K(\mathbb{C})$ is connected or not.

**Remark 3.1.** For a fixed embedding $\iota : \mathbb{Q}_\ell \hookrightarrow \mathbb{C}$ recall that we have from the work of Serre and [FKRS12] Lem. 2.8, the following isomorphisms:
\[
(3.1) \quad \pi_0(G_{\ell,K}^{\text{alg}}) \cong \pi_0(G_{\ell,K,1}^{\text{alg}}) \cong \pi_0(G_{\ell,K,1,C}^{\text{alg}}) \cong \pi_0(ST_K(A)).
\]

where $ST_K(A)$ is a maximal compact subgroup of $G_{\ell,K,1,C}^{\text{alg}} = G_{\ell,K,1,C}^{\text{alg}}(\mathbb{C})$.

In [Ser12] Chap. 8 Serre stated the algebraic Sato–Tate conjecture which was later on developed in [BK15] Conj. 2.1 and [BK16] Conj. 5.1 as follows.
Conjecture 3.2. (Algebraic Sato–Tate conjecture)
(a) There is a natural-in-$K$ reductive algebraic group $\text{AST}_K(A)$ defined over $\mathbb{Q}$ such that $\text{AST}_K(A) \in \text{Sp}(V,\psi)$ and for each prime number $\ell$ there is a natural-in-$K$ monomorphism of group schemes:

\[
\text{ast}_{\ell,K} : G_{\ell,K,1}^{\text{alg}} \hookrightarrow \text{AST}_K(A)_{\mathbb{Q}_\ell}.
\]

(b) The map \(\text{ast}_{\ell,K}\) is an isomorphism:

\[
\text{ast}_{\ell,K} : G_{\ell,K,1}^{\text{alg}} \cong \text{AST}_K(A)_{\mathbb{Q}_\ell}.
\]

Definition 3.3. The Sato–Tate group $\text{ST}_K(A)$ is a maximal compact subgroup of $\text{AST}_K(A)(\mathbb{C})$.

Remark 3.4. Assume Mumford–Tate conjecture for the abelian variety $A$. By \cite{CC}, Algebraic Sato–Tate conjecture holds for $A$. Hence under the assumption that $d : \pi_0(G_{\ell,K,1}^{\text{alg}}) \rightarrow \pi_0(\text{DL}_K(A))$ is an isomorphism and \cite{BK15} Prop. 3.5, we have the following isomorphisms:

\[
\pi_0(\text{AST}_K(A)) \cong \pi_0(\text{ST}_K(A)) \cong \pi_0(G_{\ell,K,1}^{\text{alg}}) \cong \pi_0(\text{DL}_K(A)).
\]

Remark 3.5. Assume that $A$ is fully of Lefschetz type and $C_D(\text{Sp}(V,\psi))$ is connected i.e. $\mathcal{L}(A) = C_D(\text{Sp}(V,\psi))$. Then by \cite{CC} and \cite{BK15} Cor. 9.9 the equality \(\mathcal{L}(A)\) also holds.

We will determine the component group of the twisted Lefschetz group $\text{DL}_K(A)$ in terms of the component group of $\text{DL}_K^{\text{id}}(A)$ and the Galois group $\text{Gal}(K_e/K)$. Firstly, we need to establish a relation between the component groups $\pi_0(\text{DL}_K(A))$ and $\pi_0(\text{DL}_K^{\text{id}}(A))$ knowing that $\text{DL}_K(A)^{\text{\circ}} = \text{DL}_K^{\text{id}}(A)^{\text{\circ}}$. Secondly, from \cite{BK16} Thm. 4.6 and equality \(\mathcal{L}(A)\) we have the following isomorphisms:

\[
G_{\ell,K,1}^{\text{alg}} / G_{\ell,K,1}^{\text{id}} \cong G_{\ell,K,1}^{\text{alg}} / G_{\ell,K,1}^{\text{alg}} \cong G_{\ell,K}^{\text{alg}} / G_{\ell,K}^{\text{alg}} \cong \text{Gal}(K_e/K),
\]

where $G_{\ell,K,1}^{\text{id}} = \{g \in G_{\ell,K,1}^{\text{alg}} \mid g\beta g^{-1} = \beta \quad \forall \beta \in D\}$.

3.2. The main result. The goal of this section is to prove Theorem \cite{E11}. Let us consider an abelian variety defined over a number field $K$. Firstly, observe that if $s_\ell$ is a splitting homomorphism of $\pi_\ell$ in Diagram \ref{diagram} then $d \circ s_\ell$ is a splitting homomorphism of $\pi$. If the homomorphism $d$ in Diagrams \ref{diagram} is an isomorphism and $\sigma_\ell$ is a splitting homomorphism of $j_\ell$ then, $\sigma := d^{\text{id}} \circ \sigma_\ell \circ d^{-1}$ is a splitting homomorphism of $j$. Until the end of Section \ref{section} we will work with splitting $\sigma_\ell$ rather than $s_\ell$ because in principal our work concerns non-abelian groups (cf. Appendix \ref{appendix}).
Theorem 3.6. Consider an abelian variety $A$ defined over a number field $K$. Assume that in Diagram 2 homomorphism $d$ is an epimorphism and $\sigma_\ell$ splits $j_\ell$. Then, there exists $\sigma$ that splits $j$ and there is the following direct product decomposition:

$$\pi_0(\text{DL}_K(A)) = \pi_0(\text{DL}_{\text{id}}K(A)) \times \text{Gal}(K_\ell/K).$$

Proof. Assume that $d$ is an epimorphism and that $\sigma_\ell$ splits $j_\ell$. Hence, for every $\beta \in \pi_0(\text{DL}_K(A))$ there is $\alpha \in \pi_0(\text{G}_{\text{alg}}\ell,K,1)$ such that $d(\alpha) = \beta$. Define the following map:

$$\sigma : \pi_0(\text{DL}_K(A)) \to \pi_0(\text{DL}_{\text{id}}K(A))$$

$$\beta \mapsto \sigma(\beta) := d^{\text{id}} \circ \sigma_\ell(\alpha).$$

We will prove that $\sigma(\beta)$ does not depend on the choice of $\alpha$. $\sigma$ is a well defined homomorphism which splits $j$. Indeed, if $d(\alpha) = 0$, then $\pi(d(\alpha)) = 0$. From Diagram 3 we read that $\pi_\ell(\alpha) = 0$. Hence $\alpha = j_\ell(\alpha_0)$ for some $\alpha_0 \in \pi_0(\text{G}_{\text{alg}}\ell,K,1)^{\text{id}}$. Hence $j \circ d^{\text{id}} \circ \sigma_\ell(\alpha) = j \circ d^{\text{id}} \circ \sigma_\ell(\alpha_0) = j \circ d^{\text{id}}(\pi_\ell(\alpha_0)) = d \circ j_\ell(\alpha_0) = d(\alpha) = 0$. Because $j$ is a monomorphism then $\sigma(0) = d^{\text{id}} \circ \sigma_\ell(\alpha) = 0$. Hence $\sigma(\beta)$ does not depend on $\alpha$. In the same way as above we prove that $\sigma$ is a homomorphism.

Let us verify that $\sigma$ splits $j$. Indeed from the property of Diagram 3 and the assumption that $d$ is an epimorphism, it is clear that $d^{\text{id}}$ is an epimorphism. Since
\(\sigma_\ell\) is an epimorphism, \(\sigma\) is an epimorphism as well. Moreover, from the definition of \(\sigma\) we obtain \(\sigma \circ d = d^{id} \circ \sigma_\ell\). Now observe that

\[
\sigma \circ j \circ d^{id} \circ \sigma_\ell = \sigma \circ d \circ j_\ell \circ \sigma_\ell = d^{id} \circ \sigma_\ell \circ j_\ell \circ \sigma_\ell = d^{id} \circ \sigma_\ell.
\]

Therefore, for all \(\alpha \in \pi_0(G_{l,K,1})\) we obtain that

\[
\sigma \circ j \circ d^{id} \circ \sigma_\ell(\alpha) = d^{id} \circ \sigma_\ell(\alpha),
\]

and hence, since \(d^{id} \circ \sigma_\ell\) is an epimorphism we obtain that \(\sigma \circ j = \text{id}\). This implies that \(\sigma\) splits \(j\). Thus, the homomorphism

\[
(\sigma, \pi) : \pi_0(\text{DL}_K(A)) \to \pi_0(\text{DL}^{id}_K(A)) \times \text{Gal}(K_e/K)
\]

gives the splitting (3.6).

We can deduce Theorem 1.1 from Theorem 3.6. In addition, notice that under the assumption that \(A\) is fully of Lefschetz type we have the following isomorphisms

\[
(3.8) \quad \pi_0(\text{DL}_K(A)) \simeq \pi_0(G_{l,K,1}^{\text{alg}}) \simeq \pi_0(\text{AST}_K(A)_{\mathbb{Q}_\ell}) \simeq \pi_0(\text{ST}_K(A)).
\]

Notice also that \(\pi_0(\text{DL}_K(A)_{\mathbb{Q}_\ell}) = \pi_0(\text{DL}_K(A))\). Moreover, Theorem 3.6 allows us to obtain the following corollary which enable us to determine concretely the component group of \(\text{DL}_K(A)_{\mathbb{Q}_\ell}\) in terms the component group of \(\text{DL}^{id}_K(A)\) and \(\text{Gal}(K_e/K)\).

**Corollary 3.7.** Under the hypotheses of Theorem 3.6 and the assumption that \(A\) is fully of Lefschetz type, we have the following isomorphisms:

\[
(3.9) \quad \pi_0(\text{DL}_K(A)) \simeq \pi_0(\text{ST}_K(A)) \simeq \pi_0(\text{DL}^{id}_K(A)) \times \text{Gal}(K_e/K).
\]

In particular if \(\text{DL}^{id}_K(A)\) is connected then we have

\[
(3.10) \quad \pi_0(\text{DL}_K(A)) \simeq \pi_0(\text{ST}_K(A)) \simeq \text{Gal}(K_e/K),
\]

We can reformulate the above-mentioned corollary in terms of the endomorphism algebra of a simple abelian variety:

**Corollary 3.8.** Under the hypotheses of Theorem 3.6 an the assumption that \(A\) is a simple abelian variety, fully of Lefschetz type, we have the following isomorphisms:

1. If \(A\) is of type I, II or IV - in the sense of Albert’s classification - we have:

\[
(3.11) \quad \pi_0(\text{DL}_K(A)) \simeq \pi_0(\text{ST}_K(A)) \simeq \text{Gal}(K_e/K).
\]

2. If \(A\) is of type III - in the sense of Albert’s classification - we have:

\[
(3.12) \quad \pi_0(\text{DL}_K(A)) \simeq \pi_0(\text{ST}_K(A)) \simeq \pi_0(\text{DL}^{id}_K(A)) \times \text{Gal}(K_e/K).
\]

**Proof.** Let us remark that the group \(\text{DL}^{id}_K(A)\) is connected for all simple abelian varieties of type I, II and IV and it is not connected for simple abelian varieties of type III, in the sense of Albert’s classification. For instance, see the table at the end of [Mil99] §2 for further details. Indeed, \(\text{DL}^{id}_K(A) = C_D(\text{Sp})\) which corresponds to the reductive group \(S(A)\) according to the notations of Milne. In that setting, when the group \(\text{DL}^{id}_K(A)\) is not connected we know that \(\text{DL}^{id}_K(A)\) is several copies of the orthogonal group. More precisely, recall that in type III, the dimension of \(A\) is given by \(\dim(A) = g + 2eh\), where \(h\) is the relative dimension of \(A\). Therefore, the group \(\text{DL}^{id}_K(A)\) is given by \(e\) copies of the orthogonal group \(O_{2h}\). \(\Box\)
3.3. A conjecture of Serre and further results. The purpose of this section is to illustrate the fact that we can obtain similar diagrams as Diagram 1 but with the algebraic Sato–Tate group and the Sato–Tate group of the abelian variety \( A \) introduced in Section 3.1. Similarly to (2.10) we define the following objects for every \( \tau \in \text{Gal}(K_e/K) \):

\[
\begin{align*}
\text{AST}_K^\tau(A) &:= \text{AST}_K(A) \cap \text{DL}_K^\tau(A), \\
\text{ST}_K^\tau(A) &:= \text{ST}_K(A) \cap \text{AST}_K^\tau(A)(\mathbb{C}).
\end{align*}
\]

Comparing this definition with the one of \( \text{DL}_K(A) \) we obtain:

\[
\begin{align*}
\text{AST}_K(A) &= \bigsqcup_{\tau \in \text{Gal}(K_e/K)} \text{AST}_K^\tau(A), \\
\text{ST}_K(A) &= \bigsqcup_{\tau \in \text{Gal}(K_e/K)} \text{ST}_K^\tau(A).
\end{align*}
\]

Recall from Conjecture 3.2 that, for abelian varieties \( A \) defined over a number field \( K \), it is known that for every primer \( \ell \), there exists a natural-in-\( K \)-reductive algebraic group \( \text{AST}_K(A) \subset \text{Sp}(V,\psi) \) defined over \( \mathbb{Q} \) and a natural-in-\( K \) monomorphism \( \text{ast}_{\ell,K} : G_{\ell,K,1}^{\text{alg}} \to \text{AST}_K(A)_{\mathbb{Q}} \) (see (3.2) in Conjecture 3.2). Hence we have the following commutative diagram.

\[
\begin{array}{c}
1 \longrightarrow \pi_0((G_{\ell,K,1}^{\text{alg}})^{\text{id}}) \overset{j}{\longrightarrow} \pi_0(G_{\ell,K,1}^{\text{alg}}) \overset{\pi_\ell}{\longrightarrow} \text{Gal}(K_e/K) \longrightarrow 1 \\
\downarrow d^{\text{id}} \quad \downarrow d \quad \quad \quad \downarrow \cong \\
1 \longrightarrow \pi_0(\text{AST}_{K}^{\text{id}}(A)) \overset{j}{\longrightarrow} \pi_0(\text{AST}_K(A)) \overset{\pi}{\longrightarrow} \text{Gal}(K_e/K) \longrightarrow 1
\end{array}
\]

Diagram 4.

Since \( \text{AST}_K(M)^{\circ}(\mathbb{C}) \) is a connected complex Lie group and any maximal compact subgroup of a connected complex Lie group is a connected real Lie group we have a natural isomorphism:

\[
\pi_0(\text{AST}_K(A)) \cong \pi_0(\text{ST}_K(A)).
\]

This isomorphism allow us to obtain the following commutative diagram:

\[
\begin{array}{c}
1 \longrightarrow \pi_0((G_{\ell,K,1}^{\text{alg}})^{\text{id}}) \overset{j_{\ell}}{\longrightarrow} \pi_0(G_{\ell,K}^{\text{alg}}) \overset{\pi_\ell}{\longrightarrow} \text{Gal}(K_e/K) \longrightarrow 1 \\
\downarrow d^{\text{id}} \quad \downarrow d \quad \quad \quad \downarrow \cong \\
1 \longrightarrow \pi_0(\text{ST}_{K}^{\text{id}}(A)) \overset{j}{\longrightarrow} \pi_0(\text{ST}_K(A)) \overset{\pi}{\longrightarrow} \text{Gal}(K_e/K) \longrightarrow 1
\end{array}
\]

Diagram 5.

Moreover, computations and the diagram in [BK16, p. 27] and [BK16 Thm. 11.8, (11.12)] allow us to obtain the following commutative diagram:
Serre conjectured [Ser94, 9.2 p. 386] that the bottom horizontal arrow in the Diagram 6 is an epimorphism and this conjecture can be rephrased as follows:

**Conjecture 3.9.** (J.-P. Serre) The following natural map is an epimorphism:

\[ d : \pi_0(G_{\ell, K}^{\text{alg}}) \to \pi_0(\text{AST}_K(A)) \]  

One can strengthen Serre’s Conjecture 3.9 to expect more:

**Conjecture 3.10.** (Weak Algebraic Sato–Tate Conjecture) The map (3.17) is an isomorphism.

**Remark 3.11.** It is obvious that the Algebraic Sato–Tate Conjecture 3.2 implies Weak Algebraic Sato–Tate Conjecture 3.10.

Similarly as for Theorem 3.6 we can determine \( \pi_0(\text{AST}_K(A)) \) (resp. \( \pi_0(\text{ST}_K(A)) \)) in terms of \( \text{Gal}(K_\ell/K) \) and \( \pi_0(\text{AST}^{\text{id}}_K(A)) \) (resp. \( \pi_0(\text{ST}^{\text{id}}_K(A)) \)).

**Theorem 3.12.** Assume that Serre’s Conjecture 3.9 holds and there exists \( \sigma_\ell \) that splits \( j_\ell \). Then, there is \( \sigma \) that splits \( j \) in each of Diagrams 4 and 5 and we have the following direct product decompositions:

\[ \pi_0(\text{AST}_K(A)) = \pi_0(\text{AST}^{\text{id}}_K(A)) \times \text{Gal}(K_\ell/K), \]  
\[ \pi_0(\text{ST}_K(A)) = \pi_0(\text{ST}^{\text{id}}_K(A)) \times \text{Gal}(K_\ell/K). \]

**Proof.** The proof is the same as the proof of Theorem 3.6. \( \square \)

**Corollary 3.13.** Assume that the Weak Algebraic Sato–Tate Conjecture 3.10 holds and that there exists \( \sigma_\ell \) splits \( j_\ell \). Then, we have the following direct products decompositions:

\[ \pi_0(\text{AST}_K(A)) = \pi_0(\text{AST}^{\text{id}}_K(A)) \times \text{Gal}(K_\ell/K), \]  
\[ \pi_0(\text{ST}_K(A)) = \pi_0(\text{ST}^{\text{id}}_K(A)) \times \text{Gal}(K_\ell/K). \]

**Proof.** It follows from Theorem 3.12. \( \square \)

**APPENDIX A. REMARKS ON SPLITTING EXACT SEQUENCES**

The goal of this appendix is to establish some remarks regarding the splitting behavior of an exact sequence in the case of both, arbitrary groups and abelian groups. Recall that in section 3.2 we studied this behavior for non-abelian groups.
A.1. Arbitrary groups. Consider the following exact sequence of groups with a splitting homomorphism $\sigma$ i.e. $\sigma \circ j = \text{Id}_K$.

$$1 \longrightarrow K \xleftarrow{j} G \xrightarrow{\pi} C \longrightarrow 1$$

Diagram 7.

Splitting $\sigma$ gives natural isomorphism:

$$\begin{equation}
(s, \pi) : G \rightarrow K \times C \\
g \mapsto (\sigma(g), \pi(g)).
\end{equation}
$$

(A.1)

Indeed, if $(\sigma(g), \pi(g)) = (1, 1)$ then there is $k \in K$ such that $j(k) = g$. Hence $1 = \sigma(g) = \sigma j(k) = k$. Therefore $g = 1$ so $(s, \pi)$ is a monomorphism.

On the other hand take any $(k, c) \in K \times C$ and take $g \in G$ such that $\pi(g) = c$. Then one checks that

$$\begin{equation}
(s, \pi)(j(k) g j(\sigma(g^{-1}))) = (k, c).
\end{equation}
$$

Hence $(s, \pi)$ is an epimorphism.

Observe that splitting $s$ of the form

$$1 \longrightarrow K \xleftarrow{j} G \xrightarrow{\pi} C \longrightarrow 1$$

Diagram 8.

provides the action of the group $C$ on the group $K$:

$$\begin{equation}
C \times K \rightarrow K \\
(c, k) \mapsto c(k) := j^{-1}(s(c) j(k) s(c^{-1}))
\end{equation}
$$

(A.2)

Identifying $j(K)$ with $K$ we can write this action in a simpler way:

$$c(k) := s(c) k s(c^{-1}).$$

This action leads directly to the following isomorphism:

$$\begin{equation}
(j, s) : K \ltimes C \rightarrow G \\
(k, c) \mapsto (j, s)(k, c) := j(k) s(c).
\end{equation}
$$

(A.3)

Indeed $j(k) s(c) = 1$ implies $1 = \pi(j(k) s(c)) = \pi s(c)) = c$. Hence $j(k) = 1$ so $k = 1$. Therefore $(j, s)$ is a monomorphism.

Take $g \in G$ and set $c := \pi(g)$. Observe that $\pi(g s(c^{-1}) = 1$. Hence there is $k \in K$ such that $j(k) = g s(c^{-1})$ so $j(k) s(c) = g$. Therefore $(j, s)$ is an epimorphism.
Example A.1. Let $F$ be a field. Consider the following well-known exact sequence with splitting $s$ defined as follows:

$$
1 \rightarrow \text{SL}_n(F) \rightarrow \text{GL}_n(F) \xrightarrow{\text{det}} F^* \rightarrow 1
$$

Diagram 9.

$$s(a) := \begin{bmatrix}
a & 0 & 0 & \cdots & 0 \\
0 & 1 & 0 & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & 1
\end{bmatrix}
$$

It is known that the exact sequence does not split as a direct product for $n > 1$. Hence there is no splitting homomorphism $\sigma$ of the embedding $\text{SL}(F) \rightarrow \text{GL}(F)$.

A.2. **Abelian groups.** If the group $G$ in the following exact sequence is abelian, hence $K$ and $C$ are also abelian, then the existence of the splitting homomorphism $\sigma$ is equivalent to the existence of splitting homomorphism $s$.

$$
1 \rightarrow K \xrightarrow{j} G \xrightarrow{\pi} C \rightarrow 1
$$

Diagram 10.

Indeed if $s$ is a splitting homomorphism then we define:

$$\sigma(g) := j^{-1}(g s(\pi(g^{-1}))).$$

Direct checking shows that $\sigma$ is a well-defined splitting homomorphism of $j$.

On the other hand if $\sigma$ is a splitting homomorphism of $j$ then define $s$ as follows. For $c \in C$ take $g$ such that $\pi(g) = c$. Then put:

$$s(c) := g j(\sigma(g^{-1})).$$

This definition does not depend on the choice of $g$. Indeed if $\pi(g') = c$ then

$$g j(\sigma(g^{-1})) = g' j(\sigma((g')^{-1}))$$

is equivalent to

$$j(\sigma(g^{-1}g')) = g^{-1}g'$$

which holds true because $\pi(g^{-1}g') = 1$. Hence $s$ is well defined. Direct checking shows that $s$ is a splitting homomorphism of $\pi$. 
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