PROJECTIVE DIMENSION IN FILTRATED K-THEORY

RASMUS BENTMANN

Abstract. Under mild assumptions, we characterise modules with projective resolutions of length $n \in \mathbb{N}$ in the target category of filtrated K-theory over a finite topological space in terms of two conditions involving certain $\text{Tor}$-groups. We show that the filtrated $K$-theory of any separable $C^*$-algebra over any topological space with at most four points has projective dimension 2 or less. We observe that this implies a universal coefficient theorem for rational equivariant KK-theory over these spaces. As a contrasting example, we find a separable $C^*$-algebra in the bootstrap class over a certain five-point space, the filtrated $K$-theory of which has projective dimension 3. Finally, as an application of our investigations, we exhibit Cuntz-Krieger algebras which have projective dimension 2 in filtrated K-theory over their respective primitive spectrum.

1. Introduction

A far-reaching classification theorem in [7] motivates the computation of Eberhard Kirchberg’s ideal-related Kasparov groups $KK(X; A, B)$ for separable $C^*$-algebras $A$ and $B$ over a non-Hausdorff topological space $X$ by means of $K$-theoretic invariants. We are interested in the specific case of finite spaces here. In [9][10], Ralf Meyer and Ryszard Nest laid out a theoretic framework that allows for a generalisation of Jonathan Rosenberg’s and Claude Schochet’s universal coefficient theorem [16] to the equivariant setting. Starting from a set of generators of the equivariant bootstrap class, they define a homology theory with a certain universality property, which computes $KK(X)$-theory via a spectral sequence. In order for this universal coefficient spectral sequence to degenerate to a short exact sequence, it remains to be checked by hand that objects in the range of the homology theory admit projective resolutions of length 1 in the Abelian target category.

Generalising earlier results from [3][11][15], the verification of the above-mentioned condition for filtrated $K$-theory was achieved in [2] for the case that the underlying space is a disjoint union of so-called accordion spaces. A finite connected $T_0$-space $X$ is an accordion space if and only if the directed graph corresponding to its specialisation pre-order is a Dynkin quiver of type $A$. Moreover, it was shown in [2][10] that, if $X$ is a finite $T_0$-space which is not a disjoint union of accordion spaces, then the projective dimension of filtrated $K$-theory over $X$ is not bounded by 1 and objects in the equivariant bootstrap class are not classified by filtrated $K$-theory. The assumption of the separation axiom $T_0$ is not a loss of generality in this context (see [11] §2.5).

There are two natural approaches to tackle the problem arising for non-accordion spaces: one can either try to refine the invariant—this has been done with some success in [10] and [1]; or one can hold onto the invariant and try to establish projective resolutions of length 1 on suitable subcategories or localisations of the category $\mathcal{R}(X)$, in which $X$-equivariant KK-theory is organised. The latter is the course we pursue in this note. We state our results in the next section.

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2. Statement of Results

The definition of filtrated K-theory and related notation are recalled in [3].

Proposition 1. Let $X$ be a finite topological space. Assume that the ideal $\mathcal{N}T_{\text{nil}} \subset \mathcal{N}T^+(X)$ is nilpotent and that the decomposition $\mathcal{N}T^+(X) = \mathcal{N}T_{\text{nil}} \rtimes \mathcal{N}T_{ss}$ holds. Fix $n \in \mathbb{N}$. For an $\mathcal{N}T^+(X)$-module $M$, the following assertions are equivalent:

(i) $M$ has a projective resolution of length $n$.

(ii) The Abelian group $\text{Tor}_{n+1}^{\mathcal{N}T^+(X)}(\mathcal{N}T_{ss}, M)$ is free and the Abelian group $\text{Tor}_{n+1}^{\mathcal{N}T^+(X)}(\mathcal{N}T_{ss}, M)$ vanishes.

The basic idea of this paper is to compute the Tor-groups above by writing down projective resolutions for the fixed right-module $\mathcal{N}T_{ss}$.

Let $Z_m$ be the $(m+1)$-point space on the set $\{1, \ldots, m+1\}$ such that $Y \subseteq Z_m$ is open if and only if $Y \supseteq m + 1$ or $Y = \emptyset$. A $C^*$-algebra over $Z_m$ is a $C^*$-algebra $A$ with a distinguished ideal such that the corresponding quotient decomposes as a direct sum of $m$ orthogonal ideals. Let $S$ be the set $\{1, 2, 3, 4\}$ equipped with the topology $\{\emptyset, 6, 4, 234, 234, 324, 234\}$, where we write $24 := \{2, 4\}$ etc. A $C^*$-algebra over $S$ is a $C^*$-algebra together with two distinguished ideals which need not satisfy any further conditions; see [11] Lemma 2.35.

Proposition 2. Let $X$ be a topological space with at most 4 points. Let $M = \text{FK}(A)$ for some $C^*$-algebra $A$ over $X$. Then $M$ has a projective resolution of length 2 and $\text{Tor}_2^{\mathcal{N}T^+(X)}(\mathcal{N}T_{ss}, M) = 0$.

Moreover, we can find explicit formulas for $\text{Tor}_1^{\mathcal{N}T^+(X)}(\mathcal{N}T_{ss}, M)$; for instance, $\text{Tor}_1^{\mathcal{N}T^+(\mathbb{Z}_2)}(\mathcal{N}T_{ss}, M)$ is isomorphic to the homology of the complex

\[
\begin{array}{cccccc}
3 & M(j4) & \overset{3}{\underset{k=1}{\bigoplus}} M(234 \setminus k) & (i & i + i) & M(1234) \ .
\end{array}
\]

A similar formula holds for the space $S$; see [10].

The situation simplifies if we consider rational KK($X$)-theory, whose morphism groups are given by $\text{KK}(X; A, B) \otimes \mathbb{Q}$; see [6]. This is a $\mathbb{Q}$-linear triangulated category which can be constructed as a localisation of $\text{Rf}(X)$; the corresponding localisation of filtrated K-theory is given by $A \mapsto \text{FK}(A) \otimes \mathbb{Q}$ and takes values in the category of modules over the $\mathbb{Q}$-linear category $\mathcal{N}T^+(X) \otimes \mathbb{Q}$.

Proposition 3. Let $X$ be a topological space with at most 4 points. Let $A$ and $B$ be $C^*$-algebras over $X$. If $A$ belongs to the equivariant bootstrap class $\mathcal{B}(X)$, then there is a natural short exact universal coefficient sequence

\[
\text{Ext}^1_{\mathcal{N}T^+(X) \otimes \mathbb{Q}}(\text{FK}_{+1}(A) \otimes \mathbb{Q}, \text{FK}_{+}(B) \otimes \mathbb{Q}) \to \text{KK}_{+}(X; A, B) \otimes \mathbb{Q}
\]

\[
\to \text{Hom}_{\mathcal{N}T^+(X) \otimes \mathbb{Q}}(\text{FK}_{+}(A) \otimes \mathbb{Q}, \text{FK}_{+}(B) \otimes \mathbb{Q}) .
\]

In [6], a long exact sequence is constructed which in our setting, by the above proposition, reduces the computation of $\text{KK}_{+}(X; A, B)$, up to extension problems, to the computation of a certain torsion theory $\text{KK}_{+}(X; A, B; \mathbb{Q}/\mathbb{Z})$.

The next proposition says that the upper bound of 2 for the projective dimension in Proposition 2 does not hold for all finite spaces.
Proposition 4. There is an $N^T\ast(Z_4)$-module $M$ of projective dimension 2 with free entries and $\text{Tor}_2^{N^T}(N^T_{ss}, M) \neq 0$. The module $M \otimes \mathbb{Z}/k$ has projective dimension 3 for every $k \in \mathbb{N}_{\geq 2}$. Both $M$ and $M \otimes \mathbb{Z}/k$ can be realised as the filtrated K-theory of an object in the equivariant bootstrap class $B(X)$.

As an application of Proposition 2, we investigate in §10 the obstruction term $\text{Tor}_1^{N^T}(N^T_{ss}, \text{FK}(A))$ for certain Cuntz-Krieger algebras with four-point primitive ideal spaces. We find:

Proposition 5. There is a Cuntz-Krieger algebra with primitive ideal space homeomorphic to $Z_4$ which fulfills Cuntz’s condition (II) and has projective dimension 2 in filtrated K-theory over $Z_3$. The analogous statement for the space $S$ holds as well.

The relevance of this observation lies in the following: if Cuntz-Krieger algebras had projective dimension at most 1 in filtrated K-theory over their primitive ideal space, this would lead to a strengthened version of Gunnar Restorff’s classification result [14] with a proof avoiding reference to results from symbolic dynamics.

3. Preliminaries

Let $X$ be a finite topological space. A subset $Y \subseteq X$ is called locally closed if it is the difference $U \setminus V$ of two open subsets $U$ and $V$ of $X$; in this case, $U$ and $V$ can always be chosen such that $V \subseteq U$. The set of locally closed subsets of $X$ is denoted by $\mathcal{L}\mathcal{C}(X)$. By $\mathcal{L}\mathcal{C}(X)^\ast$, we denote the set of non-empty, connected locally closed subsets of $X$.

Recall from [11] that a C*-algebra over $X$ is pair $(A, \psi)$ consisting of a $C^*$-algebra $A$ and a continuous map $\psi$: $\text{Prim}(A) \to X$. A C*-algebra $(A, \psi)$ over $X$ is called tight if the map $\psi$ is a homeomorphism. A C*-algebra $(A, \psi)$ over $X$ comes with distinguished subquotients $A(Y)$ for every $Y \in \mathcal{L}\mathcal{C}(X)$.

There is an appropriate version $\text{KK}(X)$ of bivariant K-theory for $C^*$-algebras over $X$ (see [7][11]). The corresponding category, denoted by $\mathcal{A}\mathcal{R}(X)$, is equipped with the structure of a triangulated category (see [12]); moreover, there is an equivariant analogue $\mathcal{B}(X) \subseteq \mathcal{A}\mathcal{R}(X)$ of the bootstrap class $\mathcal{B}$.

Recall that a triangulated category comes with a class of distinguished candidate triangles. An anti-distinguished triangle is a candidate triangle which can be obtained from a distinguished triangle by reversing the sign of one of its three morphisms. Both distinguished and anti-distinguished triangles induce long exact Hom-sequences.

As defined in [10], for $Y \in \mathcal{L}\mathcal{C}(X)$, we let $\text{FK}_Y(A) := K_\ast(A(Y))$ denote the $\mathbb{Z}/2$-graded K-group of the subquotient of $A$ associated to $Y$. Let $\mathcal{N}T(X)$ be the $\mathbb{Z}/2$-graded pre-additive category whose object set is $\mathcal{L}\mathcal{C}(X)$ and whose space of morphisms from $Y$ to $Z$ is $\mathcal{N}T_\ast(Y, Z)$ - the $\mathbb{Z}/2$-graded Abelian group of all natural transformations $\text{FK}_Y \Rightarrow \text{FK}_Z$. Let $\mathcal{N}T^\ast(X)$ be the full subcategory with object set $\mathcal{L}\mathcal{C}(X)^\ast$. We often abbreviate $\mathcal{N}T^\ast(X)$ by $\mathcal{N}T^\ast$.

Every open subset of a locally closed subset of $X$ gives rise to an extension of distinguished subquotients. The corresponding natural maps in the associated six-term exact sequence yield morphisms in the category $\mathcal{N}T$, which we briefly denote by $i$, $r$ and $\delta$.

A (left-)module over $\mathcal{N}T(X)$ is a grading-preserving, additive functor from $\mathcal{N}T(X)$ to the category $\text{Ab}^{\mathbb{Z}/2}$ of $\mathbb{Z}/2$-graded Abelian groups. A morphism of $\mathcal{N}T(X)$-modules is a natural transformation of functors. Left-modules over $\mathcal{N}T^\ast(X)$ are defined similarly. By $\text{Mod}(\mathcal{N}T^\ast(X))_\pi$ we denote the category of countable $\mathcal{N}T^\ast(X)$-modules.
Filtrated K-theory is the functor $\mathfrak{R}(X) \to \mathfrak{Mod}(\mathcal{N}T^*(X))$ taking a C*-algebra $A$ over $X$ to the collection $\left(K_*(A(Y))\right)_{Y \in \mathcal{L}(X)}$, with the obvious $\mathcal{N}T^*(X)$-module structure.

Let $\mathcal{N}T_{\text{nil}} \subset \mathcal{N}T^*$ be the ideal generated by all natural transformations between different objects, and let $\mathcal{N}T_{\text{ss}} \subset \mathcal{N}T^*$ be the subgroup spanned by the identity transformations $\text{id}_X$ for objects $Y \in \mathcal{L}(X)^*$. The subgroup $\mathcal{N}T_{\text{ss}}$ is in fact a subring of $\mathcal{N}T^*$ isomorphic to $\mathbb{Z}[\mathcal{L}(X)^*]$. We say that $\mathcal{N}T^*$ decomposes as semi-direct product $\mathcal{N}T^* = \mathcal{N}T_{\text{nil}} \rtimes \mathcal{N}T_{\text{ss}}$ if $\mathcal{N}T^*$ as an Abelian group is the inner direct sum of $\mathcal{N}T_{\text{nil}}$ and $\mathcal{N}T_{\text{ss}}$; see [10]. We do not know if this fails for any finite space.

We define right-modules over $\mathcal{N}T^*(X)$ as contravariant, grading-preserving, additive functors $\mathcal{N}T^*(X) \to \mathfrak{Ab}_{\mathbb{Z}/2}$. If we do not specify between left and right, then we always mean left-modules. The subring $\mathcal{N}T_{\text{ss}} \subset \mathcal{N}T^*$ is regarded as an $\mathcal{N}T^*$-right-module by the obvious action: The ideal $\mathcal{N}T_{\text{nil}} \subset \mathcal{N}T^*$ acts trivially, while $\mathcal{N}T_{\text{ss}}$ acts via right-multiplication in $\mathcal{N}T_{\text{ss}} \cong \mathbb{Z}[\mathcal{L}(X)^*]$. For an $\mathcal{N}T^*$-module $M$, we set $M_\mathfrak{ss} := M/\mathcal{N}T_{\text{nil}} \cdot M$.

For $Y \in \mathcal{L}(X)^*$ we define the free $\mathcal{N}T^*$-left-module on $Y$ by $P_Y(Z) := \mathcal{N}T(Z, Y)$ for all $Z \in \mathcal{L}(X)^*$ and similarly for morphisms $Z \to Z'$ in $\mathcal{N}T^*$. Analogously, we define the free $\mathcal{N}T^*$-right-module on $Y$ by $Q_Y(Z) := \mathcal{N}T(Z, Y)$ for all $Z \in \mathcal{L}(X)^*$. An $\mathcal{N}T^*$-left/right-module is called free if it is isomorphic to a direct sum of degree-shifted free left/right-modules on objects $Y \in \mathcal{L}(X)^*$. It follows directly from Yoneda’s Lemma that free $\mathcal{N}T^*$-left/right-modules are projective.

An $\mathcal{N}T^*$-module $M$ is called exact if the $\mathbb{Z}/2$-graded chain complexes

$$
\cdots \to M(U) \xrightarrow{Y_U} M(Y) \xrightarrow{Y\setminus U} M(Y \setminus U) \xrightarrow{Y\setminus U} M(U)[1] \to \cdots
$$

are exact for all $U, Y \in \mathcal{L}(X)$ with $U$ open in $Y$. An $\mathcal{N}T^*$-module $M$ is called exact if the corresponding $\mathcal{N}T$-module is exact (see [2]).

We use the notation $C \in \mathcal{C}$ to denote that $C$ is an object in a category $\mathcal{C}$.

In [10], the functors $F_K$ are shown to be representable, that is, there are objects $\mathcal{R}_Y \in \mathfrak{R}(X)$ and isomorphisms of functors $F_K \cong \text{KK}(X; \mathcal{R}_Y, \ldots)$. We let $F_K$ denote the stable cohomological functor on $\mathfrak{R}(X)$ represented by the same set of objects $\{\mathcal{R}_Y \mid Y \in \mathcal{L}(X)^*\}$; it takes values in $\mathcal{N}T^*$-right-modules. We warn that $\text{KK}(X; A, \mathcal{R}_Y)$ does not identify with the K-homology of $A(Y)$. By Yoneda’s lemma, we have $F_K(\mathcal{R}_Y) \cong P_Y$ and $F_K(\mathcal{R}_Y) \cong Q_Y$.

We occasionally use terminology from [10] concerning homological algebra in $\mathfrak{R}(X)$ relative to the ideal $\mathfrak{I} := \ker(F_K)$ of morphisms in $\mathfrak{R}(X)$ inducing trivial module maps on $F_K$. An object $A \in \mathfrak{R}(X)$ is called $\mathfrak{I}$-projective if $\mathfrak{I}(A, B) = 0$ for every $B \in \mathfrak{R}(X)$. We recall from [9] that $F_K$ restricts to an equivalence of categories between the subcategories of $\mathfrak{I}$-projective objects in $\mathfrak{R}(X)$ and of projective objects in $\mathfrak{Mod}(\mathcal{N}T^*(X))$. Similarly, the functor $\hat{F}_K$ induces a contravariant equivalence between the $\mathfrak{I}$-projective objects in $\mathfrak{R}(X)$ and projective $\mathcal{N}T^*$-right-modules.

4. Proof of Proposition

Recall the following result from [10].

**Lemma 1** ([10] Theorem 3.12). Let $X$ be a finite topological space. Assume that the ideal $\mathcal{N}T_{\text{nil}} \subset \mathcal{N}T^*(X)$ is nilpotent and that the decomposition $\mathcal{N}T^*(X) = \mathcal{N}T_{\text{nil}} \rtimes \mathcal{N}T_{\text{ss}}$ holds. Let $M$ be an $\mathcal{N}T^*(X)$-module. The following assertions are equivalent:

1. $M$ is a free $\mathcal{N}T^*(X)$-module.
(2) $M$ is a projective $\mathcal{NT}^*(X)$-module.

(3) $M_{ss}$ is a free Abelian group and $\text{Tor}_1^{\mathcal{NT}^*(X)}(\mathcal{NT}_{ss}, M) = 0$.

Now we prove Proposition 4. We consider the case $n = 1$ first. Choose an epimorphism $f : P \twoheadrightarrow M$ for some projective module $P$, and let $K$ be its kernel. $M$ has a projective resolution of length 1 if and only if $K$ is projective. By Lemma 1 this is equivalent to $K_{ss}$ being a free Abelian group and $\text{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, K) = 0$.

We have $\text{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, K) = 0$ if and only if $\text{Tor}_2^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, M) = 0$ because these groups are isomorphic. We will show that $K_{ss}$ is free if and only if $\text{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, M)$ is free. The extension $K \ni P \to M$ induces the following long exact sequence:

$$0 \to \text{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, M) \to K_{ss} \to P_{ss} \to M_{ss} \to 0.$$ 

Assume that $K_{ss}$ is free. Then its subgroup $\text{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, M)$ is free as well. Conversely, if $\text{Tor}_1^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, M)$ is free, then $K_{ss}$ is an extension of free Abelian groups and thus free. Notice that $P_{ss}$ is free because $P$ is projective. The general case $n \in \mathbb{N}$ follows by induction using an argument based on syzygies as above. This completes the proof of Proposition 4.

5. Free Resolutions for $\mathcal{NT}_{ss}$

The $\mathcal{NT}^*$-right-module $\mathcal{NT}_{ss}$ decomposes as a direct sum $\bigoplus_{Y \in \mathcal{LC}(X)^*} S_Y$ of the simple submodules $S_Y$ which are given by $S_Y(Y) \cong \mathbb{Z}$ and $S_Y(Z) = 0$ for $Z \neq Y$. We obtain

$$\text{Tor}_n^{\mathcal{NT}^*}(\mathcal{NT}_{ss}, M) = \bigoplus_{Y \in \mathcal{LC}(X)^*} \text{Tor}_n^{\mathcal{NT}^*}(S_Y, M).$$

Our task is then to write down projective resolutions for the $\mathcal{NT}^*$-right-modules $S_Y$. The first step is easy: we map $Q_Y$ onto $S_Y$ by mapping the class of the identity in $Q_Y(Y)$ to the generator of $S_Y(Y)$. Extended by zero, this yields an epimorphism $Q_Y \twoheadrightarrow S_Y$.

In order to surject onto the kernel of this epimorphism, we use the indecomposable transformations in $\mathcal{NT}^*$ whose range is $Y$. Denoting these by $\eta_i : W_i \to Y, 1 \leq i \leq n$, we obtain the two step resolution

$$\bigoplus_{i=1}^n QW_i \xrightarrow{(\eta_1 \eta_2 \cdots \eta_n)} Q_Y \twoheadrightarrow S_Y.$$ 

In the notation of [10], the map $\bigoplus_{i=1}^n QW_i \to Q_Y$ corresponds to a morphism $\phi : \mathcal{R}_Y \to \bigoplus_{i=1}^n \mathcal{R}_W_i$ of $3$-projectives in $\mathcal{R}(X)$. If the mapping cone $C_\phi$ of $\phi$ is again $3$-projective, the distinguished triangle $\Sigma C_\phi \to \mathcal{R}_Y \xrightarrow{\phi} \bigoplus_{i=1}^n \mathcal{R}_W_i \to C_\phi$ yields the projective resolution

$$\cdots \to Q_Y \to Q_\phi[1] \to \bigoplus_{i=1}^n QW_i[1] \to Q_Y[1] \to Q_\phi \to \bigoplus_{i=1}^n QW_i \to Q_Y \twoheadrightarrow S_Y,$$

where $Q_\phi = \text{FK}(C_\phi)$. We denote periodic resolutions like this by

$$Q_\phi \xrightarrow{\bigoplus_{i=1}^n QW_i} Q_Y \twoheadrightarrow S_Y.$$ 

If the mapping cone $C_\phi$ is not $3$-projective, the situation has to be investigated individually. We will see examples of this in 47 and 49. The resolutions we construct in these cases exhibit a certain six-term periodicity as well. However, they begin with a finite number of “non-periodic steps” (one in 47 and two in 49), which can be considered as a symptom of the deficiency of the invariant filtrated $K$-theory over non-accordion spaces from the homological viewpoint. We remark without proof
that the mapping cone of the morphism $\phi: \mathcal{R}_Y \to \bigoplus_{i=1}^n \mathcal{R}_{W_i}$ is 3-projective for every $Y \in \mathbb{L}C(X)^*$ if and only if $X$ is a disjoint union of accordion spaces.

6. Tensor Products with Free Right-Modules

Lemma 2. Let $M$ be an $\mathcal{N}T^*$-left-module. There is an isomorphism $Q_Y \otimes_{\mathcal{N}T^*} M \cong M(Y)$ of $\mathbb{Z}/2$-graded Abelian groups which is natural in $Y \in \mathcal{N}T^*$.

Proof. This is a simple consequence of Yoneda’s lemma and the tensor-hom adjunction. 

Lemma 3. Let $\Sigma \mathcal{R}_{(3)} \xrightarrow{\gamma} \mathcal{R}_{(1)} \xrightarrow{\gamma} \mathcal{R}_{(2)} \xrightarrow{\beta} \mathcal{R}_{(3)}$ be a distinguished or anti-distinguished triangle in $\mathcal{R}_\mathcal{R}(X)$, where $\mathcal{R}_{(i)} = \bigoplus_{j=1}^{m_i} \mathcal{R}_{Y_j} \oplus \bigoplus_{k=1}^{n_i} \Sigma \mathcal{R}_{Z_k}$ for $1 \leq i \leq 3$, $m_i, n_i \in \mathbb{N}$ and $Y_j, Z_k \in \mathbb{L}C(X)^*$. Set $Q_{(i)} = \mathbb{F}K(\mathcal{R}_{(i)})$. If $M = \mathbb{F}K(\mathcal{A})$ for some $\mathcal{A} \in \mathcal{R}_\mathcal{R}(X)$, then the induced sequence

$$Q_{(1)} \otimes_{\mathcal{N}T^*} M \xrightarrow{\alpha \otimes \text{id}_M} Q_{(2)} \otimes_{\mathcal{N}T^*} M \xrightarrow{\beta \otimes \text{id}_M} Q_{(3)} \otimes_{\mathcal{N}T^*} M$$

is exact.

Proof. Using the previous lemma and the representability theorem, we naturally identify $Q_{(i)} \otimes_{\mathcal{N}T^*} M \cong \mathbb{K}K(X; \mathcal{R}_{(i)}, \mathcal{A})$. Since, in triangulated categories, distinguished or anti-distinguished triangles induce long exact Hom-sequences, the sequence (2) is thus exact. 

7. Proof of Proposition 2

We may restrict to connected $T_0$-spaces. In [11], a list of isomorphism classes of connected $T_0$-spaces with three or four points is given. If $X$ is a disjoint union of accordion spaces, then the assertion follows from [2]. The remaining spaces fall into two classes:

1. all connected non-accordion four-point $T_0$-spaces except for the pseudocircle;
2. the pseudocircle (see §7.2).

The spaces in the first class have the following in common: If we fix two of them, say $X, Y$, then there is an ungraded isomorphism $\Phi: \mathcal{N}T^*(X) \to \mathcal{N}T^*(Y)$ between the categories of natural transformations on the respective filtrated K-theories such that the induced equivalence of ungraded module categories

$$\Phi^*: \mathcal{M}od^{ungr}(\mathcal{N}T^*(Y)) \to \mathcal{M}od^{ungr}(\mathcal{N}T^*(X))$$

restricts to a bijective correspondence between exact ungraded $\mathcal{N}T^*(Y)$-modules and exact ungraded $\mathcal{N}T^*(X)$-modules. Moreover, the isomorphism $\Phi$ restricts to isomorphisms from $\mathcal{N}T_{\text{ss}}(X)$ onto $\mathcal{N}T_{\text{ss}}(Y)$ and from $\mathcal{N}T_{\text{nil}}(X)$ onto $\mathcal{N}T_{\text{nil}}(Y)$. In particular, the assertion holds for $X$ if and only if it holds for $Y$.

The above is a consequence of the investigations in [12], the same kind of relation was found in [2] for the categories of natural transformations associated to accordion spaces with the same number of points. As a consequence, it suffices to verify the assertion for one representative of the first class—we choose $Z_4$—and for the pseudocircle.
7.1. Resolutions for the space $Z_3$. We refer to [10] for a description of the category $N'\mathcal{T}^\ast (Z_3)$, which in particular implies, that the space $Z_3$ satisfies the conditions of Proposition 4. Using the extension triangles from [10] (2.5), the procedure described in [5] yields the following projective resolutions induced by distinguished triangles as in Lemma 3:

\[
\begin{align*}
Q_1[1] & \longrightarrow Q_4 \longrightarrow Q_{14} \rightarrow S_{14} , \\
Q_{1234}[1] & \longrightarrow Q_1[1] \oplus Q_2[1] \oplus Q_3[1] \longrightarrow Q_4 \rightarrow S_4 ; \\
Q_{234} & \longrightarrow Q_{1234} \longrightarrow Q_1 \rightarrow S_1 ,
\end{align*}
\]

and similarly for $S_{24}$, $S_{34}$;

Next we will deal with the modules $S_{jk4}$, where $1 \leq j < k \leq 3$. We observe that there is a Mayer-Vietoris type exact sequence of the form

\[
\begin{align*}
Q_4 & \longrightarrow Q_{j4} \oplus Q_{k4} \longrightarrow Q_{jk4} .
\end{align*}
\]

**Lemma 4.** The candidate triangle $\Sigma \mathcal{R}_4 \rightarrow \mathcal{R}_{jk4} \rightarrow \mathcal{R}_{j4} \oplus \mathcal{R}_{k4} \rightarrow \mathcal{R}_4$ corresponding to the periodic part of the sequence (3) is distinguished or anti-distinguished (depending on the choice of signs for the maps in (3)).

**Proof.** We give the proof for $j = 1$ and $k = 2$. The other cases follow cyclicly permuting the indices 1, 2 and 3. We denote the morphism $\mathcal{R}_{124} \rightarrow \mathcal{R}_{14} \oplus \mathcal{R}_{24}$ by $\varphi$ and the corresponding map $Q_{14} \oplus Q_{24} \rightarrow Q_{124}$ in (3) by $\varphi^\ast$. It suffices to check that $\text{FK}(\text{Cone } \varphi)$ and $Q_4$ correspond, possibly up to a sign, to the same element in $\text{Ext}^1_{N'\mathcal{T}^\ast (Z_3)}(\text{ker}(\varphi^\ast), \text{coker}(\varphi^\ast)[1])$. We have $\text{coker}(\varphi^\ast) \cong S_{124}$ and an extension $S_{124}[1] \rightarrow Q_4 \rightarrow \text{ker}(\varphi^\ast)$. Since $\text{Hom}(Q_4, S_{124}[1]) \cong S_{124}(4)[1] = 0$ and $\text{Ext}^1(Q_4, S_{124}[1])$ because $Q_4$ is projective, the long exact Ext-sequence yields $\text{Ext}^1(\text{ker}(\varphi^\ast), \text{coker}(\varphi^\ast)[1]) \cong \text{Hom}(S_{124}[1], S_{124}[1]) \cong \mathbb{Z}$. Considering the sequence of transformations $3 \rightarrow 124 \rightarrow 1234 \rightarrow 3$, it is straight-forward to check that such an extension corresponds to one of the generators $\pm 1 \in \mathbb{Z}$ if and only if its underlying module is exact. This concludes the proof because both $\text{FK}(\text{Cone } \varphi)$ and $Q_4$ are exact.

Hence we obtain the following projective resolutions induced by distinguished or anti-distinguished triangles as in Lemma 3:

\[
\begin{align*}
Q_4 & \longrightarrow Q_{j4} \oplus Q_{k4} \longrightarrow Q_{jk4} \rightarrow S_{jk4} .
\end{align*}
\]

To summarize, by Lemma 3 $\text{Tor}^1_{N'\mathcal{T}^\ast}(S_Y, M) = 0$ for $Y \neq 1234$ and $n \geq 1$.

As we know from [10], the subset 1234 of $Z_3$ plays an exceptional role. In the notation of [10] (with the direction of the arrows reversed because we are dealing with right-modules), the kernel of the homomorphism $Q_{124} \oplus Q_{134} \oplus Q_{234} \xrightarrow{(1,1,1)} Q_{1234}$ is of the form

\[
\begin{align*}
\mathbb{Z} & \longrightarrow 0 \longrightarrow \mathbb{Z}[1] \\
\mathbb{Z}^2 & \longrightarrow \mathbb{Z} \longrightarrow 0 \longrightarrow \mathbb{Z}[1] \longrightarrow \mathbb{Z}[2] ,
\end{align*}
\]
It is the image of the module homomorphism

\[
Q_{14} \oplus Q_{24} \oplus Q_{34} \xrightarrow{egin{pmatrix} i & -i \\ 0 & 1 \\ 1 & 0 \end{pmatrix}} Q_{124} \oplus Q_{134} \oplus Q_{234},
\]

the kernel of which, in turn, is of the form

\[
\begin{array}{c}
0 \leftarrow Z[1] \\
\end{array}
\]

\[
\begin{array}{c}
\overset{\delta_{1234}}{0} \leftarrow Z[1] \\
\end{array}
\]

A surjection from \( Q_4 \oplus Q_{1234}[1] \) onto this module is given by \( \begin{pmatrix} i & i & 0 \\ 0 & 1 & 0 \end{pmatrix} \), where \( \delta_{1234} := \delta_{1234}^3 \circ r_{1234}^3 \). The kernel of this homomorphism has the form

\[
\begin{array}{c}
Z[1] \leftarrow Z[1] \\
\end{array}
\]

\[
\begin{array}{c}
\overset{\delta_{1234}}{0} \leftarrow Z[1] \\
\end{array}
\]

This module is isomorphic to \( \text{Syz}_{1234}[1] \), where \( \text{Syz}_{1234} := \ker(Q_{1234} \twoheadrightarrow S_{1234}) \).

Therefore, we end up with the projective resolution

\[
Q_4 \oplus Q_{1234}[1] \xrightarrow{\delta_{1234}} Q_{14} \oplus Q_{24} \oplus Q_{34} \xrightarrow{Q_{124} \oplus Q_{134} \oplus Q_{234}} Q_{1234} \xrightarrow{S_{1234}}.
\]

The homomorphism from \( Q_{124} \oplus Q_{134} \oplus Q_{234} \) to \( Q_4 \oplus Q_{1234}[1] \) is given by \( \begin{pmatrix} 0 & -\delta_{1234}^3 & i \delta_{1234}^3 \\ i & 1 & 0 \end{pmatrix} \), where \( \delta_{1234}^3 := \delta_1^3 \circ r_{1234}^3 \).

**Lemma 5.** The candidate triangle in \( \mathcal{R}(X) \) corresponding to the periodic part of the sequence (4) is distinguished or anti-distinguished (depending on the choice of signs for the maps in (5)).

**Proof.** The argument is analogous to the one in the proof of Lemma 4. Again, we consider the group \( \text{Ext}^1_{\mathcal{N}(T)}(Z_3^i, \ker(\varphi^*)[1]) \) where \( \varphi^* \) now denotes the map (4). We have \( \text{coker}(\varphi^*) \cong \text{Syz}_{1234} \) and an extension \( Q_4 \twoheadrightarrow \ker(\varphi^*) \twoheadrightarrow S_{1234}[1] \).

Using long exact sequences, we obtain

\[
\text{Ext}^1(\ker(\varphi^*),\text{coker}(\varphi^*))[1]) \cong \text{Ext}^1(S_{1234}[1], \text{Syz}_{1234}[1]) \cong \text{Hom}(S_{1234}[1], S_{1234}[1]) \cong \mathbb{Z}.
\]

Again, an extension corresponds to a generator if and only if its underlying module is exact. \(
\)

By the previous lemma and (4), computing the tensor product of this complex with \( M \) and taking homology shows that \( \text{Tor}^n_{\mathcal{N}(T)}(\mathcal{N}(T_{ss}), M) = 0 \) for \( n \geq 2 \) and that \( \text{Tor}^1_{\mathcal{N}(T)}(\mathcal{N}(T_{ss}), M) \) is equal to \( \text{Tor}^1_{\mathcal{N}(T)}(S_{1234}, M) \) and isomorphic to the homology of the complex (4). \( \square \)
Example 1. For the filtrated K-module with projective dimension 2 constructed in [10] §5 we get \( \text{Tor}_1^{NT^*} (NT_{ss}, M) \cong \mathbb{Z}/k \).

Remark 1. As explicated in the beginning of this section, the category \( NT^* (S) \) corresponding to the four-point space \( S \) defined in the introduction is isomorphic in an appropriate sense to the category \( NT^* (\mathbb{Z}_4) \). As has been established in [1], the indecomposable morphisms in \( NT^* (S) \) are organised in the diagram

\[
\begin{array}{cccc}
12 & 34 & r & i \\
123 & 234 & i & r \\
13 & 24 & r & i
\end{array}
\]

In analogy to [1], we have that \( \text{Tor}_1^{NT^* (S)} (NT_{ss}, M) \) is isomorphic to the homology of the complex

\[
(1) \quad M(12)[1] \oplus M(4) \oplus M(13)[1] \xrightarrow{\left(\begin{array}{ccc}
\delta & r & 0 \\
-\delta & 0 & -r
\end{array}\right)} M(34) \oplus M(1)[1] \oplus M(24) \xrightarrow{(\delta, r, i)} M(234)
\]

where \( M = FK(A) \) for some separable C*-algebra \( A \) over \( X \).

7.2. Resolutions for the pseudocircle. Let \( C_2 = \{1, 2, 3, 4\} \) with the partial order defined by \( 1 < 3, 1 < 4, 2 < 3, 2 < 4 \). The topology on \( C_2 \) is thus given by \( \{\emptyset, 3, 4, 34, 134, 234, 1234\} \). Hence the non-empty, connected, locally closed subsets are

\[
\mathcal{L}(C_2)^* = \{3, 4, 134, 234, 1234, 13, 14, 23, 24, 124, 123, 1, 2\}.
\]

The partial order on \( C_2 \) corresponds to the directed graph

\[
\begin{array}{c}
4 \\
\downarrow \\
3 \\
\downarrow \\
2 \\
\downarrow \\
1
\end{array}
\]

The space \( C_2 \) is the only \( T_0 \)-space with at most four points with the property that its order complex (see [10] Definition 2.6)) is not contractible; in fact, it is homeomorphic to the circle \( S^1 \). Therefore, by the representability theorem [10] §2.1] we find

\[
NT_*(C_2, C_2) \cong KK_*(X; R_{C_2}, R_{C_2}) \cong K_* (RC_2(C_2)) \cong K^* (S^1) \cong \mathbb{Z} \oplus \mathbb{Z}[1]
\]

that is, there are non-trivial odd natural transformations \( FK_{C_2} \Rightarrow FK_{C_2} \). These are generated, for instance, by the composition \( C_2 \xrightarrow{\delta} 1 \xrightarrow{\delta} 3 \xrightarrow{i} C_2 \). This follows from the description of the category \( NT^* (C_2) \) below. Note that \( \delta_{C_2} \circ \delta_{C_2} \) vanishes because it factors through \( r_{13} \circ i_{13} = 0 \).

Figure 1 displays a set of indecomposable transformations generating the category \( NT^* (C_2) \) determined in [1] §6.3.2], where also a list of relations generating the relations in the category \( NT^* (C_2) \) can be found. From this, it is straightforward to verify that the space \( C_2 \) satisfies the conditions of Proposition [1].
Proceeding as described in §5, we find projective resolutions of the following form (we omit explicit descriptions of the boundary maps):

\[ Q_{123}[1] \rightarrow Q_1[1] \oplus Q_2[1] \rightarrow Q_3 \rightarrow S_3, \quad \text{and similarly for } S_4; \]
\[ Q_1[1] \rightarrow Q_4 \oplus Q_4 \rightarrow Q_{134} \rightarrow S_{134}, \quad \text{and similarly for } S_{234}; \]
\[ Q_4 \rightarrow Q_{134} \rightarrow Q_{13} \rightarrow S_{13}, \quad \text{and similarly for } S_{14}, S_{23}, S_{24}; \]
\[ Q_3 \oplus Q_4 \rightarrow Q_{134} \oplus Q_{234} \rightarrow Q_{1234} \rightarrow S_{1234}; \]
\[ Q_4 \oplus Q_{123}[1] \rightarrow Q_{134} \oplus Q_{234} \rightarrow Q_{1234} \oplus Q_{13} \oplus Q_{23} \rightarrow Q_{123} \rightarrow S_{123}, \quad \text{and similarly for } S_{124}; \]
\[ Q_{234} \oplus Q_1[1] \rightarrow Q_{1234} \oplus Q_{23} \oplus Q_{24} \rightarrow Q_{123} \oplus Q_{124} \rightarrow Q_1 \rightarrow S_1, \quad \text{and similarly for } S_2. \]

Again, the periodic part of each of these resolutions is induced by an extension triangle, a Mayer-Vietoris triangle as in Lemma 4 or a more exotic (anti-)distinguished triangle as in Lemma 5 (we omit the analogous computation here).

We get \( \text{Tor}^{N^T \otimes \mathbb{Q}}_1 (S_Y, M) = 0 \) for every \( Y \in \mathbb{LC}(C_2)^* \setminus \{123, 124, 1, 2\} \), and \( \text{Tor}^{N^T \otimes \mathbb{Q}}_n (S_Y, M) = 0 \) for all \( Y \in \mathbb{LC}(C_2)^* \) and \( n \geq 2 \). Therefore,

\[ \text{Tor}^{N^T \otimes \mathbb{Q}}_1 (N^T_{ss}, M) \cong \bigoplus_{Y \in \{123, 124, 1, 2\}} \text{Tor}^{N^T \otimes \mathbb{Q}}_1 (S_Y, M). \]

The four groups \( \text{Tor}^{N^T \otimes \mathbb{Q}}_1 (S_Y, M) \) with \( Y \in \{123, 124, 1, 2\} \) can be described explicitly as in §7.1 using the above resolutions. This finishes the proof of Proposition 2.

8. Proof of Proposition 3

We apply the Meyer-Nest machinery to the homological functor \( FK \otimes \mathbb{Q} \) on the triangulated category \( \mathbb{SH}(X) \otimes \mathbb{Q} \). We need to show that every \( N^T \otimes \mathbb{Q} \) module of the form \( M = FK(A) \otimes \mathbb{Q} \) has a projective resolution of length 1. It is easy to see that analogues of Propositions 1 and 2 hold. In particular, the term \( \text{Tor}^{N^T \otimes \mathbb{Q}}_2 (N^T_{ss} \otimes \mathbb{Q}, M) \) always vanishes. Here we use that \( \mathbb{Q} \) is a flat \( \mathbb{Z} \)-module,

Figure 1. Indecomposable natural transformations in \( N^T(C_2) \)
so that tensoring with $\mathbb{Q}$ turns projective $\mathcal{N}\mathcal{T}^*$-module resolutions into projective $\mathcal{N}\mathcal{T}^* \otimes \mathbb{Q}$-module resolutions. Moreover, the freeness condition for the $\mathbb{Q}$-module $\text{Tor}_1^{\mathcal{N}\mathcal{T}^* \otimes \mathbb{Q}}(\mathcal{N}\mathcal{T}_{ss} \otimes \mathbb{Q}, M)$ is empty since $\mathbb{Q}$ is a field.

9. Proof of Proposition 4

The computations to determine the category $\mathcal{N}\mathcal{T}^*(Z_4)$ are very similar to those for the category $\mathcal{N}\mathcal{T}^*(Z_3)$ which were carried out in [10]. We summarise its structure in Figure 2. The relations in $\mathcal{N}\mathcal{T}^*(Z_4)$ are generated by the following:

- the hypercube with vertices 5, 15, 25, ..., 12345 is a commuting diagram;
- the following compositions vanish:

$$
\begin{align*}
1235 \overset{1}{\to} 12345 & \overset{r}{\to} 4, & 1245 \overset{1}{\to} 12345 & \overset{r}{\to} 3, \\
1345 \overset{1}{\to} 12345 & \overset{r}{\to} 2, & 2345 \overset{1}{\to} 12345 & \overset{r}{\to} 1, \\
1 \overset{5}{\to} 5 & \overset{i}{\to} 15, & 2 \overset{5}{\to} 5 & \overset{i}{\to} 25, & 3 \overset{5}{\to} 5 & \overset{i}{\to} 35, & 4 \overset{5}{\to} 5 & \overset{i}{\to} 45;
\end{align*}
$$

- the sum of the four maps $12345 \to 5$ via 1, 2, 3, and 4 vanishes.

This implies that the space $Z_4$ satisfies the conditions of Proposition 4.

![Figure 2. Indecomposable natural transformations in $\mathcal{N}\mathcal{T}^*(Z_4)$](image)

In the following, we will define an exact $\mathcal{N}\mathcal{T}^*$-left-module $M$ and compute $\text{Tor}_1^{\mathcal{N}\mathcal{T}^*}(S_{12345}, M)$. By explicit computation, one finds a projective resolution of the simple $\mathcal{N}\mathcal{T}^*$-right-module $S_{12345}$ of the following form (again omitting explicit formulas for the boundary maps):

$$
\begin{align*}
\bigoplus_{1 \leq i \leq 4} Q_{12345,i}^{[1]} & \longrightarrow \bigoplus_{1 \leq i \leq 4} Q_{12345}^{[1]} & \longrightarrow \bigoplus_{1 \leq j \leq 5} Q_{12345}^{[1]} \\
\bigoplus_{1 \leq i \leq 4} Q_{12345,i} & \longrightarrow Q_{12345} & \longrightarrow S_{12345}.
\end{align*}
$$

Notice that this sequence is periodic as a cyclic six-term sequence except for the first two steps.
Consider the exact $\mathcal{N}^T^*$-left-module $M$ defined by the exact sequence

\[
0 \to P_{1245} \xrightarrow{\begin{pmatrix} 1 & 1 \\ \end{pmatrix}} \bigoplus_{1 \leq i \leq 4} P_{12345 \setminus i} \xrightarrow{\begin{pmatrix} 1 & 1 & 0 & 0 \\
-1 & 0 & 1 & 0 \\
0 & 1 & -1 & 0 \\
1 & 0 & 0 & -1 \\
0 & 0 & 1 & -1 \\
\end{pmatrix}} \bigoplus_{1 \leq j < k \leq 4} P_{j k 5} \twoheadrightarrow M.
\]

We have $\bigoplus_{1 \leq i \leq 4} M(i5) \oplus M(12345)[1] \cong 0 \oplus \mathbb{Z}^3$, $\bigoplus_{1 \leq j < k \leq 4} M(j k 5) \cong \mathbb{Z}^6$, and $M(5) \oplus \bigoplus_{1 \leq i \leq 4} M(12345 \setminus i)[1] \cong \mathbb{Z}[1] \oplus \mathbb{Z}[1]^8$. Since

\[
\bigoplus_{1 \leq i \leq 4} M(i5) \oplus M(12345)[1] \to \bigoplus_{1 \leq j < k \leq 4} M(j k 5)
\]

is exact, a rank argument shows that the map

\[
\bigoplus_{1 \leq j < k \leq 4} M(j k 5) \to \bigoplus_{1 \leq i \leq 4} M(12345 \setminus i)
\]

is non-trivial; it consists precisely of the elements in

\[
\bigoplus_{1 \leq j < k \leq 4} M(j k 5) \cong \bigoplus_{1 \leq j < k \leq 4} \mathbb{Z}[id_{j k 5}]
\]

which are multiples of $(id_{j k 5})_{1 \leq j < k \leq 4}$. This shows $\text{Tor}_2^{\mathcal{N}^T^*}(S_{12345}, M) \cong \mathbb{Z}$. Hence, by Proposition 1, the module $M$ has projective dimension at least 2. On the other hand, if we abbreviate the resolution (7) for

\[
0 \to P^{(5)} \xrightarrow{\alpha} P^{(4)} \xrightarrow{\beta} P^{(3)} \to M,
\]

a projective resolution of length 3 for $M_k$ is given by

\[
0 \to P^{(5)} \xrightarrow{\begin{pmatrix} k \\
\alpha \\
\end{pmatrix}} P^{(5)} \oplus P^{(4)} \xrightarrow{\begin{pmatrix} -k & \beta \\
0 & \alpha \\
\end{pmatrix}} P^{(4)} \oplus P^{(3)} \xrightarrow{\begin{pmatrix} \beta & k \\
\end{pmatrix}} P^{(3)} \to M_k,
\]

where $k$ denotes multiplication by $k$.

It remains to show that the modules $M$ and $M_k$ can be realised as the filtrated $K$-theory of objects in $\mathcal{B}(X)$. It suffices to prove this for the module $M$ since tensoring with the Cuntz algebra $\mathcal{O}_{k + 1}$ then yields a separable $C^*$-algebra with filtrated $K$-theory $M_k$ by the Künneth Theorem.

The projective resolution (8) can be written as

\[
0 \to \text{FK}(P^2) \xrightarrow{\text{FK}(f_2)} \text{FK}(P^1) \xrightarrow{\text{FK}(f_1)} \text{FK}(P^0) \to M,
\]

because of the equivalence of the category of projective $\mathcal{N}^T^*$-modules and the category of $\mathcal{O}$-projective objects in $\mathcal{R}(X)$. Let $N$ be the cokernel of the module.
map \( \text{FK}(f_2) \). Using [10, Theorem 4.11], we obtain an object \( A \in \mathcal{B}(X) \) with \( \text{FK}(A) \cong N \). We thus have a commutative diagram of the form

\[
\begin{array}{ccc}
0 & \xrightarrow{\text{FK}(f_3)} & \text{FK}(P^2) & \xrightarrow{\text{FK}(f_1)} & \text{FK}(P^1) & \xrightarrow{\gamma} & \text{FK}(P^0) & \xrightarrow{M} . \\
\end{array}
\]

Since \( A \) belongs to the bootstrap class \( \mathcal{B}(X) \) and \( \text{FK}(A) \) has a projective resolution of length 1, we can apply the universal coefficient theorem to lift the homomorphism \( \gamma \) to an element \( f \in \text{KK}(X; A, P^0) \). Now we can argue as in the proof of [10, Theorem 4.11]: since \( f \) is 3-monic, the filtrated K-theory of its mapping cone is isomorphic to \( \text{coker}(\gamma) \cong M \). This completes the proof of Proposition 4.

10. Cuntz-Krieger Algebras with Projective Dimension 2

In this section we exhibit a Cuntz-Krieger algebra \( A \) which is a tight \( C^* \)-algebra over the space \( Z_3 \) and for which the odd part of \( \text{Tor}_1^{NT}(Z_3)(\mathcal{N} \mathcal{T}_{ss}, \text{FK}(A)) \)—denoted \( \text{Tor}_1^{odd} \) in the following—is not free. By Proposition 2 this \( C^* \)-algebra has projective dimension 2 in filtrated K-theory.

In the following we will adhere to the conventions for graph algebras and adjacency matrices from [4]. Let \( E \) be the finite graph with vertex set \( E^0 = \{v_1, v_2, \ldots, v_8\} \) and edges corresponding to the adjacency matrix

\[
\begin{pmatrix}
B_4 & 0 & 0 & 0 \\
X_1 & B_1 & 0 & 0 \\
X_2 & 0 & B_2 & 0 \\
X_3 & 0 & 0 & B_3
\end{pmatrix} := \begin{pmatrix}
3 & 2 & 0 & 0 & 0 \\
2 & 3 & 0 & 0 & 0 \\
1 & 1 & 3 & 2 & 0 \\
1 & 1 & 1 & 2 & 0 \\
1 & 1 & 0 & 3 & 2 \\
1 & 1 & 0 & 1 & 2 \\
1 & 1 & 0 & 0 & 3 & 2 \\
1 & 1 & 0 & 0 & 1 & 2
\end{pmatrix}.
\]

Since this is a finite graph with no sinks and no sources, the associated graph \( C^*(E) \) is in fact a Cuntz-Krieger algebra (we can replace \( E \) with its edge graph; see [11, Remark 2.8]). Moreover, the graph \( E \) is easily seen to fulfill condition (K) because every vertex is the base of two or more simple cycles. As a consequence, the adjacency matrix of the edge graph of \( E \) fulfills condition (II) from [5]. In fact, condition (K) is designed as a generalisation of condition (II); see, for instance, [8].

Applying [13, Theorem 4.9]—and carefully translating between different graph algebra conventions—we find that the ideals of \( C^*(E) \) correspond bijectively and in an inclusion-preserving manner to the open subsets of the space \( Z_3 \). By [11, Lemma 2.35], we may turn \( A \) into a tight \( C^* \)-algebra over \( Z_3 \) by declaring \( A(\{4\}) = I_{(v_1, v_2)}, A(\{1, 4\}) = I_{(v_1, v_2, v_3, v_4)}, A(\{2, 4\}) = I_{(v_1, v_2, v_3)} \) and \( A(\{3, 4\}) = I_{(v_1, v_2, v_3, v_4)} \), where \( I_S \) denotes the ideal corresponding to the saturated hereditary subset \( S \).

It is known how to compute the six-term sequence in K-theory for an extension of graph \( C^* \)-algebras: see [4]. Using this and Proposition 2 \( \text{Tor}_1^{\text{odd}} \) is the homology of the complex

\[
\begin{pmatrix}
1 & -1 & 0 \\
-1 & 0 & 1 \\
0 & 1 & -1
\end{pmatrix} \xrightarrow{\ker(\phi_0)} \ker(\phi_1) \xrightarrow{\ker(\phi_1)} \ker(\phi_2) .
\]

\[
\begin{pmatrix}
i & -1 & 0 \\
-1 & 0 & i \\
0 & i & -1
\end{pmatrix} .
\]
where $\phi_0 = \text{diag} \left( \left( \begin{array}{c} B_1' \ X_1' \\ 0 \ B_2' \end{array} \right) \right)$, $\left( \begin{array}{c} B_2' \ X_2' \\ 0 \ B_3' \end{array} \right)$, $\left( \begin{array}{c} B_3' \ X_3' \\ 0 \ B_4' \end{array} \right)$, $\phi_1 = \text{diag} \left( \left( \begin{array}{c} B_1'' \ X_1'' \\ 0 \ B_2'' \end{array} \right) \right)$, $\left( \begin{array}{c} B_2'' \ X_2'' \\ 0 \ B_3'' \end{array} \right)$, $\left( \begin{array}{c} B_3'' \ X_3'' \\ 0 \ B_4'' \end{array} \right)$, $\phi_2 = \left( \begin{array}{c} B_1''' \ X_1''' X_1'' \\ 0 \ B_1' \ 0 \ 0 \ 0 \ B_2' \ 0 \ 0 \ 0 \ 0 \ B_3' \end{array} \right)$,

and $B_k' = B_k'' - (\frac{1}{3} \frac{1}{3}) = (\frac{2}{3} \frac{2}{3})$ and $B_k'' = B_k''' - (\frac{1}{3} \frac{1}{3}) = (\frac{2}{3} \frac{2}{3})$ for $1 \leq j \leq 3$. We obtain a commutative diagram

\[
\begin{array}{ccc}
\ker(\phi_0) & \xrightarrow{(\mathbb{Z}^2)^{(2 \times 3)}} & \text{im}(\phi_0) \\
\phi_0 & \xrightarrow{f_K} & \phi_1 \\
\phi_1 & \xrightarrow{f} & \phi_2 \\
\phi_2 & \xrightarrow{g_K} & \text{im}(\phi_2)
\end{array}
\] (11)

where $f$ and $g$ have the block forms

\[
f = \left( \begin{array}{llllllllll}
id & 0 & -id & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & id & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & id & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & id & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & id & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & id & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & id & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & id & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & id
\end{array} \right),
\]

and $f_K := f|_{\ker(\phi_0)}$, $f_I := f|_{\text{im}(\phi_0)}$, $g_K := g|_{\ker(\phi_1)}$, $g_I := g|_{\text{im}(\phi_1)}$. Notice that $f$ and $g$ are defined in a way such that the restrictions $f|_{\ker(\phi_0)}$ and $g|_{\ker(\phi_1)}$ are exactly the maps from (10) in the identification made above.

We abbreviate the above short exact sequence of cochain complexes (11) as $K_\bullet \rightarrow Z_\bullet \rightarrow I_\bullet$. The part $H^0(Z_\bullet) \rightarrow H^0(I_\bullet) \rightarrow H^1(K_\bullet) \rightarrow H^1(Z_\bullet)$ in the corresponding long exact homology sequence can be identified with

\[
\ker(f) \xrightarrow{\phi_0} \ker(f_I) \rightarrow \frac{\ker(g_K)}{\text{im}(f_K)} \rightarrow 0.
\]

Hence

\[
\text{Tor}_1^{odd} \cong \frac{\ker(g_K)}{\text{im}(f_K)} \cong \frac{\ker(f_I)}{\phi_0(\ker(f))} \cong \frac{\ker(f) \cap \text{im}(\phi_0)}{\phi_0(\ker(f))}.
\]

We have $\ker(f) = \{(v, 0, v, 0, v, 0) \mid v \in \mathbb{Z}^2\} \subset (\mathbb{Z}^2)^{(2 \times 3)}$.

From the concrete form (11) of the adjacency matrix, we find that $\ker(f) \cap \text{im}(\phi_0)$ is the free cyclic group generated by $(1, 1, 0, 0, 1, 1, 0, 0, 1, 1, 0, 0)$, while $\phi_0(\ker(f))$ is the subgroup generated by $(2, 2, 0, 0, 2, 2, 0, 0, 2, 2, 0, 0)$. Hence $\text{Tor}_1^{odd} \cong \mathbb{Z}/2$ is not free.

Now we briefly indicate how to construct a similar counterexample for the space $S$. Consider the integer matrix

\[
\left( \begin{array}{ccc}
B_{41} & 0 & 0 \\
B_{43} & B_3 & 0 \\
B_{42} & 0 & B_2 \\
B_{41} & X_{31} & X_{31} \end{array} \right) := \left( \begin{array}{cccc}
3 & 0 & 0 & 0 \\
2 & (3) & 0 & 0 \\
2 & 0 & (3) & 0 \\
2 & 1 & 1 & 2 \end{array} \right).
\]

The corresponding graph $F$ fulfills condition (K) and has no sources or sinks. The associated graph $C^\ast$-algebra $C^\ast(F)$ is therefore a Cuntz-Krieger algebra satisfying condition (II). It is easily read from the block structure of the edge matrix that the primitive ideal space of $C^\ast(F)$ is homeomorphic to $S$. We are going to compute the
even part of $\text{Tor}^N_{T^r(S)}(\mathcal{N}T_{ss}, \text{FK}(C^*(F)))$. Since the nice computation methods from the previous example do not carry over, we carry out a more ad hoc calculation.

By Remark 1, the even part of our Tor-term is isomorphic to the homology of the complex

$$\begin{array}{c}
\text{ker} \left( B'_4 \, X'_{41} \atop 0 \, B'_3 \right) \\
\text{coker} \left( B'_3 \, X'_{31} \atop 0 \, B'_2 \right)
\end{array} \xrightarrow{-i} \begin{array}{c}
\text{ker} \left( B'_2 \, X'_{21} \atop 0 \, B'_1 \right) \\
\text{coker} \left( B'_1 \, X'_{11} \atop 0 \, B'_0 \right)
\end{array}
$$

where column-wise direct sums are taken. Here $B'_1 = B'_1 - (1,0,1) = (1,1)$ and $B'_j = B'_j - (1) = (2)$ for $2 \leq j \leq 4$. This complex can be identified with

$$\mathbb{Z} \oplus \mathbb{Z}/2 \oplus \mathbb{Z} \xrightarrow{\psi} (\mathbb{Z}/2)^2 \oplus (\mathbb{Z}/2)^2 \xrightarrow{\phi} (\mathbb{Z}/2)^3,$$

the homology of which is isomorphic to $\mathbb{Z}/2$; a generator is given by the class of $(0,1,1,0,1) \in (\mathbb{Z}/2)^2 \oplus \mathbb{Z} \oplus (\mathbb{Z}/2)^2$. This concludes the proof of Proposition 5.

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Department of Mathematical Sciences, University of Copenhagen, Universitetsparken 5, 2100 Copenhagen Ø, Denmark
E-mail address: bentmann@math.ku.dk