MONO-ATOMIC DISINTEGRATION AND LYAPUNOV EXPONENTS FOR DERIVED FROM ANOSOV Diffeomorphisms

G. Ponce, A. Tahzibi, and R. Varão

Abstract. In this paper we mainly address the problem of disintegration of Lebesgue measure and measure of maximal entropy along the central foliation of (conservative) Derived from Anosov (DA) diffeomorphisms. We prove that for accessible DA diffeomorphisms of $T^3$, atomic disintegration has the peculiarity of being mono-atomic (one atom per leaf). We further provide open and non-empty condition for the existence of atomic disintegration. Finally, we prove some new relations between Lyapunov exponents of DA diffeomorphisms and their linearization.

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1. Introduction

A diffeomorphism is called partially hyperbolic if the tangent bundle admits an invariant decomposition $TM = E^s \oplus E^c \oplus E^u$, such that all unit vectors

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\(v^\sigma \in E_x^\sigma, \sigma \in \{s, c, u\}\) for all \(x \in M\) satisfy:

\[
\|D_x f v^s\| < \|D_x f v^c\| < \|D_x f v^u\|
\]

and moreover \(\|D f|E^s\| < 1\) and \(\|D f^{-1}|E^u\| < 1\). We call \(f\) absolute partially hyperbolic if it is partially hyperbolic and for any \(x, y, z \in M\)

\[
\|D_x f v^s\| < \|D_y f v^c\| < \|D_z f v^u\|
\]

where \(v^s, v^c, v^u\) belong respectively to \(E_x^s, E_y^c\) and \(E_z^u\). In this paper when we require partial hyperbolicity, we mean absolute partial hyperbolicity.

The subbundles \(E^s\) and \(E^u\) integrate into \(f\)-invariant foliations, respectively the stable foliation, \(\mathcal{F}^s\), and the unstable foliation, \(\mathcal{F}^u\). These foliations have a nice property called absolute continuity. Among different definitions for absolute continuity of foliations we choose the following one which is suitable for our purpose: A set of full volume measure on \(M\) must intersect almost every leaf of \(\mathcal{F}^s\) (or \(\mathcal{F}^u\)) in a set of full Lebesgue measure of the leaf. Although the absolute continuity of \(\mathcal{F}^s\) and \(\mathcal{F}^u\) are mandatory for a (general) \(C^2\) partially hyperbolic diffeomorphism, this is not the case for the center foliation \(\mathcal{F}^c\) (it is not even true that there will exist such a foliation, but by [14] for all absolute partially hyperbolic diffeomorphisms of \(\mathbb{T}^3\) the center foliation exists). The center foliation might not be absolutely continuous, at least this is, in general, expected to happen for diffeomorphisms which preserves volume (see [18], [11], [6]). For many examples (some of them described below) the center foliation has atomic disintegration. In all such examples there exists a set \(A \subset M\) of full volume measure, such that the intersection of \(A\) with every leaf of \(\mathcal{F}^c\) are \(k\) points, where \(k\) is a natural number independent of the leaf (in the ergodic context). In principle for a general partially hyperbolic diffeomorphism the geometric structure of the support of disintegration measures is not clear. We do not have examples of atomic disintegration with infinitely many atoms.

There exist essentially three known category of partially hyperbolic diffeomorphisms on three-dimensional manifolds (see conjecture of Pujals in [4]). We deal here with the so called Derived from Anosov (DA) diffeomorphisms. By a DA diffeomorphism we mean a partially hyperbolic diffeomorphism \(f : \mathbb{T}^3 \to \mathbb{T}^3\) such that its linearization (see [2,2] in Preliminaries) is an Anosov diffeomorphism. Observe that by [8] the linearization of \(f\) is also partially hyperbolic in the sense that it admits three invariant sub-bundles. The other two classes of partially hyperbolic diffeomorphisms are the skew-product type and perturbations of time-one of Anosov flows (see as well Hammerlindl-Potrie [16] for a discussion and new results).

For the perturbation of a time-one map of the geodesic flow for a closed negatively curved surfaces (which is an Anosov flow), it was shown by A. Avila, M. Viana and A. Wilkinson [11] that \(\mathcal{F}^c\) has atomic disintegration or it is absolutely
continuous. For a large class of skew-product diffeomorphisms, they announced that they can prove an analogous result.

It is interesting to emphasize that (conservative) Derived from Anosov (DA) diffeomorphisms on $\mathbb{T}^3$ show a feature that is not, so far, shared with any other known partially hyperbolic diffeomorphisms on dimension three, it admits all three disintegration of volume on the center leaf, namely: Lebesgue, atomic, and, by a recent result of R. Varão [21], they can also have a disintegration which is neither Lebesgue nor atomic.

More precisely, R. Varão [21] showed that there exist Anosov diffeomorphisms with non-absolutely continuous center foliation which does not have atomic disintegration. Here we show a new behavior for DA diffeomorphisms on $\mathbb{T}^3$, and that is the existence of atomic disintegration (Theorem B). This behavior can be verified for an open class of diffeomorphisms found by Ponce-Tahzibi in [7]. In fact we prove (Theorem A): if the disintegration of Lebesgue measure is atomic, then it is in fact mono-atomic, i.e there is just one atom per leaf. We should mention that the most important part of our proof is to conclude finiteness of atoms in the case of atomic disintegration. However, for general partially hyperbolic diffeomorphisms finiteness does not imply mono-atomic disintegration. See [17] and our discussion after Theorem A below for a contrast with the skew product case.

**Theorem A** Let $f$ be a volume preserving accessible DA diffeomorphism on $\mathbb{T}^3$. If volume has atomic disintegration on center leaves, then it has one atom per leaf.

*Remark 1.1.* The accessibility hypothesis for partially hyperbolic diffeomorphisms with one dimensional center bundle implies that $f^n$ is ergodic for all $n \geq 1$. (see [10] or [3].) This is the unique place that we use accessibility, so that the accessibility hypothesis can be substituted by any other hypothesis that provides the ergodicity of all $f^n$. For instance, by Hammerlindl-Ures [9] we know that if the center exponent of $f$ is not zero almost everywhere (consequently the center exponent of $f^n$ is not zero almost everywhere, for all $n$) then it is ergodic (consequently $f^n$ is ergodic for all $n$). This observation will be important in the proof of Theorem B.

The conclusion about the number of atoms is not true for skew-products in general. Let $B := A \times \text{Id}$ where $A := \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Then arbitrarily close to $B : \mathbb{T}^3 \to \mathbb{T}^3$ there is an open set of partially hyperbolic diffeomorphisms $g$ such that $g$ is ergodic and there is an equivariant fibration $\pi : \mathbb{T}^3 \to \mathbb{T}^2$ such that the fibers are circles, $\pi \circ g = B \circ \pi$. And $g$ has positive central Lyapunov exponent, hence the central foliation is not absolutely continuous. In Ruelle and Wilkinson’s paper [17], we see that there exist $S \subset \mathbb{T}^3$ of full volume and $k \in \mathbb{N}$ such that $S$
meets every leaf in exactly \( k \) points. In Shub and Wilkinson’s example \[19\] the fibers of the fibration are invariant under the action of a finite non-trivial group and consequently in their example the number of atoms cannot be one.

In the next theorem we introduce a class of derived from Anosov diffeomorphisms where Theorem A can be applied. The work of Ponce, Tahzibi \[7\] guarantees that there exist DA diffeomorphisms satisfying these conditions.

**Theorem B** Let \( f : \mathbb{T}^3 \to \mathbb{T}^3 \) be a volume preserving, DA diffeomorphism. Suppose that its linearization \( A \) has the splitting \( T_A M = E^{su} \oplus E^{wu} \oplus E^s \) (\( su \) and \( wu \) represent strong unstable and weak unstable bundles.) If \( f \) has \( \lambda^c(x) < 0 \) for Lebesgue almost every point \( x \in \mathbb{T}^3 \), then volume has atomic disintegration on \( F^c_c f \), in fact the disintegrated measures have one atom per center leaf.

**Remark 1.2.** The key point in the proof of Theorem \[B\] is show that we get atomic disintegration. Then, we use Theorem \[A\] (because of Remark \[1.1\]) to obtain one atom per leaf.

We look at the linearization of \( f \), the linear Anosov \( A \), as a partially hyperbolic for which the center leaf \( F^c_A \) is the expanding leaf \( F^{wu}_A \).

We mention that the examples of non-absolutely continuous weak foliation of Anosov diffeomorphisms was known by Saghin-Xia \[18\] and Baraviera-Bonatti \[2\] near to geodesic flows, and by A.Gogolev near to hyperbolic automorphisms of the 3-torus \[6\]. However, we are introducing examples of non-Anosov diffeomorphisms with non-absolutely continuous central foliation and prove atomic disintegration. It is not known whether the disintegration of the Lebesgue measure can be atomic in the case of Anosov diffeomorphisms.

1.1. Maximal entropy measures and Lyapunov exponents. Lyapunov exponents are celebrated constants which are related to the entropy of invariant measures. In this paper we denote by \( \lambda^u(f) \), \( \lambda^c(f) \) and \( \lambda^s(f) \) the Lyapunov exponents of Lebesgue measure (ergodic case). When we are referring to the Lyapunov exponents of any other (ergodic) invariant measure \( \mu \) we use the subscript \( \lambda^*_\mu \). The relationship between the Lyapunov exponents of a partially hyperbolic diffeomorphism and its linearization is an interesting issue. In \[13\], the authors among other results proved a folkloric semi-rigidity property of unstable and stable Lyapunov exponents of (absolute) partially hyperbolic diffeomorphisms of \( \mathbb{T}^3 \):

**Theorem 1.3.** Let \( f \) be a \( C^2 \) conservative partially hyperbolic diffeomorphism on the 3–torus and \( A \) its linearization then

\[
\lambda^u(f, x) \leq \lambda^u(A) \text{ and } \lambda^s(f, x) \geq \lambda^s(A) \text{ for Lebesgue a.e. } x \in \mathbb{T}^3.
\]
The above theorem relates the extremal Lyapunov exponents of a non-linear and linear partially hyperbolic diffeomorphism. For the central Lyapunov exponent we do not expect such behavior. However, it is a problem in [15]:

**Problem 1.** In the context of the above theorem, suppose that $\lambda^c(f) > 0$ and $\mathcal{F}^c$ is (upper leafwise) absolutely continuous. Is it true that $\lambda^c(f) \leq \lambda^c(A)$?

The answer to the above problem is positive when $f$ is an Anosov diffeomorphism (see [15, 21]).

We emphasize that the above relations between Lyapunov exponents are valid for the volume measure. For other natural invariant measures the scenario can be different. Let $\mu_f$ be the unique maximizing measure for a partially hyperbolic diffeomorphism $f$ homotopic to Anosov $A$. In [20], R. Ures proved that if $\lambda^c(A) > 0$, i.e $A$ has weak unstable sub-bundle then $\lambda^c_{\mu_f} \geq \lambda^c(A)$.

In the following theorem we see that if the central Lyapunov exponent for Lebesgue measure of $f$ is strictly smaller than the central exponent of the linearization $A$, then either the central or the unstable exponent of the maximizing measure of $f$ is strictly larger than the corresponding exponent of $A$.

Before stating our last theorem we give one more definition. A measure is called u-Gibbs if the disintegration subordinated to the unstable foliation (corresponding to all positive Lyapunov exponents) gives conditional measures equivalent to the Lebesgue on the leaf. These measures play an important role in dynamical systems, [12].

**Theorem C** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a DA diffeomorphism with $A : \mathbb{T}^3 \to \mathbb{T}^3$ its linearization. Assume that $\lambda^c(A) > 0$ and $\lambda^c(f) < \lambda^c(A)$. Then $\mu_f$ is not u-Gibbs. Also, denoting by $\lambda^u_{\mu_f}, \lambda^c_{\mu_f}, \lambda^s_{\mu_f}$ the Lyapunov exponents of $f$ (for $\mu_f$ almost every point) we have

$$\lambda^c_{\mu_f} > \lambda^c(A) \text{ or } \lambda^u_{\mu_f} > \lambda^u(A).$$

We recall the example of Ponce-Tahzibi in [7] where $f$ has negative central Lyapunov exponent and the linearization of $f$ has positive central Lyapunov exponent. The authors began with a linear Anosov diffeomorphism $A$ with splitting $E^u \oplus E^{wu} \oplus E^s$ and after a modification they found $f$ partially hyperbolic ergodic volume preserving such that $\lambda^c(f) < 0$. The point is that, in their construction $\lambda^u_{\mu_f}(f) = \lambda^u(A)$ and by the above theorem we conclude that $\lambda^c_{\mu_f} > \lambda^c(A)$. That is, although after perturbation the central Lyapunov exponent of Lebesgue measure drops, the central exponent of maximal entropy measure increases. We describe in some more details the construction of this example on [23].

2. Preliminaries
2.1. Measurable partitions and disintegration of measures. Let \((M, \mu, \mathcal{B})\) be a probability space, where \(M\) is a compact metric space, \(\mu\) a probability measure and \(\mathcal{B}\) the borelian \(\sigma\)-algebra. Given a partition \(\mathcal{P}\) of \(M\) by measurable sets, we associate the probability space \((\mathcal{P}, \tilde{\mu}, \tilde{\mathcal{B}})\) by the following way. Let \(\pi : M \to \mathcal{P}\) be the canonical projection, that is, \(\pi\) associates a point \(x\) of \(M\) with the partition element of \(\mathcal{P}\) that contains it. Then we define \(\tilde{\mu} := \pi_* \mu\) and \(\tilde{\mathcal{B}} := \pi_* \mathcal{B}\).

**Definition 2.1.** Given a partition \(\mathcal{P}\). A family \(\{\mu_P\}_{P \in \mathcal{P}}\) is a system of conditional measures for \(\mu\) (with respect to \(\mathcal{P}\)) if

i) given \(\phi \in C^0(M)\), then \(P \mapsto \int \phi \mu_P\) is measurable;

ii) \(\mu_P(P) = 1\) \(\tilde{\mu}\)-a.e.;

iii) if \(\phi \in C^0(M)\), then \(\int_M \phi d\mu = \int_{\mathcal{P}} \left( \int_P \phi d\mu_P \right) d\tilde{\mu}\).

When it is clear which partition we are referring to, we say that the family \(\{\mu_P\}\) disintegrates the measure \(\mu\).

**Proposition 2.2.** If \(\{\mu_P\}\) and \(\{\nu_P\}\) are conditional measures that disintegrate \(\mu\), then \(\mu_P = \nu_P \tilde{\mu}\)-a.e.

**Corollary 2.3.** If \(T : M \to M\) preserves a probability \(\mu\) and the partition \(\mathcal{P}\), then \(T_* \mu_P = \mu_P \tilde{\mu}\)-a.e.

**Proof.** It follows from the fact that \(\{T_* \mu_P\}_{P \in \mathcal{P}}\) is also a disintegration of \(\mu\). \(\square\)

**Definition 2.4.** We say that a partition \(\mathcal{P}\) is measurable with respect to \(\mu\) if there exist a measurable family \(\{A_i\}_{i \in \mathbb{N}}\) and a measurable set \(F\) of full measure such that if \(B \in \mathcal{P}\), then there exists a sequence \(\{B_i\}\), where \(B_i \in \{A_i, A_i^c\}\) such that \(B \cap F = \bigcap_i B_i \cap F\).

**Proposition 2.5.** Let \((M, \mathcal{B}, \mu)\) a probability space where \(M\) is a compact metric space and \(\mathcal{B}\) is the Borel sigma-algebra. If \(\mathcal{P}\) is a continuous foliation of \(M\) by compact measurable sets, then \(\mathcal{P}\) is a measurable partition.

**Theorem 2.6** (Rokhlin’s disintegration). Let \(\mathcal{P}\) be a measurable partition of a compact metric space \(M\) and \(\mu\) a borelian probability. Then there exists a disintegration by conditional measures for \(\mu\).

In general the partition by the leaves of a foliation may be non-measurable. It is for instance the case for the stable and unstable foliations of a linear Anosov diffeomorphism. Therefore, by disintegration of a measure along the leaves of a foliation we mean the disintegration on compact foliated boxes. In principle, the conditional measures depend on the foliated boxes, however, two different foliated boxes induce proportional conditional measures. See [1] for a discussion. We define absolute continuity of foliations as follows:
Definition 2.7. We say that a foliation $F$ is absolutely continuous if for any foliated box, the disintegration of volume on the segment leaves have conditional measures equivalent to the Lebesgue measure on the leaf.

Definition 2.8. We say that a foliation $F$ has atomic disintegration with respect to a measure $\mu$ if the conditional measures on any foliated box are a sum of Dirac measures. Note that this is equivalent to saying that there is a set of $\mu$-full measure that intersects each leaf on a discrete set.

Although the disintegration of a measure along a general foliation is defined in compact foliated boxes, it makes sense to say that the foliation $F$ has a quantity $k_0 \in \mathbb{N}$ atoms per leaf. The meaning of “per leaf” should always be understood as a generic leaf, i.e. almost every leaf. That means that there is a set $A$ of $\mu$-full measure which intersects a generic leaf on exactly $k_0$ points. Let’s see that this implies atomic disintegration. Definition 2.1 shows that it only make sense to talk about conditional measures from the generic point of view, hence when restricted to a foliated box $\mathcal{B}$, the set $A \cap \mathcal{B}$ has $\mu$-full measure on $\mathcal{B}$, therefore the support of the conditional measure disintegrated on $\mathcal{B}$ must be contained on the set $A$. This implies atomic disintegration.

It is well worth to remark that the weight of an atom for a conditional measure naturally depends on the foliated box, but a point $x$ is atom independent of the foliated box where we disintegrate a measure.

2.2. Preliminaries on Partial Hyperbolicity. If $f : M \to M$ is a partially hyperbolic diffeomorphism with

$$ T_xM = E^s(x) \oplus E^c(x) \oplus E^u(x) $$

the bundles $E^s$ and $E^u$ are tangent to invariant foliations $F^s$ and $F^u$. $f$ is called accessible if for any two points $x$ and $y$ there is a piecewise smooth curve connecting $x$ to $y$ and tangent to $E^s \cup E^u$. If the central bundle is one dimensional and $f$ is volume preserving then accessibility of $f$ implies that all iterates $f^n$ are ergodic ([3], [10]).

We define the Lyapunov exponents of $f$ by

$$ \lambda^\tau(x) := \lim_{n \to \infty} \frac{1}{n} \log ||Df^n(x) \cdot v|| $$

where $v \in E^\tau$ and $\tau \in \{s, c, u\}$. If $f$ is ergodic, the stable, center and unstable Lyapunov exponents are constant almost everywhere. Otherwise they have no reason to be constant in the general case.

Any $A \in SL(3, \mathbb{Z})$ with at least one eigenvalue with norm larger than one, induces a linear partially hyperbolic diffeomorphism on $T^3$. Conversely for any
partially hyperbolic diffeomorphism \( f \) on \( \mathbb{T}^3 \), there exist a unique linear diffeomorphism \( A \), such that \( A \) induces the same automorphism as \( f \) on the fundamental group \( \pi_1(\mathbb{T}^3) \).

Let \( f : \mathbb{T}^3 \to \mathbb{T}^3 \) be a partially hyperbolic diffeomorphism. Consider \( f_* : \mathbb{Z}^3 \to \mathbb{Z}^3 \) the action of \( f \) on the fundamental group of \( \mathbb{T}^3 \). \( f_* \) can be extended to \( \mathbb{R}^3 \) and the extension is the lift of a unique linear automorphism \( A : \mathbb{T}^3 \to \mathbb{T}^3 \) which is called the linearization of \( f \). It can be proved that \( A \) is a partially hyperbolic automorphism of torus ([13]). We will say that \( f \) is derived from Anosov (DA diffeomorphism) if its linearization \( f_* \) is an Anosov diffeomorphism.

Let \( f \) be DA diffeomorphism defined as above, then by [5] we know that \( f \) is semi-conjugated to its linearization by a function \( h : \mathbb{T}^3 \to \mathbb{T}^3 \), \( h \circ f = A \circ h \). It follows from [20] that \( \mathcal{F}^c(A) = h(\mathcal{F}^c(f)) \). Moreover, there exists a constant \( K \in \mathbb{R} \) such that if \( \tilde{h} : \mathbb{R}^3 \to \mathbb{R}^3 \) denotes the lift of \( h \) to \( \mathbb{R}^3 \) we have \( \| \tilde{h}(x) - x \| \leq K \) for all \( x \in \mathbb{R}^3 \).

It is not difficult to see that in large scale \( f \) and \( A \) behaves similarly (see [8], lemma 3.6). More precisely, for each \( k \in \mathbb{Z} \) and \( C > 1 \) there is an \( M > 0 \) such that for all \( x, y \in \mathbb{R}^3 \),

\[
\| x - y \| > M \Rightarrow \frac{1}{C} < \frac{\| \tilde{f}^k(x) - \tilde{f}^k(y) \|}{\| A^k(x) - A^k(y) \|} < C.
\]

(2.1)

where \( \tilde{f} : \mathbb{R}^3 \to \mathbb{R}^3 \) is the lift of \( f \) to \( \mathbb{R}^3 \).

**Definition 2.9.** A foliation \( \mathcal{F} \) defined on a manifold \( M \) is quasi-isometric if the lift \( \tilde{\mathcal{F}} \) of \( \mathcal{F} \) to the universal cover of \( M \) has the following property: There exist positive constant \( Q \) such that for all \( x, y \) in a common leaf of \( \tilde{\mathcal{F}} \) we have

\[
d_{\tilde{\mathcal{F}}}(x, y) \leq Q\| x - y \|,
\]

where \( d_{\tilde{\mathcal{F}}} \) denotes the riemannian metric on \( \tilde{\mathcal{F}} \) and \( \| x - y \| \) is the distance on the ambient manifold of the foliation.

For absolute partially hyperbolic diffeomorphisms on \( \mathbb{T}^3 \) the stable, unstable and central foliations are quasi isometric in the universal covering \( \mathbb{R}^3 \).

**2.3. “Pathological” example.** As we remarked before, in theorem [3] one of the hypothesis is that the center Lyapunov exponent of the diffeomorphism \( f \) and of its linearization \( A \) have opposite sign. Since we have center leaf conjugacy between \( f \) and \( A \) and since in large scale the behavior of the center leaves is similar (see (2.1)), this hypothesis would imply that the asymptotic growth of the center leaves (which is a local issue) and global behavior of the center leaves of \( f \) are opposite. In [7], the authors constructed an example to show that this kind of phenomena occurs in an open set of partially hyperbolic dynamics and we briefly describe it here.
Start with the family of linear Anosov diffeomorphisms \( f_k : \mathbb{T}^3 \to \mathbb{T}^3 \) induced by the integer matrices:

\[
A_k = \begin{pmatrix}
0 & 0 & 1 \\
0 & 1 & -1 \\
-1 & -1 & k
\end{pmatrix}, k \in \mathbb{N}.
\]

This family of Anosov diffeomorphisms has two important characteristics that justify this choice. Denote by \( \lambda^s_k, \lambda^c_k, \lambda^u_k \) the three eigenvalues of \( A_k \) with \( \lambda^s_k < \lambda^c_k < \lambda^u_k \) and, for each \( k \), denote by \( E^s_k, E^c_k, E^u_k \) the stable, central and unstable fiber bundles with respect to \( A_k \). Then an easy calculation shows that

\[
\lambda^s_k \to 0, \lambda^c_k \to 1, \lambda^u_k \to \infty,
\]
as \( k \to \infty \).

Using a Baraviera-Bonatti [2] local perturbation method, for large \( k \) the authors managed to construct small perturbation of \( f_k \) and obtain partially hyperbolic diffeomorphisms \( g_k \) such that the central Lyapunov exponent of \( g_k \) is positive.

By taking the family \( g_k^{-1} \) we obtain partially hyperbolic diffeomorphisms with negative center exponent and isotopic to Anosov diffeomorphism with weak expanding subbundle. In fact any \( f := g_k^{-1} \) satisfy the desired properties. By construction of the perturbation, it comes out that the center-stable bundle of \( f \) coincides with the sum of stable and weak unstable bundle of Anosov linearizations \( f^* = A : E^{cu}(x) = E^{wu}_A(x) \oplus E^{s}_A(x) \). Moreover, for any \( x \in \mathbb{T}^3 \) we have

\[
\log J^{cs}(f) = \log J^s \circ A + \log J^{wu}(A).
\]

As both \( f \) and \( A \) are volume preserving we conclude that

\[
\int \log J^u(f, x)d\mu = \lambda^u(A)
\]

and by ergodicity \( \lambda^u(x) = \lambda^u(A) \) for \( \mu \) almost every \( x \) where \( \mu \) is any invariant probability measure.

This family of diffeomorphisms fulfills the hypothesis required in Theorems [3] and [4].

3. Proof of results

**Theorem A.** Let \( f \) be a volume preserving accessible DA diffeomorphism on \( \mathbb{T}^3 \). If the volume has atomic disintegration on the center leaves, then it has one atom per leaf.

**Proof.** Let \( h : \mathbb{T}^3 \to \mathbb{T}^3 \) be the semi-conjugacy between \( f \) and its linearization \( A \), hence \( h \circ f = A \circ h \). We can assume that \( A \) has two eigenvalues larger than one, otherwise we work with \( f^{-1} \).

Let \( \{R_i\} \) be a Markov partition for \( A \), and define \( \tilde{R}_i := h^{-1}(R_i) \). We claim that

\[
Vol \left( \bigcup \text{int } \tilde{R}_i \right) = 1. \quad (3.1)
\]
Indeed, first look at the center direction of \( A \). For simplicity we consider the center direction as a vertical foliation. This means that the rectangle \( R_i \) has two types of boundaries, the one coming from the extremes of the center foliation and the lateral ones. We call \( \partial_c R_i \) the boundary coming from these extremes of the center foliation, i.e

\[
\partial_c R_i = \bigcup_{x \in R_i} \partial(\mathcal{F}^c_x \cap R_i).
\]

Since \( h \) takes center leaves to center leaves, we conclude that the respective boundary for the \( \tilde{R}_i \) sets is \( \partial_c \tilde{R}_i = h^{-1}(\partial_c R_i) \), and since \( \bigcup_i \partial_c R_i \) is an \( A \)-invariant set, \( \bigcup_i \partial_c \tilde{R}_i \) is an \( f \)-invariant set. By ergodicity of \( f \) it follows that \( \bigcup_i \partial_c \tilde{R}_i \) has zero or full measure. Since the volume of the interior cannot be zero, then the volume of \( \bigcup_i \partial_c \tilde{R}_i \) cannot be one. Therefore it has zero measure.

By (3.1) we can consider the partition \( \hat{\mathcal{P}} = \{ \mathcal{F}^c_{R(x)} : x \in \tilde{R}_i \text{ for some } i \} \) where \( \mathcal{F}^c_{R(x)} \) denotes the connected component of \( \mathcal{F}^c_f(x) \cap R(x) \) which contains \( x \). Thus we can consider the Rokhlin disintegration of volume on the partition \( \hat{\mathcal{P}} \). Denote this system of measures by \( \{ m_x \} \), so that each \( m_x \) is supported in \( \mathcal{F}^c_f(x) \).

**Lemma 3.1.** There is a natural number \( \alpha_0 \in \mathbb{N} \), such that for almost every point, \( \mathcal{F}^c_{R(x)} \) contains exactly \( \alpha_0 \) atoms.

**Proof.** The semi-conjugacy \( h \) sends center leaves of \( f \) to center leaves of \( A \). Also, the points of the interior of the \( \tilde{R}_i \) satisfy that \( f(\mathcal{F}^c_{R(x)}) \supset \mathcal{F}^c_{R(f(x))} \), which just comes from the Markov property of the rectangles \( R_i \). This implies that

\[
f_* m_x \leq m_f(x)
\]

on \( \mathcal{F}^c_{R(f(x))} \). Given any \( \delta \geq 0 \) consider the set \( A_\delta = \{ x \in \mathbb{T}^3 : m_x(\{ x \}) > \delta \} \), that is, the set of atoms with weight at least \( \delta \). If \( x \in A_\delta \) then

\[
\delta < m_x(\{ x \}) = f_* m_x(\{ f(x) \}) \leq m_f(x)(\{ f(x) \}).
\]

Thus \( f(A_\delta) \subset A_\delta \), and by the ergodicity of \( f \) we have that \( Vol(A_\delta) \) is zero or one, for each \( \delta \geq 0 \). Note that \( Vol(A_0) = 1 \) and \( Vol(A_1) = 0 \). Let \( \delta_0 \) be the critical point for which \( Vol(A_\delta) \) changes value, i.e, \( \delta_0 = \sup\{ \delta : Vol(A_\delta) = 1 \} \). This means that all the atoms have weight \( \delta_0 \). Since \( m_x \) is a probability we have an \( \alpha_0 := 1/\delta_0 \) number of atoms as claimed. \( \square \)

**Lemma 3.2.** There is a positive number \( L_0 \) such that on almost every center leaf there is a point \( x \) such that \( B_c(x, L_0) \) (ball centered on \( x \) inside \( \mathcal{F}^c(x) \) of size \( L_0 \)) contains all the atoms of the leaf \( \mathcal{F}^c_f(x) \).

**Proof.** We divide the proof in two possible cases. The first one is the case where we have finite number of atoms on each center leaf and the second is that we have infinitely many atoms on each center leaf.
Case one (Finite atoms): Suppose we have a finite number of atoms on each center leaf. Given $L \in \mathbb{R}$, consider the following set

$$B_L = \{ x \mid B_c(x, L) \text{ contains all the atoms of } \mathcal{F}_x^c \}.$$ 

For any large enough $L$ the set $B_L$ has positive volume. To prove that $Vol(B_L) = 1$ for some large $L$, it is sufficient to prove that $f^{-n}(B_L) \subseteq B_L$ for some $n$. This is because, $f^n$ is ergodic. We do so by proving that $f^{-n}(B_c(x, L)) \subseteq B_c(f^{-n}(x), L)$, where $B_c(\xi, \rho)$ stands for the ball inside of the center leaf centered on $\xi$ and radius $\rho$ and $d_c$ is the distance on the center leaf coming from the metric when restricted to the center leaf.

It is enough to prove that $f^{-n}(B_c(x, L)) \subseteq B_c(f^{-n}(x), L)$ on the lift, since the projection is locally an isometry. We now work on the lift, but we carry the same notation as it should not make any confusion. Let $K \in \mathbb{R}$ such that

$$\|h - Id\|_{C^0} < K,$$

And $Q$ coming from the quasi-isometry property of the center foliation. Let $y, z \in B_c(x, L)$ be the extremals, i.e. $d_c(y, z) = 2L$, then

$$d_c(f^{-n}(y), f^{-n}(z)) \leq Q\|f^{-n}(y) - f^{-n}(z)\| \leq Q(\|hf^{-n}(y) - hf^{-n}(z)\| + 2K)$$

$$= Q(\|A^{-n}h(y) - A^{-n}h(z)\| + 2K)$$

$$= Q(e^{-n\lambda_{wu}(A)}\|h(y) - h(z)\| + 2K)$$

$$\leq Q(e^{-n\lambda_{wu}(A)}(\|y - z\| + 2K) + 2K)$$

$$\leq Q(e^{-n\lambda_{wu}(A)}(d_c(y, z) + 2K) + 2K) = Q(e^{-n\lambda_{wu}(A)}(2L + 2K) + 2K)$$

Since $K$ is fixed, first consider a large $L$ and then a large enough $n$ such that

$$Q(e^{-n\lambda_{wu}(A)}(2L + 2K) + 2K) < 2L.$$

For these choices of $n$ and $L$, we have $f^{-n}(B_c(x, L)) \subseteq B_c(f^{-n}(x), L)$.

Case two (Infinite Atoms): Suppose we have an infinitely many atoms on each center leaf. Let $\beta \in \mathbb{R}$ be a large number (for instance much bigger than $KQ$ where $K$ is the distance between $h$ and the identity map and $Q$ is the quasi isometric constant in the definition 2.9). Since we have a finite number of $\tilde{R}_B$, from the previous Lemma, we know that there is a number $\tau \in \mathbb{R}$ for which every center segment of size smaller then $\beta$ must contain at most $\tau$ atoms. But, since on each center leaf there are infinity many atoms, take a segment of leaf big enough so that it contains more then $\tau$ atoms. Iterate this segment backwards and it will eventually be smaller than $\beta$ but containing more than $\tau$ atoms. Indeed,

$$h \circ f^{-n} = A^{-n} \circ h$$

$$\|h(f^{-n}(x)) - h(f^{-n}(y))\| = \|A^{-n}(h(x)) - A^{-n}(h(y))\|$$
\[ \leq e^{-n\lambda^u(A)}\|h(x) - h(y)\| \]

As \( h \) is at a distance \( K \) to identity we have

\[ \|f^{-n}(x) - f^{-n}(y)\| \leq e^{-n\lambda^u(A)}\|h(x) - h(y)\| + K \leq \frac{\beta}{Q} \]

So, finally by quasi isometric property

\[ d_c(f^{-n}(x), f^{-n}(y)) \leq \beta. \]

The above contradiction implies that the number of atoms can not be infinite and now we proceed as in the previous case.

\[ \square \]

**Lemma 3.3.** The disintegration of the Lebesgue measure along the central leaves is mono-atomic, i.e there is just one atom per leaf.

**Proof.** We have a finite number of atoms on each center leaf and since the center foliation is an oriented foliation we may talk about the first atom. The set of first atom of all generic leaves is an invariant set with positive measure, therefore it has full measure. This means there is only one atom per center leaf, which concludes the proof. \[ \square \]

The above Lemma concludes the proof of the theorem. \[ \square \]

**Problem 2.** Is there any ergodic invariant measure \( \mu \) with disintegration having more than one atom on leaves?

We note once again that since the work of Ponce-Tahzibi [7] assures that the set of DA satisfying the hypothesis of the next theorem is non-empty, we prove that these diffeomorphisms have atomic disintegration.

### 3.1. A Glimpse of Pesin Theory.

Before presenting the proof of Theorem [B] we recall some basic notions of Pesin theory. Let \( f : T^3 \to T^3 \) a partially hyperbolic diffeomorphism with splitting

\[ TM = E^s \oplus E^c \oplus E^u. \]

Call \( \Gamma \) the set of regular points of \( f \), that is, the set of points \( x \in T^3 \) for which the Lyapunov exponents are well defined. Then, for each \( x \in \Gamma \) we define the Pesin-stable manifold of \( f \) at \( x \) as the set

\[ W^s(x) = \left\{ y : \limsup_{n \to \infty} \frac{1}{n} d(f^n(x), f^n(y)) < 0 \right\}. \]

The Pesin-stable manifold is an immersed sub manifold of \( T^3 \). Similarly we define the Pesin-unstable manifold at \( x \), \( W^u(x) \), using \( f^{-1} \) instead of \( f \) in the definition.

It is clear that for a partially hyperbolic diffeomorphism \( W^s(x) \) contains the stable leaf \( \mathcal{F}^s(x) \). In Theorem [B] we assume that the central Lyapunov exponent
is negative and consequently the Pesin-stable manifolds are two dimensional. By $W^c(x)$ we denote the intersection of the Pesin stable manifold $W^s(x)$ of $x$ with the center leaf $F^c_f(x)$ of $x$. These manifolds depends only measurable on the base point $x$, as it is proved in the Pesin theory. However, there is a filtration of the set of regular points by Pesin blocks: $\Gamma = \bigcup_{i \in \mathbb{N}} \Gamma_i$ such that each $\Gamma_i$ is a closed (not necessarily invariant) subset and $x \to W^c(x)$ varies continuously on each $\Gamma_i$.

**Theorem B.** Let $f: \mathbb{T}^3 \to \mathbb{T}^3$ be a volume preserving, DA diffeomorphism. Suppose its linearization $A$ has the splitting $T_A M = E^{su} \oplus E^{wu} \oplus E^s$ (su and wu represents strong unstable and weak unstable bundles.) If $f$ has $\lambda^c(x) < 0$ for Lebesgue almost every point $x \in \mathbb{T}^3$, then volume has atomic disintegration on $F^c_f$, in fact it is one atom per center leaf.

**Proof.** To begin, we prove that the size of the weak stable manifolds $W^c(x)$ is uniformly bounded for $x$ belonging to the regular set. In particular this enables us to prove that the partition (mod-0) by $W^c(x)$ is a measurable partition.

**Lemma 3.4.** The size of $\{W^c(x)\}_{\{x: \lambda^c(x) < 0\}}$ is uniformly bounded for $x \in \Gamma$. More precisely, the image of $W^c(x)$ by $h$ is a unique point.

**Proof.** Let $\tilde{f}: \mathbb{R}^3 \to \mathbb{R}^3$ and $\tilde{A}: \mathbb{R}^3 \to \mathbb{R}^3$ denote the lifts of $f$ and $A$ respectively and $\tilde{h}: \mathbb{R}^3 \to \mathbb{R}^3$ the lift of the semi-conjugacy $h$ between $f$ and $A$. Consider $\gamma \subset \tilde{W}^c(x)$, where $\tilde{W}^c(x)$ is the lift of $W^c(x)$. Thus, $\gamma$ is inside the intersection of the center manifold of $\tilde{f}$ and the Pesin-stable manifold of $\tilde{f}$ passing through $x$.

Let us show that $\tilde{h}$ collapses $\tilde{W}^c(x)$ to a unique point. If we prove that, it clearly comes out ( from the bounded distance of $h$ to identity that, the size of $W^c(x)$ is uniformly bounded. Suppose by contradiction that $h(\gamma)$ has more than one point. By semi-conjugacy $\tilde{h}(f^n(\gamma)) = A^n(\tilde{h}(\gamma))$. As $\tilde{h}(\gamma)$ is a subset of weak unstable foliation of $A$ for large $n$ the size of $A^n(h(\gamma))$ is large. On the other hand, $\gamma$ is in the Pesin stable manifold of $f$ and consequently for large $n$, the size of $f^n(\gamma)$ is very small. As $\|h - id\| \leq K$ we conclude that for large $n$ the size of $h(f^n(\gamma))$ cannot be very big. This contradiction completes the proof.

**Corollary 3.5.** The family $\{W^c(x)\}_{\{x: \lambda^c(x) < 0\}}$ forms a measurable partition.

This corollary uses the same idea of the proof of the Proposition 2.5. However, that proposition is proved for continuous foliations and we adapt the proof for the Pesin measurable lamination.

First of all we consider a new partition $\{\tilde{W}^c(x)\}$ whose elements are the closure of the elements $W^c(x)$, that is, $\tilde{W}^c(x)$ is a bounded length center segment with its extremum points. Since $h$ collapses the Pesin-stable manifolds of $f$ into points, two different elements $W^c(x)$ and $W^c(y)$ cannot have a common extrema, so that $\{\tilde{W}^c(x)\}$ is indeed a partition with compact elements. Let us prove that it is indeed a measurable partition.
Lemma 3.6. There exist a set \( \mathcal{A} \subset \mathbb{T}^3 \) and a real number \( R > 0 \), such that \( \text{Vol}(\mathcal{A}) > 0.5 \) and if \( x \in \mathcal{A} \), then \( \text{diam}^c(\mathcal{W}^c(x)) > R \).

Proof. Comes from the fact that \( \lim_{n \to \infty} \text{Vol}(\{x : \text{diam}^c(\mathcal{W}^c(x)) > 1/n\}) = 1 \).

The next lemma is inspired on the work of Ruelle, Wilkinson [17].

Lemma 3.7. Disintegration of volume on the measurable partition \( \{\mathcal{W}^c(x)\} \) is atomic.

Proof. Let \( \pi : \mathbb{T}^3 \to \mathbb{T}^3/\{\mathcal{W}^c(x)\} \) be the natural projection, \( \nu := \pi_* \text{Vol} \) and \( \Lambda \) as in Lemma 3.6. Consider \( B = \pi(\Lambda) \) and \( N \) be the minimum number of balls with diameter \( R/10 \) needed to cover \( \mathbb{T}^3 \). We also denote \( \{\eta_x\} \) the system of conditional measures of \( \text{Vol} \) on the partition \( \{\mathcal{W}^c(x)\} \) defined previously. For \( x \in \mathbb{T}^3 \) define

\[
m(x) = \inf \sum \text{diam}^c(U_i \cap F_f(x))\]

where the infimum is taken over all collections of closed balls \( U_1, \ldots, U_k \) in \( \mathbb{T}^3 \) such that \( k \leq N \) and \( \eta_x(\bigcup_{i=1}^k U_j) \geq 0.5 \).

We now define

\[
m = \text{ess sup}_{x \in B} m(x) .
\]

We claim that \( m = 0 \). Suppose, by contradiction that \( m > 0 \). Then, given \( \varepsilon > 0 \) there exist an integer \( J \) such that

\[
C \varepsilon J N < m/2,
\]

where \( C > 0 \) is such that \( d^c(f^j(y), f^j(z)) \leq C \varepsilon^j d^c(y, z) \) for all pair of points \( y, z \in \Lambda \) with \( z \in \mathcal{W}^c(y) \).
Let $\mathcal{U}$ be a cover of $\mathbb{T}^3$ by $N$ closed balls of diameter $R/10$. For $x \in \mathbb{T}^3$ such that $\pi(x) \in B$, let $U_1(x), \ldots, U_{k(x)}(x)$ such that $\eta(x) \left( \bigcup_{i=1}^{k(x)} U_i(x) \right) \geq 0.5$. Note that

$$f^i \eta_x = \eta_{f^i(x)} \Rightarrow \eta_{f^i(x)} \left( \bigcup_{i=1}^{k(x)} f^j(U_i(x)) \right) \geq 0.5. \forall i \in \mathbb{N}.$$ 

Also note that $\text{diam}^{\epsilon}(f^j(U_i(x))) \leq \alpha \epsilon^j$. By Poincaré recurrence theorem, $y = f^{j_0}(x)$

$$m(y) \leq \sum_{j=1}^{k(x)} \text{diam}(f^{j_0}(U_j(x))) \leq Ck(x) \epsilon^{j_0} \leq CN \alpha \epsilon^{j_0} < m/2.$$

Then, $m = \text{ess sup}_{x \in B} m(x) < m/2$, which is a contradiction with $m > 0$.

Hence, $m = 0$ implies that there is a sequence of closed balls with diameter going to zero and having measure greater then $0.5/N$. By Hammerlindl-Ures [9] we know that $f$ is ergodic, hence we have atomic disintegration.

Once obtained atomicity we can apply Theorem A (see Remarks 1.1 and 1.2) to get one atom per leaf.

4. Maximal entropy measure

Given a volume preserving partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$, we say that a measure $\mu$ is a maximizing entropy measure if the metric entropy with respect to the measure is equal to the topological entropy of $f$, i.e,

$$h_\mu(f) = h_{\text{top}}(f).$$

Given a volume preserving partially hyperbolic diffeomorphism $f : \mathbb{T}^3 \to \mathbb{T}^3$ that is isotopic to a linear Anosov, the maximizing entropy measure is unique and is natural, in the sense that it is just the pull-back of the volume measure by the semi-conjugacy function $h$. More specifically, R. Ures showed in [20].

**Theorem 4.1.** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be an absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic linear automorphism $A : \mathbb{T}^3 \to \mathbb{T}^3$. Then, $f$ has a unique maximizing entropy measure $\mu_f$. More precisely, if $h$ is the semi-conjugacy between $f$ and $A$, i.e, $h \circ f = A \circ h$, then

$$m = h_\mu(f)$$

where $m$ denotes the Lebesgue measure on $\mathbb{T}^3$. 
In the same article, R. Ures also showed that the center Lyapunov exponent for the maximizing measure entropy is greater or equal to the center exponent of the linear hyperbolic diffeomorphism. In the hypothesis of Theorem B this fact contrasts with what happens to the center exponent with respect to the Lebesgue measure, and with this we prove in Theorem C that $\mu_f$ cannot be $u$-gibbs.

**Theorem 4.2** (Theorem 5.1 of [20]). Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a $C^{1+\alpha}$ absolutely partially hyperbolic diffeomorphism homotopic to a hyperbolic linear automorphism $A$ with center Lyapunov exponent $\lambda_A^c > 0$. Let $\mu_f$ be the maximizing measure of $f$. Then, the center Lyapunov exponent of $\mu_f$, $\lambda^{c}_{\mu_f}$, satisfies

$$\lambda^{c}_{\mu_f} \geq \lambda_A^c.$$

To prove Theorem C we need a celebrated result of Ledrappier-Young [12].

**Theorem 4.3** (Theorem A of [12]). Let $f : M \to M$ be a $C^2$ diffeomorphism of a compact Riemannian manifold $M$ preserving a Borel probability measure $\mu$. Then $\mu$ has absolutely continuous conditional measure on unstable manifolds if, and only if,

$$h_\mu(f) = \int \sum_i \lambda^+_i(x) \dim E_i(x) d\mu(x)$$

where $a^+ := \max\{a, 0\}$.

**Theorem C.** Let $f : \mathbb{T}^3 \to \mathbb{T}^3$ be a partially hyperbolic diffeomorphism with Anosov linearization $A : \mathbb{T}^3 \to \mathbb{T}^3$. Assume that $\lambda^c_A > 0$ and $\lambda^u_f < \lambda^c_A$. Then $\mu_f$ is not $u$-gibbs. Also denoting by $\lambda^{u}_{\mu_f}, \lambda^{c}_{\mu_f}, \lambda^{s}_{\mu_f}$ the Lyapunov exponents of $f$ (for $\mu_f$ almost every point) then

$$\lambda^{c}_{\mu_f} > \lambda^c_A \text{ or } \lambda^{u}_{\mu_f} > \lambda^u_A.$$

**Proof.** By contradiction, assume that $\mu_f$ is $u$-Gibbs. Consider

$$A^+ := \{x \in M : \lambda^c(x) \text{ is defined and } \lambda^c(x) \geq \lambda_A^c\}.$$

By 4.2 we have that $\mu_f(A^+) = 1$. Since $\mu_f$ is $u$-gibbs, for some $x \in \mathbb{T}^3$ (actually for $\mu_f$ almost every $x$) the center unstable leaf $\mathcal{F}^{cu}(x)$ intersects the set $A^+$ in a set of positive leaf measure, that is,

$$m_{cu}(\mathcal{F}^{cu}(x) \cap A^+) > 0.$$

Now, define $B = \mathcal{F}^{cu}(x) \cap A^+$ and set

$$C = \bigcup_{y \in B} \mathcal{F}^s.$$

Then, for all $y \in C$ we have that $\lambda^c(y) \geq \lambda_A^c$. Now by absolute continuity property, we get that

$$m(C) > 0.$$
That is, we constructed a set of positive Lebesgue measure for which every point has center Lyapunov exponent bigger then $\lambda_c$ contradicting one of the hypothesis. So $\mu_f$ is not $u$-gibbs as we wanted to show.

Now, from Theorem [1.3] given an invariant measure $\mu$ we can write

$$h_\mu(f) = \lambda^u_\mu \gamma_1 + \lambda^c_\mu \gamma_2$$

for some constants $0 < \gamma_1, \gamma_2 \leq 1$ where

$$\gamma_1 = \gamma_2 = 1 \Leftrightarrow \mu_f \text{ is } u-\text{gibbs.}$$

Also, we know that $h_{top}(f) = h_{top}(A)$. Then, since $\mu_f$ is the maximizing entropy measure it follows that

$$\lambda^u_{\mu_f} \gamma_1 + \lambda^c_{\mu_f} \gamma_2 = \lambda^u_A + \lambda^c_A. \quad (4.1)$$

Since $\mu_f$ is not $u$-Gibbs, then $\gamma_1$ and $\gamma_2$ cannot be both 1. So

$$\lambda^u_{\mu_f} + \lambda^c_{\mu_f} \geq \lambda^u_{\mu_f} \gamma_1 + \lambda^c_{\mu_f} \gamma_2 = \lambda^u + \lambda^c.$$

So either $\lambda^c_{\mu} > \lambda^c$ or $\lambda^u_{\mu} > \lambda^u$. \hfill \Box

We end by analyzing the disintegration of the measure of maximal entropy.

**Theorem D.** Let $f : T^3 \to T^3$ be a partially hyperbolic diffeomorphism with Anosov linearization. If the disintegration of the maximizing entropy measure $\mu_f$ on the central leaves is atomic then it has exactly one atom per leaf.

**Proof.** The proof is analogous to the proof of Theorem [A] in this case we have to verify that

- $\mu_f$ is ergodic and
- the $\mu_f(\cup_i \partial_c \tilde{R}_i) = 0$ (where $\partial_c \tilde{R}_i$ as defined in the proof of Theorem [A]).

The first item was already observed by R. Ures in [20]. The second item comes from the definition of $\mu_f$. We know that the entropy maximizing measure $\mu_f$ is unique and that $h_* \mu_f = m$. Now, by the definition of the sets $\tilde{R}_i = h^{-1}(R_i)$ we have that $\partial_c \tilde{R}_i = h^{-1}(\partial_c R_i)$. Thus

$$\mu_f(\partial_c \tilde{R}_i) = \mu_f(h^{-1}(\partial_c R_i)) = h_* \mu_f(\partial_c R_i) = m(\partial_c R_i) = 0.$$

The proof follows as in the proof of Theorem [A] \hfill \Box

**References**

[1] A. Avila, M. Viana, and A. Wilkinson. Absolute continuity, lyapunov exponents and rigidity i: geodesic flows. arXiv:1110.2365v2, 2011.

[2] A. Baraviera and C. Bonatti. Removing zero lyapunov exponents. *Ergodic Theory and Dynamical Systems*, pages 1655–1670, 2003.

[3] K. Burns and A. Wilkinson. On the ergodicity of partially hyperbolic systems. *Annals of Mathematics*, 171(1):451–489, 2010.
[4] A. Wilkinson C. Bonatti. Transitive partially hyperbolic diffeomorphisms on 3-manifolds. *Topology*, 2005.
[5] J. Franks. Anosov diffeomorphisms. *Global Analysis (Proc. Sympos. Pure Math., Vol. XIV, Berkeley, Calif., 1968)* pp. 61-93 *Amer. Math. Soc.*, 1970.
[6] A. Gogolev. How typical are pathological foliations in partially hyperbolic dynamics: an example. *Israel Journal of Mathematics*, 187(1):493–502, 2012.
[7] G. Ponce and A. Tahzibi. Zero center lyapunov exponent and non-compact center leaves. *to Appear in the Proceedings of AMS*, 2013.
[8] A. Hammerlindl. Leaf conjugacies on the torus. *Ph.D. Thesis*, 2009.
[9] A. Hammerlindl and R. Ures. Ergodicity and partial hyperbolicity on the 3-torus. *Preprint. arXiv:1208.5660*, 2012.
[10] F.R. Hertz, J.R. Hertz, and R. Ures. Accessibility and stable ergodicity for partially hyperbolic diffeomorphisms with 1d-center bundle. *Inventiones Mathematicae*, 172:353–381, 2008.
[11] M. Hirayama and Y. Pesin. Non-absolutely continuous foliations. *Israel Journal of Mathematics*, 160:173–187, 2007.
[12] F. Ledrappier and L. S. Young. The metric entropy of diffeomorphisms: Part i: Characterization of measures satisfying pesin’s entropy formula. *Annals of Mathematics*, 122(3):509–539, 1985.
[13] M. Brin, D. Burago, and S. Ivanov. On partially hyperbolic diffeomorphisms of 3-manifolds with commutative fundamental group. *Modern dynamical systems and applications*, pages 307–312, 2004.
[14] M. Brin, D. Burago, and S. Ivanov. Dynamical coherence of partially hyperbolic diffeomorphisms of the 3-torus. *J. Mod. Dyn*, pages 1–11, 2009.
[15] F. Micena and A. Tahzibi. Regularity of foliations and lyapunov exponents for partially hyperbolic dynamics. *Nonlinearity*, 23:1071–1082, 2013.
[16] A. Hammerlindl R. Potrie. Pointwise partial hyperbolicity in 3-dimensional nilmanifolds. *Preprint*, 2013.
[17] D. Ruelle and A. Wilkinson. Absolutely singular dynamical foliations. *Comm. Math. Phys.*, 219:481–487, 2001.
[18] Radu Saghin and Zhihong Xia. Geometric expansion, Lyapunov exponents and foliations. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 26(2):689–704, 2009.
[19] M. Shub and A. Wilkinson. Pathological foliations and removable zero exponents. *Inventiones Mathematicae*, 2000.
[20] R. Ures. Intrinsic ergodicity of partially hyperbolic diffeomorphisms with a hyperbolic linear part. *Proc. Amer. Math. Soc.*, 140(6), 2012.
[21] R. Varão. Center foliation: absolute continuity, disintegration and rigidity. *arXiv:1302.1637*, 2013.

**DEPARTAMENTO DE MATEMÁTICA, ICMC-USP SÃO CARLOS-SP, BRAZIL.**

*E-mail address: gaponce@icmc.usp.br*

**DEPARTAMENTO DE MATEMÁTICA, ICMC-USP SÃO CARLOS-SP, BRAZIL.**

*E-mail address: tahzibi@icmc.usp.br*

**DEPARTAMENTO DE MATEMÁTICA, ICMC-USP SÃO CARLOS-SP, BRAZIL & DEPARTMENT OF MATHEMATICS, UNIVERSITY OF CHICAGO, CHICAGO, USA.**

*E-mail address: regisvarao@icmc.usp.br*