Dirac equation for massive neutrinos in a Schwarzschild-de Sitter spacetime from a 5D vacuum.

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Abstract

Starting from a Dirac equation for massless neutrino in a 5D Ricci-flat background metric, we obtain the effective 4D equation for massive neutrino in a Schwarzschild-de Sitter (SdS) background metric from an extended SdS 5D Ricci-flat metric. We use the fact that the spin connection is defined to an accuracy of a vector, so that the covariant derivative of the spinor field is strongly dependent of the background geometry. We show that the mass of the neutrino can be induced from the extra space-like dimension.

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I. INTRODUCTION

The study of wave scattering in black hole spacetimes is crucial to the understanding of the signals expected to be received by the new generation of gravitational-wave detectors in the near future\cite{1}. Since the linear perturbations of black holes are represented by fields of integral spins, the study of the scattering of wave fields are concentrated on these cases while that of the Dirac fields are thus less common, especially for the massive ones\cite{2}. The neutrino does not respond directly to electric or magnetic fields. Therefore, if one wishes to influence its orbit by forces subject to simple analysis, one has to make use of gravitational fields. In other words, one has to consider the physics of a neutrino in a curved metric. There are both numerical and analytical methods in solving the various wave equations in black hole scattering\cite{3}. Other approximated solutions can be obtained using the semi-analytic WKB approximation\cite{4}, which has been proven to be very useful and accurate in many cases like, for example, the evaluation of the quasi normal mode frequencies\cite{5}. Quasi normal modes frequencies has been obtained using approximated analytical solutions with SUSY for massless neutrinos\cite{7}.

On the other hand, the extension of 4D spacetime to $N(\geq 5)D$ manifolds is the preferred route to a unification of the interactions of particle physics with gravity. The two approaches to 5D relativity of most current interest are brane theory and induced matter theory, which are mathematically equivalent\cite{6}. The former commonly uses the warp metric in 5D, and in general leads to an extra force on a massive particle in 4D\cite{8}, where however the particle can travel on a null path in 5D\cite{9}. Similar results were found previously for induced-matter theory, which commonly uses the canonical metric in 5D to isolate the fifth force\cite{10} and examine null geodesics in 5D\cite{11}. These investigations are classical in nature, but clearly invite an examination of the corresponding picture in quantum theory. In a more recent work was examined the 4D Klein-Gordon and Dirac equation from 5D null paths\cite{12}.

In this letter we shall study the 4D Dirac equation for neutral $1/2$-spin fermions with mass (neutrinos) which are close to a non-rotating SdS black hole, but using a extended SdS, which describes a 5D vacuum on a 5D Ricci-flat metric. We shall consider that the space-like extra coordinate is noncompact.
II. THE DIRAC EQUATION FOR MASSLESS NEUTRINOS ON A 5D VACUUM

We shall consider the 5D Ricci-flat ($R_{AB} = 0$ and then the Ricci scalar $R = 0$) metric $g_{AB}$, given by the line element\[13\]

$$dS^2 = \left(\frac{\psi}{\psi_0}\right)^2 \left[ -c^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2\theta d\phi^2) \right] + d\psi^2, \quad (1)$$

where $f(r) = 1 - (2G\zeta\psi_0/(rc^2))[1 + c^2 r^3/(2G\zeta\psi_0^2)]$ is a dimensionless function, $\{t, r, \theta, \phi\}$ are the usual local spacetime spherical coordinates employed in general relativity and $\psi$ is the noncompact space-like extra dimension. Furthermore, $\psi$ and $r$ have length units, $\theta$ and $\phi$ are angular coordinates, $t$ is a time-like coordinate, $c$ denotes the speed of light, $\psi_0$ is an arbitrary constant with length units and the constant parameter $\zeta$ has units of $(mass)(length)^{-1}$.

We are concerned with a 5D spacetime as a main bundle and $U(1)$ its structural group. Einstein’s 4D spacetime is the usual base of this bundle, so, physical fields will not depend on the extra coordinate.

A. The 5D Clifford algebra for spinors

To define a 5D vacuum we shall consider a Lagrangian for a massless 5D spinor field minimally coupled to gravity

$$L = \frac{\hbar c}{2} \left[ \bar{\Psi} \gamma^A (\nabla_A \Psi) - (\nabla_A \bar{\Psi}) \gamma^A \Psi \right] + \frac{R}{2K}, \quad (2)$$

where $K = \frac{8\pi G}{c^4}$ and $\gamma^A$ are the Dirac matrices which satisfy

$$\{\gamma^A, \gamma^B\} = 2g^{AB} \mathbb{I}, \quad (3)$$

such that the covariant derivative of the spinor $\Psi$ on $\Pi$ is defined in the following form:

$$\nabla_A \Psi = \left( \partial_A + \frac{1}{8} \Gamma_A \right) \Psi, \quad (4)$$

and the spin connection is given by

$$\Gamma_A = \frac{1}{8} \left[ \gamma^b, \gamma^c \right] e^B_b \nabla_A [e_{cB}], \quad (5)$$

$$\nabla_A [e_{cB}] = \partial_A e_{cB} - \Gamma^D_{AB} e_{cD}$$

being the covariant derivative of the five-bein $e^A_a$ (the symbol $\partial_A$ denotes the partial derivative with respect to $x^A$ and $\eta_{ab} = g_{AB} e^A_a e^B_b$ denotes the 5D
Minkowsky spacetime in cartesian coordinates), which we introduce in order to generalize the well known 4D vierbein [18], but to relate the extended Schwarzschild-de Sitter metric (11) with the 5D Minkowsky spacetime written in cartesian coordinates: $dS^2 = -c^2 dt^2 + (dx^2 + dy^2 + dz^2) + d\psi^2$

$$e^c_B = \begin{pmatrix}
\left(\frac{\psi}{\psi_0}\right) c\sqrt{f(r)} & 0 & 0 & 0 & 0 \\
0 & \left(\frac{\psi}{\psi_0}\right) \frac{\sin \theta \cos \phi}{\sqrt{f(r)}} & \left(\frac{\psi}{\psi_0}\right) \frac{\sin \theta \sin \phi}{\sqrt{f(r)}} & \left(\frac{\psi}{\psi_0}\right) \frac{\cos \theta}{\sqrt{f(r)}} & 0 \\
0 & \left(\frac{\psi}{\psi_0}\right) r \cos \theta \cos \phi & \left(\frac{\psi}{\psi_0}\right) r \cos \theta \sin \phi & -\left(\frac{\psi}{\psi_0}\right) r \sin \theta & 0 \\
0 & -\left(\frac{\psi}{\psi_0}\right) r \sin \theta \sin \phi & \left(\frac{\psi}{\psi_0}\right) r \sin \theta \cos \phi & 0 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}$$

(6)

The Dirac matrices $\gamma^a$ are represented in an Euclidean space instead in a Lorentzian space, and are described by the algebra [19, 20]: $\{\gamma^a, \gamma^b\} = 2\eta^{ab}\mathbb{I}$

$$\gamma^0 = \begin{pmatrix}
-i & 0 & 0 & 0 \\
0 & -i & 0 & 0 \\
0 & 0 & i & 0 \\
0 & 0 & 0 & i
\end{pmatrix}, \quad \gamma^1 = \begin{pmatrix}
0 & 0 & -i \\
0 & i & 0 \\
0 & 0 & 0 \\
i & 0 & 0
\end{pmatrix}, \quad \gamma^2 = \begin{pmatrix}
0 & 0 & 0 & -1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
-1 & 0 & 0 & 0
\end{pmatrix}, \quad \gamma^3 = \begin{pmatrix}
0 & -i & 0 \\
0 & 0 & i \\
i & 0 & 0 \\
0 & -i & 0
\end{pmatrix},$$

(7)

such that $\gamma^4 = i\gamma^0\gamma^1\gamma^2\gamma^3$, and the $\sigma^i$

$$\sigma^1 = \begin{pmatrix}
0 & 1 \\
1 & 0
\end{pmatrix}, \quad \sigma^2 = \begin{pmatrix}
0 & -i \\
i & 0
\end{pmatrix}, \quad \sigma^3 = \begin{pmatrix}
1 & 0 \\
0 & -1
\end{pmatrix},$$

(8)

are the Pauli matrices.

The stress tensor for free massless neutrinos is

$$T^a_b = -\left\{\frac{1}{4} [\bar{\Psi}, \gamma^a \nabla_b \Psi] + \frac{1}{4} [\bar{\Psi}, \gamma^a \nabla_b \bar{\Psi}]\right\},$$

(9)

For a Ricci-flat metric, like (11), the expectation value of this tensor would be zero: $\langle T^a_b \rangle = 0$. Therefore, in a Ricci-flat metric the following condition must be hold

$$[\bar{\Psi}, \gamma^a \nabla_b \Psi] = [\gamma^a \nabla_b \bar{\Psi}, \Psi].$$

(10)
B. The Dirac equation for spinors in 5D

Finally, using the fact that $\gamma^A = e^A_a \gamma^a$, we obtain that the Dirac equation on the metric (1), when we use 3D spherical coordinates $(r, \theta, \phi)$, is

$$\gamma^0 \frac{1}{c \sqrt{f(r)}} \frac{\partial \Psi}{\partial t} + \gamma^r \left[ f(r) \right]^{1/4} \frac{\partial}{\partial r} \left[ r \left[ f(r) \right]^{1/4} \Psi \right] - \frac{\gamma^r}{r} \left( \vec{\Sigma}.\vec{L} + \mathbb{I} \right) \Psi + \gamma^4 \left[ \left( \frac{\psi}{\psi_0} \right) \frac{\partial \Psi}{\partial \psi} + \frac{2}{\psi_0} \Psi \right] = 0,$$

(11)

where $\gamma^r$ is defined as

$$\gamma^r = \gamma^1 \sin \theta \cos \phi + \gamma^2 \sin \theta \sin \phi + \gamma^3 \cos \theta,$$

(12)

and the ordinary angular momentum operators are

$$\vec{\Sigma} = \begin{pmatrix} \vec{\sigma} & 0 \\ 0 & \vec{\sigma} \end{pmatrix}, \quad \vec{L} = \vec{r} \times \vec{p},$$

(13)

so that

$$\gamma^r \left( \vec{\Sigma}.\vec{L} \right) = \gamma^1 \left( - \cos \theta \cos \phi \frac{\partial}{\partial \phi} + \sin \phi \frac{\partial}{\sin \theta \partial \phi} \right)$$

$$+ \gamma^2 \left( - \cos \theta \sin \phi \frac{\partial}{\partial \theta} - \frac{\cos \phi}{\sin \theta} \frac{\partial}{\partial \theta} \right) + \gamma^3 \sin \theta \frac{\partial}{\partial \theta}.$$

(14)

We consider the 5D Dirac equation (11). We can make the ansatz: $\Psi(t, r, \theta, \phi, \psi) = \bar{\Psi}(t, r, \theta, \phi) \Upsilon(\psi)$, such that the equation for $\Upsilon(\psi)$ is

$$\left[ \left( \frac{\psi}{\psi_0} \right) \frac{\partial \Upsilon(\psi)}{\partial \psi} + \frac{2}{\psi_0} \Upsilon(\psi) \right] = m \Upsilon(\psi),$$

(15)

$m = m_0/\psi_0$ being a separation constant.

C. The mass of neutrinos in a 5D vacuum

The solution for the equation (15) is

$$\Upsilon(\psi) = \Upsilon_0 \left( \frac{\psi}{\psi_0} \right)^{m_0 - 2},$$

(16)

where $\Upsilon_0$ is a constant of integration. On the other hand, for $m_0 < 2$ the function $\Upsilon(\psi)$ tends to 0 for $\psi \to \pm \infty$, but is divergent for $\psi \to 0$. In order to the function $\Upsilon(\psi)$ to be
real, we must ask \( m_0 \) take integer unbounded values: \( m_0 = ..., 2, 1, 0, -1, -2, ... \). It appears to be a form of quantization. In the figure (1) we have plotted \( \Upsilon(\psi) \) for different values of \( m_0 \). Notice that for even \( |m_0| \) values the function \( \Upsilon(\psi) \) is even but for odd \( |m_0| \) values the function is also odd. Finally, one could distinguish between two different cases. 

i) For \((\psi_0 > 0 \text{ and } m_0 \geq 0) \text{ or } (\psi_0 < 0 \text{ and } m_0 \leq 0)\), one obtains \( m > 0 \).

ii) For \((\psi_0 > 0 \text{ and } m_0 \leq 0) \text{ or } (\psi_0 < 0 \text{ and } m_0 \geq 0)\), the mass is negative: \( m < 0 \). The last case is a nonsense result in 4D physics and therefore should be discarded.

### III. THE INDUCED 4D DIRAC EQUATION FOR MASSIVE NEUTRINOS CLOSE TO A SDS SPACETIME

Now let us to assume that the 5D spacetime can be foliated by the family of hypersurfaces \( \{\Sigma_0 : \psi = \psi_0\} \). On every generic hypersurface \( \Sigma_0 \) the induced metric is given by the 4D line element

\[
dS_{ind}^2 = -c^2 f(r) dt^2 + \frac{dr^2}{f(r)} + r^2 (d\theta^2 + \sin^2 \theta d\phi^2),
\]

which describes a Schwarzschild-de Sitter spacetime with an equation of state \( \omega = P/(c^2 \rho) = -1 \). If we assume a static foliation of the 5D spacetime on the 4D hypersurface \( \Sigma_0 \), the 4D energy momentum tensor will be described by a perfect fluid \( T_{\alpha\beta} = e^A e^B T_{AB} = (\rho c^2 + P) u_\alpha u_\beta - P h_{\alpha\beta} \), where \( \rho(t,r) \) and \( P(t,r) \) are respectively the energy density and pressure of the induced matter. Furthermore, the 4-velocities \( u_\alpha \) are related to the 5-velocities \( U_A \) by \( u_\alpha = e^A U_A \), and \( h_{\alpha\beta} = e^A e^B g_{AB} \) are the components of the tensor metric in (17). From the relativistic point of view, observers that are on \( \Sigma_0 \) move with \( U^\psi = 0 \). From the mathematical point of view, the Campbell-Magaard theorem[14–17] serves as a ladder to go between manifolds whose dimensionality differs by one. This theorem, which is valid in any number of dimensions, implies that every solution of the 4D Einstein equations with arbitrary energy momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in vacuum. Because of this, the tensor \( \bar{T}_{\mu\nu} \) is induced as a 4D manifestation of the embedding geometry.

The Einstein field equations on \( \Sigma_0 \) for the metric in (17), read

\[
\frac{r}{dr} \frac{df}{dr} - 1 + f = -\frac{8\pi G}{c^2} r^2 \rho, \tag{18}
\]

\[
\frac{r}{dr} \frac{df}{dr} - 1 + f = \frac{8\pi G}{c^4} r^2 P. \tag{19}
\]
The resulting equation of state is
\[ P = -\rho c^2 = -\frac{3c^4}{8\pi G \psi_0^2}, \tag{20} \]
which is the equation of state for a vacuum dominated by the cosmological constant \( \Lambda_0 = 1/\psi_0^2 > 0 \), which has been induced from the extra space-like dimension.

If we take a constant foliation \( \psi = \psi_0 \neq 0 \) [to avoid a possible divergence of \( \Upsilon(\psi = \psi_0) \)] on the metric (11) we obtain the metric (17), such that the effective 4D Dirac equation is given by
\[ \gamma^0 \frac{1}{c\sqrt{f(r)}} \frac{\partial \bar{\Psi}}{\partial t} + \frac{\gamma^r [f(r)]^{1/4}}{r} \frac{\partial}{\partial r} \left[ r[f(r)]^{1/4} \bar{\Psi} \right] - \frac{\gamma^r}{r} \left( \bar{\Sigma} \bar{\Sigma} + \Pi \right) \bar{\Psi} + \gamma^4 m \bar{\Psi} = 0, \tag{21} \]
where \( \bar{\Psi} \equiv \bar{\Psi}(t, r, \theta\phi) \) and \( m < m_0/\psi_0 \) should be real (\( m_0 = 2, 1, 0, -1, -2, ... \)), for any \( \psi_0 \neq 0 \). As a consequence of this result we obtained a quantized mass \( m \). This is an important result that shows how the mass \( m \) of the neutrinos, which are close of a background 4D SdS metric, can be induced from a free massless 5D test spinors close to a Ricci-flat metric in 5D. Finally, numerical solutions for this equation were obtained in [21].

IV. FINAL COMMENTS

We have studied a formalism to describe 4D massive neutrinos which are near a SdS black hole, from a 5D Ricci-flat extended SdS metric. On this metric we define a vacuum and hence the spinors are considered as test massless non-interacting fermion fields. Physically, the background metric here employed describes a 5D extension of an usual SdS static spacetime. The energy-momentum tensor \( \bar{T}_{\mu\nu} \) can be induced in 4D as a manifestation of the embedded geometry, once we use the Campbell-Magaard theorem. This theorem is valid in any number of dimensions and implies that every solution of the 4D Einstein equations with arbitrary energy momentum tensor can be embedded, at least locally, in a solution of the 5D Einstein field equations in vacuum (i.e., at least Ricci-flat).

Of course, this result could be extended to many other applications, as the study of primordial neutrinos in the early universe. This issue will be the subject of future research. Moreover, according to the recent experimental data [22], the study of some possible consequences for the existence of overlighting neutrinos in the inflationary universe deserves a careful study.
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FIG. 1: The function $Y(\psi)$ for different values of $m_0$: a) $m_0 = 1$, b) $m_0 = 0$, c) $m_0 = -1$, d) $m_0 = -2$, e) $m_0 = -3$ and f) $m_0 = -4$.

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