Nonequilibrium dynamics in deconfined quantum critical point revealed by imaginary-time evolution

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As proposed to describe putative continuous phase transitions between two ordered phases, the deconfined quantum critical point (DQCP) goes beyond the prevalent Landau-Ginzburg-Wilson (LGW) paradigm since its critical theory is not expressed in terms of the order parameters characterizing either state, but involves fractionalized degrees of freedom and an emergent symmetry1,2. So far, great efforts have been spent on its equilibrium properties1–16, but the nonequilibrium properties therein are largely unknown. Here we study the nonequilibrium dynamics of the DQCP via the imaginary-time evolution in the two-dimensional (2D) J-Q3 model. We discover fascinating nonequilibrium scaling behaviors hinging on the process of fractionization and the dynamics of emergent symmetry associated with two length scales. Our findings not only constitute a new realm of nonequilibrium criticality in DQCP, but also offer a controllable knob by which to investigate the dynamics in strongly correlated systems.

Exotic phenomena often emerge in the 2D quantum magnetic systems17. A prominent example is the possible Landau-forbidden continuous phase transition between the Néel order and the spontaneously dimerized valence-bond solid (VBS) in the spin-1/2 Heisenberg model. To explain it, the DQCP theory was proposed by showing that the dominant fluctuating modes near this critical point are the deconfined spinons and the emergent gauge fields, and both the Néel and the VBS order parameters are composites of these deconfined objects, rather than fundamental variables1,2. To reconcile these two orders with non-compatible broken symmetries, an emergent SO(5) symmetry appears at the critical point18, demonstrating that the critical theory of DQCP can be cast into the noncompact CP1 model with the monopole fugacity as its dangerously irrelevant scaling variable1,2. Pertinent to this, an extra divergent length ξ′, which measures the spinon confinement length or the thickness of the VBS domain walls, develops, in addition to the conventional correlation length ξ2,13. And they satisfy ξ′ ∝ ξν′/ν with ν and ν′ being the corresponding critical exponents2,11. It was plausibly shown that the interplay between these two length scales may take responsibility for the anomalous equilibrium scaling behaviors near the DQCP12,13.

Near a critical point, the nonequilibrium dynamics is uniquely essential due to the critical slowing down. Extensive studies have been invested in the nonequilibrium behaviors in both classical and quantum phase transitions19,20. For the DQCP, fundamental and interesting questions arise: How does the deconfined dynamic process happen at the critical point? Is there any specific scaling behavior associated with this fractionalized procedure? How do the two length scales affect the nonequilibrium critical dynamics?

To answer these questions, we explore the nonequilibrium dynamics of the DQCP in imaginary-time direction. It was shown that the imaginary-time dynamics not only shares some universal properties with the real-time dynamics21, but also bears amenability to large-scale quantum Monte Carlo (QMC) simulations without sign-problem21. Besides, the imaginary-time evolution recently finds its application in quantum computers22. Moreover, previous studies demonstrated that the short-imaginary-time dynamics (SITD) in the LGW quantum phase transitions23 exhibits scaling behaviors in analogy to the classical short-time critical dynamics24–26, providing fruitful insights in quantum critical dynamics23.

Here we study the SITD in DQCP. The J-Q3 model is taken as an example3. Its Hamiltonian reads

\[ H = -J \sum_{(ij)} P_{ij} - Q \sum_{(ijklmn)} P_{ij}P_{kl}P_{mn}, \]

in which J > 0 and Q > 0, (ij) and (ijklmn) denote, respectively, nearest neighbors and three nearest-neighbor pairs in horizontal or vertical columns on the square lattice, and \( P_{ij} \) denotes the spin singlet operator defined as \( P_{ij} = \frac{1}{2} - S_i \cdot S_j \) with \( S \) being the spin-1/2 operator. The system favors the Néel (VBS) phase with a finite order parameter \( M \) (D) when \( q \equiv J/Q \gg 1 \) (\( q \ll 1 \)). For the SITD, the evolution of the wave function \( |\psi(\tau)\rangle \) obeys the imaginary-time Schrödinger equation \( -\frac{\partial}{\partial \tau} |\psi(\tau)\rangle = H|\psi(\tau)\rangle \) with an uncorrelated initial state23.

We first study the dynamics of two length scales. The SITD theory in the LGW phase transition with a single length scale \( \xi \) shows that for an uncorrelated initial state, \( \xi \) increases with \( \tau \) as \( \xi \propto \tau^{z} \) with \( z \) the dynamic exponent23. In the DQCP it is widely believed that \( z = 1 \)
FIG. 1. Relaxation dynamics with the VBS initial state. a, Illustration of the fractionalization process of spinons. Shown is the evolution of typical configurations for a sampled overlap $\langle \psi_{\text{left}} | \psi_{\text{right}} \rangle$ in $S = 1$ sector with two spinon strings which are initially located in nearest-neighbor sites embedded in the VBS background (not shown). b, At the critical point, in the short-time stage, the size of spinon pair, $\Lambda$, increases with the imaginary-time $\tau$ as $\Lambda \propto \tau^{0.931}$, obtained by power fitting. This exponent is close to 1, indicating that $\Lambda \propto \tau^{1/z}$.

This result is further confirmed by the finite-size scaling (FSS) collapse according to $\Lambda \propto L f(\tau L^{-2z})$ as shown in b (right). c, The average sign of the VBS order parameter $M$ is reshaped as a density of domain wall energy $\nu/\nu'$, accordingly, $q$ and thus one might take it for granted that $\xi \propto \tau^{1/z}$. To see this, we study the deconfined process from an initial state with a triplet embedded in the VBS background at the critical point (determined in the following). We find that the size of the spinon pair $\Lambda$, defined via the strings connecting the unpaired spins in the $S = 1$ sector, increases with $\tau$ as $\Lambda \propto \tau^{0.931}$ as shown in Fig. 1 a,b. This demonstrates that $\xi' \propto \tau^{1/z}$ since it was shown that $\xi' \propto \Lambda^{13}$. There follows $\xi \propto \tau^{1/z}$ with $z_u \equiv \frac{2\beta'}{\nu'}$ (subscript $u$ indicates usual length scale).

Generally, the relaxation dynamics of the DQCP is controlled by the changes of two relevant length scales $\xi' \propto \tau^{1/z}$ and $\xi \propto \tau^{1/z_u}$. For an operator $Y$ its dynamic
FIG. 2. Relaxation dynamics with the saturated Néel initial state. a, The average sign of the Néel order parameter \( I_M \) is used to estimate the critical point. For different \( L \) and fixed \( \tau L^{-1} = 1/4 \), crossing points (left) of curves of \( I_M - q \) are extrapolated to estimate the critical point as \( q_c \approx 0.671(2) \) (right), corroborating that from \( I_D \) shown in Fig. 1. b, At \( q = q_c \), in the long-time stage, \( M^2 \propto L^{-1.220} \), suggesting that \( 2\beta/\nu \approx 1.220(6) \) (Supplementary Information), while in the short-time stage, \( M^2 \propto \tau^{-0.863} \), suggesting the exponent is \( 23/\nu z_u \). FSS collapse (right) confirms the scaling form \( M^2(\tau, L) = (\tau L^{-z_u} f(\tau L^{-z_u})) \). Here, the subscript \( z \) of \( M \) means its \( z \)-component. c, In the short-time region, \( D^2 \) increases as \( D^2 \propto \tau^{-9.689} \) (left), this exponent is also close to \( (d - 2\beta/\nu)/z \) albeit with relatively large error (Supplementary Information), and \( D^2 \propto L^{-2} \) for fixed \( \tau \) (supplementary information). The whole relaxation satisfies \( D^2(\tau, L) = L^{-d} \tau^{4/3 - 2\beta/\nu} f(\tau L^{-z_u}) \), verified by FSS collapse (right). Here, the subscript \( z, x \) of \( D \) means its component with \( z \)-direction spin and \( x \)-direction bond. d, In the short-time stage, the susceptibility \( \chi \) increases as \( \chi \propto \tau^{4/3 - \nu'/\nu} \) and depends on \( L \) as \( \chi \propto L^{-1 + \nu'/\nu} \) (left) (Supplementary Information). The scaling form \( \chi(\tau, L) = L^{-1} f(\tau L^{-\xi}) \) is verified by scaling collapse (right). These results show that \( I_M \) and \( D^2 \) is controlled by the dynamics of \( \xi' \), \( M^2 \) is controlled by the dynamics of \( \xi \), and \( \chi \) is hiddenly controlled by both. Note that \( M^2 \) and \( D^2 \) exchange their scaling forms compared with those from the VBS initial state as shown in Fig. 1.

Scaling should satisfy

\[
Y(\tau, \delta, L, \{X\}) = \tau^{\tilde{z}} f(\delta \tau^{\tilde{\nu}}, \tau L^{-z_u}, \tau L^{-z_u}, \{X\} \tau^{-\tilde{z}}),
\]

in which \( s \) is the exponent related to \( Y \), \( \delta \equiv q - q_c \) with \( q_c \) the critical point, \( L \) is the lattice size, and \( \tilde{z} \) is the dynamic exponent, which can be \( z \) or \( z_u \), or their combination, depending on the dynamic process, similarly, \( \tilde{\nu} \) can be \( \nu \) or \( \nu' \). and \( \{X\} \) with its exponent \( c \) represents other possible relevant variables associated with the initial state. For saturated order and completely disordered initial states, \( X \) vanishes since these states keep invariant under scale transformation\(^{23,24} \). If \( z_u = z \), equation (2) recovers the usual single-length-scale SIDT scaling theory, in which, for instance, a dimensionless variable at \( \delta = 0 \) is a function of \( \tau L^{-z_u} \), and the order parameter scales as \( M^2 \propto \tau^{-2\beta/\nu} f(\tau L^{-z_u}) \) for a saturated initial state and \( M^2 \propto L^{-d} \tau^{-2\beta/\nu} f(\tau L^{-z_u}) \) for a disordered initial state\(^{26} \).

We then explore the dynamic scaling in model (1) from a saturated VBS state. For a dimensionless quantity \( I_D(\tau, \delta, L) \), defined as the average sign of the VBS order parameter, \( I_D \equiv \langle \text{sgn}(D(\tau)) \rangle \), we find in Fig. 1 c that for a fixed \( \tau L^{-z} \) the convergence of crossing-points of \( I_D(\tau, \delta, L) \) for large \( L \) corroborates the value of the critical point \( q_c \approx 0.671(2) \)\(^7 \), which indicates that \( \tau L^{-z} \) dominates the dynamics of \( I_D \), rather than \( \tau L^{-z_u} \). Otherwise, no convergence should be seen for fixed \( \tau L^{-z} \). This makes the scaling form of \( I_D \) remarkably different from the usual one since the relevant length scale is not the conventional \( \xi \) but the spinon confinement length \( \xi' \). A possible reason is that \( I_D \) is closely related to the domain walls separating regions with different VBS state, of which the thickness is characterized by \( \xi' \).

In equilibrium, \( D^2 \) and \( M^2 \) were shown to satisfy similar scaling forms\(^{7,10,18} \). Strikingly, here we find that their relaxation dynamics from a saturated VBS initial state are controlled by different length scales. For \( D^2 \), Fig. 1 d shows that it obeys \( D^2(\tau, L) = \tau^{-2\beta/\nu} f(\tau L^{-z_u}) \) for which \( \xi \approx \tau^{-\frac{1}{\nu}} \) dominates, similar to the usual single-length-scale case. In the short-time stage, \( D^2(\tau, L) \propto \tau^{-2\beta/\nu} \) and it is almost independent of \( L \), while its long-time limit recovers the equilibrium scaling \( D^2 \propto L^{-\frac{2\beta}{\nu}} \). The scaling form of \( D^2 \) meets these two conditions simultaneously provided that \( f(\tau L^{-z_u}) \sim (\tau L^{-z_u})^{-\nu/\nu'} \) for \( \tau \to \infty \). Otherwise, if \( \xi' \) dominates, the appearance of \( \tau L^{-z} \) will make it difficult for the scaling function to satisfy these two limits simultaneously with a simple form. The above scaling form is confirmed by rescaling collapse.
in Fig. 1d. Contrarily, for \( M^2 \), Fig. 1e shows that its short-time dynamics obeys \( M^2 \propto L^{-d+\frac{d_z}{\nu}-2\frac{1}{\nu_z}} \) (Supplementary Information), requiring its full scaling form to be \( M^2(\tau, L) = L^{-d+\frac{d_z}{\nu}-2\frac{1}{\nu}} f(\tau L^{-\nu}) \), where \( \xi' \) and \( \tau L^{-\nu} \) dominate. A possible reason for this discrepancy is that \( D^2 \) is deeply affected by the memory from the initial state, whereas \( M^2 \) only feels a disordered initial state (Supplementary Information).

More interestingly, some quantities can even show fascinating relaxation behaviors controlled by the dynamics of \( \xi \) and \( \xi' \) simultaneously. For instance, in equilibrium, the VBS domain wall energy density \( \kappa \) is found to scale as \( \kappa \propto \xi^{-1} - \xi'^{-1}(d+z-2) \) \((d+z-2 = 1)\). Here we generalized this scaling into the nonequilibrium case. As shown in Fig. 1f, \( \kappa \) relaxes according to \( \kappa(\tau, L) = \tau^{-\frac{1}{2}} \tau^{-\frac{1}{\nu}} f(\tau L^{-\nu}) \) from the saturated VBS state at \( q_c \). In the short-time stage, \( \kappa \propto \tau^{-\frac{3}{2}} \tau^{-\frac{1}{\nu}} \), while in the long-time stage, \( \kappa \) crosses over to \( \kappa \propto L^{-1+\frac{d_z}{\nu}} \) as \( f(\tau L^{-\nu}) \propto (\tau L^{-\nu})^{\frac{1}{2}+\frac{1}{\nu}} \) for \( \tau \to \infty \), recovering its equilibrium finite-size scaling \(^2^{13}27\) (Supplementary Information).

To illustrate the role played by the initial state in the dynamic relaxation dynamics, we now study the dynamic scaling with the saturated antiferromagnetic initial state. From Fig. 2a, we find that the average sign of the Néel order parameter defined as \( I_M = \langle \text{sgn}(M(\tau)) \rangle \) obeys \( I_M = f(\tau L^{-\nu}) \) similar to \( I_D \). The critical point estimated by crossing-point analyses of \( I_M \) is 0.671(2), consistent with that given by \( I_D \). Notably, we find that \( M^2 \) and \( D^2 \) exchange their scaling forms compared with the VBS initial state case. Namely, \( M^2(\tau, L) = \tau^{-\frac{d_z}{\nu}} f(\tau L^{-\nu}) \) governed by \( \xi \), and \( D^2 \) satisfies \( D^2(\tau, L) = L^{-d+\frac{d_z}{\nu}-2\frac{1}{\nu}} f(\tau L^{-\nu}) \) governed by \( \xi' \), as shown in Fig. 2b,c,d, respectively. In addition, we find in Fig. 2e that the dynamics of the susceptibility \( \chi \) at the momentum \((\frac{2\pi}{L}, 0)\) satisfies \( \chi(\tau, L) = L^{-\frac{d}{\nu}} f(\tau L^{-\frac{d}{\nu}}) \) (here \( d - z = 1 \)). An interesting phenomenon is that in the short-time stage \( \chi(\tau, L) \sim L^{-1}(\tau L^{-\frac{d}{\nu}})^{\frac{1}{2}} \), indicating a hidden interplay between the dynamics of two length scales (Supplementary Information).

The exchange of scaling forms for \( M^2 \) and \( D^2 \) exhibits an enchanting dual dynamic scaling behavior, which reflects the imprint of emergent symmetry in nonequilibrium dynamics. Its appearance can be traced back to the intertwining relation between the Néel order and the VBS order, i.e., the Néel order can be regarded as the condensation of the spinon which stays at the intersection of the VBS domain walls; while the VBS order can be regarded as the condensation of the hedgehog topological defects of the Néel phase \(^12\). In this context, \( \xi' \) is not only the typical distance of spinons which make up the Néel order parameter from the VBS background, but also measures the thickness of the VBS domain walls which are constructed by the quadrupled monopole events in the antiferromagnetic background and characterizes the VBS fluctuations. This may explain why \( M^2 \) \((D^2) \) is controlled by the dynamics of \( \xi' \) when the initial state is the saturated VBS (Néel) state.

The dual dynamic scaling also manifests itself when the initial state is a disordered state. We find in Fig. 3a,b that \( M^2 \) and \( D^2 \) evolves according to the same scaling
form $P^2(\tau, L) = L^{-d} \xi^d \xi^{-\frac{2\theta}{z}} f(\tau L^{-z})$ in which $P$ represents $M$ or $D$ and $\xi^d$ dominates their relaxation dynamics. Moreover, for this initial state, the SITD features another characteristic phenomenon dubbed the critical initial slip (CIS)\cite{23, 24}. The CIS in the usual LGW criticality shows that $C_P(\tau) \equiv L^d(P(0)P(\tau))$ obeys $C_P(\tau) \approx \tau^\theta f(\tau L^{-z})$ with $\theta > 0$ \cite{29}. In contrast, in the DQCP, we find in Fig. 3\textsuperscript{c,d} that $\xi^d$ governs the CIS behavior and $C_P$ obeys $C_P(\tau, L) \approx \tau^\theta f_c(\tau L^{-z})$ in which both $\theta$ are negative, i.e., $\theta \approx -0.516(5)$ for $M$ and $\theta \approx -0.562(23)$ for $D$ (Supplementary Information). In the LGW criticality, the reason for a positive $\theta$ is that the real critical point usually shifts towards the ordered phase compared with its mean-field value due to critical fluctuations\cite{23, 24}. However, in the DQCP, both phases are ordered phases and the fluctuations are well-matched in both sides of the critical point due to the emergent symmetry. This explains the negative $\theta$. In addition, the value of $\theta$ for $M$ is close to that for $D$, demonstrating again the dual dynamic scaling.

To conclude, our work studies the nonequilibrium dynamics in 2D DQCP and finds that the interplay between the deconfined process and the fluctuating modes can contribute striking nonequilibrium properties. Our present results could be realized in noisy-intermediate-scale quantum computers (e.g., Regetti), for which the imaginary-time evolution algorithm was proposed recently\cite{22}. Our work also provides significant instructions in the real-time dynamics of the DQCP by noting that the short-time scaling also manifests itself in real-time dynamics in various quantum systems\cite{30}. Moreover, our results can be generalized to other systems featuring exotic deconfinement dynamics and boundary criticality of DQCP.

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Methods

The imaginary-time relaxation results for the J-Q₃ model were obtained using the projector QMC method. The method is well documented in the literature. Here we only give brief overviews, particularly focusing on its application in the imaginary-time dynamics.

For a given starting state |ψ(0)⟩ at τ = 0, the imaginary-time evolution of the state obeys the Schrödinger dynamics which gives |ψ(τ)⟩ = e⁻τH|ψ(0)⟩. In the projector method, the imaginary-time evolution operator U(τ) = e⁻τH is Taylor-expanded in powers of the Hamiltonian Hⁿ. The normalization Z = ⟨ψ(0)| e⁻²τH|ψ(0)⟩ is importance sampled using the overlap of the bra and ket states as the sampling weight in the standard S² basis or other basis such as the nonorthogonal valence bond (VB) basis. The expansion power n is updated and automatically truncated to some maximum length without detectable errors. A full Monte Carlo sweep consists of local diagonal updates and global off-diagonal updates. The local diagonal updates replace unit operators with diagonal ones with appropriate acceptance rate and vice versa in the operator sequence. The global operator-loop updates used here switch the operator types from diagonal to off-diagonal or vice versa and update the corresponding states along the propagation direction with probability 1/2. Detailed balance and ergodicity are maintained. The computational consumption of a full sweep of Monte Carlo update scales as 2τN with N = Lᵈ. Expectation value for an operator O at imaginary time τ is given by ⟨O(τ)⟩ = ⟨ψ(τ)|O|ψ(τ)⟩/Z, where ⟨⋯⟩ represents nonequilibrium average.

In the projector QMC method, the initial states are realized by controlling the boundaries of the projection direction, for instance, to enforce antiferromagnetic order in the initial state, the loops touching the imaginary-time boundaries in the operator-loop updates should be fixed. Regarding the initial state required, different bases are applied. For the VBS initial state, the VB basis is used as it restricts the sampling in the singlet sector. In this basis, the initial state is expressed in terms of valence bonds, which makes it advantageous for VBS initial states. For the antiferromagnetic or disordered initial state, the simulations are carried out standard S² basis.

To study the relaxation dynamics, we investigate the imaginary-time evolution of different physical quantities under different initial conditions, including the Néel and VBS order parameters, domain wall energy density κ, spinon size Λ, and susceptibility χ.

The Néel and VBS order parameters are defined as M = ∑ᵣₙ(-1)ⁿSᵣ/N and Dₓ = ∑ᵣₙ(-1)ⁿ(Sₓ · Sᵣ₊ₓ)/N, respectively. The subscript x in Dₓ represents x-oriented bond order and x is the unit lattice vector in the x direction. The squared order parameters M² and Dₓ² are defined accordingly. In the VB basis, improved estimators for M² and Dₓ² are available, which are expressed in the context of loops formed by valence bonds. In the S² basis, we compute the squared z-component of the VBS order parameter Dₓ², which has the same scaling form of the full VBS order parameter. To ensure the convergence, we analyze the imaginary-time dependence of the order parameters until the results only fluctuate within statistical errors.

For the density of domain wall energy κ, different boundary conditions are employed in order to introduce twisted boundaries. In the background of y-oriented dimers, shifting one of the two x-edges by one lattice spacing brings a twist angle of Δφ = π. The periodic boundary condition in the x direction is broken. The energy E(Δφ) difference between the system with and without twisted boundaries defines the size-normalized domain wall energy density, namely, κ = (E(π) − E(0))/L. Analogously, rotating one of the two x-edges from y into x-oriented bond order causes Δφ = π/2 (the boundaries are illustrated in Supplementary Information). In the projector QMC method, E is simply given by the average number of non-unit operators, namely E = −⟨n⟩/2τ.

The computations of the spinon size Λ are done in the S = 1 sector with two unpaired spins in the VBS background. The unpaired spin in the bra and ket states is connected via a string, which represents a spinon. The two spinon distance is found capable in capturing the faster divergence of ξ'. The definition of Λ is associated with the lattice sites covered by the string. For the two spinons covering n₁ and n₂ sites, we compute ⟨ξ²⟩ = ∑ᵣᵡ¹⁰ᵣᵡ₂/n₁n₂ with rᵡᵡ = |rᵡ(1) − rᵡ(2)|² being the squared distance of the two points i and j located on the two strings. The spinon size Λ is then defined as Λ = ⟨ξ²⟩¹/₂.

The susceptibility χ studied here is defined via the fourier transformation of the local correlations G(i, j) = ∂(Sᵢ) / ∂Γⱼ at zero external field Γⱼ = 0, which measures the response of spin i to the external field Γⱼ at site j. In the projector QMC method, χ(q) is given by

\[ \chi(q) = \frac{\tau}{N} \left( \sum_{m=0}^{n} \frac{w(n, m)}{n + 1 - m} Mₐ(m) \sum_{p=0}^{n} M₋ₐ(p) \right) \]

\[ + \sum_{m=0}^{n} \frac{w(n, m)}{n + 1 - m} Mₐ(m)M₋ₐ(m) \],

in which Mₐ = ∑ᵣᵡSᵣₑ⁻ᵣₑ, and w(n, m) = (n⁺²⁻ⁿ⁻ᵐ)/n. Only the real parts should be calculated since χ(q) must be real-valued. Here we compute χ at the smallest nonzero momentum q = (2π/T₀) since the uniform susceptibility χ(q = 0) vanishes in the S = 0 sector.

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I. SUPPLEMENTARY INFORMATION: ADDITIONAL NUMERICAL RESULTS

A. The road to emergent symmetry

The DQCP theory shows that near the critical point of the J-Q3 model, the deconfined spinons coupled to an emergent $U(1)$ gauge field dominate the critical behaviors and the Néel order parameter $M \equiv (M_x, M_y, M_z)$ and VBS order parameter $D \equiv (D_x, D_y)$ should be understood as composites of these objects. Although these two order parameters are utterly inequivalent microscopically, an emergent SO(5) symmetry between them arises in the infrared scale to take into account the rotation in the superspin $S = (M_x, M_y, M_z, D_x, D_y)^{1-4}$. This emergent symmetry only appears at the critical point, while in the VBS phase, the system breaks this continuous symmetry into a discrete $Z_4$ symmetry$^2$. Accordingly, besides the usual correlation length $\xi$, there exists another length scale $\xi'$ characterizing the spinon confinement length or the thickness of the VBS domain walls. Scaling theories based on these two length scales were proposed to explain the anomalous scaling properties near the DQCP.

![Diagram showing the road to the emergent symmetry.](https://example.com/diagram.png)

**FIG. S1. The road to the emergent symmetry.** a, With a saturated initial VBS state, the histogram of the probability distribution of the order parameters shows that the SO(5) symmetry emerges gradually as $\tau$ increases at the critical point (after a trivial rescaling of VBS order parameter). b, Similarly, with a saturated initial Néel state, the histogram of the probability distribution of the order parameters also shows that the SO(5) symmetry emerges gradually as $\tau$ increases at the critical point.

In the imaginary-time relaxation, the energy scale flows from the ultraviolet scale down to the infrared scale.
Accordingly, the generation process of the emergent symmetry can be observed. From a saturated VBS order with vanishing correlation length, we show in Fig. S1 the dynamics of the appearance of the SO(5) symmetry at the critical point of model (1) by calculating numerically the evolution of the probability distribution of the order parameters. As displayed in Fig. S1, in the infancy stage, the microscopic dynamics dominates and the system breaks the $Z_4$ symmetry. Then the probability distribution of the superspin begins to spread. In the $D_x - M_z$ plane, the symmetry under continuous $U(1)$ of $D$ emerges gradually, although the microscopic lattice symmetry only allows $\pi/2$ rotation. Moreover, in the $D_x - M_z$ plane, another $U(1)$ symmetry mixing the $M_z$ and $D_x$ emerges gradually from an apparent symmetry-breaking initial state. By taking into account the SO(3) symmetry of $M$, we demonstrate the relaxation dynamics to the emergent SO(5) symmetry. Similarly, Fig. S1b shows the generation process of the emergent SO(5) symmetry with a saturated Neel initial state.

B. Scaling properties of the Neel and VBS order parameters

First, we show the scaling properties of $M^2$ and $D^2$ in the long-time equilibrium limit $\tau \to \infty$. In equilibrium, it was shown that at the critical point $M^2$ satisfies $M^2 \propto L^{-2\beta/\nu}$ and $D^2$ also satisfies $D^2 \propto L^{-2\beta/\nu}$ with almost the same exponents. In Fig. S2, we verify these scaling properties.

Second, we explore the dependence of the dynamics of $M^2$ on the lattice size $L$ with the initial state being the saturated VBS state. In Fig. S3, we find that in the short-time relaxation stage $M^2 \propto L^{-\beta}$ for a fixed $\tau$, confirming $M^2 \propto L^{-\beta}$. Combining the dependence of $M^2$ on $\tau$, we confirm that in the short-time stage, $M^2$ satisfies $M^2 \propto L^{-d+\beta/\nu}$. Similarly, from a saturated Neel state, the VBS order parameter satisfies $D^2 \propto L^{-d}$ for fixed $\tau$ in the short-time stage as shown in Fig. S4. Accordingly, in the short-time region, $D^2$ obeys $D^2 \propto L^{-d+\beta/\nu}$. In addition, when the initial state is the disordered state, both $M^2$ and $D^2$ are proportional to $L^{-2}$ for a fixed time in the short-time stage as shown in Fig. S5, demonstrating their short-time relaxation dynamics satisfy $M^2 \propto D^2 \propto L^{-d+\beta/\nu}$.

Third, as we discussed above, the relaxation dynamics of $M^2$ from the saturated VBS initial state satisfies $M^2 \propto L^{-d+\beta/\nu}$ in the short-time stage. We find this scaling behavior is well captured for large system sizes. However, for small systems, finite size corrections should be included. We extract the exponent of $\tau$ for different system sizes as shown in Fig. S6. It is expected this exponent will be saturated at $\frac{\beta}{\nu}$ for large $L$. Similar behavior of the exponent also happens for the dynamics of $D^2$ from the saturated Neel initial state, as shown in Fig. S6.

![Equilibrium finite-size scaling of M^2 and D^2 at the critical point](image)

FIG. S2. Equilibrium finite-size scaling of $M^2$ and $D^2$ at the critical point. For large $\tau$, the system tends to the equilibrium state and the order parameters becomes their equilibrium values. They are independent of $\tau$ and only depend on $L$. Power fitting shows that $M^2 \propto L^{-1.220}$ and $D^2 \propto L^{-1.220}$. These two exponents are just $2\beta/\nu$. They are close to each other, indicating the appearance of the emergent symmetry.

In above cases, by assuming that the short-time dynamics and the long-time equilibrium dynamics can be described by a unified scaling form, we find that the order parameter $M^2$ in the whole relaxation process obeys $M^2 = L^{-d+\beta/\nu} f(\tau L^{-z})$, and $D^2$ satisfies the similar scaling form only with a different scaling function. In these scaling forms, the scaling function $f(\tau L^{-z}) \sim \text{const}$ with $\text{const}$ being a constant when $\tau \ll L^{z}$, while $f(\tau L^{-z}) \sim (\tau L^{-z})^{\frac{\beta}{\nu}\frac{d}{2}}$ when $\tau \gg L^z$. Although this scaling form is similar in form to the conventional corre-
FIG. S3. Dependence of $M^2$ on $L$ for fixed $\tau$ in the short-time relaxation stage with the saturated VBS initial state. For different lattice sizes $L$ and $\tau = 16$, $M^2 \propto L^{-2}$. The exponent 2 is just the spatial dimension of the system.

FIG. S4. Dependence of $D^2$ on $L$ for fixed $\tau$ in the short-time relaxation stage with the saturated Néel initial state. For different lattice sizes $L$ and $\tau = 16$, $D^2 \propto L^{-2}$. The exponent 2 comes from the spatial dimension of the system.

sponding scaling theory\(^5\), they are remarkably different from each other, since in the present case, $z$ represents the scaling relation between the time and the spinon confinement length instead of the usual correlation length.

Here, we want to emphasize that although for the present range of lattice sizes the scaling form works quite well, more complicated scaling behaviors can not ruled out for large system size. Therein both $\tau L^{-z}$ and $\tau L^{-z_u}$ are possibly involved simultaneously. However, to probe these possible complex scaling behaviors, more powerful computational resources are needed. And we leave them for further studies.

C. Scaling properties of the density of the domain wall energy $\kappa$

The density of domain wall energy $\kappa$ can be calculated by the energy difference between ground states with and without domain walls\(^6\). Figure S7 illustrates two different boundary conditions of the VBS domain walls with two different twist angles. In the left panel of Fig. S7, the VBS order has a relative shift of one lattice spacing between two opposite edges. This corresponds to the twist angle $\Delta \phi = \pi$ of the VBS order parameter. In the right panel of Fig. S7, the VBS order has vertical and horizontal dimers on two opposite sides. This corresponds to the twist angle $\Delta \phi = \pi/2$ of the VBS order parameter.

In the main text, we focus on the case for $\Delta \phi = \pi$. We find that in the short-time stage, $\kappa$ changes with $\tau$ as
FIG. S5. Dependence of $M^2$ and $D^2$ on $L$ for fixed $\tau$ in the short-time relaxation stage with the completely disordered initial state. For different lattice sizes $L$ and $\tau = 8$, $D^2 \propto M^2 \propto L^{-2}$. The exponent 2 comes from the spatial dimension of the system.

FIG. S6. Dependence of the exponent of $\tau$ in $M^2$ ($D^2$) on $L$ in the short-time relaxation stage with the VBS (Néel) initial state.

FIG. S7. Boundary conditions for the calculation of the domain wall energy density $\kappa$. Boundary states with a twist angle $\Delta \phi = \pi$ and $\Delta \phi = \pi/2$ are shown in the left and right panels, respectively. The blue bonds represent the bulk state and the red bonds represent the edge state. The open boundary condition is chosen in the horizontal direction and the periodic boundary condition is imposed in the vertical direction.
\( \kappa \propto \tau^{-\frac{3}{2}} \tau^{-\frac{4}{9}} \) at the critical point and its dependence on \( L \) is very weak. This result is consistent with the dynamic generalization of the scaling relation \( \kappa \propto \xi^{-1} \xi'^{-1} \), given that \( \xi(\tau) \sim \tau^{-\xi} \) and \( \xi'(\tau) \sim \tau^{-\xi'} \). In the long-time equilibrium limit, \( \kappa \) recovers its equilibrium scaling \( \kappa \propto L^{-(1+\frac{d}{d^*})} \). These two limits impose that in the whole process, \( \kappa \) should satisfy \( \kappa(\tau,L) = \tau^{-\frac{1}{2}} \tau'^{-\frac{1}{9}} f(\tau L^{-\frac{1}{2}}) \). In the main text, this scaling form is confirmed by scaling collapse for different system sizes.

**FIG. S8.** Dependence of \( \kappa \) on \( \tau \) for fixed \( \tau L^{-\frac{d}{d^*}} \) with different boundary conditions. For \( \tau L^{-\frac{d}{d^*}} = 1/4 \), \( \kappa \) decays with \( \tau \) as \( \kappa \propto \tau^{-1.717} \) with the exponent being \( 1 + \nu' / \nu \). This relation applies for different boundary conditions, demonstrating the universal properties.

Here we check the universal properties of the scaling behaviors of \( \kappa \) with a different twist angle \( \pi / 2 \). In this case, we find that for fixed \( \tau L^{-\frac{d}{d^*}} \) with \( z = 1 \), \( \kappa \) kappa decays with \( \tau \) according to \( \kappa \propto \tau^{-\frac{1}{2}} \tau^{-\frac{1}{9}} \), as shown in Fig. S8. For comparison, we also plot the results for \( \Delta \phi = \pi \). These results show that \( \kappa \) satisfies \( \kappa(\tau,L) = \tau^{-\frac{1}{2}} \tau'^{-\frac{1}{9}} f(\tau L^{-\frac{1}{2}}) \) regardless of the boundary condition, confirming its universality.

### D. Scaling properties of the susceptibility \( \chi \)

In the main text, we show that \( \chi \) increases with \( \tau \) as \( \chi \propto \tau^{\frac{d}{d^*}} \) for a saturated Néel initial state. Here, we show that in the short-time stage, \( \chi \propto L^{-[(d-z) + \frac{d^*}{d}]} \) (here \( d - z = 1 \)) for a fixed \( \tau \), as illustrated in Fig. S9. Thus, in the short-time stage, \( \chi \propto L^{-1} \) as shown in the main text. Also, in the main text, the scaling collapse shows that \( \chi \) in the whole relaxation process satisfies \( \chi = L^{-(1+\frac{d^*}{d})} \tau^{\frac{d}{d^*}} f(\tau L^{-\frac{1}{2}}) \).

However, it was shown that for larger system size the equilibrium scaling of \( \chi \) crosses over to \( \chi \propto L^{−\frac{d}{d^*} \nu} \). This indicates that the above scaling form is not competent to describe the scaling behavior for large system sizes. We find that the full scaling form should be amended by including \( \tau L^{−z_u} \) as an additional variable in the scaling function,

\[
\chi(\tau, L) = L^{-(d-z)} f(\tau L^{-\frac{1}{2}}, \tau L^{-z_u}). \tag{S1}
\]

For small system sizes, \( \tau L^{-\frac{d}{d^*}} \) dominates and equation (S1) can be simplified as \( \chi(\tau, L) = L^{-(d-z)} f(\tau L^{-\frac{1}{2}}) \sim L^{-(d-z-\frac{d}{d^*})} f(\tau L^{-\frac{1}{2}}) \). For large system sizes, a possible crossover behavior is that equation (S1) can be approximated as \( \chi(\tau, L) = L^{-(d-z)} f(\tau L^{-\frac{1}{2}}, \tau L^{-z_u}) = L^{-(d-z)} (\tau L^{-\frac{1}{2}})^{\frac{d}{d^*}} g(\tau L^{-\frac{1}{2}}, \tau L^{-z_u}) = L^{-\frac{d}{d^*} \nu} L^{-(d-z)} h(\tau L^{-\frac{1}{2}}, \tau L^{-z_u}) \). For \( \tau \to \infty \) and \( d - z = 1 \), \( g \) tends to a constant and thus \( \chi \propto L^{-\frac{d}{d^*} \nu} \), while for small \( \tau \), \( \chi \) still satisfies \( \chi \propto L^{-(1+\frac{d^*}{d})} \). However, for the J-Q model, numerical examination of this crossover behavior is beyond our present computational capacity, and we leave it for further studies.

### E. Critical initial slip

In the main text, we shows that the time-correlation functions of the order parameters demonstrates a critical initial slip behavior with \( \theta \) being its characteristic exponent. The scaling form of \( C_P \) (\( P \) represents \( M \) or \( D \)) reads
FIG. S9. **Dependence of the susceptibility χ on L for fixed τ in the short-time stage.** For τ = 8, χ scales with L as χ ∝ L$^{-1.717}$ with the exponent being 1 + ν/ν$^\prime$.

$C_P = \tau^\theta f_C(\tau L^{-z})$. In Fig. 3, we further confirm this scaling form by showing that for fixed τL$^{-z}$, $C_P \propto \tau^\theta$ with $\theta \simeq 0.514(3)$ for $P = M$ and $\theta \simeq 0.546(7)$ for $P = D$.

Moreover, the critical initial slip also manifests itself in the scaling behavior of the order parameter$^7$–$^9$. In the usual LGW phase transitions, it was shown that in the short-time stage for an uncorrelated initial state with a small initial order parameter $P_0$, the order parameter $P$ changes as $P \sim P_0 \tau^\theta$. Scaling analyses show that $P_0$ has a dimension $x_0$ which is different from the dimension of $P$. Explicitly, $x_0 = \theta z + \frac{\beta}{\nu}$. Directly generalizing this relation to the DQCP and comparing the exponent $\beta/\nu$ and $\theta$, one concludes that $x_0 \simeq 0.1$. This small positive $x_0$ indicates that the initial order parameter $P_0$ is a weakly relevant scaling variable. Therefore, we infer that the dependence of some quantities on the initial order parameter is weak.

To examine this inference, we calculate $I_D$ for a different initial $D_0$. We find in Fig. S11 that for a fixed τL$^{-z}$ the crossing points of $I_D$ for different lattice sizes converge onto a point which is just the critical point within the error range, confirming that the dependence of the relaxation dynamics on the initial order parameter is weak.

Moreover, in the main text, we focus on the case when $P_0$ is saturated or zero. These two cases correspond to the fixed points of $P_0$. When $P_0$ is very small, $P$ changes as $P \sim P_0 \tau^\theta$; when $P_0$ is saturated, $P$ changes as $P \sim P_0 \tau^{-\beta/\nu z}$. Besides, in the usual LGW critical point, initial conditions with finite $P_0$ were also considered. In this case, the subsequent relaxation dynamics cannot be described by power functions. Instead, a complex universal
FIG. S11. Estimation of the critical from an unsaturated VBS state with $D_0 = 3/2L$. For $\tau L^{-z} = 1/8$, crossing points of $I_D - \delta$ for different $L$ converge onto $0.677(3)$. For $\tau L^{-z} = 1/4$, crossing points of $I_D - \delta$ for different $L$ converge onto $0.6774(7)$. These results corroborate the value of critical point estimated in the main text.

characteristic function\textsuperscript{11,12} was proposed. It is also instructive to study the universal characteristic function in the DQCP.
II. SUPPLEMENTARY INFORMATION: FIELD THEORY

An equivalent description of the deconfined critical point in the J-Q model is given by SU(2) QCD$_3$ where spinons in a fundamental representation interacts with SU(2) gauge bosons in an adjoint representation$^{13-16}$. The lowest energy configuration corresponds to the well-known π-flux in square lattice$^{13}$. To describe the low energy spinon moving in the background of π-flux square lattice, we define the following matrices as an explicit representation of Gamma matrices,

$$\gamma^0 = \sigma^x, \gamma^1 = -\sigma^y \tau^z, \gamma^2 = -\sigma^x, \gamma^3 = \sigma^y \tau^x, \gamma^5 = \sigma^x \tau^y, \alpha^\mu = i\gamma^\mu,$$

where $\sigma$ and $\tau$ act on sublattices and valleys, respectively. The kinetic energy of spinon is given by

$$K = \psi^\dagger p \cdot \alpha \psi, \quad p = (p_x, p_y), \quad \alpha = (\alpha^1, \alpha^2),$$

where $\psi$ ($\psi^\dagger$) is the spinon annihilation (creation) operator, and we set the spinon velocity to one. The matrices that correspond to SO(5) orders are ($\alpha^3, \alpha^5, \gamma^0 s$), among which the first two are VBS orders and the last three are AFM orders with $s$ acting on spins. It is apparent that the kinetic energy is invariant under the SO(5) rotation.

We consider the quench protocol that the initial spinon has a finite gap rendered by the background orders$^{17}$. We refer to VBS quench and AFM quench as the quench protocol where the initial state has a finite VBS and AFM order, respectively. In general we need to consider individually the VBS and AFM quenches, however, as shown in the main text, a dual dynamic scaling appears as long as the universal low-energy dynamics is considered. So we can study one of them, and the other simply follows. Let us consider the VBS quench without loss of generality.

The prequnch Hamiltonian is

$$H = K + m\psi^\dagger \alpha^5 \psi,$$

where $m$ characterizes the initial VBS order. With the help of projection operator,

$$P_\pm (m) = \frac{1}{2} (1 \pm (\hat{p} \cdot \alpha + m\alpha^5)), \quad \hat{p} = (\sin \theta \cos \phi, \sin \theta \sin \phi, \cos \psi, \theta = \arctan \frac{m}{|\hat{p}|} \text{ and } \phi = \arctan \frac{p_x}{p_z},$$

the solution for the prequnch Hamiltonian is

$$\psi_\pm (\tau) = (e^{-\xi \tau} P_+(m) + e^{\xi \tau} P_-(m)) \psi_0, \quad \xi = \sqrt{|\hat{p}|^2 + m^2},$$

where $\psi_0$ is an arbitrary initial field configuration that will be integrated over.

At time zero, the finite order is suddenly turned off, $m = 0$. We first consider the free kinetic energy as the quench effect shows up, and then discuss the effect of interactions through SU(2) gauge fields. So in the lowest order without interactions, the postqunch Hamiltonian is $K$ governing the dynamics on the Keldysh contour.

It is convenient to introduce the classical/quantum fields,

$$\psi_{c/q} = \frac{1}{\sqrt{2}} (\psi_+ \pm \psi_-), \quad \psi^\dagger_{c/q} = \frac{1}{\sqrt{2}} (\psi^\dagger_+ \mp \psi^\dagger_-),$$

where $\psi_{c/q}$ are fields evolving forwards and backwards on the Keldysh contour. And the boundary condition connected the Keldysh contour and the initial field is given by $\psi_+(0) = \psi_i(\beta), \psi_-(0) = -\psi_i(0)$, where the minus sign is due to fermionic operators. The propagator along the Keldysh contour is defined as

$$i \hat{G}(t, t') \equiv i \begin{pmatrix} G_R(t, t') & G_K(t, t') \\ G_K(t, t') & G_A(t, t') \end{pmatrix} \begin{pmatrix} \langle \psi_c(t) \psi^\dagger_c(t') \rangle \\ \langle \psi_q(t) \psi^\dagger_q(t') \rangle \end{pmatrix} = \begin{pmatrix} \langle \psi_c(t) \psi^\dagger_c(t') \rangle \\ 0 \end{pmatrix} \begin{pmatrix} \langle \psi_c(t) \psi^\dagger_c(t') \rangle \\ \langle \psi_q(t) \psi^\dagger_q(t') \rangle \end{pmatrix}. \quad (S8)$$

The retarded and advanced propagators are familiar,

$$iG_R(t, t') = \theta(t - t') \left( e^{-ip(t-t')} P_+ + e^{ip(t-t')} P_- \right), \quad iG_A(t, t') = -\theta(t' - t) \left( e^{-ip(t-t')} P_+ + e^{ip(t-t')} P_- \right). \quad (S9)$$

with $p \equiv |\hat{p}|$ and $P_{c/q} \equiv P_{c/q}(0)$. While the Keldysh propagator reads

$$iG_K(t, t') = \tanh \frac{\beta \xi}{2} (e^{-ip(t-t')} P_+ + e^{ip(t-t')} P_-) (P_+(m) - P_-(m)) (e^{-ip(t-t')} P_+ + e^{ip(t-t')} P_-). \quad (S11)$$

The VBS quench is encoded in the Keldysh propagator, and it is easy to check without the quench, $m = 0$, (S11) restores time translation symmetry and reduces to equilibrium Keldysh propagator.
In order to calculate the equal time correlation function, we need for instance the greater propagator

\[ G_+(t, t') \equiv -i(\psi_+(t)\psi_+(t')) = \frac{1}{2}(G_R(t, t') + G_A(t, t') + G_K(t, t')) , \]  

in which the subscript + means the forward Keldysh contour. With these propagators, we now can evaluate the correlation function of different orders which are collected as follows,

\[ \Phi = (D, M), \quad D = (\psi^1\alpha^1\psi, \psi^1\alpha^2\psi), \quad M = \psi^1\gamma^0s\psi. \]  

Due to the SO(5) symmetry, the correlation functions have two distinct types: one is parallel to the quenched order, and the other is orthogonal to the quenched order, where the quenched order is \( \Phi_2 = D_2 \).

Before we calculate the equal time correlation function, we notice that the equal time correlator has a short distance singularity which is commonly seen in quantum field theory. The order is a bilinear function of spinons, so it has the singularity which is commonly seen in quantum field theory. The order is a bilinear function of spinons, so it has the

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Due to the SO(5) symmetry, the correlation functions have two distinct types: one is parallel to the quenched order, and the other is orthogonal to the quenched order, where the quenched order is \( \Phi_2 = D_2 \).
A hallmark of the deconfined quantum critical point is the dynamics of deconfinement of gauge fields, and this brings a new length scale—the confinement length $\xi'$—that is distinct from the conventional correlation length scale. More explicitly, in the language of deconfined spinon, the confinement length $\xi'$ is associated with the Wilson loop

$$W(x, y) = \langle \bar{\psi}(x)e^{i\int_x^y dx'\gamma_\mu A_\mu(x')}\psi(y) \rangle \propto e^{-\frac{|x-y|}{\xi'}},$$

(S22)

where $A$ denotes the $SU(2)$ gauge field, whereas the conventional correlation length scale is associated with the spinon bilinear mass, i.e., $G(x, y) \sim e^{-|x-y|/\xi}$.

Physically, the confinement length $\xi'$ is the width of VBS domain wall or equivalently the spinon confinement length. So it is greater than the conventional correlation length $\xi$, and these two diverge at different scalings,

$$\xi \propto \delta^{-\nu}, \quad \xi' \propto \delta^{-\nu'},$$

(S23)

where $\delta = q - q_c$ is the distance to the critical point, and $\nu' > \nu$.

For the VBS quench, the parallel (VBS) correlation function in (S20) remembers the finite initial order which enters as the correlation length $\xi \sim 1/m$. In this case $\xi \propto \tau^{1/z_u}$, $z_u \equiv z\nu'/\nu$ dominates and $D^2$ behaves as

$$D^2(\tau, L) = L^{-2\beta/\nu z_u} F_D(\tau/L^{\nu'/\nu}).$$

(S24)

This equation can be transformed as

$$D^2(\tau, L) = \tau^{-2(\beta/\nu)(\nu'/\nu)} F_D(\tau/L^{\nu'/\nu}).$$

(S25)

This is just the scaling form of $D^2$ discussed in the main text. In the short-time stage, $\tau \ll L^z$, $D^2(\tau, L)$ is mainly affected by the initial condition but almost independent of $L$. In this case, $D^2(\tau, L) \propto \tau^{-2(\beta/\nu)(\nu'/\nu')} = \tau^{-2(\beta/\nu z_u)}$.

Note that here one may argue that $D^2$ can also be expressed as $D^2(\tau, L) \propto \tau^{-2(\beta/\nu' z_u)}$. In this case, $2\beta/\nu' \approx 0.881$ is smaller than one. Accordingly, the scaling law $2\beta/\nu' - 1 = \eta'$ gives a negative anomalous dimension $\eta'$, which would imply a nonunitary theory. Alternatively, to satisfy the unitarity bound, we choose to adopt $D^2(\tau, L) \propto \tau^{-2(\beta/\nu z_u)}$ in which $2\beta/\nu$ keeps intact while an additional dynamic exponent $z_u$ is introduced. We want to emphasize that the appearance of this $z_u$ does not mean that the equilibrium correlation time needs two dynamic exponents. This $z_u$ only measures the scaling relation between the age of the system $\tau$ and the correlation length $\xi$.

For the orthogonal (AFM) correlation in (S21) the effect of the initial order is washed away in the imaginary evolution but the intrinsic deconfined dynamics with length scale $\xi' \propto \tau^{1/z_u}$ is intact. Thus the correct scaling function resulted from VBS quench should be

$$M^2(\tau, L) = L^{-2\beta/\nu} F_M(\tau/L),$$

(S26)

When $\tau \ll L^z$, $M^2$ behaves as $L^{-d}$ according to the central limit theorem. Combining with the scaling requirements that $M^2 \propto \tau^{-2\beta/\nu}$, one obtain $M^2 \propto L^{-d(1/d-z\beta/\nu)}$ in the short-time stage. This is just the scaling form of $M^2$ discussed in the main text. Moreover, if one starts from a random disordered state, it is natural to expect that the initial quench protocol does not contain any useful information that can be probed by any simple observables. Therefore, both the VBS and AFM orders will have similar behaviors like the orthogonal correlation where the intrinsic deconfinement length scale $\xi' \propto \tau^{1/z_u}$ takes place and leads to (S26). This is fully consistent with the simulation starting from a disordered state in the main text.
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