POSITIVITY OF BROWN-YORK MASS WITH QUASI-POSITIVE BOUNDARY DATA

YUGUANG SHI\(^1\) AND LUEN-FAI TAM\(^2\)

Abstract. In this short note, we prove positivity of Brown-York mass under quasi-positive boundary data which generalize some previous results by the authors. The corresponding rigidity result is obtained.

1. Introduction

Let \((\Omega^n, g)\) be a compact manifold with smooth boundary \(\partial\Omega\). In this work, we always assume that \(\Omega\) is orientable. It is an interesting question to understand the relation between the geometry of \(\Omega\) in terms of scalar curvature and the intrinsic and extrinsic geometry of \(\partial\Omega\) in terms of the mean curvature. The question is closely related to the notion of quasi-local mass in general relativity. On the other hand, given an compact manifold \((\Sigma, \gamma)\) without boundary and given a smooth function \(H\) on \(\Sigma\), one of basic problems in Riemannian geometry is to study: under what kind of conditions so that \(\gamma\) is induced by a Riemannian metric \(g\) with nonnegative scalar curvature, for example, defined on \(\Omega^n\), and \(H\) is the mean curvature of \(\Sigma\) in \((\Omega^n, g)\) with respect to the outward unit normal vector? These two problems are closely related and there are no satisfactory answers yet.

In this kind of study, a result was proved by the authors which implies the positivity of Brown-York quasi-local mass \([2, 3]\), denoted by \(m_{BY}(\Sigma; \Omega, g)\). For its definition please see (2.1) below. More specifically, using the quasi-spherical metrics introduced by Bartnik \([1]\), in \([16]\) the authors proved the following:

**Theorem 1.1.** Let \((\Omega^3, g)\) be a compact, connected Riemannian manifold with nonnegative scalar curvature, and with compact mean-convex
boundary $\partial \Omega$, which consists of spheres with positive Gaussian curvature. Then,

\[
\mathbf{m}_{BY}(\Sigma_\ell; \Omega, g) \geq 0
\]

for each component $\Sigma_\ell \subset \partial \Omega$, $\ell = 1, \ldots, k$. Moreover, equality holds for some $\ell = 1, \ldots, k$ if and only if $\partial \Omega$ has only one component and $(\Omega, g)$ is isometric to a domain in $\mathbb{R}^3$.

Clearly Theorem 1.1 provides a necessary condition for a boundary data $(\Sigma, \gamma, H)$ to be the one induced by a Riemannian metric defined on the ambient manifold and with nonnegative scalar curvature. The existence of quasi-spherical metric in the proof of the theorem makes use of the fact that the mean curvature is positive at the boundary. Otherwise, it is unclear if one can construct such kind of metrics, see [1, 17]. With these facts in mind, it is natural to ask if Theorem 1.1 is still true in a more general context. In this note, we consider the problem in the situation of quasi-positive boundary data. Here a function defined on a set is said to be quasi positive if it is nonnegative and is positive somewhere. The specific results are the following:

**Theorem 1.2.** Let $(\Omega, g)$ be a compact three manifold with smooth boundary $\partial \Omega$. Let $\Sigma$ be a component of $\partial \Omega$. Assume the following:

(i) $\partial \Omega$ has nonnegative mean curvature.

(ii) $\Sigma$ has quasi positive Gaussian curvature $K$ so that the mean curvature $H \geq 0$ satisfies $H + K > 0$.

(iii) $(\Omega, g)$ has nonnegative scalar curvature.

Then

\[
\mathbf{m}_{BY}(\Sigma; \Omega, g) \geq 0.
\]

If $\partial \Omega \setminus \Sigma \neq \emptyset$, then we have

\[
\mathbf{m}_{BY}(\Sigma; \Omega, g) > 0.
\]

We also have the following rigidity result.

**Theorem 1.3.** Let $(\Omega, g)$ and $\Sigma$ be as in Theorem 1.2 if $\mathbf{m}_{BY}(\Sigma; \Omega, g) = 0$, then it is a domain in $\mathbb{R}^3$.

In case that $H = K = 0$ somewhere, we still have positivity of Brown-York mass, but the rigidity part is still unsolved. Namely, we have the following generalization:

**Theorem 1.4.** Let $(\Omega, g)$, $\Sigma$ as in Theorem 1.2 without assuming that $H + K > 0$ on $\Sigma$. Let $\Sigma$ be a component of $\partial \Omega$. Then the following are true:
(i) Suppose $\partial \Omega \setminus \Sigma \neq \emptyset$ or suppose $\partial \Omega = \Sigma$ and $\Omega$ is not homeomorphic to the open ball in $\mathbb{R}^3$. Then

$$m_{BY}(\Sigma; \Omega, g) > 0.$$ 

(ii) $$m_{BY}(\Sigma; \Omega, g) \geq 0.$$

We first remark that in case $\partial \Omega$ has quasi positive Gaussian curvature and has positive mean curvature or $\partial \Omega$ has positive Gaussian curvature and has nonnegative mean curvature the nonnegativity part of Theorem 2.1 was proved in [17] and [15] respectively. However, the rigidity part in the first instance was studied in [17] but not solved very satisfactorily. The rigidity part in the second instance was not addressed in [15]. One can see that Theorems 1.2 and 1.3 can be applied to the above mentioned two cases.

To show Theorem 1.1 we have to use the notion of quasi-spherical metric introduced by Bartnik [1]. However, if the boundary data is only quasi-positive, a parabolic equation involved in the quasi-spherical metric may be degenerated. To overcome this difficult, we adopt a careful conformal perturbation on the ambient metric $g$ so that the boundary data induced on $\Sigma$ is positive, then by Theorem 1.1 the Brown-York mass of $\Sigma$ with respect to the perturbed metric is positive. By approximation we get Theorem 1.2. We prove Theorem 1.3 in several steps. First, by Theorem 1.2 we know the boundary $\partial \Omega$ is connected. Next we prove that $\Omega$ is simply connected (see Lemma 3.2). Another main observation of the proof of Theorem 1.3 is the Proposition 3.1 which is a generalization of a result in [18]. By suitable approximations, we succeed to construct a weak solution of the inverse mean curvature flow (IMCF) in $(\Omega, g)$ with a point $p \in \Omega$ as the initial data (see Lemma 3.1 below). Note the scalar curvature of $g$ is nonnegative, we see that the Hawking mass of each slice of the weak solution of IMCF is nonnegative, combine the fact with Proposition 3.1 we know that the Hawking masses of those slices are zero, then by the same arguments in [7], we see $(\Omega, g)$ is flat. We believe that hyperbolic version of above results should be true and will plan to discuss in the future. The proof Theorem 1.4 use the result on positivity of some Brown-York type quasi-local mass in [19] with reference background being the hyperbolic space. However, the behavior of the inverse mean curvature flow is not good enough to prove the rigidity, for instance, see [13, 20].

The remaining part of the paper goes as follows: in the section 2, we prove the positivity result Theorem 1.2 in the section 3, we prove Theorem 1.3 and in section 4, we prove Theorem 1.4.
Acknowledgment: The authors would like to thank Man-Chuen Cheng for many useful discussions.

2. Positivity

Let $(\Omega, g)$ be compact three manifold with smooth boundary $\partial \Omega$. Let $\Sigma$ be a connected component of $\partial \Omega$ with induced metric $\gamma$. Suppose the Gaussian curvature of $(\Sigma, \gamma)$ is quasi positive. Then it can be $C^{1.1}$ isometrically embedded in $\mathbb{R}^3$ as a convex surface with mean curvature $H_0$ which is defined almost everywhere in $\Sigma$. Moreover,

$$\int_{\Sigma} H_0 d\sigma$$

is well-defined and is positive, see [5, 6, 17]. Here and below mean curvature is computed with respect to the unit outward normal and the mean curvature of the boundary of the unit ball in $\mathbb{R}^3$ is 2.

Define the Brown-York mass [2, 3] of $\Sigma$ in $(\Omega, g)$ by

$$(2.1) \quad m_{BY}(\Sigma; \Omega, g) = \frac{1}{8\pi} \int_{\Sigma} (H_0 - H) d\sigma.$$  

Here $H$ is the mean curvature of $\Sigma$ in $(\Omega, g)$.

Theorem 2.1. Let $(\Omega, g)$ be a compact three manifold with smooth boundary $\partial \Omega$. Let $\Sigma$ be a component of $\partial \Omega$. Assume the following:

(i) $\partial \Omega$ has nonnegative mean curvature.
(ii) $\Sigma$ has quasi positive Gaussian curvature $K$ so that the mean curvature $H \geq 0$ satisfies $H + K > 0$.
(iii) $(\Omega, g)$ has nonnegative scalar curvature.

Then

$$m_{BY}(\Sigma; \Omega, g) \geq 0.$$  

If $\partial \Omega \setminus \Sigma \neq \emptyset$, then we have

$$m_{BY}(\Sigma; \Omega, g) > 0.$$  

Proof. Let us first consider the case that $\partial \Omega = \Sigma$. We want to prove the first statement of the theorem. If the scalar curvature of $g$ is positive somewhere, one can conformally change $g$ so that the mean curvature of $\Sigma$ increases and $g$ is unchanged at the boundary. So we may assume the scalar curvature of $g$ is zero. Moreover, since $H_0 \geq 0$, we may assume that the mean curvature $H$ of $\Sigma$ is quasi positive.

Consider the compact subsets in $\Sigma$: $Z_K = \{x \in \Sigma| K = 0\}$ and $Z_H = \{x \in \Sigma| H = 0\}$. We can find open sets $U_1 \subset U_2$ in $\Sigma$ such that $Z_K \subset U_1$ and open sets $V_1 \subset V_2$ in $\Sigma$ such that $Z_H \subset V_1$. Moreover, $U_2 \cap V_2 = \emptyset$. This can be done because $Z_K \cap Z_H = \emptyset$. One
can find a smooth function $w$ on $U_2$ with $\Delta_\gamma w = -1$ and such that $w \leq -1$, say. We can extend $w$ to be a smooth function on $\Sigma$ with $w \leq 0$ such that $w = 0$ on $V_1$.

Let $\epsilon_i > 0$, and $\epsilon_i \to 0$. For each $i$, let $v_i$ be the solution of

$$
\begin{align*}
\Delta g v_i &= 0 \text{ in } \Omega; \\
v_i &= e^{\epsilon_i w} \text{ on } \partial \Omega.
\end{align*}
$$

By maximum principle, $v_i > 0$. Let $g_i = v_i^4 g$. Then it is easy to see that $g_i$ converge to $g$ in $C^\infty$ norm with respect to a background metric on $\Omega$.

Then scalar curvature of $g_i$ is also zero. The metric on $\Sigma$ induced by $g_i$ is given by $\gamma_i = e^{2\epsilon_i w} \gamma$. Hence the Gaussian curvatures $K$ and $K_i$ respect to $\gamma$, $\gamma_i$ are related by

$$
K_i = e^{-2\epsilon_i w} (K - \epsilon_i \Delta_\gamma w).
$$

For $x \in U_1$, $\Delta_\gamma w = -1$ and $K \geq 0$ which implies that $K_i > 0$. On the other hand, there is $C_1 > 0$ such that $K \geq C_1$ outside $U_1$. Hence if $i$ is large enough, then $K_i > 0$ outside $V_1$. We conclude that $K_i > 0$ provided $i$ is large enough.

The mean curvature $H_i$ of $\Sigma$ with respect to $g_i$ is given by

$$
H_i = \frac{1}{v_i^2} \left( H + \frac{4}{v_i} \frac{\partial v_i}{\partial \nu} \right)
$$

where $\nu$ is the unit outward normal with respect to $g$. Note that $v_i$ is not constant because $w$ is not constant. Moreover, $v_i = 1$ on $V_1$ and $v_i \leq 1$ on $\Sigma$. By the strong maximum principle, we conclude that

$$
\frac{\partial v_i}{\partial \nu} > 0
$$
on $V_1$. Hence at $V_1$, $H_i > 0$. On the other hand, there is a constant $C_2 > 0$ such that $H \geq C_2 > 0$ outside $V_1$. Since $v_i \to 1$ smoothly on $\Omega$, we conclude that $H_i > 0$ provided $i$ is large enough. Hence we may assume that $H_i > 0$, $K_i > 0$ for all $i$. By [16], we conclude that

$$
\int_{\Sigma} (H_{i,0} - H_i) d\sigma_i \geq 0.
$$

Here $H_{i,0}$ is the mean curvature of $(\Sigma, \gamma_i)$ when isometrically embedded in $\mathbb{R}^3$ and $d\sigma_i$ is the area element of $\gamma_i$. The fact that

$$
\int_{\Sigma} (H_{i,0} - H_i) d\sigma_i \geq 0
$$

follows from the fact that $H_i \to H$, $d\sigma_i \to \sigma$ and the following Lemma 2.1.
Next suppose $\Sigma' = \partial \Omega \setminus \Sigma$ is not empty. To prove the second statement, we may assume that the scalar curvature of $g$ is zero and the mean curvature of $\Sigma$ is quasi positive because
\[
\int_\Sigma H_0 d\sigma > 0.
\]
For each $i$, let $u_i$ be the solution of
\[
\begin{cases}
\Delta_g u_i = 0 \text{ in } \Omega; \\
u_i = e^{\frac{2}{\epsilon}u_i} \text{ on } \Sigma; \\
u_i = 1 \text{ on } \partial \Omega \setminus \Sigma.
\end{cases}
\tag{2.3}
\]

Let $h_i = u_i^4g$ and let $\beta_i$ be the induced metric on $\Sigma$. Then as before, for $i$ large enough, the mean curvature and the Gaussian curvature of $\Sigma$ are both positive with respect to $h_i$ and the mean curvature of $\partial \Omega \setminus \Sigma$ is positive. Hence we can find a smooth metric on $\Omega$ with zero scalar curvature so that the mean curvature and the Gaussian curvature of $\Sigma$ are positive. Moreover, there is a constant $a > 0$ so that the mean curvature of $\partial \Omega \setminus \Sigma$ is bounded below by $a$.

On the other hand, for any $\epsilon > 0$, let $\tilde{u}_\epsilon$ be the harmonic function on $\Omega$ so that $u_\epsilon = 1$ on $\Sigma$ and $u_\epsilon = 1 - \epsilon$ on $\partial \Omega \setminus \Sigma$. Consider the metrics $u_\epsilon^4g$. It is easy to see that one can choose $\epsilon$ small enough so that $\tilde{h} = u_\epsilon^4g$ has scalar curvature, $\tilde{h}|_{T(\Sigma)} = \gamma$ and the mean curvature of $\partial \Omega \setminus \Sigma$ is bounded in absolute value by $\frac{a}{2}$. Moreover,
\[
H_{\tilde{h}} > H_g,
\]
where $H_{\tilde{h}}, H_g$ are the mean curvature of $\Sigma$ with respect to $\tilde{h}, g$ respectively. We want to prove that
\[
\int_\Sigma (H_0 - H_{\tilde{h}}) d\sigma \geq 0.
\tag{2.4}
\]
If this is true, then the second statement follows. Note that $\tilde{h}|_{\partial \Omega, \Sigma} = (1 - \epsilon)^4 h|_{\partial \Omega, \Sigma}$. If we denote $(\Omega, (1 - \epsilon)^4 h)$ by $M_1$ and $(\Omega, \tilde{h})$ by $M_2$, then we can glue $M_1$ and $M_2$ along $\partial \Omega \setminus \Sigma$. Then we obtain a manifold with smooth boundary which are two copies of $\Sigma$. Let us denote the copy in $M_1$ by $\Sigma_1$ and the part in $M_2$ by $\Sigma_2$. Denote the resulting manifold by $\tilde{M}$ with metric $\tilde{g}$. Then the following are true:

- The metric $\tilde{g}$ is smooth away from $\partial \Omega \setminus \Sigma$ and is Lipschitz.
- The scalar curvature of $\tilde{g}$ is zero away from $\partial \Omega \setminus \Sigma$.
- The mean curvatures across $\partial \Omega \setminus \Sigma$ with respect to the normal pointing outside $M_1$ satisfies $H_- - H_+ \geq a = \frac{a}{2} > 0$.
- The mean curvature and the Gaussian curvature of $\Sigma_1$ are positive.
• The mean curvature and Gaussian curvature of $\Sigma_2$ are quasi positive so that their sum is positive.

By [9, Theorem 3.3] and Theorem 1.1, one can conclude that (2.4) is true. This completes the proof of the theorem.

\[ \square \]

**Lemma 2.1.** Let $\Sigma, \gamma, \gamma_i$ be as in the proof of the Theorem 2.1. Let $H_{i,0}$ be the mean curvature of $(\Sigma, \gamma_i)$ when it is isometrically embedded in $\mathbb{R}^3$. Then
\[
\lim_{i \to \infty} \int_\Sigma H_{i,0} d\sigma_i = \int_\Sigma H_0 d\sigma.
\]
where $d\sigma, d\sigma_i$ are the area elements with respect to $\gamma, \gamma_i$ respectively.

**Proof.** See [17, p.69–70]. \[ \square \]

3. **Rigidity**

In the section, we want to show:

**Theorem 3.1.** Let $(\Omega, g)$ and $\Sigma$ be as in Theorem 2.1, if $m_{BY}(\Sigma; \Omega, g) = 0$, then it is a domain in $\mathbb{R}^3$.

By Theorem 2.1, we conclude that $\partial \Omega = \Sigma$. Moreover, since $\Sigma$ has quasi positive Gaussian curvature, we conclude that $\Sigma$ also has quasi positive mean curvature.

We first have the following:

**Lemma 3.1.** Let $(\Omega, g), \Sigma$ be as in Theorem 3.1. For any $p \in \Omega$, there is a weak solution for the inverse mean curvature flow in $(\Omega, g)$ with $p$ as the initial data.

**Proof.** Let $U$ be a small neighborhood of $\partial \Omega$, then extend $\Omega \cup U$ to be Euclidean near infinity, the resulting metric is denoted by $\hat{g}$.

Let us consider the inverse mean curvature flow (IMCF) in $(M, \hat{g})$ with $\partial B_r(p)$ as the initial data. By Theorem 3.1 in [7], there is a weak solution $u_r$ to this IMCF with $u_r|_{\partial B_r(p)} = 0$ and
\[
|\nabla u_r|(x) \leq \sup_{\partial B_r(p) \cap B_{\rho}(x)} H_+ + \frac{C}{\rho},
\]
for any $0 < \rho \leq \sigma(x)$, here $C$ is a universal constant independent on $\rho$ and $r$, $\sigma(x)$ is defined in Definition 3.3 in [7], i.e. for any $x \in \Omega$, let $\tau(x) \in (0, \infty]$ be the supremum of radii $r$ such that $B_r(x) \subset \Omega$, and
\[
Rc \geq -\frac{1}{1000\tau^2} \text{ in } B_r(x),
\]
and there is a \( C^2 \) function \( p \) on \( B_r(x) \) such that \( p(x) = 0, \ p \geq d^2(x), \) and \( \left| \nabla p \right| \leq 3d(x), \ \nabla^2 p \leq 3g \) on \( B_r(x) \), define \( \sigma(x) = \min \{ \tau(x), d(x, \partial \Omega) \} \). Let \( \Omega' \subset \subset \Omega \) with \( \text{dist}(\partial \Omega', \partial \Omega) \) being any fixed small number and \( p \in \Omega' \). Without loss of the generality, it suffices to consider the case that \( x \in \Omega' \), so, we may assume \( \sigma(x) \geq \sigma_0 \) for any \( x \in \Omega' \), here \( \sigma_0 \) is a fixed number depends only on \( \text{dist}(\partial \Omega', \partial \Omega) \) and \((\Omega, g)\).

Let us choose \( r \) small enough so that \( \sup_{\partial B_r(p)} H \leq \frac{3}{r} \). Now, we claim that for any \( x \in \Omega' \)

\begin{equation}
|\nabla u_r|(x) \leq \frac{C}{d(x, p)},
\end{equation}

here \( C \) is a universal constant independent on \( r \), \( d(x, p) \) is the distance function to \( p \) with respect to the metric \( g \).

In fact, if \( d(x, p) \leq 4r \), then we take \( \rho = \frac{r}{2} \), here we assume \( r \leq \sigma_0 \), we get \( (3.1) \); if \( d(x, p) > 4r \), let \( \rho = \min \{ \frac{1}{2} \text{dist}(x, p), \frac{\sigma_0}{2} \} \), together with the fact \( \text{dist}(x, p) \leq \Lambda \sigma_0 \), where \( \Lambda \) is a universal constant, we still get \( (3.1) \).

On the other hand, together with Theorem 2.1 in [7] and the remarks following it, we know that by taking a subsequence of \( \{u_r\} \), denoted by \( \{u_{r_i}\} \), there is a constant \( C_i \) so that \( \{u_{r_i} - C_i\} \) converges to the weak solution of IMCF \( -\infty < u \in (\Omega', g) \) with \( p \) as the initial data. Note that the mean curvature of \( \partial B_r(p) \) is positive for all \( r \leq \delta \), we see that the level set of \( u \) in \( B_\delta(p) \subset \subset \Omega' \) cannot jump, and

\begin{equation}
|\nabla u|(x) \leq \frac{C}{d(x, p)},
\end{equation}

and \(-\infty < u \leq t_0 \), here \( t_0 \) is a universal constant. \( \square \)

Let us first recall the definition of minimizing hull in \( \Omega \). A subset \( E \) of \( \Omega \) with locally finite perimeter said to be a minimizing hull in \( \Omega \) if

\[ |\partial^* E \cap K| \leq |\partial^* F \cap K| \]

for any set \( F \subset \Omega \) with locally finite perimeter such that \( F \supset E \) and \( F \setminus E \in \Omega \) and for any compact set \( K \) with \( F \setminus E \subset K \subset \Omega \). Here \( \partial^* E, \partial^* F \) are the reduced boundaries of \( E \) and \( F \) respectively.

By the proof in Theorem 2.5 in [18], we see that for \( t \) small enough, the slice \( N_t = \partial \{ u < t \} \) of the weak IMCF in Lemma \( 3.1 \) is the boundary of a minimizing hull in \( (\Omega, g) \) with \( C^1,\alpha \) smooth and \( \int_{N_t} |A|^2 d\sigma < \infty \), and \( \mathfrak{m}_H(N_t) \geq 0 \).

Next, we want to generalize a result in [18]:

**Proposition 3.1.** Let \((\Omega, g), \Sigma\) as in Theorem [2.7]. Assume that \( \Omega \) is simply connected. Let \( E \subset \subset \Omega \) is a minimizing hull with \( C^1 \) boundary
∂E so that $\int_{\partial E} |A|^2 d\sigma < \infty$. Here $A$ is the generalized second fundamental form. Then

$$m_{BY}(\Sigma; \Omega, g) \geq m_H(\partial E).$$

**Proof.** We may assume that $m_H(\partial E) > 0$ because $m_{BY}(\Sigma; \Omega, g) \geq 0$. For any $\theta > 0$ small enough, there is $E \subset F \subset \subset \Omega$ such that

$$|\partial E|_g \leq |\partial F|_g \leq |\partial E|_g + \theta; \text{ } m_H(\partial F) \geq m_H(\partial E) - \theta > 0.$$

Moreover $\partial F$ is smooth. Let $g_i$ be as in the proof of Theorem 2.1. Let $F_i$ be the minimizing hull of $F$ w.r.t. to $g_i$. Then $F_i \subset \subset \Omega$, $\partial F_i$ is $C^{1,1}$. By [18, Theorem 3.1], we conclude that

$$m_{BY}(\Sigma; \Omega, g_i) \geq m_H(\partial F_i, g_i).$$

For any $\epsilon > 0$, then for $i$ large enough, we have

$$m_{BY}(\Sigma; \Omega, g) + \epsilon \geq m_{BY}(\Sigma; \Omega, g_i) \geq m_H(\partial F_i, g_i) \geq m_H(\partial F, g_i)$$

(since $H = 0$ on $\partial F_i \setminus \partial F$, and the same $H$ on $\partial F_i \cap \partial F$)

$$= \left( \frac{|\partial F_i|_{g_i}}{|\partial F|_{g_i}} \right)^{\frac{1}{2}} \left( 1 - \frac{1}{16\pi} \int_{\partial F_i} H^2 d\sigma_i \right) \geq \left( \frac{|\partial F_i|_{g_i}}{|\partial F|_{g_i}} \right)^{\frac{1}{2}} \left( m_H(\partial F, g) - \epsilon \right) \geq \left( \frac{|\partial F_i|_{g_i}}{|\partial F|_{g_i}} \right)^{\frac{1}{2}} (m_H(\partial E, g) - \theta - \epsilon).$$

Here we have used the fact that $g_i \to g$ in $\overline{\Omega}$ in $C^\infty$ norm.

Since

$$|\partial F_i|_{g_i} \geq (1-\epsilon)|\partial F_i|_g \geq (1-\epsilon)|\partial E|_g; |\partial F|_{g_i} \leq (1+\epsilon)|\partial F|_g \leq (1+\epsilon)(|\partial E|_g + \theta)$$

and

$$|\partial F_i|_{g_i} \leq |\partial F|_{g_i} \leq (1+\epsilon)|\partial F|_g \leq (1+\epsilon)(|\partial E|_g + \theta); |\partial F|_{g_i} \geq (1-\epsilon)|\partial F|_g \leq (1-\epsilon)|\partial E|_g$$

the result follows by letting $\epsilon \to 0$ and then $\theta \to 0$.

□

**Lemma 3.2.** Let $(\Omega, g), \Sigma$ be as in Theorem 3.1. Then $\Omega$ is simply connected.
Proof. As mentioned before, \( \partial \Omega = \Sigma \) is connected. Moreover, since the Gaussian curvature of \( \Sigma \) is quasi-positive. So \( \Sigma \) is a topological sphere. We claim that \( \Omega \) is a handle body. If this is true, then \( \Omega \) is a topological ball and simply connected.

To prove the claim, suppose \( \Omega \) is not a handle body, then by \cite[Theorem 1']\} there is an embedded minimal surface \( S \) which is either a sphere or a minimal projective space inside \( \Omega \).

**Case 1:** Suppose \( S \) is a sphere. Since \( S \) is orientable, there is a smooth unit normal vector field on \( S \) and there is an embedding \( F : S \times (-1, 1) \to \Omega \) so that \( F(\cdot, 0) = S \) and the image of \( F \) is a tabular neighborhood \( N \) of \( S \) in \( \Omega \). Then \( N \setminus S \) is a manifold with boundary which are two copies of \( S \) with two components. Hence \( \Omega \setminus S \) is a manifold with boundary which is a copy of \( S \). Let \( \Omega \setminus S \), then \( (\Omega, g) \) has nonnegative scalar curvature so that \( \partial \Omega \) is disconnected, and \( \text{m}_{BY}(\Sigma, \Omega, g) = 0 \), which is a contradiction to Theorem 2.1.

**Case 2:** Suppose \( S \) is a projective space. \( f : \mathbb{RP}^2 \to \Omega \) is an embedding. We want to construct a double cover \( p : \hat{\Omega} \to \Omega \) so that \( p^{-1}(f(\mathbb{RP}^2)) \) is a sphere. Let \( \Omega \) be the connected component containing \( \partial \Omega = \Sigma \) of this manifold. Then \( \hat{\Omega} \) has nonnegative scalar curvature so that \( \partial \hat{\Omega} \) is disconnected, and \( \text{m}_{BY}(\Sigma, \hat{\Omega}, g) = 0 \), which is a contradiction to Theorem 2.1 again.

We are ready to prove the rigidity result.
Proof of Theorem 3.1. We are going to show $(\Omega, g)$ is flat. For any $p \in \Omega$, let $N_t$ be here the slice of the weak IMCF in Lemma 3.1 with initial data $p \in \Omega$. By Proposition 3.1, we see that $m_{H}(N_t) = 0$ for sufficient small $t$. By gluing an asymptotically flat (AF) end outside $\Omega$ we get an AF manifold $(M, \hat{g})$. As in the proof of Theorem 2.5 in [18], we know that $N_t$ is also the boundary of a minimizing hull in $(M, \hat{g})$. By the uniqueness Theorem 2.2 in [7], we see that $N_t$ is also the slice of the weak IMCF in $(M, \hat{g})$ with initial data $p \in \Omega$, then by the similar arguments in [7, p.422-424], using the fact that $\Omega$ has nonnegative scalar curvature, we see that $p$ is a flat point. Hence $(\Omega, g)$ is flat, note that $\Omega$ is simply connected, we see it is a domain in $\mathbb{R}^3$. □

4. A GENERALIZATION

In this section, we want to study the case that the condition $H + K > 0$ in Theorem 2.1 is removed. Some of the previous results are still true.

Theorem 4.1. Let $(\Omega, g), \Sigma$ as in Theorem 2.1 without assuming that $H + K > 0$ on $\Sigma$. Let $\Sigma$ be a component of $\partial \Omega$. Then the following are true:

(i) Suppose $\partial \Omega \setminus \Sigma \neq \emptyset$ or suppose $\partial \Omega = \Sigma$ and $\Omega$ is not homeomorphic to the open ball in $\mathbb{R}^3$. Then

$$m_{BY}(\Sigma; \Omega, g) > 0.$$ 

(ii) $m_{BY}(\Sigma; \Omega, g) \geq 0$.

Proof. To prove (i), we may assume that the scalar curvature of $\Omega$ is zero and the mean curvature $H_g$ of $\Sigma$ is quasi positive. Suppose $\Sigma' = \partial \Omega \setminus \Sigma \neq \emptyset$. Then we can proceed as in the proof of the second part of Theorem 2.1 To be precise, let $x_0 \in \Sigma$ with $H(x_0) > 0$. Let $U$ be a neighborhood of $x_0$ in $\Sigma$ such that $H_g \geq c_0 > 0$ in $U$. Let $0 \leq \phi \leq 1$ be a cutoff function with support in $U$ so that $\phi = 1$ in a neighborhood of $x_0$. Given $\epsilon > 0$ and let $u$ be the solution of

$$\begin{cases} \Delta_g u = 0 & \text{in } \Omega \\ u = (1 - \epsilon)\phi + (1 - \phi) & \text{on } \partial \Omega. \end{cases}$$

For $\epsilon > 0$ small enough, $g_1 = u^4g$ has zero scalar curvature so that $\partial \Omega$ has positive mean curvature. Fix such an $\epsilon > 0$ and let $M_1 = (\Omega, g_1)$. On the other hand, let $v$ be the solution of

$$\begin{cases} \Delta_g v = 0 & \text{in } \Sigma \\ v = 1 & \text{in } \Sigma \\ v = (1 - \delta) & \text{on } \Sigma'. \end{cases}$$
For $\delta > 0$ small enough, $g_2 = v^4 g$ has zero scalar curvature, the mean curvature on $\Sigma$ satisfies $H_{g_2} > H_g$ and the metric on $\Sigma$ induced by $g$ and $g_2$ are the same. Moreover, the mean curvature on $\Sigma'$ satisfies $|H_{g_2}| < H_{g_1}$ with respect to the unit outward normal. Let $M_2 = (\Omega, g_2)$ and glue $M_2$ with $\tilde{M}_1 = (\Omega, (1 - \delta)^4 g_1)$ along $\Sigma'$ and argue as in the proof of Theorem 2.1 to conclude that

$$m_{BY}(\Sigma; \Omega, g) > 0$$

by Theorem 2.1.

If $\partial \Omega = \Sigma$, then $\Sigma$ is homeomorphic to the unit sphere. If $\Omega$ is not a handle body, then one can argue as in Lemma 3.2 to conclude that

$$m_{BY}(\Sigma; \Omega, g) > 0$$

by the above part.

To prove (ii), it is sufficient to consider the case that that $\partial \Omega = \Sigma$ which is a sphere because its Gaussian curvature is quasi positive. Moreover, we may assume that the scalar curvature of $\Omega$ is zero and the mean curvature $H$ of $\Sigma$ is quasi positive. As in the proof of (i), let $x_0 \in \Sigma$ with $H(x_0) > 0$. Let $U$ be an neighborhood of $x_0$ in $\Sigma$ such that $H_g \geq c_0 > 0$ in $U$. Let $0 \leq \phi \leq 1$ be a cutoff function with support in $U$ so that $\phi = 1$ in a neighborhood of $x_0$. Given $\epsilon > 0$ and let $u = u(\epsilon)$ be the solution of

$$\begin{cases}
\Delta_g u = 0 & \text{in } \Omega \\
u = (1 - \epsilon)\phi + (1 - \phi) & \text{on } \partial \Omega.
\end{cases}$$

For $\epsilon > 0$ small enough, $g(\epsilon) = u^4 g$ has zero scalar curvature so that $\partial \Omega$ has positive mean curvature. Let $\sigma(\epsilon)$ be the metric on $\Sigma$ induced by $g(\epsilon)$ and let $K(\epsilon)$ be the Gaussian curvature of $\Sigma$ with respect to $\sigma(\epsilon)$. Then

(4.1) \[ K(\epsilon) > -\kappa^2(\epsilon) \]

where $\kappa(\epsilon) > 0$, $\kappa(\epsilon) \to 0$ as $\epsilon \to 0$. Then we can isometrically embed $(\Sigma, \sigma(\epsilon))$ in $\mathbb{H}_{-\kappa^2(\epsilon)}$ as a strictly convex surface in the ball model defined in the ball

$$\{ |x| < \kappa^{-2}(\epsilon) \}.$$ 

by [14]. Moreover, we may assume the origin is inside the embedded surface. Let $H(\epsilon)$ be the mean curvature of $\Sigma$ with respect to $g(\epsilon)$ and let $H_{\kappa(\epsilon)}$ be the mean curvature when $(\Sigma, \sigma(\epsilon))$ is isometrically embedded in $\mathbb{H}_{-\kappa^2(\epsilon)}$. Then by [19], we have

$$\int_{\Sigma} (H_{\kappa(\epsilon)} - H(\epsilon)) \cosh(\kappa(\epsilon)r) d\sigma \geq 0$$
where $r$ is the distance from the origin in $\mathbb{H}_\kappa(\epsilon)$.

Observe that we can find $\epsilon_i \to 0$ such that $g(\epsilon_i) \to g$ in $C^\infty$ norm on $\Omega$. Hence intrinsic diameter of $(\Sigma, \sigma(\epsilon_i))$ is bounded by a constant independent of $i$, we conclude that $r$ is bounded by a constant independent of $i$. Also by [8, p.7152-7154], one can choose $\epsilon_i \to 0$ such that:

- $H_{\kappa(\epsilon_i)}$ are uniformly bounded from above. (Note that $H_{\kappa(\epsilon_i)} > 0$).
- If $X_i = (x^1, x^2, x^3)$ is the isometric embedding of $(\Sigma, \sigma(\epsilon_i))$, then the $C^2$ norm with respect to the fixed metric $\sigma$ are uniformly bounded.

Suppose this can be done, then

$$\lim_{i \to \infty} \inf \int_\Sigma (H_{\kappa(\epsilon_i)} - H_g) d\sigma \geq 0.$$ 

Moreover, $X_i$ converge to a $C^{1,1}$ embedding of $(\Sigma, \sigma)$ in $\mathbb{R}^3$ as a convex surface. As in [17], one can conclude that

$$\lim_{i \to \infty} \int_\Sigma H_{\kappa(\epsilon_i)} d\sigma = \int_\Sigma H_0 d\sigma.$$ 

where $H_0$ is the mean curvature as defined in section [2]. From this (ii) follows.

\[\square\]

Remark 4.1. From the theorem, if $m_{\text{BY}}(\Sigma; \Omega, g) = 0$, then $\Omega$ is homeomorphic a the unit ball in $\mathbb{R}^3$, with zero scalar curvature. Moreover, $g$ is static by the result in [4]. However, we still do not know if $g$ is flat. The rigidity part is related to an open problem whether the Brown-York mass is still nonnegative if the mean curvature is negative somewhere. In fact, if $g$ is nonflat, then by [12] one can perturbed the metric $g$ to obtain a metric $\tilde{g}$ with zero scalar curvature so that $g, \tilde{g}$ induces the same metric on $\Sigma$ so that the mean curvature $H$ with respect to $\tilde{g}$ is negative somewhere and the Brown-York mass is negative.

References

[1] Bartnik, R., Quasi-spherical metrics and prescribed scalar curvature, J. Differential Geom. 37 (1993) 31–71.
[2] Brown, J.D.; York, J.W., Quasilocal energy in general relativity, in ‘Mathematical aspects of classical field theory (Seattle, WA, 1991), Contemp. Math., 132, Amer. Math. Soc., Providence, RI, 1992, 129–142.
[3] Brown, J.D.; York, J.W., Quasilocal energy and conserved charges derived from the gravitational action, Phys. Rev. D (3) 47(4) (1993) 1407–1419.
[4] Corvino, J., Scalar curvature deformation and a gluing construction for the Einstein constraint equations, Comm. Math. Phys. 214 (2000), 137–189.
[5] Guan, P.; Li, Y.-Y., The Weyl problem with nonnegative Gauss curvature, J. Differential Geom. 39 (1994), no. 2, 331–342.

[6] Hong, J.; Zuily, C., Isometric embedding of the 2-sphere with nonnegative curvature in $\mathbb{R}^3$, Math. Z. 219 (1995), no. 3, 323–344.

[7] Huisken, G., Ilmanen, T., The inverse mean curvature flow and the Riemannian Penrose inequality, J. Differential Geom. 59 (2001), no. 3, 353–437.

[8] Lin, C.-Y.; Wang, Y.-K., On isometric embeddings into anti-de Sitter spacetimes, Int. Math. Res. Not. IMRN 2015, no. 16, 7130–7161.

[9] Mantoulidis, C.; P. Miao, P.; Tam, L.-F., Capacity, quasi-local mass, singular fill-ins, preprint, arXiv:1805.05493.

[10] Meeks, W. III; Simon, L.; Yau, S.-T., Embedded Minimal Surfaces, Exotic Spheres, and Manifolds with Positive Ricci Curvature, Annals of Mathematics, Second Series, 116, no. 3 (1982), pp. 621–659.

[11] Miao, P., Positive mass theorem on manifolds admitting corners along a hypersurface, Adv. Theor. Math. Phys. 6 (2002), no. 6, 1163–1182.

[12] Miao, P.; Shi, Y.-G.; Tam, L.-F., On geometric problems related to Brown-York and Liu-Yau quasilocal mass Comm. Math. Phys. 298 (2010), no. 2, 437–459.

[13] Neves, André Insufficient convergence of inverse mean curvature flow on asymptotically hyperbolic manifolds, J. Differential Geom. 84 (2010), no. 1, 191–229.

[14] Pogorelov, A., Extrinsic Geometry of Convex Surfaces, Translations of Mathematical Monographs 35. Providence, RI: American Mathematical Society, 1973.

[15] Shi, Y.-G.; Tam, L.-F., Some lower estimates of ADM mass and Brown-York mass, math/0406559.

[16] Shi, Y.-G.; Tam, L.-F., Positive mass theorem and the boundary behaviors of compact manifolds with nonnegative scalar curvature, J. Differential Geom. 62 (2002), 79–125.

[17] Shi, Y.-G.; Tam, L.-F., Quasi-spherical metrics and applications, Comm. Math. Phys. 260 (2004), 65–80.

[18] Shi, Y.-G.; Tam, L.-F Quasi-Local Mass and the Existence of Horizons, Comm. Math. Phys. 274, No.2, (2007), 277–295.

[19] Shi, Y.-G.; Tam, L.-F., Rigidity of compact manifolds and positivity of quasi-local mass Classical Quantum Gravity 24 (2007), no. 9, 2357–2366.

[20] Shi, Y.-G.; Zhu, J.-T. Regularity of inverse mean curvature flow in asymptotically hyperbolic manifolds with dimension 3, arXiv:1811.06158

(Yuguang Shi) Key Laboratory of Pure and Applied Mathematics, School of Mathematical Sciences, Peking University, Beijing, 100871, P. R. China
E-mail address: ygshi@math.pku.edu.cn

(Luen-Fai Tam) The Institute of Mathematical Sciences and Department of Mathematics, The Chinese University of Hong Kong, Shatin, Hong Kong, China.
E-mail address: lftam@math.cuhk.edu.hk