Abstract
We characterise the distributions generated by the boundary values of functions from Privalov spaces.

1. Introduction
We use the following notation and preliminaries. \(U\) stands for the open unit disc in \(C\) and \(T\) is its boundary, i.e. \(U = \{ z \in C \mid |z| < 1 \}\), \(T = \partial U\), and \(\Pi^{+}\) is the upper half plane, meaning \(\Pi^{+} = \{ z \in C \mid \text{Im}z > 0 \}\). For a function \(f\) holomorphic on a region \(\Omega\) we write \(f \in H(\Omega)\). \(L^p(\Omega)\) is the space of measurable functions on \(\Omega\) such that \(\int_{\Omega} |f(x)|^p dx < \infty\); \(L^p_{\text{loc}}(\Omega)\) is the space of measurable functions on \(\Omega\) such that for every compact set \(K \subset \Omega\) the following holds \(\int_{K} |f(x)|^p dx < \infty\).

Privalov spaces on \(U\) and \(\Pi^{+}\) and their properties: Privalov class, denoted with \(N^p\), \(1 < p < \infty\), consists of all functions \(f \in H(U)\) such that
\[
\sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} (\log^+ |f(re^{i\theta})|)^p d\theta < \infty.
\]

Theorem. ([8]) The function \(f\), holomorphic on \(U\), belongs to \(N^p\) if and only if for every \(\epsilon > 0\) there exist \(\delta > 0\) such that for every measurable set \(E \subset T\), satisfying \(m(E) < \delta\) the following holds
\[
\int_{E} (\log^+ |f(e^{i\theta})|)^p d\theta < \epsilon, \quad \text{for all } 0 \leq r < 1.
\]

Theorem. ([8]) The function \(f\), holomorphic on \(U\), belongs to \(N^p\) if and only if the subharmonic function \(z \mapsto (\log^+ |f(z)|)^p(z \in U)\) has a harmonic majorant.

Every function in Nevalina class, \(N(U)\), because of Fatou’s lemma, has a nontangential (radial) limit on \(T\) almost everywhere; every function in Privalov class, \(N^p(U)\), has a nontangential (radial) limit on \(T\) almost everywhere, in both cases we denote the boundary value with \(f^*(e^{i\theta}) = \lim_{r \to 1} f(e^{i\theta})\).

The class \(N^p(\Pi^{+}), p > 1\), is introduced analogously to \(N^p(U)\), and is the set of all holomorphic functions on \(\Pi^{+}\) satisfying
\[
\sup_{0 < y < \infty} \int_{-\infty}^{\infty} (\log (1 + |f(x + iy)|))^p dx < \infty.
\]

Every \(f \in N^p(\Pi^{+})\) has a nontangential limit \(f^*(x)\) almost everywhere on the real axis.

Theorem. ([12]) The set \(L\) is bounded in \(N^p(\Pi^{+})\) if and only if
i) There exist $C > 0$ such that 
\[
\int_R (\log(1 + |f^*(x)|))^p dx < C
\]
for all $f \in L$.

ii) For every $\varepsilon > 0$, exist $\delta > 0$ such that 
\[
\int_E (\log(1 + |f^*(x)|))^p dx < \varepsilon
\]
for all $f \in L$, and every Lebesgue measurable $E \subset R$ satisfying $m(E) < \delta$.

**Distributions:** $C^\infty(R^n)$ denotes the set of all complex valued functions infinitely differentiable on $R^n$; $C^\infty_c(R^n)$ is the subset of $C^\infty(R^n)$ which contains compactly supported functions. Support of the function $f$ denoted with $\text{supp} f$ is the clossure of the set $\{x: f(x) \neq 0\}$ in $R^n$. $D = D(R^n)$ denotes the space $C^\infty_c(R^n)$ in which the convergence is defined in the following way: the sequence $\{\varphi_\lambda\}$, of functions $\varphi_\lambda \in D$, converges to $\varphi \in D$ when $\lambda \to \lambda_0$ if and only if there exist compact subset of $R^n$ such that $\text{supp} \varphi_\lambda \subseteq K$ for all $\lambda$, $\text{supp} \varphi \subseteq K$, and for every $n$-tuple $\alpha$ of nonnegative integers the sequence $(D_\alpha^2(\varphi_\lambda(x)))$ converges to $(D_\alpha^2(\varphi(x)))$ uniformly on $K$ when $\lambda \to \lambda_0$.

With $D' = D'(R^n)$ is denoted the space of all continuous, linear functionals on $D$, where the continuity is in the sense: from $\varphi_\lambda \to \varphi$ in $D$ when $\lambda \to \lambda_0$, it follows that $(T, \varphi_\lambda) \to (T, \varphi)$ in $C$, when $\lambda \to \lambda_0$.

The space $D'$ is called the space of distributions. We use the convention $(T, \varphi) = T(\varphi)$ for the value of the functional $T$ acting on the function $\varphi$.

Let $\varphi \in D$ and $f(x) \in L^1_{\text{loc}}(R^n)$. Then the functional $T_f$ on $D$ defined with 
\[
(T_f, \varphi) = \int_{R^n} f(t)\varphi(t) dt, \varphi \in D,
\]
is an element in $D'$ and it is called the regular distribution generated by the function $f$.

**2. Main results**

**Theorem.** ([5]) Sufficient and necessary condition for the measurable function $\varphi(e^{it})$ defined on $T$ to coincide almost everywhere on $T$ with the boundary value $f^*(e^{it})$ of some function $f(z)$ in $N(U)$, is to exist a sequence of polynomials $\{P_n(z)\}$ such that:

i. $\{P_n(e^{i\theta})\}$ converges to $\varphi(e^{i\theta})$ almost everywhere on $T$;

ii. $\lim_{n \to \infty} \int_0^{2\pi} (|P_n(e^{i\theta})|)^p d\theta < \infty$.

**Theorem 1.** Let $T_f \in D'$ is generated by the boundary value $f^*(x)$ of a function $f(z)$ in $N^p(T^n)$. There exist sequence of polynomials $\{P_n(z)\}, z \in T^n$, and respectively $\{T_n\}, T_n \in D'$, generated by the boundary values $P_n^*(x)$ of the polynomials $P_n(z)$, i.e. $T_n = T_{P_n^*}$ such that:

i. $T_n \to T_f$ in $D'$ when $n \to \infty$;

ii. $\lim_{n \to \infty} \int_0^{2\pi} (|P_n^*(x)|)^p dx < \infty$ for every $\varphi \in D$.

**Proof.** Let the assumptions of the theorem hold. Since $f \in N^p(T^n)$, one has $f \in H(T^n)$ and there exist a constant $C > 0$ such that 
\[
f = \log(1 + |f(x + iy)|)^p dx \leq C \text{ for every } z = x + iy \in T^n.
\]

Let $\{y_n\}$ be a sequence of positive real numbers satisfying $\lim_{n \to \infty} y_n = 0$. We define a sequence of complex functions $\{F_n(z)\}$ with
\[
F_n(z) = f(z + iy_n).
\]
The functions $F_n(z)$ are holomorphic on $T^n \cup R$. Margeljan theorem implies that for arbitrary compact subset $K$ of $T^n \cup R$ with complement being connected, for the functions $F_n(z)$ there exist polynomials $P_n(z)$ such that $|F_n(z) - P_n(z)| < \varepsilon_n$ for all $z \in K$, where $\varepsilon_n > 0$ and $\varepsilon_n \to 0$ when $n \to \infty$.

In what follows we prove i. and ii.
Let \( \varphi \in D, \supp \varphi = K. \) Then

\[
|\langle T_n \varphi \rangle - \langle T'_r \varphi \rangle| = \left| \int_{-\infty}^{\infty} P_n^* (x) \varphi(x) dx - \int_{-\infty}^{\infty} f^*(x) \varphi(x) dx \right|
\]

\[
= \left| \int_{-\infty}^{\infty} [P_n^*(x) - f^*(x)] \varphi(x) dx \right| = \left| \int_K [P_n^*(x) - f^*(x)] \varphi(x) dx \right|
\]

\[
\leq M \int_K [P_n^*(x) - f^*(x)] dx \leq M \varepsilon_n m(K) \to 0
\]
when \( n \to \infty. \)

In the previous calculations we use the notation \( m(K) \) for the Lebesgue measure of the set \( K, M = \max(\varphi(x) : x \in K) \) and \( \varepsilon_n = \varepsilon + [f^*(x) - F_n(x)] \). It is obvious that \( \varepsilon_n \to 0 \) when \( n \to \infty \). The later calculation implies that \( \langle T_n, \varphi \rangle \to \langle T'_r, \varphi \rangle \) when \( n \to \infty \) for every, but fixed, \( \varphi \in D \), meaning \( T_n \to T'_r \) weakly in \( D' \). To prove the convergence in the strong topology it suffices to prove the same convergence for \( \varphi \in B \) for an arbitrary bounded set in \( D \). Choose \( B \subset D \), arbitrary bounded set. The condition of boundness implies that there exists a compact set \( K \) such that \( \supp \varphi = K, ||\varphi||_{D(K)} < M, \) for every \( \varphi \in B \). Note that the calculations at the beginning of the paragraph hold for every \( \varphi \in B \) and the new compact set chosen for the boundness condition. Hence, \( T_n \to T'_r \) in \( D' \):

(ii)

\[
\int_{-\infty}^{\infty} (\log (1 + |P_n^*(x)|^p) |\varphi(x)| dx
\]

\[
= \int_K (\log (1 + |P_n^*(x) + F_n(x) - F_n(x)|^p) |\varphi(x)| dx
\]

\[
\leq \int_K (\log (1 + |P_n^*(x) - F_n(x)| + |F_n(x)|)^p) |\varphi(x)| dx
\]

\[
\leq \int_K (\log (1 + |F_n(x)| + |P_n^*(x) - F_n(x)|)^p) |\varphi(x)| dx
\]

\[
\leq M 2^{p-1} \int_K (\log (1 + |F_n(x)|)^p) dx + M 2^{p-1} \int_K |P_n^*(x) - F_n(x)|^p dx
\]

\[
\leq MC + M \varepsilon_n^p m(K).
\]

Because \( \varepsilon_n \to 0, n \to \infty \) we get \( \int_K (\log (1 + |P_n^*(x)|)^p) |\varphi(x)| dx < C' \) meaning

\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} (\log (1 + |P_n^*(x)|)^p) |\varphi(x)| dx < \infty, \text{ for all } \varphi \in D.
\]

In the proof of ii. We use the inequalities \( |a + b| \leq |a| + |b|, \) \( \log (1 + a + b) \leq \log (1 + a) + b \), \( a, b > 0 \) and \( (a + b)^p \leq 2^{p-1} (a^p + b^p) \), \( p \geq 1. \)

**Theorem 2.** Let \( \varphi_0 \) be a locally integrable function and \( T_{\varphi_0} \in D' \) is generated by the function \( \varphi_0 \). Let there exist sequence of polynomials \( P_n(z) \) satisfying the conditions:

i. The sequence of distributions generated by the boundary values \( P_n^*(x) \) of \( P_n(z) \) converges to \( T_{\varphi_0} \) in \( D' \) when \( n \to \infty; \)

ii. \( \lim_{n \to \infty} \int_{-\infty}^{\infty} (\log (1 + P_n(x + iy))^p) |\varphi(x)| dx < \infty, \forall x + iy = x + iy \in \Pi^+, \varphi \in D. \)

There exists a function \( f \in H(\Pi^+) \) such that
\[
\int_K (\log (1 + |f(x + iy)|))^p \, dx < C < \infty, \forall \, z = x + iy \in \Pi^+,
\]
for every compact \( K \subset R \), and \( p > 0 \), and every \( \varphi \in D \) and arbitrary compact set \( K \subset R \).

\[
\lim_{y \to 0^+} \int_{-\infty}^{\infty} f(x + iy)\varphi(x) \, dx = \langle T_{\varphi^p}, \varphi \rangle.
\]

**Proof.** Let the assumptions of the theorem are fulfilled. In [3] it is proven that from i., i.e.
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} P_n(x)\varphi(x) \, dx = \int_{-\infty}^{\infty} \varphi_0(x)\varphi(x) \, dx, \quad \varphi \in D,
\]
implies the existence of \( f \in H(\Pi^+) \) such that the sequence of polynomials converges to \( f \), uniformly on arbitrary compact subsets of \( \Pi^+ \) when \( n \to \infty \).

Firstly we will prove that this function \( f \) is holomorphic and satisfies the condition
\[
\int_K (\log (1 + |f(x + iy)|))^p \, dx < C 
\]
for all \( z = x + iy \in \Pi^+ \) and arbitrary compact set \( K \subset R \).

Indeed, we use the condition ii., i.e.
\[
\lim_{n \to \infty} \int_{-\infty}^{\infty} (\log (1 + |P_n(x + iy)|))^p \, dx < C < \infty, \forall \, z = x + iy \in \Pi^+, \quad \varphi \in D.
\]

Let \( K \) be compact set. There exists \( \varphi(x) \in C_c^{\infty}(R^n), \varphi(x) = 1, \forall x \in K \). To obtain the last statement, it is enough to take characteristic function of the set \( K \) and to regularize it. Substitution of such \( \varphi \) in to ii., implies that for every \( n \in N \),
\[
\int_K (\log (1 + |P_n(x + iy)|))^p \, dx < C < \infty, \forall \, z = x + iy \in \Pi^+.
\]

Now,
\[
\int_K (\log (1 + |f(x + iy)|))^p \, dx = \int_K \lim_{n \to \infty} (\log (1 + |P_n(x + iy)|))^p \, dx 
\]
\[
\leq \lim_{n \to \infty} \int_{-\infty}^{\infty} (\log (1 + |P_n(x + iy)|))^p \, dx < C < \infty,
\]
i.e.
\[
\int_K (\log (1 + |f(x + iy)|))^p \, dx \leq C < \infty \quad \text{for arbitrary compact set} \quad K \subset R \quad \text{and every} \quad z = x + iy \in \Pi^+.
\]

It remains to be proved that \( \lim_{y \to 0^+} \int_{-\infty}^{\infty} f(x + iy)\varphi(x) \, dx = \langle T_{\varphi^p}, \varphi \rangle \), for every \( \varphi \in D \).

Let \( \varphi \in D \) and \( \text{supp} \varphi = K \subset R \). Then
\[
\lim_{y \to 0^+} \int_{-\infty}^{\infty} f(x + iy)\varphi(x) \, dx = \lim_{y \to 0^+} \int_{-\infty}^{\infty} (P_n(x + iy))\varphi(x) \, dx =
\]
\[
= \lim_{y \to 0^+} \lim_{n \to \infty} \int_{-\infty}^{\infty} (P_n(x + iy))\varphi(x) \, dx = \lim_{n \to \infty} \lim_{y \to 0^+} \int_{-\infty}^{\infty} (P_n(x + iy))\varphi(x) \, dx =
\]
\[
= \lim_{n \to \infty} \int_{-\infty}^{\infty} P_n(x + iy)\varphi(x) \, dx = \int_{-\infty}^{\infty} \varphi_0(x)\varphi(x) \, dx = \langle T_{\varphi^p}, \varphi \rangle,
\]
for every \( \varphi \in D \).

The previous equalities are obvious, except the following
\[
\lim_{y \to 0^+} \lim_{n \to \infty} \int_{-\infty}^{\infty} P_n(x + iy)\varphi(x) \, dx = \lim_{n \to \infty} \lim_{y \to 0^+} \int_{-\infty}^{\infty} P_n(x + iy)\varphi(x) \, dx \quad \text{...} \quad (*)
\]
for \( z = x + iy \in \Pi^+ \).
We will prove (\ast).

To do that we consider the sequence of functions \( \{g_n(y)\} \) defined by

\[ g_n(y) = \int_K (P_n(x + iy)\varphi(x))\,dx, \quad x + iy \in K, \]

for \( K \) compact subset of \( \Pi^+ \) such that \( z \in K \) for \( \text{Re}(z) \in K \). Because \( \{P_n(x + iy)\} \) converges to \( (x + iy) \) uniformly on \( K \), when \( n \to \infty \), one obtains that for fixed \( y \)

\[ \lim_{n \to \infty} g_n(y) = \int_K (P_n(x + iy)\varphi(x))\,dx = \int_K (f(x + iy)\varphi(x))\,dx = g(y), \]

i.e. the sequence \( \{g_n(y)\} \) converges to \( g(y) \) when \( n \to \infty \). We will prove that this convergence is uniform on \( \text{Im}K \), which will imply the statement. Indeed,

\[
0 \leq \sup_y |g_n(x + iy) - g(x + iy)| = \sup_y \left| \int_K (P_n(x + iy) - f(x + iy))\varphi(x)\,dx \right| \\
= \sup_y \left| \int_K [P_n(x + iy) - f(x + iy)]\varphi(x)\,dx \right| \\
\leq \sup_y \int_K |(P_n(x + iy) - f(x + iy))\varphi(x)|\,dx \\
\leq M \sup_y \int_K |P_n(x + iy) - f(x + iy)|\,dx.
\]

Since \( P_n(x + iy) \to f(x + iy) \) uniformly on \( K \), it follows that

\[
\int_K |P_n(x + iy) - f(x + iy)|\,dx
\]

converges to 0 uniformly on \( \text{Im}(K) \) meaning

\[
\lim_{n \to \infty} \sup_y |g_n(x + iy) - g(x + iy)| = 0.
\]

Finally, \( \lim_{n \to \infty} \sup_y |g_n(x + iy) - g(x + iy)| = 0. \)

3. Conclusion

We obtain necessary and sufficient condition for a distribution generated from an element of the Privalov class to be boundary value of analytic functions on upper half space. The boundary values are taken in the distributional sense.

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Conflicts of Interest
The authors don’t have competing for any interests

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