Newly discovered $\Xi^0_c$ resonances and their parameters

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The aim of the present article is investigation of the newly observed resonances $\Xi_c(2923)^0$, $\Xi_c(2939)^0$, and $\Xi_c(2965)^0$ which are real candidates to charm-strange baryons. To this end, we calculate the mass and pole residue of the ground-state and excited $1P$ and $2S$ spin-1/2 flavor-s Sextet baryons $\Xi^0_c$, $\Xi^0_c(1/2^-)$ and $\Xi^0_c(1/2^+)$ with quark content $c s d$, respectively. The masses and pole residues of the ground-state and excited spin-3/2 baryons $\Xi^0_c$ are found as well. Spectroscopic parameters of these particles are computed in the context of the QCD two-point sum rule method. Widths of the excited baryons are evaluated through their decays to final states $\Lambda^+_c K^-$ and $\Xi^0_c\pi$. These processes are explored by means of the full QCD light-cone sum rule method necessary to determine strong couplings at relevant vertices. Obtained predictions for the masses and widths of the four excited baryons, as well as previous results for $1P$ and $2S$ flavor-antitriplet spin-1/2 particles $\Xi^0_c$ are confronted with available experimental data on $\Xi^0_c$ resonances to fix their quantum numbers. Our comparison demonstrates that the resonances $\Xi_c(2923)^0$ and $\Xi_c(2939)^0$ can be considered as $1P$ excitations of the spin-1/2 flavor-s Sextet and spin-3/2 baryons, respectively. The resonance $\Xi_c(2965)^0$ may be interpreted as the excited $2S$ state of either spin-1/2 flavor-s Sextet or antitriplet baryon.

I. INTRODUCTION

The discovery of three new resonances $\Xi_c(2923)^0$, $\Xi_c(2939)^0$, and $\Xi_c(2965)^0$ by the LHCb collaboration is a last result of the experiments devoted to investigation of charmed and bottom baryons with different spin-parities and quark contents. Five narrow states $\Omega^0_c$ fixed in the $\Xi^+_c K^-$ invariant mass distribution, and four peaks $\Omega^0_c$ detected recently in the $\Xi^0_c K^-$ spectrum were results of previous measurements performed by LHCb.

Needless to say, that discovery of the resonances $\Omega^0_c$ stimulated numerous studies of excited charmed baryons aimed to understand their internal organizations and quantum numbers. Actually, heavy flavored baryons were already objects of theoretical analyses, in which spectroscopic parameters of the ground state and excited particles, their decay channels and strong couplings, magnetic moments and radiative decays were studied by means of different models and methods of high energy physics. New experimental information on $\Omega^0_c$, besides traditional models, gave rise to their interpretations as exotic pentaquark states. In our articles, we investigated the baryons $\Omega^0_c$ and $\Omega^0_c$, where one can find further details and references to relevant publications.

The baryons from the $\Xi^+_c = c s d$ family are another interesting objects for both experimental and theoretical analyses. Parameters of the ground-state $J^P = 1/2^+$ and $3/2^+$ baryons with the content $c s d$ were measured already and included into Particle Data Group (PDG) tables. Thus, the mass and mean lifetime of the flavor-antitriplet baryon $\Xi^0_c$ are

$$m_{\exp} = 2470.90^{+0.22}_{-0.29} \text{ MeV}, \quad \tau_{\exp} = (1.53 \pm 0.06) \times 10^{-13} \text{ s},$$

whereas for the mass of the flavor-s Sextet $J^P = 1/2^+$ ground-state particle $\Xi^0_c$ we have

$$m_{\exp} = (2579.2 \pm 0.5) \text{ MeV}. \quad (2)$$

The mass of the $J^P = 3/2^+$ baryon $\Xi_c(2645)^0$ is also known

$$m_{\exp} = 2645.56^{+0.24}_{-0.30} \text{ MeV}. \quad (3)$$

There are a few charged and neutral particles of this family listed in Ref. 31, some of which will be considered in the last section of the present work.

As we have noted above, theoretical investigations of heavy flavored baryons, including $\Xi_c$ ones, have long history. These particles were explored in the context of various quark models, by using the QCD sum rule method, by means of the Heavy Quark Effective Theory (HQET) and lattice simulations.

The discovery of three new resonances by LHCb added valuable knowledge about excited baryons $\Xi^0_c$, which together with $\Xi_c(2930)^0$ and $\Xi_c(2970)^0$ generated theoretical activities to explain their parameters. Problem is that LHCb did not inform on spins and parities of these resonances, which are important topic of continuing theoretical studies. Here, it is necessary to give some information about the resonance $\Xi_c(2930)^0$, which is relatively "old" member of this family. It was observed by the BaBar collaboration as the intermediate resonant structure in the process $B^- \to \Lambda^{+}_c \Lambda^{0}_c K^-$. Existence of $\Xi_c(2930)^0$ was confirmed recently by Belle in Ref. 32, in which the collaboration reported about its observation as a resonance in the $\Lambda^{+}_c K^-$ invariant mass spectrum in the same decay process. The mass and width of this state reported
by Belle are

\[ m = (2928.9 \pm 3.0^{+0.9}_{-1.2}) \text{MeV}, \]
\[ \Gamma = (19.5 \pm 8.4^{+5.9}_{-7.5}) \text{MeV}. \]  

Parameters of \( \Xi_c(2930)^0 \), its mass and width were calculated in the framework of different approaches \[22\] to \[38\].

The new resonances have masses and widths which do not differ considerably from ones of \( \Xi_c(2930)^0 \). For simplicity of presentation, we label parameters of \( \Xi_c(2930)^0 \), \( \Xi_c(2939)^0 \), and \( \Xi_c(2965)^0 \) by subscripts 1, 2, and 3, respectively. The masses and widths of these states are equal to \[1\]

\[ m_1 = (2923.04 \pm 0.25 \pm 0.20 \pm 0.14) \text{MeV}, \]
\[ \Gamma_1 = (7.1 \pm 0.8 \pm 1.8) \text{MeV}, \]  
\[ m_2 = (2938.55 \pm 0.21 \pm 0.17 \pm 0.14) \text{MeV}, \]
\[ \Gamma_2 = (10.2 \pm 0.8 \pm 1.1) \text{MeV}, \]  
\[ m_3 = (2964.88 \pm 0.26 \pm 0.14 \pm 0.14) \text{MeV}, \]
\[ \Gamma_3 = (14.1 \pm 0.9 \pm 1.3) \text{MeV}. \]  

These resonances immediately became object of theoretical investigations \[39\] to \[42\], in which they were studied in a rather detailed form. These states were considered mostly as conventional flavor-sextet 1P-wave baryons of different spins \[39\] to \[40\], though sextet 2S interpretation of the heaviest resonance from this list is also on agenda \[41\]. The particles \( \Xi_0^0 \) were described also as molecular \( \bar{D} \Lambda - D \Sigma \) states \[42\]. The mass and width of the excited flavor-antitriplet baryons \( \Xi_0^0 \) were calculated recently in Ref. \[38\]. Performed analysis allowed the authors to conclude that the baryon \( \Xi_0^0(1/2^-) \) with parameters \( \tilde{m} = (2922 \pm 83) \text{MeV} \) and \( \tilde{\Gamma} = (19.4 \pm 3.3) \text{MeV} \), and quantum numbers \( (1P, 1/2^-) \) may be interpreted as the state \( \Xi_c(2930)^0 \). The remaining radically excited antitriplet baryon \( \Xi_0^0(1/2^+) \) with \( m = (2922 \pm 83) \text{MeV} \) and \( \Gamma = (13.6 \pm 2.3) \text{MeV} \) can be examined as a candidate to one of three new resonances.

As is seen, various suggestions were made on structures and quantum numbers of the \( c \bar{s} d \) states, and predictions obtained by means of different methods in the context of these assumptions, sometimes, contradict to each other. Therefore, additional studies of these baryons are required to clarify situation with \( c \bar{s} d \) resonances. Before detailed analysis, there is a necessity to establish short-hand notations for different baryons to be studied in this article. First of all, we omit superscript 0 for all baryons. For the flavor-antitriplet spin-1/2 baryons, we use standard notations \( \Xi_c, \Xi_c(1/2^-) \), and \( \Xi_c(1/2^-) \) for the ground-state, first orbitally and radially excited states, respectively. The flavor-sextet spin-1/2 baryons will be presented as \( \Xi_c, \Xi_c(1/2^-), \) and \( \Xi_c(1/2^-) \) in accordance with their quantum numbers \( (1S, 1/2^+), (1P, 1/2^-), \) and \( (2S, 1/2^+) \). For the spin-3/2 baryons with \( (1S, 3/2^+), (1P, 3/2^-), \) and \( (2S, 3/2^+) \), we introduce brief notations \( \Xi_c^*, \Xi_c(3/2^-) \), and \( \Xi_c(3/2^-) \), which cannot lead to confusions.

In the present article, we explore the ground-state \( \Xi_c^* \) and excited spin-1/2 flavor-sextet baryons \( \Xi_c^*(1/2^-) \) and \( \Xi_c^*(1/2^-) \), and spin-3/2 particles \( \Xi_c^*, \Xi_c(3/2^-) \), and \( \Xi_c(3/2^-) \) by computing their masses and pole residues. These spectroscopic parameters are evaluated using the QCD sum rule method \[13\] to \[14\], in which contributions of vacuum condensates up to dimension 10 are taken into account. We determine also widths of excited baryons \( \Xi_c(1/2^-), \Xi_c^*(1/2^-), \Xi_c(3/2^-), \) and \( \Xi_c(3/2^-) \) by calculating partial widths of their strong decays to final states \( \Lambda^+_K \bar{K} \). Obtained predictions for partial widths of these decay modes will allow us to estimate full widths of \( \Xi_c(1/2^-), \Xi_c^*(1/2^-), \Xi_c(3/2^-), \) and \( \Xi_c(3/2^-) \). The strong decay processes are explored by means of the QCD light-cone sum rule (LCSR) approach \[43\].

This article is structured in the following way: In Sec. \[II\] we calculate the spectroscopic parameters of the ground state and excited baryons \( \Xi_c^*, \Xi_c(1/2^-), \) and \( \Xi_c(1/2^-) \). Here, we also evaluate the parameters of the states \( \Xi_c^*, \Xi_c(3/2^-), \) and \( \Xi_c(3/2^-) \). Results extracted from the sum rules in this section will be compared with the experimental data, but also serve as input information for the next sections. In Sec. \[III\] we derive the LCSRs for the strong couplings \( g_1 \) and \( g_2 \) describing the vertices \( \Xi_c^*(1/2^-) \Lambda^+_K \bar{K} \) and \( \Xi_c^*(1/2^-) \Lambda^+_K \bar{K} \), that are key ingredients to evaluate width of the processes \( \Xi_c^*(1/2^-) \rightarrow \Lambda^+_K \bar{K} \) and \( \Xi_c^*(1/2^-) \rightarrow \Lambda^+_K \bar{K} \). In Sec. \[IV\] we analyze the vertices \( \Xi_c^*(1/2^-) \Xi_c^* \bar{\Xi} \) and \( \Xi_c^*(1/2^-) \Xi_c^* \bar{\Xi} \), and calculate corresponding strong couplings \( g_2 \) and \( g_1 \). The partial widths of decays \( \Xi_c^*(1/2^-) \rightarrow \Xi_c^* \bar{\Xi} \) and \( \Xi_c^*(1/2^-) \rightarrow \Xi_c^* \bar{\Xi} \) are also found in this section. Section \[V\] is devoted to investigation of the decays \( \Xi_c^*(3/2^-) \rightarrow \Lambda^+_K \bar{K} \), \( \Xi_c^* \bar{\Xi} \) and \( \Xi_c(3/2^-) \rightarrow \Lambda^+_K \bar{K} \), \( \Xi_c^* \bar{\Xi} \). The last Section \[VI\] is reserved for comparison of obtained theoretical predictions with the LHCb data and, in accordance with this analysis, assignment of appropriate quantum numbers to new three LHCb resonances. This section contains also our concluding notes. Appendix contains explicit expressions some of invariant amplitudes employed to extract parameters of color-sextet spin-1/2 baryons.

### II. Masses and Pole Residues of the Baryons \( \Xi_c^* \) and \( \Xi_c \)

The sum rules required to evaluate the mass and residue of the spin-1/2 baryons \( \Xi_c^*, \Xi_c(1/2^-) \), and \( \Xi_c(1/2^-) \), and spin-3/2 baryons \( \Xi_c^*, \Xi_c(3/2^-) \), and \( \Xi_c(3/2^-) \) can be obtained from analysis of the following two-point correlation functions

\[ \Pi_{(\mu \nu)}(p) = i \int d^4 x e^{i px} \langle 0 | T \{ \eta_{(\mu)}(x) \bar{\eta}_{(\nu)}(0) \} | 0 \rangle, \]
where $\eta(x)$ and $\eta_\mu(x)$ are interpolating fields for $\Xi'_c$ and $\Xi_c$ states with spins $1/2$ and $3/2$, respectively. In the case of the flavor-sexet spin-$1/2$ baryons the current $\eta$ is given by the formula
\[
\eta = -\frac{1}{\sqrt{2}} \epsilon^{abc} \left\{ (d^T_a C_8) \gamma_5 s_c + \beta (d^T_a C \gamma_5 c_b) s_c \right. \\
\left. - \left[(c^T_a C_{sb}) \gamma_5 d_c + \beta (c^T_a C \gamma_5 s_b) d_c \right] \right\}. \tag{9}
\]
For spin-$3/2$ baryons, we use
\[
\eta_\mu = -\frac{2}{\sqrt{3}} \epsilon^{abc} \left\{ (d^T_a C_\gamma s_b) c_c + \left(s^T_a C_\gamma s_c \right) d_c \right. \\
\left. + \left(c^T_a C_\gamma d_b \right) s_c \right\}. \tag{10}
\]
In formulas for the currents $C$ is the charge conjugation matrix. The current $\eta(x)$ for the spin-$1/2$ baryons depends on an arbitrary mixing parameter $\beta$ with $\beta = -1$ corresponding to the Ioffe current.

We begin from the spin-$1/2$ baryons and first compute the mass of the ground-state particle $\Xi'_c$. For this purposes, we express the correlation function $\Pi^\text{Phys}(p)$ using the physical parameters of the ground-state particle $(1S, 1/2^+)$. Then, in the "ground-state+continuum" approximation $\Pi^\text{phys}(p)$ is given by the simple formula
\[
\Pi^\text{Phys}(p) = \frac{\langle 0| \eta_\mu \Xi'_c(p, s) \rangle \langle \Xi'_c(p, s) | \eta_\mu \rangle}{m^2 - p^2} + \cdots, \tag{11}
\]
where $m$ and $s$ are the mass and spin of $\Xi'_c$, respectively. Contributions of higher resonances and continuum states are denoted in Eq. (11) by dots. In expression for $\Pi^\text{phys}(p)$ summation over the spin $s$ is implied.

We continue our analysis by using the matrix element
\[
\langle 0| \eta_\mu \Xi'_c(p, s) \rangle = \lambda u(p, s), \tag{12}
\]
where $\lambda$ is the pole residue of $\Xi'_c$. Carrying out summation over $s$ in Eq. (11) by employing the matrix element (12) and the formula
\[
\sum_s u(p, s) \pi(p, s) = \not{p} + m, \tag{13}
\]
we get
\[
\Pi^\text{Phys}(p) = \frac{\lambda^2 (\not{p} + m)}{m^2 - p^2} + \cdots. \tag{14}
\]
The function $\Pi^\text{Phys}(p)$ contains Lorentz structures proportional to $\not{p}$ and $I$. To find the sum rule, we can employ invariant amplitudes that correspond to these structures.

The second component of our investigation is the QCD side of the sum rule $\Pi^\text{OPE}(p)$, which should be computed in the operator product expansion (OPE) with certain accuracy. To this end, one has to insert the interpolating current $\eta$ into Eq. (5) and contract the quark fields. We compute $\Pi^\text{OPE}(p)$ using light $q$ and heavy $Q$ quark $x$-space propagators, explicit expressions of which are presented below
\[
S^b_q(x) = i \frac{x \delta_{ab}}{2 \pi^2 x^4} - \frac{m_q \delta_{ab}}{4 \pi^2 x^2} + \frac{(7q) \delta_{ab}}{12 (1 - i m_q/4 q)} \left(1 - i m_q/4 q\right) x^2 \pi^2 (g^2)^2 \delta_{ab} - \frac{x^2 \pi^2 (g^2)^2 \delta_{ab}}{17.76} + \frac{m_q g_s}{32 \pi^2} G_{qab}^\mu \sigma_{\mu
u} \left[\ln\left(-\frac{x^2 \Lambda^2}{4}\right) + 2 \gamma_E + \cdots\right], \tag{15}
\]
and
\[
S^b_Q(x) = \frac{m_Q^2 \delta_{ab}}{4 \pi^2} + \frac{m_Q \sqrt{-x^2}}{\sqrt{-x^2}} K_0 \left(m_Q \sqrt{-x^2}\right) + \frac{g_s m_Q}{16 \pi^2} \int_0^1 du C_{qab}^{\mu
u} \left(u x\right) \left[\ln\left(-\frac{x^2 \Lambda^2}{4}\right) + 2 \gamma_E + \cdots\right]. \tag{16}
\]
Here, $u = 0, d$ or $s$, $\gamma_E \simeq 0.577$ is the Euler constant, and $A$ is the QCD scale parameter. We also introduce the notations $G_{qab}^{\mu
u} = G_A^{\mu
u} t_a t_b$, $G^0 = G_{a\beta} G_{a\beta}$, $A = 1, 2, \ldots, 8$, and $t_A = \lambda_A/2$, with $\lambda_A$ being the Gell-Mann matrices. The first two terms in Eq. (16) in square brackets are the free part of the heavy quark propagator in the coordinate representation, and $K_0(z)$ are the modified Bessel functions of the second kind.

After performing required calculations, for $\Pi^\text{OPE}(p)$ we get
\[
\Pi^\text{OPE}(p) = p \Pi_1^\text{OPE}(p^2) + \Pi_2^\text{OPE}(p^2). \tag{17}
\]
The function $\Pi^\text{OPE}(p)$ expressed in terms of quark-gluon degrees of freedom has the same Lorentz structure as $\Pi^\text{Phys}(p)$. By equating two representations of the correlation function, performing the Borel transformation and subtracting contributions due to higher resonances and continuum states, we extract Borel sum rules.

It is not difficult to see that the Borel transformation of $\Pi^\text{Phys}(p)$ is equal to
\[
\mathcal{B} \Pi^\text{Phys}(p) = \lambda^2 e^{-\frac{\Lambda^2}{\sqrt{2}}} (\not{p} + m). \tag{18}
\]
To derive the sum rule for $m$, it is enough to use the equality
\[
\lambda^2 e^{-\frac{\Lambda^2}{\sqrt{2}}} = \Pi_1^\text{OPE}(M^2, s_0), \tag{19}
\]
where $\Pi_1^\text{OPE}(M^2, s_0)$ is the Borel transformed and subtracted invariant amplitude $\Pi_1^\text{OPE}(p^2)$, and $M^2$ and $s_0$ are the Borel and continuum threshold parameters, respectively. The sum rule for the mass of the ground state
Then, the sum rule equalities are counted effects of the baryons $\Xi'$, of the sum rule, we use the "ground-state+excited-state" scheme. Therefore, we take into account effects of the baryons $\Xi'_c$ and $\Xi'_c(1/2^-)$, and find
\[\Pi^{\text{Phys}}(p) = \frac{\langle 0|\eta|\Xi'_c(1/2^-,p,s)\rangle\langle \Xi'_c(p,s)|\eta(0)\rangle}{m^2 - p^2} + \frac{\langle 0|\eta|\Xi'_c(1/2^+,p,s)\rangle\langle \Xi'_c(p,s)|\eta(0)\rangle}{m^2 - p^2} + \cdots,\] (21)
where $m$ and $s$ are the mass and spin of the excited state $\Xi'_c(1/2^-)$.

To simplify $\Pi^{\text{Phys}}(p)$, we employ Eq. (12) and additionally introduce the matrix element
\[\langle 0|\eta|\Xi'_c(1/2^-,p,s)\rangle = \bar{\lambda}\gamma_5\bar{u}(p,s),\] (22)
where $\bar{\lambda}$ is the pole residue of the baryon $\Xi'_c(1/2^-)$.

Performing summations over $s$ and $\bar{s}$ in Eq. (21) by employing relevant matrix elements and the formula (13), we get
\[\Pi^{\text{Phys}}(p) = \frac{\lambda^2(p+m)}{m^2 - p^2} + \frac{\bar{\lambda}^2(p - \bar{m})}{\bar{m}^2 - p^2} + \cdots.\] (23)

The Borel transformation of $\Pi^{\text{Phys}}(p)$ is equal to
\[\mathcal{B}\Pi^{\text{Phys}}(p) = \lambda^2 e^{-\frac{m^2}{M^2}}(p + m) + \bar{\lambda}^2 e^{-\frac{\bar{m}^2}{M^2}}(\bar{p} - \bar{m}).\] (24)

Then, the sum rule equalities are
\[\lambda^2 e^{-\frac{m^2}{M^2}} + \bar{\lambda}^2 e^{-\frac{\bar{m}^2}{M^2}} = \Pi^{\text{OPE}}(M^2, \bar{s}_0),\] (25)
and
\[\lambda^2 m^2 e^{-\frac{m^2}{M^2}} - \bar{\lambda}^2 \bar{m}^2 e^{-\frac{\bar{m}^2}{M^2}} = \Pi^{\text{OPE}}(M^2, \bar{s}_0).\] (26)

The first of these expressions is extracted from the structure $\sim p$, whereas the second one corresponds to terms proportional to $I$.

The derived equalities (25) and (26) contain four parameters ($m$, $\lambda$) and ($\bar{m}$, $\bar{\lambda}$) of the ground state and first orbitally excited baryon. As the mass $m$ of the ground-state baryon $\Xi'_c$, we use its value evaluated from the sum rule (20). Therefore, one has to find sum rules for the pole residue of the ground-state particle, as well as parameters ($\bar{m}$, $\bar{\lambda}$) of the excited state. Usual way to handle this problem is to act by the operator $d/d(-1/M^2)$ to Eqs. (25) and (26), and get missing equations. Then, after simple manipulations, we obtain
\[\bar{m}^2 = \frac{\Pi^{\text{OPE}}}{\Pi^{\text{OPE}} - m\Pi^{\text{OPE}}},\] \[\lambda^2 = \frac{m\Pi^{\text{OPE}}}{m + \bar{m}} + \frac{\Pi^{\text{OPE}}}{m + \bar{m}}\] (27)

Expressions written down in Eq. (27) are the QCD two-point sum rules for parameters of the ground-state and excited baryon, which can be employed to evaluate their numerical values. In these formulas, for simplicity, we do not show dependence of the functions $\Pi^{\text{OPE}}(M^2, \bar{s}_0)$ on $M^2$ and $\bar{s}_0$. The parameters of the radially excited baryon $\Xi'_c(1/2^+)$ can be extracted using $\Pi^{\text{Phys}}(p)$, in which the excited 1P state is replaced by $2S$ particle. In Eq. (27) this is equivalent to transformation $\bar{m} \to -m$, and redefinition of the residue $\lambda \to \lambda'$, where $(m', \lambda')$ are parameters of $\Xi'_c(1/2^+)$. The sum rules (20) and (27) depend on the vacuum expectations values of the different quark, gluon, and mixed operators, as well as on the masses of $s$ and $c$-quarks. Values of these universal input parameters are presented below
\[\langle \bar{q}q \rangle = -(0.24 \pm 0.01)^3 \text{GeV}^3, \langle \bar{s}s \rangle = 0.8\langle q\bar{q} \rangle,\]
\[\langle \bar{q}g_s\sigma Gq \rangle = m_0^2\langle \bar{q}q \rangle, \langle \bar{s}g_s\sigma Gs \rangle = m_0^2\langle \bar{s}s \rangle,\]
\[m_0^2 = (0.8 \pm 0.1) \text{GeV}^2,\]
\[\langle \alpha_s G^2 \rangle = (0.012 \pm 0.004) \text{GeV}^4,\]
\[m_s = 93^{+11}_{-5} \text{ MeV}, m_c = 1.27 \pm 0.2 \text{ GeV}.\] (28)

The sum rules contain also auxiliary parameters $M^2$ and $s_0$, which are not arbitrary, but should meet some restrictions. Thus, inside of working regions of these parameters convergence of the operator product expansion should be fulfilled. The dominance of the pole contribution, and prevalence of the perturbative term in the sum rules are also among constraints of computations. The extracted predictions should be stable against variations of $M^2$ and $\beta$: the latter is necessary for spin-1/2 particles. In order to explore the dependence on $\beta$, it is convenient to introduce a parameter $\cos \theta$ through $\beta = \tan \theta$.

Calculations of $\Pi^{\text{OPE}}(p)$ are performed by including into analysis nonperturbative terms till dimension 10. In computations we set $m_q = 0$, but take into account terms $\sim m_s$. Explicit expressions of the amplitudes $\Pi^{\text{OPE}}(M^2, s_0)$ and $\Pi^{\text{OPE}}(M^2, s_0)$ in simple case $m_s = 0$ can be found in Appendix.

First, we calculate the mass of the ground-state particle $\Xi'_c$. The parameters $M^2$ and $s_0$ necessary for such
TABLE I: The sum rule results for the masses and residues of the spin-1/2 flavor-sextet baryons $\Xi'_c$.

| Baryons | $\Xi'_c$ | $\Xi'_c(1/2^-)$ | $\Xi'_c(1/2^+)$ |
|---------|---------|----------------|----------------|
| $(n, \Lambda^2)$ | $(1S, \frac{1}{2}^+)$ | $(1P, \frac{1}{2}^-)$ | $(2S, \frac{1}{2}^+)$ |
| $M^2$ (GeV$^2$) | $3 - 5$ | $3 - 5$ | $3 - 5$ |
| $s_0$ (GeV$^2$) | $2.8^2 - 3.0^2$ | $3.2^2 - 3.4^2$ | $3.2^2 - 3.4^2$ |
| $m$ (MeV) | $2576 \pm 150$ | $2925 \pm 115$ | $2925 \pm 115$ |
| $\lambda \times 10^2$ (GeV$^2$) | $4.0 \pm 0.5$ | $3.9 \pm 1.3$ | $15.4 \pm 5.0$ |

Reliable predictions for physical quantities imply dominance of the pole contribution (PC) in the sum rule analysis. In the "ground-state+continuum" scheme fixed by $s_0 \sim 2.9^2$ MeV$^2$, a PC higher than 50% of the whole result leads to credible predictions for parameters of the ground-state baryon. In the "ground-state +first excited state+continuum" scheme determined by $s_0 \sim 3.3^2$ MeV$^2$, there are two particles that generate the pole contribution. In our case, these two baryons constitute 79% of the total contribution in average. In other words, excited state form approximately 25% of the whole contribution. This is less than 50% limit necessary for the ground-state or isolated excited particle. But mass and pole residue of the excited baryon are obtained from expressions, which contain contributions of the ground-state baryon as well. For these expressions, as we have noted above, PC $\approx 79\%$ which assures correctness of extracted quantities. It is worth noting that continuum threshold parameters $s_0$ and $\tilde{s}_0$ in this two schemes obey the restriction $s_0 < \tilde{s}_0$ which demonstrates the self-consistency of performed analyses. The central values of the masses of the excited baryons $\tilde{m} = m'$ are above $\sqrt{s_0}$ and below $\sqrt{\tilde{s}_0}$, as they should be.

In Fig. 2 we plot the mass of the particle $\Xi'_c(1/2^-)$ as a function of $M^2$. Here, one can see dependence of the obtained result on the Borel parameter $M^2$, which have been pictured at fixed values of the continuum threshold parameter $s_0$. The residues of the excited baryons $\Xi'_c(1/2^-)$ and $\Xi'_c(1/2^+)$ are drawn in Fig. 3 where sensitivity of $\tilde{\lambda}$ and $\lambda'$ to a choice of $M^2$ is shown. All these parameters are very stable against variation of the Borel parameter. The main part of theoretical uncertainties come from variation of the continuum threshold parameter $s_0$, which is also seen in these figures.
The similar analysis with some new technical details can be carried out for the spin-3/2 baryons $\Xi^*$ as well. Indeed, in this case, in order to find the physical side of the sum rules, we use the matrix elements

\begin{align}
(0|\eta_\mu|\Xi^*_c(p,s)) &= \lambda^* u_\mu(p,s), \\
(0|\eta_\mu|\Xi^*_c(3/2^-, p, \bar{s})) &= \tilde{\lambda}^* \tilde{u}_\mu(p, \bar{s}), \quad (30)
\end{align}

where $u_\mu(p,s)$ and $\tilde{u}_\mu(p,\bar{s})$ are the Rarita-Schwinger spinors, and perform the summation over the spins $s$ and $\bar{s}$ using the expression

\begin{equation}
\sum_s u_\mu(p,s)\pi_\nu(p,s) = -(p + m^*) F_{\mu\nu}(m^*, p), \quad (31)
\end{equation}

where

\begin{equation}
F_{\mu\nu}(m^*, p) = \left[ g_{\mu\nu} - \frac{1}{3} \gamma^\mu \gamma^\nu - \frac{2}{3 m^2} p_\mu p_\nu \\
+ \frac{1}{3 m^2} (p_\mu \gamma^\nu - p_\nu \gamma^\mu) \right]. \quad (32)
\end{equation}

Here, $m^*$ is the mass of the spin-3/2 baryons $\Xi^*_c$.

In calculations one should take into account that the interpolating current $\eta_\mu$ couples both to spin-3/2 and spin-1/2 baryons. Therefore, the sum rules contain contributions of spin-1/2 particles as well. These terms should be removed by using a special ordering of the Dirac matrices. Indeed, it is easy to demonstrate that structures $q g_{\mu\nu}$ and $q \eta_\mu$ are formed only due to contributions of spin-3/2 baryons. Therefore, to find the sum rules for the mass and residue of the ground state particle $\Xi^*_c$, and parameters of the excited baryons $\Xi^*_c(3/2^-)$, and $\Xi^*_c(3/2^+)$, we use only these structures and corresponding invariant amplitudes.

To determine the QCD side of the sum rules, the correlation function $\Pi_{\mu\nu}(p)$ has to be computed also in terms of the quark propagators. We calculate $\Pi_{\mu\nu}(p)$ by utilizing Eq. (8) and the current given by Eq. (11).

Operations to find $\Pi_{\mu\nu}(p)$ using the quark propagators in the $x$-space and calculation of the Borel transformed and subtracted invariant amplitudes are well known and were presented in the literature (for instance see [23, 26, 33, 46, 17]). Thus, we do not go into further details of these computations, and emphasize only that analysis has been performed with dimension-10 accuracy. We also note that the final results are very lengthy, so we do not present their explicit expressions here.

Results obtained for parameters of the spin-3/2 baryons $\Xi^*_c$ by utilizing Eq. (8) and the current given by Eq. (11).

| Baryons | $\Xi^*_c$ | $\Xi^*_c(3/2^-)$ | $\Xi^*_c(3/2^+)$ |
|---------|-----------|------------------|------------------|
| $m^*$ (MeV) | 2655 ± 102 | 2960 ± 67 | 2960 ± 67 |
| $\lambda^* \times 10^2$ (GeV$^3$) | 4.7 ± 0.4 | 2.4 ± 0.3 | 10.4 ± 1.4 |

TABLE II: The predictions for spectroscopic parameters of the spin-3/2 baryons $\Xi^*_c$. 

As is seen, the sum rule method employed in the present work to find masses of the spin-1/2 and -3/2 baryons $\Xi^*$ and $\Xi^*$ leads for the first orbitally and radially excited states to the same predictions. Therefore, relying only on this information, it is impossible to make assignment for three new resonances observed by the LHCb col-
laboration. To compare with LHCb experimental data, one needs to determine also widths of these particles.

III. DECAYS OF $\Xi_c(1/2^-)$ AND $\Xi'_c(1/2^+)$ TO $\Lambda_c^\pm K^-$

In this section we study the vertices $\Xi_c'(1/2^-)\Lambda_c^+ K^-$ and $\Xi_c'(1/2^+)\Lambda_c^+ K^-$, and calculate corresponding strong couplings, which are required to compute width of the decays $\Xi_c'(1/2^-) \rightarrow \Lambda_c^+ K^-$ and $\Xi_c'(1/2^+) \rightarrow \Lambda_c^+ K^-$, respectively. There are different sum rule methods to extract numerical values of these couplings. They can be calculated using both the QCD three-point and light-cone sum rule methods. But LCSR method has some advantages compared to the three-point sum rule approach in calculations of the strong couplings and form factors. The reason is that in the three-point sum rules higher orders in OPE are enhanced by powers of the heavy quark mass, and for sufficiently large masses OPE breaks down. The LCSR method does not suffer from such problems, because it is consistent with the heavy quark limit and provides more effective tools for investigations than alternative approaches \[48\]. Therefore, for analyses of the vertices $\Xi_c'(1/2^-)\Lambda_c^+ K^-$ and $\Xi_c'(1/2^+)\Lambda_c^+ K^-$, we use the QCD LCSR method.

To this end, we start from analysis of the correlation function

$$\Pi(p, q) = i \int d^3x e^{ipx} \langle K(q)\{T\{\eta_A(x)\eta(0)\}\\}0, \rangle$$

where $\eta_A(x)$ is the interpolating field for the $\Lambda_c$ baryon. The $\Lambda_c$ is the flavor-antitriplet spin-1/2 particle, and its current is given by the expression

$$\eta_A = \frac{1}{\sqrt{6}} e^{abc} \left\{ \frac{1}{2} \left( u_a^T C d_b \right) \gamma_5 c_c + 2\bar{\beta} \left( u_a^T C \gamma_5 d_b \right) c_c \\
+ \left( u_a^T C \gamma_5 d_b \right) \gamma_5 c_c + 2\bar{\beta} \left( u_a^T C \gamma_5 d_b \right) d_c \\
+ \left( c_a^T C d_b \right) \gamma_5 u_c + \bar{\beta} \left( c_a^T C \gamma_5 d_b \right) u_c \right\},$$

where $\beta$ is the arbitrary mixing parameter.

First, we write the correlation function $\Pi(p, q)$ in terms of involved baryons’ parameters, and find by this way the physical or hadronic side of the sum rule. As a result, we obtain

$$\Pi^{\text{Phys}}(p, q) = \frac{\langle 0|\eta_A|\Lambda_c^+(p,s)\rangle}{p^2 - m_{\Lambda}^2} \left( K(q)\Lambda_c^+(p,s)\Xi_c'(p',s') \right)$$

$$\times \frac{\langle \Xi_c'(p',s')|\eta(0)\rangle}{p'^2 - m^2} + \frac{\langle 0|\eta_A|\Lambda_c^-(p,s)\rangle}{p^2 - m_{\Lambda}^2} \left( K(q)\Lambda_c^-(p,s)\Xi_c'(p',s') \right)$$

$$\times \frac{\langle \Xi_c'(p',s')|\eta(0)\rangle}{p'^2 - m^2} + \frac{\langle 0|\eta_A|\Lambda_c^0(p,s)\rangle}{p^2 - m_{\Lambda}^2} \left( K(q)\Lambda_c^0(p,s)\Xi_c'(p',s') \right)$$

$$\times \frac{\langle \Xi_c'(p',s')|\eta(0)\rangle}{p'^2 - m^2} + \cdots,$$\hspace{1cm}(35)$$

where $p' = p + q$, $p$ and $q$ are the momenta of the $\Xi_c'$, $\Lambda_c$ baryons and $K$ meson, respectively. The $\Lambda_c^+$ and $\Lambda_c^-$ are baryons with quantum numbers $(1S, 1/2^+)$ and $(1P, 1/2^-)$, and masses $m_{\Lambda}$ and $m_{\bar{\Lambda}}$, respectively. The dots in Eq. (35) stand for contributions of the higher resonances and continuum states.

To continue, we introduce the matrix elements of the baryons $\Lambda_c$

$$\langle 0|\eta_A|\Lambda_c^+(p,s)\rangle = \lambda_{\Lambda}(u(p,s),$$

$$\langle 0|\eta_A|\Lambda_c^-(p,s)\rangle = \bar{\lambda}_{\Lambda}(u(p,s),$$

and also parametrize remaining unknown matrix elements in terms of the strong couplings \[24\] \[38\]

$$\langle K(q)\Lambda_c^+(p,s)|\Xi_c'(p',s')\rangle = \gamma_0 \overline{\Xi}(p, s)\gamma_5 u(p', s'),$$

$$\langle K(q)\Lambda_c^-(p,s)|\Xi_c'(p',s')\rangle = \gamma_0 \overline{\Xi}(p, s)u(p', s'),$$

$$\langle K(q)\Lambda_c^0(p,s)|\Xi_c'(1/2^-,p',s')\rangle = \gamma_0 \overline{\Xi}(p, s)u(p', s'),$$

$$\langle K(q)\Lambda_c^-(p,s)|\Xi_c'(1/2^-,p',s')\rangle = \gamma_1 \overline{\Xi}(p, s)\gamma_5 u(p', s'),$$

(37)

where $\lambda_{\Lambda}$ and $\bar{\lambda}_{\Lambda}$ are pole residues of $\Lambda_c^+$ and $\Lambda_c^-$, respectively.

Then using the matrix elements of the particles $\Xi_c'$ and $\Xi_c'(1/2^-)$, carrying out the summations over the spins $s$ and $s'$, and applying the double Borel transformation with respect $p^2$ and $p'^2$, for the phenomenological side of the sum rules, we obtain

$$\Pi^{\text{Phys}}(p^2, p'^2) = \gamma_0 \overline{\Lambda} e^{-m^2/2} e^{-\overline{m}^2/2} (p + m_{\Lambda})$$

$$\times \gamma_5 (p' + m) - \gamma_0 \overline{\Lambda} e^{-m^2/2} e^{-\overline{m}^2/2} (p + m_{\Lambda})$$

$$\times \gamma_5 (p' + m) + \gamma_1 \overline{\Lambda} e^{-\overline{m}^2/2} e^{-m^2/2} (p + m_{\Lambda})$$

$$\times \gamma_5 (p' + m) - \gamma_1 \overline{\Lambda} e^{-m^2/2} e^{-\overline{m}^2/2} (p + m_{\Lambda})$$

$$\times (p' - \overline{m})\gamma_5 (p' - \overline{m}),$$

(38)

where $M_{\text{B}}^2$ and $M_{\overline{B}}^2$ are the Borel parameters.

As is seen, Eq. (35) contains different Lorentz structures. To extract sum rules, it is convenient to reorganize these terms into structures proportional to $\gamma_0$, $\gamma_1$, and $\gamma_0\gamma_1$.
\( \gamma_5, \bar{\gamma}_5 \) and \( \gamma_5 \), and employ corresponding invariant amplitudes. The same structures appear in the QCD side of the sum rule equality, which has to be calculated using the quark propagators. After performing the double Borel transformation of \( \Pi^{\mathrm{OPE}}(p,q) \), we get
\[
\mathcal{H}^{\mathrm{OPE}}(p^2, p'^2) = \Pi^{\mathrm{OPE}}(M_1^2, M_2^2)
\]
which is a function of two Borel parameters. To proceed, it is convenient to introduce \( M^2 \) through the relation
\[
\frac{1}{M^2} = \frac{1}{M_1^2} + \frac{1}{M_2^2}. \tag{39}
\]
and use \( M_1^2 = M_2^2 = 2M^2 \) to go from the double-dispersion integral to the single integral representation by performing one of the dispersion integrals. We set \( M_1^2 = M_2^2 \) as the masses of the \( \Xi \) and \( \Lambda \) baryons are close, and uncertainties expected due to this choice are small. As a result, we get a single integral representation for \( \Pi^{\mathrm{OPE}}(M^2) \), which considerably simplifies the continuum subtraction. By equating now \( \Pi^{\mathrm{OPE}}(M^2) \) with the expression Eq. \( 35 \) and performing the continuum subtraction, we find the sum rule equality which depends on \( \Pi^{\mathrm{OPE}}(M^2, s_0) \): After the subtraction procedure the correlation function \( \Pi^{\mathrm{OPE}}(M^2, s_0) \) acquires dependence on the continuum threshold parameter \( s_0 \). The formulas necessary to carry out subtractions can be found in Appendix B of Ref. \[49\].

By equating invariant amplitudes corresponding to aforementioned Lorentz structures in both sides of the sum rule equality, one finds four equations which should be solved to determine sum rules for the strong couplings. We denote invariant amplitudes corresponding to the structures \( \gamma_5, \bar{\gamma}_5, \gamma_5 \) and \( \gamma_5 \) by \( \Pi_i^{\mathrm{OPE}}(M^2, s_0) \), where \( i = 1, 2, 3 \) and 4, respectively.

The solution of these equations for the coupling of interest \( g_1 \) is
\[
g_1 = \left\{ \Pi_1^{\mathrm{OPE}} \left[ m_K^2 \right]
\begin{align*}
&= \frac{e^2/M_1^2 \, e^2/M_2^2}{\lambda \lambda (m \mp \bar{m}) (m + \bar{m})} \left\{ \Pi_1^{\mathrm{OPE}} \left[ m_K^2 \right]
\begin{align*}
&= \frac{1}{M_1^2} + \frac{1}{M_2^2}.
\end{align*}
\end{align*}
\right.
\]

Here, \( m_K = (493.677 \pm 0.016) \) MeV is the mass of the \( K \) meson. The sum rules for the strong coupling \( g_2 \) corresponding to the vertex \( \Xi^0(1/2^+) \Lambda_K^+ K^- \) and responsible for the decay \( \Xi^0' \rightarrow \Lambda_K^+ K^- \) can be determined from Eq. \( 40 \) by replacements \( \bar{m} \rightarrow -m' \) and \( \bar{\lambda} \rightarrow \lambda' \).

In order to activate Eq. \( 40 \), it is necessary to calculate the correlation function \( \Pi^{\mathrm{OPE}}(p,q) \) and find the invariant amplitudes \( \Pi^{\mathrm{OPE}}(M^2, s_0) \). After contracting the quark fields and inserting into the obtained formula quark propagators, we get the expression which depend on the non-local matrix elements of operators \( \not{\Sigma} u \) placed between the states \( \langle K(q) \rangle \) and \( |0\rangle \). We should express the correlation function \( \Pi^{\mathrm{OPE}}(p,q) \) using the distribution amplitudes (DAs) of \( K \) meson with different quark-gluon compositions and twists. To this end, we use the expansion
\[
\not{\Sigma}_i u_i = \frac{1}{12} \Gamma_i \delta_{ab}(\not{\Sigma} u), \tag{41}
\]
where \( \Gamma_i = 1, \gamma_5, \gamma_{\mu}, i\gamma_5\gamma_{\mu}, \sigma_{\mu\nu}/\sqrt{2} \) are the Dirac matrices. These terms placed between the \( K \) meson and vacuum states generate the two-particle DAs of the leading and nonleading twists. They are defined by the expressions \[49\]
\[
(0|\overline{q}(x)\gamma_{5}(x)\gamma_{5}(-x)|K(q)) = i f_K q_u \int_0^1 du \, e^{iu} [\phi_{2:K}(u)]
\]
\[
+ \frac{1}{4} \, z^2 \phi_{4:K}(u)
\]
\[
+ \frac{1}{2} \frac{f_K}{\sqrt{M}} \int_0^1 du \, e^{iu} \phi_{3:K}(u), \tag{42}
\]
\[
(0|\overline{q}(x)i\gamma_{5}(x)\gamma_{5}(-x)|K(q)) = \frac{f_K}{3} \frac{m^2}{m_u + m_q} \int_0^1 du \, e^{iu} \phi_{3:K}(u), \tag{43}
\]
and
\[
(0|\overline{q}(x)\gamma_{5}(x)\gamma_{5}(-x)|K(q)) = -i f_K \frac{m^2}{3} \frac{m_u + m_q}{q_u \gamma_{5} - q_{\beta} \gamma_{\alpha}} \int_0^1 du \, e^{iu} \phi_{3:K}(u), \tag{44}
\]
where \( f_K = (155.72 \pm 0.51) \) MeV is the decay constant of the \( K \) meson. In expressions above \( \xi = 2u - 1, \) with \( u \) being the longitudinal momentum fraction carrying the quark in the \( K \) meson. The subscripts in DAs label the twist of these functions.

![FIG. 4: The leading twist diagram contributing to \( \Pi^{\mathrm{OPE}}(p,q) \).](image-url)
known as the leading twist contribution: the corresponding Feynman diagram is plotted in Fig. [4]. Contributions of terms containing three-particle DA of the $K$ meson generate only nonleading twist effects. In the present work, we take into account contributions due to two- and three-particle DAs including twist-4 corrections. An analytic expression for the double Borel transformed and subtracted correlation function $\Pi^{\text{OPE}}(M^2, s_0)$ is rather cumbersome, therefore we do not write it down here. From derived expression of $\Pi^{\text{OPE}}(M^2, s_0)$ one can extract invariant amplitudes required for our calculations.

The functions $\Pi^{\text{OPE}}(M^2, s_0)$ depend on DAs of $K$ meson. In numerical computations for these DAs, we have utilized models and parameters presented in Ref. [39]. Apart from DAs, the sum rules for the couplings $g_1$ and $g_2$ contain also masses of the ground state $\Lambda_c^+$ and first orbitally excited $\Lambda_c^-$ baryons for which we use their values from Ref. [27]

$$m_{\Lambda} = (2286.46 \pm 0.14) \text{ MeV},$$
$$\tilde{m}_{\Lambda} = (2592.25 \pm 0.28) \text{ MeV}. \quad (45)$$

The pole residue of $\Lambda_c^+$ denoted in Eq. [40] by $\lambda_\Lambda$ is borrowed from the work [38]

$$\lambda_\Lambda = (3.8 \pm 0.9) \times 10^{-2} \text{ GeV}^3. \quad (46)$$

The Borel and continuum threshold parameters for the decay of the baryons $\Xi'_c(1/2^-)$ and $\Xi'_c(1/2^+)$ are fixed exactly as in computations of their masses. The parameters $\beta$ and $\bar{\beta}$ in the interpolating currents of $\Xi'_c$ and $\Lambda_c$ are taken equal to each other and varied within the limits presented in Eq. [29].

Numerical calculations lead to the following predictions

$$g_1 = 0.41 \pm 0.04, \ |g_2| = 7.19 \pm 0.65. \quad (47)$$

The widths of the decays $\Xi'_c(1/2^-) \to \Lambda_c^+ K^-$ and $\Xi'_c(1/2^+) \to \Lambda_c^+ K^-$ can be obtained in terms of the strong couplings $g_1$ and $g_2$, respectively. They are determined by the formulas

$$\Gamma(\Xi'_c(1/2^-) \to \Lambda_c^+ K^-) = \frac{g_1^2}{8\pi m^2} \left[ (\tilde{m} + m_{\Lambda})^2 - m_K^2 \right] \times f(\tilde{m}, m_{\Lambda}, m_K), \quad (48)$$

and

$$\Gamma(\Xi'_c(1/2^+) \to \Lambda_c^+ K^-) = \frac{g_2^2}{8\pi m^2} \left[ (m' + m_{\Lambda})^2 - m_K^2 \right] \times f(m', m_{\Lambda}, m_K), \quad (49)$$

where the function $f(x, y, z)$ is given by the expression

$$f(x, y, z) = \frac{1}{2x} \sqrt{x^4 + y^4 + z^4 - 2x^2y^2 - 2x^2z^2 - 2y^2z^2}. \quad (50)$$

The predictions for the width of the decays $\Xi'_c(1/2^-) \to \Lambda_c^+ K^-$ and $\Xi'_c(1/2^+) \to \Lambda_c^+ K^-$ are equal to

$$\Gamma(\Xi'_c(1/2^-) \to \Lambda_c^+ K^-) = (7.3 \pm 1.4) \text{ MeV},$$
$$\Gamma(\Xi'_c(1/2^+) \to \Lambda_c^+ K^-) = (14.2 \pm 2.7) \text{ MeV.} \quad (51)$$

Theoretical ambiguities in Eq. [51] are generated by the strong couplings $g_1^2$ and $g_2^2$, and by the masses $\tilde{m}$ and $m'$ of excited baryons $\Xi'_c(1/2^-)$ and $\Xi'_c(1/2^+)$, which have been extracted in the present work. For the masses $m_{\Lambda}$, $\tilde{m}_{\Lambda}$ and $m_K$, we use their experimental values, which are known with high precision: Relevant experimental errors are very small and do not affect error estimates for decay widths.

Predictions for partial widths of these two channels can be used for further studies of the baryons $\Xi'_c(1/2^-)$ and $\Xi'_c(1/2^+)$. 

**IV. PROCESSES $\Xi'_c(1/2^-) \to \Xi'_c \pi$ AND $\Xi'_c(1/2^+) \to \Xi'_c \pi$**

Analysis of the decays $\Xi'_c(1/2^-) \to \Xi'_c \pi$ and $\Xi'_c(1/2^+) \to \Xi'_c \pi$ does not differ from our studies carried out in the previous section. The correlation function to investigate these processes is given by the expression

$$\Pi_\pi(p, q) = i \int d^4xe^{ipx} \langle \pi(q) | T \{ \eta(x) \eta(0) \} | 0 \rangle, \quad (52)$$

with the same interpolating current $\eta(x)$.

Let us consider the decay $\Xi'_c(1/2^-) \to \Xi'_c \pi$ to outline methods of analysis. For this process the physical side of the LCSR has the form

$$\Pi_\pi^{\text{phys}}(p, q) = \frac{(0|\Xi'_c(p, s))}{p^2 - m^2} \langle \pi(q) \Xi'_c(p, s) | \Xi'_c(1/2^-, p', s') \rangle \times \frac{\langle \Xi'_c(1/2^-, p', s') | \eta(0) \rangle}{p^2 - m^2} + \cdots. \quad (53)$$

Matrix elements of the ground-state and excited baryons $\Xi'_c$ and $\Xi'_c(1/2^-)$ are well known. Additionally, we model the vertex matrix element by introducing the strong coupling $g_3$

$$\langle \pi(q) \Xi'_c(p, s) | \Xi'_c(1/2^-, p', s') \rangle = g_3 \overline{m}(p, s)u(p', s'). \quad (54)$$

Then, the double Borel transformation of the correlation function is determined by the expression

$$\Pi_\pi^{\text{phys}}(p^2, q^2) = g_3 \overline{m} e^{-m^2/M_1^2} e^{-m^2/M_2^2} \times \langle \Xi'_c(p', s') | \eta(0) \rangle. \quad (55)$$

The QCD side of the LCSR is determined by the correlator $\Pi_\pi(p, q)$ expressed in terms of the $c$ and $s$ quark propagators and distribution amplitudes of the pion. The matrix elements of operators $\overline{d}(x) T^1 d(0)$ in $\Pi_\pi^{\text{OPE}}(p, q)$ can be expanded over $x^2$ and expressed by means of the
pion’s two-particle DAs of different twist. As examples, in the case of $\Gamma = i\gamma_\mu\gamma_5$ and $\gamma_5$ we get

$$\sqrt{2}(\pi^0(q)\bar{d}(x)\gamma_5\gamma_\mu\gamma_5d(0)|0) = f_\pi q_\mu \int_0^1 du e^{imx} \left[ \phi_\pi(u) + \frac{m^2 + m^2 x^2}{16} A_4(u) \right] + \frac{f_\pi m^2}{2} x u \int_0^1 du e^{imx} B_4(u),$$

and

$$2\sqrt{2} m_\pi (\pi^0(q)\bar{d}(x)\gamma_5d(0)|0) = \times f_\pi m^2 \int_0^1 du e^{imx} \phi^\alpha_{3\pi}(u),$$

where $m_\pi = 135$ MeV and $f_\pi = 131$ MeV are parameters of the pion mass. In Eq. (56) $\phi_\pi(u)$ is the twist-2 distribution amplitude, and $A_4(u)$ and $B_4(u)$ are higher-twist functions that can be expressed in terms of the pion two-particle twist-4 DAs. One of two-particle twist-3 distributions $\phi^\alpha_{3\pi}(u)$ determines the matrix element given by Eq. (57). Another twist-3 DA $\phi^\alpha_{3\pi}(u)$ corresponds to matrix element with $\sigma_\mu$ insertion. The non-local operators that appear due to a gluon field strength tensor $G_{\mu\nu}(ux)$ insertions to $d(x)\Gamma^j d(0)$ generate the pion’s three-particle distributions. Their definitions and further details were presented in Ref. [19, 22].

The leading twist DA $\phi_\pi(u)\gamma_\mu$ is expressible in terms of the Gegenbauer polynomials $C_{2n}^3/2(\xi)$

$$\phi_\pi(u,\mu^2) = 6\mu\pi \left[ \sum_{n=1,2,3...} a_{2n}(\mu^2) C_{2n}^3/2(u - \mu) \right],$$

where $\mu = 1 - u$. The coefficients $a_{2n}(\mu^2)$ depend on a scale $\mu$, as a result, $\phi_\pi(u,\mu^2)$ is a function of $\mu$. The Gegenbauer momenta $a_{2n}(\mu^2)$ at some normalization point $\mu = \mu_0$ should be fixed from phenomenological analysis or computed by employing, for example, lattice simulations.

We derive the sum rule for the coupling $g_3$ by employing invariant amplitudes corresponding to the structure $\not{p}\not{p}\gamma_5$ both in $\Pi_{\pi}^{\text{phys}}(p, q)$ and $\Pi_{\pi}^{\text{OPE}}(p, q)$, and get

$$g_3 = \frac{e\mu^2/M^2_\Lambda e^{m^2/M^2_\Lambda}}{\lambda \lambda} \Pi_{\pi}^{\text{OPE}},$$

where $\Pi_{\pi}^{\text{OPE}}$ is the relevant amplitude. In computations the Borel and continuum threshold parameters $M^2$ and $s_0$ are chosen as in Table I Parameters of the pion’s DAs are borrowed from Refs. [50, 51]. Numerical calculations of $g_3$ and the partial width of the decay $\Xi_0'(1/2^-) \rightarrow \Xi_0'\pi$ lead to results

$$g_3 = 0.14 \pm 0.02,$$

and

$$\Gamma (\Xi_0'(1/2^-) \rightarrow \Xi_0'\pi) = (1.1 \pm 0.2) \text{ MeV}. (61)$$

The similar analysis can be done in the case of the second process $\Xi_0'(1/2^+) \rightarrow \Xi_0'\pi$. The strong coupling $g_4$ can be obtained from Eq. (59) by replacements $\tilde{m} \rightarrow m'$ and $\lambda \rightarrow \lambda'$. Computations give following predictions

$$g_4 = 1.04 \pm 0.16, \quad \Gamma (\Xi_0'(1/2^+) \rightarrow \Xi_0'\pi) = (0.39 \pm 0.09) \text{ MeV}. (62)$$

Information obtained here will be used to evaluate widths of the baryons $\Xi_0'(1/2^-)$ and $\Xi_0'(1/2^+)$. The decays of the spin-3/2 baryons $\Xi_0^*(3/2^-)$ and $\Xi_0^*(3/2^+)$ to the final state $\Lambda_0^+K^-$ can be explored by a manner described above for the spin-1/2 particles. To this end, we begin from calculation of the correlation function

$$\Pi_{\mu}(p, q) = i \left[ d^4xe^{ipx} (K(q)|T\{\eta_\Lambda(x)|\Pi_{\mu}(0)\})|0 \right],$$

where $\eta_\Lambda$ is the interpolating current for spin-3/2 baryons $\Xi_0^*$ given by Eq. (11).

To calculate the phenomenological side of the sum rules $\Pi_{\pi}^{\text{phys}}(p, q)$, we write down it in the form similar to one presented in Eq. (35) with simple modifications. We also define the strong couplings $G_0(1)$ and $G_0(1)$ using the matrix elements

$$\langle K(q)\Lambda_0^+(p, s)|\Xi_0^*(p', s')\rangle = G_0\langle u(p, s)u_c(p', s')q^*\rangle,$$

$$\langle K(q)\Lambda_0^+(p, s)|\Xi_0^*(p', s')\rangle = G_0\langle u(p, s)\gamma_5 u_c(p', s')q^*\rangle,$$

$$\langle K(q)\Lambda_0^+(p, s)|\Xi_0^*(3/2^-, p', s')\rangle = G_1\bar{u}(p, s)\gamma_5 u_c(p', s')q^*,$$

$$\langle K(q)\Lambda_0^-(p, s)|\Xi_0^*(3/2^-, p', s')\rangle = G_1\bar{u}(p, s)u_c(p', s')q^*.$$

After some manipulations, for the Borel transformation of $\Pi_{\pi}^{\text{phys}}(p, q)$, we obtain the following expression

$$\langle B\Pi_{\mu}(p^2, q^2) = G_0\lambda\Lambda\lambda\Lambda e^{-m^2/M^2} e^{-m^2/M^2} (p + m\Lambda) \times (p' + m\alpha) F_{\alpha\mu}(m^*, p') q^* - G_0\lambda\Lambda\Lambda\Lambda e^{-m^2/M^2} \times e^{-m^2/M^2} (p - \tilde{m}\Lambda) (p' + m\alpha) F_{\alpha\mu}(m^*, p') q^* + G_1\lambda\Lambda\Lambda\Lambda e^{-m^2/M^2} e^{-m^2/M^2} (p + m\Lambda) (p' - \tilde{m}\alpha) \gamma_5 \times F_{\alpha\mu}(m^*, p') q^* - G_1\lambda\Lambda\Lambda\Lambda e^{-m^2/M^2} e^{-m^2/M^2} \times (p - \tilde{m}\Lambda) (p' - \tilde{m}\alpha) \gamma_5 F_{\alpha\mu}(m^*, p') q^*. (65)$$

To derive the sum rules, we use available structures in Eq. (65). The same terms are fixed in $\Pi_{\pi}^{\text{phys}}(p^2, q^2)$ and matched with ones from $\Pi_{\pi}^{\text{phys}}(p^2, q^2)$. The final expressions of the strong couplings are rather lengthy, therefore we do not write them down here.

The strong coupling required to compute the width of the decay $\Xi_0^*(3/2^-) \rightarrow \Lambda_0^+K^-$ is $G_1$. The coupling $G_2$
necessary to find the width of the process \( \Xi^+_c(3/2^-) \rightarrow \Lambda_c^+K^- \) can be obtained from the relevant sum rule after simple replacements. In numerical computations the parameters \( M^2, s_0 \) are chosen as in the corresponding mass calculations. For \( G_1 \) and \( G_2 \) our analysis leads to the following predictions (in units of \( \text{GeV}^{-1} \))

\[
G_1 = 22.0 \pm 2.6, \quad |G_2| = 3.6 \pm 0.4. \tag{66}
\]

The information gained from these studies is enough to determine the widths of the corresponding decay channels. In fact, the width of the decay \( \Xi^*_c(3/2^-) \rightarrow \Lambda_c^+K^- \) can be found using the expression

\[
\Gamma(\Xi^*_c(3/2^-) \rightarrow \Lambda_c^+K^-) = \frac{G_1^2}{24\pi m^*} \left[ (m^* - m_{\Lambda})^2 - m_K^2 \right] \times f^3(m^*, m_{\Lambda}, m_K), \tag{67}
\]

whereas for \( \Gamma(\Xi^*_c(3/2^+) \rightarrow \Lambda_c^+K^-) \), we employ

\[
\Gamma(\Xi^*_c(3/2^+) \rightarrow \Lambda_c^+K^-) = \frac{G_2^2}{24\pi m^*} \left[ (m^* + m_{\Lambda})^2 - m_K^2 \right] \times f^3(m^*, m_{\Lambda}, m_K). \tag{68}
\]

Numerical computations lead to predictions

\[
\Gamma (\Xi^*_c(3/2^-) \rightarrow \Lambda_c^+K^-) = (10.5 \pm 2.1) \text{ MeV},
\]

\[
\Gamma (\Xi^*_c(3/2^+) \rightarrow \Lambda_c^+K^-) = (35.4 \pm 7.2) \text{ MeV}. \tag{69}
\]

Exploration of the second pair of processes \( \Xi^*_c(3/2^-) \rightarrow \Xi_c^0 \pi \) and \( \Xi^*_c(3/2^+) \rightarrow \Xi_c^0 \pi \) is performed using the correlation function

\[
\Pi_{\pi \rho}(p, q) = i \int d^4xe^{ipx}\langle \pi(q)|T\{\eta(x)\pi_p(0)\}|0\rangle. \tag{70}
\]

The sum rules for \( G_3 \) and \( G_4 \) are obtained by means of the structures \( d\bar{q}q \) in the physical and QCD representation of the correlator \( \Pi_{\pi \rho}(p, q) \). Our analysis for strong couplings and partial widths yields:

\[
G_3 = 4.3 \pm 0.6,
\]

\[
\Gamma (\Xi^*_c(3/2^-) \rightarrow \Xi_c^0 \pi) = (4.0 \pm 0.8) \times 10^{-2} \text{ MeV}, \tag{71}
\]

and

\[
G_4 = 0.45 \pm 0.06,
\]

\[
\Gamma (\Xi^*_c(3/2^+) \rightarrow \Xi_c^0 \pi) = (1.7 \pm 0.4) \times 10^{-1} \text{ MeV}. \tag{72}
\]

Partial widths of processes considered here allow us to evaluate the full widths of the baryons \( \Xi^*_c(3/2^-) \) and \( \Xi^*_c(3/2^+) \) saturated by two dominant decay modes. It is clear that effect of the process \( \Xi^*_c(3/2^-) \rightarrow \Xi_c^0 \pi \) on the full width of \( \Xi^*_c(3/2^-) \) is negligible, and can be safely ignored. For width of the baryon \( \Xi^*_c(3/2^+) \), we get

\[
\Gamma' = (35.6 \pm 2.1) \text{ MeV}. \tag{73}
\]

This estimate for \( \Gamma' \), and prediction \( \tilde{\Gamma} = (10.5 \pm 2.1) \text{ MeV} \) should be compared with the LHCb data.

VI. ANALYSIS AND CONCLUDING NOTES

We have calculated the masses and pole residues of the ground state spin-1/2 and -3/2 baryons \( \Xi_c^0 \) and \( \Xi_c^* \):

\[
m_{\text{th}} = (2576 \pm 150) \text{ MeV}, \tag{74}
\]

and

\[
m^*_{\text{th}} = (2655 \pm 102) \text{ MeV}. \tag{75}
\]

Comparing obtained theoretical predictions for masses of these particles with experimental data [22] and [33], we see nice agreements between them: Theoretical errors of \( m_{\text{th}} \) and \( m^*_{\text{th}} \) are typical for this method and do not exceed allowed limits.

In the present work we have also computed the masses and widths of the spin-1/2 flavor-sextet baryons \( \Xi_c^0 \), and spin-3/2 baryons \( \Xi_c^* \) in order to compare obtained information with results of the LHCb collaboration. The masses of these particles have been extracted from two-point sum rules, whereas to calculate their widths, we have used the QCD light-cone sum rule approach.

The sum rule method is a powerful nonperturbative tool to explore features of conventional and exotic hadrons. It relies on first principles on the QCD by employing quark-gluon structure of particles under analysis, and universal vacuum expectations values of various local quark, gluon, and mixed operators. Predictions obtained in this context depend on a few auxiliary parameters of computations, which limit theoretical accuracy of investigations. Main part of uncertainties is generated by a choice of the Borel parameter \( M^2 \): its variation within allowed working region leads to ambiguities in values of extracted parameters. In this sense the mass of a hadron is most protected physical quantity the reason being in a functional form of a relevant sum rule. In fact, sum rules for the masses of hadrons are given as a ratio of correlation functions (see, for instance Eq. (27)), which reduces uncertainties and stabilize a final result.

In the present article ambiguities in the masses of the excited spin-1/2 and -3/2 baryons \( \Xi_c^0 \) and \( \Xi_c^* \) amount to \( \pm (2.2 - 3.8)\% \) of central values, which is nice accuracy for sum rule computations. In other words, the masses of the baryons may be chosen from values spanning approximately \( (120 - 220) \text{ MeV} \) region. Because, resonances discovered by LHCb have very close masses and cover narrow range of \( \sim 40 \text{ MeV} \), the sum rule method could not resolve such fine structure: its predictions are compatible with all of these resonances. Performed analysis also does not "see" resonances \( \Xi_c(2790) \) and \( \Xi_c(2815) \), because central values for masses of the excited states extracted in the present work are higher than masses of these particles. Therefore, classification of the spin-1/2 and -3/2 excited baryons \( \Xi_c^0 \) and \( \Xi_c^* \), and their possible interpretation as resonances \( \Xi_c(2923)^0 \), \( \Xi_c(2939)^0 \), and \( \Xi_c(2965)^0 \) should be done using widths of these particles, which differ from each other and have been evaluated with accuracy enough for such differentiation.
Let us note that parameters of the flavor-antitriplet spin-1/2 states \( csd \) were calculated in Ref. \[38\]. In that paper the authors considered the baryon \( \Xi_c(1/2^-) \) with parameters \( \bar{m} = (2922 \pm 83) \text{ MeV} \) and \( \bar{\Gamma} = (19.4 \pm 3.3) \text{ MeV} \) as the resonance \( \Xi_c(2930)^0 \). The radial excitation of the spin-1/2 antitriplet baryon \( \Xi_c(1/2^+) \) has the same mass but lower width

\[
\begin{align*}
m' &= (2922 \pm 83) \text{ MeV}, \\
\Gamma' &= (13.6 \pm 2.3) \text{ MeV}.
\end{align*}
\]  

This particle should be taken into account in our present analysis.

Before confronting with experimental data the full widths of the flavor-sextet baryons should be found using results of the sections \[III\] and \[IV\]. Simple analysis leads to the following estimates

\[
\bar{\Gamma} = (8.4 \pm 1.4) \text{ MeV}, \quad \Gamma' = (14.6 \pm 2.7) \text{ MeV}. \tag{77}
\]

Then, it is not difficult to see that sextet baryon \( \Xi_c(1/2^-) \), which has the mass and width

\[
\begin{align*}
\bar{m} &= (2925 \pm 115) \text{ MeV}, \\
\bar{\Gamma} &= (8.4 \pm 1.4) \text{ MeV},
\end{align*}
\]  

(78)

can be interpreted as the resonance \( \Xi_c(2923)^0 \) with parameters \( \bar{m} \).

The second resonance \( \Xi_c(2939)^0 \) may be considered as the excited spin-3/2 baryon \( \Xi_c^*(3/2^-) \)

\[
\begin{align*}
\bar{m}^* &= (2960 \pm 67) \text{ MeV}, \\
\bar{\Gamma}^* &= (10.5 \pm 2.1) \text{ MeV}.
\end{align*}
\]  

The interpretation of \( \Xi_c(2965)^0 \) with parameters \( \bar{m} \) is twofold: it may be considered as the spin-1/2 antitriplet baryon \( \Xi_c(1/2^+) \) with \( \Gamma' = (13.6 \pm 2.3) \text{ MeV} \). But one can identify it also with radially excited sextet particle \( \Xi_c^*(1/2^+) \) with the width (14.2 \pm 2.7) \text{ MeV} \). Let us note that masses of these particles within theoretical errors are compatible with the LHCb data.

Because the radially excited spin-3/2 particle \( \Xi_c^*(3/2^+) \) has the width (35.6 \pm 2.1) \text{ MeV} \), it cannot be considered as a candidate to new resonances. It is worth noting that parameters of \( \Xi_c^*(3/2^+) \) are close to the mass and width of the baryon \( \Xi_c(2970)^0 \)

\[
\begin{align*}
m &= (2970.8 \pm 0.7 \pm 0.2) \text{ MeV} \\
\Gamma &= 30.3 \pm 2.3^{+1.0}_{-1.4} \text{ MeV}.
\end{align*}
\]  

Recently the Belle collaboration determined the spin-parity of \( \Xi_c(2970) \) as \( J^P = 1/2^+ \) \[54\]. Conclusion about radially excited spin-1/2 nature of the baryon \( \Xi_c(2970) \) was made also in Ref. \[53\]. In the light of these circumstances \( \Xi_c^*(3/2^+) \) cannot be interpreted as \( \Xi_c(2970) \). Identification of two resonances \( \Xi_c(2965)^0 \) and \( \Xi_c(2970)^0 \)

seems also to be problematic, because they have significantly different decay widths.

We have noted above that new LHCb resonances were explored in Refs. \[39\] \[42\] using various suggestions on their structure and employing different computational schemes. Thus, in Ref. \[39\] these states were investigated in the context of the HQET. In this theory a \( P \)-wave baryon consists of one charm quark and light diquark, and contains one orbital excitation between light quarks (\( p \)-mode), or between a charm and light diquark (\( \lambda \)-mode). In this paper, the resonances \( \Xi_c(2923)^0, \Xi_c(2939)^0, \) and \( \Xi_c(2965)^0/\Xi_c(2970)^0 \) were interpreted as \( P \)-wave baryons \( J^P = 1/2^-, \quad J^P = 3/2^- \) and \( J^P = 3/2^- \)-containing one \( \lambda \)-mode: A difference in organization of two last resonances with \( J^P = 3/2^- \) was explained in Ref. \[39\]. Our interpretation of \( \Xi_c(2923)^0, \) and \( \Xi_c(2939)^0 \) is consistent with this picture. But in our analysis the last resonance from this list \( \Xi_c(2965)^0 \) is either 2\( S \) antitriplet or sextet spin-1/2 particle.

The new resonances were considered also in Ref. \[40\], in which first two states were interpreted as \( \lambda \)-mode baryons with \( J^P = 3/2^- \), and the last state as \( J^P = 5/2^- \). Interpretation of only the resonance \( \Xi_c(2939)^0 \) in this scheme coincides with our prediction. In Ref. \[41\] the last particle \( \Xi_c(2965)^0 \) was regarded as spin-1/2 flavor-sextet 2\( S \) baryon, which is in accord with our assignment.

New measurements by LHCb provided information on parameters of three resonances which can be considered as charm-strange baryons. In the present work we have calculated the masses and widths of four excited \( csd \) baryons with different spins. Theoretical investigations of first orbitally and radially excited spin-1/2 sextet and spin-3/2 baryons, as well as existing results for spin-1/2 flavor-antitriplet states fix parameters of six particles. We have used new resonances \( \Xi_c(2923)^0, \Xi_c(2939)^0, \) and \( \Xi_c(2965)^0, \) and known ones \( \Xi_c(2930)^0 \) and \( \Xi_c(2970)^0 \) to confront their parameters with our predictions: Obtained results have been discussed above. From brief analysis of theoretical articles it is evident that interpretations of excited \( csd \) baryons are controversial. Evidently, for comprehensive analysis of this sector of hadron spectroscopy more detailed experimental information and further investigations are required.

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\section*{Appendix: The invariant amplitudes \( \Pi_1^{OP}(M^2, s_0) \) and \( \Pi_2^{OP}(M^2, s_0) \)}
The invariant amplitudes $\Pi^{\text{OPE}}_{1}(p^2)$ used to calculate the mass and pole residue of the flavor-sextet spin-1/2 baryons after the Borel transformation and subtraction prescriptions take the following form

$$\Pi^{\text{OPE}}_{1}(M^2, s_0) = \int_{M^2}^{\infty} ds \rho^{\text{OPE}}_{1}(s) e^{-s/M^2} + \Pi_{1}(M^2), \quad (A.1)$$

where $M = m_c + m_s$. The spectral densities $\rho^{\text{OPE}}_{1}(s)$ in Eq. (A.1) are found from the imaginary part of the correlation function and encompass essential piece of $\Pi^{\text{OPE}}(p)$. The Borel transformations of remaining terms in $\Pi^{\text{OPE}}(p)$ are included into $\Pi_{1}(M^2)$, and have been calculated directly from the expression of $\Pi^{\text{OPE}}(p)$.

The functions $\rho^{\text{OPE}}_{1}(s)$ and $\Pi_{1}(M^2)$ contain components of different dimensions and have the structure

$$\rho^{\text{OPE}}_{1}(s) = \rho^{\text{pert}}_{1}(s) + \sum \rho^{\text{Dim} N}_{1}(s), \quad \Pi_{1}(M^2) = \sum \Pi^{\text{Dim} N}(M^2). \quad (A.2)$$

The $\Pi^{\text{OPE}}_{1}(M^2, s_0)$ have been computed by setting $m_q = 0$ and $m_s \neq 0$, and used to perform numerical analyses. The amplitudes $\Pi^{\text{OPE}}_{1}(M^2, s_0)$, in general, contain a few hundred terms and exist as Mathematica files. Their explicit expressions are cumbersome, therefore we provide below simplified formulas in which $m_s = 0$.

The perturbative contribution and nonperturbative terms with dimensions 3, 4, 5 and 7 in the case of the spectral density $\rho^{\text{OPE}}_{1}(s)$ are given by the expressions:

$$\rho^{\text{pert}}_{1}(s) = \frac{(5 + 2\beta + 5\beta^2)}{2048\pi^4 s^2} \left[m_c^0(8s - m_c^2) + s^3(s - 8m_c^2) + 12m_c^4s^2 \ln \left(\frac{s}{m_c^2}\right)\right],$$

$$\rho^{\text{Dim} 3}_{1}(s) = \frac{(\bar{s}s + (\bar{d}d))}{192\pi^2 s^2} m_c \left(m_c^2 - s\right)^2 \left(1 + 4\beta - 5\beta^2\right),$$

$$\rho^{\text{Dim} 4}_{1}(s) = \frac{(g_s^2 G^2)}{3072\pi^4 s^2} \left(s - m_c^2\right) \left[8m_c^2(1 + \beta + \beta^2) + s(5 + 2\beta + 5\beta^2)\right],$$

$$\rho^{\text{Dim} 5}_{1}(s) = \frac{\left(\bar{d}g_s \sigma Gd\right) + (\bar{s}g_s \sigma Gs)}{768\pi^2 s^2} m_c(\beta - 1) \left[m_c^2(7 + 11\beta) - 6s(1 + \beta)\right],$$

$$\rho^{\text{Dim} 7}_{1}(s) = \frac{(g_s^2 G^2)(\bar{s}s + (\bar{d}d))}{384\pi^2 s^2} m_c(1 - \beta^2).$$

The function $\Pi_{1}(M^2)$ is composed of the following components:

$$\Pi^{\text{Dim} 6}_{1}(M^2, s_0) = \frac{(\bar{s}s)(\bar{d}d)}{72} (11\beta^2 + 2\beta - 13) e^{-m_c^2/M^2},$$

$$\Pi^{\text{Dim} 7}_{1}(M^2, s_0) = \frac{(g_s^2 G^2)(\bar{s}s + (\bar{d}d))}{864\pi^2 m_c} (\beta^2 + \beta - 2) e^{-m_c^2/M^2},$$

$$\Pi^{\text{Dim} 8}_{1}(M^2, s_0) = -\frac{(g_s^2 G^2)^2}{27 \cdot 2^{11} \pi^4 M^4} \left[13\beta^2 + 10\beta + 13\right] e^{-m_c^2/M^2} + \frac{\left(\bar{s}g_s \sigma Gs\right)(\bar{d}d)}{288 M^4} (1 - \beta) \times \left[m_c^2(26 + 22\beta) + M^2(25 + 23\beta)\right] e^{-m_c^2/M^2},$$

$$\Pi^{\text{Dim} 9}_{1}(M^2, s_0) = \frac{\left(\bar{s}g_s \sigma Gs\right)(g_s^2 G^2)}{27 \cdot 2^{11} \pi^2 m_c M^4} (1 - \beta) \left[m_c^2(31 + 11\beta) - 2M^2(1 + \beta)\right] e^{-m_c^2/M^2},$$

$$\Pi^{\text{Dim} 10}_{1}(M^2, s_0) = 0.$$

For the spectral density $\rho^{\text{OPE}}_{2}(s)$, we get

$$\rho^{\text{pert}}_{2}(s) = \frac{m_c(13 - 2\beta - 11\beta^2)}{1536\pi^4 s} \left[m_c^6 + 9sm_c^4 - 9m_c^2s^2 - s^3 + 6m_c^2s(s + m_c^2) \ln \left(\frac{s}{m_c^2}\right)\right],$$
\[ \rho_2^{\text{Dim3}}(s) = \frac{\langle \overline{s}s \rangle + \langle \overline{d}d \rangle}{192\pi^2 s} \left( m_c^2 - s \right)^2 \left( 1 + 4\beta - 5\beta^2 \right), \]

\[ \rho_2^{\text{Dim4}}(s) = \frac{\left( g_s^2 \mathcal{G}^2 \right)}{9216\pi^4 m_c s} \left( 1 - \beta \right) \left[ \left( m_c^2 - s \right) \left( s(13 + 11\beta) + m_c^2(53 + 67\beta) \right) \right] 
+ 3m_c^2 s(11 + 13\beta) \ln \left( \frac{s}{m_c^2} \right), \]

\[ \rho_2^{\text{Dim5}}(s) = \frac{\langle \overline{d}g_s\sigma Gd \rangle + \langle \overline{s}g_s\sigma Gs \rangle}{768\pi^2 s} \left( 1 - \beta \right) \left[ m_c^2(5 + \beta) - 6s(1 + \beta) \right]. \]

The function \( \Pi_2(M^2) \) is determined by the components

\[ \Pi_2^{\text{Dim6}}(M^2, s_0) = \frac{\langle \overline{s}s \rangle \langle \overline{d}d \rangle}{24} m_c \left( 5\beta^2 + 2\beta + 5 \right) e^{-m_c^2/M^2}, \]

\[ \Pi_2^{\text{Dim7}}(M^2, s_0) = \frac{\left( g_s^2 \mathcal{G}^2 \right)}{3456\pi^2} \left( \beta^2 - 8\beta + 7 \right) e^{-m_c^2/M^2}, \]

\[ \Pi_2^{\text{Dim8}}(M^2, s_0) = \frac{\left( g_s^2 \mathcal{G}^2 \right)^2}{27 \cdot 21\pi^4 m_c M^2} \left( m_c^2 - 2M^2 \right) \left( 11 + 2\beta - 13\beta^2 \right) e^{-m_c^2/M^2} + \frac{\langle \overline{s}g_s\sigma Gs \rangle \langle \overline{d}d \rangle}{144M^4} m_c \]
\times \left[ M^2 (\beta - 1)^2 - 3m_c^2(5 + 2\beta + 5\beta^2) \right] e^{-m_c^2/M^2},

\[ \Pi_2^{\text{Dim9}}(M^2, s_0) = \frac{\langle \overline{s}g_s\sigma Gs \rangle \langle g_s^2 \mathcal{G}^2 \rangle}{27 \cdot 21\pi^4 M^4} m_c^2(\beta^2 + 28\beta - 29) e^{-m_c^2/M^2}, \]

\[ \Pi_2^{\text{Dim10}}(M^2, s_0) = 0. \]

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