Research Article

Maryam Khorshidi, Mehdi Nadjafikhah*, and Hossein Jafari

Fractional derivative generalization of Noether’s theorem

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Abstract: The symmetry of the Bagley–Torvik equation is investigated by using the Lie group analysis method. The Bagley–Torvik equation in the sense of the Riemann–Liouville derivatives is considered. Then we prove a Noether-like theorem for fractional Lagrangian densities with the Riemann-Liouville fractional derivative and few examples are presented as an application of the theory.

Keywords: Fractional derivatives, Symmetry, Fractional variational calculus, Fractional Euler–Lagrange equations, Conservation laws, Noether’s theorem

MSC: 70H33, 34K17, 70G65

1 Introduction

Fractional differentiation is a significant tool to describe and obtain mathematical model of real phenomena in various field of sciences [1, 2]. During the last four decades several analytical and numerical methods were presented for solving fractional differential equations (FDE) [1–4]. However there are some limitation for using those methods for solving different classes of FDE.

Symmetry is an important property of nature and all of the equations that are able to describe physical, biological or chemical phenomena have symmetry properties which follow from some fundamental rules [5–7].

Gazizov et al. [8], generalized the prolongation formulas for fractional derivatives and adapted the method of Lie group for symmetry analysis of FDEs.

The concept conservation laws or first integrals of the Euler-Lagrange equations is well known in Physics. The general principle relating to symmetry groups and conservation laws was first introduced by Noether (1918) [9, 10]. Riewe [11, 12] studied Euler-Lagrange equations for problems of the calculus of variations with fractional derivatives. Agrawal [13] presented extensions to traditional calculus of variations for systems containing fractional derivatives. Accordingly, the Euler-Lagrange equations were used by Frederico and Torres to prove a Noether-type theorem and Fractional Noether’s theorem in the Riesz–Caputo sense [14, 15].

The Bagley-Torvik equation formulae was originally considered in the studies on properties of real material by using fractional calculus, especially in 1/2 or 3/2 order derivatives [16, 17].

The paper is organized as follows. Section 2 includes several properties pertaining to the fractional derivative Symmetries and prolongation. Then we investigate a symmetry for Bagley-Torvik equation. In Section 3 our aim in
the present work is to discuss the fractional Euler-Lagrange equations and generalized Noether’s Theorem and prove the theorems in a different way.

2 Preliminaries

In the sequel we briefly recall some basic facts from the fractional calculus. Let $f \in L^1([a, b])$, $\alpha > 0$ and $t \in [a, b]$, the left, resp. right Riemann-Liouville integrals are defined in the following way:

$$a I_t^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{f(\tau)}{(t-\tau)^{1-\alpha}} \, d\tau,$$

resp.

$$I_b^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{f(\tau)}{(\tau-t)^{1-\alpha}} \, d\tau,$$

where $\Gamma$ is the Euler gamma function.

Left, resp. right Riemann-Liouville fractional derivative of order $n - 1 \leq \alpha < n$ is well defined for $f \in AC([a, b])$ and $t \in [a, b]$ as

$$a D_t^n f(t) = \frac{d}{dt} a I_t^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \frac{d}{dt} \int_a^t (t-\tau)^{n-\alpha-1} f(\tau) \, d\tau,$$

resp.

$$D_b^n f(t) = \frac{d}{dt} I_b^{n-\alpha} f(t) = \frac{1}{\Gamma(n-\alpha)} \int_t^b (\tau-t)^{n-\alpha-1} f(\tau) \, d\tau.$$

Applied problems require definitions of fractional derivatives allowing the utilization of physically interpretable initial conditions, so the Caputo fractional derivatives was proposed by Caputo first in his paper [18] as

$$a D_t^n f(t) = a I_t^{n-\alpha} f^n(t) = \frac{1}{\Gamma(\alpha-n)} \int_a^t (t-\tau)^{n-\alpha-1} f^n(\tau) \, d\tau,$$

and according to [1]

$$a D_t^n f(t) = a D_t^n f(t) = \frac{d^n}{dt^n} f(t), \quad \text{if } \alpha = n \in \mathbb{N},$$

$$a D_t^n f(t) = a D_t^n f(t), \quad \text{if } f^{(k)}(a) = 0 \quad k = 0, 1, 2, \ldots, n - 1,$$

$$a D_t^n f(t) = a D_t^n f(t) + \sum_{k=0}^{n-1} \frac{f^{(k)}(a)(x-a)^{k}}{\Gamma(1+k-n)}.$$

Similarly to integer-order differentiation, fractional differentiation is a linear operation [1]. Generalized Leibnitz rule gives [8]

$$D_t^n f(t)g(t) = \sum_{n=0}^{\infty} \binom{\alpha}{n} D_t^{\alpha-n} f(t) D_t^n g(t). \quad (2)$$

We refer the reader interested in supplementary information on fractional derivatives to the encyclopedia book of Samko et al. and Podlubny [1, 19].

Consider manifold $M = X \times U \simeq \mathbb{R}^p \times \mathbb{R}^q$ to be total space and whose members are $p$ independent and $q$ dependent variables. For $m \leq \alpha < m + 1$, $\alpha$-th jet space of $M$

$$M^{(\alpha)} = M^{U^1} \times \ldots \times M^{U^m} \times U^\alpha,$$

whose coordinates of $U^k$ represent the derivatives of the dependent variables with respect to $X$ of order $k = 1, 2, \ldots, m, \alpha$. 
When transformation (3) is applied to usual partial derivatives and the infinitesimal operator or generator of the group $G$ is of (4) on derivatives of $u$, after splitting (7) with respect to $\delta$, are said to be symmetry transformations of $u$ with respect to the variables $x$, $t$, $u$, $\varepsilon$. All such transformations form one parameter symmetry group $G$, that is

$$G : I \times M \rightarrow M, \quad I \subseteq \mathbb{R},$$

$$(\varepsilon, (t, x, u)) \mapsto (t + \varepsilon \xi^0(t, x, u) + O(\varepsilon^2), \quad x + \varepsilon \xi^1(t, x, u) + O(\varepsilon^2), u + \varepsilon \eta(x, u) + O(\varepsilon^2)),$$

and the infinitesimal operator or generator of the group $G$ is

$$v = \xi^0(x, t, u) \partial_x + \xi^1(x, t, u) \partial_t + \eta(x, t, u) \partial_u,$$

where $D$ denotes a total derivative.

By extending transformation (3) to the operator of Riemann-Liouville fractional $D^\alpha_t u$ and $D^\alpha_{xt} u$, we have

$$D^\alpha_{xt} \tilde{u}(\tilde{t}, \tilde{x}) = D^\alpha_t u + \varepsilon \tau^0 + O(\varepsilon^2),$$

where $\tau^0$ is given by prolongation formulae (2)

$$\tau^0 = D^\alpha_t (\eta - \xi^0 u_t - \xi^1 u_x) + \xi^0 D^\alpha_x u_t + \xi^1 D^\alpha_t u_x.$$

$\alpha \tau^0$ can be calculated in a similar operation [8]. The invariance condition causes $[\xi^0(t, x, u)]_{t=0} = 0$ [8].

If $P r^\alpha(v)$ denotes the prolongation of the operator $v$ of the $\alpha$-order derivative, the classical Lie theory can be written in a more compact form

$$P r^\alpha(v)(\Delta) |_{\Delta=0} = 0,$$

that has to extend $v$ up to appearance derivative orders. The obtained equations define all infinitesimal symmetries of $\Delta$.

**Example 2.1.** Consider the Bagley-Torvik equation [21]

$$u'' + 0 D^{3/2}_t u(t) + u(t) = 2 + 4 \sqrt{\frac{t}{\pi}} + t^2,$$

subject to the initial conditions and boundary condition as

$$u(0) = u'(0) = 0, \quad u(1) = 1, \quad u'(1) = 2.$$

By (1), it is clear $a D^{3/2}_t f(t) = a D^{3/2}_t f(t)$. Substitution (6) into (5) for $\alpha = 3/2$ yields

$$[\xi^1 \partial_x + \tau^0 + \eta - \left( \frac{2}{\sqrt{\pi}} x^{-1/2} + 2x \right) \xi^0] |_{\Delta=0} = 0.$$  (7)

After splitting (7) with respect to $u_t, u_{tt}, u_{ttt}, \cdots, D^{n-1}_{xt} u(t)$, for $n = 0, 1, \cdots$ which are considered independent, we arrive to over determined system of linear fractional differential equations. Solution of this system gives

$$\xi = c_1, \quad \eta + \eta_{xx} + D^{1/2}_x \eta = c_1 \left( \frac{2}{\sqrt{\pi}} x^{-1/2} + 2x \right),$$

so $\eta = 2c_1 x$ and $v = c_1 \partial_t + 2c_1 x \partial_u$ is a symmetry of (6).
3 Noether’s theorem for fractional problems of the calculus of variations

In this section \( t = (t_1, t_2, \ldots, t_p) \), \( u = (u^1, u^2, \ldots, u^q) \) and \( u^{(n)} \) shows the derivatives of \( u \) up to \( n \)-order, and \( m \leq \alpha, \beta < m + 1 \). Now we note an equivalent useful way to writing down the prolongation formulae. Suppose that

\[
v = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial t_i^\alpha} + \sum_{j=1}^{q} \eta^j \frac{\partial}{\partial u^j},
\]

and

\[
Q_\sigma(t, u^{(1)}) = \eta_\sigma - \sum_{i=1}^{p} \xi^i \frac{\partial u^\sigma}{\partial t_i}, \quad \sigma = 1, 2, \ldots, q,
\]

the \( q \)-tuple \( Q = (Q_1, \ldots, Q_q) \) is characteristic of the vector field \( v \), then

\[
P_{\nu^{(n)}} v = P_{\nu^{(n)}} v_Q + \sum_{i=1}^{p} \xi^i D_i,
\]

where

\[
v_Q = \sum_{\sigma=1}^{q} Q_\sigma(t, u^{(1)}) \frac{\partial}{\partial u^\sigma}, \quad P_{\nu^{(n)}} v_Q = \sum_{\sigma=1}^{q} \sum_{J} D_J Q_\sigma \frac{\partial}{\partial u^\sigma},
\]

and the summation extends to over multi–indices \( J = (j_1, \ldots, j_k) \), with \( 1 \leq j_k \leq p \) and \( k \geq 0 \) [5, 6].

**Definition 3.1.** Suppose \( f \) and \( g \) are of class \( C^1 \) in the interval \( [a, b] \), we have

\[
D^\nu_v(f, g) = -g_{\nu} D^\nu_f f + f_{\nu} D^\nu_g g,
\]

where \( t \in [a, b] \) and if \( v = 1 \), we have \( D^1_v(f, g) = d/dt(fg) \).

**Definition 3.2.** Quantity \( P(t, u, a D^\alpha u, i D^\beta_b u) \) is said to be fractional conserved quantity if and only if \( P \) is in the form

\[
P(t, u, a D^\alpha u, i D^\beta_b u) = \sum_{i=1}^{m} P^1_i (t, u, a D^\alpha u, i D^\beta_b u) P^2_i (t, u, a D^\alpha u, i D^\beta_b u).
\]

for some \( m \in \mathbb{N} \), and some functions \( P^1_i \) and \( P^2_i \), where each pair \( P^1_i \) and \( P^2_i \) satisfy

\[
D^{\gamma_i}_v (P^1_i (t, u, a D^\alpha u, i D^\beta_b u) P^2_i (t, u, a D^\alpha u, i D^\beta_b u)) = 0.
\]

with \( \gamma_i \in \{\alpha, \beta, 1\} \), \( j_1 = 1 \) and \( j_2 = 2 \) or \( j_1 = 2 \) and \( j_2 = 1 \), along all the solutions of the fractional Euler-Lagrange equations [14].

In continuation, \( \partial_{\alpha} L \) is the notation of the partial derivative of \( L \) with respect to its \( \alpha \)-th argument.

**Definition 3.3.** For \( 1 \leq \kappa \leq q \), the \( \kappa \)-th Euler operator is given by

\[
E_\kappa = \sum_j (-D)_j \frac{\partial}{\partial u^j} + \tau D^\alpha_{\beta} \frac{\partial}{\partial a D^\alpha u} + a D^\beta_{\beta} \frac{\partial}{\partial \tau D^\beta u}.
\]

For \( \alpha, \beta \in \mathbb{N} \), we have \( a D^\alpha_t = \frac{d}{dt} \) and \( i D^\beta_b = -\frac{d}{dt} \) and (9) reduces to the standard Euler-Lagrange equation [5, 13].
Theorem 3.4. Let \( L(u) \) be a functional of the form
\[
L(u) = \int_a^b L(t, u^{(n)}, a D_t^\alpha u, b D_t^\beta u) dt,
\]
defined on the set of functions \( u(t) \) which have continuous left resp. right Riemann–Liouville fractional derivative of order \( \alpha \) resp. \( \beta \) in \([a, b]\) and satisfy the boundary conditions \( u(a) = u_a \) and \( u(b) = u_b \). Then a necessary condition for \( L(u) \) to have an extreme for a given function \( u(t) \) is that \( u(t) \) satisfies the following Euler–Lagrange equation [5, 13]:
\[
E_k(L) = 0, \quad k = 1, \cdots, q.
\]

Definition 3.5. A local group of transformations \( G \) acting on \( M \subseteq C \times U \) is a variational symmetry group of (10), if whenever \( \Omega \subseteq C \), \( u = f(t) \) defined over \( C \) whose graph lies in \( M \) and \( g \in G \) is such that \( \mathcal{P} = \mathcal{T}(\mathcal{F}) = g \cdot f(\mathcal{F}) \), then
\[
\int_{\Omega} L(t, P r^{(n)} f(t), a D_t^\alpha u, b D_t^\beta u) dt = \int_{\mathcal{T}} L(\mathcal{T}, P r^{(n)} \mathcal{F}(\mathcal{T}), a D_t^\alpha \mathcal{F}, b D_t^\beta \mathcal{F}) d\mathcal{T}.
\]

Note that for each \( g \in G \), the group transformation
\[
(\mathcal{T}, \mathcal{F}) = g \cdot (t, u) = (\psi(t, u, \varepsilon), \Pi(t, u, \varepsilon)),
\]
can be regarded as a change of variables, so we can rewrite (12) in the form
\[
\int_{\mathcal{T}} L(\mathcal{T}, P r^{(n)} \mathcal{F}(\mathcal{T}), a D_t^\alpha \mathcal{F}, b D_t^\beta \mathcal{F}) d\mathcal{T} = L(t, P r^{(n)} f(t), a D_t^\alpha u, b D_t^\beta u)
\]
where the Jacobian has entries
\[
J^{ij}(t, u, \varepsilon) = D_t^i \psi_j(t, u, \varepsilon).
\]

Theorem 3.6. A connected group of transformations \( G \) acting on \( M \subseteq C \times U \) is a variational symmetry group of the functional (10), if and only if
\[
P r^{(\alpha, \beta)} v(L) + L D i v \xi = 0,
\]
for every infinitesimal generator
\[
v = \sum_{i=1}^p \xi^i \frac{\partial}{\partial t_i} + \sum_{j=1}^q \eta^j \frac{\partial}{\partial u^j},
\]
of \( G \). \( D i v \xi \) denotes the total divergence of the \( p \)-tuple \( \xi = (\xi^1, \xi^2, \ldots, \xi^p) \).

Proof. Since (12) and (13) are held for all functions \( u = f(x) \), the integrands must agree pointwise
\[
L(\mathcal{T}, P r^{(n)} \mathcal{F}(\mathcal{T}), a D_t^\alpha \mathcal{F}, b D_t^\beta \mathcal{F}) d\mathcal{T} = L(t, P r^{(n)} f(t), a D_t^\alpha u, b D_t^\beta u),
\]
for all \((t, u^{(n)}) \in M^{(n)}\). To obtain the infinitesimal version of (15), setting \( g = g_\varepsilon = \exp(\varepsilon v) \) and differentiate with respect to \( \varepsilon \), we get
\[
(P r^{(\alpha, \beta)} v(L) + L D i v \xi) d\mathcal{T} = 0,
\]
because
\[
\frac{d}{dt} [d\mathcal{T}] = D i v \xi (P r^{(1)} g_\varepsilon (t, u^{(1)})) d\mathcal{T} = 0.
\]
and
\[
\frac{d}{d\varepsilon} \bigg|_{\varepsilon=0} Pr^{(n)}[\exp(\varepsilon v)](t, u^{(n)}) = Pr^{(n)}v \bigg|_{(t, u^{(n)})}, \quad v(t, u^{(n)}) \in M^{(n)}.
\]
At \( \varepsilon = 0 \), \( g_\varepsilon \) is the identity map and we have proved the necessity of (14). To prove conversely is trivial. \( \square \)

**Theorem 3.7.** Suppose \( G \) is a variational symmetry group of the (10). Let
\[
v = \sum_{i=1}^{p} \xi^i \frac{\partial}{\partial t^i} + \sum_{j=1}^{q} \eta^j \frac{\partial}{\partial u^j},
\]
be the infinitesimal generator of \( G \), then \( Q = (Q_1, \ldots, Q_q) \) is the characteristic of a conservation law for Euler–Lagrange equations \( E(L) = 0 \), in other words, there is a \( p \)-tuple \( P(x, u^{(\alpha, \beta)}) = (P_1, P_2, \ldots, P_p) \) such that
\[
\text{Div} P = Q \cdot E(L) = \sum_{\nu=1}^{q} Q_\nu E_\nu(L).
\]

**Proof.** By substituting the prolongation formula (14) into (8), we find
\[
0 = Pr^{(\alpha, \beta)}v(L) + L \text{Div} \xi
\]
\[
= Pr^{(\alpha, \beta)}Q(L) + \sum_{i=1}^{p} \xi^i D_i L + L \sum_{i=1}^{p} D_i \xi^i
\]
\[
= Pr^{(\alpha, \beta)}Q(L) + \text{Div}(L \xi)
\]
\[
= \sum_{\alpha, J} Q_\alpha(-D)J \frac{\partial L}{\partial u^J} + \text{Div}(Q_\alpha \frac{\partial L}{\partial u^J} + L \xi)
\]
\[
= \sum_{\alpha=1}^{q} Q_\alpha E_\alpha(L) + \text{Div} P.
\]
So we have proved \( P = Q_\alpha \frac{\partial L}{\partial u^J} + L \xi \) is a conservation law for Euler–Lagrange equations. \( \square \)

**Example 3.8.** If \( v = \eta \partial u \) is variational symmetry of the functional
\[
L = \int_a^b L(t, u, u^1, \ldots, D_1^n u) dt,
\]
then
\[
Q = \eta, \quad \xi = 0,
\]
\[
Pr^{(\alpha)}v(L) + L \text{Div} \xi - Q \cdot E(L) = (\eta \partial u + \frac{\partial \eta}{\partial t} \partial u_t)
\]
\[
+ aD_1^n \eta \partial aD_1^n u)(L) - \eta(\partial_2 L - D_2 \partial_3 L + D_3 \partial_4 L) =
\]
\[
\frac{\partial \eta}{\partial t} \partial_3 L + aD_1^n \eta \partial_4 L + \eta D_1 \partial_3 L - \eta_1 D_2^n \partial_4 L =
\]
\[
D_1(\eta \partial_3 L) + aD_1^n (\partial_4 L, \eta).
\]

**Example 3.9.** If \( v = \eta \partial u \) is variational symmetry of the functional
\[
L = \int_a^b L(t, u, aD_1^n u, D_1^n u) dt,
\]
then
\[
Q = \eta, \quad \xi = 0,
\]
\[
Pr^{(\alpha)}v(L) + L \text{Div} \xi - Q \cdot E(L) = (\eta \partial u + \frac{\partial \eta}{\partial t} \partial u_t)
\]
\[
+ aD_1^n \eta \partial aD_1^n u)(L) - \eta(\partial_2 L - D_2 \partial_3 L + D_3 \partial_4 L) =
\]
\[
\frac{\partial \eta}{\partial t} \partial_3 L + aD_1^n \eta \partial_4 L + \eta D_1 \partial_3 L - \eta_1 D_2^n \partial_4 L =
\]
\[
D_1(\eta \partial_3 L) + aD_1^n (\partial_4 L, \eta).
\]
then

\[
Q = \eta, \quad \xi = 0,
\]

\[
P_r^{(\alpha, \beta)}v(L) + LD_i v \xi - Q.E(L) = (\eta \partial u + a D_t^\alpha \eta \partial_u D_t^\alpha u
\]

\[
- \partial D^\alpha L + \partial D_t^\alpha L = a D_t^\alpha \eta \partial_3 L - \eta_1 D_t^\alpha \partial_3 L - \eta_2 D_t^\alpha \partial_4 L =
\]

\[
D_t^\alpha (\partial_3 L, \eta) + D_t^\alpha (\partial_4 L, \eta).
\]

**Example 3.10.** If \(v = \xi \partial_t + \eta \partial_u\) is variational symmetry of the functional \(L = \int_a^b L(t, u, a D_t^\alpha u, \partial u) dt\), then

\[
Q = \eta - \xi \frac{\partial u}{\partial t},
\]

\[
P_r^{(\alpha, \beta)}v(L) + LD_i v \xi - Q.E(L) = \xi \partial_1 L + \xi \partial_2 L \frac{\partial u}{\partial t} + \xi \partial_3 L_a D_t^\alpha + \eta u
\]

\[
+ \xi \partial_4 L_t D_t^\alpha + \xi \partial_4 L_t D_t^\alpha \partial_3 L + \eta L_t D_t^\alpha \partial_3 L - \eta_1 D_t^\alpha \partial_3 L - \eta_2 D_t^\alpha \partial_4 L
\]

\[
+ \xi \frac{\partial u}{\partial t} D_t^\alpha \partial_3 L - \partial_3 L_a D_t^\alpha \frac{\partial u}{\partial t} + \xi \frac{\partial u}{\partial t} D_t^\alpha \partial_4 L - \partial_4 L_a D_t^\alpha \frac{\partial u}{\partial t} \xi =
\]

\[
D(L, \xi) + D^\alpha (\partial_3 L, \eta) - D^\alpha (\partial_4 L, \eta) - D^\alpha (\partial_3 L, \xi \frac{\partial u}{\partial t}) - D^\alpha (\partial_4 L, \xi \frac{\partial u}{\partial t}).
\]

### 4 Conclusion

The classical Lie point symmetry method has been investigated for the Bagley-Torvik equation in the sense of the Riemann–Liouville derivatives. The general form of Noether-like theorem for fractional Lagrangian densities is proved. Few examples are presented as an application of the theory.

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