A General Prescription for Semi-Classical Holography

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Abstract

We present a version of holographic correspondence where bulk solutions with sources localized on the holographic screen are the key objects of interest, and not bulk solutions defined by their boundary values on the screen. We can use this to calculate semi-classical holographic correlators in fairly general spacetime regions, including flat space with timelike boundaries. In AdS, the distinction between our approach and the standard Dirichlet-like approach is superficial. But in more general settings, the analytic continuation of the Dirichlet Green function does not lead to a Feynman propagator in the bulk. Our prescription avoids this problem. Furthermore, in Lorentzian signature we find an additional homogeneous mode. This is a natural proxy for the AdS normalizable mode and allows us to do bulk reconstruction. Perturbatively adding bulk interactions to these discussions is straightforward. We conclude by elevating some of these ideas into a general philosophy about mechanics and field theory. We argue that localizing sources on suitable submanifolds can be an instructive alternative formalism to treating these submanifolds as boundaries.

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1 Introduction

Immediately after the discovery of the AdS/CFT duality [1], a semi-classical holographic correspondence between the bulk and the boundary of AdS was presented in eg., [2, 3, 4, 5, 6, 7]. These papers use classical bulk physics to determine correlators and states on the boundary. One of the rudimentary quasi-tests of the duality was that objects calculated this way had the form expected in a conformal field theory (CFT). A bit later, it was noted by Hamilton, Kabat, Lifschytz and Lowe (HKLL) [8] that using the correspondence, one could also express local bulk physics (again at the semi-classical level) in terms of non-local operators in the boundary theory. This was the beginnings of the idea of bulk reconstruction.

One of the striking features of these developments is that none of them rely on the fine details of the dual theories. This is remarkable, and makes one wonder: how much of the semi-classical description of holography is actually tied to AdS? Is it possible to give a prescription for computing boundary correlators via bulk calculations in more general spacetimes (or regions of spacetime)? How about bulk reconstruction from appropriate boundary data? Of course, one cannot expect the holographic correlators one computes in a general non-AdS setting to turn out to be those of a conformal field theory (or perhaps even a local theory). But that does not answer the question whether a semi-classical holographic description can be found at all (eg., do there exist a natural set of boundary correlators that one can compute anti-holographically?), and if so, how best to formulate it.

In this note, we will observe that the general structure of the AdS/CFT correspondence uncovered in the early papers on the subject, has a very natural adaptation to more general settings. We will argue that the manner in which the semi-classical bulk physics of a region of spacetime is encoded on a boundary holographic screen in terms of sources, condensates and correlators is essentially universal in large classes of spacetimes\textsuperscript{1}. The general prescription we give will be in terms of bulk sources (localized at the screen) and homogeneous modes, instead of the usual normalizable and non-normalizable modes familiar from AdS [4, 5]. Instead of working with boundary values of bulk fields, our Euclidean correspondence will be built on bulk sources localized on the holographic screen. This means that we will work \emph{not} with Dirichlet Green function (which is defined by a vanishing condition at the screen) but with the standard Euclidean bulk-bulk Green function that dies out at infinity. In particular, this means that when analytically continued to Lorentzian signature, this will lead us to the standard Feynman propagator as we would like, for causality reasons. In Lorentzian, we also

\textsuperscript{1}We will describe the classes of spacetimes in which we expect our claims to hold, somewhat imprecisely, in the next sections. Our claims depend on the existence/uniqueness of solutions of partial differential (wave) equations, which may have some subtleties (especially in Lorentzian signature) in some situations. A very concrete example of our prescription is provided in the companion paper [14] for the case of flat space with a box boundary, but we expect that the prescription holds somewhat more generally.
find a bonus feature: the source and the Green function do not uniquely fix the solution, we also have the freedom to add a homogeneous mode that is regular everywhere. This is the analogue of the normalizable mode in AdS. From the boundary (i.e., on-screen) value of the homogeneous mode, we will also be able to reconstruct the bulk field starting with the closely related spacelike Green function \[^{[8, 11]}\] and extracting an HKLL-like kernel.

We will see that our prescription has a simple map to the standard AdS/CFT correspondence, when restricted to AdS. To understand this connection it is useful first to note that sources placed on the holographic screen will depend on the value of the holographic coordinate (let us call it \(r\)) of the screen \((r = R)\), and therefore will lead to correlators that depend on this \(R\). A crucial point is that the \(R\)-dependence of a general bulk solution is (in general) dependent on the angular harmonics\(^2\). The beautiful fact about AdS is that the angular harmonics dependence of \(R\) dies out near the boundary, and in fact the \(R\)-dependence factorizes out. This enables us to define a new class of correlators very simply, where this \(R\)-dependence can be compensated. These \(R\)-independent correlators are precisely those of a CFT. The deep reason why this happens in AdS is of course because the bulk isometries act as the conformal group on the asymptotic boundary. In practice this means that one can replace the bulk-to-boundary propagator that one typically uses in AdS (basically a type of Dirichlet Green function) with the boundary limit of the bulk-to-bulk propagator (the Euclidean Green function that dies out at infinity) as long as one keeps track of some simple \(R\)-dependent scalings.

In this note, we will merely lay out the general statements. However, to illustrate that the prescription indeed gives concrete results, in a companion paper \(^{[14]}\) we will present detailed calculations for an example of some interest: 3+1 dimensional Minkowski space, with the holographic screen chosen to be a spherical box \(\mathbb{R} \times S^2\) of finite radius \(R\). On top of serving as a detailed illustration of the implementation of the prescription, our results there include holographic correlators, Witten diagram calculations and HKLL-like smearing functions for flat space.

Let us summarize. When trying to formulate holographic correspondence in general spacetime regions, a useful object is the general solution of the bulk equations of motion with sources placed on the holographic screen. Unlike in AdS, the notion of normalizability vs non-normalizability of the solutions of the bulk field equations becomes awkward in general settings. But a natural generalization presents itself in many cases: the homogeneous and inhomogeneous pieces in the solutions of the bulk field equations with an arbitrary source at the holographic screen. This structure is general enough to allow the entire semi-classical holographic correspondence to go through in fairly general classes of spacetime regions in-

\(^2\)Note that the general solution is a sum of products of the angular and radial parts: very schematically, \(\sum_l R_l(r)Y_l(\Omega)\). This means that typically one cannot factorize out the \(r\)-dependence.
cluding flat space, and it reduces in a suitable sense to the usual prescription in the AdS case. It is also immediate to see that even though our statements are phrased in terms of free theories, perturbatively adding interactions is as straightforward as it is in AdS.

We elaborate on these ideas in the next sections. One point of view that emerges from these discussions is that it might be worthwhile on general grounds, to consider sources localized on submanifolds as a general approach to formulating dynamics. This could be of some interest even beyond holography. It serves as an alternative to the usual formalism familiar from mechanics and field theory, where we describe dynamics via boundary and/or initial value data on submanifolds, together with boundary terms and the like. Instead, here the idea is to consider boundary conditions that always die down at (possibly Euclidean) infinity, but allowing sources localized on appropriate submanifolds as a proxy for capturing the physics of initial/boundary value surfaces.

2 Euclidean Holography: Sources at the Holographic Screen

Let us start with Euclidean signature. Consider a connected region of a general spacetime, with a codimension one hypersurface as its boundary/holographic screen\(^3\). Intuitively, we will think of the holographic screen as a hypersurface defined by two properties\(^4\): (a) it must separate the spacetime into two disconnected regions, (b) it should have a smooth deformation to the empty hypersurface that respects property (a). The first condition is obvious. The second condition means that the screen belongs to a one parameter family of screens connected to the trivial (i.e., non-existent) screen. Note that this parameter can be thought of as the holographic direction. The above conditions mean that (for example) in three dimensional Euclidean space, topological spheres and infinite cylinders are acceptable holographic screens, but not infinite planes. This is consistent with the intuition that a holographic screen can be shrunk.

Using the \(d + 1\) coordinate freedoms available to us, we can work (without loss of generality) in a gauge where the bulk metric takes the form

\[
ds^2 = dr^2 + \gamma_{ab}(r, x) \, dx^a dx^b.\tag{2.1}
\]

We will take the holographic screen to be at \(r = R\). While the form of the metric above is just a coordinate choice and does not lead to any serious loss of generality, in making the above choice of screen, we are being restrictive. We are taking our boundary to be normal to the radial coordinate, but since we have already made use of all our freedoms to pick

\(^3\)We can relax these conditions (e.g., multiple disconnected boundaries) by making various trade-offs. But we will stick with this for concreteness.

\(^4\)We assume that the spacetime is topologically trivial.
Figure 1: The blue region denotes the Euclidean $d+1$-dimensional bulk $\mathcal{M}$. The $d$-dimensional boundary aka holographic screen is $\partial \mathcal{M}$, the bulk field is $\phi(r, x)$, and the source on the boundary is $J_0(R, x)$. The coordinate $r$ represents the bulk foliation and $r = R$ will be taken as the boundary. The region $\mathcal{M}$ can be a sub-region of spacetime, and the $x$ is $d$-dimensional.

We wish to holographically describe the dynamics of a bulk field $\phi(r, x)$ with the action $S_{\text{bulk}}$ around a semi-classical bulk background. We wish to do this as much as possible by analogy with AdS. In Euclidean signature, we will make the following guess for the proposal, and it will also serve as a stepping stone to the Lorentzian case, which involves some further subtleties. Without further ado, let us first state the Euclidean prescription:

- Find the general solution of the bulk wave equation with an arbitrary source $J_0(R, x)$ at the holographic screen\textsuperscript{6}. In a non-singular Euclidean geometry, this will take the form

$$\phi(r, x) = \int_{\partial \mathcal{M}} d^d x' \sqrt{\gamma(R, x')} G_E(r, x; R, x') J_0(R, x')$$

(2.2)

where the Green function $G_E(r, x; R, x')$ is the bulk-to-bulk propagator (with one local-

\textsuperscript{5}Let us emphasize the obvious here however: this choice does not mean that we are constraining ourselves to spherical symmetry.

\textsuperscript{6}More precisely, we mean that we are considering the bulk free field equations of motion, with a bulk source $J_0(R, x)\delta(r - R)$ in spacetime.
tion taken to the boundary). Note that in Euclidean signature, the wave equations are elliptic equations, and therefore we expect these solutions to exist, be regular (except perhaps at the source) and be unique\(^7\). Note also that in order to write the above form, we have implicitly assumed that the bulk fields are free. We will be able to perturbatively add interactions via generalizations of Witten diagrams etc., but truly strong coupling bulk effects are as difficult here, as they are in AdS.

A key point is that the choice of the Green function is not completely fixed without some further input, a boundary condition of some sort. We will take this condition to be that it should vanish at infinity, fixing it to be the conventional Euclidean Green function (we have incorporated this choice already in the notation with the subscript \(E\)). Our motivation for this choice is that we want the Green function to analytically continue to the bulk Feynman propagator when we analytically continue the time coordinate to Lorentzian signature. We will see in the next section that there exists physically significant candidate \(\gamma_{ab}\)'s for which this can clearly be arranged.

- Compute the bulk on-shell (semi-classical) partition function, or (in practice) the classical action, of this solution within the holographic screen. It is a straightforward fact that for a scalar two-derivative theory, this will be of the form

\[
S_{\text{bulk}} = \lim_{r \to R} \int_{\partial M} d^d x \sqrt{\gamma(r, x)} \phi(r, x) \partial_r \phi(r, x) \tag{2.3}
\]

and using (2.2) this can immediately be written as a functional of \(J_0(R, x)\). This lets us calculate correlation functions of operators \(O(x)\) dual to the sources \(J_0(R, x)\) via functional derivatives with respect to the \(J_0(R, x)\), in a manner very similar to AdS. When evaluated at zero-source, this procedure leads to a vanishing 1-point function, and a two-point function of the form

\[
\langle O(x')O(x'') \rangle = \lim_{r \to R} \int_{\partial M} d^d x \sqrt{\gamma(r, x)} \partial_r \{G_E(r, x; R, x'') G_E(r, x; R, x') \}. \tag{2.4}
\]

Even though our notation does not emphasize it, it should be kept in mind that the dual operators \(O\) and their correlators can depend on \(R\).

\(^7\)A source-less Euclidean “wave” equation is a Laplace-type equation, and by analogy with flat space, generically (i.e., in generic dimension and for generic angular quantum number) we expect it to have two kinds of solutions. One that is singular at the origin, and another that is singular at infinity. A source at a finite radius will create a perturbation that should die down at infinity, and should therefore be expressible exclusively in terms of solutions singular at the origin. Linear combinations of singular solutions can result in a finite shift in the location of the singularity. This is basically the idea behind the familiar multi-pole expansion in electrostatics. When one goes over to Lorentzian signature, it turns out that the divergent solution at infinity ceases to exist, and gets replaced by solutions that are regular everywhere. In the next section, we will see that these regular solutions play a role analogous to AdS normalizable modes.
Let us make some comments.

One might (or at least we did) worry that since we are working with sources instead of boundary values, we might run into trouble with these calculations because of potential divergences at the location of the sources. This is a false concern. A familiar example that clarifies this point is to consider the electrostatic potential due to a (uniformly) charged shell: the potential is finite at the shell. The key point is that to get a divergence at the source, we need the source to be point-like, i.e., localized in all coordinate directions\(^8\).

Another related possible confusion (to which we were again victims of) is that since the field is finite at the source, why don’t we simply view this as a Dirichlet problem where the field is held fixed at the screen? The answer is that while the field may be finite, its value now depends on the Green function as well as the source, and so it makes a difference, what is the natural \textit{a priori} data for the problem. That data is provided by the sources, and not the boundary values, in our prescription. If we want to work with Dirichlet data, we need to make sure that the Green function vanishes at the screen, which is what defines the Dirichlet Green function.

Let us also note that even though the field itself is not divergent, the boundary correlators that we calculate will have singularities (as can be easily checked for specific examples), when two operators coincide. This is physical, and should be compared to the isomorphic phenomenon in Euclidean AdS/CFT. Once one goes over to Lorentzian, these turn into null separation singularities at the boundary, which also applies in our case.

To summarize, we decree that the bulk (semi-classical) partition function in a spacetime region is defined for field configurations with sources at the boundary, and is a functional of those sources. From the perspective of the dual theory on the screen, the same source also couples to the operators dual to the bulk field. In AdS, that the boundary values of bulk fields are sources for the boundary theory is well-discussed, but it is also true (though less emphasized) that these boundary values are essentially (though not exactly) also sources for the \textit{bulk} theory. We will clarify this point momentarily. In any event, what we have done here is to reverse the logic a bit and to treat localized bulk sources instead of boundary values as the key objects. This is a small step, but it is a useful step in going to the Lorentzian signature, as we will see.

\textbf{AdS:} In this subsection, we will clarify the connection between our prescription and the standard Euclidean AdS/CFT\(^9\). In [2, 3] one seeks solutions of the bulk wave equations

\(^8\)Stated differently, remember the familiar fact that a uniform charge on an infinite plane leads to a constant electric field in the bulk. This is because as you step back from the plane, you “see” more of the plane and therefore more of the charge on it, so the field does not fall too fast. One cannot have a localized singularity in the field without the field dropping fast.

\(^9\)See [9] for a discussion of the usual AdS/CFT correspondence that is adapted to our discussion here.
that are regular in the interior. The resulting solutions are necessarily of the so-called non-normalizable variety, and at the boundary (defined by $z = 0$ in a standard choice of the Poincare patch coordinates) their behavior (for scalar fields) is of the form

$$\phi(z, x) \to z^{d-\Delta} \phi_0(x) + \ldots$$

(2.5)

where $\Delta$ is determined by the mass of the scalar $^{10}$ Note the remarkable fact that the $z$-dependence has factorized out, which is a special feature of AdS – in a general spacetime, the $z$-dependence and the angular harmonic dependence will mix and the full solution can only be written as a sum of products, not a single product.

In any event, this $\phi_0(x)$ is interpreted as the source to which the boundary operator couples. The bulk solution can be written in terms of this $\phi_0(x)$ as

$$\phi(z, x) \sim \int d^d x' \frac{z^\Delta}{(z^2 + (x - x')^2)^\Delta} \phi_0(x')$$

(2.6)

$$\equiv \int d^d x' G_{EAdS}(z, x; x') \phi_0(x')$$

(2.7)

where the $G_{EAdS}$ is usually called the bulk-to-boundary propagator in standard AdS/CFT. Our prescription on the other hand is that we should work with the bulk-to-bulk propagator itself, but with a source $J_0$ that couples to it at the boundary. We place the source close to the boundary at $z' = \epsilon$ and the solution takes the form

$$\phi(z, x) = \int d^d x' \sqrt{\gamma(z, x)} G_\Delta(z, x; \epsilon, x') J_0(\epsilon, x')$$

(2.8)

$$\sim \int d^d x' \frac{z^\Delta}{(z^2 + (x - x')^2)^\Delta} \epsilon^{\Delta-d} J_0(\epsilon, x').$$

(2.9)

The first line introduces the bulk-to-bulk Green function. In the second line we have used its explicit form that can be looked up from [10] and the fact that $\epsilon$ is small to write down its leading behavior in $\epsilon$. Altogether, this means that boundary correlators computed via functional derivatives with respect to $\phi_0(x)$ (like in standard AdS/CFT) and $J_0(\epsilon, x)$ (like we do) differ merely by a simple $\epsilon$ dependent factor that is trivially incorporated into the prescription. If we kept track of the precise numerical coefficients of the Green functions above, the precise map turns out to be

$$\phi_0(x) \leftrightarrow \frac{\epsilon^{\Delta-d}}{2\Delta - d} J_0(\epsilon, x).$$

(2.10)

Our prescription clarifies what makes AdS special: in AdS there is a specific scaling of the sources that results in correlators that are independent of the location of the screen, at least

$^{10}$We will assume here that $\Delta > d/2$ to avoid some technicalities. With a bit more nuance, we can extend the discussion all the way to the CFT unitarity bound, $\Delta = \frac{d+2}{2}$. 

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when the screen is close to the asymptotic boundary. This is a result of the conformal action of the bulk isometries on the AdS boundary. In particular, in a general spacetime/region where we do not expect conformal invariance, but nonetheless expect holography to hold in some suitable sense, it should be clear that we should expect scale dependence. It is a feature, and not a bug. Indeed, in flat space, the correlators we find are $R$-dependent\(^{11}\).

3 Lorentzian Holography: Homogeneous Solutions

We will now consider Lorentzian spacetime regions of the form (2.1). We will take the (Lorentzian) time to be one of the $a, b, \ldots$ directions. The paradigmatic example we will have in mind will be $d+1$-dimensional flat space with an Einstein-static screen at $r = R$:

$$\begin{align*}
ds^2 &= dr^2 + (-dt^2 + r^2 d\Omega_{d-1}^2) 
\end{align*}$$

(3.1)

Note that (Poincare) AdS also falls into the same structure as (2.1). We expect that the claims we make in this section will hold somewhat more generally than either of these examples. In particular, regions of spacetimes which are “flat enough” (see eg., figure 2) should satisfy them. At least in cases where a $d+1$-dimensional region is bounded by a timelike hypersurface, such that the intersection of this surface with a constant time slice (Cauchy slice) is topologically $S^{d-1}$, we expect that the claims of this section have a chance of holding\(^{12}\). See figure 2 for an example region of this type carved out in the exterior Schwarzschild geometry.

We now present the statement of the Lorentzian Prescription in the following form:

- Find the general solution of the bulk wave equation with an arbitrary source at the holographic screen. As can be checked explicitly for the special case of (3.1), this has

\(^{11}\)It would of course be interesting to see if the correlators in flat space have some other transformation (Poincare? BMS?) under which they have an invariance which becomes fruitful in the limit where $R$ goes to infinity.

\(^{12}\)This is just the statement that we are considering tube-like spacetime regions bounded by timelike holographic screens. One possible subtlety is that solutions of wave equations in such a tube-like region could leak out through the top/bottom, and this could qualitatively change the solution structure, eg., think about the Penrose diagram of de Sitter space. But at least if in the Penrose diagram, the spacetime region (together with its timelike boundary) is represented by a closed region, we expect that the region is sufficiently similar to flat space that the prescription we give will hold. Clearly, this should be viewed as a plausible sufficient condition, and it is satisfied by regions like the one in figure 2. But these conditions may not be necessary. AdS Penrose diagram does not have the same structure, but AdS is consistent with a version of our prescription. It is clearly of interest to make a more precise statement about the class of spacetime regions for which our prescription holds, this will require statements about solutions spaces of wave equations in various geometries. In practice, we will keep (3.1) in the back of our minds in the following discussions.
Figure 2: The blue region is bounded by a timelike tubular holographic screen in the exterior Schwarzschild geometry, and we expect our prescription to have analogues there.

the form

$$\phi(r, x) = \int_{\partial M} d^d x' \sqrt{-\gamma} G_L(r, x; R, x') J_0(R, x') + \phi_h(r, x)$$  \hspace{1cm} (3.2)$$

A key difference\textsuperscript{13} here from the Euclidean scenario (other than in the explicit form of the Green’s function) is the presence of the homogeneous solution $\phi_h$. The homogeneous solution is annihilated by the wave operator, but unlike in Euclidean signature where such solutions are divergent at infinity, now it is regular everywhere including at infinity (where it vanishes). This means that the general solution with source can include such a piece as well. The Green function $G_L$ is the analytic continuation of the Euclidean Green function, and is the bulk-to-bulk Feynman propagator.

- The homogeneous mode $\phi_h(r, x)$ (or equivalent data, namely its value $\phi(R, x)$ at $r = R$) is dual to a state (denoted $|\phi_h\rangle$) in the dual theory. Taking the functional derivative of the action with respect to the source gives us

$$\langle \phi_h | O(y) | \phi_h \rangle_{J_0 = 0} = \lim_{r \to R} \int_{\partial M} d^d x \sqrt{-\gamma} \partial_r \{ \phi_h(r, x) G_L(r, x; R, y) \}$$  \hspace{1cm} (3.3)$$

$$\langle \phi_h | O(y) O(z) | \phi_h \rangle_{J_0 = 0} = \lim_{r \to R} \int_{\partial M} d^d x \sqrt{-\gamma} \partial_r \{ G_L(r, x; R, y) G_L(r, x; R, z) \}$$  \hspace{1cm} (3.4)$$

Note that the presence of the homogeneous piece leads to a non-trivial 1-point function even when the source is zero. Again, we have suppressed the $R$-dependence of the left hand side.

To summarize, in the Lorentzian picture, the semi-classical partition function (or on-shell action) with the source $J_0(R, x)$ and homogeneous mode $\phi_h(R, x)$ turned on, is equal to the

\textsuperscript{13}A secondary difference is that the radial direction $r$ here labels a timelike foliation.
expectation value of the dual theory deformed by the source term, in the state $|\phi_h\rangle$, i.e.,
$\langle \phi_h | e^{i \int_M J_0(R,x) O(x)} | \phi_h \rangle$.

**AdS:** Let us briefly compare this with AdS. In Lorentzian AdS, on top of the non-normalizable mode as in the Euclidean case, we also have normalizable modes that are regular in the interior [5]. At the boundary, they behave like

$$\phi_n(z, x) \rightarrow z^\Delta \phi_n(x) + \cdots$$

(3.5)

We claim that they are the natural analogs of our homogeneous modes. In particular, the general solution in Lorentzian AdS is of the form

$$\phi(z, x) = \int d^dx' G_{LAdS}(z, x; x') \phi_0(x') + \phi_n(z, x)$$

(3.6)

where $G_{LAdS}$ is the Lorentzian AdS bulk-to-boundary Green function. An equation of a similar form can be found in eg., [5]. The parallel with (3.2) is evident.

### 4 Bulk Reconstruction

When the non-normalizable mode is turned off, in Lorentzian AdS, we have a direct map between the bulk field $\phi(z, x)$ and the boundary state dual to $\phi_n(x)$. Because of the CFT state-operator correspondence, this means that in AdS this can also be viewed as a map between bulk field and boundary operator. The question of bulk reconstruction is the question of reconstructing a bulk field given this boundary operator. To do this, we use an object called the HKLL smearing kernel [8]. We will be able to define a similar object in our more general context as well.

But before we describe this, we will briefly note a conceptual issue. In a CFT, we have the canonical state-operator correspondence, which gives a map between local operators and states. Here on the other hand, we do not even know that the holographic theory is local, let alone that it is a CFT. But nonetheless, we expect that the theory has local operators (the $O(x)$ in our notation) because we are able to define them via holography. We further know that homogeneous bulk fields $\phi_h(r, x)$ lead to boundary fields $\phi_h(R, x)$. As far as the structure of PDEs go, this data is precisely enough to write down an HKLL kernel with which we can do bulk reconstruction in analogy with AdS. For these reasons, we find it

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14 In fact, it is straightforward to build a precise parallel like we did for Euclidean sources earlier.

15 It is important to note that the reconstructed operator is to be understood as acting within correlation functions, so reconstruction at the level of operators should be taken with a grain of salt.

16 Of course their correlators need not be that of a local quantum field theory, which is indeed what is generally expected for the hologram of flat space. Let us emphasize that a non-local theory can have local operators.
plausible that there exists a class of boundary operators that are naturally associated to the fluctuations of a given semi-classical bulk background. These operators are what we will use to reconstruct bulk fields.

With this understanding, it is straightforward to adapt the approach of [8] to find our smearing kernels. We do this by using a version of Green’s theorem to express the bulk field $\phi(r, x) = \lim_{r' \to R} \int_{\partial M} d^d x' \sqrt{\gamma(r', x')} \left( \phi_h(r', x') \partial_{r'} G_S(r, x; r', x') - G_S(r, x; r', x') \partial_{r'} \phi_h(r', x') \right)$

$$= \int_{\partial M} d^d x' \sqrt{\gamma(R, x')} K(r, x; R, x') \phi_h(R, x').$$

Here in the first line, $G_S$ is the spacelike bulk-to-bulk Green function. The first line is an identity, a version of Green’s theorem. The smearing kernel makes its appearance in the second line, where we read it off based on our knowledge of the first line, and $\phi_h(R, x')$. For a second order PDE, $\phi_h(r', x')$ and $\partial_{r'} \phi_h(r', x')$ are generically independent data at $r'$, but this is where the fact that we are working with homogeneous modes plays a key role. This knowledge enables us to express the latter in terms of the former, which is necessary for the definition in the last line to make sense.

The kernels constructed this way can have subtleties in coordinate space due to divergences (see eg., the discussion in section 4 of [8]) and non-uniqueness, which lead one to interpret them as distributions (see discussion and references in eg., [12]). In practise, this means that suitably defined reconstruction kernels of the form $K(r, x; R, \ell)$ are often better defined as functions, where $\ell$ stands for the Fourier space of the boundary coordinates $x'$, see eg., [13] for an AdS discussion.

Let us also note that the mode expansion of the homogeneous solution gives us an alternative approach to constructing a bulk reconstruction kernel. We will illustrate this in the next section for flat space. It will be interesting to see if these two constructions of the smearing kernel yield identical objects in flat space [14].

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17Note that if this were not so, it would be puzzling why the bulk theory has a structure isomorphic to the one found in AdS.

18The recipe for obtaining the spacelike Green function (when it exists) is as follows. First we write down the most general Euclidean Green function. This object contains two independent constants because our differential equation is second order. We fix the first constant by demanding the correct behavior near the delta-function source. In the conventional Euclidean Green function, the second constant is fixed by demanding that the Green function vanishes at spatial infinity. If we analytically continue this object, we get the Feynman Green function as we alluded to previously. To obtain the spacelike Green’s function, we do not demand vanishing at Euclidean infinity. Instead we analytically continue, and then demand that the Green function vanishes at non-spacelike separation, which fixes the second constant and uniquely fixes the Green function as the spacelike Green function. This approach was developed in [8, 11].

19This is fairly easy to verify in AdS, because asymptotically the radial dependence factorizes.
5 Adding Interactions

Adding bulk interactions and perturbatively determining higher point functions in Euclidean space follows an isomorphic picture to what happens in AdS. Witten diagrams are adapted trivially. Witten diagrams involve bulk-to-bulk and bulk-to-boundary correlators in AdS: the former remain unchanged here, and the latter follow our discussion in a previous section relating our source/correlator prescription to the conventional AdS source/correlator prescription. Higher point boundary correlators follow. See [14] for details.

In the Lorentzian case, a similar statement can be made for bulk reconstruction as well. When the bulk theory has interactions, one can modify the bulk reconstruction procedure parallel to what is done in AdS. A key equation as far as we are concerned is, say eqn. (3.14) (see also eg., eqn. (3.15)) in [12]. Note that this equation is an identity and holds equally well in our case also. It follows immediately that we can correct the bulk reconstruction procedure order by order in the bulk coupling, in a systematic way.

These remarks demonstrate that it is straightforward to add perturbative bulk interactions in computations of correlators as well as in bulk reconstruction in our approach.

6 Example: Flat Space with $\mathbb{R} \times S^2$ Screen

The above discussion has largely been quite abstract, so let us present a concrete example where explicit calculations are possible. A simple and potentially highly interesting example is the case of Minkowski space, $M_{3+1}$. See (3.1) for the Lorentzian version of the metric, the Euclidean version has positive sign for the time part. The boundary is at radius $r = R$. In this letter, we will present a few simple results, details and more complete results can be found in the companion paper [14].

Because of the homogeneity of flat space the Euclidean, Lorentzian and Spacelike massive scalar Green functions can be written in terms of geodesic lengths [8, 11]

\[
G_E(\sqrt{\sigma}) = \frac{m}{(2\pi)^2} \frac{K_1(m\sqrt{\sigma})}{\sqrt{\sigma}}, \quad G_S(\sqrt{\sigma}) = \frac{m}{(2\pi)^2} \frac{\pi I_1(m\sqrt{\sigma})}{2\sqrt{\sigma}}
\]

where $\sigma$ is the (Euclidean/Lorentzian) geodesic length. Explicit expressions for these geodesic lengths in terms of coordinates are straightforward [14]. $K$ and $I$ are the modified Bessel functions of the first and second kind respectively. The Lorentzian (Feynman) Green function is related to the Euclidean Green function via $G_L(\sqrt{\sigma}) = iG_E(\sqrt{\sigma} + i\epsilon)$. When treating these as bulk-to-boundary propagators, we treat one of the terminal points of the geodesic segment to be at the boundary $(R, x')$. Using these it is easy to write explicit expressions for the holographic correlators and the smearing kernels, from the general formulas we have
discussed in the previous sections. The boundary correlators follow straightforwardly from (2.4) and (3.4) using the $G_E$ and $G_L$ above. The explicit formulas are messy but perfectly well-defined. We will only present the Euclidean case for the dual operators here:

$$\langle O(t', \Omega', R) O(t'', \Omega'', R) \rangle =$$

$$= -\frac{m^2}{4\pi^4} \int dt d\Omega \ m R^3 \left( \Xi' \frac{K_1(m\sqrt{\Lambda'}) K_2(m\sqrt{\Lambda'})}{\Lambda' \sqrt{\Lambda''}} + (\Lambda'', \Xi'' \leftrightarrow \Lambda', \Xi') \right)$$

To specify the notation, let us note that $d\Omega$ contains the $\sin \theta$. We also have

$$\Lambda'' = 2R^2 + (t - t'')^2 - 2R^2 (\cos \theta \cos \theta'' + \sin \theta \sin \theta'' \cos(\phi - \phi''))$$

$$\Lambda' = 2R^2 + (t - t')^2 - 2R^2 (\cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi'))$$

$$\Xi' = -1 + \cos \theta \cos \theta' + \sin \theta \sin \theta' \cos(\phi - \phi')$$

$$\Xi'' = -1 + \cos \theta \cos \theta'' + \sin \theta \sin \theta'' \cos(\phi - \phi'')$$

The Lorentzian expression is morally similar, see [14].

Using the mode expansion of the homogeneous solution

$$\phi_h(r, t, \Omega) = \sum_{l,n} \int_{\omega > m} \frac{d\omega}{2\pi} e^{-i\omega t} a_{l,n}(\omega) \frac{J_\nu(r \sqrt{\omega^2 - m^2})}{\sqrt{r}} Y_{l,n}(\Omega) + h.c.,$$

an HKLL-like smearing function in momentum space is easy to read off as

$$\tilde{K}(r, t, \Omega; R, \omega, l, n) = \frac{1}{R^2} \sqrt{R} J_\nu(r \sqrt{\omega^2 - m^2}) e^{-i\omega t} Y_{l,n}(\Omega)$$

where $\nu = \frac{1}{2} + l$. The reconstruction works via

$$\phi_h(r, t, \Omega) = \sum_{l,n} \int_{\omega > m} \frac{d\omega}{2\pi} K(r, t, \Omega; R, \omega, l, n) \tilde{\phi}_h(R, \omega, l, n) + h.c.\quad (6.5)$$

where $\tilde{\phi}_h(R, \omega, l, n) \equiv a_{l,n}(\omega) \frac{J_\nu(r \sqrt{\omega^2 - m^2})}{\sqrt{r}}$. The pre-factor $\frac{1}{R^2}$ in (6.4) arises because $\sqrt{\gamma(R, x')} = R^2 \times \sin \theta$.

We have been fairly telegraphic in this section and only presented (some of) the basic results, a systematic presentation and discussion of some physics will be given in the companion paper [14]. The only point we wish to make here is that the formalism allows explicit calculations.
7 Comments and Future Directions

We presented a general prescription for defining semi-classical holography in a fairly large class of spacetime regions.

In Euclidean geometries our discussion was quite general, but is somewhat different from some previous efforts. For example, holographic correlators in Euclidean flat space with the metric $d s_{d+1}^2 = d\rho^2 + \rho^2 d\Omega_d^2$ have been discussed in [15, 16], where a Dirichlet boundary condition was imposed at a boundary cut-off, and the boundary value was taken as the source for the dual theory. Note that this is distinct from our prescription: the key difference at the level of Green functions is that the prescription of [15, 16] uses the Dirichlet Green function (this object depends on the screen, and is defined to vanish there), while ours uses the Euclidean Green function that vanishes at spatial infinity whose analytic continuation leads to the Feynman Green function in the Lorentzian signature.

These two approaches can be viewed as two separate ways to generalize the usual holographic correspondence in AdS, to other geometries. In (Euclidean) AdS, the boundary value of the bulk field and the boundary limit of a bulk source are essentially the same object. This coincidence of two logically distinct objects in AdS means that there are two possible paths to generalize the holographic prescription to other geometries. The usual philosophy, whose concrete realization can be found in [15, 16], adopts the stance that the boundary value of the bulk field is still the object that should be viewed as the source for the dual theory, even when we are not in AdS. We have instead taken the perspective that the bulk source localized at the boundary/screen is what should be interpreted as the dual source. These two perspectives are largely indistinguishable in AdS because the $z$-dependence of the solution factorizes out near the AdS boundary.

Our prescription has a few features we find attractive:

- The Euclidean Green function we work with is not defined via a screen-dependent condition, it is a property of the theory. The Dirichlet Green function on the other hand is defined via a vanishing condition at $r = R$. Note that in AdS this issue is ameliorated because there is a canonical boundary at $z = 0$.

- Our Green function when Wick rotated to Lorentzian signature, ends up being the Feynman Green function, which is what we would like as the standard Lorentzian bulk

\[^{20}\text{As we have been careful to emphasize, they are not exactly the same object: the boundary value of the bulk field is the dual source, after multiplication by } z^{\Delta-d}. \text{ A closely related fact is that the bulk-to-boundary Green function in AdS is not quite the bulk-to-bulk propagator with one point at the boundary, but its derivative in the holographic direction (see eg., [14]). This is expected in a Dirichlet Green function [15].}\]

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propagator. The spacelike Green function that we use for bulk reconstruction is also very closely related to the Euclidean Green function, as we explained earlier.

- Our prescription has a close parallel to AdS even in the Lorentzian signature. In particular, the homogeneous mode that we find is a very natural analogue of the normalizable mode in AdS.

- The homogeneous mode allows us to do HKLL-like bulk reconstruction.

- More philosophically, holographic duality implies that there is only one underlying theory, so it is perhaps natural that the source for the boundary theory also has an interpretation as the source for the bulk theory.

There are many directions here worth developing, some of which will be presented elsewhere. These observations suggest that it is worth re-thinking mechanics and field theory in bounded regions along the lines suggested in this paper. In particular, instead of boundary terms (for well-defined variational principles etc.), it might be instructive to consider an auxiliary system with sources localized at the “boundary” in an otherwise unbounded space(time). In other words, instead of the standard particle mechanics problem with “Dirichlet” boundary conditions at the end points,

\[ S_p^{\text{D}} = \int_{t_1}^{t_2} dt \left( \frac{1}{2} \dot{q}^2 - V(q) \right), \]  

it may be interesting to consider something like

\[ S_p^{\text{aux}}[J(t_1), J(t_2)] = \int_{-\infty}^{+\infty} dt \left( \frac{1}{2} \dot{q}^2 - V(q) \right) + J(t_2)q(t_2) - J(t_1)q(t_1). \]  

Natural generalizations of this to field theory or gravity, clearly exist. Various boundary related themes of a similar flavor have recently been investigated in the context of holography [17]. A closely related question is whether these considerations can be further adapted to say something useful about cosmological backgrounds.

Let us close by presenting a highly incomplete list of open questions, some more accessible than others:

- We have only investigated scalar fields in this paper. But clearly, similar constructions must exist for gauge fields and gravitons as well, both when it comes to holographic correlator calculations, as well as for bulk reconstruction [18].

- Can we relate the flat space boundary correlators to S-matrix elements in flat space [19]? Our discussion was in coordinate space. What about correlators in momentum space or Mellin space?
• What is the significance of the state-operator correspondence for holography? We feel that this question has not been investigated with enough gravitas.

• AdS/CFT emerged in the decoupling limit of the brane and the bulk, and this is related to the conformal invariance of the duality. In our general holographic setting, there is no decoupling and there is no scale independence. So to what extent the on-screen-correlators can capture the bulk dynamics beyond the semi-classical limit is a worthy, but possibly difficult problem.

• It might be interesting introduce a weak dependence on $r$ in our sources. When the dependence becomes delta-function localized on $r = R$ it will reduce to the discussion in this paper. The introduction of $r$-dependence might be a useful way to incorporate some aspects of the previous bullet point.

• In a theory with dynamical gravity, fixing a location for the screen in some coordinates is a fishy business. Yet, this is what we have done in this paper. We believe nonetheless that our conclusions still capture some physics, certainly in scenarios where the background is approximately fixed near the screen, but also possibly in somewhat more general situations as long as the physics is semi-classical\textsuperscript{21}. But it will be useful to address these questions from a technically more solid footing than what we have attempted here.

• We have not discussed black holes at all, but it seems possible that many of the statements about black holes in AdS/CFT can be adapted here. It will be instructive to clarify the precise sense in which there are differences.

• We have also not discussed cosmology, but it is clear that correlators on spatial slices built analogously to what we have done in this paper for timelike slices, will be useful for dealing with cosmological holography. This is in analogy with the dS/CFT correspondence [20]. But perhaps more significantly, the formulation in terms of sources raises the possibility that various types of boundaries can be dealt with fairly democratically.

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\textsuperscript{21}Our results have a certain robustness to them: our prescription is based largely only on the (fairly rigid) structure of PDEs and their solutions.
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