Some generalized fractional integral Simpson’s type inequalities with applications

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Abstract: In the article, we establish a Simpson-type generalized identity containing multi-parameters and derive some new estimates for the generalized Simpson’s quadrature rule via the Raina fractional integrals. As applications, we provide several inequalities for the \( f \)-divergence measures and probability density functions.

Keywords: Simpson’s inequality; cumulative distribution function; fractional integral; \( f \)-divergence measure

Mathematics Subject Classification: 26D15, 26A51

1. Introduction

Over few years, the fractional calculus has attracted the attention of many researchers due to its has wide applications in pure and applied mathematics [1–7]. Like ordinary calculus, the fractional integral and derivative have not unique representation, with the passage of time, different authors have different representations. It is well-known that inequality is an indispensable research object in mathematics, it can give explicit error bounds for some known and some new quadrature formulae, for example, the Simpson’s inequality [8], Jensen’s inequality [9, 10], Hermite-Hadamard’s inequality [11–15] and integral inequalities [16–21]. The following inequality is well known as Simpson’s inequality which provides an error bound for the Simpson’s rule.
**Theorem 1.1.** (See [7]) Let \(a, b \in \mathbb{R}\) with \(a < b\), and \(f : [a, b] \to \mathbb{R}\) be a four times differentiable function on \((a, b)\) such that \(\|f^{(4)}\|_{\infty} = \sup_{x \in (a,b)} |f^{(4)}(x)| < \infty\). Then the inequality
\[
\left| \frac{1}{3} \left[ f(a) + f(b) \right] + \frac{2}{3} f\left( \frac{a + b}{2} \right) \right| - \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq \frac{\|f^{(4)}\|_{\infty} (b - a)^{4}}{2880}
\]
holds.

In [22], Dragomir et al. improved Theorem 1.1 to the following Theorem 1.2.

**Theorem 1.1.** (See [22]) Let \(a, b \in \mathbb{R}\) with \(a < b\), and \(f : [a, b] \to \mathbb{R}\) be a differentiable function on \((a, b)\) such that its derivative is continuous on \((a, b)\) and \(\|f''\|_{1} = \int_{a}^{b} |f''(x)| dx < \infty\). Then the inequality
\[
\left| \frac{1}{3} \left[ f(a) + f(b) \right] + \frac{2}{3} f\left( \frac{a + b}{2} \right) \right| - \frac{1}{b - a} \int_{a}^{b} f(x) dx \leq \frac{b - a}{3} \|f''\|_{1}
\]
holds.

Recently, the Simpson type inequalities have been the subject of intensive research since many important inequalities can be obtained from the Simpson inequality. The main purpose of the article is to provide several generalized fractional integral versions of the Simpson’s inequality and give their applications in \(f\)-divergence measures and probability density functions.

**2. Preliminaries and assumptions**

**Definition 2.1.** (See [23]) Let \(p \in \mathbb{R}\) with \(p \neq 0\) and \(I \subseteq (0, \infty)\) be an interval. Then the real-valued function \(f : I \to \mathbb{R}\) is said to be \(p\)-convex (concave) if the inequality
\[
f(t\sqrt[p]{x^{p} + (1 - t)y^{p}}) \leq (\geq) tf(x) + (1 - t)f(y)
\]
holds for all \(x, y \in I\) and \(t \in [0, 1]\). Moreover, if \(s \in (0, 1)\), then the function \(f\) is said to be \((s, p)\)-convex (concave) if the inequality
\[
f(\sqrt[s]{tx^{p} + (1 - t)y^{p}}) \leq (\geq) t^{s} f(x) + (1 - t)^{s} f(y)
\]
holds for all \(x, y \in I\) and \(t \in [0, 1]\).

Let \(p > 0\), \(s \in (0, 1)\) and \(f : (0, \infty) \to (0, \infty)\) be defined by \(f(x) = x^{sp}\). Then we clearly see that \(f\) is a \((s, p)\)-convex function.

**Definition 2.2.** (See [24]) Let \(\alpha > 0\), \([a, b] \subseteq \mathbb{R}\) be a finite interval and \(f : [a, b] \to \mathbb{R}\) be a real-valued function such that \(f \in L[a,b]\). Then the right-hand side and the left-hand side Riemann-Liouville fractional integrals \(J_{a}^{\alpha} f\) and \(J_{b}^{\alpha} f\) of order \(\alpha\) are defined by
\[
(J_{a}^{\alpha} f)(x) = \frac{1}{\Gamma(\alpha)} \int_{a}^{x} (x - t)^{\alpha - 1} f(t) dt \quad (x > a)
\]
and

\[\text{AIMS Mathematics}\]

Volume 5, Issue 6, 5859–5883.
\[(\mathcal{J}_b^a f)(x) = \frac{1}{\Gamma(a)} \int_x^{b} (t-x)^{a-1} f(t) dt \quad (x < b), \tag{2.2}\]

respectively.

**Definition 2.3.** The gamma function $\Gamma$, beta function $\mathbb{B}$ and the hypergeometric function $\,_{2}F_{1}$ are defined by

\[
\Gamma(x) = \int_0^{\infty} e^{-t} t^{x-1} dt \quad (x > 0),
\]

\[
\mathbb{B}(x, y) = \frac{\Gamma(x) \Gamma(y)}{\Gamma(x+y)} = \int_0^{1} t^{x-1}(1-t)^{y-1} dt \quad (x, y > 0)
\]

and

\[
\,_{2}F_{1}(a, b; c, z) = \frac{1}{\mathbb{B}(b, c-b)} \int_0^{1} t^{b-1}(1-t)^{c-b-1}(1-zt)^{-a} dt \quad (|z| < 1),
\]

respectively.

**Definition 2.4** (See [25]) The hypergeometric function $\,_{2}F_{1}$ can be given by

\[
\,_{2}F_{1}(a, [b, c]; y, x) = \frac{1}{\mathbb{B}(b, c-b)} \int_0^{y} t^{b-1}(1-t)^{c-b-1}(1-xt)^{-a} dt
\]

for $y < 1$ and $Re(c) > Re(b) > 0$.

**Definition 2.5** The incomplete beta function $\mathbb{B}(z; x, y)$ is defined by

\[
\mathbb{B}(z; x, y) = \int_0^{z} t^{x-1}(1-t)^{y-1} dt \quad (Re(x) > Re(y) > 0, 0 \leq z < 1).
\]

Raina [26] introduced a class of functions as follows

\[
\delta_{\rho, \sigma}^{\frac{\sigma'}{\sigma}}(x) = \sum_{k=0}^{\infty} \frac{\sigma(k)}{\Gamma(\rho k + \lambda)} x^k \quad (\rho, \lambda \in \mathbb{R}^+, \, x \in \mathbb{R}) \tag{2.3}
\]

where the coefficients $\sigma(k) \in \mathbb{R}^+$, $k \in \mathbb{N}_0$ form a bounded sequence. By using (2.3), in [26, 27], the authors defined the left-side and right-sided fractional integral operators

\[
(\mathcal{L}_{\rho, \sigma}^{\frac{\sigma'}{\sigma}} + w \phi)(x) = \int_a^{x} (x-t)^{\frac{\sigma'}{\sigma}-1} \delta_{\rho, \sigma}^{\frac{\sigma'}{\sigma}}[w(x-t)\phi(t)] dt \quad (x > a) \tag{2.4}
\]

and

\[
(\mathcal{L}_{\rho, \sigma}^{\frac{\sigma'}{\sigma}} - w \phi)(x) = \int_x^{b} (t-x)^{\frac{\sigma'}{\sigma}-1} \delta_{\rho, \sigma}^{\frac{\sigma'}{\sigma}}[w(t-x)\phi(t)] dt \quad (x < b), \tag{2.5}
\]

respectively, where $w \in \mathbb{R}$ and $\phi$ is a function such that the integrals on right hand sides exit. It is easy to verify that $\mathcal{L}_{\rho, \sigma}^{\frac{\sigma'}{\sigma}}(x)$ and $\mathcal{L}_{\rho, \sigma}^{\frac{\sigma'}{\sigma}}(\phi(x)$ are bounded integral operators on $L(a, b)$ if $\Re = \delta_{\rho, \sigma}^{\frac{\sigma'}{\sigma}}[w(b-a)\phi] < \infty$. In fact, if $\phi \in L(a, b)$, then we have

\[
\|\mathcal{L}_{\rho, \sigma}^{\frac{\sigma'}{\sigma}} + w \phi\|_1 \leq \Re(b-a)^1\|\phi\|_1, \quad \|\mathcal{L}_{\rho, \sigma}^{\frac{\sigma'}{\sigma}} - w \phi\|_1 \leq \Re(b-a)^1\|\phi\|_1.
\]
Before starting our main results, we introduce some notations as follows.

\[ G_1(a, b; \xi, \epsilon, \gamma) = 2F_1 \left( \frac{\xi p - \xi}{p}, \beta + \xi pk + \epsilon + 1; \xi \beta + \xi pk + \gamma + 2; \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right), \]

\[ G_2(a, b; \xi, \epsilon, \gamma) = 2F_1 \left( \frac{\xi p - \xi}{p}, \xi \beta + \xi pk + \epsilon + 1, \xi \beta + \xi pk + \gamma + 2; \frac{1}{2}; \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right), \]

\[ H_{1, \lambda}(a, b; \pi, \sigma) = 2F_1 \left( \frac{p - 1}{p}, \pi, \sigma, \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right), \]

\[ H_{2, \lambda}(a, b; \pi, \sigma) = 2F_1 \left( \frac{p - 1}{p}, \pi, \sigma; \frac{1}{2}; \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right), \]

\[ \sigma_1 = \sigma(k) \left( (\lambda f'(a)) + (1 - \lambda) f'(b) \right) \mathbb{B}(\beta + \rho k + 1, s + 1) G_1(a, b; 1, s, 0) + \lambda G_1(a, b; 1, s, 0) \]

\[ \sigma_2 = \left\{ G_2(a, b; x, 0, 0) \right\} \left\{ (2^{s+1} - 1) f'(b)^\lambda + (2^{s+1} - 1) f'(a) \right\} \]

\[ \sigma_3 = \sigma(k) \mathbb{B}(\beta + \rho k + 1, 1) \left[ (2^{s+1} + 1, s + 1) G_2(a, b; 1, 0, 0) \right] \left[ (2^{s+1} + 1, s + 1) G_2(a, b; 1, 0, 0) \right] \]

3. Main results

To establish our results for generalized Simpson’s type inequality using \((s, p)\)-convex function, we need the following lemma.

**Lemma 3.1.** Let \( I \subseteq \mathbb{R}^+ \) be an interval, \( I^\circ \) be the interval of \( I, a, b \in I \) with \( a < b, \rho, \beta > 0, p \in \mathbb{R} \) with \( p \neq 0, \) and \( g(\xi) = \sqrt[4p]{\xi} \) for \( \xi > 0. \) Then the identity

\[ \psi(t, a, b; f) = p \left[ \lambda(b^p - a^p) \right]^\rho \left\{ 2^{-\rho} \delta_{p, \beta+1} \left[ \frac{\lambda(b^p - a^p)}{4} \right]^{\rho} - u \right\} \times f \left( \sqrt[4p]{\frac{\lambda a^p + (2 - \lambda)b^p}{2}} + uf \left( \sqrt[4p]{\lambda a^p + (1 - \lambda)b^p} \right) - p \left[ \frac{\lambda a^p + (2 - \lambda)b^p}{2} \right]^{\rho} \right) \]
Proof. Integrating by parts leads to

\[
\times(\lambda a^p + (1 - \lambda)b^p) + p [\lambda(b^p - a^p)]^\beta \left\{ \left[ \frac{\lambda}{2} \right] - v \right\} f(b) - \left\{ 2^{2-\beta} \Delta_{\beta, p+1}^\beta \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right) \right] - v \right\} f(b) \]

holds for \( u, w \in \mathbb{R} \) and \( \lambda \in [0, 1] \).

Proof. Integrating by parts leads to

\[
I_1 = \int_0^1 \left\{ \left[ \frac{\lambda}{2} \right] - v \right\} \left[ (1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right] \frac{1}{\sqrt{r}}
\]

\[ \times f' \left( \sqrt{(1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p} \right) dt \]

\[ = \left[ \int_0^1 \left\{ \frac{\lambda}{2} - v \right\} \left[ (1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right] \frac{1}{\sqrt{r}} \right] f' \left( \sqrt{(1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p} \right) \left|_0^1 \right.
\]

\[ - \int_0^1 \left\{ \frac{\lambda}{2} - v \right\} \left[ (1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right] \frac{1}{\sqrt{r}} \right] f' \left( \sqrt{(1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p} \right) dt
\]

Let \( x = (1-t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \). Then \( dx = [\lambda(b^p - a^p)] dt \), \( 0 \leq t \leq 1/2 \) is equivalent to \( (\lambda a^p + (1 - \lambda)b^p) \leq x \leq \frac{\lambda a^p + (1 - \lambda)b^p}{2} \). Therefore, we get

\[
I_1 = \frac{p}{\lambda(b^p - a^p)} \left\{ 2^{2-\beta} \Delta_{\beta, p+1}^\beta \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right) \right] - v \right\} f(b) + \left( \sqrt{\lambda a^p + (1 - \lambda)b^p} \right) \left|_0^1 \right.
\]

\[ + uf' \left( \sqrt{(\lambda a^p + (1 - \lambda)b^p)} \right) - \frac{p}{[\lambda(b^p - a^p)]^{1+\beta}} \int_{\lambda a^p + (1 - \lambda)b^p}^{[\lambda a^p + (1 - \lambda)b^p]} \left[ x - (\lambda a^p + (1 - \lambda)b^p) \right]^{\beta-1}
\]

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Again integrating by parts gives

\( \lambda \left( b^p - a^p \right)^{1+\beta} \frac{1}{p} I_1 = \left[ \lambda \left( b^p - a^p \right)^{\beta} \left( \frac{x - \left( \lambda a^p + (1 - \lambda) b^p \right)}{2} \right)^p \right] \left( f \circ g \right)(x)dx \)

and

\[ \frac{\lambda \left( b^p - a^p \right)^{1+\beta}}{p} \int_1 \frac{\sqrt{\lambda a^p + (2 - \lambda) b^p}}{2} + uf \left( \sqrt{\lambda a^p + (1 - \lambda) b^p} \right) - \left( \frac{\lambda a^p + (1 - \lambda) b^p)}{2} \right) \left( f \circ g \right)(\lambda a^p + (1 - \lambda) b^p). \] (3.2)

Again integrating by parts gives

\[ I_2 = \int_1^{\frac{1}{2}} \left\{ 2^\beta \delta^\rho_{\beta+1} \left[ w \left( \frac{\lambda a^p + (1 - \lambda) b^p}{2} \right)^p \right] - v \right\} \left[ (1 - t)(\lambda a^p + (1 - \lambda) b^p) + tb^p \right]^{1+p} \]

\[ \times f' \left( \sqrt{1 - t}(\lambda a^p + (1 - \lambda) b^p) + tb^p \right) dt \]

\[ = \left[ \frac{p^\beta \delta^\rho_{\beta+1} \left[ w \left( \frac{\lambda a^p + (1 - \lambda) b^p}{2} \right)^p \right]}{\lambda(b^p - a^p)} \right] - v \right\} \left( \sqrt{1 - t}(\lambda a^p + (1 - \lambda) b^p) + tb^p \right)^{1+p} \]

\[ - \int_1^{\frac{1}{2}} \frac{p^\beta \delta^\rho_{\beta+1} \left[ w \left( \frac{\lambda a^p + (1 - \lambda) b^p}{2} \right)^p \right]}{\lambda(b^p - a^p)} \left( \sqrt{1 - t}(\lambda a^p + (1 - \lambda) b^p) + tb^p \right) dt \]

Let \( y = (1 - t)(\lambda a^p + (1 - \lambda) b^p) + tb^p \). Then one has

\[ I_2 = \left[ \frac{p^\beta \delta^\rho_{\beta+1} \left[ w \left( \frac{\lambda a^p + (1 - \lambda) b^p}{2} \right)^p \right]}{\lambda(b^p - a^p)} \right] - v \right\} \left( \sqrt{1 - t}(\lambda a^p + (1 - \lambda) b^p) + tb^p \right)^{1+p} \]

\[ - \int_1^{\frac{1}{2}} \frac{p^\beta \delta^\rho_{\beta+1} \left[ w \left( \frac{\lambda a^p + (1 - \lambda) b^p}{2} \right)^p \right]}{\lambda(b^p - a^p)} \left( \sqrt{1 - t}(\lambda a^p + (1 - \lambda) b^p) + tb^p \right) dt. \]
\[
\frac{p}{\lambda(b^p - a^p)} \left[ \left\{ \psi_{\rho,\beta+1}^{\sigma} \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right)^{p} \right] - v \right\} f(b) \right. \\
\left. - \left\{ 2^{-\beta/\rho} \psi_{\rho,\beta+1}^{\sigma} \left[ w \left( \frac{\lambda(b^p - a^p)}{4} \right)^{p} \right] - v \right\} f \left( \sqrt{\frac{\lambda a^p + (2 - \lambda)b^p}{2}} \right) \right]
\]
\[
- \int_{\lambda a^p+(1-\lambda)b^p}^{\lambda a^p+(1-\lambda)b^p} \left[ y - (\lambda a^p + (1 - \lambda)b^p)^{\beta-1} \right] \\
\times \psi_{\rho,\beta}^{\sigma} \left[ w \left( \frac{y - (\lambda a^p + (1 - \lambda)b^p)}{2} \right)^{p} \right] (f \circ g)(y) dy
\]
\[
\text{and}
\]
\[
\left[ \frac{[\lambda(b^p - a^p)]^{1+\beta}}{p} \right] I_2 = \left[ \frac{[\lambda(b^p - a^p)]^{1+\beta}}{p} \right] \left\{ \left\{ \psi_{\rho,\beta+1}^{\sigma} \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right)^{p} \right] - v \right\} f(b) \right. \\
\left. - \left\{ 2^{-\beta/\rho} \psi_{\rho,\beta+1}^{\sigma} \left[ w \left( \frac{\lambda(b^p - a^p)}{4} \right)^{p} \right] - v \right\} f \left( \sqrt{\frac{\lambda a^p + (2 - \lambda)b^p}{2}} \right) \right]
\]
\[
- \left\{ \left( \psi_{\rho,\beta}^{\sigma} \right) \left( \frac{\lambda a^p + (1 - \lambda)b^p - \psi_{\rho,\beta}^{\sigma} \left[ \frac{\lambda a^p + (1 - \lambda)b^p}{2} \right]}{\sqrt{\frac{\lambda a^p + (2 - \lambda)b^p}{2}}} \right) f \circ g \right\} (\lambda a^p + (1 - \lambda)b^p). \tag{3.3}
\]

Therefore, the desired inequality (3.1) can be obtained by adding (3.2) and (3.3). \hfill \square

Remark 3.1. From Lemma 3.1 we clearly see that

1. Lemma 3.1 reduces to Lemma 1 of [8] if \( \beta, \lambda, \sigma(0) \to 1, w \to 0, u \to \frac{1}{6} \) and \( v \to \frac{5}{6} \);
2. Lemma 3.1 leads to Lemma 2 of [8] if \( p, \beta, \lambda, \sigma(0) \to 1, w \to 0 \) and \( a \to am \);
3. Lemma 3.1 becomes Lemma 2.1 of [28] if \( p, \beta, \lambda, \sigma(0) \to 1, w \to 0; u \to \frac{1}{6}, v \to \frac{5}{6} \) and \( a \to am \).
4. Lemma 3.1 degenerates into Lemma 3 of [8] if \( \lambda, \sigma(0) \to 1, w \to 0 \) and \( p > 0 \).

Theorem 3.1. Let \( I \subseteq \mathbb{R}^+ \) be an interval and \( I^c \) be the interior of \( I, a, b \in I^c \) with \( a < b, \rho, \beta > 0, u, w \in \mathbb{R}, (s, \lambda) \in (0, 1] \times [0, 1], g(\xi) = \sqrt{\frac{\xi}{\beta}} \) for \( \xi > 0 \), and \( f : I \to \mathbb{R} \) be a differentiable function on \( I^c \) such that \( |f'| \) is \((s, p)\)-convex. Then one has

\[
|\psi(t, a; f)| \leq \frac{[\lambda(b^p - a^p)]^{1+\beta}}{(s+1)} \left\{ \left\{ \psi_{\rho,\beta+1}^{\sigma} \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right)^{p} \right] - v \right\} f(b) \right. \\
\left. + \frac{(\lambda[f'(a)] + (1 - \lambda)f'(b))(|u| - |v|)H_{2,1}(a, b; 1, s + 2) + |v|H_{1,1}(a, b; 1, s + 2)}{s+1} \right\}.
\tag{3.4}
\]

Proof. It follows from (3.1) and the \((s, p)\)-convexity of \(|f'|\) that

\[
|\psi(t, a; f)| \leq [\lambda(b^p - a^p)]^{1+\beta}[|I_1| + |I_2|], \tag{3.5}
\]

\[
|I_1| = \int_0^1 \left\{ t^{\beta/\rho} \psi_{\rho,\beta+1}^{\sigma} \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right)^{p} \right] - u \right\} [(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p]^{1-\rho}.
\]

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\[
\times f'(\left((1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p\right)^{\frac{1}{p}}) \left((1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p\right)^{\frac{1}{p}}
\]

\[
\leq \int_0^{\frac{1}{p}} \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(a^p-b^p)}{2}\right)^{\rho k}}{\Gamma(\rho k + \beta + 1)} \right\}
\times f'(\left((1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p\right)^{\frac{1}{p}}) \left((1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p\right)^{\frac{1}{p}} dt
\]

\[
= \frac{\sigma(k)|w|^k \left(\frac{\lambda(a^p-b^p)}{2}\right)^{\rho k}}{\Gamma(\rho k + \beta + 1)}
\times (\sqrt[\rho k + \beta]{\lambda a^p + (1-\lambda)b^p})^{1-p} \left[ 1 - t \frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p} \right]^{\frac{1}{\rho k}}
\times (1-t)^s \left(\frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p}\right)^{\frac{1}{\rho k}}
\times |f'(a)| + (1-\lambda)|f'(b)|
\times \Gamma(\rho k + \beta + 1)
\]

\[
\times (\sqrt[\rho k + \beta]{\lambda a^p + (1-\lambda)b^p})^{1-p} \left| (\lambda a^p + (1-\lambda)b^p) \right|^{\frac{1}{\rho k}}
\times (1-t)^s dt
\]

\[
+ |f'(b)| \int_0^{\frac{1}{p}} \left[ 1 - t \frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p} \right]^{\frac{1}{\rho k}} dt
\]

\[
+ |u| \left(\sqrt[\rho k + \beta]{\lambda a^p + (1-\lambda)b^p}\right)^{1-p} \left(\frac{\lambda a^p + (1-\lambda)b^p}{\lambda a^p + (1-\lambda)b^p}\right)^{\frac{1}{\rho k}}
\times (1-t)^s dt
\]

\[
= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left(\frac{\lambda(a^p-b^p)}{2}\right)^{\rho k}}{\Gamma(\rho k + \beta + 1)}
\times (1-t)^s \left(\frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p}\right)^{\frac{1}{\rho k}}
\times _2F_1\left(\frac{D-1}{p}, \beta + \rho k + 1, 1 + s + \beta + \rho k + 2; \frac{1}{2}, \frac{\lambda(a^p-b^p)}{\lambda a^p + (1-\lambda)b^p}\right)
\]
\[ +2F_1\left(\frac{p-1}{p}, \left[\beta + \rho k + s + 1, \beta + \rho k + s + 2; \frac{1}{2}\right], \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}\right) \]

\[ \times |f'(b)| \mathbb{E}(\beta + \rho k + s + 1, 1) + |u_1(\sqrt[p]{\lambda a^p + (1 - \lambda)b^p})^{1-p} \]

\[ \times [(\lambda f'(a)) + (1 - \lambda)|f'(b)|] \mathbb{E}(1, s + 1) \]

\[ \times_2 F_1\left(\frac{p-1}{p}, \left[1, s + 2; \frac{1}{2}\right], \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}\right) \]

\[ + |f'(b)| \mathbb{E}(s + 1, 1) F_1\left(\frac{p-1}{p}, \left[s + 1, s + 2; \frac{1}{2}\right], \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}\right) \]

(3.6)

and

\[ |I_2| = \left| \int_\frac{1}{2}^1 \left\{ P^\rho,\beta,\gamma_1 \left[ w \left(\frac{\lambda(b^p - a^p)}{2}\right)^{\rho} \right] - v \right\} \left[(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right]^{\frac{1-p}{\gamma}} \]

\[ \times f' \left(\left((1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right)^\frac{1}{\gamma}\right) \right| dt \]

\[ \leq \int_\frac{1}{2}^1 \left\{ P^\rho,\beta,\gamma_1 \left[ w \left(\frac{\lambda(b^p - a^p)}{2}\right)^{\rho} \right] + |v| \right\} \left[(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right]^{\frac{1-p}{\gamma}} \]

\[ \times f' \left(\left((1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right)^\frac{1}{\gamma}\right) \right| dt \]

\[ = \int_\frac{1}{2}^1 \left\{ \sum_{k=0}^{\infty} \sigma(k)|w|^k \left(\frac{\lambda(b^p - a^p)}{2}\right)^{\rho k} \right\} \]

\[ \times \left|(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right|^{\frac{1-p}{\gamma}} \]

\[ \times f' \left(\left((1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right)^\frac{1}{\gamma}\right) \right| dt \]

\[ \leq \int_\frac{1}{2}^1 \left\{ \sum_{k=0}^{\infty} \sigma(k)|w|^k \left(\frac{\lambda(b^p - a^p)}{2}\right)^{\rho k} \right\} \]

\[ \times \left|(\sqrt[p]{\lambda a^p + (1 - \lambda)b^p})^{1-p} \left[1 - t\frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}\right]\right|^{\frac{1-p}{\gamma}} \]

\[ \times [(1 - t)^t |\lambda f'(a)| + (1 - \lambda)|f'(b)| + t^1 |f'(b)|] \right| dt \]

\[ = \sum_{k=0}^{\infty} \sigma(k)|w|^k \left(\frac{\lambda(b^p - a^p)}{2}\right)^{\rho k} \left|(\sqrt[p]{\lambda a^p + (1 - \lambda)b^p})^{p-1} \Gamma(\rho k + \beta + 1)\right| \]

\[ + (1 - \lambda)|f'(b)| \int_\frac{1}{2}^1 b^{\rho + p k}(1 - t)^t \left[1 - t\frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}\right]\right|^{\frac{1-p}{\gamma}} \right| dt \]

\[ + |f'(b)| \int_\frac{1}{2}^1 b^{\rho + p k + s} \left[1 - t\frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}\right]\right|^{\frac{1-p}{\gamma}} \right| dt \]
Corollary 3.1. Let $I \subseteq \mathbb{R}^+$ be an interval and $I^o$ be the interior of $I$, $a, b \in I^o$ with $a < b$, $\beta > 0$, $u, w \in \mathbb{R}^+$, $s \in (0, 1)$, $p < 0$, $g(\xi) = \sqrt[k]{\xi}$ for $\xi > 0$, and $f : I \to \mathbb{R}$ be a differentiable function on $I^o$ such that $|f'|$ is $(s, p)$-convex. Then one has
Proof. Let $\beta, \lambda, \sigma(0) \to 1$ and $w = 0$. Then Corollary 3.1 follows directly from Theorem 3.1.

**Theorem 3.2.** Let $I \subseteq \mathbb{R}^+$ be an interval and $I^p$ be the interior of $I$, $a, b \in I^p$ with $a < b$, $\rho, \beta > 0$, $(s, \rho) \in (0, 1) \times [0, 1]$, $p \in \mathbb{R}$ with $p \neq 0$, $u, w, \in \mathbb{R}$, $y > 1$, $x = y/(y - 1)$, $g(\xi) = \sqrt[2]{\xi}$ for $\xi > 0$, and $f : I \to \mathbb{R}$ be a differentiable function on $I^p$ such that $|f'|^p$ is $(s, p)$-convex. Then the inequality

\[
|\psi(t, a, b; f)| \leq \frac{[\lambda(b^p - a^p)]^{1+\beta}}{(\lambda a^p + (1 - \lambda)b^p)^{\alpha(p-1)}} \left[ w \left( \frac{\lambda(b^p - a^p)}{2} \right)^p \right] \\
+ \left[ \frac{\lambda a^p + (1 - \lambda)b^p}{(\lambda a^p + (1 - \lambda)b^p)^{\alpha(p-1)}} \right]^{1/\alpha(p-1)} \left[ |d| \left[ \mathbb{E} \left( \frac{\lambda(b^p - a^p)}{2(\lambda a^p + (1 - \lambda)b^p)^p}; 1, x \frac{p - 1}{p} + 1 \right) \right]^{1/\alpha(p-1)} \\
\times \left\{ 2^{s+1} - 1 \right\} f'(b|^p) + \lambda(2^{s+1} - 1)|f'(a|^p) \right]^{1/\alpha(p-1)} \\
+ |v| \left[ \mathbb{E} \left( \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p}; 1, x \frac{p - 1}{p} + 1 \right) - \mathbb{E} \left( \frac{\lambda(a^p - b^p)}{2(\lambda a^p + (1 - \lambda)b^p)^p}; 1, x \frac{p - 1}{p} + 1 \right) \right]^{1/\alpha(p-1)} \\
\times \left\{ 2^{s+1} - 1 \right\} f'(b|^p) + \lambda|f'(a|^p) \right]^{1/\alpha(p-1)} \right]^{1/\alpha(p-1)} \\
(3.8)
\]

holds.

Proof. It follows from the $(s, p)$-convexity of $|f'|^p$ and Hölder inequality that

\[
|I| = \int_{0}^{1} \left\{ \mathbb{E} \left[ \mathbb{E} \left( \frac{\lambda(b^p - a^p)}{2} \right)^p \right] - u \right\} [(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p]^{1-p} \\
\times f' \left( \left[ (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right]^{1-p} \right) dt \\
\leq \int_{0}^{1} \left\{ \mathbb{E} \left[ \mathbb{E} \left( \frac{\lambda(b^p - a^p)}{2} \right)^p \right] + |u| \right\} [(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p]^{1-p} \\
\times f' \left( \left[ (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right]^{1-p} \right) dt
\]
\[
\begin{align*}
&= \int_0^1 \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p-a^p)}{2} \right)^{\rho k}}{\Gamma(\rho k + \beta + 1)} \rho^{\rho k + |u|} \right\} \\
&\quad \times [(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p]^{\frac{1-p}{p-1}} \\
&\quad \times \left| f' \left( \left(1 - t\left( \sqrt[\lambda a^p + (1 - \lambda)b^p\rho^p + tb^p\right)\right)^{\frac{1}{p}} \right) \right| \, dt \\
&\quad = \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p-a^p)}{2} \right)^{\rho k}}{\Gamma(\rho k + \beta + 1)} \\
&\quad \times \left\{ \int_0^{\frac{1}{2}} t^{\lambda(p+\rho k)} \left[(1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p\right]^{\frac{1-\rho p}{p-1}} \, dt \right\} \\
&\quad \times \left\{ \int_0^{\frac{1}{2}} \left[ f'(1 - t) \left( \sqrt[\lambda a^p + (1 - \lambda)b^p\rho^p + tb^p\right)^{\frac{1}{p}} \right] \, dt \right\} \\
&\quad + |u| \left\{ \int_0^{\frac{1}{2}} \left[ f'(1 - t) \left( \sqrt[\lambda a^p + (1 - \lambda)b^p\rho^p + tb^p\right)^{\frac{1}{p}} \right] \, dt \right\} \\
&\quad = \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p-a^p)}{2} \right)^{\rho k}}{\Gamma(\rho k + \beta + 1) \left( \sqrt[\lambda a^p + (1 - \lambda)b^p\rho^p + tb^p\right)^{\lambda(p-1)}} \right\} \\
&\quad \times \left[ \int_0^{\frac{1}{2}} t^{\lambda(p+\rho k)} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{\frac{1-\rho p}{p-1}} \, dt \right] \\
&\quad + \frac{|u|}{\left( \sqrt[\lambda a^p + (1 - \lambda)b^p\rho^p + tb^p\right)^{\lambda(p-1)}} \\
&\quad \times \left[ \int_0^{\frac{1}{2}} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{\frac{1-\rho p}{p-1}} \, dt \right] \right\} \\
&\quad \times \left\{ \int_0^{\frac{1}{2}} [(1 - t)^s |f(t)\beta^s| + (1 - \lambda)|f(t)\beta^s|] + t^s |f(t)\beta^s| \, dt \right\}^{\frac{1}{2}} \\
&\quad = \left\{ \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p-a^p)}{2} \right)^{\rho k}}{\Gamma(\rho k + \beta + 1) \left( \sqrt[\lambda a^p + (1 - \lambda)b^p\rho^p + tb^p\right)^{\lambda(p-1)}} \sqrt[\lambda a^p + (1 - \lambda)b^p + 1, 1} \right\}
\end{align*}
\]
\[
\times \left[ \text{I}_1 \right] = \left\{ \int_{1/2}^{1} \left\{ t^{\gamma \beta_{p+1}} \left[ \text{I}_1 \right] \right\} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right) \frac{dt}{t^{\gamma \beta_{p+1}}} \right\}
\]

\[
\left\{ \int_{1/2}^{1} \left\{ f' \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right) \right\} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right) \frac{dt}{t^{\gamma \beta_{p+1}}} \right\}
\]

\[
\times \left\{ \int_{1/2}^{1} \left\{ |t| \left[ \text{I}_1 \right] \right\} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right) \frac{dt}{t^{\gamma \beta_{p+1}}} \right\}
\]
Corollary 3.2. Let $I \subseteq \mathbb{R}^+$ be an interval and $I^p$ be the interior of $I$, $a, b \in I^p$ with $a < b$, $p < 0$, $u, v > 0$, $s \in (0, 1)$, $g(\xi) = \sqrt{\xi}$ for $\xi > 0$, $y = \frac{1}{\xi} > 1$, and $f : I \to \mathbb{R}$ be a differentiable function on $I^p$ such that $|f''|^p$ is $(s, p)$-convex. Then the inequality

$$
\left| uf(a) + (1 - v)f(b) + (v - u)f \left( \frac{\sqrt{a^p + b^p}}{2} \right) - \frac{1}{b^p - a^p} \int_{a^p}^{b^p} (f \circ g)(x)dx \right|
\leq \frac{(b^p - a^p)}{p \sqrt{a^p - b^p} \sqrt{2^{s+1}(s + 1)}} \left| u \right| \sqrt{\left( \frac{a^p - b^p}{2a^p} ; 1, \frac{xp - x + p}{p} \right)} \times \sqrt{\left| f'(b) \right|^p + (2^{s+1} - 1)\left| f'(a) \right|^p}
$$

Therefore, the desired inequality (3.9) follows from (3.5) and (3.10) together with (3.11). \hfill \Box

Corollary 3.2. Let $I \subseteq \mathbb{R}^+$ be an interval and $I^p$ be the interior of $I$, $a, b \in I^p$ with $a < b$, $p < 0$, $u, v > 0$, $s \in (0, 1)$, $g(\xi) = \sqrt{\xi}$ for $\xi > 0$, $y = \frac{1}{\xi} > 1$, and $f : I \to \mathbb{R}$ be a differentiable function on $I^p$ such that $|f''|^p$ is $(s, p)$-convex. Then the inequality

$$
\left| uf(a) + (1 - v)f(b) + (v - u)f \left( \frac{\sqrt{a^p + b^p}}{2} \right) - \frac{1}{b^p - a^p} \int_{a^p}^{b^p} (f \circ g)(x)dx \right|
\leq \frac{(b^p - a^p)}{p \sqrt{a^p - b^p} \sqrt{2^{s+1}(s + 1)}} \left| u \right| \sqrt{\left( \frac{a^p - b^p}{2a^p} ; 1, \frac{xp - x + p}{p} \right)} \times \sqrt{\left| f'(b) \right|^p + (2^{s+1} - 1)\left| f'(a) \right|^p}
$$

Therefore, the desired inequality (3.9) follows from (3.5) and (3.10) together with (3.11). \hfill \Box
\[ + \sqrt{\frac{v}{E}} \left( \frac{a^p - b^p}{a^p}; 1, \frac{xp - x + p}{p} \right) - \sqrt{\frac{v}{E}} \left( \frac{a^p - b^p}{2a^p}; 1, \frac{xp - x + p}{p} \right) \times \sqrt{(2^{1+1} - 1)|f''(b)|^p + |f'(a)|^p} \]

holds.

**Theorem 3.3.** Let \( I \subseteq \mathbb{R}^+ \) be an interval and \( f^p \) be the interior of \( I \), and \( a, b \in I^p \) with \( a < b \), \( p, \rho, \beta > 0 \), \( x \geq 1, u, w \in \mathbb{R}, (s, \lambda) \in (0, 1] \times [0, 1] \), \( g(\xi) = \sqrt{\xi} \), and \( f : I \rightarrow \mathbb{R} \) be a differentiable function on \( I^p \) such that \( |f'| \) is \((s, p)\)-convex. Then one has

\[
|\psi(t, a; f)| \leq \frac{[\lambda(b^p - a^p)]^{1+\beta}}{\lambda a^p + (1 - \lambda)b^p} \left( \sum_{k=1}^{s+1} \left( \frac{\lambda a^p + (1 - \lambda)b^p}{s+1} \right)^{\frac{1}{p}} \right) \times \frac{\lambda a^p + (1 - \lambda)b^p}{s+1} \left( \frac{\lambda a^p + (1 - \lambda)b^p}{s+1} \right) \times \frac{H_{1,2}(a, b; 1, s + 1) - H_{2,2}(a, b; 1, s + 2)}{s+1} \times \frac{f'(b)^1}{s+1} \left( H_{1,2}(a, b; s + 1, s + 2) - H_{2,2}(a, b; s + 1, s + 2) \right) + \frac{f'(b)}{s+1} \right) \}
\]

**Proof.** It follows from the \((s, p)\)-convexity of \(|f'|\) and the power-mean inequality that

\[
|I_1| = \left| \int_0^1 \left\{ x^p \rho^p \beta^{p+1} \left[ \frac{\lambda a^p + (1 - \lambda)b^p}{s+1} \right]^{\frac{1}{p}} \left( (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right)^{\frac{1}{p}} \right] \right| \times f' \left( ((1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p) \right) \right| \right| dt 
\]

\[
\leq \int_0^1 \left\{ x^p \rho^p \beta^{p+1} \left[ \frac{\lambda a^p + (1 - \lambda)b^p}{s+1} \right]^{\frac{1}{p}} \left( (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right)^{\frac{1}{p}} \right] \right| \times f' \left( ((1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p) \right) \right| \right| dt 
\]

\[
= \int_0^1 \left\{ \sum_{k=0}^{1 \to \infty} \frac{\sigma(k) |w|^k \left( (\lambda b^p - a^p) \right)^{\beta k}}{\Gamma(\rho k + \beta + 1)} \right\}^{\frac{1}{p}} \times (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right|_{x^p}^{\frac{1}{p}} 
\]
\[ \times \left( (1 - t) \left( \sqrt{\lambda a^p + (1 - \lambda)b^p} \right)^{\frac{1}{p}} \right) dt \]

\[ = \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p - a^p)}{2} \right)^{\lambda k}}{\Gamma(\rho k + \beta + 1)} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ (1 - t)(\lambda a^p + (1 - \lambda)b^p) + tb^p \right]^{1-p} dt \right\} \]

\[ = \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p - a^p)}{2} \right)^{\lambda k}}{\Gamma(\rho k + \beta + 1)} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]

\[ = \sum_{k=0}^{\infty} \frac{\sigma(k)w^k \left( \frac{\lambda(b^p - a^p)}{2} \right)^{\lambda k}}{\Gamma(\rho k + \beta + 1)} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]

\[ \times \left\{ \int_0^{\frac{1}{2}} \rho^{\beta + \rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{1-p} dt \right\} \]
\[
\begin{align*}
&\times [(1-t)^{\nu}[|\lambda f''(a)|^{\nu} + (1-\lambda)|f''(b)|^{\nu}] + t^\nu|f'(b)|^{\nu}] dt \frac{1}{\nu} \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left( \frac{\lambda a^p}{2} \right)^{\nu k}}{\Gamma(\rho k + \beta + 1) \left( \sqrt{\lambda a^p + (1-\lambda)b^p} \right)^{(x-1)(\beta-1)}} \\
&\times \left\{ \int_0^1 t^{\rho k} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1-\lambda)b^p} \right]^{\nu} dt \right\}^{\nu-1} \\
&\times \{ |\lambda f''(a)|^{\nu} + (1-\lambda)|f''(b)|^{\nu} \} \\
&\times \left\{ \int_0^1 (1-t)^{\nu} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1-\lambda)b^p} \right]^{\nu} dt \right\}^{\nu-1} \\
&\times \{ |\lambda f''(a)|^{\nu} + (1-\lambda)|f''(b)|^{\nu} \} \\
&= \sum_{k=0}^{\infty} \frac{\sigma(k)|w|^k \left( \frac{\lambda a^p}{2} \right)^{\nu k}}{\Gamma(\rho k + \beta + 1) \left( \sqrt{\lambda a^p + (1-\lambda)b^p} \right)^{(x-1)(\beta-1)}} \left\{ \mathbb{B}(\beta + \rho k + 1, 1) + \right. \\
&\left. \times_2 F_1 \left( \frac{p-1}{p}, \left[ \beta + \rho k + 1, \beta + \rho k + 2; \frac{1}{2} \right], \frac{\lambda(a^p - b^p)}{\lambda a^p + (1-\lambda)b^p} \right) \right\}^{\nu-1} \\
&\times \{ |\lambda f''(a)|^{\nu} + (1-\lambda)|f''(b)|^{\nu} \} \mathbb{B}(\beta + \rho k + 1, s + 1) \\
&\times_2 F_1 \left( \frac{p-1}{p}, \left[ \beta + \rho k + 1, s + \beta + \rho k + 2; \frac{1}{2} \right], \frac{\lambda(a^p - b^p)}{\lambda a^p + (1-\lambda)b^p} \right) + |f'(b)|^{\nu} \\
&\times_2 F_1 \left( \frac{p-1}{p}, \left[ \beta + \rho k + 1, s + \beta + \rho k + 2; \frac{1}{2} \right], \frac{\lambda(a^p - b^p)}{\lambda a^p + (1-\lambda)b^p} \right) \\
&\times \mathbb{B}(\beta + \rho k + s + 1, 1) \frac{1}{\nu} + \frac{|u|}{\left( \sqrt{\lambda a^p + (1-\lambda)b^p} \right)^{(x-1)(\beta-1)}}
\end{align*}
\]
and

\[ I_2 = \left| \int_{\frac{1}{2}}^{1} \left\{ p_{\beta} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right)^{1-p} \right\} \left| \int_{\frac{1}{2}}^{1} \left\{ f^p \left( \left( 1-t \left( \sqrt[\lambda p]{(1-\lambda)b^p} \right)^p + tb^p \right)^{\frac{1}{2}} \right) dt \right\} \right| \]

\[ = \sum_{k=0}^{\infty} \frac{\sigma(k) |w|^k \left( \frac{\lambda(b^p-a^p)}{2} \right)^{\frac{k}{2}}}{\Gamma(\rho k + \beta + 1)} \left[ (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right]^{\frac{1-p}{2}} \]

\[ \times \left\{ \int_{\frac{1}{2}}^{1} \rho^{\rho p k} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right)^{\frac{1-p}{2}} dt \right\} \]

\[ \times \left\{ \int_{\frac{1}{2}}^{1} \rho^{\rho p k} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right)^{\frac{1-p}{2}} dt \right\} \]

\[ \times \left\{ \int_{\frac{1}{2}}^{1} \rho^{\rho p k} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right)^{\frac{1-p}{2}} dt \right\} \]

\[ \times \left\{ \int_{\frac{1}{2}}^{1} \rho^{\rho p k} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right)^{\frac{1-p}{2}} dt \right\} \]

\[ \times \left\{ \int_{\frac{1}{2}}^{1} \rho^{\rho p k} \left( (1-t)(\lambda a^p + (1-\lambda)b^p) + tb^p \right)^{\frac{1-p}{2}} dt \right\} \]
\[
\left. \times f' \left( \left( 1 - t \right) \left( \sqrt[\lambda]{a^p} + (1 - \lambda)b^p \right)^p \right) \right|_0^x \right]^{1/2}
\]

\[
= \sum_{k=0}^{\infty} \frac{\sigma(k)\omega^k (\lambda(b^p - a^p))^p}{\Gamma(\rho k + \beta + 1) \left( \sqrt[\lambda]{a^p} + (1 - \lambda)b^p \right)^{(x-1)(p-1)}}
\]

\[
\times \int_1^x \beta + pk \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{\frac{p+1}{p}} dt \right]^{1-\frac{1}{2}}
\]

\[
\times \left[ (1 - t)^{\alpha} | \frac{f''(a)}{f'(a)} |^x + (1 - \lambda) | \frac{f''(b)}{f'(b)} |^x \right] dt \right]^{1/2}
\]

\[
= \sum_{k=0}^{\infty} \frac{\sigma(k)\omega^k (\lambda(b^p - a^p))^p}{\Gamma(\rho k + \beta + 1) \left( \sqrt[\lambda]{a^p} + (1 - \lambda)b^p \right)^{(x-1)(p-1)}}
\]

\[
\times \int_1^x \beta + pk \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{\frac{p+1}{p}} dt \right]^{1-\frac{1}{2}}
\]

\[
\times \left[ | \frac{f''(a)}{f'(a)} |^x + (1 - \lambda) | \frac{f''(b)}{f'(b)} |^x \right] \left[ (1 - t)^{\alpha} \left| \frac{f''(a)}{f'(a)} \right|^x + (1 - \lambda) \left| \frac{f''(b)}{f'(b)} \right|^x \right] dt \right]^{1/2}
\]

\[
\times \left[ \left( 1 - t \right)^{\alpha} \left( \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right)^{\frac{p+1}{p}} dt + \left| f''(b) \right|^x \right]^{\frac{1}{2}}
\]

\[
\times \int_1^x \beta + pk \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right]^{\frac{p+1}{p}} dt \right]^{\frac{1}{2}}
\]

\[
\times \left[ \sqrt[\lambda]{a^p} + (1 - \lambda)b^p \right]^{(x-1)(p-1)}
\]
\begin{align*}
&\times \left\{ \int_0^1 \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda) b^p} \right] \frac{1}{\sqrt{t}} \, dt \right\}^{1 - \frac{1}{2}} \\
&\times \left\{ [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \int_0^1 \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda) b^p} \right] \frac{1}{\sqrt{t}} \, dt \\
&\quad + |f'(b)|^s \int_0^1 t^\frac{1}{2} \left[ 1 - t \frac{\lambda(a^p - b^p)}{\lambda a^p + (1 - \lambda) b^p} \right] \frac{1}{\sqrt{t}} \, dt \right\}^{\frac{1}{2}} \\
&= \sum_{k=0}^{\infty} \sigma(k) w^k \left( \frac{\lambda(b^p - a^p)}{2} \right)^{\nu k} (\beta + \rho k + 1, 1) \\
&\times [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \int \left[ \lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right]^{1 - \frac{1}{2}} \\
&\quad \times \left\{ \lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right\}^{1 - \frac{1}{2}} \\
&\quad \times \left\{ [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right\}^{1 - \frac{1}{2}} \\
&\quad \times \left\{ [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right\}^{1 - \frac{1}{2}} \\
&\quad \times \left\{ [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right\}^{1 - \frac{1}{2}} \\
&\quad \times \left\{ [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right\}^{1 - \frac{1}{2}} \\
&\quad \times \left\{ [\lambda f'(a)]^s + (1 - \lambda)|f'(b)|^s \right\} \right\}^{1 - \frac{1}{2}}
\[ \Delta \left( \frac{p-1}{p}, s+1; s+2, \frac{\lambda (a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right) \]

\[ -2F_1 \left( \frac{p-1}{p}, s+1, s+2; \frac{1}{2}, \frac{\lambda (a^p - b^p)}{\lambda a^p + (1 - \lambda)b^p} \right)^\frac{1}{2}. \]  

Combining the inequalities (3.5), (3.14) and (3.15) gives the desired inequality (3.13).

\[ \square \]

**Corollary 3.3.** Let \( I \subseteq \mathbb{R}^+ \) be an interval and \( I^o \) be the interior of \( I \), \( a, b \in I^o \) with \( a < b \), \( p < 0 \), \( s \in (0, 1] \), \( u, v > 0 \), \( g(\xi) = \sqrt{\xi} \), \( y = \frac{\xi}{x_1} > 1 \) and \( f : I \to \mathbb{R} \) be a differentiable function on \( I^o \) such that \( |f''| \) is \((s, p)\)-convex. Then

\[ |uf(a) + (1-v)f(b) + (v-u)f \left( \sqrt{s} \left( \frac{a^p + b^p}{2} \right) - \frac{1}{bp - ap} \int_{ap}^{bp} (f \circ g)(x)dx \right)| \]

\[ \leq \frac{bp - ap}{p \left( \sqrt{s} \right)^{(p-1)(s-1)}} \left\{ u \left\{ \frac{ap}{ap - bp} \left( \frac{a^p - b^p}{2ap^p}; 1, \frac{1}{p} \right) \right\} \right\}^{\frac{1}{s-1}} \]

\[ \times \frac{\left| f'(a) \right|^s H_{2,1}(a, b; 1, s + 2) + |f'(b)|^s H_{2,1}(a, b; s + 1, s + 2)}{s + 1} \]

\[ + v \left\{ \frac{ap}{a^p - b^p} \left[ H_{1,1}(a, b; 1, s + 2) - H_{2,1}(a, b; 1, s + 2) \right] \right\}^{\frac{1}{s-1}} \]

\[ \times \left\{ \frac{|f'(a)|^s H_{1,1}(a, b; 1, s + 2) - H_{2,1}(a, b; s + 1, s + 2)}{s + 1} \right\}. \]  

**Proof.** Let \( \beta, \lambda, \sigma(0) \to 1 \) and \( w = 0 \). Then Corollary 3.3 follows easily from Theorem 3.3.

\[ \square \]

### 4. Applications

In this section, we provide some applications on \( f \)-divergence measures and probability density functions by using the results obtained in Section 3.

#### 4.1. \( f \)-divergence measures

Let \( \phi \) be a set, \( \mu \) be the \( \sigma \) finite measure, \( \Omega = \{ \chi(x) : \phi \to \mathbb{R}, \chi(x) > 0, \int_{\phi} \chi(x) d\mu(x) = 1 \} \) be the set of all probability densities on \( \mu \), and \( f : (0, \infty) \to \mathbb{R} \) be a real-valued function. Then the Csiszár \( f \)-divergence \( D_f(\chi, \psi) \) is defined by

\[ D_f(\chi, \psi) = \int_{\phi} \chi(x)f \left[ \frac{\psi(x)}{\chi(x)} \right] d\mu(x) \quad (\chi, \psi \in \Omega) \]  

\[ (4.1) \]
Proposition 4.1. Let $I \subseteq \mathbb{R}^+$ be an interval and $\mathcal{I}$ be the interior of $I$, $a, b \in \mathcal{I}$ with $a < b$, $s \in (0, 1]$ and $f : I \rightarrow \mathbb{R}$ be a differentiable function on $\mathcal{I}$ such that $|f''|$ is $s$-convex and $f(1) = 0$. Then

$$
\frac{1}{6} \left[ D_f(\chi, \psi) + 4 \int_0^1 \chi(x)f \left( \frac{\psi(x) + \chi(x)}{2\chi(x)} \right) d\mu(x) \right] - D_f^H(\chi, \psi) \leq \frac{(b - a)f'(a)}{6(s + 1)} \int_0^1 \chi(x) \left[ 52F_1 \left( 0, 1, s + 2, \frac{\psi(x) - \chi(x)}{\chi(x)} \right) \right] d\mu(x)
$$

$$
+ \frac{(b - a)f''(b)}{6(s + 1)} \int_0^1 \chi(x) \left[ 52F_1 \left( 0, s + 1, s + 2, \frac{\psi(x) - \chi(x)}{\chi(x)} \right) \right] d\mu(x).
$$

(4.3)

Proof. Let $\Phi_1 = \{ x \in \phi : \psi(x) > \chi(x) \}$, $\Phi_2 = \{ x \in \phi : \psi(x) < \chi(x) \}$ and $\Phi_3 = \{ x \in \phi : \psi(x) = \chi(x) \}$. Then we clearly see that inequality (4.3) holds if $x \in \Phi_3$.

For the case of $x \in \Phi_1$, taking $a, p \rightarrow 1$, $b \rightarrow \frac{\psi(x)}{\chi(x)}$, $u \rightarrow \frac{1}{6}$ and $v \rightarrow \frac{5}{6}$ in Corollary 3.1, multiplying both sides of the obtained result by $\chi(x)$ and integrating over $\Phi_1$ lead to the conclusion that

$$
\frac{1}{6} \left[ 4 \int_0^1 \chi(x)f \left( \frac{\psi(x) + \chi(x)}{2\chi(x)} \right) d\mu(x) + \int_0^1 \chi(x)f \left( \frac{\psi(x)}{\chi(x)} \right) d\mu(x) \right] - \int_{\Phi_1} \chi(x) \left[ \frac{\phi(x)}{\chi(x)} - 1 \right] d\mu(x) \leq \frac{(b - a)f''(a)}{6(s + 1)} \int_{\Phi_1} \chi(x) \left[ 52F_1 \left( 0, 1, s + 2, \frac{\chi(x) - \psi(x)}{\chi(x)} \right) \right] d\mu(x)
$$

$$
- 4_2F_1 \left( 0, \left[ 1, s + 2; \frac{1}{2} \right], \frac{\chi(x) - \psi(x)}{\chi(x)} \right) d\mu(x)
$$

$$
+ \frac{(b - a)f''(b)}{6(s + 1)} \int_{\Phi_1} \chi(x) \left[ 52F_1 \left( 0, s + 1, s + 2, \frac{\chi(x) - \psi(x)}{\chi(x)} \right) \right] d\mu(x).
$$

(4.4)

Similarly, for the case of $x \in \Phi_2$, taking $b, p \rightarrow 1$, $a \rightarrow \frac{\phi(x)}{\chi(x)}$, $u \rightarrow \frac{1}{6}$ and $v \rightarrow \frac{5}{6}$ in Corollary 3.1, multiplying both sides to the obtained result by $\chi(x)$ and integrating over $\Phi_2$, we get
Then from Corollary 3.1 we clearly see that

\[
- \int \chi(x) \left[ \frac{\int_0^x f(t)dt}{\chi(x)} \right] d\mu(x) \leq \frac{(b-a)f'(a)}{6(s+1)} \int \chi(x) \left[ 5_2F_1 \left( 0, 1, s+2, \frac{\psi(x) - \chi(x)}{\chi(x)} \right) \right] d\mu(x) \\
- 4_2F_1 \left( 0, 1, s+2; \frac{1}{2}, \frac{\psi(x) - \chi(x)}{\chi(x)} \right) d\mu(x)
\]

Therefore, the desired inequality (4.3) can be derived by adding inequalities (4.4) and (4.5) together with the triangular inequality. \(\square\)

4.2. Probability density functions

Let \(a, b \in \mathbb{R}\) with \(a < b\), \(g : [a, b] \to [0, 1]\) be the probability density function of a continuous random variable \(X\) with the cumulative distribution function \(F\) given by

\[
F(x) = P(X \leq x) = \int_a^x g(t)dt, \quad E(X) = \int_a^b t g(t)dt = b - \int_a^b F(t)dt.
\]

Then from Corollary 3.1 we clearly see that

\[
\left| \frac{1}{6} \left[ 4P \left( X \leq \frac{a+b}{2} \right) + 1 \right] - \frac{1}{b-a}(b - E(X)) \right| \\
\leq \frac{b-a}{s+1} \left\{ \left| g(a) \right| \frac{5_2F_1 \left( 0, 1, s+2, \frac{a-b}{a} \right) - 4_2F_1 \left( 0, 1, s+2; \frac{1}{2}, \frac{a-b}{a} \right) }{6} \right. \\
+ \left| g(b) \right| \frac{5_2F_1 \left( 0, s+1, s+2, \frac{a-b}{a} \right) - 4_2F_1 \left( 0, s+1, s+2; \frac{1}{2}, \frac{a-b}{a} \right) }{6} \right. \\
\right\}
\]

if \(p \to 1, u \to \frac{1}{6}\) and \(v \to \frac{5}{6}\).

5. Conclusions

We have established some new estimates for the generalized Simpson’s quadrature rule via the Raina fractional integrals by use of a Simpson-type generalized identity with multi-parameters, and discovered several inequalities for the \(f\)-divergence measures and probability density functions. Our obtained results are the improvements and generalizations of some previous known results, our ideas and approach may lead to a lot of follow-up research.
Acknowledgements

The authors would like to thank the anonymous referees for their valuable comments and suggestions, which led to considerable improvement of the article.

The work was supported by the Natural Science Foundation of China (Grant Nos. 61673169, 11971142, 11701176, 11626101, 11601485).

Conflict of interest

The authors declare no conflict of interest.

References

1. S. Rashid, F. Jarad, H. Kalsoom, et al. *On Pólya-Szegö and Čebyšev type inequalities via generalized k-fractional integrals*, Adv. Differ. Equ., 2020 (2020), 1–18.
2. S. S. Zhou, S. Rashid, F. Jarad, et al. *New estimates considering the generalized proportional Hadamard fractional integral operators*, Adv. Differ. Equ., 2020 (2020), 1–15.
3. A. Iqbal, M. Adil Khan, S. Ullah, et al. *Some new Hermite-Hadamard-type inequalities associated with conformable fractional integrals and their applications*, J. Funct. Space., 2020 (2020), 1–18.
4. S. Rashid, F. Jarad, Y. M. Chu, *A note on reverse Minkowski inequality via generalized proportional fractional integral operator with respect to another function*, Math. Probl. Eng., 2020 (2020), 1–12.
5. Y. Khurshid, M. Adil Khan, Y. M. Chu, *Conformable fractional integral inequalities for GG- and GA-convex function*, AIMS Math., 5 (2020), 5012–5030.
6. S. Rashid, M. A. Noor, K. I. Noor, et al. *Ostrowski type inequalities in the sense of generalized K-fractional integral operator for exponentially convex functions*, AIMS Math., 5 (2020), 2629–2645.
7. S. Rashid, İ. İşcan, D. Baleanu, et al. *Generation of new fractional inequalities via n polynomials s-type convexity with applications*, Adv. Differ. Equ., 2020 (2020), 1–20.
8. G. Hu, H. Lei, T. S. Du, *Some parameterized integral inequalities for p-convex mappings via the right Katugampola fractional integrals*, AIMS Math., 5 (2020), 1425–1445.
9. M. Adil Khan, M. Hanif, Z. A. Khan, et al. *Association of Jensen’s inequality for s-convex function with Csiszár divergence*, J. Inequal. Appl., 2019 (2019), 1–14.
10. M. Adil Khan, J. Pečarić, Y. M. Chu, *Refinements of Jensen’s and McShane’s inequalities with applications*, AIMS Math., 5 (2020), 4931–4945.
11. M. A. Latif, S. Rashid, S. S. Dragomir, et al. *Hermite-Hadamard type inequalities for co-ordinated convex and quasi-convex functions and their applications*, J. Inequal. Appl., 2019 (2019), 1–33.
12. M. U. Awan, N. Akhtar, S. Iftikhar, et al. *New Hermite-Hadamard type inequalities for n-polynomial harmonically convex functions*, J. Inequal. Appl., 2020 (2020), 1–12.
13. M. Adil Khan, N. Mohammad, E. R. Nwaeze, et al. *Quantum Hermite-Hadamard inequality by means of a Green function*, Adv. Differ. Equ., 2020 (2020), 1–20.
14. S. Rashid, M. A. Noor, K. I. Noor, et al. Hermite-Hadamrad type inequalities for the class of convex functions on time scale, Mathematics, 7 (2019), 1–20.

15. M. U. Awan, S. Talib, Y. M. Chu, et al. Some new refinements of Hermite-Hadamard-type inequalities involving $\Psi_k$-Riemann-Liouville fractional integrals and applications, Math. Probl. Eng., 2020 (2020), 1–10.

16. M. K. Wang, Z. Y. He, Y. M. Chu, Sharp power mean inequalities for the generalized elliptic integral of the first kind, Comput. Meth. Funct. Th., 20 (2020), 111–124.

17. S. Rashid, R. Ashraf, M. A. Noor, et al. New weighted generalizations for differentiable exponentially convex mapping with application, AIMS Math., 5 (2020), 3525–3546.

18. T. H. Zhao, M. K. Wang, Y. M. Chu, A sharp double inequality involving generalized complete elliptic integral of the first kind, AIMS Math., 5 (2020), 4512–4528.

19. Z. H. Yang, W. M. Qian, W. Zhang, et al. Notes on the complete elliptic integral of the first kind, Math. Inequal. Appl., 23 (2020), 77–93.

20. M. K. Wang, H. H. Chu, Y. M. Chu, Precise bounds for the weighted Hölder mean of the complete $p$-elliptic integrals, J. Math. Anal. Appl., 480 (2019), 1–9.

21. M. U. Awan, N. Akhtar, A. Kashuri, et al. 2D approximately reciprocal $\rho$-convex functions and associated integral inequalities, AIMS Math., 5 (2020), 4662–4680.

22. S. S. Dragomir, R. P. Agarwal, P. Cerone, On Simpson’s inequality and applications, J. Inequal. Appl., 5 (2000), 533–579.

23. Z. B. Fang, R. J. Shi, On the $(p,h)$-convex function and some integral inequalities, J. Inequal. Appl., 2014 (2014), 1–16.

24. N. Mehreen, M. Anwar, Hermite-Hadamard and Hermite-Hadamard-Fejer type inequalities for $p$-convex functions via new fractional conformable integral operators, J. Math. Comput. Sci., 19 (2019), 230–240.

25. M. A. Özarslan, C. Ustaoglu, Some incomplete hypergeometric functions and incomplete Riemann-Liouville fractional integral operators, Mathematics, 7 (2019), 1–18.

26. R. K. Raina, On generalized wright’s hypergeometric functions and fractional calculus operators, East Asian Math. J., 21 (2005), 191–203.

27. R. P. Agarwal, M. J. Luo, R. K. Raina, On Ostrowski type inequalities, Fasc. Math., 56 (2016), 5–27.

28. S. Qaisar, C. J. He, S. Hussain, A generalizations of Simpson’s type inequality for differentiable functions using $(\alpha,m)$-convex functions and applications, J. Inequal. Appl., 2013 (2013), 1–13.