Abstract

We present an analytical upper bound on the number of required vehicles for vehicle routing problems with split deliveries and any number of capacitated depots. We show that a fleet size greater than the proposed bound is not achievable based on a set of common assumptions. This property of the upper bound is proved through a dynamic programming approach. We also discuss the validity of the bound for a wide variety of routing problems with or without split deliveries.

Keywords— Split delivery; Number of vehicles; Multiple depots; Dynamic programming

1 Introduction

Vehicle routing problems constitute an important class of combinatorial optimization problems. We develop an analytical upper bound on the number of required vehicles for split delivery routing problems that are also valid for other routing problems. The upper bound is based on the following four assumptions:

- A1: Each vehicle delivers to exactly one depot.
- A2: Vehicles are homogeneous.
- A3: The aggregate load for each pair of vehicles exceeds the capacity of one vehicle.
- A4: All parameters as well as the load for each vehicle are integers.

Without loss of generality, we assume that goods are collected from the demand points and delivered to a number of depots. Note that there is an equivalence between pure distribution and pure collection since we can reverse the routes so that collection becomes distribution and vice versa (Toth and Vigo, 2014).

Assumption A1 is quite common as many routing problems assume that each route begins and ends at the same depot. It is also valid when vehicles start and end their routes at different depots, but each of them makes a delivery to exactly one depot. This includes problems where a vehicle can visit any number of nodes (depots or demand points) to pick up goods and deliver to exactly one depot. Assumption A2 is valid for a large proportion of the routing problems where
the fleet of vehicles is homogeneous. In heterogeneous routing problems, we may have different types of vehicles that can differ in capacity, variable and fixed costs, speeds, and the customers that they can access (Toth and Vigo, 2014; Koç et al., 2016). More precisely, the set of vehicles is partitioned into a number of subsets of homogeneous vehicles each corresponding to a vehicle type. In these problems, we are interested to know the number of required vehicles for each vehicle type. Given that vehicles belonging to a subset are homogeneous, we may employ the proposed upper bound for each subset of vehicles separately by assigning the total accessible demands to them.

Assumption A3 is typically valid for routing problems where the cost/time or distance matrix satisfies the triangle inequality. If the total load for any pair of vehicles is less than or equal to the capacity of each vehicle, then we can simply combine the loads of the two vehicles and use only one vehicle to deliver them. The triangle inequality implies that the new route is not longer than the sum of the lengths of the previous two routes. Note that in the case where the triangle inequality does not hold, we sometimes can obtain an equivalent instance of the routing problem that satisfies the triangle inequality. This can be done by simply replacing the actual distance between each pair of nodes with the length of a shortest path connecting them (i.e., finding the metric closure of the network).

Assumption A4 is valid for many real-world routing problems where a fractional number representing demand or capacity can be rounded to an integer. When parameters of the problem are all integers, the underlying routing problem often has an optimal solution in which the load for each vehicle is an integer. For example, Archetti et al. (2006) consider a split delivery routing problem and show that when demands and capacity of each vehicle are integers, then there exists an optimal solution in which vehicle loads are all integers. Furthermore, when split deliveries are not allowed and demands are all integers, then the vehicle loads in every feasible solution are all integers.

We formulate two maximization problems based on Assumptions A1-A4 and obtain a closed-form solution for each of them. The optimal objective value of these optimization problems give a tight upper bound for the fleet size in single-depot and multiple-depot routing problems, respectively. We refer to such a bound as “tight” in the sense that a fleet size greater than the bound is not achievable. Although, the bound itself may also not be achievable for some routing problems.

The second optimization problem represents the general case where we have any number of depots. We use a dynamic programming approach to obtain an analytical solution for this problem. The upper bound is useful as it can be employed to design solution algorithms for relevant routing problems. Furthermore, vehicle-flow (often three-index) formulations of routing problems require the maximum number of vehicles a priori (Toth and Vigo, 2014). The numbers of decision variables and constraints of these formulations are greatly affected by the upper bound on the number of vehicles used in the optimization. Consequently, it is important to specify a good upper bound to reduce the computational time. Chandran and Raghavan (2008) solve the linear programming relaxation of a mixed integer linear program to obtain a good upper bound on the number of vehicles. This upper bound is utilized to reduce the number of decision variables of a routing problem in a preprocessing approach. Archetti et al. (2011) present an upper bound on the number of vehicles for a routing problem and use this upper bound in a column generation approach to solve the problem to optimality.

Our proposed upper bound can be computed very efficiently which makes it desirable for implementation within exact methods or as a preprocessing step. Although we focus on the routing problems with split deliveries, the proposed bound is also valid for the problems where split deliveries are not allowed. This can be realized by considering the upper bound as the optimal objective value of a maximization problem with the objective function being the number of vehicles used. Allowing split deliveries can be seen as a relaxation of this maximization problem which gives a valid upper bound for the corresponding routing problem. Furthermore, we point out that our proposed upper bound is valid for routing problems in which Assumption A1 does not hold. The reason is that removing Assumption A1 provides more degrees of freedom for
the corresponding routing problem (removing a restriction on the number of deliveries that a vehicle can make). However, our proposed upper bound might not be the best possible bound that can be obtained for such routing problems based on our assumptions.

The rest of the paper is organized as follows. In Section 2 we review the notations and definitions that are used throughout the paper. The upper bounds on the number of vehicles for routing problems with single and multiple depots are developed in Section 3 and Section 4, respectively. Finally, we give some concluding remarks in Section 5.

2 Notation

We denote the set of all non-negative integers by \( \mathbb{Z}_0 \), the set of all positive integers by \( \mathbb{Z}_+ \) and the set of all integers by \( \mathbb{Z} \). For a given set \( S \), we use \( |S| \) to denote its cardinality. The floor and ceiling functions are denoted by \( \lfloor \cdot \rfloor \) and \( \lceil \cdot \rceil \), respectively. We sometimes take advantage of the inequality \( r \leq \lceil r \rceil < r + 1 \) for a real number \( r \), and refer to it as the ceiling property.

We denote the set of demand points by \( I \) and the set of potential locations for facilities (depots) by \( J \). We often use \( n \) to denote the number of potential facilities, that is, \( n = |J| \). Demand point \( i \in I \) has a demand \( d_i \in \mathbb{Z}_0 \) that must be satisfied. The total demand is denoted by \( \Delta = \sum_{i \in I} d_i \). We assume that transportation of demand from demand points to the depots is carried out by a fleet of homogeneous vehicles each with capacity \( q \in \mathbb{Z}_+ \). The capacity of depot \( i \in J \) is denoted by \( c_i \in \mathbb{Z}_+ \). We assume, without loss of generality, that \( c_1 \geq \cdots \geq c_n \).

3 Single depot

When there is only one depot (i.e., \( n = 1 \) and \( I \cap J = \emptyset \)), we may consider \( \sum_{i \in I} \lceil \frac{d_i}{q} \rceil \) as a trivial upper bound on the number of vehicles (Lee et al., 2006; Archetti and Speranza, 2008). The upper bounds \( \lceil \frac{2\Delta}{q} \rceil \) and \( 2 \lceil \frac{\Delta}{q} \rceil \) have also been proposed by Labbé et al. (1991) and Archetti et al. (2011), respectively. In this section, we develop the maximum achievable upper bound based on Assumptions A1-A4. Let \( m \) be the number of required vehicles and \( v_i \) denote the load of vehicle \( i \in \{1, \ldots, m\} \). We are interested to know the optimal objective value of the following single-depot formulation (SDF):

\[
\begin{align*}
\text{max} \quad & m \\
\text{s.t.} \quad & \sum_{i=1}^{m} v_i = \Delta \\
& v_i + v_j \geq q + 1 \quad \quad \quad \quad \quad i, j \in \{1, \ldots, m\} \text{ with } i \neq j \\
& v_i \in \{1, \ldots, q\} \quad \quad \quad \quad \quad i \in \{1, \ldots, m\} \\
& m \in \mathbb{Z}_0.
\end{align*}
\]

The objective of the SDF is to maximize the number of required vehicles, \( m \). Constraint (2) states that the total demand \( \Delta \) must be delivered to the depot by \( m \) vehicles. Constraints (3) are due to Assumptions A3 and A4. Constraints (4) and (5) specify domains of the decision variables.

Proposition 1. For \( \Delta \in \mathbb{Z}_0 \) and \( q \in \mathbb{Z}_+ \), the function \( \pi : \mathbb{Z}_0 \to \mathbb{Z}_0 \) defined by

\[
\pi(\Delta) := \begin{cases} \\
\left\lfloor \frac{\Delta}{q} \right\rfloor & \Delta \leq q \\
\left\lfloor \frac{\Delta - q}{q + 1} \right\rfloor + 1 & \Delta > q,
\end{cases}
\]

gives the optimal value of the SDF.
This implies that and therefore \( m \). Without loss of generality, we assume that vehicles are sorted such that \( \Delta > q \). It can be easily verified that Constraints (2) and (5) are satisfied by this solution. Now we use SDF \( \lfloor \frac{\Delta - q}{q+1} \rfloor \geq \alpha \) subject to \( v_{m-1} \) and \( v_m \) subject to \( v_{m-1} \geq \lfloor \frac{q+1}{2} \rfloor \). Since \( m \) and \( v_{m-1} \) take integer numbers, we can write \( v_{m-1} \geq \lfloor \frac{q+1}{2} \rfloor \) and \( v_m \geq \lfloor \frac{q+1}{2} \rfloor \) for Constraint (4), we check the inequality

\[
\Delta - \lfloor \frac{q+1}{2} \rfloor \left( \frac{\Delta - q}{q+1} \right) = q \quad i, j < m^*
\]

For Constraint (4), we check the inequality \( v_i^* \leq q \) as follows:

\[
v_i^* = \begin{cases} \left\lfloor \frac{q+1}{2} \right\rfloor \leq q & i < m^* \\ \Delta - \left\lfloor \frac{q+1}{2} \right\rfloor \left( \frac{\Delta - q}{q+1} \right) \leq \Delta - \left\lfloor \frac{q+1}{2} \right\rfloor \left( \frac{\Delta - q}{q+1} \right) = q & i = m^* 
\end{cases}
\]

The inequality \( v_i^* \geq 1 \) is satisfied since

\[
v_i^* = \begin{cases} \left\lceil \frac{q+1}{2} \right\rceil \geq 1 & i < m^* \\ \Delta - \left\lceil \frac{q+1}{2} \right\rceil \left( \frac{\Delta - q}{q+1} \right) > \Delta - \left\lceil \frac{q+1}{2} \right\rceil \left( \frac{\Delta - q}{q+1} \right) + 1 \geq 0 & i = m^* 
\end{cases}
\]

Hence Constraint (4) is also satisfied.
Multiple depots

In this section, we assume that \( n \geq 1 \) and \( \mathcal{I} \cap \mathcal{J} = \emptyset \). We introduce an optimization problem whose optimal solution provides the maximum achievable upper bound on the number of vehicles. Let the decision variable \( x_i \) represent the quantity of demand delivered to depot \( i \in \mathcal{J} \). The multi-depot formulation (MDF) can then be expressed as follows:

\[
\max \sum_{i=1}^{n} \pi(x_i) \tag{7}
\]

s.t. \( \sum_{i=1}^{n} x_i \leq \Delta \) \hspace{1cm} \( x_i \in \{0, \ldots, c_i\} \) \hspace{1cm} \( i \in \{1, \ldots, n\} \). \hspace{1cm} \( \Delta \)

Constraint (8) states that the total quantity delivered by vehicles must not exceed the total demand, \( \Delta \). Constraints (9) specify domains of the decision variables. Observe that the decision variable \( x_i \) can take the value zero. This means that it is not required that every depot has positive demand delivered. Consequently, the bound is valid whether the locations of the depots are predetermined or not.

The objective of the MDF is to maximize the number of required vehicles. The function \( \pi(x_i) \) represents the maximum number of vehicles required to deliver \( x_i \) units of demand to depot \( i \in \mathcal{J} \). Because each vehicle can deliver to exactly one depot (by Assumption A1), the objective function \( \sum_{i=1}^{n} \pi(x_i) \) properly represents the total number of required vehicles. When there is only one depot, we know that the function \( \pi(x_i) \) introduced in Section 3 gives the maximum achievable number of vehicles. Therefore, the optimal value of the MDF gives the maximum achievable bound on the number of vehicles when \( n \) depots are available. In the absence of Assumption A1, the resulting optimal value of the MDF is still a valid upper bound. However, it might not be a tight upper bound for every class of instances.

We solve the MDF using a dynamic programming approach. Let \( V_j^*(\delta_j) \) be the maximum number of vehicles required to deliver \( \delta_j \in \{0, \ldots, \Delta\} \) units of demand to depots \( 1, \ldots, j \). In other words,

\[
V_j^*(\delta_j) := \max_{x_1, \ldots, x_j} \sum_{i=1}^{j} \pi(x_i) \tag{10}
\]

s.t. \( \sum_{i=1}^{j} x_i \leq \delta_j \) \hspace{1cm} \( x_i \in \{0, \ldots, c_i\} \) \hspace{1cm} \( i \in \{1, \ldots, j\} \). \hspace{1cm} \( \delta_j \)

We refer to this formulation as the Dynamic Programming Formulation (DPF\(_j\)). We therefore obtain the optimality equation

\[
V_j^*(\delta_j) = \begin{cases} 
\max_{x_j \in \{0, \ldots, \min(c_j, \delta_j)\}} \{ \pi(x_j) \} = \pi(\min(c_j, \delta_j)) & j = 1 \\
\max_{x_j \in \{0, \ldots, \min(c_j, \delta_j)\}} \{ V_j(x_j, \delta_j) \} & j > 1,
\end{cases} \tag{13}
\]

in which \( V_j(x_j, \delta_j) := \pi(x_j) + V_{j-1}^*(\delta_j - x_j) \). We can employ Equation (13) to obtain an optimal policy for the MDF. Let us begin by introducing the function \( \theta: \mathbb{Z}_0 \to \mathbb{Z}_0 \) defined by

\[
\theta(\alpha) := \begin{cases} 
\pi(\alpha) & \alpha \leq q \\
(\pi(\alpha) - 1) \left\lceil \frac{q + 1}{2} \right\rceil + \left\lfloor \frac{q + 1}{2} \right\rfloor & \alpha > q.
\end{cases} \tag{14}
\]

This function has some interesting properties that allow us to obtain an analytical solution.
Lemma 1. For each $\alpha \in \mathbb{Z}_0$ we have $\theta(\alpha) \leq \alpha$ and $\pi(\theta(\alpha)) = \pi(\alpha)$.

Proof. We know that $\pi(0) = 0$ and $\pi(1) = 1$ for $1 \leq \alpha \leq q$. Therefore, it can be verified that the statement is true for $\alpha \leq q$. For $\alpha > q$ we have $\pi(\alpha) = \left\lfloor \frac{\alpha - q}{q + 1} \right\rfloor + 1 \geq 2$. Thus,

$$\theta(\alpha) = \left\lfloor \frac{\alpha - q}{q + 1} \right\rfloor \left\lfloor \frac{q + 1}{2} \right\rfloor + \left\lfloor \frac{q + 1}{2} \right\rfloor \geq \left\lfloor \frac{q + 1}{2} \right\rfloor + \left\lfloor \frac{q + 1}{2} \right\rfloor = q + 1.$$

Furthermore, the ceiling property implies that

$$\theta(\alpha) < \alpha - q + \left\lfloor \frac{q + 1}{2} \right\rfloor + \left\lfloor \frac{q + 1}{2} \right\rfloor = \alpha + 1.$$

Therefore, $q < \theta(\alpha) \leq \alpha$. Since $\theta(\alpha) > q$ we have $\pi(\theta(\alpha)) = \left\lfloor \frac{\theta(\alpha) - q}{q + 1} \right\rfloor + 1$. Now by considering the definition (14) we obtain

$$\pi(\theta(\alpha)) = (\pi(\alpha) - 1) + \left\lfloor \frac{q + 1}{2} - q \right\rfloor + 1$$

$$= (\pi(\alpha) - 1) + \left\lfloor \frac{q + 1}{2} + \left\lfloor \frac{q + 1}{2} \right\rfloor - q \right\rfloor$$

$$= (\pi(\alpha) - 1) + \left\lfloor \frac{1}{q + 1} \right\rfloor = \pi(\alpha) \square.$$

Lemma 2. For each $\alpha, \beta \in \mathbb{Z}_+ \quad q < \alpha < \beta - q$ we have

$$\pi(\alpha) + \pi(\beta - \alpha) \leq 1 + \pi \left( \beta - \left\lfloor \frac{q + 1}{2} \right\rfloor \right).$$

Proof. From Lemma 1, we know that $\theta(\alpha) \leq \alpha$. As $\beta - \alpha > q$, we obtain $\beta - \theta(\alpha) \geq \beta - \alpha > q$. Now by considering that $\alpha > q$ we can write

$$\pi(\alpha) + \pi(\beta - \alpha) \leq \pi(\alpha) + \pi(\beta - \theta(\alpha))$$

$$= \pi(\alpha) + \pi \left( \beta - (\pi(\alpha) - 1) \left\lfloor \frac{q + 1}{2} \right\rfloor - \left\lfloor \frac{q + 1}{2} \right\rfloor \right)$$

$$= \pi(\alpha) - (\pi(\alpha) - 1) + \left\lfloor \beta - \left\lfloor \frac{q + 1}{2} \right\rfloor - q \right\rfloor + 1$$

$$= 1 + \pi \left( \beta - \left\lfloor \frac{q + 1}{2} \right\rfloor \right) \square.$$

Note that $\beta > \alpha + q > 2q$ which follows that $\beta - \left\lfloor \frac{q + 1}{2} \right\rfloor > q$.

Lemma 3. For $\alpha \in \mathbb{Z}_0$, we have $\pi(\beta + 1) - \pi(\beta) \leq 1$.

Proof. The result follows from the ceiling property. \square

Lemma 4. For each $\alpha \in \mathbb{Z}_0$ and $\beta \in \mathbb{Z}_+ \quad \alpha \leq \beta$ we have

$$\pi(\alpha) + \pi(\beta - \alpha) \leq 1 + \pi(\beta - 1).$$

Proof. From Lemma 3 we know that $\pi(\beta) - \pi(\beta - 1) \leq 1$. Thus, the result follows for $\alpha = 0$. Now consider the case $1 \leq \alpha < q$ in which $\pi(\alpha) = 1$. Since $\pi(\cdot)$ is an increasing function and $\beta - \alpha \leq \beta - 1$, the result follows. Finally, we suppose that $q < \alpha \leq \beta$. Since the case $\alpha = \beta$ is equivalent to the case $\alpha = 0$, we assume that $q < \alpha \leq \beta - 1$.

If $\beta - \alpha \leq q$, then we obtain

$$\pi(\alpha) + \pi(\beta - \alpha) = \pi(\alpha) + 1 \leq \pi(\beta - 1) + 1.$$
For $\beta - \alpha > q$, we can utilize Lemma 2 to obtain
\[ \pi(\alpha) + \pi(\beta - \alpha) \leq 1 + \pi \left( \beta - \left\lfloor \frac{q + 1}{2} \right\rfloor \right) \leq 1 + \pi(\beta - 1). \]

**Lemma 5.** For $\alpha \in \mathbb{Z}_0$, we have $\pi(\alpha) \leq \alpha$.

**Proof.** Given that $\alpha \geq 0$, we just need to show that the function $f(\alpha) := \pi(\alpha) - \alpha$ is decreasing. We have
\[ f(\alpha + 1) - f(\alpha) = \pi(\alpha + 1) - \pi(\alpha) - 1 \leq 0, \]
which is true due to Lemma 3. \hfill \square

**Lemma 6.** If $\delta_j \leq j$, then
\[ x_i^* := \begin{cases} 1, & i = 1, \ldots, \delta_j \\ 0, & i = \delta_j + 1, \ldots, j \end{cases} \]
is an optimal solution for $\text{DPF}_j$. Furthermore, $V_j^*(\delta_j) = \delta_j$.

**Proof.** It is clear that the given solution is feasible for $\text{DPF}_j$ and $\sum_{i=1}^{j} \pi(x_i^*) = \delta_j$. The proof is complete if we show that $\delta_j$ is an upper bound for $V_j^*(\delta_j)$. According to Lemma 5,
\[ \sum_{i=1}^{j} \pi(x_i) \leq \sum_{i=1}^{j} x_i \leq \delta_j, \]
which states that $\delta_j$ is an upper bound for $V_j^*(\delta_j)$. \hfill \square

**Lemma 7.** Consider an instance of $\text{DPF}_j$ with $\delta_j \geq j$. There exists an optimal solution for this instance in which $x_i \geq 1$ for $i = 1, \ldots, j$.

**Proof.** Let $(x_1^*, \ldots, x_j^*)$ be an optimal solution of $\text{DPF}_j$. If $x_i^* \geq 1$ for $i = 1, \ldots, j$ then the proof is complete. Therefore, suppose that $x_i^* = 0$ for some $l \in \{1, \ldots, j\}$. Since the solution is optimal and $\delta_j \geq j$, there must exist some $k \in \{1, \ldots, j\}$ with $x_k^* \geq 2$. Now we construct a new feasible solution by setting $x_l^{\text{new}} = 1$ and $x_k^{\text{new}} = x_k^* - 1$. According to Lemma 3,
\[ \pi(x_l^*) + \pi(x_k^*) \leq \pi(x_l^{\text{new}}) + \pi(x_k^{\text{new}}), \]
implying that the new feasible solution is also optimal. We can repeat this procedure until we have an optimal solution in which $x_i \geq 1$ for $i = 1, \ldots, j$. \hfill \square

**Theorem 1.** For $\delta_j \geq j$, the optimal value of $\text{DPF}_j$ is given by
\[ V_j^*(\delta_j) = j - 1 + \pi(\min(\lambda_{\ell_j}, c_{\ell_j})) + \sum_{i=1}^{\ell_j - 1} (\pi(c_i) - 1), \]
where $\lambda_{\ell_j} := (\delta_j - j + 1) - \sum_{r=1}^{\ell_j - 1} (\theta(c_r) - 1)$, and $\ell_j \in \{1, \ldots, j\}$ is the largest integer such that $\lambda_{\ell_j} \geq 1$.

**Proof.** It is enough to show that the policy
\[ x_i^* := \begin{cases} \theta(c_i) & i = 1, \ldots, \ell_j - 1 \\ \theta(\min(\lambda_{\ell_j}, c_{\ell_j})) & i = \ell_j \\ 1 & i = \ell_j + 1, \ldots, j, \end{cases} \tag{15} \]
is optimal. From Lemma 1 we know that $\theta(c_i) \leq c_i$ for $i = 1, \ldots, j$, implying that the given policy (15) is feasible. We give a proof by induction on $j$ to show that it is also an optimal policy. For $j = 1$ the given policy states that $\ell_1 = 1$ and $x_1^* = \theta(\min(\delta_1, c_1))$. In addition, the
optimality equation (13) states that $V^*_j(\delta) = \pi(\min\delta, c))$. From Lemma 1 we know that $\pi(\min(\delta_1, c_1)) = \pi(\theta(\min(\delta_1, c_1)))$. Therefore, $x_1^* = \theta(\min(\delta_1, c_1))$ is an optimal policy for DPF$_1$, implying that the assertion is true for $j = 1$. Next we show that the assertion holds for $j = 2$, that is, the given policy (15) is optimal for DPF$_2$.

We have

$$V^*_2(\delta_2) = \max_{x_2 \in \{0, \ldots, \min(\delta_2, c_2)\}} V_2(x_2, \delta_2).$$

Here, we consider two possible cases for $\delta_2$ and show that in both of them the given policy (15) is optimal.

**Case 1: $\delta_2 \leq \theta(c_1)$:** In this case, Equation (15) prescribes that $\ell_2 = 1$, $x_2^* = 1$ and $x_1^* = \theta(\min(\delta_2 - 1, c_1))$. Given that $\delta_2 \leq \theta(c_1)$, Lemma 1 implies that $\delta_2 \leq c_1$ and therefore $x_1^* = \theta(\delta_2 - 1)$. Now, by utilizing Lemma 4 and Lemma 1, we have

$$V_2(x_2, \delta_2) = \pi(x_2) + \pi(\min(c_1, \delta_2 - x_2))$$

$$= \pi(x_2) + \pi(\delta_2 - x_2)$$

$$\leq 1 + \pi(\delta_2 - 1) = 1 + \pi(\theta(\delta_2 - 1)).$$

for any $x_2 \in \{0, \ldots, \min(\delta_2, c_2)\}$. Thus, $(x_1^*, x_2^*)$ is an optimal policy for DPF$_2$.

**Case 2: $\delta_2 > \theta(c_1)$:** In this case, Equation (15) prescribes that $\ell_2 = 2$, $x_2^* = \theta(c_1)$ and $x_1^* = \theta(\min(\delta_2 - \theta(c_1), c_2))$. There are three possible categories to choose $x_2$:

(i) **Choose $x_2 \leq q$:** We have

$$V_2(x_2, \delta_2) = \pi(x_2) + \pi(\min(\delta_2 - x_2, c_1))$$

$$\leq \pi(q) + \pi(c_1)$$

$$= 1 + \pi(x_1^*) \leq \pi(x_2^*) + \pi(x_1^*).$$

Hence, there is no policy better than $(x_1^*, x_2^*)$ when $x_2 \leq q$.

(ii) **Choose $q < x_2 < \delta_2 - q$:** Since $x_2 \leq c_2 \leq c_1$, the choice of $x_2 > q$ can be feasible if $c_2 > q$ and $c_1 > q$. By utilizing Lemma 2, we have

$$V_2(x_2, \delta_2) = \pi(x_2) + \pi(\min(\delta_2 - x_2, c_1))$$

$$\leq \pi(x_2) + \pi(\delta_2 - x_2)$$

$$\leq 1 + \pi\left(\delta_2 - \left\lfloor \frac{q + 1}{2}\right\rfloor\right).$$

Now it is enough to show that the right hand side of the last inequality is less than or equal to $\pi(x_1^*) + \pi(x_2^*)$. For this purpose, we first consider the case $\delta_2 - \theta(c_1) \leq q$. In this case, we obtain $x_2^* = \theta(\delta_2 - \theta(c_1))$ with $\pi(x_2^*) = 1$ because $c_2 > q$ and $\delta_2 > \theta(c_1)$. Therefore, we have $\pi(x_1^*) + \pi(x_2^*) = 1 + \pi(c_1)$.

Given that $\delta_2 - \theta(c_1) \leq q$, we conclude that

$$\pi\left(\delta_2 - \left\lfloor \frac{q + 1}{2}\right\rfloor\right) \leq \pi\left(\theta(c_1) + q - \left\lfloor \frac{q + 1}{2}\right\rfloor\right).$$

So, it is enough to show that the right hand side of the last inequality is equal to $\pi(c_1)$. First note that $c_1 > q$ which implies that $\theta(c_1) > q$ (see the proof of Lemma 1). On the other hand, $q - \left\lfloor \frac{q + 1}{2}\right\rfloor \geq 0$ for any $q \in \mathbb{Z}_+$. Therefore we have

$$\pi\left(\theta(c_1) + q - \left\lfloor \frac{q + 1}{2}\right\rfloor\right) = \pi\left((\pi(c_1) - 1)\left\lfloor \frac{q + 1}{2}\right\rfloor + q\right)$$

$$= \pi(c_1) - 1 = \pi(c_1).$$

Next we consider the case $\delta_2 - \theta(c_1) > q$, in which we have

$$\pi(x_1^*) + \pi(x_2^*) = \pi(c_1) + \pi(\min(\delta_2 - \theta(c_1), c_2)).$$
If $\delta_2 - \theta(c_1) > c_2$, then $\pi(x_1^*) + \pi(x_2^*) = \pi(c_1) + \pi(c_2)$ which is clearly an upper bound for $V_2(x_2, \delta_2)$. This implies that there is no policy better than $(x_1^*, x_2^*)$ in this case. Therefore, we assume that $q < \delta_2 - \theta(c_1) \leq c_2$ which follows that

$$\pi(x_1^*) + \pi(x_2^*) = \pi(c_1) + \pi(c_2 - \theta(c_1))$$

$$= \pi(c_1) + \pi\left( \delta_2 - (\pi(c_1) - 1) \left\lfloor \frac{q + 1}{2} \right\rfloor - \left\lfloor \frac{q + 1}{2} \right\rfloor \right)$$

$$= \pi(c_1) - (\pi(c_1) - 1) + \left\lceil \frac{\delta_2 - \left\lfloor \frac{q + 1}{2} \right\rfloor - q}{q} \right\rceil + 1$$

$$= 1 + \pi\left( \delta_2 - \left\lfloor \frac{q + 1}{2} \right\rfloor \right).$$

Note that $\delta_2 > \theta(c_1) + q > 2q$ which follows that $\delta_2 - \left\lfloor \frac{q + 1}{2} \right\rfloor > q$.

(iii) Choose $x_2 \geq \delta_2 - q$: In this case $\delta_2 - x_2 \leq q$ and because $x_2 \leq c_2 \leq c_1$ we can write

$$V_2(x_2, \delta_2) = \pi(x_2) + \pi(\min(c_1, \delta_2 - x_2))$$

$$\leq \pi(x_2) + \pi(\delta_2 - x_2) \leq \pi(x_2) + 1$$

$$\leq \pi(c_1) + 1 \leq \pi(x_1^*) + \pi(x_2^*).$$

Now, let us assume that the given policy (15) is optimal for $\text{DPF}_k$, where $k \geq 2$. We show that it is also optimal for $\text{DPF}_{k+1}$. For $j = k + 1$ we have

$$V_{k+1}(\delta_{k+1}) = \max_{x_{k+1} \in \{1, \ldots, \min(c_{k+1}, \delta_{k+1})\}} V_{k+1}(x_{k+1}, \delta_{k+1}),$$

in which $\delta_{k+1} \geq k + 1$ and

$$V_{k+1}(x_{k+1}, \delta_{k+1}) = \pi(x_{k+1}) + V_k^*(\delta_{k+1} - x_{k+1}).$$

Note that we assume $x_{k+1} \geq 1$ which is valid due to Lemma 7. From the induction hypothesis we know that Equation (15) can be used to obtain $V_k^*(\delta_{k+1} - x_{k+1})$. Let $x_1^*, \ldots, x_k^*$ be the resulting optimal policy, that is,

$$x_i^* = \begin{cases} 
\theta(c_i) & i = 1, \ldots, \ell_k - 1 \\
\theta(\min(\lambda_{\ell_k}, c_{\ell_k})) & i = \ell_k \\
1 & i = \ell_k + 1, \ldots, k,
\end{cases}$$

where $\ell_k \in \{1, \ldots, k\}$ is the largest integer such that

$$\lambda_{\ell_k} = (\delta_{k+1} - x_{k+1} - k + 1) - \sum_{r=1}^{\ell_k-1} (\theta(c_r) - 1) \geq 1.$$

Furthermore, let $z = (z_1, \ldots, z_{k+1})$ be the solution given by Equation (15) to solve $\text{DPF}_{k+1}$. The proof is completed if we can show that

$$V_{k+1}(x_{k+1}, \delta_{k+1}) \leq \sum_{i=1}^{k+1} \pi(z_i),$$

for any $x_{k+1} \in \{1, \ldots, \min(c_{k+1}, \delta_{k+1})\}$. In other words, it is enough to show that the solution $u := (x_1^*, \ldots, x_k^*, x_{k+1})$ is not better than $z$. To this end, we first define the solution $u = (u_1, \ldots, u_{k+1})$ as follows:

$$u_i := \begin{cases} 
x_i^* & i = 1, \ldots, \ell_k \\
x_{k+1} & i = \ell_k + 1 \\
1 & i = \ell_k + 2, \ldots, k + 1.
\end{cases}$$
Given this policy, we can conclude that completes the proof.

Furthermore, \(\xi\) is the remaining unmet demand that can be assigned to \(x_{\ell_k}\) and \(x_{k+1}\). Note that \(\xi \geq x_{\ell_k}^* + x_{k+1} \geq 2\). According to the induction hypothesis for \(j = 2\), Equation (15) gives an optimal policy for assigning \(\xi\) to \(x_{\ell_k}\) and \(x_{k+1}\) as follows:

\[
(w_{\ell_k}, w_{k+1}) = \begin{cases} 
(\theta(\min(\xi - 1, c_{\ell_k})), 1) & 2 \leq \xi \leq \theta(c_{\ell_k}) \\
(\theta(c_{\ell_k}), \theta(\min(\xi - \theta(c_{\ell_k}), c_{k+1}))) & \xi \geq \theta(c_{\ell_k}) + 1.
\end{cases}
\]

Given this policy, we can conclude that

\[
V_{k+1}(x_{k+1}, \delta_{k+1}) = \sum_{i=1}^{k+1} \pi(u_i) \leq \sum_{i=1}^{k+1} \pi(w_i).
\]

Furthermore, \(w\) coincides with Equation (15) for \(\text{DPF}_{k+1}\) which implies that \(w = z\). This completes the proof. □

We summarize the results obtained in this section in Table 1. This table presents the optimal value of the MDF which is the maximum achievable number of vehicles based on our assumptions.

| Case | Optimal value |
|------|--------------|
| \(0 \leq \Delta \leq n\) | \(\Delta\) |
| \(n \leq \Delta < n + q\) | \(n\) |
| \(n + q \leq \Delta \leq c_1\) | \(n - 1 + \pi(\Delta - n + 1)\) |
| \(\max(n + q, c_1) \leq \Delta\) | \(n - 1 + \pi(\min(\lambda_{\ell_n}, c_{\ell_n})) + \sum_{i=1}^{\ell_n} \pi(c_i) - 1)\) |

\(^\dagger\) \(\ell_n \leq n\) is the largest integer such that \(\lambda_{\ell_n} = (\Delta - n + 1) - \sum_{k=1}^{\ell_n} \pi(c_k) - 1\) \(\geq 1\).

5 Conclusion

In this paper, we studied the maximum number of homogeneous vehicles in node routing problems with integer splits. We assumed that all parameters as well as the vehicle loads are integers. Furthermore, each vehicle can make a delivery to exactly one depot, and the aggregate load for each pair of vehicles exceeds the capacity of one vehicle. We first obtained a closed-form for the maximum achievable number of vehicles when there is only one predetermined depot. We
then used this upper bound to develop an optimization problem whose optimal value gives the maximum achievable number of vehicles for multiple capacitated depots. The desirable characteristics of the proposed upper bound is that it is optimal in the sense that it gives the maximum achievable number of vehicles based on our assumptions for any given instance. Furthermore, they can be computed very efficiently in $O(n)$.

A future research direction for this work is to obtain stronger bounds by extending the set of assumptions. For example, we may consider a particular routing problem and obtain better upper bounds by considering more details such as routing costs. It would also be interesting to know how we can obtain better theoretical bounds on the number of vehicles if we have a heuristic bound on the optimal objective value of a routing problem. This can be helpful particularly for exact methods that seek a global optimal solution. The upper bounds obtained in this paper are also valid for node routing problems in which split deliveries are not allowed and a vehicle can deliver to any number of depots. However, these upper bounds are not the best possible bounds that can be found for those routing problems. Therefore, another research direction is to develop better theoretical bounds for those routing problems.

**Acknowledgments.** We would like to thank Mojtaba Heydar for discussing this problem with us. This study was funded by the Australian Research Council (Grant ID: IC140100032).

**References**

Archetti, C., Bianchessi, N., and Speranza, M. G. (2011). A column generation approach for the split delivery vehicle routing problem. *Networks*, 58(4):241–254.

Archetti, C. and Speranza, M. G. (2008). The split delivery vehicle routing problem: A survey. In Golden, B., Raghavan, S., and Wasil, E., editors, *The Vehicle Routing Problem: Latest Advances and New Challenges*, pages 103–122. Springer US.

Archetti, C., Speranza, M. G., and Hertz, A. (2006). A tabu search algorithm for the split delivery vehicle routing problem. *Transportation science*, 40(1):64–73.

Chandran, B. and Raghavan, S. (2008). Modeling and solving the capacitated vehicle routing problem on trees. In Golden, B., Raghavan, S., and Wasil, E., editors, *The Vehicle Routing Problem: Latest Advances and New Challenges*, pages 239–261. Springer US.

Graham, R. L., Knuth, D. E., and Patashnik, O. (1994). Integer functions. In *Concrete Mathematics: A Foundation for Computer Science, Second Edition*, chapter 3, pages 67–101. Addison-Wesley Longman Publishing Co., Inc.

Koç, Ç., Bektaş, T., Jabali, O., and Laporte, G. (2016). Thirty years of heterogeneous vehicle routing. *European Journal of Operational Research*, 249(1):1–21.

Labbé, M., Laporte, G., and Mercure, H. (1991). Capacitated vehicle routing on trees. *Operations Research*, 39(4):616–622.

Lee, C.-G., Epelman, M. A., White, C. C., and Bozer, Y. A. (2006). A shortest path approach to the multiple-vehicle routing problem with split pick-ups. *Transportation research part B: Methodological*, 40(4):265–284.

Toth, P. and Vigo, D. (2014). *Vehicle routing: problems, methods, and applications*. SIAM.