ENTROPY DECAY FOR NON-LOCAL DIFFUSION EQUATIONS WITH Ornstein-Uhlenbeck OPERATORS

ANTONIO AGRESTI, PAOLA LORETI, AND DANIELA SFORZA

Abstract. In this paper we establish the decay of the entropy for non-local in time Ornstein-Uhlenbeck equations. We show that the logarithmic Sobolev inequality for the Gaussian measure on \( \mathbb{R}^d \) is equivalent to an entropic estimate for the solution of non-local equations. We are able to give, for some class of kernels, the explicit rate for the entropy decay.

1. Introduction

In this paper we study entropy decay estimates for the following non-local in time Ornstein-Uhlenbeck flow:
\[
\frac{d}{dt} \left( u(x,t) + (k \ast u(x,\cdot))(t) \right) = \Delta u(x,t) - \alpha x \cdot \nabla u(x,t) + k(t)u_0(x),
\]
for \( t > 0 \) and \( x \in \mathbb{R}^d \).

Here, as usual, for \( k, f \in L^1_{\text{loc}}(0,\infty) \), \( k \ast f(t) := \int_0^t k(t-s)f(s)ds \) denotes the convolution between functions and \( \alpha > 0 \) is a positive constant. In the following, if it will be clear by the context, in (1.1) we suppress the dependence on \( x \in \mathbb{R}^d \), therefore we write \( u(t) \) instead of \( u(x,t) \). Non-local in time equations arise naturally in the study of viscoelasticity, heat conduction with memory and others, see e.g. [2, 7, 12, 13, 18] and the references therein, and they have been studied for a long time, see the monograph [16].

Entropic estimates are also well known. Indeed, in the case without memory (that is \( k \equiv 0 \)), it is well known that entropic estimates are equivalent to logarithmic Sobolev inequalities, see e.g. [1, Chapter 5]. Moreover, in the same book, for the case without memory, the following decay of the entropy is established
\[
\text{Ent}(u(\cdot,t)) \leq e^{-2\alpha t} \text{Ent}(u_0),
\]
where, for a positive function \( f \) such that \( f \in L^1(\gamma_\alpha) \) and \( \int_{\mathbb{R}^d} |f| d\gamma_\alpha < \infty \), the entropy is defined as
\[
\text{Ent} f := \int_{\mathbb{R}^d} f(x) \ln f(x) d\gamma_\alpha - \left( \int_{\mathbb{R}^d} f(x) d\gamma_\alpha \right) \ln \left( \int_{\mathbb{R}^d} f(x) d\gamma_\alpha \right),
\]
and
\[
d\gamma_\alpha(x) := \left( \frac{\alpha}{2\pi} \right)^\frac{d}{2} e^{-\frac{\alpha|x|^2}{2}} dx.
\]

We observe that the equation (1.1) differs from the one considered in [9]. Indeed, our choice allows us to recover the case without memory by setting \( k \equiv 0 \) which is
not possible with the memory contribution chosen in [9]. The equation (1.1) makes possible the comparison between the case with and without memory.

Our main results concern the extension of the estimates like (1.2) to equation (1.1). Indeed we obtain, under suitable assumptions on $k$, the following generalization of (1.2)

$$\text{Ent}(u(\cdot,t)) \leq s_{2\alpha}(t)\text{Ent}(u_0),$$

where $s_{2\alpha}(t)$ is defined in Subsection 2.4.

Inequalities of the form (1.4) appear in [9] for the first time in the studying of close related class of equations. We remark that, in contrast to [9, Theorem 1.1] the existence of smooth solutions it is not assumed but it is explicitly constructed. The reader should be aware of the fact that solutions to non-local in time evolution equations may show modest regularity in time, thus it is unclear to us when the results of [9] apply. Although the existence theory for (1.1) can be formulated thanks to [16], we follow the strategy to give a detailed and rigorous proof adapted to our case in order to combine the requested assumptions of the related theorems with the regularity of the solutions. This makes the proof simple and clear.

Furthermore, we remark that our results does not follows from the one [9] due to the different choice of the memory term. Our interest consists in the study of the contribution of the integral term in the analysis of the constants with a detailed study of the parameters which defines the kernel. However the class of kernels we are concerned is more regular the one in [9].

Let us point it out that our method seems to be flexible enough to replace the Ornstein-Uhlenbeck operator with a generic Markov diffusion operator (see [1]) if the underline measure space is finite. For instance, this covers the Laplace operator on closed compact manifolds. Moreover our approach may be extended to cover operators of the form

$$L_W := \Delta - \nabla W \cdot \nabla.$$

Such operators and related logarithmic type Sobolev inequalities were considered in [10] under suitable assumptions on the potential $W$. In this paper we consider the case $W(x) = \frac{2}{d}|x|^2$.

Another possible extension may be the study of the decay of the so called $\Phi$-entropy, that is

$$\text{Ent}_\Phi f := \int_{\mathbb{R}^d} \Phi(f)d\gamma_\alpha - \Phi \left( \int_{\mathbb{R}^d} fd\gamma_\alpha \right),$$

where $\Phi : \mathcal{W} \to \mathbb{R}$ is a suitable function and $f$ takes values in $\mathcal{W}$. Thus, if $\Phi(x) = x \ln x$ and $\mathcal{W} = (0, \infty)$ we recover (1.3). For other details on this topic see [1, Section 7.6].

A further study related to our line of research is the analysis of (1.1) where $k(t) = t^{-\beta}$ for some $\beta \in (0, 1)$. This will be done in a future work.

1.0.1. Structure of the paper. In Section 2 we collect basic facts on the Ornstein-Uhlenbeck operator, logarithmic Sobolev inequality and evolutionary integral equations. Section 3 is devoted to the statement and the proof of the main results, i.e. Theorems 3.1-3.2. Section 4 contains explicit examples of kernel $k$ to which our theorems apply for the stretched exponential, see Example 4.2 below. In particular, for the special case $k(t) = \nu e^{-t} \geq 0$, where $\nu > 0$, we are able to compute explicitly the function $s_{2\alpha}(t)$. 


2. Preliminaries

2.1. Notations. For any $T \in (0, \infty]$ and $p = 1, 2, \infty$ we denote by $L^p(0,T)$ the usual spaces of measurable functions $v : (0, T) \to \mathbb{R}$ such that one has

$$
\|v\|_{p,T}^p := \int_0^T |v(t)|^p \, dt < \infty, \quad p = 1, 2,
$$

$$
\|v\|_{\infty,T} := \sup_{0 \leq t \leq T} |v(t)| < \infty,
$$

respectively. We shall use the shorter notation $\|v\|_p$ for $\|v\|_{p,\infty}$, $p = 1, 2, \infty$. We denote by $L^1_{loc}(0,\infty)$ the space of functions belonging to $L^1(0, T)$ for any $T \in (0, \infty)$.

For any $\psi, v \in L^1_{loc}(0,\infty)$ the symbol $\psi * v$ stands for convolution from 0 to $t$, that is

$$
\psi * v(t) = \int_0^t \psi(t-s)v(s) \, ds, \quad t \in [0, T].
$$

In the following we denote by $W^{1,1}(0,T)$ the Banach space of functions $v \in L^1(0,T)$ such that $\dot{v} \in L^1(0,T)$ and by $W^{1,1}_{loc}(0,\infty)$ the space of functions belonging to $W^{1,1}(0,T)$ for any $T \in (0, \infty)$.

As usual, we denote the Laplace transform of a function $f \in L^1_{loc}(0,\infty)$ having sub-exponential growth (i.e. for all $\omega > 0$, $\int_0^\infty e^{-\omega t} |f(t)| dt < \infty$) by

$$
\hat{f}(\lambda) := \int_0^\infty e^{-\lambda t} f(t) dt \quad \lambda \in \mathbb{C}, \; \Re \lambda > 0.
$$

2.2. The Ornstein-Uhlenbeck operator. As in the introduction, we denote by

$$
\gamma_\alpha(x) := \left(\frac{\alpha}{2\pi}\right)^{\frac{d}{2}} e^{-\frac{\alpha |x|^2}{2}}
$$

the a Gaussian distribution on $\mathbb{R}^d$ where $d \geq 1$ is an integer. In addition, we set $d\gamma_\alpha(x) := \gamma_\alpha(x) \, dx$ the associated probability measure on $\mathbb{R}^d$. In the case $\alpha = 1$ we write $\gamma := \gamma_\alpha$.

In addition, we denote by $L^2(\gamma_\alpha)$ is the set of all measurable maps $f : \mathbb{R}^d \to \mathbb{R}$ such that $\int_{\mathbb{R}^d} |f|^2 \, d\gamma_\alpha < \infty$ endowed with the usual norm.

There are several way to introduce the Ornstein-Uhlenbeck operator on $L^2(\gamma_\alpha)$. For future convenience we follow [6] and we use the form method. Let $H^1(\gamma_\alpha)$ the set of all $f \in L^2(\gamma_\alpha)$ such $\nabla f \in L^2(\gamma_\alpha)$ (the gradient is understood in the sense of distribution) and

$$
\|f\|_{H^1(\gamma_\alpha)} := \|f\|_{L^2(\gamma_\alpha)} + \|\nabla f\|_{L^2(\gamma_\alpha)}.
$$

Define the bilinear symmetric form $\mathcal{L} : H^1(\gamma_\alpha) \times H^1(\gamma_\alpha) \to \mathbb{R}$ as

$$
\mathcal{L}_\alpha(f,g) := \int_{\mathbb{R}^d} \nabla f \cdot \nabla g \, d\gamma_\alpha.
$$

Since the bilinear form is symmetric, it is immediate to see that $\mathcal{L}_\alpha$ induces a positive self-adjoint operator $-L_\alpha$ on $L^2(\gamma_\alpha)$ such that

$$
-(L_\alpha f,g)_{L^2(\gamma_\alpha)} = \mathcal{L}_\alpha(f,g), \quad \forall g \in H^1(\gamma_\alpha), f \in D(L_\alpha),
$$

where

$$
D(L_\alpha) = \{ f \in H^1(\gamma_\alpha) : \Delta f - \alpha x \cdot \nabla f \in L^2(\gamma_\alpha) \},
$$

$$
L_\alpha f = \Delta f - \alpha x \cdot \nabla f, \quad f \in D(L_\alpha).
$$
It is well known that the Ornstein-Uhlenbeck generates an analytic semigroup $(T(t))_{t \geq 0}$ on $L^2(\gamma_\alpha)$, see e.g. [3] or [14] formula (2),
\[ (T(t)f)(x) = \left(\frac{\alpha}{2\pi(e^{2t\alpha} - 1)}\right)^{\frac{d}{2}} \int_{\mathbb{R}^d} e^{-\frac{|x|^2}{2\alpha(e^{2t\alpha} - 1)}} f(e^{t\alpha}x - y) dy, \]
x \in \mathbb{R}^d \text{ and } f \in L^2(\gamma_\alpha). \text{ Let us recall the following useful fact.}

Lemma 2.1. Let $L_\alpha$ the Ornstein-Uhlenbeck operator defined above. Then, for each $f \in D(L_\alpha)$ there exists a sequence \( \{f_j\} \subset D(L_\alpha) \cap C_0^\infty(\mathbb{R}^d) \) such that $\nabla f_j \to \nabla f$ in $L^2(\gamma_\alpha)$ and $L_\alpha f_j \to L_\alpha f$ in $D(L_\alpha)$.

Proof. Let $\phi$ be a $C_c^\infty(\mathbb{R})$ function such that, $0 \leq \phi \leq 1$, $\phi(x) = 1$ if $|x| \leq 1$ and $\phi(x) = 0$ if $|x| \geq 2$. Let $R > 0$ and $\phi_R := \phi(\cdot/R)$. Then, by Lebesgue dominated convergence, $L_\alpha(\phi_R f) \to L_\alpha f$ and $\nabla(\phi_R f) \to \nabla f$ in $L^2(\gamma_\alpha)$ as $R \to \infty$.

Since $\phi_R f$ has compact support for any $R > 0$ then the claim follows by the density of smooth function in the classical Sobolev spaces. \qed

2.3. Entropy and Logarithmic Sobolev inequality. In the mathematical literature, for a non-negative measurable function $f$ such that $\int_{\mathbb{R}^d} f \ln f \, d\gamma_\alpha < \infty$ (where $0 \ln 0 := 0$), the entropy of $f$ is defined as
\[ \text{Ent}_f := \int_{\mathbb{R}^d} f \ln f \, d\gamma_\alpha - \left( \int_{\mathbb{R}^d} f \, d\gamma_\alpha \right) \ln \left( \int_{\mathbb{R}^d} f \, d\gamma_\alpha \right). \]
Note that, by Jensen inequality applied to $x \ln x$, it follows that $\text{Ent}_f \geq 0$. Moreover, with simple computations one can readily check that
\[ \text{Ent}(cf) = c \text{Ent}(f), \]
for each $c \in (0, \infty)$ and $f$ as above.

For future convenience, let us recall the following logarithmic Sobolev inequality.

Theorem 2.2. Let $f : \mathbb{R}^d \to \mathbb{R}$ be a measurable function such that $f \in H^1(\gamma_\alpha)$ then $f^2 \ln f^2 \in L^1(\gamma_\alpha)$. Moreover,
\[ \text{Ent}(f^2) \leq \frac{2}{\alpha} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_\alpha. \]
Lastly, the constant in the above inequality is optimal.

Proof. The proof in the case $\alpha = 1$ can be found in [6]; see also [1, Proposition 5.5.1].

The case $\alpha \neq 1$ can be recover from the latter one. Indeed, let $f : \mathbb{R}^d \to \mathbb{R}$ and set $f_\alpha := f(\cdot/\sqrt{\alpha})$. One can readily check that $\int_{\mathbb{R}^d} |f| \, d\gamma_\alpha = \int_{\mathbb{R}^d} |f_\alpha|^2 \, d\gamma_\alpha$.

Now, let $f \in H^1(\gamma_\alpha)$. Simple computations shows that $f \in H^1(\gamma_\alpha)$ if and only if $f_\alpha \in H^1(\gamma)$. Thus,
\[ \text{Ent}_f = \int_{\mathbb{R}^d} f \ln f \, d\gamma_\alpha - \left( \int_{\mathbb{R}^d} f \, d\gamma_\alpha \right) \ln \left( \int_{\mathbb{R}^d} f \, d\gamma_\alpha \right) \]
\[ = \int_{\mathbb{R}^d} f_\alpha \ln f_\alpha \, d\gamma - \left( \int_{\mathbb{R}^d} f_\alpha \, d\gamma \right) \ln \left( \int_{\mathbb{R}^d} f_\alpha \, d\gamma \right) \]
\[ \leq 2 \int_{\mathbb{R}^d} |\nabla f_\alpha|^2 \, d\gamma = \frac{2}{\alpha} \int_{\mathbb{R}^d} |\nabla f(\cdot/\sqrt{\alpha})|^2 \, d\gamma(x) = \frac{2}{\alpha} \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_\alpha. \]
The optimality of the constant in the case $\alpha \neq 1$ follows by the above argument and the optimality in the case $\alpha = 1$. \qed
For future purpose it is important to reformulate the above inequality using the Fisher information, which will be denote by $I_g$, where $g$ is positive.

To define $I_g$, let $g : \mathbb{R}^d \rightarrow \mathbb{R}^+$ be such that $g \in L^2(\gamma_\alpha)$ and $\nabla g \in L^2(\gamma_\alpha)$. The Fisher information of $g$ as

$$I_g := \int_{\mathbb{R}^d} \frac{|\nabla g|^2}{g} \, d\gamma_\alpha := \lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \frac{|\nabla g|^2}{g + \varepsilon} \, d\gamma_\alpha \in [0, \infty].$$

The following Lemma gives the formulation of the logarithmic Sobolev inequality in terms of the Fisher information $I$.

**Lemma 2.3.** Let $C > 0$ be a positive constant. The following assertion are equivalent.

1. The logarithmic Sobolev inequality holds, i.e. for all $f \in H^1(\gamma_\alpha)$,

$$\text{Ent}(f^2) \leq C \int_{\mathbb{R}^d} |\nabla f|^2 \, d\gamma_\alpha.$$

2. The logarithmic inequality in the Fisher information form holds, i.e. $g \in H^1(\gamma_\alpha)$ such that $I_g < \infty$,

$$\text{Ent} g \leq \frac{C}{4} I_g.$$

**Proof.** The argument is quite simple and it is recalled in [1, p. 237]. We recall the proof of the implication $(1) \Rightarrow (2)$.

Let $\varepsilon > 0$. One can apply the inequality in $(1)$ to $f := \sqrt{g + \varepsilon}$ and one obtains

$$\int_{\mathbb{R}^d} (g + \varepsilon) \ln(g + \varepsilon) \, d\gamma_\alpha \leq \frac{1}{2\alpha} \int_{\mathbb{R}^d} \frac{|\nabla g|^2}{g + \varepsilon} \, d\gamma_\alpha + \left( \int_{\mathbb{R}^d} (g + \varepsilon) \, d\gamma_\alpha \right) \ln \left( \int_{\mathbb{R}^d} (g + \varepsilon) \, d\gamma_\alpha \right).$$

Since $g$ is positive $\int_{\mathbb{R}^d} \frac{|\nabla g|^2}{g + \varepsilon} \, d\gamma_\alpha \leq \frac{1}{2\alpha} \|\nabla g\|_{L^2(\gamma_\alpha)}$ for each $\varepsilon > 0$. In addition, the map $\varepsilon \mapsto \int_{\mathbb{R}^d} \frac{|\nabla g|^2}{g + \varepsilon} \, d\gamma_\alpha$ is non-decreasing. This implies that $\lim_{\varepsilon \to 0} \int_{\mathbb{R}^d} \frac{|\nabla g|^2}{g + \varepsilon} \, d\gamma_\alpha$ exists in $[0, \infty]$. Using Fatou’s lemma in (2.3) and the above notation one has

$$\text{Ent} g \leq \frac{1}{2\alpha} I_g,$$

for all $g$ such that $g \geq 0$ $d\gamma_\alpha$-a.e., $g \in L^2(\gamma_\alpha)$ and $\nabla g \in L^2(\gamma_\alpha)$. \hfill $\Box$

**Remark 2.4.** Theorem 2.2 and Lemma 2.3 shows that the inequality in (2) holds with $C/4$ replaced by $1/(2\alpha)$. However, in the following we will need the general statement given in Lemma 2.3.

Integration by parts formula plays an important role through the article.

**Proposition 2.5.** Let $\mathcal{U} \subset \mathbb{R}$ be an open set and let $f \in D(L_\alpha)$ and $g \in H^1(\gamma_\alpha)$ be such that $g \in \mathcal{U}$ $\gamma_\alpha$-a.e. on $\mathbb{R}^d$. Let $\Phi : \mathcal{U} \rightarrow \mathbb{R}$ be a $C^1$ function such that $\Phi(g) \in L^2(\gamma_\alpha)$ and $\Phi'(g) \in L^\infty(\gamma_\alpha)$. Then the following integration by parts formulae holds:

$$\int_{\mathbb{R}^d} L_\alpha f \Phi(g) \, d\gamma_\alpha = - \int_{\mathbb{R}^d} \Phi'(g) \nabla f \cdot \nabla g \, d\gamma_\alpha.$$

Although the proof is elementary, we include the details for completeness.
Proof. First, by Lemma 2.1 and the the fact that \(\Phi(g) \in L^2(\gamma_\alpha)\) and \(\Phi'(g) \in L^\infty(\gamma_\alpha)\), it is enough to show the equality if \(f\) is replaced by \(f \in D(L_\alpha) \cap C^\infty_\alpha(\mathbb{R}^d)\).

Choose \(R > 0\) such that \(\text{supp}(f) \subset B_R\) where \(B_R := \{|x| \leq R\}\), then

\[
\int_{\mathbb{R}^d} L_\alpha f \Phi(g) d\gamma_\alpha = \int_{B_R} L_\alpha f \Phi(g) d\gamma_\alpha
= \int_{B_R} \Delta f \Phi(g) d\gamma_\alpha - \int_{B_R} \alpha x \cdot \nabla f \Phi(g) d\gamma_\alpha
= -\int_{B_R} \Phi'(g) \nabla f \cdot \nabla \Phi d\gamma_\alpha.
\]

\[\Box\]

2.4. **Evolutionary integral equations.** This section is devoted to recall some well-known notions and results about integral equations.

Classical results for integral equations (see, e.g., [4, Theorem 2.3.5]) ensure that, for any kernel \(k \in L^1_{\text{loc}}(0, \infty)\) and any \(g \in L^1_{\text{loc}}(0, \infty)\), the problem

\[
(2.4) \quad \varphi(t) + k \ast \varphi(t) = g(t), \quad t \geq 0,
\]

admits a unique solution \(\varphi \in L^1_{\text{loc}}(0, \infty)\). Moreover, if \(g \in W^{1,1}_{\text{loc}}(0, \infty)\), then we have \(\varphi \in W^{1,1}_{\text{loc}}(0, \infty)\) too.

It is useful to recall the following result, see [11, Lemma 1.3].

**Lemma 2.6.** If \(k \in L^1_{\text{loc}}(0, \infty)\) is non-negative and non-increasing and \(g \in L^1_{\text{loc}}(0, \infty)\) is non-negative and non-decreasing, then the solution \(\varphi\) of the integral equation (2.4) satisfies

\[
(2.5) \quad 0 \leq \varphi(t) \leq g(t) \quad \text{for a.e. } t \geq 0.
\]

Given \(b \in L^1_{\text{loc}}(0, \infty)\), recall that \(b\) is a kernel of positive type if

\[
(2.6) \quad \int_0^T b \ast v(t) v(t) \, dt \geq 0, \quad \text{for any } T > 0, v \in L^2(0, T).
\]

If \(b \in L^\infty(0, \infty)\), \(b\) is of positive type if and only if

\[
(2.7) \quad \Re \hat{b}(\lambda) \geq 0 \quad \text{for any } \lambda \in \mathbb{C}, \quad \Re \lambda > 0
\]

(see, e.g., [16, p.38]).

Also, \(b\) is said to be a completely positive kernel if there exists \(k \in W^{1,1}_{\text{loc}}(0, \infty)\) non-negative and non-increasing such that

\[
(2.8) \quad b(t) + \int_0^t k(t - s) b(s) ds = 1, \quad \forall t > 0.
\]

**Lemma 2.7.** If \(b\) is a completely positive kernel, then we have

i) \(b \in W^{2,1}_{\text{loc}}(0, \infty), \quad 0 \leq b(t) \leq 1 \quad \forall t \geq 0.\)

ii) If \(k\) is the function in (2.8), then we have

\[
(2.9) \quad \hat{b}(\lambda) = \frac{1}{\lambda(1 + k(\lambda))}, \quad \Re \lambda > 0.
\]

iii) \(b\) is a kernel of positive type.
iv) For any \( u_0 \in \mathbb{R} \) and \( f \in L^1_{loc}(0, \infty) \) \( u(t) \) satisfies the identity

\[
(2.10) \quad u(t) = u_0 + b \ast f(t)
\]

if and only if the following holds

\[
\begin{cases}
\frac{d}{dt} (u + k \ast u)(t) = k(t)u_0 + f(t), \quad t > 0 \\
u(0) = u_0.
\end{cases}
\]

Proof. i) Let \( k \in W^{1,1}_{loc}(0, \infty) \) the non-negative and non-increasing function such that (2.8) holds. We can apply Lemma 2.6 with \( g(t) \equiv 1 \) to obtain \( 0 \leq b(t) \leq 1 \) for any \( t \geq 0 \).

ii) Thanks to i) and \( 0 \leq k(t) \leq k(0), \) \( t \geq 0, \) we have \( b, k \in L^{\infty}(0, \infty) \). Therefore we can take the Laplace transform of equation (2.8) to get

\[
\mathcal{L}b(\lambda)(1 + \mathcal{L}k(\lambda)) = \frac{1}{\lambda}, \quad \forall \Re \lambda > 0,
\]

and hence \( 1 + \mathcal{L}k(\lambda) \neq 0, \Re \lambda > 0, \) and (2.9) holds.

iii) Since \( b \in L^{\infty}(0, \infty) \) we will prove (2.7). Indeed, from (2.9) we deduce for \( \Re \lambda > 0 \)

\[
\Re \mathcal{L}b(\lambda) = \frac{\Re \lambda + \Re \lambda \Re \mathcal{L}k(\lambda) - \Im \lambda \Im \mathcal{L}k(\lambda)}{|\lambda(1 + \mathcal{L}k(\lambda))|^2}.
\]

Integrating by parts, we have

\[
\Re \lambda \Re \mathcal{L}k(\lambda) = \Re \lambda \int_{0}^{\infty} e^{-\Re \lambda t} \cos(\Im \lambda t)k(t) \, dt = -\int_{0}^{\infty} \partial_t(e^{-\Re \lambda t}) \cos(\Im \lambda t)k(t) \, dt = k(0) + \Im \lambda \Im \mathcal{L}k(\lambda) + \int_{0}^{\infty} e^{-\Re \lambda t} \cos(\Im \lambda t) \mathcal{L}k(t) \, dt.
\]

Thanks to \( \dot{k}(t) \leq 0 \) we note that

\[
k(0) + \int_{0}^{\infty} e^{-\Re \lambda t} \cos(\Im \lambda t) \dot{k}(t) \, dt = \int_{0}^{\infty} (e^{-\Re \lambda t} \cos(\Im \lambda t) - 1) \dot{k}(t) \, dt \geq 0,
\]

and hence

\[
\Re \lambda \Re \mathcal{L}k(\lambda) - \Im \lambda \Im \mathcal{L}k(\lambda) \geq 0,
\]

that is \( \Re \mathcal{L}b(\lambda) > 0 \) for \( \Re \lambda > 0 \). 

For future convenience let us introduce the functions \( s_{\mu}(t) \) associated to a completely positive kernel \( b \). By [16, Proposition 4.5], for each \( \mu > 0 \) there exists \( s_{\mu} \in W^{1,1}_{loc}(0, \infty) \) positive and non-increasing such that

\[
(2.11) \quad s_{\mu}(t) + \mu b \ast s_{\mu}(t) = 1 \quad \forall t > 0.
\]

Thanks to Lemma 2.7-iv), equation (2.11) can be written in the equivalent form

\[
(2.12) \quad \dot{s}_{\mu}(t) + k \ast \dot{s}_{\mu}(t) + \mu s_{\mu}(t) = 0 \quad \forall t > 0, \quad s_{\mu}(0) = 1.
\]
We will study the well-posedness of the integro-differential problem
\begin{equation}
\tag{2.13}
\begin{aligned}
\frac{d}{dt}(u + k \ast u)(t) &= L_\alpha u(t) + k(t)u_0, \quad t > 0 \\
u(0) &= u_0.
\end{aligned}
\end{equation}

where the operator $L_\alpha$ is defined by (2.1) and the dependence on the spatial variable $x \in \mathbb{R}^d$ is not explicitly indicated.

Throughout the paper we will assume on the convolution kernel $k$ the following

**Assumption 2.8.** $k \in W^{1,1}_{ac}(0, \infty) \cap L^1(0, \infty)$ is non-negative and non-increasing.

As recalled in Section 2.4 there exists a unique solution $b \in W^{2,1}_{ac}(0, \infty)$ of the integral equation
\begin{equation}
\tag{2.14}
b(t) + \int_0^t k(t - s)b(s)ds = 1, \quad \forall t > 0.
\end{equation}

According to the definition $b$ is a completely positive kernel. By Lemma 2.7-iv) for any $u_0 \in D(L_\alpha)$ we have that $u \in C^1([0, \infty); D(L_\alpha))$ is a solution of (2.13) if and only if $u \in C([0, \infty); D(L_\alpha))$ is the solution of the integral equation
\begin{equation}
\tag{2.15}
u(t) = u_0 + \int_0^t b(t - s)L_\alpha u(s)ds, \quad t \geq 0.
\end{equation}

Therefore to solve (2.13) it is sufficient to prove the well-posedness for (2.15).

**Proposition 2.9.** Let Assumption 2.8 be satisfied. Then, there exists the resolvent for (2.15), that is a family $\{S(t)\}_{t \geq 0} \subset \mathcal{L}(L^2(\gamma_\alpha))$ of linear bounded operators in $L^2(\gamma_\alpha)$ such that
\begin{enumerate}
\item $S(0) = I$ and for any $f \in L^2(\gamma_\alpha)$ the map $t \mapsto S(t)f$ is continuous;
\item for any $f \in D(L_\alpha)$ and $t \geq 0$ $S(t)f \in D(L_\alpha)$ and $L_\alpha S(t)f = S(t)L_\alpha f$;
\item for any $u_0 \in D(L_\alpha)$ we have
\begin{equation}
\tag{2.16}
S(t)u_0 = u_0 + \int_0^t b(t - s)L_\alpha S(s)u_0ds, \quad t \geq 0.
\end{equation}
\end{enumerate}

Moreover, for any $f \in D(L_\alpha)$ $t \mapsto S(t)f$ is differentiable.

In particular, for any $u_0 \in D(L_\alpha)$ $u(t) := S(t)u_0$ is the solution of (2.13).

**Proof.** By Lemma 2.7-iii) $b$ is a kernel of positive type. Since $L_\alpha$ generates an analytic semigroup (see Subsection 2.2), we can apply [16, Corollary 3.1] to have that equation (2.15) is parabolic, and hence there exists $M > 0$ such that
\[\|\hat{I - \hat{b}(\lambda)A}^{-1}\|_{\mathcal{L}(L^2(\gamma_\alpha))} \leq M \quad \forall \Re \lambda > 0.\]

Moreover, in order to apply [16, Theorem 3.1], we have to show that $b$ is 1-regular, i.e. there exists $C > 0$ such that $|\lambda \hat{b'}(\lambda)| \leq C|\hat{b}(\lambda)|$ for all $\Re \lambda > 0$. Indeed, thanks to (2.9) we have
\[\frac{\lambda \hat{b'}(\lambda)}{\hat{b}(\lambda)} = \frac{1 + (\lambda \hat{k}(\lambda))'}{1 + \hat{k}(\lambda)}.
\]

Now, also by an integration by parts we get
\[(\lambda \hat{k}(\lambda))' = \hat{k}(\lambda) - \lambda \int_0^\infty e^{-\lambda t}k(t)dt = -\hat{\lambda k}(\lambda).\]
and hence
\[ \frac{\hat{\lambda} b'(\lambda)}{\lambda b(\lambda)} = \frac{\hat{\lambda} k(\lambda) - 1}{1 + k(\lambda)}. \]

To prove the boundedness of the right hand-side, thanks to \( k \in L^1(0, \infty) \), we have \( \hat{k}(\lambda) \to 0 \) as \( |\lambda| \to \infty \) by Riemann-Lebesgue lemma. This implies that \( 1 + \hat{k}(\lambda) \) is bounded from below on \( \{ \Re \lambda > 0 \} \). In addition, integrating by parts we get
\[ |\hat{k}(\lambda)| \leq -\int_0^{\infty} t \hat{k}(t)dt = \|k\|_{L^1(0, \infty)} \quad \forall \Re \lambda > 0. \]

Therefore we have that \( b \) is 1-regular. Thus by Theorem [16, Theorem 3.1] we obtain the existence of the resolvent for the integral equation (2.15).

The last claim follows from \( b \in W^{1,1}_{loc}(0, \infty) \) and (2.16), see also [16, p. 34]. \( \square \)

The following proposition ensures the positivity of the resolvent family constructed in the previous proposition.

**Proposition 2.10.** Let Assumption 2.8 be satisfied. Then \( S(t)u_0 \geq 0 \) for each \( t \geq 0 \) and \( d\gamma_\alpha \)-a.e. on \( \mathbb{R}^d \), provided \( u_0 \geq 0 \) \( d\gamma_\alpha \)-a.e.

In particular, \( S(t)u_0 \geq \varepsilon \) provided \( u_0 \geq \varepsilon d\gamma_\alpha \)-a.e. on \( \mathbb{R}^d \).

**Proof.** Let us recall that \( L_\alpha \) generates a positive strongly continuous semigroup on \( L^2(\gamma_\alpha) \); see e.g. [1, Section 2.7.1]. Moreover, as noted above, \( c \) is a completely positive kernel. Thus the first claim follows from [15, Theorem 5].

The last claim follows by applying the first assertion to \( v_0 := u_0 - \varepsilon \geq 0 \) \( d\gamma_\alpha \)-a.e. and noticing that \( S(t)\varepsilon \equiv \varepsilon \). \( \square \)

2.4.2. **Fundamental identity and related issues.** For estimating the entropy of the solution to (1.1) or (2.15), we need a non-local version of the chain rule for the operator \( f \mapsto k \ast f \). Such type of identity turns out to be a very powerful tools in the study of non-local (in time) equations; see e.g. [8, 17] and [5].

**Lemma 2.11.** Let the Assumption 2.8 be satisfied. Let \( T > 0 \) and \( U \) be an open subset of \( \mathbb{R} \). Further, let \( \Phi \in C^1(U) \), \( u \in L^1(0, T) \) with \( u(t) \in U \) a.e. on \( (0, T) \). Suppose that the functions \( \Phi'(u), \Phi'(u)u \) and \( \Phi'(u)(k \ast u) \) belong to \( L^1(0, T) \). Then
\[
\Phi'(u(t))k \ast \dot{u}(t) = k \ast \left( \frac{d}{dt}\Phi(u) \right)(t) + [\Phi(u(0)) - \Phi(u(t))]k(t) + \int_0^t \Phi(u(t-s)) - \Phi(u(t)) - \Phi'(u(t))[u(t-s) - u(t)]k(s)ds.
\]

**Proof.** Due to the regularity assumptions, for each \( t > 0 \),
\[
\frac{d}{dt}(k \ast u)(t) = k \ast \dot{u}(t) + k(t)u(0),
\]
\[
\frac{d}{dt}(k \ast (\Phi(u)))(t) = \left( k \ast \frac{d}{dt}\Phi(u) \right)(t) + k(t)\Phi(u(0)).
\]

Thus, the claim follows by substitute the above identities in [17, Lemma 2.2]. \( \square \)

As in [8, Corollary 6.1], for convex function \( \Phi \) some terms appearing in the identity in the previous lemma can be estimated easily.
Proposition 2.12. Let the assumption of Lemma 2.11 be satisfied. Assume that \( \Phi \) is convex on \( U \). Then

\[
2.17 \quad k \ast \left( \frac{d}{dt} \Phi(u) \right)(t) \leq \Phi'(u(t))(k \ast \dot{u})(t), \quad t > 0.
\]

Proof. Recall that, by Assumption 2.8, one has \( k \geq 0 \) and \( -\dot{k} \geq 0 \) a.e. on \( (0, \infty) \). Thus each term on the right hand side of the identity of Lemma 2.11 are strictly positive by convexity of \( \Phi \). This implies the claim. \( \square \)

The following interesting comparison result is inspired by [17].

Theorem 2.13. Let the Assumption 2.8 be satisfied and let \( T > 0 \) and \( C > 0 \). Suppose \( u, v \in W_{loc}^{1,1}(]0, T[) \) satisfy \( v(0) \leq w(0) \) and

\[
\dot{v} + k \ast \dot{v} + Cv \leq 0, \quad t \in (0, T),
\]

\[
\dot{w} + k \ast \dot{w} + Cw \geq 0, \quad t \in (0, T).
\]

Then \( v \leq w \) on \( (0, T) \).

Proof. The idea is essentially given in [17, Lemma 2.6]. For the sake of completeness, let us make sketch the main step.

Let \( z := v - w \) one has \( \dot{z} + k \ast \dot{z} + Cz \leq 0 \). Applying (2.17) to the convex function \( \Phi(y) = \frac{1}{2}(y_+)^2 \), where \( y_+ := \max\{y, 0\} \),

\[
\frac{d}{dt}(z_+^2) + k \ast \left( \frac{d}{dt} z_+^2 \right)(t) \leq 2z_+ (\dot{z} + k \ast \dot{z}) \leq -2Cz z_+, \quad \text{on } (0, T).
\]

Convolving with \( b \) and applying (2.14),

\[
z_+^2 + 2Cb \ast (z z_+) \leq 0, \quad \text{on } (0, T).
\]

Since \( z_+ z = z_+^2 \) a.e. on \( (0, T) \) and \( b \) is positive (see Lemma 2.7-i)) it follows that

\[
z_+^2 \leq z_+^2 + 2Cb \ast (z_+^2) \leq z_+^2 + 2Cb \ast (z z_+) \leq 0, \quad \text{on } (0, T).
\]

Thus \( v \leq w \) on \( (0, T) \). \( \square \)

The previous result allow us to prove the following interesting result. In the case semigroup framework, such properties is referred as invariance of the measure \( \gamma_\alpha \) with respect to \( L_\alpha \) (cf. [1, p. 54]).

Lemma 2.14 (Invariance). Let \( u_0 \in L^2(\gamma_\alpha) \). Then, for each \( t \geq 0 \),

\[
\int_{\mathbb{R}^d} (S(t)u_0)(x) d\gamma_\alpha(x) = \int_{\mathbb{R}^d} u_0(x) d\gamma_\alpha(x).
\]

Proof. Recall that \( t \mapsto S(t)u_0 \in C([0, \infty); L^2(\gamma_\alpha)) \). For notational convenience we set \( u(t) := S(t)u_0 \). Thus, integrating (1.1) over \( \mathbb{R}^d \), one has

\[
\left( \frac{d}{dt} \int_{\mathbb{R}^d} S(t)u_0 d\gamma_\alpha(x) \right) + k \ast \left( \frac{d}{dt} \int_{\mathbb{R}^d} S(t)u_0 d\gamma_\alpha(x) \right) = \int_{\mathbb{R}^d} L_\alpha S(t)u_0 d\gamma_\alpha(x).
\]

Moreover, let \( \phi \in C_0^\infty(\mathbb{R}^d) \), \( 0 \leq \phi \leq 1 \) be such that \( \phi = 1 \) on \( |x| \leq 1/2 \) and \( \phi = 0 \) on \( |x| \geq 1 \). In addition, for \( R > 0 \), we set \( \phi_R := \phi(\cdot/R) \). Then, by Lebesgue dominated convergence and (2.1),

\[
\int_{\mathbb{R}^d} L_\alpha S(t)u_0 d\gamma_\alpha(x) = \lim_{R \to \infty} \int_{|x| < R} \phi_R \Delta u(t) d\gamma_\alpha - \alpha \int_{|x| < R} \phi_R x \cdot \nabla u d\gamma_\alpha
\]
where in the first equality follows by an integration by parts using that \( \phi_R |_{x=R} = 0 \) and the last one follows by the fact that \( \nabla u \in L^1(\gamma_\alpha) \), \( \| \nabla \phi_R \|_{L^\infty} \leq C/R \) for some \( C > 0 \) independent of \( R > 0 \).

The previous imply that

\[
\left( \frac{d}{dt} \int_{\mathbb{R}^d} S(t) u_0 d\gamma_\alpha (x) \right) + k \ast \left( \frac{d}{dt} \int_{\mathbb{R}^d} S(t) u_0 d\gamma_\alpha (x) \right) = 0,
\]

for any \( t > 0 \). Therefore, the claim follows by Theorem 2.13, with \( C = 0 \), \( w = 0 \) and \( v(t) = \int_{\mathbb{R}^d} S(t) u_0 d\gamma_\alpha \).

\[ \Box \]

3. Entropy decay

3.1. Statement of the main result. In this subsection we collect the main result of this paper. The first result concerns the entropy decay for (1.1).

Below, \( (S(t))_{t \geq 0} \) is the resolvent family constructed in Proposition 2.9.

**Theorem 3.1** (Decay of the Entropy). Let the Assumption 2.8 be satisfied. Then, for any \( u_0 \geq 0 \) \( d\gamma_\alpha \)-a.e. on \( \mathbb{R}^d \) such that \( u_0 \in L^2(\gamma_\alpha) \) and \( \text{Ent}(u_0) < \infty \),

\[
\text{Ent}(S(t)u_0) \leq s_{\mu_\alpha}(t) \text{Ent}(u_0), \quad \forall t > 0;
\]

where \( \mu_\alpha := 2\alpha \).

As in the semigroup theory (cf. [1, Theorem 5.2.1]) a sharp decay estimate for the entropy enables one to deduce the correct constant in the Logarithmic Sobolev inequality.

**Theorem 3.2** (Entropy decay). Assume that there exists a constant \( C > 0 \) such that for any \( u_0 \in L^2(\gamma_\alpha) \) such that \( u_0 \geq 0 \), \( u_0 \in L^1(\mathbb{R}^d) \) and \( \text{Ent}(u_0) < \infty \) it follows that

\[
\text{Ent}(S(t)u_0) \leq s_C(t) \text{Ent}(u_0), \quad \forall t > 0.
\]

Then for any \( f \in H^1(\gamma_\alpha) \), the logarithmic Sobolev inequality holds with the constant \( 4/C > 0 \), i.e.

\[
\text{Ent}(f^2) \leq \frac{4}{C} \int_{\mathbb{R}^d} |\nabla f|^2 d\gamma_\alpha.
\]

3.2. Proof of the main results. Let us begin with the proof of Theorem 3.1.

**Proof of Theorem 3.1.** By an approximation argument, we can assume that \( u_0 \geq \varepsilon \) for some \( \varepsilon > 0 \) and \( u_0 \in D(L) \). Thus, by Proposition 2.10, the solution \( u(x,t) := S(t)u_0 \) verifies \( u(x,t) \geq \varepsilon > 0 \) \( d\gamma_\alpha \)-a.e. on \( \mathbb{R}^d \).

Set \( u(t) := S(t)u_0 \). Thus \( u \in C([0, \infty); D(L_\alpha)) \cap C^1((0, \infty); L^2(\gamma_\alpha)) \). Applying the inequality in Proposition 2.12 to \( v(t) = u(x,t) \) where \( x \in \mathbb{R}^d \) is fixed, and \( \Phi(x) = x \log(x) \), we have

\[
\frac{d}{dt} \Phi(u(x,t)) + k \ast \frac{d}{dt} \Phi(u(x,\cdot))(t) \leq \Phi'(u(x,t))(\dot{u}(x,t) + k \ast \dot{u}(x,\cdot)(t)).
\]
Since \( u \geq \varepsilon > 0 \) for all \( t \in \mathbb{R}_+ \) and \( d\gamma_\alpha \)-almost all \( x \in \mathbb{R}^d \), we can integrate the above inequality and we obtain
\[
(3.1) \quad \int_{\mathbb{R}^d} \frac{d}{dt} \Phi(u(t)) + k \ast \frac{d}{dt} \Phi(u(\cdot))(t)d\gamma_\alpha \leq \int_{\mathbb{R}^d} (\Phi'(u(x,t)))(\dot{u}(t) + k \ast \dot{u}(\cdot)(t))d\gamma_\alpha
\]
\[
= \int_{\mathbb{R}^d} \Phi'(u(x,t))L_\alpha u(x,t)d\gamma_\alpha(x),
\]
where we have used (1.1). Moreover, using that \( \Phi'(u(t)) = \ln u(t) + 1 \in L^2(\gamma_\alpha) \), \( \Phi'(u(t)) = \frac{1}{u(t)} \in L^\infty(\gamma_\alpha) \), for each \( t > 0 \), by \( u \in C(\mathbb{R}_+; D(L)) \) and \( u \geq \varepsilon \), one can apply Proposition 2.5,
\[
\int_{\mathbb{R}^d} \Phi'(u(x,t))L_\alpha u(x,t)d\gamma_\alpha(x) = -\int_{\mathbb{R}^d} \Phi''(u(x,t))|\nabla u(x,t)|^2d\gamma_\alpha(x)
\]
\[
= -\int_{\mathbb{R}^d} \frac{|\nabla u(x,t)|^2}{u(x,t)}d\gamma_\alpha(x).
\]
Applying Lemma 2.3 for \( C = 2/\alpha \) one has,
\[
\int_{\mathbb{R}^d} \frac{d}{dt} \Phi(u(t)) + k \ast \frac{d}{dt} \Phi(u(\cdot))(t)d\gamma_\alpha \leq -2\alpha \text{Ent}(u(\cdot,t)).
\]
By Lemma 2.14, \( \Phi(\int_{\mathbb{R}^d} u(t)d\gamma_\alpha) = \Phi(\int_{\mathbb{R}^d} u_0d\gamma_\alpha) \). Combining (3.1) and the previous, one obtains
\[
(3.2) \quad \frac{d}{dt} \text{Ent}(u(t)) + k \ast \left( \frac{d}{dt} \text{Ent}(u) \right)(t) + 2\alpha \text{Ent}(u(t)) \leq 0.
\]
Applying Theorem 2.13 with \( C = 2\alpha =: \mu_\alpha \), \( v = \text{Ent}(u(\cdot)) \) and \( w = \text{Ent}u_0s_{\mu_\alpha} \), one obtains \( v \leq w \) on \( (0,T) \) for each \( T > 0 \). Thus,
\[
\text{Ent}(u(t)) \leq s_{\mu_\alpha}(t)\text{Ent}(u_0), \quad t > 0,
\]
which is the desired estimate. \( \square \)

Theorem 3.2 will be an easy corollary of the following non-local version of the de Bruijn’s identity (cf. [1, Section 5.2]). Roughly speaking, the de Bruijn’s type-identities describe the decay of the entropy along the solution.

**Proposition 3.3.** Let the assumption of Theorem 3.2 be satisfied. Let \( u_0 \in D(L_\alpha) \) be such that \( u_0 \geq \varepsilon > 0 \) \( d\gamma_\alpha \)-a.e. for some \( \varepsilon > 0 \). Then, setting \( u(x,t) = S_\alpha(t)u_0 \) and \( \Phi(x) := x \ln x \), one has
\[
(3.3) \quad \int_{\mathbb{R}^d} \Phi'(u(t)) (\dot{u}(t) + k \ast \dot{u}) d\gamma_\alpha
\]
\[
= \frac{d}{dt} \text{Ent}(u(t)) + k \ast \left( \frac{d}{dt} \text{Ent}(u) \right)(t)
\]
\[
+ \int_{\mathbb{R}^d} [\Phi(u(0)) - \Phi(u(t)) + \Phi'(u(t))(u(t) - u(0))]k(t)d\gamma_\alpha
\]
\[
- \int_{\mathbb{R}^d} \left( \int_0^t [\Phi(u(t-s)) - \Phi(u(t)) - \Phi'(u(t))[u(t-s) - u(t)]k(s)ds \right)d\gamma_\alpha.
\]

**Proof.** Recall that, by Proposition 2.10, \( u \geq \varepsilon \) \( d\gamma_\alpha \)-a.e. on \( \mathbb{R}^d \). Thus \( \Phi'(u(t)) \in L^2(\gamma_\alpha) \) for each \( t \geq 0 \). Moreover, if \( u_0 \in D(L_\alpha) \), by Proposition 2.9 \( u \in C^1([0,\infty); D(L_\alpha)) \).
Thus the map \( t \mapsto \text{Ent}(u(\cdot, t)) \) is differentiable by Lebesgue dominated convergence. Thus all the term appearing in the formula (3.3) are well defined.

The identity is an easy corollary of the fundamental identity, i.e. Lemma 2.11, where \( u := S(t)u_0, \mathcal{W} = (\varepsilon, \infty) \) and using that

\[
\frac{d}{dt} \int_{\mathbb{R}^d} \Phi(u(x, t))d\gamma_\alpha = \frac{d}{dt} \text{Ent}(u(t)),
\]

since \( \text{Ent}(u(t)) = \int_{\mathbb{R}^d} \Phi(u(x, t))d\gamma_\alpha - \Phi(\int_{\mathbb{R}^d} u(x, t)d\gamma_\alpha) \) and \( \int_{\mathbb{R}^d} u(x, t)d\gamma_\alpha(x) \) is constant by Lemma 2.14.

We are ready to prove Theorem 3.2.

**Proof of Theorem 3.2.** As in the proof of Theorem 3.1, we can assume that \( u_0 \geq \varepsilon \) for some \( \varepsilon > 0 \) and \( u_0 \in D(L_\alpha) \). Thus, by Proposition 2.10, the solution \( u(x, t) := S(t)u_0 \) verifies \( u(x, t) \geq \varepsilon > 0 \) \( d\gamma_\alpha \)-a.e. on \( \mathbb{R}^d \). As in the proof of Proposition 3.3 the map \( t \mapsto \text{Ent}(u(\cdot, t)) \) is differentiable.

Thus, computing (3.3) at time \( t = 0 \) one obtains,

\[
\left. \int_{\mathbb{R}^d} \Phi'(u_0) \left( \dot{u}(t) + k \ast \dot{u}(t) \right) \right|_{t = 0} d\gamma_\alpha = \left. \frac{d}{dt} \text{Ent}(u(t)) \right|_{t = 0}.
\]

To estimate the right hand side of the previous, let us note that, by assumption,

\[ \text{Ent}(u(\cdot, t)) - \text{Ent}(u_0) \leq [s_C(t) - 1] \text{Ent}(u_0), \quad t > 0. \]

Dividing for \( t > 0 \) and sending \( t \searrow 0 \) one discovers

\[
\left. \frac{d}{dt} \text{Ent}(u(\cdot, t)) \right|_{t = 0} \leq s_C(0) \text{Ent}(u_0) = -C \text{Ent}(u_0).
\]

where the last equality follows by (2.12) using the fact that \( s_C(0) = 1 \).

Thus combining (3.4) with (3.5),

\[
-C \text{Ent}(u_0) \geq \int_{\mathbb{R}^d} \Phi'(u_0) \left( \dot{u}(t) + k \ast \dot{u}(t) \right) \left|_{t = 0} \right. d\gamma_\alpha = \int_{\mathbb{R}^d} \Phi'(u_0) L_\alpha u_0 d\gamma_\alpha = - \int_{\mathbb{R}^d} \frac{\left| \nabla u_0 \right|^2}{u_0} d\gamma_\alpha;
\]

where in the last inequality we have used Proposition 2.5. Thus, the following logarithmic Sobolev inequality holds,

\[
\text{Ent} u_0 \leq \frac{1}{C} I u_0.
\]

The claim follows by Lemma (2.3).

**4. Examples**

In this section we collect two example of kernel which satisfies Assumption 2.8 and we (almost) explicitly compute the rate of decay of the entropy due to Theorem 3.1.

Let us begin with the following simple example.

**Example 4.1.** Let \( k \equiv 0 \). Then (2.12) is equivalent to

\[
\dot{s}_\mu(t) + \mu s_\mu(t) = 0, \quad \forall t > 0,
\]

and \( s_\mu(0) = 1 \). It is immediate to see that \( s_\mu(t) = e^{-\mu t} \) and thus \( s_{\mu_\alpha}(t) = e^{-2\alpha t} \) where \( \mu_\alpha = 2\alpha \) as in Theorem 3.1.
Thus Theorem 3.1 implies (1.2), which coincide with the result given in [1, Chapter 5].

The following gives an example of non-trivial kernel for which we can explicitly compute the relaxation functions.

**Example 4.2 (Stretched exponential).** In this example we study the stretched exponential \(k(t) = \nu e^{-t^\beta}\) with \(\nu > 0\) and \(\beta \in (0, 1]\), \(t > 0\).

In the case \(\beta = 1\), we can give explicitly the expression of \(s_\mu(t), \mu > 0\). To this end we will use (2.12), that is
\[
s_\mu(t) + \nu e^{-t} \ast s_\mu(t) + \mu s_\mu(t) = 0 \quad \forall t > 0, \quad s_\mu(0) = 1.
\]

Multiplying by \(e^t\), we can write
\[
e^t s_\mu(t) + \nu \int_0^t e^\tau s_\mu(\tau)d\tau + \mu e^t s_\mu(t) = 0.
\]
Set \(g(t) := e^t s_\mu(t)\). Note that \(g(0) = 1\), \(e^t s_\mu(t) = \dot{g}(t) = g(t)\) and \(\dot{g}(0) = 1 - \mu\). By the previous displays,
\[
\dot{g}(t) - g(t) + \nu g(t) - \nu - \nu \int_0^t g(\tau)d\tau + \mu g(t) = 0.
\]
Differentiating we get
\[
\ddot{g}(t) + (\mu - 1 + \nu)\dot{g}(t) - \nu g(t) = 0, \quad g(0) = 1, \quad \dot{g}(0) = 1 - \mu.
\]
Therefore \(g(t) = C_+ e^{\lambda_+ t} + C_- e^{\lambda_- t}\), where we have set
\[
\lambda_\pm := \frac{-(\mu - 1 + \nu) \pm \sqrt{(\mu - 1 + \nu)^2 + 4\nu}}{2},
\]
\[
C_+ := \frac{1 - \mu - \lambda_-}{\sqrt{(\mu - 1 + \nu)^2 + 4\nu}}, \quad C_- := \frac{\lambda_- - 1 + \mu}{\sqrt{(\mu - 1 + \nu)^2 + 4\nu}}.
\]
Since \(s_\mu(t) = e^{-t}g(t)\), we have
\[
s_\mu(t) = C_+ e^{(\lambda_+ - 1) t} + C_- e^{(\lambda_- - 1) t}, \quad t > 0.
\]
Moreover, \(\lambda_- - 1 < \mu < \lambda_+ - 1 < 0\). Thus,
\[
s_{2\alpha}(t) \sim C_+ e^{(\lambda_+ - 1) t}, \quad t \to \infty,
\]
(here \(f \sim g \) as \(t \to \infty\) means that \(\lim_{t \to \infty} \frac{f(t)}{g(t)} = 1\)).

Therefore, in the case \(k(t) = \nu e^{-t}\), the entropy of the solutions to (1.1) decays like an exponential. This behaviour is similar to the semigroup case consider in the Example 4.1, however the rate of convergence is different from the latter.

To conclude we shall prove that \(s_\mu(t) \geq e^{-t\mu}\) (here \(s_\mu\) is given by (4.1)) for each \(t \geq 0\) and \(\mu > 0\), therefore the decay of the entropy in the semigroup case is always better than the one with the memory effect.

To see this, note that \(\phi(t) := e^{t\mu} s_\mu(t)\) verifies \(\phi(0) = 1\) and \(\phi'(t) \geq 0\) for all \(t > 0\), since \(C_+ + C_- = 1\) and \(C_+ \lambda_+ + C_- \lambda_- = 1 - \mu\). Therefore, for each choice of \(\mu, \nu\),
\[
e^{-\mu t} \leq s_\mu(t), \quad t > 0.
\]
This shows that the memory effect makes the entropy estimate larger.
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Department of Mathematics Guido Castelnuovo, Sapienza University of Rome, P.le A. Moro 2, 00185 Roma, Italy.

E-mail address: agresti@mat.uniroma1.it

SBAI Department, Sapienza University of Rome, Via Antonio Scarpa, 16, 00161 Roma, Italy.

E-mail address: paola.loreti@sbai.uniroma1.it

SBAI Department, Sapienza University of Rome, Via Antonio Scarpa, 16, 00161 Roma, Italy.

E-mail address: daniela.sforza@sbai.uniroma1.it