ON POSITIVE SOLUTIONS OF INTEGRAL EQUATIONS WITH THE WEIGHTED BESSEL POTENTIALS

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ABSTRACT. This paper is devoted to exploring the properties of positive solutions for a class of nonlinear integral equation(s) involving the Bessel potentials, which are equivalent to certain partial differential equations under appropriate integrability conditions. With the help of regularity lifting theorem, we obtain an integrability interval of positive solutions and then extend the integrability interval to the whole $[1, \infty)$ by the properties of the Bessel kernels and some delicate analysis techniques. Meanwhile, the radial symmetry and the sharp exponential decay of positive solutions are also obtained. Furthermore, as an application, we establish the uniqueness theorem of the corresponding partial differential equations.

1. Introduction. Let $\alpha$ be a real number satisfying $0 < \alpha < n$. The Bessel potential of a positive function $f \in L^p(\mathbb{R}^n)$ ($1 \leq p \leq \infty$) is

$$B_\alpha(f)(x) = g_\alpha(x) * f(x) = \int_{\mathbb{R}^n} g_\alpha(x-y)f(y)dy,$$

where

$$g_\alpha(x) = \frac{1}{(4\pi)^{\frac{n}{2}} \Gamma\left(\frac{n}{2}\right)} \int_0^\infty \exp \left(-\frac{\pi}{t} |x|^2 - \frac{t}{4\pi}\right) t^{\frac{n}{2} - \alpha} \frac{dt}{t}$$

is the Bessel kernel.

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In this paper, we will consider the following non-linear integral equation involving the weighted Bessel kernel functions
\[ u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) \frac{u^p(y)}{|y|^\beta} dy, \] 
and
\[ \begin{cases} 
  u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) \frac{u^p(y)v^q(y)}{|y|^\beta} dy, \\
  v(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) \frac{v^p(y)u^q(y)}{|y|^\beta} dy, 
\end{cases} \] 
where \( 0 < \beta < \alpha < n \) and \( p, q \geq 1 \).

In recent years, there has been tremendous interest in studying integral equation(s), which are equivalent to some partial differential equation(s) in \( \mathbb{R}^n \) and also provide a special skill to investigate the global properties of corresponding differential equation(s), such as integrability and asymptotic behavior. First of all, we recount some closely related investigations and backgrounds.

As \( \beta = 0 \), the equation (1) can be reduced to the following form:
\[ u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) u^p(y) dy. \] 
By the method of moving planes in an integral form, Ma and Chen [22], combining with Sobolev embedding inequality, showed that all the positive solutions of (3) are radially symmetric and monotone decreasing about some point, and Han and Lu [8], by the regularity lift theorem, proved that all positive solutions are uniformly bounded in \( \mathbb{R}^n \) with \( u \in L^q_0(\mathbb{R}^n) \) and \( q_0 > \max\{n(p-1)/\alpha, p\} \). In addition, we know that, under certain integrability conditions, equation (3) is equivalent to the following differential equation:
\[ (I - \Delta)^{\alpha/2} u(x) = u^p(x), \] 
which has been extensively studied, as the parameters take special values. For example, when \( \alpha = 2 \) and \( p = 3 \), Coffman [4] got the uniqueness of positive solution of (4) in \( \mathbb{R}^3 \), and Mcleod and Serrin [24] extended the corresponding result to \( \mathbb{R}^n \); when \( \alpha = 2 \) and \( p = (n+2)/(n-2) \), the uniqueness theorem of (4) in bounded or unbounded annular region was given by Kwong in [11].

On the other hand, a natural extension of (3) is the following integral system:
\[ \begin{cases} 
  u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) u^p(y)v^q(y) dy, \\
  v(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) v^p(y)u^q(y) dy. 
\end{cases} \] 
Under certain appropriate decay conditions on the solutions \( (u, v) \) at infinity, system (5) is equivalent to the following partial differential equations:
\[ \begin{cases} 
  (I - \Delta)^{\alpha/2} u = u^p v^q, \\
  (I - \Delta)^{\alpha/2} v = v^p u^q. 
\end{cases} \] 
When \( \alpha = 2 \), system (6) is related to two coupled nonlinear Schrödinger equations, which model the Bose-Einstein condensate. Lin and Wei [20] gave the existence and nonexistence theorem of ground state solutions of nonlinear Schrödinger equations. Recently, for \( \alpha \in (0, n) \), Lei [14] showed that all of positive solutions to system (5) or (6) are radially symmetric and decreasing about \( x_0 \in \mathbb{R}^n \), under the conditions
u \in L^{[n(p+q-1)]/\mu}(\mathbb{R}^n) \text{ and } \mu \in (0, \alpha]; \text{ moreover, the optimal integrable intervals and sharp decay rate were also given in } [14].

Another integral system similar to (5) is the following Riesz potential type one:

\[
\begin{align*}
\begin{cases}
u(x) &= \int_{\mathbb{R}^n} u^p(y) v^q(y) \frac{1}{|x-y|^{n-\alpha}} dy, \\
v(x) &= \int_{\mathbb{R}^n} u^p(y) v^q(y) \frac{1}{|x-y|^{n-\alpha}} dy,
\end{cases}
\tag{7}
\end{align*}
\]

which is closely related to the stationary Schrödinger system as \( \alpha = 2 \). System (7) and its various weighted forms have recently attracted a great deal of attentions. The radial symmetry, optimal integrable intervals and sharp asymptotic behaviors of positive solutions to (7) were investigated successively. We refer the readers to [1, 2, 3, 9, 10, 12, 13, 15, 16, 23, 26, 27, 25, 28] and the references therein for more information about the Riesz potential type integral equations.

Inspired by the works mentioned above, this paper aims to explore the global properties, such as integrability and asymptotic behaviors etc., of positive solutions to (1) and (2). Our main results can be formulated as follows:

**Theorem 1.1.** Assume that \( u \in L^{n(p-1)/(\alpha-\beta)}(\mathbb{R}^n) \) is a positive solution of equation (1) with \( 0 \leq \beta < \alpha < n \) and \( p > (n-\beta)/(n-\alpha) \). Then

- **Integrability:**
  \( u(x) \in L^s(\mathbb{R}^n), \forall s \in [1, \infty). \)  

- **Symmetry and monotonicity:** \( u(x) \) is radially symmetric and monotone decreasing about the origin.

- **Sharp decay rate:**
  \( u(x) \asymp g_\alpha(x) \tag{9} \)

Here we use the notation \( u(x) \asymp g_\alpha(x) \) to denote that there exist two constants \( C_1 \) and \( C_2 \) such that for \( |x| \) large enough
\[
C_1 g_\alpha(x) \leq u(x) \leq C_2 g_\alpha(x).
\]

**Theorem 1.2.** Assume that \( u, v \in L^\tau(\mathbb{R}^n), \tau = n(p+q-1)/(\alpha-\beta), \) is a pair of positive solutions of weighted integral system (2). Then

- **Integrability:**
  \( u(x), v(x) \in L^s(\mathbb{R}^n), \forall s \in [1, \infty). \)  

- **Symmetry and monotonicity:** \( u(x), v(x) \) is radially symmetric and monotone decreasing about the origin.

- **Sharp decay rate for \( |x| \) large enough:**
  \( u(x) \asymp g_\alpha(x), \quad v(x) \asymp g_\alpha(x) \tag{11} \)

System (2) is an extended model of (1) and (5). When \( \alpha = 2 \), system (2), under certain integrability conditions, is also equivalent to the following system of differential equations:

\[
\begin{align*}
\begin{cases}
u(x) - \Delta u(x) &= \frac{v^p(x) u^q(x)}{|x|^\beta}, \\
u(x) - \Delta v(x) &= \frac{u^p(x) v^q(x)}{|x|^\beta}.
\end{cases}
\tag{12}
\end{align*}
\]
The system (12) arises in mathematical models from various physical phenomena, such as the incoherent solitons in nonlinear optics and the multispecies Bose-Einstein condensates in hyperfine spin states. The interested readers can consult [5, 7, 6], among numerous references, for more information.

Set \( w(x) \triangleq u(x) + v(x) \) and \( \gamma = p + q \). Noting that
\[
 w^p(y)v^q(y) + v^p(y)u^q(y) \leq 2 (u(y) + v(y))^{p+q} = 2 \ w^{p+q}(y).
\]
Therefore there exists a bounded function \( R(x) \) such that system (2) can be rewritten as
\[
 w(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) \frac{u^p(y)v^q(y) + u^q(y)v^p(y)}{|y|^\beta} dy 
= R(x) \int_{\mathbb{R}^n} g_\alpha(x-y) \frac{w^\gamma(y)}{|y|^\beta} dy,
\]
where \( R(x) \) is bounded function defined by
\[
 0 < R(x) = \frac{\int_{\mathbb{R}^n} g_\alpha(x-y) \frac{u^p(y)v^q(y) + u^q(y)v^p(y)}{|y|^\beta} dy}{\int_{\mathbb{R}^n} g_\alpha(x-y) \frac{w^\gamma(y)}{|y|^\beta} dy} \leq 2.
\]

Due to (13) and \( \max\{u(x), v(x)\} \leq u(x) + v(x) = w(x) \), by the same arguments as in the proof of Theorem 1.1, we can obtain the following results.
\[
 u(x), \ v(x) \in L^s(\mathbb{R}^n), \ \forall s \in [1, \infty] \tag{14}
\]
and for \( |x| \) sufficiently large,
\[
 \max \left\{ u(x), v(x) \right\} \leq u(x) + v(x) = w(x) \leq C g_\alpha(x). \tag{15}
\]
Therefore, to obtain Theorem 1.2, it suffices to show that \( u(x), v(x) \) is radially symmetric and monotone decreasing about the origin and, for sufficiently large \( |x| \),
\[
 u(x) \geq C g_\alpha(x), \ v(x) \geq C g_\alpha(x). \tag{16}
\]

Finally, based on the explanations above, as an application of Theorem 1.2, we will establish the following uniqueness theorem of (12).

**Theorem 1.3.** For \( 0 \leq \beta < 2 \) and \( p \leq q \), assume that \( u, v \in L^r(\mathbb{R}^n), r = n(p + q - 1)/(2 - \beta) \), is a pair of positive solutions of the system (12). Then
\[
 u(x) \equiv v(x), \ \forall x \in \mathbb{R}^n. \tag{17}
\]

This paper is organized as follows. In Section 2, we will use the regularity lifting theorem, delicate analysis techniques, the properties of Bessel potentials and the interpolation theorem to obtain the best integrable interval of (1). In Section 3, we shall prove that all of positive solutions for (1) are radially symmetry by moving plane method. The sharp decay rate of (1) will be formulated by iteration in Section 4. The important properties of integral system (2) will be discussed in Section 5. Finally, the uniqueness theorem of system (12) will be shown.

Throughout the rest of this paper, we always use the letter \( C \), sometimes with certain parameters, to denote positive constants that may vary at each occurrence but are independent of the essential variables.
2. **Integrability.** This section is concerned with the proof of the integrability of positive solution \( u(x) \) of system (1) in Theorem 1.1, which will be divided into three steps. In Step 1, we will use the regularity lifting theorem to deduce the following integrability of \( u(x) \):

\[
u(x) \in L^s(\mathbb{R}^n) \cap L^{\frac{n(\alpha-1)}{n-\alpha}}(\mathbb{R}^n), \quad \forall s > \frac{n}{n-\alpha}.
\]

(18)

In Step 2, with the help of some delicate analysis techniques, we will obtain the \( L^\infty \) estimation of \( u(x) \). In the last step, we will apply the properties of Bessel potential and \( L^p \) interpolation theorem to extend the integrability interval to the whole \([1, \infty]\).

**Step 1.** In order to show (18), we firstly introduce some necessary notions. For \( A > 0 \), set

\[
u_A(x) = \begin{cases} u(x), & \text{if } u(x) \geq A \text{ or } |x| \geq A, \\ 0, & \text{otherwise}, \end{cases}
\]

(19)

and \( w(x) = u(x) - u_A(x) \). Let \( t \) be a positive constant such that \( t > n/(n - \alpha) \). For \( f \in L^t(\mathbb{R}^n) \), define the operator \( \mathcal{J}_A(f) \) by

\[
\mathcal{J}_A(f)(x) := \int_{\mathbb{R}^n} g_\alpha(x - y) \frac{u^{n-1}_A(y) f(y)}{|y|^\beta} dy,
\]

(20)

and write

\[
\mathcal{F}(x) = \int_{\mathbb{R}^n} g_\alpha(x - y) \frac{w^p(y)}{|y|^\beta} dy.
\]

Then \( u(x) \) is a solution of the following equation:

\[
f(x) = \mathcal{J}_A(f) + \mathcal{F}(x).
\]

(21)

We claim that

\[
\mathcal{F}(x) \in L^s(\mathbb{R}^n) \cap L^\infty(\mathbb{R}^n), \quad s > \frac{n}{n-\alpha}.
\]

(22)

Firstly, we recall the following important estimate (see [29, (2.63)]) for the detail,

\[
B_\alpha(f)(x) \leq C \int_{\mathbb{R}^n} \frac{f(y)}{|x - y|^{n-\alpha}} dy \leq C I_\alpha(f)(x)
\]

(23)

where \( I_\alpha \) is the classical Riesz potential operator. Therefore, by the weighted Hardy-Littlewood-Sobolev inequality, we have

\[
\|\mathcal{F}\|_s \leq C(\alpha, n, \beta) \|w^p\|_{L^{\frac{n(\alpha-1)}{n-\alpha}}(\mathbb{R}^n)}, \quad s > \frac{n}{n-\alpha}.
\]

Next, we show that \( \mathcal{F} \in L^\infty(\mathbb{R}^n) \). By (23) and the definition of \( w(x) \), we get

\[
\mathcal{F}(x) \leq C \|w^p\|_{L^\infty} \int_{|y| \leq A} \frac{1}{|(x - y)|^{n-\alpha}} \frac{1}{|y|^\beta} dy.
\]

Denote

\[
R_\infty(x) = \int_{|y| \leq A} \frac{1}{|(x - y)|^{n-\alpha}} \frac{1}{|y|^\beta} dy.
\]

It suffices to show that \( R(x) < \infty \), which will be divided into the following two cases:

**Case 1** If \( x \in \mathbb{R}^n \setminus B_{2A}(0) \), then \( |x - y| \geq |x| - |y| \geq |y| \) and

\[
R_\infty(x) \leq \int_{|y| \leq A} \frac{1}{|y|^\beta + n - \alpha} dy \leq C(n, \alpha, \beta) A^{\alpha-\beta} < \infty.
\]
Case 2 If \( x \in B_{2A}(0) \), then, by H"older’s inequality and Young’s inequality, we have
\[
R_\infty(x) \leq \int_{|y| \leq A} \frac{1}{|y|^{n-\alpha+\beta}} dy + \int_{|x-y| \leq 3A} \frac{1}{|x-y|^{n-\alpha+\beta}} dy \leq C(A, \alpha, \beta, n).
\]
This together with (26) implies the claim holds.

Now we turn to \( \mathcal{T}_A(f) \). By the weighted Hardy-Littlewood-Sobolev inequality and H"older’s inequality, we conclude that, for \( s > n/(n-\alpha) \)
\[
\|\mathcal{T}_A(f)\|_{L^s(\mathbb{R}^n)} \leq \|u^{p-1}f\|_{\frac{ns}{s+n-\alpha}} \leq \|u_A\|_{\frac{n(n-1)}{n-\alpha}} \|f\|_s.
\]
Since \( u \in L^{n(p-1)/(\alpha-\beta)}(\mathbb{R}^n) \), this implies that there exists a real number \( A > 0 \) large enough such that \( \|u_A\|_{\frac{n(n-1)}{n-\alpha}} \leq 1/2 \) and \( \mathcal{T}_A(f) \) is a contraction map from \( L^s(\mathbb{R}^n) \) into itself, provided \( s > n/(n-\alpha) \). Therefore taking \( \mathcal{X} = L^s(\mathbb{R}^n), \mathcal{Y} = L^s(\mathbb{R}^n) \) for any \( s \in (n/(n-\alpha), \infty) \) and \( \mathcal{Z} = L^s(\mathbb{R}^n) \cap L^{n(p-1)/(\alpha-\beta)}(\mathbb{R}^n) \) in [1, Theorem 3.31], we obtain \( u \in L^s(\mathbb{R}^n) \cap L^{n(p-1)/(\alpha-\beta)}(\mathbb{R}^n) \).

Step 2. In this step, we will show that \( u \in L^\infty(\mathbb{R}^n) \). By (19)-(22), it suffices to verify that
\[
\mathcal{T}_A(u) \leq \int_{\mathbb{R}^n} \frac{u_A^p(y)}{|x-y|^{\alpha-\beta}} dy \triangleq \mathcal{D}(x) \in L^\infty(\mathbb{R}^n).
\]

Write
\[
\mathcal{D}(x) = \int_{\mathbb{R}^n} \frac{u_A^p(y)}{|x-y|^{\alpha-\beta}} dy = \int_{B_A(0)} \frac{u_A^p(y)}{|x-y|^{\alpha-\beta}} dy + \int_{\mathbb{R}^n \setminus B_A(0)} \frac{u_A^p(y)}{|x-y|^{\alpha-\beta}} dy \leq \mathcal{D}_1(x) + \mathcal{D}_2(x),
\]
where and in what follows \( B_r(x) \) denote the ball on \( \mathbb{R}^n \) centered at \( x \) with radius \( r \).

We first estimate \( \mathcal{D}_1(x) \). For \( x \in \mathbb{R}^n \setminus B_{2A}(0) \), taking \( s = (2n)/(\alpha-\beta) \), we have
\[
ps = \frac{2np}{\alpha-\beta} > \frac{n(n-1)}{\alpha-\beta} \quad \text{and} \quad \frac{1}{s'} = 1 - \frac{1}{s} = \frac{2n - (\alpha-\beta)}{2n} > \frac{n - (\alpha-\beta)}{n}
\]
and the H"older inequality leads to
\[
\mathcal{D}_1(x) \leq \int_{B_A(0)} \frac{u_A^p(y)}{|y|^{\beta+n-\alpha}} dy \leq \left( \int_{B_A(0)} |y|^{-(\alpha-\beta)s'} dy \right)^{\frac{1}{s'}} \times \left( \int_{\mathbb{R}^n} u^{ps}(y) dy \right)^{\frac{1}{s}} < \infty.
\]

For \( x \in B_{2A}(0) \), by H"older’s inequality and Young’s inequality, we conclude that
\[
\mathcal{D}_1(x) \leq \left( \int_{B_A(0)} \left[ \frac{u_A^{\frac{\alpha-\beta}{\alpha-\beta}}(y)}{|y|^{\beta+n-\alpha}} \right]^{\frac{n-\alpha+\beta}{\alpha-\beta}} dy \right)^{\frac{\beta}{n-\alpha+\beta}} \times \left( \int_{B_A(0)} \left[ \frac{u_A^{\frac{n-\alpha}{\alpha-\beta}}(y)}{|x-y|^{\alpha-\beta}} \right]^{\frac{n-\alpha+\beta}{n-\alpha}} dy \right)^{\frac{\alpha}{n-\alpha+\beta}} \leq \int_{B_A(0)} \frac{u_A^p(y)}{|y|^{\beta+n-\alpha}} dy + \int_{B_A(0)} \frac{u_A^p(y)}{|x-y|^{\beta+n-\alpha}} dy < \infty.
\]

This together with (26) implies the boundedness of \( \mathcal{D}_1(x) \).
Next, we estimate $D_2(x)$. We can write
\begin{equation}
D_2(x) = \int_{B_A(0) \cap B_A(x)} \frac{u_A^p(y)}{|x - y|^{n-\alpha}} dy + \int_{B_A(0) \cap [\mathbb{R}^n \setminus B_A(x)]} \frac{u_A^p(y)}{|x - y|^{n-\alpha}} dy
\end{equation}
(28)
\[D_2 \leq D_2.1(x) + D_2.2(x)\]

By (18), it is easy to check that
\begin{equation}
D_2.1(x) = \frac{1}{|A|} \int_{B_A(x)} \frac{u_A^p(y)}{|x - y|^{n-\alpha}} dy
\end{equation}
(29)
\[
\leq \left( \int_{B_A(x)} |x - y|^{(\alpha - n)t'} dy \right)^{\frac{1}{t'}} \times \left( \int_{B_A(x)} u^\mu(y) dy \right)^{\frac{1}{\mu}} < \infty,
\]
where $t = n/(\alpha - \beta)$ and
\[(n - \alpha)t' = (n - \alpha)\frac{t}{t - 1} = \frac{n(n - \alpha)}{n - (\alpha - \beta)} < n.
\]

By the same arguments as in (27), we have
\begin{equation}
D_2.2(x) \leq \int_{B_A(0)^c} \frac{u_A^p(y)}{|y|^{\beta + n - \alpha}} dy + \int_{B_A(x)^c} \frac{u_A^p(y)}{|x - y|^{n - \alpha + \beta}} dy
\end{equation}
(30)

Observing that as $\varepsilon \in (0, 1/(p - 1))$ and $\mu = n/[(\alpha - \beta)(1 + \varepsilon)]$, we have
\[p\mu = \frac{np}{(\alpha - \beta)(1 + \varepsilon)} > \frac{n(p - 1)}{\alpha - \beta} > \frac{n + \beta - \alpha}{n} > 1.
\]

By Hölder’s inequality and (18), we deduce that
\[\int_{B_A(0)^c} \frac{u_A^p(y)}{|y|^{\beta + n - \alpha}} dy \leq \left( \int_{B_A(0)^c} u_A^{\mu p}(y) dy \right)^{\frac{1}{\mu}} \times \left( \int_{B_A(0)^c} \frac{1}{|y|^{(\beta + n - \alpha)\mu'}} dy \right)^{\frac{1}{\mu'}} < \infty,
\]
and
\[\int_{B_A(x)^c} \frac{u_A^p(y)}{|x - y|^{\beta + n - \alpha}} dy \leq \left( \int_{B_A(x)^c} u_A^{\mu p}(y) dy \right)^{\frac{1}{\mu}} \times \left( \int_{B_A(x)^c} \frac{1}{|x - y|^{(\beta + n - \alpha)\mu'}} dy \right)^{\frac{1}{\mu'}} < \infty.
\]

This together with (24)-(30) implies that $u \in L^\infty(\mathbb{R}^n)$.

**Step 3.** We extend the integrability of $u \in L^s(\mathbb{R}^n)$ from $s \in (n/(n - \alpha), \infty]$ to $s \in [1, \infty]$. Note that $u \in L^\infty(\mathbb{R}^n)$, by the interpolation theorem, it suffices to show that $u \in L^1(\mathbb{R}^n)$. Since
\[\int_{\mathbb{R}^n} g_\alpha dx = \widehat{g_\alpha}(0) = 1,
\]
we have
\[\int_{\mathbb{R}^n} u(x) dx = \int_{\mathbb{R}^n} \int_{\mathbb{R}^n} g_\alpha(x - y) \frac{u(y)}{|y|^\beta} dy dx = \int_{\mathbb{R}^n} g_\alpha(x) dx \int_{\mathbb{R}^n} \frac{u(y)}{|y|^\beta} dy = \int_{\mathbb{R}^n} \frac{u(y)}{|y|^\beta} dy.
\]

It remains to show that
\[\int_{\mathbb{R}^n} \frac{u(y)}{|y|^\beta} dy < \infty.
\]
Write
\[ \int_{\mathbb{R}^n} u^p(y) \frac{dy}{|y|^\beta} = \int_{|y| \leq A} u^p(y) \frac{dy}{|y|^\beta} + \int_{\mathbb{R}^n \setminus B_A(0)} u^p(y) \frac{dy}{|y|^\beta} =: \mathcal{G}_1(x) + \mathcal{G}_2(x). \]

By the integrability \( u \in L^s(\mathbb{R}^n) \) for \( s \in (n/(n - \alpha), \infty) \) and the Hölder inequality, it is easy to check that \( \mathcal{G}_1(x) \) is bounded. Now we estimate \( \mathcal{G}_2(x) \). Since \( p > (n - \beta)/(n - \alpha) \), there exists \( \varepsilon > 0 \) small enough such that \( p > [(n - \beta)/(1 + \varepsilon)]/(n - \alpha) \). Taking \( \tau = n/[(n - \beta)(1 + \varepsilon)] \), we have
\[ \frac{1}{\tau} = 1 - \frac{1}{\tau} = \frac{\beta - (n - \beta)\varepsilon}{n} < \frac{\beta}{n} \text{ and } p\tau = \frac{np}{(n - \beta)(1 + \varepsilon)} > \frac{n}{n - \alpha}. \]

This combining (18) with Hölder’s inequality leads to
\[ \mathcal{G}_2(x) \leq \int_{\mathbb{R}^n \setminus B_A(0)} \frac{u^p(y)}{|y|^\beta} \frac{dy}{|y|^\beta} \leq \left( \int_{\mathbb{R}^n \setminus B_A(0)} |y|^{-\beta\tau} \frac{dy}{|y|^\beta} \right)^\frac{1}{\tau} \times \left( \int_{\mathbb{R}^n} u_p(y) \right)^\frac{1}{\tau} < \infty, \]
which completes the proof of the first part in Theorem 1.1. \( \square \)

3. Radial symmetry. In this section, we will apply the method of moving plane in integral forms introduced by Chen et al in [2, 3] to obtain the radial symmetry of positive solutions of system (1) in Theorem 1.1.

For convenience, given \( \lambda \in \mathbb{R} \), set
\[ \Sigma_\lambda \triangleq \{ x = (x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \geq \lambda \}, \]
\[ x^\lambda \triangleq (2\lambda - x_1, x_2, \ldots, x_n), \text{ and } u_\lambda(x) \triangleq u(x^\lambda). \]

For any solutions \( u \) of system (1.1), it is easy to verify that
\[ u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( \frac{u^p(y)}{|y|^\beta} - \frac{u_\lambda^p(y)}{|y|^\beta} \right) \frac{dy}{|y|^\beta} \]
\[ = \int_{\Sigma_\lambda} \left\{ \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( \frac{1}{|y|^\beta} - \frac{1}{|y|^\beta} \right) u^p(y) \right. \]
\[ + \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( u^p(y) - u_\lambda^p(y) \right) \left( \frac{1}{|y|^\beta} \right) \frac{dy}{|y|^\beta} \]
\[ \triangleq A_1(x) + A_2(x) \quad (32) \]

The proof is made up of two steps. In Step 1, to obtain the radial symmetry, we compare the value of \( u(x) \) with \( u_\lambda(x) \) and show that for sufficiently negative \( \lambda < 0 \), there holds
\[ u(x) \geq u_\lambda(x), \quad \forall x \in \Sigma_\lambda \setminus \{0\}. \quad (33) \]

In Step 2, we continuously move the plane \( x_1 = \lambda \) along \( x_1 \) direction from near negative infinity to the right as long as (33) holds. By moving this plane in this way, we finally show that the plane will stop at the origin. Now we turn our attention to Step 1.

**Step 1:** Noting that as \( x, y \in \Sigma_\lambda \), it is easy to verify that \( |x^\lambda - y| \geq |x - y| \) and \( |y^\lambda| \geq |y| \). By the monotonicity of \( g_\alpha(x) \) and (18), we know that \( A_1(x) \leq 0 \) and
weighted Hardy-Littlewood-Sobolev inequality and Hölder inequality leads to

$$\int_{\Sigma_\lambda^n} \frac{(g_\alpha(x^\lambda - y) - g_\alpha(x - y)) [u^p(y) - u_\lambda^p(y)]}{|y|^\beta} dy$$

$$\leq \int_{\Sigma_\lambda^n} \frac{p g_\alpha(x - y)}{|y|^\beta} u_\lambda^{p-1}(y) [u_\lambda - u(y)] dy,$$

where $$\Sigma_\lambda^n = \{ x \in \Sigma_\lambda : u(x) < u_\lambda(x) \}$$. Denote $$s = [n(p-1)]/\alpha - \beta$$. Invoking the weighted Hardy-Littlewood-Sobolev inequality and Hölder inequality leads to

$$\|u_\lambda - u\|_{L^s(\Sigma_\lambda^n)} \leq C_p \|u_\lambda^{p-1} - u\|_{L^{\frac{n(p-1)s}{n-\alpha s}}(\Sigma_\lambda^n)}$$

$$\leq C_p \|u_\lambda\|_{L^s(\Sigma_\lambda^n)} \|u_\lambda - u\|_{L^s(\Sigma_\lambda^n)}$$.

On the other hand, for $$u \in L^s(\mathbb{R}^n)$$, it is easy to see that $$\|u_\lambda\|_{L^s(\Sigma_\lambda^n)}$$ is sufficiently small, as $$\lambda \to -\infty$$. Therefore, we can choose $$N > 0$$ large enough, such that for any $$\lambda \leq -N$$, $$C_p \|u_\lambda\|_{L^s(\Sigma_\lambda^n)} \leq 1/2$$ and

$$\|u_\lambda - u\|_{L^s(\Sigma_\lambda^n)} \leq \frac{1}{2} \|u_\lambda - u\|_{L^s(\Sigma_\lambda^n)}$$  \hspace{1cm} (34)

which implies that $$\Sigma_\lambda^n = \{ x \in \Sigma_\lambda : u(x) < u_\lambda(x) \}$$ must be zero. This verifies (33).

**Step 2:** We continuously move $$x_1 = \lambda$$ to the right as long as (33) holds. Indeed, suppose that at $$x_1 = \lambda_0 < 0$$, we have, for any $$x \in \Sigma_{\lambda_0} \setminus \{0\}$$,

$$u(x) \geq u_{\lambda_0}(x), \text{ but } u(x) \neq u_{\lambda_0}(x).$$

Next, we will show that the plane can be moved further to the right. Precisely, there exists an $$\epsilon$$ such that

$$u(x) \geq u_\lambda(x), \text{ for } x \in \Sigma_{\lambda} - \{0\}, \text{ with } \lambda \in [\lambda_0, \lambda_0 + \epsilon)$$

By (32), we know that $$u(x) \geq u_\lambda(x)$$ in the interior of $$\Sigma_{\lambda_0}$$. Let

$$\Sigma_{\lambda_0}^u = \{ x \in \Sigma_{\lambda_0} : u(x) \leq u_{\lambda_0}(x) \}.$$

From the analysis mentioned above, it is easy to verify that the $$\Sigma_{\lambda_0}^u$$ is a zero measure set in $$\mathbb{R}^n$$ and $$\lim_{\lambda \to \lambda_0} \Sigma_{\lambda_0}^u \subset \Sigma_{\lambda_0}$$. This together with (34) and the integrability conditions $$u \in L^s(\mathbb{R}^n)$$ ensures that one can choose $$\epsilon$$ small enough such that for any $$\lambda \in [\lambda_0, \lambda_0 + \epsilon)$$

$$\|u_\lambda - u\|_{L^s(\Sigma_\lambda^n)} \leq \frac{1}{2} \|u_\lambda - u\|_{L^s(\Sigma_\lambda^n)}$$

and $$\Sigma_{\lambda}^u = \{ x \in \Sigma : u(x) < u_\lambda(x) \}$$ must be zero.

Finally, we show that the plane can’t stop before hitting the origin. On the contrary, if the plane stops at $$x_1 = \lambda_0 < 0$$, then $$u(x)$$ must be symmetric about the plane $$x_1 = \lambda_0$$, i.e.,

$$u(x) = u_{\lambda_0}(x), \text{ for } \forall x \in \Sigma_{\lambda_0}.$$  \hspace{1cm} (35)

On the other hand, noting that $$|x^{\lambda_0}| > |x|$$ for any $$x \in \Sigma_{\lambda_0}$$, by (32), we have

$$u_{\lambda_0}(x) - u(x) = \int_{\Sigma_{\lambda_0}} \left( g_\alpha(x^{\lambda_0} - y) - g_\alpha(x - y) \right) \left( \frac{w^p(y)}{|y|^\beta} - \frac{u_\lambda^{p}(y)}{|y|^{\lambda_0 + \beta}} \right) dy$$

$$= \int_{\Sigma_{\lambda_0}} \left( g_\alpha(x^{\lambda_0} - y) - g_\alpha(x - y) \right) \left( \frac{1}{|y|^\beta} - \frac{1}{|y|^{\lambda_0 + \beta}} \right) w^p(y) dy > 0,$$
which obviously contradicts with (35). Since the direction is arbitrary, we deduce that \( u \) is radially symmetric about the origin and decreasing. This completes the proof of the second part of Theorem 1.2.

4. Decay rate. This section is devoted to the proof of decay rate (9) for the positive solutions of (1) in Theorem 1.1. The proof consists of the following three propositions.

**Proposition 4.1.** Under the assumption of Theorem (1.1), there exists \( C(n, \alpha, \beta) \) such that, for sufficiently large \(|x|\),

\[
    u(x) \geq C g_\alpha(x).
\]

**Proof.** By (31) and the integrability of \( u \), it is easy to verify that for \(|x| \geq 2\),

\[
    \int_{B_1(0) \cap B|x|(x)} \frac{u^p(y)}{|y|^\beta} dy \geq C(n, \beta) > 0.
\]

Therefore, for sufficiently large \(|x|\),

\[
    u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) \frac{u^p(y)}{|y|^\beta} dy
    \geq \cdots \geq C \int_{B_1(0) \cap B|x|}(x) \frac{u^p(y)}{|y|^\beta} dy \int_0^\infty \exp \left( -\frac{\pi}{t} |x|^2 - \frac{t}{4\pi} \right) t^{\frac{n-\alpha-2}{2}} dt = C g_\alpha(x),
\]

which completes proof of Proposition 4.1.

**Proposition 4.2.** Under the assumption of Theorem (1.1), we have

\[
    \lim_{|y| \to \infty} \frac{u^p(y)}{|y|^\beta} = 0. \tag{37}
\]

**Proof.** By (31), we know that \( u^p(y)/|y|^\beta \in L^1(\mathbb{R}^n) \). Then

\[
    \lim_{R \to \infty} \int_{B_{2R}(0) \setminus B_R(0)} \frac{u^p(y)}{|y|^\beta} dy = 0. \tag{38}
\]

On the other hand, since \( u(x) \) is radially symmetric and decreasing about the origin, we have

\[
    \frac{u^p(2R)}{R^\beta} \int_{B_{2R}(0) \setminus B_R(0)} dy \leq \int_{B_{2R}(0) \setminus B_R(0)} \frac{u^p(y)}{|y|^\beta} dy.
\]

This together with (38) implies that (37) holds.

**Proposition 4.3.** As \(|x|\) is sufficiently large, there exists \( C > 0 \) such that

\[
    u(x) \leq C g_\alpha(x).
\]

**Proof.** For \( \eta \in (0, 1/2) \), let \( \{ \zeta_k : k = 0, 1, 2, \ldots \} \) be a sequence as follows:

\[
    \zeta_0 = 1, \quad \zeta_k = \eta \sum_{j=1}^k 2^{-j} = \eta(1 - \frac{1}{2k+1}), \quad k \geq 1.
\]
For every fixed $R > 0$, we can write

$$u(x) = \left( \int_{B_R(0)} + \int_{B_{\xi_1}|x|} + \int_{[\mathbb{R}^n \setminus B_R(0)] \setminus B_{\xi_1}|x|} \right) g_\alpha(x - y) \frac{u^p(y)}{|y|^{\beta}} \, dy$$

(39)

$$\Delta I(x) + II(x) + III(x)$$

Noticing that for $y \in B_R(0)$, we have

$$\lim_{|x| \to \infty} \frac{g_\alpha(x - y)}{g_\alpha(x)} = 1,$$

which together with (31) implies that

$$I(x) \leq 2g_\alpha(x) \int_{B_R(0)} \frac{u^p(y)}{|y|^{\beta}} \, dy. \quad (40)$$

Next, we turn to $II(x)$. By (37), we deduce that there exists $R_1$ large enough such that

$$0 \leq \frac{u^p((1 - \zeta_1)|x|)}{|(1 - \zeta_1)|x|^\beta} \leq \frac{1}{2}, \quad \forall (1 - \zeta_1)|x| \geq R_1.$$  

Therefore,

$$II(x) \leq \frac{u^p((1 - \zeta_1)|x|)}{|(1 - \zeta_1)|x|^\beta} \int_{B_{\zeta_1}|x|} g_\alpha(x - y) \, dy \leq \frac{1}{2} u((1 - \zeta_1)|x|) \quad (41)$$

Observing that $|y - x| \geq \zeta_1|x|$ as $y \in \{ y \mid \mathbb{R}^n \setminus B_R(0)] \setminus B_{\zeta_1}|x| \}$, and for sufficiently large $|x|$, we have

$$g_\alpha(x) \leq C \exp(-c|x|) \leq 1,$$

combining with that $g_\alpha$ is radial and decreasing, we get

$$III(x) \leq \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)}{|y|^{\beta}} \, dy. \quad (42)$$

It follows from (39)–(42) that

$$u(|x|) \leq 2g_\alpha(|x|) \int_{B_R(0)} \frac{u^p(y)}{|y|^{\beta}} \, dy + \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)}{|y|^{\beta}} \, dy + \frac{1}{2} u((1 - \zeta_1)|x|). \quad (43)$$

Similarly to (39) and (43), for $u(|1 - \zeta_k|x|)$, we have

$$u(|1 - \zeta_k|x|) = \int_{B_R(0)} g_\alpha(1 - \zeta_k - y) \frac{u^p(y)}{|y|^{\beta}} \, dy$$

$$+ \int_{B_{\zeta_k+1}|x|} g_\alpha(1 - \zeta_k - y) \frac{u^p(y)}{|y|^{\beta}} \, dy$$

$$+ \int_{\mathbb{R}^n \setminus B_R(0) \setminus B_{\zeta_k+1}|x|} g_\alpha(1 - \zeta_k - y) \frac{u^p(y)}{|y|^{\beta}} \, dy, \quad (44)$$

and

$$u(|1 - \zeta_k||x|) \leq 2g_\alpha(|1 - \zeta_k|x|) \int_{B_R(0)} \frac{u^p(y)}{|y|^{\beta}} \, dy$$

$$+ \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)}{|y|^{\beta}} \, dy + \frac{1}{2} u((1 - \zeta_{k+1})|x|), \quad k = 1, 2, \ldots \quad (45)$$

Since $g_\alpha$ is a radial and deceasing function, by the definition of $\{\zeta_k\}$, we have

$$g_\alpha(|1 - \zeta_k|x|) \leq g_\alpha(|1 - \zeta_m|x|), \quad \forall k \leq m.$$
Therefore, inserting (45) from \( k = 1 \) to \( k = m \) into (43), we deduce that, for any given integer \( m > 1 \),

\[
\begin{align*}
  u(|x|) & \leq 2g_\alpha([1 - \zeta_m]x) \left( \sum_{k=0}^{m} \frac{1}{2^k} \right) \int_{B_R(0)} \frac{u^p(y)}{|y|^\beta} \, dy \\
  & \quad + \left( \sum_{k=1}^{m} \frac{1}{2^{k-1}} \right) \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)}{|y|^\beta} \, dy + \frac{1}{2^{m+1}} u((1 - \zeta_m)|x|).
\end{align*}
\]

Since \( g_\alpha(x) \) is continuous function on \( \mathbb{R}^n \setminus \{0\} \) and \( u(x) \) is a bounded function in \( \mathbb{R}^n \), invoking that \( \lim_{m \to \infty} \zeta_m = \eta \), we get

\[
u(|x|) \leq 4g_\alpha([1 - \eta]x) \int_{B_R(0)} \frac{u^p(y)}{|y|^\beta} \, dy + 2 \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)}{|y|^\beta} \, dy,
\]

which combining with (31) implies that

\[
\begin{align*}
  u(|x|) & \leq 4g_\alpha([1 - \eta]x) \lim_{R \to \infty} \int_{B_R(0)} \frac{u^p(y)}{|y|^\beta} \, dy \\
  & \quad + 2 \lim_{R \to \infty} \int_{\mathbb{R}^n \setminus B_R(0)} \frac{u^p(y)}{|y|^\beta} \, dy = 4g_\alpha([1 - \eta]x) \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^\beta} \, dy.
\end{align*}
\]

Therefore, by the continuity of \( g_\alpha(x) \) on \( \mathbb{R}^n \setminus \{0\} \), we conclude that

\[
u(|x|) \leq \lim_{\eta \to 0} 4g_\alpha([1 - \eta]x) \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^\beta} \, dy = 4g_\alpha(x) \int_{\mathbb{R}^n} \frac{u^p(y)}{|y|^\beta} \, dy.
\]

This completes the proof of Proposition 4.3. \( \square \)

Summing up the conclusions of Propositions 4.1 and 4.3, we get (9) and completes the proof of Theorem 1.1.

5. Proof of Theorem 1.2. This section is devoted to proving Theorem 1.2. By (1.2) and (12), it suffices to obtain the symmetry and sharp decay of system (2). Therefore, we firstly discuss the radial symmetry and monotonicity of positive solutions of system (2). For simplicity, we use same notations as section 3. For any pair of solutions \((u, v)\) of system (2), it is easy to check that

\[
u_\lambda(x) - u(x) = \int_{\Sigma_\lambda} \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( \frac{u^p v^q(y)}{|y|^\beta} - \frac{u_\lambda^p v_\lambda^q(y)}{|y^\lambda|^\beta} \right) \, dy
\]

\[
= \int_{\Sigma_\lambda} \left\{ \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( \frac{1}{|y|^\beta} - \frac{1}{|y^\lambda|^\beta} \right) u^p v^q(y) \right.
\]

\[
+ \left. \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( u^p v^q(y) - u_\lambda^p v_\lambda^q(y) \right) \frac{1}{|y^\lambda|^\beta} \right\} \, dy
\]

\[
\triangleq B_1(x) + B_2(x), \quad (46)
\]
and
\[
v_\lambda(x) - v(x) = \int_{\Sigma_\lambda} \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( \frac{v^p}{|y|} - \frac{v^p u_\lambda^q(y)}{|y|^\beta} \right) dy \\
= \int_{\Sigma_\lambda} \left\{ \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( \frac{1}{|y|} - \frac{1}{|y|^\beta} \right) v^p(y) \right. \\
+ \left( g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right) \left( v^p u_\lambda^q(y) - v^p u_\lambda^q(y) \right) \frac{1}{|y|^\beta} \left\} dy \\
\triangleq C_1(x) + C_2(x). \tag{47}
\]

The proof of symmetry and monotonicity for system (2) is similar to equation (1), which is also made up of two steps. In step 1, we will compare the value of \( u(x) \) with \( u_\lambda(x) \) and show that for sufficiently negative \( \lambda < 0 \), there holds
\[
u(x) \geq u_\lambda(x), \quad \text{and} \quad v(x) \geq v_\lambda(x), \quad \forall x \in \Sigma_\lambda \setminus \{0\}. \tag{48}
\]

In step 2, we continuously move the plane \( x_1 = \lambda \) along \( x_1 \) direction from near negative infinity to the right as long as (48) holds. By moving this plane in this way, we finally show that the plane will stop at the origin. Now we turn to step 1.

**Step 1.** Since \( B_1(x) \leq 0 \) and \( |y|^\lambda > |y| \) for any \( y \in \Sigma_\lambda \), we have
\[
u_\lambda(x) - u(x) \leq \int_{\Sigma_\lambda} \left[ g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right] \frac{u^p v^q(y) - u^p v_\lambda^q(y)}{|y|^\beta} dy \\
\leq \int_{\Sigma_\lambda} \left[ g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right] \frac{u^p(y) (v^q(y) - v_\lambda^q(y))}{|y|^\beta} \\
+ \int_{\Sigma_\lambda} \left[ g_\alpha(x^\lambda - y) - g_\alpha(x - y) \right] \frac{(u^p(y) - u_\lambda^p(y))v_\lambda^q(y)}{|y|^\beta} \leq B_{2,1}(x) + B_{2,2}(x), \tag{49}
\]

where \( \Sigma_\lambda^w = \{ x \in \Sigma_\lambda \mid u(x) < u_\lambda(x) \} \) and \( \Sigma_\lambda^v = \{ x \in \Sigma_\lambda \mid v(x) < v_\lambda(x) \} \).

Set \( \tau \triangleq \frac{n(p+q-1)}{(\alpha-\beta)} \), by the weighted Hardy-Littlewood-Sobolev inequality and Hölder’s inequality, we have
\[
\| B_{2,1}(x) \|_{L^\tau(\Sigma_\lambda^w)} \leq C(\alpha, n, q) \| u^p v_\lambda^q - 1(v_\lambda - v) \|_{L^{\frac{\alpha}{\alpha-\beta}}(\Sigma_\lambda^w)} \\
\leq C(\alpha, n, q) \| v_\lambda \|_{L^{\tau(\Sigma_\lambda)}}^{q-1} \| v \|_{L^{\tau(\Sigma_\lambda)}}^{q} \| v_\lambda - v \|_{L^{\tau(\Sigma_\lambda)}}, \tag{50}
\]

and
\[
\| B_{2,2}(x) \|_{L^\tau(\Sigma_\lambda^w)} \leq C(\alpha, n, p) \| u^p - v^p \|_{L^{\tau(\Sigma_\lambda)}} \| v^p \|_{L^{\tau(\Sigma_\lambda)}}^{q} \| u_\lambda - u \|_{L^{\tau(\Sigma_\lambda)}}, \tag{51}
\]

This combining with (49) and (50), implies that
\[
u_\lambda(x) - u(x) \|_{L^{\tau}(\Sigma_\lambda^w)} \leq C(\alpha, n, q) \| v_\lambda \|_{L^{\tau(\Sigma_\lambda)}}^{q-1} \| u \|_{L^{\tau(\Sigma_\lambda)}} \| v_\lambda - v \|_{L^{\tau(\Sigma_\lambda)}} + C(\alpha, n, p) \| u_\lambda \|_{L^{\tau(\Sigma_\lambda)}}^{p-1} \| v_\lambda \|_{L^{\tau(\Sigma_\lambda)}}^{q} \| u_\lambda - u \|_{L^{\tau(\Sigma_\lambda)}}, \tag{52}
\]

On the other hand, for \( u, v \in L^\tau(\mathbb{R}^n) \), it is easy to choose \( N > 0 \) large enough, such that for any \( \lambda \leq -N < 0 \),
\[
C(\alpha, n, q) \| u \|_{L^{\tau(\Sigma_\lambda)}} \| v_\lambda \|_{L^{\tau(\Sigma_\lambda)}}^{q-1} \leq 1/4
\]
and
\[
C(\alpha, n, p) \| u \|_{L^{\tau(\Sigma_\lambda)}} \| v_\lambda \|_{L^{\tau(\Sigma_\lambda)}}^{q} < 1/4,
\]
which combining with (52), implies that
\[
\|u(\lambda) - u(x)\|_{L^r(\Sigma^u_\lambda)} \leq \frac{1}{4} \|v(\lambda) - v\|_{L^r(\Sigma^u_\lambda)} + \frac{1}{4} \|\lambda - u\|_{L^r(\Sigma^u_\lambda)}.
\]
Similarly, we have
\[
\|v(\lambda) - v(x)\|_{L^r(\Sigma^u_\lambda)} \leq \frac{1}{4} \|v(\lambda) - v\|_{L^r(\Sigma^u_\lambda)} + \frac{1}{4} \|\lambda - u\|_{L^r(\Sigma^u_\lambda)}.
\]
This together with (53) leads to that
\[
\|u(\lambda) - u(x)\|_{L^r(\Sigma^v_\lambda)} = \|v(\lambda) - v(x)\|_{L^r(\Sigma^v_\lambda)} = 0.
\]
Therefore $\Sigma^u_\lambda$ and $\Sigma^v_\lambda$ must be two zero-measure sets, which completes Step 1.

**Step 2.** We continuously move $x_1 = \lambda$ to the right as long as (48) holds. Indeed, suppose that at $x_1 = \lambda_0 < 0$, we have, for any $x \in \Sigma_{\lambda_0} \setminus \{0\}$
\[
u(x) \geq u_{\lambda_0}(x), \quad v(x) \geq v_{\lambda_0}(x), \text{ but } u(x) \neq u_{\lambda_0}(x), \quad \text{ and } v(x) \neq v_{\lambda_0}(x).
\]
Next, we will show that the plane can be moved further to the right. Precisely, there exists an $\epsilon$ depending on $n, \alpha, \beta$ and the solution $(u, v)$ itself such that
\[
u(x) \geq u_{\lambda_0}(x), \quad v(x) \geq v_{\lambda_0}(x), \quad \forall x \in \Sigma_\lambda \setminus \{0\}, \lambda \in [\lambda_0, \lambda_0 + \epsilon]
\]
Under the assumption that $v(x) \neq v_{\lambda_0}$ on $\Sigma_{\lambda_0}$, by (46) and the non-negativity of $u, v$, we have $u(x) > u_{\lambda_0}(x)$ in the interior of $\Sigma_{\lambda_0} \setminus \{0\}$.

Let
\[
\tilde{\Sigma}^u_{\lambda_0} = \{x \in \Sigma_{\lambda_0} \mid u(x) \leq u_{\lambda_0}(x)\} \text{ and } \tilde{\Sigma}^v_{\lambda_0} = \{x \in \Sigma_{\lambda_0} \mid v(x) \leq v_{\lambda_0}(x)\}.
\]
From the analysis mentioned above, it is easy to check that the $\tilde{\Sigma}^u_{\lambda_0}$ is a zero measure set in $\mathbb{R}^n$. Similarly, we also have $m\{\tilde{\Sigma}^v_{\lambda_0}\} = 0$. This together with (52) and the integrability conditions $u, v \in L^s(\mathbb{R}^n)$ ensures that one can choose $\epsilon$ small enough such that for all $\lambda$ in $[\lambda_0, \lambda_0 + \epsilon]$
\[
C(\alpha, n, p)\|u_{\lambda}\|_{L^r(\Sigma_\lambda)}^{\frac{1}{p}}\|v_{\lambda}\|_{L^r(\Sigma_\lambda)}^{\frac{1}{q}} \leq \frac{1}{4}, \quad C(\alpha, n, q)\|v_{\lambda}\|_{L^r(\Sigma_\lambda)}^{\frac{1}{p}}\|u_{\lambda}\|_{L^r(\Sigma_\lambda)}^{\frac{1}{q}} \leq \frac{1}{4},
\]
\[
C(n, \alpha, p)\|u_{\lambda}\|_{L^r(\Sigma_\lambda)}\|v_{\lambda}\|_{L^r(\Sigma_\lambda)}^{\frac{1}{q}} \leq \frac{1}{4}, \quad C(n, \alpha, q)\|v_{\lambda}\|_{L^r(\Sigma_\lambda)}\|u_{\lambda}\|_{L^r(\Sigma_\lambda)}^{\frac{1}{q}} \leq \frac{1}{4}.
\]
So (53) and (54) hold. Thus we also have
\[
\|u(\lambda) - u(x)\|_{L^r(\Sigma^u_\lambda)} = 0 \quad \text{and} \quad \|v(\lambda) - v(x)\|_{L^r(\Sigma^v_\lambda)} = 0,
\]
which implies that the measures of $\tilde{\Sigma}^u_\lambda$ and $\tilde{\Sigma}^v_\lambda$ must be zero. This verifies (56).

Finally, we show that the plane can’t stop before hitting the origin. On the contrary, if the plane stops at $x_1 = \lambda_0 < 0$, then $u(x)$ and $v(x)$ must be symmetric about the plane $x_1 = \lambda_0$, i.e.,
\[
u(x) = u_{\lambda_0}(x) \text{ and } v(x) = v_{\lambda_0}(x), \quad \forall x \in \Sigma_{\lambda_0}.
\]
On the other hand, noting that $|y_{\lambda_0}| > |y|$ for any $y \in \Sigma_{\lambda_0}$, we have
\[
u(x) - u_{\lambda_0}(x) = \int_{\mathbb{R}^n} \left[ g_\alpha(x - y) - g_\alpha(x_{\lambda_0} - y) \right]\frac{u^p(y)v^q(y)}{|y|^3} dy
\]
\[
= \int_{\Sigma_{\lambda_0}} \left[ g_\alpha(x - y) - g_\alpha(x_{\lambda_0} - y) \right] \left( \frac{u^p(y)v^q(y)}{|y|^3} - \frac{u^p_{\lambda_0}(y)v^q_{\lambda_0}(y)}{|y_{\lambda_0}|^3} \right) dy
\]
\[
= \int_{\Sigma_{\lambda_0}} \left[ g_\alpha(x - y) - g_\alpha(x_{\lambda_0} - y) \right] \left( \frac{1}{|y|^3} - \frac{1}{|y_{\lambda_0}|^3} \right) u^p(y)v^q(y) dy > 0.
\]
which obviously contradicts with (57). Since the direction is arbitrary, we derive
that \( u \) and \( v \) are radially symmetric about the origin and decreasing. Next, we turn
our attentions to obtain the sharp asymptotic behavior of system (2). By (13), it
suffices to show that there exists \( C \) satisfying, for sufficiently large \( |x| \),

\[
u(x) \geq Cg_\alpha(x), \quad v(x) \geq Cg_\alpha(x).
\]

Similar to (36), it is easy to check that for \( |x| \geq 2 \),

\[
\int_{B_1(0) \cap B_{|x|}(x)} \frac{u^p(y)v^q(y)}{|y|^\beta} \, dy \geq C(n, \beta, \alpha, p, q) > 0.
\]

and

\[
u(x) = \int_{\mathbb{R}^n} g_\alpha(x-y) u^p v^q(y) \, dy
\]

\[
= \frac{1}{(4\pi)^{n/2} \Gamma(n/2)} \int_{0}^{\infty} \int_{\mathbb{R}^n} \exp \left( -\frac{\pi}{t} |x-y|^2 \right) \frac{u^p v^q(y)}{|y|^\beta} \, dy \, t^{n-\beta/2} \, dt
\]

\[
= C \int_{0}^{\infty} \int_{B_1(0) \cap B_{|x|}(x)} \exp \left( -\frac{\pi}{t} |x-y|^2 \right) \frac{u^p v^q(y)}{|y|^\beta} \, dy \, t^{n-\beta/2} \, dt
\]

\[
\geq C \int_{B_1(0) \cap B_{|x|}(x)} \frac{u^p v^q(y)}{|y|^\beta} \, dy \int_{0}^{\infty} \exp \left( -\frac{\pi}{t} |x|^2 - \frac{1}{4\pi} t \frac{n}{2} - \frac{\beta}{2} - \frac{n-\beta}{2} \right) dt = C g_\alpha(x),
\]

This, with (13) and Theorem 1.1, completes the proof of Theorem 1.2.

6. Uniqueness. This section is devoted to the proof of (14). Note that the system
(10) is equivalent to the system (2) for \( \alpha = 2 \). It follows from Theorem 1.2 that the
positive solutions \( u \) and \( v \) are symmetric about the origin and exponential decay at
infinity. Hence,

\[
u(r) = u(|r|), \quad v(r) = v(|r|),
\]

and

\[
\lim_{r \to \infty} u(r) = \lim_{r \to \infty} v(r) = 0. \quad (58)
\]

Consequently, in polar coordinates, the positive solution \((u, v)\) of system (10) satisfy
the following differential equation:

\[
\left\{
\begin{align*}
-d\left( r^{n-1} u_r \right) &= r^{n-1} \frac{u^p v^q}{r^\beta} - u, \quad \text{in } (0, \infty), \\
-d\left( r^{n-1} v_r \right) &= r^{n-1} \frac{v^p u^q}{r^\beta} - v, \quad \text{in } (0, \infty).
\end{align*}
\right.
\]

A direct calculation shows that

\[
u_r(r) = -r^{-(n-1)} \int_{0}^{r} s^{n-1} \left( \frac{u^p v^q(s)}{s^\beta} - u(s) \right) ds,
\]

and for any \( r_0 \in [0, \infty) \),

\[
u(r) - u(r_0) = - \int_{r_0}^{r} t^{-(n-1)} \int_{0}^{t} s^{n-1} \left( \frac{u^p v^q(s)}{s^\beta} - u(s) \right) ds dt. \quad (59)
\]

Similarly,

\[
v(r) - v(r_0) = - \int_{r_0}^{r} t^{-(n-1)} \int_{0}^{t} s^{n-1} \left( \frac{v^p u^q(s)}{s^\beta} - v(s) \right) ds dt. \quad (60)
\]

We claim that

\[
u(r_0) = v(r_0). \quad (61)
\]
Indeed, if (61) does not hold, then one of the following results is true:

(1) \( u(r_0) > v(r_0) \) and (2) \( u(r_0) < v(r_0) \). \( (62) \)

If the inequality (1) in (62) holds, by the continuity of \( u(r) \), there exists \( R > 0 \) such that

\[ v(r) < u(r), \quad \forall r \in (r_0, R). \] \( (63) \)

Set

\[ R_0 = \sup \{ R > 0 : v(r) < u(r), r \in (r_0, R) \}. \]

By (58), we have

\[ u(R_0) = v(R_0), \]

and with \( r = R_0 \) in (59) and (60),

\[ u(r_0) - v(r_0) = \int_{r_0}^{R_0} \left\{ \int_{r_0}^{t} t^{-(n-1)} \int_{r_0}^{t} s^{n-1} \left( \frac{u^p v^q(s)}{s^\beta} - u(s) \right) ds \right\} dt. \]

On the other hand, note that \( p \leq q \) and (63), we conclude that

\[ u^{p-q}(s) \leq v^{p-q}(s), \quad \forall s \in (r_0, R_0) \]

and

\[ \frac{u^p v^q(s)}{s^\beta} - u(s) \leq \frac{v^p u^q(s)}{s^\beta} - v(s) \quad \forall s \in (r_0, R_0). \]

This implies that

\[ u(r_0) \leq v(r_0), \]

which contradicts with (1) in (62). Similarly, the inequality (2) is not true. Therefore, by the arbitrariness of \( r_0 \), we get \( u(r) = v(r) \) for any \( r \in (0, \infty) \). This completes the proof of Theorem 1.3.

\[ \square \]

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