A NOTE ON SPECTRAL TRIPLES ON THE QUANTUM DISK

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Abstract. By modifying the ideas from our previous paper [4], we construct spectral triples from implementations of covariant derivations on the quantum disk.

1. Introduction

Spectral triples are a key tool in noncommutative geometry [1], as they allow analytical methods in studying quantum spaces. In this note we show how, by changing the concept of a Hilbert space implementation of an unbounded derivation, one can use the techniques of our papers [4], [5] to construct meaningful, geometrical spectral triples for the quantum disk, the Toeplitz C*-algebra of the unilateral shift.

Our previous paper [4] classified unbounded derivations, covariant with respect to a natural rotation, and their implementations in Hilbert spaces obtained from the GNS construction with respect to invariant states. Surprisingly, no implementation of a covariant derivation in any GNS Hilbert space for a faithful, normal, invariant state turned out to have compact parametrices for a large class of boundary conditions. However, if we relax the concept of an implementation by allowing operators to act between different Hilbert spaces, then it turns out, as demonstrated in this paper, that there is an interesting class of examples of spectral triples that can be constructed this way using APS-type boundary conditions. Other examples of spectral triples of the Toeplitz algebra, in GNS Hilbert spaces of non-normal states, were also constructed in section 4.2 of [2] and in [3].

We review the notation and basic concepts from [4] below and, in a number of places, we use the results contained in that reference.

2. Quantum Disk

Let \( \{E_k\}_{k=0}^{\infty} \) be the canonical basis for \( \ell^2(\mathbb{Z}_{\geq 0}) \) and \( U \) be the unilateral shift defined by \( UE_k = E_{k+1} \).

Note that \( U \) is an isometry, i.e. \( U^*U = I \). Consider the Toeplitz algebra \( A = C^*(U) \), the C*-algebra generated by \( U \). This algebra is called the quantum disk. We also use the diagonal label operator

\[ KE_k = kE_k. \]

It follows that for \( a : \mathbb{Z}_{\geq 0} \to \mathbb{C} \), we have

\[ a(K)E_k = a(k)E_k. \]

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These are precisely the operators which are diagonal with respect to $\{E_k\}$. The operators $(\mathbb{K}, U)$ serve as noncommutative polar coordinates, and they satisfy the following relation:

$$\mathbb{K}U = U(\mathbb{K} + I).$$

Let $c$ be the set of $a(k)$, as above, which are convergent as $k \to \infty$. Let $c_{00}$ be the set of $a(k)$ such that there exists $k_0$ satisfying $a(k) = 0$ for $k \geq k_0$. Lastly, let $c_{00}^+$ be the set of all eventually constant functions, i.e. functions $a(k)$ such that there exists $k_0$ where $a(k)$ is constant for $k \geq k_0$.

Consider, for future reference, the following two dense $^*$-subalgebras of $A$:

$$A = \left\{ a = \sum_{n \geq 0} U^n a_n(\mathbb{K}) + \sum_{n < 0} a_n(\mathbb{K})(U^*)^{-n} : a_n(k) \in c_{00}^+, \text{ finite sums} \right\}$$

and

$$A_0 = \left\{ a = \sum_{n \geq 0} U^n a_n(\mathbb{K}) + \sum_{n < 0} a_n(\mathbb{K})(U^*)^{-n} : a_n(k) \in c_{00}, \text{ finite sums} \right\}.$$ Naturally, $A_0 \subseteq A$, and, by Proposition 3.1 in [4], $A = Pol(U, U^*)$.

3. DERIVATIONS ON QUANTUM DISK

Let $\rho : A \to A$, $0 \leq \theta < 2\pi$, be a one parameter group of automorphisms of $A$ defined by $\rho_\theta(U) = e^{i\theta}U$ and $\rho_\theta(U^*) = e^{-i\theta}U^*$. The automorphisms $\rho_\theta$ can also be written in terms of the label operator $\mathbb{K}$:

$$\rho_\theta(a) = e^{i\theta \mathbb{K}} a e^{-i\theta \mathbb{K}},$$

and they preserve $A$. By Proposition 4.2 in [4], any $d : A \to A$, covariant with respect to $\rho_\theta$:

$$\rho_\theta(d(a)) = e^{i\theta}d(\rho_\theta(a)),$$

is of the following form:

$$d(a) = [U \beta(\mathbb{K}), a],$$

where $\beta(k + 1) - \beta(k) \in c$. We use notation:

$$\lim_{k \to \infty} \beta(k + 1) - \beta(k) := \beta_\infty,$$

and below we only consider covariant derivations with $\beta_\infty \neq 0$.

4. COVARIANT IMPLEMENTATIONS ON QUANTUM DISK

We will begin by introducing the following family of states $\tau_w : A \to \mathbb{C}$ on $A$, defined by

$$\tau_w(a) = \text{tr}(w(\mathbb{K})a),$$

where $w(k) > 0$ for all $k \in \mathbb{Z}_{\geq 0}$ and

$$\sum_{k=0}^{\infty} w(k) = 1.$$

As a result of Proposition 5.4 in [4], $\tau_w$ are precisely the $\rho_\theta$-invariant, normal, faithful states on $A$. Let $H_w$ be the Hilbert space obtained by Gelfand-Naimark-Segal (GNS) construction.
on $A$ using state $\tau_w$. Since the state is faithful, $H_w$ is the completion of $A$ with respect to the inner product given by
\[(a, b)_w = \tau(w(\mathbb{K})a^*b).\]
A simple calculation leads to the following precise description: $H_w$ is the Hilbert space consisting of infinite series of operators
\[f = \sum_{n \geq 0} U^n f_n(\mathbb{K}) + \sum_{n < 0} f_n(\mathbb{K})(U^*)^{-n}\]
satisfying:
\[\|f\|_w^2 = \sum_{n \geq 0} \sum_{k \geq 0} w(k)|f_n(k)|^2 + \sum_{n < 0} \sum_{k \geq 0} w(k - n)|f_n(k)|^2 < \infty.\]
It is important to notice that $A_0 \subseteq A \subseteq H_w$ and that both $A$ and $A_0$ are dense in $H_w$.

The GNS representation map $\pi_w : A \to B(H_w)$ is given by left-hand multiplication:
\[\pi_w(a)f = af.\]

Define a one parameter group of unitary operators $U^w_\theta : H_w \to H_w$ via the formula:
\[U^w_\theta f = \sum_{n \geq 0} U^n e^{i\theta} f_n(\mathbb{K}) + \sum_{n < 0} e^{i\theta} f_n(\mathbb{K})(U^*)^{-n}.\]
It is easily seen that they are implementing $\rho_\theta$, as we have:
\[\pi_w(\rho_\theta(a)) = U^w_\theta \pi_w(a)(U^w_\theta)^{-1}.\]

Consider an additional weight, $w'(k)$, possibly different from $w(k)$, satisfying the same conditions. An operator $D : H_w \supseteq A \to H_{w'}$ is called a covariant implementation of a covariant derivation $d$ if for every $a \in A$, and for every $f \in A$ considered as an element of both $H_w$ and $H_{w'}$, we have:
\[D\pi_w(a)f - \pi_{w'}(a)Df = \pi_{w'}(d(a))f,\]
and, additionally, $D$ satisfies:
\[U^w_\theta D(U^w_\theta)^{-1} f = e^{i\theta} Df.\]
Allowing for implementations between different Hilbert spaces is the key difference between this paper and reference [4].

Exactly the same argument as in Proposition 6.1 in [4] shows that any implementation $D$ is of the form:
\[Df = U\beta(\mathbb{K})f - fU\alpha(k),\]
where $\alpha(k)$ is a sequence such that:
\[\sum_{k=0}^\infty |\beta(k) - \alpha(k)|^2 w'(k) < \infty. \tag{4.1}\]

The assumption $\beta_\infty \neq 0$ implies that $\beta(k)$ has at most finitely many zeros. Without loss of generality, we may assume that $\beta(k) \neq 0$ as this can be obtained by a bounded perturbation. Arguments in [4] show that if $\alpha(k)$ has infinitely many zeros, then $\text{Ker}(D)$ has infinite dimension, and so cannot define a spectral triple. So we assume that for every $k$ we have:
\[\alpha(k), \beta(k) \neq 0. \tag{4.2}\]
Additionally, as in [4], we write
\[ \alpha(k) = \beta(k) \frac{\mu(k+1)}{\mu(k)} \] where \( \mu(0) = 1 \). \hfill (4.3)

To set up APS-type boundary conditions on \( D \) we need the following definition. Let \( \mathcal{A}_{APS} \), \( \mathcal{A}_0 \subseteq \mathcal{A}_{APS} \subseteq \mathcal{A} \), be a subspace defined as:
\[
\mathcal{A}_{APS} = \left\{ a = \sum_{n \geq 0} U^n a_n(\mathbb{K}) + \sum_{n < 0} a_n(\mathbb{K})(U^*)^{-n} : a_n(k) \in c_{00} \text{ for } n \geq 0, a_n(k) \in c_{00}^{+} \text{ for } n < 0 \right\}.
\]

The implementation operators \( D \) are initially defined on a small domain and we introduce boundary conditions in the usual way. Define \( D_{APS} : \overline{\mathcal{A}_{APS}} \rightarrow H_{w'} \) by \( D_{APS} = \overline{D} |_{\mathcal{A}_{APS}} \), the closure of \( D \) restricted to \( \mathcal{A}_{APS} \). The following is the main technical result of this paper.

**Theorem 4.1.** Assume
\[
\frac{|\beta(k) \cdots \beta(k+n)|}{|\beta(j) \cdots \beta(j+n)|} \leq \text{const} \text{ for all } k \leq j, n \in \mathbb{Z}_{n \geq 0} \] \hfill (4.4)
and
\[
\sum_{k=0}^{\infty} \sum_{j=0}^{\infty} \frac{1}{(\max(j,k)+1)^2} \left| \frac{\mu(j)}{\mu(k)} \right|^2 w(k) \frac{1}{w'(j)} < \infty.
\] \hfill (4.5)
Assume additionally that \( \beta_{\infty} \neq 0 \), and \( 4.1, 4.2, \) and \( 4.3 \) hold. Then \( D_{APS} \) is invertible and its inverse is compact.

**Proof.** We begin by expressing the action of \( D_{APS} \) on elements of \( \mathcal{A}_{APS} \) in terms of the Fourier decomposition:
\[
D_{APS} f = \sum_{n \geq 0} U^{n+1} (D_{n} f_n(\mathbb{K})) + \sum_{n < 0} (D_{n} f_n)(\mathbb{K})(U^*)^{-n-1},
\]
where \( D_{n} \) are the closures of the following:

for \( n \geq 0 \) and \( f \in c_{00} \subseteq \ell_{w}^{2} \):
\[
D_{n} f(k) = \beta(k+n)f(k) - \beta(k) \frac{\mu(k+1)}{\mu(k)} f(k+1),
\]

for \( n < 0 \) and \( f \in c_{00}^{+} \subseteq \ell_{w_{n}}^{2} \):
\[
D_{n} f(k) = \beta(k-n-1) \frac{\mu(k-n)}{\mu(k-n-1)} f(k) - \beta(k-1) f(k-1),
\]
where we used the following notation:
\[
\ell_{w}^{2} = \left\{ f(k) : \sum_{k=0}^{\infty} |f(k)|^2 w(k) < \infty \right\}
\]
and
\[
\ell_{w_{n}}^{2} = \left\{ f(k) : \sum_{k=0}^{\infty} |f(k)|^2 w(k-n) < \infty \right\}.
\]
The above is done by direct calculation, see additionally Lemma 7.7 in [4].
It follows that, for \( n \geq 0 \), \( D_n : \ell_w^2 \supseteq c_{00} \to c_{00} \subseteq \ell_w^2 \) is invertible, with the inverse given by the formula:

\[
D_n^{-1} g(k) = \sum_{j=k}^{\infty} \frac{\beta(k) \cdots \beta(k+n-1) \mu(j)}{\beta(j) \cdots \beta(j+n)} g(j)
\]

while, for \( n < 0 \), \( \ker(D_n) = 0 \) and \( D_n : \ell_w^2 \supseteq c_{00} \to \ell_{w+1}^2 \) is invertible with the inverse given by:

\[
D_n^{-1} g(k) = \sum_{j=0}^{k} \frac{\beta(j) \cdots \beta(j-n-2) \mu(j-n-1)}{\beta(k) \cdots \beta(k-n-1)} g(j).
\]

We now show that, in each case, \( D_n^{-1} \) is a Hilbert-Schmidt operator and is thus compact. First, for \( n \geq 0 \) we have:

\[
\|D_n^{-1}\|^2_{HS} = \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \left| \frac{\beta(k) \cdots \beta(k+n-1)}{\beta(j) \cdots \beta(j+n)} \right|^2 \frac{1}{\beta(j+n)} \frac{\mu(j)}{\mu(k)}^2 \frac{w(k)}{w'(j)}.
\]

Therefore, using assumption \([4,4]\) we can estimate as follows:

\[
\|D_n^{-1}\|^2_{HS} \leq \sum_{k=0}^{\infty} \sum_{j=k}^{\infty} \frac{1}{(1+j+n)^2} \left| \frac{\mu(j)}{\mu(k)} \right|^2 \frac{w(k)}{w'(j)}.
\]

Consequently, \( \|D_n^{-1}\|^2_{HS} < \infty \) by \([4,5]\) and \( \|D_n^{-1}\|_{HS} \to 0 \) as \( n \to \infty \) by Lebesgue’s Dominated Convergence Theorem.

Secondly, for \( n < 0 \) we obtain the following expression:

\[
\|D_n^{-1}\|^2_{HS} = \sum_{k=0}^{\infty} \sum_{j=0}^{k} \left| \frac{\beta(j) \cdots \beta(j-n-2)}{\beta(k) \cdots \beta(k-n-1)} \right|^2 \frac{\mu(j-n-1)}{\mu(k-n)} \frac{w(k-n)}{w'(j-n-1)}.
\]

Changing the indices \( k-n \to k' \) and \( j-n-1 \to j' \), the order of summation, and estimating as above yields:

\[
\|D_n^{-1}\|^2_{HS} \leq \sum_{j=-n-1}^{\infty} \sum_{k=j+1}^{\infty} \frac{1}{(1+k)^2} \left| \frac{\mu(j)}{\mu(k)} \right|^2 \frac{w(k)}{w'(j)} < \infty.
\]

It follows that \( \|D_n^{-1}\|^2_{HS} \to 0 \) as \( -n \to \infty \) since this is the tail of an absolutely convergent series.

Since \( D_{APS}^{-1} \) is unitarily equivalent to a direct sum of Hilbert-Schmidt operators \( D_n^{-1} \) with Hilbert Schmidt norms approaching zero, it follows that \( D_{APS}^{-1} \) is compact. This completes the proof. \( \square \)

The main significance of this result is outlined in the following theorem. First, we introduce some notation related to spectral triples as considered in \([4]\).

Let \( H = H_w \oplus H_w \), with grading \( \Gamma|_{\ell_w} = 1 \) and \( \Gamma|_{\ell_w} = -1 \). Define a representation \( \pi : A \to \text{B}(H) \) of \( A \) in \( H \) by the formula:

\[
\pi(a) = (\pi_w(a), \pi_w(a)),
\]
and also define a quantum analog of a Dirac operator on the unit disk by:

\[ \mathcal{D} = \begin{bmatrix} 0 & D_{APS} \\ D_{APS}^* & 0 \end{bmatrix}, \]

so that \( \pi(a) \) are even and \( \mathcal{D} \) is odd with respect to grading \( \Gamma \).

**Theorem 4.2.** With the above notation, \((\mathcal{A}, H, \mathcal{D})\) forms an even spectral triple over \( \mathcal{A} \).

**Proof.** By Theorem 4.1, we have that \( D_{APS}^{-1} \) is compact and so \( \mathcal{D} \) has compact parametrices by the results in the appendix of [4]. Since \( D_{APS} \) is an implementation of a derivation \( d : \mathcal{A} \to \mathcal{A} \), the commutator \([\mathcal{D}, \pi(a)]\) is bounded for all \( a \in \mathcal{A} \). This completes the proof. \( \square \)

We conclude this paper by giving explicit examples of parameters \( w(k), w'(k), \beta(k), \) and \( \mu(k) \) that satisfy the conditions of the above theorems.

Assume \( 3 < a < 2b - 1 < c, \beta(k) = 1 + k, \) and consider the following sequences:

\[
\begin{align*}
    w'(k) &= \frac{w'(0)}{(1 + k)^a}, & \mu(k) &= \frac{1}{(1 + k)^b}, & w(k) &= \frac{w(0)}{(1 + k)^c}
\end{align*}
\]

where \( w'(0) \) and \( w(0) \) are such that \( \sum_{k=0}^{\infty} w'(k) = 1 = \sum_{k=0}^{\infty} w(k) \). Then a straightforward calculation shows that they satisfy the necessary conditions for \((\mathcal{A}, H, \mathcal{D})\) to be an even spectral triple.

**References**

[1] Connes, A., *Non-Commutative Differential Geometry*, Academic Press, 1994.

[2] Connes, A. and Moscovici, H., Transgression and the Chern character of finite-dimensional K-cycles, *Commun. Math. Phys.*, 155: 103-122, 1993.

[3] Englis, M. Falk, K. and Iochum, B., Spectral triples and Toeplitz operators, *J. Noncommut. Geom.*, 9, 1041-107, 2015

[4] Klimek, S., McBride, M., Rathnayake, S., Sakai, and Wang, H., Derivations and Spectral Triples on Quantum Domains I: Quantum Disk. *SIGMA*, 13, 075, 2017.

[5] Klimek, S., McBride, M., and Rathnayake, S., Derivations and Spectral Triples on Quantum Domains II: Quantum Annulus, to appear *Sci. Chi. Math.*, arXiv:1710.06257.

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