BILINEAR MAPS AND CENTRAL EXTENSIONS OF ABELIAN GROUPS
Arturo Magidin*

Abstract. Every nilpotent group of class at most two may be embedded in a central extension of abelian groups with bilinear cocycle. The embedding is shown to depend only on the base group. Some refinements are obtained by considering the cohomological situation explicitly.

The main result of this paper is that every nilpotent group of class at most two may be embedded into a central extension of abelian groups, in which the associated cocycle is bilinear (definitions are recalled in Section 1). The result is related to a paper of N.J.S. Hughes (see [2]), in which he establishes a one to one correspondence between the equivalence classes of central extensions of abelian groups with what he calls “regular bilinear mappings”. The proof of Theorem 2 in [2] is very similar to our proof of Theorem 2.18 in the present work.

In the first section we will recall the basic definitions relating to central extensions and the second cohomology group of two abelian groups. We will also introduce the notion of a twisted product of abelian groups, to be used later. In the second section we establish our main result and discuss some examples. In the third and final section we cast the main theorem into a cohomological setting to gain more information about the embedding we construct in Section 2

It is my very great pleasure to express my deep gratitude and indebtedness to Prof. George M. Bergman, for his ever ready advice and encouragement throughout the preparation of this paper.

Section 1. Preliminaries and notation

We will denote the identity element of a group $G$ by $e_G$, if $G$ is written multiplicatively, and by $0_G$ if it is written additively; we will omit the subscript if it is clear from context. Given a group $G$ and elements $x, y \in G$, $[x, y]$ denotes their commutator; that is

$$[x, y] = x^{-1}y^{-1}xy.$$

Given two subgroups $U$ and $V$ of $G$, $[U, V]$ denotes the subgroup of $G$ generated by all elements $[u, v]$, where $u \in U$ and $v \in V$.

For general notation and a more complete treatment of the extension problem, we direct the reader to [3], Chapter 7. We will also omit proofs and direct the reader to Rotman’s text for them. Let

$$1 \rightarrow B \xrightarrow{i} G \xrightarrow{\pi} A \rightarrow 1$$

* The author was supported by a fellowship from the Programa de Formación y Superación del Personal Académico de la UNAM, administered by the DGAPA.

AMS Classification: 20E22, 20F18 (primary); 20J05 (secondary). Keywords: central extensions, nilpotent of class two.
be a central extension (so that $B \subseteq Z(G)$) of groups with $B$ and $A$, both abelian. It
will be convenient (and the formulas clearer) if the product in $G$ (and hence in $B$) is
written additively, and the product in $A$ multiplicatively, although in general $G$ may
be nonabelian.

Recall that a factor set (or cocycle) is a map $\gamma: A \times A \to B$ that satisfies the identities:

(1.1) $\forall x, y \in A$, $\gamma(e, y) = \gamma(x, e) = 0$

(1.2) The cocycle identity holds for every $x, y,$ and $z$ in $A$:

$$\gamma(x, y) + \gamma(xy, z) = \gamma(y, z) + \gamma(x, yz).$$

Any central extension determines a factor set $\gamma$, once a transversal $\ell: A \to G$ with $\ell(e) = 0$
and $\pi \circ \ell = \text{id}_A$ has been chosen, by letting $\gamma(x, y)$ be the element of $B$ such that

$$\ell(x) + \ell(y) = \ell(xy) + \gamma(x, y).$$

Note that $\ell$ need not be (and in general will not be) a group morphism.

Conversely any factor set $\delta$ satisfying (1.1) and (1.2) determines a central extension and
a transversal. Namely, we take the set of all pairs $(a, b)$ with $a \in A$ and $b \in B$ as the
underlying set for the extension, and define the multiplication by the rule

$$(a, b) \cdot (a', b') = (aa', b + b' + \delta(a, a'))$$

and the transversal by $\ell(a) = (a, 0)$.

We note that if $\delta$ is a bilinear map from $A \times A$ to $B$, then it satisfies (1.1) and (1.2),
although the converse is not necessarily true. If the factor set $\delta$ is bilinear, we will denote
the resulting group by $A \times_{\delta} B$ and say that this group is a twisted product of $B$ by $A$.

A function $\eta: A \times A \to B$ is called a coboundary iff there is a set map $h: A \to B$ with
$h(e) = 0$ and such that for all $x$ and $y$ in $A$,

$$\eta(x, y) = h(x) + h(y) - h(xy).$$

We denote the set of all factor sets by $Z^2(A, B)$, and the set of all coboundaries by
$B^2(A, B)$. They both carry a structure of abelian groups, given by pointwise addition, and $B^2(A, B)$ is a subgroup of $Z^2(A, B)$. We define the second cohomology group of $A$
with coefficients in $B$ to be the quotient group

$$H^2(A, B) \cong Z^2(A, B)/B^2(A, B).$$

It is not hard to verify that given a map $\phi: B \to C$, we obtain an induced map

$$\phi^*: H^2(A, B) \to H^2(A, C)$$
by taking the class of the factor set $\delta: A \times A \to B$ to $\phi \circ \delta$. Dually, a map $\psi: A \to C$ induces a map
$$\psi_*: H^2(C, B) \to H^2(A, B)$$
by taking the class of the factor set $\gamma: C \times C \to B$ to the factor set $\gamma \circ (\psi \times \psi)$.

Recall also that two central extensions of $B$ by $A$, $1 \rightarrow B \xrightarrow{i} G \xrightarrow{\pi} A \rightarrow 1$ and $1 \rightarrow B \xrightarrow{i'} G' \xrightarrow{\pi'} A \rightarrow 1$ are said to be equivalent iff there exists a homomorphism $\phi: G \to G'$ making the following diagram of exact sequences commute:

$$
\begin{array}{cccccc}
1 & \longrightarrow & B & \xrightarrow{i} & G & \xrightarrow{\pi} & A & \longrightarrow & 1 \\
\| & & \| & & \downarrow{\phi} & & \| \\
1 & \longrightarrow & B & \xrightarrow{i'} & G' & \xrightarrow{\pi'} & A & \longrightarrow & 1
\end{array}
$$

By a classical theorem of Schreier, the elements of $H^2(A, B)$ are in one to one correspondence with the equivalence classes of central extensions of $B$ by $A$. The extensions which are abelian groups form a subgroup, namely $\text{Ext}(A, B)$.

We will denote the set of all equivalence classes of central extensions of $B$ by $A$, at least one of whose representatives is a twisted product, by $H^2_{\text{Bil}}(A, B)$. Note that

$$
H^2_{\text{Bil}}(A, B) \cong \left( \text{Hom}(A \otimes A, B) + B^2(A, B) \right) / B^2(A, B)
$$

$$
\cong \text{Hom}(A \otimes A, B) / \left( \text{Hom}(A \otimes A, B) \cap B^2(A, B) \right)
$$

**Remark 1.3.** There are central extensions which are not equivalent to a twisted product; that is, extensions for which the factor set associated to a transversal does not lie in $H^2_{\text{Bil}}(A, B)$. For example, let $p$ be an odd prime, and consider the extension

$$
0 \longrightarrow \mathbb{Z}/p\mathbb{Z} \xrightarrow{i} \mathbb{Z}/p^2\mathbb{Z} \xrightarrow{\pi} \mathbb{Z}/p\mathbb{Z} \longrightarrow 0
$$
given by $i(\overline{1}) = \overline{p}$ (we are viewing $\mathbb{Z}/a\mathbb{Z}$ as the integers modulo $a$, so it has the class of 1 as a distinguished generator; also note that we are writing all groups additively in this example).

As a transversal $\ell: \mathbb{Z}/p\mathbb{Z} \to \mathbb{Z}/p^2\mathbb{Z}$, we can take the map $\ell(\overline{x}) = \overline{x}$ for $0 \leq x < p$. Then, the associated factor set $\gamma$ must satisfy

$$
\ell(\overline{x}) + \ell(\overline{y}) = \ell(\overline{x+y}) + \gamma(\overline{x}, \overline{y})
$$

for every $0 \leq x, y < p$. Therefore, we have

$$
\gamma(\overline{x}, \overline{y}) = \begin{cases} 
0 & \text{if } 0 \leq x + y < p; \\
p & \text{if } p \leq x + y
\end{cases}
$$
for all $x$ and $y$, with $0 \leq x, y < p$. This is clearly not bilinear, since, for example
\[ \gamma(p-1, p-1) = p, \text{ but } \gamma(0,0) - \gamma(0,1) - \gamma(1,0) + \gamma(1,1) = 0. \]

Is $\gamma$ cohomologous to a bilinear map? The answer is no. If it were, we would be able to write $\mathbb{Z}/p^2\mathbb{Z}$ as a twisted product $\mathbb{Z}/p\mathbb{Z} \times_\delta \mathbb{Z}/p\mathbb{Z}$; an easy calculation will show that for every $(x, y)$ in such a group,
\[
(x, y)^p = \left(px, py + \frac{p(p-1)}{2}\delta(x, x)\right)
= (0, 0)
\]
contradicting our assumption that such a twisted product is isomorphic to $\mathbb{Z}/p^2\mathbb{Z}$. So $\gamma$ is not cohomologous to a bilinear map.

Section 2. Central extensions and twisted products

We are now ready to describe our result. Let $A$ and $B$ be two abelian groups, and let $G$ be a central extension of $B$ by $A$, with transversal $\ell$ and factor set $\gamma$. We will show that there exists an abelian group $L$, an embedding $\varphi: B \to L$, a twisted product $G'$ of $L$ by $A$, and an embedding $\psi: G \to G'$ such that the following diagram of exact rows commutes:

\[
\begin{array}{cccccc}
1 & \to & B & \to & G & \to & A & \to & 1 \\
\downarrow\varphi & & \downarrow\psi & & \| \\
1 & \to & L & \to & G' & \to & A & \to & 1
\end{array}
\]

This will imply that the twisted product $G'$ realizes the factor set $\varphi^*(\gamma)$. That is, that $\varphi^*(\gamma)$ is cohomologous to the bilinear factor associated to the twisted product $G'$.

In fact, the group $L$ may be taken to depend only on $B$, and not on the group $A$ or the factor set $\gamma$. In other words, given an abelian group $B$ there exists an abelian group $L$ and an embedding $\varphi: B \to L$ such that for all abelian groups $A$, the induced map
\[
\varphi^*: H^2(A, B) \to H^2(A, L)
\]
has image in $H^2_{\text{Bil}}(A, L)$.

Recall that a bilinear map $\alpha: A \times A \to B$ is said to be alternating if $\alpha(x, x) = 0$ for all $x \in A$. It is not hard to verify that this condition implies that $\alpha(x, y) = -\alpha(y, x)$, and that the converse holds if $B$ has no 2-torsion.

Suppose that we have two abelian groups $A$ and $B$, and an alternating bilinear map $\alpha: A \times A \to B$. Let the abelian group $C$, the bilinear map $\beta: A \times A \to C$, and the map $i_B: B \to C$ be the universal triple with the following universal property:
(2.4) The map $i_B: B \to C$ makes the following diagram commute:

\[
\begin{array}{ccc}
C & \xrightarrow{\beta(x,y) - \beta(y,x)} & C' \\
A \times A & \xrightarrow{i_B} & B \\
\alpha(x,y) & & \\
\end{array}
\]

(2.5) For any abelian group $C'$, any bilinear map $\beta': A \times A \to C'$, and map $\varphi': B \to C'$ such that

\[
\beta'(x,y) - \beta'(y,x) \xrightarrow{A \times A} \varphi'
\]

commutes, there exists a unique $\psi: C \to C'$ such that $\varphi' = \psi \circ i_B$ and $\beta' = \psi \circ \beta$.

**Remark 2.6.** We claim that the universal map $i_B: B \to C$ must be injective. For this, first we examine the case of $A$ finitely generated, say

\[
A \cong \mathbb{Z}^r \oplus \mathbb{Z}/a_{r+1}\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/a_{r+m}\mathbb{Z}
\]

with $a_{r+1} | \cdots | a_{r+m}$ (we write $A$ additively for the remainder of this remark). Let $g_1, \ldots, g_{r+m}$ be the canonical set of generators for $A$ under this isomorphism, and take $x, y \in A$, given by

(2.7) \quad x = k_1g_1 + \cdots + k_{r+m}g_{r+m}

(2.8) \quad y = l_1g_1 + \cdots + l_{r+m}g_{r+m}

where $k_1, \ldots, k_{r+m}, l_1, \ldots, l_{r+m} \in \mathbb{Z}$ and $0 \leq k_{r+i}, l_{r+i} < a_{r+i}$ for $i = 1, \ldots, m$. We let $C' = B$ and define

\[
\beta'(x, y) = \prod_{1 \leq i < j \leq r+m} k_jl_i \left( \alpha(g_i, g_j) \right)
\]

It is now easy to see that for $x, y \in A$ as in (2.7) and (2.8), we have

\[
\beta'(x, y) - \beta'(y, x) = \alpha(x, y)
\]

so we can set up a commutative diagram as in (2.5) with $B$ playing the role of $C'$; by the universal property of the triple $(C, i_B, \beta)$, there exists a unique map $\psi: C \to B$ with $\text{id}_B = \psi \circ i_B$. So, in particular, $i_B$ must be injective.
For the general case, we consider finitely generated subgroups $A'$ of $A$, and the restriction of $\alpha$ to $A' \times A'$. The triples $(A', B, \alpha|_{A' \times A'})$ form a directed system, with the connecting maps being the obvious inclusions. The universal object $C$ will be the limit of the universal objects $C'$ at each level, and the universal map $i_B$ will be the direct limit of the maps at each level. Since direct limits respect monomorphisms, the universal map $i_B : B \to C$ is injective, establishing the claim.

Let $G$ be a central extension of $B$ by $A$, with exact sequence

\[(2.9)\quad 1 \to B \overset{i}{\to} G \overset{\pi}{\to} A \to 1,\]

let $L$ an abelian group (written additively), and $\beta : A \times A \to L$ a bilinear map. We abuse notation slightly and denote the induced bilinear map on $G \times G$ also by $\beta$. We denote the image of an arbitrary $g \in G$ under $\pi$ by $\overline{g}$.

We want to know what conditions a set map $f : G \to L$ must satisfy in order for the map $\varphi : G \to A \times \beta L$, given by

\[(2.10)\quad \varphi(g) = \left(\overline{g}, f(g)\right),\]

to be an injective group morphism.

**Lemma 2.11.** Let $G$, $L$ and $\beta$ be as in (2.9) and the subsequent two paragraphs. A set map $f : G \to L$ makes $\varphi$, defined as in (2.10), a group morphism if and only if

\[(2.12)\quad \forall x, y \in G \quad f(x + y) = f(x) + f(y) + \beta(x, y).\]

Furthermore, such an $f$ makes $\varphi$ into an injective group morphism if and only if it also satisfies

\[(2.13)\quad f|_B : B \to L \quad \text{is a one-to-one group morphism.}\]

**Proof:** Let $x, y \in G$. Then we must have

\[
(\overline{x}, f(x)) \cdot (\overline{y}, f(y)) = (\overline{xy}, f(x) + f(y) + \beta(x, y))
= (\overline{x + y}, f(x + y))
\]

so condition (2.12) is both necessary and sufficient.

Assuming we already have condition (2.12), we want to know when $\varphi$ is injective. Clearly, if $x$ and $y$ are not congruent modulo $B$, then $\overline{x} \neq \overline{y}$ in $A$, so $\varphi(x) \neq \varphi(y)$ is guaranteed. So it is both necessary and sufficient to require that $f$ be an injective set map on $B$. Since $B$ lies in both the kernel on the right and on the left of $\beta$, this condition is equivalent to asking that $f|_B$ be an injective group morphism, giving condition (2.13).
Remark 2.14. If \( f \) satisfies condition (2.12), then

\[
\forall x \in G, \forall n \in \mathbb{Z} \quad f(nx) = nf(x) + \binom{n}{2} \beta(x, x)
\]

where \( \binom{n}{2} = \frac{n(n-1)}{2} \) for all \( n \in \mathbb{Z} \). This is easily established by induction on \( n \).

Remark 2.15. An easy calculation shows that if \( f \) satisfies (2.12) then

\[
(2.16) \quad \forall x, y \in G \quad f([x,y]) = \beta(x, y) - \beta(y, x).
\]

We want to show that for \( G \) as in (2.9) and the subsequent two paragraphs, we can always find a group \( L \) and define a set map \( f \) satisfying conditions (2.12) and (2.13). We will do this by defining \( f \) on the subgroup \( B \), making sure it satisfies (2.16), and then showing it can be extended to all of \( G \). For this, we need an extension result.

Lemma 2.17. Let \( G \) be as above, and let \( U \) and \( V \) be two subgroups of \( G \), with \( [V,U] \subseteq U \). Let \( L \) be an abelian group, and

\[
\beta: A \times A \to L
\]

be a bilinear map. Suppose we are given functions

\[
\begin{align*}
 f_U &: U \to L \\
 f_V &: V \to L
\end{align*}
\]

with \( f_U|_{U \cap V} = f_V|_{V \cap U} \), and satisfying (2.12) (That is, \( f_U \) satisfies (2.12) with \( x \) and \( y \) ranging over \( U \), and \( f_V \) with \( x \) and \( y \) ranging over \( V \)). Suppose further that \( f_U \) satisfies equation (2.16) whenever \( x \in V \) and \( y \in U \). Let \( W = \langle U,V \rangle = UV \) be the subgroup generated by \( U \) and \( V \). Finally, let \( f_W: W \to L \) be given by

\[
f_W(u + v) = f_U(u) + f_V(v) + \beta(u,v).
\]

Then \( f_W \) is an extension of \( f_U \) and \( f_V \), and satisfies (2.12) (with \( x \) and \( y \) ranging over \( W \)).

Proof: Note that since \( [G,G] \subseteq B \), and \( B \) lies in both the kernel on the right and the kernel on the left of \( \beta \), it follows that \([U,V]\) also lies in both the kernel on the right and kernel on the left of \( \beta \).

We need to check that \( f_W \) is well defined, that it indeed extends both \( f_U \) and \( f_V \), and that it satisfies (2.12).

Suppose that \( u_1 + v_1 = u_2 + v_2 \). Then \( -u_2 + u_1 = v_2 - v_1 \in U \cap V \). So in particular we must have that \( f_U(-u_2 + u_1) = f_V(v_2 - v_1) \), i.e.

\[
f_U(u_1) - f_U(u_2) + \beta(u_2, u_2) - \beta(u_2, u_1) = f_V(v_2) - f_V(v_1) + \beta(v_1, v_1) - \beta(v_2, v_1)
\]
We also have \( u_1 = u_2 + v_2 - v_1 \) and \( v_2 = -u_2 + u_1 + v_1 \). So, according to the definitions, we have

\[
f_W(u_1 + v_1) - f_W(u_2 + v_2) = f_U(u_1) + f_V(v_1) + \beta(u_1, v_1) - \left( f_U(u_2) + f_V(v_2) + \beta(u_2, v_2) \right)\]

\[
= f_U(u_1) - f_U(u_2) - \left( f_V(v_2) - f_V(v_1) \right) + \beta(u_1, v_1) - \beta(u_2, v_2)\]

\[
= \beta(v_1, v_1) - \beta(v_2, v_1) - \beta(u_2, u_2) + \beta(u_2, u_1) + \beta(u_2 + v_2 - v_1, v_1) - \beta(u_2, -u_2 + u_1 + v_1)\]

\[
= \beta(v_1, v_1) - \beta(v_2, v_1) - \beta(u_2, u_2) + \beta(u_2, u_1) + \beta(u_2, v_1) + \beta(v_2, v_1) - \beta(u_2, v_1) + \beta(u_2, u_2) - \beta(u_2, u_1) - \beta(u_2, v_1)\]

\[
= 0
\]

so \( f_W \) is well defined.

That \( f_W \) extends both \( f_U \) and \( f_V \) now follows trivially. Finally, let \( u_1 + v_1 \) and \( u_2 + v_2 \) be two elements of \( W \). Then, since \([U, V] \subseteq Z(G)\),

\[
(u_1 + v_1) + (u_2 + v_2) = (u_1 + u_2 + [v_1, u_2]) + (v_1 + v_2)
\]

with \( u_1 + u_2 + [v_1, u_2] \in U \) and \( v_1 + v_2 \in V \). So

\[
f_W((u_1 + v_1) + (u_2 + v_2)) = f_W(u_1 + u_2 + [v_1, u_2] + v_1 + v_2)\]

\[
= f_U(u_1 + u_2 + [v_1, u_2]) + f_V(v_1 + v_2) + \beta(u_1 + u_2 + [v_1, u_2], v_1 + v_2)\]

\[
= f_U(u_1) + f_U(u_2) + \beta(v_1, u_2) - \beta(u_2, v_1) - \beta(v_1, u_2) + \beta(u_1, v_1) + f_V(v_1) + f_V(v_2) + \beta(v_1, v_2) + \beta(u_1, u_2) + \beta(u_2, v_1) + \beta(u_2, v_2)\]

\[
= f_U(u_1) + f_U(u_2) + \beta(v_1, u_2) - \beta(u_2, v_1) - \beta(v_1, u_2) + \beta(u_1, v_1) + f_V(v_1) + f_V(v_2) + \beta(v_1, v_2) + \beta(u_1, v_1) + \beta(u_1, u_2) + \beta(u_2, v_2)\]

On the other hand we have

\[
f_W(u_1 + v_1) + f_W(u_2 + v_2) + \beta(u_1 + v_1, u_2 + v_2) = f_U(u_1) + f_V(v_1) + f_U(u_2) + f_V(v_2) + \beta(u_1, v_1) + \beta(u_2, v_1) + \beta(u_1, u_2) + \beta(u_2, v_2) + \beta(u_1, v_2) + \beta(u_1, u_2) + \beta(u_2, v_2) + \beta(v_1, v_2)\]

so \( f_W \) satisfies (2.12) and we are done.

Now we set up some notation. Let \( G \) be a central extension of \( B \) by \( A \), and \( H \) a proper subgroup of \( G \) with \( B \subseteq H \). Let \( L \) be an abelian divisible group, \( \beta: A \times A \to L \) a bilinear map, and \( f: H \to L \) be a set map satisfying (2.12), (2.13), and (2.16). Note that condition (2.12) is assumed only for \( x \) and \( y \) in \( H \), whereas condition (2.16) is assumed for any commutator \([x, y]\) with \( x \) and \( y \) arbitrary elements of \( G \). Finally, let \( g \in G\setminus H \) and let \( K = \langle H, g \rangle \).

We want to extend \( f \) to all of \( K \), retaining properties (2.12) (this time for all \( x \) and \( y \) in \( K \)), (2.13), and (2.16).

**Theorem 2.18.** Let \( G, H, \beta, L, f \) and \( K \) be as in the previous two paragraph. Then \( f \) can be extended to \( K \).
Proof: We want to apply Lemma 2.17, so we let $H$ play the role of $U$ and the cyclic group generated by $g$ play the role of $V$. $f_U$ will simply be our given $f$, and we need to define $f_V$. Let $I = \{a \in \mathbb{Z}_+ \mid ag \in U\}$. We have two cases.

Case 1. $I = \emptyset$. Note in particular that in this case, $g$ cannot be a torsion element of $G$. We define $f_V(g) = 0$ and, extend to every power of $g$ using the formula in Remark 2.14. Then Lemma 2.17 gives us an extension of $f$ to all of $K$.

Case 2. $I \neq \emptyset$. Let $n_0 = \min\{n \in I\}$. Then $U \cap V = \langle n_0 g \rangle$. Using Remark 2.14, we know that whatever we define $f_V(g)$ to be, it must satisfy

$$f_V(n_0 g) = n_0 f_V(g) + \binom{n_0}{2} \beta(g, g).$$

However, $n_0 g \in U$, so $f_V(n_0 g) = f_U(n_0 g)$. Solving for $f_V(g)$, we see that $f_V(g)$ must satisfy

$$n_0 f_V(g) = f_U(n_0 g) - \binom{n_0}{2} \beta(g, g);$$

so we define

$$f_V(g) = \frac{1}{n_0} \left( f_U(n_0 g) - \binom{n_0}{2} \beta(g, g) \right)$$

where by $\frac{1}{n} (\cdots)$ we mean any $n$-th root, chosen now once and for all. Then we extend $f_V$ to all of $V$ using the formula in Remark 2.14. We need to verify that this is well defined if $g$ is a torsion element, and that it agrees with $f_U$ on $U \cap V$.

First, suppose that $g$ is a torsion element. By definition of $n_0$, it follows that the order of $g$ is a multiple of $n_0$, say $k_0 n_0$. We want to verify that for every $a \in \mathbb{Z}$, $f_V(ag) = f_V((k_0 n_0 + a)g)$.

By definition,

$$f_V(ag) = a f_V(g) + \binom{a}{2} \beta(g, g)$$

$$= a \left( \frac{1}{n_0} \left( f_U(n_0 g) - \binom{n_0}{2} \beta(g, g) \right) \right) + \binom{a}{2} \beta(g, g).$$

On the other hand, we have

$$f_V((k_0 n_0 + a) g) = (k_0 n_0 + a) f_V(g) + \binom{k_0 n_0 + a}{2} \beta(g, g)$$

$$= (k_0 n_0 + a) \left( \frac{1}{n_0} \left( f_U(n_0 g) - \binom{n_0}{2} \beta(g, g) \right) \right) + \binom{k_0 n_0 + a}{2} \beta(g, g)$$

A long but straightforward calculation now shows that the values indeed agree.
As for the values on the intersection of $U$ and $V$, since the intersection is generated by $n_0 g$, we need to check the values of $f_V$ at $kn_0 g$, for $k \in \mathbb{Z}$. We have

$$f_V((kn_0)g) = kn_0 f_V(g) + \left( \frac{kn_0}{2} \right) \beta(g, g)$$

$$= k \left( f_V(n_0 g) - \left( \frac{n_0}{2} \right) \beta(g, g) \right) + \left( \frac{kn_0}{2} \right) \beta(g, g)$$

$$= kf_V(n_0 g) + \left( \frac{kn_0}{2} \right) - k \left( \frac{n_0}{2} \right) \beta(g, g)$$

$$= kf_V(n_0 g) + \left( \frac{k}{2} \right) n_0^2 \beta(g, g)$$

$$= kf_V(n_0 g) + \left( \frac{k}{2} \right) \beta(n_0 g, n_0 g)$$

$$= f_U(k(n_0 g))$$

$$= f_U((n_0 k)g)$$

so $f_U|_{U \cap V} = f_V|_{V \cap U}$. Applying Lemma 2.17, we get an extension of $f$ to all of $K$. □

**Remark 2.19.** Notice that the divisibility of $L$ was only used in defining $f_V(g)$, in the case where $I$ is nonempty. So, for example, if we know that the quotient $G/H$ has exponent $k$, then we would only need $L$ to have all $k$-th roots.

**Theorem 2.20.** Let $1 \rightarrow B \xrightarrow{i} G \xrightarrow{\pi} A \rightarrow 1$ be a central extension of abelian groups. Then there exists an abelian group $L$, a bilinear map $\beta : A \times A \rightarrow L$ and a set map $f : G \rightarrow L$ satisfying (2.12) and (2.13).

**Proof:** For the moment assume $L$ given. We will consider the set of pairs $(H, f_H)$, where $H$ is a subgroup of $G$ containing $B$, and $f_H : H \rightarrow L$ is a set function satisfying (2.12), (2.13), and (2.16). We partially order the set by inclusion on the first coordinate, and “extension” on the second. i.e. $(H, f_H) \leq (H', f'_H)$ iff $H \subseteq H'$ and $f_{H'}|_{H} = f_{H}$.

We need to verify that there is an $L$ for which this set of pairs is non-empty. For that, it suffices to show that there exists an abelian group $L$, a bilinear map $\beta : A \times A \rightarrow L$, and a set map $f : B \rightarrow L$ satisfying (2.13) and (2.16) (for $x, y \in B$, (2.12) follows from (2.13)).

To establish the existence of $L$, we consider the alternating bilinear map $\alpha(x, y) = [x, y]$. Then Remark 2.6 shows that there is an abelian group $C$, an embedding $i_B : B \rightarrow C$, and a bilinear map $\beta : A \times A \rightarrow C$ satisfying $\beta(x, y) - \beta(y, x) = [x, y]$ for all $x$ and $y$ in $G$, with the triple $(C, i_B, \beta)$ being universal.

Let $L$ be a divisible abelian group containing $B$, and $j : B \rightarrow L$ an embedding. Since $L$ is divisible, it is an injective object of the category of abelian groups, so the inclusion $i_B : B \rightarrow C$ factors through $j$; that is, there exists a map $\chi : C \rightarrow L$ (not necessarily a
monomorphism), such that \( \chi \circ i_B = j \). We now let \( \beta' = \chi \circ \beta \), and we let \( f : B \to L \) be equal to \( j \). Clearly, \( f \) satisfies (2.13). Also,

\[
    f([x, y]) = j([x, y]) = j(\alpha(x, y)) = \chi(\alpha_B(\alpha(x, y))) = \chi(\beta(x, y) - \beta(y, x)) = \beta'(x, y) - \beta'(y, x),
\]

so \( f \) satisfies (2.13) and (2.16). Therefore, the set of pairs defined in this first paragraph of this proof is non-empty, for the given \( L \).

Finally, we apply Zorn’s Lemma to this set of pairs. Theorem 2.18 implies that a maximal element in our set must have first coordinate equal to \( G \), so we are done. \( \square \)

**Corollary 2.21.** Let \( 1 \to B \xrightarrow{i} G \xrightarrow{\pi} A \to 1 \) be a central extension of \( B \) by \( A \). Then there exists an abelian group \( L \) containing \( B \), a bilinear map \( \beta : A \times A \to L \), and an embedding \( \varphi : G \to A \times_{\beta} L \) such that the following diagram commutes and has exact rows:

\[
\begin{array}{cccccc}
1 & \to & B & \xrightarrow{i} & G & \xrightarrow{\pi} & A & \to & 1 \\
\downarrow{j} & & \downarrow{\varphi} & & \parallel & & \\
1 & \to & L & \xrightarrow{i'} & A \times_{\beta} L & \xrightarrow{\pi'} & A & \to & 1
\end{array}
\]

Furthermore, \( L \) depends only on \( B \).

**Proof:** Let \( L \), \( f \) and \( \beta \) be as in Theorem 2.20. We note that the group \( L \) we obtain this way depends only on \( B \), and not on the groups \( A \) or \( G \). The map \( \varphi \) is simply given by the projection onto \( A \) in the first coordinate, and the map \( f \) in the second coordinate. It only remains to show that the diagram commutes.

Because the first component of \( \varphi \) is just \( \pi \), commutativity of the square on the right is immediate. For the square on the left, let \( b \in B \). Then \( \varphi(i(b)) = (1, b) \) by construction of \( f \), which is also the same as \( j(i'(b)) \), since \( j \) is an embedding and \( i' \) is just the inclusion of \( L \) into \( A \times_{\beta} L \). So the diagram commutes and we are done. \( \square \)

**Example 2.22.** Let \( p \) be an odd prime, and let \( G \), written multiplicatively, be the group presented by

\[
G = \langle x, y, z \mid x^p = y^p = [x, y]^p = [[x, y], x] = [[x, y], y] = [x, z] = [y, z] = e; \ z^p = [x, y] \rangle.
\]

Let \( B \) be the subgroup of \( G \) generated by \([x, y]\). Then \( B = [G, G] \) and \( A \cong G/B \) is the abelianization of \( G \), that is

\[
A = \langle x, y, z \mid x^p = y^p = z^p = [x, y] = [x, z] = [y, z] = e \rangle
\]
so $A$ is the product of three cyclic groups of order $p$.

Let $\beta: G \times G \to B$ be given by commutator map; that is, $\beta(g, g') = [g, g']$. Since $G$ is nilpotent of class two, it is not hard to verify that $\beta$ induces a well defined map on $A$, and is bilinear.

Is $G$ a twisted product? The answer is no. For supposing it were, there would exist a bilinear map $\gamma: A \times A \to B$ such that $G \cong A \times \gamma B$. Since both $A$ and $B$ are of exponent $p$, and since $p$ is odd, it follows that $G$ is also of exponent $p$; but $z$ is a $p$-th power in $G$, and non-trivial. Therefore, $G$ is not a twisted product.

Let $L$ be a divisible abelian group containing $B$; to fix ideas, let $L = \mathbb{Q}/\mathbb{Z}$, and let $j: B \to L$ be given by $j([x, y]) = \frac{1}{p}$.

We first let $f = j$ be defined only on $B$, and we want to extend $f$ to all of $G$. First we extend $f$ to $x$. Since $x^p$ is the smallest positive power of $x$ that lies in $B$, we choose a $p$-th root of $f(x^p) - \binom{p}{2} j(\beta(x, x)) = 0$.

The easiest choice to make is $f(x) = 0$, and we define $f$ accordingly. Next we extend $f$ to $y$; again, the least positive power of $y$ lying in $\langle B, x \rangle$ is $y^p$, so we choose a $p$-th root of $f(y^p) - \binom{p}{2} j(\beta(y, y)) = 0$.

so again we make the easiest choice and define $f(y) = 0$.

Finally, we extend $f$ to $z$. Here we have that the least power of $z$ which lies in $\langle B, x, y \rangle$ is $z^p$, so we pick a $p$-th root of $f(z^p) - \binom{p}{2} j(\beta(z, z)) = f(z^p) = f([x, y]) = \frac{1}{p}$.

So we define $f(z) = \frac{1}{p}$, and extend to all of $G$ using the formula in Remark 2.14.

The map $\phi: G \to G^{ab} \times j_0 \beta L$ given by $\phi(g) = (\overline{g}, f(g))$ is the desired inclusion of $G$ into a twisted product.

Note that we can embed $G$ into a simpler twisted product by replacing $L$ with the image of $f$, namely a cyclic group of order $p^2$.

**Example 2.23.** For another example, consider $\mathbb{Z}/p^2\mathbb{Z}$ as an extension of $\mathbb{Z}/p\mathbb{Z}$ by $\mathbb{Z}/p\mathbb{Z}$. We take $\mathbb{Z}/p^2\mathbb{Z}$ as the integers modulo $p^2$, so $G$ has a distinguished generator, namely $\overline{1}$, the class of $1$. Thus $B$ is the subgroup generated by $\overline{p}$. Here the map $(x, y) \mapsto [x, y]$ is trivial, so the universal map $\beta$ associated to this bilinear map is symmetric. Again, let $L = \mathbb{Q}/\mathbb{Z}$, and define $f: B \to L$ by $f(\overline{p}) = \frac{1}{p}$.
To extend $f$ to all of $G$, we pick $\overline{1}$ as an element of $G$ not in $B$; we need a $p$-th root of

$$f(\overline{p}) - \left(\begin{array}{c} p \\ 2 \end{array}\right) \beta(\overline{1}, \overline{1}) = f(\overline{p}) = \frac{1}{p}$$

so we let $f(\overline{1}) = \frac{1}{p^2}$, and extend to all of $G$ using the formula in Remark 2.14.

The map we obtain, $\varphi: \mathbb{Z}/p^2\mathbb{Z} \to \mathbb{Z}/p\mathbb{Z} \times_\beta \mathbb{Q}/\mathbb{Z}$ is given by

$$\varphi(\overline{1}) = \left(\overline{1}, \frac{1}{p^2}\right).$$

Note that this embeds $G$ into the second coordinate, and that the first coordinated of the twisted product provides no new information; also note that in fact

$$\mathbb{Z}/p\mathbb{Z} \times_\beta \mathbb{Q}/\mathbb{Z} \cong \mathbb{Z}/p\mathbb{Z} \oplus \mathbb{Z}/\mathbb{Z}.$$
Proof: Let $L$, $j$ and $f: G \to L$ be as constructed in Theorem 2.20. It suffices to show that $j \circ \gamma$ is equal to the $\beta$ of the twisted product obtained in Theorem 2.20, as elements of $H^2(A,L)$. Again we abuse notation and use $\beta$ to denote both the bilinear map from $A$ to $L$, and the induced bilinear map from $G$ to $L$.

Let $h: A \to L$ be the set map given by $h(x) = f(\ell(x))$ for every $x \in A$. Then

$$h(e) = f(\ell(e)) = f(\ell) = 0,$$

and for any $x$ and $y$ in $A$, we have

$$h(x) + h(y) - h(xy) = f(\ell(x)) + f(\ell(y)) - f(\ell(xy))$$

$$= f(\ell(x)) + f(\ell(y)) - f(\ell(x) + \ell(y) - \gamma(x,y))$$

$$= f(\ell(x)) + f(\ell(y)) - f(\ell(x) + \ell(y)) - f(-\gamma(x,y)) + \beta(\ell(x) + \ell(y), \gamma(x,y))$$

$$= f(\ell(x)) + f(\ell(y)) - f(\ell(x) + \ell(y)) + j(\gamma(x,y))$$

(Since $\gamma(x,y)$ lies in $B$, hence in the kernel on the right of $\beta$)

$$= f(\ell(x)) + f(\ell(y)) - f(\ell(x)) - f(\ell(y)) - \beta(\ell(x), \ell(y)) + j(\gamma(x,y))$$

$$= j(\gamma(x,y)) - \beta(\ell(x), \ell(y))$$

$$= j(\gamma(x,y)) - \beta(x,y).$$

(Since $\ell(x)$ is just the image of $x$)

So $\beta$ and $j \circ \gamma$ are equal in $H^2(A,L)$, as claimed.

**Theorem 3.26.** Let $B$ be an abelian group. Then there exists an abelian group $L$, and an embedding $j: B \to L$ such that

$$j^*(H^2(A,B)) \subseteq H^2_{\text{Bil}}(A,L)$$

for every abelian group $A$.

As noted before, $L$ will be a divisible abelian group by construction, with the possibility of weakening this requirement for a particular given $A$.

Example 2.23 calls our attention to the fact that the embedding obtained may very well be “uninteresting”, that is, that it may be an embedding of our given group $G$ into a direct sum (the zero element of $H^2(A,L)$). So the obvious question to ask is, under our construction, when does a central extension get embedded into a direct sum?

Let $1 \to B \xrightarrow{\delta} L \xrightarrow{p} L/B \to 1$ be the exact sequence associated to the inclusion of $B$ into a divisible abelian group $L$. We will now use several results from cohomology of groups. We refer the reader to either [1] or [4] for the details.

We have for every abelian group $A$ a long exact sequence of cohomology groups

$$0 \to H^0(A,B) \xrightarrow{j^*} H^0(A,L) \xrightarrow{\delta_0} H^0(A,L/B) \xrightarrow{\delta_0} H^1(A,B) \xrightarrow{j^*} H^1(A,L) \xrightarrow{\delta_1} H^1(A,L/B) \xrightarrow{\delta_1} H^2(A,B) \xrightarrow{j^*} H^2(A,L) \xrightarrow{\delta_2} \cdots$$
Recall that in our case, with $A$ acting trivially on $B$, $H^0(A,B)$ is just $B$, and that $H^1(A,B)$ is the group of all group morphisms from $A$ to $B$. So we see that the above exact sequence yields the exact sequence

$$0 \to B \xrightarrow{j} L \xrightarrow{p} L/B \xrightarrow{\delta_0} \text{Hom}(A,B) \xrightarrow{j^*} \text{Hom}(A,L/B) \xrightarrow{\delta_1} H^2(A,B) \xrightarrow{j^*} H^2(A,L) \xrightarrow{p^*} \cdots$$

where $\text{Hom}(A,B)$ is the set of group maps from $A$ to $B$, etc. Since $p$ is surjective, $\delta_0$ is the zero map, and we have the exact sequence

$$0 \to \text{Hom}(A,B) \xrightarrow{j^*} \text{Hom}(A,L) \xrightarrow{p^*} \text{Hom}(A,L/B) \xrightarrow{\delta_1} \text{Ext}(A,B) \xrightarrow{j^*} \text{Ext}(A,L) \xrightarrow{p^*} \cdots$$

Also from Homological Algebra, we know that the right derived functor of $\text{Hom}$ is $\text{Ext}$, that is, that we have an exact sequence

$$0 \to \text{Hom}(A,B) \xrightarrow{j^*} \text{Hom}(A,L) \xrightarrow{p^*} \text{Hom}(A,L/B) \xrightarrow{\delta_1} \text{Ext}(A,B) \xrightarrow{j^*} \text{Ext}(A,L) \xrightarrow{p^*} \cdots$$

Since $L$ is divisible, it is an injective abelian group, so $\text{Ext}(A,L) = 0$; hence $\delta_1$ is surjective onto $\text{Ext}(A,B)$, which as we know is a subgroup of $H^2(A,B)$. We conclude that the kernel of $j^*: H^2(A,B) \to H^2(A,L)$, which is the map in Theorem 3.26, is none other than $\text{Ext}(A,B)$, that is, the abelian extensions of $B$ by $A$.

So we have proven:

**Theorem 3.27.** Let $B$ be an abelian group. Then there exists an abelian group $L$, and an embedding $j: B \to L$, such that for any abelian group $A$,

$$j^*: H^2(A,B) \to H^2(A,L)$$

has image contained in $H^2_{\text{Bil}}(A,L)$, and kernel equal to $\text{Ext}(A,B)$. In other words, for every central extension $G$ of $B$ by $A$ there exists a twisted product $A \times_B L$ and an embedding $G \hookrightarrow A \times_B L$, whose first component is the projection of $G$ onto $A$. This twisted product is equivalent to a direct sum of $L$ and $A$ if and only if $G$ is abelian.

**Corollary 3.28.** Let $A$ be any abelian group, and let $B$ be divisible. Then every element of $H^2(A,B)$ is cohomologous to a bilinear factor set. That is, every central extension of $B$ by $A$ is equivalent to a twisted product of $B$ by $A$.

**Proof:** We can let $L$ in Theorem 2.20 be equal to $B$, and $j: B \to L$ be the identity of $B$. Then Theorem 3.27 says that $\text{id}_B(= \text{id}_{H^2(A,B)})$ has image in $H^2_{\text{Bil}}(A,B)$ and trivial kernel; so we have that

$$H^2(A,B) = H^2_{\text{Bil}}(A,B)$$

as claimed.

**Theorem 3.29.** If $A$ is a free abelian group, and $B$ is any abelian group, then

$$H^2(A,B) = H^2_{\text{Bil}}(A,B).$$

**Proof:** Let $x_{i \in I}$ be a basis for $A$. It is not hard to verify that we can apply the same process as in Example 2.22 sequentially to the elements $x_i$. It is not hard to verify that we will always be in Case 1 of the proof of Theorem 2.18, and thus that we may bypass the need for a divisible group in that proof.
References

[1] Brown, Kenneth S. Cohomology of Groups, 2nd Edition. Graduate texts in mathematics 87, Springer Verlag, 1994. MR:96a:20072

[2] Hughes, N.J.S. The use of bilinear mappings in the classification of groups of class 2. Proc. Amer. Math. Soc. 2 (1951) pp. 742–747. MR:13,528e

[3] Rotman, J.J. Introduction to the Theory of Groups, 4th edition. Graduate texts in mathematics 119, Springer Verlag, 1994. MR:95m:20001

[4] Weibel, Charles. Introduction to Homological Algebra. Cambridge University Press 1994. MR:95f:18001

Arturo Magidin
Department of Mathematics
970 Evans Hall
University of California at Berkeley
Berkeley, CA 94720, USA
e-mail: magidin@math.berkeley.edu