The Terwilliger algebra of Odd graphs

Qian Kong  Benjian Lv  Kaishun Wang∗

Sch. Math. Sci. & Lab. Math. Com. Sys., Beijing Normal University, Beijing, 100875, China

Abstract

In [The Terwilliger algebra of the Johnson schemes, Discrete Mathematics 307 (2007) 1621–1635], Levstein and Maldonado computed the Terwilliger algebra of the Johnson scheme $J(n, m)$ when $3m \leq n$. The distance-$m$ graph of $J(2m + 1, m)$ is the Odd graph $O_{m+1}$. In this paper, we determine the Terwilliger algebra of $O_{m+1}$ and give its basis.

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1 Introduction

Suppose $\Gamma = (X, R)$ denotes a simple connected graph with diameter $D$. For each $i \in \{0, 1, \ldots, D\}$, let $\Gamma_i(x) = \{y \in X \mid \partial(x, y) = i\}$, where $\partial(x, y)$ is the distance between $x$ and $y$. Define $E_i^* = E_i^*(x)$ to be the diagonal matrix in $\text{Mat}_X(\mathbb{C})$ with $yy$-entry

$$(E_i^*)_{yy} = \begin{cases} 1, & \text{if } y \in \Gamma_i(x), \\ 0, & \text{otherwise}. \end{cases}$$

The Terwilliger algebra $\mathcal{T}(x)$ of $\Gamma$ with respect to a given vertex $x$ is the subalgebra of $\text{Mat}_X(\mathbb{C})$ generated by the adjacency matrix $A$ of $\Gamma$ and $E_0^*, E_1^*, \ldots, E_D^*$.

Terwilliger [10] initiated the study of the Terwilliger algebra of association schemes, which has been used to study (almost) bipartite $P$- and $Q$-polynomial association schemes [2, 3], 2-homogeneous bipartite distance-regular graphs [4], Hypercubes [5], Hamming graphs [7], Johnson graphs [8], incidence graphs of Johnson geometry [6] and so on.

Let $\Omega$ be a set of cardinality $2m + 1$ and let $\binom{\Omega}{i}$ denote the set of all $i$-subsets of $\Omega$. The Odd graph $O_{m+1}$ is the graph whose vertex set is the set $X = \binom{\Omega}{m}$, where two vertices are adjacent if they are disjoint. Levstein and Maldonado [8] determined the Terwilliger algebra of the Johnson graph $J(n, m)$ when $3m \leq n$. Observe $O_{m+1}$ is the distance-$m$ graph of the Johnson graph $J(2m + 1, m)$, and they have the same Terwilliger algebra. In this paper we shall determine the Terwilliger algebra of $O_{m+1}$ (Theorem 3.5), give one of its bases (Proposition 3.6) and compute its dimension (Corollary 3.7).

∗Corresponding author.

E-mail addresses: kongqian@mail.bnu.edu.cn (Qian Kong); benjian@mail.bnu.edu.cn (Benjian Lv); wangks@bnu.edu.cn (Kaishun Wang)
2 Intersection matrix

In this section we first introduce the intersection matrix, then discover the relationship between the adjacency matrix of the Odd graph $O_{m+1}$ and the intersection matrices.

Since $O_{m+1}$ is distance-transitive with diameter $m$ (cf. [4]), the isomorphism class of $T(x)$ is independent of the choice of $x$, denoted by $T := T(x)$.

Let $V$ be a set of cardinality $v$. Let $H_{i,j}^l(v)$ be a binary matrix with rows indexed by $\binom{V}{i}$ and columns indexed by $\binom{V}{j}$, whose $yz$-entry is defined by

$$(H_{i,j}^l(v))_{yz} = \begin{cases} 
1, & \text{if } |y \cap z| = l, \\
0, & \text{otherwise.}
\end{cases}$$

This matrix is a class of intersection matrices. Observe that $H_{i,j}^l(v) \neq 0$ if and only if $\max(0, i + j - v) \leq l \leq \min(i, j)$. We adopt the convention that $H_{i,j}^l(v) = 0$ for any integer $l$ such that $l < 0$ or $l > \min(i, j)$. From [9] Proposition 4, we have

$$H_{i,j}^l(v)H_{j,k}^s(v) = \sum_{g=0}^{\min(i,k)} \sum_{h=0}^{g} \binom{g}{h} \binom{k-g}{l-h} \binom{v-g-i-k}{j+h-l-s} H_{i,k}^g(v).$$

In particular,

$$H_{i,j}^l(v)H_{j,k}^s(v) = \sum_{s=\max(0,i+j+k-l-v)}^{\min(i-l,k)} \binom{i-s}{l} \binom{v+s-i-k}{j+l} H_{i,k}^s(v).$$

**Lemma 2.1** Let $\Gamma$ be the Odd graph $O_{m+1}$ with the adjacency matrix $A$, and let $A_{i,j}$ be the submatrix of $A$ with rows indexed by $\Gamma_i(x)$ and columns indexed by $\Gamma_j(x)$. Then

$$A_{i,j} = 0 \quad (0 \leq i \leq j \leq m, \ i \neq j - 1 \text{ or } i = j \neq m),$$

$$A_{2i,2i+1} = H_{m-i,i}^0(m) \otimes H_{i,m-i}^0(m+1) \quad (0 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1),$$

$$A_{2i+1,2i+2} = H_{i,m-i-1}^0(m) \otimes H_{m-i,i+1}^0(m+1) \quad (0 \leq i \leq \lfloor \frac{m}{2} \rfloor - 1),$$

$$A_{m,m} = H_{\binom{m}{m-1},\binom{m}{m-1}}^0(m) \otimes H_{\binom{m}{i},\binom{m}{i}}^0(m+1),$$

where “$\otimes$” denotes the Kronecker product of matrices.

**Proof.** Since $O_{m+1}$ is almost bipartite, [3] is directed.

Pick $y \in \Gamma_{2i}(x)$, $z \in \Gamma_{2i+1}(x)$. Note that $\partial(x,y) = 2i$ if and only if $|x \cap y| = m - i$; $\partial(x,z) = 2i + 1$ if and only if $|x \cap z| = i$. Then $|x \cap y| = m - i$ and $|x \cap z| = i$. Suppose $y = \alpha_{m-i} \beta_i := \alpha_{m-i} \cup \beta_i$, $z = \alpha'_i \beta'_{m-i}$, where $\alpha_{m-i} \in \binom{x}{m-i}$ and $\beta_i \in \binom{\Omega(x)}{i}$, while $\alpha'_i \in \binom{\Omega(x)}{i}$ and $\beta'_{m-i} \in \binom{\Omega(x)}{m-i}$. Then

$$(A_{2i,2i+1})_{yz} = (H_{m-i,i}^0(m) \otimes H_{i,m-i}^0(m+1))_{yz} = \begin{cases} 
1, & \text{if } \alpha_{m-i} \cap \alpha'_i = \emptyset \text{ and } \beta_i \cap \beta'_{m-i} = \emptyset, \\
0, & \text{otherwise,}
\end{cases}$$

which leads to [4].

Similarly, [5] [6] hold. \qed
The Terwilliger algebra

In this section we fix \( x \in \binom{\Omega}{m} \), then consider the Terwilliger algebra \( T = T(x) \) of \( O_{m+1} \).

For \( 0 \leq i, j \leq m \), any matrix \( M \) indexed by elements in \( \Gamma_i(x) \times \Gamma_j(x) \) can be embedded into \( \text{Mat}_X(\mathbb{C}) \) by

\[
L(M)_{p(x) \times q(x)} = \begin{cases} M, & \text{if } p = i \text{ and } q = j, \\ 0, & \text{otherwise.} \end{cases}
\]

Write \( G_{i,j}(v) = \{ g | \max(0, i + j - v) \leq g \leq \min(i, j) \} \). Let

\[
\mathcal{M} = \bigoplus_{p,q=0}^m L(\mathcal{M}_{p,q}),
\]

where \( L(\mathcal{M}_{p,q}) = \{ L(M) | M \in \mathcal{M}_{p,q} \} \), and

\[
\mathcal{M}_{2i,2j} = \text{Span}\{ H_{m-i,m-j}^l(m) \otimes H_{i,i}^s(m+1) | l \in G_{m-i,m-j}(m), s \in G_{i,j}(m+1) \},
\]

\[
\mathcal{M}_{2i,2j+1} = \text{Span}\{ H_{m-i,m-j}^l(m) \otimes H_{i,m-j}^s(m+1) | l \in G_{m-i,m-j}(m), s \in G_{i,j}(m+1) \},
\]

\[
\mathcal{M}_{2i+1,2j} = \text{Span}\{ H_{i,m-j}^l(m) \otimes H_{m-i,m-j}^s(m+1) | l \in G_{i,j}(m), s \in G_{m-i,m-j}(m+1) \},
\]

\[
\mathcal{M}_{2i+1,2j+1} = \text{Span}\{ H_{i,j}^l(m) \otimes H_{m-i,m-j}^s(m+1) | l \in G_{i,j}(m), s \in G_{m-i,m-j}(m+1) \}.
\]

Note that \( \mathcal{M} \) is a vector space. By Lemma 3.1 we have \( \mathcal{M} \) is an algebra. Next we shall prove \( T = \mathcal{M} \).

Lemma 3.1 The Terwilliger algebra \( T \) is a subalgebra of \( \mathcal{M} \).

Proof. By Lemma 2.1 we have \( A \in \mathcal{M} \). Since

\[
E_{2i}^* = L(H_{m-i,m-i}^l(m) \otimes H_{i,i}^s(m+1)) \in \mathcal{M}, \quad 0 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor,
\]

\[
E_{2i+1}^* = L(H_{i,i}^l(m) \otimes H_{m-i,m-i}^s(m+1)) \in \mathcal{M}, \quad 0 \leq i \leq \left\lfloor \frac{m}{2} \right\rfloor - 1,
\]

we have \( T \subseteq \mathcal{M} \). \( \Box \)

For \( 0 \leq i, j \leq m \), let \( \mathcal{T}_{i,j} = \{ M_{i,j} | M \in T \} \), where \( M_{i,j} \) is the submatrix of \( M \) with rows indexed by \( \Gamma_i(x) \) and columns indexed by \( \Gamma_j(x) \). Since \( T \) is an algebra, each \( \mathcal{T}_{i,j} \) is a linear space. From \( \mathcal{T}E^*_j \mathcal{T} \subseteq \mathcal{T} \) we obtain \( (\mathcal{T}E^*_j \mathcal{T})_{i,k} \subseteq \mathcal{T}_{i,k} \), which implies that

\[
\mathcal{T}_{i,j} \mathcal{T}_{j,k} \subseteq \mathcal{T}_{i,k}.
\]

(12)

From \( A, E_i^* \in \mathcal{T} \), we have \( AE_{i_2}^*A\cdotsAE_{i_n}^*A \in \mathcal{T} \), which follows that

\[
A_{i_1,i_2}A_{i_2,i_3}\cdotsA_{i_{n-2},i_{n-1}}A_{i_{n-1},i_n} \in \mathcal{T}_{i_1,i_n}.
\]

(13)

Lemma 3.2 For \( i \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor - 1 \) and \( 0 \leq l \leq i \), we have

\[
H_{i,j}^l(m) \otimes H_{m-i,m-j}^s(m+1) \in \mathcal{T}_{2i+1,2j+1}.
\]
Proof. We use induction on $l$.

By (13), for $i < j$ we have $A_{2i+1,2i+2}A_{2i+2,2i+3} \cdots A_{2j,2j+1} \in \mathcal{T}_{2i+1,2j+1}$, which yields that

$$H^l_{i,j}(m) \otimes H^{m-j}_{m-i,m-j}(m+1) \in \mathcal{T}_{2i+1,2j+1}. \hspace{1cm} (14)$$

When $i = j$ we pick $I_{m}^{(m)} \otimes I_{m+i}^{(m+i)} \in \mathcal{T}_{2i+1,2j+1}$, which also satisfies (14).

Assume that $H^l_{i,j}(m) \otimes H^{m-j}_{m-i,m-j}(m+1) \in \mathcal{T}_{2i+1,2j+1}$ for $g \geq l$. By (12) and (13), for $2j+1 < m$ we obtain

$$(H^l_{i,j}(m) \otimes H^{m-j}_{m-i,m-j}(m+1))A_{2j+1,2j+2}A_{2j+2,2j+1} \in \mathcal{T}_{2i+1,2j+1}\mathcal{T}_{2j+1,2j+1} \subseteq \mathcal{T}_{2i+1,2j+1},$$

and for $2j+1 = m$ we have

$$(H^l_{i,j}(m) \otimes H^{m-j}_{m-i,m-j}(m+1))A^2_{2j+1,2j+1} \in \mathcal{T}_{2i+1,2j+1}.$$  

Then we get

$$(a_1H^{l-1}_{i,j}(m) + a_2H^l_{i,j}(m) + a_3H^{l+1}_{i,j}(m)) \otimes H^{m-j}_{m-i,m-j}(m+1) \in \mathcal{T}_{2i+1,2j+1},$$

where $a_1, a_2$ and $a_3$ are some positive integers. It follows that $H^{l-1}_{i,j}(m) \otimes H^{m-j}_{m-i,m-j}(m+1) \in \mathcal{T}_{2i+1,2j+1}$. Hence the conclusion is obtained by induction. \hfill \Box

Lemma 3.3 For $i+1 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor$ and $0 \leq l \leq i$, we have

$$H^l_{i,m-j}(m) \otimes H^{l-i-1}_{m-j,i}(m+1) \in \mathcal{T}_{2i+1,2j}.$$  

Proof. By (13) we have $A_{2i+1,2i+2}A_{2i+2,2i+3} \cdots A_{2j-1,2j} \in \mathcal{T}_{2i+1,2j}$ for $i+1 \leq j$, i.e.,

$$H^0_{i,m-j}(m) \otimes H^{l-i-1}_{m-j,i}(m+1) \in \mathcal{T}_{2i+1,2j}.$$  

Assume that $H^g_{i,m-j}(m) \otimes H^{j-i-1}_{m-j,i}(m+1) \in \mathcal{T}_{2i+1,2j}$ for $g \leq l$. Then by (12) and (13), we obtain

$$(H^l_{i,m-j}(m) \otimes H^{j-i-1}_{m-j,i}(m+1))A_{2j-1,2j}A_{2j,2j+1} \in \mathcal{T}_{2i+1,2j},$$

which gives

$$(b_1H^{l-1}_{i,m-j}(m) + b_2H^l_{i,m-j}(m) + b_3H^{l+1}_{i,m-j}(m)) \otimes H^{j-i-1}_{m-j,i}(m+1) \in \mathcal{T}_{2i+1,2j},$$

where $b_1, b_2$ and $b_3$ are some positive integers. Thus $H^{l+1}_{i,m-j}(m) \otimes H^{j-i-1}_{m-j,i}(m+1) \in \mathcal{T}_{2i+1,2j}$ and the conclusion is valid by induction. \hfill \Box

Lemma 3.4 The algebra $\mathcal{M}$ is a subalgebra of $\mathcal{T}$.

Proof. In order to prove this result, we only need to show that $\mathcal{M}_{p,q} \subseteq \mathcal{T}_{p,q}$ for $0 \leq p, q \leq m$. Write $\mathcal{M}_{p,q} = \{M^i \mid M \in \mathcal{M}_{p,q}\}$ and $\mathcal{T}_{p,q} = \{M^i \mid M \in \mathcal{T}_{p,q}\}$. Since $\mathcal{M}_{q,p} = \mathcal{M}_{p,q}$ and $\mathcal{T}_{q,p} = \mathcal{T}_{p,q}$, it suffices to prove $\mathcal{M}_{p,q} \subseteq \mathcal{T}_{p,q}$ for $p \leq q$. We use induction on $p$.

Step 1. Show $\mathcal{M}_{0,q} \subseteq \mathcal{T}_{0,q}$ ($0 \leq q \leq m$).

According to (8), (9), we get

$$\mathcal{M}_{0,2j} = \text{Span}(H^{m-j}_{m,m-j}(m) \otimes H^0_{0,j}(m+1)) \quad (0 \leq j \leq \left\lfloor \frac{m}{2} \right\rfloor),$$
and 
\[ M_{0,2j+1} = \text{Span}\{H_{m,m-j}^j(m) \otimes H_{0,m-j}^0(m+1)\} \quad (0 \leq j \leq \left\lceil \frac{m}{2} \right\rceil - 1). \]

By Lemma 2.1 and 2, we have
\[ A_{0,1}A_{1,2} \cdots A_{2j-1,2j} = c_1H_{m,m-j}^m(m) \otimes H_{0,m-j}^0(m+1) \]
and
\[ A_{0,1}A_{1,2} \cdots A_{2j,2j+1} = c_2H_{m,j}^m(m) \otimes H_{0,m-j}^0(m+1), \]
where \( c_1, c_2 \) are some positive integers. Then by (13) we have \( M_{0,2j} \subseteq T_{0,2j} \) and \( M_{0,2j+1} \subseteq T_{0,2j+1} \).

**Step 2.** Assume that \( M_{p,q} \subseteq T_{p,q} \) for \( p \leq 2i \). We will show that \( M_{2i+1,q} \subseteq T_{2i+1,q} \) and \( M_{2i+2,q} \subseteq T_{2i+2,q} \).

**Step 2.1.** Show \( M_{2i+1,q} \subseteq T_{2i+1,q} \) \((2i+1 \leq q \leq m)\).

**Case 1.** \( q = 2j + 1 \) \((i \leq j \leq \left\lceil \frac{m}{2} \right\rceil - 1)\).

By inductive hypothesis we have
\[ H_{m-i,j}^l(m) \otimes H_{m,j}^s(m+1) \in M_{2i+1,2j+1} \subseteq T_{2i+1,2j+1}, \quad l \in G_{m-i,j}(m), \quad s \in G_{i,m-j}(m+1). \]

Since \( A_{2i+1,2i} = A_{2i,2i+1} = H_{m-i,j}^l(m) \otimes H_{m-j}^s(m+1) \in T_{2i+1,2i+2} \), by (12) we have
\[ (H_{m-i,j}^l(m) \otimes H_{m-j}^s(m+1))(H_{m-i,j}^l(m) \otimes H_{m-j}^s(m+1)) \in T_{2i+1,2i}T_{2i,2j+1} \subseteq T_{2i+1,2j+1}. \]
From \( 2 \) we obtain
\[ H_{i,j}^l(m) \otimes H_{m-i,j}^s(m+1) \in T_{2i+1,2j+1}, \quad l \in G_{i,j}(m), \quad s \in G_{m-i,j}(m+1), \]
which implies \( M_{2i+1,2j+1} \subseteq T_{2i+1,2j+1} \).

**Case 2.** \( q = 2i \) \((i+1 \leq j \leq \left\lceil \frac{m}{2} \right\rceil)\).

By inductive hypothesis, we have
\[ H_{m-i,j}^l(m) \otimes H_{m-j}^s(m+1) \in M_{2i,2j} \subseteq T_{2i,2j}, \quad l \in G_{m-i,j}(m), \quad s \in G_{i,j}(m+1). \]
Thus by (12) we obtain
\[ A_{2i+1,2i}(H_{m-i,j}^l(m) \otimes H_{m-j}^s(m+1)) \in T_{2i+1,2i}T_{2i,2j} \subseteq T_{2i+1,2j}. \]
From \( 2 \), we have
\[ H_{m-i,j}^l(m) \otimes ((s + 1)H_{m-j}^s(m+1) + (i - s + 1)H_{m-j}^{s-1}(m+1)) \in T_{2i+1,2j}. \]
Since \( l \in G_{m-i,j}(m) \) and \( s \in G_{i,j}(m+1) \), by (15) and Lemma 3.2 we get
\[ H_{i,j}^l(m) \otimes H_{m-i,j}^s(m+1) \in T_{2i+1,2j+1}, \quad l' \in G_{i,j}(m), \quad s' \in G_{m-i,j}(m+1), \]
which yields \( M_{2i+1,2j} \subseteq T_{2i+1,2j} \).

**Step 2.2.** Show \( M_{2i+2,q} \subseteq T_{2i+2,q} \) \((2i+2 \leq q \leq m)\).

The proof of this step is similar to that of Step 2.1 and we omit it here.

Hence the desired result follows by induction. \( \Box \)
**Theorem 3.5** Let $\mathcal{T}$ be the Terwilliger algebra of the Odd graph $O_{m+1}$ and $\mathcal{M}$ be the algebra defined in [7]. Then $\mathcal{T} = \mathcal{M}$.

**Proof.** Combining Lemma 3.1 and Lemma 3.4, the desired result follows. \qed

Since the generating matrices of each vector space in [8]-[11] are linearly independent, we have the following result.

**Proposition 3.6** The Terwilliger algebra $\mathcal{T}$ of the Odd graph $O_{m+1}$ has a basis:

$$\{ L(H^l_{m-i,m-j}(m) \otimes H^s_{i,j}(m+1)), \ l \in G_{m-i,m-j}(m), \ s \in G_{i,j}(m+1) \}^m_{i,j=0}.$$  

**Corollary 3.7** The dimension of $\mathcal{T}$ is $\binom{m+4}{4}$.

**Proof.** By Proposition 3.6 we get

$$\dim \mathcal{T} = \sum_{i,j=0}^{m} |G_{m-i,m-j}(m)| |G_{i,j}(m+1)|$$

$$= \sum_{i,j=0}^{m} (\min(m-i,m-j) - \max(0,m-i-j) + 1)(\min(i,j) - \max(0,i+j-m-1) + 1).$$

By zigzag calculation, the desired result follows. \qed

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