Thermodynamics and Excitations of the Supersymmetric $t-J$ Model

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Abstract

The free energy of the supersymmetric $t-J$ model is expressed in terms of finite temperature excitations above thermodynamic equilibrium. This reveals that the free energy has the form of noninteracting fermions with temperature dependant excitation spectra. We also discuss the ground state and zero temperature excitations.

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1 Introduction

Strong electron correlations are believed to be important in understanding the high $T_c$ superconductors [1]. This idea is supported by the fact that the high $T_c$ compounds display antiferromagnetism in the absence of doping. The $t-J$ model describes strongly correlated electrons with antiferromagnetic exchange interactions, and has been proposed as a candidate to model high $T_c$ superconductors by Zhang and Rice [2]. The one dimensional version of the model becomes integrable [3] and supersymmetric [4] for special values of the parameters $t$ and $J$. Integrability allows the model to be solved by Bethe’s ansatz and supersymmetry leads to the construction of

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three distinct Bethe ansatz solutions. Lai [5] and Sutherland [6] each found a solution in the context of models of hard core bosons and fermions which are now understood to be equivalent to the supersymmetric $t - J$ model. Schlottman [3] applied the Bethe ansatz to the $t - J$ Hamiltonian, and found Lai’s solution describes the model when electrons are treated as dynamical objects in a background of empty sites. Sarkar [7] discovered that Sutherland’s solution applies to the $t - J$ model if one treats holes and spin down electrons as dynamical objects in a background of spin up electrons. There is a third and quite recent solution due to Essler and Korepin [8], which is similar to Sutherland’s, but interchanges the roles of the spin down electrons and the holes.

The ground state of the $t - J$ model was constructed in [3, 5, 7]. The free energy was calculated in [3]. Bares, Blatter, and Ogata [9] use both Lai’s and Sutherland’s ansatzes to give a detailed account of the ground state and excitation spectrum, including extensive numerical analysis.

The outline of the paper is as follows: in sec. 2 we define the model and take the thermodynamic limit of the the Bethe ansatz equations. In sec. 3, the bulk free energy is expressed in terms of finite temperature excitations. In these terms, the free energy has the form of a noninteracting system with temperature dependant excitation spectra. This structure has been observed in such models as the $\delta$-function bose gas [10] and the XXZ spin chain [11], here these ideas are extended to the $t - J$ model. In sec. 4 we discuss the ground state and the order one excitation spectrum. The results agree with those of [3, 9], we present them here for completeness, and to see how they arise from a different form of Bethe ansatz. In the appendix we note that the different ansatzes are related by an interchange of particle and hole rapidities.

2 Formulation

The $t - J$ model describes spin-$\frac{1}{2}$ fermions on a lattice. The on-site coulomb repulsion is taken to be infinite, so that there is no double occupancy of the sites. The constraint of no double occupancy implies that the number of empty sites (holes) is conserved, which allows one to think of the empty sites as dynamical objects. The parameter $t$ describes how easily the electrons hop from site to site and $J$ describes the strength of an antiferromagnetic exchange interaction. The Hamiltonian is

$$H = \mathcal{P} \{ \sum_{j=1}^{L} \sum_{\sigma=\uparrow, \downarrow} t (c_{j,\sigma}^\dagger c_{j+1,\sigma} + c_{j+1,\sigma}^\dagger c_{j,\sigma}) + J (\vec{\mathbf{S}}_j \cdot \vec{\mathbf{S}}_{j+1} - \frac{1}{4} n_j n_{j+1}) \} \mathcal{P} + 2 \hat{N} - L. \quad (2.1)$$

$L$ is the number of lattice sites. $c_{j,\sigma}$, and $c_{j',\sigma'}^\dagger$ are electron creation and annihilation operators which obey the anti-commutation relations

$$\{c_{j,\sigma}, c_{j',\sigma'}^\dagger\} = 0, \quad \{c_{j,\sigma}, c_{j',\sigma'}^\dagger\} = \delta_{j,j'} \delta_{\sigma,\sigma'}, \quad \sigma, \sigma' = \uparrow, \downarrow. \quad (2.2)$$
\( P \) is the operator which projects out doubly occupied sites, \( \vec{S}_j = c_j^{\dagger} \vec{\sigma}_{\sigma,\sigma'} c_{j,\sigma'} \), where \( \vec{\sigma}_{\sigma,\sigma'} \) is the vector of Pauli spin matrices, and \( n_j \) is the number of electrons at site \( j \). \( \hat{N} \) is the operator for the total number of electrons. We have added the term \( 2\hat{N} - L \) onto the definition of the Hamiltonian in \([3, 4]\), which just shifts the energy and chemical potential. We shall consider this model at the special point \( J = 2t = 2 \), where the model is integrable and supersymmetric \([4]\). Also, at this special point the Hamiltonian \([2.1]\) may be rewritten as a graded permutation operator \([7]\),

\[
H = - \sum_{j=1}^{L} \Pi_{j,j+1},
\]

where \( \Pi_{j,j+1} \) interchanges the states on neighboring sites, with a minus sign if electrons sit on both sites.

The eigenstates and spectrum of the Hamiltonian \([2.1]\) at the supersymmetric point may be found by nested Bethe ansatz, a technique introduced by Yang in \([12]\). The spectrum of the Hamiltonian \([2.1]\) and other conserved quantities are given in terms of a set of coupled algebraic equations known as Bethe ansatz equations (BAE). As mentioned in the introduction, there are three distinct sets of BAE for the model. We shall analyze the solution due to Essler and Korepin \([8]\), where the coordinates of holes and spin down electrons are taken as dynamical objects moving in a background of spin up electrons. This solution expresses the eigenvalues of the Hamiltonian and total momentum as:

\[
E = L - \sum_{l=1}^{N_\downarrow+N_h} \frac{4}{\lambda_l^2 + 1},
\]

\[
P = \sum_{l=1}^{N_\downarrow+N_h} i \log \left( \frac{\lambda_l + i}{\lambda_l - i} \right),
\]

where the \( N_h + N_\downarrow \) spectral parameters \( \lambda_l \) describe the motion of holes and spin down electrons. They must satisfy the Bethe ansatz equations

\[
\left( \frac{\lambda_l + i}{\lambda_l - i} \right)^L = \prod_{\beta=1}^{N_\downarrow+N_h} \frac{\lambda_l - \Lambda_\beta + i}{\lambda_l - \Lambda_\beta - i} \quad l = 1, \ldots, N_\downarrow+N_h,
\]

\[
1 = \prod_{l=1}^{N_\downarrow+N_h} \frac{\lambda_l - \Lambda_\beta + i}{\lambda_l - \Lambda_\beta - i} \quad \beta = 1, \ldots, N_\downarrow.
\]

\( N_\downarrow, N_\uparrow, \) and \( N_h = L - N_\downarrow - N_\uparrow \) are the number of spin up electrons, spin down electrons, and holes respectively. The \( N_\downarrow \) parameters \( \Lambda_\beta \) describe the motion of the spin down electrons relative to the holes.
The equations (2.6) and (2.7) have real and complex solutions. The complex solutions are of a special form known as strings, which may be found by fixing \(N_h\) and \(N_A\) and letting the lattice size \(L\) go to infinity. One finds complex solutions consisting of \(n\ \lambda\)'s and \(n - 1\ \Lambda\)'s in the following combinations [8]:

\[
\begin{align*}
\lambda_j &= \lambda + i(n + 1 - 2j) \quad j = 1, \ldots, n \\
\Lambda_\delta &= \lambda + i(n - 2\delta) \quad \delta = 1, \ldots, n - 1,
\end{align*}
\]

for arbitrary \(n = 1, \ldots, \infty\). The real part common to all the parameters, \(\lambda\), is known as the center of the string. Physically, string solutions are bound states in the sense that an eigenfunction with the string combinations decays exponentially with respect to the coordinates of the down spins and holes. We shall see later in this section that the string solutions (2.8) yield a complete spectrum of the hamiltonian. In a finite box, the string solutions (2.8) are not exact. We assume, along with many authors (ie. [13, 14, 15]), that the corrections vanish in the thermodynamic limit.

We wish to write BAE for the centers of the strings. Let \(\lambda^n_\alpha\) the center of the \(\alpha\)th string of length \(n\), and \(\Lambda_\beta\) be real rapidities for spin down electrons not in any bound state. Denote the number of strings of length \(n\) by \(N_n\) and the number of free spin downs by \(N_A\). Multiplying together the equations for each part of the string yields

\[
\left(\frac{\lambda^n_\alpha + in}{\lambda^n_\alpha - in}\right)_L = \prod_{m=1}^{\infty} N_n \prod_{\gamma=1}^{N_n} F_{nm}(\lambda^n_\alpha - \lambda^m_\gamma) \prod_{\beta=1}^{N_A} \frac{\lambda^n_\alpha - \Lambda_\beta + in}{\lambda^n_\alpha - \Lambda_\beta - in}
\]

(2.9)

where \(n = 1, \ldots, \infty\) and

\[
1 = \prod_{n=1}^{\infty} \prod_{\alpha=1}^{N_n} \frac{\Lambda_\beta - \lambda^n_\alpha + in}{\Lambda_\beta - \lambda^n_\alpha - in},
\]

(2.10)

where

\[
F_{nm}(x) = e\left(\frac{x}{n - m}\right) e^{2\left(\frac{x}{n - m} + 2\right)} \times \cdots \times e^{2\left(\frac{x}{n + m - 2}\right)},
\]

(2.11)

and we have defined \(e(x) = \frac{x+i}{x-i}\).

The energy and momentum in terms of the centers of the strings are

\[
E = L - \sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} \frac{4n}{(\lambda^n_\alpha)^2 + n^2},
\]

(2.12)

and

\[
P = \sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} p\left(\frac{\lambda^n_\alpha}{n}\right),
\]

(2.13)
where \( p(\lambda) = 2 \tan^{-1}(\lambda) - \pi \).

In order to count the solutions and take the thermodynamic limit of the BAE, we take the logarithm of (2.9) and (2.10):

\[
L \theta(\frac{\lambda^n}{n}) = 2 \pi I_{\alpha} + \sum_{m=1}^{\infty} \sum_{\gamma=1}^{N_m} \Theta_{nm}(\lambda_{\alpha}^n - \lambda_{\gamma}^m) + \sum_{\beta=1}^{N_\Lambda} \theta(\frac{\lambda_{\alpha}^n - \Lambda_{\beta}}{n}),
\]

(2.14)

and

\[
\sum_{n=1}^{\infty} \sum_{\alpha} \theta(\frac{\Lambda_{\beta} - \lambda_{\alpha}^n}{n}) = 2 \pi J_{\beta},
\]

(2.15)

where \( \theta(x) = 2 \tan^{-1}(x) \), and

\[
\Theta_{nm}(x) = \begin{cases} 
\theta(\frac{x}{m-n}) + 2\theta(\frac{x}{|n-m|+2}) + \cdots + 2\theta(\frac{x}{n+m-2}) & \text{if } n \neq m \\
2\theta(\frac{x}{n}) + 2\theta(\frac{x}{n+m-2}) + \cdots + 2\theta(\frac{x}{n+m-2}) & \text{if } n = m.
\end{cases}
\]

(2.16)

(\( I_{\alpha} \) and \( J_{\beta} \) are integers (half integers) arising from the choice of the branch of the logarithm. \( I_{\alpha} \) is an integer (half integer) if \((L - \sum_{m \neq n} N_m - N_\Lambda)\) is even (odd). \( J_{\beta} \) is an integer (half integer) if \((\sum_{n=1}^{\infty} N_n)\) is even (odd).)

Given a solution \( \{\lambda_{\alpha}^n\}, \{\Lambda_{\beta}\} \) define functions

\[
z_n(\lambda) \equiv \theta(\frac{\lambda}{n}) - \frac{1}{L} \sum_{m=1}^{\infty} \sum_{\gamma=1}^{N_m} \Theta_{nm}(\lambda - \lambda_{\gamma}^m) - \frac{1}{L} \sum_{\beta=1}^{N_\Lambda} \theta(\frac{\lambda - \Lambda_{\beta}}{n})
\]

(2.17)

for \( n = 1, \ldots, \infty \), and

\[
z^\lambda(\Lambda) = \frac{1}{L} \sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} \theta(\frac{\Lambda - \lambda_{\alpha}^n}{n}).
\]

(2.18)

By definition, \( L z^n(\lambda_{\alpha}) = 2 \pi I_{\alpha}^n \) and \( L z^\lambda(\Lambda_{\beta}) = 2 \pi J_{\beta} \). If \( z_n(\lambda) \) and \( z^\lambda(\Lambda) \) are monotonic functions, then specifying sets of integers \( \{I_{\alpha}^n\} \) and \( \{J_{\beta}\} \) uniquely determines the solution of (2.9) and (2.10). Define the integers corresponding to infinite spectral parameter by \( 2 \pi I_{\alpha}^\infty = L z^n(\infty) \) and \( 2 \pi J_{\beta} = L z^\lambda(\infty) \). In order to count the states correctly, we discard solutions with infinite spectral parameter, yielding the maximum allowable integers

\[
I_{\max}^n = I_{\alpha}^\infty - 1 = \frac{1}{2} (L - \sum_{m=1}^{\infty} N_m (t_{nm} - 1) - N_\Lambda - 2),
\]

(2.19)
and
\[ J_{\text{max}} = J_\infty - 1 \]  
\[ = \frac{1}{2} \left( \sum_{n=1}^{\infty} N_n - 2 \right), \]

where \( t_{nm} = 2 \min(m, n) - \delta_{nm} \). Counting solutions of the BAE (2.9) and (2.10) using the above, and taking into account that each Bethe state generates a supersymmetry multiplet of size \( d = 8S_z \), one obtains a sum for the total number of states identical to the corresponding formula found by Foerster and Karowski using Sutherland’s ansatz in [16]. There it is shown that the sum adds up to \( 3^L \), the total number of possible states. Thus there are no additional complex solutions other than (2.8) contributing to the spectrum.

We may also use equations (2.17) and (2.18) to define hole rapidities. Having assumed that \( z^n \) and \( z^\Lambda \) are monotonic functions of \( \lambda \) and \( \Lambda \), then for any unoccupied integer \( \bar{\lambda}^n \), there exist a corresponding hole rapidity \( \bar{\lambda}^n \). We will call the combined sets of \( \{\lambda^n\} \) and \( \{\bar{\lambda}^n\} \) the vacancies for parameters. Vacancy integers run through the entire allowed range of integers.

### 2.1 Thermodynamic Limit of BAE

We now take the thermodynamic limit of the Bethe ansatz equations (2.9) and (2.10), fixing the electron and magnetization densities. The solutions to the BAE become densely packed with differences between neighboring \( \lambda^\Lambda_{j+1} - \lambda^p_j \), \( \Lambda_{j+1} - \Lambda_j \sim O(1/L) \). One passes to a description of the states in terms of densities. Define particle and hole densities:

\[ p^p_n(\lambda) \equiv \lim_{L \to \infty} \frac{1}{L} \left( \lambda^p_{I_{j+1}} - \lambda^p_j \right), \]
\[ p^h_n(\lambda) \equiv \lim_{L \to \infty} \frac{1}{L} \left( \lambda^h_{I_{j+1}} - \lambda^h_j \right) \]  
\[ (2.21) \]

and

\[ \sigma^p(\Lambda) \equiv \lim_{L \to \infty} \frac{1}{L} \left( \Lambda_{\bar{I}_{j+1}} - \Lambda_{\bar{I}_j} \right), \]
\[ \sigma^h(\Lambda) \equiv \lim_{L \to \infty} \frac{1}{L} \left( \Lambda^\Lambda_{\bar{I}_{j+1}} - \Lambda^\Lambda_{\bar{I}_j} \right). \]  
\[ (2.22) \]

and the total density of vacancies is \( \rho^v_t = \rho^p_t + \rho^h_t \) and \( \sigma_t = \sigma^p + \sigma^h \). In the thermodynamic limit, the Bethe ansatz equations become integral relations between the densities:

\[ \rho^p_n(\lambda) = f_n(\lambda) - \sum_{m=1}^{\infty} \int_{-\infty}^{\infty} d\lambda' A_{nm}(\lambda - \lambda') \rho^p_m(\lambda') - \int_{-\infty}^{\infty} d\Lambda f_n(\lambda - \Lambda) \sigma^p(\Lambda) \]  
\[ (2.23) \]

and

\[ \sigma^p(\Lambda) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty} d\lambda f_n(\Lambda - \lambda) \rho^p_n(\lambda), \]  
\[ (2.24) \]
where
\[ f_n(\lambda) = \frac{1}{2\pi} \frac{d}{d\lambda} \theta(\lambda) = \frac{n}{\pi \lambda^2 + n^2} \]  
and
\[ A_{nm}(\lambda) = \begin{cases} f_{|n-m|}(\lambda) + 2f_{|n-m|+2} + \cdots + 2f_{n+m-2} & \text{if } n \neq m \\ 2f_2 + 2f_4 + \cdots + 2f_{2n-2} & \text{if } n = m. \end{cases} \]  

3 Thermodynamics

3.1 Free Energy

To calculate the free energy, we follow the techniques developed by Yang and Yang [10], Taka-hashi [13], and Gaudin [14]. The free energy is a functional of the densities:
\[ F = E - \mu N - BS_z - TS, \]  
where the energy is (all integrals beyond this point are between \(-\infty\) and \(\infty\) unless explicitly otherwise)
\[ E = L(1 - \sum_{n=1}^{\infty} \int d\lambda 4\pi f_n(\lambda)\rho_n^p(\lambda)), \]  
the number of particles is
\[ N = L(1 - \int d\lambda \sum_{n=1}^{\infty} \rho_n^p(\lambda) + \int d\Lambda \sigma^p(\Lambda)), \]  
the magnetization is
\[ 2S_z = L(1 - \sum_{n=1}^{\infty} (2n-1) \int d\lambda \rho_n^p(\lambda) - \int d\Lambda \sigma^p(\Lambda)), \]  
and the entropy is given by
\[ S = L[\int d\lambda \sum_{n=1}^{\infty} [\rho_n^t \log(\rho_n^t) - \rho_n^p \log(\rho_n^p) - \rho_n^h \log(\rho_n^h)] + \int d\Lambda [\sigma^t \log(\sigma^t) - \sigma^p \log(\sigma^p) - \sigma^h \log(\sigma^h)]]. \]  

To obtain the entropy (3.5), note that the number of states in in an interval \(d\lambda\) in one of the fermi seas is
\[ N(\lambda, d\lambda) = e^{S(\lambda)d\lambda} = \frac{[L\rho^t d\lambda]!}{[L\rho^p d\lambda]![L\rho^h d\lambda]!}. \]
This the number of ways to put \([L\rho d\lambda]\) “particles” into \([L\rho' d\lambda]\) vacancies. Using Stirling’s approximations for the factorial function at large \(N\), log \(N! \sim N \log N\), and adding up the contribution due to all types of particles, one arrives at (3.3).

We now have the free energy for an arbitrary state. To find the equilibrium value, one minimizes the free energy with respect to the densities. We take the \(\rho_p^v\) and \(\sigma_p^v\) as the independent variables, with \(\rho_t^v\) and \(\sigma_t^v\) determined from (2.23) and (2.24). Setting the variation of the free energy equal to zero yields an infinite set of coupled nonlinear integral equations for the equilibrium densities:

\[
\epsilon_n(\lambda) = \epsilon_n^0(\lambda) + T \sum_{m=1}^\infty \int d\nu A_{nm}(\lambda - \nu) \log(1 + e^{-\epsilon_m(\nu)/T}) - T \int d\Lambda f_n(\lambda - \Lambda) \log(1 + e^{-\epsilon_\Lambda(\Lambda)/T}),
\]

with \(n = 1, \ldots, \infty\), and

\[
\epsilon_\Lambda(\Lambda) = -\mu - B + T \sum_{n=1}^\infty \int d\lambda f_n(\Lambda - \lambda) \log(1 + e^{-\epsilon_n(\lambda)/T}),
\]

where we have written the equations in terms of the functions

\[
\epsilon_n \equiv T \log\left(\frac{\rho_n^h}{\rho_n^v}\right), \quad \epsilon_\Lambda \equiv T \log\left(\frac{\sigma_h}{\sigma^v}\right),
\]

and

\[
\epsilon_n^0(\lambda) = -4\pi f_n(\lambda) - B(2n - 1) + \mu.
\]

Evaluating the free energy at the minimum yields the equilibrium value

\[
\frac{F}{L} = (1 + B - \mu) - T \sum_{n=1}^\infty \int d\lambda f_n(\lambda) \log(1 + e^{-\epsilon_n(\lambda)/T}).
\]

3.2 Finite Temperature Excitations

We now construct finite temperature excitations above the equilibrium state, following the techniques developed in [10]. The idea is as follows: The equilibrium state consists of an infinite
number of fermi seas. The state is highly degenerate, so arbitrarily choose one representative. Now add a particle to one of the seas, all the other seas will change in response. The energy of such an excitation is independent of which equilibrium state one started from, and is called a finite temperature excitation.

We begin with a single particle excitation in the sea of strings of length $n'$. Consider an equilibrium state for large but finite $L$. If one inserts an additional particle into the sea at $\lambda_p$, this leads to an order one excitation above the equilibrium state. This is shown in Figure 1. The rapidities in all the seas will shift, $\lambda^n_\alpha \rightarrow \tilde{\lambda}^n_\alpha$, $\Lambda^\beta \rightarrow \tilde{\Lambda}^\beta$, with $\lambda^n_\alpha - \tilde{\lambda}^n_\alpha \sim \Lambda^\beta - \tilde{\Lambda}^\beta \sim O(1/L)$. We will describe this flow by introducing shift functions [17]:

$$S_n(\lambda|\lambda_p) \equiv \lim_{L \rightarrow \infty} \frac{\lambda^n_\alpha - \tilde{\lambda}^n_\alpha}{\lambda^{n+1}_\alpha - \lambda^n_\alpha}, \quad (3.14)$$

and

$$S_\Lambda(\Lambda|\lambda_p) \equiv \lim_{L \rightarrow \infty} \frac{\Lambda^\beta - \tilde{\Lambda}^\beta}{\Lambda^{\beta+1} - \Lambda^\beta}, \quad (3.15)$$

where the difference in the denominator is between neighboring vacancies. The energy of the excitation is

$$\Delta E = \epsilon_{n'}^0(\lambda_p) + \sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} (\epsilon_{n'}^0(\tilde{\lambda}_n^\alpha) - \epsilon_{n}^0(\lambda_{n'}^\alpha)) \quad (3.16)$$

$$\simeq \epsilon_{n'}^0(\lambda_p) - \sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} (\partial_\lambda \epsilon_{n}^0(\lambda_{n'}^\alpha)(\lambda_{n'}^\alpha - \tilde{\lambda}_n^\alpha)). \quad (3.17)$$

In the thermodynamic limit the sums become integrals

$$\Delta E = \epsilon_{n'}^0(\lambda_p) - \sum_{n=1}^{\infty} \int d\lambda (\partial_\lambda \epsilon_{n}^0(\lambda)) \theta_n(\lambda) S_n(\lambda|\lambda_p), \quad (3.18)$$

where $\theta_n(\lambda) = (1+e^{\lambda/n})^{-1}$ and $\theta_\Lambda = (1+e^{\Lambda/T})^{-1}$. In a similar way, the momentum of the excitation is determined to be

$$\Delta P = p(\tilde{\lambda}_n^\alpha) - \sum_{n=1}^{\infty} \int p'_{n}^0(\lambda) \theta_n(\lambda) S_n(\lambda|\lambda_p) \equiv k_n(\lambda_p), \quad (3.19)$$

where $p'_{n}^0(\lambda) = \partial_\lambda p(\tilde{\lambda}_n^\alpha)$. The shift functions can be found from the BAE. Consider the equations for vacancies,

$$L \theta(\frac{\lambda_n^\alpha}{n}) = 2\pi T^\alpha + \sum_{m=1}^{\infty} \sum_{\gamma=1}^{N_m} \Theta_{n,m}(\lambda_n^\alpha - \lambda_m^\gamma) + \sum_{\beta=1}^{N_\Lambda} \theta(\frac{\lambda_n^\alpha - \Lambda_\beta}{n}) \quad (3.20)$$
and

$$\sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} \theta \left( \frac{\bar{\lambda}_\beta - \lambda^n_\alpha}{n} \right) = 2\pi J,$$

(3.21)

where $I^n$ and $J$ now take on all integers values within the allowed range. The BAE after adding one particle to the $n'$ sea are

$$L\theta \left( \frac{\bar{\lambda}_n^n}{n} \right) = 2\pi (I^n + s^n) + \sum_{m=1}^{\infty} \sum_{\gamma=1}^{N_m} \Theta_{nm}(\bar{\lambda}_\alpha^n - \bar{\lambda}_\gamma^m) + \sum_{\beta=1}^{N_\Lambda} \theta \left( \frac{\bar{\lambda}_\beta^n}{n} \right) + \Theta_{nn'}(\bar{\lambda}_\alpha^n - \lambda_p^n) - \Theta_{nn'}(\bar{\lambda}_\alpha^n - \lambda_h^n)$$

(3.22)

and

$$\sum_{\alpha=1}^{N_n} \theta \left( \frac{\bar{\lambda}_\beta^n - \lambda^n_\alpha}{n} \right) + \theta \left( \frac{\bar{\lambda}_\beta^n - \lambda_p^n}{n'} \right) - \theta \left( \frac{\bar{\lambda}_\beta^n - \lambda_h^n}{n'} \right) = 2\pi (J + s^\Lambda).$$

(3.23)

Here $s^n (n \neq n')$, $s^\Lambda$ are arbitrary half integers, $s^n'$ is an integer arising from from a freedom to shift the excited state distribution by a constant. Shifting the seas by by half-integers (integer) leads to an $O(1/L)$ shift in the energy and an $O(1)$ shift in the momentum. Subtracting the shifted equations from the unshifted one and writing the differences $f(x + dx) - f(x)$ as $f'(x)dx$, one obtains

$$Lf_n(\lambda^n_\alpha)(\lambda^n_\alpha - \bar{\lambda}_n^n) = \sum_{m=1}^{\infty} \sum_{\gamma=1}^{N_m} A_{nm}(\lambda^n_\alpha - \lambda^m_\gamma)((\lambda^n_\alpha - \bar{\lambda}_n^n) - (\lambda^m_\gamma - \bar{\lambda}_\gamma^m))$$

(3.24)

$$+ \sum_{\beta=1}^{N_\Lambda} f_n(\lambda^n_\alpha - \Lambda_\beta)((\lambda^n_\alpha - \bar{\lambda}_n^n) - (\Lambda_\beta - \bar{\lambda}_\beta^n))$$

(3.25)

$$- \frac{1}{2\pi} \Theta_{nn'}(\bar{\lambda}_n^n - \lambda_p^n) - s^n$$

(3.26)

and

$$\sum_{n=1}^{\infty} \sum_{\alpha=1}^{N_n} f_n(\Lambda_\beta - \lambda^n_\alpha)((\Lambda_\beta - \bar{\lambda}_\beta^n) - (\lambda^n_\alpha - \bar{\lambda}_n^n)) - \frac{1}{2\pi} \theta \left( \frac{\bar{\lambda}_\beta^n - \lambda_p^n}{n'} \right) = -s^\Lambda$$

(3.27)

Then using (2.23), (2.24) one obtains the following integral equations for the shift functions:

$$S_n(\lambda) = -\sum_{m=1}^{\infty} \int d\nu A_{nm}(\lambda - \nu)\theta_m(\nu)S_m(\nu)$$

(3.28)

$$- \int d\Lambda f_n(\lambda - \Lambda)\theta(\Lambda)S_{\Lambda}(\Lambda)$$

$$- \frac{1}{2\pi} \Theta_{nn'}(\lambda - \lambda_p^n) - s^n$$
and
\[ S_\Lambda(\Lambda) = \sum_{n=1}^{\infty} \int d\nu f_n(\Lambda - \lambda)\theta_n(\lambda)S_n(\lambda) + \frac{1}{2\pi}\theta\left(\frac{\Lambda - \lambda_p}{n'}\right) + s^\Lambda \] (3.29)

We can now show an interesting identity. First rewrite
\[ -\frac{1}{2\pi}\Theta_{nn'}(\lambda - \lambda_p) = -\int_{-\infty}^{\lambda_p} d\nu \frac{1}{2\pi} (\partial_\nu \Theta_{nn'})(\lambda - \nu) - \frac{1}{2\pi} \Theta_{nn'}(\infty) \] (3.30)
\[ = \int_{-\infty}^{\lambda_p} d\nu A_{nn'}(\lambda - \nu) - \frac{1}{2} (2\min(n, n') - \delta_{nn'} - 1), \] (3.31)
and also
\[ \frac{1}{2\pi} \theta\left(\frac{\Lambda - \lambda_p}{n'}\right) = \int_{-\infty}^{\lambda_p} d\nu \epsilon_{n'}(\lambda - \nu) + \frac{1}{2}. \] (3.32)

Taking the derivative of the integral equations for the functions \(\epsilon_n, \epsilon_\Lambda\) ((3.7) and (3.8)), with respect to \(\lambda, \Lambda\) and noting that all of the kernels are symmetric functions, one can see by simple substitution that
\[ \Delta E = \epsilon_{n'}(\lambda_p) - B(2n' - 1) + \mu. \] (3.33)

In other words, the functions \(\epsilon_n\) may be interpreted as the dressed excitation energy above the equilibrium. Proceeding in a similar way for particle-hole excitation in the \(\Lambda\) sea, one obtains
\[ \Delta E = \epsilon_\Lambda(\Lambda_p) - B - \mu \] (3.34)
and
\[ \Delta P = - \sum_{n=1}^{\infty} \int d\lambda p' \left(\frac{\lambda}{n}\right)\theta_n(\lambda)\bar{S}_n(\lambda) \equiv k_\Lambda(\Lambda_p), \] (3.35)
where the shift functions \(\bar{S}_n(\lambda|\Lambda_p)\) and \(\bar{S}_\Lambda(\Lambda|\Lambda_p)\) satisfy
\[ \bar{S}_n(\lambda) = - \sum_{m=1}^{\infty} \int d\nu A_{nm}(\lambda - \nu)\theta_n(\nu)\bar{S}_m(\nu) \] (3.36)
\[ - \int d\lambda f_n(\lambda - \Lambda)\theta_\Lambda(\Lambda)\bar{S}_\Lambda(\Lambda) \]
\[ - \frac{1}{2\pi} \theta\left(\frac{\lambda - \lambda_p}{n}\right) - s^n \]
and
\[ \bar{S}_\Lambda(\Lambda) = \sum_{n=1}^{\infty} \int d\lambda f_n(\Lambda - \lambda)\theta_n(\lambda)\bar{S}_n(\lambda) + s^\Lambda. \] (3.37)
Hole excitations may also be constructed. The equations for the shift functions are the same as \((3.28), (3.29)\), except the inhomogeneous terms have opposite sign. This leads to the energy
\[
\Delta E = -\epsilon_n(\lambda_h) + B(2n' - 1) + \mu. \tag{3.38}
\]

Composite excitations are sums of the elementary ones, for instance a particle-hole excitation in the \(n'\) sea has shift functions
\[
S_n(\lambda|\lambda_p, \lambda_h) = S_n(\lambda|\lambda_p) + S_n(\lambda|\lambda_h), \tag{3.39}
\]

A general order one excitation above the equilibrium is of the form
\[
\Delta E = \sum_{n=1}^{\infty} \sum_k \epsilon_n(\lambda_{pk}) - \sum_{n=1}^{\infty} \sum_l \epsilon_n(\lambda_{hl}) + \sum_m \epsilon_{\Lambda}(\Lambda_{pm}) - \sum_n \epsilon_{\Lambda}(\Lambda_{hn}), \tag{3.40}
\]
with momentum
\[
\Delta P = \sum_{n=1}^{\infty} \sum_k k_n(\lambda_{pk}) - \sum_{n=1}^{\infty} \sum_l k_n(\lambda_{hl}) + \sum_m k_{\Lambda}(\Lambda_{pm}) - \sum_n k_{\Lambda}(\Lambda_{hn}). \tag{3.41}
\]

An interpretation of these results is as follows \cite{10}: Consider the functions \(\epsilon_n(\lambda, \mu, T)\) and \(\epsilon_{\Lambda}(\lambda, \mu, T)\) as temperature dependent excitation energies of a \textit{noninteracting} fermi system. If one calculates the free energy of such a system, the result is exactly the answer for the true free energy \((3.11)\) of the interacting system. There is a freedom to choose how many species of fermions represent the system, either an infinite number for equation \((3.11)\) or two for equation \((3.11)\).

4 Ground State and Excitations

4.1 Ground State

We shall obtain the ground state and excitations by taking the \(T \to 0\) limit of the thermodynamic equations. Taking the zero temperature limit of equations \((3.7)\) and \((3.8)\), we note that \(\epsilon_n \geq 0\) for \(n \geq 2\). This makes the dependence of the integral equations on \(\epsilon_n\), for \(n \geq 2\), vanish in the zero temperature limit yielding the dressed energies
\[
\epsilon_1(\lambda) = \epsilon_1^0(\lambda) + \int d\Lambda f_1(\lambda - \Lambda)\epsilon_{\Lambda}(\Lambda), \tag{4.1}
\]
\[
\epsilon_{\Lambda}(\Lambda) = -\mu + B - \int d\lambda f_1(\Lambda - \lambda)\epsilon_{1}^-(\lambda), \tag{4.2}
\]

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and
\[ \epsilon_n(\lambda) = \epsilon_n^0(\lambda) + \int d\Lambda f_n(\lambda - \Lambda)\epsilon_\Lambda(\Lambda) - \int d\nu f_{n-1}(\lambda - \nu)\epsilon_\nu^-(\nu), \]
(4.3)

where
\[ \epsilon^-(\lambda) = \begin{cases} \epsilon(\lambda) & \text{if} \ \epsilon < 0 \\ 0 & \text{if} \ \epsilon \geq 0. \end{cases} \]
(4.4)

This tells us that the ground state consists of two fermi seas: one of \(N_1\) parameters and one of \(N_\Lambda\) parameters. This contrasts with Lai’s ansatz which yields a ground state consisting of spin up electrons and and bound state singlet electron pairs.

In spectral parameter space, the seas are filled according to Figure 2. The \(N_1\) sea is filled symmetrically around \(\lambda = 0\) up to some fermi level \(\pm Q_1\). The \(N_\Lambda\) sea is filled placing particles at the maximum spectral parameter down to a fermi level \(Q_\Lambda\).

The ground state energy is
\[ \frac{E_0}{L} = 1 - \int_{-Q_1}^{Q_1} d\lambda 4\pi f_1(\lambda)\rho^p_1(\lambda), \]
(4.5)

where the densities \(\rho^p_i\) and \(\sigma^p\) satisfy the integral equations
\[ \rho^p_1(\lambda) = f_1(\lambda) - (\int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty})d\Lambda f_1(\lambda - \Lambda)\sigma^p(\Lambda) \]
(4.6)

and
\[ \sigma^p(\Lambda) = \int_{-Q_1}^{Q_1} d\lambda f_1(\Lambda - \lambda)\rho^p_1(\lambda). \]
(4.7)

The fermi boundaries \(Q_1\) and \(Q_\Lambda\) may be determined from
\[ D = \frac{N_e}{L} = 1 - \int_{-Q_1}^{Q_1} d\lambda \rho^p_1(\lambda) + (\int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty})d\Lambda \sigma^p(\Lambda) \]
(4.8)

and
\[ \frac{2S_z}{L} = 1 - \int_{-Q_1}^{Q_1} d\lambda \rho^p_1(\lambda) - (\int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty})d\Lambda \sigma^p(\Lambda). \]
(4.9)

In zero magnetic field, the magnetization density of the ground state is zero, due to a theorem by Lieb and Mattis [18]. This requires that \(N_1 = L - N_\Lambda - 1\). The number of available spaces for \(N_1\) parameters according to (2.19) is \(N_1\), so the sea is entirely filled. Thus the fermi sea boundary \(Q_1\) goes to \(\infty\) in the thermodynamic limit.

Considering the half-filled case, \(N/L = 1\), in zero magnetic field, we have \(Q_\Lambda = 0\) and \(Q_1 = \infty\). Here the equations for the ground state are easily solved by fourier transform. The result for the ground state energy is \(E_0 = 1 - 2\log 2\), in agreement with [3] and [9].
4.2 Excitation Spectrum in Zero Magnetic Field

We shall consider order one excitations spectrum in zero magnetic field. The dressed energy of the excitations is given by the functions $\epsilon_1, \epsilon_\Lambda$ satisfying (4.1) and (4.2). Consider the case of $L$ even, $N_1$ odd, and $N_\Lambda$ even. The momenta of an excitation in the $N_1$ sea is given by the zero temperature limit of (3.19),(3.28),and (3.29):

$$k_1(\lambda_p) = p(\lambda_p) - \int_{-\infty}^{\infty} dp' \nu p' S_1(\nu|\lambda_p)$$

where $S_1, S_\Lambda$ satisfy the equations

$$S_1(\lambda|\lambda_p) = -\left(\int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty}\right) d\lambda f_1(\lambda - \Lambda) S_\Lambda(\Lambda),$$

and

$$S_\Lambda(\Lambda|\lambda_p) = \int_{-\infty}^{\infty} d\lambda f_1(\lambda - \Lambda) S_1(\lambda) - \frac{1}{2\pi} \theta(\Lambda - \lambda_p).$$

Excitations in the $N_\Lambda$ sea have momentum given by the zero temperature limit of (3.35),(3.36), and (3.37):

$$k_\Lambda(\Lambda_p) = -\int_{-Q_1}^{Q_1} d\lambda p'(\lambda) \bar{S}_1(\lambda|\Lambda_p),$$

where the shift functions $\bar{S}_1, \bar{S}_\Lambda$ satisfy

$$\bar{S}_1(\lambda|\Lambda_p) = -\left(\int_{-\infty}^{-Q_\Lambda} + \int_{Q_\Lambda}^{\infty}\right) d\lambda f_1(\lambda - \Lambda) \bar{S}_\Lambda(\Lambda|\Lambda_p) + \frac{1}{2\pi} \theta(\lambda - \Lambda_p),$$

and

$$\bar{S}_\Lambda(\Lambda|\lambda_p) = \int_{-\infty}^{\infty} d\lambda f_1(\lambda - \Lambda) \bar{S}_1(\lambda|\Lambda_p).$$

For the case under consideration, all the shifts $s^1, s^\Lambda, \bar{s}^1$, and $s^\Lambda$ are zero. The momentum of adding arbitrary numbers of particles and holes to the fermi seas is

$$\Delta P = \sum_{N_1 \text{ particles}} k_1(\lambda_p) - \sum_{N_1 \text{ holes}} k_1(\lambda_h) + \sum_{N_\Lambda \text{ particles}} k_\Lambda(\Lambda_p) - \sum_{N_\Lambda \text{ holes}} k_\Lambda(\Lambda_h).$$

To see how these functions combine into excitations, first consider the less than half filled case, $D < 1$. The possible elementary processes are:

1. Create hole in $N_1$ sea: $\Delta N_1 = -1, \Delta N_\Lambda = 0$. This corresponds to adding a spin up electron, $\Delta N_e = 1 \ \Delta S = \frac{1}{2}$. The energy and momentum are

$$\Delta E = -\epsilon_1(\lambda_h), \quad \Delta P = -k_1(\lambda_h).$$
2. Create hole in $N_\Lambda$ sea: $\Delta N_1 = 0$, $\Delta N_\Lambda = -1$. This corresponds to removing a spin down electron, $\Delta N_e = -1$, $\Delta S = \frac{1}{2}$. This creates a hole in the $N_1$ sea, so the energy and momentum are

$$\Delta E = -\epsilon_1(\lambda_h) - \epsilon_\Lambda(\Lambda_h), \quad \Delta P = -k_1(\lambda_h) - k_\Lambda(\Lambda_h).$$

(4.18)

3. Transfer particle from $N_\Lambda$ sea to $N_1$ sea: $\Delta N_1 = +1$, $\Delta N_\Lambda = -1$. Removing a $\Lambda$ parameter increases holes in the $N_1$ sea by one, in which the one $\lambda$ parameter is placed. Thus, there is no parametric dependence on $\lambda$. This excitation removes 2 electrons, with no change in spin, $\Delta N_e = -2$, $\Delta S = 0$. The energy and momenta are

$$\Delta E = -\epsilon_\Lambda(\Lambda_h), \quad \Delta P = -k_\Lambda(\Lambda_h).$$

(4.19)

4. Transfer particle from $N_1$ sea to $N_\Lambda$ sea: $\Delta N_1 = -1$, $\Delta N_\Lambda = +1$. As in type 3, there is no parametric $\lambda$ dependence. This excitation adds 2 electrons, with no change in spin. The energy and momentum of the excitation are

$$\Delta E = \epsilon_\Lambda(\Lambda_p), \quad \Delta P = k_\Lambda(\Lambda_p).$$

(4.20)

5. Bind particles into bound state: $\Delta N_1 = -2$, $\Delta N_\Lambda = -1$, $\Delta N_2 = +1$. This opens up two holes in the $h+\downarrow$ sea, and one in the $\downarrow$ sea. There is only one vacancy for the bound state, so there is no parametric dependence on its location. This excitation does not change the number of electrons or the spin. The energy and momentum are

$$\Delta E = -\epsilon_1(\lambda_{h_1}) - \epsilon_1(\lambda_{h_2}) - \epsilon_\Lambda(\Lambda_h), \quad \Delta P = -k_1(\lambda_{h_1}) - k_1(\lambda_{h_2}) - k_\Lambda(\Lambda_h).$$

(4.21)

4.3 Excitations at Half Filling

In the half filled case, $D = 1$, we can solve the integral equations (4.1),(4.2),(4.10)-(4.15) for the energy and momenta. The dressed energy and momenta of an excitation in the $N_1$ sea are

$$\epsilon_1(\lambda) = \frac{-\pi}{\cosh(\frac{\pi\lambda}{2})}$$

and

$$k_1(\lambda) = -\tan^{-1}(\sinh(\frac{\pi\lambda}{2})) + \frac{\pi}{2}.$$
For an excitation in the $N_\Lambda$ sea, the dressed energy and momentum are

$$\epsilon_\Lambda(\Lambda) = R(\Lambda) - 2 \log 2$$ (4.24)

and

$$k_\Lambda(\Lambda) = \int_{-\Lambda}^{\Lambda} d\Lambda' R(\Lambda'),$$ (4.25)

where

$$R(\Lambda) = \int_{-\infty}^{\infty} \frac{e^{-i\Lambda w}}{1 + e^{2|w|}}.$$ (4.26)

The $n$-string excitations have zero energy, $\epsilon_{n>1} = 0$. The ground state has both fermi seas filled, $N_1 = L/2$, $N_\Lambda = L/2 - 1$.

These functions combine to form excitations in the following ways:

1. Create hole in $N_\Lambda$ sea: $\Delta N_1 = 0$, $\Delta N_\Lambda = -1$. This corresponds to removing a spin down electron, $\Delta N_e = -1$, $\Delta S = \frac{1}{2}$. This creates a hole in the $h+\downarrow$ sea, so the energy and momentum are

$$\Delta E = -\epsilon_1(\lambda_h) - \epsilon_\Lambda(\lambda_h), \quad \Delta P = -k_1(\lambda_h) - k_\Lambda(\lambda_h).$$ (4.27)

2. Transfer particle from $N_\Lambda$ sea to $N_1$ sea: $\Delta N_1 = +1$, $\Delta N_\Lambda = -1$. In contrast with the case away from half filling, now two holes in the $N_\Lambda$ sea are created. This excitation removes 2 electrons, with no change in spin, $\Delta N_e = -2$, $\Delta S = 0$. The energy and momenta are

$$\Delta E = -\epsilon_\Lambda(\lambda_{h_1}) - \epsilon_\Lambda(\lambda_{h_2}), \quad \Delta P = -k_\Lambda(\lambda_{h_1}) - k_\Lambda(\lambda_{h_2}).$$ (4.28)

3. Create holes in $N_1$ and $N_\Lambda$ seas: $\Delta N_1 = -1$, $\Delta N_\Lambda = -1$. This is not simply a sum of type 1 and 2 excitations for $D < 1$ because here there is no hole in the $\downarrow$ sea. This excitation has $\Delta N_e = 0, \Delta S = 1$. The energy and momentum are

$$\Delta E = -\epsilon_1(\lambda_{h_1}) - \epsilon_1(\lambda_{h_2}), \quad \Delta P = -k_1(\lambda_{h_1}) - k_1(\lambda_{h_2}).$$ (4.29)

4. Bind particles into bound state: $\Delta N_1 = -2$, $\Delta N_\Lambda = -1$, $\Delta N_2 = +1$. This opens up two holes in the $N_1$ sea, and none in the $N_\Lambda$ sea, in contrast with type 5 excitation for $D < 1$. There is only one space for the bound state, so there is no parametric dependence its location. This excitation has $\Delta N_e = 0$, $\Delta S = 0$. The energy and momentum are

$$\Delta E = -\epsilon_1(\lambda_{h_1}) - \epsilon_1(\lambda_{h_2}), \quad \Delta P = -k_1(\lambda_{h_1}) - k_1(\lambda_{h_2}).$$ (4.30)

The excitations 3 and 4 are identical to those in the XXX spin chain [19]. The results here agree with those obtained in [3] using Lai’s and Sutherland’s ansatz.
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A Relationship to Lai’s Ansatz

In this section we shall how the solutions of Essler and Korepin’s ansatz, (2.6) and (2.7), are related to solutions of Lai’s ansatz (A.1) and (A.2). A mathematical equivalence between the ansatzes has been shown in [8] and [20].

First we recall the Lai solution [3, 5] which treats the electrons as dynamical objects in a background of empty sites. The $N_{\downarrow} + N_{\uparrow}$ spectral parameters $\lambda_l$ describe the kinetic degrees of freedom, the $N_{\downarrow}$ parameters describe the motion of the spins of the electrons. The spectal parameters must satisfy the BAE:

\[
\left( \frac{\lambda_l + i}{\lambda_l - i} \right)^L = \prod_{\beta=1}^{N_{\downarrow}} \left( \frac{\lambda_l - \Lambda_\beta + i}{\lambda_l - \Lambda_\beta - i} \right) \quad l = 1, ..., N_{\downarrow} + N_{\uparrow}, \quad (A.1)
\]

\[
\prod_{l=1}^{N_{\downarrow} + N_{\uparrow}} \left( \frac{\Lambda_\beta - \lambda_l + i}{\Lambda_\beta - \lambda_l - i} \right) = -\prod_{\gamma=1}^{N_{\downarrow}} \left( \frac{\Lambda_\beta - \Lambda_\gamma + 2i}{\Lambda_\beta - \Lambda_\gamma - 2i} \right). \quad \beta = 1, ..., N_{\downarrow}. \quad (A.2)
\]

In this ansatz, there are two types of string solutions:

1. bound singlet electron pairs: composed of 2 $\lambda$ and 1 $\Lambda$ parameters:
   \[
   \lambda = \Lambda + i \quad \lambda' = \Lambda - i \quad (A.3)
   \]

2. spin bound states: take $n$ $\Lambda$ parameters and set
   \[
   \Lambda_\delta = \Lambda + i(n + 1 - 2\delta) \quad \delta = 1, ..., n. \quad (A.4)
   \]

In the thermodynamic limit one define densities $\rho, \sigma', \sigma_1$, and $\sigma_n (n = 2, ..., \infty)$ for free electrons, bound pairs, down spins, and spin bound states respectively. The thermodynamic limit of the Lai BAE is

\[
\rho'(\lambda) = f_1(\lambda) - \int d\Lambda f_1(\lambda - \Lambda) \sigma'^p(\Lambda) - \sum_{n=1}^{\infty} \int d\nu f_n(\lambda - \nu) \sigma_n^p(\nu), \quad (A.5)
\]

\[
\sigma''(\Lambda) = f_2(\Lambda) - \int d\Lambda' f_2(\Lambda - \Lambda') \sigma'^p(\Lambda') - \int d\lambda f_1(\lambda - \Lambda) \rho^p(\lambda), \quad (A.6)
\]
and
\[ \sigma_n^t(\lambda) = \int d\nu f_n(\lambda - \nu)\rho^p(\nu) - \sum_{m=1}^{\infty} \int d\nu B_{nm}(\lambda - \nu)\sigma_m^p(\nu), \] (A.7)
where
\[ B_{nm} = \begin{cases} f_{|n-m|} + 2f_{|n-m|+2} + \ldots + 2f_{n+m} & \text{if } n \neq m \\ 2f_2 + \ldots + 2f_{2n-2} + f_{2n} & \text{if } n = m \end{cases}. \] (A.8)

Now if one makes identification between the densities in the two ansätze
\[ \rho^{p,h} \leftrightarrow \rho_1^{h,p} \quad \sigma^{p,h} \leftrightarrow \sigma^{p,h} \quad \sigma_n^{p,h} \leftrightarrow \rho_n^{p,h}, \quad n = 1, \ldots, \infty, \] (A.9)
we may see that equations (A.5), (A.6), and (A.7) it coincide with (2.23) and (2.24). To show this one only needs the identity
\[ \hat{f}_n \hat{f}_m = \hat{f}_{n+m}, \] (A.10)
where have written the action of the kernel as an integral operator on some function \( g(\lambda) \):
\[ \hat{f}_n g(\lambda) \equiv \int_{-\infty}^{\infty} f_n(\lambda - \nu)g(\nu). \] (A.11)

Thus the role of particles and hole solutions are interchanged in the \( \uparrow + \downarrow \) sea in Lai’s ansatz and the \( N_1 \) sea in Essler and Korepin’s ansatz. One expects that this will hold in a finite box as well, the hole solutions of (2.7) will coincide with the particle solutions for bound pairs in Lai’s ansatz.

Using this particle-hole correspondence, one can show that the free energy obtained here is equal to that in ref. [3]. Identifying the energy functions, one can rewrite the integral equations for the \( \epsilon \) functions, (3.7) and (3.8), to coincide with the corresponding equations in Lai’s ansatz, equation (5.16) in [3].

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