JACOBIANS AMONG ABELIAN THREEFOLDS:
A GEOMETRIC APPROACH

ARNAUD BEAUVILLE AND CHRISTOPHE RITZENTHALER

Abstract. Let \((A, \theta)\) be a principally polarized abelian threefold over a perfect field \(k\), not isomorphic to a product over \(k\). There exists a canonical extension \(k'/k\), of degree \(\leq 2\), such that \((A, \theta)\) becomes isomorphic to a Jacobian over \(k'\). The aim of this note is to give a geometric construction of this extension.

1. Introduction

Let \((A, \theta)\) be a principally polarized abelian variety of dimension 3 over a field \(k\). If \(k\) is algebraically closed, \((A, \theta)\) is the Jacobian variety of a curve \(C\) (or a product of Jacobians). If \(k\) is an arbitrary perfect field the situation is more subtle (see Proposition 3.1 below): there is still a curve \(C\) defined over \(k\), but either \((A, \theta)\) is isomorphic to \(JC\), or they become isomorphic only after a quadratic extension \(k'\) of \(k\), uniquely determined by \((A, \theta)\).

Now given \((A, \theta)\), how can we decide if it is a Jacobian, and more precisely determine the extension \(k'/k\)? For \(k \subset \mathbb{C}\), a solution is given in [LRZ] in terms of modular forms. Here we propose a geometric approach, based on a construction of Recillas. We have to assume that \(A\) admits a rational theta divisor \(\Theta\), and a rational point \(a \in A(k)\) outside a certain explicit divisor \(\Sigma_A \subset A\). This guarantees that the curve \(\Theta \cap (\Theta + a)\) is smooth, and that there exists \(b \in A(k)\) such that the involution \(z \mapsto b - z\) acts freely on that curve. The quotient \(X_a\) is a non-hyperelliptic genus 4 curve; its canonical model lies on a unique quadric \(Q \subset \mathbb{P}^3\). Then for \(\text{char}(k) \neq 2\) the extension \(k'/k\) is \(k(\sqrt{\text{disc}(Q)})\) (we will give more detailed statements in §3).

The proof has two steps. We consider first the case where \((A, \theta)\) is a Jacobian, and prove that in that case the quadric \(Q\) is \(k\)-isomorphic to \(\mathbb{P}^1 \times \mathbb{P}^1\) (§2). Then we treat the case where \((A, \theta)\) is not a Jacobian, and prove that the nontrivial automorphism of the extension \(k'/k\) exchanges the two rulings of \(Q\) (§3); this is enough to prove the theorem.
2. Recillas’ construction

Throughout the paper we work over a perfect field $k$.

In this section we fix a non-hyperelliptic curve $C$ of genus 3 (that is, a smooth plane quartic curve), defined over $k$. We will denote by $K$ its canonical class. We assume that the principal polarization of $JC$ can be defined by a theta divisor $\Theta$ defined over $k$. There exists a degree 2 divisor class $D$ such that $\Theta$ is the image of $\text{Sym}^2C - D$ in $JC$; this class is unique, hence defined over $k$. Note that since $C$ is not hyperelliptic, $\Theta$ is smooth and the map $E \mapsto E - D$ induces an isomorphism of $\text{Sym}^2C$ onto $\Theta$.

We choose a point $a \in JC(k)$; we are interested in the curve $X_a := \Theta \cap (\Theta + a)$. Put $b = K + a - 2D \in JC(k)$; we have $-\Theta = \Theta + a - b$. The involution $z \mapsto b - z$ exchanges $\Theta$ and $\Theta + a$, hence induces an involution $\iota$ of $X_a$. We define a divisor $\Sigma_{JC} \subset JC$ by $\Sigma_{JC} = \Sigma'_{JC} \cup \Sigma''_{JC}$, where

$$\Sigma'_{JC} = \{2E - K \mid E \in \text{Sym}^2C\} \quad \text{and} \quad \Sigma''_{JC} = C - C.$$ 

**Proposition 2.1.** The curve $X_a := \Theta \cap (\Theta + a)$ is smooth and connected if and only if $a \in JC \setminus \Sigma_{JC}$. If this is the case, the involution $\iota$ of $X_a$ is fixed point free.

**Proof:** Throughout the paper it will be convenient to use the following notation: given a divisor class $d$ of degree 2 on $C$ with $h^0(\mathcal{O}_C(d)) = 1$, we denote by $\langle d \rangle \in \text{Sym}^2C$ the unique effective divisor in the class $d$.

Let $z \in X_a$. By [K], thm. 2, the tangent space $\mathbb{P}T_z(\Theta) \subset \mathbb{P}T_z(JC) = \mathbb{P}^2$ is identified with the line spanned by the divisor $\langle D + z \rangle \in \text{Sym}^2C$. Similarly $\mathbb{P}T_z(\Theta + a)$ is identified with the line spanned by $\langle D + z - a \rangle \in \text{Sym}^2C$; the intersection $X_a$ is singular at $z$ if and only if these two lines coincide. If this happens, then either

- the two divisors $\langle D + z \rangle$ and $\langle D + z - a \rangle$ have a common point, which implies $a \in C - C$; or
- $\langle D + z \rangle + \langle D + z - a \rangle \sim K$, which implies $a \in \Sigma'_{JC}$.

Conversely, if $a \in \Sigma'_{JC}$, we have $K + a \sim 2E$ with $E \in \text{Sym}^2C$; then $z = E - D$ is a singular point of $X_a$. If $a \sim p - q$, with $p, q \in C$, the intersection $X_a$ is reducible, equal to $(C + p - D) \cup (K - D - q - C)$.

Assume now $a \notin \Sigma_{JC}$, so that $X_a$ is smooth; the exact sequence

$$0 \rightarrow \mathcal{O}_{JC}(-\Theta - (\Theta + a)) \rightarrow \mathcal{O}_{JC}(-\Theta) \oplus \mathcal{O}_{JC}(-(\Theta + a)) \rightarrow \mathcal{O}_{JC} \rightarrow \mathcal{O}_{X_a} \rightarrow 0$$

gives $h^0(\mathcal{O}_{X_a}) = 1$, hence $X_a$ is connected. If a point $z \in X_a$ is fixed by $\iota$ it satisfies $2(D + z) \sim K + a$, which implies $a \in \Sigma'_{JC}$. ■
2.2. We assume from now on $a \notin \Sigma_{JC}$. We denote by $X_a$ the quotient curve $	ilde{X}_a/\iota$. The adjunction formula gives

$$K_{\tilde{X}_a} \sim (\Theta + (\Theta + a))|_{\tilde{X}_a} = 2\Theta^3 = 12, \quad \text{hence} \quad g(\tilde{X}_a) = 7 \quad \text{and} \quad \varrho(\tilde{X}_a) = 4.$$  

If $\text{char}(k) \neq 2$, the principally polarized abelian variety $JC$ is canonically isomorphic to the Prym variety associated to the double covering $\tilde{X}_a \rightarrow X_a$ ([3], §3.b; [R]).

We embed $\tilde{X}_a$ into $\text{Sym}^2 C \times \text{Sym}^2 C$ by $z \mapsto ((D + z), (K + a - (D + z)))$. Then $\tilde{X}_a$ is identified with $s^{-1}(|K + a|)$, where $s : \text{Sym}^2 C \times \text{Sym}^2 C \rightarrow \text{Sym}^4 C$ is the sum map. The involution $\iota$ is induced by the involution of $\text{Sym}^2 C \times \text{Sym}^2 C$ which exchanges the factors. The map $s : \tilde{X}_a \rightarrow |K + a|$ factors through $\iota$, hence induces a 3-to-1 map $t : X_a \rightarrow |K + a|$ ($\cong \mathbb{P}^1$). The fibre of $t$ above $E \in |K + a|$ parametrizes the decompositions $E = d + d'$, with $d, d'$ in $\text{Sym}^2 C$.

We now consider the involution $(d, d') \mapsto ((K - d), (K - d'))$ of $\text{Sym}^2 C \times \text{Sym}^2 C$; it maps $\tilde{X}_a$ onto $s^{-1}(|K - a|) = \tilde{X}_{-a}$ and commutes with $\iota$, hence induces an isomorphism $X_a \cong X_{-a}$. By composition with the map $X_{-a} \rightarrow |K - a|$ defined above we obtain another degree 3 map $t' : X_a \rightarrow |K - a|$.

The maps $t$ and $t'$ are defined over $k$; they define two $g^1_3$ on $X_a$, that is, two linear series of degree 3 and projective dimension 1, defined over $k$.

**Lemma 2.3.** The two $g^1_3$ defined by $t$ and $t'$ on $X_a$ are distinct.

**Proof:** Let us first observe that the degree 4 morphism $f : C \rightarrow \mathbb{P}^1$ defined by the linear system $|K + a|$ is separable. If this is not the case, we have $\text{char}(k) = 2$ and $f$ factors as $C \xrightarrow{F} C_1 \xrightarrow{g} \mathbb{P}^1$, where $C_1/k$ is the pull back of $C/k$ by the automorphism $\lambda \mapsto \lambda^2$ of $k$, $F$ is the Frobenius $k$-morphism and $g$ is separable of degree 2 (see [I], IV.2). But then $C_1$ is hyperelliptic, hence also $C$.

Assume that the two linear series are the same. By the previous observation there exists a divisor $E = p + q + r + s$ in $|K + a|$ consisting of 4 distinct points. There must exist $E' \in |K - a|$ such that $t^{-1}(E') = t'^{-1}(E)$. This means that for each decomposition $E = d + d'$ with $d, d'$ in $\text{Sym}^2 C$, we have $E' = (K - d) + (K - d')$.

Let us write $(K - p - q) = p' + q'$ and $(K - r - s) = r' + s'$, so that $E' = p' + q' + r' + s'$. We must have $E' = (K - p - r) + (K - q - s)$, so we can suppose $(K - p - r) = p' + r'$. Then $K - p - p' \sim q + q' \sim r + r'$, which implies $r' = q, q' = r$. But then we get $K - p - q - r \sim p'$ and $a \sim s - p'$, which contradicts the hypothesis $a \notin \Sigma_{JC}$.  

We can now conclude:

**Proposition 2.4.** For \( a \notin \Sigma_{JC} \), the genus 4 curve \( X_a \) is not hyperelliptic; the unique quadric \( Q \subset \mathbb{P}^3 \) containing its canonical model is smooth and split over \( k \) (that is, isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \) over \( k \)).

**Proof:** Since \( X_a \) admits a base point free \( g^1_3 \) it cannot be hyperelliptic ([ACGH], p. 13). Let us denote the two distinct \( g^1_3 \) of \( X_a \) by \( |E| \) and \( |E'| \).

We have \( E + E' \sim K_{X_a} \); by the base-point free pencil trick, the multiplication map \( H^0(X_a, E) \otimes H^0(X_a, E') \rightarrow H^0(X_a, K_{X_a}) \) is an isomorphism. Thus the canonical map of \( X_a \) is the composition of \( (t, t') : X_a \rightarrow \mathbb{P}^1 \times \mathbb{P}^1 \) and of the Segre embedding \( \mathbb{P}^1 \times \mathbb{P}^1 \cong \mathbb{P}^3 \), so the unique quadric containing the image is isomorphic to \( \mathbb{P}^1 \times \mathbb{P}^1 \).

**Remark 2.5.** If \( C \) is hyperelliptic, Proposition 2.1 still holds, with essentially the same proof. However Lemma 2.3 fails: in fact, we have \( t' = \sigma \circ t \), where \( \sigma : |K + a| \rightarrow |K_a| \) is induced by the hyperelliptic involution. Actually in that case \( X_a \) has a unique \( g^1_3 \), at least if \( \text{char}(k) \neq 2 \). Indeed \( \Theta \) has a singular point, given by the \( g^1_2 \) of \( C \); on the other hand \( JC \) is isomorphic to the Prym variety of \( \tilde{X}_a/X_a \). By [M2], §7, Thm. (c), this happens if and only if \( X_a \) admits a unique \( g^1_3 \).

2.6. The divisor \( \Sigma'_{JC} \) is equal to \( 2 \cdot \Delta \), where \( 2 \) is the endomorphism \( z \mapsto 2z \) of \( JC \) and \( \Delta \) is any symmetric theta divisor; thus it can be defined on any absolutely indecomposable principally polarized abelian threefold \((A, \theta)\), with no reference to the isomorphism \( A \cong JC \). The same holds for \( \Sigma''_{JC} \) provided \( \text{char}(k) \neq 2 \). Recall indeed that there is a canonical linear system on \( A \), denoted \( |2\theta| \), which contains the double of each symmetric theta divisor.

**Lemma 2.7.** If \( \text{char}(k) \neq 2 \), the divisor \( \Sigma''_{JC} = C - C \) is the unique divisor in \( |2\theta| \) with multiplicity \( \geq 4 \) at 0.

This is quite classical if \( k = \mathbb{C} \), see [GG]. We do not know whether it still holds when \( \text{char}(k) = 2 \).

**Proof:** The difference map \( C \times C \rightarrow C - C \) is an isomorphism outside the diagonal \( \Delta \), and contracts \( \Delta \) to 0; therefore the multiplicity of \( C - C \) at 0 is \( -\Delta^2 = 4 \).

Let us prove the unicity; we may assume \( k = \bar{k} \). We denote by \( |2\theta|_0 \) the subspace of elements of \( |2\theta| \) containing 0. The multiplicity at 0 of an element of \( |2\theta| \) is even: this follows from the “inverse formula” of [M1], p. 331. Thus we have a projective linear map \( \tau : |2\theta|_0 \rightarrow |\mathcal{O}_{\mathbb{P}^2}(2)| \) which associates to a divisor its quadratic tangent cone at 0. Since \( \dim |2\theta|_0 = 6 \)
and \( \dim \mathcal{O}_{\mathbb{P}^2}(2) = 5 \), it suffices to prove that \( \tau \) is surjective. For each \( E \in \text{Sym}^2 C \), the divisor \((\text{Sym}^2 C - E) + (\text{Sym}^2 C - (K - E))\) belongs to \(|2\theta_0|\); by [K], thm. 2, its tangent cone at 0 is twice the line in \( \mathbb{P}^2 \) spanned by \( E \). Since the double lines span the space of conics, \( \tau \) is surjective.

\[ \square \]

3. The main result

In this section we fix a principally polarized abelian threefold \((A, \theta)\) over \( k \). We assume that it is absolutely indecomposable, that is, \((A, \theta)\) is not isomorphic over \( \overline{k} \) to a product of two principally polarized abelian varieties. It is equivalent to say that the theta divisor of \( A \) is irreducible (over \( \overline{k} \)), or that \((A, \theta)_{\overline{k}}\) is isomorphic to the Jacobian of a curve [OU]. This does not imply that \((A, \theta)\) itself is a Jacobian; indeed we have [S]:

**Proposition 3.1.** There exists a curve \( C \) over \( k \) and a character \( \varepsilon_A : \text{Gal}(\overline{k}/k) \to \{\pm 1\} \), uniquely determined, such that \((A, \theta)\) is \( k \)-isomorphic to \( JC \) twisted by \( \varepsilon_A \). If \( C \) is hyperelliptic, \( \varepsilon_A \) is trivial.

3.2. In more down-to-earth terms this means the following. Let \( k' \) be the extension of \( k \) defined by the character \( \varepsilon_A \). Then:

- if \( \varepsilon_A = 1 \) (that is, \( k' = k \)), \( JC \) is \( k \)-isomorphic to \((A, \theta)\). This is the case if \( C \) is hyperelliptic.
- if \( \varepsilon_A \neq 1 \) (that is, \( k' \) is a quadratic extension of \( k \)), \( JC \) is isomorphic to \((A, \theta)\) over \( k' \) but not over \( k \). More precisely, let \( \sigma \) be the nontrivial automorphism of \( k'/k \); there exists an isomorphism \( \varphi : (A, \theta) \to JC \) such that \( \sigma \varphi = -\varphi \).

3.3. Our aim is to describe geometrically the character \( \varepsilon_A \). We will compare it to the character associated to a smooth quadric \( Q \subset \mathbb{P}^2 \) in the following way: such a quadric admits two rulings defined over \( \overline{k} \), so the action of \( \text{Gal}(\overline{k}/k) \) on these rulings provides a character \( \varepsilon_Q : \text{Gal}(\overline{k}/k) \to \{\pm 1\} \). We will describe this character in more concrete terms below.

We define the divisor \( \Sigma_A = \Sigma_A' \cup \Sigma_A'' \) on \( A \) as in [24T] we put \( \Sigma_A' = 2, \Delta \) for any symmetric theta divisor \( \Delta \); if \( \text{char}(k) \neq 2 \), \( \Sigma_A'' \) is the unique divisor in \(|2\theta|\) with multiplicity \( \geq 4 \) at 0. An alternative definition, which works in all characteristics, is as follows: we choose an isomorphism \( A \cong JC \) over \( k' \) and put \( \Sigma_A'' = \varphi^{-1}(C - C) \). Since \( C - C \) is symmetric this definition does not depend on the choice of \( \varphi \).

We assume that \( A \) admits a theta divisor \( \Theta \) defined over \( k \). If \( \Theta \) is singular, \( C \) is hyperelliptic, hence \( A \cong JC \) by Proposition [3.1]. Thus we may assume that \( \Theta \) is smooth.
We also assume that there exists a point $a \in A(k)$ outside $\Sigma_A$. The divisor $-\Theta$ is in the class of the polarization $\theta$, hence there is a unique $b \in A(k)$ such that $(-\Theta) + b = \Theta + a$; the involution $z \mapsto b - z$ exchanges $\Theta$ and $\Theta + a$.

**Theorem 3.4.** Let $X_a$ be the quotient of the curve $\Theta \cap (\Theta + a)$ by the involution $z \mapsto b - z$. Then $X_a$ is a smooth curve of genus 4, non hyperelliptic. Its canonical model lies in a smooth quadric $Q \subset \mathbb{P}^3$, and we have $\varepsilon_A = \varepsilon_Q$.

**Proof:** Following 3.2 we choose an isomorphism $\varphi : (A, \theta) \to JC$ defined over $k'$. It induces an isomorphism of $\tilde{X}_a$ onto the corresponding curve $X_{\varphi(a)} \subset JC$, hence of $X_a$ onto $X_{\varphi(a)}$. By remark 2.6 $\varphi$ maps $\Sigma_A$ onto $\Sigma_{JC}$, thus $\varphi(a) \notin \Sigma_{JC}$; then Proposition 2.4 tells us that $X_a$ is not hyperelliptic and that its canonical model is contained in a smooth quadric $Q \subset \mathbb{P}^3$ which is split over $k'$. This means that the character $\varepsilon_Q$ is trivial on the subgroup $\text{Gal}(\bar{k}/k')$ of $\text{Gal}(\bar{k}/k)$; in other words, $\varepsilon_Q$ is either trivial or equal to $\varepsilon_A$.

It remains to prove that $\varepsilon_Q$ is nontrivial when $k' \neq k$, that is, the nontrivial automorphism $\sigma$ of $k'/k$ exchanges the two rulings of $Q$, or equivalently the two $g^1_3$ of $X_a$.

We have $\sigma \varphi = -\varphi$ [2.4]. We write as before $\varphi(\Theta) = \text{Sym}^2 C - D$; we observe that $\sigma(\varphi(\Theta)) = -\varphi(\Theta)$, hence $\sigma D \sim K - D$. Recall that the maps $t : X_a \to |K + \varphi(a)|$ and $t' : X_a \to |K - \varphi(a)|$ defining the two $g^1_3$ are given by

$$
t(\tilde{z}) = \langle D + \varphi(z) \rangle + \langle K - D - \varphi(z) + \varphi(a) \rangle$$

$$
t'(\tilde{z}) = \langle K - D - \varphi(z) \rangle + \langle D + \varphi(z) - \varphi(a) \rangle$$,

where $z$ is a point of $\tilde{X}_a$ and $\tilde{z}$ its image in $X_a$.

Using $\sigma \varphi = -\varphi$ and $\sigma D \sim K - D$ we get

$$
\sigma t(\tilde{z}) = \langle K - D - \varphi(z) \rangle + \langle D + \varphi(z) - \varphi(a) \rangle = t'(\tilde{z}) ;
$$

thus $\sigma$ exchanges $t$ and $t'$, hence the two rulings of $Q$. \[\blacksquare\]

One can describe the extension $k'/k$ (hence the character $\varepsilon_Q$) using the even Clifford algebra $C^+(Q)$ [De]: its center is isomorphic to $k'$ if $k' \neq k$ and to $k \times k$ otherwise. From the description of this center (see [Bo], §9, no. 4, Remarque 2), we obtain:

**Proposition 3.5.** Assume $\text{char}(k) \neq 2$, and let $\delta \in k^*$ be the discriminant of $Q$ (well defined mod. $k^{*2}$). The extension $k'$ is isomorphic to $k(\sqrt{\delta})$.

Similarly, if $\text{char}(k) = 2$, we have $k' = k(\lambda)$ with $\lambda^2 + \lambda = \Delta$, where $\Delta$ is the pseudo-discriminant of $Q$ ([Bo], §9, exerc. 9).
Finally let us observe that the existence of a rational theta divisor is automatic when \( k \) is finite. Indeed the theta divisors in the class \( \theta \) form a torsor under \( A \); by a theorem of Lang [L], such a torsor is trivial.

References

[ACGH] E. Arbarello, M. Cornalba, P.A. Griffiths, J. Harris, Geometry of algebraic curves, Vol. I. Grundlehren der Mathematischen Wissenschaften 267, Springer-Verlag, New-York 1985.

[A] M. Atiyah : Riemann surfaces and spin structures. Ann. Sci. École Norm. Sup. (4) 4 (1971), 47–62.

[B] A. Beauville: Sous-variétés spéciales des variétés de Prym. Compositio Math. 45 (1982), no. 3, 357–383.

[Bo] N. Bourbaki : Algèbre, ch. 9. Hermann, Paris (1959).

[De] P. Deligne: Quadriques. SGA7 II, Exp. XII. Lecture Notes in Math. 340, 62–81; Springer, Berlin, 1973.

[GG] B. van Geemen, G. van der Geer : Kummer varieties and the moduli spaces of abelian varieties. Amer. J. Math. 108 (1986), no. 3, 615–641.

[H] R. Hartshorne : Algebraic geometry. Graduate Texts in Mathematics 52. Springer-Verlag, New York-Heidelberg, 1977.

[K] G. Kempf : On the geometry of a theorem of Riemann. Ann. of Math. (2) 98 (1973), 178–185.

[L] S. Lang : Abelian varieties over finite fields. Proc. Nat. Acad. Sci. U.S.A. 41 (1955), 174–176.

[LRZ] G. Lachaud, C. Ritzenthaler, A. Zykin : Jacobians among Abelian threefolds: a formula of Klein and a question of Serre. Preprint [arXiv:0802.4017] Math. Research Letters, to appear.

[M1] D. Mumford : On the equations defining abelian varieties I. Invent. Math. 1 (1966), 287–354.

[M2] D. Mumford : Prym Varieties I. Contributions to analysis, pp. 325–350. Academic Press, New York (1974).

[OU] F. Oort, K. Ueno : Principally polarized abelian varieties of dimension two or three are Jacobian varieties. J. Fac. Sci. Univ. Tokyo (IA) 20 (1973), 377–381.

[R] S. Recillas : Jacobians of curves with \( g_1 \)'s are the Prym's of trigonal curves. Bol. Soc. Mat. Mexicana (2) 19 (1974), no. 1, 9–13.

[S] J.-P. Serre : Appendix to Geometric methods for improving the upper bounds on the number of rational points on algebraic curves over finite fields, by K. Lauter. J. Algebraic Geom. 10 (2001), no. 1, 30–36.