Consideration of block structure in mathematical models of rock mass

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Abstract. For the media with periodic changes in Young’s modulus and yield strength, the problems of stress, strain and displacement distribution around single excavations of spherical and cylindrical shape are solved. Block structure of the medium is determined by the difference between the elastic modules and the yield strength in the blocks and interblock space. In each case analytical solutions are obtained. The influence of blocks quantity, difference in block properties and interlayers on the nature of changes in stresses, strains and displacements was studied. It is noted that block structure is one of the factors for forming zonal disintegration around excavations.

1. Introduction
The authors of earlier studies [1] noted that there are numerous issues related to the study of inhomogeneities and anisotropy of technological origin that are of special concern in rock mechanics. These specific characteristics of rock mass commonly result from underground mining operations producing the workings, and as the back-break effect around the blasting zone, etc. [2]. During the study of the mechanical properties and stress state of rocks near the contours of vertical shafts discussed in [3], which involved coring at a depth up to 10 m and testing the sampled rock specimens, with simultaneous measurement of stresses in wells, the authors established the peak-valley (“wavy”) pattern of the distribution of mechanical characteristics, due to the alternations of zones with elevated and lowered values of mechanical characteristics. Inhomogeneity of rocks has been addressed to in many research papers, for example in [4], where its study provided insights in the light of oil recovery process enhancement.

Along with the studies focusing on heterogeneity and anisotropy of the media, there are many works attempting to take into account the block structure of rock mass around the mine workings. These include mathematical models of rock mass structurally composed of parallelepiped-like deformable elements connected by “weakened” elastic material [5], while in other models the block medium is associated with a chain of coalesced masses connected by elastic springs and viscous elements [6-8]. In this paper, we propose to study the block model of a rock mass based on periodic functions, i.e., on the assumption that the block medium is nonuniform and is characterized by periodic changes in the elastic moduli, plasticity, elastic and yield strength, etc. Let us consider examples that illustrate this standpoint and the influence of block parameters on variations in the main characteristics of the stress-strain state, specifically stress, strain, and displacement.
2. Mathematical model and numerical results

Example 1. Suppose that a spherical mined out space has been created in a rock massif. It is assumed that the action of maximum shear stresses is a major mechanism maintaining the block structure of rock mass. Since the action of these stresses in the elastic environment is accompanied by displacements with a shear modulus $2\mu$, we assume that this modulus is a periodic function of the coordinate $r$ which is the distance from the workings’ center to the point of interest. Note that a similar assumption for variations in Young’s module $E$ was emphasized in [4]. A further objective is to estimate periodicity in the displacement module $2\mu$ with respect to the evolution of displacements in rock mass surrounding the spherical mine workings, using analytical solution that will be obtained below.

Solving the problem in a spherical coordinate system $r, \phi, x$ in the case of central symmetry is underpinned by: Cauchy relations

\[ \varepsilon_r = \frac{du_r}{dr}, \quad \varepsilon_\phi = \varepsilon_\phi = \frac{u_r}{r}, \quad \gamma_{r\phi} = \gamma_{r\phi} = \gamma_{r\phi} = 0, \quad (1) \]

equilibrium equations

\[ \sigma_r = \sigma_r, \quad \tau_{r\phi} = \tau_{r\phi} = \tau_{r\phi} = 0, \quad (2) \]

relations of Hooke’s law

\[ \varepsilon_r - \varepsilon_\phi = \frac{\sigma_r - \sigma_\phi}{2\mu}, \quad \varepsilon_r + 2\varepsilon_\phi = \frac{\sigma_r + 2\sigma_\phi}{3k}, \quad (3) \]

where $2\mu = E / (1 + \nu), \quad 3k = E / (1 - 2\nu)$.

Expressing from (3), (1) stresses $\sigma_r, \sigma_r - \sigma_\phi$ through displacements $u_r$, one obtains

\[ \sigma_r = \left( k + \frac{4}{3} \mu \right) \frac{du_r}{dr} + \left( 2k - \frac{4}{3} \mu \right) \frac{u_r}{r}, \quad \sigma_r - \sigma_\phi = 2\mu \left( \frac{du_r}{dr} - \frac{u_r}{r} \right). \quad (4) \]

Beginning from this point, we assume that the displacement modulus is a specified function $r$:

\[ 2\mu = 2\mu(r). \quad (5) \]

Next, substituting (4) under the condition (5) in (2), we find the equation for finding the displacement $u = u_r$:

\[ \frac{d^2u}{dr^2} + A(r) \left( \frac{du}{dr} - \frac{u_r}{r} \right) + 2 \left( \frac{1}{r \frac{dr}{dr} } \right) \frac{u_r}{r^2} = 0, \quad (6) \]

which is set to be

\[ F_r = 0, \quad A(r) = d \ln \left( k + \frac{4}{3} \mu \right) / dr. \quad (7) \]

Now we proceed to the following task of finding a general solution (6) under the conditions (7). Pass on from variable $r / r_0$, where $r_0$ is the radius of the mined-out space to a variable $\xi = \ln(r / r_0)$. Then from condition $r = r_0 e^{\xi}$ we obtain $\xi = 0$ for $r = r_0$. The expressions for derivatives that follow from this definition are:

\[ \frac{du}{dr} = \frac{du}{d\xi} \frac{d\xi}{dr} = \frac{1}{r} \frac{d^2u}{d\xi^2}, \quad \frac{d^2u}{dr^2} = \frac{1}{r^2} \frac{d^2u}{d\xi^2} - \frac{1}{r^2} \frac{du}{d\xi} \frac{d^2\xi}{d\xi^2}. \quad (8) \]

Upon substituting in (6), they give the following equation for finding the function $u$:
\[ \frac{1}{r^2} \left[ d^2 u \right] \left[ \frac{du}{d\xi} - \frac{du}{d\xi} \right] + \frac{1}{r^2} dA(\xi) \left[ \frac{du}{d\xi} - u \right] + \frac{2}{r^2} \left[ \frac{du}{d\xi} - u \right] = 0. \]  
\[ (9) \]

After multiplying (9) by \( r^2 \), we obtain
\[ \frac{d^2 u}{d\xi^2} - \frac{du}{d\xi} + \left( P(\xi) - \frac{du}{d\xi} - u \right) + 2 \frac{du}{d\xi} - 2u = 0, \]
from whence it follows that
\[ \frac{d^2 u}{d\xi^2} + \left( \frac{du}{d\xi} - 2u \right) + \left( P(\xi) - \frac{du}{d\xi} - u \right) = 0. \]
\[ (10) \]

The notation used here is: \( P(\xi) = \frac{d}{d\xi} \ln \left( k + \frac{4}{3} \mu \right) \).

For solving (11), an auxiliary function is introduced
\[ v = \frac{du}{d\xi} - u. \]
\[ (12) \]

Then, using this substitution, equation (11) is reduced to a first-order differential equation to determine \( v \). This will be shown below.

In fact, from (12) one finds sequentially
\[ \frac{dv}{d\xi} = \frac{dv}{d\xi} + \frac{du}{d\xi} = \frac{dv}{d\xi} + \frac{du}{d\xi}. \]
\[ (13) \]

Based on (13) from (11), one gets that
\[ \frac{dv}{d\xi} + \frac{dv}{d\xi} + \frac{du}{d\xi} - 2u + vP(\xi) = 0. \]
\[ (14) \]

Using the substitution (12) yet again, the equation for determining the function \( v \) is finally defined
\[ dv / v = -(P(\xi) + 2) d\xi. \]
\[ (15) \]

This is a first-order differential equation with separable variables. After determining the function \( v \) in (15) of (12), we find \( u \). We give the solutions to all these equations. Separating the variables in (15), one obtains
\[ dv / v = -(P(\xi) + 2) d\xi. \]
\[ (16) \]

By integrating (16) it is established that
\[ \ln \left[ \frac{v}{v_0} \right] = -2\xi - \ln \left[ k + \frac{4}{3} \mu \right] \left[ 1 + e^{\left( \frac{4}{3} \right) \xi} \right] \]
\[ (17) \]

Then
\[ v = v_0 e^{-2\xi} \left( k + \frac{4}{3} \mu \right) \left[ 1 + e^{\left( \frac{4}{3} \right) \xi} \right], \]
\[ (18) \]

where \( v_0 \) is the value of function \( v \) for \( \xi = 0 \) (i.e. at \( r = r_0 \)).

Let’s introduce the repeating pattern of periodic function \( \mu \):
\[ k + \frac{4}{3} \mu = \left( k + \frac{4}{3} \mu \right) \left( 1 + N \sin(2\pi n\xi) \right), \]
\[ (19) \]

where \( N \) is the parameter varying in the range from \(-1\) to \(+1\), i.e. \( |N| < 1 \), \( n \) is the number of “blocks” [9] in the interval of the coordinate \( \xi \) changing from 0 to 1.

Then
\[ v = v_0 e^{-3\xi} (1 + N \sin(2\pi n \xi))^{-1}. \]  \hspace{1cm} (20)

To determine \( u \), one needs to solve (12) under the condition (20). Considering (12), a general solution of the corresponding homogeneous equation is found, which is \( u = Ce^{\xi} \).

Next, the method of variation of an arbitrary constant is utilized, i.e. assuming \( C = C(\xi) \). Then for the calculation \( C(\xi) \) the differential equation is obtained
\[ C'(\xi) = \frac{v_0 e^{-3\xi}}{1 + N \sin(2\pi n \xi)}. \]  \hspace{1cm} (21)

By integrating it, one finds
\[ C(\xi) = v_0 \int_0^\xi \frac{e^{-3\xi}}{1 + N \sin(2\pi n \xi)} d\xi + u_0, \quad u = \left( v_0 \int_0^\xi \frac{e^{-3\xi}}{1 + N \sin(2\pi n \xi)} d\xi \right) e^{\xi} + u_0 e^{\xi}. \]  \hspace{1cm} (21)

To define the constants \( u_0, v_0 \) one uses the initial conditions: \( u|_{\xi=0} = u_0, \sigma_r|_{\xi=0} = 0 \). The substitution (21) in the expression for \( \sigma_r, (4) \) yields
\[ \sigma_r = \left( k + \frac{4}{3} \mu \right) \frac{1}{r} \frac{du_r}{d\xi} + \left( 2k - \frac{4}{3} \mu \right) \frac{u_r}{r} \bigg|_{r=r_0} = 0 \]
or
\[ \left( k + \frac{4}{3} \mu \right) (v_0 + u_0) + \left( 2k - \frac{4}{3} \mu \right) u_0 = 0. \]

Then \( v_0 = -\frac{3k}{k + \frac{4}{3} \mu} u_0 \). Thus, an analytical solution of the differential equation (6) with a variable shear modulus is obtained \( 2\mu \).

Figure 1 shows the dependence of the relative displacement \( u(\xi)/u_0 \) on the logarithmic coordinate \( \xi \), given that \( \nu = 0.4 \). Figure 2a shows the dependence of the radial stress \( r_0 \sigma_r / ku_0 \) on the logarithmic coordinate \( \xi \) at \( \nu = 0.4 \), while Figure 2b shows radial deformation from the logarithmic coordinate \( \xi \) at.

**Figure 1.** Dependence of the relative displacement \( u(\xi)/u_0 \) on the logarithmic coordinate \( \xi \): 1—\( n = 8 \ (\xi = 0–1), \ N = 0.9; 2—n = 5, \ N = 0.5; 3—at n = 1, \ N = 0.\)
Figure 2. Dependence of the radial stress \( r_0 \sigma_r / k u_0 \) on the logarithmic coordinate \( \xi \) (a) and radial deformation \( r_0 \varepsilon_r / k u_0 \) on the logarithmic coordinate \( \xi \) (b) at \( \nu = 0.4 : 1 \ n = 8 \ (\xi = 0–1), \ N = 0.5 ; 2—n = 5, \ N = 0.3 ; 3—n = 1, \ N = 0 \).

Example 2. Let’s consider the periodic change in the elastic strength in a rock mass. Let there be a rock mass with the workings of a cylindrical shape. For considering this case, we have: Cauchy relations

\[
\varepsilon_r = \frac{du}{dr}, \quad \varepsilon_\varphi = \frac{u}{r}, \quad \varepsilon_z = \frac{u}{r}, \quad \gamma_{r\varepsilon} = 0, \quad \gamma_{rz} = \gamma_{z\varphi} = 0, \quad (22)
\]

equilibrium equation

\[
\frac{d\sigma_r}{dr} + \frac{\sigma_r - \sigma_\varphi}{r} = 0, \quad (23)
\]

plasticity condition in the form of

\[
\tau_{\text{max}} = \frac{\sigma_r - \sigma_\varphi}{2} = \tau_S (1 + N \sin \lambda (r - r_0)), \quad (24)
\]

where \( \tau_S, \ \lambda, \ N \) are the constants, \( |N| < 1; \ r_0 \) is the radius of mine excavation.

To determine the stress, substitute (24) in (23). Then the equation is obtained written as

\[
\frac{d\sigma_r}{dr} = -\frac{2\tau_S}{r} - 2\tau_S N \sin \lambda (r - r_0). \quad (25)
\]

By integrating (25), one finds that

\[
\sigma_r = -2\tau_S \ln r - 2\tau_S N [\cos(\lambda r_0) Si(\lambda r) - \sin(\lambda r_0) Ci(\lambda r)] + C, \quad (26)
\]

where \( Si, \ Ci \) are integral sine and cosine, respectively.

It is required to evaluate the influence of parameters on the values of stresses, deformations, and displacements \( u \) around the mine workings with a radius \( r = r_0 \).

To define the constant \( C \) one uses the initial condition \( \sigma_r |_{r = r_0} = 0 \). Taking into consideration this condition

\[
\sigma_r = -2\tau_S \ln \frac{r}{r_0} - 2\tau_S N [\cos(\lambda r_0)(Si(\lambda r) - Si(\lambda r_0)) - \sin(\lambda r_0)(Ci(\lambda r) - Ci(\lambda r_0))], \quad (26)
\]

where \( Si(\lambda r) - Si(\lambda r_0) = \int_{\lambda r_0}^{\lambda r} \frac{\sin(\lambda r)}{\lambda r} d(\lambda r), \ Ci(\lambda r) - Ci(\lambda r_0) = \int_{\lambda r_0}^{\lambda r} \frac{\cos(\lambda r)}{\lambda r} d(\lambda r) \).
The stress $\sigma_\phi$ is defined by equation (24). Taking into account (26) one obtains

$$\sigma_\phi = \sigma_r - 2\tau_s - 2\tau_s N \sin(\lambda(r - r_0)).$$  \hfill (27)

To find the deformations $\varepsilon_r$ and $\varepsilon_\phi$, one needs to find the displacement $u$. To determine the displacement $u$, we have the condition of elastic volume change for a plane deformation in the form:

$$\varepsilon_r + \varepsilon_\phi = \frac{\sigma_r + \sigma_\phi}{2k},$$  \hfill (28)

where $2k = E / (1 + \nu)(1 - 2\nu)$, $E$ is Young’s module; $\nu$ is Poisson’s ratio.

By substituting (26), (27) in (28), the equation is obtained for $u$:

$$k \left( \frac{du}{dr} + \frac{u}{r} \right) = -2\tau_s \ln \frac{r}{r_0} - \tau_s N \sin \lambda(r - r_0) - 2\tau_s N [\cos(\lambda r_0)(S_i(\lambda r) - S_i(\lambda r_0)) - \sin(\lambda r_0)(C_i(\lambda r) - C_i(\lambda r_0))].$$  \hfill (29)

Equation (29) is integrated using the substitutions

$$ku = C / r,$$ \hfill (30)

where $C = C(r)$. Substituting (30) in (29) under the assumption $C = C(r)$, one gets the equation for the function $C$. Integrating it, one obtains

$$C = -\tau_s r^2 \ln \frac{r}{r_0} - \tau_s N r^2 \left[ \cos(\lambda r_0)(S_i(\lambda r) - S_i(\lambda r_0)) - \sin(\lambda r_0)(C_i(\lambda r) - C_i(\lambda r_0)) \right] + C_0$$ \hfill (31)

Then

$$ku = \frac{C_0}{r^2} - \tau_s r \ln \frac{r}{r_0} - \tau_s N r \left[ \cos(\lambda r_0)(S_i(\lambda r) - S_i(\lambda r_0)) - \sin(\lambda r_0)(C_i(\lambda r) - C_i(\lambda r_0)) \right]$$ \hfill (32)

Hence the deformations

$$k\varepsilon_\phi = \frac{ku}{r} = \frac{C_0}{r^2} - \tau_s r \ln \frac{r}{r_0} - \tau_s N \left[ \cos(\lambda r_0)(S_i(\lambda r) - S_i(\lambda r_0)) - \sin(\lambda r_0)(C_i(\lambda r) - C_i(\lambda r_0)) \right]$$ \hfill (33)

$$k\varepsilon_r = k \frac{du}{dr} = -\frac{C_0}{r^2} - \tau_s \left( \ln \frac{r}{r_0} + 1 \right) - \tau_s N r \sin \lambda(r - r_0).$$ \hfill (34)

The constant $C_0$ is derived from (32) under condition $u|_{r=r_0} = u_0$. Then $C_0 = ku_0 / r_0$. Thus, in this case, an analytical expression of the solution of the problem is obtained, which can be considered as one of the possible models of the block-structured medium. Figure 3a shows the dependencies of the relative displacement $ku(r) / \tau_s$ of the coordinate $r$; Figure 3b illustrates a radial deformation $r_0\varepsilon_r / ku_0$ of the coordinate $r$. 
Figure 3. Dependence of the relative displacement (a) and radial deformation (b) on the coordinate $r$ at $\nu = 0.3$: $1 - \nu = 0$; $2 - \nu = 0.3$; $3 - \nu = 0.7$; $4 - \nu = 0.9$.

Analysis of Figures 1–3 allows to infer that near the mine workings contour, there is an alternation of regions of tensile and compressive deformations, which may eventually serve as the cause of zonal disintegration of the rock mass around the mine workings.

3. Conclusions
The obtained analytical solutions concern the distribution of stresses, deformations, and displacements around the workings of spherical and cylindrical configuration with periodic changes in the shear modulus and yield strength. The block-structure effects on the mechanism of changes in the principal values of the stress–strain state are established. It is shown that the axial deformation near the workings contour is positive, which can entail the phenomenon of zonal disintegration.

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