In this article we provide a new finite class of elements in any Coxeter system \((W, S)\) called low elements. They are defined from Brink and Howlett’s small roots, which are strongly linked to the automatic structure of \((W, S)\). Our first main result is to show that they form a Garside shadow in \((W, S)\), i.e., they contain \(S\) and are closed under join (for the right weak order) and by taking suffixes. These low elements are the key to prove that all finitely generated Artin-Tits groups have a finite Garside family. This result was announced in a note with P. Dehornoy in *Comptes rendus mathématiques* [9] in which the present article was referred to under the following working title: *Monotonicity of dominance-depth on root systems and applications*.

The proof is based on a fundamental property enjoyed by small roots and which is our second main result; the set of small root is bipodal.

For a natural number \(n\), we define similarly \(n\)-low elements from \(n\)-small roots and conjecture that the set of \(n\)-small roots is bipodal, implying the set of \(n\)-low elements is a Garside shadow; we prove this conjecture for affine Coxeter groups and Coxeter groups whose graph is labelled by 3 and \(\infty\). To prove the latter, we extend the root poset on positive roots to a weak order on the root system and define a Bruhat order on the root system, and study the paths in those orders in order to establish a criterion to prove bipodality involving only finite dihedral reflection subgroups.

1. Introduction

In this article, we introduce and investigate the notion of a Garside shadow in a Coxeter system \((W, S)\): a Garside shadow in \((W, S)\) is a subset of \(W\) that contains \(S\) and closed under join (taken in the right weak order) and suffix. For instance \(W\) itself is a Garside shadow; see §2.2. The notion of Garside shadow is analogous to the notion of a Garside family in a monoid [7, 8].

We prove the existence of a finite Garside shadow in every Coxeter group by introducing the notion of a low element in \((W, S)\) and proving that the finite family of low elements is a Garside shadow.

An element in \(w \in W\) is low if its (left) inversion set \(N(w)\) — the set of positive roots that are sent to negative roots under \(w^{-1}\) — is the conic hull of some small roots; see §3 for a precise definition of these notions. Small roots were introduced by B. Brink and B. Howlett [6] in their work on the regularity of the language of...
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reduced words in \( W \), in which they show that the set of small roots is finite. We state now our first main result.

**Theorem 1.1.** For any Coxeter system \((W, S)\) with \( S \) finite, the set of low elements of \( W \) is a Garside shadow in \((W, S)\).

Since the intersection of a family of Garside shadows is a Garside shadow (c.f. Proposition 2.2), there is a smallest Garside shadow in \((W, S)\) and the next result follows by Theorem 1.1.

**Corollary 1.2.** For any Coxeter system \((W, S)\) with \( S \) finite, the smallest Garside shadow in \((W, S)\) is finite.

The initial motivation for the paper was a question by P. Dehornoy in the context of the associated finitely generated Artin-Tits group \( G \): *is there a finite Garside family in \( G \)?* Finite Garside families are important as they provide normal forms with nice properties and are potentially linked to the problem of decidability of the Word problem and the Conjugacy Problem in finitely generated Artin-Tits braid group; see [8, Questions 26 and 27] or [7] for more details. In particular, they provide an affirmative answer to the problem of decidability of the conjugacy problem in the Artin-Tits monoid. P. Dehornoy not only provided the initial translation of the problem in Artin-Tits braid groups into a problem about Coxeter groups, but he also provided many partial results, including examples in affine and right angled cases, which were essential for the work described here. Our Theorem 1.1 (and Corollary 1.2) answers Dehornoy’s question in the positive. Indeed, \( \sigma \) is the canonical lifting of a Coxeter group \( W \) into the associated Artin-Tits monoid \( M \), then \( A \) is a Garside shadow in \((W, S)\) if, and only if, \( \sigma(A) \) is a Garside family in \( M \); see [9, §3]. Therefore, the smallest Garside family is the copy in \( M \) of the smallest Garside shadow in \((W, S)\). Theorem 1.1, for which only a sketch of the proof was given in [9], appears as [9, Theorem 1.2].

The proof that the set of low elements is finite and closed under join is given in Proposition 3.26. The difficult part of the proof lies in the stability by taking suffixes of the set of low elements. In order to lift up this difficulty we proceed as follows in §4. On the one hand, we provide for a prefix \( w' \) of \( w \in W \) a complete description involving the Bruhat order of the rays of the cone over the inversion set of \( w' \) in function of the rays of the inversion set of \( w \); see §4.3. On the other hand, we describe a fundamental property enjoyed by small roots and which is our second main result; the set of small root is *bipodal*, see Theorem 4.18. We conclude then the proof of Theorem 1.1 in §4.5.

Along the way, in §3.3, we discuss the question of the existence of an infinite filtration of \( W \) by finite Garside subsets of \( W \) called the \( n \)-low elements, which would provide, using the dictionary in [9], an infinite filtration of bounded Garside families in Artin-Tits groups. More precisely, the set of positive roots may be ranked by the *dominance order*, which was introduced by B. Brink and B. Howlett together with small roots in [6]: for any positive root \( \beta \) there is a nonnegative integer \( n \in \mathbb{N} \) such that \( \beta \) strictly dominates exactly \( n \) positive roots; we call \( n \) the *dominance depth* of \( \beta \). Define the \( n \)-small roots to be those of dominance depth at most \( n \); see §3.2. With this definition, the small roots are precisely the 0-small roots. X. Fu [20] and the first author (unpublished, see [19]) show that the set of \( n \)-small roots is always finite for any \( n \in \mathbb{N} \). Extending the definition of low
elements, we call \textit{n-low elements} the elements in \( W \) whose left inversion sets are the conic hull of some \( n \)-small roots. We prove in Proposition \ref{prop:n-low-elements} that the set of \( n \)-low elements is finite and closed under join (taken in the right weak order). We conjecture that they are also stable under suffix.

\textbf{Conjecture 1.} If \( n \in \mathbb{N} \), the set of \( n \)-low elements is a finite Garside shadow in \((W, S)\).

This conjecture would be true if, for instance, the set of \( n \)-low elements is bipodal (Conjecture \ref{conj:bipodal}): this is discussed in \cite{2}. We show that Conjecture \ref{conj:finite} is true if \((W, S)\) is finite, is a dihedral Coxeter system, is an affine Coxeter system (Theorem \ref{thm:finite}), if \( n = 0 \) (Theorem \ref{thm:n=0}) or by the following last main result.

\textbf{Theorem 1.3.} Let \( n \in \mathbb{N} \) and assume \( S \) finite. Suppose that every entry of the Coxeter matrix of \((W, S)\) is either 1, 2, 3 or \( \infty \). Then the set of \( n \)-low elements of \( W \) is a finite Garside shadow in \((W, S)\).

This theorem is restated as Theorem \ref{thm:main} and \S 5 is dedicated to its proof. There, we extend the root poset on positive roots (see \cite[Chapter 4]{2}) to a weak order on the root system and define a Bruhat order on the root system, and study the paths in those orders in order to establish a criterion to prove bipodality involving only finite dihedral reflection subgroups of \((W, S)\) (Proposition \ref{prop:bipodal} and Corollary \ref{cor:bipodal}). The proof resolves a conjecture raised in \cite{19} that the set of \( n \)-small roots is \textit{balanced} for all \( n \), which may be phrased as a monotonicity property of the dominance depth for positive roots in any maximal dihedral reflection subgroup. We prove that this monotonicity property fails in general for finite dihedral reflection subgroups but is true in a stronger form for infinite dihedral reflection subgroups. The key part of the argument applies to a more general family of length functions on positive roots, with both the standard depth on positive roots and the dominance depth on roots as special cases, to show they are monotonic non-decreasing in the Bruhat order on roots.

2. Weak order and Garside shadows

Fix \((W, S)\) a Coxeter system with length function \( \ell : W \to \mathbb{N} \). The \textit{rank} of \( W \) is the cardinality of \( S \). The \textit{standard parabolic subgroup} \( W_I \) is the subgroup of \( W \) generated by \( I \subseteq S \). It is well-known that \((W_I, I)\) is itself a Coxeter system and that the length function \( \ell_I : W_I \to \mathbb{N} \) is the restriction of \( \ell \) to \( W_I \). Moreover, \( W_I \) is finite if and only if it contains a \textit{longest element}, which is then unique, denoted by \( w_{\alpha, I} \). In \cite{25} we discuss more general facts about reflection subgroups for which standard parabolic subgroups are an instance. We refer the reader to \cite{25, 2} for general definitions and properties of Coxeter groups.

2.1. Weak order and reduced words. We say that \( s_1 \ldots s_k \) \((s_i \in S)\) is a \textit{reduced word} for \( w \in W \) if \( w = s_1 \ldots s_k \) and \( k = \ell(w) \). For \( u, v, w \in W \), we adopt the following terminology:

- \( w = uv \) is \textit{reduced} if \( \ell(w) = \ell(u) + \ell(v) \);
- \( u \) is a \textit{prefix} of \( w \) if a reduced word for \( u \) is a prefix of a reduced word for \( w \);
- \( v \) is a \textit{suffix} of \( w \) if a reduced word for \( v \) is a suffix of a reduced word for \( w \).

Observe that if \( w = uv \) is reduced then the concatenation of any reduced word for \( u \) with any reduced word for \( v \) is a reduced word for \( w \); so in this case \( u \) is a prefix of \( w \) and \( v \) is a suffix of \( w \).
The (right) weak order is the order on \( W \) defined by \( u \leq_R v \) if \( u \) is a prefix of \( v \). Since we only consider the right weak order in this article, we only use from now on the term weak order. The weak order gives a natural orientation of the Cayley graph of \( (W, S) \): we orient an edge \( w \to ws \) if \( w \in W \) and \( s \in S \) such that \( w \leq_R ws \). Moreover, Björner [1, Theorem 8] shows that the poset \( (W, \leq_R) \) is a complete meet semilattice: for any \( A \subseteq W \), there exists an infimum \( \bigwedge A \in W \), also called the meet of \( A \), see [2] Chapter 3 for more details. A subset \( X \subseteq W \) is bounded in \( W \) if there is \( g \in W \) such that \( x \leq_R g \) for any \( x \in X \). Therefore any bounded subset \( X \subseteq W \) admits a least upper bound \( \bigvee X \) called the join of \( X \):

\[
\bigvee X = \bigwedge \{g \in W \mid x \leq_R g, \ \forall x \in X\}.
\]

The example of the infinite dihedral group is illustrated in Figure 1. When \( W \) is finite, any element \( w \in W \) is a prefix of the longest element \( w_0 \). So in this case \( W \) itself is bounded and \( (W, \leq_R) \) turns out to be a complete ortholattice, see for instance [2] Corollary 3.2.2.

2.2. Garside shadows. We are now able to discuss the notion of Garside shadows in a Coxeter system mentioned in the introduction.

**Definition 2.1.** A Garside shadow in \( (W, S) \) is a subset \( A \) of \( W \) containing \( S \) such that:

(i) \( A \) is closed under join in the weak order: if \( X \subseteq A \) is bounded, then \( \bigvee X \in A \);

(ii) \( A \) is closed under suffix: if \( w \in A \), then any suffix of \( w \) is also in \( A \).

The definition of a Garside shadow extends naturally to the standard parabolic subgroup \( W_I \) generated by \( I \subseteq S \): a Garside shadow \( A \) of \( (W_I, I) \) is a subset of \( W_I \) containing \( I \) and verifying Conditions (i)–(ii) above. Moreover, since a bounded \( X \) is necessarily finite, Condition (i) above is equivalent to the following condition:

(i') if \( u, v \in A \) and \( u \vee v \) exists then \( u \vee v \in A \).

**Proposition 2.2.** (1) The intersection of a family of Garside shadows in \( (W, S) \) is a Garside shadow in \( (W, S) \). In particular, for any \( X \subseteq W \) there is a smallest Garside shadow \( \text{Gar}_S(X) \) of \( (W, S) \) that contains \( X \).

(2) If \( I \subseteq S \) and \( X \subseteq W_I \), then \( \text{Gar}_I(X) \subseteq \text{Gar}_S(X) \).

(3) If \( I \subseteq S \) is spherical, that is \( W_I \) is finite, then \( \text{Gar}_I(I) = W_I \) is the unique Garside shadow in \( (W_I, I) \).

**Proof.** The two first statements follow easily from Definition 2.1. For the third statement, observe first that, since the longest element \( w_{o,I} \) of \( W_I \) is the unique maximal length element, \( w_{o,I} \) is the unique element of \( W_I \) that has all \( s \in I \) as prefix. So \( \bigvee I = w_{o,I} \) and therefore \( w_{o,I} \in \text{Gar}_I(I) \) by Condition (i). Moreover it is well-known that any element of \( W_I \) is prefix of \( w_{o,I} \). Hence \( W_I \subseteq \text{Gar}_I(I) \) by Condition (ii), which concludes the proof. \( \square \)

**Example 2.3.** Let \( W \) be the infinite dihedral group \( D_\infty \) generated by \( S = \{s, t\} \) with Coxeter graph:

\[
s \quad \infty \quad t
\]

Then \( S \) is not bounded, see Figure 1. So the join of \( s \) and \( t \) does not exist. Since the identity \( e \) is a suffix of \( s \) and \( t \) we have \( \hat{S} = \text{Gar}_S(S) = \{e, s, t\} \).
Example 2.4. Let $W$ be the universal Coxeter group generated by $S$; its Coxeter graph is the complete graph whose edges are labelled by $\infty$. Then any subset $\{s,t\} \subseteq S$ of cardinality 2 generates a standard dihedral parabolic subgroup as in Example 2.3 and is therefore not bounded. So $\tilde{S} = \text{Gar}_S(S) = S \cup \{e\}$.

Remark 2.5. (a) The smallest Garside shadow in $(W,S)$ is $\tilde{S} = \text{Gar}_S(S)$. The existence of a finite Garside shadow in $(W,S)$ stated in Theorem 1.1 implies that the smallest Garside shadow $\tilde{S}$ is finite, which is Corollary 1.2.

(b) We do not have a nice combinatorial characterization of the smallest Garside shadow in general. The main reason is the absence of a suitable combinatorial characterization of existence of the join in general. We give in Example 2.11 the construction of the smallest Garside shadow in the case of the affine Coxeter group of type $\tilde{A}_2$ by using a geometric characterization of the existence of the join.

(c) Let $u,v \in W_I$, where $W_I$ is the standard parabolic subgroup generated by $I \subseteq S$. It is well-known that any reduced words for $u$ or $v$ have all their letters in $I$. So any suffix of $u$ or $v$ is again an element of $W_I$. Moreover, it is not difficult to see, with the help of Proposition 2.8 below for instance, that the join $u \lor v$ is therefore an element of $W_I$. We deduce then that if $F$ is a Garside set in $(W,S)$, then $B \cap W_I$ is a Garside set in $(W_I,I)$.

We do not know however if the notion of Garside shadows closure is stable by restriction: let $I \subseteq S$ and $X \subseteq W_I$, is it true that $\text{Gar}_I(X) = \text{Gar}_S(X) \cap W_I$?

This question has an affirmative answer for standard parabolic subgroups of rank 2.

2.3. Geometric representation and root system. We recall here useful facts on root systems and reflection subgroups that will be needed to give an interpretation for the join in the weak order in 2.4.

A Coxeter system can be seen as a discrete reflection subgroup in some quadratic space $(V,B)$, where $V$ is a real vector space endowed with a symmetric bilinear form $B$. The group of linear maps that preserves $B$ is denoted by $O_B(V)$. The isotropic cone of $(V,B)$ is $Q = \{v \in V \mid B(v,v) = 0\}$. To any non-isotropic vector $\alpha \in V \setminus Q$, we associate the $B$-reflection $s_\alpha \in O_B(V)$ defined by $s_\alpha(v) = v - 2\frac{B(\alpha,v)}{B(\alpha,\alpha)}\alpha$.

Fix a geometric representation of $(W,S)$, i.e., a faithful representation of $W$ as a subgroup of $O_B(V)$ such that $S$ is mapped into a set of $B$-reflections associated to a simple system $\Delta = \{\alpha_s \mid s \in S\} (s = s_{\alpha_s})$. Recall that a simple system in $(V,B)$ is a finite subset $\Delta$ in $V$ such that:

(i) $\Delta$ is positively linearly independent: if $\sum_{\alpha \in \Delta} a_\alpha \alpha$ with $a_\alpha \geq 0$, then all $a_\alpha = 0$;

(ii) for all $\alpha, \beta \in \Delta$ distinct, $B(\alpha,\beta) \in ]-\infty,-1[ \cup \{-\cos\left(\frac{\pi}{k}\right), \, k \in \mathbb{N}_{\geq 2}\}$;

(iii) for all $\alpha \in \Delta$, $B(\alpha,\alpha) = 1$.

Note that, since $\Delta$ is positively linearly independant, the cone $\text{cone}(\Delta)$ is pointed: $\text{cone}(\Delta) \cap \text{cone}(-\Delta) = \{0\}$ (here $\text{cone}(A)$ is the set of non-negative linear combinations of vectors in $A$). Note also that if the order $m_{st}$ of $st$ is finite, then $B(\alpha_s,\alpha_t) = -\cos\left(\frac{\pi}{m_{st}}\right)$ and that $B(\alpha_s,\alpha_t) \leq -1$ if and only if the order of $st$ is infinite.

Denote by $\Phi = W(\Delta)$ the corresponding root system with base $\Delta$, which is partitioned into positive roots $\Phi^+ = \text{cone}(\Delta) \cap \Phi$ and negative roots $\Phi^- = -\Phi^+$. 


The pair $(\Phi, \Delta)$ is called a based root system. The rank of $(\Phi, \Delta)$ is the rank of $(W, S)$, i.e., $|\Delta| = |S|$. The classical geometric representation is obtained by assuming that $\Delta$ is a basis of $V$ and that $B(\alpha_s, \alpha_t) = -1$ if the order of $st$ in $W$ is infinite. For more details on geometric representations, the reader may, for instance, consult [24, §1].

There is a useful statistic on the positive root system $\Phi^+$ called the depth defined as follows:

$$\text{dp}(\beta) = \min \{ \ell(w) \mid w \in W, w(\beta) \in \Phi^- \}, \quad \beta \in \Phi^+.$$  

The next proposition is well-known, see for instance [2 Lemma 4.6.2].

**Proposition 2.6.** Let $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$. We have $\text{dp}(\alpha_s) = 1$ and

$$\text{dp}(s(\beta)) = \begin{cases} 
\text{dp}(\beta) - 1 & \text{if } B(\alpha_s, \beta) > 0 \\
\text{dp}(\beta) & \text{if } B(\alpha_s, \beta) = 0 \\
\text{dp}(\beta) + 1 & \text{if } B(\alpha_s, \beta) < 0 
\end{cases}$$

In particular, $\text{dp}(\alpha) = 1$ for any simple root $\alpha \in \Delta$.

**Remark 2.7.** In order to present examples of rank $|S| = 2, 3, 4$ easily, it is useful to consider the projective geometric representation of $W$, as in [24, 12]. Since $\Phi = \Phi^+ \sqcup \Phi^-$ is encoded by the set of positive roots $\Phi^+$, we represent $\Phi$ by an ‘affine cut’ $\tilde{\Phi}$: there is an affine hyperplane $V_1$ in $V$ transverse to $\Phi^+$, i.e., for any $\beta \in \Phi^+$, the ray $\mathbb{R}^+ \beta$ intersects $V_1$ in a unique nonzero point $\tilde{\beta}$. So $\mathbb{R} \beta \cap V_1 = \{ \tilde{\beta} \}$ for any $\beta \in \Phi$. The set of normalized roots $\hat{\Phi} = \{ \tilde{\beta} \mid \beta \in \Phi \}$ is contained in the compact set $\text{conv}(\Delta)$ and therefore admits a set $E$ of accumulation points called the set of limit roots. We have $E = \emptyset$ if and only if $W$ is finite; $E$ is a singleton if $(W, S)$ is affine and irreducible. Moreover, limit roots are in the isotropic cone $Q$ of $B$:

$$E \subseteq \hat{Q} = \{ x \in V_1 \mid B(x, x) = 0 \}.$$  

The group $W$ acts on $\hat{\Phi} \sqcup E \cup \text{conv}(E)$ componentwise: $w \cdot x = \widehat{w(x)}$. We refer the reader to [24, 12] for more details.

2.4. **Weak order and inversion sets.** In this text, we make use of the interplay between reduced words and their geometric counterparts: inversion sets.

The (left) inversion set $N(w)$ of $w \in W$ is defined by:

$$N(w) = \Phi^+ \cap w(\Phi^-) = \{ \beta \in \Phi^+ \mid \ell(s_\beta w) < \ell(w) \}.$$  

Its cardinality is well-known to be $\ell(w)$. Inversion sets allow a useful geometric interpretation of the weak order. For $A \subseteq V$ we denote $\text{cone}_\Phi(A) = \text{cone}(A) \cap \Phi$ the set of roots in $\text{cone}(A)$.

**Proposition 2.8.**

1. If $w = s_1 \ldots s_k$ is a reduced word for $w \in W$, then

$$N(w) = \{ \alpha_{s_1}, s_1(\alpha_{s_2}), \ldots, s_1 \ldots s_{k-1}(\alpha_{s_k}) \}.$$  

2. For $u, v \in W$ we have $u \leq_R v$ if and only if $N(u) \subseteq N(v)$.

3. The map $N : (W, \leq_R) \rightarrow (\mathcal{P}(\Phi^+), \subseteq)$ is a poset monomorphism.

4. If $X \subseteq W$ is bounded, then

$$N \left( \bigvee X \right) = \text{cone}_\Phi \left( \bigcup_{x \in X} N(x) \right).$$
An illustration of this proposition is given in Figure 1. The three first statements are classical and can be found in [2, Chapter 3] stated within the language of reflections instead of roots. The last statement follows from Dyer [17]; a complete proof may also be found in [23, §2.2-§2.3].

Figure 1. The weak order on the infinite dihedral group $D_\infty$: the vertices are labelled by $w \in D_\infty$ together with its inversion set $N(w)$.

**Remark 2.9.** Within the projective representation, see Remark 2.7, conic closure is replaced by convex hull and $A \subseteq \Phi^+$ is replaced by $\hat{A} \subseteq \hat{\Phi}$. So Proposition 2.8(d) translates as follows (see Figure 2 for an illustration): let $X$ be a bounded subset of $W$ and write $\tilde{N}(x)$ instead of $N(x)$. Then the join $\bigvee X$ exists and $\tilde{N}(\bigvee X) = \text{conv}\Phi(\tilde{N}(X))$. Moreover, in [23, Theorem 3.2], the authors show that $X \subseteq W$ is bounded if and only if

$$\text{conv} \left( \bigcup_{x \in X} \tilde{N}(x) \right) \cap \text{conv}(E) = \emptyset.$$ 

This is well illustrated in the case of the infinite dihedral group $D_\infty$ as in Example 2.3, in this case $E$ is always in the convex hull of $\hat{\Delta} = \{\alpha_s, \alpha_t\}$ and therefore $S = \{s, t\}$ is not bounded. A more general example of application is given in Example 2.11 to justify the construction of the smallest Garside shadow in the case $\tilde{A}_2$. This point of view provided the intuition behind the definition of low elements.

As a direct consequence, we obtain the following geometric interpretation of our terminology on reduced words.

**Corollary 2.10.** For $u, v, w \in W$, we have:

- $w = uv$ is reduced if and only if $N(w) = N(u) \cup u(N(v))$;
- $u$ is a prefix of $w$ if and only if $N(u) \subseteq N(w)$;
- $v$ is a suffix of $w$ if and only if $N(v) = v w^{-1} (N(w) \setminus N(wv^{-1}))$.

In particular, $v$ is a maximal proper suffix of $w$ (i.e., $w = sv$ is reduced with $s \in S$) if and only if $N(v) = s(N(w) \setminus \{\alpha_s\})$.

**Example 2.11** (The smallest Garside shadow in the affine group of type $\tilde{A}_2$). We show now how to obtain the smallest Garside shadow in the case $\tilde{A}_2$. In this
example, the difficulty is to show the existence or not of the join of a subset $X$ of $W$, i.e., to decide if $X$ is bounded or not in $W$, before taking suffixes. However, in the proof of Theorem 1.1 the difficulty lies in the stability by taking suffixes.

Let $(W, S)$ be the affine Coxeter system of type $\tilde{A}_2$ with Coxeter graph

```
3
1
2
```

Our aim is to show that

$$\tilde{S} = \text{Gar}_S(S) = \{e, 1, 2, 3, 12, 21, 13, 31, 23, 32, 121, 131, 232, 123, 213, 312\}.$$  

We build $\tilde{S}$ recursively:

- **Initial step:** $\tilde{S}_0 = S \cup \{e\}$;
- **Inductive step:** If $m$ is in $\mathbb{N}$, then $\tilde{S}_{m+1}$ is constituted of all the joins of bounded pairs of $\tilde{S}_m$ together with their suffixes;
- **Final step:** $\tilde{S} = \bigcup_{m \in \mathbb{N}} \tilde{S}_m$. Actually, Corollary 1.2 shows that this union is finite.

Start with $\tilde{S}_0 = \{e, 1, 2, 3\}$. The proper maximal standard parabolic subgroups of $W$ are all finite of rank 2 with Coxeter graph the edges of the above graph. So the join of $I = \{1, 2\}$ exists and $1 \vee 2 = 121 = 212$, by Proposition 2.2. By taking all the suffixes we obtain therefore $\text{Gar}_{(1, 2)}(\{1, 2\}) = W_I = \{e, 1, 2, 12, 21, 121\}$. Repeating the same argument for all other maximal standard parabolic subgroups we obtain:

$$\tilde{S}_1 = \{e, 1, 2, 3, 12, 21, 13, 31, 23, 32, 121, 131, 232\}.$$  

Now to construct $\tilde{S}_2$ we consider join of pairs of elements in $W$ that are not contained in a proper standard parabolic subgroup. For instance

$$21 \vee 23 = 21.31 = 23.13$$

is the smallest word having both 21 and 23 as prefixes. We obtain similarly

$$12 \vee 13 = 12.32 = 13.23 \quad \text{and} \quad 31 \vee 32 = 31.21 = 32.12.$$  

We claim that

$$\tilde{S} = \tilde{S}_2 = \{e, 1, 2, 3, 12, 21, 13, 31, 23, 32, 121, 131, 232, 123, 213, 312\}.$$  

It is easy to check that $\tilde{S}_2$ is closed under suffixes. In order to prove that there is no other element of $W$ obtained by joining elements of $\tilde{S}_1$, we consider the geometric interpretation of the weak order as described in Remark 2.9. Let $V$ be a real vector space of dimension 3 with bilinear form $B$ and basis $\Delta = \{\alpha_1, \alpha_2, \alpha_3\}$ such that for $1 \leq i, j \leq 3$ we have $B(\alpha_i, \alpha_j) = -\cos \left(\frac{\pi}{m_{ij}}\right)$, where $m_{ij}$ is the order of the element $ij$ in $W$. So $\Phi = W(\Delta)$ is a root system with simple system $\Delta$ for $(W, S)$ of type $\tilde{A}_2$. In Figure 2 one finds the first few normalized roots in blue and the unique limit root $\delta = \alpha_1 + \alpha_2 + \alpha_3$ is the red dot and then $E = \tilde{Q} = \{\tilde{\delta}\}$. From Remark 2.9 we know that $u \vee v$ exists if and only if $\text{conv}(\tilde{N}(u) \cup \tilde{N}(v)) \cap E = \emptyset$, and that in this case $\tilde{N}(u \vee v) = \text{conv}_\Phi(\tilde{N}(u) \cup \tilde{N}(v))$. For instance consider $N(31) = \{\alpha_3, \alpha_1 + \alpha_3\}$ and $N(32) = \{\alpha_3, \alpha_2 + \alpha_3\}$. Then

$$\text{conv}(\tilde{N}(31) \cup \tilde{N}(32)) = \text{conv}(\tilde{\alpha}_3, \tilde{\alpha}_1 + \tilde{\alpha}_3, \tilde{\alpha}_2 + \tilde{\alpha}_3)$$
does not intersect $E$ and we have:

\[
\hat{N}(31 \vee 32) = \text{conv}_\Phi(\hat{N}(31) \cup \hat{N}(32)) = \{\hat{\alpha}_3, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3, \hat{\beta}\} = \hat{N}(3121),
\]

where $\beta = s_{\alpha_3}(\alpha_1 + \alpha_2) = \alpha_1 + \alpha_2 + 2\alpha_3$.

But $\text{conv}(\hat{N}(1) \cup \hat{N}(32)) = \text{conv}(\hat{\alpha}_1, \hat{\alpha}_3, \alpha_2 + \alpha_3)$ does not intersect $E$, since the segment $[\hat{\alpha}_1, \alpha_2 + \alpha_3]$ contains $E$. Therefore $1 \vee 32$ does not exist. This is the argument that shows our claim: $\hat{S} = \hat{S}_2$. Indeed, let $u, v \in \hat{S}_1$ be such that $\{u, v\}$ is not contained in a proper standard parabolic subgroup and is different from $\{31, 32\}$, $\{12, 13\}$ and $\{21, 23\}$. Without loss of generality, we consider $u = 1u'$ reduced and $v = 32v'$ reduced. So $E \subseteq [\hat{\alpha}_1, \alpha_2 + \alpha_3] \subseteq \text{conv}(\hat{N}(u) \cup \hat{N}(v))$ and therefore $u \vee v$ does not exists. So no more elements are added to $\hat{S}_2$ than the ones we have already found.

\[\text{Figure 2. The inversion sets of } 31, 32 \text{ and } 3121 = 31 \vee 32 \text{ pictured in the projective representation (see Remark 2.7 and Remark 2.9) of the affine Coxeter group of type } \tilde{A}_2. \text{ Each blue point represents a normalized root } \hat{\gamma} \text{ which we simply label by the corresponding root } \gamma. \text{ The red dot represents the unique limit root } \delta.\]

2.5. Reflection subgroups. We end this section by recalling some useful facts about reflection subgroups that will be needed to give an interpretation of suffixes of a word in $\S\S 4.3$.

A reflection subgroup $W'$ of $W$ is a subgroup $W' = \langle s_\beta \mid \beta \in A \rangle$ generated by the reflections associated to the roots in some $A \subseteq \Phi^+$. Write

\[
\Phi_{W'} := \{\beta \in \Phi \mid s_\beta \in W'\} \quad \text{and} \quad \Delta_{W'} := \{\alpha \in \Phi^+ \mid N(s_\alpha) \cap \Phi_{W'} = \{\alpha\}\}.
\]

Then the first author shows in [13] that $\Phi_{W'}$ is a root system in $(V, B)$ with simple system $\Delta_{W'}$, called the canonical simple system of $\Phi_{W'}$. Therefore the pair $(W', S')$
is a Coxeter system, with canonical simple reflections

\[ S' = \chi(W') := \{ s_\alpha \mid \alpha \in \Delta W' \}, \]

and with corresponding positive roots \( \Phi^+_W, = \Phi_W \cap \Phi^+; \) see also \[10\] (both notions depend on \((W, S)\) and not just \(W\)).

**Remark 2.12.** Other characterizations of the canonical simple system \( \Delta_{W'} \) are that it is the unique inclusion-minimal subset \( \Gamma \) of \( \Phi^+_W \) such that \( \Phi^+_W \subseteq \text{cone}(\Gamma) \), and it is the set consisting of all representatives in \( \Phi \) of extreme rays of cone(\( \Phi^+_W \)) (see \[10\]).

Various notions attached above and below to the root system \((\Phi, \Delta)\) may be applied to the based root system \((\Phi_W, \Pi_W)\), and will be denoted by attaching \(W'\) as decoration (usually, as subscript) on the corresponding notation used for \((\Phi, \Pi)\). For example, \(\ell_{W'} : W' \to \mathbb{N}\) is the length function of \((W', S')\), and \(N_{W'}(u) := \Phi^+_W \cap u(\Phi^+_W) = N(u) \cap \Phi_{W'}\) for \(u \in W'\) where \(\ell_{W'}(u) = |N_{W'}u|\).

**Example 2.13.** Let \(\alpha, \beta \in \Phi^+\) such that \(B(\alpha, \beta) \leq -1\). Then \(\alpha \neq \beta\) and \(\Delta_{W'} = \{\alpha, \beta\}\) verifies the condition to be a simple system as in \[2.3\]. Let \(W' := \langle s_\alpha, s_\beta\rangle\) the dihedral reflection subgroup associated to \(\Delta'\) and note that \(W'\) is infinite.

A useful observation in the context of normalized roots, as in Remark \[2.7\] is that for any roots \(\alpha, \beta \in \Phi^+\), the dihedral reflection subgroup generated by \(s_\alpha, s_\beta\) is finite if and only if the line \((\hat{\alpha}, \hat{\beta}) \cap Q = \emptyset\). Otherwise the line \((\hat{\alpha}, \hat{\beta})\) intersects \(Q\) in one or two points and contains an infinite number of normalized roots, see Figure \[3\] for an illustration. We refer the reader to \[24,12\] for more details.

![Image of a diagram](image)

**Figure 3.** (\[12\] Figure 7) Projective representation of a infinite dihedral reflection subgroup \(W'\). In green are the two limit points of the root subsystem associated to \(W'\) together with the action of \(W'\).

**Example 2.14.** (Maximal dihedral reflection subgroups). A maximal rank 2 root subsystem of \(\Phi\) is a set \(\Phi'\) of the form \(\Phi' = P \cap \Phi\) where \(P\) is a plane in \(V\) intersecting \(\Phi^+\) in at least two roots. The cone spanned by \(\Phi' \cap \Phi^+\) has then a basis \(\Delta' = \{\alpha, \beta\}\) of cardinality 2 included in \(\Phi' \cap \Phi^+\), and then one has

\[ \Phi' = P \cap \Phi^+ = \text{cone}(\Delta') \cap \Phi. \]

One can show that \((\Phi', \Delta')\) is a based root subsystem of \((\Phi, \Delta)\) of rank 2. Write \(S' = \{s_\alpha, s_\beta\}\), then the dihedral reflection subgroup \(W' = \langle S'\rangle\) is a maximal dihedral reflection subgroup.

Any dihedral reflection subgroup \(W''\) of \(W\) is contained in such a maximal dihedral reflection subgroup since it has a rank 2 based root subsystem that is contained in such a maximal rank 2 root subsystem.

In the context of normalized roots, as in Remark \[2.7\] planes correspond to lines so maximal dihedral reflection subgroups corresponds to lines passing through two normalized roots. If \(P\) is such a line, then \(\hat{\Phi}' = P \cap \hat{\Phi}\) is contained in a segment.
Let $\hat{\alpha}, \hat{\beta}$ and $\hat{\Delta}' = \{\hat{\alpha}, \hat{\beta}\}$. For instance, in Figure 2 we see that $W' = \langle s_{\alpha_1 + \alpha_2}, s_{\alpha_3} \rangle$ is a (infinite) maximal dihedral reflection subgroup containing any dihedral reflection subgroup generated by two roots in the segment $[\alpha_1 + \alpha_2, \hat{\alpha}_3]$.

We state now the following sufficient condition for a reflection subgroup to be finite; this result is used in the proof of Lemma 3.21.

**Proposition 2.15.** Let $w \in W$.

1. If $W'$ is a reflection subgroup of $W$ with $\Delta_{W'} \subseteq N(w)$, then $W'$ is finite.
2. If $\alpha, \beta \in N(w)$, then $B(\alpha, \beta) > -1$.

**Proof.** To prove (1), note that Proposition 2.8(4) implies in particular that

$$\Phi_{W'} \subseteq \text{cone}_\Phi(\Delta_{W'}) \subseteq \text{cone}_\Phi(N(w)) = N(w).$$

Since $N(w)$ is finite, so is $\Phi_{W'}$. Therefore $\Phi_{W'}$ is finite. Since $W'$ acts faithfully as permutation group on $\Phi_{W'}$, $W'$ is finite.

To prove (2), assume for a contradiction that $\alpha, \beta \in \Phi$ with $B(\alpha, \beta) \leq -1$. Then $\alpha \not= \beta$ and $\Delta_{W'} = \{\alpha, \beta\}$ verifies the condition to be a simple system as in Proposition 2.3. Let $W' := \langle s_\alpha, s_\beta \rangle$ the dihedral reflection subgroup associated to $\Delta'$ and note that $W'$ is infinite. But $\Delta_{W'} \subseteq N(w)$ by assumption, contradicting (1). \hfill \square

We end this section by recalling the notion of shortest coset representatives of reflection subgroups. Let $W'$ be a reflection subgroup of $W$ with canonical generators $S'$ and based root subsystem $(\Phi_{W'}, \Delta_{W'})$ and set

$$X_{W'} := \{v \in W \mid \ell(s_\beta v) > \ell(v), \forall \beta \in \Phi_{W'}\} = \{v \in W \mid N(v) \cap \Phi_{W'} = \emptyset\}.$$ 

Any coset $W'w$ in $W$ contains an element of minimal length, which must be in $X_{W'}$. It is known (see [11, 3.3(ii),3.4]) that

**Proposition 2.16.** Let $W'$ be a reflection subgroup of $W$ and $w \in W$. Write $w = uw$ with $(u, v) \in W' \times X_{W'}$. We have:

1. $N(w) \cap \Phi_{W'} = N_{W'}(u)$;
2. $\ell(s_\beta w) < \ell(w)$ if and only if $\ell_{W'}(s_\beta u) < \ell_{W'}(u)$ for all $\beta \in \Phi_{W'}$.

This second statement of this last proposition, which will be used frequently in §§ 4 and 5 will be referred to as “functoriality of the Bruhat graph” (for inclusions of reflection subgroups; see [11,14]). It implies that the elements of $X_{W'}$ are precisely those elements $v$ of $W$ which are of minimum length in their coset $W'v$, and that any element $w \in W'$ can be uniquely expressed in the form $w = uv$ where $u \in W'$ and $v \in X_{W'}$. Accordingly, $X_{W'}$ is called the set of minimal length left coset representatives for $W'$ in $W$. Functoriality of the Bruhat graph also implies the following useful alternative characterization of $X_{W'}$:

$$X_{W'} = \{v \in W \mid \ell(s_\beta v) > \ell(v), \forall \beta \in \Delta_{W'}\}.$$ 

3. SMALL ROOTS AND LOW ELEMENTS

Let $(W, S)$ be a finitely generated Coxeter system together with a root system $\Phi$ and simple system $\Delta$ as in [2, 3]. The aim of the next two sections is to prove Theorem 1.1: the set of low elements is a finite Garside shadow in $(W, S)$. In order to do this we first review one important partial order on the set of positive roots $\Phi^+$ that lead to the definitions of small roots and low elements.
3.1. **Dominance order on roots.** Introduced by Brink and Howlett in [6], the *dominance order* is the partial order \( \preceq \) on \( \Phi^+ \) defined by:

\[
\alpha \preceq \beta \iff (\forall w \in W, \beta \in N(w) \implies \alpha \in N(w)).
\]

(we say in this case that \( \beta \) dominates \( \alpha \)). Related to the dominance order, there is a notion of dominance depth, called \( \infty \)-depth, of a positive root.

**Definition 3.1.** Let \( \beta \in \Phi^+ \) and \( n \in \mathbb{N} \).

(i) The dominance set of \( \beta \) is \( \text{Dom}(\beta) = \{\alpha \in \Phi^+ \mid \alpha \prec \beta\} \), the set of positive roots strictly dominated by \( \beta \).

(ii) The \( \infty \)-depth on \( \Phi^+ \) is defined by \( \text{dp}_\infty(\beta) = |\text{Dom}(\beta)| \).

**Remark 3.2.** (a) In the definition of \( \text{Dom}(\beta) \), the inequality is strict: \( \beta \notin \text{Dom}(\beta) \).

(b) The dominance order together with the \( \infty \)-depth \( \text{dp}_\infty(\beta) \) should not be confused with the root poset [27 §4.6] defined by using the usual depth of a root \( \text{dp}(\beta) \) from [23, see [5] for further discussions on this subject.

The following result, which is analogous to Proposition 2.6, gives a recurrence formula for \( \infty \)-depth of roots; see Proposition 5.10 for a common generalization.

**Proposition 3.3.** Let \( s \in S \) and \( \beta \in \Phi^+ \) with \( \beta \neq \alpha_s \). We have \( \text{dp}_\infty(\alpha_s) = 0 \) and

\[
\text{dp}_\infty(s(\beta)) = \begin{cases} 
\text{dp}_\infty(\beta) - 1 & \text{if } B(\alpha_s, \beta) \geq 1 \\
\text{dp}_\infty(\beta) & \text{if } B(\alpha_s, \beta) \in ]1,1[
\text{dp}_\infty(\beta) + 1 & \text{if } B(\alpha_s, \beta) \leq -1.
\end{cases}
\]

In particular, \( \text{dp}_\infty(\alpha) = 0 \) for any simple root \( \alpha \in \Delta \).

**Proof.** The first statement follows from definition: if \( \alpha_s \), which is element of \( N(s) \), dominates a positive root \( \beta \), then \( \beta \in N(s) = \{\alpha_s\} \). The second statement is a restatement of [20, Proposition 3.14]. \( \square \)

The following lemma recollects useful properties of dominance order regarding restriction to root subsystem.

**Lemma 3.4.**

(1) For any reflection subgroup \( W' \) of \( W \), dominance order \( \prec_{W'} \) on \( \Phi^+_{W'} \) is the restriction to \( \Phi^+_{W'} \) of the dominance order \( \prec \) on \( \Phi \).

(2) If \( I \subseteq S \), then the \( \infty \)-depth on \( \Phi_I^- \) is the restriction of the \( \infty \)-depth on \( \Phi^+ \) to \( \Phi_I^- \).

**Proof.** Part (1) appears in [22]; see [21, Corollary 3.3(2)] for a more detailed proof. Let us now prove (2): since any reduced word for \( w \in W_I \) has its letters in \( I \), we have \( N_I(w) = N(w) \) for any \( w \in W_I \). So by definition of dominance for \( \beta \in \Phi_I^- \) and \( \alpha \in \Phi^+ : \alpha \preceq \beta \implies \alpha \in N(s_\beta) \subseteq \Phi_I^- \). So \( \text{Dom}(\beta) \subseteq \Phi_I^- \) if \( \beta \in \Phi_I^- \) and therefore the \( \infty \)-depth on \( \Phi_I^- \) is the restriction of the \( \infty \)-depth on \( \Phi^+ \) to \( \Phi_I^- \). \( \square \)

**Remark 3.5.** The \( \infty \)-depth has a nice geometric interpretation in the context of normalized roots (see Remarks 2.7 and 2.9). Following [12], we say that \( \beta \in \Phi \) is visible from \( \alpha \in \Phi \) looking at \( Q \), where \( \alpha \neq \beta \), if the segment \([\alpha, \beta]\) has empty intersection with \( Q \) and if the half-line \([\alpha, \beta]\) starting at \( \alpha \) and passing through \( \beta \) intersects \( Q \). Then \( \alpha \prec \beta \) if and only if \( \beta \) is visible from \( \alpha \) looking at \( Q \); see [12, Proposition 5.7]. For instance, in Figure 2, \( \beta \) is visible from \( \alpha_3 \) looking at the point \( Q \) (in red). For a normalized root \( \beta \in \Phi \), define the blind cone of \( \beta \) to be the cone \( B(\beta) \) pointed in \( \beta \) and constituted of the points \( a \) such that the line \((a, \beta)\) cuts
and such that the half-line \([\hat{\beta}, a]\) starting at \(\hat{\beta}\) and passing through \(a\) does not intersect \(\hat{Q}\). So

\[
\text{Dom}(\beta) = \{\alpha \in \Phi^+ \mid \hat{\alpha} \in Bl(\beta) \setminus \{\hat{\beta}\}\}.
\]

In particular \(\text{dp}_\infty(\beta)\) is the number of normalized roots in the blind cone of \(\beta\) without counting \(\beta\):

\[
\text{dp}_\infty(\beta) = |Bl(\beta) \cap \hat{\Phi} \setminus \{\hat{\beta}\}| = |Bl(\hat{\beta}) \cap \hat{\Phi}| - 1
\]

An example of the blind cone of a positive root \(\beta\) such that \(\text{dp}_\infty(\beta) = 2\) is given in Figure 3; note that in Figure 4 the blind cone starting at the root of \(\infty\)-depth 2 is a half-line passing through \(\gamma\).

Using Remark 3.5 one gives the following examples.

**Example 3.6.** If \(W\) is finite, then \(\text{dp}_\infty(\alpha) = 0\) for all \(\alpha \in \Phi^+\). In particular any positive root dominates only itself.

**Example 3.7.** Assume, as in Example 2.3, that \(W\) is the infinite dihedral group generated by \(S = \{s, t\}\). Choose a geometric representation with \(B(\alpha_s, \alpha_t) \leq -1\), the associated projective representation is illustrated in Figure 3. Denote by \([s, t]_k\) the reduced word \(s \ldots s\) with \(k\) letters; this word ends with a \(s\) if \(k\) is odd and with a \(t\) if \(k\) is even. Denote for \(k \in \mathbb{N}\):

\[
\alpha_{s, k} = \begin{cases} 
[s, t]_k(\alpha_s) & \text{if } k \text{ is even.} \\
[s, t]_k(\alpha_t) & \text{if } k \text{ is odd.}
\end{cases}
\]

In particular \(\alpha_{s, 0} = \alpha_s\) and the \(\hat{\alpha}_{s, k}\) are the roots on the left-hand side of \(\hat{Q}\) in Figure 3. We have therefore

\[
\text{Dom}(\alpha_{s, k}) = \{\alpha \in \Phi^+ \mid \hat{\alpha} \in Bl(\alpha_{s, k}) \setminus \{\hat{\beta}\}\} = \{\alpha_{s, j} \mid 0 \leq j < k\}
\]

Hence \(\text{dp}_\infty(\alpha_{s, k}) = k\). By symmetry we can define \(\alpha_{t, k}\) using the words \([t, s]_k\). In particular the unique roots of \(\infty\)-depth equals to 0 are the simple roots \(\alpha_s\) and \(\alpha_t\).

**Example 3.8.** In Figure 4 we give the \(\infty\)-depth on the first roots for the affine Coxeter system of type \(\tilde{G}_2\).

**Example 3.9 (Affine Weyl groups).** We now give the example of an affine Coxeter system \((W, S)\), i.e., \(W\) is an affine Weyl group.

When working with an affine Weyl group, one usually uses its crystallographic root system, which we recall now. Let \(\Psi_0\) be a reduced, irreducible, crystallographic root system of a finite Weyl group \(W_0\), in a real vector space \(V_0\), with \(W_0\)-invariant positive definite scalar product \(\langle - | - \rangle\) (see [25]). Choose a positive system \(\Psi_0^+\) with simple roots \(\Delta_0\). Form a new vector space \(V\) with \(V_0\) as a codimension 1 subspace, say \(V = V_0 \oplus \mathbb{R} \delta\) and extend \(\langle - | - \rangle\) to a symmetric invariant bilinear form \(B\) on \(V\) with radical \(\mathbb{R} \delta\). In particular the isotropic cone is \(Q = \mathbb{R} \delta\). Set

\[
\Psi := \{\mu + k\delta \mid \mu \in \Psi_0, k \in \mathbb{Z}\}, \quad \Psi^+ = \{\mu + k\delta \in \Psi \mid \mu \in \Psi_0, k \geq 1 \text{ if } \mu \in \Psi_0^+\}
\]

and \(\Pi = \Delta_0 \cup \{\delta - \omega\}\) where \(\omega\) is the highest root of \(\Psi_0\). It is well known that \(\Psi\) is a standard crystallographic root system of the affine Weyl group \(W\) corresponding to \(W_0\), with \(\Psi^+\) as positive system and \(\Delta\) as corresponding simple roots (see e.g. [26] or [18]). For instance:
Figure 4. The normalized isotropic cone $\hat{Q} = \{\hat{\delta}\}$ in red, the small roots (with $\infty$-depth 0) and the $\infty$-depth on some other roots for the normalized root system of affine type $\tilde{G}_2$. The dotted red line forms the blind cone (c.f. Remark 3.5) of the root of depth 2 it emanates from.

- in Example 2.11, $W$ is the affine Weyl group of type $\tilde{A}_2$: $W_0 = S_3$, which is a finite Weyl group of type $A_2$, with $\Delta_0 = \{\alpha_1, \alpha_2\}$. Then the highest root is $\alpha_1 + \alpha_2$ and $\alpha_3 = \delta - \alpha_1 - \alpha_2$.
- in Example 3.8, $W$ is the affine Weyl group of type $\tilde{G}_2$: $W_0 = D_6$, which is a finite Weyl group of type $G_2$, with $\Delta_0 = \{\alpha, \beta\}$ and we have two choices for a finite crystallographic root system $\Psi_0$, which we give us the same based root system. Choose $\Psi_0^+ = \{\alpha, \alpha + \beta, 2\alpha + 3\beta, \alpha + 2\beta, \alpha + 3\beta, \beta\}$; the highest root is $2\alpha + 3\beta$ and $\gamma = \delta - 2\alpha - 3\beta$.

The crystallographic root system $\Psi$ is easily converted to the root system of a based root system $(\Phi, \Delta)$ in $(V, B)$ by normalizing roots to have square length 1: since the elements of $\Psi$ are non-isotropic, i.e. $\Psi \cap R\delta = \emptyset$, we define $\alpha' := \frac{1}{B(\alpha, \alpha)^{\frac{1}{2}}} \alpha$ for any $\alpha \in \Psi$; then set

$$\Phi := \{\alpha' \mid \alpha \in \Psi\}, \quad \Delta := \{\alpha' \mid \alpha \in \Pi\} \quad \text{and} \quad \Phi^+ = \text{cone}_B(\Delta) = \{\beta' \mid \beta \in \Psi^+\}.$$ 

Note that the reflection $s_\alpha = s_{ku}$ for any $k \in \mathbb{R}^*$ and $u \in V \setminus R\delta$ so $s_{\alpha'} = s_\alpha \in W$ for $\alpha' \in \Phi$, i.e., $\alpha \in \Psi$. So we may extend the notion of $\infty$-depth and dominance to the crystallographic root system $\Psi$ by setting $\alpha \preceq \beta$ if $\alpha' \preceq \beta'$ and

$$dp_{\infty}(\alpha) := dp_{\infty}(\alpha'), \quad \text{for any } \alpha, \beta \in \Psi.$$ 

It is well known and easily seen that the positive root systems of infinite maximal dihedral reflection subgroups of $W$ (see Example 2.11) are precisely the sets

$$\{\mu + k\delta, -\mu + (k + 1)\delta \mid \mu \in \Psi_0^+, k \in \mathbb{N}\}.$$
Those correspond to all the sets of normalized roots on segments in Figure 2 and Figure 4 passing through the roots on the face corresponding to the roots in \( \Psi_0 \) and the red dot \( \delta \). It is clear by the definition that \( \Psi^+ \) is the union of all positive root systems of infinite maximal dihedral reflection subgroups of \( W \). The only dominances in \( \Psi \) are thus:

\[
\mu + k\delta \succeq \mu + l\delta, \quad \text{for } k \leq l \text{ in } \mathbb{Z} \text{ and } \mu \in \Psi_0.
\]

It follows that for \( \beta = \mu + k\delta \in \Psi^+ \) we have:

\[
(\Diamond) \quad dp_\infty(\beta) = dp_\infty(\mu + k\delta) = \begin{cases} 
  k & \text{if } \mu \in \Psi^+_0 \text{ and } k \in \mathbb{N} \\
  k - 1 & \text{if } \mu \in \Psi^-_0 \text{ and } k \in \mathbb{N}^*
\end{cases}
\]

where \( \mathbb{N}^* = \mathbb{N} \setminus \{0\} \). In other words, on the segments in Figure 2 and Figure 4 representing infinite maximal dihedral reflection subgroups, \( dp_\infty \) is strictly increasing from 0 to \( \infty \) starting from a root \( \mu \in \Psi^+_0 \) to \( \delta \) (the red dot), then decreasing from \( \delta \) to \( -\mu + \delta \).

**Example 3.10.** In Figure 5 we give the \( \infty \)-depth on the first roots for the affine Coxeter system of rank 3 whose graph is in the top right corner of the figure.

**Figure 5.** The normalized isotropic cone \( \hat{Q} \) in red, the small roots (with \( \infty \)-depth 0) and the \( \infty \)-depth on some other roots for the normalized root system associated to the Coxeter graph on the top right corner. The roots in the pointed shaded red cone form the blind cone (c.f. Remark 3.5) of the root of depth 2 it is pointed on.

### 3.2. Small roots and small inversion sets.

**Definition 3.11.** Let \( \beta \in \Phi^+ \) and \( n \in \mathbb{N} \).

(i) The positive root \( \beta \) is small\(^1\) if \( \beta \) dominates no other positive root than itself, i.e., \( dp_\infty(\beta) = 0 \).

(ii) The positive root \( \beta \) is \( n \)-small if \( dp_\infty(\beta) \leq n \).

\(^1\)These roots are also called humble or elementary in the literature. We adopt here the terminology of [2]. See [2] Notes, p.130 for more details.
The set of n-small roots is denoted by $\Sigma_n(W)$, or simply by $\Sigma_n$ if there is no possible confusion. We denote by $\Sigma = \Sigma_0$ the set of small roots in $\Phi$.

**Remark 3.12.** The collection $(\Sigma_n)_{n \in \mathbb{N}}$ is a filtration of $\Phi^+$: we have

$$\Phi^+ = \bigcup_{n \in \mathbb{N}} \Sigma_n$$

and $\Delta \subseteq \Sigma = \Sigma_0 \subseteq \Sigma_1 \subseteq \cdots \subseteq \Sigma_n, \forall n \in \mathbb{N}$.

**Proposition 3.13.** If $I \subseteq S$ and $n \in \mathbb{N}$, then $\Sigma_n(W_I) \subseteq \Sigma_n(W)$.

**Proof.** The result follows from Lemma 3.4.

The next proposition gives a useful characterization of small roots, see [6, 2].

**Proposition 3.14.** Let $\beta$ in $\Phi^+ \setminus \Delta$ and $\alpha$ in $\Delta$ such that $B(\alpha, \beta) > 0$, then $\beta \in \Sigma$ if and only if $s_\alpha(\beta)$ lies in $\Sigma$ and $B(\alpha, \beta) < 1$ holds; if and only if $dp(s_\alpha(\beta)) < dp(\beta)$.

The case $n = 0$ in the following theorem is due to Brink-Howlett [6, Theorem 2.8] whereas the general case is due to Fu [20, Corollary 3.9] and [21, Theorem 3.22] and Dyer (unpublished; see [19]).

**Theorem 3.15.** For all $n \in \mathbb{N}$, the set $\Sigma_n$ is finite and does not depend of the choice of the root system. In particular $\Sigma_0 = \Phi^+$ if and only if $W$ is finite.

**Example 3.16.** If $W$ is the infinite dihedral group, as in Examples 2.3 and 3.7, then $\Sigma = \Delta = \{\alpha_s, \alpha_t\}$. For similar reason, $\Sigma = \Delta$ if $W$ is a universal Coxeter group as in Example 2.4.

**Example 3.17.** Small roots are all represented in the cases illustrated in Figures 4 and 5. In the case of the affine Coxeter group of type $\tilde{A}_2$ as in Example 2.11 and Figure 2, the small roots are the roots corresponding to proper standard parabolic subgroups: $\Sigma = \{\alpha_1, \alpha_2, \alpha_3, \alpha_1 + \alpha_2, \alpha_1 + \alpha_3, \alpha_2 + \alpha_3\}$.

The case of small roots ($n = 0$) is at the heart of the work of Brink and Howlett [6] on the automatic structure of Coxeter systems: they provide, by the mean of small inversion sets, a finite state automaton that recognizes the language of reduced words. The notion of small inversion sets will be used in §3.3 and beyond.

**Definition 3.18.** Let $n \in \mathbb{N}$, the (left) $n$-small inversion set of $w \in W$ is

$$\Sigma_n(w) = N(w) \cap \Sigma_n.$$  

We denote by $\Lambda_n(W)$ (or simply $\Lambda_n$ if there is not possible confusion) the set of all (left) $n$-small inversion sets.

**Remark 3.19.** In [2, §4.8], the authors use right inversion sets: $N(w^{-1}) \cap \Sigma$.

Theorem 3.15 has the following interesting direct consequence:

**Corollary 3.20.** The set $\Lambda_n(W)$ is finite for all $n \in \mathbb{N}^*$.

We end this discussion on small descent sets with the following lemma and proposition. Note that we do not need these results in the rest of this article; but we state them anyway since they play an important role in relation to finite state automata associated to Coxeter groups (see [19] for a multi-parameter generalization). For $n = 0$, this goes back to [6], see also [2, §4.8].

**Lemma 3.21.** Let $w \in W$, $s \in S$ and $n \in \mathbb{N}$.

1. If $\ell(sw) > \ell(w)$, then $\Sigma_n(sw) = \{\alpha_s\} \cup (\Sigma_n \cap s(\Sigma_n(w)))$.  

Proof. Under the assumptions of (1), the word $sw$ is reduced and $\alpha_s \in \Delta \subseteq \Sigma_n$. So $N(w) = \{\alpha\} \cup s(N(w))$ by Corollary 2.10 which implies the right hand side of (1) is contained in the left hand side. To prove the reverse inclusion, it suffices to show that if $\beta \in \Sigma_n(sw) = N(sw) \cap \Sigma_n$ and $\beta \neq \alpha_s$, then $s(\beta) \in \Sigma_n$, since $s(\beta) \in N(w)$ from above. Assume for a contradiction that $dp_{\infty}(s(\beta)) \geq n+1$. Then Proposition 3.3 forces $B(\alpha_s, \beta) \leq -1$ since $dp_{\infty}(\beta) \leq n$. Since $\alpha_s, \beta \in N(sw)$, this contradicts Proposition 2.15.

The proof of (2) is similar but simpler. Under these assumptions, we have $\alpha_s \in \Delta \subseteq \Sigma_n$ and $w = s(sw)$ is reduced. So $N(sw) = s(N(w) \setminus \{\alpha_s\})$ by Corollary 2.10. This implies the right hand side of (2) is contained in the left hand side. To prove the reverse inclusion, it suffices to show that if $\beta \in \Sigma_n \setminus (sw)$, then $s(\beta) \in \Sigma_n$, since $s(\beta) \in N(w)$ from above and we cannot have $s(\beta) = \alpha$. The result then follows from Proposition 3.3. \hfill \Box

Proposition 3.22. For any $n \in \mathbb{N}$, the finite set $\Lambda_n$ is the inclusion-minimum subset of $\Sigma_n$ such that

(2) If $n \geq 1$ and $\ell(sw) < \ell(w)$, then $\Sigma_{n-1}(sw) = \Sigma_{n-1} \cap s(\Sigma_n(w))$. 

Proof. Suppose $A := \Sigma_n(w)$ for some $w \in W$. Then for $s \in S$, one has $\ell(sw) > \ell(w)$ if and only if $\alpha_s \notin N(w)$, so and only if $\alpha_s \notin A$. In that case $\Sigma_n(sw) = \{\alpha\} \cup (s(A) \cap \Sigma_n)$ which is completely determined by $A$ and $s$. The result is clear from this using Lemma 3.21. \hfill \Box

Remark 3.23. Following [2] p.119–120], we define for $n \in \mathbb{N}$ a $n$-canonical automaton that recognizes the language of reduced words: the set of states is $\Lambda_n(W)$; for each $A \in \Lambda_n(W)$ and $s \in S$ such that $\alpha_s \in \Delta \setminus A$ we put a transition:

$$A \xrightarrow{s} \{\alpha\} \cup (s(A) \cap \Sigma_n).$$

Note that the 0-canonical automaton is the canonical automaton described in [2] p.120 and in [19] §4.3. However if $n > 0$, the canonical automata defined and studied in [19] §4.3 are different than the one we define above.

3.3. Low elements. We are now ready to define low elements.

Definition 3.24. Let $n \in \mathbb{N}$. An element $w \in W$ is $n$-low if $N(w) = \text{cone}_\phi(A)$ for some $A \subseteq \Sigma_n$, or equivalently if $N(w) = \text{cone}_\phi(\Sigma_n(w))$. We denote by $L_n(W)$ the set of $n$-low elements in $W$.

A 0-low element is simply called a low element and $L_0(W)$ is denoted by $L(W)$.

Remark 3.25. The collection $(L_n(W))_{n \in \mathbb{N}}$ is a filtration of $W$: one has

$$W = \bigcup_{n \in \mathbb{N}} L_n(W), \text{ and } L_0(W) \subseteq L_1(W) \subseteq \ldots \subseteq L_n(W) \subseteq \ldots, \quad \forall n \in \mathbb{N}.$$

The following proposition shows part of Theorem 1.1.

Proposition 3.26. Let $n \in \mathbb{N}$.

(1) We have $S \cup \{\epsilon\} \subseteq L_n(W)$.

(2) The map $\Sigma_n : L_n(W) \rightarrow \Lambda_n(W)$ is injective.

(3) The set $L_n(W)$ is finite and closed under join, i.e., if $X \subseteq L_n(W)$ is bounded then $\vee X \in L_n(W)$. 

Proof. For the first statement: $e \in L_n(W)$ since $N(e) = \emptyset = \text{cone}_\Phi(\emptyset)$; moreover $N(s) = \{\alpha_s\} = \text{cone}_\Phi(\alpha_s)$ and $\alpha_s \in \Delta \subseteq \Sigma_0 \subseteq \Sigma_n$. For (2), let $w, w' \in L_n(W)$ such that $\Sigma_n(w) = \Sigma_n(w')$, then $N(w) = \text{cone}_\Phi(\Sigma_n(w)) = \text{cone}_\Phi(\Sigma_n(w)) = N(w')$.

By Proposition 2.8, $N$ is injective and so $w = w'$, which implies that $\Sigma$ is injective. It remains to show (3). Finiteness of $L_n(W)$ holds by (2) and Corollary 3.20. Now let $X \subseteq L_n(W)$ bounded. For each $x \in X$, we have by definition that $N(x) = \text{cone}_\Phi(\Sigma_n(x))$ and $\bigvee X$ exists. The fact that $L_n(W)$ is closed under join follows now from the definition and Proposition 2.8.

$$N \left( \bigvee X \right) = \text{cone}_\Phi \left( \bigcup_{x \in X} \text{cone}_\Phi(\Sigma_n(x)) \right) = \text{cone}_\Phi \left( \bigcup_{x \in X} \Sigma_n(x) \right),$$

and $\bigcup_{x \in X} \Sigma_n(x) \subseteq \Lambda_n$. So $\bigvee X \in L_n(W)$. \hfill \halmos

**Example 3.27.** If $W$ is the infinite dihedral group $\mathcal{D}_\infty$, as in Examples 2.3, 3.7 and 3.16, then $\Sigma = \Delta = \{\alpha_s, \alpha_t\}$. Therefore the conic hull of the two small roots contains $\Phi^+$ which is infinite in this case.

Hence $\text{cone}_\Phi(\Delta) = \Phi^+$ cannot be an inversion set. So $L(W) = S \cup \{e\}$.

**Example 3.28.** If $(W, S)$ is a universal Coxeter system, as in Examples 2.4 and 3.16, then $\Sigma = \Sigma_0 = \Delta$ and therefore $L(W) = S \cup \{e\}$. Indeed, the conic hull of two small roots is the positive root system for an infinite dihedral group, a standard parabolic subgroup of rank 2 as in Examples 3.16 and 3.27 hence we cannot have an inversion set that arises from the conic hull of more than one small root.

**Example 3.29.** In the case of the affine Coxeter group of type $\hat{A}_2$ as in Examples 2.11 and 3.17 and Figure 2, it is not difficult to see, using the same techniques as in Example 2.11, that

$$L(\hat{A}_2) = \{e, 1, 2, 3, 12, 21, 13, 31, 23, 32, 121, 131, 232, 1232, 2313, 3121\}.$$

In both these examples and more, the map $\Sigma$ is bijective, which leads us to state the following conjecture.

**Conjecture 2.** The map $\Sigma_n : L_n(W) \rightarrow \Lambda_n(W)$ is a bijection.

**Remark 3.30.** In the examples above, the set of low elements $L(W)$ is also the smallest Garside shadow $\tilde{S}$ from Remark 2.5. But it is not true in general. If $(W, S)$ is of affine type $\tilde{G}_2$ (see Figure 4), then $w = s_\alpha s_\gamma s_\beta$ is a low element since its inversion set is $N(w) = \text{cone}_\Phi(\alpha, \gamma, \nu)$, where $s_\alpha(\gamma) = \gamma$ and $\nu = s_\alpha s_\gamma(\beta)$; but is not in $\tilde{S}$ since we cannot obtain $w$ by join or suffix closure starting from $S$.

4. Low elements form a finite Garside shadow

The aim of this section is to finish the proof of Theorem 1.1. In regard to Proposition 3.26(3), we just have to show that the set $L(W)$ of low elements is closed under taking suffix. In order to do this, we first give a description of the rays of the cone over the inversion set of a suffix of $w \in W$ as a function of rays of $\text{cone}(N(w))$. Then we prove that the set of small roots is bipodal: if a small root is a positive and non-simple root of a maximal dihedral reflection subgroup $W'$, then the simple system $\Delta_{W'}$ is constituted of two small roots.
4.1. Bruhat order. The Bruhat order on $W$ is the partial order $\leq$ arising as the reflexive, transitive closure of the relation $\rightarrow$ on $W$ defined by $x \rightarrow y$ if there is $\beta \in \Phi^+$ such that $y = s_\beta x$ with $\ell(x) < \ell(y)$. We say that a pair $x \leq y$ is a covering in the Bruhat order, which we denote by $x \lessdot y$, if for any $z \in W$ such that $x \leq z \leq y$ then we have $z = x$ or $z = y$. The chain property of Bruhat order implies that $x \lessdot y$ is a covering if $x < y$ and $\ell(y) = \ell(x) + 1$, i.e., if there is $\beta \in \Phi^+$ such that $y = s_\beta x$ and $\ell(y) = \ell(x) + 1$. It is known that $u \leq_R v$ implies that $u \leq v$. We refer the reader to [2, Chapter 2] for more details. The following well-known proposition can be found for instance as [2, Proposition 2.2.7].

**Proposition 4.1** (Lifting property). Let $s \in S$ and $u < v$ in $W$ such that $sv < v$ and $u \lessdot sv$, then $u \leq sv$ and $su \leq sv$.

The next corollary is used in the proof of Theorem 4.10 below.

**Corollary 4.2.** Let $s \in S$, $x \in W$ and $\gamma \in \Phi^+$ such that $sx < x$ and $s_x < x$. Then either $\gamma = \alpha_s$ or $ss_x x < s_x x$.

**Proof.** Since $s \in S$ we have $\ell(ss_x x) = \ell(s_x x) \pm 1$. So either $ss_x x < s_x x$ and we are done, or $s_x x < ss_x x$. In this last case, we have $\ell(ss_x x) = \ell(s_x x) + 1 = \ell(x)$ since $s_x x < x$. Moreover Proposition 4.1 implies, with $u = s_x x$ and $v = x$, that $ss_x x < x$. Since $ss_x x$ and $x$ have the same length, we have $x = ss_x x$, forcing $s = s_x$, i.e., $\gamma = \alpha_s$. □

**Example 4.3.** Let $W$ be a dihedral group $D_m$ ($m \in \mathbb{N}_{\geq 2} \cup \{\infty\}$) generated by $S = \{s, t\}$ with Coxeter graph:

```
   s   m   t
```

We follow the notations introduced in Example 3.7 for $[s, t]_k$, $[t, s]_k$, $\alpha_{s, k}$ and $\alpha_{t, k}$ for $k \in \mathbb{N}$. We illustrate in Figure 6 the Hasse diagrams for the Bruhat order on dihedral groups with the edges representing the covering $x < y$ labelled by the root $\beta$ such that $y = s_\beta x$.

The above example of dihedral group leads to the following lemma, which will be used to prove Theorem 4.10 below (recall the definition of maximal dihedral reflection subgroup in Example 2.14).

**Lemma 4.4.** Let $s \in S$, $\beta \in \Phi^+$ and $x \in W$ such that $sx < x$. Denote $W' = \langle s, s_\beta \rangle$, a reflection subgroup with simple system $\Delta_{W'}$, and positive root system $\Phi^+_{W'}$. The following assertion are equivalent:

1. $ss_\beta x < s_\beta sx < sx < x$;
2. $W'$ is a maximal dihedral reflection subgroup, $\Delta_{W'} = \{\alpha_s, \beta\}$, $\ell(s(s_\beta x) = \ell(x) - 3$ and there is $\gamma \in \Phi^+_{W'} \setminus \{\alpha_s\}$ such that $s_x x < x$.

**Proof.** Assume (1) to be true. Note that $W'$ is of rank 2 since $\alpha_s \neq \beta$ by (1). Let $W''$ be the maximal dihedral reflection subgroup containing $W'$. Write $x = uv$ with $u \in W''$ and $v \in X_{W''}$ as in Proposition 2.16. Proposition 2.16(2) shows that the map $\psi_v : (W'', \leq_{W''}) \mapsto (W''v, \leq)$, $z \mapsto vz$, is an order preserving bijection. Therefore, even though the inverse map may not be order preserving, (1) forces $ss_\beta su \leq_{W''} s_\beta ssu \leq_{W''} su \leq_{W''} u$. 
But since the graph of the Bruhat order on $W''$ is as in Example 4.3 and Figure 6, the only possible chains of covering with repetition of the same reflection is one involving the two canonical generators of $W''$. So $S'' = \{s, s_\beta\}$ must be the canonical generators of $W''$. In other words, $W' = W''$ and $\Delta_{W'} = \{\alpha_s, \beta\}$. 

**Figure 6.** The Hasse diagram of the Bruhat order on the finite dihedral group $D_m$. For $D_\infty$, the Hasse diagram is the diagram with infinite vertices obtained by not considering the top part of the above diagram. The labels for the interior edges are as follows: the ‘parallel’ red ones corresponds to the label $\alpha_s$ and the ‘parallel’ blue ones to the label $\alpha_t$. All the labels on the other edges are distinct.
Moreover, since each Bruhat interval of length 2 (or more) in dihedral groups have two coatoms (see Figure 6), there is \( \gamma \in \Phi_{W'} \setminus \{ \alpha \} \) such that \( s \beta su <_W s, u <_W u \). Applying the order preserving map \( \psi \), we obtain \( s \beta sx < s, x < x \). Since by assumption \( \ell(s \beta sx) = \ell(x) - 2 \), we have \( s \beta sx < s, x < x \).

Assume now (2) to be true. As in (1) we use the decomposition \( x = wu \) and the map \( \psi \). Observe that, and \( \ell(ss \beta sx) = \ell(x) - 3 = \ell(sx) - 2 \) since \( sx < x \). We must have \( s \beta sx < sx \), since otherwise \( \ell(ss \beta sx) \geq \ell(s \beta sx) - 1 \geq \ell(sx) \). So \( s \beta su <_W su <_W u \), since \( S' = \{ s, s \beta \} \), and \( s, u <_W u \). This implies that \( s \beta su <_W s, u <_W u \) since \( s, u \) and \( su \) are the two coatoms of the length 2 interval \( [s \beta su, u] \) in \( (W', \leq_W) \).

If \( s \beta su >_W su \), this would force \( s \beta su <_W s \beta su \) since \( S' = \{ s, s \beta \} \). Therefore, \( s \beta su \in \{ su, s, u \} \) since there is only two atoms above \( s \beta su \) in \( (W', \leq_W) \). Since \( s \beta \neq s \), we have \( s \beta su = s, u \) and hence \( s \beta sx = s, x \) contrary to \( s, x < x \) and \( \ell(ss \beta sx) = \ell(x) - 3 \).

Hence \( s \beta su <_W s, su \) and therefore \( ss \beta sx <_W s, su <_W u \). Applying \( \psi \) shows \( ss \beta sx < \beta \beta sx < sx < x \). Since \( \ell(ss \beta sx) = \ell(x) - 3 \), it follows that \( ss \beta sx < s \beta sx < sx < x \).

4.2. Bases of inversion sets. Using the Bruhat order, the first author (M.D.) [15, 16, 17] describes the rays of the cone over an inversion set. If \( C \) is a cone such that \( C \cap -C = \{ 0 \} \), an extreme ray of \( C \) is a ray \( \mathbb{R}^+ \alpha \) where \( \alpha \in C \setminus \{ 0 \} \) such that if \( \beta, \gamma \in C \) and \( \mathbb{R}^+ \alpha \subseteq \text{cone}(\beta, \gamma) \), then \( \beta \in \mathbb{R}^+ \alpha \) or \( \gamma \in \mathbb{R}^+ \alpha \). In that case, any non-zero \( \alpha \in F \) is called a representative of \( F \).

Definition 4.5. Let \( w \in W \), the base \( N^i(w) \) of the inversion set \( N(w) \) is the set of representatives of \( \text{cone}(N(w)) \) constituted of roots:

\[
N^1(w) = \{ \beta \in \Phi^+ | \mathbb{R}^+ \beta \text{ is an extreme ray of } \text{cone}(N(w)) \}.
\]

The following proposition is [17] Lemma 1.7(a).

Proposition 4.6. Let \( w \in W \), then the base of \( N(w) \) is:

\[
N^1(w) = \{ \beta \in \Phi^+ | \ell(s \beta w) = \ell(w) - 1 \} = \{ \beta \in \Phi^+ | s \beta w < w \}.
\]

The concept of base of inversion sets has the following interesting consequences for the weak order and \( n \)-low elements that are worth mentioning, but are not necessary to prove Theorem 1.1.

Corollary 4.7. Let \( x, y \in W \) and \( \beta \in \Phi \).

1. If \( \ell(s \beta (x \vee y)) = \ell(x \vee y) + 1 \), then either \( \ell(s \beta x) = \ell(x) + 1 \) or \( \ell(s \beta y) = \ell(y) + 1 \).

2. If \( x \parr y \) exists and \( \ell(s \beta (x \parr y)) = \ell(x \parr y) - 1 \), then either \( \ell(s \beta x) = \ell(x) - 1 \) or \( \ell(s \beta y) = \ell(y) - 1 \).

Proof. We prove only (2), the proof of (1) being similar using [17] Lemma 1.7(b)]. Proposition 2.8 and Proposition 4.6 with \( w = x \) and \( w = y \) respectively give

\[
N(x \parr y) = \text{cone}_\Phi(N^1(x) \cup N^1(y)).
\]

So by definition of the base of an inversion set, we have \( N^1(x \parr y) \subseteq N^1(x) \cup N^1(y) \). This is equivalent to the desired conclusion, by Proposition 4.6.

Proposition 4.8. Let \( u, v \in W \) both be \( n \)-low elements.

1. For \( x \in W \), one has \( u \preceq_W x \) if and only if \( N^1(u) \subseteq \Sigma_n(x) \).
The proof of this theorem uses the following lemma.

**Lemma 4.11.** Let \( s \in S \) and \( x \in W \) such that \( sx \triangleleft x \), then:

1. \( \{ \beta \in N^1(sx) \mid s_\beta sx < ss_\beta sx \} = \{ \beta \in \Phi^+ \mid s_\beta sx < ss_\beta sx \text{ and } s_\beta sx \triangleleft sx \} = s(N^1(x) \setminus \{ \alpha_s \}); \)

2. \( \{ \beta \in N^1(sx) \mid s_\beta sx > ss_\beta sx \} = \{ \beta \in \Phi^+ \mid ss_\beta sx < s_\beta sx < sx \triangleleft x \} = \{ \beta \in f_s(N^1(x) \setminus \{ \alpha_s \}) \mid ss_\beta sx < s_\beta sx < sx \triangleleft x \} = \{ \beta \in f_s(N^1(x) \setminus \{ \alpha_s \}) \mid \ell(s_\beta x) = \ell(x) - 3 \}. \)

**Proof.** Recall that, for all \( w \in W \), \( \ell(sw) = \ell(w) \pm 1 \) since \( s \in S \); so either \( sw \triangleleft w \) or \( w \triangleleft sw \) and \( sw \triangleleft w \) if and only if \( \ell(sw) = \ell(w) - 1 \). This shows the first equalities in both (1) and (2) by taking \( w = s_\beta sx \).

For the proof of (1), suppose first that \( \gamma \in N^1(x) \setminus \{ \alpha_s \} \) and write \( \beta = s(\gamma) \). Observe that \( s_\beta sx = s_{s(\gamma)}x = ss_\gamma x \) since \( s_\beta = s(s(\gamma)) = ss_\gamma s \). We show that \( \beta \) is in the left hand side of (1), that is:

\[ s_\beta sx = ss_\gamma x \triangleleft x \] and \( s_\beta sx = ss_\gamma x \triangleleft ss_\beta sx = s_\gamma x, \]

(2) The join of \( u \) and \( v \) exists (i.e. \( \{ u, v \} \) is bounded in \( W \)) if and only if there is some \( A \in \Lambda_n \) with \( N^1(u) \cup N^1(v) \subseteq A \).

**Proof.** We prove (1). If \( v \triangleleft_R x \), then \( N(u) \subseteq N(x) \). Since \( u \) is \( n \)-low we have \( N^1(u) \subseteq \Sigma_n(u) = N(u) \cap \Sigma_\ell = \Sigma_n(x) \). Conversely, if \( N^1(u) \subseteq \Sigma_n(x) \), then \( N^1(u) \subseteq N(x) \) and so \( N(u) = \text{cone}_R(N^1(u)) \subseteq N(x) \) i.e. \( u \triangleleft_R x \) by Proposition 2.8. Part (2) follows from (1) and the definition of \( \Lambda_n \). \( \square \)

**Remark 4.9.** If \( S \) is finite, the proposition gives rise to an algorithm to determine whether two elements \( u, w \) of \( W \) have a join in weak order, since \( \Sigma_n \) and \( \Lambda_n \) can be effectively computed. In general (for possibly infinite \( S \)) one may find a finite subset \( I \subseteq S \) such that \( u, w \in W_I = \langle I \rangle \). Using Proposition 2.8, for instance, one easily sees that \( u, w \) have a join in \( W \) if and only if they have one in \( W_I \), and that if so, the joins of \( u \) and \( w \) in \( W_I \) and \( W \) coincide.

4.3. Inversion sets of suffixes. We give here a useful description of the base of the inversion set of a suffix of \( w \in W \) obtained from the description of the base of \( N(w) \). We deduce from there a sufficient condition for a suffix of a \( n \)-low element to be \( n \)-low.

For each \( s \in S \), we define a function \( f_s : \Phi^+ \setminus \{ \alpha_s \} \rightarrow \Phi^+ \) as follows. Let \( \beta \in \Phi^+ \setminus \{ \alpha_s \} \). Then the dihedral reflection subgroup \( \langle s, s_\beta \rangle \) is contained in a unique maximal dihedral reflection subgroup \( W' \), as defined in Example 2.14. We have necessarily \( \alpha_s \in \Delta_{W'} \) since \( \alpha_s \in \Delta \), which is a basis of \( \text{cone}(\Phi^+) \). Define \( f_s(\beta) \) to be the other element of \( \Delta_{W'} \) i.e. \( \Delta_{W'} = \{ \alpha_s, f_s(\beta) \} \). Equivalently, \( f_s(\beta) \) is defined by the conditions \( f_s(\beta) \in \Phi^+ \) and \( (\mathbb{R}\alpha_s + \mathbb{R}\beta) \cap \Phi^+ = \text{cone}(\alpha_s, f_s(\beta)) \cap \Phi^+ \).

**Theorem 4.10.** Let \( s \in S \) and \( x \in W \).

1. If \( sx \) is a suffix of \( x \), i.e., \( sx \triangleleft x \), then
   \[ N^1(sx) \subseteq s(N^1(x) \setminus \{ \alpha_s \}) \cup f_s(N^1(x) \setminus \{ \alpha_s \}) . \]

2. If \( x \) is a suffix of \( sx \), i.e., \( x \triangleleft sx \), then
   \[ N^1(sx) = \{ \alpha_s \} \cup s \{ \beta \in N^1(x) \mid s_\beta x < ss_\beta x \} . \]

The proof of this theorem uses the following lemma.
by Proposition 4.6. By Proposition 4.6 again we have $s_x x \prec x$. Since $s_x x \prec x$ and $\gamma \neq \alpha_s$ we have $s s_x x \prec s_x x$ by Corollary 4.2 proving the first statement of ($\star$). The second statement follows from the following computation:

$$\ell(s_\beta s x) = \ell(s s_\gamma x) = \ell(s_\gamma x) - 1 = \ell(x) - 2 = \ell(s x) - 1.$$ 

This shows the right hand side of (1) is contained in the left hand side. For the reverse inclusion, let $\beta$ be in the left hand side of (1) and set $\gamma := s(\beta)$. Since $\gamma \neq \alpha_s$ it will suffice to show that $\gamma \in N^1(x)$. But we have by assumption on $\beta$ that $\ell(s, x) = \ell(ss_\beta s x) = 1 + \ell(s_\beta s x) = \ell(s x) = \ell(x) - 1$, and the result follows.

Now for the proof of (2), the first equality follows from the discussion at the beginning of this proof. The second and third equalities follows from Lemma 4.4 and the definition of $f_s$, since there is $\gamma \in N^1(x) \setminus \{\alpha_s\}$, i.e., $s_x x \prec x$, and $\gamma$ is in the positive root system spanned by $\Delta_W' = \{\alpha_s, \beta\}$. □

**Proof of Theorem 4.10.** Part (1) is a direct consequence of Lemma 4.11 since

$$N^1(s x) = \{\beta \in N^1(s x) | s_\beta s x > s s_\beta s x\} \cup \{\beta \in N^1(s x) | s_\beta s x < s s_\beta s x\}.$$ 

Item (2) follows easily by replacing $x$ by $s x$ in Lemma 4.11(1). □

**Corollary 4.12.** Let $n \in \mathbb{N}$ and $x \in L_n(W)$ be a $n$-low element. Let $s \in S$ such that $s x$ is a suffix of $x$. If $f_s(N^1(x) \setminus \{\alpha_s\}) \subseteq \Sigma_n$, then $s x \in L_n(W)$.

**Proof.** We have, by definition of $n$-low elements, to show that $N^1(s x) \subseteq \Sigma_n$. Let $\gamma \in N^1(s x)$. By Theorem 4.10(1), either $\gamma \in f_s(N^1(x) \setminus \{\alpha_s\})$ or $\gamma = s(\beta)$ where $\beta \in N^1(x) \setminus \{\alpha_s\}$. In the first case, $\gamma \in \Sigma_n$ by assumption. In the second case, if $\gamma \notin \Sigma_n$, we have $d_\infty(\beta) \leq n$ and $d_\infty(s(\beta)) > n$. By Proposition 2.15 this implies that $B(\alpha_s, \beta) \leq -1$. But $\alpha_s, \beta \in N^1(x) \subseteq N(x)$, so this contradicts Proposition 2.15(2). □

The next result is immediate from Corollary 4.12 and Proposition 3.26(\beta).

**Corollary 4.13.** Let $n \in \mathbb{N}$. Suppose that for every $s \in S$ and $\beta \in \Phi^+ \setminus \{\alpha_s\}$, one has $f_s(\beta) \in \Sigma_n$ whenever $\beta \in \Sigma_n$. Then the set $L_n(W)$ of $n$-low elements of $W$ is a finite Garside shadow.

4.4. **Bipodality.** We present here a property called bipodality and show that if the set $\Sigma_n$ of $n$-small roots is bipodal, then $\Sigma_n$ meets the hypothesis of Corollary 4.13 and so $L_n(W)$ is a finite Garside shadow.

**Definition 4.14.** A subset $A \subseteq \Phi^+$ of positive roots is bipodal if for any maximal dihedral reflection subgroup $W'$ with canonical simple system $\Delta_{W'}$ and positive root system $\Phi^+_{W'}$, we have the following property:

$$A \cap (\Phi^+_{W'} \setminus \Delta_{W'}) \neq \emptyset \implies \Delta_{W'} \subseteq A.$$ 

**Remark 4.15.** In the context of normalized roots, as in Remark 2.7, $A$ is bipodal if for any maximal segment $[\alpha, \beta] = \Phi \cap P$ ($P$ is some line containing at least two normalized roots) such that $A \cap (\alpha, \beta) \neq \emptyset$ the endpoints $\alpha, \beta$ are elements of $A$.

**Proposition 4.16.** Let $n \in \mathbb{N}$. If the set $\Sigma_n$ of $n$-small roots is bipodal, then the set $L_n(W)$ of $n$-low elements is a finite Garside shadow.
Proof. By Proposition 3.26, it is enough to show that if $s \in S$ and $x \in L_n(W)$ such that $sx$ is a suffix of $x$, then $sx \in L_n(W)$. By Corollary 1.13, we have to show that if $\beta \in \mathcal{N}(x) \setminus \{\alpha_s\}$, then $f_s(\beta) \in \Sigma_n$. Since $x \in L_n(W)$ and $\beta$ is in the base of its inversion set, we have $\beta \in \Sigma_n$. Let $W'$ be the dihedral reflection subgroup generated by the reflection in the simple system $\Delta_{W'} = \{\alpha_s, f_s(\beta)\}$; by definition of $f_s$, this reflection subgroup is maximal. So either $f_s(\beta) = \beta$ and we are done, or $\beta \in \Phi^+_W \setminus \Delta_{W'}$. This forces $f_s(\beta) \in \Sigma_n$ since $\Sigma_n$ is bipodal by assumption. □

In regards of the above proposition, it is now time to state our second conjecture, that would imply, if true, Conjecture 1.

**Conjecture 3.** The set of $n$-small roots $\Sigma_n$ is bipodal for any $n \in \mathbb{N}$.

This conjecture is obviously true for finite Coxeter groups, since all roots are small (see Example 3.6); it is also true for infinite dihedral groups (see Example 3.7) and for affine Coxeter systems as shown in the following theorem.

**Theorem 4.17.** Let $n \in \mathbb{N}$ and assume $(W, S)$ is an affine Weyl group. Then:

1. $\Sigma_n$ is bipodal;
2. $L_n(W)$ is a finite Garside shadow in $(W, S)$.

Proof. By Proposition 4.16 one just has to show (1). Assume without loss of generality that $(W, S)$ is irreducible. We use the notations and results of Example 3.9.

Let $\gamma$ be a root in the positive crystallographic root system $\Psi$ such that $\gamma' \in \Sigma_n$. Let $W'$ be a maximal dihedral reflection subgroup with simple system $\Delta_{W'} = \{\alpha', \beta'\}$, with $\alpha', \beta' \in \Psi^+$, and positive root subsystem $\Phi^+_W$, such that $\gamma' \in \Phi^+_W \setminus \Delta_{W'}$. By symmetry between $\alpha$ and $\beta$, one just has to show that $d_{p\infty}(\alpha) \leq d_{p\infty}(\gamma)$, which would imply $\alpha' \in \Sigma_n$.

Since $\Phi^+_W = \text{cone}_\delta(\Delta_{W'})$, $\Psi$ is crystallographic and $\gamma' \notin \Delta_{W'}$, there is $a, b \in \mathbb{N}^*$ such that $\gamma = a\alpha + b\beta$. By the description of $\Psi^+$ given in Example 3.9 we have $\alpha = \alpha_0 + k\delta$ and $\beta = \beta_0 + l\delta$, for some $k, l \in \mathbb{N}$ and $\alpha_0, \beta_0 \in \Psi_0$. So

$$\gamma = a\alpha_0 + b\beta_0 + (ak + bl)\delta.$$ 

By the description of $\Psi^+$ again, we must have $\gamma_0 := a\alpha_0 + b\beta_0 \in \Psi_0$ since $\delta$ is not in the linear span of $\Psi_0$ but $\gamma_0 = a\alpha_0 + b\beta_0$ is. So by Equation (3) in Example 3.9, we have:

$$d_{p\infty}(\gamma) = d_{p\infty}(\gamma_0 + (ak + bl)\delta) = \left\{ \begin{array}{ll}
ak + bl & \text{if } \gamma_0 \in \Psi^+_0 \\
ak + bl - 1 & \text{if } \gamma_0 \in \Psi^-_0.
\end{array} \right.$$ 

and

$$d_{p\infty}(\alpha) = d_{p\infty}(\alpha_0 + n\delta) = \left\{ \begin{array}{ll}
k & \text{if } \alpha_0 \in \Psi^+_0 \\
k - 1 & \text{if } \alpha_0 \in \Psi^-_0.
\end{array} \right.$$ 

So $d_{p\infty}(\alpha) \leq d_{p\infty}(\gamma)$ unless $\gamma_0 \in \Psi^-_0$, $\alpha_0 \in \Psi^+_0$ and $l = 0$. In this case $\beta = \beta_0$ must be a positive root in $\Psi_0^+$ and therefore $\gamma_0 = a\alpha_0 + b\beta_0 \in \Psi_0^+$ contradicting $\gamma_0 \in \Psi^-_0$. In conclusion, $d_{p\infty}(\alpha) \leq d_{p\infty}(\gamma)$, $\alpha' \in \Sigma_n$ and $\Sigma_n$ is bipodal. □

### 4.5. Bipodality of small roots and proof of Theorem 1.1

As seen in Proposition 4.16, in order to show Theorem 1.1 it is enough to show that the set of small roots $\Sigma$ is bipodal.

**Theorem 4.18.** The set of small roots $\Sigma$ is bipodal.

This theorem is the direct consequence of the three following technical lemmas.
Lemma 4.19. The set $\Sigma$ is bipodal if and only if the following condition holds:

$(\forall)$ for any maximal dihedral reflection subgroup $W'$ of $W$, $\gamma \in \Sigma \cap (\Phi^+_W \setminus \Delta_W)$ and simple root $\alpha \in \Delta \setminus \Delta_{W'}$ with $B(\alpha, \gamma) > 0$, one has $-1 < B(\alpha, \beta) < 1$ for all $\beta \in \Delta_{W'}$.

Remark 4.20. Observe that the condition $B(\alpha, \gamma) > 0$ is equivalent to say that $\gamma$ is on the positive side of the hyperplane $H_\alpha = \ker(v \mapsto B(\alpha, v))$ and that the condition $-1 < B(\alpha, \beta) < 1$ is equivalent to say that the dihedral reflection subgroup generated by $s_\alpha, s_\beta$ is finite, which is equivalent to $(\mathbb{R} \alpha + \mathbb{R} \beta) \cap \Phi$ is finite, or equivalently, $(\mathbb{R} \alpha + \mathbb{R} \beta) \cap Q = \{0\}$; see [12] for more details. So the condition $(\forall)$ has the following geometric interpretation: for any small root $\gamma \in \Sigma$, simple root $\alpha \in \Delta$ such that $\gamma$ is on the positive side of the hyperplane $H_\alpha$ and maximal dihedral reflection subgroup $W'$ of $W$ such that $\gamma \in (\Phi^+_W \setminus \Delta_W)$, one has $(\mathbb{R} \alpha + \mathbb{R} \beta) \cap \Phi$ finite (which translates in the language of normalized roots to: the line passing through $\hat{\alpha}$ and $\hat{\beta}$ contains a finite number of normalized roots).

Proof. Suppose $\Sigma$ is bipodal. Let $W'$ be a maximal dihedral reflection subgroup of $W$, and $\gamma \in \Sigma \cap (\Phi^+_W \setminus \Pi_{W'})$. Since $\Sigma$ is bipodal, $\Delta_{W'} \subseteq \Sigma$. Now let $\alpha \in \Delta \setminus \Delta_{W'}$ with $B(\alpha, \gamma) > 0$. Abbreviate $s = s_\alpha$ and $W'' := sW's$ (a maximal dihedral reflection subgroup of $W$). Note that $\alpha \notin \Phi^+_W$, for otherwise $\alpha \in \Pi_{W'}$ since $\alpha \in \Delta$. Hence $\Delta_{W''} = s(\Delta_{W'})$ and $s(\gamma) \in s(\Phi^+_W) = \Phi^+_W$. Proposition 3.3 forces $0 \leq dp_\infty(s(\gamma)) \leq dp_\infty(\gamma) = 0$, so $s(\gamma) \in \Sigma \cap (\Phi^+_W \setminus \Delta_{W'})$. Since $\Sigma$ is bipodal, this implies $\Delta_{W''} \subseteq \Sigma$. Now let $\beta \in \Delta_{W''}$, so $s(\beta) \in \Delta_{W''}$ and $dp_\infty(\beta) = dp_\infty(s(\beta)) = 0$ by above. By Proposition 3.3 if $B(\alpha, \beta) \leq -1$, then $dp_\infty(s(\beta)) = dp_\infty(\beta) + 1$, whereas if $B(\alpha, \beta) \geq 1$, then $dp_\infty(s(\beta)) = dp_\infty(\beta) + 1$. In either case, we have a contradiction, so $-1 < B(\alpha, \beta) < 1$. Hence $(\forall)$ holds, completing the proof of the “only if” direction.

Now we assume $(\forall)$. We show that for any maximal dihedral reflection subgroup $W'$ of $W$ and any $\gamma \in \Sigma \cap (\Phi^+_W \setminus \Delta_{W'})$, one has $\Delta_{W'} \subseteq \Sigma$, by induction on $dp(\gamma)$ using Proposition 2.6. This will obviously imply that $\Sigma$ is bipodal.

Choose $\alpha \in \Delta$ with $B(\alpha, \gamma) > 0$. Since $B(\alpha, \beta) \leq 0$ for all $\alpha, \beta \in \Delta$, we must have $\gamma \notin \Delta$. So $dp(\gamma) \geq 3$ by Proposition 2.6. Set as above $s = s_\alpha$ and $W' := sW's$. By Proposition 2.6 we have $dp(s(\gamma)) = dp(\gamma) - 1$, so $s(\gamma) \in \Sigma$ by Proposition 3.14. We distinguish the cases $\alpha \in \Delta_{W'}$ and $\alpha \notin \Delta_{W'}$.

First assume $\alpha \in \Delta_{W'}$, so $W'' = W'$. If $s(\gamma) \in \Delta_{W'}$, then $\Delta_{W'} = \{\alpha, \gamma\} \subseteq \Sigma$ as required. On the other hand, if $s(\gamma) \notin \Delta_{W'}$, then $s(\gamma) \in \Sigma \cap (\Phi^+_W \setminus \Delta_{W'})$ with $dp(s(\gamma)) < dp(\gamma)$. Hence $\Delta_{W'} \subseteq \Sigma$ by induction.

Now consider the case $\alpha \notin \Delta_{W'}$. Then $\Delta_{W'} = s(\Delta_{W'})$, $\Phi^+_{W'} = s(\Phi^+_W)$ and $s(\gamma) \in \Sigma \cap (\Phi^+_W \setminus \Delta_{W'})$ with $dp(s(\gamma)) < dp(\gamma)$. By induction, $\Delta_{W'} \subseteq \Sigma$. But by $(\forall)$, for any $\beta \in \Delta_{W'}$, we have $-1 < B(\alpha, \beta) < 1$ and hence $dp_\infty(\beta) = d_\infty(s_\alpha(\beta)) = 0$ since $s_\alpha(\beta) \in \Delta_{W'}$. Hence $\Delta_{W'} \subseteq \Sigma$ as required to complete the proof of the “if” direction.

Lemma 4.21. Assume that $\Sigma$ is bipodal whenever $(W, S)$ is of rank $|S| = 3$. Then $\Sigma$ is bipodal for all $(W, S)$.

Proof. We regard the condition $(\forall)$ in Lemma 4.19 as a condition on the based root system $(\Phi, \Delta)$: assume that $(\forall)$ holds for each rank three root subsystem $(\Psi, \Pi)$ of $(\Phi, \Delta)$. Let $W'$ be any rank 2 maximal dihedral reflection subgroup of $W$,
γ ∈ Σ ∩ (Φ_W \ \Delta_W) and α ∈ Δ \ \Delta_W with B(α, γ) > 0 and β ∈ Δ_W. We have to show that −1 < B(α, β) < 1.

Define the reflection subgroup G := ⟨W', s_α⟩ of W. Since Φ_W ∪ {α} spans a subspace of V of dimension three and Φ_W ∪ {α} ⊆ Φ_G, the rank of G has to be at least 3. Since G is generated by three reflections, it is of rank at most three by [13 Corollary 3.11]. So the rank of G is 3.

Set Ψ := Φ_G and let Π := Δ_G, so (Ψ, Π) is a rank three based root subsystem of (Φ, Δ) (but not a standard parabolic root subsystem in general). Note that W' is a maximal dihedral reflection subgroup of G and that the set Ψ_W', of positive roots of W' in Ψ is Ψ_W' = Φ_W' and the canonical set of simple roots Δ_W', with respect to (Ψ, Π) is Π_W' = Δ_W'. Further, γ ∈ Σ(G) ∩ (Ψ_W' \ Π_W') by Lemma 3.4 and α ∈ Π \ Π_W' since any simple root lies in the canonical simple system of any root subsystem containing it. Hence by (∇) for (Ψ, Π), which holds by assumption since Ψ is of rank three, it follows that −1 < B(α, β) < 1 for β ∈ Π_W' = Δ_W'.  

**Lemma 4.22.** If (W, S) has rank 3, then Σ is bipodal.

**Proof.** Assume that (W, S) is of rank 3. Let Γ be the Coxeter graph of (W, S), with set of vertices Δ = {α_1, α_2, α_3}. We denote S = {s_1, s_2, s_3} with s_i = s_α_i. The support of a root β = α_1α_1 + α_2α_2 + α_3α_3 is the set supp(β) = {α_i ∈ Δ | α_i ≠ 0}. We denote by Γ(β) the full subgraph of Γ on the vertex set supp(β).

The proof proceeds essentially by systematically listing all possible (Γ; γ, W') satisfying the conditions γ ∈ Σ and W' is a maximal dihedral reflection subgroup such that γ ∈ Φ_W', Δ_W, and checking that Δ_W' ⊆ Σ in each case.

If Γ is not connected, i.e., Φ is reducible, then Φ is the disjoint union of a simple root and of the root system of a dihedral standard parabolic subgroup, for which the set of small roots is bipodal. It is then easy to see that Σ is bipodal.

So from now on, we suppose that Γ is a connected graph.

Assume that γ is not of full support, i.e., supp(γ) = I ⊆ Δ. So γ ∈ Φ_W' and therefore γ is in a facet of cone(Δ). Since γ ∈ Φ_W' \ Δ_W', then γ is in the relative interior of cone(Δ_W'). This forces Δ_W' ⊆ Φ_W' ⊆ Φ_I, since Δ_W' ⊆ cone(Δ_W') ⊆ cone(Δ). So we are in the case of a dihedral standard parabolic subgroup. But in this case we know that Δ_W' ⊆ Σ(W_I). So Δ_W' ⊆ Σ by Proposition 3.13.

So from now on γ is assumed to be of full support, i.e., Γ(γ) = Γ.

By Brink’s characterization of the support of small roots [5, Lemma 4.1], the support of any small root contains no cycle and no edge with infinite label. We may therefore assume that the Coxeter graph is of the form

```
    α_1  m  α_2  n  α_3
```

where m, n ∈ N and m ≥ n ≥ 3.

We already know Σ is bipodal if W is finite (by Example 3.6) or affine (by Theorem 4.17). So the classification of affine and finite Coxeter groups (see for instance [29]) forces m > n ≥ 4, or m ≥ 7 and n = 3.

It is easy now to list for each of these Γ the small roots γ of full support. This may be done either by simple direct calculation using Proposition 3.3 or by using [5], where the small roots are recursively determined for all finite rank Coxeter systems; see especially [5, Propositions 4.7 and 6.7]. For each γ, all possible maximal dihedral
reflection subgroups $W'$ are obtained by specifying the canonical simple system $\Delta_{W'} = \{\mu_1, \mu_2\}$, which is obtained by inspection since $\gamma = a\mu_1 + b\mu_2$ with $a, b \geq 1$. Observe that $\text{supp}(\mu_i) \neq \{\alpha_1, \alpha_3\}$ since there is no root with full support in the subgroup generated by $s_1, s_3$.

To complete the proof, we list below all the possible $\gamma$ and $W'$ (by specifying $\Delta_{W'}$), writing $c_p := 2\cos \frac{p\pi}{2}$ for all $p \in \mathbb{N}_{\geq 2}$. In each case, each element of $\Delta_{W'}$ lies in a dihedral finite standard parabolic root subsystem of $\Phi$ and so is small as required.

- If $m > n \geq 4$, the only possible $\gamma$ is $c_m\alpha_1 + \alpha_2 + c_n\alpha_3 = s_1s_3(\alpha_2)$ and then $\Delta_{W'}$ is either $\{\alpha_1, s_3(\alpha_2) = \alpha_2 + c_n\alpha_3\}$ or $\{\alpha_3, s_1(\alpha_2) = \alpha_2 + c_m\alpha_1\}$.
- If $m \geq 7$ and $n = 3$, there are three possible choices for $\gamma$.
  1. $\gamma = \alpha_1 + c_m\alpha_2 + c_n\alpha_3 = s_3s_2(\alpha_1)$; in this case $\Delta_{W'}$ is:
     - either $\{\alpha_3, s_2(\alpha_1) = \alpha_1 + c_m\alpha_2\}$ or $\{\alpha_1, s_3(\alpha_2) = \alpha_2 + \alpha_3\}$.
  2. $\gamma = c_m\alpha_1 + \alpha_2 + c_n\alpha_3 = s_1s_2(\alpha_3)$; in this case $\Delta_{W'}$ is:
     - either $\{\alpha_3, s_1(\alpha_2) = \alpha_2 + c_m\alpha_1\}$ or $\{\alpha_1, s_2(\alpha_3) = \alpha_2 + \alpha_3\}$.
  3. $\gamma = (c_m^2 - 1)\alpha_1 + c_m\alpha_2 + c_n\alpha_3 = s_1s_3s_2(\alpha_1)$; in this case $\Delta_{W'}$ is:
     - either $\{\alpha_3, s_1s_2(\alpha_1) = (c_m^2 - 1)\alpha_1 + c_m\alpha_2\}$ or $\{\alpha_1, s_3(\alpha_2) = \alpha_2 + \alpha_3\}$.

\[ \square \]

5. Weak and Bruhat order on root systems and bipodality

Let $(\Phi, \Delta)$ be a based root system in the quadratic space $(V, B)$ with Coxeter system $(W, S)$. In the preceding sections we conjectured that:

- The set $L_n(W)$ of $n$-low elements in $W$ is a finite Garside shadow in $(W, S)$ for all $n \in \mathbb{N}$ (Conjecture 1).
- The set $\Sigma_n$ of $n$-small roots is bipodal for all $n \in \mathbb{N}$ (Conjecture 3).

In view of Proposition 4.18 we know that Conjecture 1 is implied by Conjecture 3. We know therefore that Conjectures 1 and 3 are true in the following cases:

- if $n = 0$ by Theorem 1.1 and Theorem 4.18
- if $W$ is finite, dihedral or an affine Weyl group by Theorem 4.17 and Examples 4.6 and 4.7
- if the Coxeter graph of $(W, S)$ has labels 3 or $\infty$, by Theorem 5.1 in [5]

The aim of this section is to show Conjecture 1 and Conjecture 3 hold also in the following cases.

**Theorem 5.1.** Suppose that $(W, S)$ is a Coxeter system such that its Coxeter graph has all edges labelled by 3 or $\infty$, i.e., all entries of the Coxeter matrix of $(W, S)$ lie in $\{1, 2, 3, \infty\}$. Then for each $n \in \mathbb{N}$:

1. $\Sigma_n(W)$ is a bipodal subset of $\Phi^+$;
2. $L_n(W)$ is a finite Garside shadow in $(W, S)$.

In order to show bipodality, we will show that $\Sigma_n$ enjoys a stronger property that we define now.

**Definition 5.2.** A subset $A \subseteq \Phi^+$ is balanced if for all $\gamma \in A$ and all maximal dihedral reflection subgroups $W'$ of $W$ with $\gamma \in \Phi_{W'}$, the following holds: if $\beta \in \Phi_{W'}^+$, such that $\ell_{W'}(s_\beta) < \ell_{W'}(s_\gamma)$ then $\beta \in A$. 

This definition goes back to Edgar’s thesis [19] (in the case of the standard length function). Since the canonical simple reflections $s_\alpha, s_\beta$ of $W'$ are of length $\ell_{W'}(s_\alpha) = \ell_{W'}(s_\beta) = 1$, it is easy to see that a balanced set of roots is necessarily a bipodal set of roots.

**Remark 5.3.** (a) In the language of normalized roots, as in Remark 2.7, a maximal dihedral subgroup $W'$ with canonical simple system $\Delta_{W'} = \{\alpha, \beta\}$ corresponds to a “maximal” segment $[\alpha, \beta] \cap \hat{\Phi}$; maximality holds in the sense that there are no other roots on the line $([\alpha, \beta]$ but the ones in $[\alpha, \beta] \cap \hat{\Phi} = \hat{\Phi}_{W'}$. If $W'$ is finite (infinite), one may index $\Phi_{W'}^+$ as $\{\alpha_1 = \beta_n, \alpha_2 = \beta_{n-1}, \ldots, \alpha_n = \beta_1\}$ (resp., $\{\alpha_1, \alpha_2, \ldots, \alpha_n, \ldots, \beta_n, \ldots, \beta_2, \beta_1\}$); see Figure 3 in the order corresponding to that in which a point moving along the line segment $[\alpha, \beta]$ from $\beta$ to $\alpha$ passes through the associated normalized roots. One has $l_{W'}(\alpha_i) = l_{W'}(\beta_i) = 2i-1$ provided $1 \leq 2i-1 \leq |\Phi_{W'}^+|$. Set $a_i := dp_\infty(\alpha_i)$ and $b_i = dp_\infty(\beta_i)$. Then $\Sigma_n$ is balanced if and only if (for each $W'$), $1 \leq 2i-1 \leq |\Phi_{W'}^+|$ and $\min(a_i, b_i) \leq n$ imply that $a_j, b_j \leq n$ for all $j$ with $1 \leq j < i$.

(b) It is not true that a bipodal set of roots is balanced. Assume $(W, S)$ to be of type $G_2$ as in Figure 4, then $\Sigma$ is bipodal but not balanced. For the maximal (finite) segment $[\hat{\gamma}, s_{\beta'} \hat{\alpha}] \cap \Phi$ has $\infty$-depth values $(0, 0, 1, 0, 1, 0)$, where, in the notation of (a), $b_3 = 0$ but $b_2 = 1 \not\in 0$. Hence the conjecture, mentioned in [19], that $\Sigma_n$ is always balanced is false (even for $n = 0$).

Theorem 5.1 is a consequence of the following proposition, as we show now.

**Proposition 5.4.** Let $W'$ be a dihedral reflection subgroup of $W$, and $\gamma, \delta \in \Phi_{W'}^+$.  

1. If $W'$ is infinite and $\ell_{W'}(\gamma) < \ell_{W'}(s_\delta)$, then $dp_\infty(\gamma) < dp_\infty(\delta)$.
2. If $W'$ is finite and there exists $x \in W'$ such that $\ell(x) = \ell(\gamma) + 2\ell_{W'}(x)$, then $dp_\infty(\gamma) \leq dp_\infty(\delta)$.

**Remark 5.5.** In the notation of Remark 5.3 (1) says that (for infinite $W'$), $a_i, b_j < a_i, b_j$ if $1 \leq j < i$, while (2) says that (for finite $W$), $a_i < b_i$ and $b_{i-1} < a_i$ if $3 \leq 2i-1 \leq |\Phi_{W'}^+|$.

The proof of the proposition involves a study of an analogue for $\Phi$ of the weak order of $W$, and is postponed to the end of this section. For now, assuming the above proposition, let us give the proof of the theorem.

**Proof of Theorem 5.1.** We shall show that $\Sigma_n$ is balanced, and hence it is bipodal. We begin the argument assuming just that $(W, S)$ is a finite rank Coxeter system. Let $\gamma \in \Sigma_n$ and $W'$ be a (maximal) dihedral reflection subgroup of $W$ such that $\gamma \in \Phi_{W'} \setminus \Delta_{W'}$. Write $\Delta_{W'} = \{\alpha, \beta\}$. We wish to show that $\alpha, \beta \in \Sigma_n$.

In any case, $\ell_{W'}(s_\alpha), \ell_{W'}(s_\beta) < \ell_{W'}(s_\gamma)$. If $W'$ is infinite, it follows from Proposition 5.4(1) that $dp_\infty(\alpha), dp_\infty(\beta) < dp_\infty(\gamma) \leq n$ and $\alpha, \beta \in \Sigma_n$ as required. Otherwise, $W'$ is finite. It follows from Proposition 5.4(2) and Remark 5.5 that $dp_\infty(s_\beta(\alpha)) \geq dp_\infty(s_\alpha), dp_\infty(s_\alpha(\beta)) \geq dp_\infty(s_\beta)$ and either $dp_\infty(s_\alpha) \geq dp_\infty(s_\beta(\alpha))$ or $dp_\infty(s_\alpha) \geq dp_\infty(s_\beta(\beta))$. Hence it would suffice to show that $dp_\infty(s_\beta(\alpha)) \geq dp_\infty(s_\gamma)$ and $dp_\infty(s_\alpha(\beta)) \geq dp_\infty(s_\gamma)$. We do not know if this holds in general. However, we now make the special assumptions on the Coxeter matrix in Theorem 5.1 and show instead that one has $s_\alpha = s_\alpha(\beta) = s_{\beta}(\alpha)$.

An argument due to Tits (see [4] Ch IV, §4, Ex 4(d)], [5] Proposition 1.3] or [2 Theorem 4.5.3]) shows that a finite subgroup $H$ of $W$ must be conjugate to a
reflection subgroup of a finite standard parabolic subgroup of \( W \). A minor elaboration of Tits’ argument shows that if \( H \) is a reflection subgroup, and its roots span a subspace of dimension \( r \), then the standard parabolic subgroup may be taken to be of rank \( r \). This also follows from the well known fact that, in a finite Coxeter group, the parabolic closure \( H'' \) of a rank \( r \) reflection subgroup \( H' \) (i.e. the inclusion-minimal parabolic subgroup \( H'' \) containing \( H' \)) has as its root system the set of all roots in the linear span of the root system of \( H' \).

Applying the previous paragraph with \( H = W' \) shows that \( W' \) is conjugate to a subgroup of a finite dihedral standard parabolic subgroup of \( W \). By assumption, this subgroup must be of order 4 or 6 with two or three positive roots respectively. In particular, either \( B(\alpha, \beta) = 0 \) (which is impossible since then \( \Phi_{W'} \setminus \Delta_{W'} = \emptyset \)) contradicting \( \gamma \in \Phi_{W'} \setminus \Delta_{W'} \) or \( B(\alpha, \beta) = -\frac{1}{2} \) and \( \gamma = \alpha + \beta = s_\alpha(\beta) = s_\beta(\alpha) \) as required.

Proposition 5.4 has also the following noteworthy consequence, which was observed in the proof above.

**Corollary 5.6.** Let \( n \in \mathbb{N} \). The set \( \Sigma_n \) of \( n \)-small roots is bipodal if and only for any pair of roots \( \alpha, \beta \in \Phi^+ \) such that \( W' := \langle s_\alpha, s_\beta \rangle \) is a finite maximal dihedral reflection subgroup with canonical simple system \( \Delta_{W'} = \{ \alpha, \beta \} \), \( B(\alpha, \beta) \neq 0 \), and \( s_\alpha(\beta) \in \Sigma_n \), one has \( \alpha \in \Sigma_n \).

**5.1. Length and depth functions on \( \Phi \).**

**Definition 5.7.** Fix any subset \( X \subseteq \mathbb{R}^+ := \{ \lambda \in \mathbb{R} \mid \lambda > 0 \} \). The \( X \)-length on \( \Phi \) is the function \( d_X : \Phi \to \mathbb{Z} \) defined as follows: let \( \beta \in \Phi^+ \), then
\[
d_X(\pm \beta) = \pm|\{ \alpha \in N(s_\beta) \mid B(\alpha, \beta) \in X \}|.
\]

**Example 5.8.** Note that if \( \beta \in \Phi^+ \) and \( \alpha \in N(s_\beta) \), then \( B(\alpha, \beta) > 0 \), since otherwise \( s_\beta(\alpha) = \alpha - 2B(\alpha, \beta)\beta \in \Phi^+ \) contrary to \( \alpha \in N(s_\beta) \). So if \( X = \mathbb{R}^+ \) the \( \mathbb{R}^+ \)-length on \( \beta \) corresponds to the usual length of the associated reflection \( s_\beta \):
\[
d_{\mathbb{R}^+}(\beta) = |N(s_\beta)| = \ell(s_\beta) = 2d_\beta(\beta) - 1, \quad \beta \in \Phi^+.
\]

**Remark 5.9.** (a) For a reflection subgroup \( W' \) of \( W \), \( d_{W', X} : \Phi_{W'} \to \mathbb{N} \) denotes the function attached to \( (\Phi_{W'}, \Delta_{W'}, X) \) in the same way as \( d_X \) is attached to \( (\Phi, \Delta, X) \).

(b) The function \( d_X \) depends only on \( X \cap \{ B(\gamma, \beta) \mid \gamma, \beta \in \Phi \} \). Also, \( d_{\cup_{i \in I} X_i}(\alpha) = \sum_i d_{X_i}(\alpha) \) (where the right hand side has only finitely many non-zero terms for fixed \( \alpha \in \Phi \)) if the subsets \( X_i \subseteq \mathbb{R}^+ \) for \( i \in I \) are pairwise disjoint. It follows that the functions \( d_X \) for arbitrary subsets of \( \mathbb{R} \) are determined by the functions \( d_X \) for order coideals of \( \mathbb{R} \): such an order coideal \( X \) is either \( \emptyset, \mathbb{R}^+ \) or an open or closed ray \( [a, \infty] \) or \( ]a, \infty[ \) for some \( a > 0 \).

**Proposition 5.10.** Let \( X \subseteq \mathbb{R}^+ \) and set \( -X := \{-c \mid c \in X \} \). Let \( \beta \in \Phi \) and \( \alpha \in \Delta \). Then \( d_X(\alpha) \) is equal to 1 if 1 is in \( X \), and to 0 otherwise. Further,
\[
d_X(s_\alpha(\beta)) = \begin{cases} d_X(\beta) - 2, & \text{if } B(\alpha, \beta) \in X \\
d_X(\beta), & \text{if } B(\alpha, \beta) \in \mathbb{R} \setminus (X \cup -X) \\
d_X(\beta) + 2, & \text{if } B(\alpha, \beta) \in -X. \end{cases}
\]

**Proof.** The first claim is immediate from the definitions. It is easy to check that the displayed formula is equivalent to any of its variants given by replacing \( \beta \) by
\( \pm \beta \) or \( \pm s_\alpha(\beta) \). Hence, without loss of generality, we may (and do) assume that \( \beta \in \Phi^+ \) and \( B(\beta, \alpha) \geq 0 \). If \( \beta = \alpha \) or \( B(\beta, \alpha) = 0 \), the conclusion follows readily. Otherwise, \( \gamma := s_\alpha(\beta) \in \Phi^+ \) and \( s_\beta = s_\alpha s_\gamma s_\alpha \) with \( \ell(s_\beta) = \ell(s_\gamma) + 2 \). Hence
\[
N(s_\beta) = \{ \alpha \} \cup s_\alpha(N(s_\gamma)) \cup \{ s_\alpha s_\gamma(\alpha) \}
\]
by Proposition 2.8. One has
\[
B(s_\alpha s_\gamma(\alpha), \beta) = B(s_\alpha s_\gamma(\alpha), -\beta) = B(s_\beta(\alpha), -\beta) = B(\alpha, \beta).
\]
For any \( \gamma' \in N(s_\gamma) \) we have \( B(\gamma', \gamma) \in X \) if and only if \( B(s_\alpha(\gamma'), \beta) \in X \), since
\[
B(s_\alpha(\gamma'), \beta) = B(\gamma', s_\alpha(\beta)) = B(\gamma', \gamma).
\]
Let \( c \) denote 1 if \( B(\alpha, \beta) \in X \) and 0 otherwise. Now, since \( B(\alpha, \beta) > 0 \), the desired conclusion follows:
\[
d_X(\beta) = |\{ \rho \in N(s_\beta) \mid B(\rho, \beta) \in X \}| = |\{ \gamma' \in N(s_\gamma) \mid B(s_\alpha(\gamma'), \beta) \in X \}| + 2c
\]
\[
= |\{ \gamma' \in N(s_\gamma) \mid B(\gamma', \gamma) \in X \}| + 2c = d_X(\gamma) + 2c.
\]
\[\square\]

**Corollary 5.11.**

1. Let \( w \in W \) and \( \beta \in \Phi \). Then \( d_X(w(\beta)) \) and \( d_X(\beta) \) are integers of the same parity, and \(-2\ell(w) \leq d_X(w(\beta)) - d_X(\beta) \leq 2\ell(w)\)

2. \( d_X(\beta) \) is an odd integer or an even integer for all \( \beta \in \Phi \) according as to whether \( 1 \in X \) or \( 1 \notin X \).

**Proof.** Part (1) is proved by induction on \( \ell(w) \) using Proposition 5.10. Part (2) follows from (1) and the description of \( d_X(\alpha) \) for \( \alpha \in \Delta \) in Proposition 5.10, since every root \( \beta \) is in the \( W \)-orbit of some simple root \( \alpha \).

Thanks to Proposition 5.10, we can define \( X \)-depth on \( \Phi \).

**Definition 5.12.** Let \( X \subseteq \mathbb{R}^+ \), the \( X \)-depth on \( \Phi^+ \) is the function \( d_{X} : \Phi^+ \rightarrow \mathbb{N} \) defined as follows:
\[
d_{X}(\beta) = \begin{cases} 
\frac{d_{X}(\beta) - 1}{2}, & \text{if } 1 \in X \\
\frac{d_{X}(\beta)}{2}, & \text{if } 1 \notin X.
\end{cases}
\]

The following proposition follows from Proposition 5.10 together with Proposition 2.6 and Proposition 3.3.

**Proposition 5.13.**

1. If \( X = \mathbb{R}^+ \) then \( d_{X}(\beta) = d_p(\beta) - 1 \) for all \( \beta \in \Phi^+ \).

2. If \( X = [1, +\infty) \) then \( d_{X}(\beta) = d_{\infty}(\beta) \) for all \( \beta \in \Phi^+ \).

**Remark 5.14.** Sometimes, the depth of a positive root is defined so that the simple roots equal have depth 0 and not 1 as in this text. In this case, then the equality would be \( d_{X}(\beta) = d_p(\beta) \) in (1) above.

**Example 5.15.** We discuss some examples of the functions \( d_X : \Phi \rightarrow \mathbb{R} \), restricting to the case in which \( X \) is an order ideal of \( \mathbb{R}^+ \). The claims are readily checked using Proposition 5.10.

(a) Fix \( \alpha \in \Phi^+ \). If \( W \) is finite and \( X = [1, \infty[ \), then \( d_X(\alpha) = 0 \). If \( X = [1, \infty[ \), then \( d_X(\alpha) = 1 \). If \( X = \mathbb{R}^+ \), then \( d_X(\alpha) = \ell(s_\alpha) \).

(b) If \( W \) is a finite simply laced Weyl group then for any order coideal \( X \) of \( \mathbb{R}^+ \), one has \( d_X = d_Y \) where \( Y \) is either \( [1, \infty[ \), \( [1, \infty[ \), or \( \mathbb{R}^+ \).

(c) Suppose \( W \) is of type \( B/C_n \), with positive roots
\[
\Phi^+ = \{ \frac{1}{\sqrt{2}}(e_i \pm e_j) \mid 1 \leq i < j \leq n \} \cup \{ e_i \mid 1 \leq i \leq n \}.
\]
Let $\alpha \in \Phi^+$. If $1 \not\in X$, then $d_X(\alpha) = 0$. If $1 \in X$ but $\frac{1}{\sqrt{2}} \not\in X$, then $d_X(\alpha) = 1$. If $\frac{1}{2} \in X$, then $d_X = d_{\Phi^+}$, which is given by $d_{\Phi^+}((e_i - e_j)/\sqrt{2}) = 2j - 2i - 1$ and $d_{\Phi^+}((e_i + e_j)/\sqrt{2}) = 4n - 2i - 2j + 1$ if $1 \leq i < j \leq n$ and $d_{\Phi^+}(e_i) = 2n - 2i + 1$ for $1 \leq i \leq n$. In the remaining case ($\frac{1}{2} \not\in X$ but $\frac{1}{\sqrt{2}} \in X$), one checks that $d_X((e_i - e_j)/\sqrt{2}) = 1$ and $d_X((e_i + e_j)/\sqrt{2}) = 3$ if $1 \leq i < j \leq n$, while $d_X(e_i) = 2n - 2i + 1$ for $1 \leq i \leq n$.

5.2. **The weak order on $\Phi$.** If $X = \mathbb{R}^+$, we abbreviate $d_{\Phi^+}$ by $d$ and call it the *standard length function on $\Phi$. We saw in Example 5.8 that for any $\beta \in \Phi^+$, we have

$$d(\pm \beta) = \pm \ell(s_\beta) = \pm (\ell(\beta) - 1).$$

Proposition 5.10 implies that if $\beta \in \Phi$ and $\alpha \in \Delta$, then

$$d(s_n(\beta)) > d(\beta) \iff d(s_n(\beta)) = d(\beta) + 2 \iff B(\alpha, \beta) < 0.$$

**Definition 5.16.** Let $\prec$ be the relation on $\Phi$ defined by $\alpha \prec \beta$ if there is $\gamma \in \Delta$ such that $B(\gamma, \alpha) < 0$ and $\beta = s_\gamma(\alpha)$ both hold. We define the *weak order on $\Phi$* to be the partial order $\preceq$ on $\Phi$ obtained as the reflexive transitive closure of the relation $\prec$.

Since $d(\beta)$ is odd for all $\beta \in \Phi$, (3) implies that $\prec$ is the covering relation of $\preceq$. Thus, $\alpha \preceq \beta$ holds in $\Phi$ if and only if there exist $n \in \mathbb{N}$ and $\beta_0, \ldots, \beta_n \in \beta$ with

$$\alpha = \beta_0 \prec \beta_1 \prec \ldots \prec \beta_n = \beta.$$

Since $d(\beta_i) = d(\beta_{i+1}) + 2$ for all $i$ from 1 to $n$, this implies that

$$d(\beta) = d(\alpha) + 2n, \quad n = \frac{1}{2}(d(\beta) - d(\alpha)).$$

**Remark 5.17.** (a) If $\alpha \prec \beta$, then $\beta = \alpha + c\gamma$ where $c = -2B(\alpha, \gamma) > 0$, so $\gamma$ and $c$ are uniquely determined by $\alpha$ and $\beta$.

(b) In [2], the set $\Phi^+$ partially ordered by restriction of $\preceq$ is called the *root poset.*

Note that this partial order on $\Phi^+$ is not in general the order obtained by restricting weak order on $W$ to $T = \{s_\beta | \beta \in \Phi\}$ and transferring to $\Phi^+$ via the standard bijection $\beta \mapsto s_\beta$. It is easy to see that if $\beta \in \Phi^+$, then $-\beta \preceq \beta$. The map $\beta \mapsto -\beta$ for $\beta \in \Phi$ is an order-reversing bijection of $\Phi$ with itself in weak order. Note also that if $\beta \preceq \alpha$ where $\alpha \in \Delta$, then $\beta = -\alpha$.

5.3. **The root category.** The relationship between the weak orders on $\Phi$ and $W$ is clarified by the introduction of a certain category $\mathcal{C}$, which we shall call the *root category.*

First, let $\mathcal{C}' = G$ denote the transformation groupoid of $W$ on $\Phi$; this is the category (in fact, a groupoid) such that:

$$\text{ob}(G) = \Phi \text{ and } \text{mor}_G(\alpha, \beta) = \{(\beta, w, \alpha) | w \in W, w(\alpha) = \beta\},$$

with composition defined by $(\gamma, v, \beta)(\beta, w, \alpha) = (\gamma, vw, \alpha)$. For $\beta \in V$, $c \in \mathbb{R}$ and $\bullet$ denoting one of the symbols $\equiv$, $\prec$, $\leq$, $\geq$ or $\succ$, define $V^\bullet_\beta := \{\alpha \in V | B(\alpha, \beta) \bullet c\}.$

**Proposition 5.18.** (1) There is a subcategory $\mathcal{C}$ of $\mathcal{C}'$ with all objects of $\mathcal{C}'$, and only those morphisms $(\beta, w, \alpha)$ in $\mathcal{C}'$ with $N(w) \subseteq V^\prec_\beta$.

(2) There is a duality (contravariant involutive automorphism) of $\mathcal{C}$ given by $\alpha \mapsto -\alpha$ on objects and $(\beta, w, \alpha) \mapsto (-\alpha, w^{-1}, -\beta)$ on morphisms.
(3) For two morphisms \((\gamma, v, \beta)\) and \((\beta, w, \alpha)\) of \(C\), one has \(\ell(vw) = \ell(v) + \ell(w)\) and hence \(v \leq_R vw\) in right weak order \(\leq_R\) on \(W\).

(4) Let \((\gamma, x, \alpha)\) be a morphism in \(C\) and \(v \in W\) with \(v \leq_R x\). Set \(w := v^{-1}x\) and \(\beta := w(\alpha) = v^{-1}(\gamma)\). Then \((\gamma, v, \beta)\) and \((\beta, w, \alpha)\) are morphisms in \(C\) and \((\gamma, v, \beta)(\beta, w, \alpha) = (\gamma, x, \alpha)\).

Proof. Consider two composable morphisms \((\gamma, v, \beta)\) and \((\beta, w, \alpha)\) of \(C\) with both \(N(v) \subseteq V^0_\beta\) and \(N(w) \subseteq V^0_\alpha\). Then \(v(N(w)) \subseteq v(V^0_\beta) = V^0_{\ell(\beta)} = V^0_\gamma\). Therefore \(v(N(w)) \cap -N(v) = \emptyset\). By Corollary 2.10 we get \(\ell(vw) = \ell(v) + \ell(w)\), so \(v \leq_R vw\), and \(N(vw) = N(v) \cup v(N(w)) \subseteq V^0_\gamma\). This implies (1) and (3). Part (2) is readily proved using the fact that \(N(w^{-1}) = -w^{-1}(N(w))\). For (4), note that we have \(N(x) = N(v) \cup v(N(w))\) by Corollary 2.10 again, so \(N(v) \subseteq N(x) \subseteq V^0_\gamma\) and \(N(w) \subseteq v^{-1}(N(x)) \subseteq v^{-1}(V^0_\gamma) = V^0_{\ell^{-1}(\gamma)} = V^0_\beta\).

Corollary 5.19. Let \(\alpha, \beta \in \Phi\). Then there is a natural bijective correspondence between the set of maximal chains from \(\alpha\) to \(\beta\) in \((\Phi, \leq)\) and the set of reduced expressions of elements \(w \in W\) such that \((\beta, w, \alpha) \in \text{mor}_C(\alpha, \beta)\).

Proof. Consider \(\gamma \prec \gamma'\) in \((\Phi, \leq)\). Write \(\gamma' = s_\beta(\gamma)\) with \(\beta \in \Delta\). Then \((\gamma', s_\beta, \gamma)\) is in \(\text{mor}(C)\), by the definitions. On the other hand, if \((\gamma', s_\beta, \gamma)\) is in \(\text{mor}(C)\) with \(\beta \in \Delta\), then \(\gamma \prec \gamma'\) in \((\Phi, \leq)\).

Given a maximal chain \(c : \alpha = \alpha_0 \prec \cdots \prec \alpha_n = \beta\) in \((\Phi, \leq)\), let \(\beta_i \in \Delta\) with \(\alpha_i = s_\beta_i(\alpha_{i-1})\). By the above, \((\alpha_i, s_\beta_i, \alpha_{i-1})\) is in \(\text{mor}(C)\) for \(i = 1, \ldots, n\) with composite \((\alpha_n, s_\beta_n, \alpha_{n-1}) \cdots (\alpha_1, s_\beta_1, \alpha_0) = (\beta, w, \alpha) \in \text{mor}(C)\) where \(w := s_\beta_n \cdots s_\beta_1 \in W\). We have \(n = \ell(w)\) by Proposition 5.18, and we attach to \(c\) the above reduced expression defining \(w\).

On the other hand, consider a morphism \((\beta, w, \alpha)\) in \(C\) and a reduced expression \(w = s_\beta \cdots s_\beta_1\), where \(\beta_i \in \Delta\). Let \(w_i := s_\beta \cdots s_\beta_1\) and \(\alpha_i := w_i(\alpha)\) for \(i = 0, \ldots, n\). From Proposition 5.18 again, we see that \(c : \alpha = \alpha_0 \prec \cdots \prec \alpha_n = \beta\) is a maximal chain from \(\alpha\) to \(\beta\) in \((\Phi, \leq)\).

It is clear that the maps of the two previous paragraphs define inverse bijections as required.

Proposition 5.20. For all \(\alpha, \beta \in \Phi\) and \(w \in W\), one has \((\beta, w, \alpha) \in \text{mor}(C)\) if and only if \(\beta = w(\alpha)\) and \(d(\beta) = d(\alpha) + 2\ell(w)\) hold.

Proof. Suppose \((\beta, w, \alpha) \in \text{mor}(C)\). Then by the proof of Corollary 5.19 there is a maximal chain from \(\alpha\) to \(\beta\) in \((\Phi, \leq)\) of length \(\ell(w)\). By (6), this maximal chain necessarily has length \(\frac{1}{2}(d(\beta) - d(\alpha))\), proving that \(d(\beta) = d(\alpha) + 2\ell(w)\).

On the other hand, suppose that \(\beta = w(\alpha)\) where \(d(\beta) = d(\alpha) + 2\ell(w)\). Choose a reduced expression \(w = s_\beta \cdots s_\beta_i\), where each \(\beta_i \in \Delta\). Let \(w_i := s_\beta \cdots s_\beta_i\) and \(\alpha_i := w_i(\alpha)\) for \(i = 0, \ldots, n\). Then \(d(\alpha_i) \leq d(\alpha) + 2\ell(w_i) = d(\alpha) + 2i\) and \(d(\alpha_i) = d(w_i w^{-1} \beta) \geq d(\beta) - 2\ell(w_i w^{-1}) = d(\beta) - 2(\ell(w) - \ell(w_i)) = d(\alpha) + 2i\). Hence \(d(\alpha_i) = d(\alpha) + 2i\) and it follows from \(\alpha = \alpha_0 \prec \cdots \prec \alpha_n = \beta\) is a maximal chain from \(\alpha\) to \(\beta\) in \((V, \leq)\). By the proof of Corollary 5.19, \((\beta, w, \alpha) \in \text{mor}(C)\).
in \((\Phi, \subseteq)\). Let \(\beta_i \in \Delta\) with \(\alpha_i = s_{\beta_i}(\alpha_{i-1})\). Then \((\alpha_i, s_{\beta_i}, \alpha_{i-1})\) is in \(\text{mor}(C)\) for \(i = 1, \ldots, n\) with composite
\[
(\alpha_n, s_{\beta_n}, \alpha_{n-1}) \cdots (\alpha_1, s_{\beta_1}, \alpha_0) = (\beta, w, \alpha) \in \text{mor}(C)
\]
where \(w := s_{\beta_n} \cdots s_{\beta_1} \in W\). We have \(n = \ell(w)\) by Proposition \ref{prop:length}.

For \(i := 1, \ldots, n\), let \(c_i := B(\beta_i, \alpha_i) \in \mathbb{R}^+\),
\[
w_i := s_{\beta_i} \cdots s_{\beta_1} \quad \text{and} \quad \gamma_i := ww_i^{-1}(\beta_i) = s_{\beta_n} \cdots s_{\beta_{i+1}}(\beta_i) \in \Phi^+.
\]
We call \(L(p) := n\) the length of \(p\) and \(L_W(p) := w\) the \(W\)-length of \(p\). Define the simple root label \(\Delta(p) := (\beta_n, \ldots, \beta_1) \in \Delta^n\), the root label \(\Phi(p) = (\gamma_n, \ldots, \gamma_1) \in \Phi^n\) and the numerical label \(c(p) := (c_n, \ldots, c_1) \in (\mathbb{R}^+)^n\). We record the following simple relationships amongst these invariants for reference.

\[
\begin{align*}
(7) \quad & w = s_{\beta_n} \cdots s_{\beta_1} = s_{\gamma_n} \cdots s_{\gamma_1}, \quad n = \ell(w), \quad N(w) = \{\gamma_1, \ldots, \gamma_n\}. \\
(8) \quad & \alpha_i = w_i(\alpha) = w_iw_i^{-1}(\beta), \quad i = 0, \ldots, n \\
(9) \quad & \beta_i = w_iw_i^{-1}(\gamma_i), \quad c_i = B(\alpha_i, \beta_i) = B(\beta_i, \gamma_i), \quad i = 1, \ldots, n. \\
(10) \quad & s_{\beta} = wsw^{-1}, \quad d(\beta) = d(\alpha) + 2\ell(w). \\
(11) \quad & N(s_{\beta}) = N(w) \cup w(N(s_\alpha)) \cup -s_{\beta}(N(w)) \text{ if } \alpha \in \Phi^+. \\
(12) \quad & w(N(s_\alpha)) = \{\beta\} \text{ if } \alpha \in \Delta.
\end{align*}
\]

**Corollary 5.21.** Consider \(\alpha \leq \beta\) in \((\Phi, \subseteq)\). Fix a path \(p\) from \(\alpha\) to \(\beta\) and write its numerical label as \(\Delta(p) = (c_1, \ldots, c_n)\). Also let \((\beta, w, \alpha)\), where \(w \in W\), be in \(\text{mor}_C(\alpha, \beta)\) (e.g. \(w = L_W(p)\) is the \(W\)-length of \(p\)). Then for any \(X \subseteq \mathbb{R}^+\),
\[
|\{\gamma \in N(w) \mid B(\gamma, \beta) \in X\}| = |\{i \mid 1 \leq i \leq n, c_i \in X\}| = \frac{d_X(\beta) - d_X(\alpha)}{2}.
\]
In particular, the two cardinalities depend only on \(\alpha\) and \(\beta\), and for any \(\alpha \leq \beta\) in \(\Phi\), one has \(d_X(\alpha) \leq d_X(\beta)\).

**Proof.** The second equality in the displayed equation follows from Proposition \ref{prop:length} by induction on \(L(p)\). To prove the first equality in that equation, one may, by the proof of Corollary \ref{cor:length}, assume without loss of generality that \(w = L_W(p)\). Then the first equality is a consequence of \((7)\) and \((9)\). The final statement of the corollary follows from the displayed equation. \(\square\)

**Remark 5.22.** (a) The corollary refines (with simpler proof) unpublished results of the first author (M. D) used in his proof of \ref{prop:length} and related results.

(b) The corollary shows that, for all \(X \subseteq \mathbb{R}^+\), the function \(d_X\) is monotonic non-decreasing with respect to weak order on \(\Phi\).

### 5.5. Length functions and Bruhat order on root systems

In view of Remark \ref{rem:length}, we assume now that \(X\) is an order coideal of \(\mathbb{R}^+\): \(X\) is either \(\emptyset\), \(\mathbb{R}^+\) or an open or closed ray \([a, \infty[\) or \(]a, \infty]\) for some \(a > 0\).

The following lemma (in the case \(X = [1, \infty[\)) is needed for the proof of Proposition \ref{prop:length}.

**Lemma 5.23.**

1. Let \((\alpha, w, \alpha')\) be a morphism in \(C\). Then
\[
d_X(s_{\alpha}(\beta)) - d_X(\beta) \geq d_X(s_{\alpha'}(\beta')) - d_X(\beta')
\]
for all \(\beta, \beta' \in \Phi\) such that \(\beta = w(\beta')\) and \(B(\alpha, \beta) \leq 0\).
(2) Let $x \in \Phi$, $y \in \Phi^+$ with $B(x, y) \leq 0$. Then $d_X(s_y(y)) \geq d_X(y)$ with strict inequality if $B(x, y) \in -X$.

Proof. Note that in (1), $x = w(x')$ so $B(x, y) = B(x', y')$ for $x, y$ as there. By a predicate, we mean here a function which attaches to each element of its domain a truth value. Consider the predicate which attaches to any morphism $(x, y, x')$ in $C$ the truth value of the second sentence in (1). It is easy to check that this predicate is true on identity morphisms and that if it is true on each of two composable morphisms in $C$, it is true on their composite. Since every (non-identity) morphism in $C$ is a composite of morphisms $(x, y, x')$ with $y \in S$, we may assume without loss of generality that $w = s_{\gamma}$ for some $\gamma \in \Delta$.

Rearranging the equation in (1), we see now that it holds if and only if

$$d_X(s_x(x)) - d_X(s_x(s_x(x))) \geq d_X(x) - d_X(s_x(s_x(x)))$$

for all $x, y \in \Phi$ and $\gamma \in \Delta$ with $B(x, y) \leq 0$ and $B(\gamma, y) > 0$. We have $s_x(x) = y + c\alpha$ where $c := -2B(x, y) \geq 0$, so $B(\gamma, s_x(x)) = B(\gamma, y) + cB(\gamma, \alpha) \geq B(\gamma, y)$. Then Equation (13) follows by computing both sides by Proposition 5.10 using $B(\gamma, s_x(x)) \geq B(\gamma, y)$. This completes the proof of (1).

Now let $x, y$ be as in (2). Since $x \in \Phi^+$, we may choose a morphism $(x, y, x')$ in $C$ with $x' \in \Delta$, and set $\beta = w^{-1}(x)$. By (1), $d_X(s_x(x)) - d_X(x) \geq d_X(s_x(x')) - d_X(x')$. But since $B(x', x') \leq 0$, Proposition 5.10 implies that $d_X(s_x(x')) - d_X(x') \geq 0$ with strict inequality if $B(x', x') \in -X$ i.e. if $B(x, y) \in -X$. This proves (2).

The remaining results of this subsection are not required in the proof of Proposition 5.3. They provide a more conceptual interpretation of Lemma 5.23(2) as a monotonicity property of the length functions $d_X$ on $\Phi$, in terms of a second partial order $\leq'$ on $\Phi$ which we now define.

**Definition 5.24.** (i) Let $\leq'$ be the relation on $\Phi$ defined by $x \leq' y$ if there is $\gamma \in \Phi^+$ such that $c := B(\gamma, y) > 0$ and $\beta = s_x(y)$. This implies that $\beta = x - c\gamma$, so $\gamma$ and $c$ are uniquely determined by $x$ and $\beta$.

(ii) Define a preorder (reflexive transitive relation) $\leq'$ on $\Phi$ as the reflexive, transitive closure of the relation $\leq'$.

If $x \leq' y$, then $x - y \in \text{cone}(\Phi)$. Since $\text{cone}(\Phi) \cap - \text{cone}(\Phi) = \{0\}$, it follows that $\leq'$ is anti-symmetric i.e. it is a partial order, which we call the Bruhat order $\leq'$ on $\Phi$.

**Remark 5.25.** (a) Note that $\leq'$ is not the covering relation of $\leq'$ in general.

(b) The Bruhat order is stronger than weak order, in the sense that $x \leq y$ implies $x \leq' y$, for any $x, y \in \Phi$.

(c) The assertions of (1.7)(2), excluding those in the last sentence there, still hold with “weak order” replaced by “Bruhat order” and “$\leq$” replaced by “$\leq'$”.

(d) Analogues of weak and Bruhat orders on $W$-orbits on $V$ will be discussed more systematically in relation to orders on $W$ by the first author (M.D.) elsewhere.

Length functions have the following interesting behavior in relation to the Bruhat order on $\Phi$.

**Theorem 5.26.** Let $W'$ be a reflection subgroup of $W$, and $\leq'$ (resp., $\leq'$) denote Bruhat order on $\Phi$ (resp., $\Phi_{W'}$).

(1) If $x, y \in \Phi$ and $x \leq y$, then $d_X(x) \leq d_X(y)$.
Proof of Theorem 5.26. Part (1) follows from Lemma 5.23(2) and the definition of Bruhat order \( \leq \) on \( \Phi \). Part (3) follows from (2) by taking \( \beta := -\beta' \), so \( \beta \leq W, \beta' \) since \( \beta' \in \Phi_W^+ \), and dividing the resultant inequalities in (2) by 2.

It remains to prove (b). By definition of \( \leq_W, \) it is sufficient to do this in the case that \( \beta \leq_W \beta' \) i.e. when \( \beta' = s_\alpha(\beta) \) for some \( \alpha \in \Phi^+_W \) with \( \langle \alpha, \beta \rangle < 0 \). This immediately implies \( \beta \leq \beta' \), so \( \beta \leq \beta' \). One has \( 0 \leq d_{W',X}(\beta') - d_{W,X}(\beta') \) by (1) applied to \( W' \). To complete the proof of (2), we shall show by induction on \( n \in \mathbb{N} \) that for all reflection subgroups \( W' \) of \( W \) and all \( \beta \in \Phi_W, \) and \( \alpha \in \Phi_W^+ \) with \( B(\alpha, \beta) < 0 \) and \( d(\alpha) = 2n + 1 \), one has

\[
\tag{14} d_{W',X}(s_\alpha(\beta)) - d_{W,X}(s_\alpha(\beta)) \leq d_X(s_\alpha(\beta)) - d_X(\beta).
\]

If \( n = 0 \), then \( \alpha \in \Delta \). Since also \( \alpha \in \Phi_W^+ \), we have \( \alpha \in \Delta_W^+ \). Equality holds in (14), by computing both sides using Proposition 5.10 applied to \( W' \) and \( W \).

Assume now that \( n > 0 \). Choose \( \gamma \in \Delta \) so \( \alpha' := s_\gamma(\alpha) < \alpha \). Then \( d(\alpha') = 2n - 1 > 0 \) so \( \alpha' \in \Phi^+ \). Define the reflection subgroup \( W'' := s_\gamma W s_\gamma \) of \( W \) and the root \( \beta' := s_\gamma(\beta) \in \Phi \). One has \( \alpha' \in \Phi_W^+ \) and \( \beta' \in \Phi_W^+ \) with \( B(\alpha', \beta') < 0 \) and \( d(\alpha') = 2n - 1 \). By induction, we have \( d_{W',X}(s_{\alpha'}(\beta')) - d_{W,X}(\beta') \leq d_X(s_{\alpha'}(\beta')) - d_X(\beta') \).

That is,

\[
\tag{15} d_{W',X}(s_\gamma(\beta)) - d_{W,X}(s_\gamma(\beta)) \leq d_X(s_\gamma(\beta)) - d_X(s_\gamma(\beta)).
\]

We now consider two cases as follows. First, suppose that \( \gamma \in \Phi_W^+ \). Then \( W'' = W' \) and \( \gamma \in \Delta_W^+ \). By Proposition 5.10 applied to both \( W \) and \( W' \), one has \( d_{W',X}(s_\gamma(\beta')) - d_{W,X}(\beta') = d_X(s_\gamma(\beta')) - d_X(\beta') \) for all \( \beta' \in \Phi_W^+ \). Taking \( \beta' := s_\gamma(\beta) \) and \( \beta := \beta \) in turn in this, we see that (14) follows from (15).

The second and final case is that in which \( \gamma \not\in \Phi_W^+ \). Since \( \gamma \in \Delta \setminus \Delta_W^+ \), we have \( \Delta_W'' = s_\gamma(\Delta_W) \). It follows that the map \( \beta'' \mapsto s_\gamma(\beta'') \) defines a bijection \( \Phi_W^+ \to \Phi_W^+ \). This bijection is an order isomorphism \( (\Phi_W^+, \leq_W) \to (\Phi_W^+, \leq_W) \) in the corresponding weak orders, restricts to bijections \( \Delta_W^+ \to \Delta_W'' \) and \( \Phi_W^+ \to \Phi_W^+ \), and preserves bilinear forms in the sense \( B(s_\gamma(\beta''), s_\gamma(\beta''')) = B(\beta'', \beta''') \) for all \( \beta'', \beta''' \in \Phi_W^+ \). It follows that \( d_{W'',X}(s_\gamma(\beta'')) = d_{W,X}(\beta'') \) for all \( \beta'' \in \Phi_W^+ \). Taking \( \beta'' \) equal to \( s_\alpha(\beta) \) and to \( \beta \) in turn shows that the left hand sides of (15) and (14) are equal. On the other side, by (13), the right hand side of (15) is less than or equal to the right hand side of (14). Hence (14) also follows from (15) in this case. This completes the proof of (2) and of the theorem. \( \square \)

5.6. Consequences for \( \infty \)-depth and dominance order. Henceforward, we take \( X = [1, \infty) \). Recall that for \( \alpha \in \Phi^+ \) we have:

\[
d_{[1, \infty)}(\alpha) = 2d_{\infty}(\alpha) + 1 = 2|\text{Dom}(\alpha)| + 1,
\]

where \( \text{Dom}(\alpha) = \{ \beta \in \Phi^+ | \beta \prec \alpha \} \). Dominance order and \( \infty \)-depth are related to \( (\Phi, \leq) \) as follows.

Proposition 5.27. Let \( \beta \in \Phi^+ \). Choose a path \( \alpha_0 < \alpha_1 < \ldots < \alpha_n = \beta \) in \( (\Phi, \leq) \)

where \( \alpha_0 \in \Delta \). Write \( c(p) := (c_n, \ldots, c_1) \) and \( \Phi(p) = (\gamma_n, \ldots, \gamma_1) \). Then

(1) \( \text{Dom}(\beta) = \{ \gamma_i | 1 \leq i \leq n, c_i \geq 1 \} \)

(2) \( d_{\infty}(\beta) = |\{ i | 1 \leq i \leq n, c_i \geq 1 \}| \).
(3) If $(\beta, y, \alpha)$ is in mor$(C)$ and $\alpha \in \Delta$, then
\[ \text{Dom}(\beta) = \{ \gamma \in N(y) \mid B(\gamma, \beta) \geq 1 \}. \]

(4) $\text{Dom}(\beta) = y(\text{Dom}(\alpha)) \cup \{ \gamma \in N(y) \mid B(\gamma, \beta) \geq 1 \}$ if $(\beta, y, \alpha) \in \text{mor}(C)$ and $\alpha \in \Phi^+$. Write $\Delta$.

(5) $\text{Dom}(\beta) = \{ \gamma \in N(s_\beta) \mid B(\gamma, \beta) \geq 1, \ell(s_\gamma) < \ell(s_\beta) \}$. Write $\Delta$.

Proof. We prove $(1)$ using the facts listed in 5.4. Write $\Delta(p) = (\beta_n, \ldots, \beta_1)$. Suppose $1 \leq i \leq n$ and $c_i \geq 1$. Then $B(\gamma_i, \beta) = c_i \geq 1$. Also, since $\gamma_i = \beta_n \ldots \beta_i$, we have
\[ \ell(s_\gamma) \leq 2(n - i) + 1 < 2n + 1 = \ell(s_\beta). \]

This implies that $\gamma_i \prec \beta$. On the other hand, let $\gamma \in \Phi^+$ with $\gamma \prec \beta$. Set $w := L_W(p)$, so $w(\alpha_0) = \beta$. Hence $s_\alpha w^{-1}(\beta) = -\alpha_0$, we have $\gamma \in N(ws_\alpha) = N(w) \cup \{ \beta \} = \{ \gamma_n, \ldots, \gamma_1, \beta \}$. Therefore $\gamma = \gamma_i$ for some $1 \leq i \leq n$, with $1 \leq B(\gamma_i, \beta) = c_i$. Part $(2)$ follows by taking cardinalities in the equality in $(1)$.

To prove $(3)$ and $(4)$, we may suppose without loss of generality that $\alpha = \alpha_0$ and $y = L_W(p)$; so $\beta = y(\alpha_0)$. Then for $(3)$ $N(y) = \{ s_{\gamma_1}, \ldots, s_{\gamma_n} \}$ and $B(\gamma_i, \beta) = c_i$ for $i = 1, \ldots, n$, so $(3)$ follows from $(1)$. For $(4)$ choose a morphism $(\alpha, z, \delta)$ in $C$ with $\delta \in \Delta$. Then $(\beta, y, z) = (\beta, y, \alpha)(\alpha, z, \delta)$ in mor$(C)$. By $(5)$,
\[ \text{Dom}(\beta) = \{ \gamma \in N(\gamma y) \mid B(\beta, \gamma) \geq 1 \} \text{ and } \text{Dom}(\alpha) = \{ \gamma \in N(\alpha y) \mid B(\beta, \gamma) \geq 1 \}. \]

But $N(\gamma y) = N(\gamma) \cup y(N(\gamma))$ and for $\gamma \in N(\gamma)$, one has $B(\gamma, \alpha) = B(y(\gamma), \beta)$, so $(4)$ follows.

Finally, we prove $(5)$. Let $y = L_W(p)$. Then $\beta = y(\alpha_0)$ and $s_\beta = y s_{\alpha_0} y^{-1}$ where $\ell(s_\beta) = 2\ell(y) + 1$. Hence $N(s_\beta)$ is the disjoint union $N(s_\beta) = N(\gamma) \cup \{ \beta \} \cup -\gamma$. By $(3)$, it will suffice to show that if $\gamma \in \{ \beta \} \cup -\gamma(N(\gamma))$, then either $\ell(s_\gamma) > \ell(s_\gamma)$ or $B(\gamma, \beta) < 1$. This is trivial if $\gamma = \beta$. Otherwise, write $\gamma = -s_\beta(\gamma')$ where $\gamma' \in N(\gamma)$. Then $\ell(s_\gamma') < \ell(s_\beta)$ and $B(\gamma, \beta) = B(\gamma', \beta)$. If $B(\gamma', \beta) < 1$, then $B(\gamma, \beta) < 1$ also. If $B(\gamma', \beta) \geq 1$, then $\gamma' \prec \beta$ by $(5)$.

The following facts can be easily deduced from the definition of dominance order, see 6.2 [Lemma 2.2]: the action of $W$ preserves the dominance order so $-\gamma = s_\beta(\gamma') \prec s_\beta(\beta) = \beta$; multiplication by $-1$ reverses the dominance order so $\beta \prec \gamma$ and in this case $dp(\beta) < dp(\gamma)$. Therefore by Example 5.8 we obtain $\ell(s_\beta) < \ell(s_\gamma)$ as required.

5.7. Proof of Proposition 5.4 Assume first for the proof of 5.4 (1) that $W$ is infinite. We use here Proposition 5.27. Write $\Delta_W = \{ \alpha, \beta \}$. Note that the inner product of any two roots in $\Phi_{W^\vee}$ is greater than or equal to $1$ in absolute value, since $W^\vee$ is infinite. There is some $\rho \in \Delta_W$, with $B(\rho, \delta) > 0$. Interchanging $\alpha$ and $\beta$ if necessary, we assume without loss of generality that $B(\alpha, \delta) > 0$. If also $B(\alpha, \gamma) > 0$, then $\alpha \leq \gamma < \delta$ since $1 = \ell_W(s_\alpha) \leq \ell_W(s_\gamma) < \ell_W(s_\delta)$. Hence Dom$(\delta) \supseteq \text{Dom}(\gamma) \cup \{ \gamma \}$ and $\text{dp}_\infty(\delta) = |\text{Dom}(\delta)| \geq |\text{Dom}(\gamma)| + 1 > \text{dp}_\infty(\gamma)$. The other case is that $B(\alpha, \gamma) \leq 1$. Let $\gamma' := s_\alpha(\gamma)$. Then $\text{dp}_\infty(\gamma) > \text{dp}_\infty(\gamma')$ by Lemma 5.23 (2). We have $B(\alpha, \gamma') > 0, B(\alpha, \delta) > 0$ and $\ell_W(s_\gamma) = \ell_W(s_\gamma) + 2 \leq \ell_W(s_\delta)$, so $\alpha < \gamma' < \delta$. This gives either $\gamma' = \delta$ or else $\text{dp}_\infty(\gamma') < \text{dp}_\infty(\delta)$, by arguing as before but with $\gamma$ replaced by $\gamma'$. In either case, $\text{dp}_\infty(\gamma) < \text{dp}_\infty(\gamma') \leq \text{dp}_\infty(\delta)$ as required.
Now assume for the proof of [5.4(2)] that $W'$ is finite. By induction on $\ell_{W'}(x)$, it is sufficient to prove the assertion there in the special case that $x = s_\alpha$ where $\alpha \in \Delta_{W'}$. But then the hypotheses $\delta = s_\alpha \gamma$ with $\ell_{W'}(s_\delta) = \ell_{W'}(s_\gamma) + 2$ imply that $B(\gamma, \alpha) < 0$ and so $dp_\infty(\gamma) \leq dp_\infty(s_\alpha(\gamma)) = dp_\infty(s_\delta)$ as required, by Lemma [5.23(2)] again. This finishes the proof of Proposition 5.4.

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(Christophe Hohlweg) Université du Québec à Montréal, LACIM et Département de Mathématiques, CP 8888 Succ. Centre-Ville, Montréal, Québec, H3C 3P8, Canada

E-mail address: hohlweg.christophe@uqam.ca

URL: http://hohlweg.math.uqam.ca

Department of Mathematics, 255 Hurley Building, University of Notre Dame, Notre Dame, Indiana 46556, U.S.A.

E-mail address: dyer.1@nd.edu