The Second Order Estimate for Fully Nonlinear Uniformly Elliptic Equations without Concavity Assumption

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Abstract

Investigating for interior regularity of viscosity solutions to the fully nonlinear elliptic equation

\[ F(x, u, \nabla u, \nabla^2 u) = 0, \]

we establish the interior \( C^{1+1} \) continuity under the assumptions that \( F \) is uniformly elliptic, Hölder continuous and satisfies the natural structure conditions of fractional order, but without the concavity assumption of \( F \). These assumptions are weaker and the result is stronger than that of Caffarelli and Wang[1], Chen[2].

1. Introduction

The study of solvability problems for the second order fully nonlinear uniformly elliptic equation, i.e., the existence, uniqueness and regularity of solutions with Dirichlet boundary data can be divided into two classes naturally. The first one is about classical solvability, i.e., solution \( u \in C^2 \). The related results are rich and systematic, See [3], [4], [5] and therein. All of them is obtained under the assumption of concavity for the equation with respect to their arguments. The second is without the concave condition of \( F \). In this case, we must seek for solution in generalized sense. The suitable one is viscosity solution[6]. But in this case the results are incomplete. Because the regularity is a key stone for existence and uniqueness, we prove the interior \( C^{1+1} \) continuity without the concavity assumption of \( F \) in this paper and the existence and uniqueness results will be in next one. Our assumptions are \( F(x, z, p, X) \in C^\beta (\beta > 0) \) of its arguments only, weaker than those in Caffarelli and Wang[1] (\( \beta = 1 \)) and Chen [2] (\( \beta > 1/2 \)). And the result, \( u \in C^{1+1} \) is stronger than theirs (\( u \in C^{1+\alpha}, 0 < \alpha \) and small ). This paper is organized in five sections: The second is preliminaries, statements for conditions and the main results. The third is general comparison principle. For solution \( u(x) (x \in \Omega, \Omega \subset \mathbb{R}^n) \), we investigate the general conditions for \( u(x) - u(y) - \Phi(x, y) \) takes maximum in a \( \mathbb{R}^{2n} \) domain \( Q \subset \Omega \times \Omega \), where \( \Phi(x, y) \in C^2(Q) \). If these conditions are violate, together with the assumption
\[ u(x) - u(y) - \Phi \|_{Q} \leq 0, \]
we have \( u(x) - u(y) - \Phi \leq 0, \) and the useful estimation \( u(x) - u(y) \leq \Phi \) follows. The fourth is Hölder and Lipschitz continuity. The Hölder and Lipschitz estimates are obtained by selecting suitable \( Q \) and \( \Phi \). Although these results are not new, we prove them here for the sake of applications of the results of section 3.

The last section is \( C^{1+1} \) estimate. Having got the interior Lipschitz estimate for \( u(x) \), we can conclude that there exist Caffarelli points such that in small neighborhood of them \( u(x) \) can be separated into linear and second order parts. We estimate the lipschitz coefficient of \( u(x) \) subtracting linear part in suitable neighborhood of Caffarelli point by selecting suitable comparison function \( \Phi(x, y) \). By this way we get a fundamental lemma: the lipschitz coefficient is diminishing in certain constant ratio accompanied with diminishing of radius of spherical neighborhood in suitably constant ratio.

The \( C^{1+\alpha} \) estimate and \( C^{1+1} \) in small part follows from this fundamental lemma. And the \( C^{1+1} \) estimate in the whole region follows by putting the estimates in small parts altogether.

2. Preliminaries

Let \( \Omega \) be a bounded domain in \( \mathbb{R}^n \) with the boundary \( \partial \Omega \in C \). We consider the regularity problem for solutions of the equation

\[
(2.1) \quad F(x, u, \nabla u, \nabla^2 u) = 0 \quad \text{in} \quad \Omega
\]

where \( F(x, u, p, X) \) is a function on \( \Gamma = \Omega \times \mathbb{R} \times \mathbb{R}^n \times S^n \), \( S^n \) the space of \( n \times n \) symmetric matrices equipped with usual order. We assume that \( F(x, u, p, X) \) is uniformly elliptic in the following sense

\[
(2.2) \quad \lambda \text{Tr}(X - Y) \leq F(x, z, p, Y) - F(x, z, p, X) \leq \Lambda \text{Tr}(X - Y)
\]

for all \( (x, z, p, X) \in \Gamma, \ Y \in S^n \) and \( X \geq Y \), where \( \lambda \) and \( \Lambda \) are positive constants with \( \lambda \leq \Lambda \). Assume that \( F \) is monotone with respect to \( z \)

\[
(2.3) \quad F(x, z, p, X) - F(x, w, p, X) \leq 0
\]

\( \forall z \leq w \). Furthermore, we suppose that there exist positive constants \( \mu \) and \( \beta \) with \( \beta \leq 1 \) such that

\[
(2.4) \quad |F(x, z, p, X) - F(y, w, q, X)| \leq \mu |(x - y)(1 + |p| + |q|) + \frac{|p - q|}{1 + |p| + |q|}^\beta \\
\cdot (1 + |p|^2 + |q|^2 + ||X||)
\]

\( \forall (x, z, p, X) \in \Gamma \) and \( (y, w, q, X) \in \Gamma \). \( (2.4) \) is called natural structural condition for \( F \) of fractional order (order \( \beta \), where \( 0 < \beta \leq 1 \)).

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Now we give the definition of viscosity solutions and main theorem.

**Definition 1.** Let $u$ be an upper semi-continuous (resp. lower semi-continuous) function in $\Omega$. $u$ is said to be a viscosity subsolution (resp. supersolution) of (2.1) if for all $\phi \in C^2(\Omega)$ the following inequality

$$F(x_0, u(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \leq 0$$

(resp. $F(x, u(x_0), \nabla\phi(x_0), \nabla^2\phi(x_0)) \geq 0$)

holds at each local maximum (resp. minimum) point $x_0 \in \Omega$ of $u - \phi$.

**Definition 2.** $u \in C(\Omega)$ is said to be a viscosity solution of (2.1) if $u$ is both a viscosity subsolution and a supersolution.

**Theorem 2.1.** Assume $F(x, z, p, X)$ satisfies the conditions (2.2) − (2.4) and $u$ is a viscosity solution of (2.1), then $u \in C^{1+1}(\Omega)$.

In this paper, we don’t study any boundary value problem for (2.1). About the existence and uniqueness for solution of boundary value problems, we shall discuss them in a next paper.

### 3. General Comparison Principle

For suitably selected regular non-negative function $\Phi(x, y)$ and a bounded domain $Q \subset \mathbb{R}^{2n}$, we suppose that $u(x) - u(y) - \Phi(x, y)$ takes a positive maximum value at an interior point $(\bar{x}, \bar{y}) \in Q$, then

$$u(x) - u(y) - \Phi(x, y) \leq u(\bar{x}) - u(\bar{y}) - \Phi(\bar{x}, \bar{y}) > 0$$

For simplicity, we omit the upper bar on $x, y$ in the following, i.e., write $(x, y) = (\bar{x}, \bar{y})$, thus

(3.1) \quad u(x) - u(y) > \Phi(x, y) \geq 0.

In particular we have, $x \neq y$. From [6], there exist $X, Y \in \mathbb{S}^n$ such that

(3.2) \quad F(x, u(x), \Phi_x(x, y), X) - F(y, u(y), -\Phi_y(x, y), -Y) \leq 0

and

(3.3) \quad \begin{pmatrix} X & 0 \\ 0 & Y \end{pmatrix} \leq \begin{pmatrix} \Phi_{xx} & \Phi_{xy} \\ \Phi_{yx} & \Phi_{yy} \end{pmatrix}

From above inequality, we have

(3.4) \quad \begin{pmatrix} X + Y & X - Y \\ X - Y & X + Y \end{pmatrix} \leq \begin{pmatrix} Z_1 & Z \\ Z & Z_2 \end{pmatrix},

where

(3.5) \quad Z_1 = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})^2\Phi,

\quad Z_2 = (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})^2\Phi,

\quad Z = (\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(\frac{\partial}{\partial x} + \frac{\partial}{\partial y})\Phi.
A consequence of (3.4) are

\[(3.6) \quad X + Y - Z_1 \leq 0, \quad X + Y - Z_2 \leq 0.\]

∀ \(\sigma \in \mathbb{R}\) and \(\xi \in \mathbb{R}^n\), multiplying (3.4) from left and right by \((\sigma\xi, \xi)\) and \((\sigma\xi, \xi)^T\) respectively, we get

\[\sigma^2 < (X + Y - Z_1)\xi, \xi > + 2\sigma < (X - Y - Z)\xi, \xi > + < (X + Y - Z_2)\xi, \xi > \leq 0.\]

Hence, for \(\xi \in \mathbb{R}^n\) with \(|\xi| = 1\),

\[< (X - Y - Z)\xi, \xi >^2 \leq < (X + Y - Z_1)\xi, \xi > < (X + Y - Z_2)\xi, \xi > .\]

i.e.,

\[||X - Y - Z||^2 \leq ||X + Y - Z_1|| ||X + Y - Z_2|| \leq ||X + Y - Z_1||^2 + ||Z_2 - Z_1|| ||X + Y - Z_1||.\]

This inequality implies that

\[||X - Y - Z|| \leq ||X + Y - Z_1|| + ||Z_2 - Z_1||^{1/2} ||X + Y - Z_1||^{1/2} \leq C_1[|Tr(X + Y - Z_1)| + ||Z_2 - Z_1||^{1/2}|Tr(X + Y - Z_1)|^{1/2}].\]

Thus,

\[||X|| + ||Y|| \leq ||X + Y|| + ||X - Y|| \leq ||Z_1|| + ||Z|| + ||X + Y - Z_1|| + ||X - Y - Z|| \leq C_1[||Z_1|| + ||Z|| + 2|Tr(X + Y - Z_1)| + ||Z_2 - Z_1||^{1/2}|Tr(X + Y - Z_1)|^{1/2}].\]

Taking a positive parameter \(\omega\) to be determined later. From the above inequality, we have

\[\omega(||X|| + ||Y||) \leq C_1\omega(||Z_1|| + ||Z|| + 2|Tr(X + Y - Z_1)|| + \frac{\lambda}{2}|Tr(X + Y - Z_1)| + \frac{2C_1\omega^2}{\lambda}||Z_2 - Z_1||.\]

On the other hand, by (2.2) and (3.6), we have

\[\lambda|Tr(X + Y - Z_1)| = \lambda Tr(Z_1 - X - Y) \leq F(x, u(x), \Phi_x, X) - F(x, u(x), \Phi_x, Z_1 - Y).\]
Applying (3.2) and (2.3), we have
\[ F(x, u(x), \Phi_x, X) \leq F(y, u(y), -\Phi_y, -Y) \]
\[ \leq F(y, u(x), -\Phi_y, -Y). \]
Separating \(Z_1\) into \(Z_1^+ + Z_1^-\) where \(Z_1^+ \geq 0, Z_1^- \leq 0\) and by (2.4), we obtain
\[ -F(x, u(x), \Phi_x, X) \leq -F(x, u(x), X) + \Lambda \text{Tr} Z_1^+ - \lambda \text{Tr} Z_1^- \]
Combining the above inequalities and applying (2.4), we have
\[ \lambda |\text{Tr}(X + Y - Z_1)| \]
\[ \leq C_2 \{ ||Z_1|| + [(1 + A)|x - y| + \frac{|Z_0|}{1 + A}] \}^\beta \]
(3.8)
\[ -(1 + A^2 + ||X|| + ||Y||) \}, \]
where
\[ Z_0 = \Phi_x + \Phi_y = (\frac{\partial}{\partial x} + \frac{\partial}{\partial y})\Phi, \]
(3.9)
\[ A = |Z_0| + \{(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})\Phi\}. \]
Taking the parameter \(\omega\) to be
\[ \omega = C_2 [(1 + A)|x - y| + \frac{|Z_0|}{1 + A}]^\beta. \]
Combining (3.7), (3.8) and (3.10), we have
\[ \frac{\lambda}{2} - 2C_1 \omega)|\text{Tr}(X + Y - Z_1)| \]
\[ \leq C \{ ||Z_1|| + (1 + A^2 + ||Z||)\omega + ||Z_2 - Z_1||\omega^2 \}. \]
As \(\omega\) is small, \(\omega \ll 1\), (3.11) represent an upper estimate for \(|\text{Tr}(X + Y - Z_1)|\).
We still need a lower estimate for \(|\text{Tr}(X + Y - Z_1)|\).
Let \(P\) be a \(n \times n\) diagonal matrix with \(0 < P \leq I\), where \(I\) is unit matrix. Since \(\text{Tr}\) is invariant under coordinate rotation, denote \(S\) be coordinate rotation matrix, which is symmetric and satisfies \(S^2 = I\), then
\[ -\text{Tr}[SPS(Z_2 - Z_1)] \]
\[ = -\text{Tr}[S^2PS(Z_2 - Z_1)S] = -\text{Tr}[PS(Z_2 - Z_1)S] \]
\[ = \text{Tr}[PS(Z_1 - X - Y)S] - \text{Tr}[PS(Z_2 - X - Y)S] \]
Applying (3.6), we get
\[ S(Z_1 - X - Y)S \geq 0, S(Z_2 - X - Y)S \geq 0 \]
Since $P$ is diagonal and $0 < P \leq I$, we have
\[ PS(Z_1 - X - Y)S \leq S(Z_1 - X - Y)S, \quad PS(Z_2 - X - Y)S \geq 0. \]

Hence
\[ -Tr[SPS(Z_2 - Z_1)] \leq Tr[S(Z_1 - X - Y)S] = Tr(Z_1 - X - Y) = |Tr(X + Y - Z_1)| \]

Setting
\[ \Upsilon = \frac{(x - y) \otimes (x - y)}{|x - y|^2} \]

it satisfies $0 \leq \Upsilon \leq I$, and selecting $S$ such that $SPS = \frac{1}{1+\varepsilon}(\Upsilon + \varepsilon I)$ and let $\varepsilon \to 0$, we have
\[ (3.12) \quad -Tr[\Upsilon(Z_2 - Z_1)] \leq |Tr(X + Y - Z_1)|. \]

(3.12) is a lower estimate for $Tr(X + Y - Z_1)$ which are needed. Combining (3.11) and (3.12), we have
\[ (3.13) \quad -(\frac{\lambda}{2} - 2C_1\omega)Tr(\Upsilon Z_2) \leq C[||Z_1|| + (1 + A^2 + ||Z||)\omega + ||Z_2||\omega^2]. \]

which is a necessary condition for $u(x) - u(y) - \Phi(x, y)$ takes positive maximum value in $Q$. If (3.13) is violate, i.e.
\[ (3.14) \quad -(\frac{\lambda}{2} - 2C_1\omega)Tr(\Upsilon Z_2) > C[||Z_1|| + (1 + A^2 + ||Z||)\omega + ||Z_2||\omega^2], \]

then $u(x) - u(y) - \Phi(x, y)$ cannot take positive maximum value in $Q$, hence \forall $(x, y) \in Q$, we have
\[ (3.15) \quad u(x) - u(y) - \Phi(x, y) \leq [u(x) - u(y) - \Phi(x, y)]|\partial Q \]

The above all are general discuss.

4. Hölder and Lipschitz continuity

**Theorem 4.1** Let $u$ be a viscosity solution to (2.1) in $\Omega$ and suppose that (2.2)-(2.4) hold, then there exists a constant $\alpha \in (0, 1)$, such that $u$ is Hölder continuous with exponent $\alpha$ in $\Omega$ and the following estimate holds \forall $x, y \in \Omega$,
\[ (4.1) \quad |u(x) - u(y)| \leq \frac{C}{d^\alpha} |x - y|\alpha, \]

where $d = \min[d(x, \partial \Omega), d(y, \partial \Omega)]$ and $C$ is a constant depending only on $n, \lambda, \Lambda, \mu, \beta$ and $\sup_{x \in \Omega}|u(x)|$. 


Proof: Let $x_0 \in \Omega$ and $R > 0$ such that $B(x_0, 2R) \subset \Omega$. Without loss of generality, suppose $x_0$ is the origin and $\sup_{B_{2R}}|u(x)| = 1$. Let $\alpha < 1$ and $K > 1$ be two constants to be chosen and taking the domain

$$Q = \{(x, y) \in \Omega ||x|^2 + |y|^2 < R^2\}$$

and the function

$$\Phi(x, y) = \frac{2}{R^2}(|x + y|^2 + |x - y|^2) + K\frac{|x - y|^\alpha}{R^\alpha}$$

and setting

$$w = u(x) - u(y) - \Phi(x, y),$$

then we have $w \leq 0$ on $\partial Q$. We claim that $w \leq 0$ in $Q$. If it is not true, there exists a positive maximum value of $w$ in $Q$ at $(x, y)$ and by (3.1) and (3.2), we have

$$|x - y| \leq \left( \frac{|u(x) - u(y)|}{K} \right)^{1/\alpha} R \leq \left( \frac{2}{K} \right)^{1/\alpha} R,$$

or

$$K \left( \frac{|x - y|}{R} \right)^\alpha \leq 2. \tag{4.3}$$

It is easy to see

$$|Z_0| \leq CR^{-1},$$

$$\frac{\alpha K}{R^{\alpha}} |x - y|^{\alpha - 1} \leq A \leq C(\frac{R^{-1} + \alpha K}{R^{\alpha}}) |x - y|^{\alpha - 1}$$

where $Z_0$ and $A$ are defined by (3.9). From (3.10) and (4.3), we have

$$\omega \leq C \left| \frac{|x - y|}{R} \right| + \alpha K \left( \frac{|x - y|}{R} \right)^{\alpha} + (\alpha K)^{-1} \left( \frac{|x - y|}{R} \right)^{1 - \alpha} \beta \leq C(\alpha + \alpha^{-1} K^{-1/\alpha}) \beta = o(1),$$

as taking $\alpha$ small first and then taking $K$ large. By the definition of (3.5), we get

$$||Z_1|| \leq CR^{-2}, Z = 0,$$

$$||Z_2|| \leq CR^{-2} + C\alpha KR^{-\alpha} |x - y|^{\alpha - 2}.$$

Hence, for our $\Phi$, by applying (4.3), (3.13) becomes

$$\frac{4\alpha(1 - \alpha)K}{R^\alpha} |x - y|^\alpha + O(R^{-2}) \leq C|R^{-2} + (\alpha KR^{-\alpha} |x - y|^{\alpha - 1})^2 \omega + \alpha KR^{-\alpha} |x - y|^{\alpha - 2} \omega^2|,$$

$$\leq C|R^{-2} + \alpha^2 KR^{-\alpha} |x - y|^{\alpha - 2} \omega + \alpha KR^{-\alpha} |x - y|^{\alpha - 2} \omega^2|.$$ 

(4.4) can not hold when we take $\alpha$ small first and then take $K$ large. Thus $u(x) - u(y) - \Phi$ cannot takes positive maximum value in $Q$. Combining estimate for $|u(y) - u(x) - \Phi|_{\partial Q}$, we have

$$|u(x) - u(y)| \leq \Phi = \frac{2}{R^2}(|x + y|^2 + |x - y|^2) + K\frac{|x - y|^\alpha}{R^\alpha}.$$


Especially,

$$|u(x) - u(0)| \leq \frac{4}{R^2}|x|^2 + K \frac{|x|^\alpha}{R^\alpha} \leq (4 + K) \frac{|x|^\alpha}{R^\alpha},$$

By coordinate translation, \( \forall \ y \) such that \(|y| < R\), we have

$$|u(x + y) - u(y)| \leq (4 + K) \frac{|x|^\alpha}{R^\alpha},$$

The theorem follows by substituting \( x + y \) for \( x \).

**Theorem 4.2** Let \( u \) be a viscosity solution of (2.1) in \( \Omega \) and suppose that (2.2) – (2.4) hold, then \( u \) is locally Lipschitz continuous and satisfies the following estimate

(4.5) $$|u(x) - u(y)| \leq \frac{C}{d}|x - y|.$$  

**Proof:** Let \( x_0 \in \Omega \) and \( R > 0 \) such that \( B(x_0, 2R) \subset \Omega \). Without loss of generality, suppose \( x_0 \) is the origin and \( \sup_{B_{2R}}|u(x)| = 1 \). Consider the function

(4.6) $$\Phi(x, y) = \frac{2}{R^2}(|x + y|^2 + |x - y|^2) + 4\left[1 - \frac{1}{4}(\frac{K|x - y|}{R})^\gamma\right] \frac{K|x - y|}{R}$$

in the domain

$$Q = \{(x, y) \in \Omega||x|^2 + |y|^2 < R^2, K|x - y| < R\}.$$  

where \( \gamma(\leq \alpha \beta) \) and \( K(\geq 1) \) are positive constants to be chosen. Obviously, we have \( w = u(x) - u(y) - \Phi(x, y) \leq 0 \) on \( \partial Q \). If \( w \) takes positive maximum in \( Q \) at \((x, y)\), by (3.1), (4.1) and (4.6), we have

$$2\frac{|x + y|^2}{R^2} \leq \Phi \leq u(x) - u(y) \leq \frac{C|x - y|^\alpha}{R^\alpha},$$

hence

$$|x + y| \leq CR(\frac{|x - y|}{R})^{\alpha/2}$$

and

$$K (\frac{|x - y|}{R}) \leq \Phi \leq u(x) - u(y) \leq \frac{C|x - y|^\alpha}{R^\alpha},$$

$$K \left(\frac{|x - y|}{R}\right)^{1-\alpha} \leq C, \quad K \frac{|x - y|^\alpha}{R} \leq C, \quad K \frac{|x - y|}{R} \leq CK^{-\alpha}.$$  

Thus

$$K \left(\frac{|x - y|}{R}\right) \leq CK^{-\frac{\alpha}{\alpha - 1}}(\frac{|x - y|}{R})^{\frac{1}{2}},$$

and the corresponding \( Z_0, A \) and \( \omega \) defined by (3.9) and (3.10) have the following estimates.

$$|Z_0| = O(R^{-1}(\frac{|x - y|}{R})^{\frac{\alpha}{2}}), \quad \frac{K}{R} \leq A = O(\frac{K}{R}),$$

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\[ \omega = O((1 + A)|x - y| + \frac{|Z_0|}{1 + A})^{\beta} \]
\[ = O(K^{-\alpha \beta} \frac{|x - y|}{R})^{\alpha \beta} = o(1) \]
as \( K \) is large. And
\[ ||Z_1|| = O(R^{-2}), \ Z = 0, \ ||Z_2|| = O(K^{-\alpha \beta}) \]
\[ -Tr(\mathcal{Y}Z_2) = O(R^{-2}) + (1 + \gamma) \frac{K^2}{R^2} \frac{|x - y|}{R}^{\alpha \beta} K \frac{R}{R|x - y|} \gamma - 1, \]
where \( Z, Z_1, Z_2 \) are defined by (3.5).

Substituting the above estimates into (3.13), we obtain
\[ (4.7) \]
\[ [O(R^{-2}) + (1 + \gamma) \frac{K^2}{R^2} \frac{|x - y|}{R}]^{\gamma - 1} \]
\[ \leq C[O(R^{-2}) + K^{-\alpha \beta} \frac{|x - y|}{R})^{\alpha \beta} K \frac{R}{R|x - y|} \]
\[ + K^{-\alpha \beta} \frac{|x - y|}{R}^{\alpha \beta} K \frac{R}{R|x - y|}. \]

Since \( \frac{|x - y|}{R} \leq C \), then \( (\frac{|x - y|}{R})^{\gamma - 1} \geq C \). If we take \( \gamma = \alpha \beta \) and \( K \) large enough, (4.7) cannot hold. This means
\[ |u(x) - u(y)| \leq \Phi(x, y). \]
In particular, we have
\[ |u(x) - u(0)| \leq \Phi(x, 0) \]
\[ \leq 4|x|^2 + \frac{4K|x|}{R} \leq 4(1 + K) \frac{|x|}{R}. \]
Applying coordinate translation, \( \forall \ y \) such that \( |y| < K \) we have
\[ |u(x + y) - u(y)| \leq 4(1 + K) \frac{|x|}{R}. \]
The theorem is proved by substituting \( x + y \) for \( x \). \( \Box \)

5. \( C^{1+1} \) interior estimate

**Definition 3.** \( u \) is said in \( C^{1+1}(\Omega) \), if \( Du(x) \) exists \( \forall x \in \Omega \) and moreover, \( Du(x) \) satisfies Lipschitz condition for all closed subset \( \tilde{\Omega} \subseteq \Omega \), and, for all line segment \( \overline{xy} \in \tilde{\Omega}, \)
\[ (5.1) \]
\[ |Du(x) - Du(y)| \leq C|x - y|, \]
where \( C \) depends only on \( n, \lambda, \Lambda, \mu, \beta \) and \( \text{dist}(\tilde{\Omega}, \partial \Omega) \).
If (5.1) is replaced by H"older condition and
\begin{equation}
|Du(x) - Du(y)| \leq C|x - y|^\alpha (0 < \alpha < 1),
\end{equation}
we call \(u \in C^{1+\alpha}(\Omega)\).

The following is a covering theorem.

**Theorem 5.1** For any given sufficiently small positive constant \(\theta\) and \(\forall B(x_0, R_0) \subset \Omega\) with \(R_0 \leq \theta\), \(u \in C^{1+1}(B(x_0, \xi R_0))\) where \(\xi\) is a constant \(\xi \in (0, 1)\), then \(u \in C^{1+1}(\Omega)\).

**Proof:** For any \(\tilde{\Omega} \subset \subset \Omega\) and all \(x, y \in \tilde{\Omega}\) with \(xy \in \tilde{\Omega}\), we cover \(xy\) by a finite set of spheres \(\{B(z_i, \xi R_i)\}_{0 \leq i \leq m-1} \subset \tilde{\Omega}\), with \(B(z_i, R_i) \subset \tilde{\Omega}\), where \(R_i \leq \theta\) with \(x_0 = x, x_m = y\) and \(x_i x_{i+1} \in B(z_i, \xi R_i)\) \((0 \leq i \leq m - 1)\). Hence we have
\begin{equation}
|Du(x) - Du(y)| = | \sum_{i=0}^{m-1} [Du(x_i) - Du(x_{i+1})] |
\end{equation}
\begin{equation}
\leq \sum_{i=0}^{m-1} |Du(x_i) - Du(x_{i+1})| \leq C \sum_{i=0}^{m-1} |x_i - x_{i+1}| = C|x - y|.
\end{equation}
The theorem is proved. \(\square\)

Fix a \(B(x_0, R_0)\) such that \(\overline{B(x_0, R_0)} \subset \Omega\) and by section 4, we see that there exists a constant \(M_1\) such that \(\forall x, y \in B(x_0, R_0)\)
\begin{equation}
|u(x) - u(y)| \leq M_1|x - y|.
\end{equation}
Without loss of generality we assume \(\theta \leq M_1\), otherwise substituting \(M_1\) by \(\max(\theta, M_1)\). We consider a function \(v(y)\) in \(B(0, 1)\) as follows
\begin{equation}
v(y) = \frac{u(x_0 + R_0 y) - u(x_0)}{M},
\end{equation}
\begin{equation}
\tilde{M} = \text{osc}_{x \in B(x_0, R_0)} u(x)
+ R_0 \left[1 + \sup_{|p| \leq M, x \in B(x_0, R_0)} |F(x, u(x), p, 0)|\right].
\end{equation}
By the definition of viscosity solution it is easy to see in \(B(0, 1)\),
\begin{equation}
|v| \leq 1, \ v \in S(f), \ |f| \leq 1,
\end{equation}
where \(S(f)\) denotes the class of viscosity solutions to elliptic equation related to Pucci’s extremal operator (see [7]).

It is well known that (see[8]) the set \(Y_E\) of points \(\{y_1\}\) in \(B(0, 1)\) satisfying the following inequality
\begin{equation}
|v(y) - v(y_1)| < \langle a, y - y_1 \rangle \leq |E| |y - y_1|^2.
\end{equation}
\(\forall y \in B(0, 1)\) has density \(\frac{|Y_E|}{|B(0, 1)|}\) greater than \(1 - E^{-1}\), where the constant \(E\) is suitably large, i.e., \(E\) is bounded below by a constant \(E_0\) depending only on
Moreover, \( n, \lambda, \Lambda, \mu \) and unbounded above. And \( \Gamma \) satisfies \( 0 < \Gamma < 1 \) and depends only on \( n, \lambda, \Lambda, \mu \) as well. All \( y_1 \in Y_E \) are called Caffarelli points in \( B(0, 1) \).

In the following we always fix a large \( E \) for our study. Taking any point \( y_1 \in Y_E \) and scaling back as \( x_1 \), we have \( x_1 \in B(x_0, R_0) \) such that \( \forall \ x \in B(x_0, R_0) \)

\[
|u(x) - u(x_1) - < a, x - x_1>| \leq \frac{E}{R_0^2} M |x - x_1|^2 \leq \frac{C_1 EM_1}{R_0} |x - x_1|^2,
\]

where

\[
C_1 = 3 + \sup_{|p| \leq M} \sup_{x \in B(x_0, R_0)} |F(x, u(x), p, 0)|.
\]

The inequality (5.7) is called Caffarelli expansion of \( u(x) \) in \( B(x_0, R_0) \) and \( x_1 \) is called a Caffarelli point in \( B(x_0, R_0) \), and (5.7) is valid under the restriction \( \theta \leq M_1 \), hence \( R_0 \leq \theta \leq M_1 \), and the constant vector \( a \) can be determined by dividing (5.7) by \( |x - x_1| \), and then let \( x \to x_1 \) in any fixed direction \( l = \frac{x - x_1}{|x - x_1|} \).

We have

\[
< Du(x_1), l > - < a, l > = 0,
\]

or \( a = Du(x_1) \). This means that at any Caffarelli point \( x_1 \), \( Du(x_1) \) exists. Moreover,

\[
|Du(x_1)| \leq \lim_{y \to x_1} \frac{|u(x_1) - u(y)|}{|x_1 - y|} \leq M_1.
\]

Since Caffarelli point of \( B(x_0, R_0) \) is always a Caffarelli point of \( B(y_0, S_0) \), where \( B(y_0, S_0) \subset B(x_0, R_0) \). And the inverse is not true. So that talk about Caffarelli point, we must point out its related sphere simultaneously. Let \( x_1 \) be a Caffarelli point of \( B(x_0, R_0) \) such that \( B(x_1, \sqrt{2}R_1) \subset B(x_0, R_0) \) and assume that

\[
\forall x, y \in B(x_1, \sqrt{2}R_1). \text{ Applying (5.3) and (5.9) we have the Lipschitz coefficient } K \leq 2M_1.
\]

We estimate the Lipschitz coefficient in the following region of \( R^{2n} \).

\[
Q : \{ x, y | J \equiv \left( \frac{|x + y - 2x_1|}{2R_1} \right)^2 + \left( \frac{|x - y|}{2\epsilon R_1} \right)^2 + e^{-2m(1+2\delta)\sigma} < 1 \}
\]

where \( \epsilon, \sigma \) are small positive constants and only \( \epsilon \) is depending on \( E \), \( \delta \) is a constant, \( \delta \in [\frac{1}{2}, \frac{3}{2}] \), \( m(>1) \) is a constant independent of \( E \). It is easy to show that

\[
Q \subset \{ x, y | (\frac{|x + y - 2x_1|}{2R_1})^2 + (\frac{|x - y|}{2R_1})^2 < 1 \}
\]

\[
\subset \{ x | |x - x_1| < \sqrt{2}R_1 \} \times \{ y | |y - x_1| < \sqrt{2}R_1 \}
\]
Take the comparison function $\Phi(x, y)$ to be

$$
\Phi(x, y) = K J^{\delta/2} (1 + \varphi) |x - y| - \Psi(|x - y|),
$$

where $\Psi$ is a $C^2$ function to be determined which satisfies

$$
|\Psi'(|x - y|)| \leq \varphi, \quad \Psi''(|x - y|) = O\left(\frac{1}{|x - y|}\right).
$$

We investigate the conditions for

$$
|u(x) - u(y) - < Du(x_1), x - y > \mid \leq \Phi(x, y), \quad \forall x, y \in Q
$$

On \( \partial \Omega \), applying (5.10), (5.12) and (5.13), we see that (5.14) is satisfied.

In $Q$ we have

$$
|Z_0| = |\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\pm Du(x_1) + \Phi\right)|
$$

$$
= 4KJ^{\delta/2} - 1 \frac{|x + y - 2x_1|}{R_i^2} \Big[(1 + \varphi) |x - y| - \Psi(|x - y|)\Big]
$$

$$
\leq 4(1 + \varphi)KJ^{\delta/2} \frac{|x - y|}{R_1}
$$

$$
= O\left(K \epsilon \left(\frac{|x - y|}{2\epsilon R_1}\right)^{1 - \frac{1}{2}\ln \frac{1}{\epsilon}}\right).
$$

$$
A = |Z_0| + |\left(\frac{\partial}{\partial x} + \frac{\partial}{\partial y}\right)\left(\pm < Du(x_1), x - y > + \Phi\right)| = O(1).
$$

$$
\omega = C_2(1 + A)|x - y| + \frac{|Z_0|}{1 + A}^\beta
$$

$$
= O\left((\epsilon R_1 + K \epsilon) \left(\frac{|x - y|}{2\epsilon R_1}\right)^{1 - \frac{1}{2}\ln \frac{1}{\epsilon}}\right)
$$

$$
= O\left(K \epsilon^\beta \left(\frac{|x - y|}{2\epsilon R_1}\right)^{\beta(1 - \frac{1}{2}\ln \frac{1}{\epsilon})}\right) = o(1).
$$

When $\epsilon$ is small and we restrict $R_1$ by

$$
R_1 \leq K.
$$

It is easy to calculate

$$
|Z_1| = O\left(\frac{K}{R_1^2} J^{\delta/2 - 1} |x - y|\right),
$$

$$
|Z| = O\left(\frac{K}{R_1} J^{\delta/2}\right),
$$

$$
|Z_2| = O\left(K J^{\delta/2} \frac{1}{|x - y|}\right),
$$

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by using (5.13).

\[(5.16)\]

\[-Tr(\mathcal{Y} Z_2) \geq 4KJ^{3/2}\Psi''(|x - y|) + O(KJ^{3/2-1}\frac{\sigma}{\ln E} \frac{1}{2\epsilon R_1}(\frac{|x - y|}{2\epsilon R_1})^{2m-1}).\]

The validity of (5.16) is due to

\[
\frac{\delta}{2} (1 - \frac{\delta}{2})J^{\frac{1}{2}} - 2Tr[\mathcal{Y}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)(\frac{|x - y|}{2\epsilon R_1})^{\frac{2m}{\ln E}}]
\]

\[
\otimes[(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})(\frac{|x - y|}{2\epsilon R_1})^{\frac{2m}{\ln E}}][(1 + \varphi)|x - y| - \Psi(|x - y|)] \geq 0,
\]

and

\[
(\frac{\partial}{\partial x} - \frac{\partial}{\partial y})^2 \Psi = 2\Psi\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)\Psi = 2\Psi'\frac{(x - y)}{|x - y|} - 2\Psi''\frac{(x - y) \otimes (x - y)}{|x - y|^2}
\]

\[-Tr[\mathcal{Y}\left(\frac{\partial}{\partial x} - \frac{\partial}{\partial y}\right)^2 \Psi] = 4\Psi''(|x - y|),\]

We select \(\Psi(|x - y|)\) such that

\[
\Psi''(|x - y|) = \begin{cases} \frac{\varphi}{3|x - y|^2 \ln\frac{2\epsilon R_1}{|x - y|}} \frac{2m}{\ln E}[(m + 1)\sigma(1 + 2\delta) \ln E]^{-1}, & \text{when } 0 < |x - y| \leq 2\epsilon R_1 E^{-m\sigma(1+2\delta)}, \\ \frac{\varphi}{3|x - y|^{3/2}} \ln\frac{2\epsilon R_1}{|x - y|} + \sigma(1 + 2\delta) \ln E)^{-\frac{1}{2}}[(m + 1)\sigma(1 + 2\delta) \ln E]^{-\frac{1}{2}}, & \text{when } 2\epsilon R_1 E^{-m\sigma(1+2\delta)} \leq |x - y| \leq 2\epsilon R_1, \end{cases}
\]

Integrating the above expression we have

\[
\Psi'(|x - y|) = \begin{cases} \frac{\varphi}{3 |x - y|^2 \ln\frac{2\epsilon R_1}{|x - y|}} \frac{2m}{\ln E}[(m + 1)\sigma(1 + 2\delta) \ln E]^{-1}, & \text{when } 0 < |x - y| \leq 2\epsilon R_1 E^{-m\sigma(1+2\delta)}, \\ \frac{\varphi}{3 |x - y|^{3/2}}[(m + 1)\sigma(1 + 2\delta) \ln E]^{-1} + 2 \ln\frac{2\epsilon R_1}{|x - y|} + \sigma(1 + 2\delta) \ln E)^{1/2}[(m + 1)\sigma(1 + 2\delta) \ln E]^{-1/2}, & \text{when } 2\epsilon R_1 E^{-m\sigma(1+2\delta)} \leq |x - y| \leq 2\epsilon R_1, \end{cases}
\]

Therefore

\[
\Psi(|x - y|) = \int_0^{|x - y|} \Psi'(t)dt.
\]

Hence (5.13) is valid by the explicit expression of \(\Psi, \Psi'\).

We want to prove that

\[(5.17)\]

\[-Tr(\mathcal{Y} Z_2) \geq KJ^{3/2}\Psi''(|x - y|)\]

under suitable conditions. (5.17) follows from

\[
J^{\delta/2-1}\frac{\sigma}{\ln E} \frac{1}{2\epsilon R_1} \left(\frac{|x - y|}{2\epsilon R_1}\right)^{\frac{2m}{\ln E} - 1} \ll J^{\delta/2}\Psi''(|x - y|),
\]

13
or

\[ I = \frac{3}{\varphi \ln E} \frac{1}{2 \epsilon R_1} \left( \frac{|x - y|}{2 \epsilon R_1} \right)^{2 \sigma} - 1 \frac{1}{\Psi''(|x - y|)} \ll J, \]

\[ = \left( \frac{|x + y - 2x_1|}{2R_1} \right)^2 + \left( \frac{|x - y|}{2 \epsilon R_1} \right)^{2 \sigma} \ln E \theta^2 \Psi''(\theta), \]

In the interval

\[ 2 \epsilon R_1 E^{-m(1+2\delta)} \leq |x - y| \leq 2 \epsilon R_1, \]

\[ I = \frac{3}{\varphi \ln E} \frac{1}{2 \epsilon R_1} \left( \frac{|x - y|}{2 \epsilon R_1} \right)^{2 \sigma} - 1 \frac{1}{\Psi''(|x - y|)} \ll \frac{(m + 1) \sigma (1 + 2 \delta) \ln E}{\sigma} \]

\[ \cdot \left[ (m + 1) \sigma (1 + 2 \delta) \ln E \right]^\frac{1}{2} \]

\[ \leq \frac{3}{\varphi} \sigma^2 (m + 1) (1 + 2 \delta) \left( \frac{|x - y|}{2 \epsilon R_1} \right)^{2 \sigma} \ll \left( \frac{|x - y|}{2 \epsilon R_1} \right)^{2 \sigma}, \]

if the following condition is true

(5.18) \( (m + 1) \sigma^2 \ll 1. \)

In the interval \( 0 < |x - y| < 2 \epsilon R_1 E^{-m(1+2\delta)}, \)

\[ I = \frac{3}{\varphi \ln E} \frac{1}{2 \epsilon R_1} \left( \frac{|x - y|}{2 \epsilon R_1} \right)^{2 \sigma} - 1 \frac{1}{\Psi''(|x - y|)} \ll \frac{(m + 1) \sigma (1 + 2 \delta) \ln E}{\sigma} \]

\[ \cdot \left[ 2 \epsilon R_1 E^{-m(1+2\delta)} \right]^\frac{2 \sigma}{m} \left[ (m + 1) \sigma (1 + 2 \delta) \ln E \right] \]

\[ = \frac{6}{\varphi} \sigma^2 (1 + 2 \delta) e^{-2m(1+2\delta) \sigma} \ll e^{-2m(1+2\delta) \sigma}, \]

if the condition (5.18) is true. Hence, (5.17) is valid under the restriction (5.18).

It is easy to prove that

\[ \|Z_1\| \ll -\operatorname{Tr}(Y Z_2) \]

when

\[ \sigma \epsilon^2 \ln E \ll 1; \]

\[ \|Z_2\| \omega^2 \ll -\operatorname{Tr}(Y Z_2) \]

when

\[ \epsilon^{2\beta} \sigma \ln E \left( \frac{|x - y|^{2\beta}}{2 \epsilon R_1} \right)^{2\beta} \left[ \frac{1}{\Psi''(\theta)} \right] \ll \epsilon^{2\beta} \sigma \ln E \ll 1; \]

\[ \|Z\| \ll -\operatorname{Tr}(Y Z_2) \]
when
\[ \epsilon^\beta \sigma \ln E \ll 1; \]
\[ \omega \ll -\text{Tr}(\Upsilon Z_2) \]

when
\[ \epsilon^{1+\beta} R_1 \sigma \ln E \ll 1. \]

Taking \( \epsilon = (\ln E)^{-\frac{2}{\sigma}} \), then the estimates for \( \|Z_1\|, \|Z_2\|, \|Z\|, \omega \) are all true. Hence we have the following lemma.

**Lemma 5.2** The estimates (5.14) is valid when (5.15) and (5.18) are true.

**Proof:** The lemma is proved since (3.14) is true by the above estimates. \( \square \)

(5.14) implies that
\[
|u(x) - u(y) - < Du(x_1), x - y >| \leq \\
(1 + \varphi)K\left(\frac{|x + y - 2x_1|}{2R_1}\right)^2 + \left(\frac{|x - y|}{2R_1}\right)^2 e^{-2m(1+2\delta)\sigma} \frac{1}{2} |x - y|,
\]
\( \forall x, y \in Q \). Taking constants \( m, \sigma \) such that \( m\sigma \) is large and \( m\sigma^2 \) is small, hence (5.18) is valid. Taking \( |x - y| \) small such that
\[
|u(x) - u(y) - < Du(x_1), x - y >| \leq e^{-2m(1+2\delta)\sigma}
\]
then (5.19) implies that
\[
|u(x) - u(y) - < Du(x_1), x - y >| \leq \\
(1 + \varphi)K\left[\frac{|x + y - 2x_1|}{2R_1}\right]^2 + 2e^{-2m(1+2\delta)\sigma} \frac{1}{2} |x - y|,
\]
\( \forall x, y \in Q \) satisfies (5.20). We relax the restriction (5.20) for \( x, y \) and assume \( x, y \) satisfying
\[
\forall x, y \in \tilde{Q}: \{x, y\left(\frac{|x + y - 2x_1|}{2R_1}\right)^2 + \left(\frac{|x - y|}{2R_1}\right)^2 < 1 - e^{-2m(1+2\delta)\delta}\},
\]
interpolating line segment \( xy \) by \( z_i = \frac{x + y}{2} + \frac{i}{2M}(x - y) \), where \( i = -M, -M + 1, \cdots, 0, \cdots, M - 1, M \). By applying (5.21), we have
\[
|u(z_i) - u(z_{i-1}) - < Du(x_1), z_i - z_{i-1} >| + |u(z_{i+1}) - u(z_i) - < Du(x_1), z_{i+1} - z_i >| \\
\leq (1 + \varphi)K\left[(U + V)^\frac{1}{2} + (U - V)^\frac{1}{2}\right] \frac{|x - y|}{2M},
\]
where
\[
U = \left(\frac{|x + y - 2x_1|}{2R_1}\right)^2 + \left(\frac{2i + 1}{2M}\right)^2 \left(\frac{|x - y|}{2R_1}\right)^2 + 2e^{-2m(1+2\delta)\sigma},
\]
\[ V = \frac{2i + 1}{M} < \frac{x + y - 2x_1}{2R_1}, \frac{x - y}{2R_1} >. \]

Since \(0 < \frac{\delta}{2} \leq \frac{3}{4} \leq 1\), we have
\[
(U + V)^{\frac{\delta}{2}} + (U - V)^{\frac{\delta}{2}} \leq 2U^{\frac{\delta}{2}}
\]
\[
\leq 2\left[\frac{|x + y - 2x_1|^2}{2R_1} + \frac{|x - y|^2}{2R_1} + 2e^{-2m(1+2\delta)}\right]^{\frac{\delta}{2}}
\]

Summing for \(i = 0, 1, \cdots, M - 1\), we have
\[
|u(x) - u(y) - < Du(x_1), x - y >| \leq (1 + \varphi)K\left[\frac{|x + y - 2x_1|^2}{2R_1} + \frac{|x - y|^2}{2R_1} + \frac{\xi^2}{1 - \xi^2}\right] |x - y|,
\]
\[
\forall x, y \in \tilde{Q} : \{ x, y | \left(\frac{|x + y - 2x_1|^2}{2R_1} \right) + \left(\frac{|x - y|^2}{2R_1} \right) < 1 - \xi^2 \} \text{ where}
\]
\[
2e^{-2m(1+2\delta)} \leq 2e^{-3m\sigma} \equiv \xi^2 \ll 1.
\]
\[
\forall \delta \in [\frac{1}{7}, \frac{3}{16}].
\]

Now we state and prove the fundamental lemma on decreasing of Lipschitz coefficient in certain constant ratio accompanied with decreasing of radius of spherical region in suitable constant ratio.

**Remark 5.1** Let \(y \to x\) in (5.22), approximately we have
\[
|Du(x) - Du(x_1)| \leq (1 + \varphi + \varepsilon)\left(\frac{|x - x_1|}{R_1}\right)^{\delta}, \text{ for } \varepsilon = \left(\frac{\xi R_1}{|x - x_1|}\right),
\]
\[
\delta = \frac{2}{3} \text{ is a sharp estimate. But Eq.}(5.24) \text{ is valid only in the spherical shell } \xi R_1 \leq |x - x_1| \leq R_1, \text{ and not for the sphere } |x - x_1| \leq R_1, \text{ so that, Eq.}(5.24) \text{ and (5.22) are reasonable.}
\]

**Lemma 5.3** Let \(x_1\) be a Caffarelli point of \(B(x_0, R_0)\) satisfying the Lipschitz condition (5.10), \(\forall x, y \in B(x_1, \sqrt{2}R_1) \subset B(x_0, R_0)\). Under the restriction (5.15), \(\forall x_2 \in B(x_1, \xi R_1), \exists \text{ Caffarelli point } y_1 \text{ of } B(x_1, R_1)\text{ such that } y_1 \in B(x_1, \xi R_1), \text{ and } |y_1 - x_2| \leq \frac{\xi}{3} R_1, \text{ we have}
\]
\[
|Du(y_1) - Du(x_1)| \leq \eta K,
\]
\[
|u(x) - u(y) - < Du(y_1), x - y >| \leq \zeta |x - y|,
\]
\[
\forall x, y \text{ such that } \{ x, y | \frac{x + y}{2} - x_2 |^2 + \left(\frac{|x - y|}{2}\right)^2 \leq \frac{R_1^2}{9} \}, \text{ where } \eta \text{ is a small constant and } \zeta \text{ is a constant, } \zeta < 1. \text{ Both } \eta, \zeta \text{ depend on } \xi \text{ only when we fixed } \delta \text{ to be } \delta = \frac{2}{3}.
\]

**Proof:** Since \(x_2 \in B(x_1, \xi R_1)\), in \(B(x_2, \frac{\xi}{3} R_1) \cap B(x_1, \xi R_1)\), \(\exists \text{ Caffarelli point of } B(x_1, R_1)\), because of \(B(x_2, \frac{\xi}{3} R_1) \cap B(x_1, \xi R_1) \supset B(x_2 + \frac{x_1 - x_2}{|x_1 - x_2|} \frac{\xi}{3} R_1)\) and
the density of sphere $B(x_2 + \frac{x_1 - x_2}{|x_1 - x_2|} \xi, \frac{\xi}{\sqrt{2}} R_1)$ with respect to $B(x_1, R_1)$ satisfying $(\frac{\xi}{\sqrt{2}})^n \gg E^{-\Gamma}$ (If this condition is not satisfied, since $\xi$ and $\Gamma$ are independent of $E$, we substitute $E$ by a large one, such that this condition is satisfied.) We denote one of Caffarelli point by $y_1$. Hence $y_1 \in B(x_1, R_1)$, $|y_1 - x_2| \leq \frac{\xi}{\sqrt{2}} R_1$ and moreover $Du(y_1)$ exists. Take $y = y_1$ in (5.22) and then divide it by $|x - y|$ and let $x \to y_1$, we have

$$|Du(y_1) - Du(x_1)| \leq (1 + \varphi)(2\xi^2)^\frac{4}{9} K$$

$$= (1 + \varphi)(2\xi^2)^\frac{4}{9} K \equiv \eta K,$$

this is (5.25). Since

$$\{x, y||x + y - x_2|^2 + (\frac{|x-y|}{2})^2 \leq \frac{R_1^2}{9}\}$$

$$\subset \{x, y||x + y - x_1|^2 + (\frac{|x-y|}{2})^2 \leq \frac{R_1^2}{4}\} \subset \tilde{Q},$$

applying (5.22) and (5.25) we have

$$u(x) - u(y) - <Du(y_1), x-y> \leq [\eta + (1 + \varphi)(\frac{1}{4} + \xi_1)\eta^\frac{1}{4}] K|x-y| \leq \zeta K|x-y|.$$

where we denote

$$(5.27) \quad \zeta = \eta + (1 + \varphi)(\frac{1}{4} + 2\xi^2)^\frac{1}{4} (1 + 9\xi^2)^\frac{1}{4} + \eta(\frac{1}{9} + 2\xi^2)^{-\frac{1}{4}}.$$

We take a little larger $\zeta$ over our necessary for the sake of next lemma. Since $\xi, \eta$ are small, we have

$$\zeta \approx (1 + \varphi)(\frac{1}{4})^\frac{1}{4} = [1 + \frac{1}{2}(\sqrt{2} - 1)]\frac{1}{\sqrt{2}} = \frac{1}{2}(1 + \frac{1}{\sqrt{2}}) < 1,$$

hence we can take $\xi$ small such that $\zeta < 1$.

The lemma is proved completely. □

**Remark 5.2** The only reason for approximating the general point $x_2$ by Caffarelli point $x_1, y_1, \ldots$ is that there exists first derivative on Caffarelli points.

**Lemma 5.4** We have $u \in C^{1+\alpha}(B(x_0, \frac{\xi}{\sqrt{2}} R_0))$, where

$$(5.28) \quad \alpha = \frac{\ln \zeta}{\ln \frac{1}{2}}$$

**Proof:** Fixed a $x_2 \in (B(x_0, \frac{\xi}{\sqrt{2}} R_0))$. Take a Caffarelli point $x_1$ of $(B(x_0, \frac{\xi}{\sqrt{2}} R_0))$ such that $|x_1 - x_2| \leq \frac{\xi}{\sqrt{2}} R_0$. Take $R_1 = (1 - \xi)\frac{1}{\sqrt{2}} R_0$ and $K = 2M_1$. Denote
$R^{(0)} = R_1, R^{(k)} = \frac{R_k}{3^k}, K^{(0)} = K, K^{(k)} = \zeta^k K (k = 0, 1, 2, \cdots)$. Denote $x_1 \equiv y_0$, take Caffarelli point $y_1$ of $B(y_0, R^{(0)})$ such that $y_1 \in B(y_0, \xi R^{(0)}) \cap B(x_2, \frac{2}{3} R^{(0)})$, hence $|y_1 - x_2| < \frac{2}{3} R^{(0)} = \xi R^{(1)}$, i.e. $y_1 \in B(x_2, \xi R^{(1)})$, take Caffarelli point $y_2$ of $B(y_1, R^{(1)})$ such that $y_2 \in B(y_1, \xi R^{(1)}) \cap B(x_2, \frac{2}{3} R^{(1)}) \cdots$. In general, take Caffarelli point $y_k$ of $B(y_{k-1}, R^{(k-1)})$ such that $y_k \in B(y_{k-1}, \xi R^{(k-1)}) \cap B(x_2, \frac{2}{3} R^{(k-1)})$, $\forall k = 1, 2, \cdots$.

Since we restrict $R_0$ to be $\leq M_1$, we have

$$R^{(k)} = \frac{R_k}{3^k} \leq \frac{R_0}{3^{k} \sqrt{2}} \leq \frac{M_1}{3^k} \leq K \zeta^k = K^{(k)} (k = 0, 1, 2, \cdots),$$

i.e. (5.15) is valid for all $k = 0, 1, 2, \cdots$.

We prove by induction that

(5.29) \[ |Du(y_k) - Du(y_{k-1})| \leq K^{(k)}, \]

(5.30) \[ |u(x) - u(y) - < Du(y_k), x-y > | \leq K^{(k)} |x-y|, \]

$\forall x, y$ such that \( \{|x+u - x_2|^2 + (|x-y|)^2 \leq (R^{(k)})^2\}, \forall k = 1, 2, \cdots \)

When $k = 1$, (5.29) follows from (5.25) and $\eta K \leq \zeta K = K^{(1)}$, (5.30) follows from (5.26).

When (5.29) and (5.30) are valid for $k - 1$, substituting $y_0 = x_1, R^{(0)} = R_1, K^{(0)} = K = 2M_1$ by $y_{k-1}, R^{(k-1)}, K^{(k-1)}$ in lemma 5.2, 5.3. Since $R^{(k)} \leq K^{(k)}$, hence lemma 5.2 is valid and lemma 5.3 is valid by using $y_k, R^{(k)}$ and $K^{(k)}$ instead of $y_1, R^{(1)} = \frac{R_0}{3}, K^{(1)} = \zeta K$. (5.29) and (5.30) follow for $k$ by applying lemma 5.3 and $\eta K^{(k-1)} \leq \zeta K^{(k-1)} = K^{(k)}$. i.e. (5.29) and (5.30) are valid $\forall k = 1, 2, \cdots$.

Applying (5.29) we have

$$\sum_{k=1}^{\infty} |Du(y_k) - Du(y_{k-1})| \leq \sum_{k=1}^{\infty} K \zeta^k < \infty.$$

Hence the series

$$\sum_{k=1}^{\infty} [Du(y_k) - Du(y_{k-1})]$$
converges, denote its limit by $\tilde{a} - Du(x_1)$. \forall x, y satisfies $\{x, y|x \frac{x+y}{2} - x_2| \leq (R^{(k)})^2\}$, applying (5.30) we have
\[
\left| u(x) - u(y) - \langle \tilde{a}, x - y \rangle \right| \leq \left| u(x) - u(y) - \langle Du(y^k), x - y \rangle \right| + \left| Du(y^k) - \tilde{a}, x - y \rangle \right|
\]
\[
\leq K^{(k)}|x - y| + \left( \sum_{l=k+1}^{\infty} K\zeta^l\right)|x - y| = \frac{1 + \zeta}{1 - \zeta}K^{(k)}|x - y|
\]
(5.31)
\[
= \frac{1 + \zeta}{1 - \zeta}K\zeta^k|x - y| = \frac{1 + \zeta}{1 - \zeta}K\left(\frac{R^{(k)}}{R^{(0)}}\right)^\alpha |x - y|
\]
\[
\leq \frac{1 + \zeta}{1 - \zeta}K\left(\frac{3R^{(k+1)}}{(1 - \zeta)R^{(0)}}\right)^\alpha |x - y| \leq CR^\alpha|x - y|,
\]
when we denote
\[
R = (\left| x + y \right| - x_2^2 + (\left| y_2 \right|^2)^\frac{1}{2})\)
(5.32)
\[
C = \frac{1 + \zeta}{1 - \zeta}2M_1\left(\frac{3}{1 - \zeta}\right)^\alpha R_0^{-\alpha},
\]
and assume
\[
R^{(k+1)} \leq R \leq R^{(k)}(k = 0, 1, 2, \cdots)
\]
Put the case $k = 0, 1, 2, \cdots$ together, we obtain that (5.31) is true
\[
\forall 0 \leq \left(\left| \frac{x + y}{2} - x_2 \right| + \left\| \frac{x - y}{2} \right\|^2\right)^\frac{1}{2} \leq R \leq R^{(1)} = R_1 = \frac{1 - \xi}{3\sqrt{2}}R_0.
\]
Putting $y = x_2$ and dividing (5.31) by $|x - x_2|$, then let $x \to x_2$, we have $Du(x_2)$ exists and $\tilde{a} = Du(x_2)$. Hence we have
(5.33) \[
|u(x) - u(y) - \langle Du(x_2), x - y \rangle| \leq CR^\alpha|x - y|,
\]
$\forall x, y$ satisfies $\{x, y|\left(\left| \frac{x + y}{2} - x_2 \right| + \left\| \frac{x - y}{2} \right\|^2\right)^\frac{1}{2} \leq R \leq R_1 = \frac{1 - \xi}{3\sqrt{2}}R_0\}$
(5.33) implies that
\[
|Du(x) - Du(x_2)| \leq C|x - x_2|\alpha, \forall x, x_2 \in B(x_0, \frac{\xi}{\sqrt{2}}R_0)
\]
Hence
\[
u \in C^{1+\alpha}(B(x_0, \frac{\xi}{\sqrt{2}}R_0))
\]
The lemma is proved.
\[
\square
\]
Now we study the case that Lipschitz coefficient contains a factor $R^\gamma$, where $\gamma \in (0, 1)$.}

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**Lemma 5.5** \(\forall x_2 \in B(x_0, \frac{\xi}{\sqrt{2}} R_0)\), if constant \(\gamma \in (0, 1)\) and positive constant \(H, S \leq \frac{(1 - \xi)}{3\sqrt{2}} R_0\) exist such that we have

(5.34) \[|u(x) - u(y)| - < Du(x_2), x - y > | \leq HR^\gamma |x - y|,\]

\(\forall x, y\) satisfies \(\{x, y|(|\frac{x+y}{2} - x_2|^2 + (\frac{|x-y|}{2})^2)^\frac{1}{2} \leq R \leq S\}\)

Then in case

(5.35) \[\gamma \leq 1 - \alpha,\]

where \(\alpha\) is defined by (5.28). \(\exists\) constant \(\bar{H}(> H)\) and \(\tilde{S}(< S)\) such that we have

(5.36) \[|u(x) - u(y)| - < Du(x_2), x - y > | \leq HR^{\gamma + \alpha} |x - y|,\]

\(\forall x, y\) satisfies \(\{x, y|(|\frac{x+y}{2} - x_2|^2 + (\frac{|x-y|}{2})^2)^\frac{1}{2} \leq R \leq \tilde{S}\}\).

**Proof:** Denote \(R^{(0)} = \min\{S, H\frac{1}{\sqrt{\gamma}}\}, K^{(0)} = HR^{(0)}\gamma\), where the meaning of \(R^{(0)}\) is different from that in lemma 5.4.

Let \(y_0 \in B(x_2, \xi R^{(0)})\) be a Caffarelli point of \(u(x)\) in \(B(x_2, R^{(0)})\). In the region \(B(y_0, (1 - \xi) R^{(0)})\), the lemma 5.2 for estimate \(|u(x) - u(y)| - < Du(y_0), x - y > |\) is also valid in the present case, this is because of (5.15)

\[R^{(0)} \leq H(R^{(0)})^\gamma = K^{(0)}\]

is true. Take \(K = H(R^{(0)})^\gamma, \delta = \frac{1}{2} + \gamma\). From (5.22),

\(\forall x, y \in \bar{Q}: \{x, y|(|\frac{x+y-2y_0}{2R^{(0)}}|^2 + (\frac{|x-y|}{2R^{(0)}})^2 \leq 1 - \xi^2\}\),

we have

\[|u(x) - u(y)| - < Du(y_0), x - y > | \leq (1 + \varphi)H(R^{(0)})^\gamma [\left|\frac{x+y-2y_0}{2R^{(0)}}\right|^2 + (\frac{|x-y|}{2R^{(0)}})^2 + 2\xi^2] \leq |x - y|.\]

Hence we have for all Caffarelli point \(y_1\) of \(B(y_0, R^{(0)})\) satisfying \(y_1 \in B(y_0, \xi R^{(0)}) \cap B(x_2, \frac{\xi}{\sqrt{2}} R^{(0)})\), we have

(5.37) \[|Du(y_1) - Du(y_0)| \leq (1 + \varphi)H(R^{(0)})^\gamma (2\xi^2)^\frac{1}{2} \leq (1 + \varphi)K^{(0)}(2\xi^2)^\frac{1}{2} = \eta K^{(0)},\]

\(\forall x, y\) satisfying \(\{x, y|(|\frac{x+y}{2} - x_2|^2 + (\frac{|x-y|}{2})^2 < (\frac{R^{(0)}}{3})^2\}\), we have

\[|u(x) - u(y)| - < Du(y_1), x - y > | \leq (1 + \varphi)H(R^{(0)})^\gamma \left(\frac{1}{9} + 2\xi^2\right) [\left\{\frac{1}{4} + 2\xi^2\right\}\leq \eta K^{(0)}] |x - y|\]

(5.38) \[\leq \zeta K^{(0)} |x - y|,\]

where \(\zeta\) is defined by (5.27).
(5.37) and (5.38) are the similar relation of lemma 5.3 in the present case. Define \( R^{(k)} = \frac{R^{(0)}}{3^k}, K^{(k)} = H(R^{(k)})^\gamma \zeta^k (k = 1, 2, 3, \ldots) \). Take Caffarelli point \( y_k \) of \( B(y_k - 1, B^{(k-1)}) \) such that \( y_k \in B(y_k - 1, B^{(k-1)}) \cap B(x_2, \frac{3}{2} R^{(k-1)}) \).

First we verify (5.15) in the present case. Applying (5.35) we have

\[
3^1 - \gamma \zeta \geq 3^0 \zeta = \frac{3 \ln 1}{3 \ln \zeta} = \frac{1}{\zeta} = 1,
\]
or it is

\[
\frac{1}{3^k} \leq \frac{\zeta^k}{(3^k)^\gamma}.
\]

Hence

\[
R^{(k)} = \frac{R^{(0)}}{3^k} \leq K^{(0)} \frac{H(R^{(0)})^\gamma}{3^k} \leq H \left( \frac{R^{(0)}}{3^k} \right)^\gamma \zeta^k = H(R^{(k)})^\gamma \zeta^k = K^{(k)},
\]
i.e. (5.15) is valid in the present case.

Then we prove (5.29) and (5.30) by induction. When \( k = 1 \), applying (5.37) we have

\[
|Du(y_1) - Du(y_0)| \leq \eta K^{(0)} = \frac{\eta}{\zeta} 3^\gamma K^{(1)} \leq \frac{3\eta}{\zeta} K^{(1)} \leq K^{(1)}.
\]

Applying (5.38) we have

\[
|u(x) - u(y) - < Du(y_1), x - y > | \leq \frac{K^{(0)}}{3^1} |x - y| = K^{(1)} |x - y|,
\]

\( \forall x, y \) satisfies \( \{ x, y ||x + y - 2y_1 - x_1 ||^2 + (\frac{x - y}{2})^2 < \left( \frac{R^{(0)}}{3} \right)^2 = (R^{(1)})^2 \} \)
i.e. (5.29) and (5.30) are valid for \( k = 1 \).

Suppose (5.29) and (5.30) are valid for \( k - 1 \). Since \( K^{(k)} = H(R^{(k)})^\gamma \zeta^k \), take \( \delta = 1 + \gamma \) and apply (5.22),

\[
|u(x) - u(y) - < Du(y_k), x - y > | \leq (1 + \varphi) K^{(k)} \left[ \left( \frac{|x + y - 2y_k|}{2R^{(k)}} \right)^2 + \left( \frac{|x - y|}{2R^{(k)}} \right)^2 + 2\zeta^2 \right]^{\frac{1}{\zeta} + \frac{\gamma}{\zeta}} |x - y|,
\]

\( \forall x, y \) satisfies

\[
\{ x, y \left( \frac{|x + y - 2y_k|}{2R^{(k)}} \right)^2 + \left( \frac{|x - y|}{2R^{(k)}} \right)^2 < 1 - \zeta^2 \}.
\]

Hence applying (5.39) we have

\[
|Du(y_{k+1}) - Du(y_k)| \leq (1 + \varphi) K^{(k)} (2\zeta^2)^{\frac{1}{\zeta} + \gamma} \leq (1 + \varphi) K^{(k)} (2\zeta^2)^{\frac{1}{\zeta}}
\]

\[
= \eta K^{(k)} = \frac{\eta}{\zeta} 3^\gamma K^{(k+1)} \leq \frac{3\eta}{\zeta} K^{(k+1)} \leq K^{(k+1)}.
\]
And

\[ |u(x) - u(y) - < Du(y_{k+1}), x - y > | \]
\[ \leq (1 + \varphi)[K^{(k)}(1/9 + 2\xi^2) + (1/4 + 2\xi^2) + \eta K^{(k)}]|x - y| \]
\[ = \frac{\zeta}{3^k} K^{(k)}|x - y| = K^{(k+1)}|x - y|, \]

\( \forall x, y \) satisfies \( \{ x, y ||x - x_2|^2 + |y - x_2|^2 < R \leq R^{(k)} \} \).

Hence (5.29) and (5.30) are valid for all \( k = 1, 2, \ldots \).

Since \( Du(x) \in C, \forall x \in B(x_0, \sqrt{2}R_0) \). Applying (5.29) and (5.30) we have

\[ |u(x) - u(y) - < Du(x_2), x - y > | \]
\[ \leq |u(x) - u(y) - < Du(y_k), x - y > | + [ \sum_{l=k+1}^{\infty} K^{(l)}]|x - y| \]
\[ \leq K^{(k)}|x - y| + [ \sum_{l=k+1}^{\infty} K^{(l)}]|x - y| \]
\[ = H[(R^{(k)})^\gamma \zeta^k + \sum_{l=k+1}^{\infty} (R^{(l)})^\gamma \zeta^l]|x - y| \]
\[ = \frac{1 + \frac{\zeta}{3^k}}{1 - \frac{\zeta}{3}} H(R^{(k)})^\gamma \zeta^k |x - y| \]
\[ = \frac{1 + \frac{\zeta}{3^k}}{1 - \frac{\zeta}{3}} H(R^{(k)})^\gamma \frac{(R^{(k)})^\alpha}{R^{(0)}} |x - y|. \]

In the region \( R^{(k+1)} \leq (|x - x_2|^2 + |y - x_2|^2)^{1/2} = R \leq R^{(k)} \), we have

\[ R^{(k)} = 3R^{(k+1)} \leq 3R. \]

Hence

\[ |u(x) - u(y) - < Du(x_2), x - y > | \leq \tilde{H} R^{\gamma + \alpha} |x - y|, \]

where

\[ \tilde{H} = \frac{1 + \frac{\zeta}{3^k}}{1 - \frac{\zeta}{3}} 3^{\gamma + \alpha} H(R^{(0)})^{-\alpha}. \]

Putting \( k = 1, 2, 3, \ldots \) together we have

\[ |u(x) - u(y) - < Du(x_2), x - y > | \leq \tilde{H} R^{\gamma + \alpha} |x - y|, \]

\( \forall x, y \) satisfies \( \{ x, y ||x - x_2|^2 + |y - x_2|^2 \leq R \leq \tilde{S} \}, \)

\[ \tilde{S} = \frac{R^{(0)}}{3} \leq \frac{1 - \frac{\zeta}{3}}{3} \min\{ S, H^{1-\gamma} \}. \]
The lemma is proved. □

**Lemma 5.6** ∀(x₀, R₀) ⊂ Ω, we have

\[ u(x) \in C^{1+1}(B(x₀, \frac{\xi}{\sqrt{2}}R₀)), \]

where ξ is defined by (5.23).

**Proof:** Since α and ζ are defined by (5.28) and (5.27), substituting ζ by a little large one such that ζ < 1 still valid and \( \frac{1}{\alpha} \) is a positive integer M. Applying lemma 5.4 once we obtain (5.33). Then applying lemma 5.5 successively.

Denote

\[ R_j^{(0)} = \min\{S_j, H_j^{\frac{1}{1-j\alpha}}\}, \quad j = 1, 2, \cdots, M. \]

where \( S_j, H_j \) are the value of \( S, H \) in lemma 5.5 corresponding to \( \gamma = j\alpha \).

Applying lemma 5.4 we have

\[ S_1 = \frac{1 - \xi}{3\sqrt{2}}R₀, \]
\[ H_1 = \frac{1 + \zeta}{1 - \zeta}2M_1\left(\frac{3}{1 - \xi}\right)\left(\frac{1 - \xi}{\sqrt{2}}R₀\right)^{-\alpha}. \]

Since \( M_1 \geq R₀ \), it is easy to obtain that

\[ H_1 \geq S_1^{1-\alpha}. \]

Hence applying (5.42) we have

\[ R_1^{(0)} = S_1. \]

Applying lemma 5.5 we have

\[ S_{j+1} = \frac{1 - \xi}{3} \min\{S_j, H_j^{\frac{1}{1-j\alpha}}\}. \]

\[ H_{j+1} = \frac{1 + \zeta}{1 - \frac{3}{3\alpha}}(j+1)\alpha H_j(R_j^{(0)})^{-\alpha} \geq H_j(R_j^{(0)})^{-\alpha}. \]

We prove by induction that

\[ H_j \geq S_j^{1-j\alpha} \]

is true, then by (5.42) we have \( R_j^{(0)} = S_j \).

When \( j = 1 \), (5.46) is true by (5.43). If (5.46) is true for \( j \), applying (5.45) we have

\[ H_{j+1} \geq H_j(R_j^{(0)})^{-\alpha} \geq S_j^{1-j\alpha}S_j^{-\alpha} = S_j^{1-(j+1)\alpha}, \]

i.e. (5.46) is valid for \( j \) substitute by \( j + 1 \).
Hence (5.46) is true ∀j = 1, 2, · · · , M. Applying (5.44) we have

\[ S_{j+1} = \frac{1 - \xi}{3} S_j (j = 1, 2, \cdots, M - 1). \]

Hence

\[ S_{j+1} = \left( \frac{1 - \xi}{3} \right)^j S_1 = \left( \frac{1 - \xi}{3} \right)^{j+1} \frac{R_0}{\sqrt{2}} \]

Especially we have

\[ S_M = R_{M}^{(0)} = \left( \frac{1 - \xi}{3} \right)^M \frac{R_0}{\sqrt{2}} = [1 + o(1)] e^{-\frac{\ln 3}{\sqrt{2}}} \frac{R_0}{\sqrt{2}} \approx \frac{(\ln 3)^2}{\sqrt{2} R_0} \]

when ξ is small. \( \frac{(\ln 3)^2}{\sqrt{2} R_0} \) is a constant < 1 and is independent of ξ. Hence

\[ |u(x) - u(y) - \langle Du(x_2), x - y \rangle| \leq \tilde{C} \frac{R}{R_0} |x - y| \]

∀x, y ∈ B(x_0, R_{M}^{(0)}).

Since \( R_{M}^{(0)} > \xi \frac{R}{\sqrt{2}}, \) hence we have ∀x_2, x_3 ∈ B(x_0, ξ \frac{R}{\sqrt{2}} R_0),

\[ |Du(x_3) - Du(x_2)| \leq \tilde{C} \frac{|x_3 - x_2|}{R_0} \]

The lemma is proved. \( \square \)

Our main theorem 2.1.

Proof: Theorem 2.1 follows by applying theorem 5.1 and lemma 5.7. \( \square \)

A final remark After we proved the regularity result \( u \in C^{1+1}(\Omega), \) a natural question has arisen. Is the regularity result the best possible or not? If it is yes, we must give an example to show that the solution of (2.1) under the conditions (2.2)-(2.4)) cannot belong to \( C^2(\Omega). \) More exactly, the solution cannot possesses more regularity when it is lack of concavity assumption, so that more regularity assumptions on \( F \) has no effect. Hence we present the following problem.

An open problem Let the equation (2.1) satisfying the following assumptions:

(a) \( F(x, z, p, X) \) is sufficiently smooth with respect to its arguments \( x, z, p, X. \)
(b) Assumption (2.2).
(c) \( F \) is strict monotone with respect to \( z \) in the following meaning

\[ F(x, z, p, X) + K(w - z) \leq F(x, w, p, X), \]
(d) $F$ satisfies the natural structure condition

$$|F| + (1 + |p|)|F_p| + |F_z| + \frac{1}{1 + |p|}|F_x| \leq \mu(1 + |p|^2 + |X|),$$

where $\lambda, \Lambda, K, \mu$ are positive constants.

Can you construct an example to show that the solution $u(x)$ of (2.1) under the assumptions (a)-(d) does not belong to $C^2(\Omega)$?

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