PERIODICITY AND STABILITY ANALYSIS OF IMPULSIVE NEURAL NETWORK MODELS WITH GENERALIZED PIECEWISE CONSTANT DELAYS

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Abstract. In this paper, the global exponential stability and periodicity are investigated for impulsive neural network models with Lipschitz continuous activation functions and generalized piecewise constant delay. The sufficient conditions for the existence and uniqueness of periodic solutions of the model are established by applying fixed point theorem and the successive approximations method. By constructing suitable differential inequalities with generalized piecewise constant delay, some sufficient conditions for the global exponential stability of the model are obtained. The methods, which does not make use of Lyapunov functional, is simple and valid for the periodicity and stability analysis of impulsive neural network models with variable and/or deviating arguments. The results extend some previous results. Typical numerical examples with simulations are utilized to illustrate the validity and improvement in less conservatism of the theoretical results. This paper ends with a brief conclusion.

1. Introduction. Multi-variable feedback systems can exert the retroactive effect on very different time scales. Exemplifying by the extremes, according to the date of the information that is used for feedback, this action can define: (a) a continuous process or (b) one discrete process. In case (a), the growth rates of the variables are feed-backed at each instant, let’s say in real time. While, in case (b) there is a set of isolated dates, for example, a succession of instants in which the information is taken, in order to feedback the period between two consecutive sequence elements.

Normally and for mathematical modeling purposes, in case (a) differential equations are used and in case (b), if there is no other dynamics effect between the feedback times, difference equations can be used to express the essence of the dynamics. There are processes (real-world systems, such as some biotechnology-based ones) that can not be categorized into types (a) or (b), as they combine characteristics of both types of scales among other particular effects. This leads to the use of hybrid type equations, for example the impulsive differential equations with

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**piecewise constant argument** (IDEPCA), that were first considered by Wiener and Lakshmikantham [37] in 2000, and differential equations with piecewise constant argument without impulsive effect (DEPCA) were considered by Shah and Wiener [30] and Wiener [35] in 1983; and has been developed by many other authors. We highlight the book of J. Wiener [36], pioneer of DEPCA, that collects much of the research done in DEPCA. In this case, DEPCA of generalized type were discussed extensively in [1,6–13,28,29].

When scales are mixed these feedback systems can be visualized as control systems, in that, one scale represents the intrinsic of the process and the other is external intervention. However, based on internal parameters. As an example, mentioned in Busenberg and Cooke [3], is the case of the stabilization of hybrid control systems with feedback delay, where a hybrid system is one with a continuous plant with a discrete (sampled) controller. What scale is the internal and what is the control, whether continuous or discrete? That depends on the attributes and simplifications of modeling on the process, being the most usual, to represent the intrinsic process with the continuous time scale and to reflect the intervention from the external environment to the system with the discrete scale.

Note that, either as a feedback system or as a system under control, the questions of interest usually refer to the behavior of the variables in the long term, in particular looking for specific patterns according to values in the space of feasible parameters. For reasons of practical necessity for the modeled processes, the most recurrently sought after behavior is stability. For example, this is seen as convergence to a steady state, or towards dynamic cycles.

As far as the present work is concerned, we are interested in systems of \( n \)-variables in time \( x(t) = (x_1(t), \ldots, x_n(t)) \), \( t \geq 0 \), with hybrid type feedback, i.e. in which to the properly continuous retroaction, that is, an ordinary differential system \( x'(t) = F(t, x(t)) \), is added another action \( G(\cdot, \beta(\cdot), x(\beta(\cdot))) \) of constant type during intervals of time \( I_k = [t_k, t_{k+1}) \), \( k \geq 0 \), whose edge points are a predetermined sequence of times \( \{t_k\} \), this from internal information obtained in said sequence and \( J_k \) are the abrupt changes of the states at the impulsive moments \( t_k \).

Hence

\[
\begin{align*}
x'(t) &= F(t, x(t)) + G(t, \beta(t), x(\beta(t))), \quad t \neq t_k, \quad (1a) \\
\Delta x|_{t_k} &= J_k(x(t_k)), \quad k \in \mathbb{N}, \quad (1b)
\end{align*}
\]

where the timer is given by \( \beta(t) = t_k \) if \( t \in I_k \).

In 1988, Chua and Yang [16] proposed a novel class of information-processing systems called cellular neural networks (CNNs). The study of the stability of CNNs and DCNNs (delayed CNNs) is known to be an important problem in theory and applications. Many essential features of these networks, such as qualitative properties of stability, periodicity, oscillation, and convergence issues have been investigated by many authors, see [2, 4, 5, 11, 17–27, 31, 32, 38, 39, 41, 42] and the references cited therein.

Huang et al. [19] were the first to consider a cellular neural network defined by (1a) with \( F(\cdot, x(\cdot)) = 0 \) and where the \( i \)-th component of \( G(\cdot, \beta(\cdot), x(\beta(\cdot))) \) is given by

\[
- a_i([t]) + \sum_{j=1}^{n} b_{ij}([t]) g_j(x_j([t])) + d_i([t]), \quad (2)
\]
i.e. $\beta(\cdot)$ is the greatest integer function. In this case, $x'(t)$ depends during all the interval $[n, n+1]$, $n$ an integer number, only of the value of functions defined at instant $n$. So, equations type (1), with $\beta(\cdot)$ a constant type function, are named differential equation with generalized piecewise constant delays (DEGPCD). Note that the scientific mathematical community around the DEGPCD with impulsive effect (IDEGPCD) is very limited. See [1, 15].

In the present work, we will consider a case of the IDEGPCD system (1a)-(1b) of more linear nature, but also combining information of the instant with information of the past and impulsive effect. This is, (1a)-(1b) with:

$$
x'(t) = -A(t)x(t) + B(t)f(x(t)) + C(t)g(x(\beta(t))) + D(t), \quad t \neq t_k, \quad (3a)
$$

$$
\Delta x|_{t_k} = J_k(x(t^-_k)), \quad k \in \mathbb{N}, \quad (3b)
$$

where $A(t) = \text{diag}\{a_i(t)\}$, $B = \{b_{ij}(t)\}$ and $C = \{c_{ij}(t)\}$ are real $n \times n$-matrix functions and $D(t) = \{d_i(t)\}$ is $n \times 1$-matrix real function, $J_k = \{J_{ik}\}$ represents the impulsive effects. Eqs. (3a)-(3b) can model many processes that are controlled with internal information to the system recorded every certain time intervals. For instance, in Population Ecology, if $(x_1(\cdot), \cdots, x_n(\cdot))$ summarizes the abundance of a metapopulation of $n$-patches, for a given $i \in \{1, \cdots, n\}$, with respect to the $i$-th patch, the terms $a_{ii}(\cdot)$ can be interpreted as an intrinsic per capita decreasing rate, $b_{ij}f_j(x(\cdot))$ as internal growth when $j = i$ and as migration when $i \neq j$ and $d_i(\cdot)$ as a flow of individuals out or towards the system. While $c_{ij}g_j(x(\cdot))$ can be thought of as a conservation measure, this is, an external flux of individuals towards the $i$-th patch according to an evaluation of the abundance in the $j$-th at the past instant $\beta(t)$. Moreover, $J_k$ is the net dispersal rate between the patches. Besides, the dispersal behavior of populations between patches occurs only at the impulsive instants $t_k, k \in \mathbb{N}$.

Notice that, to know information about the behavior of solutions of the IDEGPCD system (3a)-(3b), as a mathematical problem it has a historical evolution, we can point out that:

1. In 2010, M. U. Akhmet et al. [1] applied the linearization method and obtained some sufficient conditions for global exponential stability of the equilibrium point and the globally asymptotically stable periodic solution.
2. In 2016, T.H. Yu, D.Q. Cao, S.Q. Liu and H.T. Chen [40], trying to generalize to systems (3a) with periodic matrices (periodic coefficients) and without impulsive effects (3b), they established a parametric condition (which involves Lipschitz constants of the coordinates of the functions $f(\cdot)$ and $g(\cdot)$) for global exponential stability of a unique periodic solution.

The novelty of our work is to present new sufficient conditions (that relax those given in [1]) ensuring existence, uniqueness and global exponential stability of periodic solutions for impulsive neural network models with generalized piecewise constant delays (in short, the ICNN models with IDEGPCD system). The method is given by the traditional and tailored route of a: Gronwall-type inequality, fixed point theorem and Green’s function.

This paper is organized as follows. In Section 2, we focus on some preliminary results which will be used in the existence and stability of a unique $\omega$-periodic solution of the ICNN models with IDEGPCD system. In Section 3, we establish several criteria for the existence and uniqueness of the solutions for the ICNN models with
IDEGPCD system (3a)-(3b). Here, a new IDEGPCD’s Gronwall-type inequality is very useful. In Section 4, we derive some sufficient conditions for the global exponential stability of a unique $\omega$-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b). In Section 5, two examples and the numerical simulations are given to demonstrate the validity of our results. The conclusions are drawn in Section 6.

2. Preliminaries. In this section, we will focus on presenting some preliminary concepts and propositions, which will be used in the proofs about existence and stability of a unique $\omega$-periodic solution of the ICNN models with IDEGPCD system.

The system under study is a ICNN model with generalized piecewise constant delay. Where, the state of the $i$-th, $1 \leq i \leq n$, neuron at time $t > 0$ is given by:

$$x_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} \left[b_{ij}(t)f_j(x_j(t)) + c_{ij}(t)g_j(x_j(\beta(t)))\right] + d_i(t), \quad t \neq t_k, \quad (4a)$$

$$\Delta x_i|_{t_k} = J_{ik}(x_i(t^-_k)), \quad i = 1, 2, ..., n, \quad k \in \mathbb{N}, \quad (4b)$$

with $1 \leq i \leq n$, where:

- The constant delay of generalized type is determined by a strictly increasing unbounded sequence of times $\{t_k\}$ and the function $\beta(\cdot)$ defined by $\beta(t) = t_k$ if $t \in I_k = [t_k, t_{k+1})$ and $\beta(t) = t_{k+1} - t_k, \quad k \in \mathbb{N}$.
- The positive function $a_i(\cdot)$ denotes the relative rate with which the $i$-th unit resets its potential to the resting state when isolated from other units and inputs. So in (4a), it represents an exponential decay.
- The measure of activation of continuous type (resp. piecewise constant type) of the $j$-th neuron to its incoming potentials is given at any time by the function $f_j(x_j(\cdot))$ (resp. $g_j(x_j(\beta(\cdot)))$).
- The function $b_{ij}(\cdot)$ (resp. $c_{ij}(\cdot)$) denotes the synaptic connection weight of continuous type (resp. piecewise type) of the unit $j$ on the unit $i$.
- For each neuron, there is an activation flow from outside the system. It is represented by the function $d_i(\cdot)$ for the $i$-th one.
- $\Delta x_i(t_k)$ denotes $x_i(t_k^-) - x_i(t_k^+)$, where $x_i(t_k^+) = \lim_{h \to 0^+} x_i(t_k + h)$. Moreover, the numbers $x_i(t_k^-)$ and $x_i(t_k)$ are, respectively, the states of the $i$-th unit before and after impulse perturbation at the moment $t_k, \quad k \in \mathbb{N}$, and represents the abrupt change of the state $J_{ik}(x_i(t^-_k))$ at the impulsive moment $t_k$.

For reasons of convenience, certain assumptions are formulated below, which will be convened when necessary.

(L) Lipschitz Condition: The activation functions $f_j$ and $g_j$ with $f_j(0) = 0$, $g_j(0) = 0$, $0 \leq j \leq n$, satisfy

$$|f_j(u) - f_j(v)| \leq L^f_j|u - v|, \quad |g_j(u) - g_j(v)| \leq L^g_j|u - v|,$$

The impulsive operator $J_{ik}$ with $0 \leq i \leq n, \quad k \in \mathbb{N}$, satisfies

$$|J_{ik}(u) - J_{ik}(v)| \leq L^I_{ik}|u - v|,$$

for some constants $L^f_j, L^g_j, L^I_{ik} > 0$ and for all $u, v \in \mathbb{R}$. 
Given a pair

against Proposition 1. Integral Representation:

construction of an equivalent integral equation, which we give in the proposition

\( x \) function

Definition 2.1. \[9,11,37\] allows us to define a solution of the IDEGPCD system (4a)-(4b). A natural extension of the original definition of a solution of IDEPCA

\( R \)

only if their coordinates satisfy on

is a solution of the IDEGPCD system (4a)-(4b) in the sense of Definition 2.1 if and

where

\( \Phi(\cdot) \) is an indexer defined by

\( i(t) = k \) if \( t \in I_k = [t_k, t_{k+1}) \) and \( \vartheta_k = t_{k+1} - t_k, k \in \mathbb{N} \).

First, we prove the existence and uniqueness of solutions of the IDEGPCD system (4a)-(4b). A natural extension of the original definition of a solution of IDEPCA [9,11,37] allows us to define a solution of the IDEGPCD system.

**Definition 2.1.** A function \( x \) is a solution of the IDEGPCD system (4a)-(4b) in \( \mathbb{R}^+ = [0, \infty) \) if

i) \( x(t) \) is continuous for \( t \in \mathbb{R}^+ \) with the possible exception of the points \( t = t_k, k \in \mathbb{N} \).

ii) \( x(t) \) is right continuous and has left-hand limits at the points \( t = t_k, k \in \mathbb{N} \).

iii) \( x(t) \) is differentiable and satisfies (4a) for any \( t \in \mathbb{R}^+ \), with the possible exception of the points \( t = t_k, k \in \mathbb{N} \), where one-sided derivatives exist.

iv) \( x(t_k) \) satisfies (4b), \( k \in \mathbb{N} \).

To study the nonlinear IDEGPCD system, we will use the approach based on the construction of an equivalent integral equation, which we give in the proposition that follows:

**Proposition 1. Integral Representation:** Given a pair \( (\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n \), a function \( x = (x_1(\cdot), \ldots, x_n(\cdot)) : \mathbb{R}^+ \to \mathbb{R}^n \) such that \( x(\tau) = x_0 = (x_1(\tau), \ldots, x_n(\tau))^T \)

is a solution of the IDEGPCD system (4a)-(4b) in the sense of Definition 2.1 if and only if their coordinates satisfy on \( \mathbb{R}^+ \) the following set of integral equations

\[
x_i(t) = e^{-\int_0^t a_i(s)ds}x_i(\tau) + \int_\tau^t e^{-\int_s^t a_i(u)du} \left[ \sum_{j=1}^n b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^n c_{ij}(s)g_j(x_j(\beta(s))) + d_i(s) \right] ds
\]

\[
+ \sum_{k=i(\tau)+1}^{i(t)} e^{-\int_{t_k}^{t} a_i(s)ds} J_{ik}(x_i(t_k^-)), \quad i \in \{1, \ldots, n\},
\]

(5)

or

\[
x(t) = \Phi(t, \tau)x_0 + \int_\tau^t \Phi(t, s) [B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s)] ds
\]

\[
+ \sum_{k=i(\tau)+1}^{i(t)} \Phi(t, t_k)J_k(x(t_k^-)), \quad t \in \mathbb{R}^+,
\]

(6)

where \( \Phi(t) \) is a fundamental solution of \( u'(t) = -A(t)u(t) \) and \( \Phi(t, s) = \Phi(t)\Phi^{-1}(s) \).

We omit the proof of this assertion, since it can be proved in the same way as Proposition 2. in [9] and Proposition 2.1 in [11].

As in [9], one of the main tools used is a global solution of a Gronwall-type inequality with deviating argument. This allows us to study a variety of perturbations with a growth of exponential order. Using the same technique of [9], we have the following results:
Lemma 2.2. IDEGPCD’s Gronwall Inequality: Let $u : [0, \infty) \to [0, \infty)$ be a function such that $u$ is continuous with possible points of discontinuity of the first kind at $t = t_k$, $k \in \mathbb{N}$, satisfying the inequality

$$u(t) \leq \alpha + \int_{\tau}^{t} [\eta_1(s)u(s) + \eta_2(s)u(\beta(s))] \, ds + \sum_{k=i(\tau)+1}^{i(t)} \varrho_k u(t_{k-})^k, \quad (7)$$

where $\alpha \geq 0$ and $\eta_i : \mathbb{R}^+ \to \mathbb{R}^+$, $i \in \{1, 2\}$, is a piecewise continuous functions and $\varrho_k$ is non-negative real constant, $k \in \mathbb{N}$. Then:

1. If $t \geq \tau$,

$$u(t) \leq \alpha \prod_{k=i(\tau)+1}^{i(t)} (1 + \varrho_k) \cdot \exp \left( \int_{\tau}^{t} [\eta_1(s) + \eta_2(s)] \, ds \right) . \quad (8)$$

2. If $0 \leq t \leq \tau$,

$$u(t) \leq \alpha \prod_{k=i(\tau)+1}^{i(\tau)} (1 - \varrho_k)^{-1} \cdot \exp \left( \int_{t}^{\tau} \left[ \eta_1(s) + \frac{\eta_2(s)}{1 - \sigma} \right] \, ds \right) , \quad (9)$$

where

$$\sigma := \max_{1 \leq k \leq i(\tau)} \int_{t_k}^{t_{k+1}} [\eta_1(s) + \eta_2(s)] \, ds \leq \kappa < 1 \quad \text{and} \quad \varrho_k < 1, \quad 1 \leq k \leq i(\tau).$$

3. Existence and uniqueness of solutions. In this section, sufficient conditions that govern the network parameters and the activation functions are established for the existence and global exponential stability of a unique solution of the ICNN models with IDEGPCD system (4a)-(4b).

We need to have the global unique existence of solutions of the nonlinear IDEGPCD (4a)-(4b). The following proposition provides conditions of unique existence of solutions on differentiability intervals. One can easily see that the IDEGPCD system (4a)-(4b) has the form of the DEGPCD system without impulsive effect within the intervals $[t_i, t_{i+1})$, $i \in \mathbb{N}$, then using the same technique of Gronwall-type inequality with piecewise constant argument (see [7], [8] and [28]), we have the following results.

Proposition 2. Let us assume that the conditions (L) and (E) are satisfied. Then, given an initial condition $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot, \tau, x_0) = (x_1(\cdot), ..., x_n(\cdot))^T$ of the ICNN models with IDEGPCD system (4a)-(4b) on $[t_{i(\tau)}, t_{i(\tau)+1})$ such that $x(\tau) = x_0 = (x_0^1, ..., x_0^n)^T$. The previous proposition assures the existence and uniqueness of solutions in a local sense. The following theorem provides the existence of a unique solution when the initial moment is an arbitrary positive real number $\tau$.

Theorem 3.1. Let us assume that the conditions (L) and (E) are satisfied. Then, given an initial condition $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot, \tau, x_0) = (x_1(\cdot), ..., x_n(\cdot))^T$ of the ICNN models with IDEGPCD system (4a)-(4b) in the sense of Definition 2.1 such that $x(\tau) = x_0 = (x_0^1, ..., x_0^n)^T$. 

**Proof.** Fix $\tau \in \mathbb{R}^+$, then $\tau \in I_i(\tau) = [t_i(\tau), t_i(\tau)+1)$. Use Proposition 2 with $x(\tau) = x_0$ to obtain the unique solution $x(t) = x(t, \tau, x_0)$ on $I_i(\tau)$. Then apply the impulse condition (4b) to evaluate uniquely

$$x(t_i(\tau)+1, \tau, x_0) = x(t_i(\tau)+1, \tau, x_0) + J_i(\tau)(x(t_i(\tau)+1, \tau, x_0)).$$

Now, on the interval $I_i(\tau)+1 = [t_i(\tau)+1, t_i(\tau)+2)$ the solution satisfies the DEGPCD system

$$\frac{dy_i(t)}{dt} = -a_i(t)y_i(t) + \sum_{j=1}^{n} \left[ b_{ij}(t)f_j(y_j(t)) + c_{ij}(t)g_j(y_j(\beta(t))) \right] + d_i(t), \quad i = 1, 2, \ldots, n.$$

The IDEGPCD system has a unique solution $y(t, t_i(\tau)+1, x(t_i(\tau)+1, \tau, x_0))$ by Definition 2.1 of the solution of the IDEGPCD system (4a)-(4b), $x(t, \tau, x_0) = y(t, t_i(\tau)+1, x(t_i(\tau)+1, \tau, x_0))$ on $I_i(\tau)+1 = [t_i(\tau)+1, t_i(\tau)+2)$. The mathematical induction completes the proof.

\[ \square \]

From Theorem 3.1, we can derive the following particular results:

**Corollary 1.** Let us assume the condition (L) and the inequality

$$\max_{1 \leq i \leq n} \left\{ \frac{\exp(\bar{a}_i \vartheta) - 1}{\bar{a}_i} \left( \sum_{j=1}^{n} L_j^f |b_{ij}| + L_j^g |c_{ij}| \right) \right\} < 1,$$

(10)

where $\bar{a}_i = \sup_{t \in \mathbb{R}^+} a_i(t)$, $\bar{b}_{ij} = \sup_{t \in \mathbb{R}^+} |b_{ij}(t)|$, $\bar{c}_{ij} = \sup_{t \in \mathbb{R}^+} |c_{ij}(t)|$ and $\vartheta = \max_{1 \leq k \leq i(\tau)} (t_k+1 - t_k)$. Then, given $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot) = x(\cdot, \tau, x_0) = (x_1(\cdot), \ldots, x_n(\cdot))^T$ of the IDEGPCD system (4a)-(4b) in the sense of Definition 2.1 such that $x(\tau) = x_0 = (x_0^1, \ldots, x_0^n)^T$.

**Corollary 2.** Let us assume the condition (L), that $a_i(t) \equiv a_i$, $b_{ij}(t) \equiv b_{ij}$, $c_{ij}(t) \equiv c_{ij}$, $d_i(t) \equiv d_i$ and the inequality

$$\max_{1 \leq i \leq n} \left\{ \frac{\exp(a_i \vartheta) - 1}{a_i} \left( \sum_{j=1}^{n} L_j^f |b_{ij}| + L_j^g |c_{ij}| \right) \right\} < 1,$$

(11)

where $\vartheta = \max_{1 \leq k \leq i(\tau)} (t_k+1 - t_k)$. Then, given $(\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n$, there exists a unique solution $x(\cdot) = x(\cdot, \tau, x_0) = (x_1(\cdot), \ldots, x_n(\cdot))^T$ of the IDEGPCD system (4a)-(4b), but with the respective constant coefficients, in the sense of Definition 2.1 such that $x(\tau) = x_0 = (x_0^1, \ldots, x_0^n)^T$.

**Remark 1.** When $a_i(t) \equiv a_i$, $b_{ij}(t) \equiv b_{ij}$, $c_{ij}(t) \equiv c_{ij}$, $d_i(t) \equiv d_i$, the ICNN models with IDEGPCD system (4a)-(4b) can be reduced to the following IDEGPCD system with constant coefficients:

$$x_i'(t) = -a_i x_i(t) + \sum_{j=1}^{n} \left[ b_{ij} f_j(x_j(t)) + c_{ij} g_j(x_j(\beta(t))) \right] + d_i, \quad t \neq t_k,$$

(12a)

$$\Delta x_i|_{t_k} = J_k(x_i(t_k^-)), \quad i = 1, 2, \ldots, n, \quad k \in \mathbb{N}.$$

(12b)

In Ref. [1], the IDEGPCD system (12a)-(12b) with constant coefficients is considered and a sufficient condition guaranteeing the existence and uniqueness of the solutions is derived. Compared with the results obtained in [1], our conditions in Corollary 2 are less restrictive than those in Ref. [1], because those works didn’t have a global IDEGPCD’s Gronwall-type inequality. It is in this point that our
results generalize and improve those in [1]. In Section 5, Example 2 is represented
to show the effectiveness of our result.

When the impulsive jumps of ICNN models with IDEGPCD system (4b) are
absent, ICNN models with IDEGPCD system (4a) reduces to the following non-
impulsive system:

\[ x'_i(t) = -a_i(t)x_i(t) + \sum_{j=1}^{n} \left[ b_{ij}(t)f_j(x_j(t)) + c_{ij}(t)g_j(x_j(\beta(t))) \right] + d_i(t), \quad t \in \mathbb{R}, \quad (13) \]

with \(1 \leq i \leq n\).

Applying our results to CNN models with DEGPCD system (13) without impul-
sive effects, we have:

Corollary 3. Let us assume that the conditions (L) and (E) are satisfied. Then,
given an initial condition \((\tau, x_0) \in \mathbb{R}^+ \times \mathbb{R}^n\), there exists a unique solution \(x(\cdot) = x(t, \tau, x_0) = (x_1(\cdot), ..., x_n(\cdot))^T\) of the CNN models with DEGPCD system (13), such
that \(x(\tau) = x_0 = (x_0^1, ..., x_0^n)^T\).

Remark 2. Corollary 3 reduces to the results of Chiu et al. [14, Theorem 3.1].

Remark 3. The CNN models with DEGPCD system (13) with periodic coefficients
and without impulsive effects (4b) is neither more nor less than the DEGPCD
system (1) in [40]. But the paper never had the conditions to guarantee the existence
and uniqueness of the solutions of the DEGPCD system (13).

4. Existence and stability of periodic solutions. In this section, we will give
the sufficient conditions for existence and global exponential stability of the
\(\omega\)-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b). Here we
assume the Periodicity Condition:

(P) Periodicity Condition: There exists \(\omega > 0\) such that:
1) \(a_i(\cdot) > 0, b_{ij}(\cdot), c_{ij}(\cdot)\) and \(d_i(\cdot)\) are continuously periodic functions in \(\mathbb{R}^+\)
with a common period \(\omega\).
2) There exists \(p \in \mathbb{N}\), for which the sequences \(\{t_k\}_{k \in \mathbb{N}}\) and \(\{J_k\}_{k \in \mathbb{N}}\), satisfies
the \((\omega, p)\) condition, that is
\(t_{k+p} = t_k + \omega\) and \(J_{k+p} = J_k\). \quad (14)

Remark 4. Note that \((\omega, p)\) condition is a discrete relation, which moves the
interval \(I_k\) into \(I_{k+p}\). Then we have the following consequence: For \(t \in [t_k, t_{k+1})\),
\(\beta(t + \omega) = \beta_{i(t+\omega)} = \beta_{i(t)+p} = \beta_{i(t)} + \omega = \beta(t) + \omega\).

For \(\omega > 0\), let \(PC_\omega\) be the set of all \(n\)-vector piecewise continuous function \(x(t)\)
with points of discontinuity of the first kind at \(t = t_i, i \in \mathbb{N}\), periodic in \(t\) of period
\(\omega\). Then \((PC_\omega, \|\cdot\|)\) is a Banach space with the supremum norm
\(\|x\| = \max_{1 \leq i \leq n} \|x_i\| = \max_{1 \leq i \leq n} \left[ \sup_{t \in \mathbb{R}^+} |x_i(t)| \right] = \max_{1 \leq i \leq n} \left[ \sup_{t \in [\tau, \tau+\omega]} |x_i(t)| \right].\)
4.1. **Existence of periodic solutions.** In this subsection, we will establish the sufficient conditions for existence and uniqueness of the \( \omega \)-periodic solutions of the ICNN models with IDEGPCD system (4a)-(4b).

Before giving our main result of this subsection, we need to establish some definitions and elementary facts:

**Definition 4.1.** For each \( t, s \in [\tau, \tau + \omega] \), the Green’s function for the ICNN models with IDEGPCD system (4a)-(4b) is given by \( G(t, s) = \text{diag}\{G_i(t, s)\}, \; i = 1, ..., n \), where

\[
G_i(t, s) = \begin{cases} 
\frac{\exp(\int_{\tau}^{t+s} a_i(u)du)}{\exp(\int_{\tau}^{t} a_i(u)du) - 1} \exp(\int_{t}^{s} a_i(\kappa)d\kappa), & \tau \leq t \leq \tau + \omega, \\
\frac{1}{\exp(\int_{\tau}^{t+s} a_i(u)du) - 1} \exp(\int_{t}^{s} a_i(\kappa)d\kappa), & \tau \leq t \leq s \leq \tau + \omega.
\end{cases}
\] (15)

Note that \( G \) and \( G_i \) are bi \( \omega \)-periodic, i.e., \( G(t + \omega, s + \omega) = G(t, s) \) and the denominator in \( G_i(t, s) \) for \( i = 1, ..., n \), is not zero since we have assumed that \( a_i(c) > 0 \) for some \( c \in [\tau, \tau + \omega] \). Observe that \( G_i(t, s) \) has maximum and minimum values:

\[
\frac{1}{\exp(\int_{\tau}^{t+s} a_i(\kappa)d\kappa) - 1} \leq G_i(t, s) \leq \frac{\exp(\int_{\tau}^{t+s} a_i(\kappa)d\kappa)}{\exp(\int_{\tau}^{t+s} a_i(\kappa)d\kappa) - 1}, \; t \leq s \leq t + \omega.
\]

For the sake of convenience, we adopt the following notations:

\[
c_G := \max_{t, s \in [\tau, \tau + \omega]} |G(t, s)| = \max_{1 \leq i \leq n} \frac{\exp(\int_{\tau}^{t+s} a_i(\kappa)d\kappa)}{\exp(\int_{\tau}^{t+s} a_i(\kappa)d\kappa) - 1} \quad \text{and} \quad a_*(t) = \min_{1 \leq i \leq n} a_i(t).
\]

Using Definition 4.1, Remark 4 and bi \( \omega \)-periodicity of the Green’s function, we have the following lemma.

**Lemma 4.2.** Suppose that the condition (P) holds and \( x(t) \) is a solution of the ICNN models with IDEGPCD system (4a)-(4b), then \( x(t) \in \text{PC}_\omega \) if and only if

\[
x(t) = \int_{\tau}^{t+\omega} G(t, s) \left[ B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s) \right] ds + \sum_{i(t)+p} G(t, t_k)J_k(x(t_k^+)),
\] (16)

where \( G(t, s) \) is the Green’s function.

In particular, we have

\[
x_i(t) = \int_{\tau}^{t+\omega} G_i(t, s) \left[ \sum_{j=1}^{n} b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^{n} c_{ij}(s)g_j(x(\beta(s))) + d_i(s) \right] ds + \sum_{k=i(t)+1}^{i(t)+p} G_i(t, t_k)J_{ik}(x(t_k^+)), \quad 1 \leq i \leq n.
\] (17)

**Proof.** Sufficiency. First, we will prove that if Eq.(16) has a unique solution, then \( x(t) \in \text{PC}_\omega \).

Indeed, by the condition (P) and bi \( \omega \)-periodicity of the Green’s function, \( x(t + \omega) \)
is a solution of Eq. (16)

\[ x(t + \omega) = \int_{\tau + \omega}^{\tau + 2\omega} G(t + \omega, s) [B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s)] ds + \sum_{k=1}^{i(\tau)+p} G(t + \omega, t_k) J_k(x(t_k^-)) \]

Then, \( x(t + \omega) = x(t) \) and \( x(t) \) is an \( \omega \)-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b). This proves the sufficiency.

**Necessity.** Let \( x(t) \in PC_\omega \) be a solution of the ICNN models with IDEGPCD system (4a)-(4b), \( \Phi(t) \) is a fundamental solution of \( u'(t) = -A(t)u(t) \) and \( I_n = \text{diag}(1, 1, ..., 1) \). By Proposition 1, we have

\[ x(t) = \Phi(t, \tau)x_0 + \int_{\tau}^{t} \Phi(t, s) [B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s)] ds + \sum_{k=1}^{i(\tau)+p} \Phi(t, t_k) J_k(x(t_k^-)), \quad \tau, t \in \mathbb{R}^+. \]

Since \( x(\tau) = x_0 = x(\tau + \omega) \), we get

\[ x_0 = (I_n - \Phi(\tau + \omega, \tau))^{-1} \int_{\tau}^{\tau+\omega} \Phi(\tau + \omega, s) [B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s)] ds + \sum_{k=1}^{i(\tau)+p} \Phi(\tau + \omega, t_k) J_k(x(t_k^-)). \quad (18) \]

A substitution of (18) into (5) yields

\[ x(t) = \Phi(t, \tau) \left( (I_n - \Phi(\tau + \omega, \tau))^{-1} \int_{\tau}^{\tau+\omega} \Phi(\tau + \omega, s) [B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s)] ds + \sum_{k=1}^{i(\tau)+p} \Phi(\tau + \omega, t_k) J_k(x(t_k^-)) \right) + \int_{\tau}^{t} \Phi(t, s) [B(s)f(x(s)) + C(s)g(x(\beta(s))) + D(s)] ds + \sum_{k=1}^{i(\tau)+p} \Phi(t, t_k) J_k(x(t_k^-)). \quad (19) \]

Let \( E = \left( (\Phi^{-1}(\tau)\Phi(\tau + \omega))^{-1} - I_n \right)^{-1} = \text{diag} \left\{ \frac{1}{e^{\int_{\tau+\omega}^{\tau+2\omega} A_r(\tau) dr} - 1}, ..., \frac{1}{e^{\int_{\tau+\omega}^{\tau+2\omega} A_n(\tau) dr} - 1} \right\} \). Since \( (I_n + E) = \left( (\Phi^{-1}(\tau)\Phi(\tau + \omega))^{-1} E = (I_n - \Phi^{-1}(\tau)\Phi(\tau + \omega))^{-1} \right) \), (19) becomes
where operator $T$ for $i \in \mathbb{N}$ is defined as:

$$T = 1$$

The following theorem is our main result of this subsection:

This completes the proof of Lemma 4.2. 

Now, define the operator $T : PC_\omega \to PC_\omega$ by

$$(Tx)(t) = ((Tx)_1(t), (Tx)_2(t), ..., (Tx)_n(t))^T,$$

where

$$(Tx)_i(t) = \int_\tau^{\tau+\omega} G_i(t, s) \left[ \sum_{j=1}^n b_{ij}(s)f_j(x_j(s)) + \sum_{j=1}^n c_{ij}(s)g_j(x_j(\beta(s))) 
+ d_i(s) \right] ds + \sum_{k=i(\tau)+1}^{i(\tau)+p} G_i(t, t_k)J_{ik}(x_i(t_k^-)),$$

for $i = 1, 2, ..., n$. By Lemma 4.2, it is easy to verify that $x = x(t)$ is an $\omega$-periodic solution of the ICNN models with IDEGPCD system \((4a)-(4b)\) if and only if the operator $T$ has one fixed point in $PC_\omega$.

The following theorem is our main result of this subsection:
Theorem 4.3. If \((L), (E)\) and \((P)\) are satisfied and
\[
c_G \cdot \max_{1 \leq i \leq n} \left\{ \left[ \int_{\tau}^{\tau+\omega} \left( \sum_{j=1}^{n} \mathcal{L}^f_j |b_{ij}(s)| + \mathcal{L}^g_j |c_{ij}(s)| \right) ds \right] + \sum_{k=(\tau)+1}^{i(\tau)+p} L^L_{ik} \right\} < 1.
\]
(20)

Then the ICNN models with IDEGPCD system \((4a)-(4b)\) has a unique \(\omega\)-periodic solution.

Proof. Define the operator \(T\) in \(PC_\omega\) by \(T : PC_\omega \to PC_\omega\) such that if \(\varphi \in PC_\omega\), then
\[
(T \varphi)_i(t) = \int_{\tau}^{\tau+\omega} G_i(t, s) \left[ \sum_{j=1}^{n} b_{ij}(s) f_j(\varphi_j(s)) + \sum_{j=1}^{n} c_{ij}(s) g_j(\varphi_j(\beta(s))) + d_i(s) \right] ds + \sum_{k=(\tau)+1}^{i(\tau)+p} G_i(t, t_k) J_{ik}(\varphi_i(t_k)),
\]
for \(i = 1, 2, \ldots, n\).

Take subset \(PC^*_\omega \subset PC_\omega\),
\[
PC^*_\omega = \left\{ \varphi | \varphi \in PC_\omega, \| \varphi - \varphi_0 \| \leq \frac{\varphi_0}{1 - \varrho} \right\},
\]
where \(\varrho := c_G \cdot \left( \max_{1 \leq i \leq n} \left\{ \left[ \int_{\tau}^{\tau+\omega} \left( \sum_{j=1}^{n} \mathcal{L}^f_j |b_{ij}(s)| + \mathcal{L}^g_j |c_{ij}(s)| \right) ds \right] + \sum_{k=(\tau)+1}^{i(\tau)+p} L^L_{ik} \right\} \right)\).

\(\varphi = c_G D \omega, D = \max_{1 \leq i \leq n} \left( \sup_{t \in \mathbb{R}^+} |d_i(t)| \right)\) and \((\varphi_0)_i(t) = \int_{\tau}^{\tau+\omega} G_i(t, s) d_i(s) ds, i = 1, \ldots, n\).

Then \(PC^*_\omega\) is a closed convex subset of \(PC_\omega\) and
\[
|((\varphi_0)_i(t)| = \left| \int_{\tau}^{\tau+\omega} G_i(t, s) d_i(s) ds \right| \leq c_G D \omega = \varphi < \infty.
\]

So,
\[
\| \varphi_0 \| = \max_{1 \leq i \leq n} \left[ \sup_{t \in \mathbb{R}^+} |((\varphi_0)_i(t)| \right] \leq \varphi.
\]

Then, for an arbitrary \(\varphi \in PC^*_\omega\), we have
\[
\| \varphi \| \leq \| \varphi - \varphi_0 \| + \| \varphi_0 \| \leq \frac{\varphi_0}{1 - \varrho} + \varphi = \frac{\varphi}{1 - \varrho}.
\]
We first prove that the mapping $T$ is self-mapping from $PC^*_\omega$ to $PC^*_\omega$. In fact, for any \( \varphi \in PC^*_\omega \), we have

\[
\|T\varphi - \varphi_0\| = \max_{1 \leq i \leq n} \|(T\varphi)_i(\cdot) - (\varphi_0)_i(\cdot)\|
\leq \max_{1 \leq i \leq n} \left\{ \int_{\tau}^{\tau+\omega} \max_{t \in [\tau, \tau+\omega]} |G_i(t, s)| \left[ \sum_{j=1}^{n} |b_{ij}(s)||f_j(\varphi_j(s))| + \sum_{j=1}^{n} |c_{ij}(s)||g_j(\varphi_j(\beta(s)))| \right] ds \\
+ \sum_{k=1}^{l(\tau)+p} \max_{t \in [\tau, \tau+\omega]} |G_i(t, t_k)| |J_{ik}(\varphi_i(t_k^-))| \right\}
\leq c_G \cdot \max_{1 \leq i \leq n} \left\{ \left[ \int_{\tau}^{\tau+\omega} \left( \sum_{j=1}^{n} b_{ij}(s)|L^f_j|\varphi_j(s) + L^g_j|c_{ij}(s)| \right) ds \right] + \sum_{k=1}^{l(\tau)+p} L^f_{ik} \right\} \|\varphi\|
\leq \frac{\varrho}{1 - \varrho} = \frac{\psi^{*}}{1 - \varrho}.
\]

That is $T\varphi \in PC^*_\omega$. Secondly, we prove that $T$ is a contraction mapping. For any $\psi, \varphi \in PC^*_\omega$, by using (L), we can deduce

\[
\|T\psi - T\varphi\| = \max_{1 \leq i \leq n} \|(T\psi)_i(\cdot) - (T\varphi)_i(\cdot)\|
\leq \max_{1 \leq i \leq n} \left\{ \int_{\tau}^{\tau+\omega} \max_{t \in [\tau, \tau+\omega]} |G_i(t, s)| \left[ \sum_{j=1}^{n} |b_{ij}(s)||f_j(\psi_j(s)) - f_j(\varphi_j(s))| + \sum_{j=1}^{n} |c_{ij}(s)||g_j(\psi_j(\beta(s)) - g_j(\varphi_j(\beta(s)))| \right] ds \\
+ \sum_{k=1}^{l(\tau)+p} \max_{t \in [\tau, \tau+\omega]} |G_i(t, t_k)| |J_{ik}(\psi_i(t_k^-)) - J_{ik}(\varphi_i(t_k^-))| \right\}
\leq c_G \cdot \max_{1 \leq i \leq n} \rho_i \cdot \|\psi - \varphi\|,
\]

where

\[
\rho_i := \int_{\tau}^{\tau+\omega} \left( \sum_{j=1}^{n} L^f_j|b_{ij}(s)| + L^g_j|c_{ij}(s)| \right) ds + \sum_{k=1}^{l(\tau)+p} L^f_{ik}.
\]

Finally, we have

\[
\|T\psi - T\varphi\| \leq c_G \cdot \max_{1 \leq i \leq n} \rho_i \cdot \|\psi - \varphi\|.
\]
From the condition (20), it follows that \( T \) is contraction mapping in \( PC_\omega^m \). Therefore, the mapping \( T \) possesses a unique fixed point \( \varphi^* \in PC_\omega \) such that \( T \varphi^* = \varphi^* \). So, by Lemma 4.2, \( \varphi^* \) is the unique \( \omega \)-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b). The proof is completed.

From Theorem 4.3, we can derive the following results.

**Corollary 4.** If \((L),(P)\) and (10) are satisfied and
\[
\max_{1 \leq i \leq n} \left\{ \frac{\exp(a_i)}{\exp(\bar{a}_i)} - 1 \left[ \sum_{j=1}^{n} \left( L_j^f b_{ij} + L_j^g c_{ij} \right) \right] \right\} < 1, \tag{21}
\]
where, \( \bar{a}_i = \sup_{t \in \mathbb{R}^+} a_i(t) \), \( \underline{a}_i = \inf_{t \in \mathbb{R}^+} a_i(t) > 0 \), \( \bar{b}_{ij} = \sup_{t \in \mathbb{R}^+} |b_{ij}(t)| \) and \( \bar{c}_{ij} = \sup_{t \in \mathbb{R}^+} |c_{ij}(t)| \).

Then the ICNN models with IDEGPCD system (4a)-(4b) has a unique \( \omega \)-periodic solution.

**Corollary 5.** For \( a_i(t) \equiv a_i > 0 \), \( b_{ij}(t) \equiv b_{ij} \) and \( c_{ij}(t) \equiv c_{ij} \) constants, if \((L),(P)\) and (11) are satisfied and
\[
\max_{1 \leq i \leq n} \left\{ \frac{\exp(a_i)}{\exp(\bar{a}_i)} - 1 \left[ \sum_{j=1}^{n} \left( L_j^f b_{ij} + L_j^g c_{ij} \right) \right] \right\} < 1. \tag{22}
\]

Then the ICNN models with IDEGPCD system (12a)-(12b) (the respective constant coefficients) has a unique \( \omega \)-periodic solution.

Note that Corollary 5 has been obtained in [1] by M. U. Akhmet et al. under stronger conditions. See the Remark 9.

Applying our results to CNN models with DEGPCD system (13) without impulsive effects, we have:

**Corollary 6.** If \((L),(P)\) and (10) are satisfied and
\[
\max_{1 \leq i \leq n} \left\{ \frac{\exp(\bar{a}_i)}{\exp(\underline{a}_i)} - 1 \left[ \sum_{j=1}^{n} \left( L_j^f b_{ij} + L_j^g c_{ij} \right) \right] \right\} < 1, \tag{23}
\]
where, \( \bar{a}_i = \sup_{t \in \mathbb{R}^+} a_i(t) \), \( \underline{a}_i = \inf_{t \in \mathbb{R}^+} a_i(t) > 0 \), \( \bar{b}_{ij} = \sup_{t \in \mathbb{R}^+} |b_{ij}(t)| \) and \( \bar{c}_{ij} = \sup_{t \in \mathbb{R}^+} |c_{ij}(t)| \).

Then the CNN models with DEGPCD system (13) has a unique \( \omega \)-periodic solution.

**Remark 5.** Corollary 6 reduces to the results of Chiu et al. [14, Corollary 3].

**Corollary 7.** If \((L),(P)\) and (11) are satisfied and
\[
\max_{1 \leq i \leq n} \left\{ \frac{\exp(a_i)}{\exp(\underline{a}_i)} - 1 \left[ \sum_{j=1}^{n} \left( L_j^f b_{ij} + L_j^g c_{ij} \right) \right] \right\} < 1. \tag{24}
\]

Then the CNN models with DEGPCD system (13) with the respective constant coefficients has a unique \( \omega \)-periodic solution.
4.2. Global exponential stability of the periodic solution. The following result will obtain sufficient conditions for the global exponential stability of the ω-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b). Here we assume the Stability Condition:

(S) Stability Condition: There exists $\rho \in \mathbb{R}^+$, such that

$$a_\ast(t) - K(t) - L^I_{i(t)} \geq \rho > 0, \quad t \in \mathbb{R}^+,$$

where $a_\ast(t) = \min_{1 \leq i \leq n} a_i(t)$, $L^I_{i(t)} = \max_{i(\tau)+1 \leq k \leq i(t)} \frac{\ln(1+L^J_{ik})}{\sigma_k}$, $L^J_k = \max_{1 \leq i \leq n} L^J_{ik}$ and

$$K(t) = \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n L^J_j |h_{ij}(t)| + \sum_{j=1}^n L^J_j |c_{ij}(t)| \exp \left( \int_{\beta(t)}^t a_\ast(s) ds \right) \right].$$

Theorem 4.4. If (L), (E), (P), (S) and (20) are satisfied. Then the ω-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b) is globally exponentially stable.

To prove Theorem 4.4, we need the following lemma.

Lemma 4.5. If (L) and (E) are satisfied, then the solutions $\varphi$ and $\psi$ of the IDEGPCD system (4a)-(4b) satisfy for all $t \geq \tau$ the inequality

$$|\varphi(t) - \psi(t)| \leq |\varphi(\tau) - \psi(\tau)| \exp \left( - \int_{\tau}^t \lambda(s) ds \right),$$

where $\lambda(t) = a_\ast(t) - K(t) - L^I_{i(t)}$, $a_\ast(t) = \min_{1 \leq i \leq n} a_i(t)$, $L^I_{i(t)} = \max_{i(\tau)+1 \leq k \leq i(t)} \frac{\ln(1+L^J_{ik})}{\sigma_k}$, $L^J_k = \max_{1 \leq i \leq n} L^J_{ik}$ and

$$K(t) = \max_{1 \leq i \leq n} K_i(t)$$

$$= \max_{1 \leq i \leq n} \left[ \sum_{j=1}^n L^J_j |h_{ij}(t)| + \sum_{j=1}^n L^J_j |c_{ij}(t)| \exp \left( \int_{\beta(t)}^t a_\ast(s) ds \right) \right].$$

Proof. Suppose that $\varphi(t) = (\varphi_1, ..., \varphi_n)^T$ and $\psi(t) = (\psi_1, ..., \psi_n)^T$ are arbitrary solutions of the IDEGPCD system (4a)-(4b). Let $y(t) = \varphi(t) - \psi(t)$ and by (4a)-(4b) it follows that $y(t)$ satisfies

$$\begin{cases}
\dot{y}(t) = -A(t)y(t) + B(t) \left[ f(y(t) + \psi(t)) - f(\psi(t)) \right] + C(t) \left[ g(y(\beta(t)) + \psi(\beta(t))) - g(\psi(\beta(t))) \right], \\
\Delta y|_{t=t_k} = J_k (y(t^-_k) + \psi(t^-_k)) - J_k (\psi(t^-_k)), \quad k \in \mathbb{N}.
\end{cases}$$

By Proposition 1, it can be proved that

$$y(t) = \Phi(t, \tau)y(\tau) + \int_\tau^t \Phi(t, s) R(s, y(s)) ds + \sum_{k=\tau}^{i(t)} \Phi(t, t_k) J_k (y(t^-_k)),$$

where

$$R(s, y(s)) := B(s) \left[ f(y(s) + \psi(s)) - f(\psi(s)) \right] + C(s) \left[ g(y(\beta(s)) + \psi(\beta(s))) - g(\psi(\beta(s))) \right].$$
and

\[ \mathcal{J}_k(y(t^-_k)) := J_k(y(t^-_k) + \psi(t^-_k)) - J_k(\psi(t^-_k)). \]

Notice that (L) implies that

\[ |\mathcal{R}_i(s, y(s))| \leq \left( \sum_{j=1}^{n} \mathcal{L}^i_j |b_{ij}(s)||y_j(s)| + \sum_{j=1}^{n} \mathcal{L}^j_i |c_{ij}(s)||y_j(\beta(s))| \right), \]

\[ |\mathcal{R}(s, y(s))| \leq \max_{1 \leq i \leq n} \left( \sum_{j=1}^{n} \mathcal{L}^i_j |b_{ij}(s)||y_j(s)| + \sum_{j=1}^{n} \mathcal{L}^j_i |c_{ij}(s)||y_j(\beta(s))| \right) \]

and

\[ |\mathcal{J}_k(y(t^-_k))| \leq \mathcal{L}^i_k |y(t^-_k)|. \]

By (27), we can deduce that \( v_i(t) = e^{\int_{\tau}^{t} a_*(s)ds}y_i(t) \) satisfies

\[ |v_i(t)| \leq |\varphi_0(\tau) - \psi_0(\tau)| + \int_{\tau}^{t} \left[ \sum_{j=1}^{n} \mathcal{L}^i_j |b_{ij}(s)||v_j(s)| + \sum_{j=1}^{n} \mathcal{L}^j_i |c_{ij}(s)||v_j(\beta(s))|e^{\int_{\tau}^{s} a_*(\kappa) d\kappa} \right] ds + \sum_{k=i(\tau)+1}^{i(t)} \mathcal{L}^i_k |v_i(t^-_k)|, \]

or

\[ |v(t)| \leq |\varphi_0(\tau) - \psi_0(\tau)| + \max_{1 \leq i \leq n} \int_{\tau}^{t} \left[ \sum_{j=1}^{n} \mathcal{L}^i_j |b_{ij}(s)||v(s)| + \sum_{j=1}^{n} \mathcal{L}^j_i |c_{ij}(s)||v(\beta(s))|e^{\int_{\tau}^{s} a_*(\kappa) d\kappa} \right] ds + \sum_{k=i(\tau)+1}^{i(t)} \mathcal{L}^i_k |v(t^-_k)|, \]

for any finite \( t \in [\tau, \infty) \).

Hence, by Lemma 2.2 of the IDEGPCD’s Gronwall inequality implies

\[ |v(t)| \leq |\varphi_0(\tau) - \psi_0(\tau)| \prod_{k=i(\tau)+1}^{i(t)} \left( 1 + \mathcal{L}^i_k \right) \exp \left\{ \max_{1 \leq i \leq n} \int_{\tau}^{t} \sum_{j=1}^{n} \mathcal{L}^i_j |b_{ij}(s)| + \sum_{j=1}^{n} \mathcal{L}^j_i |c_{ij}(s)| \exp \left( \int_{\tau}^{t} a_*(\kappa) d\kappa \right) \right\}. \]

Then, we obtain

\[ |\varphi_0(t) - \psi_0(t)| \prod_{k=i(\tau)+1}^{i(t)} \left( 1 + \mathcal{L}^i_k \right) \exp \left\{ - \int_{\tau}^{t} a_*(s) ds + \max_{1 \leq i \leq n} \int_{\tau}^{t} \mathcal{K}_i(s) ds \right\}, \]

or

\[ |\varphi_0(t) - \psi_0(t)| \leq |\varphi_0(\tau) - \psi_0(\tau)| \exp \left\{ - \int_{\tau}^{t} a_*(s) ds + \int_{\tau}^{t} \max_{1 \leq i \leq n} \left( \mathcal{K}_i(s) + \frac{\ln(1 + \mathcal{L}^i_k)}{\vartheta_k} \right) ds \right\}, \]

and the statement (26) follows.
Proof of Theorem 4.4. According to Theorem 4.3, we know that the ICNN models with IDEGPCD system (4a)-(4b) has an $\omega$-periodic solution $x^*(t) = (x_1^*(t), ..., x_n^*(t))^T$ with initial value $x^*(\tau) = (x_1^*(\tau), ..., x_n^*(\tau))^T$.

Suppose that $x(t) = (x_1(t), ..., x_n(t))^T$ is an arbitrary solution of the ICNN models with IDEGPCD system (4a)-(4b) with initial value $x(\tau) = (x_1(\tau), ..., x_n(\tau))^T$.

Consider the change of variables

$$z(t) = x(t) - x^*(t) = (x_1(t) - x_1^*(t), ..., x_n(t) - x_n^*(t))^T.$$ 

Then it follows from system (4a)-(4b) that

$$\begin{cases}
\frac{dz_i(t)}{dt} = -a_i(t)z_i(t) + \sum_{j=1}^{n} \left[ b_{ij}(t)\tilde{f}_j(z_j(t)) + c_{ij}(t)\tilde{g}_j(z_j(\beta(s))) \right], \quad t \neq t_k, \\
\Delta z_i|_{t=t_k} = \tilde{J}_{ik}(z_i(t^-_k)), \quad i = 1, 2, ..., n, \quad k \in \mathbb{N},
\end{cases}$$

where

$$\tilde{f}_j(z_j(t)) = f_j(z_j(t) + x_j^*(t)) - f_j(x_j^*(t)),$$

$$\tilde{g}_j(z_j(\beta(t))) = g_j(z_j(\beta(t)) + x_j^*(\beta(t))) - g_j(x_j^*(\beta(t))),$$

and

$$\tilde{J}_{ik}(z_i(t^-_k)) = J_k(z_i(t^-_k) + x_i^*(t^-_k)) - J_k(x_i^*(t^-_k)).$$

Now, by Proposition 1, it can be proved that

$$z_i(t) = e^{-\int_{\tau}^{t} a_i(s)ds} z_i(\tau) + \int_{\tau}^{t} e^{-\int_{s}^{t} a_i(u)du} \left[ \sum_{j=1}^{n} b_{ij}(s)\tilde{f}_j(z_j(s)) + \sum_{j=1}^{n} c_{ij}(s)\tilde{g}_j(z_j(\beta(s))) \right] ds + \sum_{k=\tau}^{n(t)+1} e^{-\int_{k}^{t} a(s)ds} \tilde{J}_{ik}(z_i(t^-_k)),$$

and similar to the proof of Lemma 4.5, we obtain

$$\max_{1 \leq i \leq n} |z_i(t)| \leq \max_{1 \leq i \leq n} |z_i(\tau)| \exp \left( -\int_{\tau}^{t} [a_i(s) - K_i(s) - L_i'(s)] ds \right).$$

Thus, from the assumption of (25), we can conclude that the $\omega$-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b) is globally exponentially stable and this completes the proof of Theorem 4.4.

It should be pointed that, because of the complexity of the results, the problem of finding appropriate parameters is a difficult task. Therefore, in order to easily check the applicability of the results, we will give corollaries as follows.

**Corollary 8.** If (L) and (10) are satisfied, then the solutions $\varphi$ and $\psi$ of the IDEGPCD system (4a)-(4b) satisfy for all $t \geq \tau$ the inequality

$$|\varphi(t) - \psi(t)| \leq |\varphi(\tau) - \psi(\tau)| \exp \left( -(\bar{a}_* - \bar{K}_* - \bar{L}_i'(0))(t - \tau) \right)$$

where

$$a_* = \inf_{t \in \mathbb{R}^+} a_*(t), \quad L_i'(0) = \max_{1 \leq i \leq n} \max_{1 \leq k \leq n} \frac{\ln(1 + \bar{L}_i')}{\theta_k}, \quad L_k = \max_{1 \leq i \leq n} \bar{L}_i,$$

$$\bar{K}_i = \max_{1 \leq i \leq n} K_i = \max_{1 \leq i \leq n} \sum_{j=1}^{n} \left( \bar{L}_j^i |\bar{b}_{ij}| + \bar{L}_j^i |\bar{c}_{ij}| \exp(\bar{a}_* \cdot \theta) \right), \quad \bar{L}_k = \max_{1 \leq i \leq n} \bar{L}_i,$$

and

$$\bar{a}_* = \sup_{t \in \mathbb{R}^+} a_*(t), \quad \theta = \sup_{k \in \mathbb{N}} \theta_k.$$
Corollary 9. If the assumptions (L), (10) and
\[ a_* - \tilde{K}^* - \mathcal{L}_{ij}^f \geq \rho > 0, \quad t \in \mathbb{R}^+ \] 
are satisfied, then the ICNN models with IDEGPCD system (4a)-(4b) is globally exponentially stable.

Corollary 10. If the assumptions of Corollary 4 and (29) are satisfied. Then the \( \omega \)-periodic solution of the ICNN models with IDEGPCD system (4a)-(4b) is globally exponentially stable.

Corollary 11. If the assumptions of Corollary 5 are satisfied and
\[ a_* - K^* - \mathcal{L}_{ij}^f \geq \rho > 0, \quad t \in \mathbb{R}^+ \] 
where \( a_* = \inf_{1 \leq i \leq n} a_i \) and \( K^* = \max_{1 \leq i \leq n} \sum_{j=1}^{n} (\mathcal{L}_{ij}^f |b_{ij}| + \mathcal{L}_{ij}^g |c_{ij}| \exp(\mathcal{L}_{ij}^f \cdot \theta)) \). Then the \( \omega \)-periodic solution of the ICNN models with IDEGPCD system (12a)-(12b) (the respective constant coefficients) is globally exponentially stable.

Remark 6. Corollary 11 generalizes corresponding results obtained by M. U. Akhmet et al. [1] under complicated and stronger conditions. If we are only interested in the forward continuation, i.e. only for \( t \geq \tau \), then we will need only the forward condition (L), (11) with \( \vartheta = \tau - t_{\xi(t)} \) and (30). See Example 2.

As a direct consequence of the method, we have the following result without impulsive effects.

Corollary 12. If the assumptions of Corollary 6 are satisfied and
\[ a_* - K^* - \mathcal{L}_{ij}^f \geq \rho > 0, \quad t \in \mathbb{R}^+ \] 
Then the \( \omega \)-periodic solution of the CNN models with DEGPCD system (13) is globally exponentially stable.

Remark 7. Corollary 12 reduces to the results of Chiu et al. [14, Theorem 5.2].

Remark 8. T.H. Yu et al. [40] had similar results of Corollary 12 for the stability criteria of the CNN models with DEGPCD system (13) without impulsive effects which are independent on the upper bound of the adjacent element difference of the discontinuous switching moments. And the authors said that “different from the discussions in previous results, by the novel differential inequality with piecewise constant arguments, it is not necessary to establish any relationship between the norms of the states with/without piecewise constant arguments to estimate the derivative of the Lyapunov function. Thus, the method developed in this paper is more powerful than the methods used in early works.”

If his affirmations had been correct, we can ignore the effects of deviation arguments and the change of constancy for the arguments. This implies that all DEPCA’s results can be applied to DEPCAC’s results of generalized type. Obviously, that is not true. See [7, Remark 2].

Corollary 13. If the assumptions of Corollary 7 are satisfied and
\[ a_* - K^* \geq \rho > 0, \quad t \in \mathbb{R}^+ \] 
Then the \( \omega \)-periodic solution of the CNN models with DEGPCD system (13) with the respective constant coefficients is globally exponentially stable.
5. Examples and simulations. In this section, we give two examples with numerical simulations to illustrate the effectiveness of the proposed method and results.

Example 1. Consider the following non-autonomous 3-dimensional impulsive cellular neural network models with generalized piecewise constant delay:

\[
\begin{align*}
\begin{cases}
    x'_1 &= -a_1(t)x_1 + b_{12}(t)f_2(x_2) + c_{13}(t)g_3(x_3(\beta(t))) + d_1(t), \\
    x'_2 &= -a_2(t)x_2 + b_{23}(t)f_3(x_3) + c_{21}(t)g_1(x_1(\beta(t))) + d_2(t), \\
    x'_3 &= -a_3(t)x_3 + b_{31}(t)f_1(x_1) + c_{32}(t)g_2(x_2(\beta(t))) + d_3(t),
\end{cases}
\end{align*}
\]

(33a)

\[
\begin{align*}
\begin{cases}
    \Delta x_1|_{t_k} &= J_{1k}(x_1(t_k^-)), \\
    \Delta x_2|_{t_k} &= J_{2k}(x_2(t_k^-)), \\
    \Delta x_3|_{t_k} &= J_{3k}(x_3(t_k^-)),
\end{cases}
\end{align*}
\]

(33b)

where

\[
\begin{align*}
a_1(t) &= 0.85 - 0.05 \cos(4t), \\
a_2(t) &= 0.7 + 0.1 \sin(4t), \\
a_3(t) &= 0.67 - 0.03 \sin(4t), \\
b_{12}(t) &= 0.3 + 0.25 \sin(4t), \\
b_{23}(t) &= 0.2 - 0.15 \cos(4t), \\
b_{31}(t) &= 0.5 - 0.15 \sin(4t), \\
c_{13}(t) &= 0.2 + 0.15 \cos(4t), \\
c_{21}(t) &= 0.3 - 0.15 \sin(4t), \\
c_{32}(t) &= 0.4 - 0.15 \cos(4t), \\
d_{1}(t) &= 4 + 0.2 \cos(4t), \\
d_{2}(t) &= 3 + 0.2 \sin(4t), \\
d_{3}(t) &= 2 + 0.2 \cos(4t).
\end{align*}
\]

The output functions are

\[
\begin{align*}
f_1(x_1(t)) &= \tanh(x_1(t)/6), \\
f_2(x_2(t)) &= \tanh(x_2(t)/10), \\
f_3(x_3(t)) &= \tanh(x_3(t)/8), \\
g_1(x_1(\beta(t))) &= \frac{|x_1(\beta(t)) + 1/4| - |x_1(\beta(t)) - 1/4|}{10}, \\
g_2(x_2(\beta(t))) &= \frac{|x_2(\beta(t)) + 1/4| - |x_2(\beta(t)) - 1/4|}{10}, \\
g_3(x_3(\beta(t))) &= \frac{|x_3(\beta(t)) + 1/4| - |x_3(\beta(t)) - 1/4|}{8}.
\end{align*}
\]

The impulsive functions are

\[
\begin{align*}
J_{1k}(x_1(t_k^-)) &= x_1(t_k^-)/4, \\
J_{2k}(x_2(t_k^-)) &= x_2(t_k^-)/5, \\
J_{3k}(x_3(t_k^-)) &= x_3(t_k^-)/6,
\end{align*}
\]

and \( \beta(t) = t_i \) if \( t_i \leq t < t_{i+1}, i \in \mathbb{N}, t_i = \pi(i - 1)/2 \).

We can easily obtain that the maximal distance \( \theta = \theta_k = t_{k+1} - t_k = \pi/2 \), \( L_1^f = L_3^f = 1/6 \), \( L_2^f = L_4^f = 0.1 \), \( L_5^g = L_6^g = 0.125 \), \( L_7^g = L_8^g = 0.2 \), \( L_9^g = L_2^g = 0.25 \), \( a_\pi = 0.7 \), \( a_\alpha = 0.6 \), \( L_k^g = 0.25 \) and \( \{t_i\}_{i \in \mathbb{N}}, \{J_k\}_{i \in \mathbb{N}} \) satisfy the \((\pi/2, 1)\) condition.

Moreover,

| \( \max_{t \in \mathbb{R}^+} \) | \( a_1(\cdot) \) | \( a_2(\cdot) \) | \( a_3(\cdot) \) | \( d_1(\cdot) \) | \( d_2(\cdot) \) | \( d_3(\cdot) \) |
|-----------------|--------|--------|--------|--------|--------|--------|
| \( \min_{t \in \mathbb{R}^+} \) | 0.9    | 0.8    | 0.7    | 4.2    | 3.2    | 2.2    |

| \( \max_{t \in \mathbb{R}^+} \) | \( b_{12}(\cdot) \) | \( b_{23}(\cdot) \) | \( b_{31}(\cdot) \) | \( c_{13}(\cdot) \) | \( c_{21}(\cdot) \) | \( c_{32}(\cdot) \) |
|-----------------|--------|--------|--------|--------|--------|--------|
| \( \min_{t \in \mathbb{R}^+} \) | 0.55   | 0.35   | 0.65   | 0.35   | 0.45   | 0.55   |

| \( \max_{t \in \mathbb{R}^+} \) | \( b_{12}(\cdot) \) | \( b_{23}(\cdot) \) | \( b_{31}(\cdot) \) | \( c_{13}(\cdot) \) | \( c_{21}(\cdot) \) | \( c_{32}(\cdot) \) |
|-----------------|--------|--------|--------|--------|--------|--------|
| \( \min_{t \in \mathbb{R}^+} \) | 0.05   | 0.05   | 0.35   | 0.05   | 0.15   | 0.25   |

It follows that:
Proposition 2. See [7] and [28].

One can see that all conditions (L), (P), (10), (21) and (29) in Corollary 10 are satisfied. Therefore, the ICNN models with IDEGPCD system (33a)-(33b) has a unique globally exponentially stable \( \pi/2 \)-periodic solution. The fact can be seen by simulation in Figs 1.

Let us simulate a solution of the ICNN models with IDEGPCD system (33a)-(33b) with initial condition \( x_1(0) = x_0^1, x_2(0) = x_0^2 \) and \( x_3(0) = x_0^3 \). Since the IDEGPCD system (33a)-(33b) is of generalized type, the numerical analysis has a specific character and it should be described more carefully. One will see that this algorithm is in full accordance with the approximations made in the proof of Proposition 2. See [7] and [28].

\[
\begin{align*}
\max_{1 \leq k \leq i(\tau)} \left\{ \frac{\exp(\bar{a}_1(t_{k+1} - t_k)) - 1}{\bar{a}_1} \left( \mathcal{L}_2^f \bar{b}_{12} + \mathcal{L}_3^f \bar{c}_{13} \right) \right\} & \approx 0.4926 < 1, \\
\max_{1 \leq k \leq i(\tau)} \left\{ \frac{\exp(\bar{a}_2(t_{k+1} - t_k)) - 1}{\bar{a}_2} \left( \mathcal{L}_4^g \bar{c}_{21} + \mathcal{L}_6^g \bar{b}_{23} \right) \right\} & \approx 0.3141 < 1, \\
\max_{1 \leq k \leq i(\tau)} \left\{ \frac{\exp(\bar{a}_3(t_{k+1} - t_k)) - 1}{\bar{a}_3} \left( \mathcal{L}_4^f \bar{b}_{31} + \mathcal{L}_5^f \bar{c}_{32} \right) \right\} & \approx 0.6859 < 1.
\end{align*}
\]

\[
\frac{\exp(\bar{a}_1\omega)}{\exp(\bar{a}_1\omega) - 1} \left[ \left( \mathcal{L}_2^f \bar{b}_{12} + \mathcal{L}_3^f \bar{c}_{13} \right) \omega + \mathcal{L}_1^f \right] \approx 0.775 < 1,
\]

\[
\frac{\exp(\bar{a}_2\omega)}{\exp(\bar{a}_2\omega) - 1} \left[ \left( \mathcal{L}_4^g \bar{c}_{21} + \mathcal{L}_6^g \bar{b}_{23} \right) \omega + \mathcal{L}_2^g \right] \approx 0.8009 < 1,
\]

\[
\frac{\exp(\bar{a}_3\omega)}{\exp(\bar{a}_3\omega) - 1} \left[ \left( \mathcal{L}_4^f \bar{b}_{31} + \mathcal{L}_5^f \bar{c}_{32} \right) \omega + \mathcal{L}_3^f \right] \approx 0.9409 < 1.
\]

\[
\bar{K}_1 = \mathcal{L}_2^f \bar{b}_{12} + \mathcal{L}_3^f \bar{c}_{13} \exp(\bar{a}_* \cdot \theta) \approx 0.3177,
\]

\[
\bar{K}_2 = \mathcal{L}_4^g \bar{c}_{21} + \mathcal{L}_6^g \bar{b}_{23} \exp(\bar{a}_* \cdot \theta) \approx 0.2126,
\]

\[
\bar{K}_3 = \mathcal{L}_4^f \bar{b}_{31} + \mathcal{L}_5^f \bar{c}_{32} \exp(\bar{a}_* \cdot \theta) \approx 0.4386,
\]

\[
\mathcal{L}^f_{i(t)} = \max_{i(\tau)+1 \leq k \leq i(t)} \frac{\ln(1 + \mathcal{L}_k^f)}{\theta_k} \approx 0.14205.
\]

By (28), we obtain \( \bar{K}^* = \max_{1 \leq i \leq 4} \bar{K}_i \approx 0.4386 \). Then

\[
\bar{a}_* - \bar{K}^* - \mathcal{L}^f_{i(t)} \approx 0.01929 > 0.
\]
We start with the interval \( I_0 = [t_1, t_2] \): that is, \([0, \pi/2)\). On this interval, the ICNN models with IDEGPCD system (33a)-(33b) has the form

\[
\frac{dx(t)}{dt} = - \begin{pmatrix}
0.85 - 0.05 \cos(4t) & 0 & 0 \\
0 & 0.7 + 0.1 \sin(4t) & 0 \\
0 & 0 & 0.67 - 0.03 \sin(4t)
\end{pmatrix}
\begin{pmatrix}
x_1(0) \\
x_2(0) \\
x_3(0)
\end{pmatrix}
+ \begin{pmatrix}
0 & 0.3 + 0.25 \sin(4t) & 0 \\
0 & 0 & 0.2 - 0.15 \cos(4t) \\
0.5 - 0.15 \sin(4t) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
tanh(x_1(t)/6) \\
tanh(x_2(t)/10) \\
tanh(x_3(t)/8)
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0.2 + 0.15 \cos(4t) \\
0.3 - 0.15 \sin(4t) & 0 & 0 \\
0 & 0.4 - 0.15 \cos(4t) & 0
\end{pmatrix}
\begin{pmatrix}
\frac{|x_1(0) + 1| - |x_1(0) - 1|}{6} \\
\frac{|x_2(0) + 1| - |x_2(0) - 1|}{10} \\
\frac{|x_3(0) + 1| - |x_3(0) - 1|}{8}
\end{pmatrix}
+ \begin{pmatrix}
4 + 0.2 \cos(4t) \\
3 + 0.2 \sin(4t) \\
2 + 0.2 \cos(4t)
\end{pmatrix},
\]

(34)

For this reason, we will arrange approximations in the following way. Consider the sequence of equations

\[
\frac{dx^{(m+1)}(t)}{dt} = - \begin{pmatrix}
0.85 - 0.05 \cos(4t) & 0 & 0 \\
0 & 0.7 + 0.1 \sin(4t) & 0 \\
0 & 0 & 0.67 - 0.03 \sin(4t)
\end{pmatrix}
\begin{pmatrix}
x_1^{(m)}(0) \\
x_2^{(m)}(0) \\
x_3^{(m)}(0)
\end{pmatrix}
+ \begin{pmatrix}
0 & 0.3 + 0.25 \sin(4t) & 0 \\
0 & 0 & 0.2 - 0.15 \cos(4t) \\
0.5 - 0.15 \sin(4t) & 0 & 0
\end{pmatrix}
\begin{pmatrix}
tanh(x_1^{(m)}(t)/6) \\
tanh(x_2^{(m)}(t)/10) \\
tanh(x_3^{(m)}(t)/8)
\end{pmatrix}
+ \begin{pmatrix}
0 & 0 & 0.2 + 0.15 \cos(4t) \\
0.3 - 0.15 \sin(4t) & 0 & 0 \\
0 & 0.4 - 0.15 \cos(4t) & 0
\end{pmatrix}
\begin{pmatrix}
\frac{|x_1^{(m)}(0) + 1| - |x_1^{(m)}(0) - 1|}{6} \\
\frac{|x_2^{(m)}(0) + 1| - |x_2^{(m)}(0) - 1|}{10} \\
\frac{|x_3^{(m)}(0) + 1| - |x_3^{(m)}(0) - 1|}{8}
\end{pmatrix}
+ \begin{pmatrix}
4 + 0.2 \cos(4t) \\
3 + 0.2 \sin(4t) \\
2 + 0.2 \cos(4t)
\end{pmatrix},
\]

(35)

where \( m = 1, 2, 3, \ldots \) with \( x_1^{(0)}(t) = x_1^0, x_2^{(0)}(t) = x_2^0, x_3^{(0)}(t) = x_3^0 \). We evaluate the solutions \( x^{(m)}(t) \), by using MATLAB 7.14, and stop the iterations at \( (x_1^{(1000)}(t), x_2^{(1000)}(t), x_3^{(1000)}(t)) \). Then, we assign \( x_1(t) = x_1^{(1000)}(t), x_2(t) = x_2^{(1000)}(t), x_3(t) = x_3^{(1000)}(t) \) on the interval \([0, \pi/2)\). Next, a similar operation is done on the interval \([\pi/2, \pi)\). That is, we construct the sequence \((x_1^{(m)}, x_2^{(m)}, x_3^{(m)})\) of solutions.
again for the ICNN models with IDEGPCD system (33a)-(33b)

\[
\frac{dx^{(m+1)}(t)}{dt} = -\begin{pmatrix}
0.85 - 0.05 \cos(4t) & 0 & 0 \\
0 & 0.7 + 0.1 \sin(4t) & 0 \\
0 & 0 & 0.67 - 0.03 \sin(4t)
\end{pmatrix} \times 
\begin{pmatrix}
x_1^{(m)}(\pi/2) \\
x_2^{(m)}(\pi/2) \\
x_3^{(m)}(\pi/2)
\end{pmatrix} 
+ 
\begin{pmatrix}
0 & 0.3 + 0.25 \sin(4t) & 0 \\
0 & 0.2 - 0.15 \cos(4t) & 0 \\
0.5 - 0.15 \sin(4t) & 0 & 0
\end{pmatrix} \times 
\begin{pmatrix}
tanh(x_1^{(m)}(t/6)) \\
tanh(x_2^{(m)}(t/10)) \\
tanh(x_3^{(m)}(t/8))
\end{pmatrix} 
+ 
\begin{pmatrix}
0 & 0 & 0.2 + 0.15 \cos(4t) \\
0.3 - 0.15 \sin(4t) & 0 & 0 \\
0 & 0.4 - 0.15 \cos(4t) & 0
\end{pmatrix} \times 
\begin{pmatrix}
\left| x_1^{(m)}(\pi/2) + 1 - x_1^{(m)}(\pi/2) \right| \\
\left| x_2^{(m)}(\pi/2) + 1 - x_2^{(m)}(\pi/2) \right| \\
\left| x_3^{(m)}(\pi/2) + 1 - x_3^{(m)}(\pi/2) \right|
\end{pmatrix} 
+ 
\begin{pmatrix}
4 + 0.2 \cos(4t) \\
3 + 0.2 \sin(4t) \\
2 + 0.2 \cos(4t)
\end{pmatrix}.
\]

(36)

where \(x_1^{(m)}(\pi/2), x_2^{(m)}(\pi/2), x_3^{(m)}(\pi/2)\) are still unknown and

\[
x_1^0(t) = x_1^{(1000)}(\pi/2) = 5/4 \cdot x_1^{(1000)}(\pi/2^-), \\
x_2^0(t) = x_2^{(1000)}(\pi/2) = 6/5 \cdot x_2^{(1000)}(\pi/2^-), \\
x_3^0(t) = x_3^{(1000)}(\pi/2) = 7/6 \cdot x_3^{(1000)}(\pi/2^-).
\]

Then, we reassign \(x_1(t) = x_1^{(1000)}(t), x_2(t) = x_2^{(1000)}(t), x_3(t) = x_3^{(1000)}(t)\) on \([\pi/2, \pi]\). Proceeding in this way, one can obtain a simulation which demonstrates the asymptotic property. Specifically, the simulation result with several random initial points is shown in Fig. 1a.

The numerical simulations, showing the convergence of the \(\pi/2\)-periodic solution of the ICNN models (33a)-(33b) with and without impulses, are given in Figs. 1 and Figs. 2.

![Fig. 1a. Some trajectories uniformly convergent to the unique exponentially stable \(\pi/2\)-periodic solution of the ICNN models with IDEGPCD system (33).](image-url)
Fig. 1b. Phase plots of state variable \((x_1, x_2, x_3)\) in the ICNN models with IDEGPCD system (33) with the initial condition \((7, 6, 3)\).

Fig. 1c. Phase plots of state variable \((x_1, x_2, x_3)\) in the ICNN models with IDEGPCD system (33) with the initial condition \((6.7897, 6.0565, 4.6992)\).

Fig. 1d. Phase plots of state variable \((t, x_1, x_2)\) in the ICNN models with IDEGPCD system (33).
Fig. 1e. Phase plots of state variable \((t, x_1, x_3)\) in the ICNN models with IDEGPCD system (33).

Fig. 1f. Phase plots of state variable \((t, x_2, x_3)\) in the ICNN models with IDEGPCD system (33).

Numerical simulations confirm that the proposed conditions in our results are effective for the ICNN models with IDEGPCD system (33a)-(33b).

Fig. 2a. \(\pi/2\)-periodic solution of the CNN models with DEGPCD system (33a) for \(t \in [0, 6\pi]\) with the initial value \((4.9228, 4.5238, 3.6121)\).
For the numerical simulation, we show transient behavior of the CNN models with DEGPCD system (33a) without impulses.

Fig. 2b. Trajectories uniformly convergent to the unique exponentially stable \( \pi/2 \)-periodic solution of the CNN models with DEGPCD system (33a) with the initial value \((5.0, 4.3, 3.65)\).

**Example 2.** Let \( a_1(t) \equiv 1.3, a_2(t) \equiv 1.2, b_{11}(t) \equiv 0.25, b_{12}(t) \equiv 0.3, b_{21}(t) = c_{21}(t) \equiv 0.2, b_{22}(t) = c_{11}(t) = c_{22}(t) \equiv 0.125, c_{12}(t) \equiv 0.4, \) and \( d_1(t) \equiv 4, d_2(t) \equiv 3 \). Then the ICNN models with IDEGPCD system (4a)-(4b) reduces to the following system:

\[
\frac{dx(t)}{dt} = -\begin{pmatrix} 1.3 & 0 \\ 0 & 1.2 \end{pmatrix} \begin{pmatrix} x_1(t) \\ x_2(t) \end{pmatrix} + \begin{pmatrix} 0.25 & 0.3 \\ 0.2 & 0.125 \end{pmatrix} \begin{pmatrix} \tanh\left(\frac{x_1(t)}{4}\right) \\ \tanh\left(\frac{x_2(t)}{8}\right) \end{pmatrix} + \begin{pmatrix} 0.125 & 0.4 \\ 0.2 & 0.125 \end{pmatrix} \begin{pmatrix} \tanh\left(\frac{x_1(2t)}{8}\right) \\ \tanh\left(\frac{x_2(2t)}{4}\right) \end{pmatrix} + \begin{pmatrix} 4 \\ 3 \end{pmatrix},
\]

\[ (37a) \]

\[
\Delta x|_{t=k} = \begin{pmatrix} J_{1k} \left( x_1(k^-) \right) \\ J_{2k} \left( x_2(k^-) \right) \end{pmatrix} = \begin{pmatrix} (-1)^k \frac{x_1(k^-)}{8} \\ (-1)^k \frac{x_2(k^-)}{6} \end{pmatrix}, \quad k \in \mathbb{N}.
\]

\[ (37b) \]
By simple calculation, we can see that: the maximal distance $\theta = \partial_k = t_{k+1} - t_k = 0.5$, $k \in \mathbb{N}$, $\mathcal{L}_{2}^{l} = \mathcal{L}_{2}^{g} = 0.25$, $\mathcal{L}_{2}^{l} = \mathcal{L}_{2}^{l} = 0.125$, $\mathcal{L}_{2}^{l} = 1/6$, $a_* = 1.2$, $L_{2}^{l} = 1/6$. Moreover, the sequences $\{t_k\}_{k \in \mathbb{N}}$ and $\{J_k\}_{k \in \mathbb{N}}$, satisfy the $(1, 2)$ condition.

It follows that:

(a)  
\[
\max_{1 \leq i \leq 2} \left\{ \frac{1}{a_i} \left[ \exp(a_i \max_{k \in \mathbb{N}} (t_{k+1} - t_k)) - 1 \right] \left( \sum_{j=1}^{2} \mathcal{L}_{j}^{l} |b_{ij}| + \mathcal{L}_{j}^{g} |c_{ij}| \right) \right\} \approx 0.1518 < 1,
\]

(b)  
\[
\frac{\exp(a_1 \omega)}{\exp(a_1 \omega) - 1} \left[ \sum_{j=1}^{2} \left( \mathcal{L}_{j}^{l} |b_{1j}| + \mathcal{L}_{j}^{g} |c_{1j}| \right) \omega + 2 \cdot \mathcal{L}_{1}^{l} \right] \approx 0.9364 < 1.
\]

and  
\[
\frac{\exp(a_2 \omega)}{\exp(a_2 \omega) - 1} \left[ \sum_{j=1}^{2} \left( \mathcal{L}_{j}^{l} |b_{2j}| + \mathcal{L}_{j}^{g} |c_{2j}| \right) \omega + 2 \cdot \mathcal{L}_{2}^{l} \right] \approx 0.8258 < 1.
\]

By (30), we obtain $K^* = \max_{1 \leq i \leq 2} K_i \approx 0.3214$. Then  
\[
a_* - K^* - \mathcal{L}_{i(t)}^{l} \approx 0.5702 > 0.
\]

In this case, we can easily verify that all conditions (L), (P), (11), (22) and (30) of Corollary 11 are satisfied. Thus, according to Corollary 11, the ICNN models with IDEGPCD system (37) with constant coefficients has a unique 1-periodic solution and all other solution of system (37) converge exponentially to it as $t \to \infty$.

The numerical simulations, showing the convergence of the 1-periodic solution of the ICNN models (37) with and without impulses, are given in Figs. 3 and Figs. 4.

Fig. 3a. Some trajectories uniformly convergent to the unique 1-periodic solution of the ICNN models with IDEGPCD system (37).
Fig. 3b. Exponential convergence of two trajectories towards a 1-periodic solution of the ICNN models with IDEGPCD system (37). Initial conditions: (i) (3, 6) in red and (ii) (4, 6) in blue.

Fig. 3c. Phase plots of state variable $(t, x_1, x_2)$ in the ICNN models with IDEGPCD system (37).

Remark 9. Note that the simulation illustrates that all trajectories uniformly converge to the unique 1-periodic solution of the ICNN models with IDEGPCD system (37).

This conclusion cannot be derived by applying the main results of [1], because the following conditions are not satisfied:

\[(C3)\]
\[
\theta \cdot \left( \sum_{i=1}^{2} a_i + \sum_{i=1}^{2} \sum_{j=1}^{2} |b_{ji}| \mathcal{L}_j^f + \sum_{i=1}^{2} \sum_{j=1}^{2} |c_{ji}| \mathcal{L}_j^g \right) \approx 1.4188 > 1.
\]

\[(C4)\]
\[
\theta \cdot \left\{ \sum_{i=1}^{2} \sum_{j=1}^{2} |c_{ji}| \mathcal{L}_j^g + \left( \sum_{i=1}^{2} a_i + \sum_{i=1}^{2} \sum_{j=1}^{2} |b_{ji}| \mathcal{L}_j^f \right) \right\} \approx 5.5738 > 1.
\]
(\(B\)) \(a_\ast - \sum_{i=1}^{2} \sum_{j=1}^{2} |b_{ji}| L'_j - B \left[ \sum_{i=1}^{2} \sum_{j=1}^{2} |c_{ji}| L'_j \right] - \frac{\ln(1+1)}{\bar{y}} \) is not well defined, because \(\zeta\) is a positive constant defined in [1].

By simple calculation, we can see that

\[
B := \{1 - \theta \cdot \left( \sum_{i=1}^{2} \sum_{j=1}^{2} |c_{ji}| L'_j + \left( \sum_{i=1}^{2} a_i + \sum_{i=1}^{2} \sum_{j=1}^{2} |b_{ji}| L'_j \right) \right) \times \left( 1 + \theta \cdot \left( \sum_{i=1}^{2} \sum_{j=1}^{2} |c_{ji}| L'_j \right) e^{\theta \cdot \left( \sum_{i=1}^{2} a_i + \sum_{i=1}^{2} \sum_{j=1}^{2} |b_{ji}| L'_j \right)} \right) \}^{-1}
\]

\[\approx -0.2186 < 0.\]

Thus, the results in [1] cannot be used to obtain the exponential stability of the ICNN models with IDEGPCD system (37).

This confirms that the proposed conditions in our results are effective for the ICNN models with IDEGPCD system (37).

Now, for the numerical simulation, we show transient behavior of the CNN models with DEGPCD system (37a) without impulses.

Fig. 4a. Unique asymptotically stable solution of the CNN models with DEGPCD system (37a).

Fig. 4b. Unique asymptotically stable solution of the CNN models with DEGPCD system (37a).
Initial conditions: (i) (3, 6) in red and (ii) (4, 6) in blue.

Fig. 4c. Some trajectories uniformly convergent to the unique asymptotically stable solution of the CNN models with DEGPCD system (37a).

6. Conclusions. In the paper, the existence, uniqueness and global exponential stability of periodic solutions for the impulsive cellular neural networks with generalized piecewise constant delay have been investigated. We used the new method, which is different from those of the previous literature, to study the existence and global stability of periodic solution for general non-autonomous impulsive cellular neural network with piecewise constant argument. Several new sufficient conditions have been derived for checking the global exponential stability and the existence of periodic solution for the considered system based on IDEPCAG’s integral inequality of Gronwall type, fixed point theorem and Green’s function. In addition, two examples with numerical simulations are given to show the effectiveness of the proposed method and results. The obtained results have been shown to be more general and less restrictive than the previous results derived in [1] and [14].

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