On non–dissipative and dissipative qubit manifolds

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The trajectories of a qubit dynamics over the two–sphere are shown to be geodesics of certain Riemannian or physically–sound Lorentzian manifolds, both in the non–dissipative and dissipative formalisms, when using action–angle variables. Several aspects of the geometry and topology of these manifolds (qubit manifolds) have been studied for some special physical cases.
I. INTRODUCTION

Since its conception, one of the paradigms of quantum mechanics is the two–level system. Its role in almost all the fields of physics is difficult to overemphasize, going, for instance, from high energy physics and parity–violating chiral molecules to macroscopic quantum phenomena, as shown by Feynman in his elegant and pedagogical dynamical theory of the Josephson effect. During the past years, the interest in two–level systems has increased considerably due to its applicability in quantum computation under the name of qubits. Interestingly, not only for physicists but also for mathematicians the qubit can be used to explore and test sophisticated theories, in particular using geometrical concepts. Specifically, due to the usual decomposition of the scalar product between two states in the associated Hilbert space, \( \mathcal{H} \), in its real and imaginary parts, both Riemannian and symplectic structures can be introduced in \( \mathcal{H} \), which turns out to be the basis for the geometrization of quantum mechanics. Although the probabilistic aspects of the theory, including the uncertainty principle and related facts are due to the Riemannian structure, the whole quantum dynamics can be formulated as a pure classical theory by defining a symplectic structure over the projective Hilbert space, \( \mathcal{P}(\mathcal{H}) \), taken as a Kähler manifold, which is the quantum phase space where the dynamics takes place. These interesting points and subsequent extensions were made by Kibble and other authors, respectively (for a very readable introduction to geometric structures in quantum mechanics see, for example, [13]).

For our purposes, let us briefly sketch this quantum–classical equivalence for a qubit. If \( \mathcal{H} \) is a two–dimensional Hilbert space and \( |\Psi\rangle \in \mathcal{H} \) is a normalized qubit, then \( |\Psi\rangle \in S^3 \). By the celebrated first Hopf fibration \( S^3 \to S^2 \) we can gauge out the global phase and arrive at the Bloch sphere representation. This map can be understood as a composition, \( \Pi = \Xi \circ \Omega \), where \( \Omega : S^3 \subset \mathbb{C}^2 \to \mathbb{C}P^1 \) links an element of \( \mathbb{C}^2 \) to its equivalence class and \( \Xi : \mathbb{C}P^1 (= \mathbb{C} \cup \infty) \to S^2 \) is given by the stereographic projection. It can be shown that the Hopf map can be written in terms of the Pauli matrices as \( \Pi (|\Psi\rangle \in S^3) = (\langle \Psi|\hat{\sigma}_x|\Psi\rangle, \langle \Psi|\hat{\sigma}_y|\Psi\rangle, \langle \Psi|\hat{\sigma}_z|\Psi\rangle) \in S^2 \), where \( \langle \Psi|\hat{\sigma}_x|\Psi\rangle^2 + \langle \Psi|\hat{\sigma}_y|\Psi\rangle^2 + \langle \Psi|\hat{\sigma}_z|\Psi\rangle^2 = 1 \). Thus, from the first Hopf map, quantum and classical mechanics may be embedded in the same formulation. Specifically, for the qubit case, the Strocchi map is exactly the Hopf map previously described. After defining appropriate canonical, action–angle variables \((I, \Phi)\)
on $S^2$, a classical Hamiltonian function can be derived. In fact, one can prove that the Schrödinger dynamics on $\mathcal{H}$ corresponds to a Hamiltonian dynamics defined by the symplectic form $\Omega = d\Phi \wedge dI$ on $S^2$. Thus, $S^2$, taken as a symplectic manifold, can be regarded as the quantum phase space for a qubit.

In the two–dimensional case, the normalized qubit state can be expanded as $|\Psi\rangle = a_1|1\rangle + a_2|2\rangle$, where $a_j = |a_j|e^{i\phi_j} \in \mathbb{C}$. Let us define the pair of action–angle variables as $I \equiv |a_1|^2 - |a_2|^2$ and $\Phi \equiv \phi_1 - \phi_2$. Then, a general Hamiltonian operator $\hat{H} = \sum_i A_i \hat{\sigma}_i$, where $\hat{\sigma}_i$ are the Pauli matrices and $A^i \in \mathbb{R}$, can be mapped into the Hamiltonian function

$$H_0 = 2\langle \Psi | \hat{H} | \Psi \rangle$$

$$= \sqrt{1 - I^2} \left( 2A_x \cos \Phi + 2A_y \sin \Phi \right) + 2A_z I$$

(1)

where $H_0$ is a generalized Meyer–Miller-Stock-Thoss Hamiltonian [19, 20], widely used in molecular physics (see [21] and references therein). Notice that, within this canonical formulation, the variables $I, \Phi$ play the role of generalized momentum and position, respectively. Therefore, after a time re-scaling $t' \rightarrow 2t$, the solutions of $i\partial_t |\Psi\rangle = \hat{H} |\Psi\rangle$ ($\hbar = 1$) are the same as those of $\dot{I} = -\partial H_0 / \partial \Phi$ and $\dot{\Phi} = \partial H_0 / \partial I$ (the new time variable is again denoted as $t$).

Thus, the qubit can be taken as a classical particle moving on the surface of $S^2$, as stated before. It is well known that the motion of classical particles can be geometrized according to the following theorems [22]:

**Theorem 1.** A point mass confined to a smooth Riemannian manifold moves along geodesic lines.

**Theorem 2.** In the case where there is a potential energy, it can be shown that the trajectories of the equations of dynamics are also geodesics in a certain Riemannian metric.

Therefore, one could ask whether similar theorems hold for the case of a qubit. For example, if it is taken as a free classical particle moving over $S^2$, then Hamilton’s equations derived from $H_0$ have to be the same as that of the geodesics of $S^2$ written in action–angle coordinates. Although the qubit trajectories coincide with the geodesics of $S^3$, in this brief article we show that this is not the case for $S^2$, as also pointed out by Kryukov [23]. However, these trajectories are shown to be geodesics in a certain Riemannian metric. In this sense, we extend the previous theorems to the isolated qubit, which can be considered as a paradigmatic example in quantum mechanics. Moreover, as the Euler characteristic of the manifolds whose geodesics are the qubit trajectories (qubit manifolds) is zero, it will be
shown that they can also be endowed with a Lorentzian metric, whose physical interpretation is briefly discussed. In addition, these results will also be extended, when possible, to non-isolated qubits by means of an effective Hamiltonian description which includes dissipative terms due to the presence of an environment [24].

II. RIEMANNIAN QUBIT MANIFOLDS

Let us start by defining a Riemannian qubit manifold.

Definition 1. Let $\mathcal{M}$ be a two-dimensional connected, compact and orientable Riemannian $C^n$–manifold ($n \geq 2$) and let $H_0(u, v)$ be a Hamiltonian function for a qubit, where $(u, v)$ are any pair of coordinates used to represent $H_0$. If $\ddot{u} = -\frac{d}{dt} \left( \frac{\partial H_0}{\partial v} \right) = f(u, v, \dot{u}, \dot{v})$ and $\ddot{v} = \frac{d}{dt} \left( \frac{\partial H_0}{\partial u} \right) = g(u, v, \dot{u}, \dot{v})$ coincide with the geodesics of $\mathcal{M}$, then $\mathcal{M}$ is said to be a qubit manifold.

Proposition 1. No qubit manifold exists such that $(A_i, A_z) \neq (0, 0)$ ($i = x$ or $y$) or $A_z \neq 0$ and $(A_x, A_y) \neq (0, 0)$.

Proof. The corresponding equations of motion issued from $H_0$ are, in action–angle coordinates covering the region $I \in (-1, 1)$ and $\Phi \in [0, 2\pi]$,

$$\ddot{I} = 2\sqrt{1 - I^2} (A_x \sin \Phi - A_y \cos \Phi)$$
$$\ddot{\Phi} = -\frac{2I}{\sqrt{1 - I^2}} (A_y \sin \Phi + A_x \cos \Phi) + 2A_z.$$  \hfill (2)

Therefore,

$$\ddot{I} + \frac{I}{1 - I^2} \dot{I}^2 + \frac{1 - I^2}{I} \dot{\Phi} \left( \dot{\Phi} - 2A_z \right) = 0$$
$$\ddot{\Phi} + \dot{I} \dot{\Phi} \frac{I^2 + 1}{I(I^2 - 1)} + \frac{2\dot{I}A_z}{I(1 - I^2)} = 0.$$  \hfill (3)

Thus, if the latter pair of equations is likely to describe the geodesics of $\mathcal{M}$ then, by comparing them with the geodesic equation

$$\ddot{x}^\mu + \Gamma^\mu_{\nu\delta} \dot{x}^\nu \dot{x}^\delta = 0,$$  \hfill (4)

it has to be $A_z = 0$. Thus, no qubit manifold exist such that $(A_i, A_z) \neq (0, 0)$ ($i = x$ or $y$) or $A_z \neq 0$ and $(A_x, A_y) \neq (0, 0)$ in action–angle coordinates. Moreover, by defining a new pair of coordinates, $u = f(I, \Phi)$ and $v = g(I, \Phi)$, a new $A_z$–term linear in $\dot{u}$ and $\dot{v}$ appears.
Therefore, no qubit manifold exist such that \((A_i, A_z) \neq (0,0)\) \((i = x \text{ or } y)\) or \(A_z \neq 0\) and \((A_x, A_y) \neq (0,0)\). □

In the following, the qubit manifold corresponding to the case \(A_i \neq 0 \ (i = x \text{ or } y)\) and \(A_z = 0\) will be denoted by \(\mathcal{M}_x\) or \(\mathcal{M}_y\). If \(A_x \neq 0, A_y \neq 0\) and \(A_z = 0\), it will be denoted by \(\mathcal{M}_{xy}\). Finally, in the \((A_x, A_y) = (0,0)\) and \(A_z \neq 0\) case, it will be denoted by \(\mathcal{M}_z\).

Although an easy but lengthy calculation shows that no metric connection exist on \(\mathcal{M}_x, \mathcal{M}_y\) nor \(\mathcal{M}_{xy}\) such that it is diagonal neither in action–angle \((I, \Phi)\) nor spherical \((\theta, \Phi)\) coordinates \((I = \cos \theta)\), the following propositions can be stated.

Proposition 2. The metric

\[
ds^2 = -\frac{1}{2I} \frac{1 + I^2}{1 - I^2} dI^2 + 2 \sqrt{\frac{1 + I^2}{I(1 - I^2)(I^2 - 1)}} d\Phi dI + \frac{\sqrt{I^2 - 1}}{I} d\Phi^2 \tag{5}
\]

is a metric for \(\mathcal{M}_x, \mathcal{M}_y\) and \(\mathcal{M}_{xy}\) in action–angle coordinates.

Proof. If \(A_z = 0\), by comparing Eqs. (3) with Eqs. (4), the only nonvanishing connection coefficients are shown to be given by

\[
\begin{align*}
\Gamma^I_{II} &= \frac{I}{1 - I^2} \\
\Gamma^I_{\Phi\Phi} &= (\Gamma^I_{II})^{-1} \\
\Gamma^\Phi_{I\Phi} &= \frac{I}{2I} \frac{I^2 + 1}{I^2 - 1} \tag{6}
\end{align*}
\]

Let us impose that these are the Levi–Civita connections coefficients for a metric of the general form \(ds^2 = f(I)dI^2 + 2g(I)dI d\Phi + h(I)d\Phi^2\). Then, using the well–known relation between the Christoffel symbols and the metric coefficients, \(\Gamma^\lambda_{\mu\nu} = \frac{1}{2}g^{\lambda\rho}(g_{\rho\mu,\nu} + g_{\rho\nu,\mu} - g_{\mu\nu,\rho})\), we arrive at

\[
\begin{align*}
\frac{1}{2f} \frac{df}{dI} + \frac{1}{g} \frac{dI}{dI} &= \frac{I}{1 - I^2} \\
\frac{1}{2f} \frac{dh}{dI} &= \frac{1 - I^2}{I} \\
\frac{1}{2h} \frac{dh}{dI} &= \frac{I}{2I} \frac{I^2 + 1}{I^2 - 1}. \tag{7}
\end{align*}
\]

whose solutions are given by

\[
\begin{align*}
f(I) &= -\frac{1}{2I} \frac{1 + I^2}{1 - I^2} \\
g(I) &= \sqrt{\frac{1 + I^2}{I(1 - I^2)(I^2 - 1)}} \\
h(I) &= \frac{\sqrt{I^2 - 1}}{I}. \tag{8}
\end{align*}
\]
Then, the desired result is proved. □

**Proposition 3.** There exists a pair of coordinates \((\bar{I}, \bar{\Phi})\) such that

\[
ds^2 = 2d\bar{I}d\bar{\Phi} + h(\bar{I})d\bar{\Phi}^2
\]

is a metric for \(\mathcal{M}_x\), \(\mathcal{M}_y\) and \(\mathcal{M}_{xy}\).

**Proof.** This result follows from the definition of the new pair of coordinates \((\bar{I}, \bar{\Phi})\), given by

\[
\bar{\Phi} = \Phi + \int \frac{g(I) - \sqrt{-\det h(I)}}{h(I)} dI
\]

\[
\bar{I} = \int \sqrt{-\det} dI,
\]

where \(\det = f(I)h(I) - g^2(I)\). □

Now it is interesting and illustrative to analyze the case for only \(A_z \neq 0\). Thus,

**Proposition 4.** The qubit manifold \(\mathcal{M}_z\) is the compact flat cylinder \([-1, 1] \times S^1\).

**Proof.** In this case, the dynamics is given by

\[
\dot{I} = 0
\]

\[
\dot{\Phi} = 2A_z
\]

or, in terms of spherical coordinates,

\[
\ddot{\theta} + \dot{\theta}^2 \cot \theta = 0
\]

\[
\ddot{\Phi} = 0.
\]

If the latter pair of equations describe the geodesics for some metric then, by comparing again them with Eqs. \(\text{(11)}\), the non-vanishing connection coefficients are given by

\[
\Gamma_{\theta\theta}^\theta = \cot \theta \quad \text{and} \quad \Gamma_{ij}^\Phi = 0.
\]

Using the fact that any two dimensional Riemannian metric can be locally recast as \(ds^2 = du^2 + f(u, v)dv^2\), let us look for a metric of the general form \(ds^2 = f(\theta, \Phi)d\theta^2 + d\Phi^2\). If the metricity condition of the connection is imposed, the differential equation which has to be fulfilled is \(\frac{1}{2} \frac{\partial f}{\partial \theta} = \cot \theta\). After simple manipulations, its solution is given by \(f(\theta) = \sin^2 \theta\). Then, the corresponding metric is

\[
ds^2 = \sin^2 \theta d\theta^2 + d\Phi^2
\]
or, in action–angle coordinates,

\[ ds^2 = dI^2 + d\Phi^2 \]  \hspace{1cm} (15)

which describes a flat cylinder embedded in \( \mathbb{R}^3 \). □

Notice that this result is consistent with the fact that the solution of Eqs. (11) is a straight line in the \((I, \Phi)\)–plane which, after identifying \( \Phi(0) \) with \( \Phi(2\pi) \) becomes a flat cylinder (see Fig. (1)).

![Figure 1](image-url)

**Figure 1.** Geodesic of \( \mathcal{M}_z \) (red line) in the \((I, \Phi)\)–plane with constant \( I_0 \).

**Proposition 5.** The qubit manifolds \( \mathcal{M}_x, \mathcal{M}_y \) and \( \mathcal{M}_{xy} \) are the compact and curved cylinders with \([-1, 1] \times S^1\) topology.

**Proof.** On one hand, it can be shown by Proposition 3 that the metric for \( \mathcal{M}_z \), which is a compact and flat cylinder, can be recast as \( ds^2 = 2d\bar{I}d\bar{\Phi} + d\Phi^2 \). On the other hand, a straightforward calculation shows that the scalar curvature corresponding to the metric of the form \( ds^2 = 2d\bar{I}d\bar{\Phi} + h(\bar{I})d\Phi^2 \) is given by \( R = h''(\bar{I}) \), where a prime denotes \( d/d\bar{I} \). Therefore, the qubit manifolds \( \mathcal{M}_x, \mathcal{M}_y \) and \( \mathcal{M}_{xy} \) can be taken to be curved cylinders. □

We note that any qubit manifold is conformally flat (in fact, any two–dimensional Riemannian manifold is conformally flat [25]). Although this is not evident for the \( \mathcal{M}_x, \mathcal{M}_y \) nor \( \mathcal{M}_{xy} \) cases, this can be proved by inspection for the \( \mathcal{M}_z \) qubit manifold since it corresponds to a flat cylinder.
III. EXTENSION TO OPEN–SYSTEM DYNAMICS

In a previous work, a geometrical description of a Caldeira–Legget–like open system dynamics for a qubit has been developed [24], showing that the effective open–system dynamics driven by the Hamiltonian

\[ H = H_0 + \frac{1}{2} \sum_i (p_i^2 + x_i^2 \omega_i^2) - \Phi \sum_i c_i x_i + \sum_i \Phi^2 c_i^2 \]  

(16)

where the oscillator mass has been taken to be one and \( c_i \) are the system–bath coupling constants, can be described by [21, 26]

\[ H_t = H_0 + 2\gamma \dot{\Phi} - \xi \Phi, \]  

(17)

where \( \gamma \) stands for a friction constant and \( \xi \) is a stochastic Gaussian process representing a noisy environment (for technical details see [24] and references therein). We remark that the effective Hamiltonian function, \( H_t \), is not a conserved quantity (notice the \( t \) index in \( H_t \)). After a second time derivative, the corresponding equations of motion issued from \( H_t \) can be written as

\[ \ddot{I} + \frac{I}{1 - I^2} \dot{I}^2 + \frac{1 - I^2}{I} \Phi \left( \dot{\Phi} - 2A_z \right) \]

\[ - 2\gamma \left( \frac{I^2 + 1}{I(I^2 - 1)} \dot{I} \dot{\phi} + \frac{2A_z I}{I} \right) + \dot{\xi}(t) = 0 \]

\[ \ddot{\Phi} + \dot{I} \dot{\Phi} - \frac{I^2 + 1}{I(I^2 - 1)} + \frac{2I A_z}{I(1 - I^2)} = 0. \]  

(18)

These equations (or the existence of \( H_t \)) motivate the following definition:

**Definition 2.** Let \( \mathcal{M}^\gamma \) be a two–dimensional connected, compact and orientable Riemannian \( C^n \)–manifold \( (n \geq 2) \) and let \( H_t \equiv H_t(u, v, \dot{u}, \dot{v}) \) be an effective Hamiltonian function for a dissipative qubit. The pair \((u, v)\) refers to any pair of coordinates used to represent \( H_0 \). If \( \dot{u} = -\frac{d}{dt} \left( \frac{\partial H_t}{\partial \dot{v}} \right) = f(u, v, \dot{u}, \dot{v}) \) and \( \dot{v} = \frac{d}{dt} \left( \frac{\partial H_t}{\partial u} \right) = g(u, v, \dot{u}, \dot{v}) \) coincide with the geodesics of \( \mathcal{M}^\gamma \), then \( \mathcal{M}^\gamma \) is said to be a dissipative qubit manifold.

As carried out in the non–dissipative description, the dissipative qubit manifold corresponding to the case \( A_i \neq 0 \) \((i = x \text{ or } y)\) and \( A_z = 0 \) will be denoted by \( \mathcal{M}^\gamma_i \). In the \((A_x, A_y) \neq (0, 0)\) and \( A_z = 0 \) case, it will be denoted by \( \mathcal{M}^\gamma_{xy} \). Finally, in the \((A_x, A_y) = (0, 0)\) and \( A_z \neq 0 \) case, it will be denoted by \( \mathcal{M}^\gamma_z \). The only way this dynamics could correspond to a geodesic motion is when noisy terms are not included.
Proposition 6. No dissipative qubit manifold exists such that \((A_i, A_z) \neq (0, 0)\) \((i = x\) or \(y)\) and \(A_z \neq 0\) and \((A_x, A_y) \neq (0, 0)\).

Proof. Similar to Proposition 1. Compare Eqs. \([18]\) (without the term of the time derivative of the noise) with the geodesic equation. □

Proposition 7. No dissipative qubit manifold exists such that \(A_i \neq 0\) \((i = x\) or \(y)\) and \(A_z = 0\).

Proof. If Eqs. \([18]\) (without the noisy term) are likely to describe the geodesics of \(M^\gamma_x\), \(M^\gamma_y\) or \(M^\gamma_{xy}\), then the corresponding connection coefficients are given by Eqs. \([6]\) together with \(\Gamma^{I}_{I\Phi} = 2\gamma \Gamma^{\phi}_{I\Phi}\). Thus, the differential equations one has to solve are Eqs. \([7]\) together with \(\frac{1}{f} \frac{df}{dt} + \frac{1}{2g} \frac{dg}{dt} = -\gamma \frac{f^2 + 1}{f(f^2 - 1)}\). The incompatibility of these equations proves the required result. □

In spite of these negative results, we have the following

Proposition 8. The dissipative qubit manifold \(M^\gamma_z\) is the compact flat cylinder with \([-1, 1] \times S^1\) topology.

Proof. In this case, the corresponding dissipative dynamics is given by
\[
\dot{I} = -\gamma \dot{\Phi} \\
\dot{\Phi} = 2A_z
\]
(19)
or, in terms of spherical coordinates,
\[
\ddot{\theta} + \dot{\theta}^2 \cot \theta = 0 \\
\ddot{\Phi} = 0
\]
(20)
which coincide with Eqs. \([12]\). Therefore, the required result follows from Proposition 3. □

The main difference with the non–dissipative case is that, in the \(\gamma \neq 0\) situation, the geodesic does not lie in the same plane for all \(t\). This can be shown by noting that the solutions of Eqs. \([19]\) give place to \(I(t) = -\gamma \Phi(t) + I(0) + \gamma \Phi(0)\), which becomes an helix after identifying \(\Phi(0)\) with \(\Phi(2\pi)\) (see Fig. \([2]\)).

IV. LORENTZIAN QUBIT MANIFOLDS

Extending some of the previous results to the Lorentzian case, far from being a purely mathematical generalization, can be physically justified. In particular, introducing a
Lorentzian signature in $\mathcal{M}_z$ seems to be rather natural since, in this case, $H_0 = 2A_z I$ and $I$ is a generalized momentum, $p_I$. Then, by taking $2A_z = c$, the Hamiltonian function can be recast as $H_0 = p_I c$, which is precisely the dispersion relation of a massless particle. Although there are global obstructions for a manifold to admit a Lorentzian metric, this photon–like particle will be shown to propagate in two–dimensional Minkowski space with the cylinder topology. As in the Riemannian case, dissipation will be included by adding the term $2\gamma \dot{\Phi} \dot{\Phi}$ to $H_0$.

The obstructions for a manifold to admit a Lorentzian metric are reflected in the following theorem [27]:

**Theorem 3.** A manifold admits a Lorentzian metric if and only if it is noncompact or has zero Euler characteristic.

Therefore, remembering that the only dissipative qubit manifold is $\mathcal{M}^\gamma_2$, the following results can be stated:

**Proposition 9.** Any non–dissipative or dissipative qubit manifold admits a Lorentzian metric.

**Proof.** As the cylinder has zero Euler characteristic then, by Propositions 4, 5 and Theorem 3, this result is straightforward. □

**Proposition 10.** $\mathcal{M}_2$ and $\mathcal{M}^\gamma_2$ are qubit Lorentzian manifolds with the compact flat cylinder $[-1, 1] \times S^1$ topology.
Proof. Any two-dimensional Lorentzian metric can be locally recast as \( ds^2 = du^2 - f(u,v)dv^2 \). Then, by adapting the procedure employed in Propositions 4 and 8 to the Lorentzian case, we reach that

\[
ds^2 = d\Phi^2 - dI^2
\]

is a Lorentzian metric for \( M_z \) and \( M'_z \). \( \square \)

V. CONCLUSIONS AND FUTURE WORK

In this work, we have applied a geometrization of quantum mechanics using the first Hopf fibration to show that the trajectories of a qubit dynamics over the two-sphere are geodesics in certain Riemannian or physically-sound Lorentzian metrics, which turned to be flat and curved cylinders. In addition, by including dissipative terms to the dynamics by means of a Caldeira–Legget–like coupling to the environment, the previous findings for the simplest dissipative qubit have been generalized. Extension of these results to deal with two-qubit entanglement on \( S^4 \) and, in general, with the dynamics of n–level systems on \( \mathbb{C}P^{n-1} \) is currently in progress.

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