A Compound Poisson Convergence Theorem for Sums of \(m\)-Dependent Variables

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Abstract We prove the Simons–Johnson theorem for sums \(S_n\) of \(m\)-dependent random variables with exponential weights and limiting compound Poisson distribution \(\text{CP}(s, \lambda)\). More precisely, we give sufficient conditions for \(\sum_{k=0}^{\infty} e^{hk} |P(S_n = k) - \text{CP}(s, \lambda)\{k\}| \to 0\) and provide an estimate on the rate of convergence. It is shown that the Simons–Johnson theorem holds for the weighted Wasserstein norm as well. The results are then illustrated for \(N(n; k_1, k_2)\) and \(k\)-runs statistics.

Keywords Poisson distribution · Compound Poisson distribution · \(M\)-dependent variables · Wasserstein norm · Rate of convergence

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1 Introduction

Simons and Johnson [18] established an interesting result that the convergence of the binomial distribution to the limiting Poisson law can be much stronger than in total variation. Indeed, they proved that if \(S_n = X_1 + X_2 + \cdots + X_n\) has binomial distribution with parameters \(n, \ p = \lambda / n\) and \(g(x)\) satisfies \(\sum_{k=0}^{\infty} g(k)\text{Pois}(\lambda)\{k\} < \infty\), then
\[ \sum_{k=0}^{\infty} g(k) |P(S_n = k) - \text{Pois}(\lambda)[k]| \to 0, \ n \to \infty, \tag{1} \]

where here and henceforth \( \text{Pois}(\lambda) \) denotes Poisson distribution with mean \( \lambda \). The above result was then extended to the case of independent and nonidentically distributed indicator variables by Chen [6]; see also Barbour et al. [1] and Borisov and Ruzankin [4] for a comprehensive study in this direction. That similar results hold for convolutions on measurable Abelian group was proved in Chen [7], see also Chen and Roos [8]. Dasgupta [9] showed that to some extent, the binomial distribution in (1) can be replaced by a negative binomial distribution. Wang [20] later extended Simons and Johnson’s result in (1) to the case of nonnegative integer-valued random variables and compound Poisson limit, under the condition that \( P(X_i = k)/P(X_i > 0) \) does not depend on \( i \) and \( n \).

All the above-mentioned works deal with sums of independent random variables only. Moreover, the essential step in the proofs lies in establishing an upper bound for the ratio \( P(S_n = k)/\text{Pois}(\lambda)[k] \) or making similar assumptions on the measures involved. The case of dependent random variables is notably less investigated. In Čekanavičius [5], the result in (1) was proved for the Markov binomial distribution with \( g(k) = e^{hk} \). The possibility to switch from dependent random variables to independent ones was considered in Ruzankin [16]. However, results from Ruzankin [16] are of the intermediate type, since their estimates usually contain expectations of the unbounded functionals of the approximated random variables \( X_1, \ldots, X_n \), which still need to be estimated.

In this paper, we prove the Simons–Johnson theorem with exponential weights and for the sums of \( m \)-dependent random variables and limiting compound Poisson distribution. The main result contains also estimates on the rate of convergence. A sequence of random variables \( \{X_k\}_{k \geq 1} \) is called \( m \)-dependent if, for \( 1 < s < t < \infty, \ t - s > m \), the sigma-algebras generated by \( X_1, \ldots, X_s \) and \( X_t, X_{t+1}, \ldots \) are independent. Though the main result is proved for \( 1 \)-dependent random variables, it is clear, by grouping consecutive summands, that one can reduce the sum of \( m \)-dependent variables to the sum of \( 1 \)-dependent ones. We exemplify this possibility by considering \((k_1, k_2)\)-events and \( k \)-runs.

We consider henceforth the sum \( S_n = X_1 + X_2 + \cdots + X_n \) of nonidentically distributed \( 1 \)-dependent random variables concentrated on nonnegative integers. We denote distribution and characteristic function of \( S_n \) by \( F_n(x) \) and \( \hat{F}_n(it) \), respectively. Note that we include imaginary unit in the argument of \( \hat{F}_n \), a notation traditionally preferred over \( \hat{F}_n(t) \) when conjugate distributions are applied. We define \( j \)th factorial moment of \( X_k \) by \( \nu_j(k) = E[X_k(X_k - 1) \cdots (X_k - j + 1)], \ k = 1, 2, \ldots, n; \ j = 1, 2, \ldots \). Let

\[
\Gamma_1 = ES_n = \sum_{k=1}^{n} v_1(k), \quad \Gamma_2 = \frac{1}{2} (\text{Var} S_n - ES_n) \\
= \frac{1}{2} \sum_{k=1}^{n} (v_2(k) - v_1^2(k)) + \sum_{k=2}^{n} \text{Cov}(X_{k-1}, X_k). \]

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Formally,
\[ \hat{F}_n(it) = \exp\{\Gamma_1(e^{it} - 1) + \Gamma_2(e^{it} - 1)^2 + \cdots\}. \] (2)

It is clear that Poisson limit occurs only if \( \Gamma_1 \to \lambda, \Gamma_2 \to 0, \) and other factorial cumulants also tend to zero. Similar arguments apply for compound Poisson limit as well.

Next, we introduce compound Poisson distribution \( CP(s, \lambda) = CP(s, \lambda_1, \ldots, \lambda_s), \) where \( s \geq 1 \) is an integer. Let \( N_i \) be independent Poisson random variables with parameters \( \lambda_i \geq 0, i = 1, 2, \ldots, s. \) Then, \( CP(s, \lambda) \) is defined as the distribution of \( N_1 + 2N_2 + 3N_3 + \cdots + sN_s \) with characteristic function
\[ \hat{CP}(s, \lambda)(it) = \exp\left\{ \sum_{m=1}^{s} \lambda_m(e^{itm} - 1) \right\} = \exp\left\{ \sum_{j=1}^{s} (e^{it} - 1)^j \sum_{m=j}^{s} \binom{m}{j} \lambda_m \right\}. \] (3)

Note also that \( N_1 + 2N_2 + \cdots + sN_s \equiv Y_1 + Y_2 + \cdots + Y_N, \)

where the \( Y_j \) are independent random variables with \( P(Y_1 = j) = \lambda_j / (\sum_{i=1}^{s} \lambda_i), \) for \( 1 \leq j \leq s \) and \( N \sim \text{Pois}(\sum_{i=1}^{s} \lambda_i). \) It is clear that when \( s = 1, \) \( CP(1, \lambda) = \text{Pois}(\lambda), \) the distribution of \( N \) in this case.

Let \( M \) be a signed measure concentrated on nonnegative integers. The total variation norm of \( M \) is denoted by \( \| M \| = \sum_{m=0}^{\infty} |M(m)|. \) Properties of the norm are discussed in detail in Shiryaev [17], pp. 359–362. The total variation norm is arguably the most popular metric used for estimation of the accuracy of approximation of discrete random variables. The Wasserstein (or Kantorovich) norm is defined as \( \| M \|_W = \sum_{m=0}^{\infty} \left| \sum_{k=0}^{m} M(k) \right|. \) For other expressions of \( \| M \| \) and \( \| M \|_W, \) one can consult appendix A1 in Barbour et al. [2].

2 The Main Results

Henceforth, we assume that all random variables are uniformly bounded from above, that is, \( X_i \leq C_0, 1 \leq i \leq n. \) Here, \( C_0 \geq 1 \) is some absolute constant. First, we formulate sufficient conditions for compound Poisson limit with exponential weights.

**Theorem 2.1** Let \( X_i \) be nonidentically distributed \( l \)-dependent random variables concentrated on nonnegative integers, \( X_i \leq C_0, 1 \leq i \leq n. \) Let \( F_n(x) \) denote the distribution of \( S_n = X_1 + X_2 + \cdots + X_n \) and let \( CP(s, \lambda) \) be defined by (3). Let \( s \geq 1 \) be an integer, \( \lambda_j \geq 0, 1 \leq j \leq s, \) and \( h \geq 0 \) be fixed numbers. If, as \( n \to \infty, \)
\[
\max_{1 \leq j \leq n} \nu_1(j) \to 0, \quad (4)
\]
\[
\frac{1}{m!} \sum_{j=1}^{n} v_m(j) \to \sum_{l=m}^{s} \binom{l}{m} \lambda_l, \quad m = 1, 2, \ldots, s; \quad (5)
\]
\[
\sum_{j=1}^{n} v_{s+1}(j) \to 0, \quad (6)
\]
\[
\sum_{j=2}^{n} |\text{Cov}(X_{j-1}, X_j)| \to 0, \quad (7)
\]

then
\[
\sum_{k=0}^{\infty} e^{hk} |F_n\{k\} - \text{CP}(s, \lambda)\{k\}| \to 0. \quad (8)
\]

**Remark 2.1** (i) Assumption \( C_0 \geq 1 \) is not restrictive. Indeed, \( X_i < 1 \) is equivalent to the trivial case \( X_i \equiv 0 \), since we assume that \( X_i \) is concentrated on integers.

(ii) Technical assumption that all random variables are uniformly bounded significantly simplifies all proofs. Probably it can be replaced by some more general uniform smallness conditions for the tails of distributions.

(iii) Conditions for convergence to compound Poisson distribution can be formulated in various terms. In Theorem 2.1, we used factorial cumulants. Observe that such approach allows natural comparison of the characteristic functions due to the exponential structure of \( \text{CP}(s, \lambda)\{t\} \).

(iv) Assumptions (4)--(7) are sufficient for convergence, but not necessary. For example, consider the case \( s = 2 \) and compare (2) and (3). The convergence then implies \( \Gamma_1 \to \lambda_1 + 2\lambda_2 \) and \( \Gamma_2 \to \lambda_2 \). If we assume, in addition (4), then the last condition is equivalent to
\[
\frac{1}{2} \sum_{j=1}^{n} v_2(j) + \sum_{j=2}^{n} \text{Cov}(X_{j-1}, X_j) \to \lambda_2,
\]
and is more general than the assumptions \( \sum_{1}^{n} v_2(j)/2 \to \lambda_2 \) and (7).

Observe that we can treat (1) as a weighted total variation norm with increasing weights. A natural question that arises is the following: is it possible to extend this result to stronger norms? If we consider the Wasserstein norm, then the answer is affirmative, see Lemma 4.8 below. Let \( F_n(k) = F_n([0, k]) \) and \( \text{CP}(s, \lambda)(k) = \text{CP}(s, \lambda)\{[0, k]\} \) denote the corresponding distribution functions. For exponentially weighted Wasserstein norm, we have the following inequality:
\[
\sum_{k=0}^{\infty} e^{hk} |F_n\{k\} - \text{CP}(s, \lambda)\{k\}| \leq \frac{1}{e^h - 1} \sum_{k=0}^{\infty} e^{hk} |F_n\{k\} - \text{CP}(s, \lambda)\{k\}|, \quad (9)
\]
provided the left-hand side is finite and $h > 0$. We see that, though Wasserstein norm (which corresponds to the case $h = 0$) is stronger than the total variation norm, the weighted Wasserstein norm is bounded from above by the correspondingly weighted total variation norm. Consequently, from (9) and Theorem 2.1, the following corollary immediately follows.

**Corollary 2.1** Let $\lambda_1 \geq 0, \ldots, \lambda_s \geq 0$, and $s \geq 1$ be an integer. Assume conditions (4)–(7) are satisfied. Then, for fixed $h > 0$,

$$
\sum_{k=0}^{\infty} e^{hk} | F_n(k) - CP(s, \lambda)(k) | \to 0. \quad (10)
$$

Indeed, Theorem 2.1 follows from more general Theorem 2.2 given below. Assuming $\max_j \nu_1(j)$ to be small, but not necessarily converging to zero, we obtain estimates of remainder terms. Let

$$
a = a(h, C_0) = e^{hC_0(2+h)\sqrt{C_0}}, \psi = \exp \left\{ \max (4a^2 \Gamma_1, \sum_{m=1}^{s} \lambda_m(e^{hm} + 1)) \right\}, \quad (11)
$$

$$
K_1 = \psi \sqrt{\pi + 1}(e^h + 1)^s(s + 1 + 4a^2 \Gamma_1),
$$

$$
K_2 = \psi \sqrt{\pi + 1}(s + 1 + 4a^2 \Gamma_1) \frac{e^{hC_0(e^h + 1)^s+1}}{(s + 1)!},
$$

$$
K_3 = 16\psi a^4 \sqrt{\pi + 1}(5 + 6a^2 \Gamma_1), \quad K_4 = 4\psi a^3 \sqrt{\pi + 1}(1.1 + a^2 \Gamma_1).
$$

Let us denote henceforth $\nu_1^{(n)} = \max_{1 \leq j \leq n} \nu_1(j)$, for simplicity. We are ready to state the main result of this paper.

**Theorem 2.2** Let $s \geq 1$ be an integer, $h \geq 0$, $\lambda_j \geq 0$, $1 \leq j \leq s$, and let $a^2 \nu_1^{(n)} \leq 1/100$. Then,

$$
\sum_{k=0}^{\infty} e^{hk} | F_n(k) - CP(s, \lambda)(k) | \leq K_1 \sum_{m=1}^{s} \frac{1}{m!} \sum_{j=1}^{n} \nu_m(j) - \sum_{l=m}^{s} \binom{l}{m} \lambda_l 
\quad + K_2 \sum_{j=1}^{n} \nu_{s+1}(j) + K_3 \sum_{j=1}^{n} \nu_1^2(j) 
\quad + K_4 \sum_{j=2}^{n} |\text{Cov}(X_{j-1}, X_j)|. \quad (12)
$$

We next illustrate the results for the cases $s = 1$ and $s = 2$, which are of particular interest. Note here the corresponding limiting distributions are as follows:

$$
\hat{\text{Pois}}(\lambda)(it) = \exp \{ \lambda(e^{it} - 1) \}, \quad \hat{\text{CP}}(2, \lambda)(it) = \exp \{ \lambda_1(e^{it} - 1) + \lambda_2(e^{2it} - 1) \}.
$$

The following corollary is immediate from (12).
Corollary 2.2 Let \( a^2 \nu_1^{(\nu)} \leq 1/100 \). Assume \( h \geq 0 \), \( \lambda, \lambda_1 \) and \( \lambda_2 \) are positive reals. Then,

\[
(i) \quad \sum_{k=0}^{\infty} e^{hk} |F_n(k) - \text{Pois}(\lambda)\{k\}| \\
\leq C_1(h, \lambda) \exp\{4a^2 \Gamma_1\} \left\{ |\Gamma_1 - \lambda| + \sum_{j=1}^{n} \nu_2(j) + \sum_{j=1}^{n} \nu_1^2(j) \\
+ \sum_{j=2}^{n} |\text{Cov}(X_{j-1}, X_j)| \right\},
\]

(13)

\[
(ii) \quad \sum_{k=1}^{n} e^{hk} |F_n(k) - \text{CP}(2, \lambda)\{k\}| \\
\leq C_2(h, \lambda_1, \lambda_2) \exp\{4a^2 \Gamma_1\} \left\{ |\Gamma_1 - \lambda_1 - 2\lambda_2| + \sum_{j=1}^{n} \nu_2(j) - 2\lambda_2 \right\} \\
+ \sum_{j=1}^{n} \nu_3(j) + \sum_{j=1}^{n} \nu_1^2(j) + \sum_{j=2}^{n} |\text{Cov}(X_{j-1}, X_j)| \right\}.
\]

(14)

Note here the constants \( C_1 \) and \( C_2 \) depend on \( h, \lambda, \lambda_1 \) and \( \lambda_2 \) only.

Remark 2.2 (i) Applying (9), we can obtain the estimate for exponentially weighted Wasserstein norm, similar to Theorem 2.2.

(ii) Let us consider the sum of independent Bernoulli variables, \( W = \xi_1 + \cdots + \xi_n \), where \( P(\xi_i = 1) = 1 - P(\xi_i = 0) = p_i \). Assume that, for some fixed \( \lambda > 0 \), the parameter \( p_i \) satisfies \( \sum_{k=1}^{n} p_i = \lambda \) and \( \sum_{k=1}^{n} p_i^2 \to 0 \), as \( n \to \infty \). Then, putting \( h = 0 \) in (13), we obtain an estimate for total variation metric as

\[
\sum_{k=0}^{\infty} |P(W = k) - \text{Pois}(\lambda)\{k\}| \leq C_3 \sum_{j=1}^{n} p_j^2,
\]

if \( n \) is sufficiently large. Observe that this estimate is of the right order.

We next show that Simons–Johnson result holds for convergence associated with \((k_1, k_2)\)-events and \( k \)-runs, which have applications in statistics. For example, the number of \( k \)-runs have been used to develop certain nonparametric tests for randomness. See [13] for more details.

3 Some Examples

In examples below, we assume \( \lambda, \lambda_1, \lambda_2 \) and \( h \geq 0 \) are some absolute constants.

1. Number of \((k_1, k_2)\)-events. Consider a sequence of independent Bernoulli trials with the same success probability \( p \). We say that \((k_1, k_2)\)-event has occurred if \( k_1 \)
consecutive failures are followed by \( k_2 \) consecutive successes. Such sequences can be meaningful in biology (see Huang and Tsai [12], p. 126), or in agriculture, since sequences of rainy and dry days have impact on the yield of raisins (see [10], p. 1698).

More formally, let \( \eta_j \) be independent Bernoulli \( \text{Be}(p) \) \((0 < p < 1)\) variables and \( Z_j = (1 - \eta_{j-m+1}) \cdots (1 - \eta_{j-k_2})\eta_{j-k_2+1} \cdots \eta_{j-1}\eta_j, \ j = m, m+1, \ldots, n, \) where \( m = k_1 + k_2 \) and \( k_1 > 0 \) and \( k_2 > 0 \) are fixed integers. Then, \( N(n; k_1, k_2) = Z_m + Z_{m+1} + \cdots + Z_n \) denotes the number of \((k_1, k_2)\)-events in \( n \) Bernoulli trials. We denote the distribution of \( N(n; k_1, k_2) \) by \( H \). It is well known that \( N(n; k_1, k_2) \) has limiting Poisson distribution, see Huang and Tsai [12] and Vellaisamy [19]. Note also that \( Z_1, Z_2, \ldots \) are \( m \)-dependent. Consequently, the results of previous section cannot be applied directly. However, one can group the summands in the following natural way:

\[
N(n; k_1, k_2) = (Z_m + Z_{m+1} + \cdots + Z_{2m-1}) + (Z_{2m} + Z_{2m+1} + \cdots + Z_{3m-1}) + \cdots = X_1 + X_2 + \ldots
\]

Here, each \( X_j \), with probable exception of the last one, contains \( m \) summands. Let \( K \) and \( \delta \) be the integer and fractional parts of \((n - m + 1)/m\), respectively, so that

\[
K = \left\lfloor \frac{n-m+1}{m} \right\rfloor, \quad \frac{n-m+1}{m} = K + \delta, \quad 0 \leq \delta < 1,
\]

and \( a(p) = (1 - p)^{k_1} p^{k_2} \). Then, considering the structure of new variables \( X_j \) we see that, for \( j = 1, \ldots, K \)

\[
X_j = \begin{cases} 1, \text{ with probability } ma(p), \\
0, \text{ with probability } 1 - ma(p), \end{cases} \quad X_{K+1} = \begin{cases} 1, \text{ with probability } \delta ma(p), \\
0, \text{ with probability } 1 - \delta ma(p). \end{cases}
\]

Consequently, \( v_2(j) = v_2(K + 1) = 0, \ v_1(j) = ma(p), \ v_1(K + 1) = \delta a(p), \ \Gamma_1 = (n - m + 1)a(p) \) and we obtain, checking for nonzero products,

\[
\mathbb{E}(X_1 X_2) = a^2(p)(m + (m - 1) + (m - 2) + \cdots + 1) = \frac{a(p)^2 m (m + 1)}{2},
\]

\[
\mathbb{E}(X_K X_{K+1}) = \frac{\delta m (\delta m + 1) a^2(p)}{2}.
\]

Therefore,

\[
\text{Cov}(X_{j-1}, X_j) = -\frac{m (m - 1) a^2(p)}{2}, \quad \text{Cov}(X_K, X_{K+1}) = \frac{a^2(p) \delta m (\delta m + 1 - 2m)}{2},
\]

for \( j = 1, 2, \ldots, K \). Consequently, if \((n - m + 1)a(p) \to \lambda, \) then

\[
\sum_{j=0}^{\infty} e^{\lambda j} \left| \text{H}[j] - \text{Pois} (\lambda) [j] \right| \to 0.
\]
Indeed, we have \( a(p) = o(1) \) and

\[
\sum_{j=2}^{K+1} |\text{Cov}(X_{j-1}, X_j)| \leq \frac{Km(m-1)a^2(p) + a^2(p)\delta m(2m-1 - \delta m)}{2} \\
\leq (Km + \delta m)a^2(p)m = (n - m + 1)a^2(p) \to 0,
\]

\[
\sum_{j=1}^{K+1} v_1^2(j) \leq a(p)\Gamma_1 \to 0.
\]

Using (13) of Corollary 2.2, we see that (8) holds with CP(1, \( \lambda \)).

2. **Statistic of k-runs.** Let \( \eta_i, 1 \leq i \leq n + k - 1 \), be independent Bernoulli \( \text{Be}(p) \) \((0 < p < 1)\) variables and let \( Z_j = \eta_j \eta_{j+1} \cdots \eta_{j+k-1} \). Then, \( S = Z_1 + Z_2 + \cdots + Z_n \) is called k-runs statistic. Runs statistics are important in reliability theory \((m \text{ consecutive } k \text{ out of } n \text{ failure system})\) and quality control (see, for discussion, Wang and Xia [21]). Approximations of 2 or k-runs statistic (including the case of different probabilities \( p_j \)) by various distributions have been considered in numerous papers, see Röllin [15] and Wang and Xia [21] and the references therein. As in the previous example, we switch from \( k \)-dependent case to 1-dependent one by grouping \( k \) consecutive summands as \( X_1 = Z_1 + \cdots + Z_k, X_2 = Z_{k+1} + \cdots + Z_{2k} \) and so on. Note that such a grouping is not unique. For example, it is possible to group \((k - 1)\) consecutive summands. Let \( K \) denote the integer part of \((n/k)\), where \( k \) is fixed. Next, we apply Corollary 2.2. It is obvious that \( \Gamma_1 = np^k, v_2(K+1) = o(1), \) and \( E(X_K X_{K+1}) = o(1) \) as \( n \to \infty \). For \( j = 2, \ldots, K \), we have \( \text{E}(X_{j-1} X_j) \leq \text{C}(k) p^{k+1} \) and \( v_2(j) \leq \text{C}(k) p^{k+1} \). Indeed, in both the cases, at least two of \( Z_i \)'s must be equal to unity. Next, note that

\[
\sum_{j=2}^{K} |\text{Cov}(X_{j-1}, X_j)| \leq \sum_{j=2}^{K} \text{E}(X_{j-1} X_j) + \sum_{j=2}^{K} v_1(j-1)v_1(j) \leq \text{C}(k)np^{k+1}.
\]

Consequently, if \( np^k \to \lambda \), then (8) holds for \( F_n = \mathcal{L}(S) \) with limiting Pois(\( \lambda \)) distribution.

3. **Convergence to CP(2, \( \lambda \)).** By slightly modifying 2-runs, we construct an example of 1-dependent summands with limiting compound Poisson distribution. Let \( \eta_i \sim \text{Be}(p), (0 < p < 1, i = 1, \ldots, n+1) \) and \( \xi_j \sim \text{Be}(\overline{p}), (0 < \overline{p} < 1, j = 1, \ldots, n) \) be two sequences of independent Bernoulli variables (any \( \xi_j \) and \( \eta_i \) are also independent). Let \( X_1 = \eta_1 \eta_2 + 2\xi_1(1 - \eta_1 \eta_2), X_2 = \eta_2 \eta_3 + 2\xi_2(1 - \eta_2 \eta_3), X_3 = \eta_3 \eta_4 + 2\xi_3(1 - \eta_3 \eta_4) \) and so on. Let \( S = X_1 + \cdots + X_n \). It is obvious that \( X_1, X_2, \ldots, X_n \) are 1-dependent random variables. Moreover,

\[
v_1(j) = p^2 + 2\overline{p}(1 - p^2), \quad v_2(j) = \text{E}(X_j(X_j - 1)) = 2\overline{p}(1 - p^2), \quad v_3(j) = 0, \quad |\text{Cov}(X_1, X_2)| \leq \text{E}(X_1 X_2) + v_1(1)v_1(2) \leq \text{C}(p^3 + p^2\overline{p} + \overline{p}^2).
\]
Let \( np^2 \to \lambda_1 \) and \( n\overline{p} \to \lambda_2 \), as \( n \to \infty \). Then,
\[
\Gamma_1 = n\nu_1(1) \to \lambda_1 + 2\lambda_2, \quad \sum_{j=1}^{n} \nu_2(j) \to 2\lambda_2, \quad \sum_{j=1}^{n} \nu_1^2(j) \to 0,
\]
\[
\sum_{j=2}^{n} |\text{Cov}(X_{j-1}, X_j)| \to 0.
\]

Therefore, it follows from (14) that
\[
\sum_{k=0}^{\infty} e^{kh} |P(S = k) - CP(2, \lambda)[k]| \to 0,
\]
leading to strong compound Poisson convergence.

4 Proofs

It is not difficult to observe that the weighted sum in Theorem 2.2 can be treated as the total variation of some conjugate measure. Indeed,
\[
\sum_{k=0}^{\infty} e^{kh} |F_n[k] - CP(s, \lambda)[k]| = \|M\|.
\]

Here, \( M[k] = e^{kh} (F_n[k] - CP(s, \lambda)[k]) \). For estimation of \( \|M\| \), we apply the characteristic function method. Observe that
\[
\hat{M}(it) = \hat{F}_n(it + h) - \overline{CP(s, \lambda)(it + h)}.
\]

We need to estimate \( |\hat{M}(it)| \). Therefore, the crucial step in the proof, is expansion of \( \hat{F}_n(it + h) \) in moments of \( S_n \). The essential tool for this is Heinrich [11] representation of \( \hat{F}_n(it) \) as a product of \( n \) functions. For Heinrich’s representation, we need some additional notations. Let \( \{U_k\}_{k \geq 1} \) be a sequence of arbitrary real or complex-valued random variables. Also, let \( \hat{E}(U_1) = E(U_1) \) and, for \( k \geq 2 \), define
\[
\hat{E}(U_1, U_2, \ldots, U_k) = E(U_1U_2 \ldots U_k) - \sum_{j=1}^{k-1} \hat{E}(U_1, \ldots, U_j)E(U_{j+1} \ldots U_k).
\]

Then, it is obvious that \( \hat{E}(X_{k-1}, X_k) = \text{Cov}(X_{k-1}, X_k) \).

We require the following two lemmas from Heinrich [11].

**Lemma 4.1** ([11]) Let \( U_1, U_2, \ldots, U_k \) be 1-dependent complex-valued random variables with \( E|U_m|^2 < \infty \), \( 1 \leq m \leq k \). Then,
\[
|\hat{E}(U_1, U_2, \ldots, U_k)| \leq 2^{k-1} \prod_{m=1}^{k} (E|U_m|^2)^{1/2}.
\]

For a complex number \( z \), let
\[
w(z) = \max_{1 \leq k \leq n} \sqrt{E|e^{czX_k} - 1|^2}, \quad K = \{z : w(z) \leq 1/6\}.
\]
Lemma 4.2 ([11]) Let $X_1, X_2, \ldots, X_n$ be a sequence of $I$-dependent random variables. Then for each $z \in \mathcal{K}$, the following product representation holds:

$$
E(e^{zS_n}) = \varphi_1(z)\varphi_2(z) \cdots \varphi_n(z).
$$

Here, $\varphi_1(z) = E(e^{zX_1})$ and for $k = 2, \ldots, n$,

$$
\varphi_k(z) = 1 + E(e^{zX_k} - 1) + \sum_{j=1}^{k-1} \frac{\hat{E}((e^{zX_j} - 1), (e^{zX_{j+1}} - 1), \ldots, (e^{zX_k} - 1))}{\varphi_j(z)\varphi_{j+1}(z) \cdots \varphi_{k-1}(z)},
$$

(15)

Further,

$$
|\varphi_k(z) - 1| \leq |E(e^{zX_k} - 1)| + \frac{2(E|e^{zX_k-1} - 1|^2E|e^{zX_k} - 1|^2)^{1/2}}{1 - 4u(z)},
$$

(16)

for $z \in \mathcal{K}$ and $1 \leq k \leq n$.

In addition, we use the following notation: $u = it + h$, $Y_j = \exp(itX_j) - 1$, $\Psi_{jk} = \hat{E}(Y_j, \ldots, Y_k)$. We use symbol $\theta$ to denote a real or a complex number satisfying $|\theta| \leq 1$. Assume $\nu_j(k) = 0$ and $X_k = 0$ for $k \leq 0$. Moreover, $\nu_j(k) = 0$ if $X_j < k$ and $\nu_1(n) = \max_{1 \leq j \leq n} \nu_1(j)$, as earlier. The primes denote the derivatives with respect to $t$.

Lemma 4.3 The following relations hold for all $t$, $k = 1, \ldots, n$, and an integer $s \geq 1$:

$$
|Y_k| \leq e^{hC_0}(2 + h)X_k, \quad |Y_k|^2 \leq a^2X_k, \quad E|Y_k| \leq a\nu_1(k), \quad E|Y_k|^2 \leq a^2\nu_1(k),
$$

(17)

$$
|Y_k'| \leq e^{hC_0}X_k, \quad |Y_k'|^2 \leq e^{2hC_0}C_0X_k, \quad E|Y_k'| \leq \frac{a}{2}\nu_1(k), \quad E|Y_k'|^2 \leq \frac{a^2}{4}\nu_1(k),
$$

(18)

$$
EY_k = \sum_{m=1}^{s} \frac{v_m(k)}{m!}(e^u - 1)^m + \theta e^{hC_0}(e^h + 1)^{r+1}\frac{v_{s+1}(k)}{(s + 1)!},
$$

(19)

$$
EY_k' = i \sum_{m=1}^{s} \frac{v_m(k)}{(m - 1)!}e^u(e^u - 1)^{m-1} + \theta e^{hC_0}(e^h + 1)^{s}\frac{v_{s+1}(k)}{s!}.
$$

(20)

Proof Since $|\exp(it(X_k - j))| = 1$, we have

$$
|Y_k| \leq e^{hX_k}|e^{itX_k - 1}| + e^{hX_k} - 1 \leq e^{hX_k} \left(|e^{it(X_k-1)}| + |e^{it(X_k-2)}| + \cdots + 1 \right)|e^{it} - 1| + hX_k e^{hX_k} \leq e^{hC_0}X_k |e^{it}| + 1 + hX_k e^{hC_0}
$$

$$
\leq e^{hC_0}(2 + h)X_k.
$$
Other relations of (17) now follow. The proof of (18) is obvious. For the proof of (19), we apply Bergström [3] identity

$$\alpha^N = \sum_{m=0}^{s} \binom{N}{m} \beta^{N-m}(\alpha - \beta)^m + \sum_{m=s+1}^{N} \binom{m-1}{s} \alpha^{N-m}(\alpha - \beta)^{s+1} \beta^{m-s-1},$$

(21)

which holds for any numbers \(\alpha, \beta\) and \(s = 0, 1, 2, \ldots, N\). Let \(\binom{j}{k} = 0\), for \(k > j\). Then, (21) holds for all \(s = 0, 1, \ldots\). We apply (21) with \(N = X_k\), \(\alpha = e^u\) and \(\beta = 1\). Then,

$$Y_k = \sum_{m=1}^{s} \binom{X_k}{m} (e^u - 1)^m + \sum_{m=s+1}^{X_k} \binom{m-1}{s} e^{u(X_k-m)} (e^u - 1)^{s+1}.$$ (22)

Using the results

$$\sum_{m=s+1}^{N} \binom{m-1}{s} = \binom{N}{s+1}, \quad |e^u| = e^h,$$

we obtain

$$\sum_{m=s+1}^{X_k} \binom{m-1}{s} |e^{u(X_j-m)}| \leq e^{hC_0} \binom{X_j}{s+1}.$$ 

The proof of (19) now follows by finding the mean of \(Y_k\) in (22) and using the definition of \(v_j(k)\).

For the proof of (20), we once again apply (21) to obtain

$$Y'_k = iX_k e^{uX_k} = iX_k e^{u(X_k-1)}$$

$$= iX_k e^{\left\{ \sum_{m=0}^{s-1} \binom{X_k-1}{m} (e^u - 1)^m + (e^u - 1)^s \sum_{m=s}^{X_k-1} \binom{m-1}{s-1} e^{u(X_k-1-m)} \right\}}.$$ 

The rest of the proof is the same as that of (19) and, therefore, omitted. \(\Box\)

**Lemma 4.4** Let \(a^2 v_1^{(n)} \leq 0.01\). Then, for \(k = 4, \ldots, n\) and \(j = 1, \ldots, k - 3\),

$$|\Psi_{jk}| \leq 250a^4 \left( \frac{1}{5} \right)^{k-j} \sum_{l=0}^{3} v_1^2(k-l),$$

$$|\Psi'_{jk}| \leq 125a^4(k-j+1) \left( \frac{1}{5} \right)^{k-j} \sum_{l=0}^{3} v_1^2(k-l).$$
and for \( k = 2, \ldots, n; \ j = 1, \ldots, k - 1, \)
\[
|\Psi_{jk}| \leq 5a^2 \left( \frac{1}{5} \right)^{k-j} [v_1(k-1) + v_1(k)],
\]
\[
|\Psi'_{jk}| \leq (2.5)a^2(k-j+1) \left( \frac{1}{5} \right)^{k-j} [v_1(k-1) + v_1(k)].
\]

**Proof** From Lemma 4.1 and (17), we have
\[
|\Psi_{jk}| \leq 2^{k-j} \prod_{l=j}^{k} \sqrt{a^2 v_1(l)} \leq 2^{k-j} (0.1)^{k-j-3} a^4 \sqrt{v_1(k)} v_1(k-1) v_1(k-2) v_1(k-3)
\]
and the estimates for \( \Psi_{jk} \) follow. Similarly,
\[
|\Psi'_{jk}| \leq \sum_{i=j}^{k} \sqrt{E(Y_j, \ldots, Y_i', \ldots, Y_k)} \leq \sum_{i=j}^{k} 2^{k-j} \sqrt{E|Y'_i|^2} \prod_{l \neq i}^{k} \sqrt{E|Y_l|^2}
\]
\[
\leq 2^{k-j-1}(k-j+1) \prod_{i=j}^{k} \sqrt{a^2 v_1(i)}
\]
and hence, the remaining two estimates follow. \( \square \)

**Lemma 4.5** Let \( a^2 v_1^{(n)} < 0.01 \) and \( s \geq 1 \) be an integer. Then, for \( k = 1, 2, \ldots, n \) and \( t \in \mathbb{R} \),
\[
|\varphi_k(u) - 1| \leq \frac{a^2}{6} [10v_1(k-1) + 13v_1(k)],
\]
\[
|\varphi_k(u) - 1| \leq \frac{1}{25}, \quad \frac{1}{|\varphi_k(u)|} \leq \frac{10}{9}, \quad \varphi_k(u) = 1 + \sum_{m=1}^{s} \frac{v_m(k)}{m!} (e^u - 1)^m + \theta \left\{ \frac{e^{h C_0} (e^h + 1)^{s+1} v_{s+1}(k)}{(s+1)!} \right\}
\]
\[
+(3.53)a^4 \sum_{l=0}^{3} v_1^2(k-l) + (1.8)a^3 [v_1^2(k-1) + v_1^2(k)]
\]
\[
+(1.8)a^3 |\text{Cov}(X_{k-1}, X_k)|, \quad \varphi'_k(u) \leq 2a^2[v_1(k-1) + v_1(k)], \quad |\varphi'_k(u)| \leq 0.04, \quad (23)
\]

\( \square \)
\[ \varphi'_k(u) = i \sum_{m=1}^{s} \frac{v_m(k)}{(m-1)!} (e^u - 1)^{m-1} e^u + \theta \left\{ \frac{e^{hC_0} (e^h + 1)^{s+1}(k)}{s!} \right\} + (8.2)a^4 \sum_{l=0}^{3} v_1^2(k-l) + 2.6a^3 [v_1^2(k-1) + v_1^2(k)] \]
\[ + (2.6)a^3 |\text{Cov}(X_{k-1}, X_k)|. \] (26)

**Proof** Further on, we assume that \( k \geq 4 \). For smaller values of \( k \), all the proofs indeed become shorter. For brevity, we omit the argument \( u \), whenever possible. First note that for all \( t \in \mathbb{R}, u \in K \). Indeed,
\[ w(u) = \max_j \sqrt{E|Y_j|^2} \leq \max_j \sqrt{a^2v_1(j)} \leq \frac{1}{10}. \]

Consequently, by (16) and (17)
\[ |\varphi_k - 1| \leq E|Y_k| + \frac{2 \left( E|Y_{k-1}|^2 E|Y_k|^2 \right)^{1/2}}{1 - 4w(u)} \]
\[ \leq av_1(k) + \frac{10}{3} (a^4v_1(k-1)v_1(k))^{1/2} \]
\[ \leq \frac{a^2v_1(k)}{2} + \frac{5a^2}{3} [v_1(k-1) + v_1(k)] \]
\[ = \frac{a^2}{6} [10v_1(k-1) + 13v_1(k)]. \]
Using the assumption and noting that \( 1/|\varphi_k| \leq 1/(1 - |\varphi_k - 1|) \), we obtain (23). By (15)
\[ \varphi_k = 1 + EY_k + \frac{\Psi_{k-1,k}}{\varphi_{k-1}} + \frac{\Psi_{k-2,k}}{\varphi_{k-2}\varphi_{k-1}} + \sum_{j=1}^{k-3} \frac{\Psi_{j,k}}{\varphi_j \cdots \varphi_{k-1}}. \] (27)

Using Lemma 4.4, it follows that
\[ \sum_{j=1}^{k-3} \frac{|\Psi_{j,k}|}{\varphi_j \cdots \varphi_{k-1}} \leq 250a^4 \sum_{l=0}^{3} v_1^2(k-l) \sum_{j=1}^{k-3} \left( \frac{10}{9} \right)^{k-j} \left( \frac{1}{5} \right)^{k-j} \]
\[ \leq (3.53)a^4 \sum_{l=0}^{3} v_1^2(k-l). \] (28)
Similarly, we have from (17)

\[
|\hat{E}(Y_{k-1}, Y_k)| \leq E|Y_{k-1}, Y_k| + E|Y_{k-1}|E|Y_k| \leq a^2E X_{k-1}X_k + a^2v_1(k-1)v_1(k) = a^2\text{Cov}(X_{k-1}, X_k) + a^22v_1(k-1)v_1(k) \leq a^2[\text{Cov}(X_{k-1}, X_k)] + a^2[v_1^2(k-1) + v_1^2(k)].
\]

Due to the trivial estimate \( a \geq 2 \),

\[
\frac{|\Psi_{k-1,k}|}{|\varphi_{k-1}|} \leq \frac{5a^3}{9}\left(|\text{Cov}(X_{k-1}, X_k)| + [v_1^2(k-1) + v_1^2(k)]\right). \tag{29}
\]

By assumption, \( v_1^{(n)} \leq 1/400 \) and

\[
|\hat{E}(Y_{k-2}, Y_{k-1}, Y_k)| \leq \left[e^{hC_0}(2 + h)^3\right] \left(E(X_{k-2}X_{k-1}X_k) + v_1(k-2)E(X_{k-1}X_k)
\right.

\[
+ E(X_{k-2}X_{k-1})v_1(k) + v_1(k-2)v_1(k-1)v_1(k))
\]

\[
\leq \left[e^{hC_0}(2 + h)^3\right] \left(E(X_{k-1}X_k)(C_0 + 1/400)
\right.

\[
+ v_1(k-1)v_1(k)(C_0 + 1/400)\right)
\]

\[
\leq \frac{401a^3}{400} (E(X_{k-1}X_k) + v_1(k-1)v_1(k)) \leq \frac{401a^3}{400}(|\text{Cov}(X_{k-1}, X_k)| + [v_1^2(k-1) + v_1^2(k)]).
\]

Therefore,

\[
\frac{|\Psi_{k-2,k}|}{|\varphi_{k-2}\varphi_{k-1}|} \leq \left(\frac{10}{9}\right)^2 \frac{401a^3}{400} \left(|\text{Cov}(X_{k-1}, X_k)| + [v_1^2(k-1) + v_1^2(k)]\right). \tag{30}
\]

The proof of (24) now follows by combining the last estimate with (28), (29), (27) and (19). We prove (25) by induction. We have

\[
\varphi'_k = EY_k' + \sum_{j=1}^{k-1} \frac{\Psi'_{jk}}{\varphi_{j} \cdots \varphi_{k}} - \sum_{j=1}^{k-1} \frac{\Psi_{jk}}{\varphi_{j} \cdots \varphi_{k}} \sum_{m=j}^{k-1} \frac{\varphi'_{m}}{\varphi_{m}}.
\]

Applying Lemma 4.4 and using (17) and (18), we then get

\[
|\varphi'_k| \leq \frac{a^2}{4}v_1(k) + \frac{5}{2}a^2[v_1(k-1) + v_1(k)]\sum_{j=1}^{k-1} \left(\frac{2}{9}\right)^{k-j} (k-j+1)
\]

\[
+ 5a^2[v_1(k-1) + v_1(k)]\frac{10}{9}0.04 \sum_{j=1}^{k-1} (k-j)\left(\frac{2}{9}\right)^{k-j}
\]

\( \odot \) Springer
\[ \leq a^2[v_1(k - 1) + v_1(k)]\left(\frac{1}{4} + \frac{80}{49} + \frac{4}{49}\right) \]
\[ \leq 2a^2[v_1(k - 1) + v_1(k)]. \]

The proof of (26) is similar to the proof of (24). We have
\[ |\varphi'_k - \mathbb{E}Y'_k| \leq \sum_{j=1}^{k-3} \left(\frac{10}{9}\right)^{k-j} |\Psi'_{jk}| + \sum_{j=1}^{k-3} \left(\frac{10}{9}\right)^{k-j} |\Psi_{jk}|(k - j)\left(\frac{2}{45}\right) \]
\[ + \sum_{j=k-2}^{k-1} \left(\frac{10}{9}\right)^{k-j} |\Psi'_{jk}| + \sum_{j=k-2}^{k-1} \frac{|\Psi_{jk}|}{|\varphi_j \cdots \varphi_{k-1}|}(k - j)\left(\frac{2}{45}\right). \]
(31)

Applying Lemma 4.4, we prove that
\[ \sum_{j=1}^{k-3} \left(\frac{10}{9}\right)^{k-j} |\Psi'_{jk}| + \sum_{j=1}^{k-3} \left(\frac{10}{9}\right)^{k-j} |\Psi_{jk}|(k - j)\left(\frac{2}{45}\right) \leq (8.2)a^4 \sum_{l=0}^{3} v^2_l(k - l). \]
(32)

From (29) and (30), it follows that
\[ \sum_{j=k-2}^{k-1} \frac{|\Psi_{jk}|}{|\varphi_j \cdots \varphi_{k-1}|}(k - j)\left(\frac{2}{45}\right) \leq (0.135)a^3(|\text{Cov}(X_{k-1}, X_k)| + v^2_1(k - 1) + v^2_1(k)). \]
(33)

Taking into account (17) and (18), we obtain
\[ |\hat{E}(Y'_{k-1}, Y_k)| \leq \mathbb{E}|Y'_{k-1}Y_k| + \mathbb{E}|Y'_{k-1}|\mathbb{E}|Y_k| \]
\[ \leq e^{2hC_0}(2 + h)\mathbb{E}(X_{k-1}X_k) + \frac{a^2}{2} v_1(k - 1)v_1(k) \]
and
\[ \frac{10|\Psi'_{k-1,k}|}{9} \leq \frac{10a^2}{9} \left(|\text{Cov}(X_{k-1}, X_k)| + [v_1(k - 1) + v_1(k)]\right). \]
(34)

Similarly,
\[ \left(\frac{10}{9}\right)^2 |\Psi'_{k-2,k}| \leq (1.86)a^3(|\text{Cov}(X_{k-1}, X_k)| + [v_1(k - 1) + v_1(k)]). \]

Combining the last estimate with (31)–(34) and (20), we complete the proof of (26). \[\square\]
Let now
\[ A(u) = \sum_{k=1}^{n} \ln \varphi_k(u) = \sum_{k=1}^{n} \sum_{j=1}^{\infty} \frac{(-1)^{j+1}(\varphi_k(u) - 1)^j}{j}. \] (35)

**Lemma 4.6** Let \( a^2 v_1^{(n)} \leq 1/100 \). Then for all \( t \in \mathbb{R} \),

\[ |A| \leq 4a^2 \Gamma_1, \quad |A'| \leq 4a^2 \Gamma_1, \] (36)

\[ A = \sum_{m=1}^{s} \frac{(e^u - 1)^m}{m!} \sum_{k=1}^{n} v_m(k) + \theta \left\{ \frac{e^{hC_O}(e^h + 1)^s+1}{(s+1)!} \sum_{k=1}^{n} v_{s+1}(k) \right\} + 24a^4 \sum_{k=1}^{n} v_1^2(k) + (1.8)a^3 \sum_{k=2}^{n} |\text{Cov}(X_{k-1}, X_k)| \right\}, \] (37)

\[ A' = i \sum_{m=1}^{s} \frac{v_m(k)}{(m-1)!} (e^u - 1)^{m-1} e^u + \theta \left\{ \frac{e^{hC_O}(e^h + 1)^s}{s!} \sum_{k=1}^{n} v_{s+1}(k) \right\} + (51.4)a^4 \sum_{k=1}^{n} v_1^2(k) + (2.6)a^3 \sum_{k=2}^{n} |\text{Cov}(X_{k-1}, X_k)| \right\}. \] (38)

**Proof** Using the first estimate in (23), we have \(|\varphi_k - 1| \leq 0.04\). Therefore,

\[ |A| \leq \sum_{k=1}^{n} |\varphi_k - 1| \sum_{j=1}^{\infty} (0.04)^{j-1} \leq \left( \frac{1}{0.96} \right) \sum_{k=1}^{n} a^2 [10v_1(k-1) + 13v_1(k)] \leq 4a^2 \Gamma_1. \]

Similarly,

\[ |A'| \leq \sum_{k=1}^{n} |\varphi'_k| \|\varphi_k\| \leq \frac{10}{9} \sum_{k=1}^{n} |\varphi_k| \leq \frac{20a^2}{9} \sum_{k=1}^{n} [v_1(k-1) + v_1(k)] \leq 4a^2 \Gamma_1. \]

From Lemma 4.5, it follows

\[ |\varphi_k - 1|^2 \leq \frac{a^4}{36} \left( 10v_1(k-1) + 13v_1(k) \right)^2 \leq \frac{a^4}{36} \left( 230v_1^2(k-1) + 299v_1^2(k) \right) \]

and

\[ \sum_{k=1}^{n} \sum_{j=2}^{\infty} |\varphi_k - 1|^{j-2} j \leq \frac{1}{2} \sum_{k=1}^{n} |\varphi_k - 1|^2 \sum_{j=2}^{\infty} (0.04)^{j-2} \leq (7.66) a^4 \sum_{k=1}^{n} v_1^2(k). \]
Consequently,
\[ A = \sum_{k=1}^{n} (\varphi_k - 1) + (7.66)\theta a^4 \sum_{k=1}^{n} v_1^2(k) \]
and (37) follows from Lemma 4.5 and the rough estimate \( a^3 \leq a^4/2 \), since \( a \geq 2 \).

For the proof of (38), note that
\[ A' = \sum_{k=1}^{n} \varphi'_k + \sum_{k=1}^{n} \frac{\varphi'_k}{\varphi_k} (1 - \varphi_k) \]
and applying a slightly sharper estimate than in Lemma 4.5, namely \( 1/|\varphi| \leq 25/24 \), we obtain
\[ \sum_{k=1}^{n} \left| \frac{\varphi'_k}{\varphi_k} \right| |1 - \varphi_k| \leq \frac{25a^4}{72} \sum_{k=1}^{n} [v_1(k - 1) + v_1(k)][10v_1(k - 1) + 13v_1(k)] \leq 16a^4 \sum_{k=1}^{n} v_1^2(k). \]

Now, it remains to apply (26) to complete the proof. \( \square \)

**Lemma 4.7** Let \( M \) be a finite variation measure concentrated on integers and \( \sum_{k} |k| |M(k)| < \infty \). Then,
\[ \sum_{k=-\infty}^{\infty} |M(k)| \leq \left( 1 + \frac{1}{2\pi} \right)^{1/2} \left( \int_{-\pi}^{\pi} |\hat{M}(ir)|^2 + |\hat{M}'(ir)|^2 dt \right)^{1/2}. \]

Lemma 4.7 is a special case of the Lemma from Presman [14].

In the following lemma, we assume that \( M(k) = M([0, k]) = \sum_{j=0}^{k} M\{j\} \).

**Lemma 4.8** Let \( g(k) > 0 \) be an increasing function and \( M \) be a measure, both defined on nonnegative integers. Assume \( \sum_{k=0}^{\infty} M\{k\} = 0 \) and \( \sum_{k=0}^{\infty} kg(k)|M(k)| < \infty \). Then,
\[ \sum_{k=0}^{\infty} g(k)|M(k)| \leq \sum_{k=0}^{\infty} kg(k)|M(k)|. \]

If \( g(k) = e^{hk} \), for some \( h > 0 \), then
\[ \sum_{k=0}^{\infty} e^{hk}|M(k)| \leq \frac{1}{e^h - 1} \sum_{k=0}^{\infty} e^{hk}|M(k)|. \]
Proof We have
\[
\sum_{k=0}^{\infty} g(k)|M(k)| = \sum_{k=0}^{\infty} g(k) \left| \sum_{j=0}^{k} M[j] \right| = \sum_{k=0}^{\infty} g(k) \left| \sum_{j=k+1}^{\infty} M[j] \right|
\leq \sum_{k=0}^{\infty} g(k) \sum_{j=k+1}^{\infty} |M[j]| = \sum_{j=1}^{\infty} |M[j]| \sum_{k=0}^{j-1} g(k) \leq \sum_{j=0}^{\infty} jg(j)|M[j]|.
\]
When \(g(k) = e^{hk}, \ h > 0\), we simply note that
\[
\sum_{k=0}^{j-1} e^{hk} = \frac{e^{hj} - 1}{e^h - 1} < \frac{e^{hj}}{e^h - 1}.
\]

\[\square\]

Proof of Theorem 2.2 Let \(M\) defined by
\[
M[k] = e^{hk}(F_n[k] - CP(s, \lambda)[k])
\]
be a (signed) measure. Then,
\[
\hat{M}(it) = \sum_{k=0}^{\infty} (F_n[k] - CP(s, \lambda)[k])e^{hk+itk} = \hat{F}_n(it + h) - \hat{CP}(s, \lambda)(it + h).
\]
Applying Lemma 4.7, we obtain
\[
\sum_{k=0}^{\infty} e^{hk}|F_n[k] - CP(s, \lambda)[k]| \leq \sqrt{\pi + 1} \left( \sup_{|r|\leq\pi} |\hat{M}(ir)| + \sup_{|r|\leq\pi} |\hat{M}'(ir)| \right). \tag{39}
\]
Let \(A\) be defined as in (35) and
\[
B = \sum_{j=1}^{s} \lambda_j(e^{ju} - 1) = \sum_{m=1}^{s}(e^u - 1)^m \sum_{j=m}^{s} \binom{j}{m} \lambda_j.
\]
Then, \(|B| \leq \sum_{j=1}^{s} \lambda_j(e^{jh} + 1)\). Also, \(\hat{M}(it) = e^A - e^B\) and
\[
|\hat{M}(it)| \leq |e^A - e^B| \leq \psi |A - B|, \tag{40}
|\hat{M}'(it)| \leq |A'||e^A - e^B| + |e^B||A' - B'| \leq \psi |A'||A - B| + \psi |A' - B'|. \tag{41}
\]
where \( \psi \) is defined in (11). The inequality \( |e^A - e^B| \leq \psi |A - B| \) follows from the fact that if the real part of a complex number \( \text{Re}z \leq 0 \), then

\[
|e^z - 1| \leq \left| \int_0^1 ze^{\tau z} d\tau \right| \leq |z| \int_0^1 \exp\{\tau \text{Re}z\} d\tau \leq |z|.
\]

Indeed, if \( \text{Re}(A - B) < 0 \), then

\[
|e^A - e^B| = |e^B||e^{A-B} - 1| \leq |e^B||A - B| \leq \psi |A - B|.
\]

If \( \text{Re}(B - A) \leq 0 \), then

\[
|e^A - e^B| = |e^A||1 - e^{B-A}| \leq |e^A||A - B| \leq \psi |A - B|.
\]

The proof is now completed by combining (40), (41) with (39) and using Lemma 4.6.

\[\square\]

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