Higher-Order Topological Phases on Quantum Fractals

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Electronic materials harbor a plethora of exotic quantum phases, ranging from unconventional superconductors to non-Fermi liquids, and more recently topological phases of matter. While these quantum phases in integer dimensions are well characterized by now, their presence in fractional dimensions remain vastly unexplored. Here we theoretically show that a special class of crystalline, namely higher-order topological phases that via an extended bulk-boundary correspondence feature robust gapless modes on lower dimensional boundaries, such as corners and hinges, can be found on a representative family of fractional materials, quantum fractals. To anchor this general proposal, we demonstrate realizations of second-order topological insulators and superconductors, respectively supporting charged and neutral Majorana corner modes, on planar Sierpenski carpet and triangle fractals. These predictions can be experimentally tested on designer electronic fractal materials, as well as on various highly tunable metamaterial platforms, such as photonic and acoustic lattices.

Introduction. Crystals are ubiquitous in nature, manifesting discrete reflection, rotational, and translational symmetries. On the other hand, quasicrystals and fractals are paradigmatic examples of noncrystalline materials. While the former are projections of higher-dimensional crystals on lower-dimensional branes, realized by completely tiling the physical space in an aperiodic fashion, thereby exhibiting local discrete, often crystal forbidden, rotational symmetries [1–3], fractals by contrast display a fourth type of symmetry, self-similarity, resulting in pattern repetition over many scales [4]. Fractals appear at macroscale (coastline and trees), as well as at microscales, with recently engineered electronic Sierpenski triangle in designer materials opening a paradigm of quantum fractals [5]. Despite being embedded in integer d-dimensional space, fractals are characterized by Hausdorff or fractal dimension (dfrac) with fractional dfrac < d. Therefore, when combined with geometry and topology of electronic wavefunction, quantum fractals give rise to a rich, still vastly unexplored, landscape of topology in fractional dimensions [5–13].

Here we explore this territory by focusing on a newly emerged family of crystalline, higher-order topological (HOT) phases, and show realizations of both HOT insulators and superconductors on Sierpinski carpet and glued Sierpinski triangle fractals [Figs. 1 and 3]. In general, HOT phases via an extended bulk-boundary correspondence host robust topological modes on lower-dimensional boundaries, such as corners and hinges, characterized by respective codimensions d̂ = d and d − 1 [14–49]. As such, a HOT phase of order n can be constructed from its conventional first-order counterpart by systematically introducing n number of suitable discrete symmetry breaking Wilson-Dirac masses that partially gap out the edge or surface states, for example, with d̂ = 1, leaving the modes residing on boundaries with d̂ = n gapless [18–25]. We show that this principle is operative on fractal lattices as well. In particular, when the global shape of these two fractals are tailored in such a way that four corners reside along the inversion axes of the second-order Wilson-Dirac mass, both HOT insulators and superconductors support robust corner modes [Figs. 1 and 3]. Moreover, the HOT insulators possess quantized quadrupole moment Qxy = 0.5, which becomes origin independent in the thermodynamic limit, indicating their intrinsic nature [Fig. 2]. By contrast, Qxy in HOT superconductors exhibit a significant origin dependence, and are thus possibly extrinsic in nature.

Model. To outline the general protocol of engineering HOT phases, here we consider its paradigmatic example on a square lattice, captured by the Hamiltonian operator \( \hat{h} = \hat{h}_1 + \hat{h}_2 \), where

\[
\begin{align*}
\hat{h}_1 &= t \left[ \sin(k_x a) \sigma_3 \tau_1 + \sin(k_y a) \sigma_0 \tau_2 \right] + M(k) \sigma_0 \tau_3, \\
\hat{h}_2 &= g \left[ \cos(k_x a) - \cos(k_y a) \right] \sigma_1 \tau_1.
\end{align*}
\]
The uniform first-order Wilson-Dirac mass
\[
M(k) = m_0 + 2t_0 - t_0 [\cos(k_x a) + \cos(k_y a)]
\] (2)
preserves all discrete symmetries. Two sets of the Pauli matrices \(\{\sigma_\mu\}\) and \(\{\tau_\mu\}\) respectively operate on the spin and orbital indices, with \(\mu = 0, \cdots, 3\). Hereafter we set the lattice spacing \(a = 1\). Only in the parameter regime \(-2 < m_0/t_0 < 4\), \(\hat{h}_1\) features two counter-propagating one dimensional edge modes with \(d_e = 1\) for opposite spin projections, thereby yielding a first-order quantum spin Hall insulator. Otherwise, the system is a trivial or normal insulator, devoid of any topological edge states [70].

The second-order Wilson-Dirac mass \(\hat{h}_2\) anticommutes with \(\hat{h}_1\). It thus acts as a mass to one-dimensional counter propagating edge modes of \(\hat{h}_1\) by causing hybridization between them. Naturally, \(\hat{h}_2\) gaps out the edge modes, however, only partially as it assumes the profile of a domain-wall mass flipping sign four times under \(2\pi\) rotation and vanishing along the diagonal (11) directions. As a result, when the corners of a square lattice reside along its diagonals, four corner modes with \(d_e = 2\) get pinned therein, following the spirit of generalized Jackiw-Rebbi mechanism [51]. We then realize a second-order topological insulator. These modes appear at zero energy due to both unitary and antiunitary particle-hole symmetries of \(\hat{h}\), respectively generated by \(C = \sigma_2 \tau_1\) and \(\Theta = \sigma_3 \tau_3 K\), where \(K\) is the complex conjugation, as \([\hat{h}, C] = [\hat{h}, \Theta] = 0\) [77].

Fractal HOT insulators. This mechanism is not restricted to the square lattice. If we maintain the symmetry of the model and cleave the system such that four corners are placed along the inversion axes of the HOT Wilson-Dirac mass, it can support corner localized zero-energy modes. To extend the jurisdiction of this model beyond the realm of topological crystals, we consider a real space version of \(\hat{h}\), given by \(H = H_1 + H_2\), with
\[
H_1 = \sum_{j \neq k} \frac{F(r_{jk})}{2} c_j^\dagger \left[ -it(\sigma_3 \tau_1 \cos \phi_{jk} + \sigma_0 \tau_2 \sin \phi_{jk}) - t_0 \sigma_0 \tau_3 \right] c_k + \sum_{j} c_j^\dagger (m_0 + 2t_0) \sigma_0 \tau_3 c_j,
\]
\[
H_2 = g \sum_{j \neq k} \frac{F(r_{jk})}{2} c_j^\dagger (\cos 2\phi_{jk}) \sigma_1 \tau_1 c_k,
\]
and \(c_j = [c_{j\alpha}\tau_\sigma, c_{j\beta}\tau_\sigma, c_{j\alpha\tau}, c_{j\beta\tau}]^T\). Here \(c_{j\sigma\tau}\) is the electron annihilation operator at site \(j\) with spin projection \(\sigma = \uparrow, \downarrow\) and on orbital \(\tau = \alpha, \beta\). The azimuthal angle between the \(j\)th and \(k\)th lattice sites, respectively located at \(r_j\) and \(r_k\), is \(\phi_{jk}\), measured with respect to the horizontal direction. In order to ensure that the sites in any noncrystalline lattice remain well connected we replace the nearest-neighbor hopping probabilities by a long range one, described by the rotationally invariant function
\[
F(r_{jk}) = \exp \left(1 - \frac{|r_j - r_k|}{r_0}\right),
\] (4)
Here \(r_0\) is the decay length, typically set to be the nearest-neighbor distance. In principle, this generalized model for HOT insulator can be implemented on any noncrystalline systems, such as fractals, amorphous materials [52] and quasicrystals [19, 23, 24], as well as on a regular square lattice. Here we focus on the former most system and scrutinize the possibility of realizing HOT insulators with corner modes on quantum fractals.

Results obtained on Sierpinski carpet fractal with \(d_{soc} = \ln(8)/\ln(3) \approx 1.89\) are shown in Fig. 1 depicting four near (due to finite system size) zero energy modes, which are well separated from the rest of the spectra. The spatial distribution of the corresponding local density of states (LDOS) shows that these modes are highly localized at four outer corners, while the inner corners are devoid of any such mode, in contrast to Ref. [8]. This observation strongly suggest a possible realization of electronic HOT insulator on Sierpinski carpet fractal.

To anchor this claim, we compute the quadrupole mo-
ment \((Q_{xy})\) for the fractal HOT insulators \([52, 54]\). To proceed, we first evaluate

\[
 n = \text{Re} \left[ -\frac{i}{2\pi} \text{Tr} \left( \ln \left\{ U^\dagger \exp \left[ 2\pi i \sum_r \hat{q}_{xy}(r) \right] U \right\} \right) \right],
\]

(5)

where \(\hat{q}_{xy}(r) = x y \hat{n}(r)/L^2\), \(\hat{n}(r)\) is the number operator at \(r = (x, y)\) of an open boundary system of linear dimension \(L\) in each direction, and \(U\) is constructed by column-wise arranging the eigenvectors for the negative energy filled states. The quadrupole moment is then defined as \(Q_{xy} = n - n_{al}\) (modulo 1), where \(n_{al} = (1/2) \sum_r x y / L^2\) represents \(n\) in the atomic limit and at half filling. The results are displayed in Fig. 2(a). We compute \(Q_{xy}\) for all origin choices. When the HOT insulator supports corner modes, for most of the origin choices \(Q_{xy}\) is quantized to 0.5 within the numerical accuracy. But, in any finite system there always exist a few origin choices for which \(Q_{xy} = 0\), despite the presence of the corner modes. Such an origin dependence can be quantified by \(F_r\), measuring the fraction of all origin choices for which corner modes corroborate quantized \(Q_{xy} = 0.5\). As the generation number \(f\) or number of lattice sites \(N\) is increased, \(F_r \to 1\) in the thermodynamic limit, corresponding to \(f \to \infty\) or \(N \to \infty\) [Fig. 2(b)]. Therefore, in the thermodynamic limit \(Q_{xy}\) becomes origin independent and stands as a bona fide order parameter for HOT insulators on quantum fractals. Thus, the HOT insulator on Sierpinski carpet fractal is intrinsic in nature.

Ultimate origin independence of \(Q_{xy}\) allows us to construct a global phase diagram in the \((n_{al}, |g|)\) plane [Fig. 2(c)]. It supports two topologically distinct phases: (a) fractal HOT insulator with \(Q_{xy} = 0.5\) and (b) trivial insulator with \(Q_{xy} = 0\). Small and moderate \((|g|)\) values of \(n_{al}\) and \(|g|\) are conducive to HOT (trivial) insulator. Only, but the entire fractal HOT insulator phase supports four zero energy corner modes. The stability of the fractal HOT insulator can be established from the scaling of the gap between corner modes with the closest finite energy states (not corner localized), respectively shown in red and blue in Fig. 2(a)[inset], with the generation and site numbers. This gap remains finite as we approach the thermodynamic limit [Fig. 2(d)], in turn assuring that corner modes are separated by a finite gap, thereby yielding stability to the fractal HOT insulator.

Next we investigate the possibility of realizing HOT insulators on a glued Sierpinski triangle fractal. In order to obtain four outer corners along the inversion axes of the second-order Wilson-Dirac mass, we glue two Sierpinski triangle fractals, each being a right angled triangle, slightly different from its known geometry [41]. Consequently, the corresponding fractal dimension is \(d_{frac} = \ln(6)/\ln(\sqrt{3}) \approx 1.72\). Numerical diagonalizations reveal that the number of zero energy modes can depend on the generation number [Fig. 3(a)]. In the sixth generation there are altogether sixteen such modes [Fig. 3(b)], well separated from the other nearby states [Fig. 3(d)].

The LDOS of zero energy modes predominantly occupy four outer corners in a system with open boundaries [Fig. 3(c)], qualitatively similar to the situation in Sierpinski carpet fractal. But, in contrast, the LDOS also displays subdominant localization at the inner shared naked corners, that are devoid of other neighboring sites. The manifold of the zero energy modes, however, does not fragment between the outer and inner naked corners. In a periodic system, the number of zero energy modes remains unchanged and the corresponding LDOS appears only at the inner corners [Fig. 3(d)]. Additionally, the LDOS weakly spreads over the inner edges making \(\pi/4\) angle with the horizon, since the Wilson-Dirac mass vanishes in that direction [Fig. 3(c),(d)].

The HOT insulators with outer and naked inner corner modes on glued Sierpinski triangle fractals possess quantized \(Q_{xy} = 0.5\), which slowly becomes origin independent as we approach the thermodynamic limit [Fig. 2(b)]. The slowness of \(F_r \to 1\) possibly stems from the inner edges at \(\pi/4\) angle, which always absorb a tiny fraction of the LDOS associated with the zero energy modes. The global phase diagram of this system in the \((n_{al}, |g|)\) plane is qualitatively similar to the one from Fig. 2(c) [45].

Fractal HOT superconductors. Continuing the journey through the territory of HOT phases on quantum fractals, next we search for HOT superconductors on Sierpinski carpet and glued Sierpinski triangle fractals. In principle, with suitable choices of Hermitian matrices and the corresponding spinor, which includes both electron and hole like components (Nambu doubling), \(\hat{h}\) can also describe a second-order topological superconductor [Eq. 1]. Namely, the quantity appearing with \(t\) describes a \(p\)-wave pairing, the term proportional to \(g\) re-
resents a $d_x^2-y^2$ pairing, and $M(k)$ gives rise to a Fermi surface when $-2 < m_0/t_0 < 4$. The resulting $p+id$ pairing is a prominent candidate for HOT superconductors that supports four corner localized Majorana zero modes [61][61]. Naively it is, therefore, tempting to conclude that quantum fractals harbor HOT superconductors based on the results shown in Figs. 4[4][4] which, however, encounters a few fundamental shortcomings.

Primarily, the Hamiltonian $\hat{h}$ does not reveal any microscopic origin of the $p+id$ pairing nor it unveils any potential material platform where such pairing can be realized. Even more importantly, when we extend $\hat{h}$ to a real space hopping Hamiltonian [Eq. (3)], the pairing terms (proportional to $t$ and $g$) become infinitely long-ranged connecting all the sites with decaying amplitude of the Cooper pairs [Eq. (1)], which is unphysical. And finally, the notion of a Fermi surface in the absence of an underlying translational symmetry, as in fractals, becomes moot. To circumvent these limitations we search for a suitable material platform where on site or local pairings can give rise to HOT superconductors, which do not strictly rely on a sharp Fermi surface. A class of systems that satisfies all these realistic requisite features is second-order Dirac insulator, whose normal state is described by the Hamiltonian $\hat{h}$ [Eq. (1)]. To accommodate superconducting orders in this system, we Nambu double the spinor. The Hamiltonian then reads as $\hat{h}_{\text{Nam}} = \eta_1 \hat{h}_1 + \eta_0 \hat{h}_2$. The newly introduced Pauli matrices $\{\eta_\mu\}$ with $\mu = 0, \cdots, 3$ operate on the particle-hole index. Here we focus only on the local or on site pairings which are oblivious to the underlying lattice structure, and thus possess natural immunity against the lack of crystalline order. Due to the Pauli exclusion principle, the number of such pairings is restricted to be six, which is exactly the number of purely imaginary four-dimensional Hermitian matrices.

The local second-order topological superconductor can be unambiguously identified from its requisite symmetries. For example, it must anticommute with the Dirac kinetic energy, captured by the terms proportional to $t$ in $\hat{h}_{\text{Nam}}$, such that the pairing represents a topological Nambu-Dirac mass. In addition, it must commute with the first-order Wilson-Dirac mass, so that the boundary modes of this pairing are not uniformly gapped. Finally, it must anticommute with the second-order Wilson-Dirac mass such that the Majorana edge modes are gapped, but only partially, producing localized zero energy Majorana modes at four corners, when they reside along the (11) directions. These constraints select a unique candidate for the second-order topological superconductor, for which the effective single particle Bogoliubov de-Gennes Hamiltonian reads

$$\hat{h}_{\text{pair}} = \Delta (\eta_1 \cos \phi + \eta_2 \sin \phi) \sigma_1 \tau_2.$$  \hspace{1cm} (6)

Here $\Delta$ is the pairing amplitude and $\phi$ is the U(1) superconducting phase. The Nambu Hamiltonian $\hat{h}_{\text{Nam}}^\text{total} = \hat{h}_{\text{Nam}} + \hat{h}_{\text{pair}}$ can be implemented on any fractal lattice following Eq. (5). Without loss of generality, we set $\phi = 0$.

The resulting energy spectra and LDOS corresponding to the near zero energy modes are qualitatively similar to the ones shown in Figs. 1 and 3 on the Sierpinski carpet and glued Sierpinski triangle fractals, respectively [55]. These observations confirm realization of HOT superconductors on quantum fractals. Furthermore, to attribute the resulting corner modes solely to the paired state, we choose the normal state to be topologically trivial. However, the quadrupole moment associated with a second-order topological superconductor is found to be $Q_{xy} = 0.5$ for a very few origin choices and there is no clear indication of $F_x \to 1$ in the thermodynamic limit, due to strong interband scattering. Therefore, in likelihood the fractal HOT superconductors, in contrast to their insulating counterparts, are extrinsic in nature. Still the spectral gap between (near) zero energy corner modes and other closest to zero energy (not corner localized) states approaches a finite value in the thermodynamic limit [Fig. 2(b)]. So, extrinsic fractal HOT superconductors and their hallmark corner modes are stable. These outcomes remain qualitatively unaltered even when the normal state is a fractal HOT insulator.

Summary and discussions. Here we construct a concrete path to theoretically harness HOT phases on a family of fractional materials, quantum fractals, and demonstrate their realizations on Sierpinski carpet and glued Sierpinski triangle fractals. While the HOT insulators are intrinsic in nature, their superconducting cousins are possibly extrinsic. Nonetheless, the HOT paired state in a second-order Dirac insulator is energetically most favored among all symmetry allowed local pairings over a wide parameter range [55]. This procedure can be generalized to identify HOT phases on fractals with different geometries, as well as on higher-dimensional fractals [1]. Furthermore, by stacking planar HOT fractals in the out of plane direction one can also construct HOT semimetals in a hybrid dimension. These exciting possibilities, inhabiting the landscape of topological quantum fractals, will be systematically explored in the future following our general principle of construction.

Electronic fractal materials, such as the ones recently engineered in designer [5] and molecular [8] compounds, constitute the ideal platform where our proposed fractal HOT insulators and superconductors can be realized in experiments. In these quantum fractals, while the insulating HOT phases can be unveiled by designing appropriate hopping elements, their pairing counterparts should become energetically favored upon chemical doping. Our predicted fractal HOT insulators can also be tailored on various classical metamaterials, such as photonic [50] and phononic or acoustic [57] [55] lattices, with longer range coupling between the photonic waveguides and microwave resonators, respectively. Topolectric circuits constitute yet another promising platform where our predictions can be tested [69] [68], especially given that quasicrystalline quadrupole insulators have already been realized therein [61]. For practical purposes, it should be noted that it is not necessary for the hopping
amplitudes to be sufficiently long ranged [Eq. (4)]. As long as all the sites on fractal lattices stay connected, all our findings remain qualitatively unchanged.

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