AFFINE TYPE A CRYSTAL STRUCTURE ON TENSOR PRODUCTS OF RECTANGLES, DEMAZURE CHARACTERS, AND NILPOTENT VARIETIES

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Abstract. Answering a question of Kuniba, Misra, Okado, Takagi, and Uchiyama, it is shown that certain Demazure characters of affine type A, coincide with the graded characters of coordinate rings of closures of conjugacy classes of nilpotent matrices.

1. Introduction

In [9, Theorem 5.2] it was shown that the characters of certain level one Demazure modules of type $A_n^{(1)}$, when decomposed as linear combinations of irreducible characters of type $A_{n-1}$, have coefficients given by Kostka-Foulkes polynomials in the variable $q = e^{-\delta}$ where $\delta$ is the null root. The key steps in the proof are that

1. The Demazure crystals are isomorphic to tensor products of classical $\hat{sl}_n$ crystals indexed by fundamental weights [10].
2. The generating function over these crystals by weight and energy function is equal to the generating function over column-strict Young tableaux by weight and charge [22].
3. The Kostka-Foulkes polynomial is the coefficient of an irreducible $sl_n$-character in the tableau generating function [1] [17].

The main result of this paper is that for a Demazure module of arbitrary level whose lowest weight is a multiple of one of those in [9], the corresponding coefficient polynomial is the Poincaré polynomial of an isotypic component of the coordinate ring of the conjugacy class of a nilpotent matrix. The Poincaré polynomial is the $q$-analogue of the multiplicity of an irreducible $gl(n)$-module in a tensor product of irreducible $gl(n)$-modules in which each factor has highest weight given by a rectangular (that is, a multiple of a fundamental) weight. These polynomials, which possess many properties generalizing those of the Kostka-Foulkes, have been studied extensively using algebro-geometric and combinatorial methods [8] [13] [24] [25] [30] [31].

The connection between the Demazure modules and the nilpotent adjoint orbit closures can be explained as follows. Let $X_\mu$ be the Zariski closure of the conjugacy class of the nilpotent Jordan matrix with block sizes given by the transpose partition $\mu^t$ of $\mu$, that is,

$$X_\mu = \{ A \in gl_n(\mathbb{C}) \mid \dim \ker A^i \geq \mu_1 + \cdots + \mu_i \text{ for all } i \}.$$

Lusztig gave an embedding of the variety $X_\mu$ as an open dense subset of a $P$-stable Schubert variety $Y_\mu$ in $\hat{SL}_n/P$, where $P \cong SL_n$ is the parabolic subgroup given by “omitting the reflection $r_0$” [19]. The desired level $l$ Demazure module, viewed as an $sl_n$-module, is isomorphic to the dual of the space of global sections
The proof of the main result entails generalizations of the three steps in the proof of \cite{9, Theorem 5.2}. First, the methods of \cite{9} may be used to show that the Demazure crystal is isomorphic to a classical \( \hat{\mathfrak{sl}}_n \) crystal \( B \) that is a tensor product of crystals of the form \( B^{k,l} \) (notation as in \cite{7}). We call \( B^{k,l} \) a rectangular crystal since it is indexed by the weight \( l\Lambda_k \) that corresponds to the rectangular partition with \( k \) rows and \( l \) columns.

Second, it is shown that the crystal \( B \) is indexed by sequences of Young tableaux of rectangular shape equipped with a generalized charge map. In particular, we give explicit descriptions in terms of tableaux and the Robinson-Schensted-Knuth correspondence, of

- The zero-th crystal raising operator \( \tilde{e}_0 \) acting on \( B \), which involves the generalized cyclage operators of \cite{24} on LR tableaux and a promotion operator on column-strict tableaux.
- The combinatorial \( R \)-matrices on a tensor product of the form \( B^{k_1,l_1} \otimes B^{k_2,l_2} \), which are given by a combination of the generalized automorphism of conjugation \cite{24} and the energy function.
- The energy function, which equals the generalized charge of \cite{13, 24}.

Moreover, it is shown that every generalized cocyclage relation \cite{24} on LR tableaux may be realized by \( \tilde{e}_0 \). The formula for the corresponding change in the energy function by \( \tilde{e}_0 \) was known \cite{22} in the case that all rectangles are single rows or all are single columns.

Third, it must be shown that the tableau formula coincides with the Poincaré polynomial. This was accomplished in \cite{24}, where it is shown that the tableau formula satisfies a defining recurrence of Weyman \cite{31, 32} for the Poincaré polynomials that is closely related to Morris’ recurrence for Kostka-Foulkes polynomials \cite{21}.

As an application of the formula for the energy function, we give a very simple proof of a monotonicity property for the Poincaré polynomials (conjectured by A. N. Kirillov) that extends the monotonicity property of the Kostka-Foulkes polynomials that was proved by Han \cite{4}.

Thanks to M. Okado for pointing out the reference \cite{26} which has considerable overlap with this paper and \cite{24}.

2. Notation and statement of main result

2.1. Quantized universal enveloping algebras. For this paper we only require the following three algebras: \( U_q(\mathfrak{sl}_n) \subset U'_q(\hat{\mathfrak{sl}}_n) \subset U_q(\hat{\mathfrak{sl}}_n) \).

Let us recall some definitions for quantized universal enveloping algebras taken from \cite{3} and \cite{4}. Consider the following data: a finitely generated \( \mathbb{Z} \)-module \( P \) (weight lattice), a set \( I \) (index set for Dynkin diagram), elements \( \{ \alpha_i \mid i \in I \} \) (basic roots) and \( \{ h_i \in P^* = \text{Hom}_\mathbb{Z}(P, \mathbb{Z}) \mid i \in I \} \) (basic coroots) such that \( \langle h_i, \alpha_j \rangle_{i,j \in I} \) is a generalized Cartan matrix, and a symmetric form \( \langle \cdot, \cdot \rangle : P \times P \rightarrow \mathbb{Q} \) such that \( (\alpha_i, \alpha_i) \in \mathbb{Z} \) is positive, \( \langle h_i, \lambda \rangle = 2 (\alpha_i, \lambda)/(\alpha_i, \alpha_i) \) for \( i \in I \) and \( \lambda \in \mathbb{Q} \otimes P \).

This given, let \( U_q(\hat{\mathfrak{sl}}_n) \) be the quantized universal enveloping algebra, the \( \mathbb{Q}(q) \)-algebra with generators \( \{ e_i, f_i \mid i \in I \} \) and \( \{ q^h \mid h \in P^* \} \) and relations as in \cite{3} Section 2}.
For \( U_q(\hat{sl}_n) \), let \( I = \{0, 1, 2, \ldots, n-1\} \) be the index set for the Dynkin diagram, \((a_{ij})_{i,j \in I}\) the Cartan matrix of type \( A_{n-1}^{(1)} \), \( P \) the free \( \mathbb{Z} \)-module with basis \( \{\Lambda_i \mid i \in I\} \cup \{\delta\} \) (fundamental weights) and let \( P^* \) have dual basis \( \{h_i \mid i \in I\} \cup \{d\} \). Define the elements \( \{\alpha_i \in P \mid i \in I\} \) by

\[
\alpha_i = \delta_0 + \sum_{j \in I} a_{ij}\Lambda_j,
\]

so that \((<h_i, \alpha_j>)_{i,j \in I}\) is the Cartan matrix of type \( A_{n-1}^{(1)} \) and \( \delta = \sum_{i=0}^{n-1} \alpha_i \). Define the symmetric \( \mathbb{Q} \)-valued form \((\cdot, \cdot)\) by \((\alpha_i, \alpha_j) = a_{ij}\) for \( i, j \in I \), \((\alpha_i, \Lambda_0) = \delta_i \) for \( i \in I \), and \( (\Lambda_0, \Lambda_0) = 0 \). Then the quantized universal enveloping algebra for this data is \( U_q(\hat{sl}_n) \). Let \( P^+ = \mathbb{Z}_+ \delta \oplus \bigoplus_{i \in I} \mathbb{Z}_+ \Lambda_i \) be the dominant weights. For \( \Lambda \in P^+ \) let \( \mathcal{V}(\Lambda) \) be the irreducible integrable highest weight \( U_q(\hat{sl}_n) \)-module of highest weight \( \Lambda \), \( \mathcal{B}(\Lambda) \) its crystal graph, and \( u_\Lambda \in \mathcal{B}(\Lambda) \) the highest weight vector.

For \( U_q(\hat{sl}_n) \), let \( I \) and \( (a_{ij}) \) be as above, but instead of \( P \) use the “classical weight lattice” \( P_{cl} = P/\mathbb{Z}\delta = \bigoplus_{i \in I} \mathbb{Z}\Lambda_i \) (where by abuse of notation the image of \( \Lambda_i \) in \( P_{cl} \) is also denoted \( \Lambda_i \)). The basic coroots \( \{h_i \mid i \in I\} \) form a \( \mathbb{Z} \)-basis of \( P_{cl}^* \). The basic roots are \( \{\alpha_i \mid i \in I\} \) where \( \alpha_i \) denotes the image of \( \alpha_i \) in \( P_i \) for \( i \in I \).

Note that the basic roots are linearly dependent. The pairing and symmetric form are induced by those above. The algebra for this data is denoted \( U_q^*(\hat{sl}_n) \) and may be viewed as a subalgebra of \( U_q(\hat{sl}_n) \) since its generators are a subset of those of \( U_q(\hat{sl}_n) \) and its relations map to relations. Let \( P_{cl}^+ = \bigoplus_{i \in I} \mathbb{Z}_+ \Lambda_i \). For \( \Lambda \in P^+ \), the \( U_q(\hat{sl}_n) \)-module \( \mathcal{V}(\Lambda) \) is a \( U_q^*(\hat{sl}_n) \)-module by restriction and weights are taken modulo \( \delta \).

For \( U_q(sl_n) \), let \( J = \{1, 2, \ldots, n-1\} \subset I \) be the index set for the Dynkin diagram, \((a_{ij})_{i,j \in J}\) the restriction of the above Cartan matrix to \( J \times J \), and \( P_{cl} = P_{cl}/\mathbb{Z}\Lambda_0 \) the weight lattice. The basic coroots \( \{h_i \mid i \in J\} \) form a \( \mathbb{Z} \)-basis of \( P_{cl}^* \). The basic roots are \( \{\alpha_i \mid i \in J\} \). The algebra for this data is \( U_q(sl_n) \), which can be viewed as a subalgebra of \( U_q(\hat{sl}_n) \). Let \( P_{cl}^+ = \bigoplus_{i \in J} \mathbb{Z}_+ \Lambda_i \) be the dominant integral weights. For \( \lambda \in P_{cl}^+ \) let \( \mathcal{V}(\lambda) \) be the irreducible \( U_q(sl_n) \)-module of highest weight \( \lambda \), and \( \mathcal{B}(\lambda) \) its crystal graph.

Denote by \( W \) the Weyl group of the algebra \( \hat{sl}_n \) and by \( \overline{W} \) that of \( sl_n \). \( W \) is the subgroup of automorphisms of \( P \) generated by the simple reflections \( \{r_i \mid i \in I\} \) where

\[
r_i(\lambda) = \lambda - <h_i, \lambda> \alpha_i.
\]

Let \( \overline{Q} = \bigoplus_{i \in J} \mathbb{Z}\alpha_i \subset P_{cl} \). \( W \) acts faithfully on the affine subspace \( X = \Lambda_0 + \overline{Q} \subset P_{cl} \). For \( \lambda \) in \( \overline{Q} \) write \( \tau_\lambda \) for the map \( X \rightarrow X \) given by translation by \( \lambda \). Let \( \theta = \sum_{i \in J} i\alpha_i \in P_{cl} \). Then the action of \( r_i \) on \( X \) for \( i \in J \) is given by

\[
r_i(\lambda) = \lambda - <h_i, \lambda> \alpha_i,
\]

for \( \lambda \in X \) and \( r_0 = \tau_\theta r_\theta \), where

\[
r_\theta = r_1 \ldots r_{n-2} r_{n-1} r_{n-2} \ldots r_1
\]

is the reflection through the hyperplane orthogonal to \( \theta \). Then \( W \cong \overline{W} \rtimes \overline{Q} \) where the element \( \lambda \in \overline{Q} \) acts by \( \tau_\lambda \).

For \( \Lambda \in P^+ \) and \( w \in W \) the Demazure module of lowest weight \( w\Lambda \) is defined by \( \mathcal{V}_w(\Lambda) = U_q(b)v_{w\Lambda} \) where \( v_{w\Lambda} \) is a generator of the (one dimensional) extremal
weight space in \( V(\Lambda) \) of weight \( w\Lambda \) and \( U_q(b) \) is the subalgebra of \( U_q(\widehat{sl}_n) \) generated by the \( e_i \) and \( h \) in \( P^* \).

2.2. **Main result.** Let \( \mu \) be a partition of \( n \). The coordinate ring \( \mathbb{C}[X_\mu] \) of the nilpotent adjoint orbit closure \( X_\mu \) has a graded \( sl_n \)-action induced by matrix conjugation on \( X_\mu \). For \( \lambda \in P_\text{cl}^+ \), define the Poincaré polynomial of the \( \lambda \)-th isotypic component of \( \mathbb{C}[X_\mu] \), by

\[
K_{\lambda, \mu}(q) = \sum_{d \geq 0} q^d \dim \text{Hom}_{sl_n}(V^\lambda, \mathbb{C}[X_\mu], d)
\]

where \( \mathbb{C}[X_\mu]_d \) is the homogeneous component of degree \( d \).

Partitions with at most \( n \) parts are projected to dominant integral weights of \( sl_n \) by \( \mu \mapsto \text{wt}_{sl}(\mu) \), where

\[
\text{wt}_{sl}(\mu) = n - 1 \sum_{i=1}^{n} (\mu_i - \mu_{i+1})\Lambda_i \in P_\text{cl}^+.
\]

**Remark 1.** Warning: this is not the Kostka polynomial, but a generalization; see [31]. In the special case that \( \lambda \) is a partition of \( n \) with at most \( n \) parts then \( K_{\text{wt}_{sl}(\lambda), \mu}(q) = \tilde{K}_{\lambda, \mu}(q) \) which is a renormalization of the Kostka-Foulkes polynomial with indices \( \lambda^t \) and \( \mu \).

**Theorem 2.** Let \( l \) be a positive integer, \( \mu \) a partition of \( n \), and \( w_\mu \in W \) the translation by the antidominant weight \(-w_0\text{wt}_{sl}(\mu^t) \in P_\text{cl} \), where \( w_0 \) is the longest element of \( W \). Then

\[
e^{-l\Lambda_0} \text{ch}_{w_\mu}(l\Lambda_0) = \sum_{\lambda} K_{\text{wt}_{sl}(\lambda), \mu}(q) \text{ch}_{w_\mu}(\lambda)
\]

where \( \lambda \) runs over the partitions of the multiple \( ln \) of \( n \) with at most \( n \) parts.

3. **Crystal structure on tensor products of rectangles**

The goal of this section is to give explicit descriptions of the classical \( \widehat{sl}_n \) crystal structure on tensor products of rectangular crystals and their energy functions. This is accomplished by translating the theory of \( sl_n \) crystals and classical \( \widehat{sl}_n \) crystals in [6] [7] [11] [21] into the language of Young tableaux and the Robinson-Schensted-Knuth (RSK) correspondence.

3.1. **Crystals.** This section reviews the definition of a weighted crystal [6] and gives the convention used here for the tensor product of crystals.

A \( P \)-weighted \( I \)-crystal is a a weighted \( I \)-colored directed graph \( B \), that is, a set equipped with a weight function \( \text{wt} : B \rightarrow P \) and directed edges colored by the set \( I \), satisfying the following properties.

(C1) There are no multiple edges; that is, for each \( i \in I \) and \( b, b' \in B \) there is at most one edge colored \( i \) from \( b \) to \( b' \).

If such an edge exists, this is denoted \( b' = \bar{f}_i(b) \) or equivalently \( b = \bar{e}_i(b') \), by abuse of the notation of a function \( B \rightarrow B \). It is said that \( \bar{f}_i(b) \) is defined or equivalently that \( \bar{e}_i(b') \) is defined, if the edge exists.

\[
\phi_i(b) = \max \{ m \in \mathbb{Z}_+ \mid \bar{f}_i^m(b) \text{ is defined} \}
\]

\[
\epsilon_i(b) = \max \{ m \in \mathbb{Z}_+ \mid \bar{e}_i^m(b) \text{ is defined} \}
\]
If \( \tilde{f}_i(b) \) is defined then \( \text{wt}(\tilde{f}_i(b)) = \text{wt}(b) - \alpha_i \). Equivalently, \( \text{wt}(\tilde{c}_i(b)) = \text{wt}(b) + \alpha_i \).

(C3) \( \langle h_i, \text{wt}(b) \rangle = \phi_i(b) - \epsilon_i(b) \).

If \( B_j \) is a \( P \)-weighted \( I \)-crystal for \( 1 \leq j \leq m \), the Cartesian product \( B_m \times \cdots \times B_1 \) can be given a crystal structure as follows; this crystal is denoted \( B = B_m \otimes \cdots \otimes B_1 \). The convention used here is opposite that in much of the literature but is convenient for the tableau combinatorics used later. Let \( b_j \in B_j \) and \( b = b_m \otimes \cdots \otimes b_1 \in B \).

The weight function on \( B \) is given by

\[
\text{wt}(b) = \sum_{j=1}^{m} \text{wt}(b_j).
\]

The root operators \( \tilde{f}_i \) and the functions \( \phi_i \) are defined by the “signature rule”. Given \( b \in B \) and \( i \in I \), construct a biword (sequence of pairs of letters) consisting of \( \phi_i(b_j) \) copies of the biletter \((\downarrow)\) and \( \epsilon_i(b_j) \) copies of the biletter \((\uparrow)\) for all \( j \), sorted in weakly increasing order by the order \((\downarrow) < (\uparrow)\) if \( j > j' \) and \((\downarrow) < (\uparrow)\).

This biword is now repeatedly reduced by removing adjacent biletters whose lower letters are \(+\) in that order. If \(+\) and \(-\) are viewed as left and right parentheses then this removes matching pairs of parentheses. At the end one obtains a biword whose lower word has the form \(-^{s}+^{t}\). Then define \( \phi_i(b) = s \) and \( \epsilon_i(b) = t \). If \( s > 0 \) (resp. \( t > 0 \)) let \( j_- \) (resp. \( j_+ \)) be the upper letter corresponding to the rightmost \(-\) (resp. leftmost \(+\)) in the reduced biword, and define

\[
\tilde{f}_i(b) = b_m \otimes \cdots \otimes b_{1+j_-} \otimes \tilde{f}_i(b_{j_-}) \otimes b_{-1+j_-} \otimes \cdots \otimes b_1,
\]

respectively,

\[
\tilde{c}_i(b) = b_m \otimes \cdots \otimes b_{1+j_+} \otimes \tilde{c}_i(b_{j_+}) \otimes b_{-1+j_+} \otimes \cdots \otimes b_1.
\]

A morphism \( g : B \to B' \) of \( P \)-weighted \( I \)-crystals is a map \( g \) that preserves weights and satisfies \( g(\tilde{f}_i(b)) = \tilde{f}_i(g(b)) \) for all \( i \in I \) and \( b \in B \), that is, if \( \tilde{f}_i(b) \) is defined then \( \tilde{f}_i(g(b)) \) is, and the above equality holds.

It is easily verified that the \( P \)-weighted \( I \)-crystals form a tensor category.

We only require the following kinds of crystals.

1. The crystal graphs of integrable \( U_q(\hat{sl}_n) \)-modules are \( P \)-weighted \( I \)-crystals and are called \( \hat{sl}_n \)-crystals.
2. The crystal graphs of \( U'_q(\hat{sl}_n) \)-modules that are either integrable or are finite-dimensional and have a weight space decomposition, are \( P_{\text{cl}} \)-weighted \( I \)-crystals and are called classical \( \hat{sl}_n \)-crystals.
3. The crystal graphs of integrable \( U'_q(sl_n) \)-modules are \( \mathcal{T}_{\text{cl}} \)-weighted \( J \)-crystals and are called \( sl_n \)-crystals.

3.2. Crystal reflection operator and Weyl group action. Let \( B \) be an \( sl_n \) crystal and \( i \in J \). Write \( p = \phi_i(b) - \epsilon_i(b) \). Define

\[
\tilde{r}_i(b) = \begin{cases} 
\tilde{f}_i^p(b) & \text{if } p > 0 \\
b & \text{if } p = 0 \\
\tilde{c}_i^{-p}(b) & \text{if } p < 0
\end{cases}
\]

The Weyl group \( \mathbb{W} \) acts on \( B \) by \( r_i(b) = \tilde{r}_i(b) \) for \( i \in J \). It is obvious that \( \tilde{r}_i \) is an involution and that \( \tilde{r}_i \) and \( \tilde{r}_j \) commute for \( |i - j| > 1 \), but not at all obvious
that the $\tilde{r}_i$ satisfy the other braid relation. A combinatorial proof of this fact is indicated in [13] for the action of $\overline{W}$ on an irreducible $sl_n$ crystal.

3.3. Irreducible $sl_n$ crystals. Let $\lambda = (\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_n) \in \mathbb{Z}_+^n$ be a partition of length at most $n$. Let $V^\lambda$ be the irreducible $U_q(sl_n)$-module of highest weight $\text{wt}_q(\lambda)$ and $B^\lambda$ its crystal. In [1], the structure of the $sl_n$ crystal $B^\lambda$ is determined explicitly. The crystal $B^\lambda$ may be indexed by the set $\text{CST}(\lambda)$ of column-strict tableaux of shape $\lambda$ with entries in the set $[n] = \{1, 2, \ldots, n\}$. The combinatorial construction yielding the action of the crystal operators $\tilde{e}_i$ and $\tilde{f}_i$ for $1 \leq i \leq n-1$ on tableaux was already known. In a 1938 paper, in the course of proving the Littlewood-Richardson rule, G. de B. Robinson gave a form of the Robinson-Schensted-Knuth (RSK) correspondence which is defined by giving the value of the map on $sl_n$-highest weight vectors and then extending it by via canonical sequences of raising operators $\tilde{f}_i$ [23, Section 5]; see also [20, I.9] where Robinson’s proof is cleaned up.

Suppose first that $\lambda = (1)$ so $B^\lambda$ is the crystal of the defining representation of $sl_n$. This crystal is indexed by the set $[n] = \{1, 2, \ldots, n\}$ and $\tilde{f}_i(j)$ is defined if and only if $j = i$ and in that case $\tilde{f}_i(i) = i+1$.

Next consider the tensor product $(B^{(1)})^\otimes m$. It may be indexed by words $u = u_m \ldots u_1$ of length $m$ in the alphabet $[n]$, where $u_j \in [n]$. Its $sl_n$ crystal structure is defined by the signature rule. This case of the signature rule is given in [13].

Now it is necessary to introduce notation for Young tableaux.

Some definitions are required. The Ferrers diagram $D(\lambda)$ is the set of pairs of integers $D(\lambda) = \{(i,j) \in \mathbb{Z}_+^2 \mid 1 \leq j \leq \lambda_i\}$. A skew shape $D = \lambda/\mu$ is the set difference of the Ferrers diagrams $D(\lambda)$ and $D(\mu)$ of the partitions $\lambda$ and $\mu$. If $D$ and $E$ are skew shapes such that $D$ has $c$ columns and $E$ has $r$ rows, then define the skew shape

$$D \otimes E = \{(i+r,j) \mid (i,j) \in D\} \cup \{(i,j+c) \mid (i,j) \in E\}.$$  

In other words, $D \otimes E$ is the union of a translate of $D$ located to the southwest of a translate of $E$.

A tableau of (skew) shape $D$ is a function $T : D \to [n] = \{1, 2, \ldots, n\}$, and is depicted as a partial matrix whose $(i,j)$-th entry is $T(i,j)$ for all $(i,j) \in D$. Denote by shape($T$) the domain of $T$. The tableau $T$ is said to be column-strict if $T(i,j) \leq T(i,j+1)$ for all $(i,j) \in D$ and $T(i,j) < T(i+1,j)$ for all $(i,j), (i+1,j) \in D$. Let CST($D$) be the set of column-strict tableaux of shape $D$.

The content of a tableau is the sequence

$$\text{content}(T) = (m_1(T), \ldots, m_n(T))$$

where $m_i(T)$ is the number of occurrences of the letter $i$ in $T$. Let CST($D, \gamma$) denote the set of column-strict tableaux of shape $D$ and content $\gamma$. The (row-reading) word of the tableau $T$ is the word given by word($T$) = $\cdots \gamma^2 \gamma^1$ where $\gamma^i$ is the word obtained by reading the $i$-th row of $T$ from left to right. Say that the word $u$ fits the skew shape $D$ if there is a column-strict tableau (necessarily unique) whose row-reading word is $u$.

Remark 3. Let $D$ be a skew shape, $u$ a word in the alphabet $[n]$ and $1 \leq i \leq n-1$. It is well-known and easy to verify that if $\tilde{e}_i(u)$ is defined, then $u$ fits $D$ if and only if $\tilde{e}_i(u)$ does. This given, if $T$ is a column-strict tableau of shape $D$ and $\tilde{e}_i(\text{word}(T))$
is defined, then let \( \tilde{e}_i(T) \) be the unique column-strict tableau of shape \( D \) such that word(\( \tilde{e}_i(T) \)) = \( e_i(\text{word}(T)) \). The same can be done for \( f_i \).

Thus the set \( \text{CST}(D) \) is an \( sl_n \) crystal; call it \( B^D \). When \( D = D(\lambda) \) this is the crystal \( B^\lambda \).

3.4. Tensor products of irreducible \( sl_n \) crystals and RSK. Let \( D \) be a skew diagram and \( B^D \) the \( sl_n \) crystal defined in the previous section. The RSK correspondence yields a combinatorial decomposition of \( B \) into irreducible \( sl_n \) crystals. The RSK map can be applied to tensor products of irreducible \( sl_n \) crystals. The goal of this section is to review a well-known parametrizing set for the multiplicity space of such a tensor product, by what we shall call Littlewood-Richardson (LR) tableaux.

Let \( \eta = (\eta_1, \eta_2, \ldots, \eta_m) \) be a sequence of positive integers summing to \( n \) and \( R = (R_1, R_2, \ldots, R_m) \) a sequence of partitions such that \( R_i \) has \( \eta_i \) parts, some of which may be zero. Let \( A_1 \) be the first \( \eta_1 \) numbers in \([n]\), \( A_2 \) the next \( \eta_2 \), and so on. Recall the skew shape \( R_m \otimes \cdots \otimes R_1 \), embedded in the plane so that \( A_i \) gives the set of row indices for \( R_i \). Let \( \gamma = (\gamma_1, \ldots, \gamma_n) \in \mathbb{Z}^n_+ \) where \( \gamma_i \) is the length of the \( i \)-th row of \( R_m \otimes \cdots \otimes R_1 \), that is, \( \gamma \) is obtained by juxtaposing the partitions \( R_i \) through \( R_m \). The tensor product crystal may be viewed as a skew crystal:

\[
B^R := B^{R_m} \otimes \cdots \otimes B^{R_1} \cong B^{R_m} \otimes \cdots \otimes B^{R_1}.
\]

Let \( b \in B^R \) and write \( b = b_m \otimes \cdots \otimes b_1 \) where \( b_i \in \text{CST} (R_i) \). Let \( v_r \) for the weakly increasing word (of length \( \gamma_r \)) comprising the \( r \)-th row of \( b \), for \( 1 \leq r \leq n \). The word of \( b \) regarded as a skew column-strict tableau, is given by \( \text{word}(b) = v_n \cdots v_1 \).

Recall Knuth’s equivalence on words [22]. Say that a skew shape is normal (resp. antinormal) if it has a unique northwest (resp. southeast) corner cell [3]. A normal skew shape is merely a translation in the plane of a partition shape, and an antinormal shape is the 180-degree rotation of a normal shape. For any word \( v \), there is a unique (up to translation) column-strict tableau \( P(v) \) of normal shape such that \( v \) is Knuth equivalent to the word of \( P(v) \). There is also a unique (up to translation) skew column-strict tableau \( P_\gamma(v) \) of antinormal shape such that \( v \) is Knuth equivalent to the word of \( P_\gamma(v) \).

The tableau \( P(v) \) may be computed by Schensted’s column-insertion algorithm [21]. For a subinterval \( A \subset [n] \) and a (skew) column-strict tableau \( T \), define \( T|_A \) to be the skew column-strict tableau obtained by restricting \( T \) to \( A \), that is, removing from \( T \) the letters that are not in \( A \). Define the pair of column-strict tableaux \( (P(b), Q(b)) \) by

\[
\text{shape}(Q(b)|_r) = \text{shape}(P(v_r \cdots v_1)) \quad \text{for all } 0 \leq r \leq n
\]

By definition \( \text{shape}(P(b)) = \text{shape}(Q(b)) \). It is easy to show that \( Q(b) \) is column-strict and of content \( \gamma \). This gives an embedding

\[
B^R \hookrightarrow \bigcup_\lambda \text{CST}(\lambda) \times \text{CST}(\lambda, \gamma)
\]

(3.1)

\[
b \mapsto (P(b), Q(b))
\]
It is well-known that this is a map of $\mathfrak{sl}_n$ crystals. That is, if $g$ is any of $\tilde{e}_i$, $\tilde{f}_i$, or $\tilde{r}_i$ for $i \in J$, then

$$\begin{align*}
P(g(b)) &= g(P(b)) \\
Q(g(b)) &= Q(b)
\end{align*}$$

For the case $g = \tilde{r}_i$ this fact is in [18].

Let us describe the image of the map (3.1). For the partition $\lambda = (\lambda_1, \ldots, \lambda_n)$ and a permutation $w$ in the symmetric group $S_n$, the key tableau $\text{Key}(w\lambda)$ of content $w\lambda$, is the unique column-strict tableau of shape $\lambda$ and content $w\lambda$. In the above notation for the sequence of partitions $R$, for $1 \leq j \leq m$ let $Y_j = \text{Key}(R_j)$ in the alphabet $A_j$.

Say that a word $u$ in the alphabet $[n]$ is $R$-LR (short for $R$-Littlewood-Richardson) if $P(u|A_i) = Y_j$ for all $1 \leq j \leq m$, where $u|A_i$ is the restriction of the word $u$ to the subalphabet $A_j \subset [n]$. Say that a (possibly skew) column-strict tableau is $R$-LR if its row-reading word is $R$. Denote by $\text{LRT}(\lambda; R)$ the $R$-LR tableaux of partition shape $\lambda$ and $\text{LRT}(R) = \bigcup_{\lambda} \text{LRT}(\lambda; R)$.

The following theorem is essentially a special case of [32, Theorem 1] which is a strong version of the classical rule of Littlewood and Richardson [16].

**Theorem 4.** The map (3.1) gives a bijection

$$B^R \cong \bigcup_{\lambda} \text{CST}(\lambda) \times \text{LRT}(\lambda; R).$$

3.5. **Kostka crystals.** In the case that $m = n$ and $R_i = (\gamma_i)$ for $\gamma \in \mathbb{Z}^+_n$, we call $B^R$ a Kostka crystal. The set $\text{LRT}(\lambda; R)$ is merely the set $\text{CST}(\lambda, \gamma)$.

Let $R = (R_1, \ldots, R_m)$ be a sequence of rectangles as usual. Define the Kostka crystal $B^{\text{rows}(R)}$ by the sequence of one row partitions $\text{rows}(R)$ whose $r$-th partition is given by $(\gamma_r)$ where $\gamma_r$ is the length of the $r$-th row of the skew shape $R_m \otimes \cdots \otimes R_1$. Letting $b \in B = B^R$ and $v_r$ the $r$-th row of $b$, there is the obvious $\mathfrak{sl}_n$ crystal embedding

$$B^R \hookrightarrow B^{\text{rows}(R)}$$

$$b \mapsto v_n \otimes \cdots \otimes v_1.$$ 

In fact, the RSK correspondence (3.3) may be defined using the commutativity of the diagram in (3.2) for $g = \tilde{f}_i$ and all $i \in J$, and giving its values on the $\mathfrak{sl}_n$-highest weight elements in $B$. Suppose $b \in B$ is such that word$(b)$ is of $\mathfrak{sl}_n$-highest weight, that is, $\epsilon_i(b) = 0$ for all $i \in J$. Such words are said to possess the lattice property. In this case content$(\text{word}(b))$ must be a partition, say $\lambda$, and $P(b) = P(\text{word}(b)) = \text{Key}(\lambda)$. For the recording tableau, write word$(b) = v_n \otimes \cdots \otimes v_1$ where $v_r$ is the $r$-th row of $b$ viewed as a tableau of the skew shape $R_m \otimes \cdots \otimes R_1$. Then $Q(b)$ is the column-strict tableau of shape equal to content$(b)$, whose $i$-th row contains $m$ copies of the letter $j$ if and only if the word $v_j$ contains $m$ copies of the letter $i$, for all $i$ and $j$. In particular, if elements $b, b' \in B$ admit the same sequences of raising operators, then $P(b) = P(b')$.

3.6. **Rectangle-switching bijections.** From now on we consider only crystals $B^R$ where $R_j = (\mu_j^\eta_j)$ is the partition with $\eta_j$ rows and $\mu_j$ columns for $1 \leq j \leq m$.

Consider the case $m = 2$, with $R = (R_1, R_2)$, $r_1 R = (R_2, R_1)$, $B = B^R = B^{R_2} \otimes B^{R_1}$ and $r_1 B = B^{R_1} \otimes B^{R_2}$. Since the $U_q(\mathfrak{sl}_n)$-module $V^{R_2} \otimes$
$V^{R_1} \cong V^{R_1} \otimes V^{R_2}$ is multiplicity-free, it follows that there is a unique $sl_n$ crystal isomorphism

$$\sigma_{R_2, R_1} : B^{R_2} \otimes B^{R_1} \cong B^{R_1} \otimes B^{R_2}.$$  

It is defined explicitly in terms of the RSK correspondence as follows. By the above multiplicity-freeness, for any partition $\lambda$,

$$|\text{LRT}(\lambda; (R_1, R_2))| = LRT(\lambda; (R_2, R_1)) \leq 1.$$

Thus there is a unique bijection

$$\tau = \tau_{\lambda; (R_2, R_1)} : \text{LRT}(\lambda; (R_1, R_2)) \to \text{LRT}(\lambda; (R_2, R_1)).$$

For later use, extend $\tau$ to a bijection from the set of $(R_1, R_2)$-LR words to the set of $(R_2, R_1)$-LR words by

$$P(\tau(u)) = \tau(P(u))$$

$$Q(\tau(u)) = Q(u)$$

where $Q(u)$ is the standard column-insertion recording tableau (that is, $Q(u) = Q(u)$ with $u$ regarded as a word in the tensor product $(B^{(1)})^\otimes N$ where $N$ is the length of $u$).

$\tau$ is the rectangular generalization of an automorphism of conjugation. $\sigma$ is defined by the commutative diagram

$$B^{R_1 \otimes R_2} \xrightarrow{\text{RSK}} \bigcup_\lambda \text{CST}(\lambda) \times \text{LRT}(\lambda; (R_1, R_2))$$

$$\sigma \downarrow \quad \quad \downarrow \text{id} \times \tau$$

$$B^{R_2 \otimes R_1} \xrightarrow{\text{RSK}} \bigcup_\lambda \text{CST}(\lambda) \times \text{LRT}(\lambda; (R_2, R_1))$$

In other words, for all $b \in B^{R_2} \otimes B^{R_1}$,

$$P(\sigma(b)) = P(b)$$

$$Q(\sigma(b)) = \tau(Q(b)).$$

(3.6)

Now let $R = (R_1, \ldots, R_m)$ and $B = B^R$ the tensor product of rectangular crystals. Let $w \in S_m$ be a permutation in the symmetric group on $m$ letters. Write $wB = B^{wR}$ where $wR$ is the sequence of rectangles $wR = R_{w^{-1}(1)} \otimes \cdots \otimes R_{w^{-1}(m)}$.

Write $\sigma_{R, R_j}$ for the action of the above $sl_n$ crystal isomorphism at consecutive tensor positions in $wB = \cdots \otimes B^{R_i} \otimes B^{R_j} \otimes \cdots$. Then the isomorphisms $\sigma_{R, R_j}$ satisfy a Yang-Baxter identity

$$(\sigma_{B_j, B_k} \otimes \text{id}) \circ (\text{id} \otimes \sigma_{B_k, B_k}) \circ (\sigma_{B_i, B_k} \otimes \text{id})$$

$$= (\text{id} \otimes \sigma_{B_i, B_j}) \circ (\sigma_{B_i, B_k} \otimes \text{id}) \circ (\text{id} \otimes \sigma_{B_j, B_k})$$

This is a consequence of the corresponding difficult identity for bijections $\tau_{R_i, R_j}$ on recording tableaux, defined and conjectured in [13] and proven in [24, Theorem 9 (A5)]. By composing maps of the form $\sigma_{R, R_j}$, it is possible to well-define $sl_n$ crystal isomorphisms

$$(3.7) \quad \quad \sigma_{R, wR} : B \to wB$$

such that $\sigma_{R, vwR} = \sigma_{wR, vwR} \circ \sigma_{R, wR}$ for $v, w \in S_m$. These bijections satisfy

$$P(\sigma_{R, wR}(b)) = P(b)$$

$$Q(\sigma_{R, wR}(b)) = \tau_{R, wR}(Q(b))$$

(3.8)
whose entries are obtained from those of $b_{i}^\ast$ that (skew) tableau with its row-reading word. 

**Proof.** Let $\psi$ for $\tau$ in $\mathfrak{sl}_n$ by a jeu-de-taquin on the two row skew tableau with word $b_{i+1}b_{j}$. 

3.7. $\hat{\mathfrak{sl}}_n$ crystal structure on rectangular crystals. Suppose the sequence $R$ consists of a single rectangular partition with $k$ rows and $l$ columns, so that $B^{R} = B^{k,l}$. In $\mathfrak{sl}_n$ the existence of a unique classical $\hat{\mathfrak{sl}}_n$ crystal structure on $B^{k,l}$ was proved. The $\mathfrak{sl}_n$ crystal structure has already been described in Section 3.3. Using the properties of the perfect crystal $B^{k,l}$ given in $\mathfrak{sl}_n$, an explicit tableau construction for $\hat{\mathfrak{sl}}_n$ is presented.

The Dynkin diagram of $\hat{\mathfrak{sl}}_n$ admits the rotation automorphism that sends $i$ to $i + 1$ modulo $n$. It follows that there is a bijection $\psi : B^{k,l} \to B^{k,l}$ such that the following diagram commutes for all $i \in I$:

$$
\begin{array}{ccc}
B^{k,l} & \xrightarrow{\psi} & B^{k,l} \\
\downarrow & & \downarrow \\
\tilde{f}_i & \xrightarrow{\psi} & \tilde{f}_{i+1}
\end{array}
$$

(3.9)

where subscripts are taken modulo $n$. Of course it is equivalent to require that $\psi$ satisfy the diagram with $\tilde{e}_i$ replacing $\tilde{f}_i$.

**Lemma 6.** $\psi$ is unique and rotates content in the sense that for all $b \in B^{k,l}$

$$
m_{i+1}(\psi(b)) = m_i(b)
$$

(3.10)

for all $0 \leq i \leq n - 1$ where $m_0$ is equal to $m_n$ by convention.

**Proof.** Let $b$ be the $\mathfrak{sl}_n$-highest weight vector in $B^{k,l}$, given explicitly by $\mathrm{Key}((l^k))$. By the definition of $\psi$ and the connectedness of $B^{k,l}$ it is enough to show that $\psi(b)$ is uniquely determined. By definition $\epsilon_i(b) = 0$ for all $i \in J$. Recall from $\mathfrak{sl}_n$ that for all $b' \in B^{k,l}$, $\sum_{i \in I} \epsilon_i(b') \geq l$, that $b'$ is said to be minimal if equality holds, and that for any sequence of numbers $(a_0, \ldots, a_{n-1}) \in \mathbb{Z}_+^n$ that sum to $l$, there is a unique minimal vector $b'$ such that $\epsilon_i(b') = a_i$ for all $i \in I$.

These facts imply that $\epsilon_0(b) = l$. From the definition of $\psi$ it follows that $\epsilon_i(\psi(b)) = l\delta_{i1}$. Thus $\psi(b)$ is minimal and hence uniquely defined.

Next it is shown that $\psi$ is uniquely defined by a weaker condition than $\{3.9\}$.

**Lemma 7.** $\psi$ is uniquely defined by (3.9) for $1 \leq i \leq n - 2$ and (3.10).

**Proof.** Let $b \in B^{k,l}$, $\tilde{b} = b|_{[2,n-1]}$ the restriction of $b$ to the subinterval $[2, n - 1]$, $b' = \psi(b)$, and $\tilde{b}' = b'|_{[3,n]}$. By abuse of notation we shall occasionally identify a (skew) tableau with its row-reading word.

If $u$ is a word or tableau and $p$ is an integer, denote by $u + p$ the word or tableau whose entries are obtained from those of $u$ by adding $p$. The first goal is to show that $\tilde{b}$ and $\tilde{b}' - 1$ are Knuth-equivalent, that is, $P(\tilde{b}) = P(\tilde{b}' - 1)$. By the assumption
on \( \psi \), \( \hat{b} \) admits a sequence of lowering operators \( e_{i_1} \ldots e_{i_p} \), for \( 2 \leq i_j \leq n - 2 \), if and only if \( \hat{b}' - 1 \) admits the sequence \( e_{i_1} \ldots e_{i_p} \). Since \( \hat{b} \) and \( \hat{b}' - 1 \) are words in the alphabet \([2, n - 1]\)
this proves that \( P(\hat{b}) = P(\hat{b}' - 1) \), by Robinson’s characterization of the RSK map (see section \([3.3]\)).

Now the shape of the tableau \( b' \) is a rectangle, so its restriction \( \hat{b}' = b'_{[3,n]} \) to a final subinterval of \([n]_k\), has antinormal shape. Hence \( \hat{b}' - 1 \) (and hence \( \hat{b}' \)) is uniquely determined by \( b \).

It only remains to show that the subtableau \( b'|_{[1,2]} \) is uniquely specified. Its shape must be the partition shape given by the complement in the rectangle \((l^k)\) with the shape of \( b'_{[3,n]} \). Now \( b'|_{[1,2]} \) is a column-strict tableau of partition shape and contains only ones and twos, so it has at most two rows and is therefore uniquely determined by its content. But its content is specified by \((3.10)\).

The map \( \psi \) is explicitly constructed by exhibiting a map that satisfies the conditions in Lemma \([3.4]\).

The following operation is Schützenberger’s promotion operator, which was defined on standard tableaux but has an obvious extension to column-strict tableaux \([3.24]\). Let \( D \) be a skew shape and \( b \in \text{CST}(D) \). The promotion operator applied to \( b \) is computed by the following algorithm.

1. Remove all the letters \( n \) in \( b \), which removes from \( D \) a horizontal strip \( H \) (skew shape such that each column contains at most one cell).
2. Slide (using Schützenberger’s jeu-de-taquin \([3.28]\)) the remaining subtableau \( b|_{[n-1]} \) to the southeast into the horizontal strip \( H \), entering the cells of \( H \) from left to right.
3. Fill in the vacated cells with zeros.
4. Add one to each entry.

The resulting tableau is denoted \( \text{pr}(b) \in \text{CST}(D) \) and is called the promotion of the tableau \( b \).

**Proposition 8.** The map \( \psi \) of \((3.9)\) is given by \( \text{pr} \).

**Proof.** \( \text{pr} \) is content-rotating (satisfies \((3.10)\)) and satisfies \((3.9)\) for \( 1 \leq i \leq n - 2 \), since \( \text{pr}(t)|_{[2,n]} = P_\triangledown(t|_{[n-1]}) + 1 \) and \( P_\triangledown \) commutes with \( sl_n \) crystal operators. By Lemma \([3.4]\), \( \text{pr} = \psi \).

In light of \((3.9)\), the operators \( \bar{e}_0 \) and \( \bar{f}_0 \) on \( B^{k,l} \) are given explicitly by

\[
\bar{e}_0 = \text{pr}^{-1} \circ \bar{e}_1 \circ \text{pr} \\
\bar{f}_0 = \text{pr}^{-1} \circ \bar{f}_1 \circ \text{pr}.
\]

**Remark 9.** Consider again the map \( \text{pr} \) on \( b \in B^{k,l} \). The tableau \( \hat{b} = b|_{[n-1]} \) has partition shape \( \lambda := (t^{k-1}, l - p) \), where \( p = m_n(b) \). Its row-reading word has Schensted tableau pair \( \mathbb{P}(\hat{b}) = \hat{b} \) and \( \mathbb{Q}(\hat{b}) = \text{Key}(\lambda) \). Let \( \hat{b}' := \text{pr}(b)|_{[2,n]} = P_\triangledown(b|_{[n-1]}) \),
which has antinormal shape and whose complementary shape inside the rectangle \((l^k)\) must be a single row of length \( p \), that is, \( \text{shape}(\hat{b}') = (l^k)/p \). So \( \mathbb{P}(\hat{b}') = \mathbb{P}(\hat{b}) \), and \( \mathbb{Q}(\hat{b}') \) is a column-strict tableau of shape \( \lambda \) and content \((l - p, l^{k-1})\), that is,
\[ Q(\hat{b}') = \text{Key}(w_0\lambda) \] where \( w_0 \) is the longest element of the symmetric group \( S_k \). So \( P(\hat{b}') = P(\hat{b}) \) and \( Q(\hat{b}') = w_0Q(\hat{b}) \).

Let \( D \) be the skew shape \( R_m \otimes \cdots \otimes R_1 \), \( B = B^R \) be the tensor product of rectangular crystals, \( b = b_m \otimes \cdots \otimes b_1 \in B \). Note that the operator \( \text{pr} \) on \( \text{CST}(D) = B \) may be described by

\[
\text{pr}(b) = \text{pr}(b_m) \otimes \cdots \otimes \text{pr}(b_1)
\]

By the definition of \( \tilde{\epsilon}_0 \) on a rectangular crystal and the signature rule, it follows that

\[
\tilde{\epsilon}_0 = \text{pr}^{-1} \circ \tilde{\epsilon}_1 \circ \text{pr}
\]

(3.12)

\[
\tilde{f}_0 = \text{pr}^{-1} \circ \tilde{f}_1 \circ \text{pr}
\]

as operators on \( B = B^R \).

**Example 10.** Let \( n = 7 \), \( R = ((2,2),(3,3,3),(3,3)) \), and \( b \in B^R \) given by the following skew tableau of shape \( R_3 \otimes R_2 \otimes R_1 \):

\[
\times \times \times \times \times \times 1 1 \\
\times \times \times \times \times \times 2 2 \\
\times \times \times 1 1 3 \\
\times \times \times 2 3 4 \\
\times \times \times 3 4 5 \\
2 4 6 \\
3 5 7
\]

The element \( \text{pr}(b) \) is given by

\[
\times \times \times \times \times \times 2 2 \\
\times \times \times \times \times \times 3 3 \\
\times \times \times 2 2 4 \\
\times \times \times 3 4 5 \\
\times \times \times 4 5 6 \\
1 3 5 \\
4 6 7
\]

The signature for calculating \( \tilde{\epsilon}_1 \) on \( \text{pr}(b) \) is

\[
3 2 2 1 1 \\
- + + + +
\]

So \( \tilde{\epsilon}_1 \) must be applied to the second tensor position. Then \( \tilde{\epsilon}_1(\text{pr}(b)) \) equals

\[
\times \times \times \times \times \times 2 2 \\
\times \times \times \times \times \times 3 3 \\
\times \times \times 1 2 4 \\
\times \times \times 3 4 5 \\
\times \times \times 4 5 6 \\
1 3 5 \\
4 6 7
\]
Finally $\tilde{c}_0(b) = \text{pr}^{-1}(\tilde{c}_1(\text{pr}(b)))$ is given by:

\[
\begin{array}{cccccccc}
\times & \times & \times & \times & \times & 1 & 1 \\
\times & \times & \times & \times & \times & 2 & 2 \\
\times & \times & 1 & 3 & 3 \\
\times & \times & 2 & 4 & 4 \\
\times & \times & 3 & 5 & 7 \\
2 & 4 & 6 \\
3 & 5 & 7 \\
\end{array}
\]

3.8. Action of $\tilde{c}_0$ on the tableau pair. In this section an algorithm is given to compute the tableau pair $(P(\tilde{c}_0(b)), Q(\tilde{c}_0(b)))$ of $\tilde{c}_0(b)$ directly in terms of the tableau pair $(P(b), Q(b))$ of $b$. In light of (3.12) and (3.2) with $g = \tilde{e}_1$, it is enough to give $P(\text{pr}(b))$ and $Q(\text{pr}(b))$ in terms of $P(b)$ and $Q(b)$.

$\text{pr}(b) = \text{pr}(b_{m}) \otimes \cdots \otimes \text{pr}(b_{1})$ can be constructed by applying Remark $\mathbb{H}$ to each tensor factor. The element $b$ is regarded as a skew tableau of shape $R_{m} \otimes \cdots \otimes R_{1}$. Let $\tilde{b} = b_{|n-1|}$ and write $\tilde{b} = \cdots \otimes \tilde{b}_{j} \otimes \cdots$, so that $\tilde{b}_{j} = b_{j}|n-1|$. Let $D_{j} = (\mu_{j}^{-1} + \eta_{j}, \mu_{j} - m_{n}(b_{j}))$ denote the shape of $\tilde{b}_{j}$, so that $\tilde{b}$ has shape $D_{m} \otimes \cdots \otimes D_{1}$.

Next, let $\tilde{b}_{j}'$ be the tableau of skew shape $D_{j}' = (\mu_{j}^{n}/(m_{n}(b_{j}))$ obtained from $\tilde{b}_{j}$ as in Remark $\mathbb{H}$. Write $\tilde{b}' = \tilde{b}_{m}' \otimes \cdots \otimes \tilde{b}_{1}'$, a skew column-strict tableau of shape $D' = D_{m}' \otimes \cdots \otimes D_{1}'$. Finally, $\text{pr}(b)$ is obtained by adjoining zeros to $\tilde{b}'$ at the vacated positions of the shape $R_{m} \otimes \cdots \otimes R_{1}$ that are not in $D'$, and then adding one to each entry.

Since $\text{word}({\tilde{b}}) = \text{word}(b)|_{n-1}$,

$$P(\tilde{b}) = P(\text{word}(\tilde{b})) = P(\text{word}(b)|_{n-1}) = P(\text{word}(b))|_{n-1} = P(b)|_{n-1}. $$

In other words, $P(\tilde{b})$ is obtained from $P(b)$ by removing the letters $n$, which occupy a horizontal strip (call it $H$). It is well-known that $Q(\tilde{b})$ is obtained from $Q(b)$ by reverse column insertions at the cells of $H$ starting with the rightmost cell of $H$ and proceeding to the left, ejecting a weakly increasing word $v$ of length $m_{n}(b)$. Another way to say this is that there is a unique weakly increasing word $v$ of length $m_{n}(b)$ such that $P(vQ(\tilde{b})) = Q(b)$. So the content of $v$ is the difference of the contents of $Q(b)$ and $Q(\tilde{b})$. Since $Q(b) \in \text{LRT}(R)$, it follows that $v$ is the weakly increasing word comprised of $m_{n}(b_{j})$ copies of the maximum letter of $A_{j}$ for all $j$.

In light of Remarks $\mathbb{H}$ and $\mathbb{I}$

$$Q(1 + \tilde{b}') = Q(\tilde{b}') = w_{0}^{R}Q(\tilde{b}),$$

where $w_{0}^{R}$ is the automorphism of conjugation corresponding to the longest element in the Young subgroup $S_{A_{1}} \times \cdots \times S_{A_{m}}$ in the symmetric group $S_{n}$. Recall that $1 + \tilde{b}' = \text{pr}(b)|_{2,n}$. Let $H_{1}$ be the skew shape given by the difference of the shapes of $P(\text{pr}(b))$ and $P(\text{pr}(b)|_{2,n})$. It is well-known that $Q(\text{pr}(b)|_{2,n})$ is obtained from $Q(\text{pr}(b))$ by reverse row insertions at the cells of $H_{1}$ starting from the rightmost and proceeding to the left. Let $u$ be the ejected word. Then using an argument similar to that above, $P(Q(\text{pr}(b)|_{2,n})u) = Q(\text{pr}(b))$ and $u$ is the weakly increasing word comprised of $m_{n}(b_{j})$ copies of the minimal letter of $A_{j}$ for all $j$. Since $u$ and $v$ are both weakly increasing words it is easy to calculate directly that $u = w_{0}^{R}v$. 

Therefore

\[ Q(b) = P(vQ(\hat{b})) \]
\[ Q(pr(b)) = P(Q(1 + \hat{b}^t)u) \]
\[ = P((w_0^RQ(\hat{b}))(w_0^Rv)). \]

**Remark 11.** In summary, the tableau pair \((P(pr(b)), Q(pr(b)))\) is constructed from the tableau pair \((P(b), Q(b))\) by the following steps. Let \(P = P(b)\) and \(Q = Q(b)\).

1. Let \(H\) be the horizontal strip given by the positions of the letters \(n\) in \(P\).
2. Let \(v\) be the weakly increasing word and \(\hat{Q}\) the tableau such that \(\text{shape}(\hat{Q}) = \text{shape}(Q) - H\), such that \(Q = P(v\hat{Q})\). These may be produced by reverse column insertions on \(Q\) at \(H\) from right to left.
3. Then \(Q(pr(b)) = P((w_0^R\hat{Q})(w_0^Rv)).\)
4. Let \(H_1\) be the horizontal strip \(\text{shape}(Q(pr(b))) - \text{shape}(\hat{Q})\).
5. Let \(P_1\) be the column-strict tableau given by adjoining to \(P|_{[n-1]}\) the letters \(n\) at the cells of \(H_1\).
6. \(P(pr(b)) = \text{pr}(P_1)\).

By [24, Proposition 15],

\[ (w_0^R\hat{Q})(w_0^Rv) = \chi_R^{-m_n(b)}(v\hat{Q}) \]

where \(\chi_R\) is the LR analogue of the right circular shift of a word by positions defined in [24].

All of these steps are invertible, so a description of \(\text{pr}^{-1}\) is obtained as well.

**Example 12.** Continuing the previous example, the image of \(b\) under the map \((3.3)\) is given by the tableau pair \(P = P(b)\) and \(Q = Q(b)\):

\[
\begin{align*}
P &= 1 1 1 1 1 1 3 3 \\
    & 2 2 2 2 2 2 4 6 \\
    & 3 3 3 3 3 4 5 7 \\
    & 4 4 4 \\
    & 5 5 5 7 \\
    & 6 6 \\
    & 7 7 \\
\end{align*}
\]

\[
\begin{align*}
Q &= 4 4 4 5 6 . \\
    & 5 5 5 7 \\
    & 6 6 \\
    & 7 7 \\
\end{align*}
\]

Then \(H\) is the skew shape given by the single cell \((7, 1)\), \(v = 7\), \(w_0^Rv = 6\),

\[
\begin{align*}
\hat{Q} &= 3 4 5 7 \\
    & 4 5 6 \\
    & 5 7 5 7 \\
    & 6 7 \\
\end{align*}
\]

and

\[
\begin{align*}
\hat{Q}(pr(b)) &= 3 4 5 7 \\
    & 4 5 6 \\
    & 5 7 \\
    & 7 .
\end{align*}
\]
3.9. The R-cocyclage and $\bar{e}_0$. In [24] the R-cocyclage relation was defined on the set $\text{LRT}(R)$. In the Kostka case this is a weak version of the dual of the cyclage poset of Lascoux and Schützenberger [18]. It is shown that every covering $R$-cocyclage relation, realized as recording tableaux, is induced by $\bar{e}_0$ on some element of $B^R$. Theorem 13. Let $ux$ be an $R$-LR word with $x$ a letter. Then there is an element $b \in B^R$ such that $Q(b) = P(ux)$ and $Q(\bar{e}_0(b)) = P(\chi_R(ux))$, provided that the cell $s = \text{shape}(P(ux))/\text{shape}(P(u))$ is not in the $n$-th row. In particular, every $R$-cocyclage covering relation is realized by the action of $\bar{e}_0$ in this way.

Proof. By definition (see [24]), every covering relation in the $R$-cocyclage has the form that $P(v)$ covers $P(\chi_R(v))$ where $v$ is an $R$-LR word. It follows from [24, Proposition 23] that if $P(ux) \rightarrow P(\chi_R(ux))$ is an $R$-cocyclage then $x > 1$. If $s$ is in the $n$-th row, then by the column-strictness of $P(ux)$ and the fact that all letters are in the set $[u]$, $x = 1$. So no $R$-cocyclage covering relations are excluded by the restriction that $s$ not be in the $n$-th row.

Let $x'u' = \chi_R(ux)$, that is, $x' = w_0^R x$ and $u' = w_0^R u$. Since $P(u') = P(w_0^R u) = w_0^R P(u)$ and the automorphism of conjugation $w_0^R$ preserves shape, without loss of generality it may be assumed that $u$ is the row-reading word of a column-strict tableau $U$ of partition shape $\lambda = \lambda - s$ where $\lambda = \text{shape}(P(ux))$. Now a skew tableau $t$ has partition shape if and only if $Q(t) = \text{Key}(u')$, so $u'$ is the word of the column-strict tableau $U' = w_0^R U$.

Let $Q = P(Ux)$, $\lambda = \text{shape}(Q)$, $s = (t, \lambda_t) = \lambda - \hat{\lambda}$, and $w = r_1 r_2 \ldots r_{t-1}$. Define $b \in B^R$ by $\mathbb{P}(b) = \text{Key}(w \lambda)$ and $Q(b) = Q$; such a $b$ exists since $Q \in \text{LRT}(\lambda; R)$ and (3.3) is a bijection. It must be shown that $Q(\bar{e}_0(b)) = P(x'u')$. This shall be verified by applying the formula (3.12) and Remark 11.

Now $\lambda_t > \lambda_n$ since $s = (t, \lambda_t)$ is a corner cell and $t < n$. The horizontal strip $H$ given by the cells of $\text{Key}(w \lambda)$ containing the letter $n$, is merely the $n$-th row of the shape $\lambda$. Since $Q \in \text{CST}(\lambda)$ (and all tableaux are in the alphabet $[u]$), the subtableau given by the first $\lambda_n$ columns of $Q$ is equal to $\text{Key}((\lambda_n^0))$. Let $Q_r$ be the rest of $Q$ and $R'$ the sequence of rectangles obtained by removing the first $\lambda_n$ columns from each of the rectangles in $R$. Since $Q$ is $R$-LR and the column-reading word of $Q$ equals that of $\text{Key}((\lambda_n^0))Q_r$, it follows that $Q_r$ is $R'$-LR. Let $y$ be the minimal element of the last interval $A_n$. In the notation of Remark 11 since $Q_r$ is $R'$-LR, it follows that $v = n^x_n$, $Q = \text{Key}((\lambda_n^{n-1}, 0))Q_r$, $w_0^R v = y^\lambda_n$, and $w_0^R Q = \text{Key}((\lambda_n^{n-1}, 0, \lambda_n^{-1+n_m}))Q_r$. So

$$Q(\mathbb{P}(b)) = P((w_0^R Q)(w_0^R v)) = \text{Key}((\lambda_n^{n-1}, 0, \lambda_n^{-1+n_m}))Q_r y^\lambda_n.$$  

(3.14)

The right hand side is a column-strict tableau of shape $(\lambda_1 + \lambda_n, \lambda_2, \ldots, \lambda_{n-1}, 0)$, so that the horizontal strip $H_1 = (\lambda_1 + \lambda_n)/(\lambda_1)$ is entirely in the first row,
and \( \mathbb{P}(\text{pr}(b)) \) is formed from \( 1 + \text{Key}(w(\lambda_1, \ldots, \lambda_{n-1}, 0)) \) by pushing the first row to the right by \( \lambda_n \) cells and placing 1’s in the vacated positions. The tableau \( \text{Key}(w(\lambda_1, \ldots, \lambda_{n-1}, 0)) \) contains \( \lambda_t \) ones. Hence \( \mathbb{P}(\text{pr}(b)) \) contains \( \lambda_t \) twos in its first row, that is, \( \epsilon_1(\mathbb{P}(\text{pr}(b))) = \lambda_t > 0 \).

So \( \mathbb{P}(\text{pr}(b)) \) admits \( \tilde{e}_0 \), which changes the letter 2 at the cell \( (1, m_n(b) + 1) \) to a 1. By (3.2),

\[
\mathbb{P}(\tilde{e}_1(\text{pr}(b))) = \tilde{e}_1(\mathbb{P}(\text{pr}(b))) \\
\mathbb{Q}(\tilde{e}_1(\text{pr}(b))) = \mathbb{Q}(\text{pr}(b)).
\]

Now \( \text{pr}^{-1} \) is applied, reversing the algorithm in Remark 14 starting with the tableau pair \( \mathbb{P}(\tilde{e}_1(\text{pr}(b))) \) and \( \mathbb{Q}(\tilde{e}_1(\text{pr}(b))) \), writing \( H'_1 \) and \( H' \) for the analogues of \( H_1 \) and \( H \). By Remark 14 and direct calculation,

\[
1 + \mathbb{P}(\text{pr}^{-1}(\tilde{e}_1(\text{pr}(b))))|_{|n-1\rangle} = P(\mathbb{P}(\tilde{e}_1(\text{pr}(b))))|_{|2,n\rangle} = 1 + \text{Key}(w(\lambda_1, \ldots, \lambda_{n-1}, 0) - \{s\}).
\]  

In particular \( H'_1 = H_1 \cup \{s\} \). The reverse row insertions on \( \mathbb{Q}(\tilde{e}_1(\text{pr}(b))) = \mathbb{Q}(\text{pr}(b)) \) at \( H_1 \subset H'_1 \) merely remove the \( \lambda_n \) copies of \( y \) from the first row by (3.14). The final reverse row insertion (at the cell \( s \)) stays within the subtableau \( Q_r \), and changes it to the tableau \( Q_r^- \) say, and ejects the letter \( x \), since since \( x \) is obtained from \( Q \) by reverse row insertion at \( s \) and \( Q = \text{Key}(\lambda_n)Q_r \). The result of the reverse row insertion on \( \mathbb{Q}(\tilde{e}_1(\text{pr}(b))) \) at \( H'_1 \) is

\[
Q_1 = \text{Key}(\lambda_n, 0, \lambda_n^{-1}\eta_m)Q_r^-,
\]

with ejected word \( xy^{\lambda_n} \). Writing \( U' = \text{Key}(\lambda_n)Q_r' \) and using the fact that \( Q_r \) is \( R' \)-LR,

\[
\mathbb{Q}(\tilde{e}_0(b)) = \mathbb{Q}(\text{pr}^{-1}(\tilde{e}_1(\text{pr}(b)))) \\
= P((y_0 w^{R} x y^{\lambda_n}) (w_0^R \text{Key}(\lambda_n^{-1}\eta_m)) Q_r^-) \\
= P(x' \text{Key}(\lambda_n) Q_r') \\
= P(x' U').
\]

**Remark 14.** Suppose \( B^R \) is a Kostka crystal with \( R_j = (\gamma_j) \) for all \( 1 \leq j \leq n \). Then Theorem 13 shows that every covering relation in the cyclage poset \( \bigcup_{\lambda} \text{CST}(\lambda; \gamma) \) is realized in the recording tableaux by \( \tilde{e}_0 \) acting on some \( b \in B^R \).

**Remark 15.** Let \( M = \max_i \mu_i \), the maximum number of columns among the rectangles \( R_i \) in \( R \). Suppose \( b \in B^R \) is an \( sl_n \)-highest weight vector such that \( \mathbb{Q}(b) \) has shape \( \lambda \), such that \( \lambda_1 > M \). Then \( \mathbb{Q}(b) \rightarrow \mathbb{Q}(\tilde{e}_0(b)) \) is a covering relation in the \( R \)-cocyclage.

To see this, let us adopt the notation of the proof of Theorem 13. Let \( s = (t, \lambda_t) \) be the corner cell in the last column of \( \lambda \). Then \( \lambda_t = \lambda_1 \) so that \( w_\lambda = \lambda \). By the choice of \( b \), \( \mathbb{P}(b) = \text{Key}(\lambda) = \text{Key}(w_\lambda) \).

To apply Theorem 13 it must be shown that \( t < n \). Suppose not. Then \( \lambda = (\lambda_t^+) \) and \( \mathbb{Q}(b) \) is an \( R \)-LR tableau of shape \( (\lambda_t^+)^n \). Since the total of the heights of the rectangles in \( R \) is \( n \), it follows that all of the rectangles in \( R \) must have exactly \( \lambda_1 \) columns, contradicting the assumption that \( \lambda_1 > M \).
Apply the reverse row insertion on $Q(b)$ at the cell $s$, obtaining the column-strict tableau $U$ of shape $\lambda - \{s\}$ and ejecting the letter $x$. Then $P(Ux) \rightarrow P(\chi_R(Ux))$ is an $R$-cocyclage, by [24, Remark 17].

Example 16. Continuing the example, $\bar{e}_0(b)$ is computed below. $Q(\bar{e}_1(pr(b))) = Q(pr(b))$ and

$$P(\bar{e}_1(pr(b))) = \begin{array}{cccc}
1 & 1 & 2 & 2 \\
3 & 3 & 3 & 3 \\
4 & 4 & 4 & 4 \\
5 & 5 & 5 & 5 \\
6 & 6 & 6 & 7 \\
\end{array}.$$  

Applying $pr^{-1}$ to the tableau pair of $\bar{e}_1(pr(b))$ and denoting by $P'_1$, $\hat{Q}'$, $v'$ the corresponding tableaux and word, we obtain

$$P'_1 = \begin{array}{cccc}
1 & 1 & 1 & 2 & 7 \\
2 & 2 & 2 & 3 \\
3 & 3 & 3 & 7 \\
4 & 4 & 4 & 7 \\
5 & 5 & 6 & 7 \\
6 & 6 & 7 \\
\end{array}, \quad w'^R_0 \hat{Q}' = \begin{array}{cccc}
3 & 4 & 5 \\
4 & 5 & 6 \\
5 & 5 & 7 \\
6 & 7 \\
\end{array},$$

$L = 36$, $v' = 57$,

$$\hat{Q}' = \begin{array}{cccc}
1 & 1 & 3 & 6 \\
2 & 2 & 4 & 7 \\
3 & 3 & 5 \\
4 & 4 & 6 \\
5 & 7 \\
6 & 6 & 7 \\
\end{array}, \quad Q(\bar{e}_0(b)) = \begin{array}{cccc}
1 & 1 & 3 & 6 \\
2 & 2 & 4 & 7 \\
3 & 3 & 5 \\
4 & 4 & 6 \\
5 & 5 & 7 \\
6 & 7 \\
\end{array}.$$

and finally

$$\hat{Q}' = \begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 3 \\
3 & 3 & 3 \\
4 & 4 & 4 \\
5 & 5 & 7 \\
6 \\
7 \\
\end{array}, \quad \hat{Q}' = \begin{array}{cccc}
1 & 1 & 1 & 2 \\
2 & 2 & 2 & 3 \\
3 & 3 & 3 \\
4 & 4 & 4 \\
5 & 5 & 7 \\
6 \\
7 \\
\end{array}.$$  

Remark 17. Suppose that $R = (R_1, \ldots, R_m)$ is such that each $R_j$ has a common number of columns, say $l$. Then $B^R$, being the tensor product of perfect crystals $B^{n_j,t}$ of level $l$, is perfect of level $l$ and therefore connected by [6]. In this case, more is true. Using Remark [3], every element can be connected to the unique $sl_n$-highest weight vector in $B^R$ of charge zero, by applying last column $R$-cocyclages on the recording tableau.

For $R$ a general sequence of rectangles, $B^R$ is still connected. However it is not necessarily possible to use $\bar{e}_0$ to connect every $sl_n$-component to the zero energy component in such a way that the energy always drops by one. For example,
take \( n = 3, \ R = ((2), (1), (1)) \) and the \( sl_3 \)-component with \( Q \)-tableau \( \begin{array}{cc} 1 & 1 \\ 3 & \end{array} \). The applications of \( \tilde{e}_0 \) on the five elements of this component that admit \( \tilde{e}_0 \), all produce elements in the component with \( Q \)-tableau \( \begin{array}{cc} 1 & 1 \\ 3 & \end{array} \) which has the same energy.

3.10. Rectangle-switching bijections and \( \tilde{e}_0 \).

**Proposition 18.** Let \( R_1 \) and \( R_2 \) be rectangles. Then the rectangle-switching bijection

\[
\sigma_{R_2, R_1} : B^{R_2} \otimes B^{R_1} \to B^{R_1} \otimes B^{R_2}
\]

is an isomorphism of classical \( \widehat{sl}_n \) crystals.

**Proof.** Since \( \sigma = \sigma_{R_2, R_1} \) is known to be an isomorphism of \( sl_n \)-crystals, it only remains to show that \( \sigma \) commutes with \( \tilde{e}_0 \). Let \( b \in B^{R_2} \otimes B^{R_1} \). By the bijectivity of (3.13) it is enough to show that

\[
\begin{align*}
\mathbb{P}(\tilde{e}_0(\sigma(b))) &= \mathbb{P}(\sigma(\tilde{e}_0(b))) \\
\mathbb{Q}(\tilde{e}_0(\sigma(b))) &= \mathbb{Q}(\sigma(\tilde{e}_0(b))).
\end{align*}
\]

(3.16)

Consider first the process in passing from \( (\mathbb{P}(b), \mathbb{Q}(b)) \) to \( (\mathbb{P}(\tilde{e}_0(b)), \mathbb{Q}(\tilde{e}_0(b))) \). Let \( H \) and \( H_1 \) be the horizontal strips, \( v \) the weakly increasing word and \( \hat{Q} \) the tableau as in Remark 11. Let \( b' = \sigma(b) \), and let \( H', H'_1, v', \hat{Q}' \) the analogous objects in passing from \( (\mathbb{P}(b'), \mathbb{Q}(b')) \) to \( (\mathbb{P}(\tilde{e}_0(b')), \mathbb{Q}(\tilde{e}_0(b'))) \).

Let \( \lambda = \text{shape}(\mathbb{P}(b)) \). Observe that \( \mathbb{P}(b') = \mathbb{P}(\sigma(b)) = \mathbb{P}(b) \) by (3.13), so that \( \text{shape}(\mathbb{P}(b')) = \lambda \) and \( H' = H \). This implies that the increasing words \( v' \) and \( v \) have the same length \( m_\lambda(b) \) and \( \text{shape}(\hat{Q}') = \text{shape}(\hat{Q}) \); call their common shape \( \hat{\lambda} \).

Now \( P(v' \hat{Q}') = \mathbb{Q}(b) \) is the unique element in \( \text{LRT}(\lambda; (R_2, R_1)) \) and \( P(v \hat{Q}) = \mathbb{Q}(b) \) is the unique element in \( \text{LRT}(\lambda; (R_1, R_2)) \). So

\[
\begin{align*}
P(\tau(v \hat{Q})) &= \tau P(v \hat{Q}) \\
&= \tau \mathbb{Q}(b) = \mathbb{Q}(b') = P(v' \hat{Q}').
\end{align*}
\]

(3.17)

On the other hand, consider the word \( v \hat{Q}', \) identifying \( \hat{Q} \) with its row-reading word. The tableau \( \mathbb{Q}(v \hat{Q}) \) has shape \( \lambda \). Let \( p \) and \( q \) be the number of cells in \( \hat{Q} \) and \( Q \) respectively. Since \( \hat{Q} \) is a column-strict tableau of shape \( \hat{\lambda} \), \( \mathbb{Q}(v \hat{Q})|_{[p]} \) is the rowwise standard tableau of shape \( \hat{\lambda} \), the unique standard tableau of shape \( \lambda \) in which \( i + 1 \) is located immediately to the right of \( i \) provided \( i + 1 \) is not in the first column. Now \( \mathbb{Q}(v \hat{Q})|_{[p]} \) is filled from left to right by the numbers \( p + 1 \) through \( q \), since it records the column-insertion of the weakly increasing word \( v \). The same argument applies to \( \mathbb{Q}(v' \hat{Q}') \), so that \( \mathbb{Q}(v \hat{Q}) = \mathbb{Q}(v' \hat{Q}') \). By this, (3.17), and (3.3), it follows that \( \tau(v \hat{Q}) = v' \hat{Q}' \). Write \( R' = (R_2, R_1), w_0^{R'} \) and \( \chi_{R'} \) for the corresponding constructions for \( R' \). By [24, Theorem 16],

\[
(w_0^{R'} \hat{Q}')(w_0^{R'} v') = \chi_{R'}^{-m_\lambda(b)}(v' \hat{Q}') = \chi_{R'}^{-m_\lambda(b)}(\tau(v \hat{Q})) = \tau(\chi_{R'}^{-m_\lambda(b)}(v \hat{Q})) = \tau((w_0^R \hat{Q})(w_0^R v)).
\]
Applying the $P$ tableau part of (3.5) to the word $(w_0^R Q)(w_0^R v)$,
\[
\tau(Q(\tilde{e}_0(b))) = \tau(P((w_0^R \hat{Q})(w_0^R v))) \\
= P(\tau((w_0^R \hat{Q})(w_0^R v))) \\
= P((w_0^R \hat{Q}')(w_0^R v')) \\
= Q(\tilde{e}_0(b')).
\]
By this and the $Q$ tableau part of (3.4) for the word $\tilde{e}_0(b)$,
\[
Q(\sigma(\tilde{e}_0(b))) = \tau(Q(\tilde{e}_0(b))) \\
= Q(\tilde{e}_0(b')) \\
= Q(\tilde{e}_0(\sigma(b))
\]
This proves the equality of the $Q$-tableaux in (3.16).
For the $P$-tableaux, let us recall the process that leads from $(P(b), Q(b))$ to $(P(\tilde{e}_0(b)), Q(\tilde{e}_0(b)))$ and from $(P(b'), Q(b'))$ to $(P(\tilde{e}_0(b')), Q(\tilde{e}_0(b')))$ where $b' = \sigma(b)$. Recall that $P(b') = P(b)$. Clearly they have the same restriction to the alphabet $[n - 1]$. On the other hand, since it has been shown that the tableaux $Q(\tilde{e}_0(b))$ and $Q(\tilde{e}_0(b'))$ have the same shape, it follows that the horizontal strips $H_1$ and $H'_1$ coincide. So $P(\tilde{e}_0(b))$ and $P(\tilde{e}_0(b'))$ coincide when restricted to the alphabet $[2, n]$. Since both tableaux also have the same partition shape they must coincide. This, together with the $P$-tableau part of (3.6) applied to $\tilde{e}_0(b)$, shows that
\[
P(\tilde{e}_0(\sigma(b))) = P(\tilde{e}_0(b')) \\
= P(\tilde{e}_0(b)) = P(\sigma(\tilde{e}_0(b))).
\]

3.11. Energy function. In this section it is shown that the energy function of the classical $\tilde{s}_n$ crystal $B^R$ is given by the generalized charge of $\tilde{P}$ on the $Q$-tableau. The definition of energy function follows [1] and [23].

Consider the unique classical $\tilde{s}_n$ crystal isomorphism
\[
\sigma : B^{R_2} \otimes B^{R_1} \cong B^{R_1} \otimes B^{R_2} \\
b_2 \otimes b_1 \mapsto b_1' \otimes b_2'.
\]
An energy function $H : B^{R_2} \otimes B^{R_1} \to \mathbb{Z}$ is a function that satisfies the following axioms:

(H1) $H(f_i(b)) = H(b)$ for all $i \in J$ and $b \in B^{R_2} \otimes B^{R_1}$ such that $f_i(b)$ is defined, and similarly for $\tilde{f}_i$.

(H2) For all $b = b_2 \otimes b_1 \in B^{R_2} \otimes B^{R_1}$ such that $\tilde{e}_0(b)$ is defined,
\[
H(\tilde{e}_0(b)) - H(b) = \begin{cases} 
1 & \text{if } \epsilon_0(b_2) \leq \phi_0(b_1) \text{ and } \epsilon_0(b'_1) \leq \phi_0(b'_2) \\
-1 & \text{if } \epsilon_0(b_2) > \phi_0(b_1) \text{ and } \epsilon_0(b'_1) > \phi_0(b'_2) \\
0 & \text{otherwise.}
\end{cases}
\]
If $H' : B^{R_1} \otimes B^{R_2} \to \mathbb{Z}$ is defined in the same way then $H' \circ \sigma = H$.

Lemma 19. $B^{R_2} \otimes B^{R_1}$ is connected.

Proof. Without loss of generality assume that $\mu_1 \geq \mu_2$. Let
\[
\gamma(R) = (\mu_1^{R_1}, \mu_2^{R_2})
\]
be the partition whose Ferrers diagram is obtained by placing the shape $R_1$ atop $R_2$. Let $Q_R$ be the unique element of the singleton set $\text{LRT}(\gamma(R); (R_1, R_2))$. Observe that $\gamma(R)$ is the only shape admitting an $R$-LR tableau that has at most $\mu_1$ columns. Let $v_R \in B^R$ be the unique $sl_n$-highest weight vector of weight $\gamma(R)$; it is given explicitly by $v_R = Y_2 \otimes Y_1$ in the notation of the definition of $R$-LR, and satisfies $\mathbb{P}(v_R) = \text{Key}(\gamma(R))$ and $\mathbb{Q}(v_R) = Q_R$.

It is shown that every element $b \in B^R$ is connected to $v_R$. First, using $sl_n$-raising operators, it may be assumed that $b$ is an $sl_n$-highest weight vector. If $\mathbb{Q}(b)$ has at most $\mu_1$ columns then $b = v_R$. Otherwise $\mathbb{Q}(b)$ has more than $\mu_1$ columns, and Remark 15 applies. But $\bar{e}_0(b)$ is closer to $v_R$ in the sense that $\mathbb{Q}(\bar{e}_0(b))$ has one fewer cells to the right of the $\mu_1$-th column than $\mathbb{Q}(b)$ does \cite[Proposition 38]{24}. Proposition 38, so induction finishes the proof. 

By Lemma 14, $H$ is uniquely determined up to a global additive constant. $H$ is normalized by the condition

$$H(v_R) = 0$$

with $v_R$ as in Lemma 13. Equivalently,

$$H(\text{Key}(R_2) \otimes \text{Key}(R_1)) = \min(\eta_1, \eta_2) \min(\mu_1, \mu_2),$$

where $\text{Key}(R_i)$ is the highest-weight vector in $B^{R_i}$ and the value of $H$ is the size of the rectangle $R_1 \cap R_2$.

For a tableau $Q \in \text{LRT}(\lambda; (R_1, R_2))$, define $d_{R_1, R_2}(Q)$ to be the number of cells in the shape of $Q$ that are strictly east of the $\mu_1\mu_2$-th column.

**Proposition 20.** Let $R = (R_1, R_2)$ be a pair of rectangles. Then for all $b \in B^R$,

$$H(b) = d_{R_1, R_2}(\mathbb{Q}(b)).$$

**Proof.** Follows immediately from the proof of Lemma 13. 

Now consider $R = (R_1, \ldots, R_m)$ with $B_j = B^{R_j}$ and $B = B^R = B_m \otimes \cdots \otimes B_1$. The energy function for $B$ is given as follows. Denote by $H_{i,w} : wB \rightarrow \mathbb{Z}$ given by the value of the energy function $H_{(wB)^{i+1}, (wB)}$ at the $(i+1)$-st and $i$-th tensor positions (according to our convention). Recall the isomorphisms of classical $\hat{sl}_n$-crystals \cite{34, 37}. Define the cyclic permutation $w_{i,j} = r_{i+1}r_{i+2} \cdots r_{j-1}$ for $1 \leq i < j \leq m$. Define $\mathcal{E}_R : B \rightarrow \mathbb{Z}$ by

$$E^j_R(b) = \sum_{1 \leq j \leq m} \sum_{1 \leq i < j} H_{i,w_{i,j}}(\sigma_{R,w_{i,j}}R(b)).$$

Call the inner sum $E^{(j)}_R(b)$.

The following version of \cite[Lemma 5.1]{24} holds for $E_R$ with no additional difficulty.

**Lemma 21.** Let $B = B^R$ where $R_j = (\mu_j^{l_j})$ and $l \geq \mu_j$ for all $j$. Suppose $\bar{e}_0(b) > \mu_j$ for all $j$. Write $\bar{e}_0(b) = \cdots \otimes b_{k+1} \otimes \bar{e}_0(b_k) \otimes b_{k-1} \cdots$. Then $k > 1$ and $E^{(j)}_R(\bar{e}_0(b)) = E^{(j)}_R(b) - \delta_{j,k}$. 


Example 22. Let \( b \) be as in the running example. Then

\[
\begin{align*}
\times \times \times \times \times \times 1 & 1 \\
\times \times \times \times \times 2 & 2 \\
\times \times \times 1 & 3 & 3 \\
\tau_2(b) = \times & \times & \times & 2 & 4 & 4 \\
1 & 3 & 5 \\
2 & 4 & 6 \\
3 & 5 & 7
\end{align*}
\]

Write \( \tau_2(b) = b'_2 \otimes b'_1 \otimes b_1 \). Now \( P(\text{word}(b_2 \otimes b_1)) \) has shape \((4, 3, 3, 2, 1)\) so that \( d_1(b) = 1 \). \( P(\text{word}(b_3 \otimes b_2)) \) has shape \((4, 4, 3, 2, 2)\) so that \( d_2(b) = 2 \). Finally \( P(\text{word}(b'_3 \otimes b_1)) \) has shape \((3, 3, 2, 2)\) so that \( d_1(\tau_2(b)) = 0 \). So \( E_R(b) = 1 + 2 + 0 = 3 \).

3.12. Energy and generalized charge. Define the map \( E_R : \text{LRT}(R) \rightarrow \mathbb{Z} \) by \( E_R(Q) = E_R(b) \) for any \( b \in B^R \) such that \( Q(b) = Q \). This map is well-defined since \( E_R \) is constant on \( sl_n \)-components and the map \( (3.3) \) is a bijection. It follows immediately from the definitions that

\[
E_R(Q) = \sum_{1 \leq i < j \leq m} d_{R_i, R_j}(\tau_{R_{i+1}, R_j} \circ \cdots \circ \tau_{R_{i-1}, R_j})(Q).
\]

The Kostka case of the following result was first proven by K. Kilpatrick and D. White. In the further special case that \( \mu \) is a partition it was shown in [22] that \( E_R(Q) \) is the charge. Now in the Kostka case the generalized charge statistic \( \text{charge}_R \) specializes to the formula of charge in [15].

Theorem 23. \( \text{charge}_R = E_R \) on \( \text{LRT}(R) \).

Proof. Let \( Q \in \text{LRT}(R) \) of shape \( \lambda \), say. It will be shown by induction on the number of rectangles \( m \) and then on \( \text{charge}_R(Q) \), that that \( E_R : \text{LRT}(R) \rightarrow \mathbb{Z} \) satisfies the intrinsic characterization of \( \text{charge}_R \) by the properties (C1) through (C4) [24, Theorem 21]. Let \( M = \max_{j} \mu_j \).

First, (C2) need only be checked when \( Q' <_R Q \) is a last column \( R \)-cocykale, and (C4) need only be verified when \( \lambda_1 = M \). To see this, if \( \lambda_1 > M \) then there is a last column \( R \)-cocykale \( Q' <_R Q \) and in this case \( \text{charge}_R(Q') = \text{charge}_R(Q) - 1 \). Otherwise \( \lambda_1 = M \). If (C3) does not apply, then one may apply (C4) several times to switch the widest rectangle closer to the beginning of the sequence \( R \) and then apply (C3), which decreases the number of rectangles \( m \).

(C1) is trivial. For (C2), let \( Q' <_R Q \) be a last-column \( R \)-cocykale with \( \text{shape}(Q) = \lambda \) such that \( \lambda_1 > M \). Let \( P = \text{Key}(\lambda) \) and \( b \) such that \( P(b) = \text{Key}(\lambda) \) and \( Q(b) = Q \) as in Theorem 13 and Remark 13. By the proof of Theorem 13 in this case, \( \epsilon_0(b) = \epsilon_1(\mathbb{P}(\text{pr}(b))) = \lambda_i = \lambda > M \). By Lemma 21,

\[
E_R(Q(\tilde{x}_0(b))) = E_R(\tilde{x}_0(b)) = E_R(b) - 1 = E_R(Q(b)) - 1 = E_R(Q) - 1.
\]

However, \( Q(\tilde{x}_0(b)) = Q' \) by Remark 13, so \( E_R(Q') = E_R(Q) - 1 \), and (C2) has been verified.

To check (C3), let \( \hat{R} = (R_2, \ldots, R_m) \) and \( \hat{Q} = Q|_{[\tau_1 + 1, n]} \). Then \( Q \) consists of \( \text{Key}(R_i) \) sitting atop \( \hat{Q} \) and \( \hat{Q} \in \text{LRT}(\hat{R}) \). It follows that

\[
d_{R_1, R_j}(\tau_{R_2, R_j} \tau_{R_3, R_j} \cdots \tau_{R_{j-1}, R_j} Q) = 0
\]
for all $j > 1$. Therefore
\[
E_R(Q) = \sum_{1 \leq i < j \leq m} d_{R_i, R_j}(\tau_{R_{i+1}, R_j} \cdots \tau_{R_{j-1}, R_j} Q)
\]
\[
= \sum_{2 \leq i < j \leq m} d_{R_i, R_j}(\tau_{R_{i+1}, R_j} \cdots \tau_{R_{j-1}, R_j} Q)
\]
\[
= E_R(\hat{Q}),
\]
which verifies (C3).

For (C4), the proof may be reduced to the case $m = 3$. By abuse of notation we suppress the notation for the sequence of rectangles, writing $\tau_p(Q)$ for the operator that acts on the restriction of an LR tableau to the $p$-th and $(p + 1)$-st alphabets, and similarly for the function $d_p$. Write $w_{i,j} := \tau_{i+1} \tau_{i+2} \cdots \tau_{j-1}$ for $1 \leq i < j \leq m$.

Fix $1 \leq p \leq m - 1$ and $1 \leq i < j \leq m$. Write
\[
d'_{i,j} := d_i(w_{i,j} \tau_p Q)
\]
\[
d_{i,j} := d_i(w_{i,j} Q),
\]
The value $d'_{i,j}$ is computed using a case by case analysis.

1. $i < p + 1$. In this case it is clear that $d'_{i,j} = d_{i,j}$.
2. $i = p + 1$. Then $w_{p+1,j} \tau_p = \tau_p w_{p+1,j}$ and
\[
d'_{p+1,j} = d_{p+1}(\tau_p w_{p+1,j} Q).
\]
3. $p = i$ and $p + 1 < j$. Then
\[
w_{p,j} \tau_p = \tau_{p+1} w_{p+1,j} \tau_p = \tau_p w_{p+1,j} Q
\]
so that
\[
d'_{p,j} = d_p(\tau_p w_{p+1,j} Q).
\]
4. $p = i$ and $p + 1 = j$. Here $w_{i,j}$ is the identity, and
\[
d'_{p,p+1} = d_p(\tau_p Q) = d_p(Q) = d_{p,p+1}.
\]
5. $i < p$ and $p + 1 < j$. Then $w_{i,j} \tau_p = \tau_{p+1} w_{i,j}$ so that
\[
d'_{i,j} = d_i(\tau_{p+1} w_{i,j} Q) = d_i(w_{i,j} Q) = d_{i,j}.
\]
since the restriction of $w_{i,j} Q$ to the $i$-th and $i + 1$-st subalphabets is not affected by $\tau_{p+1}$.
6. $i < p$ and $j = p + 1$. Then $w_{i,p+1} \tau_p = w_{i,p}$ and
\[
d'_{i,p+1} = d_i(w_{i,p} Q) = d_{i,p}.
\]
7. $i < p$ and $j = p$. Then $w_{i,p} \tau_p = w_{i,p+1}$ and
\[
d'_{i,p} = d_i(w_{i,p+1} Q) = d_{i,p+1}.
\]
8. $j < p$. In this case it is clear that $d_{i,j} = d'_{i,j}$.

Based on these computations, the difference in energies $E_{\tau_p R}(\tau_p Q) - E_R(Q)$ is given as follows. In cases 1, 4, and 5, and 8, $d'_{i,j} = d_{i,j}$ so these terms cancel. The sum
of the terms in cases 6 and 7 cancel. So it is enough to show that the sum of terms in 2 and 3 cancel, that is,

\[ 0 = \sum_{j > p+1} (d_{p+1}(\tau_{p}w_{p+1,j}Q) - d_{p+1}(w_{p+1,j}Q)) + \sum_{j > p+1} (d_{p}(\tau_{p+1}\tau_{p}w_{p+1,j}Q) - d_{p}(w_{p,j}Q)). \]

Rewriting \( d_{p}(w_{p,j}Q) = d_{p}(\tau_{p}w_{p+1,j}Q) \), observe that without loss of generality it may be assumed that \( m = 3 \) and it must be shown that

\[ (3.18) \quad 0 = d_{2}(\tau_{1}\tau_{2}Q) - d_{2}(Q) + d_{1}(\tau_{2}\tau_{1}Q) - d_{1}(Q). \]

Recall that in verifying (C4) it may be assumed that \( \lambda_1 = M \). In this case [24, Remark 39] applies. Say \( \mu_k = \lambda_1 \). Then \( d_k(Q) = 0 \) and if \( k > 1, d_{k-1}Q = 0 \) There are three cases, namely \( k = 1, k = 2, \) or \( k = 3 \). If \( k = 3 \) then all four terms in (3.18) are zero. If \( k = 1 \) then the first and fourth terms are zero and the second and third agree, while if \( k = 2 \) then the second and third are zero and the first and fourth agree.

Corollary 24.

\[ \sum_{b \in B} e^{\text{wt}(b)} q^{E_{R}(b)} = \sum_{\lambda} \text{ch} V^{\text{wt}(\lambda)} K_{\lambda; R}(q), \]

where \( \lambda \) runs over partitions of length at most \( n \).

Proof. The equality follows immediately from Theorem 23, the weight-preserving bijection \{b\}, and [24, Theorem 11].

4. Tensor product structure on Demazure crystals

The tensor product structure for the Demazure crystals, is a consequence of an inhomogeneous version of \[10, \text{Theorem 2.3}\] that uses Lemma 21.

Theorem 25. Let \( \mu = (\mu_1, \ldots, \mu_m) \) be a partition of \( n \). Then

\[ B_{w_{\mu_{i}}}(\Lambda_{0}) \cong B^{\mu_{m},l} \otimes \cdots \otimes B^{\mu_{1},l} \otimes u_{l\Lambda_{0}} \]

as classical \( \widehat{\mathfrak{sl}}_{n} \) crystals, where the affine \( \widehat{\mathfrak{sl}}_{n} \) Demazure crystal is viewed as a classical \( \mathfrak{sl}_{n} \)-crystal by composing its weight function \( \text{wt} \) with the projection \( P \rightarrow P_{cl} \). Moreover, if \( v \mapsto b \otimes u_{l\Lambda_{0}} \) then

\[ \langle d, l\Lambda_{0} - \text{wt}(v) \rangle = E_{R}(b) \]

where the left hand side is the distance along the null root \( \delta \) of \( v \) from the highest weight vector \( u_{l\Lambda_{0}} \in V(l\Lambda_{0}) \) and \( R \) is defined by \( R_{j} = (l^{v_{j}}) \) for \( 1 \leq j \leq m \).

5. Proof of Theorem 2

Theorem 2 follows from Theorem 25 and Corollary 24.
6. Generalization of Han’s monotonicity for Kostka-Foulkes polynomials

The following monotonicity property for the Kostka-Foulkes polynomials was proved by G.-N. Han [4]:

\[ K_{\lambda,\mu}(q) \leq K_{\lambda \cup \{a\},\mu \cup \{a\}}(q) \]

where \( \lambda \cup \{a\} \) denotes the partition obtained by adding a row of length \( a \) to \( \lambda \).

Here is the generalization of this result for the polynomials \( K_{\lambda,R}(q) \) that was conjectured by A. N. Kirillov.

**Theorem 26.** Let \( R \) be a dominant sequence of rectangles and \((k^m)\) another rectangle. Then

\[ K_{\lambda,R}(q) \leq K_{\lambda \cup (k^m); R \cup (k^m)}(q) \]

where \( \lambda \cup (k^m) \) is the partition obtained by adding \( m \) rows of size \( k \) to \( \lambda \) and \( R \cup (k^m) \) is any dominant sequence of rectangles obtained by adding the rectangle \((k^m)\) to \( R \).

**Proof.** Write \( R = (R_1,\ldots,R_t) \), \( R_0 = (k^m) \), and \( R^+ = (R_0,R_1,\ldots,R_t) \). Define the map \( i_R : \text{LRT}(\lambda;R) \to \text{LRT}(\lambda \cup (k^m); R^+) \) by \( i_R(Q) = P((Q + m)\,Y_0) \) where \( Y_0 = \text{Key}(k^m) \). Since the letters of \( Y_0 \) are smaller than those of \( Q + m \), it follows that \( \text{shape}(i_R(Q)) = \lambda \cup (k^m) \). Moreover \( i_R(Q) \) is \( R^+\text{-LR} \) since it is Knuth equivalent to a shuffle of \( Y_0 \) and the tableau \( Q + m \), which is \( R\text{-LR} \) in the alphabet \([m+1,n+m]\). Thus the map \( i_R \) is well-defined. Let \( B \) represent the union of the zero-th and first subalphabets for \( w_{0,j}R^+ \) and let \( Y \) be the key tableau for the first subalphabet of \( w_{0,j}R^+ \). Then

\begin{align}
\text{d}_0(w_{0,j}i_R(Q)) &= d_0(w_{0,j}QY_0) = d_0((w_{0,j}Q)Y_0) = d_0((w_{0,j}Q)Y_0)|_B = 0,
\end{align}

(6.1)

by the Knuth invariance of \( d_0 \), the fact that \( w_{0,j} \) doesn’t touch letters in the zero-th subalphabet, the definition of \( d_0 \), the fact that \( w_{0,j}Q \in \text{LRT}(w_{0,j}R) \), and direct calculation of the shape of \( P(Y_0) \) combined with Proposition 2. If \( i > 0 \) then

\begin{align}
\text{d}_i(w_{i,j}i_R(Q)) &= d_i((w_{i,j}i_R(Q))|_{[m+1,m+n]})
\end{align}

(6.2)

\begin{align}
&= d_i(w_{i,j}(i_R(Q)|_{[m+1,m+n]}))
&= d_i(w_{i,j}Q).
\end{align}

From (6.1) and (6.2) it follows that \( E_{R^+}(i_R(Q)) = E_R(Q) \). \( \square \)

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