The intermediate and spin-liquid phase of the half-filled honeycomb Hubbard model

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We obtain the phase-diagram of the half-filled honeycomb Hubbard model with density matrix embedding theory, to address recent controversy at intermediate couplings. We use clusters from 2-12 sites and lattices at the thermodynamic limit. We identify a paramagnetic insulating state, with possible hexagonal cluster order, competitive with the antiferromagnetic phase at intermediate coupling. However, its stability is strongly cluster and lattice size dependent, explaining controversies in earlier work. Our results support the paramagnetic insulator as being metastable, rather than a true, intermediate phase, in the thermodynamic limit.

PACS numbers: 71.10.-w, 71.10.Fd, 71.30.+h, 75.10.Kt, 71.20.-b, 71.15.-m

Recently, there has been much debate over the zero-temperature phases of the Hubbard honeycomb model at half-filling. The accepted picture for many years was that for small interactions \( U \), the system is semi-metallic (SM) with a Dirac cone in the density of states [1]; at large \( U \), the system is antiferromagnetically long-range ordered (AFM) [2, 3]. However, Meng et al. proposed recently that an additional phase appears at intermediate couplings, arising from strong quantum fluctuations due to the low coordination number [4]. This phase has been further suggested to be a gapped spin liquid [5–8]. If present, this would be extremely significant, as spin liquids are not known to exist at half-filling without frustration [9–11], and would greatly advance the search for experimental realizations of 2D spin-liquids, both in correlated honeycomb materials as well as optical lattices [12, 13].

A large number of numerical methods have been applied to study the honeycomb Hubbard model [1–4, 11, 14–31]. The work by Meng et al. used zero-temperature auxiliary-field (determinant) quantum Monte Carlo (AFQMC) [4]. These results were viewed with particular confidence because AFQMC has no sign problem in this model and thus correlations were treated “exactly”, the only errors arising from using a finite size lattice of 648 sites. Subsequent to this, many reports of an intermediate phase have appeared using quantum cluster methods [32], such as Cluster Dynamical Mean-Field Theory (CDMFT) with exact diagonalization (ED) (Liebsch [24] and He et al. [26]) and quantum Monte Carlo (CT-QMC) solvers (Wu et al.) [28], and the Variational Cluster Approximation (VCA) [27]. While these various quantum cluster methods differ in some details, they share a unifying Green function and self-energy functional formulation [33, 34], thus we refer to them collectively as Green function cluster (GFC) methods.

Despite these initial reports, there now appears increasing evidence that there may in fact be no spin-liquid phase [11, 18, 20]. The strongest hint is from Sorella et al. [20], who repeated Meng’s AFQMC calculation with a larger lattice of 2592 sites. They found that at the increased lattice size, the region for an intermediate phase shrunk significantly, suggesting weak (if any) evidence for a spin-liquid phase. Subsequently, a further VCA calculation by Hassan and Senechal, in contrast to earlier quantum cluster calculations, also found no spin-liquid phase [11].

These conflicting reports raise important questions both for the physics of the honeycomb Hubbard model, and the numerical methods used to study it. Is there an intermediate phase, and is it a spin-liquid? If not, what is observed in calculations which see an intermediate phase - is the system “close” to a spin-liquid or some other state? Why do some quantum cluster calculations observe an intermediate phase and others not? And how do the various numerical approximations, such as finite lattice size in AFQMC, or finite cluster size in GFC methods, bias the calculations? These are the main questions we target in this work.

To answer these questions, we use a different quantum cluster method to those used previously - the recently proposed Density Matrix Embedding Theory (DMET) [35–38]. Note that DMET is not a variant of the GFC methods discussed above. In principle, GFC methods complement AFQMC, because they work in the thermodynamic limit, but only treat a finite range of correlations determined by the impurity cluster size. In practice, however, solving the impurity problem within GFC methods involves further numerical approximations, such as bath discretization, or analytic continuation. Both require careful treatment to avoid affecting the physics. In contrast, DMET is a quantum cluster method without bath discretization error by construction, and which yields a quantum impurity problem where spectral functions can be practically computed without analytic continuation. This allows us to study the honeycomb Hubbard model as a function of cluster correlation length and at the thermodynamic limit with no further extraneous
nearest-neighbour spin-spin correlation, \( U \) bers are from ter shapes in our study, for \( N \)bers derived from the and GFC methods \([39–42]\), is that the DMET bath is purity plus bath problem, and is exact in the weak- and numerical approximations.

DMET maps a large lattice problem to a quantum impurity plus bath problem, and is exact in the weak- and strong-coupling limits, as well as the limit of infinite cluster size. The main conceptual difference between DMET and GFC methods \([39–42]\), is that the DMET bath is derived from the entanglement of the quantum state of the impurity cluster, rather than its single-particle Green function. DMET has several important technical advantages. If one is interested in thermodynamic properties, no Green functions need be computed; only the ground-state quantum impurity problem is solved, a significant computational savings. Further, since the DMET bath is defined from a Schmidt decomposition, the bath is the same size as the impurity with no bath discretization error, in contrast to the infinite bath in DMFT. Typically, an \( n \)-site cluster \((n > 1)\) DMET calculations yields similar physics to an \( n \)-site CDMFT that is converged with respect to the bath parametrization, even though the DMET calculations require only a fraction of the cost \([35–38]\).

We now describe the DMET method briefly; for more details we refer to the original references \([35–37]\). We first choose an impurity cluster, cut from the underlying lattice. The underlying lattice is finite, but can trivially be made very large: in this study, we use lattices with more than 90,000 sites. The “environment” lattice sites outside the cluster are then replaced by a bath. In ground-state DMET, the bath is defined with the help of a model lattice ground-state wavefunction \(|\Psi^{(0)}\rangle\). In this study, \(|\Psi^{(0)}\rangle\) is a (possibly spin-broken) Slater determinant (ground-state) of a non-interacting lattice Hamiltonian \(h + u\), where \(h\) is the hopping matrix, and \(u\) is a frequency-independent one-particle operator acting in each cluster cell on the lattice, analogous to a cluster self-energy. For ground-state DMET, the Schmidt decomposition of \(|\Psi^{(0)}\rangle\) between the cluster and environment defines a bath space \(|\beta^{(0)}\rangle\), of the same size as the impurity space. The quantum impurity Hamiltonian \(H'\) is then obtained by projecting a model lattice Hamiltonian \(H_{\text{lat}}\) onto the quantum impurity plus bath space,

\[
H' = PH_{\text{lat}}P = P(h + U_{\text{cluster}} + u)P
\]

where \(U_{\text{cluster}}\) indicates that in \(H_{\text{lat}}\), Hubbard interactions are present only on the cluster sites, as in CDMFT, while the one-particle operator \(u\) is used for lattice sites outside the cluster, i.e. there are no interactions in the bath. Solving for the ground-state of \(H'\) is a many-body problem with twice the degrees of freedom as the impurity cluster. The resulting quantum impurity wavefunction yields expectation values, such as energies and correlation functions that approximate those of the original lattice problem. \(u\) is adjusted to minimize the difference between the single-particle density matrix \(\langle a_i^\dagger a_j \rangle\) of \(H'\) and the ground-state of the model lattice Hamiltonian \(h + u\) projected to the impurity plus bath space, analogous to the self-consistent update of the self-energy in DMFT.

To obtain spectra in DMET a modified procedure is used, where the ground-state bath space \(|\beta^{(0)}\rangle\) is augmented to reproduce dynamical properties. For example, to compute the local single-particle Green function, we consider the response vector of the model lattice wavefunction,

\[
|\Psi^{(1)}(\omega)\rangle = \frac{1}{\omega + \mu - h + u + i0}a_\alpha^{(1)}|\Psi^{(0)}\rangle
\]

and obtain a set of additional bath states from the Schmidt decomposition of \(|\Psi^{(1)}(\omega)\rangle\), leading to a total bath space \(|\beta^{(0)} \oplus \beta^{(1)}(\omega)\rangle\). The cluster Green function is determined by solving, at each frequency, the response problem for \(H_{\text{lat}}\) projected into the impurity plus dynamical bath space. Although the bath space is finite, it changes with frequency, thus the finite impurity model produces a continuous spectrum along the real axis without artificial broadening.

We now turn to applying DMET to the honeycomb Hubbard model. To identify different phases we monitor several quantities. The SM phase is characterized by a vanishing single-particle gap and a Dirac cone at low energies in the single-particle density of states \(A(\omega)\). The AFM phase is characterized by non-vanishing staggered magnetization, \(M = \frac{i}{2}\sum_{\langle j\rangle} (-1)^{\alpha+\beta}(n_{ij} - n_{ji})\), a non-zero single-particle gap (which grows with increasing \(U\)), and a vanishing spin-gap. The proposed intermediate gapped phase is identified by a non-vanishing single-particle and spin-gap without long-range AFM order. We

![Figure 1](image-url)
The staggered magnetization appears at the same point as the single-particle gap (Fig. 2). Meng et al. [37] found that the Dirac cone in the SM phase. To do so, we restrict our DMET calculations to parameter values that are consistent with the bath. We use an exact diagonalization solver; for smaller lattice size, with the bath we use an exact diagonalization solver; for larger lattice size, with the bath we use a density matrix renormalization group (DMRG) solver [43], keeping up to 2000 states, sufficient for quasi-exactness. It is important to distinguish cluster size \( N_c \) from the size of the lattice \( (2L)^2 \) in which the cluster is embedded. To study finite-size scaling effects of the underlying lattice, we use embedding lattices with \( L \times L \) supercells \( (2L)^2 \) sites, with \( L = 12 - 216 \) (up to over 90,000 sites). The smaller \( L \) calculations allow direct comparison to finite AFQMC calculations: for comparison, Meng et al. used up to \( L = 18 \); Sorella et al. used up to \( L = 36 \).

We start by scanning the phase diagram as a function of \( U \), allowing spontaneous antiferromagnetism to develop. The single-particle gap, magnetization, and density of states for \( N_c = 2 - 12 \) are shown in Fig. 2. We see at small \( U \) the single particle gap vanishes and the density of states displays a low-energy Dirac cone, clearly demonstrating that the system is in the SM phase. As we increase \( U \) beyond a critical \( U_{\text{AFM1}} \) an AFM solution to the DMET equations appears and a gap opens in the spectral function (Fig. 2). Note that \( U_{\text{AFM1}} \) is the earliest point at which the AFM solution can be found but this does not indicate a thermodynamic phase transition at this point, which requires a more detailed examination of the energies, discussed below. At \( U_{\text{AFM1}} \), the single-particle gap and magnetization vanish simultaneously. Fitting \( M = |U - U_{\text{AFM1}}|^{\beta} \) to the \( N_c = 6 \) data gives a critical exponent \( \beta = 0.72 \), compared with \( 0.80 \pm 0.04 \) from the AFQMC calculations of Sorella et al., and \( \beta = 1 \) from mean-field. \( U_{\text{AFM1}} \) shows significant cluster and lattice size dependence, as seen in Fig. 2; for \( N_c = 6 \), \( L = 12 - 216 \), \( U_{\text{AFM1}} \) decreases from 3.320 to 3.198, while at \( L = 216 \), for \( N_c = 2 - 12 \), \( U_{\text{AFM1}} \) increases from 2.92 to 3.2. Our \( U_{\text{AFM1}} = 3.2 \) for \( N_c = 12 \), \( L = 216 \) is somewhat lower than the recent AFQMC result \( U_{\text{AFM1}} = 3.869 \) of Sorella et al. [20]. The lattice size dependence of \( U_{\text{AFM1}} \) gives an estimate of the finite size error in Sorella et al.’s \( L = 36 \) calculations; from \( L = 36 - 216 \), \( U_{\text{AFM1}} \) decreases by \( \sim 0.01 \).

A quantitative check of the accuracy of our calculations is to directly compare the energies for a given finite size lattice with the exact AFQMC energies for the same lattice size. This comparison for total energies is shown in Fig. 4. We have compared over only a limited range of \( U \) as this is where we had access to the QMC data [20]. However, theoretically and numerically it is known that the error in the DMET energies vanishes exactly at weak and strong coupling; the largest errors are near phase transitions [35], the region tested in Fig. 4. Even in this challenging region, the cluster DMET total energies appear extremely good, as expected from earlier benchmarks [35, 36, 38]; with \( N_c = 12 \) the energies are within 0.2% of the exact results. The more sensitive kinetic-energy derivative \( dT/dU \) reported by Meng et al. (and obtained from Sorella et al.) is also shown in Fig. 5. Here we find that although Meng et al. argued that the two changes in curvature in \( dT/dU \) near \( U = 3.5, 4.3 \) indicate an intermediate phase, we also observe two changes in curvature in our cluster DMET curve, at similar places, with just a SM and AFM phase, showing this is not a good diagnostic for an intermediate phase.

We now consider the intermediate \( U \) region and look for evidence of possible metastable intermediate phases. To do so, we restrict our DMET calculations to paramagnetic phases and increase \( U \). Interestingly, as \( U \) is increased, we observe two kinds of paramagnetic transitions depending on cluster shape: for \( N_c = 2, 4, 8 \) there is a first-order transition to a gapped phase, while for
$N_c = 6, 12$, there is a second-order transition: the gap opens continuously. In either case, we find a paramagnetic insulating phase (PMI) at large $U$. This finding of a PMI phase is not unexpected; preceding 6-site GFC calculations have identified a similar phase, typically identified as a Mott insulator. We can compare the $N_c = 6$ DMET PMI gaps directly to earlier 6-site paramagnetic GFC calculations (see [1] for a summary). The agreement at large $U$ is good, but DMET does not produce the spurious gap seen in some GFC calculations at small $U$ that has led to erroneous conclusions that there is no SM phase [11]. Note further, that we find the paramagnetic transition $U_{PM}$ is very sensitive to cluster size, as seen in Fig. 3.

Hassan and Senechal conjectured that the purported spin-liquid phase in the honeycomb Hubbard model is in fact the PMI phase in a quantum cluster calculation [11]. We now examine this conjecture. To identify whether the PMI phase is a true intermediate phase, we must check its stability in the presence of antiferromagnetism. In the square lattice, antiferromagnetism order appears at infinitesimal $U$ before the PMI phase is reached. The honeycomb Hubbard model differs in this regard because AFM order develops at finite $U_{AFM1}$, which could in principle be larger than $U_{PM}$. In Fig. 5 we show the energy of the AFM phase relative to the SM/PMI phases as a function of cluster and lattice size. Here, we find that for our cluster sizes the transition between SM and AFM does not in fact occur at $U_{AFM1}$, rather there is a first-order coexistence region between $U_{AFM1}, U_{AFM2}$, where $U_{AFM2}$ is strongly cluster size (and to a lesser degree lattice size) dependent (Fig. 5). As the cluster size increases, the first-order coexistence region decreases, and it appears that in the limit of infinite cluster size the true AFM transition is second-order (Fig. 5). Further, in all instances we have studied, $U_{AFM2} < U_{PM}$ at the same cluster size. This means that there is in fact no stable intermediate paramagnetic phase. However, for the special cluster sizes $N_c = 6, 12$, the PMI phase is particularly competitive, and for these clusters, $U_{AFM2}$ is very close to $U_{PM}$. Since locating $U_{PM}$ precisely is itself difficult due to its continuous nature, a small uncertainty in $U_{PM}$ could then yield $U_{PM} < U_{AFM2}$, and a conclusion that there is a true paramagnetic intermediate phase. In addition, the very small difference between $U_{AFM2}$ and $U_{PM}$ provides a basis to explain the conflicting observations, in both AFQMC and in GFC calculations, of an intermediate phase near the antiferromagnetic transition: small changes in calculation parameters can selectively stabilize the PMI phase. Our arguments therefore support the interpretation of Hassan et al. that observations of the “intermediate” phase can be identified with the PMI state in quantum cluster calculations.

What is the nature of the metastable paramagnetic...
and SM phase, although this region appears to vanish with $U_L$.

The transition is first order for $N_c = 2, 4, 8$ and second order for $N_c = 6, 12$. Note that for $N_c = 6, 12$, $U_{PM}$ is very close to $U_{AFM}$ indicating that the PM order can be stabilized by introducing modest frustration. Finally our work demonstrates the potential of the DMET methodology, which allows computation of both energies and spectral functions at the thermodynamic limit, without incurring additional numerical artifacts. We gratefully acknowledge S. Sorella and Y. Otsuka for providing the energy data from their QMC calculations.

In conclusion, we have carried out cluster DMET calculations, as a function of cluster and lattice size, to elucidate the phase diagram of the half-filled Hubbard honeycomb model. Our detailed calculations find that at intermediate couplings, there is a metastable paramagnetic insulating phase, that is very competitive with the antiferromagnetic phase. This insulating phase displays an associated hexagonal cluster order. The closeness of the two phases at intermediate couplings means that small changes in calculational details can significantly affect their relative stability, and this explains the large number of conflicting results regarding the intermediate phase. It further seems likely that the intermediate paramagnetic phase can be stabilized by introducing modest frustration.

We gratefully acknowledge S. Sorella and Y. Otsuka for providing the energy data from their QMC calculations. This work was funded by the US Department of Energy, primarily through DE-SC0008624 and DE-SC0010530. Additional funding was provided through the DOE-CMCSN program DE-SC0006613.

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