ON THE THEORY OF VISCOELASTICITY FOR MATERIALS WITH DOUBLE POROSITY

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Abstract. In this paper the linear theory of viscoelasticity for Kelvin-Voigt materials with double porosity is presented and the basic partial differential equations are derived. The system of these equations is based on the equations of motion, conservation of fluid mass, the effective stress concept and Darcy’s law for materials with double porosity. This theory is a straightforward generalization of the earlier proposed dynamical theory of elasticity for materials with double porosity. The fundamental solution of the system of equations of steady vibrations is constructed by elementary functions and its basic properties are established. Finally, the properties of plane harmonic waves are studied. The results obtained from this study can be summarized as follows: through a Kelvin-Voigt material with double porosity three longitudinal and two transverse plane harmonic attenuated waves propagate.

1. Introduction. The concept of porous media is used in many areas of applied science (e.g., biology, biophysics, biomechanics) and engineering. There are a number of theories which describe mechanical properties of porous materials. The general 3D theory of consolidation for materials with single porosity was formulated in [7]. One important generalization of this theory that has been studied extensively started with the work [3], where a fissured porous medium is modeled as two completely overlapping flow regions: one representing the porous matrix, and the other the fissure network. The theory of consolidation for elastic materials with double porosity was presented in [6, 34, 55]. This theory unifies the earlier proposed models of Barenblatt for porous media with double porosity [3] and Biot for porous media with single porosity [7].

With regard to the Aifantis’ quasi-static theory, the cross-coupled terms were included in the equations of fluid mass conservation in [35, 36]. The phenomenological equations of the quasi-static theory for double porosity media were established and the method to determine the relevant coefficients was presented in [4, 5, 39]. The governing system of equations for an anisotropic material with double porosity was obtained in [56].

In the governing equations of the above mentioned theories of poroelasticity the inertial term was neglected and quasi-static problems were investigated (see [4 - 7, 34 - 36, 39, 56]). On the other hand, the inertial effect plays a pivotal role in the investigation of various problems of vibrations and wave propagation through double porosity media. Therefore, it is important to study a full dynamic model.
Recently, the linear theory of elastodynamics for the anisotropic nonhomogeneous materials with double porosity was considered in [44], and the uniqueness and stability of solution of the initial-boundary value problem were proved. The boundary value problems in the quasi-static and full dynamical cases of the theory of elasticity for double porosity materials were investigated by using the potential method (boundary integral method) and the theory of singular integral equations in [49, 50, 54]. The fundamental solutions of the system of equations of this theory were constructed by elementary functions in [48, 51, 52]. The basis properties of the plane harmonic waves were established and uniqueness theorems in the coupled linear theory of elasticity for solids with double porosity were proved in [13].

The double porosity model would consider the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity. The extensive review of the results in the theory of bone poroelasticity can be found in the survey papers [16, 17, 41]. In reality, bone is a viscoelastic material with triple porosity [16, 17, 33, 41]. There are three levels of bone porosity within cortical bone and within the trabeculae of cancellous bone, all containing a fluid. The pore sizes in cortical bone are of approximately three discrete magnitudes: the largest pore size (approx. 50 µm diameter) is associated with the vascular porosity, the second largest pore size (approx. 0.3 µm diameter) is associated with the lacunar-canalicular porosity, and the smallest pore size (approx. 10 nm diameter) is in the collagen-apatite porosity [16, 17, 41]. In the double porosity models the bone fluid pressure in the vascular porosity and the bone fluid pressure in the lacunar-canalicular porosity are considered, and the movement of the bone fluid in the collagen-apatite porosity is neglected.

The classical theories of viscoelasticity was initiated by Maxwell, Meyer, Boltzmann, and studied by Voigt, Kelvin, Zaremba, Volterra and others. An account of the historical developments of the theory of viscoelasticity as well as references to various contributions may be found in the books [1, 12, 26, 33], papers [2, 20-25, 27] and references therein.

In the last decade, several mechanical theories of viscoelasticity and thermoviscoelasticity for Kelvin-Voigt materials were formulated. A nonlinear theory for a viscoelastic composite as a mixture of a porous elastic solid and a Kelvin-Voigt material was developed in [29]. A linear variant of this theory was presented in [40]. Some exponential decay estimates of solutions of equations of steady vibrations in the theory of viscoelasticity for Kelvin-Voigt materials were obtained in [11]. A theory of thermoviscoelastic composites modelled as interacting Cosserat continua was developed in [30]. In [31], the basic equations of the nonlinear theory of thermoviscoelasticity for Kelvin-Voigt materials with voids were established. Recently, the theory of thermoviscoelasticity for Kelvin-Voigt microstretch composite materials was presented in [38]. The main results in the theories of viscoelasticity and thermoviscoelasticity of differential and integral types were obtained in the series of papers [14, 15, 18, 19, 32, 42, 43, 45 - 47, 53].

In this paper the linear theory of viscoelasticity for Kelvin-Voigt materials with double porosity is presented and the basic partial differential equations are derived. The system of these equations is based on the equations of motion, conservation of fluid mass, the effective stress concept and Darcy’s law for material with double porosity. This theory is a straightforward generalization of the dynamical theory for materials with double porosity. The full dynamical system able to describe the deformation in single porosity media was developed in [8 - 10].
Kelvin-Voigt material with double porosity consists of the following equations:

\[ \sum_{l=1,2,3} \hat{u}_l \] and the dot denotes differentiation with respect to time with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3), and the fissure fluid pressures, respectively.

The equations of motion for materials with double porosity \[13, 44, 49, 52\]. The fundamental properties of plane harmonic waves are studied. The results obtained from this study can be summarized as follows: a Kelvin-Voigt material with double porosity admits three longitudinal and two transverse plane harmonic attenuated waves.

2. Basic equations. Let \( x = (x_1, x_2, x_3) \) be a point of the Euclidean three-dimensional space \( \mathbb{R}^3 \), let \( t \) denote the time variable, \( t \geq 0 \), \( \mathbf{u}(x,t) \) is the displacement vector in the solid, \( \mathbf{u} = (\dot{u}_1, \dot{u}_2, \dot{u}_3) \); \( \hat{p}_1(x,t) \) and \( \hat{p}_2(x,t) \) are the pore and fissure fluid pressures, respectively.

We assume that the subscripts preceded by a comma denote partial differentiation with respect to the corresponding Cartesian coordinate, repeated indices are summed over the range (1,2,3), and the dot denotes differentiation with respect to \( t \).

The governing system of field equations in the linear theory of viscoelasticity for Kelvin-Voigt material with double porosity consists of the following equations:

a) The equations of motion \[8 - 10\]

\[ t_{lj,j} = \rho (\ddot{u}_l - F'_l(j)), \quad l = 1, 2, 3, \]  

(1)

where \( t_{lj} \) are the components of the total stress tensor, \( \rho \) is the reference mass density, \( \rho > 0 \), \( F' = (F'_1, F'_2, F'_3) \) is the body force per unit mass.

b) The equations of fluid mass conservation \[35, 36\]

\[ \text{div} \mathbf{v}^{(1)} + \dot{\gamma}_1 + \beta_1 \varepsilon_{rr} + \gamma (\hat{p}_1 - \hat{p}_2) = 0, \]

(2)

and

\[ \text{div} \mathbf{v}^{(2)} + \dot{\gamma}_2 + \beta_2 \varepsilon_{rr} - \gamma (\hat{p}_1 - \hat{p}_2) = 0, \]

(3)

where \( \mathbf{v}^{(1)} \) and \( \mathbf{v}^{(2)} \) are the fluid flux vectors for the pores and fissures, respectively; \( \varepsilon_{ij} \) are the components of the strain tensor,

\[ \varepsilon_{lj} = \frac{1}{2} (\ddot{u}_{l,j} + \ddot{u}_{j,l}), \quad l, j = 1, 2, 3, \]

(4)

\( \beta_1 \) and \( \beta_2 \) are the effective stress parameters, \( \gamma \) is the internal transport coefficient (leakage parameter) and corresponds to a fluid transfer rate respecting the intensity of flow between the pores and fissures, \( \gamma > 0 \); \( \zeta_1 \) and \( \zeta_2 \) are the increments of fluid (volumetric strain) in the pores and fissures, respectively, and defined by

\[ \zeta_1 = \alpha_1 \hat{p}_1 + \alpha_{12} \hat{p}_2, \quad \zeta_2 = \alpha_{21} \hat{p}_1 + \alpha_2 \hat{p}_2, \]

(5)

where \( \alpha_1 \) and \( \alpha_2 \) measure the compressibilities of the pore and fissure systems, respectively; \( \alpha_{12} \) and \( \alpha_{21} \) are the cross-coupling compressibilities for fluid flow at the interface between the two pore systems at a microscopic level \[35, 36\]. However, the coupling effect \((\alpha_{12} \text{ and } \alpha_{21})\) is often neglected in the literature \[6, 34, 55\]. In the following we assume that \( \beta_1^2 + \beta_2^2 > 0 \) (the case \( \beta_1 = \beta_2 = 0 \) is too simple to be considered).

(6)

c) The equations of effective stress concept (extending Terzaghi’s effective stress concept to double porosity) \[4, 5, 35, 36, 55\]

\[ t_{lj} = t'_{lj} - (\beta_1 \hat{p}_1 + \beta_2 \hat{p}_2) \delta_{lj}, \quad l, j = 1, 2, 3, \]

(6)

where

\[ t'_{lj} = 2\mu \varepsilon_{lj} + \lambda \varepsilon_{rr} \delta_{lj} + 2\mu^* \dot{\varepsilon}_{lj} + \lambda^* \dot{\varepsilon}_{rr} \delta_{lj} \]
are the components of effective stress tensor, $\lambda, \mu, \lambda^*$ and $\mu^*$ are the constitutive coefficients, $\delta_{ij}$ is the Kronecker’s delta.

**d) The Darcy’s law for materials with double porosity** [4, 5, 39]

$$v^{(1)} = -\frac{1}{\mu} (\kappa_1 \text{grad} \hat{p}_1 + \kappa_2 \text{grad} \hat{p}_2) - \rho_1 s^{(1)},$$

$$v^{(2)} = -\frac{1}{\mu} (\kappa_{21} \text{grad} \hat{p}_1 + \kappa_2 \text{grad} \hat{p}_2) - \rho_2 s^{(2)},$$

(7)

where $\mu$ is the fluid viscosity, $\kappa_1$ and $\kappa_2$ are the macroscopic intrinsic permeabilities associated with matrix and fissure porosity, respectively; $\kappa_{12}$ and $\kappa_{21}$ are the cross-coupling permeabilities for fluid flow at the interface between the matrix and fissure phases; $\rho_1$, $s^{(1)}$ and $\rho_2$, $s^{(2)}$ are the densities of fluid and the external forces (such as gravity) for the pore and fissure phases, respectively. The cross-coupling terms of (7) with coefficients $\kappa_{12}$ and $\kappa_{21}$ are considered by several authors [4, 5, 39]. However, the latter coupling effect ($\kappa_{12}$ and $\kappa_{21}$) is neglected in the literature [6, 34, 55].

Substituting equations (4) - (7) into (1) - (3), we obtain the following system of equations of motion in the full coupled linear theory of viscoelasticity for Kelvin-Voigt materials with double porosity expressed in terms of the displacement vector $\hat{u}$ and pressures $\hat{p}_1$ and $\hat{p}_2$:

$$\mu \Delta \hat{u} + (\lambda + \mu) \text{grad div} \hat{u} + \mu^* \Delta \hat{u} + (\lambda^* + \mu^*) \text{grad div} \hat{u}$$

$$-\beta_1 \text{grad} \hat{p}_1 - \beta_2 \text{grad} \hat{p}_2 = \rho (\hat{u} - F'),$$

$$k_1 \Delta \hat{p}_1 + k_{12} \Delta \hat{p}_2 - \alpha_1 \hat{p}_1 - \alpha_{12} \hat{p}_1 - \gamma (\hat{p}_1 - \hat{p}_2) - \beta_1 \text{div} \hat{u} = -\rho_1 \text{div} s^{(1)},$$

$$k_{21} \Delta \hat{p}_1 + k_2 \Delta \hat{p}_2 - \alpha_{21} \hat{p}_1 - \alpha_2 \hat{p}_2 + \gamma (\hat{p}_1 - \hat{p}_2) - \beta_2 \text{div} \hat{u} = -\rho_2 \text{div} s^{(2)},$$

(8)

where $\Delta$ is the Laplacian operator, $k_j = \frac{k_j}{\mu'}$ ($j = 1, 2$), $k_{12} = \frac{\kappa_{12}}{\mu'}$, $k_{21} = \frac{\kappa_{21}}{\mu'}$.

If the body force $F'$ and the external forces $s^{(1)}$ and $s^{(2)}$ are assumed to be absent, and the displacement vector $\hat{u}$ and the pressures $\hat{p}_1$ and $\hat{p}_2$ are postulated to have a harmonic time variation, that is,

$$\{\hat{u}, \hat{p}_1, \hat{p}_2\} (\mathbf{x}, t) = \text{Re} \left[ \{u, p_1, p_2\} (\mathbf{x}) e^{-i\omega t} \right],$$

then from the system (8) we obtain the following system of frequency-domain equations describing plane waves under the full coupled linear theory of viscoelasticity for Kelvin-Voigt materials with double porosity

$$\mu_1 \Delta u + (\lambda_1 + \mu_1) \text{grad div} u - \beta_1 \text{grad} p_1 - \beta_2 \text{grad} p_2 + \rho \omega^2 u = 0,$$

$$\left( k_1 \Delta + a_1 \right) p_1 + \left( k_{12} \Delta + a_{12} \right) p_2 + i \omega \beta_1 \text{div} u = 0,$$

$$\left( k_{21} \Delta + a_{21} \right) p_1 + \left( k_2 \Delta + a_2 \right) p_2 + i \omega \beta_2 \text{div} u = 0,$$

(9)

where $\lambda_1 = \lambda - i\omega \lambda^*$, $\mu_1 = \mu - i\omega \mu^*$, $a_j = i\omega \alpha_j - \gamma$, $a_{lj} = i\omega \alpha_{lj} + \gamma$ ($l, j = 1, 2$); $\omega$ is the oscillation frequency, $\omega > 0$. Obviously, neglecting inertial effect in the first equation of (8), from (9) we obtain the system of homogeneous equations of steady vibrations in the full coupled linear quasi-static theory of viscoelasticity for solids with double porosity.

We introduce the second order matrix differential operators with constant coefficients:
1) \( A(D_x) = (A_{ij}(D_x))_{5 \times 5} \), \( A_{ij}(D_x) = (\mu_1 \Delta + \rho \omega^2) \delta_{ij} + (\lambda_1 + \mu_1) \frac{\partial^2}{\partial x_i \partial x_j} \),

\[
A_{l,m+3}(D_x) = -\beta_m \frac{\partial}{\partial x_l}, \quad A_{m+3;l}(D_x) = i \omega \beta_m \frac{\partial}{\partial x_l},
\]

\( A_{44}(D_x) = k_1 \Delta + a_1, \quad A_{45}(D_x) = k_2 \Delta + a_{12}, \quad A_{54}(D_x) = k_2 \Delta + a_2, \quad A_{55}(D_x) = k_2 \Delta + a_2, \quad m = 1, 2, \quad l, j = 1, 2, 3. \)

\[ A^{(o)}(D_x) = \begin{pmatrix} A_{ij}^{(o)}(D_x) \end{pmatrix}_{5 \times 5}, \quad A_{ij}^{(o)}(D_x) = \mu_1 \Delta \delta_{ij} + (\lambda_1 + \mu_1) \frac{\partial^2}{\partial x_i \partial x_j}, \]

\[
A_{44}^{(o)}(D_x) = k_1 \Delta, \quad A_{45}^{(o)}(D_x) = k_2 \Delta, \quad A_{54}^{(o)}(D_x) = k_2 \Delta, \quad A_{55}^{(o)}(D_x) = k_2 \Delta, \quad m = 1, 2, \quad l, j = 1, 2, 3.
\]

It is easily seen that the system (9) can be written as \( A(D_x)U(x) = 0 \), where \( U = (u, p_1, p_2) \) is a five-components vector function and \( x \in \mathbb{R}^3 \). The matrix differential operator \( A^{(o)}(D_x) \) is called the principal part of the operator \( A(D_x) \).

**Definition 2.1.** The operator \( A(D_x) \) is said to be elliptic if \([28]\)

\[
\det A^{(o)}(\xi) \neq 0,
\]

where \( \xi = (\xi_1, \xi_2, \xi_3), |\xi| \neq 0. \)

Obviously, we have

\[
\det A^{(o)}(\xi) = \mu_0^2 \mu_0 k |\xi|^{10},
\]

where \( \mu_0 = \lambda_1 + 2 \mu_1, \quad k = k_1k_2 - k_2k_1 \). Hence, \( A(D_x) \) is an elliptic differential operator if and only if

\[
\mu_1 \mu_0 k \neq 0. \quad (10)
\]

**Definition 2.2.** The fundamental solution of system (9) (the fundamental matrix of operator \( A(D_x) \)) is the matrix \( \Gamma(x) = (\Gamma_{ij}(x))_{5 \times 5} \) satisfying condition (in the class of generalized functions) \([28]\)

\[
A(D_x)\Gamma(x) = \delta(x)J,
\]

where \( \delta(x) \) is the Dirac delta, \( J = (\delta_{ij})_{5 \times 5} \) is the unit matrix, \( x \in \mathbb{R}^3. \)

In the next section the matrix \( \Gamma \) is constructed in terms of elementary functions and some of its basic properties are established.

3. **Fundamental solution of the system of steady vibrations equations.**

We consider the system of nonhomogeneous equations

\[
\mu_1 \Delta u + (\lambda_1 + \mu_1) \text{ grad } u + i \omega \beta_1 \text{ grad } p_1 + i \omega \beta_2 \text{ grad } p_2 + \rho \omega^2 u = f,
\]

\[
(k_1 \Delta + a_1) p_1 + (k_2 \Delta + a_2) p_2 - \beta_1 \text{ div } u = f_1,
\]

\[
(k_2 \Delta + a_1) p_1 + (k_2 \Delta + a_2) p_2 - \beta_2 \text{ div } u = f_2,
\]

where \( f \) is a three-components vector function, \( f_1 \) and \( f_2 \) are scalar functions on \( \mathbb{R}^3. \) As one may easily verify, the system (12) may be written in the form

\[
A^T(D_x)U(x) = F(x),
\]

where \( A^T \) is the transpose of matrix \( A, \) \( F = (f, f_1, f_2) \) is a five-components vector function and \( x \in \mathbb{R}^3. \)
Applying the operator div to (12) from system (12) we obtain

\[(\mu_0 \Delta + \rho \omega^2) \text{div} \, \mathbf{u} + i \omega \beta_1 \Delta p_1 + i \omega \beta_2 \Delta p_2 = \text{div} \, \mathbf{f},\]
\[(k_1 \Delta + a_1) p_1 + (k_{21} \Delta + a_{21}) p_2 - \beta_1 \text{div} \, \mathbf{u} = f_1,\]
\[(k_{12} \Delta + a_{12}) p_1 + (k_2 \Delta + a_2) p_2 - \beta_2 \text{div} \, \mathbf{u} = f_2.\]  

From (14) we have

\[B(\Delta) \mathbf{V}(\mathbf{x}) = \mathbf{\varphi}(\mathbf{x}),\]  

where \( \mathbf{V} = (\text{div} \, \mathbf{u}, p_1, p_2), \, \mathbf{\varphi} = (\varphi_1, \varphi_2, \varphi_3) = (\text{div} \, \mathbf{f}, f_1, f_2) \) and

\[
B(\Delta) = (B_{ij}(\Delta))_{3 \times 3} = \begin{pmatrix}
\mu_0 \Delta + \rho \omega^2 & i \omega \beta_1 \Delta & i \omega \beta_2 \Delta \\
-\beta_1 & k_1 \Delta + a_1 & k_{21} \Delta + a_{21} \\
-\beta_2 & k_{12} \Delta + a_{12} & k_2 \Delta + a_2
\end{pmatrix}_{3 \times 3}.
\]

We introduce the notation

\[\Lambda_1(\Delta) = \frac{1}{k \mu_0} \det \, B(\Delta).\]

It is easily seen that \( \Lambda_1(\tau) = 0 \) is a cubic algebraic equation and there exist three roots \( \tau_1^2, \tau_2^2 \) and \( \tau_3^2 \) (with respect to \( \tau \)). Then we have

\[\Lambda_1(\Delta) = (\Delta + \tau_1^2)(\Delta + \tau_2^2)(\Delta + \tau_3^2).\]

The system (15) implies

\[\Lambda_1(\Delta) \mathbf{V} = \mathbf{\Phi},\]  

where

\[\mathbf{\Phi} = (\Phi_1, \Phi_2, \Phi_3), \quad \Phi_j = \frac{1}{k \mu_0} \sum_{l=1}^{3} B_{lj}^* \varphi_l, \quad j = 1, 2, 3\]  

and \( B_{lj}^* \) is the cofactor of element \( B_{lj} \) of the matrix \( B \).

Now applying the operator \( \Lambda_1(\Delta) \) to (12) and taking into account (16), we obtain

\[\Lambda_2(\Delta) \mathbf{u} = \mathbf{F}_1,\]  

where \( \Lambda_2(\Delta) = \Lambda_1(\Delta)(\Delta + \tau_4^2), \, \tau_4^2 = \frac{\rho \omega^2}{\mu_1} \) and

\[\mathbf{F}_1 = \frac{1}{\mu_1} [\Lambda_1(\Delta) \mathbf{f} - (\lambda_1 + \mu_1) \text{grad} \, \Phi_1 - i \omega \beta_1 \text{grad} \, \Phi_2 - i \omega \beta_2 \text{grad} \, \Phi_3].\]  

On the basis of (16) and (18) we get

\[\Lambda(\Delta) \mathbf{U}(\mathbf{x}) = \psi(\mathbf{x}),\]  

where \( \psi = (\mathbf{F}_1, \Phi_2, \Phi_3) \) is a five-components vector and

\[\Lambda(\Delta) = (\Lambda_{ij}(\Delta))_{5 \times 5}, \quad \Lambda_{11}(\Delta) = \Lambda_{22}(\Delta) = \Lambda_{33}(\Delta) = \Lambda_{2}(\Delta), \quad \Lambda_{44}(\Delta) = \Lambda_{55}(\Delta) = \Lambda_1(\Delta), \quad \Lambda_{ij}(\Delta) = 0, \quad l, j = 1, 2, \ldots, 5, \quad l \neq j.\]

We introduce the notations

\[n_{j1}(\Delta) = -\frac{1}{k \mu_1 \mu_0} \left[ (\lambda_1 + \mu_1) B_{1j}^*(\Delta) + i \omega \beta_1 B_{j2}^*(\Delta) + i \omega \beta_2 B_{j3}^*(\Delta) \right],\]
\[n_{jl}(\Delta) = \frac{1}{k \mu_0} B_{jl}^*(\Delta), \quad j = 1, 2, 3, \quad l = 2, 3.\]
In view of (17) and (21), from (19) it follows that
\[
F_1 = \left[ \frac{1}{\mu_1} \Lambda_1(\Delta) I + n_{11}(\Delta) \text{grad div} \right] f + n_{21}(\Delta) \text{grad} f_1 + n_{31}(\Delta) \text{grad} f_2,
\]
(22)
\[
\Phi_m = n_{1m}(\Delta) \text{div} f + n_{2m}(\Delta) f_1 + n_{3m}(\Delta) f_2, \quad m = 2, 3,
\]
where \( I = (\delta_{ij})_{3 \times 3} \) is the unit matrix.

Thus, from (22) we have
\[
\psi(\mathbf{x}) = \mathbf{L}^T (\mathbf{D}_x) \mathbf{F}(\mathbf{x}),
\]
(23)
where
\[
\mathbf{L}(\mathbf{D}_x) = (L_{ij}(\mathbf{D}_x))_{5 \times 5}, \quad L_{ij}(\mathbf{D}_x) = \frac{1}{\mu_1} \Lambda_1(\Delta) \delta_{ij} + n_{11}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j},
\]
\[
L_{l,m+2}(\mathbf{D}_x) = n_{1m}(\Delta) \frac{\partial}{\partial x_l}, \quad L_{m+2:l}(\mathbf{D}_x) = n_{m1}(\Delta) \frac{\partial}{\partial x_l},
\]
\[
L_{m+2:4}(\mathbf{D}_x) = n_{m2}(\Delta), \quad L_{m+2:5}(\mathbf{D}_x) = n_{m3}(\Delta),
\]
l, \( j = 1, 2, 3, m = 2, 3 \).

By virtue of (13) and (23), from (20) it follows that \( \mathbf{A} \mathbf{U} = \mathbf{L}^T \mathbf{A}^T \mathbf{U} \). It is obvious that \( \mathbf{L}^T \mathbf{A}^T = \mathbf{L} \) and, hence,
\[
\mathbf{A}(\mathbf{D}_x) \mathbf{L}(\mathbf{D}_x) = \Lambda(\Delta).
\]
(25)
We assume that \( \tau^2_l \neq \tau^2_j \), where \( l, j = 1, 2, 3, 4 \) and \( l \neq j \). Let
\[
\mathbf{Y}(\mathbf{x}) = (Y_{lm}(\mathbf{x}))_{5 \times 5}, \quad Y_{11}(\mathbf{x}) = Y_{22}(\mathbf{x}) = Y_{33}(\mathbf{x}) = \sum_{j=1}^{4} \eta_{2j} \gamma_j(\mathbf{x}),
\]
\[
Y_{44}(\mathbf{x}) = Y_{55}(\mathbf{x}) = \sum_{j=1}^{3} \eta_{1j} \gamma_j(\mathbf{x}), \quad Y_{lm}(\mathbf{x}) = 0,
\]
l \( \neq m \), \( l, m = 1, 2, \ldots, 5 \),
where
\[
\gamma_j(\mathbf{x}) = -\frac{e^{i\tau_j |\mathbf{x}|}}{4\pi |\mathbf{x}|}
\]
(27)
is the fundamental solution of Helmholtz’ equation, i.e., \( (\Delta + \tau^2_j) \gamma_j(\mathbf{x}) = \delta(\mathbf{x}) \) and
\[
\eta_{1m} = \prod_{l=1, l \neq m}^{3} \left( \tau^2_l - \tau^2_m \right)^{-1}, \quad \eta_{2j} = \prod_{l=1, l \neq j}^{4} \left( \tau^2_l - \tau^2_j \right)^{-1},
\]
m \( = 1, 2, 3 \), \( j = 1, 2, 3, 4 \).

Lemma 3.1. The matrix \( \mathbf{Y} \) is the fundamental solution of operator \( \Lambda(\Delta) \), that is,
\[
\Lambda(\Delta) \mathbf{Y}(\mathbf{x}) = \delta(\mathbf{x}) \mathbf{J},
\]
(28)
where \( \mathbf{x} \in \mathbb{R}^3 \).

Proof. It suffices to show that \( Y_{11} \) and \( Y_{44} \) are the fundamental solutions of operators \( \Lambda_2(\Delta) \) and \( \Lambda_1(\Delta) \), respectively, i.e.,
\[
\Lambda_2(\Delta) Y_{11}(\mathbf{x}) = \delta(\mathbf{x})
\]
(29)
and

\[ \Lambda_1(\Delta)Y_{44}(x) = \delta(x). \]

Taking into account the equalities

\[
\begin{align*}
\eta_{11} + \eta_{12} + \eta_{13} &= 0, \\
\eta_{12}(\tau_2^2 - \tau_3^2) + \eta_{13}(\tau_1^2 - \tau_3^2) &= 0, \\
\eta_{13}(\tau_1^2 - \tau_2^2)(\tau_2^2 - \tau_3^2) &= 1, \\
(\Delta + \tau_2^2)\gamma_j(x) &= \delta(x) + (\tau_2^2 - \tau_j^2)\gamma_j(x), \\
l, j &= 1, 2, 3, \quad x \in \mathbb{R}^3,
\end{align*}
\]

we have

\[
\begin{align*}
\Lambda_1(\Delta)Y_{44}(x) &= (\Delta + \tau_2^2)(\Delta + \tau_3^2) \sum_{j=1}^{3} \eta_{1j} \left[ \delta(x) + (\tau_2^2 - \tau_j^2)\gamma_j(x) \right] \\
&= (\Delta + \tau_2^2)(\Delta + \tau_3^2) \sum_{j=2}^{3} \eta_{1j} (\tau_1^2 - \tau_j^2)\gamma_j(x) \\
&= (\Delta + \tau_3^2) \sum_{j=2}^{3} \eta_{1j} (\tau_1^2 - \tau_j^2) \left[ \delta(x) + (\tau_2^2 - \tau_j^2)\gamma_j(x) \right] \\
&= (\Delta + \tau_3^2)\gamma_3(x) = \delta(x).
\end{align*}
\]

Equation (29) is proved quite similarly. \[\square\]

We introduce the matrix

\[ \Gamma(x) = L(D_x)Y(x). \tag{30} \]

Using identities (25) and (28) from (30) we get

\[ A(D_x)\Gamma(x) = A(D_x)L(D_x)Y(x) = \Lambda(\Delta)Y(x) = \delta(x)J. \]

Hence, \( \Gamma(x) \) is the solution of (11). We have thereby proved the following theorem.

**Theorem 3.2.** If the condition (10) is satisfied, then the matrix \( \Gamma(x) \) defined by (30) is the fundamental solution of system (9), where the matrices \( L(D_x) \) and \( Y(x) \) are given by (24) and (26), respectively.

Obviously, each element \( \Gamma_{lj}(x) \) of the matrix \( \Gamma(x) \) is represented in the following form

\[ \Gamma_{lj}(x) = L_{lj}(D_x)Y_{11}(x), \quad \Gamma_{lm}(x) = L_{lm}(D_x)Y_{44}(x), \] \[ l = 1, 2, \cdots, 5, \quad j = 1, 2, 3, \quad m = 4, 5. \tag{31} \]

**Remark 1.** In the cases \( \lambda^* = \mu^* = 0 \) and \( \lambda^* = \mu^* = k_{12} = k_{21} = \alpha_{12} = \alpha_{21} = \rho = 0 \), the matrix \( \Gamma(x) \) is constructed in [52] and [48], respectively.

**Remark 2.** On the basis of operator \( L(D_x) \) and (30) we can obtain the Galerkin type representation of solution of system (9).

**Remark 3.** Obviously, the operator \( A(D_x) \) is not self adjointed. It is possible to construct the fundamental solution of adjoined operator in a quite similar manner.

**Remark 4.** The matrix \( \Gamma(x) \) is constructed by 4 metaharmonic functions (solutions of the Helmholtz’ equation) \( \gamma_m(m = 1, 2, 3, 4) \) (see (27)).
4. Basic properties of fundamental solution. Theorem 3.2 leads to the following results.

**Theorem 4.1.** Each column of the matrix $\Gamma(x)$ is a solution of homogeneous equation

$$A(D_x)U(x) = 0$$

at every point $x \in \mathbb{R}^3$ except the origin.

**Theorem 4.2.** If condition (10) is satisfied, then the fundamental solution of the system

$$A^{(o)}(D_x)U(x) = 0$$

is the matrix $\Psi(x) = (\Psi_{ij}(x))_{5 \times 5}$, where

$$\Psi_{ij}(x) = \frac{1}{\mu_1} \left( \Delta \delta_{ij} - \frac{\lambda_1 + \mu_1}{\mu_0} \frac{\partial^2}{\partial x_i \partial x_j} \right) \gamma_2(x)$$

$$\Psi_{44}(x) = \frac{k_2}{k} \gamma_1(x), \quad \Psi_{45}(x) = -\frac{k_{12}}{k} \gamma_1(x),$$

$$\Psi_{54}(x) = -\frac{k_{21}}{k} \gamma_1(x), \quad \Psi_{55}(x) = \frac{k_1}{k} \gamma_1(x),$$

$$\Psi_{lm} = \Psi_{ml} = 0, \quad R_{lj}(D_x) = -\frac{\partial^2}{\partial x_l \partial x_j} - \delta_{lj},$$

$$\chi' = -\frac{\lambda_1 + 3\mu_1}{8\pi \mu_1 \mu_0}, \quad \mu' = -\frac{\lambda_1 + \mu_1}{8\pi \mu_1 \mu_0}, \quad l, j = 1, 2, 3, \quad m = 4, 5.$$

It is easy to verify that $R(D_x) = (R_{lj}(D_x))_{3 \times 3} = \text{curl curl}$. Obviously, theorem 4.2 leads to the following result.

**Corollary 1.** The relations

$$\Psi_{ij}(x) = O \left( \frac{1}{|x|} \right), \quad \Psi_{mn}(x) = O \left( \frac{1}{|x|} \right)$$

hold in the neighborhood of the origin, where $l, j = 1, 2, 3$ and $m, n = 4, 5$.

We shall use the following lemma.

**Lemma 4.3.** If condition (10) is satisfied, then

$$\Delta n_{11}(\Delta) = -\frac{1}{\mu_1} \Lambda_1(\Delta) + \frac{1}{k \mu_0} (\Delta + \tau_1^2) B_{11}^*,$$

$$n_{21}(\Delta) = \frac{i\omega}{k \mu_0} (\Delta + \tau_1^2) \left[ \beta_1(k_2 \Delta + a_2) - \beta_2(k_{12} \Delta + a_{12}) \right],$$

$$n_{31}(\Delta) = -\frac{i\omega}{k \mu_0} (\Delta + \tau_1^2) \left[ \beta_1(k_{21} \Delta + a_{21}) - \beta_2(k_1 \Delta + a_1) \right].$$

**Proof.** Taking into account the equalities (21) and

$$(\mu_0 \Delta + \rho \omega^2) B_{11}^*(\Delta) + i\omega \beta_1 \Delta B_{12}^*(\Delta) + i\omega \beta_2 \Delta B_{13}^*(\Delta) = \det B$$

we have

$$\Delta n_{11}(\Delta) = -\frac{1}{k \mu_1 \mu_0} \left[ \det B - (\mu_1 \Delta + \rho \omega^2) B_{11}^*(\Delta) \right]$$

$$= -\frac{1}{\mu_1} \Lambda_1(\Delta) + \frac{1}{k \mu_0} (\Delta + \tau_1^2) B_{11}^*.$$
The formulae (34)_2 and (34)_3 are proven in a quite similar manner.

We introduce the notations
\[
\begin{align*}
&d^{(m)}_{11} = -\frac{1}{k\mu_0\tau_m^2}\eta_m B_{11}^*(\tau_m^2), \quad d^{(4)}_{11} = \frac{1}{\rho\omega^2}, \\
n_{q1} = \eta_2m\eta_{q1}(\tau_m^2), \quad d^{(m)}_{q1} = \eta_1m\eta_{1q}(\tau_m^2), \\
n_{qr} = \eta_1m\eta_{qr}(\tau_m^2), \quad m = 1, 2, 3, \quad q, r = 2, 3.
\end{align*}
\]

(35)

On the basis of lemma 4.3 we can rewrite the fundamental solution \(\Gamma(x)\) in the simple form for \(x \neq 0\). We have the following result.

**Theorem 4.4.** If \(x \neq 0\), then
\[
\begin{align*}
\Gamma_{ij}(x) &= \sum_{m=1}^{3} d^{(m)}_{11} \gamma_m \delta_{ij}(x) + d^{(4)}_{11} R_{ij} \gamma_4(x), \\
\Gamma_{lq+2}(x) &= \sum_{m=1}^{3} d^{(m)}_{1q} \gamma_m \delta(l, j), \quad \Gamma_{q+2j}(x) = \sum_{m=1}^{3} d^{(m)}_{q1} \gamma_m \delta(l, j), \\
\Gamma_{q+2,r+2}(x) &= \sum_{m=1}^{3} d^{(m)}_{qr} \gamma_m(x), \quad l, j = 1, 2, 3, \quad q, r = 2, 3.
\end{align*}
\]

**Proof.** Let \(x \neq 0\). It is easy to verify that
\[
\Delta \gamma_m(x) = -\tau_m^2 \gamma_m(x), \quad \delta_{ij} \gamma_m(x) = -\frac{1}{\tau_m^2} \left( \frac{\partial^2}{\partial x_i \partial x_j} - R_{ij} \right) \gamma_m(x),
\]
\[
\begin{align*}
l, j = 1, 2, 3, \quad m = 1, 2, 3, 4.
\end{align*}
\]

On the second hand from (34)_1 it follows that
\[
n_{11}(\tau_m^2) - \frac{1}{\mu_1 \tau_m^2} \Lambda_1(\tau_m^2) = -\frac{1}{k\mu_0\tau_m^2}(\tau_m^2 - \tau_m^2)B_{11}^*(\tau_m^2), \quad m = 1, 2, 3, 4.
\]

(38)

By virtue of (24), (26), (37) and (38) from (31) we have
\[
\begin{align*}
\Gamma_{ij}(x) &= \left[ \frac{1}{\mu_1} \Lambda_1(\Delta) \delta_{ij} + n_{11}(\Delta) \frac{\partial^2}{\partial x_i \partial x_j} \right] \sum_{m=1}^{4} \eta_{2m} \gamma_m(x) \\
&= \sum_{m=1}^{4} \eta_{2m} \left[ \frac{1}{\mu_1} \Lambda_1(\tau_m^2) \delta_{ij} + n_{11}(\tau_m^2) \frac{\partial^2}{\partial x_i \partial x_j} \right] \gamma_m(x) \\
&= \sum_{m=1}^{4} \eta_{2m} \left[ -\frac{1}{\mu_1 \tau_m^2} \Lambda_1(\tau_m^2) \left( \frac{\partial^2}{\partial x_i \partial x_j} - R_{ij} \right) + n_{11}(\tau_m^2) \frac{\partial^2}{\partial x_i \partial x_j} \right] \gamma_m(x) \\
&= \sum_{m=1}^{4} \eta_{2m} \left[ -\frac{1}{k\mu_0\tau_m^2}(\tau_m^2 - \tau_m^2)B_{11}^*(\tau_m^2) \frac{\partial^2}{\partial x_i \partial x_j} + \frac{1}{\mu_1 \tau_m^2} \Lambda_1(\tau_m^2) R_{ij} \right] \gamma_m(x).
\end{align*}
\]

(39)

On the basis of (35) and identities
\[
\begin{align*}
(\tau_i^2 - \tau_j^2)\eta_{2j} &= \eta_j, \quad j = 1, 2, 3, \\
\eta_{2m} \Lambda_1(\tau_m^2) &= \begin{cases} 0 & \text{for } m = 1, 2, 3 \\ 1 & \text{for } m = 4 \end{cases}
\end{align*}
\]
from (39) we obtain
\[ \Gamma_{lj}(x) = \sum_{m=1}^{3} d_{11}^{(m)} \frac{\partial^2}{\partial x_l \partial x_j} \gamma_m(x) + \sum_{m=1}^{4} \frac{1}{\mu^m} \eta_{2m} \Lambda_l(-\tau_m^2) R_{lj} \gamma_4(x) \]
\[ = \sum_{m=1}^{3} d_{11}^{(m)} \gamma_m, lj(x) + d_{11}^{(4)} R_{lj} \gamma_4(x). \]

The other formulae of (36) can be proven quite similarly. □

**Theorem 4.5.** The relations
\[ \Gamma_{lj}(x) = O \left( |x|^{-1} \right), \quad \Gamma_{mq}(x) = O \left( |x|^{-1} \right), \quad (40) \]
\[ \Gamma_{mj}(x) = O (1), \quad \Gamma_{jm}(x) = O (1) \]
hold in the neighborhood of the origin, where \( l, j = 1, 2, 3, m, q = 4, 5. \)

**Lemma 4.6.** If condition (10) is satisfied, then
\[ \sum_{m=1}^{3} d_{11}^{(m)} = -\frac{1}{\rho \omega^2}, \quad \sum_{m=1}^{3} \tau_m^2 d_{11}^{(m)} = -\frac{1}{\mu_0}. \quad (41) \]

**Proof.** It is easy to verify that
\[ B_{11}^*(-\tau_m^2) = k \tau_m^4 + (a_{12} k_{21} + a_{21} k_{12} - a_1 k_2 - a_2 k_1) \tau_m^2 + a, \]
\[ \tau_1^2 \tau_2^2 \tau_3^2 = \frac{\rho \omega^2 a}{k \mu_0}, \quad (42) \]
where \( a = a_1 a_2 - a_1 a_2. \) By virtue of (42) we obtain
\[ \sum_{m=1}^{3} \frac{1}{\tau_m^2} \eta_{1m} B_{11}^*(-\tau_m^2) = \frac{B_{11}^*(-\tau_1^2)}{\tau_1^2 (\tau_2^2 - \tau_1^2) (\tau_2^2 - \tau_3^2)} + \frac{B_{11}^*(-\tau_2^2)}{\tau_2^2 (\tau_1^2 - \tau_2^2) (\tau_3^2 - \tau_2^2)} \]
\[ + \frac{B_{11}^*(-\tau_3^2)}{\tau_3^2 (\tau_1^2 - \tau_3^2) (\tau_2^2 - \tau_3^2)} = \frac{a}{\tau_1^2 \tau_2^2 \tau_3^2} = \frac{k \mu_0}{\rho \omega^2}. \]

Hence, from (35) we have
\[ \sum_{m=1}^{3} d_{11}^{(m)} = -\frac{1}{k \mu_0} \sum_{m=1}^{3} \frac{1}{\tau_m^2} \eta_{1m} B_{11}^*(-\tau_m^2) = -\frac{1}{\rho \omega^2}. \]

Similarly, by virtue of (42) we obtain
\[ \sum_{m=1}^{3} \eta_{1m} B_{11}^*(-\tau_m^2) = \frac{B_{11}^*(-\tau_1^2)}{(\tau_2^2 - \tau_1^2) (\tau_2^2 - \tau_3^2)} + \frac{B_{11}^*(-\tau_2^2)}{(\tau_1^2 - \tau_2^2) (\tau_3^2 - \tau_2^2)} \]
\[ + \frac{B_{11}^*(-\tau_3^2)}{(\tau_1^2 - \tau_3^2) (\tau_2^2 - \tau_3^2)} = k. \]

Finally, from (35) we get
\[ \sum_{m=1}^{3} \tau_m^2 d_{11}^{(m)} = -\frac{1}{k \mu_0} \sum_{m=1}^{3} \eta_{1m} B_{11}^*(-\tau_m^2) = -\frac{1}{\mu_0}. \]
Now we can establish the singular part of the matrix $\Gamma (x)$ in the neighborhood of the origin.

**Theorem 4.7.** The relations

$$\Gamma_{lj} (x) - \Psi_{lj} (x) = \text{const} + O (|x|)$$  \hspace{1cm} (43)

hold in the neighborhood of the origin, where $l, j = 1, 2, \cdots, 5$.

**Proof.** Let $x \neq 0$. In view of (32) and (36) we obtain

\[
\Gamma_{lj} (x) - \Psi_{lj} (x) = \frac{\partial^2}{\partial x_l \partial x_j} \left[ \sum_{m=1}^{3} d_{11}^{(m)} \gamma_m (x) \right] - \frac{1}{\mu_0} \gamma_2 (x) + R_{lj} \left[ \frac{1}{\rho \omega^2} \gamma_4 (x) + \frac{1}{\mu_1} \gamma_2 (x) \right]
\]

(44)

for $l, j = 1, 2, 3$. In the neighborhood of the origin from (27) have

\[
\gamma_m (x) = - \frac{1}{4\pi |x|} \sum_{n=0}^{\infty} \left( \frac{i\tau_{lm} |x|^n}{n!} \right) = \gamma_1 (x) - \frac{i\tau_{lm}}{4\pi} - \tau_{m}^2 \gamma_2 (x) + \tilde{\gamma}_m (x),
\]

(45)

where $\tilde{\gamma}_m (x) = - \frac{1}{4\pi |x|} \sum_{n=3}^{\infty} \left( \frac{i\tau_{lm} |x|^n}{n!} \right), m = 1, 2, 3, 4$. Obviously,

\[
\tilde{\gamma}_m (x) = O (|x|^2), \quad \tilde{\gamma}_{m,j} (x) = O (|x|), \quad \tilde{\gamma}_{m,lj} (x) = \text{const} + O (|x|),
\]

(46)

\[
\text{On the basis of (45) from (44) we get}
\]

\[
\sum_{m=1}^{3} d_{11}^{(m)} \gamma_m (x) - \frac{1}{\mu_0} \gamma_2 (x) = \sum_{m=1}^{3} d_{11}^{(m)} \left[ \gamma_1 (x) - \frac{i\tau_{lm}}{4\pi} \right] + \left( \sum_{m=1}^{3} \tau_{m}^2 d_{11}^{(m)} + \frac{1}{\mu_0} \right) \gamma_2 (x).
\]

(47)

By virtue of equalities (41) from (47) it follows that

\[
\sum_{m=1}^{3} d_{11}^{(m)} \gamma_m (x) - \frac{1}{\mu_0} \gamma_2 (x) = - \frac{1}{\rho \omega^2} \gamma_1 (x) - \frac{i\tau_4}{4\pi} \sum_{m=1}^{3} \tau_{m} d_{11}^{(m)} + \sum_{m=1}^{3} d_{11}^{(m)} \tilde{\gamma}_m (x).
\]

(48)

Similarly, from (45) we have

\[
\frac{1}{\rho \omega^2} \gamma_4 (x) + \frac{1}{\mu_1} \gamma_2 (x) = \frac{1}{\rho \omega^2} \left[ \gamma_1 (x) - \frac{i\tau_4}{4\pi} - \tau_1^2 \gamma_2 (x) + \tilde{\gamma}_4 (x) \right] + \frac{1}{\mu_1} \gamma_2 (x)
\]

\[
= \frac{1}{\rho \omega^2} \gamma_1 (x) - \frac{i\tau_4}{4\pi} \rho \omega^2 + \frac{1}{\rho \omega^2} \tilde{\gamma}_4 (x).
\]

(49)

Taking into account (46), (48), (49) and $\Delta \gamma_1 (x) = 0 (x \neq 0)$ from (44) we obtain

\[
\Gamma_{lj}^{(s)} (x) - \Psi_{lj} (x) = - \frac{1}{\rho \omega^2} \left( \frac{\partial^2}{\partial x_l \partial x_j} - R_{lj} \right) \gamma_1 (x) + \sum_{m=1}^{3} d_{11}^{(m)} \gamma_m (x) + \frac{1}{\rho \omega^2} \tilde{\gamma}_4 (x)
\]

\[
= - \frac{1}{\rho \omega^2} \Delta \gamma_1 (x) + \text{const} + O (|x|) = \text{const} + O (|x|), \quad l, j = 1, 2, 3.
\]

The other formulae of (43) can be proven in a quite similar manner. \qed
Thus, on the basis of corollary 1 and theorem 4.7 the matrix Ψ(ξ) is the singular part of the fundamental solution Φ(ξ) in the neighborhood of the origin (see (33), (40) and (43)).

5. Plane harmonic waves. In this section we assume that \( k_{12} = k_{21} \) and \( \alpha_{12} = \alpha_{21} \) and these values are denoted by \( k_3 \) and \( \alpha_3 \), respectively. We suppose that the following inequalities are true

\[
\mu^* > 0, \quad \lambda^* + 2\mu^* > 0, \quad k_1 > 0, \quad k_1k_2 > k_3^2, \\
\alpha_1 > 0, \quad \alpha_1\alpha_2 > \alpha_3^2, \quad \gamma > 0.
\]

(50)

Let us suppose that plane harmonic waves corresponding to a wave number \( \xi \) and to an angular frequency \( \omega \) propagate in the \( x_1 \)-direction through the porous solid with double porosity. Then displacement vector and pressures have the following form

\[
\hat{u}(x, t) = C e^{i(\xi x_1 - \omega t)}, \quad \hat{p}_j(x, t) = C_{j+3} e^{i(\xi x_1 - \omega t)},
\]

(51)

where \( C = (C_1, C_2, C_3) \) is a constant vector, \( C_4 \) and \( C_5 \) are constant values, \( \omega > 0 \), \( j = 1, 2 \).

On the basis of (51), from the system of homogeneous equations

\[
\mu \Delta \hat{u} + (\lambda + \mu) \text{grad div} \hat{u} + \mu^* \Delta \hat{u} + (\lambda^* + \mu^*) \text{grad div} \hat{u}
\]

\[
- \beta_1 \text{grad} \hat{p}_1 - \beta_2 \text{grad} \hat{p}_2 - \rho \hat{u} = 0,
\]

\[
k_1 \Delta \hat{p}_1 + k_3 \Delta \hat{p}_2 - \alpha_1 \hat{p}_1 - \alpha_3 \hat{p}_1 - \gamma (\hat{p}_1 - \hat{p}_2) - \beta_1 \text{div} \hat{u} = 0,
\]

\[
k_3 \Delta \hat{p}_1 + k_2 \Delta \hat{p}_2 - \alpha_3 \hat{p}_1 - \alpha_2 \hat{p}_2 + \gamma (\hat{p}_1 - \hat{p}_2) - \beta_2 \text{div} \hat{u} = 0,
\]

it follows that

\[
\{[\mu_1 + (\lambda_1 + \mu_1) \delta_{11}] \xi^2 - \rho \omega^2 \} C_1 + i \beta_1 \xi \delta_{11} C_4 + i \beta_2 \xi \delta_{11} C_5 = 0,
\]

\[
\beta_1 \omega \xi C_1 + (k_1 \xi^2 - a_1) C_4 + (k_3 \xi^2 - a_3) C_5 = 0,
\]

(52)

\[
\beta_2 \omega \xi C_1 + (k_3 \xi^2 - a_3) C_4 + (k_2 \xi^2 - a_2) C_5 = 0,
\]

where \( a_3 = i \omega_3 + \gamma, \ l = 1, 2, 3 \).

By virtue of (52) for \( C_1, C_4 \) and \( C_5 \) we have the system

\[
(\mu_0 \xi^2 - \rho \omega^2) C_1 + i \beta_1 \xi C_4 + i \beta_2 \xi C_5 = 0,
\]

\[
\beta_1 \omega \xi C_1 + (k_1 \xi^2 - a_1) C_4 + (k_3 \xi^2 - a_3) C_5 = 0,
\]

(53)

\[
\beta_2 \omega \xi C_1 + (k_3 \xi^2 - a_3) C_4 + (k_2 \xi^2 - a_2) C_5 = 0.
\]

From (53) it follows that

\[
L(\xi^2) C_l = 0, \quad l = 1, 4, 5,
\]

(54)

where

\[
L(\xi^2) = \mu_0 k' \xi^6 - (\mu_0 r + \rho \omega^2 k + i \omega r_1) \xi^4 + (\mu_0 a + \rho \omega^2 r + i \omega r_2) \xi^2 - \rho \omega^2 a
\]

and

\[
a = a_1 a_2 - a_3^2, \quad k' = k_1 k_2 - k_3^2, \quad r = a_1 k_2 + a_2 k_1 - 2 a_3 k_3,
\]

\[
r_1 = k_2 \beta_1^2 + k_1 \beta_2^2 - 2 k_3 \beta_1 \beta_2, \quad r_2 = a_2 \beta_1^2 + a_1 \beta_2^2 - 2 a_3 \beta_1 \beta_2.
\]

(55)
If $\xi$ is a solution of the equation
\[ L(\xi^2) = 0, \]  
then (54) has non-trivial solutions $(C_1, C_4, C_5)$.

Similarly, from (52) we have the following equation for $C_2$ and $C_3$
\[ T(\xi^2) C_j = 0, \quad j = 2, 3, \]  
where $T(\xi^2) = \mu_1 \xi^2 - \rho \omega^2$.

If $\xi$ is a solution of the equation
\[ T(\xi^2) = 0, \]  
then (58) has non-trivial solutions $C_2$ and $C_3$.

The relations (57) and (59) will be called the dispersion equations of longitudinal and transverse plane harmonic waves, respectively. Obviously, (59) is the dispersion equation of transverse plane harmonic waves in the classical theory of viscoelasticity for Kelvin-Voigt materials without porosity (see [45, 46]) and, consequently, transverse waves in the theory of viscoelasticity for Kelvin-Voigt materials with double porosity have the same characteristics as the corresponding waves in the classical theory of viscoelasticity.

Obviously, if $\xi > 0$, then the corresponding plane wave has constant amplitude, and if $\xi$ is complex with $\text{Im} \xi > 0$, then the plane wave is attenuated as $x_1 \to +\infty$.

Let $\xi_1^2, \xi_2^2, \xi_3^2$ and $\xi_4^2$ be roots of the equation
\[ L(\xi) = 0 \]  
and $T(\xi) = 0$, respectively. From (59) it follows that $\xi_4^2 = \rho \omega^2 \mu_1^{-1}$. Obviously, $\xi_1, \xi_2, \xi_3$ and $\xi_4$ are the wave numbers of longitudinal and transverse plane harmonic waves, respectively.

We denote the longitudinal plane wave with wave number $\xi_j$ ($j = 1, 2, 3$) by $P_j$ (P-primary), and the transverse horizontal and vertical plane waves with wave number $\xi_4$ by $SH$ and $SV$, respectively (S-secondary).

In what follows we use the following result.

**Lemma 5.1.** If the condition (50) is satisfied, then the equation (60) has not a positive root.

**Proof.** On the basis of (56) we have
\[ r = -\gamma k_0 + i \omega \alpha_4, \quad r_2 = -\gamma \beta_3^2 + i \omega \beta_3, \quad a = -\omega^2 \alpha - i \omega \gamma \alpha_0, \]  
where
\[ \alpha = \alpha_1 \alpha_2 - \alpha_3^2, \quad \alpha_0 = \alpha_1 + \alpha_2 + 2 \alpha_3, \quad \alpha_4 = \alpha_1 k_2 + \alpha_2 k_1 - 2 \alpha_3 k_3, \]
\[ k_0 = k_1 + k_2 + 2 k_3, \quad \beta_0 = \beta_1 + \beta_2, \quad \beta_3 = \alpha_1 \beta_2^2 + \alpha_2 \beta_1^2 - 2 \alpha_3 \beta_1 \beta_2. \]  
By virtue of (50), from (56) and (62) we get
\[ k' > 0, \quad k_0 > 0, \quad \alpha > 0, \quad \alpha_0 > 0, \quad \alpha_4 > 0, \quad \beta_3 > 0. \]  

On the other hand, keeping in mind (56), (61) and (62) we obtain
\[ \mu_0 r + \rho \omega^2 k' + i \omega r_1 = \rho \omega^2 k' + \omega^2 \mu_2 \alpha_4 - \gamma \lambda_2 k_0 + i \omega (\lambda_2 \alpha_4 + \mu_2 \gamma k_0 + r_1), \]
\[ \mu_0 a + \rho \omega^2 r + i \omega r_2 = -\omega^2 (\alpha \lambda_2 + \mu_2 \gamma \alpha_0 + \rho \gamma k_0 + \beta_3) + i \omega (\omega^2 \alpha \mu_2 + \rho \omega^2 \alpha_4 - \gamma \alpha_0 \lambda_2 - \gamma \beta_3^2), \]  
where $\lambda_2 = \lambda + 2 \mu, \mu_2 = \lambda^* + 2 \mu^*$. 

It is easy to see that (60) is an equation with complex coefficients. Let $\xi_0$ be the real root of (60). Separating real and imaginary parts in (60), on the basis of (55), (61) and (64) we obtain the following system

$$ k'\lambda_2\xi_0^2 - (\rho\omega^2k' + \omega^2\mu_2\alpha_4 - \gamma\lambda_2k_0)\xi_0^2 $$
$$ -\omega^2(\alpha\lambda_2 + \mu_2\gamma\alpha_0 + \rho\gamma k_0 + \beta_3)\xi_0 + \alpha\rho\omega^4 = 0, $$
$$ k'\mu_2\xi_0^2 + (\lambda_2\alpha_4 + \mu_2\gamma k_0 + r_1)\xi_0^2 $$
$$ -\omega^2\rho\mu_2 + \rho\omega^2\alpha_4 - \gamma\alpha_0\lambda_2 - \gamma\beta_0^2)\xi_0 - \rho\omega^2\alpha_0\gamma = 0. $$

As one may easily verify, the system (65) may be written in the form

$$(\lambda_2\xi_0 - \rho\omega^2) (k'\xi_0^2 + \gamma k_0\xi_0 - \omega^2) = \omega^2\xi_0 [\mu_2(\alpha_4\xi_0 + \gamma\alpha_0) + \beta_3],$$

$$(\lambda_2\xi_0 - \rho\omega^2)(\alpha_4\xi_0 + \alpha_0\gamma) + \mu_2\xi_0 (k'\xi_0^2 + \gamma k_0\xi_0 - \omega^2) = -\xi_0(r_1\xi_0 + \gamma\beta_0^2).$$

It is obvious that $(\lambda_2\xi_0 - \rho\omega^2)\xi_0 \neq 0$ and, hence, from the system (66) it follows that

$$(k'\xi_0^2 + \gamma k_0\xi_0 - \omega^2) [\mu_2 (k'\xi_0^2 + \gamma k_0\xi_0 - \omega^2) + r_1\xi_0 + \gamma\beta_0^2]$$
$$ = -\omega^2 [\mu_2(\alpha_4\xi_0 + \gamma\alpha_0) + \beta_3] (\alpha_4\xi_0 + \alpha_0\gamma),$$

and we have

$$ c_1\xi_0^4 + c_2\xi_0^3 + c_3\xi_0^2 + c_4\xi_0 + c_5 = 0, $$

where

$$ c_1 = \mu_2 k', \quad c_2 = 2\mu_2 k'k_0\gamma, $$
$$ c_3 = \mu_2 \gamma^2 k_0^2 + \gamma (r_1 k_0 + k'\beta_0^2) + \omega^2 \mu_2 (\alpha_4^2 - 2k'\alpha), $$
$$ c_4 = \omega^2 (\beta_3\alpha_4 - \alpha r_1) + 2\omega^2\gamma \mu_2 (\alpha_4\alpha_0 - k_0\alpha) + \gamma^2 k_0\beta_0^2, $$
$$ c_5 = \omega^2 \mu_2 (\alpha_4^2\omega^2 + \alpha_0^2\gamma^2) + \omega^2\gamma (\alpha_0\beta_3 - \alpha\beta_0^2). $$

Taking into account (50), (63) and

$$ \alpha_4^2 - 2k'\alpha = 2(k_1k_2 + k_3^2)\alpha_3^2 - 4k_3(k_1\alpha_2 + k_2\alpha_1)\alpha_3 + k_1^2\alpha_2^2 + k_3^2\alpha_1^2 + 2k_3^2\alpha_1\alpha_2 $$
$$ = \frac{2}{k_1k_2 + k_3^2} [(k_1k_2 + k_3^2)\alpha_3 - k_3(k_1\alpha_2 + k_2\alpha_1)]^2 $$
$$ + \frac{k'}{k_1k_2 + k_3^2} [(k_1\alpha_2 - k_2\alpha_1)^2 + 2k'\alpha_1\alpha_2] > 0, $$

$$ \beta_3\alpha_4 - \alpha r_1 = k_1(\beta_1\alpha_2 - \beta_2\alpha_3)^2 + k_2(\beta_1\alpha_3 - \beta_2\alpha_1)^2 $$
$$ - 2k_3(\beta_1\alpha_2 - \beta_2\alpha_3)(\beta_1\alpha_3 - \beta_2\alpha_1) \geq 0 $$
$$ \alpha_4\alpha_0 - k_0\alpha = k_1(\alpha_2 + \alpha_3)^2 + k_2(\alpha_1 + \alpha_3)^2 - 2k_3(\alpha_1 + \alpha_3)(\alpha_2 + \alpha_3) > 0 $$
$$ \alpha_0\beta_3 - \alpha\beta_0^2 = [\beta_1(\alpha_2 + \alpha_3) - \beta_2(\alpha_1 + \alpha_3)]^2 \geq 0 $$

from (68) we obtain

$$ c_j > 0, \quad j = 1, 2, \cdots, 5, $$

i.e. (67) is the equation with positive coefficients and it has not a positive root.

Hence, the longitudinal and the transverse plane wave numbers $\xi_1, \xi_2, \xi_3$ and $\xi_4$ are complex values. We assume that Im$\xi_j > 0$ ($j = 1, 2, 3, 4$). The amplitude change of a decaying plane wave can be expressed as

$$ C'_l = C_l e^{-\text{Im}\xi_j^l x_1}, \quad j = 1, 2, 3, \quad l = 1, 4, 5. $$
In this expression the amplitude $C'_l$ is the reduced amplitude after the wave has traveled a distance $x_1$ from the initial location, $C_l$ is the unattenuated amplitude of the propagating wave at the same location, the quantity $\text{Im} \xi_j$ is the attenuation coefficient of the wave traveling in the $x_1$-direction.

Lemma 5.1 leads to the following result.

**Theorem 5.2.** If condition (50) is satisfied, then through a Kelvin-Voigt material with double porosity five plane harmonic waves propagate:

i) three longitudinal plane waves $P_1, P_2, P_3$ with wave numbers $\xi_1, \xi_2$ and $\xi_3$, respectively; these are attenuated waves as $x_1 \to +\infty$, and

ii) two transverse plane waves $SH$ and $SV$ with wave number $\xi_4$; these are attenuated waves as $x_1 \to +\infty$.

**Remark 5.** In the quasi-static theory (the inertial term is neglected in the first equation of (8)), plane harmonic waves have the same properties (these properties are given in theorem 5.2) as in the above considered dynamical theory (the values $c_1, c_2, \cdots, c_5$ are independent on the density $\rho$).

**Remark 6.** The basic properties of plane harmonic waves in elastic materials with double porosity and viscoelastic materials with voids are established in [13, 49] and [43, 46], respectively.

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