IMPROVED RESTRICTION ESTIMATE FOR HYPERBOLIC SURFACES IN $\mathbb{R}^3$

CHU-HEE CHO AND JUNGJIN LEE

Abstract. Recently, L. Guth improved the restriction estimate for the surfaces with strictly positive Gaussian curvature in $\mathbb{R}^3$. In this paper we extend his restriction estimate to the surfaces with strictly negative Gaussian curvature.

1. Introduction

Let $S$ be a smooth compact hypersurface with boundary in $\mathbb{R}^d$, which has a surface measure $d\sigma$. The Fourier transform of the measure $f d\sigma$ is written as

$$\hat{f}d\sigma(x) = \int_S e^{2\pi i x \cdot w} f(w) d\sigma(w).$$

The restriction problem posed by Stein [13] is to find $(p, q)$ for which the adjoint restriction estimate

$$\|\hat{f}d\sigma\|_q \leq C\|f\|_{L^p(S)} \quad (1.1)$$

holds for all $f \in C^\infty_c(S)$, where the constant $C$ may depend on $p, q, d, S$ but not on $f$. This problem is connected to questions about the convergence of Fourier summation methods such as the Bochner-Riesz conjecture and local smoothing conjecture. Also, there is a fundamental relation between the restriction problem and the Kakeya problem. Moreover, the restriction problem is associated with the analysis of linear PDE such as the Helmholtz equation, Schrödinger equation, wave equation and the Korteweg-de Vries equation. See [3, 15].

For several decades, a fair amount of work was devoted to this problem (particularly when $S$ is an elliptic surface such as the unit sphere and paraboloid). After Bourgain [2] combined a multiscale analysis approach with his Kakeya estimate, Bourgain’s methods were developed over the years; see [12][17][18]. Especially, from the analysis of $L^2$ bilinear variants of the problem, Wolff [20] and Tao [16] obtained the $L^2$ bilinear restriction theorem for the cones and paraboloids respectively, which made a significant progress on the restriction problems. On the other hand, Bennet, Carbery and Tao [1], using the heat-flow method, obtained the multilinear Kakeya theorem and the multilinear restriction theorem. (Later, Guth [6, 8] gave an alternative proof of the multilinear Kakeya theorem.) After several years, Bourgain and Guth [4] found a new way to apply the multilinear restriction theorem to the restriction problem, and they obtained some improvements. Recently, Guth [7] further developed it in $\mathbb{R}^3$ by adapting the polynomial partitioning. (It is a method that has brought some important results about overlapping lines in incidence geometry; see [9][10].)

In [7], Guth considered the restriction estimate for surfaces with strictly positive Gaussian curvature. The aim of this paper is to extend Guth’s restriction estimate to the case of quadratic surfaces with strictly negative Gaussian curvature in $\mathbb{R}^3$. The following is our main result.

Date: November 4, 2016.

2010 Mathematics Subject Classification: Primary 42B20.
Theorem 1.1. Let $S$ be a compact quadratic surface with strictly negative Gaussian curvature in $\mathbb{R}^3$. Then, for $p > 3.25$ and $p = q$, the estimate (1.1) is valid.

Stein [14] verified that the estimate (1.1) holds for $q \geq 4$ and $\frac{2}{q} \leq 1 - \frac{1}{p}$. The best previously known result due to Lee [11] and Vargas [19] was $q > 10/3$ and $\frac{2}{q} < 1 - \frac{1}{p}$. By interpolating Theorem 1.1 with the previous result, the $(p, q)$-range is extended to $q > 3.25$, $\frac{2}{q} < 1 - \frac{1}{3p}$ and $\frac{2}{q} < 1 - \frac{1}{p}$.

Our proof is based on Guth’s arguments in [7]. The key ingredients in his arguments are a broad function, polynomial partitioning, induction and bilinear estimate. Roughly speaking, the polynomial partitioning and induction are used to reduce a 3-dimensional restriction problem to an essentially 2-dimensional one. The broad function is exploited for a bilinear approach to the derived 2-dimensional problem. We will modify the definition of broad function and the related bilinear estimates. As mentioned in [11] and [19], we need a stronger separation condition to obtain bilinear restriction estimates for hyperbolic surfaces than that for elliptic ones. Accordingly, our broad function will be defined to involve such strong separation condition. Then, it is possible to have the same bilinear estimates as in [7].

The paper is organized as follows. In next section, we prepare the proof of our result by giving an elementary proposition about a wave packet decomposition. In section 3, we define a broad function, and reduce a Fourier restricted function to its broad function. In section 4, we prove the main part. We trim down the problem by using a polynomial partitioning and induction arguments, and then we bilinearly approach the remaining part.

Throughout the paper we use $C$ to denote positive constants $\geq 1$ which may be different at each occurrence. We denote $A \lesssim B$ or $A = O(B)$ to mean $A \leq CB$, and $A \sim B$ to mean $C^{-1}B \leq A \leq CB$. We denote the number of members of a set $A$ by $\#A$.

2. Wave packet decomposition

In this section we recall a wave packet decomposition which has been a fundamental tool in restriction problems.

By a suitable translation and linear transformation we may set $S$ as the hyperbolic paraboloid defined by

$$S = \{(\xi_1, \xi_2, \xi_1 \xi_2) \in \mathbb{R}^3 : (\xi_1, \xi_2) \in D(1)\},$$

where $D(1)$ is the unit square centered at the origin. Let us define the extension operator $Ef$ by

$$Ef(x) = \int_S e^{2\pi i x \cdot \xi} f(\xi) d\sigma(\xi) \sim \int_{D(1)} e^{2\pi i (x_1 \xi_1 + x_2 \xi_2 + x_3 \xi_1 \xi_2)} \tilde{f}(\xi_1, \xi_2) d\xi_1 d\xi_2$$

where $\tilde{f}(\xi_1, \xi_2) = f(\xi_1, \xi_2, \xi_1 \xi_2) \frac{d\sigma(\xi_1, \xi_2)}{d\xi_1 d\xi_2}$.

We decompose $S$ into caps $\Omega$ of diameter $R^{-1/2}$. Let $n(\Omega)$ be the unit normal vector to $S$ at the center of $\Omega$. Let $\delta > 0$ be a small parameter. For each cap $\Omega$, we define $T(\Omega)$ to be the set of cylindrical tubes $T$ of radius $R^{1/2+\delta}$ which are parallel to $n(\Omega)$ and cover a ball $B(R)$ of radius $R > 1$ with finite overlap. If $T \in T(\Omega)$ then $v(T)$ indicates $n(\Omega)$, and $\omega(T)$ denotes the center of $\Omega$. We define $T = \bigcup T(\Omega)$.

We use the following standard wave packet decomposition. This is a simple modification of Proposition 2.6 in [7]. (We can find a similar decomposition in [11, Lemma 2.2] and in [19, section 3].)
Proposition 2.1 (Wave packet decomposition). Let \( R \gg 1 \) and let \( B(R) \) be a ball of radius \( R \). If \( f \in L^2(S) \), then for each tube \( T \in \mathbb{T} \) there exists a function \( f_T \) satisfying the following conditions:

1. If \( T \in \mathbb{T}^{(1)}(\Omega) \) then supp \( f_T \subset 3\Omega \).
2. If \( x \in B(R) \backslash T \) then \( |E f_T(x)| \leq R^{-1000} \|f\|_{L^2(S)} \).
3. For any \( x \in B(R) \), \( |E f(x) - \sum_{T \in \mathbb{T}} E f_T(x)| \leq R^{-1000} \|f\|_{L^2(S)} \).
4. If \( T_1, T_2 \in \mathbb{T}(\Omega) \) and \( T_1, T_2 \) are disjoint, then \( \int f_{T_1} f_{T_2} \leq R^{-1000} \int |f|^2 \).
5. \( \sum_{T \in \mathbb{T}(\Omega)} \int |f_T|^2 \leq \int |f|^2 \).
6. Let \( \tau \subset S \) be a cap of radius \( > 10R^{-1/2} \) and \( f_{\tau} := f \chi_{\tau} \). Then for any \( T' \subset \mathbb{T} \) and any \( \omega \in S \),

\[
\| \sum_{T \in \mathbb{T}^{(1)}(\omega)(T) \in \tau} f_T \|_{L^2(B(\omega,R^{-1/2}) \cap S)} \lesssim \|f\|_{L^2(10B(\omega,R^{-1/2}) \cap S)}.
\]

The proof will be given in Appendix.

3. Reduction and the broad function

In this section we reduce the restriction estimate to a problem of obtaining good localized estimates for some regularized (adjoint) restriction operator.

As in [2], by the Stein-Nikishin factorization theorem, it suffices to show (1.1) for \( q > 3.25 \) and \( p = \infty \). Furthermore, by Tao’s \( \epsilon \)-removal lemma it is reduced to showing the following:

Theorem 3.1. Let \( p_0 = 3.25 \). For any \( R \geq 1 \) the estimate

\[
\|E f\|_{L^{p_0}(B(R))} \leq C R^q \|f\|_{L^\infty(S)}
\]

is valid for all \( f \) on \( S \), all \( 0 < \epsilon \ll 1 \) and all ball \( B(R) \) of radius \( R \).

By translation invariance we may assume that \( B(R) \) is centered at the origin.

Fix \( R \gg 1 \); in the case \( R \sim 1 \), it is easy to see (3.1). First, we take a large dyadic number

\[
K = K(\epsilon) \text{ with } \lim_{\epsilon \to 0} K(\epsilon) = \infty \text{ (we may set } K \sim e^{10} \text{).}
\]

We divide \( D(1) \) into \( K^2 \) squares \( \bar{\tau} \) of sidelength \( K^{-1} \) whose sides are parallel to standard unit vectors \( e_1 \) and \( e_2 \). Let \( L_{\|e_1\|} \) denote the \( K \) strips of width \( K^{-1} \) such that their center lines are parallel to \( e_1 \in \mathbb{R}^2 \) and they are composed of the squares \( \bar{\tau} \). \( L_{\|e_2\|} \) are similar strips but their center lines are parallel to \( e_2 \). Let \( \tau := \{(\xi_1,\xi_2,\xi_3 e_2) : (\xi_1, \xi_2) \in \bar{\tau}\} \). Then the surface \( S \) is covered by the \( K^2 \) caps \( \tau \) of diameter \( \sim K^{-1} \). Set \( f_{\tau} = \chi_{\tau} f \).

For \( \alpha \in (0, 1) \), we define an \( \alpha \)-broad point of \( E f \) to be the point \( x \) at which

\[
\max_\tau |E f_{\tau}(x)| + \max_{L=1_{e_1} \text{ or } 1_{e_2}} \left| \sum_{\tau : \tau \subseteq L} E f_{\tau}(x) \right| \leq \alpha |E f(x)|.
\]

If \( A_\alpha \) is the set of all \( \alpha \)-broad points of \( E f \), then we define an \( \alpha \)-broad function \( B_\alpha[E f] \) by

\[
B_\alpha[E f](x) = E f(x) \chi_{A_\alpha(x)}.
\]

Then for given \( x \in B(R) \), there exist \( \tau \) and \( L \) such that

\[
|E f(x)| \leq |B_\alpha[E f](x)| + \alpha^{-1} \left( |E f_{\tau}(x)| + \sum_{\tau : \tau \subseteq L} E f_{\tau}(x) \right).
\]

From this we have that for any \( x \in B(R) \),

\[
|E f(x)|^{p_0} \lesssim \left| B_\alpha[E f](x) \right|^{p_0} + \alpha^{-p_0} \left( \sum_{\tau} |E f_{\tau}(x)|^{p_0} + \sum_{L} \left| \sum_{\tau : \tau \subseteq L} E f_{\tau}(x) \right|^{p_0} \right).
\]
By integrating over \(B(R)\),
\[
\int_{B(R)} |Ef(x)|^p \lesssim \int_{B(R)} |B_\alpha[Ef](x)|^p + \alpha^{-p_0} \left( \sum_\tau \int_{B(R)} |Ef_\tau(x)|^p + \sum_{L = L_{\|\xi\|_1} \text{ or } L_{\|\xi\|_2}} \int_{B(R)} \|\sum_{\tau: \tau \in L} Ef_\tau(x)|^p \right).
\]
(3.2)

We first deal with the summation parts of the above inequality. For this we use an inductive argument on \(R\); we assume that for any \(1 \leq r \leq R/2\), the estimate (3.1) holds for all \(f\), all \(0 < \epsilon \ll 1\) and all balls \(B(R)\).

By using the induction hypothesis we can prove the following estimates by scaling.

**Lemma 3.2.**
\[
\|Ef_\tau\|_{L^{p_0}(B(R))} \leq CC_r K^{-2 + \frac{1}{p_0}} R^r \|f_\tau\|_{L^\infty(S)}, \quad (3.3)
\]
\[
\left\| \sum_{\tau: \tau \in L} Ef_\tau \right\|_{L^{p_0}(B(R))} \leq CC_r K^{-1 + \frac{1}{p_0}} R^r \left\| \sum_{\tau: \tau \in L} f_\tau \right\|_{L^\infty(S)}. \quad (3.4)
\]

**Proof.** We first show (3.3). By translation we may assume that \(\bar{\tau}\) is centered at the origin. By abuse of notation, we use \(f\) instead of \(\hat{f}\). Then \(Ef_\tau\) is supported in the square of sidelength \(K^{-1}\) with center at the origin. By scaling \((\xi_1, \xi_2) \to (K^{-1}\xi_1, K^{-1}\xi_2)\),
\[
Ef_\tau(x_1, x_2, x_3) = K^{-2}[Ef^K_\tau](K^{-1}x_1, K^{-1}x_2, K^{-2}x_3)
\]
where \(f^K_\tau = f_\tau(K^{-1}\xi_1, K^{-1}\xi_2)\). Note that \(f^K_\tau\) is supported in the unit square. By a change of variables,
\[
\|[Ef^K_\tau](K^{-1}, K^{-1}, K^{-2})\|_{L^{p_0}(B(R))} = K^{4/p_0} \|Ef^K_\tau\|_{L^{p_0}(T)}
\]
where \(T\) is a tube of dimensions \(R/K \times R/K \times R/K^2\). From the above equations, we have
\[
\|Ef_\tau\|_{L^{p_0}(B(R))} \leq K^{-2 + \frac{1}{p_0}} \|Ef^K_\tau\|_{L^{p_0}(T)}.
\]
Since \(\|Ef^K_\tau\|_{L^{p_0}(T)} \leq \|Ef^K_\tau\|_{L^{p_0}(B(R/2))}\), we can apply the induction hypothesis. Thus,
\[
\|Ef_\tau\|_{L^{p_0}(B(R))} \leq CC_r K^{-2 + \frac{1}{p_0}} \|f_\tau\|_{L^\infty(S)} = CC_r K^{-1 + \frac{1}{p_0}} \|f_\tau\|_{L^\infty(S)}.
\]

Now we prove (3.4). Let \(L = L_{\|\xi\|_1}\); when \(L = L_{\|\xi\|_2}\) the argument below is similar. By translation we may assume that the center line of \(L\) is \(e_1\). Let \(f_L = \sum_{\tau: \tau \in L} Ef_\tau\). Taking a rescaling \((\xi_1, \xi_2) \to (\xi_1, K^{-1}\xi_2)\), we have
\[
Ef_L(x_1, x_2, x_3) = K^{-1}[Ef^K_L](x_1, K^{-1}x_2, K^{-1}x_3)
\]
where \(f^K_L = f_L(K^{-1}\xi_1, K^{-1}\xi_2)\). We can see that \(f^K_L\) is supported in \([-1, 1]^2\). By changing of variables,
\[
\|[Ef^K_L](\cdot, K^{-1}, K^{-1})\|_{L^{p_0}(B(R))} = K^{2/p_0} \|Ef^K_L\|_{L^{p_0}(L^*)}
\]
where \(L^*\) is a tube of dimensions \(R \times R/K \times R/K\). Thus, combining these, we have
\[
\|Ef_L\|_{L^{p_0}(B(R))} \leq K^{-1 + \frac{2}{p_0}} \|Ef^K_L\|_{L^{p_0}(L^*)}.
\]
Cover \(L^*\) with two balls of radius \(\frac{3}{4}R\). Since \(\|Ef_L\|_{L^{p_0}(L^* \cap B(\frac{3}{4}R))} \leq \|Ef_L\|_{L^{p_0}(B(\frac{3}{4}R))}\), we can apply the induction hypothesis to each ball. So, we obtain
\[
\|Ef_L\|_{L^{p_0}(B(R))} \leq CC_r R^r K^{-1 + \frac{2}{p_0}} \|f^K_L\|_{L^\infty(S)} = CC_r R^r K^{-1 + \frac{2}{p_0}} \|f_L\|_{L^\infty(S)}.
\]
\[\square\]
Let us set $\alpha = K^{-\epsilon}$. After raising both sides in (3.3) to the $p_0$th power, we sum these over $\tau$. Since the number of caps $\tau$ is $K^2$, we have

$$K^{O(\epsilon)} \sum_{\tau} \int_{B(R)} |Ef_\tau(x)|^{p_0} \leq C_\epsilon^{p_0} (CK^{-2p_0+6+O(\epsilon)}) \|f\|_{L^\infty(S)}^{p_0}.$$  

Since $p_0 > 3$ and $\lim_{\epsilon \to 0} K(\epsilon) = \infty$, we can take a sufficiently small $\epsilon > 0$ such that

$$CK^{-2p_0+6+O(\epsilon)} \leq (1/3)^{p_0}.$$  

So, it gives

$$K^{O(\epsilon)} \sum_{\tau} \int_{B(R)} |Ef_\tau(x)|^{p_0} \leq 3^{-p_0} C_\epsilon^{p_0} \|f\|_{L^\infty(S)}^{p_0}, \quad (3.5)$$

Similarly, we raise both sides in (3.4) to the $p_0$th power, and sum these over $L$. Then, since the number of strips $L$ is $K$, we have

$$K^{O(\epsilon)} \sum_{L} \int_{B(R)} \left| \sum_{\tau+L} E\hat{f}_\tau(x) \right|^{p_0} \leq C_\epsilon^{p_0} (CK^{-p_0+3+O(\epsilon)}) \|f\|_{L^\infty(S)}^{p_0}.$$  

From $p_0 > 3$ and $\lim_{\epsilon \to 0} K(\epsilon) = \infty$, it is possible to take a sufficiently small $\epsilon > 0$ so that

$$CK^{-p_0+3+O(\epsilon)} \leq (1/3)^{p_0}.$$  

Then,

$$K^{O(\epsilon)} \sum_{L} \int_{B(R)} \left| \sum_{\tau+L} E\hat{f}_\tau(x) \right|^{p_0} \leq 3^{-p_0} C_\epsilon^{p_0} \|f\|_{L^\infty(S)}^{p_0}, \quad (3.6)$$

To show (3.1), by (3.2), (3.5) and (3.6) it suffices to prove

$$\|B_{K^{-\epsilon}}[Ef]\|_{L^{p_0}(B(R))} \leq 3^{-1} C_\epsilon R^\epsilon \|f\|_{L^\infty(S)}. \quad (3.7)$$

This immediately follows from the following.

**Theorem 3.3.** Let $R \gg 1$. For any $0 < \epsilon \ll 1$, there exists $\delta_2 \in (0, \epsilon)$ and $K = K(\epsilon)$ with $\lim_{\epsilon \to 0} K(\epsilon) = \infty$ such that if for any $\omega \in S$,

$$\int_{B(\omega, R^{-1/2}) \cap S} |f|^2 \leq 1, \quad (3.8)$$

then

$$\int_{B(R)} |B_{K^{-\epsilon}}[Ef]|^{p_0} \leq C_\epsilon R^\epsilon R^{\delta_2} \left( \int_{S} |f|^2 \right)^{3/2 + \epsilon}. \quad (3.9)$$

Here, $C_\epsilon$ is independent of $R$ and $f$.

Indeed, the implication from Theorem 3.3 to (3.7) can be proven as follows. We may assume that $\|f\|_{L^\infty(S)} \leq 1$ by normalization. Then, it is easy to see that $\int_{B(\omega, R^{-1/2}) \cap S} |f|^2 \leq \|f\|_{L^\infty(S)}^2$ for any $\omega \in S$. From (3.8) it follows that $\int_{S} |f|^2 \leq \sum_{\Omega} \int_{\Omega} |f|^2 \leq 1$. Combining this with the above estimate we have

$$\|B_{K^{-\epsilon}}[Ef]\|_{L^{p_0}(B(R))} \leq C_\epsilon^{1/p_0} R^{2\epsilon/p_0}. \quad (3.10)$$

Since $\epsilon > 0$ is arbitrary, this gives (3.7). Now, it remains to prove Theorem 3.3. This will be done in the next section.

**Remark 3.4.** The broad function defined in this paper is different from that in [7]. This new broad function guarantees that the bilinear operator in Lemma 4.5 has a stronger separation condition than that in [7].
4. Proof of Theorem 3.3

This section is devoted to prove Theorem 3.3. We first mention a polynomial partitioning which is a technique recently applied to some problems in incidence geometry. For a function \( f \), we define the zero set of \( f \) by

\[
Z(f) = \{ x : f(x) = 0 \}.
\]

For a polynomial \( P \), we say that a polynomial \( P \) is non-singular if it satisfies \( \nabla P(x) \neq 0 \) for each point \( x \) in \( Z(P) \). It is known that non-singular polynomials are dense in the vector space of polynomials on \( \mathbb{R}^n \) of degree at most \( M \). The following is a polynomial partitioning involving non-singular polynomials.

**Theorem 4.1** (Polynomial partitioning for non-singular polynomials, [7]). Assume that a non-negative function \( f \in L^1(\mathbb{R}^n) \) is given. Then for each \( M = 1, 2, \cdots \), there exists a non-zero polynomial \( P \) of degree at most \( M \) such that

\[
\mathbb{R}^n \setminus Z(P) = \bigcup_{i=1}^{O(M^3)} O_i
\]

and all \( \int_{O_i} f \) are comparable. Moreover, the polynomial \( P \) is a product of non-singular polynomials.

Now we prove Theorem 3.3. To begin with, let us set

\[
\delta = \epsilon^2, \quad \delta_1 = \epsilon^4 \quad \text{and} \quad \delta_2 = \epsilon^6. \tag{4.1}
\]

Then we have the relation \( \epsilon \gg \delta \gg \delta_1 \gg \delta_2 \). (This relation plays a crucial role to close the induction below.)

For fixed \( R \gg 1 \), we analyze \( B_\alpha[\mathbb{E}f] \). First, we apply Theorem 4.1 with

\[
M = R^{\delta_1} \tag{4.2}
\]

to \( \chi_{B(R)}|B_\alpha[\mathbb{E}f]|^{p_0} \). Then there exists a non-zero polynomial \( P \) of degree at most \( M \) such that

\[
\mathbb{R}^n \setminus Z(P) = \bigcup_{i=1}^{O(M^3)} O_i
\]

and for each \( i \), it satisfies

\[
\int_{B(R) \cap O_i} |B_\alpha[\mathbb{E}f]|^{p_0} \sim M^{-3} \int_{B(R)} |B_\alpha[\mathbb{E}f]|^{p_0}. \tag{4.3}
\]

Let us define the wall \( W \) by the \( R^{1/2+\delta} \)-neighborhood of \( Z(P) \) and the cell \( O'_i \) by \( O_i \setminus W \). Then we have

\[
\int_{B(R)} |B_\alpha[\mathbb{E}f]|^{p_0} = \sum_i \int_{B(R) \cap O'_i} |B_\alpha[\mathbb{E}f]|^{p_0} + \int_{B(R) \cap W} |B_\alpha[\mathbb{E}f]|^{p_0}. \tag{4.4}
\]

To estimate the above we will use two kinds of induction. The first one is an induction on the scale \( R \). We assume that for any \( 1 \leq r \leq R/2 \), Theorem 3.3 is true. If \( R = 1 \) then it is easy to see that the estimate (3.9) holds. The other one is an induction on \( \|f\|_{L^2(S)} \). We assume that for all \( g \) with \( \|g\|_{L^2(S)} \leq \frac{1}{2}\|f\|_{L^2(S)} \), Theorem 3.3 is true. If \( \|g\|_{L^2(S)} \leq R^{-1000} \) then we can easily obtain (4.9).
4.1. **Cell estimate.** We consider the contribution of the summation part of the right side of (4.4). To deal with this part we will use the second induction. Suppose that this summation part dominates the other term. Then,

\[ \int_{B(R)} |B_\alpha[Ef]|_{p_0} \lesssim \sum_i \int_{B(R) \cap O'_i} |B_\alpha[Ef]|_{p_0}. \] (4.5)

**Lemma 4.2.** Assume (4.3) and (4.5). Then there exists a subcollection \( \mathcal{I} \) with cardinality \( O(M^3) \) such that for all \( i \in \mathcal{I} \),

\[ \int_{B(R) \cap O'_i} |B_\alpha[Ef]|_{p_0} \sim M^{-3} \int_{B(R)} |B_\alpha[Ef]|_{p_0}. \] (4.6)

**Proof.** For convenience, let \( X_i := \int_{B(R) \cap O'_i} |B_\alpha[Ef]|_{p_0} \) and \( A := M^{-3} \int_{B(R)} |B_\alpha[Ef]|_{p_0} \), and let \( N \) be the number of cells \( O_i \). Then from (4.3) we see that there exists a constant \( C_1 \geq 1 \) such that for each \( i \),

\[ X_i \leq C_1 A, \] (4.7)

and from (4.5) we see that there exists a constant \( C_2 \geq 1 \) such that

\[ NA \leq C_2 \sum_{i=1}^N X_i. \] (4.8)

Let \( c_* \) be a small positive number which will be chosen later. Suppose that there are \( \lambda N \) cells, \( \lambda \in [0, 1] \), such that \( X_i \geq c_* A \) for all \( i \). Then it suffices to show \( \lambda \sim 1 \). In (4.8) we decompose \( \sum X_i \) into two parts as follows:

\[ C_2^{-1} NA \leq \sum_{X_i \geq c_* A} X_i + \sum_{X_i < c_* A} X_i. \]

By (4.7), it is bounded by

\[ \leq \lambda NC_1 A + \sum_{X_i < c_* A} (1-\lambda)X_i \]

\[ \leq \lambda NC_1 A + (1-\lambda)c_* A. \]

By dividing the above by \( NA \), we have \( C_2^{-1} \leq C_1 \lambda + (1-\lambda)c_* \). By rearranging we have \( \lambda \geq \frac{C_2^{-1} - c_*}{c_* - c_*} \) provided \( 0 < c_* \leq \frac{1}{2C_2} \). It means \( \lambda \sim 1 \). □

We rewrite (4.6) as

\[ \int_{B(R)} |B_\alpha[Ef]|_{p_0} \sim M^3 \int_{B(R) \cap O'_i} |B_\alpha[Ef]|_{p_0}. \] (4.9)

We will apply the second induction hypothesis to the above. For this we need several lemmas for restricting the wavepackets \( f_T \) to those with \( T \) passing through \( O'_i \). We decompose \( f \) into the wave packets on \( B(R) \). By (3) of Proposition 2.1 we may set

\[ f = \sum_{T \in \mathcal{T}} f_T. \] (4.10)

Then, \( f_\tau \) can be written as

\[ f_\tau = \sum_{T \in \mathcal{T}, \omega(T) \in \tau} f_T. \]
For each \( i \) and \( \tau \), let us define \( f_{\tau,i} \) and \( f_i \) by
\[
f_{\tau,i} = \sum_{T \in \mathbb{T} : \omega(T) \in \tau} f_T \quad \text{and} \quad f_i = \sum_{\tau} f_{\tau,i}
\]
respectively. We will consider the wave packets \( f_T \) with \( T \cap O'_i \neq \emptyset \). Let \( \mathbb{T}_i(\Omega) \) be the subcollection defined by
\[
\mathbb{T}_i(\Omega) = \{ T \in \mathbb{T}(\Omega) : T \cap O'_i \neq \emptyset \},
\]
and let \( \mathbb{T}_i = \bigcup_{\Omega} \mathbb{T}_i(\Omega) \).

**Lemma 4.3.** For \( x \in O'_i \),
\[
|B_{\alpha}[Ef](x)| \lesssim |B_{4\alpha}[Ef_i](x)| + R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)}.
\]

**Proof.** Using \((4.10)\) we decompose \( Ef \) as
\[
Ef = \sum_{T \in \mathbb{T}} Ef_T = \sum_{T \in \mathbb{T}_i} Ef_T + \sum_{T \in \mathbb{T}_i \setminus \mathbb{T}_i} Ef_T.
\]
From (2) of Proposition \ref{Proposition 2.1}, it follows that for \( x \in O'_i \),
\[
Ef(x) = Ef_i(x) + O(R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)}).
\]

Now it suffices to show that if \( x \in O'_i \) is an \( \alpha \)-broad point of \( Ef \) then \( x \) is also a \( 4\alpha \)-broad point of \( Ef_i \). We may assume \( |Ef_i(x)| \geq R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)} \); otherwise, from \((4.12)\) we have
\[
|B_{\alpha}[Ef](x)| \leq |Ef(x)| \leq |Ef_i(x)| + O(R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)}) \lesssim R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)},
\]
which satisfies \((4.11)\). Since \( x \in O'_i \) is an \( \alpha \)-broad point of \( Ef \), we have that for any cap \( \tau \),
\[
|Ef_{\tau,i}(x)| \leq |Ef_{\tau,i}(x)| + O(R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)})
\]
\[
\leq \alpha |Ef(x)| + O(R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)}).
\]

From \((4.12)\) it is bounded by \( \alpha |Ef_i(x)| + O(R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)}) \). So, for large \( R \), it implies that for any \( \tau \),
\[
|Ef_{\tau,i}(x)| \leq 2\alpha |Ef_i(x)|.
\]
Similarly, for any \( L = L_{\|e_1\|} \) or \( L_{\|e_2\|} \), we have that for any \( x \in O'_i \),
\[
\sum_{\tau : \tau \subseteq L} Ef_{\tau,i}(x) \leq \sum_{\tau : \tau \subseteq L} Ef_{\tau}(x) + O(R^{-900} \sum_{\tau : \tau \subseteq L} \|f_{\tau}\|_{L^2(S)})
\]
\[
\leq \alpha |Ef(x)| + O(R^{-900} \sum_{\tau : \tau \subseteq L} \|f_{\tau}\|_{L^2(S)})
\]
\[
\leq \alpha |Ef_i(x)| + O(R^{-900} \sum_{\tau} \|f_{\tau}\|_{L^2(S)})
\]
\[
\leq 2\alpha |Ef_i(x)|.
\]
From these it follows that for any \( \alpha \)-broad point \( x \in O'_i \),
\[
\max_{\tau} |Ef_{\tau,i}(x)| + \max_{L = L_{\|e_1\|} \text{ or } L_{\|e_2\|}} \sum_{\tau : \tau \subseteq L} Ef_{\tau,i}(x) \leq 4\alpha |Ef_i(x)|.
\]
We raise both sides in (4.11) to the $p_0$th power and integrate it over $B(R) \cap O'_i$. Then
\[
\int_{B(R) \cap O'_i} |B_{\alpha}[Ef]|^{p_0} \lesssim \int_{B(R) \cap O'_i} |B_{4\alpha}[Efi]|^{p_0} + K^2 R^{-2000} \sum_{\tau} \|f_\tau\|_{L^2(S)}^{p_0} 
\]
\[
\lesssim \int_{B(R)} |B_{4\alpha}[Efi]|^{p_0} + K^2 R^{-2000} \sum_{\tau} \|f_\tau\|_{L^2(S)}^{p_0}, \tag{4.13}
\]

From (6) in Proposition 2.1, we have
\[
\int_{B(\omega, R^{-1/2}) \cap S} |f_{\tau,i}|^2 \lesssim \int_{10B(\omega, R^{-1/2}) \cap S} |f_\tau|^2 \lesssim 1.
\]

So, to apply the second induction hypothesis to (4.13), it remains to show
\[
\|f_\tau\|_{L^2(S)} \lesssim \frac{1}{2} \|f\|_{L^2(S)}. \tag{4.14}
\]

We first prove the following lemma by using the geometric fact that if $P$ is a non-zero polynomial of degree $M$ then the algebraic surface $Z(P)$ intersects a line in at most $M$ points.

**Lemma 4.4.**
\[
\sum_i \int |f_{\tau,i}|^2 \lesssim M \int |f_\tau|^2 + R^{-900} \|f_\tau\|_{L^2(S)}^2. \tag{4.15}
\]

**Proof.** From (1) of Proposition 2.1 we have that for each $i$,
\[
\int |f_{\tau,i}|^2 \lesssim \int \left| \sum_{\Omega: \Omega \cap \tau \neq \emptyset} \sum_{T \in \mathcal{T}_i(\Omega)} f_T \right|^2 
\]
\[
\lesssim \sum_{\Omega: \Omega \cap \tau \neq \emptyset} \sum_{T \in \mathcal{T}_i(\Omega)} \int |f_T|^2.
\]

From (4) of Proposition 2.1 it follows that for each $i$,
\[
\int |f_{\tau,i}|^2 \lesssim \sum_{\Omega: \Omega \cap \tau \neq \emptyset} \sum_{T \in \mathcal{T}_i(\Omega)} \int |f_T|^2 + R^{-950} \|f_\tau\|_{L^2(S)}^2.
\]

By summing over $i$,
\[
\sum_i \int |f_{\tau,i}|^2 \lesssim \sum_{\Omega: \Omega \cap \tau \neq \emptyset} \sum_i \sum_{T \in \mathcal{T}_i(\Omega)} \int |f_T|^2 + R^{-900} \|f_\tau\|_{L^2(S)}^2 
\]
\[
\lesssim \sum_{\Omega: \Omega \cap \tau \neq \emptyset} \sum_{T \in \mathcal{T}_i(\Omega)} \sum_{i: O_i \cap T' \neq \emptyset} \int |f_T|^2 + R^{-900} \|f_\tau\|_{L^2(S)}^2.
\]

We observe that each tube $T \in \mathcal{T}$ intersects $O_i$ at most $(M+1)$ times because a line can cross $Z(P)$ at most $M$ times. It makes
\[
\sum_i \int |f_{\tau,i}|^2 \lesssim M \sum_{\Omega: \Omega \cap \tau \neq \emptyset} \sum_{T \in \mathcal{T}_i(\Omega)} \int |f_T|^2 + R^{-900} \|f_\tau\|_{L^2(S)}^2.
\]

From (1) and (4) of Proposition 2.1, we can finally obtain (4.15). \hfill $\square$

We sum (4.15) over $\tau$, and then we use the pigeonhole principle to select an $i_0 \in \mathcal{I}$ such that
\[
\sum_{\tau} \int |f_{\tau,i_0}|^2 \lesssim M^{-2} \sum_{\tau} \int |f_\tau|^2 + M^{-3} R^{-900} \sum_{\tau} \|f_\tau\|_{L^2(S)}^2. \tag{4.16}
\]
Since $S$ is covered by caps $\tau$, it means that $\|f_{i_0}\|^2_{L^2(S)} \leq (CM^{-2} + M^{-3}R^{-900})\|f\|^2_{L^2(S)}$. Thus, by (4.2) we have $\|f\|^2_{L^2(S)}$ for sufficiently large $R$. Now we apply the second induction hypothesis to (4.13) with $i = i_0$. Then it gives that

$$\int_{B(R) \cap O'_0} |B_0[Ef_{i_0}]|^{p_0} \lesssim C_\epsilon R^\epsilon R^{\delta_2} \left( \sum_\tau |f_\tau| \right)^{3/2+\epsilon} + K^2 R^{-2000} \sum_\tau \|f_\tau\|^{p_0}_{L^2(S)}.$$  

By substituting this in (4.3), one has

$$\int_{B(R)} |B_\alpha[Ef]|^{p_0} \lesssim C_\epsilon M^3 R^\epsilon R^{\delta_2} \left( \sum_\tau |f_\tau| \right)^{3/2+\epsilon} + K^2 R^{-2000} \sum_\tau \|f_\tau\|^{p_0}_{L^2(S)}.$$  

By (4.10), it is bounded by

$$\lesssim C_\epsilon (M^{-2\epsilon} R^{\delta_2}) \left( \sum_\tau |f_\tau| \right)^{3/2+\epsilon} + K^2 R^{-1000} \sum_\tau \|f_\tau\|^{p_0}_{L^2(S)} \lesssim C_\epsilon R^\epsilon (M^{-2\epsilon} R^{\delta_2} + K^2 R^{-1000}) \left( \sum_\tau |f_\tau| \right)^{3/2+\epsilon},$$

where we used the estimate $\|f\|^2_{L^2(S)} \lesssim 1$, (which follows from the condition (3.8): $\|f\|^2_{L^2(S)} \lesssim \sum_\Omega |f_\Omega|^2 \leq 1$).

From (4.2) and (4.1), one has $M^{-2\epsilon} R^{\delta_2} = R^{-2\epsilon^5+\epsilon^6}$. Since the exponent of $R$ is negative, we have $CR^{-2\epsilon^5+\epsilon^6} + CK^2 R^{-1000} \leq 1/2$ for sufficiently large $R$. Thus we obtain

$$\int_{B(R)} |B_\alpha[Ef]|^{p_0} \leq 2^{-1} C_\epsilon R^\epsilon \left( \sum_\tau |f_\tau| \right)^{3/2+\epsilon}.$$  

4.2. Wall estimate. Now we suppose that the integral $\int_{B(R) \cap W} |B_\alpha[Ef]|^{p_0}$ dominates the other term in (4.14). Then it suffices to prove

$$\int_{B(R) \cap W} |B_\alpha[Ef]|^{p_0} \leq 2^{-1} C_\epsilon R^\epsilon R^{\delta_2} \left( \sum_\tau |f_\tau| \right)^{3/2+\epsilon}.$$  

We split the wave packets $f_\tau$ into transverse ones and tangent ones to the wall $W$. We first cover $B(R)$ with $O(R^{\delta_5})$ balls $B_j$ of radius $R^{1-\delta}$. (Later we will use an inductive argument to each $B_j$ to estimate the transversal part.)

We define the collection $T_j^0$ of tangential tubes to be the collection of all tubes $T \in \mathbb{T}$ such that $T \cap W \cap B_j \neq \emptyset$ and if $z$ is any non-singular point of $Z(P)$ in $2B_j \cap 10T$, then

$$\angle(v(T), T_z Z) \leq R^{-1/2+2\delta},$$

where $T_z Z$ is the tangent plane of $Z(P)$ at a point $z$. We also define the collection $T \in \mathbb{T}$ of transversal tubes $T_j^0$ to be the collection of all tubes such that $T \cap W \cap B_j \neq \emptyset$ and there exists a non-singular point $z$ of $Z(P)$ in $2B_j \cap 10T$ so that

$$\angle(v(T), T_z Z) > R^{-1/2+2\delta}.$$  

If $I$ is a subcollection of the caps $\tau$, we define $f_I$ by

$$f_I := \sum_{\tau \in I} f_\tau,$$
and set
\[ f^T_{\tau,j} := \sum_{\tau \in \tau_j} f_T, \quad f^I_{\tau,j} := \sum_{\tau \in \tau_j} f^I_{\tau,j} \text{ and } f^T_{\tau,j} := \sum_{\tau \in \tau_j} f^T_{\tau,j}, \]
and similarly define \( f^T_{\tau,j}, f^I_{\tau,j} \) and \( f^T_{\tau,j} \).

We will consider a bilinear form of \( Ef \) under a certain separation condition. For \( A, B \subset \mathbb{R}^2 \), let \( \text{dist}_{\xi_i}(A, B) := \text{dist}(\text{proj}_{\xi_i}A, \text{proj}_{\xi_i}B) \), where \( \text{proj}_{\xi_i} \) is a projection to \( \xi_i \)-axis. We define the bilinear operator \( \text{Bil}(Ef) \) as
\[
\text{Bil}(Ef) := \sum_{(\tau_1, \tau_2): \text{dist}_{\xi_i}(\tau_1, \tau_2) \geq \frac{1}{4}K^{-1}, \text{dist}_{\xi_2}(\tau_1, \tau_2) \geq \frac{1}{4}K^{-1}} |Ef_{\tau_1}|^{1/2}|Ef_{\tau_2}|^{1/2}.
\]
By using the definition of broad point we can decompose \( B_\alpha[Ef] \) as follows.

**Lemma 4.5.** Let \( \alpha = K^{-\epsilon} \). Then, for any \( x \in B_j \cap W \),
\[
|B_\alpha[Ef](x)| \lesssim \sum_{I} |B_{C\alpha}(Ef^I_{\tau,j}(x))| + K^{100}\text{Bil}(Ef^I_j)(x) + R^{-900}\sum_{\tau} \|f_{\tau}\|_{L^2(S)}, \tag{4.18}
\]
where \( I \) runs over all subcollections consisting of caps \( \tau \).

**Proof.** Suppose that \( x \in B_j \cap W \) is an \( \alpha \)-broad point of \( Ef \). We assume that
\[
|Ef(x)| \geq CR^{-900}\sum_{\tau} \|f_{\tau}\|_{L^2(S)}; \tag{4.19}
\]
otherwise, it trivially gives (4.18).

Let \( I \) be the collection of caps \( \tau \) satisfying
\[
|Ef^I_{\tau,j}(x)| \leq K^{-100}|Ef_{\tau}(x)|. \tag{4.20}
\]
Consider the complement \( I^c \). We will say that caps \( \tau_1 \) and \( \tau_2 \) are strong-separated if \( \text{dist}_{\xi_1}(\tau_1, \tau_2) \geq \frac{1}{2}K^{-1} \) and \( \text{dist}_{\xi_2}(\tau_1, \tau_2) \geq \frac{1}{2}K^{-1} \). If \( I^c \) has two strong-separated caps \( \tau_1, \tau_2 \), then one has
\[
|Ef(x)| \leq K^{100}|Ef^I_{\tau,j}(x)|^{1/2}|Ef^I_{\tau,j}(x)|^{1/2}, \tag{4.21}
\]
since \( |Ef(x)| < K^{100}|Ef^I_{\tau,j}(x)| \) for all \( \tau \in I^c \).

We now suppose that \( I^c \) does not have any strong-separated pair of caps. We claim that there exists a strip \( L \) of width \( \leq 8K^{-1} \) which is parallel to \( e_1 \) or \( e_2 \) and contains all \( \bar{\tau} \) for \( \tau \in I^c \).

Let us prove the claim. By abusing notations, we identify a cap \( \tau \) with the projected cap \( \bar{\tau} \). Fix a cap \( \tau_0 \in I^c \). Let \( A_j = \{ \tau \in I^c: \text{dist}_{\xi_1}(\tau_0, \tau) < \frac{1}{4}K^{-1} \} \) for \( j = 1, 2 \), and let \( A_0 = \{ \tau \in I^c: \text{dist}_{\xi_1}(\tau_0, \tau) < \frac{3}{4}K^{-1} \} \) and \( \text{dist}_{\xi_2}(\tau_0, \tau) < \frac{3}{4}K^{-1} \). Then since every \( \tau \in I^c \) is not strong-separated to \( \tau_0 \), one has \( I^c = A_1 \cup A_2 \). Observe that if \( \tau_1 \in A_1 \setminus A_0 \) and \( \tau_2 \in A_2 \setminus A_0 \) then \( \tau_1 \) and \( \tau_2 \) are strong-separated. Thus, one has that \( A_1 \setminus A_0 = \emptyset \) or \( A_2 \setminus A_0 = \emptyset \) by the supposition. If \( A_1 \setminus A_0 \) is nonempty, we can take a strip of width \( 8K^{-1} \) and of being parallel to \( e_2 \) which contains both \( A_0 \) and \( A_1 \). For a case of \( A_2 \setminus A_0 \neq \emptyset \) we can take a similar strip of being parallel to \( e_1 \). Therefore, we have the claim.
Let \( \hat{I}^c \) be the collection of caps \( \tau \) with \( \hat{r} \cap \hat{L} \neq \emptyset \), and \( \hat{I} = I \setminus \hat{I}^c \). Since \( x \) is an \( \alpha \)-broad point of \( Ef \), we have

\[
|Ef(x)| \leq \left| \sum_{\tau \in I} Ef_\tau(x) \right| + \left| \sum_{\tau \in I^c} Ef_\tau(x) \right| \\
\leq \left| \sum_{\tau \in I} Ef_\tau(x) \right| + 16 \max_{L=L_{|e|1} \text{ or } L_{|e|2}} \left| \sum_{\tau: \tau \subset L} Ef_\tau(x) \right| \\
\leq |Ef_\hat{I}(x)| + 16\alpha|Ef(x)|.
\]

If \( K \) is large enough, then one has \( 0 < 16\alpha < 1/2 \). So, by rearranging this we get

\[
|Ef(x)| \lesssim |Ef_\hat{I}(x)|. \tag{4.22}
\]

Now we decompose \( f_\hat{I} \) into \( f_\hat{I} = \sum_{\tau \in I} \sum_{T: T(\omega(T)) \subset \tau} \hat{f}_\tau \). By (2) in Proposition \ref{prop:2.1} we can ignore \( Ef_T \) with \( T \cap (B_j \cap W) = \emptyset \), and so we have

\[
\left| \sum_{T: T(\omega(T)) \subset \tau \cap B_j \cap W = \emptyset} Ef_T(x) \right| \lesssim R^{-990} \| f_\tau \|_{L^2(S)}.
\]

Each tube \( T \) intersecting \( B_j \cap W \) is contained in either \( T_j^5 \) or \( T_j^4 \). So, for each \( \tau \in \hat{I}' \),

\[
Ef_\tau(x) = Ef_{\hat{I},j}^\sharp(x) + Ef_{\hat{I},j}^\flat(x) + O(R^{-990} \| f_\tau \|_{L^2(S)}).
\tag{4.23}
\]

By summing over \( \tau \in \hat{I} \),

\[
|Ef_\hat{I}(x)| \leq |Ef_{\hat{I},j}^\sharp(x)| + \sum_{\tau \in I} |Ef_{\hat{I},j}^\flat(x)| + O(R^{-990} \sum_{\tau} \| f_\tau \|_{L^2(S)}).
\]

From (4.20) and \( I' \subset I \) it follows that

\[
\sum_{\tau \in I} |Ef_{\hat{I},j}^\flat(x)| \leq K^{-98} |Ef(x)|. \tag{4.24}
\]

Inserting this into the previous inequality, we obtain

\[
|Ef_\hat{I}(x)| \leq |Ef_{\hat{I},j}^\sharp(x)| + K^{-98} |Ef(x)| + O(R^{-990} \sum_{\tau} \| f_\tau \|_{L^2(S)}),
\]

and by (4.19),

\[
|Ef_\hat{I}(x)| \leq |Ef_{\hat{I},j}^\flat(x)| + K^{-98} |Ef(x)| + CR^{-90} |Ef(x)|.
\]

We combine this with (4.22) and rearrange it. Then it follows that

\[
|Ef_\hat{I}(x)| \lesssim |Ef_{\hat{I},j}^\sharp(x)|. \tag{4.25}
\]

To prove \( \| B_\alpha[\hat{f}](x) \| \lesssim \| B_{C\alpha}[E_{\hat{I},j}^\sharp](x) \| \), it remains to show that if \( x \in B_j \cap W \) is an \( \alpha \)-broad point of \( Ef \) then \( x \) is also a \( C\alpha \)-broad point of \( E_{\hat{I},j}^\sharp \). Let \( \tau \in \hat{I}' \) be given. By (4.23) we have

\[
|Ef_{\hat{I},j}^\sharp(x)| \leq |Ef_{\hat{I},j}^\sharp(x)| + \| Ef_{\hat{I},j}^\flat(x) \| + O(R^{-990} \| f_\tau \|_{L^2(S)}).
\]

From (2) in Proposition \ref{prop:2.1} we have that

\[
|Ef_{\hat{I},j}^\flat(x)| \leq |Ef_\tau(x)| + O(R^{-990} \sum_{\tau} \| f_\tau \|_{L^2(S)}) \text{ for } x \in B_j \cap W.
\]

So it follows that

\[
|Ef_{\hat{I},j}^\sharp(x)| \leq |Ef_\tau(x)| + \| Ef_{\hat{I},j}^\flat(x) \| + O(R^{-990} \| f_\tau \|_{L^2(S)}).
\]
Since \( x \) is an \( \alpha \)-broad point of \( Ef \), we have
\[
|Ef_{\tau,j}^\sharp(x)| \leq \alpha|Ef(x)| + |Ef_{\tau,j}^\flat(x)| + O(R^{-990}\|f_\tau\|_{L^2(S)}).
\]
From (4.20), (4.19) and \( \alpha = K^{-\tau} \), we have
\[
|Ef_{\tau,j}^\flat(x)| + O(R^{-990}\|f_\tau\|_{L^2(S)}) \leq (K^{-100} + CR^{-90})|Ef(x)| \leq \alpha|Ef(x)|
\]
for large \( R \). By substituting this in the previous one, we obtain
\[
|Ef_{\tau,j}^\sharp(x)| \leq 2\alpha|Ef(x)|.
\]

(4.26)

Now, let \( \Lambda = L_{\|\xi_1\|} \) or \( L_{\|\xi_2\|} \). By (4.23),
\[
\left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_{\tau,j}^\sharp(x) \right| \leq \left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_{\tau,j}(x) \right| + \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} |Ef_{\tau,j}^\flat(x)| + O(R^{-990} \sum_\tau \|f_\tau\|_{L^2(S)}).
\]

If \( \Lambda \) is parallel to \( \hat{L} \) then one has \( |\sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_\tau(x)| \leq |\sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_\tau(x)| \). If \( \Lambda \) is perpendicular to \( \hat{L} \) then \( |\sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_\tau(x)| \leq |\sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_\tau(x)| + 16 \max_\tau |Ef_\tau(x)| \). Thus, it gives
\[
\left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_{\tau,j}^\sharp(x) \right| \leq \left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_{\tau,j}(x) \right| + 16 \max_\tau |Ef_\tau(x)| + \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} |Ef_{\tau,j}^\flat(x)| + O(R^{-990} \sum_\tau \|f_\tau\|_{L^2(S)}).
\]

(4.27)

Since \( x \) is an \( \alpha \)-broad point of \( Ef \), one has
\[
\left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_\tau(x) \right| + 16 \max_\tau |Ef_\tau(x)| \leq 16\alpha|Ef(x)|.
\]

By (4.24) and (4.19) we have
\[
\sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} |Ef_{\tau,j}^\flat(x)| + O(R^{-990} \sum_\tau \|f_\tau\|_{L^2(S)}) \leq (K^{-98} + CR^{-90})|Ef(x)| \leq \alpha|Ef(x)|
\]
for large \( R \). Inserting these two estimates into (4.27), it follows that
\[
\left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_{\tau,j}^\sharp(x) \right| \leq 20\alpha|Ef(x)|.
\]

To the sum of (4.26) and the above estimate, we apply (4.25). Then,
\[
|Ef_{\tau,j}^\sharp(x)| + \left| \sum_{\tau \in \mathcal{I} : \tau \subset \Lambda} Ef_{\tau,j}^\flat(x) \right| \leq C\alpha|Ef_{\tau,j}^\sharp(x)|.
\]

Thus, \( x \) is a \( C\alpha \)-broad point of \( Ef_{\tau,j}^\flat \) and so we have
\[
|B_a[Ef](x)| \lesssim |B_{Ca}[Ef_{\tau,j}^\flat](x)|.
\]

By combining this and (4.21) we have
\[
|B_a[Ef](x)| \lesssim |B_{Ca}[Ef_{\tau,j}^\flat](x)| + K^{100}|Ef_{\tau,j}^\flat(x)|^{1/2}|Ef_{\tau,j}^\flat(x)|^{1/2} + R^{-990} \sum_\tau \|f_\tau\|_{L^2(S)}.
\]

In this estimate, two strong-separated caps \( \tau_1, \tau_2 \) and \( I \) depend on \( x \in B_j \cap W \). To have independently we replace \( |B_{Ca}[Ef_{\tau,j}^\flat](x)| + K^{100}|Ef_{\tau,j}^\flat(x)|^{1/2}|Ef_{\tau,j}^\flat(x)|^{1/2} \) with \( \sum_I |B_{Ca}[Ef_{\tau,j}^\flat](x)| + K^{100} \text{Bil}(Ef_{\tau,j}^\flat)(x) \). Then we obtain (4.18).
From Lemma 4.5 it follows that

\[
\int_{B(R) \cap W} |B_\alpha [Ef]|^{p_0} \leq C_\epsilon \left( \sum_{j,I} \int_{B_j \cap W} |B_{C\alpha} [Ef_{I,j}]|^{p_0} + K^{400} \sum_j \int_{B_j \cap W} \text{Bil}(Ef_j)^{p_0} + R^{-850} \sum_\tau \|f_\tau\|^{p_0}_{L^2(S)} \right).
\]

Now we will consider the transversal part \( \sum_{j,I} \int_{B_j \cap W} |B_{C\alpha} [Ef_{I,j}]|^{p_0} \) and the tangential part \( \sum_j \int_{B_j \cap W} \text{Bil}(Ef_j)^{p_0} \), respectively. (The last error term \( R^{-850} \sum_\tau \|f_\tau\|^{p_0}_{L^2(S)} \) is trivially bounded by \( R^\epsilon \left( \sum_\tau \int_S |f_\tau|^2 \right)^{3/2+\epsilon} \) for a sufficiently large \( R \); for instance, we can use an estimate \( \|f\|_{L^2(S)} \lesssim 1 \) which follows from (3.3). For the transversal part we will utilize the induction on scale \( R \), and for the tangential part we will directly estimate it by using the bilinear method in \([11,19]\).

4.2.1. Estimate for the transversal part. We claim

\[
\sum_{j,I} \int_{B_j} |B_{C\alpha} [Ef_{I,j}]|^{p_0} \leq C_\epsilon R^\delta \int_S |f_\tau|^2 \left( \sum_\tau \int_S |f_\tau|^2 \right)^{3/2+\epsilon}. \tag{4.28}
\]

To prove this we use the inductive argument on \( R \). From (6) in Proposition 2.1 we can see that \( \|f_{I,j}\|_{L^2(S)} \lesssim \|f_{(\omega,R^{-1/2}\cap S)}\|_{L^2(S)} \). Using the induction hypothesis we have

\[
\int_{B_j} |B_{C\alpha} [Ef_{I,j}]|^{p_0} \leq C_\epsilon R^{(1-\delta)(\epsilon+\delta_2)} \left( \sum_\tau \int_S |f_{\tau,j}|^2 \right)^{3/2+\epsilon}.
\]

By summing these over \( j \),

\[
\sum_j \int_{B_j} |B_{C\alpha} [Ef_{I,j}]|^{p_0} \leq C_\epsilon R^{(1-\delta)(\epsilon+\delta_2)} \left( \sum_\tau \sum_j \int_S |f_{\tau,j}|^2 \right)^{3/2+\epsilon}. \tag{4.29}
\]

Now we estimate \( \sum_j \int_S |f_{\tau,j}|^2 \). For this we use the following geometric lemma about the transverse tubes.

**Lemma 4.6** (Lemma 3.5 in \([7]\)). There is a nonnegative constant \( C \) such that each tube \( T \in \mathcal{T} \) belongs to at most \( M^\delta = R^{\delta_1} \) different sets \( \mathcal{T}_j \).

Using this we can obtain the following lemma.

**Lemma 4.7.** For each \( \tau \),

\[
\sum_j \int |f_{\tau,j}|^2 \lesssim M^\delta \int |f_\tau|^2 + R^{-900} \|f_\tau\|^2_{L^2(S)}. \tag{4.30}
\]

**Proof.** From (1) in Proposition 2.1

\[
\int |f_{\tau,j}|^2 \lesssim \left\| \sum_{\Omega : \Omega \cap \tau \neq \emptyset} \sum_{T \in \mathcal{T}_j(\Omega)} f_T \right\|^2 \lesssim \sum_{\Omega : \Omega \cap \tau \neq \emptyset} \left( \sum_{T \in \mathcal{T}_j(\Omega)} |f_T|^2 \right).
\]
From (4) in Proposition 2.1, we see
\[ \int |f^{I,j}_{\tau}|^2 \lesssim \sum_{\Omega \in \mathcal{T} \neq \emptyset} \sum_{T \in \mathcal{T}_j(\Omega)} \int |f_T|^2 + R^{-900} \|f_T\|^2_{L^2(S)} \]
and by summing over \( j \),
\[ \sum_j \int |f^{I,j}_{\tau}|^2 \lesssim \sum_{\Omega \in \mathcal{T} \neq \emptyset} \sum_j \sum_{T \in \mathcal{T}_j(\Omega)} \int |f_T|^2 + R^{-900} \|f_T\|^2_{L^2(S)} \]
\[ \lesssim \sum_{\Omega \in \mathcal{T} \neq \emptyset} \sum_{T \in \mathcal{T}(\Omega) : \omega(T) \in \tau} \int |f_T|^2 + R^{-900} \|f_T\|^2_{L^2(S)}. \]
By Lemma 4.6,
\[ \sum_j \int |f^{I,j}_{\tau}|^2 \lesssim M^C \sum_{\Omega \in \mathcal{T} \neq \emptyset} \sum_{T \in \mathcal{T}(\Omega)} \int |f_T|^2 + R^{-900} \|f_T\|^2_{L^2(S)} \]
\[ \lesssim M^C \sum_{T \in \mathcal{T}(\Omega) : \omega(T) \in \tau} \int |f_T|^2 + R^{-900} \|f_T\|^2_{L^2(S)}. \]
By (4) of Proposition 2.1, we obtain (4.30).

We plug (4.30) into (4.29). Then we have
\[ \sum_j \int_{B_j} |B_{C\alpha}[E f^{I,j}_T]|^{p_0} \leq C_\epsilon (R^{1-\delta} (\epsilon+\delta_2) M^C + R^{-1000}) \left( \sum_{\tau} \int_S |f_\tau|^2 \right)^{3/2+\epsilon}, \]
and by summing over \( I \),
\[ \sum_{j,I} \int_{B_j} |B_{C\alpha}[E f^{I,j}_T]|^{p_0} \leq C_\epsilon (R^{\epsilon+\delta_2} M^C R^{-\delta (\epsilon+\delta_2)} + R^{-1000}) \left( \sum_{\tau} \int_S |f_\tau|^2 \right)^{3/2+\epsilon}, \]
because the number of subcollections \( I \) is at most \( 2^{K^2} \) which can be absorbed in \( C_\epsilon \). From (4.61) and (4.12), we have \( M^C R^{-\delta (\epsilon+\delta_2)} = R^{C \epsilon^3-\epsilon^2} \leq R^{-\epsilon}/2 \). For a sufficiently large \( R \) we obtain (4.28).

4.2.2. Estimate for the tangential part. Until now we reduced the problem by using the inductive argument. In this subsection we will directly estimate the remaining part. The key ingredient is the following geometric estimate.

Lemma 4.8 (Lemma 3.6 in [7]). For each \( j \), the number of different \( \Omega \) with \( T_j^\phi \cap T(\Omega) \neq \emptyset \) is at most \( R^{1/2+O(\delta)} \).

This lemma implies that \( T_j^\phi \) is made up of tubes in only \( R^{1/2+O(\delta)} \) different directions. To prove (4.17) we will show that
\[ K^{400} \sum_{j} \int_{B_j \cap W} \text{Bil}(E f^{I,j}_T)^{p_0} \leq C_\epsilon R^\epsilon R^\delta_2 \left( \sum_{\tau} \int_S |f_\tau|^2 \right)^{3/2}. \]
Since the number of cubes \( B_j \) is \( \lesssim R^{C \delta} \), there is a cube \( B_j \) such that
\[ K^{400} \sum_{j} \int_{B_j \cap W} \text{Bil}(E f^{I,j}_T)^{p_0} \leq C_\epsilon R^{C \delta} \int_{B_j \cap W} \text{Bil}(E f^{I,j}_T)^{p_0}, \]
where $K^{400}$ can be absorbed in $C_e$. Since $R^{C_5} \leq R^4$ by \((4.1)\), it suffices to show that for each $j$,

$$
\int_{B_j \cap W} \text{Bil}(Ef_j)^{p_0} \leq C e^{R^{C_5}} \left( \sum_\tau |f_\tau|^2 \right)^{3/2}.
$$

(4.31)

Decompose $B_j \cap W$ into finer cubes $Q$ of side-length $R^{1/2}$. We define $T_{1,Q}^1$ and $T_{2,Q}^2$ by

$$
T_{1,Q}^1 = \{ t \in \mathbb{T}_2^1 : t \cap Q \neq \emptyset, \omega(t) \in \tau_1 \},
$$

$$
T_{2,Q}^2 = \{ t \in \mathbb{T}_2^1 : t \cap Q \neq \emptyset, \omega(t) \in \tau_2 \}.
$$

We first show the orthogonality among the bilinear wave packets $Ef_{T_1}Ef_{T_2}$ for $T_1 \in T_{1,Q}^1$ and $T_2 \in T_{2,Q}^2$. We can see that the tubes in $T_{1,Q}^1 \cup T_{2,Q}^2$ are contained in a $O(R^{1/2+\delta})$-neighborhood of some tangent plane. So, the orthogonal property observed in the proof of the 2-dimensional restriction theorem can be obtained.

**Lemma 4.9.** Let us set $F_T = Ef_T$. Suppose that $\tau_1$ and $\tau_2$ satisfy the condition that for any $(\xi_1, \zeta_1) \in \tau_1$ and any $(\xi_2, \zeta_2) \in \tau_2$,

$$
|\xi_1 - \xi_2| \gtrsim K^{-1} \quad \text{and} \quad |\zeta_1 - \zeta_2| \gtrsim K^{-1}.
$$

Then for any $Q$ intersecting $B_j \cap W$,

$$
\int \left| \sum_{T_1 \in T_{1,Q}^1} \sum_{T_2 \in T_{2,Q}^2} F_{T_1}F_{T_2} \right|^2 \lesssim R^{C_5} \sum_{T_1 \in T_{1,Q}^1} \sum_{T_2 \in T_{2,Q}^2} \int |F_{T_1}F_{T_2}|^2.
$$

(4.33)

**Proof.** One can write as

$$
\int \left| \sum_{T_1 \in T_{1,Q}^1} \sum_{T_2 \in T_{2,Q}^2} F_{T_1}F_{T_2} \right|^2 = \sum_{T_1,T_1' \in T_{1,Q}^1} \sum_{T_2,T_2' \in T_{2,Q}^2} \langle F_{T_1}F_{T_2}, F_{T_1'}F_{T_2'} \rangle.
$$

By Parseval’s identity,

$$
\langle F_{T_1}F_{T_2}, F_{T_1'}F_{T_2'} \rangle = \langle \hat{F}_{T_1} \ast \hat{F}_{T_2}, \hat{F}_{T_1'} \ast \hat{F}_{T_2'} \rangle.
$$

We now consider the supports of $\hat{F}_{T_1} \ast \hat{F}_{T_2}$. Recall that $Ef$ can be written as $(f_T d\sigma_S)^\vee$. By (1) of Proposition 2.1 we have $Ef_T = (\chi_{\mathcal{Q}^1} f_T d\sigma_S)^\vee$ provided $T \in \mathbb{T}(\Omega)$. So, $\hat{F}_{T_1}$ is supported in the $O(R^{-1/2})$-neighborhood of $\omega(T)$. Let $\omega(J_j) = (\xi_j, \zeta_j, \xi_j, \zeta_j)$ for $j = 1, 2$. If the above equation does not vanish, then the following relations are satisfied:

$$
\xi_1 + \xi_2 = \xi_1' + \xi_2' + O(R^{-1/2}),
$$

$$
\zeta_1 + \zeta_2 = \zeta_1' + \zeta_2' + O(R^{-1/2}),
$$

$$
\xi_1\zeta_1 + \xi_2\zeta_2 = \xi_1'\zeta_1' + \xi_2'\zeta_2' + O(R^{-1/2}).
$$

(4.34)

From these relations it follows that if $(\xi_1, \zeta_1), (\xi_2, \zeta_2)$ and $(\xi_1', \zeta_1')$ are given, then $(\xi_2', \zeta_2')$ is determined as follows:

$$
(\xi_2', \zeta_2') = (\xi_1 + \xi_2 - \xi_1', \zeta_1 + \zeta_2 - \zeta_1') + O(R^{-1/2}).
$$

(4.35)

So, if we set $\omega'_2 = \omega(T_1) + \omega(T_2) - \omega(T_1')$, then

$$
\sum_{T_1,T_1' \in T_{1,Q}^1} \sum_{T_2,T_2' \in T_{2,Q}^2} \langle F_{T_1}F_{T_2}, F_{T_1'}F_{T_2'} \rangle = \sum_{T_1,T_1' \in T_{1,Q}^1} \sum_{T_2 \in T_{2,Q}^2} \sum_{T_2' \in T_{2,Q}^2} \langle F_{T_1}F_{T_2}, F_{T_1'}F_{T_2'} \rangle.
$$

Note that the number of choice of $T_2'$ is $O(1)$, because $T_2'$ passes through $Q$. 

Now we restrict \((\xi_1, \zeta_1)\) when \((\xi'_1, \zeta'_1)\) and \((\xi_2, \zeta_2)\) are given. For this we insert (4.35) into (4.34). By rearranging this, we obtain

\[
(\xi_1 - \xi'_1)(\zeta_2 - \zeta'_1) + (\zeta_1 - \zeta'_1)(\xi_2 - \xi'_1) = O(R^{-1/2}). \tag{4.36}
\]

Let \(\ell(T_1', T_2)\) be the line passing through \((\xi'_1, \zeta'_1)\) and of direction normal to the vector \((\zeta_2 - \zeta'_1, \xi_2 - \xi'_1)\). Then from (4.36), it follows that if \((\xi'_1, \zeta'_1)\) and \((\xi_2, \zeta_2)\) are given, then \((\xi_1, \zeta_1)\) is contained in an \(O(R^{-1/2})\)-neighborhood of the line \(\ell(T_1', T_2)\). Thus, it implies

\[
\sum_{T_1, T_1' \in \mathbb{T}_{1,Q}^\circ} \sum_{T_2, T_2' \in \mathbb{T}_{2,Q}^\circ} \langle F_{T_1} F_{T_2}, F_{T_1'} F_{T_2'} \rangle = \sum_{T_1' \in \mathbb{T}_{1,Q}^\circ} \sum_{T_2 \in \mathbb{T}_{2,Q}^\circ} \sum_{T_1 \in \mathbb{T}_{1,Q}^\circ: \text{dist}(\omega(T_1), \ell(T_1', T_2)) \leq R^{-1/2}} \sum_{T_2' \in \mathbb{T}_{2,Q}^\circ: |\omega(T_2') - \omega(T_2)| \leq R^{-1/2}} \langle F_{T_1} F_{T_2}, F_{T_1'} F_{T_2'} \rangle.
\]

We see that all tube segments \(T \cap B(R)\) for \(T \in \mathbb{T}_{1,Q}^\circ \cup \mathbb{T}_{2,Q}^\circ\) are contained in the \(R^{1/2 + C\delta}\)-neighborhood of some plane. So, all directions \(v(T)\) for \(T \in \mathbb{T}_{1,Q}^\circ \cup \mathbb{T}_{2,Q}^\circ\) are also contained in the \(R^{-1/2 + C\delta}\)-neighborhood of a plane \(\pi\) passing through the origin. If \(\omega(T) = (\xi, \zeta, \zeta')\) then we can see

\[
v(T) = \frac{1}{\sqrt{\xi^2 + \zeta^2 + 1}}(\xi, \zeta, -1).
\]

So, we can correspond \(v(T), T \in \mathbb{T}_{1,Q}^\circ \cup \mathbb{T}_{2,Q}^\circ\), to a point \((\zeta, \xi, -1)\).

\[
\pi
\]

\[
-1
\]

\[
(\zeta, \xi, -1)
\]

By considering the mapping

\[
\frac{1}{\sqrt{\xi^2 + \zeta^2 + 1}}(\zeta, \xi, -1) \to (\xi, \zeta),
\]

it follows that \((\xi_1, \zeta_1)\) is in the \(O(R^{-1/2 + \delta})\)-neighborhood of the line \(l\) passing through \((\xi'_1, \zeta'_1)\) and \((\xi_2, \zeta_2)\). On the other hand, \((\xi_1, \zeta_1)\) obeys (4.36). Thus, \((\xi_1, \zeta_1)\) is contained in the intersection between the \(R^{-1/2 + C\delta}\)-neighborhood of \(l\) and the \(O(R^{-1/2})\)-neighborhood of \(\ell(T_1', T_2)\).

Now we consider the directions of \(l\) and \(\ell(T_1', T_2)\). The direction of \(l\) is normal to the vector \((\zeta_2 - \zeta'_1, -\xi_2 + \xi'_1)\), and that of \(\ell(T_1', T_2)\) is normal to \((\zeta_2 - \zeta'_1, \xi_2 - \xi'_1)\). The condition (4.32) guarantees that the two vectors \((\zeta_2 - \zeta'_1, -\xi_2 + \xi'_1)\) and \((\zeta_2 - \zeta'_1, \xi_2 - \xi'_1)\) are transverse. This means that the directions of \(l\) and \(\ell(T_1', T_2)\) are also transverse, and thus \((\xi_1, \zeta_1)\) is contained in
a disc of radius $R^{-1/2+C\delta}$. Since all tubes are passing through $Q$, we conclude that the number of choice of $T_1$ is $O(R^{-C\delta})$.

Accordingly, it follows that

$$\sum_{T_1,T_1' \in \mathbb{T}_{1,Q}^*} \sum_{T_2,T_2' \in \mathbb{T}_{2,Q}^*} \langle F_{T_1} F_{T_2}, F_{T_1'} F_{T_2'} \rangle \lesssim R^{C\delta} \sum_{T_1 \in \mathbb{T}_{1,Q}^*} \sum_{T_2 \in \mathbb{T}_{2,Q}^*} \int |F_{T_1} F_{T_2}|^2,$$

which is (4.33). □

Due to the orthogonality we can obtain the following estimate.

**Lemma 4.10.** Let $\tau_1$ and $\tau_2$ be as in Lemma 4.9. Then for any $Q$ intersecting $B_j \cap W$,

$$\int_Q |Ef_{\tau_1,j}^\rho|^2 |Ef_{\tau_2,j}^\rho|^2 \lesssim R^{C\delta} R^{-1/2} \left( \sum_{T_1 \in \mathbb{T}_{1,Q}^*} \|f_{T_1}\|_{L^2(S)}^2 \right) \left( \sum_{T_2 \in \mathbb{T}_{2,Q}^*} \|f_{T_2}\|_{L^2(S)}^2 \right) + O(R^{-990} \|f\|_{L^2(S)}^2). \quad (4.37)$$

**Proof.** By (3) of Proposition 2.1, we have that for $k=1,2$,

$$Ef_{\tau_k,j} = \sum_{T \in \mathbb{T}_{k,Q}^*} Ef_T + O(R^{-990} \|f\|_{L^2(S)}).$$

Substituting this in the left-hand side of (4.37) and applying Lemma 4.9 we get

$$\int_Q |Ef_{\tau_1,j}^\rho|^2 |Ef_{\tau_2,j}^\rho|^2 \lesssim R^{C\delta} \sum_{T_1 \in \mathbb{T}_{1,Q}^*} \sum_{T_2 \in \mathbb{T}_{2,Q}^*} \int |Ef_{T_1} Ef_{T_2}|^2 + O(R^{-1500} \|f\|_{L^2(S)}^4), \quad (4.38)$$

where we use a trivial estimate $\|Ef\|_{L^2(B(R))} \lesssim R^C \|f\|_{L^2(S)}$ (or (4.32) below). Now it suffices to show that

$$\int |Ef_{T_1} Ef_{T_2}|^2 \lesssim R^{-1/2} \|f_{T_1}\|_{L^2(S)}^2 \|f_{T_2}\|_{L^2(S)}^2. \quad (4.39)$$

By the Plancherel theorem and (1) of Proposition 2.1

$$\int |Ef_{T_1} Ef_{T_2}|^2 = \int |f_{T_1} d\sigma_{\Omega_1} * f_{T_2} d\sigma_{\Omega_2}|^2.$$ 

From the condition (4.32), we see that the unit normal vectors $n(\Omega_1)$ and $n(\Omega_2)$ are transverse, and thus the translations of $\Omega_1$ meet $\Omega_2$ transversally. From this observation it follows that $\|d\sigma_{\Omega_1} * d\sigma_{\Omega_2}\|_\infty \lesssim R^{-1/2}$. Using this we have

$$\|f_{T_1} d\sigma_{\Omega_1} * f_{T_2} d\sigma_{\Omega_2}\|_\infty \leq CR^{-1/2} \|f_{T_1}\|_{L^\infty(S)} \|f_{T_2}\|_{L^\infty(S)}.$$

On the other hand, by Young’s inequality it gives

$$\|f_{T_1} d\sigma_{\Omega_1} * f_{T_2} d\sigma_{\Omega_2}\|_1 \leq \|f_{T_1}\|_{L^1(S)} \|f_{T_2}\|_{L^1(S)}.$$

By interpolating these two estimates,

$$\|f_{T_1} d\sigma_{\Omega_1} * f_{T_2} d\sigma_{\Omega_2}\|_2 \lesssim R^{-1/4} \|f_{T_1}\|_{L^2(S)} \|f_{T_2}\|_{L^2(S)},$$

and so

$$\|Ef_{T_1} Ef_{T_2}\|_2 \lesssim R^{-1/4} \|f_{T_1}\|_{L^2(S)} \|f_{T_2}\|_{L^2(S)},$$

which is (4.39). □
Now we sum (4.37) over Q. For this we use the way of dealing with two dimensional Kakeya set (see [3]). By simple calculation we know that if $T_1 \in \mathbb{T}_{1,Q}$ and $T_2 \in \mathbb{T}_{2,Q}$ then
\[
\int \chi_{T_1} \chi_{T_2} \sim KR^{3/2 + \delta},
\] (4.40)
because we have $|v(T_1) - v(T_2)| \sim K^{-1}$ by (4.32).

Inserting $K^{-1}R^{-3/2} \int \chi_{T_1} \chi_{T_2}$ into the right-hand side of (4.37), we have
\[
\int_Q |E f_{\tau_1,j}^a|^2 |E f_{\tau_2,j}^a|^2 \lesssim K^{-1} R^{C_6} R^{-2} \sum_{T_1 \in \mathbb{T}_{1,j}} \sum_{T_2 \in \mathbb{T}_{2,j}} \|f_{T_1}\|_{L^2(S)}^2 \|f_{T_2}\|_{L^2(S)}^2 \int KQ \chi_{T_1} \chi_{T_2} + O(R^{-1500}\|f\|_{L^2(S)}^4).
\]
Using $\int \chi_{T_1} \chi_{T_2} \lesssim R^{C_6} \int_{KQ} \chi_{T_1} \chi_{T_2}$ we rewrite this as
\[
\int_Q |E f_{\tau_1,j}^a|^2 |E f_{\tau_2,j}^a|^2 \lesssim K^{-1} R^{C_6} R^{-2} \sum_{T_1 \in \mathbb{T}_{1,j}} \sum_{T_2 \in \mathbb{T}_{2,j}} \|f_{T_1}\|_{L^2(S)}^2 \|f_{T_2}\|_{L^2(S)}^2 \int_{KQ} \chi_{T_1} \chi_{T_2} + O(R^{-1500}\|f\|_{L^2(S)}^4)
\]
where $\mathbb{T}_{1,j} = \{T \in \mathbb{T}_j : \omega(T) \in \tau_1\}$ and $\mathbb{T}_{2,j} = \{T \in \mathbb{T}_j : \omega(T) \in \tau_2\}$.

We sum the above estimate over Q with $Q \cap B_j \cap W \neq \emptyset$, and insert (4.40). Then
\[
\int_{B_j \cap W} |E f_{\tau_1,j}^a|^2 |E f_{\tau_2,j}^a|^2 \lesssim K^{C} R^{C_6} R^{-2} \sum_{T_1 \in \mathbb{T}_{1,j}} \sum_{T_2 \in \mathbb{T}_{2,j}} \|f_{T_1}\|_{L^2(S)}^2 \|f_{T_2}\|_{L^2(S)}^2 \int \chi_{T_1} \chi_{T_2} + O(R^{-1000}\|f\|_{L^2(S)}^4)
\]
\[
\lesssim K^{C} R^{C_6} R^{-1/2} \left( \sum_{T_1 \in \mathbb{T}_{1,j}} \|f_{T_1}\|_{L^2(S)}^2 \right) \left( \sum_{T_2 \in \mathbb{T}_{2,j}} \|f_{T_2}\|_{L^2(S)}^2 \right) + O(R^{-1000}\|f\|_{L^2(S)}^4)
\]
\[
\lesssim K^{C} R^{C_6} R^{-1/2} \|f_{\tau_1,j}\|_{L^2(S)}^2 \|f_{\tau_2,j}\|_{L^2(S)}^2 + O(R^{-900}\|f\|_{L^2(S)}^4).
\]
Here, the (5) of Proposition [24] is used in the last line. So, it implies
\[
\int_{B_j \cap W} \text{Bil}(E f_{\tau}^a) \lesssim K^{C} R^{C_6} R^{-1/2} \sum_{T_1} \|f_{\tau_1,j}\|_{L^2(S)}^2 \sum_{T_2} \|f_{\tau_2,j}\|_{L^2(S)}^2 + K^2 R^{-900}\|f\|_{L^2(S)}^4
\]
\[
\lesssim K^{C} R^{C_6} R^{-1/2} \left( \sum_{\tau} \|f_{\tau,j}\|_{L^2(S)}^2 \right)^2 + K^2 R^{-900}\|f\|_{L^2(S)}^4.
\]
Therefore, we obtain
\[
\|\text{Bil}(E f_{\tau}^a)\|_{L^4(B_j \cap W)} \lesssim C_\circ \left( R^{C_6} R^{-1/8} \left( \sum_{\tau} \|f_{\tau,j}\|_{L^2(S)}^2 \right)^{1/2} + R^{-200}\|f\|_{L^2(S)} \right).
\] (4.41)

On the other hand, by Plancherel’s theorem it follows that
\[
\int_{R/2}^{R/2} \int_{D(R)} |E f(x', x_3)|^2 dx' dx_3 \lesssim R \|f\|_{L^2(S)}.
\]
This is written as
\[ \|Ef\|_{L^2(B(R))} \lesssim R^{1/2}\|f\|_{L^2(S)}. \]  
\[ (4.42) \]

By the Cauchy-Schwarz inequality and the above estimate,
\[ \|\text{Bil}(Ef^j)\|^2_{L^2(B_j \cap \Omega)} \lesssim K^C \sum_{\tau_1} \sum_{\tau_2} \int_{B(R)} |Ef^j_{\tau_1,j}| |Ef^j_{\tau_2,j}| \lesssim K^{C'} R \sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)}. \]

Thus,
\[ \|\text{Bil}(Ef^j)\|_{L^2(B_j \cap \Omega)} \lesssim C_\epsilon R^{1/2}(\sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)})^{1/2}. \]  
\[ (4.43) \]

We now interpolate \[ (4.41) \] with \[ (4.43) \]. Then it follows that for all \( 3 \leq p \leq 4, \)
\[ \|\text{Bil}(Ef^j)\|_{L^p(B_j \cap \Omega)} \lesssim C_\epsilon R^{3\delta} R^{\frac{5}{4p} - \frac{3}{4}} (\sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)})^{1/2} + O(R^{-100}\|f\|_{L^2(S)}). \]  
\[ (4.44) \]

Let us consider \( \sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)}. \) We utilize a geometric estimate for the tangential tubes. By (4) of Proposition 2.1
\[ \sum_{\tau} \int_{S} |f^j_{\tau,j}|^2 \lesssim \sum_{T \in \mathcal{T}_j} \int_{T} |f^j| \lesssim \sum_{\tau} \|f^j\|_{L^2(S)}^2. \]

By Lemma 4.8 there is an \( \Omega \) such that
\[ \sum_{\tau} \int_{S} |f^j_{\tau,j}|^2 \lesssim R^{1/2 + C\delta} \sum_{T \in \mathcal{T}_j(\Omega)} \int_{T} |f^j| \lesssim \sum_{\tau} \|f^j\|_{L^2(S)}^2 \]
\[ \lesssim R^{1/2 + C\delta} \sum_{T \in \mathcal{T}(\Omega)} \int_{T} |f_{\tau,j}| \lesssim \sum_{\tau} \|f^j_{\tau,j}\|_{L^2(S)}^2. \]

Thus, by (5) of Proposition 2.1 we have
\[ \sum_{\tau} \int_{S} |f^j_{\tau,j}|^2 \lesssim R^{1/2 + C\delta} \int_{\Omega} |f| \lesssim \sum_{\tau} \|f^j_{\tau,j}\|_{L^2(S)}^2. \]

From \( f^j_{\Omega} |f|^2 \lesssim 1, \) we obtain
\[ \sum_{\tau} \int_{S} |f^j_{\tau,j}|^2 \lesssim R^{1/2 + C\delta} R^{-1} = R^{-1/2 + C\delta}. \]  
\[ (4.45) \]

By raising both sides of \[ (4.44) \] to the \( p \)th power,
\[ \|\text{Bil}(Ef^j)|_{L^p(B_j \cap \Omega)} \lesssim C_\epsilon R^{C\delta} R^{\frac{5}{4p} - \frac{3}{4}} (\sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)})^{p/2} + R^{-300}\|f\|_{L^2(S)^p}. \]

Using \[ (4.45) \] we have the estimate
\[ \left( \sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)} \right)^{p/2} = \left( \sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)} \right)^{(p-3)/2} \left( \sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)} \right)^{3/2} \]
\[ \lesssim R^{C\delta} R^{\frac{3}{4p} - \frac{3}{4}} \left( \sum_{\tau} |f^j_{\tau,j}|^2_{L^2(S)} \right)^{3/2}. \]
Therefore, it gives
\[
\|\text{Bil}(Ef_j^2)\|_{L^p(B_j \cap W)} \lesssim C_\varepsilon R^{C_5} R^{\frac{3p}{2}-p} \left( \sum_{\tau} \|f_{\tau}\|_{L^2(S)}^2 \right)^{3/2} + R^{-300} \|f\|_{L^2(S)}^p.
\]

which we used the estimate \( \|f\|_{L^2(S)} \lesssim 1 \). By taking \( p = p_0 \) we finally obtain (4.31).

5. Appendix

In this section, we prove the wave packet decomposition.

5.1. Proof of Proposition 2.1. We first define bump functions. Let \( \phi : \mathbb{R}^2 \rightarrow \mathbb{R} \) be a nonnegative Schwartz function such that \( \hat{\phi} \) is supported in a disc \( D(0, \frac{3}{2}) \) and \( \sum_{j \in \mathbb{Z}^2} \phi(x - j) = 1 \). Let \( \psi : \mathbb{R}^2 \rightarrow \mathbb{R} \) is a nonnegative smooth function that is equal to 1 on a disc \( D(0, 2) \) and supported in \( D(0, 3) \). For a disc \( D \) we define \( a_D \) to be an affine map from the unit disc to \( D \), and let \( \phi_D = \phi \circ a_D^{-1} \) and \( \psi_D = \psi \circ a_D^{-1} \).

By translating we may assume that \( B(R) \) is centered at the origin. We define \( f_\Omega := f \chi_\Omega \) and \( \tilde{f}_\Omega := \tilde{f} \chi_\Omega \) where \( \tilde{\Omega} = \{ \xi' \in \mathbb{R}^2 : (\xi', \xi_3) \in \Omega \} \). Then we have \( \tilde{f}_\Omega(\xi') = J(\xi') f_\Omega(\xi) \) where \( J(\xi_1, \xi_2) := \frac{d(\xi_1, \xi_2)}{d\xi_1 d\xi_2} \). Since \( |J| \sim 1 \) on \( B(R) \), we may identify \( \tilde{f}_\Omega \) with \( f_\Omega \). For each \( T \in \mathcal{T}(\Omega) \) we define \( \tilde{f}_T \) by

\[
\tilde{f}_T := \psi_T(\phi_{D_T} * \tilde{f}_\Omega),
\]

where \( D_T = \{ x' \in \mathbb{R}^2 : (x', 0) \in T \} \). From this definition it follows that \( \tilde{f}_T \) is supported in \( 3 \Omega \). We define \( f_T \) on \( S \) by \( f_T(\xi') = J(\xi') f_T(\xi', \xi_1, \xi_2) \). Then \( f_T \) has Property (1).

Consider Property (2). For \( T \in \mathcal{T}(\Omega) \), we write

\[
Ef_T(x', x_3) = \int_{Q(1)} e^{2\pi i (x' \cdot \xi' + x_3 \xi_3)} \tilde{f}_T(\xi') d\xi'
= \int e^{2\pi i x' \cdot \xi'} \Psi_{x_3}(\xi') (\phi_{D_T} * \tilde{f}_\Omega)(\xi') d\xi'
= \Psi_{x_3} (\phi_{D_T} \tilde{f}_\Omega)(x')
\]

where \( \Psi_{x_3}(\xi') := e^{2\pi i (\xi_1 x_3)} \psi_{x_3}(\xi') \).

If \( (\omega_1, \omega_2) \) be the center of \( \tilde{\Omega} \) then we can see that the normal vector \( n(\Omega) \) is parallel to \( (\omega_2, \omega_1, -1) \), so the tubes \( T \in \mathcal{T}(\Omega) \) are written as

\[
T = \{ (x', x_3) : |x' - x'_T + x_3(\omega_2, \omega_1)| \lesssim R^{1/2+\delta} \}
\]

where \( x'_T \) is the center of \( D_T \).

Using integrating by parts, we can obtain that for \( (x', x_3) \in \mathbb{R}^2 \times [-10R, 10R] \),

\[
|\Psi_{x_3}(x')| \leq C_M |\tilde{\Omega}| (1 + R^{-1/2} |x'_T + x_3(\omega_2, \omega_1)|)^{-M}, \quad \forall M > 0.
\]

By inserting this into (5.2) we have that for \( (x', x_3) \in \mathbb{R}^2 \times [-10R, 10R] \),

\[
|Ef_T(x', x_3)| \leq C_M |\tilde{\Omega}| (1 + R^{-1/2} |x'_T + x_3(\omega_2, \omega_1)|)^{-M} \int |\phi_{D_T} \tilde{f}_\Omega(y')| dy'
\leq C_M |\tilde{\Omega}| |D_T|^{1/2} (1 + R^{-1/2} |x'_T + x_3(\omega_2, \omega_1)|)^{-M} \|\tilde{f}_\Omega\|_2
\leq C_M R^{-1/2+\delta} (1 + R^{-1/2} |x'_T + x_3(\omega_2, \omega_1)|)^{-M} \|f\|_{L^2(\tilde{\Omega})}
\]

(5.4)
for any $M > 0$. Using this and \((5.3)\) we have that for $x \in B(R) \setminus T$,
\[
|Ef(x)| \leq C_M R^{-1/2+\delta} R^{-\delta M} \|f\|_{L^2(\Omega)} \lesssim R^{-10000} \|f\|_{L^2(\Omega)}
\]
provided $M > 0$ is sufficiently large. Thus we have Property (2).

Let $\bar{T}(\Omega)$ be the collection of cylindrical tubes of radius $R^{1/2+\delta}$ which are parallel to $n(\Omega)$ and cover $\mathbb{R}^3$ with finite overlap. By the definition \((5.1)\) it follows that
\[
\tilde{f} = \sum_{\Omega} \sum_{T \in \bar{T}(\Omega)} \tilde{f}_T.
\]
Since $Ef$ is a linear operator, we have
\[
Ef(x) = \sum_{\Omega} \sum_{T \in \bar{T}(\Omega)} Ef_T(x).
\]
If $x \in B(R)$ we can restrict $T \in \bar{T}(\Omega)$ to $T(\Omega)$ in the above summation, because by \((5.4)\) the contribution of $Ef_T$ for $T \in \bar{T} \setminus T$ is negligible. Thus, we have Property (3).

Consider Property (4). Let $T_1$ and $T_2$ be two disjoint tubes in $\bar{T}(\Omega)$. We see that $| \int f_{T_1} f_{T_2} | \leq | \int \tilde{f}_{T_1} \tilde{f}_{T_2} |$. We will use that $\tilde{f}_{T_1}$ is essentially Fourier supported in $2D_T$. By Parseval’s identity,
\[
\int \tilde{f}_{T_1} \tilde{f}_{T_2} = \int (\psi_{T_1}^\vee \ast (\phi_{D_T} \tilde{f}_{T_1}))(\ast \phi_{D_T} \tilde{f}_{T_2}) \leq \| \tilde{f}_{T_1} \|_2 \| \tilde{f}_{T_2} \|_2 
\]
\[
\leq \| \tilde{f}_{T_1} \|_2 \| \tilde{f}_{T_2} \|_2.
\]
Since $\phi$ and $\psi$ are smooth bump functions, we have $| \phi_D(x') | \lesssim (1 + \text{dist}(x', D))^{-4000}$ and $| \psi_D(x') | \lesssim (1 + R^{-1/2}|x'|)^{-4000}$. Since $T_1$ and $T_2$ are disjoint, the distance between $D_{T_1}$ and $D_{T_2}$ is at least $1/4 R^{1/2+\delta}$, from which we have $| \int (\psi_{T_1}^\vee \ast \phi_{D_{T_1}})(\phi_{D_{T_2}} \tilde{f}_{T_2}) | \lesssim R^{-2000}$. Using the Riemann-Lebesgue lemma and Hölder’s inequality we have $\| \tilde{f}_{T_1} \|_2 \lesssim \| f \|_{L^2(\Omega)}$. Thus by inserting these estimates into \((5.5)\) we have Property (4).

Consider Property (5). It easily follows from Property (4). Indeed, by (4),
\[
\sum_{T \in \bar{T}(\Omega)} \int_{\Omega} | f_T |^2 \lesssim \int_{\Omega} \sum_{T \in \bar{T}(\Omega)} | f_T |^2 + R^{-9000} \int_{\Omega} | f |^2.
\]
Since $\int_{\Omega} | f_T |^2 \lesssim \int_{\Omega} | f_{T_1} |^2$, we have Property (5).

Consider Property (6). Let us denote by $V = \{ x' \in \mathbb{R}^2 : (x', x_3) \in B(\omega, R^{-1/2}) \cap S \}$. By the relation between $f$ and $\tilde{f}$ it suffices to show
\[
\| \sum_{T \in \bar{T} : \omega(T) \in \tau} \tilde{f}_T \|_{L^2(\tilde{V})} \lesssim \| \tilde{f} \|_{L^2(\tilde{V})}
\]
where $\tilde{V} = \{ x' \in \mathbb{R}^2 : (x', x_3) \in 10B(\omega, R^{-1/2}) \cap S \}$. By Property (1),
\[
\| \sum_{T \in \bar{T} : \omega(T) \in \tau} \tilde{f}_T \|_{L^2(B)} \leq \| \sum_{\Omega : 3\delta \cap 3\delta \neq \emptyset} \sum_{T \in \bar{T}(\Omega)} \tilde{f}_T \|_{L^2(B)}^2.
\]
Since $\Omega$ is finitely overlapped, we have
\[
\| \sum_{\Omega : 3\delta \cap 3\delta \neq \emptyset} \sum_{T \in \bar{T}(\Omega)} \tilde{f}_T \|_{L^2(B)}^2 \lesssim \sum_{\Omega : 3\delta \cap 3\delta \neq \emptyset} \| \sum_{T \in \bar{T}(\Omega)} \tilde{f}_T \|_{L^2(B)}^2.
\]
By Property (4) this is bounded by
\[ \lesssim \sum_{\Omega:3 \bar{\Omega} \cap V \cap \bar{\tau} \neq \emptyset} \sum_{T \in T'(\Omega)} \| \tilde{f}_T \|_2^2 + R^{-900} \sum_{\Omega:3 \bar{\Omega} \cap V \cap \bar{\tau} \neq \emptyset} \| \tilde{f} \|_{L^2(\bar{\Omega})}^2. \]

Now we replace $T'$ with $T$. Then the above is bounded by
\[ \sum_{\Omega:3 \bar{\Omega} \cap V \cap \bar{\tau} \neq \emptyset} \sum_{T \in T(\Omega)} \| \tilde{f}_T \|_2^2 + R^{-900} \sum_{\Omega:3 \bar{\Omega} \cap V \cap \bar{\tau} \neq \emptyset} \| \tilde{f} \|_{L^2(\bar{\Omega})}^2. \]

By using (4) again, this is
\[ \lesssim \sum_{\Omega:3 \bar{\Omega} \cap V \cap \bar{\tau} \neq \emptyset} \sum_{T \in T(\Omega)} \| \tilde{f}_T \|_2^2 + R^{-900} \sum_{\Omega:3 \bar{\Omega} \cap V \cap \bar{\tau} \neq \emptyset} \| \tilde{f} \|_{L^2(\bar{\Omega})}^2. \]

Thus we have Property (6).

REFERENCES

[1] J. Bennett, A. Carbery, and T. Tao, On the multilinear restriction and Kakeya conjectures, Acta mathematica 196 (2006), no. 2, 261–302.
[2] J. Bourgain, Besicovitch type maximal operators and applications to Fourier analysis, Geometric and Functional analysis 1 (1991), no. 2, 147–187.
[3] ________ Harmonic analysis and combinatorics: how much may they contribute to each other, Mathematics: Frontiers and Perspectives (2000), 13–32.
[4] J. Bourgain and L. Guth, Bounds on oscillatory integral operators based on multilinear estimates, Geometric and Functional Analysis 21 (2011), no. 6, 1239–1295.
[5] A. Córdoba, Geometric fourier analysis, Annales de l’institut fourier, 1982, pp. 215–226.
[6] L. Guth, The endpoint case of the Bennett–Carbery–Tao multilinear Kakeya conjecture, Acta mathematica 205 (2010), no. 2, 263–286.
[7] ________, A restriction estimate using polynomial partitioning, Journal of the American Mathematical Society (2015), http://dx.doi.org/10.1090/jams827.
[8] ________, A short proof of the multilinear Kakeya inequality, Mathematical Proceedings of the Cambridge Philosophical Society 158 (2015), no. 1, 147–153.
[9] L. Guth and N. Katz, On the Erdős distinct distance problem in the plane, Annals of Mathematics 181 (2015), no. 2, 155–190.
[10] H. Kaplan, J. Matoušek, and M. Sharir, Simple proofs of classical theorems in discrete geometry via the Guth–Katz polynomial partitioning technique, Discrete & Computational Geometry 48 (2012), no. 3, 499–517.
[11] S. Lee, Bilinear restriction estimates for surfaces with curvatures of different signs, Transactions of the American Mathematical Society 358 (2006), no. 8, 3511–3533.
[12] A. Moyua, A. Vargas, and L. Vega, Restriction theorems and maximal operators related to oscillatory integrals in $\mathbb{R}^3$, Duke mathematical journal 96 (1999), no. 3, 547–574.
[13] E.M. Stein, Some problems in harmonic analysis, Harmonic analysis in euclidean spaces, 1979, pp. 3–20.
[14] ________, Oscillatory integrals in Fourier analysis, Beijing lectures in harmonic analysis, 1986, pp. 307–355.
[15] T. Tao, From rotating needles to stability of waves: Emerging connections between combinatorics, analysis and PDE, Notices of the AMS 48 (2001), no. 3, 294–303.
[16] ________, A sharp bilinear restriction estimate for paraboloids, Geometric and functional analysis 13 (2003), no. 6, 1359–1384.
[17] T. Tao and A. Vargas, A bilinear approach to cone multipliers I. Restriction estimates, Geometric and functional analysis 10 (2000), no. 1, 185–215.
[18] T. Tao, A. Vargas, and L. Vega, A bilinear approach to the restriction and Kakeya conjectures, Journal of the American Mathematical Society 11 (1998), no. 4, 967–1000.
[19] A. Vargas, *Restriction theorems for a surface with negative curvature*, Mathematische Zeitschrift **249** (2005), no. 1, 97–111.

[20] T. Wolff, *A sharp bilinear cone restriction estimate*, Annals of Mathematics **153** (2001), no. 3, 661–698.

School of Mathematical Sciences, Seoul National University, Seoul 08826, Republic of Korea

E-mail address: akilus@snu.ac.kr

Department of Mathematical Sciences, School of Natural Science, Ulsan National Institute of Science and Technology, UNIST-gil 50, Ulsan 44919, Republic of Korea

E-mail address: jungjinlee@unist.ac.kr