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Existence results for some integro-differential equations with state-dependent nonlocal conditions in Fréchet Spaces

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Abstract: In this work, we present existence of mild solutions for partial integro-differential equations with state-dependent nonlocal local conditions. We assume that the linear part has a resolvent operator in the sense given by Grimmer. The existence of mild solutions is proved by means of Kuratowski’s measure of noncompactness and a generalized Darbo fixed point theorem in Fréchet space. Finally, an example is given for demonstration.

Keywords: Integro-differential equations; finite delay; mild solution; state-dependent delay; semigroup theory; resolvent operator; Kuratowski measure of noncompactness; state-dependent delay; nonlocal initial conditions

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1 Introduction

In this work, we study existence of mild solutions for the following partial functional integrodifferential equation

$$
\begin{align*}
\dot{\vartheta}(t) &= A\vartheta(t) + \int_{0}^{t} Y(t-s)\vartheta(s)ds + F(t, \vartheta_{\rho(t, \vartheta)}, \vartheta_{\tau(t, \vartheta)}), \quad t \in \mathbb{R}^{+} = [0, +\infty), \\
\vartheta_{0} &= h(\zeta(\vartheta), \vartheta) \in \mathcal{C}([-r, 0], X),
\end{align*}
$$

(1)

where $A : D(A) \subset X \to X$ is the infinitesimal generator of $C_0$-semigroup $(T(t))_{t \geq 0}$ on a Banach space $X$, for $t \geq 0$, $Y(t)$ is a closed linear operator with domain $D(Y) \supset D(A)$, $F : \mathbb{R}^{+} \times \mathcal{C} \to X$, $h : \mathbb{R}^{+} \times \mathcal{C}(\mathbb{R}^{+}, \mathcal{C}(\mathbb{R}^{+}, X) \to \mathcal{C}$, $\zeta : \mathcal{C}(\mathbb{R}^{+}, X) \to \mathbb{R}^{+}$, $\rho : \mathbb{R}^{+} \times \mathcal{C} \to \mathbb{R}^{+}$ are suitable functions that satisfy appropriate conditions which will be describe in the sequel. For any continuous function $\vartheta$ defined on $[-r, +\infty)$ and any $t \in [0, +\infty)$, we denote by $\vartheta_{t}$ the element of $\mathcal{C}$ defined by $\vartheta_{t}(\theta) = \vartheta(t + \theta)$ for $\theta \in [-r, 0]$.

In the past few decades, interest in a variety of problems such as delay differential equations (DDEs), retarded differential equations (RDEs), neutral delay differential equations (NDDEs) has expanded. The
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main areas of application are biological science, economics, materials science, medicine, public health and robotics ([13, 16, 17]).

The theory of integro-differential equations has become an active area of investigation due to their applications in the fields such as engineering, mechanics, physics, chemistry, biology, economics, ecology and so on. One can see [22] and references therein. In various works the problem of existence of solutions of the Cauchy problem for integro-differential equations has been studied; we refer the reader to books [1, 15, 18] and to papers [2, 3]. In addition, the nonlinear integro-differential equations with resolvent operator serve as an abstract formulation of partial integro-differential equations that arise in many physical phenomena.

On the other hand, nonlocal conditions are known to make a much better description of real models than classical initial ones. Byszewski’s work [7] provides the first result as well as the physical significance for nonlocal issues. It then generated increased interest in many nonlocal issues regarding differential equations. Some basics outcomes on nonlocal issues are obtained see [8, 9, 22] and the references therein for additional commentary and citations. Hernandez and O’Regan’s work [15] proposes the concept of state-dependent nonlocal conditions which generalizes many nonlocal conditions. Recently, with this class of conditions, Hernandez studied the existence and uniqueness of solution for a general class of abstract differential equations with state dependent delay (see [14]) and in [4], Benchohra and al discussed the existence of mild solutions for non-linear fractional integrodifferential equations. Motivated by the previously mentionned works, in this paper we will extend some such results of mild solutions for the following abstract integro-differential system with state-dependent delays:

In our current paper, we will investigate the existence of solutions for the previously mentioned integro-differential system since this problem still has not been considered in the literature. The main contributions of this paper are summarized as follows:

1. The study of integro-differential equations via measure of noncompactness in the form (1) is an untreated topic in the literature and this is an additional motivation for writing this paper.

2. We establish some sufficient conditions for the nonlocal existence by means of Darbo fixed-point Theorem via the noncompactness measure in Fréchet space.

3. The results are established with the use of the theory of resolvent operator in the sense of Grimmer.

The structure of this work is as follows: Sect. 2, is preliminaries on some basic definitions, lemmas and notations. Sect. 3 is focused upon existence of mild solution of Eq. (1). In Sect. 4, we provide a concrete example to illustrate the efficiency of our results. The last section is devoted to our conclusions.

2 Notations and preliminaries

In this section, we give basic concepts, Definitions and Lemmas which will be used in the sequel, to obtain the main results.

Let $J := [0, T]$ where $T > 0$. A measurable function $\vartheta : J \rightarrow X$ is Bochner integrable if and only if $\|\vartheta\|$ is Lebesgue integrable.

By $L(X)$ we denote the Banach space of bounded linear operators from $X$ into $X$, with norm

$$
\|M\|_{L(X)} = \sup_{\|\vartheta\| = 1} \|M(\vartheta)\|.
$$

Let $L^1(J, X)$ denote the Banach space of measurable functions $\vartheta : J \rightarrow X$ which are Bochner integrable, endowed with the norm

$$
\|\vartheta\|_{L^1} = \int_{0}^{a} \|\vartheta(t)\| dt.
$$

Let $C(J, X)$ be the Banach space of continuous functions from $J$ into $X$, furnished with the norm

$$
\|\vartheta\|_{\infty} = \sup\{\|\vartheta(t)\| : t \in J\}.
$$
Finally, we introduce the space $C(R^+)$ which is the Fréchet space of all continuous functions $\theta : R^+ \rightarrow X$, equipped with the family seminorms
\[
\|\theta\|_n = \sup_{t \in [0, n]}\|\theta(t)\|; \ n \in N
\]
and the distance
\[
d(\theta_1, \theta_2) = \sum_{n=1}^{\infty} 2^{-n} \frac{\|\theta_1 - \theta_2\|_n}{1 + \|\theta_1 - \theta_2\|_n}; \ \theta_1, \theta_2 \in C(R^+).
\]
We recall some basic results about the resolvent operators for the following linear homogeneous equation
\[
\begin{aligned}
\dot{\theta}(t) &= A\theta(t) + \int_0^t Y(t-s)\theta(s)ds \quad \text{for } t \geq 0, \\
\theta(0) &= \theta_0 \in \mathcal{Y},
\end{aligned}
\]  
(2)
where $A$ and $Y(t)$ are closed linear operators on $X$. In the following $\mathcal{Y}$ represents the Banach space $D(A)$ equipped with the graph norm $\|\theta\|_\mathcal{Y} := \|A\theta\| + \|\theta\|$ for $\theta \in \mathcal{Y}$. $C(R^+, \mathcal{Y})$, $\mathcal{L}(\mathcal{Y}, X)$ stand for the space of all continuous functions from $R^+$ into $X$ and the set of all bounded linear operators from $\mathcal{Y}$ into $X$, respectively.

**Definition 2.1** ([12]). A resolvent operator for Eq. (2) is a bounded linear operator valued function $R(t) \in \mathcal{L}(X)$ for $t \geq 0$, having the following properties:
(i) $R(0) = I$ and $\|R(t)\| \leq N e^{\beta t}$ for some constants $N$ and $\beta$.
(ii) For all $\theta \in \mathcal{Y}$, $R(t)\theta$ is strongly continuous for $t \geq 0$.
(iii) For $\theta \in \mathcal{Y}$, $R(\cdot)\theta \in C^1(R^+; X) \cap C(R^+; \mathcal{Y})$ and
\[
R'(t)\theta = AR(t)\theta + \int_0^t Y(t-s)R(s)\theta ds
= R(t)A\theta + \int_0^t R(t-s)Y(s)\theta ds, \quad \text{for } t \geq 0.
\]

In order to get the existence of the resolvent operators, we assume the following assumptions:
(H$_1$) The operator $A$ is the infinitesimal generator of a strongly continuous semigroup $(T(t))_{t \geq 0}$ on $X$.
(H$_2$) For all $t \geq 0$, $Y(t)$ is a closed linear operator from $D(A)$ to $X$. For any $\theta \in \mathcal{Y}$, the map $t \mapsto Y(t)\theta$ is bounded, differentiable and the derivative $t \mapsto Y'(t)\theta$ is bounded and uniformly continuous on $R^+$. 

The following theorem provides adequate conditions to ensure that the resolvent operator for the system (2) exists.

**Theorem 2.1** ([12]). Assume that (H$_1$)-(H$_2$) hold. Then there exists a unique resolvent operator to the Cauchy problem (2).

In what follows, we give some results for the existence of solutions for the following integro-differential equation.
\[
\begin{aligned}
\dot{\theta}(t) &= A\theta(t) + \int_0^t Y(t-s)\theta(s)ds + l(t) \quad \text{for } t \geq 0, \\
\theta(0) &= \theta_0 \in X.
\end{aligned}
\]  
(3)
where $l : [0, +\infty] \rightarrow X$ is a continuous function.

**Definition 2.2** ([12]). A continuous function $\theta : [0, +\infty] \rightarrow X$ is said to be a strict solution of the Eq. (3) if
1. \( \mathcal{D} \in \mathcal{C}([0, +\infty], X) \cap \mathcal{C}([0, +\infty], Y) \).
2. \( \mathcal{D} \) satisfies Eq. (3) for \( t \geq 0 \).

**Theorem 2.2** ([12]). Assume that hypotheses \((H_1)\) and \((H_2)\) hold. If \( \mathcal{D} \) is a strict solution of the Eq. (3), then the variation of constant formula holds

\[
\mathcal{D}(t) = R(t)\mathcal{D}_0 + \int_0^t R(t-s)I(s)ds \quad \text{for } t \geq 0.
\]

We recall the following definition of the notion of a sequence of measures of noncompactness [10, 11].

**Definition 2.3.** The Kuratowski measure of noncompactness \( \alpha(\cdot) \) defined for a bounded subset \( \mathcal{D} \) of a Banach space \( X \) is given by

\[
\alpha(\mathcal{D}) = \inf \left\{ \varepsilon > 0 : \mathcal{D} = \bigcup_{i=1}^k \mathcal{D}_i \text{ and diam} (\mathcal{D}_i) \leq \varepsilon \right\}.
\]

**Theorem 2.3.** Let \( \mathcal{M}_\mathcal{F} \) be the family of all nonempty and bounded subsets of a Fréchet space \( \mathcal{F} \). A family of functions \( \{ \Phi_n \}_{n \in \mathbb{N}} \) where \( \Phi_n : \mathcal{M}_\mathcal{F} \to [0, +\infty) \) is said to be a family of measures of noncompactness in the real Fréchet space \( \mathcal{F} \) if it satisfies the following conditions for all \( \mathcal{D}_1, \mathcal{D}_2 \in \mathcal{M}_\mathcal{F} \):

(i) \( \mathcal{D}_1 \) is pre-compact if and only if \( \Phi_n(\mathcal{D}_1) = 0 \) for all \( n \in \mathbb{N} \).

(ii) \( \Phi_n(\mathcal{D}_1) \leq \Phi_n(\mathcal{D}_2) \), where \( \mathcal{D}_1 \subseteq \mathcal{D}_2 \) for all \( n \in \mathbb{N} \).

(iii) \( \Phi_n(\mathcal{D}) = \Phi_n(\overline{\mathcal{D}}) = \Phi_n(\mathcal{D}) \) for all \( n \in \mathbb{N} \), where \( \overline{\mathcal{D}} \) and \( \overline{\mathcal{D}} \) are the closure and convex hull of \( \mathcal{D} \), respectively.

(iv) If \( \{ \mathcal{D}_i \}_{i=1}^\infty \) is a sequence of closed sets of \( \mathcal{M}_\mathcal{F} \) such that \( \mathcal{D}_{i+1} \subset \mathcal{D}_i, \ i = 1, \cdots, \) and if \( \lim_{i \to \infty} \Phi_n(\mathcal{D}_i) = 0 \), then for each \( n \in \mathbb{N} \), the intersection set \( \mathcal{D}_\infty = \bigcap_{i=1}^\infty \mathcal{D}_i \) is nonempty.

**Example 2.1.** Let \( \mathcal{F} = \mathcal{C}(\mathbb{R}^+) \). For \( \mathcal{D} \in \mathcal{M}_\mathcal{F} \), \( y \in \mathcal{D}, \ n \in \mathbb{N} \) and \( \varepsilon > 0 \), let us denote by \( \omega^n(y, \varepsilon) \) the modulus of continuity of the function \( y \) on the interval \( [0, n] \) that is

\[
\omega^n(y, \varepsilon) = \sup \left\{ ||y(t) - y(s)||, \ t, s \in [0, n], |t-s| \leq \varepsilon \right\}.
\]

Further, let us put

\[
\omega^n(\mathcal{D}, \varepsilon) = \sup \left\{ \omega^n(y, \varepsilon) : y \in \mathcal{D} \right\},
\]

\[
\omega^n_0(\mathcal{D}) = \lim_{\varepsilon \to 0^+} \omega^n(\mathcal{D}, \varepsilon),
\]

\[
\overline{\omega}^n(\mathcal{D}) = \sup_{t \in [0, n]} \alpha(\mathcal{D}(t)) = \sup_{t \in [0, n]} \alpha \left( \{ y(t) : y \in \mathcal{D} \} \right),
\]

and

\[
\alpha_n(\mathcal{D}) = \omega^n_0(\mathcal{D}) + \overline{\omega}^n(\mathcal{D}).
\]

The family of mapping \( \{ \alpha_n : n \in \mathbb{N} \} \) where \( \alpha_n : \mathcal{M}_\mathcal{F} \to [0, +\infty) \) is a family of measure of noncompactness (see [20, 21]).

**Remark 2.1.** Notice that if the set \( \mathcal{D} \) is equicontinuous, then \( \omega^n_0(\mathcal{D}) = 0 \).

In the sequel, we need the useful following results for the computation of \( \alpha(\cdot) \).

**Definition 2.4.** A nonempty subset \( \mathcal{D} \subset \mathcal{F} \) is said to be bounded if for \( n \in \mathbb{N} \), there exists \( M_n > 0 \) such that

\[
\|y\|_n \leq M_n, \quad \text{for each } y \in \mathcal{D}.
\]
Lemma 2.4 ([6]). If $D$ is a bounded subset of a Banach space $X$, then for each $\varepsilon > 0$ there is a sequence $\{\delta_k\}_{k=1}^{\infty} \subset D$ such that

$$a(D) \leq 2a(\{\delta_k\}_{k=1}^{\infty}) + \varepsilon.$$ 

Lemma 2.5 ([19]). If $\{\delta_k\}_{k=1}^{\infty} \subset L^1(\mathbb{R}^+, F)$ is uniformly integrable, then $a(\{\delta_k(t)\}_{k=1}^{\infty})$ is measurable and

$$a\left(\left\{ \int_0^t \delta_k(r)dr \right\}_{k=1}^{\infty} \right) \leq 2\int_0^t a(\{\delta_k(r)\}_{k=1}^{\infty})dr,$$ 

t $\geq 0$

where $a$ is a Kuratowski measure of noncompactness on $F$.

Definition 2.5. Let $D$ be a nonempty subset of a Fréchet space $W$, and let $\Pi : D \to W$ be a continuous operator which transforms bounded subsets into bounded ones. One says $\Pi$ satisfies the Darbo condition with constants $\{q_n : n \in \mathbb{N}\}$ with respect to a family of measures of noncompactness $\{\Phi_n : n \in \mathbb{N}\}$, if

$$\Phi_n(\Pi(D)) \leq q_n \Phi_n(D)$$

for each bounded set $D \subset D$ and $n \in \mathbb{N}$. If $q_n < 1$, $n \in \mathbb{N}$ then $\Pi$ is called a contraction with respect to $\{\Phi_n : n \in \mathbb{N}\}$.

The following generalization of the classical Darbo fixed point Theorem for Fréchet spaces, plays an important role in the proof of our main result.

Theorem 2.6 ([10, 11]). Let $D$ be a nonempty, bounded, closed, and convex subset of a Fréchet space $F$ and let $\Theta : D \to D$ be a continuous mapping. Suppose that $\Theta$ is a contraction with respect to a family of measures of noncompactness $\{\Phi_n, n \in \mathbb{N}\}$. Then $\Theta$ has at least one fixed point in the set $D$.

3 The Main Results

In this section, we establish and prove the existence of mild solution for the system (1). First we give the definition of the mild solution for (1).

Definition 3.1. A function $\vartheta \in C([-r, +\infty); X)$ is called the mild solution of the system (1), if

(i) $\vartheta_0 = h(\vartheta(t), \vartheta)$, for all $t \in [-r, 0]$,

(ii) for each $t \in \mathbb{R}^+$, $\vartheta(t)$ satisfies the following integral equation:

$$\vartheta(t) = R(t)h(\vartheta(t), \vartheta(0)) + \int_0^t R(t-s)f(s, \vartheta(s), \vartheta)ds.$$ 

In this work, we will work under the following assumptions:

(A1) There exists a constant $M_0 > 1$ such that $\|R(t)\|_{L(0)} \leq M_0$ for every $t \in \mathbb{R}^+$.

(A2) (i) The function $t \to F(t, \vartheta)$ is measurable on $\mathbb{R}^+$ for each $\vartheta \in \mathcal{C}$, and the function $\vartheta \to F(t, \vartheta)$ is continuous on $\mathcal{C}$ for a.e. $t \in \mathbb{R}^+$.

(ii) There exists a function $f \in L^{\infty}_{\text{loc}}(\mathbb{R}^+, \mathbb{R}^+)$ and a continuous nondecreasing function $\Xi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$\|F(t, \vartheta)\| \leq f(t)\Xi(\|\vartheta\|_{\infty})$$

for a.e. $t \in \mathbb{R}^+$ and each $\vartheta \in \mathcal{C}$.

(iii) For each bounded set $D \subset \mathcal{C}$ and for each $t \in [0, n]$, $n \in \mathbb{N}$, we have

$$a(F(t, D)) \leq f(t) \sup_{\epsilon \to 0} a(D(s)),$$

where $a$ is a measure of noncompactness on the Banach space $X$. 
(A3) (i) For each $n \in \mathbb{N}$, there exists $J_n > 0$ such that
\[ \| h(\zeta(\vartheta), \vartheta) \| \leq J_n (1 + \| \vartheta \|) \text{ for each } \vartheta \in C([-r, \infty), X). \]

(ii) For each $n \in \mathbb{N}$, there exists $\delta_n > 0$ such that
\[ a(\zeta(\vartheta), D) \leq \delta_n \sup_{t \in [-r, n]} a(D(t)) \]
for any bounded $D \subset C([-r, +\infty), X)$.

(A4) For each $n \in \mathbb{N}$, there exists $q_n > 0$ such that
\[ M_0 \left[ J_n (1 + q_n) + \bar{T}_n \Xi(q_n) \right] \leq q_n, \]
where for $n \in \mathbb{N}$
\[ \bar{T}_n := \int_0^n f(s) \, ds. \]

Define on $C([-r, +\infty); X)$ the family of measures of noncompactness by
\[ \alpha_n(D) = \omega_n^0(D) + \sup_{t \in [-r, n]} \alpha(D(t)), \]
and
\[ D(t) = \{ \vartheta(t) \in X; \vartheta \in D \}, \quad t \in [-r, n]. \]

Remark 3.1. Notice that if the set $D$ is equicontinuous, then $\omega_n^0(D) = 0$.

Now, we present the main result of this section.

Theorem 3.1. Assume that $(H_1)$-$\text{H}_2$), $(A_1)$-$\text{A}_4$ hold and for each $n \in \mathbb{N}$,
\[ M_0 \left[ \delta_n + 2 \bar{T}_n \right] < \frac{1}{2}. \]

Then the nonlocal state dependent problem (1) has at least one mild solution.

Proof. We consider the operator $\Theta$ on $C([-r, +\infty); X)$ defined by
\[ \Theta(\vartheta)(t) = \begin{cases} h(\zeta(\vartheta), \vartheta), & t \in [-r, 0], \\ R(t)h(\zeta(\vartheta), \vartheta)(0) + \int_0^t R(t-s)F(s, \vartheta(\rho(s, \vartheta))) \, ds, & t \in \mathbb{R}^+. \end{cases} \]

(6)

By Definition 3.1, it is easy to see that the mild solution of nonlocal problem (1) is equivalent to the fixed point of the operator $\Theta$ defined by (6).

We define the ball $B_{q_n} = B(0, q_n) = \{ \vartheta \in C([-r, +\infty), X) : \| \vartheta \| \leq q_n \}$. 
Firstly, we claim that $\Theta(B_{q_0}) \subset B_{q_0}$. In fact, for any $n \in \mathbb{N}$, and each $\vartheta \in B_{q_0}$, and $t \in [0, n]$, by $(A_1)-(A_4)$, we have

$$
\|\Theta(t)\| \leq \|R(t)h(\vartheta(\vartheta), \vartheta(0))\| + \int_0^t \|R(t-s)F(s, \vartheta(\vartheta(\vartheta), \vartheta(0)))\| ds
$$

$$
\leq M_0\|h(\vartheta(\vartheta), \vartheta(0))\| + M_0 \int_0^t \|F(s, \vartheta(\vartheta(\vartheta), \vartheta(0)))\| ds
$$

$$
\leq M_0 J_n(1 + \|\vartheta\|_n) + M_0 \int_0^t \|F(s, \vartheta(\vartheta(\vartheta), \vartheta(0)))\| ds
$$

$$
\leq M_0 [J_n(1 + q_n) + \vartheta(X)\|q_n\|_n]
$$

Thus $\|\Theta(\vartheta)\| \leq q_n$.

This proves that $\Theta$ transforms the ball $B_{q_0}$ into $B_{q_0}$. We complete the proof in the following steps.

**Step 1:** $\Theta : B_{q_0} \rightarrow B_{q_0}$ is continuous.

Let $\{\vartheta^k\}_{k \in \mathbb{N}}$ be a sequence such that $\vartheta^k \rightarrow \vartheta$ in $B_{q_0}$. Then for each $t \in [0, n]$, we have

$$
\|\Theta(\vartheta^k)(t) - \Theta(\vartheta)(t)\| \leq M_0\|h(\vartheta^k(\vartheta, \vartheta(0)), \vartheta^k(\vartheta(0))) - h(\vartheta(\vartheta), \vartheta(0))\| + M_0 \int_0^t \|F(s, \vartheta^k(\vartheta, \vartheta(0))) - F(s, \vartheta(\vartheta))\| ds.
$$

Since $\vartheta^k \rightarrow \vartheta$ as $k \rightarrow \infty$, the Lebesgue dominated convergence theorem implies that

$$
\|\Theta(\vartheta^k) - \Theta(\vartheta)\| \rightarrow 0 \text{ as } k \rightarrow +\infty.
$$

Thus $\Theta$ is continuous.

**Step 2:** $\Theta(B_{q_0})$ is bounded.

Since $\Theta(B_{q_0}) \subset B_{q_0}$ and $B_{q_0}$ is bounded, then $\Theta(B_{q_0})$ is bounded.

**Step 3:** For each equicontinuous subset $K$ of $B_{q_0}$, $a_n(\Theta(K)) \leq k_n a_n(K)$.

From Lemmas 2.4 and 2.5, for any equicontinuous set $K \subset B_{q_0}$, and any $\epsilon > 0$, there exists a sequence $\{\vartheta^k\}_{k=1}^\infty \subset K$, such that for all $t \in [0, n]$, we have

$$
a((\Theta(K))(t)) = a(\left\{ R(t)h(\vartheta(\vartheta), \vartheta(0)) + \int_0^t R(t-s)F(s, \vartheta(\vartheta(\vartheta), \vartheta(0))) ds : \vartheta \in K \right\})
$$

$$
\leq a(\left\{ R(t)h(\vartheta(\vartheta), \vartheta(0)) : \vartheta \in K \right\}) + a(\left\{ \int_0^t R(t-s)F(s, \vartheta(\vartheta(\vartheta), \vartheta(0))) ds : \vartheta \in K \right\})
$$

$$
\leq 2a(\left\{ R(t)h(\vartheta^k(\vartheta, \vartheta(0)), \vartheta^k(\vartheta(0))) \right\}_{k=1}^\infty) + 2a(\left\{ \int_0^t R(t-s)F(s, \vartheta^k(\vartheta, \vartheta(0))) ds \right\}_{k=1}^\infty) + \epsilon
$$

$$
\leq 2a(\left\{ \|R(t)\|_{\mathcal{L}(X)} \right\}_{k=1}^\infty) \left\{ h(\vartheta(\vartheta), \vartheta^k(\vartheta, \vartheta(0))) \right\}_{k=1}^\infty) + 4 \int_0^t \left\{ \|R(t-s)\|_{\mathcal{L}(X)} \right\}_{k=1}^\infty) \left\{ F(s, \vartheta^k(\vartheta, \vartheta(0))) \right\}_{k=1}^\infty) ds + \epsilon
$$

$$
\leq 2M_0 \delta_n \sup_{t \in [0,n]} a(\left\{ \vartheta^k(\vartheta, \vartheta(0)) \right\}_{k=1}^\infty) + 4M_0 \int_0^t f(s) a(\left\{ \vartheta^k(\vartheta, \vartheta(0)) \right\}_{k=1}^\infty) ds + \epsilon
$$

$$
\leq 2M_0 \delta_n a_n(K) + 4M_0 \tilde{t}_n a_n(K) + \epsilon
$$

$$
\leq \left(2M_0 [\delta_n + 2 \tilde{t}_n] \right) a_n(K) + \epsilon.
$$

Since $\epsilon > 0$ is arbitrary, then
4 An example

In this section, an example is provided to illustrate the obtained theory.

We consider the following integro-differential equation with state-dependent nonlocal initial conditions:

\[
\begin{aligned}
\frac{\partial}{\partial t} z(t, \xi) &= \left[ 2 \frac{\partial^2 z(t, \xi)}{\partial \xi^2} \right] + \int_0^t \Gamma(t-s) \left( 2 \frac{\partial^2 z(s, \xi)}{\partial \xi^2} \right) ds + G(t)z(1 - \rho(z(t, \xi)), \xi), \quad t \in [0, \infty], \xi \in [0, \pi], \\
H(z_0(t, \xi)) &= 0, \quad \theta \in [-r, 0], \xi \in [0, \pi],
\end{aligned}
\]

where \( \Gamma : \mathbb{R}_+ \rightarrow \mathbb{R} \) is bounded uniformly continuous, continuously differentiable and \( \Gamma' \) is bounded uniformly continuous, \( \rho \in C([0, r]), \, B \in C([0, r], \mathbb{R}), \, q \in C([-r, +\infty]), \mathbb{R}^+) \), \( G(t) \) is a continuous function from \( \mathbb{R}^+ \) to \( \mathbb{R} \).

Let \( X = \mathcal{Y} = L^2([0, \pi], \mathbb{R}) \). We define the linear operator \( A : D(A) \subset X \rightarrow X \) by

\[
D(A) = \{ z \in X : z, z' \text{ are absolutely continuous, } z'' \in X, \, z(0) = z(\pi) = 0 \}
\]

Then

\[
AZ = \sum_{n=1}^{\infty} n^2 \langle z, e_n \rangle e_n, \quad z \in D(A),
\]

where \( e_n(z) = (\frac{z}{\pi})^{1/2} \sin(nz), \, 0 \leq z \leq \pi, \, n = 1, 2, \cdots \) is the orthonormal set of eigenvectors of \( A \). It is known that \( A \) is the infinitesimal generator of a analytic semigroup \( T(t)_{t \geq 0} \) in \( X \), which is given by

\[
T(t)z = \sum_{n=1}^{\infty} e^{-n^2 t} \langle z, e_n \rangle e_n, \quad z \in X.
\]

Let \( Y(t) : D(A) \subset X \rightarrow X \) be the operator defined by \( Y(t)l = \Gamma(t)Al \) for \( t \geq 0 \) and \( l \in D(A) \).

Set

\[
\begin{aligned}
\vartheta(t, \xi) &= z(t, \xi), \quad t \in [0, +\infty[, \xi \in [0, \pi], \\
F(t, \phi)(\xi) &= G(t)\phi(\xi), \quad t \in [0, +\infty[, \xi \in [0, \pi], \phi \in X, \\
h(t, u) &= B(u(\theta)), \quad t \in [0, +\infty[, \theta \in [-r, 0].
\end{aligned}
\]

Then with these settings system (10) can be written in the abstract form

\[
\begin{aligned}
\vartheta'(t) &= \mathcal{A} \vartheta(t) + \int_0^t Y(t-s)\vartheta(s)ds + F(t, \vartheta_{\rho(t, s)}), \quad t \in \mathbb{R}^+ = [0, +\infty), \\
\vartheta_0 &= h(\vartheta(0), \theta) \in C([-r, 0], X).
\end{aligned}
\]

Furthermore, we can check that the assumptions of Theorem 3.1 hold. Consequently, the problem (10) has at least one mild solution on \([-r; +\infty)\).
5 Conclusion

In this paper, we consider a class of integro-differential system with state-dependent delay and nonlocal conditions in a Frechet space. More precisely, using the generalized Darbo fixed point theorem associated with the theory of measures of noncompactness and the theory of the resolvent operator, we established a new set of conditions which guarantees the existence of a mild solution for the considered system. In further works, we intend to extend the obtained results for stochastic integro-differential systems.

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