Large-time asymptotic behavior of the infinite system of harmonic oscillators on the half-line

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Abstract

The mixed initial-boundary value problem for infinite one-dimensional chain of harmonic oscillators on the half-line is considered. We study the large time behavior of solutions and derive the dispersive bounds.

Key words and phrases: one-dimensional system of harmonic oscillators on the half-line, mixed initial-boundary value problem, Fourier–Laplace transform, Puiseux expansion, dispersive estimates

1 Introduction

We consider the infinite system of harmonic oscillators on the half–line:

\[ \ddot{u}(x, t) = (\nu^2 \Delta_L - m^2)u(x, t), \quad x \in \mathbb{N}, \quad t > 0, \]  

with the boundary condition (as \( x = 0 \))

\[ \ddot{u}(0, t) = \nu^2(u(1, t) - u(0, t)) - m^2u(0, t) - \kappa u(0, t) - \gamma \dot{u}(0, t), \quad t > 0, \]  

and with the initial condition (as \( t = 0 \))

\[ u(x, 0) = u_0(x), \quad \dot{u}(x, 0) = v_0(x), \quad x \geq 0. \]  

Here \( u(x, t) \in \mathbb{R}, \ \nu > 0, \ \kappa, \gamma \geq 0, \ \Delta_L \) denotes the second derivative on \( \mathbb{Z} \):

\[ \Delta_L u(x) = u(x + 1) - 2u(x) + u(x - 1), \quad x \in \mathbb{Z}. \]

If \( \gamma = 0 \), then formally the system (1.1)–(1.2) is Hamiltonian with the Hamiltonian functional

\[ H(u, \dot{u}) := \frac{1}{2} \sum_{x \geq 0} \left( |\dot{u}(x, t)|^2 + \nu^2|u(x + 1, t) - u(x, t)|^2 + m^2|u(x, t)|^2 \right) + \frac{1}{2}\kappa|u(0, t)|^2. \]  

We assume that the initial data \( Y_0(x) = (u_0(x), v_0(x)) \) belong to the Hilbert space \( \mathcal{H}_{\alpha,+} \), \( \alpha \in \mathbb{R} \), defined below.

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Definition 1.1. (i) $\ell^2_{\alpha, +} \equiv \ell^2_{\alpha, +}(\mathbb{Z}_+), \alpha \in \mathbb{R},$ is the Hilbert space of sequences $u(x)$, $x \geq 0$, with norm $\|u\|^2_{\alpha, +} = \sum_{x \geq 0} |u(x)|^2 x^{2\alpha} < \infty,$ $\langle x \rangle := (1 + x^2)^{1/2}$.

(ii) $\mathcal{H}_{\alpha, +} = \ell^2_{\alpha, +} \otimes \ell^2_{\alpha, +}$ is the Hilbert space of pairs $Y = (u, v)$ of sequences equipped with norm $\|Y\|^2_{\alpha, +} = \|u\|^2_{\alpha, +} + \|v\|^2_{\alpha, +} < \infty$.

On the coefficients $m, \kappa, \nu, \gamma$ of the system we impose condition $\mathbf{C}$ or $\mathbf{C}_0$.

$\mathbf{C}$ If $\gamma \neq 0$, then $m$ or $\kappa$ is not zero.
In addition, if $\gamma \in (0, \nu)$ and $m = 0$, then $\kappa \neq 2(\nu^2 - \gamma^2)$;
if $\gamma \in (0, (\sqrt{m^2 + 4\nu^2} - m)/2]$ and $m \neq 0$, then $\kappa \neq \nu^2 - \gamma^2 \pm \sqrt{(\nu^2 - \gamma^2)^2 - m^2\gamma^2}$.
If $\gamma = 0$, then $\kappa \in (0, 2\nu^2)$.

$\mathbf{C}_0$ $\gamma = 0$ and $\kappa = 2\nu^2$ or $\gamma = \kappa = 0$ and $m \neq 0$.

The main objective of the paper is to prove that for any initial data $Y_0 \in \mathcal{H}_{\alpha, +}$ with $\alpha > 3/2$, the solution $Y(t) = (u(\cdot, t), \hat{u}(\cdot, t))$ of the system obeys the following bound
\begin{equation}
\|Y(t)\|_{-\alpha, +} \leq C\langle t \rangle^{-\beta/2}\|Y_0\|_{\alpha, +}, \quad t \in \mathbb{R},
\end{equation}
where $\beta = 3$ if $\mathbf{C}$ holds, and $\beta = 1$ if $\mathbf{C}_0$ holds. We specify the behavior of the solutions as $t \to \infty$ in Theorem 2.4.

For the solutions of the linear discrete Schrödinger and Klein–Gordon equations in the whole space, the dispersive estimates of the type (1.5) were obtained by Shaban and Vainberg [10], Komech, Kopylova and Kunze [8] and Pelinovsky and Stefanov [9]. The wave operators for the discrete Schrödinger operators were studied by Cuccagna [1]. In [4], we considered the linear Hamiltonian system consisting of the discrete Klein–Gordon field coupled to a particle and obtained the similar results on the long-time behavior for the solutions. In [5], the considered model (1.1)–(1.3) was studied with random initial data $Y_0 \in \mathcal{H}_{\alpha, +}$ with $\alpha < -3/2$. In this paper, the model is studied with initial data from the space $\mathcal{H}_{\alpha, +}$ with $\alpha > 3/2$, and the long time asymptotics of the solutions are constructed.

2 Main Results

The existence and uniqueness of the solutions to the problem (1.1)–(1.3) was proved in [5].

Theorem 2.1. Let $\gamma, \kappa, m \geq 0,$ $\nu > 0,$ and let $Y_0 \in \mathcal{H}_{\alpha, +}, \alpha \in \mathbb{R}$. Then the problem (1.1)–(1.3) has a unique solution $Y(t) \in C(\mathbb{R}, \mathcal{H}_{\alpha, +})$. The operator $U(t) : Y_0 \to Y(t)$ is continuous on $\mathcal{H}_{\alpha, +}$. Moreover, there exist constants $C, B < \infty$ such that $\|U(t)Y_0\|_{\alpha, +} \leq Ce^{B|t|}\|Y_0\|_{\alpha, +}, \quad t \in \mathbb{R}$. For $Y_0 \in \mathcal{H}_{0, +}$, the following bound holds,
\begin{equation}
H(Y(t)) + \gamma \int_0^t |\hat{u}(0, s)|^2 \, ds = H(Y_0), \quad t \in \mathbb{R},
\end{equation}
where $H(Y(t))$ is defined in (1.4).
Theorem 2.3. Let \( Y_0 \in \mathcal{H}_{\alpha,+} \) and \( \alpha > 3/2 \). Then
\[
\|U_0(t)Y_0\|_{\alpha,+} \leq C(t)^{-3/2}\|Y_0\|_{\alpha,+}, \quad t \in \mathbb{R}.
\]

We state the results concerning the solutions of the problem (2.3)–(2.5).

Lemma 2.2. (see Lemma 2.7 in [3]) Assume that \( \alpha \in \mathbb{R} \). Then for any \( Y_0 \in \mathcal{H}_{\alpha,+} \), there exists a unique solution \( Z(t) \equiv (z(\cdot,t),\dot{z}(\cdot,t)) \in \mathcal{C}(\mathbb{R},\mathcal{H}_{\alpha,+}) \) to the mixed problem (2.3)–(2.5); the operator \( U_0(t) : Y_0 \mapsto Z(t) \) is continuous on \( \mathcal{H}_{\alpha,+} \). Furthermore, the following bound holds,
\[
\|U_0(t)Y_0\|_{\alpha,+} \leq C(t)\sigma\|Y_0\|_{\alpha,+},
\]
with some constants \( C = C(\alpha), \sigma = \sigma(\alpha) < \infty \).

The proof of Lemma 2.2 is based on the following formula for the solution \( Z(x,t) = (Z^0(x,t),Z^1(x,t)) \equiv (z(x,t),\dot{z}(x,t)) \) of the problem (2.3)–(2.5):
\[
Z^i(x,t) = \sum_{j=0,1} \sum_{x' \geq 1} G^{ij}_{t,+}(x,x')Y^j_0(x'), \quad x \in \mathbb{Z}_+,
\]
where the Green function \( G_{t,+}(x,x') \) is
\[
G^{ij}_{t,+}(x,x') := G^{ij}_t(x-x') - G^{ij}_t(x+x'), \quad G^{ij}_t(x) \equiv \frac{1}{2\pi} \int_T e^{-ix\theta} \hat{G}^{ij}_t(\theta) \, d\theta,
\]
and
\[
(\hat{G}^{ij}_t(\theta))_{i,j=0}^1 = \begin{pmatrix}
\cos \phi(\theta)t \\
-\phi(\theta) \sin \phi(\theta)t
\end{pmatrix}
\begin{pmatrix}
\frac{\sin \phi(\theta)t}{\phi(\theta)^t} \\
\frac{\cos \phi(\theta)t}{\phi(\theta)^t}
\end{pmatrix}, \quad \phi(\theta) = \sqrt{\nu^2(2-2\cos \theta) + m^2}.
\]
In particular, \( \phi(\theta) = 2\nu|\sin(\theta/2)| \) if \( m = 0 \). We see that \( Z(0,t) \equiv 0 \) for any \( t \), since \( G^{ij}_t(-x) = G^{ij}_t(x) \). For the solutions of the problem (2.3)–(2.5), the following bound is true.

Theorem 2.3. Let \( Y_0 \in \mathcal{H}_{\alpha,+} \) and \( \alpha > 3/2 \). Then
\[
\|U_0(t)Y_0\|_{\alpha,+} \leq C(t)^{-3/2}\|Y_0\|_{\alpha,+}, \quad t \in \mathbb{R}.
\]
This theorem is proved in Appendix B.

To formulate the main result, introduce the following notations.

(i) Denote by \(G_j^i(y,t), j=0,1\), the following function
\[
G_j^i(y,t) := \left( G_i^j(1+y), G_i^j(1+y) \right) = \left( G_i^j(1-y) - G_i^j(1+y), G_i^j(1-y) - G_i^j(1+y) \right), \quad y \in \mathbb{Z}, \ t \in \mathbb{R}. \tag{2.15}
\]

(ii) Let \(G^j(y), j=0,1\), stand for the vector valued function defined as
\[
G^j(y) = \int_0^{+\infty} N(s)G_i^j(y,-s) \, ds = \int_0^{+\infty} N(s)G_i^j(y,-s) \, ds, \quad y \in \mathbb{Z}, \quad j=0,1, \tag{2.16}
\]
where the function \(N(s)\) is introduced in (3.11)–(3.13).

(iii) Denote by \(U'_0(t)\) the operator adjoint to \(U_0(t)\):
\[
\langle Y, U'_0(t)\Psi \rangle_+ = \langle U_0(t)Y, \Psi \rangle_+ , \quad Y \in \mathcal{H}_{\alpha,+}, \quad \Psi \in \mathcal{S} = \mathcal{S}(\mathbb{Z}_+)^2 , \quad t \in \mathbb{R}, \tag{2.17}
\]
where \(\mathcal{S}(\mathbb{Z}_+)\) denotes the class of rapidly decreasing sequences on \(\mathbb{Z}_+ = \{ x \in \mathbb{Z} : x \geq 0 \}\).

Using the Green function \(G_t^j\), we rewrite \(U'_0(t)\Psi\) in the form
\[
(U'_0(t)\Psi)^j(y) = \sum_{i=0}^1 \sum_{x \geq 0} G_t^j(x,y)\Psi(x), \quad t \in \mathbb{R}, \quad y \in \mathbb{Z}_+, \quad j=0,1.
\]

In particular, \(G_0^0(y,t) = (U'_0(t)Y_0)(y)\) with \(Y_0(x) = (\delta_{1x}, 0)\) (see (2.15)), where \(\delta_{1x}\) denotes the Kronekker symbol.

(iv) Denote by \(K^j(x,y), j=0,1, x \in \mathbb{N}, y \in \mathbb{Z}\), vector-valued functions of a form
\[
K^j(x,y) = \int_0^{+\infty} K(x,s) \left( U'_0(-s)G^j \right)(y) \, ds = \int_0^{+\infty} \int_0^{+\infty} K(x,s)N(s)G_i^j(y,-s) \, ds \, d\tau, \quad x \in \mathbb{N}, \ y \in \mathbb{Z}, \tag{2.18}
\]
where \(K(x,s)\) is defined in (3.5).

(v) Define an operator \(\Omega : \mathcal{H}_{\alpha,+} \rightarrow \mathcal{H}_{-\alpha,+}, \ \alpha > 3/2\), by the rule
\[
\Omega : Y \rightarrow Y + r^2 \left( \langle Y(\cdot), K^j(x,\cdot) \rangle_+ - \langle Y(\cdot), K^k(x,\cdot) \rangle_+ \right). \tag{2.19}
\]

Here we put \(K^j(x,y)\mid_{x=0} := G^j(y), \ y \in \mathbb{Z}\). The properties of the functions \(K^j\) and the operator \(\Omega\) are specified in Remark 4.4. The main result of the paper is the following theorem.

**Theorem 2.4.** Let \(Y_0 \in \mathcal{H}_{\alpha,+}, \ \alpha > 3/2\), and conditions \(C\) or \(C_0\) hold. Then the following assertions are fulfilled.

(i) \(U(t)Y_0 = \Omega(U_0(t)Y_0) + r(t)\), where \(\|r(t)\|_{-\alpha,+} \leq C(t)^{-\beta/2}\|Y_0\|_{\alpha,+}\), \(\beta = 3\) if \(C\) holds and \(\beta = 1\) if \(C_0\) holds, \(\Omega\) is a bounded operator defined by (2.19).

(ii) The solution of the problem (1.1)–(1.3) obeys the bound (1.5).

This theorem is proved in Section 4. The behavior of the solutions with the initial data from the space \(\mathcal{H}_{\alpha,+}\) is discussed in Remark 4.5. If conditions \(C\) and \(C_0\) are not fulfilled, then the bound (1.5) for any initial data from \(\mathcal{H}_{\alpha,+}\) is incorrect, see Remark 4.6.
3 Fourier–Laplace transform

In this section, we study the properties of the solutions \( q(x,t) \) to the problem (2.6)–(2.9) using the Fourier–Laplace transform.

**Definition 3.1.** Let \( |q(t)| \leq Ce^{Bt} \). The Fourier–Laplace transform of \( q(t) \) is given by the formula
\[
\tilde{q}(\omega) = \int_{0}^{+\infty} e^{i\omega t} q(t) \, dt, \quad \Im \omega > B.
\] (3.1)

The Gronwall inequality implies standard a priori estimate for the solutions \( q(x,t), x \geq 1 \). In particular, there exist constants \( A, B < \infty \) such that
\[
\sum_{x \in \mathbb{N}} (|q(x,t)|^2 + |\dot{q}(x,t)|^2) \leq Ce^{Bt} \quad \text{as} \quad t \to +\infty.
\]

Hence, the Fourier–Laplace transform of the solutions \( q(x,t) \) to the problem (2.6), (2.8) with respect to \( t \)-variable, \( q(x,t) \to \tilde{q}(x,\omega) \), exists at least for \( \Im \omega > B \) and satisfies the following equation
\[
(-\nu^2 \Delta_L + m^2 - \omega^2)\tilde{q}(x,\omega) = 0, \quad x \in \mathbb{N}, \quad \Im \omega > B.
\] (3.2)

We construct the solution of (3.2). We first note that the Fourier transform of the operator
\[-\nu^2 \Delta_L + m^2\] is the operator of multiplication by the function
\[
\phi^2(\theta) = \nu^2(2 - 2\cos \theta) + m^2.
\]
Thus, \(-\nu^2 \Delta_L + m^2\) is a self-adjoint operator and its spectrum is absolutely continuous and coincides with the range of \( \phi^2(\theta) \), i.e., with the segment \([m^2, m^2 + 4\nu^2]\).

**Lemma 3.2.** (see Lemma 2.1 in [8]) Denote \( \Lambda := [-\sqrt{4\nu^2 + m^2}, -m] \cup [m, \sqrt{4\nu^2 + m^2}] \). For given \( \omega \in \mathbb{C} \setminus \Lambda \), the equation
\[
\nu^2(2 - 2\cos \theta) = \omega^2 - m^2
\] (3.3)
has the unique solution \( \theta(\omega) \) in the domain \( \{ \theta \in \mathbb{C} : \Im \theta > 0, -\pi < \Re \theta \leq \pi \} \). Moreover, \( \theta(\omega) \) is an analytic function in \( \mathbb{C} \setminus \Lambda \).

Since we seek the solution \( q(\cdot,t) \in \ell^2_{\alpha,+} \) with some \( \alpha \), \( \tilde{q}(x,\omega) \) has a form
\[
\tilde{q}(x,\omega) = \tilde{q}(0,\omega)e^{i\theta(\omega)x}, \quad x \geq 0.
\]
Introduce a function \( \tilde{K}(x,\omega) = e^{i\theta(\omega)x} \). Applying the inverse Fourier–Laplace transform with respect to \( \omega \)-variable, we write the solution \( q(x,t) \) of the problem (2.6), (2.8) in the form
\[
(q(x,t), \dot{q}(x,t)) = \int_{0}^{t} K(x,t-s)(q(0,s), \dot{q}(0,s)) \, ds, \quad x \in \mathbb{N}, \quad t > 0,
\] (3.4)
where
\[
K(x,t) = \frac{1}{2\pi} \int_{-\infty+iy}^{+\infty+iy} e^{-i\omega t} \tilde{K}(x,\omega) \, d\omega, \quad \tilde{K}(x,\omega) = e^{i\theta(\omega)x}, \quad x \in \mathbb{N}, \quad t > 0,
\] (3.5)
with some \( \mu > 0 \). The following theorem was proved in [5].
**Theorem 3.3.** For any \( \alpha < -3/2 \), the following bound holds,

\[
\sum_{x \in \mathbb{N}} \langle x \rangle^{2\alpha} |K(x, t)|^2 \leq C(1 + t)^{-3} \quad \text{for} \quad t > 0.
\]

(3.6)

In particular,

\[
|K(1, t)| \leq C(1 + t)^{-3/2}, \quad t > 0.
\]

(3.7)

To estimate \( q(0, t) \), we use (3.4) and rewrite Eqn (2.7) in the form

\[
\ddot{q}(0, t) = -(\kappa + \nu^2 + m^2) q(0, t) - \gamma \dot{q}(0, t) + \nu^2 \int_0^t K(1, t-s) q(0, s) \, ds + \nu^2 z(1, t), \quad t > 0.
\]

(3.8)

At first, we study the solutions of the corresponding homogeneous equation

\[
\ddot{q}(0, t) = -(\kappa + \nu^2 + m^2) q(0, t) - \gamma \dot{q}(0, t) + \nu^2 \int_0^t K(1, t-s) q(0, s) \, ds, \quad t > 0,
\]

(3.9)

with the initial data

\[
q(0, t)|_{t=0} = u_0(0) =: q_0, \quad \dot{q}(0, t)|_{t=0} = v_0(0) =: p_0.
\]

(3.10)

Applying the Fourier–Laplace transform to the solutions \( q(0, t) \) of (3.9), we obtain

\[
\tilde{q}(0, \omega) = \tilde{N}(\omega) (-i\omega q_0 + q_0 \gamma + p_0) \quad \text{for} \quad \Im \omega > B,
\]

(3.11)

where, by definition, \( \tilde{N}(\omega) := [\tilde{D}(\omega)]^{-1} \) and

\[
\tilde{D}(\omega) := -\omega^2 + \kappa + \nu^2 + m^2 - i\omega \gamma - \nu^2 \tilde{K}(1, \omega), \quad \tilde{K}(1, \omega) = e^{i\theta(\omega)}, \quad \omega \in \mathbb{C}.
\]

(3.12)

The properties of the functions \( \tilde{D}(\omega) \) and \( \tilde{N}(\omega) \) are studied in Appendix A. In particular, we prove that \( \tilde{N}(\omega) \) is an analytic function in the upper half-space. Denote

\[
N(t) = \frac{1}{2\pi} \int_{-\infty + i\mu}^{+\infty + i\mu} e^{-i\omega t} \tilde{N}(\omega) \, d\omega, \quad t \geq 0, \quad \text{with some} \ \mu > 0.
\]

(3.13)

The following theorem is proved in Appendix A.

**Theorem 3.4.** let condition \( C \) or \( C_0 \) hold. Then

\[
|N^{(k)}(t)| \leq C(1 + t)^{-\beta/2}, \quad t \geq 0, \quad k = 0, 1, 2
\]

(3.14)

where \( \beta = 3 \) if \( C \) holds and \( \beta = 1 \) if \( C_0 \) holds.

**Corollary 3.5.** Denote by \( S(t) \) a solving operator of the Cauchy problem (3.9), (3.10). Then the variation constants formula gives the following representation for the solution of the problem (3.8), (3.10):

\[
\begin{pmatrix} q(0, t) \\ \dot{q}(0, t) \end{pmatrix} = S(t) \begin{pmatrix} q_0 \\ p_0 \end{pmatrix} + \int_0^t S(\tau) \begin{pmatrix} 0 \\ \nu^2 z(1, t-\tau) \end{pmatrix} \, d\tau, \quad t > 0.
\]

Evidently, \( S(0) = I \), and the matrix \( S(t) \) has a form \( \begin{pmatrix} \tilde{N}(t) + \gamma \tilde{N}(t) & N(t) \\ \tilde{N}(t) + \gamma \tilde{N}(t) & \tilde{N}(t) \end{pmatrix} \). Moreover, \( |S(t)| \leq C(1 + t)^{-\beta/2} \), by Theorem 3.4.
4 Asymptotic behavior of \( Y(t) \) as \( t \to \infty \)

Set \( q^{(0)}(x, t) = q(x, t) \), \( q^{(1)}(x, t) = \dot{q}(x, t) \), \( x \in \mathbb{Z}_+ \).

**Proposition 4.1.** Let \( Y_0 \in \mathcal{H}_{\alpha,+} \), \( \alpha > 3/2 \), condition \( C \) or \( C_0 \) hold, and \( q(0, t) \) be a solution of the problem (3.8), (3.10). Then

\[
q^{(j)}(0, t) = \nu^2 \langle U_0(t)Y_0, G^j \rangle_+ + r_j(t), \quad t > 0, \quad |r_j(t)| \leq C(t)^{-\beta/2}\|Y_0\|_{\alpha,+}, \quad j = 0, 1, \quad (4.1)
\]

where the functions \( G^j \) are defined in (2.10), the number \( \beta \) is introduced in Theorem 2.4.

**Proof** Corollary 3.5 and the bound (3.14) imply that

\[
q^{(j)}(0, t) = \nu^2 \int_0^t N^{(j)}(\tau) z(1, t - \tau) d\tau + O((1 + t)^{-\beta/2}), \quad t > 0, \quad j = 0, 1.
\]

Moreover, the bounds (2.14) and (3.14) give

\[
\left| \int_0^{+\infty} N^{(j)}(\tau) z(1, t - \tau) d\tau \right| \leq C \int_0^{+\infty} (\tau)^{-\beta/2}(t - \tau)^{-3/2} d\tau \leq C(t)^{-\beta/2}\|Y_0\|_{\alpha,+}.
\]

This implies the representation (4.1), since by (2.15) and (2.11), we have

\[
z(1, t - \tau) = \langle U_0(t)Y_0(\cdot), G^0(\cdot, -\tau) \rangle_+.
\]

**Remark 4.2.** Now we list the properties of the functions \( G^j_1(y, t) \) and \( G^j(y) \), \( j = 0, 1 \).

(i) By (2.12) and (2.13), the function \( G^j_1(y, t) \) is odd w.r.t. \( y \in \mathbb{Z} \). Then the function \( G^j \) is also odd. Formulas (2.13) and the Parseval identity give

\[
\|G^0_1(\cdot, t)\|_0^2 = C \int_0^\pi \left( \cos^2(\phi(t)) + \frac{\sin^2(\phi(t))}{\phi^2(t)} \right) \sin^2(\theta) d\theta \leq C < \infty. \quad (4.2)
\]

Here \( \| \cdot \|_0 \) denotes norm in \( \ell^2 \times \ell^2 \).

(ii) Let condition \( C \) or \( C_0 \) hold. Since \( G^0_1(y, t) = U^0_0(t)(\delta_{1x}, 0) \), then for any \( \alpha > 3/2 \),

\[
\|U^0_0(t)G^j\|_{-\alpha,+} \leq \int_0^{+\infty} |N^{(j)}(s)|\|U^0_0(t - s)(\delta_{1x}, 0)\|_{-\alpha,+} ds \leq C(t)^{-\beta/2}, \quad (4.3)
\]

due to the bound (2.14), because the action of the group \( U^0_0(t) \) coincides with action of the group \( U_0(t) \), up to order of the components. Therefore, for \( \alpha > 3/2 \),

\[
|\langle U_0(t)Y_0, G^j \rangle_+| \leq \|Y_0\|_{\alpha,+}\|U^0_0(t)G^j\|_{-\alpha,+} \leq C(t)^{-\beta/2}\|Y_0\|_{\alpha,+}. \quad (4.4)
\]

(iii) Let condition \( C \) hold. Since \( U^0_0(t)(y, -s) = G^0_1(y, t - s) \), then the bounds (3.14) and (4.2) yield

\[
\sup_{t \in \mathbb{R}}\|U^0_0(t)G^j(\cdot)\|_0 \leq \sup_{t \in \mathbb{R}} \int_0^{+\infty} |N^{(j)}(s)|\|G^0_1(\cdot, t - s)\|_0 ds \leq C \int_0^{+\infty} |N^{(j)}(s)| ds < \infty. \quad (4.5)
\]
Lemma 4.3. Let $Y_0 \in \mathcal{H}_{\alpha,+}$, $\alpha > 3/2$, and condition C or $C_0$ hold. Then the solution $q(x,t)$ of the problem (2.4)–(2.9) with $x \geq 1$, admits the following representation

$$q^{(j)}(x,t) = \nu^2 \langle U_0(t)Y_0, K^j(x,\cdot) \rangle_+ + r_j(x,t), \quad j = 0,1, \quad t > 0,$$

(4.6)

where $K^j$ is introduced in (2.18). $\|r_j(\cdot,t)\|_{\alpha,+} \leq C\langle t \rangle^{-\beta/2} \|Y_0\|_{\alpha,+}$. Here, by definition, $\|r\|^2_{\alpha,+} := \sum_{x \in \mathbb{N}} (x^{-2\alpha} \|r(x)\|^2)$. 

**Proof** At first, by (3.4) and (4.1), we have

$$q^{(j)}(x,t) = \int_0^t K(x,t-s) \langle U_0(s)Y_0, G^j(\cdot) \rangle_+ ds + r_j'(x,t), \quad x \in \mathbb{N},$$

(4.7)

where $\|r_j'(\cdot,t)\|_{\alpha,+} \leq C\langle t \rangle^{-\beta/2}$. Indeed, (11.1) and (3.6) give

$$\|r_j'(\cdot,t)\|_{\alpha,+} = \left\| \int_0^t K(\cdot,t-s)r_j(s) ds \right\|_{\alpha,+} \leq \int_0^t \|K(\cdot,t-s)\|_{\alpha,+} r_j(s) ds,$$

$$\leq C \int_0^t (1 + t - s)^{-3/2}(1 + s)^{-\beta/2} ds \leq C_1\langle t \rangle^{-\beta/2}.$$

Second, the first term in the r.h.s. of (4.7) has a form (see (2.18))

$$\nu^2 \int_0^t K(x,s) \langle U_0(t-s)Y_0, G^j(\cdot) \rangle_+ ds = \nu^2 \langle U_0(t)Y_0, K^j(x,\cdot) \rangle_+ + r_j''(x,t),$$

(4.8)

where, by definition, $r_j''(x,t) := -\nu^2 \int_t^{+\infty} K(x,s) \langle U_0(t-s)Y_0, G^j(\cdot) \rangle_+ ds$. The bounds (3.6) and (4.4) yield

$$\|r_j''(\cdot,t)\|_{\alpha,+} \leq \nu^2 \int_t^{+\infty} \|K(\cdot,s)\|_{\alpha,+} |\langle U_0(t-s)Y_0, G^j(\cdot) \rangle_+| ds \leq C\langle t \rangle^{-\beta/2} \|Y_0\|_{\alpha,+}.$$  

(4.9)

Hence, the bounds (4.7)–(4.9) imply (4.6) with $r_j(x,t) = r_j'(x,t) + r_j''(x,t)$. 

**Remark 4.4.** (i) Set $K(0,\omega) := 1$. Then, formally, $K(0,t) = \delta_{0t}$. Hence, we can put $K^j(0,y) = G^j(y), \ y \in \mathbb{Z}$. Then, the representation (4.11) follows from (4.6).

(ii) By Remark 4.2 and definition (2.18), the function $K^j(x,y)$ is odd w.r.t. $y \in \mathbb{Z}$. Furthermore, formulas (2.18), (3.6) and (4.3) imply that for $\alpha > 3/2$,

$$\left\| \langle U_0(t)K^j(x,\cdot) \rangle_{\alpha,+} \right\|_{\alpha,+} \leq \int_0^{+\infty} \|K(x,s)\|_{\alpha,+} \|U_0(t-s)G^j\|_{\alpha,+} ds < C\langle t \rangle^{-\beta/2}.$$  

Hence, for $\alpha > 3/2$,

$$\|\langle U_0(t)Y_0, K^j(x,\cdot) \rangle_{\alpha,+} \|_{\alpha,+} = \|\langle Y_0(\cdot), U_0(t)K^j(x,\cdot) \rangle_{\alpha,+} \|_{\alpha,+} \leq C\langle t \rangle^{-\beta/2} \|Y_0\|_{\alpha,+}, \ t \in \mathbb{R}.(4.10)$$

In particular, the operator $\Omega$ introduced in (2.19) is bounded.

(iii) If condition C holds, then $\|K^j(x,\cdot)\|_0 \in \mathcal{H}_{-\alpha,+}$ with $\alpha > 3/2$, since

$$\left\| \|K^j(x,\cdot)\|_0 \right\|_{\alpha,+} \leq \int_0^{+\infty} \|K(x,s)\|_{\alpha,+} \|U_0(t-s)G^j\|_0 ds < \infty$$

by virtue of (2.18), (3.6) and (4.3). Therefore, (2.19) implies that for $Y \in \mathcal{H}_{0,+}$,

$$\|\Omega Y\|_{\alpha,+} \leq \|Y\|_{\alpha,+} + \nu^2 \sum_{j=0,1} \|\langle Y(\cdot), K^j(x,\cdot) \rangle_{\alpha,+} \|_{\alpha,+} \leq C\|Y\|_{0,+}.$$
Proof of Theorem 2.4 The item (i) follows from the representations (2.2), (4.11) and (4.6). Further, definition (2.19), the bounds (2.14), (4.1) and (4.10) give
\[ \|\Omega(U_0(t)Y_0)\|_{-a,+} \leq \|U_0(t)Y_0\|_{-a,+} + \nu^2 \sum_{j=0}^\infty \|\langle U_0(t)Y_0, K_j(x,\cdot)\rangle_+\|_{-a,+} \]
\[ \leq C(t)^{-\beta/2}\|Y_0\|_{a,+}, \quad \alpha > 3/2. \quad (4.11) \]
Thus, the bound (1.5) follows from the part (i) of Theorem 2.4 and the bound (4.11).

Remark 4.5. Let condition C hold. From the proofs of Lemmas 4.1 and 4.3 we see that the remainders \( r_j(t) \) and \( r_j(x,t) \) in decompositions (4.1) and (4.6) are estimated by \( z(1,t) \),
\[ |r_j(t)| \leq C_1|Y_0(0)|\langle t\rangle^{-3/2} + C_2 \int_t^{+\infty} \langle \tau\rangle^{-3/2}|z(1,t-\tau)|d\tau. \]
\[ \|r_j(\cdot,t)\|_{-a,+} \leq C_1|Y_0(0)|\langle t\rangle^{-3/2} + C_2 \int_0^t \langle t-s\rangle^{-3/2}ds \int_s^{+\infty} \langle \tau\rangle^{-3/2}|z(1,s-\tau)|d\tau \]
\[ + C_3 \int_t^{+\infty} \langle s\rangle^{-3/2}ds \int_0^{+\infty} \langle \tau\rangle^{-3/2}|z(1,t-s-\tau)|d\tau. \]
Hence, if \( \sup_{t\in \mathbb{R}}|z(1,t)| =:M_0 < \infty \), then \( U(t)Y_0 = \Omega(U_0(t)Y_0) + r(x,t) \), where
\[ \|r(\cdot,t)\|_{-a,+} \leq C_1|Y_0(0)|\langle t\rangle^{-3/2} + C M_0(\nu^2)^{-1/2} \leq C(t)^{-1/2}. \]
For instance, if initial data \( Y_0(x) \) are such that \( \hat{u}_{\text{odd}}(\theta), \hat{v}_{\text{odd}}(\theta)/\phi(\theta) \in L^1(\mathbb{T}) \), where \( Y_{\text{odd}}(x) = (u_{\text{odd}}(x),v_{\text{odd}}(x)) \) is defined in (3.2), then \( |z(1,t)| \leq C < \infty \). In particular, this is true if \( m \neq 0 \) and \( Y_0 \in \mathcal{H}_{a,+} \).

Remark 4.6. If conditions C and \( C_0 \) are not fulfilled, then the bound (1.5) for any initial data \( Y_0 \in \mathcal{H}_{a,+} \) is incorrect. Indeed, if \( m = \kappa = 0 \), then \( \bar{N}(\omega) \) has a simple pole at zero, and any constant is a solution of the system (1.1)–(1.2). If \( \gamma = 0 \) and \( \kappa > 2\nu^2 \), then there exists a number \( \omega_0 > \sqrt{4\nu^2 + m^2} \) such that \( \bar{D}(\omega_0) = 0 \), and \( \bar{N}(\omega) \) has simple poles at the points \( \omega = \pm \omega_0 \), see Remark A.3 below. Therefore, a function of the form \( u(x,t) = e^{i\theta(x)\omega} \sin(\omega_0 t) \) is a solution of the system, where \( \theta(\omega) \) is the solution of (3.3), \( \Re\theta(\omega_0) = \pi, \Im\theta(\omega_0) > 0 \). If one of the following two conditions holds: (1) \( m = 0, \kappa = 2(\nu^2 - \gamma^2) \) and \( \gamma \in (0,\nu) \), or (2) \( m \neq 0, \kappa = \nu^2 - \gamma^2 \pm \sqrt{(\nu^2 - \gamma^2)^2 - m^2}\gamma^2 \) and \( \gamma \in (0,\sqrt{m^2 + 4\nu^2 - m}/2] \), then there exist points \( \omega_0 \in \Lambda \setminus \Lambda_0 \) such that \( \bar{D}(\omega_0 - i0) = 0 \) (see item (iv) of Lemma A.2). We denote \( \theta_+ := \lim_{\varepsilon \to +0} \theta(\omega_0 + i\varepsilon), \theta_+ \in \mathbb{R} \). Then the function of the form \( u(x,t) = \sin(\theta_+ x + \omega t) \) is a solution of the system (1.1)–(1.2).

Appendix A: Properties of \( \bar{D}(\omega) \) and \( \bar{N}(\omega) \) for \( \omega \in \mathbb{C} \)

Let \( \Lambda := [-\sqrt{4\nu^2 + m^2},-m] \cup [m,\sqrt{4\nu^2 + m^2}] \). \( \Lambda_0 = \{\pm m, \pm \sqrt{4\nu^2 + m^2} \} \) denotes the set of the “spectral edges”. We first list the properties of the function \( e^{i\theta(\omega)} \) for \( \omega \in \mathbb{C} \setminus \Lambda \), \( \omega \in \Lambda \setminus \Lambda_0 \), and \( \omega \in \Lambda_0 \).

Let \( \omega \in \mathbb{C} \setminus \Lambda \). Then \( \Im\theta(\omega) > 0 \) and \( e^{i\theta(\omega)} \) is an analytic function. Moreover, by (3.3) and the condition \( \Im\theta(\omega) > 0 \), we have
\[ |e^{i\theta(\omega)}| \leq C|\omega|^{-2} \quad \text{as} \quad |\omega| \to \infty. \]
For $\omega \in \Lambda \setminus \Lambda_0$, put $\theta(\omega \pm i0) = \lim_{\varepsilon \to +0} \theta(\omega \pm i\varepsilon)$. Since $\overline{\theta(\omega)} = -\theta(\bar{\omega})$ for $\omega \in \mathbb{C} \setminus \Lambda$, then $e^{i\theta(\omega-i0)} = e^{i\theta(\omega+i0)}$ for $\omega \in \Lambda \setminus \Lambda_0$.

We study the behavior of $e^{i\theta(\omega)}$ near the points in the set $\Lambda_0$. From Eqn (3.3) we have
\[ e^{i\theta(\omega)} = \cos \theta(\omega) + i \sin \theta(\omega) = 1 - \frac{1}{2\nu^2}(\omega^2 - m^2) + \frac{i}{2\nu^2} \sqrt{(\omega^2 - m^2)(4\nu^2 + m^2 - \omega^2)} \quad \text{(A.2)} \]
for $\omega \in \mathbb{C} \setminus \Lambda$. The Taylor expansion implies
\[ e^{i\theta(\omega)} = 1 + \frac{i}{\nu} \sqrt{\omega^2 - m^2} - \frac{1}{2\nu^2}(\omega^2 - m^2) - \frac{i}{8\nu^3}(\omega^2 - m^2)^{3/2} + \ldots \quad \text{as } \omega \to \pm m + i0, \quad \text{(A.3)} \]
where $\omega \in \mathbb{C}_+ := \{\omega \in \mathbb{C} : \Im \omega > 0\}$, $\Im \sqrt{\omega^2 - m^2} > 0$. Here $\text{sgn}(\Re \sqrt{\omega^2 - m^2}) = \text{sgn}(\Re \omega)$ for $\omega \in \mathbb{C}_+$. This choice of the branch of the complex root $\sqrt{\omega^2 - m^2}$ follows from the condition $\Im \theta(\omega) > 0$. Similarly,
\[ e^{i\theta(\omega)} = -1 + \frac{i}{\nu} \sqrt{\omega^2 + 4\nu^2 - \omega^2} + \frac{1}{2\nu^2}(m^2 + 4\nu^2 - \omega^2) - \frac{i}{8\nu^3}(m^2 + 4\nu^2 - \omega^2)^{3/2} + \ldots \quad \text{(A.4)} \]
as $\omega \to \pm \sqrt{m^2 + 4\nu^2}, \quad \omega \in \mathbb{C}_+$. Here the branch of the complex root $\sqrt{m^2 + 4\nu^2 - \omega^2}$ is chosen so that $\text{sgn}(\Re \sqrt{m^2 + 4\nu^2 - \omega^2}) = \text{sgn}(\Re \omega)$ that follows from the condition $\Im \theta(\omega) > 0$. If $m = 0$, then (A.2) and the Taylor expansion imply
\[ e^{i\theta(\omega)} = 1 + \frac{i\omega}{\nu} - \frac{\omega^2}{2\nu^2} - \frac{i\omega^3}{8\nu^3} + \ldots \quad \text{as } \omega \to 0, \quad \text{(A.5)} \]
and $e^{i\theta(\omega)} = -1 + i\sqrt{4\nu^2 - \omega^2}/\nu + \ldots$ as $\omega \to \pm 2\nu$, $\omega \in \mathbb{C}_+$.

**Lemma A.1.** (i) $\tilde{N}(\omega)$ is meromorphic for $\omega \in \mathbb{C} \setminus \Lambda$.
(ii) $|\tilde{N}(\omega)| = O(|\omega|^{-2})$ as $|\omega| \to \infty$.
(iii) $\tilde{D}(\omega) \neq 0$ for all $\omega \in \mathbb{C}_+ = \{\omega \in \mathbb{C} : \Im \omega > 0\}$.
(iv) If $\gamma = 0$, then $\tilde{D}(\omega) \neq 0$ for any $\omega \in \mathbb{C}_- = \{\omega \in \mathbb{C} : \Im \omega < 0\}$.

**Proof** The first assertion of the lemma follows from the analyticity of $\tilde{D}(\omega)$ for $\omega \in \mathbb{C} \setminus \Lambda$. The assertion (ii) follows from (3.12) and (A.1). To prove the third assertion, we assume opposite that $\tilde{D}(\omega_0) = 0$ for some $\omega_0 \in \mathbb{C}_+$. Hence, the function $u_s(x,t) = e^{i\theta(\omega_0)x}e^{-i\omega_0t}, \quad x \geq 0, \quad t \geq 0$, is a solution of the problem (1.1)–(1.2) with the initial data $Y_s = e^{i\theta(\omega_0)x}(1,-i\omega_0)$. Therefore, the Hamiltonian (1.3) is
\[ H(u_s(\cdot,t),\dot{u_s}(\cdot,t)) = e^{2t\Im \omega_0}H(Y_s) \quad \text{for any } t > 0, \quad \text{where } H(Y_s) > 0. \]
Since $\Im \omega_0 > 0$ and $Y_s \in H_{0,+}$, this exponential growth contradicts the energy estimate (2.1). Hence, $\tilde{D}(\omega) \neq 0$ for any $\omega \in \mathbb{C}_+$. If $\gamma = 0$, then $\tilde{D}(\omega) \neq 0$ for $\omega \in \mathbb{C}_+$.

**Lemma A.2.** Let the condition C or $C_0$ hold. Then $\tilde{D}(\omega) \neq 0$ for $\omega \in \mathbb{R} \setminus \Lambda$, $\tilde{D}(\omega \pm i0) \neq 0$ for $\omega \in \Lambda \setminus \Lambda_0$. \qed
Proof (i) Let $\omega \in \mathbb{R}$ and $|\omega| > \sqrt{4\nu^2 + m^2}$. Then $\Re \theta(\omega) = \pm \pi$. Therefore, 
\[
\tilde{D}(\omega) = -\omega^2 + \kappa + \nu^2 + m^2 - i\omega \gamma + \nu^2 e^{-\Im \theta(\omega)} \quad \text{with} \quad \Im \theta(\omega) > 0.
\]
Hence, $\Im \tilde{D}(\omega) \neq 0$ iff $\gamma \neq 0$. On the other hand, $\Re \tilde{D}(\omega) = \kappa - 2\nu^2$ for $\omega = \pm \sqrt{4\nu^2 + m^2}$, and $\Re \tilde{D}(\omega_1) < \Re \tilde{D}(\omega_2)$ if $|\omega_1| > |\omega_2| \geq \sqrt{4\nu^2 + m^2}$. In particular, $\Re \tilde{D}(\omega) \to -\infty$ as $|\omega| \to \infty$. Hence, for $|\omega| > \sqrt{4\nu^2 + m^2}$, $\Re \tilde{D}(\omega) \neq 0$ iff $\kappa \leq 2\nu^2$. Therefore, for such values of $\omega$, $\tilde{D}(\omega) \neq 0$ iff either $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \leq 2\nu^2$.

(ii) Let $m \neq 0$ and $\omega \in (-m,m)$. Then, $\Re \theta(\omega) = 0$. Hence, 
\[
\Re \tilde{D}(\omega) = -\omega^2 + \kappa + \nu^2 + m^2 - \nu^2 e^{i\theta(\omega)} > \kappa \quad \text{for} \quad |\omega| < m,
\]
and $\Re \tilde{D}(\pm m) = \kappa$. Therefore, $\tilde{D}(\omega) \neq 0$ for any $|\omega| < m$, since $\kappa \geq 0$.

(iii) Let $\omega \in (-\sqrt{4\nu^2 + m^2}, -m) \cup (m, \sqrt{4\nu^2 + m^2})$. Then $\Re \theta(\omega + i0) \in (-\pi, 0) \cup (0, \pi)$ and $\Im \theta(\omega + i0) = 0$. Moreover, $\text{sign}(\text{sign}(\theta(\omega + i0))) = \text{sign} \omega$. Hence, for $m \neq 0$,
\[
\Im \tilde{D}(\omega + i0) = -\omega \gamma - \nu^2 \sin \theta(\omega + i0) = -\text{sign}(\omega) \left(|\omega| \gamma + \sqrt{\omega^2 - m^2 \nu^2} \right) 
eq 0.
\]
If $m = 0$, then $D(\omega + i0) = \kappa - \omega^2/2 - i\omega \left(\gamma + \sqrt{\nu^2 - \omega^2}/4\right) \neq 0$ for any $\kappa, \gamma \geq 0$.

(iv) Since $\tilde{D}(\omega - i0) = D(\omega + i0) = 2i\omega \gamma$, for $\omega \in \Lambda \setminus \Lambda_0$, then 
\[
\tilde{D}(\omega - i0) = -\omega^2 + \kappa + \nu^2 + m^2 - \nu^2 \cos \theta(\omega + i0) + i\nu^2 \sin \theta(\omega + i0) - i\omega \gamma
\]
\[
= \kappa - (\omega^2 - m^2)/2 + i \left(\text{sign}(\omega) \frac{1}{2} \sqrt{\omega^2 - m^2 \nu^2} \sqrt{4\nu^2 + m^2 - \omega^2 - \omega \gamma} \right)
\]
for $\omega \in \Lambda \setminus \Lambda_0$. Hence, $\tilde{D}(\omega - i0) = 0$ for $\omega \in \Lambda \setminus \Lambda_0$ iff 
\[
\kappa = (\omega^2 - m^2)/2 \quad \text{and} \quad \sqrt{\omega^2 - m^2 \nu^2} \sqrt{4\nu^2 + m^2 - \omega^2} = 2|\omega| \gamma, \quad \omega^2 \in (m^2, m^2 + 4\nu^2).
\]
Then, $\gamma \neq 0$. Put $P := \omega^2 - m^2$. Hence, $P$ is a solution of the following equation
\[
P^2 + 4P(\gamma^2 - \nu^2) + 4m^2 \gamma^2 = 0, \quad P \in (0, 4\nu^2).
\]
If $m = 0$, then Eqn (A.7) has a unique solution $P = 4(\nu^2 - \gamma^2) \in (0, 4\nu^2)$ iff $\gamma < \nu$. Then, $\kappa = (\omega^2 - m^2)/2 = P/2 = 2(\nu^2 - \gamma^2)$ by the first equation in (A.6). Thus, if $m = 0$, $\kappa = 2(\nu^2 - \gamma^2)$ and $\gamma \in (0, \nu)$, then there exist two points $\omega = \pm \omega_* = \pm 2\sqrt{\nu^2 - \gamma^2} \in \Lambda \setminus \Lambda_0$ such that $\tilde{D}(\omega_* - i0) = 0$.

If $m \neq 0$, then (A.7) has a solution iff $(\gamma^2 - \nu^2)^2 - m^2 \gamma^2 \geq 0$ and $\gamma \in (0, \nu)$, which is equivalent to the conditions $\gamma^2 + m\gamma - \nu^2 \leq 0$ and $\gamma \in (0, \nu)$, which coincides with the inequality $\gamma \in \left(0, \frac{\sqrt{m^2 + 4\nu^2} - m}{2}\right]$. Therefore, if $m \neq 0$ and $\gamma \in \left(0, \frac{\sqrt{m^2 + 4\nu^2} - m}{2}\right]$, then Eqn (A.7) has solutions
\[
P = 2(\nu^2 - \gamma^2) \pm 2\sqrt{(\nu^2 - \gamma^2)^2 - m^2 \gamma^2} \in (0, 4\nu^2).
\]
Hence, $\kappa = (\omega^2 - m^2)/2 = P/2 = \nu^2 - \gamma^2 \pm \sqrt{(\nu^2 - \gamma^2)^2 - m^2 \gamma^2}$.

Thus, there are points $\omega_* \in \Lambda \setminus \Lambda_0$, in which $\tilde{D}(\omega_* - i0) = 0$, iff $\gamma \neq 0$ and one of the following conditions is fulfilled: (1) $m = 0$, $\kappa = 2(\nu^2 - \gamma^2)$ and $\gamma \in (0, \nu)$; (2) $m \neq 0$, $\kappa = \nu^2 - \gamma^2 \pm \sqrt{(\nu^2 - \gamma^2)^2 - m^2 \gamma^2}$ and $\gamma \in \left(0, \frac{\sqrt{m^2 + 4\nu^2} - m}{2}\right]$. These values of $\kappa, m, \gamma$ are eliminated by the condition C.
**Remark A.3.** If condition C holds, then $\tilde{D}(\omega) \neq 0$ for $\omega \in \Lambda_0$, because $\tilde{D}(\pm \sqrt{4\nu^2 + m^2}) = \kappa - 2\nu^2 \mp i\gamma \sqrt{4\nu^2 + m^2}$ and $\tilde{D}(\pm m) = \kappa \mp i\gamma m$. If condition C is not satisfied, then there are points $\omega \in \mathbb{R}$, in which $\tilde{D} = 0$. For example, $\tilde{D}(0) = 0$ in the case $m = \kappa = 0$. If $\gamma = \kappa = 0$, then $\tilde{D}(\pm m) = 0$. If $\gamma = 0$ and $\kappa = 2\nu^2$, then $\tilde{D}(\pm \sqrt{m^2 + 4\nu^2}) = 0$. If $\gamma = 0$ and $\kappa > 2\nu^2$, then $\exists \omega_0 > \sqrt{4\nu^2 + m^2}$ such that $\tilde{D}(\pm \omega_0) = 0$, and $\tilde{D}'(\omega_0) = -2\omega_0(\kappa - \nu^2)/(2\kappa + m^2 - \omega_0^2) < 0$.

Now we study the asymptotic behavior of $\tilde{D}(\omega)$ and $\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1}$ near the points $\omega \in \Lambda_0$. In the neighborhood of the points $\omega = \pm \sqrt{4\nu^2 + m^2}$ we use the representation (A.4) and obtain

$$
\tilde{D}(\omega) = \kappa - 2\nu^2 \mp i\gamma \sqrt{4\nu^2 + m^2} - i\nu(4\nu^2 + m^2 - \omega^2)^{1/2} + \ldots
$$

as $\omega \to \pm \sqrt{4\nu^2 + m^2}$, $\omega \in \mathbb{C}_+$. Therefore, if $\gamma \neq 0$ or $\gamma = 0$ and $\kappa \neq 2\nu^2$, then

$$
\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1} = C_1 + i C_2 \sqrt{4\nu^2 + m^2 - \omega^2} + \ldots, \quad \omega \to \pm \sqrt{4\nu^2 + m^2}, \quad \omega \in \mathbb{C}_+,
$$

where $C_1 = (\kappa - 2\nu^2 \mp i\gamma \sqrt{4\nu^2 + m^2})^{-1}$ and $C_2 = \nu C_1^2$. If $\gamma = 0$ and $\kappa = 2\nu^2$, then

$$
(\tilde{D}(\omega))^{-1} = \frac{i}{\nu}(4\nu^2 + m^2 - \omega^2)^{-1/2} + \frac{1}{2\nu^2} + \ldots, \quad \omega \to \pm \sqrt{4\nu^2 + m^2}.
$$

In the neighborhood of the points $\omega = \pm m$ we apply (A.3) (if $m \neq 0$) and obtain

$$
\tilde{D}(\omega) = \kappa + im\gamma - i\nu \sqrt{\omega^2 - m^2} - i(\omega \mp m)\gamma - \frac{1}{2}(\omega^2 - m^2) + \ldots, \quad \omega \to \pm m, \quad \omega \in \mathbb{C}_+.
$$

(A.9)

In the case when $m = 0$, (A.5) yields

$$
\tilde{D}(\omega) = \kappa - i\omega(\gamma + \nu) - \frac{1}{2}\omega^2 + \frac{i}{8\nu} \omega^3 + \ldots, \quad \omega \to 0.
$$

(A.10)

Suppose that $m\gamma \neq 0$ or $\kappa \neq 0$. Then, by virtue of (A.9) and (A.10), we obtain

$$
\tilde{N}(\omega) = (\tilde{D}(\omega))^{-1} = \begin{cases} 
1/\kappa + i\omega(\gamma + \nu)/\kappa^2 + \ldots, & \omega \to 0, \quad m = 0, \\
C_3 + i C_4 (\omega^2 - m^2)^{1/2} + \ldots, & \omega \to \pm m, \quad m \neq 0,
\end{cases} \quad \omega \in \mathbb{C}_+,
$$

where $C_3 = (\kappa + im\gamma)^{-1}$ and $C_4 = \nu C_3^2$. If $\gamma = \kappa = 0$ and $m \neq 0$, then

$$
\tilde{N}(\omega) = \frac{i}{\nu}(\omega^2 - m^2)^{-1/2} - \frac{1}{2\nu^2} + \ldots, \quad \omega \to \pm m, \quad \omega \in \mathbb{C}_+.
$$

If $\kappa = m = 0$, then $\tilde{N}(\omega) = i\omega^{-1}/(\gamma + \nu) - 1/(2(\gamma + \nu)^2) + \ldots$ as $\omega \to 0$.

Since $\tilde{N}(\omega) = (\tilde{D}(\omega) - 2i\omega\gamma)^{-1}$ for $\omega \in \mathbb{C}_-$, then the expansion for $\tilde{N}(\omega)$ as $\omega \to \omega_0$ ($\omega_0 \in \Lambda_0$, $\omega \in \mathbb{C}_-$) can be constructed using (A.8) and (A.9). In particular,

$$
\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0) = O(|\omega^2 - \omega_0^2|^{j/2}) \quad \text{as} \quad \omega \to \omega_0, \quad \omega_0 \in \Lambda_0,
$$

(A.11)

where $j = 1$ if the condition C is satisfied, and $j = -1$ if the condition C_0 is satisfied.
Proof of Theorem 3.4 Using Lemma [A.1] we vary the integration contour in the right hand side of (3.13):

\[
N(t) = - \frac{1}{2\pi} \int_{|\omega|=R} e^{-i\omega t} \tilde{N}(\omega) \, d\omega, \quad t > 0, \tag{A.12}
\]

where \( R \) is chosen enough large such that \( \tilde{N}(\omega) \) has no poles in the region \( \mathbb{C}_- \cap \{|\omega| \geq R\} \). Note that if \( \gamma = 0 \), then \( \tilde{N}(\omega) \) has no poles in \( \mathbb{C}_- \) by Lemma [A.1] (iv). Denote by \( \sigma_j \) the poles of \( \tilde{N}(\omega) \) in \( \mathbb{C}_- \) (if they exist). By Lemmas [A.1] and [A.2] there exists a \( \delta > 0 \) such that \( \tilde{N}(\omega) \) has no poles in the region \( \Im \omega \in [-\delta, 0) \). Hence, we can rewrite \( N(t) \) as

\[
N(t) = -i \sum_{j=1}^{\kappa} \text{Res}_{\omega=\sigma_j} \left[ e^{-i\omega t} \tilde{N}(\omega) \right] - \frac{1}{2\pi} \int_{\Lambda_\varepsilon} e^{-i\omega t} \tilde{N}(\omega) \, d\omega, \quad t > 0,
\]

where \( \varepsilon \in (0, \delta) \), the contour \( \Lambda_\varepsilon \) surrounds segments of \( \Lambda \) and belongs to an \( \varepsilon \)-neighborhood of \( \Lambda \) (\( \Lambda_\varepsilon \) is oriented anticlockwise). Passing to a limit as \( \varepsilon \to 0 \), we obtain

\[
N(t) = \frac{1}{2\pi} \int_{\Lambda} e^{-i\omega t} \left( \tilde{N}(\omega + i0) - \tilde{N}(\omega - i0) \right) \, d\omega + o(t^{-N})
\]

\[
= \sum_{\pm} \sum_{j=1}^{2} \frac{1}{2\pi} \int_{\Lambda} e^{-i\omega t} P^\pm_j(\omega) \, d\omega + o(t^{-N}), \quad t \to +\infty, \quad \text{with any } N > 0.
\]

Here \( P^\pm_j(\omega) := \zeta^\pm_j(\omega)(\tilde{N}(\omega + i0) - \tilde{N}(\omega - i0)) \), \( j = 1, 2 \), where \( \zeta^\pm_j(\omega) \) are smooth functions such that \( \sum_{\pm,j} \zeta^\pm_j(\omega) = 1 \), \( \omega \in \mathbb{R} \), supp \( \zeta^\pm_1 \subset O(\pm m) \), supp \( \zeta^\pm_2 \subset O(\pm \sqrt{4\nu^2 + m^2}) \) (\( O(a) \) denotes a neighborhood of the point \( \omega = a \)). In the case \( m = 0 \), instead of \( \zeta^\pm_1 (P^\pm_1) \) we introduce the function \( \zeta_1 \) (respectively, \( P_1 \)) with supp \( \zeta_1 \subset O(0) \). Then, ([A.11]) implies the bound (3.14) with \( k = 0 \). Here we use the following estimate (with \( j = \pm 1 \))

\[
\left| \int_{\mathbb{R}} \zeta(\omega)e^{-i\omega t}(a^2 - \omega^2)^{j/2} \, d\omega \right| \leq C(1 + t)^{-1-j/2} \quad \text{as } t \to +\infty, \quad j \text{ is odd}, \tag{A.13}
\]

where \( \zeta(\omega) \) is a smooth function, and \( \zeta(\omega) = 1 \) for \( |\omega - a| \leq \delta \) with some \( \delta > 0 \) (see, for example, [11] Lemma 2). The bound (3.14) with \( k = 1, 2 \) can be proved by a similar way.

Remark A.4. If conditions C and \( C_0 \) are not fulfilled, then \( N(t) \) does not decay as \( t \to \infty \). For example, if \( \kappa = m = 0 \), then \( \tilde{N}(\omega) \) has a simple pole at zero. Calculating the residue of \( \tilde{N}(\omega) \) at the point \( \omega = 0 \), we obtain \( N(t) = (\gamma + \nu)^{-1} \), \( t \to \infty \).

If \( \gamma = 0 \) and \( \kappa > 2\nu^2 \), then there exists a number \( \omega_0 > \sqrt{4\nu^2 + m^2} \) such that \( \tilde{N}(\omega) \) has simple poles at the points \( \omega = \pm \omega_0 \). Calculating the residue of \( e^{-i\omega t} \tilde{N}(\omega) \) at these points, we obtain \( N(t) \sim C \sin(\omega_0 t) + O(t^{-3/2}) \) as \( t \to \infty \).
Appendix B: Proof of Theorem 2.4

Consider the mixed initial-boundary value problem (2.3)–(2.5). Without loss of generality, we assume that \( u_0(0) = v_0(0) = 0 \). Write \( Z(x, t) = (Z^0(x, t), Z^1(x, t)) \equiv (z(x, t), \dot{z}(x, t)) \), \( Y_0(x) = (u_0(x), v_0(x)) \). The solution of problem (2.3)–(2.5) can be represented as the restriction of the solution to the Cauchy problem with odd initial data on the half-line,

\[
Z^i(x, t) = \sum_{y \in \mathbb{Z}} G^i_t(x - y) Y^j_{\text{odd}}(y), \quad x \geq 0, \quad i = 0, 1,
\]

where \( G_t(x) \) is defined in (2.12) and (2.13), and, by definition,

\[
Y_{\text{odd}}(x) = Y_0(x) \quad \text{for } x > 0, \quad Y_{\text{odd}}(0) = 0, \quad Y_{\text{odd}}(x) = -Y_0(-x) \quad \text{for } x < 0.
\]

To prove Theorem 2.3 we first consider the following Cauchy problem for the discrete Klein–Gordon equation in the whole line,

\[
\begin{aligned}
&\ddot{u}(x, t) - (\nu^2 \Delta_L - m^2) u(x, t), \quad t \in \mathbb{R}, \quad x \in \mathbb{Z}, \\
&u(x, t)|_{t=0} = u_0(x), \quad \dot{u}(x, t)|_{t=0} = v_0(x).
\end{aligned}
\]

(B.3)

By \( \ell^2_\alpha \equiv \ell^2_\alpha(\mathbb{Z}) \), \( \alpha \in \mathbb{R} \), we denote the Hilbert space of sequences with the norm \( \|u\|_{\ell^2_\alpha}^2 = \sum_{x \in \mathbb{Z}} |u(x)|^2 (x)^{2\alpha} < \infty \). Let \( \mathcal{H}_\alpha := \ell^2_\alpha \otimes \ell^2_\alpha \) be the Hilbert space of pairs \( Y = (u, v) \) with the norm \( \|Y\|_{\ell^2_\alpha}^2 = \sum_{x \in \mathbb{Z}} \langle x \rangle^{2\alpha} (|u(x)|^2 + |v(x)|^2) < \infty \).

It is well-known (see for instance, [3]), that for any \( Z_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha \), there exists a unique solution \( W(t)Z_0 \in C(\mathbb{R}, \mathcal{H}_\alpha) \) to the problem (B.3). Moreover, there exist constants \( C, \sigma = \sigma(\alpha) < \infty \) such that the following bound holds,

\[
\|W(t)Z_0\|_{\mathcal{H}_\alpha} \leq C(t)^{\sigma}\|Z_0\|_{\mathcal{H}_\alpha}, \quad t \in \mathbb{R}, \quad \alpha \in \mathbb{R}.
\]

(B.4)

Lemma B.1. Let \( Z_0 \equiv (u_0, v_0) \in \mathcal{H}_\alpha \) with \( \alpha > 5/2 \). If \( \hat{Z}_0(0) = \hat{Z}_0(\pi) = 0 \), then

\[
\|W(t)Z_0\|_{-\alpha} \leq C(t)^{-3/2}\|Z_0\|_{\mathcal{H}_\alpha}, \quad t \in \mathbb{R}.
\]

(B.5)

Otherwise, \( \|W(t)Z_0\|_{-\alpha} \leq C(t)^{-1/2}\|Z_0\|_{\mathcal{H}_\alpha}, \quad t \in \mathbb{R} \).

Below we outline the proof of this lemma.

By the bound (B.4), the Laplace–Fourier transform of the solution \( u(x, t) \) with respect to \( t \)-variable exists at least for \( 3\omega > 0 \) and satisfies equation (3.2) for \( x \in \mathbb{Z}, \; 3\omega > 0 \). Let \( u \) be a solution of the equation \( (-\nu^2 \Delta_L + m^2 - \omega^2) \nu = f \) with \( f \in \ell^2 \). Define the resolvent operator \( R_\omega \) as \( u = R_\omega f = (-\nu^2 \Delta_L + m^2 - \omega^2)^{-1} f \).

Applying the inverse Fourier–Laplace transform with respect to \( \omega \)-variable, we write the solution \( u(x, t) \) of the problem (B.3) in the form

\[
u(x, t) = \frac{1}{2\pi} \int_{3\omega=\mu} e^{-i\omega t} R_\omega (v_0(x) - i\omega u_0(x)) \, d\omega, \quad x \in \mathbb{Z}, \quad t > 0, \quad \mu > 0.
\]

(B.6)

To derive the asymptotic behavior of \( u(x, t) \), we first study the properties of the operator \( R_\omega \) for \( \omega \in \mathbb{C} \), see [6, 10, 8]. To formulate them, we denote by \( B(\alpha, \alpha') = \mathcal{L}(\ell^2_\alpha, \ell^2_{\alpha'}) \) the
space of bounded linear operators from $\ell^2_{x}$ to $\ell^2_{y}$.

**I.** For $\omega \in \mathbb{C} \setminus \Lambda$, the resolvent $R_\omega$ is the integral operator with the kernel $R_\omega(x, y)$, $x, y \in \mathbb{Z}$, and by the Cauchy Residue Theorem, we have

$$R_\omega(x, y) = \frac{1}{2\pi i} \int_{\gamma} e^{-i\theta(x-y)} \frac{e^{i\theta|\omega-\theta|}}{\nu^2(2 - 2\cos \theta) + m^2 - \omega^2} d\theta = \frac{i e^{i\theta|\omega-\theta|}}{2\nu^2 \sin(\theta(\omega))}, \quad \omega \in \mathbb{C} \setminus \Lambda,$$

where $\theta(\omega)$ is defined in Lemma 3.2. Therefore, for $\omega \in \mathbb{C} \setminus \Lambda$, the resolvent $R_\omega$ is an analytic operator-valued function in the complex $\omega$-plane with the cut along the intervals $\Lambda$. Moreover, the sequence $\{e^{-i\theta|\omega|}\}$, $x \in \mathbb{Z}$, is exponentially decaying as $|x| \to \infty$. Hence for $\omega \in \mathbb{C} \setminus \Lambda$, $R_\omega$ is a bounded operator in $\ell^2(\mathbb{Z})$.

**II.** Write $\theta(\omega \pm i\varepsilon) := \lim_{\varepsilon \to 0} \theta(\omega \pm i\varepsilon)$. For $\omega \in \Lambda \setminus \Lambda_0$ and $x, y \in \mathbb{Z}$, the following pointwise limit exists $R_{\omega \pm i\varepsilon}(x, y) \to R_{\omega \pm i0}(x, y)$ as $\varepsilon \to 0$. Moreover, $|\theta(\omega \pm i\varepsilon)| \leq C(\omega)$ and $|\sin(\theta(\omega \pm i\varepsilon))| > 0$ for $\omega \in \Lambda \setminus \Lambda_0$. Hence, $|R_{\omega \pm i\varepsilon}(x, y)| \leq C(\omega)$ for $\omega \in \Lambda \setminus \Lambda_0$. Therefore, for any $\alpha > 1/2$ and $\omega \notin \Lambda_0$, have

$$\sum_{x, y \in \mathbb{Z}} |R_{\omega \pm i\varepsilon}(x, y) - R_{\omega \pm i0}(x, y)|^2 \langle x \rangle^{-2\alpha} \langle y \rangle^{-2\alpha} \to 0, \quad \varepsilon \to 0,$$

by the Lebesgue dominated convergence theorem. Thus, for $\omega \in \Lambda \setminus \Lambda_0$, the resolvent $R_{\omega \pm i\varepsilon}$ converges to $R_{\omega \pm i0}$ as Hilbert–Schmidt operator in the space $B(\alpha, -\alpha)$, $\alpha > 1/2$. Moreover, $\theta(\omega) = -\theta(\bar{\omega})$ for $\omega \in \mathbb{C} \setminus \Lambda$. Hence, $R_{\omega \pm i\varepsilon}(x, y) = R_{\omega \pm i0}(x, y)$ for $\omega \in \Lambda \setminus \Lambda_0$, $x, y \in \mathbb{Z}$.

**III.** The operator $R_{\omega \pm i0}$ diverges near points $\omega \in \Lambda_0$ because $\sin \theta(\omega \pm i0)$ vanishes in these points. Using formula (B.7) and decompositions (A.3)–(A.5), we obtain a formal Puiseux expansion of $R_\omega$ as $\omega \to \omega_0$, $\omega \in \mathbb{C} \setminus \Lambda$, $\omega_0 \in \Lambda_0$. Indeed, for $\omega \to \pm m$ ($m \neq 0$, $\omega \in \mathbb{C}_+$), we have

$$R_\omega(x, y) = \frac{i}{2\nu} (\omega^2 - m^2)^{-1/2} - \frac{i}{2\nu^2} |x-y| - \frac{i \omega}{16\nu^3} (4|x-y|^2 - 1)(\omega^2 - m^2)^{1/2} + \ldots,$$

where $\Im(\omega^2 - m^2) > 0$. In particular, if $m = 0$, then

$$R_\omega(x, y) = \frac{i}{2\nu} - \frac{1}{2\nu^2} |x-y| - \frac{i \omega}{16\nu^3} (4|x-y|^2 - 1) + \ldots, \quad \omega \to 0.$$

For $\omega \to \pm \sqrt{4\nu^2 + m^2}$, $\omega \in \mathbb{C}_+$,

$$R_\omega(x, y) = (-1)^{|x-y|} \left( \frac{i}{2\nu} (4\nu^2 + m^2 - \omega^2)^{-1/2} + \frac{1}{2\nu^2} |x-y| \right) - \frac{i}{16\nu^3} (4|x-y|^2 - 1) \sqrt{4\nu^2 + m^2 - \omega^2} + \ldots.$$  

(B.9)

Since

$$\sum_{x, y \in \mathbb{Z}} \langle x \rangle^{-2\alpha} |x-y|^{2p} \langle y \rangle^{-2\alpha} < \infty$$

for $\alpha > \frac{1}{2} + p$, with any $p = 0, 1, 2, \ldots$,

$$\|x-y\|^p f(y) \|_{-\alpha} \leq C \|f\|_\alpha, \quad f \in \ell^2_\alpha, \quad \alpha > \frac{1}{2} + p, \quad p = 0, 1, 2, \ldots.$$  

(B.10)

Applying these estimates to the terms in the expansions (B.8) and (B.9), we come to the following result.
Lemma B.2. (see [3, Lemma 3.2]) Let \( f \in \ell^2_{\alpha}, \ \alpha > 5/2 \). Then for \( \omega \rightarrow \pm m, \ \omega \in \mathbb{C} \backslash \Lambda \), we have

\[
(R_{\omega}f)(x) = \frac{i\hat{f}(0)}{2\nu\sqrt{\mu^2 - m^2}} - \frac{1}{2\nu^2} \sum_{y \in \mathbb{Z}} |x - y|f(y) + \sqrt{\mu^2 - m^2} r_{\omega}^1 f,
\]

and for \( \omega \rightarrow \pm \sqrt{4\nu^2 + m^2}, \ \omega \in \mathbb{C} \backslash \Lambda \),

\[
(R_{\omega}f)(x) = \frac{i(-1)^{\nu}\hat{f}(\pi)}{2\nu\sqrt{4\nu^2 + m^2 - \omega^2}} + \frac{1}{2\nu^2} \sum_{y \in \mathbb{Z}} (-1)^{|x-y|} |x - y|f(y) + \sqrt{4\nu^2 + m^2 - \omega^2} r_{\omega}^2 f,
\]

where the remainder terms have the form \( r_{\omega}^1 f = \sum_{k=0}^{2} b_k(\omega) \sum_{y \in \mathbb{Z}} |x-y|^k f(y), \ b_k(\omega) = O(1) \) as \( \omega \rightarrow \pm m \) and \( b_k(\omega) = O(1) \) as \( \omega \rightarrow \pm \sqrt{4\nu^2 + m^2} \). In particular, \( \|r_{\omega}^1 f\|_{-\alpha} \leq C\|f\|_{\alpha} \).

Now Lemma B.1 follows from the equality (B.6) and Lemma B.2, using arguments similar to the proof Theorem 3.4 and technique of the paper [3].

Proof of the bound (2.14). Using the representation (B.1) and formula (B.6), we rewrite the solution of the problem (2.3)–(2.5) in the form

\[
z(x, t) = \frac{1}{2\pi} \int_{\mathbb{R}} e^{-i\omega t} R_{\omega} f_{\text{odd}} d\omega, \quad x \in \mathbb{Z}^+, \quad t > 0, \quad \mu > 0,
\]

where \( f_{\text{odd}}(x) := v_{\text{odd}}(x) - i\omega u_{\text{odd}}(x) \) (see (B.2)). Applying arguments similar to the proof of Theorem 3.4 we obtain

\[
z(x, t) = \frac{1}{2\pi} \int_{\Lambda} e^{-i\omega t} (R_{\omega+0} - R_{\omega-0}) f_{\text{odd}} d\omega = \frac{1}{\pi} \int_{\Lambda} e^{-i\omega t} \Im (R_{\omega+0} f_{\text{odd}}) d\omega. \tag{B.11}
\]

Let \( Y_0 \in \mathcal{H}_{\alpha, +} \) with \( \alpha > 3/2 \). Then, \( Y_{\text{odd}} \in \mathcal{H}_\alpha \) and \( \hat{f}_{\text{odd}}(0) = \hat{f}_{\text{odd}}(\pi) = 0 \). We want to apply Lemma B.2 to the function \( f_{\text{odd}}(x) \), but with \( \alpha > 3/2 \) instead of \( \alpha > 5/2 \), using the oddness of \( f_{\text{odd}}(x) \). Note that for \( k = 1, 2 \),

\[
\left| \sum_{y \in \mathbb{Z}} |x-y|^k f_{\text{odd}}(y) \right| \leq 2|x| \sum_{y \geq 1} |y| f_0(y),
\]

where \( f_0(x) := v_0(x) - i\omega u_0(x), \ x \in \mathbb{Z}^+ \). Therefore, applying the Cauchy–Bunyakovskii inequality, we obtain for \( \alpha > 3/2 \),

\[
\|r_{\omega}^1 f_{\text{odd}}\|_{-\alpha, +} \leq C \sum_{k=0}^{2} \left( \sum_{y \in \mathbb{Z}} |x-y|^k f_{\text{odd}}(y) \right)_{-\alpha, +} \leq C_1 \left( \sum_{x \in \mathbb{Z}^+} \langle x \rangle^{-2\alpha} x^2 \left( \sum_{y \in \mathbb{Z}^+} |y| f_0(y) \right)^2 \right) \leq C_2 \sum_{y \in \mathbb{Z}^+} \langle y \rangle^{-\alpha} |y| \cdot \langle y \rangle^\alpha |f_0(y)| \leq C\|f_0\|_{\alpha, +}.
\]

Thus, in the neighborhood of the singular points \( \omega_0 \in \Lambda_0 \) the following estimate holds

\[
\|\Im R_{\omega+0} f_{\text{odd}}\|_{-\alpha, +} \leq C|\omega^2 - \omega_0^2|^{1/2} \|f_0\|_{\alpha, +}, \ \omega \rightarrow \omega_0, \quad \alpha > 3/2, \ \omega_0 \in \Lambda_0, \ \omega \in \mathbb{R}.
\]

Now the estimate (2.14) follows from the equality (B.11), estimate (B.12) and Lemma 10.2 from [7], which is a generalization of the estimate (A.13).
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