IMPROVED REGULARITY OF HARMONIC DIFFEOMORPHIC EXTENSIONS ON QUASIHYPHERBOLIC DOMAINS∗

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Abstract  Let $X$ be a Jordan domain satisfying certain hyperbolic growth conditions. Assume that $\varphi$ is a homeomorphism from the boundary $\partial X$ of $X$ onto the unit circle. Denote by $h$ the harmonic diffeomorphic extension of $\varphi$ from $X$ onto the unit disk. We establish the optimal Orlicz-Sobolev regularity and weighted Sobolev estimate of $h$. These generalize the Sobolev regularity of $h$ in [A. Koski, J. Onninen, Sobolev homeomorphic extensions, J. Eur. Math. Soc. 23 (2021) 4065–4089, Theorem 3.1].

Key words  Poisson extension; Orlicz-Sobolev homeomorphisms; weighted Sobolev homeomorphisms; quasihyperbolic domains

2010 MR Subject Classification  58E20; 46E35; 30C62

1 Introduction

A planar Jordan curve $\Gamma$ is a simple closed curve, i.e., a non-self-intersection continuous curve, in the plane $\mathbb{R}^2$. The famous Jordan Curve Theorem asserts that each planar Jordan curve $\Gamma$ divides the plane into two regions (an “interior” and an “exterior”), with $\Gamma$ being the common boundary. Every region with a Jordan curve as its boundary is called a Jordan domain. The Jordan-Schoenflies theorem is a sharpening of the Jordan curve theorem. Fix two Jordan domains $X \subset \mathbb{R}^2$ and $Y \subset \mathbb{R}^2$. The ensuing theorem states that every homeomorphism $\varphi$ from

∗Received April 25, 2021; revised June 14, 2022. The authors were partially supported by the Young Scientist Program of the Ministry of Science and Technology of China (2021YFA1002200). The first author was supported by National Natural Science Foundation of China (12101226). The second author was supported by Shandong Provincial Natural Science Foundation (ZR2021QA032), and partially supported by the National Natural Science Foundation of China (12101362).
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the boundary $\partial X$ of $X$ onto $\partial Y$ has a homeomorphic extension $\Phi : \mathbb{R}^2 \to \mathbb{R}^2$. It is then natural to ask: what good extensions $\Phi$ can we find?

Let $Y$ be a bounded convex Jordan domain in the plane. Assume that $\varphi$ is a homeomorphism from the unit circle $S$ onto the boundary $\partial Y$ of $Y$. Denote by $h$ the complex-valued Poisson extension of $\varphi$, that is,

$$ h(z) = \frac{1}{2\pi} \int_S \frac{1 - |z|^2}{|z - \xi|^2} \varphi(\xi) \, d\xi $$

for all $z \in \mathbb{D}$. Then the Radó-Kneser-Choquet-Lewy theorem states that $h$ is a real analytic diffeomorphism from $\mathbb{D}$ onto $Y$; see [3]. Notice that derivatives of $h$ are not always uniformly bounded. Therefore, it is natural to study the Sobolev regularity of $h$. With, in addition, the $C^1$-smoothness of $\partial Y$, Astala et al. [1] showed that the square integrability of the derivative of $h$ on $\mathbb{D}$ is comparable to the fact that

$$ \int_{\partial Y} \int_{\partial Y} |\log |\varphi^{-1}(\xi) - \varphi^{-1}(\eta)|| \, d\xi |d\eta| < \infty. \quad (1.1) $$

The condition (1.1) does not automatically hold. In fact, Verchota [21] showed that the harmonic homeomorphisms of $\mathbb{D}$ onto itself need to be in the Sobolev space $W^{1,p}(\mathbb{D}, \mathbb{D})$ for $p < 2$, but not in $W^{1,2}$. Iwaniec et al. [10] established more delicate estimates, which provided a weak type $L^2$-estimate and estimates in Orlicz classes near $L^2(\mathbb{D})$ for the gradient of $h$. Later Xu et al. [15, 22] generalized these results to the case in which $Y$ is an internal chord-arc Jordan domain. Their extension is a composition with respect to the harmonic mapping $h$ onto $\mathbb{D}$. Moreover, they studied connections between the $W^{1,p}(\mathbb{D}, Y)$ regularity of an extension for $p \geq 1$, a double integral condition on $\varphi^{-1}$ like (1.1), and the internal $p$-Douglas condition

$$ \int_S \int_S \frac{(\lambda_Y(\varphi(\xi), \varphi(\eta)))^p}{|\xi - \eta|^p} |d\xi| |d\eta| < \infty. $$

Here $\lambda_Y(y_1, y_2)$ is the internal metric on $Y$, which is defined as the infimum of lengths of all rectifiable curves in $Y$ joining $y_1$ to $y_2$.

In all of the above mentioned works, harmonic extensions are defined on the unit disk $\mathbb{D}$. By the Riemann mapping theorem we can define the complex-valued harmonic extension on a Jordan domain $X$. For those $X$ has $s$-hyperbolic growth for $s \in (0, 1)$ (see Definition 2.3), Koski and Onninen recently proved in [16, Theorem 3.2] that the harmonic extension has $W^{1,p}(X, \mathbb{D})$ regularity for all $p < 1 + s$. In this paper, we explore the regularity under different function spaces. The first main result is pertaining to the estimate for the derivatives of $h$ in Orlicz classes near $L^{1+s}(X)$.

**Theorem 1.1** Let $X$ be a Jordan domain with $s$-hyperbolic growth for $s \in (0, 1)$ and let $\varphi : \partial X \to \partial D$ be a homeomorphism. Let $h : X \to \mathbb{D}$ be the harmonic extension of $\varphi$. Then $h \in W^{1,p}(X, \mathbb{D})$, where $\Phi(t) = t^{1+s} \log^\lambda(e + t)$ for all $\lambda < -1$.

Our second main result concerns the weighted $W^{1,1+s}$ estimate of $h$. Denote by $d(z, \partial X)$ the Euclidean distance between $z$ and the boundary $\partial X$ of a Jordan domain $X$.

**Theorem 1.2** Let $X$ be a Jordan domain with $s$-hyperbolic growth for $s \in (0, 1)$ and let $\varphi : \partial X \to \partial D$ be a homeomorphism. The harmonic extension $h : X \to \mathbb{D}$ of $\varphi$ satisfies that

$$ \int_X |Dh(z)|^{1+s} \log^\lambda(e + \frac{1}{d(z, \partial X)}) \, dz < \infty $$

for all $\lambda < -1$. © Springer
In Theorem 1.2, a certain integral of the harmonic extension is finite when the parameter \( \lambda \) is strictly less than \(-1\). We will show in Example 4.1 that this integral might be infinite if \( \lambda = -1 \). Moreover the sharp bound \(-1\) of \( \lambda \) is independent of the degree \( s \) of hyperbolic growth. The above discussion on \( \lambda \) of Theorem 1.2 is also applied to the \( \lambda \) in Theorem 1.1. In Corollary 3.4, we will obtain analogous Theorems 1.1 and 1.2 on more general domains.

There are different extension methods aside from Poisson extensions. Let \( Y \) be a quasidisk. Koskela-Koski-Onninen [13] found a \( W^{1,p}_{1,loc}(\mathbb{R}^2, \mathbb{R}^2) \) homeomorphic extension \( \varphi : S \rightarrow \partial Y \) for all \( p < 2 \). They used the bi-Lipschitz characterization of quasidisks by Rohde [19] and an extension result of quasidisks by Tukia [20]. Guo, Xu et al. [6–8, 23] studied extension problems on cardioid-type domains. From cardioid geometry they decomposed domains into countable pieces, and extended each piece by piecewise affine mappings. Recently, Koski and Onninen [17] introduced a new method which is independent of the geometry of domains. By hyperbolic geodesics they decomposed \( Y \) into a sequence of pieces. Then they extended each piece, and the resulting extension \( \Phi : \mathbb{R}^2 \rightarrow \mathbb{R}^2 \) is the sum of extensions on pieces.

The rest of this paper is organized as follows: in Section 2, we present preliminaries about the hyperbolic metric, the quasihyperbolic metric, and Young functions. In Section 3 we show the proofs of Theorem 1.1 and Theorem 1.2, and study these two theorems on more general domains. We construct counter-examples in Section 4 to show the optimality of the preceding results.

Notation For \( i = 1, 2, \cdots \), denote that \( \log(i)(x) = \log(\cdots(\log(\log(x)))\cdots) \) is an \( i \)-iterated logarithmic function and that \( e_i = \exp(\cdots(\exp(\exp))\cdots) \) is an \( i \)-iterated exponent.

2 Preliminaries

In this section we review the hyperbolic metric, the quasihyperbolic metric, and Young functions.

Definition 2.1 The hyperbolic metric \( j_D \) on \( D \) is defined by

\[
j_D(z_1, z_2) = \inf_{\gamma} \int_\gamma \frac{2|dz|}{1 - |z|^2},
\]

where the infimum is taken from all rectifiable arcs \( \gamma \) joining \( z_1 \) and \( z_2 \) in \( D \).

It is well-known that

\[
j_D(z_1, z_2) = \log \left( \frac{|1 - \bar{z}_1 z_2| + |z_1 - z_2|}{|1 - \bar{z}_1 z_2| - |z_1 - z_2|} \right).
\]

Let \( \Omega \) be a simply connected domain. By the Riemann mapping theorem, there is a conformal mapping \( f \) from \( D \) onto \( \Omega \). Afterwards, the hyperbolic metric \( j_\Omega \) on \( \Omega \) is defined as

\[
j_\Omega(z_1, z_2) := j_D(f^{-1}(z_1), f^{-1}(z_2)).
\]

The good definition of \( j_\Omega \) comes from the fact that \( j_D \) is invariant under Möbius transformations. Usually we do not have the explicit formula of the above \( f \). Therefore we cannot explicitly calculate a formula for \( j_\Omega \), like the one for \( j_D \) is found in (2.2). Next, we recall a useful substitute for \( j_\Omega \), whose definition is analogous to (2.1).
Definition 2.2 The quasihyperbolic metric $h_{\Omega}$ on a domain $\Omega$ is defined as

$$h_{\Omega}(z_1, z_2) = \inf_{\gamma} \int_{\gamma} \frac{1}{d(z, \partial \Omega)} |dz|,$$

(2.4)

where the infimum is taken from all rectifiable arcs $\gamma$ joining $z_1$ and $z_2$ in $\Omega$, and $d(z, \partial \Omega)$ is the euclidean distance between $z$ and $\partial \Omega$.

In order to study the quasiconformal homogeneity, Gehring and Palka [4] introduced a quasihyperbolic metric; this metric has a number of applications. For example, Jones [11] used this metric to study the extension theorems of BMO functions. The domains onto which conformal mappings map the unit disk are global Hölder continuous, are explicitly characterized by an analytic condition on this metric; see Becker-Pommerenke’s work [2]. We also recommend an expert survey on this metric by Koskela [12].

We show how $h_{\Omega}$ works as a substitute for $j_{\Omega}$. A corollary of the Koebe distortion theorem ([18, Corollary 1.4]) states that a conformal mapping $g : D \to \mathbb{C}$ satisfies that

$$|g'(z)| \approx \frac{d(g(z), \partial g(D))}{d(z, \partial D)}$$

(2.5)

for all $z \in D$. Therefore, when $\Omega$ is simply connected, from (2.1), (2.3) and (2.4), the Riemann mapping theorem implies that

$$h_{\Omega}(z_1, z_2) \approx j_{\Omega}(z_1, z_2)$$

(2.6)

for all $z_1, z_2 \in \Omega$.

By the quasihyperbolic metric, the domains with hyperbolic growth are defined.

Definition 2.3 Let $\Omega$ be a planar domain. Fix a point $z_0 \in \Omega$. We say that $\Omega$ satisfies $s$-hyperbolic growth for $s \in (0, 1)$ if

$$h_{\Omega}(z_0, z) \leq \left( \frac{d(z_0, \partial \Omega)}{d(z, \partial \Omega)} \right)^{1-s}$$

(2.7)

holds for all $z \in \Omega$.

Recall that $h_{\Omega}(x, y) \geq |\log \frac{d(x, \partial \Omega)}{d(y, \partial \Omega)}|$ for all $x \in \Omega$ and $y \in \Omega$; see [4, Lemma 2.1]. Hence the requirement (2.7) makes sense. Relaxing the power of distance as in (2.7), Gotoh [5] in 2000 studied geometric properties of these generalized domains. Their relations to quasiconformal mappings were explored by Hencl et al. in [9] and [14].

Let us show some examples on domains with the hyperbolic growth condition. John disks are an important research object in geometric analysis. It is easy to check that a $c$-John disk $\Omega$ satisfies $h_{\Omega}(z_0, z) \leq c^{-1} \log \frac{1}{d(z, \partial \Omega)}$ for $c \geq 1$. Second, the degree of an outer-cusp will determine the power of distance in (2.7). For example, the domain

$$\Omega = \{(x, y) : |y| \leq x^{1/s}, \ x \in [0, 1]\} \cup \{(x, y) : (x-1)^2 + y^2 \leq 1\},$$

with $s \in (0, 1)$, has an $s$-hyperbolic growth condition.

We next provide some estimates related to conformal mappings onto domains with $s$-hyperbolic growth. These are useful for proofs of Theorem 1.1 and Theorem 1.2. Let $\Omega$ be a such domain, and let $g : D \to \Omega$ be a conformal mapping. Then taking (2.6), (2.3) and (2.2) together implies that

$$h_{\Omega}(g(0), g(x)) \approx j_{\Omega}(g(0), g(x)) \approx j_D(0, x) \approx \log \frac{1}{1 - |x|}$$
for all $x \in \mathbb{D}$. In addition of (2.7), it follows that
\[
d(g(x), \partial \Omega) \lesssim \log \frac{1}{|x|}.
\] (2.8)

Furthermore, (2.5) implies that
\[
|g'(x)| \lesssim \frac{1}{1 - |x|} \log \frac{1}{1 - |x|}.
\] (2.9)

A function $\Phi : [0, \infty) \rightarrow [0, \infty)$ is a Young function if it is a continuous, increasing and convex function satisfying $\Phi(0) = 0$.

Theorem 3.1

In this section, we show the proofs of Theorems 1.1 and 1.2. We also obtain analogies of these two theorems in more generalized domains. Before the proof of Theorem 1.1, we need the following theorem:

Let $\Phi(t) = t^\alpha \log^\lambda (e + t)$ with $\lambda \in \mathbb{R}$ and the function
\[
\Psi(t) = t^\alpha \log^{\sigma_1} (e + t) \log^{\sigma_2} (e + t) \cdots \log^{\sigma_n} (e + t),
\]
with $\sigma_i \in \mathbb{R}, i = 1, \ldots, n$ are Young functions satisfying the $\Delta_2$-condition. The Orlicz-Sobolev space $W^{1,\Phi}(X, \mathbb{C})$ contains all measurable functions $f : X \rightarrow \mathbb{C}$ in the Orlicz space $L^\Phi$ such that the weak derivatives of $f$ are also in the $L^\Phi$ space. Here $f \in L^\Phi$ means that $\int_X \Phi(|f|) < \infty$.

3 Proofs of Main Results

In this section, we show the proofs of Theorems 1.1 and 1.2. We also obtain analogies of these two theorems in more generalized domains. Before the proof of Theorem 1.1, we need the following theorem:

Theorem 3.1 Let $X$ be a Jordan domain, and denote by $g : \mathbb{D} \rightarrow X$ a conformal map onto $X$. Let $\Phi$ be a Young function satisfying the $\Delta_2$-condition. Suppose that the condition
\[
\sup_{w \in \partial X} \int_{\partial \mathbb{D}} \Phi \left( \frac{1}{|g'(z)||w - z|} \right) |g'(z)|^2 \, dz \leq M < \infty
\] (3.1)
holds. Then the harmonic extension $h : X \rightarrow \mathbb{D}$ of any boundary homeomorphism $\varphi : \partial X \rightarrow \partial \mathbb{D}$ lies in the Orlicz-Sobolev space $W^{1,\Phi}(X, \mathbb{C})$.

Proof Let $g : \mathbb{D} \rightarrow X$ be a conformal mapping. As in [16] by Koski-Ommen, without loss of generality, we may assume that $h \circ g$ is smooth up to the boundary. Hence,
\[
|\psi'(e^{it})| \leq \frac{1}{\sqrt{1 - e^{2it}}} |\varphi'(e^{it})| + \frac{1}{\sqrt{1 - e^{2it}}} |\varphi'(e^{it})| dt.
\] (3.2)

A change of variable and the estimate (3.2) implies that
\[
\int_X \Phi(|h_\varphi(z)|) \, dz = \int_{\partial \mathbb{D}} \Phi \left( \frac{|(h \circ g)(z)|}{|g'(z)|} \right) |g'(z)|^2 \, dz 
\leq \int_{\partial \mathbb{D}} \Phi \left( \int_0^{2\pi} |\varphi'(e^{it})| \, dt \right) |g'(z)|^2 \, dz.
\]
Note that \(\int_0^{2\pi} |\psi'(e^{it})| dt = 2\pi\). By Jensen’s inequality and the \(\Delta_2\)-condition of \(\Phi\), it follows from the Fubini theorem that
\[
\int_X \Phi(|h_\zeta(\xi)|) d\xi \lesssim \int_D \Phi \left( \int_0^{2\pi} \frac{|\psi'(e^{it})|}{2\pi |g'(z)||z - e^{it}|} dt \right) |g'(z)|^2 dz
= \int_D \Phi \left( \int_0^{2\pi} \frac{|\psi'(e^{it})|}{\int_0^{2\pi} |\psi'(e^{it})| dt |g'(z)||z - e^{it}|} dt \right) |g'(z)|^2 dz
\leq \int_D \phi \left( \frac{1}{|g'(z)||z - e^{it}|} \right) |\psi'(e^{it})| |g'(z)|^2 dz
= \frac{1}{2\pi} \int_0^{2\pi} |\psi'(e^{it})| \left( \int_D \Phi \left( \frac{1}{|g'(z)||z - e^{it}|} \right) dz \right) dt \leq M < \infty.
\]
Hence, under the condition (3.2), we have the harmonic extension \(h \in W^{1,\Phi}(D, \mathbb{C})\).

**Proof of Theorem 1.1** From the estimate (2.9), we know that, for any \(z \in D\),
\[
|g'(z)| \leq \frac{C}{(1 - |z|) \log^{1/(1-s)} \left( \frac{1}{1-|z|} \right)}.
\]
Note that \(\Phi(t) = t^p \log^\lambda (e + t)\) with \(p = 1 + s\) and \(\lambda < -1\) is a Young’s function satisfying the \(\Delta_2\)-condition. Since \(2 - p = 1 - s\), we obtain that
\[
\Phi \left( \frac{1}{|g'(z)||w - z|} \right) |g'(z)|^2 = \frac{|g'(z)|^{2-p}}{|w - z|^p} \log^\lambda \left( e + \frac{1}{|g'(z)||w - z|} \right) \log^\lambda \left( e + \frac{(1-|z|) \log^{1/(1-s)} \left( \frac{1}{1-|z|} \right)}{|w - z|} \right)
\leq C \frac{\log^\lambda \left( e + \frac{(1-|z|) \log^{1/(1-s)} \left( \frac{1}{1-|z|} \right)}{|w - z|} \right)}{(1 - |z|)^{2-p} |w - z|^p \log 1 - \log |z|}.
\]
Hence it is enough to prove that the quantity
\[
\int_D \log^\lambda \left( e + \frac{(1-|z|) \log^{1/(1-s)} \left( \frac{1}{1-|z|} \right)}{|w - z|} \right) dz = \int_D F(z) dz = \int_{\mathbb{D} \setminus \frac{1}{2} \mathbb{D}} F(z) dz + \int_{\frac{1}{4} \mathbb{D}} F(z) dz
\]
is finite, as then applying the rotational symmetry will imply that the estimate (3.1) holds for all \(\omega\). Notice that \(F(z)\) is bounded on the set \(\frac{1}{2} \mathbb{D}\), and hence \(\int_{\frac{1}{4} \mathbb{D}} F(z) dz < \infty\). Thus it is sufficient to show that \(\int_{\mathbb{D} \setminus \frac{1}{2} \mathbb{D}} F(z) dz < \infty\). Towards this end, we follow the idea of Koski-Onninen [16] to divide \(\mathbb{D} \setminus \frac{1}{2} \mathbb{D}\) into three pieces: \(S_1, S_2\) and \(S_3\). More precisely,
\[
S_1 = \{ 1 + re^{i\theta} : r \in [0, 3/4], \theta \in [3\pi/4, 5\pi/4] \},
S_2 = \{ (x, y) \in \mathbb{D} : y \in [-1/\sqrt{2}, 1/\sqrt{2}], x \in [1 - |y|, 1] \},
S_3 = \{ r e^{i\theta} : r \in [1/2, 1], \theta \in [\pi/4, 7\pi/4] \}.
\]
Therefore it suffices to check that \(\int_{S_1} F < \infty\), \(\int_{S_2} F < \infty\) and \(\int_{S_3} F < \infty\).

On the set \(S_1\), we have the estimate \(1 - |z| \approx |1 - z|\). Hence we may apply polar coordinates around the point \(z = 1\) to find that
\[
\int_{S_1} F(z) dz \lesssim\int_{S_1} \frac{\log^\lambda \left( e + \log^{1/(1-s)} \left( \frac{1}{1-|z|} \right) \right)}{|1 - z|^2 \log \frac{1}{|1 - z|}} dz \lesssim\int_{3\pi/4}^{5\pi/4} \int_0^{3/4} \frac{\log^\lambda \left( \log (1/r) \right)}{r \log (1/r)} dr d\theta < \infty.
\]
On the set \( S_3 \), we have that \(|1 - z|\) is bounded away from zero. Hence \( \frac{1}{|1 - z|^p} \) in the expression \( F \) has a positive constant upper bound. Notice that
\[
\log^\lambda \left( e + \frac{(1 - |z|) \log^{1/(1-s)} 1}{1 - |z|} \right) \leq \log^\lambda(e) = 1.
\]
We have the estimate that
\[
\int_{S_3} F(z)dz \lesssim \int_{S_3} \frac{1}{(1 - |z|)^{2-p}} \log \frac{1}{1 - |z|} dz \lesssim \int_{\pi/4}^{\pi/4} \int_0^1 (1-r)^2 p \, dr \, d\theta < \infty,
\]
where the second inequality is from an estimate \( \log^\lambda \left( \frac{1}{1 - |z|} \right) \lesssim 1 \).

On the set \( S_2 \), since on each \( R_\theta \) \( |1 - z| \) is comparable to the size of the angle \( \theta \) and \( |1 - z| < |1 - z| \), we obtain the following estimate:
\[
F(z) \lesssim \frac{\log^\lambda \left( e + \frac{(1-|z|) \log^{1/(1-s)} 1}{1 - |z|} \right)}{(1 - |z|)^{2-p} \log \frac{1}{|\theta|} }, \quad z \in R_\theta.
\]
By the property that \( 1 - \rho(\theta) \) is comparable to \( |\theta| \), we estimate that
\[
\int_{S_2} F(z)dz \lesssim \int_{-\pi/4}^{\pi/4} \int_{-\pi/4}^{\pi/4} \log^\lambda \left( e + \frac{(1-\rho(\theta)) \log^{1/(1-s)} 1}{|\theta|} \right) \frac{1}{(1 - |z|)^{2-p}} dr \, d\theta
\]
\[
\approx \int_{-\pi/4}^{\pi/4} \log^\lambda \left( e + \log^{1/(1-s)} \frac{1}{|\theta|} \right) \frac{1}{(1 - \rho(\theta))^{1-p}} d\theta
\]
\[
\lesssim \int_{-\pi/4}^{\pi/4} \log^\lambda \left( \log \left( \frac{1}{|\theta|} \right) \right) d\theta < \infty.
\]
Summing up the integrations over the sets \( S_1, S_2 \) and \( S_3 \), we finish the proof. \( \square \)

Towards obtaining the proof of Theorem 1.2, we state by giving the following lemma:

**Lemma 3.2** Let \( X \) be a Jordan domain with \( s \)-hyperbolic growth for \( s \in (0,1) \). For any conformal mapping \( g : \mathbb{D} \to X \), we have that
\[
\int_{\mathbb{D}} \frac{|g'(z)|^{1-s}}{|1 - z|^{1+s}} \log^\lambda \log \left( \frac{1}{1 - |z|} \right) dz < \infty
\]
for all \( \lambda < -1 \).

**Proof** The inequality found in (2.9) states that \( |g'(z)| \lesssim \frac{1}{|1 - z|} \log^\lambda \left( \frac{1}{1 - |z|} \right) \) for \( z \in \mathbb{D} \). Hence, replacing \( |g'(z)| \) by this upper bound implies that
\[
\frac{|g'(z)|^{1-s}}{|1 - z|^{1+s}} \log^\lambda \log \left( \frac{1}{1 - |z|} \right) \lesssim \frac{1}{|1 - z|^{1+s}(1 - |z|)^{1-s}} \log^{-1} \left( \frac{1}{1 - |z|} \right) \log^\lambda \log \left( \frac{1}{1 - |z|} \right).
\]
To complete the proof, it is enough to check that the following integral is finite:
\[
\int_{\mathbb{D} \setminus \frac{1}{4} \mathbb{D}} \frac{1}{|1 - z|^{1+s}(1 - |z|)^{1-s}} \log^{-1} \left( \frac{1}{1 - |z|} \right) \log^\lambda \log \left( \frac{1}{1 - |z|} \right) dz =: \int_{\mathbb{D} \setminus \frac{1}{4} \mathbb{D}} G(z) dz.
\]
In sequel, we let $\lambda < -1$. As in Koski-Onnninen’s work [16], we divide $\mathbb{D} \setminus \frac{1}{2} \mathbb{D}$ into subsets $S_1$, $S_2$ and $S_3$. Here

$$S_1 = \{1 + re^{i\theta} : r \in [0, 3/4], \theta \in [3\pi/4, 5\pi/4]\},$$

$$S_2 = \{(x, y) \in \mathbb{D} : y \in [-1/\sqrt{2}, 1/\sqrt{2}], x \in [1 - |y|, 1]\},$$

$$S_3 = \{re^{i\theta} : r \in [1/2, 1], \theta \in [\pi/4, 7\pi/4]\}.$$  

Afterwards, it suffices to check that $\int_{S_1} G < \infty$, $\int_{S_2} G < \infty$ and $\int_{S_3} G < \infty$.

Notice that $|1 - z| \approx 1 - |z|$ for $z \in S_1$. Hence, the polar coordinates around the point $z = 1$ imply that

$$\int_{S_1} G(z) dz \approx \int_1^{5\pi/4} \int_{3/4}^{\pi/4} \frac{1}{|1 - z|^2} \log^{-1}(\frac{1}{|1 - z|}) \log^\lambda \log(\frac{1}{1 - z}) dz$$

On the set $S_3$, the modulus $|1 - z|$ is bounded away from zero. Therefore, by bounding the term $|1 - z|^{-1 - s}$ in the integrand $G$ and applying polar coordinates around the origin, we have that

$$\int_{S_3} G(z) dz \lesssim \int_3^{\pi/4} \int_{1/2}^{1} \frac{r}{|1 - r|^{1 - s}} \log^{-1}(\frac{1}{1 - r}) \log^\lambda \log(\frac{1}{1 - r}) dr d\theta < \infty.$$  

Finally, we check that $\int_{S_2} G < \infty$. Applying the triangle inequality $1 - |z| \leq |1 - z|$, by $\log^{-1}(\frac{1}{|1 - z|})$ we can replace the term $\log^{-1}(\frac{1}{|1 - z|})$ in the integrand $G$. Afterwards, we use the polar coordinates $(r, \theta)$ around the origin. The modulus $|1 - z|$ is comparable to the size of the angle $\theta$. For each angle $\theta$, the modulus $r = |z|$ ranges from $\rho(\theta)$ to 1, and $1 - \rho(\theta) = \frac{\sin(\pi/4 + \theta) - \sin(\pi/4)}{\sin(\pi/4 + \theta)}$ is comparable to $|\theta|$. By these estimates, we obtain that

$$\int_{S_2} G(z) dz \lesssim \int_{S_2} \frac{1}{(1 - |z|)^{1-s}} \frac{1}{|1 - z|^{1+s}} \log^{-1}(\frac{1}{|1 - z|}) \log^\lambda \log(\frac{1}{1 - z}) dz$$

$$\lesssim \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta^{1+s}} \log^{-1}(\frac{1}{|\theta|}) \log^\lambda \log(\frac{1}{|\theta|}) d\theta \int_{\rho(\theta)}^{1} \frac{1}{1 - r} dr$$

$$\approx \int_{-\pi/4}^{\pi/4} \frac{1}{|\theta|} \log^{-1}(\frac{1}{|\theta|}) \log^\lambda \log(\frac{1}{|\theta|}) d\theta < \infty.$$  

The proof is complete. $\square$

**Proof of Theorem 1.2** Take a conformal mapping $g : \mathbb{D} \to \mathbb{X}$. Then (2.8) shows that $d(g(z), \partial \mathbb{X}) \lesssim \log^{-\lambda}(\frac{1}{|1 - |z||})$ for all $z \in \mathbb{D}$. Together with a change of variables $w = g(z)$, we estimate that

$$\int_{\mathbb{X}} |h'(w)|^{1+s} \log^\lambda(e + \frac{1}{d(w, \partial \mathbb{X})})dw = \int_{\mathbb{D}} |(h \circ g)'(z)|^{1+s} |g'(z)|^{1-s} \log^\lambda(e + \frac{1}{d(g(z), \partial \mathbb{X})})dz$$

$$\lesssim \int_{\mathbb{D}} |(h \circ g)'(z)|^{1+s} |g'(z)|^{1-s} \log^\lambda(\frac{1}{1 - |z|})dz. \ (3.3)$$  

Next we look for an upper bound for the rightmost integral of (3.3).

Notice that $h \circ g$ is a harmonic mapping, which can be expressed as the Poisson extension $\frac{1}{2\pi} \int_{S} \frac{1}{|z - \xi|^2} \psi(\xi) d\xi$. Here $\psi = \varphi \circ g|_{S}$ is a self-homeomorphism on $S$. Basically, $\psi$ can be seen
as a self-homeomorphism (then a strictly monotonic function) on the interval $[0, 2\pi]$. Hence the derivative $\psi'$ exists a.e. on $S$ and
\[
\int_0^{2\pi} |\psi'(e^{i\theta})|\,dt < \infty. \tag{3.4}
\]
Furthermore, by differentiating the Poisson integral of $h \circ g$, we obtain, for all $z \in D$, the following pointwise estimate on the derivative of $h \circ g$:
\[
|(h \circ g)'(z)| = \left| \int_0^{2\pi} \psi'(e^{i\theta}) \frac{e^{i\theta}}{z - e^{i\theta}}\,d\theta \right| \leq \int_0^{2\pi} \left| \psi'(e^{i\theta}) \right| \,d\theta.
\]
By this estimate and Minkowski's inequality, we estimate that
\[
\int_D |(h \circ g)'(z)|^{1+s} |g'(z)|^{1-s} \log^\lambda \log(\frac{1}{1-|z|})\,dz
\leq \left( \int_0^{2\pi} \left| \psi'(e^{i\theta}) \right| \left( \int_D \frac{|g'(z)|^{1-s}}{|z - e^{i\theta}|^{1+s}} \log^\lambda \log(\frac{1}{1-|z|})\,dz \right)^{1/(1+s)} \,d\theta \right)^{1+s}. \tag{3.5}
\]
The composition of (3.3) with (3.5) implies that
\[
\int_X |h'(w)|^{1+s} \log^\lambda (e + \frac{1}{d(w, \partial X)})\,dw
\leq \left( \int_0^{2\pi} \left| \psi'(e^{i\theta}) \right| \left( \int_D \frac{|g'(z)|^{1-s}}{|z - e^{i\theta}|^{1+s}} \log^\lambda \log(\frac{1}{1-|z|})\,dz \right)^{1/(1+s)} \,d\theta \right)^{1+s}. \tag{3.6}
\]
By the rotation invariance and Lemma 3.2, we have that
\[
\int_D \frac{|g'(z)|^{1-s}}{|z - e^{i\theta}|^{1+s}} \log^\lambda \log(\frac{1}{1-|z|})\,dz = \int_D \frac{|g'(z)|^{1-s}}{1-|z|^{1+s}} \log^\lambda \log(\frac{1}{1-|z|})\,dz < \infty
\]
for all $t \in [0, 2\pi]$. Therefore, with the addition of (3.4), we derive from (3.6) that
\[
\int_X |h'(w)|^{1+s} \log^\lambda (e + \frac{1}{d(w, \partial X)})\,dw \leq \left( \int_0^{2\pi} \left| \psi'(e^{i\theta}) \right|\,d\theta \right)^{1+s} < \infty. \tag{3.7}
\]
The proof is complete. $\square$

In Theorems 1.1 and 1.2, we obtained the regularity of harmonic extension on domains satisfying the condition (2.7). Now we relax these domains by replacing the upper bound \((\frac{d(z, \partial X)}{d(z, \partial X)})^{1-s}\) in (2.7) by \((\frac{1}{d(x, \partial X)})^{1-s} \log^\sigma (e + \frac{1}{d(x, \partial X)})\). We discuss analogous regularity on these domains.

**Corollary 3.3** Let $X$ be a Jordan domain satisfying that $h_X(x_0, x) \leq (\frac{1}{d(x, \partial X)})^{1-s} \log^\sigma (e + \frac{1}{d(x, \partial X)})$ for $\sigma \geq -1$. Let $\varphi : \partial X \to S$ be a homeomorphism and let $h : X \to D$ be its harmonic extension. Suppose that $\Phi(t) = t^{1+s} \log^\lambda (e + t)$. Then
\[
\int_X \Phi(|Dh(z)|)\,dz < \infty \tag{3.8}
\]
and
\[
\int_X |Dh(z)|^{1+s} \log^\lambda (e + \frac{1}{d(z, \partial X)})\,dz < \infty, \tag{3.9}
\]
whenever $\lambda < -1 - \sigma$.
Proof The proof is analogous to that for Theorems 1.1 and 1.2, hence we only sketch the process. Let \( g : \mathbb{D} \to X \) be a conformal mapping. Under the assumption \( h_X(x_0, x) \leq \Psi(\frac{1}{d(x, \partial X)}) \) with \( \Psi(t) = t^{1-s}\log^s(e + t) \), the same arguments as those for (2.8) and (2.8) imply that
\[
\begin{align*}
  d(g(z), \partial X) &\leq \log \frac{1}{1 - |z|} \log \frac{1}{1 - |z|} \\
  |g'(z)| &\approx \frac{d(g(z), \partial X)}{1 - |z|} \leq \log \frac{1}{1 - |z|} \log \frac{1}{1 - |z|} \log \frac{1}{1 - |z|} \\
\end{align*}
\]
for all \( z \in \mathbb{D} \). Here we used the fact that \( \Psi^{-1}(t) \approx t^{\frac{1}{\lambda}} \log t^{\frac{1}{\lambda}}(t) \). Afterwards, the estimate (3.8) is obtained by applying (3.11) to the proof of Theorem 1.1.

Applying (3.10) and (3.11) in order, from a change of variable \( w = g(z) \) and Minkowski’s inequality, we obtain that
\[
\int_X |h'(w)|^{1+s} \log^\lambda(e + \frac{1}{d(w, \partial X)}) \, dw
\leq \int_{\mathbb{D}} |g'(z)|^{1-s} \log^\lambda \log(\frac{1}{1 - |z|}) \, dz
\leq \int_{\mathbb{D}} \frac{1}{1 - |z|} \log(\frac{1}{1 - |z|}) \log^\lambda \log(\frac{1}{1 - |z|}) \, dz.
\]
Analogously to the proof of Lemma 3.2, the lowest integral in (3.12) is finite whenever \( \lambda < -1 - \sigma \), which gives the estimate (3.9). The proof is complete.

Here we give a more generalized variant of Corollary 3.3. Let \( \sigma = (\sigma_1, \ldots, \sigma_n) \) be a vector with \( \sigma_i \in \mathbb{R} \) for all \( i = 1, \ldots, n \). For \( s \in \mathbb{R} \), denote
\[
\Psi_{1-s, \sigma}(t) = t^{1-s} \log^{\sigma_1}(e + t) \log^{\sigma_2}(e_2 + t) \cdots \log^{\sigma_n}(e_n + t).
\]

Corollary 3.4 Let \( X \) be a Jordan domain satisfying that \( h_X(x_0, x) \leq \Psi_{1-s, \sigma}(\frac{1}{d(x, \partial X)}) \) with \( s \in (0, 1) \) and \( \sigma_i \geq -1 \) for all \( i = 1, \ldots, n \). Let \( \varphi : \partial X \to S \) be a homeomorphism and let \( h : X \to \mathbb{D} \) be its harmonic extension. Then, for \( \lambda = (\lambda_1, \ldots, \lambda_n) \), we have that
\[
\int_X \Psi_{1+s, \lambda}(|Dh(z)|) \, dz < \infty
\]
and
\[
\int_X |Dh(z)|^{1+s} \Psi_{0, \lambda}(\frac{1}{d(\varphi(z), \partial X)}) \, dz < \infty,
\]
whenever \( \lambda_n < -1 - \sigma_n \) and \( \lambda_i < -1 - \sigma_i \) for all \( i = 1, \ldots, n - 1 \).

The proof of Corollary 3.4 is similar to that of Corollary 3.3. The only thing to be noticed is that for \( \Psi_{1-s, \sigma} \), as in (3.13), we estimate its inverse as
\[
\Psi_{1-s, \sigma}^{-1}(t) \approx t^{\frac{1}{\lambda}} \log t^{\frac{1}{\lambda}}(e + t) \log t^{\frac{1}{\lambda}}(e_2 + t) \cdots \log t^{\frac{1}{\lambda}}(e_n + t).
\]

4 Counter-examples Related to Optimal Regularities

In this section, we provide an example to show the sharpness of ranges on \( \lambda \) in Theorems 1.1 and 1.2, and also an example for those in Corollary 3.4.

Example 4.1 For any \( s \in (0, 1) \), there is a Jordan domain \( X \) with \( s \)-hyperbolic growth and a homeomorphism \( \varphi : \partial X \to \partial \mathbb{D} \) which do not admit a homeomorphic extension \( h : X \to \mathbb{D} \) with \( \int_X |Dh|^{1+s} \, dx < \infty \) or \( \int_X |Dh|^{1+s} \log \frac{1}{d(x, \partial X)} < \infty \).
Construction of Example 4.1  Our construction is analogous to that by Koski-Onninen [16]. Start with a graph \( \{(x, |x|^s) : x \in [-1, 1]\} \). Then complete this graph to obtain a smooth Jordan curve except for the above cusp point. Denote by \( \mathbb{X} \) the bounded Jordan domain enclosed by the preceding curve. The above description on \( \partial \mathbb{X} \) determines that \( \mathbb{X} \) has \( s \)-hyperbolic growth. In sequel we only care about \( \varphi \) and \( h \) in a neighbourhood of the cusp point.

\[
\text{Figure 1: The construction}
\]

Let us define a homeomorphism \( \varphi : \partial \mathbb{X} \to \partial \mathbb{D} \). Divide the cusp graph by a sequence of line segments parallel to the \( x \)-axis. Denote by \( p_k^- \) the left intersection point between the \( k \)-line segment and the graph, and by \( p_k^+ \) the right one; see Figure 1. We require that the \( y \)-coordinates of \( p_{k-1}^+ \) are \( \sum_{k=2}^{\infty} \epsilon_k = k^{-2} \) be the difference between the \( y \)-coordinates of \( p_{k-1}^- \) and \( p_k^+ \) for \( k = 2, 3, \cdots \). We afterwards divide the unit disk \( \mathbb{D} \cap \{ (x, y) : x \geq 0 \} \). To do this we use a sequence of line segments parallel to the \( y \)-axis. Denote by \( a_k^+ \) the upper intersection point between the \( k \)-line segment and \( \partial \mathbb{D} \), and by \( a_k^- \) the lower one. Let \( d_k = \log \frac{1}{1+s \log(1+k)} \) be the length of the line segment between \( a_k^- \) and \( a_k^+ \). By the constant speed, we define \( \varphi \) to map the arc of the cusp graph between \( p_k^- \) and \( p_{k-1}^- \) onto the arc of the unit circle between \( a_k^- \) and \( a_{k-1}^- \).

Assume that \( h \) is a homeomorphism extension of \( \varphi \) with Sobolev regularity. To finish the construction of Example 4.1, it suffices to check that

\[
\int_{\mathbb{X}} |Dh|^{1+s} \log^{-1}(e + |Dh|) = \infty \quad \text{and} \quad \int_{\mathbb{X}} |Dh|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial \mathbb{X})}) = \infty.
\]

Let \( S_k \) be the subset of the original cusp domain between line segments \( p_{k-1}^- p_k^- \) and \( p_k^- p_k^+ \). By Fubini’s theorem and the ACL property of Sobolev functions, we have that

\[
\int_{S_k} |Dh| \gtrsim \sum_{j \geq k+1} \epsilon_j \int_{-y^{1/s}}^{y^{1/s}} \left| \frac{\partial h(x, y)}{\partial x} \right| dxdy \gtrsim \sum_{j \geq k+1} \epsilon_j d_k dy = \epsilon_k d_k.
\]  \hspace{1cm} (4.1)

Here we only care about the sufficiently large \( k \). Note that \( |S_k| \approx \epsilon_k \left( \sum_{j=k}^{\infty} \epsilon_j \right)^{1/s} \approx k^{-2-\frac{s}{2}} \). Hence (4.1) implies that

\[
\int_{S_k} |Dh| \gtrsim k^{\frac{s}{2}} \log^{-1} \log(k).
\]  \hspace{1cm} (4.2)

The function \( t^{1+s} \log^{-1}(e + t) \) is convex and increasing whenever \( t \gg 1 \). Therefore, Jensen’s
inequality and the estimate (4.2) in order imply that
\[ \int_{S_k} |Dh|^{1+s} \log^{-1}(e + |Dh|) \geq |S_k| \left( \int_{S_k} |Dh| \right)^{1+s} \log^{-1}(e + \int_{S_k} |Dh|) \]
\[ \geq |S_k| \left( \sum_{j \leq k} \log^{\frac{k}{1+s}}(k) \right)^{1+s} \log^{-1}(k \log^{\frac{k}{1+s}}(k)) \]
\[ \approx \frac{1}{k \log(k) \log \log(k)}. \]

Afterwards,
\[ \int_{\cup_{k=1}^{\infty} S_k} |Dh|^{1+s} \log^{-1}(e + |Dh|) = \sum_{k=1}^{\infty} \int_{S_k} |Dh|^{1+s} \log^{-1}(e + |Dh|) \]
\[ \geq \sum_{k=1}^{\infty} \frac{1}{k \log(k) \log \log(k)} = \infty. \]

Since \( \bigcup_{k=1}^{\infty} S_k \subset X \), we obtain that \( \int_X |Dh|^{1+s} \log^{-1}(e + |Dh|) = \infty. \)

We next check that \( \int_X |Dh|^{1+s} \log^{-1} \left( e + \frac{1}{d(z, \partial X)} \right) = \infty. \) By Hölder’s inequality, we have that
\[ \int_{S_k} |Dh| = \int_{S_k} |Dh(z)| \log^{\frac{k}{1+s}}(e + \frac{1}{d(z, \partial X)} \log^{\frac{k}{1+s}}(e + \frac{1}{d(z, \partial X)}) dz \]
\[ \leq \left( \int_{S_k} |Dh(z)|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial X)}) dz \right)^{\frac{1}{1+s}} \left( \int_{S_k} \log^k(e + \frac{1}{d(z, \partial X)}) dz \right)^{\frac{1}{s}}. \]
Therefore,
\[ \int_{S_k} |Dh|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial X)}) \geq \left( \int_{S_k} |Dh| \right)^{1+s} \left( \int_{S_k} \log^s(e + \frac{1}{d(z, \partial X)}) \right)^{-s}. \quad (4.3) \]
Notice that
\[ \int_{S_k} \log^\frac{k}{1+s}(e + \frac{1}{d(z, \partial X)}) dz \approx \int \sum_{j \leq k+1} \sum_{j \leq k+1} \left( \int_0^{y^1/s} \log^\frac{k}{1+s}(e + \frac{1}{y^{1/s} - x}) dx \right) dy \]
\[ \approx \int \sum_{j \leq k+1} y^1/s \log^1/s(e + y^{-1/s}) dy \]
\[ \approx \frac{1}{k^{2+s}} \log^{1/s}(k). \]

Therefore, with the addition of (4.1), we obtain from (4.3) that
\[ \int_{S_k} |Dh|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial X)}) dz \geq (\epsilon_k d_k)^{1+s} \left( \frac{1}{k^{2+s}} \log^1/s(k) \right)^{-s} \approx \frac{1}{k \log(k) \log \log(k)}. \]

Afterwards,
\[ \int_{\bigcup_{k=1}^{\infty} S_k} |Dh|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial X)}) = \sum_{k=1}^{\infty} \int_{S_k} |Dh|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial X)}) \]
\[ \geq \sum_{k=1}^{\infty} \frac{1}{k \log(k) \log \log(k)} = \infty. \]

Since \( \bigcup_{k=1}^{\infty} S_k \subset X \), we obtain that \( \int_X |Dh|^{1+s} \log^{-1}(e + \frac{1}{d(z, \partial X)}) = \infty. \)

The following example is to show the sharpness of \( \lambda_i \) in Corollary 3.4.

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Example 4.2 Let $\Psi_{1-s,\sigma}$ be as in (3.13). We construct a Jordan domain $X$ satisfying that $h_X(x_0, x) \leq \Psi_{1-s,\sigma}(\frac{1}{d(z, \partial X)})$ with $s \in (0, 1)$. There is a homeomorphism $\varphi : \partial X \to \mathbb{S}$ which does not admit a homeomorphic extension $h : X \to \mathbb{D}$ with $\int_X \Psi_{1+s,\lambda}(|Dh|) < \infty$ or $\int_X |Dh|^{1+s}(z)\Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz < \infty$ when $\lambda = (\lambda_1, \ldots, \lambda_n)$ with $\sigma_i + \lambda_i = -1$ for any $i = 1, \ldots, n$.

Construction of Example 4.2 The construction is analogous to that of Example 4.1. In sequel, we sketch the process and estimates, so here we leave the detailed arguments to the interested reader. We follow the notation in Example 4.1. By the cusp graph $(x, \Psi_{-s,\sigma}(1/|x|)) : x \in [-1, 1]$, we define a Jordan domain $X$. We calculate that $X$ satisfies

$$\int_X \Psi_{1+s,\lambda}(|Dh|) = \infty \text{ and } \int_X |Dh|^{1+s}(z)\Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz = \infty,$$

with $\sigma_i + \lambda_i = -1$ for any $i = 1, \ldots, n$. On a piece $S_k \subset X$, we estimate that

$$\int_{S_k} |Dh| \geq \epsilon_k d_k \text{ and } |S_k| \approx \epsilon_k \Psi_{1-s,\sigma}(1/d(z, \partial X)).$$

Therefore,

$$\int_{S_k} \Psi_{1+s,\lambda}(|Dh|) \geq |S_k| \Psi_{1+s,\lambda}(\int_{S_k} |Dh|) \geq \frac{1}{k} \log^{\sigma_1+\lambda_1}(k) \cdots \log^{\sigma_n+\lambda_n}(k) \log^{-1}(k).$$

Finally, $\int_X \Psi_{1+s,\lambda}(|Dh|) \geq \sum_{k=1}^{\infty} \frac{1}{k} \log^{\sigma_1+\lambda_1}(k) \cdots \log^{\sigma_n+\lambda_n}(k) \log^{-1}(k) = \infty$ whenever $\sigma_i + \lambda_i = -1$ for any $i = 1, \ldots, n$.

As in (4.3), we estimate that

$$\int_{S_k} |Dh|^{1+s} \Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz \geq \left( \int_{S_k} |Dh| \right)^{1+s} \left( \int_{S_k} \Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz \right)^{-s}. \quad (4.5)$$

Moreover, $\int_{S_k} \Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz \approx \Psi_{2-s,\sigma} \approx \frac{1}{s} \Psi_{2-s,\sigma}(k)$. Here $\frac{s+\lambda}{s} = (\frac{s+1}{s}, \ldots, \frac{s+n}{s})$. Together with the estimate $\int_{S_k} |Dh|$ as in (4.4), we derive from (4.5) that

$$\int_{S_k} |Dh|^{1+s}(z)\Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz \geq \frac{1}{k} \log^{\sigma_1+\lambda_1}(k) \cdots \log^{\sigma_n+\lambda_n}(k) \log^{-1}(k).$$

Finally, $\int_X |Dh|^{1+s}(z)\Psi_{0,\lambda}(\frac{1}{d(z, \partial X)})dz \geq \sum_{k=1}^{\infty} \frac{1}{k} \log^{\sigma_1+\lambda_1}(k) \cdots \log^{\sigma_n+\lambda_n}(k) \log^{-1}(k) = \infty$ whenever $\sigma_i + \lambda_i = -1$ for any $i = 1, \ldots, n$.

Acknowledgements The authors would like to thank Prof. Chang-Yu Guo for a careful reading of the manuscript and for many suggestions.
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