The construction of a partially regular solution to the Landau–Lifshitz–Gilbert equation in $\mathbb{R}^2$

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Received 29 December 2004, in final form 8 August 2005
Published 16 September 2005
Online at stacks.iop.org/Non/18/2681

Recommended by Weinan E

Abstract

We establish a framework to construct a global solution in the space of finite energy to a general form of the Landau–Lifshitz–Gilbert equation in $\mathbb{R}^2$. Our characterization yields a partially regular solution, smooth away from a two dimensional locally finite Hausdorff measure set. This construction relies on approximation by discretization, using the special geometry to express an equivalent system whose highest order terms are linear and the translation of the machinery of linear estimates on the fundamental solution from the continuous setting into the discrete setting. This method is quite general and accommodates a larger class of geometries involving targets that are closed smooth surfaces.

Mathematics Subject Classification: 35M10, 35Q99

1. Introduction

Micromagnetics, a model based on the work of Landau and Lifshitz, offers a description of magnetic behaviour in ferromagnetic materials. This continuum model is based on finding critical points of the micromagnetic energy associated to the magnetic moment represented by the field $u : \Omega \to S^2$:

$$E[u] = \int_{\Omega} |\nabla u|^2 + \kappa \int_{\Omega} \phi(u) + \int_{\mathbb{R}^3} |\nabla M|^2 \, dx - 2 \int_{\Omega} h_{\text{ext}} \cdot u,$$

where $\Omega$ is the region occupied by the ferromagnet, and all the physical constants have been normalized to be 1. These four terms are known as the exchange, anisotropic, magnetostatic and external energies, respectively. The exchange energy penalizes spatial variation over macroscopic distances. The anisotropic term favours special directions of the magnetization. The magnetostatic energy represents the work required to build up the macroscopic body by bringing its magnetic moments from infinity to their eventual position; $M$ is nonlocal and is defined by $\Delta M = \text{div} \ u$ where $u$ is extended to be 0 outside of $\Omega$, and the equation is to be understood in the sense of distributions. Finally, the external energy favours magnetization
aligned with an external applied field. The first variation of the energy with respect to $u$ is called the effective field:

$$h_{\text{eff}} = -\frac{\delta E[u]}{\delta u}.$$

Constrained to the set of candidate solutions $\{u : \Omega \to S^2\}$, the resulting dynamic equations associated to this energy are, in dimensionless form, given by

$$\partial_t u = u \land h_{\text{eff}},$$

where $\land$ denotes the wedge product in $\mathbb{R}^3$. To incorporate the Gilbert damping law, whose origin lies in the observation that such systems reach equilibrium and must have decreasing energy over time, a dissipative term can be added on, resulting in the following equation:

$$\partial_t u = u \land h_{\text{eff}} - \alpha u \land (u \land h_{\text{eff}}).$$

In the physical setting, the magnetization configuration is a compromise that results from the competition of all of these terms to minimize the energy. However, many relevant features of solutions to the full equation are captured by retaining the term associated to the exchange energy in which case $h_{\text{eff}} = \Delta u$. For the question we address here, in particular, the other terms can be considered lower order terms.

In this paper, we consider an analytically interesting variant of the problem which results from enlarging the class of target spaces beyond $S^2$, in particular, to smooth closed surfaces $\mathcal{N}$ embedded in $\mathbb{R}^3$. Although this variant does not hold common currency in micromagnetics, this is potentially a reasonable means of incorporating anisotropic contributions. Let $\mathcal{N}$ be a smooth closed surface embedded in $\mathbb{R}^3$ with normal vector $\nu$. The Cauchy problem for this generalized Landau–Lifshitz–Gilbert (LLG) equation is the problem of finding $u : \mathbb{R}^2 \times \mathbb{R} \to \mathcal{N}$, given initial data $f : \mathbb{R}^2 \to \mathcal{N}$, satisfying

$$\left\{ \begin{array}{l}
\partial_t u = \nu(u) \land \Delta u - \alpha \nu(u) \land (\nu(u) \land \Delta u), \\
u(x, 0) = f,
\end{array} \right.$$

(1.1)

where $\alpha > 0$. We will refer to the first term as the Schrödinger term (this terminology will be clarified presently) and the second as the damping term. When only the damping term is present, this equation is the harmonic map heat flow problem, which will later be a focus of discussion. Although standard usage of LLG refers to (1.1) in the special case $\mathcal{N} = S^2$, we adopt here a usage to accommodate this general class of target spaces. There are several standard forms of LLG which are valid for smooth solutions. Using the vector identity $\xi \land (\xi \land v) = -v + (v, \xi)\xi$ which holds for $v$ a unit vector, we can write LLG as

$$\partial_t u = \nu(u) \land \Delta u + \alpha (\nu(u) \cdot \nabla u)\nu(u).$$

(1.2)

The next two formulations are easy consequences of (1.1) and (1.2):

$$\partial_t u = \nu(u) \land \Delta u + \alpha \nabla (\nu(u) \cdot \nabla u)\nu(u),$$

(1.3)

$$\partial_t u + \alpha \nu(u) \land \partial_t u = (1 + \alpha^2)\Delta u + \alpha (\nu(u) \cdot \nabla u)\nu(u).$$

(1.4)

There is an intrinsic interpretation of the LLG as a specific instance of a damped Schrödinger map. This can be done by realizing that two geometric facts are being employed when writing the LLG: the metric given by the inner product in $\mathbb{R}^3$ and the complex structure on the tangent space. A damped Schrödinger map is given by $u : \mathbb{R}^d \times \mathbb{R} \to (\mathcal{N}, g, J)$, where $\mathcal{N}$ is a Kähler manifold equipped with an almost complex structure $J$ and a metric $g$ giving rise to covariant differentiation $D$, satisfying

$$\partial_t u = J(u)D\partial u + \alpha D\partial u.$$
In the case of LLG, $J = v(u) \wedge$ and $D\partial u = \Delta u - (\Delta u, u)u$. Our attempt to generalize the setting of the LLG beyond $S^2$ target must first respect these geometric properties which are crucial to the well-posedness of the equation. An easy check verifies that $v(u) \wedge$ satisfies the properties of a complex structure acting on $T_uN$. Our choice of target space to include closed surfaces with sufficient smoothness is driven by our need to define a frame over $N$ to establish a derivative formulation of the equation.

The question that we address in this paper is whether global solutions exist in the class of finite Dirichlet energy. The main result in this paper is the following theorem.

**Theorem 1.1.** Let $\mathcal{N}$ be a smooth closed surface embedded in $\mathbb{R}^3$. For any $f \in H^1(\mathbb{R}^2, N)$, there exists a solution to (1.1) which is smooth away from a singular set that has a closed set of locally finite two dimensional Hausdorff measure with respect to the parabolic metric.

Global weak solutions, even partially regular ones, have been shown to exist for such equations in the case $\mathcal{N} = S^2$. The techniques developed, however, often exploit the special geometry of $S^2$ and side-step the natural difficulties that the general problem poses. Consequently, these techniques are not easily adaptable to the setting that we consider here. One category of results showing existence of weak solutions has made crucial use of the fact that LLG can be written in divergence form in the case $\mathcal{N} = S^2$; this in turn gives rise to a definition of weak solution. Amongst these is the work of Alouges and Soyeur [1] who have shown that in three dimensions energy bounds are sufficient for the existence of such a weak solution. Guo and Hong [4] carried through the argument that Struwe in [9] employed for the harmonic map heat flow to exhibit a Struwe solution, a partially regular solution that satisfies an energy inequality and is smooth away from a finite set of point singularities. In this dimension, Struwe shows that the existence of smooth solutions in addition to energy estimates is sufficient for the characterization of such a partially regular solution. In the case of the sphere, the problematic term $u \wedge \Delta u$ does not preclude energy estimates since multiplication by $u$ and $\Delta u$ eliminates this term altogether. Wu, in [11], further clarified the local existence that was used by Guo and Hong using a Galerkin approximation. It is worthwhile to note that uniqueness in the class of partially regular solutions is open; our partially regular solution may be different from the Struwe solution. Through private communication, we have learned that Melcher in [8] has recently attained a characterization of partially regular solution similar to ours for $d = 3$ and $\mathcal{N} = S^2$ using a Ginzburg–Landau approximation.

In the context of existing results, the main difficulties are already apparent. The nonlinearity in the highest order $v(u) \wedge \Delta u$ term poses the most apparent obstacle. At first glance, it renders unavailable the machinery of partial regularity theory established for semilinear equations: linear estimates on the fundamental solution (e.g. $L^p - L^q$, Strichartz-type) and regularity arguments based on the inversion of the fundamental solution. An additional problem that this term poses for general target $\mathcal{N}$ is that the local existence of smooth solutions is not just an immediate consequence of parabolic theory. Crucial tools that are lost include the maximum principle as well as a Bochner identity. Lastly, the divergence structure of the equation is lost when we depart from the sphere.

Our method is based on approximations of the problem that arise from discretizing the domain using a uniform spatial grid. The idea is to show that a sequence of solutions to these approximations converges to a solution of the original equation as the grid-size approaches zero. The idea of approximating systems via discretization is not new; in fact, one of the earliest works investigating (1.1) for the case $\alpha = 0$, $\mathcal{N} = S^2$ [10] uses discrete approximations to show local existence. However, extending its scope to partial regularity is new. While the techniques developed here are demonstrated for the specific setting of the LLG equations, the generality of these techniques should be emphasized. The key ingredients used are (i) the
construction of a suitable discretization of the system; (ii) establishing a suitable linearization of the discrete system; (iii) the construction of the discrete fundamental solution of the resulting linear operator and deriving appropriate linear mixed space–time estimates. The setting that enables the discrete constructions described by (i) and (iii) are the subject of section 2.

The advantages of this method address the difficulties of the problem. By discretizing the system, existence of solutions to the resulting ODE is immediate once we attain uniform bounds; there is no need to appeal to heavy machinery to conclude that smooth solutions exist. The transformation which results in a linear highest order term opens the door to the arsenal of linear estimates and methods. The explicit construction of the discrete fundamental solution and the resulting linear estimates permit higher derivative bounds that have no dependence on the special structure of the sphere.

Before employing the proposed method to construct partially regular solutions for LLG, we illustrate in section 3 this method for the harmonic map heat flow problem, which involves only the damping term. The same characterization of a partially regular solution has been achieved by other methods (e.g. [3]), but we consider this excursion to be worthwhile since it is a setting in which we can isolate the difficulties that just one term poses (makes use of ingredients (i) and (iii) above) before adding on the separate difficulty involving the Schrödinger term. In section 4, our main result is proved.

2. The discrete setting

2.1. Notation and definitions

Consider a uniform grid on \( \mathbb{R}^d \) determined by the lines \( x_i = j_i h, i = 1, 2, \ldots, d \) where \( j_i \) are integers. For vector valued functions \( u^h \) defined on the grid, the definitions of scalar product and discrete \( L^p \) norms that will be used are

\[
(u^h, v^h)_{L^2_h} = h^d \sum_j u^h_j v^h_j,
\]

\[
\|u^h\|_{L^p_h} = \left( h^d \sum_j |u^h_j|^p \right)^{1/p},
\]

where \( u^h_j = u^h_{j_1 h, \ldots, j_d h} = u(j_1 h, \ldots, j_d h) \) and \( u^h_{j_i+1} = u(j_1 h, \ldots, j_i h + h, \ldots, j_d h) \). The basic difference operations are

\[
D_{+i} u^h_j = \frac{u^h_{j_i+1} - u^h_j}{h}, \quad D_{-i} u^h_j = \frac{u^h_j - u^h_{j_i-1}}{h}, \quad D_{0i} u^h_j = \frac{u^h_{j_i+1} - u^h_{j_i-1}}{2h}
\]

and

\[
\delta_{+i}^2 u^h_j = D_{+i} D_{-i} u^h_j = D_{-i} D_{+i} u^h_j = \frac{1}{h} (D_{+i} - D_{-i}) u^h_j = \frac{u^h_{j_i+1} - 2u^h_j + u^h_{j_i-1}}{h^2}.
\]

The discrete Laplacian \( \Delta_h \) can be defined as

\[
\Delta_h = \sum_{i=1}^d \delta_{+i}^2.
\]

For some multi-index \( \alpha = (\alpha_1^+, \alpha_1^-, \ldots, \alpha_d^+, \alpha_d^-) \), \( \alpha_i^+, \alpha_i^- \in \mathbb{N} \), we will use the notation

\[
|\alpha| = \sum_{i=1}^d (\alpha_i^+ + \alpha_i^-)
\]
The discrete Sobolev spaces are defined as
\[ W^{k,p}_h \triangleq \left\{ u^h \mid \| u^h \|_{W^{k,p}_h} = \sum_{|\alpha| \leq k} \| D^\alpha u^h \|_{L^p_h} < \infty \right\}, \]
with \( H^k_h = W^{k,2}_h \). Their homogeneous counterparts are defined by
\[ \dot{W}^{k,p}_h \triangleq \left\{ u^h \mid \| u^h \|_{\dot{W}^{k,p}_h} = \sum_{|\alpha| = k} \| D^\alpha u^h \|_{L^p_h} < \infty \right\}, \]
with \( \dot{H}^k_h = \dot{W}^{k,2}_h \).

Notable imperfections of discretization. Discretization introduces some imperfection which are important to highlight. The common operations of product differentiation and integration by parts have a slight shift. The forms of product differentiation that will be used are the following:
\[
\begin{align*}
D^\alpha_i (u^h v^h)_j & = u^h_j D^\alpha_i v^h_j + v^h_j D^\alpha_i u^h_j \\
& = v^h_j D^\alpha_i u^h_j + u^h_j D^\alpha_i v^h_j \\
& = \frac{1}{2} [ (u^h_{j+1} + u^h_j) D^\alpha_i v^h_j + (v^h_{j+1} + v^h_j) D^\alpha_i u^h_j ].
\end{align*}
\]
For functions \( u^h, v^h \) with decay both belonging to \( L^2_h \), the summation by parts formula—the discrete analogue of integration by parts—is given by
\[
(u^h, D^\alpha_i v^h)_{L^2_h} = - (D^\alpha_i u^h, v^h)_{L^2_h}.
\]

2.2. Basic inequalities and some useful tools

Most of the inequalities that will be used have long been established. Amongst these are Holder’s, Minkowski’s and the \( L^p \) interpolation inequalities which translate perfectly to the discrete setting. Discrete versions of the Sobolev embedding theorems were established by Ladyzhenskaya [6] using interpolation operators. In particular, we will make use of the following discrete Sobolev-interpolation inequality:
\[
\| D^j u^h \|_{L^q} \leq C \| u^h \|_{L^r}^{1-\theta} \| D^k u^h \|_{L^p}^\theta, \quad \frac{1}{r} - \frac{j}{n} = \frac{1}{q} - \frac{k}{n} + \theta \left( \frac{1}{q} - \frac{1}{p} \right). \tag{2.1}
\]

The interpolation operators that supply the proof for the above inequalities are useful, and we mention one which will play an immediate and a later role. Confining our attention to the case \( d = 2 \) for the remainder of this section, associate with \( u \) (which is only defined on the uniform grid in the plane) the unique interpolated polynomial \( p_n \), which matches the value of \( u \) at each grid point and is of the form \( a_0 + a_1 x + a_2 y + a_3 xy \) in each square. Specifically, for \( j \) indexing pairs \((j_1, j_2)\),
\[
p_n = \sum_j (a_j + D_{j_1} u_j (x - j_1 h) + D_{j_2} u_j (y - j_2 h) + D_{j_1} D_{j_2} u_j (x - j_1 h)(y - j_2 h)) \chi_{\square_j} (x, y)
\]
where \( \chi_{\square}(x, y) \) denotes the characteristic function on the square \( K_j \). Across any one side of a square \( p_h \) is linear, so we have that \( p_h \) is continuous across interfaces. \( p_h \) is an \( H^1 \)-interpolant associated to \( u \). Of particular importance are the equivalence of the norms \( \| p_h \|_{L^p} \) and \( \| u^h \|_{L^p} \) as well as \( \| \nabla p_h \|_{L^p} \) and \( \| D^1 u^h \|_{L^p} \) for any \( p \geq 1 \). Equipped with this interpolant, we can prove the following propositions. The first is a localized Sobolev-interpolation inequality.

**Proposition 2.1.** Let \( \Omega \subset \mathbb{R}^2 \) and \( \zeta \in C^\infty_0(\Omega) \). Then

\[
\|u^h\|^2 \zeta \|_{L^p_0(\mathbb{R}^2)} \leq C\|u^h\|_{L^p_0(\Omega)}\|D^1(u^h\zeta)\|_{L^p_0(\mathbb{R}^2)}, \quad \frac{s + 1}{2s} = \frac{1}{p} + \frac{1}{q}.
\]

**Remark 2.2.** Without the \( \zeta \), the quantity of interest is \( \|u^h\|^2 \|_{L^p_0(\mathbb{R}^2)} \), and the proposed inequality is given by (2.1) for \( n = 2, r = 4s, j = 0, k = 1, \theta = \frac{1}{2} \).

**Proof.** This inequality follows from the equivalence of the norms \( \| p_h \|_{L^p} \) and \( \| u^h \|_{L^p} \) as well as \( \| \nabla p_h \|_{L^p} \) and \( \| D^1 u^h \|_{L^p} \) for any \( p \geq 1 \) and the equivalent statement in the continuous setting, as proved in appendix A.

**Proposition 2.3.** If \( \|u^h\|^2 \|_{H^1_0(\mathbb{R}^2)} < C \) independent of \( h \), then there is a subsequence \( \{ p_{h_n} \} \) that converges strongly in \( H^1(\Omega) \) for \( \Omega \) compact in \( \mathbb{R}^2 \).

**Proof.** Notice that the boundedness of \( p_h \) in \( H^2(\Omega) \) immediately gives weak convergence in \( H^1(\Omega) \). To show strong \( H^1 \) convergence, we employ Ascoli–Arzela. In particular, we must show that \( \| \nabla p_h \|_{L^2} \) are bounded and equicontinuous in \( L^2(\Omega) \).

We have \( L^2 \) bounds on \( \nabla p_h \) by construction. To show equicontinuity, we must show that \( \forall \epsilon > 0, \exists \delta = (\delta_1, \delta_2) = (k_1h, k_2h) \) such that \( \| \nabla p_h(\cdot + \delta_1, \cdot + \delta_2) - \nabla p_h \|_{L^2} < c|\delta| \). By the equivalence of the norms \( \| p_h \|_{L^p} \) and \( \| u^h \|_{L^p} \), it suffices to show this for \( k_1, k_2 \in \mathbb{Z} \).

We have

\[
\| \nabla p_h(\cdot + \delta_1, \cdot + \delta_2) - \nabla p_h \|_{L^2}^2 \leq 2\left( \sum_{j_1, j_2} \int_{\Omega} (D_{j_1} u_{j_1 + k_1, j_2} - D_{j_1} u_{j_1, j_2})^2 + (D_{j_2} D_{j_1} u_{j_1 + k_1, j_2} - D_{j_2} D_{j_1} u_{j_1, j_2})^2 \right)
\]

To bound \( I \), we can write the difference \( D_{j_1} u_{j_1 + k_1, j_2} - D_{j_1} u_{j_1, j_2} \) as

\[
D_{j_1} u_{j_1 + k_1, j_2} - D_{j_1} u_{j_1, j_2} = \sum_{k=1}^{k_1} (D_{j_1} u_{j_1 + k, j_2} - D_{j_1} u_{j_1, j_2}).
\]
Since \((a_1 + \cdots + a_n)^2 \leq 2n(a_1^2 + \cdots + a_n^2)\), we have the bound
\[
|D_{+1} u_{j_1+k_1, j_2} - D_{+1} u_{j_1, j_2}|^2 \leq 2k_1 \sum_{k=1}^{k_1} |D_{+1} u_{j_1+k, j_2} - D_{+1} u_{j_1+k, j_2}|^2.
\]
Similarly, we have the bound
\[
|D_{+2} u_{j_1+j_2+k_2} - D_{+1} u_{j_1, j_2}|^2 \leq 2k_2 \sum_{k=1}^{k_2} |D_{+1} u_{j_1+j_2+k} - D_{+1} u_{j_1+j_2+k}|^2.
\]
Using the given information that \(\|u^h\|_{B_1} < C\), we have in particular that
\[
\|\Delta u\|_{L_2}^2 = \sum_{j_1, j_2} |D_{+1} u_{j_1, j_2} - D_{-1} u_{j_1, j_2}|^2 + |D_{+2} u_{j_1, j_2} - D_{-2} u_{j_1, j_2}|^2 < C,
\]
which allows us to conclude that
\[
I = \sum_{j_1, j_2} \int_{\square_{j_1, j_2}} |D_{+1} u_{j_1+k_1, j_2} - D_{+1} u_{j_1, j_2}|^2 + |D_{+2} u_{j_1+j_2+k_2} - D_{+2} u_{j_1+j_2}|^2
\leq h^2 \sum_{j_1, j_2} |D_{+1} u_{j_1+k_1, j_2} - D_{+1} u_{j_1, j_2}|^2 + |D_{+2} u_{j_1+j_2+k_2} - D_{+2} u_{j_1+j_2}|^2
\leq h^2 [2(k_1 - 1) \sum_{j_1, j_2} \sum_{k=1}^{k_1} |D_{+1} u_{j_1+k_1, j_2} - D_{+1} u_{j_1+k_1-1, j_2}|^2
+ 2(k_2 - 1) \sum_{j_1, j_2} \sum_{k=1}^{k_2} |D_{+1} u_{j_1+j_2+k_2} - D_{+1} u_{j_1+j_2+k_2-1}|^2]
\leq 2h^2 [(k_1 - 1)k_1 C + (k_2 - 1)k_2 C]
\leq 2Ch^2(k_1^2 + k_2^2)
\leq c|\delta|^2.
\]
By assumption, we also have that \(\sum_{j_1, j_2} h^2 |D_{+1} D_{+2} u_{j_1, j_2}|^2 \leq C\), so
\[
II \leq \frac{h^3}{3} \sum_{j_1, j_2} |D_{+1} D_{+2} u_{j_1+k_1, j_2} - D_{+1} D_{+2} u_{j_1, j_2}|^2
+ |D_{+1} D_{+2} u_{j_1+j_2+k_2} - D_{+1} D_{+2} u_{j_1+j_2}|^2
\leq c_1 h^4 \left( k_1(k_1 - 1) \frac{C}{h^2} + k_2(k_2 - 1) \frac{C}{h^2} \right)
\leq \tilde{c} |\delta|^2.
\]

2.3. Constructing discrete approximations
We now construct semi-discrete approximations for LLG. The method that we adopt is that of finite-differencing on a uniform spatial grid with gridsize \(h\). This method reduces the problem to a system of ODEs with the unknowns being \(\{u^h_j\}\), the value of the function at every grid point. We seek a construction that yields a stable scheme. A natural way to achieve this is to start by discretizing the energy rather than by discretizing the equation. Explicitly, the Dirichlet energy can be discretized to yield the discrete energy \(E^h[u]\):
\[
E^h[u^h] = \frac{1}{2} h^2 \sum_j \frac{1}{2} (|D_{+a} u^h_j|^2 + |D_{-a} u^h_j|^2).
\]
where $j$ indexes over the grid pairs $(j_1, j_2)$. The discrete analogue of $\delta E[u]/\delta u$ is

$$\frac{\delta E^h[u^h]}{\delta u^h_j} = -\frac{u^h_{j+1} - 2u^h_j + u^h_{j-1}}{h^2} = -\Delta^h u^h_j.$$  

The discretization of the system associated to this energy is then given by

$$\begin{cases}
\partial_t u^h_j(t) = \nu_j \wedge \Delta^h u^h_j + \alpha (\Delta^h u^h_j + \lambda^h_j \nu_j), \\
u^h_j(t = 0) = f^h_j,
\end{cases} \tag{2.2}$$

where $\nu_j$ denotes the unit normal vector to $N$ at $u^h_j(t)$ and Lagrange’s multiplier $\lambda^h_j(t)$ is given by $\lambda^h_j(t) = -\Delta^h u^h_j(t) \cdot \nu_j(t)$.

We also have the discrete equivalent of (1.3):

$$\alpha \partial_t u^h_j(t) - \nu_j \wedge \partial_t u^h_j = (1 + \alpha^2)(\Delta^h u^h_j + \lambda^h_j v_j). \tag{2.3}$$

This discretization yields energy bounds. In particular, since $\nu_j$ is orthogonal to $\partial_t u^h_j$, we can multiply (2.3) by $\partial_t u^h_j$ and sum across indices $j$ to get

$$\partial_t E^h[u^h(t)] = \left( \partial_t u^h, \frac{\delta E^h}{\delta u^h} \right)_{L^2_h} = -\alpha \| \partial_t u^h \|^2_{L^2_h}. \tag{2.4}$$

Since $\alpha > 0$, we can integrate on any time interval $[0, T]$ to get the discrete energy bound:

$$E^h[u^h(T)] \leq E^h[f^h].$$

### 2.4. Constructing discrete fundamental solutions

Our method of proof is based on linear estimates. Although not immediately apparent, the relevant linear equation to consider is the linear damped Schrödinger equation. Since we will also be looking at the harmonic map heat flow, we need to consider the linear heat equation.

We now construct fundamental solutions for the discrete analogues of these equations. The fundamental solution of the discrete heat equation is given by $\Phi^h_j(t) = (\Phi^h_j)_j$, where $\Phi^h_j$ solves the system

$$\begin{cases}
\partial_t \Phi^h_j(t) = \Delta^h \Phi^h_j, \\
\Phi^h_j(t = 0) = \frac{1}{h^d} \delta^h_{k-j},
\end{cases}$$

where $\delta^h_{k-j} = 1$ for $k = j$ and 0 elsewhere. The notation is consistent with the definition of the weighted $h$ norms that is being used and yields $\|(1/h^d)\delta^h_x\|_{L^1(R^d)} = 1$. An explicit description of the fundamental solution follows from the observation that the discrete Laplacian $\Delta^h u^h_j = \sum_{i=1}^d \frac{1}{h} (D_{+i} - D_{-i}) u^h_j$ can be written as the sum of operators,

$$\Delta^h u^h_j = \sum_{i=1}^d \frac{1}{h} (D_{+i} - D_{-i}) u^h_j.$$  

As in the continuous case, the description of the fundamental solution in higher dimension rests on the one dimensional problem of solving

$$\begin{cases}
\partial_t u^h_{+j}(t) = \frac{1}{h} D^+ u^h_{+j} - \frac{1}{h} D^- u^h_{+j}, \\
u^h_{+j}(t = 0) = \frac{1}{h} \delta^h_{j+1},
\end{cases} \tag{2.5}$$
The fundamental solution associated to the operator \((\partial / \partial t) - (1/h)D^+\) is given by \(K^h_+\), where

\[
(K^h_+)_j(t) = \begin{cases} 
\frac{1}{h} \left( \frac{t}{h^2} \right)^{-j} e^{-t/h^2} & \text{if } j_i \leq 0, \\
0 & \text{otherwise}
\end{cases}
\]

and that associated with \((\partial / \partial t) + (1/h)D^-\) is given by \(K^h_-\), where

\[
(K^h_-)_j(t) = \begin{cases} 
\frac{1}{h} \left( \frac{t}{h^2} \right)^j e^{-t/h^2} & \text{if } j_i \geq 0, \\
0 & \text{otherwise}
\end{cases}
\]

Since the operators \(D^+\) and \(D^-\) commute, the fundamental solution associated to their sum is just the discrete convolution of the individual fundamental solutions. The solution to (2.5) is thus given by

\[
\Phi^h_j(t) = \sum_{k \geq j} \frac{1}{h} \left( \frac{t}{h^2} \right)^k \frac{(t/h^2)^k}{(k)!} e^{-2t/h^2}.
\]

The fundamental solution of the discrete heat operator in \(\mathbb{R}^d\), \(\Phi^h_j\), is then just the product of \(\Phi^h_j\) over \(i = 1, \ldots, d\).

Similarly, the fundamental solution of the discrete damped Schrödinger equation, denoted by \(U^\alpha,h = (U^\alpha,h_j)\), solves the system

\[
\begin{align*}
\partial_t U^\alpha,h_j(t) &= (\alpha + i) \Delta^h U^\alpha,h_j, \\
U^\alpha,h_j(t = 0) &= \frac{1}{h^d} \delta^h.
\end{align*}
\]

The construction proceeds exactly as in the case of the heat equation. The following proposition summarizes these constructions.

**Proposition 2.4.**

1. \(\Phi^h = (\Phi^h_j)\), the fundamental solution for the \(d\)-dimensional discrete heat equation, is given by

\[
\Phi^h_j(t) = \sum_{k \geq j} \frac{1}{h} \left( \frac{t}{h^2} \right)^k \frac{(t/h^2)^k}{(k)!} e^{-2t/h^2}.
\]

2. \(U^\alpha,h = (U^\alpha,h_j)\), the fundamental solution for the \(d\)-dimensional discrete damped Schrödinger equation, is given by

\[
U^\alpha,h_j(t) = \sum_{k \geq j} \frac{(\alpha + i)(t/h^2)^k}{h} \frac{(t/h^2)^k}{(k)!} e^{-2(\alpha + i)t/h^2}.
\]
2.5. Discrete linear estimates

Having constructed the discrete fundamental solutions, it is possible to derive discrete analogues of the well-known $L^p - L^q$ estimates.

**Proposition 2.5 (Lp - Lq estimates).** Let $K^h = \Phi^h$ or $U_{\alpha,h}$, $\alpha > 0$ from proposition 2.4. Then the following estimates hold:

\[
(1) \quad \|K^h(t)f^h\|_{L^p_h(\mathbb{R}^d)} \leq C \frac{\|f^h\|_{L^q_h}}{t^{d/(2(1/q - 1/p))}},
\]

\[
(2) \quad \|D^1K^h(t)f^h\|_{L^p_h(\mathbb{R}^d)} \leq C \frac{\|f^h\|_{L^q_h}}{t^{(1/2)+d/(2(1/q - 1/p))}}.
\]

**Proof.** We will show the derivation of both estimates by taking $K^h = \Phi^h$. Since $\alpha > 0$, the same derivation holds in the case that $K^h = U_{\alpha,h}$.

These estimates will follow upon obtaining suitable $L^1_h$ and $L^\infty_h$ estimates on $\Phi^h$ and $D^1\Phi^h$. Specifically, if the bounds

\[
\|\Phi^h(t)\|_{L^1_h} \leq 1, \quad \|\Phi^h(t)\|_{L^\infty_h} \leq \frac{1}{t^{d/2}}
\]

hold, then by Hausdorff–Young we have

\[
\|\Phi^h(t)f^h\|_{L^q_h} \leq \|\Phi^h(t)\|_{L^1_h} \|f^h\|_{L^q_h},
\]

where $1 + (1/p) = 1/r + 1/q$. By interpolation, $\|\Phi^h(t)\|_{L^q_h} \leq (\|\Phi^h(t)\|_{L^1_h})^{1-\theta}(\|\Phi^h(t)\|_{L^\infty_h})^{\theta}$ for $\theta = 1 - (1/r) = 1 - (1 + 1/p - 1/q) = 1/q - 1/p$, so $\|\Phi^h(t)\|_{L^q_h} \leq 1/t^{d/(2(1/q - 1/p))}$, from which (1) follows. Similarly, (2) follows from the estimates

\[
\|D^1\Phi^h(t)\|_{L^1_h} \leq \frac{1}{t^{1/2}}, \quad \|D^1\Phi^h(t)\|_{L^\infty_h} \leq \frac{1}{t^{1/2+d/2}}.
\]

(1) To show $\|\Phi^h(t)\|_{L^1_h} \leq 1$, it suffices to get an estimate on the kernel of the one-dimensional problem, $w^h_{\jmath}(t)$, given by (2.6) which is a discrete convolution of two functions in $L^1_h$. Since the $L^1_h$ bound of a convolution of two $L^1_h$ functions is bounded by the product of the two $L^1_h$ bounds, we have

\[
\|w^h_{\jmath}(t)\|_{L^1_h} = \sum_{k \geq j_h} \frac{1}{h} \frac{t/h^2}{(k-j_\jmath)!} \frac{(t/h^2)^k}{(k)!} e^{-t/h^2} \leq \frac{e^{-2t/h^2}}{h} \sum_{k \geq j_h} \frac{1}{h^2} \frac{(t/h^2)^k}{(k-j_\jmath)!} \frac{(t/h^2)^k}{(k)!} \leq \frac{e^{-2t/h^2}}{h} \frac{1}{h^2} \frac{(t/h^2)^k}{(k)!} \leq \frac{e^{-2t/h^2}}{h^2} e^{2t/h^2} \leq 1.
\]

Since $w^h_{\jmath}(t)$ is a convolution of two $L^1_h$ functions (in $j_\jmath$), the $L^\infty_h$ bound of $w^h_{\jmath}(t)$ will be the product of the $L^\infty_h$ bound of any one of the convolved pair with the $L^1_h$ bound of the
other. Using Stirling’s formula, and re-indexing with $k$, we have
\[
\|w_{j}(t)\|_{L_{h}^{\infty}} \leq \left\| \frac{1}{h} \left( \frac{t}{h^2} \right)^k e^{-t/h^2} \right\|_{L_{h}^{L_{h}}} \leq \frac{1}{h} C k^k e^{-t/h^2} \left\| w_{j}(t) \right\|_{L_{h}^{\infty}} \leq C \left( \frac{1}{h \sqrt{k}} \right)^k e^{-t/h^2} \left\| w_{j}(t) \right\|_{L_{h}^{\infty}}.
\]

Writing $x = t/h$, and letting $f(k) = (x/k)^k e^{-x+k}$, we have
\[
\frac{\partial f}{\partial k} = \left( \left( \frac{x}{k} \right)^k \right)' e^{-x+k} + \left( \frac{x}{k} \right)^k e^{-x+k} = \left( \frac{x}{k} \right)^k (\ln x - 1) e^{-x+k} + \left( \frac{x}{k} \right)^k e^{-x+k} = e^{-x+k} \left( \frac{x}{k} \right)^k \ln x.
\]

Thus $f$ is maximum when $\partial f / \partial k$ equals 0 which occurs when $k = x = t/h^2 \Rightarrow 1/h = \sqrt{k/t}$. Returning to the $L_{h}^{\infty}$ bound of $w_{j}(t)$, we have
\[
\left\| w_{j}(t) \right\|_{L_{h}^{\infty}} \leq C \left( \frac{1}{h \sqrt{k}} f(k) \right)_{L_{h}^{\infty}} \leq C \left( \frac{1}{h \sqrt{k}} \right)^k f(x) = C \frac{1}{\sqrt{t}}.
\]

(2) To get gradient bounds on the fundamental solution, we express
\[
D_{\alpha} \Phi^{\delta}(t) = D_{\alpha} w_{j}^{\delta}(t) \prod_{i \neq \alpha} w_{i}^{\delta}(t).
\]

For $L_{h}^{\infty}$ bounds on the gradient, we use the fact that the gradient will only ‘hit’ one term in the convolved pair and that the other is in $L_{h}^{1}$. Again, let $x = t/h$ and $f(k) = (x/k)^k e^{-x+k}$. Since $f(k)$ achieves its maximum at $k = x$,
\[
|D_{\alpha} w_{j}^{\delta}(t)| \leq \left| \frac{1}{h^2} \left( \frac{t}{h^2} \right)^k f(k) \left( \frac{1}{h^2} \frac{1}{h} + 1 \right) \right| \leq f(x) \frac{k}{k+1} \frac{1}{h} \leq C \frac{1}{h}.
\]

Using the $L_{h}^{\infty}$ bound given by (2.11) for each of the other $d-1$ $w_{j}^{\delta}(t)$ terms in the product, we get
\[
\left\| D_{\alpha} \Phi^{\delta}(t) \right\|_{L_{h}^{\infty}} \leq C \sup_{1 \leq i \leq d} \left\| (D_{\alpha} w_{i}^{\delta}(t)) \prod_{j \neq i} w_{j}^{\delta}(t) \right\|_{L_{h}^{\infty}} \leq C \frac{1}{t^{1/2}}.
\]

Therefore,
\[
\left\| D_{\delta} \Phi^{\delta}(t) \right\|_{L_{h}^{\infty}} \leq C \left( \frac{1}{t^{1/2}} \right)^{d/2}.
\]

Employing again the fact that the $L_{h}^{1}$ bound of a convolution is bounded by the product of each of the $L_{h}^{1}$ bounds in the convolved pair (and hence $\left\| D_{\alpha} w_{j}^{\delta}(t) \right\|_{L_{h}^{1}} \leq C / h \leq C / t^{1/2}$ for each $\alpha$), and that each of the $w_{j}^{\delta}(t)$ is an $L_{h}^{1}$ function, we conclude that
\[
\left\| D_{\delta} \Phi^{\delta}(t) \right\|_{L_{h}^{1}} \leq C \sup_{1 \leq i \leq d} \left\| (D_{\alpha} w_{i}^{\delta}(t)) \prod_{j \neq i} w_{j}^{\delta}(t) \right\|_{L_{h}^{1}} \leq C \frac{1}{t^{1/2}}.
\]
3. Harmonic map heat flow

The first problem on which we apply the machinery that has been developed so far is the Cauchy problem for the harmonic map heat flow problem in two dimensions. This is the problem of finding \( u \), given \( f : \mathbb{R}^2 \rightarrow N \) such that

\[
\begin{aligned}
\partial_t u &= \Delta u + (\nabla v(u) \cdot \nabla u)v(u), \\
\tau(0) &= f.
\end{aligned}
\]

(3.1)

\( N \subset \mathbb{R}^3 \) is taken to be a closed smooth surface. The semi-discrete approximation for this equation follows the construction in section 2.3 and is given by

\[
\begin{aligned}
\partial_t u^h_j (t) &= \Delta^h u^h_j (t) + \lambda^h_j(t) v_j(t), \\
u^h_j(0) &= f^h.
\end{aligned}
\]

(3.2)

where \( v_j = v(u^h_j) \) and Lagrange’s multiplier \( \lambda^h_j(t) \) is given by

\[
\lambda^h_j(t) = -\Delta^h u^h_j(t) \cdot v_j(t).
\]

In what follows, the letter \( C \) will designate a generic constant that may depend on \( N \) but is independent of the gridsize \( h \) or any solution unless specified explicitly like \( C(E^h[f^h]) \).

Estimates on \( \lambda^h_j \). For the purpose of attaining bounds on \( u^h_j \) solving (3.2), it will be useful to preface the technical detail with several observations concerning \( \lambda^h_j \). In the continuous setting, there is an explicit expression for Lagrange’s multiplier due to the fact that \( \partial_i u \in T_u N \). This permits the following manipulation:

\[
-\partial_i u \cdot v(u) = \partial_i ( -\partial_i u \cdot v(u)) + (\partial_i u \cdot \partial_i v(u)) = \nabla v(u) \cdot \nabla u.
\]

Thus,

\[
|\lambda^h_j| = |(\nabla v(u) \cdot \nabla u)| \leq C|\nabla u|^2. \tag{3.3}
\]

In the discrete setting, difference derivatives lose their tangency which robs \( \lambda^h_j \) of as clean an expression. Writing \( a^h_j = D_{+h} u^h_j \cdot v_j \), we can attempt the same manipulation as in the continuous case to get the following expression for \( \lambda^h_j \):

\[
\lambda^h_j = D_{-} a^h_j - (D_{-} u \cdot D_{-} v_j).
\]

The first observation is that even though this expression has a local defect, we can still establish a discrete analogue of (3.3).

**Observation 1:**

\[
|\lambda^h_j| \leq C(|D_{+h} u_j|^2 + |D_{-h} u_j|^2). \tag{3.4}
\]

Expressing

\[
\lambda^h_j = -\frac{1}{h} \{(D_{+h} u^h_j - D_{-h} u^h_j) \cdot v_j \},
\]

we need only show that for fixed index \( k \),

\[
\left|\frac{1}{h} (D_{+h} u^h_j \cdot v_j)\right| = |(u^h_{j+1} - u^h_j) \cdot v_j| \leq C|D_{+h} u^h_j|^2.
\]

Fix some \( \delta > 0 \) and choose \( h \) less than \( \delta \). Then if \( |u^h_{j+1} - u^h_j| \geq \delta \), we have

\[
|(u^h_{j+1} - u^h_j) \cdot v_j| \leq \frac{|u^h_{j+1} - u^h_j|}{\delta} |u^h_{j+1} - u^h_j|,
\]
in which case we can take $C = 1/\delta$. We can additionally consider $|(u^h_{j+1} - u^h_j) \cdot v_j| > 0$ since the inequality is trivial otherwise. It suffices then to consider for this case that $\mathcal{N}$ is not locally flat. Due to the smoothness of $\mathcal{N}$, we can thus express $\mathcal{N}$ locally as the graph of some quadratic function, so there is a ball about every point such that $\mathcal{N}$ behaves quadractically. Due to the compactness of $\mathcal{N}$, there is a finite subcover of these balls. Choose $\delta$ to be the minimum of these radii. The interpretation of $(u^h_{j+1} - u^h_j) \cdot v_j$ as the projection of $u^h_{j+1} - u^h_j$ onto $v_j$ immediately yields the given inequality.

**Observation 2:** For $\delta > 0$, choose $h$ such that for $k = 1, 2, |u^h_{j+1} - u^h_j| < \delta, |u^h_k - u^h_{j-1}| < \delta$. Then

$$|D^1\lambda^h_j| = O(|D^1u^h_j|^3, |D^1u^h_j||D^2u^h_j|),$$

(3.5)

where $l$ indexes over next to nearest neighbours of $j$. This follows from the calculation

$$D_{nk}\lambda^h_j = D_{nk}(\Delta^h u^h_j \cdot v_j)$$

$$= \frac{1}{h}(D_{nk}(D_{nk} u^h_j \cdot v_j) - D_{n-1}(D_{nk} u^h_j \cdot v_j))$$

$$= C \frac{1}{h} ((D_{nk} - D_{n-1})(h D_n u^h_j)^2)$$

$$= O(|D^1u^h_j|^3, |D^1u^h_j||D^2u^h_j|),$$

$l$ indexes over next to nearest neighbours of $j$.

### 3.1. A global smooth solution for small energy data

The first result that will be shown is that under the assumption that the initial data have small total energy, there exists a smooth global solution. While the result in this section can be seen as a corollary to the proof of the finite energy data case, this assumption permits the ‘fastest’ path to our goal. The process is laid out here to fix ideas and to motivate the line of reasoning underlying some of the extra steps needed in the next section. The result that we show here can be improved to show that the solution that we find is in fact smooth. The procedure that is followed in the proof here can be re-applied by bootstrapping to higher derivative estimates on the discrete solution and then constructing higher order interpolants which converge to a smooth solution.

For our purposes, recall the $H^1$ interpolant $p_h$ associated to a sequence $\{u^h_j\}$:

$$p_h = \sum_j (u_j + D_1 u_j (x - j_1 h) + D_2 u_j (y - j_2 h) + D_1 D_2 u_j (x - j_1 h)(y - j_2 h)) \chi_{\square_j}(x, y).$$

The result which we will show is the following theorem.

**Theorem 3.1.** There is a constant $\epsilon_0$ such that for $f^h \in H^1_0(\mathbb{R}^2)$ and $\sup_h E^h[f^h] < \epsilon_0$, the associated sequence of $H^1$-interpolants $\{p_h\}$ converges strongly in $H^1$ to $p$, which solves (3.1) in the sense of distributions.

**Proof.** The proof of this result is a two-step process of attaining appropriate bounds on the sequence $u^h$ and using these bounds to construct interpolants that converge to the solution claimed.
Step 1: Attaining bounds. The only quantities on which we have bounds immediately are energy and $\partial_t u^h$. Namely,

$$E^h[u^h(T)] + \|\partial_t u^h\|^2_{L^2([0,T],L^2)} \leq E^h[f^h].$$

(3.6)

Define

$$\mathcal{F}^h = \{u^h = \{u^h_j\}_{j=1}^T | t^{1/2}\|D^2 u^h(t)\|_{L^2} < \infty \forall t\}.$$ 

The higher derivative bounds that we aim for here are uniform-in-$h$ bounds in (i) $L^2(H^1_0)$, (ii) $\mathcal{F}^h$ and (iii) $C(H^1_0)$.

(i) $L^2(H^1_0)$: To get bounds on terms involved in $\|D^2 u^h\|_{L^2}$, it suffices to bound $\|D^4 u^h\|_{L^2}$ since the discrete Laplacian controls all second difference derivatives. This can be seen since the operators $D_i$, $D_{-j}$ commute. Specifically, a term like $\|D_i D_{-j} u^h\|_{L^2}$ can be written as

$$\|D_i D_{-j} u^h\|^2_{L^2} = (D_i D_{-j} u^h, D_i D_{-j} u^h)_{L^2} = -(D_{-j} D_i u^h, D_{-j} u^h)_{L^2} = -(D_{-j} D_{-j} u^h, D_{-j} u^h)_{L^2} = (D_{-j} D_i u^h, D_{-j} u^h)_{L^2} \leq \frac{1}{4}\|\Delta^2 u^h\|^2_{L^2}.$$ 

Using observation 1 and (2.1) for $j = 1, r = 4, p = q = 2$, and the global energy bound (3.6),

$$\|\Delta^2 u^h\|_{L^2} \leq C\left(\|\partial_t u^h(t)\|_{L^2}^2 + \left(h^2 \sum_j |\lambda_j|^2 \right)^{1/2}\right)$$

$$\leq C\left(\|\partial_t u^h(t)\|_{L^2}^2 + \|D^4 u^h(t)\|^2_{L^2} \right)$$

$$\leq C\left(\|\partial_t u^h(t)\|_{L^2}^2 + \|D^4 u^h(t)\|^2_{L^2} + E^h[f^h]^{1/2} \|D^2 u^h(t)\|_{L^2} \right)$$

$$\leq C\left(\|\partial_t u^h(t)\|_{L^2}^2 + E^h[f^h]^{1/2} \|D^2 u^h(t)\|_{L^2} \right)$$

Now choosing $\epsilon_0$ sufficiently small, we can absorb the $C(\epsilon_0^{1/2})\|D^2 u^h\|_{L^2}$ term into the left-hand side to get

$$\|\Delta^2 u^h\|_{L^2} \leq C\left(\|\partial_t u^h(t)\|_{L^2} \right).$$

Now integrating in time, we can make use of (3.6) again to conclude that

$$\|\Delta^2 u^h\|_{L^2([0,T],L^2)} \leq C\left(\|\partial_t u^h(t)\|_{L^2([0,T],L^2)} \right) \leq C(E^h[f^h]^{1/2}) \leq C(\epsilon_0).$$

(ii) $\mathcal{F}^h$: For each index $j$, $u^h_j$ satisfies

$$u^h_j(t) = \Phi^h_j(t) f^h + \int_0^t \Phi^h_j(t-s) \lambda^h_j(s) v(u^h_j(s)) \, ds.$$ 

To get bounds involving $D^2 u^h$ bounds, we can take difference derivatives of the equation and then gain a derivative by using the estimates on the derivative of the fundamental solution. Using the estimate given by (2.10), once with $p = q = 2$ and another
with $p = 2$, $q = p' = 4/3$, 
\[
\|D^2 u^h(t)\|_{L^2_h} \leq \frac{\|D^1 f^h\|_{L^2_h}}{t^{1/2}} + C \int_0^t \frac{1}{(t-s)^{3/4}} \|D^1(\lambda^h(s)v(u^h(s)))\|_{L^2_h} ds.
\]
Observations 1 and 2 on $\lambda^h_j$ together with the smoothness of $v$ yield the inequality 
\[
D^1(\lambda^h_j v_j) \leq C(|D^1 \lambda^h_j| + |\lambda^h_j||D^1 v_j|) \leq C(|D^1 u^h|^3 + |D^1 u^h||D^2 u^h|),
\]
where $l$ indexes over next to nearest neighbours of $j$. We can now apply (2.1) in addition to the repeated use of the discrete Holder and Minkowski inequalities to get
\[
\|D^1(\lambda^h v)\|_{L^2} \leq C(|D^1 u^h|_{L^2} \|D^2 u^h\|_{L^2_h} + \|D^1 u^h\|_{L^2_h}^3).
\]
Proceeding,
\[
t^{1/2} \|D^2 u^h(t)\|_{L^2_h} \leq C(\|D^1 f^h\|_{L^2_h})
\]
\[
+ C \int_0^t \frac{1}{(t-s)^{3/4}} \|D^1 u^h(s)\|_{L^2_h} \|D^2 u^h(s)\|_{L^2_h} + \|D^1 u^h\|_{L^2_h}^3 ds
\]
\[
\leq CE^h|f^h|^{1/2} + C \int_0^t \frac{1}{(t-s)^{3/4}} \|D^1 u^h(s)\|_{L^2_h}^{1/2} \|D^2 u^h(s)\|_{L^2_h}^{3/2}
\]
\[
+ \|D^1 u^h(s)\|_{L^2_h}^{3/2} \|D^2 u^h(s)\|_{L^2_h}^{3/2} ds
\]
\[
\leq C \left(\sup_{t \geq 0} E^h[u^h(t)]\right) t^{1/2} \int_0^t \frac{1}{(t-s)^{3/4}} \|D^2 u^h(s)\|_{L^2_h}^{3/2} ds
\]
\[
\leq C\epsilon_0^{1/2} + C(\epsilon_0) t^{1/2} \int_0^t \frac{1}{(t-s)^{3/4}} \frac{1}{s^{3/4}} \|D^2 u^h(s)\|_{L^2_h}^{3/2} ds.
\]
Letting $y(t) = \sup_{0 \leq s \leq t} s^{1/2} \|D^2 u^h(s)\|_{L^2_h}$,
\[
y(t) \leq \epsilon_0^{1/2} + c(\epsilon_0) y(t)^{3/2} t^{1/2} \int_0^t \frac{1}{(t-s)^{3/4}} \frac{1}{s^{3/4}} ds.
\]
Noting that
\[
t^{1/2} \int_0^t \frac{1}{t-s} \frac{1}{s^{3/4}} ds \leq t^{1/2} \left(\frac{2}{t}\right)^{3/4} \int_0^t \frac{1}{s^{3/4}} ds \leq C
\]
and
\[
t^{1/2} \int_0^t \frac{1}{t-s} \frac{1}{s^{3/4}} ds \leq t^{1/2} \left(\frac{2}{t}\right)^{3/4} \int_0^t \frac{1}{(t-s)^{3/4}} ds \leq C,
\]
we have that $y(t)$ satisfies
\[
y(t) \leq C\epsilon_0^{1/2} + C(\epsilon_0) y(t)^{3/2}
\]
for all $t \in [0, \infty)$. The function $L(y) = y - C(\epsilon_0)^{\delta/3}$ has one root at $y = 0$ and another positive root, reaching its maximum at $y = 4/9C(\epsilon_0)^2$. Now we can choose $\epsilon_0$ so that $C\epsilon_0^{1/2} < L(4/9C(\epsilon_0)^2)$. Since $t \mapsto y(t)$ is continuous and $y(0) = 0$, we can conclude that $y(t)$ is bounded, uniform in $h$.

(iii) $C(\Phi^h_1)$: The energy bound (3.6) immediately implies that $u^h \in L^\infty(\Phi^h_1)$. We can do better using the bounds that we have just attained in $\Phi^h$. On the interval $[\delta, \infty)$, $\Phi^h$ has uniformly bounded derivatives and $\lambda^h(t)v(u^h(t))$ is bounded in $L^\infty_0$ since $u^h \in \Phi^h$, so it
Recall that, for \((x, y)\) suffices to show that \(u(t)\) continuously achieves its initial data in \(H^1_h\). We have the following expression for \(t \in [0, \delta], \delta > 0\):

\[
\|u(t) - f\|_{H^1} \leq \|\Phi^h(t) f^h - f^h\|_{H^1} + \int_0^t \|\Phi^h(t-s)\lambda^h(s)\|_{H^1}^2 \, ds \\
\leq \|\Phi^h(t) f^h - f^h\|_{H^1} + \int_0^t \left(\frac{1}{(t-s)^{\gamma/2}} \|D^1 u^h(s)\|_{L^2}^2 \right) \, ds \\
\leq \|\Phi^h(t) f^h - f^h\|_{H^1} + \int_0^t \left(\frac{1}{(t-s)^{\gamma/2}} \|D^2 u^h(s)\|_{L^2} \|D^1 u^h(s)\|_{L^2} \right) \, ds \\
\leq \|\Phi^h(t) f^h - f^h\|_{H^1} + \int_0^t \frac{1}{(t-s)^{\gamma/2}} \|D^1 u^h(s)\|_{L^2}^3 \, ds \\
\to 0 \quad \text{as } t \to 0.
\]

**Step 2: Convergence of the interpolants.** We have shown that there exists, for each \(h\), sequences of solutions \(u^h = \{u^h(t)\}\) such that

\[
u^h \in L^2(\mathcal{H}^2_h) \cap \mathcal{F}^h \cap C(\mathcal{H}^1_h)
\]

with uniform bounds in \(h\). Associate with each sequence \(\{u^h(t)\}\) the \(H^1\) interpolant \(p_h(t)\). Then for \(\phi \in C^\infty(\mathbb{R} \times \mathbb{R}^2)\), we will show the following statement:

\[
- \int_0^\infty \int_{\mathbb{R}^2} p_h \cdot \Delta \phi + p_h \cdot \Delta \phi + (\nabla p_h \cdot \nabla (p_h)) u^h \cdot \phi \, dt = \int_{\mathbb{R}^2} p_h(0) \cdot \phi(0) + O(h).
\]

(3.7)

Recall that, for \((x, y) \in \square, j = (j_1, j_2)\), the \(H^1\) interpolant is given by

\[
p_h(x, y, t) = u^h(t) + D_{11} u^h(t)(x - j_1 h) + D_{21} u^h(t)(y - j_2 h) \\
+ D_{12} D_{21} u^h(t)(x - j_1 h)(y - j_2 h).
\]

For \(j = (j_1, j_2)\), the notation \(\int_{\square_j}\) will denote \(\int_{j_1 h}^{(j+1) h} \int_{j_2 h}^{(j+1) h}\). We will also slightly abuse the notation \(u^h\) (which is only defined on grid points) to play the dual role of the simple function which takes the value of \(u^h\) in \(\square_j\). Since \(u^h \in L^2(\mathcal{H}^2_h) \cap L^\infty(\mathcal{H}^1_h)\), the following are immediate:

(i) \[
\int_0^\infty \left(\int_{\mathbb{R}^2} p_h \cdot \Delta \phi - \sum_j \int_{\square_j} u^h_j \cdot \Delta \phi \right) = O(h),
\]

(ii) \[
\int_0^\infty \left(\int_{\mathbb{R}^2} p_h \cdot \Delta \phi - \sum_j \int_{\square_j} u^h_j \cdot \Delta \phi \right) = O(h).
\]

From (i),

\[
\int_0^\infty \int_{\square_j} u^h_j \cdot \Delta \phi = - \int_0^\infty \int_{\square_j} \partial_x u^h_j \cdot \phi + \sum_j \int_{\square_j} f^h(0) \cdot \phi(0).
\]
Using the equation (3.2) for $\partial_t u^h_j$, what remains to be shown is that

(iii) $\int_0^\infty \sum_j \int_{\square_j} \Delta^h u^h_j \cdot \phi - p_h \cdot \Delta \phi = O(h)$,

(iv) $\int_0^\infty \sum_j \int_{\square_j} \lambda^h_j v_j \cdot \phi + (\nabla p_h \cdot \nabla (p_h)) v(p_h) \cdot \phi = O(h)$.

To show (iii), it suffices to show that

$$\int_0^\infty \sum_j \int_{\square_j} \Delta^h u^h_j \cdot \phi = \int_0^\infty \sum_j \int_{\square_j} u^h_j \cdot \Delta \phi + O(h),$$

$$\int_0^\infty \sum_j \int_{\square_j} \Delta^h u^h_j \cdot \phi = -\int_0^\infty \sum_j \int_{\square_j} \frac{D_{a^h_j} u^h_j - D_{-i} u^h_j h}{h} \cdot \phi \cdot \phi$$

$$= \int_0^\infty \sum_j \int_{\square_j} u^h_j \cdot \phi(x) - 2 \phi(x_i) + \phi(x_i - h) \frac{h^2}{h^2}$$

$$= \int_0^\infty \sum_j \int_{\square_j} u^h_j \cdot \Delta \phi + O(h).$$

What remains is (iv). Associate with $\phi$ the step function $\bar{\phi} = \sum_j \phi_j x_{\square_j}$. We treat this term by splitting the integral into two parts:

$$I = \int_0^\infty \sum_j \int_{\square_j} (\lambda^h_j v_j + (\nabla p_h \cdot \nabla (p_h)) v(p_h)) \cdot (\phi - \bar{\phi})$$

and

$$II = \int_0^\infty \sum_j \int_{\square_j} (\lambda^h_j v_j + (\nabla p_h \cdot \nabla (p_h)) v(p_h)) \cdot \bar{\phi}.$$

Since $u^h \in L^2(H_0^2)$, $I = O(h)$. The reason behind the splitting is really to negotiate the local defect regarding $\lambda^h_j$. Fixing $j$, $\lambda^h_j$ does not come close to cancelling out the relevant terms in $\nabla p_h(jh) \cdot \nabla v(p_h(jh))$. However, just as it was seen in observation 1 that the local defect did not prevent us from attaining a global statement that was satisfactorily close to the continuous case, we can also show that by considering the sum of the two terms in $II$ as two sums the terms that result have a satisfactory cancellation. The only thing that prevented us before from switching the spatial sum and integral was $\phi$. Replacing $\phi$ by $\bar{\phi}$ resolves this immediate problem. Keeping in mind that the terms in $\nabla p_h(jh) \cdot \nabla v(p_h(jh))$ that require cancellation are of the form $D_{a^h_j} u^h_j \cdot (v_j)_j$, we consider the sum $\sum_j \lambda^h_j v_j \cdot \phi^h_j$.

Letting $a^h_j = (D_{ik} u^h_j \cdot v_j)$,

$$\sum_j \lambda^h_j v_j \cdot \phi^h_j = -\sum_j \lambda^h_j v_j \cdot \phi^h_j$$

$$= -\sum_j (D_{a^h_j} u^h_j \cdot \phi^h_j)$$

$$= -\sum_j (D_{a^h_j} u^h_j \cdot \phi^h_j).$$
This term is already in a form that the cancellation is apparent; the local defect in $\lambda^h_j$ has essentially been transferred to a local defect in $\nu$ and $\phi$, but these we can handle because they are smooth.

\[ II = \int_0^\infty \int_{\Omega_j} \sum_j \phi_j \cdot (\lambda_j^h v_j + (D_{si} u_i^h)(v_i) v(p_h)) + O(h) \]
\[ = \int_0^\infty h^2 \sum_j \lambda_j^h v_j \cdot \phi_j + (D_{si} u_i^h \cdot (v_i) v_j) \cdot \phi_j + O(h) \]
\[ = \int_0^\infty h^2 \left\{ - \sum_j (D_{si} u_i^h \cdot D_{si} v_j) v_{j+1} \cdot \phi_{j+1} + \sum_j (D_{si} u_i^h \cdot (v_i) v_j) v_j \cdot \phi_j \right\} + O(h) \]
\[ = O(h). \]

(i)–(iv) together entail (3.7). With this expression, we can split the time interval into subintervals $[0, \delta)$ and $[\delta, \infty)$. On $[\delta, \infty)$, we have uniform $H^2_h$ which implies from proposition 2.3 that there exists a subsequence $\{h_k\}$ such that $p_{h_k}(t) \xrightarrow{H^1} p(t)$. For any $h_k$, then, we can use (i)–(iv) and the fact that $u^h \in C([0, \delta); H^1_h)$ to show that
\[ \int_0^\infty \int_{\mathbb{R}^2} p_{h_k} \cdot \Delta \phi + p_{h_k} \cdot (\nabla p_{h_k} \cdot \nabla v(p_{h_k})) v(p_{h_k}) \cdot \phi \]
\[ = \left( \int_0^\delta + \int_\delta^\infty \right) \int_{\mathbb{R}^2} p_{h_k} \cdot \Delta \phi + p_{h_k} \cdot (\nabla p_{h_k} \cdot \nabla v(p_{h_k})) v(p_{h_k}) \cdot \phi \]
\[ \to \int_0^\infty \int_{\mathbb{R}^2} p \cdot \Delta \phi + p \cdot (\nabla p \cdot \nabla v(p)) v(p) \cdot \phi, \]
from which we can conclude that $p$ solves (3.1) in the sense of distributions. □

3.2. A partially regular solution for finite energy data

Now we consider the discrete system only assuming $H^1$ data. While we do not have total small energy, our initial data are still of finite energy. Locally, then, the energy of the initial data can be made small. What is needed is a localized version of the ‘small energy ⇒ regularity’ statement in the last section. Achieving such a statement will allow us to estimate the size of the concentration sets, which in turn will allow us to extend the associated interpolants across these sets in such a way that they will converge as $h \to 0$ in the sense of distributions to a solution of the continuous problem. We proceed now to show the following result.

**Theorem 3.2.** For any $f \in H^1(\mathbb{R}^2, N)$, there exists a global, partially regular solution to (3.1), smooth away from a closed set of locally finite two-dimensional Hausdorff measure set with respect to the parabolic metric.

The core statement to be shown is the following lemma.

**Lemma 3.3.** There is an $\epsilon_0 > 0$ such that if $E^h[u^h(t_0); B_R(x_0)] \leq \epsilon_0$ then $\exists \delta$, independent of $h$ and $\tau \in (t_0, t_0 + \delta R^2)$ such that
\[ \sup_{\tau \leq t \leq t_0 + \delta R^2} \| D^2 u^h(t) \|_{L^2(B_{\delta R}(x_0))} \leq C. \]

Notation: $E^h[u^h(t); B_R(x_0)]$ denotes the energy of $u^h(t)$ indexing over $j = (j_1, j_2)$ such that $j h \in B_R(x_0)$. 


Proof. The goal is to achieve uniform $H^2_\alpha$ bounds in a reduced parabolic cylinder. We will pass through the following intermediate stages to achieve this end:

(i) show a local energy inequality;
(ii) show $u^h \in L^2(H^2_\alpha)$ in a reduced parabolic cylinder;
(iii) show $D^1 u^h \in L'(L^2_\alpha)$ for $r < \infty$ in a reduced parabolic cylinder;
(iv) show $u^h \in L^\infty(H^2_\alpha)$ in a reduced parabolic cylinder.

Proceeding,

(i) For any $\epsilon > 0, \sup_h E^h[u^h(t_0); B_{2R}(x_0)] \leq \epsilon/2$, \exists \delta such that $\forall t, t_0 \leq t \leq t_0 + \delta R^2$,

\[
\sup_h E^h[u^h(t); B_R(x_0)] + \|\partial_t u^h\|_{L^2((t_0,t); L^2_\alpha(B_R(x_0)))}^2 < \epsilon.
\] (3.8)

Proof. Denote by $\zeta$ the cutoff function that equals 1 for $j \in B_{2R}(x_0)$ and 0 for indices outside of $B_{2R}(x_0)$ with $|D^1 \zeta| \leq k/R$. Multiplying (3.2) by $(\partial u^h/\partial t)\zeta^2$ and summing over $j$,

\[
h^2 \sum_j \left| \frac{\partial u^h_j}{\partial t} \right|^2 \zeta^2 = h^2 \sum_j \Delta u^h_j \zeta^2
\]

\[
= h^2 \sum_j D_{\alpha j} D_{-\alpha j} \frac{\partial u^h_j}{\partial t} \zeta^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \frac{\partial u^h_j}{\partial t} \zeta^2 \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

\[
= \left( D_{\alpha j} D_{-\alpha j} \left( \frac{\partial u^h_j}{\partial t} \zeta^2 \right) \right)^2
\]

Since the same expression holds with $D_{\alpha j}$ replacing $D_{-\alpha}$, we make use of the total energy inequality to get

\[
\frac{\partial}{\partial t} (\zeta^2 E^h[u^h(t)]) + h^2 \sum_j \left| \frac{\partial u^h_j}{\partial t} \right|^2 \zeta^2 \leq \frac{1}{R^2} E^h[u^h(t_0)] \leq \frac{1}{R^2} E^h[f^h].
\]

Integrating from $t_0$ to $t$,

\[
E^h[u^h(t); B_R(x_0)] + \int_{t_0}^t \left| \frac{\partial u^h_j}{\partial t} \right|^2 \zeta^2 \leq E^h[u^h(t_0); B_{2R}(x_0)] + C \frac{t - t_0}{R^2} E^h[f].
\]

In particular, choosing $\delta = C \epsilon/2 E^h[f^h]$, we get the desired result.
(ii) $L^2(H_0^1)$: For $\varepsilon > 0$, there exists $\delta$ such that

$$E^h[u^h(t_0); B_2R(x_0)] < \varepsilon \Rightarrow \sup_h \|D^2u^h\|_{L^2([0,\Delta_1+\delta R^2],L^2(R^2))} \leq C(\varepsilon). \quad (3.9)$$

**Proof.** To get bounds on terms involved in $\|D^2u^h\|_{L^2}$, it suffices to bound $\|\Delta^h u^h\|_{L^2}$ since the discrete Laplacian controls all second difference derivatives. For any cutoff function $\zeta$ and $u^h_j$ solving (3.2), $u^h_j \zeta_j$ satisfies the following equation:

$$\Delta^h(u^h_j \zeta_j) = \frac{1}{2}(\zeta_{j+1} + 2\zeta_j + \zeta_{j-1}) \Delta^h u^h_j + 2(D_1u^h_j \zeta_j + D_1^\perp u^h_j)(D_1^i u^h_j + D_1^\perp u^h_j) + \frac{1}{2}(u^h_{j+1} + u^h_j + u^h_{j-1}) \Delta^h \zeta_j$$

$$= \frac{1}{2}(\zeta_{j+1} + 2\zeta_j + \zeta_{j-1}) \Delta^h u^h_j + 2(D_1u^h_j \zeta_j + D_1^\perp u^h_j)(D_1^i u^h_j + D_1^\perp u^h_j) + \frac{1}{2}(u^h_{j+1} + u^h_j + u^h_{j-1}) \Delta^h \zeta_j$$

$$= I + II + III + IV.$$ 

This is just the discrete analogue of the expression

$$\Delta(u \zeta) = \Delta u \zeta + \nabla u \nabla \zeta + u \Delta \zeta = (\partial_2 u - \nabla u \cdot \nabla v(u)) \zeta + 2 \nabla u \nabla \zeta + u \Delta \zeta.$$ 

Choose $\zeta$ such that it equals 1 at indices $j$ such that $j \in B_R(x_0)$ and 0 outside of $B_R(x_0)$ with $|D^1\zeta| \leq (C/R)$ and $|\Delta^h \zeta| \leq (C/R^2)$. Then

- $\|I\|_{L^2(R^2)} \leq C|\Delta^h u^h(t)|_{L^2(R^2)}$
- $\|II\|_{L^2(R^2)} \leq C\|D^1u^h(t)\|^2_{L^2(R^2)}$
- $\|III\|_{L^2(R^2)} \leq C\|[D^1\zeta]|_{L^1(R^2)} \leq C(\varepsilon)$
- $\|IV\|_{L^2(R^2)} \leq C\|\Delta^h \zeta\|_{L^2(R^2)} \leq C(\varepsilon)$

where $III$ is bounded by using the local energy inequality that we attained in (i). Now integrating on an interval of length $\delta R^2$, we can see that $III$ and $IV$ are lower order terms in the sense that

- $\|III\|_{L^2([\delta_0,\delta_1+\delta R^2],L^2(R^2))} \leq C(\varepsilon)\delta R$
- $\|IV\|_{L^2([\delta_0,\delta_1+\delta R^2],L^2(R^2))} \leq C\delta$.

We plan to use the local space–time bound on $\partial_2 u^h$ from (i) to handle $I$. However, $I$ poses an obstacle since there is a ‘missing’ $\zeta$ term which prevents the step that we used in the small energy case:

$$\|D^1u^h(t)\|_{L^2} \leq \|D^1u^h(t)\|^2_{L^2}.$$ 

The extra work that is needed is performed in proposition 2.1 corresponding to $s = 1$, $p = q = 2$. Getting estimates on $D^2(u^h \zeta)$ and denoting the lower order terms, $l.o.t. = \|III\|_{L^2(R^2)} + \|IV\|_{L^2(R^2)}$, we have

$$\|D^2u^h(t)\|_{L^2(R^2)} \leq C\|\Delta(u^h(t)\zeta)\|_{L^2(R^2)}$$

$$\leq C\|[D^1u^h(t)]^2\|_{L^2(R^2)} + \|\partial_2 u^h(t)\|_{L^2(R^2)} + l.o.t.$$ 

$$\leq C\|D^1u^h(t)\|_{L^2(R^2)}\|D^1u^h(t)\|_{L^2(R^2)} + \|\partial_2 u^h(t)\|_{L^2(R^2)} + l.o.t.$$ 

$$\leq C(\varepsilon)\|D^2(u^h(t)\zeta)\|_{L^2(R^2)} + \|\partial_2 u^h(t)\|_{L^2(R^2)} + l.o.t.$$
For $\epsilon$ small enough, we can absorb the $\|D^2(u^h_1)(t)\|_{L^2_0(\mathbb{R}^2)}$ term on the right-hand side of the equation into the left to get
\[
\|D^2u^h(t)\|_{L^2_0(\mathbb{R}^2)} \lesssim C\|\tilde{u}_0\|_{L^2_0(\mathbb{R}^2)} + l.o.t.
\]
We can now take $L^2$ in time on the time interval $[t_0, t_0 + \delta R^2]$ and use (i) to get
\[
\|D^2u^h\|_{L^2([t_0, t_0 + \delta R^2], L^2_0(\mathbb{R}^2))} \lesssim C(\epsilon).
\]
We can now start at $t\in [t_0, t_0 + \delta R^2]$ and use (ii) to get
\[
\|D^2u^h\|_{L^2([t_0, t_0 + \delta R^2], L^2_0(\mathbb{R}^2))} \lesssim C(\epsilon).
\]
\[
\text{(iii) } L^\prime(W_{h}^{1,4}) : \text{For } \epsilon > 0, \exists \delta \text{ and } \tau \in (t_0, t_0 + \delta R^2) \text{ such that for } E^h[u^h(t_0); B_{\delta R}(x_0)] < \epsilon \text{ and } r > 1,
\]
\[
\sup_{h} \|D^4u^h\|_{L^\prime([\tau, \tau + \delta R^2], L^2_0(B_{\delta R}(x_0)))} \lesssim C R^2 \|u^h\|_{H^2_r} E^h[f^h].
\] (3.11)

**Proof.** To get the bounds that we want for any $r > 1$, we cannot hope to start our time interval at $t_0$ since all we are assuming is that $u^h(t_0) \in H^1_h$. However, from (ii), we can find $\delta_0$ such that
\[
\|D^2u^h\|_{L^2([t_0, t_0 + \delta_0 R^2], L^2_0(\mathbb{R}^2))} \lesssim C.
\]
For any $t_1 \in (t_0, t_0 + \delta_0 R^2)$, then, there exists $\tau \in (t_0, t_1)$ such that
\[
\|D^2u^h(\tau)\|_{L^2_0(\mathbb{R}^2)} \lesssim C \frac{1}{(t_1 - t_0)}.
\]
We can now start at $t = \tau$ with $H^2_r$ initial data.

For any choice of $\zeta$, $s^0_j$ solving (3.2), $u^h_j$ solves the equation:
\[
\partial_t(u^h_j\zeta_j) - \Delta^h(u^h_j) = \left(\frac{1}{2}(\zeta_{j+1} + 2\zeta_j + \zeta_{j-1}) + 2(D_{j\zeta} u^h_j + D_{-j\zeta} u^h_j) - \frac{1}{2}(u^h_{j+1} + u^h_{j-1})\Delta^h\zeta_j\right) + \lambda h^2
\]
(3.12)

Now we choose $\zeta$ to be the cutoff function, $1$ in a ball of radius $R$ supported in $B_{2R}(x_0)$. By Duhamel’s, we have
\[
\|D^4(u^h(t)\zeta)\|_{L^2_0(\mathbb{R}^2)} \lesssim \|\Phi^h(t)(D^4(u^h(\tau)\zeta))\|_{L^2_0(\mathbb{R}^2)} + \int_\tau^t \frac{1}{1 - s} \|D^4u^h(s)\|_{L^2_0(\mathbb{R}^2)} ds\]

Using proposition 2.1 for $s = 2, p = 4, q = 2$, we get
\[
\|D^4(u^h(t)\zeta)\|_{L^2_0(\mathbb{R}^2)} \lesssim \|\Phi^h(t - \tau)(D^4(u^h(\tau)\zeta))\|_{L^2_0(\mathbb{R}^2)} + \int_\tau^t \frac{1}{1 - s} \|D^4u^h(s)\|_{L^2_0(\mathbb{R}^2)} ds\]

Now taking $L^\prime$ in time of both sides, we know by Young’s inequality that we can take $\|D^2u^h\|_{L^2_0(B_{\delta R}(x_0))}$ in $L^2$ and $\|D^4(u^h(s)\zeta)\|_{L^2_0}$ in $L^\prime$; the first term is bounded by $C(\epsilon)$ by (ii) and the second is exactly the quantity that we are trying to bound. This entire product can be absorbed into the left-hand side. The linear term $\|\Phi^h(t)(D^4(u^h(\tau)\zeta))\|_{L^2_0}$ is in $L^\prime$ since $u^h$
Taking \( H^2 \) at \( t = \tau \). Therefore,
\[
\|D^1(u^h(\tau)\xi)\|_{L^2(\tau, t_0 + \delta R^2; L^p(\mathbb{R}^2))} \leq C(\|u^h(\tau)\|_{H^2}, E^h[|f^h|]).
\]

(iv) \( L^\infty(H^2) \): For \( \varepsilon > 0, 3\delta \) and \( \tau \in (t_0, t_0 + \delta R^2) \) such that
\[
E^h[u^h(t_0); B_{R}(x_0)] < \varepsilon \Rightarrow \sup_{\tau \leq t \leq t_0 + \delta R^2} \|D^2u^h(t)\|_{L^p(\mathbb{R}^2)} \leq C(\|u^h(\tau)\|_{H^2}, E^h[|f^h|]).
\]

(13.13)

Proof. As in (iii), we can use the \( L^2(H^2) \) bounds to find \( \tau \) such that \( u^h(\tau) \in H^2 \). For second derivative estimates on \( u^h_\xi \), we need to differentiate (3.12) and use the estimates (2.10) by taking \( p = 4, q = p' = 4/3 \). Using observations 1 and 2 on \( \lambda_h^j \) and the smoothness of \( v \),
\[
D^1(\lambda_h^j v) \lesssim |D^1(\lambda_h^j)| + |\lambda_h^j||D^1v| \lesssim |D^1u^h|^3 + |D^1u^h||D^2u^h|.
\]

To avoid cluttering the basic idea with overwhelming technical detail, we highlight the crucial term:
\[
\|D^2(u^h(t)\xi)\|_{L^2(\mathbb{R}^2)} \leq \|\Phi^h(t - \tau)D^2(u^h(\tau)\xi)\|_{L^2(\mathbb{R}^2)}
\]
\[
+ \int_{\tau}^{t} \frac{1}{(t - s)^{3/4}} \|\xi D^1(\lambda_h^j(s)v(s))\|_{L^2(\mathbb{R}^2)} \, ds + C(R, E^h[|f^h|])
\]
\[
\leq C\|D^2(u^h(\tau)\xi)\|_{L^2(\mathbb{R}^2)} + C\int_{\tau}^{t} \frac{1}{(t - s)^{3/4}} \|D^1u^h(s)\|^2 \|D^2u^h(\tau)\|_{L^2(\mathbb{R}^2)} \, ds + C(R, E^h[|f^h|])
\]
\[
\leq C\|D^2(u^h(\tau)\xi)\|_{L^2(\mathbb{R}^2)} + C\int_{\tau}^{t} \frac{1}{(t - s)^{3/4}} \|D^1u^h(s)\|_{L^2(\mathbb{R}^2)} \|D^2u^h(\tau)\|_{L^2(\mathbb{R}^2)} \, ds + C(R, E^h[|f^h|])
\]
\[
\leq C\|D^2(u^h(\tau)\xi)\|_{L^2(\mathbb{R}^2)} + C\int_{\tau}^{t} \frac{1}{(t - s)^{3/4}} \|D^1u^h(s)\|_{L^2(\mathbb{R}^2)} \|D^2u^h(\tau)\|_{L^2(\mathbb{R}^2)} \, ds + C(R, E^h[|f^h|]).
\]

Now we can take \( L^\infty \) in time of this expression by taking \( \|D^2(u^h(\tau)\xi)\|_{L^2(\mathbb{R}^2)} \) in \( L^\infty \) and \( \|D^1u^h(s)\|_{L^2(\mathbb{R}^2)} \) in \( L^r \) for \( r > 4 \). From this we can conclude \( L^\infty(H^2) \) bounds in the parabolic cylinder \( B_{R}(x_0) \times [\tau, t_0 + \delta R^2] \).

Proof (Theorem 3.2). The construction of a solution to (3.1) begins, as before, at the level of the discrete approximations. Existence for any fixed \( h \) is not an issue. Moreover, since our initial data are of finite energy, we know from the energy inequality that any solution remains of finite energy. However, this energy may concentrate which in turn invalidates an immediate argument for the convergence of the interpolants. As seen in the case of small energy initial data, the convergence argument depended on having control on the second derivatives or uniform control on the gradient. What lemma 3.3 provides is local control of these quantities. The key extra step that is needed is an estimate of the size of the energy concentration set. From this we can conclude that the limit of the \( H^1 \) interpolants can be extended across the concentration sets in such a way that it is a partially regular solution of the harmonic map heat flow problem.

Step 1: Attaining bounds. This is the content of lemma 3.3.
Step 2: Estimating the size of the concentration set. By invariance of the equation \((3.2)\) by time translation, it suffices to estimate the concentration set on the interval \([0, 1]\). Fix \(\epsilon_0\) and \(\delta\) of theorem 3.2. Since \(\delta\) is independent of \(h\), it makes sense, for fixed \(R_0\), to partition \([0, 1]\) into intervals \(I_0, I_1, \ldots, I_{2/\delta R_0^2}\) of length \(\delta R_0^2/2\). Denote \(T_j\) to be the base time of \(I_j\). Define \(\beta: [0, 1] \rightarrow \{0, \ldots, 2/\delta R_0^2\}\), assigning to every \(t\) in \([0, 1]\) the index \(j\) such that \(t \in I_j\). For arbitrary \(R\) and any point \((x_0, t_0) \in \mathbb{R}^2 \times (0, 1]\) denote by \(P_R(x_0, t_0)\) the parabolic cylinder
\[
P_R(x_0, t_0) = \left\{ (x, t) \bigg| x \in B_R(x_0), t \in \left[ t_0 - \frac{\delta R_0^2}{2}, t_0 + \frac{\delta R_0^2}{2} \right] \right\}.
\]
Let \(E^h[u^h(t), P_R(x, t)]\) denote the energy of \(u^h(t)\) for \((jh, t) \in P_R(x, t)\). We introduce the singular set:
\[
\Sigma_1 = \bigcap_{R > 0} \{(x, t) \in \mathbb{R}^2 \times \mathbb{R}_+ : \liminf_{h \to 0} E^h[u^h, P_R(x, t)] > \epsilon_0 \}.
\]
Then \((x_k, t_k) \in \Sigma\) implies, by the proof of lemma 3.3, that for every \(R \leq R_0\) and every \(h\) sufficiently small,
\[
E^h\left[u^h\left(t_k - \frac{\delta R_0^2}{2}\right), B_{2R_k}(x_k)\right] > \frac{\epsilon_0}{2}.
\]
For any \((x_k, t_k)\), associate an \(R_k \leq R_0\) satisfying
\[
t_k - \frac{\delta R_0^2}{2} = T_{\beta(t_k)}.
\]
Then
\[
E^h[u^h(T_{\beta(t_k)}), B_{2R_k}(x_k)] > \frac{\epsilon_0}{2}.
\]
Now these concentration balls have been ‘snapped to grid’ associated to \(R_0\). At any time slice, a direct consequence of the energy bound is that there is a finite number of balls of radius \(2R\) in which the energy exceeds \(\epsilon_0/2\), independent of \(R\). Specifically, there can be no more than \(E^h[\mathcal{J}^h]/(2\epsilon_0)\) such balls at any time slice for any \(R\). The balls \(B_{2R_k}\) at time \(T_j\) cover the bases of the concentration cylinders in \(I_j\) in the sense of Vitali, so by Vitali’s covering lemma we can take these balls to be disjoint. Now adding up the contribution of all the concentration cylinders over the \(2/(\delta R_0^2)\) intervals, we have
\[
\Sigma_k Vol(C_k) \leq \frac{1}{2} \delta R_0^2 \Sigma_k R^2_k \leq \frac{1}{2} \delta R_0^2 \sum_{k=0}^{2/\delta R_0^2} E^h[\mathcal{J}^h] \frac{2}{\delta R_0^2} \leq \frac{\epsilon_0}{\delta R_0^2}.
\]
Thus
\[
\Sigma_k R_k^2 \leq \frac{E^h[\mathcal{J}^h]}{2\delta \epsilon_0}.
\]
We can conclude that the concentration set \(\Sigma\) has locally finite two-dimensional Hausdorff measure with respect to the parabolic metric. Show that \(\Sigma\) is closed is a direct consequence of lemma 3.3. Take any \((x_0, t_0) \notin \Sigma\). Then there exists \(R > 0\) such that \(E^h[u^h, P_R(x_0, t_0)] \leq \epsilon_0\) for an unbounded sequence of \(h\). We can then find some cylinder around \((x_0, t_0)\) in which we have \(L^\infty(H^2_\delta)\) bounds, which gives a local bound on the energy density away from \(\Sigma\).

Step 3: Convergence of the interpolants. For \((x_0, t_0) \in \Sigma^c\), there exists \(R > 0\) such that for some unbounded sequence \(h\),
\[
E^h[u^h, P_R(x_0, t_0)] \leq \epsilon_0.
\]
Equivalentely,
\[ E^h[u^h(T_0), B_{2R}(x_0)] \leq \frac{\epsilon_0}{2}, \]
so by lemma 3.3, \( \|D^2u_h\|_{L^2} \leq C \) in a uniform neighbourhhood \( Q \) of \((x_0, t_0)\). Associate with each sequence \( u_h^j \) the \( H^1 \) interpolant \( p^h \). From proposition 2.3, \( p^h \to p \) in \( H^1(Q) \). By the argument given in the global small energy case we have, for \( \zeta \in C_0^\infty(Q) \), that
\[ \int_Q \partial_t p \cdot \zeta + \nabla p \cdot \nabla \zeta - (\nabla p \cdot \nabla \nu(p))\nu(p) \cdot \zeta = 0. \]
Off \( \Sigma \), then, \( p \) solves (3.1).

To show that \( p \) solves the equation on \( \mathbb{R}_+ \times \mathbb{R}^2 \), fix a compact set \( Q \subset \subset \mathbb{R}_+ \times \mathbb{R}^2 \). As in the covering argument above, we can, for fixed \( R \), partition up the time interval occupied by \( Q \) into intervals of length \( \delta R^2 \). Without loss of generality, we take the total time interval to be of unit length. Given \( R > 0 \), let \( \{Q_{i,l} = P_R(x_{i,l}, t_l)\} \) be a covering of \( \Sigma \cap Q \) by parabolic cylinders. Denote by \( N_l \) the number of parabolic cylinders in each time interval \( I_l \). By the energy inequality, \( N_l \) is bounded by some finite number \( N_0 \) independent of \( R \), so the number of such cylinders is less than \( N_0/\delta R^2 \).

Consider smooth cutoff functions \( 0 \leq \eta_1, \eta_2 \leq 1 \) identically 0 in \( B_1(0) \) and 1 outside of \( B_2(0) \) and \( |\nabla \eta_1| \leq C/R \). Define the cutoff function:
\[ \eta_R = \sum_{i=1}^{1/\delta R^2} \sum_{l=1}^{N_l} \eta_1 \left( \frac{x - x_{i,l}}{R} \right) \eta_2 \left( \frac{t - t_l}{\delta R^2} \right). \]
Let \( \phi \in C_0^\infty(Q) \). Then setting \( \zeta = \phi \eta_R \),
\[ 0 = \int_Q \{ \partial_t p \cdot \phi + \nabla p \cdot \nabla \phi - (\nabla p \cdot \nabla \nu(p))\nu(p) \cdot \phi \} \eta_R + L, \]
where \( L \) can be bounded by
\[
|L| \leq \int_Q |\nabla p| \left| \nabla \left( \sum_{i=1}^{1/\delta R^2} \sum_{l=1}^{N_l} \eta_1 \right) \right| \eta_2 \left( \frac{t - t_l}{\delta R^2} \right) \left| \nabla \phi \right| \eta_R \ dx \ dt
\leq C \sum_{i=1}^{1/\delta R^2} \sum_{l=1}^{N_l} \frac{1}{R^2} \int_{Q_{i,l}} |\nabla p| \ dx \ dt
\leq C \sum_{i=1}^{1/\delta R^2} \sum_{l=1}^{N_l} \frac{1}{R^2} \left( \int_{Q_{i,l}} |\nabla p|^2 \right)^{1/2} \left( \int_{Q_{i,l}} \ dx \ dt \right)^{1/2}
\leq C \delta^{1/2} \sum_{i=1}^{1/\delta R^2} \sum_{l=1}^{N_l} R^2 \left( \int_{Q_{i,l}} |\nabla p|^2 \right)^{1/2}
\leq C \delta^{1/2} \left( \sum_{i=1}^{N_0} \sum_{l=1}^{1/\delta R^2} R^2 \right)^{1/2} \left( \sum_{i=1}^{N_0} \sum_{l=1}^{1/\delta R^2} \int_{Q_{i,l}} |\nabla p|^2 \right)^{1/2}
\leq CN_0^{1/2} \left( \sum_{i=1}^{N_0} \sum_{l=1}^{1/\delta R^2} \int_{Q_{i,l}} |\nabla p|^2 \right)^{1/2}.
\]
Taking $R \to 0$, $|L| \to 0$ by absolute continuity of the Lebesgue integral so that $p$ weakly solves (3.1) in $\mathbb{R}^+ \times \mathbb{R}^2$. □

4. Landau–Lifshitz–Gilbert equation

We proceed to prove our main result, theorem 1.1. The construction of a solution to this problem will follow the same path as that for the harmonic map heat flow. The semi-discrete approximation for LLG is given by (2.2). The methods of the previous section heavily depend on estimates for the linear equation. One additional ingredient is needed for the LLG: a reformulation of (2.2) that yields a system that is linear in its highest order term. In the continuous setting, this reformulation is achieved by showing that the coordinates of $u$ solving (1.1) for $\alpha = 0$ onto a moving frame on $\mathcal{N}$ satisfy a system whose highest order term is a linear Schrödinger term [2]. The concept of frames customized to the needs of this section is briefly reviewed in appendix B. For our discrete setting, we achieve such a formulation by replacing derivatives of $u$ with projections onto the tangent space of difference derivatives.

4.1. The linearized discrete system

The linearized discrete system depends on the existence of a moving frame $(e_1, e_2)$ on $\mathcal{N}$. In the absence of a global frame on $\mathcal{N}$, a moving frame can exist by constructing a pullback frame which depends on a map $u$ taking values in $\mathcal{N}$. A moving frame which is finite energy and satisfies a Coulomb gauge can be constructed so long as $\|\nabla u\|_{L^2(B_r)} \leq \gamma_0$, where $\gamma_0$ only depends on $\mathcal{N}$. Since the energy is scale invariant and our equation translation invariant, $B_r^2$ can be replaced by any ball $B_r(x_0)$ in this small energy condition. Consider now a sequence $\{u^h_j\}_j$ for fixed $h$. If this sequence is of finite discrete energy, then we can construct an interpolant $p^h$ which is $H^1$. Given $\gamma_0$, there exists a radius $r$ such that $\|\nabla p^h\|_{L^2(B_r(x_0))} < \gamma_0$. On $B_r(x_0)$ and fixed $h$, then, we can find a moving frame $e = (e_1, e_2)$ such that $\|\nabla e\|_{L^2(B_r)} < \infty$. To clarify the presentation, we preface the derivation of the linearized discrete system with some notation and observations.

**Notation:** Choose $x_0, r$ and fix $h$ small enough so that we can construct $\{e_1, e_2\}$ as above which is defined on $B_r(x_0)$. Introduce the complex notation:

$$i : TN \to TN, \quad ie_1 = e_2, \quad ie_2 = -ie_1.$$  

A consequence of discretization is that difference derivatives lose their tangency. In particular, difference derivatives of $u_j$ do not lie in $T_{u_j} \mathcal{N}$. To navigate through the messy consequences of this defect, we separate out the tangent and the orthogonal portions. Denote

$$D_{+k}u^h_j = v^{+k}_j + w^{+k}_j, \quad D_{-k}u^h_j = v^{-k}_j + w^{-k}_j,$$

where $v^{+k}_j, v^{-k}_j \in T_{u_j} \mathcal{N}$ and $w^{+k}_j, w^{-k}_j \perp T_{u_j} \mathcal{N}$. The tangent portion can be further decomposed as follows. Letting $e = e^1$ and $q_j^{+k} = q_j^{+k,1} + iq_j^{+k,2}, q_j^{-k} = q_j^{-k,1} + iq_j^{-k,2}$,

$$v_j^+ = P_{T_{u_j} \mathcal{N}}(D_{+k}u_j) = q_j^{+k} e_j, \quad v_j^- = P_{T_{u_j} \mathcal{N}}(D_{-k}u_j) = q_j^{-k} e_j,$$

where $P_{T_{u_j} \mathcal{N}}$ denotes the orthogonal projection onto $T_{u_j} \mathcal{N}$. The orthogonal portion can be written as

$$w_j^{+k} = a_j^{+k} v_j, \quad v_j^{-k} = a_j^{-k} v_j.$$

$\mathcal{N}$ is a smooth embedded surface so that the unit normal $v$ is a smooth function. Let $c_v = \|\nabla v\|_{L^\infty}$. 


Observations: Fix $h$, $R$ and construct the frame $e$ defined on $B_R(x_0)$ as above. Consider $j$ indexing in the domain $B_R(x_0)$:

1. $|v_j^{+k}| = |q_j^{+k}|$ and $|v_j^{-k}| = |q_j^{-k}|$.
2. $a_j^{+k} = D_{sk}u_j^h$, $v_j = O(h|D_{sk}u_j^h|^2)$. Similarly, $a_j^{-k} = O(h|D_{sk}u_j^h|^2)$.
3. $\lambda_j^k = O((D^1u_j^h)^2)$.
4. For suitable $\epsilon$, $|u_{j+1}^h - u_j^h|^2 < \epsilon$,

$$|D_{sk}u_j^h|^2 = O(|v_j^{+k}|^2), \quad |D_{sk}u_j^h|^2 = O(|v_j^{-k}|^2).$$

This follows from the following manipulation:

$$|D_{sk}u_j^h|^2 \leq 2(|v_j^{+k}|^2 + |v_j^{-k}|^2) = 2(|v_j^+|^2 + |a_j^+|^2) \leq 2(|v_j^+|^2 + C|hD_{sk}u_j^h|^2) \leq 2(|v_j^+|^2 + C|u_{j+1}^h - u_j^h|^2|D_{sk}u_j^h|^2),$$

where we have used (1) in the last line. Using a similar argument, for suitable $\epsilon$, $|u_{j+1}^h - u_j^h|^2 < \epsilon \Rightarrow$

$$|D_{sk}u_j^h|^2 = O(|q_j^{+k}|^2), \quad |D_{sk}u_j^h|^2 = O(|q_j^{-k}|^2),$$

$$a_j^{+k} = O(|q_j^{+k}|), \quad a_j^{-k} = O(|q_j^{-k}|).$$

5. Since $v_j = v(u_j^h)$:

$$\partial_t v_j = \nabla_v \partial_t u_j = O(|\partial_t u_j^h|), \quad D_k v_j = O(|D_k u_j^h|).$$

Lemma 4.1. Let $u^h$ satisfy $\|D^1 u^h\|_{B_R(x_0)} \leq \gamma_0$. Choose $h$ small enough so that the condition in observation 4 is satisfied:

$$|u_{j+1}^h - u_j^h|^2 < \epsilon, \quad |u_{j+1}^h - u_{j-1}^h|^2 < \epsilon.$$  

Fix $h$, and construct the frame $(e^1, e^2) = (e, v \wedge e)$ defined on $B_R(x_0)$ as above. Choose $\tilde{R}$ so that for $j\in B_{\tilde{R}}(x_0)$, nearest neighbours still lie within $B_R(x_0)$. Consider $j$ indexing in the domain $B_{\tilde{R}}(x_0)$. Then $q_j = q_j^{+k}$ or $q_j^{-k}$ satisfies

$$\partial_t q_j = i\Delta^h q_j + \Delta^k q_j + F^h(q_j, D^1 q_j, e_j),$$

where $F^h(q, D^1 q, e) = O(|q|^3, |q||D^1 q|)$ and $l$ indexes over nearest neighbours of $j$.

Proof. Let $u_j = u_j^h$, $q_j = q_j^{+k}$ and $v_j = v_j^{+k}$. To get an equation in terms of $q_j$, take $D_{sk}$ of (2.2), and then take the inner product of the resulting expression with $e_j$. Let

$$I = D_{sk}(\partial_t u_j) \cdot e_j = \partial_t(D_{sk} u_j) \cdot e_j = II + III + IV,$$

where

$$II = (v_j \wedge D_{sk} \Delta^h u_j^h) \cdot e_j,$$

$$III = (D_{sk} v_j \wedge D_{sk}^2 u_j^h) \cdot e_j,$$

$$IV = D_{sk}(\Delta^h u_j^h + \lambda_j^k v_j) \cdot e_j.$$

$$I = \partial_t(D_{sk} u_j) \cdot e_j = \partial_t(v_j + a_j v_j) \cdot e_j = \partial_t(q_j e_j) \cdot e_j + F_1 = \partial_t q_j + F_1.$$
where \( F_1 = c_i a_j (v_j \wedge \Delta^h u_j + \Delta^h u_j) \cdot e_j \) belongs to the nonlinear term. We need to show that the magnitude of \( F_1 \) is of the order claimed. Writing \((v_j \wedge \Delta^h u_j) \cdot e_j = (v_j \wedge D_{-i} v_j) \cdot e_j + (v_j \wedge D_{-i} w_j) \cdot e_j\), we consider each term in turn:

\[
|\langle v_j \wedge D_{-i} v_j \rangle \cdot e_j \rangle| = |\langle v_j \wedge (D_{-i} q_j e_j + q_{j-1} D_{-i} e_j) \rangle \cdot e_j | \\
\leq |\langle D_{-i} q_j (v_j \wedge e_j) \rangle \cdot e_j | + |q_{j-1} (v_j \wedge D_{-i} e_j) \cdot e_j | \\
= |\langle D_{-i} q_j \rangle | + |q_{j-1} (e_{j-1} \wedge D_{-i} v_j) \cdot e_j | \\
= |\langle D_{-i} q_j \rangle | + |q_{j-1}| |c_j D_{-i} u_j| \\
= O(\|D^1 q_i\| \| q_i \|^2),
\]

\[
|\langle v_j \wedge D_{-i} w_j \rangle \cdot e_j | = |\langle v_j \wedge D_{-i} (a_j v_j) \rangle \cdot e_j | \\
= |\langle v_j \wedge a_{j-1} D_{-i} (v_j) \rangle \cdot e_j | \\
= |c_i a_{j-1} (v_j \wedge D_{-i} u_j) \cdot e_j | \\
= O(\|h|D_{-i} u_j|^2 |D_{-i} u_j|) \\
= O(\|(u_j - u_{j-1})|D_{-i} u_j|^2) \\
= O(\|q_i \|^2).
\]

Similarly, \( |\Delta^h u_j \cdot e_j| = |\langle D_{-i} (v_j + w_j) \rangle \cdot e_j | = O(\|D^1 q_i\| \| q_i \|^2) \). Multiplying both terms by \( c_i a_j = O(\|q_i \|) \), we have \( F_1 = O(\|q_i \|^3 \| q_i \| \|D^1 q_i\|) \).

\( II = (v_j \wedge D_{-i} \Delta^h u_j) \cdot e_j = (v_j \wedge \Delta^h v_j + v_j \wedge \Delta^h w_j) \cdot e_j = (v_j \wedge \Delta^h v_j) \cdot e_j + F_2 \).

Writing \( \Delta^h v_j = \Delta^h (q_j e_j) = \Delta^h q_j e_j + (1/h)(q_{j+1} D_{-i} e_j - q_{j-1} D_{-i} e_j) \), we can express the first term in \( II \) as

\( (v_j \wedge \Delta^h v_j) \cdot e_j = (v_j \wedge \Delta^h q_j e_j) \cdot e_j + F_3 = \Delta^h q_j (v_j \wedge e_j) \cdot e_j = i \Delta^h q_j + F_3 \).

Altogether, then, we have

\( II = i \Delta^h q_j + F_2 + F_3, \)

\( |F_2| = \|v_j \wedge \Delta^h w_j\| \cdot e_j | \\
= \frac{1}{h} |\langle v_j \wedge (D_{-i} u_j - D_{-i} w_j) \rangle \cdot e_j | \\
= \frac{1}{h} |\langle v_j \wedge a_{j+1} D_{-i} u_j - v_j \wedge a_{j-1} D_{-i} u_j \rangle \cdot e_j | \\
= \frac{1}{h} |\langle v_j \wedge a_{j+1} v_j^{(i)} - v_j \wedge a_{j-1} v_j^{(i)} \rangle \cdot e_j | \\
= O \left( \frac{1}{h} \| D^1 u_j \| \| v_j^{(i)} \| \| D^1 u_j \| \| v_j^{(i)} \| \right) \\
= O(\|q_i \|^3),
\]

\( |F_3| = \left| \langle v_j \wedge \frac{1}{h} (q_{j+1} D_{-i} e_j - q_{j-1} D_{-i} e_j) \rangle \cdot e_j | \\
\leq \frac{1}{h} |q_{j+1} (v_j \wedge D_{-i} e_j) + q_{j-1} (v_j \wedge D_{-i} e_j) | \\
= \frac{1}{h} |q_{j+1} (D_{-i} v_j \wedge e_{j+1}) + q_{j-1} (D_{-i} v_j \wedge e_{j-1}) | \\
= O(\|q_i \| \|D^1 q_i\|) \).
Term \(III\) only contributes to the nonlinear term and after some manipulation, we have
\[
|III| = |D_{jk} v_j \wedge D_{jk}^2 u_j|
\]
\[
\lesssim c_v \left( D_{kk} u_j \wedge \frac{1}{h} D_{kk} u_{j+1} \right)
\]
\[
= c_v \frac{1}{h} (D_{kk} u_{j+1} \wedge D_{kk} u_{j+1})
\]
\[
\lesssim \frac{1}{h} \left( (v_{j+1}^- + w_{j+1}^-) \wedge (v_{j+1}^+ + w_{j+1}^+) \right)
\]
\[
\lesssim \frac{1}{h} \left( v_{j+1}^+ \wedge w_{j+1}^+ \wedge v_{j+1}^+ \right)
\]
\[
\lesssim \frac{1}{h} \left( q_{j+1}^- a_{j+1}^+(i e_{j+1}) + a_{j+1}^- q_{j+1}^+(i e_{j+1}) \right)
\]
\[
= O(|q_{j+1}^3|).
\]

Lastly,
\[
IV = D_{sk} (\Delta^h u_j + \lambda_{j+1}^h v_j) \cdot e_j = (\Delta^h v_j \cdot + \Delta^h w_j + \lambda_{j+1}^h D_{sk} v_j) \cdot e_j = \Delta^h q_j + F_4 + F_5,
\]
where
\[
|F_4| = \frac{1}{h} \left( q_{j+1}^- a_{j+1}^- \left. - i e_{j+1} \right) + a_{j+1}^- q_{j+1}^+ (i e_{j+1}) \right)
\]
\[
= O(|q_{j+1}| |D^1 q_j|)
\]
and
\[
|F_5| = |(\Delta^h w_j + \lambda_{j+1}^h D_{sk} v_j) \cdot e_j|
\]
\[
= \frac{1}{h} \left( a_{j+1}^- D_{sk} v_j - a_{j+1}^- D_{sk} u_j \right) \cdot e_j + \lambda_{j+1}^h D_{sk} v_j \cdot e_j
\]
\[
\lesssim \frac{1}{h} \left( a_{j+1}^- D_{sk} v_j - a_{j+1}^- D_{sk} u_j \right) \cdot e_j + \lambda_{j+1}^h D_{sk} v_j \cdot e_j
\]
\[
\lesssim \frac{1}{h} \left( q_{j+1}^- a_{j+1}^- \left. - i e_{j+1} \right) + a_{j+1}^- q_{j+1}^+ (i e_{j+1}) \right)
\]
\[
= O(|q_{j+1}| |D^1 q_j|, |q_{j+1}^3|).
\]

4.2. Proof of main result

Equipped with lemma 4.1, we can follow the steps illustrated for the harmonic map heat flow to prove our main result. Given \( N \), let \( \gamma_0 \) be the quantity associated to \( N \) such that a finite energy moving frame can be defined whenever
\[
E^h[u_h, B_R(x_0 + 0)] \leq \gamma_0.
\]

**Lemma 4.2.** There is a \( \tilde{h} < \epsilon_0 \leq \gamma_0 \) and \( \tilde{h} \) such that if \( E^h[u_h(t_0); B_R(x_0)] \leq \epsilon_0 \) then \( \exists \delta \) independent of \( h \) and \( \tau \in (t_0, t_0 + \delta R^2) \),
\[
\sup_{t \in [0, t_0 + \delta R^2]} \| D^2 u_h(t) \|_{L^2(t_0; B_R(x_0))} \leq C.
\]

**Proof.** From (2.3), the uniform bounds on energy and \( \| \partial_t u_h \|_{L^2} \) are immediate. Multiplying (2.3) by \( \partial_t u_h \), summing over \( j \) and integrating in \( t \), we have
\[
E^h[u_h(t)] + \| \partial_t u_h \|_{L^2(L^2)} \leq E^h[f^h].
\]
We will achieve $L^\infty(H^1_b)$ bounds in the incremental way that was illustrated for the harmonic map heat flow:

(i) a local energy inequality follows exactly as in the proof of 3.8 using (2.3);
(ii) $L^2(H^1_b)$: also using (2.3), nothing changes in the proof of this estimate except that $\partial_t u_j$ has to be replaced by $\partial_t u_j^b - v_j \wedge \delta_R u_j^b$;
(iii) $W^{1,4}_h$: the statement that we will show is:

for $\epsilon > 0$, $\exists \delta$ and $\tau \in (t_0, t_0 + \delta R^2)$ such that for $r > 1$ and $u^b$ solving (2.2).

$E^b[u^b(t); B_{\epsilon R}(x_0)] < \epsilon \Rightarrow \| D^1 u^b \|_{L^1([\tau, t_0 + \delta R^2] \cap [u^b B_{\epsilon R}(x_0))]} \leq c(\|u^b(\tau)\|_{H^1_b}; E^b[f^b])$.

**Proof.** We can use the $L^2(H^1_b)$ bounds to show that for any $t_1 \in (t_0, t_0 + \delta R^2)$, there exists $\tau \in (t_0, t_1)$ such that

$$\| D^2 u^b(\tau) \|_{L^2_b(H^1(x_0))} \leq \frac{C}{(t_1 - t_0)}.$$  

We can now start at $t = \tau$ with $H^1_b$ initial data. Since the intention is to use linear estimates on the fundamental solution, we will resort to the linearized discrete system given by lemma 4.1. Choose $\hat{h}$ so that the conditions of the lemma are satisfied, and consider $h \leq \hat{h}$. Choose $\zeta$ to be the spatial cutoff function, $1$ in a ball of radius $R$ supported in $B_{2R}(x_0)$. Then for $q_j = q_j^k$, $q_j^k$ solves the equation as follows:

$$\partial_t(q_j \zeta_j) - (1 + i)\Delta_h(q_j \zeta_j) = (1 + i)(\frac{1}{2}(\zeta_{j+1} + 2\zeta_j + \zeta_{j-1}) - 2(\zeta_{j+1} + \zeta_{j-1}))F^h(q_{j}, D^1 q_j, \epsilon_i)$$

$$- 2(D_{\alpha\beta} \zeta_j + D_{-\alpha} \zeta_j)(D_{\alpha\beta} q_j + D_{-\alpha} q_j) - \frac{1}{4}(q_{j+1} + 2q_j + q_{j-1})\Delta_h \zeta_j.$$  

Applying Duhamel,

$$\|q(t)\zeta\|_{L^1_b(H^1)} \leq \Delta \sum_k \|q^k(t)\zeta\|_{L^1_b(H^1)} + \|q^{-k}(t)\zeta\|_{L^1_b(H^1)}$$

$$\lesssim \|u^b(t - \tau)q(\tau)\|_{L^1_b(H^1)} + \int_{\tau}^t \frac{1}{(t - s)^{1/2}} \{\|q(s)\|^3$$

$$+ \|D^1 q(s)\|_{L^1_b(H^1)}\} \|D^1 \zeta\|_{L^1_b(H^1)} + \|q(s)\|_{L^1_b(H^1)} \|D^2 \zeta\|_{L^1_b(H^1)}$$

$$+ \|D^1 q(s)\|_{L^1_b(H^1)} + \|q(s)\|_{L^1_b(H^1)} \|D^1 \zeta\|_{L^1_b(H^1)} + \|q(s)\|_{L^1_b(H^1)} \|\Delta_h \zeta\|_{L^1_b(H^1)} \} ds.$$  

By definition $q_j^{k, a} = D_{ik} u_j^b \cdot e^a$, so

$$D_{ik} q_j^{k, a} = O(\|D^2 u_j^b\|, \|D^1 u_j\|^2).$$

The $\|D^1 u_j\|^2$ term can be seen as the error that appears from using $q$ to estimate $D^1 u^b$; if it were not there, we would immediately have that $\|D^1 q\| = O(\|D^2 u^b\|)$, and then exploit the $L^2(L^2)$ bounds on $u^b$ to attain bounds on $q \zeta$ from the above integral inequality. Instead, we keep in mind that the use of $q$ was purely to make use of the linear estimates; it is $D^1 u^b \zeta$ that we need to bound. So long as $q$ satisfactorily controls $D^1 u^b$, we can achieve the desired bounds. This control is a consequence of observation 4; namely,

$$\|D_{ik} u_j^b\|^2 = O(\|q_j^{k, a}\|^2), \quad \|D_{-ik} u_j^b\|^2 = O(\|q_j^{k, a}\|^2).$$
Since \( \delta R \approx 1 \), therefore, it suffices to attain uniform, local \( L^2 \)-bounds in a reduced parabolic cylinder, bounding the second term is easily bounded. Integrating on a time interval of \( \delta R^2 \) and using the \( L^2 \)-bounds, the second term is easily bounded. Therefore,

\[
\|D^1(u^h \xi)\|_{L^2([\tau, \tau+1]; L^2(\mathbb{R}^2))} \lesssim C(\|u^h(\tau)\|_{L^2}, E^h [f^h]).
\]

(iv) \( L^\infty(H^2_0) \)

**Proof.** Since

\[
D^1 q_j^{*,a} = O(\|D^2 u^j\|, \|D^1 u_j\|^2)
\]

and we have from (iii) that \( u^h \) has uniform \( L^2(W^1_0) \) bounds in a reduced parabolic cylinder, it suffices to attain uniform, local \( L^\infty(W^1_0) \) bounds on \( q_\xi \):

\[
\|D^1 q(t) \xi\|_{L^\infty(\mathbb{R}^2)} \lesssim \|\delta^h (t - \tau)(D^1 q(t) \xi)\|_{L^\infty(\mathbb{R}^2)} + \int_{\tau}^{t'} \frac{1}{(t-s)^{3/4}} \left[ \|q(s)\|^3 + \|D^1 q(s)\| \|D^1 \xi\|_{L^\infty(\mathbb{R}^2)} \right] ds
\]

\[
+ C(R, E^h [f^h])
\]

\[
\lesssim \|D^1 q(t) \xi\|_{L^\infty(\mathbb{R}^2)} + \int_{\tau}^{t'} \frac{1}{(t-s)^{3/4}} \left[ \|q(s)\|_{L^2(\mathbb{R}^2)} \|q(s)\|^2 \|D^1 \xi\|_{L^2(\mathbb{R}^2)} \right]
\]

\[
+ C(R, E^h [f^h]) ds
\]

\[
\lesssim \|D^1 q(t) \xi\|_{L^\infty(\mathbb{R}^2)} + \int_{\tau}^{t'} \frac{1}{(t-s)^{3/4}} \left[ \|q(s)\|_{L^2(\mathbb{R}^2)} \|q(s)\|^2 \|D^1 \xi\|_{L^2(\mathbb{R}^2)} \right]
\]

\[
+ \|q(s)\|_{L^2(\mathbb{R}^2)} \|q(s)\|^2 \|D^1 \xi\|_{L^2(\mathbb{R}^2)} ds + C(R, E^h [f^h]) ds
\]

\[
\lesssim \|D^1 q(t) \xi\|_{L^\infty(\mathbb{R}^2)} + \int_{\tau}^{t'} \frac{1}{(t-s)^{3/4}} \left[ \|q(s)\|_{L^2(\mathbb{R}^2)} \|D^1 q(s)\| \|D^1 \xi\|_{L^\infty(\mathbb{R}^2)} \right] ds
\]

\[
+ C(R, E^h [f^h]) ds
\]
Taking $L^\infty$ in time of both sides, we have by Young’s inequality that we can take $\|q(s)\|_{L^4([t_0; t; \mathbb{R}^2])}$ in $L^r$ for any $r > 4$ to achieve the bound

$$
\|D^1(u^h \xi)\|_{L^\infty([t_0; t; \mathbb{R}^2], L^\infty([\mathbb{R}^2]))} \leq C(\|u^h\|_H^2, E^h[f^h]).
$$

\[\square\]

**Proof (theorem 1.1).**

**Step 1: Attaining bounds.** This is the content of lemma 4.2.

**Step 2: Estimating the size of the concentration set.** There is no change in this argument from the case of the harmonic map heat flow.

**Step 3: Convergence of the interpolants.** The argument has been illustrated in the case of the harmonic map heat flow. The only work that has to be done to accommodate for the presence of the Schrödinger term is to show that on a compact set $Q$ for which we have uniform $L^\infty(H^2_h)$ bounds the interpolant $p_h$ converges. For $\phi \in C^\infty_0(Q)$, we need to show that

$$
\int_Q \partial_t p^h \cdot \phi + \nabla p^h \cdot \nabla \phi - \nabla v(p^h) \cdot \nabla v(p^h) + \sum_i \{(v(p^h) \wedge \partial_i p^h) \cdot \partial_i \phi

- (\partial_i v(p^h) \wedge \partial_i p^h) \cdot \phi \} = O(h).
$$

Expanding the $\partial_t p^h \cdot \phi$ term, collecting terms in $\partial_t u^h$ and using the $L^2(L^2_h)$ bound on $\partial_t u^h$, the only term that is not already $O(h)$ contains the following terms:

$$
\partial_t u^h(t) \cdot \phi = (u^h \wedge \Delta u^h) \cdot \phi + (\Delta^h u^h(t) + \lambda^h v) \cdot \phi.
$$

It was shown in the harmonic map heat flow case that the terms involving the damping term have the correct cancellation. It remains to show that

$$
\int_Q \sum_i (D_{vi}(v_i \wedge D_{-i} u^h_j) - D_{vi} v_i \wedge D_{vi} u^h_j) \cdot \phi

+ (v(p^h) \wedge \partial_i p^h) \cdot \partial_i \phi - (\partial_i v(p^h) \wedge \partial_i p^h) \cdot \phi

= O(h).
$$

The uniform $L^\infty(H^2_h)$ bounds resolve most of the terms. In addition, the following manipulation

$$
\int_Q \sum_i D_{vi}(v_i \wedge D_{-i} u^h_j) \cdot \phi + (v(p^h) \wedge \partial_i p^h) \cdot \partial_i \phi

= \int_Q \sum_i ((v_j - v(p^h)) \wedge D_{-i} u^h_j) \cdot \phi - v(p^h) \wedge (h \Delta^h u^h_j) \cdot \phi + O(h)
$$

combined with these bounds permits us to conclude that $p_h$ converges in $Q$. The argument to show that $p$ solves the equation on $\mathbb{R}_+ \times \mathbb{R}^2$ proceeds with no change from the harmonic map heat flow case.

A consequence of the proof for theorem 1.1 is the special case for small energy initial data.

**Corollary 4.3.** There is a constant $\epsilon_0$ such that for $f \in H^1(\mathbb{R}^2, \mathcal{N})$, $E[f] < \epsilon_0$, there exists a smooth global solution to (1.1).
Acknowledgments

This work was part of a dissertation completed at the Courant Institute. The author would like to thank Fanghua Lin for suggesting the problem that led to this work, Jalal Shatah for his guidance, and the two anonymous referees whose suggestions contributed to the readability of this document. This work was partially supported by NSF grant DMS-0402788.

Appendix A

Let $\Omega \subseteq \mathbb{R}^2$ and $\xi \in C_0^\infty(\Omega)$. Then

$$\|f^2 \xi\|_{L^2(\mathbb{R}^2)} \leq C(s) \|f\|_{L^p(\Omega)} \|\nabla(f \xi)\|_{L^q(\mathbb{R}^2)}; \quad \frac{s + 1}{2s} = \frac{1}{p} + \frac{1}{q}. \quad (A.1)$$

**Proof.** We first take the easy case $q < 2$. Then by Hölder and Sobolev embedding,

$$\|f\|_{L^p(\Omega)} \|f \xi\|_{L^r(\mathbb{R}^2)} \leq C \|f\|_{L^p(\Omega)} \|\xi\|_{L^q(\mathbb{R}^2)},$$

the last inequality holding iff $(1/q) - (1/2) > 0$ iff $q < 2$. $p$, $q$ and $s$ are related as follows:

$$\frac{1}{p} = \frac{1}{2s} - \frac{1}{r} = \frac{1}{2s} - \frac{1}{q} + \frac{1}{2}.$$

This is the inequality that we want.

Now consider the case $q > 2$. The proof follows in the same manner as that for the normal Gagliardo–Sobolev–Nirenberg inequalities with the help of the following identity:

$$\partial_k(f^{2s} \xi^s) = 2sf^{2s-1} \partial_k(f \xi) f^{2s} - f^{2s} \partial_k(\xi^s).$$

We can write

$$|\partial_1(f^{2s}(x_1, x_2)\xi^s(x_1, x_2))|$$

$$\leq 2s \int_{-\infty}^{x_1} |\partial_1(f(y_1, x_2)\xi(y_1, x_2)) f^{2s-1}(y_1, x_2)\xi^{s-1}(y_1, x_2)| \, dy_1$$

$$+ \int_{-\infty}^{x_1} |f^{2s}(y_1, x_2)\partial_1(\xi^s(y_1, x_2))| \, dy_1$$

and a similar expression holding $x_1$ fixed and integrating on the interval $(-\infty, x_2)$. Multiplying these two expressions together, and integrating, we have

$$\|f^2 \xi\|_{L^2}^{2s} = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |f^{2s}(x_1, x_2)\xi^s(x_1, x_2)|^2 \, dx_1 \, dx_2$$

$$\leq C(s) \left\{ \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla(f \xi)||f^{2s-1}||\xi^{s-1}| \, dx_1 \, dx_2 \right)^2 + \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla(f \xi)||f^{2s-1}||\xi^{s-1}| \, dx_1 \, dx_2 \right) \left( \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} |\nabla(\xi^s)||f^{2s} \, dx_1 \, dx_2 \right)^2 \right\}$$

$$\leq C(s) \|\nabla(f \xi)\|_{L^q}^2 \|f\|_{L^p}^2 \|f^2 \xi^{s-1}\|_{L^q}^2.$$

$\square$
Appendix B

Fix $\mathcal{N}$ to be an embedded, closed surface without boundary. A moving frame on $\mathcal{N}$ is a function $e : p \in \mathcal{N} \rightarrow e(p)$, where $e(p)$ defines an orthonormal basis of $T_p\mathcal{N}$. If $e$ is defined for all $p \in \mathcal{N}$ then $\mathcal{F}$ is a global frame for $\mathcal{N}$. The assumption that $\mathcal{N}$ has a global frame is inordinately restrictive and excludes simple examples such as $S^2$. One potential remedy, borrowed from harmonic map literature [5], is to isometrically embed $\mathcal{N}$ into a larger manifold that does have a global frame and in such a way that the equation on the larger manifold has $\mathcal{N}$ as an invariant submanifold. Such larger manifolds can be constructed so long as $\mathcal{N}$ is sufficiently smooth. For the case where $(\mathcal{N}, g, J)$ is Kähler, however, our requirement for an embedding has the additional requirement that the larger manifold must also be equipped with a complex structure such that its restriction on $\mathcal{N}$ is $J$. For the case where $\mathcal{N}$ is compact, this additional requirement makes such a construction impossible. A theorem from minimal surface theory states that every complex submanifold of a Kähler manifold is minimal, which excludes compact manifolds [7].

The remedy that will work for our case is to isometrically embed $\mathcal{N}$ into a larger manifold. A pullback frame is defined as follows. Let $e_0$ be a frame at $\mathcal{N}$, consisting of all pairs $(u, e)$, where $u \in \mathcal{N}$ and $e = (e_1, e_2)$ is an orthonormal basis of $T_u\mathcal{N}$. Assume $u \in H^1(B^2, \mathcal{N})$ is smooth for $B^2$ the unit ball in $\mathbb{R}^2$. Then $u$ can be lifted to a map $(u, e) : B^2 \rightarrow \mathcal{F}$. There are an infinite number of such lifts and can be fixed with a choice of gauge. The pullback of the tangent bundle of $\mathcal{N}$ by $u$ is the bundle $u^*\mathcal{F} = \{ (x, (e_1, e_2)) : B^2 \times \mathbb{R}^2 | (e_1, e_2) \text{ is an orthonormal basis of } T_{u(x)}\mathcal{N} \}$. If $u$ is a smooth map, then it is possible to construct a section of $u^*\mathcal{F}$ which is also smooth. If $\mathcal{N}$ admits a global frame, then a pullback frame for $u$ automatically exists, given by $\{ e_1(u(x)), e_2(u(x)) \}$. However, the converse is not true. The pullback frame is a frame on the domain and might be different at $u(x_1)$ and $u(x_2)$, even when the two values coincide on $\mathcal{N}$.

The construction of such a pullback frame is most apparent in the case $u : I \subset \mathbb{R} \rightarrow \mathcal{N}$. Let $e_0$ be a frame at $T_{u(0)}\mathcal{N}$. Then we can solve for $e$ by parallel transport (i.e. $e$ satisfying $D_t e = 0$) which gives the following ODE:

\[
\left\{
\begin{array}{l}
\partial_t e + (e \cdot (v(u))), v(x) = 0, \\
e(0) = e_0.
\end{array}
\right.
\]  

(A.2)

From this construction it is apparent that the frame reflects the geometry of $u$, since $e$ and $u$ have the same regularity. This particular choice of gauge is called the Coulomb gauge. A frame $(e_1, e_2)$ satisfying the Coulomb gauge must satisfy the following conservation law:

\[
\partial_t (\partial_t e_1, e_2) + \partial_t (\partial_t e_1, e_2) = 0.
\]

The construction of such a frame is given in [5]. It is shown that if $u \in H^1(B^2, \mathbb{R}^2)$ then there exists $\gamma_0$ depending only on $\mathcal{N}$ such that if $\|Vu\|_{L^2(B^2)} \leq \gamma_0$ then there exists a finite energy moving frame that satisfies the Coulomb gauge.

References

[1] Alouges F and Soyeur A 1992 On global weak solutions for Landau–Lifshitz equations: existence and nonuniqueness Nonlinear Anal. 18 1071–84
[2] Chang N-H, Shatah J and Uhlenbeck K 2000 Schrödinger maps Commun. Pure Appl. Math. 53 590–602
[3] Chen Y M and Struwe M 1989 Existence and partial regularity results for the heat flow for harmonic maps Math. Z. 201 83–103
[4] Guo B and Hong M-C 1993 The Landau–Lifshitz equation of the ferromagnetic spin chain and harmonic maps Calc. Var. Partial Diff. Eqns 1 311–34
[5] Hélein F 2002 *Harmonic Maps, Conservation Laws and Moving Frames (Cambridge Tracts in Mathematics)* vol 150, 2nd edn (Cambridge: Cambridge University Press) Translated from the 1996 French, with a foreword by J Eells

[6] Ladyzhenskaya O A 1985 *The Boundary Value Problems of Mathematical Physics (Applied Mathematical Sciences)* vol 49 (New York: Springer) Translated from the Russian by J Lohwater

[7] Blaine Lawson H Jr 1980 *Lectures on Minimal Submanifolds*, vol I (*Mathematics Lecture Series*) vol 9, 2nd edn (Wilmington: Publish or Perish)

[8] Melcher C Existence of partially regular solutions for Landau–Lifshitz equations in $\mathbb{R}^3$ *Commun. Partial Diff. Eqns* submitted

[9] Struwe M 1985 On the evolution of harmonic mappings of Riemannian surfaces *Comment. Math. Helv.* 60 558–81

[10] Sulem C-L, Sulem C and Bardos C 1986 On the continuous limit for a system of classical spins *Commun. Math. Phys.* 107 431–54

[11] Wu X 2000 Two dimensional Landau–Lifshitz equations in micromagnetism *PhD Dissertation, New York University*