Gamma distributed random variables and their semi–quantum operators

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Abstract. We first introduce the joint semi–quantum operators of a finite family of random variables having finite moments of all orders. We then use the semi–quantum operators to characterize the one–dimensional Gamma and Gaussian distributions in terms of their commutators.

1. Introduction

A series of papers, see [1], [2], [9], and [10], have been written by various authors to characterize certain classes of random variables or properties of their probability distributions in terms of their quantum operators. A special attention has been given to the Meixner class because of the Lie Algebra structure enjoyed by the vector space spanned by the identity and quantum operators. Using the commutators of the quantum operators, a recursive way of computing the moments have been used in [8] and [9]. Moreover, one paper has been written about the $q$–commutators of the quantum operators of some class of random variables, see [4].

In this paper we propose a study of random variables in terms of their semi–quantum operators, which are obtained by splitting the preservation operators in half and adding one half to the creation operators and the other half to the annihilation operators. A particular subclass of the Meixner random variables, namely the Gamma distributed and Gaussian (which can be viewed as limits of the Gamma distributed) random variables, can be characterized in an elegant way using the semi–quantum operators. We observe a Lie Algebra structure for the vector space spanned by the identity and semi–quantum operators of these random variables.

The paper is structured as follows:
In section 2 we present a minimal background on the quantum operators, and use them to define the semi–quantum operators. In section 3 we characterize the one–dimensional Gamma and Gaussian random variables in terms of the commutators of their semi–quantum operators. The proof is done via the Szegő–Jacobi parameters. Finally, in section 4 we show how the moments, of a random variable, can be recovered, in the one–dimensional case, from the commutator of the semi–quantum operators, and use these moments to compute the distribution of the random variable, via the Laplace transform.
2. Background

Let $X_1, X_2, \ldots, X_d$ be $d$–random variables defined on the same probability space $(\Omega, \mathcal{F}, P)$ and having finite moments of all orders, i.e., for all $i \in \{1, 2, \ldots, d\}$, and all $p > 0$,

$$E[|X_i|^p] < \infty,$$

where $E$ denotes the expectation.

For each non–negative integer $n$, we define the space $F_n$ of all polynomial random variables of the form $P(X_1, X_2, \ldots, X_d)$, where $P$ is a polynomial of $d$ variables, with complex coefficients, of total degree less than or equal to $n$. Since $X_1, X_2, \ldots, X_d$ have finite moments of all orders, each space $F_n$, for $n \geq 0$, is contained in $L^2(\Omega, \mathcal{F}, P)$. Therefore, we have:

$$\mathbb{C} = F_0 \subset F_1 \subset F_2 \subset \cdots \subset L^2(\Omega, \mathcal{F}, P).$$

We also define the space $F_{-1} := \{0\}$ (the null space). Moreover, since for all $n \geq 0$, $F_n$ is finite dimensional, $F_n$ is a closed subspace of $L^2(\Omega, \mathcal{F}, P)$. Therefore, we can orthogonalize the spaces $\{F_n\}_{n \geq 0}$, and define, for all $n \geq 0$:

$$G_n := F_n \ominus F_{n-1},$$

that means $G_n$ is the orthogonal complement of $F_{n-1}$ into $F_n$. We also define $G_{-1} := \{0\}$.

We introduce now the spaces:

$$F = \bigcup_{n=0}^{\infty} F_n,$$

$$= \sum_{n=0}^{\infty} G_n,$$

and

$$H = \oplus_{n=0}^{\infty} G_n.$$

The distinction between $F$ and $H$ is the following: $F$ is the algebraic sum of the vector spaces $\{G_n\}_{n \geq 0}$, that means every element in $F$ is a sum of only finitely many non–zero elements of $\bigcup_{n \geq 0} G_n$, while $H$ is the orthogonal direct sum (a Hilbert space) of the orthogonal closed vector subspaces $\{G_n\}_{n \geq 0}$ of $L^2(\Omega, \mathcal{F}, P)$. We call $F$ the space of polynomial random variables generated by $X_1, X_2, \ldots, X_d$, and $H$ the chaos space generated by $X_1, X_2, \ldots, X_d$. For each $n \geq 0$, we call $G_n$ the $n$–th homogenous chaos space generated by $X_1, X_2, \ldots, X_d$, and each element $P(X_1, X_2, \ldots, X_d)$ a homogenous polynomial of degree $n$.

We are going to view now each random variable $X_1, X_2, \ldots, X_d$ as a multiplication linear operator. For each $i \in \{1, 2, \ldots, d\}$, we consider the multiplication operator: $M_{X_i} : F \to F$ defined by:

$$M_{X_i} P(X_1, X_2, \ldots, X_d) := X_i P(X_1, X_2, \ldots, X_d).$$

To keep the notation simple we denote $M_{X_i}$ by $X_i$.

The following lemma can be easily checked using the symmetry of the multiplication operators: $X_1, X_2, \ldots, X_d$.

**Lemma 2.1** For all $i \in \{1, 2, \ldots, d\}$ and all $n \geq 0$, we have:

$$X_i G_n \perp G_k,$$

for all $k \neq n-1, n$, and $n+1$, where “$\perp$” means “orthogonal to”.


Let $i \in \{1, 2, \ldots, d\}$ be fixed. Since, for all $n \geq 0$,
\[
G_n \subset H = G_0 \oplus G_1 \oplus \cdots,
\]
but $X_i G_n \perp G_k$, for all $k \notin \{n - 1, n, n + 1\}$, we have:
\[
G_n \subset G_{n-1} \oplus G_n \oplus G_{n+1}.
\]
Thus, for all $f \in G_n$, there exists and are unique $f_{n-1} \in G_{n-1}$, $f_n \in G_n$, and $f_{n+1} \in G_{n+1}$, such that:
\[
X_i f := f_{n-1} + f_n + f_{n+1}.
\]
We define the following three linear operators:
\[
D_n^- (i) : G_n \rightarrow G_{n-1},
\]
\[
D_n^- (i) f := f_{n-1},
\]
\[
D_n^0 (i) : G_n \rightarrow G_n,
\]
\[
D_n^0 (i) f := f_n,
\]
and
\[
D_n^+ (i) : G_n \rightarrow G_{n+1},
\]
\[
D_n^+ (i) f := f_{n+1}.
\]
Since $D_n^- (i)$ maps $G_n$ into $G_{n-1}$, it associates to a homogenous polynomial of degree $n$ a homogenous polynomial of degree $n - 1$ and so, it looses one unit in the degree. $D_n^- (i)$ is called an annihilation operator. Similarly, since $D_n^0 (i)$ maps $G_n$ into $G_n$, it preserves the degree of a homogenous polynomial. $D_n^0 (i)$ is called a preservation operator. Finally, since $D_n^+ (i)$ maps $G_n$ into $G_{n+1}$, it increases the degree of a homogenous polynomial by one unit. $D_n^+ (i)$ is called a creation operator.

Lemma 2.1 becomes now:
\[
X_i |_{G_n} = D_n^- (i) + D_n^0 (i) + D_n^+ (i).
\]
So far, we have defined the annihilation, preservation, and creation operators only on each individual space $G_n$, $n \geq 0$. Since $F = \sum_{n \geq 0} G_n$, we can extend the definition of these operators, in a linear way, to the whole space $F$. Namely, let:
\[
a^-(i), a^0(i), a^+(i) : F \rightarrow F,
\]
be defined in the following way: if $f \in F$, there exist and are unique $f_0 \in G_0$, $f_1 \in G_1$, $f_2 \in G_2$, $\cdots$, with only finitely many of them being non–zero, such that:
\[
f := f_0 + f_1 + f_2 + \cdots.
\]
For such an $f$, we define:

$$a^{-}(i)f := D_{0}^{-}(i)f_{0} + D_{1}^{-}(i)f_{1} + D_{2}^{-}(i)f_{2} + \cdots,$$  \hspace{2cm} (20)

$$a^{0}(i)f := D_{0}^{0}(i)f_{0} + D_{1}^{0}(i)f_{1} + D_{2}^{0}(i)f_{2} + \cdots,$$  \hspace{2cm} (21)

and

$$a^{+}(i)f := D_{0}^{+}(i)f_{0} + D_{1}^{+}(i)f_{1} + D_{2}^{+}(i)f_{2} + \cdots.$$  \hspace{2cm} (22)

We call $a^{-}(i)$ an annihilation operator, $a^{0}(i)$ a preservation operator, and $a^{+}(i)$ a creation operator.

We call $\{a^{-}(i)\}_{1\leq i \leq d}$, $\{a^{0}(i)\}_{1\leq i \leq d}$, and $\{a^{+}(i)\}_{1\leq i \leq d}$ the joint quantum operators generated by $X_{1}$, $X_{2}$, $\ldots$, $X_{d}$.

Lemma 2.1 becomes now:

**Theorem 2.2** For all $i \in \{1, 2, \ldots, d\}$, we have:

$$X_{i} = a^{-}(i) + a^{0}(i) + a^{+}(i),$$  \hspace{2cm} (23)

where the domain of the linear operators $X_{i}$, $a^{-}(i)$, $a^{0}(i)$, and $a^{+}(i)$ is understood to be the space $F$ of all polynomial random variables.

For all $i \in \{1, 2, \ldots, d\}$, $a^{0}(i)$ is a symmetric operator, while $(a^{-}(i))^{*} = a^{+}(i)$, meaning, that for all $f$ and $g$ in $F$, we have:

$$\langle a^{0}(i)f, g \rangle := \langle f, a^{0}(i)g \rangle$$  \hspace{2cm} (24)

and

$$\langle a^{-}(i)f, g \rangle := \langle f, a^{+}(i)g \rangle,$$  \hspace{2cm} (25)

where, for all $u$ and $v$ in $F$, we define:

$$\langle u, v \rangle := E \left[ u(X_{1}, X_{2}, \ldots, X_{d})v(X_{1}, X_{2}, \ldots, X_{d}) \right].$$  \hspace{2cm} (26)

For every $i \in \{1, 2, \ldots, d\}$, we define the semi–annihilation operator:

$$U_{i} := a^{-}(i) + \frac{1}{2}a^{0}(i)$$  \hspace{2cm} (27)

and the semi–creation operator:

$$V_{i} := a^{+}(i) + \frac{1}{2}a^{0}(i).$$  \hspace{2cm} (28)

We can see that for all $i \in \{1, 2, \ldots, d\}$, we have:

$$V_{i} = U_{i}^{a}$$  \hspace{2cm} (29)

and

$$X_{i} = U_{i} + V_{i}.$$  \hspace{2cm} (30)

We call $\{U_{i}\}_{1\leq i \leq d}$ and $\{V_{i}\}_{1\leq i \leq d}$ the joint semi–quantum operators of $X_{1}$, $X_{2}$, $\ldots$, $X_{d}$. 
Proposition 2.3 For all $i$ and $j$ in $\{1, 2, \ldots, d\}$, we have:

(i) $[U_i, X_j]$ is a self-adjoint operator.

(ii) 
\[
[U_i, X_j] = [U_j, X_i].
\] (31)

(iii) 
\[
[X_j, V_i] = [X_i, V_j].
\] (32)

Proof.

(i) We have:
\[
[U_i, X_j]^* = [X_j, V_i] = [X_j, X_i - U_i] = 0 - [X_j, U_i] = [U_i, X_j].
\]

(ii) Since, as it was shown in [1] and [2], we have:
\[
[a^{-}(i), a^{-}(j)] = 0,
\] (33)
\[
[a^{-}(i), a^{0}(j)] = [a^{-}(j), a^{0}(i)],
\] (34)
\[
[a^{-}(i), a^{+}(j)] + [a^{0}(i), a^{0}(j)] + [a^{+}(i), a^{-}(j)] = 0,
\] (35)

and
\[
[a^{0}(i), a^{+}(j)] = [a^{0}(j), a^{+}(i)],
\] (36)

we have:
\[
[U_i, X_j] - [U_j, X_i] = \left[ a^{-}(i) + \frac{1}{2} a^{0}(i), a^{-}(j) + a^{0}(j) + a^{+}(j) \right] - \left[ a^{-}(j) + \frac{1}{2} a^{0}(j), a^{-}(i) + a^{0}(i) + a^{+}(i) \right] = [a^{-}(i), a^{-}(j)] - [a^{-}(j), a^{-}(i)]
\]
\[
+ \frac{3}{2} \left( [a^{-}(i), a^{0}(j)] - [a^{-}(j), a^{0}(i)] \right)
\]
\[
+ [a^{-}(i), a^{+}(j)] + [a^{0}(i), a^{0}(j)] + [a^{+}(i), a^{-}(j)]
\]
\[
+ \frac{1}{2} \left( [a^{0}(i), a^{+}(j)] - [a^{0}(j), a^{+}(i)] \right)
\]
\[
= 0.
\]

(iii) follows by taking the adjoint in both sides of (ii).

In the particular case, when $d = 1$, we have only one random variable $X_1$, one annihilation operator $a^{-}(1)$, one preservation operator $a^{0}(1)$, and one creation operator $a^{+}(1)$, which we can call simply $X$, $a^{-}$, $a^{0}$, and $a^{+}$. Since for each $n \geq 0$, $F_n$ is spanned by: $1$, $X$, $X^2$, $\ldots$, $X^n$, the
codimension of $F_{n-1}$ into $F_n$ is at most 1. Thus, for each $n \geq 0$, the dimension of the $n$–th chaos space $G_n$ is at most 1. More precisely, if the support of the probability distribution $\mu$ of the random variable $X$ is not a finite set, then for all $n \geq 0$, the dimension of $G_n$ is equal to 1. On the other hand, if $X$ takes on only finitely many values $x_1, x_2, \ldots, x_k$, with positive probability, then using a Lagrange interpolation polynomial, one can see that for every function $f : \mathbb{R} \to \mathbb{C}$, $f(X)$ is equal almost surely with a polynomial random variable $P(X)$, where $P$ has degree at most $k - 1$. Thus, we have:

$$F_{n-1} = F_n = F_{n+1} = \cdots.$$  \hfill (37)

Therefore, in this case, if “dim” denotes the dimension, he have:

$$\dim(G_n) = \begin{cases} 1 & \text{if } n \leq k - 1 \\ 0 & \text{if } n \geq k \end{cases}. \hfill (38)$$

For every $n \geq 0$, if $G_n \neq \{0\}$, since $\dim(G_n) = 1$, there exists a unique polynomial $f_n$ of degree $n$ with leading coefficient 1, such that $f_n(X) \in G_n$. We call $f_n$ the $n$–th orthogonal polynomial generated by $X$. Since $XG_n \subset G_{n-1} \oplus G_n \oplus G_{n+1}$, there exist $\alpha_n$ and $\omega_n$ real numbers, such that:

$$X f_n(X) = f_{n+1}(X) + \alpha_n f_n(X) + \omega_n f_{n-1}.$$ \hfill (39)

For $n = 0$, we can define $f_{-1} := 0$ and $\omega_0 := 0$. In the above equation, the coefficient of $f_{n+1}$ is 1, since the coefficient of $x^{n+1}$ must be 1 in both sides of it. The numbers $\{\alpha_n\}_{n \geq 0}$ and $\{\omega_n\}_{n \geq 1}$ are called the Szegő–Jacobi parameters of the random variable $X$. In this case, for all $n \geq 0$, we have:

$$a^- f_n(X) = \omega_n f_{n-1}(X),$$ \hfill (40)

$$a^0 f_n(X) = \alpha_n f_n(X),$$ \hfill (41)

and

$$a^+ f_n(X) = f_{n+1}(X).$$ \hfill (42)

3. Gamma distributed random variables

In this section we present a characterization, in the one–dimensional case, of the Gamma distributions in terms of the semi–quantum operators. The Gamma distributed random variables, form a sub–class of the Meixner class, see [6]. It is known that a random variable $X$, having finite moments of all orders, whose Szegő–Jacobi parameters are of the form:

$$\alpha_n = \alpha n + \alpha_0,$$ \hfill (43)

and

$$\omega_n = \beta n^2 + (t - \beta)n,$$ \hfill (44)

for all $n \geq 1$, where $\alpha$, $\beta$, $t$, and $\alpha_0$ are fixed real numbers, such that: $\alpha^2 = 4\beta$ and $t > 0$, is a shifted Gamma distributed random variable with shift parameter $2t/\alpha$ and scaling parameter
function is:

\[ f(x) = \frac{2^{2t/\alpha}}{\alpha^{2t/\alpha} \Gamma(2t/\alpha)} x^{(2t/\alpha)-1} e^{-2x/\alpha} \chi_{(0,\infty)}, \]

if \( \alpha \neq 0 \).

If \( \alpha = 0 \), then \( X \) is a Gaussian random variable. Since the case \( \alpha = 0 \), can be understood as a limit of the case \( \alpha \neq 0 \), as \( \alpha \to 0 \), we will call the Gaussian random variables also Gamma random variables, but we should keep in mind that this is true only in a limit sense.

We have the following proposition:

**Proposition 3.1** Let \( X \) be a random variable having finite moments of all orders. Then \( X \) is a Gamma (or Gaussian) random distributed random variable if and only if the real vector space spanned by its semi-quantum operators \( U \) and \( V \), and the identity operator \( I \), equipped with the bracket given by the commutator \( [\cdot, \cdot] \) form a Lie Algebra.

**Proof.** (\( \Rightarrow \)) Let us suppose that \( X \) is a Gamma distributed random variable. Then, there exists \( \alpha, \alpha_0 \), and \( t \) real numbers, such that its Szegő–Jacobi parameters are: \( \alpha_n = \alpha n + \alpha_0 \) and \( \omega_n = (1/4)\alpha^2 n^2 + [t - (\alpha^2/4)]n \). If, for all \( n \geq 0 \), we denote by \( f_n \), the orthogonal monic polynomial of degree \( n \), generated by \( X \), then we have:

\[
[U, V] f_n = \left[ a^- + \frac{1}{2} a^0, a^+ + \frac{1}{2} a^0 \right] f_n
\]

\[
= \left[ a^-, a^+ \right] f_n + \frac{1}{2} \left[ a^-, a^0 \right] f_n + \frac{1}{2} \left[ a^0, a^+ \right] f_n
\]

\[
= (\omega_{n+1} - \omega_n) f_n + \frac{1}{2} (\alpha_n - \alpha_{n-1}) \omega_n f_{n-1} + \frac{1}{2} (\alpha_{n+1} - \alpha_n) f_{n+1}
\]

\[
= \left( \frac{\alpha^2 n + t}{2} \right) f_n + \frac{1}{2} \alpha \omega_n f_{n-1} + \frac{1}{2} \alpha f_{n+1}
\]

\[
= \frac{\alpha}{2} \left[ \omega_n f_{n-1} + \frac{1}{2} (\alpha n + \alpha_0) f_n \right] + \frac{\alpha}{2} \left[ f_{n+1} + \frac{1}{2} (\alpha n + \alpha_0) f_n \right] + \left( t - \frac{\alpha \alpha_0}{2} \right) f_n
\]

\[
= \frac{\alpha}{2} \left( a^- + \frac{1}{2} a^0 \right) f_n + \frac{\alpha}{2} \left( a^+ + \frac{1}{2} a^0 \right) f_n + \left( t - \frac{\alpha \alpha_0}{2} \right) I f_n.
\]

Thus, we have:

\[
[U, V] = \frac{\alpha}{2} U + \frac{\alpha}{2} V + \left( t - \frac{\alpha \alpha_0}{2} \right) I.
\]

Therefore, the commutator \([U, V]\) can be expressed as a linear combination of \( U \), \( V \), and \( I \), and so the space \( W = \mathbb{R}U + \mathbb{R}V + \mathbb{R}I \) is a Lie Algebra.

(\( \Leftarrow \)) Let us suppose now that the space \( W = \mathbb{R}U + \mathbb{R}V + \mathbb{R}I \), equipped with the commutator \([\cdot, \cdot]\) is a Lie Algebra. That means, there exist \( c, d, \) and \( e \) real numbers, such that:

\[
[U, V] = cU + dV + eI.
\]

Since we have:

\[
[U, X] = [U, U + V]
\]

\[
= [U, V]
\]

\[
= [U, V]
\]

\[
= [U, V]
\]
and we know from Proposition 2.3 that \([U, X]\) is self-adjoint, we conclude that:

\[
c = d.
\]

(46)

Thus, we have:

\[
[U, V] f_n = \frac{1}{2} (\alpha_{n+1} - \alpha_n) f_{n+1} + (\omega_{n+1} - \omega_n) f_n + \frac{1}{2} (\alpha_n - \alpha_{n-1}) \omega_n f_{n-1}
\]

\[
= cX f_n + eI f_n
\]

\[
= cf_{n+1} + (c\alpha_n + e) f_n + c\omega_n f_{n-1}.
\]

Since \(f_{n-1}, f_n, \) and \(f_{n+1}\) are linearly independent, being orthogonal, from the above equality, we conclude that, for all \(n \geq 0\), we have:

\[
\frac{1}{2} (\alpha_{n+1} - \alpha_n) = c
\]

(47)

and

\[
\omega_{n+1} - \omega_n = c\alpha_n + e.
\]

(48)

Thus, for all \(n \geq 1\), we have:

\[
\alpha_n = \alpha_0 + \sum_{k=0}^{n-1} (\alpha_{k+1} - \alpha_k)
\]

\[
= \alpha_0 + \sum_{k=0}^{n-1} 2c
\]

\[
= 2cn + \alpha_0.
\]

This implies that since \(\omega_0 := 0\), for all \(n \geq 1\), we have:

\[
\omega_n = \sum_{k=0}^{n-1} (\omega_{k+1} - \omega_k)
\]

\[
= \sum_{k=0}^{n-1} (c\alpha_k + e)
\]

\[
= c \sum_{k=0}^{n-1} (2ck + \alpha_0) + ne
\]

\[
= 2c^2 \frac{(n-1)n}{2} + c\alpha_0 n + ne
\]

\[
= c^2 n^2 + (c\alpha_0 + e - c^2) n.
\]

Thus \(X\) is a Gamma distributed random variables with \(\alpha := 2c, \beta := c^2, \) and \(t = c\alpha_0 + e, \) since \(\alpha^2 = 4\beta.\)

\[
\square
\]

Observe, that formula (45) can be written as:

\[
[U, X] = cX + eI.
\]

(49)
4. Deriving the Gamma distributions from the semi–quantum operators

Let $X$ be a random variable having finite moments of all orders, such that, if $U$ and $V$ denotes its semi–annihilation and semi–creation operators, the real vector space spanned by $U$, $V$, and $I$, where $I$ is the identity operator, equipped with the commutator bracket $[\cdot, \cdot]$, form a Lie Algebra. Since as a multiplication operator $X = U + V$, and $[U, X]$ is a symmetric operator, we have seen in the previous section that there exist two real numbers $b$ and $\alpha$, such that:

$$[U, X] = bX + \alpha I. \quad (50)$$

Let $\phi := 1$ be the constant polynomial equal to 1.

For all non–negative integers $n$, using Leibniz rule for commutators, we have:

$$E[X^{n+1}] = \langle(U + V)X^n \phi, \phi \rangle = \langle UX^n \phi, \phi \rangle + \langle VX^n \phi, \phi \rangle = \langle [U, X^n] \phi, \phi \rangle + \langle X^n [U, \phi] + \langle X^n \phi, U \phi \rangle = \sum_{i=0}^{n-1} \langle X^i [U, X] X^{n-1-i} \phi, \phi \rangle + \langle X^n [a^- + (1/2)a^0] \phi, \phi \rangle + \langle X^n \phi, [a^- + (1/2)a^0] \phi \rangle. \quad (51)$$

Since $a^- \phi = 0$ and $a^0 \phi = E[X] \phi$, we obtain:

$$E[X^{n+1}] = \sum_{i=0}^{n-1} \langle X^i (bX + \alpha I) X^{n-1-i} \phi, \phi \rangle + \frac{1}{2} \langle X^n E[X] \phi, \phi \rangle + \frac{1}{2} \langle X^n \phi, E[X] \phi \rangle = nbE[X^n] + n\alpha E[X^{n-1}] + E[X] E[X^n]. \quad (52)$$

If we define:

$$c := E[X], \quad (53)$$

then we obtain the following recursive formula for the moments of $X$:

$$E[X^{n+1}] = nbE[X^n] + n\alpha E[X^{n-1}] + cE[X^n], \quad (54)$$

for all $n \geq 0$.

Let $K := b + \alpha + c$. We can see from (52), by induction on $n$, that for all $n \geq 0$, we have:

$$|E[X^n]| \leq K^n n!. \quad (55)$$

The growth condition (53) allows us to define the Laplace transform of $X$:

$$\varphi(s) = E[\exp(sX)], \quad (56)$$

and differentiate its Taylor series term by term, for any $s$ in a neighborhood of 0.

Multiplying both sides of (52) by $s^n/n!$ and summing up from $n = 0$ to $\infty$, we obtain:

$$\sum_{n=0}^{\infty} \frac{s^n}{n!} E[X^{n+1}] = b s \sum_{n=1}^{\infty} \frac{s^{n-1}}{(n - 1)!} E[X^n] + \alpha s \sum_{n=1}^{\infty} \frac{s^{n-1}}{(n - 1)!} E[X^{n-1}] + c \sum_{n=0}^{\infty} \frac{s^n}{n!} E[X^n].$$

This is equivalent to:

$$\varphi'(s) = b s \varphi'(s) + \alpha s \varphi(s) + c \varphi(s). \quad (57)$$
Therefore, for all $s$ in a small neighborhood $V$ of 0, such that $1 - bs > 0$ and $\varphi(s) > 0$ (since $\varphi$ is continuous and $\varphi(0) = 1 > 0$), we have:

$$\frac{\varphi'(s)}{\varphi(s)} = \frac{\alpha s + c}{-bs + 1}.$$  \hspace{1cm} (56)

We distinguish now between two cases:

**Case 1:** If $b \neq 0$, then we can rewrite the previous equation as:

$$\frac{\varphi'(s)}{\varphi(s)} = -\frac{\alpha}{b} + \left(\frac{\alpha}{b} + c\right) \cdot \frac{1}{-bs + 1}.$$  \hspace{1cm} (57)

Introducing now the constants: $\beta := -\alpha/b$ and $\gamma := -\beta + c$, we obtain:

$$\frac{d}{ds} \ln(\varphi(s)) = \frac{d}{ds} \left[\beta s - \gamma \ln(-bs + 1)\right],$$

for all $s$ in $V$. Thus, for all $s \in V$, we obtain:

$$E[\exp(sX)] := A \exp(\beta s(-bs + 1)^{-\gamma/\beta},$$

where $A$ is a positive real number.

It is not hard to see that the random variables that have a Laplace transform of the form (59) are exactly the shifted and re-scaled Gamma distributed random variables.

**Case 2.** If $b = 0$, then equation (56) becomes:

$$\frac{\varphi'(s)}{\varphi(s)} = \alpha s + c.$$  \hspace{1cm} (60)

It is clear now that:

$$E[\exp(sX)] = B \exp\left(\frac{\alpha^2}{2}s^2 + cs\right),$$

where $B$ is a positive real number.

It is easy to see that the random variables, having a Laplace transform of the form (61), are exactly the Gaussian random variables.

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