Steiner Variations on Random Surfaces

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Abstract

Ambartzumian et.al. suggested that the modified Steiner action functional had desirable properties for a random surface action. However, Durhuus and Jonsson pointed out that such an action led to an ill-defined grand-canonical partition function and suggested that the addition of an area term might improve matters. In this paper we investigate this and other related actions numerically for dynamically triangulated random surfaces and compare the results with the gaussian plus extrinsic curvature actions that have been used previously.
There has been considerable recent interest in the theory and simulation of random surfaces, inspired both by string theory and the study of membranes in solid state physics. The Polyakov partition function \( Z \) for a string embedded in euclidean space with a fixed intrinsic area worldsheet discretizes to

\[
Z = \sum_T \int \prod_{i=1}^{N-1} dX_i^\mu \exp(-S_g),
\]

where \( \sum_T \) is a sum over different triangulations of the worldsheet with the same number of nodes \( N \) which is the discrete analog of the path integration over the intrinsic metric in the continuum. The discretized action, \( S_g \), is just a simple gaussian

\[
S_g = \frac{1}{2} \sum_{<ij>} (X_i^\mu - X_j^\mu)^2,
\]

where the variables \( X_i \) live on the nodes of the triangulation and the sum \( <ij> \) runs over its edges. To a solid state physicist this would be the action for a fluid (because of the sum over triangulations) membrane with gaussian interactions and no self-avoidance. The difficulties encountered in analytical calculations for string theories in physical dimensions are mirrored in simulations of equ.(1) which generate very crumpled surfaces from which it is impossible to take a continuum limit. Variations on the gaussian action, such as using the sum of the area of the triangles making up the surface or the sum of their edge lengths, also fail to generate a smooth continuum limit.

The addition of an extrinsic curvature term, or “stiffness”, to the gaussian action which originally arose in the context of biological membranes and QCD has been mooted as a possible resolution of these problems. One discretization of this has shown particular promise,

\[
S_e = \sum_{\Delta_i,\Delta_j} (1 - n_i \cdot n_j),
\]

where the normals \( n_i, n_j \) are on adjacent triangles \( \Delta_i, \Delta_j \). The strategy is to examine the phase structure of the gaussian plus extrinsic curvature action \( S_g + \lambda S_e \) (henceforward GPEC) as the coupling \( \lambda \) is varied to see if any continuous transitions are present at which one might define a non-trivial continuum theory. The case of fixed, or “tethered”, surfaces where there is no sum over triangulations offers some hope because there does appear to be a second order transition between a low \( \lambda \) crumpled phase and a large \( \lambda \) smooth phase for these when there is no self-avoidance. Initial work on dynamical surfaces with modestly sized meshes tended to support a similar picture, with a peak in the specific heat that grew on this because the increase in the peak height tails off with increasing mesh size. More recent simulations with higher statistics and larger meshes have cast doubt on this. However, it was pointed out in that the data might also be construed as indicating a rapid crossover rather than a true transition or even the presence of a low mass bound state like that in the 2d \( O(3) \) model.

This rather confused picture for GPEC actions naturally prompts the question of whether discrete models with more clearly cut phase transitions exist. It was suggested in that an action based on the modified Steiner functional might be a suitable alternative candidate as it possessed the nice geometrical property of being subdivision invariant as well as retaining a desirable stiffening effect on the surfaces. The action was

\[
S_{steiner} = \frac{1}{2} \sum_{<ij>} |X_i^\mu - X_j^\mu| \theta(\alpha_{ij}),
\]

where \( \theta(\alpha_{ij}) = |\pi - \alpha_{ij}| \) and \( \alpha_{ij} \) is the angle between the embedded neighbouring triangles with common link \( <ij> \). This is essentially a coarse discretization of the absolute value of the trace of the second fundamental form of the surface. However it was observed in that an action of this form ran into problems with the entropy of vertices in smooth configurations and failed to give a well-defined grand canonical partition function. Our previous microcanonical simulation of such an action had shown that it produced smooth surfaces but in this case of course it is not clear how to obtain the continuum limit, where the problems of would presumably resurface.

\[1\] These have been proposed as models of polymerized membranes in solid state physics.
We had observed in [13] that varying the coupling \( \lambda \) in an action of the form

\[
S_1 = \frac{1}{2} \sum_{<ij>} |X_i^\mu - X_j^\mu| + \lambda \sum_{<ij>} |X_i^\mu - X_j^\mu| \theta(\alpha_{ij})
\]  

(5)

would allow one to employ the approach used in our earlier simulations of GPEC actions, namely hunting for continuous transitions at which to define the continuum theory. We chose the first, edge-length, term in \( S_1 \) because of its simplicity and similarity to the second Steiner term. The authors of [14] pointed out it disappears in the sum because it is perpendicular to \( n \). Thus, we considered \( n \) pick, where the subscript 2 indicated in Fig.1. One could take

\[
S_2 = \sum_\Delta |\Delta| + \lambda \sum_{<ij>} |X_i^\mu - X_j^\mu| \theta(\alpha_{ij}),
\]  

(6)

where \( |\Delta| \) is the area of a triangle, might cure the entropy problems of \( S_{\text{steiner}} \) by virtue of the first term being coercive enough to compete with the entropy of smooth surfaces. \( S_1 \) might also be expected to effect a cure for the same reason, but it is no longer subdivision invariant. It is not clear that this is a prerequisite for a well-behaved action in any case as the GPEC action does not share this property either. If we do not insist on subdivision invariance we could also envisage combining a gaussian term with \( S_{\text{steiner}} \)

\[
S_3 = \frac{1}{2} \sum_{<ij>} (X_i - X_j)^2 + \lambda \sum_{<ij>} |X_i^\mu - X_j^\mu| \theta(\alpha_{ij}).
\]  

(7)

The arguments presented in [12] suggested that \( S_{\text{steiner}} \) was essentially a discretization of the absolute value of the trace of the second fundamental form, at least for surfaces embedded in three dimensions. Making use of the original definition of the trace of the second fundamental form

\[
K = K_\alpha \approx \partial^\mu X^\mu \cdot \partial_\alpha n_\mu
\]  

(8)

we could also attempt a direct discretization in the three dimensional case. Various possibilities are indicated in Fig.1. One could take

\[
|K| = \sum_{<ij>} |(X_i - X_j) \cdot (n_1 - n_{2,3})|.
\]  

(9)

where the subscript 2, 3 denotes a choice of either normal \( n_2 \) or \( n_3 \) in the discretization of \( \partial n \). If we pick \( n_2 \) it disappears in the sum because it is perpendicular to \( X_i - X_j \), whereas a coarser discretization of the derivative given by choosing \( n_3 \) retains both of the normals \( n_1 \) and \( n_3 \). We can then pair these alternative discretizations with either edge-length, area or gaussian terms to give the further possibilities

\[
S_4 = \frac{1}{2} \sum_{<ij>} |X_i^\mu - X_j^\mu| + \lambda \sum_{<ij>} |K|
\]

\[
S_5 = \sum_\Delta |\Delta| + \lambda \sum_{<ij>} |K|
\]

\[
S_6 = \frac{1}{2} \sum_{<ij>} (X_i - X_j)^2 + \lambda \sum_{<ij>} |K|.
\]  

(10)

In this paper we carry out a qualitative exploration of the phase structure of \( S_1, S_2, S_3, S_4, S_5 \) and \( S_6 \) to see if there is any justification for expecting a well-behaved theory. In order to do this we measured the standard repertoire of intrinsic and extrinsic variables. We included a local factor in the measure for compatibility with our earlier simulations which can be exponentiated to give

\[
S_m = \frac{d}{2} \sum_i \log(q_i),
\]  

(11)

where \( q_i \) is the number of neighbours of point \( i \), and \( d = 3 \) dimensions. If we denote the general form of actions \( S_1, \ldots, S_6 \) by \( S_\alpha + \lambda S_\beta \) we thus simulated \( S_\alpha + \lambda S_\beta + S_m \). We measured \( < S_m > \) and the

\footnote{Choosing \( X_j - X_k \) instead of \( X_i - X_j \) gives identical results}
mean maximum number of neighbours \(<\max(q_i)\)> to get some idea of the behaviour of the intrinsic geometry. The extrinsic geometry was observed by measuring \(<S_\lambda>\) and its associated specific heat

\[ C_\beta = \frac{\lambda^2}{N} \left( <S^2_\beta> - <S_\beta>^2 \right) \] (12)

as well as the gyration radius \(X^2\), a measure of the mean size of the surface as seen in the embedding space,

\[ X^2 = \frac{1}{9N(N-1)} \sum_{ij} (X^i_j - X^j_i)^2 q_i q_j. \] (13)

Note that the sum for \(X^2\) is over all pairs of \(X\)’s on the mesh. The expectation value of the first term in the action \(<S_\alpha>\) was also measured for completeness, although unlike the GPEC actions we can no longer use its value to confirm that equilibration has occurred.

The simulation used a Monte Carlo procedure which we have described in some detail elsewhere [16]. It first goes through the mesh moving the \(X\)’s, carrying out a Metropolis accept/reject at each step, and then goes through the mesh again carrying out the “flip” moves on the links, again applying a Metropolis accept/reject at each stage. The entire procedure constitutes a sweep. Due to the correlated nature of the data, a measurement was taken every tenth sweep and binning techniques were used to analyse the errors. We carried out 10K thermalization sweeps followed by 30K or 50K measurement sweeps for each data point. The acceptance for the \(X\) move was monitored and the size of the shift was adjusted to maintain an acceptance of around 50 percent. The acceptance for the flip move was also measured, but in this case there is nothing to adjust, so as for GPEC actions this dropped with increasing \(\lambda\) (but was still appreciable even for quite large \(\lambda\)).

We now move on to consider the numerical results in more detail for action \(S_1\) (edge-length plus Steiner term) on a 72-node surface. We can see from Table 1

| \(\lambda\) | sweeps | \(S_\alpha\) | \(S_m\) | \(S_\beta\) | \(C_\beta\) | \(X^2\) | \(\max(q_i)\) |
|---|---|---|---|---|---|---|---|
| 0.000 | 30K | 213.08(0.20) | 119.67(0.01) | 382.66(0.35) | 0.00(0.00) | 4.21(0.04) | 16.31(0.02) |
| 0.050 | 30K | 120.48(0.16) | 120.37(0.01) | 35.11(0.15) | 0.64(0.01) | 1.64(0.02) | 15.57(0.02) |
| 0.100 | 30K | 120.77(0.01) | 124.43(0.08) | 1.13(0.01) | 0.98(0.01) | 15.21(0.02) |
| 0.150 | 30K | 71.80(0.12) | 94.16(0.06) | 1.40(0.01) | 0.71(0.01) | 14.89(0.02) |
| 0.200 | 30K | 61.43(0.16) | 75.84(0.06) | 1.64(0.02) | 0.58(0.01) | 14.60(0.03) |
| 0.250 | 30K | 54.40(0.19) | 63.45(0.05) | 1.81(0.02) | 0.50(0.01) | 14.35(0.03) |
| 0.300 | 30K | 49.91(0.50) | 54.34(0.12) | 2.03(0.10) | 0.47(0.03) | 14.09(0.06) |
| 0.350 | 30K | 50.68(1.66) | 49.85(0.44) | 2.31(0.14) | 0.60(0.10) | 13.69(0.16) |
| 0.400 | 30K | 83.55(1.04) | 37.23(0.13) | 1.90(0.11) | 2.43(0.06) | 10.45(0.04) |
| 0.450 | 30K | 81.87(0.56) | 32.84(0.08) | 1.58(0.05) | 2.34(0.03) | 10.27(0.02) |
| 0.500 | 30K | 79.34(0.61) | 29.72(0.08) | 1.39(0.03) | 2.19(0.04) | 10.20(0.01) |
| 0.550 | 30K | 76.99(1.09) | 27.50(0.15) | 1.35(0.03) | 2.08(0.07) | 10.20(0.01) |

Table 1
Results for \(S_1\), \(N=72\)

that there does, indeed, appear to be a transition to a smooth phase at large \(\lambda\), as inspection of Figs.2,3 confirms. We can see in Fig.2 that there is a peak in the associated specific heat \(C_\beta\) at around \(\lambda = 3.25\) where we also observe a sharp jump in the measured size of the surface as given by \(X^2\) which is plotted in Fig.3. The smooth nature of the large \(\lambda\) phase is corroborated by the snapshot of a surface at \(\lambda = 5\) in Fig.4. For small \(\lambda\) the surfaces are crumpled as can be seen in Fig.5. The behaviour in the crumpled phase is, however, different from that seen with GPEC actions. Firstly, at \(\lambda = 0\) the surfaces are larger than those seen with a purely gaussian action, presumably because the edge length action is less confining than the gaussian action. Secondly, and rather strikingly, \(X^2\) decreases with increasing \(\lambda\) up to the putative phase transition point, where it increases again. This can be understood by noting that the Steiner term, \(S_\beta\), is not scale invariant like the extrinsic curvature. There are thus two ways of decreasing the value of \(S_\beta\), one can either make the surface smaller to decrease the \(|X_i - X_j|\) terms, or make the surface smoother to decrease the \(\theta\) terms. It is evident that the first occurs for \(\lambda\) up to the phase transition, as can be confirmed by looking at \(S_\alpha\), which is a measure of the sum of \(|X_i - X_j|\) terms - this decreases. The
The behaviour of $X^2$ in the crumpled phase appears to be a generic feature of all of the actions containing Steiner terms as it is also observed for $S_2$ (area + Steiner) and $S_3$ (gaussian + Steiner) actions. The Steiner term $S_\beta$ appears to correlate well with the extrinsic curvature, as can be seen in Table 1, where its value is high in the crumpled low $\lambda$ phase (where the extrinsic curvature, which we have not tabulated, is also high) and drops off as $\lambda$ increases and the surfaces become smoother. The behaviour of the intrinsic observables $S_m$ and $\text{max}(q_i)$ is very similar to that of the GPEC actions: $S_m$ increases smoothly through the phase transition and $\text{max}(q_i)$ decreases, indicating that the intrinsic geometry becomes slightly more regular at larger $\lambda$.

$S_2$ (area plus Steiner term) gives the results in Table 2. Again we have a smooth phase at large $\lambda$ with a transition to a crumpled phase at lower $\lambda$. Snapshots of surfaces in the smooth phase look similar to that produced by $S_1$ in Fig. 4 and the low $\lambda$ phase is, if anything, even spikier than that of $S_1$. In fact, at $\lambda = 0$ the surfaces appear to be unstable to the formation of very large spikes, appearing as a collection of thin whiskers emanating from a central point. We have not included the values for $\lambda = 0$ in Table 2 and Figs. 2, 3 as $X^2$ and $S_{\beta}$ are very large. As we can see from Table 2:

| $\lambda$ | sweeps | $S_\alpha$ | $S_m$ | $S_\beta$ | $S_{\beta}$ | $X^2$ | $\text{max}(q_i)$ |
|-----------|--------|-----------|-------|-----------|-----------|------|----------------|
| 1.000     | 30K    | 35.23(0.09)| 121.08(0.01)| 142.72(0.02)| 1.28(0.00)| 1.64(0.02)| 14.90(0.02) |
| 1.250     | 30K    | 29.76(0.01)| 121.34(0.00)| 122.66(0.02)| 1.50(0.00)| 1.47(0.00)| 14.60(0.00) |
| 1.500     | 30K    | 25.79(0.04)| 121.58(0.00)| 107.23(0.01)| 1.76(0.01)| 1.34(0.01)| 14.40(0.01) |
| 1.750     | 30K    | 23.55(0.05)| 121.88(0.01)| 94.74(0.05)| 1.95(0.00)| 1.43(0.02)| 14.02(0.02) |
| 2.000     | 30K    | 24.58(0.71)| 122.33(0.05)| 82.04(0.64)| 2.01(0.43)| 1.87(0.13)| 13.39(0.10) |
| 2.250     | 30K    | 36.87(0.82)| 123.45(0.04)| 62.48(0.50)| 2.06(0.24)| 1.80(0.12)| 11.51(0.07) |
| 2.500     | 30K    | 41.70(0.23)| 123.87(0.01)| 51.70(0.12)| 2.49(0.11)| 1.50(0.02)| 10.73(0.02) |
| 3.000     | 30K    | 10.95(0.17)| 123.99(0.00)| 43.95(0.02)| 1.37(0.01)| 1.50(0.02)| 10.56(0.00) |
| 4.000     | 30K    | 34.63(0.11)| 124.02(0.00)| 35.89(0.03)| 1.40(0.02)| 1.80(0.01)| 10.52(0.00) |
| 5.000     | 30K    | 29.09(0.09)| 124.00(0.00)| 30.98(0.03)| 1.60(0.01)| 2.22(0.01)| 10.54(0.00) |

Table 2
Results for $S_2$, $N = 72$

The $X^2$ values for $S_2$ even with non-zero $\lambda$ are larger for a given coupling than for $S_1$, suggesting the spikiness persists away from $\lambda = 0$. We can also see from the table that $X^2$ decreases at small $\lambda$ as does $S_\alpha$ for precisely the reasons indicated above for $S_1$. It is clear that the transition is taking place at a lower $\lambda$ value than for $S_1$, as both the peak in the specific heat and the jump in $X^2$ are at $\lambda \simeq 2.25$, rather than $\lambda \simeq 3.25$. At first sight this might appear rather surprising as we have just seen that the crumpled phase is spikier for $S_2$ rather than $S_1$, so we might expect to have to work harder to escape from it and thus have a phase transition at larger $\lambda$. However, the dimensions of the area and edge length terms are different so we cannot directly attach significance to the numerical value of the Steiner coupling. We can see in Fig. 1 that the peak in the specific heat for $S_2$ appears to be considerably stronger than for $S_1$.

$S_3$ (gaussian plus Steiner term) gives the results shown in Table 3, where we can see that the behaviour is similar to $S_2$. Although $S_3$ is not subdivision invariant there is still a larger peak in the specific heat than for $S_1$ as can be seen clearly in Fig. 2. The peak is at a larger $\lambda$ ($\simeq 3$) value than for $S_2$, which is perhaps surprising as one might have expected that it would be easier to escape from the crumpled phase of $S_3$ because it is less “spiky” than that of $S_2$. On the other hand, as can be seen in Fig. 3, the $X^2$ values of $S_3$ are more similar to $S_1$ than $S_2$. The characteristic decrease in $X^2$ for small $\lambda$ is also present for $S_3$. Again the values for $\lambda = 0$ are omitted, this time because they are identical to the GPEC action at $\lambda = 0$.

\[3\] It is worth remarking that the sharpness of the peak in the specific heat should not be regarded as the criterion for a good candidate continuum theory as models which do have second order transitions on fixed lattices, such as Ising or Potts models, have higher order transitions on dynamical lattices with well behaved continuum limits. The GPEC action \[\square\] and $S_1$ may be similar.
We might naively expect $S_4$, $S_5$, $S_6$ to behave in a similar fashion to $S_1$, $S_2$, $S_3$ respectively, but a glance at Fig.6 and Fig.7 where we show snapshots of 72-node surfaces generated by $S_1$ and $S_5$ respectively at $\lambda = 5$ reveal that this is not the case. It is clear for both of these that $S_\beta$ is failing to smooth out the surfaces at large $\lambda$, for $S_4$ (and $S_5$ too, which we have not shown) because of large numbers of “back to back” triangles and for $S_5$ because it fails to prevent the surface degenerating into thin whiskers, similar to those observed for $S_2$ at $\lambda = 0$. In Table 4.

| $\lambda$ | sweeps | $S_\alpha$ | $S_m$ | $S_\beta$ | $C_\beta$ | $X2$ | $\max(q_i)$ |
|----------|--------|-----------|-------|-----------|---------|------|-------------|
| 0.500    | 30K    | 65.08(0.02) | 120.33(0.00) | 106.39(0.02) | 0.38(0.00) | 1.24(0.00) | 15.61(0.00) |
| 1.000    | 30K    | 44.86(0.03) | 120.81(0.00) | 123.70(0.01) | 0.91(0.00) | 0.98(0.00) | 15.15(0.00) |
| 1.500    | 30K    | 33.52(0.04) | 121.23(0.00) | 97.39(0.01) | 1.31(0.01) | 0.82(0.00) | 14.72(0.00) |
| 2.000    | 30K    | 26.84(0.06) | 121.59(0.01) | 79.02(0.01) | 1.63(0.00) | 0.74(0.01) | 14.36(0.01) |
| 2.500    | 30K    | 22.91(0.05) | 121.97(0.01) | 66.65(0.03) | 2.04(0.01) | 0.71(0.01) | 13.96(0.01) |
| 2.950    | 30K    | 30.14(0.45) | 123.11(0.03) | 51.72(0.28) | 6.45(0.20) | 1.49(0.03) | 12.22(0.06) |
| 3.000    | 30K    | 32.54(0.24) | 123.32(0.02) | 49.42(0.17) | 6.77(0.19) | 1.69(0.02) | 11.87(0.05) |
| 3.050    | 30K    | 36.73(1.16) | 123.68(0.07) | 45.95(0.65) | 5.39(0.43) | 2.04(0.09) | 11.24(0.14) |
| 4.000    | 30K    | 41.32(0.14) | 124.31(0.00) | 32.61(0.01) | 1.41(0.01) | 2.48(0.01) | 10.11(0.01) |
| 4.500    | 30K    | 38.61(0.09) | 124.32(0.00) | 29.81(0.01) | 1.24(0.01) | 2.21(0.01) | 10.09(0.00) |
| 5.000    | 30K    | 36.94(0.17) | 124.31(0.00) | 27.88(0.02) | 1.30(0.01) | 2.21(0.01) | 10.11(0.00) |

Table 3
Results $S_3$, $N = 72$

the numerical results for $S_4$ show that despite the obvious roughness of the surfaces at large $\lambda$, $S_\beta$ which is supposed to discretize the absolute value of the trace of the second fundamental form, is decreasing with increasing $\lambda$. The numerical results for $S_5$, $S_6$ are similar to this - although $S_3$ decreases with increasing $\lambda$ the surfaces obviously remain rough. We have not pursued simulations of $S_4$, $S_5$, $S_6$ on larger surfaces because of the obvious pathologies seen in these results.

For the choice of $n_2$ in the discretization of $|K|$ in equ.(9) the nature of the pathology can be pinned down by looking at $S_\beta$ without the modulus sign for the various surfaces. Summing up the various contributions on the smooth surfaces generated by $S_1$, $S_2$, $S_3$ gives a much larger total than summing up the contributions on surfaces generated by $S_4$, $S_5$, $S_6$ where we see a large degree of cancellation between terms with alternating signs. The reason for this is clear if we examine equ.(9): there is an ambiguity in the direction of $n_1$ if it is forced to be perpendicular to $X_i - X_j$. It can be at either $+90^\circ$ or $-90^\circ$ and the Monte Carlo will pick both with equal probability, leading to normals reversing direction from triangle to triangle and the non-smooth surfaces that are observed with $S_4$, $S_5$, $S_6$. This problem does not arise for $S_1$, $S_2$, $S_3$ where the Steiner term is minimized when the angle between adjacent triangles is $180^\circ$ and there is no possibility of confusion in the orientation of the normals. Choosing $n_3$ in the discretization of $|K|$ in equ.(9), which retains both normals, might be expected to give better results as it is more difficult to generate crumpled configurations with both $n_1$ and $n_3$ perpendicular to $X_i - X_j$. Nonetheless, this alternate discretization also fails to produce a smooth phase. Examining $S_3$ without the modulus sign again reveals a large degree of cancellation due to alternating signs in the surfaces generated by $S_4$, $S_5$, $S_6$ (which is not seen for the smooth surfaces produced by $S_1$, $S_2$, $S_3$).

The simulations described in this paper can be regarded as a qualitative study of the feasibility of using alternatives to the GPEC action for random surfaces. We have found that $S_1$ and $S_3$ are
not subdivision invariant and $S_2$, which is, all seem to be good candidates for further exploration. $S_4$, $S_5$ and $S_6$ on the other hand, fail to discretize the trace of the second fundamental form properly. It thus appears that the choice of “stiffness” term essentially determines at least the qualitative behaviour of the surface at the crumpling transition. Similar conclusions have been drawn by one of the authors for various versions of the extrinsic curvature combined with area, gaussian or tethering terms on fixed random surfaces [17]. The crumpled phase for the various actions, particularly for $S_2$ at very small $\lambda$, does display variations. The interesting question of whether the subdivision invariance of $S_2$ has an effect on the random surface model as it approaches the continuum limit has not been investigated in this paper because we have made no attempt to carry out finite size scaling for $S_1, S_2, S_3$ at the transition. This would also allow a more quantitative comparison with the GPEC actions and establish whether the properties of the transition for $S_1, S_2, S_3$ are more clear cut than those seen in [10], [11]. These issues, and the equally important question of whether the string tension and mass gap scale, are currently being addressed in a much larger simulation.

The comparison with the results of the GPEC action will also shed some light on the still murky problem of universality for random surfaces [18]. As we have seen, there is a wide choice of possible terms and discretizations for a random surface action and the question of whether they lead to the same continuum theory has not been clearly answered. In the longer run the best approach may be to use the Monte Carlo renormalization group, perhaps starting on rigid meshes because of the computational complexity of dynamical meshes. This would then allow consideration of all possible terms up to a given weight. We are currently exploring the feasibility of this.

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Figure Captions

Fig. 1. The edges and normals that are involved in the discretization of the the $S_\beta$ term in $S_4$, $S_5$ and $S_6$.

Fig. 2. The specific heat $C_\beta$ for actions $S_1, S_2, S_3$.

Fig. 3. The gyration radii $X_2$ for actions $S_1, S_2, S_3$.

Fig. 4. A snapshot of a mesh generated by $S_1$ with $\lambda = 5$.

Fig. 5. A snapshot of a mesh generated by $S_1$ with $\lambda = 0$.

Fig. 6. A snapshot of a mesh generated by $S_4$ with $\lambda = 5$.

Fig. 7. A snapshot of a mesh generated by $S_5$ with $\lambda = 5$. 