The polytopes in a Poisson hyperplane tessellation

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Abstract

For a stationary Poisson hyperplane tessellation $X$ in $\mathbb{R}^d$, whose directional distribution satisfies some mild conditions (which hold in the isotropic case, for example), it was recently shown that with probability one every combinatorial type of a simple $d$-polytope is realized infinitely often by the polytopes of $X$. This result is strengthened here: with probability one, every such combinatorial type appears among the polytopes of $X$ not only infinitely often, but with positive density.

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1 Introduction

Imagine a system $H$ of hyperplanes in Euclidean space $\mathbb{R}^d$ ($d \geq 2$) that induces a tessellation $T_H$ of $\mathbb{R}^d$. This means that any bounded subset of $\mathbb{R}^d$ meets only finitely many hyperplanes of $H$ and that the components of $\mathbb{R}^d \setminus \bigcup_{H \in H} H$ are bounded. The closures of these components are then convex polytopes which cover $\mathbb{R}^d$ and have pairwise no common interior points. The set of these polytopes is denoted by $T_H$. We impose the additional assumption that the hyperplanes of $H$ are in general position; then each polytope of $T_H$ is simple, that is, each of its vertices is contained in precisely $d$ facets. The polytopes appearing in $T_H$ may be rather boring; they could, for example, all be parallelepipeds. However, if the hyperplanes of $H$ have sufficiently many different directions, one can imagine that quite different shapes of polytopes appear in $T_H$. Is it possible that every combinatorial type of a simple $d$-polytope is realized in $T_H$? This can be achieved in a much stronger sense.

In fact, suppose that $\hat{X}$ is a stationary and isotropic Poisson hyperplane process in $\mathbb{R}^d$ (explanations are found in [7], for example). Its hyperplanes are almost surely in general position and induce a random tessellation of $\mathbb{R}^d$, denoted by $X$. The general character of the polytopes in $X$ was recently investigated in [4]. For example, it was shown there that almost surely (a.s.) the translates of the polytopes in $X$ are dense in the space of convex bodies in $\mathbb{R}^d$ (with the Hausdorff metric). Another result was that a.s. the polytopes of $X$ realize every combinatorial type of a simple $d$-polytope infinitely often. In the following, we improve the latter result considerably, replacing ‘infinitely often’ by ‘with positive density’.

In the subsequent definition, $B_n$ is the ball in $\mathbb{R}^d$ with center at the origin and radius $n \in \mathbb{N}$, and $\lambda_d$ denotes Lebesgue measure in $\mathbb{R}^d$. Further, $1_A$ is the indicator function of $A$.

Definition 1. Let $T$ be a tessellation of $\mathbb{R}^d$, and let $A$ be a translation invariant set of polytopes in $\mathbb{R}^d$. We say that $A$ appears in $T$ with density $\delta$ if

$$\liminf_{n \to \infty} \frac{1}{\lambda_d(B_n)} \sum_{P \in T, P \subseteq B_n} 1_A(P) = \delta.$$
With this definition, we prove below that in a Poisson hyperplane tessellation in $\mathbb{R}^d$ which is stationary and isotropic (that is, has a motion invariant distribution), almost surely every combinatorial type of a simple $d$-polytope appears with positive density. The actual result will, in fact, be more general: it is sufficient that the Poisson hyperplane tessellation is stationary and that its directional distribution, a measure on the unit sphere, is not zero on any nonempty open set and is zero on any great subsphere. The precise theorem is formulated in the next section.

2 Explanations

We work in the $d$-dimensional Euclidean space $\mathbb{R}^d$ ($d \geq 2$) with its usual scalar product $\langle \cdot, \cdot \rangle$. By $\lambda_d$ we denote its Lebesgue measure, by $o$ its origin, by $B^d$ its unit ball (with $nB^d =: B_n$), and by $\mathbb{S}^{d-1}$ its unit sphere. The space of hyperplanes in $\mathbb{R}^d$, with its usual topology, is denoted by $H$, and $\mathcal{B}(H)$ is the $\sigma$-algebra of Borel sets in $H$. Hyperplanes in $\mathbb{R}^d$ are often written in the form $H(u, \tau) = \{x \in \mathbb{R}^d : \langle x, u \rangle \leq \tau\}$ with $u \in \mathbb{S}^{d-1}$ and $\tau \in \mathbb{R}$.

We assume that $\hat{X}$ is a stationary Poisson hyperplane process in $\mathbb{R}^d$, thus, a Poisson point process in the space $H$ of hyperplanes, with the property that its distribution is invariant under translations (we refer, e.g., to [7] for more details). The intensity measure $\hat{\Theta}$ of $\hat{X}$ is defined by

$$\hat{\Theta}(A) = \mathbb{E} \hat{X}(A) \quad \text{for } A \in \mathcal{B}(H).$$

Here $\mathbb{E}$ denotes expectation, and we write $(\Omega, \mathcal{A}, \mathbb{P})$ for the underlying probability space. It is assumed that $\hat{\Theta}$ is locally finite and not identically zero. That $\hat{X}$ is a Poisson process includes that

$$\mathbb{P}(\hat{X}(A) = k) = e^{-\hat{\Theta}(A)} \frac{\hat{\Theta}(A)^k}{k!} \quad \text{for } k \in \mathbb{N}_0,$$

for any $A \in \mathcal{B}(H)$ with $\hat{\Theta}(A) < \infty$.

Since $\hat{X}$ is stationary, the measure $\hat{\Theta}$ has a decomposition

$$\hat{\Theta}(A) = \hat{\gamma} \int_{\mathbb{S}^{d-1}} \int_{-\infty}^{\infty} 1_A(H(u, \tau)) d\tau \varphi(du)$$

for $A \in \mathcal{B}(H)$ (see [7], Theorem 4.4.2 and (4.33)). The number $\hat{\gamma} > 0$ is the intensity of $\hat{X}$, and $\varphi$ is a finite, even Borel measure on the unit sphere. It is called the spherical directional distribution of $\hat{X}$. For any such measure $\varphi$ and any number $\hat{\gamma} > 0$, there exists a stationary Poisson hyperplane process in $\mathbb{R}^d$ with these data, and it is unique up to stochastic equivalence.

The hyperplane process $\hat{X}$ induces a random tessellation of $\mathbb{R}^d$, which we denote by $X$. As usual, a random tessellation is formalized as a particle process; we refer again to [7].

Since we are considering only simple processes, it is convenient to identify such a process, which by definition is a counting measure, with its support, which is a locally finite set. In particular, a realization of $\hat{X}$ is also considered as a set of hyperplanes, and a realization of $X$ is considered as a set of polytopes. The notations $\hat{X}(\{H\}) = 1$ and $H \in \hat{X}$ for a hyperplane $H$, for example, are therefore used synonymously.

The combinatorial type of a polytope $P$ in $\mathbb{R}^d$ is the set of all polytopes in $\mathbb{R}^d$ that are combinatorially isomorphic to $P$. Now we can formulate our result.
Theorem 1. Let \( X \) be a tessellation of \( \mathbb{R}^d \) that is induced by a stationary Poisson hyperplane process \( \tilde{X} \) with spherical directional distribution \( \varphi \). Suppose that the support of \( \varphi \) is the whole unit sphere \( S^{d-1} \) and that \( \varphi \) assigns measure zero to each great subsphere of \( S^{d-1} \). Then, with probability one, each combinatorial type of a simple \( d \)-polytope appears with positive density in \( X \).

Theorem 1 implies, trivially, that under its assumptions almost surely each combinatorial type of a simple \( d \)-polytope appears infinitely often in \( X \). When the latter fact was proved, among other results, in [4], a tool was a strengthened version of the Borel–Cantelli lemma, due to Erdős and Rényi [3] (see also [5, p. 327]). When the note [4] was submitted, an anonymous referee wrote “that the use of ergodicity of the mosaic could lead to a possibly shorter alternative proof”, and he/she briefly indicated a possible approach. After thorough consideration, we preferred the more elementary Borel–Cantelli lemma. However, reconsideration revealed that ergodicity, applied in a different way, might lead to a stronger result, as far as the occurrence of combinatorial types is concerned. This is carried out in the following.

3 Proof

Let \( X \) satisfy the assumptions of Theorem 1. Under the only assumption that the spherical directional distribution of the stationary Poisson hyperplane tessellation \( X \) is zero on every great subsphere, it was shown in [7, Thm. 10.5.3] that \( X \) is mixing and hence ergodic. This requires a few explanations. To model \( X \) as a point process, we consider the space \( \mathcal{K} \) of convex bodies (nonempty, compact, convex subsets) in \( \mathbb{R}^d \) with the Hausdorff metric. By \( \mathcal{B}(\mathcal{K}) \) we denote the \( \sigma \)-algebra of Borel sets in \( \mathcal{K} \). Let \( N_s(\mathcal{K}) \) be the set of simple, locally finite counting measures on \( \mathcal{B}(\mathcal{K}) \) and \( \mathcal{N}_s(\mathcal{K}) \) its usual \( \sigma \)-algebra (for details see, e.g., [7, Sect. 3.1]). As underlying probability space \( (\Omega, \mathcal{A}, \mathbb{P}) \), on which \( X \) is defined, we can use \( (\mathcal{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X) \), where \( \mathbb{P}_X \) is the distribution of \( X \). For \( t \in \mathbb{R}^d \), a bijective map \( T_t : \eta \mapsto T_t \eta \) of \( \mathcal{N}_s(\mathcal{K}) \) onto itself is defined by

\[
(T_t \eta)(B) := \eta(B - t), \quad B \in \mathcal{B}(\mathcal{K}), \ \eta \in \mathcal{N}_s(\mathcal{K}).
\]

Since \( X \) is stationary, we have

\[
\mathbb{P}_X(T_t A) = \mathbb{P}_X(A) \quad \text{for} \quad A \in \mathcal{N}_s(\mathcal{K}),
\]

thus \( T_t \) induces a measure preserving map of \( \mathcal{N}_s(\mathcal{K}) \) into itself. Let \( \mathcal{T} := \{T_t : t \in \mathbb{R}^d \} \). As shown in [7, Thm. 10.5.3], the dynamical system \( (\mathcal{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X, \mathcal{T}) \) is mixing, that is,

\[
\lim_{||t|| \to \infty} \mathbb{P}_X(A \cap T_t B) = \mathbb{P}_X(A) \mathbb{P}_X(B)
\]

holds for all \( A, B \in \mathcal{N}_s(\mathcal{K}) \). It follows that the system is ergodic, which means that \( \mathbb{P}_X(A) \in \{0, 1\} \) for all \( A \in \mathcal{T} := \{A \in \mathcal{N}_s(\mathcal{K}) : T_t A = A \text{ for all } t \in \mathbb{R}^d \} \). Therefore, the ‘Individual Ergodic Theorem for \( d \)-dimensional Shifts’ yields the following.

Proposition 1. Let \( f \) be an integrable random variable on \( (\mathcal{N}_s(\mathcal{K}), \mathcal{N}_s(\mathcal{K}), \mathbb{P}_X) \). Then

\[
\lim_{n \to \infty} \frac{1}{\lambda_d(B_n)} \int_{B_n} f(T_t \omega) \lambda(dt) = \mathbb{E} f
\]

holds for \( \mathbb{P}_X \)-almost all \( \omega \in \mathcal{N}_s(\mathcal{K}) \).
We refer to Daley and Vere–Jones \cite{DaleyVereJones} Proposition 12.2.II for a more general formulation (with hints to proofs of more general results in Tempel’man \cite{Tempelman}). However, we have already incorporated into our Proposition \[1\] the information that in our case \((N_s(K), N_s(K), P_X, T)\) is ergodic, which yields that the limit is equal to the expectation of \(f\).

We apply this Proposition in the following way. First we choose a center function \(c\) on \(\mathcal{K}\); for example, let \(c(K)\) denote the circumcenter of \(K \in \mathcal{K}\), which is the center of the smallest ball containing \(K\). Let \(A \in \mathcal{B}(\mathcal{K})\) be a translation invariant Borel set of convex bodies. Given any bounded Borel set \(B \in \mathcal{B}(\mathbb{R}^d)\), we define

\[
f(B, \omega) := \sum_{K \in X(\omega), c(K) \in B} 1_A(K)
\]

for \(\omega \in \Omega\), where we use \((\Omega, \mathcal{A}, P) = (N_s(\mathcal{K}), N_s(\mathcal{K}), P_X)\) as the underlying probability space. Then \(f(B, \cdot)\) is measurable, and \(f(B + t, \omega) = f(B, T^{-t} \omega)\) for \(t \in \mathbb{R}^d\). The following generalizes an approach of Cowan \cite{Cowan} in the plane (“Tricks with small disks”). Assuming that \(n > 1\), we have

\[
\int_{B_{n-1}} f(B_1 + t, \omega) \lambda_d(dt)
\]

\[
= \sum_{K \in X(\omega)} \int_{\mathbb{R}^d} 1\{t \in B_{n-1}\} 1\{K \in A\} 1\{c(K) \in B_1 + t\} \lambda_d(dt).
\]

Since

\[
1\{t \in B_{n-1}\} 1\{c(K) \in B_1 + t\} \leq 1\{t \in -B_1 + c(K)\} 1\{c(K) \in B_n\},
\]

we get

\[
\int_{B_{n-1}} f(B_1 + t, \omega) \lambda_d(dt)
\]

\[
\leq \sum_{K \in X(\omega)} \int_{\mathbb{R}^d} 1\{t \in -B_1 + c(K)\} 1\{K \in A\} 1\{c(K) \in B_n\} \lambda_d(dt)
\]

\[
= \lambda_d(B_1) f(B_n, \omega).
\]

Similarly,

\[
\int_{B_{n+1}} f(B_1 + t, \omega) \lambda_d(dt)
\]

\[
\geq \sum_{K \in X(\omega)} \int_{\mathbb{R}^d} 1\{t \in -B_1 + c(K)\} 1\{K \in A\} 1\{c(K) \in B_n\} \lambda_d(dt)
\]

\[
= \lambda_d(B_1) f(B_n, \omega).
\]

We conclude that

\[
\frac{\lambda_d(B_{n-1})}{\lambda_d(B_n)} \frac{1}{\lambda_d(B_{n-1})} \int_{B_{n-1}} f(B_1, T^{-t} \omega) \lambda_d(dt)
\]

\[
\leq \frac{\lambda_d(B_1)}{\lambda_d(B_n)} f(B_n, \omega)
\]

\[
\leq \frac{\lambda_d(B_{n+1})}{\lambda_d(B_n)} \frac{1}{\lambda_d(B_{n+1})} \int_{B_{n+1}} f(B, T^{-t} \omega) \lambda_d(dt).
\]
By the Proposition, the lower and the upper bound converge, for \( n \to \infty \), almost surely to
\[
\lim_{n \to \infty} \frac{1}{\lambda_d(B_n)} f(B_n, \cdot) = \frac{\mathbb{E} f(B_1, \cdot)}{\lambda_d(B_1)}.
\] (1)

Now we assume in addition that there is a constant \( D > 0 \) such that all convex bodies
\( K \in A \) satisfy \( \text{diam} K \leq D \), where \( \text{diam} \) denotes the diameter. The center function \( c \) satisfies
\( c(K) \in K \), hence if \( c(K) \in B_{n-D} \) (with \( n > D \)) and \( \text{diam} K \leq D \), then \( K \subset B_n \). It follows
that, for \( n > D \),

\[
\frac{\lambda_d(B_{n-D})}{\lambda_d(B_n)} \frac{1}{\lambda_d(B_{n-D})} \sum_{K \in X} 1_A(K) 1\{c(K) \in B_{n-D}\}
\]

\[
\leq \frac{1}{\lambda_d(B_n)} \sum_{K \in X, K \subset B_n} 1_A(K)
\]

\[
\leq \frac{1}{\lambda_d(B_n)} \sum_{K \in X} 1_A(K) 1\{c(K) \in B_n\}.
\]

As \( n \to \infty \), the lower and the upper bound converge a.s. to the right side of (1), hence
a.s. we have
\[
\delta(X, A) := \lim_{n \to \infty} \frac{1}{\lambda_d(B_n)} \sum_{K \in X, K \subset B_n} 1_A(K) = \frac{1}{\lambda_d(B^d)} \mathbb{E} \sum_{K \in X, c(K) \in B^d} 1_A(K).
\] (2)

Now we consider the special case where \( A_D \) is the set of polytopes that are combinatorially
isomorphic to a given simple \( d \)-polytope \( P \) and have diameter at most \( D \), for some fixed
number \( D > 0 \). We remark that (2) shows that
\[
\delta(X, A_D) = \frac{1}{\lambda_d(B^d)} \mathbb{E} \sum_{K \in X, c(K) \in B^d} 1\{K \in A_D\},
\] (3)

It remains to show that
\[
\mathbb{E} \sum_{K \in X, c(K) \in B^d} 1\{K \in A_D\} > 0.
\] (4)

For this, we use an argument from [4], which we recall for completeness.

Without loss of generality, we can assume that \( c(P) = o \). Let \( F_1, \ldots, F_m \) be the facets of
\( P \). We denote by \( B(x, \varepsilon) \) the ball with center \( x \) and radius \( \varepsilon > 0 \), set \([B(x, \varepsilon)]_H := \{H \in \mathcal{H} : H \cap B(x, \varepsilon) \neq \emptyset\}\), and define
\[
A_j(P, \varepsilon) := \bigcap_{v \in \text{vert} F_j} [B(v, \varepsilon)]_{H_j}, \quad j = 1, \ldots, m,
\]

where \( \text{vert} \) denotes the set of vertices. Each hyperplane from \( A_j(P, \varepsilon) \) is said to be \( \varepsilon \)-close to
\( F_j \). A polytope \( Q \) is said to be \( \varepsilon \)-close to \( P \) if it has \( m \) facets \( G_1, \ldots, G_m \) and, after suitable
renumbering, the affine hull of \( G_j \) is \( \varepsilon \)-close to \( F_j \), for \( j = 1, \ldots, m \). Since \( P \) is simple and
\( c(P) = o \), we can choose numbers \( D, \varepsilon_0 > 0 \) such that for \( 0 < \varepsilon \leq \varepsilon_0 \), the following is true:

- the sets \( A_1(P, \varepsilon), \ldots, A_m(P, \varepsilon) \) are pairwise disjoint, and any hyperplanes \( H_j \in A_j(P, \varepsilon) \),
  \( j = 1, \ldots, m \), are the facet hyperplanes of a polytope \( Q \) that is \( \varepsilon \)-close to \( P \).
Any polytope $Q$ that is $\varepsilon$-close to $P$ satisfies the following:
- $Q$ is combinatorially isomorphic to $P$,
- $Q \subset P + B^d$,
- $\operatorname{diam} Q \leq D$,
- $c(Q) \in B^d$.

That this can be achieved by suitable choices of $D$ and $\varepsilon_0$, follows from easy continuity considerations and the fact that $P$ is simple.

Now we define

$$C(P, \varepsilon) := \{ H \in \mathcal{H} : H \cap (P + B^d) \neq \emptyset, \ H \notin A_j(P, \varepsilon) \text{ for } j = 1, \ldots, m \}$$

and consider the event $E(P, \varepsilon)$ defined by

$$\tilde{X}(A_j(P, \varepsilon)) = 1 \text{ for } j = 1, \ldots, m \text{ and } \tilde{X}(C(P, \varepsilon)) = 0.$$ 

Let $0 < \varepsilon \leq \varepsilon_0$. The following was proved in [4]:
- If the event $E(P, \varepsilon)$ occurs, then some polytope $Q$ of the tessellation $X$ is $\varepsilon$-close to $P$ and hence satisfies $Q \in A_D$ and $c(Q) \in B^d$.
- The event $\mathbb{P}(E(P, \varepsilon))$ has positive probability.

Now it follows that

$$\mathbb{E} \sum_{K \in X, c(K) \in B^d} 1\{ K \in A_D \} \geq \mathbb{P}(E(P, \varepsilon)) > 0,$$

which proves [4].

The result is that $\delta(X, A_D) > 0$ a.s. This implies, in particular, that with probability one the polytopes of the combinatorial type of $P$ appear in $X$ with positive density. Since there are only countably many combinatorial types, it also holds with probability one that each combinatorial type of a simple $d$-polytope appears in $X$ with positive density.

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