DOMINATION OF OPERATORS IN THE NON-COMMUTATIVE SETTING

TIMUR OIKHBERG AND EUGENIU SPINU

Abstract. We consider majorization problems in the non-commutative setting. More specifically, suppose $E$ and $F$ are ordered normed spaces (not necessarily lattices), and $0 \leq T \leq S$ in $B(E, F)$. If $S$ belongs to a certain ideal (for instance, the ideal of compact or Dunford-Pettis operators), does it follow that $T$ belongs to that ideal as well? We concentrate on the case when $E$ and $F$ are $C^*$-algebras, preduals of von Neumann algebras, or non-commutative function spaces. In particular, we show that, for $C^*$-algebras $A$ and $B$, the following are equivalent: (1) at least one of the two conditions holds: (i) $A$ is scattered, (ii) $B$ is compact; (2) if $0 \leq T \leq S : A \to B$, and $S$ is compact, then $T$ is compact.

1. Preliminaries

1.1. Introduction. Following [45, Definition II.1.2], we say that a real Banach space $Z$ is an ordered Banach space (OBS for short) if it is equipped with a positive cone $Z_+$, closed in the norm topology. Throughout, we assume that $Z_+$ is proper (or pointed) — that is, $Z_+ \cap (-Z_+) = \{0\}$. The positive cone of an OBS $Z$ is called generating if $Z_+ - Z_+ = Z$. Equivalently (see [6], [8]), there exists $G_Z$ (the generating constant of $Z$) so that, for any $z \in Z$, there exist $a, b \in Z_+$ so that $z = a - b$, and $\max\{\|a\|, \|b\|\} \leq G_Z \|z\|$. Abusing the notation slightly, we call such OBSs generating. We say that an OBS $Z$ is normal if there exists $N_Z$ (the normality constant of $Z$) so that $\|z\| \leq N_Z(\|a\| + \|b\|)$ whenever $a \leq z \leq b$. By [8, Section 1.1] or [6], $Z$ is normal iff its dual $Z^*$ is generating, and vice versa.

In the current article we consider the following question. Suppose $0 \leq T \leq S$ are operators acting between two ordered Banach spaces, and $S$ belongs to a certain class of operators (say, compact or Dunford-Pettis). Does this imply...
that $T$ belongs to the same class? This question is usually referred as the Domination Problem. For arbitrary ordered normed spaces, the set-up may be too general to obtain meaningful results. In the (rather restrictive) setting of operators between Banach lattices, the Domination Problem has been widely investigated (see e.g. [2], [3], [16], [47], [25], [30], [51]).

We concentrate on the non-commutative version of the Domination Problem. More specifically, we consider the case when the domain and/or range of the operators involved is either a $C^*$-algebra, its dual or predual, or a non-commutative function space. We refer the reader to e.g. [18], or to the survey article [41], for the definition of the latter. Here, we only briefly outline the basic properties of such spaces.

Suppose a von Neumann algebra $A$ is equipped with a normal faithful semifinite trace $\tau$. An operator $x$ is called $\tau$-measurable if it is (i) closed and densely defined; (ii) affiliated with $A$, in the sense that $ux = xu$ for any unitary $u \in A'$; and (iii) for some $c > 0$, the spectral projection $\chi_{(c,\infty)}(|x|)$ has finite trace. On the set $\tilde{A}$ of $\tau$-measurable operators, we define the generalized singular value function: for $x \in A$ and $t \geq 0$, $\mu_x(t) = \inf \{ \|xe\| : e \in P(A), \tau(e^\perp) \leq t \}$ (see e.g. [41], [23] for other formulae for $\mu_x(\cdot)$). Here and below, $P(A)$ stands for the set of all projections in $A$.

Now suppose $E$ is a linear subset of $\tilde{A}$, complete in its own norm $\| \cdot \|_E$. We say that $E$ is a non-commutative function space if:

1. $L_1(\tau) \cap A \subset E \subset L_1(\tau) + A$.
2. For any $x \in E$ and $a, b \in A$, we have $axb \in E$, and $\|axb\|_E \leq \|a\| \|x\|_E \|b\|$.

$E$ is called symmetric if, whenever $x \in E$, $y \in \tilde{A}$, and $\mu_y \leq \mu_x$, then $y \in E$, with $\|y\|_E \leq \|x\|_E$. Following [22], we say that $E$ is strongly symmetric if, in addition, for any $x, y \in E$, with $y \ll x$, we have $\|y\|_E \leq \|x\|_E$. Here, $\ll$ refers to the Hardy-Littlewood domination: for any $\alpha > 0$, $\int_0^\alpha \mu_y(t) dt \leq \int_0^\alpha \mu_x(t) dt$. It is known that, as in the commutative case, $y \ll x$ iff there exists an operator $T$, contractive both on $A$ and $A_* = L_1(\tau)$, so that $y = Tx$ [17]. We say that $E$ is fully symmetric if it is strongly symmetric and, for any $x \in E$ and $y \in \tilde{A}$, we have $y \in E$ whenever $y \ll x$.

A non-commutative function space is said to be order continuous if, for any sequence $x_n \downarrow 0$, we have $\lim_n \|x_n\| = 0$. Emulating the proof of [37, Proposition 1.a.8], one proves that this is equivalent to requiring that, for any net $x_\alpha \downarrow 0$, $\lim_\alpha \|x_\alpha\| = 0$. 

Note that, if $-a \leq b \leq a$ for $a, b \in \tilde{A}$, then $\mu_b \leq \mu_a$. Indeed, pick $t \in \mathbb{R}$ and $\lambda > \mu_a(t)$. Set $e = \chi_{[0, \lambda]}(a)$. Then $\tau(e^\perp) \leq t$. Furthermore, $eae \geq ebe \geq -eae$, hence $\mu_b(t) \leq \|ebe\| \leq \|eae\| \leq \lambda$. Taking the infimum over $\lambda$, we obtain $\mu_b \leq \mu_a$.

Consequently, if $a, b \in \mathcal{E}$ satisfy $-a \leq b \leq a$, then $\|b\| \leq \|a\|$. Therefore, $\mathcal{E}$ is normal with constant 2. It is also easy to see that $\mathcal{E}$ is generating with constant 2. Consequently, the duals of $\mathcal{E}$ of all orders are both generating and normal.

Many symmetric non-commutative function spaces arise from their commutative analogues. Indeed, suppose $\tau$ is a normal faithful semi-finite trace on a von Neumann algebra $A$. It is known that if $A$ has no atomic projections, then the range of $\tau$ (denoted by $\Omega = \Omega_{\tau}$) is $[0, \tau(1)]$ (with $\tau(1) < \infty$), or $[0, \infty)$. On the other hand, if $A$ is atomic (that is, any projection has a minimal sub-projection), then $\Omega_{\tau}$ is either $\{0, 1, \ldots, n\}$ or $\mathbb{Z}_+ = \{0, 1, 2, \ldots\}$. Suppose $\mathcal{E}$ is a symmetric function space (in the sense of e.g. [35]) on $\Omega$. We can define the corresponding non-commutative function space $\mathcal{E}(\tau)$, consisting of those $x \in \tilde{A}$ for which the norm $\|x\|_{\mathcal{E}(\tau)} = \|\mu_x\|_{\mathcal{E}}$ is finite. By [31], this procedure yields a Banach space. It is well known (see e.g. [18], [22], [41]) that many properties of the function space $\mathcal{E}$ (for instance, being reflexive or order continuous) pass to the non-commutative space $\mathcal{E}(\tau)$.

In the discrete case ($\mathcal{E}$ is a symmetric sequence space on $\mathbb{N}$, and $\tau$ is the canonical trace on $B(H)$), the construction above produces a non-commutative symmetric sequence space (often referred to as a Schatten space), denoted by $\mathcal{S}_\mathcal{E}(H)$ (instead of $\mathcal{E}(\tau)$). When $H = \ell_2$ ($H = \ell^n_2$), we write $\mathcal{S}_\mathcal{E}$ (resp. $\mathcal{S}^n_\mathcal{E}$) instead of $\mathcal{S}_\mathcal{E}(H)$. For properties of Schatten spaces, the reader is referred to e.g. [27], [46]. We must note that any separable symmetric non-commutative sequence space arises from a sequence space [27, Section III.6].

Note that a symmetric function (or sequence) space is separable iff it is order continuous. Indeed, symmetric function spaces are order complete, and, for such spaces, separability implies order continuity [37, Proposition I.a.7]. On the other hand, it is well known that any non-negative function is a limit (a.e.) of an increasing sequence of simple functions. Thus, by [35, Theorem II.4.8], any order continuous symmetric function space is separable. Furthermore, by [35, Theorem II.4.10 and its Corollary], such spaces are fully symmetric (equivalently, they are interpolation spaces between $L_1$ and $L_\infty$). Some non-commutative generalizations of these results are contained in [21].
Surprisingly, the non-commutative Domination Problem has attracted little attention so far. The connections between domination and irreducibility (for maps between von Neumann algebras) were studied in [24]. In [40], domination of linear functionals on Banach ∗-algebras was used to obtain automatic continuity results. Domination of completely positive compact operators has recently been investigated in [20].

The paper is structured as follows. First (Section 1), we prove some preliminary results about the properties of positive operators, order intervals, and positive solids. In Subsection 1.2, we establish some basic facts about non-commutative function spaces. In Subsection 1.3, we investigate compact C∗-algebras, characterizing them in terms of compactness of order intervals. We also show that a C∗-algebra is compact iff it is hereditary in its enveloping algebra. Subsection 1.4 deals with the positive analogues of the Schur Property. In Subsection 1.5, we study compactness of order intervals in preduals of von Neumann algebras.

Our main results are contained in Section 2. In Subsection 2.1, we investigate whether an operator to or from a non-commutative function space, dominated by a compact operator, must itself be compact. Subsection 2.2 is devoted to the same question for C∗-algebras. In Subsection 2.3, we consider domination by compact multiplication operators on C∗-algebras. In Subsection 2.4, we tackle domination properties of Dunford-Pettis Schur multipliers. Subsection 2.5 is devoted to the domination properties of weakly compact operators.

Other classes of operators are considered in Section 3. In Subsection 3.1, we show that complete positivity and decomposability are not preserved under domination. Subsection 3.2 demonstrates that operator systems have too little structure to meaningfully consider domination.

Throughout the paper, we use standard Banach space results and notation. If a is a (closed densely defined) operator, a∗ refers to the adjoint of a. The same notation is used in preduals of von Neumann algebras. If E is a Banach space, E∗ refers to its dual. Similar notation is used for the predual, and for the conjugate of an operator between Banach spaces. B(E) stands for the unit ball of E. If S is a subset of an ordered Banach space, we denote by S+ the intersection of S with the positive cone. We denote by E∗∗ the Köthe dual of a non-commutative symmetric function space E (see e.g. [18], [41] for the definition and the basic properties of Köthe duals).
1.2. Compactness and positivity in Schatten spaces. To work with Schatten spaces, we need to introduce some notation. Denote the canonical basis in \( \ell_2 \) by \( (e_k) \). Let \( P_n \) be the orthogonal projection onto \( \text{span}[e_1, \ldots, e_n] \), and \( P^\perp_n = 1 - P_n \). For convenience, set \( P_0 = 0 \). If \( \mathcal{E} \) is a non-commutative symmetric sequence space, let \( Q_n \) be the projection on \( \mathcal{E} \), defined via \( Q_n x = P_n x P_n \). Similarly, let \( R_n x = P^\perp_n x P^\perp_n \).

**Lemma 1.2.1.** Suppose \( \mathcal{E} \) is a non-commutative symmetric sequence space on \( B(\ell_2) \), \( Z \) is an ordered normed space, and \( T : \mathcal{E} \to Z \) is a positive operator. Then, for any \( x \in \mathcal{E}_+ \), \( \|T(x - R_n x - Q_n x)\|^2 \leq 16N_Z \|T(Q_n x)\| \|T(R_n x)\| \), where \( N_Z \) is the normality constant of \( Z \). If \( Z \) is a non-commutative symmetric function space, then \( \|T(x - R_n x - Q_n x)\|^2 \leq 4\|T(Q_n x)\| \|T(R_n x)\| \).

**Proof.** For \( t \in \mathbb{R} \setminus \{0\} \), consider \( U(t) = tP_n + t^{-1}P^\perp_n \), and \( V(t) = tP_n - t^{-1}P^\perp_n \). These operators are self-adjoint and invertible, hence \( x(t) = U(t)xU(t) \) and \( y(t) = V(t)xV(t) \) are positive elements of \( \mathcal{E} \). An elementary calculation shows that \( x(t) = t^2Q_n x + t^{-2}R_n x + (x - Q_n x - R_n x) \), and \( y(t) = t^2Q_n x + t^{-2}R_n x - (x - Q_n x - R_n x) \). Let \( a(t) = t^2Q_n x + t^{-2}R_n x \), and \( b = x - Q_n x - R_n x \). By the above, \( -a(t) \leq b \leq a(t) \). Therefore, for any \( t \),

\[
\frac{1}{2N_Z} \|Tb\| \leq \|Ta(t)\| \leq t^2\|TQ_n x\| + t^{-2}\|TR_n x\|.
\]

Taking \( t = \|TR_n x\|^{1/4}/\|TQ_n x\|^{1/4} \), we obtain the desired inequality. If, in addition, \( Z \) is a non-commutative symmetric function space, then \( \|Tb\| \leq \|Ta(t)\| \).

**Corollary 1.2.2.** Suppose \( \mathcal{E} \) is a non-commutative symmetric sequence space on \( B(\ell_2) \), \( Z \) is a normal OBS, and \( T : \mathcal{E} \to Z \) is a positive operator. Then

\[
\|T(I - Q_n)\| \leq \|TR_n\| + 16N_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}.
\]

If \( Z \) is a non-commutative symmetric function space, then \( \|T(I - Q_n)\| \leq \|TR_n\| + 8N_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2} \).

**Proof.** We prove the corollary for general \( Z \) (the case of \( Z \) being a non-commutative function space follows with minor modifications). Lemma 1.2.1 shows that, for \( x \geq 0 \),

\[
\|T(I - R_n - Q_n)x\| \leq 4N_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2}\|x\|.
\]

A polarization argument implies \( \|T(I - R_n - Q_n)\| \leq 16N_Z^{1/2}\|TR_n\|^{1/2}\|TQ_n\|^{1/2} \). By the triangle inequality, \( \|T(I - Q_n)\| \leq \|TR_n\| + \|T(I - R_n - Q_n)\| \).
Lemma 1.2.3. Suppose $\tau$ is a normal faithful semi-finite trace on a von Neumann algebra $A$, and a strongly symmetric non-commutative function space $E$ is order continuous. Suppose, furthermore, that $x$ is an element of $A$, and a sequence of projections $p_n \in A$ decreases to $0$ in the strong operator topology. Then $\lim_n \|xp_n\| = \lim_n \|p_nx\| = \lim_n \|p_nxp_n\| = 0$.

Specializing to Schatten spaces, we obtain:

Corollary 1.2.4. Suppose $E$ is an order continuous symmetric sequence space. Then, for every $x \in S_E$, $\lim_n \|x - Q_nx\| = 0$.

Proof. By [18, Section 3], $S_E$ is order continuous iff $E$ is order continuous. It suffices to show that, for $x \in B(S_E)_+$, and $\varepsilon \in (0,1)$, $\|x - Q_nx\| < \varepsilon$ for $n$ sufficiently large. This follows from the estimate $\|x - Q_nx\| = \|P_n^+xP_n + xP_n^⊥\| \leq \|P_n^+x\| + \|xP_n^⊥\|$ and Lemma 1.2.3.

Lemma 1.2.5. Suppose $E$ is an order continuous symmetric sequence space, not containing $\ell_1$, and $S : S_E \to Z$ is compact ($Z$ is a Banach space). Then $\lim_n \|S|_{R_n(S_E)}\| = 0$.

Proof. Suppose not. By Corollary 1.2.4, we have $\lim_n \|(I - Q_n)x\| = 0$. A standard approximation argument yields a sequence $0 = n_0 < n_2 < \ldots$ with the property that for each $k$ there exists $x_k \in S_E$, so that $\|x_k\| = 1$, and $(P_{n_k} - P_{n_{k-1}})x_k(P_{n_k} - P_{n_{k-1}}) = x_k$, and $\|Sx_k\| > c > 0$. By compactness, the sequence $(Sx_k)$ must have a convergent subsequence $(Sx_{k_i})$. Then $\lim_{N} N^{-1}\|\sum_{i=1}^{N} Sx_{k_i}\| > 0$, while $\lim_{N} N^{-1}\|\sum_{i=1}^{N} x_{k_i}\| = 0$.

Next we describe the Schatten spaces not containing $\ell_1$.

Proposition 1.2.6. Let $E$ be a separable symmetric sequence space. For any infinite-dimensional Hilbert space $H$, the following are equivalent:

1. $E$ contains a copy of $\ell_1$.
2. $E$ contains a lattice copy of $\ell_1$ positively complemented.
3. $S_E(H)$ contains a positively complemented copy of $\ell_1$ spanned by a disjoint positive sequence.
4. $S_E(H)$ contains a copy of $\ell_1$.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are trivial. To show (2) $\Rightarrow$ (3), observe that $S_E(H)$ contains $E$ as a diagonal subspace, which is positively
complemented. (4) ⇒ (1) follows directly from [7, Corollary 3.2]. To prove (1) ⇒ (2), apply a “gliding hump” argument to show that \( \mathcal{E} \) contains disjoint vectors \((x_i)_i\), equivalent to the canonical basis of \( \ell_1 \). Then \( X = \text{span}[|x_i| : i \in \mathbb{N}] \) is a sublattice of \( \mathcal{E} \), lattice isomorphic to \( \ell_1 \). By [39, Theorem 2.3.11], \( X \) is positively complemented.

For a subset \( M \subset X_+ \) (\( X \) is an OBS), define the positive solid of \( M \):

\[
\text{PSol}(M) = \{ x \in X_+, \text{ such that } 0 \leq x \leq y \text{ and } y \in M \}.
\]

**Lemma 1.2.7.** If \( \mathcal{E} \) is an order continuous non-commutative symmetric sequence space, and \( M \subset \mathcal{E} \) is relatively compact, then \( \text{PSol}(M) \) is relatively compact. In particular, any order interval in an order continuous non-commutative symmetric sequence space is compact.

For the proof, we need two technical results.

**Lemma 1.2.8.** Suppose \( \mathcal{E} \) and \( M \) are as in Lemma 1.2.7. Then there exists a projection \( p \) with separable range, so that \( M = pM \).

**Proof.** The set \( M \) must contain a countable dense subset \( S \). The elements of \( M \) are compact operators, hence, for any \( x \in S \), there exists a projection \( p_x \) with separable range, so that \( p_x x p_x = x \). Then \( p = \bigvee_{x \in S} p_x \) has the desired properties. ■

**Lemma 1.2.9.** Suppose \( \mathcal{E} \) is an order continuous non-commutative symmetric sequence space on \( B(\ell_2) \), and \( M \) is relatively compact subset of \( \mathcal{E} \). Then \( \lim_n \| R_n |M\| = 0 \).

**Proof.** For every \( \varepsilon > 0 \) there are \( x_1, \ldots, x_k \) in \( M \) such that for every \( x \in M \) there is an \( 1 \leq i \leq k \) such that \( \| x - x_i \| < \varepsilon/2 \). Pick \( N \in \mathbb{N} \) such that \( \| R_n x_i \| < \varepsilon/2 \) for every \( n > N \) and \( 1 \leq i \leq k \). Hence, \( \| R_n x \| \leq \| R_n x_i \| + \| R_n \| \| x - x_i \| < \varepsilon \) for every \( x \in M \) and \( n > N \).

**Proof of Lemma 1.2.7.** By Lemma 1.2.8, we can restrict ourselves to spaces on \( B(\ell_2) \). As \( Q_n \) is a finite rank projection, it suffices to show that, for any \( \varepsilon \in (0, 1) \), there exists \( n \in \mathbb{N} \) so that \( \| (I - Q_n)x \| < \varepsilon \) for any \( x \in \text{PSol}(M) \). To this end, write \( (I - Q_n)x = (x - Q_n x - R_n x) + R_n x \). Reasoning as in the proof of Lemma 1.2.1, we observe that

\[
-(t^2 Q_n x + t^{-2} R_n x) \leq x - Q_n x - R_n x \leq t^2 Q_n x + t^{-2} R_n x
\]
for any \( t > 0 \), hence \( \| x - Q_n x - R_n x \| \leq t^2 \| Q_n x \| + t^{-2} \| R_n x \| \). Taking \( t = \| R_n x \|^{1/2}/\| Q_n x \|^{1/2} \), we obtain \( \| x - Q_n x - R_n x \| \leq 2 \| R_n x \|^{1/2} \| Q_n x \|^{1/2} \).

By scaling, we can assume that \( \sup_{y \in M} \| y \| = 1 \). By Lemma 1.2.9, there exists \( n \in \mathbb{N} \) so that \( \| R_n y \| < \varepsilon^2/16 \) for any \( y \in M \). For any \( x \in \text{PSol}(M) \), there exists \( y \in M \) so that \( 0 \leq x \leq y \), hence \( 0 \leq R_n x \leq R_n y \). By the above, \( \| x - Q_n x - R_n x \| \leq 2 \| R_n y \|^{1/2} \| Q_n x \|^{1/2} < \varepsilon/2 \), hence

\[
\|(I - Q_n)x\| = \| x - Q_n x - R_n x \| + \| R_n x \| \leq \frac{\varepsilon}{2} + \frac{\varepsilon^2}{16} < \varepsilon.
\]

Recall that if \( Z \) is an OBS, and \( x \in Z_+ \), the order interval \([0, x] \) is the set \( \{ y \in Z_+ : y \leq x \} \).

**Corollary 1.2.10.** Suppose \( E \) is a fully symmetric non-commutative sequence space. Then \( E \) is order continuous if and only if any order interval in \( E \) is compact.

**Lemma 1.2.11.** Suppose \( E \) is a fully symmetric non-commutative function or sequence space, which is not order continuous. Then there exists a positive complete isomorphism \( j : \ell_\infty \rightarrow E \).

**Proof.** In the notation of [22, Section 6], there exists \( x \in E_+ \setminus E^{an} \). Moreover, there exists a sequence of mutually orthogonal projections \( e_i \in A \) (\( i \in \mathbb{N} \)), so that \( \inf_i \| e_i x e_i \| > 0 \). The map \( y \mapsto \sum_i e_i y e_i \) is contractive in \( A \), and in its predual, hence \( \sum_i e_i y e_i \preceq y \), for any \( y \in A + A_* \). Due to \( A \) being fully symmetric, \( \sum_i e_i x e_i \in E \), and \( \| \sum_i e_i x e_i \| \leq \| x \| \). Therefore, the map

\[
j : \ell_\infty \rightarrow E : (\alpha_i) \mapsto (\sum_i \alpha_i e_i)(\sum_i e_i x e_i) = \sum_i \alpha_i e_i x e_i
\]

has the desired properties.

**Proof of Corollary 1.2.10.** Note that an order interval \([0, x] \) is closed. If \( E \) is order continuous, an application of Lemma 1.2.7 to \( M = \{ x \} \) shows the compactness of \([0, x] \). If \( E \) is not order continuous, then, for \( x \) as in Lemma 1.2.11, \([0, x] \) is not (relatively) compact.

1.3. **Compactness of order intervals in \( C^*\)-algebras.** In this subsection, we investigate the compactness of order intervals in \( C^*\)-algebras, and obtain a new description of compact \( C^*\)-algebras.

First we introduce some definitions. We say that an element \( a \) of a Banach algebra \( A \) is **multiplication compact** if the map \( A \rightarrow A : b \mapsto aba \) is compact.
Combining [57], [58], we see that, for an element \(a\) of a \(C^*\)-algebra \(A\), the following are equivalent:

1. \(a\) is multiplication compact.
2. The map \(A \rightarrow A : b \mapsto ab\) is weakly compact.
3. The map \(A \rightarrow A : b \mapsto ba\) is weakly compact.
4. The map \(A \rightarrow A : b \mapsto aba\) is weakly compact.

By [56], there exists a faithful representation \(\pi : A \rightarrow B(H)\) so that \(a\) is multiplication compact iff \(\pi(a)\) is a compact operator on \(H\). If, in addition, \(A\) is an irreducible \(C^*\)-subalgebra of \(B(H)\), then \(a \in A\) is multiplication compact iff \(a\) is a compact operator [55].

Suppose \(A\) is a \(C^*\)-subalgebra of \(B(H)\), where \(H\) is a Hilbert space. For \(x \in B(H)\) we define an operator \(M_x : A \rightarrow B(H) : a \mapsto x^*ax\).

**Lemma 1.3.1.** For an element \(a\) of a \(C^*\)-algebra \(A\), the following are equivalent.

1. \(a\) is multiplication compact.
2. The operator \(M_a\) is compact.
3. The operator \(M_a\) is weakly compact.

**Proof.** (2) \(\Rightarrow\) (3) is trivial. To show (1) \(\Rightarrow\) (2), recall that \(a\) is multiplication compact iff the map \(A \rightarrow A : b \mapsto ab\) is weakly compact. Passing to the adjoint, we see that the last statement holds iff the map \(A \rightarrow A : b \mapsto ba^*\) is weakly compact, or equivalently, iff \(a^*\) is multiplication compact. By [10], this implies the compactness of \(M_a\).

To prove (3) \(\Rightarrow\) (1), note that \(M_a^{**}\) takes \(b \in A^{**}\) to \(a^*ba\). We identify \(M_a^{**}\) with \(M_a\), acting on \(A^{**}\). Write \(a = cu\), where \(c = (aa^*)^{1/2}\), and \(u\) (respectively, \(u^*\)) is a partial isometry from \((\ker a)^\perp = (\ker c)^\perp\) to \(\text{ran} a = \text{ran} c\) (from \(\text{ran} a^* = \text{ran} c^*\) to \((\ker a^*)^\perp = (\ker c^*)^\perp\)). Then \(M_a = M_uM_c\), and \(M_u\) is an isometry on \(\text{ran} (M_c) \subset A^{**}\). Writing \(M_c = M_u^{-1}M_a\), we conclude that \(M_c\) is weakly compact. However, \(M_c x = cx\), hence, by the remarks preceding the lemma, \(c\) is multiplication compact. The operator \(S : A^{**} \rightarrow A^{**} : b \mapsto aba\) can be written as \(S = UM_lV\), where \(Vb = ub\) and \(Ub = bu\). Then \(S\) is weakly compact, and therefore, \(a\) is multiplication compact.

Multiplication compactness of elements of a \(C^*\)-algebra can be described in terms of compactness of order intervals.
Proposition 1.3.2. For a positive element $a$ of a $C^*$-algebra $A$, the following are equivalent:

1. $a$ is multiplication compact.
2. $a^\alpha$ is multiplication compact for any $\alpha > 0$.
3. The order interval $[0,a]$ is compact.
4. The order interval $[0,a]$ is weakly compact.

Proof. The implications (2) $\Rightarrow$ (1) and (3) $\Rightarrow$ (4) are immediate. To establish (1) $\Rightarrow$ (2), pick a faithful representation $\pi$ so that $a$ is multiplication compact if and only if $\pi(a)$ is compact, and note that the compactness of $\pi(a)\alpha = \pi(a\alpha)$. For (2) $\Rightarrow$ (3), assume $\|a\| = 1$. By [13, Lemma I.5.2], for any $x \in [0,a]$ there exists $u \in B(A)$, so that $x^{1/2} = ua^{1/4}$, hence $x = a^{1/4}u^*ua^{1/4}$. In particular, $[0,a] \subset M_{a^{1/4}}(B(A))$. If $a$ is multiplication compact, then so is $a^{1/4}$. Therefore, $[0,a]$ is compact.

To prove (4) $\Rightarrow$ (1), suppose $a$ is not multiplication compact. Then $a^{1/2}$ is not multiplication compact, hence $M_{a^{1/2}}(B(A))$ is not relatively compact. Note that any element $x \in B(A)$ can be written as $x = x_1 - x_2 + i(x_3 - x_4)$, with $x_1, x_2, x_3, x_4 \in B(A)_+$. Thus, $M_{a^{1/2}}(B(A)_+)$ is not relatively weakly compact. However, $[0,a] \supset M_{a^{1/2}}(B(A)_+)$. Indeed, if $0 \leq y \leq 1$, then $0 \leq a^{1/2}ya^{1/2} \leq a$. Therefore, $[0,a]$ is not relatively weakly compact.

These results allow us to obtain new characterizations of compact $C^*$-algebras. Recall that a Banach algebra is called compact (or dual) if all of its elements are multiplication compact. By [1], compact $C^*$-algebras are precisely the algebras of the form $A = (\sum_{i \in I} K(H_i))_{c_0}$, where each $H_i$ is a complex Hilbert space, and $K(H)$ denotes the space of compact operators on $H$. Several alternative characterizations of compact $C^*$-algebras can be found in [14, 4.7.20].

Proposition 1.3.3. For a $C^*$-algebra $A$, the following four statements are equivalent.

1. $A$ is compact.
2. For any $c \in A_+$, the order interval $[0,c]$ is compact.
3. For any $c \in A_+$, the order interval $[0,c]$ is weakly compact.
4. For any relatively compact $M \subset A_+$, $PSol(M)$ is relatively compact.

Proof. The implications (4) $\Rightarrow$ (2) $\Rightarrow$ (3) are immediate.

(3) $\Rightarrow$ (1): by Proposition 1.3.2, any positive $a \in A$ is multiplication compact. By [10, Corollary 10.4], the map $A \rightarrow A : x \mapsto axb$ is compact for any
These projections belong to \( A \times I \) implemented by \( p \). Denote the corresponding spectral projections by \( a_{ij}^{1/4} \) for \( i, j = 1 \ldots n \). Without loss of generality, we can assume \( 0 \leq a_{ij}^{1/4} \leq 1 \). In other words, for every \( \delta > 0 \), there exists \( \epsilon > 0 \), such that, for any \( a_{ij}^{1/4} \), we have \( \|a_{ij}^{1/4} - b_{ij}\| < \epsilon/4 \). Then

\[
\|a_{ij}^{1/4} u^* u a_{ij}^{1/4} - b_{ij}\| \leq \|(a_{ij}^{1/4} - a_{ij}^{1/4}) u^* u a_{ij}^{1/4}\|
\]

\[
+ \|a_{ij}^{1/4} u^* (a_{ij}^{1/4} - a_{ij}^{1/4})\| + \|a_{ij}^{1/4} u^* a_{ij}^{1/4} - b_{ij}\| < \epsilon.
\]

Recall that a \( C^\ast \)-subalgebra \( A \) of a \( C^\ast \)-algebra \( B \) is called hereditary if, for any \( a \in A_+ \), we have \( \{b \in B : 0 \leq b \leq a\} \subset A \).

**Proposition 1.3.4.** A \( C^\ast \)-algebra \( A \) is a hereditary subalgebra of \( A^{**} \) if and only if \( A \) is a compact \( C^\ast \)-algebra.

**Proof.** If \( A \) is compact, then it is an ideal in \( A^{**} \) [57]. It is well known (see e.g. [9, Proposition II.5.3.2]) that any ideal in a \( C^\ast \)-algebra is hereditary.

Now suppose \( A \) is a hereditary subalgebra of \( A^{**} \). By [14, Exercise 4.7.20], it suffices to show that, for any \( a \in A_+ \), any non-zero point of the spectrum of \( a \) is an isolated point. Suppose, for the sake of contradiction, that there exists \( a \in A_+ \) whose spectrum contains a strictly positive non-isolated point \( \alpha \). In other words, for every \( \delta > 0 \), \( ((\alpha - \delta, \alpha) \cup (\alpha, \alpha + \delta)) \cap \sigma(a) \neq \emptyset \). Without loss of generality, we can assume \( 0 \leq a \leq 1 \). Thus, we can find countably many mutually disjoint non-empty subsets \( S_i \) of \((\alpha/2, \infty) \cap \sigma(a)\). Denote the corresponding spectral projections by \( p_i \) (that is, \( p_i = \chi_{S_i}(a) \)). These projections belong to \( A^{**} \). Furthermore, \( p_i \leq (\inf S_i)^{-1}a \), hence, by the hereditary property, these projections belong to \( A \).

Now consider the linear map \( T : A \to A : x \mapsto axa \). Then \( T^{**} \) is also implemented by \( x \mapsto axa \). If \( 0 \leq x \leq 1 \), then \( axa \leq a^2 \), hence \( axa \in A \). Therefore, \( T^{**} \) takes \( A^{**} \) to \( A \). By Gantmacher’s Theorem (see e.g. [4,
Theorem 5.23), $T$ is weakly compact. However, $T$ is an isomorphism on the copy of $c_0$, spanned by the projections $p_i$, leading to a contradiction.

**Remark 1.3.5.** The above result was independently proved in [5], using a different method.

1.4. **Positive Schur Property.** Compactness of order intervals in Schatten spaces. An OBS $X$ is said to have the Positive Schur Property (PSP) if every weakly null positive sequence is norm convergent to 0 and $X$ has the Super Positive Schur Property (SPSP) if every positive weakly convergent sequence is norm convergent. Clearly, the Schur Property implies the SPSP, which, in turn, implies the PSP. Note that, if $X$ has the SPSP, then, by the Eberlein-Smulian Theorem, any weakly compact subset of $X_\pi$ is compact.

The PSP and SPSP of Banach lattices have been investigated earlier. By [52], the Schur Property and the PSP coincide for atomic Banach lattices. In [33], it is shown that $\ell_1$ is the only symmetric sequence space with the Schur Property (by Remark 1.4.7 below, the symmetry assumption is essential). [34] gives a criterion for the PSP of Orlicz spaces.

**Lemma 1.4.1.** Suppose $E$ is a symmetric sequence space, and $(A_n)$ is a positive bounded sequence in $S_E$ without a convergent subsequence. Then there exist a subsequence $(A_{n_k})$ and $c > 0$ such that $\|R_k A_{n_k}\| > c$ for every $k$.

**Proof.** Assume there is no such subsequence, that is

$$\lim_{m} \sup_{n} \|R_m A_n\| = 0.$$ 

Applying Lemma 1.2.1 when $T$ is the identity operator, we obtain the inequality

$$\|A_n - Q_m A_n\| \leq \|A_n - Q_m A_n - R_m A_n\| + \|R_m A_n\| \leq 2 \|Q_m A_n\|^{\frac{1}{2}} \|R_m A_n\|^{\frac{1}{2}} + \|R_m A_n\|.$$ 

Thus, $\lim_{m} \sup_{n} \|A_n - Q_m A_n\| = 0$. However, $Q_m$ is a finite rank map, hence the set $(A_n)$ is relatively compact, a contradiction.

**Proposition 1.4.2.** Suppose $E$ is a separable symmetric sequence space. Let $(A_n)$ be a weakly null positive sequence in $S_E(H)$, which contains no convergent subsequences. Then there exists $c > 0$ with the property that, for any $\varepsilon \in (0, 1)$, there exist sequences $1 = n_1 < n_2 < \ldots$ and $0 = m_0 < m_1 < \ldots$, so that $\inf_k \|A_{n_k}\| > c$, and

$$\sum_{k} \|A_{n_k} - (P_{m_k} - P_{m_{k-1}}) A_{n_k} (P_{m_k} - P_{m_{k-1}})\| < \varepsilon.$$
Consequently, the sequence \((A_{n_k})\) is equivalent to a disjoint sequence of positive finite dimensional operators.

**Proof.** By the separability (equivalently, order continuity) of \(E\), there exists a projection \(p \in B(H)\) with separable range, so that \(pA_kp = A_k\) for any \(k\). Thus, it suffices to prove our proposition in \(S_E\).

Furthermore, the order continuity of \(E\) implies that the finite rank operators are dense in \(S_E\). It is easy to see that, for any rank 1 operator \(u\), \(\lim_n \|u - Q_nu\| = 0\). Thus, \(\lim_n \|x - Q_nx\| = 0\) for any \(x \in E\).

By scaling, we can assume \(\sup_n \|A_n\| = 1\). Applying Lemma 1.4.1, and passing to a subsequence if necessary, we may assume that \(\|R_nA_n\| > c\), for some positive number \(c\). We construct the sequences \((n_k)\) and \((m_k)\) recursively. Set \(n_1 = 1\) and \(m_0 = 0\). As noted above, there exists \(m_1 > m_0\) so that \(\|A_{n_1} - P_{m_1}A_{n_1}P_{m_1}\| < \varepsilon/2\).

Suppose we have already selected \(0 = m_0 < m_1 < \ldots < m_j\) and \(1 = n_1 < n_2 < \ldots < n_j\) so that, for \(1 \leq j \leq k\),

\[
\|A_{n_k} - (P_{m_k} - P_{m_{k-1}})A_{n_k}(P_{m_k} - P_{m_{k-1}})\| < 2^{-j}\varepsilon.
\]

As \(Q_m\) is a finite rank operator for any \(m\), and the sequence \((A_n)\) is weakly null, \(\lim_n \|Q_mA_n\| = 0\). Consequently, there exists \(n_{k+1} > n_k\) so that \(\|Q_{m_k}A_{n_{k+1}}\| < 2^{-(k+1)-4}\varepsilon^2\). Then

\[
\begin{align*}
\|A_{n_{k+1}} - R_{m_k}A_{n_{k+1}}\| & \leq \|A_{n_{k+1}} - R_{m_k}A_{n_{k+1}} - Q_{m_k}A_{n_{k+1}}\| + \|Q_{m_k}A_{n_{k+1}}\| \\
& \leq 2\|Q_{m_k}A_{n_{k+1}}\|^{1/2}\|R_{m_k}A_{n_{k+1}}\|^{1/2} + \|Q_{m_k}A_{n_{k+1}}\| < 2^{-(k+2)}\varepsilon.
\end{align*}
\]

Now find \(m_{k+1}\) so that \(\|R_{m_k}A_{n_{k+1}} - Q_{m_{k+1}}R_{m_k}A_{n_{k+1}}\| < 2^{-(k+2)}\varepsilon\).

**Proposition 1.4.3.** For any Hilbert space \(H\), \(S_1(H)\) has the SPSP.

**Proof.** It suffices to consider the case of infinite dimensional \(H\). Suppose \(A_0, A_1, A_2, \ldots\) are positive elements of \(S_1(H)\), and \(A_n \to A_0\) weakly. Then there exist projections \(p_0, p_1, p_2, \ldots\) with separable range, so that \(p_iA_ip_i = A_i\) for every \(i\). Then \(p = \bigvee_{i \geq 0} p_i\) has separable range, and \(pA_ip = A_i\) for every \(i\). Thus, we can assume that \(H = \ell_2\).

By Lemma 1.4.1 there exist \(c > 0\) and a subsequence such that \(\|R_{k}A_{n_k}\| > c\). Since \(R_m \geq R_k\) when \(m \leq k\), we have \(\text{tr}(R_mA_{n_k}) > c\) for every \(k\). On the other hand we can always pick \(m\) such that \(\text{tr}(R_mA) = \|R_mA\| < c\). This contradicts \(A_n \to A_0\) weakly.

\[\blacksquare\]
Proposition 1.4.4. Suppose $E$ is a strongly symmetric sequence space, and $H$ is an infinite dimensional Banach space. Then the following are equivalent:

1. $E = \ell_1$.
2. $E$ has the Schur Property.
3. $E$ has the PSP.
4. $E$ has the SPSP.
5. $S_E(H)$ has the PSP.
6. $S_E(H)$ has the SPSP.

Proof. (1) $\Rightarrow$ (2) is well known. The implications (2) $\Rightarrow$ (4) $\Rightarrow$ (3), (6) $\Rightarrow$ (4), and (6) $\Rightarrow$ (5) $\Rightarrow$ (3) are obvious. (1) $\Rightarrow$ (6) follows from Proposition 1.4.3.

(3) $\Rightarrow$ (1). Assume that basis $(e_n)$ of $E$ is not equivalent to the canonical basis of $\ell_1$. By symmetry, $(e_n)$ contains no subsequence equivalent to the canonical basis of $\ell_1$. By Rosenthal’s dichotomy, the sequence $(e_n)$ is weakly null, which contradicts the PSP.

We complete this section by (partially) describing Banach lattices possessing various versions of the Schur Property.

Proposition 1.4.5. Any Banach lattice $E$ with the SPSP is atomic.

Recall that a Banach lattice is called atomic if it is the band generated by its atoms.

Proof. Clearly, a Banach lattice with the SPSP cannot contain a lattice copy of $c_0$. Theorems 2.4.12 and 2.5.6 of [39] show that $E$ is a KB-space. In particular, $E$ is order continuous. By [37, Proposition 1.a.9], without loss of generality, we may assume $E$ is atomless and has a weak unit. Therefore, by [37, Theorem 1.b.4], there exists an atomless probability measure space $(\Omega, \mu)$, so that $L_\infty(\mu) \subset E \subset L_1(\mu)$. Suppose, furthermore, that $e \in E_+ \setminus \{0\}$. Find $S \subset \Omega$ of finite measure, so that $e\chi_S > \alpha \chi_S$ for some positive number $\alpha$. By the proof of [11, Proposition 2.1], there exists a weakly null sequence $(f_n)$, so that $|f_n| = 1$ $\mu$-a.e. on $S$, $f_n = 0$ on $\Omega \setminus S$, and $f_n \to 0$ in $\sigma(L_\infty(\mu), L_1(\mu))$. Letting $e_n = e + f_n$, we conclude that $e_n \geq 0$ for every $n$, and $e_n \to e$ weakly, but not in norm.

Proposition 1.4.6. For any order continuous Banach lattice $E$ the SPSP, the PSP, and the Schur Property are equivalent.
Proof. Proposition 1.4.5 implies $E$ is atomic. Therefore the result follows from the fact that the lattice operations are weakly sequentially continuous, see [39, Proposition 2.5.23].

Remark 1.4.7. An order continuous atomic Banach lattice with the Schur Property need not be isomorphic to $\ell_1$, even as a Banach space. Indeed, suppose $(E_n)$ is a sequence of finite dimensional lattices. Then $E = (\sum_{n=1}^{\infty} E_n)\ell_1$ has the Schur Property. If, for instance, $E_n = \ell^n_2$, $E$ is not isomorphic to $\ell_1$. We do not know of any Banach lattice with the Schur Property which is not isomorphic to an $\ell_1$ sum of finite dimensional spaces.

1.5. Compactness of order intervals in preduals of von Neumann algebras. Following [49, Definition III.5.9], we say that a von Neumann algebra $\mathcal{A}$ is atomic if every projection in $\mathcal{A}$ has a minimal subprojection. Note that $\mathcal{A}$ is atomic if it is isomorphic to $(\sum_{i\in I} B(H_i))\ell_\infty(I)$, for some index set $I$, and collection of Hilbert spaces $(H_i)_{i\in I}$. Indeed, any von Neumann algebra of the above form is atomic. To prove the converse, note that an atomic algebra must be of type $I$. Moreover, it can be written as $\mathcal{A} = (\sum_{j\in J} \mathcal{A}_j)\ell_\infty(J)$, where $\mathcal{A}_j$ is an atomic algebra of type $I_j$. By [49, Theorem V.1.27] (see also [32, Theorem 6.6.5] and [9, III.1.5.3]), $\mathcal{A}_j$ is isomorphic to $\mathcal{C}_j\overline{\otimes}B(H_j)$, where $\mathcal{C}_j$ is the center of $\mathcal{A}_j$. Denote the set of all minimal projections in $\mathcal{C}_j$ by $F_j$. Then the elements of $F_j$ are mutually orthogonal, and their join equals the identity of $\mathcal{C}_j$. Thus, $\mathcal{C}_j$ is isomorphic to $\ell_\infty(F_j)$. Alternatively, one could use [9, III.1.5.18] and its proof to show that $\mathcal{C}_j$ is an $\ell_\infty$ space.

Theorem 1.5.1. For a von Neumann algebra $\mathcal{A}$, the following are equivalent:

1. $\mathcal{A}$ is an atomic von Neumann algebra.
2. $\mathcal{A}_*$ has the SPSP.
3. All order intervals in $\mathcal{A}_*$ are compact.

Remark 1.5.2. Note that the predual of any von Neumann algebra has the PSP. Indeed, suppose $(f_n)$ is a sequence of positive elements of $\mathcal{A}_*$, converging weakly to 0. Then $\|f_n\| = \langle f_n, 1 \rangle$, hence $\lim_n \|f_n\| = \lim_n \langle f_n, 1 \rangle = 0$.

Also, any order interval in the predual of a von Neumann algebra is weakly compact. Indeed, suppose $f$ is a positive element of $\mathcal{A}_*$. Then $[0, f]$ is convex and closed. For any $g \in [0, f]$ and $a \in \mathcal{A}$, Cauchy-Schwarz Inequality [49, Proposition I.9.5] yields $|g(a)|^2 \leq g(1)g(a^*a) \leq f(1)f(a^*a)$. By [49, Theorem III.5.4], $[0, f]$ is relatively weakly compact.
To prove Theorem 1.5.1, we need to determine when $A_*$ contains an order copy of $L_1(0,1)$, complemented via a positive projection.

**Proposition 1.5.3.** For a von Neumann algebra $A$, the following statements hold:

1. If $A$ is atomic, then $A_*$ does not contain $L_1(0,1)$ isomorphically.
2. If $A$ is not atomic, then there exists an isometric order isometry $j : L_1(0,1) \rightarrow A_*$, and a positive projection $P : A_* \rightarrow \text{ran}(j)$.

**Proof.** (1) Note that, for any Hilbert space $H$, $S_1(H)$ does not contain $L_1(0,1)$ isomorphically. Indeed, otherwise, by the separability argument, we would be able to embed $L_1(0,1)$ into $S_1$. This, however, is impossible, by e.g. [20]. To finish the proof of (1), recall that, if $A$ is atomic, then it can be identified with $(\sum_i B(H_i))_\infty$, and $A_*$ is isometric to $(\sum_i S_1(H_i))_1$.

(2) We can write $A = A_I \oplus A_{-I}$, where $A_I$ has type $I$, and $A_{-I}$ has no type $I$ components (that is, it is a direct sums of von Neumann algebras of types $II$ and $III$). Either $A_I$ is not atomic, or $A_{-I}$ is non-trivial.

If $A_I$ is not an atomic von Neumann algebra, write $A_I = (\sum_{s \in S} A_s)_{\ell_\infty(S)}$, with $A_s = C_s \overline{\otimes} B(H_s)$ ($C_s$ is the center of $A_s$). By [49, Theorem III.1.18], $C_s$ is isomorphic to $L_\infty(\nu_s)$, for some locally finite measure $\nu_s$. Consequently, $A_s$ contains $L_1(\nu_s) \otimes S_1(H_s)$ as a positively and completely contractively complemented subspace. As $A_I$ is not an atomic von Neumann algebra, then $\nu_s$ is not a purely atomic measure, for some $s$. By the above, $A_s$ contains $L_1(\nu_s) \otimes S_1(H_s)$ as a positively and completely contractively complemented subspace. Furthermore, $L_1(\nu_s)$ is complemented in $L_1(\nu_s) \otimes S_1(H_s)$ via a positive projection $Q$: just pick a rank one projection $e \in B(H_s)$, and set $Q(x) = (I_{L_1(\nu_s)} \otimes e)x(I_{L_1(\nu_s)} \otimes e)$. Finally, $L_1(\nu_s)$ contains a positively complemented copy of $L_1(0,1)$. Indeed, we can represent $L_1(\nu_s)$ a direct sum of spaces $L_1(\sigma_i)$, where $\sigma_i$ is a finite measure. Since $\nu_s$ is not purely atomic, the same is true for $L_1(\sigma_i)$, for some $i$. By [49, Theorem III.1.22] (or [32, Theorem 9.4.1]), $L_1(\nu_s)$ contains a positively complemented copy of $L_1(0,1)$.

Now suppose $A_{-I}$ is non-trivial. By the reasoning of [38, Page 217], $A_{-I}$ contains a von Neumann subalgebra $B$, isomorphic to the hyperfinite $II_1$ factor $R$. Furthermore, there exists a normal contractive projection (conditional expectation) $P : A_{-I} \rightarrow B$. By [49, Theorem III.3.4], $P$ is positive. Consequently, $A_*$ contains a copy of $R_*$, complemented via a positive contractive projection.
Let $\mu$ be the “canonical” measure on the Cantor set $\Delta$, defined as follows: represent $\Delta = \{0, 1\}^\mathbb{N}$, and write $\mu = \nu^\mathbb{N}$, where the measure $\nu$ on $\{0, 1\}$ satisfies $\nu(0) = \nu(1) = 1/2$. For $\alpha = (i_1, \ldots, i_n) \in I = \{0, 1\}^{<\mathbb{N}}$, define the function $f_\alpha$ by setting $f_\alpha(j_1, j_2, \ldots) = \prod_{k=1}^n \delta_{i_k,j_k}$ (here, $\delta_{i,j}$ stands for Kronecker’s delta). Note that $f_\alpha$ and $f_\beta$ have disjoint supports if $\alpha$ and $\beta$ are different bit strings of the same length. Moreover, $f_\alpha = f_{(\alpha,0)} + f_{(\alpha,1)}$. Clearly, $L_1(\mu)$ is the closed linear span of the functions $f_\alpha$. Subdividing $(0,1)$ appropriately, one can also construct an isometric order isomorphism between $L_1(\mu)$ and $L_1(0,1)$.

It therefore suffices to show that there exists an order isometry $J : L_1(\mu) \to \mathcal{R}_*$, so that the range of $J$ is the range of a positive projection. To prove this, let $\Delta_n = \{0,1\}^n$, and denote by $\mu_n$ the product of $n$ copies of $\nu$. In this notation, $L_1(\mu_n)$ is isometric to $\ell_2^n$. We can also identify $L_1(\mu_n)$ with $\text{span}\{f_\alpha : |\alpha| = n\}$. Let $i_n$ be the formal identity $L_1(\mu_{n-1}) \to L_1(\mu_n)$ (taking $f_\alpha$ to itself, when $|\alpha| \leq n$).

For $n \in \mathbb{N}$, consider the map $j_n : M_{2^{n-1}} \to M_{2^{n}} : x \mapsto x \otimes M_2$. Denote by $\text{Tr}_n$ the normalized trace on $M_{2^n}$, and by $M_{2^n}^*$ the dual of $M_{2^n}$ defined using $\text{Tr}_n$. Then $j_n : M_{2^{n-1}}^* \to M_{2^n}^*$ is an isometry. Furthermore, the diagonal embedding $u_n : L_1(\mu_n) \to M_{2^n}^*$ is an isometry, and $u_n i_n = j_n u_{n-1}$. We can view both $M_{2^{n-1}}^*$ and $L_1(\mu_n)$ as subspaces of $M_{2^n}^*$. Furthermore, for any $n$ there exist positive contractive unitary projections $p_n : M_{2^n}^* \to L_1(\mu_n)$ and $q_n : M_{2^n}^* \to M_{2^{n-1}}^*$ (the “diagonal” and “averaging” projections, respectively).

We then have $p_n j_n = i_n p_{n-1}$.

It is well known (see e.g. [44, Theorem 3.4]) that $\mathcal{R}_*$ can be viewed as $\bigcup_n M_{2^n}^*$. Moreover, for any $n$ there exists a positive contractive unital projection $\tilde{q}_n : \mathcal{R}_* \to M_{2^n}^*$ (with $\tilde{q}_n | M_{2^{n-k}}^* = q_{n+1} \cdots q_n$). Now identify $L_1(\mu)$ with $\bigcup_n L_1(\mu_n)$, and define the projection $P : \mathcal{R}_* \to J(L_1(\mu))$ by setting $P|_{M_{2^n}^*} = q_n$.

**Proof of Theorem 1.5.1.** If (1) holds, then $\mathcal{A} = (\sum_i B(H_i))_\infty$, hence $\mathcal{A}_* = (\sum_i \mathcal{S}_1(H_i))_1$. (2) and (3) follow from Propositions 1.4.4 and 1.2.7, respectively.

Now suppose $\mathcal{A}$ is not atomic. By Proposition 1.5.3, $\mathcal{A}_*$ contains a (positively and contractively complemented) lattice copy of $L_1(0,1)$. To finish the proof, note that $L_1(0,1)$ fails the SPSP, and has non-compact order intervals. Indeed, let $f = 1$, and $f_n = 1 + r_n$, where $r_1, r_2, \ldots$ are Rademacher functions. Then $f_n \to f$ weakly, but not in norm. This witnesses the failure of the SPSP. Moreover, $f_n/2 \in [0,1]$, hence the order interval $[0,1]$ is not compact.

2. Main results on majorization
2.1. **Compact operators on non-commutative function spaces.** First we consider maps from ordered Banach spaces into Schatten spaces.

**Proposition 2.1.1.** Suppose $E$ is a separable symmetric sequence space, $H$ is a Hilbert space, $A$ is a generating OBS, and $0 \leq T \leq S : A \to \mathcal{S}_E(H)$ (not necessary linear). If $S$ is compact, then $T$ is compact.

**Proof.** It is enough to show $T(B(A)_+)$ is relatively compact. Thus follows from Lemma 1.2.7, since $T(B(A)_+)(B(E)) \subseteq \text{PSol}(S(B(A)_+))$. ■

For operators into Schatten spaces, we have:

**Proposition 2.1.2.** Suppose $E$ is a separable symmetric sequence space, and $H$ is a Hilbert space.

1. If $E$ does not contain $\ell_1$, and operators $T$ and $S$ from $\mathcal{S}_E(H)$ to a normal OBS $Z$ satisfy $0 \leq T \leq S$, then the compactness of $S^*$ implies the compactness of $T^*$.

2. Conversely, suppose $E$ contains $\ell_1$, and a Banach lattice $Z$ is either not atomic, or not order continuous. Then there exist $0 \leq T \leq S : \mathcal{S}_E(H) \to Z$ so that $S$ is compact, but $T$ is not.

**Proof.** (1) By [36, Theorem 1.c.9], $\mathcal{E}^*$ is separable. Now apply Proposition 2.1.1.

(2) By [51], there exist $0 \leq \tilde{T} \leq \tilde{S} : \ell_1 \to Z$ so that $\tilde{S}$ is compact, but $\tilde{T}$ is not. By Proposition 1.2.6, there exists a lattice isomorphism $j : \ell_1 \to \mathcal{S}_E$, and a positive projection $P$ from $\mathcal{S}_E$ onto $j(\ell_1)$. Then the operators $T = \tilde{T}j^{-1}P$ and $S = \tilde{S}j^{-1}P$ have the desired properties. ■

Finally we deal with operators on general non-commutative function spaces.

**Proposition 2.1.3.** Suppose $E$ is a strongly symmetric non-commutative function space, such that $\mathcal{E}^*$ is not order continuous. Suppose, furthermore, that a symmetric non-commutative function space $F$ contains non-compact order intervals. Then there exist $0 \leq T \leq S : E \to F$, so that $S$ has rank 1, and $T$ is not compact.

Note that many spaces $F$ contain non-compact order ideals. Suppose, for instance, that $F$ arises from a von Neumann algebra $A$ that is not atomic, and is equipped with a normal faithful semifinite trace $\tau$. Using the type decomposition, we can find a projection $p \in A$ with a finite trace. Then the interval $[0, p]$ is not compact. Indeed, [49, Proposition V.1.35] allows us
to construct a family of projections \( (p_{ni}) \) \( (n \in \mathbb{N}, 1 \leq i \leq 2^n) \), so that (i) \( p = p_{11} + p_{12} \), and \( p_{ni} = p_{n+1,2i-1} + p_{n+1,2i} \) for any \( n \) and \( i \), and (ii) all projections \( p_{ni} \) are equivalent. Then the family \( q_n = \sum_{i=1}^{2^n-1} p_{n,2i} \) is a sequence in \([0, p]\), with no convergent subsequences.

Note that, for fully symmetric non-commutative sequence spaces, order continuity is fully described by Corollary \ref{corollary:ordered_algebras}.

**Lemma 2.1.4.** Suppose \( \mathcal{E} \) is a strongly symmetric non-commutative function space, so that \( \mathcal{E}^\times \) is not order continuous. Then there exists an isomorphism \( j : \ell_1 \to \mathcal{E} \), so that both \( j \) and \( j^{-1} \) are positive, and \( j(\ell_1) \) is the range of a positive projection.

**Proof.** By \cite{18}, \( \mathcal{E}^\times \) is fully symmetric. By Lemma \ref{lemma:ordered_algebras}, there exists \( x \in \text{B}(\mathcal{E}^\times)_+ \), and a sequence of mutually orthogonal projections \( (e_i) \), so that \( (\alpha_i) \mapsto \sum \alpha_i e_i x e_i \) determines a positive embedding of \( \ell_\infty \) into \( \mathcal{E}^\times \). For each \( i \), find \( y_i \in \mathcal{E}_+ \) so that \( e_i y_i e_i = y_i \), \( \|y_i\| < 2\|e_i x e_i\|^{-1} \), and \( \langle e_i x e_i, y_i \rangle = 1 \). The map \( j : \ell_1 \to \mathcal{E} : (\alpha_i) \mapsto \sum \alpha_i y_i \) determines a positive isomorphism. Furthermore, define \( U : \mathcal{E} \to \ell_1 : y \mapsto (\langle e_i x e_i, y \rangle)_i \). Clearly, \( U \) is a bounded positive map, and \( U j = I_{\ell_1} \). Therefore, \( j U \) is a positive projection onto \( j(\ell_1) \). □

**Proof of Proposition 2.1.3.** In view of Lemma 2.1.4, it suffices to construct \( 0 \leq T \leq S : \ell_1 \to \mathcal{F} \), so that \( S \) has rank 1, and \( T \) is not compact. Pick \( y \in \mathcal{F} \), so that \([0, y]\) is not compact. Then find a sequence \( (y_i) \subset [0, y] \), without convergent subsequences. Denote the canonical basis of \( \ell_1 \) by \( (\delta_i) \). Let \( \delta_i^* \) be the biorthogonal functionals in \( \ell_\infty \). Following \cite{51}, define \( S \) and \( T \) by setting \( S \delta_i = y_i \), and \( T \delta_i = y_i \). In other words, for \( a = (\alpha_i) \in \ell_1 \), \( Sa = \langle 1, a \rangle y \), and \( Ta = \sum_i \langle \delta_i^*, a \rangle y_i \). It is easy to see that rank \( S \) = 1, and \( 0 \leq T \leq S \). Moreover, \( T(\text{B}(\ell_1)) \) contains the non-compact set \( \{y_1, y_2, \ldots\} \), hence \( T \) is not compact. □

### 2.2. Compact operators on \( \text{C}^* \)-algebras and their duals.

In this section, we determine the \( \text{C}^* \)-algebras \( \mathcal{A} \) with the property that every operator on \( \mathcal{A} \), dominated by a compact operator, is itself compact. First we introduce some definitions. Let \( \mathcal{A} \) be a \( \text{C}^* \)-algebra, and consider \( f \in \mathcal{A}^* \). Let \( e \in \mathcal{A}^{**} \) be its support projection. Following \cite{29}, we call \( f \) atomic if every non-zero projection \( e_1 \leq e \) dominates a minimal projection (all projections are assumed to “live” in the enveloping algebra \( \mathcal{A}^{**} \)). Equivalently, \( f \) is a sum of pure positive functionals. We say that \( \mathcal{A} \) is scattered if every positive functional is atomic. By \cite{28}, \cite{29}, the following three statements are equivalent: (i) \( \mathcal{A} \) is scattered;
(ii) $\mathcal{A}'' = (\sum_{i \in I} B(H_i))_{\infty}$; (iii) the spectrum of any self-adjoint element of $\mathcal{A}$ is countable. Consequently (see [14, Exercise 4.7.20]), any compact $C^*$-algebra is scattered. In [53], it is proven that a separable $C^*$-algebra has separable dual if and only if it is scattered.

The main result of this section is:

**Theorem 2.2.1.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras, and $E$ is a generating OBS.

1. Suppose $\mathcal{A}$ is a scattered. Then, for any $0 \leq T \leq S : E \to \mathcal{A}^*$, the compactness of $S$ implies the compactness of $T$.
2. Suppose $\mathcal{B}$ is a compact. Then, for any $0 \leq T \leq S : E \to \mathcal{B}$, the compactness of $S$ implies the compactness of $T$.
3. Suppose $\mathcal{A}$ is not scattered, and $\mathcal{B}$ is not compact. Then there exist $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, so that $S$ has rank 1, while $T$ is not compact.

From this, we immediately obtain:

**Corollary 2.2.2.** Suppose $\mathcal{A}$ and $\mathcal{B}$ are $C^*$-algebras. Then the following are equivalent:

1. At least one of the two conditions holds: (i) $\mathcal{A}$ is scattered, (ii) $\mathcal{B}$ is compact.
2. If $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, and $S$ is compact, then $T$ is compact.

It is easy to see that a von Neumann algebra is scattered if an only if it is finite dimensional if and only if it is compact. This leads to:

**Corollary 2.2.3.** If von Neumann algebra $\mathcal{A}$ and $\mathcal{B}$ are infinite dimensional, then there exist $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, so that $S$ has rank 1, while $T$ is not compact.

We establish similar results about preduals of von Neumann algebras.

**Lemma 2.2.4.** (1) Suppose $\mathcal{A}$ is an atomic von Neumann algebra, and $E$ is a generating OBS. Then $0 \leq T \leq S : E \to \mathcal{A}_*$, where $S$ is a compact operator, implies $T$ is compact.

2. Suppose $\mathcal{A}$ is a von Neumann algebra, and $\mathcal{A}_I, \mathcal{A}_{II}, \mathcal{A}_{III}$ are its summands of type I, II, and III, respectively. Suppose, furthermore, that one of the three statements is true: (i) $\mathcal{A}_I$ is not atomic, (ii) $\mathcal{A}_{II}$ is not empty, (iii) $\mathcal{A}_{III}$ is non-empty, and has separable predual. Then there exists $0 \leq T \leq S : \mathcal{A}_* \to \mathcal{A}_*$, so that $S$ is compact, and $T$ is not.
Proof. (1) The weak compactness of $S$ implies, by Theorem 2.5.1, the weak compactness of $T$. By Theorem 1.5.1, $A_\star$ has the SPSP, hence $T(B(E)_+)$ is relatively compact. Thus, $T(B(E))$ is relatively compact as well, hence $T$ is compact.

(2) It suffices to show that there exists an order isomorphism $j : L_1(0,1) \to A_\star$, so that there exists a positive projection $P$ onto $\text{ran } (j)$. Indeed, by [51], there exist operators $0 \leq T_0 \leq S_0 : L_1(0,1) \to L_1(0,1)$, so that $S_0$ is compact, and $T_0$ is not. Then $T = jT_0j^{-1}P$ and $S = jS_0j^{-1}P$ have the desired properties. The existence of $j$ and $P$ as above follows from the proof of Proposition 1.5.3.

To establish Theorem 2.2.1, we need a series of lemmas.

Lemma 2.2.5. Suppose $A$ is a $C^*$-algebra for which $A_\star$ has non-compact order intervals, and a Banach lattice $E$ is not order continuous. Then there exist $0 \leq T \leq S : A \to E$, so that $S$ has rank 1, while $T$ is not compact.

Proof. By [39, Theorem 2.4.2], there exists $y \in E_+$, and normalized elements $y_1,y_2,\ldots \in [0,y]$ with disjoint supports. By our assumption there exist $\psi \in A_\star^+$ and a sequence $(\phi_i) \subset [0,\psi]$ which does not have convergent subsequences. By Alaoglu’s theorem we may assume $\phi_i \to \phi$ in weak* topology. Define two operators via

$$Sx = \psi(x)y \text{ and } Tx = \phi(x)y + \sum_{n=1}^{\infty} (\phi_n - \phi)(x)y_n.$$ 

Note that $T$ is well defined: $(\phi_n - \phi)(x) \to 0$ for all $x$, hence

$$\| \sum_{n=m+1}^{k} (\phi_n - \phi)(x)y_n \| \leq \sup_{m > n} |(\phi_m - \phi)(x)||y|| \xrightarrow{n \to \infty} 0.$$

Moreover, for any $x > 0$ and $N \in \mathbb{N}$ we have

$$\phi(x)y + \sum_{n=1}^{N} (\phi_n - \phi)(x)y_n = \phi(x)(y - \sum_{n=1}^{N} y_n) + \sum_{n=1}^{N} \phi_n(x)y_n \geq 0,$$

and

$$\psi(x)y - \phi(x)y - \sum_{n=1}^{N} (\phi_n - \phi)(x)y_n =$$

$$\psi(x)y - \sum_{n=1}^{N} \phi_n(x)y_n - \phi(x)(y - \sum_{n=1}^{N} y_n) \geq$$

$$(\psi(x) - \phi(x))(y - \sum_{n=1}^{N} y_n).$$
By sending $N$ to infinity, we obtain that $0 \leq Tx \leq Sx$ for every $x > 0$. Clearly, rank $S = 1$. It remains to show that $T^*$ is not compact. Note that there exist norm one $f_1, f_2, \ldots \in E^*$ so that $f_n(y_m) = \delta_{nm}$. It is easy to see that $T^*f = f(y)\phi + \sum_{n=1}^{\infty} f(y_n)(\phi_n - \phi)$, hence $T^*f_m = (f_m(y) - 1)\phi + \phi_m$. The sequence $(T^*f_m)$ has no convergent subsequences, since if it had, $(\phi_m)$ would have a convergent subsequence, too. This rules out the compactness of $T^*$.

Corollary 2.2.6. Suppose a $C^*$-algebra $\mathcal{B}$ is not compact, and $\mathcal{A}^*$ has non-compact order intervals. Then there exist $0 \leq T \leq S : \mathcal{A} \to \mathcal{B}$, so that $S$ has rank 1, while $T$ is not compact.

Proof. By Lemma 2.2.5, it suffices to show that $\mathcal{B}$ contains a Banach lattice which is not order continuous. By [14, Exercise 4.7.20], $\mathcal{B}$ contains a positive element $b$, whose spectrum contains a positive non-isolated point. Then the abelian $C^*$-algebra $\mathcal{B}_0$, generated by $b$, is not order continuous. Indeed, suppose $\alpha > 0$ is not an isolated point of $\sigma(a)$. Then there exist disjoint subintervals $I_i = (\beta_i, \gamma_i) \subset (\alpha/2, 3\alpha/2)$, so that $\delta_i = (\beta_i + \gamma_i)/2 \in \sigma(b)$ for every $i \in \mathbb{N}$. For each $i$, consider the function $\sigma_i$, so that $\sigma_i(\beta_i) = \sigma_i(\gamma_i) = 0$, $\sigma_i((\beta_i + \gamma_i)/2) = 1$, and $\sigma_i$ is defined by linearity elsewhere. Then the elements $y_i = \sigma_i(b)$ belongs to $\mathcal{B}_0$, are disjoint and normalized, and $y_i \leq y = 2\alpha^{-1}b$.

Proof of Theorem 2.2.1. (1) If $\mathcal{A}$ is scattered, then $\mathcal{A}^{**}$ is atomic. Now invoke Lemma 2.2.4(1).

(2) By assumption, $M = S((\mathcal{B}(E)_+)$ is relatively compact, and $T((\mathcal{B}(E)_+) \subset \text{PSol}(M)$. By Proposition 1.3.3, $T((\mathcal{B}(E)_+)$ is relatively compact.

(3) Combine Theorem 1.5.1 with Corollary 2.2.6.

2.3. Comparisons with multiplication operators. Suppose $\mathcal{A}$ is a $C^*$-subalgebra of $B(H)$, where $H$ is a Hilbert space. For $x \in B(H)$ we define an operator $M_x : \mathcal{A} \to B(H) : a \mapsto x^*ax$. In this section, we study domination of, and by, multiplication operators, in relation to compactness. First, record some consequences of the results from Section 1.3.

Proposition 2.3.1. Suppose $x$ is an element of a $C^*$-algebra $\mathcal{A}$.

1. If $M_x$ is weakly compact, and $0 \leq T \leq M_x : \mathcal{A} \to \mathcal{A}$, then $T$ is compact.

2. If $0 \leq M_x \leq S : \mathcal{A} \to \mathcal{A}$, and $S$ is weakly compact, then $M_x$ is compact.

Proof. By passing to the second adjoint if necessary, we can assume $\mathcal{A}$ is a von Neumann algebra. Note that $[0, x^*x] = M_x(B(\mathcal{A})_+)$. Indeed, if $a \in B(\mathcal{A})_+$,
then \(0 \leq a \leq 1\), hence \(0 \leq M_x a \leq M_x 1 = x^* x\), hence \(M_x a \in [0, x^* x]\). Next show that any \(b \in [0, x^* x]\) belongs to \(M_x a \in [0, x^* x]\). By [15, p. 11], there exists \(v \in B(A)\) so that \(b^{1/2} = vc\), where \(c = (x^* x)^{1/2}\). Write \(x = uc\), where \(u\) is a partial isometry from \((\ker x)^\perp\) onto \(\text{ran} x\). Then \(c = u^* x = x^* u\), and therefore, \(b = M_x(uv^* vu^*)\).

Therefore, \(M_x\) is (weakly) compact if and only if the interval \([0, x^* x]\) is (weakly) compact. By Proposition 1.3.2, the compactness and weak compactness of \([0, x^* x]\) are equivalent. To establish (1), suppose \(0 \leq T \leq M_x\), and \(M_x\) is weakly compact. Then \(T(B(A)_+)\) is relatively compact, as a subset of \([0, x^* x]\). Thus, \(T\) is compact. (2) is established similarly. 

If the “symbol” \(x\) of the operator \(M_x\) comes from the ambient \(B(H)\), we obtain:

**Proposition 2.3.2.** Suppose \(A\) is an irreducible \(C^*\)-subalgebra of \(B(H)\), \(x \in B(H)\), \(M_x : A \to B(H)\) is compact, and \(0 \leq T \leq M_x\). Then \(T\) is compact.

**Proposition 2.3.3.** Suppose \(A\) is an irreducible \(C^*\)-subalgebra of \(B(H)\), \(S : A \to B(H)\) is compact, \(x \in B(H)\), and \(0 \leq M_x \leq S\). Then \(M_x\) is compact.

**Remark 2.3.4.** The irreducibility of \(A\) is essential here. Below we construct an abelian \(C^*\)-subalgebra \(A \subset B(H)\), and operators \(x_1, x_2 \in B(H)\), so that \(0 \leq M_{x_1} \leq M_{x_2}, M_{x_2}\) is compact, while \(M_{x_1}\) is not (here, \(M_{x_1}\) and \(M_{x_2}\) are viewed as taking \(A\) to \(B(H)\)). By [51], there exist operators \(0 \leq R_1 \leq R_2 : C[0, 1] \to C[0, 1]\) so that \(R_2\) is compact, and \(R_1\) is not. Let \(\lambda\) be the usual Lebesgue measure on \([0, 1]\), and let \(j : C[0, 1] \to B(L_2(\lambda))\) be the diagonal embedding (taking a function \(f\) to the multiplication operator \(\phi \mapsto \phi f\)). By [42, Theorem 3.11], \(R_1\) and \(R_2\) are completely positive. Thus, by Stinespring Theorem, these operators can be represented as \(R_i(f) = V_i^* \pi_i(f) V_i\) \((i = 1, 2)\), where \(\pi_i : C[0, 1] \to B(H_i)\) are representations, and \(V_i \in B(L_2(\lambda), H_i)\). Let \(H = L_2(\lambda) \oplus_2 H_1 \oplus_2 H_2\). Then \(\pi = j \oplus \pi_1 \oplus \pi_2 : C[0, 1] \to B(H)\) is an isometric representation. Let \(A = \pi(C[0, 1])\). Furthermore, consider operators \(x_1\) and \(x_2\) on \(H\), defined via

\[
x_1 = \begin{pmatrix} 0 & 0 & 0 \\ V_1 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad x_2 = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ V_2 & 0 & 0 \end{pmatrix}.
\]

Then, for any \(f \in C[0, 1]\), \(jR_i(f) = x_i^* \pi(f)x_i\). Considering \(M_{x_1}\) and \(M_{x_2}\) as operators on \(A\), we see that \(0 \leq M_{x_1} \leq M_{x_2}, M_{x_2}\) is compact, and \(M_{x_1}\) is not.
The following lemma establishes a criterion for compactness of $M_x$. This result may be known to experts, but we could not find any references in the literature.

**Lemma 2.3.5.** Suppose $A$ is an irreducible $C^*$-subalgebra of $B(H)$, and $c \in B(H)$. Then $c^* B(A)_+ c$ is a relatively compact set if and only if $c$ is a compact operator.

**Proof.** By polar decomposition, it suffices to consider the case of $c \geq 0$. Indeed, write $c = du$, where $d = (cc^*)^{1/2}$, and $u$ is a partial isometry from $(\ker c)^\perp = \text{ran } c^*$ to $(\ker c^*)^\perp = \text{ran } c$. Then $M_c = M_d M_d$, and $M_d = M_d^* M_c$ (here, we abuse the notation slightly, and allow $M_u$ and $M_u^*$ to act on $B(H)$). Therefore, the sets $c^* B(A)_+ c = M_c(B(A)_+)$ and $dB(A)_+ d = M_d(B(A)_+)$ are compact simultaneously.

If $c$ is compact, then, by [56], $cB(B(H))c$ is relatively compact. The set $cB(A)_+ c$ is also relatively compact, since it is contained in $cB(B(H))c$.

Now suppose $c$ is not compact. By scaling, we can assume that the spectral projection $p = \chi_{(1, \infty)}(c)$ has infinite rank. We shall show that, for every $n \in \mathbb{N}$, there exist $a_1, \ldots, a_n \in B(A)_+$ so that $\|c(a_i - a_j)c\| > 1/3$ for $i \neq j$. Note first that there exist mutually orthogonal unit vectors $\xi_1, \ldots, \xi_n$ in $\text{ran } p$, so that $\langle \xi_i, \xi_j \rangle = \langle c\xi_i, c\xi_j \rangle = 0$ whenever $i \neq j$. Indeed, if $\sigma(c) \cap (1, \infty)$ is infinite, then there exist disjoint Borel sets $E_i \subset (1, \infty)$ ($1 \leq i \leq n$), so that $\sigma(c) \cap E_i \neq \emptyset$.

Then we can take $\xi_i \in \chi_{E_i}(c)$. On the other hand, if $\sigma(c) \cap (1, \infty)$ is finite, then for some $s \in \sigma(c) \cap (1, \infty)$, the projection $q = \chi_{\{s\}}(c)$ has infinite rank. Then we can take $\xi_1, \ldots, \xi_n \in \text{ran } q$.

Let $\eta_i = c\xi_i/\|c\xi_i\|$ (by construction, these vectors are mutually orthogonal). As $A$ is irreducible, its second commutant is $B(H)$. By Kaplansky Density Theorem (see e.g. [13, Theorem I.7.3]), $B(A)_+$ is strongly dense in $B(B(H))_+$. Thus, for every $1 \leq i \leq n$ there exist $a_i \in B(A)_+$ so that $\|a_i \eta_k\| < 1/3$ for $i \neq k$, and $\|a_i \eta_i - \eta_i\| < 1/3$. Consider $b_i = c a_i c \in c(B(A)_+)c$. For $i \neq j$,

$$\|b_i - b_j\| \geq \langle c(a_i - a_j)c\xi_i, \xi_i \rangle = \|c\xi_i\|^2 \langle (a_i - a_j)\eta_i, \eta_i \rangle > \frac{2}{3} \cdot \frac{1}{3} = \frac{1}{3}.$$

As $n$ is arbitrary, we conclude that $c(B(A)_+)c$ is not relatively compact. □

**Proof of Proposition 2.3.2.** Suppose $x \in B(H)$ is such that $M_x : A \to B(H)$ is compact. By polar decomposition, we can assume that $x \geq 0$. Then $xB(A)_+ x$
is relatively compact, and therefore, By Lemma 2.3.5, $x$ is a compact operator. By Proposition 1.3.2, $[0, x^2]$ is compact. But $T(B(A)_+) \subset [0, x^2]$, hence $T(B(A)_+)$ is relatively compact. By polarization, $T(B(A))$ is compact. ■

To prove Proposition 2.3.3, we need a technical result.

**Lemma 2.3.6.** Suppose $z \in B(H)$, and $x, y \in [0, 1_H]$. Then $zz^* \geq zxyxz^*$.

**Proof.** Note that $zz^* - zxyxz^* = z(x - x^2)z^* + zx(1 - y)xz^*$, and both terms on the right are positive. ■

**Proof of Proposition 2.3.3.** As in the proof of Proposition 2.3.2, we can assume that $x \geq 0$, and that $p = \chi_{(1, \infty)}(x)$ is a projection of infinite rank. It suffices to show that there exist $a_0 \geq a_1 \geq \ldots \geq a_n \in B(A)_+$ so that $\|x(a_{k-1} - a_k)x\| > 2/3$ for $1 \leq k \leq n$. Indeed, if $S$ is compact, then there exist $u_1, \ldots, u_m \in B(H)$, so that for every $a \in B(A)_+$ there exists $j \in \{1, \ldots, m\}$ so that $\|Sa - u_j\| < 1/3$. By the pigeon-hole principle, if $n > m$, there exist $i < j$ in $\{1, \ldots, n\}$ and $k$ in $\{1, \ldots, m\}$, so that $\max\{\|Sa_i - u_k\|, \|Sa_j - u_k\|\} < 1/3$. However, $\|Sa_i - Sa_j\| \geq \|x(a_i - a_j)x\| > 2/3$, leading to a contradiction.

Imitating the proof of Proposition 2.3.2, we use the spectral decomposition of $x$ to find mutually orthogonal unit vectors $\xi_1, \ldots, \xi_n$ in ran $p$, so that (i) $x^k\xi_i$ is orthogonal to $x^\ell\xi_j$ for any $i \neq j$, and $k, \ell \in \{0, 1, \ldots\}$, and (ii) for any $i$, $1 = \|\xi_i\| \leq \|x^2\xi_i\| \leq \|x^2\xi_i\| \leq \ldots$. To construct $a_0, \ldots, a_n$, let $c = (2/3)^{1/(2n+1)}$, and let $\eta_i = x\xi_i/\|x\xi_i\|$. By Kaplansky Density Theorem, for $0 \leq k \leq n$ there exist $b_k \in B(A)_+$, so that

$$b_k\eta_i = \begin{cases} c\eta_i & 1 \leq i \leq n - k \\ 0 & i > n - k \end{cases}$$

(we can take $b_n = 0$). Let $a_0 = b_0$, $a_1 = b_0b_1b_0$, $a_2 = b_0b_1b_2b_1b_0$, etc. By Lemma 2.3.6, $a_0 \geq a_1 \geq \ldots \geq a_n$. Furthermore,

$$a_k\eta_i = \begin{cases} c^{2k-1}\eta_i & 1 \leq i \leq n - k \\ 0 & i > n - k \end{cases},$$

and therefore,

$$\|x(a_{k-1} - a_k)x\| \geq \langle x(a_{k-1} - a_k)x\xi_{n-k+1}, \xi_{n-k+1} \rangle$$

$$= \langle (a_{k-1} - a_k)\eta_{n-k+1}, \eta_{n-k+1} \rangle = c^{2k-1} > \frac{2}{3}.$$  

Therefore, the sequence $(a_k)_{k=0}^n$ has the desired properties. ■
2.4. Dunford-Pettis Schur multipliers. Recall that a map $T : S_E \to S_E$ is called a Schur (or Hadamard) multiplier if it can be written in the coordinate form, as $(Tx)_{ij} = \phi_{ij} x_{ij}$. The infinite matrix $\phi$ is called the symbol of $T$, which we denote by $S_{\phi}$. The main goal of this section is to prove:

**Theorem 2.4.1.** Suppose $0 \leq S_{\phi} \leq S_\psi$ are Schur multipliers from $S_1$ to $S_E$ ($E$ is a symmetric sequence space). If $S_\psi$ is Dunford-Pettis, then the same is true for $S_{\phi}$.

Recall that an operator is called Dunford-Pettis if it maps weakly null sequences to norm null ones. Equivalently, it carries relatively weakly compact sets to relatively norm compact sets. The reader is referred to e.g. [4, Section 5.4] for more information.

The proof relies on several technical lemmas, which may be known to experts.

**Lemma 2.4.2.** A bounded sequence $(x_n)$ in $S_1$ is weakly null if and only if the following two conditions are satisfied: (1) $\lim_m \sup_n \| R_m x_n \| = 0$, and (2) for every $m$, $\lim_n \| Q_m x_n \| = 0$.

*Proof.* Suppose first $(x_n)$ is weakly null. As $Q_m$ has finite rank, (2) must be satisfied. If (1) fails, then one can assume, by passing to a subsequence, that there exists $c > 0$, and a sequence $n_1 < n_2 < \ldots$, so that, for every $k$, $\| Q_{n_k+1} R_{n_k} x_k \| > c$, while $\| R_{n_k+1} x_k \| + \| Q_{n_k} x_k \| < 10^{-k} c$. Consider the block-diagonal truncation $P : S_1 \to S_1 : x \mapsto \sum_k Q_{n_k+1} R_{n_k} x$. Clearly, $P$ is contractive. Letting, for every $k$, $y_k = Q_{n_k+1} R_{n_k} x_k$, we see that $\| Px_k - y_k \| < 10^{-k} c$. Thus, for every sequence $(\alpha_k)$,

$$\| \sum_k \alpha_k x_k \| \geq \| \sum_k \alpha_k y_k \| - \sum_k |\alpha_k| \cdot 10^{-k} c > \frac{c}{2} \sum_k |\alpha_k|.$$ 

Thus, the sequence $(x_k)$ is equivalent to the canonical basis of $\ell_1$, hence not weakly null.

Now suppose (1) and (2) are satisfied for a bounded sequence $(x_n)$, and show that, for any $f \in B(\ell_2)$, $\lim_n f(x_n) = 0$. Indeed, otherwise, by passing to a subsequence, and by scaling, we can assume that $\sup_n \| x_n \| \leq 1$, and there exists $f \in B(\ell_2)$ so that $\inf_n |f(x_n)| > c$. Pick $m$ so that $\sup_n \| R_m x_n \| < c/5$. Note that there exists $M > m$ so that $\|(I - Q_M)(I - R_m)f\| < c/5$. Indeed,

$$(I - Q_M)(I - R_m)f = P_{M}^\perp f P_m + P_m f P_{M}^\perp.$$

For a fixed $m$, $B(\ell_2)P_m$ is isomorphic to a Hilbert space. For every $y \in B(\ell_2)P_m$, $P_M^1y \to 0$, hence $\lim_M P_M^1 f P_m = 0$. Similarly, $\lim_M P_m f P_M^1 = 0$.

Finally, pick $N$ so that, for $n > N$, $\|Q_M x_n\| < c/5$. As

$$
\langle f, x_n \rangle = \langle f, (R_m + (I - R_m)Q_M + (I - Q_M)(I - R_m))x_n \rangle \\
= \langle f, R_m x_n \rangle + \langle (I - R_m)f, Q_M x_n \rangle + \langle (I - Q_M)(I - R_m)f, x_n \rangle,
$$

we have, for $n > N$,

$$
c < |\langle f, x_n \rangle| \leq \|R_m x_n\| + 2\|Q_M x_n\| + \|(I - Q_M)(I - R_m)f\| < \frac{4c}{5},
$$
a contradiction.

**Corollary 2.4.3.** An operator $T : S_1 \to X$ is Dunford-Pettis if and only if, for every $i$, the restrictions of $T$ to span$[E_{ij} : j \in \mathbb{N}]$ and span$[E_{ji} : j \in \mathbb{N}]$ are compact.

Proof. Suppose the restrictions of $T$ to span$[E_{ij} : j \in \mathbb{N}]$ and span$[E_{ji} : j \in \mathbb{N}]$ are compact, and $(x_n)$ is a weakly null sequence in $S_1$. We have to show that, for every $c > 0$, $\|Tx_n\| < c$ for $n$ large enough. Without loss of generality, assume $T$ is a contraction, and $\sup_n \|x_n\| \leq 1$. Find $M > m$ so that $\sup_n \|R_m x_n\| < c/4$, and

$$
\|T\|_{\text{span}[E_{ij} : j > M]} + \|T\|_{\text{span}[E_{ji} : j > M]} < \frac{c}{4M}.
$$

Find $N \in \mathbb{N}$ so that $\sup_{n > N} \|Q_M x_n\| < c/4$. Thus, for $n > N$, $\|Tx_n\| < 3c/4$.

Conversely, suppose $T$ is Dunford-Pettis, but its restriction to span$[E_{ij} : j \in \mathbb{N}]$ is not compact. Then there exist $n_1 < n_2 < \ldots$, and $\alpha_j \in \mathbb{C}$, so that the vectors $x_k = \sum_{j=n_k+1}^{n_{k+1}} \alpha_j E_{ij}$, so that $\|x_k\| = 1$, and $\limsup_k \|Tx_k\| > 0$. However, the sequence $(x_k)$ is weakly null, while the sequence $(Tx_k)$ is not norm null, yielding a contradiction. The restrictions to span$[E_{ji} : j \in \mathbb{N}]$ are handled similarly.

Specializing the previous result to Schur multipliers, we immediately obtain:

**Corollary 2.4.4.** A Schur multiplier with the symbol $\phi$, acting from $S_1$ to $S_E$, is Dunford-Pettis if and only if, for any $i$, $\lim_j \phi_{ij} = \lim_j \phi_{ji} = 0$.

Proof. By Corollary 2.4.3, $S_\phi : S_1 \to S_E$ is Dunford-Pettis iff, for every $i$, the restrictions of $S_\phi$ to span$[E_{ij} : j \in \mathbb{N}]$ and span$[E_{ji} : j \in \mathbb{N}]$ are compact. By the definition, $S_\phi$ maps $E_{ij}$ to $\phi_{ij} E_{ij}$. It is well known that, for any $E$, span$[E_{ij} : j \in \mathbb{N}] \subset S_E$ is isometric to $\ell_2$, via an isometry sending the matrix
units $E_{ij}$ to the elements of the orthonormal basis. Thus, $S_{\phi}|_{\text{span}\{E_{ij}: j \in \mathbb{N}\}}$ is compact iff $\lim j \phi_{ij} = 0$. Similarly, $S_{\phi}|_{\text{span}\{E_{ji}: j \in \mathbb{N}\}}$ is compact iff $\lim j \phi_{ji} = 0$.

**Lemma 2.4.5.** Suppose $c > 0$ and $m \in \mathbb{N}$ satisfy $(mc)^2 > m + 1$. Suppose, furthermore, that $C$ and $D$ are positive matrices, with entries $C_{ij}$ and $D_{ij}$ ($0 \leq i, j \leq m$), respectively, so that $\max_i, j \{\max \{|C_{ij}|, |D_{ij}|\}\} \leq 1$, $|C_{0j}| > c$ for $1 \leq j \leq m$, and $|D_{ij}| < 10^{-2(i+j)}$ for $i \neq j$. Then the inequality $C \leq D$ cannot hold.

**Proof.** Suppose, for the sake of contradiction, that $D \geq C$. Then, for any vector $\xi \in \ell_2^{m+1}$,

$$\|D^{1/2}\xi\|^2 = \langle D^{1/2}\xi, D^{1/2}\xi \rangle = \langle D\xi, \xi \rangle = \langle C\xi, \xi \rangle = \|C^{1/2}\xi\|^2,$$

hence there exists a contraction $U$ so that $UD^{1/2}\xi = C^{1/2}\xi$. Thus, $C = D^{1/2}U^*UD^{1/2}$. By [59, Lemma 1.21], the block matrix $\begin{pmatrix} D & C \\ C & D \end{pmatrix}$ is positive.

Denote the canonical basis in $\ell_2^{m+1}$ by $(e_i)_{i=0}^m$. Consider the vector $\xi = \begin{pmatrix} \xi_1 \\ \xi_2 \end{pmatrix} \in \ell_2^{2(m+1)}$, where $\xi_1 = \alpha e_0$, and $\xi_2 = -\sum_{i=1}^m \omega_i e_i$. Here, $\omega_i = C_{i0}/|C_{i0}|$, and $\alpha = mc$. By the above,

$$0 \leq \begin{pmatrix} D & C \\ C & D \end{pmatrix} \xi, \xi \geq \langle D\xi_1, \xi_1 \rangle + \langle D\xi_2, \xi_2 \rangle + 2 \text{Re}(C\xi_1, \xi_2).$$

Note that $\langle D\xi_1, \xi_1 \rangle = \alpha^2 D_{00} \leq \alpha^2$, and

$$\langle D\xi_2, \xi_2 \rangle \leq \sum_{i=1}^m D_{ii} + 2 \sum_{1 \leq i < j \leq m} |D_{ij}| \leq m + 2 \sum_{1 \leq i < j \leq m} 10^{-2(i+j)} < m + 1.$$ 

On the other hand,

$$\langle C\xi_1, \xi_2 \rangle = -\alpha \sum_{i=1}^m C_{i0} \cdot \frac{C_{i0}}{|C_{i0}|} < -\alpha mc.$$

Returning to (2.1), we see that

$$\begin{pmatrix} D & C \\ C & D \end{pmatrix} \xi, \xi \leq \alpha^2 + m + 1 - 2\alpha mc < 0,$$

a contradiction.

**Proof of Theorem 2.4.1.** We say that an infinite matrix $\phi$ is formally positive if each of its finite submatrices is positive. By [42, Theorem 3.7], $S_{\sigma} \succ 0$ iff $\sigma$ is formally positive.
Suppose, for the sake of contradiction, that $0 \leq S_\phi \leq S_\psi$, where $S_\psi$ is Dunford-Pettis, while $S_\phi$ is not. We can assume that $S_\psi$ is contractive, hence, for any $(i,j)$, $\max\{|\phi_{ij}|,|\psi_{ij}|\} \leq 1$. Corollary 2.4.4 shows that, for any $i$, $\lim_{j \to \infty} \psi_{ij} = 0$. By rearranging rows and columns if necessary, we can assume the existence of $n_0 < n_1 < n_2 < \ldots$, so that $|\phi_{n_0 n_k}| > c > 0$. Passing to further subsequence, we obtain $|\psi_{n_i n_j}| < 10^{-2(i+j)}$ for $i \neq j$.

Now select $m$ so that $mc > 4(m + 1)$, and define matrices $C$ and $D$, with entries $C_{ij} = \phi_{n_i n_j}$ and $D_{ij} = \psi_{n_i n_j}$ ($0 \leq i, j \leq m$), respectively. As noted above, the matrices $C$ and $D$ are positive. By Lemma 2.4.5, we cannot have $C \leq D$. Thus, a contradiction.

2.5. Weakly compact operators. In this section, we show that, under certain conditions, weak compactness is inherited under domination. First consider operators on $C^*$-algebras and their duals.

Theorem 2.5.1. Suppose $E$ is an OBS, and $\mathcal{A}$ is a $C^*$-algebra, $S$ is a weakly compact operator, and one of the following holds:

1. $E$ is generating, and $0 \leq T \leq S : E \to \mathcal{A}^*$.
2. $E$ is normal, and $0 \leq T \leq S : \mathcal{A} \to E$.

Then $T$ is weakly compact.

Note that, for commutative $\mathcal{A}$, this theorem follows from [50], and the order continuity of $\mathcal{A}^*$.

Proof. (1) Suppose, for the sake of contradiction, that $T(B(E)_+) \not\subset$ is not weakly compact. By Pfitzner’s Theorem [43], there exist a bounded sequence $(a_n) \subset \mathcal{A}$ of positive pairwise orthogonal elements, a sequence $(\phi_n) \subset B(E)_+$, and $c > 0$, such that $T\phi_n(a_n) > c$. Therefore, $S\phi_n(a_n) > c$, which contradicts the weak compactness of $S(B(E))$ (once again, by Pfitzner’s Theorem).

(2) Apply part (1) to $0 \leq T^* \leq S^*$.

Remark 2.5.2. Theorem 2.5.1 fails for operators from duals of $C^*$-algebras to $C^*$-algebras, even in the commutative setting. Indeed, by [4, Theorem 5.31], there exist $0 \leq T \leq S : \ell_1 \to \ell_\infty$, so that $S$ is weakly compact, whereas $T$ is not.

For operators to or from general Banach lattices, we have:

Theorem 2.5.3. Suppose either (i) $A$ is a generating OBS, and $B$ is order continuous Banach lattice, or (ii) $A$ is a Banach lattice with order continuous
dual, and \( B \) is an normal OBS. If \( 0 \leq T \leq S : A \to B \), and \( S \) is weakly compact, then \( T \) is weakly compact as well.

**Proof.** The proof of \((i)\) is contained in the first few lines of the proof of [4, Theorem 5.31]. \((ii)\) follows by duality.

Next we obtain a partial generalization of the above results for non-commutative function spaces. In the discrete case, we obtain a characterization of order continuous Banach lattices.

**Proposition 2.5.4.** Suppose \( \mathcal{E} \) is a symmetric sequence space, containing a copy of \( \ell_1 \), \( H \) is an infinite dimensional Hilbert space, and \( X \) is a Banach lattice. Then the following are equivalent:

1. If \( 0 \leq T \leq S : \mathcal{S}_\mathcal{E}(H) \to X \), and \( S \) is weakly compact, then \( T \) is weakly compact.
2. \( X \) is order continuous.

**Proof.** \((2) \Rightarrow (1)\) follows from Theorem 2.5.3.

\((1) \Rightarrow (2): \) By Proposition 1.2.6 \( \mathcal{S}_\mathcal{E}(H) \) contains a positive disjoint sequence, that spans a positively complemented copy of \( \ell_1 \). Hence, the result follows from [4, Theorem 5.31].

Now consider domination by a weakly compact operator for non-commutative function spaces.

Recall that a non-commutative symmetric function space \( \mathcal{E} \) is said to have the **Fatou Property** (sometimes referred to as the **Beppo Levi Property**) if for any norm-bounded increasing net \( (x_i) \subset \mathcal{E}_+ \), there exists \( x \in \mathcal{E} \) so that \( x_i \uparrow x \), and \( \|x\| = \sup_i \|x_i\| \). In the commutative setting, any symmetric space with the Fatou Property is order complete.

We say that a non-commutative function space \( \mathcal{E} \) is a **KB space** if any increasing norm bounded sequence in \( \mathcal{E} \) is norm-convergent. Equivalently, \( \mathcal{E} \) is order continuous, and has the Fatou Property (see [21]). Furthermore, the following are equivalent: (i) \( \mathcal{E} \) is a KB space, (ii) \( \mathcal{E} \) is weakly sequentially complete, and (iii) \( \mathcal{E} \) contains no copy of \( c_0 \). It is clear from [18] that, if \( \mathcal{E} \) is symmetric KB function space, then the same is true of \( \mathcal{E}(\tau) \).

The following result is contained in [18, Section 5].

**Proposition 2.5.5.** Suppose \( \mathcal{E} \) is a non-commutative strongly symmetric function space. Then:
(1) $\mathcal{E}^\times$ is strongly symmetric.
(2) $\mathcal{E}^\times$ coincides with $\mathcal{E}^*$ if and only if $\mathcal{E}$ is order continuous. In this case, for every $f \in \mathcal{E}^*$ there exists a unique $y \in \mathcal{E}^\times$ so that $f(x) = \tau(xy)$, for any $x \in \mathcal{E}$.
(3) $\mathcal{E}$ coincides with $\mathcal{E}^{\times\times}$ if and only if $\mathcal{E}$ has the Fatou Property.

**Proposition 2.5.6.** Suppose $\mathcal{E} = \mathcal{E}(\tau)$ is a non-commutative strongly symmetric $KB$ function space, $X$ a generating OBS, and $0 \leq T \leq S : X \to \mathcal{E}$, with $S$ weakly compact. Then $T$ is weakly compact as well.

**Proof.** By [18, Section 5], any positive element $\phi \in \mathcal{E}^{**} = (\mathcal{E}^\times)^*$ can be written as $\phi(f) = \tau(af) + \psi(f)$, where $a \in \mathcal{E}$ is positive, and $\psi$ is a positive singular functional. The canonical embedding of $\mathcal{E}$ into its double dual takes $a$ to the linear functional $f \mapsto \tau(fa)$.

$S$ is weakly compact, hence $S^{**}(X) \subset \mathcal{E}$. A normal functional cannot dominate a singular one, hence $T^{**}(\mathcal{B}(X^{**})_+) \subset \mathcal{E}$. As noted in Section 1.1, $X^{**}$ is a generating OBS, hence $T^{**}(\mathcal{B}(X^{**})) \subset \mathcal{E}$. Therefore, $T$ is weakly compact.

Alternatively, one can prove the above result using the characterization of $\sigma(\mathcal{F}, \mathcal{F})$-compact sets given in [19, Proposition 6.2].

**Remark 2.5.7.** Note that the assumptions of Proposition 2.5.6 are stronger than those of its commutative counterpart – Theorem 2.5.3. For instance, the statement of Theorem 2.5.3(i) holds when the range space is order continuous. Propositions 2.5.6 is proved under the additional assumption of the Fatou property. One reason for this is that much more is known about order continuous Banach lattices (see e.g. [39, Section 2.4]). One useful characterization states that a Banach lattice $\mathcal{E}$ is order continuous iff it is an ideal in its second dual. No such description seems to be known in the non-commutative setting.

3. Miscellaneous results

3.1. 2-positivity and decomposability: negative results. In this section we consider stronger versions of positivity, such as 2-positivity and indecomposability, as well as the appropriate notions of domination. We show that these properties are not, in general, inherited by the dominated operator.

**Proposition 3.1.1.** (a) There are $0 \leq T \leq c_S$, acting on $M_2$, so that $S$ is completely positive, but $T$ is not 2-positive.
(b) There are \( 0 \leq T \leq_{c} S \), acting on \( M_3 \), so that \( S \) is completely positive, but \( T \) is not decomposable.

For the definition and basic properties of decomposable maps, see e.g. [48]. Note that part (b) is optimal in the sense that any positive map from \( M_2 \) to \( M_3 \) is decomposable [54].

In the proof below, we use the notation \( E_{ij} \) for the matrix with 1 in the \((i,j)\) position, and 0’s elsewhere.

**Proof.** (a) Define \( T(a) = a^{t} \), and \( S(a) = \text{tr}(a)1 \) \((\text{tr}(\cdot) \) stands for the canonical trace on \( M_2 \)). Clearly, \( T \geq 0 \), and \( S \) is completely positive. Indeed, consider \( a = \sum_{i,j=1}^{n} E_{ij} \otimes a^{(ij)} \in M_n(M_2) \geq 0 \) (here, \( a^{(ij)} = (a_{k\ell}^{ij})_{k,\ell=1}^{2} \in M_2 \)). Passing to submatrices, we see that for \( k = 1, 2 \), the \( n \times n \) matrix \( a'_k = (a_{kk}^{(ij)}) \) is positive. Thus, \((I_{M_n} \otimes S)a = (a'_1 + a'_2) \otimes (E_{11} + E_{22}) \geq 0 \).

The fact that \( T \) is not 2-positive is folklore: just apply \( I_{M_2} \otimes T \) to \( \sum_{i,j=1}^{2} E_{ij} \otimes E_{ij} \). To establish that \( S - T \geq_{cp} 0 \), note that \((S - T)(a) = uu^{*} \), where \( u = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \).

(b) Define

\[
U \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & -a_{12} & -a_{13} \\ -a_{21} & a_{22} & -a_{23} \\ -a_{31} & -a_{32} & a_{33} \end{pmatrix},
\]

\[
V \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{33} & 0 & 0 \\ 0 & a_{11} & 0 \\ 0 & 0 & a_{22} \end{pmatrix},
\]

\[
W \begin{pmatrix} a_{11} & a_{12} & a_{13} \\ a_{21} & a_{22} & a_{23} \\ a_{31} & a_{32} & a_{33} \end{pmatrix} = \begin{pmatrix} a_{11} & 0 & 0 \\ 0 & a_{22} & 0 \\ 0 & 0 & a_{33} \end{pmatrix}.
\]

Let \( T = U + V \), and \( S = V + 2W \). By [48], \( T \) is positive, but not decomposable. On the other hand, the maps \( V \) and \( W \) are completely positive, hence so is \( S \). Furthermore, \( S - T = I \) (the identity map on \( M_3 \)), hence it is completely positive as well.

For powers of operators, we get:

**Proposition 3.1.2.** There are \( 0 \leq T \leq_{c} S \), acting on \( M_2 \), so that \( S \) is completely positive, while \( T \) is not 2-positive, and \( T = T^2 \).
Proof. Define $T(a) = (a + a^t)/2$, and $S(a) = (\text{tr}(a)1 + a)/2$. As in the proof of Proposition 3.1.1, we can establish the inequalities $0 \leq S$ and $0 \leq T \leq cS$. Clearly, $T = T^2$. To show that $T$ is not 2-positive, consider $x = \sum_{i,j=1}^{2} E_{ij} \otimes E_{ij} \in M_2 \otimes M_2$. $x$ can be identified with the $4 \times 4$ matrix

$$
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.
$$

Then

$$(I_{M_2} \otimes T)(x) = \frac{1}{2} \begin{pmatrix}
2 & 0 & 0 & 1 \\
0 & 0 & 1 & 0 \\
0 & 1 & 0 & 0 \\
1 & 0 & 0 & 2
\end{pmatrix},$$

which is not positive.

Remark 3.1.3. It is not clear whether we can strengthen Proposition 3.1.1(b) to make the powers of $T$ (not just $T$ itself) non-decomposable. The operator $T$ presented in the proof of Proposition 3.1.1(b) will not work, since $T^2$ is completely positive. Indeed, [48] shows that $T = U + \mu V$ is not decomposable for $\mu \geq 1$. However, $U^2 = I$, and $UV = VU = V$. Thus, $T^2 = I + 2\mu V + \mu^2 V^2$, which is completely positive.

3.2. A remark on operator systems. In previous section, we were working with non-commutative function spaces, or with $C^*$-algebras. This brief section shows that general operator systems have too few positive elements for any results about domination and inheritance of properties.

Recall that an operator system is a subspace of $B(H)$, closed under conjugation. It is unital if it contains $1$. If $A$ and $B$ are operator systems, and $T : A \to B$, we say that $T$ is positive ($T \geq 0$) if $Ta \geq 0$ for any $a \geq 0$. Moreover, $T$ is completely positive ($T \geq_c 0$) if $T \otimes I_{M_n} \geq 0$ for every $n$. Write $T \geq S$ ($T \geq_c S$) if $T - S \geq 0$ (resp. $T - S \geq_c 0$).

It turns out that little can be said about domination in operator systems. More precisely, there exists a unital operator system $A$, and a rank 1 $S \in B(A)$, so that $I_A \leq_c S$. $A$ may be chosen to be infinite dimensional, and even non-separable. We describe the construction of $A$ and $S$ below.

Suppose $X \subset B(H)$ is an operator system (not necessarily unital). Using “Paulsen’s trick”, define $A$ as the set of all block matrices on $H \oplus_2 H$, 

\[ 
\begin{pmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{pmatrix}.
\]
of the form $\begin{pmatrix} \lambda & x \\ y & \lambda \end{pmatrix}$, where $\lambda \in \mathbb{C}$, and $x, y \in X$. It is easy to see that $\begin{pmatrix} \lambda & x \\ y & \lambda \end{pmatrix} \geq 0$ iff $x = y^*$, and $\lambda \geq \|x\|$. Set $S \begin{pmatrix} \lambda & x \\ y & \lambda \end{pmatrix} = 2 \begin{pmatrix} \lambda & 0 \\ 0 & \lambda \end{pmatrix} = 2\lambda 1_{H \oplus 2H}$.

**Proposition 3.2.1.** In the above notation, $S \geq c I_A$.

**Proof.** It suffices to observe that

$$(S - I_A) \begin{pmatrix} \lambda & x \\ y & \lambda \end{pmatrix} = \begin{pmatrix} \lambda & -x \\ -y & \lambda \end{pmatrix} = u \begin{pmatrix} \lambda & x \\ y & \lambda \end{pmatrix} u,$$

with $u = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$.

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**References**

[1] J. Alexander, ‘Compact Banach algebras’, *Proc. London Math. Soc.* 18 (1968), 1–18.

[2] C.D. Aliprantis and O. Burkinshaw, ‘Positive compact operators on Banach Lattices’, *Math Z.* 174 (1980), 289–298.

[3] C.D. Aliprantis and O. Burkinshaw, ‘On weakly compact operators on Banach lattices’, *Proc. Amer. Math. Soc.* 274 (1981), 573–578.

[4] C.D. Aliprantis and O. Burkinshaw, *Positive operators*, Springer, Dordrecht (2006).

[5] M. Almus, D. Blecher, and C. Read, ‘Ideals and hereditary subalgebras in operator algebras’, preprint.

[6] T. Ando, ‘On fundamental properties of a Banach space with a cone’, *Pacific J. Math.* 12 (1962), 1163–1169.

[7] J. Arazy, ‘Basic Sequences, embeddings, and the uniqueness of the symmetric structure in unitary matrix spaces’, *J. Funct. Anal.* 40 (1980), 302–340.

[8] C. Batty and D. Robinson, ‘Positive One-Parameter Semigroups on Ordered Banach Spaces’, *Acta Applicandae Mathematicae* 1 (1984), 221–296.

[9] B. Blackadar, *Operator algebras*, Springer-Verlag, Berlin (2006).

[10] M. Bresar and Yu. Turovskii, ‘Compactness conditions for elementary operators’, *Studia Math.* 178 (2007), 1–18.

[11] Z. Chen and A. Wickstead, ‘Positive weak compactness of solid hulls in Banach lattices’, *Indag. Math.* 9 (1998), 187–196.

[12] V. Chilin and F. Sukochev, ‘Weak convergence in non-commutative symmetric spaces’, *J. Operator Theory* 31 (1994), 35–65.

[13] K. Davidson, *C*-algebras by example*, Amer. Math. Soc., Providence, RI (1996).

[14] J. Dixmier, *C*-algebras, North Holland, Amsterdam (1977).

[15] J. Dixmier, *von Neumann algebras*, North Holland, Amsterdam (1981).

[16] P. G. Dodds and D.H. Fremlin, ‘Compact operators in Banach lattices’, *Israel J. Math.* 34 (1979), 287–320.

[17] P. Dodds, T. Dodds, and B. de Pagter, ‘Fully symmetric operator spaces’, *Int. Eq. Op. Theory* 15 (1992) 942–972.
[18] P. Dodds, T. Dodds, and B. de Pagter, ‘Non-commutative Köthe duality’, Trans. Amer. Math. Soc. 339 (1993) 717–750.

[19] P. Dodds and B. de Pagter, ‘Non-commutative Yosida-Hewitt theorems and singular functionals in symmetric spaces of $\tau$-measurable operators’, in Vector measures, integration and related topics, Oper. Theory Adv. Appl. 201 (2010), 183–198.

[20] P. Dodds and B. de Pagter, ‘Completely positive compact operators on non-commutative symmetric spaces’, Positivity 14 (2010) 665–679.

[21] P. Dodds and B. de Pagter, ‘Properties $(u)$ and $(V^*)$ of Pelczynski in symmetric spaces of $\tau$-measurable operators’, Positivity 14 (2011) 571–594.

[22] P. Dodds and B. de Pagter, ‘The non-commutative Yosida-Hewitt Theorem revisited’, Trans. Amer. Math. Soc., to appear.

[23] T. Fack and H. Kosaki, ‘Generalized $s$-numbers of $\tau$-measurable operators’, Pacific J. Math. 123 (1986), 269–300.

[24] D. Farenick, ‘Irreducible positive linear maps on operator algebras’, Proc. Amer. Math. Soc. 124 (1996), 3381–3390.

[25] J. Flores, F. L. Hernandez, and P. Tradacete, ‘Powers of operators dominated by strictly singular operators’, Q. J. Math. 59 (2008), 321–334.

[26] Y. Friedman, ‘Subspace of $LC(H)$ and $C_p$', Proc. Amer. Math. Soc. 53 (1975), 117–122.

[27] I. C. Gohberg and M. G. Krein, Introduction to the theory of linear nonselfadjoint operators, Amer. Math. Soc., Providence, RI (1969).

[28] T. Huruya, ‘A Spectral Characterization of a class of $C^*$-algebras’, Sci. Rep. Niigata Univ. Ser. A 15 (1978), 21–24.

[29] H. Jensen, ‘Scattered $C^*$-algebras’, Math. Scand. 41 (1977), 308–314.

[30] N. Kalton and P. Saab, ‘Ideal properties of regular operators between Banach lattices’, Illinois J. Math. 29 (1985), 382–400.

[31] N. Kalton and F. Sukochev, ‘Symmetric norms and spaces of operators’, J. Reine Angew. Math. 621 (2008), 81–121.

[32] R. Kadison and J. Ringrose, Fundamentals of the theory of operator algebras II, Amer. Math. Soc., Providence, RI (1997).

[33] A. Kaminska and M. Mastylo, ‘The Dunford-Pettis property for symmetric spaces’, Canad. J. Math. 52 (2000), 789–803.

[34] A. Kaminska and M. Mastylo, ‘The Schur and (weak) Dunford-Pettis properties in Banach lattices’, J. Aust. Math. Soc. 73 (2002), 251–278.

[35] S. Krein, Yu. Petunin, and E. Semenov, Interpolation of linear operators, Translated from the Russian by J. Szücs, Translations of Mathematical Monographs, 54, Amer. Math. Soc., Providence, RI (1984).

[36] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces I, Springer-Verlag, Berlin (1977).

[37] J. Lindenstrauss and L. Tzafriri, Classical Banach spaces II, Springer-Verlag, Berlin (1979).

[38] J. Marcolino Nhany, ‘La stabilité des espaces $L_p$ non-commutatifs’, Math. Scand. 81 (1997), 212–218.

[39] P. Meyer-Nieberg, Banach lattices, Springer-Verlag, Berlin (1991).

[40] T. Murphy, ‘Continuity of positive linear functionals on Banach $*$-algebras’, Bull. London Math. Soc. 1 (1969), 171–173.

[41] B. de Pagter, ‘Non-commutative Banach function spaces’, Positivity, Trends Math., Birkhäuser, Basel (2007), 197–227.
[42] V. Paulsen, *Completely bounded maps and operator algebras*, Cambridge University Press (2002).
[43] H. Pfitzner, ‘Weak compactness in the dual of a $C^*$-algebra is determined commutatively’, *Math. Ann.* 298 (1994), 349–371.
[44] G. Pisier, ‘$L_p$ spaces’, *Asterisque* (1998).
[45] H. H. Schaefer, *Banach lattices and positive operators*, Springer, Berlin (1974).
[46] B. Simon, *Trace ideals and their applications. Second edition*, Amer. Math. Soc., Providence, RI (2005).
[47] E. Spinu, ‘Dominated inessential operators’, *J. Math. Anal. Appl.* 383 (2011), 250–264.
[48] E. Stormer, ‘Decomposable positive maps on $C^*$-algebras’, *Proc. Amer. Math. Soc.* 86 (1982), 402–404.
[49] M. Takesaki, *Theory of operator algebras I*, Springer-Verlag, New York (2003).
[50] A. Wickstead, ‘Extremal structure of cones of operators’, *Quart. J. Math. Oxford Ser.* (2) 126 (1981), 239–253.
[51] A. Wickstead, ‘Converses for the Dodds-Fremlin and Kalton-Saab theorems’, *Math. Proc. Cambridge Philos. Soc.* 120 (1996), 175–179.
[52] W. Wnuk, ‘A note on the positive Schur property’, *Glasgow Math. J.* 31 (1989), 169–172.
[53] P. Wojtaszczyk, ‘On linear properties of separable conjugate spaces of $C^*$-algebras’, *Studia Math.* 52 (1974), 143–147.
[54] S. Woronowicz, ‘Positive maps of low dimensional matrix algebras’, *Rep. Math. Phys.* 10 (1976), 165–183.
[55] K. Ylinen, ‘Compact and finite dimensional elements of normed algebras’, *Ann. Acad. Sci. Fenn. Ser. A I Math.*, 408 (1968).
[56] K. Ylinen, ‘A note on the compact elements of $C$-algebras’, *Proc. Amer. Math. Soc.* 35 (1972), 305–306.
[57] K. Ylinen, ‘Weakly compact continuous elements of $C^*$-algebras’, *Proc. Amer. Math. Soc.* 52 (1975), 323–326.
[58] K. Ylinen, ‘Dual $C^*$-algebras, weakly semi-completely continuous elements, and the extreme rays of the positive cone’, *Ann. Acad. Sci. Fenn. Ser. A I Math.* 599 (1975).
[59] X. Zhan, *Matrix inequalities*, Springer-Verlag, New York (2002).

DEPT. OF MATHEMATICS, UNIVERSITY OF ILLINOIS AT URBANA-CHAMPAIGN, URBANA IL 61801, USA
E-mail address: oikhberg@illinois.edu

DEPT. OF MATHEMATICAL AND STATISTICAL SCIENCES, UNIVERSITY OF ALBERTA EDMONTON, ALBERTA T6G 2G1, CANADA
E-mail address: espinu@ualberta.ca