Convergence Rate of Euler-Maruyama Scheme for SDDEs of Neutral Type

Yanting Ji,* Jianhai Bao,† Chenggui Yuan ‡

Abstract

In this paper, we are concerned with convergence rate of Euler-Maruyama (EM) scheme for stochastic differential delay equations (SDDEs) of neutral type, where the neutral term, the drift term and the diffusion term are allowed to be of polynomial growth. More precisely, for SDDEs of neutral type driven by Brownian motions, we reveal that the convergence rate of the corresponding EM scheme is one half; Whereas for SDDEs of neutral type driven by jump processes, we show that the best convergence rate of the associated EM scheme is close to one half.

AMS subject Classification: 65C30, 60H10.

Key Words: stochastic differential delay equation of neutral type, polynomial condition, Euler scheme, convergence rate, jump processes.

1 Introduction

There is numerous literature concerned with convergence rate of numerical schemes for stochastic differential equations (SDEs). Under a Hölder condition, Gyöngy-Rásonyi [8] provided a convergence rate of EM scheme; Under the Khasminskii-type condition, Mao [11] revealed that the convergence rate of the truncated EM method is close to one half; Sabanis [16] recovered the classical rate of convergence (i.e., one half) for the tamed Euler schemes, where, for the SDE involved, the drift coefficient satisfies a one-side Lipschitz condition and a polynomial Lipschitz condition, and the diffusion term is Lipschitzian. There is also some literature on convergence rate of numerical schemes for stochastic functional differential equations (SFDEs). For example, under a log-Lipschitz condition, Bao et al. [4] studied convergence rate of EM approximation for a range of SFDEs driven by jump processes; Bao-Yuan [3] investigated convergence rate of EM approach for a class of SDDEs, where the drift and diffusion coefficients are allowed to be polynomial growth with respect to the delay variables; Gyöngy-Sabanis [7] discussed rate of almost sure convergence of Euler approximations for SDDEs under monotonicity conditions.

*Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK, mathjyt@gmail.com
†School of Mathematics and Statistics, Central South University, China, jianhaibao13@gmail.com
‡Department of Mathematics, Swansea University, Singleton Park, SA2 8PP, UK, C.Yuan@swansea.ac.uk
Increasingly real-world systems are modelled by SFDEs of neutral type, as they represent systems which evolve in a random environment and whose evolution depends on the past states and derivatives of states of the systems through either memory or time delay. In the last decade, for SFDEs of neutral type, there are a large number of papers on, e.g., stochastic stability (see, e.g., [5][12][13], on large fluctuations (see, e.g., [1]), on large deviation principle (see, e.g., [4]), on transportation inequality (see, e.g., [2]), to name a few.

So far, the topic on numerical approximations for SFDEs of neutral type has also been investigated considerably. For instance, under a global Lipschitz condition, Wu-Mao [18] revealed convergence rate of the EM scheme constructed is close to one half; under a log-Lipschitz condition, Jiang et al. [9] generalized [20] by Yuan and Mao to the neutral case; under the Khasminskii-type condition, following the line of Yuan-Glover [21], [15][22] studied convergence in probability of the associated EM scheme; for preserving stochastic stability under the global Lipschitz condition, where, in particular, the neutral term is contractive. Consider the following SDDE of neutral type

\[ d\{X(t) - X^2(t - \tau)\} = \{aX(t) + bX^3(t - \tau)\}dt + cX^2(t - \tau)dB(t), \quad t \geq 0, \quad (1.1) \]

in which \(a, b, c \in \mathbb{R}, \tau > 0\) is some constant, and \(B(t)\) is a scalar Brownian motion. Observe that all the neutral, drift and diffusion coefficients in (1.1) are highly linear with respect to the delay variable so that the existing results on convergence rate of EM schemes associated with SFDEs of neutral type cannot be applied to the example above. So in this paper we intend to establish the theory on convergence rate of EM scheme for a class of SDDEs of neutral type, where, in particular, the neutral term is of polynomial growth, so that it covers more interesting models.

Throughout the paper, the shorthand notation \(a \lesssim b\) is used to express that there exists a positive constant \(c\) such that \(a \leq cb\), where \(c\) is a generic constant whose value may change from line to line. Let \((\Omega, \mathcal{F}, \mathbb{P})\) be a complete probability space with a filtration \((\mathcal{F}_t)_{t \geq 0}\) satisfying the usual conditions (i.e., it is right continuous and \(\mathcal{F}_0\) contains all \(\mathbb{P}\)-null sets). For each integer \(n \geq 1\), let \((\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)\) be an \(n\)-dimensional Euclidean space. For \(A \in \mathbb{R}^n \otimes \mathbb{R}^m\), the collection of all \(n \times m\) matrices, \(\|A\|\) stands for the Hilbert-Schmidt norm, i.e., \(\|A\| = (\sum_{i=1}^{m} |Ae_i|^2)^{1/2}\), where \((e_i)_{i \geq 1}\) is the orthogonal basis of \(\mathbb{R}^m\). For \(\tau > 0\), which is referred to as delay or memory, \(\mathcal{C} := C([-\tau, 0]; \mathbb{R}^n)\) means the space of all continuous functions \(\phi: [-\tau, 0] \mapsto \mathbb{R}^n\) with the uniform norm \(\|\phi\|_\infty := \sup_{-\tau \leq \theta \leq 0} |\phi(\theta)|\). Let \((B(t))_{t \geq 0}\) be a standard \(m\)-dimensional Brownian motion defined on the probability space \((\Omega, \mathcal{F}, (\mathcal{F}_t)_{t \geq 0}, \mathbb{P})\).

To begin, we focus on an SDDE of neutral type on \((\mathbb{R}^n, \langle \cdot, \cdot \rangle, |\cdot|)\) in the form

\[ d\{X(t) - G(X(t - \tau))\} = b(X(t), X(t - \tau))dt + \sigma(X(t), X(t - \tau))dB(t), \quad t > 0 \quad (1.2) \]

with the initial value \(X(\theta) = \xi(\theta)\) for \(\theta \in [-\tau, 0]\), where \(G: \mathbb{R}^n \mapsto \mathbb{R}^n, b: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^n, \sigma: \mathbb{R}^n \times \mathbb{R}^n \mapsto \mathbb{R}^{n \times m}\).

We assume that there exist constants \(L > 0\) and \(q \geq 1\) such that, for \(x, y, \overline{x}, \overline{y} \in \mathbb{R}^n\),
(A1) \(|G(y) - G(\bar{y})| \leq L(1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|;\)

(A2) \(|b(x, y) - b(\bar{x}, \bar{y})| + \|\sigma(x, y) - \sigma(\bar{x}, \bar{y})\| \leq L|x - \bar{x}| + L(1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|,\) where \(\| \cdot \|\) stands for the Hilbert-Schmidt norm;

(A3) \(|\xi(t) - \xi(s)| \leq L|t - s|\) for any \(s, t \in [-\tau, 0].\)

Remark 1.1. There are some examples such that (A1) and (A2) hold. For instance, if 
\(G(y) = y^2, b(x, y) = \sigma(x, y) = ax + y^3\) for any \(x, y \in \mathbb{R}\) and some \(a \in \mathbb{R},\) then both (A1) and (A2) hold by taking \(V(y, \bar{y}) = 1 + \frac{3}{2}y^2 + \frac{3}{2}\bar{y}^2\) for arbitrary \(y, \bar{y} \in \mathbb{R}.\)

By following a similar argument to [12, Theorem 3.1, p.210], (1.2) has a unique solution \(\{X(t)\}\) under (A1) and (A2). In the sequel, we introduce the EM scheme associated with (1.2). Without loss of generality, we assume that \(h = T/M = \tau/m \in (0, 1)\) for some integers \(M, m > 1.\) For every integer \(k = -m, \cdots, 0,\) set \(Y_h^{(k)} := \xi(kh),\) and for each integer \(k = 1, \cdots, M - 1,\) we define
\[
Y_h^{(k+1)} - G(Y_h^{(k+1-m)}) = Y_h^{(k)} - G(Y_h^{(k-m)}) + b(Y_h^{(k)}, Y_h^{(k-m)})h + \sigma(Y_h^{(k)}, Y_h^{(k-m)})\Delta B_h^{(k)}, \quad (1.3)
\]
where \(\Delta B_h^{(k)} := B((k+1)h) - B(kh).\) For any \(t \in [kh, (k+1)h),\) set \(Y_h(t) := Y_h^{(k)}.\) To avoid the complex calculation, we define the continuous-time EM approximation solution \(Y(t)\) as below: For any \(\theta \in [-\tau, 0],\) \(Y(\theta) = \xi(\theta),\) and
\[
Y(t) = G(\overline{Y}(t - \tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t b(\overline{Y}(s), \overline{Y}(s - \tau))ds \quad (1.4)
\]
\[+ \int_0^t \sigma(\overline{Y}(s), \overline{Y}(s - \tau))dB(s), \quad t \in [0, T].\]

A straightforward calculation shows that the continuous-time EM approximate solution \(Y(t)\) coincides with the discrete-time approximation solution \(\overline{Y}(t)\) at the grid points \(t = nh.\)

The first main result in this paper is stated as below.

**Theorem 1.1.** Under the assumptions (A1)-(A3),
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t) - Y(t)|^p\right) \lesssim h^{p/2}, \quad p \geq 2. \quad (1.5)
\]
So the convergence rate of the EM scheme (i.e., (1.4)) associated with (1.2) is one half.

Next, we move to consider the convergence rate of EM scheme corresponding to a class of SDDEs of neutral type driven by pure jump processes. More precisely, we consider an SDDEs of neutral type
\[
d\{X(t) - G(X(t - \tau))\} = b(X(t), X(t - \tau))dt + \int_U g(X(t - \tau), X((t - \tau) -), u)\tilde{N}(du, dt) \quad (1.6)
\]
with the initial data \(X(\theta) = \xi(\theta), \theta \in [-\tau, 0].\) Herein, \(G\) and \(b\) are given as in (1.2), \(g : \mathbb{R}^n \times \mathbb{R}^n \times U \mapsto \mathbb{R}^m,\) where \(U \in \mathcal{B}(\mathbb{R});\) \(\tilde{N}(dt, du) := N(dt, du) - dt\lambda(du)\) is the compensated
Poisson measure associated with the Poisson counting measure \( N(dt, du) \) generated by a stationary \( F_t \)-Poisson point process \( \{ p(t) \}_{t \geq 0} \) on \( \mathbb{R} \) with characteristic measure \( \lambda(\cdot) \), i.e.,

\[
N(t, U) = \sum_{s \in D(p), s \leq t} I_U(p(s)) \text{ for } U \in \mathcal{B}(\mathbb{R}); \quad X(t) := \lim_{\tau \downarrow 0} X(s).
\]

We assume that \( b \) and \( G \) such that (A1) and (A2) with \( \sigma \equiv 0_{n \times m} \) therein. We further suppose that there exist \( L_0, r > 0 \) such that for any \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \) and \( u \in U \),

(A4) \[ |g(x, y, u) - g(\bar{x}, \bar{y}, u)| \leq L_0(|x - \bar{x}| + (1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|)|u|^r \quad \text{and} \quad |g(0, 0, u)| \leq |u|^r, \]

where \( q \geq 1 \) is the same as that in (A1).

(A5) \[ \int_U |u|^p \lambda(du) < \infty \text{ for any } p \geq 2. \]

Remark 1.2. The jump coefficient may also be highly non-linear with respect to the delay argument; for example, \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \) and \( q \geq 1 \), \( g(x, y, u) = (x + y^q)u \) satisfies (A5).

By following the procedures of (1.3) and (1.4), the discrete-time EM scheme and the continuous-time EM approximation associated with (1.6) are defined respectively as below:

\[ Y_{h}^{(n+1)} - Y_{h}^{(n+1-m)} = Y_{h}^{(n)} - G(Y_{h}^{(n-m)}) + b(Y_{h}^{(n)}, Y_{h}^{(n-m)})h + g(Y_{h}^{(n)}, Y_{h}^{(n-m)}, u) \Delta \tilde{N}_{nh}, \]

where \( \Delta \tilde{N}_{nh} := \tilde{N}((n+1)h, U) - \tilde{N}(nh, U) \), and

\[
Y(t) = G(\bar{Y}(t - \tau)) + \xi(0) - G(\xi(-\tau)) + \int_0^t b(\bar{Y}(s), \bar{Y}(s - \tau))ds + \int_0^t \int_U g(\bar{Y}(s - \tau), \bar{Y}((s - \tau)-), u) \tilde{N}(du, ds), \tag{1.8}
\]

where \( \bar{Y} \) is defined similarly as in (1.4).

Our second main result in this paper is presented as follows.

**Theorem 1.2.** Under(A1)-(A5) with \( \sigma \equiv 0_{n \times m} \) therein, for any \( p \geq 2 \) and \( \theta \in (0, 1) \)

\[
\mathbb{E} \left( \sup_{0 \leq t \leq T} |X(t) - Y(t)|^p \right) \lesssim h^{\frac{1}{2}}. \tag{1.9}
\]

So the best convergence rate of EM scheme (i.e., (1.8)) associated with (1.6) is close to one half.

Remark 1.3. By a close inspection of the proof for Theorem 1.2, the conditions (A4) and (A5) can be replaced by: For any \( p > 2 \) there exists \( K_p, K_0 > 0 \) and \( q > 1 \) such that

\[
\int_U |g(x, y, u)|^p \lambda(du) \leq K_p(1 + |x|^p + |y|^q);
\]

\[
\int_U |g(x, y, u) - g(\bar{x}, \bar{y}, u)|^p \lambda(du) \leq K_p[|x - \bar{x}|^p + (1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|^p];
\]

\[
\int_U |g(x, y, u)|^2 \lambda(du) \leq K_0(1 + |x|^2 + |y|^q);
\]

\[
\int_U |g(x, y, u) - g(\bar{x}, \bar{y}, u)|^2 \lambda(du) \leq K_0[|x - \bar{x}|^2 + (1 + |y|^q + |\bar{y}|^q)|y - \bar{y}|^2]
\]

for any \( x, y, \bar{x}, \bar{y} \in \mathbb{R}^n \).
2 Proof of Theorem 1.1

The lemma below provides estimates of the $p$-th moment of the solution to (1.2) and the corresponding EM scheme, alongside with the $p$-th moment of the displacement.

Lemma 2.1. Under (A1) and (A2), for any $p \geq 2$ there exists a constant $C_T > 0$ such that

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |X(t)|^p \right) \vee \mathbb{E}\left( \sup_{0 \leq t \leq T} |Y(t)|^p \right) \leq C_T, \tag{2.1}
\]

and

\[
\mathbb{E}\left( \sup_{0 \leq t \leq T} |\Gamma(t)|^p \right) \lesssim h^{p/2}, \tag{2.2}
\]

where $\Gamma(t) := Y(t) - \overline{Y}(t)$.

Proof. We focus only on the following estimate

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |Y(s)|^p \right) \leq C_T \tag{2.3}
\]

for some constant $C_T > 0$ since the uniform $p$-th moment of $X(t)$ in a finite time interval can be done similarly. From (A1) and (A2), one has

\[
|G(y)| \lesssim 1 + |y|^{1+q}, \tag{2.4}
\]

and

\[
|b(x, y)| + \|\sigma(x, y)\| \lesssim 1 + |x| + |y|^{1+q} \tag{2.5}
\]

for any $x, y \in \mathbb{R}^n$. By the Hölder inequality, the Burkhold-Davis-Gundy (B-D-G) inequality (see, e.g., [12, Theorem 7.3, p.40]), we derive from (2.4) and (2.5) that

\[
\mathbb{E}\left( \sup_{-\tau \leq s \leq t} |Y(s)|^p \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E}\left( \sup_{-\tau \leq s \leq t} |\overline{Y}(s)|^{p(1+q)} \right)
\]

\[
+ \int_0^t \{\mathbb{E}[|Y(s)|^p] + \mathbb{E}[|\overline{Y}(s - \tau)|^{p(1+q)}]\} ds
\]

\[
\lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E}\left( \sup_{-\tau \leq s \leq t - \tau} |Y(s)|^{p(1+q)} \right)
\]

\[
+ \int_0^T \mathbb{E}\left( \sup_{-\tau \leq r \leq s} |Y(r)|^p \right) ds,
\]

where we have used $Y(kh) = \overline{Y}(kh)$ in the last display. This, together with Gronwall's inequality, yields that

\[
\mathbb{E}\left( \sup_{0 \leq s \leq t} |Y(s)|^p \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)} + \mathbb{E}\left( \sup_{0 \leq s \leq (t-\tau) \vee 0} |X(s)|^{p(1+q)} \right),
\]
which further implies that
\[ E\left( \sup_{0 \leq t \leq \tau} |X(t)|^p \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)}, \]
and that
\[ E\left( \sup_{0 \leq t \leq 2\tau} |X(t)|^p \right) \lesssim 1 + E\|\xi\|_\infty^{p(1+q)} + \left( \sup_{0 \leq t \leq \tau} |X(t)|^{p(q+1)} \right) \lesssim 1 + \|\xi\|_\infty^{p(1+q)^2}. \]
Thus (2.3) follows from an inductive argument.

Employing Hölder’s inequality and BDG’s inequality, we deduce from (1.4) and (2.5) that
\[ E\left( \sup_{0 \leq t \leq T} |\Gamma(t)|^p \right) \lesssim \sup_{0 \leq k \leq M-1} \left\{ h^{p-1}E \int_{kh}^{(k+1)h} |b(\overline{Y}(s), \overline{Y}(s-\tau))| \, ds \right\} + h^{\frac{p}{2}-1} \left\{ \int_{kh}^{(k+1)h} \|\sigma(\overline{Y}(s), \overline{Y}(s-\tau))\|^p \, ds \right\} \leq h^{\frac{p}{2}} \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left( 1 + E|\overline{Y}(s)|^p + E|\overline{Y}(s-\tau)|^{p(q+1)} \right) \, ds \right\} \lesssim h^\frac{p}{2}, \]
where in the last step we have used (2.3). The desired assertion is therefore complete. \( \Box \)

With Lemma 2.1 in hand, we are now in the position to finish the proof of Theorem 1.1.

Proof of Theorem 1.1. We follow the Yamada-Watanabe approach (see, e.g., [3]) to complete the proof of Theorem 1.1. For fixed \( \kappa > 1 \) and arbitrary \( \varepsilon \in (0, 1) \), there exists a continuous non-negative function \( \varphi_{\kappa\varepsilon}(\cdot) \) with the support \( [\varepsilon/\kappa, \varepsilon] \) such that
\[ \int_{\varepsilon/\kappa}^{\varepsilon} \varphi_{\kappa\varepsilon}(x) \, dx = 1 \quad \text{and} \quad \varphi_{\kappa\varepsilon}(x) \leq \frac{2}{x \ln \kappa}, \quad x > 0. \]
Set
\[ \phi_{\kappa\varepsilon}(x) := \int_{0}^{x} \int_{0}^{y} \varphi_{\kappa\varepsilon}(z) \, dz \, dy, \quad x > 0. \]
It is readily to see that \( \phi_{\kappa\varepsilon}(\cdot) \) such that
\[ x - \varepsilon \leq \phi_{\kappa\varepsilon}(x) \leq x, \quad x > 0. \quad (2.6) \]
Let
\[ V_{\kappa\varepsilon}(x) = \phi_{\kappa\varepsilon}(|x|), \quad x \in \mathbb{R}^n. \quad (2.7) \]
By a straightforward calculation, it holds
\[ (\nabla V_{\kappa \varepsilon})(x) = \phi'_{\kappa \varepsilon}(|x|)|x|^{-1} x, \quad x \in \mathbb{R}^n \]
and
\[ (\nabla^2 V_{\kappa \varepsilon})(x) = \phi'_{\kappa \varepsilon}(|x|)(|x|^2 I - x \otimes x)|x|^{-3} + |x|^{-2} \phi''_{\kappa \varepsilon}(|x|) x \otimes x, \quad x \in \mathbb{R}^n, \]
where \( \nabla \) and \( \nabla^2 \) stand for the gradient and Hessian operators, respectively, \( I \) denotes the identical matrix, and \( x \otimes x = xx^* \) with \( x^* \) being the transpose of \( x \in \mathbb{R}^n \). Moreover, we have
\[ |(\nabla V_{\kappa \varepsilon})(x)| \leq 1 \quad \text{and} \quad \|(\nabla^2 V_{\kappa \varepsilon})(x)\| \leq 2n \left( 1 + \frac{1}{\ln \kappa} \right) \frac{1}{|x|} \mathbb{1}_{[\varepsilon/\kappa, \varepsilon]}(|x|), \quad (2.8) \]
where \( \mathbb{1}_A(\cdot) \) is the indicator function of the subset \( A \subset \mathbb{R}_+ \).

For notation simplicity, set
\[ Z(t) := X(t) - Y(t) \quad \text{and} \quad \Lambda(t) := Z(t) - G(X(t - \tau)) + G(Y(t - \tau)). \quad (2.9) \]
In the sequel, let \( t \in [0, T] \) be arbitrary and fix \( p \geq 2 \). Due to \( \Lambda(0) = 0 \in \mathbb{R}^n \) and \( V_{\kappa \varepsilon}(0) = 0 \), an application of Itô’s formula gives
\[ V_{\kappa \varepsilon}(\Lambda(t)) = \int_0^t \langle (\nabla V_{\kappa \varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle ds + \frac{1}{2} \int_0^t \text{trace}\{ (\Gamma_2(s))^*(\nabla^2 V_{\kappa \varepsilon})(\Lambda(s))\Gamma_2(s) \} ds \]
\[ + \int_0^t \langle \nabla (V_{\kappa \varepsilon})(\Lambda(s)), \Gamma_2(s) dB(s) \rangle \]
\[ =: I_1(t) + I_2(t) + I_3(t), \]
where
\[ \Gamma_1(t) := b(X(t), X(t - \tau)) - b(\bar{Y}(t), \bar{Y}(t - \tau)) \quad (2.10) \]
and
\[ \Gamma_2(t) := \sigma(X(t), X(t - \tau)) - \sigma(\bar{Y}(t), \bar{Y}(t - \tau)). \]
Set
\[ V(x, y) := 1 + |x|^q + |y|^q, \quad x, y \in \mathbb{R}^n. \quad (2.11) \]
According to (2.11), for any \( q \geq 2 \) there exists a constant \( C_T > 0 \) such that
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} V(X(t - \tau), \bar{Y}(t - \tau))^q \right) \leq C_T. \quad (2.12) \]
Noting that
\[ X(t) - \bar{Y}(t) = \Lambda(t) + \Gamma(t) + G(X(t - \tau)) - G(\bar{Y}(t - \tau)), \quad (2.13) \]
and using Hölder’s inequality and B-D-G’s inequality, we get from (2.3) and (A1)-(A2) that

\[
\Theta(t) := E\left(\sup_{0 \leq s \leq t} |I_1(s)|^p \right) + E\left(\sup_{0 \leq s \leq t} |I_3(s)|^p \right)
\]

\[
\lesssim \int_0^t \{E|\Lambda(s)|^p + E||\Gamma(s)||^p\} ds
\]

\[
\lesssim \int_0^t E|X(s) - \overline{Y}(s)|^p ds + \int_{s=t}^{t-\tau} E(V(X(s), \overline{Y}(s))^p|X(s) - \overline{Y}(s)|^p) ds
\]

\[
\lesssim \int_0^t E\{|\Lambda(s)|^p + |\Gamma(s)|^p\} ds + \int_{s=t}^{t-\tau} E(V(X(s), \overline{Y}(s))^p|X(s) - \overline{Y}(s)|^p) ds.
\]

Also, by Hölder’s inequality, it follows from (2.2), (A3) and (2.12) that

\[
\Theta(t) \lesssim \int_0^t \{E|\Lambda(s)|^p + E|\Gamma(s)|^p + (EV(X(s-\tau), \overline{Y}(s-\tau))^{2p})^{1/2}
\]

\[
\times ((E|Z(s-\tau)|^{2p})^{1/2} + (E|\Gamma(s-\tau)|^{2p})^{1/2}) ds
\]

\[
\lesssim \int_0^t \{E|\Lambda(s)|^p + E|\Gamma(s)|^p + (E|Z(s-\tau)|^{2p})^{1/2} + (E|\Gamma(s-\tau)|^{2p})^{1/2}\} ds
\]

\[
\lesssim \int_0^t \{E|\Lambda(s)|^p + (E|Z(s-\tau)|^{2p})^{1/2} + h^{p/2}\} ds.
\]

In the light of (2.8)-(2.13), we derive from (A1) that

\[
E\left(\sup_{0 \leq s \leq t} |I_2(s)|^p \right)
\]

\[
\lesssim E\left(\int_0^t ||(\nabla^2 \overline{V}_{\kappa\ell})(\Lambda(s))||^{p}||\Gamma_2(s)||^{2p} ds \right)
\]

\[
\lesssim E\left(\int_0^t \frac{1}{|\Lambda(s)|^p}\{|X(s) - \overline{Y}(s)|^{2p} + V(X(s-\tau), \overline{Y}(s-\tau))^{2p}
\]

\[
\times (|X(s-\tau) - \overline{Y}(s-\tau)|^{2p})I_{[\varepsilon/\kappa, \varepsilon]}(|\Lambda(s)|)\} ds \right)
\]

\[
\lesssim E\left(\int_0^t \frac{1}{|\Lambda(s)|^p}\{|\Lambda(s)|^{2p} + |\Gamma(s)|^{2p} + |G(X(s-\tau)) - G(\overline{Y}(s-\tau))|^{2p}
\]

\[
+ V(X(s-\tau), \overline{Y}(s-\tau))^{2p}(|X(s-\tau) - \overline{Y}(s-\tau)|^{2p})\}I_{[\varepsilon/\kappa, \varepsilon]}(|\Lambda(s)|)\} ds \right)
\]

\[
\lesssim E\left(\int_0^t \frac{1}{|\Lambda(s)|^p}\{|\Lambda(s)|^{2p} + |\Gamma(s)|^{2p}
\]

\[
+ V(X(s-\tau), \overline{Y}(s-\tau))^{2p}(|X(s-\tau) - \overline{Y}(s-\tau)|^{2p})\}I_{[\varepsilon/\kappa, \varepsilon]}(|\Lambda(s)|)\} ds \right)
\]

\[
\lesssim E\left(\int_0^t \{\varepsilon^{-p}h^{p} + \varepsilon^{-p}(E(|Z(s-\tau)|^{4p}))^{1/2}\} ds, \right)
\]

8
Thus, Gronwall’s inequality gives that
\[ \mathbb{E}\left( \sup_{0 \leq s \leq t} |\Lambda(s)|^p \right) \lesssim \varepsilon^p \]
\[ + \mathbb{E}\left( \sup_{0 \leq s \leq t} |V_{\nu s}(\Lambda(s))| \right) \]
\[ \lesssim \varepsilon^p + \Theta(t) + \mathbb{E}\left( \sup_{0 \leq s \leq t} |I_3(s)|^p \right) \]
\[ \lesssim \varepsilon^p + \int_0^t \{ h^{p/2} + \varepsilon^{-p}h^p + \mathbb{E}\left( \sup_{0 \leq r \leq s} |\Lambda(r)|^p \right) \]
\[ + (\mathbb{E}(|Z(s - \tau)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(s - \tau)|^{4p}))^{1/2} \} \, ds. \]

Thus, Gronwall’s inequality gives that
\[ \mathbb{E}\left( \sup_{0 \leq s \leq t} |\Lambda(s)|^p \right) \lesssim \varepsilon^p + h^{p/2} + \varepsilon^{-p}h^p \]
\[ + \int_0^{(t-\tau)v_0} \{ (\mathbb{E}(|Z(s)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(s)|^{4p}))^{1/2} \} \, ds \quad (2.17) \]
\[ \leq h^{p/2} + \int_0^{(t-\tau)v_0} \{ (\mathbb{E}(|Z(s)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(s)|^{4p}))^{1/2} \} \, ds, \]
by choosing \( \varepsilon = h^{1/2} \) and taking \( |Z(t)| \equiv 0 \) for \( t \in [-\tau, 0] \) into account. Next, by (A1) and (2.12) it follows from Hölder’s inequality that
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} |Z(t)|^p \right) \lesssim \mathbb{E}\left( \sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + \mathbb{E}\left( \sup_{-\tau \leq t \leq T-\tau} |G(X(t)) - G(\bar{Y}(t))|^p \right) \]
\[ \lesssim \mathbb{E}\left( \sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + \mathbb{E}\left( \sup_{-\tau \leq t \leq T-\tau} (V(X(t), \bar{Y}(t))^p |X(t) - \bar{Y}(t)|^p) \right) \]
\[ \lesssim \mathbb{E}\left( \sup_{0 \leq t \leq T} |\Lambda(t)|^p \right) + h^{p/2} + \mathbb{E}\left( \sup_{0 \leq t \leq (T-\tau)v_0} |Z(t)|^{2p} \right)^{1/2}. \]

Substituting (2.17) into (2.18) yields that
\[ \mathbb{E}\left( \sup_{0 \leq t \leq T} |Z(t)|^p \right) \lesssim h^{p/2} + \left( \mathbb{E}\left( \sup_{0 \leq t \leq (T-\tau)v_0} |Z(t)|^{2p} \right) \right)^{1/2} \]
\[ + \int_0^{(T-\tau)v_0} \{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(t)|^{4p}))^{1/2} \} \, dt. \]

Hence, we have
\[ \mathbb{E}\left( \sup_{0 \leq t \leq \tau} |Z(t)|^p \right) \lesssim h^{p/2}, \]
and
\[ \mathbb{E}\left( \sup_{0 \leq t \leq 2\tau} |Z(t)|^p \right) \lesssim h^{p/2} + \left( \mathbb{E}\left( \sup_{0 \leq t \leq \tau} |Z(t)|^{2p} \right) \right)^{1/2} \]
\[ + \int_0^\tau \{ (\mathbb{E}(|Z(t)|^{2p}))^{1/2} + \varepsilon^{-p}(\mathbb{E}(|Z(t)|^{4p}))^{1/2} \} \, dt \]
\[ \lesssim h^{p/2} \]
by taking $\varepsilon = h^{1/2}$. Thus, the desired assertion follows from an inductive argument.

3 Proof of Theorem 1.2

Hereinafter, $(X(t))$ is the strong solution to (1.6) and $(Y(t))$ is the continuous-time EM scheme (i.e., (1.8)) associated with (1.6).

The lemma below plays a crucial role in revealing convergence rate of the EM scheme.

Lemma 3.1. Under (A1)-(A5) with $\sigma \equiv 0_{n \times m}$ therein,

$$
\mathbb{E}\left(\sup_{0 \leq t \leq T} |X(t)|^p\right) \vee \mathbb{E}\left(\sup_{0 \leq t \leq T} |Y(t)|^p\right) \leq C_T, \quad p \geq 2
$$

(3.1)

for some constant $C_T > 0$ and

$$
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p\right) \lesssim h, \quad p \geq 2,
$$

(3.2)

where $\Gamma(t) := Y(t) - \overline{Y}(t)$.

Proof. By carrying out a similar argument to that of [12, Theorem 3.1, p.210], (1.6) admits a unique strong solution $\{X(t)\}$ according to [17, Theorem 117, p.79]. On the other hand, since the proof of (3.1) is quite similar to that of (2.1), we here omit its detailed proof.

In the sequel, we aim to show (3.2). From (A4), it follows that

$$
|g(x, y, u)| \leq C(1 + |x| + |y|^{1+q}|u|^r), \quad x, y \in \mathbb{R}^n, \quad u \in U
$$

(3.3)

for some $C > 0$. Applying B-D-G’s inequality (see, e.g., Lemma [14, Theorem 1]) and Hölder’s inequality, we derive that

$$
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Gamma(t)|^p\right) \lesssim \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \mathbb{E}\left(\int_{kh}^{t} b(\overline{Y}(s), \overline{Y}(s-\tau))ds \right)^p \right\}^{\frac{1}{p}}
$$

$$
+ \int_{U} \mathbb{E}\left|g(\overline{Y}(s), \overline{Y}(s-\tau), u)\right|^p \lambda(du) ds
$$

$$
\lesssim \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left(h^{p-1}\mathbb{E}|b(\overline{Y}(s), \overline{Y}(s-\tau))|p
$$

$$
+ \int_{U} \mathbb{E}|g(\overline{Y}(s), \overline{Y}(s-\tau), u)|^p \lambda(du) ds \right\}
$$

$$
\lesssim \sup_{0 \leq k \leq M-1} \left\{ \int_{kh}^{(k+1)h} \left(1 + \mathbb{E}\left(\sup_{-\tau \leq r \leq s} |Y(r)|^{p(1+q)}\right)\right)
$$

$$
\times \left(h^{p-1} + \int_{U} |u|^p \lambda(du) ds \right)
$$

$$
\lesssim h^p + h
$$

$$
\lesssim h,
$$

10
Proof of Theorem 1.2.

We follow the idea of the proof for Theorem 1.1 to complete the proof. So (3.2) follows as required.

Next, we go back to finish the proof of Theorem 1.2.

**Proof of Theorem 1.2.** We follow the idea of the proof for Theorem 1.1 to complete the proof. Set

$$
\Gamma_3(t,u) := g(X(t),X(t-\tau),u) - g(\overline{Y}(t),\overline{Y}(t-\tau),u).
$$

Applying Itô's formula as well as the Lagrange mean value theorem to $V_{\kappa\varepsilon}(\cdot)$, defined by (2.7), gives that

$$
V_{\kappa\varepsilon}(\Lambda(t)) = \int_0^t \langle (\nabla V_{\kappa\varepsilon})(\Lambda(s)), \Gamma_1(s) \rangle ds
$$

$$
+ \int_0^t \int_U \{ V_{\kappa\varepsilon}(\Lambda(s) + \Gamma_3(s)) - V_{\kappa\varepsilon}(\Lambda(s)) \} \lambda(du) ds
$$

$$
+ \int_0^t \int_U \{ V_{\kappa\varepsilon}(\Lambda(s-)) + \Gamma_3(s-)) - V_{\kappa\varepsilon}(\Lambda(s-)) \} \tilde{N}(du,ds)
$$

where (3.3) in the third step, and (3.1) and (A5) that

$$
\begin{align*}
Y(t) := & \sum_{i=1}^3 \mathbb{E} \left( \sup_{0 \leq s \leq t} |J_i(s)|^p \right) \lesssim \int_0^t \mathbb{E} \{ |\Lambda(s)|^p + |\Gamma(s)|^p \} ds \\
&+ \int_{t-\tau}^{t-\tau} \mathbb{E} \{ V(X(s), Y(s))^p |X(s) - Y(s)|^p \} ds,
\end{align*}
$$

where $V(\cdot, \cdot)$ is introduced in (2.11). Observe from Hölder's inequality that

$$
\begin{align*}
\mathbb{E} & \{ V(X(s), Y(s))^p |X(s) - Y(s)|^p \} \\
\lesssim & (\mathbb{E} V(X(s), Y(s))^{p(1+\theta)/\theta})^{\theta/(1+\theta)} (\mathbb{E} |X(s) - Y(s)|^{p(1+\theta)})^{(1+\theta)/(1+\theta)} \\
\lesssim & (\mathbb{E} V(X(s), \overline{Y}(s))^{p(1+\theta)/\theta})^{\theta/(1+\theta)} (\mathbb{E} |Z(s)|^{p(1+\theta)} + \mathbb{E} |\Gamma(s)|^{p(1+\theta)})^{(1+\theta)/(1+\theta)} \\
\lesssim & (\mathbb{E} (1 + |X(s)|)^{p(1+\theta)/\theta})^{\theta/(1+\theta)} (\mathbb{E} |Z(s)|^{p(1+\theta)} + \mathbb{E} |\Gamma(s)|^{p(1+\theta)})^{(1+\theta)/(1+\theta)} \\
\lesssim & (\mathbb{E} |Z(s)|^{p(1+\theta)/\theta})^{\theta/(1+\theta)} (\mathbb{E} |\Gamma(s)|^{p(1+\theta)})^{(1+\theta)/(1+\theta)} \\
\lesssim & h^{1+\theta} + (\mathbb{E} |Z(s)|^{p(1+\theta)/\theta})^{1/(1+\theta)}, \quad \theta > 0,
\end{align*}
$$

where we have used (A2) with $\sigma \equiv 0_{n \times m}$ and (3.3) in the third step, and (3.1) and (A5) in the last two steps, respectively. So (3.2) follows as required. \hfill \Box
in which we have used \([3.1]\) in the penultimate display and \([3.2]\) in the last display, respectively. So we arrive at
\[
\Upsilon(t) \lesssim h^{1+\sigma} + \int_0^t \mathbb{E}\{|\Lambda(s)|^p + |\Gamma(s)|^p\}ds + \int_{-\tau}^{t-\tau} (\mathbb{E}|Z(s)|^{p(1+\theta)})^{-\frac{1}{1+\sigma}}ds.
\]
This, together with \([2.6]\) and \([3.2]\), implies
\[
\mathbb{E}\left(\sup_{0 \leq s \leq t} |\Lambda(t)|^p\right) \lesssim \epsilon^p + \mathbb{E}\left(\sup_{0 \leq s \leq t} V_{\kappa}(\Lambda(s))\right)
\]
\[
\lesssim \epsilon^p + h^{1+\sigma} + \int_0^t \mathbb{E}\{|\Lambda(s)|^p + |\Gamma(s)|^p\}ds + \int_{-\tau}^{t-\tau} (\mathbb{E}|Z(s)|^{p(1+\theta)})^{-\frac{1}{1+\sigma}}ds
\]
\[
\lesssim h^{1+\sigma} + \int_0^t \mathbb{E}|\Lambda(s)|^pds + \int_{-\tau}^{t-\tau} (\mathbb{E}|Z(s)|^{p(1+\theta)})^{-\frac{1}{1+\sigma}}ds
\]
by taking \(\epsilon = h^{\frac{1}{1+\sigma}}\) in the last display. Using Gronwall’s inequality, due to \(Z(\theta) = 0\) for \(\theta \in [-\tau, 0]\), one has
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p\right) \lesssim h^{1+\sigma} + \int_{0}^{(T-\tau)\vee 0} (\mathbb{E}|Z(s)|^{p(1+\theta)})^{-\frac{1}{1+\sigma}}ds.
\]
Next, observe from \((A1)\) and Hölder’s inequality that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq T} |Z(t)|^p\right) \lesssim \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p\right) + \mathbb{E}\left(\sup_{-\tau \leq t \leq T-\tau} |G(X(t)) - G(\Upsilon(t))|^p\right)
\]
\[
\lesssim \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p\right) + \mathbb{E}\left(\sup_{-\tau \leq t \leq T-\tau} (V(X(t), \Upsilon(t))^p) |X(t) - \Upsilon(t)|^p\right)
\]
\[
\lesssim \mathbb{E}\left(\sup_{0 \leq t \leq T} |\Lambda(t)|^p\right)
\]
\[
+ \left\{1 + \mathbb{E}\left(\sup_{-\tau \leq t \leq T} |X(t)|^{\frac{p(1+\theta)}{\theta}}\right) + \left(\mathbb{E}\left(\sup_{-\tau \leq t \leq T} |Y(t)|^{\frac{p(1+\theta)}{\theta}}\right)\right)^\frac{\theta}{1+\sigma}\right\}^{\frac{\theta}{1+\sigma}}
\]
\[
\times \left\{\mathbb{E}\left(\sup_{-\tau \leq t \leq T} |Z(t)|^{p(1+\theta)}\right) + \mathbb{E}\left(\sup_{-\tau \leq t \leq T} |\Gamma(t)|^{p(1+\theta)}\right)\right\}^{\frac{1}{1+\sigma}}
\]
\[
\lesssim h^{1+\sigma} + \left(\mathbb{E}\left(\sup_{0 \leq t \leq (T-\tau)\vee 0} |Z(t)|^{p(1+\theta)}\right)\right)^{\frac{1}{1+\sigma}},
\]
where in the last step we have utilized \([3.1]\) and \([3.2]\). So we find that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq \tau} |Z(t)|^p\right) \lesssim h^{1+\sigma},
\]
which, in addition to \((3.5)\), further yields that
\[
\mathbb{E}\left(\sup_{0 \leq t \leq 2\tau} |Z(t)|^p\right) \lesssim h^{1+\sigma} + \left(\mathbb{E}\left(\sup_{0 \leq t \leq \tau} |Z(t)|^{p(1+\theta)}\right)\right)^{\frac{1}{1+\sigma}}
\]
\[
\lesssim h^{\frac{1}{1+\theta} + \frac{1}{1+\sigma}}
\]
\[
\lesssim h^{\frac{1}{1+\theta} + \frac{1}{1+\sigma}}.
\]
Thus, the desired assertion follows from an inductive argument.

References

[1] Appleby, John A. D., Appleby-Wu, H., Mao, X., On the almost sure running maximum of solutions of affine neutral stochastic functional differential equations, arXiv:1310.2349v1.

[2] Bao, J., Wang, F.-Y., Yuan, C., Transportation cost inequalities for neutral functional stochastic equations, Z. Anal. Anwend., 32 (2013), 457–475.

[3] Bao,J., Yuan, C., Convergence rate of EM scheme for SDDEs. Proc. Amer. Math. Soc., 141 (2013), 3231–3243.

[4] Bao, J., Böttcher, B., Mao, X., Yuan, C., Convergence rate of numerical solutions to SFDEs with jumps, J. Comput. Appl. Math., 236 (2011), 119–131.

[5] Bao, J., Hou, Z., Yuan, C., Stability in distribution of neutral stochastic differential delay equations with Markovian switching, Statist. Probab. Lett., 79 (2009), 1663–1673.

[6] Bao, J., Yuan, C., Large deviations for neutral functional SDEs with jumps, Stochastics, 87(2015), 48-70.

[7] Gyöngy, I., Sabanis, S., A Note on Euler Approximations for Stochastic Differential Equations with Delay, Appl. Math. Optim., 68 (2013), 391–412.

[8] Gyöngy, I., Rásonyi, M., A note on Euler approximations for SDEs with Hölder continuous diffusion coefficients. Stochastic Process. Appl., 121(2011),2189-2200.

[9] Jiang, F., Shen, Y., and Wu, F., A note on order of convergence of numerical method for neutral stochastic functional differential equations. Commun. Nonlinear Sci. Numer. Simul., 17 (2012) 1194-1200.

[10] Li,X., Cao, W., On mean-square stability of two-step Maruyama methods for nonlinear neutral stochastic delay differential equations, Appl. Math. Comput., 261 (2015), 373–381.

[11] Mao, X., Convergence rates of the truncated Euler-Maruyama method for stochastic differential equations, J. Comput. Appl. Math., 296 (2016), 362–275.

[12] Mao, X., Stochastic differential equations and applications. Second Edition. Horwood Publishing Limited, 2008. Chichester.

[13] Mao, X., Shen, Y., Yuan, C., Almost surely asymptotic stability of neutral stochastic differential delay equations with Markovian switching, Stochastic Process. Appl., 118 (2008), 1385–1406.

[14] Marinelli, C., Röckner, M., On Maximal inequalities for purely discountinuous martingales in infinite dimensional, Séminaire de Probabilités XLVI, Lecture Notes in Mathematics 2123 (2014), 293-316.

[15] Milosevic, M., Highly nonlinear neutral stochastic differential equations with time-dependent delay and the Euler-Maruyama method, Math. Comput. Modelling, 54 (2011), 2235–2251.

[16] Sabanis, S., A note on tamed Euler approximations, Electron. Commun. Probab., 18 (2013), 1–10.

[17] Situ, R., Theory of stochastic differential equations with jumps and applications. Mathematical and Analytical Techniques with Applications to Engineering, Springer, 2005. New York.

[18] Wu, F., Mao, X., Numerical solutions of neutral stochastic functional differential equations. SIAM J. Numer. Anal., 46 (2008), 1821-1841.

[19] Yu, Z., Almost sure and mean square exponential stability of numerical solutions for neutral stochastic functional differential equations, Int. J. Comput. Math., 92 (2015), 132–150.
[20] Yuan, C., Mao, X., A note on the rate of convergence of the Euler-Maruyama method for stochastic differential equations, *Stoch. Anal. Appl.*, 26 (2008), 325–333.

[21] Yuan, C., Glover, W., Approximate solutions of stochastic differential delay equations with Markovian switching, *J. Comput. Appl. Math.*, 194 (2006), 207–226.

[22] Zhou, S., Fang, Z., Numerical approximation of nonlinear neutral stochastic functional differential equations, *J. Appl. Math. Comput.*, 41 (2013), 427–445.

[23] Zong, X., Wu, F. and Huang, C., Exponential mean square stability of the theta approximations for neutral stochastic differential delay equations. *J. Comput. Appl. Math.*, 286 (2015), 172-185.

[24] Zong, X., Wu, F., Exponential stability of the exact and numerical solutions for neutral stochastic delay differential equations, *Appl. Math. Model.*, 40 (2016), 19–30.