Variational Minimizing Parabolic Orbits for the
2-Fixed Center Problems

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\textbf{ABSTRACT:} Using variational minimizing methods, we prove the existence of an odd symmetric parabolic orbit for the 2-fixed center problems with weak force type homogeneous potentials.

\textbf{KEY WORDS:} 2-Fixed Center Problems, Odd Symmetric Parabolic Orbits, Variational Minimizers.

\textbf{AMS Subject Classification:} 34C15, 34C25.

1 Introduction and Main Results

Sitnikov \cite{1} and Moser \cite{2} and Mathlouthi \cite{3} and Souissi \cite{4} and Zhang \cite{5} etc. studied the model for the circular restricted 3-body problems: two mass points of equal mass \( m_1 = m_2 = \frac{1}{2} \) move in the plane of their circular orbits such that the center of masses is at rest, and the third small mass which does not influence the motion of the first two ones moves on the line perpendicular to the plane containing the first two mass points and going through the center of mass.

Let \( z(t) \) be the coordinate of the third mass point, then \( z(t) \) satisfies

\[
\ddot{z}(t) + \alpha \frac{z(t)}{(|z(t)|^2 + |r|^2)^{\alpha/2+1}} = 0.
\] (1)

Zhang \cite{5} used variational minimizing method to prove:

\textbf{Theorem 1.1} For the equation (1) with \( 0 < \alpha < 2 \), there exists one odd parabolic
or hyperbolic orbit.

The 2-fixed center problem is an old problem studied by Euler[6-8] etc. ([9], [10], [11], [12]): For two masses $1 - \mu$ and $\mu$ fixed at $q^1 = (-\mu, 0)$ and $q^2 = (1 - \mu, 0)$, the problem is to study the motion $q(t) = (x(t), y(t))$ of the third body with mass $m_3 > 0$. Here, we consider the motion of the third body attracted by the 2-fixed center masses with general homogeneous potentials, then it satisfies the following equation:

$$\ddot{q}(t) + \frac{\partial V(q)}{\partial q} = 0,$$

(2)

$$V(q) = -\frac{1 - \mu}{|q - q^1|^\alpha} - \frac{\mu}{|q - q^2|^\alpha}.$$

(3)

For $\mu = 1/2$, we study the existence for the motion $q(t) = (x(t), y(t))$ of the third body satisfying $(x(-t), y(-t)) = (-x(t), -y(t))$, here we use variational minimizing method to prove:

**Theorem 1.2** For (2) – (3) with $\mu = \frac{1}{2}$ and $0 < \alpha < 2$, there exists an odd symmetrical parabolic-type unbounded orbit.

## 2 Truncation Functional and Its Minimizing Critical Points

In order to find parabolic-type orbit of (2) – (3), firstly, we restrict $t \in [-n, n]$ and find solutions of (2) – (3), then let $n \to +\infty$ to get the parabolic-type orbit. Noticing the symmetry of the equation, we can find the odd solutions of the following ODE:

$$\ddot{q}(t) = \frac{\partial U(q)}{\partial q},$$

(4)

$$U(q) = \frac{1/2}{|q - q^1|^\alpha} + \frac{1/2}{|q - q^2|^\alpha}.$$

(5)

We define the functional:

$$f(q) = \int_{-n}^{n} \left( \frac{1}{2} |\dot{q}(t)|^2 + \frac{1/2}{|q - q^1|^\alpha} + \frac{1/2}{|q - q^2|^\alpha} \right) dt,$$

(6)

where

$$q \in H_n = \{ q(t) = (x(t), y(t)) : x, y \in W^{1,2}[-n, n]; q(-t) = -q(t), q(t) \neq q^i, t \in [-n, n] \}.$$ 

(7)
Since \( \forall q \in H_n, q(0) = 0 \), then by the famous Hardy-Littlewood-Polya’s inequality ([9], inequality 256), for \( \forall q \in H_n \), we have an equivalent norm:

\[
\|q\|_n = \left( \int_{-n}^{n} |q(t)|^2 dt \right)^{1/2}.
\]

**Remark** Here we didn’t assume \( q(-n) = q(n) = 0 \) since we want to get the parabolic-type orbit satisfying

\[
\max_{t \in \mathbb{R}} |q(t)| = +\infty,
\]

\[
\min_{t \in \mathbb{R}} |\dot{q}(t)| = 0.
\]

We didn’t assume the periodic property for \( q(t) \) since we need non-periodic odd test function in order to get Lemma 2.6.

**Lemma 2.1** \( f(q) \) is weakly lower semi-continuous (w.l.s.c.) on the closure \( \bar{H}_n \) of \( H_n \).

**Proof:** (i). It is well-known that the norm and its square are w.l.s.c.

(ii). \( \forall \{q_m\} \subset H_n \), if \( q_m \rightharpoonup q \in H_n \) weakly, then by compact embedding theorem, we have the uniformly convergence:

\[
\max_{-n \leq t \leq n} |q_m(t) - q(t)| \to 0,
\]

as \( m \to +\infty \), so

\[
\int_{-n}^{n} \frac{1}{|q_m - q|^\alpha} dt \to \int_{-n}^{n} \frac{1}{|q - q|^\alpha} dt, i = 1, 2,
\]

as \( m \to +\infty \). Hence

\[
\lim_{m \to \infty} f(q_m) \geq f(q).
\]

(iii). \( \forall \{q_m\} \subset H_n \), if \( q_m \rightharpoonup q \in \partial H_n \) weakly, let

\[
S = \{ t_0 \in [-n, n] : q(t_0) = q_1(t_0), or, q_2(t_0) \}
\]

(1). The Lebesgue measure of \( S \) is zero, then \( U(q_m(t)) \to U(q(t)) \) almost everywhere, then by Fatou’s Lemma, \( \int_{-n}^{n} U(q)dt \) is w.l.s.c., it is well-known that the norm and its square are w.l.s.c., so \( f(q) \) is w.l.s.c.
(2). The Lebesgue measure of $S:L(S) > 0$, then

$$\int_{-n}^{n} U(q) dt = +\infty, f(q) = +\infty,$$

then by compact embedding theorem, we have the uniformly convergence on $S$:

$$\max_{-n \leq t \leq n} |q_m(t) - q(t)| \to 0,$$

as $m \to +\infty$, so on $S$, we have the uniformly convergence:

$$\int_{-n}^{n} \frac{1}{|q_m - q|^\alpha} dt \to +\infty, i = 1, or, 2,$$

as $m \to +\infty$. Hence

$$\int_{-n}^{n} U(q_m(t)) dt \to +\infty$$

$$\lim_{m \to \infty} f(q_m) = +\infty \geq f(q).$$

**Lemma 2.2** $f$ is coercive on $H_n$.

**Proof:** From the definition of $f(q)$ and Hardy-Littlewood’s inequality, it is clear that the coercivity holds ($f(q) \to +\infty, \|q\| \to +\infty$).

**Lemma 2.3** (Tonelli, [13], [14]) Let $X$ be a reflexive Banach space, $M \subset X$ be a weakly closed subset, $f : M \to R \cup \{+\infty\}$, but $f(x)$ is not always $+\infty$, suppose $f$ is weakly lower semi-continuous and coercive ($f(x) \to +\infty, \|x\| \to +\infty$), then $f$ attains its infimum on $M$.

**Lemma 2.4** (Palais’s Symmetry Principle [15]) Let $G$ be a finite or compact group, $\sigma$ be an orthogonal representation of $G$, let $H$ be a real Hilbert space, $f : H \to R$ satisfying

$$f(\sigma \cdot x) = f(x), \forall \sigma \in G, \forall x \in H.\]$$

Let

$$F \triangleq \{x \in H | \sigma \cdot x = x, \forall \sigma \in G\}.$$
Then the critical point of \( f \) in \( F \) is also a critical point of \( f \) in \( H \).

**Lemma 2.5** \( f(q) \) attains its infimum on \( \bar{H}_n \), the minimizer \( \tilde{q}_{\alpha,n}(t) \) is an odd solution.

**Proof:** Since we had proved Lemmas 2.1-2.2, so in order to apply for Lemma 2.3, we need to apply for Lemma 2.4 to prove that the critical point of \( f(q) \) on \( H_n \) is the odd solution of (4) – (5): We define groups \( G_1 = \{I_{2\times 2}, -I_{2\times 2}\} \); \( G_2 = \{1, -1\} \) and their actions:

\[
\sigma_1 \cdot q(t) = I_{2\times 2}q(t), \\
\sigma_2 \cdot q(t) = -I_{2\times 2}q(t); \\
\tilde{\sigma}_1 \cdot q(t) = q(t), \\
\tilde{\sigma}_2 \cdot q(t) = q(-t).
\]

Then it’s easy to prove that \( f(q) \) is invariant under \( \sigma_1, \sigma_2, \tilde{\sigma}_1, \tilde{\sigma}_2, \sigma_i \cdot \tilde{\sigma}_j, \tilde{\sigma}_j \cdot \sigma_i \) and the fixed point set of the group actions for \( G_1 \times G_2 \) is just \( H_n \), so we can apply for Palais’s Symmetrical Principle.

In order to get the parabolic type solution, we need to prove that

\[
\tilde{q}_{\alpha,n}(t) \to \tilde{q}_{\alpha}(t)
\]

when \( n \to \infty \), and \( \tilde{q}_{\alpha}(t) \) has the properties:

\[
\max_{t \in \mathbb{R}} |\tilde{q}_{\alpha}(t)| = +\infty, \\
\min_{t \in \mathbb{R}} |\tilde{q}_{\alpha}(t)| = 0.
\]

In order for that, we need some furthermore Lemmas:

**Lemma 2.6** There exist constants \( c > 0 \) and \( 0 < \theta < 1 \) independent of \( n \) such that the variational minimizing value \( a_n \) for \( f(q) \) on \( \bar{H}_n \) satisfies \( a_n \leq cn^\theta \).

**Proof:** (i). If \( \tilde{q}(t) = (\tilde{x}, \tilde{y}) \in H_n \) is located on y-axis, then we choose a special odd function defined by

\[
\tilde{x} = 0, \quad \tilde{y} = t^\beta, \quad t \in [-n, n],
\]
where
\[ \frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha}, \]
l, m are odd numbers and \((l, m) = 1\). Then
\[
f(\tilde{q}(t)) = \frac{1}{2} \int_0^\beta t^{2(\beta-1)} dt + \int_0^n \left[ \frac{1}{|t^{2\beta} + \frac{1}{4}|^{\alpha/2}} + \frac{1}{|t^{2\beta} - \frac{1}{4}|^{\alpha/2}} \right] dt
\]
\[ \leq \frac{\beta^2}{2\beta - 1} n^{2\beta - 1} + \frac{2}{1 - \alpha \beta} n^{1-\alpha \beta}. \]

Now we define
\[ \theta = \max(2\beta - 1, 1 - \alpha \beta), \quad (8) \]
\[ c = \frac{\beta^2}{2\beta - 1} + \frac{2}{1 - \alpha \beta} > 0. \quad (9) \]

When
\[ \frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha}, \]
then
\[ 2\beta - 1 > 0, \quad 1 - \alpha \beta > 0 \]
and \(0 < \theta < 1\). Hence we have
\[ f(\tilde{q}) \leq cn^\theta. \]

(ii). If \(\tilde{q}(t) = (\tilde{x}, \tilde{y})\) is not on y-axis, we choose a special odd function on \(t\) defined by
\[ \tilde{x}(t) = t^{\beta}, \tilde{y}(t) = 0, \quad t \in [-n, n], \]
where
\[ \frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha}, \]
l, m are odd numbers and \((l, m) = 1\). Then, we have
\[
f(\tilde{q}(t)) \leq \int_0^n \beta^2 t^{2(\beta-1)} dt + \int_0^n \left[ \frac{1}{|t^{2\beta} + \frac{1}{4}|^{\alpha/2}} + \frac{1}{|t^{2\beta} - \frac{1}{4}|^{\alpha/2}} \right] dt
\]
\[ \leq \frac{\beta^2}{2\beta - 1} n^{2\beta - 1} + \left[ \frac{1}{1 - \alpha \beta} n^{1-\alpha \beta} + \int_0^n \frac{1}{|t^{2\beta} - \frac{1}{4}|^{\alpha/2}} dt \right]. \]

Now we estimate
\[ \int_0^n \frac{1}{|t^{2\beta} - \frac{1}{2}|^{\alpha/2}} dt. \]
Let \[ t^\beta - \frac{1}{2} = \tau^\beta, \]
then \( t > \tau \) and
\[
dt = \left(\frac{\tau}{t}\right)^{\beta-1}d\tau
\]
also
\[
\int_0^n \frac{1}{|t^\beta - \frac{1}{2}|^\alpha} dt < \int_{(-\frac{1}{2})^{\frac{1}{2}}}^{(n^\beta - \frac{1}{2})^{\frac{1}{2}}} \tau^{-\alpha} d\tau
< \frac{1}{1 - \alpha \beta} [n^{1-\alpha \beta} - (\frac{1}{2})^{-\frac{1}{2}(1-\alpha \beta)}].
\]
Define \( \theta = \max\{2\beta - 1, 1 - \alpha \beta\} \),
\[
c = \frac{\beta^2}{2\beta - 1} + \frac{3}{1 - \alpha \beta} > 0.
\]
When \( \frac{1}{2} < \beta = \frac{l}{m} < \frac{1}{\alpha} \),
then \( 2\beta - 1 > 0, 1 - \alpha \beta > 0 \)
and \( 0 < \theta < 1 \). Hence we also have
\[
f(\tilde{q}) \leq cn^\theta.
\]
Furthermore, for our minimizer, we have

**Lemma 2.7** Let \( \tilde{q}_{\alpha,n} \) be critical points corresponding to the minimizing critical values
\( a_n = \min_{H_n} f(q) \), then \( \|\tilde{q}_{\alpha,n}\|_\infty \to +\infty \), when \( n \to +\infty \).

**Proof:** By the definition of \( f(\tilde{q}_{\alpha,n}) \) and Lemma 2.6, we have
\[
\begin{align*}
    cn^\theta & \geq f(\tilde{q}_{\alpha,n}) \\
    & \geq \int_0^n \frac{1}{|(x + \frac{1}{2})^2 + y^2|^{\alpha/2}} + \frac{1}{|(x - \frac{1}{2})^2 + y^2|^{\alpha/2}} dt.
\end{align*}
\]
We notice that
\[
(x + \frac{1}{2})^2 + y^2 \leq 2(x^2 + y^2) + \frac{5}{4}
\]
\[(x - \frac{1}{2})^2 + y^2 \leq (x^2 + y^2) + \frac{1}{4},\]

so

\[
cn^\theta \geq \int_0^n \frac{dt}{2 (\|\tilde{q}_{\alpha,n}\|_\infty^2 + \frac{5}{4})^{\alpha/2}} + \int_0^{\beta(n)} \frac{dt}{(\|\tilde{q}_{\alpha,n}\|_\infty^2 + \frac{1}{4})^{\alpha/2}}
\]

Hence

\[
\|\tilde{q}_{\alpha,n}\|_\infty^2 \to +\infty,
\]

as \(n \to +\infty\).

**Lemma 2.8** \(I_a^b |\tilde{q}_{\alpha,n}|^2 dt\) is uniformly bounded on any compact set \([a, b] \subset \mathbb{R}\).

**Proof:** Since the system is autonomous, so for any given \(\alpha, n\), along the solution \(\tilde{q}_{\alpha,n}(t)\), the energy \(h(t)\) is conservative, i.e., a constant \(h = h(\alpha, n)\):

\[
\frac{1}{2} |\dot{\tilde{q}}_{\alpha,n}|^2 - \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} - \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} = h.
\]

By the above energy relationship and the definition of the functional \(f\), we have

\[
f(\tilde{q}_{\alpha,n}) = \int_{-\infty}^{\infty} \left( \frac{1}{2} |\dot{\tilde{q}}_{\alpha,n}|^2 + \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} \right) dt
\]

\[
= \int_{-\infty}^{\infty} \left( \frac{1}{2} |\dot{\tilde{q}}_{\alpha,n}|^2 - \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} - \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} \right) dt + 2 \int_{-\infty}^{\infty} \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} dt
\]

\[
= 2nh + 2 \int_{-\infty}^{\infty} \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} dt.
\]

By Lemma 2.6, we have

\[
cn^\theta \geq 2nh + 2 \int_{-\infty}^{\infty} \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} dt,
\]

and

\[
h \leq \frac{c}{2} n^{\theta - 1} - \frac{1}{n} \int_{-n}^{n} \left( \frac{1/2}{|\tilde{q}_{\alpha,n} - q_1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n} - q_2|^\alpha} \right) dt \leq \frac{c}{2} n^{\theta - 1}
\]

(12)
When $n$ is large enough, $|\tilde{q}_{\alpha,n}(t) - q^i|$ has uniformly positive lower bound, that is, 
\[ \min_{a \leq t \leq b} |\tilde{q}_{\alpha,n}(t) - q^i| \geq c > 0, \]
then we have
\[
\int_a^b \frac{1}{2} |\dot{\tilde{q}}_{\alpha,n}|^2 = h(b - a) + \int_a^b \left[ \frac{1/2}{|\tilde{q}_{\alpha,n} - q^1|^\alpha} + \frac{1/2}{|\tilde{q}_{\alpha,n} - q^2|^\alpha} \right] dt \\
\leq \frac{c}{2} (b - a) + c^{-\alpha} (b - a).
\]

(2). There exist $i_0 = 1$ or $2$ and a sequence $t_n \subset [a, b]$ such that 
$\tilde{q}_{\alpha,n}(t_n) \to q^{i_0}$, then since $0 < \alpha < 2$, the potential is weak force potential, so when 
$n$ is large, we have
\[
\int_a^b \frac{1}{2} |\dot{\tilde{q}}_{\alpha,n}|^2 dt \leq \frac{c}{2} (b - a) + M.
\]

3 Proof of Theorem 1.2

By $\tilde{q}_{\alpha,n}(0) = 0$ and Cauchy-Schwarz inequality and Lemma 2.8 we have
\[
|\tilde{q}_{\alpha,n}(t)| = |\int_0^t \dot{\tilde{q}}_{\alpha,n}(s) ds| \leq (b - a)^{1/2} [\int_a^b |\dot{\tilde{q}}_{\alpha,n}|^2 ds]^{1/2} \leq M_1,
\]
so we have

(i). $\{\tilde{q}_{\alpha,n}\}$ is uniformly bounded on any compact set of $R$.

By Cauchy-Schwarz inequality and Lemma 2.8 we have
\[
|\tilde{q}_{\alpha,n}(t_2) - \tilde{q}_{\alpha,n}(t_1)| = |\int_{t_1}^{t_2} \dot{\tilde{q}}_{\alpha,n}(s) ds| \leq [\int_a^b |\dot{\tilde{q}}_{\alpha,n}|^2 ds]^{1/2} (t_2 - t_1)^{1/2} \leq M_2 (t_2 - t_1)^{1/2},
\]
so we have

(ii). $\{\tilde{q}_{\alpha,n}\}$ is uniformly equi-continuous on any $[a, b] \subset R$.

Now we can apply Ascoli-Arzelà Theorem, we know $\{\tilde{q}_{\alpha,n}\}$ has a sub-sequence converging uniformly to a limit $\tilde{q}_\alpha(t)$ on any compact set of $R$, and $\tilde{q}_\alpha(t)$ is a solution of (2.2). By the energy conservation law and Lemmas 2.7-2.8, we have
\[
h = \frac{1}{2} |\dot{\tilde{q}}_\alpha|^2 - \frac{1}{2} \left( \frac{1}{|\tilde{q}_\alpha - q^1|^\alpha} + \frac{1}{|\tilde{q}_\alpha - q^2|^\alpha} \right) = 0.
\]

Then by Corollary 2.3 of [20], we have
\[
\frac{1}{2} |\dot{q}_\alpha|^2 = \frac{1/2}{|\tilde{q}_\alpha - q^1|^\alpha} + \frac{1/2}{|\tilde{q}_\alpha - q^2|^\alpha} \geq \left[2^{\alpha+2}\right]^2 |\tilde{q}_\alpha|^2 + \frac{1}{2} - \alpha/2.
\]

(13)

Now we claim

(a).

\[
\max_{t \in \mathbb{R}} |\tilde{q}_\alpha(t)| = +\infty.
\]

(14)

In fact, if \( \exists \beta > 0 \) such that

\[
|\tilde{q}_\alpha| < \beta, \forall t \in \mathbb{R}.
\]

By (13), \( \exists \gamma > 0 \) such that

\[
|\dot{\tilde{q}}_\alpha| > \gamma, \forall t \in \mathbb{R}.
\]

Then when \( n \) is large, we have

\[
|\tilde{q}_{\alpha,n}| > \gamma, \forall t \in \mathbb{R}.
\]

\[
\int_{-n}^{n} |\tilde{q}_{\alpha,n}|^2 > 2n\gamma^2,
\]

which is a contradiction.

Now by (13) we have

(b).

\[
\min_{t \in \mathbb{R}} |\tilde{q}_\alpha(t)| = 0.
\]

(15)

4 Acknowledgements

The authors sincerely thank the referee for his/her many valuable comments and remarks which helped us revising the paper, we aslo thank the supports of NSF of China and a research fund for the Doctoral program of higher education of China.

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