Neutrino oscillations: what is magic about the “magic” baseline?

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**Abstract**

Physics interpretation of the “magic” baseline, $L_{\text{magic}}$, that can play important role in future oscillation experiments is given. The “magic” baseline coincides with the refraction length, $l_0$. The latter, in turn, approximately equals the oscillation length in matter at high energies. Therefore at the baseline $L = l_0$ the oscillation phase is $2\pi$, and consequently, the “solar” amplitude of oscillations driven by $\theta_{12}$ and $\Delta m_{21}^2$ vanishes. As a result, in the lowest order (i) the interference of amplitudes in the $\nu_e - \nu_\mu$ ($\nu_\tau$) transition probability is absent; (ii) dependence of the probability on the CP-phase, $\delta$, as well as on $\theta_{12}$ and $\Delta m_{21}^2$ disappears. Corrections to the equality $L_{\text{magic}} = l_0$ are estimated. Effect of changing density is considered and two new magic trajectories are identified for neutrinos that cross the core of the Earth. Other magic baselines associated with zeros of the atmospheric amplitude are discussed.

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1 Introduction

It was observed some time ago that at the baseline

\[ L_{\text{magic}} = \frac{2\pi}{\sqrt{2} G_F n_e}, \]

where \( G_F \) is the Fermi coupling constant and \( n_e \) is the electron number density, the analytic formula for the \( \nu_e - \nu_\mu \) oscillation probability in matter (in 3\( \nu \)-mixing context) takes a very simple form \[1\]. The probability does not depend on the CP violation phase, \( \delta \), as well as on the mixing angle, \( \theta_{12} \), and the mass splitting, \( \Delta m^2_{21} \), of the 1-2 sector. Therefore, neutrino oscillation experiments with the baseline \( L = L_{\text{magic}} = (7300 - 7600) \) km will allow one to perform clean measurements of \( \theta_{13} \) resolving degeneracy with the phase \( \delta ]1, 2] \). That was discussed in a number of recent recent publications in some details \[3\]. The baseline \( L_{\text{magic}} \) depends only on matter density and does not depend on neutrino energy and oscillation parameters. For this reason it was termed the “magic” baseline in \[4\].

Apparently the “magic” baseline defined in \[1\] coincides with the refraction length, \( l_0 \), introduced by Wolfenstein almost 30 years ago \[5\]:

\[ L_{\text{magic}} \equiv l_0. \]

This is not accidental coincidence and in what follows we will explain the reason behind the equality \[2\]. The explanation is given in secs. 2 and 3. Secs. 4 and 5 contain some more advanced material, in particular, discussion of various corrections to the equality \[2\].

2 Magic baseline and refraction length

Recall that the refraction length, \( l_0 \), has been defined as the distance over which an additional phase difference, \( \phi_m \), acquired by neutrinos due to interactions with matter, equals \( 2\pi \):

\[ \phi_m = V l_0 = 2\pi. \]

Here \( V \equiv \sqrt{2} G_F n_e \) is the difference of potentials for \( \nu_e \) and \( \nu_\mu \) in usual matter, so that \( l_0 \) is a characteristic of medium relevant for \( \nu_e - \nu_\mu \) mixing.

The refraction length enters the MSW-resonance condition \[6\],

\[ l_0 \cos 2\theta = l_\nu, \]

where \( l_\nu \equiv 4\pi E/\Delta m^2 \) is the vacuum oscillation length and \( \theta \) is the vacuum mixing angle. Inverse quantity, \( 1/l_0 \), determines the eigenfrequency of medium, and for small mixing the condition \[1\] means that the eigenfrequency of medium coincides with the eigenfrequency of neutrino system, \( 1/l_\nu \). For large mixing (strongly coupled system) there is a shift of resonance frequency given by \( \cos 2\theta \).
The ratio $l_0/l_\nu$ determines modifications of the mixing angle, $\theta_m$, and oscillation length in matter $l_m$:

$$\sin 2\theta_m = \sin 2\theta \left[ (\cos 2\theta - l_\nu/l_0)^2 + \sin^2 2\theta \right]^{1/2}.$$  

$$l_m = l_\nu \left[ (\cos 2\theta - l_\nu/l_0)^2 + \sin^2 2\theta \right]^{1/2}. \quad (6)$$

According to (6), at low energies (below resonance) one has $l_m \approx l_\nu$; with increase of energy the length $l_m$ first increases, it reaches maximal value, $l_{m\text{max}} = l_0/\sin 2\theta$, at $l_\nu = l_0/\cos 2\theta$ (i.e., above the resonance: $E_{\text{max}} = E_R/\cos^2 2\theta$) \(^1\), and then decreases approaching $l_0$ from above. In the non-resonance channel, $l_m$ increases with $E$ approaching $l_0$ from below. Thus, in the case of large densities of matter or large energies of neutrinos, when $l_\nu \gg l_0$, we obtain

$$l_m \approx l_0. \quad (7)$$

In other words, in the matter dominating case the oscillation length in matter approximately equals the refraction length. This is the key point of physics interpretation of the magic baseline in the next section.

### 3 Where does the magic baseline come from?

Let us consider oscillations of three mixed neutrinos $\nu_f \equiv (\nu_e, \nu_\mu, \nu_\tau)^T$ in the matter. The vacuum mixing matrix, $U_{PMNS}$, that relates $\nu_f$ and the mass eigenstates $\nu = (\nu_1, \nu_2, \nu_3)^T$, $\nu_f = U_{PMNS}\nu$, can be parametrized as

$$U_{PMNS} = U_{23}I_3U_{13}I_3 U_{12}. \quad (8)$$

Here $U_{ij} = U_{ij}(\theta_{ij})$ performs rotation in the $ij$-plane by the angle $\theta_{ij}$ and $I_3 \equiv \text{diag}(1,1,e^{i\delta})$ is the matrix of CP-violation phase.

The $\nu_\mu \rightarrow \nu_e$ transition probability can be represented as

$$P(\nu_\mu \rightarrow \nu_e) = \left| \cos \theta_{23} A_S e^{i\delta} + \sin \theta_{23} A_A \right|^2 \quad (9)$$

(see some details in sec. 4). Here $A_S$ is the solar amplitude that in the lowest order approximation (up to corrections of the order $\Delta m_{21}^2/\Delta m_{31}^2$, and $\sin^2 \theta_{13}$) depends on the solar neutrino oscillation parameters: $\Delta m_{21}^2$, $\theta_{12}$. $A_A$ is the atmospheric amplitude that depends on the atmospheric neutrino oscillation parameters, $\Delta m_{31}^2$, $\theta_{13}$. The CP-violation effects are given by interference of the two amplitudes in (9).

Let us consider the constant density medium that, in fact, is a very good approximation for neutrinos propagating in the mantle of the Earth. Up to the phase factor we obtain for the uniform medium

$$A_S = \sin 2\theta_{12}^m \sin \frac{\delta_{S}^m}{2}, \quad (10)$$

\(^1\)I am grateful to J. Kersten for correcting the maximal value.
where $\theta_{12}^m$ is the 1-2 mixing angle \cite{5} and $\phi_S^m$ is the oscillation phase in matter:

$$\phi_S^m = \frac{2\pi L}{l_m}, \quad (11)$$

the oscillation length, $l_m$, is given in \cite{5}. Apparently $A_S$ is a square root of the usual oscillation probability. (Similar expression can be written for $A_A$ with substitution 1-2 parameters by 1-3 oscillation parameters.)

Let us consider neutrinos with energies $E > (0.5 - 1)$ GeV relevant for accelerator experiments and atmospheric neutrino studies. For these energies $l_m(\Delta m_{21}^2) \equiv 4\pi E/\Delta m_{21}^2 \gg l_0$, and therefore the solar oscillation mode is in the matter dominating regime when

$$l_m(\Delta m_{21}^2) \approx l_0, \quad (12)$$

Correspondingly, the phase of oscillations equals

$$\phi_S^m \approx \frac{2\pi L}{l_0}, \quad (13)$$

and at the baseline $L = l_0$ we obtain

$$\phi_S^m = 2\pi, \quad A_S = 0. \quad (14)$$

The solar amplitude vanishes, and consequently,

$$P(\nu_\mu \rightarrow \nu_e) \approx |\sin \theta_{23} A_A|^2. \quad (15)$$

So, for high energies and $L = l_0$ the “magic” properties are reproduced:

- the interference and dependence on the phase $\delta$ disappear;
- the dependence on solar oscillation parameters disappear too.

Summarizing, the “magic” length is nothing but the refraction length, at least in the lowest order in small parameters. For high energies at the distance $L = l_0$, the phase of “solar” oscillation amplitude becomes $2\pi$. The amplitude vanishes, and therefore dependence of probability on CP-phase disappears. Similar consideration is valid for the $\nu_e - \nu_\tau$ channel with substitution $\sin \theta_{23} \rightarrow \cos \theta_{23}$ and $\cos \theta_{23} \rightarrow -\sin \theta_{23}$ in \cite{9}.

4 Correction to the equality $L_{\text{magic}} = l_0$

In the previous section we have neglected terms of the order $\sin^2 \theta_{13}$ and $\Delta m_{21}^2/\Delta m_{31}^2$ which can be as large as 3%. Also we have taken $l_m = l_0$, but at relatively low energies the deviation from this equality can be significant. Notice that the 3% correction to the refraction length of 7300 km equals 220 km which may be non-negligible for selection of a detector.
place, if high precision measurements of parameters are planned. One way to proceed is to perform numerical search of the baseline when the probability has the weakest dependence on $\delta$ [2]. Here we present some analytic consideration.

We will define the magic baseline as the distance over which the “solar” amplitude vanishes:

$$A_S(L_{\text{magic}}) = 0.$$  \hspace{1cm} (16)

In the constant density approximation (and in the lowest order in $\sin \theta_{13}$) this leads to the condition $L_{\text{magic}} = l_m$, and for high energies $l_m \approx l_0$. There are two types of corrections to the equality $L_{\text{magic}} \approx l_0$: (i) due deviation of $l_m$ from the asymptotic value, $l_0$, and (ii) due to dependence of $A_S$ on $\sin \theta_{13}$. We consider these corrections in order.

1). Corrections due to $l_m \neq l_0$: For $l_\nu \gg l_0$ we have

$$L_{\text{magic}} = l_m \approx l_0 \left(1 + \cos 2\theta_{12} \frac{l_0}{l_\nu}\right) = l_0 \left(1 + \cos 2\theta_{12} \frac{\Delta m_{21}^2}{2E}\right) \tag{17}$$

in the resonance channel. The correction is positive and equals $\approx 0.1/E$ (GeV). For a constant density medium, with increase of energy the correction and the magic baseline become smaller. However, in the case of realistic Earth density profile an average density along a trajectory increases with a length of trajectory, or equivalently, with $|\cos \Theta_\nu|$, where $\Theta_\nu$ is the zenith angle of the trajectory. Therefore $l_0$ decreases. So, for the Earth the magic trajectory does not change significantly with the energy being at $|\cos \Theta_\nu| \approx 0.6$. For the non-resonance channel the correction is negative (minus sign in Eq. (17)).

2). Corrections due to 1-3 mixing. To evaluate these corrections we need to give precise definitions of the amplitudes $A_S$ and $A_A$ in (9) and find their dependence on the oscillation parameters. The evolution of the flavor states $\nu_f$ is described by the Hamiltonian

$$H = U_{PMNS} \text{Diag}(0, \Delta_{21}, \Delta_{31}) U_{PMNS}^\dagger + \hat{V}, \tag{18}$$

where $\hat{V} = \text{diag}(V,0,0)$, $\Delta_{21} \equiv \Delta m_{21}^2/2E$ and $\Delta_{31} \equiv \Delta m_{31}^2/2E$. Let us define the neutrino propagation basis, $\tilde{\nu} = (\nu_e, \tilde{\nu}_2, \tilde{\nu}_3)^T$, through

$$\nu_f = U_{23} I_\delta \tilde{\nu}. \tag{19}$$

The Hamiltonian $\tilde{H}$ that describes oscillations of $\tilde{\nu}$ can be obtained from (18), (8) and (19):

$$\tilde{H} = U_{13} U_{12} \text{Diag}(0, \Delta_{21}, \Delta_{31}) U_{12}^\dagger U_{13}^\dagger + \hat{V},$$

or explicitly,

$$\tilde{H} = \begin{pmatrix} c_{13}^2 s_{12}^2 \Delta_{12} + V + s_{13}^2 \Delta_{13} & c_{13}s_{12}c_{12}\Delta_{12} & s_{13}c_{13}\Delta_{13} - s_{13}s_{13}s_{2}^2 \Delta_{12} \\ \vdots & c_{12}^2 \Delta_{12} & -s_{13}s_{12}c_{12}\Delta_{12} \\ \vdots & \vdots & c_{13}^2 \Delta_{13} + s_{13}s_{12}^2 \Delta_{12} \end{pmatrix}, \tag{20}$$
where $c_{13} \equiv \cos \theta_{13}$, $s_{13} \equiv \sin \theta_{13}$, etc.. Introducing the evolution matrix in the propagation basis as

$$\tilde{S} = | | A_{ij} ||, \quad i, j = e, 2, 3,$$

it is straightforward to find that the $\nu_\mu - \nu_e$ transition probability has the form (9) with

$$A_S = A_{e2}, \quad A_A = A_{e3}. \quad (22)$$

That is, the solar amplitude is given by the amplitude of transition $\tilde{\nu}_2 \rightarrow \nu_e$ and the atmospheric amplitude coincides with the $\tilde{\nu}_3 \rightarrow \nu_e$ transition amplitude. In the limit $\theta_{13} \rightarrow 0$ the state $\tilde{\nu}_3$ becomes the mass eigenstate $\nu_3$ and decouples from the rest of neutrino system, whereas $\tilde{\nu}_2$ becomes the combination of $\nu_\mu$ and $\nu_\tau$ that mixes with $\nu_e$ with the solar oscillation parameters. Therefore in this limit $A_{e2} = A_{e2}(\Delta m^2_{21}, \theta_{12})$ as we discussed in sec. 3.

For energies below the 1-3 resonance ($E \sim 1$ GeV), when the 33-element of the Hamiltonian dominates, the corrections can be calculated immediately. Performing an additional rotation of the neutrino basis by $U_{13}(\theta_{13})$ and then making block-diagonalization of the obtained Hamiltonian, we obtain that the heaviest state decouples and the two others form a $2\nu-$ system with the mixing angle $\theta_{12}$ and modified potential:

$$V \rightarrow V - s_{13}^2 V \left(1 + \frac{V}{\Delta_{13}}\right). \quad (23)$$

Correspondingly, the magic length is modified as

$$L_{\text{magic}} \approx l_0 \left[1 + \cos 2\theta_{12} \frac{l_0}{l_\nu} + s_{13}^2\right]. \quad (24)$$

Notice that the block-diagonalization removes the imaginary part of the corrections to the solar amplitude (see discussion below).

Let us present an estimation of the $s_{13}$-corrections that are valid in whole the energy range including the 1-3 resonance, and also take into account the imaginary part of the solar amplitude. We perform an additional 1-3 rotation of the neutrino basis that vanishes the 1-3 element of the Hamiltonian (20):

$$\tilde{\nu} = U_{13}(\theta_{13}^m) \nu_m. \quad (25)$$

Here $\nu_m \equiv (\nu_{1m}, \tilde{\nu}_2, \nu_{3m})$, and the angle is given by

$$\tan 2\theta_{13}^m = \frac{2\tilde{H}_{e3}}{H_{33} - H_{ee}}, \quad (26)$$

with $\tilde{H}_{ij}$ being the $ij$-element of the Hamiltonian (20). In the new basis, the Hamiltonian has the form

$$H^m = \begin{pmatrix}
H_{11}^m & \cos(\theta_{13}^m - \theta_{13})a_{12} & 0 \\
\vdots & H_{22}^m & \sin(\theta_{13}^m - \theta_{13})a_{12} \\
\vdots & \vdots & H_{33}^m
\end{pmatrix}, \quad (27)$$

$$...$$
where

\[ a_{12} \equiv s_{12} c_{12} \Delta_{12}, \]

\[
H_{11}^m = \cos^2 \theta_{13}^m \tilde{H}_{ee} - \sin 2\theta_{13}^m \tilde{H}_{e3} + \sin^2 \theta_{13}^m \tilde{H}_{33},
\]

\[
H_{33}^m = \sin^2 \theta_{13}^m \tilde{H}_{ee} + \sin 2\theta_{13}^m \tilde{H}_{e3} + \cos^2 \theta_{13}^m \tilde{H}_{33}.
\]

Since we are looking for the corrections due to the 1-3 mixing the terms \( \propto \Delta_{12} \) can be omitted in the diagonal elements, \( H_{11}^m, H_{33}^m \), and consequently we obtain

\[
H_{11}^m = V \cos \theta_{13}^m \cos \theta_{13}^m, \quad H_{33}^m = V \cos \theta_{13}^m \sin \theta_{13}^m.
\]

Here we used expression for \( \Delta_{13} \) in terms of mixing angle in matter that can be obtained from (26):

\[ \Delta_{13} = \frac{V \sin 2\theta_{13}^m}{\sin 2(\theta_{13}^m - \theta_{13})}. \]

In the propagation basis, the transition \( \nu_e \rightarrow \tilde{\nu}_2 \) proceeds in two different ways: \( \nu_e \rightarrow \nu_{1m} \rightarrow \tilde{\nu}_2 \) and \( \nu_e \rightarrow \nu_{3m} \rightarrow \tilde{\nu}_2 \). Therefore the “solar” amplitude can be written as

\[ A_{e2} = \cos \theta_{13}^m A_{12}^m + \sin \theta_{13}^m A_{32}^m, \]

where \( A_{12}^m \) and \( A_{32}^m \) are the amplitudes of \( \nu_{1m} \rightarrow \tilde{\nu}_2 \) and \( \nu_{3m} \rightarrow \tilde{\nu}_2 \) transitions correspondingly. Then in the lowest approximation using the Hamiltonian (27) we obtain for (32):

\[
A_{e2} \approx 2a_{12} \left[ \cos \theta_{13}^m \cos(\theta_{13}^m - \theta_{13}) \frac{\sin(H_{11}^m L/2)}{H_{11}^m} e^{-iH_{11}^m L/2} + \sin \theta_{13}^m \sin(\theta_{13}^m - \theta_{13}) \frac{\sin(H_{33}^m L/2)}{H_{33}^m} e^{-iH_{33}^m L/2} \right].
\]

For energies much below the 1-3 resonance we have \( \sin \theta_{13}^m \approx \sin \theta_{13} \ll 1 \). Therefore the second term in (33) is strongly suppressed, furthermore \( H_{11}^m \approx \tilde{H}_{ee} \approx V \), and consequently, the amplitude is reduced to the one considered in sec. 3 is recovered. For energies above the 1-3 resonance, \( \cos \theta_{13}^m \rightarrow 0 \) and the first term in (32) vanishes. Now \( H_{33}^m \approx \tilde{H}_{ee} \approx V \) and again we recover the result of sec. 3.

Minimal value of \( A_{e2} \) corresponds to \( H_{11}^m L \approx H_{33}^m L \approx 2\pi \) when both terms in (33) are close to zero. (We confirm this by explicit calculation.) Then introducing small quantities

\[ \epsilon_1 = \frac{1}{2} H_{11}^m L - \pi, \quad \epsilon_3 = \frac{1}{2} H_{33}^m L - \pi \]

(\( \epsilon_i \ll \pi \)), and taking the first terms of expansions of sines and exponents in (33) around \( \pi \) we obtain:

\[
A_{e2} \approx 2a_{12} \left[ \cos \theta_{13}^m \cos(\theta_{13}^m - \theta_{13}) (1 - i\epsilon_1) \left( \frac{L}{2} - \frac{\pi}{H_{11}^m} \right) + \sin \theta_{13}^m \sin(\theta_{13}^m - \theta_{13}) (1 - i\epsilon_3) \left( \frac{L}{2} - \frac{\pi}{H_{33}^m} \right) \right].
\]
The imaginary part contains an additional power of small parameters \( \epsilon_i \).

Using explicit expressions for the \( H_{11}^m \) and \( H_{33}^m \) we obtain from (35) the real and imaginary parts of the amplitude:

\[
A_e^{(R)} = c_{13}s_{12}c_{12}\Delta_{12} \left( L - \frac{2\pi}{Vc_{13}^2} \right),
\]

\[
A_e^{(I)} = -\frac{1}{2}c_{13}s_{12}c_{12}\Delta_{12}V \left[ \left( L - \frac{2\pi}{V} \right)^2 + \frac{4\pi^2}{V^2}\tan^2\theta_{13} \right].
\]

Let us analyze these results.

1). The real part of the amplitude vanishes if

\[
L = \frac{2\pi}{Vc_{13}^2}
\]

that coincides with the baseline obtained in (38).

2). The imaginary part is always non-zero with minimum

\[
|A_e^{(I)}|_{\min} = 2\pi^2s_{13}^2s_{12}c_{12}\Delta_{12}
\]

at \( L = 2\pi/V \).

3). At the baseline that corresponds to zero real part (38), we obtain from (37) that the correction to the minimal value (39) is of the order \( s_{13}^4 \): 

\[
|A_e^{(I)}| = |A_e^{(I)}|_{\min}(1 + s_{13}^2).
\]

This means that the baseline (38) provides a minimum of the total amplitude in the order \( s_{13}^2 \), and therefore it can be identified with the magic length. When corrections due to 1-3 mixing are included, the solar amplitude does not vanish exactly, and therefore the magic properties are satisfied only approximately.

4). The amplitudes (36, 37) do not depend on the mixing angle in matter, and therefore the results are valid in whole energy range including the 1-3 resonance region and the region above the resonance.

5 The case of non-constant density

In the case of non-constant potential (density) along the neutrino trajectory, the refraction length can be defined by the condition

\[
\int_0^{l_0} dx V(x) = 2\pi.
\]

7
In the lowest order, for the "solar" amplitude in the matter dominating case one can obtain the following expression (see [7] for details)

\[ A_{S} = \frac{1}{2} \sin 2\theta_{12} \Delta_{12} \int_{0}^{L} dx \exp \left( -i \int_{x}^{L} dy V(y) \right). \] (41)

Consequently, the magic baseline can be found from the condition

\[ \int_{0}^{L_{\text{magic}}} dx \exp \left( i \int_{0}^{x} dy V(y) \right) = 0. \] (42)

The double integration takes into account the change of both oscillation length and mixing angle along the trajectory.

Let us consider the three-layer profile with constant potentials (densities) \( V_{m}, V_{c} \) and \( V_{m}, \) and baselines \( L_{m}, L_{c} \) and \( L_{m}. \) The densities and baselines in the first and the third layers coincide. To a good approximation that corresponds to profile along the neutrino trajectories that cross the core of the earth with \( V_{m} \) and \( V_{c} \) being the potentials in the mantle and the core correspondingly.

Performing integration in eq. (41) we find

\[ A_{S} = \sin 2\theta_{m} \left[ \sin \left( \frac{\phi_{c}}{2} + \phi_{m} \right) - \left( 1 - \frac{V_{m}}{V_{c}} \right) \sin \frac{\phi_{c}}{2} \right], \] (43)

where

\[ \sin 2\theta_{m} \approx \sin 2\theta_{12} \frac{\Delta_{12}}{V_{m}} \] (44)

is the mixing angle in the mantle, and

\[ \phi_{m} = V_{m} L_{m}, \quad \phi_{c} = V_{c} L_{c} \] (45)

are the phases acquired in the mantle and the core. The sum \( \phi_{c} + 2\phi_{m} \) is the total oscillation phase. The first term in (43) corresponds to the amplitude of pure adiabatic transition. It depends on the mixing angle at the surface of the earth and on the total phase. The second term is the correction due to the adiabaticity violation at the border between the mantle and the core. So, in the presence of the adiabaticity violation the amplitude is not determined by total phase.

According to (43) the magic baseline is determined by the condition

\[ \sin \left( \frac{\phi_{c}}{2} + \phi_{m} \right) = \left( 1 - \frac{V_{m}}{V_{c}} \right) \sin \frac{\phi_{c}}{2}. \] (46)

For constant density along whole trajectory \( (V_{c} = V_{m}) \) or in the adiabatic case, eq. (46) would lead to

\[ \frac{\phi_{c}}{2} + \phi_{m} = \pi k; \quad k = 1, 2, 3 \ldots . \] (47)
The violation of adiabaticity modifies this condition, and apparently, the bigger the difference of the potentials $V_m$ and $V_c$ the stronger the deviation from (47).

In the case of the Earth profile, the baselines $L_m$ and $L_c$ are correlated: they are determined by the zenith angle, $\Theta_\nu$, of neutrino trajectory:

$$L_m = R|\cos \Theta_\nu| - L_c/2, \quad L_c = 2\sqrt{R^2 \cos^2 \Theta_\nu - (R^2 - R_c^2)}.$$

(48)

Here $R = 6370$ km is the radius of the Earth and $R_c = 3486$ km is the radius of the core. Using eqs. (45), (48) we obtain that the condition (46) is satisfied for

$$|\cos \Theta_\nu|_{\text{magic}} \approx 0.88, \quad \text{and} \quad |\cos \Theta_\nu|_{\text{magic}} \approx 0.98.$$  

(49)

(For this estimation we took the potentials according to the average densities in the mantle and in the core $\rho_m = 4.5$ g/cm$^3$ and $\rho_c = 11.5$ g/cm$^3$.) Thus, there are two magic trajectories for neutrinos that cross the core. They correspond to the baselines $L = 2R|\cos \Theta_\nu|$, $L_{\text{magic}} = 11210$ km and $L_{\text{magic}} = 12485$ km. For these baselines the total oscillations phases, $\phi_c + 2\phi_m$, equal $\sim 4\pi$ and $\sim 6\pi$.

These new magic baseline could of interest for the atmospheric neutrino studies.

6 More magic baselines

The interference, and consequently, dependence on the CP-phase vanish also when $A_A = 0$. In this case the probability is determined by the solar amplitude:

$$P(\nu_e \to \nu_e) \approx |\cos \theta_{23} A_S|^2.$$  

(50)

Now the probability is determined by the solar parameters and can be used for measurements of $\theta_{12}$ as well as and $\theta_{23}$.

In the constant density approximation, up to the phase factor, the atmospheric amplitude equals

$$A_A = \sin 2\theta_{13}^m \sin \frac{\phi_A^m}{2}.$$  

(51)

(The phase factor $\exp(-i\Delta_{13} L/2)$ is omitted in (51). This factor should be restored if both solar and atmospheric amplitudes are non-zero.) So, the condition $A_A = 0$ gives the “magic” baseline (integer of the oscillation length in matter)

$$L_{\text{magic}} = nl_\nu[(\cos 2\theta_{13} - l_\nu/l_0)^2 + \sin^2 2\theta_{13}]^{-1/2},$$

(52)

$n = 1, 2, \ldots$ and here $l_\nu = 4\pi E/\Delta m_{13}^2$.

An interesting range of energies is below 3-4 GeV where the amplitude $A_S$ is not suppressed too strongly by matter effects. Here, however, the “magic” baseline strongly depends
on neutrino energy. Therefore, a narrow energy neutrino beam should be employed or recon-
struction of the neutrino energy should be done to suppress the interference and dependence
of the probability on $\delta$. For $E_{\nu} = 1$ GeV the magic baselines are 1080 km ($n = 1$), 2070 km
($n = 2$), 3250 km ($n = 3$), etc.. For $L = 3700$ km that maximizes the solar amplitude, the
“magic” energies are at 1.1 GeV, 1.5 GeV and 2.5 GeV.

7 Conclusion

The magic baseline (in the first approximation) is an integer of the refraction length. At
high energies the latter approximately equals the oscillation length in matter. Therefore at
the “magic” baseline the phase of oscillations driven by the solar mass splitting is $2\pi$, and
consequently the solar amplitude vanishes in the transition probability. The interference
of amplitudes, and consequently, dependence of probability on the CP-phase disappear.
Defining the magic baseline as the distance that corresponds to vanishing solar amplitude,
we have estimated various corrections to the equality $L_{\text{magic}} = l_0$. The magic lengths in the
non-uniform medium were discussed, and it is found that two additional magic trajectories
exist for neutrinos crossing the core. We also discussed features of the magic baselines that
correspond to zeros of the atmospheric amplitude.

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References

[1] V. Barger, D. Marfatia and K. Whisnant, Phys. Rev. D 65 (2002) 073023
arXiv:hep-ph/0112119.

[2] P. Huber and W. Winter, Phys. Rev. D 68 (2003) 037301 arXiv:hep-ph/0301257.

[3] see e.g., P. Huber, M. Lindner, M. Rolinec and W. Winter, Phys. Rev. D 74 (2006)
073003 arXiv:hep-ph/0606119;
A. Blondel, A. Cervera-Villanueva, A. Donini, P. Huber, M. Mezzetto and P. Strolin,
Acta Phys. Polon. B 37 (2006) 2077 arXiv:hep-ph/0606111;
A. Donini, E. Fernandez-Martinez, D. Meloni and S. Rigolin, Nucl. Phys. B 743 (2006) 41 [arXiv:hep-ph/0512038];

P. Huber, M. Lindner, M. Rolinec and W. Winter, Phys. Rev. D 73 (2006) 053002 [arXiv:hep-ph/0506237];

R. Gandhi, P. Ghoshal, S. Goswami, P. Mehta and S. Uma Sankar, [arXiv:hep-ph/0506145];

R. Gandhi, P. Ghoshal, S. Goswami, P. Mehta and S. Uma Sankar, Phys. Rev. D 73 (2006) 053001 [arXiv:hep-ph/0411252];

E. K. Akhmedov, R. Johansson, M. Lindner, T. Ohlsson and T. Schwetz, JHEP 0404 (2004) 078 [arXiv:hep-ph/0402175];

W. Winter, Phys. Lett. B 613 (2005) 67 [arXiv:hep-ph/0411309].

[4] P. Huber, J. Phys. G 29 (2003) 1853 [arXiv:hep-ph/0210140].

[5] L. Wolfenstein, Phys. Rev. D 17 (1978) 2369.

[6] S. P. Mikheyev and A. Yu. Smirnov, Yad. Fiz. 42 (1985) 1441, [Sov. Jour. Nucl. Phys. 42 (1985) 913].

[7] E. K. Akhmedov, M. Maltoni and A. Y. Smirnov, Phys. Rev. Lett. 95 (2005) 211801 [arXiv:hep-ph/0506064].