STRONG APPROXIMATIONS FOR NONCONVENTIONAL SUMS WITH APPLICATIONS TO LAW OF ITERATED LOGARITHM AND ALMOST SURE CENTRAL LIMIT THEOREM

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Abstract. We improve, first, a strong invariance principle from [9] for nonconventional sums of the form
\[ \sum_{n=1}^{[Nt]} F(X(n), X(2n), ..., X(\ell n)) \] (normalized by $1/\sqrt{N}$) where $X(n), n \geq 0$’s is a sufficiently fast mixing vector process with some moment conditions and stationarity properties and $F$ satisfies some regularity conditions. Applying this result we obtain next a version of the law of iterated logarithm for such sums, as well as an almost sure central limit theorem. Among motivations for such results are their applications to multiple recurrence for stochastic processes and dynamical systems.

1. Introduction

In this paper we study almost sure limit theorems for nonconventional sums of the form
\[ \Xi(t) = \sum_{1 \leq n \leq t} (F(X(n), X(2n), ..., X(\ell n)) - \bar{F}) \]
where $X(n), n \geq 0$’s is a sufficiently fast mixing vector valued process with some moment conditions and stationarity properties, $F$ is a continuous function with polynomial growth and certain regularity properties, $\bar{F} = \int Fd(\mu \times \cdots \times \mu)$ and $\mu$ is the distribution of $X(0)$. The name ”nonconventional” comes from [4] where ergodic theorems for averages $N^{-1}\Xi(N)$ were studied in the case when $X(n) = X(n, x) = T^n x$ with $T$ being a measure preserving ergodic transformation. We observe that the topic of nonconventional ergodic theorems was extensively studied during the last 30 years.

Recently the setup of nonconventional sums was studied from the probabilistic point of view and the strong law of large numbers, the functional central limit theorem and a version of the strong invariance principle (called also a strong approximation theorem) were obtained in [8], [10] and [9], respectively. In this paper we partially improve the strong invariance principle from [9] which enable us both to obtain a better than in [9] version of the law of iterated logarithm and, moreover, to derive an almost sure central limit theorem for sums $\Xi(t)$. We will show in this
paper that the sum $\Xi(t)$ can be approximated as $t \to \infty$ with an error term of order $t^{2-\gamma}$, $\gamma > 0$ by certain Gaussian process $G(t)$ having, in general, dependent increments. This will enable us to obtain the law of iterated logarithm and the almost sure central limit by certain modifications of familiar proofs. We observe that in [2] strong approximations were obtained only for certain components of the sum $\Xi(t)$ which did not allow to derive the corresponding result for the whole sum. On the other hand, terms $X(q(n))$ with nonlinear $q(n)$ were considered in [3] while we do not deal with them here.

One of motivations for nonconventional limit theorems comes from multiple recurrence problems. Let, for instance, $F(x_1, ..., x_\ell) = x_1 \cdots x_\ell$ and $X(n) = \mathbb{I}_A(\xi_n)$ where $\mathbb{I}_A$ is the indicator of a set $A$ and $\xi_n$ is either a dynamical system $\xi_n = \xi_n(\omega) = T^n \omega$ or a Markov chain. Then $\Xi(N) = \sum_{n=1}^{N} F(X(n), ..., X(\ell n))$ is the number of events $\{ \xi_n \in A, \xi_{2n} \in A, ..., \xi_{\ell n} \in A \}$ for $n \leq N$, and so our results describe limiting behavior of such quantities. This also can be described as a number of arithmetic progressions of length $\ell$ starting at 0 whose difference is between 1 and $N$ and such that at any positive time belonging to such progression the process $\xi$ is contained in $A$.

As in [3], [6] and [10] our results hold true when, for instance, $X(n) = T^nf$ where $f = (f_1, ..., f_\ell)$, $T$ is a mixing subshift of finite type, a hyperbolic diffeomorphism (see [11]) or an expanding transformation taken with a Gibbs invariant measure, as well, as in the case when $X(n) = f(\xi_k)$, $f = (f_1, ..., f_\ell)$ where $\xi_n$ is a Markov chain satisfying the Doeblin condition (see, for instance, [2]) considered as a stationary process with respect to its invariant measure. Furthermore, our results are applicable to other dynamical systems such as the Gauss map of the interval (see, for instance, [7] or [6]) and a large class of transformations having a spectral gap of their transfer operator which ensure their fast mixing properties. On the probabilistic side our results work for Markov processes having transition operators with a spectral gap, in particular, for Ornstein-Uhlenbeck type processes.

2. PRELIMINARIES AND MAIN RESULTS

As in [3] we deal with the setup which consists of a $\varphi$-dimensional stochastic process $\{X(n), n = 0, 1, ...\}$ on a probability space $(\Omega, \mathcal{F}, P)$ and of a family of $\sigma$-algebras $\mathcal{F}_{kl} \subset \mathcal{F}$, $-\infty \leq k \leq l \leq \infty$ such that $\mathcal{F}_{kl} \subset \mathcal{F}_{k'l'}$ if $k' \leq k$ and $l' \geq l$. The dependence between two sub $\sigma$-algebras $\mathcal{G}, \mathcal{H} \subset \mathcal{F}$ is measured often via the quantities

\begin{equation}
\varphi_{q,p}(\mathcal{G}, \mathcal{H}) = \sup \{|E[g|\mathcal{G}] - E[g]|_p : g \text{ is } \mathcal{H} - \text{ measurable and } \|g\|_q \leq 1\},
\end{equation}

where the supremum is taken over real functions and $\| \cdot \|_p$ is the $L^p(\Omega, \mathcal{F}, P)$-norm.

Then more familiar $\alpha, \rho, \phi$ and $\psi$-mixing (dependence) coefficients can be expressed in the form (see [2], Ch. 4)

\begin{align*}
\alpha(\mathcal{G}, \mathcal{H}) &= \frac{1}{2} \varphi_{\infty,1}(\mathcal{G}, \mathcal{H}), \\
\rho(\mathcal{G}, \mathcal{H}) &= \varphi_{2,2}(\mathcal{G}, \mathcal{H}) \\
\phi(\mathcal{G}, \mathcal{H}) &= \frac{1}{2} \varphi_{\infty,\infty}(\mathcal{G}, \mathcal{H}) \text{ and } \psi(\mathcal{G}, \mathcal{H}) = \varphi_{1,\infty}(\mathcal{G}, \mathcal{H}).
\end{align*}

The relevant quantities in our setup are

\begin{equation}
\varphi_{q,p}(n) = \sup_{k \geq 0} \varphi_{q,p}(\mathcal{F}_{k+n,\infty})
\end{equation}
and accordingly

\[
\alpha(n) = \frac{1}{4} w_{\infty, 1}(n), \quad \rho(n) = w_{2, 2}(n), \quad \phi(n) = \frac{1}{2} w_{\infty, \infty}(n) \quad \text{and} \quad \psi(n) = w_{1, \infty}(n).
\]

Our assumptions will require certain speed of decay as \( n \to \infty \) of both the mixing rates \( w_{q, p}(n) \) and the approximation rates defined by

\[
\beta_p(n) = \sup_{m \geq 0} \| X(m) - E(X(m) | F_{m-n, m+n}) \|_p.
\]

In what follows we can always extend the definitions of \( F_{kl} \) given only for \( k, l \geq 0 \) to negative \( k \) by defining \( F_{kl} = F_{0l} \) for \( k < 0 \) and \( l \geq 0 \). Furthermore, we do not require stationarity of the process \( X(n), n \geq 0 \) assuming only that the distribution of \( X(n) \) does not depend on \( n \) and the joint distribution of \( \{ X(n), X(n') \} \) depends only on \( n - n' \) which we write for further references by

\[
X(n) \overset{d}{\sim} \mu \quad \text{and} \quad (X(n), X(n')) \overset{d}{\sim} \mu_{n-n'}, \quad \text{for all} \quad n, n'
\]

where \( Y \overset{d}{\sim} Z \) means that \( Y \) and \( Z \) have the same distribution.

Next, let \( F = F(x_1, \ldots, x_\ell), \quad x_j \in \mathbb{R}^p \) be a function on \( \mathbb{R}^{p\ell} \) such that for some \( \iota, K > 0, \kappa \in (0, 1] \) and all \( x_i, y_i \in \mathbb{R}^p, i = 1, \ldots, \ell, \)

\[
|F(x_1, \ldots, x_\ell) - F(y_1, \ldots, y_\ell)| \leq K \{ 1 + \sum_{j=1}^{\ell} |x_j|^\iota + \sum_{j=1}^{\ell} |y_j|^\iota \} \sum_{j=1}^{\ell} |x_j - y_j|^\kappa
\]

and

\[
|F(x_1, \ldots, x_\ell)| \leq K \{ 1 + \sum_{j=1}^{\ell} |x_j|^\iota \}.
\]

The above assumptions allow us to consider, for instance, functions \( F \) polynomially dependent on their arguments, in particular, the product function \( F(x_1, \ldots, x_\ell) = x_1 \cdots x_\ell \). To simplify formulas we assume a centering condition

\[
\bar{F} = \int F(x_1, \ldots, x_\ell) \, d\mu(x_1) \cdots d\mu(x_\ell) = 0
\]

which is not really a restriction since we always can replace \( F \) by \( F - \bar{F} \).

For each \( \theta > 0 \) set

\[
\gamma_\theta^0 = \|X\|^\theta = E|X(n)|^\theta = \int \|x\|^\theta \, d\mu.
\]

Our main result relies on

2.1. **Assumption.** With \( d = (\ell - 1)\varphi \) there exist \( p, q \geq 1 \) and \( \delta, m > 0 \) with \( \delta < \kappa - \frac{2}{p} \) satisfying

\[
\sum_{n=0}^{\infty} n w_{q, p}(n) < \infty, \quad \sum_{r=0}^{\infty} r^{\frac{16}{p} + \delta} \beta^\delta_q(r) < \infty, \quad \gamma_m < \infty, \quad \gamma_{2q(\iota + 2)} < \infty \quad \text{with} \quad \frac{1}{2 + \delta} \geq \frac{1}{p} + \frac{\iota + 2}{m} + \frac{\delta}{q}.
\]
A reader willing to avoid some visual technicalities may be advised to consider bounded Lipschitz continuous functions $F$ and exponentially fast decaying $\beta_q(n)$ as $n \to \infty$ but since we mostly rely in this paper on estimates from [10] and [9] this would not matter much here.

As in [10] a crucial part of our approach is the representation of $F = F(x_1, \ldots, x_\ell)$ in the form
\begin{equation}
F = F_1(x_1) + \cdots + F_\ell(x_1, x_2, \ldots, x_\ell)
\end{equation}
where for $i < \ell$,
\begin{align*}
F_i(x_1, \ldots, x_i) &= \int F(x_1, x_2, \ldots, x_\ell) \, d\mu(x_{i+1}) \cdots d\mu(x_\ell) \\
&\quad - \int F(x_1, x_2, \ldots, x_\ell) \, d\mu(x_i) \cdots d\mu(x_\ell)
\end{align*}
and
\begin{align*}
F_\ell(x_1, x_2, \ldots, x_\ell) &= F(x_1, x_2, \ldots, x_\ell) - \int F(x_1, x_2, \ldots, x_\ell) \, d\mu(x_\ell)
\end{align*}
which ensures, in particular, that
\begin{equation}
\int F_i(x_1, x_2, \ldots, x_{i-1}, x_i) \, d\mu(x_i) \equiv 0 \quad \forall \quad x_1, x_2, \ldots, x_{i-1}.
\end{equation}
These enable us to write
\begin{equation}
\Xi(t) = \sum_{i=1}^{\ell} \Psi_i(it)
\end{equation}
where for $1 \leq i \leq \ell$,
\begin{equation}
\Psi_i(t) = \sum_{1 \leq n \leq t/i} F_i(X(n), X(2n), \ldots, X(in)).
\end{equation}

As in [15] we will write $Z(t) \ll a(t)$ a.s. for a family of random variables $Z(t), t \geq 0$ and a positive function $a(t), t \geq 0$ if $\limsup_{t \to \infty} |Z(t)/a(t)| < \infty$ almost surely (a.s.)

2.2. Theorem. Suppose that Assumption 2.1 holds true. Then without changing its distribution the process $\Xi(t), t \geq 0$ given by (1.1) can be redefined on a richer probability space where there exists an $\ell$-dimensional Gaussian process $G(t) = (G_1(t), \ldots, G_\ell(t))$ with stationary independent increments having covariances $EG_i(s)G_j(t) = D_{ij}s \wedge t$, for some nonnegatively definite matrix $D = (D_{ij}, 1 \leq i, j \leq \ell)$ and such that for some constant $\gamma > 0$,
\begin{equation}
\Xi(t) - Q(t) \ll t^{1/2-\gamma} \quad \text{a.s.}
\end{equation}
where $Q(t) = \sum_{j=1}^{\ell} G_j(jt)$ is a Gaussian process having, in general, dependent increments (see [10]).

As in [9] we will rely on the representation (2.10) but in [9] we were able to obtain strong invariance principles only for each $\Psi_i$ separately while Theorem 2.2 provides a strong invariance principle for the original process $\Xi$. The proof of Theorem 2.2 will rely on strong approximation of the vector process $\Psi(t) = (\Psi_1(t), \ldots, \Psi_\ell(t))$ by the vector Gaussian process $\tilde{G}(t)$ which unlike [9] cannot be done via the Skorokhod embedding and we will have to verify conditions of another approach from [16] (see also [15] and references there).
Let

\[ R(s, t) = EQ(s)Q(t) = \sum_{1 \leq i, j \leq \ell} D_{ij}((is) \wedge (jt)) \]

be the covariance function of the Gaussian process \( Q(t), t \geq 0 \). Observe that \( R(rs, rt) = rR(s, t) \) for any \( r > 0 \). This together with (2.12) and (2.13) enable us to rely on the law of iterated logarithm for Gaussian processes from [14] which is formulated in terms of the reproducing kernel Hilbert space \( H \) corresponding to the kernel \( R(s, t) \). Namely, we obtain

2.3. Corollary. Let \( K = \{ h \in H : \| h \|_H R(1, 1) \leq 1 \} \) where \( \| \cdot \|_H \) denotes the norm in \( H \). Then the sequence of random functions

\[ f_n(t) = \Xi(nt)(2R(n, n) \ln \ln n)^{-1/2}, \quad n \geq 3 \]

is a.s. equicontinuous and its set of limit points is a.s. contained in \( K \).

In order to conclude that the set of limit points of the sequence \( f_n \) coincides with \( K \) we have to impose at least some nondegeneracy conditions on the pair of \( F \) and the process \( X(n), n \geq 0 \) but this question is not quite clear yet. It seems natural to expect that for generic (in some sense) pairs of \( F \) and \( X(n), n \geq 0 \) the set of limit points of the sequence \( f_n \) will coincide with \( K \).

Next, we describe our results concerning an a.s. central limit theorem. For each \( t \in [0, 1] \) and \( n \in \mathbb{N} \) set

\[ Q_n(t) = n^{-1/2}\Xi([nt])(1 + [nt] - nt) + n^{-1/2}\Xi([nt] + 1)(nt - [nt]) \]

which produces a random element of the space \( \mathbb{C}[0, 1] \) of continuous functions on \( [0, 1] \) considered with the supremum norm topology. Denote also by \( \eta_Q \) the probability distribution of the process \( Q(t), t \in [0, 1] \) on \( \mathbb{C}[0, 1] \) while, as usual, by \( \delta_x \) with \( x \in \mathbb{C}[0, 1] \) we denote the unit mass at \( x \).

2.4. Theorem. Under Assumption 2.1 and notations above

\[ \lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} k^{-1} \delta_{Q_k} = \eta_Q \quad \text{a.s.} \]

where the limit is taken in the sense of weak convergence of measures on \( \mathbb{C}[0, 1] \).

This result can be rephrased saying that outside of a single probability zero set

\[ \lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} k^{-1} \delta_{\phi(Q_k)} = \phi(\eta_Q) \quad \text{a.s.} \]

for all measurable functions \( \phi \) on \( \mathbb{C}[0, 1] \) which are continuous \( \eta_Q \)-a.s., where \( \phi(\eta_Q) \) is the image measure of \( \eta_Q \) under \( \phi \). Letting, in particular, \( \phi_1(x) = x(1) \) we obtain that a.s. as \( n \to \infty \),

\[ \lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} k^{-1} \delta_{\Xi(k)k^{-1/2}} = \text{distribution of } Q(1). \]

Our proof of Theorem 2.4 will follow the scheme of [3] (see also [11]) deriving first an almost sure central limit theorem for the Gaussian process \( Q \) from Theorem 2.2 and then relying on the strong approximation result 2.12 in place of the Skorokhod embedding (representation) employed in [3] which does not work here.
2.5. Remark. An analogy of Corollary 1.5 from [3] concerning the arcsine distribution can also be obtained in our circumstances. Namely, let
\[ L_n = n^{-1} \# \{ k : \Xi(k) > 0, 1 \leq k \leq n \} \]
and define \( \phi : C[0,1] \to \mathbb{R} \) by \( \phi(x) = \text{Lebesgue measure of} \{ u \in [0,1]: x(u) > 0 \} \).
It is easy to see that \( \phi \) is \( \eta_Q \)-a.s. continuous and it follows from Theorem 2.4 that with probability one
\[
\lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} k^{-1} \delta_{\phi(Q_k)} = \phi(\eta_Q)
\]
where the right hand side of (2.19) can be interpreted as an analogy of the arcsine law corresponding to the Gaussian process \( Q \) in the same sense as the usual arcsine law corresponds to the standard Brownian motion. Now, with probability one
\[
\lim_{n \to \infty} |\phi(Q_n) - L_n| = 0
\]
and employing an analogy of Lemma 2.12 from [3] (see Lemma 4.2 below) we obtain that with probability one
\[
\lim_{n \to \infty} \frac{1}{\ln n} \sum_{k=1}^{n} k^{-1} \delta_{L_k} = \phi(\eta_Q).
\]

2.6. Remark. In addition to almost sure central limit theorem large deviations from the limit were studied, as well (see [13] and [5]). These results were also based on strong approximations but it was crucial there to rely for that on the Skorokhod embedding into the Brownian motion where rather specific estimates are available. Namely, for these large deviations estimates the result of Theorem 2.2 in the form (2.12) does not suffice since now we have to show that for each \( \varepsilon > 0 \),
\[
\lim_{t \to \infty} \frac{1}{\ln t} \ln \mathbb{P} \{ \sup_{0 \leq s \leq t} |\Xi(s) - Q(s)| > \varepsilon \sqrt{t} \} = -\infty.
\]
We can do this for each \( |\Psi_i(s) - G_i(s)| \) separately obtaining \( G_i \) via the Skorokhod embedding as in [9] but for \( \Xi \) we need a multidimensional version of strong approximations for which estimates of the form (2.21) do not seem to be directly available though they should be possible under appropriate conditions. For instance, (2.21) would hold true if for any \( m \in \mathbb{N}, t > 0 \) and some \( C, \varepsilon > 0 \),
\[
E \sup_{0 \leq s \leq t} |\Xi(s) - Q(s)|^m \leq C t^{m(\frac{1}{2} - \varepsilon)}.
\]
The latter inequality can be obtained via estimates of the form
\[ P\{ |A_n - B_n| \geq \alpha_n \} \leq \alpha_n, \]
where \( A_n = \Psi_i(n) - \Psi_i(n-1) \) and \( B_n = G_i(n) - G_i(n-1) \), which appear in proofs of usual multidimensional strong approximation theorems (cf., for instance, Section 2.4 in [12]).

3. Strong approximations

The following result which is a part of Corollary 3.6 from [10] will be a basis of estimates here.
3.1. Proposition. Let $\mathcal{G}$ and $\mathcal{H}$ be $\sigma$-subalgebras on a probability space $(\Omega, \mathcal{F}, P)$, $X$ and $Y$ be $d$-dimensional random vectors and $f = f(x, \omega)$ be collections of random variables that are continuously (or separable) dependent on $x \in \mathbb{R}^d$ for almost all $\omega$, measurable with respect to $\mathcal{H}$ and satisfy
\[
\|f(x, \omega) - f(y, \omega)\|_q \leq C_1 (1 + |x|^\alpha + |y|^\alpha) |x - y|^\kappa
\]
and
\[
\|f(x, \omega)\|_q \leq C_2 (1 + |x|^\alpha).
\]
Set $\tilde{f}(x, \omega) = E(f(x, \cdot)|\mathcal{G})$ and $g(x) = E(f(x, \omega)$.

(i) Assume that $q \geq p$, $1 \geq \kappa > \theta > \frac{d}{p}$ and $\frac{1}{2} + \frac{1}{m} + \frac{\kappa}{q}$. Then
\[
\|E(f(X, \cdot)|\mathcal{G}) - g(X)\|_a \leq c \left( 1 + \|X\|^{a+2}_m \right) (\tau_{q,p}(\mathcal{G}, \mathcal{H}) + \|X - E(X|\mathcal{G})\|_q^\delta)
\]
where $c = c(\iota, \kappa, \theta, p, q, a, \delta, d) > 0$ depends only on the parameters in brackets.

(ii) Furthermore, let $x = (v, z)$ and $X = (\Pi, Y)$, where $\Pi$ and $Y$ are $d_1$ and $d - d_1$-dimensional random vectors, respectively, and let $f(x, \omega) = f(v, z, \omega)$ satisfy the conditions above in $x = (v, z)$. Set $\tilde{g}(v) = E[f(v, Y(\cdot), \omega)]$. Then
\[
\|\tilde{f}(X(\omega), \omega) - \tilde{g}(Y(\omega), \omega) - g(X) + g(Y)\|_a
\]
\[
\leq c \tau_{q,p}(\mathcal{G}, \mathcal{H}) (1 + \|X\|^{a+2}_m + \|Y\|^{a+2}_m) \|X - Y\|_q^\delta
\]
where $c = c(\iota, \kappa, \theta, p, q, a, \delta, d) > 0$ depends only on the parameters in brackets.

Observe that the conditions of Proposition 3.1 are satisfied in all our applications below in view of (2.36) and (2.37). We will rely on the following result which in a close form appears as Theorem 1.3 in [16] (see also [15] and references there).

3.2. Theorem. Let $\{M(n), \mathcal{G}_n\}_{n=1}^\infty$, $M(n) = (M_1(n), ..., M_\ell(n))$ be a square-integrable sequence of $\mathbb{R}^\ell$-valued martingale differences on a probability space $(\Omega, \mathcal{G}, P)$. Define conditional covariance matrices $\sigma(n) = \sigma(n))_{1 \leq i,j \leq \ell}$ by
\[
\sigma_{ij}(n) = E(M_i(n)M_j(n)|\mathcal{G}_{n-1})
\]
and set $\Sigma(n) = \sum_{k=1}^n \sigma(k)$. Suppose that there exists a covariance matrix $D$ and a constant $\gamma \in (0, 1)$ such that
\[
\Sigma(n) - nD \ll n^{1-\gamma} \text{ a.s. or } E\|\Sigma(n) - nD\| = O(n^{1-\gamma}),
\]
where $\|\cdot\|$ is the Euclidean matrix norm, and
\[
\sum_{n=1}^\infty n^{\gamma-1} E\left( \|M(n)\|_{\mathcal{G}}^2 \|M(n)\|_{\mathcal{G}}^2 \right) < \infty.
\]
Then without changing its distribution the sequence $\{M(n), n \geq 1\}$ can be redefined on a richer probability space where there exists a sequence $\{Y(n), n \geq 1\}$ of independent identically distributed (i.i.d.) random vectors with the covariance matrix $D$ and an $\ell$-dimensional Brownian motion $Z(t)$, $t \geq 0$ with the covariance matrix $D$ such that
\[
\sum_{k \leq n} (M(k) - Y(k)) \ll n^{1/2-\gamma} \text{ and }
\]
(3.7) \[ \sum_{k \leq t} M(k) - Z(t) \ll t^{\frac{1}{2} - \kappa} \]

where \( \kappa > 0 \) does not depend on \( n \) or on \( t \).

Actually, only (3.6) is obtained in [10] but, in fact, (3.7) follows from (3.6) by a standard argument based on the Kolmogorov extension theorem (see Section 2.4 in [12]). We observe also that if the original probability space \((\Omega, \mathcal{F}, P)\) is already rich enough to have a uniformly distributed random variable independent of the sequence \(\{M(n), n \geq 1\}\) then the sequence \(\{Y(n), n \geq 1\}\) and the process \(\{Z(t), t \geq 0\}\) can already be constructed on \((\Omega, \mathcal{F}, P)\) and there is no need in enrichment and in redefinitions.

Our Theorem 2.2 will follow immediately from the following result.

3.3. Proposition. Suppose that Assumption 2.1 holds true. Then without changing its distributions the vector process \(\Psi(t) = (\Psi_1(t), ..., \Psi_\ell(t)), t \geq 0\) (with \(\Psi_i\) the same as in (2.9)) can be redefined on a richer probability space where there exists an \(M\) and \(R\) (3.11)

Next, set \( (3.8) \quad \Psi(t) - G(t) \ll t^{\frac{1}{2} - \gamma} \text{ a.s.} \)

Proof. First, we recall the block construction from [9]. We will use the following notations from [9]

(3.9) \[ F_{i,r,n}(x_1, x_2, \ldots, x_{i-1}, \omega) = E(F(x_1, x_2, \ldots, x_{i-1}, X(n))|\mathcal{F}_{n-r,n+r}), \]

\[ X_r(n) = E(X(n)|\mathcal{F}_{n-r,n+r}), \quad Y_i(q(n)) = F_i(X(q_1(n)), \ldots, X(q_\ell(n))) \]

and \( Y_i(j) = 0 \) if \( j \neq q_i(n) \) for any \( n \), \( Y_{i,r}(q_i(n)) = F_{i,r,q_i(n)}(X_r(q_i(n))) \)

Next, we fix some positive numbers \(4\eta < 2\theta < \tau < \delta/4\) where \(\delta\) is the same as in Assumption 2.1. Now, we introduce pairs of "big" and "small" increasing blocks (somewhat differently than in [9]) defining for each \(i\) random variables \(V_i(j)\) and \(W_i(j)\) inductively so that

(3.10) \[ V_i(1) = Y_{i,1}(q_i(1)), \quad W_i(1) = Y_{i,1}(q_i(2)), \quad a(1) = 0, b(1) = 1 \text{ and for } j > 1, \]

\[ a(j) = b(j - 1) + [(j - 1)^\theta], \quad b(j) = a(j) + [j^\gamma], \quad r(j) = [j^{\beta}], \]

\[ V_i(j) = \sum_{a(j) < n \leq b(j)} Y_{i,r}(n) \text{ and } W_i(j) = \sum_{b(j) < n \leq a(j+1)} Y_{i,r}(n). \]

Next, set

(3.11) \[ R_i(m) = \sum_{j = m+1}^{\infty} E(V_i(j)|\mathcal{G}_m) \]

and \( M_i(m) = V_i(m) + R_i(m) - \frac{R_i(m-1)}{\mathcal{G}_m} = F_{-\infty,b(m)}+r(m) \). In the same way as in Section 3 of [9] we see that \((M_i(m), \mathcal{G}_m)_{m \geq 1}\) is a martingale differences sequence for each \(i = 1, ..., \ell\), and so \(M(m) = (M_1(m), ..., M_\ell(m))\), \(m \geq 1\) is a vector martingale differences sequence. Now, proceeding similarly to Section 3 in [9] we obtain that for some \(\varepsilon > 0\),

(3.12) \[ \|\Psi(t) - \sum_{1 \leq j \leq n(t)} M(j)\| \ll t^{\frac{1}{2} - \varepsilon} \text{ a.s} \]
where \( \nu(t) = \max\{ j : b(j) + [j^\theta] \leq t \} \) and \( \| \cdot \| \) is the Euclidean norm in \( \mathbb{R}^\ell \).

In order to complete the proof of Proposition 3.3 it remains to verify the conditions of Theorem 3.2 for the vector martingale differences sequence \( (M(m), \mathcal{G}_m)_{m \geq 1} \). In Section 4 of [9] we showed relying on Proposition 3.1 that for each \( i = 1, ..., \ell \),

\[
(3.13) \quad E\Psi_i^2(t) - \sum_{1 \leq l \leq \nu(t)} M_i^2(l) \ll t^{1-\varepsilon} \quad \text{a.s}
\]

for some \( \varepsilon > 0 \) independent of \( t \). Essentially, the same proof shows that for any \( i, j = 1, ..., \ell \),

\[
(3.14) \quad E\Psi_i(t)\Psi_j(t) - \sum_{1 \leq l \leq \nu(t)} M_i(l)M_j(l) \ll t^{1-\varepsilon} \quad \text{a.s}
\]

for some \( \varepsilon > 0 \) independent of \( t \). Existence and a description of the limit

\[
(3.15) \quad \lim_{t \to \infty} t^{-1}E\Psi_i(t)\Psi_j(t) = D_{ij}
\]

was provided by Proposition 4.1 from [10]. In fact, it turns out that

\[
(3.16) \quad |E\Psi_i(t)\Psi_j(t) - D_{ij}| \leq C_0
\]

for some \( C_0 > 0 \) independent of \( t \) which is actually hidden inside of the proof in [10], though, of course, a bound \( C_0t^{1-\varepsilon} \), \( \varepsilon > 0 \) in the right hand side of (3.16) would suffice for our purposes, as well. The corresponding arguments are essentially contained at the end of Section 4 in [9] but for readers’ convenience we explain them here too.

Recall, that according to Proposition 4.1 of [10] (see also Lemma 4.4 there) the limit in (3.15) can be written in the form

\[
(3.17) \quad D_{ij} = \frac{v}{ij} \sum_{u=-\infty}^{\infty} a_{ij}(u, 2u, ..., vu)
\]

where \( v \) is the greatest common divisor of \( i \) and \( j \) with \( i = vi' \), \( j = vj' \) and \( i', j' \) being coprime and

\[
(3.18) \quad a_{ij}(u, 2u, ..., vu) = \int F_i(x_1, ..., x_i)F_j(y_1, ..., y_j) \prod_{\sigma \in (i, 2i, ..., vi')} d\mu(x_\sigma) \prod_{\sigma' \in (j, 2j', ..., vj')} d\mu(y_{\sigma'}) \prod_{\eta=1}^{v} d\mu_n(x_{\eta}y_{\eta}) \prod_{\eta'=1}^{v} d\mu_n(x_{\eta'}y_{\eta'})
\]

where \( d\mu_n(x, y) = \delta_{x,y}d\mu(x) \) is the measure supported by the diagonal.

Next, we have

\[
(3.19) \quad E\Psi_i(t)\Psi_j(t) = \sum_{0 \leq n \leq \ell \cap 0 \leq n' \leq t/j} b_{ij}(n, n')
\]

where

\[
(3.19) \quad b_{ij}(n, n') = E\Phi_i(X(n), ..., X(in))\Phi_j(X(n'), ..., X(jn')).
\]

Suppose that \( |in - jn'| = m \leq \frac{1}{4} \max(n, n') \) and let, for instance, \( in - jn' = m \). Then \( \ell n \geq n' \geq (n - m)/\ell \), and so \( \max(n, n') \leq \ell n \) and \( m \leq n/4 \). It follows that

\[
\min(in, jn') - \max((i-1)n, (j-1)n') = \min(n - m, n') \geq (n - m)/\ell \geq \frac{3\max(n, n')}{4\ell^2}.
\]
The same argument works for the case \(jn' - in = m\), as well. Hence we can apply Proposition 3.1(ii) with
\[
\mathcal{G} = \mathcal{F}_{-\infty, \min\{mn, m'n'\} - \frac{1}{3\ell^2} \max\{n, n'\}}, \quad \mathcal{H} = \mathcal{F}_{\min\{mn, m'n'\} - \frac{1}{3\ell^2} \max\{n, n'\}},
\]
\[
\Pi = (X(n), ..., X((i-1)n); X(n'), ..., X((j-1)n')) \text{ and } \Upsilon = (X(in), X(jn'))
\]
which yields that
\[
|b_{ij}(n, n') - \int EF_i(X(n), ..., X((i-1)n), x)F_j(X(n'), ..., X((j-1)n')) \nu(xk_{j+1}, ..., x, y_{j+1} + 1, ..., y|) \leq \varepsilon_k, k' \text{ independent of } n, n'.
\]
for some \(C_1 > 0\) independent of \(n\) and \(n'\).

We proceed by induction dealing with the case \(|in - jn'| = m > \frac{1}{4\ell^2} \max\{n, n'\}\) by the argument below. Suppose that we already proved that for some \(k < i\) and \(k' < j\),
\[
|b_{ij}(n, n') - \int EF_i(X(n), ..., X(kn), x_{k+1}, ..., x_i)F_j(X(n'), ..., X(k'n', y_{k'1+1}, ..., y_j)) \nu(x_{k+1}, ..., x, y_{k'1+1} + 1, ..., y_j)| \leq \varepsilon_k, k' \text{ independent of } n, n'.
\]
If \(|kn - k'n'| \leq \frac{1}{4\ell^2} \max\{n, n'\}\) then in the same way as in (3.20) we obtain that
\[
|b_{ij}(n, n') - \int EF_i(X(n), ..., X((k-1)n), x, ..., x_i)F_j(X(n'), ..., X((k'-1)n'), y_{k'1}, ..., y_{j-1})) \nu(x_{k+1}, ..., x, y_{k'1} + 1, ..., y_{j-1})| \leq \varepsilon_k, k' \text{ independent of } n, n'.
\]
for some \(C_2 > 0\) independent of \(n\) and \(n'\). On the other hand, if \(|kn - k'n'| > \frac{1}{4\ell^2} \max\{n, n'\}\) then
\[
\max\{kn, k'n'\} \geq \max\{\min\{kn, k'n'\}, \max\{(k-1)n, (k'-1)n'\}\} + \frac{1}{4\ell^2} \max\{n, n'\},
\]
and we can apply again Proposition 3.1(ii) with
\[
\mathcal{G} = \mathcal{F}_{-\infty, \max\{kn, k'n'\} - \frac{1}{12\ell^2} \max\{n, n'\}}, \quad \mathcal{H} = \mathcal{F}_{\max\{kn, k'n'\} - \frac{1}{12\ell^2} \max\{n, n'\}},
\]
\[
\Pi = (X(n), ..., X((k-1)n); X(n'), ..., X((k'-1)n'); X(\min\{kn, k'n'\}))
\]
and \(\Upsilon = X(\max\{kn, k'n'\})\) to obtain that
\[
|b_{ij}(n, n') - \int EF_i(X(n), ..., X((k-1)n), U_{kn}, x_{k+1}, ..., x_i)F_j(X(n'), ..., X((k'-1)n'), y_{k'1}, ..., y_{j-1})) \nu(U_{\max\{kn, k'n'\}}, x_{k+1}, ..., x_i, y_{k'1} + 1, ..., y_{j-1})| \leq \varepsilon_k, k' \text{ independent of } n, n'.
\]
for some \(C_3 > 0\) independent of \(n\) and \(n'\) where \(U_{\min\{kn, k'n'\}} = X(\min\{kn, k'n'\})\). In particular, if \(k = i\) and \(k' = j\) we obtain from (2.9) that
\[
|b_{ij}(n, n')| \leq C_3(\nu, \frac{1}{12\ell^2} \max\{n, n'\}) + \beta_q(\frac{1}{12\ell^2} \max\{n, n'\})
\]
This together with the above induction argument yields that
\[
|b_{ij}(n, n') - a_{ij}(u, 2u, ..., vv)| \leq C_4(\nu, \frac{1}{12\ell^2} \max\{n, n'\}) + \beta_q(\frac{1}{12\ell^2} \max\{n, n'\})
\]
for some \(C_4 > 0\) independent of \(n\) and \(n'\) provided \(ni - jn' = vu\) and \(v\) is the greatest common divisor of \(i\) and \(j\) with \(a_{ij}\) defined by (3.18).

It is not difficult to see (the explanation can be found in the proof of Lemma 4.4 of [10]) that the number of integer solutions of \(in - jn' = vu\) with \(in, jn' \leq t\) can differ from \([vt^{ij}]^{-1}\) by at most a constant independent of \(t\). This together
with (3.17)–(3.19), (3.25) and Assumption 2.1 yields (3.10). Taking into account (3.14) we conclude that the condition (3.4) of Theorem 3.2 is satisfied for the vector martingale differences sequence \( \{M(j), 1 \leq j \leq \nu(t)\} \) constructed above.

Next, we verify the condition (3.5). By the Cauchy–Schwarz and the Chebyshev inequalities

\[
A(n) = \frac{1}{n} \mathbb{E} \left[ |M(n)|^2 \mathbb{I}_{\{ |M(n)|^2 \geq n^{1-\gamma} \}} \right]
\leq \frac{1}{n} \mathbb{E} \left[ |M(n)|^4 \right]^{1/2} \left( \mathbb{P} \left\{ |M(n)|^2 \geq n^{1-\gamma} \right\} \right)^{1/2}
\leq \frac{1}{n} \mathbb{E} \left[ |M(n)|^4 \right].
\]

Now

\[
\|M(n)\| \leq \sum_{i=1}^{t} |M_i(n)| \leq \sum_{i=1}^{t} (|V_i(n)| + |R_i(n)| - |R_i(n) - 1|)
\leq \sum_{i=1}^{t} \sum_{a(n) < d < b(n)} |Y_{i,r(n)}| + \sum_{i=1}^{t} (|R_i(n)| - |R_i(n) - 1|).
\]

Since \((a_1 + \cdots + a_k)^4 \leq k^3(a_1^4 + \cdots + a_k^4)\), we conclude from here relying on (2.6), Assumption 2.1 and the Hölder inequality that for any \(n \geq 1\),

\[
\mathbb{E} \left[ |M(n)|^2 \right] \leq C_5 n^4
\]

where \(C_5 > 0\) is independent of \(n\), and so

\[
A(n) \leq C_5 n^{4-2(1-\gamma)} \leq C_5 n^{\delta-2(1-\gamma)}.
\]

Since \(\delta < 1\) we can choose \(\gamma > 0\) so small that \(\delta - 2(1-\gamma) < -1\) whence \(\sum_{n=1}^{\infty} A(n) < \infty\) and the condition (3.5) holds true completing the proof of Proposition 3.3. \(\square\)

Finally, (2.12) follows from (3.8) in view of (2.10) concluding the proof of Theorem 2.2. \(\square\)

4. Almost sure central limit theorem

We start the proof of Theorem 2.4 with the following

4.1. Proposition. Set

\[
Q_u^{(s)} = s^{-1/2} Q(us) = s^{-1/2} \sum_{j=1}^{t} G_j(ujs), \quad u \in [0, 1]
\]

and define random measures on \(\mathbb{C}[0, 1]\) by

\[
\zeta(t) = \frac{1}{\ln t} \int_{1}^{t} \frac{ds}{s} \delta_{Q_u^{(s)}}(\omega), \quad t > 1
\]

where \(\delta_{Q_u^{(s)}}(\omega)\) is the unit mass concentrated on the element \(Q_u^{(s)}(\omega) \in \mathbb{C}[0, 1]\). Set also

\[
\nu_n(\omega) = \frac{1}{\ln n} \sum_{k=1}^{n} k^{-1} \delta_{Q_u^{(s)}}, \quad n \geq 2.
\]

Then with probability one

\[
\lim_{t \to \infty} \zeta(t) = \eta_Q \text{ and } \lim_{n \to \infty} \nu_n = \eta_Q
\]

where, again, the limit is taken in the sense of weak convergence of measures on \(\mathbb{C}[0, 1]\).
Proof. In the same way as in [3] we show first that with probability one the measures \( \{ \zeta_t, \ t \geq 0 \} \) are tight observing that the estimates for the Brownian motion in [3] go through for our process \( Q \), as well, since it is a linear combination of linearly time changed Brownian motions.

Next, as in [3] we define \( \phi : \mathbb{C}[0,1] \rightarrow \mathbb{R} \) by
\[
\phi(x) = E^\ell(x_{\alpha_1}x_1 + \cdots + x_{\alpha_k}x_k)
\]
for some \( k \in \mathbb{N}, \ 0 < u_1 < \cdots < u_k \leq 1 \) and \( \alpha_1, ..., \alpha_k \in \mathbb{R} \). We want to show that
\[
\lim_{t \to \infty} \int_{\mathbb{C}[0,1]} \phi(x) \zeta_t(\omega)(dx) = \int_{\mathbb{C}[0,1]} \phi(x) d\eta_Q(x).
\]
Set
\[
\Phi_s = \exp \left( i(\alpha_1 \sqrt{u_1}R_{u_1} + \cdots + \alpha_k \sqrt{u_k}R_{u_k}) \right)
\]
where \( R_s = Q_1^{(s)} \). Then
\[
\int_{\mathbb{C}[0,1]} \phi(x) \zeta_t(\omega)(dx) = \frac{1}{\ln t} \int_1^t \frac{ds}{s} \Phi_s
\]
since \( \sqrt{u}R_{us} = \sqrt{u}Q_1^{(us)} = Q_u^{(s)} \). Next, observe that the laws of \( \{ R_s, s > 0 \} \) and \( \{ R_{sh}, s > 0 \} \) (as processes) coincide for each \( h > 0 \) since both processes are Gaussian with zero mean and in view of Theorem 2.2, the covariance function
\[
ER_{sh}R_{th} = h^{-1} EQ^{(s)}_hQ^{(t)}_h = h^{-1} \sum_{i,j=1}^t E \zeta_i(hs)G_j(hs) = s \sum_{i,j=1}^t D_{ij}(i \wedge j)
\]
does not depend on \( h \). It follows from here and the definition (4.7) that if we set \( \hat{\Phi}_s = \Phi_{in,s} \) then the laws of \( \{ \hat{\Phi}_s, s \in \mathbb{R} \} \) and \( \{ \Phi_{s+h}, s \in \mathbb{R} \}, h \in \mathbb{R} \) coincide. Then by the Birkhoff ergodic theorem together with the fact that the tail \( \sigma \)-algebra \( \cap_{t > \sigma} \{ G_1(u), G_2(u), ..., G_l(u); u > t \} \) is trivial (as for an \( \ell \)-dimensional Brownian motion) we conclude that with probability one
\[
\lim_{n \to \infty} \frac{1}{n} \int_1^n \frac{ds}{s} \Phi_s = \lim_{n \to \infty} \frac{1}{n} \int_0^n \Phi_{t+u} du = E \Phi_0 = E \Phi_1.
\]
A simple comparison of integrals \( \int_{1}^{e^t} \) and \( \int_{1}^{[e^t]} \) as in [3] shows also that
\[
\lim_{t \to \infty} \frac{1}{t} \int_1^{e^t} \frac{ds}{s} \Phi_s = E \Phi_1.
\]
But by (4.6) and (4.7),
\[
E \Phi_1 = E \exp \left( i(\alpha_1 Q^{(1)}_{u_1} + \cdots + \alpha_k Q^{(1)}_{u_k}) \right) = \int_{\mathbb{C}[0,1]} \phi(x) d\eta_Q(x)
\]
since \( Q^{(1)}_u = Q(u) \). Hence, (4.6) follows from (4.1), (4.8), (4.11) and (4.12). Since (4.11) holds true for any \( \phi \) defined by (4.5) then relying on tightness of the family \( \{ \zeta_t, t \geq \varepsilon \} \) we obtain the first limit in (4.13). The second limit there holds true by the same arguments as in [3] which are just estimates for the Brownian motion valid in our case of a linear combination of linearly time changed Brownian motions, as well. \( \square \)
In order to derive Theorem 2.4 from Proposition 3.3 we will rely on the following result which appears in [3] as Lemma 2.12.

4.2. Lemma. Let \( \Pi, \Psi : \mathbb{R}^+ \to \mathbb{C}[0,1] \) or \( \Pi, \Psi : \mathbb{N}^+ \to \mathbb{C}[0,1] \) be measurable \( \mathbb{C}[0,1] \)-valued stochastic processes such that, respectively,

\[
\lim_{s \to \infty} ||\Pi_s - \Psi_s||_{\mathbb{C}[0,1]} = 0 \quad \text{or} \quad \lim_{n \to \infty} ||\Pi_n - \Psi_n||_{\mathbb{C}[0,1]} = 0
\]

where \( ||\cdot||_{\mathbb{C}[0,1]} \) is the supremum norm on \( \mathbb{C}[0,1] \). Then for all bounded, uniformly continuous functions \( f : \mathbb{C}[0,1] \to \mathbb{R} \),

\[
\lim_{t \to \infty} \left( \frac{1}{\ln t} \int_1^t \frac{ds}{s} f(\Pi_s) \right) = 0 \quad \text{or} \quad \lim_{n \to \infty} \left( \frac{1}{\ln n} \sum_{k=1}^n k^{-1} f(\Pi_k) \right) = 0,
\]

respectively.

In order to prove Theorem 2.4 it suffices in view of Proposition 4.1 and Lemma 4.2 to show that

\[
\lim_{n \to \infty} ||Q^{(n)} - Q_n||_{\mathbb{C}[0,1]} = 0 \quad \text{a.s.}
\]

where \( Q^{(n)} \) and \( Q_n \) are defined by (2.11) and (2.15), respectively. For each \( t \in [0,1] \), \( n \in \mathbb{N} \) define \( \Psi^{(n)}(t) = (\Psi_1^{(n)}, ..., \Psi_\ell^{(n)}) \) and \( G^{(n)}(t) = (G_1^{(n)}, ..., G_\ell^{(n)}) \) where for \( j = 1, ..., \ell \),

\[
\Psi_j^{(n)}(t) = n^{-1/2} \Psi_j(j[n]t)(1 + [nt] - nt) + n^{-1/2} \Psi_j(j[n]t + j)(nt - [nt]),
\]

\[
G_j^{(n)}(t) = n^{-1/2} G_j(j[n]t)(1 + [nt] - nt) + n^{-1/2} G_j(j[n]t + j)(nt - [nt])
\]

and \( H_j^{(s)}(t) = s^{-1/2} G_j(tjs) \) where \( \Psi_j \) and \( G_j \) are the same as in (2.11) and Theorem 2.2, respectively. Then in order to obtain (4.16) it suffices to show that for each \( j = 1, ..., \ell \),

\[
\lim_{n \to \infty} ||H_j^{(n)} - G_j^{(n)}||_{\mathbb{C}[0,1]} = 0 \quad \text{a.s.}
\]

\[
\lim_{n \to \infty} ||G_j^{(n)} - \Psi_j^{(n)}||_{\mathbb{C}[0,1]} = 0 \quad \text{a.s.}
\]

Now, in the same way as in [3] for each \( \varepsilon > 0 \) by the Doob martingale inequality,

\[
P\{ ||H_j^{(n)} - G_j^{(n)}||_{\mathbb{C}[0,1]} \geq \frac{1}{2} \varepsilon \sqrt{n} \}
\leq n \sum_{k=0}^{n-1} P\{ \sup_{k \leq t \leq k+1} |G_j(jt) - G_j(jk)| \geq \frac{1}{2} \varepsilon \sqrt{n} \}
\leq C \varepsilon^{-6} n^{-2}.
\]

This together with the Borel–Cantelli lemma yields (4.19). Finally,

\[
||G_j^{(n)} - \Psi_j^{(n)}||_{\mathbb{C}[0,1]} \leq n^{-1/2} \max_{1 \leq k \leq n+1} |G_j(jk) - \Psi_j(jk)|
\]

and the right hand side of (4.21) converges with probability one to 0 as \( n \to \infty \) in view of Theorem 2.2, completing the proof of Theorem 2.4.

4.3. Remark. A slightly different method of proof of the almost sure central limit theorem from [11] can also be adapted to our nonconventional setup.
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