COUNTABLE SUCCESSOR ORDINALS AS GENERALIZED ORDERED TOPOLOGICAL SPACES

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Abstract. We prove the following Main Theorem: Assume that any continuous image of a Hausdorff topological space $X$ is a generalized ordered space. Then $X$ is homeomorphic to a countable successor ordinal (with the order topology).

The converse trivially holds.

1. Introduction and Main Theorem

All topological spaces are assumed to be Hausdorff. Remind that $L$ is a Linearly Ordered Topological Space (LOTS) whenever there is a linear ordering $\leq^L$ on the set $L$ such that a basis of the topology $\lambda^L$ on $L$ consists of all open convex subsets. A convex set $C$ in a linear ordering $M$ is a subset of $M$ with the property: for every $x < z < y$ in $M$, if $x, y \in C$ then $z \in C$. The above topology, denoted by $\lambda^L$ is called an order topology. Since the order $\leq^L$ defines the topology $\lambda^L$ on $L$ (but not vice-versa), we denote also by $(L, \leq^L)$ the structure including the topology $\lambda^L$. A topological space $(X, \tau^X)$ is called a Generalized Ordered Space (GO-space) whenever $(X, \tau^X)$ is homeomorphic to a subspace of a LOTS $(L, \lambda^L)$, that is $\tau^X = \lambda^L|_X := \{U \cap X : U \in \lambda^L\}$ (see [2]).

Evidently, every LOTS, and thus any GO-space, is a Hausdorff topological space, but not necessarily separable or Lindelöf. The Sorgenfrey line $Z$ is an example of a GO-space, which is not a LOTS, and such that every subspace of $Z$ is separable and Lindelöf (see [4]).

By definition, every subspace of a GO-space is also a GO-space. In this article, in a less traditional manner, we say that a space $X$ is a hereditarily GO-space if every continuous image of $X$ (in particular, $X$ itself) is a GO-space.

Main Theorem 1.1. Every hereditarily GO-space is homeomorphic to a countable successor ordinal, considered as a LOTS. The converse obviously holds.

This result is closely related to the following line of research: characterize Hausdorff topological spaces $X$ such that all continuous images of $X$ have the topological property $\mathcal{P}$. All questions listed below for concrete $\mathcal{P}$ are still open.

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Problem 1.2.

1. Characterize Hausdorff spaces such that all continuous images of $X$ are regular.
2. Characterize Hausdorff spaces $X$ such that all continuous images of $X$ are normal. (This was partially solved by W. Fleissner and R. Levy in [5, 6].)
3. Characterize Hausdorff spaces such that all continuous images of $X$ are realcompact. (This question has been formulated in [1].)
4. Characterize Hausdorff spaces such that all continuous images of $X$ are paracompact.
5. Characterize Hausdorff spaces such that all continuous images of $X$ are monotonically normal. (This is related to “Niekel Conjecture” answered positively by M. E. Rudin [12].)

□

Remark 1.3. Let $b$ be the minimal cardinality of unbounded subsets of $\omega$. Recently M. Bekkali and S. Todorčević proved the following relevant result: Continuous zero-dimensional images of a compact LOTS of weight less than $b$ is itself a LOTS [3, Theorem 4.2].

Note also a recent paper [14], which studies topological properties $\mathcal{P}$ that are reflectable in small continuous images. For instance, a GO-space $X$ is hereditarily Lindelöf iff all continuous images of $X$ have countable pseudocharacter [14].

As a special case of Main Theorem 1.1, we obtain the following fact.

Corollary 1.4. Assume that any continuous image of a Hausdorff topological space $X$ is a LOTS. Then $X$ is homeomorphic to a countable successor ordinal.

In order to make this paper widely readable, we have tried to give self-contained and elementary proofs, even when our results could be deduced from more general theorems. By these reasons, and for the readers’ convenience, we include a separate and short proof of Corollary 1.4 in Section 2.

In our paper, a GO-structure, formally, is a 4-tuple $\langle X, \tau^X, L, \leq_L \rangle$ where

1. $X \subseteq L$,
2. $\langle L, \leq_L \rangle$ is a linear ordering with the order topology $\lambda^L$; and
3. the order $\leq^X$ on $X$ is the restriction $\leq^L | X$ of $\leq_L$ to $X$, and $\tau^X$ is the topology $\lambda^L | X := \{ U \cap X : U \in \lambda \}$

Any GO-space $X$ can be written under the above structure. In [7], [10] GO-space are denoted by $\langle X, \tau^X, \leq^X \rangle$ where $\leq^X$ is the restriction of $\leq_L$ to $X$. It is easy to see that the following are equivalent.

1. $\langle X, \tau^X, \leq^X \rangle$ is a GO-space and
2. $\lambda^X \subseteq \tau^X$ and $\tau^X$ has a base consisting of convex sets.

Many times we denote $\langle X, \tau^X, L, \leq^L \rangle$ by $\langle X, \tau^X, L, \lambda_L^X \rangle$.

The proof of Main Theorem 1.1, is organized as follows. In §3, we present the basic facts on LOTS and GO-spaces. In particular, Proposition 3.4 shows that if $\langle X, \tau^X, L, \leq^L \rangle$ is a GO-structure, then we may assume that $L$ is a complete ordering and that $X$ is topologically dense in $\langle L, \lambda_L^X \rangle$. In §4.1 we show that any hereditarily GO-space satisfies c.c.c. property: every family of pairwise disjoint nonempty open set is countable (Lemma 4.1). In §4.2 we prove that any hereditarily GO-space has no countable closed and relatively discrete subset (Corollary 4.5): we recall that a subset $D$ of a space $Y$ is relatively discrete whenever there is a family $\mathcal{U}_D := \{ U_d : d \in D \}$ of open subsets of $Y$ such that $D \cap U_d = \{ d \}$ for every $d$. 


Proposition 4.8 shows that a hereditarily GO-space is a subspace of a scattered linear order (with the order topology). Finally, in §4.4 we conclude the proof of Main Theorem.

2. Elementary proof of Main Theorem for LOTS: Corollary 1.4

We assume that the reader is familiar with the properties of LOTS and give a self-contained proof of Corollary 1.4. To prove the result, we need some preliminaries.

Fact 1. Let \( \langle L, \leq^L \rangle \) be a LOTS. If \( \langle L, \leq^L \rangle \) is a scattered ordering then \( \langle L, \lambda^L \rangle \) is a scattered topological space.

The converse does not hold: consider the lexicographic sum \( N := \sum_{r \in \mathbb{Q}} \mathbb{Z}_r \) of copies \( \mathbb{Z}_r \) of the integers \( \mathbb{Z} \) over the rational chain \( \mathbb{Q} \). \( \langle N, \leq^N \rangle \) is not a scattered ordering but \( \langle N, \lambda^N \rangle \) is a scattered linear ordering, \( \lambda^N \) is a discrete space (this example will be used again in Part (1) of Remark 3.3).

The next fact is used implicitly in the arguments that follow.

Fact 2. Let \( \langle L, \leq^L \rangle \) be a LOTS and \( F \) be a closed subspace of \( \langle L, \lambda^L \rangle \). Then the induced topology \( \lambda^L \upharpoonright F \) on \( F \) is the order topology on \( F \) defined by the restriction \( \leq^L \upharpoonright F \upharpoonright F \).

Fact 3. Let \( \langle L, \leq^L \rangle \) be a scattered linear ordering then \( \langle L, \lambda^L \rangle \) is 0-dimensional (i.e. \( \langle L, \lambda^L \rangle \) has a base consisting of clopen sets).

Proof. The proof uses the fact that if \( \langle L, \leq^L \rangle \) is a scattered linear ordering, then the Dedekind completion \( \langle L^c, \leq^{L^c} \rangle \) of \( \langle L, \leq^L \rangle \) is also a scattered linear ordering.

Fact 4. Let any continuous image of a Hausdorff topological space \( Y \) is a LOTS. Then \( Y \) satisfies c.c.c.. In particular, \( \omega_1 \) and \( \omega_1^* \) are not order-embeddable in \( Y \).

Fact 5. Let \( Y \) be a 0-dimensional LOTS. If \( D \) is a countable closed and discrete subset of \( Y \) then \( D \) is a continuous image of \( Y \).

Proof. Let \( \{U_d \}_{d \in D} \) be clopen subsets of \( Y \) such that \( U_d \cap D = \{d\} \) for \( d \in D \). Fix \( d_0 \in D \). Let \( \approx \) be the equivalence relation on \( Y \) defined by \( x \approx y \) whenever there is \( d \in D \setminus \{d_0\} \) such that \( x, y \in U_d \), or \( x, y \notin U := \bigcup\{U_d : d \in D \setminus \{d_0\}\} \). Then \( Y/\approx \) is a continuous image of \( Y \), \( Y/\approx \) is Hausdorff and \( D \) is homeomorphic to \( Y/\approx \).

Fact 6. Let \( D \) be a countable and discrete space. Then \( D \) has a continuous image which is not homeomorphic to a LOTS.

Proof. Consider \( \omega \) as a discrete space. Let \( \mathcal{U} \) on \( \omega \) be a non-principal ultrafilter on \( \omega \) and let * be a new element with \( * \notin \omega \). We equip \( \mathbb{N}^* := \omega \cup \{\ast\} \) with the topology induced from the Stone–Cech compactification \( \beta \mathbb{N} \). It is well-known that \( \mathbb{N}^* \) is not a LOTS: this is so because \( \mathbb{N}^* \) is separable and its topology does not have a countable base [4, §3.6]. Evidently, countable \( \mathbb{N}^* \) is a continuous image of \( \omega \).

As a consequence of Facts 4–6, we have the following result.

Fact 7. Let \( \langle X, \leq^X, \lambda^X \rangle \) be a scattered linear ordering. Assume that any continuous image of \( X \) is a LOTS. Then the following holds

1. If \( x \) is in the topological closure of a nonempty subset \( A \) in \( X \), then there is a countable monotone sequence of elements of \( A \) converging to \( x \).

2. Every monotone sequence \( \{x_n\}_{n \in \omega} \) converges. In particular, there exist both minimum and maximum in \( \langle X, \leq \rangle \).

Now we are in a position to finish the argument.
Proof of Main Theorem (for LOTS). Let \( \equiv \) be the equivalence on \( \langle X, \leq^X, \lambda^X \rangle \) defined by:

\[
  x \equiv y \text{ if and only if } \left\{ \begin{array}{ll}
  x \leq y & \text{and } [x, y] \text{ is a scattered linear ordering} \\
  x \leq y & \text{and } [y, x] \text{ is a scattered linear ordering}
  \end{array} \right.
\]

Note that each \( \equiv \)-class is closed for the topology \( \lambda^X \) and each \( \equiv \)-class is convex and scattered for the order \( \leq^X \). Moreover, \( X/\equiv \), denoted by \( X_1 \), is a LOTS.

Note that there are no consecutive classes in \( X_1 \), and thus \( X_1 \) is order-dense. Also the map \( \varphi : X \to X/\equiv \), preserving supremum and infimum, is increasing and continuous.

Let \( N \) be a linear ordering and \( N^c \) its Dedekind completion. Recall that a cut is a member of \( N^c \setminus (L \cup \{\min(N^c), \max(N^c)\}) \). For instance, in the chain \( \mathbb{Q} \) of rationals, the cuts are the irrationals.

**Case 1.** The set \( \Gamma := \langle X_1^c \setminus X_1 \rangle \) of cuts of \( X_1 \) has no consecutive elements and \( \Gamma \) is topologically dense in \( X_1 \).

So, \( \Gamma \) is order-dense and \( \Gamma \) has no first and no last element. Note that, in that case, \( X_1 \) is 0-dimensional. Let \( c \in \Gamma \). Let \( \langle c_\alpha \rangle_{\alpha < \lambda} \) be a strictly increasing sequence cofinal in \( (-\infty, c) \) with \( \lambda \) regular. By Fact 4, \( \lambda = \omega \). Since \( c \in \Gamma \), \( D := \{x_\alpha : \alpha \in \omega\} \) is a countable discrete and closed subset of \( X_1 \), which contradicts Fact 6. So, Case 1 does not occur.

**Case 2.** The set \( \Gamma := \langle X_1^c \setminus X_1 \rangle \) of cuts of \( X_1 \) has two consecutive elements, or \( \Gamma \) is not topologically dense in \( X_1 \).

Then there is a nonempty open interval \( (u, v) \) of \( X \) such that \( (u, v) \cap \Gamma = \emptyset \). We set \( X_1' := [u, v] \). So \( X_1' \) is connected, infinite and order-dense. Also \( X_1' \) is a continuous image of \( X_1 \). Let \( X_2 \) be the quotient space of \( X_1' \), obtained by identification of \( u \) and \( v \). Obviously, \( X_2 \) is connected. Since for every \( t \in X_2 \) the set \( X_2 \setminus \{t\} \) is connected, \( X_2 \) is not a LOTS.

We have proved that \( |X_1| = 1 \), that is: \( X \) is a scattered linear ordering. By Fact 1, \( X \) is a scattered topological space. Moreover, \( X \) satisfies c.c.c., and thus \( \omega_1 \) and \( \omega_1^* \) are not order-embeddable in the scattered linear ordering \( X \). In particular, the space \( X \) has only countably many isolated points. Also the space \( X \) has no infinite and discrete subset. Hence the linear ordering \( X \) is complete and thus the space \( X \) is compact. We have proved that \( X \) is a countable compact and scattered space, that is, \( X \) is homeomorphic to \( \alpha + 1 \) for some \( \alpha < \omega_1 \).

\[ \square \]

3. Basic facts on LOTS and GO-spaces

Let \( S \) be a set, \( \mathcal{U} \subseteq \mathcal{P}(S) \) and \( T \subseteq S \). We set \( \mathcal{U}/T = \{U \cap T : U \in \mathcal{U}\} \). Recall that a linear ordering \( \langle M, \leq^M \rangle \) is complete whenever every subset \( A \) of \( M \) has the supremum \( \sup^M(A) \) and the infimum \( \inf^M(A) \). In particular, there exist both the maximum \( \max(M) \) and the minimum \( \min(M) \) in \( M \).

Let \( N \) be a linear ordering. The Dedekind completion of \( N \), denoted by \( N^c \), is a complete linear ordering containing \( N \), such that \( N^c \) is minimal with respect to this property. That is,

\begin{align*}
  (D1) & \ N^c \text{ is a complete chain,} \\
  (D2) & \text{for every } x, y \in N; \text{ if } x <^N y \text{ then } x <^{N^c} y,
\end{align*}
Thus, a dense-in-itself subset of \( N \), \( N \) embeddable in \( \langle N, \leq, \rangle \), and let \( \langle N, \leq \rangle \) be a linear ordering. We say that \( \langle a, b \rangle \) are consecutive in \( N \) whenever \( a, b \in N, a < b \) and \( (a, b)^N = \emptyset \). So, for any linear ordering \( N \):

1. If \( \langle a', b' \rangle \) are consecutive in \( N \) then \( \langle a', b' \rangle \) are consecutive in \( N^c \), and
2. If \( \langle a', b' \rangle \) are consecutive in \( N \) then \( a', b' \in N \) and \( \langle a', b' \rangle \) are consecutive in \( N \).

The following fact is well-known.

**Proposition 3.1.** Let \( N \) be a LOTS.

1. \( \langle N, \lambda^N \rangle \) is a compact space if and only if \( \langle N, \leq^N \rangle \) is order complete.
2. \( \langle N, \lambda^N \rangle \) is a connected space if and only if \( \langle N, \leq^N \rangle \) has no cuts and no consecutive elements.

Next we introduce some basic notions on scattered linear orders and scattered spaces. Let \( N \) be a linear ordering. We say that \( N \) is order-dense, or dense if between two elements of \( N \) there is a member of \( N \). Notice that for any dense linear order \( N \), the rational chain \( \mathbb{Q} \) is order-embeddable in \( N \). A linear order \( N \) is called order-scattered or simply scattered, whenever the rational chain \( \mathbb{Q} \) is not embeddable in \( N \). For example, \( \omega \) and its converse ordering \( \omega^*_1 \) are scattered linear ordering (by the definition, \( \langle \omega^*_1, \leq \rangle \) is the ordering \( \langle \omega_1, \geq \rangle \)).

A space \( Y \) is dense-in-itself if \( Y \) is nonempty and has no isolated point. A dense-in-itself and closed subspace \( Y \) of a space \( Z \) is called a perfect subspace of \( Z \). A space \( X \) is called topologically-scattered, or simply scattered (space), whenever \( X \) does not contain a perfect subspace, that is, every nonempty subset \( A \) of \( X \) with the induced topology has an isolated point in \( A \). We state other well-known facts about LOTS. For completeness we include the proofs.

**Proposition 3.2.** Let \( N \) be a LOTS.

1. The following hold.
   a. If \( \langle N, \leq^N \rangle \) is a scattered linear ordering then \( \langle N, \lambda^N \rangle \) is topologically scattered.
   b. Assume that \( \langle N, \leq^N \rangle \) is a complete chain, i.e., by Proposition 3.1(1), \( \langle N, \lambda^N \rangle \) is a compact space. Then the following are equivalent.
      i. \( \langle N, \lambda^N \rangle \) is a scattered topological space.
      ii. \( \langle N, \leq^N \rangle \) is a scattered linear ordering.
   c. If \( \langle N, \leq^N \rangle \) is order-scattered then \( \langle N, \lambda^N \rangle \) is 0-dimensional.
2. If \( N \) has only countably many isolated points then \( N \) is countable.

**Proof.**
1. a. Assume that \( \langle N, \lambda^N \rangle \) is not a scattered space. Let \( D \subseteq N \) be a dense-in-itself subset of \( N \). Then \( \langle D, \leq^N \mid D \rangle \) contains an order-dense subset, and thus \( N \) is not a scattered chain.
   b. Suppose that \( \langle N, \lambda^N \rangle \) is compact and that \( \langle N, \leq^N \rangle \) is not order-scattered. Let \( S \subseteq N \) be an order-dense subset of \( N \), and let \( T \) be its topological closure in \( \langle N, \lambda^N \rangle \). Then \( T \) has no isolated points, i.e. \( T \) is dense-in-itself. By compactness, \( T \) is compact and thus \( T \) is perfect. Hence \( \langle N, \lambda^N \rangle \) is not a scattered space.
We prove a little bit more. We have \( N = N^c \). Let \( M \) be a linear ordering and let \( M^c \) be its the Dedekind completion. The following are equivalent: (i) \( M \) is a scattered chain, (ii) \( Q \) is not order-embeddable in \( M \), (iii) \( Q \) is not order-embeddable in \( M^c \), and (iv) \( M^c \) is a scattered chain. Now since \( \langle M^c, \leq_{M^c} \rangle \) is a complete chain, \( \langle M^c, \lambda^{M^c} \rangle \) is a compact space. Therefore the previous items are equivalent to each of the following (v) \( Q \) is not order-embeddable in \( M^c \), and (vi) \( \langle M^c, \lambda^{M^c} \rangle \) is a scattered space.

(2) In the proof of Part (1), we have seen that if \( N \) is a scattered chain, then its Dedekind completion \( N^c \) is a scattered chain and thus \( N^c \), considered as a LOTS, is compact and topologically-scattered. Therefore \( N^c \) is 0-dimensional, and thus \( N \) is also 0-dimensional.

(3) By the hypothesis, the set \( S := \text{Iso}(N) \) of isolated points in \( N \) is countable. Since \( N \) is a scattered space, \( S \) is topologically dense in \( N \). Since \( S \) is a chain, by the proof of Part (1), the chain \( S^c \) is scattered. We claim that \( S^c \) is countable. This is so, because if \( S^c \) is an uncountable scattered chain, then \( \omega_1 \) or \( \omega_1^* \) is order-embeddable in \( S^c \) and thus the same holds for \( S \), and thus, \( S \) is uncountable contradicting our assumptions.

Next, since \( S^c \) is countable, \( N \) must be countable. Indeed, there exists a continuous (increasing) map \( f \) from \( N \) into \( S^c \) such that \( |f^{-1}(x)| \leq 2 \) for any \( x \in N \). (That is, the case if \( N := \omega + 1 + 1 + \omega^* \), and thus \( S := \text{Iso}(N) = \omega + \omega^* \) and \( S^c = \omega + 1 + \omega^* \).)

\[ \square \]

**Remark 3.3.** (1) Proposition 3.2(1)(a) is not reversible. As an example, consider the lexicographic sum \( N := \sum_{q \in \mathbb{Q}} \mathbb{Z}_q \) of copies \( \mathbb{Z}_q \) of (the the chain of integers) \( \mathbb{Z} \), indexed by the chain of rationals \( \mathbb{Q} \). Then \( N \) is a non-scattered linear ordering, but \( N \) is a topological discrete LOTS and thus \( N \) is a scattered topological space.

(2) Recall that the Dedekind completion of \( M := (0, 1) \cap \mathbb{Q} \) is \( M^c = [0, 1]^\mathbb{R} \). On the other hand, \( M \) is topologically dense in the Cantor set \( 2^\omega \) (considered as a subset of \( \mathbb{R} \)).

(3) Let \( X := \{1/n : n > 0 \} \cup \{-1\} \subset \mathbb{R} := L \). Then \( \langle X, \tau^X \rangle \) is compact, but \( \langle X, \tau^X \rangle \) is infinite and discrete. \[ \square \]

- Let \( \langle X, \tau^X, L, \leq^L \rangle \) be a GO-space. Then \( \lambda^L \downarrow X \subseteq \tau^X \).

Indeed, let \( a < b \) in \( X \). So \( (a, b)^X \in \lambda^X \) and thus, by the definition, \( (a, b)^X = (a, b)^N \cap X \in \tau^X \).

The following result is well-known. For completeness we include its proof.

**Proposition 3.4.** Let \( \langle X, \tau^X \rangle \) be a GO-space. Let \( \langle L, \leq^L \rangle \) be such that \( \langle X, \tau^X, L, \leq^L \rangle \) is a GO-space. Without loss of generality we may assume that \( L \) satisfies:

\begin{enumerate}
  \item[(H1)] \( X \) is topologically dense in \( \langle L, \lambda^L \rangle \);
  \item[(H2)] \( \langle L, \leq^L \rangle \) is a complete linear ordering.
\end{enumerate}

**Proof.** The proof follows from the following two facts.

**Fact 1.** Let \( \langle N, \leq^N \rangle \) be a linear ordering and \( \langle N^c, \leq^{N^c} \rangle \) be its Dedekind completion. Then \( \lambda^N = \tau^{N^c} \downarrow N \). That is, the order topology \( \lambda^N \) is the induced topology of \( \tau^{N^c} \) on \( N \).
Proof. Let \( c \in N^c \). Then \( c \) is a cut if and only if \( c \not\in N \cup \{ \min(N), \max(N) \} \) and \( c \) has no predecessor nor a successor in \( N^c \). Obviously, \( \lambda^N \subseteq \tau^N \restriction N \). Next let \((a, b)^N \in \lambda^N \) be an open convex set in \( (N^c, \leq^N) \). So \( a, b \in N^c \). If \( a \not\in N \) then \( a = \inf\{\{a' \in N : a' > a\}\} \) and if \( b \not\in N \) then \( b = \inf\{\{b' \in N : b' < b\}\} \). So \((a, b)^N \cap N \) is an union of open convex sets \((a', b')^N \) where \( a', b' \in N \).

Fact 2. Let \( (X, \tau^X, N, \leq^N) \) be a GO-structure such that \( (N, \tau^N) \) is a complete ordering. Let \( L \) be the topological closure of \( X \) in the space \( (N, \lambda^N) \). Then \( \tau^X = \lambda^L \restriction X \). That is, \((X, \tau^X, L, \leq^N | L)\) is a GO-structure.

Proof. It suffices to show that for every \( a < b \) in \( N \) there are \( a' < b' \) in \( L \) such that \((*)\): \((a', b')^L \cap X = (a, b)^N \cap X \). Fix \( a < b \) in \( N \). If \( a \in X \) \( b \in X \) set \( a' = a \) \( b' = b \). Next suppose that \( a \in N \setminus L \). Since \( N \) is a complete ordering and \( N \setminus L \) is open in \( L \), there is a (maximal) open convex set \((a, a')^N \) in \( N \) such that \( a \in (a, a')^N \), \((a, a')^N \cap L = \emptyset \) and \( a, a' \in L \); and we set \( a' = a \).

Similarly, suppose that \( b \in N \setminus L \). Again, since \( N \) is complete and \( N \setminus L \) is open in \( L \), there is a (maximal) open convex set \((\beta, \beta')^N \) such that \( b \in (\beta, \beta')^N \), \((\beta, \beta')^N \cap L = \emptyset \) and \( \beta, \beta' \in L \); and we set \( b' = \beta' \). Now obviously \((a', b')^L \) is as required in \((*)\).

Now let \((X, \tau^X, N, \leq^N)\) be a GO-structure. By Fact 1 we may assume that \( N \) is a complete ordering. Finally, the result follows from Fact 2.

Remark 3.5. (1) In general \( \lambda^X \not\subseteq \tau^X \). For example, consider \( L = \omega_1 \) and let \( \text{Lim} \) be the set of all countable limit ordinals. We set \( X = \omega_1 \setminus \text{Lim} \). We have:
(i) \( \tau^X \) is the discrete topology,
(ii) \( Y \) is topologically dense in \( (\omega_1, \lambda^{\omega_1}) \), and
(iii) \( X \) is order-isomorphic to \( \omega_1 \) and thus \( (Y, \lambda^{\omega_1}) \) is homeomorphic to the ordinal space \( \omega_1 \).

(2) The one-point compactification of an uncountable discrete space is not a GO-space (by Lemma 4.1), and there is a countable space which is not a GO-space (by Lemma 4.4).

For completeness we recall the proof of the following fact.

Proposition 3.6. [10, Lemma 6.1] Let \((X, \tau^X, L, \leq^L)\) be a GO-structure satisfying \((H1)\) and \((H2)\). So \((X, \leq^L \restriction X)\) is a linear ordering.

(1) If \((X, \tau^X)\) is a compact space then \( X = L \) and \( \tau^X = \lambda^X \).

(2) If \((X, \tau^X)\) is a connected space then \( \tau^X = \lambda^X \).

Proof. (1) Since \( X \) is compact then \( X \) is closed in \( (L, \lambda^L) \). By \((H1)\), \( X = L \) and thus \( \tau^X = \lambda^X \).

(2) Next suppose that \((X, \tau^X)\) is connected. Then \( X \), considered as the LOTS \((X, \lambda^X)\), has no consecutive point and no cut because for each final subset of \((X, \leq^L \restriction X)\) is closed in \((X, \tau^X)\). Therefore, \( X \setminus \{ \min(L), \max(L) \} = L \setminus \{ \min(L), \max(L) \} \) and thus \( \tau^X = \lambda^X \).

4. Proof of Main Theorem

As one of the main parts of Main Theorem 1.1 Proposition 4.8 implies that it suffices to assume that \((L, \leq)\) is a scattered linear order. To prove this result, we use Corollary 4.5 (in §4.2) which says that a GO-space does not contain an infinite countable relatively discrete closed subset.
Recall that \( \langle X, \tau \rangle \) is a GO-space means that \( \langle X, \tau, L, \leq \rangle \) is a GO-structure. So, we have \( X \subseteq L \). Also for simplicity \( \langle X, \tau \rangle \) is denoted by \( X \). In the sequel, by Proposition 3.4, we assume that the GO-structure \( \langle X, \tau, L, \leq \rangle \) satisfies properties (H1) and (H2).

### 4.1. A hereditarily GO-space satisfies c.c.c. property

**Lemma 4.1.** Assume that \( \langle X, \tau, L, \leq \rangle \) is a 0-dimensional hereditarily GO-structure. Then

1. \( X \) satisfies c.c.c. property.
2. The linear orderings \( \omega_1 \) and \( \omega_1^* \) are not order-embeddable in \( X \).

**Proof.** (1) Since \( \langle X, \tau \rangle \) is 0-dimensional, any nonempty open subset of \( \langle X, \tau \rangle \) contains a clopen convex subset of the form \( (a, b)^X := (a, b)^L \cap X \) where \( a, b \in L \). By contradiction, assume that \( \{ U_i : i \in I \} \) is an uncountable family of pairwise nonempty clopen convex subsets of \( X \). So each \( U_i \) is of the form \( (a_i, b_i)^X \) with \( a_i, b_i \in L \).

Fix \( i_0 \in I \). Let \( U := \bigcup \{ U_i : i \in I \setminus \{ i_0 \} \} \) and \( \sim \) be the equivalence relation on \( X \) defined as follows: \( x \sim y \) if and only if \( x, y \in Y \setminus U \) or there is \( i \in I \) such that \( x, y \in U_i \). Denote by \( X' \) the set \( X/\sim \) and by \( f : X \to X' \) the quotient map. We endow \( X' \) with the quotient topology \( \tau' \) on \( X' \). So \( V' \in \tau' \) if and only if \( f^{-1}[V'] \in \tau \). In particular, \( u_i := f[U_i] \) is an isolated point in \( X' \) for any \( i \in I \setminus \{ i_0 \} \).

Setting \( u = f[X \setminus U] \) we have \( X' = \{ u \} \cup \bigcup \{ u_i : i \in I \setminus \{ i_0 \} \} \) and the set

\[
\{ \{ u_i \} : i \in I \setminus \{ i_0 \} \} \cup \{ X' \setminus \{ u_i \} : i \in I \setminus \{ i_0 \} \}
\]

is a subbase of a topology \( \tau'' \) on \( X' \) satisfying:

1. \( \tau'' \subseteq \tau' \) and thus \( f : \langle X, \tau \rangle \to \langle X', \tau'' \rangle \) is continuous,
2. \( \langle X', \tau'' \rangle \) is compact (this follow from the definition of \( \tau'' \)), and
3. \( \langle X', \tau'' \rangle \) is homeomorphic to the one-point compactification of the uncountable discrete set. This is so because \( u \) is the unique accumulation point of \( \langle X', \tau'' \rangle \).

We show that \( \langle X', \tau'' \rangle \) is not a GO-space. By contradiction, suppose that \( \langle X', \tau'' , L', \leq' \rangle \) is a GO-structure. Since \( \langle X', \tau'' \rangle \) is compact, by Proposition 3.6(1), \( L' = X' \) and \( \tau'' = \lambda' := \lambda \leq' \). So it suffices to prove that

4. \( \langle X', \tau'' \rangle : = \langle X', \lambda' \rangle \) is not a LOTS.

Assume that \( \langle L', \leq' \rangle \) is a chain. Hence, for instance, \( (-\infty, u)^L' \) is uncountable. Consider any \( y \in (-\infty, u) \) such that \( (-\infty, y)^L' \) is infinite. By the definition, \( (-\infty, y)^L' \) is infinite, discrete, closed and thus compact, that contradicts the fact that \( u \notin (-\infty, y)^L' \). We have proved that \( X \) satisfies c.c.c. property.

(2) follows from Part (1).

Now remind the classical result which is due to Mazurkiewicz and Sierpiński (for example, see [8, Theorem 17.11], [13, Ch. 2, Theorem 8.6.10]).

**Lemma 4.2.** Every topologically scattered compact and countable space is homeomorphic to a countable and successor ordinal space.
4.2. A hereditarily GO-space has no countable closed and relatively discrete subsets

Lemma 4.3. Let $M$ be an order-scattered LOTS. If $M$ contains a closed and countable relatively discrete subset $D$, then $D$ is a continuous image of $M$.

Proof. First we introduce a new definition. Let $Y$ be a topological space. For a family $\mathcal{Y}$ of pairwise disjoint subsets of $Y$, we denote by $\text{acc}(\mathcal{Y})$ the set of accumulation points of $\mathcal{Y}$. By definition, $x \in \text{acc}(\mathcal{Y})$ if and only if for every neighborhood $W$ of $x$ the set $\{V \in \mathcal{Y} : V \cap W \neq \emptyset\}$ is infinite. So if $Z \subseteq Y$, $\text{acc}(Z) = \text{acc}\{\{x\} : x \in Z\}$.

We say that a subset $D$ of a space $Y$ is strongly discrete whenever $\mathcal{U}_D$ satisfies $\text{acc}(D) = \text{acc}(\mathcal{U}_D)$. The next result is well-known. For completeness we recall its proof.

Fact 1. Let $M$ be a 0-dimensional LOTS and $D \subseteq M$. If $D$ is relatively discrete then $D$ is strongly discrete.

Proof. For $d \in D$ let $U_d := (a_d, b_d)^L$ be a clopen convex set such that $D \cap U_d = \{d\}$. Note the following property $(\ast)$: if $d < d'$ then $x < x'$ for every $x \in U_d$ and $x' \in U_{d'}$. We set $\mathcal{U}_D = \{U_d : d \in D\}$. Obviously, $\text{acc}(D) \subseteq \text{acc}(\mathcal{U}_D)$. Conversely, let $x \in \text{acc}(\mathcal{U}_D)$ and $V$ be a neighborhood of $x$. We may assume that $V$ is of the form $(a, b)$ with $a < b$ in $M$. Therefore, by $(\ast)$, there are infinitely many $d_i$ such that $d_i \in U_{d_i} \subseteq V$, and thus $x \in \text{acc}(\mathcal{U}_D) \subseteq \text{acc}(D)$. $\square$

Since $\langle M, \leq^M \rangle$ is an order-scattered LOTS, by Proposition 3.2(2), $\langle M, \leq^M \rangle$ is 0-dimensional. By Fact 1, let $\mathcal{U}_D := \{U_d : d \in D\}$ be a family of clopen convex subsets of $M$ such that $U_d \cap D = \{d\}$ for $d \in D$ and $\text{acc}(D) = \text{acc}(\mathcal{U}_D)$. Since $D$ is closed, $\text{acc}(D) = \emptyset$ and thus $\bigcup \mathcal{U}_D$ is a clopen subset of $M$.

Let $d_0 \in D$ be fixed and $\mathcal{U}_D \setminus \{d_0\} = \{U_d : d \in D \setminus \{d_0\}\}$. So $\bigcup \mathcal{U}_D \setminus \{d_0\} = \bigcup \mathcal{U}_D \setminus U_{d_0}$ is a clopen subset of $M$. Let $\approx$ be the equivalence relation on $M$ defined by $x \approx y$ whenever $x, y \in U_D$ for some $d \in D \setminus \{d_0\}$, or $x, y \in M \setminus \bigcup \mathcal{U}_D \setminus \{d_0\}$. For each $x \in M$ there is an unique $f(x) \in D$ such that $f(x) \approx x$. It is easy to check that the mapping $f : M \to D$ is onto and that $f$ is continuous: this is so, because $f^{-1}(d)$ is clopen in $M$ for any $d \in D$. $\square$

Our next result, which is apparently well-known, strengthens Fact 6 from the proof of Corollary 1.4. Consider again $\omega$ as a discrete space. Let $\mathcal{U}$ on $\omega$ be a non-principal ultrafilter on $\omega$ and let $\ast$ be a new element with $\ast \notin \omega$. The space $\mathbb{N}^* := \omega \cup \{\ast\}$ is equipped with the topology induced from the Stone–Čech compactification $\beta\mathbb{N}$.

Lemma 4.4. The countable space $\mathbb{N}^*$ is not a GO-space.

Proof. Recall that the character $\chi(X)$ of the infinite topological space $X$ is the supremum of cardinalities of minimal local neighborhood bases of all points in $X$.

Fact 1. Let $\langle X, \tau^X, \lambda^L \rangle$ be a GO-structure satisfying (H1) and (H2). If $X$ is countable then $\chi(\langle X, \tau^X \rangle) = \aleph_0$.

Proof. Since $X$ is a countable subset of $L$ and $X$ is topologically dense in $L$, the chain $L$ is order-embeddable in the segment $[0, 1]$ of $\mathbb{R}$. Since $\mathbb{Q} \cap [0, 1] \subseteq [0, 1]$ we have $\chi([0, 1]) = \aleph_0$. So, $\aleph_0 \leq \chi(X) \leq \chi(L) \leq \chi([0, 1]) = \aleph_0$. $\square$

Now it suffices to remind a well-known fact that $\chi(\mathbb{N}^*) > \aleph_0$ (see [4, 3.6.17]).

Therefore, $\mathbb{N}^*$ is not a GO-space. $\square$

As a consequence of the above results 4.3 and 4.4, we have:
Corollary 4.5. Let $X$ be a hereditarily GO-space. Then $X$ does not contain an infinite countable relatively discrete closed subset. □

4.3. A hereditarily GO-space comes from a scattered linear ordering

Let GO-structure $\langle X, \tau^X, L, \leq^L \rangle$ satisfies the conditions:

(H1) $X$ is topologically dense in $\langle L, \lambda^L \rangle$.

(H2) $\langle L, \leq^L \rangle$ is complete, meaning that $\langle L, \lambda^L \rangle$ is a compact LOTS.

Hence, by Proposition 3.2(1)(b),

(H3) For any $u < v$ in $L$: $[u, v]^L$ is order-scattered if and only if $[u, v]^L$ is topologically-scattered.

Let $\langle X, \tau \rangle$ be a hereditarily GO-space, meaning that $\langle X, \tau^X, L, \leq^L \rangle$ is a hereditarily GO-structure satisfying (H1)–(H3). For simplicity denote the space $\langle X, \tau^X \rangle$ by $X$. We shall show that $\langle L, \leq^L \rangle$ is order-scattered (Proposition 4.8), and thus, by Proposition 3.2(1)(b), $\langle L, \lambda^L \rangle$ is topologically–scattered. Therefore $X$, as subset of $L$, is also topologically–scattered.

Let $\equiv^L$ be the equivalence relation on $L$ defined as follows. For $x, y \in L$, we set $x \equiv^L y$ if $x \leq y$ and $[x, y]^L$ is an order-scattered subset of $L$, or $y \leq x$ and $[y, x]^L$ is an order-scattered subset of $L$. Note that, by (H3), in the definition of $\equiv^L$ we have: $[u, v]^L$ is order-scattered if and only if $[u, v]^L$ is topologically-scattered.

Now, each equivalence class is an order-scattered and convex subset of the LOTS $\langle L, \leq \rangle$ and each equivalence class is closed in $\langle L, \lambda \rangle$. ($\equiv^L$ is standard (see the proof of Theorem 19.26, in [9])).

We denote $L/\equiv^L$ by $L_1$ and by $\pi: L \to L_1$ the projection map. So $\pi$ is increasing and thus $\pi$ induces a linear order $\leq_1$ on $L_1$.

Lemma 4.6. The linear ordering $\langle L_1, \leq_1 \rangle$ has the following properties.

(1) $L_1$ is a complete linear ordering and the quotient topology on $L_1$ is the order topology $\lambda_1 := \lambda^{\leq_1}$.

(2) $L_1$ is order-dense, i.e. $L_1$ has no consecutive elements.

(3) $L_1$ is a compact and dense-in-itself space.

(4) $L_1$ is a connected space.

Proof. (1) This part follows from the fact that $L$ is complete and $\pi$ is increasing and onto.

(2)–(3) Notice first that $\langle L_1, \leq_1 \rangle$ is a complete chain, and thus $\langle L_1, \lambda_1 \rangle$ is compact. Secondly, there are no consecutive $\equiv^L$-classes in $L$, this is so because the union of two consecutive $\equiv^L$-classes is an $\equiv^L$-class. Hence $L_1$ has no consecutive elements.

Therefore, $\langle L_1, \leq_1 \rangle$ is a dense chain and thus $L_1$ is dense-in-itself.

(4) By Part (2), $L_1$ has no consecutive elements. Also since $L_1$ is a complete chain, $L_1$ has no cuts. So, by Proposition 3.6, $L_1$ is a connected. □

Now we recall that $X \subseteq L$ and that for $x < y$ in $L$:

(1) $[x, y]^L$ is a scattered subspace of $\langle L, \lambda \rangle$ if and only if $[x, y]^L$ is a scattered subchain of $\langle L, \leq \rangle$, and

(2) if $[x, y]^L$ is a scattered subspace of $L$ then $[x, y]^X := [x, y]^L \cap X$ is a scattered subspace of $X$ (but not vice-versa).
Now the relation \( \equiv^L \) induces an equivalence relation \( \equiv^X \) on \( X \), setting for \( x, y \in X \):

\[
x \equiv^X y \text{ if and only if } x \equiv^L y.
\]

We denote by \( X_1 \) the space \( X/\equiv^X \).

**Lemma 4.7.** The following hold for \( (H2) \):

1. The continuous inclusion embedding \( X \subseteq L \) induces a continuous inclusion embedding \( X_1 \subseteq L_1 \).
2. \( X_1 \) is a topologically dense subset of \( L_1 \) for the topology \( \lambda_1 \) (on \( L_1 \)).
3. \( X_1 \) considered as subordering of \( L_1 \) has no consecutive points.
4. \( \langle X_1, \tau_1, L_1, \leq_1 \rangle \) is a GO-structure.

**Proof.**

(1) Let \( \pi : L \to L_1 \) be the projection map. Then \( \pi[X] := X_1 \subseteq L_1 \) and the embedding \( X_1 \subseteq L_1 \) is continuous.

(2) Since \( X \) is a topologically dense subset of \( L \), \( X_1 \) is topologically dense in \( L_1 \).

(3) Since \( X_1 \) is topologically dense in \( L_1 \), if \( a_1 < b_1 \) are consecutive elements in \( X_1 \) then \( a_1 < b_1 \) are also consecutive in \( L_1 \). This contradicts Lemma 4.6(2).

(4) follows from the definitions. \qed

We have seen that \( \langle X_1, \tau_1, L_1, \leq_1 \rangle \) is a GO-structure with the properties \((H1)\), \((H2)\), and the properties \((1)-(4)\) of Lemma 4.6 and \((1)-(3)\) of Lemma 4.7.

**Proposition 4.8.** Let \( \langle X, \tau, L, \leq \rangle \) be a hereditarily GO-space satisfying \((H1)\) and \((H2)\). Then \( \langle L, \leq \rangle \) is a scattered chain.

**Proof.** Now, with the above notations of §4.3, we consider the GO-space \( \langle X_1, \tau^{X_1}, L_1, \leq^{L_1} \rangle \) instead of the GO-space \( \langle X, \tau^X, L, \leq^L \rangle \). Let

\[
\Gamma_1 = L_1 \setminus (X_1 \cup \{\min(L_1), \max(L_1)\}).
\]

That is, \( \Gamma_1 \) is the set of cuts of the chain \( \langle X_1, \leq^{X_1} \rangle \) considered as linear subordering order of \( \langle L_1, \leq^{L_1} \rangle \). An obvious characterization of the elements of \( \Gamma_1 \) is stated in the following fact.

**Fact 1.** We have that \( \gamma \in \Gamma_1 \) if and only if \( \gamma \) defines two nonempty clopen sets, namely \((\alpha, \gamma)X_1^1 \) and \((\gamma, +\infty)X_1^1 \), for the induced topology \( \tau^{X_1} \).

Therefore \( \langle X_1, \tau^{X_1} \rangle \) is a connected space if and only if \( \Gamma_1 \) is empty and \( X_1 \) has no consecutive elements. \qed

To prove Proposition 4.8, we distinguish two cases, and in fact we prove that \( |X_1| = 1 \). This implies that \( \langle L, \leq^L \rangle \) is order-scattered.

**Case 1.** \( \Gamma_1 \) has no consecutive elements and \( \Gamma_1 \) is topologically dense in \( L_1 \) for the order topology \( \lambda_1 \).

So the set \( \Gamma_1 \) is a dense linear order and every nonempty open convex subset of \( L_1 \) contains a cut. Since \( X_1 \) is topologically dense in \( L_1 \), any nonempty open convex subset of \( L_1 \) contains a cut and thus, in that case,

- \( \langle X_1, \tau_1 \rangle \) is 0-dimensional.

Let \( c \in \Gamma_1 \). Since \( \Gamma_1 \) has no first element, let \( \langle c_\alpha : \alpha < \lambda \rangle \) be a cofinal strictly increasing sequence in \((\alpha, \infty)c^{X_1} \) where \( \lambda \) is an infinite regular cardinal, that is, for every \( x \in (\alpha, \infty)c^{X_1} \) there is \( \alpha \) such that \( x \leq c_\alpha \).

**Fact 2.** \( \lambda = \omega \).
Proof. If not then \( \omega_1 \) is order-embeddable in \( \Gamma_1 \). Choose \( x_\alpha \in (c_\alpha, c_{\alpha+1}) \cap X_1 \) for any \( \alpha \), then \( \langle x_\alpha : \alpha < \lambda \rangle \) is a sequence in \( X_1 \), order-isomorphic to \( \omega_1 \). Since \( \langle X_1, \tau_1 \rangle \) is 0-dimensional (but not necessarily an interval space), this contradicts Lemma 4.1(2). We have proved that \( \lambda = \omega \). \( \square \)

Keeping the same notations, we have (\(*\)) the set \( D := \{ x_\alpha : \alpha \in \omega \} \) is countable and relatively discrete. Also since \( c \) is a cut, (**): \( D \) is closed in \( X_1 \). (\(*\)) together with (**) contradicts Corollary 4.5. Therefore, Case 1 does not occur.

Case 2. Not Case 1.

This implies that either \( \Gamma_1 := L_1 \setminus \langle X_1 \cup \{ \min(L_1), \max(L_1) \} \rangle \) has two consecutive elements in \( L_1 \) or \( \Gamma_1 \) is not topologically dense in \( L_1 \). Anycase, there is an infinite open interval \( (u, v) \) of \( L_1 \) with \( u < v \) in \( L_1 \) that does not contain a member of \( \Gamma_1 \). Recall that we have the following properties.

(P1) The elements of \( L_1 \) are exactly the \( =_\Gamma \)-classes of \( L \), and
(P2) \( (L_1, \leq_{1}) \) is a dense linear order.

Since \( (u, v) \) is infinite, we may assume that \( u, v \notin \Gamma_1 \). So, we have the additional properties.

(P3) \( [u, v]^{L_1} \cap \Gamma_1 = \emptyset \), and thus \( [u, v]^{L_1} = [u, v]^{X_1} \).
(P4) \( [u, v]^{L_1} \) is infinite.
(P5) \( X_1 \), and thus \( [u, v]^{X_1} \), has no consecutive elements (Lemma 4.7(3)).
(P6) Hence, by (P3), (P5) and Proposition 3.1(2), \( [u, v]^{X_1} \) is a connected space.

Next we prove that \( |X_1| = 1 \). By contradiction, assume that \( |X_1| > 1 \). We consider the equivalence relation \( \equiv_{X_1} \) on \( X_1 \) which identifies all elements of \( (-\infty, u) \cup [v, +\infty) \). So \( X_1 \equiv_{X_1} \), denoted by \( X_2 \), is a continuous image of \( X_1 \). Also \( X_2 \) is a connected space and, by (P4), \( X_2 \) is an infinite continuous image of \( X_1 \).

Fact 3. Let \( \langle Y, \tau^Y, M, \leq^M \rangle \) be a GO-structure such that \( \langle Y, \tau^Y \rangle \) is connected.

Then, for every \( y \in Y \): if \( y \) is not the minimum nor the maximum of \( Y \) (if they exist) then the subspace \( Y \setminus \{ y \} \) is not a connected space.

Proof. By Proposition 3.6(2), \( \tau^Y = \lambda^Y \). Let \( y \in Y \) be such that \( y \) is not the minimum nor the maximum of \( Y \). Set \( U = (\infty, y)^M \) and \( V = (y, \infty)^M \). By the definition, \( U \) and \( V \) are open subsets of \( M \). Hence \( U \cap Y \) and \( V \cap Y \) define a partition of \( Y \) into two nonempty open sets of \( Y \). We have proved Fact 3. \( \square \)

We claim that

Fact 4. For every \( x \in X_2 \), the subspace \( X_2 \setminus \{ x \} \) is connected.

Proof. First recall that \( X_2 \) is a connected space. Also we can define \( X_2 \) as follows: \( X_2 \) is the quotient of \( [u, v]^{X_1} \) identifying \( u \) and \( v \). Fact 4 follows from the claim that \( [u, v]^{X_1} \) is a connected interval subspace of \( X_1 \). \( \square \)

Now, from (P6) it follows that: \( X_2 \) is connected, and by Fact 4: for any \( x \in X_2 \) the space \( X_2 \setminus \{ x \} \) is connected. Hence, by Fact 3, the space \( X_2 \) is not a GO-space. So \( X_1 \) and thus \( X \) is not a GO-space.

In other words, by (P1) and (P2), if \( X \) (or equivalently \( L \)) has more than one \( \equiv_\Gamma \)-class, then \( X_2 \) is a continuous image \( X \) and \( X_2 \) is not a GO-space. This contradicts the fact that \( X \) is a hereditarily GO-space. We have proved that \( |X_1| = 1 \).

Further, \( |X_1| = 1 \) means that \( X_1 \) consists of exactly one \( \equiv_{X_1} \)-class, or, equivalently, there is the unique \( \equiv_{\Gamma} \)-class in \( L \). Since for \( x < y \) in \( X \): \( x \equiv_{\Gamma} y \) if and only if \( [x, y]^{L_1} \) is a scattered chain, the chain \( L \) is scattered. \( \square \)
4.4. End of the proof of Main Theorem

Let \( \langle X, \tau, L, \leq \rangle \) be a hereditarily GO-space. We prove first that \( X \) is countable. By Proposition 4.8, \( \langle L, \leq \rangle \) is a scattered chain. Since \( \langle L, \lambda \rangle \) is compact and topologically-scattered, by Lemma 4.1, \( \langle X, \tau \rangle \) satisfies c.c.c. property, so \( X \) has only countably many isolated points. Denote by \( \text{Iso}(Y) \) the set of isolated points of \( Y \). Since \( X \) is topologically dense in \( L \), we have \( \text{Iso}(L) = \text{Iso}(X) \) and thus \( \text{Iso}(L) = \aleph_0 \). Therefore, by Proposition 3.2(3), \( L \), and thus \( X \) is also countable.

Next, by Lemma 4.3(3), the space \( X \) does not contain a countable relatively discrete set. Since \( X \) is countable, \( X \) is closed under supremum and infimum in \( L \), and thus \( X \), as a linear order, is complete. We have seen that \( L \) and \( X \) are compact and countable. Finally, since \( X \) is topologically-scattered, by Lemma 4.2, \( X \) is homeomorphic to the LOTS \( \alpha + 1 \) where \( \alpha \) is a countable ordinal.

We have proved Main Theorem 1.1.

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