$W_{1+\infty}$ constraints for the hermitian one-matrix model

Rui Wang$^a$, Ke Wu$^a$, Zhao-Wen Yan$^b$, Chun-Hong Zhang$^c$, Wei-Zhong Zhao$^a$

$^a$School of Mathematical Sciences, Capital Normal University, Beijing 100048, China
$^b$School of Mathematical Sciences, Inner Mongolia University, Hohhot 010021, China
$^c$School of Mathematics and Statistics, North China University of Water Resources and Electric Power, Zhengzhou 450046, China

Abstract

We construct the multi-variable realizations of the $W_{1+\infty}$ algebra such that they lead to the $W_{1+\infty}$ $n$-algebra. Based on our realizations of the $W_{1+\infty}$ algebra, we derive the $W_{1+\infty}$ constraints for the hermitian one-matrix model. The constraint operators yield not only the $W_{1+\infty}$ algebra but also the closed $W_{1+\infty}$ $n$-algebra.

Keywords: Conformal and $W$ Symmetry, Matrix Models, $n$-algebra

1 Introduction

The Virasoro constraints for the matrix models have attracted remarkable attention [1]-[5]. Since the $W_{1+\infty}$ algebra can be generated by the higher order differential operators with respect to the eigenvalues of the matrices, an approach to derive a large class of constraint equations for matrix models at finite $N$ was proposed in Ref.[6]. These constraints are associated with the higher order differential operators of $W_{1+\infty}$ algebra, where the well-known Virasoro constraints are associated with the first order differential operators. However, it seems rather nontrivial to write down the constraints explicitly. The Ding-Iohara-Miki (DIM) algebra is a quantum deformation of the toroidal algebra with two central charges [7]-[10], which has attracted much interest from physical and mathematical points of view. It was found that the Ward identities in the network matrix models can be described in terms of this symmetry [11][12].

The elliptic generalization of hermitian matrix model is known to be associated with the 4d $\mathcal{N} = 1$ $U(N)$ gauge theory on $S^3 \times S^1$ [13]-[15]. The $q$-Virasoro constraints for this matrix model have been derived by the insertion of the $q$-Virasoro generators under the contour integral [15], where the $q$-Virasoro generators are constructed in terms of $q$-derivatives within the $q$-calculus.

\footnote{Corresponding author: zhaowz@cnu.edu.cn}
and the corresponding $q$-Virasoro algebra is a special case of a more general elliptic deformation of the Virasoro algebra [16]. Since 3-algebra has recently been found useful in the Bagger-Lambert-Gustavsson (BLG) theory of M2-branes [17, 18], the applications of $n$-algebra have aroused much interest [19]-[25]. More recently it was found that there are the generalized $q$-$W_{\infty}$ constraints for the elliptic matrix model [26]. Although these constraint operators do not yield the closed algebras, by applying the strategy of carrying out the action of the operators on the partition function as done in Ref.[15], the ($n$-)commutators of the constraint operators lead to the generalized $q$-$W_{\infty}$ algebra and $n$-algebra, respectively. For the elliptic matrix model, its partition function also satisfies the constraints from the elliptic DIM algebra [11]. In this letter, we focus on the hermitian one-matrix model and derive its $W_{1+\infty}$ constraints. We show that the derived constraint operators yield the $W_{1+\infty}$ $n$-algebra.

2 The multi-variable realizations of $W_{1+\infty}$ algebra and its $n$-algebra

Let us first recall the $W_{1+\infty}$ algebra [27]

$$\left[W_{r_{m_1}}^{r_{m_2}}, \left[W_{r_{m_2}}^{r_{m_3}}, \ldots, W_{r_{m_n}}^{r_{m_n}} \right] \right] = \left( \sum_{k=0}^{r_{m_1}-1} C_{r_{m_1}-1}^{k} A_{m_2+r_{m_2}-1}^{k} - \sum_{k=0}^{r_{m_2}-1} C_{r_{m_2}-1}^{k} A_{m_3+r_{m_3}-1}^{k} \right) W_{m_1+r_{m_2}-1-k}^{r_{m_1}+r_{m_2}-1-k} \right), \tag{1}$$

where $A_{n}^{k} = \left\{ \begin{array}{ll}
n(n-1)\cdots(n-k+1), & k \leq n, \\
0, & k > n,
\end{array} \right.$

Its single variable realization is given by

$$W_{m}^{r} = z^{m+r-1} \frac{d^{r-1}}{dz^{r-1}}, \quad r \in \mathbb{Z}_+, m \in \mathbb{Z}, \tag{2}$$

which not only yields [11], but also leads to the $W_{1+\infty}$ $n$-algebra [28]

$$\left[W_{r_{m_1}}^{r_{m_2}}, \ldots, W_{r_{m_n}}^{r_{m_n}} \right] := \epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}} W_{m_{1}m_{2}}^{r_{m_{1}}r_{m_{2}}} \cdots W_{m_{n}m_{n}}^{r_{m_{n}}r_{m_{n}}}$$

$$\epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}} := \epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}} \sum_{\alpha_{1}=0}^{\beta_{1}} \sum_{\alpha_{2}=0}^{\beta_{2}} \cdots \sum_{\alpha_{n-1}=0}^{\beta_{n-1}} C_{\beta_{1}}^{\alpha_{1}} C_{\beta_{2}}^{\alpha_{2}} \cdots C_{\beta_{n-1}}^{\alpha_{n-1}} A_{m_{2}+\cdots+m_{n}}^{\alpha_{1}+\cdots+\alpha_{n-1}-1} A_{m_{1}+\cdots+m_{n}}^{\alpha_{1}+\cdots+\alpha_{n-1}-1}$$

$$\epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}} = \epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}} \sum_{\alpha_{1}=0}^{\beta_{1}} \sum_{\alpha_{2}=0}^{\beta_{2}} \cdots \sum_{\alpha_{n-1}=0}^{\beta_{n-1}} C_{\beta_{1}}^{\alpha_{1}} C_{\beta_{2}}^{\alpha_{2}} \cdots C_{\beta_{n-1}}^{\alpha_{n-1}} A_{m_{2}+\cdots+m_{n}}^{\alpha_{1}+\cdots+\alpha_{n-1}-1} A_{m_{1}+\cdots+m_{n}}^{\alpha_{1}+\cdots+\alpha_{n-1}-1} \epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}}, \tag{3}$$

where $\epsilon_{i_{1}i_{2}\cdots i_{n}}^{j_{1}j_{2}\cdots j_{n}} = \det \begin{pmatrix} \delta_{i_{1}j_{1}} & \cdots & \delta_{i_{1}j_{n}} \\ \vdots & \ddots & \vdots \\ \delta_{i_{n}j_{1}} & \cdots & \delta_{i_{n}j_{n}} \end{pmatrix}$ and $\beta_{k} = \left\{ \begin{array}{ll}
r_{i_{1}} - 1, & k = 1, \\
\sum_{j=1}^{k} r_{i_{j}} - k - \sum_{i=1}^{k-1} \alpha_{i}, & 2 \leq k \leq n - 1.\end{array} \right.$
Since the associativity of the product of the operators \( \mathcal{O} \) holds, the \( n \)-algebra \( \mathfrak{A} \) with \( n \) even satisfies the generalized Jacobi identity (GJI) \[19\]

\[\epsilon_{i_1 \cdots i_{2n-1}}^{i_1 \cdots i_{2n-1}}[[A_{i_1}, A_{i_2}, \cdots, A_{i_n}], A_{i_{n+1}}, \cdots, A_{i_{2n-1}}] = 0.\] (4)

When \( n \) is odd, it satisfies the generalized Bremner identity (GBI) \[29, 30\]

\[\epsilon_{i_1 \cdots i_{3n-3}}^{i_1 \cdots i_{3n-3}}[[A, B_{i_1}, \cdots, B_{i_n}], [B_{i_{n+1}}, \cdots, B_{i_{2n-1}}], B_{i_{2n}}, \cdots, B_{i_{3n-3}}] = \epsilon_{i_1 \cdots i_{3n-3}}^{i_1 \cdots i_{3n-3}}[[A, [B_{i_1}, \cdots, B_{i_n}], B_{i_{n+1}}, \cdots, B_{i_{2n-2}}], B_{i_{2n-1}}, \cdots, B_{i_{3n-3}}].\] (5)

Hence the \( W_{1+\infty} \) \( n \)-algebra with \( n \) even is a generalized Lie algebra (or higher order Lie algebra). A remarkable property of \( \mathfrak{A} \) is that there are the following subalgebras

\[ [W_{m_1}^{n+1}, W_{m_2}^{n+1}, \ldots, W_{m_{2n}}^{n+1}] = \prod_{1 \leq j < k \leq 2n} (m_k - m_j) W_{m_1+m_2+\cdots+m_{2n}}^{n+1}, \] (6)

and

\[ [W_{m_1}^{n+1}, \ldots, W_{m_{2n+1}}^{n+1}] = 0. \] (7)

A well-known multi-variable realization of \( \mathfrak{A} \) is

\[ \bar{W}_m^r = \sum_{i=1}^{N} z_i^{m+r-1} \frac{\partial^{r-1}}{\partial z_i^{r-1}}, \quad r \in \mathbb{Z}_+, m \in \mathbb{Z}. \] (8)

However the generators \( \bar{W}_m^r \) do not yield the nontrivial \( n \)-algebra except for the null \((2nN+1)\)-algebra \[28\]

\[ [\bar{W}_{m_1}^{n+1}, \bar{W}_{m_2}^{n+1}, \ldots, \bar{W}_{m_{2nN+1}}^{n+1}] = 0. \] (9)

Note that it is not only determined by the superindex of the generators, but also the number of variables \( N \).

In order to construct the multi-variable realization of \( W_{1+\infty} \) \( n \)-algebra, let us introduce the Euler operator

\[ O_E = \sum_{i=1}^{N} z_i \frac{\partial}{\partial z_i} \] (10)

and the Lassalle operators \[31\]

\[ O_L^A = \sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} - \frac{2}{\alpha} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z_i - z_j} \frac{\partial}{\partial z_i}, \] (11)
\[ O_B^B = \sum_{i=1}^{N} \left( \frac{\partial^2}{\partial z_i^2} + \frac{\beta}{z_i} \frac{\partial}{\partial z_i} + \gamma \sum_{j=1}^{N} \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_i} \right) \]  

(12)

then we have the commutation relations

\[ [O_L^{A,B}, O_E] = 2O_L^{A,B}. \]  

(13)

It is known that the Hamiltonians of the \( A_{N-1} \) and \( B_N \)-Calogero models are \[ \hat{H}_C^A = \frac{1}{2} \sum_{i=1}^{N} (-\frac{\partial^2}{\partial z_i^2} + \omega z_i^2) + V^A, \]  

(14)

and

\[ \hat{H}_C^B = \frac{1}{2} \sum_{i=1}^{N} (-\frac{\partial^2}{\partial z_i^2} + \omega z_i^2) + V^B, \]  

(15)

where the potentials \( V^A \) and \( V^B \) are given by \( V^A = \frac{1}{2} \sum_{i=1}^{N} \sum_{j \neq i} \frac{a(a-1)}{(z_i - z_j)^2} \), \( V^B = \frac{N}{2} \sum_{i=1}^{N} \frac{b(b-1)}{z_i^2} + a(a-1) \sum_{i=1}^{N} \sum_{j \neq i} \frac{z_i^2 + z_j^2}{(z_i - z_j)^2} \), respectively, the constants \( a \) and \( b \) are the coupling parameters, \( \omega \) is the strength of the external harmonic well.

By performing a similarity transformation and removing the ground state from the Hamiltonian, we obtain

\[ H_C^A = (\Psi_g^A)^{-1} (\hat{H}_C^A - E_g^A) \Psi_g^A \]

\[ = \sum_{i=1}^{N} \left( -\frac{1}{2} \frac{\partial^2}{\partial z_i^2} + \omega z_i \frac{\partial}{\partial z_i} \right) - a \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z_i - z_j} \frac{\partial}{\partial z_i} \]  

(16)

and

\[ H_C^B = (\Psi_g^B)^{-1} (\hat{H}_C^B - E_g^B) \Psi_g^B \]

\[ = \sum_{i=1}^{N} \left( -\frac{1}{2} \frac{\partial^2}{\partial z_i^2} - \frac{b}{z_i} \frac{\partial}{\partial z_i} + \omega z_i \frac{\partial}{\partial z_i} \right) - 2a \sum_{i=1}^{N} \sum_{j \neq i} \frac{z_i}{z_i^2 - z_j^2} \frac{\partial}{\partial z_i} \]  

(17)

where \( \Psi_g^A \) and \( \Psi_g^B \) are the ground state wave functions, \( E_g^A \) and \( E_g^B \) are the ground state energies.

Then in terms of the Euler and Lassalle operators, the Hamiltonians \[ \text{[10]} \] and \[ \text{[11]} \] can be rewritten in a unified fashion \[ \text{[34]} \]

\[ H_C^{A,B} = \omega O_E - \frac{1}{2} O_L^{A,B}, \]  

(18)

where the parameters in \( O_L^{A,B} \) \[ \text{[11]} \] and \[ \text{[12]} \] take \( \alpha = -1/a \), \( \beta = 2b \) and \( \gamma = 4a \).
Let us take the operators

\[
\hat{W}_m^r = \left(\frac{O_E + N}{2}\right)^{r-1}(O_L^A + V^A)^m
\]

\[
= \left(\frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} z_i^r\right)^{r-1} \left(\sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} - 2 \frac{N}{\alpha} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \frac{1}{n!}(z_i - z_j)^m\right),
\]

where \(r \in \mathbb{Z}_+, m \in \mathbb{N}\) and we take \(\alpha = \frac{1}{n(\alpha-1)}\) in (11). We then obtain the algebra

\[
[W_{m_1}^{r_1}, W_{m_2}^{r_2}] = \left(\sum_{k=0}^{r_2-1} \sum_{m_{k1}}^{r_1-1} C_{r_2-1}^k m_1^k - \sum_{k=0}^{r_1-1} C_{r_1-1}^k m_2^k\right)W_{m_1+m_2}^{r_1+r_2-1-k}.
\]

When particularized to the \(r_1 = r_2 = 2\) case in (20), it gives the Witt algebra

\[
[W_{m_1}^2, W_{m_2}^2] = (m_1 - m_2)W_{m_1+m_2}^2.
\]

By replacing the generators \(W_m^r \rightarrow \hat{W}_m^r = -\sum_{n=-r+1}^{+\infty} \frac{m_1^r}{(n+r+1)} W_n^r\) in the commutation relation (1), after some simple calculation, it gives the commutation relation (20). Hence we also call (20) the \(W_{1+\infty}\) algebra. It should be noted that not as the case of (1), (20) contains the operators \(\hat{W}_m^r\) only for \(m \geq 0\) and \(r \geq 1\). Precisely speaking (20) is a subalgebra of the \(W_{1+\infty}\) algebra.

By direct calculation of the \(n\)-commutator of (19), it gives the \(W_{1+\infty}\) \(n\)-algebra

\[
[W_{m_1}^{r_1}, W_{m_2}^{r_2}, \ldots, W_{m_n}^{r_n}] = (-1)^{\frac{n(n-1)}{2}} \sum_{\alpha_1=0}^{r_1} \sum_{\alpha_2=0}^{r_2} \cdots \sum_{\alpha_n=0}^{r_n} C_{\alpha_1}^{\beta_1} C_{\alpha_2}^{\beta_2} \cdots C_{\alpha_n}^{\beta_n}
\]

\[
\cdot m_{1j}^{\alpha_1} m_{i_2}^{\alpha_2} \cdots m_{i_n}^{\alpha_n-1} W_{m_1+m_2+\cdots+m_n}^{r_1+r_2-1-n-\alpha_1-\cdots-\alpha_n-1}.
\]

When \(n\) is even, it is a generalized Lie algebra.

As the case of (3), we can show that there are the following subalgebras

\[
[W_{m_1}^{n+1}, W_{m_2}^{n+1}, \ldots, W_{m_{2n}}^{n+1}] = \prod_{1 \leq j < k \leq 2n} (m_k - m_j)W_{m_1+m_2+\cdots+m_{2n}}^{n+1},
\]

and

\[
[W_{m_1}^{n+1}, \ldots, W_{m_{2n+1}}^{n+1}] = 0,
\]

where we take the scaled generators \(\hat{W}_m^{n+1} \rightarrow \Lambda^{-1} \hat{W}_m^{n+1}\), and the scaling coefficient \(\Lambda\) is given by \(\Lambda = (-1)^n \sum_{(\alpha_1, \alpha_2, \ldots, \alpha_{2n-1}) \in S_{2n-1}} C_{\beta_1}^{\alpha_1} C_{\beta_2}^{\alpha_2} \cdots C_{\beta_2n-1}^{\alpha_{2n-1}}\) with \(\beta_1 = n, \beta_k = kn - \sum_{i=1}^{k-1} \alpha_i, 2 \leq k \leq 2n - 1\).
It should be noted that the operators $\hat{W}_m^r$ are not the conserved operators, i.e., $[\hat{W}_m^r, \hat{H}_C^A] \neq 0$. For the Calogero model (14) without the harmonic potential, its conserved operators are constructed by the recursive definition \[35\]

\[\hat{W}_m^1 = \sum_{j,k} (L^m)_{jk}, \quad \hat{W}_m^{r+1} = \frac{1}{2} (m+r) \sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} + \frac{\partial}{\partial z_i} \beta \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \frac{z_i^2}{z_i^2 - z_j^2}, \quad r \geq 2, m \geq -r+1, \tag{25}\]

and $\hat{W}_m^1 = \sum_{j,k} (L^m)_{jk}$, the Lax operator $L_{jk}$ is given by $L_{jk} = -i \delta_{jk} \frac{\partial}{\partial z_j} + (1 - \delta_{jk}) \frac{z_k}{z_i - z_k}$, $i = \sqrt{-1}$. These conserved operators constitute the $W_1^{1+\infty}$ algebra

\[\hat{W}_m^1, \hat{W}_n^1 \mid (r_2 - 1)m_1 - (r_1 - 1)m_2 \mid \hat{W}_m^{r_1+1} \hat{W}_n^{r_2-2} + \cdots \tag{26}\]

where $\cdots$ indicates the lower-order terms corresponding to the quantum effect. When $r_1 = r_2 = 2$ in (26), it gives the Witt algebra (21). It should be pointed out that these conserved operators do not yield the closed $n$-algebra.

We have presented a realization of $W_1^{1+\infty}$ algebra in terms of the Euler and Lassalle operators and the potential of the $A_{N-1}$-Calogero model. Let us turn to introduce another realization

\[\hat{W}_m^r = \frac{O_E + N}{2} (O_L^B + V_B)^m \tag{27}\]

where $r \in \mathbb{Z}^+$, $m \in \mathbb{N}$ and $O_E = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} z_i$, $O_L^B = \sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} \frac{\beta}{2} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} \frac{z_i^2}{z_i^2 - z_j^2}$. Straightforward calculation shows that the operators (27) also yield the $W_1^{1+\infty}$ algebra (20) and $n$-algebra (22).

3 $W_1^{1+\infty}$ constraints for the hermitian one-matrix model

The partition function of the hermitian one-matrix model is

\[Z_N(t) = \int dM \exp \left( \sum_{k=0}^{\infty} t_k \text{Tr} M^k \right), \tag{28}\]

where $t = \{t_k | k \in \mathbb{N}\}$, $M$ is an $N \times N$ hermitian matrix and $dM$ is the Haar measure

\[dM = \prod_{i=1}^{N} dM_i \prod_{i<j} d(\text{Re} M_{ij}) d(\text{Im} M_{ij}), \tag{29}\]

which is invariant under the gauge transformation $M \rightarrow U M U^\dagger$, and $U$ is a $U(N)$ matrix. In terms of the eigenvalues, the integral can be rewritten as

\[Z_N(t) = \int d^N z \Delta(z)^2 \exp(U(t)), \tag{30}\]
where $\Delta(z) = \prod_{1 \leq i < j \leq N} (z_i - z_j)$ and $U(t) = \sum_{k=0}^{\infty} t_k \sum_{i=1}^{N} z_i^k$. An approach to derive the $W_{1+\infty}$ constraints for this matrix model was proposed in Ref. [6].

The following identity has been used there

$$\int d^N z \Delta \cdot D_{n,m}(\exp(U)\Delta) - (D_{n,m}^\dagger \Delta) \cdot \exp(U)\Delta = 0,$$

where $D_{n,m} = \sum_{i=1}^{N} z_i^n \frac{\partial}{\partial z_i^m}$ and $D_{n,m}^\dagger = (-1)^m \sum_{i=1}^{N} \frac{\partial}{\partial z_i^m} z_i^n$, $m, n \in \mathbb{Z}_+$, whose Lie algebras are isomorphic to the $W_{1+\infty}$ algebra [1]. The derived $W_{1+\infty}$ constraints are

$$\bar{\mathcal{W}}^s(P)Z_N(t) = 0, \quad s \geq 2,$$

with

$$\bar{\mathcal{W}}^s(P) = \frac{1}{s} : Q_s[j(P)] + \frac{d}{dj}(P) : - (-1)^s \frac{1}{s} Q_s^\dagger \frac{d}{dj}(P) :$$

$$= \sum_{n=-s+1}^{\infty} \bar{\mathcal{W}}^s_n P^{-n-s},$$

where $j(P) := \sum_{k=1}^{\infty} k t_k P^{k-1} = \partial_P U(P)$, $\frac{d}{dj}(P) := \sum_{k=0}^{\infty} P^{-k-1} \frac{\partial}{\partial P}$, the function $Q_s[f] := (\frac{\partial}{\partial P} + f(P))^s \cdot 1$, $Q_s^\dagger[f] := (-\frac{\partial}{\partial P} + f(P))^s \cdot 1$, the normal ordering $:\ :$ means we put the differential operator $j(P)$ before $\frac{d}{dj}(P)$ and the subscript “−” is the projection to the negative powers of $P$.

When $s = 2$, (33) reduces to

$$\bar{\mathcal{W}}^2(P) = (\frac{d}{dj}(P))^2 + (j \frac{d}{dj}(P)) - \sum_{n=-1}^{\infty} \bar{\mathcal{W}}^2_n P^{-n-2},$$

where the operators $\bar{\mathcal{W}}^2_n$ are given by

$$\bar{\mathcal{W}}^2_n = \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_{k+n}} + \sum_{k=0}^{n} \frac{\partial}{\partial t_{k}} \frac{\partial}{\partial t_{n-k}}, \quad n \geq -1,$$

which satisfy [21]. Thus we have the Virasoro constraints

$$\bar{\mathcal{W}}^2_n Z_N(t) = 0.$$  

(36)

Taking $s = 3$ in (33), we obtain the constraints

$$\bar{\mathcal{W}}^3_n Z_N(t) = 0,$$

(37)

where

$$\bar{\mathcal{W}}^3_n = \sum_{k=0}^{\infty} k t_k \bar{\mathcal{W}}^2_{n+k} - (n + 2) \bar{\mathcal{W}}^2_n, \quad n \geq -2.$$  

(38)
It is difficult to write down the operators $\hat{W}_n^r$ explicitly from (33). A conjecture is that the constraints (32) with $s > 2$ are reducible to the Virasoro constraints [6].

Let us focus on the partition function (30) and insert the operators (19) under the integral as done in Ref.[15]. Then we have

$$
\int d^N z \hat{W}_m^r(\Delta(z)^2 \exp(U(t))) = 0, \quad r \in \mathbb{Z}_+, m \in \mathbb{N}.
$$

From the insertion of $\hat{W}_0^2 = \frac{1}{2} \sum_{i=1}^{N} \frac{\partial}{\partial z_i} z_i$, we obtain the identity

$$
\int d^N z \hat{W}_0^2(\Delta(z)^2 \exp(U(t))) = \int d^N z \frac{1}{2} \sum_{k=0}^{\infty} k t_k(\sum_{i=1}^{N} z_i^k) + N^2(\Delta(z)^2 \exp(U(t)))
$$

$$
= \mathcal{O}_E \int d^N z \Delta(z)^2 \exp(U(t)) = 0,
$$

where

$$
\mathcal{O}_E = \frac{1}{2} \sum_{k=0}^{\infty} k t_k \frac{\partial}{\partial t_k} + \frac{1}{2} \frac{\partial^2}{\partial t_0^2}.
$$

For the operator $\hat{W}_1^1 = \sum_{i=1}^{N} \frac{\partial^2}{\partial z_i^2} - \frac{2}{\alpha} \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{z_i - z_j}$, we have the action

$$
\hat{W}_1^1(\Delta(z)^2 \exp(U(t))) = [(2 - \frac{2}{\alpha}) \sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{(z_i - z_j)^2} + (\frac{1}{\alpha} - 1) \sum_{k=0}^{\infty} k(k-1)t_k(\sum_{i=1}^{N} z_i^{k-2})
$$

$$
+ (2 - \frac{1}{\alpha}) \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} k t_k(\sum_{i=1}^{N} z_i^l)(\sum_{j=1}^{N} z_j^{k-2-l})
$$

$$
+ \sum_{k,l=0}^{\infty} k t_k t_l(\sum_{i=1}^{N} z_i^{k+l-2})(\Delta(z)^2 \exp(U(t))).
$$

Since the action of any differential operator with respect to the variables $t$ on (30) cannot generate the term $\sum_{i=1}^{N} \sum_{j \neq i} \frac{1}{(z_i - z_j)^2}$ in (42), we insert the operator $\hat{W}_1^1$ with $\alpha = 1$ under the integral. Then we have

$$
\int d^N z \hat{W}_1^1(\Delta(z)^2 \exp(U(t)))|_{\alpha=1} = \int d^N z \left[ \sum_{k,l=0}^{\infty} k t_k t_l(\sum_{i=1}^{N} z_i^{k+l-2})
$$

$$
+ \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} k t_k(\sum_{i=1}^{N} z_i^l)(\sum_{j=1}^{N} z_j^{k-2-l})(\Delta(z)^2 \exp(U(t)))
$$

$$
= \mathcal{O}_L \int d^N z \Delta(z)^2 \exp(U(t)) = 0,
$$

8
where
\[ O_L = \sum_{k,l=0}^{\infty} klt_{k,l} \frac{\partial}{\partial t_{k+l-2}} + \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} klt_k \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_{k-l-2}}. \] (44)

Note that the operators with the same expressions as (41) and (44) have also been presented for the Gaussian hermitian model [36]. Since the constraint operators \( O_E \) and \( O_L \) are associated with the (Euler) Lassalle operators, here we call (40) and (43) the Euler and Lassalle constraints, respectively. The commutation relation between \( O_L \) and \( O_E \) is
\[ [O_L, O_E] = -O_L. \] (45)

By means of (40) and (43), for the case of operators \( W^r_m \) with \( \alpha = 1 \), we may derive the constraints from (39)
\[ W^r_m Z_N(t) = 0, \] (46)
where the constraint operators are given by
\[ W^r_m = O^r_m O^{-1}_E, \quad r \in \mathbb{Z}_+, m \in \mathbb{N}. \] (47)

By direct calculation of the commutator of (47), we obtain
\[ [W^r_{m_1}, W^r_{m_2}] = (\sum_{k=0}^{r_1} C_{r_1-1}^k m_1^k - \sum_{k=0}^{r_2} C_{r_2-1}^k m_2^k) W^{r_1+r_2-1-k}_{m_1+m_2}, \] (48)
which is isomorphic to the \( W_{1+\infty} \) algebra (20).

From (46), we have the Virasoro constraints
\[ W^2_m Z_N(t) = 0, \quad m \in \mathbb{N}, \] (49)
where the constraint operators are given by
\[ W^2_m = (\sum_{k,l=0}^{\infty} klt_{k,l} \frac{\partial}{\partial t_{k+l-2}} + \sum_{k=2}^{\infty} \sum_{l=0}^{k-2} klt_k \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_{k-l-2}})^m (\frac{1}{2} \sum_{k=0}^{\infty} klt_k \frac{\partial}{\partial t_k} + \frac{1}{2} \frac{\partial^2}{\partial t_0^2}), \] (50)
which also satisfy (21).

Here we should like to draw attention to the fact that both the Virasoro constraint operators (50) and (51) subject to the relation (21) and annihilate the partition function \( Z_N(t) \) (30). However they are completely the different operators. We have mentioned previously that the constraints (32) for the hermitian one-matrix model seem to be reducible to the Virasoro constraints. Unlike that case, we observe that the \( W_{1+\infty} \) constraints (46) are indeed reducible.
to the Euler and Lassalle constraints. An intriguing property of the constraint operators \([47]\) is that they yield the closed \(n\)-algebra

\[
[W_{m_1}^r, W_{m_2}^r, \ldots, W_{m_n}^r] = \sum_{i_1 i_2 \cdots i_n} \frac{\beta_1}{\alpha_1!} \frac{\beta_2}{\alpha_2!} \cdots \frac{\beta_{n-1}}{\alpha_{n-1}!} C^\alpha_1 C^\alpha_2 \cdots C^\alpha_{n-1} \cdot m_{i_1}^\alpha_1 m_{i_2}^\alpha_2 \cdots m_{i_n}^\alpha_{n-1} W_{m_1+m_2+\ldots+m_n}^{r_1+\ldots+r_n-(n-1)-\alpha_1-\ldots-\alpha_{n-1}},
\]

which is isomorphic to the \(W_{1+\infty}\) \(n\)-algebra \([22]\).

When particularized to the Virasoro constraint operators in \([51]\), it gives the null 3-algebra

\[
[W_{m_1}^2, W_{m_2}^2, W_{m_3}^2] = 0.
\] (52)

For the well-known Virasoro constraint operators \([35]\), it can be shown by direct calculation that they do not yield any closed \(n\)-algebra.

Let us consider the insertions of the operators \(\tilde{W}_m^r\) \([24]\) under the integral \([30]\). For this case, we have

\[
\int d^N z \ W_m^r (\Delta(z)^2 \exp(U(t))) = 0, \quad r \in \mathbb{Z}_+, m \in \mathbb{N}.
\] (53)

Although the integral will vanish under the insertions of \(\tilde{W}_m^r\), unfortunately, we can not derive the corresponding \(W_{1+\infty}\) constraints from \([53]\).

Making the changes of variables \(z_i \rightarrow \sqrt{z_i}\) in the Euler operator \(O_E\), Lassalle operator \(O_L^B\) and potential \(V^B\), then we have \(O_E \rightarrow 2O_E, \ O_L^B \rightarrow \tilde{O}_L^B = 4 \sum \frac{\partial}{\partial z_i} + (2 + 2\beta) \sum \frac{\partial}{\partial z_i} + 2\gamma \sum_{i=1}^{N} \frac{z_i}{z_i - z_j} \frac{\partial}{\partial z_j}\) and \(V^B \rightarrow \tilde{V}^B = \sum_{i=1}^{N} \frac{b(b-1)}{2z_i} + a(a-1) \sum_{i=1}^{N} \frac{z_i^2}{(z_i-z_j)^2}.

Let us introduce the operators similar to \([27]\)

\[
\tilde{W}_m^r = (O_E + N)^{-1} (\tilde{O}_L^B + \tilde{V}^B)^m
\]

\[
= \sum_{i=1}^{N} \frac{\partial}{\partial z_i} (z_i)^{-1} (4 \sum_{i=1}^{N} \frac{\partial}{\partial z_i} z_i + (2\beta - 6) \sum_{i=1}^{N} \frac{\partial}{\partial z_i} - 8 \sum_{i=1}^{N} \sum_{j \neq i} \frac{\partial}{\partial z_i} z_i - z_j)^m,
\] (54)

where we take \(b = 0\) and \(\gamma = a(1-a) = -4\). It should be noted that the operators \([54]\) also yield the \(W_{1+\infty}\) algebra \([20]\) and \(n\)-algebra \([22]\).

Inserting \(\tilde{W}_m^r\) \([54]\) under the integral \([30]\), we may derive

\[
\tilde{W}_m^r Z_N(t) = 0,
\] (55)

where the constraint operators are

\[
\tilde{W}_m^r = \tilde{O}_L^m (2O_E)^{-1}, \quad r \in \mathbb{Z}_+, m \in \mathbb{N},
\] (56)
the operator $\tilde{O}_L$ is given by

$$\tilde{O}_L = (2\beta - 2) \sum_{k=0}^{\infty} kt_k \frac{\partial}{\partial t_{k-1}} + 4 \sum_{k=1}^{\infty} \sum_{l=0}^{k-1} kt_k \frac{\partial}{\partial t_l} \frac{\partial}{\partial t_{k-l-1}} + 4 \sum_{k,l=0}^{\infty} klt_{k+l-1} \frac{\partial}{\partial t_{k+l-1}},$$  

(57)

which satisfies $\tilde{O}_L Z_N(t) = 0$.

It can be shown that the constraint operators (56) yield the same $W_{1+\infty} (n)$-algebras as the cases of (47).

4 Summary

In terms of the Euler and Lassalle operators and the potentials of the $(A_{N-1})B_N$-Calogero models, we have presented the multi-variable differential operator realizations of the $W_{1+\infty}$ algebra. These operator realizations lead to the $W_{1+\infty}$ algebra and nontrivial $n$-algebra. It should be noted that these operators are not the conserved operators for the Calogero model. Based on the Lax operator of the Calogero model, the conserved operators of this system which yield the $W_{1+\infty}$ algebra have been constructed in Ref.[35]. However this type realization does not lead to the closed $n$-algebra. Therefore the higher algebraic structures still deserve further study for the Calogero model.

We have reinvestigated the hermitian one-matrix model. From the insertions of our realizations of the $W_{1+\infty}$ algebra under the integral, we have derived the $W_{1+\infty}$ constraints, which are different from the constraints presented in Ref.[6]. The remarkable property of the derived constraint operators is that they yield not only the $W_{1+\infty}$ algebra but also the closed $W_{1+\infty}$ $n$-algebra. The higher algebraic structures should provide new insight into the matrix models.

Acknowledgment

This work is supported by the National Natural Science Foundation of China (Nos. 11875194, 11871350 and 11605096).

References

[1] A. Mironov, A. Morozov, On the origin of Virasoro constraints in matrix models: Lagrangian approach, Phys. Lett. B 252 (1990) 47.
[2] F. David, Loop equations and non-perturbative effects in two-dimensional quantum gravity, Mod. Phys. Lett. A 5 (1990) 1019.

[3] J. Ambjørn, Yu. Makeenko, Properties of loop equations for the hermitian matrix model and for two-dimensional quantum gravity, Mod. Phys. Lett. A 5 (1990) 1753.

[4] H. Itoyama, Y. Matsuo, Noncritical Virasoro algebra of the $d < 1$ matrix model and the quantized string field, Phys. Lett. B 255 (1991) 202.

[5] R. Dijkgraaf, H.L. Verlinde, E.P. Verlinde, Loop equations and Virasoro constraints in nonperturbative 2D quantum gravity, Nucl. Phys. B 348 (1991) 435.

[6] H. Itoyama, Y. Matsuo, $W_{1+\infty}$-type constraints in matrix models at finite $N$, Phys. Lett. B 262 (1991) 233.

[7] J. Ding, K. Iohara, Generalization and deformation of Drinfeld quantum affine algebras, Lett. Math. Phys. 41 (1997) 181, arXiv:q-alg/9608002.

[8] K. Miki, A $(q, \gamma)$ analog of the $W_{1+\infty}$ algebra, J. Math. Phys. 48 (2007) 123520.

[9] B. Feigin, E. Feigin, M. Jimbo, T. Miwa, E. Mukhin, Quantum continuous $gl_{\infty}$: semi-infinite construction of representations, Kyoto J. Math. 51 (2011) 337, arXiv:1002.3100.

[10] H. Awata, B. Feigin, A. Hoshino, M. Kanai, J. Shiraishi, S. Yanagida, Notes on Ding-Iohara algebra and AGT conjecture, arXiv:1106.4088.

[11] A. Mironov, A. Morozov, Y. Zenkevich, Ding-Iohara-Miki symmetry of network matrix models, Phys. Lett. B 762 (2016) 196, arXiv:1603.03547.

[12] H. Awata, H. Kanno, T. Matsumoto, A. Mironov, A. Morozov, A. Morozov, Y. Ohkubo, Y. Zenkevich, Explicit examples of DIM constraints for network matrix models, JHEP 07 (2016) 103, arXiv:1604.08366.

[13] A. Gadde, W. Yan, Reducing the 4d index to the $S^3$ partition function, JHEP 12 (2012) 003, arXiv:1104.2592.

[14] F.A. Dolan, H. Osborn, Applications of the superconformal index for protected operators and $q$-hypergeometric identities to $N = 1$ dual theories, Nucl. Phys. B 818 (2009) 137, arXiv:0801.4947.
[15] A. Nedelin, M. Zabzine, $q$-Virasoro constraints in matrix models, JHEP 03 (2017) 098, arXiv:1511.03471.

[16] J. Shiraishi, H. Kubo, H. Awata, S. Odake, A quantum deformation of the Virasoro algebra and the Macdonald symmetric functions, Lett. Math. Phys. 38 (1996) 33, arXiv:q-alg/9507034.

[17] J. Bagger, N. Lambert, Modeling multiple M2’s, Phys. Rev. D 75 (2007) 045020, arXiv:hep-th/0611108.

J. Bagger, N. Lambert, Gauge symmetry and supersymmetry of multiple M2-branes, Phys. Rev. D 77 (2008) 065008, arXiv:0711.0955.

J. Bagger, N. Lambert, Comments on multiple M2-branes, JHEP 02 (2008) 105, arXiv:0712.3738.

[18] A. Gustavsson, Algebraic structures on parallel M2-branes, Nucl. Phys. B 811 (2009) 66, arXiv:0709.1260.

[19] J.A. de Azcárraga, J.M. Izquierdo, $n$-ary algebras: a review with applications, J. Phys. A: Math. Theor. 43 (2010) 293001, arXiv:1005.1028.

[20] P. Richmond, Higher derivative BLG: Lagrangian and supersymmetry transformations, JHEP 09 (2012) 090, arXiv:1207.1208.

[21] B. Estienne, N. Regnault, B.A. Bernevig, $D$-algebra structure of topological insulators, Phys. Rev. B 86 (2012) 241104(R), arXiv:1202.5543.

[22] T. Neupert, L. Santos, S. Ryu, C. Chamon, C. Mudry, Noncommutative geometry for three-dimensional topological insulators, Phys. Rev. B 86 (2012) 035125, arXiv:1202.5188.

[23] M.R. Chen, S.K. Wang, K. Wu, W.Z. Zhao, Infinite-dimensional 3-algebra and integrable system, JHEP 12 (2012) 030, arXiv:1201.0417.

[24] K. Hasebe, Higher dimensional quantum Hall effect as $A$-class topological insulator, Nucl. Phys. B 886 (2014) 952, arXiv:1403.5066.

[25] A.S. Arvanitakis, Higher spins from Nambu-Chern-Simons theory, Commun. Math. Phys. 348 (2016) 1017, arXiv:1511.01482.
[26] R. Wang, K. Wu, J. Yang, C.H. Zhang, W.Z. Zhao, Generalized $q$-$W_\infty$ constraints for the elliptic hermitian matrix model, Phys. Lett. B 783 (2018) 241.

[27] A. Cappelli, C.A. Trugenberger, G.R. Zemba, Infinite symmetry in the quantum Hall effect, Nucl. Phys. B 396 (1993) 465, [arXiv:hep-th/9206027].

[28] C.H. Zhang, L. Ding, Z.W. Yan, K. Wu, W.Z. Zhao, On $W_{1+\infty}$ $n$-algebra, [arXiv:1606.07570v3].

[29] T. Curtright, X. Jin, L. Mezincescu, Multi-operator brackets acting thrice, J. Phys. A: Math. Theor. 42 (2009) 462001, [arXiv:0905.2759].

[30] C. Devchand, D. Fairlie, J. Nuyts, G. Weingart, Ternutator identities, J. Phys. A: Math. Theor. 42 (2009) 475209, [arXiv:0908.1738].

[31] M. Lassalle, Generalized Hermite polynomials, C. R. Acad. Sci. Paris série I 313 (1991) 579;
M. Lassalle, Generalized Laguerre polynomials, C. R. Acad. Sci. Paris série I 312 (1991) 725.

[32] B. Sutherland, Quantum many-body problem in one dimension: ground state, J. Math. Phys. 12 (1971) 246.

[33] T. Yamamoto, Multicomponent Calogero model of $B_N$-type confined in harmonic potential, Phy. Lett. A. 208 (1995) 293, [arXiv:cond-mat/9508012].

[34] A. Nishino, H. Ujino, M. Wadati, Symmetric Fock space and orthogonal symmetric polynomials associated with the Calogero model, Chaos, Solitons & Fractals 11 (2000) 657, [arXiv:cond-mat/9803284].

[35] K. Hikami, M. Wadati, Integrable systems with long-range interactions, $W_\infty$ algebra, and energy spectrum, Phys. Rev. Lett. 73 (1994) 1191;
K. Hikami, M. Wadati, Infinite symmetry of the spin systems with inverse square interactions, J. Phys. Soc. Jpn. 62 (1993) 4203.

[36] A. Morozov, Sh. Shakirov, Generation of matrix models by $\hat{W}$-operators, JHEP 04 (2009) 064, [arXiv:0902.2627].