ON PROXIMAL FINENESS OF TOPOLOGICAL GROUPS IN THEIR RIGHT UNIFORMITY

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Abstract. A uniform space $X$ is said to be proximally fine if every proximally continuous map on $X$ into a uniform is uniformly continuous. We supply a proof that every topological group which is functionally generated by its precompact subsets is proximally fine with respect to its right uniformity. On the other hand, we show that there are various permutation groups $G$ on the integers $\mathbb{N}$ that are not proximally fine with respect to the topology generated by the sets $\{g \in G : g(A) \subset B\}$, $A, B \subset \mathbb{N}$.

1. Introduction

A function $f : X \to Y$ between two uniform spaces is said to be proximally continuous if for every bounded uniformly continuous function $g : Y \to \mathbb{R}$, the composition function $g \circ f : X \to \mathbb{R}$ is uniformly continuous; the reals $\mathbb{R}$ being equipped with the usual metric. A uniform space $X$ is said to be proximally fine if every proximally continuous function defined on $X$ is uniformly continuous. It is well-known that metric spaces and all products of metric uniform spaces are proximally fine; however, there are many uniform spaces that are not proximally fine although they are topologically well-behaved (some may be locally compact and even discrete). We refer the reader to Hušek’s recent paper [12] for more information about proximally fineness of general uniform spaces. We are interested here in the fine proximal condition of the topological groups when these spaces are endowed with their right uniformity. This subject seems to have no specific literature, although there are some questions which could have been naturally addressed in this sitting, such as the Itzkowitz problem for metric and/or locally compact groups [13] (the link between Itzkowitz’s problem and proximity theory was made later in [6]).

In view of the close relationship between the right uniformity of topological groups and the system of neighborhoods of their identity, it is reasonable to expect that the proximal fineness of a given topological group $G$ is satisfied provided that

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a not too much restrictive condition is imposed on the topology of $G$. This feeling is heightened by Corollary 2.6 in this note asserting that it is sufficient to assume that $G$ is functionally generated (in Arkhangel’skii’s sense) by its precompact subsets. In fact, this is still true under much less restrictive conditions (Theorem 2.4 below). Let us mention that one part of Theorem 2.4 was asserted without proof in [5]. The main subject of this note can be examined in the context of $G$-sets (or $G$-spaces) without significantly altering its essence, so the main result is stated and established in a somewhat more general form (Theorem 2.5).

Some examples of non-proximally fine groups are given in Section 3. To do that, we consider the topology $\tau$ on $\mathbb{N}^\mathbb{N}$ of uniform convergence when (the target set) $\mathbb{N}$ is endowed with the Samuel uniformity. We show in Corollary 3.4 that for any permutation group $H$ on $\mathbb{N}$ for which all but finitely many orbits are finite and uniformly bounded, the only group topology on $H$ which finer than $\tau$ and proximally fine is the discrete topology. The question of whether there is an abelian (or at least SIN) group which is non-proximally fine group is left open.

2. Main result

For undefined terms we refer to the books [8] and [15]. One of the tools used here is Katetov’s extension theorem of uniformly continuous bounded real-valued functions [14]. The following statement is also well-known (see[8]); its proof is outlined here for the sake of completeness and because it can be adapted to show a result in the same spirit that will be used in the proof Theorem 2.5. As usual, if $(X,\mathcal{U})$ is a uniform space, where $\mathcal{U}$ is the uniform neighborhoods of the diagonals of $X$, then for $A \subseteq X$ and $U \in \mathcal{U}$, $U[A]$ stands for the set of $y \in X$ such that $(x, y) \in U$ for some $x \in A$.

**Proposition 2.1.** Let $f : X \to Y$, where $(X,\mathcal{U})$ and $(Y,\mathcal{V})$ are two uniform spaces. Then the following are equivalent:

1. $f : X \to Y$ is proximally continuous,
2. for every $A \subseteq X$ and $V \in \mathcal{V}$, there exists $U \in \mathcal{U}$ such that $f(U[A]) \subseteq V[f(A)]$.

**Proof.** To show that 1) implies 2), let $d$ be a bounded uniformly continuous pseudometric on $Y$ such that $(x, y) \in V$ whenever $d(x, y) < 1$ ([8]) and consider

\[1\]The author is indebted to Professor Michael D. Rice for his interest in this result from [5] and its proof, which motivated the present work.
the uniformly continuous function $\phi$ on $Y$ defined by $\phi(y) = d(y, f(A))$. Since $\phi \circ f$ is uniformly continuous, there is $U \in \mathcal{U}$ such that $(x, y) \in U$ implies $|\phi(f(x)), \phi(f(y))| < 1$. Then $f(U[A]) \subset V[f(A)]$.

For the converse, let $\phi : Y \to \mathbb{R}$ be a bounded uniformly continuous function and let $\varepsilon > 0$. Define $V = \{(x, y) \in Y \times Y : |\phi(x) - \phi(y)| < \varepsilon/2\}$. Then $V \in \mathcal{V}$ and since $\phi$ is bounded, there is a finite set $F \subset Y$ so that $Y = V[F]$. Let $A_z = f^{-1}(W[z])$, $z \in F$, and choose $U \in \mathcal{U}$ such that $f(U[A_z]) \subset W[f(A_z)]$ for each $z \in F$. Then $|\phi \circ f(x) - \phi \circ f(y)| \leq \varepsilon$ for each $(x, y) \in U$. □

Let us say that a topological group $G$ is proximally fine if the uniform space $(G, \mathcal{U}_r)$ is proximally fine, where $\mathcal{U}_r$ is the right uniformity of $G$. It is equivalent to say that $(G, \mathcal{U}_l)$ is proximally fine, where $\mathcal{U}_l$ is the left uniformity of $G$. Recall that a basis of $\mathcal{U}_r$ (respectively, $\mathcal{U}_l$) is given by the sets of the form $\{(g, h) \in G \times G : gh^{-1} \in V\}$ (respectively, $\{(g, h) \in G \times G : g^{-1}h \in V\}$) as $V$ runs over the set $\mathcal{V}(e)$ of neighborhoods of the unit $e$ of $G$.

**Proposition 2.2.** Let $G$ be a topological group, $(Y, \mathcal{U})$ a uniform space and let $f : G \to Y$ be a function. For $g \in G$, let $\psi_g : G \to Y$ be the function defined by $\psi_g(h) = f(gh)$. Then, the following are equivalent:

1. $f$ is uniformly continuous, $G$ being equipped with the right uniformity,
2. the function $\psi : g \in G \to \psi_g \in Y^G$ is continuous when $Y^G$ is endowed with the uniformity of uniform convergence on $G$.

**Proof.** To show that (1) implies (2), suppose that $f$ is right uniformly continuous and let $U \in \mathcal{U}$. There is a neighborhood $V$ of $e$ such that $xy^{-1} \in V$ implies $(f(x), f(y)) \in U$. Let $g \in G$. Then $Vg$ is a neighborhood of $g$ in $G$ and for each $h \in Vg$ and $x \in G$ we have $hx(gx)^{-1} \in V$ hence $(f(hx), f(gx)) \in U$, that is, $(\psi_h(x), \psi_g(x)) \in U$.

To show that (2) implies (1), let $U \in \mathcal{U}$ and choose $V \in \mathcal{V}(e)$ such that $(\psi_g(x), \psi_e(x)) \in U$ for every $g \in V$ and $x \in G$. Then, for every $g, h \in G$ such that $h \in Vg$ we have $(\psi_{hg^{-1}}(x), \psi_e(x)) \in U$ for each $x \in G$, equivalently, $(f(hx), f(gx)) \in U$, for every $x \in G$. □

It is possible to expand substantially the framework of the starting topic of this note without major changes as follows: Let $G$ be a topological group and let $X$ be a $G$-set, that is, $X$ is a nonempty set for which there is a map $*: G \times X \to X$ satisfying $(gh) * x = g * (h * x)$ for every $g, h \in G$ and $x \in X$. The function $*$ is
called a left action of $G$ on $X$. To simplify, write $gx$ in place of $g \ast x$ and $UA$ in place of $U \ast A$ if $U \subset G$ and $A \subset X$. No topology will be required on $X$. Let $(Y, \mathcal{V})$ be a uniform space and $f : X \to Y$ a function. It is consistent with the definitions given above to say that $f$ is right uniformly continuous if for each $V \in \mathcal{V}$, there is $U \in \mathcal{V}(e)$ such that $(f(gx), f(hx)) \in V$ for each $x \in X$, whenever $gh^{-1} \in U$. Similarly, the function $f$ is said to be right proximally continuous if for each bounded uniformly continuous function $\phi : Y \to \mathbb{R}$, the function $\phi \circ f$ is right uniformly continuous. It is easy to check (see the proof of Proposition 2.2) that $f$ is right uniformly continuous iff the function $\psi : g \in G \to f(gx) \in Y$ is continuous when $Y^X$ is endowed with the uniform convergence. Similarly, a simple adaptation of the proof of Proposition 2.1 shows that $f$ is right proximally continuous iff for each $A \subset GX$ and $V \in \mathcal{V}$, there is $U \in \mathcal{V}(e)$ such that $f(UA) \subset V[f(A)]$.

The following properties are required for Theorem 2.5; we are formulating them separately to reduce the proof to its essential components. Let $X$ be a $G$-set and $f : X \to Y$ a right proximally continuous, as defined above. Then, for each $A \subset G$:

(c1) for every $x \in X$, the function $g \in G \to f(gx) \in Y$ is continuous,

(c2) if $\psi|_A$ is right uniformly continuous, then $\psi|_{\overline{A}}$ is right uniformly continuous,

(c3) if $a \in A$ is a point of continuity of $\psi|_A$, then $a$ is a point of continuity of $\psi|_{\overline{A}}$.

We check the validity of these properties for the benefit of the reader. Let $V \in \mathcal{V}$. For every $x \in X$ and $g \in G$, there is $U \in \mathcal{V}(e)$ such that $f(Vgx) \subset V[f(gx)]$. Since $Vg$ is a neighborhood of $g$ in $G$, (c1) holds. Property (c2) and (c3) follows from (c1). For, if $x \in X$, $U \in \mathcal{V}(e)$ are such that $(f(ax), f(bx)) \in V$ for every $a, b \in A$ with $a \in UUb$, then (c1) implies that $f(gx), f(gx)) \in V^2$ for each $g, h \in \overline{A}$ such that $g \in Uh$. Taking $x$ arbitrary in $X$ gives (c2). For (c3), let $U$ be an open neighborhood of the unit in $G$ such that $(f(gx), f(ax)) \in V$ for ever $g \in Ua \cap A$ and $x \in X$. Since $Ua \cap \overline{A} \subset \overline{Ua \cap A}$, it follows from (c1) that for every $g \in Ua \cap \overline{A}$ and $x \in X$, $(f(gx), f(ax)) \in V^2$.

To establish Theorem 2.5 we also need the next key lemma; this is a well-known tool in the theory of proximity spaces (most often with $W^4$ instead of $W^3$).

**Lemma 2.3.** Let $X$ be a set and $W$ be a symmetric binary relation on $X$. Then, for every infinite cardinal $\eta$ and for every sequence $(x_n, y_n)_{n<\eta} \subset X \times X$ such that
$(x_n, y_n) \notin W^3$ for each $n < \eta$, there is a cofinal set $A \subset \eta$ such that $(x_n, y_m) \notin W$ for every $n, m \in A$.

Proof. Replacing $\eta$ by its cofinality $\text{cf}(\eta)$, we may suppose that $\eta$ is regular. Let $M \subset \eta$ be a maximal set satisfying $(x_n, y_m) \notin W$ for every $n, m \in M$. If $M$ is cofinal in $\eta$, the proof is finished, so suppose that $M$ is not cofinal in $\eta$. For each $j \in M$, let $A_j = \{ n < \eta : (x_n, y_j) \in W \}$, $B_j = \{ n < \eta : (x_j, y_n) \in W \}$, $C_j = A_j \cup B_j$ and $C = \cup_{j \in M} C_j$. The maximality of $M$ implies that $\eta \subset M \cup C$. Since $\eta$ is regular, there is $j \in M$ such that $C_j$ is cofinal in $\eta$, therefore $A_j$ or $B_j$ is cofinal in $\eta$. We suppose that it is $A_j$, the other case is similar. Let $n, m \in A_j$. Then $(x_n, y_j) \in W$ and $(x_m, y_j) \in W$, hence $(x_n, y_m) \notin W$ since $(x_m, y_m) \notin W^3$ (recall that $W$ is symmetric). Similarly, $(x_m, y_n) \notin W$. □

We will now specify a few topological concepts that will be used in what follows. The first is a variant of Herrlich’s notion of radial spaces [11]. Radial spaces were characterized by A.V. Arhangel’skii [3] as follows: A space $X$ is radial if and only if for each $x \in X$ and $A \subset X$ such that $x \in \overline{A}$, there is $B \subset A$ of regular cardinality $|B|$ such that $x \in \overline{C}$ for every $C \subset B$ having the same cardinality as $B$. Let us say that a subset $A$ of $X$ is relatively o-radial in $X$ if for every collection $(O_i)_{i \in I}$ of open sets in $X$ and $x \in A$ such that

$$x \in \bigcup_{i \in I} O_i \cap \overline{A} \setminus \bigcup_{i \in I} \overline{O_j},$$

there is a set $J \subset I$ of regular cardinality such that $x \in \bigcup_{j \in J} \overline{O_j}$ whenever $L \subset J$ and $|L| = |J|$. If the set $J$ can always be chosen countable, then $A$ is said to be relatively o-Malykhin in $X$. All closures are taken in $X$.

Every almost metrizable (in particular, Čech-Complete) group is o-Malykhin (in itself). More generally, every inframetrizable group [15] is o-Malykhin, see [6].

Following Arhangel’skii [2], the space $X$ is said to be strongly functionally generated (respectively, functionally generated) by a collection $\mathcal{M}$ of subsets of $X$ if for every discontinuous function $f : X \to \mathbb{R}$, there exists $A \in \mathcal{M}$ such that the restriction $f|_A : A \to \mathbb{R}$ of $f$ to the subspace $A$ of $X$ is discontinuous (respectively, has no continuous extension to $X$).

The following is the main result of this note. The statement corresponding to the case (2) was asserted (without proof) in [5].

**Theorem 2.4.** Let $G$ be a topological group satisfying at least one of the following:
(1) $G$ is functionally generated by the sets $\overline{A} \subset G$ such that $AA^{-1}$ is relatively $o$-radial in $G$,
(2) $G$ is strongly functionally generated by the sets $A \subset G$ such that $A$ is relatively $o$-radial in $G$.

Then $G$ is proximally fine.

According to Proposition 2.2 and keeping the above notations, Theorem 2.4 is obtained from the following general result by considering the left action of $G$ on itself.

**Theorem 2.5.** Let $G$ be a topological group and suppose that for each discontinuous bounded function $\alpha : G \to \mathbb{R}$, there is a set $A \subset G$ having at least one of the following conditions:

(1) $\alpha|_A$ has no continuous extension to $G$ and $AA^{-1}$ is relatively $o$-radial in $G$,
(2) $\alpha|_A$ is discontinuous at some point of $A$ and $A$ is relatively $o$-radial in $G$.

Let $X$ be a $G$-set, $(Y, V)$ a uniform space and let $f : X \to Y$ be a right proximally continuous. Then $f : G \to X$ is right uniformly continuous.

**Proof.** We have to show that the function $\psi : g \in G \to \psi(g) \in Y^X$ (where $\psi(g)(x) = f(gx)$) is continuous. We proceed by contradiction by supposing that $\psi$ is not continuous. Then, there is a bounded uniformly continuous function $\theta : Y^X \to \mathbb{R}$ such that $\theta \circ \psi$ is not continuous (see [8]). Let $A \subset G$ satisfying at least one of the conditions (1) and (2) with respect to the function $\theta \circ \psi$. In case (1), there is no compatible uniformity on $G$ making uniformly continuous the function $\theta \circ \psi|_A$; for, otherwise, Katetov’s theorem would give us a continuous extension of $\theta \circ \psi|_A$. In particular, $\psi|_A$ is not right uniformly continuous. As remarked above, it follows from (c2) that $\psi|_A$ is not right uniformly continuous. There is then an open and symmetric $W \in \mathcal{U}$ such that for every $V \in \mathcal{V}(e)$, there exist $a_V \in V$, $g_V \in A$ and $x_V \in X$ satisfying $a_V g_V \in A$ and

\[
(2.1) \quad (f(a_V g_V x_V), f(g_V x_V)) \notin W.
\]

For each $V \in \mathcal{V}(e)$, let $h_V = g_V x_V$ and define

$$O_V = \{g \in G : (f(g h_V), f(a_V h_V)) \in W\}.$$ 

Since the functions $g \in G \to f(g h_V) \in Y$, $V \in \mathcal{V}(e)$, are continuous (c1), each $O_V$ is open in $G$. Since $a_V \in O_V \cap Ag_V^{-1} \subset AA^{-1}$ and $a_V \in V$ for each $V \in \mathcal{V}(e)$,
it follows that
\[(2.2) \quad e \in \bigcup_{V \in \mathcal{V}(e)} O_V \cap (AA^{-1}).\]

We also have \(e \notin \overline{O}_V\), for each \(V \in \mathcal{V}(e)\). Indeed, otherwise, there exists \(g \in O_V\) such that \((f(gh_V), f(h_V)) \in W\), hence \((f(a_V h_V), f(h_V)) \in W^2\) which contradicts (2.1). Since \(AA^{-1}\) is relatively o-radial in \(G\), in view of (2.2), there is a set \(\Gamma \subset \mathcal{V}(e)\) of regular cardinal such that for each set \(I \subset \Gamma\) of the same cardinal as \(\Gamma\), we have
\[(2.3) \quad e \in \bigcup_{V \in I} \{g \in G : (f(gh_V), f(a_V h_V)) \in W\} \cap (AA^{-1}).\]

By Lemma 2.3 and (2.1), there is \(I \subset \Gamma\) such that \(|I| = |\Gamma|\) (since \(|\Gamma|\) is regular) and \((f(a_V h_V), f(h_V)) \notin W^2\) for every \(U, V \in I\). Since \(f\) is right proximally continuous, there exists \(V \in \mathcal{V}(e)\) such that
\[(2.4) \quad f(V \{h_U : U \in I\}) \subset W[f(\{h_U : U \in I\})].\]

By (2.3) applied to \(I\), there is \(U_1 \in I\) such that \(V \cap O_{U_1} \neq \emptyset\). Let \(g \in V\) be such that \((f(gh_{U_1}), f(a_{U_1} h_{U_1})) \in W\) and by (2.4) let \(U_2 \in I\) be so that \((f(gh_{U_1}), f(h_{U_2})) \in W\). It follows that \((f(a_{U_1} h_{U_1}), f(h_{U_2})) \in W^2\), which is a contradiction. Therefore, \(\psi\) is continuous in case (1).

In case (2), \(A\) is o-radial in \(G\) and \(\theta \circ \psi_{|A}\) is discontinuous at some point \(a \in A\). Since \(\theta\) is continuous, \(\psi_{|A}\) is necessarily discontinuous at \(a\). It follows from the property (c3) that \(\psi_{|A}\) is discontinuous at \(a\). Let \(W \in \mathcal{U}\) be symmetric and open such that for every \(V \in \mathcal{V}(e)\), there exist \(a_V \in V\) and \(x_V \in G\), such that \(a_V a \in A\) and \((f(a_V x_V), f(x_V)) \notin W^6\). Taking \(h_V = a x_V\) for each \(V \in \mathcal{V}(e)\), we have
\[e \in \bigcup_{V \in \mathcal{V}(e)} \{g \in G : (f(gh_V), f(h_V)) \in W\} \cap (AA^{-1}).\]

It is easy to see that \(AA^{-1}\) is relatively o-radial in \(G\), therefore the proof can be continued and concluded in the same way as in the first case. It should be noted that Katetov’s theorem was not used in this case.

Let \(A \subset G\), where \(G\) is a topological group. It is proved in [6] that \(AA^{-1}\) is relatively o-Malykhin in \(G\), provided that \(A\) is left and right precompact. Thus Theorem 2.4 yields:

**Corollary 2.6.** Every topological group \(G\) which is functionally generated by the collection of its precompact subsets is proximally fine.
In view of the role played by locally compact groups in many areas of mathematics, it is worth mentioning the following particular case of Corollary 2.6.

**Corollary 2.7.** Every locally compact topological group is proximally fine.

Recall that the group $G$ is said to SIN (or with small invariant neighborhoods of the identity) if its left uniformity and right uniformity are equal. The group $G$ is said to be FSIN (or functionally balanced) if every bounded left uniformly continuous function $f : G \to \mathbb{R}$ is right uniformly continuous. The group $G$ is said to be *strongly* FSIN if every every real-valued uniformly continuous function on $G$ is left uniformly continuous. The question whether every FSIN group is SIN is called Itzkowits problem and is still open. We refer the reader to [5] for more information; see also [16] for a very recent contribution to this topic. The corollary of Theorem 2.4 that every FSIN group is SIN provided that it is strongly functionally generated by its relatively o-radial subsets has already been stated (implicitly and without proof) in [5]. This is supplemented by the following:

**Corollary 2.8.** Every FSIN group $G$ which is functionally generated by the sets $\mathcal{A} \subset G$ such that $\mathcal{A}A^{-1}$ is relatively o-radial in $G$ is a SIN group.

To conclude this section, we would like to take this opportunity to comment on the parenthesized question of [5, Question 6] whether every bounded topological group is FSIN. The answer is of course no, since FSIN is a hereditary property (by Katetov’s theorem) and every group is isomorphic both algebraically and topologically to a subgroup of a bounded group [10] (see also [9]).

3. **Examples**

In this section, we give some examples of non-proximally fine Hausdorff topological groups and examine their behavior towards the FSIN property. The set of positive integers is denoted by $\mathbb{N}$ and $\mathcal{U}$ is the Samuel uniformity of the uniform discrete space $\mathbb{N}$. The uniformity $\mathcal{U}$ is sometimes called the precompact reflection of the uniform discrete space $\mathbb{N}$. A basis of $\mathcal{U}$ is given by the sets $\cup_{i \leq n} A_i \times A_i$, where $A_1, \ldots, A_n$ is a partition of the integers. Let $\mathbb{N}^\mathbb{N}$ be endowed with the uniformity $\mathcal{V}$ of uniform convergence when $\mathbb{N}$ (the target space) is equipped with the uniformity $\mathcal{U}$. Let $G$ denote the permutation group of the set $\mathbb{N}$ of positive integers and let $S$ be the normal subgroup of $G$ given by finitary permutations $g \in G$; that is, $g \in S$ iff the set $\text{supp}(g) = \{ x \in \mathbb{N} : g(x) \neq x \}$ is finite. It is easy
to see that \( G \) (hence \( S \)) is a topological group when equipped with the topology \( \tau \) induced by the uniformity \( \mathcal{V} \). More precisely, \( G \) is a non-Archimedean group, since a basis of neighborhoods of its unit is given by the subgroups of \( G \) of the form

\[ H_\pi = \{ g \in G : g(A_i) = A_i, i = 1, \ldots, n \}, \]

where \( \pi = \{ A_1, \ldots, A_n \} \) is a finite partition of \( \mathbb{N} \).

The so-called natural Polish topology \( \tau_0 \) on \( G \), given by pointwise convergence, is coarser than \( \tau \) and for every partition \( \pi \) of \( \mathbb{N} \) the set \( H_\pi \) is \( \tau_0 \)-closed. This is to say that \( \tau_0 \) is a cotopology for \( \tau \) in the sense of [1]; in particular, \( (G, \tau) \) is submetrizable and Baire. In what follows, unless otherwise stated, the groups \( G \) and \( S \) will be systematically considered under the topology \( \tau \).

For later use, we check that the quotient group \( G/S \) (with the quotient topology) is Hausdorff, that is, \( S \) is closed in \( G \). Let \( g \in G \setminus S \). A simple induction allows to construct an infinite set \( A \subset \mathbb{N} \) such that \( g(A) \subset \mathbb{N} \setminus A \) (alternatively, \( A \) is obtained from Lemma 2.3 applied to the set \( \{ (x, g(x)) : x \in \text{supp}(g) \} \)).

Let \( \pi = \{ A, \mathbb{N} \setminus A \} \). Then \( gH_\pi \) is a \( \tau \)-neighborhood of \( g \) and for each \( h \in H_\pi \), \( gh(A) = g(A) \subset \mathbb{N} \setminus A \). Hence \( gH_\pi \cap S = \emptyset \).

Recall that for a topological group \( H \), the lower uniformity \( \mathcal{U}_l \land \mathcal{U}_r \) on \( H \) is called the Roelcke uniformity and has base consisting of the sets \( \{ (x, y) \in H \times H : x \in V y V \}, V \in \mathcal{V}(e) \) (see [15]). Every (right) proximally fine group is proximally fine with respect to the Roelcke uniformity; therefore, the following shows in a strong way that the groups \( G \) and \( S \) are not proximally fine:

**Proposition 3.1.** Let \( k \in \mathbb{N} \) and \( \phi : G \to \mathbb{N} \) be the evaluation function \( \phi(g) = g(k) \), \( \mathbb{N} \) being equipped with the discrete uniformity.

1. The function \( \phi : G \to \mathbb{N} \) is left uniformly continuous and right proximally continuous. In particular, \( \phi \) is Roelcke-proximally continuous.
2. If \( \mathbb{N} \) is endowed with the uniformity \( \mathcal{U} \), then \( \phi \) is right uniformly continuous. Conversely, if \( \mathcal{V} \) is uniformity on \( \mathbb{N} \) such that the restriction of \( \phi|_S : S \to (\mathbb{N}, \mathcal{V}) \) is right uniformly continuous, then \( \mathcal{V} \subset \mathcal{U} \).

In particular, \( G \) and \( S \) are not Roelcke proximally fine.

**Proof.** 1) Clearly, \( \phi \) is left uniformly continuous with respect to the natural topology \( \tau_0 \); since \( \tau_0 \subset \tau \), \( \phi \) is left uniformly continuous. To show that \( \phi \) is right proximally continuous, let \( L \subset G \) and put \( \tau_0 = \{ A, \mathbb{N} \setminus A \}, \) where \( A = \{ g(k) : g \in L \} \), and let us verify that \( \phi(H_\pi L) \subset \phi(L) \). Proposition 2.1 will then conclude the
proof. Let \( h_0 \in H_{\pi_0} \) and \( g_0 \in L \). Then \( h_0(g_0(k)) \in \{g(k) : g \in L\} \), hence we can write \( h_0(g_0(k)) = g(k) \) for some \( g \in L \), thus \( \phi(h_0g_0) \in \phi(L) \).

2) \( \phi : G \to (\mathbb{N}, U) \) is right uniformly continuous, since for every partition \( \pi = \{A_1, \ldots, A_n\} \) of \( \mathbb{N} \), we have \( (g(k), h(k)) \in \bigcup_{i \leq n} A_i \times A_i \) provided that \( hg^{-1} \in H_\pi \).

For the converse, suppose that \( V \not\subset U \) and let \( V \in V \setminus U \). We will check that for any partition \( \pi = \{A_1, \ldots, A_n\} \) of \( \mathbb{N} \), there are \( g, h \in S \) having \( g \in H_\pi h \) and \( (\phi(g), \phi(h)) \not\in V \). We may suppose that \( A_1 = \{k\} \) and that \( A_2 \) contains two elements \( a \) and \( b \) such that \( (a, b) \not\in V \). Define \( g \in S \) by \( g(k) = a \), \( g(a) = k \) and \( g(x) = x \) for \( x \not\in \{k, a\} \). Define also \( h \in S \) by \( h(k) = b \), \( h(a) = k \), \( h(b) = a \) and \( h(x) = x \) otherwise. Then \( gh^{-1} \in H_\pi \), but \( (\phi(g), \phi(h)) \not\in V \). \( \square \)

For a subgroup \( H \) of \( G \) and \( L \subset \mathbb{N} \), let \( H(L) \) stand for the pointwise stabilizers of \( L \) in \( H \) (that is, the set of \( h \in H \) such that \( h(x) = x \) for all \( x \in L \)). The following extremal property of \( \tau \) shows that the above examples are somehow optimal.

**Proposition 3.2.** Let \( H \) be subgroup of \( G \) and \( F \subset \mathbb{N} \) a finite set. Let \( \tau_1 \) be a group topology on \( H \) and for each \( k \in F \), let \( \phi_k : g \in H \to g(k) \in \mathbb{N} \), where \( \mathbb{N} \) is endowed with the discrete uniformity.

1. If for each \( k \in F \), \( \phi_k \) is right proximally continuous, then \( \tau_1 \) is coarser than \( \tau_1 \) on \( H_{(\mathbb{N}\setminus HF)} \).
2. If for each \( k \in F \), \( \phi_k \) is right uniformly continuous, then \( \tau_1 \) is discrete on \( H_{(\mathbb{N}\setminus HF)} \).
3. If \( \tau_0|_H \subset \tau_1 \) and \((H, \tau_1)\) is strongly FSIN, then \( \tau_1 \) is discrete on \( H_{(\mathbb{N}\setminus HF)} \).

**Proof.** 1) We first show that for a given \( A \subset \mathbb{N} \), there is a \( \tau_1 \)-neighborhood \( V_A \) of the unit (in \( H \)) such that \( f(A \cap HF) \subset A \) for every \( f \in V_A \). For each \( k \in F \), define \( L_k = \{g \in H : g(k) \in A\} \). According to Proposition 2.1, there is a \( \tau_1 \)-neighborhood \( V_A \) of the unit such that for every \( k \in F \), \( \phi_k(V_AL_k) \subset \phi_k(L_k) \). Let \( n \in A \cap HF \) and \( f \in V_A \). Choose \( g \in H \) and \( k \in F \) such that \( g(k) = n \). Then \( g \in L_k \), hence \( \phi_k(fg) \in \phi_k(L_k) \) and thus \( f(n) \in A \). This show that \( f(A \cap HF) \subset A \) for every \( f \in V_A \). It follows that for any partition \( \pi = \{A_1, \ldots, A_1\} \) of \( \mathbb{N} \), there is a \( \tau_1 \)-neighborhood of the unit in \( H_{(\mathbb{N}\setminus HF)} \), namely \( V = H_{(\mathbb{N}\setminus HF)} \cap V_{A_1} \cap \ldots \cap V_{A_n} \), such that \( V \subset H_\pi \). Since \( \tau_1 \) is a group topology, it follows that \( \tau \) is coarser than \( \tau_1 \) on \( H_{(\mathbb{N}\setminus HF)} \).

2) Suppose that \( \tau_1 \) is not discrete on \( H_{(\mathbb{N}\setminus HF)} \) and let \( V \) be \( \tau_1 \)-neighborhood of the unit in \( H \). We will show that there are \( g, h \in H \) and \( k \in F \) such that
Let \( H \) be a subgroup of \( G \), \( m \in \mathbb{N} \) and \( L \subset \mathbb{N} \) such that \(|L \cap K| \leq m\) for each orbit \( K \) of the action of \( H \) on \( \mathbb{N} \). Then \( H_{(L)} \in \tau \).

Proof. There is a finite partition \( A_0, \ldots, A_m \) of \( \mathbb{N} \) such that \( A_0 = \mathbb{N} \setminus L \) and \(|A_i \cap K| \leq 1\) for each \( 1 \leq i \leq m \) and every orbit \( K \). Then \( H_L \subset H_{(L)} \). Indeed, if \( f \in H_L \) and \( x \in K \cap A_i \) with \( i \geq 1 \), then \( f(x) \in K \) and \( f(x) \in A_i \), thus \( f(x) = x \) since \(|A_i \cap K| \leq 1\).

Corollary 3.4. Let \( H \) be a subgroup of \( G \) for which all but finitely many orbits are finite and uniformly bounded. Then the discrete topology is the only group topology on \( H \) that is both proximally fine and finer than \( \tau_H \).

Proof. If \( \tau_1 \) is a proximally fine group topology on \( H \) finer than \( \tau \), then every function \( \phi_k \) is right uniformly continuous (with respect to \( \tau_1 \)). It follows from Proposition 3.2(2) that \( H_{(\mathbb{N} \setminus HF)} \) is \( \tau_1 \)-discrete for some finite set \( F \subset \mathbb{N} \) such that the cardinals of all orbits \( H_n \), \( n \notin F \), are finite and uniformly bounded. By Lemma 3.3, \( H_{(\mathbb{N} \setminus HF)} \) is \( \tau \)-open hence \( \tau_1 \)-open, consequently, \( \tau_1 \) is discrete.

Similarly, the next result follows from Proposition 3.2(3) and Lemma 3.3.

Corollary 3.5. Let \( H \) be subgroup of \( G \) for which all but finitely many orbits are finite and uniformly bounded. If \( (H, \tau) \) strongly FSIN, then \( (H, \tau) \) is discrete. Moreover, if \( H \) has finitely many orbits and \( (H, \tau_0) \) is strongly FSIN, then \( H \) is a (closed) discrete subgroup of \( (G, \tau_0) \).

It follows Proposition 3.1 that \( G \) and \( S \) are not strongly FSIN, hence not SIN, but this does not allows us to conclude that \( G \) and \( S \) are not FSIN, because none of the functions \( \phi_k \), \( k \in \mathbb{N} \), is bounded. For a topological group \( H \), let \( R(H) \), respectively \( U(H) \), stand for the real Banach spaces of bounded right uniformly continuous and of bounded right and left uniformly continuous functions on \( H \).
Proposition 3.6. The groups G, S and G/S are not FSIN. Moreover, the density character of the quotient Banach space $R(G/S)/U(G/S)$ is at least $2^\omega$.

Proof. We shall exhibit, in (1) below, a real-valued bounded function which is left uniformly continuous on G but not right uniformly continuous when restricted to S. It will follows that G and S are not FSIN. As for G/S, our strategy is as follows: For each nonprincipal ultrafilter $p$ on $\mathbb{N}$, we shall give in (2) a bounded right uniformly continuous $\phi_p$ defined on G which is not left uniformly continuous. This function is in addition constant on every coset of $G/S$. Then, we show that for each bounded left uniformly continuous function $\psi$ on G, we have $||\psi_p + \psi_q + \psi|| \geq 1$ for any distinct nonprincipal ultrafilters $p$, $q$. This will imply that the Banach space $R(G/S)/U(G/S)$ contains a uniformly discrete set of the same cardinality as $\beta\mathbb{N}\setminus\mathbb{N}$, since the quotient map $G \to G/S$ is both left and right uniformly continuous.

(1) Let $\chi : G \to \{0, 1\}$ be the function defined by $\chi(f) = 1$ if $f(1) \leq f(2)$ and $\chi(f) = 0$ otherwise. Then, clearly, $\chi$ is left uniformly continuous. Let us show that it is not right uniformly continuous on $S$. Let $A_1, \ldots, A_n$ be a partition of $\mathbb{N}$. We may suppose that $A_1 = \{a, b\}$ with $a \neq b$. Define $f$ and $g$ on $S$ by $f(1) = g(2) = a$, $f(2) = g(1) = b$, and $f(x) = g(x)$ for $x \notin \{1, 2\}$. Then $f^{-1}(A_i) = g^{-1}(A_i)$ for each $i = 1, \ldots, n$, but $\chi(f) \neq \chi(g)$.

(2) Let $p$ be a nontrivial ultrafilter on $\mathbb{N}$ and fix an infinite $A \subset \mathbb{N}$ such that $\mathbb{N} \setminus A$ is infinite. Let $\psi_p : G \to \{0, 2\}$ be the function given by $\psi_p(f) = 2$ if $f^{-1}(A) \in p$. Clearly, the function $\psi_p$ is bounded and right uniformly continuous. To show that $\psi_p$ is constant on every coset of $S$, let $q \in G$ and $f \in S$. Then, for every $B \subset \mathbb{N}$, $g^{-1}(B) \setminus \text{supp}(f) \subset (gf)^{-1}(B)$. Thus, taking $B = A$ if $g^{-1}(A) \in p$ or $B = \mathbb{N} \setminus A$ if not, we get that $\psi_p(gf) = \psi_p(g)$.

Let $p$ and $q$ be two distinct nonprincipal ultrafilters on $\mathbb{N}$ and let us verify that $||\psi_p + \psi_q + \psi|| \geq 1$ for each bounded left uniformly continuous $\psi : G \to \mathbb{R}$. It will follows that the quotient $R(G/S)/U(G/S)$ contains norm 1 discrete copy of $\beta\mathbb{N}\setminus\mathbb{N}$. Let $\varepsilon > 0$ and $\pi = \{B_1, \ldots, B_n\}$ be a partition of $\mathbb{N}$ such that $|\psi(f) - \psi(g)| < \varepsilon$ for every $f, g \in G$ such that $g \in fH_\pi$. We may suppose that $B_1 = C \cup D$ with $C \in p$, $D \in q$ and $C \cap D = \emptyset$. Write again $D = D_1 \cup D_2$, where $D_1$ and $D_2$ are infinite, disjoint and $D_1 \in q$. Finally, let $\{E, F, K\}$ be a partition of $\mathbb{N} \setminus A$, with $E$ and $F$ infinite and $|K| = |\mathbb{N} \setminus B_1|$. There are certainly $f, g \in G$ such that $f(C) = A$, $f(D) = E \cup F$, $g(C) = F$, $g(D_1) = E$, and $g(B_2) = K$.
The fact that the Hausdorff group $G/S$ is not discrete was established and used by Banakh et al. in [4] to answer a question by Dikranjan in [7]. Knowing that the symmetric group $G$ and its finitary subgroup $S$ are highly nonabelian (their centers are trivial) and taking Corollary 3.5 into account, we are naturally led to conclude by asking the following:

**Question 3.7.** Is there a Hausdorff topological group that is abelian (or at least SIN) and non-proximally fine?

**References**

[1] J.M. Aarts, J. de Groot, R.H. McDowell, *Cotopology for metrizable spaces*, Duke Math. J. 37 (1970) 291–295.
[2] A.V. Arkhangel’ski˘ı, Topological Function spaces, Vol. 78, Kluwer Academic, Dordrecht, 1992.
[3] A.V. Arhangel’ski˘ı, *Some properties of radial spaces*, Math. Notes Russ. Acad. Sci. 27 (1980) 50–54.
[4] T. Banakh, I. Guran and I. Protasov, *Algebraically determined topologies on permutation groups*, Topology Appl. 159 (2012) 2258–2268.
[5] A. Bouziad and J.-P. Troallic, *Problems about the uniform structures of topological groups*, in Open Problems in Topology II. Ed. Elliott Pearl. Amsterdam: Elsevier, 2007. 359–366.
[6] A. Bouziad and J.-P. Troallic, *Left and right uniform structures on functionally balanced groups*, Topology Appl. 153, no 13, (2006) 2351–2361.
[7] D. Dikranjan and A. Giordano Bruno, *Arnautov’s problems on semitopological isomorphisms*, Appl. Gen. Topol. 10 (1) (2009) 85–119.
[8] R. Engelking, General Topology, Heldermann, Berlin, 1989.
[9] H. Fuhr and W. Roelcke, *Contributions to the theory of boundedness in uniform spaces and topological groups*, Note di Matematica, vol. 16, no. 2 (1996) 189–226.
[10] S. Hartman and J. Mycielski, *On the Imbedding of Topological Groups into Connected Topological Groups*, Colloq. Math. 5 (1958) 167–169.
[11] H. Herrlich, *Quotienten geordneter Räume und Folgenkonvergenz*, Fund. Math. 61 (1967) 79–81.
[12] M. Hušek, *Ordered sets as uniformities*, Topol. Algebra Appl. 6 (2018), no. 1, 67–76.
[13] G. L. Itzkowitz, *Continuous measures, Baire category, and uniform continuity in topological groups*, Pacific J. Math. 54 (1974) 115–125.
[14] M. Katetov, *On real-valued functions in topological spaces*, Fund. Math. 38 (1951) 85–91.
[15] W. Roelcke and S. Dierolof, Uniform structures in topological groups and their quotients, McGraw-Hill, New York, 1981.
[16] M. Shlissberg, *Balanced and functionally balanced P-groups*, Topol. Algebra Appl. 6, no. 1, (2018) 53–59.
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