Probability Distribution of (Schwämmle and Tsallis) Two-parameter Entropies
and the Lambert W-function

Somayeh Asgarani * and Behrouz Mirza †

Department of Physics, Isfahan University of Technology, Isfahan 84156-83111, Iran

Abstract

We investigate a two-parameter entropy introduced by Schwämmle and Tsallis and obtain its probability distribution in the canonical ensemble. The probability distribution is given in terms of the Lambert W-function which has been used in many branches of physics, especially in fractal structures. Also, extensivity of \( S_{q,q'} \) is discussed and a relationship is found to exist between the probabilities of a composite system and its subsystems so that the two-parameter entropy, \( S_{q,q'} \), is extensive.

1 Introduction

It is known that some physical systems cannot be described by the Boltzmann-Gibbs (BG) statistical mechanics. Within a long list showing power-law behaviours, we may mention diffusion [1], turbulence [2], transverse momentum distribution of hadron jets in \( e^+ e^- \) collisions [3], thermalization of heavy quarks in collisional process [4], astrophysics [5], solar neutrinos [6], and among others [7, 8, 9, 10, 11]. Such systems typically have long-range interactions, long-time memory, and multifractal or hierarchical structures. To overcome at least some of these difficulties, Tsallis proposed a generalized entropic form [12, 13, 14], namely

\[
S_q = k \frac{1 - \sum_{i=1}^{\omega} p_i^q}{q - 1},
\]

where, \( k \) is a positive constant and \( \omega \) is the total number of microscopic states. It is clear that the q-entropy \( (S_q) \) recovers the usual BG-entropy \( (S_{BG} = -k \sum_{i=1}^{\omega} p_i \ln p_i) \) in the limit \( q \to 1 \). By defining q-logarithm

\[
\ln_q x \equiv \frac{x^{1-q} - 1}{1-q} \quad (\ln_1 x = \ln x),
\]

*email: sasgarani@ph.iut.ac.ir
†email: b.mirza@cc.iut.ac.ir
the entropic form, $S_q$, can be written as

$$S_q = k \frac{1 - \sum_{i=1}^{\omega} p_i^q}{q - 1} = k \sum_{i=1}^{\omega} p_i \ln_q \frac{1}{p_i}. \tag{3}$$

Hence, in the case of equiprobability, $p_i = \frac{1}{\omega}$, the well-known Boltzmann law is recovered in the limit $q \to 1$. The q-entropy also satisfies the relevant properties of entropy like expansibility, composability, Lesche-stability \[17\] and concavity (for $q > 0$). The inverse function of the q-logarithm is called q-exponential \[15\] and is given by

$$\exp_q x \equiv \left[1 + (1-q)x\right]^\frac{1}{1-q} \quad (\exp_1 x = \exp x). \tag{4}$$

Recently in \[18\], the two-parameter logarithm $\ln_{q,q'}(x)$ and exponential $\exp_{q,q'}(x)$ were defined which recovered q-logarithm and q-exponential, in the limit $q \to 1$ or $q' \to 1$. So, the two-parameter entropy, similar to Eq. (1), can be defined as

$$S_{q,q'} \equiv \sum_{i=1}^{\omega} p_i \ln_{q,q'} \frac{1}{p_i} = \frac{1}{1-q'} \sum_{i=1}^{\omega} p_i \left[ \exp \left( \frac{1-q'}{1-q} (p_i^{q-1} - 1) \right) - 1 \right]. \tag{5}$$

The above entropy for the whole range of the space parameter does not fulfill all the necessary properties of a physical entropy. Therefore, it may not be appropriate for describing the physical systems, but useful for solving optimization problems. In this paper, we will find the probability distribution $p_i$ for the two-parameter entropy $S_{q,q'}$ \[17\], when canonical constraints are imposed on the system. As a result, it will be shown that the probability distribution is expressed in terms of the Lambert W-function \[19, 20, 21, 22\], also called omega function, which is an analytical function of $z$ defined over the hole complex $z$-plane, as the inverse function of

$$z = We^W. \tag{6}$$

Using this function, it is possible to write the series of infinite exponents in a closed form \[23\]

$$z^{-z} = -\frac{W(-\ln z)}{\ln z}. \tag{7}$$

The above equation shows that W-function can be used to express self-similarity in some fractal structures. This has been recently shown in a number of studies. Banwell and Jayakumar \[24\] showed that W-function describes the relation between voltage, current and resistance in a diode, Packel and Yuen \[25\] applied the W-function to a ballistic projectile in the presence of air resistance.

Other applications of the W-function include those in statistical mechanics, quantum chemistry, combinatorics, enzyme kinetics, vision physiology, engineering of thin films, hydrology, and the analysis of algorithms \[26, 27, 28\]. $S_q$ may become extensive in cases where there are correlations between subsystems. Hence, a point to be discussed will be the extensivity of $S_{q,q'}$ \[29\].

This paper is organized as follows. In Sec. 2, we will review the Generalized two parameter entropy, $S_{q,q'}$ and its properties. In Sec. 3, the probability distribution $p_i$ of the two-parameter entropy, $S_{q,q'}$, will be obtained in the canonical formalism. In Sec. 4, assuming that the two-parameter entropy be extensive, we will develop a relationship holding between probability in the composite system and probabilities of subsystems and in Sec. 5, we will have a conclusion.
2 Generalized two parameter entropy, $S_{q,q'}$

In this section, we will review the procedure for finding the two-parameter entropy, $S_{q,q'}$ [18]. As we know, it is possible to define two composition laws, the generalized q-sum and q-product [16], defined as follows

$$x \oplus_q y \equiv x + y + (1-q)xy$$
$$x \otimes_q y \equiv \left( x^{1-q} + y^{1-q} - 1 \right)^{\frac{1}{1-q}}$$

Using the definition of q-logarithm (Eq. (2)) and q-exponential (Eq. (4)), the above relations can be rewritten as

$$\ln_q(xy) = \ln_q x \oplus_q \ln_q y,$$  \hspace{1cm} (10)
$$\ln_q(x \otimes_q y) = \ln_q x + \ln_q y.$$  \hspace{1cm} (11)

Very recently in [18], Eqs. (10) and (11) were generalized by defining a two-parameter logarithmic function, denoted by $\ln_{q,q'} x$, which satisfies the equation

$$\ln_{q,q'}(x \otimes_q y) = \ln_{q,q'} x \oplus_{q'} \ln_{q,q'} y.$$  \hspace{1cm} (12)

Assuming $\ln_{q,q'} x = g(\ln_q x) = g(z)$ and using $x = y$ in Eq. (12), then applying some general properties of a logarithm function like

$$\ln_{q,q'} 1 = 0,$$  \hspace{1cm} (13)
$$\frac{d}{dx} \ln_{q,q'} x|_{x=1} = 1,$$  \hspace{1cm} (14)

the two-parameter generalized logarithmic function will be given as

$$\ln_{q,q'} x = \frac{1}{1-q'} \left[ \exp \left( \frac{1-q'}{1-q} (x^{1-q} - 1) \right) - 1 \right] = \ln_q e^{\ln_{q'} x},$$  \hspace{1cm} (15)

and its inverse function will be defined as a two-parameter generalized exponential, $\exp_{q,q'} x$

$$\exp_{q,q'} x = \left\{1 + \frac{1-q}{1-q'} \ln[1 + (1-q')x] \right\}^{1-q}. $$  \hspace{1cm} (16)

The entropy can be constructed based on the two-parameter generalization of the standard logarithm

$$S_{q,q'} \equiv k \sum_{i=1}^{\omega} p_i \ln_{q,q'} \frac{1}{p_i} = \frac{k}{1-q'} \sum_{i=1}^{\omega} p_i \left[ \exp \left( \frac{1-q'}{1-q} (p_i^{q'-1} - 1) \right) - 1 \right],$$  \hspace{1cm} (17)

which, in the case of equiprobability ($p_i = \frac{1}{\omega}$ $\forall i$), $S_{q,q'} = k \ln_{q,q'} \omega$.

The above entropy is Lesche-stable and some properties such as expansibility and concavity are satisfied if certain restrictions are imposed on $(q, q')$. 


3 Finding probability distribution in the canonical ensemble

In this section, we are interested in maximizing the entropy $S_{q,q'}$ under the constraints

$$\sum_{i=1}^{\omega} p_i - 1 = 0 ,$$  
(18)
$$\sum_{i=1}^{\omega} p_i \varepsilon_i - E = 0 .$$  
(19)

These constraints are added to the entropy with Lagrange multipliers to construct the entropic functional

$$\Phi_{q,q'}(p_i, \alpha, \beta) = S_{q,q'} + \alpha \left( \sum_{i=1}^{\omega} p_i - 1 \right) + \beta \sum_{i=1}^{\omega} p_i (\varepsilon_i - E) = 0 .$$  
(20)

To reach the equilibrium state, the entropic functional $\Phi_{q,q'}$ should be maximized, namely

$$\frac{\partial \Phi_{q,q'}(p_i, \alpha, \beta)}{\partial p_i} = 0 \Rightarrow \ln_{q,q'} \frac{1}{p_i} + p_i \frac{\partial \ln_{q,q'}(\frac{1}{p_i})}{\partial p_i} + \alpha + \beta (\varepsilon_i - E) = 0 .$$  
(21)

Using the definition of the two-parameter logarithm (Eq. (15)) and after some calculations, we get

$$\exp \left( \frac{1 - q'}{1 - q} (p_i q^{-1} - 1) \right) \left( 1 - (1 - q') p_i q^{-1} \right) = 1 - (1 - q') \left( \alpha + \beta (\varepsilon_i - E) \right) .$$  
(22)

To solve the above equation and to find $p_i(\varepsilon_i)$, one may define

$$z_i \equiv \frac{1 - q'}{1 - q} (p_i q^{-1} - 1) \Rightarrow 1 - (1 - q') p_i q^{-1} = (q - 1) z_i + q' ,$$  
(23)

and so, Eq. (22) can be rewritten as

$$(q' + (q - 1) z_i) \exp(z_i) = \gamma_i ,$$  
(24)

with the definition

$$\gamma_i \equiv 1 - (1 - q') \left( \alpha + \beta (\varepsilon_i - E) \right) .$$  
(25)

Solving the above equation gives us

$$z_i = W \left[ \frac{q'}{q - 1} \gamma_i \right] + \frac{q'}{1 - q} ,$$  
(26)

where, $W(z)$ is the Lambert $W$-function [19, 20, 21, 22]. From Eqs. (26) and (23), the probability distribution is given by:

$$p_i = \frac{1}{Z_q} \left\{ 1 + (1 - q) W \left[ \frac{q'}{q - 1} \gamma_i \right] \right\}^{\frac{1}{q - 1}} ,$$  
(27)

where,

$$Z_q = \sum_{i=1}^{\omega} \left\{ 1 + (1 - q) W \left[ \frac{q'}{q - 1} \gamma_i \right] \right\}^{\frac{1}{q - 1}} .$$  
(28)
Figure 1: Normalized probability distributions are depicted for different values of $q$ and $q'$ in (a) and (b), respectively. It is also shown that for $q = 1.1$ and $q' = 0.9$, the plots approach to normalized Gaussian.

In Eq. (25), $\beta$ is entered as an inverse of pseudo-temperature, but it may be interesting to write the probability distribution in terms of a deformed q-exponential which is more similar to the Boltzmann probability distribution. So, $\gamma$ in Eq. (22) can be written as

$$
\gamma_i = (1 - \alpha(1 - q'))\left(1 - \frac{\beta(1 - q')}{1 - \alpha(1 - q')} (\varepsilon_i - E)\right) \equiv (1 - \alpha(1 - q'))\left[\exp_q(-\beta_q(\varepsilon_i - E))\right]^{1-q},
$$

where, $\beta_q$ may be defined as the inverse of the pseudo-temperature

$$
\beta_q \equiv \frac{1}{k_B T_q} \equiv \frac{\beta}{1 - \alpha(1 - q')}.
$$

The probability distribution can be written in a better form

$$
p_i = \frac{1}{Z_q} \left\{ \exp_q \left[ W \left( \frac{e^{q' \varepsilon_i}}{q-1} (1 - \alpha(1 - q')) (\exp_q(-\beta_q(\varepsilon_i - E)))^{1-q} \right) \right] \right\}^{-1}.
$$

We can assume the energy level, $\varepsilon_i$, as a quadratic function of the variable $x_i$. The continuous normalized probability distribution as a function of $x$ can then be rewritten as:

$$
p(x) = \frac{\left\{1 + (1 - q)W \left[ \frac{e^{q' \varepsilon_i}}{q-1} (1 - (1 - q')(\alpha + \beta(x^2 - E)))\right] \right\}^{1-q}}{\int_{-\infty}^{\infty} \left\{1 + (1 - q)W \left[ \frac{e^{q' \varepsilon_i}}{q-1} (1 - (1 - q')(\alpha + \beta(x^2 - E)))\right] \right\}^{1-q} dx}.
$$

This is illustrated in Fig. 1. Comparing these plots with the normalized Guassian distribution shows that in the limit $q, q' \rightarrow 1$, the plots approximate the Guassian function as expected.

It is also possible to repeat the procedure with the energy constraint $\sum_{i=1}^{W} p_i^q \varepsilon_i = E$. But in that case, the equation which maximizes the entropy, is not solvable.

### 4 How to interpret the entropy $S_{q,q'}$ extensive?

Extensivity, together with concavity, Lesche stability, and finiteness of the entropy production per time, increases the suitability of an entropy. But, is BG-entropy the only extensive one? In Ref. [29], a question is raised as to whether entropy $S_q$ is extensive or not? The answer is meaningful only if
a composition law is specified, otherwise, it is tacitly assumed that subsystems are independent. Special correlations can be mathematically constructed such that by the multiplication of subsystems, probabilities. (b) shows an example of the correlated A and B.

Table 1: (a) shows the independence of two equal two-state subsystems A and B. The joint probabilities are given by the multiplication of subsystems, probabilities. (b) shows an example of the correlated A and B.

| A \ B | 1     | 2     | 1     | 2     |
|-------|-------|-------|-------|-------|
| 1     | $p_{11}^{A+B} = p^2$ | $p_{12}^{A+B} = p(1 - p)$ | $p_{11}^{A+B} = 2p - 1$ | $p_{12}^{A+B} = 1 - p$ |
| 2     | $p_{21}^{A+B} = p(1 - p)$ | $(1 - p)^2$ | $p_{22}^{A+B} = 1 - p$ | $p_{22}^{A+B} = 0$ |
| p     | 1 - p | 1     | p     | 1     |

(a) (b)

S\text{BG}(A + B) = S\text{BG}(A) + S\text{BG}(B),  \hspace{1cm} (33)

S_q(A + B) = S_q(A) + S_q(B) + (1 - q)S_q(A)S_q(B),  \hspace{1cm} (34)

However, (b) shows a special correlation between subsystems which leads to non-extensivity for the BG-entropy and extensivity for the Tsallis entropy at $q = 0$.

$$S_0(A + B) = S_0(A) + S_0(B).$$  \hspace{1cm} (35)

As can be seen in Table, (b), one of the states of the composite system appears with the zero probability and so, the number of effective states is $w_{\text{eff}}^{A+B} = 3$, which is not equal to $w^{A+B} = w^A \times w^B = 4$. This simple model can be improved to describe non-ergodic systems, where not all the states are accessible. In the following, along the lines of what is done in [29][30], we will find a relation between the probabilities of the composite system (joint probabilities) and the probabilities of subsystems (marginal probabilities), such that the two-parameter entropy $S_{q,q'}$ is extensive.

Now, we are interested in finding the extensivity condition for the two-parameter entropy. Consider $N$ subsystems $(A_1, A_2, \ldots, A_N)$, each with the probability $p_{is}$ $(s$ is related to that system). The probabilities in the composite system, $p_{i_1,i_2,\ldots,i_N}^{A_1+A_2+\ldots+A_N}$, should satisfy the condition

$$\sum_{i_1,i_2,\ldots,i_N} p_{i_1,i_2,\ldots,i_N}^{A_1+A_2+\ldots+A_N} = 1. \hspace{1cm} (36)$$

The marginal probability related to the system, $s$, is defined as

$$p_{is}^{A_s} = \sum_{i_1,i_2,\ldots,i_N} p_{i_1,i_2,\ldots,i_N}^{A_1+A_2+\ldots+A_N}. \hspace{1cm} (37)$$

It may be interesting to find the condition which makes the entropy $S_{q,q'}$ extensive. In other words, we want to know the relationship between the probability in the composite system and probabilities
of the subsystems when the entropy $S_{q,q'}$ is extensive. Let us consider the relation

$$\frac{1}{p_{A_1+A_2+\ldots+A_N}} = \exp_{q,q'} \left( \sum_{i,j,k} \frac{1}{p_{i,j}} \ln q_{i,j} + \phi_{i,j,k} \right),$$

(38)

where, $\phi_{i,j,k}$ is set to ensure Eq. (36). The above equation can be rewritten in a different form

$$\frac{1}{p_{A_1+A_2+\ldots+A_N}} = \frac{1}{p_{A_1}} \otimes_{q,q'} \frac{1}{p_{A_2}} \otimes_{q,q'} \ldots \otimes_{q,q'} \frac{1}{p_{A_N}} \otimes_{q,q'} \exp_{q,q'} (\phi_{i,j,k}),$$

(39)

with the definition of $\otimes_{q,q'}$-product [18]

$$x \otimes_{q,q'} y \equiv \exp_{q,q'} (\ln q_{x} + \ln q_{y}).$$

(40)

A nonzero function $\phi_{i,j,k}$ is related to the existence of the correlation in the system, because in the case of independent subsystems ($q,q' \to 1$), the $\otimes_{q,q'}$-product becomes the usual product and $\phi_{i,j,k} = 0$. In Ref. [30], the values of $\phi_{i,j,k}$, making the entropy $S_q$ extensive, are obtained for two equal two-state subsystems. In the case of equiprobability, Eq. (39) save for the function $\phi_{i,j,k}$, shows the generalized multiplication of the number of states of the subsystems, which may be defined as the effective number of states

$$w_{eff}^{A+B} = w_{eff}^{A_1} \otimes_{q,q'} w_{eff}^{A_2} \otimes_{q,q'} \ldots \otimes_{q,q'} w_{eff}^{A_N} \exp_{q,q'} (\phi_{i,j,k}),$$

(41)

The entropy of a composite system similar to Eq. (17) can be defined as follows

$$S_{q,q'} \left( \sum_{s=1}^{N} A_s \right) = \frac{k}{p_{A_1+A_2+\ldots+A_N}} \ln q_{A_s} \left[ \exp_{q,q'} \left( \sum_{s=1}^{N} \frac{1}{p_{A_s}} \ln q_{A_s} + \phi_{i,j,k} \right) \right].$$

(42)

Using Eq. (38), the entropy can be written as

$$S_{q,q'} \left( \sum_{s=1}^{N} A_s \right) = k \sum_{i,j,k} p_{A_1+A_2+\ldots+A_N} \ln q_{A_s} \left[ \exp_{q,q'} \left( \sum_{s=1}^{N} \frac{1}{p_{A_s}} \ln q_{A_s} + \phi_{i,j,k} \right) \right]$$

$$= k \sum_{i,j,k} p_{A_1+A_2+\ldots+A_N} \ln q_{A_s} \left[ \exp_{q,q'} \left( \sum_{s=1}^{N} \frac{1}{p_{A_s}} \ln q_{A_s} + \phi_{i,j,k} \right) \right]$$

$$= k \sum_{i,j,k} p_{A_1+A_2+\ldots+A_N} \ln q_{A_s} \left[ \exp_{q,q'} \left( \sum_{s=1}^{N} \frac{1}{p_{A_s}} \ln q_{A_s} + \phi_{i,j,k} \right) \right]$$

$$= k \sum_{i,j,k} p_{A_1+A_2+\ldots+A_N} \ln q_{A_s} \left[ \exp_{q,q'} \left( \sum_{s=1}^{N} \frac{1}{p_{A_s}} \ln q_{A_s} + \phi_{i,j,k} \right) \right]$$

$$= \sum_{i,j,k} S_{q,q'} (A_s) + k \sum_{i,j,k} p_{A_1+A_2+\ldots+A_N} \phi_{i,j,k},$$

(43)

where, the definition of marginal probability Eq. (37) is used in the last line. Eq. (43) ensures extensivity of $S_{q,q'}$ if the constraint

$$\sum_{i,j,k} p_{A_1+A_2+\ldots+A_N} \phi_{i,j,k} = 0,$$

(44)

is satisfied. In other words, assuming the above constraint will be equivalent to the existence of extensivity. According to Eqs. (15), (16), and (38), for the probability of a composite system, we
get

$$p_{i_1,i_2,...,i_N}^{A_1+A_2+...+A_N} = \left\{ 1 + \frac{1 - q}{1 - q'} \ln \left[ 1 - N + (1 - q')\phi_{i_1,i_2,...,i_N} - \sum_{s=1}^{N} \exp \left[ \frac{1 - q'}{1 - q} (p_{s}^{q-1} - 1) \right] \right] \right\}^{\frac{1}{q-1}}. \quad (45)$$

It is clear that in the limit $q \to 1$, the proposed relation of probabilities is recovered [29, 30]

$$p_{i_1,i_2,...,i_N}^{A_1+A_2+...+A_N} = \left[ 1 - N + (1 - q')\phi_{i_1,i_2,...,i_N} + \sum_{s=1}^{N} (p_{s})^{q'-1} \right]^{\frac{1}{q'-1}}. \quad (46)$$

where, in the limit $q' \to 1$, the usual product of probabilities, $p_{i_1,i_2,...,i_N}^{A_1+A_2+...+A_N} = \prod_{s=1}^{N} p_{s}$, is given, which describes the case of independent subsystems.

## 5 Conclusion

In this paper, a special set of two-parameter entropies [18] were maximized in the canonical ensemble by the energy constraint $\sum_{i=1}^{\omega} p_{i} \varepsilon_{i} = E$. We expected that the probability distribution, $p_{i}(\varepsilon_{i})$, can be expressed in terms of the generalized two-parameter exponential defined in [18]. But unexpectedly, solution of the related equation took the form of the Lambert function which has been used in many branches including statistical mechanics, quantum chemistry, enzyme kinetics, and thin films, among others. The Lambert function can also describe some fractal structures because infinite exponents can be written in terms of the Lambert function (Eq. 7). This suggests that the two-parameter entropies are probably related to the fractal structures in a phase space which may be a subject for future study. Also, assuming extensive $S_{q,q'}$, the probability of a composite system was given in terms of probabilities of the subsystems.
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