Parameter restrictions in a non-commutative geometry model do not survive standard quantum corrections

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Abstract

We have investigated the standard one-loop quantum corrections for a particularly simple non-commutative geometry model containing fermions interacting with a unique abelian gauge field and a unique scalar through Yukawa couplings. In this model there are certain relations among the different coupling constants quite similar to the ones appearing in the Connes-Lott version of the standard model. We find that it is not possible to implement those relations in a renormalization-group invariant way.
1 Introduction

There seems to be a growing consensus that non-commutative geometry (NCG from now on) (cf.\cite{1,2} for general reviews) is one of the most important developments in mathematics in the recent years. In addition to that, Connes and Lott (\cite{3}) have invented a mechanism somewhat similar to the old Kaluza-Klein idea (but using discrete internal spaces instead, in such a way that the Higgs field is interpreted as a sort of gauge field in the internal direction), to show that the standard $SU(3) \times SU(2) \times U(1)$ model of strong and electroweak interactions (SM in the sequel) appears naturally in this framework. The fascinating point is that, when interpreted in this way, not all the parameters of the SM are free, but have to obey certain restrictions. In particular, the Higgs mass in terms of the top mass (neglecting all other fermion masses) is given by:

$$M_H = c(x)m_t$$ \hspace{1cm} (1)

Where $c(x)$ is a constant, depending on the parameter $x$, which measures the splitting of the trace between the leptonic and the quark sector ($-1 \leq x \leq 1$). Actually, if the total Hilbert space of the fermions in the Connes-Lott derivation is $H = H_{leptons} \bigoplus H_{quarks}$, then the total NCG Yang-Mills functional is a convex combination:

$$\frac{1+x}{2}(YM)_{leptons} \bigoplus \frac{1-x}{2}(YM)_{quarks}$$ \hspace{1cm} (2)

To be specific\cite{4},

$$c(x)^2 = 3 - \frac{9x^2 - 24x + 15}{10x^2 - 34x + 28}$$ \hspace{1cm} (3)

As far as one can tell with our limited knowledge of quantum field theory, the only consistent way of imposing constraints among different coupling constants in a given model, is in a renormalization group invariant way. This is almost always a consequence of the Ward identities corresponding to some symmetry of the underlying action principle, although this is not always necessary, as in Zimmermann’s examples of ”reduction of coupling constants” (cf., for example,\cite{5}).

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We could, of course, impose those restrictions as defining the "physical" renormalization scheme, even if they do not hold for general and sensible renormalization schemes. But this would be most "unnatural", from the quantum field theory point of view, and besides, we do not see a compelling reason in the Connes-Lott construction to do that.

It is then obviously of great interest to investigate whether those restrictions are indeed first integrals of the renormalization-group flow. Were that the case, it would point out either to a "hidden" symmetry of the standard model, overlooked until now, or else to some reduction mechanism of the Zimmermann type. In either case it would have been most remarkable.

Although our main interest lies in the SM, there are many technical complications (coming essentially from the chiral character of the model), which make the computation of the one-loop renormalization group of the model a nontrivial matter. We have decided, in view of that, to study first a toy model, in which we have succeeded to include some relations among the parameters quite similar to the ones appearing in the Connes-Lott version of the SM.

2 The Non-commutative geometry model

Using suitable generalizations of the Connes-Lott construction one can obtain a limited type of lagrangians only. The fact that the SM is among them is already quite remarkable. The toy model we are going to discuss for the remaining of this paper is obtained by choosing as our manifold the product space of $\mathbb{Z}_2$ and (euclidean) spacetime $M$, including a trivial one-dimensional vector bundle on each piece. This is actually the simplest nontrivial (in the NCG sense) and non-pathological model available (anomalous models result, for example, when one considers the set theoretical union of spacetime with discrete sets of points).

The basic tool in NCG is the K-cycle, which in the commutative case is equivalent to having a gauge field coupled to massless fermions. In order to obtain Higgs fields, one needs product K-cycles, as in our toy model, were we took the product of Dirac’s K-cycle with the K-cycle giving the geometry of $\mathbb{Z}_2$, which contains in embryonic form both the Higgs and the Yukawa couplings. Details can be obtained from [2]; our present model can
be actually obtained by putting $Z = W = \phi_2 = 0$ and $\phi^*_1 = \phi_1$ in the computation of the NCG version of the SM in Section 7 of that paper.

In the model there are $N$ species of fermions, $\psi_i$, all with the same $U(1)$ charge with respect to an abelian gauge field $A_\mu$, and with different Yukawa couplings $g_i$ with a single scalar field, of mass $2M$, and self-coupling $\lambda$.

The Lagrangian (after rotating to Minkowski space) is:

$$L = -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} + \frac{1}{2} \partial_\mu \phi \partial^\mu \phi + \frac{1}{2} M^2 \phi^2 - \lambda \frac{\lambda^4}{24} \phi^4 + \sum_{j=1}^{N} i \bar{\psi}_j \gamma^\mu (\partial_\mu - eA_\mu) \psi_j - g_j \bar{\psi}_j \psi_j \phi$$

(4)

With regard to the analogy with the standard model, it seems more adequate to think of the above fermions as “leptons” instead of “quarks”, because color enters NCG through Poincare duality, which is trivial for the algebra $C^2 \otimes C^\infty(M)$.

In our model the restrictions on the allowed values of the different coupling constants arise just because our product K-cycle contains information about the fermion mass spectrum only \footnote{Let us point out that there is a mass non-degeneracy condition implying that $N \geq 2$}, the only remaining freedom being the scale of the $U(1)$ connection; that is, the charge.

To be specific, the parameters above must obey three different relations among themselves:

$$\sum_{i=1}^{N} g_i^2 = \frac{Ne^2}{2}$$

(5)

$$\lambda = 6Ne^2 \frac{trm^4_1}{(trm^2)^2}$$

(6)

$$M^2 = \frac{trm^4_1}{trm^2}$$

(7)

The fermionic mass spectrum is determined by the diagonal matrix $m$, and $m^2_\perp = m^2 - (trm^2)/N$, where the parameters $m_i$ and the Yukawa couplings must obey the relation:

$$g_i^2 = \frac{Ne^2 m_i^2}{2trm^2}$$

(8)
When one of the fermions (for example, the first) is much heavier than all the others, the second equation reduces to:

\[ \lambda = 6(N - 1)e^2 \]  

(9)

and the third one reduces in turn to

\[ NM^2 = (N - 1)m_1^2 \]  

(10)

The last two relations have close analogous in the Connes-Lott version of the SM(cf. [3, 2]). The first one has an analogous also, although apparently it has never been explicitly written in the literature. Let us write it here for completeness. In terms of the parameter \( x \) we introduced in the first paragraph, it yields:

\[ \frac{1 + x}{2} \sum_{i=\text{leptons}} g_i^2 + \frac{3(1 - x)}{2} \sum_{i=\text{quarks}} g_i^2 = \frac{N(2 - x)e^2}{4\sin\theta_W^2} \]  

(11)

3 The one-loop beta functions of the model

Our toy model is obviously anomaly free and renormalizable. Besides, it is non-chiral, so that we can freely use dimensional regularization without having to worry about how to define \( \gamma_5 \). The one loop beta functions in the MS scheme are easily shown to be:

\[ \beta_{g_i} = \frac{1}{16\pi^2} (5g_i^3 - 6e^2g_i + 2g_i \sum_{j \neq i} g_j^2) \]  

(12)

\[ \beta_e = \frac{N}{12\pi^2} e^3 \]  

(13)

\[ \beta_{m_i} = \frac{3m_i}{16\pi^2} (-2e^2 + g_i^2) \]  

(14)

\[ \beta_{M^2} = \frac{M^2}{16\pi^2} (\lambda + 4 \sum_i g_i^2) \]  

(15)

3 There is some controversy on this point ([4, 6]).

4 To perform perturbative computations one has to take into account that in our model \( \langle \phi \rangle \neq 0 \)
\[ \beta_\lambda = \frac{1}{16\pi^2}(-48 \sum_i g_i^4 + 3\lambda^2 + 8\lambda \sum_i g_i^2) \]  

(16)

Let us make two technical comments here, for the sake of completeness: first, the formal Ward identities coming from both the discrete spontaneously broken symmetry and the \( U(1) \) gauge symmetry of our model hold automatically in our substraction scheme. Secondly, equations (5) and (6) yield \( \lambda \sim e^2 \) and at least one Yukawa \( g_j \sim e \); this means that contributions of the order \( g_i\lambda^2 \) and \( M^2\lambda^2 \) constitute higher order corrections, and are thus neglected here.

Now let us look at the physical implications of these values for the beta functions of the model.

It is easily seen from the renormalization group equations for a generic coupling constant (including masses), \( g \)

\[ \frac{d g(\mu)}{d\mu} = \beta(g(\mu)) \]  

(17)

that the conditions the \( \beta \)-functions of the model ought to obey in order for the NCG conditions be first integrals of the above differential equations are:

\[ 2 \sum_i g_i \beta_{g_i} = Ne\beta_e \]  

(18)

\[ \beta_\lambda = \frac{12N}{(trm^2)^2}(trm_1^4(e\beta_e - 2e^2 \sum_i \frac{m_i\beta_{m_i}}{trm^2}) + 2e^2 \sum_i m_i \beta_{m_i}(m_i^2 - \frac{trm^2}{N}))) \]  

(19)

\[ \beta_{M^2} = \sum_i \frac{2m_i\beta_{m_i}}{trm^2}(-\frac{trm_1^4}{trm^2} + 2m_i^2 - \frac{2trm^2}{N}) \]  

(20)

The fact that these equations are not satisfied identically in parameter space means that if one imposes them at one scale \( \mu_0 \), the flow takes the system away from the constraint surface.

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5 Actually, the dimensionless counterterms of the unbroken phase are the same as their counterparts in the broken phase.
4 Conclusions

What can we conclude from the preceding analysis? One thing, at least, seems to be clear: there is no underlying hidden symmetry in our model, nor a Zimmermann-like mechanism of reduction of coupling constants, because in both cases the constraint surface would remain invariant under the renormalization group flow.

One can always argue, however, that the only physically admissible renormalization scheme is the one preserving the constraints coming from NCG; which is certainly technically possible, because we have less restrictions than coupling constants.

In a different, and perhaps deeper, vein, it can certainly be maintained that the structure of the non-commutative geometry underlying the model requires a drastic change in the standard quantization rules based upon replacing Poisson (or Dirac) brackets by commutators of operators acting in some linear space.

Our present work, being rooted in ordinary quantum mechanics is unable to rule out such a possibility.

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