Maximal singular integral operators acting on noncommutative $L_p$-spaces

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Abstract

In this paper, we study the boundedness theory for maximal Calderón–Zygmund operators acting on noncommutative $L_p$-spaces. Our first result is a criterion for the weak type $(1, 1)$ estimate of noncommutative maximal Calderón–Zygmund operators; as an application, we obtain the weak type $(1, 1)$ estimates of operator-valued maximal singular integrals of convolution type under proper regularity conditions. These are the first noncommutative maximal inequalities for families of truly non-positive linear operators. For homogeneous singular integrals, the strong type $(p, p)$ $(1 < p < \infty)$ maximal estimates are shown to be true even for rough kernels. As a byproduct of the criterion, we obtain the noncommutative weak type $(1, 1)$ estimate for Calderón–Zygmund operators with integral regularity condition that is slightly stronger than the Hörmander condition; this provides somewhat an affirmative answer to an open question in the noncommutative Calderón–Zygmund theory.

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1 Introduction and main results

Motivated by quantum mechanics, operator algebra, noncommutative geometry and quantum probability, noncommutative harmonic analysis gains rapid development recently and many fundamental works appeared (see e.g. [6, 13, 14, 22, 24–29, 35, 38, 41, 47, 48]). Due to the noncommutativity, many real variable tools or methods such as maximal functions, stopping times etc. are not available, which imposes numerous difficulties in developing noncommutative theory. The semi-commutative (or operator-valued) harmonic analysis seems to be the easiest noncommutative theory, but it requires novel ideas, insights or techniques. Moreover, together with various transference techniques (see e.g. [6, 20, 29, 37, 41] and references therein), it has many exciting applications in other research fields or plays important roles in the more sophisticated noncommutative setting where the explicit expressions or the estimates of the kernels are absent. For instance, motivated by the theory of noncommutative martingales [30, 31, 43], Mei [35] developed systematically the theory of operator-valued Hardy spaces and BMO spaces which incidentally solved an open question in matrix-valued harmonic analysis arising from prediction theory. Mei’s theory is also used to develop harmonic analysis on group von Neumann algebras [27, 29, 41] and quantum tori (or quantum Euclidean space) [6, 14, 33, 47, 48]. Based on Cuculescu’s maximal weak (1,1) estimate for noncommutative martingales [10], Parcet [39] formulated a kind of noncommutative Calderón–Zygmund decomposition and established the weak type (1,1) estimate for operator-valued Calderón–Zygmund singular integrals, which finds its unexpected application in the complete resolution of the Nazarov–Peller conjecture arising from the perturbation theory [5]. For more related results on weak type (1, 1) estimates and noncommutative Calderón–Zygmund decomposition we refer the reader to [1, 19, 21, 32, 36].

In the present paper, we study semi-commutative Calderón–Zygmund theory but focus on the maximal singular integral operators. In the commutative case, the maximal function or operator \( Mf = \sup_n |T_n f| \) for a sequence of linear operators \((T_n)\) acting on some function \(f\) is an instrumental tool in the real variable theory, and is the main tool to obtain the pointwise convergence result. But it usually requires much more ideas or tools in estimating maximal inequalities \(Mf\) (resp. pointwise convergence) than estimating inequalities of \(T_n f\) uniformly (resp. norm convergence). For instance, the norm convergence of the Dirichlet series is equivalent to the boundedness of Hilbert transform; but the corresponding pointwise convergence is guaranteed by the famous Carleson’s maximal theorem which was obtained around 40 years later. In the noncommutative setting, since the maximal function is not available any more (see [30]), the maximal inequality is much more difficult to get. The first two non-trivial maximal inequalities go back 1970s in the last century, that are, Yeadon’s maximal weak type (1,1) estimate for ergodic averages (see [49]) and Cuculescu’s one for conditional expectations mentioned previously. However, it took around 30 years to obtain the noncommutative \(L_p\)-maximal inequalities along the line of Cuculescu and Yeadon’s theorems; the formulation of \(L_p\)-maximal inequalities was not possible
until the appearance of Pisier’s vector-valued noncommutative $L_p$-spaces (see [42]) which required the full strength of the operator space theory. Indeed, the $L_p(\Omega)$-norm of the maximal function $Mf$ must be understood as the $L_p(\Omega; \ell_\infty)$-norm of the sequence $(T_n f)_n$ in the noncommutative case. Motivated by Pisier’s definition of $\ell_\infty$-valued noncommutative $L_p$-spaces for hyperfinite algebra and the noncommutative martingale inequalities in the seminal paper [43], Junge [23] in 2002 extended Pisier’s definition for general algebras and Doob’s $L_p$-maximal inequality for noncommutative martingales with argument based on Hilbert module theory. A few years later, Junge and Xu [30] obtained $L_p$-maximal inequalities for ergodic averages; the key tool was a noncommutative analogue of Marcinkiewicz interpolation theorem for families of positive maps which allows to deduce results from Cuculescu and Yeadon’s inequalities.

Even though the noncommutative maximal inequalities for ergodic averages and conditional expectations have now been established, there appear a lot of difficulties to obtain maximal inequalities for other families of linear operators. For instance, Mei [35] has to invent an ingenious idea to show the noncommutative Hardy–Littlewood maximal inequalities based on Junge’s noncommutative Doob’s inequalities; in [20], Hong et al exploited the group structure or probabilistic method to show the noncommutative maximal ergodic inequalities associated with the action of groups of polynomial growth; since the lack of estimates of Fourier multipliers on group algebras, the first author and his collaborators [22] invented some quantum semigroup and analyze carefully its difference with Fourier multiplier to establish the maximal inequalities and show the pointwise convergence of noncommutative Fourier series. We refer the reader to the above mentioned papers and references therein for more results on noncommutative maximal inequalities.

As far as we know, except some non-sharp results for Bochner–Riesz means in [6, 22], there does not exist in the literature any other non-trivial noncommutative maximal inequalities for families of non-positive linear operators, such as the truncated Calderón–Zygmund operators and Dirichlet means.

In this paper, we will establish the noncommutative maximal inequalities for families of truncated operator-valued singular integrals.

Recall that a (standard) Calderón–Zygmund operator (abbreviated as CZO) $T$ is a singular integral operator in $\mathbb{R}^d$ mapping test functions to distributions associated with a (standard) Calderón–Zygmund kernel $k : \mathbb{R}^d \times \mathbb{R}^d \setminus \{(x, x) : x \in \mathbb{R}^d\} \to \mathbb{C}$ in the sense that

$$ Tf(x) = \int_{\mathbb{R}^d} k(x, y) f(y) \, dy, \quad (1.1) $$

whenever $f \in C_c^\infty(\mathbb{R}^d)$ a test function and $x \notin \text{supp } f$, which admits a bounded extension on $L_2(\mathbb{R}^d)$. A Calderón–Zygmund kernel $k$ is called standard if it satisfies the size condition

$$ |k(x, y)| \leq \frac{C}{|x - y|^d} \quad (1.2) $$
for \(x, y \in \mathbb{R}^d\), where and in the sequel, \(C\) is a positive numerical constant, and the \(\gamma\)-Lipschitz regularity condition with \(\gamma \in (0, 1]\),

\[
|k(x, y) - k(x, z)| + |k(y, x) - k(z, x)| \leq \frac{C|y - z|^\gamma}{|x - y|^{d+\gamma}} \quad (1.3)
\]

for \(x, y, z \in \mathbb{R}^d\) satisfying \(|x - y| \geq 2|y - z|\). The distance \(|x - y|\) between \(x\) and \(y\) is taken with respect to \(\ell_2\) metric throughout the paper. The simplest example of CZO is the Hilbert transform (or Riesz transform). It is well-known that a standard Calderón–Zygmund operator \(T\) admits a bounded extension and thus is well-defined on \(L_p\) for all \(1 \leq p < \infty\); moreover, the same conclusions hold still true if the \(\gamma\)-Lipschitz condition is weakened to some \(L_q\)-integral regularity condition with \(q \in (0, +\infty)\) (see for instance [17])

\[
\sum_{m=1}^{\infty} \delta_q(m) < \infty, \quad (1.4)
\]

where

\[
\delta_q(m) = \sup_{R > 0, y \in \mathbb{R}^d \atop |v| \leq R} \left(2^m R^{d(q-1)} \int_{2^m R \leq |x - y| \leq 2^{m+1} R} |k(x, y + v) - k(x, y)|^q \, dx \right)^{\frac{1}{q}}.
\]

(1.5)

If \(k\) satisfies the Lipschitz smoothness condition, then \(\delta_q(m) \leq C_d 2^{-m\gamma}\) (\(C_d\) is a constant depending on dimension \(d\)) for any \(q > 0\), which has nice decay property; in view of (1.4), the Hörmander condition

\[
\int_{|x - y| \geq 2|y - z|} |k(x, y) - k(x, z)| \, dx < \infty, \quad \forall y, z \in \mathbb{R}^d
\]

can be restated as \(L_1\)-integral regularity condition \(\sum_{m=1}^{\infty} \delta_1(m) < \infty\). In this paper, our results will be concerned with the case \(q = 2\) which is slightly stronger than the Hörmander condition. Indeed, by the Hölder inequality, \(\delta_1(m) \leq C_d \delta_2(m)\).

Let \(\mathcal{M}\) be a von Neumann algebra equipped with a normal semi-finite faithful trace (abbreviated as n.s.f) \(\tau\) and \(\mathcal{N} = L_\infty(\mathbb{R}^d) \otimes \mathcal{M}\) be the von Neumann algebra tensor product equipped with tensor trace \(\varphi = \int \otimes \tau\). Then we can define the associated noncommutative \(L_p\)-spaces \(L_p(\mathcal{M})\) and \(L_p(\mathcal{N})\). The latter can be identified as the space of \(L_p(\mathcal{M})\)-valued \(p\)-th integrable functions on \(\mathbb{R}^d\). From semi-commutative Calderón–Zygmund theory [35, 39], any Calderón–Zygmund operator \(T\) with kernel satisfying the size and the Hörmander conditions admits completely bounded extension and thus is well-defined on \(L_p(\mathcal{N})\) for all \(1 < p < \infty\); moreover if the Hörmander condition is strengthened to the \(\gamma\)-Lipschitz condition, Parcet [39] showed that the extension \(T \otimes id_\mathcal{M}\) is of weak type \((1, 1)\) and thus well-defined on \(L_1(\mathbb{R}^d; L_1(\mathcal{M}))\) (see also [1]) and left the sufficiency of Hörmander condition as an open question. For
simplifying the notation, the extension $T \otimes id_{\mathcal{M}}$ will be still denoted as $T$, admitting the following expression

$$Tf(x) = \int_{\mathbb{R}^d} k(x, y) f(y) \, dy,$$

whenever $f$ is a $L_1(\mathcal{M}) \cap L_\infty(\mathcal{M})$-valued compactly supported measurable function and $x \notin \text{supp } f$ which is the support of $f$ as an operator-valued function in $\mathbb{R}^d$.

Given a Calderón–Zygmund operator $T$ with kernel $k$. For any $\varepsilon > 0$, we define the associated truncated singular integrals $T_\varepsilon f$ by

$$T_\varepsilon f(x) = \int_{|x-y| > \varepsilon} k(x, y) f(y) \, dy.$$  

It is well-known that $T_\varepsilon$’s are Calderón–Zygmund operators with kernels satisfying the same conditions as those of $k$.

Our first result is a criterion for the noncommutative weak type $(1, 1)$ estimate of maximal Calderón–Zygmund operators; the reader is referred to Sect. 2 for the notions of $\ell_\infty$-valued noncommutative $L_p$ spaces $L_p(\mathcal{N}; \ell_\infty)$.

**Theorem 1.1** Let $T$ be a Calderón–Zygmund operator defined as (1.6) associated to a kernel satisfying (1.2) and (1.4) with $q = 2$ and let $T_\varepsilon$ be defined as (1.7). Assume that there exists one $p_0 \in (1, \infty)$ such that $(T_\varepsilon)_{\varepsilon > 0}$ is of strong type $(p_0, p_0)$, that is,

$$\| (T_\varepsilon f)_{\varepsilon > 0} \|_{L_{p_0}(\mathcal{N}; \ell_\infty)} \leq C \| f \|_{p_0}, \quad \forall f \in L_{p_0}(\mathcal{N}).$$

Then $(T_\varepsilon)_{\varepsilon > 0}$ is of weak type $(1, 1)$, that is, for any $f \in L_1(\mathcal{N})$ and $\lambda > 0$, there exists a projection $e \in \mathcal{N}$ such that

$$\sup_{\varepsilon > 0} \| e(T_\varepsilon f) e \|_{\infty} \leq \lambda \quad \text{and} \quad \varphi(e^{-1}) \leq C_d \lambda^{-1} \| f \|_1.$$  

One is able to formulate the weak type $(1, 1)$ estimate (1.9) in a similar way as the strong type (1.8) if using the $\ell_\infty$-valued weak $L_p$ space $\Lambda_{p, \infty}(\mathcal{N}; \ell_\infty)$ (for its definition see (2.1)).

In the process of showing Theorem 1.1, we observe that the argument essentially works also for Calderón–Zygmund operator itself with $L_2$-integral regularity condition. As mentioned previously, this solves partially a conjecture (see e.g. [39, Page 575], that is, the pointwise $\gamma$-Lipschitz condition can be weakened to some integral regularity condition.

**Theorem 1.2** Let $T$ be defined as (1.6) associated to a kernel satisfying (1.2) and (1.4) with $q = 2$. Assume that $T$ is bounded on $L_{p_0}(\mathcal{N})$ for some $p_0$ with $1 < p_0 < \infty$. Then for any $f \in L_1(\mathcal{N})$,

$$\| Tf \|_{L_1(\mathcal{N})} \leq C_d \| f \|_1.$$  

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Remark 1.3  After the preliminary version of the paper was completed, the first author was kindly told by Javier Parcet that Theorem 1.2 has been discovered independently by him, Léonard Cadilhac and José M Conde-Alonso, see [3, Theorem A].

In the commutative case, that is $\mathcal{M} = \mathbb{C}$, under the assumption that the kernel satisfies the $\gamma$-Lipschitz regularity condition, conclusion (1.9), as well as assumption (1.8), can be deduced from the following Cotlar inequality: for all $0 < s < 1$,

$$
\sup_{\varepsilon > 0} |T_{\varepsilon} f(x)| \leq C_{d, \gamma, s} \left[ \sup_{\varepsilon > 0} M_{\varepsilon}(|f|)(x) + \sup_{\varepsilon > 0} (M_{\varepsilon}(|T_{\varepsilon} f|^s)(x))^{\frac{1}{s}} \right],
$$

(1.10)

where $M_{\varepsilon}$ is the Hardy–Littlewood averaging operator

$$
M_{\varepsilon} g(x) = \frac{1}{\varepsilon^d} \int_{|x-y| \leq \varepsilon} g(y) dy,
$$

(1.11)

and $C_{d, \gamma, s}$ is a constant depending on $d$, $\gamma$ and $s$. The Cotlar inequality (1.10) in turn follows from some pointwise localized argument (see e.g. [16, Theorem 4.2.4]). But it seems impossible that the above pointwise estimate (1.10) and its proof admit noncommutative analogues. On the other hand, in the commutative case, the $L_2$-integral regularity condition seems not sufficient for a weak Cotlar inequality, that is, (1.10) with $s = 1$, which would be enough for strong type $(p, p)$ estimate for maximal CZOs for all $1 < p < \infty$. In conclusion, under the conditions (1.2) and (1.4) with $q = 2$, it is not clear at all whether (1.8) holds for any $p_0 \in (1, \infty)$.

However, when the CZO is of convolution type, that is, $k(x, y) = k(x - y)$, a noncommutative variant of the weak Cotlar inequality—(1.10) with $s = 1$—in terms of norms can be verified under the $\gamma$-Lipschitz regularity condition (see (5.1)). This might be known to experts, but we will formulate it rigorously.

It is well-known (see e.g. [15]) that under the size condition (1.2) and the $\gamma$-Lipschitz condition (1.3), $T$ is a standard CZO of convolution type if and only if its associated kernel $k$ satisfies additionally the cancellation condition

$$
\sup_{0 < r < R < \infty} \left| \int_{r < |x| < R} k(x) dx \right| < \infty.
$$

(1.12)

It is also known from classical Calderón–Zygmund theory [15] that (1.12) implies that there exists a subsequence $(\varepsilon_j)_{j \in \mathbb{N}} \subset (0, +\infty)$ with $\varepsilon_j \to 0$ as $j \to +\infty$ such that

$$
\lim_{j \to +\infty} \int_{\varepsilon_j < |x| \leq 1} k(x) dx \text{ exists}
$$

(1.13)

and for any $f \in L_p(\mathbb{R}^d)$ with $1 \leq p < \infty$,

$$
T_{\varepsilon_j} f(x) \to T f(x) \text{ a.e., as } j \to +\infty.
$$

(1.14)

We state our second result of this paper.
Theorem 1.4 Let $T$ be a CZO of convolution type with associated kernel satisfying (1.2), (1.3) and (1.12). Then we have the following conclusions.

(i) For $1 < p < \infty$, $(T_\varepsilon)_{\varepsilon>0}$ is of strong type $(p, p)$.
(ii) $(T_\varepsilon)_{\varepsilon>0}$ is of weak type $(1, 1)$.
(iii) For any $f \in L_p(\mathcal{N})$ with $1 \leq p < \infty$,

$$T_{\varepsilon_j} f \xrightarrow{\text{b.a.u.}} Tf \text{ as } j \to +\infty,$$

where $(\varepsilon_j)_{j \in \mathbb{N}}$ is the subsequence that appeared in (1.13). In other words, the principle value $Tf$ exists in the sense of b.a.u. for any $f \in L_p(\mathcal{N})$ with $1 \leq p < \infty$.

Here b.a.u. denotes the noncommutative analogue of the notion of almost everywhere convergence and we refer the reader to Definition 7.1 in the last section for information.

As mentioned previously, even in the commutative case, it is unclear whether Theorem 1.4 (i) holds if the $\gamma$-Lipschitz condition (1.3) is weakened to some integral regularity condition (1.4). However, if the kernel has further homogeneous property, that is, has the form $\Omega(x)/|x|^d$ where $\Omega$ is a homogeneous function defined on $\mathbb{R}^d \setminus \{0\}$ with degree zero

$$\Omega(\lambda x') = \Omega(x') \quad \forall \lambda > 0 \quad \text{and} \quad x' \in S^{d-1} \quad (1.15)$$

and integrable on the sphere $S^{d-1}$ with mean value zero

$$\int_{S^{d-1}} \Omega(x') d\sigma(x') = 0,$$

then any regularity condition is not necessary for all the strong type $(p, p)$ estimates of $(T_{\Omega, \varepsilon})_{\varepsilon>0}$, where

$$T_{\Omega, \varepsilon} f(x) = \int_{|x-y|>\varepsilon} \frac{\Omega(x-y)}{|x-y|^d} f(y) dy. \quad (1.16)$$

Instead, we only need some integrability condition. For $s \geq 0$, we say that $\Omega \in L(\log^+ L)^s(S^{d-1})$ if and only if

$$\int_{S^{d-1}} |\Omega(\theta)||\log(2 + |\Omega(\theta)|)|^s d\theta < \infty,$$

where $d\theta = d\sigma(\theta)$ denotes the sphere measure of $S^{d-1}$. When $s = 1$, we use the convention $L \log^+ L(S^{d-1}) \triangleq L(\log^+ L)^1(S^{d-1})$. It is well-known that the following inclusion relations hold:

$$L_2(S^{d-1}) \subset L(\log^+ L)^2(S^{d-1}) \subset L \log^+ L(S^{d-1}) \subset L_1(S^{d-1}).$$
For the endpoint case $p = 1$, even in the commutative setting, it is still a conjecture that the maximal homogeneous singular integral operator with rough kernel is of weak type $(1, 1)$. Thus at the moment, we are content with getting results by imposing the so-called $L_2$-Dini assumption on $\Omega$

$$\int_0^1 \frac{\omega_2(s)}{s} ds < \infty,$$

(1.17)

where

$$\omega_2(\delta) = \sup_{0<|\alpha|<\delta} \left( \int_{S^{d-1}} |\Omega(\theta) - \Omega(\theta + \alpha)|^2 d\theta \right)^{1/2}.$$

Now we state the third result of this paper.

**Theorem 1.5** Let $T_\Omega$ be a homogeneous singular integral operator with $\Omega$ integrable on $S^{d-1}$ with mean value zero. Then we have the following conclusions.

(i) Let $\Omega \in L^{\log+} L(S^{d-1})$. Then $(T_{\Omega, \varepsilon})_{\varepsilon > 0}$ is of strong type $(p, p)$ with $1 < p < \infty$.

(ii) Let $\Omega \in L_2(S^{d-1})$ satisfy (1.17). Then $(T_{\Omega, \varepsilon})_{\varepsilon > 0}$ is of weak type $(1, 1)$.

(iii) Let $\Omega \in L^{\log+} L(S^{d-1})$. Then for any $f \in L_p(N)$ with $1 < p < \infty$

$$T_{\Omega, \varepsilon} f \xrightarrow{b.a.u.} T_\Omega f \quad \text{as} \quad \varepsilon \to 0.$$

Moreover if $\Omega \in L_2(S^{d-1})$ satisfies (1.17), then the above b.a.u. convergence holds for $f \in L_1(N)$.

Theorems 1.4 and 1.5, as Mei’s and Parcet’s aforementioned results in the first paragraph, together with the transference techniques (see e.g. [20]), have applications in other related topics such as ergodic theory. Indeed, given a trace-preserving automorphic action $\alpha$ of $\mathbb{R}^d$ on $M$, then $\alpha$ extends to an isometric automorphism on $L_p(M)$ for all $0 < p < \infty$. Let $f \in L_p(M)$ with $1 \leq p < \infty$. Let $k$ be a complex-valued measurable function defined on $\mathbb{R}^d \setminus \{0\}$ satisfying the assumptions in Theorem 1.5 (depending on the case $p = 1$ or $p > 1$), the result in [20] implies that the element of the operator $T_k$ induced by $k$ acting on $f \in L_p(M)$ exists as a principle value in the sense of b.a.u. convergence, that is,

$$T_k f = \lim_{\varepsilon \to 0} \int_{|x| > \varepsilon} k(x) \alpha_x f dx, \quad \text{b.a.u.}$$

If $k$ satisfies the assumptions in Theorem 1.4, then the above limit has to be taken along $(\varepsilon_j)_j$. We will take care of all this in a forthcoming paper.

Let us briefly describe the strategy of the proof of Theorem 1.1. As mentioned after Theorem 1.2, the commutative approaches depending heavily on the existence of maximal function and pointwise estimates such as the Cotlar inequality (1.10) or a Calderón–Zygmund argument (see e.g. [15, Page 297]) do not work in the noncommutative setting. Thus we have to provide a new approach. We start with two reductions.
The first one is to reduce a general CZO to two selfadjoint ones (see Lemma 3.1); this reduction is not difficult but essential in our argument in dealing with noncommutative maximal estimates. The second one is to reduce the desired maximal estimate of \((T_\varepsilon)_{\varepsilon>0}\) to that of the lacunary sequence \((T_j)_{j \in \mathbb{Z}}\) (see (3.3)) via the noncommutative Hardy–Littlewood maximal inequalities; this reduction, which should be known to harmonic analysts but seems to be exploited in the present context for the first time, plays a key role in the present noncommutative setting, see Sect. 3.

With these two reductions, we give our main efforts to the weak type \((1,1)\) maximal estimate of lacunary sequence of truncated Calderón–Zygmund operators. In this step, there are two new ingredients. The first one is a new noncommutative Calderón–Zygmund decomposition communicated to us by Cadilhac [2], see Theorem 2.5 and [3, Page 7]. This decomposition, compared to Parcet’s one [39], admits a great advantage that the off-diagonal part of the good function vanishes (see Remark 2.6); recall that in his long paper [39], in order to deal with this part, Parcet had to exploit a pseudo-localization principle which constitutes a major part of that paper (see [1] for a simplified proof of this principle). In our present case for noncommutative maximal estimate, it is not clear at all whether there exists a pseudo-localization principle. The second one is a uniform estimate of the sequence of truncated singular integrals (see Lemma 4.4), which enables us to obtain the maximal estimates.

The rest of the paper is organized as follows. In Sect. 2, we review some facts on the \(\ell_\infty\)-valued noncommutative \(L_p\) spaces as well as the noncommutative Calderón–Zygmund decomposition found by Cadilhac. Sections 3 and 4 are devoted to the proof of Theorem 1.1. In Sect. 5 (resp. Sect. 6), we give the proof of the maximal inequalities stated in Theorem 1.4 (resp. Theorem 1.5). In the last section, we show the noncommutative pointwise convergence results stated in Theorems 1.4 and 1.5.

**Notation** Throughout the paper we write \(X \lesssim Y\) for nonnegative quantities \(X\) and \(Y\) to mean that there is some inessential constant \(C > 0\) such that \(X \leq CY\) and we write \(X \approx Y\) to imply that \(X \lesssim Y\) and \(Y \lesssim X\).

## 2 Preliminaries

### 2.1 Noncommutative \(L_p\)-spaces

Let \(\mathcal{M}\) be a semifinite von Neumann algebra equipped with a \(n.s.f.\) trace \(\tau\). Denote by \(\mathcal{M}_+\) the positive part of \(\mathcal{M}\) and let \(S_{\mathcal{M}_+}\) be the set of all \(x \in \mathcal{M}_+\) whose support projection satisfying \(\tau(\text{supp } x) < \infty\). Let \(S_{\mathcal{M}}\) be the linear span of \(S_{\mathcal{M}_+}\). Then \(S_{\mathcal{M}}\) is a \(w^*\)-dense \(*\)-subalgebra of \(\mathcal{M}\). Let \(0 < p < \infty\). For any \(x \in S_{\mathcal{M}}\), \(|x|^p \in S_{\mathcal{M}}\) and we set

\[
\|x\|_p = \left(\tau(|x|^p)\right)^{1/p}, \quad x \in S_{\mathcal{M}}.
\]

Here \(|x| = (x^*x)^{1/2}\) is the modulus of \(x\). We define the noncommutative \(L_p\)-space associated with \((\mathcal{M}, \tau)\) as the completion of \((S_{\mathcal{M}}, \|\cdot\|_p)\) and it is denoted by \(L_p(\mathcal{M})\).
For convenience, we set $L_\infty(M) = M$ equipped with the operator norm $\| \cdot \|_M$. Let $L_p(M)_+$ denote the positive part of $L_p(M)$.

Suppose that $M \subset B(H)$ acts on a separable Hilbert space $H$. Let $M'$ be the commutant of $M$. A closed densely defined operator on $H$ is said to be affiliated with $M$ when it commutes with every unitary operator $u$ in $M'$. If $x$ is a densely defined selfadjoint operator on $H$ and $x = \int X_\lambda \, d\gamma_\lambda(x)$ is its corresponding spectral decomposition, then the spectral projection $\int T \, d\gamma_\lambda(x)$ will be simply denoted by $\chi_T(x)$, where $T$ is a measurable subset of $\mathbb{R}$. A closed and densely defined operator affiliated with $M$ is called $\tau$-measurable if there exists $\lambda > 0$ such that

$$\tau(\chi(\lambda, \infty)(|x|)) < \infty.$$  

We denote the set of the $\ast$-algebra of $\tau$-measurable operators by $L_0(M)$. For $1 \leq p < \infty$, the weak $L_p$-space $L_{p, \infty}(M)$ is defined as the set of all $x$ in $L_0(M)$ with finite quasi-norm

$$\|x\|_{p, \infty} = \sup_{\lambda > 0} \lambda \tau(\chi(\lambda, \infty)(|x|))^{1/p}.$$  

We refer the reader to [12, 44] for a detailed exposition of noncommutative $L_p$-spaces.

### 2.2 Vector-valued noncommutative $L_p$-spaces

We first recall the column space. Let $(\Sigma, \mu)$ be a measure space. The column space $L_p(M; L^2_\Sigma)$ consists of the operator-valued functions $f$ with finite norm for $p \geq 1$ (quasi-norm for $0 < p < 1$)

$$\|f\|_{L_p(M; L^2_\Sigma)} = \left\| \left( \int_\Sigma \lambda d\gamma_\lambda(\lambda) \right)^{1/2} \right\|_p.$$  

We refer the reader to [44] for precise definition and related properties of the Hilbert valued operator spaces. The most important property for our purpose is the following Hölder type inequality (see e.g. [35, Proposition 1.1]).

**Lemma 2.1** Let $0 < p, q, r \leq \infty$ be such that $1/r = 1/p + 1/q$. Then for any $f \in L_p(M; L^2_\Sigma)$ and $g \in L_q(M; L^2_\Sigma)$

$$\left\| \int_\Sigma f^*(\omega) g(\omega) d\mu(\omega) \right\|_r \leq \left\| \int_\Sigma |f(\omega)|^2 d\mu(\omega) \right\|_p \left\| \int_\Sigma |g(\omega)|^2 d\mu(\omega) \right\|_q^{1/2}.$$  

As we mentioned in the introduction, a fundamental objective of this paper is the $\ell_\infty$-valued noncommutative $L_p$ spaces $L_p(M; \ell_\infty)$ introduced by Pisier [42] and Junge [23]. Given $1 \leq p \leq \infty$, we define $L_p(M; \ell_\infty)$ as the space of all
sequences \( x = (x_k)_{k \geq 1} \) in \( L_p(\mathcal{M}) \) which admits a factorization of the form: there exist \( a, b \in L_{2p}(\mathcal{M}) \) and a bounded sequence \( y = (y_k)_{k \geq 1} \subset L_{\infty}(\mathcal{M}) \) such that

\[
x_k = ay_kb, \quad k \geq 1.
\]

The norm of \( x \) in \( L_p(\mathcal{M}; \ell_{\infty}) \) is defined as

\[
\|x\|_{L_p(\mathcal{M}; \ell_{\infty})} = \inf \left\{ \|a\|_{2p} \sup_k \|y_k\|_{\infty} \|b\|_{2p} \right\},
\]

where the infimum is taken over all factorizations of \( x \) as above. It is easy to verify that \( L_p(\mathcal{M}; \ell_{\infty}) \) is a Banach space equipped with the norm \( \| \cdot \|_{L_p(\mathcal{M}; \ell_{\infty})} \). As usual, the norm of \( x \) in \( L_p(\mathcal{M}; \ell_{\infty}) \) is conventionally denoted by \( \|x\|_{L_p(\mathcal{M}; \ell_{\infty})} \). However, we should point out that \( \sup_{k \geq 1}^+ x_k \) is just a notation since \( \sup_{k \geq 1}^+ x_k \) does not make any sense in the noncommutative setting. We just use this notation for convenience. More generally, for any index set \( I \), the space \( L_p(\mathcal{M}; \ell_{\infty}(I)) \) can be defined similarly.

**Remark 2.2** Let \( x = (x_k)_{k \in I} \) be a sequence of selfadjoint operators in \( L_p(\mathcal{M}) \). Then \( x \in L_p(\mathcal{M}; \ell_{\infty}) \) iff there is a positive operator \( a \in L_p(\mathcal{M}) \) such that \(-a \leq x_k \leq a\) for all \( k \in I \), and moreover,

\[
\|\sup_{k \in I}^+ x_k\|_p = \inf \left\{ \|a\|_p : a \in L_p(\mathcal{M})_+, \, -a \leq x_k \leq a, \, \forall \, k \in I \right\}.
\]

Besides the strong maximal norm which corresponds to the \( L_p \)-norm of a maximal function, we now turn to the weak maximal norm (see e.g. [18]) which corresponds to the weak \( L_p \) norm of a maximal function. Let \( I \) be an index set. Given a family \( (x_k)_{k \in I} \) in \( L_p(\mathcal{M}) \) with \( 1 \leq p < \infty \), we define

\[
\|(x_k)_{k \in I}\|_{\Lambda_{p,\infty}(\mathcal{M}; \ell_{\infty}(I))} = \sup_{\lambda > 0} \lambda \inf_{e \in \mathcal{P}(\mathcal{M})} \left\{ \left( \tau'(e^{-1}) \right)^{\frac{1}{p}} : \|ex_k e\|_{\infty} \leq \lambda \text{ for all } k \in I \right\},
\]

(2.1)

where \( \mathcal{P}(\mathcal{M}) \) is the set of all projections in \( \mathcal{M} \). Finally, we set the quasi-Banach space \( \Lambda_{p,\infty}(\mathcal{M}; \ell_{\infty}(I)) \) to be the set of all sequences \( x = (x_k)_{k \in I} \in L_{p,\infty}(\mathcal{M}) \) such that its \( \Lambda_{p,\infty}(\mathcal{M}; \ell_{\infty}(I)) \) quasi-norm is finite. We will omit the index set \( I \) when it will not cause confusions.

**Remark 2.3** Let \( x = (x_k)_k \) be a sequence of selfadjoint operators in \( L_p(\mathcal{M}) \). As in Remark 2.2, there is a similar characterization of \( \Lambda_{p,\infty}(\mathcal{M}; \ell_{\infty}) \) quasi-norm for a sequence of selfadjoint operators \( x = (x_k)_k \),

\[
\|(x_k)_k\|_{\Lambda_{p,\infty}(\mathcal{M}; \ell_{\infty})} = \sup_{\lambda > 0} \lambda \inf_{e \in \mathcal{P}(\mathcal{M})} \left\{ \left( \tau'(e^{-1}) \right)^{\frac{1}{p}} : -\lambda \leq ex_k e \leq \lambda \text{ for all } k \right\}.
\]

(2.2)
Indeed, this follows from
\[-\lambda \leq e_k e \leq \lambda \iff |e_k e| \leq \lambda \iff \|e_k e\|_{\infty} \leq \lambda\] (2.3)
for any \(k\).

**Remark 2.4** The equivalence relationships in (2.3) would not be true in general if \(\lambda\) is replaced by a positive operator. More precisely, considering a positive operator \(g\) and a selfadjoint operator \(f\) such that \(-g \leq f \leq g\), it is not true that \(|f| \leq g\) in general. For instance, consider
\[f = \begin{pmatrix} 4 & 0 \\ 0 & -4 \end{pmatrix}, \quad g = \begin{pmatrix} 5 & 3 \\ 3 & 5 \end{pmatrix};\]
then it is easy to see that \(-g \leq f \leq g\), while \(g - |f|\) is not positive.

### 2.3 Noncommutative Calderón–Zygmund decomposition

The noncommutative Calderón–Zygmund decomposition is based on Cuculescu’s construction for the standard dyadic martingales. Let us recall briefly the related notions. Given an integer \(k \in \mathbb{Z}\), \(Q_k\) stands for the set of dyadic cubes of side length \(2^{-k}\), \(\sigma_k\) be the \(\sigma\)-algebra generated by \(Q_k\) and \(N_k = L_\infty(\mathbb{R}^d, \sigma_k, dx) \otimes \mathcal{M}\) be the associated von Neumann subalgebra of \(\mathcal{N}\), where \(\mathcal{N} = L_\infty(\mathbb{R}^d) \otimes \mathcal{M}\) was given in the introduction. Then it is well-known that \((N_k)_{k \in \mathbb{Z}}\) is a sequence of increasing von Neumann subalgebras such that the union is weak* dense in \(\mathcal{N}\), and thus forms a filtration with the associated conditional expectations \((E_k)_{k \in \mathbb{Z}}\) defined as
\[E_k(f) = \sum_{Q \in Q_k} f_Q \chi_Q, \forall f \in L_1(\mathcal{N})\]
where \(\chi_Q\) is the characteristic function of \(Q\) and \(f_Q\) denotes the mean of \(f\) over \(Q\)
\[f_Q = \frac{1}{|Q|} \int_Q f(y) \, dy.\]

Here \(|Q|\) denotes the volume of \(Q\).

Let \(1 \leq p \leq \infty\). A sequence \((f_k)_{k \in \mathbb{Z}} \subset L_p(\mathcal{N})\) will be called a \(L_p\)-martingale if it satisfies \(E_{k-1}(f_k) = f_{k-1}\); in this case, the martingale difference is defined as 
\[d f_k = f_k - f_{k-1}.\]

For further convenience, we make some conventions or definitions. Denote by \(Q\) the set of all standard dyadic cubes in \(\mathbb{R}^d\). The notation \(dist(x, Q)\) means the distance between \(x\) and \(Q\). For all \(x \in \mathbb{R}^d\), denote by \(Q_{x,k}\) the unique cube in \(Q_k\) containing \(x\) and \(c_{x,k}\) denotes its centre. Let \(Q \in Q\) and \(i\) be any odd positive integer, \(iQ\) stand for the cube with the same center as \(Q\) such that \(\ell(iQ) = i \ell(Q)\), where \(\ell(Q)\) denotes the side length of \(Q\).
As in [39], the noncommutative Calderón–Zygmund decomposition will be constructed for functions in the class

\[ \mathcal{N}_{c,+} = \left\{ f : \mathbb{R}^d \to \mathcal{M} \cap L_1(\mathcal{M}) \mid f \geq 0, \text{supp} f \text{ is compact} \right\}, \]

which is dense in \( L_1(\mathcal{N})_+ \). Recall that \( \text{supp} \) means the support of \( f \) as an operator-valued function in \( \mathbb{R}^d \), which is different from its support projection as an element of a von Neumann algebra.

Cuculescu’s construction [10]. Let \( f \in \mathcal{N}_{c,+} \) and \( \lambda > 0 \). Denote \( f_k = E_k(f) \). It was shown in [39, Lemma 3.1] that there exists \( m_\lambda(f) \in \mathbb{Z} \) such that \( f_k \leq \lambda 1_\mathcal{N} \) for all \( k \leq m_\lambda(f) \). Now by adopting Cuculescu’s construction [10] to \( (f_k)_{k \in \mathbb{Z}} \) relative to the dyadic filtration \( (\mathcal{N}_k)_{k \in \mathbb{Z}} \), one can find a sequence of decreasing projections \( (q_k)_{k \in \mathbb{Z}} \) defined recursively by \( q_k = 1_\mathcal{N} \) for \( k \leq m_\lambda(f) \) and for \( k > m_\lambda(f) \)

\[ q_k = q_k(f, \lambda) = \chi(0, \lambda](q_{k-1} f_k q_{k-1}) \]

such that

- \( q_k \) commutes with \( q_{k-1} f_k q_{k-1} \);
- \( q_k \) belongs to \( \mathcal{N}_k \) and \( q_k f_k q_k \leq \lambda q_k \);
- the following estimate holds

\[ \varphi \left( 1_\mathcal{N} - \bigwedge_{k \in \mathbb{Z}} q_k \right) \leq \frac{\|f\|_1}{\lambda}. \]

In the present semi-commutative setting, \( q_k \) admits the following expression

\[ q_k = \sum_{Q \in \mathcal{Q}_k} q_Q \chi_Q, \]

where \( q_Q \) is a projection in \( \mathcal{M} \) with

\[ q_Q = \begin{cases} 1_\mathcal{M} & \text{if } k \leq m_\lambda(f), \\ \chi(0, \lambda](q_{\widehat{Q}} f_{\widehat{Q}} q_{\widehat{Q}}) & \text{if } k > m_\lambda(f). \end{cases} \]

Here \( \widehat{Q} \) is the dyadic father of \( Q \). Accordingly these projections satisfy

\[ q_Q \leq q_{\widehat{Q}}, \quad q_Q \text{ commutes with } q_{\widehat{Q}} f_{\widehat{Q}} q_{\widehat{Q}}, \quad q_Q f_{\widehat{Q}} q_Q \leq \lambda q_Q. \quad (2.4) \]

Then one can define the sequence \( (p_k)_{k \in \mathbb{Z}} \) of disjoint projections by \( p_k = q_{k-1} - q_k \) such that

\[ \sum_{k \in \mathbb{Z}} p_k = 1_\mathcal{N} - q = q^\perp \quad \text{with} \quad q = \bigwedge_{k \in \mathbb{Z}} q_k. \]
One can then express the projections $p_k$ as

$$p_k = \sum_{Q \in Q_k} (q_Q - q_Q^\Delta) \chi_Q \triangleq \sum_{Q \in Q_k} p_Q \chi_Q. \quad (2.5)$$

The following version of noncommutative Calderón–Zygmund decomposition was communicated to us by Cadilhac [2] (see also [3, Page 3–7]).

**Theorem 2.5** Fix $f \in \mathcal{N}_{c,+}$, $\lambda > 0$ and $s \in \mathbb{N}$. Let $(q_k)_{k \in \mathbb{Z}}$ and $(p_k)_{k \in \mathbb{Z}}$ be the two sequences of projections appeared in the above Cuculescu’s construction. Then there exist a projection $\zeta \in \mathcal{N}$ defined by

$$\zeta = \left( \bigvee_{Q \in \mathcal{Q}} p_Q \chi_{(2s+1)Q} \right), \quad (2.6)$$

and a decomposition of $f$,

$$f = g + b = g + b_d + b_{\text{off}} \quad (2.7)$$

such that the following assertions hold.

(i) $\varphi(1_{\mathcal{N}} - \zeta) \leq (2s + 1)^d \frac{\|f\|_1}{\lambda}$, where $1_{\mathcal{N}}$ stands for the unit element in $\mathcal{N}$.

(ii) $g$ is positive and $b$ is selfadjoint in $\mathcal{N}$; moreover, $\|g\|_1 \leq \|f\|_1$ and $\|g\|_\infty \leq 2^d \lambda$.

(iii) $b_d = \sum_{n \in \mathbb{Z}} b_{d,n}$, where

$$b_{d,n} = p_n(f - f_n)p_n. \quad (2.8)$$

Each $b_{d,n}$ satisfies the cancellation conditions: for $Q \in \mathcal{Q}_n$, $\int_Q b_{d,n} = 0$; and for all $x, y \in \mathbb{R}^d$ such that $y \in (2s + 1)Q_{x,n}$, $\zeta(x)b_{d,n}(y)\zeta(x) = 0$. Furthermore, $\sum_{n \in \mathbb{Z}} \|b_{d,n}\|_1 \leq 2 \|f\|_1$.

(iv) $b_{\text{off}} = \sum_{n \in \mathbb{Z}} b_n$, where

$$b_n = p_n(f - f_n)q_n + q_n(f - f_n)p_n. \quad (2.9)$$

Each $b_n$ satisfies: for $Q \in \mathcal{Q}_n$, $\int_Q b_n = 0$; and for all $x, y \in \mathbb{R}^d$ such that $y \in (2s + 1)Q_{x,n}$, $\zeta(x)b_n(y)\zeta(x) = 0$.

**Remark 2.6** The details of the proof of Theorem 2.5 can be found in [3] and Theorem 2.5 is one of the main results there. The new input in Cadilhac’s decomposition (2.7) is to replace the maximum $i \lor j = \max(i, j)$ in Parcet’s decomposition (see [39, (3.3)]) by the minimum $i \land j = \min(i, j)$; one can then check easily that $g_{\text{off}}$ vanishes.

**Remark 2.7** To simplify our calculations in this paper (see e.g. (4.15)), we will fix $s = 4[\sqrt{d}]$ in the rest of the paper, where $[l]$ denotes the integer part of $l$. 

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3 Proof of Theorem 1.1: two reductions

To prove Theorem 1.1, we give the two reductions in the present section.

3.1 Reduction to real Calderón–Zygmund kernel

Given a CZO $T$ and its truncated ones $T_{\varepsilon}$ with kernel $k$, the real and imaginary parts of the kernel are denoted by $\text{Re}(k)$ and $\text{Im}(k)$ respectively; accordingly, the corresponding operators are denoted by

$$
\text{Re}(T_{\varepsilon})f(x) = \int_{|x-y|>\varepsilon} \text{Re}(k)(x, y) f(y) dy
$$

and

$$
\text{Im}(T_{\varepsilon})f(x) = \int_{|x-y|>\varepsilon} \text{Im}(k)(x, y) f(y) dy.
$$

Due to the following observation, we are able to assume that all the kernels are real, which will play an essential role in establishing the noncommutative maximal estimates.

Lemma 3.1 Let $T$ be a CZO with associated kernel $k$ satisfying (1.2) and (1.4) with $q = 2$ and let $T_{\varepsilon}$ be defined as (1.7). Then we have following basic facts.

(i) Both $\text{Re}(k)$ and $\text{Im}(k)$ satisfy (1.2) and (1.4) with $q = 2$.

(ii) If $(T_{\varepsilon})_{\varepsilon>0}$ is of strong type $(p_0, p_0)$ for some $p_0 \in (1, \infty)$, then both $(\text{Re}(T_{\varepsilon}))_{\varepsilon>0}$ and $(\text{Im}(T_{\varepsilon}))_{\varepsilon>0}$ are of strong type $(p_0, p_0)$.

(iii) If $(\text{Re}(T_{\varepsilon}) f)_{\varepsilon>0}$ and $(\text{Im}(T_{\varepsilon}) f)_{\varepsilon>0}$ are of weak type $(1, 1)$, so is $(T_{\varepsilon} f)_{\varepsilon>0}$.

Proof

(i) Note that $\max \{|\text{Re}(k)|, |\text{Im}(k)|\} \leq |k|$, it is easy to verify that $\text{Re}(k)$ and $\text{Im}(k)$ satisfy the conditions (1.2) and (1.4) with $q = 2$.

(ii) We only consider $(\text{Re}(T_{\varepsilon}))_{\varepsilon>0}$ since the argument for $(\text{Im}(T_{\varepsilon}))_{\varepsilon>0}$ is similar. Let $f \in L_{p_0}(\mathcal{N})$. To estimate $L_{p_0}(\mathcal{N}; \ell_\infty)$-norm of $(\text{Re}(T_{\varepsilon}) f)_{\varepsilon>0}$, by the triangle inequalities and the facts that $L_{p_0}$-norm of $\text{Re}(f)$ and $\text{Im}(f)$ are dominated by $\|f\|_{p_0}$, it suffices to assume $f = f^*$. Then the claim follows from $\text{Re}(T_{\varepsilon}) f = \frac{1}{\varepsilon} (T_{\varepsilon} f + (T_{\varepsilon} f)^*)$ for any $\varepsilon > 0$ and the fact that the adjoint map is an isometric isomorphism on $L_{p_0}(\mathcal{N}; \ell_\infty)$.

(iii) This assertion follows from $T_{\varepsilon} f = \text{Re}(T_{\varepsilon}) f + i \text{Im}(T_{\varepsilon}) f$ for any $\varepsilon$ and the quasi-norm of $\Lambda_{1, \infty}(\mathcal{N}; \ell_\infty)$.

3.2 Reduction to the maximal estimates of lacunary sequence

In this subsection, we reduce the study of $(T_{\varepsilon})_{\varepsilon>0}$ to its lacunary sequence.

Let $\phi$ be a smooth radial nonnegative function on $\mathbb{R}^d$ such that $\text{supp } \phi \subset \{x \in \mathbb{R}^d : 1/2 \leq |x| \leq 2\}$ and $\sum_{i \in \mathbb{Z}} \phi_i(x) = 1$ for all $x \in \mathbb{R}^d \setminus \{0\}$, where $\phi_i(x) = \phi(2^i x / \sqrt{d})$. 

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This $\phi$ will be fixed in the whole paper. Consequently, for a reasonable $f$,

$$T_\varepsilon f(x) = \sum_{i \in \mathbb{Z}} \int_{|x-y| > \varepsilon} k(x, y) \phi_i(x - y) f(y) dy.$$ 

For convenience, let $k_i(x, y)$ denote the kernel $k(x, y) \phi_i(x - y)$ and set $\Delta_i = \{ x \in \mathbb{R}^d : 2^{-i-1} \sqrt{d} \leq |x| \leq 2^{-i+1} \sqrt{d} \}$. Then supp $\phi_i \subset \Delta_i$. Note that

$$\int_{|x-y| > \varepsilon} k_i(x, y) f(y) dy$$

vanishes if the intersection of $x + \Delta_i$ and $\{ y \in \mathbb{R}^d : |x-y| > \varepsilon \}$ is empty. This implies $2^{-i+1} \sqrt{d} > \varepsilon$ and thus $i < \log_2 \left( \frac{2 \sqrt{d}}{\varepsilon} \right)$ by a simple calculation. Set $j_\varepsilon \triangleq \lceil \log_2 \left( \frac{2 \sqrt{d}}{\varepsilon} \right) \rceil$. With these conventions and observations, we may write $T_\varepsilon f(x)$ as

$$T_\varepsilon f(x) = \sum_{i: i \leq j_\varepsilon - 1} \int_{\mathbb{R}^d} k_i(x, y) f(y) dy + \int_{|x-y| \geq \varepsilon} k_{j_\varepsilon}(x, y) f(y) dy. \quad (3.1)$$

For convenience, let us write the second term above as $T_{\varepsilon, j_\varepsilon} f(x)$.

**Proposition 3.2** Let $T$ be a CZO with kernel $k$ satisfying the assumptions in Theorem 1.1. Then

(i) $(T_{\varepsilon, j_\varepsilon})_{\varepsilon > 0}$ is of weak type $(1, 1)$. More precisely, for any $\lambda > 0$ and $f \in L_1(N)$, there exists a projection $\eta \in N$ such that

$$\sup_{\varepsilon > 0} \| \eta(T_{\varepsilon, j_\varepsilon} f) \|_\infty \lesssim \lambda \quad \text{and} \quad \varphi(\eta^\perp) \lesssim \frac{\| f \|_1}{\lambda};$$

(ii) $(T_{\varepsilon, j_\varepsilon})_{\varepsilon > 0}$ is of strong type $(p, p)$ for all $1 < p \leq \infty$.

**Proof** (i) By the quasi-triangle inequality, we can assume that $f$ is positive; by Lemma 3.1, $k$ can be assumed to be real. Applying the size condition (1.2) of the kernel $k$ as well as the support of $\phi_{j_\varepsilon}$, we find that for any $\varepsilon > 0$

$$T_{\varepsilon, j_\varepsilon} f(x) \leq \int_{|x-y| > \varepsilon} |k_{j_\varepsilon}(x, y)| f(y) dy \leq \int_{2^{-j_\varepsilon-1} \sqrt{d} \leq |x-y| \leq 2^{-j_\varepsilon+1} \sqrt{d}} |k_{j_\varepsilon}(x, y)| f(y) dy$$

$$\leq \int_{2^{-j_\varepsilon-1} \sqrt{d} \leq |x-y| \leq 2^{-j_\varepsilon+1} \sqrt{d}} \phi_{j_\varepsilon}(x-y) \frac{\phi_{j_\varepsilon}(x-y)}{|x-y|^d} f(y) dy \lesssim M_{2^{-j_\varepsilon+1} \sqrt{d}} f(x),$$

where $M_r$ is the Hardy–Littlewood averaging operator. On the other hand, note that

$$T_{\varepsilon, j_\varepsilon} f(x) \geq - \int_{|x-y| > \varepsilon} |k_{j_\varepsilon}(x, y)| f(y) dy.$$
thus one gets $-M_{2^{-j_0+1}} \sqrt{d} f(x) \lesssim T_{\varepsilon, j_0} f(x)$. Hence,

$$-M_{2^{-j_0+1}} \sqrt{d} f(x) \lesssim T_{\varepsilon, j_0} f(x) \lesssim M_{2^{-j_0+1}} \sqrt{d} f(x).$$

(3.2)

We now appeal to Mei’s noncommutative Hardy–Littlewood maximal weak type $(1, 1)$ inequality [35]: there exists a projection $\eta \in \mathcal{N}$ such that for any $\varepsilon > 0$

$$\varphi(\eta) \lesssim \frac{\|f\|_1}{\lambda}$$

and $\eta M_{2^{-j_0+1}} \sqrt{d} f \eta \leq \lambda$.

We then deduce that for any $\varepsilon > 0$,

$$-\lambda \leq -\eta M_{2^{-j_0+1}} \sqrt{d} f \eta \lesssim \eta T_{\varepsilon, j_0} f \eta \lesssim \eta M_{2^{-j_0+1}} \sqrt{d} f \eta \leq \lambda.$$

This, by Remark 2.3, gives the desired weak type $(1, 1)$ maximal estimate of $(T_{\varepsilon, j_0})_{\varepsilon > 0}$.

(ii) By noting Remark 2.2, the strong type $(p, p)$ estimate of $(T_{\varepsilon, j_0})_{\varepsilon > 0}$ is a consequence of (3.2) and Mei’s noncommutative Hardy–Littlewood maximal strong type $(p, p)$ $(1 < p \leq \infty)$ inequality [35].

By the quasi-triangle inequality in $A_{1, \infty}(\mathcal{N}, \ell_\infty)$ and Proposition 3.2, to establish Theorem 1.1, it suffices to prove the desired maximal weak type $(1, 1)$ result of the first term on the right-hand side of (3.1). For convenience, denote the operator $T_j$ by

$$T_j f(x) = \sum_{i: i < j} \int_{\mathbb{R}^d} k_i(x, y) f(y) dy.$$

(3.3)

**Theorem 3.3** Let $T$ be a CZO with kernel $k$ satisfying the assumptions in Theorem 1.1 and let $T_j$ be defined as (3.3). Then the sequence of linear operators $(T_j)_{j \in \mathbb{Z}}$ is of weak type $(1, 1)$. More precisely, for any $f \in L_1(\mathcal{N})$ and $\lambda > 0$, there exists a projection $e \in \mathcal{N}$ such that

$$\sup_{j \in \mathbb{Z}} \|e(T_j f)e\|_{\infty} \lesssim \lambda \quad \text{and} \quad \varphi(e) \lesssim \frac{\|f\|_1}{\lambda}.$$

4 **Proof of Theorem 1.1: maximal estimate of lacunary sequence $(T_j)_{j}$**

In this section, we will complete the proof of Theorem 1.1 by showing Theorem 3.3.

According to Lemma 3.1, we suppose that the kernel $k$ is real. By decomposing any element into linear combination of four positive elements and recalling that $\mathcal{N}_{r^+}$ is dense in $L_1(\mathcal{N})_+$ and by the standard density argument, it suffices to show the maximal weak type $(1, 1)$ estimate for $f \in \mathcal{N}_{r^+}$. Now fix $f \in \mathcal{N}_{r^+}$ and $\lambda \in (0, +\infty)$. By translation, we may assume for simplicity $m_\lambda(f) = 0$ (see e.g. [39, Remark 3.3]).
By the noncommutative Calderón–Zygmund decomposition in Theorem 2.5, we can decompose \( f = g + b \). Thus it suffices to find a projection \( e \in \mathcal{N} \) such that

\[
\forall j \in \mathbb{Z}, \quad \max \left\{ \| e T_j g e \|_\infty, \| e T_j b e \|_\infty \right\} \leq \lambda \quad \text{and} \quad \varphi(e_\perp) \lesssim \frac{\| f \|_1}{\lambda},
\]

which will follow from, by setting \( e = e_1 \land e_2 \), the existence of two projections \( e_1 \) and \( e_2 \) such that

\[
\begin{align*}
\sup_{j \in \mathbb{Z}} \| e_1 T_j g e_1 \|_\infty &\leq \lambda \quad \text{and} \quad \varphi(e_1^\perp) \lesssim \frac{\| f \|_1}{\lambda}, \\
\sup_{j \in \mathbb{Z}} \| e_2 T_j b e_2 \|_\infty &\leq \lambda \quad \text{and} \quad \varphi(e_2^\perp) \lesssim \frac{\| f \|_1}{\lambda}.
\end{align*}
\]

### 4.1 Estimate for the good function \( g \): (4.1)

The following lemma will be used in estimating \((T_j g)_{j \in \mathbb{Z}}\).

**Lemma 4.1** The sequence of operators \((T_j)_{j \in \mathbb{Z}}\) is of strong type \((p_0, p_0)\).

**Proof** From (3.1), for any \( j \in \mathbb{Z} \), there exists one \( \varepsilon = \varepsilon_j \) such that \( j = j_\varepsilon \) and

\[
T_j = T_{j_\varepsilon} - T_{j_\varepsilon, j_\varepsilon}.
\]

Note that the strong type \((p_0, p_0)\) of \((T_{j_\varepsilon})_{j_\varepsilon \in \mathbb{Z}}\) (resp. \((T_{j_\varepsilon, j_\varepsilon})_{j_\varepsilon \in \mathbb{Z}}\)) follows from the corresponding assumption (resp. conclusion (ii)) in Theorem 1.1 (resp. Proposition 3.2). Thus by the triangle inequality, we finish the proof.

Now we are ready to prove estimate (4.1).

**Proof of estimate (4.1).** The desired maximal weak type \((1, 1)\) estimate for the diagonal part of the good function can be deduced from conclusion (ii) in Theorem 2.5 and Lemma 4.1. Indeed, since \( g \) is positive stated in conclusion (ii) in Theorem 2.5 and \( k \) is real-valued, \( T_j g \) is selfadjoint. Applying Lemma 4.1 and Remark 2.2, we can find a positive operator \( a \in L_{p_0}(\mathcal{N}) \) such that for any \( j \in \mathbb{Z} \),

\[
-a \leq T_j g \leq a \quad \text{and} \quad \| a \|_{p_0} \lesssim \| g \|_{p_0}.
\]

Set \( e_1 = \chi_{(0, \lambda]}(a) \). Then for any \( j \in \mathbb{Z} \),

\[
-\lambda \leq -e_1 a e_1 \leq e_1 T_j g e_1 \leq e_1 a e_1 \leq \lambda.
\]

On the other hand, by the Chebyshev and Hölder inequalities,

\[
\varphi(e_1^\perp) = \varphi(\chi_{(\lambda, \infty)}(a)) \leq \frac{\| a \|_{p_0}}{\lambda^{p_0}} \lesssim \frac{\| g \|_{p_0}}{\lambda^{p_0}} \leq \frac{\| g \|_1 \| g \|_{p_0}^{-1}}{\lambda^{p_0}} \lesssim \frac{\| f \|_1}{\lambda},
\]

where in the last inequality we used conclusion (ii) stated in Theorem 2.5. This, by Remark 2.3, provides the desired estimate for \((T_j g)_{j \in \mathbb{Z}}\). \( \square \)
4.2 Estimate for the bad function $b$: (4.2)

Before giving the proof of (4.2), we first introduce two lemmas. For $i, n \in \mathbb{Z}$ and $x, y \in \mathbb{R}^d$, we define

$$k_{i,n}(x,y) = (k_i(x,y) - k_i(x, cy_n)). \quad (4.3)$$

Lemma 4.2 For $i < n - 1$ and $1 \leq q < \infty$, the following estimate holds

$$\sup_{y \in \mathbb{R}^d} \int_{\mathbb{R}^d} |k_{i,n}(x,y)|^q dx \lesssim 2^{id(q-1)}(\delta_q^q(n-i) + \delta_q^q(n-i+1) + 2^q(i-n)), \quad (4.4)$$

where $\delta_q$ was defined in (1.5).

Proof To see this, note that

$$\int_{\mathbb{R}^d} |k_{i,n}(x,y)|^q dx \lesssim \int_{\mathbb{R}^d} |k(x,y) - k(x, cy_n)|^q |\phi_i(x-y)|^q dx + \int_{\mathbb{R}^d} |k(x, cy_n)|^q |\phi_i(x-y) - \phi_i(x- cy_n)|^q dx.$$

By applying the mean value theorem, we get

$$\int_{\mathbb{R}^d} |k(x, cy_n)|^q |\phi_i(x-y) - \phi_i(x- cy_n)|^q dx \lesssim \int_{A_{i,y,n}} |k(x, cy_n)|^q 2^{id} |y- cy_n|^q dx, \quad (4.5)$$

where $A_{i,y,n}$ denotes the support set of $\phi_i(x-y) - \phi_i(x- cy_n)$. In the following we show that there exist two constants $C_{1,d}$ and $C_{2,d}$ such that

$$A_{i,y,n} \subset \{ x \in \mathbb{R}^d : C_{1,d} 2^{-i} \leq |x- cy_n| \leq C_{2,d} 2^{-i} \}. \quad (4.6)$$

Indeed, fixing $y \in \mathbb{R}^d$, it suffices to consider those $x$ such that $\phi_i(x-y) - \phi_i(x- cy_n) \neq 0$. By the support of $\phi_i$, at least one of the following conditions hold

1. $2^{-i-1} \sqrt{d} \leq |x-y| \leq 2^{-i+1} \sqrt{d}$;
2. $2^{-i-1} \sqrt{d} \leq |x- cy_n| \leq 2^{-i+1} \sqrt{d}$.

If (2) holds, then we get $|x- cy_n| \approx 2^{-i}$. If (1) holds, then by the definition of $c_{y,n}$, we see that $|y- cy_n| \leq 2^{-n-1} \sqrt{d}$. Moreover, since $i < n - 1$,

$$|y- cy_n| \leq 2^{-i-2} \sqrt{d} < \frac{1}{2} |x-y|.$$
Thus we deduce that
\[ |x - c_{y,n}| \leq |x - y| + |y - c_{y,n}| < \frac{3}{2} |x - y| < 2^{-i} \cdot 3\sqrt{d}. \]

On the other hand,
\[ |x - c_{y,n}| \geq |x - y| - |y - c_{y,n}| > \frac{1}{2} |x - y| > 2^{-i-2} \sqrt{d}. \]

Therefore, in this case we still have \( |x - c_{y,n}| \approx 2^{-i} \), where the constant only depends on the dimension \( d \). Thus we obtain (4.6).

Finally, by the smoothness condition (1.5) and size condition (1.2) of the kernel \( k \), the support of \( \phi_i \) and (4.6), we find
\[
\int_{\mathbb{R}^d} |k_{i,n}(x, y)|^q dx \lesssim \int_{2^{-i+n+1} \leq \sqrt{d} \leq 2^{-i+n+2} \leq \sqrt{d}} |k(x, y) - k(x, c_{y,n})|^q dx
+ \int_{2^{-i+n+1} \leq \sqrt{d} \leq 2^{-i+n+2} \leq \sqrt{d}} |k(x, y) - k(x, c_{y,n})|^q dx
+ \int_{A_{i,y,n}} |k(x, c_{y,n})|^q 2^q |y - c_{y,n}|^q dx
\lesssim 2^{d(q-1)}(\delta_2^q (n-i) + \delta_2^q (n-i+1) + 2^{q(i-n)}),
\]
since \( |y - c_{y,n}| \leq 2^{-n-1} \sqrt{d} \). This gives the desired estimate. \( \square \)

**Lemma 4.3** Let \( h \in N_{c,+} \). Then for \( i < n - 1 \), the following estimates hold for any \( Q \in Q_n \),
\[
\int_{\mathbb{R}^d} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1(M)}^{\frac{1}{2}} dx
\lesssim \left( (\delta_2^q (n-i) + \delta_2^q (n-i+1) + 2^{q(i-n)})^2 \tau(p_Q) \varphi(h_Q \chi_Q) \right)^{\frac{1}{2}}. \quad (4.7)
\]

**Proof** Fix \( i < n - 1 \) and \( Q \in Q_n \). Let \( x \in \mathbb{R}^d \). By the same argument as in (4.6), we have
\[
\int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy = \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \chi_{B_Q}(x),
\]
where \( B_Q \subset \{ x : C_1 d \cdot 2^{-i} \leq dist(x, Q) \leq C_2 d \cdot 2^{-i} \} \) for some constants \( C_1,d \) and \( C_2,d \).
Now we apply the H"older and Cauchy–Schwarz inequalities to get
\[
\int_{\mathbb{R}^d} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1_2(M)}^\frac{1}{2} \, dx \\
\leq \int_{\mathbb{R}^d} \left\| p_Q \chi_B(x) \right\|_{L^1_2(M)} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1_2(M)} \, dx \\
\leq \left( \int_{\mathbb{R}^d} \left\| p_Q \chi_B(x) \right\|_{L^1_2(M)} \, dx \right)^\frac{1}{2} \left( \int_{\mathbb{R}^d} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1_2(M)} \, dx \right)^\frac{1}{2}.
\]
Since $i < n - 1$ and $\ell(Q) = 2^{-n}$, it is easy to verify that
\[
\left( \int_{\mathbb{R}^d} \left\| p_Q \chi_B(x) \right\|_{L^1_2(M)} \, dx \right)^\frac{1}{2} \lesssim 2^{-\frac{id}{2}} (\tau(p_Q))^\frac{1}{2}. \tag{4.8}
\]
We then consider the other term. The Fubini theorem implies
\[
\int_{\mathbb{R}^d} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1_2(M)} \, dx \\
= \tau \int_Q \left( \int_{\mathbb{R}^d} |k_{i,n}(x, y)|^2 \, dx \right) p_Q h(y) p_Q dy.
\]
By Lemma 4.2, this gives rise to
\[
\int_{\mathbb{R}^d} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1_2(M)} \, dx \\
\lesssim 2^{id} (\delta_2^2 (n - i) + \delta_2^2 (n - i + 1) + 2^{2(i-n)}) \varphi(hp_Q \chi_Q). \tag{4.9}
\]
Finally, exploiting (4.8) and (4.9), we find
\[
\int_{\mathbb{R}^d} \left\| \int_Q |k_{i,n}(x, y)|^2 p_Q h(y) p_Q dy \right\|_{L^1_2(M)} \, dx \\
\lesssim \left( \tau(p_Q) \right)^\frac{1}{2} \left( (\delta_2^2 (n - i) + \delta_2^2 (n - i + 1) + 2^{2(i-n)}) \varphi(hp_Q \chi_Q) \right)^\frac{1}{2} \\
= \left( (\delta_2^2 (n - i) + \delta_2^2 (n - i + 1) + 2^{2(i-n)}) \tau(p_Q) \varphi(hp_Q \chi_Q) \right)^\frac{1}{2}.
\]
This is the desired estimate (4.7). \qed

Let us start to prove (4.2). Using the projection $\zeta$ constructed in (2.6), we decompose $T_j b$ in the following way
\[ T_j b = \zeta^\perp T_j b \zeta^\perp + \zeta T_j b \zeta^\perp + \zeta^\perp T_j b \zeta + \zeta T_j b \zeta. \]
Then taking \( e_{2,1} = \zeta \), we are reduced to finding a projection \( e_{2,2} \) such that

\[
\sup_{j \in \mathbb{Z}} \| e_{2,2} \xi T_j b \xi e_{2,2} \|_{\infty} \leq \lambda \quad \text{and} \quad \varphi(e_{2,2}^\perp) \lesssim \frac{\| f \|_1}{\lambda}.
\]  

(4.10)

Indeed, taking \( e_2 = e_{1,2} \wedge e_{2,2} \), then we obtain for any \( j \in \mathbb{Z} \)

\[
\| e_2 T_j b e_2 \|_{\infty} \leq \| e_{2,2} \xi T_j b \xi e_{2,2} \|_{\infty} \leq \lambda.
\]

Together with (4.10), conclusion (i) in Theorem 2.5 and recalling that \( s = 4[\sqrt{d}] \) announced in Remark 2.7, we get

\[
\varphi(e_2^\perp) \lesssim \| f \|_1.
\]

Thus we get estimate (4.2).

For each \( i \in \mathbb{Z} \), define the operator \( S_i \) as

\[
S_i h(x) = \int_{\mathbb{R}^d} k_i(x, y) h(y) dy,
\]  

(4.11)

then \( T_j = \sum_{i: i < j} S_i \).

**Lemma 4.4** The following estimate holds:

\[
\sum_{i \in \mathbb{Z}} \| \zeta S_i b \xi \|_1 \lesssim \| f \|_1.
\]  

(4.12)

Once we obtain Lemma 4.4, we can prove (4.10) as follows. Set \( e_{2,2} = \chi_{(0, \lambda]} \left( \sum_{i \in \mathbb{Z}} |\zeta S_i b \xi| \right) \). Recall that \( b \) is selfadjoint from Theorem 2.5 and the kernel \( k_i \) is real. Then for any \( j \in \mathbb{Z} \)

\[-\lambda \leq -e_{2,2} \sum_{i \in \mathbb{Z}} |\zeta S_i b \xi| e_{2,2} \leq e_{2,2} \xi T_j b \xi e_{2,2} \leq e_{2,2} \sum_{i \in \mathbb{Z}} |\zeta S_i b \xi| e_{2,2} \leq \lambda; \]

moreover, by the Chebyshev inequality and (4.12), we find

\[
\varphi(e_{2,2}^\perp) \leq \sum_{i \in \mathbb{Z}} \| \zeta S_i b \xi \|_1 \lesssim \frac{\| f \|_1}{\lambda}.
\]

This gives the desired estimate (4.10) thanks to Remark 2.3.

Now we start to prove Lemma 4.4.

**Proof of Lemma 4.4** By noticing \( b = b_d + b_{\text{off}} \) and using the Minkowski inequality, it suffices to show

\[
\sum_{i \in \mathbb{Z}} \| \zeta S_i b_d \xi \|_1 \lesssim \| f \|_1
\]  

(4.13)
and
\[ \sum_{i \in \mathbb{Z}} \| \xi S_i b_{d \xi} \|_1 \lesssim \| f \|_1. \] (4.14)

**Proof of estimate (4.13).** Since \( m_n(f) = 0 \), we have \( p_n = 0 \) for all \( n \leq 0 \). By conclusion (iii) in Theorem 2.5, \( b_d = \sum_{n=1}^{\infty} b_{d,n} \) where \( b_{d,n} = p_n(f - f_n)p_n \) as in (2.8).

We claim that \( \xi S_i b_{d,n} \xi = 0 \) unless \( i < n - 1 \). Fix one \( x \in \mathbb{R}^d \). Recalling that \( s = 4\sqrt{d} \) in Remark 2.7 and by the second cancellation property of \( b_{d,n} \)-conclusion (iii) in Theorem 2.5, one has
\[ \zeta(x) S_i b_{d,n}(x) \zeta(x) = \int_{\mathbb{R}^d} k_i(x, y) \chi_{y \notin (8[\sqrt{d}] + 1) Q_{x,n}}(\xi) b_{d,n}(y) \zeta(x) dy. \] (4.15)

Indeed, recalling the definition of \( k_i \), it suffices to consider those \( y \) in the integral above such that \( \phi_i(x - y) \neq 0 \) and \( y \notin (8[\sqrt{d}] + 1) Q_{x,n} \). Note that the support of \( \phi_i \) implies \( 2^{-i-1}\sqrt{d} \leq |x - y| \leq 2^{-i+1}\sqrt{d} \); while \( y \notin (8[\sqrt{d}] + 1) Q_{x,n} \) implies that \( |x - y| > 4\sqrt{d} \cdot 2^{-n} \). Thus \( \zeta(x) S_i b_{d,n}(x) \zeta(x) \) may be equal to 0 unless \( 2^{-i+1}\sqrt{d} > 4\sqrt{d} \cdot 2^{-n} \), that is, \( i < n - 1 \). This is precisely the claim.

Taking these observations into consideration, we deduce that
\[ \sum_{i \in \mathbb{Z}} \| \xi S_i b_d \xi \|_1 \leq \sum_{n=1}^{\infty} \sum_{i : i < n - 1} \left\| \int_{\mathbb{R}^d} k_i(x, y) b_{d,n}(y) dy \right\|_{L_1(M)}. \]

Furthermore, by applying the first cancellation property of \( b_{d,n} \)-conclusion (iii) stated in Theorem 2.5, we get
\[ \sum_{i \in \mathbb{Z}} \| \xi S_i b_d \xi \|_1 \leq \sum_{n=1}^{\infty} \sum_{i : i < n - 1} \left\| \int_{\mathbb{R}^d} k_{i,n}(x, y) b_{d,n}(y) dy \right\|_{L_1(M)} \triangleq \| F_1 \|_1, \]

where \( k_{i,n}(x, y) \) was given in (4.3). Therefore, it is enough to show
\[ \| F_1 \|_1 \lesssim \| f \|_1. \] (4.16)

To see this, by using the Fubini theorem and the Minkowski inequality, we arrive at
\[ \| F_1 \|_1 = \tau \int_{\mathbb{R}^d} \left( \sum_{n=1}^{\infty} \sum_{i : i < n - 1} \left\| \int_{\mathbb{R}^d} k_{i,n}(x, y) b_{d,n}(y) dy \right\| \right) dx \]
\[ = \sum_{n=1}^{\infty} \sum_{i : i < n - 1} \int_{\mathbb{R}^d} \left\| \int_{\mathbb{R}^d} k_{i,n}(x, y) b_{d,n}(y) dy \right\|_{L_1(M)} dx \]
\[ \leq \sum_{n=1}^{\infty} \sum_{i : i < n - 1} \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} |k_{i,n}(x, y)| \| b_{d,n}(y) \|_{L_1(M)} dy dx. \]
Now exploiting (4.4) in Lemma 4.2 and the Fubini theorem, we find

$$\|F_1\|_1 \leq \sum_{n=1}^{\infty} \sum_{i: i < n-1} \int_{\mathbb{R}^d} \|b_{d,n}(y)\|_{L_1(\mathcal{M})} \int_{\mathbb{R}^d} |k_{i,n}(x, y)| \, dx \, dy$$

$$\lesssim \sum_{n=1}^{\infty} \sum_{i: i < n-1} [\delta_1(n - i) + \delta_1(n - i + 1) + 2^{-n+i}] \int_{\mathbb{R}^d} \|b_{d,n}(y)\|_{L_1(\mathcal{M})} \, dy$$

$$\leq \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [2\delta_1(m) + 2^{-m}] \int_{\mathbb{R}^d} \|b_{d,n}(y)\|_{L_1(\mathcal{M})} \, dy$$

$$\lesssim \sum_{n=1}^{\infty} \sum_{m=1}^{\infty} [2\delta_2(m) + 2^{-m}] \|b_{d,n}\|_{L_1(\mathcal{N})},$$

where in the last inequality we used the Hölder inequality. Now we use (1.4) with $q = 2$ and conclusion (iii) announced in Theorem 2.5 to get

$$\|F_1\|_1 \lesssim \sum_{n=1}^{\infty} \|b_{d,n}\|_1 \lesssim \|f\|_1,$$

which establishes (4.16). This completes the proof of estimate (4.13).

**Proof of estimate (4.14).** By the assumption $m_\lambda(f) = 0$, conclusion (iv) in Theorem 2.5 yields $b_{off} = \sum_{n=1}^{\infty} b_n$, where $b_n = p_n(f - fn)q_n + q_n(f - fn)p_n$ as in (2.9). On the other hand, applying the similar argument in dealing with the diagonal term $b_d$ (see (4.15)), we can deduce that $\zeta S_i b_n \zeta = 0$ unless $i < n - 1$. Consequently,

$$\sum_{i \in \mathbb{Z}} \|\zeta S_i b_{off} \zeta\|_1 \leq \sum_{n=1}^{\infty} \sum_{i: i < n-1} \left\|\int_{\mathbb{R}^d} k_i(x, y) b_n(y) \, dy\right\|_{L_1(\mathcal{M})}.$$

Using the cancellation property of $b_n$ announced in Theorem 2.5, we obtain

$$\sum_{i \in \mathbb{Z}} \|\zeta S_i b_{off} \zeta\|_1 \leq \sum_{n=1}^{\infty} \sum_{i: i < n-1} \sum_{Q \in Q_n} \left\|\int_Q k_{i,n}(x, y) b_n(y) \, dy\right\|_{L_1(\mathcal{M})} \triangleq \|F_2\|_1.$$

Apparently, the proof of estimate (4.14) would be completed if the following estimate were verified:

$$\|F_2\|_1 \lesssim \|f\|_1. \quad (4.17)$$
To show (4.17), by using the Fubini theorem, we clearly have

\[ \|F_2\|_1 = \tau \int_{\mathbb{R}^d} \sum_{n=1}^{\infty} \sum_{i:i < n-1} \sum_{Q \in \mathcal{Q}_n} \left| \int_Q k_{i,n}(x, y) b_n(y) dy \right| dx \]

\[ = \sum_{n=1}^{\infty} \sum_{i:i < n-1} \sum_{Q \in \mathcal{Q}_n} \left( \int_Q k_{i,n}(x, y) b_n(y) dy \right)_{L^1(\mathcal{M})} dx. \]

The Minkowski inequality and the definition of \( b_n \) imply that \( \|F_2\|_1 \) can be controlled by the sum of the following four terms

\[ \|F_{2,1}\|_1 \triangleq \sum_{n=1}^{\infty} \sum_{i:i < n-1} \sum_{Q \in \mathcal{Q}_n} \left( \int_Q k_{i,n}(x, y) p_Q f(y) q_Q dy \right)_{L^1(\mathcal{M})} dx, \]

\[ \|F_{2,2}\|_1 \triangleq \sum_{n=1}^{\infty} \sum_{i:i < n-1} \sum_{Q \in \mathcal{Q}_n} \left( \int_Q k_{i,n}(x, y) q_Q f(y) p_Q dy \right)_{L^1(\mathcal{M})} dx, \]

\[ \|F_{2,3}\|_1 \triangleq \sum_{n=1}^{\infty} \sum_{i:i < n-1} \sum_{Q \in \mathcal{Q}_n} \left( \int_Q k_{i,n}(x, y) p_Q f_n(y) q_Q dy \right)_{L^1(\mathcal{M})} dx, \]

\[ \|F_{2,4}\|_1 \triangleq \sum_{n=1}^{\infty} \sum_{i:i < n-1} \sum_{Q \in \mathcal{Q}_n} \left( \int_Q k_{i,n}(x, y) q_Q f_n(y) p_Q dy \right)_{L^1(\mathcal{M})} dx. \]

Therefore, this leads to show

\[ \max \left\{ \|F_{2,1}\|_1, \|F_{2,2}\|_1, \|F_{2,3}\|_1, \|F_{2,4}\|_1 \right\} \lesssim \|f\|_1. \]

Consider \( \|F_{2,1}\|_1 \) firstly. For convenience, we set

\[ G_{i,n,Q}(x) \triangleq \int_Q k_{i,n}(x, y) p_Q f(y) q_Q dy. \]

We now treat with \( \|G_{i,n,Q}(x)\|_{L^1(\mathcal{M})} \). We use Lemma 2.1 to find that

\[ \|G_{i,n,Q}(x)\|_{L^1(\mathcal{M})} \]

\[ \leq \left( \left( \int_Q |k_{i,n}(x, y)|^2 p_Q f(y) p_Q dy \right)^{\frac{1}{2}} \right)^{\frac{1}{2}} \left( \left( \int_Q q_Q f(y) q_Q dy \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}_{L^1(\mathcal{M})} \]

\[ = \left( \int_Q |k_{i,n}(x, y)|^2 p_Q f(y) p_Q dy \right)^{\frac{1}{2}} \left( \left( \int_Q q_Q f(y) q_Q dy \right)^{\frac{1}{2}} \right)^{\frac{1}{2}}_{L^1(\mathcal{M})}. \]

(4.18)
Now applying the properties (2.4) to the second term above, we get
\[
\|G_{i,n,Q}(x)\|_{L^1(M)} \leq \left\| \int_{Q} |k_{i,n}(x,y)|^2 p_Q f(y) p_Q dy \right\|_{L^1(M)} |Q|^{-\frac{1}{2}}.
\]

This provides us with the estimate
\[
\|F_{2,1}\|_1 \leq \sum_{n=1}^{\infty} \sum_{i : i < n-1} \sum_{Q \in Q_n} \left\| \int_{Q} |k_{i,n}(x,y)|^2 p_Q f(y) p_Q dy \right\|_{L^1(M)} |Q|^{-\frac{1}{2}} dx.
\]

Now we apply (4.7) stated in Lemma 4.3, (1.4) with \( q = 2 \) and the Fubini theorem to deduce that
\[
\|F_{2,1}\|_1 \leq \sum_{n=1}^{\infty} \sum_{i : i < n-1} \sum_{Q \in Q_n} (\delta_2(n-i) + \delta_2(n-i+1) + 2^{i-n}) \left( \tau(p_Q) \varphi(f p_Q \chi_Q) |Q|^{-\frac{1}{2}} \right)
\]
\[
\leq \sum_{n=1}^{\infty} \sum_{Q \in Q_n} \sum_{m=1}^{\infty} (2\delta_2(m) + 2^{-m}) \left( \tau(p_Q) \varphi(f p_Q \chi_Q) |Q|^{-\frac{1}{2}} \right)
\]
\[
\leq \sum_{n=1}^{\infty} \sum_{Q \in Q_n} \left( \tau(p_Q) \varphi(f p_Q \chi_Q) |Q|^{-\frac{1}{2}} \right).
\]

Furthermore, the Cauchy–Schwarz inequality implies
\[
\sum_{Q \in Q_n} \left( \tau(p_Q) \varphi(f p_Q \chi_Q) |Q|^{-\frac{1}{2}} \right)^2 \leq \left( \sum_{Q \in Q_n} \tau(p_Q) |Q|^{-\frac{1}{2}} \right)^{\frac{1}{2}} \left( \sum_{Q \in Q_n} \varphi(f p_Q \chi_Q) \right)^{\frac{1}{2}}
\]
\[
= \left( \lambda \varphi(p_n) \right)^{\frac{1}{2}} \left( \varphi(f p_n) \right)^{\frac{1}{2}}.
\]

Then applying the Cauchy–Schwarz inequality once more, we finally get
\[
\|F_{2,1}\|_1 \leq \sum_{n=1}^{\infty} \left( \lambda \varphi(p_n) \right)^{\frac{1}{2}} \left( \varphi(f p_n) \right)^{\frac{1}{2}}
\]
\[
\leq \left( \sum_{n=1}^{\infty} \lambda \varphi(p_n) \right)^{\frac{1}{2}} \left( \sum_{n=1}^{\infty} \varphi(f p_n) \right)^{\frac{1}{2}}
\]
\[
= \left( \lambda \varphi(1-q) \right)^{\frac{1}{2}} \left( \varphi(f(1-q)) \right)^{\frac{1}{2}} \leq \|f\|_1.
\]

This finishes the estimate of \( \|F_{2,1}\|_1 \).

The same arguments work also for other three terms \( \|F_{2,2}\|_1, \|F_{2,3}\|_1 \) and \( \|F_{2,4}\|_1 \) by Lemma 4.3 and noting \( \varphi(f_n p_n) = \varphi(f p_n) \). Thus we finish the proof of estimate (4.14).
Remark 4.5 One can see in the proof of the diagonal estimate (4.13), the $L_1$-integral condition $\sum_{m=1}^{\infty} \delta_1(m) < \infty$ is enough. While in the proof of the off-diagonal estimate (4.14), we need the $L_2$-integral condition $\sum_{m=1}^{\infty} \delta_2(m) < \infty$.

Remark 4.6 We would like to point out that the estimate of $\|F_{2,1}\|_1$, in particular estimate (4.18), is partially motivated by Cadilhac’s note [2].

Remark 4.7 It is worth pointing out that by showing the weak type $(1, 1)$ of $(T_j)_{j \in \mathbb{Z}}$, we see that Theorem 1.1 still holds true if we weaken the assumption of the strong type $(p_0, p_0)$ of $(T_{\varepsilon})_{\varepsilon > 0}$ to the weak type $(p_0, p_0)$ of $(T_{\varepsilon})_{\varepsilon > 0}$. The details are left to the interested readers.

5 Proof of Theorem 1.4: maximal inequalities

In this section, we prove the maximal inequalities announced in Theorem 1.4. By Theorem 1.1, the weak type $(1, 1)$ result is a consequence of the strong type $(p, p)$ of $(T_{\varepsilon})_{\varepsilon > 0}$. While to show the strong type $(p, p)$ of $(T_{\varepsilon})_{\varepsilon > 0}$, as mentioned in the introduction, it suffices to show the following Cotlar inequality in terms of norm: for all $f \in L^p(\mathcal{N})_+$ with $1 < p < \infty$,\[
\left\| \sup_{\varepsilon > 0} T_{\varepsilon} f \right\|_p \lesssim \left\| \sup_{\varepsilon > 0} M_{\varepsilon}(T f) \right\|_p + \left\| \sup_{\varepsilon > 0} M_{\varepsilon} f \right\|_p. \tag{5.1}
\]

Indeed, by Mei’s noncommutative Hardy–Littlewood maximal inequalities [35] and the fact that $T$ is bounded on $L^p(\mathcal{N})$ under the kernel conditions (1.2), (1.3) and (1.12) (see e.g. [39, Theorem A]), (5.1) implies\[
\left\| \sup_{\varepsilon > 0} T_{\varepsilon} f \right\|_p \lesssim \|T f\|_p + \|f\|_p \lesssim \|f\|_p, \quad \forall f \in L^p(\mathcal{N}).
\]

We need the following lemma, which should be well-known to noncommutative analysts, see e.g. [6, Theorem 4.3].

Lemma 5.1 Let $\psi$ be a non-negative radial function on $\mathbb{R}^d$ such that $\psi(x) \lesssim \frac{1}{(1+|x|)^{d+\delta}}$ for $x \in \mathbb{R}^d$ with some $\delta > 0$. Let $\psi_{\varepsilon}(x) = \frac{1}{\varepsilon^d} \psi\left(\frac{x}{\varepsilon}\right)$ for $x \in \mathbb{R}^d$ and $\varepsilon > 0$. Let $1 < p \leq \infty$. Then\[
\left\| \sup_{\varepsilon > 0} \psi_{\varepsilon} * f \right\|_p \lesssim \left\| \sup_{\varepsilon > 0} M_{\varepsilon} f \right\|_p, \quad \forall f \in L^p(\mathcal{N}).
\]

Now we are ready to prove the Cotlar inequality (5.1). The idea is inspired by the classical one as presented in [15, Theorem 5.3.4].

Proof of estimate (5.1) Let $\varphi$ be a smooth, radial, radially decreasing non-negative function which is supported in the ball $B(0, \frac{1}{2})$ and $\int_{\mathbb{R}^d} \varphi(x) dx = 1$. We will use the notation $\varphi_{\varepsilon}(x) = \frac{1}{\varepsilon^d} \varphi\left(\frac{x}{\varepsilon}\right)$ for $x \in \mathbb{R}^d$. Fix $f \in L^p(\mathcal{N})_+$, now we express $T_{\varepsilon} f$ as\[
T_{\varepsilon} f = \varphi_{\varepsilon} * (T f) + (T_{\varepsilon} f - \varphi_{\varepsilon} * (T f)). \tag{5.2}
\]
By Lemma 5.1, we trivially have
\[ \| \sup_{\varepsilon > 0} \varphi_\varepsilon * (T f) \|_p \lesssim \| \sup_{\varepsilon > 0} M_\varepsilon (T f) \|_p. \]

On the other hand, we rewrite the difference in (5.2) as
\[ T_\varepsilon f(x) - \varphi_\varepsilon * (T f) = \left[ \lim_{t \to 0} (k \chi_{|.| > \varepsilon} - \varphi_\varepsilon * k \chi_{|.| > t}) \right] * f. \]

It is easy to verify that (see e.g. [15, Theorem 5.3.4])
\[ \lim_{t \to 0} |k \chi_{|.| > \varepsilon} - \varphi_\varepsilon * k \chi_{|.| > t}| \lesssim \psi_\varepsilon, \]
where
\[ \psi(x) = \frac{1}{(1 + |x|)^d + \gamma} \]
with \( \gamma \) being the regularity index of the kernel. By decomposing \( k \chi_{|.| > \varepsilon} - \varphi_\varepsilon * k_\varepsilon \) into a linear combination of four positive functions, one gets
\[ \| \sup_{\varepsilon > 0} (T_\varepsilon f - \varphi_\varepsilon * (T f)) \|_p \lesssim \| \sup_{\varepsilon > 0} \left[ \lim_{t \to 0} |k \chi_{|.| > \varepsilon} - \varphi_\varepsilon * k \chi_{|.| > t}| \right] * f \|_p \]
\[ \lesssim \| \sup_{\varepsilon > 0} \psi_\varepsilon * f \|_p \lesssim \| \sup_{\varepsilon > 0} M_\varepsilon f \|_p. \]

Combining the above estimates, we get the desired Cotlar inequality (5.1).

6 Proof of Theorem 1.5: maximal inequalities

In this section, we prove the maximal inequalities stated in Theorem 1.5. We start with the strong type \((p, p)\) estimate of \((T_{\Omega, \varepsilon})_{\varepsilon > 0}\). Our strategy is the method of rotation which is inspired by the classical one (see [4] or [15, Section 5.2]).

6.1 Strong type \((p, p)\) estimate of \((T_{\Omega, \varepsilon})_{\varepsilon > 0}\)

For convenience, we denote \(k_{\Omega}(x) \triangleq \Omega(x)/|x|^d\). Without loss of generality, we may assume that \(\Omega\) (and thus \(k_{\Omega}\)) is real-valued. Since \(\Omega\) on \(S^{d-1}\) can be decomposed into its even and odd parts,
\[ \Omega_\varepsilon(x) = \frac{1}{2} (\Omega(x) + \Omega(-x)), \quad \Omega_\sigma(x) = \frac{1}{2} (\Omega(x) - \Omega(-x)), \]
(6.1)
it suffices to consider \(\Omega\) with the cases of being odd and even, respectively. We begin with the case of being odd.
6.1.1 Case: $\Omega$ is odd

As in the commutative case, the method of rotation will play a crucial role in the study of $T_\Omega$ and $(T_{\Omega,\epsilon})_{\epsilon>0}$ (see [15]). We first study the directional Hilbert transforms. Given a unit vector $\theta$ in $\mathbb{R}^d$, the directional Hilbert transform in the direction $\theta$ is defined as:

$$H_\theta f(x) = p.v. \frac{1}{\pi} \int_{\mathbb{R}} f(x - t\theta) \frac{dt}{t}. \quad (6.2)$$

Likewise, we define the associated directional truncated Hilbert transform for $\epsilon > 0$,

$$H_{\theta,\epsilon} f(x) = \frac{1}{\pi} \int_{|t|>\epsilon} f(x - t\theta) \frac{dt}{t}. \quad (6.3)$$

**Lemma 6.1** Let $H_\theta$ and $H_{\theta,\epsilon}$ be defined as in (6.2) and (6.3), respectively. Then for all $f \in L_p(\mathcal{N})$ with $1 < p < \infty$

$$\|H_\theta f\|_p \lesssim \|f\|_p \quad \text{and} \quad \sup_{\epsilon>0} \|H_{\theta,\epsilon} f\|_p \lesssim \|f\|_p.$$

The above result should be known to experts, and for the sake of completeness, we give a sketch of the proof.

**Proof** Let $e_1 = (1, 0, \cdots, 0)$ be the unit vector. Then it is easy to see that the operator $H_{e_1}$ can be written as $H \otimes id_{L_\infty(\mathbb{R}^{d-1})}$ the tensor product of the usual Hilbert transform with identity map on functions on $\mathbb{R}^{d-1}$. Thus, $H_{e_1}$ is bounded on $L_p(\mathcal{N})$ with norm equal to the completely bounded norm of the usual Hilbert transform on $L_p$ (see e.g. [39, Theorem A]). Observing that for any orthogonal matrix $A$, the following identity holds

$$H_{A(e_1)} f(x) = H_{e_1} (f \circ A)(A^{-1}x). \quad (6.4)$$

This shows that the $L_p$ boundedness of $H_\theta$ can be reduced to that of $H_{e_1}$, which proves the first inequality.

Next, we consider the second inequality. Clearly, identity (6.4) is also valid for $H_{\theta,\epsilon}$. Consequently, it suffices to show that $(H_{e_1,\epsilon})_{\epsilon>0}$ is of strong type $(p, p)$. Let $f \in L_p(\mathcal{N})$. Without loss of generality, we may assume that $f$ is positive. Fixing $x_2, \ldots, x_d \in \mathbb{R}$, we consider $f(\cdot, x_2, \ldots, x_d)$ as a function in $L_p(L_\infty(\mathbb{R}) \otimes \mathcal{M})_+$. By Theorem 1.4, we know that for $1 < p < \infty$

$$\left\|\sup_{\epsilon>0} H_\epsilon f(\cdot, x_2, \ldots, x_d)\right\|_{L_p(L_\infty(\mathbb{R}) \otimes \mathcal{M})} \lesssim \|f(\cdot, x_2, \ldots, x_d)\|_{L_p(L_\infty(\mathbb{R}) \otimes \mathcal{M})}.$$

This implies that, by Remark 2.2, there exists a positive function $F(\cdot, x_2, \cdots, x_d) \in L_p(L_\infty(\mathbb{R}) \otimes \mathcal{M})$ such that for any $\epsilon > 0$

$$-F(x) \leq (H_\epsilon \otimes id_{L_\infty(\mathbb{R}^{d-1})}) f(x) = H_{e_1,\epsilon} f(x) \leq F(x),$$

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Then it is easy to see that
\[ \|F\|_{L^p(N)} \lesssim \|f\|_{L^p(N)}. \]

Therefore, we conclude that \((H_{\ell_1,\varepsilon})_{\varepsilon > 0}\) is of strong type \((p, p)\). \qed

**Proposition 6.2** If \(\Omega\) is odd and integrable over \(S^{d-1}\), then \((T_{\Omega,\varepsilon})_{\varepsilon > 0}\) is of strong type \((p, p)\) for all \(1 < p < \infty\).

**Proof** By switching to polar coordinates and the fact that \(\Omega\) is odd, we get the following identities:
\[
\int_{|y| > \varepsilon} k_\Omega(y) f(x - y) dy = \int_{S^{d-1}} \Omega(\theta) \int_{\varepsilon}^{\infty} f(x - r\theta) \frac{dr}{r} d\theta
= -\int_{S^{d-1}} \Omega(\theta) \int_{\varepsilon}^{\infty} f(x + r\theta) \frac{dr}{r} d\theta,
\]
where the second equality is a consequence of the first one via the change of variables \(\theta \rightarrow -\theta\) based on the fact that \(\Omega\) is an odd function. Thus we obtain
\[
\int_{|y| > \varepsilon} k_\Omega(y) f(x - y) dy = \frac{1}{2} \int_{S^{d-1}} \Omega(\theta) \int_{\varepsilon}^{\infty} \left(\frac{f(x - r\theta) - f(x + r\theta)}{r}\right) dr d\theta
= \frac{\pi}{2} \int_{S^{d-1}} \Omega(\theta) H_{\theta,\varepsilon} f(x) d\theta.
\]

Now by the triangle inequality and Lemma 6.1, we get that
\[
\left\| \sup_{\varepsilon > 0}^+ T_{\Omega,\varepsilon} f \right\|_p \lesssim \int_{S^{d-1}} |\Omega(\theta)| \left\| \sup_{\varepsilon > 0}^+ H_{\theta,\varepsilon} f \right\|_p d\theta \lesssim \|f\|_p.
\]
Therefore, we obtain the announced result. \qed

**6.1.2 Case: \(\Omega\) is even**

In this case, we need the directional Hardy–Littlewood maximal inequality.

**Lemma 6.3** Let \(1 < p \leq \infty\) and \(\theta\) be a unit vector in \(\mathbb{R}^d\). Then for any \(f \in L^p(N)\),
\[
\left\| \sup_{\varepsilon > 0}^+ f_{\theta,\varepsilon} \right\|_p \lesssim \|f\|_p,
\]
where
\[
f_{\theta,\varepsilon}(x) = \frac{1}{2\varepsilon} \int_{|r| \leq \varepsilon} f(x - r\theta) dr.
\]
This result can be obtained by the same argument as that for directional Hilbert transform, which should be also known to experts. We omit the details.

**Proposition 6.4** Let $\Omega$ be an even function on $S^{d-1}$ with mean value zero and $\Omega \in L \log^+ L(S^{d-1})$. Then for any $f \in L_p(\mathcal{N})$ with $1 < p < \infty$,

$$\left\| \sup_{\varepsilon > 0}^+ T_{\Omega, \varepsilon} f \right\|_p \lesssim \| f \|_p.$$ 

**Proof** The proof here is inspired by the classical one [15, Theorem 5.2.11]. Let $\phi$ a smooth radial function such that $\phi(x) = 0$ for $|x| \leq \frac{1}{4}$, $\phi(x) = 1$ for $|x| \geq \frac{3}{4}$, and $0 \leq \phi(x) \leq 1$ for all $x \in \mathbb{R}^d$. Fix $f \in L_p(\mathcal{N})$; without loss of generality, we may assume that $f$ is positive. We define the smoothly truncated singular integral as

$$\tilde{T}_{\Omega, \varepsilon} f(x) = \int_{\mathbb{R}^d} k_{\Omega}(y) \phi\left(\frac{y}{\varepsilon}\right) f(x - y)dy.$$ 

By the support of $\phi$, we deduce that

$$\tilde{T}_{\Omega, \varepsilon} f(x) - T_{\Omega, \varepsilon} f(x) = \int_{\frac{\varepsilon}{4} \leq |y| \leq \frac{3\varepsilon}{4}} k_{\Omega}(y) \phi\left(\frac{y}{\varepsilon}\right) f(x - y)dy$$

$$\leq \int_{S^{d-1}} |\Omega(\theta)| \left[\frac{4}{\varepsilon} \int_{\frac{\varepsilon}{4}}^{\frac{3\varepsilon}{4}} f(x - r\theta)dr\right]d\theta$$

$$\lesssim \int_{S^{d-1}} |\Omega(\theta)| f_{\theta, \varepsilon}(x)d\theta.$$ 

On the other hand, we have

$$- \int_{S^{d-1}} |\Omega(\theta)| f_{\theta, \varepsilon}(x)d\theta \lesssim \tilde{T}_{\Omega, \varepsilon} f(x) - T_{\Omega, \varepsilon} f(x).$$

Thus by Lemma 6.3, one gets

$$\left\| \sup_{\varepsilon > 0}^+ (\tilde{T}_{\Omega, \varepsilon} f - T_{\Omega, \varepsilon} f) \right\|_p \lesssim \| \Omega \|_1 \left\| \sup_{\varepsilon > 0}^+ f_{\theta, \varepsilon} \right\|_p \lesssim \| f \|_p.$$ 

Hence, to finish the proof, by rewriting

$$T_{\Omega, \varepsilon} f(x) = T_{\Omega, \varepsilon} f(x) - \tilde{T}_{\Omega, \varepsilon} f(x) + \tilde{T}_{\Omega, \varepsilon} f(x),$$

it suffices to consider the required strong type $(p, p)$ estimate for the smoothly truncated singular operator $(\tilde{T}_{\Omega, \varepsilon})_{\varepsilon > 0}$.

The identity property of the Riesz transforms $- \sum_{j=1}^d R_j^2 = I$ will play a key role. Let $s_j$ (resp. $k_j$) be the $j$-th Riesz transform of $k_{\Omega, \phi}$ (resp. $k_{\Omega}$). The identity property implies
\[
\tilde{T}_{\Omega, \varepsilon} f(x) = \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d} k_{\Omega} \left( \frac{y}{\varepsilon} \right) \varphi \left( \frac{y}{\varepsilon} \right) f(x - y) dy = - \left( \sum_{j=1}^{d} \frac{1}{\varepsilon^d} s_j \left( \frac{\cdot}{\varepsilon} \right) \ast R_j f \right)(x),
\]

where in the second equality we also used the homogeneity of \( R_j \). Therefore, we are able to decompose \( T_{\Omega, \varepsilon} f(x) \) as

\[
-\tilde{T}_{\Omega, \varepsilon} f(x) = \sum_{j=1}^{d} \int_{\mathbb{R}^d} \frac{1}{\varepsilon^d} s_j \left( \frac{x - y}{\varepsilon} \right) R_j f(y) dy
\]

\[
\triangleq N_{1, \varepsilon} f(x) + N_{2, \varepsilon} f(x) + N_{3, \varepsilon} f(x),
\]

where

\[N_{1, \varepsilon} f(x) = \sum_{j=1}^{d} \frac{1}{\varepsilon^d} \int_{|x - y| \leq \varepsilon} s_j \left( \frac{x - y}{\varepsilon} \right) R_j f(y) dy,
\]

\[N_{2, \varepsilon} f(x) = \sum_{j=1}^{d} \frac{1}{\varepsilon^d} \int_{|x - y| > \varepsilon} \left[ s_j \left( \frac{x - y}{\varepsilon} \right) - k_j \left( \frac{x - y}{\varepsilon} \right) \right] R_j f(y) dy,
\]

\[N_{3, \varepsilon} f(x) = \sum_{j=1}^{d} \frac{1}{\varepsilon^d} \int_{|x - y| > \varepsilon} k_j \left( \frac{x - y}{\varepsilon} \right) R_j f(y) dy.
\]

We first deal with \( N_{2, \varepsilon} f \). It follows from the definitions of \( s_j \) and \( k_j \) that

\[s_j(z) - k_j(z) = c_d \lim_{\varepsilon \to 0} \int_{|y| \leq \varepsilon} k_{\Omega}(y) (\varphi(y) - 1) \frac{z_j - y_j}{|z - y|^{d+1}} dy
\]

\[= c_d \int_{|y| \leq \frac{3}{4}} k_{\Omega}(y) (\varphi(y) - 1) \left\{ \frac{z_j - y_j}{|z - y|^{d+1}} - \frac{z_j}{|z|^{d+1}} \right\} dy
\]

whenever \(|z| \geq 1\). By using the mean value theorem, we see that the last expression above can be controlled by

\[c_d \int_{|y| \leq \frac{3}{4}} |k_{\Omega}(y)| \frac{|y|}{|y|^{d+1}} dy = c_d' \|\Omega\|_1 |z|^{-(d+1)},
\]

whenever \(|z| \geq 1\). By applying this estimate, we get that the j-th term in \( N_{2, \varepsilon} f(x) \) is bounded by

\[c_d' \|\Omega\|_1 \frac{2d}{e^d} \int_{|x - y| > \varepsilon} \frac{|R_j f(y)| dy}{\left( \frac{|x - y|}{\varepsilon} \right)^{d+1}} \leq c_d' \frac{2\|\Omega\|_1}{2-d^2 e^d} \int_{\mathbb{R}^d} \frac{|R_j f(y)| dy}{\left( 1 + \frac{|x - y|}{\varepsilon} \right)^{d+1}}.
\]
Clearly, the j-th term in $N_{2,\varepsilon} f(x)$ also has the lower bound

$$-c_d \frac{2\|\Omega\|_1}{2-\delta_{ed}} \int_{\mathbb{R}^d} \frac{|R_j f(y)|dy}{(1 + \frac{|x-y|}{\varepsilon})^{d+1}}.$$  

Applying Lemma 5.1 and the fact that the Riesz transform $R_j$ is bounded on $L_p(\mathbb{N})$, we get

$$\|\sup_{\varepsilon>0} N_{2,\varepsilon} f\|_p \lesssim \sum_{j=1}^d \|R_j f\|_p = \sum_{j=1}^d \|R_j f\|_p \lesssim \|f\|_p. \quad (6.6)$$

Next we consider the term $N_{3,\varepsilon} f$. It is already shown in [15, Theorem 5.2.11] that for any $1 \leq j \leq d$

$$k_j(x) = \Omega_j(x)/|x|^d,$$

where $\Omega_j$ is some homogeneous integrable odd function on $\mathbb{S}^{d-1}$ satisfying

$$\|\Omega_j\|_1 \leq c_d \left( \int_{\mathbb{S}^{d-1}} |\Omega(\theta)| \log^+ |\Omega(\theta)| d\theta + 1 \right),$$

where $\log^+ s = \log s$ if $s \geq 1$ and $\log^+ s = 0$ if $0 \leq s < 1$. Consequently,

$$N_{3,\varepsilon} f(x) = \sum_{j=1}^d \int_{|x-y|>\varepsilon} \frac{\Omega_j(x-y)}{|x-y|^d} R_j f(y) dy.$$

Now we use Theorem 6.2 and the fact that the Riesz transform is bounded on $L_p(\mathbb{N})$ to conclude that

$$\left\|\sup_{\varepsilon>0} N_{3,\varepsilon} f\right\|_p \lesssim \sum_{j=1}^d \|R_j f\|_p \lesssim \|f\|_p. \quad (6.7)$$

Finally, we turn our attention to the term $N_{1,\varepsilon} f$. By [15, Theorem 5.2.10], for each $1 \leq j \leq d$, there exists a non-negative homogeneous of degree zero function $g_j$ on $\mathbb{R}^d$ satisfying

$$|s_j(x)| \leq g_j(x) \quad \text{when } |x| \leq 1 \quad (6.8)$$

and

$$\int_{\mathbb{S}^{d-1}} |g_j(\theta)| d\theta \leq c_d \left( \int_{\mathbb{S}^{d-1}} |\Omega(\theta)| \log^+ |\Omega(\theta)| d\theta + 1 \right). \quad (6.9)$$
Therefore, we get
\[
N_{1,\varepsilon} f(x) \lesssim \sum_{j=1}^{d} \frac{1}{\varepsilon^d} \int_{|z| \leq \varepsilon} |s_j(z)||R_j f(x-z)|dz \\
= \sum_{j=1}^{d} \frac{1}{\varepsilon^d} \int_0^{\varepsilon} \int_{S^{d-1}} |s_j(r\theta)||R_j f(x-r\theta)|r^{d-1}d\theta dr \\
\leq \sum_{j=1}^{d} \int_{S^{d-1}} |g_j(\theta)| \frac{1}{\varepsilon^d} \int_0^{\varepsilon} |R_j f(x-r\theta)|r^{d-1}dr d\theta \\
\leq \sum_{j=1}^{d} \int_{S^{d-1}} |g_j(\theta)| \frac{1}{\varepsilon} \int_0^{\varepsilon} |R_j f(x-r\theta)|dr d\theta.
\]

where the second inequality follows from the homogeneity of \(s_j\) and (6.8). On the other hand, it is easy to verify that
\[
-\sum_{j=1}^{d} \int_{S^{d-1}} |g_j(\theta)| \frac{1}{\varepsilon} \int_0^{\varepsilon} |R_j f(x-r\theta)|dr d\theta \lesssim N_{1,\varepsilon} f(x).
\]

Hence by Lemma 6.3, (6.9) and the \(L_p\) boundedness of \(R_j\), we arrive at
\[
\|\sup_{\varepsilon>0}^{+} N_{1,\varepsilon} f\|_p \lesssim \sum_{j=1}^{d} \|R_j f\|_p = \sum_{j=1}^{d} \|R_j f\|_p \lesssim \|f\|_p. \tag{6.10}
\]

Finally, combining (6.6), (6.7) and (6.10), we get the desired strong type \((p, p)\) estimate of \((\tilde{T}_{\Omega, \varepsilon})_{\varepsilon>0}\). This completes the proof. \(\Box\)

Combining Proposition 6.2 and Proposition 6.4, we get Theorem 1.5 (i).

6.2 Weak type \((1, 1)\) estimate of \((T_{\Omega, \varepsilon})_{\varepsilon>0}\)

Lemma 6.5 The kernel \(k_{\Omega}\) given in Theorem 1.5 satisfies the smoothness condition (1.4) with \(q = 2\). More precisely,
\[
\sum_{m=1}^{\infty} \delta_2(m) \lesssim \int_0^{1} \frac{\omega_2(s)}{s} ds + \|\Omega\|_{L_2(S^{d-1})}.
\]
Proof Let $R > 0$, $|u| \leq R$ and $m \in \mathbb{N}_+$. Then

\[
\int_{2^m R \leq |x| \leq 2^{m+1} R} |k_\Omega(x + u) - k_\Omega(x)|^2 dx \\
\lesssim \int_{2^m R \leq |x| \leq 2^{m+1} R} |\Omega(x + u) - \Omega(x)|^2 |x + u|^{-2d} dx \\
+ \int_{2^m R \leq |x| \leq 2^{m+1} R} |\Omega(x)|^2 |x + u|^{-2d} - |x|^{-2d} dx \\
\triangleq J_1(m, R) + J_2(m, R).
\]  

(6.11)

We first consider $J_1(m, R)$. Since $|u| \leq R$ and $2^m R \leq |x| \leq 2^{m+1} R$, we see that $2^m R \leq |x + u| \leq 2^{m+2} R$. By a change of variable $x = r\theta$, then using the fact that the kernel $\Omega$ is of zero homogeneity, we get

\[
J_1(m, R) \lesssim (2^m R)^{-2d} \int_{2^m R}^{2^{m+1} R} r^{-d-1} \int_{S^{d-1}} |\Omega(\theta + u/r) - \Omega(\theta)|^2 d\theta dr \lesssim \omega_2^2(2^{-m})(2^m R)^{-d}.
\]

For the second term $J_2(m, R)$, using the mean value theorem, it is not difficult to get that

\[
J_2(m, R) \lesssim \|\Omega\|_{L_2(S^{d-1})}^2 2^{-m}(2^m R)^{-d}.
\]

Hence combining the estimates of $J_1(m, R)$ and $J_2(m, R)$, we see that $\delta_2(m) \lesssim \omega_2(2^{-m}) + 2^{-m/2} \|\Omega\|_{L_2(S^{d-1})}$. Therefore we obtain that

\[
\sum_{m=1}^{\infty} \delta_2(m) \lesssim \int_0^1 \frac{\omega_2(s)}{s} ds + \|\Omega\|_{L_2(S^{d-1})},
\]

where we used the fact that $\omega_2(s)$ is monotone increasing. \hfill \Box

Finally, we are ready to show the weak type $(1, 1)$ estimate of $(T_{\Omega, \epsilon})_{\epsilon > 0}$.

Proof of Theorem 1.5 (ii). The proof of Theorem 1.5 (ii) is similar to that of Theorem 1.1. So we shall be short and only indicate the necessary modifications here. Lemma 6.5 ensures that $k_\Omega$ satisfies the required regularity condition needed in Theorem 1.1. However $k_\Omega$ does not satisfy the size condition (1.2). So all the estimates related to the size condition (1.2) should be modified to adapt the kernel $k_\Omega$.

The first modification is that when reducing to the lacunary sequence in (3.1), Mei’s Hardy–Littlewood maximal inequalities should be replaced by the generalized ones for rough kernels

\[
\|(M_{\Omega, r} f)_{r > 0}\|_{A_1, \infty(N; \ell_\infty)} \lesssim \|f\|_1, \quad \|(M_{\Omega, r} f)_{r > 0}\|_{L_p(N; \ell_\infty)} \lesssim \|f\|_p
\]

(6.12)
for $1 < p \leq \infty$, where

$$M_{\Omega,r} f(x) = \frac{1}{r^d} \int_{|x-y| \leq r} \Omega(x-y) f(y) dy.$$ 

These maximal inequalities (6.12) have been established by the second author in [32] for the rough kernels which include the $L_2$-Dini assumption in Theorem 1.5 (ii).

The second modification is in the proof of Lemma 4.2. When giving an estimate of (4.5), since $k_\Omega$ is homogeneous of degree $-d$ and the support $A_{x,y,n}$ is contained in an annulus, we can make a change of variables to polar coordinate to get our required estimate with the size bound appeared in (1.2) replaced by $\|\Omega\|_{L_q(S^{d-1})}$.

For the rest of proof, using the same argument as done for Theorem 1.1, we complete the proof of Theorem 1.5 (ii).

\section{7 Proof of pointwise convergence results}

In this section, we deal with the noncommutative pointwise convergence results stated in Theorem 1.4 and Theorem 1.5. To this end, we recall first the definition of bilaterally almost uniform convergence which can be viewed as the noncommutative analogue of almost everywhere convergence. The following definition of almost uniform convergence was introduced by Lance [34].

\begin{definition}
Let $M$ be a von Neumann algebra equipped with a n.s.f trace $\tau$. Let $x_k, x \in L_0(M)$.

(i) $(x_k)$ is said to converge bilaterally almost uniformly (b.a.u. in short) to $x$ if for any $\delta > 0$, there exists a projection $e \in M$ such that

$$\tau(e^\perp) < \delta \quad \text{and} \quad \lim_{k \to \infty} \|e(x_k - x)e\|_\infty = 0.$$ 

(ii) $(x_k)$ is said to converge almost uniformly (a.u. in short) to $x$ if for any $\delta > 0$, there exists a projection $e \in M$ such that

$$\tau(e^\perp) < \delta \quad \text{and} \quad \lim_{k \to \infty} \|(x_k - x)e\|_\infty = 0.$$ 

\end{definition}

Obviously, $x_k \xrightarrow{a.u.} x$ implies $x_k \xrightarrow{b.a.u.} x$. Note that in the commutative case on probability space, both convergences in Definition 7.1 are equivalent to the usual almost everywhere convergence due to the Egorov theorem.

In the following, we prove conclusion (iii) in Theorem 1.4, while the treatment of conclusion (iii) stated in Theorem 1.5 is similar.

\begin{proof}[Proof of Theorem 1.4 (iii)] Let $(\varepsilon_j)_{j \in \mathbb{N}}$ be the subsequence appeared in (1.13). Let $f \in L_p(N)$. It is already known from the uniform boundedness principle that $T_{\varepsilon_j} f \to Tf$ in $L_p(N)$ (in $L_{1,\infty}(N)$ if $p = 1$). Thus we only need to show the b.a.u convergence of $(T_{\varepsilon_j} f)_j$ as $j \to \infty$, since then the limit must be $Tf$ (see e.g. [6]). Then it is known

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that equivalently (see e.g. [7, Proposition 1.3]) it suffices to show that for any \( \delta > 0 \), there exists a projection \( e \in \mathcal{N} \) such that

\[
\varphi(e^\perp) < \delta \quad \text{and} \quad \|e(T_{\varepsilon \ell} f - T_{\varepsilon k} f)e\|_\infty \to 0 \quad \text{as} \quad \ell, k \to \infty. \tag{7.1}
\]

First we show that (7.1) holds for \( h \in C_c^\infty(\mathbb{R}^d) \otimes S_M \). Indeed, we have

\[
\lim_{\ell, k \to +\infty} \|T_{\varepsilon \ell} h - T_{\varepsilon k} h\|_\infty = 0. \tag{7.2}
\]

To see this, assume that \( h = \sum_i \varphi_i \otimes m_i \) with finite sum of \( i \), where \( \varphi_i \in C_c^\infty(\mathbb{R}^d) \) and \( m_i \in S_M \). Then for \( \varepsilon \ell < \varepsilon k \), one has

\[
T_{\varepsilon \ell} h(x) - T_{\varepsilon k} h(x) = \sum_i \int_{\varepsilon \ell < |y| \leq \varepsilon k} k(y) \varphi_i(x - y) dy \otimes m_i
\]

\[
= \sum_i \int_{\varepsilon \ell < |y| \leq \varepsilon k} k(y)(\varphi_i(x - y) - \varphi_i(x)) dy \otimes m_i
\]

\[
+ \sum_i \int_{\varepsilon \ell < |y| \leq \varepsilon k} k(y) \varphi_i(x) dy \otimes m_i,
\]

which tends to 0 as \( \ell, k \to +\infty \) by simple calculation using the size/cancellation condition (1.2)/(1.13) of the kernel.

Fix \( \delta > 0 \). By the density of \( C_c^\infty(\mathbb{R}^d) \otimes S_M \) in \( L^p(\mathcal{N}) \), for each \( n \geq 1 \), there exists \( g_n \in C_c^\infty(\mathbb{R}^d) \otimes S_M \) such that \( \|f - g_n\|_p \leq \frac{\delta}{2^n p} \). Now applying that \( (T_{\varepsilon j})_{j \in \mathbb{N}} \) is of weak type \((p, p)\) \((1 \leq p < \infty)\), there exists a projection \( e_n \in \mathcal{N} \) such that

\[
\sup_j \|e_n T_{\varepsilon j} (f - g_n)e_n\|_\infty < \frac{1}{n} \quad \text{and} \quad \varphi(e_n^\perp) < n^p \|f - g_n\|_p^p \leq \frac{\delta}{2^n}.
\]

Let \( e = \wedge_n e_n \). Then we have

\[
\varphi(e^\perp) < \delta \quad \text{and} \quad \sup_j \|e T_{\varepsilon j} (f - g_n)e\|_\infty < \frac{1}{n}, \forall n \geq 1. \tag{7.3}
\]

The uniform convergence (7.2) implies that for any \( n \geq 1 \), there exists a positive constant \( N_n \) such that for any \( \ell, k > N_n \)

\[
\|T_{\varepsilon \ell} g_n - T_{\varepsilon k} g_n\|_\infty < \frac{1}{n}.
\]

Finally together with (7.3), for any \( \ell, k > N_n \),

\[
\|e(T_{\varepsilon \ell} f - T_{\varepsilon k} f)e\|_\infty \leq \|T_{\varepsilon \ell} g_n - T_{\varepsilon k} g_n\|_\infty + \|e T_{\varepsilon \ell} (f - g_n)e\|_\infty + \|e T_{\varepsilon k} (f - g_n)e\|_\infty < \frac{3}{n},
\]

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which yields

$$\lim_{\ell, k \to \infty} \| e(T_{\ell f} - T_{k f}) e \|_{\infty} = 0.$$ 

This completes the proof. \(\square\)

**Remark 7.2** It is worth pointing out that at the moment of writing we have no idea how to strengthen the b.a.u. convergence in Theorems 1.4 and 1.5 to a.u. convergence. One reason is that Calderón–Zygmund operators are not positive.

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**Data availability** Data sharing not applicable to this article as no datasets were generated or analysed during the current study.

**Declarations**

**Conflict of interest** On behalf of all authors, the corresponding author states that there is no conflict of interest.

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