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Classical entanglement in polarization metrology

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Abstract

Quantum approaches relying on entangled photons have been recently proposed to increase the efficiency of optical measurements. We demonstrate here that, surprisingly, the use of classical light with entangled degrees of freedom can also bring outstanding advantages over conventional measurements in polarization metrology. Specifically, we show that radially polarized beams of light allow to perform real-time single-shot Mueller matrix polarimetry. Our results also indicate that quantum optical procedures requiring entanglement without non-locality can be actually achieved in the classical optics regime.

Keywords: entanglement and quantum nonlocality, applied classical electromagnetism, polarization

1. Introduction

In the last years, quantum information theory taught us that the use of entangled photons offers the unique advantage over classical light of providing more information in metrology applications, imaging and, more generally, in optical measurements [1–6]. However,
entanglement is not necessarily a signature of the quantum mechanical nature of a system. Indeed, one can distinguish between two types of entanglement: a) entanglement between spatially separated systems (inter-system entanglement) and b) entanglement between different degrees of freedom (DoFs) of a single system (intra-system entanglement) [7, 8]. Inter-system entanglement, or ‘nonlocal entanglement’, can occur only in bona fide quantum systems and may yield to nonlocal statistical correlations. Conversely, intra-system entanglement, or ‘local entanglement’, may also appear in classical systems and cannot generate nonlocal correlations [9]. As an example, photon pairs from atomic cascades [10] show nonlocal entanglement. In contrast, local entanglement can be found, e.g., between spatial and spin DoFs in single neutrons [11]. In classical optics, local entanglement between polarization and spatial DoFs of the same beam has been lately demonstrated in radially and azimuthally polarized beams of light [12–18]. Hereafter, we shall denote the occurrence of local entanglement in classical systems by ‘classical entanglement’ [8].

Recently, it has been suggested that quantum computing tasks requiring entanglement but not nonlocality can be efficiently accomplished in the classical optical regime [15, 22, 23]. However, when and how classical entanglement can be exploited in lieu of nonlocal entanglement to improve techniques of optical metrology, still remain largely open questions [24, 25].

In this work, we address some of these central issues by illustrating a representative example of the peculiar entanglement between polarization and spatial modes in radially polarized beams of light and its possible use for enhancing the efficiency of Mueller matrix polarimetry. The underlying idea is simple: in a conventional Mueller matrix measurement setting [26–29], an either transmissive or scattering material sample (the object) is illuminated with a light beam (the probe) prepared in, at least, four different polarization states in a temporal sequence. From the analysis of the polarization of the light transmitted or scattered, the optical properties of the object can be inferred. In the alternative setting we propose here, the object is probed only once with one light beam of radial polarization, as opposed to four differently polarized beams. Then, the light transmitted or reflected by the object is analyzed both in polarization and in spatial DoFs by means of suitable polarization and spatial mode selectors. In our setting, the polarization DoFs of the beam are used to actually probe the object and the spatial DoFs are used to post-select the polarization state of the light: this is the main idea presented in this work. This scheme, in principle, outperforms conventional ones because the radially polarized beam carries all polarizations at once in a classically entangled state, thus providing for a sort of ‘polarization parallelism’. For all practical applications where the optical properties of the sample change rapidly with time, our method presents an advantage over conventional Mueller matrix polarimetry [26, 27, 30–35]. However, in practice, the detection setup required by our scheme is more involved than a conventional polarimetry one and is, therefore, potentially more sensitive to measurement errors.

Although our conclusions here are strictly valid only for optical elements that do not alter significantly the spatial structure of the probe beam, this is not a serious restriction. For example, all optical elements routinely used on an optical bench like single-mode fibers, retardation plates, birefringent prisms, optical rotators, etc, fall in this category. Another requirement for the validity of our scheme is that the polarization properties of the sample must

5 In the literature this phenomenon is also refereed to as ‘structural inseparability’ [19, 20] and ‘nonquantum entanglement’ [21].
be homogeneous over the beam cross section. Considering that an ordinary radially polarized beam of light may be prepared with a waist of the order of hundreds of μm, such a requirement does not represent an actual limitation.

2. Jones vectors, entanglement and radially polarized beams

Consider a monochromatic beam of light of angular frequency ω, propagating along the z-axis of a Cartesian reference frame (x, y, z) and polarized in the (x, y) plane. In the paraxial approximation [36], the electric field can be written as

\[ E(r, t) = \text{Re} \left[ E(r) \exp \left( i(kz - \omega t) \right) \right], \]

with \( \psi(r) \) denoting the spatial mode of the beam, \( e_x, e_y, \) and \( e_z \) being unit vectors in the x, y and z directions, respectively. In the expression above, \( r = xe_x + ye_y + ze_z \) stands for the position vector, and the two complex numbers \( A_0 \) and \( A_1 \) represent the amplitudes of the electric field along the x- and y-axis, respectively. A convenient vector notation for fields of the form (1) was introduced by R C Jones in the 1940s [27, 37]:

\[ E(r) = \begin{bmatrix} A_0 \\ A_1 \end{bmatrix} \psi(r). \]  

(2)

With this notation, the identification of the polarization and the spatial DoFs, represented by the Jones vector \( [A_0, A_1]^T \) and the scalar field \( \psi(r) \), respectively, becomes straightforward.

For a field of the form (2), polarization and spatial DoFs are said to be separable [13] because the expression of \( E(r) \) is factorizable in the product of a single space independent vector, and a single scalar field. This mathematical property reflects the absence of physical coupling between polarization and spatial DoFs. However, in general, polarization and spatial DoFs can be coupled and when this happens the factorizable representation (2) is no longer valid. Consider, for example, the electric field of a beam of light nonuniformly polarized in the (x, y) plane, which can be expressed as

\[ E(r) = A_{00}e_x\psi_{10}(r) + A_{01}e_y\psi_{01}(r) + A_{10}e_x\psi_{10}(r) + A_{11}e_y\psi_{01}(r), \]

(3)

where \( \psi_{mn}(r) \), with \( m, n \in \{0, 1, 2, \ldots\} \), is the Hermite–Gauss (HG) solution of the paraxial wave equation of order \( N = m + n \) [36] and \( A_{ij} \) denotes a complex amplitude of the field, with \( i, j \in \{0, 1\} \). In the Jones notation, (3) takes either of the two following forms:

\[ E(r) = \begin{bmatrix} A_{00}\psi_{10}(r) + A_{01}\psi_{01}(r) \\ A_{10}\psi_{10}(r) + A_{11}\psi_{01}(r) \end{bmatrix} \]

(4a)

\[ = \begin{bmatrix} A_{00} \\ A_{10} \end{bmatrix} \psi_{10}(r) + \begin{bmatrix} A_{01} \\ A_{11} \end{bmatrix} \psi_{01}(r). \]

(4b)

By writing the electric field as in (4a), it follows that the beam has a nonuniform polarization pattern because the Jones vector varies with the position vector \( r \). On the other hand, when \( E(r) \) is written in the form (4b), it appears evident that polarization and spatial coupling is absent.

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6 Please note that here \( \psi(r) = \psi(x, y; z) \) is a paraxial field, so \( x, y \) are the Cartesian coordinates in the transverse plane and \( z \) must be considered as a varying parameter.
DoFs are now *nonseparable*, or entangled, because one needs two coordinate-independent Jones vectors and two independent scalar fields, $\psi_{10}$ and $\psi_{01}$, to represent the electric field.

Mathematically speaking, occurrence of entanglement requires an expression to be written as the sum of tensor products of two or more vectors belonging to different vector spaces. This is precisely what occurs in (4b), where we have a polarization vector space and a spatial vector space, which can be represented by the polarization Poincaré sphere [27] and the first-order spatial modes Poincaré sphere [38], respectively, shown in Figure 1. This qualitative discussion may be made more quantitative by considering a radially polarized beam of light as a specific example that can be represented by (3),

$$E (r) = \frac{1}{\sqrt{2}} \left[ e, \psi_{10} (r) + e, \psi_{01} (r) \right].$$

whose characteristics are illustrated in Figure 2. One can obtain (5) from (3) by putting $A_{00} = 1/\sqrt{2} = A_{11}$ and $A_{01} = 0 = A_{10}$. This suggests, as shown in great detail in [13, 19], that it is possible to represent a radially polarized beam by an abstract four-dimensional vector, hereforth denoted with the ket $|E\rangle \triangleq [A_{00}, A_{01}, A_{10}, A_{11}]$, living in a four-dimensional two-qubit Hilbert space: $|E\rangle \in H_2 = H_{\text{pol}} \otimes H_{\text{spa}}$. Here $H_{\text{pol}} = \text{span} \{e, e\}$ denotes the polarization-qubit space and $H_{\text{spa}} = \text{span} \{\psi_{10} (r), \psi_{01} (r)\}$ indicates the spatial-qubit space. We identify the standard basis for the polarization qubit with horizontal and vertical polarization states, and the standard basis for the spatial qubit with HG modes of order $n + m = 1$, namely

$$|0\rangle_{\text{pol}} \triangleq e_x, \quad |1\rangle_{\text{pol}} \triangleq e_y, \quad (6a)$$

$$|0\rangle_{\text{spa}} \triangleq \psi_{10} (r), \quad |1\rangle_{\text{spa}} \triangleq \psi_{01} (r). \quad (6b)$$

Although throughout this work we deal with classical fields only, the quantum notation adopted here is very convenient to reveal the analogies between classical optics and quantum information theory. We can write a complete orthonormal basis of $H_2$ in the form of a tensor product: $|i, j\rangle = |i\rangle_{\text{pol}} \otimes |j\rangle_{\text{spa}}$ with $i, j \in \{0, 1\}$, where the first index $i$ marks the polarization qubit and the second index $j$ the spatial one, namely

7 The HG-modes $\psi_{mn} (r)$ are independent in the sense that they are orthonormal with respect to the spatial scalar product defined as $\int \int \psi_{mn} (r) \psi_{m'n'} (r) \, dx dy = \delta_{nn'} \delta_{mm'}$, with $m, m', n, n' \in \{0, 1, 2, \ldots \}$. 

---

**Figure 1.** Schematic visualization of (a) the polarization Poincaré sphere representation of the binary Hilbert space $H_{\text{pol}}$ [27] and (b) Poincaré sphere representation of first-order spatial modes Hilbert space $H_{\text{spa}}$ [38].
With this notation we can represent the electric field (3) as a vector in \( \mathcal{H}_2 \):

\[
|E\rangle = A_{00} |0, 0\rangle + A_{01} |0, 1\rangle + A_{10} |1, 0\rangle + A_{11} |1, 1\rangle,
\]

and a radially polarized beam can be described by the ket

\[
|E\rangle = \frac{1}{\sqrt{2}} (|0, 0\rangle + |1, 1\rangle).
\]  

This representation for the radially polarized beam is formally equivalent (isomorphic) to a Bell state of two qubits [39]. Hence, the polarization and spatial DoF may be treated as two qubits that are classically entangled. However, the key difference between (9) and a bona fide quantum optical Bell state is that in the latter the two qubits are encoded in the polarization DoFs of two separated photons (inter-system entanglement), while in (9) the two qubits are encoded in the polarization and the spatial binary DoFs of a single light beam (intra-system entanglement).
3. Stokes parameters and the Liouville representation of a quantum state

The representations (3) and (8) for the electric field of a light beam are adequate as long as one is concerned with detection schemes that can resolve both polarization and spatial DoFs. When this is not the case, as in conventional Mueller matrix polarimetry, it becomes necessary to introduce a more general representation, namely the 4×4 coherency matrix \( \rho \) of the field, defined in terms of the electric field amplitudes \( A_{ij} \) as

\[
|E \rangle \langle E| = \rho = \begin{bmatrix}
A_{00}A_{00}^* & A_{00}A_{01}^* & A_{00}A_{10}^* & A_{00}A_{11}^* \\
A_{01}A_{00}^* & A_{01}A_{01}^* & A_{01}A_{10}^* & A_{01}A_{11}^* \\
A_{10}A_{00}^* & A_{10}A_{01}^* & A_{10}A_{10}^* & A_{10}A_{11}^* \\
A_{11}A_{00}^* & A_{11}A_{01}^* & A_{11}A_{10}^* & A_{11}A_{11}^*
\end{bmatrix}.
\] (10)

Of course, at this stage (10) still contains the same amount of information as (8). As a specific example, the coherency matrix of the radially polarized beam (9) can be simply written as

\[
\rho = \frac{1}{2} \begin{bmatrix}
1 & 0 & 0 & 1 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
1 & 0 & 0 & 1
\end{bmatrix}.
\] (11)

Suppose now to have a detection scheme that is not capable of resolving the spatial DoFs. In this case, (8) would furnish redundant information about the spatial DoFs that is not at our disposal. However, a proper representation of the beam can then be obtained from (10) by tracing out the unobservable spatial DoFs. In this manner one obtains the reduced 2×2 polarization coherency matrix \( \rho_{\text{pol}} \) that encodes all the available information about the polarization of the light beam, irrespective of the spatial DoFs:

\[
\rho \rightarrow \rho_{\text{pol}} = \text{tr}_{\text{spa}}(\rho)
\]

\[
= \sum_{i=0}^{1} \text{tr}_{\text{spa}}(i | \rho | i)_{\text{spa}}
\]

\[
= AA^\dagger,
\] (12)

where \( \text{tr}_{\text{spa}}(\ldots) \) denotes the trace with respect to the spatial DoFs and the spatial kets \( |0\rangle_{\text{spa}} \) and \( |1\rangle_{\text{spa}} \) are defined in (6b). Here

\[
A = \begin{bmatrix}
A_{00} & A_{01} \\
A_{10} & A_{11}
\end{bmatrix},
\] (13)

is a 2×2 matrix whose elements are the coefficients \( A_{ij} \) of the ket expansion (8). In a similar manner, one can calculate the reduced 2×2 spatial coherency matrix \( \rho_{\text{spa}} \) that encodes all the available information about the spatial modes of the light beam, irrespective of the polarization:

\[
\rho_{\text{spa}} = \text{tr}_{\text{pol}}(\rho)
\]

\[
= (A^\dagger A)^T,
\] (14)

where \( \text{tr}_{\text{pol}}(\ldots) \) denotes the trace with respect to the polarization DoFs.
From the definition (12) it follows that $\rho_{\text{pol}}$ is Hermitian and positive semidefinite. Therefore, it admits a Liouville representation of the form

$$\rho_{\text{pol}} = \frac{1}{2} \sum_{\mu=0}^{3} S_{\mu} \sigma_{\mu},$$  \hspace{1cm} (15)pol

where the coefficients $S_{\mu}$ are real numbers and the set $\{\sigma_{\mu}\}_{\mu=0}^{3}$ of $2 \times 2$ Hermitean matrices forms a complete basis of observables [40]. The factor $1/2$ in front of (15) is conventional. In classical polarization optics, the coefficients $S_{\mu}$ are known as the Stokes parameters of the beam [41] and the basis set $\{\sigma_{\mu}\}_{\mu=0}^{3}$ is constituted by the four Pauli matrices

$$\sigma_{0} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \hspace{0.5cm} \sigma_{1} = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix},$$

$$\sigma_{2} = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \hspace{0.5cm} \sigma_{3} = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix},$$  \hspace{1cm} (16)

which are orthogonal with respect to the scalar product defined as $\sigma_{\mu} \sigma_{\nu} = 2 \delta_{\mu \nu}$. From this property and the definition (15) it follows that

$$S_{\mu} = \text{tr} \left( \rho_{\text{pol}} \sigma_{\mu} \right).$$  \hspace{1cm} (17)

For the radially polarized beam (9), $A = I_{2}/\sqrt{2}$ and $\rho_{\text{pol}} = I_{2}/2 = \rho_{\text{spa}}$, where $I_{2}$ denotes the $2 \times 2$ identity matrix. In the language of classical polarization optics, this means that the radially polarized beam is completely unpolarized: $S = [S_{0}, S_{1}, S_{2}, S_{3}]^{T} = [1, 0, 0, 0]^{T}$. This observation may appear confusing because (4a) and figure 2 show that a radially polarized beam possesses a well defined local, i.e. defined at each point $(x, y)$ of its transverse spatial profile, Jones vector given by

$$E(r) = \frac{1}{\sqrt{2}} \begin{bmatrix} \psi_{10}(r) \\ \psi_{01}(r) \end{bmatrix}. $$  \hspace{1cm} (18)

However, $\rho_{\text{pol}}$ is obtained from $\rho$ after tracing out the spatial DoFs. From a physical point of view, this corresponds to measuring the global Stokes parameters of the beam, as a whole, with ‘bucket’ detectors that integrate the intensity of light over all the cross section of the beam. A similar situation is encountered for photon-pairs in a Bell state: Although the polarization of the two-photon state is perfectly defined (pure state), each of the two photons, when observed separately, appears as completely unpolarized (mixed state) [42].

4. Mueller matrix polarimetry

Typically, in a conventional polarimetry setup, an either transmissive or scattering material sample (the object) is illuminated with a light beam (input beam) that, as a result of the interaction with the object, emerges transformed (output beam). In this section we will study how radially polarized beams transform under the action of a polarization-affecting optical element, having in mind the final goal of measuring the Mueller matrix of the latter. From a mathematical point of view, here we consider local linear transformations of the form $T_{\text{pol}} \otimes T_{\text{spa}}$, namely transformations that act on each DoF separately, where $T_{d}: \mathcal{H}_{d} \rightarrow \mathcal{H}_{d}$ with $d \in \{\text{pol, spa}\}$ is a $2 \times 2$ complex matrix known as Jones matrix in polarization optics. As in
this work we are concerned with optical elements affecting polarization DoFs solely, henceforth we assume \( T_{\text{pol}} = I_2 \), and we will omit the subscript ‘pol’ in \( T_{\text{pol}} \).

Under the action of \( T \), the generic ket (8) transforms as

\[
|E\rangle \rightarrow |E'\rangle = (T \otimes I_2) |E\rangle = \sum_{i,j=0}^{1} A'_{ij} |i, j\rangle,
\]

with \( A' := TA \). The transformation (19) links the amplitudes \( A_{ij} \) of the input beam to the amplitudes \( A'_{ij} \) of the output beam. However, in real-world experiments intensities, rather than amplitudes, are measured. Therefore, it becomes necessary to specify the type of intensity measurements actually performed upon the output beam. According to whether the detectors are or are not insensitive to the spatial DoFs of the beam, one deals with either (a) single-DoF polarimetry or (b) two-DoF polarimetry. Case (a) coincides with the conventional Mueller matrix polarimetry, while case (b) gives the novel detection scheme that we propose here. Let us shortly review case (a) first.

4.1. Single-DoF polarimetry

From (19) and the definition (12) it follows that \( \rho_{\text{pol}} \) transforms under \( T \) as

\[
\rho_{\text{pol}} \rightarrow \rho'_{\text{pol}} = A'A'^* = TAAT^*.
\]

Suppose we prepare sequentially the input beam in four different polarization states labeled by the index \( \alpha \in \{0, 1, 2, 3\} \). For example, \( \alpha = 0 \) may denote horizontal polarization, \( \alpha = 1 \) vertical polarization, \( \alpha = 2 \) diagonal polarization and \( \alpha = 3 \) left-circular polarization. Then, in the Liouville representation (15) and by using the definition (17), the transformation (20) can be written, for each different input beam labeled by the index \( \alpha \), as

\[
S_{\mu}(\alpha) = \sum_{\nu=0}^{3} M_{\mu\nu} S_{\nu}(\alpha), \quad \mu, \alpha \in \{0, 1, 2, 3\},
\]

where \( S_{\nu}(\alpha) \) and \( S_{\mu}'(\alpha) \) denote the Stokes parameters of the input and output beams, respectively, and the 16 real numbers

\[
M_{\mu\nu} = \frac{1}{2} \text{tr} \left( \sigma_\mu T \sigma_\nu T^* \right),
\]

are the (unknown) elements of the sought Mueller matrix \( M \). Then, (21) may be seen as a linear system of 16 equations and 16 unknowns that can be easily solved, for example, by defining the two \( 4 \times 4 \) matrices \( V \) and \( V' \) as: \( [V]_{\mu\nu} := S_{\nu}(\alpha) \) and \( [V']_{\mu\nu} := S'_{\mu}(\alpha) \). This permits (21) to be rewritten in the simple matrix form \( V' = MV \), and the Mueller matrix \( M \) can be finally evaluated as

\[
M = V'V^{-1},
\]

providing that \( \det(V) \neq 0 \).

---

8 When the object is a depolarizing optical element, then (22) must be replaced with \( M_{\mu\nu} = \text{tr}(\overline{\sigma_\mu T \sigma_\nu T})/2 \), where the overline symbol denotes average over a stochastic set [27]. For the sake of clarity, in the remainder we will consider only nondepolarizing optical elements.
This is the essence of conventional Mueller matrix polarimetry [43]. Of course, in a situation where experimental errors may occur, the simple linear inversion algorithm (23) often does not suffice and more sophisticated inversion methods must be used instead [28, 29]. However, the lesson to be learned here is that conventional Mueller matrix polarimetry needs the input beam to be sequentially prepared in, at least, four different polarization states to gain complete information about the object. Conversely, we are going to show soon how the same amount of information can be obtained by probing the object only once with a radially polarized beam.

4.2. Two-DoF polarimetry

We now consider a detection scheme that is capable of resolving both the polarization and the spatial DoFs. The complete coherency matrix (10) can also be written in a Liouville form similar to (15) as

$$\rho = \frac{1}{4} \sum_{\mu,\lambda=0}^{3} S_{\mu \lambda} \left( \sigma_\mu \otimes \sigma_\lambda \right),$$

(24)

where we have defined the two-DoFs Stokes parameters as

$$S_{\mu \lambda} = \text{tr} \left[ \rho \left( \sigma_\mu \otimes \sigma_\lambda \right) \right].$$

(25)

These quantities are the classical optics analogue of the two-photon Stokes parameters introduced in [44, 45]. However, while in [45] the two polarization qubits are encoded in two separated photons, in our case the polarization qubit and the spatial qubit are encoded in the same radially polarized beam of light. Therefore, the two-DoFs Stokes parameters give the intrabeam correlations between polarization and spatial DoFs [46]. In order to measure these correlations, one needs a detection scheme capable of resolving both DoFs. Such an experimental apparatus will be studied in the next section. For the radially polarized beam represented by (11), the two-DoFs Stokes parameters take the particularly simple form

$$S_{\mu \lambda} = \lambda_\mu \delta_{\mu \lambda}, \quad \text{where} \quad \{\lambda_\mu\}_{\mu=0}^{3} = \{1, 1, -1, 1\}.$$  

(26)

From (19) it follows that, under the action of $T$, (11) transforms as

$$\rho \rightarrow \rho' = (T \otimes I_3) \rho (T^\dagger \otimes I_3) = \frac{1}{4} \sum_{\mu=0}^{3} \lambda_\mu \left( T \sigma_\mu T^\dagger \right) \otimes \sigma_\mu.$$  

(27)

Substituting (27) into (25) yields

$$S'_{\mu \lambda} = \text{tr} \left[ \rho' \left( \sigma_\mu \otimes \sigma_\lambda \right) \right] = \frac{1}{4} \sum_{\alpha=0}^{3} \lambda_\alpha \text{tr} \left( \sigma_\mu T \sigma_\alpha T^\dagger \right) \text{tr} \left( \sigma_\alpha \sigma_\lambda \right) = M_{\mu \lambda} \lambda_\lambda,$$

(28)

Here we are using the standard properties of the direct product of matrices: $(A \otimes B)(C \otimes D) = AC \otimes BD$ and $\text{tr} (A \otimes B) = \text{tr} (A) \text{tr} (B)$. 

9
where (22) has been used in the last line. Since either \( \lambda_\nu = 1 \) for \( \nu \in \{0, 1, 3\} \) or \( \lambda_\nu = -1 \) for \( \nu = 2 \), from (28) it follows that the two-DoF Stokes parameters furnish a direct measure of the Mueller matrix elements:

\[
M_{\mu\nu} = \begin{cases} 
-S'_{\mu\nu}, & \text{for } \nu = 2, \\
S'_{\mu\nu}, & \text{for } \nu \neq 2.
\end{cases}
\]  

(29)

This shows that the Mueller matrix of an object can be obtained from the measurement of the 16 two-DoFs Stokes parameters \( S'_{\mu\nu} \), with a single radially polarized input beam, allowing the performance of single-shot full polarimetry. We remark that the above derivation relies upon the assumption that the optical properties of the object do not vary over the cross section of the input beam, namely that \( T \) is independent of \( x \) and \( y \).

This interesting result can be understood as an effect of post-selection on an entangled state. In a single-DoF polarimetry setup, the polarization state of the input beam is preselected before the interaction with the object, as shown in (21). Consequently, the object can be probed by only a single polarization state at a time. Vice versa, in our two-DoF polarimetry scheme the polarization state of the input beam is post-selected after the interaction with the object via the two-DoF correlation measurements. Therefore, the object is probed, at once, by all possible polarization states carried by the radially polarized beam. This magic is made possible by the entangled structure (5) or (9) of the beam: projecting the output beam on a specific spatial mode uniquely determines, \textit{a posteriori}, the polarization of the input beam, which may be either linear, diagonal or circular as shown in figure 2.

5. Real-time single-shot Mueller matrix polarimetry

In this section we propose a feasible experimental scheme for real-time single-shot Mueller matrix polarimetry. The measurement setup is illustrated in figures 3–5. The procedure we present here is an extension of the conventional polarization measurement (CPM) technique to beams of light with coupled polarization and spatial DoFs. According to (29), the fundamental quantities to estimate are the sixteen two-DoF Stokes parameters \( S'_{\mu\nu} \) that contain all the information about the Mueller matrix.

The procedure is as follows: a radially polarized beam of light (the probe) is sent through a material sample (the object) whose Mueller matrix has to be determined. Then, the idea is to first project the beam transmitted across the sample onto the four independent spatial modes \( \{\psi_{00}, \psi_{01}, \psi_{+}, \psi_{-}\} \) (also denoted \( \{\psi_{0}, \psi_{\pm}, \psi_{\mp}\} \)).\footnote{Diagonal/antidiagonal spatial modes are defined as \( \psi_{\pm} = (\psi_{0} \pm i \psi_{1})/\sqrt{2} \) and left/right-circular spatial modes are written as \( \psi_{\mp} = (\psi_{0} - i \psi_{1})/\sqrt{2} \) and \( \psi_{\mp} = (\psi_{0} + i \psi_{1})/\sqrt{2} \), respectively. Similarly, diagonal/antidiagonal polarization states are defined as \( e_{\pm} = (e_{x} \pm e_{y})/\sqrt{2} \) and left/right-circular polarization ones as \( e_{\mp} = (e_{x} \pm e_{y})/\sqrt{2} \) and \( e_{\mp} = (e_{x} - i e_{y})/\sqrt{2} \), respectively.}

These projections post-select the four independent polarization states \( \{e_{x}, e_{y}, e_{+}, e_{-}\} \), as explained in detail in subsection 5.1. It is important to stress that we perform these operations without acting directly on the polarization DoFs, which are analyzed only in a subsequent stage. For this, the light transmitted by the sample is split into three identical beams of equal intensity, which then are sent to mode converters (MCs) and mode beam splitters (MBSs). The three MCs, denoted as A, B and C in figure 3, define in which basis \( (\psi_{1}, \psi_{r}) \), \( (\psi_{\mp}, \psi_{\mp}) \) or \( (\psi_{0}, \psi_{0}) \) the incoming beam is going to be...
measured. MC A, which transforms the modes $\psi_L(0)$ into $(\psi_{01}, \psi_{10})$, is made of a $\pi/2$-converter, rotated by the angle $\theta = \pi/4$ with respect to the horizontal axis (see appendix A). MC B, which transforms the modes $(\psi_{+}, \psi_{-})$ into $(\psi_{01}, \psi_{10})$, is made of a $\pi$-converter, rotated by the angle $\theta = \pi/8$ with respect to the horizontal axis (see appendix A). MC C is made of empty space and does not change the modes. It can be shown [47] that a $\pi$-MC can be physically realized with two identical cylindrical lenses separated by a distance $2f$ equal to two focal lengths ($2f$ CL), as shown in figure 4(a). Similarly, a $\pi/2$-MC is made from two identical cylindrical lenses separated by a distance $\sqrt{2}f$ ($\sqrt{2}f$ CL), as illustrated in figure 4(b). Each MC is coupled to a MBS, which splits up a beam into its $\psi_{10}$ and $\psi_{01}$ spatial components. The MBS is made of a modified Mach–Zehnder interferometer (MZ) with an extra mirror in one
arm and a half-wave plate (HWP) in the other arm, followed by another HWP in one output port, as shown in figure 4 (d) (see appendix B and [48, 49]). As a result of these transformations, the MBS placed behind MC A splits up the incoming beam into its circular spatial components, $\psi_L$ and $\psi_R$. MC B splits the beam into its diagonal and antidiagonal spatial components $\psi_+\text{ and }\psi_-$ and MC C into the $\psi_{10}$ and $\psi_{01}$ components. By selecting one of the outputs of MBS A and of MBS B and the two outputs of MBS C, one has access to the four spatial modes $\{\psi_{10}, \psi_{01}, \psi_+, \psi_-\}$. With this operation, we have physically acted only upon the spatial modes of the beam, and we will now analyze their polarization. This corresponds to a post-selection of the polarization state of the probe beam. This is possible thanks to the entanglement between the polarization and the spatial DoFs.

The light exiting an output port of each MBS can be either directly detected, or sent through a CPM setup shown in figure 5. A beam entering the CPM is split into three beams of equal intensity. Each beam passes through a polarization converter (PC) denoted A, B and C in figure 3. They are made from, respectively, a) a quarter-wave plate (QWP) whose fast axis is tilted by $\pi/4$ with respect to the horizontal direction; b) a HWP with the fast axis rotated by $\pi/8$ with respect to the horizontal direction; and c) empty space. Each PC is coupled to a polarizing beam splitter (PBS), which splits a beam into its horizontal $e_x$ and vertical $e_y$ polarization components. The three combinations A, B and C of PCs and PBSs project the entering beams onto three mutually unbiased pairs of polarization states, namely, a) horizontal/vertical: $\{e_x, e_y\}$; b) diagonal/antidiagonal: $\{e_+, e_-\}$; and c) left/right-circular: $\{e_L, e_R\}$, respectively. The intensity $I_{\alpha\beta}$ of the light projected in the state $e_{\psi_{\alpha\beta}}$ is recorded by a photo-detector that is
identified with the same pair of indices \( \alpha \beta \): \( \alpha, \beta \in \{0, ..., 3\} \). Here the first index \( \alpha \) marks the polarization qubit and the second index \( \beta \) the spatial one, as shown in figure 1.

When four independent spatial modes are sorted \( \psi_0, \psi_1, \psi_2, \psi_3 \equiv \{ , , , \} \) by proper combinations of MCs and MBSs, and four independent polarization states \( \{e_0, e_1, e_2, e_3\} \) are selected by convenient sequences of PCs and PBSs, then the sixteen two-DoF Stokes parameters \( \mu_{\alpha \beta} \) can be entirely determined.

Please note that the scheme described above works equally well for other cylindrical vector beams, such as, e.g., azimuthally polarized optical beams. It is the Liouville form of the probe beam (24) that determines the relation between the measurement outcomes and the Mueller matrix elements. Hence, for example, in the case of probing the sample with azimuthally polarized light, one needs to use \( \{\lambda_{10}^{10}, 0, 0, 0\} = \{1, -1, -1, -1\} \) in (26), which subsequently alters also the equations (27) to (29). As in the strong focusing regime a nondesirable longitudinal electric field component occurs in a radially polarized beam of light, it might be advisable in some situations to use azimuthally polarized light.

In an ideal situation, the minimal number of detectors needed for this measurement is clearly 16. However, in real-world experiments where uncontrollable losses may occur, a maximal number of 10 additional detectors (colored gray in figures 3 and 5) may be used to ensure proper normalization of all measured intensities. It should be noticed that also in conventional Mueller matrix polarimetry at least 16 independent intensity measurements are required. Since in a conventional scheme the probe beam is not processed by so many optical
elements as in our case, the losses are less than in the proposed scheme. However, we would like to stress that this is a purely technical limitation that eventually can be overcome. Conversely, the major advantage in our scheme is that we can perform the 16 measurements at the same time, thus providing a ‘real-time’, and potentially fast, Mueller matrix determination.

5.1. Determining the two-DoF Stokes parameters

In the remainder of this section, we will illustrate explicitly the two-DoF Stokes parameters measurement process putting particular emphasis on the post-selection technique. For the sake of clarity, we will consider only the case of nondepolarizing (unknown) transmitting objects. More information about single- and two-qubit operations can be found in appendix A.

The radially polarized input beam (9) can be represented by

\[
|E\rangle = \frac{1}{\sqrt{2}} \left( |e_+\rangle |\psi_{10}\rangle + |e_-\rangle |\psi_{01}\rangle \right).
\]

(30)

where the two-qubit basis (7) has been recast here in a more suggestive form. After interacting with the object, characterized by the (unknown) Jones matrix \(T\), the state of the input beam is transformed according to:

\[
|E\rangle \rightarrow |E'\rangle = \frac{1}{\sqrt{2}} \left[ (T|e_+\rangle) |\psi_{10}\rangle + (T|e_-\rangle) |\psi_{01}\rangle \right].
\]

(31)

Using the decompositions of a radially polarized beam shown in figure 2, the state \(|E'\rangle\) of the transmitted beam can also be written in the diagonal and circular mode bases as:

\[
|E'\rangle = \frac{1}{\sqrt{2}} \left[ (T|e_+\rangle) |\psi_+\rangle + (T|e_-\rangle) |\psi_-\rangle \right]
\]

(32a)

\[
= \frac{1}{\sqrt{2}} \left[ (T|e_+\rangle) |\psi_{+}\rangle + (T|e_-\rangle) |\psi_{-}\rangle \right].
\]

(32b)

The three combinations of MCs and MBSs project the state \(|E'\rangle\) onto the four independent modes \(|\psi_{10}\rangle, |\psi_{01}\rangle, |\psi_{+}\rangle, |\psi_{-}\rangle\). These are two-step operations: first a MC transforms the state \(|E'\rangle\) into a chosen basis, then a MBS projects the transformed state into the ‘linear’ basis \(|\psi_{01}\rangle, |\psi_{+}\rangle\). For example, from (A.5) and (32a) it follows that the transformation performed by MC \(B\) (first step) produces

\[
\text{MC } B \rightarrow U_{e}(\pi/8)|E'\rangle = \frac{-i}{\sqrt{2}} \left[ (T|e_+\rangle) |\psi_{10}\rangle + (T|e_-\rangle) |\psi_{01}\rangle \right].
\]

(33)

Then, in the second step the MBS projects the state onto \(\psi_{01}\) and the result is:

\[
\text{MBS} \rightarrow \langle \psi_{01} | U_{e}(\pi/8) |E'\rangle = \frac{-i}{\sqrt{2}} T|e_-\rangle.
\]

(34)

These projections provide post-selection of the four input polarization states \(e_+, e_-, e_+, e_-\), according to

\[
\langle \psi_{10} | E' \rangle = \frac{1}{\sqrt{2}} T|e_+\rangle,
\]

(35a)
\begin{equation}
\langle \psi_{01} | E' \rangle = \frac{1}{\sqrt{2}} T | e_1 \rangle,
\tag{35b}
\end{equation}

\begin{equation}
\langle \psi_{s1} | E' \rangle = \frac{1}{\sqrt{2}} T | e_- \rangle,
\tag{35c}
\end{equation}

\begin{equation}
\langle \psi_{L1} | E' \rangle = \frac{1}{\sqrt{2}} T | e_y \rangle,
\tag{35d}
\end{equation}

where irrelevant overall phase factors have been omitted. The states (35a–d) exiting four different ports of the three MBSs, are then analyzed by CPMs that allow evaluation of all the elements of the Jones matrix $T$. Consider, for example, (35a). When this state is sent through a CPM, the following intensities can be measured:\footnote{As a technical remark, it should be noticed that the post-selected set of polarization vectors $\{ e_+, e_- , e_y, e_L \}$ does not coincide with the analyser basis $\{ e_+, e_- , e_y , e_L \}$. However, this is not a problem as long as both sets of vectors are linearly independent.}

\begin{equation}
I_{00} \equiv \left| \langle e_+ | \langle \psi_{01} | E' \rangle \right|^2 = \frac{1}{2} \left| \langle e_+ | T | e_+ \rangle \right|^2,
\tag{36a}
\end{equation}

\begin{equation}
I_{10} \equiv \left| \langle e_+ | \langle \psi_{s1} | E' \rangle \right|^2 = \frac{1}{2} \left| \langle e_+ | T | e_- \rangle \right|^2,
\tag{36b}
\end{equation}

\begin{equation}
I_{20} \equiv \left| \langle e_+ | \langle \psi_{L1} | E' \rangle \right|^2 = \frac{1}{2} \left| \langle e_+ | T | e_y \rangle \right|^2,
\tag{36c}
\end{equation}

\begin{equation}
I_{30} \equiv \left| \langle e_L | \langle \psi_{01} | E' \rangle \right|^2 = \frac{1}{2} \left| \langle e_L | T | e_+ \rangle \right|^2.
\tag{36d}
\end{equation}

When the 16 intensities $I_{i,j}$ are measured, eventually the 16 two-DoF Stokes parameters $S_{\mu \nu}$ can be determined according to the formulas (A.13) given in appendix A.2.

6. Conclusions

In this work we have shown how to exploit classical entanglement in polarization metrology, by using radially polarized beams of classical light to perform real-time single-shot Mueller matrix measurements. Our main result is that the Mueller matrix elements are simply proportional to the two-DoF Stokes parameters that quantify the intrabeam correlations between polarization and spatial DoFs of a radially polarized beam. The novelty of our approach is that while the speed of conventional Mueller matrix measurements is limited by the need to probe the sample four times in sequence with light of different polarization, in our setting the four probes are made in parallel via a single radially polarized beam of light. In conclusion, we have established a novel two-DoF polarimetry scheme, which is the classical wave analogue of two-photon polarimetry [45]. Our results generalize and extend to the classical optics regime some already known techniques of quantum metrology [4, 50, 51]. Last but not least, our work furnishes another clear proof that optical measurements requiring entanglement but not nonlocality may be accomplished by using classical light.
Appendix A. Qubit operations

A.1. Single-DoF operations

Consider the single-qubit two-dimensional Hilbert space \( \mathcal{H}_1 = \text{span}\{ |0\rangle, |1\rangle \} \), where the standard basis states \( |0\rangle \) and \( |1\rangle \) are defined as the eigenstates of the \( \sigma_3 \) Pauli matrix in (16), irrespective of the specific DoF encoding the qubit. All the results obtained in this appendix are indeed equally valid for both polarization and spatial qubits, as defined by (6a) and (6b). Similarly, the basis vectors \( \{ |+\rangle, |-\rangle \} \) and \( \{ |L\rangle, |R\rangle \} \) are defined as the eigenstates of the remaining two Pauli matrices \( \sigma_1 \) and \( \sigma_2 \), respectively, where

\[
|+\rangle = \frac{|0\rangle + |1\rangle}{\sqrt{2}}, \quad
|-\rangle = \frac{|0\rangle - |1\rangle}{\sqrt{2}}, \quad (A.1a)
\]

\[
|L\rangle = \frac{|0\rangle + i|1\rangle}{\sqrt{2}}, \quad
|R\rangle = \frac{|0\rangle - i|1\rangle}{\sqrt{2}}. \quad (A.1b)
\]

Rotatable \( \pi \)- and \( \pi/2 \)-converters permit any transformation between these basis vectors [49]. According to [52], the unitary matrices representing \( \pi \)- and \( \pi/2 \)-converters can be written as:

\[
U_\pi = e^{-i \pi/2} \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}, \quad
U_{\pi/2} = e^{-i \pi/4} \begin{bmatrix} 1 & 0 \\ 0 & i \end{bmatrix}, \quad (A.2)
\]

where the conventional overall phase factors are fixed by the condition that, for the polarization qubit, the fast axes of both HWP and QWP are horizontal. The unitary matrix \( U_\varphi(\theta) \), with \( \varphi \in \{ \pi, \pi/2 \} \) for a \( \varphi \)-converter rotated by an angle \( \theta \), is given by

\[
U_\varphi(\theta) = D(\theta) U_\pi D(-\theta), \quad (A.3)
\]

where \( D(\theta) \) denotes the standard \( 2 \times 2 \) rotation matrix:

\[
D(\theta) = \begin{bmatrix} \cos \theta & -\sin \theta \\ \sin \theta & \cos \theta \end{bmatrix}. \quad (A.4)
\]

For example, we can use (A.3) to transform the vectors (A.1a) and (A.1b) into the standard basis, as follows:

\[
U_\pi (\pi/8)|+\rangle = -i|0\rangle, \quad
U_\pi (\pi/8)|-\rangle = -i|1\rangle,
U_{\pi/2}(\pi/4)|L\rangle = |0\rangle, \quad
U_{\pi/2}(\pi/4)|R\rangle = -i|1\rangle. \quad (A.5)
\]

Consider now the four basis states \( \{ |0\rangle, |1\rangle, |+\rangle, |L\rangle \} \) that we conveniently relabel as \( \{ |0\rangle, |1\rangle, |2\rangle, |3\rangle \} \). From these states we can build the four linearly independent projection matrices \( \mu_\nu \equiv |\mu\rangle \langle \nu| \):

\[
E_0 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix} = \frac{\sigma_0 + \sigma_3}{2}, \quad
E_1 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} = \frac{\sigma_0 - \sigma_3}{2},
E_2 = \begin{bmatrix} 1 & 1 \\ 2 & 2 \end{bmatrix} = \frac{\sigma_0 + \sigma_1}{2}, \quad
E_3 = \begin{bmatrix} 1 & -1 \\ 2 & 2 \end{bmatrix} = \frac{\sigma_0 + \sigma_2}{2}. \quad (A.6)
\]

These relations can be inverted to give \( \sigma_0 = E_0 + E_1 \), \( \sigma_1 = -E_0 - E_1 + 2E_2 \), \( \sigma_2 = -E_0 - E_1 + 2E_3 \), \( \sigma_3 = E_0 - E_1 \) or, formally,
\[ E_\mu = \sum_{a=0}^{3} g_{\mu a} \sigma_a \quad \text{and} \quad \sigma_\mu = \sum_{a=0}^{3} f_{\mu a} E_a \quad (\mu = 0, 1, 2, 3), \quad \text{(A.7)} \]

where, from the orthogonality of the Pauli matrices it follows that \( g_{\mu a} = \text{tr} (E_a \sigma_\mu) / 2 \). The coefficients \( f_{\mu a} \) can be found by noticing that the two \( 4 \times 4 \) matrices \( F \) and \( G \) defined as \( [G]_{\mu a} = g_{\mu a} \) and \( [F]_{\mu a} = f_{\mu a} \), are connected by the simple relation \( F = G^{-1} \), which implies \( f_{\mu a} = [G^{-1}]_{\mu a} \).

From an operational point of view, the projector \( E_2 \) can be physically implemented with a \( \pi \)-converter followed by the projector \( E_0 \):

\[ E_2 = U_\pi^{\dagger} (\pi/8) E_0 U_\pi (\pi/8), \quad \text{(A.8)} \]

where (A.5) has been used. Similarly, the projector \( E_3 \) can be realized with a \( \pi/2 \)-converter followed by the projector \( E_0 \):

\[ E_3 = U_{\pi/2} (\pi/4) E_0 U_{\pi/2} (\pi/4). \quad \text{(A.9)} \]

Finally, the single-DoF Stokes parameters \( S_\mu = \text{tr} (\rho_d \sigma_\mu) \), with \( \rho_d \) denoting the single-DoF 2 \( \times \) 2 coherency matrix and \( d \in \{ \text{pol, spa} \} \), can be expressed in the basis \( \{ E_a \} \) as

\[ S_\mu = \sum_{a=0}^{3} f_{\mu a} I_a, \quad \text{where} \quad I_a = \text{tr} (\rho_d E_a). \quad \text{(A.10)} \]

For example, for polarization qubits \( I_a \) denotes the intensity of light polarized in the state \( |\alpha\rangle \).

### A.2. Two-DoF operations

The mathematical apparatus developed in appendix A.1 can be used to express the two-DoF Stokes parameters \( S_{\mu \nu} \) in terms of measurable intensities of light. To this end, it is enough to rewrite (25) in terms of the projection matrices \( \{ E_\mu \} \) as:

\[
S_{\mu \nu} = \text{tr} \left[ \rho (\sigma_\mu \otimes \sigma_\nu) \right] = \sum_{a,\beta=0}^{3} f_{\mu a} f_{\nu \beta} I_{\alpha \beta} = \sum_{a,\beta=0}^{3} (F \otimes F)_{\mu a,\beta} I_{\alpha \beta}, \quad \text{(A.11)}
\]

where \( I_{\alpha \beta} \equiv \text{tr} [(\rho (E_\alpha \otimes E_\beta))] \) denotes the intensity measured by the detector labeled by the pair of indices \( \alpha, \beta \) in figure 3, and (A.6) has been used for both the polarization and the spatial qubits. The last row of (A.11) furnishes a straightforward way to calculate the coefficients \( f_{\mu a} f_{\nu \beta} \). However, a more efficient formula can be obtained by defining the ‘intensity matrix’ \( I \) via the relation \( [I]_{\alpha \beta} = I_{\alpha \beta} \). Then, from (A.11) it follows that

\[ S_{\mu \nu} = [FIF^T]_{\mu \nu}, \quad \text{(A.12)} \]

From (A.12) one obtains, for example, \( S_{00} = I_{00} + I_{01} + I_{10} + I_{11} \) and \( S_{33} = -I_{00} - I_{01} + 2I_{02} + I_{10} + I_{11} - 2I_{12} \). The expression for \( S_{00} \) originates directly from the relation \( E_0 \otimes E_0 + E_0 \otimes E_1 + E_1 \otimes E_0 + E_1 \otimes E_1 = I_4 \), where \( I_4 \) denotes the 4 \( \times \) 4 identity matrix. A complete list of the two-DoF Stokes parameters \( S_{\mu \nu} \) expressed in terms of the intensities \( I_{\alpha \beta} \) is given below:
\[
S_{00} = I_{00} + I_{01} + I_{10} + I_{11}, \\
S_{01} = -I_{00} - I_{01} + 2I_{02} - I_{10} - I_{11} + 2I_{12}, \\
S_{02} = -I_{00} - I_{01} + 2I_{03} - I_{10} - I_{11} + 2I_{13}, \\
S_{03} = I_{00} - I_{01} + I_{10} - I_{11}, \\
S_{10} = -I_{00} - I_{01} - I_{10} - I_{11} + 2(I_{20} + I_{21}), \\
S_{11} = I_{00} + I_{01} - 2I_{02} + I_{10} + I_{11} + 2(I_{12} + I_{20} + I_{21} - 2I_{22}), \\
S_{12} = I_{00} + I_{01} - 2I_{03} + I_{10} + I_{11} - 2(I_{13} - I_{20} + I_{22} - 2I_{23}), \\
S_{13} = -I_{00} - I_{01} - I_{10} + I_{11} + 2I_{20} - 2I_{21}, \\
S_{20} = -I_{00} - I_{01} - I_{10} - I_{11} + 2(I_{10} + I_{31}), \\
S_{21} = I_{00} + I_{01} - 2I_{02} + I_{10} + I_{11} - 2(I_{12} + I_{30} + I_{31} - 2I_{32}), \\
S_{22} = I_{00} + I_{01} - 2I_{03} + I_{10} + I_{11} + 2(I_{13} + I_{30} + I_{31} - 2I_{33}), \\
S_{23} = -I_{00} - I_{01} - I_{10} + I_{11} + 2I_{30} - 2I_{31}, \\
S_{30} = I_{00} + I_{01} - I_{10} - I_{11}, \\
S_{31} = -I_{00} - I_{01} + 2I_{02} + I_{10} + I_{11} - 2I_{12}, \\
S_{32} = -I_{00} - I_{01} + 2I_{03} + I_{10} + I_{11} - 2I_{13}, \\
S_{33} = I_{00} - I_{01} - I_{10} + I_{11}. \\
\tag{A.13}
\]

**Appendix B. MBS**

We consider the experimental realization of a MBS displayed in figure 4(d). To this end we use a right-handed coordinate system attached to the beam whose direction of propagation always coincides with the z-axis. On each reflection the handedness of the spatial modes is inverted and the phase difference between x- and y-polarization components is shifted by $\pi$. This means that each reflection maps the coordinate $x$ onto $-x$ and the polarization vector $e_x$ onto $-e_x$. Hence, a mirror is described by the following transformation of the Jones vector:

\[
\begin{bmatrix}
E_x(r) \\
E_y(r)
\end{bmatrix}
\rightarrow
i
\begin{bmatrix}
-E_x(\bar{r}) \\
E_y(\bar{r})
\end{bmatrix},
\tag{B.1}
\]

where $\bar{r} = -xe_x + ye_y + Ze_z$. Accordingly, a symmetric 50/50 beam splitter acts on the input field $[E_x(r), E_y(r)]^T$ as follows:

\[
\begin{bmatrix}
E_{1x}(r) \\
E_{1y}(r) \\
E_{2x}(r) \\
E_{2y}(r)
\end{bmatrix}
\rightarrow
\frac{1}{\sqrt{2}}
\begin{bmatrix}
E_{1x}(r) - iE_{2x}(\bar{r}) \\
E_{1y}(r) + iE_{2y}(\bar{r}) \\
E_{2x}(r) - iE_{1x}(\bar{r}) \\
E_{2y}(r) + iE_{1y}(\bar{r})
\end{bmatrix},
\tag{B.2}
\]

where the subscripts 1 and 2 denote the two ports of the BS and, e.g., $E_x(r)$ denotes the electric field of the beam entering port 1. The HWP with its fast optical axis aligned parallel to the horizontal direction is, according to (A.2), described by the transformation.
As our proposal uses solely first-order spatial modes, let us consider the input fields $E_1^{\text{in}}(r) = e_x \psi_{10}(r) + e_y \psi_{01}(r)$ and $E_2^{\text{in}}(r) = e_x \psi_{10}(r) + e_y \psi_{01}(r)$, which are transmitted across port 1. The horizontally and vertically polarized HG modes $E_1^{\text{in}}(r) = e_x \psi_{10}(r)$ and $E_2^{\text{in}}(r) = e_y \psi_{01}(r)$ are transmitted across port 2. The MBS splits a radially polarized beam $E_1^{\text{in}}(r) = [e_x \psi_{10}(r) + e_y \psi_{01}(r)]/\sqrt{2}$ into its components $E_1^{\text{out}}(r) = e_x \psi_{10}(r)$ and $E_2^{\text{out}}(r) = e_y \psi_{01}(r)$.

As our proposal uses solely first-order spatial modes, let us consider the input fields $E_1^{\text{in}}(r) = (A_{00} \psi_{10}(r) + A_{01} \psi_{01}(r)) e_x + (A_{10} \psi_{10}(r) + A_{11} \psi_{01}(r)) e_y$ and $E_2^{\text{in}}(r) = 0$. The MBS transforms these input fields into the output fields

$$
\begin{bmatrix}
E_x(r) \\
E_y(r)
\end{bmatrix} \rightarrow i \begin{bmatrix}
-E_x(r) \\
E_y(r)
\end{bmatrix},
$$

(B.3)

As our proposal uses solely first-order spatial modes, let us consider the input fields $E_1^{\text{in}}(r) = A_{00} \psi_{10}(r) + A_{01} \psi_{01}(r)$ and $E_2^{\text{in}}(r) = A_{10} \psi_{10}(r) + A_{11} \psi_{01}(r)$, which are transmitted across port 1. The horizontally and vertically polarized HG modes $E_1^{\text{in}}(r) = e_x \psi_{10}(r)$ and $E_2^{\text{in}}(r) = e_y \psi_{01}(r)$ are transmitted across port 2. The MBS splits a radially polarized beam $E_1^{\text{in}}(r) = [e_x \psi_{10}(r) + e_y \psi_{01}(r)]/\sqrt{2}$ into its components $E_1^{\text{out}}(r) = e_x \psi_{10}(r)$ and $E_2^{\text{out}}(r) = e_y \psi_{01}(r)$.

As our proposal uses solely first-order spatial modes, let us consider the input fields $E_1^{\text{in}}(r) = (A_{00} \psi_{10}(r) + A_{01} \psi_{01}(r)) e_x + (A_{10} \psi_{10}(r) + A_{11} \psi_{01}(r)) e_y$ and $E_2^{\text{in}}(r) = 0$. The MBS transforms these input fields into the output fields

$$
E_1^{\text{out}}(r) = - (A_{00} e_x + A_{10} e_y) \psi_{10}(r),
$$

(B.4)

and

$$
E_2^{\text{out}}(r) = (A_{01} e_x + A_{11} e_y) \psi_{01}(r).
$$

(B.5)
as can be shown by successively applying the transformations of each element of the MBS described above. Furthermore, the fact was used that the HG mode $\psi_{10}$ changes sign upon reflection, i.e. $\psi_{10}(\vec{r}) = -\psi_{10}(\vec{r})$, whereas this is not the case for the HG mode $\psi_{01}$, i.e. $\psi_{01}(\vec{r}) = \psi_{01}(\vec{r})$. The action of the MBS in several input beams $E_{i}^{in}$ is visualized in figure B1.

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