THE ORTHOGONALITY AND qKZB-HEAT EQUATION FOR TRACES OF $U_q(g)$-INTERTWINERS

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Dedicated to the 90-th birthday of Izrail Moiseevich Gelfand

Abstract. In our previous paper [EV2], to every finite dimensional representation $V$ of the quantum group $U_q(g)$, we attached the trace function $F^V(\lambda, \mu)$, with values in $\text{End} V[0]$, obtained by taking the (weighted) trace in a Verma module of an intertwining operator. We showed that these trace functions satisfy the Macdonald-Ruijsenaars and the qKZB equations, their dual versions, and the symmetry identity. In this paper we show that the trace functions satisfy the orthogonality relation and the qKZB-heat equation. For $g = sl_2$, this statement is the trigonometric degeneration of a conjecture from [FV2], proved in [FV2] for the 3-dimensional irreducible $V$.

We also establish the orthogonality relation and qKZB-heat equation for trace functions obtained by taking traces in finite dimensional representations (rather than Verma modules). If $g = sl_n$ and $V = S^{k\alpha} \mathbb{C}^n$, these functions are known to be Macdonald polynomials of type $A$. In this case, the orthogonality relation reduces to the Macdonald inner product identities, and the qKZB-heat equation coincides with the q-Macdonald-Mehta identity, proved by Cherednik [Ch1].

1. Introduction

This paper is motivated by the previous papers [FTV1, FTV2, FV1, FV2, EV2, EV3, EV2]. Let $V_1, \ldots, V_n$ be finite dimensional representations of $U_q(sl_2)$. In the papers [FTV1, FTV2], G. Felder, V. Tarasov, and the second author introduced the function $u_{V_1, \ldots, V_n}(\lambda, \mu, \tau, p, q, z_1, \ldots, z_n)$ with values in $\text{End}((V_1 \otimes \ldots \otimes V_n)[0])$, which is a q-deformation of a conformal block of the $sl_2$-Wess-Zumino-Witten conformal field theory on an elliptic curve with $n$ punctures. Namely, $u$ is defined by a q-deformed version of the explicit integral representation of conformal blocks on an elliptic curve. If $n = 1$, and $V_1 = L_{2m}$ is the representation with highest weight $2m$ (of dimension $2m + 1$), then the function $u$ is independent of $z_1$ and scalar valued; it is denoted by $u_m(\lambda, \mu, \tau, p, q)$. 

It is shown in [FTV1, FTV2] that the function $u$ satisfies the qKZB difference equations, their dual version, and the symmetry with respect to the permutation $(\lambda, \tau) \leftrightarrow (\mu, p)$. Later G. Felder and the second author showed in [FV2] that for $m=1$ the function $u_m$ satisfies the orthogonality relation and the qKZB-heat equation. The latter is a q-deformation of the usual Knizhnik-Zamolodchikov-Bernard (KZB) heat equation for conformal blocks. However, for $m > 1$ the qKZB-heat equation still remains a conjecture, and the higher rank case is yet to be worked out.

In order to understand the mysterious nature of the function $u$ and in particular to answer the above questions, we proposed in [EV2] to study the representation theoretic meaning of $u$. More specifically, since conformal blocks on an elliptic curve are known to be (weighted) traces of products of intertwining operators for affine algebras, we conjectured that the function $u$ is obtained by taking a similar trace for the quantum affine algebra, and checked this conjecture in the trigonometric limit. More precisely, to every collection of finite dimensional representations $V_1,..., V_n$ of the quantum group $U_q(g)$, we attached the trace function $F^{V_1,...,V_n}(\lambda, \mu)$, which depends on two complex weights $\lambda, \mu$ of $g$ and the parameter $q$, and takes values in $\text{End}(V_1 \otimes ... \otimes V_n)[0]$, the endomorphism algebra of the zero weight space of $V$. This function is by definition a suitably renormalized weighted trace in a Verma module of a product of intertwining operators. The main results of [EV2] are that the trace functions satisfy the Macdonald-Ruijsenaars and the qKZB equations, their dual versions, and the symmetry identity, and also that for $g = sl_2$ and $V = L_{2m}$, the function $F^V$ (up to simple renormalizations) coincides with the limit of $u_m$ as $p$ and $\tau$ go to infinity.

In this paper we continue the study of the trace functions for $U_q(g)$ (for $n = 1$), and show that they satisfy the orthogonality relation and the qKZB-heat equation. We also define the integral transform with kernel $F^{V}(\mu, -\lambda)$. In the special case $V = \mathbb{C}$, this integral transform specializes to the usual Fourier transform, while for $g = sl_n$, $V = S^{kn}\mathbb{C}^n$ its symmetrized version is the Cherednik’s difference Fourier transform for type $A$ [Ch2]. The orthogonality relation can be understood as the statement that the inverse to the integral transform with kernel $F^V(\mu, -\lambda)$ is the integral transform with the kernel $F^V(\lambda, \mu)Q_V(-\mu - \rho)$ (where $Q$ is the contracted fusion operator, see [EV2]), which generalizes the standard “self-duality” property of the Fourier and Cherednik transforms. For $g = sl_2$, these results are the trigonometric degenerations of the statements conjectured in [FV2], and proved in the case when $V$ is 3-dimensional.

We also establish the orthogonality relation and qKZB-heat equation for trace functions obtained by taking traces in finite dimensional representations (rather than Verma modules). In the case $g = sl_n$ and $V = S^{kn}\mathbb{C}^n$, these functions are Macdonald polynomials of type $A$ [M], the orthogonality relation reduces to the Macdonald inner product identities, and the qKZB-heat equation coincides with the q-Macdonald-Mehta identity, proved by Cherednik [Ch1].

These results can be easily generalized to the case $n > 1$, using the techniques of [EV2]. We will not discuss this generalization.
The structure of the paper is as follows. In Section 2 we recall the basics about quantum groups, the definition of trace functions from [EV2], and formulate our main results – the orthogonality and the q-KZB heat equation. We also reformulate the orthogonality relation as an inversion formula for a generalized Fourier transform, similarly to how it was done in [EV2] for the function $u_m$. Finally, we state the self-adjointness of Macdonald-Ruijsenaars operators, and the qKZB heat equation with integration over a real cycle. The rest of the paper is devoted to the proof of these statements. Namely, in Section 3 we recall from [EV3] the definition of the dynamical Weyl group and the dynamical Weyl group symmetry for trace functions; this fact is vital for the proof of the main results. In Section 4 we prove the orthogonality relation and the qKZB-heat equation for traces in finite dimensional modules, using the techniques of the paper [EK1]; besides of their independent interest, these results are used in the proof of the main results of Section 2. In Section 5, we show that the integrals considered in Section 2 are independent on the Weyl chamber in which the cycle of integration is situated; this is an important technical ingredient in the proof. In Section 6 we prove the orthogonality relation. In Section 7 we prove the qKZB heat equation. In Section 8 we prove the properties of integral transforms. Finally, in Section 9 we prove the self-adjointness of Macdonald-Ruijsenaars operators, and the qKZB heat equation with integration over a real cycle.

We expect that the techniques of this paper can be extended to the case when a finite dimensional Lie algebra $\mathfrak{g}$ is replaced with the affine Lie algebra $\tilde{\mathfrak{g}}$, similarly to how the results of [EV2] were extended to the affine case in [ESV]. This would allow us to prove the orthogonality and qKZB-heat equation for trace functions of quantum affine algebras, and finally show (at least for $m = 1$) that the function $u_m$ from [EV2] is the trace function for $U_q(\hat{\mathfrak{sl}_2})$ (up to renormalizations). This is a subject of future research.

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2. Trace functions for $U_q(\mathfrak{g})$

2.1. Lie algebras and quantum groups. Let $\mathfrak{g}$ be a simple Lie algebra over $\mathbb{C}$ with root space decomposition $\mathfrak{g} = \mathfrak{h} \oplus (\oplus_{\alpha \in \Sigma} \mathfrak{g}_\alpha)$ where $\Sigma \subset \mathfrak{h}^*$ is the set of roots.

Fix a system of simple roots $\alpha_1, ..., \alpha_r$. Let $\Sigma_{\pm}$ be the set of positive (negative) roots. Let $n_\pm = \oplus_{\alpha \in \Sigma_{\pm}} \mathfrak{g}_\alpha$. Then $\mathfrak{g} = n_+ \oplus \mathfrak{h} \oplus n_-.$

Let $(\ , \ )$ be an invariant bilinear form on $\mathfrak{g}$. The form gives rise to a natural identification $\mathfrak{h} \rightarrow \mathfrak{h}^*$, which we will sometimes use to make no distinction between $\mathfrak{h}$ and $\mathfrak{h}^*$. 

This identification allows us to define a scalar product on $\mathfrak{h}^*$. We assume that the form is normalized so that $(\alpha, \alpha) = 2$ for short roots. We use the same notation $(,)$ for the pairing $\mathfrak{h} \otimes \mathfrak{h}^* \to \mathbb{C}$.

We denote by $\mathfrak{h}_\mathbb{R}$ (resp. $\mathfrak{h}_\mathbb{R}^*$) the real subspace of $\mathfrak{h}$ (resp. $\mathfrak{h}^*$). The form $(,)$ is positive definite on $\mathfrak{h}_\mathbb{R}$.

We use the notation:

- $Q = \oplus_{i=1}^{r} \mathbb{Z} \alpha_i$ - the root lattice;
- $Q_+ = \oplus_{i=1}^{r} \mathbb{Z}_{\geq 0} \alpha_i$;
- $Q^\vee = \oplus_{i=1}^{r} \mathbb{Z} \alpha_i^\vee$ - the dual root lattice, where $\alpha_i^\vee = 2\alpha/(\alpha, \alpha)$;
- $P = \{ \lambda \in \mathfrak{h} \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z} \}$ - the weight lattice;
- $P_+ = \{ \lambda \in \mathfrak{h} \mid (\lambda, \alpha_i^\vee) \in \mathbb{Z}_{\geq 0} \}$ - the cone of dominant integral weights;
- $\omega_i \in P_+$ - fundamental weights: $(\omega_i, \alpha_j^\vee) = \delta_{ij}$;
- $\rho = \frac{1}{2} \sum_{\alpha \in \Sigma_+} \alpha = \sum_{i=1}^{r} \omega_i$.

Define a partial order on $\mathfrak{h}$ putting $\mu < \lambda$ if $\lambda - \mu \in Q_+$.

A vector $\lambda$ in $\mathfrak{h}_\mathbb{R}$ or $\mathfrak{h}_\mathbb{R}^*$ will be called big if $| (\lambda, \alpha_i) | >> 0$ for $i = 1, \ldots, r$. More precisely, a statement holds for big $\lambda$ if there exists a constant $K$ (possibly depending on some previously fixed data) such that it holds for $\lambda$ satisfying the inequalities $| (\lambda, \alpha_i) | > K$ for all $i$.

Let $s_i : \mathfrak{h}^* \to \mathfrak{h}^*$ denote the $i$-th simple reflection, defined by

$$s_i \lambda = \lambda - (\alpha_i^\vee, \lambda) \alpha_i.$$ 

Let $\mathbb{W}$ be the Weyl group, generated by $s_1, \ldots, s_r$. For an element $w \in \mathbb{W}$, denote by $l(w)$ the length of the minimal (reduced) presentation of $w$ as a product of generators $s_1, \ldots, s_r$.

We will also consider the “dot” action of the Weyl group on $\mathfrak{h}^*$ defined by

$$w \cdot v = w(v + \rho) - \rho.$$

Let $(a_{ij})$ be the Cartan matrix of $\mathfrak{g}$. Let $d_i$ be the relatively prime positive integers such that $(d_i a_{ij})$ is a symmetric matrix. Let $e_i, f_i, h_i$ be the Chevalley generators of $\mathfrak{g}$.

Let $\kappa$ be a purely imaginary number, $\text{Im} \ k < 0$, and $q = e^{\frac{2\pi i}{\kappa}}$, $0 < q < 1$. For any operator $A$, we denote $e^{\frac{2\pi i}{\kappa} A}$ by $q^A$.

Let $U_q(\mathfrak{g})$ be the Drinfeld-Jimbo quantum group corresponding to $\mathfrak{g}$. Namely, $U_q(\mathfrak{g})$ is a Hopf algebra with generators $E_i, F_i, i = 1, \ldots, r$, $q^h, h \in \mathfrak{h}$, with relations:

$$q^{x+y} = q^x q^y \text{ for } x, y \in \mathfrak{h}, q^0 = 1, \quad q^h E_j q^{-h} = q^{a_{ij}(h)} E_i, \quad q^h F_j q^{-h} = q^{-a_{ij}(h)} F_i, \quad E_i F_j - F_j E_i = \delta_{ij} q^{d_i h_i} - q^{-d_i h_i} q^{d_i} - q^{-d_i},$$

$$\sum_{k=0}^{1-a_{ij}} (-1)^k \binom{1-a_{ij}}{k} q_i q^{1-a_{ij} - k} E_i^{1-a_{ij} - k} E_j E_i^k = 0, \quad i \neq j.$$
with highest weight \( \mu \) and we used the notation
\[
\begin{aligned}
\left[ \begin{array}{c}
\mu \\
\kappa
\end{array} \right]_{q} &= \frac{[\mu]_{q}!}{[\kappa]_{q}! [\mu - \kappa]_{q}!}, \\
[n]_{q}! &= [1]_{q} \cdot [2]_{q} \cdots [n]_{q}, \\
[n]_{q} &= \frac{q^{n} - q^{-n}}{q - q^{-1}}.
\end{aligned}
\]

where \( q \) is a formal power series in \( q \).

The comultiplication \( \Delta \), antipode \( S \), and counit \( \epsilon \) in \( U_{q}(\mathfrak{g}) \) are given by
\[
\begin{aligned}
\Delta(E_{i}) &= E_{i} \otimes q^{d_{i}h_{i}} + 1 \otimes E_{i}, \\
\Delta(F_{i}) &= F_{i} \otimes 1 + q^{-d_{i}h_{i}} \otimes F_{i}, \\
\Delta(q^{h}) &= q^{h} \otimes q^{h}, \\
S(E_{i}) &= -E_{i} q^{-d_{i}h_{i}}, \\
S(F_{i}) &= -q^{d_{i}h_{i}} F_{i}, \\
S(q^{h}) &= q^{-h}, \\
\epsilon(E_{i}) &= \epsilon(F_{i}) = 0, \\
\epsilon(q^{h}) &= 1.
\end{aligned}
\]

2.2. Intertwiners and trace functions. Let \( M_{\mu} \) be the Verma module over \( U_{q}(\mathfrak{g}) \) with highest weight \( \mu \) and highest weight vector \( x_{\mu} \). Let \( V \) be a finite dimensional representation of \( U_{q}(\mathfrak{g}) \), and \( v \in V \) a vector of weight \( \mu_{v} \). It is well known that for generic \( \mu \) there exists a unique intertwining operator \( \Phi_{\mu}^{v} : M_{\mu} \rightarrow M_{\mu - \mu_{v}} \otimes V \) such that \( \Phi_{\mu}^{v} x_{\mu} = x_{\mu - \mu_{v}} \otimes v + l.o.t. \) (here \( l.o.t. \) denotes the lower order terms, i.e. the terms of smaller weight in the first component). It is useful to consider the “generating function” of such operators, \( \Phi_{\mu}^{V} \in \text{Hom}_{\mathbb{C}}(M_{\mu}, \otimes_{\mu} M_{v} \otimes V \otimes V^{*}) \), defined by
\[
\Phi_{\mu}^{V} = \sum_{v \in B} \Phi_{\mu}^{v} \otimes v^{*},
\]
where the summation is over a homogeneous basis \( B \) of \( V \), and \( v^{*} \) are elements of the dual basis.

For \( v \in V \) consider
\[
\Psi^{v}(\lambda, \mu) = \text{Tr}|_{M_{\mu}}(\Phi_{\mu}^{v} q^{2\lambda}),
\]
a formal power series in \( V[0] \otimes q^{2(\lambda, \mu)} \mathbb{C}[[q^{-2(\lambda, \alpha_{1})}, ..., q^{-2(\lambda, \alpha_{r})}]] \). This series converges (in a suitable region of values of parameters) to a function of the form \( q^{2(\lambda, \mu)} f(\lambda, \mu) \), where \( f \) is a rational function in \( q^{2(\lambda, \alpha_{i})} \) and \( q^{2(\mu, \alpha_{i})} \), which is a finite sum of products of functions of \( \lambda \) and functions of \( \mu \) (ESu, EV2). This function is called the trace function.

The universal trace function with values in \( V[0] \otimes V^{*}[0] \) is the function
\[
\Psi^{V}(\lambda, \mu) = \sum_{v \in B} \Psi^{v}(\lambda, \mu) \otimes v^{*}.
\]
We have \( \Psi^{V}(\lambda, \mu) = \text{Tr}(\Phi_{\mu}^{V} q^{2\lambda}) \). We will consider the universal trace function as a function of \( \lambda, \mu \) with values in \( \text{End}(V[0]) \).

**Example 1.** Let \( \mathfrak{g} = sl_{2} \). In this case we represent weights by complex numbers, so that the unique fundamental weight corresponds to 1. Let \( V = L_{2} \) be the irreducible
3-dimensional representation. Then

\[ \Psi^V(\lambda, \mu) = \frac{q^{\lambda\mu}}{1 - q^{-2\lambda}} \left( 1 + (q^2 - q^{-2}) \frac{q^{-2\lambda}}{(1 - q^{2\mu})(1 - q^{-2(\lambda - 1)})} \right). \]

(Since \(V[0]\) is 1-dimensional, we view \(\Psi^V\) as a scalar function).

Let \(V, W\) be finite dimensional representations of \(U_q(\mathfrak{g})\). The fusion matrix is the operator \(J_{WV}(\mu) : W \otimes V \to W \otimes V\) defined by the formula

\[
(\Phi^w_{\mu - \mu_v} \otimes 1) \Phi^v_{\mu} = \Phi^v_{J_{WV}(\mu)(w \otimes v)},
\]

see [EV1]. The exchange matrix \(R_{VW}(\lambda)\) is defined by the formula

\[
R_{VW}(\lambda) := J_{VW}(\lambda)^{-1} R_{WV}^{21} J_{WV}^{21}(\lambda),
\]

where \(R_{WV}\) is the R-matrix acting on \(W \otimes V\).

The universal fusion matrix \(J(\lambda)\) takes values in a completion of \(U_q(\mathfrak{g}) \otimes U_q(\mathfrak{g})\) and gives \(J_{WV}(\lambda)\) when evaluated in representations \(V, W\). The universal exchange matrix is defined by the formula \(R(\lambda) := J(\lambda)^{-1} R^{21} J^{21}(\lambda)\), where \(R\) is the universal R-matrix of \(U_q(\mathfrak{g})\).

If \(J(\lambda) = \sum_i a_i \otimes b_i\) is the universal fusion matrix, set \(Q(\lambda) = \sum_i S^{-1}(b_i)a_i\), where \(S\) is the antipode. This sum defines an operator \(Q_V(\lambda) : V \to V\) invertible for generic \(\lambda\).

Let

\[
\delta_q(\lambda) = \prod_{\alpha > 0} (q^{(\lambda, \alpha)} - q^{-(\lambda, \alpha)});
\]

be the Weyl denominator.

Introduce the renormalized trace function

\[
F^V(\lambda, \mu) = \delta_q(\lambda) \Psi^V(\lambda, -\mu - \rho) Q_V^{-1}(- \mu - \rho).
\]

Let \(V^*\) be the space dual to \(V\) with the \(U_q(\mathfrak{g})\) module structure defined by the antipode. We have the following symmetry property [EV2],

\[
F^{V^*}(\lambda, \mu)^* = F^V(\mu, \lambda),
\]

where the values of both functions are regarded as linear operators on \(V[0]\).

The following lemma describes the location of poles of \(F^V(\lambda, \mu)\).

**Lemma 2.1.** The poles of the function \(F^V(\lambda, \mu)\) with respect to \(\lambda\) are simple, and the divisor of poles is contained in the union of hyperplanes defined by \([\lambda, \alpha) - k(\alpha, \alpha)/2]_q = 0\, where \(k = 1, 2, \ldots, N\), with \(N = N(V)\, and \(\alpha \in \Sigma_+\).

**Proof.** The lemma follows from Proposition 6.3 in [EV2].

**Example 2.** Let \(\mathfrak{g} = sl_2\) and let \(V = L_2\) be the 3-dimensional irreducible representation. Then

\[
F^V(\lambda, \mu) = q^{-\lambda \mu} \frac{q^{2(\lambda + \mu)} - q^{2\lambda - 2} - q^{2\mu - 2} + 1}{(1 - q^{2\lambda - 2})(1 - q^{2\mu - 2})}.
\]

Notice that the renormalized trace function is symmetric in \(\lambda\) and \(\mu\).
More generally, let $V = L_{2m}$. Then according to [EV2], formula (7.20),

$$F^V(\lambda, \mu) = q^{-\lambda \mu} \prod_{j=1}^{m} \frac{q^{2\mu - 2j} - 1}{q^{-2\mu + 2j} - q^{-2m}} \times \sum_{l=0}^{m} q^{2m + l(l-1)/2} (q - q^{-1})^l \frac{[m+l]!(m-l)!}{[l]!q[m-l]!} \prod_{j=1}^{l} (1 - q^{-2(\mu+j)}) \prod_{j=1}^{l} (1 - q^{-2(\lambda-j)})^2,$$

This function is symmetric in $\lambda$ and $\mu$, although it is not obvious from the formula.

**Example 3.** Let $g = sl_n$, $V = S^{kn} \mathbb{C}^n$. In this case, $V[0]$ is 1-dimensional, and $F^V(\lambda, \mu)$ is the Macdonald function studied in [ES] and [Cha].

2.3. **Main results.** For $\xi \in \mathfrak{h}_R^*$ consider the imaginary subspace $C_\xi = \xi + i\mathfrak{h}_R^*$ in $\mathfrak{h}^*$. This subspace is invariant with respect to translations by $\kappa Q^\vee$. The quotient $C_\xi / \kappa Q^\vee$ is a torus. The theorems formulated below involve integration of functions over $C_\xi$ and over the torus $C_\xi / \kappa Q^\vee$. The integration will be performed with respect to translation invariant measures. Abusing notation, we will denote translation invariant measures on $C_\xi$ and $C_\xi / \kappa Q^\vee$ by the same symbol $d\lambda$. The normalization of these measures is as follows. The measure $d\lambda$ on $C_\xi / \kappa Q^\vee$ is normalized by the condition

$$\int_{C_\xi / \kappa Q^\vee} d\lambda = 1,$$

while the measure $d\lambda$ on $C_\xi$ is normalized so that

$$\int_{C_\xi} q^{-(\lambda,\lambda)} d\lambda = 1.$$

This agreement is convenient because it allows one to get rid of normalization constants. It will be kept throughout the paper. We warn the reader that with these normalizations, the direct image of the measure $d\lambda$ on $C_\xi$ is not equal to the measure $d\lambda$ on $C_\xi / \kappa Q^\vee$, but is only proportional to it with some proportionality coefficient $C$, whose value is easy to compute but irrelevant to us.

We now state the main results of this paper. Notice that the function $q^{2(\lambda,\mu)} F^V(\lambda, \mu)$, considered as a function of $\lambda$ (or as a function of $\mu$) and being restricted to $C_\xi$, is $\kappa Q^\vee$ periodic.

**Theorem 2.2.** (Orthogonality)

Let $V$ be a finite dimensional $U_q(g)$ module. Assume that $\mu, \nu, \xi \in \mathfrak{h}_R^*$ are such that $\mu - \nu$ belongs to the weight lattice $P$ and $\xi$ is big. Then

$$\int_{C_\xi / \kappa Q^\vee} F^V(\mu, -\lambda) F^V(\lambda, \nu) d\lambda = \delta_{\mu,\nu} Q^{-1}_V(-\mu - \rho).$$
**Theorem 2.3.** *(gKZB-heat Equation)*

Let $V$ be a finite dimensional $U_q(\mathfrak{g})$ module. Assume that $\mu, \nu, \xi$ lie in $\mathfrak{h}_R^*$, and $\xi$ is big. Then

$$\int_{C_\xi} F^V(\mu, -\lambda) F^V(\lambda, \nu) \, q^{-\langle \lambda, \lambda \rangle} \, d\lambda = q^{(\mu,\mu)+(\nu,\nu)} F^V(\mu, \nu).$$

**Remark 1.** If $V$ is the trivial representation, then $F^V(\lambda, \mu) = q^{-2\langle \lambda, \mu \rangle}$, and Theorems 2.2 and 2.3 are obvious.

**Remark 2.** For $\mathfrak{g} = \mathfrak{sl}_2$ the trace functions are given explicitly in Example 2 of Section 2.2. Even in that case, Theorem 2.2 and especially Theorem 2.3 are rather nontrivial integral identities. More specifically, as seen from Example 2 above, the function $F^V$ for $V = L_{2m}$ is a sum of $m + 1$ products. So the left hand side of formulas in Theorems 2.2 and 2.3 is a sum of $(m + 1)^2$ integrals, while the right hand side is a single product (or zero) in Theorem 2.2 and a sum of $m + 1$ products in Theorem 2.3.

Moreover, a careful computation shows that each of the $(m + 1)^2$ individual integrals on the left hand side of Theorem 2.3 is non-elementary, and an elementary answer on the right hand side is obtained only as a result of cancellation.

### 2.4. Integral transforms.

For $\xi, \eta \in \mathfrak{h}_R^*$ consider the imaginary subspace $C_\xi = \xi + \mathfrak{h}_R^* + \mathfrak{h}_R^*$ in $\mathfrak{h}_R^*$ and the real subspace $D_\eta = i\eta + \mathfrak{h}_R^*$.

Let $\mathcal{S}(C_\xi)$ and $\mathcal{S}(D_\eta)$ be the Schwartz spaces of functions on $C_\xi$ and $D_\eta$ respectively. Introduce the spaces $\mathcal{S}_q(C_\xi) = \{ \phi : C_\xi \rightarrow \mathbb{C} \mid q^{-2\langle \xi, \lambda \rangle} \phi(\lambda) \in \mathcal{S}(C_\xi) \}$, $\mathcal{S}_q(D_\eta) = \{ \phi : D_\eta \rightarrow \mathbb{C} \mid q^{2\langle \xi, \mu \rangle} \phi(\mu) \in \mathcal{S}(E_\eta) \}$. Obviously, these spaces are canonically isomorphic to $\mathcal{S}_q(C_\xi), \mathcal{S}_q(D_\eta)$. The modified Fourier transform $f(\lambda) \mapsto \hat{f}(\mu) := \int_{C_\xi} q^{2\langle \lambda, \mu \rangle} f(\lambda) \, d\lambda$ defines an isomorphism $\mathcal{S}_q(C_\xi) \rightarrow \mathcal{S}_q(D_\eta)$. The inverse transform $g(\mu) \mapsto \check{g}(\lambda)$ is given by the formula $\check{g}(\lambda) = \int_{D_\eta} q^{-2\langle \lambda, \mu \rangle} g(\mu) \, d\mu$. This fixes uniquely a normalization of the Lebesgue measure $d\mu$ on $D_\eta$, which will be used from now on.

For a finite dimensional $U_q(\mathfrak{g})$ module $V$ consider the function $F^V(\lambda, \mu)$. Consider two integral transformations

(3) $$K_{\text{im}}^V : \mathcal{S}_q(C_\xi) \otimes V[0] \rightarrow \mathcal{S}_q(D_\eta) \otimes V[0], \quad f(\lambda) \mapsto \int_{C_\xi} F^V(\mu, -\lambda) \, f(\lambda) \, d\lambda,$$

and

(4) $$K_{\text{re}}^V : \mathcal{S}_q(D_\eta) \otimes V[0] \rightarrow \mathcal{S}_q(C_\xi) \otimes V[0], \quad f(\lambda) \mapsto \int_{D_\eta} F^V(\lambda, \mu) \, Q(-\mu - \rho) \, f(\mu) \, d\mu$$

**Theorem 2.4.** Assume that $\xi \in \mathfrak{h}_R$ is big and $\eta \in \mathfrak{h}_R$ is generic. Then the integral transforms are well defined, continuous in the Schwartz topology, and are inverse to each other,

$$K_{\text{im}}^V K_{\text{re}}^V = \text{Id}, \quad K_{\text{re}}^V K_{\text{im}}^V = \text{Id}.$$

The proof of Theorem 2.4 occupies Section 8.

2.5. Self-adjointness of the Macdonald-Ruijsenaars operators. One of the basic facts of Macdonald’s theory is that Macdonald’s operators are self-adjoint. It turns out that a similar statement holds for the Macdonald-Ruijsenaars operators introduced in [EV2].

Namely, consider the algebra $D$ of scalar difference operators. This algebra is generated by meromorphic functions $y(\lambda)$ on $\mathfrak{h}^*$ and elements $T_\beta$ with defining relation

$$T_\beta y(\lambda) = y(\lambda + \beta)T_\beta.$$ 

The algebra $D$ acts the space of functions on $\mathfrak{h}^*$: functions act by multiplication, and $T_\beta f(\lambda) = f(\lambda + \beta)$.

Let $V$ be a finite dimensional $U_q(g)$-module, and consider the algebra $D_V = D \otimes \text{End} V[0]$. Define an anti-homomorphism $D_V \rightarrow D_V^*$ as follows: $\phi(\lambda)^* = \phi(-\lambda)$ for scalar functions, $T_\beta^* = T_\beta$, and $A \rightarrow A^*$ for $A \in \text{End} V^*[0]$. It is obvious that for $L \in D_V$, the operator $L^*$ is the formal adjoint of $L$ with respect to the inner product $\langle f, g \rangle := \int (f(\lambda), g(-\lambda))d\lambda$. (We are not specifying the contour of integration since it is not important for computing the formal adjoint).

Let $D_{U,V} \in D_V$ be the Macdonald-Ruijsenaars operator corresponding to the representation $U$ of $U_q(g)$ (see [EV2]). Namely, $D_{U,V}$ is the difference operator acting on functions of $\lambda$ with values in $V[0]$, which is defined by the formula

$$(D_{U,V}f)(\lambda) = \sum_\nu \text{Tr}[U|\nu|(R_{UV}(-\lambda - \rho))]f(\lambda + \nu).$$

According to [EV2], these operators commute for different $U$ (i.e., form a quantum integrable system), and the trace function $F^V(\lambda, \mu)$ is their common eigenfunction.

**Theorem 2.5.** One has $D_{U,V}^* = D_{U,V^*}$.

2.6. qKZB-heat equation with integration over a real cycle.

**Theorem 2.6.** One has

$$\int_{D_\mu} F^V(\lambda, \mu)Q(-\mu - \rho)F^V(\mu, \nu)d\mu = q^{-(\lambda, \lambda)-(\nu, \nu)}F^V(\lambda, \nu).$$

The proof of this theorem is given in Section 9.

3. Trace Functions and the Dynamical Weyl Group

3.1. The dynamical Weyl group, [TV, EV3, STV]. Recall that a nonzero vector in a $U_q(g)$-module is said to be singular if it is annihilated by $E_i$ for all $i$.

Let $w = s_{i_1} \ldots s_{i_l}$ be a reduced decomposition of $w \in \mathbb{W}$. Set $\alpha^l = \alpha_{i_l}$ and $\alpha^j = (s_{i_l} \ldots s_{i_{j+1}})(\alpha_{i_j})$ for $j = 1, \ldots, l - 1$. For $\mu \in \mathfrak{h}^*$ let $n_j = \frac{2(\mu+\rho, \alpha_j)}{\langle \alpha_j, \alpha_j \rangle}$. For a dominant $\mu \in P_+$, the numbers $n_j$ are positive integers. Let $d^i = d_{i_j}$ (where $d_i$ are the symmetrizing
numbers). It is known that the collection of pairs of integers \((n_1, d^1), \ldots, (n_k, d^k)\) and the product \(f_{\alpha_1}^{n_1} \cdots f_{\alpha_i}^{n_i}\) do not depend on the reduced decomposition.

Define a vector \(x_{w,\mu} \in M_\mu\) by

\[
(5) \quad x_{w,\mu} = \frac{f_{\alpha_1}^{n_1}}{[n_1]_{q^d_1^1}} \cdots \frac{f_{\alpha_i}^{n_i}}{[n_i]_{q^d_i^i}} x_\mu.
\]

This vector is singular. It does not depend on the reduced decomposition.

Let \(V\) be a finite dimensional \(U_q(\mathfrak{g})\) module, and \(w \in \mathbb{W}\). According to [TV, EV3, STV] there exists a unique operator \(A_{w,V}(\mu) \in \text{End}(V)\) which rationally depends on \(q^{2(\mu,\alpha_i)}\) and has the following properties.

Let \(\mu \in P_+\) be a big vector. Let \(u \in V[\nu]\) for some \(\nu \in \mathfrak{h}^*\). Then

\[
(6) \quad \Phi^\mu_{\nu} x_{w,(\mu)} = x_{w,(\mu-\nu)} \otimes A_{w,V}(\mu) u + l.o.t.
\]

The collection of operators \(\{A_{w,V}(\mu)\}_{w \in \mathbb{W}}\) is called the dynamical Weyl group.

The operators of the dynamical Weyl group preserve the weight decomposition of \(V\) and satisfy the cocycle condition. Namely, if \(w_1, w_2 \in \mathbb{W}, l(w_1 w_2) = l(w_1) + l(w_2)\), then

\[
(7) \quad A_{w_1 w_2,V}(\mu) = A_{w_1,V}(w_2 \cdot \mu) A_{w_2,V}(\mu).
\]

Moreover, according to [EV3], on the subspace \(V[0]\) this equation is satisfied without the assumption \(l(w_1 w_2) = l(w_1) + l(w_2)\).

Let \(A_{w,V}(\lambda) := A_{w,V}(-\lambda-\rho)\). The trace function \(F^V(\lambda, \mu)\) has the following symmetry property with respect to the dynamical Weyl group [EV3],

\[
(8) \quad F^V(\lambda, \mu) = A_{w,V}(w^{-1} \lambda) F^V(w^{-1} \lambda, w^{-1} \mu) A_{w,V^*}(w^{-1} \mu)^*
\]

for any \(w \in \mathbb{W}\).

### 3.2. Intertwiners of finite dimensional modules

Let \(L_\mu\) be an irreducible finite dimensional representation with of \(U_q(\mathfrak{g})\) with highest weight \(\mu\). The intertwining operator \(\Phi^\mu_\nu : M_\mu \to M_\mu \otimes V\), descends to an operator \(\Phi^\nu_\mu : L_\mu \to L_\mu \otimes V\). We define the corresponding trace functions by

\[
\Psi^\nu_\mu(\lambda) = \text{Tr}_{L_\mu}(\Phi^\nu_\mu q^{2\lambda}), \quad \Psi^\nu_\mu(\lambda) = \sum_{\nu \in B} \Psi^\nu_\mu(\lambda) \otimes \nu^*
\]

and the corresponding renormalized trace function as

\[
F^\nu_\mu(\lambda) = \delta_q(\lambda) \Psi^V_{-\mu-\nu}(\lambda) Q^{-1}_V(\mu - \rho)\).
\]

If we regard universal trace functions as linear operators on \(V[0]\), then we have the following generalized Weyl character formulas [EV3]. For a big dominant integral weight \(\mu\), we have

\[
\Psi^V_\mu(\lambda) = \sum_{w \in \mathbb{W}} (-1)^w \Psi^V(\lambda, w \cdot \mu) A_{w,V}(\mu),
\]
and for a big anti-dominant integral weight $\mu$ we have

\begin{equation}
F^V_\mu(\lambda) = \sum_{w \in W} (-1)^w F^V(\lambda, w\mu) (A_{w,V}(\mu)^{-1})^*.
\end{equation}

Here $(-1)^w$ denotes the sign of the element $w$. The terminology is motivated by the fact that for $V = \mathbb{C}$ these formulas reduce to the usual Weyl character formula.

4. Orthogonality and the qKZB-heat equation for finite dimensional modules

4.1. Statement of results.

**Theorem 4.1.** *(Orthogonality)*

Let $\mu, \nu$ be big dominant integral weights. Let $V$ be a finite dimensional $U_q(\mathfrak{g})$ module, $V^*$ the dual module. Let $v \in V[0]$ and $v_* \in V^*[0]$ be arbitrary vectors. Consider the intertwiners $\Phi^v_\mu : L_\mu \to L_\mu \otimes V$, $\Phi^{v_*}_\nu : L_\nu \to L_\nu \otimes V^*$ and the corresponding trace functions $\Psi^v_\mu(\lambda)$, $\Psi^{v_*}_\nu(\lambda)$. Let $\xi \in \mathfrak{h}_R^*$ be a big vector. Then

\[
\frac{1}{|W|} \int_{C_\xi/\kappa Q^v} \delta_q(\lambda) \delta_q(-\lambda) (\Psi^v_\mu(\lambda), \Psi^{v_*}_\nu(-\lambda)) \, d\lambda = \delta_{\mu,\nu} (Q_V(\mu)v, v_*) .
\]

Here $|W|$ is the number of elements in the Weyl group and $(\cdot, \cdot)$ is the pairing of vectors and covectors.

**Theorem 4.2.** *(qKZB-heat equation)*

Under the assumptions of Theorem 4.1 one has

\[
\frac{1}{|W|} \int_{C_\xi} \delta_q(\lambda) \delta_q(-\lambda) (\Psi^v_\mu(\lambda), \Psi^{v_*}_\nu(-\lambda)) \, q^{-(\lambda,\lambda)} \, d\lambda = \delta_q(-\mu - \rho) \, q^{(\mu+\rho,\mu+\rho)+(\nu+\rho,\nu+\rho)} (Q_V(\mu)v, \Psi^{v_*}_\nu(-\mu - \rho)) .
\]

4.2. Proof of Theorem 4.1. The function $\Psi^{v_*}_\nu(-\lambda)$ is the trace in $L_\nu$ of the operator $A := \Phi^{v_*}_\nu q^{-2\lambda}$, acting from $L_\nu$ to $L_\nu \otimes V^*$. Therefore, the same function can be computed as the trace of the dual operator $A^*$. The operator $A^*$ can be written as $q^{2\lambda}(\Phi^{v_*}_\nu)^*$.

To interpret $(\Phi^{v_*}_\nu)^*$ as an intertwiner, let us use the operation of left dual, $W \to ^*W$, on representations of $U_q(\mathfrak{g})$. Namely, $^*W$ is the usual dual of $W$ as a vector space, with the action of $U_q(\mathfrak{g})$ defined by the formula $\pi_{^*W}(a) = \pi_W(S^{-1}(a))^*$ (as opposed to $\pi_W(S(a))^*$ for the right dual $W^*$). Also, the intertwining operator $L_\nu \to Y \otimes L_\nu$ which sends the lowest weight vector $x_\nu^{}$ to $y \otimes x_\nu^{} + \ldots$ for some $y \in Y$ will be denoted by $\Phi^y_\nu$.

With these definition, we see that $(\Phi^{v_*}_\nu)^*$ is an intertwiner $V \otimes L_\nu \to ^*L_\nu$, which can also be viewed (by swapping $V$) as an intertwiner $^*L_\nu \to ^*V \otimes L_\nu$. Thus, $(\Phi^{v_*}_\nu)^* = \Phi^{v^*}_\nu$, where $v^*$ is the highest weight of $^*L_\nu$. Thus, using the cyclic property of the trace, we find that $\Psi^{v_*}_\nu(-\lambda) = \text{Tr}(\Phi^{v^*}_\nu q^{2\lambda})$. 

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This shows that the expression $(\Psi^\nu_\mu(\lambda), \Psi^\nu_\mu(-\lambda))$ can be represented in the form 
\[ \text{Tr}|_{L_\mu \otimes L_\nu}(X(q^{2\lambda} \otimes q^{2\lambda})) \]
where $X \in \text{End}(L_\mu \otimes L_\nu^*)$ is the composition of the tensor product $\Phi^\nu_\mu \otimes \Phi^\nu_\nu^*$ with the contraction $V \otimes \ast V \rightarrow \mathbb{C}$.

Now observe that $X$ is an intertwining operator. Therefore, writing $L_\mu \otimes L_\nu^*$ as a direct sum $\oplus_\beta H^\beta_{\mu \nu^*} \otimes L_\beta$ (where $H^\beta_{\mu \nu^*}$ are the multiplicity spaces), we can represent $X$ in the form $X = \oplus_\beta X_\beta \otimes 1_{L_\beta}$, where $X_\beta \in \text{End}(H^\beta_{\mu \nu^*})$. Hence,
\[ (\Psi^\nu_\mu(\lambda), \Psi^\nu_\mu(-\lambda)) = \sum_\beta \text{Tr}(X_\beta)\chi_\beta(q^{2\lambda}), \]
where $\chi_\beta$ is the character of the representation $L_\beta$. Therefore, by the Weyl orthogonality formula for characters we get
\[ \frac{1}{|W|} \int_{C_\mu \otimes_q Q^\nu} \delta_q(\lambda) \delta_q(-\lambda) (\Psi^\nu_\mu(\lambda), \Psi^\nu_\mu(-\lambda)) d\lambda = \delta_{\mu,\nu} \text{Tr}(X_0). \]

This immediately implies the theorem for $\mu \neq \nu$ (as in this case $H^0_{\mu \nu^*} = 0$ and hence $\text{Tr}(X_0) = 0$). Thus, it remains to settle the case $\mu = \nu$.

If $\mu = \nu$ then $H^0_{\mu \nu^*}$ is one dimensional, and $X_0$ is a number. So we need to compute this number.

To compute $X_0$, pick $y \in L_\nu$, $f \in \ast L_\nu$, and look at $\text{cont}(X(y \otimes f))$, where we denote by $\text{cont} : L_\nu \otimes \ast L_\nu \rightarrow \mathbb{C}$ the contraction operator. It is easy to see that $\text{cont}(X(y \otimes f)) = X_0 f(y)$. On the other hand, it is easy to check from the definition of $X$ that $\text{cont}(X(y \otimes f)) = af(y)$, where $a$ is found from the equation
\[ (1 \otimes (\ast))(\Phi^\nu_\nu^* \otimes 1)\Phi^\nu_\nu = a \cdot 1_{L_\nu} \]
Thus, $X_0 = a$, and it can be found in terms of the fusion matrix. Namely, we find
\[ (1 \otimes (\ast))\Phi^J_{\nu^*,\nu}(v_* \otimes v) = X_0 \cdot 1_{L_\nu}, \]
hence $X_0$ is the contraction of $J_{\nu^*,\nu}(v_* \otimes v)$, which equals $(S(Q)(\nu)v, v_*)$. However, as follows from Proposition 2.13 of [EV2], $Q = S(Q)$ on the zero weight subspace. This implies Theorem 4.1

Remark. Let $g = sl_n$, and $V$ be the q-deformation of the representation $S^{kn} \mathbb{C}^n$. In this case, $V[0]$ is one-dimensional (so $\Psi^\nu_\mu$ can be viewed as a scalar function and is independent of $v$ up to scaling), and the weights $\nu$ for which the operators $\Phi^\nu_\nu$ exist are those of the form $\mu + k\rho$, where $\mu$ is a dominant integral weight. Moreover, as was shown in [EK2], the function $\Psi_{kp}(\lambda)$ is given by an explicit product formula, while $\Psi_{\mu+k\rho}(\lambda)/\Psi_{kp}(\lambda)$ is the Macdonald polynomial with highest weight $\mu$. Thus, using Proposition 41 from [EV3] (the determinant formula for $Q$), we obtain the Macdonald inner product identities for Macdonald’s polynomials of type $A$ (see e.g. [EK3] for the formulation). Another (more complicated) representation theoretic proof of these identities was given in [EK3].
4.3. A remark on theta functions and the Kostant identity. Let \( f \) be a smooth function on \( C_\xi \), which is periodic with respect to the lattice \( \kappa Q^\vee \). The function \( f \) can be decomposed into a Fourier series with respect to the basis \( q^{2(\lambda,\beta)} \), \( \beta \in P \).

Let \( \gamma(\lambda) \) denote the theta-function

\[
\gamma(\lambda) = \sum_{\beta \in P} q^{\beta^2} q^{2(\lambda,\beta)}.
\]

The following lemma is standard.

**Lemma 4.3.** We have

\[
\int_{C_\xi} f(\lambda) q^{-\lambda^2} \, d\lambda = \int_{C_{\xi}/\kappa Q^\vee} f(\lambda) \gamma(\lambda) \, d\lambda.
\]

**Proof.** We have

\[
\int_{C_\xi} f(\lambda) q^{-\lambda^2} \, d\lambda = C \int_{C_{\xi}/\kappa Q^\vee} f(\lambda) \sum_{\chi \in \kappa Q^\vee} q^{-(\lambda+\chi)^2} \, d\lambda.
\]

It follows from comparison of Fourier coefficients that

\[
C \sum_{\chi \in \kappa Q^\vee} q^{-(\lambda+\chi)^2} = \gamma(\lambda).
\]

Thus the lemma is proved. \( \square \)

Recall that the quantum dimension, \( \dim_q L_\nu \), of the representation \( L_\nu \), is the number \( \chi_\nu(q^{2\rho}) \). One has \( \dim_q L_\nu = \prod_{\alpha \in \Sigma^+_+} [(\alpha, \nu + \rho)]_q = \delta_q(\nu + \rho)/\delta_q(\rho) \).

**Theorem 4.4.** (Kostant, [Kos]) One has

\[
\gamma(\lambda) = K \sum_{\beta \in P_+} q^{(\beta,\beta+2\rho)} \chi_\beta(q^{2\lambda}) \dim_q(L_\beta),
\]

where \( K = \prod_{\alpha \in \Sigma^+_+} (1 - q^{2(\alpha,\rho)}) \).

4.4. Proof of Theorem 4.2. The proof follows the ideas of [EKi1].

By Lemma 4.3, the statement of Theorem 4.2 is equivalent to the equality

\[
\frac{1}{|W|} \int_{C_{\xi}/\kappa Q^\vee} \delta_q(\lambda) \delta_q(-\lambda) \left( \Psi^\nu_\lambda(\lambda), \Psi^{\nu^*}_\lambda(-\lambda) \right) \gamma(\lambda) \, d\lambda
= \delta_q(-\mu - \rho) q^{(\mu+\rho,\mu+\rho)+(\nu+\rho,\nu+\rho)} \left( Q_V(\mu)\nu, \Psi_{\nu^*}^{\nu^*}(-\mu - \rho) \right).
\]

Using the argument and notation of the proof of Theorem 4.4, and also the Kostant identity, we can rewrite the left hand side of this equation in the form

\[
LHS = \frac{K}{|W|} \int_{C_{\xi}/\kappa Q^\vee} \delta_q(\lambda) \delta_q(-\lambda) \sum_{\beta \in P_+} \text{Tr}(X_\beta) \chi_\beta(q^{2\lambda}) \times \sum_{\beta' \in P_+} q^{(\beta',\beta'+2\rho)} \chi_{\beta'}(q^{2\lambda}) \dim_q(L_{\beta'}) \, d\lambda
\]
Thus by the Weyl orthogonality theorem for characters, we have

$$LHS = K \sum_{\beta \in P_+} \text{Tr}(X_\beta) q^{(\beta, \beta + 2\rho)} \dim_q(L_\beta).$$

Indeed, the only nonzero contributions come from the case $\beta' = \beta^*$, while the expressions $(\beta, \beta + 2\rho)$ and $\dim_q(L_\beta)$ are invariant under the transformation $\beta \to \beta^*$.

Let $u$ be the Drinfeld element of $U_q(\mathfrak{gl}_n)$ \cite{Dr}. Namely, $u$ is an element of a completion of $U_q(\mathfrak{g})$ defined by the formula $u = \sum S(b_i) a_i$, where $\sum a_i \otimes b_i$ is the universal R-matrix $R$ of $U_q(\mathfrak{g})$. Drinfeld showed that $u$ acts on $L_\beta$ as $q^{-(\beta, \beta + 2\rho)} q^{2\rho}$, and that $uS(u)^{-1} = q^{4\rho}$. Thus, $S(u)^{-1}$ acts in $L_\beta$ by $q^{(\beta, \beta + 2\rho)} q^{2\rho}$, and hence $q^{(\beta, \beta + 2\rho)} \dim_q L_\beta = \text{Tr}|_{L_\beta}(S(u)^{-1})$. Therefore, we get

$$LHS = K \sum_{\beta \in P_+} \text{Tr}(X_\beta \otimes S(u)^{-1})|_{L_\beta} = K \text{Tr}(X \Delta(S(u)^{-1})).$$

Now, as was also shown by Drinfeld \cite{Dr}, one has $\Delta(u) = (R^{21} R)^{-1}(u \otimes u)$. Hence, $\Delta(S(u)^{-1}) = R^{21} R (S(u)^{-1} \otimes S(u)^{-1})$ (we use that $(S \otimes S)(R) = R$). Substituting this into the formula for the LHS, and remembering the definition of $X$, we get

$$LHS = K(\cdot) \otimes \text{Tr}|_{L_\mu \otimes^* L_\nu} \left( (\tilde{\Phi}_\mu^\nu \otimes \tilde{\Phi}_\nu^\mu) R^{21} R (S(u)^{-1} \otimes S(u)^{-1}) \right)$$

(the trace takes values in $V \otimes^* V$, and $(\cdot)$ denotes the contraction $V \otimes^* V \to \mathbb{C}$).

To compute this trace, let us look at the trace in one of the factors, i.e.

$$\text{Tr}|_{L_\nu}((1 \otimes \tilde{\Phi}_\nu^\mu) R^{21} R (1 \otimes S(u)^{-1})) = q^{(\mu, \nu + 2\rho)} T,$$

$$T := \text{Tr}|_{L_\nu}((1 \otimes \tilde{\Phi}_\nu^\mu) R^{21} R (1 \otimes q^{2\rho})).$$

It is easy to show that if $Y, Z$ are $U_q(\mathfrak{g})$ modules, $\dim Z < \infty$, and $\Phi \in \text{End}_{U_q(\mathfrak{g})}(Y \otimes Z)$, then $\text{Tr}|_Z(\Phi(1 \otimes q^{2\rho}))$ is an intertwining operator $Y \to Y$. Therefore, $T$ is an intertwiner $L_\mu \to L_\mu \otimes^* V$, and the operator

$$B := (1 \otimes (\cdot))(\tilde{\Phi}_\mu^\mu \otimes 1) T : L_\mu \to L_\mu$$

is a scalar.

Now, we have

$$K^{-1} q^{-(\mu, \mu + 2\rho) - (\nu, \nu + 2\rho)} LHS = \text{Tr}(B q^{2\rho}) = B \dim_q(L_\mu) = B \frac{\delta_q(\mu + \rho)}{\delta_q(\rho)} = B \frac{\delta_q(-\mu - \rho)}{\delta_q(-\rho)},$$

and

$$K = q^{2\rho^2} \delta_q(-\rho).$$

Therefore, after a simple calculation we find that the statement of Theorem 4.2 is equivalent to the identity

$$B = (Q_V(\mu)v, \Psi_\nu^\mu(-\mu - \rho)).$$

So it remains to prove formula (10).
To prove (10), let us apply the operator $T$ to the highest weight vector $x_\mu$ of $L_\mu$. We have
\[ R(x_\mu \otimes y) = x_\mu \otimes q^{\mu+y}. \]
Therefore,
\[ T x_\mu = \text{Tr}|_{L_\nu}(\Phi_\nu^* q^{2(\mu + \rho)}) x_\mu = x_\mu \otimes \text{Tr}|_{L_\nu}(\Phi_\nu^* q^{2(\mu + \rho)}) + l.o.t. \]
It follows from the proof of Theorem 4.1 that
\[ \text{Tr}|_{L_\nu}(\Phi_\nu^* q^{2(\mu + \rho)}) = \Psi_\nu^*(-\mu - \rho). \]
Therefore,
\[ T x_\mu = x_\mu \otimes \Psi_\nu^*(-\mu - \rho) + l.o.t., \]
i.e., $T = \Phi_\mu \Psi_\nu^*(-\mu - \rho)$. Thus, it follows from the definition of $B$ (as in the proof of Theorem 4.1) that
\[ B = (Q_V(\mu) v, \Psi_\nu^*(-\mu - \rho)). \]

Remark. As we mentioned, in the case $g = sl_n$, $V = S^{kn} \mathbb{C}^n$, the functions $\Psi_\nu^*$ are, up to normalization, Macdonald polynomials (of type A). Theorem 4.2 in this case coincides with Cherednik’s q-deformation of the Macdonald-Mehta identity [Ch1], and the proof we gave is the same as given in [EK1].

4.5. Reformulation of Theorems 4.1 and 4.2. Let us now reformulate Theorems 4.1 and 4.2 in terms of renormalized trace functions $F_\mu^V(\lambda)$.

Theorem 4.5. (Orthogonality)

Let $\mu, \nu$ be big dominant integral weights. Let $V$ be a finite dimensional $U_q(g)$ module, $V^*$ the dual module. Consider the trace functions $F_\mu^V(\lambda)$, $F_\nu^V(\lambda)$ as linear operators on the corresponding zero weight subspaces. Let $\xi \in h^*$ be a big vector. Then
\[ \frac{1}{|W|} \int_{C_\xi / h^* Q^V} F_\nu^V(-\lambda)^* F_\mu^V(\lambda) d\lambda = \delta_{\mu,\nu} Q_V^{-1}(-\mu - \rho). \]

Here $F_\nu^V(\lambda)^*$ denotes the operator on $V$ dual to the operator $F_\nu^V(\lambda)$ on $V^*$.

Theorem 4.6. (qKZB-heat equation)

Under the assumptions of Theorem 4.5, one has
\[ \frac{1}{|W|} \int_{C_\xi} F_\nu^V(-\lambda)^* F_\mu^V(\lambda) q^{-(\mu,\lambda)} d\lambda = q^{(\mu,\nu) + (\nu,\nu)} F_\nu^V(\mu)^*. \]

Corollary of Theorem 4.6. Under the conditions of Theorem 4.6 we have
\[ F_\nu^V(\mu)^* = F_\mu^V(\nu). \]

Indeed, let us interchange $\mu$ and $\nu$ in the formula of Theorem 4.6 change $\lambda$ to $-\lambda$, and take the dual operators to the operators on the left and right hand sides of the formula. Then the left hand side will remain the same, while in the right hand side the operator $F_\nu^V(\mu)^*$ will be replaced by $F_\mu^V(\nu)$. 

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Remark. For $\mathfrak{g} = sl_n$, $V = S^{kn} \mathbb{C}^n$, this corollary reduces to the Macdonald’s symmetry identity.

4.6. Proof of Theorem 4.5. From the definition of $F^V_\mu(\lambda)$ we have

\begin{equation}
\delta_q(\lambda) \Psi^\mu_\nu(\lambda) = F^V_{-\mu-\rho}(\lambda) Q_V(\mu)v .
\end{equation}

Substituting this into Theorem 4.1 we get

\[
\frac{1}{|W|} \int_{C_\xi/KQ^\vee} ( F^V_{-\mu-\rho}(\lambda) Q_V(\mu)v, F^V_{-\nu-\rho}(\lambda) Q_V(\nu)v_*) d\lambda = \delta_{\mu\nu} (Q_V(\mu)v, v_*) .
\]

The integrand can be written as

\[
(Q_V^*(\nu)^* F^V_{-\nu-\rho}(\lambda)^* F^V_{-\mu-\rho}(\lambda) Q_V(\mu)v, v_*) .
\]

By Proposition 2.13 in [EV2] we have $Q_V^*(\nu)|_{V[0]} = Q_V(\nu)|_{V[0]}$. Thus

\[
\frac{1}{|W|} \int_{C_\xi/KQ^\vee} ( Q_V(\nu)^* F^V_{-\nu-\rho}(\lambda)^* F^V_{-\mu-\rho}(\lambda) Q_V(\mu)v, v_*) d\lambda = \delta_{\mu\nu} (Q_V(\mu)v, v_*) .
\]

That gives Theorem 4.5.

4.7. Proof of Theorem 4.6. Using (11) the formula of Theorem 4.2 can be written as

\[
\frac{1}{|W|} \int_{C_\xi/KQ^\vee} ( Q_V^*(\nu)^* F^V_{-\nu-\rho}(\lambda)^* F^V_{-\mu-\rho}(\lambda) Q_V(\mu)v, v_*) q^{-(\lambda,\lambda)} d\lambda = q^{(\mu,\mu)+(\nu,\nu)} (Q_V^*(\nu)^* F^V_{-\nu-\rho}(\lambda)^* Q_V(\mu)v, v_*) .
\]

That gives Theorem 4.6.

5. Independence of integrals on the choice of a Weyl chamber

5.1. Statement of the result. For $\lambda \in \mathfrak{h}^*$ write $\lambda = x + iy$ with $x, y \in \mathfrak{h}_\mathbb{R}^*$. Let $g(\lambda)$ be a holomorphic function on $\mathfrak{h}^*$. We say that $g(\lambda)$ is rapidly decaying in the imaginary direction if for any positive $s, r$ there exists $C_{s, r} > 0$ such that

\[ |g(x + iy)| \leq C_{s, r} (1 + |y|)^{-s} \]

as long as $|x| \leq r$. An example of such a function is $g(\lambda) = q^{-(\lambda, \lambda)}$.

Let $g(\lambda)$ be a holomorphic function on $\mathfrak{h}^*$ which is invariant with respect to the standard Weyl group action and rapidly decaying in the imaginary direction. Let $\xi \in \mathfrak{h}_\mathbb{R}^*$ be a big vector. Consider the integral

\begin{equation}
I(\xi) = \int_{C_\xi} g(\lambda) F^V(\lambda, \mu)^* F^V(\lambda, \nu) d\lambda .
\end{equation}

It is clear that the integral does not depend on the choice of the big vector $\xi$ as long as the vector $\xi$ belongs to the same Weyl chamber.

Theorem 5.1. Under the above conditions the integral $I(\xi)$ does not depend on the choice of the Weyl chamber containing $\xi$. 

Theorem 5.1 is proved in the next three subsections.

We will also need the following modification of Theorem 5.1. Let \( g(\lambda) \) be a holomorphic function on \( h^*/\kappa Q^\vee \) which is invariant with respect to the standard Weyl group action. Let \( \xi \in h^*_R \) be a big vector. Consider the integral

\[
J(\xi) = \int_{C_\xi/\kappa Q^\vee} g(\lambda) F^V(-\lambda, \mu)^* F^V(\lambda, \nu) \, d\lambda.
\]

As before, it is clear that the integral does not depend on the choice of the big vector \( \xi \) as long as the vector \( \xi \) belongs to the same Weyl chamber.

**Theorem 5.2.** Under the above conditions the integral \( J(\xi) \) does not depend on the choice of the Weyl chamber containing \( \xi \).

The proof of Theorem 5.2 is completely parallel to the proof of Theorem 5.1 and is omitted.

### 5.2. The \( sl_2 \) case.

First of all, let us prove Theorem 5.1 for \( g = sl_2 \) and \( V = L_{2m} \), the irreducible representation with highest weight \( 2m \) (\( m \in \mathbb{Z}_+ \)). In this case, the theorem reduces to the equality

\[
\int_{\text{Re}\lambda = -a} g(\lambda) F(-\lambda, \mu) F(\lambda, \nu) \, d\lambda = \int_{\text{Re}\lambda = a} g(\lambda) F(-\lambda, \mu) F(\lambda, \nu) \, d\lambda,
\]

for large enough \( a > 0 \), which (due to rapid decay of \( g \) in the imaginary direction) is equivalent to the statement that the sum of residues of the integrand in the strip \( |\text{Re}(\lambda)| < a \) is zero.

**Remark.** Here for brevity we write \( F \) instead of \( F^V \). We also do not distinguish between \( F \) and \( F^* \), since \( V[0] \) is 1-dimensional. By Lemma 2.1 the poles of \( F(\lambda, \mu) \) with respect to \( \lambda \) are simple, and located at the points \( 1, 2, ..., m \) and their translates by \( \kappa \mathbb{Z} \). Therefore, the poles of \( G(\lambda) := F(-\lambda, \mu) F(\lambda, \nu) \) are simple as well and located at \( \pm k + \kappa \mathbb{Z}, \ k = 1, ..., m \). Since the function \( G \) is quasi-periodic with period \( \kappa \) (i.e. it is multiplied by a constant under the shift by \( \kappa \)), the cancellation of residues (and hence the theorem) follows from

**Proposition 5.3.** The residue of the function \( G(\lambda) \) at \( \lambda = k \) equals minus its residue at \( \lambda = -k \).

The rest of the subsection is the proof of this proposition.

Consider the function \( \tilde{\Psi}(\lambda, \mu) := \Psi(\lambda, \mu) \prod_{i=0}^{m-1} (q^{m-i} - q^{-m+i}) \).

Using formula (7.19) for \( Q(\mu) \) in [EV2], we have

\[
F(\lambda, \mu) = (q^\lambda - q^{-\lambda}) \tilde{\Psi}(\lambda, -\mu - 1) Q^{-1}(-\mu - 1) =
\]

\[
(q^\lambda - q^{-\lambda}) \tilde{\Psi}(\lambda, -\mu - 1) Q^{-1}(-\mu - 1) \prod_{i=0}^{m-1} \frac{1}{q^{-(\mu+1+i)} - q^{\mu+1+i}} =
\]
\[(q^\lambda - q^{-\lambda})\tilde{\Psi}(\lambda, -\mu - 1)q^{2m} \prod_{j=1}^{m} \frac{q^{-2\mu-2j} - 1}{q^{-2\mu-2j+2} - q^{-2m}} \prod_{i=0}^{m-1} \frac{1}{q^{-(\mu+1+i)} - q^{\mu+1+i}} = \]

\[C(q)(q^\lambda - q^{-\lambda})\tilde{\Psi}(\lambda, -\mu - 1) \prod_{j=1}^{m} \frac{1}{q^{-\mu-j+1+m} - q^{-\mu+j-1-m}}.\]

Changing \(\mu\) to \(-\mu\) in this formula, we get

\[F(\lambda, -\mu) = C(q)(q^\lambda - q^{-\lambda})\tilde{\Psi}(\lambda, \mu - 1) \prod_{j=1}^{m} \frac{1}{q^{-\mu-j+1+m} - q^{-\mu+j-1-m}}.\]

Now recall the following result.

**Theorem 5.4.** The function \(\tilde{\Psi}\) is holomorphic in \(\mu\), and satisfies the quasi-

\[\text{invariance (or resonance) conditions} \]

\[\tilde{\Psi}(\lambda, k) = \tilde{\Psi}(\lambda, -k - 2), \quad k = 1, \ldots, m.\]

This result and the above formulas, after a short calculation, lead to the identity

\[\text{Res}_{\mu=k} F(\lambda, \mu) = \]

\[F(\lambda, -k) \ln(q^{-2}) \prod_{j=1, j \neq m+1-k}^{m} \frac{1}{q^{-k-j+1+m} - q^{k+j-1-m}} \prod_{j=1}^{m} (q^{k-j+1+m} - q^{-(k-j+1+m)}).\]

Now, by the symmetry of \(F\), we have

\[F(-\lambda, \mu)F(\lambda, \nu) = F(\mu, -\lambda)F(\nu, \lambda).\]

Therefore,

\[\text{Res}_{\lambda=k} (F(-\lambda, \mu)F(\lambda, \nu)) = \text{Res}_{\lambda=k} (F(\nu, \lambda))F(\mu, -k) = \]

\[F(\nu, -k) \ln(q^{-2}) \prod_{j=1, j \neq m+1-k}^{m} \frac{1}{q^{-k-j+1+m} - q^{k+j-1-m}} \prod_{j=1}^{m} (q^{k-j+1+m} - q^{-(k-j+1+m)})F(\mu, -k) = \]

\[F(\nu, -k)\text{Res}_{\lambda=k} (F(\mu, \lambda)) = -F(\nu, -k)\text{Res}_{\lambda=-k} (F(\mu, -\lambda)) = -\text{Res}_{\lambda=-k} (F(-\lambda, \mu)F(\lambda, \nu)).\]

Proposition 5.3 and Theorem 5.1 for \(\text{sl}_2\) are proved.
5.3. **Proof of Theorem 5.1 in the dominant case.** Theorem 5.1 claims that for any two big weights \( \xi, \eta \in \mathfrak{h}_R \), one has \( I(\xi) = I(\eta) \). In this subsection we will establish the following special case of Theorem 5.1: if \( \xi \) is a big dominant weight, and \( s_j \) a simple reflection, then \( I(\xi) = I(s_j \xi) \). We will refer to this case as the dominant case.

Thus, let us take a big dominant \( \xi \). So we have \( (\xi, \alpha_j) = a >> 0 \). We may assume, without loss of generality, that \( (\xi, \alpha_j) \ll (\xi, \alpha) \) for all \( \alpha \in \Sigma_+, \alpha \neq \alpha_j \) (this situation may be achieved by moving \( \xi \) within its Weyl chamber).

Let us decompose \( \mathfrak{h}^\ast \) as \( \mathbb{C} \alpha_j \langle \langle \mathbb{C} \alpha_j \rangle \rangle \), and write \( \lambda \in \mathfrak{h}^\ast \) accordingly as a pair \( (\lambda_j, \lambda_j^\perp) \), where \( \lambda_j := (\lambda, \alpha_j^\perp) \in \mathbb{C} \). Then the cycle \( C_\xi \) gets represented as a product \( L_+ \times C_\xi' \), where \( L_+ = (\xi, \alpha_j^\perp) + i\mathbb{R} \subset \mathbb{C} \). Similarly, since \( s_j \lambda = (-\lambda_j, \lambda_j^\perp) \), we have \( C_{s_j \xi} = L_- \times C_\xi' \), where \( L_- = -(\xi, \alpha_j^\perp) + i\mathbb{R} \). This means that the integral with respect to \( d\lambda \) over either cycle is representable as an iterated integral. By Fubini’s theorem, we are free to choose in which order to compute the iterated integral. We will choose it so that we integrate first with respect to the scalar variable \( \lambda_j \).

Thus, to show that the integral over \( C_\xi \) is equal to the integral over \( C_{s_j \xi} \), it is sufficient to move the contour of integration with respect to \( \lambda_j \) from \( L_+ \) to \( L_- \) and show that the sum of residues at poles of the integrand that we pass on the way is zero. Then the equality is guaranteed by the residue theorem, given that the function \( g \) is rapidly decaying.

So let us study the poles of the integrand.

**Proposition 5.5.** Let \( \lambda = (\lambda_j, \lambda_j^\perp) \). Fix \( \lambda_j^\perp \) and let \( \lambda_j \) and \( \lambda_j^\perp \). Then all the poles of \( G(z) \) between the lines \( \text{Re}(z) = -a \) and \( \text{Re}(z) = a \) are simple, and located at the points \( z = k + l\kappa \), where \( l \) is an integer, and \( k = \pm 1, \ldots, \pm N \).

**Proof.** The proposition follows from the fact that \( (\xi, \alpha_j) \ll (\xi, \alpha) \) for positive roots \( \alpha \neq \alpha_j \), and from Lemma 2.1. \( \square \)

The function \( G \) is \( \kappa \)-quasi-periodic. Therefore, to prove Theorem 5.1 it suffices to establish

**Proposition 5.6.** The residue of the function \( G(z) \) at \( z = k \) equals minus its residue at \( z = -k \), \( k=1,\ldots,m \).

The rest of the subsection is occupied by the proof of this proposition.

Clearly, it suffices to assume that \( \mu, \nu \) are generic. Let us restrict the representations \( M_\mu, M_\nu \) and \( V \) to the Hopf subalgebra \( U_q^{\mathfrak{sl}_2}(\mathfrak{g}) \) generated by \( e_j, f_j \) and \( q^{\pm h_j} \). Then \( M_\mu, M_\nu \) will decompose in a direct sum of Verma modules over \( U_q^{\mathfrak{sl}_2}(\mathfrak{g}) \), and \( V \) will decompose into a direct sum of finite dimensional representations of \( U_q^{\mathfrak{sl}_2}(\mathfrak{g}) \). Therefore, it is easy to see that the matrix elements of the expression \( F^{V^\ast}(\mu)F^V(\lambda, \nu) \) are (infinite) linear combinations of matrix elements of similar expressions for \( U_q^{\mathfrak{sl}_2}(\mathfrak{g}) \), with coefficients independent of \( z \) (i.e. dependent only on \( \lambda_j^\perp \)). Hence, the statement in question reduces to the case \( \mathfrak{g} = \mathfrak{sl}_2 \), where it was proved in the previous subsection. Proposition 5.6 and Theorem 5.1 in the dominant case are proved.
5.4. **Proof of Theorem 5.1 in the general case.** Now we will prove Theorem 5.1 in general, using that it is true in the dominant case. This is done using the dynamical Weyl group. Namely, by formula (8), we have

\[(14) \quad F^\vee (\lambda, \mu)^* F^\vee (\lambda, \nu) =
\]

\[A_{w,V}(w^{-1} \mu) F^\vee (w^{-1} \lambda, w^{-1} \mu)^* A_{w,V'}(w^{-1} \lambda) F^\vee (w^{-1} \lambda, w^{-1} \nu) A_{w,V'}(w^{-1} \nu)^*.
\]

By Proposition 21 of [EV3], on \( V[0] \) one has \( A_{w,V'}(-w^{-1} \lambda)^* = A_{w^{-1},V}(\lambda) \). Therefore, the product of two \( A \) operators in the middle is \( A_{w^{-1},V}(\lambda) A_{w,V}(w^{-1} \lambda) \), which is 1 by the cocycle condition (7). Therefore, formula (14) can be rewritten in the form

\[(15) \quad F^\vee (\lambda, \mu)^* F^\vee (\lambda, \nu) =
\]

\[A_{w,V}(w^{-1} \mu) F^\vee (w^{-1} \lambda, w^{-1} \mu)^* F^\vee (w^{-1} \lambda, w^{-1} \nu) A_{w,V'}(w^{-1} \nu)^*.
\]

Now let \( \xi \) be a big dominant weight, and \( s_j \) a simple reflection. We have shown in the previous subsection that

\[\int_{C_{\xi}} g(\lambda) F^\vee (\lambda, \mu)^* F^\vee (\lambda, \nu) d\lambda = \int_{C_{s_j \xi}} g(\lambda) F^\vee (\lambda, \mu)^* F^\vee (\lambda, \nu) d\lambda,
\]

Therefore, using formula (15), we find, \( \forall w \in \mathbb{W} \)

\[A_{w,V}(w^{-1}, \mu) \int_{C_{\xi}} g(\lambda) F^\vee (\lambda, \mu)^* F^\vee (w^{-1} \lambda, w^{-1} \nu) d\lambda \cdot A_{w,V'}(w^{-1} \nu) =
\]

\[A_{w,V}(w^{-1}, \mu) \int_{C_{s_j \xi}} g(\lambda) F^\vee (\lambda, \mu)^* F^\vee (w^{-1} \lambda, w^{-1} \nu) d\lambda \cdot A_{w,V'}(w^{-1} \nu).
\]

Canceling the \( A \)-operators and changing \( w^{-1} \lambda \) to \( \lambda \), \( w^{-1} \mu \) to \( \mu \), and \( w^{-1} \nu \) to \( \nu \), we get

\[\int_{C_{w \xi}} g(\lambda) F^\vee (\lambda, \mu)^* F^\vee (\lambda, \nu) d\lambda = \int_{C_{w s_j \xi}} g(\lambda) F^\vee (\lambda, \mu)^* F^\vee (\lambda, \nu) d\lambda,
\]

which implies Theorem 5.1 in full generality, as any element of \( \mathbb{W} \) can be written as a product of simple reflections.

6. **Proof of Theorem 2.2**

6.1. **Reduction to big dominant integral \( \mu, \nu \).** Let \( V \) be a finite dimensional \( U_q(\mathfrak{g}) \) module. From now till the end of this section fix \( \beta \in P \). Let \( \mu \in \mathfrak{h}^* \) be generic, and \( \nu = \mu - \beta \).

**Lemma 6.1.** The integral

\[\int_{C_{\xi} / \kappa Q'} F^\vee (\mu, -\lambda) \ F^\vee (\lambda, \nu) \ d\lambda
\]

(for big \( \xi \)) is a rational function of the variables \( q^{2(\mu, \alpha_i)} \).
Proof. As explained in Section 2, the matrix elements of the function $F(\lambda, \mu)$ are representable as finite sums $\sum_{j=1}^{m} q^{-2(\lambda, \mu)} f_j(\lambda) g_j(\mu)$, where $f_j(\nu)$ and $g_j(\nu)$ are rational functions of the variables $q^{2(\nu, \alpha)}$. This immediately implies the statement. \qed

**Corollary 6.2.** Theorem 2.2 is true if it is true for all big anti-dominant integral $\mu$.

**Proof.** The corollary follows from the fact that a rational function of $q^{2(\mu, \alpha)}$ is determined by its values at big anti-dominant integral $\mu$. \qed

6.2. **Proof of Theorem 2.2.** We continue the proof of Theorem 2.2. As was shown in the previous subsection, we may assume that $\mu$ is big anti-dominant integral (then so is $\nu$, since $\beta$ is fixed and belongs to $P$).

Substituting the generalized Weyl character formula (9) into the left hand side of the formula of Theorem 4.5 we get

$$LHS = \frac{1}{|W|} \sum_{w, y \in W} (-1)^{w} (-1)^{y} \int_{C_{\xi} / \kappa Q^{\vee}} A_{y, V}(\mu)^{-1} F^{*}(\lambda, y \mu)^{*} F^{V}(\lambda, w \nu)(A_{w, V^{*}}(\nu)^{-1})^{*} d\lambda.$$ (16)

Using the symmetry (8) and the cocycle condition (7), we find

$$A_{y, V}(\mu)^{-1} F^{*}(\lambda, y \mu)^{*} F^{V}(\lambda, w \nu)(A_{w, V^{*}}(\nu)^{-1})^{*} = A_{y, V}(\mu)^{-1} F^{V}(\lambda, w \nu)(A_{w, V^{*}}(\nu)^{-1})^{*} A_{w, V}(\mu)^{-1} F^{*}(\lambda, y \mu)^{*} = A_{y, V}(\mu)^{-1} A_{w, V}(w^{-1} \mu)^{*} F^{V}(w^{-1} \lambda, \nu) = A_{y, V}(\mu)^{-1} A_{w, V}(w^{-1} \mu)^{*} F^{V}(w^{-1} \lambda, \nu) = A_{w^{-1}, V}(\mu)^{-1} F^{*}(\lambda, w^{-1} \mu)^{*} F^{V}(w^{-1} \lambda, \nu).$$ (17)

Now we will use Theorem 5.2. By this theorem and the last formula, equality (16) can be written as

$$LHS = \sum_{w \in W} (-1)^{w} A_{w, V}(\mu)^{-1} \int_{C_{\xi} / \kappa Q^{\vee}} F^{V}(\lambda, w \mu)^{*} F^{V}(\lambda, \nu) d\lambda.$$ (18)

Now we have

**Lemma 6.3.** If $w \in W$ is not the identity element, then

$$\int_{C_{\xi} / \kappa Q^{\vee}} F^{V}(\lambda, w \mu)^{*} F^{V}(\lambda, \nu) d\lambda = 0.$$

**Proof.** It is easy to see that the integral tends to zero as $\xi$ tends to infinity inside the dominant Weyl chamber. At the same time the integral does not change its value under variation of $\xi$. Hence, the integral is zero if $\xi$ is in the dominant chamber. It follows from Theorem 5.2 that the same is true for all other Weyl chambers. The lemma is proved. \qed
By Lemma 6.3 and formula (13), the left hand side of the formula of Theorem 4.3 is given by the formula

\[ \text{LHS} = \int_{C_\xi/\kappa Q^*_\nu} F^V(-\lambda, \mu)^* F^V(\lambda, \nu) \, d\lambda. \]

Using the symmetry (2) we see that Theorem 4.3 implies Theorem 2.2. Thus Theorem 2.2 is proved.

7. Proof of Theorem 2.3

7.1. A polynomiality lemma.

Lemma 7.1. For generic \( \nu \in \mathfrak{h}^* \) and big anti-dominant integral \( \mu \), the function \( F^V(-\lambda)^* F^V(\lambda, \nu) \) is a Laurent polynomial of \( q^{2(\lambda, \alpha_i)} \) multiplied by \( q^{2(\lambda, -\nu)} \).

Proof. It suffices to check that the same statement is true for the function

\[ (\Psi^\nu(\lambda, \nu), \Psi_\mu^\nu(-\lambda))\delta_\theta(-\lambda)\delta_\theta(\lambda) \]

for big dominant integral \( \mu \). Arguing as in the proof of Theorem 4.1, we find that

\[ (\Psi^\nu(\lambda, \nu), \Psi_\mu^\nu(-\lambda)) = \text{Tr}[(1 \otimes (.,) \otimes 1)(\Phi^\nu_\nu \otimes \Phi^\nu_\mu^*) (q^{2\lambda} \otimes q^{2\lambda})], \]

where the trace is taken in the tensor product \( M_{\nu} \otimes^* L_\mu \). This tensor product decomposes as \( M_{\nu} \otimes^* L_\mu = \oplus H^\beta_{\mu \nu} M_{\beta} \). Since the operator \( X = (1 \otimes (.,) \otimes 1)(\Phi_\nu^\nu \otimes \Phi_\mu^\nu^*) \) is an intertwiner from \( M_{\nu} \otimes^* L_\mu \) to itself, as in the proof of Theorem 4.1, we find that \( (\Psi^\nu(\lambda, \nu), \Psi_\mu^\nu(-\lambda)) \), as a function of \( \lambda \), is a finite linear combination of characters of Verma modules \( M_{\beta} \). Hence \( \delta_\theta(\lambda)(\Psi^\nu(\lambda, \nu), \Psi_\mu^\nu(-\lambda)) \) is equal to \( q^{2(\lambda, -\nu)} \) times a Laurent polynomial. This implies the statement of the lemma. \( \square \)

7.2. A modification of Theorem 2.3

Proposition 7.2. For a big anti-dominant integral \( \mu \), one has

\[ q^{-\nu, \nu} \text{LHS} = \frac{1}{|\mathbb{W}|} \int_{C_\xi} F^V_\mu(-\lambda)^* \sum_{w \in W} (-1)^w F^V(\lambda, w\nu)(A_{w, V_*}(\nu)^{-1})^* q^{-\lambda, \lambda} \, d\lambda = q^{\langle \mu, \mu \rangle + \langle \nu, \nu \rangle} F^V_\mu(\nu)^*. \]

Proof. Let \( F^V_0(\lambda, \nu) := F^V(\lambda, \nu)q^{2(\lambda, \nu)} \). We have

\[ q^{-\nu, \nu} \text{LHS} = \frac{1}{|\mathbb{W}|} \int_{C_\xi} F^V_\mu(-\lambda)^* \sum_{w \in W} (-1)^w F^V_0(\lambda, w\nu)(A_{w, V_*}(\nu)^{-1})^* q^{-(\lambda + w\nu, \lambda + w\nu)} \, d\lambda = \]

\[ \frac{1}{|\mathbb{W}|} \int_{C_\xi + \text{Re}(w\nu)} \sum_{w \in W} (-1)^w F^V_\mu(-\lambda + w\nu)^* F^V_0(\lambda - w\nu, w\nu)(A_{w, V_*}(\nu)^{-1})^* q^{-(\lambda, \lambda)} \, d\lambda. \]

By Lemma 4.1 the cycle of integration \( C_{\xi + \text{Re}(w\nu)} \) may be replaced with \( C_\xi \), and furthermore the integral is a rational function of \( q^{2\langle \nu, \alpha_i \rangle} \). Therefore, it suffices to check the
proposition in the case when \( \nu \) is big anti-dominant integral. But in this case, the result follows from Theorem 4.6 and the generalized Weyl character formula (9).

\[ \square \]

**Corollary 7.3.** For big anti-dominant integral \( \mu \) and generic \( \nu \)

\[
\frac{1}{|W|} \sum_{w,y \in W} (-1)^w (-1)^y \int_{C_\xi} A_{x,y}(\mu)^{-1} F_{V^*}(-\lambda, y\mu)^* F_V(\lambda, w\nu)(A_{w,v^*}(\nu)^{-1})^* q^{-(\lambda,\lambda)} d\lambda =
\]

\[
q^{(\mu,\mu)+(\nu,\nu)} \sum_{z \in W} (-1)^z A_{z,v}(\mu)^{-1} F_{V^*}(\nu, z\mu)^*.
\]

(20)

The corollary follows by applying the generalized Weyl character formula (9) to formula (19).

### 7.3. Proof of Theorem 2.3.

Arguing as in the proof of Theorem 2.2 (using Theorem 5.1 and formula (17)), we can rewrite identity (20) in the form

\[
q^{-(\mu,\mu)-(\nu,\nu)} \sum_{z \in W} (-1)^z \int_{C_\xi} A_{z,v}(\mu)^{-1} F_{V^*}(-\lambda, z\mu)^* F_V(\lambda, \nu) q^{-(\lambda,\lambda)} d\lambda =
\]

\[
\sum_{z \in W} (-1)^z A_{z,v}(\mu)^{-1} F_{V^*}(\nu, z\mu)^*.
\]

(21)

This identity has been established for big anti-dominant integral \( \mu \). However, we will now use it to prove Theorem 2.3, which is equivalent to the statement that this identity holds for generic complex \( \mu \) and, furthermore, term-by-term.

The proof is based on the following lemma. Let \( \mu \in \mathfrak{h}^* \) be a fixed weight, which does not belong to the pole divisor of \( F_V(\lambda, \mu) \).

**Lemma 7.4.** The function

\[
G(\mu, \nu) := \int_{C_\xi} F_{V^*}(-\lambda, \mu)^* F_V(\lambda, \nu) q^{-(\lambda,\lambda)-(\mu,\mu)-(\nu,\nu)+2(\mu,\nu)} d\lambda.
\]

is holomorphic in \( \nu \) for \( \text{Re} \ \nu \) big anti-dominant, and \( \kappa Q^\mu \)-periodic in \( \nu \); that is, \( G \) is a holomorphic function of \( z_j := q^{-2(\nu,\alpha_j)} \) in the region \( 0 < |z_j| < q^{dN} \), where \( N \) is as in Lemma 2.1, and \( d \) is the ratio of squared lengths of long and short roots. Furthermore, \( G \) is meromorphic in the region \( |z_j| < q^{dN} \), and its Laurent coefficients are rational functions of \( q^{2(\mu,\alpha_i)} \).

**Proof.** By Theorem 5.1, we may assume that \( \xi \) lies in the dominant chamber. By Lemma 2.1 the trace function \( F_v(\lambda, \mu) \) may be written as

\[
F_v(\lambda, \mu) = q^{-2(\lambda,\mu)} \frac{\tilde{F}_v(\lambda, \mu)}{\prod_{\alpha \in \Sigma^+} \prod_{k=1}^N (1 - q^{2(\lambda,\alpha) - k(\alpha,\alpha)})},
\]

where \( \tilde{F}_v(\lambda, \mu) \) is a trigonometric polynomial in \( \lambda \) whose coefficients are rational functions of \( q^{2(\mu,\alpha_i)} \).
Therefore, we have
\[ G(\mu, \nu) = \int_{C_\xi} q^{-\lambda - \mu + \nu} \frac{H(\lambda)}{\Pi_{\alpha \in \Sigma_+} (1 - q^{2(a,\lambda)})} d\lambda, \]
where \( H \) is a trigonometric polynomial in \( \lambda \) whose coefficients are rational in \( q^{2(\mu,\alpha)} \) and \( q^{2(\nu,\alpha)} \).

Now make a change of variable \( \eta = \lambda - \mu + \nu \). Then we get
\[ G(\mu, \nu) = \int_{C_\xi - \text{Re}(\mu - \nu)} q^{-(\eta, \eta)} \frac{H(\eta + \mu - \nu)}{\Pi_{\alpha \in \Sigma_+} (1 - q^{2(a,\eta + \mu - \nu)})} d\eta. \]
The fractions \( \frac{1}{1 - q^{2(a,\eta + \mu - \nu)} \pm k(a,\alpha)} \) in the integrand can be expanded into geometric series for \( \text{Re}(\alpha_i, \eta + \mu - \nu) > 0 \):
\[ \frac{1}{1 - q^{2(a,\eta + \mu - \nu)} \pm k(a,\alpha)} = \sum_{m \geq 0} q^{m(2(a,\eta + \mu - \nu) \pm k(a,\alpha))}. \]
The inequalities \( \text{Re}(\alpha_i, \eta + \mu - \nu) > 0 \) are satisfied on the cycle of integration, as \( \xi \) is a big dominant weight. Thus we can expand the integrand in a series and then integrate the series term-wise. This shows that we can expand \( G(\mu, \nu) \) in a Laurent series with respect to \( z_j = q^{-2(\nu,\alpha_j)} \), which is convergent for small enough \( z_j \), i.e., for big antidominant \( \text{Re} \nu \) (in fact, it is easy to see that \( |z_j| < q^{dN} \) is sufficient for convergence). The coefficients of this series are integrals of trigonometric polynomials of \( \eta \) with coefficients rational in \( q^{2(\mu,\alpha)} \) against the measure \( q^{-(\eta, \eta)} d\eta \). Thus they are rational functions in \( q^{2(\mu,\alpha)} \), and the lemma is proved.

Now we can finish the proof of Theorem 2.3. Lemma 7.4 implies that both sides of identity (21) admit Laurent expansions with respect to the variables \( z_j = q^{-2(\nu,\alpha_j)} \) for \( |z_j| < q^{dN} \). Let \( \beta \) be a fixed integral weight, and \( C_{\beta,l}(\mu), C_{\beta,r}(\mu) \) be the coefficients of \( q^{-2(\nu,\beta)} \) in this expansion on the left, respectively right hand side of identity (21). Let also \( C_{\beta,l}^1(\mu), C_{\beta,r}^1(\mu) \) be the coefficients of the same term in the series on the left and right hand sides of (21) given exclusively by the terms with the Weyl group element \( z \) equal to 1. It is easy to see that if \( \mu \) is a big anti-dominant integral weight, then \( C_{\beta,l}(\mu) = C_{\beta,l}^1(\mu), C_{\beta,r}(\mu) = C_{\beta,r}^1(\mu) \) (i.e., terms with \( z \neq 1 \) contribute only to very high terms of the expansion). On the other hand, since identity (21) is known to hold for big anti-dominant integral \( \mu \), we have for such \( \mu \): \( C_{\beta,l}(\mu) = C_{\beta,r}(\mu) \). Therefore, we conclude that for big anti-dominant integral \( \mu \), one has \( C_{\beta,l}(\mu) = C_{\beta,r}(\mu) \). But by Lemma 7.4, both \( C_{\beta,l}^1(\mu) \) and \( C_{\beta,r}^1(\mu) \) are rational functions of \( q^{2(\mu,\alpha)} \). Therefore, the equality \( C_{\beta,l}(\mu) = C_{\beta,r}(\mu) \) holds for generic \( \mu \), and hence the terms on the left and right sides of (21) corresponding to \( z = 1 \) are equal:
\[ q^{-(\mu,\mu) - (\nu, \nu)} \int_{C_\xi} F^{V^*}(-\lambda, \mu)^* F^V(\lambda, \nu) q^{-(\lambda, \lambda)} d\lambda = F^{V^*}(\nu, \mu)^*. \]
This implies Theorem 2.3 as $F^*V(\nu, \mu) = FV(\mu, \nu)$.

8. Integral Transforms

In this section we prove Theorem 2.4.

8.1. Proof that the integral transforms are well defined and continuous. For any vectors $v \in V$ and $v_\ast \in V^\ast$, we have

$$(F^V(\lambda, \mu)v, v_\ast) = \sum_j f_j(\lambda) g_j(\mu)$$

for suitable functions $f_j(\lambda), g_j(\lambda)$. Each of these functions has the form

$$P(\lambda) \prod_{\alpha \in \Sigma_+} \prod_{k=1}^N (1 - q^{2(\lambda, \alpha) - k(\alpha, \alpha)})$$

where $P(\lambda)$ is a Laurent polynomial in variables $q^{2(\alpha_i, \lambda)}$, whose Newton polyhedron is contained in the Newton polyhedron of the denominator (this follows from [EST], Proposition 2.2).

For any vectors $v \in V$ and $v_\ast \in V^\ast$, the function $(Q(-\lambda - \rho)v, v_\ast)$ also has such a form (again, from [EST]).

Lemma 8.1. Let a function $f(\lambda)$ be of the form (23). Let $\xi \in \h_R^\ast$ be big. Then all derivatives of the restriction of the function $f(\lambda)$ to $C_\xi$ or $D_\eta$ are bounded from above.

Proof. The statement about the restriction to $C_\xi$ is obvious, since in this case the function is periodic with respect to a lattice. So we need to prove only the statement about restriction to $D_\eta$. For this purpose, take $\lambda = i\eta + y, y \in \h_R$. Then the binomial factor $(1 - q^{2(\lambda, \alpha) - k(\alpha, \alpha)})$ from formula (23) satisfies the following lower bound:

$$|1 - q^{2(\lambda, \alpha) - k(\alpha, \alpha)}| \geq C(1 + q^{2(y, \alpha) - k(\alpha, \alpha)})$$

for some $C = C(\eta) > 0$ (This follows from the elementary estimate

$$|1 - ae^{i\theta}| \geq (1 + a)|\sin(\theta/2)|$$

if $a > 0$). This lower bound implies the claim, since it implies that the denominator of (23) is bounded from below by a polynomial with the same Newton polyhedron and positive coefficients. \qed

Lemma 8.1 implies that the integral transforms of Theorem 2.4 are well defined and continuous. Indeed, let for example $a \in S_\eta(C_\xi) \otimes V[0]$, $a(\lambda) = \sum a_m(\lambda)v_m$, where $v_m$ is a basis of $V[0]$. Then the components of the vector function $K_{V^\ast}a$ are finite linear combinations of functions of the form

$$b(\mu) = g(\mu) \int_{C_\xi} q^{2(\lambda, \mu)} f(\lambda)a(\lambda) d\lambda$$

for suitable functions $f(\lambda), g(\lambda)$.
where \( f \) and \( g \) are of the form (23). By Lemma 8.1 the multiplication operators by \( f \) and \( g \) preserve the spaces of Schwartz functions and are continuous. So the fact that \( K_{\text{Im}}^V \) is well defined and continuous follows from the classical fact that the Fourier transform is a continuous isomorphism of the Schwartz space to itself. The same proof applies to \( K_{\text{Re}}^V \).

8.2. Proof that \( K_{\text{Im}}^V K_{\text{Re}}^V = \text{Id} \).

**Proposition 8.2.** We have

\[
K_{\text{Im}}^V K_{\text{Re}}^V = \text{Id}.
\]

**Proof.** We will use the following standard lemma from Fourier analysis.

Let \( f, g \) be smooth functions on \( \mathbb{R}^n \times \mathbb{R}^n \) such that \( f(x, y) = e^{i(x,y)} f_0(x, y) \), \( g(x, y) = e^{i(x,y)} g_0(x, y) \), and \( f_0, g_0 \) are periodic with respect to the lattice \( (2\pi \mathbb{Z})^n \) in the first variable. Assume that for any \( y, z \in \mathbb{R}^n \) such that \( y - z \in (2\pi \mathbb{Z})^n \), one has

\[
\int_{(\mathbb{R}/2\pi \mathbb{Z})^n} f(x, y) g(-x, z) dx = \delta_{y, y-z}.
\]

Define integral transforms \( K_1 : C^\infty_0(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n) \), \( K_2 : \mathcal{S}(\mathbb{R}^n) \rightarrow C^\infty(\mathbb{R}^n) \) by

\[
(K_1 a)(x) = \int_{\mathbb{R}^n} g(-x, z) a(z) dz, \quad (K_2 b)(y) = \int_{\mathbb{R}^n} f(x, y) b(x) dx
\]

(here \( C^\infty_0 \) is the space of functions with compact support).

**Lemma 8.3.** Under these conditions, \( K_1 \) takes values in \( \mathcal{S}(\mathbb{R}^n) \), and \( K_2 \circ K_1 = \text{Id} \) (for an appropriate normalization of \( dx \) and \( dz \)).

The proof of the lemma is standard, using Fourier series expansions, and will be omitted.

Lemma 8.3 and Theorem 2.2 imply Proposition 8.2 since matrix elements of \( F^V(\lambda, \mu) \), \( \lambda \in C_\xi, \mu \in D_\eta \), are of the same type as the functions \( f(x, y), g(x, y) \) considered above.

8.3. Proof that \( K_{\text{Re}}^V K_{\text{Im}}^V = \text{Id} \).

**Lemma 4.** Let \( X, Y \) be vector spaces and \( A : X \rightarrow Y \) and \( B : Y \rightarrow X \) linear operators. Assume that the operator \( A \) is injective, and \( AB = \text{Id} \). Then \( BA = \text{Id} \).

**Proof.** For every \( y \in Y \) we have \( A(BA y - y) = 0 \). Since \( A \) is injective, we have \( BA = \text{Id} \).

**Lemma 4** shows that in order to finish the proof of Theorem 2.4 it is enough to prove the following

**Proposition 8.5.** Under the conditions of Theorem 2.4 the operator \( K_{\text{Im}}^V \) is injective.
Proof. First of all, using the dynamical Weyl group symmetry, we can reduce the problem to the case when \(\xi\) is anti-dominant.

Suppose that \(a \in S_\eta(C_\xi)\), and \(K_{\text{fin}} a = \int_{C_\xi} F^V(\mu, -\lambda) a(\lambda) d\lambda = 0\). As we know (see e.g. formula (22)), the function \(F^V\) is representable in the form \(F^V(\mu, \nu) = q^{-2(\mu, \nu)} \frac{F^V(\mu, \nu)}{Z(\nu)}\), where \(Z(\nu)\) is a product of binomial terms, and \(\tilde{F}\) is a trigonometric polynomial in \(\nu\). This means,

(24) \[
\int_{C_\xi} \hat{q}^{-2(\mu, \lambda)} \tilde{F}^V(\mu, -\lambda) b(\lambda) = 0,
\]

where \(b(\lambda) = Z(-\lambda) a(\lambda) \in S_\eta(C_\xi)\) (as \(Z\) is periodic).

The function \(\tilde{F}^V(\mu, -\lambda)\) can be written in the form

(25) \[
\tilde{F}^V(\mu, -\lambda) = 1 + \sum_{\beta \in L} E_\beta(\mu) q^{2(\beta, \lambda)},
\]

where \(L \subset Q_+\setminus\{0\}\) is a finite set, and \(E_\beta(\mu)\) are bounded rational functions of \(q^{2(\mu, \alpha_i)}\) on \(D_\eta\).

Using equation (25), equation (24) can be written in the form of a difference equation:

(26) \[
\hat{b}(\mu) = -\sum_{\beta \in L} E_\beta(\mu) \hat{b}(\mu + \beta).
\]

If \(\xi\) is sufficiently big, \(\hat{b}(\mu + \beta)\), decays rapidly with \(\beta \to \infty\) in \(Q_+\) (since \(\hat{b} \in S_\xi(D_\eta)\)). Thus, solving equation (24) recursively, and using that \(E_\beta(\mu)\) are bounded, we find that for sufficiently big \(\xi\), any solution \(\hat{b} \in S_\xi(D_\eta)\) of the difference equation (26) must be zero. The proposition is proved. \(\square\)

9. Proofs of Theorems 2.5 and 2.6

9.1. Proof of Theorem 2.5

Recall that the trace functions are eigenfunctions of \(D_{U,V}\), namely

\[
D^{(\lambda)}_{U,V} F(\lambda, \nu) = \chi_U(q^{-2\nu}) F(\lambda, \nu).
\]

Now let \(\mu - \nu \in P\). Then by Theorem 2.2

\[
\int_{C_\xi/\kappa Q^\nu} F(\mu, -\lambda) [D^{(\lambda)}_{U,V} - (D^{(\lambda)}_{U,V})^*] F(\lambda, \nu) d\lambda =
\]

\[
\chi_U(q^{-2\nu}) \int_{C_\xi/\kappa Q^\nu} F(\mu, -\lambda) F(\lambda, \nu) d\lambda - \chi_U(q^{-2\mu}) \int_{C_\xi/\kappa Q^\nu} F(\mu, -\lambda) F(\lambda, \nu) d\lambda =
\]

\[
(\chi_U(q^{-2\nu}) - \chi_U(q^{-2\mu})) \delta_{\mu \nu} Q^{-1}(\mu - \rho) = 0,
\]

because either \(\delta_{\mu \nu} = 0\) or \((\chi_U(q^{-2\nu}) - \chi_U(q^{-2\mu})) = 0\).
By Lemma 8.5 this implies that
\[(\mathcal{D}_{U,V}^{(\lambda)} - (\mathcal{D}_{U,V}^{(\lambda)})^*)F(\lambda, \nu) = 0.\]

This means that
\[\int_{C_\xi \cap \kappa Q^V} q^{-2(\lambda, \mu)}(\mathcal{D}_{U,V}^{(\lambda)} - (\mathcal{D}_{U,V}^{(\lambda)})^*)F(\lambda, \nu) = 0.\]

By Lemma 8.5 this means that
\[(\mathcal{D}_{U,V} - \mathcal{D}_{U,V}^*)q^{2(\lambda, \nu)} = 0,
\]
which easily implies
\[\mathcal{D}_{U,V} = \mathcal{D}_{U,V}^*.\]

The theorem is proved.

9.2. Proof of Theorem 2.6. By Theorem 2.3 we have
\[\int_{C_\xi} F^V(\mu, -\lambda)F^V(\lambda, \nu)q^{-(\lambda, \lambda)}d\lambda = q^{(\mu, \mu) + (\nu, \nu)}F^V(\mu, \nu).\]

This is equivalent to saying that for a fixed \(\nu\), one has
\[K_{\text{Im}}^V(q^{-(\lambda, \lambda)}F^V(\lambda, \nu)) = q^{(\nu, \nu)}(q^{(\mu, \mu)}F^V(\mu, \nu))\]

Applying \(K_{\text{Re}}\) to both sides, and using Theorem 2.4, we obtain
\[q^{-(\lambda, \lambda)}F^V(\lambda, \nu) = \int_{D_n} F^V(\lambda, \mu)Q(-\mu - \rho)q^{(\nu, \nu)}F^V(\mu, \nu),\]
as desired.

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