WELL-POSEDNESS OF INITIAL-BOUNDARY VALUE PROBLEMS
FOR A REACTION-DIFFUSION EQUATION

A. ALEXANDROU HIMONAS, DIONYSSIOS MANTZAVINOS∗ & FANGCHI YAN

Abstract. A reaction-diffusion equation with power nonlinearity formulated either on the half-line or on the finite interval with nonzero boundary conditions is shown to be locally well-posed in the sense of Hadamard for data in Sobolev spaces. The result is established via a contraction mapping argument, taking advantage of a novel approach that utilizes the formula produced by the unified transform method of Fokas for the forced linear heat equation to obtain linear estimates analogous to those previously derived for the nonlinear Schrödinger, Korteweg-de Vries and “good” Boussinesq equations. Thus, the present work extends the recently introduced “unified transform method approach to well-posedness” from dispersive equations to diffusive ones.

1. Introduction and Results

We establish local well-posedness of the following reaction-diffusion equation with power nonlinearity formulated on the half-line with a nonzero Dirichlet boundary condition:

\[ u_t - u_{xx} = |u|^{p-1}u, \quad x \in (0, \infty), \ t \in (0, T), \]
\[ u(x, 0) = u_0(x), \quad x \in [0, \infty), \]
\[ u(0, t) = g_0(t), \quad t \in [0, T], \]

where \( p = 2, 3, 4, \ldots, \) and \( T < 1. \)

The Sobolev spaces \( H^s_x(0, \infty) \) for the initial datum \( u_0 \) and \( H^\frac{2s+1}{4}_t(0, T) \) for the boundary datum \( g_0 \) of the above initial-boundary value problem (IBVP) are obtained as restrictions of their whole line counterparts according to the general definition

\[ H^s(\Omega) = \{ f : f = F|_{\Omega} \text{ with } F \in H^s(\mathbb{R}) \}, \quad \Omega \subset \mathbb{R}. \]

Furthermore, we shall see that the correspondence \( s \leftrightarrow \frac{2s+1}{4} \) between the regularity (in \( x \)) of \( u_0 \) given in (1.1b) and the regularity (in \( t \)) of \( g_0 \) specified in (1.1c) is determined by the linear part of equation (1.1a) (that is, the linear heat equation) via two independent directions: (i) the space regularity of the linear version of IBVP (1.1) with zero initial data, and (ii) the time regularity of the linear heat initial value problem (IVP) with data in \( H^s_x(\mathbb{R}) \). In the latter direction, especially remarkable is the fact that the time regularity of the solution of the linear heat IVP is described by the space \( H^\frac{2s+1}{4}_t(0, T) \), which is the one also associated with the linear Schrödinger IVP (see, for example, [KPV, FHM1]). This is rather surprising, taking into account the sharp contrast between the diffusive nature of the linear heat equation and the dispersive character of the linear Schrödinger equation.
The reaction-diffusion equation (1.1a) has been studied extensively and from various points of view, see, for example, [A, W, GK, G, BCG, EGKP, MM, CDW, Gal], the books [Sm, Hu] and the references therein. All of these works are concerned either with the IVP or with IBVPs formulated with a zero boundary condition. In the case of the half-line IBVP (1.1), this corresponds to taking $g_0 \equiv 0$. On the contrary, here we consider the case of nonzero Dirichlet boundary conditions.

In fact, the main objective of this work is to take advantage of a novel approach for the well-posedness of nonlinear evolution equations, which relies on the new solution formulae produced by the unified transform method of Fokas [F1, F2] for the forced linear counterparts of these equations and which was originally developed for dispersive equations such as the nonlinear Schrödinger [FHM1], the Korteweg-de Vries [FHM2] and the “good” Boussinesq [HM] equations. That is, the purpose of this work is to provide the extension of the aforementioned new approach from dispersive equations to diffusive ones.

For “smooth” data ($s > \frac{1}{2}$), local well-posedness for the reaction-diffusion IBVP (1.1) will be established in the following sense.

**Theorem 1.1** (Well-posedness on the half-line for “smooth” data). Suppose $\frac{1}{2} < s < \frac{3}{2}$, $u_0 \in H^s_x(0, \infty)$ and $g_0 \in H^{2s+1}_t (0, T)$ with the compatibility condition $u_0(0) = g_0(0)$. Furthermore, for

$$
\| (u_0, g_0) \|_D = \| u_0 \|_{H^s_x(0, \infty)} + \| g_0 \|_{H^{2s+1}_t (0, T)} ,
$$

$$
T^* = \min \left\{ T, \frac{1}{p^2 (2c_{s,p})^{2p} \| (u_0, g_0) \|_D^{2(p-1)}} \right\}, \quad c_{s,p} > 0,
$$

define the space $X$ on the half-line by

$$
X = C([0, T^*]; H^s_x(0, \infty)) \cap C([0, \infty); H^{2s+1}_t (0, T^*) ,
$$

$$
\| u \|_X = \sup_{t \in [0, T^*]} \| u(t) \|_{H^s_x(0, \infty)} + \sup_{x \in [0, \infty)} \| u(x) \|_{H^{2s+1}_t (0, T^*)}.
$$

Then, for $\frac{2p-1}{2} \in \mathbb{N}$ there exists a unique solution $u \in X$ to the reaction-diffusion IBVP (1.1), which satisfies the estimate

$$
\| u \|_X \leq 2c_{s,p} \|(u_0, g_0)\|_D .
$$

Furthermore, the data-to-solution map $\{u_0, g_0\} \mapsto u$ is locally Lipschitz continuous.

For “rough” data ($s < \frac{1}{2}$), we refine our solution space by intersecting $X$ with the space

$$
C^\alpha([0, T]; L^p_x(0, \infty)) = \left\{ u \in C([0, T]; L^p_x(0, \infty)) : \sup_{t \in [0, T]} t^\alpha \| u(t) \|_{L^p_x(0, \infty)} < \infty \right\} , \quad \alpha = \frac{1}{p} \left( \frac{1}{2} - b \right) , \quad \frac{2s+1}{4} < b < \frac{1}{2} .
$$

(1.2)

In this case, our result reads as follows.

**Theorem 1.2** (Well-posedness on the half-line for “rough” data). Suppose $\frac{1}{2} - \frac{1}{p} < s < \frac{1}{2}$, $u_0 \in H^s_x(0, \infty)$ and $g_0 \in H^{2s+1}_t (0, T)$, and for $\|(u_0, g_0)\|_D$ as in Theorem 1.1 and lifespan $T^*$ given by

$$
T^* = \min \left\{ T, \frac{1}{(2p+2)^\frac{1}{2} (2c_{s,p})^{\frac{1}{2}} \|(u_0, g_0)\|_D^2} \right\}, \quad c_{s,p} > 0,
$$

...
define the space $Y$ on the half-line by

$$Y = C([0, T^*]; H^2_x(0, \infty)) \cap C([0, T^*]; H_t^{2\alpha+1}_t(0, T^*)) \cap C([0, T^*]; L^p(0, \infty)).$$

Then, the reaction-diffusion IBVP (1.1) has a unique solution $u \in Y$, which satisfies the estimate

$$\|u\|_Y \leq 2c_{s,p} \|(u_0, g_0)\|_D.$$ 

Furthermore, the data-to-solution map $\{u_0, g_0\} \mapsto u$ is locally Lipschitz continuous.

Theorems 1.1 and 1.2 will be established by showing that the iteration map induced by an appropriate solution representation of the forced linear counterpart of the nonlinear problem is a contraction in a suitably chosen solution space. In the case of the nonlinear IBVP (1.1), the associated linear problem is the forced linear heat IBVP:

$$u_t - u_{xx} = f(x, t), \quad x \in (0, \infty), \quad t \in (0, T), \quad (1.3a)$$

$$u(x, 0) = u_0(x), \quad x \in [0, \infty), \quad (1.3b)$$

$$u(0, t) = g_0(t), \quad t \in [0, T]. \quad (1.3c)$$

Recall that we consider nonzero boundary conditions; hence, the linear IBVP (1.3) cannot be converted into an IVP via the reflection method. Moreover, the classical sine transform solution formula for this problem is not convenient for the purpose of estimates due to its oscillatory nature. Thus, a significant obstacle is present already at the very beginning of the contraction mapping argument, namely at the stage of simply specifying the iteration map. This stands in stark contrast with the case of the IVP, for which the linear problem is solved by means of a straightforward application of the Fourier transform.

A novel approach was recently introduced for the well-posedness of IBVPs involving nonlinear evolution equations. This approach bypasses the absence of Fourier transform in the IBVP setting by utilizing the unified transform method (UTM) of Fokas for the explicit solution of forced linear evolution IBVPs [F1, F2]. The new approach has already been implemented for the nonlinear Schrödinger, the Korteweg-de Vries and the “good” Boussinesq equations on the half-line [FHM1, FHM2, HM]. These three IBVPs share two things in common: (i) they concern dispersive equations, and (ii) they are formulated on the half-line. The present work extends the UTM approach to IBVP well-posedness in two different directions: (i) from dispersive to diffusive equations, and (ii) from the half-line to the finite interval.

In the case of the forced linear IBVP (1.3), UTM yields the solution formula (2.5). Theorems 1.1 and 1.2 will be established by deriving the linear estimates necessary for showing that the iteration map obtained by setting $f = |u|^{p-1}u$ in (2.5) has a unique fixed point in the relevant solution spaces. In fact, the analysis of the half-line problem (1.3) can also be exploited to show well-posedness of the following reaction-diffusion IBVP on the finite interval:

$$u_t - u_{xx} = |u|^{p-1}u, \quad x \in (0, \ell), \quad t \in (0, T), \quad (1.4a)$$

$$u(x, 0) = u_0(x) \in H^s_x(0, \ell), \quad x \in [0, \ell), \quad (1.4b)$$

$$u(0, t) = g_0(t) \in H^{2\alpha+1}_t(0, T), \quad t \in [0, T], \quad (1.4c)$$

$$u(\ell, t) = h_0(t) \in H^{2\alpha+1}_t(0, T), \quad t \in [0, T], \quad (1.4d)$$

where $p = 2, 3, 4, \ldots$, and $T < 1$. In particular, we shall prove the following result.
Theorem 1.3 (Well-posedness on a finite interval for “smooth” data). Suppose \( \frac{1}{2} < s < \frac{3}{2} \), \( u_0 \in H^s_x(0, \ell) \), \( g_0 \in H^{2s+1}_t (0, T) \) and \( h_0 \in H^{2s+1}_t (0, T) \) with the compatibility conditions \( u_0(0) = g_0(0) \) and \( u_0(\ell) = h_0(0) \). Furthermore, for

\[
\| (u_0, g_0, h_0) \|_D = \| u_0 \|_{H^s_x(0, \ell)} + \| g_0 \|_{H^{2s+1}_t (0, T)} + \| h_0 \|_{H^{2s+1}_t (0, T)},
\]

\[
T^* = \min \left\{ T, \frac{1}{p^2 (2c_{s,p})^{2p} \| (u_0, g_0, h_0) \|_D^{2(p-1)}} \right\}, \quad c_{s,p} > 0,
\]

define the space \( X \) on the interval by

\[
X = C([0, T^*]; H^s_x(0, \ell)) \cap C([0, \ell]; H^{2s+1}_t (0, T^*)),
\]

\[
\| u \|_X = \sup_{t \in [0, T^*]} \| u(t) \|_{H^s_x(0, \ell)} + \sup_{x \in [0, \ell]} \| u(x) \|_{H^{2s+1}_t (0, T^*)}.
\]

Then, for \( \frac{p-1}{2} \in \mathbb{N} \) there exists a unique solution \( u \in X \) to the reaction-diffusion IBVP (1.4), which satisfies the estimate

\[
\| u \|_X \leq 2c_{s,p} \| (u_0, g_0, h_0) \|_D.
\]

Furthermore, the data-to-solution map \( \{ u_0, g_0, h_0 \} \mapsto u \) is locally Lipschitz continuous.

For “rough” data \( (s < \frac{1}{2}) \), in analogy to the half-line we refine our solution space by additionally including the space

\[
C^\alpha([0, T]; L^p_x(0, \ell)) = \left\{ u \in C([0, T]; L^p_x(0, \ell)) : \sup_{t \in [0, T]} t^\alpha \| u(t) \|_{L^p_x(0, \ell)} < \infty \right\}
\]

with \( \alpha \) as in (1.2). In this case, we have the following result.

Theorem 1.4 (Well-posedness on the finite interval for “rough” data). Suppose \( \frac{1}{2} - \frac{1}{p} < s < \frac{1}{2} \), \( u_0 \in H^s_x(0, \ell) \), \( g_0 \in H^{2s+1}_t (0, T) \) and \( h_0 \in H^{2s+1}_t (0, T) \), and for \( \| (u_0, g_0, h_0) \|_D \) as in Theorem 1.3 and lifespan \( T^* \) given by

\[
T^* = \min \left\{ T, \frac{1}{(2p+2)\alpha^{\frac{1}{\alpha}} (2c_{s,p})^{\frac{1}{2p}} \| (u_0, g_0, h_0) \|_D^{\frac{p-1}{2p}}} \right\}, \quad c_{s,p} > 0,
\]

define the space \( Y \) on the interval by

\[
Y = C([0, T^*]; H^s_x(0, \ell)) \cap C([0, \ell]; H^{2s+1}_t (0, T^*)) \cap C^\alpha([0, T^*]; L^p_x(0, \ell))
\]

\[
\| u \|_Y = \sup_{t \in [0, T^*]} \| u(t) \|_{H^s_x(0, \ell)} + \sup_{x \in [0, \ell]} \| u(x) \|_{H^{2s+1}_t (0, T^*)} + \sup_{t \in [0, T^*]} t^\alpha \| u(t) \|_{L^p_x(0, \ell)}.
\]

Then, the reaction-diffusion IBVP (1.4) has a unique solution \( u \in Y \), which satisfies the estimate

\[
\| u \|_Y \leq 2c_{s,p} \| (u_0, g_0, h_0) \|_D.
\]

Furthermore, the data-to-solution map \( \{ u_0, g_0, h_0 \} \mapsto u \) is locally Lipschitz continuous.
Theorems 1.3 and 1.4 for the reaction-diffusion IBVP (1.4) on the finite interval will be established by deriving appropriate estimates for the forced linear IBVP

\begin{align}
  u_t - u_{xx} &= f(x,t), \quad x \in (0, \ell), \quad t \in (0, T), \\
  u(x,0) &= u_0(x), \quad x \in [0, \ell], \\
  u(0,t) &= g_0(t), \quad t \in [0, T], \\
  u(\ell,t) &= h_0(t), \quad t \in [0, T].
\end{align}

(1.7a)

Actually, thanks to the linear estimates obtained for the half-line problem (1.3), it will be sufficient to estimate the finite interval problem (1.7) in the reduced case \( f = u_0 = g_0 = 0 \).

We note that apart from the UTM approach to IBVP well-posedness there are also other approaches in the literature, namely the works of Colliander, Kenig and Holmer [CK, H1, H2] as well as of Bona, Sun and Zhang [BSZ] for the Korteweg-de Vries and the nonlinear Schrödinger equations on the half-line. Furthermore, for a different treatment of linear and nonlinear evolution IBVPs that combines UTM with inverse scattering techniques we refer the reader to Fokas and Pelloni [FP], Pelloni [P1, P2], Fokas, Its and Sung [FIS], Fokas and Lenells [FL, LF1, LF2], Fokas and Spence [FS], Deconinck, Pelloni and Sheils [DPS], Sheils and Smith [SS], and the references therein.

Finally, we would like to emphasize that one could possibly study the IBVPs considered in this work via classical techniques developed specifically for diffusive equations with data in Lipschitz spaces (see, for example, [A, W, G, Smi]), although we were not able to find the results obtained in this work in the classical literature. However, as mentioned earlier, our objective here is different, namely to demonstrate that the reaction-diffusion equation (1.1a) can be included in the theory originally developed for dispersive equations like NLS [FHM1], KdV [FHM2] and “good” Boussinesq [HM], which relies on employing UTM and the \( L^2 \)-boundedness of the Laplace transform (see Lemma 3.1) for deriving novel linear estimates along rays in the complex Fourier space.

**Structure of the paper.** The UTM solution formulae for the forced linear heat equation on the half-line and on the finite interval are derived in Section 2. Estimates for the linear heat IVP are obtained in Section 3, including those that motivate the refined solution spaces of Theorems 1.2 and 1.4. The corresponding estimates for the forced linear heat equation on the half-line are derived in Section 4 and are subsequently employed in Section 5 for carrying out the contraction mapping argument leading to Theorems 1.1 and 1.2. Finally, the forced linear heat equation on the finite interval is estimated in Section 6 and the resulting estimates yield Theorems 1.3 and 1.4 via a contraction mapping argument analogous to that of Section 5.

## 2. Linear Solution Formulae via UTM

In this section, we employ UTM in order to derive the solution formulae (2.5) and (2.10) for the forced linear IBVPs (1.1) and (1.4) on the half-line and on the finite interval respectively. The derivations presented here can be found in more detail in [F2].

### 2.1. The half-line

Let \( \tilde{u} \) satisfy the adjoint equation of the linear heat equation, i.e. \( \tilde{u}_t + \tilde{u}_{xx} = 0 \). Multiplying this equation by \( u \) and adding it to the forced linear heat equation (1.3a) multiplied by \( \tilde{u} \), we arrive at the divergence form \((\tilde{u} u)_t - (\tilde{u} u_x - \tilde{u}_x u)_x = \tilde{u} f\). Setting \( \tilde{u}(x,t) = e^{-ikx+k^2t} \) and integrating with
respect to $x$ and $t$ yields the *global relation*

$$e^{k^2t} \check{u}(k, t) = \check{u}_0(k) - \left[\check{g}_1(k^2, t) + i k \check{g}_0(k^2, t)\right] + \int_{t'=0}^{t} e^{k^2t'} \tilde{f}(k, t') dt', \quad \text{Im}(k) \leq 0,$$

(2.1)

where

$$\check{u}(k, t) = \int_{x=0}^{\infty} e^{-ikx} u(x, t) dx, \quad \check{g}_j(k^2, t) = \int_{t'=0}^{t} e^{k^2t'} \partial_x^j u(0, t') dt', \quad j = 0, 1,$$

and the half-line Fourier transforms $\check{u}_0, \tilde{f}$ are defined analogously to $\check{u}$. Note that these transforms are analytic for $\text{Im}(k) < 0$ thanks to the boundedness of $e^{ikx}$ for all $x > 0$ and a Paley-Wiener-type theorem (see, for example, Theorem 7.2.4 in [S]). Inverting (2.1), we obtain

$$u(x, t) = \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \check{u}_0(k) dk + \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \int_{t'=0}^{t} e^{k^2t'} \tilde{f}(k, t') dt' dk$$

$$- \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \left[\check{g}_1(k^2, t) + i k \check{g}_0(k^2, t)\right] dk,$$

(2.2)

The integral representation (2.2) involves the unknown Neumann boundary value $u_\pm(x, 0)$ through the term $\check{g}_1$. However, it turns out that this term can be eliminated. Indeed, since $x \geq 0$ and $t \geq t'$, the exponential $e^{ikx-k^2(t-t')} \leq 0$ whenever $0 \leq \text{Im}(k) \leq |\text{Re}(k)|$. Thus, thanks to the analyticity and exponential decay in $k$, Cauchy’s theorem implies

$$u(x, t) = \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \check{u}_0(k) dk + \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \int_{t'=0}^{t} e^{ikx-k^2t'} \tilde{f}(k, t') dt' dk$$

$$- \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} \left[\check{g}_1(k^2, t) + i k \check{g}_0(k^2, t)\right] dk,$$

(2.3)

where $\partial D^+$ is the positively oriented boundary of the region $D^+ = \{k \in \mathbb{C} : \text{Im}(k) \geq |\text{Re}(k)|\}$, as shown in Figure 2.1.

![Figure 2.1. The regions $D^\pm$ and their positively oriented boundaries $\partial D^\pm$.](image)

Moreover, the transformation $k \mapsto -k$ applied to the global relation (2.1) implies the identity

$$e^{k^2t} \check{u}(-k, t) = \check{u}_0(-k) - \left[\check{g}_1(k^2, t) - i k \check{g}_0(k^2, t)\right] + \int_{t'=0}^{t} e^{k^2t'} \tilde{f}(-k, t') dt', \quad \text{Im}(k) \geq 0.$$

Solving for $\check{g}_1$ and inserting the resulting expression in (2.3) yields the solution of IBVP (1.3) as

$$u(x, t) = \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \check{u}_0(k) dk - \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} \check{u}_0(-k) dk$$

$$+ \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \int_{t'=0}^{t} e^{k^2t'} \tilde{f}(k, t') dt' dk - \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} \int_{t'=0}^{t} e^{k^2t'} \tilde{f}(-k, t') dt' dk$$

$$- \frac{i}{\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} k \check{g}_0(k^2, t) dk,$$

(2.4)
where we have made crucial use of the fact that \( \int_{k \in \partial D^+} e^{ikx} \tilde{u}(-k,t) dk = 0 \) due to analyticity and exponential decay of the integrand in \( D^+ \). Finally, exploiting once again analyticity and exponential decay in \( D^+ \), we observe that \( \int_{k \in \partial D^+} e^{ikx-k^2t} \int_{t' = t} e^{k^2t'} u(0,t') dt' dk = 0 \). Hence, the solution formula (2.4) can be written in the following convenient form for deriving linear estimates:

\[
\begin{align*}
\frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \tilde{u}_0(k) dk &- \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} \tilde{u}_0(-k) dk \\
+ \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \int_{t' = 0}^t e^{k^2t'} f(k,t') dt' dk &- \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} \int_{t' = 0}^t e^{k^2t'} f(-k,t') dt' dk \\
- \frac{i}{\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} k\tilde{g}_0(k^2,T) dk & \cdot \tilde{g}_0(k^2,T) \triangleq \int_{t = 0}^T e^{k^2t} g_0(t) dt. \quad (2.5)
\end{align*}
\]

2.2. The finite interval

Integrating the divergence form \((\tilde{u}u)_t - (\tilde{u}u_x - \tilde{u}_x u)_x = \tilde{u}f\) with respect to \( x \) and \( t \), we obtain the global relation

\[
e^{k^2t} \tilde{u}(k, t) = \tilde{u}_0(k) + e^{-ik\ell} \left[ h_1(k^2, t) + ik\tilde{h}_0(k^2, t) \right] - \left[ \tilde{g}_1(k^2, t) + ik\tilde{g}_0(k^2, t) \right] \\
+ \int_{t' = 0}^t e^{k^2t'} \tilde{f}(k,t') dt', \quad k \in \mathbb{C}, \quad (2.6)
\]

where \( \tilde{u}(k, t) = \int_{\mathbb{R}} e^{-ikx} u(x, t) dx \) with \( \tilde{u}_0 \) and \( \tilde{f} \) defined similarly, and

\[
\tilde{g}_j(k^2, t) = \int_{t' = 0}^t e^{k^2t'} \partial_x^j u(0,t') dt', \quad \tilde{h}_j(k^2, t) = \int_{t' = 0}^t e^{k^2t'} \partial_x^j u(\ell,t') dt', \quad j = 0, 1.
\]

Note that the above spatial transforms make sense for all \( k \in \mathbb{C} \) since \( x \in [0, \ell] \). Inverting (2.6) gives rise to the integral representation

\[
\begin{align*}
\frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \tilde{u}_0(k) dk &+ \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \int_{t' = 0}^t e^{k^2t'} \tilde{f}(k,t') dt' dk \\
- \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \left[ \tilde{g}_1(k^2, t) + ik\tilde{g}_0(k^2, t) \right] dk \\
+ \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ik(x-\ell)-k^2t} \left[ \tilde{h}_1(k^2, t) + ik\tilde{h}_0(k^2, t) \right] dk. \quad (2.7)
\end{align*}
\]

The unknown Neumann values \( u_x(0,t) \) and \( u_x(\ell, t) \), which are contained in \( \tilde{g}_1 \) and \( \tilde{h}_1 \), can be eliminated from the above representation similarly to the half-line case. In particular, observe that the exponentials \( e^{ikx-k^2(t-t')} \) and \( e^{ik(x-\ell)-k^2(t-t')} \) are bounded in \( \{\text{Im}(k) \geq 0\} \setminus D^+ \) and \( \{\text{Im}(k) \leq 0\} \setminus D^- \) respectively, where \( D^+ \) is defined as in the half-line and \( D^- = \{k \in \mathbb{C} : \text{Im}(k) < -|\text{Re}(k)|\} \). Hence, using Cauchy’s theorem we can deform the contours of integration in the second and third integrals of (2.7) to the positively oriented boundaries \( \partial D^\pm \) of \( D^\pm \) to obtain

\[
\begin{align*}
\frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \tilde{u}_0(k) dk &+ \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx-k^2t} \int_{t' = 0}^t e^{k^2t'} \tilde{f}(k,t') dt' dk \\
- \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx-k^2t} \left[ \tilde{g}_1(k^2, t) + ik\tilde{g}_0(k^2, t) \right] dk \\
- \frac{1}{2\pi} \int_{k \in \partial D^-} e^{ik(x-\ell)-k^2t} \left[ \tilde{h}_1(k^2, t) + ik\tilde{h}_0(k^2, t) \right] dk. \quad (2.8)
\end{align*}
\]
Under the transformation \( k \mapsto -k \), the global relation (2.6) yields the additional identity
\[
e^{k^2 t} \tilde{u}(-k, t) = \tilde{u}_0(-k) + e^{ikt} \left[ \tilde{h}_1(k^2, t) - i k \tilde{h}_0(k^2, t) \right] - e^{-ikt} \left[ \tilde{g}_1(k^2, t) - i k \tilde{g}_0(k^2, t) \right] + \int_{t'}^t e^{k^2 t'} f(-k, t') dt', \quad k \in \mathbb{C}. \tag{2.9}
\]
Solving (2.6) and (2.9) for \( \tilde{g}_1 \) and \( \tilde{h}_1 \), and noting that the terms \( \tilde{u}(k, t) \) and \( \tilde{u}(-k, t) \) involved in the resulting expressions yield zero contribution when inserted in (2.8) thanks to analyticity and exponential decay, we obtain the solution of IBVP (1.7) in the form
\[
u(x, t) = S\left[u_0, g_0, h_0; f\right](x, t) = \frac{1}{2\pi} \int_{k \in \mathbb{R}} e^{ikx - k^2 t} \tilde{u}_0(k) dk - \frac{1}{2\pi} \int_{k \in \partial D^+} e^{ikx - k^2 t} \left[ e^{ikt} \tilde{u}_0(k) - e^{-ikt} \tilde{u}_0(-k) \right] dk - \frac{1}{2\pi} \int_{k \in \partial D^-} e^{ikx - k^2 t} \left[ e^{ikt} \tilde{u}_0(k) - e^{-ikt} \tilde{u}_0(-k) \right] dk + \frac{1}{2\pi} \int_{k \in \partial D^+} \int_{t'}^t e^{k^2 t'} f(k, t') dt' dk - \frac{1}{2\pi} \int_{k \in \partial D^-} \int_{t'}^t e^{k^2 t'} f(k, t') dt' dk - \frac{1}{2\pi} \int_{k \in \partial D^+} \int_{t'}^t e^{k^2 t'} f(k, t') dt' dk - \frac{1}{2\pi} \int_{k \in \partial D^-} \int_{t'}^t e^{k^2 t'} f(k, t') dt' dk \tag{2.10}
\]
where, similarly to the half-line, we have used analyticity and exponential decay in \( D^\pm \) to replace the transforms \( \tilde{g}_0(k^2, t) \) and \( \tilde{h}_0(k^2, t) \) by
\[
\tilde{g}_0(k^2, T) = \int_{t=0}^T e^{k^2 t} g_0(t) dt, \quad \tilde{h}_0(k^2, T) = \int_{t=0}^T e^{k^2 t} h_0(t) dt.
\]

3. Linear IVP Estimates

In this section, we write the forced linear IBVP (1.1) as a combination of simpler IVPs and IBVPs that involve only one piece of data at a time and hence are easier to handle than the full problem. Then, we proceed to the estimation of the IVP component problems.

3.1. Decomposition into simpler problems

Let \( U_0 \in H_x^s(\mathbb{R}) \) be an extension of the initial datum \( u_0 \in H_x^s(0, \infty) \) such that
\[
\|U_0\|_{H_x^s(\mathbb{R})} \leq c \|u_0\|_{H_x^s(0, \infty)} \quad \text{for } s \geq 0, \quad c \geq 1. \tag{3.1}
\]
Furthermore, since we are working with power nonlinearities, let the forcing of (1.3) be of the form \( f(x, t) = \prod_{j=1}^p f_j(x, t) \) with the whole line extension \( F(x, t) = \prod_{j=1}^p F_j(x, t) \) defined as follows:

(i) For \( s > \frac{1}{2} \), \( F_j \in C([0, T]; H_x^s(\mathbb{R})) \) are extensions of \( f_j \in C([0, T]; H_x^s(0, \infty)) \) such that
\[
\sup_{t \in [0, T]} \|F_j(t)\|_{H_x^s(\mathbb{R})} \leq c \sup_{t \in [0, T]} \|f_j(t)\|_{H_x^s(0, \infty)}, \quad c \geq 1, \quad 1 \leq j \leq p.
\]
Then, using the algebra property in $H^s_x(\mathbb{R})$ we have
\[
\sup_{t \in [0,T]} \|F(t)\|_{H^s_x(\mathbb{R})} \leq c_{s,p} \prod_{j=1}^{p} \sup_{t \in [0,T]} \|f_j(t)\|_{H^s_x(\mathbb{R})}. \tag{3.2}
\]

(ii) For $s \leq \frac{1}{2}$, the functions $F_j$ are the extensions of the functions $f_j$ by zero outside $(0, \infty)$ so that by Hölder’s inequality
\[
\|F\|_{C^{0,s}(0,T;L^p_x(\mathbb{R}))} \leq \prod_{j=1}^{p} \|f_j\|_{C^{0,s}(0,T;L^p_x(\mathbb{R}))}. \tag{3.3}
\]

For above-defined $U_0$ and $F$, we write IBVP (1.3) as the superposition of the following problems:

I. The homogeneous linear IVP
\[
U_t - U_{xx} = 0, \quad x \in \mathbb{R}, \ t \in (0, T), \tag{3.4a}
\]
\[
U(x, 0) = U_0(x) \in H^s_x(\mathbb{R}), \quad x \in \mathbb{R}, \tag{3.4b}
\]
which can be solved via the whole line Fourier transform $\hat{U}(\xi, t) = \int_{x \in \mathbb{R}} e^{-i\xi x} U(x, t) \, dx$ to yield
\[
U(x, t) = S[U_0; 0](x, t) = \frac{1}{2\pi} \int_{\xi \in \mathbb{R}} e^{i\xi x - \xi^2 t} \hat{U}_0(\xi) d\xi. \tag{3.5}
\]

II. The forced linear IVP with zero initial condition
\[
W_t - W_{xx} = F(x, t), \quad x \in \mathbb{R}, \ t \in (0, T), \tag{3.6a}
\]
\[
W(x, 0) = 0, \quad x \in \mathbb{R}, \tag{3.6b}
\]
whose solution is found via the whole line Fourier transform $\hat{W}(\xi, t) = \int_{x \in \mathbb{R}} e^{-i\xi x} W(x, t) \, dx$ as
\[
W(x, t) = S[0; F](x, t) = \frac{1}{2\pi} \int_{\xi \in \mathbb{R}} \int_{t' = 0}^{t} e^{i\xi x - \xi^2 (t-t')} \hat{F}(\xi, t') dt' d\xi \tag{3.7a}
= \int_{t' = 0}^{t} S[F(\cdot, t'); 0] (x, t - t') dt'. \tag{3.7b}
\]

III. The homogeneous linear IBVP with zero initial condition
\[
u_t - \nu_{xx} = 0, \quad x \in (0, \infty), \ t \in (0, T), \tag{3.8a}
\]
\[
\nu(x, 0) = 0, \quad x \in [0, \infty), \tag{3.8b}
\]
\[
u(0, t) = g_0(t) - U(0, t) \equiv G_0(t), \quad t \in [0, T], \tag{3.8c}
\]
with solution $\nu = S[0, G_0; 0]$ given by the UTM formula (2.5) after setting $u_0 = f = 0$.

IV. The homogeneous linear IBVP with zero initial condition
\[
u_t - \nu_{xx} = 0, \quad x \in (0, \infty), \ t \in (0, T), \tag{3.9a}
\]
\[
u(x, 0) = 0, \quad x \in [0, \infty), \tag{3.9b}
\]
\[
u(0, t) = W(0, t), \quad t \in [0, T], \tag{3.9c}
\]
with solution $\nu = S[0, W_{|x=0}; 0]$ given by the UTM formula (2.5) after setting $u_0 = f = 0$.

In summary, the UTM solution (2.5) of the forced linear IBVP (1.3) has been expressed as
\[
S[u_0, g_0; f] = S[U_0; 0] \bigg|_{x \in (0, \infty)} + S[0; F] \bigg|_{x \in (0, \infty)} + S[0, G_0; 0] - S[0, W_{|x=0}; 0], \tag{3.10}
\]
where the four quantities on the right-hand side are the solutions of problems (3.4), (3.6), (3.8) and (3.9) respectively. In the remaining of the current section, we shall obtain estimates for the first
two components of the superposition (3.10), namely for the linear IVPs (3.4) and (3.6). Then, in Section 4 we shall derive corresponding estimates for the linear IBVPs (3.8) and (3.9).

3.2. Sobolev-type estimates for the homogeneous linear IVP

We begin the analysis of the components of (3.10) by estimating the solution of the homogeneous linear IVP (3.4) in Sobolev spaces.

**Theorem 3.1 (Sobolev-type estimates for the homogeneous linear IVP).** The solution \( U = S[U_0; 0] \) of the linear heat IVP (3.4) given by formula (3.5) admits the estimates

\[
\|U\|_{C([0,T];H^s_x(\mathbb{R}))} \leq \|U_0\|_{H^s_x(\mathbb{R})}, \quad s \in \mathbb{R},
\]

\[
\|U\|_{C([0,T];H^{s+\frac{1}{2}}_x(0,T))} \leq c_s \|U_0\|_{H^s_x(\mathbb{R})}, \quad -\frac{1}{2} \leq s < \frac{3}{2}.
\]

**Remark 3.1.** The time estimate (3.12) essentially dictates the space for the boundary datum \( g_0 \) of the half-line problem (1.1). It is interesting to note that, despite the diffusive nature of the heat equation, the Sobolev exponent \( \frac{2s+1}{s+1} \) is precisely the one appearing in the corresponding estimate for the (dispersive) linear Schrödinger equation (see [KPV] and [FHM1]).

**Proof of Theorem 3.1.** The solution formula (3.5) combined with the definition of the \( H^s_x(\mathbb{R}) \)-norm imply the space estimate (3.11) for all \( s \in \mathbb{R} \) and all \( t \geq 0 \):

\[
\|U(t)\|_{H^s_x(\mathbb{R})} = \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s |e^{-\xi^2 t} \hat{U}_0(\xi)|^2 d\xi \leq \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s |\hat{U}_0(\xi)|^2 d\xi = \|U_0\|_{H^s_x(\mathbb{R})}^2.
\]

For continuity in time, we note that for any sequence \( \{t_n\} \subset [0, T] \) converging to \( t \in [0, T] \) we have

\[
\|U(t_n) - U(t)\|_{H^s_x(\mathbb{R})} = \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s \left|(e^{-\xi^2 t_n} - e^{-\xi^2 t}) \hat{U}_0(\xi)\right|^2 d\xi \lesssim \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s \left|\hat{U}_0(\xi)\right|^2 d\xi < \infty.
\]

Thus, by the dominated convergence theorem we infer that \( \lim_{n \to \infty} \|U(t_n) - U(t)\|_{H^s_x(\mathbb{R})}^2 = 0 \).

The proof of the time estimate (3.12) is more involved. Letting \( m = \frac{2s+1}{s+1} \) and noting that for \(-\frac{1}{2} \leq s < \frac{3}{2}\) we have \( 0 \leq m < 1 \), we employ the physical space definition of the Sobolev \( H^m_t(0,T) \)-norm:

\[
\|U(x)\|_{H^m_t(0,T)} = \|U(x)\|_{L^2_x(0,T)} + \|U(x)\|_m, \quad 0 \leq m < 1,
\]

where for \( 0 < m < 1 \) the fractional norm \( \|\cdot\|_m \) is defined by

\[
\|U(x)\|_m^2 = \int_{t_1=0}^T \int_{t_2=0}^T \frac{|U(x,t_1) - U(x,t_2)|^2}{|t_1 - t_2|^{1+2m}} dt_1 dt_2 \lesssim \int_{t=0}^T \int_{z=0}^{T-t} \frac{|U(x,t+z) - U(x,t)|^2}{z^{1+2m}} dz dt.
\]

The norm \( \|U(x)\|_{L^2_x(0,T)} \). We have

\[
\|U(x)\|_{L^2_x(0,T)} \lesssim \int_0^T \left[ \int_{\xi=0}^1 e^{-\xi^2 t} \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right) d\xi \right]^2 dt
\]

\[
+ \int_0^T \left[ \int_{\xi=1}^{\infty} e^{-\xi^2 t} \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right) d\xi \right]^2 dt.
\]

By the Cauchy-Schwarz inequality, we find

\[
(3.14a) \lesssim \int_0^T \left( \int_{\xi=0}^1 e^{-2\xi^2 t} (1 + \xi^2)^{-s} d\xi \right) \|U_0\|_{H^s_x(\mathbb{R})}^2 dt \lesssim c_s T \|U_0\|_{H^s_x(\mathbb{R})}^2, \quad s \in \mathbb{R}.
\]

(3.15)
Next, making the change of variable $\xi = \sqrt{\tau}$ we have

\[
(3.14b) = \int_{t=0}^{T} \left( \int_{\tau=1}^{\infty} e^{-\tau t} \left| \hat{U}_0(-\sqrt{\tau}) + |\hat{U}_0(\sqrt{\tau})| \right| 2/\sqrt{\tau} d\tau \right)^2 dt \lesssim \|\mathcal{L}\{\phi\}\|_{L^2_T(0,\infty)}^2,
\]

where $\mathcal{L}\{\phi\}(t) = \int_{\tau=0}^{\infty} e^{-\tau t} \phi(\tau)d\tau$ is the Laplace transform of the function

\[
\phi(\tau) = \begin{cases} 
\tau^{-\frac{1}{2}}(|\hat{U}_0(-\sqrt{\tau})| + |\hat{U}_0(\sqrt{\tau})|), & \tau \geq 1, \\
0, & 0 \leq \tau < 1.
\end{cases}
\]

(3.17)

**Lemma 3.1** ($L^2$-boundedness of the Laplace transform). Suppose $\phi \in L^2_T(0,\infty)$. Then, the map

\[
\mathcal{L} : \phi \mapsto \int_{\tau=0}^{\infty} e^{-\tau t} \phi(\tau)d\tau
\]

is bounded from $L^2_T(0,\infty)$ into $L^2_T(0,\infty)$ with

\[
\|\mathcal{L}\{\phi\}\|_{L^2_T(0,\infty)} \leq \sqrt{\pi} \|\phi\|_{L^2_T(0,\infty)}.
\]

A proof of Lemma 3.1 is available in [FHMI1]. Upon employing this lemma for $\phi$ given by (3.17), (3.16) yields

\[
(3.14b) \lesssim \left\| \tau^{-\frac{1}{2}}(|\hat{U}_0(-\sqrt{\tau})| + |\hat{U}_0(\sqrt{\tau})|) \right\|_{L^2_T(0,\infty)}^2 \lesssim \int_{\tau=1}^{\infty} \tau^{-1}|\hat{U}_0(-\sqrt{\tau})|^2 d\tau + \int_{\tau=1}^{\infty} \tau^{-1}|\hat{U}_0(\sqrt{\tau})|^2 d\tau.
\]

Thus, letting $\xi = -\sqrt{\tau}$ and $\xi = \sqrt{\tau}$ in the first and the second integral respectively, we obtain

\[
(3.14b) \lesssim \int_{|\xi|\geq 1} |\xi|^{-1}|\hat{U}_0(\xi)|^2 d\xi \leq \|U_0\|^2_{H^s_H^s(\mathbb{R})} \leq \|U_0\|^2_{H^s_H^s(\mathbb{R})}, \quad s \geq -\frac{1}{2}.
\]

(3.18)

Combining estimates (3.15) and (3.18), we find

\[
\|U(x,t+z) - U(x,t)\|_{L^2_T(0,T)} \leq \|U_0\|_{H^s_H^s(\mathbb{R})}, \quad s \geq -\frac{1}{2}, \quad x \in \mathbb{R}.
\]

(3.19)

**The fractional norm $\|U(x)\|_m$.** Recall that now $-\frac{1}{2} < s < \frac{3}{2}$. Starting from (3.5), we compute

\[
|U(x,t+z) - U(x,t)| \lesssim \int_{\xi \in \mathbb{R}} e^{-\xi^2 t}(1 - e^{-\xi^2 z}) |\hat{U}_0(\xi)| d\xi
\]

\[
= \int_{\xi=0}^{1} e^{-\xi^2 t}(1 - e^{-\xi^2 z}) \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right) d\xi + \int_{\xi=1}^{\infty} e^{-\xi^2 t}(1 - e^{-\xi^2 z}) \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right) d\xi.
\]

Hence, from the definition of the fractional Sobolev norm we have

\[
\|U(x)\|_{m}^2 \lesssim \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left[ \int_{\xi=0}^{1} e^{-\xi^2 t}(1 - e^{-\xi^2 z}) \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right) d\xi \right]^2 dzdt
\]

\[
+ \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left[ \int_{\xi=1}^{\infty} e^{-\xi^2 t}(1 - e^{-\xi^2 z}) \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right) d\xi \right]^2 dzdt.
\]

(3.20a)

(3.20b)

For any $s \in \mathbb{R}$, the Cauchy-Schwarz inequality implies

\[
(3.20a) \lesssim T \|U_0\|^2_{H^s_H^s(\mathbb{R})} \int_{\xi=0}^{1} (1 + \xi^2)^{-s} \int_{z=0}^{T} \frac{(1 - e^{-\xi^2 z})^2}{z^{1+2m}} dzd\xi.
\]

Moreover,

\[
\int_{z=0}^{T} \frac{(1 - e^{-\xi^2 z})^2}{z^{1+2m}} dz \lesssim \xi^{4m} \int_{\xi=0}^{\infty} \frac{(1 - e^{-\xi^2})^2}{\xi^{1+2m}} d\xi \lesssim \xi^{4m},
\]

(3.21)
since the $\zeta$-integral above converges for $0 < m < 1$. Hence,

$$
(3.20a) \lesssim T \|U_0\|_{H^2_x(\mathbb{R})}^2 \int_{\xi=0}^{1} (1 + \xi^2)^{-s} \xi^{4m} d\xi = c_s T \|U_0\|_{H^2_x(\mathbb{R})}^2, \quad -\frac{1}{2} < s < \frac{3}{2}.
$$

(3.22)

Furthermore, letting $\xi = \sqrt{\tau}$ we have

$$
(3.20b) \lesssim \int_{z=0}^{T} \frac{1}{z^{1+2m}} \|\mathcal{L} \{\phi\} (z)\|_{L^2_t(0,\infty)}^2 dz,
$$

where

$$
\phi(\tau) = \begin{cases}
\tau^{-\frac{1}{2}} (1 - e^{-\tau z}) (|\hat{U}_0(-\sqrt{\tau})| + |\hat{U}_0(\sqrt{\tau})|), & \tau \geq 1, \\
0, & 0 \leq \tau < 1.
\end{cases}
$$

Thus, by the Laplace transform bound of Lemma 3.1 we find

$$
(3.20b) \lesssim \int_{\xi=1}^{\infty} \xi^{-1} \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right)^2 \int_{z=0}^{T} \frac{(1 - e^{-\xi^2z})^2}{z^{1+2m}} dz d\xi.
$$

Hence, using estimate (3.21) for the $z$-integral, we obtain

$$
(3.20b) \lesssim \int_{\xi=1}^{\infty} \xi^{4m-1} \left( |\hat{U}_0(-\xi)| + |\hat{U}_0(\xi)| \right)^2 d\xi \lesssim \|U_0\|^2_{H^2_x(\mathbb{R})}, \quad -\frac{1}{2} < s < \frac{3}{2}.
$$

(3.23)

Note that since $\xi \geq 1$ we have $\xi^{2s} \lesssim (1 + \xi^2)^s$ even for negative values of $s$.

Combining estimates (3.22) and (3.23), we deduce

$$
\|U(x)\|_m \lesssim \|U_0\|_{H^2_x(\mathbb{R})}, \quad -\frac{1}{2} < s < \frac{3}{2}, \quad x \in \mathbb{R}.
$$

(3.24)

In turn, estimates (3.19) and (3.24) combined with the definition (3.13) yield estimate (3.12).

Continuity with respect to $x$ follows from estimating the $H^{2+1}_{\xi} (0, T)$-norm of the difference $U(x_n) - U(x)$ as above for any sequence $\{x_n\} \to x$ and then using the dominated convergence theorem to show that this norm vanishes in the limit $n \to \infty$.

3.3. Sobolev-type estimates for the forced linear IVP

Next, we shall obtain estimates in Sobolev spaces for the second component of the superposition (3.10), namely the forced linear IVP (3.6). As we will see below, the lack of algebra property in $H^s_x(\mathbb{R})$ when $s < \frac{1}{2}$ forces us to estimate the solution of problem (3.6) in a different way than for $s > \frac{1}{2}$, giving rise to the space

$$
C^\alpha([0,T]; L^p_x(\mathbb{R})) = \left\{ u \in C([0,T]; L^p_x(\mathbb{R})) : \sup_{t \in [0,T]} \ell^\alpha \|u(t)\|_{L^p_x(\mathbb{R})} < \infty \right\}
$$

(3.25)

and thereby introducing the space $C^\alpha([0,T]; L^p_x(0,\infty))$ in the “rough” data solution space $Y$ of Theorem 1.2. We note that spaces of the type (3.25) also appear in the analysis of the reaction-diffusion IVP [W].

**Theorem 3.2** (Sobolev-type estimates for the forced linear IVP). For $F = \prod_{j=1}^{p} F_j$, the solution $W = S[0; F]$ of the forced linear heat IVP (3.6) given by formula (3.7) admits the space estimates

$$
\|W\|_{C([0,T]; H^s_x(\mathbb{R}))} \lesssim T \prod_{j=1}^{p} \|F_j\|_{C([0,T]; H^s_x(\mathbb{R}))}, \quad s > \frac{1}{2},
$$

(3.26a)

$$
\|W\|_{C([0,T]; H^s_x(\mathbb{R}))} \lesssim \sqrt{T} \prod_{j=1}^{p} \|F_j\|_{C^\alpha([0,T]; L^p_x(\mathbb{R}))}, \quad 0 \leq s < \frac{1}{2},
$$

(3.26b)
and the time estimates

\[
\|W\|_{C(\mathbb{R}; H^s_x(\mathbb{R}))} \lesssim \sqrt{T} \prod_{j=1}^p \|F_j\|_{C([0,T]; H^s_x(\mathbb{R}))}, \quad \frac{1}{2} < s < \frac{3}{2},
\]

\[
\|W\|_{C(\mathbb{R}; H^{s+\frac{1}{4}}_x(\mathbb{R}))} \lesssim \sqrt{T} \prod_{j=1}^p \|F_j\|_{C^\alpha([0,T]; L^p_x(\mathbb{R}))}, \quad -\frac{1}{2} \leq s < \frac{1}{2}.
\]

**Proof of Theorem 3.2.** *Space estimate (3.26a).* Using Minkowski’s integral inequality and estimate (3.11), we have

\[
\|W(t)\|_{H^s_x(\mathbb{R})} \leq \int_{t'=0}^t \|S[F(t')0](t-t')\|_{H^s_x(\mathbb{R})} dt' \leq T \|F\|_{C([0,T]; H^s_x(\mathbb{R}))}, \quad s \in \mathbb{R},
\]

which implies the space estimate (3.26a) via the algebra property in $H^s_x(\mathbb{R})$ for $s > \frac{1}{2}$.

Regarding continuity in time, for any sequence \(\{t_n\} \subset [0, T]\) converging to $t \in [0, T]$ we have

\[
\|W(t_n) - W(t)\|_{H^s_x(\mathbb{R})} \leq \int_{t'=0}^T \left| \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s |\chi(0,t_n)(t') e^{-\xi^2(t_n-t')} - \chi(0,t)(t') e^{-\xi^2(t-t')}|^2 \right| d\xi dt'.
\]

The \(\xi\)-integral in the above inequality can be bounded above by \(\|F(t')\|_{H^s_x(\mathbb{R})}\) so, in turn, we find

\[
\|W(t_n) - W(t)\|_{H^s_x(\mathbb{R})} \lesssim T \sup_{t \in [0,T]} \|F(t)\|_{H^s_x(\mathbb{R})} < \infty.
\]

Hence, using the dominated convergence theorem for the \(t\)- and \(\xi\)-integrals, we conclude that \(\lim_{n \to \infty} \|W(t_n) - W(t)\|_{H^s_x(\mathbb{R})} = 0\).

**Space estimate (3.26b).** The proof of this estimate is more complicated because it concerns smaller values of $s$ for which the algebra property is not available. Instead, starting from the first inequality in (3.28), applying Minkowski’s integral inequality and taking sup in $\xi$, we have

\[
\|W(t)\|_{H^s_x(\mathbb{R})} \lesssim \int_{t'=0}^t \left( \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s e^{-2(t-t')\xi^2} d\xi \right)^{\frac{1}{2}} \|F(t')\|_{L^1_x(\mathbb{R})} dt'.
\]

Letting $z = 2(t-t')\xi^2$ and noting that \((a+b)^s \leq 2^s (a^s + b^s)\) for $a, b, s \geq 0$, we find

\[
\left( \int_{\xi \in \mathbb{R}} (1 + \xi^2)^s e^{-2(t-t')\xi^2} d\xi \right)^{\frac{1}{2}} \lesssim \left[ (t-t')^{-\frac{1}{4}} \right] \int_{z=0}^\infty e^{-z} dz + (t-t')^{-s-\frac{1}{4}} \int_{z=0}^\infty e^{-z} z^{s-\frac{1}{2}} dz
\]

\[
\lesssim (t-t')^{-\frac{1}{4}} + (t-t')^{-\frac{s+\frac{3}{4}}{2}} \lesssim (t-t')^{-\frac{1}{4} - \frac{s}{2}}, \quad s \geq 0,
\]

where we have also used the fact that $0 \leq t - t' \leq t \leq T < 1$ and $0 \leq \frac{1}{4} \leq \frac{1}{4} + \frac{s}{2}$ for $s \geq 0$.

Returning to (3.29), we have

\[
\|W(t)\|_{H^s_x(\mathbb{R})} \lesssim \int_{t'=0}^t (t-t')^{-\frac{1}{4} - \frac{s}{2}} \|F(t')\|_{L^1_x(\mathbb{R})} dt' \leq \sup_{t \in [0,T]} \left( t^p \|F(t)\|_{L^1_x(\mathbb{R})} \right) \int_{t'=0}^t (t-t')^{-\frac{1}{4} - \frac{s}{2}} \gamma dt'
\]

for some $\gamma$ to be specified. Hence, using the estimate

\[
\int_{t'=0}^t (t-t')^{-\beta_1} (t')^{-\beta_2} dt' = t^{1-\beta_1-\beta_2} \int_{\eta=0}^1 (1-\eta)^{-\beta_1} \eta^{-\beta_2} d\eta \simeq t^{1-\beta_1-\beta_2}, \quad \beta_1, \beta_2 < 1,
\]

for $\beta_1 = \frac{1}{4} + \frac{s}{2}$, $\beta_2 = p\gamma$ and then applying Hölder’s inequality after setting $F = \prod_{j=1}^p F_j$, we get

\[
\|W(t)\|_{H^s_x(\mathbb{R})} \lesssim t^{\frac{1}{4} - \frac{s}{2}} \gamma \prod_{j=1}^p \|F_j\|_{C^\gamma([0,T]; L^p_x(\mathbb{R}))}, \quad s < \frac{3}{2}, \quad \gamma < \frac{1}{p}.
\]
Finally, choosing $\gamma = \alpha = \frac{1}{2} \left( \frac{3}{2} - b \right)$ with $\frac{2b+1}{b} < b < \frac{3}{2}$ we have $\frac{1}{2} \left( \frac{3}{2} - s \right) - p\gamma = b - \frac{2b+1}{b} > \frac{1}{2}$.

Hence, recalling that $0 \leq t < 1$, we deduce the space estimate (3.26b). Observe that the condition $b < \frac{1}{2}$ (which ensures that $\alpha > 0$) combined with the condition $\frac{2b+1}{b} < b$ further restricts $s < \frac{1}{2}$.

**Time estimate (3.27a).** For $m = \frac{2b+1}{b}$ with $m \in [0, 1) \iff s \in \left[ -\frac{1}{2}, \frac{3}{2} \right)$, we employ the physical space definition (3.13) of the $H^m_t(0,T)$-norm. Thus, we need to estimate $\|W(x)\|_{L^2_t(0,T)}$ and $\|W(x)\|_m$.

Starting from the Duhamel representation (3.7b) and combining Minkowski’s integral inequality with the time estimate (3.12), we find

$$\|W(x)\|_{L^2_t(0,T)} \leq \int_0^T \left\| S[F(\cdot, t'); 0](x, t-t') \right\|_{L^2_t(0,T)} dt' \leq T \sup_{t \in [0,T]} \|F(t)\|_{H^s_x(R)}.$$ 

(3.31)

Moreover, writing

$$W(x, t+z) - W(x, t) = \int_{t'=0}^{t} \left[ S[F(\cdot, t'); 0](x, t+z - t') - S[F(\cdot, t'); 0](x, t - t') \right] dt'$$

$$+ \int_{t'=t}^{t+z} S[F(\cdot, t'); 0](x, t + z - t') dt'$$

we have

$$\|W(x)\|^2_m \lesssim \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left( \int_{t'=0}^{t} \left\| S[F(\cdot, t'); 0](x, t+z - t') - S[F(\cdot, t'); 0](x, t - t') \right\| dt' \right)^2 dz dt$$

(3.32a)

$$+ \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left( \int_{t'=t}^{t+z} S[F(\cdot, t'); 0](x, t + z - t') dt' \right)^2 dz dt.$$ 

(3.32b)

By Minkowski’s integral inequality and estimate (3.12), we find

$$\text{(3.32a)} \leq \left( \int_{t'=0}^{T} \left\| S[F(\cdot, t'); 0](x, t-t') \right\|_{m} dt' \right)^2 \lesssim \left( T \sup_{t \in [0,T]} \|F(t)\|_{H^s_x(R)} \right)^2.$$ 

(3.33)

Furthermore, recalling the representation (3.7a) we have

$$\text{(3.32b)} \leq \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left\| \frac{1}{2\pi} \int_{\xi \in R} e^{i\xi x} \int_{t'=t}^{t+z} e^{-\xi^2(t+t'-t')} \hat{F}(\xi, t') dt' d\xi \right\|_{L^2_x(R)}^2 dz dt$$

Hence, the Sobolev embedding theorem for $s > \frac{1}{2}$ implies

$$\text{(3.32b)} \leq \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \left\| \frac{1}{2\pi} \int_{\xi \in R} e^{i\xi x} \int_{t'=t}^{t+z} e^{-\xi^2(t+t'-t')} \hat{F}(\xi, t') dt' d\xi \right\|^2_{H^s_x(R)} dx dz dt$$

$$\lesssim \int_{t=0}^{T} \int_{z=0}^{T-t} \frac{1}{z^{1+2m}} \int_{\xi \in R} \left( 1 + \xi^2 \right)^s \left( \int_{t'=t}^{t+z} \left| \hat{F}(\xi, t') \right| dt' \right)^2 d\xi dz dt,$$

and Minkowski’s integral inequality further yields

$$\text{(3.32b)} \leq \int_{t=0}^{T} \int_{z=0}^{T-t} \sup_{t' \in [t, t+z]} \left\| F(t') \right\|_{H^s_x(R)}^2 \cdot z^{1-2m} dz dt \lesssim T^{3-2m} \sup_{t \in [0,T]} \left\| F(t) \right\|_{H^s_x(R)}^2,$$

(3.34)

after recalling that $m < 1$. Combining (3.32), (3.33), (3.34) and the fact that $T < 1$, we deduce

$$\|W(x)\|_m \lesssim \sqrt{T} \sup_{t \in [0,T]} \|F(t)\|_{H^s_x(R)}.$$ 

(3.35)
Overall, estimates (3.31) and (3.35) together with the definition (3.13) imply
\[ \|W(x)\|_{H^s_T(0,T)} \lesssim \sqrt{T} \sup_{t \in [0,T]} \|F(t)\|_{H^s_T(\mathbb{R})}, \quad \frac{1}{2} < s < \frac{3}{2}, \]
from which we deduce the time estimate (3.27a) via the algebra property in \( H^s_T(\mathbb{R}) \) since \( s > \frac{1}{2} \).

**Time estimate (3.27b).** As for time estimate (3.27a), we restrict \(-\frac{1}{2} \leq s < \frac{3}{2}\) and use the norm (3.13), which involves \( \|W(x)\|_{L^2_T(0,T)} \) and \( \|W(x)\|_m \).

Starting from formula (3.7a) and taking sup in \( \xi \), we have
\[ \|W(x)\|_{L^2_T(0,T)} \lesssim \int_0^T \left[ \int_{t'=0}^t (t - t')^{-\frac{1}{2}} \|F(t')\|_{L^2_T(\mathbb{R})} \right]^2 dt. \]
Thus, using (3.30) with \( \beta_1 = \beta_2 = \frac{1}{2} \) and Hölder’s inequality in \( x \), we obtain
\[ \|W(x)\|_{L^2_T(0,T)} \lesssim \sqrt{T} \sup_{t \in [0,T]} \|F(t)\|_{L^s_T(\mathbb{R})} \lesssim \sqrt{T} \prod_{j=1}^{p} \sup_{t \in [0,T]} \left( t^{\frac{1}{2p}} \|F_j(t)\|_{L^s_T(\mathbb{R})} \right). \] (3.36)
For the fractional norm \( \|W(x)\|_m \), we use formula (3.7a) to write
\[ \|W(x)\|_m^2 \lesssim \int_0^T \left[ \int_{t'=0}^t \left( \int_{z=0}^{x} e^{-\xi^2(t-t') \xi^2 m} d\xi \right) \|F(t')\|_{L^1_T(\mathbb{R})} dt' \right]^2 dt. \] (3.37a)
Minkowski’s integral inequality together with the bound (3.21) imply
\[ \text{(3.37a)} \lesssim \int_0^T \left[ \int_{t'=0}^t (t - t')^{-\frac{s}{2} - \frac{3}{4}} \|F(t')\|_{L^1_T(\mathbb{R})} dt' \right]^2 dt. \]
Estimating the \( \xi \)-integral as in the proof of estimate (3.26b) and recalling that \( m = \frac{2\alpha - 1}{4} \), we find
\[ \text{(3.37a)} \lesssim \int_0^T \left[ \int_{t'=0}^t (t - t')^{-\frac{s}{2} - \frac{3}{4}} \|F(t')\|_{L^1_T(\mathbb{R})} dt' \right]^2 dt. \]
Note that the singularity at \( t' = t \) is integrable provided that \( s < \frac{1}{2} \). Then, using the bound (3.30) with \( \beta_1 = \frac{\alpha}{2} + \frac{3}{4} \) and \( \beta_2 = 1 - \beta_1 \), we obtain
\[ \text{(3.37a)} \lesssim \int_0^T \left( \sup_{t' \in [0,t]} (t')^{-(\frac{s}{2} + \frac{3}{4})} \|F(t')\|_{L^1_T(\mathbb{R})} \right)^2 dt \leq T \left( \sup_{t \in [0,T]} t^{\frac{1}{2}} \left( \|F(t)\|_{L^1_T(\mathbb{R})} \right)^2 \right). \] (3.38)
For the term (3.37b), taking sup in \( \xi \) and noting that \( \int_{\xi \in \mathbb{R}} e^{-\xi^2(t+z-t')} d\xi \simeq (t + z - t')^{-\frac{1}{2}} \) we have
\[ \text{(3.37b)} \lesssim \int_0^T \int_{z=0}^{t-t} \frac{1}{z^{1+2m}} \left( \int_{t'=t}^{t+z} (t + z - t')^{-\frac{1}{2}} \|F(t')\|_{L^1_T(\mathbb{R})} dt' \right)^2 dz dt \]
\[ \leq \int_0^T \int_{z=0}^{t-t} \frac{1}{z^{1+2m}} \left( \int_{t'=t}^{t+z} (t + z - t')^{-\frac{1}{2}} \left( t' - t \right)^{-(\frac{1}{2} - b)} dt' \right)^2 \left( \sup_{t' \in [t, t+z]} \left( t' - t \right)^{\frac{1}{2} - b} \|F(t')\|_{L^1_T(\mathbb{R})} \right)^2 dz dt \]
for some \( b \) to be determined. Then, (3.30) with \( \beta_1 = \frac{1}{2}, \beta_2 = \frac{1}{2} - b \) and \( b > -\frac{1}{2} \) implies
\[ \text{(3.37b)} \lesssim T^{2(b-m)+1} \left( \sup_{t \in [0,T]} t^{\frac{1}{2} - b} \|F(t)\|_{L^1_T(\mathbb{R})} \right)^2, \quad b > m, b < \frac{1}{2}. \] (3.39)
Altogether, and since $m = \frac{2\alpha + 1}{p} \geq 0$, estimate (3.39) is valid for $\frac{2\alpha + 1}{p} < b < \frac{1}{p}$ and $-\frac{3}{2} < s < \frac{1}{2}$. Furthermore, since $t \leq T < 1$, using Hölder’s inequality estimates (3.38) and (3.39) combine to

$$\|W(x)\|_m \lesssim \sqrt{T} \prod_{j=1}^{p} \left( \sup_{t \in [0,T]} t^{\frac{1}{p} - b} \|F_j(t)\|_{L^p_x(\mathbb{R})} \right),$$

(3.40)

which, together estimate (3.36) and the fact that $b \geq 0$ for $m \geq 0$, implies estimate (3.27b).

Continuity in $x$ follows by estimating in the same way as above the $H^m_t(0,T)$-norm of the difference $W(x_n) - W(x)$ for any sequence $\{x_n\} \to x$ and then employing the dominated convergence theorem to deduce that this norm vanishes in the limit $n \to \infty$.

3.4. Additional estimates

The Sobolev-type estimates of Theorem 3.2 introduce the need for estimating the solutions of the linear IVPs (3.4) and (3.6) in the space $C^\alpha([0,T];L^p_x(\mathbb{R}))$ defined by (3.25).

**Theorem 3.3** ($C^\alpha([0,T];L^p_x(\mathbb{R}))$-estimate for the homogeneous linear IVP). Suppose $\frac{1}{2} - \frac{1}{p} \leq s < \frac{1}{2}$. Then, the solution $U = S[U_0;0]$ of the linear heat IVP (3.4) given by formula (3.5) admits the estimate

$$\|U\|_{C^\alpha([0,T];L^p_x(\mathbb{R}))} \lesssim T^\alpha \|U_0\|_{H^s_x(\mathbb{R})}.$$  

(3.41)

**Proof of Theorem 3.3.** The Sobolev inequality $\|f\|_{L^p(\mathbb{R})} \leq c \|f\|_{H^\frac{1}{2} - \frac{1}{p} \mathbb{R}}$, $2 \leq p < \infty$, (cf. [LP]) implies

$$\|U\|_{C^\alpha([0,T];L^p_x(\mathbb{R}))} = \sup_{t \in [0,T]} \left( t^n \|U(t)\|_{L^p_x(\mathbb{R})} \right) \lesssim \sup_{t \in [0,T]} \left( t^n \|U(t)\|_{H^\frac{1}{2} - \frac{1}{p} \mathbb{R}} \right).$$

Moreover, for $s > \frac{1}{2} - \frac{1}{p}$ the space estimate (3.11) yields $\|U(t)\|_{H^\frac{s - \frac{1}{2}}{p} \mathbb{R}} \leq \|U(t)\|_{H^\frac{1}{2} - \frac{1}{p} \mathbb{R}}$, which gives in turn estimate (3.41). Note that the restriction $s < \frac{1}{2}$ comes from the definition (1.2) of $\alpha$ and is not required in the current proof. Continuity in time follows via the dominated convergence theorem as in Theorem 3.1.

We complete this section with the estimation of the forced linear IVP (3.6).

**Theorem 3.4** ($C^\alpha([0,T];L^p_x(\mathbb{R}))$-estimate for the forced linear IVP). For $F = \prod_{j=1}^{p} F_j$, the solution $W = S[F;0]$ of the forced linear heat IVP (3.6) given by formula (3.7) satisfies the estimate

$$\|W\|_{C^\alpha([0,T];L^p_x(\mathbb{R}))} \lesssim T^\alpha \prod_{j=1}^{p} \|F_j\|_{C^\alpha([0,T];L^p_x(\mathbb{R}))}.$$  

(3.42)

**Proof of Theorem 3.4.** The Sobolev inequality $\|f\|_{L^p(\mathbb{R})} \leq c \|f\|_{H^\frac{1}{2} - \frac{1}{p} \mathbb{R}}$, $2 \leq p < \infty$, implies

$$\|W\|_{C^\alpha([0,T];L^p_x(\mathbb{R}))} \lesssim \sup_{t \in [0,T]} \left( t^n \|W(t)\|_{H^\frac{1}{2} - \frac{1}{p} \mathbb{R}} \right),$$

from which we can obtain the desired estimate using the space estimate (3.26b) for $s = \frac{1}{2} - \frac{1}{p}$ (this restricts $p \geq 2$) and the fact that $\alpha > 0$ and $T < 1$. Continuity in time follows via the dominated convergence theorem as in Theorem 3.2.
4. Linear IBVP Estimates on the Half-Line

Having completed the estimation of IVPs (3.4) and (3.6), we turn our attention to the remaining two linear problems involved in the decomposition of Subsection 3.1, namely the reduced IBVPs (3.8) and (3.9). These two problems are of the same type, since (i) they are both homogeneous and with zero initial datum; (ii) their boundary data $G_0$ and $W|_{x=0}$ both vanish at $t = 0$ for $s > \frac{1}{2}$; (iii) their boundary data $G_0$ and $W|_{x=0}$ belong to $H^{\frac{2s+1}{4}}_t(0, T)$. Indeed, thanks to the time estimates (3.12) and (3.27a) and the extension inequalities (3.1), (3.2) and (3.3), we have

$$\|G_0\|_{H^{\frac{2s+1}{4}}_t(0, T)} \lesssim \|u_0\|_{H^s_2(0, \infty)} + \|g_0\|_{H^{\frac{2s+1}{4}}_t(0, T)}, \quad 0 \leq s < \frac{3}{2}, \quad (4.1a)$$

$$\|W|_{x=0}\|_{H^{\frac{2s+1}{4}}_t(0, T)} \lesssim \sqrt{T} \prod_{j=1}^p \sup_{t \in [0, T]} \|f_j(t)\|_{H^s_2(0, \infty)}, \quad \frac{1}{2} \leq s < \frac{3}{2}, \quad (4.1b)$$

$$\|W|_{x=0}\|_{H^{\frac{2s+1}{4}}_t(0, T)} \lesssim \sqrt{T} \prod_{j=1}^p \|f_j\|_{C^\alpha([0, T]; L^p_s(0, \infty))}, \quad -\frac{1}{2} \leq s < \frac{1}{2}. \quad (4.1c)$$

Hence, IBVPs (3.8) and (3.9) can be treated as specific cases of the problem

$$u_t - u_{xx} = 0, \quad x \in (0, \infty), \ t \in (0, T), \quad (4.2a)$$

$$u(x, 0) = 0, \quad x \in [0, \infty), \quad (4.2b)$$

$$u(0, t) = g_0(t) \in H^{\frac{2s+1}{4}}_t(0, T), \ t \in [0, T], \quad (4.2c)$$

with $g_0(0) = 0$ for $s > \frac{1}{2}$, whose solution is given by the UTM formula (2.5) with $u_0 = f = 0$.

4.1. An IBVP with a compactly supported boundary datum

For the purpose of obtaining Sobolev-type estimates, it would be convenient if formula (2.5) involved the Fourier transform of $g_0$ instead of the time transform $\tilde{g}_0$. Note that this would indeed be the case if $g_0$ were compactly supported in $[0, T]$. Recalling that $T < 1$, this observation motivates the “embedding” of IBVP (4.2) inside the pure IBVP

$$v_t - v_{xx} = 0, \quad x \in (0, \infty), \ t \in (0, 2), \quad (4.3a)$$

$$v(x, 0) = 0, \quad x \in [0, \infty), \quad (4.3b)$$

$$v(0, t) = g(t) \in H^{\frac{2s+1}{4}}_t(\mathbb{R}), \ t \in [0, 2], \quad (4.3c)$$

where the boundary datum $g \in H^{\frac{2s+1}{4}}_t(\mathbb{R})$ is an extension of $g_0$ such that $\text{supp}(g) \subset (0, 2)$. In particular, if $0 \leq s < \frac{1}{2}$ then $g$ is simply the extension of $g_0 \in H^{\frac{2s+1}{4}}_t(0, T)$ by zero outside $(0, T)$, while if $\frac{1}{2} < s \leq \frac{3}{2}$ then $g$ is defined along the lines of [FHM1] as

$$g(t) = \begin{cases} E_\theta(t), & t \in (0, 2), \\ 0, & t \in (0, 2)^c \end{cases},$$

where $E_\theta(t) = \theta(t)E(t)$ with $\theta \in C^\infty_0(\mathbb{R})$ a smooth cut-off function such that $|\theta(t)| \leq 1$ for all $t \in \mathbb{R}$, $\theta(t) = 1$ for $|t| \leq 1$ and $\theta(t) = 0$ for $|t| \geq 2$, and $E \in H^{\frac{2s+1}{4}}_t(\mathbb{R})$ an extension of $g_0 \in H^{\frac{2s+1}{4}}_t(0, T)$ such that $\|E\|_{H^{\frac{2s+1}{4}}_t(\mathbb{R})} \leq c \|g_0\|_{H^{\frac{2s+1}{4}}_t(0, T)}$. In both cases, by construction we have $\text{supp}(g) \subset (0, 2)$ and

$$\|g\|_{H^{\frac{2s+1}{4}}_t(\mathbb{R})} \leq c_s \|g_0\|_{H^{\frac{2s+1}{4}}_t(0, T)}, \quad 0 \leq s < \frac{3}{2}, \ s \neq \frac{1}{2}. \quad (4.4)$$
By the definition of $g$ as an extension of $g_0$, the pure IBVP (4.3) restricted on $(0, \infty) \times (0, T)$ becomes IBVP (4.2). Therefore, the solution of the latter problem can be estimated via the solution of former one, which is given by the UTM formula (2.5) as

$$v(x, t) = S[0, g; 0](x, t) = \frac{1}{\pi} \int_{k=0}^{\infty} e^{ika^2 kx + ik^2 t} k\hat{g}(k^2) dk + \frac{1}{\pi} \int_{k=0}^{\infty} e^{iakx - ik^2 t} k\hat{g}(-k^2) dk$$

(4.5)

with $a = e^{i\tau}$ and $\hat{g}(\tau) = \int_{R} e^{-irt} g(t) dt$ being the Fourier transform of $g$.

### 4.2. Sobolev-type estimates

We begin with the estimation of the pure IBVP (4.3) in Sobolev spaces, which reveals to us the correct space for the boundary datum, namely the space $H^{2s+1}_s(0, T)$.

**Theorem 4.1 (Sobolev-type estimates for the pure IBVP).** The solution $v = S[0, g; 0]$ of the pure IBVP (4.3) given by the UTM formula (4.5) admits the space and time estimates

$$\|v\|_{C([0, 2]; H^s_0(0, \infty))} \leq c_s \|g\|_{H^{2s+1}_s(\mathbb{R})}, \quad s \geq 0,$$

(4.6)

$$\|v\|_{C(0, \infty); H^{2s+1}_s(0, 2)} \leq c_s \|g\|_{H^{2s+1}_s(\mathbb{R})}, \quad s \in \mathbb{R}.$$  

(4.7)

**Remark 4.1.** The space estimate (4.6) for the pure IBVP (4.3) is central in our analysis as it motivates the space $H^{2s+1}_s(0, T)$ for the boundary datum $g_0$ of the nonlinear IBVP (1.1). Recall that another source of motivation for the boundary data space is the time regularity (3.12) of the homogeneous linear IVP solution.

**Proof of Theorem 4.1.** To establish the space estimate (4.6), we write $v = v_1 + v_2$ with

$$v_1(x, t) = \int_{k=0}^{\infty} e^{i\gamma_1 kx} G_1(k, t) dk, \quad G_1(k, t) = \frac{1}{\pi} e^{ik^2 t} k\hat{g}(k^2), \quad \gamma_1 = a^3 = e^{i\frac{3\pi}{4}},$$

(4.8)

$$v_2(x, t) = \int_{k=0}^{\infty} e^{i\gamma_2 kx} G_2(k, t) dk, \quad G_2(k, t) = \frac{1}{\pi} e^{-ik^2 t} k\hat{g}(-k^2), \quad \gamma_2 = a = e^{i\frac{\pi}{4}}.$$  

(4.9)

The estimation of $v_1$ and $v_2$ is entirely analogous. Thus, we only give the details for $v_1$. We employ the physical space definition of the $H^s_0(0, \infty)$-norm:

$$\|v_1(t)\|_{H^s_0(0, \infty)} = \sum_{j=0}^{[s]} \|\partial_x^j v_1(t)\|_{L^2_x(0, \infty)} + \|\partial_x^{[s]} v_1(t)\|_{\beta}, \quad s = [s] + \beta \geq 0, \quad 0 \leq \beta < 1,$$

(4.10)

where $[\cdot]$ denotes the floor function and the fractional norm $\|\cdot\|_{\beta}$ is defined by

$$\|v_1(t)\|_{\beta} = \left( \int_0^{\infty} \int_0^{\infty} \frac{|v_1(x + z, t) - v_1(x, t)|^2}{z^{1+2\beta}} dz dx \right)^{\frac{1}{2}}, \quad 0 < \beta < 1.$$

There are three cases to consider: (i) $[s] = 0$ and $\beta \neq 0$; (ii) $\beta = 0$; (iii) $[s] \neq 0$ and $\beta \neq 0$.

*(i) The case $[s] = 0$ and $\beta \neq 0$. Then, $s = \beta \in (0, 1)$ and we need to estimate $\|v_1(t)\|_{L^2_x(0, \infty)}$ and $\|v_1(t)\|_{\beta}$. The first norm will be estimated together with the $L^2_x(0, \infty)$-norms of the higher derivatives of $v_1$ in case (ii). For the second norm, we have

$$\|v_1(t)\|_{\beta}^2 \leq \int_0^{\infty} \int_0^{\infty} \frac{1}{z^{1+2\beta}} \left( \int_{k=0}^{\infty} \left| e^{i\gamma_1 k(x+z)} - e^{i\gamma_1 kx} \right| G_1(k, t) dk \right)^2 dz dx$$

and use the following lemma.
Lemma 4.1 ([FHM2], Lemma 2.1). If \( \gamma = \gamma_R + i \gamma_I \) with \( \gamma_I > 0 \), then
\[
|e^{i\gamma k x} - e^{i\gamma k z}| \leq \sqrt{2} \left| 1 + \frac{\|\mathbf{r}\|}{\gamma} \right| |e^{-\gamma_I k x} - e^{-\gamma_I k z}| \quad \forall k, x, z \geq 0.
\]

Employing Lemma 4.1 with \( \gamma = \gamma_1 = e^{i\beta x} \) and subsequently making the change of variables \( \sqrt{2} x \rightarrow x \), \( \sqrt{2} z \rightarrow z \), we obtain
\[
\|v_1(t)\|_2^2 \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-k x} (1 - e^{-k z}) |G_1(k, t)| dk \right)^2 dx dz.
\]
We identify the \( k \)-integral in (4.2) as the Laplace transform of \( Q_{z,t}(k) = (1 - e^{-k z}) |G_1(k, t)| \).

Proceeding as in case (i), we obtain
\[
\|v_1(t)\|_2^2 \lesssim \int_{\mathbb{R}} \int_{\mathbb{R}} \left( \int_{\mathbb{R}} e^{-k x} (1 - e^{-k z}) |G_1(k, t)| dk \right)^2 dx dz.
\]
with the last inequality due to estimate (3.21) and the definition (4.8) of \( G_1 \). Then, making the change of variable \( \tau = k \) we obtain
\[
\|v_1(t)\|_2^2 \lesssim \int_{\mathbb{R}} |\tau| \frac{2^{\beta + 1}}{z^{1+2\beta}} |\hat{g}(\tau)|^2 d\tau \lesssim \|g\|_{H^\frac{2\beta + 1}{2}(\mathbb{R})}, \quad 0 < \beta < 1,
\]
which is the desired estimate for \( v_1 \) and which indicates that the optimal value of \( m \) is indeed \( \frac{2\beta + 1}{2} \).

(ii) The case \( \beta = 0 \). Differentiating formula (4.8) and taking the \( L^2 \)-norm, we have
\[
\|\partial_x v_1(t)\|_{L^2(0, \infty)} \leq \left\| \int_{\mathbb{R}} e^{-\frac{2\beta}{k} k x} |k|^{j+1} |\hat{g}(k^2)| dk \right\|_{L^2(0, \infty)}.
\]
Therefore, exploiting once again the boundedness of the Laplace transform in \( L^2 \) and estimating as in case (i) above, we find
\[
\|\partial_x^j v_1(t)\|_{L^2(0, \infty)} \lesssim \|k|^{j+1} \hat{g}(k^2)\|_{L^2(0, \infty)} \lesssim \|g\|_{H^\frac{2\beta + 1}{2}(\mathbb{R})}^2, \quad j = 0, 1, \ldots, [s],
\]
so that
\[
\|v_1(t)\|_{H^\frac{2\beta + 1}{2}(\mathbb{R})} \lesssim \|g\|_{H^\frac{2\beta + 1}{2}(\mathbb{R})}^2, \quad [s] \geq 0.
\]

(iii) The case \( [s] \neq 0 \) and \( \beta \neq 0 \). We now have \( s = [s] + \beta \) with \( [s] \geq 1 \) and \( 0 < \beta < 1 \). Thanks to our earlier work in cases (i) and (ii), it suffices to estimate the fractional norm \( \|\partial_x^{[s]} v_1(t)\|_\beta \).

Proceeding as in case (i), we obtain
\[
\|\partial_x^{[s]} v_1(t)\|_\beta^2 \lesssim \int_{\mathbb{R}} k^2 |k|^{[s] + \beta} |\hat{g}(k^2)|^2 dk = \left\| k^{[s] + \beta} \hat{g}(k^2) \right\|_{L^2(0, \infty)}^2,
\]
which can be estimated like the corresponding term in case (ii) to yield
\[
\|\partial_x^{[s]} v_1(t)\|_\beta \lesssim \|g\|_{H^\frac{2\beta + 1}{2}(\mathbb{R})}, \quad [s] \geq 1.
\]

Estimates (4.11), (4.12) and (4.13) combined with the definition (4.10) imply
\[
\|v_1(t)\|_{H^\frac{s}{2}(0, \infty)} \leq c_s \|g\|_{H^\frac{2\beta + 1}{2}(\mathbb{R})}, \quad s \geq 0, \ t \in [0, 2],
\]
which is the space estimate (4.6) for \( v_1 \). As noted earlier, this estimate can be established for \( v_2 \) in the exact same way. Hence, the proof of the space estimate (4.6) for \( v \) is complete. Continuity in time follows via the dominated convergence theorem as in Theorem 5 of [FHM1].

The time estimate (4.7) is easier to establish. In particular, the change of variable \( k = \sqrt{\tau} \) turns formula (4.5) into

\[
v(x, t) \sim \int_{\tau=0}^{\infty} e^{ia^3 \sqrt{x \tau + i \tau}} \mathcal{F}(\tau) d\tau + \int_{\tau=-\infty}^{0} e^{ia\sqrt{-x \tau + i \tau}} \mathcal{F}(\tau) d\tau.
\]

Therefore, since the \( x \)-exponentials involved above are bounded by 1 for \( x \geq 0 \), we obtain

\[
\|v(x)\|_{H^s_\tau(\mathbb{R})} \lesssim \int_{\tau=0}^{\infty} (1 + \tau^2) \frac{2s+1}{4} e^{-\sqrt{2\tau x}} |\mathcal{F}^{\tau}(\tau)|^2 d\tau + \int_{\tau=-\infty}^{0} (1 + \tau^2) \frac{2s+1}{4} e^{-\sqrt{2\tau x}} |\mathcal{F}^{\tau}(\tau)|^2 d\tau
\]

\[
\leq \int_{\tau=0}^{\infty} (1 + \tau^2) \frac{2s+1}{4} |\mathcal{F}^{\tau}(\tau)|^2 d\tau + \int_{\tau=-\infty}^{0} (1 + \tau^2) \frac{2s+1}{4} |\mathcal{F}^{\tau}(\tau)|^2 d\tau = \|g\|_{H^s_\tau(\mathbb{R})}^2.
\]

Continuity in \( x \) follows via the dominated convergence theorem as in Theorem 5 of [FHM1]. ■

4.3. Additional estimate

We complete the analysis of IBVP (4.3) with the estimation in the space \( C^\alpha([0, T]; L^p_\tau(0, \infty)) \).

**Theorem 4.2** \( C^\alpha([0, T]; L^p_\tau(0, \infty))-\text{estimate for the pure IBVP}. \) If \( \frac{1}{2} - \frac{1}{p} < s < \frac{1}{2} \), then the solution \( v = S[0, g; 0] \) of the pure IBVP (4.3) given by the UTM formula (4.5) admits the estimate

\[
\|v\|_{C^\alpha([0, T]; L^p_\tau(0, \infty))} \lesssim T^\alpha \|g\|_{H^s_\tau(\mathbb{R})}.
\]

**Proof of Theorem 4.2.** We decompose \( v = v_1 + v_2 \) as in the proof of Theorem 4.1 and estimate \( v_1 \) and \( v_2 \) separately. Using Minkowski’s integral inequality and the fact that \( x \geq 0 \), we find

\[
\|v_1(t)\|_{L^p_\tau(0, \infty)} \lesssim \int_{k=0}^{\infty} \left( \int_{x=0}^{\infty} \left| e^{ia^3 k x + ik^2 t} \mathcal{F}^{\tau}(k^2) \right|^p dx \right)^{\frac{1}{p}} dk
\]

\[
= \int_{k=0}^{\infty} k |\mathcal{F}(k^2)| \left( \int_{x=0}^{\infty} e^{-\frac{\sqrt{p} k x}{2}} dx \right)^{\frac{1}{p}} dk \lesssim \int_{k=0}^{\infty} \left( 1 + k^2 \right)^{\frac{1}{2}} |\mathcal{F}(k^2)| dk.
\]

Thus, letting \( k = \sqrt{\tau} \) and applying the Cauchy-Schwarz inequality in \( \tau \), we obtain

\[
\|v_1(t)\|_{L^p_\tau(0, \infty)} \lesssim \left( \int_{\tau=0}^{\infty} \frac{1}{\tau^{1/p}} \left( 1 + \tau^2 \right)^{\frac{2s+1}{4}} d\tau \right)^\frac{1}{2} \|g\|_{H^s_\tau(\mathbb{R})} \lesssim \|g\|_{H^s_\tau(\mathbb{R})}^{s - \frac{1}{2} - \frac{1}{p}}, \quad s > \frac{1}{2} - \frac{1}{p},
\]

from which we deduce estimate (4.15) for \( v_1 \). The estimation of \( v_2 \) is entirely analogous. Note that the restriction \( s < \frac{1}{2} \) appearing in the hypothesis of the theorem originates from the condition on \( b \) involved in (1.2) and is not needed in the present proof. ■

4.4. Summary of linear IBVP estimates on the half-line

In view of Theorems 3.1-3.4, 4.1, 4.2 and inequalities (3.1)-(3.3), (4.1), (4.4), the decomposition (3.10) yields the following estimates for the forced linear IBVP (1.3).
**Theorem 4.3 (Estimates for the forced linear heat on the half-line).** For \( f = \prod_{j=1}^{p} f_j \), the solution \( u = S[u_0, g_0; f] \) of the forced linear heat IBVP (1.3) satisfies the space estimates

\[
\|S[u_0, g_0; f]\|_{C([0,T];H^s_x(0,\infty))} \leq c_{s,p} \left( \|u_0\|_{H^s_x(0,\infty)} + \|g_0\|_{H^s_t} \right) + \sqrt{T} \prod_{j=1}^{p} \|f_j\|_{C([0,T];H^s_x(0,\infty))}, \quad \frac{1}{2} < s < \frac{3}{2},
\]

\[
\|S[u_0, g_0; f]\|_{C([0,T];H^s_x(0,\infty))} \leq c_{s,p} \left( \|u_0\|_{H^s_x(0,\infty)} + \|g_0\|_{H^s_t} \right) + \sqrt{T} \prod_{j=1}^{p} \|f_j\|_{C^\alpha([0,T];L^p_x(0,\infty))}, \quad 0 \leq s < \frac{1}{2},
\]

the time estimates

\[
\|S[u_0, g_0; f]\|_{C([0,\infty);H^s_x(0,\infty))} \leq c_{s,p} \left( \|u_0\|_{H^s_x(0,\infty)} + \|g_0\|_{H^s_t} \right) + \sqrt{T} \prod_{j=1}^{p} \|f_j\|_{C([0,T];H^s_x(0,\infty))}, \quad \frac{1}{2} < s < \frac{3}{2},
\]

\[
\|S[u_0, g_0; f]\|_{C([0,\infty);H^s_x(0,\infty))} \leq c_{s,p} \left( \|u_0\|_{H^s_x(0,\infty)} + \|g_0\|_{H^s_t} \right) + \sqrt{T} \prod_{j=1}^{p} \|f_j\|_{C^\alpha([0,T];L^p_x(0,\infty))}, \quad 0 \leq s < \frac{1}{2},
\]

and the \( L^p \)-estimate

\[
\|S[u_0, g_0; f]\|_{C^\alpha([0,T];L^p_x(0,\infty))} \leq c_{s,p} \left( \|u_0\|_{H^s_x(0,\infty)} + \|g_0\|_{H^s_t} \right) + T^\alpha \prod_{j=1}^{p} \|f_j\|_{C^\alpha([0,T];L^p_x(0,\infty))}, \quad \frac{1}{2} - \frac{1}{p} < s < \frac{1}{2},
\]

where \( c_{s,p} > 0 \) is a constant that depends only on \( s \) and \( p \).

5. Local Well-Posedness on the Half-Line

The linear estimates of Theorem 4.3 for the forced linear heat IBVP (1.3) will now be employed for showing well-posedness via contraction mapping of the (nonlinear) reaction-diffusion IBVP (1.1). We shall begin with the case of “smooth” data \( (s > \frac{1}{2}) \), which corresponds to Theorem 1.1, and then proceed to the case of “rough” data \( (s < \frac{1}{2}) \), which corresponds to Theorem 1.2.

5.1. Proof of Theorem 1.1

We shall establish local well-posedness in the sense of Hadamard, i.e. we shall show existence, uniqueness and continuous dependence of the solution on the initial and boundary data.

I. Existence and uniqueness. For \( f = |u|^{p-1}u = u^p, \frac{p-1}{2} \in \mathbb{N} \), the forced linear IBVP solution \( S[u_0, g_0; f] \) given by (2.5) induces the following iteration map for the reaction-diffusion IBVP (1.1):

\[
u \mapsto \Phi u = \Phi_{u_0,g_0} u \doteq S[u_0, g_0; u^p].\]
Existence and uniqueness in the case of “smooth” data \((s > \frac{1}{2})\) will be established by showing that the above map is a contraction in the space

\[
X = C([0, T^*]; H_x^s(0, \infty)) \cap C([0, \infty); H_t^{2s+1}(0, T^*)),
\]

(5.1)

where the lifespan \(T^* \in (0, T]\) is to be determined.

**Showing that \(u \mapsto \Phi u\) is onto \(X\).** Let \(B(0, \varrho) = \{u \in X : \|u\|_X \leq \varrho\}\) be a ball centered at 0 with radius \(\varrho = 2c_{s,p} \|(u_0, g_0)\|_D\), where \(c_{s,p} > 0\) is the constant appearing in Theorem 4.3 and the data norm \(\|\cdot\|_D\) is defined by

\[
\|(u_0, g_0)\|_D = \|u_0\|_{H_x^s(0, \infty)} + \|g_0\|_{H_t^{2s+1}(0, T^*)},
\]

(5.2)

For \(u \in B(0, \varrho)\) and \(\frac{1}{2} < s < \frac{3}{2}\), the space estimate (4.16a) and the time estimate (4.17a) imply

\[
\|\Phi u\|_X \leq \sup_{t \in [0, T^*]} \|S[u_0, g_0; u^p](t)\|_{H_x^s(0, \infty)} + \sup_{x \in [0, \infty)} \|S[u_0, g_0; u^p](x)\|_{H_t^{2s+1}(0, T^*)} \leq c_{s,p} \left(\|(u_0, g_0)\|_D + \sqrt{T^*} \|u\|_X^p\right) \leq \frac{\varrho}{2} + c_{s,p} \sqrt{T^*} \varrho^p.
\]

Thus, for \(T^* \in (0, T]\) such that

\[
\frac{\varrho}{2} + c_{s,p} \sqrt{T^*} \varrho^p \leq \varrho \iff T^* \leq \frac{1}{(2c_{s,p})^2 \|u_0\|_D^{2(p-1)}}
\]

(5.3)

the map \(u \mapsto \Phi u\) is onto the ball \(B(0, \varrho)\).

**Showing that \(u \mapsto \Phi u\) is a contraction in \(X\).** We shall show that

\[
\|\Phi u_1 - \Phi u_2\|_X \leq \frac{1}{2} \|u_1 - u_2\|_X \quad \forall u_1, u_2 \in B(0, \varrho) \subset X.
\]

(5.4)

Noting that \(\Phi u_1 - \Phi u_2 = S[0, 0; u_1^p - u_2^p]\) and using estimates (4.16a) and (4.17a), we obtain

\[
\|\Phi u_1 - \Phi u_2\|_X \leq c_{s,p} \sqrt{T^*} \sup_{t \in [0, T^*]} \|(u_1^p - u_2^p)(t)\|_{H_x^s(0, \infty)}, \quad \frac{1}{2} < s < \frac{3}{2}.
\]

Thus, employing the identity

\[
u_1^p - u_2^p = (u_1^{p-1} + u_1^{p-2}u_2 + \ldots + u_1u_2^{p-2} + u_2^{p-1})(u_1 - u_2)
\]

(5.5)

and the algebra property in \(H_x^s(0, \infty)\), we deduce

\[
\|\Phi u_1 - \Phi u_2\|_X \leq p c_{s,p} \sqrt{T^*} \varrho^{p-1} \|u_1 - u_2\|_X, \quad \frac{1}{2} < s < \frac{3}{2}.
\]

Hence, for \(T^*\) such that

\[
p c_{s,p} \sqrt{T^*} \varrho^{p-1} \leq \frac{1}{2} \iff T^* \leq \frac{1}{p^2 (2c_{s,p})^2 \|u_0\|_D^{2(p-1)}}
\]

(5.6)

the contraction inequality (5.4) is satisfied.

Since \(p > 1\), condition (5.3) is weaker than condition (5.6). Therefore, for lifespan \(T^*\) given by

\[
T^* = \min \left\{ T, \frac{1}{p^2 (2c_{s,p})^2 \|u_0\|_D^{2(p-1)}} \right\}, \quad c_{s,p} > 0,
\]

(5.7)

the contraction mapping theorem implies that the map \(u \mapsto \Phi u\) has a unique fixed point in \(B(0, \varrho)\). Equivalently, the integral equation (1.1) has a unique solution \(u \in B(0, \varrho) \subset X\). The proof of existence and uniqueness is complete.
II. Continuity of the data-to-solution map. We complete Hadamard well-posedness by showing that the data-to-solution map \( H^s_\varepsilon(0, \infty) \times H^{2s+\frac{1}{2}}_t(0, T) \ni (u_0, g_0) \mapsto u \in X \) is continuous.

Let \((u_0, g_0)\) and \((u_{02}, g_{02})\) be two pairs of data in the data space \( D \) that lie inside a ball \( B(\varrho, \delta) \subset D \) of radius \( \delta > 0 \) centered at a distance \( \varrho \) from the origin. Denote by \( u_1 = \Phi_{u_0, g_0, u_1} \) and \( u_2 = \Phi_{u_{02}, g_{02}, u_2} \) the corresponding solutions to the reaction-diffusion IBVP (1.1), and by \( T_{u_1} \) and \( T_{u_2} \) the lifespans of those solutions given according to (5.7). Since max \( \{ ||(u_0, g_0)||_D, ||(u_2, g_2)||_D \} \leq \varrho + \delta \) and \( p > 1 \), it follows that

\[
\min\{T_{u_1}, T_{u_2}\} \geq \min \left\{ T, \frac{1}{p^2(2c_{s,p})^{2p}(\varrho + \delta)^{2(p-1)}} \right\} \geq T_c.
\]

Hence, both \( u_1 \) and \( u_2 \) are guaranteed to exist for \( 0 \leq t \leq T_c \). Replacing \( T^* \) with the common lifespan \( T_c \) in (5.1) gives rise to the space

\[ X_c = C([0, T_c]; H^s_\varepsilon(0, \infty)) \cap C([0, \infty); H^{2s+\frac{1}{2}}_t(0, T_c)). \]

Observe that \( X_{u_1}, X_{u_2} \subset X_c \) with the spaces \( X_{u_1} \) and \( X_{u_2} \) defined by (5.1) with \( T^* \) replaced by \( T_{u_1} \) and \( T_{u_2} \) respectively. We shall now determine the radius \( \varrho_c \) of a ball \( B(0, \varrho_c) \subset X_c \) such that \( u_1, u_2 \in B(0, \varrho_c) \) and

\[ ||u_1 - u_2||_{X_c} \leq 2c_{s,p} ||(u_0, g_0) - (u_{02}, g_{02})||_D. \]  

(5.8)

Since \( u_1 \) and \( u_2 \) are obtained as fixed points of the maps \( u \mapsto \Phi u \) and \( u \mapsto \Phi u \) in \( X_{u_1} \) and \( X_{u_2} \) respectively, we have

\[ ||u_1 - u_2||_{X_c} = ||\Phi u_1 - \Phi u_2||_{X_c} = ||S[u_1 - u_2, g_0, g_2; u_1^p - u_2^p]||_{X_c}. \]

Thus, estimates (4.16a) and (4.17a) together with identity (5.5) and the algebra property in \( H^s_\varepsilon(0, \infty) \) imply

\[ ||u_1 - u_2||_{X_c} \leq \frac{c_{s,p}}{1 - c_{s,p}\sqrt{T_c}p\scriptstyle\hat{=}p^{-\frac{1}{p-1}}} ||(u_0, g_0) - (u_{02}, g_{02})||_D \]  

(5.9)

provided that \( c_{s,p}\sqrt{T_c}p\scriptstyle\hat{=}p^{-\frac{1}{p-1}} < 1 \), which is satisfied for \( \varrho_c = (2c_{s,p}\sqrt{T_c}p)^{-\frac{1}{p-1}} \). For this choice of \( \varrho_c \), inequality (5.9) becomes the Lipschitz condition (5.8). The proof of Theorem 1.1 is complete.

5.2. Proof of Theorem 1.2

I. Existence and uniqueness. For \( s < \frac{1}{2} \) (“rough” data), we refine the solution space \( X \) defined by (5.1) by intersecting it with the space \( C^0([0, T^*]; L^p_\varepsilon(0, \infty)) \), i.e. we replace \( X \) by

\[ Y = C([0, T^*]; H^s_\varepsilon(0, \infty)) \cap C([0, \infty); H^{2s+\frac{1}{2}}_t(0, T^*)) \cap C^0([0, T^*]; L^p_\varepsilon(0, \infty)), \]  

(5.10)

where the lifespan \( T^* \in (0, T] \) is to be determined. In analogy to the case \( s > \frac{1}{2} \) of “smooth” data, existence and uniqueness of solution to the reaction-diffusion IBVP (1.1) will be established by showing that the iteration map

\[ u \mapsto \Phi u = \Phi_{u_0, g_0} u \owns S[u_0, g_0; |u|^{p-1} u] \]

is a contraction in \( Y \). Note that since for \( s < \frac{1}{2} \) the forcing \( f = |u|^{p-1} u \) appears in the linear estimates of Theorem 4.3 under the \( L^p_\varepsilon(0, \infty) \)-norm instead of the \( H^s_\varepsilon(0, \infty) \)-norm, the restriction \( p-\frac{1}{2} \in \mathbb{N} \) that was present for \( s > \frac{1}{2} \) is not needed anymore, i.e. \( p = 2, 3, 4, \ldots \).

Showing that \( u \mapsto \Phi u \) is onto \( Y \). Let \( B(0, \varrho) = \{ u \in Y : ||u||_Y \leq \varrho \} \) be a ball centered at 0 with radius \( \varrho = 2c_{s,p} ||(u_0, g_0)||_D \), where \( c_{s,p} > 0 \) is the constant appearing in Theorem 4.3 and the norm
\(\|\cdot\|_D\) is defined by (5.2). For \(u \in B(0, g)\) and \(\frac{1}{2} - \frac{1}{p} < s < \frac{1}{2}\), using the space estimate (4.16b), the time estimate (4.17b) and the \(L^p\)-estimate (4.18) with \(f_j = u\), we have
\[
\|\Phi u\|_Y = \sup_{t \in [0,T^*]} \|S[u_0, g_0; |u|^{p-1}u](t)\|_{L^p_x(0,\infty)} + \sup_{x \in [0,\infty)} \|S[u_0, g_0; |u|^{p-1}u](x)\|_{H^s_t(x,\infty)}
\]
\[
+ \|S[u_0, g_0; |u|^{p-1}u]\|_{C^0([0,T^*];L^p_x(0,\infty))}
\]
\[
\leq c_{s,p} \left( \|u(0,0)\|_D + (T^*)^\alpha \|u\|_{C^0([0,T^*];L^p_x(0,\infty))} \right) \leq \frac{\varrho}{2} + c_{s,p} (T^*)^\alpha \varrho^p,
\]
where we have also used the fact that \(\sqrt{T^*} < (T^*)^\alpha\) since \(T^* \leq T < 1\) and \(\alpha < \frac{1}{2}\). Hence for \(T^* \in (0, T]\) satisfying
\[
\frac{\varrho}{2} + c_{s,p} (T^*)^\alpha \varrho^p \leq \varrho \iff T^* \leq \frac{1}{(2c_{s,p})^\frac{1}{\alpha}} \|(u_0, g_0)\|_{D}^{\frac{1}{\alpha}} (5.11)
\]
the map \(u \mapsto \Phi u\) is onto the ball \(B(0, \varrho)\).

**Showing that \(u \mapsto \Phi u\) is a contraction in \(Y\).** We shall show that
\[
\|\Phi u_1 - \Phi u_2\|_Y \leq \frac{1}{2} \|u_1 - u_2\|_Y \quad \forall u_1, u_2 \in B(0, \varrho) \subset Y. \quad (5.12)
\]

**Lemma 5.1.** For any \(v, w \in \mathbb{C}\) and any \(p \geq 1\), we have
\[
\|v|^{p-1}v - |w|^{p-1}w\| \leq c_p \left( \|v|^{p-1} + |w|^{p-1} \right) |v - w|, \quad c_p = 2^{p+1}p > 0.
\]

The proof of Lemma 5.1 is given at the end of the section. Employing this lemma with \(v = u_1\) and \(w = u_2\) and then applying Hölder’s inequality, we find
\[
\|u_1|^{p-1}u_1(t) - u_2|^{p-1}u_2(t)\|_{L^p_x(0,\infty)} \leq c_p \left( \|u_1(t)|^{p-1} + \|u_2(t)|^{p-1} \right) \|u_1(t) - u_2(t)\|_{L^p_x(0,\infty)}.
\]
Combining this inequality with estimates (4.16b), (4.17b) and (4.18), we obtain
\[
\|\Phi u_1 - \Phi u_2\|_Y \leq 2^{p+2}c_{s,p} (T^*)^\alpha \varrho^{p-1} \|u_1 - u_2\|_Y.
\]

Hence, for \(T^* \in (0, T]\) such that
\[
2^{p+2}c_{s,p} (T^*)^\alpha \varrho^{p-1} \leq \frac{1}{2} \iff T^* \leq \frac{1}{(2c_{s,p})^\frac{1}{\alpha}} \|(u_0, g_0)\|_{D}^{\frac{1}{\alpha}} (5.13)
\]
the contraction inequality (5.12) is satisfied.

Overall, for lifespan \(T^*\) given by
\[
T^* = \min \left\{ T, \frac{1}{(2c_{s,p})^\frac{1}{\alpha}} \|(u_0, g_0)\|_{D}^{\frac{1}{\alpha}} \right\}, \quad c_{s,p} > 0,
\]
the Banach fixed point theorem implies a unique solution \(u \in B(0, \varrho) \subset Y\) for the reaction-diffusion IBVP (1.1). The proof of existence and uniqueness is complete.

**II. Continuity of the data-to-solution map.** This part of the proof is entirely analogous to that of Subsection 5.1 for the case of “smooth” data. The proof of Theorem 1.2 is complete.

**Proof of Lemma 5.1.** Let \(f(z) = |z|^{p-1}z\) with \(z = x + iy\) and define
\[
\varphi(t) = f(w + t(v - w)) = f_1(w + t(v - w)) + if_2(w + t(v - w)) = \varphi_1(t) + i\varphi_2(t),
\]
where \(w = x_1 + iy_1, v = x_2 + iy_2\) and \(0 \leq t \leq 1\). The mean value theorem for \(\varphi_1\) and \(\varphi_2\) implies
\[
|v|^{p-1}v - |w|^{p-1}w = f(v) - f(w) = \varphi(1) - \varphi(0) = \varphi'(\tau_1) + i\varphi'_2(\tau_2)
\]
for some \(\tau_1, \tau_2 \in (0, 1)\).
for some \( \tau_1, \tau_2 \in [0, 1] \). In turn, \( |v|^{p-1}v - |w|^{p-1}w \leq |\varphi_1'(\tau_1)| + |\varphi_2'(\tau_2)| \) hence, it suffices to estimate \( |\varphi_1'(\tau_1)| \) and \( |\varphi_2'(\tau_2)| \). We compute \( \varphi_1'(t) = \frac{\partial f_1}{\partial x}(w + t(v - w)) \cdot (x_2 - x_1) + \frac{\partial f_1}{\partial y}(w + t(v - w)) \cdot (y_2 - y_1) \) and \( \varphi_2'(t) = \frac{\partial f_2}{\partial x}(w + t(v - w)) \cdot (x_2 - x_1) + \frac{\partial f_2}{\partial y}(w + t(v - w)) \cdot (y_2 - y_1) \), where

\[
\frac{\partial f_1}{\partial x} = |x|^{p-1} \left[ 1 + \frac{(p-1)x^2}{x^2+y^2} + i \frac{(p-1)y}{x^2+y^2} \right] = \frac{\partial f_1}{\partial y} + i \frac{\partial f_2}{\partial y} \quad \text{and} \quad \frac{\partial f_2}{\partial y} = |x|^{p-1} \left[ \frac{(p-1)x^2}{x^2+y^2} + i \left( 1 + \frac{(p-1)y^2}{x^2+y^2} \right) \right] = \frac{\partial f_1}{\partial y} + i \frac{\partial f_2}{\partial y}.
\]

Thus, for any \( 0 < t < 1 \) we have

\[
|\varphi_1'(t)| \leq |w + t(v - w)|^{p-1} \left[ 1 + (p - 1)(x_2 - x_1) + (p - 1)(x_2 - x_1) \right] \\
\leq p |(1-t)w + tw|^{p-1} |x_2 - x_1| \leq p |(w + v)|^{p-1} |v - w|
\]

and, noting that \( |w + v|^{p-1} \leq 2^p \left(|w|^{p-1} + |v|^{p-1}\right) \) for \( p \geq 1 \), we deduce \( |\varphi_1'(t)| \leq p \cdot 2^p \left(|w|^{p-1} + |v|^{p-1}\right) |v - w| \) for all \( t \in [0, 1] \). Similarly, we have \( |\varphi_2'(t)| \leq p \cdot 2^p \left(|w|^{p-1} + |v|^{p-1}\right) |v - w| \) for all \( t \in [0, 1] \). These two estimates combined with the mean value inequality above imply the inequality of Lemma 5.1.

6. Linear IBVP Estimates on the Finite Interval

The analysis of the forced linear heat equation on the finite interval can be simplified significantly by exploiting the results of Theorem 4.3 for this equation on the half-line. In particular, let \( U_0 \in H^s_x(\mathbb{R}) \) be a whole line extension of the initial data \( u_0 \in H^s_x(0, \ell) \) of the interval IBVP (1.7) such that \( \|U_0\|_{H^s_x(\mathbb{R})} \leq c \|u_0\|_{H^s_x(0, \ell)} \). For \( s > \frac{1}{2} \), use a similar argument to extend the forcing \( f \) of (1.7) by \( F \in C([0,T]; H^s_x(\mathbb{R})) \) such that \( \|F\|_{C([0,T]; H^s_x(\mathbb{R}))} \leq c \|F\|_{C([0,T]; H^s_x(\mathbb{R}))} \), while for \( s \leq \frac{1}{2} \) simply extend \( f \) by zero. Subsequently, restrict \( U_0 \) and \( F \) to the half-line to obtain initial data and forcing for the half-line IBVP (1.3). Then, the solution \( S[u_0, g_0, h_0; f] \) of the interval IBVP (1.7) can be expressed as

\[
S[u_0, g_0, h_0; f] = S[U_0|x\in[0,\ell]|, g_0; F|x\in[0,\ell]|]_{x\in[0,\ell]} + S[0, 0, u_0; 0],
\]

where \( S[U_0|x\in[0,\ell]|, g_0; F|x\in[0,\ell]|] \) is the solution of the half-line IBVP (1.3) restricted on \( [0, \ell] \) and \( S[0, 0, u_0; 0] \) satisfies the reduced interval IBVP

\begin{align*}
0 &- u_{xx} = 0, & x &\in (0, \ell), \ t \in (0, T), \\
0 &- u(x, 0) = 0, & x &\in [0, \ell], \\
0 &- u(0, t) = 0, & t &\in [0, T], \\
0 &- u(\ell, t) = h_0(t) - S[U_0|x\in[0,\ell]|, g_0; F|x\in[0,\ell]|](\ell, t) \equiv w_0(t), & t &\in [0, T].
\end{align*}

Since the half-line solution \( S[U_0|x\in[0,\ell]|, g_0; F|x\in[0,\ell]|] \) was estimated in the preceding sections, we only need to estimate the reduced IBVP (6.2).

Note that for all \( 0 \leq s < \frac{3}{2} \), \( s \neq \frac{1}{2} \), the boundary datum \( w_0 \) of IBVP (6.2) belongs to \( H^\frac{2s+1}{2}(0, T) \) due to the fact that both \( h_0 \) and \( S[U_0|x\in[0,\ell]|, g_0; F|x\in[0,\ell]|] \) belong to \( H^\frac{2s+1}{2}(0, T) \). In particular, combining the extension inequalities stated above and the half-line estimates (4.17), we have

\[
\|w_0\|_{H^\frac{2s+1}{2}(0, T)} \leq c_{s,p} \left( \|u_0\|_{H^s_x(0, \ell)} + \|g_0\|_{H^\frac{2s+1}{2}(0, T)} + \|h_0\|_{H^s_x(0, \ell)} \right) + \sqrt{T} \prod_{j=1}^p \|f_j\|_{C([0,T]; H^\frac{m+1}{2}(0, \ell))}, \quad \frac{1}{2} < s < \frac{3}{2}.
\]
and
\[ \|w_0\|_{H^{\frac{2m+1}{2}}_t \times (0,T)} \leq c_s, p \left( \|u_0\|_{H^2_0(\ell)} + \|g_0\|_{H^{\frac{2m+1}{2}}_t \times (0,T)} + \|h_0\|_{H^{\frac{2m+1}{2}}_t \times (0,T)} + \sqrt{T} \prod_{j=1}^{p} \|f_j\|_{C^\infty([0,T];L^p_2(0,\ell))} \right), \quad 0 \leq s < \frac{1}{2}. \] (6.3b)

The analysis can be further simplified by “embedding” IBVP (6.2) inside the pure IBVP
\[ v_t - v_{xx} = 0, \quad x \in (0, \ell), \quad t \in (0, 2), \] (6.4a)
\[ v(x, 0) = 0, \quad x \in [0, \ell], \] (6.4b)
\[ v(0, t) = 0, \quad t \in [0, 2], \] (6.4c)
\[ v(\ell, t) = h(t) \in H^{\frac{2m+1}{2}}_t (\mathbb{R}), \quad t \in [0, 2], \] (6.4d)
where \( h \) is an extension of \( w_0 \) constructed analogously to the extension \( g \) of Section 4 so that \( \text{supp}(h) \subset (0, 2) \) and
\[ \|h\|_{H^{\frac{2m+1}{2}}_t (\mathbb{R})} \leq c_s \|w_0\|_{H^{\frac{2m+1}{2}}_t (0,T)}, \quad 0 \leq s < \frac{3}{2}, \quad s \neq \frac{1}{2}. \] (6.5)

Indeed, IBVP (6.4) restricted on \((0, \ell) \times (0, T)\) becomes IBVP (6.2). Therefore, IBVP (6.2) can be estimated via IBVP (6.4), whose solution is given by the UTM formula (2.10) as
\[
v(x, t) = S[0, 0, h; 0](x, t) = \frac{1}{\pi} \int_{k=0}^{\infty} \frac{e^{iakx + ik^2}}{e^{iakt} - e^{-iakt}} \tilde{h}(k^2) dk + \frac{1}{\pi} \int_{k=0}^{\infty} e^{iakx - ik^2} \tilde{h}(k^2) dk \]
\[+ \frac{1}{\pi} \int_{k=0}^{\infty} \frac{e^{-iakx + ik^2}}{e^{iakt} - e^{-iakt}} \tilde{h}(-k^2) dk + \frac{1}{\pi} \int_{k=0}^{\infty} e^{-iakx - ik^2} \tilde{h}(-k^2) dk \] (6.6)
with \( a = e^{\frac{\pi}{2}} \) and the contours \( \partial D^\pm \) depicted in Figure 2.1. Observe that, similarly to the half-line, we have exploited the compact support of \( h \) in order to replace in formula (6.6) the time transform \( \tilde{h}_0 \) by the Fourier transform of \( h \) on the whole line.

Starting from formula (6.6), we shall establish the following Sobolev estimates for IBVP (6.4).

**Theorem 6.1 (Sobolev-type estimates for the pure IBVP).** The solution \( v = S[0, 0, h; 0] \) of the pure IBVP (6.4) given by the UTM formula (6.6) satisfies the space and time estimates
\[ \|v\|_{C([0,2];H^{s}_x(0,\ell), H^{s}_t \times (0,T))} \leq c_s \|h\|_{H^{s+1}_x(\mathbb{R})}, \quad s \geq 0; \] (6.7)
\[ \|v\|_{C([0,\ell];H^{s+1}_x(0,2))} \leq c_s \|h\|_{H^{s+1}_x(\mathbb{R})}, \quad s \in \mathbb{R}. \] (6.8)

**Proof of Theorem 6.1.** The main ideas behind the proof are the same with those that form the backbone of the proof in the case of the half-line (see Theorem 4.1). However, extra care is now needed when handling the exponential differences in the denominators of the solution formula (6.6).

We begin with the space estimate (6.7). Grouping together the first with the third and the second with the fourth term of (6.6), we write \( v = v_1 + v_2 \) where
\[ v_1(x, t) = \int_{k=0}^{\infty} \frac{e^{i\gamma_1 kx} - e^{-i\gamma_1 kx}}{e^{i\gamma_1 kt} - e^{-i\gamma_1 kt}} G_1(k, t) dk, \quad \gamma_1 = a^3, \quad G_1(k, t) = \frac{1}{\pi} e^{ik^2 t} \tilde{h}(k^2), \]
\[ v_2(x, t) = \int_{k=0}^{\infty} \frac{e^{i\gamma_2 kx} - e^{-i\gamma_2 kx}}{e^{i\gamma_2 kt} - e^{-i\gamma_2 kt}} G_2(k, t) dk, \quad \gamma_2 = a, \quad G_2(k, t) = \frac{1}{\pi} e^{-ik^2 t} \tilde{h}(-k^2). \]
As the estimation of $v_1$ and $v_2$ is analogous, we only provide the details for $v_1$. We employ the physical space definition of the $H^s_x(0, \ell)$-norm:

$$
\|v_1(t)\|_{H^s_x(0, \ell)} = \sum_{j=0}^{[s]} \| \partial_x^j v_1(t) \|_{L^2_x(0, \ell)} + \| \partial_x^{[s]} v_1(t) \|_{\beta}, \quad s = [s] + \beta \geq 0, \ 0 \leq \beta < 1,
$$

(6.10)

where $[\cdot]$ denotes the floor function and the fractional norm $\| \cdot \|_{\beta}$ is defined by

$$
\|v_1(t)\|_{\beta} = \left( \int_0^\ell \int_0^{\ell-x} \frac{|v_1(x+z, t) - v_1(x, t)|^2}{z^{1+2\beta}} \, dz \, dx \right)^{\frac{1}{2}}, \quad 0 < \beta < 1.
$$

(i) The case $[s] = 0$. Since $s = \beta \in [0, 1)$, we only need to estimate $\|v_1(t)\|_{\beta}$ and $\|v_1(t)\|_{L^2_2(0, \ell)}$. For the fractional norm $\|v_1(t)\|_{\beta}$, we write

$$
v_1(x, t) = J_1(x, t) + J_2(x, t)
$$

(6.11)

with

$$
J_1(x, t) = \int_0^1 \frac{e^{i\gamma_1 kx} - e^{-i\gamma_1 kx}}{e^{i\gamma_1 k t} - e^{-i\gamma_1 k t}} G_1(k, t) \, dk, \quad J_2(x, t) = \int_{k=1}^\infty \frac{e^{i\gamma_1 kx} - e^{-i\gamma_1 kx}}{e^{i\gamma_1 k t} - e^{-i\gamma_1 k t}} G_1(k, t) \, dk.
$$

For $J_1$, we have

$$
\|J_1(t)\|_{\beta} \leq \left( \int_0^\ell \int_0^{\ell-z} \frac{1}{z^{1+2\beta}} \left( \int_{k=0}^1 \frac{(e^{i\gamma_1 k z} - 1)(e^{i\gamma_1 k(x+\ell)} - e^{-i\gamma_1 k(x+z-\ell)})}{e^{2i\gamma_1 k t} - 1} G_1(k, t) \right)^2 \, dx \, dz \right)^{\frac{1}{2}}
$$

and noting that $x + \ell \geq 0$ and $x + z - \ell = x - (\ell - z) \leq 0$ and applying Minkowski’s integral inequality in $k$. Hence, using Lemma 4.1, integrating in $x$ and employing estimate (3.21), we obtain

$$
\|J_1(t)\|_{\beta} \leq \sqrt{\ell} \int_0^1 \frac{|G_1(k, t)|}{1 - e^{-\sqrt{2k} \ell}} \left( \int_0^\ell \frac{(1 - e^{-\sqrt{2k} z})^2}{z^{1+2\beta}} \, dz \right)^{\frac{1}{2}} \, dk \leq \int_0^1 \frac{k^{\beta + 1}}{1 - e^{-\sqrt{2k} \ell}} k^{\frac{\beta}{2}} |h(k^2)| \, dk.
$$

Then, applying the Cauchy-Schwarz inequality and noting that $\int_0^1 \frac{k^{\frac{\beta}{2}}}{1 - e^{-\sqrt{2k} \ell}} \, dk < \infty$ since the singularity at $k = 0$ is removable and the domain of integration is compact, we deduce

$$
\|J_1(t)\|_{\beta} \lesssim \|h\|_{L^2_1(\mathbb{R})} \|h\|_{H^{\frac{\beta + 1}{4}}_l(\mathbb{R})}, \quad 0 < \beta < 1.
$$

(6.12)

Regarding $J_2$, writing $J_2 = J_{21} + J_{22}$ with

$$
J_{21}(x, t) = \int_{k=1}^\infty e^{i\alpha \lambda k x} K_1(k, t) \, dk, \quad K_1(k, t) = \frac{G_1(k, t)}{e^{i\alpha \lambda k t} - e^{-i\alpha \lambda k t}},
$$

$$
J_{22}(x, t) = \int_{k=1}^\infty e^{-i\alpha \lambda k(x-\ell)} K_2(k, t) \, dk, \quad K_2(k, t) = \frac{G_1(k, t)}{1 - e^{2i\alpha \lambda k \ell}}
$$

and employing once again Lemma 4.1, we infer

$$
\|J_{21}(t)\|_{\beta} \lesssim \int_0^\ell \frac{1}{z^{1+2\beta}} \left( \int_{k=1}^\infty e^{-\sqrt{2k} z} \left(1 - e^{-\sqrt{2k} z}\right) |K_1(k, t)| \, dk \right)^2 \, dz.
$$
Hence, combining Lemma 3.1 for the $L^2$-boundedness of the Laplace transform with estimate (3.21), we find

$$
\|J_{21}(t)\|_{2}^{2} \lesssim \int_{k=1}^{\infty} \frac{k^2 |\hat{h}(k^2)|^2 k^{2\beta}}{(e^{\frac{\sqrt{2}}{2} kt} - e^{-\frac{\sqrt{2}}{2} kt})^2} dk \lesssim \int_{k=1}^{\infty} \frac{k^2 |\hat{h}(k^2)|^2 k^{2\beta}}{(e^{\frac{\sqrt{2}}{2} kt} - e^{-\frac{\sqrt{2}}{2} kt})^2} dk,
$$

(6.14)

with the last inequality due to the fact that $|e^{ia^3 k t} - e^{-ia^3 k t}| \geq e^{\frac{\sqrt{2}}{2} kt} - e^{-\frac{\sqrt{2}}{2} kt}$ for $k \geq 0$.

Next, let $\psi(k) = e^{\frac{\sqrt{2}}{2} kt} - e^{-\frac{\sqrt{2}}{2} kt}$ and note that $\psi(k) > 0$ for $k \geq 1$ (importantly, $\psi(k) = 0$ only for $k = 0$, i.e. the integrand of (6.14) is non-singular). Moreover, observe that $\psi$ is infinitely differentiable and, in particular, $\psi$ is increasing on $[1, \infty)$. Hence, $0 < e^{\frac{\sqrt{2}}{2} \ell} - e^{-\frac{\sqrt{2}}{2} \ell} \leq e^{\frac{\sqrt{2}}{2} kt} - e^{-\frac{\sqrt{2}}{2} kt}$ for $k \geq 1$. Thus, back to (6.14), we have

$$
\|J_{21}(t)\|_{2}^{2} \lesssim \int_{k=1}^{\infty} \frac{k^2 |\hat{h}(k^2)|^2 k^{2\beta}}{(e^{\frac{\sqrt{2}}{2} kt} - e^{-\frac{\sqrt{2}}{2} kt})^2} dk \lesssim \int_{k=1}^{\infty} \frac{k^{1+2\beta} |\hat{h}(k^2)|^2 k^{2\beta}}{H_t^{\frac{2\beta+1}{4}}(\mathbb{R})}.
$$

(6.15)

Concerning $J_{22}$, we note that $\|J_{22}(t)\|_{\beta} = \|\tilde{J}_{22}(t)\|_{\beta}$ where $\tilde{J}_{22}(x,t) = J_{22}(\ell - x, t)$. Therefore,

$$
\|J_{22}(t)\|_{\beta}^{2} = \|\tilde{J}_{22}(t)\|_{\beta}^{2} \lesssim \int_{x=0}^{t} \int_{z=0}^{t-x} \left( \int_{k=1}^{\infty} e^{-\frac{\sqrt{2}}{2} k x} |e^{ia^3 k z} - 1| |K_{2}(k,t)| dk \right)^{2} dz dx
$$

and employing once again Lemma 4.1 we infer

$$
\|J_{22}(t)\|_{\beta}^{2} \lesssim \int_{z=0}^{\ell} \frac{1}{z^{1+2\beta}} \int_{k=1}^{\infty} e^{-\frac{\sqrt{2}}{2} k x} (1 - e^{-\frac{\sqrt{2}}{2} k z}) |K_{2}(k,t)| dk \|^{2} dz.
$$

L_{2}([0,\infty), R)

The Laplace transform Lemma 3.1, estimate (3.21) and the fact that $|1 - e^{2ia^3 k t}| \geq 1 - e^{-\sqrt{2} k t}$ for all $k \geq 0$ imply

$$
\|J_{22}(t)\|_{\beta}^{2} \lesssim \int_{z=0}^{\sqrt{2} \ell} \frac{1}{z^{1+2\beta}} \|Q_z,t\|_{L_{2}([0,\infty), R)}^{2} dz \lesssim \int_{k=1}^{\infty} \frac{k^2 |\hat{h}(k^2)|^2}{(1 - e^{-\sqrt{2} k t})^{2}} k^{2\beta} dk.
$$

Similarly to the argument used earlier for $J_{21}$, for $k \geq 1$ we have $0 < 1 - e^{-\sqrt{2} k t} \leq 1 - e^{-\sqrt{2} k t}$. Hence,

$$
\|J_{22}(t)\|_{\beta}^{2} \lesssim \int_{k=1}^{\infty} \frac{k^2 |\hat{h}(k^2)|^2}{(1 - e^{-\sqrt{2} k t})^{2}} k^{2\beta} dk \lesssim \int_{k=1}^{\infty} \frac{k^{1+2\beta} |\hat{h}(k^2)|^2 k^{2\beta}}{H_t^{\frac{2\beta+1}{4}}(\mathbb{R})}.
$$

(6.16)

Overall, combining estimates (6.12), (6.15) and (6.16) with the writing (6.11), we find

$$
\|v_{1}(t)\|_{\beta} \lesssim \|h\|_{H_{t}^{\frac{2\beta+1}{4}}(\mathbb{R})}, \quad s = \beta \in (0,1).
$$

(6.17)

The norm $\|v_{1}(t)\|_{L_{2}([0,\ell), R)}$ will also be estimated using the splitting (6.11). In particular, for $J_{1}$ we employ Lemma 4.1 to infer

$$
\|J_{1}(t)\|_{L_{2}([0,\ell), R)}^{2} \lesssim \int_{x=0}^{\ell} \left( \int_{k=0}^{1} \frac{e^{\frac{\sqrt{2}}{2} k x} - e^{-\frac{\sqrt{2}}{2} k x}}{|e^{ia^3 k \ell} - e^{-ia^3 k \ell}|} k |\hat{h}(k^2)| dk \right)^{2} dx.
$$

(6.18)

The ratio of exponentials involved in the $k$-integral is bounded by 1 for all $k \geq 0$. Thus, applying also Cauchy-Schwarz in $k$, we obtain

$$
\|J_{1}(t)\|_{L_{2}([0,\ell), R)}^{2} \lesssim \int_{x=0}^{\ell} \left( \int_{\tau=0}^{1} |\hat{h}(\tau)|^{2} d\tau \right) dx \lesssim \|h\|_{H_{t}^{\frac{2\beta+1}{4}}(\mathbb{R})}, \quad s \geq -\frac{1}{2}.
$$

(6.19)
Concerning \( J_2 \), similarly to \( (6.18) \) we have
\[
\| J_2(t) \|_{L^2_s(0,\ell)}^2 \lesssim \int_{x=0}^{\ell} \left( \frac{1}{\pi} \int_{k=1}^{\infty} \frac{e^{\frac{1}{2}k^2} e^{-\frac{1}{2}k^2} k^2}{e^i k^2 e^{-i k^2} k^2} \right)^2 dx.
\]

Note that \( |e^{i k^2} - e^{-i k^2}| \geq e^{\frac{1}{2}k^2} e^{-\frac{1}{2}k^2} \) and, furthermore,
\[
\frac{1}{e^{\frac{1}{2}k^2} e^{-\frac{1}{2}k^2}} \leq \frac{e^{-\frac{1}{2}k^2}}{1 - e^{-2\ell}}, \quad k \geq 1.
\]

Using this bound and the fact that \( e^{-\frac{1}{2}k^2} < 1 \) for all \( k \geq 1 \), we find
\[
\| J_2(t) \|_{L^2_s(0,\ell)}^2 \lesssim \int_{x=0}^{\ell} \left( \int_{k=1}^{\infty} \left( e^{\frac{1}{2}k^2} e^{-\frac{1}{2}k^2} e^{-\frac{1}{2}k^2} k^2 \right)^2 dx \right)^2 dx
\]
\[
\lesssim \int_{x=0}^{\ell} \left( \int_{k=1}^{\infty} e^{-k^2} k^2 \right)^2 dx \lesssim \int_{x=0}^{\ell} \left( \int_{k=1}^{\infty} e^{-k^2} k^2 \right)^2 dx.
\]

Therefore, by the Laplace transform Lemma 3.1 we get
\[
\| J_2(t) \|_{L^2_s(0,\ell)}^2 \lesssim \int_{1}^{\infty} k^2 \hat{h}(k^2)^2 dk = \int_{\tau=0}^{\infty} \hat{h}(\tau)^2 d\tau \lesssim \| h \|_{H^{\beta+1}_t((R))}^2, \quad s \geq 0.
\]

Combining \( (6.19) \) and \( (6.21) \), we obtain
\[
\| v_1(t) \|_{L^2_s(0,\ell)} \lesssim \| h \|_{H^{\beta+1}_t((R))}, \quad s \geq 0.
\]

(ii) The case \( |s| > 0 \). Now \( s = |s| + \beta \) with \( |s| \in N \setminus \{0\} \) and \( \beta \in [0,1) \). Thus, according to definition \( (6.10) \) we need to estimate the fractional norm \( \| \partial_{x}^{[s]} v_1(t) \|_{H\beta} \) and also the \( L^2 \)-norm \( \| \partial_{x}^{s} v_1(t) \|_{L^2_s(0,\ell)} \) for all integers \( 0 \leq j \leq |s| \). Both of those norms can be handled in exactly the same way as the norms \( \| v_1(t) \|_{H\beta} \) and \( \| v_1(t) \|_{L^2_s(0,\ell)} \) that were estimated in case (i) above, eventually yielding the space estimate \( (6.7) \) for \( v_1 \) for all \( s \geq 0 \). As noted earlier, the estimation of \( v_2 \) is similar.

Concerning the time estimate \( (6.8) \), making the change of variable \( k = \sqrt{\tau} \) in formula \( (6.6) \) we infer that the temporal Fourier transform of \( v \) is given by
\[
\hat{v}(x,\tau) \simeq \begin{cases} 
  e^{ia^3 \sqrt{\tau}x} - e^{-ia^3 \sqrt{\tau}x} \hat{h}(\tau), & \tau \geq 0, \\
  e^{ia \sqrt{\tau}x} - e^{-ia \sqrt{\tau}x} \hat{h}(\tau), & \tau < 0.
\end{cases}
\]

Hence, by the definition of the \( H^{2\beta+1}_t((R)) \)-norm it follows that
\[
\| v(x) \|_{H^{2\beta+1}_t((R))}^2 \lesssim \int_{\tau=0}^{\infty} \left( 1 + \tau^2 \right)^{\beta+1} \left( e^{ia^3 \sqrt{\tau}x} - e^{-ia^3 \sqrt{\tau}x} \hat{h}(\tau) \right)^2 d\tau
\]
\[
+ \int_{\tau=-\infty}^{0} \left( 1 + \tau^2 \right)^{\beta+1} \left( e^{ia \sqrt{\tau}x} - e^{-ia \sqrt{\tau}x} \hat{h}(\tau) \right)^2 d\tau.
\]
Lemma 4.1 and the fact that \( k \geq 0 \) imply that the ratios of exponentials in the above integrals are bounded by 1. Therefore,

\[
\|v(x)\|_{H_1^2} \lesssim \int_{\tau = 0}^{2x+1} (1 + \tau^2) \frac{2x+1}{4} \left[ \hat{h}(\tau) \right]^2 d\tau + \int_{\tau = -\infty}^{0} (1 + \tau^2) \frac{2x+1}{4} \left[ \hat{h}(\tau) \right]^2 d\tau = \|h\|_{H_1^2}^2 \frac{2x+1}{4} (\mathbb{R}),
\]

which implies the time estimate (6.8) since \( H_1^2 (\mathbb{R}) \) is the restriction of \( H_1^2 (\mathbb{R}) \) on \( (0, T) \). As for the half-line, continuity in \( t \) and \( x \) follows along the lines of Theorem 5 of [FHM1].

We complete the analysis of IBVP (6.4) with the estimation in \( C^\alpha ([0, T]; L_x^p (0, \ell)) \).

**Theorem 6.2** (\( C^\alpha ([0, T]; L_x^p (0, \ell)) \)-estimate for the pure IBVP). If \( \frac{1}{2} < s < \frac{1}{2} \), then the solution \( v = S[0, 0, h; 0] \) of the pure IBVP (6.4) given by the UTM formula (6.6) admits the estimate

\[
\|v\|_{C^\alpha ([0, T]; L_x^p (0, \ell))} \lesssim T^\alpha \|h\|_{H_1^2} \frac{2x+1}{4} (\mathbb{R}).
\]

**Proof of Theorem 6.2.** We write \( v = v_1 + v_2 \) as in the proof of Theorem 6.1 and note that \( v_1 \) and \( v_2 \) can be estimated in the same way; hence, we only provide the proof for \( v_1 \). By Minkowski’s integral, we have

\[
\|v_1(t)\|_{L_x^p (0, \ell)} \lesssim \int_{k=0}^{1} k \left| \hat{h}(k^2) \right| \left( \int_{x=0}^{\ell} \left| \frac{e^{ia3kx} - 1}{e^{ia3kx} - e^{-ia3kx}} \right| \frac{1}{p} dx \right)^{\frac{1}{p}} dk + \int_{k=1}^{\infty} k \left| \hat{h}(k^2) \right| \left( \int_{x=0}^{\ell} \left| \frac{e^{ia3kx} - e^{-ia3kx}}{e^{ia3kx} - e^{-ia3kx}} \right| \frac{1}{p} dx \right)^{\frac{1}{p}} dk.
\]

For (6.24a), we bound the ratio of exponentials involved in the \( x \)-integral by 1 and then apply Cauchy-Schwarz in \( k \) to obtain

\[
(6.24a) \lesssim \ell^\frac{1}{p} \int_{k=0}^{1} k \left| \hat{h}(k^2) \right| dk \lesssim \|h\|_{H_1^2} \frac{2x+1}{4} (\mathbb{R}), \quad s \geq -\frac{1}{2}.
\]

For (6.24b), we employ the bound (6.20) and note that \( e^{-\sqrt{2kx}} \leq 1 \) for \( k, x \geq 0 \) to infer

\[
(6.24b) \lesssim \int_{k=1}^{\infty} k \left| \hat{h}(k^2) \right| \left( \int_{x=0}^{\ell} e^{\frac{\sqrt{2k(\ell-x)p}}{p}} dx \right)^{\frac{1}{p}} dk \lesssim \int_{k=1}^{\infty} k^{\frac{1}{2} - \frac{1}{p}} \left| \hat{h}(k^2) \right| dk.
\]

Then, letting \( k = \sqrt{\tau} \) and using Cauchy-Schwarz, we get

\[
(6.24b) \lesssim \left( \int_{\tau=1}^{\infty} \tau^{\frac{1}{p} - 1} (1 + \tau^2) \frac{2x+1}{4} d\tau \right)^{\frac{1}{2}} \|h\|_{H_1^2} \frac{2x+1}{4} (\mathbb{R}) \lesssim \|h\|_{H_1^2} \frac{2x+1}{4} (\mathbb{R}),
\]

provided that \( s > \frac{1}{2} - \frac{1}{p} \). Combining estimates (6.25) and (6.26) for this range of \( s \), we obtain\n
\[
\|v_1(t)\|_{L_x^p (0, \ell)} \lesssim \|h\|_{H_1^2} \frac{2x+1}{4} (\mathbb{R}),
\]

which implies estimate (6.23) for \( v_1 \) since \( \alpha > 0 \). Note that the restriction \( s < 1 \) comes from the condition on \( b \) in the definition (1.2) of \( \alpha \) and not from the present proof.

Overall, in view of Theorems 6.1 and 6.2, inequalities (6.3) and (6.5), and Theorem 4.3 for the half-line IBVP (1.3), the superposition (6.1) yields the following result for the forced linear IBVP (1.7) on the finite interval.
**Theorem 6.3** (Estimates for the forced linear heat on the finite interval). For \( f = \prod_{j=1}^{P} f_j \), the solution \( u = S[u_0, g_0, h_0; f] \) of the forced linear heat IBVP (1.7) admits the space estimates

\[
\| S[u_0, g_0, h_0; f] \|_{C([0,T];H^s_x(0,\ell))} \leq c_{s,p} \left( \| u_0 \|_{H^s_x(0,\ell)} + \| g_0 \|_{H^{2s+1}_t(0,T)} + \| h_0 \|_{H^{2s+1}_t(0,T)} \right) + \sqrt{T} \prod_{j=1}^{P} \| f_j \|_{C([0,T];H^s_x(0,\ell))}, \quad \frac{1}{2} < s < \frac{3}{2},
\]

(6.27a)

and the \( L^p \)-estimate

\[
\| S[u_0, g_0, h_0; f] \|_{C^\alpha([0,T];L^p_x(0,\ell))} \leq c_{s,p} \left( \| u_0 \|_{H^s_x(0,\ell)} + \| g_0 \|_{H^{2s+1}_t(0,T)} + \| h_0 \|_{H^{2s+1}_t(0,T)} \right) + \sqrt{T} \prod_{j=1}^{P} \| f_j \|_{C^\alpha([0,T];L^p_x(0,\ell))}, \quad 0 \leq s < \frac{1}{2},
\]

(6.28a)

and

\[
\| S[u_0, g_0, h_0; f] \|_{C([0,T];H^{2s+1}_t(0,T))} \leq c_{s,p} \left( \| u_0 \|_{H^s_x(0,\ell)} + \| g_0 \|_{H^{2s+1}_t(0,T)} + \| h_0 \|_{H^{2s+1}_t(0,T)} \right) + \sqrt{T} \prod_{j=1}^{P} \| f_j \|_{C([0,T];H^s_x(0,\ell))}, \quad \frac{1}{2} < s < \frac{3}{2},
\]

(6.27b)

where \( c_{s,p} > 0 \) is a constant that depends only on \( s \) and \( p \).

**Local well-posedness on the finite interval.** The linear estimates of Theorem 6.3 allow us to carry out exactly the same contraction mapping argument with that of Section 5 for the half-line in order to establish Theorems 1.3 and 1.4 for the local well-posedness of the nonlinear IBVP (1.4) on the finite interval. Indeed, the only modification required in the proofs of Section 5 is the replacement of the solution spaces \( X \) and \( Y \) and of the data norm \( \| \cdot \|_{D} \) by their finite interval counterparts as stated in Theorems 1.3 and 1.4.

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