ADJOINT $L$-FUNCTIONS FOR $\text{GL}(3)$ AND $U_{2,1}$
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Abstract. We show that the finite part of the adjoint $L$-function (including contributions from all non-archimedean places, including ramified places) is holomorphic in $\text{Re}(s) \geq 1/2$ for a cuspidal automorphic representation of $\text{GL}_3$ over a number field. This improves the main result of [H18]. We obtain more general results for twisted adjoint $L$-functions of both $\text{GL}_3$ and quasisplit unitary groups. For unitary groups, we explicate the relationship between poles of twisted adjoint $L$-functions, endoscopy, and the structure of the stable base change lifting.

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1. Introduction

1.1. Motivation and Background. Let $F$ be a number field and $A$ be its ring of adeles. Let $H$ be a connected, reductive $F$-group, isomorphic to $\text{GL}_3$ over the algebraic closure $\overline{F}$ of $F$. The $L$-group $^L H$ of $H$ is then the semi-direct product of $\text{GL}_3(\mathbb{C})$ with a Galois or Weil group, depending on which form of the $L$-group we consider (cf. [Bo]). Differentiating the action of $^L H$ on its own identity

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component by conjugation, we obtain an action on \( \mathfrak{gl}_3(\mathbb{C}) \). The subspace \( \mathfrak{sl}_3(\mathbb{C}) \) is preserved. We denote the action of \( L_H \) on \( \mathfrak{sl}_3(\mathbb{C}) \) by \( \text{Ad} \).

In this paper we study the twisted adjoint \( L \)-function \( L(s, \pi, \text{Ad} \times \chi) \) where \( \chi \) is a Hecke character. This work is motivated by the following simple conjecture regarding the untwisted \( L \)-function in the split case.

**Conjecture 1.1.** Let \( \pi \) be an irreducible cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}) \), where \( \mathbb{A} \) is the adele ring of a global field. Let \( \text{Ad} \) denote the adjoint representation of \( \text{GL}_n(\mathbb{C}) \) on \( \mathfrak{sl}_n(\mathbb{C}) \). Then the global Langlands \( L \)-function \( L(s, \pi, \text{Ad}) \) is entire.

In the case \( n = 2 \), Conjecture 1.1 was proved by Gelbart-Jacquet [GeJ], generalizing a method of Shimura [Shim]. A different proof was given in [JZ]. Flicker [F] proved that under a certain assumption about ratios of Hecke \( L \) series (see “(Ass; E, \omega)”, [F, p. 233]), Conjecture 1.1 holds if \( \pi \cong \otimes_v \pi_v \) with at least one \( \pi_v \) supercuspidal. More precisely, he proved that for such \( \pi \), the twisted adjoint \( L \)-function \( L(s, \pi, \text{Ad} \times \chi) \) is entire unless \( \chi \) is nontrivial and \( \pi \cong \pi \otimes \chi \). This proof was based on a trace formula. Our approach is based on an integral representation, which was pioneered in [G], and a method of ruling out poles which was pioneered in [GJ].

Note that the action of \( \text{GL}_n \) on the space of \( n \times n \) matrices may be regarded as the tensor product of the standard representation with its own dual. It decomposes as the direct sum of the identity matrix equipped with trivial action. It follows that

\[
L(s, \pi, \text{Ad} \times \chi) = \frac{L(s, \pi \times \overline{\pi} \times \chi)}{L(s, \chi)},
\]

where \( \overline{\pi} \) is the contragredient of \( \pi \). From this perspective, Conjecture 1.1 may be viewed as saying that \( L(s, \pi \times \overline{\pi} \times \chi) \) should be “evenly divisible” by \( L(s, \chi) \). More concretely, it says that \( L(s, \pi \times \overline{\pi}) \) must vanish at each zero of \( \zeta(s) \), to at least the same order. (Cf. [F, p. 234].)

As explained in [JngR], Conjecture 1.1 is expected to be related to the conjecture that \( \zeta_K(s)/\zeta_F(s) \) is entire for a field extension \( K/F \). As explained in the introduction of [BG], since \( \zeta_K(s)/\zeta_F(s) \) is a product of Artin \( L \)-functions, the latter conjecture is a consequence of the Artin conjecture. This relationship is one display in Flicker’s conditional result. For more details about this relationship, see [JngR].

In the case when the global field is \( \mathbb{Q} \), Conjecture 1.1 may also be viewed from the point of view of the Selberg class. We recall that the Selberg class is a class of meromorphic functions \( \mathbb{C} \to \mathbb{C} \) introduced by Selberg in [Se]. For \( \pi \) a cuspidal automorphic representation of \( \text{GL}_n(\mathbb{A}) \), the finite \( L \)-function \( L_f(s, \pi \times \overline{\pi}) \) will be an element of the Selberg class, unless \( \pi \) is a counterexample to the Ramanujan conjecture. It then would follow from the conjectures in [Se] that \( L_f(s, \pi \times \overline{\pi}) \) must be the product of the Riemann zeta function and another element of the Selberg class (cf. remark ii at the bottom of p. 370 of [Se]). In other words it would follow that \( L_f(s, \pi, \text{Ad}) \) is itself an element of the Selberg class. But then it must be entire since elements of the Selberg class have no poles except possibly at one.

### 1.2. Main results

For \( n > 2 \) the adjoint \( L \)-function of \( \text{GL}_n \) is not accessible via the Langlands-Shahidi method. Some information can be obtained through the relationship (1.1) with the better-understood Rankin-Selberg \( L \)-function. In particular, one can get a global functional equation of the adjoint \( L \)-function \( L(s, \pi, \text{Ad} \times \chi) \) by the global functional equations of \( L(s, \pi \times \overline{\pi} \times \chi) \) and \( L(s, \chi) \). To obtain further information of the adjoint \( L \)-function in the case \( n = 3 \), our main tool here is an integral representation pioneered in [G] together with a method of ruling out poles pioneered in [GJ].

Ginzburg’s local zeta integral is based on an embedding of \( \text{SL}_3 \) into a split exceptional group of type \( G_2 \). The argument was adapted in [H12] to apply to special quasisplit unitary group \( \text{SU}_{2,1} \) (which depends on a fixed quasicrystal extension of \( F \)), which also embed into \( G_2 \). Let \( \mathcal{H} \) be one of these groups, regarded as a subgroup of \( G_2 \). Then Ginzburg’s global zeta integral is given by

\[
Z(\varphi, f_s) = \int_{\mathcal{H}(F) \backslash \mathcal{H}(\mathbb{A})} \varphi(g)E(g, f_s) \, dg,
\]
where \( \varphi \) is a cuspidal automorphic form on \( \text{GL}_3(\mathbb{A}) \) or \( \text{U}_{2,1}(\mathbb{A}) \) in a fixed irreducible cuspidal automorphic representation \( \pi \) and \( E(g, f_s) \) is an Eisenstein series on \( G_2 \). See [G, H18] or §5 for more details. In the split case (when \( H = \text{SL}_3 \)), among other things, Ginzburg proved that the above local zeta integral is Eulerian
\[
Z(\varphi, f_s) = \prod_v Z^*(W_v, f_{s,v}),
\]
with
\[
Z^*(W_v, f_{s,v}) = L(3s, \chi_v)L(6s-2, \chi_v^2)L(9s-3, \chi_v^3) \int_{N_2(F_v) \backslash H(F_v)} W_v(g) f_s(w_{\beta} g) dg,
\]
where \( \chi = \otimes_v \chi_v \) is a character of \( F^\times \backslash \mathbb{A}^\times \) which is part of the datum of the Eisenstein series, \( W_v \) is the Whittaker function associated with local component of \( \varphi \), \( N_2 \) is certain subgroup of \( H = \text{SL}_3 \), and \( w_\beta \) is the Weyl element associated with the long simple root \( \beta \) of \( G_2 \). Ginzburg also showed that at unramified places, the local zeta integral \( Z^*(W_v, f_{s,v}) \) represents the adjoint \( L \)-function \( L(s, \pi_v, \text{Ad} \times \chi_v) \).

It is worth noting that both \( Z(\varphi, f_s) \) depends only on the restriction of \( \varphi \) to \( \text{SL}_3 \), and hence that one could, in principle begin with an automorphic representation of \( \text{SL}_3(\mathbb{A}) \) instead of \( \text{GL}_3(\mathbb{A}) \). However, the unfolding argument in [G] results in a particular Whittaker integral of \( \varphi \). Thus, one must restrict attention to representations which are globally generic with respect to this particular character. Now, each cuspidal automorphic representation of \( \text{SL}_3(\mathbb{A}) \) may be obtained as one of the components in the restriction of some cuspidal automorphic representation of \( \text{GL}_3(\mathbb{A}) \) (unique up to twist). As explained in [BG, p.120], the phenomenon that the integral \( Z(\varphi, f_s) \) only depends on its restriction to \( \text{SL}_3(\mathbb{A}) \) is a reflection of the fact that the adjoint representation of \( \text{GL}_3(\mathbb{C}) \) factors through \( \text{PGL}_3(\mathbb{C}) \), the \( L \)-group of \( \text{SL}_3 \). Similarly, the local zeta integral \( Z^*(W_v, f_{s,v}) \) also only depends on \( W_v|_{\text{SL}_3(F_v)} \).

To obtain the holomorphy of \( L(s, \pi, \text{Ad} \times \chi) \), on one hand, one needs to analyze the properties of the global integral \( Z(\varphi, f_s) \); on the other hand, one needs to analyze the local zeta integral \( Z^*(W_v, f_{s,v}) \) at every ramified place \( v \). Because of the functional equation, it suffices to rule out poles in the half plane \( \text{Re}(s) \geq \frac{1}{2} \). After the pioneering work of Ginzburg [G] and Ginzburg-Jiang [GJ], some progress in this direction was obtained in [H18]. The goal of our first main result in this paper is to extend [H18, Theorem 6.1], which treats the partial \( L \)-function \( L^S(s, \pi, \text{Ad} \times \chi) = \prod_{v \in S} L(s, \pi_v, \text{Ad} \times \chi_v) \) attached to a finite set \( S = S(\pi, \chi) \) of places of \( F \). Here \( S(\pi, \chi) \) contains the set of all infinite places \( S_\infty \) and all of the ramified places of \( \pi \) and \( \chi \). The main result in [H18] states that for \( \chi = 1 \) or \( \chi \) unitary and \( \pi \not= \pi \times \chi \), the function \( L^S(s, \pi, \text{Ad} \times \chi) \) has no poles on the region \( \text{Re}(s) \geq \frac{1}{2} \). However, during the referee process for this paper, a gap in the proof of the main theorem in [H18] was pointed out. In this paper, we first explain how to close this gap, and then extend [H18, Theorem 6.1] to the finite part \( L \)-function \( L_f(s, \pi, \text{Ad} \times \chi) = L^S(S, \pi, \text{Ad} \times \chi) \):

**Theorem 1.2** (Theorem 5.4). Let \( \chi \) be a unitary Hecke character of \( F^\times \backslash \mathbb{A}^\times \) and \( \pi \) be an irreducible cuspidal automorphic representation of \( \text{GL}_3(\mathbb{A}) \), then the finite part of the adjoint \( L \)-function \( L_f(s, \pi, \text{Ad} \times \chi) \) is holomorphic on the region \( \text{Re}(s) \geq 1/2 \), except for a simple pole at \( \text{Re}(s) = 1 \) when \( \chi \) is non-trivial and \( \pi \cong \pi \times \chi \) (which forces \( \chi \) to be cubic).

As mentioned above, to prove Theorem 1.2, we need to analyze Ginzburg’s local zeta integral at places \( v \in S - S_\infty \), and compare the local zeta integral with the corresponding local \( L \)-function. More precisely, we need to show that for each place \( v \in S - S_\infty \), there exists a “test” Whittaker function \( W_v \) of \( \pi_v \) and a “test” section \( f_{s,v} \in I(s, \chi_v) \) such that the local zeta integral \( Z^*(W_v, f_{s,v}) \) “detects” all poles of \( L(s, \pi_v, \text{Ad} \times \chi_v) \) on the region \( \text{Re}(s) \geq 1/2 \), i.e., the quotient \( Z^*(W_v, f_{s,v})/L(s, \pi_v, \text{Ad} \times \chi_v) \) has no zeros on the region \( \text{Re}(s) \geq \frac{1}{2} \). The main obstacle here is that, at this time, we can not rule out the existence of a place \( v \in S - S_\infty \) such that \( \chi_v \) is unramified \( \pi_v \) is both ramified and non-tempered. In this case, \( \pi_v \) is of the form \( \text{Ind}_{B_3}^{\text{GL}_3(F_v)}([\alpha] \otimes \mu_2 \otimes \lceil -\alpha \rceil) \), where \( B_3 \) is the upper triangular Borel subgroup of \( \text{GL}_3(F_v) \), \( \alpha \) is a real number with \( 0 < \alpha < 1/2 \), and \( \mu_2 \) is a ramified character of \( F_v^\times \). It turns out that the required “test” Whittaker function comes from a certain new form of \( \pi_v \). More precisely, let \( \psi_v \) be an unramified additive character, and let \( W_\psi \in W(\pi_v, \psi_v) \) be
the Whittaker function associated with a new vector of $\pi_v$ with respect to the group

$$K_v = \begin{pmatrix} \mathfrak{o} & \mathfrak{o} & \mathfrak{p}^{-c} \\ \mathfrak{p}^c & 1 + \mathfrak{p}^c & \mathfrak{o} \\ \mathfrak{p}^c & \mathfrak{p}^c & \mathfrak{o} \end{pmatrix},$$

where $\mathfrak{o}$ is the ring of integers of $F_v$, $\mathfrak{p}$ is the maximal ideal of $\mathfrak{o}$ and $c$ is the conductor of $\pi_v$ in the sense of [JPSS81]. Here a new vector of $\pi_v$ with respect to $K_v$ is a nonzero vector in the space $\pi_v^{K_v}$. It is known that $W_v$ is unique (up to scalar). The main local ingredient to attack Theorem 1.2 is the following Whittaker function formula for $W_v$:

**Theorem 1.3** (See Theorem 3.5). We have $W_v(1) \neq 0$ and

$$W_v(\text{diag}(\varpi^m, 1, \varpi^{-m})) = \frac{q^{-2m}}{t_1} (t_1^{2m+1} - t_1^{-(2m+1)}) W_v(1),$$

where $\varpi$ is a fixed uniformizer of $F_v$, $q$ is the number of the residue field of $F_v$ and $t_1 = |\varpi|^p$.

In fact, Theorem 3.5 is slightly general than the above Theorem 1.3. After Theorem 1.3, a careful choice of test section $f_{x,v} \in I(s, \chi_v)$ (See §4 for the details) will give the desired property, i.e., $Z^\ast(W_v, f_{x,v})/L(s, \pi_v, \text{Ad} \times \chi_v)$ has no zeros in the region $\text{Re}(s) \geq 1/2$.

**Remark 1.4.** After [H18], Buttcane and Zhou [BuZh, Theorem 2.4] were able to show that the adjoint $L$-function of a Maass form for $\text{SL}_3(\mathbb{Z})$ must be entire. Their argument relies on a simple comparison between where the Gamma factor can have poles and where the Riemann zeta function can have zeros. Our result gives an extension of theirs to Maass forms for congruence subgroups of $\text{SL}_3(\mathbb{Z})$.

**Remark 1.5.** Theorem 1.2 shows that $L_f(s, \pi, \text{Ad})$ has no poles in the region $\text{Re}(s) \geq \frac{1}{2}$. To prove Conjecture 1.1 in the case $n = 3$, one still needs to analyze Ginzburg’s local zeta integral $Z(W_v, f_{x,v})$ for any archimedean place $v$. In this direction, recently F. Tian is able to show the meromorphic continuation, local functional equation at archimedean places in [Ti18b]. Moreover, the local gamma factors for principal series representations at archimedean places is explicitly computed in [Ti18a, Ti18c].

The second part of this paper is devoted to the nonsplit case, i.e., when $H = \text{SU}_{2,1}$ is a quasisplit unitary group attached to a quadratic extension $E$ of $F$, and $\pi$ is a globally generic, irreducible cuspidal automorphic representation of $\text{U}_{2,1}(\hat{A})$. In this case, we have not performed a careful analysis on local zeta integrals at ramified primes, because there is still a lot to say about the simpler question of when the partial $L$ function attached to all finite unramified places will have a pole. As mentioned, the adaptation of Ginzburg’s integral representation to apply in nonsplit case was carried out in [H12]. The adaptation of Ginzburg-Jiang’s method of ruling out poles was carried out in [H18], with interesting results.

The key to Ginzburg and Jiang’s result is a “first term identity” relating residues of Eisenstein series attached to different parabolics. Using such an identity, they showed that if the global integral from [G] has a double pole, then a global integral involving an Eisenstein series on the other maximal parabolic subgroup of $G_2$ must be nonzero. They then showed that this global integral unfolds to a period integral which vanishes on every cuspidal representation of $\text{GL}_3(\mathbb{A})$. When this argument is adapted to the nonsplit case the resulting period, which is on $\text{SU}_{1,1}$, does not vanish on every cuspidal representation of $\text{U}_{2,1}(\mathbb{A})$. Rather, it fails to vanish, by earlier work of Gelbart, Rogawski, and Soudry [GeRoSo93], precisely on the image of an endoscopic lift constructed in [Ro]. By carefully combining this information with information about the relationship between Rogawski’s liftings and the stable base change lift of Kim and Krishnamurthy [KK1, KK2], as well as results about the image of that lift obtained by Ginzburg, Rallis and Soudry by the method of functorial descent [GRS11], we obtain the following main result.

**Theorem 1.6** (see Theorem 6.8). Take $\pi$ a globally generic irreducible cuspidal automorphic representation of the quasisplit unitary group $\text{U}_{2,1}(\mathbb{A})$ attached to a quadratic extension $E$ of $F$. Let $\chi_{E/F}$ be the quadratic character attached to $E/F$ by class field theory. Then $L^S(s, \pi, \text{Ad})$ is holomorphic
and nonvanishing at $s = 1$, while $L^S(s, \pi, \text{Ad} \times \chi_{E/F})$ can have at most a double pole. More precisely, we have three sets of equivalent conditions.

1. The following are equivalent:
   a. $L^S(s, \pi, \text{Ad} \times \chi_{E/F})$ is holomorphic and nonvanishing at $s = 1$
   b. the stable base change lift, $\text{sbc}(\pi)$, of $\pi$ is cuspidal
   c. $\pi$ is not endoscopic
   d. $\pi$ has a nonzero $SU_{1,1}$ period

2. The following are equivalent:
   a. $L^S(s, \pi, \text{Ad} \times \chi_{E/F})$ has a simple pole at $s = 1$
   b. $\text{sbc}(\pi)$ is the isobaric sum of a character and a cuspidal representation of $GL_2(\mathbb{A}_E)$
   c. $\pi$ is an endoscopic lift from $U_{1,1}(\mathbb{A}) \times U_1(\mathbb{A})$, but not from $U_1(\mathbb{A}) \times U_1(\mathbb{A}) \times U_1(\mathbb{A})$.

3. The following are equivalent:
   a. $L^S(s, \pi, \text{Ad} \times \chi_{E/F})$ has a double pole at $s = 1$
   b. $\text{sbc}(\pi)$ is the isobaric sum of three characters of $\mathbb{A}_E^\times$
   c. $\pi$ is an endoscopic lift from $U_1(\mathbb{A}) \times U_1(\mathbb{A}) \times U_1(\mathbb{A})$.

The twist by $\chi_{E/F}$ is of special importance for at least two reasons. First, in the nonsplit case the local zeta integral $Z^*(W_v, f_v, s)$ is equal to $L(s, \pi_v, \text{Ad} \times \chi_{E/F}, v(\chi_v))$ as opposed to $L(s, \pi_v, \text{Ad} \times \chi_v)$. (See [H12].) Second, the analogue of (1.1) in the nonsplit case relates $L(s, \pi, \text{Ad} \times \chi)$ to the Asai $L$-function $L(s, \text{sbc}(\pi), \text{Asai})$ of the stable base change lift $\text{sbc}(\pi)$ ([KK1, KK2]) of $\pi$. Recall that $\text{sbc}(\pi)$ is an automorphic representation of $GL_3(\mathbb{A}_E)$. The twist by $\chi_{E/F}$ is of particular importance because for a cuspidal representation $\Pi$ of $GL_2(\mathbb{A}_E)$, the $L(s, \Pi, \text{Asai}) L(s, \Pi, \text{Asai} \times \chi_{E/F})$ is equal to the Rankin-Selberg convolution $L$-function of $\Pi$ with the automorphic representation of $GL_3(\mathbb{A}_E)$ obtained by composing $\Pi$ with the nontrivial element of Gal($E/F$). (cf [GRS11, p.317].) We also obtain results which characterize when an adjoint $L$ function twisted by a character other than $\chi_{E/F}$ can have a pole. See theorems 6.9, 6.11 and proposition 6.14.

Because it relies on an embedding of SL$_3$ into G$_2$, Ginzburg’s integral representation does not generalize to GL$_n$ for general $n$. However, it was shown in [BG] that a certain global integral in the split exceptional group F$_4$ represents the adjoint $L$-function of GL$_4$, and some evidence was presented in [GH08] that a certain global integral in the exceptional group E$_8$ represents the adjoint $L$-function of GL$_5$. In both cases, it appears possible to adapt the construction to apply to quasi-split unitary groups as well. It is expected similar results could be done for adjoint representations of GL$_4$ and GL$_5$ by studying these integral representations.

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2. Local zeta integral for the adjoint representation of GL$_3$ and U$_{2,1}$

In this section, we review Ginzburg’s local zeta integral [G] for the adjoint representation of GL$_3$ and prove the local functional equation for these local zeta integrals. Ginzburg’s construction was extended to the U$_{2,1}$ case in [H12]. Let $F$ be a local field in this section. If $F$ is a non-archimedean local field, let $\mathfrak{o}$ be the ring of integers of $F$ and let $q$ be the number of the residue field of $F$. We also fix a uniformizer $\varpi$ of $F$ when $F$ is non-archimedean.

The local zeta integrals defined by Ginzburg involve the unique split exceptional group of type G$_2$. We denote this group G$_2$ and realize it as a set of $8 \times 8$ matrices as in [H12]. Let $B = TU$ be the upper triangular Borel subgroup of G$_2$ under the realization in [H12] with torus $T$ and maximal unipotent subgroup $U$. 
The group $G_2$ has two simple roots $\alpha, \beta$, where $\alpha$ is the short root and $\beta$ is the long root. Then the set of positive roots of $G_2$ is \{\alpha, \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\alpha + 2\beta\}. We denote the set of all roots by $\Phi$ and the set of positive roots by $\Phi^\dagger$. For a root $\delta$, we denote $U_\delta$ the corresponding root space of $\delta$ and $\mathfrak{g}_\delta : \mathfrak{g}_a \rightarrow U_\delta$ a choice of isomorphism. We let $X_\delta = d\mathfrak{g}_\delta(1)$. (Here, $d\mathfrak{g}_\delta$ is the differential.) We assume that $X_\alpha = E_{12} + E_{14} + E_{35} - E_{46} - E_{56} - E_{78}$, $X_\beta = E_{23} - E_{67}$, and that the family $\{x_\delta : \delta \in \Phi\}$ is chosen as in $[G]$, where $E_{ij} \in \text{Mat}_{8 \times 8}(F)$ is the matrix such that its $i$-th row and $j$-th column is 1 and zero elsewhere. (See also $[\text{Re}e]$). For a root $\delta$, denote $w_\delta(t) = x_\delta(t)x_{-\delta}(-t^{-1})x_\delta(t)$ and $w_\delta = w_\delta(1)$. Let $h_\delta(t) = w_\delta(t)w_\delta^{-1}$. For $a, b \in F^\times$, we denote $h(a, b) = h_\alpha(ab)h_\beta(a^2b)$. Then $T(F) = \{h(a, b) : a, b \in F^\times\}$, and $U = \prod_{\delta \in \Phi} U_\delta$.

We briefly recall some facts about $G_2$ and the particular realization of it given in $[\text{H}12]$. For more details see $[\text{H}12]$ and references therein. The group $G_2$ can be realized as the fixed points of an order three automorphism of $\text{Spin}_8$. The embedding into eight dimensional matrices in $[\text{H}12]$ is obtained by embedding into $\text{Spin}_8$ and then projecting to $\text{SO}_8$. This projection is actually injective on $G_2$. Thus a symmetric bilinear form is preserved. For the realization in $[\text{H}12]$ it is the form attached to the matrix $J$ with ones on the diagonal running from top right to lower left, and zeros elsewhere. The standard representation of $\text{SO}_8$ does not restrict to an irreducible representation of $G_2$. It decomposes as a one dimensional space on which $G_2$ acts trivially and a copy of the seven dimensional “standard” representation of $G_2$. In $[\text{H}12]$, the invariant one dimensional space is spanned by $v_0 := [0 \ 0 \ 0 \ 1 \ -1 \ 0 \ 0 \ 0]$. The seven dimensional “standard” representation of $G_2$ supports an invariant quadratic form $Q$. (In $[\text{H}12]$, it’s obtained by restricting the form induced by $J$ to the orthogonal complement of $v_0$.) The vectors satisfying $Q(v) = c$ form a single $G_2(F)$-orbit for each $c \in F^\times$. The stabilizer of such a vector is isomorphic to $\text{SL}_3$, either over $F$, or over a quadratic extension depending on $c$. For example, the stabilizer $H_\rho$ of $v_\rho := [1 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0 \ 0]$ is isomorphic to $\text{SL}_3$ over the smallest extension of $F$ in which $\rho$ is a square. Indeed, if $\rho = \tau^2$ then $h_\beta(\frac{1}{\sqrt{\tau}})x_\alpha(\frac{1}{\sqrt{\tau}})x_{-\alpha}(-\tau), v_\rho = [0 \ 0 \ 0 \ -\tau \ -\tau \ 0 \ 0 \ 0]$. (In fact, $x_\alpha(\frac{1}{\sqrt{\tau}})x_{-\alpha}(-\tau), v_\rho = [0 \ 0 \ 0 \ -\tau \ -\tau \ 0 \ 0 \ 0]$, and then acting by $h_\beta(\frac{1}{\sqrt{\tau}})$ has no effect. The reason for including $h_\beta(\frac{1}{\sqrt{\tau}})$ will be clear in a moment.) The stabilizer of this vector in $\mathfrak{g}_2$ is the image of the embedding $i : \mathfrak{sl}_3 \rightarrow \mathfrak{g}_2$ given by

$$i \begin{pmatrix} a & b & c \\ d & e & f \\ g & h & -a - e \end{pmatrix} = \begin{pmatrix} a + e & a & -f & -c \\ d & b & 0 & c \\ e & 0 & f & 0 \\ -h & -e - b & -g & -a - e \end{pmatrix}.$$  

A general element of $\mathfrak{h}_\rho$ is given by

$$\rho a \quad T_1 \quad -pd \quad pe \quad pe \quad d \quad 0 \quad -f$$

$$h \quad 0 \quad a \quad a \quad 0 \quad -d \quad -e$$

$$-pl \quad -h \quad \rho a \quad 0 \quad 0 \quad -a - pe \quad -d$$

$$-pl \quad -h \quad \rho a \quad 0 \quad 0 \quad -a - pe \quad -d$$

$$k \quad 0 \quad pl \quad h \quad h \quad -l \quad -T_1 \quad -a$$

$$0 \quad -k \quad phi \quad pl \quad pl \quad h \quad -pa \quad -T_1$$

Conjugating by $h_\beta(\frac{1}{\sqrt{\tau}})x_\alpha(\frac{1}{\sqrt{\tau}})x_{-\alpha}(-\tau)$ yields the image under the injection $i$ of the matrix

$$\begin{pmatrix} -a \tau + T_1 & e \tau - d & -\frac{\sqrt{\tau}}{d} \\ 2lp - 2h \tau & 2a \tau & -e \tau - d \\ -2k \tau & 2lp + 2h \tau & -a \tau - T_1 \end{pmatrix}.  \quad (2.1)$$
(this is where \( h_3(\frac{1}{2}) \) is needed) which is a general element of \( SL_3(F) \) if \( \tau \in F \). If \( \tau \notin F \), then (2.1) is a general element of

\[
\left\{ X \in sl_3(F(\tau)) : X \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} + \begin{pmatrix} -1 & 1 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix} X = 0 \right\},
\]

where \(-\) denotes the action of the nontrivial element of \( \text{Gal}(H_\tau/H_{12}) \) which is defined by

\[
\chi \begin{pmatrix} 1 & r & t \\ 0 & 1 & r \tau \\ 0 & 0 & 1 \end{pmatrix} \cdot x = \chi \begin{pmatrix} 1 & r & t \\ 0 & 1 & r \tau \\ 0 & 0 & 1 \end{pmatrix} x 
\]

where \( x \) is a general element of \( \text{Gal}(H_\tau/H_{12}) \). In [H12], the group \( U_{2,1} \) is defined (relative to a choice of quadratic extension) using the matrix \( G(\begin{pmatrix} 1 & 1 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}) \), i.e.,

\[
U_{2,1} = \left\{ g \in GL_3(F(\tau)) : \begin{pmatrix} 1 & r & t \\ 0 & 1 & r \tau \\ 0 & 0 & 1 \end{pmatrix} g \right\}.
\]

For consistency with [H12] we compose conjugation by \( \text{diag}(1,1,-1) \) in \( GL_3 \) with \( i \) followed conjugation by \( (h_3(\frac{1}{2}),x_\alpha,\chi_\alpha) \) in \( G_2 \) to obtain an injection \( SU_{2,1} \hookrightarrow G_2 \) in the case \( \tau \notin F \). In the case \( \tau \in F \) we obtain an injection \( SL_3 \hookrightarrow G_2 \), which is slightly different from the one used in [G].

Let \( P = MU^\alpha \) be the parabolic subgroup of \( G_2 \) such that \( U^\alpha \) is contained in the Levi \( M \cong GL_2 \) of \( P \). Here \( U^\alpha \) is the unipotent radical of \( P \), which is the product of the root subgroups of \( \beta, \alpha + \beta, 2\alpha + \beta, 3\alpha + \beta, 3\beta \). Then \( B^\beta := P \cap H_\rho \) is a Borel subgroup of \( H_\rho \). We denote its unipotent radical \( U_\rho \) and its maximal torus \( T_\rho \).

2.1. **Induced representations.** Let \( N_{2,\rho} = U_\rho \cap w_\beta P w_\beta^{-1} \). Then the image of \( N_{2,\rho}(F) \) in \( GL_3(F(\tau)) \) is

\[
\left\{ \begin{pmatrix} 1 & r \tau & t \tau + r^{2} \beta \\ 0 & 1 & r \tau \\ 0 & 0 & 1 \end{pmatrix} , r, t \in F \right\}.
\]

In the case \( \tau \in F \), this simplifies to

\[
\left\{ \begin{pmatrix} 1 & r & t \\ 0 & 1 & r \\ 0 & 0 & 1 \end{pmatrix} , r, t \in F \right\}.
\]

This is a little different from [G], where \( N_2 \) is

\[
\left\{ \begin{pmatrix} 1 & r & t \\ 0 & 1 & -r \\ 0 & 0 & 1 \end{pmatrix} , r, t \in F \right\}.
\]

The reason for the difference is the extra conjugation by \( \text{diag}(1,1,-1) \).

The Levi subgroup \( M \) is generated by elements \( x_\alpha(r), x_{-\alpha}(r), h(a,b) \). Note that \( h_\alpha(r) = h(1/r, r^2) \).

We consider the isomorphism \( M \rightarrow GL_2(F) \) defined by

\[
x_\alpha(r) \mapsto \begin{pmatrix} 1 & r \\ 0 & 1 \end{pmatrix},
\]

\[
h(a,b) \mapsto \begin{pmatrix} a & b \\ 0 & a \end{pmatrix}.
\]

For \( m \in M \), we define \( \det(m) \) using the isomorphism \( M \cong GL_2 \).

Let \( \delta_\rho \) be the modulus character of \( P \). One can check that \( \delta_\rho(m) = |\det(m)|^3 \) for \( m \in M \). Let \( \chi \) be a character of \( F^\times \), we consider the normalized induced representation

\[
I(s,\chi) = \text{Ind}_{G_2}^{G_2}(\chi_{s-1/2}),
\]

where \( \chi_s \) is the character of \( M \) defined by \( \chi(\det(m))\delta_\rho^s(m) \). Note that

\[
\chi_s(h(a,b)) = \chi(a^2b)|a^2b|^{3s}.
\]

Let \( \tilde{\omega} = w_\beta w_\alpha w_\beta w_\alpha w_\beta \). Then \( \tilde{\omega} \) represents the unique Weyl element such that \( \tilde{\omega}(\alpha) > 0 \) but \( \tilde{\omega}(U^\alpha) \) is in the opposite of \( U \). Consider the standard intertwining operator

\[
M_{\tilde{\omega}} : I(s,\chi) \rightarrow I(1-s,\chi^{-1})
\]

which is defined by

\[
M_{\tilde{\omega}}(f_s)(g) = \int_{U^s} f_s(\tilde{\omega}ug)du.
\]
By the general theory of intertwining operators, $M_{\tilde{s}}$ is absolutely convergent for $\text{Re}(s) \gg 0$ and can be meromorphically continued to all $s \in \mathbb{C}$.

2.2. **An exact sequence for induced representations of $G_2$.** For each $\rho \in F^\times$, we have the double coset decomposition $G_2 = Pw_\beta H_\rho \cup PH_\rho$, see [H12, Lemma 3]. The subgroup $T_0 := \{h(a,1) : a \in GL_1\}$ of $T$ is contained in $H_\rho$ for all $\rho$ and is the maximal $F$-split subtorus of $T_\rho$ when $\rho$ is not a square. We also have

$$H_\rho \cap w_\beta P w_\beta^{-1} = N_{2,\rho} \cdot T_0,$$

for all $\rho$. See [H12, pp. 198-199]. Moreover we have the relation

$$w_\beta h(a,1) = h(1,a)w_\beta. \tag{2.2}$$

On the other hand, recall that $H_\rho \cap P$ is the Borel subgroup $B_\rho$ of $H_\rho$. Mackey’s theory gives us an exact sequence of $H_\rho$-modules:

$$0 \to \text{ind}_{N_{2,\rho} \cdot T_0}^{H_\rho} (\chi_\rho^n) \to I(s,\chi) \to \text{n-Ind}_{B_\rho}^{H_\rho} (\chi_\rho) \to 0, \tag{2.3}$$

where ind means compact induction, n-Ind means non-normalized induction, $\chi_\rho^n$ is the character on $N_{2,\rho} \cdot T_0$ defined by

$$\chi_\rho^n(nh(a,1)) = \chi_\rho(h(a,1)) = \chi(a)|a|^{3s}, \quad (n \in N_{2,\rho}, a \in GL_1)$$

and $\chi_\rho$ is viewed as character on $B_\rho$ by restriction, since $B_\rho \subset P$. Here the embedding

$$\text{ind}_{N_{2,\rho} \cdot T_0}^{H_\rho} (\chi_\rho^n) \hookrightarrow I(s,\chi)$$

is defined as follows. For $f_\rho \in \text{ind}_{N_{2,\rho} \cdot T_0}^{H_\rho} (\chi_\rho^n)$, then the corresponding $\tilde{f}_\rho \in I(s,\chi)$ is defined by

$$\tilde{f}_\rho(muw_\beta g) = \chi_\rho(m)f_\rho(g),$$

where $m \in M, u \in U^\alpha, g \in H_\rho$, and

$$\tilde{f}_\rho(h) = 0, \text{ if } h \notin Pw_\beta H_\rho.$$

By (2.2), $\tilde{f}_\rho$ is well-defined.

2.3. **The local zeta integrals.** For $\rho$ in $F^\times$ fix $\tau$ with $\tau^2 = \rho$, and let $\tilde{H}_\rho$ be $GL_3$ when $\rho$ is a square and the quasi-split unitary group $U_{2,1}$ otherwise. Let $\pi$ be an irreducible generic representation of $\tilde{H}_\rho(F)$ and $\chi$ be a quasi-character of $F^\times$. Let $\psi$ be a nontrivial additive character of $F$.

The maximal unipotent subgroup $U_\rho$ is given by

$$\left\{ \left( \begin{array}{ccc} 1 & x+y\tau & \frac{x^2-y^2\rho}{2} + w\tau \\ 0 & 1 & -x+y\tau \\ 0 & 0 & 1 \end{array} \right) \right\},$$

If $\tau \in F$, this is simply a parametrization of the standard maximal unipotent of $SL_3$. Let $\psi_\rho : U_\rho(F) \to \mathbb{C}^\times$ be the character given by

$$\psi_\rho \left( \frac{1}{1} x+y\tau \frac{x^2-y^2\rho}{2} + w\tau \right) = \psi(x). \tag{2.4}$$

Note that $\psi_\rho|_{N_{2,\rho}} = 1$. Given $W \in W(\pi,\psi), f_\rho \in I(s,\chi)$, the local zeta integral of [G], [H12] is

$$Z(W, f_\rho) = \int_{N_{2,\rho}\backslash H_\rho(F)} W(g)f_\rho(w_\beta g)dg.$$

Observe that if $\tilde{f}_\rho \in \text{Ind}_{H_\rho}^{\tilde{H}_\rho}(\chi_\rho)$ is given in terms of $f_\rho \in \text{ind}_{N_{2,\rho} \cdot T_0}^{H_\rho} (\chi_\rho)$ as in section 2.2 then

$$Z(W, \tilde{f}_\rho) = \int_{N_{2,\rho}\backslash H_\rho(F)} W(g)f_\rho(g)dg.$$

Theorem 5.1 of [H18] states that for each $s_0$, there exists a choice of data $W, f_\rho$ (depending on $s_0$) such that $Z(W, f_\rho)$ is holomorphic and nonvanishing at $s_0$. However, there is a gap in the proof of
this theorem in the archimedean case, because it is never shown that $f_{s_0} \to Z(W, f_{s_0})$ is a continuous function of $f_{s_0} \in I(s_0, \chi)$, when $s_0$ is outside the domain of convergence. In the split case, it is shown in [Ti18a, Ti18b] that $Z$ is, in fact, continuous in both of its arguments. The technique, which was pioneered in [So95], should extend to the non-split case as well. In fact, the non-split case is easier, because the rank of the maximal split torus is only one. Here, we content ourselves with sketching how to close the gap in [H18].

**Lemma 2.1.** For any fixed $s_0 \in \mathbb{C}$ and $W$ in the Whittaker model of some irreducible unitary representation of $SU_{2,1}(\mathbb{R})$, the function $f_{s_0} \to Z(W, f_{s_0})$ is a continuous function $I(s_0, \chi) \to \mathbb{C}$.

**Proof.** Factoring the Haar measure on $SU_{2,1}(\mathbb{R})$ using the Iwasawa decomposition, we write the local zeta integral as

$$\int_K \int_0^\infty \int_{-\infty}^\infty f_x(w_{3\beta}x_\beta(r))W(d(t))\psi(tr)t^{3s-3} \, dt \, dk,$$

where $d(t) = \begin{pmatrix} 1 & t \\ 0 & 1 \end{pmatrix}$ in $SU_{2,1}(\mathbb{R})$. Arguing as in [So95, Lemma 1, p.197], or [Ti18b, Lemma 4.1], we show that continuity of $Z$ follows from continuity of the inner integral

$$\int_0^\infty W(t)t^{3s-3}Jf_x(t) \, dt, \quad Jf_x(t) := \int_{-\infty}^\infty f_x(w_{3\beta}x_\beta(r))\psi(tr)dr.$$

Arguing as in [JS88], we may write

$$W(t) = \sum_{\xi \in X} \varphi_\xi(t)\xi(t)$$

where $X$ is a finite set of finite functions on the maximal split torus, and, for each $\xi \in X$, $\varphi_\xi$ is a Schwartz function on $\mathbb{R}$.

Recall that a finite function on the multiplicative group of positive reals is of the form $t \to t^u(\log t)^n$ for some complex number $u$ and non-negative integer $n$. Hence it suffices to prove continuity of the mapping

$$f \mapsto \int_0^\infty Jf_x(t)\varphi(t)t^u(\log t)^n \, dt,$$

for any complex number $u$, nonnegative integer $n$, and Schwartz function $\varphi$. Since $Jf_x$ is just the Jacquet integral of an embedded $SL_2$ we have an asymptotic expansion as in [W92, 15.2]. This can be formulated as follows. Let $(z_n)_{n=1}^\infty$ be the sequence of complex numbers obtained by numbering the elements of $\{3s_0 + 2k : k = 0, 1, 2, \ldots\} \cup \{1 - 3s_0 + 2k : k = 0, 1, 2, \ldots\}$ in increasing order of real part. Then there is a sequence $(a_k)_{k=0}^\infty$ of continuous linear functionals $I(s_0, \chi) \to \mathbb{C}$ and a sequence $(A_k)_{k=1}^\infty$ of continuous functions $I(s_0, \chi) \to (0, \infty)$ such that for any non-negative integer $N$,

$$|Jf_{s_0}(t) - \sum_{k=0}^N a_k(f_{s_0})t^{z_k}| \leq A_{N+1}(f_{s_0})t^{z_{N+1}}, \quad \forall t < 1.$$

Let $Z(\varphi, u, n)$ be defined by

$$Z(\varphi, u, n) = \int_0^\infty \varphi(t)t^u(\log t)^n \, dt.$$

for $u > 0$ and by meromorphic continuation elsewhere. (Notice that $Z(\varphi u, 0)$ is a standard Tate zeta factor, while $Z(\varphi, u, n)$ is its $n$th derivative.) Then

$$\int_0^\infty Jf_x(t)\varphi(t)t^u(\log t)^n \, dt - \sum_{k=0}^n a_k(f_{s_0})Z(\varphi, u + z_k, n) + E(f_{s_0}),$$

where

$$|E(f_{s_0})| \leq A_{N+1}(f_{s_0})\int_0^\infty \Re(\varphi(t)t^{\Re(u+z_{N+1})}(\log t)^n) \, dt,$$

provided we choose $N$ sufficiently large to ensure that this last integral is convergent. Since each of the functions $a_k, k = 0, \ldots, N$ and $A_{N+1}$ tends to zero with $f_{s_0}$, it is now clear that (2.5) is continuous.

If $F$ is non-archimedean, we can prove a stronger form of the non-vanishing result.
Lemma 2.2. Let $F$ be non-archimedean. The local zeta integral $Z(W, f_s)$ is absolutely convergent for $\text{Re}(s) \gg 0$ and can be meromorphically continued to a rational function of $q^s$. Moreover, there exist choices of data $W, f_s$ such that $Z(W, f_s)$ is a nonzero constant.

The meromorphic continuation of the local zeta integral $Z(W, f_s)$ at the archimedean places in the split case is proved in [Ti18b].

Proof. The first statement follows from the Bruhat decomposition and the asymptotic behavior of $W$ on the torus element. We next consider the “moreover” part. To simplify the notation, we only deal with the split case, i.e., when $H_\rho \cong \text{SL}_3$. Let $K_3^m = (1 + \text{Mat}_{3 \times 3}(\mathbb{F}^m)) \cap \text{SL}_3(F)$ be the standard level $m$ congruence subgroup of $\text{SL}_3$. Recall that $B_3$ is the upper triangular Borel subgroup of $\text{GL}_3(F)$.

In this case $T_\rho \cong (F^\times)^2$. For each $a \in F^\times$, the group $T_\rho$ contains the element $h(a, 1)$, which is identified with $\text{diag}(a, 1, a^{-1})$ in $\text{SL}_3(F)$. Note that $h(a, b) \in T$ is identified with $\text{diag}(a, b, a^{-1}b^{-1}) \in \text{SL}_3(F)$.

Consider the following function on $\text{SL}_3(F)$:

$$f^m_s(g) = \begin{cases} 
0, & \text{if } g \notin (B_3 \cap \text{SL}_3(F))K_3^m \\
\chi'_s(nh(a, 1))\phi_1(b)\phi_2(x), & \text{if } g = nh(a, 1)n(x)h(1, b)k, n \in N_2, k \in K_3^m,
\end{cases}$$

where

$$n(x) = \begin{pmatrix} 1 & x & 0 \\ 0 & 1 & 0 \end{pmatrix} \in \text{SL}_3(F), \phi_1 \in \mathcal{S}(F^\times), \phi_2 \in \mathcal{S}(F).$$

To make $f^m_s$ well-defined, we need to require that $\phi_1, \chi'_s$ are constant on $K_3^m \cap T_\rho$, and $\phi_2$ is invariant under the translation of $\varpi^m a_F$. We have $f^m_s \in \text{ind}_{T_\rho}^{H_\rho} \chi'_s$. We now compute $Z(W, f^m_s)$. We assume $m$ is large enough such that $W(gk) = W(g)$ for $k \in K_3^m$. Then factoring the Haar measure on $H_\rho$ using the Iwasawa decomposition yields

$$Z(W, f^m_s) = c_1 \int_{(F^\times)^2 \times F} W(h(a, 1)n(x)h(1, b)) f^m_s(h(a, 1)n(x)h(1, b)) dx \frac{d^a d^b}{|a^3 b^2|}$$

$$= c_1 \int_{(F^\times)^2 \times F} W(n(ax)h(a, b))\chi'_s(h(1, a))\phi_1(b)\phi_2(x) dx \frac{d^a d^b}{|a^3 b^2|}$$

$$= c_1 \int_{(F^\times)^2} W(h(a, b))\phi_2(a)\phi_1(b)\chi'_s(h(1, a)) \frac{d^a d^b}{|a^3 b^2|},$$

where $c_1 = \text{Vol}(B_\rho(\chi))K_3^m$ and $\hat{\phi}_2(a) = \int_F \psi(ax)\phi_2(x) dx$.

Assume $m$ is sufficiently large that $a \mapsto \chi'_s(h(1, a))$ is trivial on $1 + \mathbb{F}^m$, and choose $\phi_1, \phi_2$ such that $\hat{\phi}_1 = \hat{\phi}_2$ is the characteristic function of $1 + \mathbb{F}^m$, we get $Z(W, f^m_s) = c_1 \text{vol}(1 + \mathbb{F}^m)^2 W(1)$. Clearly, this is constant and $W$ may be chosen so that it is nonzero. This concludes the proof. \[\Box\]

2.3.1. Dependence on $\psi$. In this section we discuss the dependence of the local Ginzburg zeta integral on the choice of additive character $\psi$. Let $\hat{F}$ be the Pontryagin dual of $F$. Then $\hat{F}$ is isomorphic to $F$, but not canonically: indeed if $\psi$ is any fixed nontrivial element of $\hat{F}$, then every other element of $\hat{F}$ is of the form $\psi^a(x) := \psi(ax)$ for some $a \in F$. The formula (2.4) gives an injection $\hat{F} \hookrightarrow \hat{U}_\rho(F)$. If $\pi$ is an irreducible $\rho$-generic representation of $\hat{H}_\rho(F)$, write $W(\pi, \psi_\rho)$ for the Whittaker model of $\pi$.

Lemma 2.3. Fix $a, \rho \in F^\times$. Then

$$(\psi^a)_\rho(u) = \psi_\rho(h(a, 1)uh(a, 1)^{-1}), \quad \forall u \in U_\rho(F).$$

Proof. Direct computation. \[\Box\]

Corollary 2.4. If $\pi$ is $\psi_\rho$ generic then it is $(\psi^a)_\rho$-generic for every $a$, and left translation by $h(a, 1)$ is an isomorphism $W(\pi, \psi_\rho) \rightarrow W(\pi, (\psi^a)_\rho)$.

Proposition 2.5. If $f_s \in I(s, \chi), W \in W(\pi, \psi_\rho)$, and $a \in F^\times$, then there exists $W' \in W(\pi, (\psi^a)_\rho)$ such that $Z(W', f_s) = \chi_s(h(1, a))^{-1} Z(W, f_s)$. 


Proof. Indeed $W'(g) := W(h(a,1)g)$ is an element of $W(\pi, (\psi^s)_T)$. A change of variable in the integral defining $Z(W, f_s)$, together with the identity $w_\beta h(a,1) = h(1,a)w_\beta$ shows that $Z(W', f_s) = \chi_s(h(1,a))^{-1}Z(W, f_s)$.

2.4. The local functional equation. In this subsection, we assume that $F$ is a non-archimedean local field. Let $\pi$ be an irreducible generic representation of $\tilde{H}_\rho$. It is known that $\pi|_{H_\rho}$ has finite length, see [GeK].

Lemma 2.6. Except for a finite number of $q^{-s}$, we have

$$\text{Hom}_{H_\rho}(n\cdot \text{Ind}_{B_{\rho}}^{H_\rho}(\chi_s), \pi) = 0.$$  

Proof. By the Frobenius reciprocity law, we have

$$\text{Hom}_{H_\rho}(n\cdot \text{Ind}_{B_{\rho}}^{H_\rho}(\chi_s), \pi) = \text{Hom}_{T_0}(\chi_s, \pi|_{U_\rho}).$$

Since $U_\rho$ is the maximal unipotent subgroup of the upper triangular Borel subgroup of both $H_\rho$ and $\tilde{H}_\rho$, we have $\dim \pi|_{U_\rho} < \infty$. The assertion follows.

Proposition 2.7. Excluding a finite number of $q^{-s}$, we have

$$\dim \text{Hom}_{H_\rho}(I(s, \chi), \pi) \leq 1.$$  

Proof. By the exact sequence (2.3) and Lemma 2.6, it suffices to show that

$$\dim \text{Hom}_{H_\rho}(\text{ind}_{N_2,\rho,T_0}^{H_\rho}(\chi'_s), \pi) \leq 1$$

except for a finite number of $q^{-s}$.

By Frobenius reciprocity law [BZ, Proposition 2.29], we have

$$\text{Hom}_{H_\rho}(\text{ind}_{N_2,\rho,T_0}^{H_\rho}(\chi'_s), \pi) = \text{Hom}_{N_2,\rho,T_0}(\chi'_s, \pi) = \text{Hom}_{T_0}(\chi'_s, \pi|_{N_2,\rho}),$$

where $\pi|_{N_2,\rho}$ is the Jacquet module. The Jacquet module $\pi|_{N_2,\rho}$ can be viewed as a representation of $T_0 \cdot U_\rho$. Since $N_2,\rho$ acts trivially on $\pi|_{N_2,\rho}$, we know that $\pi|_{N_2,\rho}$ can be viewed a representation of $T_0 \times U_\rho/N_2,\rho \cong \text{GL}_1 \times F$, where the action of $\text{GL}_1 \cong F^\times$ on $F$ is given by multiplication. As a representation of $F \cong U_\rho/N_2,\rho$, $\pi|_{N_2,\rho}$ is smooth. Denote $\sigma = \pi|_{N_2,\rho}$ and $V_\sigma$ the space of $\sigma$. Thus we have $S(F), V_\sigma = V_\sigma$. From the isomorphism induced by the Fourier transform $S(\tilde{F}) \cong S(F)$, we get $V_\sigma = S(\tilde{F}).V_\sigma$, i.e., $\sigma$ is smooth as a $S(\tilde{F})$-module. Thus by [BZ, Proposition 1.14] there exists a unique sheaf $\mathcal{V}$ on $S(\tilde{F})$ such that $\mathcal{V}_c \cong V_\sigma$, where $\mathcal{V}_c$ denotes the compact support sections in $\mathcal{V}$.

The action of $F^\times$ on $\tilde{F}$ has two orbits. Let $\psi$ be a nontrivial additive character of $F$, and let $O = \{\psi_a, a \in F^\times\}$. Then $O$ is the open orbit of the action of $F^\times$ on $\tilde{F}$, and its complement is the trivial character on $F$. We have the usual short exact sequence

$$0 \to \mathcal{V}_c(O) \to \mathcal{V}_c \to \mathcal{V}_c(0) \to 0,$$

where 0 denotes the zero character. Consider the element $\psi$ in $O$, which may be identified with the character $\psi$ of $U_\rho$ given in (2.4). The stalk of the sheaf $\mathcal{V}$ at the point $\psi$ is given by

$$(\pi|_{N_2,\rho})_{U_\rho/N_2,\rho,\psi} = \pi_{U_\rho,\psi} \cong C_\psi,$$

since $\pi$ is an irreducible representation of $\tilde{H}_\rho$ and $\psi$ is a generic character of the maximal unipotent subgroup $U_\rho$ of a Borel subgroup in $\tilde{H}_\rho$. The stabilizer of $\mathcal{V}$ at the point $\psi$ in $\text{GL}_1$ is $\{1\}$. Thus by [BZ, Proposition 2.23], we get

$$\mathcal{V}_c(O) = \text{ind}_{\text{GL}_1}(\mathcal{C}_\psi).$$

Similarly, we have $\mathcal{V}_c(0) = \pi_{U_\rho}$ which has finite dimension. Thus we get the short exact sequence

$$0 \to \text{ind}_{\text{GL}_1}(\mathcal{C}_\psi) \to \pi|_{N_2,\rho} \to \pi_{U_\rho} \to 0.$$

Since $\pi_{U_\rho}$ has finite dimension, after excluding a finite number of $q^s$, we have

$$\text{Hom}_{T_0}(\chi'_s, \pi|_{N_2,\rho}) = \text{Hom}_{\text{GL}_1}(\chi'_s, \text{ind}_{\text{GL}_1}(\mathcal{C}_\psi)) = \text{Hom}(\chi'_s, \mathcal{C}_\psi).$$
Since $\chi'_s$ and $C_{\psi}$ have dimension 1, we get
\[ \text{Hom}(\chi'_s, C_{\psi}) \]
has dimension 1.

**Corollary 2.8.** There exists a rational function $\gamma(s, \pi, \chi, \psi)$ of $q^s$ such that
\[ Z(W, M_{\psi}(f_s)) = \gamma(s, \pi, \chi, \psi)Z(W, f_s), \]
for all $W \in W(\pi, \psi), f_s \in I(s, \chi)$.

**Proof.** Since both $(W, f_s) \mapsto Z(W, f_s)$ and $(W, f_s) \mapsto Z(W, M_{\psi}(f_s))$ define a bilinear form on $\text{Hom}_{H_o}(I(s, \chi) \otimes \pi, 1)$. By Proposition 2.7, such bilinear form is unique up to a scalar. Thus there is factor $\gamma(s, \pi, \chi, \psi)$ such that $Z(W, M_{\psi}(f_s)) = \gamma(s, \pi, \chi, \psi)Z(W, f_s)$ for all $W \in W(\pi, \psi), f_s \in I(s, \chi)$. Let $W \in W(\pi, \psi), f^m_s \in I(s, \chi)$ be as in the proof of Lemma 2.2. Then $Z(W, f^m_s)$ is a non-zero constant, say $c$, by Lemma 2.2. Thus we have $\gamma(s, \pi, \chi, \psi) = c^{-1}Z(W, M_{\psi}(f^m_s))$, which is a rational function of $q^s$ by Lemma 2.2 again. 

Based on the relationship between the $L$-functions, it is reasonable to define the local gamma factors.

**Definition 2.9.** When $\rho$ is a square (so $H_{\rho} \cong \text{SL}_3$), let the twisted local adjoint gamma factor be given by
\[ \gamma(s, \pi, \text{Ad} \times \chi, \psi) = \frac{\gamma(s, (\chi \pi) \times \tilde{\pi}, \psi)}{\gamma(s, \chi, \psi)}, \]
where $\gamma(s, \pi, \chi, \psi)$ is the local Rankin-Selberg gamma factor defined by Jacquet-Piatetski-Shapiro-Shalika [JPSS83] and $\gamma(s, \chi, \psi)$ is the local gamma factor of Tate. When $\rho$ is a nonsquare (so $H_{\rho}$ is a unitary group), let the twisted local $\text{Ad}'$ gamma factor be given by
\[ \gamma(s, \pi, \text{Ad}' \times \chi, \psi) = \frac{\gamma(s, \text{Asai}(\pi), \chi \times \chi, \psi)}{\gamma(s, \chi, \psi)}, \]
where the denominator is the Tate gamma factor and the numerator is defined by the Langlands-Shahidi method.

Here $\text{Ad}'$ is defined as in [H18]. Following [H18] we define $\text{Ad}'$ to be $\text{Ad}$ in the split case. Then the unramified computations from [H12] show that in both the split and nonsplit cases,
\begin{equation}
\gamma(s, \pi, \chi, \psi) = \frac{\gamma(3s - 1, \pi, \text{Ad}' \times \chi, \psi)}{\gamma(3s - 2, \chi, \psi), \gamma(6s - 3, \chi^2, \psi)\gamma(9s - 5, \chi^3, \psi)},
\end{equation}
where the individual gamma factors on the right hand side are defined as in [JPSS83] or the Langlands-Shahidi method. (For $\text{GL}_3$ factors in the denominator, either of these other definitions reduces to the one in Tate’s thesis.) It is natural to expect that the local gamma factors in Corollary 2.8 is essentially the same as the local gamma factors defined in Definition 2.9, i.e., (2.6) should be true for all irreducible generic representation $\pi$ of $\tilde{H}_\rho$.

**Remark 2.10.** Over archimedean local field, the local functional equation of Ginzburg’s local zeta integral is proved in [Ti18a, Ti18b, Ti18c] recently. Moreover, it is verified in [Ti18a, Ti18c] that the local gamma factors for principal series representation of $\text{GL}_3(\mathbb{R})$ obtained from the local functional equation satisfies (2.6).

### 3. A Whittaker Function Formula for $\text{GL}_3$

In this section, we develop a Whittaker function formula for certain ramified induced representation of $\text{GL}_3$ over a $p$-adic field, see Theorem 3.5. This Whittaker function formula will be used to compute the Ginzburg’s local zeta integral in a special case, see Proposition 4.1, which is the main ingredient in the proof of one of our main theorem, Theorem 5.4.

In this section, let $F$ be a non-archimedean local field, $\mathfrak{o}$ the ring of integers of $F$, $\mathfrak{p}$ the maximal ideal of $\mathfrak{o}$, and $\varpi$ a fixed generator of $\mathfrak{p}$. Let $q = |\mathfrak{o}/\mathfrak{p}|$. 
3.1. Certain subgroups of $GL_3$. Let $B_3 = T_3 U_3$ be the upper triangular Borel subgroup of $GL_3(F)$ with diagonal torus $T_3$ and upper triangular unipotent subgroup $U_3$. Let $K = GL_3(o)$. Denote

$$w = \begin{pmatrix} 1 & 1 \\ & 1 \end{pmatrix},$$

and

$$u(x, y, z) = \begin{pmatrix} 1 & x & z \\ & 1 & y \\ & & 1 \end{pmatrix}, \hat{u}(x, y, z) = \begin{pmatrix} 1 & x & 1 \\ & z & y \\ & & 1 \end{pmatrix},$$

for $x, y, z \in F$.

For a nonnegative integer $n \geq 0$, we consider the subgroup

$$K'_n = \left( \begin{array}{ccc} o & o & o \\ o & o & o \\ p^n & p^n & 1 + p^n \end{array} \right)^\times$$

of $GL_3(F)$. Here and in the following, for a subset $A \subset \text{Mat}_{3\times3}(F)$ which is closed under multiplication, $A^\times$ is used to denote the subset $A^\times := \{a \in A : a^{-1} \text{ exists and } a^{-1} \in A\}$ of $A$. It’s clear that $A^\times$ is a group, whenever it is nonempty.

Given an irreducible smooth generic complex representation $(\pi, V)$ of $GL_3(F)$, we consider the $K'_n$-fixed subspace $V'(n) = V^{K'_n}$ of $V$. Denote $c = c_\pi = \min\{n|V'(n) \neq 0\}$, which is called the conductor of $\pi$. By [JPSS81], the space $V'(c)$ has dimension 1. A nonzero vector of $V'(c)$ is called a new form or new vector of $\pi$. Let $\psi$ be an unramified additive character of $F$. It is known that the epsilon factor $\epsilon(s, \pi, \psi)$ of $\pi$ has the form $Cq^{-c_\pi s}$, where $C \in \mathbb{C}^\times$, see [JPSS81, §5]. It is worth to note that there was an error in [JPSS81], which was fixed in [J12] and [Ma13].

We consider a variant of the above notions. Denote

$$\epsilon_n = \begin{pmatrix} 1 \\ \epsilon_n \end{pmatrix} \in GL_3(F),$$

and $K_n = \epsilon_n K'_n \epsilon_n^{-1}$. In matrix form, we have

$$K_n = \left( \begin{array}{ccc} o & o & p^{-n} \\ o & 1 + p^n & o \\ p^n & p^n & o \end{array} \right)^\times.$$

Given an irreducible smooth generic complex representation $(\pi, V)$ of $GL_3(F)$, denote $V(n) = V^{K_n}$, the subspace of $V$ which is fixed by $K_n$. Then $V(n) = \pi(\epsilon_n)V'(n)$. In particular we have $\dim V(\epsilon_n) = 1$.

**Remark 3.1.** Let $E/F$ be an unramified quadratic extension of $p$-adic fields and $U_{2,1}$ be the unitary group with 3 variables associated with $E/F$ realized by the matrix $w \in GL_3$. Let $o_E$ be the ring of integers of $E$ and $p_E$ be the maximal ideal of $o_E$. In [M13], Miyauchi developed a theory of local new forms for $U_{2,1}$ using the group

$$\left( \begin{array}{ccc} o_E & o_E & p_E^{-n} \\ p_E & 1 + p_E^n & o_E \\ p_E^{-n} & p_E & o_E \end{array} \right)^\times \cap U_{2,1}.$$

The group $K_n$ we choose is inspired by the above group considered by Miyauchi.

**Lemma 3.2.** Let $n \geq 1$.

1. Let $x, y \in F$. If $\hat{u}(x, 0, 0) \in B_3 K_n$, then $x \in p^n$. Similarly, if $\hat{u}(0, y, 0) \in B_3 K_n$, then $y \in p^n$.

2. Given $r \in F$, the element $w'(r) := \begin{pmatrix} 1 & 1 \\ r & 1 \end{pmatrix}$ is not in $B_3 K_n$. 


The function \( f(x,0,0) \) is for some \( n \) for all diag(\( \cdot \)) functional on \( \prod_3 \). We assume that the representation is stable and its limit is independent on the choice of the sequence \( F \). Then \( k \) " stands for stable integral. The linear map \( f(\text{diag}(a_1,a_2,a_3)) \) is equivalent to that \( f \). Let \( U^1 \subset U^2 \subset \ldots \) be a sequence of open compact subgroups of \( U_3 \) such that \( \bigcup_{k=1}^{\infty} U^k = U_3 \). For \( f \in I(\mu) \), the sequence of integrals
\[
\int_{U^k} f(wu)\psi^{-1}(u)du, k \geq 1
\]
is stable and its limit is independent on the choice of the sequence \( \{U^k, k \geq 1\} \), see [Sh, Proposition 3.2] and [CS, Corollary 1.8]. Denote
\[
\int_{U_3}^{st} f(wu)\psi^{-1}(u)du := \lim_{k \to \infty} \int_{U^k} f(wu)\psi^{-1}(u)du,
\]
where "\( st \)" stands for stable integral. The linear map \( f \mapsto \int_{U_3}^{st} f(wu)\psi^{-1}(u)du \) is a Whittaker functional on \( I(\mu) \).

In the rest of this section, we assume that \( \mu_1, \mu_3 \) are unramified and \( \mu_2 \) is ramified with conductor \( c \geq 1 \). Let \( t_i = \mu_i(\omega), i = 1, 3 \). By [GoJ, Theorem 3.4, page 36], we have \( \epsilon(s, I(\mu), \psi) = \prod_{i=1}^{3} \epsilon(s, \mu_i, \psi) \). Since \( \mu_1, \mu_3 \) are unramified, and \( \mu_2 \) has conductor \( c \), we have \( \epsilon(s, I(\mu), \psi) = Cq^{-cs} \) for some \( C \in \mathbb{C}^\times \). Thus \( c \) is also the conductor of the representation \( I(\mu) \). Consequently, we have \( \dim I(\mu)^{K_c} = 1 \).

We consider the following function \( f \) on \( GL_3(F) \). We require that \( \text{supp}(f) \subset B_3 K_c \) and
\[
f(\text{diag}(a_1,a_2,a_3)uK) = \mu_1(a_1)\mu_2(a_2)\mu_3(a_3)|a_1/a_3|, \forall a_1,a_2,a_3 \in F^\times, u \in U_3, k \in K_c.
\]
The function \( f \) is well-defined and right \( K_c \)-invariant. Thus \( f \) is a new form of \( I(\mu) \), i.e., \( f \in I(\mu)^{K_c} \).

In the following, we fix a nontrivial unramified character \( \psi \) of \( F \), and we consider the Whittaker function \( W_f \) associated with the new form \( f \):
\[
W_f(g) = \int_{U_3}^{st} f(wug)\psi^{-1}(u)du.
\]

**Lemma 3.4.** Let \( a_i \in F^\times \) with \( a_i \in \omega^{n_i} \) for \( i = 1, 2, 3 \). If \( W_f(\text{diag}(a_1,a_2,a_3)) \neq 0 \), then \( n_1 \geq n_2 \geq n_3 \).
Proof. For any \( x \in \mathfrak{o} \), we have
\[
\text{diag}(a_1, a_2, a_3) u(x, 0, 0) = u(a_1a_2^{-1}x, 0, 0) \text{diag}(a_1, a_2, a_3).
\]
Since \( u(x, 0, 0) \in K_c \) and \( W_f \) is right \( K_c \)-invariant, we have
\[
W_f(\text{diag}(a_1, a_2, a_3)) = \psi(a_1a_2^{-1}x) W_f(\text{diag}(a_1, a_2, a_3)), \forall x \in \mathfrak{o}.
\]
If \( W_f(\text{diag}(a_1, a_2, a_3)) \neq 0 \), we must have \( \psi(a_1a_2^{-1}x) = 1 \) for all \( x \in \mathfrak{o} \). Since \( \psi \) is assumed to be unramified, we have \( a_1a_2^{-1} \in \mathfrak{o} \). This shows that \( n_1 \geq n_2 \). Similarly, we can show that \( n_2 \geq n_3 \). □

Theorem 3.5. (1) We have \( W_f(1) \neq 0 \).

(2) Let \( a \in \mathbb{Z}^m \mathfrak{o}^3, b \in \mathbb{Z}^n \mathfrak{o}^3 \) with \( m, n \geq 0 \). We have
\[
W_f(\text{diag}(a, 1, b)) = \frac{q^{-(m+n)}}{t_1-t_3}(t_1t_3)^{-n}(t_1^{m+n+1} - t_3^{m+n+1}) W_f(1).
\]

Remark 3.6. We consider the embedding of \( \text{GL}_2 \) into \( \text{GL}_3 \) by
\[
\begin{pmatrix} a & b \\ c & d \end{pmatrix} \mapsto \begin{pmatrix} a & b \\ c & d \end{pmatrix}.
\]
Let \( \sigma \) be the unramified representation \( \text{Ind}_{B_2}^{\text{GL}_2}(\mu_1 \otimes \mu_3) \) of \( \text{GL}_2(F) \), where \( B_2 \) is the upper triangular subgroup of \( \text{GL}_2(F) \). Let \( W^\sigma \) be the unramified Whittaker function of \( \sigma \) normalized by \( W^\sigma(1) = 1 \). Then by Shintani’s formula for \( \text{GL}_2 \) [Shin], the above theorem says that
\[
W_f(\text{diag}(a, 1, a^{-1})) = W^\sigma(\text{diag}(a, a^{-1})) W_f(1).
\]
This property is what we need in the proof of the holomorphy of the finite part of the adjoint \( L \)-function for \( \text{GL}_3 \) on the region \( \text{Re}(s) \geq 1/2 \), see Proposition 4.1 and Theorem 5.4.

Remark 3.7. In [M14], Miyauchi developed a formula of ramified Whittaker functions for \( \text{GL}_n \) (in particular for \( \text{GL}_3 \)), associated with local new forms defined by the group \( K'_n \). But Miyauchi’s formula does not meet our purpose. It seems that there is no apparent relationship between our formula and Miyauchi’s formula, although \( K_n \) is conjugate to \( K'_n \).

We will prove Theorem 3.5 in the next section. The proof is by brute force computation. It would be interesting to know whether a proof which is more conceptual, and perhaps more amenable to generalization, can be found. As a preparation of the proof of the above theorem, we record the following useful lemma

Lemma 3.8. In the integrals appearing in the following, we fix the measure \( dx \) on \( F \) such that \( \text{vol}(\mathfrak{o}) = 1 \), and thus \( \text{vol}(\mathfrak{o}^c) = \text{vol}(\mathfrak{o}) - \text{vol}(\mathfrak{p}) = 1 - q^{-1} \).

(1) Let \( k \) be an integer, then
\[
\int_{p^k - p^{k+1}} \psi^{-1}(x) dx = \begin{cases} q^{-k}(1 - q^{-1}), & k \geq 0 \\ -1, & k = -1 \\ 0, & k \leq -2. \end{cases}
\]

(2) Let \( i \) be an integer. We have
\[
\int_{\mathfrak{o}^c} \mu_2(1 + \mathfrak{o}^i x) dx = \begin{cases} 1 - q^{-1} & i \geq c \\ -q^{-1} & i = c - 1, \\ 0, & i < c - 1. \end{cases}
\]

Proof. The first statement follows from
\[
\int_{p^k} \psi^{-1}(x) dx = \begin{cases} q^{-k}, & k \geq 0, \\ 0, & k < 0. \end{cases}
\]
The second statement can be proved in the same manner if \( i > 0 \). We consider the case \( i \leq 0 \). If \( i < 0 \), we have
\[
1 + \mathfrak{o}^i x = \mathfrak{o}^i(x + \mathfrak{o}^{-i}).
\]
The set \( \{ x + \varpi^{-i} : x \in \mathfrak{o}^\times \} \) is exactly \( \mathfrak{o}^\times \). Thus \( \int_{\mathfrak{o}^\times} \mu_2(1 + \varpi^i x) du = 0 \). If \( i = 0 \), we have

\[
\int_{\mathfrak{o}^\times} \mu_2(1 + x) dx = \int_{\mathfrak{o}^\times} \mu_2(1 + x) dx - \int_{\mathfrak{o}^\times} \mu_2(1 + x) dx = \int_{\mathfrak{o}} \mu_2(1 + x) dx - \int_{\mathfrak{p}} \mu_2(1 + x) dx.
\]

Here \( \int_{\mathfrak{o}} \mu_2(x) dx \) is understood as \( \sum_{k \geq 0} \int_{\mathfrak{o} \cap \mathfrak{p}^k} \mu_2(x) dx \), which is convergent to 0 since \( \mu_2 \) is ramified. Thus \( \int_{\mathfrak{o}^\times} \mu_2(1 + x) dx = -\int_{\mathfrak{p}} \mu_2(1 + x) dx \), which is \(-q^{-1}\) if \( c = 1 \) and 0 if \( c > 1 \). The assertion follows.

### 3.3. Proof of Theorem 3.5

We are going to compute

\[
W_f(\text{diag}(a,1,b)) = \int_{\mathfrak{U}} f(w \text{diag}(a,1,b)) \psi^{-1}(u) du,
\]

with \(|a| = q^{-m}, |b| = q^n\). We first fix the measure \( du \). For \( u = (x,y,z), x,y,z \in F \), we take \( du = dx dy dz \), where \( dx, dy, dz \) are additive measures on \( F \) such that Vol(\( \mathfrak{o} \)) = 1.

We can write the above integral as

\[
W_f(\text{diag}(a,1,b)) = \int_{\mathfrak{U}} f(wu(x,y,z) \text{diag}(a,1,b)) \psi^{-1}(x+y) dx dy dz.
\]

Since

\[
u(x,y,z) \text{diag}(a,1,b) = \text{diag}(b,1,a)w(x,y,z), \]

we get

\[
W_f(\text{diag}(a,1,b)) = \mu_1(b) \mu_3(a) |b/a| \int_{\mathfrak{U}} f(wu(a^{-1}x,by,a^{-1}bz)) \psi^{-1}(x+y) dx dy dz.
\]

Let \( U^k = \{ u(x,y,z) : x,y \in \mathfrak{p}^{-k}, z \in \mathfrak{p}^{-2k} \} \). Consider the integral

\[
I^k = I^k(m,n) = \int_{U^k} f(wu(a^{-1}x,by,a^{-1}bz)) \psi^{-1}(x+y) dx dy dz.
\]

Since \( \text{diag}(\mathfrak{o}^\times,1,\mathfrak{o}^\times) \subseteq K_c \), the right hand side of the above integral only depends on \( m,n,k \), which justifies the notation \( I^k(m,n) \). We then have

\[
W_f(\text{diag}(a,1,b)) = q^{n+m} t_1^{-n} t_3^n \lim_{k \to \infty} I^k.
\]

To compute \( I^k \), we will frequently use the following identity

\[
u(x,y,z) = \text{diag}(-1/(z-xy), 1-(xy)/z, z)
\]

\[
\cdot u \left( \frac{x(z-xy)}{z}, \frac{zy}{z-xy}, -z+x+y \right) u \left( -\frac{y}{-z-xy}, \frac{x}{z}, \frac{1}{z} \right),
\]

if \( z \neq 0, z-xy \neq 0 \). Take \( x = y = 0, z = \varpi^{-c} \) in Eq. (3.4), we get

\[
u(0,0,\varpi^{-c}) = \text{diag}(-\varpi^{-c},1,\varpi^{-c})u(0,0,-\varpi^{-c})u(0,0,\varpi^{-c}).
\]

Since \( u(0,0,\varpi^{-c}), u(0,0,\varpi^{-c}) \in K_c \), we can get

\[
f(w) = (t_1 t_3^{-1}) q^{-2} c
\]

from the above equation.

We now start to compute the integral \( I^k \). In the following computation, \( k \) is sufficiently large.

We have

\[
I^k = \int_{\mathfrak{p}^{-2k} \times \mathfrak{p}^{-k}} \int_{\mathfrak{p}^{-k}} f(wu(a^{-1}x,0,a^{-1}b(z-xy)u(0,yb,0))) \psi^{-1}(x+y) dy dx dz
\]

\[
= \int_{\mathfrak{p}^{-2k} \times \mathfrak{p}^{-k}} \int_{\mathfrak{p}^{-n}} f(wu(a^{-1}x,0,a^{-1}b(z-xy)u(0,yb,0))) \psi^{-1}(x+y) dy dx dz
\]

\[
+ \int_{\mathfrak{p}^{-2k} \times \mathfrak{p}^{-k}} \int_{\mathfrak{p}^{-k} \times \mathfrak{p}^{-n}} f(wu(a^{-1}x,0,a^{-1}b(z-xy)u(0,yb,0))) \psi^{-1}(x+y) dy dx dz
\]

\[
= I_1^k + I_2^k.
\]
Since \( yb \in \sigma \), we have \( u(0, yb, 0) \in K_c \), for \( y \in p^n \). By changing variable on \( z \), we have

\[
I_1^k = \int_{p^{-2k} \times p^{-k}} \int_{p^n} f(wu(a^{-1}x, 0, a^{-1}b(z - xy)u(0, by, 0))\psi^{-1}(x + y)dydx dz
\]

\[
= \int_{p^{-2k} \times p^{-k}} \int_{p^n} f(wu(a^{-1}x, 0, a^{-1}bz)u(0, by, 0))\psi^{-1}(x + y)dydx dz
\]

\[
= q^{-n} \int_{p^{-2k} \times p^{-k}} \int_{p^n} f(wu(a^{-1}x, 0, a^{-1}bz))\psi^{-1}(x)dx dz
\]

\[
= q^{-n} \int_{p^{-2k} \times p^{-k}} \left( \int_{p^m} f(wu(a^{-1}x, 0, a^{-1}bz))dx \right) dz
\]

\[
+ q^{-n} \int_{p^{-2k}} \left( \int_{p^{-n} - p^m} f(wu(a^{-1}x, 0, a^{-1}bz))\psi^{-1}(x)dx \right) dz
\]

\[
: = I_3^k + I_4^k.
\]

Since for \( x \in p^m, a^{-1}x \in \sigma \), we get

\[
I_3^k = q^{-m-n} \int_{p^{-2k}} f(wu(0, 0, a^{-1}bz))dz
\]

\[
= q^{-m-n} \left( \int_{p^m+n-c} f(wu(0, 0, a^{-1}bz))dz \right.
\]

\[
+ \sum_{i=1}^{2k+m+n-c} \left. \int_{p^{m+n-c-i} - p^{m+n-c-i+1}} f(wu(0, 0, a^{-1}bz))dz \right).
\]

Since \( f \) is \( K_c \)-invariant, by Eq. (3.5), we have

\[
\int_{p^{m+n-c}} f(wu(0, 0, a^{-1}bz))dz = q^{-m-n}(t_1t_3^{-1}q^{-1})^c.
\]

Note that for \( z \in p^{m+n-c-i} - p^{m+n-c-i+1} \), we have \( ab^{-1}z^{-1} \in \mathbb{C}^{n+l} \sigma \). By Eq. (3.4), we have \( wu(0, 0, a^{-1}bz) = \text{diag}(-ab^{-1}z^{-1}, 1, a^{-1}bz)u(0, 0, -a^{-1}bz)u(0, 0, ab^{-1}z^{-1}) \).

For \( i \geq 1 \), we have \( u(0, 0, ab^{-1}z^{-1}) \in K_c \) and

\[
f(wu(0, 0, a^{-1}bz)) = \mu_1(ab^{-1}z^{-1})\mu_3(a^{-1}bz^{-1})|ab^{-1}z^{-1}|^2 = (t_1t_3^{-1}q^{-2})^{c+i}.
\]

Thus

\[
I_3^k = q^{-2(m+n)} \left( (t_1t_3^{-1}q^{-1})^c + (1 - q^{-1}) \sum_{i=1}^{2k+m+n-c} (t_1t_3^{-1}q^{-1})^{c+i} \right).
\]

We next consider \( I_4^k \). We have

\[
I_4^k = q^{-n} \int_{p^m+n-c} \left( \int_{p^{-2k} - p^m} f(wu(a^{-1}x, 0, a^{-1}bz))\psi^{-1}(x)dx \right) dz
\]

\[
+ q^{-n} \int_{p^{-2k}} \left( \int_{p^{-n} - p^m} f(wu(a^{-1}x, 0, a^{-1}bz))\psi^{-1}(x)dx \right) dz
\]

\[
: = I_5^k + I_6^k.
\]

For \( z \in p^{m+n-c} \), we have \( a^{-1}bz \in p^{-c} \) and thus \( u(0, 0, a^{-1}bz) \in K_c \). We get

\[
I_5^k = q^{-m-2n+c} \int_{p^{-k} - p^m} f(wu(a^{-1}x, 0, 0))\psi^{-1}(x)dx.
\]

We have the identity

\[
uwu(a^{-1}x, 0, 0) = \text{diag}(1, -ax^{-1}, a^{-1}x)u(0, -a^{-1}x, 1)u'(ax^{-1}),
\]

see Lemma 3.2(2) for the definition of \( u'(r) \). By Lemma 3.2(2), we get \( f(wu(a^{-1}x, 0, 0)) = 0 \) and thus \( I_5^k = 0 \).
We next consider $I_6^k$. We have

$$I_6^k = q^{-n} \sum_{i=1}^{2k+m+n-c} \int_{p_{m+n-c-i-1}^m p_{m+n-c-i+1}^m} (t_1 t_3^{-1} q^{-2})^{c+i} \left( \min \{i, k+m \} \right) \left( \sum_{j=1}^{k+m} \int_{p_{m-j-1}^m p_{m-j+1}^m} \psi^{-1}(x) dx \right) dz.$$  

By Eq. (3.4) again, we have

$$wu(a^{-1}x, 0, a^{-1}bz) = \text{diag}(-ab^{-1}z^{-1}, 1, a^{-1}bz)u(a^{-1}x, 0, -a^{-1}bz)\hat{u}(0, b^{-1}xz^{-1}, ab^{-1}z^{-1}).$$

For $z \in p_{m+n-c-i-1}^m p_{m+n-c-i+1}^m, x \in p_{m-j-1}^m p_{m-j+1}^m$, we have $ab^{-1}z^{-1} \in \omega^{-i} \alpha \kappa$ and $b^{-1}xz^{-1} \in \omega^{c+i-j}$. Since $i \geq 1$, we get $\hat{u}(0, 0, ab^{-1}z^{-1}) \in K_c$. Thus

$$f(wu(a^{-1}x, 0, a^{-1}bz)) = (t_1 t_3^{-1} q^{-2})^{c+i} f(\hat{u}(0, b^{-1}xz^{-1}, 0)).$$

By Lemma 3.2 (1), if $i < j$, we get $f(wu(a^{-1}x, 0, a^{-1}bz)) = 0$. If $i \geq j \geq 1$, we have

$$f(wu(a^{-1}x, 0, a^{-1}bz)) = (t_1 t_3^{-1} q^{-2})^{c+i}.$$  

Thus, we get

$$I_6^k = q^{-n} \sum_{i=1}^{2k+m+n-c} \int_{p_{m+n-c-i-1}^m p_{m+n-c-i+1}^m} (t_1 t_3^{-1} q^{-2})^{c+i} \left( \min \{i, k+m \} \right) \left( \sum_{j=1}^{k+m} \int_{p_{m-j-1}^m p_{m-j+1}^m} \psi^{-1}(x) dx \right) dz.$$  

By Lemma 3.8(1), we have $\int_{p_{m-j-1}^m p_{m-j+1}^m} \psi^{-1}(x) = 0$ if $j > m+1$. Then in the above expression, $j$ satisfies the condition $1 \leq j \leq \min \{i, m+1 \}$. The double sum in the expression of $I_6^k$ then can be divided into 3 parts:

$$\sum_{i=1}^{2k+m+n-c} \sum_{j=1}^{m+1} \sum_{i=m+1}^{2k+m+n-c} \sum_{j=m+1}^{m+1}.$$  

Thus we get

$$I_6^k = I_6^k = q^{-2m-2n}(1-q^{-1})^2 \sum_{i=1}^{m} (t_1 t_3^{-1} q^{-1})^{c+i} \sum_{j=1}^{i} q^j$$

$$+ q^{-2m-2n}(1-q^{-1})^2 \sum_{i=m+1}^{2k+m+n-c} (t_1 t_3^{-1} q^{-1})^{c+i} \sum_{j=1}^{m} q^j$$

$$- q^{-m-2n}(1-q^{-1}) \sum_{i=m+1}^{2k+m+n-c} (t_1 t_3^{-1} q^{-1})^{c+i}$$

$$= q^{-2m-2n}(1-q^{-1}) \sum_{i=1}^{m} (t_1 t_3^{-1} q^{-1})^{c+i} (q^i - 1)$$

$$- q^{-2m-2n}(1-q^{-1}) \sum_{i=m+1}^{2k+m+n-c} (t_1 t_3^{-1} q^{-1})^{c+i}.$$
From the above calculation, Eq. (3.7) and Eq. (3.8), we get

\begin{equation}
I_k^1 = I_k^5 + I_k^4
\end{equation}

\begin{align}
&= q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{m} (t_1t_3^{-1}q^{-1})^{c+i}(q^i - 1) \\
&+ q^{-2m-2n}(t_1t_3^{-1}q^{-1})^c
\end{align}

\begin{align}
&+ q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{m} (t_1t_3^{-1}q^{-1})^{c+i} \\
&= q^{-2m-2n}(t_1t_3^{-1}q^{-1})^c + q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{m} (t_1t_3^{-1}q^{-1})^{c+i}
\end{align}

which is independent of $k$ as long as $k$ is sufficiently large. To emphasize the dependence on $m, n$, we will write the above expression as $I_1(m, n)$.

We next consider

\[ I_k^2 = \int_{p^{-2k} \times p^{-k}} f(u(a^{-1}x, 0, a^{-1}b(z - xy))u(0, by, 0))\psi^{-1}(x + y)dydx dz. \]

We have

\[ I_k^2 = \int_{p^{-2k} \times p^{-k}} f(u(a^{-1}x, by, a^{-1}bz))\psi^{-1}(x + y)dydx dz \\
= \int_{p^{-2k} \times (p^{-k} - p^n)} \int_{p^{-k}} f(u(a^{-1}x, by, a^{-1}bz))\psi^{-1}(x + y)dydx dz \\
= \int_{p^{-2k} \times (p^{-k} - p^n)} \int_{p^{-k}} f(u(a^{-1}x, by, a^{-1}bz))\psi^{-1}(x + y)dx dy dz \\
+ \int_{p^{-2k} \times (p^{-k} - p^n)} \int_{p^{-k}} f(u(a^{-1}x, by, a^{-1}bz))\psi^{-1}(x + y)dx dy dz \\
: = I_k^7 + I_k^9. \]

The calculation of $I_k^7$ is the same as that of $I_4$, Eq. (3.8). We only write the result here

\begin{equation}
I_k^7 = q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{n} (t_1t_3^{-1}q^{-1})^{c+i}(q^i - 1) \\
- q^{-2m-2n}(1 - q^{-1}) \sum_{i=n+1}^{2k+m+n-c} (t_1t_3^{-1}q^{-1})^{c+i}.
\end{equation}

We next compute

\[ I_k^8 = \int_{p^{-2k} \times (p^{-k} - p^n)} f(u(a^{-1}x, by, a^{-1}bz))\psi^{-1}(x + y)dx dy dz. \]

Using Eq. (3.4) we get

\begin{align}
I_k^8 = & \sum_{l=-\infty}^{2k+m+n-c} \int_{p^{m+n-c-l} - p^{m+n-c-i+1}} \sum_{j=1}^{k+n} \int_{p^n - p^{n-j+1}} \sum_{i=1}^{k+m} \int_{p^{m-i} - p^{m-i+1}} \\
& \cdot \psi^{-1}(x + y) \cdot \mu_1(ab^{-1}(z - xy)^{-1}) \mu_2(1 - (xy)/z) \mu_3(a^{-1}bz)|ab^{-1}(z - xy)^{-1}| \cdot |a^{-1}bz|^{-1} \\
& \cdot f \left( \left( -\frac{ay}{z - xy}, \frac{a}{bz} \right) \right) dx dy dz.
\end{align}

For $z \in p^{m+n-c-l} - p^{m+n-c-i+1}$, $y \in p^{n-j} - p^{n-j+1}$, $x \in p^{m-i} - p^{m-i+1}$, we have

\[ ab^{-1}z^{-1} \in \omega^{c+l+1}a \times, b^{-1}xz^{-1} \in \omega^{c+l-i}, (xy)/z \in \omega^{c+l-i-j}a \times. \]

For fixed $x, y$, to make sure the integral of $\mu_2(1 - (xy)/z)$ with respect to $z$ is non-zero, we need $l \geq i + j - 1$, see Lemma 3.8 (2) (one can check that the other quantities in $I_k^8$ involving $z$ only depends
on the absolute value of $z$). If $l \geq i + j - 1$, we have $ay/(z - xy) \in p^c, ab^{-1}z^{-1} \in p^c, b^{-1}xz^{-1} \in p^c$. Moreover, we have

$$ab^{-1}(z - xy)^{-1} \in \mathbb{C}^+ | o^\infty.$$  

Thus we get

$$I_8^k = \sum_{j=1}^{k+n} \int_{p^{n-j}p^{m-j+1}} \psi^{-1}(y)dy \sum_{i=1}^{k+m} \int_{p^{m-i}p^{n-i+1}} \psi^{-1}(x)dx \cdot \left(\sum_{l=i+j}^{2k+n-m-c} q^{-(m+n)}(1 - q^{-1})(t_1t_3^{-1}q^{-1})^{c+l} - q^{-1}(t_1t_3^{-1}q^{-1})^{c+i+j-1}\right).$$

Similar to the calculation of $I_6^k$, the right side of the double sum on $i, j$ in Eq.(3.11) can be divided into the following 4 parts

$$\sum_{i=1}^{m} \sum_{j=1}^{n}, \sum_{i=1}^{m} \sum_{j=m+1}^{n}, \sum_{i=m+1}^{n} \sum_{j=1}^{n}, \sum_{i=m+1}^{n} \sum_{j=m+1}^{n}.$$  

The term corresponding to $\sum_{i=1}^{m} \sum_{j=1}^{n}$ is

$$q^{-2m-2n}(1 - q^{-1})^2 \sum_{i=1}^{m} \sum_{j=1}^{n} q^{i+j} \left(\sum_{l=i+j}^{2k+n-m-c} (1 - q^{-1})(t_1t_3^{-1}q^{-1})^{c+l} - q^{-1}(t_1t_3^{-1}q^{-1})^{c+i+j-1}\right).$$

The second term is

$$-q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{m} q^{i+n} \left(\sum_{l=i+n+1}^{2k+n-m-c} (1 - q^{-1})(t_1t_3^{-1}q^{-1})^{c+l} - q^{-1}(t_1t_3^{-1}q^{-1})^{c+i+n}\right).$$

The third term is

$$-q^{-2m-2n}(1 - q^{-1}) \sum_{j=1}^{n} q^{i+m} \left(\sum_{l=i+m+1}^{2k+n-m-c} (1 - q^{-1})(t_1t_3^{-1}q^{-1})^{c+l} - q^{-1}(t_1t_3^{-1}q^{-1})^{c+i+m}\right).$$

The last term is

$$q^{-m-n} \left(\sum_{l=m+n+2}^{2k+n-m-c} (1 - q^{-1})(t_1t_3^{-1}q^{-1})^{c+l} - q^{-1}(t_1t_3^{-1}q^{-1})^{c+m+n+1}\right).$$

We consider the integral $I_2^k = I_6^k + I_8^k$. From Eq.(3.10-3.15), the coefficient of the term involves $k$ is

$$q^{-2m-2n}(1 - q^{-1}) \left(1 - q^{-1}\right)^2 \sum_{i=1}^{m} \sum_{j=1}^{n} q^{i+j} - (1 - q^{-1}) \sum_{i=1}^{m} q^{i+n} - (1 - q^{-1}) \sum_{j=1}^{n} q^{i+m} + q^{m+n} - 1,$$
which is zero. Thus $I_2^k$ is in fact independent on $k$ as long as $k$ is sufficiently large. We then get

(3.16)

\[ I_2^k = I_7^k + I_8^k \]

\[ = q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{n} (t_1 t_3^{-1} q^{-1})^{c+i} (q^i - 1) \]

\[ + q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{n} (t_1 t_3^{-1} q^{-1})^{c+i} \]

\[ - q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{m} \sum_{j=1}^{n} q^{i+j} \left( \sum_{l=1}^{i+j-1} (1 - q^{-1}) (t_1 t_3^{-1} q^{-1})^{c+l} + q^{-1} (t_1 t_3^{-1} q^{-1})^{c+i+j-1} \right) \]

\[ - q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{m} (1 - q^{-1}) (t_1 t_3^{-1} q^{-1})^{c+i} - q^{-1} (t_1 t_3^{-1} q^{-1})^{c+i+n} \]

\[ - q^{-2m-2n}(1 - q^{-1}) \sum_{i=1}^{n} (1 - q^{-1}) (t_1 t_3^{-1} q^{-1})^{c+i} - q^{-1} (t_1 t_3^{-1} q^{-1})^{c+i+m} \]

\[ + q^{-m-n} \sum_{l=1}^{m+n+1} (1 - q^{-1}) (t_1 t_3^{-1} q^{-1})^{c+l} - q^{-1} (t_1 t_3^{-1} q^{-1})^{c+m+n+1} \].

To emphasize the dependence on $m, n$, we denote the above expression by $I_2(m, n)$. Let $I(m, n) = \lim_{k \to \infty} I^k(m, n)$. By Eq. (3.6), we have

\[ I(m, n) = I_1(m, n) + I_2(m, n), \]

where the right hand side was given by Eq. (3.9) and Eq. (3.16).

A simple calculation shows that

\[ W_f(1) = I(0, 0) = I_1(0, 0) + I_2(0, 0) = (t_1 t_3^{-1} q^{-1}) (1 - t_1 t_3^{-1} q^{-1}). \]

Since $\mu_1 \mu_3^{-1} \neq | \pm |$ (irreducibility condition), we have $t_1 t_3^{-1} q^{-1} \neq 1$. Thus $W_f(1) \neq 0$.

By a careful symbolic calculation, one can get that

\[ I(m, n) = q^{-2m-2n} \frac{1 - (qX)^{m+n+1}}{1 - qX} W_f(1), \]

where $X = t_1 t_3^{-1} q^{-1}$. Now Theorem 3.5 follows from Eq. (3.3).

4. Calculation of Ginzburg’s local zeta integral in a special case

In this section, as in Section 3, we still let $F$ be non-archimedean local field, $\mathfrak{o}$ be the ring of integers of $F$, $\varpi$ be a uniformizer of $F$, and $q$ be the number of the residue field of $F$. We also fix a nonzero square $\rho$, and identify the group $H_\rho$ with $SL_3$.

As in Section 3.2, we now consider the induced representation $\pi = Ind_{B_3}^{GL_3(F)}(\mu_1 \otimes \mu_2 \otimes \mu_3)$ of $GL_3(F)$ with $\mu_1, \mu_3$ unramified, and $\mu_2$ ramified with conductor $c$. We further require that $\mu_1 \mu_3 = 1$. Let $t_1 = \mu_1(\varpi)$. Let $W_c \in W(\pi, \psi)$ be the Whittaker function defined by (3.1). We know that $W_c$ is right invariant under the group $K_c \subset GL_3(F)$. By Lemma 3.4 and Theorem 3.5 we have $W_c(1) \neq 0$ and

\[ W_c(\text{diag}(\varpi^m, 1, \varpi^{-m})) = \begin{cases} \frac{q^{-2m}}{t_1-t_3} (t_1^{2m+1} - t_1^{-(2m+1)}) W_c(1), & m \geq 0, \\ 0, & m < 0. \end{cases} \]

Let $\chi$ be an unramified quasi-character of $F^\times, s \in \mathbb{C}$. Recall that, by the exact sequence (2.3), the induced representation $I(s, \chi)$ has a subspace $Ind_{N_2, \rho, GL_1}^{H_\rho}(\chi_s)$.
For a subset $A \subset F$, we denote $\text{ch}_A$ the characteristic function of $A$. Consider the following function $f^c_s$ on $H_\rho(F) \cong \text{SL}_3(F)$:

$$f^c_s(g) = \begin{cases} 
0, & \text{if } g \notin B_3K_c \cap H_\rho(F), \\
\chi'_s(nh(a, 1))\text{ch}_{1+p^c}(b)\text{ch}_s(x), & \text{if } g = nh(a, 1)h(1, b)k, n \in N_{2, p}, k \in K_c \cap H_\rho(F),
\end{cases}$$

where

$$n(x) = \begin{pmatrix} 1 & x & 0 \\
1 & 0 & 1 \\
0 & 0 & 1 \end{pmatrix} \in H_\rho(F).$$

Note that for $n \in N_{2, p}$, if $nh(a, 1)h(1, b) \in B_3 \cap K_c \cap \text{SL}_3(F)$, we can get $a \in \mathfrak{o}^\times$, $b \in 1 + \mathfrak{p}^c$, $x \in \mathfrak{o}$, and hence $\chi'_s(nh(a, 1))\text{ch}_{1+p^c}(b)\text{ch}_s(x) = 1$. Thus $f^c_s$ is well-defined. Note that $f^c_s$ is right $K_c$-invariant. By definition, $f^c_s \in \text{ind}_{N_{2, p} \cap \text{GL}_3}(\chi'_s)$ and hence defines an element $\tilde{f}^c_s \in I(s, \chi)$ by the exact sequence (2.3).

**Proposition 4.1.** We have

$$Z(W_c, \tilde{f}^c_s) = D_c \frac{1 + \chi(\varpi)q^{-(3s-1)}}{(1 - t_1^2\chi(\varpi)q^{-(3s-1)})(1 - t_1^{-2}\chi(\varpi)q^{-(3s-1)})},$$

where $D_c = \text{vol}(K_c)\text{vol}(1 + \mathfrak{p}^c)W_c(1)$.

**Proof.** We have

$$Z(W_c, \tilde{f}^c_s) = \int_{N_{2, p} \cap H_\rho(F)} W_c(g)f^c_s(g)dg.$$

Since $\text{supp}(f^c_s) \subset B_3K_c$ and both $W_c$ and $f^c_s$ are left $N_{2, p}$-invariant and right $K_c$-invariant, we get

$$Z(W_c, \tilde{f}^c_s) = \text{vol}(K_c)\int_{F^\times} \int_{1 + \mathfrak{p}^c} \int_{\rho} W_c(h(a, 1)n(x)h(1, b))\chi'_s(h(a, 1))|a|^{-3}dxd^*b^*a$$

$$= \text{vol}(K_c)\text{vol}(1 + \mathfrak{p}^c)\int_{F^\times} W_c(h(a, 1))\chi(a)|a|^{3s-3}d^*a.$$

By (4.1), we get

$$Z(W_c, \tilde{f}^c_s) = D_c \sum_{m \geq 0} \frac{q^{-2m}}{t_1^{-1} - t_1^{2m+1}}(t_1^{2m+1} - t_1^{-(2m+1)})(\chi(\varpi)q^{-(3s-3)})^m$$

$$= D_c \frac{1}{t_1^{-1} - t_1^{2m+1}}(\chi(\varpi)q^{-(3s-3)})^m$$

$$= D_c \frac{1 + \chi(\varpi)q^{-(3s-1)}}{(1 - t_1^2\chi(\varpi)q^{-(3s-1)})(1 - t_1^{-2}\chi(\varpi)q^{-(3s-1)})}.$$

This concludes the proof. \hfill \Box

5. **Holomorphy of Adjoint $L$-function for $GL_3$**

In this section, let $F$ be a global field and $A$ be the ring of adeles of $F$. Let $\pi = \otimes \pi_v$ be an irreducible cuspidal automorphic representation of $GL_3(A)$. Let $\chi = \otimes \chi_v$ be a unitary Hecke character of $F^\times \setminus A^\times$. Then one can consider the twisted adjoint $L$-function

$$L(s, \pi, \text{Ad} \times \chi) = \frac{L(s, (\pi \otimes \pi) \times \pi)}{L(s, \chi)}.$$

For a fixed $\pi$ and $\chi$, let $S = S(\pi, \chi)$ be the finite set of places consisting of all archimedean places and all finite places $v$ such that either $\pi_v$ or $\chi_v$ is ramified. The partial twisted adjoint $L$-function $L^S(s, \pi, \text{Ad} \times \chi)$ is defined by

$$L^S(s, \pi, \text{Ad} \times \chi) = \prod_{v \notin S} L(s, \pi_v, \text{Ad} \times \chi_v) = \prod_{v \notin S} \frac{L(s, (\chi_v \otimes \pi_v) \times \pi_v)}{L(s, \chi_v)}.$$

After the pioneering work of Ginzburg [G], and Ginzburg-Jiang [GJ], the following result was obtained:
Theorem 5.1 (Theorem 6.1 of [H18]). The partial twisted adjoint $L$-function $L^\Sigma(s, \pi, \text{Ad} \times \chi)$ has no poles in the half plane $\text{Re}(s) \geq \frac{1}{2}$, except possibly for a simple pole at $\text{Re}(s) = 1$ when $\chi$ is non-trivial and $\pi \cong \pi \otimes \chi$. (This forces $\chi$ to be cubic.) Every other pole of the complete $L$-function $L(s, \pi, \text{Ad} \times \chi)$ in $\text{Re}(s) \geq \frac{1}{2}$ is a zero of the Hecke $L$-function $L(s, \chi)$ and a pole of $\prod_{v \in S} L_v(s, \pi_v, \text{Ad} \times \chi_v)$. 

The analogous result for quasisplit unitary groups was also obtained. We want to extend the above result to $L_f(s, \pi, \text{Ad} \times \chi)$, where $L_f(s, \pi, \text{Ad} \times \chi) = \prod_{v \in S_\infty} L(s, \pi_v, \text{Ad} \times \chi_v)$ is the finite part of the $L$-function, where $S_\infty$ is the set of infinite places of $F$. In order to do this, we must treat all of the infinite places $v \in S - S_\infty$, i.e., either $\pi_v$ or $\chi_v$ is ramified. The cases where $\pi_v$ is ramified can be split up according to whether $\pi_v$ is tempered or non-tempered. In the non-tempered case, we are able to exploit the classification of unitary representations, to say that a representation which is ramified, unitary, and non-tempered is of a fairly specific form. In the tempered case, the arguments which we use for $GL_3$ work equally well in the unitary group case, and we therefore record them in this generality.

Lemma 5.2. Let $v$ be a place of $F$, $\rho$ an element of $F^\times$, and $\pi_v$ an irreducible generic tempered representation of $H_\rho(F_v)$. Then $L(s, \pi_v, \text{Ad} \times \chi_v)$ has no poles in $\text{Re}(s) > 0$.

Proof. If $H_\rho$ is isomorphic to $SL_3$ over $F_v$, then $L(s, \pi_v, \text{Ad} \times \chi_v) = L(s, \pi_v \times (\pi_v \otimes \chi_v))/L(s, \chi_v)$. Since $L(s, \chi_v)^{-1}$ has no poles at all, it suffices to prove that $L(s, \pi_v \times \pi_v \times \chi_v)$ has no poles in $\text{Re}(s) > 0$. This may be deduced from [JPSS83, Proposition 8.4, page 451], since $(\pi_v \otimes \chi_v)$ is again tempered.

If $H_\rho$ is not split over $F_v$, then it determines a quadratic extension field $F_\nu(F_v)$ and $\chi_v(\psi)\otimes\chi_v(\nu)$, where $sbc$ denotes the stable base change lift of Kim and Krishnamurthy [KK1]. The twisted $\text{Ad}$ $L$-function $L(s, \text{sbc}(\pi_v), \text{sbc}(\chi_v), \text{Ad} \times \chi_v)$ may also be realized via the $\text{Ad}$ $L$-function of the twist: $L(s, \text{sbc}(\pi_v) \otimes \chi_v, \text{Ad} \times \chi_v)$ where $\chi_v$ is any character of $F_\nu(F_v)$ whose restriction to $F_\nu^\times$ is $\chi_v(\psi)\otimes\chi_v(\nu)$. Thus it suffices to show that the stable base change lift of a tempered representation is again tempered, and that the $\text{Ad}$ $L$-function of a tempered representation has no pole in $\text{Re}(s) > 0$.

The fact that the local stable base change lift of a generic tempered representation is again tempered is proved for quasisplit $U_{2n}$ in Proposition 8.6 of [KK2]. The argument adapts to $U_{2n+1}$ in a straightforward manner, using the results of [KK1]. Holomorphy of the $\text{Ad}$ $L$-function for tempered representations in $\text{Re}(s) > 0$ then follows from Proposition 7.2 of [Sha90].

Proposition 5.3. Fix a finite place $v \in S - S_\infty$ of $F$. Let $\pi_v$ be a non-tempered irreducible unitary representation of $GL_3(F_v)$ and $\psi_v$ be an additive character of $F_v$. Define $\psi_{\pi_v, v} : U_{\pi_v}(F_v) \to \mathbb{C}^\times$ by (2.4) and assume that $\pi_v$ is $\psi_{\pi_v, v}$-generic. Let $I(s, \chi_v)$ be the induced representation of $G_2(F_v)$ defined in §2.1. Then there exists a Whittaker function $W_v \in \mathcal{W}(\pi_v, \psi_{\pi_v, v})$ and a standard section $f_{s,v} \in I(s, \chi_v)$ such that

$$L(3s, \chi_v)L(6s - 2, \chi_v^2)L(9s - 3, \chi_v^3)\frac{Z(W_v, f_{s,v})}{L(3s - 1, \pi_v, \text{Ad} \times \chi_v)}$$

has no zeros on the region $\text{Re}(s) \geq \frac{1}{2}$.

Proof. By Proposition 2.5, it suffices to treat the case when $\psi$ is also unramified. By the classification of unitary representation of $GL_3$, [JPSS79, §6], the representation has the form $\pi_v = \eta_v \otimes \sigma_v$, where $\eta_v$ is a unipotent character of $F_v^\times$ and $\sigma_v$ is of the form

$$\text{Ind}_{B_3(F_v)}^{GL_3(F_v)}(1 \otimes \mu_2 \otimes | |^{-\alpha}),$$

where $\mu_2$ is a unipotent character of $F_v^\times$ and $\alpha$ is a real number with $0 < \alpha < \frac{1}{2}$.

Note that $L(s, \pi_v, \text{Ad} \times \chi_v) = L(s, \sigma_v, \text{Ad} \times \chi_v)$ and for $W_v \in \mathcal{W}(\pi_v, \psi_{\pi_v, v})$, $W_v|_{SL_3(F_v)}$ is a Whittaker function for $\sigma_v$. Since Ginzburg’s integral $Z(W_v, f_{s,v})$ only depends on $W_v|_{SL_3(F_v)}$, we can assume that $\eta_v = 1$, i.e., $\pi_v = \sigma_v$.

Note that the assumption $v \in S - S_\infty$ implies that either $\mu_2$ is ramified or $\chi_v$ is ramified.
We first consider the case when $\mu_2$ is unramified. If $\chi_v$ is ramified, from the equation $L(s, \pi_v, \text{Ad} \times \chi_v) = L(s, (\chi_v \pi_v) \otimes \bar{\pi_v})/L(s, \chi_v)$ and [JPSS83, Theorem 3.1], one can check that $L(s, \pi_v, \text{Ad} \times \chi_v) = 1$. Thus the Claim follows from the fact that one can find $W_v, f_{s,v}$ such that $Z(W_v, f_{s,v})$ is a nonzero constant, see Lemma 2.2.

We next consider the case when $\mu_2$ is ramified and $\chi_v$ is also ramified. In this case by [CPS, Theorem 4.1], one can check that

$$L(s, \pi_v, \text{Ad} \times \chi_v) = L(s, \chi_v^{-1}) |^\alpha L(s, \chi_v^{-1}) |^{-\alpha} L(s, \chi_v |^{-2\alpha}).$$

Note that if $\chi_v \mu_2$ and $\chi_v \mu_2^{-1}$ are ramified, then $L(s, \pi_v, \text{Ad} \times \chi_v) = 1$ and thus it has no pole. If either $\chi_v \mu_2$ or $\chi_v \mu_2^{-1}$ is unramified, then $L(s, \pi_v, \text{Ad} \times \chi_v)$ has no pole on the region $\text{Re}(s) \geq 1/2$. Because $\chi_v$ and $\mu_2$ are unitary and $\alpha < 1/2$. Hence $L(s, \pi_v, \text{Ad} \times \chi_v)$ has no pole in the region $\text{Re}(s) \geq 1/2$. The assertion then also follows from the nonvanishing of the local Ginzburg’s integrals, see Lemma 2.2.

Finally, we consider the case when $\mu_2$ is ramified and $\chi_v$ is unramified. In this case, by [CPS, Theorem 4.1] again, we have

$$L(s, \pi_v, \text{Ad} \times \chi_v) = L(s, \chi_v)^2 L(s, \chi_v |^{-2\alpha}).$$

By Proposition 4.1, we can take $W_v \in \mathcal{W}(\pi_v, \psi_v)$ and a standard section $f_{s,v} \in I(s, \chi_v)$ such that

$$Z(W_v, f_{s,v}) = \frac{1 + \chi_v(\omega)q^{-3s-1}}{(1 - \chi_v(\omega)|\omega|^2q^{-3s-1})(1 - \chi_v(\omega)|\omega|^{-2\alpha}q^{-3s-1})} = \frac{1 - \chi_v(\omega)^2q^{-6s-2}}{(1 - \chi_v(\omega)q^{-3s-1})(1 - \chi_v(\omega)|\omega|^2q^{-3s-1})(1 - \chi_v(\omega)|\omega|^{-2\alpha}q^{-3s-1})}.$$

Thus

$$\frac{Z(W_v, f_{s,v})}{L(3s-1, \pi_v, \text{Ad} \times \chi_v)} = \frac{1}{L(3s-1, \chi_v) L(6s-2, \chi_v^2)},$$

and

$$L(3s, \chi_v) L(6s-2, \chi_v^2) L(9s-3, \chi_v^3) Z(W_v, f_{s,v}) = L(3s-1, \pi_v, \text{Ad} \times \chi_v).$$

Replacing $s$ by $s - 1$, we can assume that $\pi$ is unitary. We first show that $L_f(s, \pi_v, \text{Ad} \times \chi_v)$ is holomorphic on the region $\text{Re}(s) \geq 1/2$, except for a simple pole at $\text{Re}(s) = 1$ when $\chi$ is non-trivial and $\pi \cong \pi \otimes \chi$ (which forces $\chi$ to be cubic).

**Theorem 5.4.** Let $\chi$ be a unitary Hecke character of $F^\times \setminus \mathbb{A}^\times$ and $\pi$ be an irreducible cuspidal automorphic representation of $\text{GL}_3(\mathbb{A})$, then the finite part of the adjoint $L$-function $L_f(s, \pi, \text{Ad} \times \chi)$ is holomorphic on the region $\text{Re}(s) \geq 1/2$, except for a simple pole at $\text{Re}(s) = 1$ when $\chi$ is non-trivial and $\pi \cong \pi \otimes \chi$ (which forces $\chi$ to be cubic).

**Proof.** Note that we can write $\pi = |\det|^\zeta$ with $\zeta \in C$ and $\pi_0$ unitary. From the relation $L_f(s, \pi, \text{Ad} \times \chi) = L_f(s, (\chi \otimes \chi) \times \bar{\pi})/L_f(s, \chi)$, we can get $L_f(s, \pi, \text{Ad} \times \chi) = L(s, \pi_0, \text{Ad} \times \chi)$. Replacing $\pi$ by $\pi_0$ if necessary, we can assume that $\pi$ is unitary.

We first show that $L_f(s, \pi_v, \text{Ad} \times \chi_v)$ is holomorphic for $\text{Re}(s) \geq 1/2$.

Let $S = S(\pi, \chi)$. Then we can write $L_f(s, \pi, \text{Ad} \times \chi) = \prod_{v \in S} L(s, \pi_v, \text{Ad} \times \chi_v) \cdot L^S(s, \pi, \text{Ad} \times \chi)$. By Theorem 5.1, it suffices to consider $L(s, \pi_v, \text{Ad} \times \chi_v)$ for every place $v \in S - S_{\infty}$. Note that each $\pi_v$ is unitary since $\pi$ is cuspidal. Note that if $\pi_v$ is tempered, $L(s, \pi_v, \text{Ad} \times \chi_v)$ has no pole on $\text{Re}(s) > 0$ by Lemma 5.2.
We next assume that \( \pi_v \) is non-tempered. Given a global section \( f_s \in I(s, \chi) \), one can consider the normalized Eisenstein series

\[
E(g, f_s) = L(3s, \chi)L(6s - 2, \chi^2)L(9s - 3, \chi^3) \sum_{\lambda \in P(F) \backslash G_2(F)} f_s(\lambda g).
\]

Given a cusp form \( \varphi \) in the space of \( \varphi \), which is assumed to correspond to a pure tensor, and a pure tensor \( f_s = \otimes f_{s,v} \in I(s, \chi) \), Ginzburg [G] defined the global integral

\[
Z(\varphi, f_s) = \int_{\text{SL}_3(F) \backslash \text{SL}_3(\mathbb{A})} \varphi(g)E(g, f_s)dg,
\]

and showed that it is Eulerian:

\[
Z(\varphi, f_s) = \prod_v Z^*(W_v, f_{s,v}),
\]

where \( Z^*(W_v, f_{s,v}) = L(3s, \chi_v)L(6s - 2, \chi_v^2)L(9s - 3, \chi_v^3)Z(W_v, f_{s,v}) \), \( W_v \) is the \( v \)-th component of the Whittaker function of \( \varphi \). In [G], it is also showed that if \( v \) is a place such that \( \pi_v, \chi_v \), and \( \psi_v \) are all unramified, then \( Z^*(W_v, f_{s,v}) = L(3s - 1, \pi_v, \text{Ad} \times \chi_v) \) for spherical \( W_v, f_{s,v} \). (Both [G] and [H12] only treat the case when \( \chi \) is trivial, but the extension to nontrivial \( \chi \) is direct.) The same identity holds up to an exponential factor at places when \( \pi_v \) and \( \chi_v \) are unramified but \( \psi_v \) is ramified. Thus we get

\[(5.1)\]

\[
Z(\varphi, f_s) = \chi_s(h(1, a)) \prod_{v \in S_\infty} Z^*(W_v, f_{s,v}) \cdot \prod_{v \in S - S_\infty} Z^*(W_v, f_{s,v}) \cdot L^S(3s - 1, \pi, \text{Ad} \times \chi) = \chi_s(h(1, a)) \prod_{v \in S_\infty} Z^*(W_v, f_{s,v}) \cdot \prod_{v \in S - S_\infty} \frac{Z^*(W_v, f_{s,v})}{L(3s - 1, \pi_v, \text{Ad} \times \chi_v)} \cdot L_f(3s - 1, \pi, \text{Ad} \times \chi),
\]

for a certain idèle \( a \) determined by the finite places \( v \) such that \( \psi_v \) is ramified and \( \pi_v \) and \( \chi_v \) are not. After the work of [GJ], it is shown in [H18] that for a flat section \( f_s \in I(s, \chi) \), \( Z(\varphi, f_s) \) has no pole on the region \( \text{Re}(s) \geq 1/2 \) except for a possible simple pole at \( \text{Re}(s) = 2/3 \) which can occur only when \( \chi \) is cubic. By Eq.(5.1), the non-vanishing results of the local zeta integrals \( Z^*(W_v, f_{s,v}) \) [H18, Theorem 5.1] and Proposition 5.3, we obtained that \( L_f(3s - 1, \pi, \text{Ad} \times \chi) \) has no pole on the region \( \text{Re}(s) \geq 1/2 \) except for a possible simple pole at \( s = 2/3 \) in the case when \( \chi \) is cubic.

Thus \( L_f(s, \pi, \text{Ad} \times \chi) \) has no pole on the region \( \text{Re}(s) \geq 1/2 \) except for a possible simple pole at \( s = 1 \) in the case \( \chi \) is cubic. By Proposition 3.6 of [JS81], combined with [JPS83, Proposition 8.4, page 451], \( L_f(s, \pi, \text{Ad} \times \chi) \) has a pole (which is simple when it exists) at \( s = 1 \) if and only if \( \chi \pi \cong \pi \). By Tate’s thesis or the original work of Hecke, \( L_f(s, \chi) \) has a pole (which is simple when it exists) at \( s = 1 \) if and only if \( \chi \) is trivial (in which case \( L_f(s, \chi) \) is just the Dedekind zeta function of \( F \)).

It follows that \( L_f(s, \pi, \text{Ad} \times \chi) \) has a pole at \( s = 1 \) if and only if \( \pi \cong \chi \pi \) (which implies \( \chi^3 = 1 \)), and \( \chi \) is nontrivial.

\[\square\]

6. Further discussion of poles in the nonsplit case

For the remainder of the paper we devote our attention to a detailed study of the poles of \( L^S(s, \pi, \text{Ad} \times \chi) \) when \( \rho \) is a non-square. Here \( S \) is a finite set of places, containing all Archimedean places, such that \( \pi_v \) and \( \chi_v \) are unramified for \( v \notin S \). By [H18], theorem 6.4, \( L^S(s, \pi, \text{Ad} \times \chi) \) has no poles in \( \text{Re}(s) \geq \frac{1}{2} \), and may have a pole at \( s = 1 \) only if \( \chi \) is nontrivial cubic or \( \chi \) is quadratic and \( \pi \) is distinguished with respect to a group \( H_\rho' \), which we may think of as \( \text{SL}_2 \), or more suggestively as \( \text{SU}_{1,1} \) embedded into \( \text{SU}_{2,1} \). See [H18] for details. (In view of Lemma 5.2, this information about the poles of \( L^S(s, \pi, \text{Ad} \times \chi) \) remains true even if we remove places \( v \) such that \( \pi_v \) is tempered from \( S \).) Let \( H_\rho' \) denote the quasisplit unitary group \( U_{1,1} \), embedded into \( H_\rho \) so that \( H_\rho' \) is the derived group. Determinant maps \( U_{1,1} \) to \( U_1 \) and we can choose a splitting to write \( H_\rho' \) as the internal semidirect product of \( H_\rho' \) and a subgroup isomorphic to \( U_1 \). Factoring the Haar measure on \( H_\rho \) accordingly
shows that any representation distinguished with respect to $H'_{\rho}$ must also support the period

$$\varphi \mapsto \int_{H'_C(F) \backslash H'_C(\mathbb{A})} \varphi(h) \eta^{-1}(\det h) \, dh$$

for some character $\eta$ of $U_1(\mathbb{A})$ (in which case we say that $\pi$ is $(H'_\rho, \eta)$-distinguished). It is proved in [GeRoSo93] that this forces the $L$-packet $\{\{\pi\}\}$ of $\pi$ to be the image, under the endoscopic transfer(s) constructed in [Ro], of some $L$-packet $\rho = \rho_2 \times \rho_1$ of $U_{1,1} \times U_1$, (where $\rho_2$ is an $L$-packet of representations of $U_{1,1}(\mathbb{A})$ and $\rho_1$ is a character of $U_1(\mathbb{A}) \cong k_F^\times$). The $L$-homomorphism to which this transfer is attached depends on some choices; varying the choice varies the packet elements of $L$ transfers to $\{\{\pi\}\}$, without changing the overall image of transfer.) It is then proved in [GeRoSo97] that if $\pi$ is $(H'_\rho, \eta)$-distinguished, then at least one of the $L$-packets $\rho = \rho_2 \times \rho_1$ which transfers to $\{\{\pi\}\}$ satisfies $\rho_1 = \eta$.

Thus, any pole of $L^S(s, \pi, \text{Ad}^j \times \chi)$ at $s = 1$, when $\chi$ is either trivial or quadratic, indicates that $\pi$ is endoscopic. In order to complete our treatment of the nonsplit case, we would like to address the case when $\chi$ is cubic, and to study poles in the case when $\pi$ is assumed to be endoscopic. Further investigation requires that we study Rogawski’s transfer(s) in more detail.

6.1. Weil forms of $L$-groups.

6.1.1. A technical point regarding $L$-groups. For purposes of discussing Rogawski’s transfer(s) the finite Galois form of the $L$-group will not suffice; we must consider the Weil form. We briefly explain the reason. We may realize the finite Galois form of $L^{U_{2,1}}$ as in [H12] and [H18], as $GL_3(\mathbb{C}) \rtimes \text{Gal}(E/F)$, with the nontrivial element of $\text{Gal}(E/F)$ acting by $g \mapsto i g^{-1}$. Here $i g = J^1 g J$, as in [H12, H18]. This automorphism of $GL_3(\mathbb{C})$ preserves the subgroup subgroup

$$(6.1) \qquad \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in GL_2(\mathbb{C}) \times GL_1(\mathbb{C}) \subset GL_3(\mathbb{C})$$

of $GL_3(\mathbb{C})$, and hence we obtain a subgroup of $L^{U_{2,1}}$ which is the semidirect product of $GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ and $\text{Gal}(E/F)$.

The finite Galois form of the $L$-group of $U_{1,1} \times U_1$ is also a semidirect product of $GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ and $\text{Gal}(E/F)$, but it may not be identified with this subgroup of $L^{U_{2,1}}$. Indeed, the nontrivial element of $\text{Gal}(E/F)$ acts on $L(U_{1,1} \times U_1)$ by an automorphism of order two. As described in [Bo], this automorphism must satisfy certain conditions. The identity component of $L(U_{1,1} \times U_1)$ inherits, from its definition via based root data, a choice of split maximal torus and Borel subgroup. We may fix the isomorphism $L(U_{1,1} \times U_1)^0 \rightarrow GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ so that they consist of the diagonal and upper triangular elements, respectively. The automorphism must map these to themselves in a manner determined by duality and the action of $\text{Gal}(\mathbb{F}/F)$ on the maximal torus of $U_{1,1} \times U_1$.

The last condition, given in [Bo], 1.2, states that for a general reductive group there must be a set $\{x_\alpha\}$ of representatives for the root subgroups attached to the simple roots, whose elements are permuted amongst themselves. In our case, where there is only one simple root, this last condition means that the action on the standard maximal unipotent subgroup of $GL_2(\mathbb{C})$ must be trivial. Since $g \mapsto i g^{-1}$ does not act trivially on the maximal unipotent subgroup of (6.1) it follows that the semidirect product of $GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ and $\text{Gal}(E/F)$, which sits naturally inside of $L^{U_{2,1}}$, may not be identified with $L(U_{1,1} \times U_1)$.

6.1.2. Definition of the Weil forms of the $L$-group. The Weil form of the $L$-group of $U_{2,1}$ is the semidirect product of $GL_3(\mathbb{C})$ and the Weil group, $W_F$ of $F$, (see [Tate1]) with the action implicit in the semidirect product being defined using the canonical mapping $W_F/W_E \rightarrow \text{Gal}(E/F)$. Thus, elements of $W_E$ commute with $GL_3(\mathbb{C})$ while elements of $W_F \times W_E$ act by $g \mapsto i g^{-1}$. The Weil form of the $L$-group of $U_{1,1} \times U_1$ is defined similarly with $GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ replacing $GL_3(\mathbb{C})$. We may take the involution of $GL_2(\mathbb{C}) \times GL_1(\mathbb{C})$ to be

$$(g, a) \mapsto \begin{pmatrix} 1 & 0 \\ -1 & 1 \end{pmatrix} t g^{-1} \begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}, a^{-1} = (\det g^{-1} \cdot g, a^{-1})$$
6.1.3. Weil forms and the Satake parametrization. We recall the parametrization of unramified representations in [Bo] which is suited to dealing with Weil forms of L-groups. It is based on L-homomorphisms from the Weil group $W_F \to \text{L}^G$. We do not need the full definition of an L-homomorphism, only the notion of an unramified L-homomorphism of $W_F$ for $F$ local nonarchimedean.

In this case $W_F$ is a dense subgroup of $\text{Gal}(\overline{F}/F)$ and comes equipped with a homomorphism $\text{ord}_{W_F}: W_F \to \mathbb{Z}$. An unramified L-homomorphism $W_F \to G'(\mathbb{C}) \times W_F$ is a homomorphism which sends $w \in W_F$ to $(t^{\text{ord}_{W_F}(w)}, w)$ for some semisimple element $t$ of $G'(\mathbb{C})$. Note that $t$ must be $\text{Gal}(\overline{F}/F)$-fixed in order for this to be a homomorphism, and, since we work up to conjugacy, we may assume $t$ is in $T'(\mathbb{C})$. Thus conjugacy classes of unramified L-homomorphisms are in bijection with Galois-fixed elements of $T'(\mathbb{C})$, which correspond to unramified characters as usual.

6.1.4. Weil forms and L-functions. Let $F$ be nonarchimedean and local. We briefly recall the definition of $L(s, \varphi)$ for $\varphi: W_F \to \text{GL}(V)$ a finite dimensional representation of $W_F$.

The kernel of $\text{ord}_{W_F}$ is a normal subgroup of $W_F$ called the inertia group. We denote it $I_F$. As $I_F$ is normal, its fixed subspace $V^{I_F}$ is $W_F$-invariant. We define

$$L(s, \varphi) := \det(I - q^{-s}\varphi(w)|_{V^{I_F}})^{-1}, \quad \text{for } w \in W_F \text{ with } \text{ord}_{W_F}(w) = 1.$$ (The expression on the right-hand-side is independent of the choice of $w$.)

This then permits us to define $L(s, \pi, r)$ for $\pi$ an unramified representation and $r$ a finite dimensional representation of $G'(\mathbb{C}) \times W_F$. Indeed $\pi$ is attached to an unramified L-homomorphism $\varphi_t(w) = (t^{\text{ord}_{W_F}(w)}), w)$, with $t \in T'(\mathbb{C})^{W_F}$ and $L(s, \pi, r)$ is defined as

$$L(s, \pi, r) = L(s, r \circ \varphi_t) = \det(I - q^{-s}r(t)r(w)|_{V^{I_F}})^{-1}, \quad \text{for } w \in W_F \text{ with } \text{ord}_{W_F}(w) = 1.$$

6.2. Adjoint representations. For $H = U_{2,1}$ (resp. $U_{1,1}$) the action of $H^*$ on itself by conjugation determines an action on $\mathfrak{sl}_2(\mathbb{C})$ (resp. $\mathfrak{sl}_2(\mathbb{C})$) which we denote $\text{Ad}$. Since each $w \in W_F \setminus W_E$ acts on $\text{GL}_3(\mathbb{C})$ by $g \mapsto g^{-1}$, it will act on $\mathfrak{sl}_3(\mathbb{C})$ by $X \mapsto -tX$. Note that this conflicts with the notation of [H12].

Let $\text{Ad}'$ denote the representation where $\text{GL}_3(\mathbb{C})$ acts by conjugation and each $w \in W_F \setminus W_E$ acts by $X \mapsto tX$. This is the representation denoted $\text{Ad}$ in [H12]. It can also be described as the twist of $\text{Ad}$ by the quadratic character $\chi_{E/F}$ attached to $E/F$.

6.3. Base Change and Automorphic Induction. Base change for the quadratic extension $E/F$ is the functorial lifting attached to the L-homomorphism

$$bc: \text{L}(\text{GL}_n) = \text{GL}_n(\mathbb{C}) \times W_F \to \text{L}((\text{Res}_{E/F}\text{GL}_n) = (\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \times W_F$$

which sends $w \in W_F$ to itself, and $g \in \text{GL}_n(\mathbb{C})$ to $(g, g) \in (\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C}))$. Automorphic induction for the quadratic extension $E/F$ is the functorial lifting attached to the L-homomorphism

$$\text{AI}_{E/F}: bc((\text{Res}_{E/F}\text{GL}_n) = (\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \times W_F \to L((\text{GL}_{2n}) = \text{GL}_{2n}(\mathbb{C}) \times W_F$$

$$\text{AI}_{E/F}(g_1, g_2) = \begin{pmatrix} g_1 \\ g_2 \end{pmatrix}, \quad \text{AI}_{E/F}(w) = \begin{pmatrix} I_n \\ I_{2n} \end{pmatrix}, \quad \text{for } w \in W_F \setminus W_E$$

Both of these cases of functoriality are proved (in greater generality) in [AC].

6.4. Stable Base Change and its image. The stable base change lifting of Kim and Krishnamurthy has already been mentioned a couple of times. It lifts globally generic automorphic representations of the quasisplit group $U_n$ (Elsewhere in the paper, we denoted $U_3$ by $U_{2,1}$ and $U_2$ by $U_{1,1}$ to emphasize the quasisplit nature. In the case of general $n$ this notation seems cumbersome.) attached to a quadratic extension $E/F$ to automorphic representations of $\text{Res}_{E/F}\text{GL}_n$. The $L$-group of $U_n$ is $\text{GL}_n(\mathbb{C}) \times W_F$, and $W_F \setminus W_E$ acts on $\text{GL}_n(\mathbb{C})$ by a nontrivial involution which we denote $g \mapsto g^*$. There is some freedom to the choice of involution by it must preserve the torus and the borel and permute a collection of elements $\{x_a\}$ as in [Bo, §1.2]. We can take $g^* = qg^{-1}$ when $n$ is odd but not when $n$ is even (cf. §6.1.1). In the even case we can take $g^* = g_0d_0^{-1}g^{-1}d_0$, where $d_0$ is a diagonal matrix with alternating 1’s and −1’s on the diagonal.
The $L$-group of $\text{Res}_{E/F} \text{GL}_n$ is $(\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \rtimes W_F$, and $W_F \setminus W_E$ acts by permuting the factors. Stable base change is the functorial lifting which corresponds to the $L$-homomorphism

$$\text{sbc}(g, w) = (g, t g^{-1}, w).$$

It is closely related to the lifting from $U_n$ to $\text{Res}_{E/F} U_n$ which is considered in [Ro] (where it is called “base change”).

6.4.1. **Asai representations.** The representation of $\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})$ on $\text{Mat}_{n \times n}(\mathbb{C})$ by $(g_1, g_2).X = g_1 X g_2$ is irreducible. Thus it has two distinct extensions to a representation of $(\text{GL}_n(\mathbb{C}) \times \text{GL}_n(\mathbb{C})) \rtimes \text{Gal}(E/F)$, the finite Galois form of the $L$-group of $\text{Res}_{E/F} \text{GL}_n$. We denote them Asai$^\pm$, such that $\text{Asai}^\pm(w).X = \pm_i X$ for $w \in W_F \setminus W_E$.

It is perhaps more conventional to define the Asai representation using the usual transpose as opposed to the lower transpose. The two representations are isomorphic, with an isomorphism given by $X \mapsto XJ$. In the usual Asai representation, the sign is $+$. 

6.4.2. **Asai of stable base change.** It is readily verified that

$$\text{Asai}^+ \circ \text{sbc} = \begin{cases} \text{Ad}' \oplus \mathbb{1}, & \text{if } n \text{ odd}, \\ \text{Ad} \oplus \chi_{E/F}, & \text{if } n \text{ even}, \\ \end{cases} \quad \text{Asai}^- \circ \text{sbc} = \begin{cases} \text{Ad} \circ \chi_{E/F}, & \text{if } n \text{ odd}, \\ \text{Ad}' \oplus \mathbb{1}, & \text{if } n \text{ even}, \\ \end{cases}$$

where $\mathbb{1}$ is the one dimensional trivial representation, and $\text{Ad}$ and $\text{Ad}'$ are defined as in [H18]. Thus if we let $(-)^k = +$ for even and $-$ for odd, then

$$(6.2) \quad L^S(s, \text{sbc}(\pi), \text{Asai}^-(n)) = L^S(s, \chi_{E/F})L^S(s, \pi, \text{Ad}) \quad L^S(s, \text{sbc}(\pi), \text{Asai}^-(n+1)) = \xi^S(s)L^S(s, \pi, \text{Ad}').$$

6.4.3. **Image of stable base change.** If $\pi$ is a globally generic automorphic representation of $U_n$, we denote the stable base change lift by $\text{sbc}(\pi)$. The image of this lifting has been characterized in [GRS11], Theorem 11.2.

**Theorem 6.1** (Ginzburg-Rallis-Soudry). An automorphic representation of $\text{GL}_n(\mathbb{A}_E)$ is in the image of $\text{sbc}$ if and only if it is an isobaric sum of distinct cuspidal representations $\tau_1 \oplus \cdots \oplus \tau_r$ of $\text{GL}_n(\mathbb{A}_E)$, such that $L^S(s, \tau, \text{Asai}^-(n+1))$ has a pole at $s = 1$ for all $i$ (this pole is necessarily simple).

**Remark 6.2.** As explained in [GRS11], $L^S(s, \pi, \text{Asai}^+)L^S(s, \pi, \text{Asai}^-)$ is the partial Rankin-Selberg convolution $L$-function $L^S(s, \pi \times \pi \circ \text{Fr})$ of $\pi$ and the representation obtained by composing $\pi$ with the nontrivial element $\text{Fr}$ of $\text{Gal}(E/F)$. It follows that at most one of $L^S(s, \pi, \text{Asai}^+)$, and $L^S(s, \pi, \text{Asai}^-)$ may have a pole at $s = 1$. Moreover, since both are nonvanishing on the line $\text{Re}(s) = 1$ by theorem 5.1 of [Sha81], it follows that if either has a pole then $\pi \circ \text{Fr} = \overline{\pi}$. This implies that the central character of $\pi$ is trivial on $\mathbb{A}_E^\times \subset \mathbb{A}_F^\times$. Finally, if $\overline{\pi} \cong \pi \circ \text{Fr}$, then either $\text{Asai}^+$ or $\text{Asai}^-$ has a pole.

6.5. **Rogawski’s liftings.**

6.5.1. **Two families of $L$-homomorphisms.** We describe two families of $L$-homomorphisms introduced in [Ro].

**Proposition 6.3** (Rogawski, pp. 52-53). Fix $\mu : E^\times \setminus \mathbb{A}_E^\times \to \mathbb{C}^\times$ satisfying $\mu|_{\mathbb{A}_E^\times} = \chi_{E/F}$, and fix $w_0 \in W_F \setminus W_E$. There is a unique $L$-homomorphism $\xi^{(2,1)}_{w_0, \mu} : L(U_{1,1} \times U_1) \to L(U_{2,1})$ such that

$$\xi^{(2,1)}_{w_0, \mu} \begin{pmatrix} a & b \\ c & d \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \xi^{(2,1)}_{w_0, \mu}(w_0) = \begin{pmatrix} 1 & 1 \\ & -1 \end{pmatrix} w_0,$$

and

$$\xi^{(2,1)}_{w_0, \mu}(w) = \begin{pmatrix} \mu(w) \\ & 1 \end{pmatrix} w, \quad w \in W_E.$$
There is a unique $L$-homomorphism $\xi_{w_0,\mu}^{(1,1)} : L(U_1 \times U_1) \to L U_{1,1}$ such that

$$\xi_{w_0,\mu}^{(1,1)}(a, b) = \begin{pmatrix} a & b \\ \end{pmatrix}, \quad \xi_{w_0,\mu}(w_0) = \begin{pmatrix} 1 & -1 \\ & 1 \end{pmatrix} w_0,$$

and

$$\xi_{w_0,\mu}^{(1,1)}(w) = \begin{pmatrix} \mu^{-1}(w) & \mu^{-1}(w) \\ & \mu^{-1}(w) \end{pmatrix} w, \quad w \in W_E.$$

6.5.2. Rogawski’s Lifts. Each of the $L$-homomorphisms $\xi_{w_0,\mu}^{(2,1)}$ determines a conjectural functorial transfer map, taking automorphic $L$-packets on $U_{1,1}(\mathbb{A}) \times U_1(\mathbb{A})$ to automorphic $L$-packets on $U_{2,1}(\mathbb{A})$. Likewise, each of the $L$-homomorphisms $\xi_{w_0,\mu}^{(1,1)}$ determines a conjectural functorial transfer map, taking automorphic $L$-packets on $U_1(\mathbb{A}) \times U_1(\mathbb{A})$ to automorphic $L$-packets on $U_{1,1}(\mathbb{A})$. The existence of these transfer maps is proved in [Ro]. Varying the choice of $w_0$ does not change the transfer mapping. Varying the choice of $\mu$ permutes the elements of the image.

An automorphic representation of $U_{1,1}(\mathbb{A})$ or of $U_{2,1}(\mathbb{A})$ is said to be endoscopic if it is in the image of the Rogawski liftings.

Applying the lifting attached to $\xi_{w_0,\mu}^{(2,1)}$ to a packet obtained from $\xi_{w_0,\mu}^{(1,1)}$ gives a packet on $U_{2,1}(\mathbb{A})$. This construction is functorial. We describe the associated $L$-homomorphism.

Define

$$\xi_{w_0,\mu}^{(1,1,1)}(a, b, c) = \begin{pmatrix} a & b & c \\ \end{pmatrix}, \quad \xi_{w_0,\mu}^{(1,1,1)}(w) = m(w)w, \quad m(w) := \begin{cases} I, w \in W_E, \\
J, w \notin W_E. \end{cases}$$

(Here $I$ is the identity matrix and $J$ is the matrix with ones on the diagonal from top right to lower left and zeros elsewhere.)

Lemma 6.4. Take $\eta_1, \eta_2, \eta_3$ three automorphic characters of $U_1(\mathbb{A})$. Let $\pi_1$ be the representation of $U_{1,1}(\mathbb{A})$ obtained from $\eta_1 \otimes \eta_3$ using the Rogawski lifting attached to $\xi_{w_0,\mu}^{(1,1)}$ and let $\pi$ be the representation of $U_{2,1}(\mathbb{A})$ obtained from $\eta_1 \otimes \eta_2$ using the Rogawski lifting attached to $\xi_{w_0,\mu}^{(2,1)}$. Then $\pi$ is the weak functorial lift of $\eta_1 \otimes \eta_2 \otimes \eta_3$ relative to the $L$-homomorphism $\xi_{w_0,\mu}^{(1,1,1)}$.

Remark 6.5. Note that $\xi_{w_0,\mu}^{(1,1)}$ and $\xi_{w_0,\mu}^{(2,1)}$ depend on the choice of $w_0$ and $\mu$, but $\xi_{w_0,\mu}^{(1,1,1)}$ does not.

Proof. Let $v$ be a finite unramified place. Each of our representations is determined by an unramified $L$-homomorphism from $W_{F_v}$ into the relevant $L$-group. Fix $w$ an element of $W_{F_v}$ of norm 1. Then each unramified $L$-homomorphism from $W_{F_v}$ is determined by its image on $w$. We regard $W_{F_v}$ as a subgroup of $W_E$ by some choice of embedding as in [Tat1]. Let us refer to this image as the Satake parameter of the representation. At a split place, the Satake parameter of $\eta_i$ is $(t_i, w)$. From considering the isomorphism $W_{F_v}/W_{F_w} \cong \text{Gal}(E/F)$ we see that $w \in W_E$ if and only if $E$ splits over $v$. When this is not the case, $t_i$ must be 1 for all $i$.

First assume $v$ is split. Then the Satake parameter of $\pi_1$ is $\left(\begin{smallmatrix} t_1\mu^{-1}(w) & t_2 \\ t_3\mu^{-1}(w) & t_3 \end{smallmatrix}\right) w$. Hence, the Satake parameter of $\eta_1 \otimes \eta_2$ is $\left(\begin{smallmatrix} (t_1\mu^{-1}(w)) & (t_2) \\ (t_3\mu^{-1}(w)) & (t_3) \end{smallmatrix}\right) w$, and that of $\pi$ is

$$\begin{pmatrix} t_1\mu^{-1}(w) \\ t_2 \\ t_3\mu^{-1}(w) \end{pmatrix} \begin{pmatrix} \mu(w) \\ 1 \\ \mu(w) \end{pmatrix} w = \begin{pmatrix} t_1 \\ t_2 \\ t_3 \end{pmatrix} w.$$

At an inert place, the Satake parameter of $\pi_1$ is $\left(\begin{smallmatrix} 1 & -1 \\ 1 & 1 \end{smallmatrix}\right) w$, that of $\pi_1 \times \eta_2$ is $\left(\begin{smallmatrix} 1 & -1 \end{smallmatrix}\right) w$, and that of $\pi$ is

$$\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix} w = Jw.$$

For $\pi_1$ a cuspidal representation of $U_{1,1}(\mathbb{A})$ and $\eta$ a character of $U_1(\mathbb{A})$ we denote the corresponding Rogawski lift by $\text{Rog}_{w_0,\mu}^{(2,1)}(\pi_1 \otimes \eta)$. The lift attached to three characters $\eta_1, \eta_2, \eta_3$ is denoted $\text{Rog}_{w_0,\mu}^{(1,1,1)}(\eta_1 \otimes \eta_2 \otimes \eta_3)$.
6.5.3. Rogawski liftings and stable base change. It’s clear from the formulae for \( \xi_{w_0, \mu}^{(2, 1)} \) and \( \xi^{(1, 1, 1)} \) that

\[
\text{sbc}(\text{Rog}^{(2, 1)}(\sigma \otimes \eta)) = (\text{sbc}(\sigma) \otimes \text{sbc}(\eta)), \quad \text{sbc}(\text{Rog}^{(1, 1, 1)}(\eta_1 \otimes \eta_2 \otimes \eta_3)) = \text{sbc}(\eta_1) \otimes \text{sbc}(\eta_2) \otimes \text{sbc}(\eta_3).
\]

6.5.4. Rogawski liftings and descent. For globally generic representations, the combination of the Kim-Krishnamurthy lifting and the method of descent gives an alternate construction of Rogawski’s endoscopic liftings, and a generalization. A representation is endoscopic if and only if its stable base change is non-cuspidal. In this situation, descent may be applied to each summand of the stable base change, and the original representation is the endoscopic lift of the collection of representations thus obtained. See [GRS11], section 11.3.

Recall that the isobaric summands of the stable base change of a cusp form are all distinct. From this we may deduce that if \( \text{sbc}(\eta_1) \), \( \text{sbc}(\eta_2) \) and \( \text{sbc}(\eta_3) \) are not distinct, then \( \text{Rog}^{(1, 1, 1)}(\eta_1 \otimes \eta_2 \otimes \eta_3) \) is not cuspidal.

These endoscopic liftings have also been studied by the trace formula method in [Mok].

6.5.5. Adjoint \( L \)-functions of Rogawski lifts. We regard \( \text{Ad} \) as an action of \( \text{GL}_3(\mathbb{C}) \times W_F \) on \( \frak{s}_3 \mathbb{C} \) by composing with the canonical projection to the finite Galois form. Thus \( w \in W_F \) acts by \( X \mapsto -\iota X \) if \( w \notin W_E \) and trivially if \( w \in W_E \). In this section we consider \( \text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)} : L(U_{1, 1} \times U_1) \to \text{GL}(\frak{s}_3 \mathbb{C}) \).

Let

\[
X \left( \begin{array}{ccc}
x_1 & x_2 & u_1 \\
x_3 & -x_1 & u_2 \\
\end{array} \right), \quad \left( \begin{array}{c} v_1 \\
v_2 \\
\end{array} \right) = \left( \begin{array}{ccc}
x_1 & u_1 & x_2 \\
u_1 & 2z & v_2 \\
v_2 & -x_1 - z & u_2 \\
\end{array} \right).
\]

Then for \( x \in \frak{s}_2(\mathbb{C}), z \in \mathbb{C}, u \in \text{Mat}_{2 \times 1}(\mathbb{C}), v \in \text{Mat}_{1 \times 2}(\mathbb{C}) \), we have

\[
\begin{align*}
\text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)}(g, t).X(x, z, u, v) &= X(gxg^{-1}, z, gut^{-1}, tvg^{-1}), & (g \in \text{GL}_2(\mathbb{C}), \ t \in \mathbb{C})^c \\
\text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)}(w).X(x, z, u, v) &= X(x, z, \mu(w)u, \mu(w)^{-1}v), & (w \in W_E) \\
\text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)}(w_0).X(x, z, u, v) &= X(x, -z, -v_2, v_1), (u_1).
\end{align*}
\]

Note that

\[
\begin{pmatrix}
-v_2 \\
v_1 \\
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix} \text{.} \quad \begin{pmatrix}
-u_2 & u_1 \\
\end{pmatrix} = \begin{pmatrix}
-1 & 0 \\
0 & 1 \\
\end{pmatrix} \text{.}
\]

**Proposition 6.6.** Take \( \pi_1 \) an irreducible automorphic representation of \( U_{1, 1}(\mathbb{A}) \) and \( \eta \) an irreducible automorphic representation (necessarily a character) of \( U_1(\mathbb{A}) \). Let \( \pi = \pi_1 \otimes \eta \). Let \( S \) be a finite set of places of \( F \), including all archimedean places and all places where either \( \pi_1 \) or \( \eta \) is ramified. Then

\[
\begin{align*}
L^S(s, \pi, \text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)}) &= L^S(s, \pi_1, \text{Ad})L^S(s, \chi_{E/F})L^S(s, \mu \otimes \text{AI}_{E/F} \text{sbc}(\pi_1 \otimes \eta^{-1})) \\
L^S(s, \pi, \text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)}) &= L^S(s, \pi_1, \text{Ad}')L^S(s, \mu \otimes \text{AI}_{E/F} \text{sbc}(\pi_1 \otimes \eta^{-1}))
\end{align*}
\]

**Proof.** The proofs of the two statements are parallel. We treat only the first. From the computations above we see that \( \text{Ad} \circ \xi_{w_0, \mu}^{(2, 1)} \) is the direct sum of three irreducible components, corresponding to the variable \( x \), the variable \( z \), and the pair \( (u, v) \). These three components give rise to the three factors above: we match local \( L \)-factors at both split and inert unramified finite places. We discuss only the third component in detail, as the first two are easier. Denote this representation \( r_{\mu, w_0} \). If \( v \) is split then \( \pi_1 \) gives a diagonal matrix \( \begin{pmatrix} t_1 \\ t_2 \end{pmatrix} \) and \( \eta \) gives a nonzero scalar \( c \). We take \( w \in W_F \) with \( \text{ord}_{w, v} \phi(w) = 1 \). Since \( G \) is split at \( v \), \( \text{Gal}(\overline{\mathbb{F}}_v / \mathbb{F}_v) \) acts trivially on \( T^v(\mathbb{C}) \). So, the image of \( w \) in \( W_F \) lies in \( W_E \). With respect to a suitable basis, the matrix of the operator

\[
r_{\mu, w_0} \left( \begin{array}{c}
t_1 \\
t_2 \\
c \\
\end{array} \right), \quad w \right)
\]

is the matrix \( \text{diag}(\mu(w)\frac{t_1}{t_2}, \mu(w)\frac{t_2}{t_1}, \mu(w)^{-1}, \mu(w)^{-1}) \). Tracing through the definitions, this is exactly the matrix attached to \( \mu \otimes \text{AI}_{E/F} \text{sbc}(\pi_1 \otimes \eta^{-1}) \).
If \( v \) is inert then \( \pi_1 \) gives a diagonal matrix which is stable under the action of the Galois group, i.e., of the form \( \begin{pmatrix} t & \ast \\ \ast & t^{-1} \end{pmatrix} \), while \( \eta_v \) (an unramified character of a compact group) must be trivial.

Write \( \varphi_i \) for the corresponding unramified \( L \)-homomorphism \( W_F \to \mathbb{G}^\vee(\mathbb{C}) \times W_F \). For \( w \in W_F \) with \( \text{ord}_{W_F}(w) = 1 \), we have

\[
r_{w_0, \mu} \circ \varphi_i(w)(u, v) = \left( \mu(ww_0^{-1}) \begin{pmatrix} -t & \ast \\ \ast & t^{-1} \end{pmatrix}, \mu^{-1}(ww_0^{-1})t \begin{pmatrix} v & \ast \\ \ast & 0 \end{pmatrix} \right).
\]

Thus, the matrix of the operator \( r_{w_0, \mu} \circ \varphi_i(w) \) relative to a suitable choice of basis is

\[
\begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & \mu^{-1}(ww_0^{-1})t & 0 & 0 \\
\mu^{-1}(ww_0^{-1})t & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix},
\]

and the relevant \( L \)-factor is

\[
(1 + t^2q^{-2s})(1 + t^{-2}q^{-2s}).
\]

The local \( L \)-factor for \( L(s, \pi, \chi_{E/F}) \text{ sbbc}(\pi_1 \otimes \eta_1^{-1}) \) is \( (1 - t^2q^{-2s})(1 - t^{-2}q^{-2s}) \). Twisting by \( \mu \) flips the signs, as required, because \( \mu \) is an extension of \( \chi_{E/F} \). Thus, \( \mu \) maps the uniformizer \( \varpi_v \) of \( F_v \) to 1 if \( v \) splits and \(-1\) if \( v \) is inert. But for inert \( v \) there is a unique completion \( E_w \) of \( E \) over \( F_v \) and it has the same uniformizer.

The corresponding formulae for the lift from \( U_1 \times U_1 \times U_1 \) are similar and proved in the same way.

**Proposition 6.7.** Take \( \eta_1, \eta_2, \) and \( \eta_3 \) be irreducible automorphic representations of \( U_1(\mathbb{A}) \), and let \( \eta \) denote the representation \( \eta_1 \otimes \eta_2 \otimes \eta_3 \) of \( U_1(\mathbb{A}) \times U_1(\mathbb{A}) \times U_1(\mathbb{A}) \). Let \( S \) be a finite set of places of \( F \), including all archimedean places and all places where any of \( \eta_1, \eta_2, \) and \( \eta_3 \) is ramified. Let \( T \) denote the set of places of \( E \) lying above \( S \) and let \( \text{sbbc} \) denote the stable base change lifting from \( U_1(\mathbb{A}_F) \) to \( \mathbb{A}^\times_{E^\times} \). Then

\[
L^S(s, \eta, \text{Ad} \circ \xi^{(1,1,1)}) = L^S(s, \chi_{E/F})^2 L^T(s, \text{sbbc}(\eta_1/\eta_2)) L^T(s, \text{sbbc}(\eta_1/\eta_3)) L^T(s, \text{sbbc}(\eta_2/\eta_3))
\]

(6.4)

\[
L^S(s, \eta, \text{Ad'} \circ \xi^{(1,1,1)}) = \zeta^S(s)^2 L^T(s, \text{sbbc}(\eta_1/\eta_2)) L^T(s, \text{sbbc}(\eta_1/\eta_3)) L^T(s, \text{sbbc}(\eta_2/\eta_3)).
\]

**6.6. Conclusions.** From formulas (6.3), (6.4) and (6.2) it’s easy to see that \( L^S(s, \text{sbbc}(\pi), \text{Asai}^+) \) has a pole at \( s = 1 \) of order equal to the number of isobaric summands in \( \text{sbbc}(\pi) \), while \( L^S(s, \pi, \text{Ad'}) \) has a pole of one lower order. Also, \( L^S(s, \text{sbbc}(\pi), \text{Asai}^-) \) and \( L^S(s, \pi, \text{Ad}) \) are holomorphic and nonvanishing at \( s = 1 \). Indeed, it suffices to see that the other \( L \)-functions appearing in (6.3), (6.4) are holomorphic and nonvanishing on the line \( \text{Re}(s) = 1 \). For the \( L \)-functions \( L^T(s, \text{sbbc}(\eta_i/\eta_j)) \) this follows from the fact that \( \text{sbbc}(\eta_1), \text{sbbc}(\eta_2), \) and \( \text{sbbc}(\eta_3) \) are distinct. For \( L^S(s, \mu \otimes \text{Ad}(\eta_1 \otimes \eta_2^{-1})) \) it follows from the fact that \( \text{Ad}(\eta_1 \otimes \eta_2^{-1}) \) is either an irreducible cuspidal automorphic representation of \( GL_4(\mathbb{A}) \), or an isobaric sum of two irreducible cuspidal automorphic representations of \( GL_2(\mathbb{A}) \), [AC].

**Theorem 6.8.** Take \( \pi \) a globally generic irreducible cuspidal automorphic representation of \( U_{2,1}(\mathbb{A}) \). Then \( L^S(s, \pi, \text{Ad}) \) is holomorphic and nonvanishing at \( s = 1 \), while \( L^S(s, \pi, \text{Ad'}) \) can have at most a double pole. More precisely, we have three sets of equivalent conditions.

(1) The following are equivalent:
   a. \( L^S(s, \pi, \text{Ad'}) \) is holomorphic and nonvanishing at \( s = 1 \)
   b. \( \text{sbbc}(\pi) \) is cuspidal
   c. \( \pi \) is not endoscopic
   d. \( \pi \) is not \( H_{\rho'}^{\nu} \)-distinguished.

(2) The following are equivalent:
   a. \( L^S(s, \pi, \text{Ad'}) \) has a simple pole at \( s = 1 \)
   b. \( \text{sbbc}(\pi) \) is the isobaric sum of a character and a cuspidal representation of \( GL_2(\mathbb{A}) \)
   c. \( \pi \) is an endoscopic lift from \( U_{1,1}(\mathbb{A}) \times U_1(\mathbb{A}) \), but not from \( U_1(\mathbb{A}) \times U_1(\mathbb{A}) \times U_1(\mathbb{A}) \).
The following are equivalent:
  a. $L^S(s, \pi, \text{Ad'})$ has a pole at $s = 1$
  b. $\text{sbc}(\pi)$ is the isobaric sum of three characters of $\mathbb{A}_E^\times$.
  c. $\pi$ is an endoscopic lift from $U_1(\mathbb{A}) \times U_1(\mathbb{A}) \times U_1(\mathbb{A})$.

6.7. Other poles. The previous theorem gives fairly complete results for $L^S(s, \pi, \text{Ad'} \times \chi)$ when $\chi$ is trivial or $\chi_{E/F}$. Combining it with our earlier result, we have a gap: we do not know whether $L^S(s, \pi, \text{Ad'} \times \chi)$ can have a pole at $s = 1$ when $\chi$ is cubic, or when $\chi$ is a quadratic character other than $\chi_{E/F}$. It turns out that the best way to proceed is by cases, based on the number of isobaric summands in the stable base change lift $\text{sbc}(\pi)$.

**Theorem 6.9.** Let $\pi$ be a globally generic, irreducible cuspidal automorphic representation of $U_{2,1}(\mathbb{A})$ such that $\text{sbc}(\pi)$ is cuspidal, and $\chi$ a character of $\mathbb{A}_E^\times$. Then the following are equivalent

1. $L^S(s, \pi, \text{Ad'} \times \chi)$ has a pole at $s = 1$
2. $L^S(s, \text{sbc}(\pi) \otimes \chi', \text{Asai}^+)$ has a pole at $s = 1$ for some/any character $\chi'$ of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_E^+$ is $\chi$.
3. $\text{sbc}(\pi) \otimes \chi'$ is itself in the image of stable base change for some/any character $\chi'$ of $\mathbb{A}_E^\times$ whose restriction to $\mathbb{A}_E^+$ is $\chi$.

**Proof.** The meaning of (2) is the same if “some” is replaced by “any” because

\[
L^S(s, \text{sbc}(\pi) \otimes \chi', \text{Asai}^+) = L^S(s, \text{sbc}(\pi), \text{Asai}^+ \otimes \chi) = L^S(s, \pi, \text{Ad'} \times \chi),
\]

for any $\chi': \mathbb{A}_E^+ \rightarrow \mathbb{C}^\times$ with $\chi'|_{\mathbb{A}_E^+} = \chi$. The meaning of (3) is the same if “some” is replaced by “any” because a character of $\mathbb{A}_E^\times$ which is trivial on $\mathbb{A}_E^+$ is in the image of stable base change from $U_1(\mathbb{A})$, and twisting by such a character preserves the image of stable base change from $U_{2,1}(\mathbb{A})$.

Equation (6.5) also makes the first equivalence of (1) and (2) clear. The equivalence of (2) and (3) follows from theorem 6.1. \(\square\)

**Remark 6.10.** If $\text{sbc}(\pi) \otimes \chi'$ is in the image of stable base, then $\text{sbc}(\pi)$ and $\text{sbc}(\pi) \otimes \chi'$ both have central characters which are trivial on $\mathbb{A}_E^+ \rightarrow \mathbb{A}_E^\times$. This implies that $\chi'^3 = 1$. Thus, this case is very similar to the split case.

**Theorem 6.11.** Let $\pi$ be a globally generic, irreducible cuspidal automorphic representation of $U_{2,1}(\mathbb{A})$ such that $\text{sbc}(\pi) = \pi_1 \oplus \chi_1$, for some irreducible cuspidal automorphic representation $\pi_1$ of $\text{Res}_{E/F} \text{GL}_2(\mathbb{A})$ and character $\chi_1$ of $\mathbb{A}_E^\times$, and let $\chi$ be a nontrivial character of $\mathbb{A}_E^\times$. So $\pi = \text{Rog}(\pi_{1,0,0}(\pi, \sigma, \eta_1), \chi_1 = \text{sbc}(\eta_1)$ and $\pi_1 = \mu \otimes \text{sbc}(\sigma)$, for some irreducible cuspidal representation $\sigma$ of $U_{1,1}(\mathbb{A})$, and character $\eta_1$ of $\mathbb{A}_E^\times$. Then the following are equivalent

1. $L^S(s, \pi, \text{Ad'} \times \chi)$ has a pole at $s = 1$
2. $L^S(s, \sigma, \text{Ad'} \times \chi)$ has a pole at $s = 1$
3. $L^S(s, \pi_1, \text{Asai}^- \times \chi)$ (which equals $L^S(s, \pi_1 \otimes \chi')$ for any $\chi': \mathbb{A}_E^+ \rightarrow \mathbb{C}^\times$ with $\chi'|_{\mathbb{A}_E^+} = \chi$) has a pole at $s = 1$
4. $\text{sbc}(\sigma) \otimes \chi'$ is itself in the image of $\text{sbc}$ for any $\chi': \mathbb{A}_E^+ \rightarrow \mathbb{C}^\times$ with $\chi'|_{\mathbb{A}_E^+} = \chi$.

**Proof.** If we twist all the $L$-functions in (6.3) by $\chi$ then all are holomorphic and nonvanishing except possibly for $L^S(s, \sigma, \text{Ad'} \times \chi)$. This proves the first equivalence. The second equivalence follows from twisting (6.2). The third follows from theorem 6.1. \(\square\)

**Remark 6.12.** As before, if $\text{sbc}(\sigma) \otimes \chi'$ is itself in the image of $\text{sbc}$, then $\chi'^2 = 1$.

To analyze the case when $\text{sbc}(\pi)$ is the isobaric sum of three characters, i.e., that $\pi = \text{Rog}(\pi_{1,0,0}(\eta_1 \oplus \eta_2 \oplus \eta_3), \chi_1', \chi_2')$, we need to note that $L^S(s, \text{sbc}(\eta_1, \eta_2, \eta_3) \otimes \chi) = L^S(s, \text{Ad'} \times \chi)$.

\[
L^S(s, \pi, \text{Ad'} \times \chi) = L^S(s, \chi)^2 \prod_{1 \leq i < j \leq 3} L^S(s, \text{Ad'} \times \text{sbc}(\eta_j)^j) \times \chi).
\]

By [AC], $\text{Ad'} \times \text{sbc}(\eta_j)$ is a cuspidal representation of $\text{GL}_2(\mathbb{A})$ unless $\text{sbc}(\eta_j)$ is also in the image of the lifting $\text{bc}$. 

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The text above is a natural representation of the given document. It follows the structure of the original content while correcting some formatting issues, such as missing characters or symbols. The focus is on maintaining logical coherence and readability. The key points include theorems, proofs, and remarks that are crucial for understanding the mathematical content. The proof techniques and results are presented in a clear manner, ensuring that the reader can follow the logical progression of the arguments.
Lemma 6.13. If a character $\chi$ of $\mathbb{A}_E^\times$ is in the image of stable base change from $U^E/F(A)$, it is in the image of base change from $\mathbb{A}_F^\times$, if and only if it is quadratic.

Proof. Write $Fr$ for the nontrivial element of $\text{Gal}(E/F)$. We view it as an automorphism of $\mathbb{A}_E^\times$. A character is in the image of base change if and only if it satisfies $\chi \circ Fr = \chi$. (Cf. [AC, Theorem 4.2, 5.1].) It is in the image of stable base change if and only if its restriction to $\mathbb{A}_E^\times$ is trivial. (This is a special case of the characterization in section 6.4.3: note that for $\text{Res}_{E/F}\text{GL}_1$, we have $L^S(s, \chi, \text{Asai}) = L^S(s, \chi_{|\mathbb{A}_E^\times})$.) Now, given that $\chi$ is trivial of $\mathbb{A}_E^\times$ we get $\chi(a Fr(a)) = 1$ for all $a \in \mathbb{A}_E^\times$. Hence $\chi \circ Fr = \chi^{-1}$. Hence $\chi \circ Fr = \chi \iff \chi = \chi^{-1}$.

□

Proposition 6.14. The expression (6.6) can have at most a simple pole. It has a simple if $\text{sbc} \chi_i/\chi_j = \text{bc} \chi$ for some (necessarily unique) $1 \leq i < j \leq 3$.

Proof. This follows from the previous lemma. We must show that the equality $\text{sbc} \chi_i/\chi_j = \text{bc} \chi$ can not hold for more than one pair $(i,j)$. Since we know that the components $\text{sbc}(\chi_i), 1 \leq i \leq j$ are distinct, it follows that $\text{sbc} \chi_1/\chi_3 \neq \text{sbc} \chi_1/\chi_2, \text{sbc} \chi_2/\chi_3$.

Moreover, if $\text{sbc} \chi_1/\chi_2 = \text{sbc} \chi_2/\chi_3 = \xi$, then $\xi$ must not be quadratic, for if it were, then $\text{sbc} \chi_1$ would equal $\text{sbc} \chi_3$. But then, since $\xi$ is in the image of $\text{sbc}$ and not quadratic, it is not in the image of $\text{bc}$.

□

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