Combining nondeterminism, probability, and termination: equational and metric reasoning

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Abstract. We study monads resulting from the combination of nondeterministic and probabilistic behaviour with the possibility of termination, which is essential in program semantics. Our main contributions are presentation results for the monads, providing equational reasoning tools for establishing equivalences and distances of programs.

Keywords: Monads · Nondeterminism · Probability · Termination

1 Introduction
In the theory of programming languages the categorical concept of monad is used to handle computational effects [33,34]. As main examples, the non-empty finite powerset monad ($\mathcal{P} : \text{Set} \rightarrow \text{Set}$) and the finitely supported probability distribution monad ($\mathcal{D} : \text{Set} \rightarrow \text{Set}$) are used to handle nondeterministic and probabilistic behaviours, respectively. The non-empty convex sets of probability distributions monad ($\mathcal{C} : \text{Set} \rightarrow \text{Set}$) has been identified in several works (see, e.g., [41,22,21,32,31,25,28,13,23,29]) as a convenient way to handle the combination of nondeterminism and probability. Liftings of these monads to the category of (1-bounded) metric spaces have been investigated using the technical machinery of Hausdorff and Kantorovich metric liftings: ($\hat{\mathcal{P}} : \text{1Met} \rightarrow \text{1Met}$), ($\hat{\mathcal{D}} : \text{1Met} \rightarrow \text{1Met}$) and more recently ($\hat{\mathcal{C}} : \text{1Met} \rightarrow \text{1Met}$) [16,6,29]. The category $\text{1Met}$ is a natural setting when it is desirable to switch from the concept of program equivalence to that of program distance.

Monads are tightly connected with equational theories. Mathematically, this connection emerges from the notion of Eilenberg-Moore (EM) algebras. For every monad $M$ there is a category $\text{EM}(M)$ of $M$-algebras and, in many interesting cases, this can be presented by (i.e., proved isomorphic to) a well-known category of algebras (in the standard sense of universal algebra). For example, $\text{EM}(\mathcal{P})$ is isomorphic to the category of semilattices and semilattice-homomorphisms. This is the mathematical fact underlying the ubiquity of semilattices in mathematical treatments of nondeterminism and is the basis of several advanced techniques for reasoning about nondeterministic programs (e.g., bisimulation up-to techniques [11].) Other important examples include the presentations of the monads $\mathcal{D}$ and $\mathcal{C}$ by convex (a.k.a. barycentric) algebras [40,19,24] and convex semilattices [13,14]. Recently, presentation results have been obtained also for the $\text{1Met}$ variants of these monads, $\hat{\mathcal{P}}$, $\hat{\mathcal{D}}$ and $\hat{\mathcal{C}}$, using the framework of quantitative algebras and quantitative theories of [26,27,4,3,2]. These monads are presented...
by the quantitative equational theories of quantitative semilattices, quantitative convex algebras and quantitative convex semilattices, respectively \cite{26,29}.

These presentation results provide equational methods for reasoning about equivalences and distances of programs whose semantics is modelled as a transition system (i.e., a coalgebra) of type $\text{States} \rightarrow F(\text{States})$, $F \in \{P, D, C, \hat{P}, \hat{D}, \hat{C}\}$ (or, if labels $L$ are considered, of type $\text{States} \rightarrow (F(\text{States}))^L$). However these functors may not be appropriate for all modelling purposes. Indeed, note that for all six functors above, the final $F$–coalgebra has the singleton set as carrier, which means that all states of an $F$–coalgebra are behavioural equivalent. Usually, what is needed is some kind of \textit{behavioural observation} such as a termination state. This is generally achieved by using the functor $F + 1$ (where $+$ and $1$ are the coproduct and the terminal object, respectively): a state can either transition to $F(\text{States})$ or terminate. Even if $F$ is a monad and the functor $F + 1$ is similar to $F$, separate work is needed to answer questions such as: is there a monad having $F + 1$ as underlying functor? How is this related to the monad $F$? What is its presentation? For some specific cases the answers are well–known. For instance, the functor $P + 1$ (possibly empty finite powerset) carries a monad structure which is presented by semilattices with bottom (i.e., semilattices with a designated element $\star$, representing termination, such that $x \oplus \star = x$).

**Contributions.** The main results of our work are the following.

1. We describe in Section 3 a \textbf{Set} monad whose underlying functor is $C + 1$ and prove that it is presented by the theory of convex semilattices extended with the axioms bottom ($x \oplus \star = x$) and black–hole ($x +_p \star = \star$, see \cite{30,38}). Transition systems of type $C + 1$ are well–known in the literature as (simple) convex Segala systems \cite{36,10,37} and are widely used to model the semantics of nondeterministic and probabilistic programs. Hence, this result provides equational reasoning methods for an important class of systems.

2. The black–hole axiom annihilates probabilistic termination, thus it is not appropriate in all modelling situations. So, we investigate in Section 4 a monad $C^\downarrow$ presented by the weaker theory of convex semilattices with bottom (but without black–hole). This equational theory has already found applications in the study of trace semantics of probabilistic nondeterministic programs in \cite{13}.

3. In an attempt to find a \textbf{1Met} monad $M$ on the Hausdorff–Kantorovich metric lifting of the \textbf{Set} functor $\hat{C} + 1$, we prove in Section 5.1 some negative results. First, no such $M$ exists having as multiplication the same operation of the \textbf{Set} monad $C + 1$. Secondly, $M$ cannot be presented by the quantitative theory of convex semilattices with bottom and black–hole, since this theory is trivial.

4. In Section 5.2 we identify the \textbf{1Met} monad $\hat{C}^\downarrow$ which is the Hausdorff–Kantorovich metric lifting of the \textbf{Set} monad $C^\downarrow$ of point (2) and is presented by the quantitative equational theory of convex semilattices with bottom.

We conclude with some examples of applications of our results to program equivalences and distances in Section 6. Full proofs of the results presented in this paper and additional background material are available in the Appendix.
2 Background

We present some definitions and results regarding monads. We assume the reader is familiar with basic concepts of category theory (see, e.g., [1]).

Definition 1 (Monad). Given a category $\mathbb{C}$, a monad on $\mathbb{C}$ is a triple $(M, \eta, \mu)$ composed of a functor $M : \mathbb{C} \to \mathbb{C}$ together with two natural transformations: a unit $\eta : \text{id}_{\mathbb{C}} \Rightarrow M$, where $\text{id}_{\mathbb{C}}$ is the identity functor on $\mathbb{C}$, and a multiplication $\mu : M^2 \Rightarrow M$, satisfying $\mu \circ \eta M = \mu \circ M \eta = \text{id}_M$ and $\mu \circ M \mu = \mu \circ \mu M$.

If $\mathbb{C}$ has coproducts, $A_1, A_2, B \in \mathbb{C}$, $f_1 : A_1 \to B$ and $f_2 : A_2 \to B$, we denote with $[f_1, f_2] : A_1 + A_2 \to B$ the unique morphism such that $f_1 = [f_1, f_2] \circ \text{inl}$ and $f_2 = [f_1, f_2] \circ \text{inr}$ where $\text{inl} : A_1 \to A_1 + A_2$ and $\text{inr} : A_2 \to A_1 + A_2$ are the canonical injections. We denote with $1_\mathbb{C}$ the terminal object of $\mathbb{C}$, if it exists.

Proposition 1. Let $\mathbb{C}$ be a category having coproducts and a terminal object. The $\mathbb{C}$ monad $+1$ is defined as the triple $(+, 1_\mathbb{C}, \eta^1, \mu^1)$ whose functor $(+ + 1_\mathbb{C})$ is defined on objects as $A \mapsto A + 1_\mathbb{C}$ and on arrows as $f \mapsto [\text{inl} \circ f, \text{inr}]$, with unit $\eta^1 = \text{inl}$ and with multiplication $\mu^1 = [[\text{inl}, \text{inr}]]$.

Monads can be combined together using monad distributive laws.

Definition 2 (Monad distributive law). Let $(M, \eta, \mu)$ and $(\hat{M}, \hat{\eta}, \hat{\mu})$ be two monads on $\mathbb{C}$. A natural transformation $\lambda : M \hat{M} \Rightarrow \hat{M}M$ is called a monad distributive law of $M$ over $\hat{M}$ if it satisfies the equations $\lambda \circ M \hat{\eta} = \hat{\eta}M$, $\lambda \circ \hat{\eta}M = \hat{M} \eta$, $\lambda \circ \mu M = M \mu \circ \lambda M$ and $\lambda \circ \hat{M} \mu = \hat{\mu}M \circ M \lambda \circ \lambda M$.

Proposition 2. If $\lambda : M \hat{M} \Rightarrow \hat{M}M$ is a monad distributive law, then $(\hat{M}, \eta, \mu)$ is a monad with $\hat{M} = \hat{M}M$, $\eta = \hat{\eta} \circ \eta$ and $\mu = (\hat{\mu} \circ \mu) \circ \hat{M} \lambda M$.

Corollary 1. Let $\mathbb{C}$ have coproducts and a terminal object and $M : \mathbb{C} \to \mathbb{C}$ be a monad. Then there is a $\mathbb{C}$ monad structure $(M(+1), \eta^M(+1), \mu^M(+1))$ on the functor $M(+1)$, given by Proposition 2 using the monad distributive law $\iota : M + 1 \Rightarrow M(+1)$ defined as $\iota_X = [\text{inl}, \eta^M_{X+1} \circ \text{inr}]$.

A monad $M$ has an associated category of $M$–algebras.

Definition 3 ($M$–algebras). Let $(M, \eta, \mu)$ be a monad on $\mathbb{C}$. An algebra for $M$ (or $M$–algebra) is a pair $(A, \alpha)$ where $A \in \mathbb{C}$ is an object and $\alpha : M(A) \to A$ is a morphism such that $\alpha \circ \eta_A = \text{id}_A$ and $\alpha \circ M \alpha = \alpha \circ \mu_A$ hold. Given two $M$–algebras $(A, \alpha)$ and $(A', \alpha')$, an $M$–algebra morphism is an arrow $f : A \to A'$ in $\mathbb{C}$ such that $f \circ \alpha = \alpha' \circ M(f)$. The category of $M$–algebras and their morphisms, denoted $\text{EM}(M)$, is called the Eilenberg-Moore category for $M$.

\footnote{$f \circ g = G_2 f \circ g F_1 = g F_2 \circ G_1 f$, for $f : F_1 \Rightarrow F_2, g : G_1 \Rightarrow G_2$ ([35], Lemma 1.4.7).}
Definition 4 (Monad map). Let \((M, \eta, \mu)\) and \((\widehat{M}, \hat{\eta}, \hat{\mu})\) be two monads. A natural transformation \(\sigma : M \Rightarrow \widehat{M}\) is called a monad map if it satisfies the laws
\[
\hat{\eta} = \sigma \circ \eta \quad \text{and} \quad \sigma \circ \mu = \hat{\mu} \circ (\sigma \circ \sigma)
\] (see footnote 7).

Proposition 3 (Theorem 6.3[8]). If \(\sigma : M \Rightarrow \widehat{M}\) is a monad map, then \(U^\sigma = (A, \alpha) \mapsto (A, \alpha \circ \alpha)\) is a functor \(\mathbb{EM}(\widehat{M}) \to \mathbb{EM}(M)\).

2.1 Monads on Set and equational theories

In this section, we restrict attention to monads on the category \(\textbf{Set}\), which has coproducts (disjoint unions) and a terminal object (the singleton set \(\{\ast\}\)). Hence, by Proposition 1 the \(+\ 1\) monad, also referred to as the termination monad, is well defined in \(\textbf{Set}\). When no confusion arises, we omit explicit mentioning of the injections, and write for example \((f + 1)(x) = x\) for \(x \in X\) and \((f + 1)(\ast) = \ast\).

We now introduce the \(\textbf{Set}\) monad \(C\) of non–empty finitely generated convex sets of finitely supported probability distributions. This requires a number of definitions and notations regarding sets and probability distributions.

A probability distribution (respectively, subdistribution) on a set \(X\) is a function \(\varphi : X \to [0,1]\) such that \(\sum_{x \in X} \varphi(x) = 1\) (respectively, \(\sum_{x \in X} \varphi(x) \leq 1\). The support of \(\varphi\) is defined as \(\text{supp}(\varphi) = \{x \in X \mid \varphi(x) \neq 0\}\). In this paper, we only consider probability distributions with finite support, so we just refer to them as distributions. The Dirac distribution \(\delta_x\) is defined as \(\delta_x(x') = 1\) if \(x' = x\) and \(\delta_x(x') = 0\) otherwise. We often denote a distribution having \(\text{supp}(\varphi) = \{x_1, \ldots, x_n\}\) by the expression \(\sum_{i=1}^{n} p_i x_i\), with \(p_i = \varphi(x_i)\). We denote with \(\mathcal{D}(X)\) the set of finitely supported probability distributions on \(X\). This becomes a \(\textbf{Set}\) functor by defining, for any \(f : X \to Y\) in \(\textbf{Set}\), the arrow \(\mathcal{D}(f) : \mathcal{D}(X) \to \mathcal{D}(Y)\) as \(\mathcal{D}(f)(\varphi) = (y \mapsto \sum_{x \in f^{-1}(y)} \varphi(x))\).

Given a set \(S \subseteq \mathcal{D}(X)\) of distributions, we denote with \(\text{cc}(S)\) the convex closure of \(S\), i.e., the set of distributions \(\varphi\) that are convex combinations \(\sum_{i=1}^{n} p_i \varphi_i \in S\). Clearly \(S \subseteq \text{cc}(S)\). We say that a convex set \(S \subseteq \mathcal{D}(X)\) is finitely generated if there exists a finite set \(S' \subseteq \mathcal{D}(X)\) such that \(S = \text{cc}(S')\). The finite set \(S'\) is referred to as a base of \(S\). We denote with \(\mathcal{C}(X)\) the set \(\{S \subseteq \mathcal{D}(X) \mid S \neq \emptyset \text{ and } S\text{ is convex and finitely generated}\}\). This can be turned into a \(\textbf{Set}\) functor by defining for every \(f : X \to Y\) the arrow \(\mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y)\) as \(\mathcal{C}(f)(S) = \{\mathcal{D}(f)(\varphi) \mid \varphi \in S\}\). We are now ready to define a monad on the \(\textbf{Set}\) functor \(\mathcal{C}\) (see [13]).

Definition 5 (Monad \(C\)). The non-empty finitely generated convex sets of distributions \(\textbf{Set}\) monad is the triple \((\mathcal{C}, \eta^C, \mu^C)\) consisting of the functor \(\mathcal{C}\), unit \(\eta^C_X(x) = \{\delta_x\}\) and multiplication defined, for any \(S \in \mathcal{C}(X)\), as
\[
\mu^C_X(S) = \bigcup_{\varphi \in \text{WMS}(\varphi)} \text{WMS}(\varphi) = \{\sum_{i=1}^{n} p_i \varphi_i \mid \text{for each } 1 \leq i \leq n, \varphi_i \in S_i\}
\]
where, for any \(\varphi \in \mathcal{D}(X)\) of the form \(\sum_{i=1}^{n} p_i S_i\), with \(S_i \in \mathcal{C}(X)\), the weighted Minkowski sum operation \(\text{WMS} : \mathcal{D}(X) \to \mathcal{C}(X)\) is defined as above.
As a consequence of Proposition 2 and Corollary 1 there is also a Set monad $(\mathcal{C}(\cdot + 1), \eta^{\mathcal{C}(+1)}, \mu^{\mathcal{C}(+1)})$ on the composition of $\mathcal{C}$ and $+1$.

**Proposition 4 (Monad $\mathcal{C}(+1)$).** There is a Set monad $(\mathcal{C}(\cdot + 1), \eta^{\mathcal{C}(+1)}, \mu^{\mathcal{C}(+1)})$.

**Remark 1.** There is a bijective correspondence between distributions $\varphi$ on $X + 1$ and subdistributions $\varphi'$ on $X$ (i.e., $\varphi(x) = \varphi'(x)$ for $x \in X$ and $\varphi(*) = 1 - \sum_{x \in X} \varphi'(x)$). We will use this identification and often refer to the monad $\mathcal{C}(\cdot + 1)$ as the non-empty finitely generated convex sets of subdistributions monad.

In [13], presentations theorems for the monads $\mathcal{C}$ and $\mathcal{C}(+1)$ are given in terms of the equational theories of convex semilattices and pointed convex semilattices, respectively. We assume the reader is familiar with the basic notions of universal algebra such as: signature, algebras for a signature, homomorphisms, etc.

**Definition 6 (Convex Semilattices).** The theory $\text{Th}_{CS}$ of convex semilattices has signature $\Sigma_{CS} = \{\oplus\} \cup \{+\}_{p \in (0, 1]}$ and the following axioms:

- (A) $x \oplus (y \oplus z) = (x \oplus y) \oplus z$ (AP) $(x + y) +_p z = x + p(y +_p (1 - p) z)$
- (C) $x \oplus y = y \oplus x$ (Cp) $x +_p y = y +_p (1 - p)x$
- (I) $x \oplus x = x$ (Ip) $x +_p x = x$
- (D) $x +_p (y \oplus z) = (x +_p y) \oplus (x +_p z)$

A convex semilattice is an $\Sigma_{CS}$-algebra satisfying the equations in $\text{Th}_{CS}$. We let $\mathbf{A}(\text{Th}_{CS})$ denote the category of convex semilattices and their homomorphisms.

**Definition 7 (Pointed convex semilattices).** The theory $\text{Th}_{CS}^\ast$ of pointed convex semilattices has signature $\Sigma_{CS}^\ast = \{\oplus\} \cup \{+\}_{p \in (0, 1]} \cup \{\ast\}$ and the same axioms of the theory of convex semilattices. We denote with $\mathbf{A}(\text{Th}_{CS}^\ast)$ the category of pointed convex semilattices and their homomorphisms.

The presentation theorems for $\mathcal{C}$ and $\mathcal{C}(+1)$ in [13] can now be formally stated as the following isomorphisms of categories.

**Proposition 5.**

1. The theory $\text{Th}_{CS}$ of convex semilattices is a presentation of the monad $\mathcal{C}$, i.e., $\text{EM}(\mathcal{C}) \cong \mathbf{A}(\text{Th}_{CS})$.

2. The theory $\text{Th}_{CS}^\ast$ of pointed convex semilattices is a presentation of the monad $\mathcal{C}(+1)$, i.e., $\text{EM}(\mathcal{C}(+1)) \cong \mathbf{A}(\text{Th}_{CS}^\ast)$.

The isomorphism in Proposition 5 is given by a pair of functors

$$P : \text{EM}(\mathcal{C}(+1)) \to \mathbf{A}(\text{Th}_{CS}^\ast) \quad P^{-1} : \mathbf{A}(\text{Th}_{CS}^\ast) \to \text{EM}(\mathcal{C}(+1))$$

with $P(f) = f$ (identity on morphisms) and $P(A, \alpha) = (A, \oplus^\alpha, \{+_p\}_{p \in (0, 1], \ast^\alpha})$ where, for all $a_1, a_2 \in A$: $\ast^\alpha = \alpha(\{\delta_\ast\}), a_1 \oplus^\alpha a_2 = \alpha(cc\{\delta_{a_1}, \delta_{a_2}\})$, and $a_1 +_p^\alpha a_2 = \alpha(\text{wMS}(p \{\delta_{a_1}\} + (1 - p) \{\delta_{a_2}\})).$

Using the above presentation result and the well-known fact that, for any monad $(M, \eta, \mu)$, the free $M$-algebra generated by $X$ is $(M(X), \mu_X)$, we can identify (up to isomorphism) the free pointed convex semilattices.
Proposition 6. The free pointed convex semilattice on $X$ is (up to isomorphism) \((C(X+1), \oplus^{C(+1)}, \{ _p^+ \}_{p \in (0, 1)}, *_{\text{plots}}(+1))\) with, for all $S_1, S_2 \in C(X+1)$: $*_{\text{plots}}(+1) = \{ \delta_x \}$, $S_1 \oplus^{C(+1)} S_2 = aC(S_1 \cup S_2)$, $S_1 +^C(+) S_2 = \text{wMS}(pS_1 + (1 - p)S_2)$.

2.2 Monads on $1\text{Met}$ and quantitative equational theories

In this section, we focus on monads on the category $1\text{Met}$ of 1–bounded metric spaces and non–expansive maps.

Definition 8 (Category $1\text{Met}$). A 1–bounded metric space is a pair $(X, d)$ with $X$ a set and $d : X \times X \to [0, 1]$ such that $d(x, y) = 0$ if and only if $x = y$, $d(x, y) = d(y, x)$, and $d(x, y) \leq d(x, z) + d(z, y)$, for all $x, y, z \in X$. A function $f : X \to Y$ between two 1–bounded metric spaces $(X, d_X)$ and $(Y, d_Y)$ is non–expansive if $d_Y(f(x_1), f(x_2)) \leq d_X(x_1, x_2)$ for all $x_1, x_2 \in X$. We denote with $1\text{Met}$ the category of 1–bounded metric spaces and non–expansive maps.

Since we only work with 1–bounded metric spaces, we often refer to them simply as metric spaces. The category $1\text{Met}$ has a terminal object (the space $1 = (\{ \ast \}, d_1)$, with $d_1(\ast, \ast) = 0$) and coproducts defined as $(X_1, d_1) + (X_2, d_2) = (X_1 + X_2, d_1 + d_2)$ where $X + Y$ denotes disjoint union and $(d_1 + d_2)(y, w) = d_i(y, w)$, if $y, w \in X_i$, for $i \in \{1, 2\}$ and $(d_1 + d_2)(y, w) = 1$ otherwise. Hence, by Proposition 11 the (+1) monad $\text{Set}_1$ is well defined in $1\text{Met}$.

We now introduce the $1\text{Met}$ monad $\mathcal{C}$, which is the Hausdorff–Kantorovich metric lifting of the Set monad $\mathcal{C}$ and which has been introduced in [29 §4].

Definition 9 (Kantorovich Lifting). Let $(X, d)$ be a 1–bounded metric space. The Kantorovich lifting of $d$ is a 1–bounded metric $K(d)$ on $\mathcal{D}(X)$, the collection of finitely supported distributions on $X$, defined for any pair $\varphi_1, \varphi_2 \in \mathcal{D}(X)$ as:

$$K(d)(\varphi_1, \varphi_2) = \inf_{\omega \in \text{Coup}(\varphi_1, \varphi_2)} \sum_{(x_1, x_2) \in X \times X} \omega(x_1, x_2) \cdot d(x_1, x_2)$$

where $\text{Coup}(\varphi_1, \varphi_2)$ is defined as the collection of couplings of $\varphi_1$ and $\varphi_2$, i.e., $\text{Coup}(\varphi_1, \varphi_2) = \{ \omega \in \mathcal{D}(X \times X) \mid \mathcal{D}(\pi_1)(\omega) = \varphi_1 \text{ and } \mathcal{D}(\pi_2)(\omega) = \varphi_2 \}$ where $\pi_1 : X_1 \times X_2 \to X_1$ and $\pi_2 : X_1 \times X_2 \to X_2$ are the projection functions.

Definition 10 (Hausdorff Lifting). Let $(X, d)$ be a 1–bounded metric space. The Hausdorff lifting of $d$ is a 1–bounded metric $H(d)$ on $\text{Comp}(X, d)$, the collection of non–empty compact subsets of $X$ (with respect to the standard open–ball topology defined by the metric $d$) defined for any pair $X_1, X_2 \in \text{Comp}(X, d)$ as:

$$H(d)(X_1, X_2) = \max \{ \sup_{x_1 \in X_1} \inf_{x_2 \in X_2} d(x_1, x_2), \sup_{x_2 \in X_2} \inf_{x_1 \in X_1} d(x_1, x_2) \}.$$
Hence, for every metric space \((X, d) \in \mathbf{1Met}\), the collection of non-empty compact sets of finitely supported probability distributions can be endowed with the Hausdorff–Kantorovich lifted metric \(H(K(d))\), which we write \(HK(d)\). Since all elements of \(\mathcal{C}(X)\) are compact, we obtain that \((\mathcal{C}(X), HK(d))\) is a 1–bounded metric space. This leads to the definition of the \(\mathbf{1Met}\) monad \(\mathcal{C}\) of \([29, \S 4]\).

**Definition 11 (Monad \(\mathcal{C}\)).** The monad \((\mathcal{C}, \eta^c, \mu^c)\) on \(\mathbf{1Met}\) is defined as follows. The functor \(\mathcal{C}\) is defined as mapping objects \((X, d)\) to \((\mathcal{C}(X), HK(d))\) and morphisms \(f : X \to Y\) to \(\mathcal{C}(f) = \mathcal{C}(f)\) (i.e., as the \(\mathbf{Set}\) functor \(\mathcal{C}\)). The unit \(\eta^c\) and the multiplication \(\mu^c\) are defined as for the \(\mathbf{Set}\) monad \(\mathcal{C}\) (Definition 7).

As a consequence of Proposition 2 and Corollary 4 there is also a \(\mathbf{1Met}\) monad \((\mathcal{C}(\cdot + 1), \eta^{c(1)}, \mu^{c(1)})\) on the composition of \(\mathcal{C}\) and \(+1\).

**Proposition 7 (Monad \(\mathcal{C}(\cdot + 1)\)).** There is a \(\mathbf{1Met}\) monad \((\mathcal{C}(\cdot + 1), \eta^{c(1)}, \mu^{c(1)})\).

Following Remark 1 we refer to \(\mathcal{C}\) (respectively \(\mathcal{C}(\cdot + 1)\)) as the \(\mathbf{1Met}\) monad of non-empty finitely generated convex sets of distributions (respectively subdistributions) with the Hausdorff–Kantorovich metric.

A main result of [29] is a presentation result for \(\mathcal{C}\), based on the recently introduced notions of quantitative algebras and quantitative equational theories introduced in [29] (see also [27,132]). This framework is, roughly speaking, adapting many usual concepts of universal algebra to deal with quantitative algebras, which are structures \((A, \{f^A\}_{f \in \Sigma}, d)\) where \((A, \{f^A\}_{f \in \Sigma})\) is a set with interpretations for the function symbols of a given signature \(\Sigma\) and \(d\) is a 1–bounded metric such that \((A, d) \in \mathbf{1Met}\) and, for every \(f \in \Sigma\) the map \(f^A\) is non-expansive with respect to the metric \(d\). The familiar concept of equation of universal algebra is replaced by that of quantitative inference \(\{x_i =_\epsilon y_i\}_{i \in I} \vdash s =_\epsilon t\) where \(\epsilon, \epsilon_i \in [0, 1]\) and \(s, t\) are terms over \(\Sigma\). A quantitative algebra \((A, \{f^A\}_{f \in \Sigma}, d)\) satisfies a quantitative inference if, for all interpretations \(\iota(x) \in A\) of the variables \(x\) as elements of \(A\), the following implication holds:

\[
\text{if } d(\iota(x_i), \iota(y_i)) \leq \epsilon_i \text{ for all } i \in I, \text{ then } (\iota(s), \iota(t)) \leq \epsilon.
\]

The apparatus of equational logic is replaced by a similar apparatus (see [1] §3) for deriving quantitative inferences from a set of generating quantitative inferences (axioms). Soundness and completeness theorems then provide the link between the semantics of quantitative inferences (Equation 2 above) and derivability in the deductive apparatus.

**Definition 12 (Quantitative Theory of Convex Semilattices).** The quantitative equational theory \(\mathbf{QTh}_{CS}\) of convex semilattices has signature \(\Sigma_{CS} = \{\{\oplus\} \cup \{+\}_{p \in (0, 1)}\}\) and is defined as the set of quantitative inferences derivable by the following axioms, stated for arbitrary \(p, q \in (0, 1)\) and \(\epsilon_1, \epsilon_2 \in [0, 1]\):

\[\text{This results can be straightforwardly adapted to obtain a presentation of } \mathcal{C}(\cdot + 1) \text{ too.}\]
Proposition 8. 1. The quantitative theory $\text{QTh}^{c}_{CS}$ of convex semilattices is a presentation of the monad $\mathcal{C}$, i.e., $\text{EM}(\mathcal{C}) \cong \text{QA}(\text{QTh}^{c}_{CS})$.

2. The quantitative theory $\text{QTh}^{c}_{CS}$ of pointed convex semilattices is a presentation of the monad $\mathcal{C}(\dag 1)$, i.e., $\text{EM}(\mathcal{C}(\dag 1)) \cong \text{QA}(\text{QTh}^{c}_{CS})$.

The isomorphism in Proposition 8 is given by a pair of functors

$\hat{P} : \text{EM}(\mathcal{C}(\dag 1)) \to \text{QA}(\text{QTh}^{c}_{CS})$, $\hat{P}^{-1} : \text{QA}(\text{QTh}^{c}_{CS}) \to \text{EM}(\mathcal{C}(\dag 1))$ (3)

whose definition is similar to the corresponding $\text{Set}$ isomorphism $P, P^{-1}$ from [I], namely $\hat{P}(A, d, \alpha) = (A, \oplus^\alpha, \{+^\alpha_p\}_{p \in (0, 1)}, \star^\alpha, d)$. As in the $\text{Set}$ case, we can use the presentation to identify the free quantitative pointed convex semilattice.

Proposition 9. The free quantitative pointed convex semilattice on $(X, d)$ is (up to isomorphism) $(\mathcal{C}(X + 1), \oplus^{C(1)}, \{+^C_p\}_{p \in (0, 1)}, \star^{C(1)}, HK(d + d_1))$, with operations interpreted as in Proposition 6.

3 Set monad $\mathcal{C} + 1$ and its presentation

The functor $\mathcal{C} + 1$ maps a set $X$ to the set of non-empty finitely generated convex sets of distributions on $X$ plus an additional element which we denote as $\star \in 1$. Equivalently, by seeing this additional element as representing the empty set of distributions on $X$, $\mathcal{C}(X) + 1$ is the set of (possibly empty) finitely generated convex sets of distributions on $X$. This is the functor of convex Segala systems [36], which have been widely studied in the literature (see, e.g., [10, 37] for an overview) as models of nondeterministic and probabilistic programs.
In this section, we investigate a monad whose underlying functor is $C + 1$. Following common practice, the monad $(C+1, \eta^{C+1}, \mu^{C+1})$ often be simply denoted by $C+1$, as its underlying functor. Our main result regarding the monad $C+1$ is a presentation theorem based on the equational theory of convex semilattices with bottom and black-hole. The black-hole axiom has already been investigated in the context of convex algebras in [38].

**Definition 13 (Theory $Th^{BH}_{C^S}$).** Let $Th^*_{C^S}$ be the equational theory of pointed convex semilattices. We let $\perp$ (bottom) and $BH$ (black hole) denote the sets of equations $\perp = \{ x \oplus \ast = x \}$ and $BH = \{ x + p \ast = \ast \mid p \in (0, 1) \}$, respectively. The equational theory of convex semilattices with bottom and black–hole, denoted by $Th^{BH}_{C^S}$, is the theory generated by the set of equations $Th^*_{C^S} \cup \perp \cup BH$.

Our monad on the functor $C + 1$ is defined using Proposition[2] i.e., we exhibit a monad distributive law of type $\gamma : C(+1) \Rightarrow (+1)C$ and this gives a monad structure $(C+1, \eta^{C+1}, \mu^{C+1})$ on the composite functor $(+1)C$, i.e., on $C+1$.

**Definition 14.** For every set $X$, the map $\gamma_X : C(X + 1) \to C(X) + 1$ is defined as follows, for any $S \in C(X) + 1$

$$\gamma_X(S) = \begin{cases} \{ \varphi \mid \varphi \in S \text{ and } \varphi(\ast) = 0 \} & \text{if } \exists \varphi \in S \text{ s.t. } \varphi(\ast) = 0 \\ \ast & \text{o/w} \end{cases}$$

The above definition can be easily understood as follows. By viewing elements $S \in C(X + 1)$ as non–empty finitely generated convex sets of subdistributions on $X$, the map $\gamma_X$ maps $S$ to its subset (which is convex) consisting of full probability distributions (i.e., assigning probability 0 to $\ast \in 1$). It is possible, however, that such subset is empty, and therefore not in $C(X)$. In this case (second clause in the definition) $S$ is mapped by $\gamma_X$ to the element $\ast \in 1$. This two–cases analysis can be further simplified if we view the element $\ast \in 1$ as representing the empty set $\emptyset$ so that we can simply write:

$$S \xrightarrow{\gamma_X} \{ \varphi \in S \mid \varphi \text{ is a full probability distribution, i.e., } \varphi(\ast) = 0 \}$$

**Lemma 1.** The family of maps $\gamma_X : C(X + 1) \to C(X) + 1$, for $X \in \text{Set}$, is a monad distributive law of the monad $C$ over the monad $+1$.

**Proof (Sketch).** To verify the commuting diagrams of Definition[2] we use the fact that $\gamma_X$ commutes with the operations of union and weighted Minkowski sum, which are used in the definition of the multiplication of the monad $C$. 

As a corollary, we obtain from Proposition[2] a monad structure $(C+1, \eta^{C+1}, \mu^{C+1})$ where $\eta^{C+1} = \eta^C$ and $\mu^{C+1}_X = \mu^{C+1}_X \circ (\mu^{C}_X + 1) \circ (\gamma_{C,X} + 1)$. Explicitly, $\eta^{C+1}_X(x) = \{ \delta_x \}$ and, for $S \in C(C(X) + 1) + 1$, if we let $\ast$ denote the element of the outer 1 and $\ast$ denote the element of the inner 1:

$$\mu^{C+1}_X(S) = \begin{cases} \mu^{C}_X(\gamma_{C(X)}(S)) & S \neq \ast \text{ and } \gamma_{C(X)}(S) \neq \ast \\ \ast & \text{o/w} \end{cases}$$

(4)
We are now ready to state the main result of this section.

**Theorem 1.** The monad $C + 1$ is presented by the equational theory of convex semilattices with bottom and black-hole, i.e., $\text{EM}(C + 1) \cong A(\text{Th}_{C_{BS}}^{⊥, BH})$.

The rest of this section outlines the proof of Theorem 1 above. Our first technical step is to prove that the distributive law $\gamma : C(+1) \Rightarrow C + 1$ of Lemma 1 is also a monad map (see Definition 4) between the monads $C(+1)$ and $C + 1$.

**Lemma 2.** $\gamma : C(+1) \Rightarrow C + 1$ is a monad map.

Using Proposition 3 the monad map $\gamma$ can be turned into a functor $U^\gamma : \text{EM}(C + 1) \rightarrow \text{EM}(C(+1))$ between the Eilenberg-Moore categories of the two monads.

**Lemma 3.** There is a functor $U^\gamma : \text{EM}(C+1) \rightarrow \text{EM}(C(+1))$ defined on objects by $(A, \alpha) \mapsto (A, \alpha \circ \gamma_A)$ and acting as identity on morphisms. This functor is an embedding, i.e., it is fully faithful and injective on objects.

As a consequence of the above lemma, the category $\text{EM}(C+1)$ can be viewed as a full subcategory of $\text{EM}(C(+1))$. Similarly, there is an embedding $\iota : A(\text{Th}_{C_{BS}}^{⊥, BH}) \rightarrow A(\text{Th}_{C_{BS}}^*)$ of the category of convex semilattices with bottom and black-hole (see Definition 13) into the category of all pointed convex semilattices. This is simply the functor that “forgets” that elements of $A(\text{Th}_{C_{BS}}^{⊥, BH})$ satisfy the additional axioms $\bot$ and $BH$. Hence, $A(\text{Th}_{C_{BS}}^{⊥, BH})$ is a full subcategory of $A(\text{Th}_{C_{BS}}^*)$.

Recall that by the presentation of the monad $C(+1)$ (Proposition 5), we have the isomorphism $P : \text{EM}(C(+1)) \cong A(\text{Th}_{C_{BS}}^*) : P^{-1}$ of Equation (1). In order to prove $\text{EM}(C + 1) \cong A(\text{Th}_{C_{BS}}^{⊥, BH})$, which is the statement of Theorem 1, we show that the functors $P$ and $P^{-1}$, when restricted to the subcategories $\text{EM}(C + 1)$ and $A(\text{Th}_{C_{BS}}^{⊥, BH})$, respectively, are isomorphisms of type:

$$P : \text{EM}(C + 1) \rightarrow A(\text{Th}_{C_{BS}}^{⊥, BH}) \quad P^{-1} : A(\text{Th}_{C_{BS}}^{⊥, BH}) \rightarrow \text{EM}(C + 1).$$

This amounts to prove the following technical result.

**Lemma 4.** 1. Given $(A, \alpha) \in \text{EM}(C + 1)$, which is embedded via $U^\gamma$ to $(A, \alpha \circ \gamma_A) \in \text{EM}(C(+1))$, the pointed convex semilattice $P((A, \alpha \circ \gamma_A))$ satisfies the $\bot$ and $BH$ equations, and therefore it belongs to the subcategory $A(\text{Th}_{C_{BS}}^{⊥, BH})$.

2. Given any $A \in A(\text{Th}_{C_{BS}}^{⊥, BH})$, which is embedded via $\iota$ to $A \in A(\text{Th}_{C_{BS}}^*)$, the Eilenberg-Moore algebra $P^{-1}(A) \in \text{EM}(C(+1))$ is in the image of $U^\gamma$, and therefore it belongs to the subcategory $\text{EM}(C + 1)$.

As a result, we obtain the desired presentation of the monad $C + 1$ as a restriction of the presentation of the monad $C(+1)$.
4 Set monad $C\downarrow$ and its presentation

In the previous section, we explored a monad structure on the Set functor $C + 1$ which is natural and important, in the context of program semantics, since it models (simple) convex Segala systems \[36\,10\,37\]. The presentation we obtained for $C + 1$ is in terms of convex semilattices with bottom and black-hole. The latter axiom imposes a strong equation $(x + _p \ast) = \ast$ which is not necessarily adequate in all modelling situations (think, for example, about the equation $\text{nil} + _p P = \text{nil}$ in a probabilistic process algebra). For this reason, in this section we consider the equational theory of convex semilattices with bottom (i.e., without the black-hole axiom), which has already appeared as relevant in the study of probabilistic program equivalences (see, e.g., \[13\] for applications in trace semantics) and investigate the monad which is presented by this theory.

**Definition 15 (Theory $\text{Th}_\downarrow^\ast CS$).** Let $\text{Th}_\downarrow^\ast CS$ denote the theory of pointed convex semilattices (Definition 7) and let $\bot$ be the equation set \{ $x \oplus \ast = x$ \}. We denote with $\text{Th}_\downarrow^\bot CS$ the theory generated by the set of equations $\text{Th}_\downarrow^\ast CS \cup \bot$ and refer to it as the theory of convex semilattices with bottom.

Before formally introducing the monad presented by the theory $\text{Th}_\downarrow^\bot CS$, we develop some useful intuitions. First, the theory $\text{Th}_\downarrow^\bot CS$ is the quotient of $\text{Th}_\downarrow^\ast CS$ obtained by adding the $\bot$ axiom. Recall from Proposition 5 that $\text{Th}_\downarrow^\ast CS$ gives a presentation of the monad $C(+1)$, which maps a set $X$ to the collection $C(X + 1)$ of non-empty finitely generated convex sets of distributions on $X$. Hence the $\bot$ axiom can be understood as the restriction of $C(X + 1)$ to sets of distributions containing the $\delta_\ast$ distribution (equivalently, the subdistribution with mass 0). Furthermore:

**Lemma 5.** The following equality is derivable in $\text{Th}_\downarrow^\bot CS$, for any $p, q \in (0, 1)$: $x + _p y = (x + _p y) \oplus ((x + _q \ast) + _p y) \oplus (\ast + _p y)$.

**Proof.** We have $x + _p y \downarrow (x \oplus \ast) + _p y \overset{D}{=} (x + _p y) \oplus (\ast + _p y)$. Then we can use the identity $z \oplus w = z \oplus w \oplus (z + _q w)$, which is valid in all convex semilattices, and obtain the desired equality.

Lemma 5 can be understood as stating that, under the theory $\text{Th}_\downarrow^\bot CS$, if a convex set contains a subdistribution $\varphi$ with $\varphi(x) = p$, then it also contains any subdistribution $\psi$ defined as $\varphi$ except that $\psi(x) < p$ (equivalently, $\psi(x) = qp$ or $\psi(x) = 0$). This leads to the following notion of $\bot$–closed convex sets.

**Definition 16 ($\bot$–closed convex set).** Let $X$ be a set and let $S \in C(X + 1)$.

We say that $S$ is $\bot$–closed if $\{ \psi \in D(X + 1) \mid \forall x \in X, \psi(x) \leq \varphi(x) \} \subseteq S$ for all $\varphi \in S$. We denote with $C\downarrow(X) \subseteq C(X + 1)$ the collection of non-empty finitely generated $\bot$–closed convex sets of subdistributions on $X$. 

Lemma 6. The family of functions \( EM \) is a monad map between the monads \( \mathcal{C} \). The structure of the proof of Theorem 4 is similar to that of Theorem 1. In the rest of this section, we outline the main steps. First, we show that the operation \( K_X \) of \( \bot \)-closure can be seen as a monad map from \( \mathcal{C}(+1) \) to \( \mathcal{C} \).

**Theorem 2.** Let \( X \) be a set. Then \( K_X \) is the \( \bot \)-closure operator, i.e., for any \( S \in \mathcal{C}(+1) \), \( K_X(S) \) is the smallest \( \bot \)-closed set containing \( S \).

It follows from Theorem 2 that, by restricting its codomain to the image, \( K_X \) defines a surjective function of type \( \mathcal{C}(+1) \to \mathcal{C}(X) \), which we also denote by \( K_X \). We can now define a functor and a monad on \( \mathcal{C}(X) \), with unit given by \( K \circ \eta \) and multiplication given by \( \mu \) restricted to \( \bot \)-closed sets.

**Definition 18 (Monad \( \mathcal{C} \)).** The non-empty finitely generated \( \bot \)-closed convex sets of subdistributions monad \( (\mathcal{C}, \eta, \mu) \) on \( \text{Set} \) is defined as follows. The functor \( \mathcal{C} \) maps a set \( X \) to \( \mathcal{C}(X) \) and maps a morphism \( f : X \to Y \) to \( \mathcal{C}(f) : \mathcal{C}(X) \to \mathcal{C}(Y) \) defined as the restriction of \( \mathcal{C}(f + 1) \) to \( \bot \)-closed sets. The unit is defined as \( \eta \) the unique pointed convex semilattice homomorphism extending \( f \) to its \( \bot \)-closure. The multiplication \( \mu \) is defined as the restriction of \( \mu \) to \( \bot \)-closed sets.

**Theorem 3.** The triple \( (\mathcal{C}, \eta, \mu) \) is a monad.

We can now state the main result of this section.

**Theorem 4.** The monad \( \mathcal{C} \) is presented by the equational theory of convex semilattices with bottom, i.e., \( \text{EM}(\mathcal{C}) \cong \text{A}(\text{Th}_{\mathcal{C}}(\bot)) \).

The structure of the proof of Theorem 4 is similar to that of Theorem 1. In the rest of this section, we outline the main steps. First, we show that the operation \( K_X \) of \( \bot \)-closure can be seen as a monad map from \( \mathcal{C}(+1) \) to \( \mathcal{C} \).

**Lemma 6.** The family of functions \( K_X : \mathcal{C}(X + 1) \to \mathcal{C}(X) \), for \( X \in \text{Set} \), is a monad map between the monads \( \mathcal{C}(+1) \) and \( \mathcal{C} \).

Using Proposition 3, the monad map \( K \) gives a functor \( U^K : \text{EM}(\mathcal{C}) \to \text{EM}(\mathcal{C}(+1)) \) between the Eilenberg-Moore categories.
Lemma 7. There is a functor $U^K : \text{EM}(C^i) \rightarrow \text{EM}(C(+1))$ defined on objects by $(A, \alpha) \mapsto (A, \alpha \circ K_A)$ and acting as identity on morphisms. This functor is an embedding, i.e., it is fully faithful and injective on objects.

As a consequence of the above lemma, the category $\text{EM}(C^i)$ can be viewed as a full subcategory of $\text{EM}(C(+1))$. Analogously, we have an embedding of categories $\iota : \text{A}(\text{Th}_{\bot CS}) \rightarrow \text{A}(\text{Th}_{\star CS})$ of the category of convex semilattices with bottom (see Definition 15) into the category of pointed convex semilattices, defined as the functor which forgets that the $\bot$–axiom is satisfied. Hence $\text{A}(\text{Th}_{\bot CS})$ is a full subcategory of $\text{A}(\text{Th}_{\star CS})$.

Let $P : \text{EM}(C(+1)) \cong \text{A}(\text{Th}_{\star CS}) : P^{-1}$ be the isomorphisms of categories from Proposition 5. We prove $\text{EM}(C^i) \cong \text{A}(\text{Th}_{\bot CS})$ (i.e., Theorem 4) by showing that the functors $P$ and $P^{-1}$, when respectively restricted to the subcategories $\text{EM}(C^i)$ and $\text{A}(\text{Th}_{\bot CS})$, are isomorphisms of type:

\[ P : \text{EM}(C^i) \rightarrow \text{A}(\text{Th}_{\bot CS}) \quad P^{-1} : \text{A}(\text{Th}_{\bot CS}) \rightarrow \text{EM}(C^i). \]

This amounts to prove the following result.

Lemma 8. 1. Given any $(A, \alpha) \in \text{EM}(C^i)$, which is embedded via $U^K$ to $(A, \alpha \circ K_A) \in \text{EM}(C(+1))$, the pointed convex semilattice $P((A, \alpha \circ K_A))$ satisfies the $\bot$ equation, and therefore it belongs to $\text{A}(\text{Th}_{\bot CS})$.

2. Given any \( \delta \in \text{A}(\text{Th}_{\bot CS}) \), which is embedded via $\iota$ to $\delta \in \text{A}(\text{Th}_{\star CS})$, the Eilenberg-Moore algebra $P^{-1}(\delta) \in \text{EM}(C(+1))$ is in the image of $U^K$, and therefore it belongs to the subcategory $\text{EM}(C^i)$.

5 Results about monads on $1\text{Met}$

In Section 3 we investigated a $\text{Set}$ monad whose underlying functor is $C + 1$ and obtained its presentation in terms of convex semilattices with bottom and black–hole. In Section 4, we investigated the $\text{Set}$ monad $C^i$ and proved that it is presented by the theory of convex semilattices with bottom. In this section, we investigate similar questions but in the category $1\text{Met}$ of 1–bounded metric spaces. First, take the functor $\hat{C} + 1$, which is the $1\text{Met}$ lifting of the functor $C + 1$ obtained with the Hausdorff–Kantorovich lifting. Is there a $1\text{Met}$ monad whose underlying functor is $\hat{C} + 1$? We do not answer the question in full generality, yet we provide some negative results by showing that any such monad:

1. cannot have a multiplication defined as the one of the $\text{Set}$ monad $C + 1$, and

2. cannot be presented by the quantitative theory of convex semilattices with bottom and black–hole, since this theory is trivial.

Secondly, the question is to find the $1\text{Met}$ monad presented by the quantitative theory of convex semilattices with bottom. In this case, we are successful and we show that this monad is exactly the lifting of the $\text{Set}$ monad $C^i$ to $1\text{Met}$ via the Hausdorff-Kantorovich distance.
5.1 Negative results on monad structures on the functor \( \hat{C} + \hat{1} \)

Recall from Section 3 that the \( \text{Set} \) monad \( C + 1 \) is obtained from the distributive law \( \gamma : C(+1) \Rightarrow C + 1 \). We now show that \( \gamma \) fails to be non-expansive when its domain and codomain are equipped with the Hausdorff–Kantorovich lifted metrics. This implies that most of the machinery developed in Section 3 to obtain the \( \text{Set} \) monad \( C + 1 \) and its presentation is not applicable in \( 1\text{Met} \).

\[ \text{Lemma 9.} \quad \text{There is a metric space} \ (X, d) \ \text{such that} \ \hat{\gamma}(X, d) \ : (C(X + 1), HK(d + d_1)) \rightarrow (C(X) + 1, HK(d + d_1)), \ \text{defined as the} \ \text{Set} \ \text{function} \ \gamma_X \ \text{from Definition 12}, \ \text{is not non-expansive.} \]

\[ \text{Proof.} \ \text{Let} \ X \ \text{be non–empty and take} \ (X, d) \ \text{with} \ d \ \text{the discrete metric. Consider} \ S_1, S_2 \in C(X + 1) \ \text{defined as} \ S_1 = \{ \frac{1}{2} x + \frac{1}{2} \} \ \text{and} \ S_2 = \{ \delta_x \}. \ \text{Then} \ HK(d + d_1)(\gamma_X(S_1), \gamma_X(S_2)) = HK(d + d_1)(\delta_x) = 1 > 0 = HK(d)(S_1, S_2). \]

As the multiplication \( \mu^{C+1} \) of the \( \text{Set} \) monad \( C + 1 \) is defined using \( \gamma \) (see Equation 11), this counterexample can be adapted to show that also the multiplication \( \mu^{C+1} \) is not non-expansive. Hence, no \( 1\text{Met} \) monad \( M \) whose underlying functor is \( \hat{C} + \hat{1} \) can have a multiplication \( \mu^M \) which, once the metric is forgotten, coincides with \( \mu^{C+1} \). Furthermore, no such monad \( M \) can be presented by the quantitative theory of convex semilattices with bottom and black–hole \( (\text{QTh}_{CS}^{\downarrow, BH}) \). This is because \( \text{QTh}_{CS}^{\downarrow, BH} \) is trivial in the sense that the quantitative inference \( \emptyset \vdash x =_0 y \), expressing that all elements are at distance 0, is derivable from the axioms or, equivalently, that any quantitative algebra in \( \text{QA}(\text{QTh}_{CS}^{\downarrow, BH}) \) has the singleton metric space \( \hat{1} \) as carrier.

\[ \text{Definition 19.} \ \text{The quantitative theory} \ \text{QTh}_{CS}^{\downarrow, BH} \ \text{of quantitative convex semilattices with bottom and black hole has the signature} \ \Sigma_{CS}^* \ \text{of pointed convex semilattices and is generated by the set of quantitative inferences} \ \text{QTh}_{CS}^\downarrow \cup \text{QH} \cup \text{BH}, \ \text{with} \ \downarrow = \{ \emptyset : x \downarrow * =_0 * \} \ \text{and} \ \text{QH} = \{ \emptyset : x + p * =_0 * \mid p \in (0, 1) \}. \]

\[ \text{Theorem 5.} \ \text{Any quantitative equational theory} \ \text{QTh} \ \text{containing} \ \text{QTh}_{CS}^{\downarrow, BH} \ \text{and} \ \text{BH} \ \text{is trivial, i.e., the quantitative inference} \ \emptyset \vdash x =_0 y \ \text{is derivable in} \ \text{QTh}. \]

5.2 The lifting of \( C^\downarrow \) to metric spaces and its presentation

The \( \text{Set} \) monad \( C^\downarrow \) (Definition 18) is obtained using \( K \circ \eta^{C(+1)} \) as unit, where \( K_X : C(X + 1) \rightarrow C(X) \) is the operation of \( \downarrow \)–closure (see Definition 14), and as multiplication the restriction of \( \mu^{C(+1)} \) to \( \downarrow \)–closed sets. We can give a similar definition in the category \( 1\text{Met} \) using the unit and multiplication of the \( 1\text{Met} \) monad \( \hat{C}(+\hat{1}) \) and the natural transformation \( K \), provided the latter exists in \( 1\text{Met} \), i.e., it is non–expansive.

\[ \text{Lemma 10.} \ \text{For any metric space} \ (X, d), \ \text{the function} \ \hat{K}_X(d) : (C(X + 1), HK(d + d_1)) \rightarrow (C(X + 1), HK(d + d_1)), \ \text{defined as the} \ \text{Set} \ \text{function} \ K_X \ \text{from Definition 17}, \ \text{is non-expansive.} \]
Proof. By Definition 17, \( K_X \) is the unique pointed semilattice homomorphism extending \( f : X \to C(X+1) \), with \( f(x) = cc(\{\delta_x, \delta_\star\}) \). The function \( \hat{f} : (X,d) \to (C(X+1), HK(d + d_1)) \), defined as \( f \) on \( X \), is easily seen to be an isometry, and thus non-expansive. Hence, \( \hat{f} \) is a morphism in \( 1\text{Met} \). Now, recall that \( ((C(X+1), HK(d + d_1)) \) is the free quantitative pointed convex semilattice on \( (X,d) \) and since the unique extension of \( \hat{f} \) is also a pointed convex semilattice homomorphism, its action on sets must coincide with \( K_X \). Hence, \( \hat{K}_{(X,d)} \) is the unique quantitative pointed convex semilattice homomorphism extending \( \hat{f} \). Therefore, \( \hat{K}_{(X,d)} \) is a morphism in \( 1\text{Met} \), which means that it is non-expansive.

Based on Lemma 10 and on \( \hat{C}(+\hat{1}) \) being a monad in \( 1\text{Met} \), we can obtain the following result in a way similar to Theorem 4.

**Definition 20 (Monad \( \hat{C} \) in \( 1\text{Met} \)).** The monad \( (\hat{C}, \eta^\hat{C}, \mu^\hat{C}) \) in \( 1\text{Met} \) is defined as follows. The functor \( \hat{C} \) maps a metric space \( (X,d) \) to \( (C(\hat{C}(X)), HK(d + d_1)) \). The action of the functor on arrows, the unit and the multiplication are defined as those of the \( \text{Set} \) monad \( C \).

**Theorem 6.** The triple \( (\hat{C}, \eta^\hat{C}, \mu^\hat{C}) \) is a \( 1\text{Met} \) monad.

We now introduce the quantitative equational theory of quantitative convex semilattices with bottom and state the main result of this section.

**Definition 21.** The quantitative equational theory \( Q\text{Th}_{CS}^\perp \) of quantitative convex semilattices with bottom is the quantitative equational theory generated by the set of quantitative inferences \( Q\text{Th}_{CS}^\bullet \cup \perp \), with \( \perp = \{ \vdash x \oplus \star = 0 \} \).

**Theorem 7.** The monad \( \hat{C} \) is presented by the quantitative equational theory of quantitative convex semilattices with bottom, i.e., \( EM(\hat{C}) \cong QA(Q\text{Th}_{CS}^\perp) \).

The proof of Theorem 7 is similar to that of Theorem 4. First, we identify \( EM(\hat{C}) \) and \( QA(Q\text{Th}_{CS}^\perp) \) as full subcategories of \( EM(\hat{C}(+\hat{1})) \) and \( QA(Q\text{Th}_{CS}^\perp) \), respectively. Then, we obtain the isomorphism \( EM(\hat{C}) \cong QA(Q\text{Th}_{CS}^\perp) \) by restricting the isomorphism \( EM(\hat{C}(+\hat{1})) \cong QA(Q\text{Th}_{CS}^\perp) \) of Proposition 8.

### 6 Examples of applications

The results presented in this paper can be summarised as follows:

| Set Monad | Eq. Theory |
|-----------|------------|
| \( C(X+1) \) | \( Th^CS \) |
| \( C(X) + 1 \) | \( Th^{BH} \) |
| \( C^\perp(X) \) | \( Th^{\perp BS} \) |

| 1\text{Met} Monad | Quantitative Eq. Theory |
|-------------------|-------------------------|
| \( C(X+1) \) | \( Q\text{Th}_{BS}^\perp \) |
| trivial | \( Q\text{Th}_{BS}^{BH} \) |
| \( C^\perp(X) \) | \( Q\text{Th}_{CS}^\perp \) |
The equational theories $\text{Th}_{CS}^*$, $\text{Th}_{CS}^{BH}$ and $\text{Th}_{CS}^\perp$, or closely related variants, have appeared in several works on mathematical formalisations of semantics of programming languages combining probability and nondeterminism, with applications including: SOS process algebras \cite{17,20,9}, axiomatisations of bisimulation (e.g., \cite{30,7}), up–to techniques \cite{13}. The functor $C + 1$, defining the well–known class of convex Segala systems \cite{36,37,10,28}, has also been considered in many works. The value of our contribution is to have established the mathematical foundation for unifying, modifying and extending several of these works. The goal of this section is to illustrate the general usefulness of our results by means of some simple examples.

We start by introducing a minimalistic process algebra with both nondeterministic and probabilistic choice. Process terms are defined by the grammar:

$$P ::= \text{nil} | a.P | P_1 \oplus P_2 | P_1 \mp_p P_2$$

for $p \in (0, 1)$. We let $\text{Proc}$ denote the set of all process terms. Intuitively, $\text{nil}$ is the terminating process, $a.P$ does an $a$–action and then behaves as $P$, and $\oplus$ and $\mp_p$ are (convex) nondeterministic and probabilistic choice operators, respectively. We assume that $a.\ (\text{a notation})$ has binding priority over the other language operators and, for $n \geq 0$, we define $a^n.P$ inductively as $a^0.P = P$ and $a^{n+1}.P = a.(a^n.P)$. For the sake of simplicity, we just consider a single action label $a$. Variants with multiple labels can be easily given. The transition function is defined as a map $\tau : \text{Proc} \to \mathcal{T}(\text{Proc}, \Sigma_{CS}^{\ast})$ (written as $\iff$ when used with infix notation) inductively defined in Figure 1 assigning to each process term $P$ a term $t$ in the signature of pointed convex semilattices built from process terms. If $\tau(P) = t$ we say that $t$ is the continuation of $P$.

The interesting point about this definition is that, depending on the equational theory applied to continuations, we obtain different transition semantics (i.e., Set coalgebras):

$$\tau_F : \text{Proc} \to F(\text{Proc})$$

for $F \in \{C(+1), C + 1, C^1\}$. For instance, by choosing the theory $\text{Th}_{CS}^{BH}$ of convex semilattices with bottom and black–hole we obtain $F = C + 1$, which is the well–studied functor of convex Segala systems \cite{36}, and our process algebra can be considered as the core of the calculus of \cite{30} (see also \cite{7}). Adopting this semantics, the continuation of $\text{nil}$ is the element $\ast \in 1$ (which can be identified with the emptyset $\emptyset$, see remark after
Definition [14] and the semantics of the process terms $P_1 = a^2 . \text{nil} \boxplus \text{nil}$ and $P_2 = a^2 . \text{nil} \boxplus a . (a^2 . \text{nil} \boxplus a . (a . \text{nil} \boxplus \text{nil}))$ can be depicted as in Figure 2. As customary, we omit the dotted probabilistic arrow when the probability is 1 and only depict some of the reached distributions in the convex set (the red arc indicates their convex closure).

We can now reason on process behaviours using standard definitions. For instance, the following is one way (see, e.g., [37] §2) for a detailed exposition) of defining behavioural equivalence coalgebraically.

**Definition 22.** Let $F$ be a Set endofunctor, $c : X \to F(X)$ a coalgebra for $F$, $\mathcal{R} \subseteq X \times X$ an equivalence relation, $X/\mathcal{R}$ the collection of $\mathcal{R}$-equivalence classes and $\pi_X : X \to X/\mathcal{R}$ the quotient map. We say that $\mathcal{R}$ is a behavioural equivalence if for all $(x, y) \in R$ it holds that $(F(\pi_X) \circ c)(x) = (F(\pi_X) \circ c)(y)$.

If we let $F = C + 1$, the above definition coincides with Segala’s convex bisimulation equivalence (see [36][10][37]). By applying Definition 22 to the coalgebra $\tau_{C+1} : \text{Proc} \to C(\text{Proc}) + 1$, defined by the operational semantics with continuations taken modulo the equational theory $\text{Th}_{C^\ast}^{\models BH}$, we can prove that $P_1$ and $P_2$ are behaviourally equivalent by exhibiting the equivalence relation $\mathcal{R}$ defined as the least equivalence relation including the following pairs:

$$\{(P_1, P_2), (a^2 . \text{nil}, a^2 . \text{nil}), (a^2 . \text{nil}, P3), (a . \text{nil}, a . \text{nil}), (a . \text{nil}, a . \text{nil} \boxplus \text{nil}), (\text{nil}, \text{nil})\}$$

with $P_3$ defined as $a^2 . \text{nil} \boxplus a . (a . \text{nil} \boxplus \text{nil})$. To prove that $\mathcal{R}$ is indeed a behavioural equivalence we need to show that the equality in Definition 22 holds. Here our equational apparatus becomes valuable. Since the monad $C + 1$ is presented by the theory $\text{Th}_{C^\ast}^{\models BH}$ of convex semilattices with bottom and black-hole, this amounts to prove that, for all $(P, Q) \in \mathcal{R}$, the continuations $\tau(P)$ and $\tau(Q)$ can be equated in the theory $\text{Th}_{C^\ast}^{\models BH}$ extended with the set of axioms $\{P = Q \mid P \mathcal{R} Q\}$. For instance, for the pair $(P_1, P_2) \in \mathcal{R}$ this means showing the following derivation:

$$\begin{align*}
\tau(P_1) &= a^2 . \text{nil} \boxplus * \\
&= a^2 . \text{nil} \\
&= a^2 . \text{nil} \boxplus a^2 . \text{nil} \\
&= a^2 . \text{nil} \boxplus P_3 \\
&= \tau(P_2) \\
&= \text{def}\ \text{of}\ \tau
\end{align*}$$

Hence, equational reasoning can be used in program equivalence proofs.

Note that in the above derivation the bottom axiom is used but the black-hole axiom is not. Indeed, the two process terms $P_1$ and $P_2$ are also equated in the weaker theory $\text{Th}_{C^\ast}^{\models}$ of convex semilattices with bottom ($F = C^\ast$), but not in the even weaker theory $\text{Th}_{C^\ast}^{\models}$ of pointed convex semilattices ($F = C(X + 1)$) which does not include the bottom axiom. Similarly, the two process terms $\text{nil}$ and $(\text{nil} \boxplus P)$ are equated, for all $P$, when $\text{Th}_{C^\ast}^{\models BH}$ is used ($F = C(X + 1$, i.e., Segala system) but not in $\text{Th}_{C^\ast}^{\models}$ or $\text{Th}_{C^\ast}^{\models}$. Which choice of functor $F \in \{C(-1), C + 1, C^\ast\}$ is best suited in a specific modelling situation is of course be-
yond the scope of this paper. But once the choice is made, appropriate equational theories are automatically provided by our results.

Furthermore, our results also provide quantitative equational theories for metric reasoning when the chosen functor is \( F \in \{ C(+)1, C^1 \} \). Following standard ideas, it is possible to endow the set \( \text{Proc} \) of process terms with the bisimulation metric using the machinery of Hausdorff–Kantorovich liftings (\cite{15,5,18}).

**Definition 23.** Let \( F \in \{ C(+)1, C^1 \} \) be a functor on \( \text{Set} \) and let \( \tau : X \rightarrow F(X) \) be a coalgebra for \( F \). A \( 1 \)-bounded metric \( d : X \times X \rightarrow [0,1] \) is a bisimulation metric if for all \( x, y \in X \) it holds that \( \Delta(d)(\tau_F(x), \tau_F(y)) \leq d(x,y) \), for \( \Delta(d) = HK(d + d_1) \). The bisimilarity metric on \( X \), denoted by \( d_{\sim} \), is defined as the point–wise infimum of all bisimulation metrics.

Once \( \text{Proc} \) is endowed with \( d_{\sim} \), the transition function \( \tau_F : (\text{Proc}, d_{\sim}) \rightarrow ˆF(\text{Proc}, d_{\sim}) \), for \( ˆF \in \{ \hat{C}(+)1, \hat{C}^1 \} \) is non–expansive and so we obtain a \( 1\text{Met} \) coalgebra on \( \text{Proc} \) terms. For a simple example, consider the quantitative theory \( \text{QTh}_{CS} \) of pointed convex semilattices (i.e., \( ˆF = \hat{C}(+)1 \)) and the two process terms \( Q_1 = \text{nil} \uplus \frac{1}{3} \text{a.nil} \) and \( Q_2 = \text{nil} \uplus \frac{1}{3} \text{a.nil} \). These two processes are not equivalent (Definition \[22\]) but it is possible to prove that \( d_{\sim}(Q_1,Q_2) \leq \frac{1}{4} \), using the metric \( d \) defined as \( d(Q_1,Q_2) = d(Q_2,Q_1) = \frac{1}{4} \) and as the discrete metric on all other pairs. We have to show that \( \Delta(d)(\tau(Q_1), \tau(Q_2)) \leq \frac{1}{4} \). The deductive apparatus of the quantitative equational theory \( \text{QTh}_{CS} \) makes this easy:

\[
\Delta(d)(\tau(Q_1), \tau(Q_2)) = \Delta(d)(\ast + \frac{1}{2} \text{a.nil}, \ast + \frac{1}{2} \text{a.nil}) \quad \text{Definition of } \tau \\
= \Delta(d)((\ast + \frac{1}{2} \ast) + \frac{1}{3} \text{a.nil}, \ast + \frac{1}{4} \text{a.nil}) \quad \text{Ip axiom} \\
= \Delta(d)((\ast + \frac{1}{3} \ast) + \frac{1}{4} \text{a.nil}, \ast + \frac{1}{4} \text{a.nil}) \quad \text{Ap axiom} \\
\leq \frac{1}{4} \cdot 0 + \frac{1}{3} \cdot \frac{1}{4} = \frac{1}{4}
\]

where the last inequality is deduced using the (K) inference rule (see Definition \[12\]), using as premises \( \Delta(d)(\ast, \ast) = 0 \) and

\[
\Delta(d)(\ast + \frac{1}{3} \text{a.nil}, \text{a.nil}) \quad \text{Ip axiom} = \Delta(d)((\ast + \frac{1}{4} \text{a.nil}, \text{a.nil}) + \frac{1}{4} \text{a.nil}) \leq \frac{1}{3}
\]

where, again, the inequality is derived by the (K) inference rule, using as premises \( \Delta(d)(\ast, \text{a.nil}) \leq 1 \) and \( \Delta(d)(\text{a.nil}, \text{a.nil}) = 0 \).

### 7 Conclusions and future work

A main line for future research is the development of compositional verification techniques, along the lines of the illustrative examples presented in Section \[6\]. This includes the axiomatization of behavioral equivalences and distances in expressive probabilistic programming languages, with features such as recursion and parallel composition.

A technical question left open in this work (Section \[5.1\]) is the following: is there a monad structure on the \( 1\text{Met} \) functor \( \hat{C} + 1 \)?
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A Appendix to background

In this section, we recall and expand some definitions given in Section 2 in order to help the reader understand the more technical proofs in the remainder of the appendix. In particular, we draw the commutative diagrams corresponding to some definitions; we use single arrows to represent morphisms and double arrows to represent natural transformations.

Monad

Given a category $\mathbf{C}$, a monad on $\mathbf{C}$ is a triple $(M, \eta, \mu)$ composed of a functor $M : \mathbf{C} \to \mathbf{C}$ together with two natural transformations: a unit $\eta : \text{id} \Rightarrow M$, where $\text{id}$ is the identity functor on $\mathbf{C}$, and a multiplication $\mu : M^2 \Rightarrow M$, satisfying the two laws $\mu \circ \eta M = \mu \circ M \eta = \text{id}_C$ and $\mu \circ M \mu = \mu \circ \mu M$.

$$\begin{array}{ccc}
M & \xrightarrow{M \eta} & M^2 \\
\downarrow{\mu} & & \downarrow{\eta M} \\
M & \xrightarrow{\mu} & M
\end{array} \quad \begin{array}{ccc}
M^3 & \xrightarrow{M \mu} & M^2 \\
\downarrow{\mu} & & \downarrow{\mu} \\
M^2 & \xrightarrow{\mu} & M
\end{array}$$

Monad Distributive Law

Let $(M, \eta, \mu)$ and $(\hat{M}, \hat{\eta}, \hat{\mu})$ be two monads on $\mathbf{C}$. A natural transformation $\lambda : M \hat{M} \Rightarrow \hat{M} M$ is called a monad distributive law of $M$ over $\hat{M}$ if it satisfies the laws $\lambda \circ M \hat{\eta} = \hat{\eta} M$, $\lambda \circ \eta \hat{M} = \hat{M} \eta$, $\lambda \circ \mu \hat{M} = \hat{M} \mu \circ \lambda M \circ M \lambda$ and $\lambda \circ M \hat{\mu} = \hat{\mu} M \circ \hat{M} \lambda \circ \lambda \hat{M}$, i.e., it makes $\lambda$ and $\mu$ commute.

$$\begin{array}{ccc}
M & \xrightarrow{M \hat{\eta}} & M\hat{M} \\
\lambda \downarrow & & \hat{\eta} M \downarrow \lambda \\
\hat{M} M & \xrightarrow{\hat{\eta} M} & \hat{M}
\end{array} \quad \begin{array}{ccc}
M M\hat{M} & \xrightarrow{\mu \hat{\eta}} & M\hat{M} \leftarrow \hat{M} M \\
\lambda M \downarrow & & \hat{\eta} M \downarrow \lambda M \\
\hat{M} M M & \xrightarrow{\lambda \hat{M}} & \hat{M} \hat{M} M
\end{array}$$

$M$-algebra

Let $(M : \mathbf{C} \to \mathbf{C}, \eta, \mu)$ be a monad. An algebra for $M$ is a pair $(A, \alpha)$ where $A \in \mathbf{C}$ is an object and $\alpha : M(A) \to A$ is a morphism such that $\alpha \circ \eta A = \text{id}_A$ and $\alpha \circ M \alpha = \alpha \circ \mu A$ hold.

$$\begin{array}{ccc}
A & \xrightarrow{\eta_A} & MA \\
\downarrow{id_A} & & \downarrow{\alpha} \\
A & \xrightarrow{\alpha} & A
\end{array} \quad \begin{array}{ccc}
M^2 A & \xrightarrow{\mu_A} & MA \\
\downarrow{M(\alpha)} & & \downarrow{\alpha} \\
M A & \xrightarrow{\alpha} & MA
\end{array} \quad \begin{array}{ccc}
MA & \xrightarrow{M(f)} & MA' \\
\downarrow{\alpha} & & \downarrow{\alpha'} \\
MA & \xrightarrow{\alpha} & A'
\end{array}$$

Given two $M$–algebras $(A, \alpha)$ and $(A', \alpha')$, a $M$–algebra morphism is an arrow $f : A \to A'$ in $\mathbf{C}$ such that $f \circ \alpha = \alpha' \circ M(f)$.
Monad Map

Let \((M, \eta, \mu)\) and \((\hat{M}, \hat{\eta}, \hat{\mu})\) be two monads, a natural transformation \(\sigma : M \Rightarrow \hat{M}\) is called a \textit{monad map} if it satisfies the equations \((\text{12})\) \(\hat{\eta} = \sigma \circ \eta\) and \((\text{13})\) \(\sigma \circ \mu = \hat{\mu} \circ (\sigma \circ \sigma)\).

\[
\begin{array}{c}
\text{id}_C \xrightarrow{\eta} M \\
\downarrow \hat{\eta} \quad \quad \quad \quad \downarrow \sigma \\
\hat{M} \quad \quad \quad \quad \quad \quad M
\end{array}
\tag{12}
\]

\[
\begin{array}{c}
M^2 \xrightarrow{\sigma \circ \sigma} \hat{M}^2 \\
\downarrow \mu \quad \quad \quad \quad \downarrow \hat{\mu} \\
\hat{M} \quad \quad \quad \quad \quad \quad M
\end{array}
\tag{13}
\]

Monad \(\cdot + 1\)

The \textit{termination} monad on \textbf{Set} is the triple \((\cdot + 1, \eta^+, \mu^+)\) defined as in Proposition \textbf{1}. For objects \(X\) in \textbf{Set}, the functor \(\cdot + 1\) maps \(X\) to the coproduct \(X + 1\), i.e., the disjoint union of the sets \(X\) and \(1 = \{\ast\}\). For arrows \(f : X \rightarrow Y\) in \textbf{Set}, the functor \(\cdot + 1\) maps \(f\) to \(f + 1 : X + 1 \rightarrow Y + 1\), defined as \(f + 1 = [\text{inl} \circ f, \text{inr}]\). The unit \(\eta^+_X : X \rightarrow X + 1\) is \(\eta^+_X(x) = \text{inl}(x)\) and the multiplication \(\mu^+_X : (X + 1) + 1 \rightarrow X + 1\) is defined as \(\mu^+_X = [\text{inl}, \text{inr}], \text{inr}\). If clear from the context, we may omit explicit mentioning of the injections, and write for example \((f + 1)(x) = x\) for \(x \in X\) and \((f + 1)(\ast) = \ast\). For the unit \(\eta^+_X : X \rightarrow X + 1\) we write \(\eta^+_X(x) = x\). For the multiplication \(\mu^+_X : (X + 1) + 1 \rightarrow X + 1\) we let \(*\) denote the element of the outer 1 and \(\ast\) denote the element of the inner 1, and we write \(\mu^+_X(x) = x\) for \(x \in X\), \(\mu^+_X(\ast) = \ast\) and \(\mu^+_X(\ast) = \ast\).

Monad \(C(+1)\)

The \textit{finitely generated non-empty convex powerset of subdistributions} monad \((C(+1), \eta^{C(+1)}, \mu^{C(+1)})\) in \textbf{Set} is defined as follows. Given an object \(X\) in \textbf{Set}, \(C(X)\) is the collection of non-empty finitely generated convex sets \(\text{WMS}(\text{set}(\text{conv}(\text{distr}(X))))\). Given an arrow \(f : X \rightarrow Y\) in \textbf{Set}, the arrow \(C(f) : C(X + 1) \rightarrow C(Y + 1)\) is defined as \(C(\cdot + 1)(f)(S) = \{U(f + 1)(\varphi) \mid \varphi \in S\}\). The unit \(\eta^{C(+1)}_X : X \rightarrow C(X + 1)\) is defined as \(\eta^{C(+1)}_X(x) = \{\delta_x\}\). The multiplication \(\mu^{C(+1)}_X : C(C(X + 1)) \rightarrow C(X + 1)\) is defined using Proposition \textbf{2} and the distributive law in Corollary \textbf{1}. We have that for any \(S \in C(C(X + 1) + 1))\)

\[
\mu^{C(+1)}_X(S) = \mu^{C}_{X + 1}(\bigcup_{\Phi \in S} \{\Phi^*\}) = \bigcup_{\Phi \in S} \text{WMS}(\Phi^*)
\tag{14}
\]

where, if we let \(*\) denote the element of the outer 1 and \(\ast\) denote the element of the inner 1, for any \(\Phi \in D(C(X + 1) + 1)\) we define \(\Phi^* \in D(C(X + 1))\) as

\[
\Phi^* = D(\iota_{X + 1}) \circ CC(\mu^{C(+1)}_{X})(\Phi) = \left( \sum_{U \in C(X + 1) \setminus \{\delta_x\}} \Phi(U)U \right) + (\Phi(\{\delta_x\}) + \Phi(\ast))\{\delta_x\}.
\]

The isomorphism \(\kappa : C(X + 1) \rightarrow \mathcal{T}(X, \Sigma^{C+S}_{\text{thCS}})\)
As explained in Proposition 5, the presentation of the monad \( C(+1) \) in terms of the theory \( Th_{CS}^* \) of pointed convex semilattices (Proposition 5) implies that the free pointed convex semilattice generated by \( X \) is isomorphic to the pointed convex semilattice \( (C(X + 1), \oplus, +_p, \{\delta_i\}) \) where for all \( S_1, S_2 \in C(X + 1) \), \( S_1 \oplus S_2 = cc(S_1 \cup S_2) \) (convex union), \( S_1 +_p S_2 = wms(p S_1 + (1 - p) S_2) \) (weighted Minkowski sum), and the distinguished element is \( \{\delta_i\} \). In other words, the set \( T(X, \Sigma_{CS}^*)_{Th_{CS}^*} \) of pointed convex semilattice terms modulo the equational theory \( Th_{CS}^* \) can be identified with the set \( C(X + 1) \) of non-empty, finitely generated convex sets of finitely supported probability subdistributions on \( X \). The isomorphism is a simple variant of the isomorphism described in [14] for the theory of convex semilattices (without a point). It is given by \( \kappa : C(X + 1) \rightarrow T(X, \Sigma_{CS}^*)_{Th_{CS}^*} \) defined as \( \kappa(S) = [\bigoplus_{\varphi \in UB(S)} ( +_{x \in supp(\varphi)} \varphi(x) x ) ]_{Th_{CS}^*} \), where \( \bigoplus_{i \in I} x_i \) and \( +_{i \in I} p_i x \) are respectively notations for the binary operations \( \oplus \) and \( +_p \) extended to operations of arity \( I \), for \( I \) finite (see, e.g., [39, 12]), and where \( UB(S) \) is the unique base of \( S \) defined as follows. Given a finitely generated convex set \( S \subseteq D(X) \), there exists one minimal (with respect to the inclusion order) finite set \( UB(S) \subseteq D(X) \) such that \( S = cc(UB(S)) \). The finite set \( UB(S) \) is referred to as the \textit{unique base} of \( S \) (see, e.g., [14]). The distributions in \( UB(S) \) are convex-linear independent, i.e., if \( UB(S) = \{\varphi_1, \ldots, \varphi_n\} \), then for all \( i \), \( \varphi_i \notin cc(\{\varphi_j | j \neq i\}) \). We remark that the equation \( x \oplus y = x \oplus y \oplus (x +_p y) \), which explicitly expresses closure under taking convex combinations, is derivable from the theory of convex semilattices (see, e.g., [14] Lemma 14), and that this derivation critically uses the distributivity axiom (D).

The Functors \( P^{-1} \) and \( \tilde{P}^{-1} \)

We describe the action of the functor \( \tilde{P}^{-1} \) explicitly (see [29]), the action of \( P^{-1} \) is similar but forgets about metrics and non-expansiveness. Given a quantitative pointed convex semilattice \( A = ((A, \oplus, +_p, \star), d) \in QTh_{CS}^* \), \( P^{-1}(A) \) is the \( C(+1) \)-algebra \( ((A, d), \alpha) \) where \( \alpha : C(A + 1) \rightarrow A \) is defined by

\[
\alpha(S) = \bigoplus_{\varphi \in B, x \in supp(\varphi)} \varphi(x) x,
\]

where \( B \) is any base for \( S \) (i.e.: \( S = cc(B) \)). Both functors act as identity on morphisms.

B Proofs of Section 3

We first prove two useful lemmas, showing that \( \gamma \) commutes with the operations of (possibly infinite) union and of weighted Minkowski sum, respectively. Since we are dealing with (generally not convex) unions, we consider the function \( \gamma \) as defined (see Definition 14) on arbitrary subsets of \( S \subseteq D(X + 1) \), rather than convex subsets \( S \in C(X + 1) \). So the generalised \( \gamma \) maps an arbitrary set \( S \subseteq D(X + 1) \) to its subset of full probability distributions (i.e., such that \( \varphi(\star) = 0 \) if such set is nonempty, and to \( \star \in 1 \) otherwise.)
Lemma 11. For any family of non-empty sets \( \{ S_i \subseteq D(X + 1) \}_{i \in I} \), we have

\[
\gamma_X \left( \bigcup_{i \in I} S_i \right) = \left\{ \begin{array}{ll}
\bigcup_{i \in I, \gamma_X(S_i) \neq \star} \gamma_X(S_i) & \exists i \in I, \gamma_X(S_i) \neq \star \\
\ast & \text{otherwise}
\end{array} \right.,
\]

Proof. It is clear that if \( \gamma_X(S_i) = \ast \) for all \( i \in I \), then all distributions \( \varphi \) in \( \bigcup_{i \in I} S_i \) are not full (i.e., \( \varphi(\ast) > 0 \)), thus \( \gamma_X(\bigcup_i S_i) = \ast \). Now, suppose \( \exists i \in I, \gamma_X(S_i) \neq \ast \), or equivalently, there is at least one full distribution in \( \bigcup_{i \in I} S_i \). Then, \( \gamma_X(\bigcup_i S_i) \) is, by definition, the union of all full distributions in each \( S_i \). Finally, since there are no full distributions in \( S_i \) if and only if \( \gamma_X(S_i) = \ast \), we obtain

\[
\gamma_X \left( \bigcup_{i \in I} S_i \right) = \bigcup_{i \in I, \gamma_X(S_i) \neq \ast} \gamma_X(S_i)
\]

\( \square \)

Lemma 12. For any \( \Phi \in DC(X + 1) \), we have

\[
\gamma_X(\text{WMS}(\Phi)) = \left\{ \begin{array}{ll}
\text{WMS}(\mathcal{D}(\gamma_X)(\Phi)) & \forall U \in \text{supp}(\Phi), \gamma_X(U) \neq \ast \\
\ast & \text{otherwise}
\end{array} \right.,
\]

Proof. Note that if there exists \( V \in \text{supp}(\Phi) \) with \( \gamma_X(V) = \ast \) (i.e., if \( V \) does not contain full distributions) then all distributions \( \sum_{U \in \text{supp}(\Phi)} \Phi(U) \cdot d_U \in \text{WMS}(\Phi) \) are not full, i.e., \( \gamma_X(\text{WMS}(\Phi)) = \ast \). This proves the second condition of the lemma. Now, for the first one, assume that \( \forall U \in \text{supp}(\Phi), \gamma_X(U) \neq \ast \). We then have the following derivation.

\[
\gamma_X(\text{WMS}(\Phi)) = \gamma_X \left\{ \sum_{U \in \text{supp}(\Phi)} \Phi(U) \cdot d_U : d_U \in U \right\}
\]

\[
= \left\{ \sum_{U \in \text{supp}(\Phi)} \Phi(U) \cdot d_U : d_U \in U \text{ is full}, \text{i.e., } d_U(\ast) = 0 \right\}
\]

\[
= \left\{ \sum_{U \in \text{supp}(\Phi)} \Phi(U) \cdot d_U : d_U \in \gamma_X(U) \right\}
\]

\[
= \text{WMS}(\mathcal{D}(\gamma_X)(\Phi))
\]

where the second equality holds because \( \sum_{U \in \text{supp}(\Phi)} \Phi(U) \cdot d_U \) is full if and only if all \( d_U \in U \) are full. \( \square \)

We are now ready to prove the results of Section \( \S \)

Proof of Lemma \( \| \)
We show that the family
\[
\gamma_X(\{\varphi_i \mid i \in I\}) = \begin{cases} 
\{\varphi_i \mid i \in I, \varphi_i(\ast) = 0\} & \exists i, \varphi_i(\ast) = 0 \\
o/w \end{cases}
\]
is a natural transformation. First, \(\gamma_X\) is well-typed because when \(S \in C(X + 1)\), there is a finite set \(\mathcal{UB}(S)\) of distributions satisfying \(cc(\mathcal{UB}(S)) = S\). Then, if \(S\) contains at least one full distribution, one can verify that
\[
\gamma_X(S) = cc\{\varphi \in \mathcal{UB}(S) \mid \varphi(\ast) = 0\},
\]
thus \(\gamma_X(S)\) is a convex and finitely generated subset of \(D(X)\). Second, \(\gamma\) is natural by the following derivation, for any \(f : X \to Y\) and \(S = \{\varphi_i \mid i \in I\} \in C(X + 1)\):
\[
(C(f) + 1)(\gamma_X(S)) = \begin{cases} 
C(f) \{\varphi_i \mid \varphi_i(\ast) = 0\} & \exists i, \varphi_i(\ast) = 0 \\
o/w \end{cases}
\]
\[
= \begin{cases} 
\{\varphi_i \mid \varphi_i(\ast) = 0\} & \exists i, \varphi_i(\ast) = 0 \\
o/w \end{cases}
\]
\[
= \gamma_Y(C(f + 1)(S))
\]
The last equality holds because \(f(x) \neq \ast\) for any \(x \in X\), since \(\ast \in 1\) is assumed to not belong to \(Y\).

We need to show that \(\gamma : C(\cdot + 1) \Rightarrow C + 1\) is a monad distributive law, i.e., that it satisfies the two commuting diagrams of (7) and (8).

First we show that (15) commutes.
\[
\begin{array}{ccc}
C X & \xrightarrow{C(\text{inl})} & C(X + 1) \\
\downarrow\text{inl} & & \downarrow\gamma_X \\
C X + 1 & \xrightarrow{\eta_X + 1} & X + 1
\end{array}
\]

For the L.H.S. (Left Hand Side), note that \(C(\text{inl})\) maps distributions \(\varphi \in C(X)\) to the corresponding “full distribution” \(\varphi \in C(X + 1)\), i.e., such that \(\varphi(\ast) = 0\). We have
\[
\gamma_X(C(\text{inl})(\{\varphi_i \mid i \in I\})) = \gamma_X(\{\varphi_i \mid i \in I\}) = \{\varphi_i \mid i \in I\}
\]
where the last equality holds as \(\{\varphi_i \mid i \in I\}\) only contains full distributions. For the R.H.S., take an element \(\omega \in X + 1\). If \(\omega = x \in X\), it is first sent by \(\eta_X^X + 1\) to \(\{\delta_x\}\) and then it is sent to \(\{\delta_x\}\) by \(\gamma_X\), and indeed we have \(\eta_X^X + 1(x) = \{\delta_x\}\). If \(\omega = \ast \in 1\), then it is first sent to \(\{\delta_\ast\}\) and then to \(\ast\), and we have \(\eta_X^X + 1(\ast) = \ast\).
Finally, we show that (16) commutes:

\[
\begin{align*}
CC(\mathcal{X} + 1) & \xrightarrow{\mu_{\gamma X}^1} \mathcal{C}(\mathcal{X} + 1) \xleftarrow{\gamma_{X+1}} \mathcal{C}((\mathcal{X} + 1) + 1) \\
C(\mathcal{C}(\mathcal{X} + 1)) & \xrightarrow{\gamma_{X+1}} \mathcal{C}(\mathcal{X} + 1) + 1 \\
CCX & \xrightarrow{\mu_{\gamma X}^1} Cx + 1 \xleftarrow{\mu_{\gamma X}^1} (Cx + 1) + 1
\end{align*}
\]

We first prove the L.H.S. of (16). Starting with a convex set \( S = \{ \Phi_i \}_{i \in I} \subseteq CC(\mathcal{X} + 1) \), the right-then-down path yields, by definition of \( \mu^C \) (Definition 5) and applying Lemma 11:

\[
\gamma_X(\mu_{\gamma X}^1(S)) = \gamma_X \left( \bigcup_{i} \text{WMS}(\Phi_i) \right) = \left\{ \bigcup_{i \mid \gamma_X(\text{WMS}(\Phi_i)) \neq *} \gamma_X(\text{WMS}(\Phi_i)) \right\} \exists i, \gamma_X(\text{WMS}(\Phi_i)) \neq * \quad \text{o/w}
\]

Then, by Lemma 12, we derive that \( \gamma_X(\text{WMS}(\Phi_i)) \neq * \) if and only if \( \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \) and we can rewrite the function above as:

\[
\left\{ \bigcup_{i \mid \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq *} \text{WMS}(\mathcal{D}(\gamma_X)(\Phi_i)) \right\} \exists i, \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \quad \text{o/w}
\]

Taking the down-then-right path, we have the following derivation.

\[
\begin{align*}
S & \xrightarrow{C(\gamma_X)} \{ \mathcal{D}(\gamma_X)(\Phi_i) \} \\
& \xrightarrow{\gamma_{X+1}} \{ \mathcal{D}(\gamma_X)(\Phi_i) \mid \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \} \exists i, \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \quad \text{o/w}
\]

where the equality in the derivation holds as

\[
* \notin \text{supp}(\mathcal{D}(\gamma_X)(\Phi_i)) \text{ if and only if } \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq *.
\]

Hence, the L.H.S commutes.
For the R.H.S. of (16), let \( S = \{ \varphi_i \}_{i \in I} \in C((X + 1) + 1) \). In the sequel, \(*\) will denote the element of the innermost \( 1 \) and \( \star \) the element of the outermost \( 1 \). Taking the top arrow, the morphism \( C(\mu_{X+1}) \) identifies both stars together by sending \( S \) to \( \{ \bar{\varphi}_i \}_{i \in I} \) where, again omitting injections, we let
\[
\bar{\varphi}_i = (\varphi_i(*) + \varphi_i(*)) \star + \sum_{x \in X} \varphi_i(x)x.
\]
Applying \( \gamma_X \) to \( \{ \bar{\varphi}_i \}_{i \in I} \) then leads to
\[
\{ \star \forall i, \star \in \text{supp}(\bar{\varphi}_i) \}
\]
Note that for every \( i \) it holds
\[
\star \in \text{supp}(\bar{\varphi}_i) \iff (\star \in \text{supp}(\varphi_i) \text{ or } \star \in \text{supp}(\varphi_i))
\]
and that, if there exists a \( \bar{\varphi}_i \) such that \( \star \notin \text{supp}(\bar{\varphi}_i) \), then \( \{ \bar{\varphi}_i | \star \notin \text{supp}(\bar{\varphi}_i) \} = \{ \varphi_i | \star \notin \text{supp}(\bar{\varphi}_i) \} \). We then derive that the left-then-down path gives
\[
\gamma_X \circ C(\mu_{X+1})(S) = \{ \star \forall i, \star \in \text{supp}(\varphi_i) \text{ or } \star \in \text{supp}(\varphi_i) \}
\]
Taking the down-then-left path, we have the following chain
\[
S \xrightarrow{\gamma_X \ni 1} \begin{cases}
\{ \varphi_i | \star \notin \text{supp}(\varphi_i) \} & \text{o/w}
\end{cases}
\]
\[
\xrightarrow{\gamma_X \ni 1} \begin{cases}
\{ \varphi_i | \star \notin \text{supp}(\varphi_i) \} & \text{o/w}
\end{cases}
\]
where (A) is the condition
\[
(\forall i, \star \in \text{supp}(\varphi_i)) \text{ or } (\exists i, \star \notin \text{supp}(\varphi_i) \text{ and } \forall i(\star \notin \text{supp}(\varphi_i) \Rightarrow \star \in \text{supp}(\varphi_i))).
\]
Condition (A) is equivalent to
\[
\forall i(\star \in \text{supp}(\varphi_i) \text{ or } \star \in \text{supp}(\varphi_i))
\]
and we thereby conclude that the R.H.S. commutes.
We conclude that \( \gamma : C(\cdot + 1) \Rightarrow C + 1 \) is a distributive law. \( \square \)

**Proof of Lemma 2**
We need to show that \( \gamma \) is a monad map from \( C(\cdot + 1) \) to \( C + 1 \). First, the unit diagram in (17) commutes because both units send \( x \in X \) to \( \{ \delta_x \} \) and \( \gamma_X \{ \delta_x \} = \{ \delta_x \} \) by definition.

\[
\begin{array}{ccc}
X & \xrightarrow{\eta_X^{C(\cdot + 1)}} & C(X + 1) \\
& \downarrow{\gamma_X} & \downarrow{\gamma_X} \\
& C(X + 1) & \xrightarrow{\eta_{C(\cdot + 1)}}
\end{array}
\]

Then, it is left to show that the following diagram commutes.

\[
\begin{array}{ccc}
C(C(X + 1) + 1) & \xrightarrow{\gamma(C(X + 1) + 1)} & C(C(X + 1)) + 1 \\
& \downarrow{\mu_X^{C(\cdot + 1)}} & \downarrow{\mu_x^{C(\cdot + 1)}} \\
C(X + 1) & \xrightarrow{\gamma_X} & C(X + 1) + 1
\end{array}
\]

Let \( \star \) denotes the element of the innermost \( 1 \) and \( \ast \) the element of the outermost \( 1 \) and consider an arbitrary \( S = \{ \Phi_i \}_{i \in I} \in C(C(X + 1) + 1) \). We have the following derivation for the top path.

\[
S^{\gamma_{C(X + 1)}} \begin{cases}
\{ \Phi_i \mid \ast \not\in \text{supp}(\Phi_i) \} & \exists i, \ast \not\in \text{supp}(\Phi_i) \\
\ast & \text{o/w}
\end{cases}
\]

\[
c_{(\gamma_X)^{\cdot + 1}} \begin{cases}
\{ D(\gamma_X)(\Phi_i) \mid \ast \not\in \text{supp}(\Phi_i) \} & \exists i, \ast \not\in \text{supp}(\Phi_i) \\
\ast & \text{o/w (i.e., } \forall i, \ast \in \text{supp}(\Phi_i) \text{ )}
\end{cases}
\]

Then, by applying \( \mu_X^{C(\cdot + 1)} \) as defined in Equation 4 (Section 3), we obtain that the right-then-down path yields:

\[
\begin{cases}
\mu_X^{C(\cdot + 1)}(\gamma_{C(X)} \{ D(\gamma_X)(\Phi_i) \mid \ast \not\in \text{supp}(\Phi_i) \} ) & \exists i, \ast \not\in \text{supp}(\Phi_i) \text{ and } \\
\ast & \gamma_{C(X)}(\{ D(\gamma_X)(\Phi_i) \mid \ast \not\in \text{supp}(\Phi_i) \} ) \neq \ast \text{ o/w}
\end{cases}
\]

Note that the condition

\[
\gamma_{C(X)}(\{ D(\gamma_X)(\Phi_i) \mid \ast \not\in \text{supp}(\Phi_i) \} ) \neq \ast
\]
holds if there exists some \( i \) such that \( \mathcal{D}(\gamma_X)(\Phi_i) \) is a full distribution, which is the case if and only if \( \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \). Hence we can rewrite as follows:

\[
\begin{align*}
\mu_X^\mathcal{C}(\gamma_C(\Phi)) \left( \left\{ \mathcal{D}(\gamma_X)(\Phi_i) \mid * \notin \text{supp}(\Phi_i) \right\} \right) & \quad \exists i, \left( * \notin \text{supp}(\Phi_i) \right) \\
& \quad \text{and} \ \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \\
& \quad \text{o/w}
\end{align*}
\]

Finally, by applying the definition of \( \mu_X^\mathcal{C} \) and of \( \gamma_C(\Phi) \) we have the equality:

\[
\mu_X^\mathcal{C}(\gamma_C(\Phi)) \left( \left\{ \mathcal{D}(\gamma_X)(\Phi_i) \mid * \notin \text{supp}(\Phi_i) \right\} \right) =
\]

\[
\bigcup \{ \text{WMS}(\mathcal{D}(\gamma_X)(\Phi_i)) \mid * \notin \text{supp}(\Phi_i) \ \text{and} \ \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \}
\]

which allows us to rewrite as follows:

\[
\begin{align*}
\left\{ \bigcup \{ \text{WMS}(\mathcal{D}(\gamma_X)(\Phi_i)) \mid * \notin \text{supp}(\Phi_i) \ \text{and} \ \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \} \right\} & \quad \exists i, \left( * \notin \text{supp}(\Phi_i) \right) \\
& \quad \text{and} \ \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \\
& \quad \text{o/w}
\end{align*}
\]

Let us now consider the down-then-right path of diagram (18). By applying \( \mu_X^{\mathcal{C}(+1)} \) as given in (14), we obtain:

\[
\mu_X^{\mathcal{C}(+1)}(S) = \bigcup \{ \text{WMS}(\Phi_i^*) \mid i \in I \}.
\]

Then \( \gamma_X \) gives, by applying Lemma 11 and Lemma 12

\[
\gamma_X(\mu_X^{\mathcal{C}(+1)}(S)) = \gamma_X \left( \bigcup \{ \text{WMS}(\Phi_i^*) \mid i \in I \} \right)
\]

\[
= \left\{ \bigcup \{ \gamma_X(\text{WMS}(\Phi_i^*)) \mid * \notin \text{supp}(\Phi_i^*) \} \right\} \quad \exists i, \gamma_X(\text{WMS}(\Phi_i^*)) \neq * \\
& \quad \text{o/w}
\]

\[
= \left\{ \bigcup \{ \text{WMS}(\mathcal{D}(\gamma_X)(\Phi_i^*)) \mid * \notin \text{supp}(\Phi_i^*) \ \text{and} \ \forall U \in \text{supp}(\Phi_i^*), \gamma_X(U) \neq * \} \right\} \quad \exists i, \forall U \in \text{supp}(\Phi_i^*), \gamma_X(U) \neq * \\
& \quad \text{o/w}
\]

We then conclude that diagram (18) commutes, as the following equality holds:

\[
\{ \Phi_i^* \mid \forall U \in \text{supp}(\Phi_i^*), \gamma_X(U) \neq * \} = \{ \Phi_i \mid * \notin \text{supp}(\Phi_i) \ \text{and} \ \forall U \in \text{supp}(\Phi_i), \gamma_X(U) \neq * \}
\]

For the left-to-right inclusion (\( \subseteq \)), suppose that \( \Phi_i^* \) is such that \( \forall U \in \text{supp}(\Phi_i^*), \gamma_X(U) \neq * \). Note that \( \Phi_i^* = \Phi_i \) because \( * \notin \text{supp}(\Phi_i) \). Indeed, it is not possible that \( * \in \text{supp}(\Phi_i) \) as this leads to a contradiction because it implies that \( \{ \delta_* \} \in \)}
Therefore, it is left to show that \((A, \alpha)\) from these properties, we conclude that we have to show two diagrams commute.

\[ \Phi^*_A \] is proved by the following equational reasoning steps.

For the first point (1), we need to show that the pointed convex semilattice \(P((A, \alpha \circ \gamma_A))\), corresponding to \((A, \alpha)\), satisfies the bottom and black-hole axioms. This is proved by the following equational reasoning steps.

\[
x \oplus \star = (\alpha \circ \gamma_A)(\{\delta_x, \delta_*\}) = \alpha(\delta_x) = (\alpha \circ \gamma_A)(\{\delta_x, \delta_*\}) = x \oplus x = x.
\]

\[
x + \mathcal{P} \star = (\alpha \circ \gamma_A)(p \delta_x) = \alpha(\star) = (\alpha \circ \gamma_A)(\delta_x, \delta_x) = \star \oplus \star = \star.
\]

For the second point (2), let \((A, \alpha)\) be the \((\mathcal{C}(\cdot + 1))\)-algebra corresponding via \(P^{-1}\) to a pointed convex semilattice \(A' = (A, \oplus, +, \mathcal{P}, \star)\) satisfying bottom and black-hole, i.e., \((A, \alpha) = P^{-1}(A)\). This \((\mathcal{C}(\cdot + 1))\)-algebra has two key properties. First, \(\alpha(S) = \alpha(\{\delta_x\})\) whenever \(\forall \varphi \in \mathcal{S}, \star \in \text{supp}(\varphi)\). Indeed, we have \(\alpha(S) = \bigoplus_{\varphi \in \mathcal{S} \cap \mathcal{B}(\star)} \varphi^A\) and since each term of this sum contains \(\star\), the whole sum equals to \(\star\), due to the black-hole axiom. Second, when \(S\) contains at least one full distribution, we have \(\alpha(S) = \alpha(\mathcal{C}(\mathcal{I}^{A+1})(\gamma_A(S)))\). This follows from the following derivation.

\[
\alpha(S) = \bigoplus_{\varphi \in \mathcal{S} \cap \mathcal{B}(\star)} \varphi^A
\]

\[
= \bigoplus_{\varphi \in \mathcal{S} \cap \mathcal{B}(\star) : \varphi(\star) = 0} \varphi^A
\]

\[
= \alpha(\{\delta_x, \delta_*\}) = \alpha(\mathcal{C}(\mathcal{I}^{A+1})(\gamma_A(S)))
\]

From these properties, we conclude that \(\alpha = \alpha' \circ \gamma_A\), where \(\alpha' : \mathcal{C}A + 1 \to A = [\alpha \circ \mathcal{C}(\mathcal{I}^{A+1}), \alpha(\{\delta_x\})]\).

Therefore, it is left to show that \((A, \alpha')\) is a \((\mathcal{C} + 1)\)-algebra to conclude that \((A, \alpha)\) is in the image of \(U^\gamma\).

We have to show two diagrams commute.

\[
A \xrightarrow{\alpha'(\delta_x)} \mathcal{C}A + 1 \quad \text{id}_A \quad \alpha' \quad \downarrow
\]

(19)
Diagram (19) commutes since $x$ is sent to $\{\delta_x\}$ and $\alpha'(\delta_x) = \alpha \{\delta_x\} = x$.

For Diagram (20), let $S \in C(CA+1)+1$. We need to distinguish three cases.

First, if $S = \ast$ then both path send $S$ to $\alpha \{\delta_x\}$.

Second, if $\forall \Phi \in S, \ast \in supp(\Phi)$, then the down-then-right path sends $S$ to $\alpha \{\delta_x\}$ again and taking the right-then-down path and denoting $S = \{\Phi_i + p_i \ast\}_i$, we have (we omit all inclusions so $\alpha'(U) = \alpha(U)$ when $U \neq \ast \in CA+1$).

Third, if $S$ contains at least one full distribution, the down-then-right path first sends $S = \{\Phi_i \mid i \in I\}$ to $\{\Phi_i \mid \Phi_i(\ast) = 0\}$. Then, this is sent to

$$\alpha' (\mu_A^{C+1} \{\Phi_i \mid \Phi_i(\ast) = 0\}) = \alpha (\mu_A^{C+1} \{\Phi_i \mid \Phi_i(\ast) = 0\}).$$
On the right-then-down path, we have the following derivation.

\[
\{\Phi_i + p_i^*\} \xrightarrow{C^{(\alpha')^+}} \left\{ \sum_{U \in \text{supp}(\Phi_i)} \Phi_i(U)\alpha(U) + p_i\alpha^* \mid i \in I \right\}
\]

\[
\xrightarrow{\alpha'} \alpha \left\{ \sum_{U \in \text{supp}(\Phi_i)} \Phi_i(U)\alpha(U) + p_i\alpha^* \mid i \in I \right\}
\]

\[
= \alpha \left( C(\alpha + 1) \left\{ \sum_{U \in \text{supp}(\Phi_i)} \Phi_i(U)U + p_i\delta_i \mid i \in I \right\} \right)
\]

\[
= \alpha \left( \mu^{C(\mu)}_A \left\{ \sum_{U \in \text{supp}(\Phi_i)} \Phi_i(U)U + p_i\delta_i \mid i \in I \right\} \right)
\]

\[
= \alpha \left( \mu^{C(\mu)}_A \left\{ \Phi_i \mid p_i = 0 \right\} \right)
\]

The last equality holds because \(\alpha\) satisfies the bottom and black-hole axioms.

\[\Box\]

C Proofs for Section 4

We first prove the following useful lemma.

Lemma 13. Let \(\varphi \in D(X + 1)\). Then \(K_X(\{\varphi\})\) is the smallest \(\bot\)-closed set containing \(\varphi\) and \(K_X(\{\varphi\}) = \{\psi \in D(X + 1) \mid \forall x \in X, \psi(x) \leq \varphi(x)\}\).

Proof. We first prove that for any \(\varphi \in D(X + 1)\),

\[K_X(\varphi) = \{\psi \in D(X + 1) \mid \forall x \in X, \psi(x) \leq \varphi(x)\}.\]

By \(K_X\) being a homomorphism, we obtain:

\[
K_X(\varphi) = K_X \left( \text{WMS} \left( \sum_{x \in \text{supp}(\varphi)} \varphi(x) \{\delta_x\} \right) \right)
\]

\[
= \text{WMS} \left( \sum_{x \in \text{supp}(\varphi)} \varphi(x) K_X(\{\delta_x\}) \right)
\]

\[
= \text{WMS} \left( \sum_{x \in \text{supp}(\varphi)} \varphi(x) \{p x + (1 - p) \ast \mid p \in [0, 1]\} \right)
\]

\[
= \left\{ \sum_{x \in \text{supp}(\varphi)} \varphi(x) (p x + (1 - p) \ast) \mid \forall x, p_x \in [0, 1] \right\}
\]

\[
= \{\psi \in D(X + 1) \mid \forall x \in X, \psi(x) \leq \varphi(x)\}
\]
The last equality follows because the $p_x$s are chosen independently in $[0, 1]$, thus any distribution with weight lower than $\varphi$ at all $x \in X$ can be obtained by choosing the right $p_x$s.

This set is clearly $\perp$–closed, and any $\perp$–closed containing $\varphi$ must contain $K_X\{\varphi\}$ by definition. Thus, $K_X\{\varphi\}$ is the smallest $\perp$–closed set containing $\varphi$. \hfill $\Box$

As a corollary and using the definition of $\perp$–closure we derive the following:

**Lemma 14.** A set $S \in \mathcal{C}(X + 1)$ is $\perp$–closed if and only if for any $\varphi \in S$, $K_X(\{\varphi\}) \subseteq S$.

**Proof of Theorem 2**

Let $S \in \mathcal{C}(X + 1)$. For each $\varphi \in S$, $K_X(\{\varphi\}) \subseteq \text{cc}(K_X(S) \cup K_X(\{\varphi\})) = K_X(\text{cc}(S \cup \{\varphi\})) = K_X(S)$. Hence, Lemma 14 implies that $K_X(S)$ is $\perp$–closed, i.e., $K_X(S) \in \mathcal{C}'(X)$. Moreover, as for all $\varphi \in S$ it holds that $\varphi \in K_X(\{\varphi\})$, by the same reasoning, we derive that $S \subseteq K_X(S)$. To prove minimality, let $S'$ be a $\perp$–closed such that $S \subseteq S'$. We need to show that $K_X(S) \subseteq S'$. Let $\{\varphi_1, \ldots, \varphi_n\}$ be a basis for $S$ (i.e., $S = \text{cc}(\bigcup_{i=1}^n \{\varphi_i\})$). By Lemma 14 as $S'$ is $\perp$–closed and $\{\varphi_1, \ldots, \varphi_n\} \subseteq S \subseteq S'$, we derive that $K_X(\{\varphi_i\}) \subseteq S'$ for any $\varphi_i$ in the basis. Thus, by convexity of $S'$ and by $K_X$ being a homomorphism, we conclude that

$$K_X(S) = K_X\left(\text{cc}\left(\bigcup_{i=1}^n \{\varphi_i\}\right)\right) = \text{cc}\left(\bigcup_{i=1}^n K_X(\{\varphi_i\})\right) \subseteq S'.$$

Hence, $K_X(S)$ is the smallest $\perp$–closed set containing $S$.

We can now prove that $S$ is $\perp$–closed if and only if $S = K_X(S)$. Suppose that $S$ is $\perp$–closed. Then, by (21), we have $K_X(S) \subseteq S$. Since we already knew that $S \subseteq K_X(S)$, we conclude that $S = K_X(S)$. For the converse implication, if $S = K(S)$ then, as $K_X(S)$ is $\perp$–closed, we have that $S$ is $\perp$–closed. \hfill $\Box$

**Proof of Theorem 3**

To show that the triple $(\mathcal{C}', \eta^\mathcal{C}', \mu^\mathcal{C}')$ is a monad, we first prove some useful commuting property of $K$ with respect to (arbitrary) unions and with respect to the multiplication of the monad $\mathcal{C}(\cdot + 1)$.

**Lemma 15.** Let $S \in \mathcal{C}(X + 1)$. Then $K_X(S) = \bigcup_{\varphi \in S} K_X(\{\varphi\})$.

**Proof.** Let $S = \text{cc}(\bigcup_{i=1}^n \{\varphi_i\}) \in \mathcal{C}(X + 1)$. By Theorem 2 $K_X(S)$ is $\perp$–closed, and thus by Lemma 14 we derive that for all $\varphi \in K_X(S)$, $K_X(\{\varphi\}) \subseteq K_X(S)$. As $S \subseteq K_X(S)$, we have that $\bigcup_{\varphi \in S} K_X(\{\varphi\}) \subseteq K_X(S)$. For the converse inclusion, using Theorem 2 we find that it is enough to show $\bigcup_{\varphi \in S} K_X(\{\varphi\})$ is $\perp$–closed as it clearly contains $S$. First, we show it is convex. Let $\psi_1, \psi_2 \in \bigcup_{\varphi \in S} K_X(\{\varphi\})$. 

By Lemma [13] there are \( \theta_1, \theta_2 \in S \) such that \( \forall x, \psi_1(x) \leq \theta_1(x) \) and \( \forall x, \psi_2(x) \leq \theta_2(x) \). We want to prove that
\[
p \cdot \psi_1 + (1 - p) \cdot \psi_2 \in \bigcup_{\varphi \in S} K_X(\{ \varphi \})
\]
which by Lemma [13] is equivalent to showing that there is an \( \varphi \in S \) such that
\[
\forall x, p \cdot \psi_1 + (1 - p) \cdot \psi_2(x) \leq \varphi(x).
\]
This holds by taking \( \varphi = p \cdot \theta_1 + (1 - p) \cdot \theta_2 \), as
\[
\forall x, p \cdot \psi_1 + (1 - p) \cdot \psi_2(x) \leq p \cdot \theta_1 + (1 - p) \cdot \theta_2(x)
\]
and, since \( S \) is convex, we indeed have \( p \cdot \theta_1 + (1 - p) \cdot \theta_2 \in S \). Next, for any \( \psi \in \bigcup_{\varphi \in S} K_X(\{ \varphi \}) \), there exists \( \varphi \in S \) such that \( \psi \in K_X(\{ \varphi \}) \). We infer (using Lemma [13]) that \( K_X(\{ \psi \}) \subseteq K_X(\{ \varphi \}) \subseteq \bigcup_{\varphi \in S} K_X(\{ \varphi \}) \). Therefore, \( \bigcup_{\varphi \in S} K_X(\{ \varphi \}) \) is \( \perp \)-closed.

**Lemma 16.**

1. Let \( S = \{ \Phi_i \}_{i \in I} \in C(C(X + 1) + 1) \). Then
\[
K_X(\mu_X^{C(1)}(S)) = \bigcup_i WMS\left( \sum_{\{K_X(U) \mid U \in \text{supp}(\Phi_i^*)\}} (\sum_{U \in K_X^{-1}(K_X(U))} \Phi_i^*(U)) K_X(U) \right)
\]

2. Let \( S = \{ \Phi_i \}_{i \in I} \in C(C^1(X) + 1) \). Then
\[
K_X(\mu_X^{C(1)}(S)) = \mu_X^{C(1)}(S)
\]

**Proof.**

1. It follows from Lemma [15] that
\[
K_X(\mu_X^{C(1)}(S)) = \bigcup_i K_X(WMS(\Phi_i^*))
\]
and, as \( K_X \) is a homomorphism with respect to weighted Minkowski sums (i.e., the interpretation of \( +_p \) on \( C(X + 1) \)),
\[
K_X(WMS(\Phi_i^*)) = WMS\left( \sum_{\{K_X(U) \mid U \in \text{supp}(\Phi_i^*)\}} (\sum_{U \in K_X^{-1}(K_X(U))} \Phi_i^*(U)) K_X(U) \right).
\]

2. Let \( S = \{ \Phi_i \}_{i \in I} \in C(C^1(X) + 1) \). By applying property 1 we have
\[
K_X(\mu_X^{C(1)}(S)) = \bigcup_i WMS\left( \sum_{\{K_X(U) \mid U \in \text{supp}(\Phi_i^*)\}} (\sum_{U \in K_X^{-1}(K_X(U))} \Phi_i^*(U)) K_X(U) \right).
\]
For each \( \Phi_i \in S \) and for each \( U \in \text{supp}(\Phi_i^*) \), either \( U \in C^1(X) \), and is thereby \( \perp \)-closed by definition, or \( U = \{ \delta_x \} \), which is \( \perp \)-closed. Hence, each
$U \in \text{supp}(\Phi^*_i)$ is $\bot$-closed, and by Theorem 2 we derive that $K_X(U) = U$ for any $U \in \text{supp}(\Phi^*_i)$. From this we derive

$$\bigcup_i \text{WMS}(\sum_{K_X(U) \mid U \in \text{supp}(\Phi^*_i)} \Phi^*_i(U)K_X(U)) = \bigcup_i \text{WMS}(\sum_{U \in \text{supp}(\Phi^*_i)} \Phi^*_i(U)U)$$

which is in turn equal to $\bigcup_i \text{WMS}(\Phi^*_i)$. By definition, this is $\mu^{C^{(i+1)}}_X(S)$.

We can now show that the triple $(C^i, \eta^C^i, \mu^C^i)$ is a monad.

It is immediate to verify that given any $f : X \to Y$ and $S \in C^i(X)$, it holds that $C^i(f)(S) \in C^i(Y)$, i.e., that $C^i(f)$ has indeed the correct type, and that $C^i$ is a functor. Then, we check the typing of the unit and multiplication. For any set $X$ and $x \in X$, the set $\eta^C^i_X(x) = K_X(\delta_x)$ is $\bot$-closed by Lemma 13. For any $S \in C^i(C^i(X))$, by Lemma 15 it holds $K_X(\mu^{C^{(i+1)}}_X(S)) = \mu^{C^{(i+1)}}_X(S)$, which in turn is equal to $\mu^C^i_X(S)$ by definition of $\mu^C^i$ and as $S$ is $\bot$-closed. Hence, the set $\mu^C^i_X(S)$ is $\bot$-closed by Theorem 2.

The unit $\eta^C^i$ is natural, as for any function $f : X \to Y$ we have

$$C(f + 1)(K_X(\delta_x)) = C(f + 1)(\{px + (1-p) \star \mid p \in [0,1]\})$$
$$= \{pf(x) + (1-p) \star \mid p \in [0,1]\}$$
$$= K_Y(\{\delta_{f(x)}\}).$$

The naturality of $\mu^C^i$ follows from the naturality of $\mu^{C^{(i+1)}}$ as the multiplication maps and the functors are defined similarly.

Second, we show that the unit diagram 5 commutes, namely, for any set $X$, $\mu^C^i_X \circ C^i(\eta^C^i_X) = \text{id}_{C^i(X)} \circ \mu^C^i_X \circ \eta^C^i_X$. For the L.H.S., let $S = \bigcup_i \{\varphi_i\}$, we
have

\[
\begin{align*}
\mu_X^i (C^i(\eta_X^i))(S) \\
= \mu_X^i (C(\eta_X^i + 1)) (S) \\
= \mu_X^i (\cup_i \{ D(\eta_X^i + 1)(\varphi_i) \}) \\
= \bigcup \text{WMS} \left( (D(\eta_X^i + 1)(\varphi_i))^* \right) \\
= \bigcup \text{WMS} \left( \left( \left( \sum_{x \in \text{supp}(\varphi_i)} \varphi_i(x) \cdot (p_x x + (1 - p_x) \ast) \ | \ \forall x \in \text{supp}(\varphi_i), p_x \in [0, 1] \right) + \left(1 - \sum_{x \in \text{supp}(\varphi_i)} \varphi_i(x) \right)^* \right) \ast \right) \\
= \bigcup \text{WMS} \left( \left( \sum_{x \in \text{supp}(\varphi_i)} \varphi_i(x) \cdot (p_x x + (1 - p_x) \ast) \ | \ \forall x \in \text{supp}(\varphi_i), p_x \in [0, 1] \right) + \left(1 - \sum_{x \in \text{supp}(\varphi_i)} \varphi_i(x) \right) \ast \right) \\
= \bigcup \left\{ \left( \sum_{x \in \text{supp}(\varphi_i)} \varphi_i(x) \cdot (p_x x + (1 - p_x) \ast) \right) + \left(1 - \sum_{x \in \text{supp}(\varphi_i)} \varphi_i(x) \right) \ast \ | \ \forall x \in \text{supp}(\varphi_i), p_x \in [0, 1] \right\} \\
= \bigcup \left\{ \left( \sum_{x \in \text{supp}(\varphi_i)} q_x x \right) + \left(1 - \sum_{x \in \text{supp}(\varphi_i)} q_x \right) \ast \ | \ \forall x \in \text{supp}(\varphi_i), 0 \leq q_x \leq \varphi_i(x) \right\}
\end{align*}
\]

Since it is clear that each \( \varphi_i \) is in this set, we infer that \( S \subseteq \mu_X^i (C(\eta_X^i))(S) \). For the other inclusion, we have that \( S \) is \( \bot \)-closed, and thus if \( \varphi_i \in S \) then whenever \( 0 \leq q_x \leq \varphi_i(x) \) it holds that \( \left( \sum_{x \in \text{supp}(\varphi_i)} q_x x \right) + \left(1 - \sum_{x \in \text{supp}(\varphi_i)} q_x \right) \ast \in S \). For the R.H.S., we have

\[
\begin{align*}
\mu_X^i (\eta_X^i C^i(X))(S) \\
= \mu_X^i \{ p \cdot S + (1 - p) \ast \ | \ p \in [0, 1] \} \\
= \bigcup_{p \in [0, 1]} \text{WMS} \left( (p \cdot S + (1 - p) \ast)^* \right) \\
= \bigcup_{p \in [0, 1]} \text{WMS} \left( p \cdot S + (1 - p) \ast \right) \\
= \bigcup_{p \in [0, 1]} \left\{ p \cdot \varphi + (1 - p) \ast \varphi \ | \ \varphi \in S \right\} \\
= S
\end{align*}
\]

The last equality holds because is \( \bot \)-closed.
Finally, we need to show that the associativity diagram \( \square \) commutes. Again, this holds merely from the fact that \( \mu^{C^1} \) is defined exactly as \( \mu^{C(+1)} \), and the diagram commutes for \( \mu^{C(+1)} \) by monadicity of \( C(+1) \).

### C.1 Proof Theorem 3

#### Proof of Lemma 6

The fact that each \( K_X \) is well-typed was proven in Theorem 2. We now show that \( K \) is natural, i.e., \( C(f + 1)(K_X(S)) = K_Y(C(f + 1)(S)) \). Take \( S = \bigcup_i \{ \varphi_i \} \in C(X + 1) \) and \( f : X \to Y \). Then by applying first Lemma 15 and then Lemma 13 we get

\[
C(f + 1)(K_X(S)) = C(f + 1) \left( \bigcup_i K_X(\{ \varphi_i \}) \right) = \bigcup_i \{ \psi \mid \psi \in D(X + 1) \text{ and } \forall x \in X, \psi(x) \leq \varphi_i(x) \} = \bigcup_i \{ D(f + 1)(\psi) \mid \psi \in D(X + 1) \text{ and } \forall x \in X, \psi(x) \leq \varphi_i(x) \}
\]

On the other side, by the same properties of \( K \) we have:

\[
K_Y(C(f + 1)(S)) = K_Y \left( \bigcup_i \{ D(f + 1)(\varphi_i) \} \right) = \bigcup_i K_Y(\{ D(f + 1)(\varphi_i) \}) = \bigcup_i \{ \theta \mid \theta \in D(Y + 1) \text{ and } \forall y \in Y, \theta(y) \leq (D(f + 1)(\varphi_i))(y) \}
\]

We prove that for any \( \varphi_i \in S \), the sets

1. \( \{ D(f + 1)(\psi) \mid \psi \in D(X + 1) \text{ and } \forall x \in X, \psi(x) \leq \varphi_i(x) \} \)
2. \( \{ \theta \mid \theta \in D(Y + 1) \text{ and } \forall y \in Y, \theta(y) \leq (D(f + 1)(\varphi_i))(y) \} \)

coincide. For \( 1 \leq i \), let \( D(f + 1)(\psi) \) with \( \psi \in D(X + 1) \) and \( \forall x \in X, \psi(x) \leq \varphi_i(x) \). Then \( D(f + 1)(\psi) \in D(Y + 1) \) and for each \( y \in Y \), \( D(f + 1)(\psi)(y) = \sum_{x \in f^{-1}(y)} \psi(x) \leq \sum_{x \in f^{-1}(y)} \varphi_i(x) = (D(f + 1)(\varphi_i))(y) \). Hence, \( D(f + 1)(\psi) \) is in the second set. For \( 2 \leq i \), let \( \theta \in D(Y + 1) \) with \( \forall y \in Y, \theta(y) \leq (D(f + 1)(\varphi_i))(y) \). Then, as \( f^{-1} \) partitions \( X \), we can assign to each \( x \) a probability value \( p_x \leq \varphi_i(x) \) such that for all \( y \), \( \sum_{x \in f^{-1}(y)} p_x = \theta(y) \). This in turn gives the probability distribution \( \psi \in D(X + 1) \) defined as \( \psi(x) = p_x \) and \( \psi(*) = 1 - \sum p_x \), which indeed satisfies that \( \forall x \in X, \psi(x) \leq \varphi_i(x) \). As \( \theta = D(f + 1)(\psi) \), we conclude that \( \theta \) is in the second set.
Next, we show commutativity of the monad map diagrams. First, the unit diagram \([12]\) commutes, because for any set \(X\) and \(x \in X\),

\[
(K_X \circ \eta_X^{(1+)}) (x) = K_X (\{ \delta_x \}) = \{ px + (1 - p) \ast p \in [0, 1] \} = \eta_X^{(1)} (x).
\]

Second, we need to show \([13]\) commutes, i.e. \(K \circ \mu^{(1+)} = \mu^{(1+)} \circ (K \circ K)\).

Let \(S \in C(C(X + 1) + 1)\). By Lemma \([19]\), applying the L.H.S. yields

\[
K_X (\mu_X^{(1+)} (S)) = \bigcup_{\Phi \in S} \text{WMS} \left( \sum_{K_X (U) U \in \text{supp}(\Phi)} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U) \right)
\]

For the R.H.S., we first apply \(K \circ K\), which yields:

\[
S \overset{C(K_X + 1)}{\longrightarrow} \bigcup_{\Phi \in S} \left( \sum_{K_X (U) U \in \text{supp}(\Phi), U \neq \ast} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U) \right) + \Phi^*(\ast)
\]

By Lemma \([15]\), the latter is equal to

\[
\bigcup_{\Phi \in S \ast} A_{\Phi} \text{ with } A_{\Phi} = K_{C(X + 1)} \left( \sum_{K_X (U) U \in \text{supp}(\Phi)} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U) \right) + \Phi^*(\ast)
\]

Then we apply \(\mu_X^{(1+)}\) and we get by Lemma \([16]\)

\[
\bigcup_{\Phi \in S \ast} \text{WMS}(\Theta^*)
\]

We first prove that \([22]\) is included in \([23]\). Note that, by definition of \(\Phi^*\),

\[
\sum_{\{K_X (U) U \in \text{supp}(\Phi^*)\} \overline{\{K_X (U) U \in \text{supp}(\Phi), U \neq \ast, U \neq \{\delta_u\}\}} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U)
\]

\[
= \left( \sum_{\{K_X (U) U \in \text{supp}(\Phi), U \neq \ast, U \neq \{\delta_u\}\}} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U) \right) + (\Phi(\{\delta_u\}) + \Phi(\ast)) \{\delta_u\}
\]

\[
= \left( \sum_{\{K_X (U) U \in \text{supp}(\Phi), U \neq \ast\}} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U) \right) + \Phi(\ast)^* \in A_{\Phi}
\]

Hence, as

\[
\left( \sum_{\{K_X (U) U \in \text{supp}(\Phi), U \neq \ast\}} \left( \sum_{U \in K_X^{-1}(K_X (U))} \Phi^*(U) \right) K_X (U) \right) + \Phi(\ast)^* \in A_{\Phi}
\]
we derive that there is a $\Theta^* \in A_{\Phi}$ such that
\[
\sum_{\{K_X(U)\mid U \in \text{supp}(\Theta^*)\}} \left( \sum_{U \in K_X^{-1}(K_X(U))} \Phi^*(U) \right) K_X(U) = \Theta^*
\]
Then we conclude that (22) is included in (23). For the converse inclusion, let $\theta \in (23)$. Then $\theta \in \text{WMS}(\Theta^*)$ for some $\Theta \in A_{\Phi}$ and for some $\Phi \in S$. By definition of $A_{\Phi}$, the distribution $\Theta \in D(C(X+1)+1)$ is such that, by Lemma 13, for each $U' \in C(X+1)$ it holds
\[
\Theta(U') \leq \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi), U \neq \{\delta,\} \}} \Phi(U) K_X(U) \right) \Theta(U') \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi), U \neq \{\delta,\} \}} \Phi^*(U) \right)
\]
By definition, $\Theta^*$ equals
\[
\left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi), U \neq \{\delta,\} \}} \Theta(K_X(U)) K_X(U) \right) + \left( 1 - \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi), U \neq \{\delta,\} \}} \Theta(K_X(U)) \right) \right) \delta_*
\]
By definition of $\Phi^*$, this is equivalent to
\[
\Theta^* = \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi^*), U \neq \{\delta,\} \}} \Theta(K_X(U)) K_X(U) \right) + \left( 1 - \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi^*), U \neq \{\delta,\} \}} \Theta(K_X(U)) \right) \right) \delta_*
\]
Then, as $\theta \in \text{WMS}(\Theta^*)$, there are distributions \( \{\varphi_{K_X(U)} \mid U \in \text{supp}(\Phi) \text{ and } U \neq \{\delta,\} \} \) such that
\[
\theta = \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi^*), U \neq \{\delta,\} \}} \Theta(K_X(U)) \cdot \varphi_{K_X(U)} \right) + \left( 1 - \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi^*), U \neq \{\delta,\} \}} \Theta(K_X(U)) \right) \right) \cdot \delta_*
\]
By (24) and the definition of $\Phi^*$, it holds that for each $K_X(U)$ such that $U \in \text{supp}(\Phi^*)$ and $U \neq \{\delta,\}$,
\[
\Theta(K_X(U)) \leq \sum_{U \in K_X^{-1}(K_X(U))} \Phi^*(U)
\]
Hence, for each $x$ it holds that $\theta(x) \leq \psi(x)$, for $\psi$ the distribution
\[
\left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi^*), U \neq \{\delta,\} \}} \Phi^*(U) \right) \cdot \varphi_{K_X(U)}
\]
\[
+ \left( 1 - \left( \sum_{x} \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi), U \neq \{\delta,\} \}} \Phi^*(U) \right) \cdot \varphi_{K_X(U)} \right) \cdot \delta_* \right) 
\]
As $K_X(\{\delta_*\}) = \{\delta_*\}$, we have
\[
\psi \in \text{WMS} \left( \sum_{\{K_X(U)\mid U \in \text{supp}(\Phi^*)\}} \Phi^*(U) K_X(U) \right)
\]
and thus, as \( \theta(x) \leq \psi(x) \) for all \( x \), by Lemma \[L3\] we derive that \( \theta \) is in the set

\[
K_X \left( \text{WMS} \left( \sum_{(K_X(U)U \in \text{supp}(\Phi^*))} \left( \sum_{U \in K_X^{-1}(K_X(U))} \Phi^*(U) \right) K_X(U) \right) \right)
\]

By \( K_X \) being a homomorphism and by \( K_X(K_X(U)) = U \) (Theorem \[2\]), we have

\[
K_X \left( \text{WMS} \left( \sum_{(K_X(U)U \in \text{supp}(\Phi^*))} \left( \sum_{U \in K_X^{-1}(K_X(U))} \Phi^*(U) \right) K_X(U) \right) \right)
\]

This set is included in \[22\], so we have proved that for each \( \theta \in \[23\] \), \( \theta \in \[22\].

**Proof of Lemma \[8.1\]**

Let \((A, \alpha) \in \text{EM}(C^1)\) and let \((A, \alpha \circ K_A) \in \text{EM}(C(+1))\) be its embedding via \( U^K \). By definition of the pointed convex semilattice \( P((A, \alpha \circ K_A)) \) and by definition of \( K \), for any \( a \in A \) it holds

\[
a = \alpha \circ K_A(a) = (\alpha \circ K_A)\{\delta_a, \delta_*\} = \alpha(\alpha A(\{\delta_a, \delta_*\})) = \alpha \circ \alpha A(a) = a.
\]

Hence, the pointed convex semilattice \( P((A, \alpha \circ K_A)) \) satisfies the \( \perp \) equation, and therefore it belongs to \( A(\text{Th}_{C^1}) \).

**Proof of Lemma \[8.2\]**

Let \( A \in A(\text{Th}_{C^1}) \), which is embedded via \( \iota \) to \( A \in A(\text{Th}_{C^1}) \). We want to show that \( P^{-1}(A) = (A, \alpha) \in \text{EM}(C(+1)) \) is in the image of \( U^K \), i.e., that there exists an algebra \( (A, \alpha') \in \text{EM}(C^1) \) such that \( \alpha = \alpha' \circ K_A \). We show that by taking as \( \alpha' \) the restriction of \( \alpha \) to \( \perp \)-closed sets, i.e., by letting \( \alpha' = \alpha|_{C^1(A)} \), we have that \( (A, \alpha') \in \text{EM}(C^1) \) and \( \alpha = \alpha' \circ K_A \).

We first prove that \( (A, \alpha') \) is a \( C^1 \)-algebra, i.e., that \[9\] and \[10\] commute. Note that, by definition of \( P \) and by \( A \) satisfying the bottom axiom, for each \( a \in A \) the following equation holds.

\[
\alpha(\text{cc}\{\delta_a, \delta_*\}) = a \oplus^{A} \star = a = a \oplus^{A} a = \alpha(\text{cc}\{\delta_a, \delta_*\}) = \alpha\{\delta_a\}
\]

Then the unit diagram commutes by

\[
\alpha|_{C^1(A)}(\eta_{C^1}(a)) = \alpha(\text{cc}\{\delta_a, \delta_*\}) \overset{(1)}{=} \alpha\{\delta_a\} = a,
\]
where the last equality holds because \( \alpha \) is a \( C(\cdot + 1) \)-algebra. The multiplication diagram commutes because it is a restriction of the multiplication diagram for \((A, \alpha)\).

It is left to show that \( \alpha|_{C^i} \circ K_A = \alpha \), i.e., that \( \alpha(K(S)) = \alpha(S) \) for any \( S \in C(A + 1) \).

We need the following lemma generalizing Lemma 5.

**Lemma 17.** The following equation is derivable in \( \text{Th}^\perp_{CS} \):

\[
\bigoplus_{0 \leq i \leq n} p_i x_i = \bigoplus_{F \subseteq \{1, \ldots, n\}} \left( \bigoplus_{i \in F} \left( p_i x_i + (1 - \sum_{i \in F} p_i) \ast \right) \right)
\]

**Proof.** First, we note that, by iterating the distributivity axiom (D), we derive in the theory of convex semilattices that:

\[
\bigoplus_{1 \leq i \leq k} p_i \left( t_i^1 \oplus \ldots \oplus t_i^n \right) = \bigoplus_{(t_1, \ldots, t_k) \in \{(t_1, \ldots, t_k) \mid t_i \in \{t_i^1, \ldots, t_i^n\}\}} \left( \bigoplus_{1 \leq i \leq k} p_i t_i \right)
\]

and this law can be alternatively written as follows, whenever for each \( i \) we have a set of terms \( S_i \):

\[
\bigoplus_{1 \leq i \leq k} p_i \left( \bigoplus_{t \in S_i} t \right) = \bigoplus_{f \in \{f : \{1, \ldots, k\} \rightarrow \mathcal{T}(X, \Sigma \ast_{CS}) \mid f(i) \in S_i\}} \left( \bigoplus_{1 \leq i \leq k} p_i f(i) \right)
\]

where \( \{f : \{1, \ldots, k\} \rightarrow \mathcal{T}(X, \Sigma \ast_{CS}) \mid f(i) \in S_i\} \) is the set of functions choosing one term in each \( S_i \).

By the bottom axiom, we have

\[
\bigoplus_{0 \leq i \leq n} p_i x_i = \bigoplus_{0 \leq i \leq n} p_i (x_i \oplus \ast)
\]

Then by applying the iterated version of the distributivity axiom (D) shown above we have

\[
\bigoplus_{0 \leq i \leq n} p_i (x_i \oplus \ast) = \bigoplus_{f \in \{f : \{1, \ldots, n\} \rightarrow \mathcal{T}(X, \Sigma \ast_{CS}) \mid f(i) \in \{x_i, \ast\}\}} \left( \bigoplus_{0 \leq i \leq n} p_i f(i) \right)
\]

which can be in turn proved equal to

\[
\bigoplus_{F \subseteq \{1, \ldots, n\}} \left( \bigoplus_{i \in F} \left( p_i x_i + (1 - \sum_{i \in F} p_i) \ast \right) \right).
\]

\( \square \)

Moreover, we have the following characterization of \( K_X(S) \), which explicits a finite base for the set.
Lemma 18. Let $S = cc(\bigcup_{0 \leq i \leq n} \{ \varphi_i \}) \in C(X + 1)$. Then

$$K_X(S) = cc\left( \bigcup_{0 \leq i \leq n} \bigcup_{F \subseteq \text{supp}(\varphi_i)} \{ \varphi_{i|F} \} \right)$$

where for any $\varphi$ and $F \subseteq \text{supp}(\varphi)$ we define

$$\varphi_{i|F} = \left( \sum_{x \in F} \varphi(x)x \right) + (1 - \sum_{x \in F} \varphi(x)) \ast .$$

Proof. We first prove that for any $\varphi$,

$$K_X(\{ \varphi \}) = cc\left( \bigcup_{F \subseteq \text{supp}(\varphi)} \{ \varphi_{i|F} \} \right).$$

By Lemma 13, this is equivalent to proving

$$\{ \psi \in D(X + 1) \mid \forall x \in X, \psi(x) \leq \varphi(x) \} = cc\left( \bigcup_{F \subseteq \text{supp}(\varphi)} \{ \varphi_{i|F} \} \right).$$

We first prove the right-to-left set inclusion. For any $F \subseteq \text{supp}(\varphi)$, $\varphi_{i|F}$ is such that for all $x \in X$,

$$\varphi_{i|F}(x) = \begin{cases} \varphi(x) & x \in F \\ 0 & \text{o/w} \end{cases} \leq \varphi(x).$$

Then by $K_X(\{ \varphi \})$ being convex we conclude the $\supseteq$ inclusion. For the converse inclusion, note that for any $p \in [0, 1]$, $\psi_1, \psi_2 \in D(X + 1)$ and $x \in X$, we have

$$p\psi_1(x) + (1 - p)\psi_2(x) \leq \max\{\psi_1(x), \psi_2(x)\}.$$

Hence, all convex combinations of elements in $\{ \varphi_{i|F} : F \subseteq \text{supp}(\varphi) \}$ have less weight at $x$ than $\varphi(x)$ for any $x \in X$. This implies the $\subseteq$ inclusion.

Then, for $S = cc(\bigcup_{0 \leq i \leq n} \{ \varphi_i \})$, as $K_X$ commutes over convex union (by being a homomorphism) and by $cc(\bigcup_i S_i) = cc(\bigcup_i cc(S_i))$, we conclude that

$$K_X(S) = cc\left( \bigcup_{0 \leq i \leq n} K_X(\{ \varphi_i \}) \right) = cc\left( \bigcup_{0 \leq i \leq n} \bigcup_{F \subseteq \text{supp}(\varphi_i)} \{ \varphi_{i|F} \} \right) = cc\left( \bigcup_{0 \leq i \leq n} \bigcup_{F \subseteq \text{supp}(\varphi_i)} \{ \varphi_{i|F} \} \right).$$

$\square$
Now, let $S = cc(\bigcup_{0 \leq i \leq n} \{\varphi_i\}) \in C(X + 1)$, with $\{\varphi_i\}$ the unique base for $S$. By applying the lemmas and the definition of $\alpha$, we can now show that $\alpha(K_X(S)) = \alpha(S)$:

$$\alpha(S) = \left( \bigoplus_{0 \leq i \leq n} \left( \bigoplus_{a \in \text{supp}(\varphi_i)} \varphi_i(a)a \right) \right)^\Delta$$

$$= \left( \bigoplus_{0 \leq i \leq n} \left( \bigoplus_{F \in \text{supp}(\varphi)} \left( \left( \bigoplus_{a \in F} \varphi_i(a)a \right) + (1 - (\bigoplus_{a \in F})^\Delta) \right) \right) \right)^\Delta$$

$$= \alpha(K_X(S)).$$

## D Proofs for Section 5

### D.1 Proofs for Section 5.1

We prove that the multiplication $\mu_{C+1}$ of the Set monad $C+1$ is non–expansive.

**Lemma 19.** Given a metric space $(X,d)$, the function $\mu_{C+1} : (C(C(X) + 1) + 1, HK(HK(d) + d_1) + d_1) \rightarrow (C(X) + 1, HK(d) + d_1)$ is not non-expansive.

**Proof.** We give a counterexample to non-expansiveness. Let $X$ be endowed with the discrete metric and take $S_1 = \{1 \frac{1}{2} \delta_x \} + \frac{1}{2} \bigstar$ and $S_2 = \delta_{(\delta_x)}$. Then we obtain

$$HK(d) + d_1(\mu_{C+1}(X,d)(S_1), \mu_{C+1}(X,d)(S_2)) = HK(d) + d_1(\bigstar, \{\delta_x\}) = 1$$

and

$$HK(HK(d) + d_1(S_1, S_2) = \frac{1}{2}$$

\[ \square \]

**Proof of Theorem 5**

The deductive system of quantitative equational logic is the one of [4, §3].

Let $p \in (0, 1)$. From the axioms $\vdash x =_0 x$ and $\vdash x =_1 \bigstar$, we derive by the (K) rule $\vdash x +_p \bigstar =_{1-p} x +_p x$. By $(I_p)$, we have $\vdash x +_p x =_0 x$, so by triangular inequality we derive $\vdash x +_p \bigstar =_{(1-p)} x$. Now, by the $BH_Q$ axiom ($\vdash x +_p \bigstar =_0 \bigstar$), symmetry and triangular inequality, we have $\vdash \bigstar =_{(1-p)} x$. Since $p \in (0, 1)$ was arbitrary, we have equivalently derived that $\vdash \bigstar =_p x$ belongs to $QTh$. For any $y$, we analogously obtain $\vdash \bigstar =_p y$. Then, by symmetry and triangular inequality we derive $\vdash x =_p y$ for all $p \in (0, 1)$, and by (Max) we have $\vdash x =_\epsilon y$ for all $\epsilon > 0$. We conclude by applying (Arch) $\{x =_\epsilon y\}_{\epsilon > 0} \vdash x =_0 y$. 


D.2 Proofs for Section 5.2

Proof of Lemma 10

By definition, \( K_X \) is the unique pointed semilattice homomorphism extending \( f : X \to C(X + 1) \), with \( f(x) = cc(\{\delta_x, \delta_x\}) \). We first show that the function \( \hat{f} : (X, d) \to (C(X + 1), HK(d) + 1) \), defined as \( f \) on \( X \), is an isometry, i.e.,
\[
(HK(d) + 1)(\hat{f}(x), \hat{f}(y)) = d(x, y).
\]
To see this, note that
\[
f(x) = \{px + (1-p) \ast | p \in [0, 1]\} \quad f(y) = \{py + (1-p) \ast | p \in [0, 1]\}
\]
and that for every \( p, q \in [0, 1] \),
\[
K(d)(px + (1-p) \ast, qy + (1-q) \ast) \geq p \cdot d(x, y)
\]
Indeed, if \( p \leq q \) then
\[
K(d)(px + (1-p) \ast, qy + (1-q) \ast) = p \cdot d(x, y) + (q-p) \cdot (d(\ast, y)) + (1-q) \cdot d(\ast, \ast)
\]
\[
= p \cdot d(x, y) + (q-p)
\]
\[
\geq p \cdot d(x, y)
\]
and if \( p = q + r > q \) then
\[
K(d)(px + (1-p) \ast, qy + (1-q) \ast) = q \cdot d(x, y) + (p-q) \cdot (d(x, \ast)) + (1-q) \cdot d(\ast, \ast)
\]
\[
= q \cdot d(x, y) + (p-q)
\]
\[
= q \cdot d(x, y) + r
\]
\[
\geq q \cdot d(x, y) + r \cdot d(x, y)
\]
\[
= p \cdot d(x, y).
\]
We derive by (25) that for any \( \varphi = px + (1-p) \ast \in f(x) \) it holds
\[
\inf_{\psi \in f(y)} K(d)(\varphi, \psi) = K(d)(px + (1-p) \ast, py + (1-p) \ast) = p \cdot d(x, y)
\]
and thus
\[
\sup_{\varphi \in f(x)} \inf_{\psi \in f(y)} K(d)(\varphi, \psi) = K(d)(\delta_x, \delta_y) = d(x, y).
\]
Symmetrically, we obtain
\[
\sup_{\psi \in f(y)} \inf_{\varphi \in f(x)} K(d)(\varphi, \psi) = K(d)(\delta_x, \delta_y) = d(x, y).
\]
We can now conclude
\[
(HK(d) + 1)(f(x), f(y)) = \max\{ \sup_{\varphi \in f(x)} \inf_{\psi \in f(y)} K(d)(\varphi, \psi), \sup_{\psi \in f(y)} \inf_{\varphi \in f(x)} K(d)(\varphi, \psi)\}
\]
\[
= d(x, y).
\]
Since \( \hat{f} \) is an isometry, it is non-expansive, and thus a morphism in \( 1\text{Met} \). Given a metric space \((X, d)\), the metric space \(((\mathcal{C}(X + 1), HK(d) + 1)\) equipped with the operations of convex union, weighted Minkowski sum and \( \{\delta_\ast\} \) (respectively interpreting \( \oplus, +_p, \) and \( * \)) is the free quantitative pointed convex semilattice on \((X, d)\). As \(((\mathcal{C}(X + 1), HK(d) + 1)\) is free, there is a unique quantitative pointed convex semilattice homomorphism extending \( \hat{f} \). It follows from the uniqueness of \( K_X \) and the definition of \( \hat{K}_{(X, d)} \) that \( \hat{K}_{(X, d)} \) is the unique quantitative pointed convex semilattice homomorphism extending \( \hat{f} \). Hence, as \( \hat{K}_{(X, d)} \) is a morphism in \( 1\text{Met} \), it is non-expansive.

**Proof of Theorem 6**

First, we prove that \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \) are natural transformations in \( 1\text{Met} \), i.e., that the naturality diagrams commute and that for any \((X, d)\), \( \eta_{(X, d)}^{\hat{c}^\perp} \) and \( \mu_{(X, d)}^{\hat{c}^\perp} \) are non-expansive. As the unit \( \eta\hat{c}^\perp \) and multiplication \( \mu\hat{c}^\perp \) are respectively defined as the unit \( \eta\hat{c}^\perp \) and multiplication \( \mu\hat{c}^\perp \) of the \( \text{Set} \) monad \( \hat{c}^\perp \), and as we know that the naturality diagrams commute for \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \), we derive that they also commute for \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \). As \( \eta_{(X, d)}^{\hat{c}^\perp} = \hat{K}_{(X, d)} \circ \eta_{(X, d)}^{\hat{c}^\perp+1} \) is non-expansiveness of \( \eta_{(X, d)}^{\hat{c}^\perp} \) follows directly from non-expansiveness of \( \hat{K}_{(X, d)} \) (Lemma 10) and non-expansiveness of \( \eta_{(X, d)}^{\hat{c}^\perp+1} \). As \( \mu\hat{c}^\perp \) is defined as \( \mu\hat{c}^\perp \), which in turn is the restriction of \( \mu\hat{c}^\perp \) to \( \perp \)-closed sets, and as \( \mu\hat{c}^\perp \) is defined as \( \mu\hat{c}^\perp \), we have that \( \mu\hat{c}^\perp \) is the restriction of \( \mu\hat{c}^\perp \) to metric spaces whose sets are \( \perp \)-closed. Then non-expansiveness of \( \mu_{(X, d)}^{\hat{c}^\perp} \) follows from non-expansiveness of \( \mu_{(X, d)}^{\hat{c}^\perp} \). To conclude, it remains to verify that \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \) satisfy the monad laws \( \text{(5)} \) and \( \text{(6)} \). This follows as \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \) are respectively defined as the unit \( \eta\hat{c}^\perp \) and multiplication \( \mu\hat{c}^\perp \) of the \( \text{Set} \) monad \( \hat{c}^\perp \), and as we know that monad laws \( \text{(5)} \) and \( \text{(6)} \) hold for \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \), we derive that the laws also hold for \( \eta\hat{c}^\perp \) and \( \mu\hat{c}^\perp \).

**Proof of Theorem 7**

The structure of the proof of Theorem 4 is very similar to that of Theorem 4 and is based on the following technical lemmas.

**Lemma 20.** The family \( \hat{K}_{(X, d)} : ((\mathcal{C}(X + 1), HK(d)) \rightarrow (\hat{\mathcal{C}}(X), HK(d)) \) is a monad map from the monad \( (\hat{\mathcal{C}}(\perp + 1)) \) to the monad \( \hat{c}^\perp \).

**Proof.** By Lemma 10 \( \hat{K}_{(X, d)} \) is non-expansive, so it is a morphism in \( 1\text{Met} \). As \( \hat{K}_{(X, d)} \) is defined as \( K_X \) on \( X \), by Lemma 6 it satisfies the monad map laws \( \text{(12)} \) and \( \text{(13)} \).

**Lemma 21.** There is a functor \( U^K : \text{EM}(\hat{\mathcal{C}}) \rightarrow \text{EM}(\hat{\mathcal{C}}(\perp + 1)) \) defined on objects by \( (A, d, \alpha) \mapsto ((A, d, \alpha \circ \hat{K}_{(A, d)})) \) and acting as identity on morphisms which is an embedding.
Proof. The fact that $U^K$ is a functor follows from Lemma 20 and Proposition 3. Fully faithfulness follows as $U^K$ acts like the identity on morphisms. Injectivity on objects follows from surjectivity of $K_{X,d}$ for any metric space $(X,d)$, which in turn follows from surjectivity of $K_X$ (Theorem 2). Indeed, if $U^K(((A,d),\alpha)) = U^K((A',d'),\alpha')$ then $A = A'$, $d = d'$, and $\alpha \circ \hat{K}_{(A,d)} = \alpha' \circ \hat{K}_{(A,d)}$, which in turn implies by surjectivity of $\hat{K}_{(A,d)}$ that $\alpha = \alpha'$.

And lastly we obtain the isomorphism of the two categories $\mathbf{EM}(\hat{C}^i)$ and $\mathbf{QA}(\mathcal{QTh}_{CS})$ by restricting the isomorphisms of $\mathbf{EM}(\hat{C}(+1))$ and $\mathbf{QA}(\mathcal{QTh}_{CS})$ witnessing the presentation of the $\mathbf{1Met}$ monad $\hat{C}(+1)$ with the theory of quantitative pointed convex semilattices. This amounts to prove the following two points:

**Lemma 22.** The following hold:

1. Given any $((A,d),\alpha) \in \mathbf{EM}(\hat{C}^i)$, which is embedded via $U^K$ to $((A,d),\alpha \circ \hat{K}_{(A,d)}) \in \mathbf{EM}(\hat{C}(+1))$, the quantitative pointed convex semilattice $P(((A,d),\alpha \circ \hat{K}_{(A,d)}))$ satisfies the $\bot_Q$ quantitative equation, and therefore it belongs to $\mathbf{QA}(\mathcal{QTh}_{CS})$.

2. Given $\AA \in \mathbf{QA}(\mathcal{QTh}_{CS})$, which is embedded via $\iota$ to $\AA \in \mathbf{QA}(\mathcal{QTh}_{CS})$, the Eilenberg-Moore algebra $P^{-1}(\AA) \in \mathbf{EM}(\hat{C}(+1))$ belongs to the subcategory $\mathbf{EM}(\hat{C}^i)$, i.e., it is in the image of $U^K$.

Proof. For item (1), let $((A,d),\hat{\alpha}) \in \mathbf{EM}(\hat{C}^i)$. Then $(A,\alpha) \in \mathbf{EM}(\hat{C}^i)$, where $\alpha$ is $\hat{\alpha}$ seen as a $\mathbf{Set}$ function, and by Lemma S1 we know that the pointed convex semilattice $P(((A,\alpha \circ K_A))$ satisfies the $\bot$ equation. By definition of $\hat{P}$, the interpretation of the pointed convex semilattice operations in $\hat{P}(((A,d),\alpha \circ \hat{K}_{(A,d,a)}))$ is the same as in $P(((A,\alpha \circ K_A))$, thus $\hat{P}(((A,d),\alpha \circ \hat{K}_{(A,d,a)}))$ satisfies the $\bot_Q$ quantitative equation as well.

For item (2), let $\AA = (A,\oplus^A,+,p^A,\ast^A,d) \in \mathbf{QA}(\mathcal{QTh}_{CS})$, which we see as a quantitative pointed convex semilattice via the embedding $\iota$, and let $\hat{P}^{-1}(\AA) = ((A,d),\hat{\alpha}) \in \mathbf{EM}(\hat{C}(+1))$. We show that $((A,d),\hat{\alpha})$ is in the image of $U^K$ by proving that $\hat{\alpha} = \hat{\alpha}|_{\mathcal{C}^i(A)} \circ \hat{K}_{(A,d)}$ and $((A,d),\hat{\alpha}|_{\mathcal{C}^i(A)}) \in \mathbf{EM}(\hat{C}^i)$, with $\hat{\alpha}|_{\mathcal{C}^i(A)} : (\mathcal{C}^i(A),HK(d)) \to (A,d)$ defined as the restriction of $\hat{\alpha}$ to $\bot$-closed sets.

First, note that $(A,\oplus^A,+,p^A,\ast^A)$ is a pointed convex semilattice. By the definition of $\hat{P}^{-1}$, we have that $P^{-1}((A,\oplus^A,+,p^A,\ast^A)) = (A,\alpha)$, where $\alpha$ is $\hat{\alpha}$ seen as a $\mathbf{Set}$ function. By the proof of Lemma S2, we know that $\alpha = \alpha|_{\mathcal{C}^i(A)} \circ K_A$, with $(A,\alpha|_{\mathcal{C}^i(A)}) \in \mathbf{EM}(\hat{C}^i)$. As $\hat{K}_{(A,d)}$ is defined as $K_A$, we derive from $\alpha = \alpha|_{\mathcal{C}^i(A)} \circ K_A$ in $\mathbf{Set}$ that $\hat{\alpha} = \hat{\alpha}|_{\mathcal{C}^i(A)} \circ \hat{K}_{(A,d)}$ in $\mathbf{1Met}$. From $(A,\alpha|_{\mathcal{C}^i(A)}) \in \mathbf{EM}(\hat{C}^i)$ we derive that $((A,d),\hat{\alpha}|_{\mathcal{C}^i(A)})$ satisfies the laws [3] and [10] for $\mathcal{C}^i$-algebras. Moreover, as $\hat{\alpha}$ is non-expansive, also its restriction $\hat{\alpha}|_{\mathcal{C}^i(A)}$ is non-expansive. Hence, $((A,d),\hat{\alpha}|_{\mathcal{C}^i(A)}) \in \mathbf{EM}(\hat{C}^i)$.  

\[\square\]