On six-valued logics of evidence and truth expanding Belnap-Dunn four-valued logic *

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Abstract

The main aim of this paper is to introduce the logics of evidence and truth \( LET^+_K \) and \( LET^+_F \) together with a sound, complete, and decidable six-valued deterministic semantics for them. These logics extend the logics \( LET_K \) and \( LET_F \) with rules of propagation of classicality, which are inferences that express how the classicality operator \( \odot \) is transmitted from less complex to more complex sentences, and vice-versa. The six-valued semantics here proposed extends the 4 values of Belnap-Dunn logic with 2 more values that intend to represent (positive and negative) reliable information. A six-valued non-deterministic semantics for \( LET_K \) is obtained by means of Nmatrices based on swap structures, and the six-valued semantics for \( LET^+_K \) is then obtained by imposing restrictions on the semantics of \( LET_K \). These restrictions correspond exactly to the rules of propagation of classicality that extend \( LET_K \). The logic \( LET^+_F \) is obtained as the implication-free fragment of \( LET^+_K \). We also show that the 6 values of \( LET^+_K \) and \( LET^+_F \) define a lattice structure that extends the lattice \( L_4 \) defined by the Belnap-Dunn four-valued logic with the 2 additional values mentioned above, intuitively interpreted as positive and negative reliable information. Finally, we also show that \( LET^+_K \) is Blok-Pigozzi algebraizable and that its implication-free fragment \( LET^+_F \) coincides with the degree-preserving logic of the involutive Stone algebras.

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1
Introduction

Logics of evidence and truth (LETs) are paracomplete (i.e. paraconsistent and paraconsistent) logics that extend the logic of first-degree entailment (FDE), also known as Belnap-Dunn four-valued logic \([7, 8, 28]\), with a unary operator \(\circ\) that recovers excluded middle and explosion for sentences in its scope by means of the following inferences:

1. \(\circ A, A, \neg A \vdash B\),
2. \(\circ A \vdash A \lor \neg A\).

LETs stemmed from the logics of formal inconsistency (LFI s, see e.g. \([16, 17]\)), which, in turn, are a further development of the seminal work of da Costa on paraconsistency (see e.g. \([25, 26]\)). In the latter, a sentence \(\circ A\) means that \(A\) is ‘well-behaved’; in LFI s, that \(A\) is ‘consistent’.\(^1\) LETs, however, have been conceived with a specific purpose, viz., to formalize the deductive behavior of positive and negative evidence, which can be conclusive or non-conclusive (see e.g. \([2, 21, 17, 19]\)). Thus, according to the intended intuitive interpretation, a sentence \(\circ A\) means that there is conclusive evidence for \(A\). It is assumed that sentences for which the evidence available is considered conclusive behave classically, and \(\circ\) is called a classicality operator. LETs can also be interpreted as information-based logics, which are logics suitable for processing information in the sense of taking a database as a set of premises and drawing conclusions from these premises in a sensible way. In this case, a sentence \(\circ A\) means that the information \(A\), positive or negative, is reliable.

The inferences 1 and 2 above added to FDE define the logic \(LET^-\), which is a sort of minimal logic of evidence and truth. Kripke models for a first-order version of \(LET^-\) have been investigated in \([17]\). The logic \(LET_K\), introduced in \([19]\), extends the logic \(FDE^-\) with 1 and 2. \(FDE^-\) is FDE with a classical implication \([34, 44]\). Sound and complete non-deterministic valuation semantics for sentential LETs can be found in \([19, 20, 49]\), and Kripke-style semantics in \([1, 50]\). It can be proven that none of the LETs studied so far can be characterized by a single finite logical matrix – that is, they are not deterministically finitely-valued. Our main aim here is to obtain a finite-valued non-deterministic semantics for \(LET_K\), as well as to introduce the sentential logics \(LET^-\) and \(LET^-\) together with a sound, complete, and decidable six-valued semantics for them. These logics extend, respectively, \(LET_K\) and \(LET^-\) with rules of propagation of classicality, which are inference rules that express how the operator \(\circ\) is transmitted from less complex to more complex sentences, and vice-versa.

The remainder of this paper is structured as follows. In Section 1 we explain the motivation for extending FDE with the operator \(\circ\) and the respective rules, adding thus two more scenarios to the four scenarios of \(FDE^-\). In Section 2, a natural deduction system for \(LET_K\) is presented, together with a sound, complete, and decidable six-valued semantics based on Nmatrices. Section 3 introduces the logic \(LET^+\) together with a sound, complete, and decidable deterministic six-valued semantics. The latter is obtained by means of restrictions imposed on the non-deterministic six-valued seman-

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\(^1\) In da Costa’s work, the notation is a bit different, it is written as \(A^\circ\) instead of \(\circ A\). The notation \(\circ A\) has been introduced by Carnielli and Marcos in \([18]\).

\(^2\) Parts of Section 1 have already appeared in \([47]\).
tics for \( LET_K \), and such restrictions correspond exactly to the rules of propagation of classicality added to \( LET_K \). In this section we also show that \( LET_K^+ \) is algebraizable in the sense of Blok and Pigozzi [9]. In Section 3 we will show how the underlying algebra of the logical matrices of \( LET_K \), presented as a three-dimensional twist-structure, can be generalized to twist-algebras based on arbitrary Boolean algebras. In this section we also discuss the lattice \( L_6 \) and the semi-lattice \( A_6 \) as expansions of the well known lattices \( L_4 \) and \( A_4 \), defined by the four values of \( FDE \) in Belnap [7]. In Section 4 we turn to the logic \( LET_F \), and show how the results obtained up to that point can be extended to \( LET_F^+ \). This analysis will raise an interesting relation between \( LET_F^+ \) and the degree-preserving logic associated with a variety of algebras known as involutive Stone algebras.

1 Six scenarios instead of four

Belnap in [7] introduced a four-valued semantics for \( FDE \) designed to represent the information stored in a possibly inconsistent and incomplete database. The semantic values of \( FDE \), represented here by \( T_0, F_0, b, \) and \( n \), allow the following four scenarios to be expressed with respect to a given sentence \( A \):

(i) \( v(A) = T_0 \): \( A \) holds and \( \neg A \) does not hold (only positive information \( A \));
(ii) \( v(A) = F_0 \): \( \neg A \) holds and \( A \) does not hold (only negative information \( A \));
(iii) \( v(A) = b \): both \( A \) and \( \neg A \) hold (contradictory information about \( A \));
(iv) \( v(A) = n \): neither \( A \) nor \( \neg A \) holds (no information about \( A \)).

As already said, \( LETs \) admit an interpretation in terms of positive and negative information, and in this case \( \circ A \) means that the information \( A \), positive or negative, is reliable, and when \( \circ A \) does not hold, it means that there is no reliable information about \( A \). As a consequence, \( LETs \) are are able to express six scenarios with respect to a sentence \( A \): (i) to (iv) above when \( \circ A \) does not hold, plus the following two, represented here by the semantic values \( T \) and \( F \):

(v) \( v(A) = T \): \( \circ A \) and \( A \) hold: reliable information that \( A \) is true;
(vi) \( v(A) = F \): \( \circ A \) and \( \neg A \) hold: reliable information that \( A \) is false.

\( LETs \) thus establish a distinction that cannot be established within \( FDE \), for when the values \( T_0 \) or \( F_0 \) are assigned to \( A \) in \( FDE \) this does not specify whether such information is or is not reliable, which is precisely the difference between scenarios (i) and (v), and (ii) and (vi) above.

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\(^3\)Note that the semantics introduced by Dunn [28] in terms of subsets of \( \{0,1\} \) contains essentially the same idea. Although the interpretation in terms of information was spelled out by Belnap in [7], the corresponding conceptual and technical ideas were worked out by Dunn as well. On the historical background of \( FDE \) and its interpretation in terms of information, from Dunn’s dissertation [27] to the papers published in the 1970s [2, 3, 28], see Dunn [30].

\(^4\)For a more detailed discussion of the notions of evidence and information, see [48]. Concerning the intuitive interpretation of \( LETs \) in terms of information, see [47].
2 The logic $LET_K$

We start with the logic $LET_K$, which extends the logic $FDE^\rightarrow$ with the operator $\circ$ and the respective rules. $FDE^\rightarrow$ is $FDE$ with a classical implication added, and can also be defined just by adding $A \lor (A \rightarrow B)$ to Nelson’s logic $N4$. From the semantical point of view, $FDE^\rightarrow$ admits a four-valued semantics that extends in a natural way the semantics of $FDE$ (see [44, p. 1036]). As far as we know, the logic $FDE^\rightarrow$ appeared for the first time in Pynko [43, p. 70], under the name $IDM4$. It appears in Hazen et al. [34, p. 165] under the name $FDE^\rightarrow$, which we adopt here.

As already mentioned in the Introduction, $LET_K$ is not (deterministically) finitely-valued. It admits a Kripke-style semantics [50] and a valuation semantics [19]. Since $LET_K$ is paradefinite, the semantic values of literals $p$ and $\neg p$ are totally independent of each other, which is precisely the point of the four scenarios of $FDE$.

In what follows we will see a natural deduction system for $LET_K$, together with an adequate valuation semantics. Then we will propose a six-valued semantics for $LET_K$, based on $N$matrices, with the semantic values $T, T_0, b, n, F_0, F$, which correspond to the six scenarios expressed by the $LET$s just seen above. We will see that the six-valued semantics combines these six scenarios in a sensible way but, as expected, it is non-deterministic.

From here on, consider a denumerable set $\mathcal{V}$ of propositional variables, and let $For(\Theta)$ be the algebra of formulas freely generated by $\mathcal{V}$ over a propositional signature $\Theta$.

**Definition 2.1.** Consider the propositional signature $\Sigma = \{\land, \lor, \rightarrow, \neg, \circ\}$. A natural deduction system over $\Sigma$ for the logic $LET_K$ is given by the following inference rules:

\[
\begin{align*}
A & \quad B & \frac{}{A \land B} \land I \\
A \land B & \frac{A \land B}{A} \land E
\end{align*}
\]

\[
\begin{align*}
A \lor B & \frac{A \lor B \quad A}{A} \lor I \\
A \lor B & \frac{B \quad B \lor B}{A \lor B} \lor E
\end{align*}
\]

\[
\begin{align*}
\neg A & \frac{\neg (A \land B)}{\neg (A \land B)} \neg \land I \\
\neg A & \frac{\neg (A \lor B)}{\neg (A \lor B)} \neg \lor I
\end{align*}
\]

5 Valuation semantics are bivalued and possibly non-deterministic semantics introduced and investigated in the 1970s onwards by da Costa, Loparic, and Alves (see e.g. [3, 26, 40, 41, 42]). The underlying idea of valuation semantics is to 'mirror' the axioms and rules in terms of the semantic values 1 and 0. On the historical and conceptual features of valuation semantics, see [2, Sect. 6].
A deduction of \( A \) from a set of premises \( \Gamma \) is defined as usual for natural deduction systems. We write \( \Gamma \vdash \) LET \( K \) \( A \) to denote that there is one of such deductions. The rules \( \lor E \), \( \neg \land E \), and \( \to I \) are called improper, while the others are proper. The axiom \( \to \text{CL} \) is neither a proper nor an improper inference rule.

## 2.1 Valuation semantics for \( \text{LET}_K \)

A sound and complete bivalued non-deterministic semantics (a valuation semantics) for \( \text{LET}_K \) is given below (cf. [19, Sects. 3.3.1-2]):

**Definition 2.2.** A bivaluation for \( \text{LET}_K \) is a function \( \rho : \text{For}(\Sigma) \rightarrow \{0, 1\} \) satisfying the following properties:

1. \( \rho(A \land B) = 1 \) iff \( \rho(A) = 1 \) and \( \rho(B) = 1 \);
2. \( \rho(A \lor B) = 1 \) iff \( \rho(A) = 1 \) or \( \rho(B) = 1 \);
3. \( \rho(A \rightarrow B) = 1 \) iff \( \rho(A) = 0 \) or \( \rho(B) = 1 \);
4. \( \rho(\neg \neg A) = 1 \) iff \( \rho(A) = 1 \);
5. \( \rho(\neg(A \land B)) = 1 \) iff \( \rho(\neg A) = 1 \) or \( \rho(\neg B) = 1 \);
6. \( \rho(\neg(A \lor B)) = 1 \) iff \( \rho(\neg A) = 1 \) and \( \rho(\neg B) = 1 \);
7. \( \rho(\neg(A \rightarrow B)) = 1 \) iff \( \rho(A) = 1 \) and \( \rho(\neg B) = 1 \);
8. if \( \rho(\circ A) = 1 \), then: \( \rho(\neg A) = 1 \) iff \( \rho(A) = 0 \).

The semantical consequence relation \( \models^2_{\text{LET}_K} \) of \( \text{LET}_K \) with respect to bivaluations is defined as follows: \( \Gamma \models^2_{\text{LET}_K} A \) if and only if, for every bivaluation \( \rho \) for \( \text{LET}_K \), if \( \rho(B) = 1 \) for every \( B \in \Gamma \) then \( \rho(A) = 1 \).

**Remark 2.3.**

1. The bivalued semantics for \( \text{LET}_K \) makes it clear that the semantic values of \( \neg A \) and \( \circ A \) are not functionally determined by the value of \( A \). This is in line with the idea of the six scenarios expressed by \( \text{LET}s \) presented in Section [1], where \( A \) holds is
to be read $\rho(A) = 1$, and the fact that $LETK$ is not (deterministically) finite-valued. In Section 2.2 a six-valued non-deterministic semantics for $LETK$ will be considered, in which the truth-values are triples formed by the values of $A$, $\neg A$, and $\circ A$ in a given bivalence.

(2) The 2-element Boolean algebra with domain $2 = \{0, 1\}$ will be denoted by $B_2$, and its operations will be denoted by $\sim$ (Boolean complement), $\cap$ (infimum), and $\cup$ (supremum). The implication will be defined as usual as $a \Rightarrow b = \neg a \cup b$. It is well known that $B_2$ is the generator of the variety of Boolean algebras, which implies that a given equation in the signature of Boolean algebras holds in $B_2$ iff it holds in every Boolean algebra (this fact will be used later in the proof of Theorem 4.3).

**Proposition 2.4.** A function $\rho : \text{For}(\Sigma) \to 2$ is a bivaluation for $LETK$ if and only if it satisfies the following properties, expressed in the language of Boolean algebras:

\begin{align*}
(v1)' & \quad \rho(A \land B) = \rho(A) \cap \rho(B); \\
(v2)' & \quad \rho(A \lor B) = \rho(A) \lor \rho(B); \\
(v3)' & \quad \rho(A \to B) = \rho(A) \Rightarrow \rho(B) = \sim \rho(A) \cup \rho(B); \\
(v4)' & \quad \rho(\neg A) = \rho(A); \\
(v5)' & \quad \rho(\neg(A \land B)) = \rho(\neg A) \cup \rho(\neg B); \\
(v6)' & \quad \rho(\neg(A \lor B)) = \rho(\neg A) \cap \rho(\neg B); \\
(v7)' & \quad \rho(\neg A \to B)) = \rho(A) \cap \rho(\neg B); \\
(v8)' & \quad \rho(\circ A) \leq \rho(A) \cup \rho(\neg A) \quad \text{and} \quad \rho(A) \cap \rho(\neg A) \cap \rho(\circ A) = 0.
\end{align*}

It is straightforward to see that clauses $(v1)'-(v7)'$ correspond to clauses $(v1)-(v7)$ of Definition 2.2. Concerning the clause $(v8)'$, note that it corresponds to the rules $PEM^*$ and $EXP^*$.

### 2.1.1 Soundness and completeness

The proofs of soundness and completeness of $LETK$ with respect to the semantics above can be found in [19]. However, in order to keep this paper as self-contained as possible, we will provide the main ideas behind these proofs, which will be adapted and extended to the system $LETK$ to be studied in the following sections.

**Definition 2.5.** Let $\Delta$ be a set of formulas over $\Sigma$, and let $F$ be a formula over $\Sigma$. The set $\Delta$ is said to be $F$-saturated in $LETK$ if: (1) $\Delta \not\models_{LETK} F$; and (2) if $A \not\in \Delta$ then $\Delta, A \vdash_{LETK} F$.

**Remark 2.6.** From a very general result for logic systems due to Lindenbaum and Łoś (see, for example, [53, Theorem 22.2] or [10, Theorem 2.2.6]), the following property holds in $LETK$ (in fact, it holds in any Tarskian and finitary logic)\(^5\).

\(^5\)Recall that a logic $L$ defined over a language $\mathcal{L}$ and with a consequence relation $\vdash$ is said to be Tarskian if it satisfies the following properties, for every set of formulas $\Gamma \cup \Delta \subseteq \mathcal{L}$: (i) if $A \in \Gamma$ then $\Gamma \vdash A$; (ii) if $\Gamma \vdash A$ and $\Gamma \subseteq \Delta$ then $\Delta \vdash A$; and (iii) if $\Delta \vdash A$ and $\Gamma \vdash B$ for every $B \in \Delta$ then $\Gamma \vdash A$. The logic $L$ is said to be finitary if the following holds: if $\Gamma \vdash A$ then there exists a finite subset $\Gamma_0$ of $\Gamma$ such that $\Gamma_0 \vdash A$. 

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If $\Gamma \not\vdash_{LET_K} A$, then there exists a set $\Delta$ such that $\Gamma \subseteq \Delta$ and $\Delta$ is $A$-saturated in $LET_K$.

**Proposition 2.7.** Let $\Delta$ be an $F$-saturated set in $LET_K$. Then:

1. $A \in \Delta$ iff $\Delta \vdash_{LET_K} A$;
2. $A \land B \in \Delta$ iff $A \in \Delta$ and $B \in \Delta$;
3. $A \lor B \in \Delta$ iff $A \in \Delta$ or $B \in \Delta$;
4. $A \rightarrow B \in \Delta$ iff $A \notin \Delta$ or $B \in \Delta$;
5. $\neg(A \land B) \in \Delta$ iff $\neg A \in \Delta$ or $\neg B \in \Delta$;
6. $\neg(A \lor B) \in \Delta$ iff $\neg A \in \Delta$ and $\neg B \in \Delta$;
7. $\neg(A \rightarrow B) \in \Delta$ iff $A \in \Delta$ and $\neg B \in \Delta$;
8. $\neg\neg A \in \Delta$ iff $A \in \Delta$;
9. If $\circ A \in \Delta$ then: either $A \in \Delta$ or $\neg A \in \Delta$, but not both.

**Proof.** Left to the reader (see [19]).

**Corollary 2.8.** Let $\Delta$ be a set of formulas which is $F$-saturated in $LET_K$. Let $\rho_\Delta : For(\Sigma) \to 2$ be the characteristic function of $\Delta$, that is: for every formula $A$, $\rho_\Delta(A) = 1$ iff $\Delta \vdash_{LET_K} A$ (by Proposition 2.7(1)). Then, $\rho_\Delta$ is a bivaluation for $LET_K$.

**Proof.** It is an immediate consequence of Proposition 2.7 items (2)-(9).

**Theorem 2.9** (Soundness and completeness of $LET_K$ w.r.t. bivaluation semantics).

For every set of formulas $\Gamma \cup \{A\} \subseteq For(\Sigma)$: $\Gamma \vdash_{LET_K} A$ iff $\Gamma \models_{2}^{LET_K} A$.

**Proof.** We will present only an sketch of the proof (see details in [19]).

"Only if" part (soundness): Let $\rho$ be a bivaluation for $LET_K$. It is immediate to see that, for every instance $A$ of an axiom of $LET_K$, $\rho(A) = 1$. On the other hand, for every proper rule of $LET_K$ (i.e., for every rule which does not discharge hypotheses), if $\rho(A) = 1$ for every premise of the rule then $\rho(B) = 1$ for the consequence of the rule. Finally, if $B$ is the consequence of an improper rule (that is, a rule which depends on previous derivations in which some hypotheses are discharged), it is also immediate to see, by induction hypothesis and by definition of $\rho$, that $\rho(B) = 1$ provided that $\rho$ satisfies all the assumptions of the rule.

"If" part (completeness): Suppose that $\Gamma \not\vdash_{LET_K} A$. By Remark 2.6 there exists a set $\Delta$ such that $\Gamma \subseteq \Delta$ and $\Delta$ is $A$-saturated in $LET_K$. By Corollary 2.8 the characteristic function $\rho_\Delta$ of $\Delta$ is a bivaluation for $LET_K$ such that $\rho_\Delta(B) = 1$ for every $B \in \Gamma$, but $\rho_\Delta(A) = 0$. This shows that $\Gamma \not\models_{2}^{LET_K} A$.

### 2.2 A six-valued (non-deterministic) semantics for $LET_K$

In this section we will present a six-valued non-deterministic semantics for $LET_K$ based on $N$matrices. The latter are defined from the above bivalued semantics for $LET_K$ (Definition 2.2) by means of swap structures, as described below.
A non-deterministic matrix (Nmatrix, for short) is a multialgebra together with a non-empty subset of its domain, which is the set of designated values. A multialgebra is an algebraic structure equipped with at least one multioperation. The latter is an operation that returns, for each input, a non-empty set of values instead of a single value. Nmatrices are a generalization of logical matrices in which each entry of the matrices interpreting the connectives can produce a non-empty set of possible semantic values. The valuations defined by means of Nmatrices choose an arbitrary value on such sets, returning thus a single semantic value for each formula. Nmatrices were formally introduced by Avron and Lev [5, 6], but there are several earlier examples in the literature of the use of non-deterministic matrices (see e.g. [35, 36, 37, 46]).

A swap structure for \( LET_K \) is a multialgebra whose domain is formed by triples \((z_1, z_2, z_3)\), called snapshots, over a Boolean algebra. Each snapshot represents a three-dimensional semantic value in which the first coordinate \( z_1 \) represents the (one-dimensional) semantic value of a formula \( A \) in a given bivaluation \( \rho \), while the other coordinates \( z_2 \) and \( z_3 \) represent the semantic values of \( \neg A \) and \( \circ A \), respectively, in this same bivaluation \( \rho \). A swap structure for a given logic \( L \) yields a non-deterministic matrix in which the set of designated values is formed by the snapshots \( z \) such that \( z_1 = 1 \), which means that the formula in the position \( z_1 \) holds, or ‘is true’. Let us recall the formal notion of Nmatrices, as introduced in Avron [5, 6]:

**Definition 2.10.** Let \( \Theta \) be a propositional signature. A non-deterministic matrix (Nmatrix, in short) is a structure \( M = (M, D, O) \) such that \( M \) and \( D \) are non-empty sets (of truth-values and designated truth-values, respectively) such that \( D \subseteq M \). \( O \) is a map which assigns, to each \( n \)-ary connective \( \# \) of \( \Theta \), a function \( O(\#) : M^n \rightarrow \wp(M) \setminus \{\emptyset\} \). A (legal) valuation over \( M \) is a function \( v : For(\Theta) \rightarrow M \) such that \( v(\#(A_1, \ldots, A_n)) \in O(\#)(v(A_1), \ldots, v(A_n)) \) for every formula \( \#(A_1, \ldots, A_n) \) over \( \Theta \), where \( \# \) is a \( n \)-ary connective. The consequence relation \( \models_M \) induced by \( M \) is defined as follows: given a set of formulas \( \Gamma \cup \{A\} \), \( \Gamma \models_M A \) if and only, for any valuation \( v \) over \( M \), \( v(A) \in D \) provided that \( v(B) \in D \) for every \( B \in \Gamma \).

**Remark 2.11.** Besides Nmatrices, another interesting non-deterministic semantical framework for non-classical logics is the so-called possible-translations semantics (PTS), introduced by Carnielli in [13, 14]. In [13], Theorem 35, it was shown that Nmatrix semantics can be described by means of a suitable PTS. Conversely, in Theorem 37 it was shown that PTS for a structural logic can be described by Nmatrices semantics. Of course this does not mean that they are similar as decision procedures: for instance, as shown in [4], da Costa’s logic \( C_1 \) cannot be characterized by a single finite Nmatrix, while admitting a finite PTS that provides a decision procedure for it (see [16], Chapter 6). It is worth noting that \( C_1 \) also admits a characterization by a finite-valued restricted swap structures semantics, i.e., a finite Nmatrix generated by a swap structure where the valuations are restricted, thus producing an alternative decision procedure for \( C_1 \) and for every \( C_n \), see [24].

Now, from the bivalued semantics of \( LET_K \) (Definition 2.2), a six-valued swap structure can be defined in a natural way, yielding a six-valued Nmatrix for \( LET_K \). Recall from Remark 2.3(2) the two-element Boolean algebra \( B_2 = (\{0, 1\}, \neg, \cup, \Rightarrow, \sim, 0, 1) \). We denote by \( 2^3 \) the set of triples \( z = (z_1, z_2, z_3) \) over \( 2 \).
Definition 2.12. The Nmatrix $\mathcal{M}_{\text{LET}_K} = (B_{\text{LET}_K}, D, \mathcal{O})$ for $\text{LET}_K$ over the Boolean algebra $B_2$ is defined as follows: its domain is the set
\[ B_{\text{LET}_K} = \{ z \in 2^3 : z_3 \leq z_1 \lor z_2 \text{ and } z_1 \land z_2 \land z_3 = 0 \} \]That is, $B_{\text{LET}_K} = \{ T, T_0, b, n, F_0, F \}$, where
\[ T = (1, 0, 1) \quad n = (0, 0, 0) \]
\[ T_0 = (1, 0, 0) \quad F_0 = (0, 1, 0) \]
\[ b = (1, 1, 0) \quad F = (0, 1, 1) \]
The set of designated elements of $\mathcal{M}_{\text{LET}_K}$ is $D = \{ z \in B_{\text{LET}_K} : z_1 = 1 \} = \{ T, T_0, b \}$, while $\text{ND} = \{ z \in B_{\text{LET}_K} : z_1 \neq 1 \} = \{ F, F_0, n \}$ is the set of non-designated truth-values. For $\# \in \{ \land, \lor, \neg, \circ \}$ the multioperations $\mathcal{O}(\#) = \#$ are defined as follows, for every $z$ and $w$ in $B_{\text{LET}_K}$:

(i) $z \land w = \{ u \in B_{\text{LET}_K} : u_1 = z_1 \land w_1 \text{ and } u_2 = z_2 \lor w_2 \}$;
(ii) $z \lor w = \{ u \in B_{\text{LET}_K} : u_1 = z_1 \lor w_1 \text{ and } u_2 = z_2 \land w_2 \}$;
(iii) $z \Rightarrow w = \{ u \in B_{\text{LET}_K} : u_1 = z_1 \Rightarrow w_1 \text{ and } u_2 = z_2 \land w_2 \}$;
(iv) $\neg z = \{ u \in B_{\text{LET}_K} : u_1 = z_2 \text{ and } u_2 = z_1 \}$;
(v) $\circ z = \{ u \in B_{\text{LET}_K} : u_1 = z_3 \}$.

Remark 2.13.

(i) The domain $B_{\text{LET}_K}$ above does not contain the triples $(0, 0, 1)$ and $(1, 1, 1)$. This is justified as follows. A snapshot $z$ in $B_{\text{LET}_K}$ represents a triple $(\rho(A), \rho(\neg A), \rho(\circ A))$ for some formula $A$ and bivaluation $\rho$ for $\text{LET}_K$. The restrictions $z_3 \leq z_1 \lor z_2$ and $z_1 \land z_2 \land z_3 = 0$ comply with the rules $PEM^\circ$ and $EXP^\circ$ and the clause (v8) of Definition 2.2, which do not allow bivaluations $\rho$ such that $\rho(A) = \rho(\neg A) = 0, \rho(\circ A) = 1$, or $\rho(A) = \rho(\neg A) = \rho(\circ A) = 1$.

(ii) As anticipated in Remark 2.3(1), the snapshots in $B_{\text{LET}_K}$ are able to express simultaneously the value of $A$, $\neg A$, and $\circ A$ in a given bivaluation. That is, they are able to express (and to combine, as we will see in the item (iii) below) the six scenarios described by $\text{LET}_K$s (recall Section 1) as being themselves semantic values: $T$ and $F$ represent, respectively, reliable information $A$ and $\neg A$; $T_0$ and $F_0$ represent information $A$ and $\neg A$, respectively, but not marked as reliable; and $b$ and $n$ represent the scenarios with contradictory information and no information at all about $A$ or $\neg A$ (note that the values $T$ and $F$ of $FDE$ have become here $T_0$ and $F_0$).

(iii) The multioperations of Definition 2.12 can also be presented as follows:

| (1) | $(z_1, z_2, z_3) \land (w_1, w_2, w_3) = (z_1 \lor w_1, z_2 \lor w_2, \_)$; |
| (2) | $(z_1, z_2, z_3) \lor (w_1, w_2, w_3) = (z_1 \land w_1, z_2 \land w_2, \_)$; |
| (3) | $(z_1, z_2, z_3) \Rightarrow (w_1, w_2, w_3) = (z_1 \Rightarrow w_1, z_1 \land w_2, \_)$; |
| (4) | $\neg(z_1, z_2, z_3) = (z_2, z_1, \_)$; |

Note that $B_{\text{LET}_K}$, as well as its multioperations, can be defined \textit{mutatis mutandis} over an arbitrary Boolean algebra besides $B_2$. This will be done in Section 4 Definition 4.1.
Here, ‘*’ means that the respective coordinate can be filled arbitrarily with 0 or 1, provided that the resulting triple belongs to $B_{\text{LET}_K}$, i.e., the triple is a snapshot for $\text{LET}_K$. To illustrate how these multioperations combine the six scenarios expressed by $\text{LET}$s, consider a sentence $A \land B$ in a scenario such that there is positive information $A$ marked as reliable and negative information $B$ not marked as reliable. This would be expressed by a bivaluation $\rho$ such that:

$\rho(A) = 1$, $\rho(\neg A) = 0$, $\rho(\circ A) = 1$ and $\rho(B) = 0$, $\rho(\neg B) = 1$, $\rho(\circ B) = 0$,

and the corresponding snapshots

$(1, 0, 1) = T$ and $(0, 1, 0) = F_0$.

From Definition 2.2 (bivaluations), we have that

$\rho(A \land B) = 0$, $\rho(\neg (A \land B)) = 1$, and $\rho(\circ (A \land B))$ is arbitrary,

which is expressed by the triple $(0, 1, *)$, and the latter is exactly what is given by the operation $\hat{\ast}$ applied to the triples $(1, 0, 1)$ and $(0, 1, 0)$. In terms of the six semantic values, we have that when $v(A) = T$ and $v(B) = F_0$, $v(A \land B)$ is either $F$ or $F_0$.

2.2.1 Non-deterministic tables for $\text{LET}_K$

The multioperations (1)-(5) above can also be described by means of the following non-deterministic tables:

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\text{\hat{\ast}} & T & T_0 & b & n & F_0 & F \\
\hline
T & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
T_0 & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
b & b & b & b & F, F_0 & F, F_0 & F, F_0 \\
\hline
n & n & n & F, F_0 & n & F, F_0 & F, F_0 \\
\hline
F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
F & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\lor & T & T_0 & b & n & F_0 & F \\
\hline
T & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 \\
\hline
T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 \\
\hline
b & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 \\
\hline
n & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 \\
\hline
F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
F & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\approx & T & T_0 & b & n & F_0 & F \\
\hline
T & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
T_0 & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
b & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
n & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 \\
\hline
F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
F & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|c|c|c|c|c|}
\hline
\neg & T & T_0 & b & n & F_0 & F \\
\hline
T & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
T_0 & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
b & T, T_0 & T, T_0 & b & n & F, F_0 & F, F_0 \\
\hline
n & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 & T, T_0 \\
\hline
F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
F & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 & F, F_0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\circ & T & T_0 \\
\hline
T & T, T_0 \\
\hline
T_0 & T, T_0 \\
\hline
b & b \\
\hline
n & n \\
\hline
F_0 & F, F_0 \\
\hline
F & F, F_0 \\
\hline
\end{array}
\]

\[
\begin{array}{|c|c|}
\hline
\rightarrow & T & T_0 \\
\hline
T & T, T_0 \\
\hline
T_0 & T, T_0 \\
\hline
b & b \\
\hline
n & n \\
\hline
F_0 & F, F_0 \\
\hline
F & F, F_0 \\
\hline
\end{array}
\]
For simplicity, we write $X$ and $X, Y$ instead of $\{X\}$ and $\{X, Y\}$. Recall that $D$ and $ND$ denote, respectively, the set of designated and non-designated values.

2.2.2 Soundness and completeness of the six-valued semantics

The six-valued semantics above, as expected, is equivalent to the bivalued semantics of Definition 2.2. Recall that, if $z \in B_{LET}$ then $z_i$ denotes the $i$th-coordinate of the triple $z$. Then:

**Proposition 2.14.** For every valuation $v$ over the Nmatrix $\mathcal{M}_{LET}$, the mapping $\rho_v: \text{For}(\Sigma) \to 2$ given by $\rho_v(A) = v(A)1$ is a bivaluation for $LET_K$ such that: $\rho_v(A) = 1$ iff $v(A) \in D$, for every formula $A$.

**Proof.** Let $A, B \in \text{For}(\Sigma)$. Since $v$ is a valuation over $\mathcal{M}_{LET}$, it satisfies the following:

$$v(\# A) \in \tilde{v}(A) \quad \text{for} \# \in \{-, \circ\}$$

$$v(A \# B) \in v(A) \tilde{v}(B) \quad \text{for} \# \in \{\land, \lor, \rightarrow\}$$

Thus, by definition of $\rho_v$, by Definition 2.12 and by the fact that $v$ is a valuation over $\mathcal{M}_{LET}$, $\rho_v(-A) = v(-A)1 = v(A)2$ and $\rho_v(\circ A) = v(\circ A)1 = v(A)3$. Moreover:

$$\rho_v(A \land B) = v(A \land B)1 = v(A)1 \cap v(B)1 = \rho_v(A) \cap \rho_v(B)$$

$$\rho_v(A \lor B) = v(A \lor B)1 = v(A)1 \lor v(B)1 = \rho_v(A) \lor \rho_v(B)$$

$$\rho_v(A \rightarrow B) = v(A \rightarrow B)1 = v(A)1 \Rightarrow v(B)1 = \rho_v(A) \Rightarrow \rho_v(B)$$

Hence, by Remark 2.3(2), $\rho_v$ satisfies clauses (v1)-(v3). On the other hand, $\rho_v(-A) = v(-A)2 = v(A)1 = \rho_v(A)$ and so $\rho_v$ satisfies (v4). In addition,

$$\rho_v(-A \land B) = v(-(A \land B))1 = v(A \land B)2 = v(A)2 \lor v(B)2 = \rho_v(-A) \lor \rho_v(-B)$$

and so $\rho_v$ satisfies clause (v5), by Remark 2.3(2). Analogously, it is proven that $\rho_v$ satisfies clauses (v6) and (v7). Finally, since $v(A) \in B_{LET}$ then $v(A)3 \leq v(A)1 \lor v(A)2$ and $v(A)1 \cap v(A)2 \cap v(A)3 = 0$. That is, $\rho_v(\circ A) \leq \rho_v(A) \lor \rho_v(-A)$ and $\rho_v(\circ A) \land \rho_v(-A) \leq \rho_v(A) \land \rho_v(-A)$ $\land$ $\rho_v(\circ A) = 0$. By Remark 2.3(2), this means that $\rho_v$ satisfies (v8). This shows that $\rho_v$ is a bivaluation for $LET_K$ such that, by definition, $\rho_v(A) = 1$ iff $v(A) \in D$, for every formula $A$.

**Theorem 2.15** (Soundness of $LET_K$ w.r.t. the Nmatrix $\mathcal{M}_{LET}$).

For every set of formulas $\Gamma \cup \{A\} \subseteq \text{For}(\Sigma)$: $\Gamma \vdash_{LET_K} A$ implies that $\Gamma \models_{\mathcal{M}_{LET}} A$.

**Proof.** Suppose that $\Gamma \vdash_{LET_K} A$. By Theorem 2.9, $\Gamma \models_{\mathcal{M}_{LET}} A$. Let $v$ be a valuation over the Nmatrix $\mathcal{M}_{LET}$ such that $v(B) \in D$ for every $B \in \Gamma$, and let $\rho_v$ be the bivaluation for $LET_K$ defined from $v$ as in Proposition 2.14. Since $\rho_v(B) = 1$ for every $B \in \Gamma$ then $\rho_v(A) = 1$; given that $\Gamma \models_{\mathcal{M}_{LET}} A$. From this $v(A) \in D$, showing that $\Gamma \models_{\mathcal{M}_{LET}} A$.

The completeness of $LET_K$ w.r.t. the six-valued Nmatrix semantics will be proved in a similar way, based once again on Theorem 2.9.
Proposition 2.16. For every bivaluation \( \rho \) for \( \text{LET}_K \) the mapping \( v_\rho : \text{For}(\Sigma) \rightarrow \text{Bi}_{\text{LET}_K} \) given by \( v_\rho(A) = (\rho(A), \rho(\neg A), \rho(\circ A)) \) is a valuation over the Nmatrix \( \mathcal{M}_{\text{LET}_K} \) such that: \( v_\rho(A) \in D \) iff \( \rho(A) = 1 \), for every formula \( A \).

Proof. Clearly \( v_\rho(A) \in \text{Bi}_{\text{LET}_K} \), hence the function is well-defined. Let \( A, B \in \text{For}(\Sigma) \). Then, if \( a = \rho(\circ(A \land B)) \) we have:

\[
\begin{align*}
v_\rho(A \land B) &= (\rho(A \land B), \rho(\neg(A \land B)), a) \\
                 &= (\rho(A) \land \rho(B), \rho(\neg A) \lor \rho(\neg B), a) \in v_\rho(A) \land v_\rho(B).
\end{align*}
\]

The cases of \( \lor \) and \( \rightarrow \) are proved analogously. Concerning negation, let \( b = \rho(\circ\neg A) \). Then:

\[
\begin{align*}
v_\rho(\neg A) &= (\rho(\neg A), \rho(\neg\neg A), b) \\
               &= (\rho(\neg A), \rho(A), b) \in \neg v_\rho(A).
\end{align*}
\]

Finally, if \( a = \rho(\circ\neg A) \) and \( b = \rho(\circ\circ A) \) then

\[
v_\rho(\circ A) = (\rho(\circ A), a, b) \in \circ \circ v_\rho(A).
\]

This shows that \( v_\rho \) is a bivaluation for \( \text{LET}_K \) such that, by definition, \( v_\rho(A) \in D \) iff \( \rho(A) = 1 \), for every formula \( A \).

Theorem 2.17 (Completeness of \( \text{LET}_K \) w.r.t. the Nmatrix \( \mathcal{M}_{\text{LET}_K} \)).

For every set of formulas \( \Gamma \cup \{A\} \subseteq \text{For}(\Sigma) \): \( \Gamma \models_{\text{LET}_K} A \) iff \( \Gamma \vdash_{\text{LET}_K} A \).

Proof. Assume that \( \Gamma \models_{\text{LET}_K} A \), and let \( \rho \) be a bivaluation for \( \text{LET}_K \) such that \( \rho(B) = 1 \) for every \( B \in \Gamma \). Let \( v_\rho \) be defined from \( \rho \) as in Proposition 2.16. Then, \( v_\rho \) is a valuation over \( \mathcal{M}_{\text{LET}_K} \) such that \( v_\rho(B) \in D \), for every \( B \in \Gamma \). By hypothesis, \( v_\rho(A) \in D \), whence \( \rho(A) = 1 \). This shows that \( \Gamma \models_{\text{LET}_K} A \). By completeness of \( \text{LET}_K \) w.r.t. bivaluations, \( \Gamma \vdash_{\text{LET}_K} A \).

Corollary 2.18. The six-valued Nmatrix \( \mathcal{M}_{\text{LET}_K} \) provides a decision procedure for \( \text{LET}_K \).

Proof. It is well known that finite Nmatrices, like the one above, provide decision procedures. This is because the number of sentential variables is finite, the number of bifurcated lines is always finite, and clearly there is no loop in the procedure.

3 Adding propagation rules to \( \text{LET}_K \): the logic \( \text{LET}_K^+ \)

In a broad sense, propagation of classicality is how classical behavior propagates from less complex to more complex sentences, and vice-versa. The \( \text{LET} \)s investigated so far (\( \text{LET}_J, \text{LET}_F, \text{LET}_K, \text{LET}_F \)) enjoy the following property:

Proposition 3.1. Let \( L \in \{ \text{LET}_F, \text{LET}_J, \text{LET}_K, \text{LET}_F \} \) and \( \Gamma = \{ \circ\neg^n A_1, \ldots, \circ\neg^n m A_m \} \), for \( n_i \geq 0 \) (where \( \circ\neg^n \), \( n_i \geq 0 \), represents \( n_i \) occurrences of negations before the formula \( A_i \)). Then, for any formula \( B \) formed with \( A_1, \ldots, A_m \) over the signature \( \{\neg, \land, \lor, \rightarrow\} \) (and \( \{\neg, \land, \lor\} \) in the case of \( \text{LET}_F \) and \( \text{LET}_F \)), \( \Gamma \vdash_L B \lor \neg B \), and \( \Gamma, B, \neg B \vdash_L C \). That is, \( B \) behaves classically in this context.
classical logic is appropriate to express the deductive behavior of reliable information. Are, rather, making the weaker claim that people reason with reliable information as if it were true, and so in the realist sense of the notion of truth that underlies the standard interpretation of classical logic. We of preservation of truth, does not mean, of course, that reliable information is being \textit{identified} with truth, in the realist sense of the notion of truth that underlies the standard interpretation of classical logic. We are, rather, making the weaker claim that people reason with reliable information as if it were true, and so classical logic is appropriate to express the deductive behavior of reliable information.

\textit{Proof.} This result is proved for \textit{LET} in \cite[Fact 31]{1}, and for \textit{LET} in \cite[Proposition 7]{2}. Proofs for \textit{LET}_s and \textit{LET}_K can be obtained similarly.

However, although in the \textit{LET}s studied so far the classical behavior is transmitted from less complex to more complex formulas, the classicality operator \(\circ\) is not; that is, the inferences \(\varphi \vdash \varphi \land \varphi\) and \(\varphi, \varphi \equiv \varphi (\varphi \equiv \varphi)\) (\(\# \in \{\land, \lor, \Rightarrow\}\)) for example, do not hold. This should be clear in \textit{LET}_K from the fact that there is no introduction rule for \(\circ\), but the reader can also check this from the six-valued matrices presented in Section \ref{2.2} which explicitly display the non-deterministic behavior of \(\circ\). But to rigorously express the idea of dividing the sentences of the language into two groups, which is an essential point of \textit{LET}s (as well as \textit{LFI}s and da Costa’s \(C_n\) hierarchy) it would be desirable for the classicality operator \(\circ\) to be transmitted as well.

We find in the literature two ways of establishing the propagation of \(\circ\), but neither fits the intended interpretation of \textit{LET}s. In da Costa’s \(C_n\) hierarchy, the well-behavior of \(A\) (originally represented by \(A^0\)) propagates as follows:

\begin{enumerate}
    \item \(\circ A \land \circ B \vdash \circ (A \# B)\), for \(\# \in \{\land, \lor, \Rightarrow\}\).

Although \(\circ\) fits with Proposition \(\circ, 1\) above, it places a too strong condition on the propagation of classicality. Indeed, we will see that it is not always necessary that both \(\circ A\) and \(\circ B\) hold for concluding \(\circ (A \# B)\). On the other hand, in the logic \(Cilo\), an \textit{LFI} investigated in \cite[pp. 116ff.]{3}, consistency propagates as follows:

\begin{enumerate}
    \item \(\circ A \lor \circ B \vdash \circ (A \# B)\), for \(\# \in \{\land, \lor, \Rightarrow\}\).

The condition for propagation in \(\circ\) however, is too weak in the sense that it allows one to conclude \(\circ (A \# B)\) in circumstances where it should not be concluded. Let us illustrate this by taking a look at how \(\circ\) should be transmitted over \(\lor\).

Recall that \(\circ A \land \circ A\) and \(\circ A \land \neg A\) are intended to mean that the information conveyed, respectively, by \(A\) and by \(\neg A\), is considered reliable. In addition, positive and negative reliable information behaves like truth and falsity in classical logic. As a consequence, from \(\circ A \land \circ A\) one should be able to infer that \(A \lor B\) is also reliable for any \(B\), no matter whether \(\circ B\) holds or not, and so \(\circ (A \lor B) \land (A \lor B)\) holds. For if this were not the case, that is, if both \((A \lor B)\) and \((A \lor B)\) held, \(A\) could not hold, since \(\neg (A \lor B)\) implies \(\neg A\). On the other hand, from \(\circ A \land \neg A\), it cannot be inferred that \(\neg (A \lor B)\) is reliable, because to conclude \(\neg (A \lor B)\) both \(\neg A\) and \(\neg B\) are required. Hence, from \(\circ A \land \neg A\), \(\circ (A \lor B)\) cannot be inferred. This suggests the validity of the following inferences:

\begin{enumerate}
    \item \(\circ A \land \circ B \vdash \circ (A \lor B)\),
    \item \(\circ B \vdash \circ (A \lor B)\),
\end{enumerate}

and so on. The point of these inferences is that positive (resp. negative) reliable information behaves like truth (resp. falsity) in classical logic: in order to have a \(A \lor B\) false, we need both \(A\) and \(B\) false, but \(A\) true is enough to conclude \(A \lor B\) true. In

\footnote{It is to be noted that the fact that reliable information is subjected to classical logic, and so to the rules of preservation of truth, does not mean, of course, that reliable information is being \textit{identified} with truth, in the realist sense of the notion of truth that underlies the standard interpretation of classical logic. We are, rather, making the weaker claim that people reason with reliable information as if it were true, and so classical logic is appropriate to express the deductive behavior of reliable information.}
the following, in order to define the system $\text{LET}_K^+$, we will extend this line of reasoning to the connectives $\neg$, $\land$, and $\to$.

**Definition 3.2.** For any formula $A$, let $A^T \overset{\text{def}}{=} \circ A \land A$ and $A^F \overset{\text{def}}{=} \circ A \land \neg A$.

A natural deduction system for $\text{LET}_K^+$ is obtained by adding the following axiom and rules to the system of $\text{LET}_K$ (Definition 2.1):

\[
\begin{align*}
&\dfrac{}{\circ A} \quad \circ \neg A \\
&\dfrac{(A \land B)^T}{A^T} \quad [I\land T] \\
&\dfrac{(A \land B)^F}{A^F} \quad [I\land F] \\
&\dfrac{(A \lor B)^T}{A^T} \quad [I\lor T] \\
&\dfrac{(A \lor B)^F}{A^F} \quad [I\lor F] \\
&\dfrac{(A \rightarrow B)^T}{A^T} \quad [I\rightarrow T] \\
&\dfrac{(A \rightarrow B)^F}{A^F} \quad [I\rightarrow F] \\
&\dfrac{(A \land B)^T}{A} \quad [E\land T] \\
&\dfrac{(A \land B)^F}{A} \quad [E\land F] \\
&\dfrac{(A \lor B)^T}{A} \quad [E\lor T] \\
&\dfrac{(A \lor B)^F}{A} \quad [E\lor F] \\
&\dfrac{(A \rightarrow B)^T}{A} \quad [E\rightarrow T] \\
&\dfrac{(A \rightarrow B)^F}{A} \quad [E\rightarrow F] \\
\end{align*}
\]

A deduction of $A$ from a set of premises $\Gamma$, $\Gamma \vdash_{\text{LET}_K^+} A$, is defined as usual for natural deduction systems.

**Proposition 3.3.** The following rules are derived in $\text{LET}_K^+$:

\[
\begin{align*}
&\dfrac{}{\circ A} \quad \circ \neg A \\
&\dfrac{}{\circ \neg A} \quad \circ \neg A \\
&\dfrac{}{\circ \neg A} \quad \circ \neg A \\
\end{align*}
\]

**Proof.** The rules above are obtained in a few steps from $\text{EXP}^\circ$, $\text{PEM}^\circ$, and $I\circ$. ■
3.1 Valuation semantics for $\text{LET}_{K}^{+}$

The rules and the axiom of $\text{LET}_{K}^{+}$ added to $\text{LET}_{K}$ induce, in a very natural way, a bivalued semantics, which will be described in what follows (as we shall see in Proposition 3.16, the definition below can be drastically simplified when expressed in the language of Boolean algebras).

**Definition 3.4** (Valuation semantics for $\text{LET}_{K}^{+}$). A bivaluation $\rho : \text{For}(\Sigma) \rightarrow 2$ for $\text{LET}_{K}$ (Definition 2.2) satisfying, in addition, the following clauses:

1. $\rho(\circ A) = 1$;
2. $\rho(\circ \lnot A) = \rho(\lnot A)$;
3. If $\rho(\circ A) = \rho(A) = 1$ and $\rho(\circ B) = \rho(B) = 1$ then $\rho(\circ(A \land B)) = 1$;
4. If $\rho(\circ A) = \rho(\lnot A) = 1$ then $\rho(\circ(A \land B)) = 1$;
5. If $\rho(\circ B) = \rho(\lnot B) = 1$ then $\rho(\circ(A \land B)) = 1$;
6. If $\rho(\circ(A \land B)) = \rho(A) = \rho(B) = 1$ then $\rho(\circ A) = \rho(\circ B) = 1$;
7. If $\rho(\circ(A \land B)) = 1$, and either $\rho(\lnot A) = 1$ or $\rho(\lnot B) = 1$, then:
   - either $\rho(\circ A) = \rho(\lnot A) = 1$ or $\rho(\circ B) = \rho(\lnot B) = 1$;
8. If $\rho(\circ A) = \rho(A) = 1$ then $\rho(\circ(A \lor B)) = 1$;
9. If $\rho(\circ B) = \rho(B) = 1$ then $\rho(\circ(A \lor B)) = 1$;
10. If $\rho(\circ A) = \rho(\lnot A) = 1$ and $\rho(\circ B) = \rho(\lnot B) = 1$ then $\rho(\circ(A \lor B)) = 1$;
11. If $\rho(\circ(A \lor B)) = 1$, and either $\rho(\circ A) = 1$ or $\rho(\circ B) = 1$, then:
    - either $\rho(\circ A) = \rho(A) = 1$ or $\rho(\circ B) = \rho(B) = 1$;
12. If $\rho(\circ(A \lor B)) = 1$ and $\rho(\circ A) = \rho(B) = 0$ then $\rho(\circ A) = \rho(\circ B) = 1$;
13. If $\rho(\circ A) = \rho(\lnot A) = 1$ then $\rho(\circ(A \rightarrow B)) = 1$;
14. If $\rho(\circ B) = \rho(B) = 1$ then $\rho(\circ(A \rightarrow B)) = 1$;
15. If $\rho(A) = 1$ and $\rho(\circ B) = \rho(\lnot B) = 1$ then $\rho(\circ(A \rightarrow B)) = 1$;
16. If $\rho(\circ(A \rightarrow B)) = 1$, and either $\rho(A) = 0$ or $\rho(B) = 1$, then:
    - either $\rho(\circ A) = \rho(\lnot A) = 1$ or $\rho(\circ B) = \rho(\lnot B) = 1$;
17. If $\rho(\circ(A \rightarrow B)) = 1$ and $\rho(A) = \rho(\lnot B) = 1$ then $\rho(\circ B) = 1$.

Now we shall prove the soundness and completeness of $\text{LET}_{K}^{+}$ w.r.t. the bivalued semantics above. The definition of $F$-saturated sets in $\text{LET}_{K}^{+}$ is analogous to the one for $\text{LET}_{K}$ (Definition 2.4).

**Proposition 3.5.** Let $\Delta$ be an $F$-saturated set in $\text{LET}_{K}^{+}$. Then, it is a closed theory (that is, $A \in \Delta$ iff $\Delta \models_{\text{LET}_{K}^{+}} A$) and it satisfies properties (2)-(9) of Proposition 2.7 plus the following:

1. $\circ \circ A \in \Delta$;
2. $\circ A \in \Delta$ iff $\lnot \circ A \in \Delta$;
3. If $\circ A \in \Delta$, $A \in \Delta$, $\circ B \in \Delta$ and $B \in \Delta$, then $\circ(A \land B) \in \Delta$;
(vp4') If \( \circ A \in \Delta \) and \( \neg A \in \Delta \), then \( \circ (A \land B) \in \Delta \);

(vp5') If \( \circ B \in \Delta \) and \( \neg B \in \Delta \), then \( \circ (A \land B) \in \Delta \);

(vp6') If \( \circ (A \land B) \in \Delta \), \( A \in \Delta \) and \( B \in \Delta \), then \( \circ A \in \Delta \) and \( \circ B \in \Delta \);

(vp7') If \( \circ (A \land B) \in \Delta \), and either \( A \in \Delta \) or \( \neg B \in \Delta \), then:
   
either \( \circ A \in \Delta \) and \( \neg A \in \Delta \) or \( \circ B \in \Delta \) and \( \neg B \in \Delta \);

(vp8') If \( \circ A \in \Delta \) and \( A \in \Delta \), then \( \circ (A \lor B) \in \Delta \);

(vp9') If \( \circ B \in \Delta \) and \( B \in \Delta \), then \( \circ (A \lor B) \in \Delta \);

(vp10') If \( \circ A \in \Delta \), \( \neg A \in \Delta \), \( B \in \Delta \) and \( \neg B \in \Delta \), then \( \circ (A \lor B) \in \Delta \);

(vp11') If \( \circ (A \lor B) \in \Delta \), and either \( A \in \Delta \) or \( B \in \Delta \), then:
   
either \( \circ A \in \Delta \) and \( A \in \Delta \) or \( \circ B \in \Delta \) and \( B \in \Delta \);

(vp12') If \( \circ (A \lor B) \in \Delta \), \( A \in \Delta \) and \( B \notin \Delta \), then \( \circ A \in \Delta \) and \( \circ B \in \Delta \);

(vp13') If \( \circ A \in \Delta \) and \( \neg A \in \Delta \), then \( \circ (A \rightarrow B) \in \Delta \);

(vp14') If \( \circ B \in \Delta \) and \( B \in \Delta \), then \( \circ (A \rightarrow B) \in \Delta \);

(vp15') If \( A \in \Delta \), \( \circ B \in \Delta \) and \( \neg B \in \Delta \), then \( \circ (A \rightarrow B) \in \Delta \);

(vp16') If \( \circ (A \rightarrow B) \in \Delta \), and either \( A \in \Delta \) or \( B \in \Delta \), then:
   
either \( \circ A \in \Delta \) and \( \neg A \in \Delta \) or \( \circ B \in \Delta \) and \( B \in \Delta \);

(vp17') If \( \circ (A \rightarrow B) \in \Delta \), \( A \in \Delta \) and \( \neg B \in \Delta \), then \( \circ B \in \Delta \).

Proof. It is immediate to see that an \( F \)-saturated set is a closed theory in any Tarskian logic. The proof of items (2)-(9) follows from \( \text{LET}_K \). The proofs of items vp1' to vp17' follow easily from the axiom and rules added to \( \text{LET}_K \) in Definition 3.2. Details are left to the reader.

Corollary 3.6. Let \( \Delta \) be a set of formulas which is \( F \)-saturated in \( \text{LET}_K^r \). Let \( \rho_\Delta : \text{For}(\Sigma) \rightarrow 2 \) be the characteristic function of \( \Delta \), that is: for every formula \( A \), \( \rho_\Delta (A) = 1 \) iff \( A \in \Delta \) (iff \( \Delta \vdash \text{LET}_K^r A \), by Proposition 3.5). Then, \( \rho_\Delta \) is a bivaluation for \( \text{LET}_K^r \).

Proof. It is an immediate consequence of Proposition 3.5.

Theorem 3.7 (Soundness and completeness of \( \text{LET}_K^r \) w.r.t. bivaluation semantics). For every set of formulas \( \Gamma \cup \{ A \} \subseteq \text{For}(\Sigma) \): \( \Gamma \vdash \text{LET}_K^r A \) iff \( \Gamma \equiv^2 \text{LET}_K^r A \).

Proof. 'Only if' part (soundness): By Theorem 2.9, it suffices proving that any bivaluation for \( \text{LET}_K^r \) satisfies the axiom and rules of Definition 3.2. But this is immediate, taking into account the other properties inherited from bivaluations for \( \text{LET}_K \). The details are left to the reader.

'If' part (completeness): It is also an extension of the proof of completeness of \( \text{LET}_K \) w.r.t. bivaluations. Thus, suppose that \( \Gamma \not\vdash \text{LET}_K^r A \). As observed in Remark 2.6, being \( \text{LET}_K^r \) a Tarskian and finitary logic, there exists a set \( \Delta \) such that \( \Gamma \subseteq \Delta \) and \( \Delta \) is \( A \)-saturated in \( \text{LET}_K^r \). By Corollary 3.6, the characteristic function \( \rho_\Delta \) of \( \Delta \) is a bivaluation for \( \text{LET}_K^r \) such that \( \rho_\Delta (B) = 1 \) for every \( B \in \Gamma \), but \( \rho_\Delta (A) = 0 \). This shows that \( \Gamma \not\vdash^2 \text{LET}_K^r A \).

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3.2 A six-valued semantics for $\text{LET}^*_K$

Let us recall the Nmatrix for $\text{LET}_K$ introduced in Section 2.2. The snapshots $z = (z_1, z_2, z_3)$ represent triples of the form $(\rho(A), \rho(\neg A), \rho(\circ A))$, for a bivaluation $\rho$ for $\text{LET}_K$ and a formula $A$. With this in mind, and taking into account the definition of the six snapshots of $\text{BLET}_K$, in what follows we will see how the clauses of Definition 3.4 impose restrictions on the multioperators of the Nmatrix of $\text{LET}_K$, which turn out to be deterministic in $\text{LET}^*_K$.

**Proposition 3.8.** Clauses (vp3)-(vp7) of Definition 3.4 impose the following restrictions to the multioperator $\tilde{\cdot}$ of the Nmatrix of $\text{LET}_K$:

- $(vp3)$: $T \wedge T = T$;
- $(vp4)$-$(vp5)$: $F \wedge z = z \wedge F = F$, for every $z$;
- $(vp6)$: $z \wedge w = T$ implies that $z = w = T$. Hence, $T \wedge T_0 = T_0 \wedge T = T_0 \wedge T_0 = T_0$;
- $(vp7)$: $z \wedge w = F$ implies that $z = F$ or $w = F$. Hence, $b \wedge n = n \wedge b = F_0 \wedge z = z \wedge F_0 = F_0$, for $z \neq F$.

This shows that $\tilde{\cdot}$ is deterministic in $\text{LET}^*_K$ and can be defined as

- $(i)$ $(z_1, z_2, z_3) \tilde{\cdot} (w_1, w_2, w_3) = (z_1 \wedge w_1, z_2 \wedge w_2, (z_1 \wedge z_3 \wedge w_1 \wedge w_3) \cup (z_2 \wedge z_3) \cup (w_2 \wedge w_3))$.

Concerning $\tilde{\vee}$, clauses (vp8)-(v12) impose the following restrictions:

- $(vp8)$-$(vp9)$: $T \vee z = z \vee T = T$, for every $z$;
- $(vp10)$: $F \vee F = F$;
- $(vp11)$: $z \vee w = T$ implies that $z = T$ or $w = T$. Hence, $b \vee n = n \vee b = T_0 \vee z = z \vee T_0 = T_0$, for $z \neq T$;
- $(vp12)$: $z \vee w = F$ implies that $z = w = F$. Hence, $F_0 \vee F_0 = F_0 \vee F = F \vee F = F_0$.

Thus, $\tilde{\vee}$ turns out to be deterministic in $\text{LET}^*_K$ and can be defined as

- $(ii)$ $(z_1, z_2, z_3) \tilde{\vee} (w_1, w_2, w_3) = (z_1 \vee w_1, z_2 \vee w_2, (z_2 \vee z_3 \vee w_2 \vee w_3) \cup (z_1 \vee z_3) \cup (w_1 \vee w_3))$.

With respect to $\tilde{\rightarrow}$, clauses (vp13)-(v17) impose the following restrictions:

- $(vp13)$: $F \rightarrow z = T$, for every $z$;
- $(vp14)$: $z \rightarrow T = T$, for every $z$;
- $(vp15)$: $z \rightarrow F = F$, for $z \in D$. Hence, $T \rightarrow F = T_0 \rightarrow F = F \rightarrow F = F$;
- $(vp16)$: $z \rightarrow w = T$ implies that $z = F$ or $w = T$. Hence, $n \rightarrow w = F_0 \rightarrow w = T_0$, for $w \neq T$; and $n \rightarrow T_0 = T_0$, for $z \in D$;
- $(vp17)$: $z \rightarrow w = F$ implies that $w = F$. Hence, $z \rightarrow F_0 = F_0$, for $z \in D$.

Thus, $\tilde{\rightarrow}$ also turns out to be deterministic, defined as

- $(iii)$ $(z_1, z_2, z_3) \tilde{\rightarrow} (w_1, w_2, w_3) = (z_1 \rightarrow w_1, z_1 \rightarrow w_2, (z_1 \rightarrow w_3) \cup (z_2 \rightarrow z_3) \cup (w_1 \rightarrow w_3))$.

Given the clause (vp2), the third coordinate of the snapshot will not be changed by the operation $\tilde{\cdot}$. For this reason, the negation $\tilde{\cdot}$ in $\text{LET}^*_K$ also turns out to be deterministic: $\tilde{\neg} T = F$, $\tilde{\neg} F = T$, $\tilde{\neg} T_0 = F_0$, and $\tilde{\neg} F_0 = T_0$. That is, it is defined as

- $(iv)$ $\tilde{\neg} (z_1, z_2, z_3) = (z_1, z_2, z_3)$.

Finally, the classicality operator $\tilde{\circ}$ of $\text{LET}^*_K$, by virtue of (vp1), which fixes the value of the third coordinate, also becomes deterministic: for $z = T$ or $z = F$, $\tilde{\circ} z = F$; otherwise $\tilde{\circ} z = F$. That is, it is defined as:
(v) \( \delta(z_1, z_2, z_3) = (z_3, \sim z_3, 1) \).

The reasoning above yields the following deterministic six-valued matrix for \( LET_K^+ \), obtained by applying the corresponding restrictions to the six-valued \( N \) matrix for \( LET_K \):

**Definition 3.9.** Let \( M_6 \) be the six-valued logical matrix with domain \( B_{LET_K} \), the set of designated values \( D = \{ T, T_0, b \} \), and the operations given by the tables below.

| \( \wedge \) | \( T \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
|---|---|---|---|---|---|---|
| \( T \) | \( T \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
| \( T_0 \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
| \( b \) | \( b \) | \( b \) | \( F_0 \) | \( F_0 \) | \( F \) |
| \( n \) | \( n \) | \( n \) | \( F_0 \) | \( n \) | \( F \) |
| \( F_0 \) | \( F_0 \) | \( F_0 \) | \( F_0 \) | \( F_0 \) | \( F \) |
| \( F \) | \( F \) | \( F \) | \( F \) | \( F \) | \( F \) |

| \( \lor \) | \( T \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
|---|---|---|---|---|---|---|
| \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) | \( T \) |
| \( T_0 \) | \( T_0 \) | \( T \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) |
| \( b \) | \( T_0 \) | \( b \) | \( T_0 \) | \( b \) | \( b \) | \( b \) |
| \( n \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) |
| \( F_0 \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F_0 \) | \( F_0 \) |
| \( F \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F_0 \) | \( F_0 \) |

| \( \to \) | \( T \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
|---|---|---|---|---|---|---|
| \( T \) | \( T \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
| \( T_0 \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F \) |
| \( b \) | \( T_0 \) | \( b \) | \( n \) | \( F_0 \) | \( F_0 \) |
| \( n \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) |
| \( F_0 \) | \( T_0 \) | \( F_0 \) | \( F_0 \) | \( F_0 \) | \( F_0 \) | \( F_0 \) |
| \( F \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) | \( T_0 \) |

Remark 3.10.

(i) Let \( A_{LET_K} \) be the six-valued multialgebra underlying the \( N \) matrix \( M_{LET_K} \) for \( LET_K \) and \( A_6 \) the six-valued algebra underlying the matrix \( M_6 \). Since every algebra is a multialgebra in which each entry of each multioperator returns a singleton set, it is immediate to see that \( A_6 \) is a submultialgebra of \( A_{LET_K} \). Assume now that \( \#_6 \) and \( \#_{LET_K} \) denote, respectively, the interpretation of the connective \( \# \) in the multialgebras \( A_6 \) and \( A_{LET_K} \). Thus, \( \#_6 z \in \#_{LET_K} z \) and \( z \#_{6} w \in z \#_{LET_K} w \) for every \( z, w \in B_{LET_K} \), \( \# \in \{ \neg, \circ \} \) and \( \#' \in \{ \land, \lor, \to \} \). Therefore, any valuation over the matrix \( M_6 \) is a valuation over the \( N \) matrix \( M_{LET_K} \).

(ii) Recall from Definition 3.2 that \( A_T \overset{def}{=} \circ A \land A \) and \( A_F \overset{def}{=} \circ A \land \neg A \), for every formula \( A \). Accordingly, let \( z^T \overset{def}{=} \circ z \land z \) and \( z^F \overset{def}{=} \circ z \land \neg z \), for every \( z \in B_{LET_K} \), where the operations correspond to \( M_6 \). It is easy to see that, for every \( z \in B_{LET_K} \): \( z^T = T \) if \( z = T \), and \( z^T = F \) otherwise; and \( z^F = T \) if \( z = F \), and \( z^F = F \) otherwise. Clearly, for every valuation \( v \) over the logical matrix \( M_6 \), \( v(A^T) = v(A)^T \) and \( v(A^F) = v(A)^F \).
3.3 Soundness and completeness of the six-valued semantics of $LET_K^+$

The next task is to prove the soundness and completeness of $LET_K^+$ w.r.t. the semantics given by $M_6$. The proof is obtained by adapting to $LET_K^+$ the proof of Theorems 2.13 and 2.17. The definition of $F$-saturated sets in $LET_K^+$ is analogous to the one for $LET_K$ (recall Definition 2.5).

**Proposition 3.11.** For every valuation $v$ over the matrix $M_6$ the mapping $\rho_v : \text{For}(\Sigma) \to 2$ given by $\rho_v(A) = v(A)_1$ is a bivaluation for $LET_K^+$ such that: $\rho_v(A) = 1$ iff $v(A) \in D$, for every formula $A$.

**Proof.** As noted in Remark 3.10(i), any valuation over the matrix $M_6$ is a valuation over the Nmatrix $M_{LET_K}$. Thus, given a valuation $v$ over $M_6$, the function $\rho_v$ defined as above is a bivaluation for $LET_K^+$, by Proposition 2.14. It remains to prove that $\rho_v$ satisfies clauses (vp1)-(vp17) from Definition 3.4. Thus, let $A, B \in \text{For}(\Sigma)$. Then $\rho_v(\circ A) = v(\circ A)_3 = (\circ v(A))_3 = 1$, by definition of $M_6$. This shows that $\rho_v$ satisfies (vp1). Using again the definition of $M_6$, $\rho_v(\circ{-}A) = v(-A)_3 = (\neg v(A))_3 = v(A)_3 = \rho_v(\circ A)$. In order to prove that $\rho_v$ satisfies (vp3), suppose that $\rho_v(\circ A) = \rho_v(A) = 1$ and $\rho_v(B) = \rho_v(B) = 1$. Then $v(A)_3 = v(A)_1 = 1$ and $v(B)_3 = v(B)_1 = 1$, that is, $v(A) \land v(B) = T$. Hence, $v(A \land B) = v(A) \land v(B) = T \land T = T$. From this, $\rho_v(\circ(A \land B)) = v(A \land B)_3 = 1$. For (vp4), suppose that $\rho_v(\circ A) = \rho_v(\neg A) = 1$. Then $v(A)_3 = v(A)_2 = 1$, that is, $v(A) = F$. Hence, $v(A \land B) = v(A) \land v(B) = F \land v(B) = F$, for every $B$. From this, $\rho_v(\circ(A \land B)) = v(A \land B)_3 = 1$. Analogously it is proved that $\rho_v$ satisfies (vp5). For (vp7), suppose that $\rho_v(\circ(A \land B)) = 1$, and either $\rho_v(\neg A) = 1$ or $\rho_v(\neg B) = 1$. Then, $v(A \land B)_3 = 1$ and $v(A)_2 = 1$ or $v(B)_2 = 1$, hence $v(A \land B)_2 = 1$. Then, $v(A \land B) = F$ and $v(A \land B) = v(A) \land v(B) = F$. That is, either $v(A)_3 = v(A)_2 = 1$ or $v(B)_3 = v(B)_2 = 1$. Hence, $\rho_v(\circ A) = \rho_v(\neg A) = 1$ or $\rho_v(\circ B) = \rho_v(\neg B) = 1$ and so $\rho_v$ satisfies (vp7). For (vp8), suppose that $\rho_v(\circ A) = \rho_v(A) = 1$. Then, $v(A)_3 = v(A)_1 = 1$, that is, $v(A) = T$. Then, $v(A \lor B) = T$ and so $v(A \lor B)_3 = 1$, for any $B$. That is, $\rho_v(\circ(A \lor B)) = 1$, for any $B$, and so $\rho_v$ satisfies (vp8). Clause (vp9) is proved analogously. For (vp16), assume that $\rho_v(\circ(A \rightarrow B)) = 1$, and either $\rho_v(A) = 0$ or $\rho_v(B) = 1$. Then, $v(A \rightarrow B)_3 = 1$, and either $v(A)_1 = 0$ or $v(B)_1 = 1$, that is, $v(A \rightarrow B)_1 = 1$. This means that $v(A \rightarrow B) = T$, and so either $v(A) = F$ or $v(B) = T$. That is, either $v(A)_3 = v(A)_2 = 1$ or $v(B)_3 = v(B)_2 = 1$. This means that either $\rho_v(\circ A) = \rho_v(\neg A) = 1$ or $\rho_v(\circ B) = \rho_v(\neg B) = 1$. Hence, $\rho_v$ satisfies clause (vp16). The rest of the clauses are proved by similar arguments. This shows that $\rho_v$ is a bivaluation for $LET_K^+$ such that, by definition, $\rho_v(A) = 1$ iff $v(A) \in D$, for every formula $A$.

**Theorem 3.12 (Soundness of $LET_K^+$ w.r.t. the six-valued logical matrix $M_6$).**

For every set of formulas $\Gamma \cup \{A\} \subseteq \text{For}(\Sigma)$: $\Gamma \vdash_{LET_K^+} A$ implies that $\Gamma \vDash_{M_6} A$.

**Proof.** Assume that $\Gamma \vdash_{LET_K^+} A$. By Theorem 3.7, $\Gamma \vdash_{LET_K}^2 A$. Now, let $v$ be a valuation over the matrix $M_6$ such that $v(B) \in D$ for every $B \in \Gamma$, and let $\rho_v$ be the bivaluation for $LET_K$ defined from $v$ as in Proposition 3.11. Hence $\rho_v(B) = 1$ for every $B \in \Gamma$ and so $\rho_v(A) = 1$, since $\Gamma \vdash_{LET_K}^2 A$. From this it follows that $v(A) \in D$. Therefore, $\Gamma \vDash_{M_6} A$. ■
Remark 3.13. It is worth noting that clauses (vp6), (vp7), (vp11), (vp12), (vp16) and (vp17) are equivalent, by contraposition, to the following ones:

(vp6)' If \( \rho(A) = \rho(B) = 1 \) and either \( \rho(A) = 0 \) or \( \rho(B) = 0 \), then \( \rho(A \land B) = 0 \);

(vp7)' If either \( \rho(\neg A) = 1 \) or \( \rho(\neg B) = 1 \); either \( \rho(A) = 0 \) or \( \rho(A) = 0 \);

and either \( \rho(B) = 0 \) or \( \rho(B) = 0 \), then \( \rho(A \land B) = 0 \);

(vp11)' If either \( \rho(A) = 1 \) or \( \rho(B) = 1 \); either \( \rho(A) = 0 \) or \( \rho(A) = 0 \);

and either \( \rho(B) = 0 \) or \( \rho(B) = 0 \), then \( \rho(A \land B) = 0 \);

(vp12)' If \( \rho(A) = \rho(B) = 0 \) and either \( \rho(A) = 0 \) or \( \rho(B) = 0 \), then \( \rho(A \lor B) = 0 \);

(vp16)' If either \( \rho(A) = 0 \) or \( \rho(B) = 1 \); either \( \rho(A) = 0 \) or \( \rho(A) = 0 \);

and either \( \rho(B) = 0 \) or \( \rho(B) = 0 \), then \( \rho(A \rightarrow B) = 0 \);

(vp17)' If \( \rho(A) = \rho(\neg B) = 1 \) and \( \rho(B) = 0 \), then \( \rho(A \rightarrow B) = 0 \).

This reformulation of the above-mentioned clauses will be useful for the proof of completeness of LET\(_K^\rho\) w.r.t. \( \mathcal{M}_6 \). Moreover, in Proposition 3.16 a more compact characterization of bivaluations for LET\(_K^\rho\) will be given.

Proposition 3.14. For every bivaluation \( \rho \) for LET\(_K^\rho\) the mapping \( v_\rho : \text{For}(\Sigma) \to \mathcal{B}_{LETK} \) given by \( v_\rho(A) = (\rho(A), \rho(\neg A), \rho(\circ A)) \) is a valuation over the matrix \( \mathcal{M}_6 \) such that: \( v_\rho(A) \in D \) iff \( \rho(A) = 1 \), for every formula \( A \).

Proof. It is clear that \( v_\rho(A) \in \mathcal{B}_{LETK} \), hence the function is well-defined. Let us prove now that \( v_\rho \) is a valuation over \( \mathcal{M}_6 \). Thus, let \( A, B \in \text{For}(\Sigma) \) (in what follows, recall clauses (v1)-(v8) and (vp1)-(vp17) from Definitions 2.2 and 3.4, as well as clauses (vp6)', (vp7)', (vp11)', (vp12)', (vp16)' and (vp17)' from Remark 3.13.

Conjunction: Suppose that \( v_\rho(A) = v_\rho(B) = T \). Then, \( \rho(\circ A) = \rho(A) = 1 \) and \( \rho(B) = \rho(B) = 1 \) and so, by (vp3) and (v1), \( \rho(\circ(\circ A B)) = \rho((A \land B)) = 1 \). Hence, \( v_\rho(A \land B) = T = v_\rho(A) \land v_\rho(B) \). Now, suppose that \( v_\rho(A) = T_0 \) and \( v_\rho(B) \in \{T, T_0\} \). Then, \( \rho(A) = 1 \), \( \rho(\neg A) = \rho(\circ A) = 0 \), \( \rho(B) = 1 \) and \( \rho(\neg B) = 0 \). By (v1), (v5) and (vp6)', \( \rho(\circ(\circ A B)) = \rho((A \land B)) = 1 \). Hence, \( v_\rho(A \land B) = T_0 = v_\rho(A) \land v_\rho(B) \). Analogously we prove that, if \( v_\rho(B) = T_0 \) and \( v_\rho(A) \in \{T, T_0\} \) then \( v_\rho(A \land B) = T_0 = v_\rho(A) \land v_\rho(B) \). Assume now that \( v_\rho(A) = b \) and \( v_\rho(B) \in D \). Then, \( \rho(A) = \rho(\neg A) = 1 \), \( \rho(\circ A) = 0 \) and \( \rho(B) = 1 \). By (v1), (v5) and (v8), \( \rho(\circ(\circ A B)) = \rho((A \land B)) = 1 \) and \( \rho(\circ(\circ A B)) = 0 \). That is, \( v_\rho(A \land B) = b = v_\rho(A) \land v_\rho(B) \). Analogously we prove that, if \( v_\rho(B) = b \) and \( v_\rho(A) \in D \) then \( v_\rho(A \land B) = b = v_\rho(A) \land v_\rho(B) \). Now, assume that \( v_\rho(A) = n \) and \( v_\rho(B) \in \{T, T_0, n\} \). Then, \( \rho(A) = \rho(\neg A) = \rho(\circ A) = 0 \) and \( \rho(B) = 0 \). Hence, by (v1) and (v5) and (v8), \( \rho(\circ(\circ A B)) = \rho((A \land B)) = 0 \). That is, \( v_\rho(A \land B) = n = v_\rho(A) \land v_\rho(B) \). Analogously we prove that, if \( v_\rho(B) = n \) and \( v_\rho(A) \in \{T, T_0\} \) then \( v_\rho(A \land B) = n = v_\rho(A) \land v_\rho(B) \). Suppose now that \( v_\rho(A) = F_0 \) and \( v_\rho(B) = F \). Then \( \rho(A) = 0 \), \( \rho(\neg A) = 1 \), \( \rho(\circ A) = 0 \) and: either \( \rho(\neg B) = 0 \) or \( \rho(\circ B) = 0 \). Hence, by (v1), (v5) and (vp7)', \( \rho(\circ(\circ A B)) = \rho((A \land B)) = 1 \) and \( \rho(\circ(\circ A B)) = 0 \). That is, \( v_\rho(A \land B) = F_0 = v_\rho(A) \land v_\rho(B) \). Analogously we prove that, if \( v_\rho(B) = n \) and \( v_\rho(A) = b \) then \( v_\rho(A \land B) = F_0 = v_\rho(A) \land v_\rho(B) \). Finally, suppose that either \( v_\rho(A) = F \)
or \( v_p(B) = F \). Then, either \( \rho(\neg A) = \rho(\neg A) = 1 \) or \( \rho(\neg B) = \rho(\neg B) = 1 \) and so, by (vp4), (vp5) and (v5), \( \rho(A \land B) = \rho(\neg(A \land B)) = 1 \). Therefore \( v_p(A \land B) = F = v_p(A) \land v_p(B) \).

**Disjunction:** Suppose that either \( v_p(A) = T \) or \( v_p(B) = T \). Then, either \( \rho(\neg A) = \rho(\neg A) = 1 \) or \( \rho(\neg B) = \rho(\neg B) = 1 \) and so, by (vp8), (vp9) and (v2), \( \rho(A \lor B) = \rho(\neg(A \lor B)) = 1 \). Therefore \( v_p(A \lor B) = T = v_p(A) \lor v_p(B) \). Suppose now that \( v_p(A) = T_0 \) and \( v_p(B) \neq T \). Then \( \rho(A) = 1, \rho(\neg A) = 0, \rho(\neg A) = 0 \) and: either \( \rho(B) = 0 \) or \( \rho(\neg B) = 0 \). By (v2),(v6) and (vp11) it follows that \( \rho(A \land B) = 1, \rho(\neg(A \land B)) = 0 \) and \( \rho(A \lor B) = 0 \). That is, \( v_p(A \lor B) = T_0 = v_p(A) \lor v_p(B) \). Analogously we prove that, if \( v_p(B) = T_0 \) and \( v_p(A) \neq T \) then \( v_p(A \lor B) = T_0 = v_p(A) \lor v_p(B) \). Now, assume that \( v_p(A) = b \) and \( v_p(B) \in \{b, F_0, F\} \). Then, \( \rho(A) = \rho(\neg A) = 1, \rho(\neg A) = 0 \) and \( \rho(B) = 1 \). Hence, by (v2), (v6) and (v8), \( \rho(A \lor B) = \rho(\neg(A \lor B)) = 1 \) and \( \rho(A \lor B) = 0 \). That is, \( v_p(A \lor B) = b = v_p(A) \lor v_p(B) \). Analogously we prove that, if \( v_p(B) = b \) and \( v_p(A) \in \{F, F_0\} \) then \( v_p(A \lor B) = b = v_p(A) \lor v_p(B) \). Suppose now that \( v_p(A) = b \) and \( v_p(B) = n \). Then \( \rho(A) = \rho(\neg A) = 1, \rho(\neg A) = 0 \) and \( \rho(B) = \rho(\neg B) = 0 \). Then, by (v2), (v6) and (vp11) \( \rho(A \lor B) = 1, \rho(\neg(A \lor B)) = 0 \) and \( \rho(A \lor B) = 0 \). That is, \( v_p(A \lor B) = F_0 = v_p(A) \lor v_p(B) \). Analogously we prove that, if \( v_p(B) = n \) and \( v_p(A) = b \) then \( v_p(A \lor B) = T_0 = v_p(A) \lor v_p(B) \). Assume now that \( v_p(A) = n \) and \( v_p(B) \in \text{ND} \). Then, \( \rho(A) = \rho(\neg A) = \rho(\neg A) = 0 \) and \( \rho(B) = 0 \). By (v2), (v6) and (v8), \( \rho(A \lor B) = \rho(\neg(A \lor B)) = 0 \) and \( \rho(A \lor B) = 1 \). We prove that, if \( v_p(B) = n \) and \( v_p(A) \in \{F, F_0\} \) then \( v_p(A \lor B) = F_0 = v_p(A) \lor v_p(B) \). Finally, suppose that \( v_p(A) = v_p(B) = F \). Then, \( \rho(A) = 0, \rho(\neg A) = \rho(\neg A) = 1 \) and \( \rho(B) = 0 \). Hence, \( v_p(A \lor B) = F = v_p(A) \lor v_p(B) \).

**Implication:** Suppose that \( v_p(A) = F \). Then, \( \rho(A) = 0, \rho(\neg A) = 1 \) and \( \rho(A) = 1 \). Using (v3), (v7) and (vp13) it follows that, for any \( B \), \( \rho(A \land B) = 1, \rho(\neg(A \land B)) = 0 \) and \( \rho(A \land B) = 1 \). That is, \( v_p(A \land B) = T = v_p(A) \land v_p(B) \). Analogously (but now by using (vp14)) it is proven that, if \( v_p(B) = T \) then \( v_p(A \land B) = T = v_p(A) \land v_p(B) \) for any \( A \).

Suppose now that \( v_p(A) = F_0 \) and \( v_p(B) \neq T \). Then \( \rho(A) = 0, \rho(\neg A) = 1, \rho(A) = 0 \) and: either \( \rho(B) = 0 \) or \( \rho(\neg B) = 0 \). By (v3),(v7) and (vp16) it follows that \( \rho(\neg(A \land B)) = 0 \) and \( \rho(\neg(A \land B)) = 0, \rho(\neg(A \land B)) = 0 \) and \( \rho(\neg(A \land B)) = 1 \). That is, \( v_p(A \land B) = T_0 = v_p(A) \lor v_p(B) \). Analogously we prove that, if \( v_p(B) = T_0 \) then \( v_p(A \land B) = T_0 = v_p(A) \lor v_p(B) \). Assume now that \( v_p(B) = b \) and \( v_p(A) \in \text{ND} \). Then, \( \rho(B) = \rho(\neg B) = 1, \rho(\neg B) = 0 \) and \( \rho(A) = 1 \). By (v3), (v7) and (v8), \( \rho(A \lor B) = \rho(\neg(A \lor B)) = 1 \) and \( \rho(\neg(A \lor B)) = 1 \). Therefore, \( v_p(A \lor B) = F_0 = v_p(A) \lor v_p(B) \). Suppose now that \( v_p(B) = F \) and \( v_p(A) \in \text{ND} \). Then, \( \rho(B) = 0, \rho(\neg B) = 1, \rho(\neg B) = 1 \) and \( \rho(A) = 1 \). By (v3), (v7) and (vp15), \( \rho(A \lor B) = 0, \rho(\neg(A \lor B)) = 1 \) and \( \rho(\neg(A \lor B)) = 1 \). That is, \( v_p(A \lor B) = F = v_p(A) \lor v_p(B) \).
Finally, suppose that \( v_p(A) = n \) and \( v_p(B) \in \{ b, n, F_0, F \} \). Then, \( \rho(A) = \rho(\neg A) = \rho(\circ A) = 0 \) and: either \( \rho(\circ B) = 0 \) or \( \rho(B) = 0 \). By (v3), (v7) and (vp16), \( \rho(A \rightarrow B) = 0 \), \( \rho(\neg(A \rightarrow B)) = 1 \) and \( \rho(\circ(A \rightarrow B)) = 0 \). That is, \( v_p(A \rightarrow B) = T_0 = T_0 \rightarrow v_p(B) \).

**Negation:** Let \( v_p(A) = (\rho(A), \rho(\neg A), \rho(\circ A)) \). Then, by (v4) and (vp2):

\[
\begin{align*}
v_p(\neg A) &= (\rho(\neg A), \rho(\neg A), \rho(\neg A)) \\
&= (\rho(\neg A), \rho(A), \rho(\circ A)) = \neg v_p(A).
\end{align*}
\]

**Classicality:** Let \( v_p(A) = (\rho(A), \rho(\neg A), \rho(\circ A)) \). Then, by (vp1):

\[
\begin{align*}
v_p(\circ A) &= (\rho(\circ A), \rho(\neg A), \rho(\circ A)) \\
&= (\rho(\circ A), \rho(\neg A), 1) = \delta v_p(A).
\end{align*}
\]

This shows that \( v_p \) is a valuation over the matrix \( \mathcal{M}_6 \) such that, for every formula \( A \), \( v_p(A) \in D \) iff \( \rho(A) = 1 \).

---

**Theorem 3.15** (Completeness of \( \text{LET}_K^* \) w.r.t. the six-valued logical matrix \( \mathcal{M}_6 \)).

For every set of formulas \( \Gamma \cup \{ A \} \subseteq \text{For}(\Sigma) \): \( \Gamma \models_{\mathcal{M}_6} A \) implies that \( \Gamma \models_{\text{LET}_K^*} A \).

**Proof.** Assume that \( \Gamma \models_{\mathcal{M}_6} A \), and let \( \rho \) be a bivaluation for \( \text{LET}_K^* \) such that \( \rho(B) = 1 \) for every \( B \in \Gamma \). Let \( v_p \) be defined as in Proposition 3.14. Then, \( v_p \) is a valuation over \( \mathcal{M}_6 \) such that \( v_p(B) \in D \), for every \( B \in \Gamma \). By hypothesis, \( v_p(A) \in D \), whence \( \rho(A) = 1 \). This shows that \( \Gamma \models_{\text{LET}_K^*} A \). By Theorem 3.7, \( \Gamma \models_{\text{LET}_K^*} A \).

As announced before, the definition of bivaluations for \( \text{LET}_K^* \) can be drastically simplified in terms of Boolean operators:

**Proposition 3.16.** Let \( \rho \) be a bivaluation for \( \text{LET}_K \). Then, \( \rho \) is a bivaluation for \( \text{LET}_K^* \) iff it satisfies, in addition, (vp1) and (vp2) (from Definition 3.4) plus the following properties, expressed in the language of Boolean algebras:

\[
\begin{align*}
(v9) & \quad \rho(\circ(A \wedge B)) = a \cup b \cup c, \text{ where } \\
a &= \rho(A) \cap \rho(\circ A) \cap \rho(\circ B), & b &= \rho(\neg A) \cap \rho(\circ A), & c &= \rho(\neg B) \cap \rho(\circ B); \\
(v10) & \quad \rho(\circ(A \vee B)) = a' \cup b' \cup c', \text{ where } \\
a' &= \rho(\neg A) \cap \rho(\circ A) \cap \rho(\neg B), & b' &= \rho(\neg A) \cap \rho(\circ A), & c' &= \rho(B) \cap \rho(\circ B); \\
(v11) & \quad \rho(\circ(A \rightarrow B)) = a'' \cup b'' \cup c'', \text{ where } \\
a'' &= \rho(A) \cap \rho(\neg B) \cap \rho(\circ B), & b'' &= \rho(\neg A) \cap \rho(\circ A), & c'' &= \rho(B) \cap \rho(\circ B).
\end{align*}
\]

**Proof.** It follows by a tedious but straightforward verification.

---

### 3.4 \( \text{LET}_K^* \) is Blok-Pigozzi algebraizable

In this section we show that \( \text{LET}_K^* \) has enough expressive power to be algebraizable in the general sense proposed by Blok and Pigozzi in [9] (see also [32]).

First, note that in \( \text{LET}_K^* \) a bi-implication \( A \leftrightarrow B \) defined as \( (A \rightarrow B) \wedge (B \rightarrow A) \) does not preserve logical equivalence through the connectives: for instance, \( T_0 \leftrightarrow b \) gets the designated value \( b \), but \( \neg T_0 \leftrightarrow \neg b = F_0 \leftrightarrow b = F_0 \), which is non-designated.
In $N_4$ and $N_4^i$ an ‘equivalence’ operator which preserves logical equivalence through the connectives (that is, defines a logical congruence) is defined as follows: $A \equiv B \overset{\text{def}}{=} (A \leftrightarrow B) \land (\neg A \leftrightarrow \neg B)$. In terms of its (two-dimensional) twist structures, this means that two pairs are equal when the respective coordinates coincide. But in $LET_K$, we are dealing with three-dimensional twist structures, that is, the snapshots are triples instead of pairs. Since we have twist operators to ‘read’ each coordinate of the snapshots ($\tilde{\circ}$ ‘reads’ the second coordinate, while $\circ$ ‘reads’ the third one) an appropriate notion of ‘equivalence’ (representing identity between triples) in $LET_K$ should be the following:

$$A \equiv B \overset{\text{def}}{=} (A \leftrightarrow B) \land (\neg A \leftrightarrow \neg B) \land (\circ A \leftrightarrow \circ B).$$

The table of the interpretation $\bar{\varepsilon}$ of this connective in $M_6$ is as follows:

|   |   | T | T | T | T |
|---|---|---|---|---|---|
| T | T | F | F | F | F |
| T₀ | F | T₀ | F₀ | n | F₀ | F |
| b | F | F₀ | b | n | F₀ | F |
| n | F | n | n | T₀ | n | F |
| F₀ | F | F₀ | F₀ | n | T₀ | F |
| F | F | F | F | F | F | F |

From the table above, it is immediate to prove the following relevant properties of $\equiv$:

**Proposition 3.17.** The following properties hold in $M_6$:

1. For every $z, w \in B_{LET_K}$, $(z \equiv w) \in D$ iff $z = w$.
2. For every formulas $A$ and $B$, and for every valuation $v$ over $M_6$: $v(A \equiv B) \in D$ iff $v(A) = v(B)$.
3. $\models_{M_6} (A \equiv A)$ for every formula $A$.
4. $\models_{M_6} (A \equiv B) \models_{M_6} (B \equiv A)$ for every formulas $A$ and $B$.
5. $\models_{M_6} (A \equiv B), (B \equiv C) \models_{M_6} (A \equiv C)$ for every formulas $A$, $B$ and $C$.
6. $(A \equiv B), (B \equiv C) \models_{M_6} (A \equiv # B)$ for every formulas $A$ and $B$ and $# \in \{\land, \lor, \rightarrow\}$.
7. $(A \equiv B), (C \equiv D) \models_{M_6} (A \equiv C \equiv # B \equiv D)$ for every formulas $A$, $B$, $C$ and $D$ and $# \in \{\land, \lor, \rightarrow\}$.
8. $(A \equiv (A \rightarrow A)) \models_{M_6} A$, and $A \models_{M_6} (A \equiv (A \rightarrow A))$ for every formula $A$.

**Proof.** Item (1) is immediate from the truth-table for $\equiv$ displayed above. An analytical proof can be done as follows: For every $z, w \in B_{LET_K}$ let $z \leftrightarrow w \overset{\text{def}}{=} (z \rightarrow w) \land (w \rightarrow z)$. Then, $(z \leftrightarrow w) = (z_1 \rightarrow w_1) \land (w_1 \rightarrow z_1)$ and so $z \leftrightarrow w \in D$ iff $z_1 = w_1$. From this, $\neg \neg z \leftrightarrow \neg \neg w \in D$ iff $z_2 = w_2$, and $\circ \circ z \leftrightarrow \circ \circ w \in D$ iff $z_3 = w_3$. This means that $(z \equiv w) \in D$ iff $z_i = w_i$ for $i = 1, 2, 3$, iff $z = w$. Item (2) follows from (1) and from the fact that $v(A \equiv B) = (v(A) \equiv v(B))$ for every formulas $A$ and $B$, and every valuation $v$ over $M_6$. Items (3)-(7) are immediate from (2), taking into account (for items (6) and (7)) that the operations in $M_6$ are functional, that is, deterministic. For (8), let $v$ be a valuation
over $\mathcal{M}_6$ such that $v(A \equiv (A \rightarrow A)) \in D$. By (2), $v(A) = v(A \rightarrow A) = v(A) \rightarrow v(A)$. But $z \rightarrow z \in D$ for every $z \in B_{LET_K}$. Hence, $v(A) \in D$, showing that $(A \equiv (A \rightarrow A)) \in \mathcal{M}_6$. Conversely, let $v$ be a valuation over $\mathcal{M}_6$ such that $v(A) \in D$. Hence, $v(A \rightarrow A) = v(A) \rightarrow v(A) = v(A)$, by definition of $\rightarrow$ (since $z \rightarrow z = z$ for every $z \in D$). By (2), $v(A \equiv (A \rightarrow A)) \in D$. This shows that $A \equiv_{\mathcal{M}_6} (A \equiv (A \rightarrow A))$.

**Theorem 3.18.** The logic $LET^+_K$ is algebraizable in the sense of Blok-Pigozzi.

**Proof.** Let $p_1$ and $p_2$ be two different propositional variables, and consider the sets

$$\Delta(p_1, p_2) = \{(p_1 \equiv p_2)\} \quad \text{and} \quad E(p_1) = \{(p_1, (p_1 \rightarrow p_1))\}.$$  

By items (3)-(8) of Proposition 3.17, the sets $\Delta(p_1, p_2)$ and $E(p_1)$ show that the logic $LET^+_K$, presented by means of $\mathcal{M}_6$, is algebraizable in the sense of Blok-Pigozzi. Indeed, conditions (3)-(8) of Proposition 3.17 are exactly the requirements for $\Delta(p_1, p_2)$ and $E(p_1)$ stated in Theorem 4.7 of [9] for a given logic being algebraizable. [11] 

This result, combined with the family of twist models for $LET^+_K$ and the relationship of $LET^+_K$ with involutive Stone algebras, to be studied in the following two sections, opens interesting possibilities for future research of $LET^+_K$ from the perspective of abstract algebraic logic.

## 4 Twist models for $LET^+_K$

In Section 3.2, we have seen how the restrictions imposed on the Nmatrix of $LET_K$ by the rules of propagation of classicality yield a (deterministic) six-valued semantics for $LET^+_K$, the matrix $\mathcal{M}_6$. $\mathcal{A}_{LET_K}$ is the six-valued multialgebra underlying the Nmatrix $\mathcal{M}_{LET_K}$, which was presented as a swap structure in Definition 2.12. In order to comply with the axiom and rules of propagation of classicality of $LET^+_K$, $\mathcal{A}_{LET_K}$ becomes the algebra $\mathcal{A}_6$ for $LET^+_K$. The latter is the underlying algebra of the logical matrix $\mathcal{M}_6$.

Twist structures are special cases of swap structures: while the latter can be multi-algebras, the former are algebras, based on operations instead of multioperations, and so the respective semantics are deterministic. [10] In what follows, the algebra $\mathcal{A}_6$ will be presented as a *three-dimensional twist structure*. We will show how $\mathcal{A}_6$ can be generalized to twist algebras generated by arbitrary Boolean algebras. This will produce a class of twist-valued models for $LET^+_K$, one for each Boolean algebra, which characterizes $LET^+_K$. Moreover, it will be proved that these models are, indeed, bounded lattices in which suprema and infima are respectively given by the operators $\land$ and $\lor$ and the top and bottom elements are given by $T$ and $F$.

---

[10] Observe that conditions (6) and (7) of Proposition 3.17 depend on the signature of the given logic. In the terminology of [2], it can be said that $\Delta(p_1, p_2)$ is a system of equivalence formulas, while $E(p_1)$ (written as $p_1 \equiv (p_1 \rightarrow p_1)$) is a system of defining equations for the deductive system generated by $\mathcal{M}_6$, that is, $LET^+_K$.

[11] For a more detailed discussion of swap and twist structures, see Coniglio et al. [23], in particular sections 9.3 and 9.4.
Definition 4.1 (Twist structures for \( \text{LET}_K^+ \)).

Let \( B = (B, \land, \lor, \neg, 0, 1) \) be a Boolean algebra. The twist structure for \( \text{LET}_K^+ \) induced by \( B \) is the algebra \( \mathcal{T}_B = (B^{\mathcal{T}_B}, \land, \lor, \neg, 0, 1) \) over \( \Sigma \) such that

\[
B^{\mathcal{T}_B}_\text{LET}_K^+ = \{ z \in B^3 : z_3 \leq z_1 \cup z_2 \text{ and } z_1 \cap z_2 \cap z_3 = 0 \}
\]

and the operations are defined as follows:

(i) \( (z_1, z_2, z_3) \land \ (w_1, w_2, w_3) = (z_1 \land w_1, z_2 \land w_2, (z_1 \cap z_2 \cap w_1 \cap w_2) \lor (z_1 \land w_1 \cap z_2 \land w_2)) \),

(ii) \( (z_1, z_2, z_3) \lor \ (w_1, w_2, w_3) = (z_1 \lor w_1, z_2 \lor w_2, (z_1 \lor z_2 \lor w_1 \lor w_2) \lor (z_1 \lor z_2 \lor w_1 \lor w_2)) \),

(iii) \( \neg \ (z_1, z_2, z_3) = (\neg z_1, \neg z_2, \neg z_3) \),

(iv) \( \delta \ (z_1, z_2, z_3) = (z_3, \neg z_1, \neg z_2) \).

Note that \( \mathcal{T}_B \) is exactly \( A_6 \). Each twist structure \( \mathcal{T}_B \) for \( \text{LET}_K^+ \) naturally induces a logical matrix \( M(B) = (\mathcal{T}_B, D_B) \) where \( D_B = \{ z \in B^{\mathcal{T}_B}_\text{LET}_K^+ : z_1 = 1 \} \). Let \( \text{Mat}(\text{LET}_K^+) \) be the class of logical matrices of the form \( M(B) \), and let \( \text{M}(\text{LET}_K^+) \) be the associated consequence relation. Notice that \( \mathcal{M}(\mathcal{T}_B) = \mathcal{M}_6 \).

In Definition 2.12, the multioperations of the multialgebra \( A_\text{LET}_K^+ \) were presented in the framework of a non-deterministic swap structure in which the third coordinate of each snapshot of \( \text{LET}_K^+ \) is not functionally determined from the input(s). On the other hand, the tables of \( \text{LET}_K^+ \) presented in Definition 3.9 make it clear that in \( A_6 \), the third coordinate of snapshots, obtained by applying the respective operation, is functionally determined from the input(s). Note, in addition, that a snapshot is ‘classical’ – that is, it belongs to \( \{ T, F \} \) – exactly when the third coordinate is 1. Thus, in the case of \( \land \), the output \( z \land w \) is ‘classical’ exactly when: (i) \( z = w = T \), or (ii) \( z = F, \text{ or } w = F \). This yields the item (i) above. By analogous reasoning for \( \lor \) and \( \neg \), we obtain items (ii) and (iii) above. Given the equivalence between \( \land A \) and \( \land \neg A \), the negation \( \neg \) in \( \text{LET}_K^+ \) is deterministic, and defined by item (iv). Finally, the classicality operator \( \delta \) of \( \text{LET}_K^+ \), given clause (vp1) of Definition 3.3, also becomes deterministic, and so defined by item (v). Indeed, by (vp1), \( \rho(\land A) = 1 \). This, together with Proposition 3.3, implies that \( \rho(\land A) = \rho(\land A) \). Hence, by (vp1), \( \rho(\land A) = \rho(\land A) \).

Remark 4.2 (Decidability in \( \text{LET}_K^+ \) is reduced to decidability in \( \text{CPL} \)).

Let \( A(p_1, \ldots, p_k) \) be a formula over the signature of \( \text{LET}_K^+ \) depending at most on the propositional variables \( p_1, \ldots, p_k \), and let \( v \) be a valuation over a matrix \( M(B) \) for \( \text{LET}_K^+ \). Since \( v(p_i) \) is in \( B^3 \), its value can be represented by 3 new propositional variables \( p_i^j \), each one representing \( v(p_i) \), the \( j \)-th projection of \( v(p_i) \) for \( 1 \leq j \leq 3 \). Let \( V_B^k = \{ p_i^j : 1 \leq i \leq k \text{ and } 1 \leq j \leq 3 \} \) be the set of such new propositional variables. Hence, the formal expression for \( v(A) \) (that is, the first coordinate of \( v(A) \)) can be represented by a term \( \tau_A \) in \( \text{For}_3^k(\Sigma_{BA}) \), the language generated by the set of variables \( V_B^k \) over the signature \( \Sigma_{BA} = \{ \land, \lor, \neg, 1, 0 \} \) of Boolean algebras. For instance, given \( A = (p_1 \land \neg p_2) \lor \rho p_1 \) and \( B = \neg ((p_1 \land \neg p_2) \lor \rho p_1) \), then \( v(A) \) and \( v(B) \) are represented, respectively, by the terms \( \tau_A = (p_1^1 \land p_2^2 \lor p_2^3) \) and \( \tau_B = (p_1^1 \lor p_2^2) \lor \rho p_1 \) in \( \text{For}_3^k(\Sigma_{BA}) \), given that \( v((p_1 \land \neg p_2) \lor \rho p_1) = (v(p_1) \lor v(p_2) \lor \rho p_1) = (v(p_1) \land v(p_2) \lor \rho p_1) \) and \( v((p_1 \land \neg p_2) \lor \rho p_1) = (v(p_1) \lor v(p_2) \lor \rho p_1) = (v(p_1) \land v(p_2) \lor \rho p_1) \). The fact that every \( v(p_i) \) is a snapshot instead of an
any arbitrary triple in $B^3$ is represented by the term $\bar{\tau}_k = \tau_1 \cap \ldots \cap \tau_k$ such that $\tau_i = (p_i \Rightarrow (p_i \cup p_i^2)) \cap (p_i \cap p_i^2 \Rightarrow p_i^2)$ for $1 \leq i \leq k$.

Now, suppose that $\Gamma = A$ such that $\Gamma = \{A_1, \ldots, A_n\}$ is non-empty and all these formulas depend on $p_1, \ldots, p_k$. Let $B = A_1 \land \ldots \land A_n$, and let $v$ be a valuation over $\mathcal{M}_B$ (that is, $v : V \to B \subseteq \mathcal{B}_\mathcal{A}$). Then, $v(B) \in \mathcal{B}_\mathcal{A}$ implies that $v(A) \in D$ or, equivalently, $v(B) = 1$ implies that $v(A) = 1$. This means that, for every homomorphism $h : \text{For}_G^k(\Sigma_{BA}) \to B_2$ such that $h(p_i) = v(p_i)$ for $1 \leq i \leq k$ and $1 \leq j \leq 3$ for a valuation $v$, $h(\tau_B) = 1$ implies that $h(\tau_A) = 1$ or, equivalently, $h(\tau_B \Rightarrow \tau_A) = 1$. If $h$ is defined as above from a function $v : V \to 2^B$ then $v(p_i) \in B \subseteq \mathcal{B}_\mathcal{A}$ for $1 \leq i \leq k$ iff $h(\tau_i) = 1$. Then, for every homomorphism $h : \text{For}_G^k(\Sigma_{BA}) \to B_2$, $h(\bar{\tau}_k) = 1$ (i.e., $(h(p_i), h(p_i^2), h(p_i^3)) \in B \subseteq \mathcal{B}_\mathcal{A}$) for $1 \leq i \leq k$ implies that $h(\tau_B \Rightarrow \tau_A) = 1$. Equivalently, $h(\tau_k) \Rightarrow (\tau_B \Rightarrow \tau_A)$ for every $k$. In other words, $\Gamma = A$ if and only if $B_2$ validates the equation $(\bar{\tau}_k \Rightarrow (\tau_B \Rightarrow \tau_A)) = \top$ in the language of Boolean algebras. The later is equivalent to saying that the formula $\bar{\tau}_k \Rightarrow (\tau_B \Rightarrow \tau_A)$ is a tautology in CPL (expressed in the signature $\Sigma_{BA}$).

**Theorem 4.3** (Soundness and completeness of $LET^*_K$ w.r.t. $\mathcal{M}(LET^*_K)$). For every set of formulas $\Gamma \subseteq \Sigma; \Gamma \vdash_{LET^*_K} A$ iff $\Gamma \vdash_{\mathcal{M}(LET^*_K)} A$.

**Proof.** (Left to right - Soundness): Suppose that $\Gamma \vdash_{LET^*_K} A$, and assume that $\Gamma = \{A_1, \ldots, A_n\}$ is non-empty (the proof for the case $\Gamma = \emptyset$ is analogous but easier). Assume that every formula in $\Gamma \cup \{A\}$ depends at most on the propositional variables $p_1, \ldots, p_k$. By Theorem 4.12, $\Gamma \models_{\mathcal{M}_B} A$. By Remark 4.2 (and using the notation established therein) it follows that $B_2$ validates the equation $(\bar{\tau}_k \Rightarrow (\tau_B \Rightarrow \tau_A)) = \top$ in the language of Boolean algebras, where $B = A_1 \land \ldots \land A_n$. By Remark 2.3, any Boolean algebra $\mathcal{B}$ validates the equation $(\bar{\tau}_k \Rightarrow (\tau_B \Rightarrow \tau_A)) = \top$. That is, for every homomorphism $h : \text{For}_G^k(\Sigma_{BA}) \to \mathcal{B}$, $h(\bar{\tau}_k \Rightarrow (\tau_B \Rightarrow \tau_A)) = 1$ or, equivalently, $h(\tau_k) \leq h(\tau_B \Rightarrow \tau_A)$. Now, let $\mathcal{B}$ be a Boolean algebra and let $v$ be a valuation over the matrix $\mathcal{M}(\mathcal{B})$ such that $v(B) \in \mathcal{B}$. Let $h : \text{For}_G^k(\Sigma_{BA}) \to \mathcal{B}$ be a homomorphism such that $h(\tau_i) = v(\tau_i)$ for $1 \leq i \leq k$ and $1 \leq j \leq 3$. Then $h(\tau_i) = 1$, since $v(\tau_i) \in \mathcal{B}$ for $1 \leq i \leq k$. From this, $h(\tau_B \Rightarrow \tau_A) = 1$, that is, $h(\tau_B \Rightarrow \tau_A) = 1$. But $h(\tau_B) = 1$, given that $v(B) \in \mathcal{B}$ (which means that $v(B) = 1$). From this we conclude that $h(\tau_A) = 1$. This means that $v(A) = 1$, i.e., $v(A) \in \mathcal{B}$. This shows that $\Gamma \models_{\mathcal{M}(\mathcal{B})} A$ for every $\mathcal{B}$, hence $\Gamma \models_{\mathcal{M}(LET^*_K)} A$.

(Right to left - Completeness): Suppose that $\Gamma \models_{\mathcal{M}(LET^*_K)} A$. Then, in particular, $\Gamma \models_{\mathcal{M}(B_2)} A$. But $\mathcal{M}(B_2)$ is $\mathcal{M}_B$, hence $\Gamma \models_{\mathcal{M}_B} A$. By Theorem 4.13, $\Gamma \vdash_{LET^*_K} A$. 

**Remark 4.4.** Recall that, in addition to the usual order-theoretic definition, a lattice can be equivalently defined as an algebra $(L, \cap, \cup)$ such that:

1. $a \cap a = a = a \cup a$;
2. $a \cap b = b \cap a$ and $a \cup b = b \cup a$;
3. $a \cap (b \cap c) = (a \cap b) \cap c$ and $a \cup (b \cup c) = (a \cup b) \cup c$; and
4. $a \cap (a \cup b) = a = a \cup (a \cap b)$, for every $a, b, c \in L$

and so the partial order is defined as: $a \leq b$ iff $a = a \cap b$ (iff $b = a \cup b$).
Theorem 4.5. For every Boolean algebra $B$ the twist structure $T_B$ is a bounded lattice in which the infimum and supremum are given by $\land$ and $\lor$, respectively, and $T = (1, 0, 1)$ and $F = (0, 1, 1)$ are the top and bottom elements (where 1 and 0 are the top and bottom elements of $B$).

Proof. It will be shown that, for every Boolean algebra $B$, the algebra $T_B$ is such that $\land$ and $\lor$ satisfy conditions (1)-(4) of Remark 4.4.

(1) Let $\# \in \{\land, \lor\}$. Given that $B$ satisfies condition (1), it is clear that $(z \# z)_i = z_i$ for $i = 1, 2$. On the other hand, $(z \# z)_3 = (z_1 \land z_3) \lor (z_2 \land z_3) = z_3 \land (z_1 \lor z_2) = z_3$. Hence, $z \# z = z$ for every $z$ and $\# \in \{\land, \lor\}$.

(2) Clearly $z \# w = w \# z$ for every $z, w$ and $\# \in \{\land, \lor\}$, by the very definitions and by the fact that $B$ satisfies condition (2).

(3) Let us first prove that $z \land (w \land u) = (z \land w) \land u$. Observe that $(z \land (w \land u))_i = ((z \land w)_i) \land u_i$, for $i = 1, 2$, by definition of $\land$ and the fact that $B$ satisfies condition (3).

Now, let $a = (w \land u)_3 = (w_1 \land w_3 \land u_1 \land u_3) \lor (w_2 \land w_3) \lor (u_2 \land u_3)$ and $b = (z \land w)_3 = (z_1 \land z_3 \land w_1 \land w_3) \lor (z_2 \land z_3) \lor (w_2 \land w_3)$. Hence,

$$c = (z \land (w \land u))_3 = (z_1 \land z_3 \land (w_1 \land u_1) \land a) \lor (z_2 \land z_3) \lor ((w_2 \land u_2) \land a),$$

$$d = (z \land w \land u)_3 = ((z_1 \land w_1) \land b \land u_1 \land u_3) \lor ((z_2 \land w_2) \land b) \lor (u_2 \land u_3).$$

By using an automatic prover for tautologies in CPL it is immediate to check that the formula $(A_c \Rightarrow A_d) \cap (A_d \Rightarrow A_c)$ is a tautology in CPL (expressed in the signature $\Sigma_{BA}$), where $A_c$ and $A_d$ are the propositional formulas in $For^3_2(\Sigma_{BA})$, respectively obtained from the terms $c$ and $d$ by replacing $z_1, w_2, j$ by the propositional variables $p_3^1, p_3^2, p_3^3$, for $1 \leq j \leq 3$ (recalling Remark 4.2 and the notation established therein). This means that the equation $A_c = A_d$ holds in $B_2$ and so it holds in every Boolean algebra $B$, as observed in Remark 2.3(2). That is, $(z \land (w \land u))_3 = ((z \land w) \land u)_3$ for every $B$ and every $z, w, u$ in $B^{L_{ETK}}_B$. Therefore, $z \land (w \land u) = (z \land w) \land u$ for every $B$ and every $z, w, u \in B^{L_{ETK}}_B$. The proof of associativity of $\land$ is obtained by similar arguments.

(4) Observe that, by the definition of $\land$ and $\lor$, $z_i = (z \lor (z \land w))_i$ for every $z, w$ and $i = 1, 2$, given that $B$ has property (4). Now, let $a = (z \lor (z \land w))_3 = (z_1 \lor z_3 \land w_1 \land w_3) \lor (z_2 \land z_3) \lor (w_2 \land w_3)$ and $b = (z \lor (z \land w))_3 = (z_2 \lor z_3 \land w_1 \land w_3) \lor (z_1 \land z_3) \lor (w_1 \land w_3)$. As we have done in item (3), let $A_b$ be the propositional formula in $For^3_2(\Sigma_{BA})$ obtained by replacing $z_1$ and $w_3$ in the expression $b$ by the propositional variables $p_3^1$ and $p_3^2$, for $1 \leq j \leq 3$. By using an automatic prover for tautologies in CPL it can be checked that, in this case, the formula $A = (p_3^1 \Rightarrow A_b) \cap (A_b \Rightarrow p_3^3)$ is not a tautology in CPL (expressed in the signature $\Sigma_{BA}$). However, the only rows in which $A$ gets the value 0 is when the triple $p_1 = (p_3^1, p_3^2, p_3^3)$ gets the value $(0, 0, 1)$ or when the triple $p_2 = (p_3^1, p_3^2, p_3^3)$ gets the value $(1, 1, 1)$, and these triples do not correspond to snapshots in $B^{L_{ETK}}_B$.

Consider then the formula $B = (p_3^2 \Rightarrow A)$ (where $p_3^2$ is defined as in Remark 4.2 by taking $k = 2$). By the previous considerations, it follows that $B$ is a tautology. That is, the equation $(p_3^2 \Rightarrow A) \equiv \tau$ in the language of Boolean algebras holds in $B_2$. By Remark 2.3(2), that equation holds in any Boolean algebra $B$. That is, for every homomorphism $h : For^3_2(\Sigma_{BA}) \rightarrow B, h(p_3^2 \Rightarrow A) = 1$ or, equivalently, $h(p_3^2) \leq h(A)$. Given a Boolean algebra $B$, let $z = (z_1, z_2, z_3)$ and $w = (w_1, w_2, w_3)$ in $B^{L_{ETK}}_B$, and
Let \( h : \text{For}_{2}(\Sigma_{BA}) \rightarrow \mathcal{B} \) be a homomorphism such that \( h(p_{1}^{j}) = z_{j} \) and \( h(p_{2}^{j}) = w_{j} \) for \( 1 \leq j \leq 3 \). Then, \( h(p_{3}) = 1 \) and so \( h(A) = 1 \). This means that \( z_{3} = b = (z \triangledown (z \wedge w))_{3} \), hence \( z = z \wedge (z \wedge w) \) for every \( z, w \in B_{\text{LET}_{K}}^{S} \). In a similar way it can be proved that \( z = z \wedge (z \wedge w) \) for every \( z, w \in B_{\text{LET}_{K}}^{S} \).

This shows that the twist structure \( \mathcal{T}_{\mathcal{B}} \) is a lattice, for every Boolean algebra \( \mathcal{B} \). Clearly, \( z \wedge T = z = z \triangledown F \), for every \( z \in B_{\text{LET}_{K}}^{S} \). Therefore \( \mathcal{T}_{\mathcal{B}} \) is a bounded lattice with top and bottom elements given by \( T \) and \( F \), respectively.

**Proposition 4.6.** The order in the lattice \( \mathcal{T}_{\mathcal{B}} \) is given as follows:

\[
(z_{1}, z_{2}, z_{3}) \leq (w_{1}, w_{2}, w_{3}) \text{ if and only if:} \\

z_{1} \leq w_{1}, \quad z_{2} \geq w_{2}, \quad z_{2} \cap z_{3} \geq w_{2} \cap w_{3}, \quad \text{and} \quad z_{3} \leq (z_{1} \cap w_{3}) \cup z_{2}.
\]

**Proof.** By definition of the order \( \leq \) in \( B_{\text{LET}_{K}}^{S} \) induced by the algebraic lattice structure of \( \mathcal{T}_{\mathcal{B}} \), and according to Theorem 4.3, it follows that \( (z_{1}, z_{2}, z_{3}) \leq (w_{1}, w_{2}, w_{3}) \) iff \( z_{1} \leq w_{1}, z_{2} \geq w_{2}, \text{ and } (z_{2} \cap z_{3}) \geq (w_{2} \cap w_{3}) \).

By taking infimum w.r.t. \( z_{2} \) in both sides of (*) we get that \( z_{2} \cap z_{3} = (z_{2} \cap z_{1} \cap z_{3} \cap w_{3}) \cup (z_{2} \cap z_{3}) \cup (w_{2} \cap w_{3}). \)

Given that \( z_{2} \cap z_{1} \cap z_{3} = 0 \) and \( z_{2} \cap w_{2} = w_{2} \) (since \( z_{2} \geq w_{2} \)), this implies that \( z_{2} \cap z_{3} = (z_{2} \cap z_{3}) \cup (w_{2} \cap w_{3}). \) That is, \( z_{2} \cap z_{3} \geq w_{2} \cap w_{3}. \) Moreover, by considering the latter relation in equation (**) we get that \( z_{3} = (z_{1} \cap z_{3} \cap w_{3}) \cup (z_{2} \cap z_{3}) = z_{3} \cap ((z_{1} \cap w_{3}) \cup z_{2}). \) This means that \( z_{3} \leq (z_{1} \cap w_{3}) \cup z_{2} \). The proof of the converse is analogous (but a little easier).

**4.1 On the lattice structure of \( \text{LET}^{F \rightarrow}_{K} \)***

In Belnap [7] we find two lattice-orderings defined by the four semantic values of \( FDE \) (and by the four values of \( FDE^{\sim} \) as well), called \( \textbf{A4} \) and \( \textbf{L4} \). The lattice \( \textbf{A4} \) has \( n \) at the bottom and \( b \) at the top:

\[
\begin{array}{c}
T \\
\downarrow \\
F \\
\downarrow \\
\uparrow \\
\uparrow
\end{array}
\]

The partial order \( a \leq b \) is read as ‘\( a \) approximates the information in \( b \)’. The underlying idea is that the amount of information grows from bottom to top, in the sense that \( T \) and \( F \) convey more information than \( n \), and \( b \) conveys more information than both \( T \) and \( F \). In Belnap’s words,

None is at the bottom because it gives no information at all; and Both is at the top because it gives too much (inconsistent) information [8, p. 39].

This order is clear if we think of the values \( n \), \( F \), \( T \), and \( b \) as subsets of \( \{0, 1\} \), respectively, \( \emptyset \), \( \{0\} \), \( \{1\} \), and \( \{0, 1\} \), and \( \leq \) as the relation of inclusion \( \subseteq \). Note that: (i)
positive and negative information, represented by the values $T$ and $F$ assigned to a sentence $A$, are on a par in this order, and (ii) it is assumed that a contradiction $A \land \neg A$ not only does contain information but also contains the highest amount of information on $A$. This is in line with the already mentioned notion of information as meaningful data, which considers false information as information (see e.g. [29, 31]), and is the notion of information that underlies the interpretation of $FDE$ worked out by Belnap and Dunn, as well as the intended interpretation of $LET$s in terms of information.

The logical lattice $L_4$, on the other hand, has $F$ at the bottom and $T$ at the top. The join operation is given by $\bar{\lor}$ and the meet by $\bar{\land}$. It is represented by the following diagram:

```
    T
   /\  \\
  n  b  \\
     \  \\
    F
```

The partial order of $L_4$ can be defined as follows. Think of the values $T, F, b, n$ as pairs $(a_1, a_2)$, where $a_1$ and $a_2$ represent, respectively, the values of formulas $A$ and $\neg A$ in a given bivaluation. Thus, $T, F, b, n$ are represented, respectively, by the pairs $(1, 0), (0, 1), (1, 1), (0, 0)$. Now, the order is given as follows: $(a_1, a_2) \leq (b_1, b_2)$ if and only if $a_1 \leq b_1$ and $a_2 \geq b_2$. This order is informally explained by Belnap as follows:

[T]he worst thing is to be told something is false, simpliciter. You are better off (it is one of our hopes) in either being told nothing about it, or in being told both that it is true and also that it is false; while of course best of all is to be told it is true, with no muddying of the waters [7, p. 42].

Now, let us call $L_6$ the lattice obtained by extending $L_4$ to six values. The values $T$ and $F$ of $FDE$ become $T_0$ and $F_0$, and we add the values $T$ and $F$ of $LET_K$ as, respectively, a new top and a new bottom. The order of $L_6$ has been given in Proposition [4.6]. The lattice structure of $L_6$ can be displayed as follows:

```
    T
   /\  \\
  T_0  \\
     \  \\
    F
```

29
It should be observed that the six-valued lattice above is exactly the lattice \( T_{B_2} \) with the order defined as in Proposition 4.6.

**Remark 4.7.** The lattice structure of \( L_6 \), together with its negation (and expanded by a suitable implication), has already appeared in the context of relevance logic in Routley (later Sylvan) \[51\], p. 224]. According to this author, this structure (also called *crystal lattice*) was first proposed by Meyer.\(^{12}\) As mentioned by Brady in \[10\], pp. 65-66], this six-valued lattice structure together with its negation and implication characterizes, as a logical matrix in which every element other than \( F \) is designated, the finitely axiomatizable relevance system \( \text{CL} \) (see Sylvan et al. \[52\], p. 114]). The implication \( \rightarrow \) of the crystal lattice is such that \( A \rightarrow (B \rightarrow A) \) is not a valid schema, hence the logic \( \text{CL} \) is different from the \( \circ \)-free fragment of \( \text{LET}_K^r \). For more results about the crystal lattice see \[39\].

It is worth noting that, in the diagram of \( L_6 \) above (including its negation), \( L_4 \) corresponds to the inner diamond. The order of \( L_6 \) can be explained by modifying the quotation above from Belnap: the worst thing is to be told something has been *conclusively* established as false, which is the value \( F \); not too bad is to be false but not conclusively false, the value \( F_0 \). To be told something is true, although not conclusively, the value \( T_0 \), is better, but the best of all is to be told it is conclusively true, which is the value \( T \).

Finally, it is worth mentioning that if we lay on its side the \( L_6 \) lattice, with \( n \) at the bottom, we obtain a meet-semilattice – call it \( A_6 \) – that fits the idea of the approximation lattice \( A_4 \): the amount of information grows from bottom to top, but from the nodes \( T_0 \) and \( F_0 \), the new information can be that the information already available is reliable, and in this case we get the values \( T \) and \( F \) respectively, or that conflicting information is obtained, and in this case \( T_0 \) and \( F_0 \) collapse in the value \( b \).

The order of \( A_6 \) can be represented by the relation of inclusion \( \subseteq \). Consider the set \( \{0, 1, c\} \), where 0 and 1 mean, respectively, negative and positive information, and \( c \), together with 0 or 1, means that the respective information is reliable. The six values \( n, T_0, F_0, T, F, b \) correspond, respectively, to the following sets: \( \emptyset, \{1\}, \{0\}, \{1, c\}, \{0, c\}, \{1, 0\} \). Note that the sets \( \{0, 1, c\} \) and \( \{c\} \) have been dropped from the powerset of \( \{0, 1, c\} \): the former because it cannot be that positive and negative information together are reliable, and the latter because \( c \) only makes sense together with either positive or negative information. The meets are given by \( \cap \) and the (existing) joins by \( \cup \), both operations restricted to the given domain of six subsets of \( \{0, 1, c\} \). Taking this into account, \( \{0, c\} \cap \{1, c\} \) is the empty set, rather than \( \{c\} \), since the latter does not belong to the domain of the meet-semilattice. Analogously, note that the join of \( T \) and \( F \), as well as the joins of \( T \) and \( b \), and also \( b \) and \( F \), do not exist in the meet-semilattice. Then, the least element is \( n \), but there is no greatest element.

\(^{12}\)In the context of order theory this lattice was introduced, possibly for the first time in the literature by means of a Haase diagram, in \[38\], fig. 11, p. 613. We thank Rodolfo Ertola for pointing out this fact to us.
Note that the diagram above can be interpreted bottom-up as stages of a database with respect to a sentence \( A \). In the bottom there is no information about \( A \), neither positive nor negative, which corresponds to the semantic value \( n \) assigned to \( A \). From this stage, there are two alternatives: either positive information \( A \) or negative information \( \neg A \) is obtained, that is, \( A \) is assigned, respectively, \( T_0 \) and \( F_0 \). Now, in each case, two alternatives are possible: either the information that \( A \) (\( \neg A \)) is reliable is obtained, yielding the value \( T (F) \), or contradictory information is obtained, and so \( b \) is assigned to \( A \) (and to \( \neg A \) as well).

5 Extending a minimal \( LET \): the logic \( LET_F^+ \)

We have already mentioned the logic \( LET_F^- \), a minimal logic of evidence and truth that extends \( FDE \) with the classicality operator \( \circ \) and the rules \( EXP^c \) and \( PEM^c \). The logic \( LET_F^+ \) is the extension of \( LET_F^- \) with the axiom \( \circ \circ A \) and the rules of propagation of classicality for \( \lor \), \( \land \), and \( \neg \) taken from Definition 3.2. It can also be defined as the \( \rightarrow \)-free fragment of \( LET_K^+ \).

In this brief section we start by \( LET_F^- \), which admits a valuation semantics and a (non-deterministic) sound and complete six-valued semantics based on swap-structures. We then move to \( LET_F^+ \), which, like \( LET_K^+ \), is semantically characterized by a six-valued logical matrix, as well as by a class of logical matrices based on twist structures.

Thus, consider the propositional signature \( \Sigma_1 = \{\land, \lor, \neg, \circ\} \). A natural deduction system for \( LET_F^- \) is obtained by dropping rules \( \rightarrow I \), \( \rightarrow E \), \( \rightarrow CL \), \( \neg \rightarrow I \), and \( \neg \rightarrow E \) from Definition 2.1. A bivalued semantics for \( LET_F^- \) is obtained by dropping clauses (v3) and (v7) from Definition 2.2. An Nmatrix \( M_{LET_F^-} \) for \( LET_F^- \) is obtained by dropping clause (iii) of Definition 2.12, which produces the six-valued (non-deterministic) truth-tables obtained just by dropping the table for implication of \( LET_K \) displayed in Subsection 2.2.1. As expected, the syntactic consequence (\( \vdash_{LET_F^-} \)) and the semantic consequence, defined by either the bivalued semantics (\( \models^2_{LET_F^-} \)) or by the six-valued semantics (\( \models_{M_{LET_F^-}} \)), are equivalent:

**Theorem 5.1.** \( \Gamma \vdash_{LET_F^-} A \) if and only if \( \Gamma \models^2_{LET_F^-} A \) if and only if \( \Gamma \models_{M_{LET_F^-}} A \).

The proof of Theorem 5.1 can be easily adapted from the proofs of Theorems 2.9, 2.15, and 2.17. The Nmatrix \( M_{LET_F^-} \), of course, provides a decision procedure for \( LET_F^- \).

We now turn to the logic \( LET_F^+ \). Recall from Remark 3.10 that \( A_6 \) denotes the six-valued algebra underlying the matrix \( M_6 \) of \( LET_K^+ \). Let \( A_6^1 \) be the six-valued algebra underlying the matrix \( M_6^1 \) obtained from \( M_6 \) by removing the implication operator.
Later, M. de Galego changed her name to M. Sagastume.

\[\rightarrow\] Consider the valuation semantics for \(LET^+_F\) obtained from the one for \(LET^+_K\) by removing the clauses for implication \(\rightarrow\). It is easy to see, by adapting the corresponding proofs for \(LET^+_K\), that

**Theorem 5.2.** \(\Gamma \vdash_{LET^+_F} A\) if and only if \(\Gamma \models^{2}_{LET^+_F} A\) if and only if \(\Gamma \models^{1}_{LET^+_F} A\).

To generalize: given a twist structure \(T_B\) for \(LET^+_K\) induced by a Boolean algebra \(B\) (recall Definition 4.1), let \(T_B^I\) be its implication-free reduct to \(\Sigma_1\). Let \(M^I(B)\) be the induced Nmatrix as in the case of \(LET^+_K\). Clearly, \(T_B^I\) is exactly \(A_0^I\), while \(M^I(B_2) = M^I_6\). Let \(Mat(LET^+_F)\) be the class of logical matrices of the form \(M^I(B)\), and let \(\models_{Mat(LET^+_F)}\) be the associated consequence relation. Then,

**Theorem 5.3.** \(\Gamma \vdash_{LET^+_F} A\) if and only if \(\Gamma \models_{Mat(LET^+_F)} A\).

### 5.1 \(LET^+_F\) and involutive Stone Algebras

By convenience, in this subsection we will consider that \(LET^+_K\) and \(LET^+_F\) are defined over a signature containing the constants \(\top\) and \(\bot\).

As mentioned in Remark 4.7, the lattice structure \(\mathbf{L}_6\) expanded by negation (that is, the De Morgan \(\{\land, \lor, \neg\}\)-reduct of \(A_6\)) presented in Subsection 4.1 coincides with (the De Morgan reduct of) the so-called Meyer’s crystal lattice.

In this subsection it will be shown that the crystal lattice also appears in a different algebraic context. Indeed, a curious and unexpected close relationship between \(LET^+_F\) and a variety of algebras known as **Involutive Stone Algebras** (ISAs, for short) can be established. Involutive Stone algebras are De Morgan algebras with an additional unary operator \(\nabla\) satisfying some specific equations. That is, ISAs are algebras defined over the signature \(\{\land, \lor, \neg, \nabla, \top, \bot\}\). The variety of ISAs was introduced by Cignoli and de Galego\footnote{Later, M. de Galego changed her name to M. Sagastume.} in 21 in the context of Lukasiewicz-Moisil algebras. In 22 they prove that the variety of ISAs is generated by \(S_6\), a 6-element ISA whose \(\{\land, \lor, \neg\}\)-reduct coincides with the lattice \(\mathbf{L}_6\) (plus negation) of \(A_6\) displayed in Subsection 4.1 that is, (the De Morgan reduct of) the crystal lattice. In \(S_6\) the \(\nabla\) operator is given by \(\nabla(a) = T\) if \(a \neq F\), and \(\nabla(F) = F\). Observe that, in the implication-free reduct \(A_0^I\) of \(A_6\), \(\nabla\) can be defined by means of the formula \(\nabla A \overset{\text{df}}{=} A \lor \neg\neg A\). On the other hand, it is clear that the formula \(\diamond A \overset{\text{df}}{=} \neg\nabla A \lor \neg\nabla A\) defines in \(S_6\) the operator \(\diamond\). This shows that the six-valued algebra \(S_6\) is equivalent in expressive power to the algebra \(A_0^I\) for \(LET^+_F\).

Cantú and M. Figallo studied in 11 the logic-preserving degrees of truth of the variety of ISAs, which is defined as follows (taking into account that \(S_6\) generates the variety): \(\Gamma \models^{2}_{S_6} A\) iff either \(v(A) = 1\), for every valuation \(v\) over \(S_6\), or there exist \(A_1, \ldots, A_n \in \Gamma\) such that \(v(A_1) \land \ldots \land v(A_n) \leq v(A)\), for every valuation \(v\) over \(S_6\). They prove in their Theorem 5.5 that the logic \(\text{Six}\) generated by the 4 matrices over \(S_6\) with the set of designated values \(\{n, T_0, T\}\), \(\{F_0, n, T_0, T\}\), \(\{T_0, T\}\) and \(\{T\}\) coincides with the logic-preserving degrees of truth of the variety of ISAs. On the other hand, in 13, Proposition 4.1, it was shown that \(\text{Six}\) can be characterized by the logical matrix over \(S_6\) with set of designated values \(\{b, T_0, T\}\). Based on this, in 12 Cantú and M. Figallo apply a general method introduced by Avron and his collaborators to give a cut-free
Stone Algebras can also be decided by means of standard 2-valued truth-tables. The same holds for Proposition 5.4.

Remark 5.5.

(1) It is worth noting that \( LET^+_F \) is not the degree-preserving expansion of the logic Six obtained by adding an implication. Indeed, the logic of \( LET^+_F \) is not the logic-preserving degrees of truth of \( A_6 \) (the expansion of \( S_6 \) with the implication \( \rightarrow \)) given that, for instance, \( \neg(A \rightarrow B) \not\models LET^+_F A \), but \( \neg(A \rightarrow B) \not\models A_6 \), where \( \Gamma \models \neg A_6 \) iff, either \( v(A) = 1 \), for every valuation \( v \) over \( A_6 \), or there exist \( A_1, \ldots, A_n \in \Gamma \) such that \( v(A_1) \cap \ldots \cap v(A_n) \neq v(A) \), for every valuation \( v \) over \( A_6 \). In fact, it is enough to consider a valuation \( v \) such that \( v(A) = b \) and \( v(B) = n \) (which is perfectly possible when \( A \) and \( B \) are two propositional variables). In this case, \( v(\neg(A \rightarrow B)) = v(A \rightarrow B) = b \rightarrow n = n \neq b = v(A) \), hence \( \neg(A \rightarrow B) \not\models \neg A_6 \).

(2) The algebraization of \( LET^+_F \) obtained in Section 3.4 shows that there is an additional difference with its implication-free fragment \( LET_F \). Indeed, as shown in Proposition 4.2, the logic Six is not algebraizable, although it is selfextensional. By Proposition 5.4 the same holds for \( LET^+_F \). On the other hand \( LET^+_F \), despite being algebraizable, it is not selfextensional: indeed, \( \neg(A \rightarrow B) \) is equivalent to \( A \land \neg B \) but \( \neg(\neg(A \rightarrow B)) \neq (A \land \neg B) \). In fact, while the former is equivalent to \( A \rightarrow B \), the latter is equivalent to \( A \land B \), and clearly these formulas are inequivalent in \( LET^+_F \). Despite these differences, it would be interesting to analyze \( LET^+_F \) and \( LET_F \) with relation to involutive Stone algebras.

(3) In [33] was proposed the study of expansions of De Morgan algebras by means of an operator \( \circ \) which, among several properties, is able to simultaneously recover explosion and excluded middle w.r.t. the De Morgan negation (that is, a classicality operator). They show that the degree-preserving logic associated to these algebras coincides, up to language, with Six (and so with \( LET^+_F \) by Proposition 5.4). This means that the algebraic conditions required in [33] for the operator \( \circ \) turn out to be equivalent to the propagation conditions required for \( \circ \) in \( LET^+_F \) (it is worth noting that no implication connective was considered in [33]). Observe that the purpose of [33] is closely related to that of \( LETs \) [10, 20, 40], that is, to define logics based on \( FDE \), the logical counterpart of De Morgan algebras, expanded with a classicality operator; however, in [33] this is done from an algebraic perspective.

We close the paper by pointing out that, by Remark 4.2, decidability in \( LET^+_F \) can be reduced to checking validity in classical propositional logic \( CPL \); in particular, the same holds for \( LET_F \). The latter means that the degree-preserving logic of Involutitive Stone Algebras can also be decided by means of standard 2-valued truth-tables.

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As observed above, \( LET^+_F \) was expanded with the definable constants \( \bot \) and \( \top \).
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References
[1] H. Antunes, W. Carnielli, A. Kapsner, and A. Rodrigues. Kripke-style models for logics of evidence and truth. Axioms, 9(3), 2020. URL https://www.mdpi.com/2075-1680/9/3/100.
[2] H. Antunes, A. Rodrigues, W. Carnielli, and M.E. Coniglio. Valuation semantics for first-order logics of evidence and truth. Journal of Philosophical Logic, 2022. doi: 10.1007/s10992-022-09662-8.
[3] Alves, E. H. Lógica e inconsistência: um estudo dos cálculos $C_n$, $1 \leq n \leq \omega$ (Logic and inconsistency: A study of the calculi $C_n$, $1 \leq n \leq \omega$, in Portuguese) Masters thesis, FFLCH, State University of São Paulo, 1976.
[4] A. Avron. Non-deterministic semantics for logics with a consistency operator. International Journal of Approximate Reasoning, 45(2):271–287, 2007.
[5] A. Avron and I. Lev. Canonical propositional Gentzen-type systems. In Proceedings of the First International Joint Conference on Automated Reasoning (IJCAR '01), pages 529–544. Springer, 2001.
[6] A. Avron and I. Lev. Non-deterministic multiple-valued structures. Journal of Logic and Computation, 15(3):241–261, 2005.
[7] N. D. Belnap. How a computer should think. In G. Ryle, editor, Contemporary Aspects of Philosophy. Oriel Press, 1977. Reprinted in New Essays on Belnap-Dunn Logic, Springer, 2019, pages 35–55.
[8] N. D. Belnap. A useful four-valued logic. In G. Epstein and J. M. Dunn (Eds.), Modern uses of multiple valued logics. In G. Ryle, editor, Contemporary Aspects of Philosophy. Dordrecht: D. Reidel, 1977. Reprinted in New Essays on Belnap-Dunn Logic, Springer, 2019, pages 55–77.
[9] W. J. Blok and D. Pigozzi. Algebraizable logics. In Memoirs of the American Mathematical Society (vol. 77). American Mathematical Society, Providence, RI, USA, 1989.
[10] R. Brady. Depth relevance of some paraconsistent logics. Studia Logica, 43(1-2): 63–73, 1984.
[11] L. M. Cantú and M. Figallo. On the logic that preserves degrees of truth associated to involutive Stone algebras. *Logic Journal of the IGPL*, 28(5):1000–1020, 2020.

[12] L. M. Cantú and M. Figallo. Cut-free sequent-style systems for a logic associated to involutive Stone algebras. *Journal of Logic and Computation*, 2022. URL https://doi.org/10.1093/logcom/exac061

[13] W. Carnielli. Many-valued logics and plausible reasoning. In: *Proceedings of the XX International Congress on Many-Valued Logics*, University of Charlotte, USA, pages 328–335. IEEE Computer Society, 1990.

[14] W. Carnielli. Possible-Translations Semantics for Paraconsistent Logics. In D. Batens, C. Mortensen, G. Priest, and J. P. Van Bendegem, editors, *Frontiers of Paraconsistent Logic: Proceedings of the I World Congress on Paraconsistency*, pages 149–163. Baldock: Research Studies Press, King’s College Publications, 2000.

[15] W. Carnielli and M. E. Coniglio. Splitting Logics. In S. Artemov, H. Barringer, A. Garcez, L. Lamb, and J. Woods, editors, *We Will Show Them! Essays in Honour of Dov Gabbay, vol. 1*, pages 389–414. College Publications, 2005.

[16] W. Carnielli and M. E. Coniglio. *Paraconsistent Logic: Consistency, Contradiction and Negation*, volume 40 of Logic, Epistemology, and the Unity of Science series. Springer, 2016.

[17] W. Carnielli, M. E. Coniglio, and J. Marcos. Logics of formal inconsistency. In D. Gabbay and F. Guenthner, editors, *Handbook of Philosophical Logic*, volume 14, pages 1–93, Amsterdam, 2007. Springer-Verlag.

[18] W. Carnielli and J. Marcos. A taxonomy of C-systems. In W. Carnielli, M. E. Coniglio, and I.M. L. D’Ottaviano, editors, *Paraconsistency: The Logical Way to the Inconsistent*. Marcel Dekker, New York, 2002.

[19] W. Carnielli and A. Rodrigues. On the philosophy and mathematics of the logics of formal inconsistency. In J.-Y. Beziau et al., editor, *New Directions in Paraconsistent Logic – Springer Proceedings in Mathematics & Statistics 152*, pages 57–88. Springer India, 2015.

[20] W. Carnielli and A. Rodrigues. An epistemic approach to paraconsistency: a logic of evidence and truth. *Synthese*, 196:3789–3813, 2017. doi: 10.1007/s11229-017-1621-7. URL https://rdcu.be/ctJRQ

[21] R. Cignoli and M. S. de Gallego. The lattice structure of some Lukasiewicz algebras. *Algebra Universalis*, 13:315–328, 1981.

[22] R. Cignoli and M. S. de Gallego. Dualities for some De Morgan algebras with operators and Lukasiewicz algebras. *Journal of the Australian Mathematical Society (Series A)*, 34:377–393, 1983.
[23] M. E. Coniglio, A. Figallo-Orellano, and A. C. Golzio. Non-deterministic algebraization of logics by swap structures. *Logic Journal of IGPL*, 28:1021–1059, 2018.

[24] M. E. Coniglio and G. V. Toledo. Two Decision Procedures for da Costa’s $C_n$ Logics Based on Restricted Nmatrix Semantics. *Studia Logica*, 110(3):601–642, 2022.

[25] N. C. A. da Costa. *Sistemas Formais Inconsistentes*. Curitiba: Editora da UFPR (1993), 1963.

[26] N. C. A. da Costa. On the theory of inconsistent formal systems. *Notre Dame Journal of Formal Logic*, XV, number 4(4):497–510, 1974.

[27] J. M. Dunn. The Algebra of Intensional Logics. Ph.D. Dissertation, University of Pittsburgh, 1966. Published as Vol. 2 in the Logic PhDs series by College Publications, London, 2019.

[28] J. M. Dunn. Intuitive semantics for first-degree entailments and ‘coupled trees’. *Philosophical Studies*, 29:149–168, 1976. Reprinted in *New Essays on Belnap-Dunn Logic*, Springer, 2019, pages 21–34.

[29] J. M. Dunn. Information in computer science. In P. Adriaans and J. van Benthem, editors, *Philosophy of Information. Volume 8 of Handbook of the Philosophy of Science*, pages 581–608. Elsevier, 2008.

[30] J. M. Dunn. Two, three, four, infinity: The path to the four-valued logic and beyond. In Omori and Wansing, editors, *New Essays on Belnap-Dunn Logic*, pages 77–97. Springer, 2019.

[31] J. Fetzer. Information: Does it have to be true? *Minds and Machines*, 14:223–229, 2004.

[32] J. M. Font. *Abstract Algebraic Logic: An Introductory Textbook*. College Publications, London, 2016.

[33] J. Gomes, V. Greati, S. Marcelino, J. Marcos and U. Rivieccio. On Logics of Perfect Paradefinite Algebras. In M. Ayala-Rincon and E. Bonelli, editors, *Proceedings of the 16th Logical and Semantic Frameworks with Applications (LSFA 2021)*. Volume 357 of *Electronic Proceedings in Theoretical Computer Science*, pages 56–76, 2022.

[34] A. Hazen and F. Pelletier. K3, L3, LP, RM3, A3, FDE, M: How to make many-valued logics work for you. In H. Omori and H. Wansing, editors, *New Essays on Belnap-Dunn Logic*. Springer, 2019.

[35] Ju. Ivlev. *Tablutznoe postrojenie propozicionalnoj modalnoj logiki* (Truth-tables for systems of propositional modal logic, in Russian). Vest. Mosk. Univ., Seria Filosofia, 1973.
[36] Ju. Ivlev. A semantics for modal calculi. *Bulletin of the Section of Logic*, 17(3/4): 114–121, 1988.

[37] T. Kearns. Modal semantics without possible worlds. *The Journal of Symbolic Logic*, 46:77–86, 1981.

[38] F. Klein-Barmen. Grundzüge Der Theorie Der Verbände. *Mathematische Annalen*, 111(1):596–621, 1935.

[39] R. L. Kramer and R. D. Maddux. Relation algebras of Sugihara, Belnap, Meyer, and Church. *Journal of Logical and Algebraic Methods in Programming*, 117, 2020, 100604. URL [https://doi.org/10.1016/j.jlamp.2020.100604](https://doi.org/10.1016/j.jlamp.2020.100604).

[40] A. Loparic. A semantical study of some propositional calculi. *The Journal of Non-Classical Logic*, 3(1):73–95, 1986.

[41] A. Loparic and E. Alves. The semantics of the systems $Cn$ of da Costa. In A. Arruda, N. da Costa, and A. Sette, editors, *Proceedings of the Third Brazilian Conference on Mathematical Logic*, pages 161–172. São Paulo: Sociedade Brasileira de Lógica, 1980.

[42] A. Loparic and N. da Costa. Paraconsistency, paracompleteness and valuations. *Logique et Analyse*, 106:119–131, 1984.

[43] S. Marcelino and U. Rivieccio. Logics of involutive Stone algebras. *Soft Computing*, 26(7):3147–3160, 2022.

[44] H. Omori and H. Wansing. 40 years of FDE: An introductory overview. *Studia Logica*, 105:1021–1049, 2017.

[45] A.P. Pynko. Functional completeness and axiomatizability within Belnap’s four-valued logic and its expansions. *Journal of Applied Non-Classical Logics*, 9:61–105, 1999.

[46] N. Rescher. Quasi-truth-functional systems of propositional logic. *The Journal of Symbolic Logic*, 27(1):1–10, 1962.

[47] A. Rodrigues and H. Antunes. First-order logics of evidence and truth with constant and variable domains. *Logica Universalis*, 16(3):419–449, 2022. URL doi: 10.1007/s11787-022-00306-8.

[48] A. Rodrigues and W. Carnielli. On Barrio, Lo Guercio, and Szmuc on logics of evidence and truth. *Logic and Logical Philosophy*, 31(2):313–338, 2022. URL [https://doi.org/10.12775/LLP.2022.009](https://doi.org/10.12775/LLP.2022.009).

[49] A. Rodrigues, J. Bueno-Soler, and W. Carnielli. Measuring evidence: a probabilistic approach to an extension of Belnap-Dunn logic. *Synthese*, 198(22):5451–5480, 2020.
[50] A. Rodrigues, M. E. Coniglio, H. Antunes, J. Bueno-Soler, and W. Carnielli. Paraconsistency, evidence, and abduction. In L. Magnani, editor, Handbook of Abductive Cognition. Springer, Cham, 2022. URL https://doi.org/10.1007/978-3-030-68436-5_27-1

[51] R. Routley. Alternative semantics for quantified first degree relevant logic. Studia Logica, 38(2):211–231, 1979.

[52] R. Sylvan, R. Meyer, R. Brady, C. Mortensen, and V. Plumwood. The Algebraic Analysis of Relevant Affixing Systems. In R. Brady, editor, Relevant logics and their rivals. A Continuation of the Work of R. Sylvan, R. Meyer, V. Plumwood and R. Brady, Volume II, pages 72—140. Volume 59 of Western Philosophy Series. Ashgate Publishing Limited, Aldershot, 2003.

[53] R. Wójcicki. Lectures on Propositional Calculi. Ossolineum, Wrocaw, Poland, 1984. URL http://www.ifispan.waw.pl/studialogica/wojcicki/papers.html