SHARP CAPACITY ESTIMATES IN S-JOHN DOMAINS

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Abstract. It is well-known that several problems related to analysis on s-John domains can be unified by certain capacity lower estimates. In this paper, we obtain general lower bounds of p-capacity of a compact set E and the central Whitney cube $Q_0$ in terms of the Hausdorff q-content of E in an s-John domain $\Omega$. Moreover, we construct several examples to show the essential sharpness of our estimates.

1. Introduction

Recall that a bounded domain $\Omega \subset \mathbb{R}^n$ is a John domain if there is a constant $C$ and a point $x_0 \in \Omega$ so that, for each $x \in \Omega$, one can find a rectifiable curve $\gamma : [0,1] \to \Omega$ with $\gamma(0) = x$, $\gamma(1) = x_0$ and with

$$Cd(\gamma(t), \partial \Omega) \geq l(\gamma([0,t]))$$

for each $0 < t \leq 1$. F. John used this condition in his work on elasticity [13] and the term was coined by Martio and Sarvas [18]. Smith and Stegenga [21] introduced the more general concept of s-John domains, $s \geq 1$, by replacing (1.1) with

$$(1.2) \quad Cd(\gamma(t), \partial \Omega) \geq l(\gamma([0,t]))^s.$$ 

The condition (1.1) is called a “twisted cone condition” in literature. Thus condition (1.2) should be called a “twisted cusp condition”.

In the last twenty years, s-John domains has been extensively studied in connection with Sobolev type inequalities; see [3, 11, 9, 14, 17, 21]. In particular, Buckley and Koskela [3] have shown that a simply connected planar domain which supports a Sobolev-Poincaré inequality is an s-John domain for an appropriate $s$. Smith and Stegenga have shown that an s-John domain $\Omega$ is a $p$-Poincaré domain, provided $s < \frac{n}{n-1} + \frac{p-1}{n}$. In particular, if $s < \frac{n}{n-1}$, then $\Omega$ is a $p$-Poincaré domain for all $1 \leq p < \infty$. These results were further generalized to the case of $(q,p)$-Poincaré domains in [11, 14, 17]. Recall that a bounded domain $\Omega \subset \mathbb{R}^n$, $n \geq 2$, is said to be a $(q,p)$-Poincaré domain if there exists a constant $C_{q,p} = \ldots$
for all \( u \in C^\infty(\Omega) \). Here \( u_\Omega = \int_\Omega u(x)\,dx \). When \( q = p \), \( \Omega \) is termed a \( p \)-Poincaré domain and when \( q > p \) we say that \( \Omega \) supports a Sobolev-Poincaré inequality.

The recent studies \([1, 5, 7]\) on mappings of finite distortion have generated new interest in the class of \( s \)-John domains. In particular, uniform continuity of quasiconformal mappings onto \( s \)-John domains was studied in \([4, 6]\).

The proofs for both types of problems rely on certain capacity estimates for subsets of \( s \)-John domains. To be more precise, for the problem related to Sobolev-Poincaré inequalities, one uses the idea of Maz’ya \([19, 20]\) to reduce the problem to capacity estimates of the form

\[
\text{Cap}_p(E, Q_0, \Omega) \geq \psi(|E|),
\]

where \( Q_0 \) is the fixed Whitney cube containing the (John) center \( x_0 \) and \( E \) is an admissible subset of \( \Omega \) disjoint from \( Q_0 \); for \((1.3)\), \( \psi(t) = Ct^{p/q} \), see also \([8, 17]\). Here, by admissible we mean that \( E \) is an open set so that \( \partial E \cap \Omega \) is a smooth submanifold. As for the uniform continuity of quasiconformal mappings onto \( s \)-John domains, one essentially needs a capacity estimate of the form

\[
\text{Cap}_n(E, Q_0, \Omega) \geq \psi(\text{diam } E),
\]

where \( E \) is a continuum in \( \Omega \) disjoint from the central Whitney cube \( Q_0 \); see \([4]\). Thus one could expect that a more general capacity estimate of the form

\[
(1.4) \quad \text{Cap}_p(E, Q_0, \Omega) \geq \psi(\mathcal{H}_\infty^q(E))
\]

holds in certain \( s \)-John domains \( \Omega \), where \( E \) is a compact set in \( \Omega \) disjoint from the central Whitney cube \( Q_0 \) and \( \mathcal{H}_\infty^q(E) \) is the Hausdorff \( q \)-content of \( E \). We confirm this expectation by showing the following result.

**Theorem 1.1.** Let \( \Omega \subset \mathbb{R}^n, n \geq 2 \), be an \( s \)-John domain. For \( 0 < \varepsilon < 1 \), \( 1 \leq p \leq n \) and \( q \geq s(n-1) + 1 - p + \varepsilon \), there exists a positive constant \( C(n, p, q, s, \varepsilon) \) such that

\[
\text{Cap}_p(E, Q_0, \Omega) \geq C(n, p, q, s, \varepsilon) \left( \mathcal{H}_\infty^q(E) \right)^{\frac{s(n-1)+1-p+\varepsilon}{q}},
\]

whenever \( E \subset \Omega \) is a compact set disjoint from \( Q_0 \).

**Remark 1.2.** If \( p = n \), \( q = 1 \) and \( E \subset \Omega \) is a continuum, then \((1.5)\) reduces to the estimate

\[
\text{Cap}_n(E, Q_0, \Omega) \geq C(n, s, \varepsilon)(\text{diam } E)^{(n-1)(s-1)+\varepsilon}.
\]
The restriction becomes $1 \geq (s - 1)(n - 1) + \varepsilon$, which is equivalent to $s \leq 1 + \frac{p-\varepsilon}{n-1}$. The range for $s$ is essentially sharp, see [6].

If $q = n$, then (1.5) reduces to the estimate

$$\text{Cap}_p(E, Q_0, \Omega) \geq C(n, s, \varepsilon)|E|^{\frac{(n-1)s+1-p+\varepsilon}{n}}.$$ 

The restriction becomes $s \leq 1 + \frac{p-\varepsilon}{n-1}$. Note that $1 + \frac{p}{n-1} > \frac{n}{n-1} + \frac{p-1}{n}$. This implies that if $s < 1 + \frac{p}{n-1}$, then $\Omega$ is a $p$-Poincaré domain. The range for $s$ is sharp, see [11].

The estimate in Theorem 1.1 is essentially sharp in the sense that the exponent of $H_\infty^q(E)$ in (1.5) cannot be made strictly smaller than $\frac{s(n-1)+1-p}{q}$; see Example 1.1 below.

Our second result shows that the requirement $q \geq s(n-1) + 1 - p + \varepsilon$ is essentially sharp in the sense that there exists an $s$-John domain $\Omega \subset \mathbb{R}^n$ such that no estimate of the form as in (1.4) holds in $\Omega$ whenever $q < \min\{s(n-1) + 1 - p, n\}$. This is somewhat surprising since the estimate in (1.5) does not degenerate when $q < s(n-1) + 1 - p$.

**Theorem 1.3.** Fix $1 \leq p \leq n$. There exists an $s$-John domain $\Omega \subset \mathbb{R}^n$ such that there is a sequence of compact sets $E_j$ in $\Omega$ with the following properties:

- Each $E_j$ is disjoint from the central Whitney cube $Q_0$;
- $H_\infty^q(E_j)$ is bounded from below uniformly by a positive constant and $\text{Cap}_p(E_j, Q_0, \Omega) \to 0$ as $j \to \infty$, whenever $q < \min\{(n-1)s + 1 - p, n\}$.

It would be interesting to know whether one can obtain an estimate of the form as in (1.4) when $q = (n-1)s + 1 - p$.

When $q < \min\{(n-1)s + 1 - p, \log_2(2^n - 1)\}$, the $s$-John domain $\Omega$ constructed in Theorem 1.3 is in fact Gromov hyperbolic in the quasihyperbolic metric. This is very surprising, since it was proven in [4] that for all Gromov hyperbolic $s$-John domains $\Omega$, an estimate of the form as in (1.4) holds when $p = n$, $q = 1$ and $E \subset \Omega$ is a continuum. Our example shows that one can not replace the assumption being a continuum by just being compact, and still obtain the estimate for all $s$-John domains. For definitions and examples of Gromov hyperbolic domains, we refer to the beautiful monograph [2].

2. Preliminary results

For an increasing function $\tau : [0, \infty) \to [0, \infty)$ with $\tau(0) = 0$, we denote by $H_\infty^\tau$ the Hausdorff $\tau$-content: $H_\infty^\tau(E) = \inf \sum_i \tau(r_i)$, where the infimum is taken over all coverings of $E \subset \mathbb{R}^n$ with balls $B(x_i, r_i)$, $i = 1, 2, \ldots$. When $\tau(t) = ts$ for some $0 < s < \infty$, we write $H_\infty^s = H_\infty^s$. 
For disjoint compact sets $E$ and $F$ in the domain $\Omega$, we denote by $\text{Cap}_p(E, F, \Omega)$ the $p$-capacity of the pair $(E, F)$:

$$\text{Cap}_p(E, F, \Omega) = \inf_u \int_{\Omega} |\nabla u(x)|^p dx,$$

where the infimum is taken over all continuous functions $u \in W^{1,p}_{\text{loc}}(\Omega)$ which satisfy $u(x) \leq 0$ for $x \in E$ and $u(x) \geq 1$ for $x \in F$.

Let $\Omega$ be a bounded domain in $\mathbb{R}^n$, $n \geq 2$. Then $W_\lambda = W(\Omega)$ denotes a Whitney decomposition of $\Omega$, i.e. a collection of closed cubes $Q \subset \Omega$ with pairwise disjoint interiors and having edges parallel to the coordinate axes, such that $\Omega = \bigcup_{Q \in W} Q$, the diameters of $Q \in W$ belong to the set $\{2^{-j} : j \in \mathbb{Z}\}$ and satisfy the condition $\text{diam}(Q) \leq \text{dist}(Q, \partial \Omega) \leq 4 \text{diam}(Q)$.

For $j \in \mathbb{Z}$ we define $W_j = \{Q \in W: \text{diam}(Q) = 2^{-j}\}$.

The following lemma is well-known, see for instance [15, Lemma 2.8].

**Lemma 2.1.** Fix $1 \leq p < \infty$. Let $B_1, B_2, \ldots$ be balls or cubes in $\mathbb{R}^n$, $a_j \geq 0$ and $\lambda > 1$. Then

$$\| \sum a_j \chi_{\lambda B_j} \|_p \leq C(n, p, q, s, \varepsilon) \sum a_j \| \chi_{B_j} \|_p.$$

3. Main proofs

**Proof of Theorem 1.1.** The proof is a combination of several well-known arguments; in particular [8, Proof of Theorem 9] and [11, Proof of Theorem 5.9]. For any compact set $E \subset \Omega$ such that $E \cap Q_0 = \emptyset$, where $Q_0$ is the central cube that contains the John center $x_0$, we fix a test function $u$ for $\text{Cap}_p(E, Q_0, \Omega)$, i.e. $u$ is a continuous function in $W^{1,p}_{\text{loc}}(\Omega)$ so that $u \geq 1$ on $E$ and $u \leq 0$ on $Q_0$. We may assume that $\text{diam} \Omega = 1$.

For each $x \in E$, we may fix an $s$-John curve $\gamma$ joining $x$ to $x_0$ in $\Omega$ and define $P(x)$ to be the collection of Whitney cubes that intersect $\gamma$. Thus $Q(x) \in P(x)$ will be the Whitney cube containing the point $x$. We next divide our compact set $E$ into the good part and the bad part according to the range of $u_Q$. Let $G = \{x \in E : u_{Q(x)} \leq \frac{1}{2}\}$ and $B = E \setminus G$.

**Claim 1:** for $1 \leq p \leq n$ and $q \geq s(n-1) + 1 - p + \varepsilon$, there exists a positive constant $C(n, p, q, s, \varepsilon)$ such that

$$\int_{\Omega} |\nabla u(x)|^p dx \geq C(n, p, q, s, \varepsilon) \left( \mathcal{H}_s^q(B) \right)^{\frac{q(n-1)+1-p+\varepsilon}{q}}.$$

**Proof of Claim 1:** Fix $1 \leq p \leq n$, $q \geq s(n-1) + 1 - p + \varepsilon$ and set $\Delta = \frac{2}{s}$. Let $Q_i, i = 1, \ldots, m$ be those Whitney cubes that intersect $B$. Fix one such Whitney cube $Q_{i_0}$ and let $x_{i_0}$ be its center. Let $Q_{i_0}^j, j = 1, \ldots, k$
be the Whitney cubes in $P(x_{i_0})$ with $Q^k_{i_0} = Q_{i_0}$. The standard chaining argument involving Poincaré inequality \[21\] gives us the estimate

$$1 \lesssim \sum_{j=1}^k \text{diam } Q^j_{i_0} \int_{Q^j_{i_0}} |\nabla u(y)| dy.$$ 

Hölder’s inequality implies

$$1 \lesssim \left( \sum_{j=0}^k r_j^{(1-\kappa)p/(p-1)} \right)^{(p-1)/p} \left( \sum_{j=0}^k r_j^{\kappa p-n} \int_{Q^j_{i_0}} |\nabla u|^p \right)^{1/p},$$

where $r_j = \text{diam } Q^j_{i_0}$ and $\kappa = \frac{s + p - 1 - \Delta}{sp}$. Using the s-John condition, one can easily conclude

$$\sum_{j=0}^k r_j^{(1-\kappa)p/(p-1)} < C.$$ 

Therefore,

\begin{equation}
\sum_{j=0}^k r_j^{\kappa p-n} \int_{Q^j_{i_0}} |\nabla u|^p \geq C,
\end{equation}

where the constant $C$ depends only on $p$, $n$, $\Delta$ and the constant from the s-John condition.

By the s-John condition $Cr_j \geq |x - y|^{s}$, for $y \in Q^j_{i_0}$, and since $\kappa p - n < 0$ according to our choice $p \leq n$, we obtain

$$r_j^{\kappa p-n} \lesssim |x - y|^{s(\kappa p-n)}$$

for $y \in Q^j_{i_0}$. For $y \in Q^j_{i_0} \cap (2^{j+1}Q_{i_0} \setminus 2^jQ_{i_0})$, we have $|x - y| \approx 2^j r_k$ and hence for such $y$,

\begin{equation}
r_j^{\kappa p-n} \lesssim (2^j r_k)^{s(\kappa p-n)}.
\end{equation}

Combining \(3.2\) with \(3.3\) leads to

$$1 \lesssim \sum_{j=0}^k r_j^{\kappa p-n} \int_{Q^j_{i_0}} |\nabla u|^p \lesssim (r_k)^{s(\kappa p-n)} \int_{Q_{i_0}} |\nabla u|^p$$

$$+ \sum_{j=0}^k (2^j r_k)^{s(\kappa p-n)} \int_{(2^{j+1}Q_{i_0} \setminus 2^jQ_{i_0}) \cap \Omega} |\nabla u|^p$$

$$\lesssim \sum_{l=0}^{\lfloor \log r_k \rfloor + 1} (2^l r_k)^{s(\kappa p-n)} \int_{2^l Q_{i_0} \cap \Omega} |\nabla u|^p.$$ 

On the other hand,

$$\sum_{l=0}^{\lfloor \log r_k \rfloor + 1} (2^l r_k)^{\Delta} < r_k^{\Delta} \sum_{l=-\infty}^{\lfloor \log r_k \rfloor + 1} 2^{l \Delta} < C.$$
Combining the above two estimates, we conclude that there exists an $l$ (depending on $\Delta$ and hence $\varepsilon$) such that
\[
(2^lr_k)\Delta \lesssim (2^lr_k)^{s(np-n)} \int_{2^Q_0 \cap \Omega} |\nabla u|^p.
\]
It follows that,
\[
\int_{\Omega \cap 2^Q_0} |\nabla u|^p \gtrsim (2^lr_k)^{s(np-n)+\Delta} = (2^lr_k)^{s(n-1)+1-p+\varepsilon}.
\]
In other words, there exists an $R_x \geq d(x, \partial \Omega)/2$ with
\[
\left(\int_{\Omega \cap B(x, R_x)} |\nabla u|^p\right) \lesssim \left(\int_{\Omega \cap B(x, R_x)} |\nabla u|^p\right)^{\frac{s(n-1)+1-p+\varepsilon}{q}} \gtrsim R_x^q.
\]
Applying the Vitali covering lemma to the covering $\{B(x, R_x)\}_{x \in E}$ of the set $\mathcal{B}$, we can select pairwise disjoint balls $B_1, \ldots, B_k, \ldots$ such that $\mathcal{B} \subset \bigcup_{i=1}^\infty 5B_i$. Let $r_i$ denote the radius of the ball $B_i$. Then
\[
\mathcal{H}^q_\infty(\mathcal{B}) \leq \sum_{i=1}^\infty (\text{diam } 5B_i)^q = 5^q \sum_{i=1}^\infty r_i^q \lesssim \sum_{i=1}^\infty \left(\int_{\Omega \cap B_i} |\nabla u|^p\right)^{\frac{s(n-1)+1-p+\varepsilon}{q}}.
\]
The desired capacity estimate follows by noticing the elementary inequality
\[
\sum_i a_i^b \lesssim \left(\sum_i a_i^b\right)^b, \quad b \geq 1.
\]

Claim 2: for $n - q < p \leq n$ and $0 < \varepsilon < p + q - n$,
\[
(3.4) \quad \int_{\Omega} |\nabla u(x)|^p dx \geq C(p, q, n, \varepsilon) \left(\mathcal{H}^q_\infty(\mathcal{G})\right)^{\frac{n-p+\varepsilon}{q}}.
\]

Proof of Claim 2: Fix $n - q < p \leq n$ and $0 < \varepsilon < p + q - n$. Our aim is to show that
\[
(3.5) \quad \int_{2Q(x)} |\nabla u(x)|^p dx \geq C(p, s, n) \mathcal{H}^s_\infty(\mathcal{G} \cap Q(x))
\]
for any $n - p < s \leq n$. We adapt the argument from [11, Proof of Theorem 5.9].

Fix $n - p < s \leq n$. For $y \in \mathcal{G}$, $u_{Q(y)} \leq \frac{1}{2}$. For $x \in \mathcal{G} \cap Q(y)$, write $Q_i = Q(x, r_i)$, where $r_i = 2^{-i-1} \text{diam } Q(y)$. Then
\[
u(x) = \lim_{i \to \infty} u_{Q_i} = \lim_{i \to \infty} \int_{Q_i} u.
\]
Now
\[
\frac{1}{2} \leq |u(x) - u_{Q_0}| \leq \sum_{i \geq 0} |u_{Q_i} - u_{Q_{i+1}}|.
\]
Since by the Poincaré inequality
\[ |u_{Q_i} - u_{Q_{i+1}}| \leq C(n) r_i \left( r_i^{-s} \int_{Q_i} |\nabla u|^p \right)^{\frac{1}{p}}, \]
we obtain that
\[ \frac{1}{2} \leq \sum_{i=1}^{\infty} C(n) r_i \left( r_i^{-s} \int_{Q_i} |\nabla u|^p \right)^{\frac{1}{p}} \leq C(p, s, n) (\text{diam } Q(y))^{\frac{p+q-n}{p}} \sup_{0<t\leq \text{diam } Q(y)} \left( t^{-s} \int_{Q(x,t)} |\nabla u|^p \right)^{\frac{1}{p}} \leq C(p, s, n) \sup_{0<t\leq \text{diam } Q(y)} \left( t^{-s} \int_{Q(x,t)} |\nabla u|^p \right)^{\frac{1}{p}}. \]
Thus, for each \( x \in \mathcal{G} \cap Q(y) \), there is a cube \( Q(x, t_x) \) such that \( t_x \leq \text{diam } Q(y) \) and that
\[ t_x^s \leq C(p, s, n) \int_{Q(x,t_x)} |\nabla u|^p. \]

By Vitali we can find pairwise disjoint cubes \( Q_1, Q_2, \ldots \) as above such that \( \mathcal{G} \cap Q(y) \subset \bigcup 5Q_i \). Then
\[ \mathcal{H}^s_\infty (\mathcal{G} \cap Q(y)) \leq C(p, s, n) \sum_{i=1}^{\infty} \int_{Q_i} |\nabla u|^p \leq C(p, s, n) \int_{2Q(y)} |\nabla u|^p. \]
Thus the proof of (3.5) is complete.

We next show that for \( n - q < p \leq n \) and for fixed \( 0 < \varepsilon < p + q - n \), the following estimate holds.
\[ (3.6) \quad \int_{2Q(x)} |\nabla u(x)|^p dx \geq C(p, q, n, \varepsilon) \left( \mathcal{H}^q_\infty (\mathcal{G} \cap Q(x)) \right)^{\frac{n-p+\varepsilon}{q}} \]
Let \( \varepsilon > 0 \) be as above. We set \( s = n - p + \varepsilon \). Then \( s < q \). Now (3.6) follows from (3.5) and the trivial estimate
\[ \left( \mathcal{H}^q_\infty (E) \right)^{\frac{1}{q}} \lesssim \mathcal{H}^s_\infty (E). \]
Taking into account the sub-additivity of Hausdorff \( q \)-content and concavity of the function \( t \mapsto t^{\frac{n-p+\varepsilon}{q}} \), (3.4) follows immediately from (3.6) and Lemma 2.1.

\[ \Box \]
4. Examples

Example 4.1. We will use the standard “rooms and corridors” type domains. This type of domains consists of a central cube shaped room along with an infinite disjoint collection of cube shaped rooms which are connected to the central room by narrow cylindrical corridors; see Figure 1.

For each \( j \in \mathbb{N} \), the attached cube shaped room \( E_j \) is of edge length \( r_j \) and the narrow cylindrical corridor is of radius \( r^s_j \) and height \( r_j \). We can ensure that the rooms and corridors are pairwise disjoint by requiring the sequence \( \{r_j\}_{j \in \mathbb{N}} \) to decrease to zero sufficiently rapidly.

It is clear that \( \Omega \) is an \( s \)-John domain.

![Figure 1. The standard “room and corridors” type domain](image)

For \( s < \frac{p+q-1}{n-1} \), we may choose \( \varepsilon > 0 \) such that \( q \geq s(n-1)+1-p+\varepsilon \). Then it is easy to obtain the following estimate:

\[
\text{Cap}_p(E_j, Q_0, \Omega) \leq C r_j^{(n-1)s-p+1} \leq C \mathcal{H}_\infty^q(E_j)^{\frac{(n-1)s-p+1}{q}}
\]

Noticing that \( r_j \to 0 \) as \( j \to \infty \), this implies that the exponent of \( \mathcal{H}_\infty^q(E) \) in Theorem 1.1 is essentially best possible.

Example 4.2. Fix \( p \in [1, n], n \geq 2 \). There exists an \( s \)-John domain \( \Omega \) in \( \mathbb{R}^n \) such that there is a sequence of compact sets \( E_j \) in \( \Omega \) with the following two properties:
Each $E_j$ is disjoint from the central Whitney cube $Q_0$;
- $\mathcal{H}_\infty^q(E_j)$ is bounded from below uniformly by a positive constant and $\text{Cap}_p(E_j, Q_0, \Omega) \to 0$ as $j \to \infty$, whenever $n - 1 \leq q < \min\{(n - 1)s + 1 - p, n\}$.

The idea of the construction of such an $s$-John domain is the following: we first construct a John domain $\Omega_0$ such that the number $N_j$ of Whitney cubes of size (comparable to) $r_j = 2^{-j}$ in $\Omega_0$ is approximately $2^{qj}$. We then build a “room and $s$-passage” $Q_s$ in each Whitney cube $Q \subset \Omega_0$ and $Q \neq Q_0$, where $Q_0$ is the central Whitney cube containing the John center; see Figure 3. If the Whitney cube $Q$ is of edge length $4r_j$, then the attached room shaped cube is of side length $r_j$ and the corresponding $s$-passage is of radius $r_j^s$ and height $r_j$.

![Figure 2. “room and $s$-passage” type replacement](image)

Let $E_j$ be the union of all the room shaped cube of edge length $r_j$. Then we have the following trivial upper estimate

$$\text{Cap}_p(E_j, Q_0, \Omega) \leq CN_j \cdot r_j^{(n-1)s-p+1} \leq Cr_j^{(n-1)s-p-q+1}.$$ 

Thus $\text{Cap}_p(E_j, Q_0, \Omega) \to 0$ whenever $q < (n - 1)s - p + 1$. On the other hand, noting that all the cubes in $E_j$ are well separated, to estimate the Hausdorff $q$-content, one has to cover each such cube by a ball of the same size (since otherwise the ball will intersects two cubes and substantially increases the radius). Thus we have

$$\mathcal{H}_\infty^q(E_j) \geq CN_j \cdot r_j^q \geq C.$$ 

To construct a John domain with the desired property, one essentially needs to construct a John domain $\Omega_0$ such that $\text{dim}_M(\partial \Omega_0) = q$ when $q \in [n - 1, n)$, where $\text{dim}_M$ denotes the upper Minkowski dimension. With this understood, one can select certain Von Koch type curve as the boundary of a John domain; see [10, Proposition 5.2] for the detailed construction of such a John domain $\Omega_0$. It is clearly that the “room and $s$-passage” type replacement described above turns $\Omega_0$ into an $s$-John domain $\Omega$. In fact, $\text{dim}_M(\partial \Omega_0) = \text{dim}_M(\partial \Omega) = q$. For these facts, see [10] Proposition 5.11 and Proposition 5.16.
Example 4.3. Fix $1 \leq p \leq n$. There exists an $s$-John domain, which is Gromov hyperbolic in the quasihyperbolic metric, such that there is a sequence of compact sets $E_j$ in $\Omega$ with the follow properties:

- Each $E_j$ is disjoint from the central Whitney cube $Q_0$;
- $H^q_\infty(E_j)$ is bounded from below uniformly by a positive constant and $\text{Cap}_p(E_j, Q_0, \Omega) \to 0$ as $j \to \infty$, whenever $q < \min\{(n - 1)s + 1 - p, \log_2(2^n - 1)\}$.

We first give a detailed construction of the $s$-John domain $\Omega$ in the plane with the desired properties. Fix $1 \leq p \leq 2$. We first consider the case $q = \log_2 3$. The $s$-John domain $\Omega$ will be constructed by an inductive process. In the first step, we have a unit cube $Q$ and four “room and $s$-passage” type “legs” as in Figure 3. The “$s$-passage” $R_1$ is a rectangle of length $2^{-1}$ and width $2^{-s-1}$ and the “room” $Q_1$ is a cube of edge-length $2^{-1}$. In the second step, we attach at each of the three corners of $Q_1$ a “room and $s$-passage” type “legs”. The “$s$-passage” $R_2$ is a rectangle of length $2^{-2}$ and width $2^{-2s-1}$ and the “room” $Q_2$ is a cube of edge-length $2^{-2}$. In general at step $j$, we have $4 \cdot 3^{j-1}$ “room and $s$-passage” type “legs”, where the “$s$-passage” $R_j$ is a rectangle of length $2^{-j}$ and width $2^{-js-1}$ and the “room” $Q_j$ is a cube of edge-length $2^{-j}$. It is easy to check that, with our choices of parameters, there is no overlap in our construction. Moreover, $\Omega$ is an $s$-John domain that is Gromov hyperbolic in the quasihyperbolic metric (since $\Omega$ is simply connected).

![Figure 3](image.png)

Figure 3. The $s$-John domain $\Omega \subset \mathbb{R}^2$

We choose $E_j$ to be the union of all the cubes at step $j$, i.e. the collection of $4 \cdot 3^{j-1}$ (disjoint) cubes of edge-length $2^{-j}$. Noting that all the cubes at step $j$ are well separated, to estimate the Hausdorff $q$-content, one has to cover each such cube by a ball of the same size.
(since otherwise the ball will intersects two cubes and substantially increases the radius). Note also that \( q = \log_2 3 \) and so it follows that
\[
\mathcal{H}^q_\infty (E_j) \geq C 4 \cdot 3^{j-1} \cdot 2^{-qj} = C.
\]
On the other hand,
\[
\text{Cap}_p(E_j, Q_0, \Omega) \leq C 4 \cdot 3^{j-1} \cdot 2^{-j(s-p+1)} \leq C 2^{-j(s-p-q+1)}.
\]
If \( q < s - p + 1 \), then \( \text{Cap}_p(E_j, Q_0, \Omega) \to 0 \) as \( j \to \infty \) as desired.

Next we consider the case \( q < \min\{ s - p + 1, \log_2 3 \} \). This case is easier and we only need to delete some “room and s-passage” type “legs” from the previous construction. To be more precise, we choose \( k_j \in \mathbb{N} \) to be an integer such that \( k_j - 1 \leq 2^{qj} \leq k_j \). The construction of the desired s-John domain can be proceeded in a similar way. In the first step, we have a unit cube \( Q \) and \( k_1 \) “room and s-passage” type “legs” as in the previous construction. The “s-passage” \( R_1 \) is a rectangle of length \( 2^{-1} \) and width \( 2^{-s-1} \) and the “room” \( Q_1 \) is a cube of edge-length \( 2^{-1} \). In the second step, we fix \( k_2 \) corners of all the cubes of edge-length \( 2^{-1} \) in step 1, and attach at each corner a “room and s-passage” type “legs”. The “s-passage” \( R_2 \) is a rectangle of length \( 2^{-2} \) and width \( 2^{-2s-1} \) and the “room” \( Q_2 \) is a cube of edge-length \( 2^{-2} \). In general at step \( j \), we have \( k_j \) “room and s-passage” type “legs”, where the “s-passage” \( R_j \) is a rectangle of length \( 2^{-j} \) and width \( 2^{-js-1} \) and the “room” \( Q_j \) is a cube of edge-length \( 2^{-j} \).

Let \( E_j \) be the union of all the cubes at step \( j \), i.e. the collection of \( k_j \) (disjoint) cubes of edge-length \( 2^{-j} \). It is clear that
\[
\mathcal{H}^q_\infty (E_j) \geq C k_j \cdot 2^{-qj} \geq C.
\]
On the other hand, we have
\[
\text{Cap}_p(E_j, Q_0, \Omega) \leq C k_j \cdot 2^{-j(s-p+1)} \leq C 2^{-j(s-p-q+1)}.
\]
If \( q < s - p + 1 \), then \( \text{Cap}_p(E_j, Q_0, \Omega) \to 0 \) as \( j \to \infty \) as desired.

We can construct similar examples in \( \mathbb{R}^n, n \geq 3 \). Fix \( 1 \leq p \leq n \). Consider the difficult case \( q = \log_2 (2^n - 1) \). The s-John domain \( \Omega \) will be constructed in a similar manner as before. In the first step, we have a unit cube \( Q \) and \( 2^n \) “room and s-passage” type “legs”. The “s-passage” \( R_1 \) is a cylinder of height \( 2^{-1} \) and radius \( 2^{-s-1} \) and the “room” \( Q_1 \) is a cube of edge-length \( 2^{-1} \). In the second step, we attach at each of the \( 2^n - 1 \) corners of \( Q_1 \) a “room and s-passage” type “legs”. The “s-passage” \( R_2 \) is a cylinder of height \( 2^{-2} \) and radius \( 2^{-2s-1} \) and the “room” \( Q_2 \) is a cube of edge-length \( 2^{-2} \). In general at step \( j \), we have \( 2^n \cdot (2^n - 1)^{j-1} \) “room and s-passage” type “legs”, where the “s-passage” \( R_j \) is a cylinder of height \( 2^{-j} \) and radius \( 2^{-js-1} \) and the “room” \( Q_j \) is a cube of edge-length \( 2^{-j} \). It is easy to check that, with our choices of parameters, there is no overlap in our construction. Moreover, \( \Omega \) is an
s-John domain that is Gromov hyperbolic in the quasihyperbolic metric. Indeed, one can easily verify that every quasihyperbolic geodesic triangle in $\Omega$ is $\delta$-thin for some $\delta < \infty$.

We choose $E_j$ to be the union of all the cubes at step $j$, i.e. the collection of $2^n \cdot (2^n - 1)^{j-1}$ (disjoint) cubes of edge-length $2^{-j}$. Note that $q = \log_2(2^n - 1)$ and we obtain that

$$\mathcal{H}_\infty^q(E_j) \geq C 2^n \cdot (2^n - 1)^{j-1} \cdot 2^{-qj} = C.$$

On the other hand,

$$\text{Cap}_p(E_j, Q_0, \Omega) \leq C 2^n \cdot (2^n - 1)^{j-1} \cdot 2^{-j[(n-1)s-p+1]} \leq C 2^{-j[(n-1)s-p-q+1]}.$$

If $q < (n-1)s-p+1$, then $\text{Cap}_p(E_j, Q_0, \Omega) \to 0$ as $j \to \infty$ as desired.

The case $q < \log_2(2^n - 1)$ can be proceeded as in the planar case by deleting the extra number of “room and $s$-passage” type “legs” and we leave the simple verification to the interested readers.

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