Abstract  We consider the two-dimensional one-component plasma (jellium) of mobile pointlike particles with the same charge $e$, interacting pairwisely by the logarithmic Coulomb potential and immersed in a fixed neutralizing background charge density. Particles are in thermal equilibrium at the inverse temperature $\beta$, the only relevant dimensionless parameter is the coupling constant $\Gamma \equiv \beta e^2$. In the bulk fluid regime and for any value of the coupling constant $\Gamma = 2 \times \text{integer}$, Šamaj and Percus [J. Stat. Phys. 80, 811–824 (1995)] have derived an infinite sequence of sum rules for the coefficients of the short-distance expansion of particle pair correlation function. In the context of the equivalent fractional quantum Hall effect, by using specific methods of quantum geometry Haldane [PRL 107, 116801 (2011) and arXiv:1112.0990v2] derived a self-dual relation for the Landau-level guiding-center structure factor. In this paper, we establish the relation between the guiding-center structure factor and the pair correlation function of jellium particles. It is shown that the self-dual formula, which provides an exact relation between the pair correlation function and its Fourier component, comes directly from the short-distance symmetry of the bulk jellium. The short-distance symmetry of pair correlations is extended to the semi-infinite geometry of a rectilinear plain hard wall with a fixed surface charge density, constraining particles to a half-space. The symmetry is derived for the original jellium model as well as its simplified version with no background charge (charged wall surface with “counter-ions only”). The obtained results are checked at the exactly solvable free-fermion coupling $\Gamma = 2$.

Keywords  Coulomb fluids · Jellium · Logarithmic interaction · Sum rules
1 Introduction

The study of statistical mechanics of classical (i.e. non-quantum) systems of particles interacting pairwisely via the Coulomb potential is of primary interest in many branches of condensed matter and soft matter physics.

In Gauss units and with the vacuum dielectric constant $\varepsilon = 1$, the Coulomb potential $\phi$ at point $r$ of an infinite Euclidean space of dimension $d$, induced by a unit charge at the origin $0$, is defined as the solution of the $d$-dimensional Poisson equation

$$\Delta \phi(r) = -s_d \delta(r)$$ (1.1)

supplemented with the boundary condition of vanishing electric field at infinity. Here, $s_d = 2\pi^{d/2}/\Gamma(d/2)$ ($\Gamma$ denotes the Gamma function \[6\]) is the surface area of the $d$-dimensional unit sphere. In particular,

$$\phi(r) = \begin{cases} -\ln(r/L) & \text{if } d = 2, \\ \frac{r^{2-d}}{d-2} & \text{otherwise}, \end{cases}$$ (1.2)

where $r = |r|$ and a free length scale $L$ fixes the zero of the two-dimensional (2D) Coulomb potential; for simplicity, we set $L = 1$. The Fourier component of such potential exhibits the characteristic form $1/k^2$ with singularity at $k = 0$ which keeps many generic properties of three-dimensional (3D) Coulomb systems with $1/r$ interaction potential. The interaction energy of two charges $q$ and $q'$ at the respective positions $r$ and $r'$ is given by $qq'\phi(|r-r'|)$. The 2D Coulomb system can be represented as parallel infinite charged lines interacting in 3D which are perpendicular to the given surface and as such are of practical interest in the field of polyelectrolytes.

From among numerous types of Coulomb models we shall concentrate on the so-called one-component plasma (OCP), or jellium, which represents a reasonable simplification of realistic systems of atomic nuclei and electrons. Jellium consists of mobile pointlike particles with equivalent (say elementary) charge $e$, immersed in a uniform neutralizing background charge density. The system is considered to be in thermal equilibrium at the temperature $T$, or the inverse temperature $\beta = 1/(k_B T)$. As for any Coulomb system, its thermal equilibrium is exactly solvable in the high-temperature region within the linear Debye-Hückel or nonlinear Poisson-Boltzmann mean-field theories \[3\]. The long-range tail of the Coulomb potential implies exact constraints (sum rules) on the moments of bulk particle correlation functions, see review \[15\]. In any spatial dimension, these sum rules include the zeroth- and second-moment Stillinger-Lovett conditions \[21,22\] and the fourth-moment (compressibility) condition \[2,23,24\]. In 2D, the sixth-moment condition was derived in Ref. \[14\]. Another kind of sum rule, relating two lowest-order coefficients of the short-distance expansion of the correlation function was derived for the 3D jellium by Jancovici \[9\].

In this paper, we shall concentrate on the 2D OCP. Its thermodynamics and particle correlation functions depend only on the coupling constant $\Gamma = \frac{1}{\beta}$. The study of statistical mechanics of classical (i.e. non-quantum) systems of particles interacting pairwisely via the Coulomb potential is of primary interest in many branches of condensed matter and soft matter physics.
\(\beta e^2\), the particle density scales appropriately distances. Besides the high-temperature Debye-Hückel limit \(\Gamma \to 0\), the 2D jellium is exactly solvable also at \(\Gamma = 2\) by mapping onto free fermions [10]. In the bulk regime, the two-body correlation functions decay exponentially at asymptotically large distances for \(\Gamma \to 0\) while the decay is Gaussian at \(\Gamma = 2\). The solvable cases involve also semi-infinite [11] or fully finite geometries, see reviews [4, 5, 13].

The extension of the 3D relation between the first two coefficients of the short-distance expansion of the correlation function [9] to 2D is nontrivial. For a sequence of the coupling constants \(\Gamma = 2 \times \text{integer}\), using a mapping of the 2D jellium onto a one-dimensional many-component anticommuting field theory on a discrete chain, a symmetry of the bulk two-body correlations with respect to a complex transformation of particle coordinates leads to a functional relation which implies an infinite sequence of the relations among the coefficients of the short-distance expansion of the correlation function [19]. Since only every second relation of the sequence is effective, this result does not provide an explicit form of the correlation function but strictly restricts its possible functional forms. The short-distance symmetry was extended to multi-particle correlations in Ref. [20].

In the fractional quantum Hall effect [16], the Hall conductance exhibits plateaux indexed via filling fractions of Landau levels. The partition function of the 2D OCP is formally the normalization factor of Laughlin’s proposal of the wave function [4]. In the context of the fractional quantum Hall effect, using specific methods of quantum geometry Haldane derived a self-dual relation for the Landau-level guiding-center structure function [7, 8].

In this paper it is shown that the guiding-center structure function is related to the two-body density of the 2D jellium. For any coupling \(\Gamma = 2 \times \text{integer}\), the counterpart of the self-dual formula comes directly from the short-distance symmetry of the pair correlations for the bulk 2D jellium. The self-dual formula relates the pair correlation function and its Fourier component. Another new result is the extension of the short-distance symmetry of pair correlations to the semi-infinite geometry of a charged rectilinear plain hard wall (line), constraining charged particles to a half-space. The extension is worked out for the original 2D jellium as well as its simplified version with no background charge, namely the charged line with “counter-ions only”. The obtained results are checked at the exactly solvable free-fermion coupling \(\Gamma = 2\).

The paper is organized as follows. In Sect. 2, we present within the canonical ensemble the definition of thermodynamic quantities of the 2D OCP constrained to an arbitrary domain and derive the general symmetry relation for two-body densities. The short-distance symmetry of the pair correlation function in the bulk regime is recapitulated in Sect. 3. Sect. 4 deals with the consequences of the studied symmetry in the Fourier space. The correlation-function counterpart of the guiding-center structure function from the fractional quantum Hall effect is defined and the self-dual relation between its Fourier and Euclidean pictures is derived. In Sect. 5, the short-distance symmetry of pair correlations is extended to the semi-infinite geometry of a rectilinear plain hard wall charged by a fixed line charge density. The original jellium model as well as its simplified version with no bulk background charge den-
sity (system with surface charge density and “counter-ions only”) are studied. The obtained results, valid for any \( \Gamma = 2 \times \text{integer} \), are checked at the exactly solvable free-fermion coupling \( \Gamma = 2 \). A short recapitulation is given in the concluding Sect. [9].

2 General symmetry

Let \( N \) pointlike mobile particles \( j = 1, 2, \ldots, N \) of charge \( e \) be constrained to a 2D domain \( A \) of surface \( |A| \) by plain hard walls (lines) which are located at the domain boundary \( \partial A \) and may carry a uniform line-charge density \( \sigma \) \( (\sigma = 0 \text{ in the case of neutral boundaries}) \). The domain points \( r = (x, y) \) will be often written in the complex notation

\[
z = x + iy, \quad \bar{z} = x - iy.
\]

A background charge density \( \rho_b \) is distributed uniformly over \( A \). The condition of overall charge neutrality reads as

\[
\rho_b |A| + Ne + \sigma |\partial A| e = 0.
\]

In this paper, we consider only infinite and semi-infinite domains with \( N \to \infty \) for which it holds that \( |\partial A|/|A| \to 0 \). Consequently, \( \rho_b = -en \) with \( n = N/|A| \) being the mean density of particles.

The dielectric constant of the walls \( \varepsilon_w \) is considered to be the same as the one \( \varepsilon \) of the medium in which the particles are immersed, say \( \varepsilon_w = \varepsilon = 1 \), so there are no image forces. Two particles at positions \( r \) and \( r' \) interact by the 2D Coulomb energy \(-e^2 \ln |r - r'|\). Let the electrostatic potential induced by uniform surface \( \rho \) and line \( \sigma \) charge densities at position \((z, \bar{z}) \in A\) be denoted by \( v(z, \bar{z}) \); the corresponding one-body Boltzmann factor at the inverse temperature \( \beta \) is \( w(z, \bar{z}) \equiv \exp[-\beta v(z, \bar{z})] \).

Due to the presence of the rigid neutralizing background, the jellium system is studied in the canonical ensemble. The partition function is given by

\[
Z_N = \frac{1}{N!} \int_A \prod_{j=1}^N d^2 r_j w(z_j, \bar{z}_j) \prod_{(j<k)=1}^N |z_j - z_k|^\Gamma, \tag{2.3}
\]

where \( \Gamma = \beta e^2 \) is the coupling constant. We study the special cases of \( \Gamma = 2\gamma \) where \( \gamma = 1, 2, \ldots \) is a (positive) integer. The canonical averaging is defined as

\[
\langle \cdots \rangle = \frac{1}{Z_N \cdot N!} \int_A \prod_{j=1}^N d^2 r_j w(z_j, \bar{z}_j) \prod_{(j<k)=1}^N |z_j - z_k|^\Gamma \cdots . \tag{2.4}
\]

The microscopic total number density of particles at point \( r \) is given by \( \hat{n}(r) = \sum_{j=1}^N \delta(r - r_j) \). At the one-particle level, one defines the average number density

\[
n(r) = \langle \hat{n}(r) \rangle. \tag{2.5}
\]
At the two-particle level, one introduces the two-body density
\[ n^{(2)}(\mathbf{r}, \mathbf{r}') = \left( \sum_{j \neq k} \delta(\mathbf{r} - \mathbf{r}_j)\delta(\mathbf{r}' - \mathbf{r}_k) \right) = \langle \hat{n}(\mathbf{r})\hat{n}(\mathbf{r}') \rangle - n(\mathbf{r})\delta(\mathbf{r} - \mathbf{r}') \] (2.6)
and the (truncated) pair correlation function
\[ b(\mathbf{r}, \mathbf{r}') = \frac{n^{(2)}(\mathbf{r}, \mathbf{r}')}{n(\mathbf{r})n(\mathbf{r}')} - 1 \] (2.7)
which vanishes at asymptotically large distances \(|\mathbf{r} - \mathbf{r}'| \to \infty\).

To derive the general symmetry of interest for two-body densities, in analogy with Ref. [19], we write down explicitly their integral representation from the definition (2.6):
\[
\frac{n^{(2)}(\mathbf{r}_1, \mathbf{r}_2)}{w(\mathbf{r}_1)w(\mathbf{r}_2)|\mathbf{r}_1 - \mathbf{r}_2|^2} = \frac{1}{2Z_N^2(N - 2)!} \int_{\Lambda} \prod_{j=3}^{N} d^2r_j w(z_j, \tilde{z}_j) \times |z_1 - z_j|^2 |z_2 - z_j|^2 \prod_{(j<k)=1}^{N} |z_j - z_k|^2. \] (2.8)

For every particle index \( j = 3, \ldots, N \), the product \(|z_1 - z_j|^2 |z_2 - z_j|^2 = (z_1 - z_j)(\tilde{z}_1 - \tilde{z}_j)(z_2 - z_j)(\tilde{z}_2 - \tilde{z}_j)\) is invariant with respect to the following transformation of particle coordinates
\[ z'_1 = z_2, \quad z'_1 = z_1, \quad z'_2 = z_1, \quad z'_2 = z_2. \] (2.9)

In the center-of-mass basis
\[ \mathbf{R} = \frac{1}{2} (\mathbf{r}_1 + \mathbf{r}_2), \quad \mathbf{r} = \mathbf{r}_1 - \mathbf{r}_2, \] (2.10)
this transformation takes the form
\[ \mathbf{R}' = \mathbf{R}, \quad \mathbf{r}' = i\mathbf{z} \times \mathbf{r}, \] (2.11)
where \( \mathbf{z} = \mathbf{x} \times \mathbf{y} \) is the unit vector perpendicular to the considered \((x, y)\) plane. Since the rhs of Eq. (2.8) is invariant with respect to the transformation (2.9), one arrives at the symmetry relation
\[
\frac{n^{(2)}(z_1, \tilde{z}_1, z_2, \tilde{z}_2)}{w(z_1, \tilde{z}_1)w(z_2, \tilde{z}_2) [(z_1 - \tilde{z}_2)(\tilde{z}_1 - z_2)]^2} = \frac{n^{(2)}(z_2, \tilde{z}_1, z_1, \tilde{z}_2)}{w(z_2, \tilde{z}_1)w(z_1, \tilde{z}_2) [(z_2 - \tilde{z}_1)(\tilde{z}_2 - z_1)]^2}. \] (2.12)

where the two-body function \( w(z, \tilde{z}_k) \) with \( j \neq k \) is the obvious generalization of the one-body Boltzmann factor \( w(z \to z_j, \tilde{z} \to \tilde{z}_k) \). The relation (2.12) can be simplified to the one
\[
\frac{n^{(2)}(z_1, \tilde{z}_1, z_2, \tilde{z}_2)}{w(z_1, \tilde{z}_1)w(z_2, \tilde{z}_2)} = (-1)^3 \frac{n^{(2)}(z_2, \tilde{z}_1, z_1, \tilde{z}_2)}{w(z_2, \tilde{z}_1)w(z_1, \tilde{z}_2)}. \] (2.13)

The practical realization of this symmetry formula depends on the form of \( \Lambda \)-domain, which manifests itself via the specific form of the one-body Boltzmann factor \( w(z, \tilde{z}) \) and the coordinate dependence of the two-body density.
3 Bulk regime

For an infinite Euclidean surface, the mean density of particles is constant, \( n(r) = n \) and \( \rho_b = -en \), so the electroneutrality constraint is local. Let us assume a circular dependence of the electrostatic potential induced by the homogeneous (infinite) background, i.e. the corresponding one-body energy of particles is given by the radial Poisson equation

\[
\frac{1}{r} \frac{d}{dr} \left[ r \frac{dn(r)}{dr} \right] = 2\pi ne. \tag{3.1}
\]

The solution of this differential equation is \( u(r) = \pi ner^2/2 \), so the one-body Boltzmann factor reads as

\[
w(z, \bar{z}) = \exp \left( -\frac{1}{2} \gamma \pi nz \bar{z} \right). \tag{3.2}
\]

The particle density \( n \) is the parameter which only scales appropriately the distances. We define the length

\[
a = \frac{1}{\sqrt{\gamma \pi n}} \tag{3.3}
\]

and express all distances in units of \( a \); in those units, \( n = 1/(\pi \gamma) \). The two-body generalization of the one-body Boltzmann factor \( w(z, \bar{z}) = \exp(-nz \bar{z}) \) reads as

\[
w(z_j, \bar{z}_k) = \exp (-z_j \bar{z}_k). \tag{3.4}
\]

It is natural to expect that the statistical mean values of an infinite system do not depend on the gauge of the background potential, so that the radius-dependent potential \( u(r) \) induces the uniform particle density; the mathematical formalism which is beyond this phenomenon was developed in Ref. [20]. The two-body densities are translational invariant, i.e., they depend on the distance between the two points: \( n^{(2)}(r, r') = n^{(2)}(|r - r'|) \) and \( h(r, r') = h(|r - r'|) \).

The two-body density \( n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) = n^{(2)}(|z_1 - z_2|) \) is at small distances \( |z_1 - z_2| \to 0 \) proportional to the interaction Boltzmann factor of the two particles \( |z_1 - z_2|^{2\gamma} \) and the remaining part of its short-distance expansion is analytic in \( |z_1 - z_2|^2 \) [19]:

\[
n^{(2)}(|z_1 - z_2|) = [(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)]^{\gamma} \sum_{j=0}^{\infty} a_j(\gamma) [(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)]^j, \tag{3.5}
\]

where the expansion coefficients \( \{a_j\} \) depend on \( \gamma \). Considering the symmetry relation (2.12) with the generalized Boltzmann factor [34], one obtains the
Denoting the particle distance as \( r = \sqrt{(z_1 - z_2)(\bar{z}_1 - \bar{z}_2)} \), the preceding two relations can be written in a simpler way as

\[
n^{(2)}(r) = r^{2\gamma} \sum_{j=0}^{\infty} a_j(\gamma) r^{2j}, \tag{3.7}
\]

\[
\sum_{j=0}^{\infty} a_j(\gamma) r^{2j} = e^{-r^2} \sum_{j=0}^{\infty} a_j(\gamma) (-1)^j r^{2j}. \tag{3.8}
\]

Since the function \( n^{(2)}(ir) \) of the purely imaginary distance \( ir \) is well defined in terms of the small-\( r \) expansion as follows

\[
n^{(2)}(ir) = (ir)^{2\gamma} \sum_{j=0}^{\infty} a_j(\gamma) (ir)^{2j}, \tag{3.9}
\]

the relation (3.8) is equivalent to the one

\[
n^{(2)}(r) = (-1)^\gamma e^{-r^2} n^{(2)}(ir). \tag{3.10}
\]

If applied twice, it is an identity. The analogous formula for the pair correlation function reads as

\[
h(r) = -1 + (-1)^\gamma e^{-r^2} + (-1)^\gamma e^{-r^2} h(ir). \tag{3.11}
\]

The relation (3.8) can be reexpressed as

\[
e^{r^2/2} \sum_{j=0}^{\infty} a_j(\gamma) r^{2j} = e^{-r^2/2} \sum_{j=0}^{\infty} a_j(\gamma) (-r^2)^j
\]

\[
= e^{(ir)^2/2} \sum_{j=0}^{\infty} a_j(\gamma) (ir)^{2j}. \tag{3.12}
\]

Consequently,

\[
e^{r^2/2} \sum_{j=0}^{\infty} a_j(\gamma) r^{2j} = \sum_{j=0}^{\infty} b_j(\gamma) r^{4j} \tag{3.13}
\]
with some new expansion coefficients \{b_j\}. We conclude that possible short-
distance functional forms of \(n^{(2)}(r)\) reduce themselves to

\[
n^{(2)}(r) = e^{-r^2 / 2}r^{2\gamma} \sum_{j=0}^{\infty} b_j(\gamma)r^{4j}.
\]  

(3.14)

The pair correlation function is expressible as

\[
h(r) = -1 + e^{-r^2 / 2}r^{2\gamma} \sum_{j=0}^{\infty} c_j(\gamma)r^{4j},
\]

(3.15)

where \(c_j(\gamma) = b_j(\gamma)/n^2\). In particular, for the exactly solvable \(\gamma = 1\) with \(h(r) = -e^{-r^2}\) it is easy to verify that

\[
c_j(1) = \frac{1}{(2j + 1)!} \frac{1}{2^{2j}}.
\]

(3.16)

The pair correlation function \(h(r)\) should vanish as \(r \to \infty\), i.e.

\[
\lim_{r \to \infty} e^{-r^2 / 2}r^{2\gamma} \sum_{j=0}^{\infty} c_j(\gamma)r^{4j} = 1.
\]

(3.17)

Since it holds that

\[
\lim_{r \to \infty} e^{-r^2 / 2}r^{2\gamma} \sum_{j=0}^{\infty} \frac{1}{(2j + \gamma)!} \frac{2}{2^{2j + \gamma}}r^{4j} = 1,
\]

(3.18)

the coefficients with asymptotically large indices should go to

\[
\lim_{j \to \infty} c_j(\gamma) = \frac{1}{(2j + \gamma)!2^{2j + \gamma}}.
\]

(3.19)

Note that for \(\gamma = 1\) the \(c\)-coefficients (3.16) are in fact equal to this asymptotic prediction. We subtract from \(c_j(\gamma)\) their asymptotic values in order to ensure the series convergence of \(h(r)\):

\[
h(r) = e^{-r^2 / 2}r^{2\gamma} \sum_{j=0}^{\infty} \left[ c_j(\gamma) - \frac{1}{(2j + \gamma)!2^{2j + \gamma}} \right] r^{4j} + f(r),
\]

(3.20)

where

\[
f(r) = e^{-r^2 / 2}r^{2\gamma} \sum_{j=0}^{\infty} \frac{1}{(2j + \gamma)!2^{2j + \gamma}} r^{4j} - 1.
\]

(3.21)

The explicit form of the function \(f(r)\) depends on whether \(\gamma\) is odd or even. In particular, for \(\gamma = 2g + 1\) \((g = 0, 1, 2, \ldots)\) one has

\[
f(r) = -e^{-r^2} - e^{-r^2 / 2} \sum_{j=0}^{g-1} \frac{1}{(2j + 1)!2^{2j}} r^{2(2j+1)}
\]

(3.22)
and for $\gamma = 2g$ ($g = 1, 2, \ldots$)
\begin{equation}
    f(r) = e^{-r^2} - e^{-r^2/2} \sum_{j=0}^{g-1} \frac{1}{(2j)!2^{2j-1}}r^{2j}.
    \tag{3.23}
\end{equation}

To simplify the notation, one defines
\begin{equation}
    d_j(\gamma) \equiv c_j(\gamma) - \frac{1}{(2j + \gamma)!2^{2j+\gamma-1}}.
    \tag{3.24}
\end{equation}

4 Self-dual relation

In this paper, the 2D Fourier transform $\tilde{f}(q)$ of the function $f(r)$ is defined by
\begin{equation}
    f(r) = \frac{1}{2} \int \frac{d^2 q}{2\pi} \tilde{f}(q)e^{-iqr}, \quad \tilde{f}(q) = 2 \int \frac{d^2 r}{2\pi} h(r)e^{iqr}.
    \tag{4.1}
\end{equation}

The prefactors $1/2$ and $2$ are introduced to simplify layout of final formulae.

Our next goal is to formulate the bulk symmetry relations, derived in the previous Sect. 3, in the Fourier space. Using (3.11) it holds that
\begin{equation}
    e^{q^2/4}\tilde{h}(q) = 2 \int \frac{d^2 r}{2\pi} h(r)e^{q^2/4+iqr}
    = 2 \int \frac{d^2 r}{2\pi} \left[ -e^{q^2/4} + (-1)^\gamma h(i\gamma) \right] e^{q^2/4}. \tag{4.2}
\end{equation}

This expression allows us to write
\begin{equation}
    e^{q^2/4}\tilde{h}(q) = (-1)^\gamma + \tilde{g}(q), \tag{4.3}
\end{equation}

where
\begin{align*}
    \tilde{g}(q) &= 2 \int \frac{d^2 r}{2\pi} \left[ (-1)^\gamma h(i\gamma) - e^{q^2/4} \right] e^{q^2/4}
    = 2 \int \frac{d^2 r}{2\pi} \left[ (-1)^\gamma h(i\gamma - q/2) - e^{(r+iq/2)^2} \right] e^{q^2/4}. \tag{4.4}
\end{align*}

The function $\tilde{g}(q)$ is related to the guiding-center structure factor $\hat{s}(q) - \hat{s}_\infty$ \cite{7,8} as follows $\tilde{g}(q) = \gamma^2[\hat{s}(q) - \hat{s}_\infty]$. As $q \to \infty$, $\tilde{g}(q)$ goes evidently to 0, since the correlation function $h$ vanishes for large distances and also $e^{(r+iq/2)^2}$ goes to 0 when $q \to \infty$. It is evident from Eq. (4.3) that this fact permits one to determine the large-$q$ asymptotic behavior of $h(q)$ for any $\gamma = 1, 2, \ldots$:
\begin{equation}
    \tilde{h}(q) = (-1)^\gamma e^{-q^2/4} + o\left(e^{-q^2/4}\right). \tag{4.5}
\end{equation}

For the exactly solvable $\gamma = 1$ case \cite{10}, which corresponds to
\begin{equation}
    h(r) = -e^{-r^2}, \quad \tilde{h}(q) = -e^{-q^2/4}, \tag{4.6}
\end{equation}
we have trivially \( \tilde{g}(q) = 0 \).

Haldane [7,8] has shown by using specific methods of quantum geometry that the guiding-center structure factor satisfies a self-dual relation between its Fourier component and the direct picture in the 2D Euclidean space. In what follows, we shall rederive this self-dual relation for the related \( g \)-function in a more direct way by using the representation (4.4) which can be expressed as follows

\[
(-1)^\gamma \tilde{g}(q) = I(q) - 2(-1)^\gamma \int \frac{d^2r}{2\pi} e^{(r+iq/2)^2} e^{-r^2},
\]

where \( I(q) \) is the integral

\[
I(q) = 2 \int \frac{d^2r}{2\pi} h(\mathbf{r} - \mathbf{q}/2) e^{-r^2}.
\]

This integral can be manipulated in the following way

\[
I(q) = \int \frac{d^2q'}{2\pi} \tilde{h}(\mathbf{q}') e^{i\mathbf{q}' \cdot \mathbf{q}/2} + \frac{1}{\gamma} \int \frac{d^2q'}{2\pi} e^{i\mathbf{q}' \cdot \mathbf{q}/2} \frac{1}{2} e^{\mathbf{q}'^2/4}.
\]

Inserting here the representation (4.3), one ends up with

\[
I(q) = \frac{1}{2} \int \frac{d^2q'}{2\pi} \tilde{g}(\mathbf{q}') e^{i\mathbf{q}' \cdot \mathbf{q}/2} + \frac{1}{2} (-1)^\gamma \int \frac{d^2q'}{2\pi} e^{i\mathbf{q}' \cdot \mathbf{q}/2}.
\]

Considering this relation in (4.7), one gets

\[
(-1)^\gamma \tilde{g}(q) = \frac{1}{2} \int \frac{d^2q'}{2\pi} \tilde{g}(\mathbf{q}') e^{i\mathbf{q}' \cdot \mathbf{q}/2} + \frac{1}{2} (-1)^\gamma \int \frac{d^2q'}{2\pi} e^{i\mathbf{q}' \cdot \mathbf{q}/2} - 2(-1)^\gamma \int \frac{d^2r}{2\pi} e^{i\mathbf{r} \cdot \mathbf{q} - \mathbf{q}/2^2}. \tag{4.11}
\]

The last two terms are proportional to the Dirac \( \delta(q) \) and they cancel with one another. Thus one arrives at the self-dual relation for the \( g \)-function

\[
(-1)^\gamma \tilde{g}(q) = \frac{1}{2} \int \frac{d^2q'}{2\pi} \tilde{g}(\mathbf{q}') e^{i\mathbf{q}' \cdot \mathbf{q}/2}.
\]

This self-dual formula relates the Fourier and Euclidean pictures of the \( g \)-function in the following way

\[
\tilde{g}(q) = (-1)^\gamma g(r = q/2).
\]

Next aim is to incorporate the general symmetry formulae (3.20)–(3.24) for the pair correlation function \( h(r) \) into the ones formulated in the Fourier space for the function \( \tilde{g}(q) \) in such a way that the self-dual relation (4.13) be automatically satisfied. The procedure depends on whether \( \gamma \) is an odd or even integer.
4.1 $\gamma = 2g + 1$

Let us first treat the case $\gamma = 2g + 1$ ($g = 1, 2, \ldots$) for which it holds that

$$h(r) + e^{-r^2} = e^{-r^2/2} \sum_{j=0}^{\infty} d_j r^{2(1+j)}/2^j.$$ 

(4.14)

One goes to the Fourier space by writing $d^2 r = r dr d\phi$ and using the definition of the Bessel function [6]

$$\int_0^{2\pi} \frac{d\varphi}{2\pi} e^{jq r \cos \varphi} = J_0(q r),$$

(4.15)

to obtain

$$\tilde{h}(q) + e^{-q^2/4} = \sum_{j=0}^{\infty} d_j 2^{(1+j)+g} \int_0^\infty dt e^{-t} t^{1+2(j+g)} J_0 \left(2\sqrt{\frac{q^2}{2} t} \right)$$

$$-4 \sum_{j=0}^{g-1} \frac{1}{(2j+1)!} \int_0^\infty dt e^{-t} t^{1+2j} J_0 \left(2\sqrt{\frac{q^2}{2} t} \right).$$

(4.16)

Applying the relation

$$\int_0^\infty dt e^{-t} L_j(2\sqrt{rt}) = e^{-r} n! L_j(r)$$

(4.17)

with $\{L_j(r)\}_{j=0}^{\infty}$ being the standard Laguerre polynomials [6], with regard to the definition (4.3) one gets the following representation of the $g$-function:

$$\tilde{g}(q) = e^{-q^2/4} \sum_{j=0}^{\infty} d_j 2^{(1+j)+g} [1 + 2(j + g)]! L_{1+2(j+g)}(q^2/2)$$

$$-4e^{-q^2/4} \sum_{j=0}^{g-1} L_{1+2j}(q^2/2).$$

(4.18)

This representation automatically fulfills the self-dual relation (4.12). Indeed, inserting into the equality

$$\frac{1}{2} \int \frac{d^2 q'}{2\pi} \tilde{g}(q') e^{i q' q/2} = \frac{1}{4} \int_0^\infty dt \tilde{g}(\sqrt{t}) J_0 \left(\frac{q}{2\sqrt{t}} \right),$$

(4.19)

the representation (4.18) and using the formula [6]

$$\int_0^\infty dt e^{-t} L_{1+2n}(2t) J_0(2\sqrt{rt}) = -e^{-r} L_{1+2n}(2r)$$

(4.20)
one recovers the self-dual relation (4.12) with \((-1)^\gamma = -1\). The representation (4.18) is in fact the most general series representation of \(\tilde{g}(q)\) which satisfies the self-dual relation.

The Laguerre polynomials satisfy the orthogonality relations \([6]\)
\[
\int_0^\infty dr \, e^{-r} L_n(r) L_m(r) = \delta_{nm} = \int_0^\infty dq \, q e^{-q^2/2} L_n(q^2/2) L_m(q^2/2). \quad (4.21)
\]

Multiplying the representation of \(\tilde{g}(q)\) (4.18) with \(q e^{-q^2/4} L_{2j}(q^2/2)\) \((j = 1, 2, \ldots)\) and integrating over \(q\) from 0 to \(\infty\), these orthogonality relations imply an infinite sequence of zero integrals
\[
\int_0^\infty dq \, q e^{-q^2/4} \tilde{g}(q) L_{2j}(q^2/2) = 0 \quad \text{for } j = 0, 1, 2, \ldots \quad (4.22)
\]
On the other hand, the multiplication of (4.18) with \(q e^{-q^2/4} L_{2j+1}(q^2/2)\) \((j = 1, 2, \ldots)\) and the consequent integration over \(q\) implies that
\[
\int_0^\infty dq \, q e^{-q^2/4} \tilde{g}(q) L_{2j+1}(q^2/2) = -4 \quad \text{for } j = 0, 1, \ldots, g - 1,
\]
\[
\int_0^\infty dq \, q e^{-q^2/4} \tilde{g}(q) L_{2j+1}(q^2/2) = d_j - g 2^{2(j+1)} (2j + 1)! \quad \text{for } j = g, g + 1, \ldots \quad (4.23)
\]

4.2 \(\gamma = 2g\)

If \(\gamma = 2g\) \((g = 1, 2, \ldots)\), one has
\[
h(r) - e^{-r^2} = e^{-r^2/2} \sum_{j=0}^\infty d_j 4^{(j+g)} - e^{-r^2/2} \sum_{j=0}^{g-1} \frac{1}{(2j)!2^{2j-1}} L_{4j}. \quad (4.24)
\]
The Fourier transform of this equation implies
\[
\tilde{g}(q) = e^{-q^2/4} \sum_{j=0}^\infty \frac{d_j 2^{1+2(j+g)} [2(j+g)]! L_{2(j+g)}(q^2/2)}{L_{2j}(q^2/2)}
\]
\[
-4 e^{-q^2/4} \sum_{j=0}^{g-1} L_{2j}(q^2/2). \quad (4.25)
\]

As before, this representation automatically fulfills the self-dual relation (4.12) with \((-1)^\gamma = -1\). This can be shown by inserting the representation (4.25) into (4.19) and by using the formula \([6]\)
\[
\int_0^\infty dt \, e^{-t} L_{2n}(2t) J_0(2\sqrt{rt}) = +e^{-r} L_{2n}(2r); \quad (4.26)
\]
note the plus sign in comparison with (4.20).
Multiplying the representation of $\tilde{g}(q)$ \[^{4.25}\text{(4.25)}\] with $q e^{-q^2/4} L_{2j+1}(q^2/2)$ ($j = 1, 2, \ldots$) and integrating over $q$, the orthogonality relations \[^{4.21}\text{(4.21)}\] imply an infinite sequence of zero integrals

$$\int_0^\infty dq q e^{-q^2/4} \tilde{g}(q) L_{2j+1}(q^2/2) = 0 \quad \text{for } j = 0, 1, 2, \ldots \quad (4.27)$$

The multiplication of \[^{4.25}\text{(4.25)}\] with $q e^{-q^2/4} L_{2j}(q^2/2)$ ($j = 1, 2, \ldots$) and the consequent integration over $q$ leads to

$$\int_0^\infty dq q e^{-q^2/4} \tilde{g}(q) L_{2j}(q^2/2) = -4 \quad \text{for } j = 0, 1, \ldots, g - 1,$$

$$\int_0^\infty dq q e^{-q^2/4} \tilde{g}(q) L_{2j}(q^2/2) = d_{j-g} 2^{j+1}(2j)! \quad \text{for } j = g, g + 1, \ldots \quad (4.28)$$

5 Semi-infinite geometry

Let us now consider the 2D geometry of the plain hard wall in the half-space $x < 0$ and the charged particles constrained to the complementary half-space $x \geq 0$. The system is infinite in the $y$-direction, $y \in (-\infty, \infty)$. The wall surface at $x = 0$ is charged by a fixed “line” charge density $-e\sigma$. The two-body density $n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2)$ is translationally invariant along the $y$-axis, i.e. it depends on $|y_1 - y_2|$. As concerns the $x$-axis, taking into account the particle interchangeability, in the center-of-mass basis \[^{2.11}\text{(2.11)}\] the two-body density depends on $(x_1 + x_2)/2$ and $|x_1 - x_2|$. Thus

$$n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) \equiv n^{(2)} \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right). \quad (5.1)$$

Like in the bulk case, the two-body density is proportional to the interaction Boltzmann factor of the two particles $|z_1 - z_2|^\gamma$ when $|z_1 - z_2| \to 0$. The remaining part of the short-distance expansion is assumed to be analytic in small quantities $(x_1 - x_2)^2$ and $(y_1 - y_2)^2$:

$$n^{(2)} \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) = \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{\gamma} \times \sum_{j,k=0}^{\infty} a_{jk} \left( \frac{x_1 + x_2}{2} \right)^j (x_1 - x_2)^2k (y_1 - y_2)^{2k}. \quad (5.2)$$

The expansion coefficients $a_{jk}$ depend on the $x$-component of the center-of-mass which is not small but can be any positive number.

Under the transformation of particle coordinates \[^{2.9}\text{(2.9)}\], the coordinate combination

$$\frac{1}{2} (x_1 + x_2) = \frac{1}{4} (z_1 + \bar{z}_1 + z_2 + \bar{z}_2) \quad (5.3)$$
remains invariant while
\[ x_1 - x_2 = \frac{1}{2} (z_1 + \bar{z}_1 - z_2 - \bar{z}_2) \rightarrow \frac{1}{2} (z'_2 + \bar{z}'_1 - z'_1 - \bar{z}'_2) = i(y'_2 - y'_1), \]
\[ y_1 - y_2 = \frac{1}{2i} (z_1 - \bar{z}_1 - z_2 + \bar{z}_2) \rightarrow \frac{1}{2i} (z'_2 - \bar{z}'_1 - z'_1 + \bar{z}'_2) = i(x'_1 - x'_2). \]

The symmetry relation (2.13) then takes the form
\[ n^{(2)} \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) \rightarrow \frac{1}{2} \left( z_1 - \bar{z}_1 - z_2 + \bar{z}_2 \right) \rightarrow \frac{1}{2i} \left( z'_2 - \bar{z}'_1 - z'_1 + \bar{z}'_2 \right) = i(y'_2 - y'_1), \]
\[ y_1 - y_2 = \frac{1}{2i} (z_1 - \bar{z}_1 - z_2 + \bar{z}_2) \rightarrow \frac{1}{2i} (z'_2 - \bar{z}'_1 - z'_1 + \bar{z}'_2) = i(x'_1 - x'_2). \]

Two kinds of OCP Coulomb systems will be considered. The first “dense” one is the standard jellium with the fixed bulk and wall surface charge densities. The second “sparse” model corresponds to the special case of the OCP with no bulk background charge density, i.e. the neutral system of counterions to the charged wall surface. For each of these models, we start with a general theory valid for any positive integer \( \gamma \) and then verify the obtained results at the free fermion point \( \gamma = 1 \).

5.1 OCP

5.1.1 General theory

We consider the standard 2D OCP with a fixed volume background charge density \(-en\) in the half-space \( x > 0 \) and line charge density \(-e\sigma\) at the wall surface \( x = 0 \). The electrostatic potential induced by the background charge density, given by the Poisson equation
\[ \frac{d^2 u(x)}{dx^2} = 2\pi ne, \]
reads as \( u_1(x) = \pi ne x^2 \). The potential induced by the line charge density is \( u_2(x) = \pi \sigma ex \). The corresponding one-body Boltzmann factor is
\[ w(z, \bar{z}) = \exp \left[ -\Gamma \pi n \left( \frac{z + \bar{z}}{2} \right)^2 - \Gamma \pi \sigma \left( \frac{z + \bar{z}}{2} \right) \right]. \]
Expressing all distances in units of length \( a \) (3.3), the generalized Boltzmann factor reads as
\[ w(z_j, \bar{z}_k) = \exp \left[ -\frac{1}{2} (z_j + \bar{z}_k)^2 - \gamma \pi \sigma (z_j + \bar{z}_k) \right], \]
where \( \sigma \) is the dimensionless line charge density \( \sigma / \sqrt{\gamma \pi n} \), in units of \( \gamma \pi n = 1 \).

The symmetry formula (2.13) leads to the relation
\[ n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) = (-1)^\gamma e^{-|z_1 - z_2|^2} n^{(2)}(z_2, \bar{z}_1, z_1, \bar{z}_2). \]
The equivalent formula (5.5) implies that
\[
n^{(2)} \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) = (-1)^\gamma e^{-[(x_1-x_2)^2+(y_1-y_2)^2]} 
\times n^{(2)} \left( \frac{x_1 + x_2}{2}, i(y_2 - y_1), i(x_1 - x_2) \right). \tag{5.10}
\]

Note that the last two equations do not depend explicitly on \( \sigma \). Introducing the auxiliary function
\[
f \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) = e^{\frac{1}{2}[x_1 - x_2]^2 + (y_1 - y_2)^2} 
\times n^{(2)} \left( \frac{x_1 + x_2}{2}, i(y_2 - y_1), i(x_1 - x_2) \right), \tag{5.11}
\]
the symmetry relation (5.10) can be rewritten as
\[
f \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) = (-1)^\gamma f \left( \frac{x_1 + x_2}{2}, i(y_2 - y_1), i(x_1 - x_2) \right). \tag{5.12}
\]

The short-distance expansion of the two-body density (5.2) leads to a similar expansion for the \( f \)-function:
\[
f \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) = \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{\gamma} 
\times \sum_{j,k=0}^{\infty} b_{jk} \left( \frac{x_1 + x_2}{2} \right) (x_1 - x_2)^j(y_1 - y_2)^{2k} \tag{5.13}
\]
with some other expansion coefficients \( b_{jk} \) which depend on the \( x \)-component of the center-of-mass of the two particles. Eq. (5.12) then implies the following relation between the coefficients \( b_{jk} \):
\[
b_{jk} \left( \frac{x_1 + x_2}{2} \right) = (-1)^{j+k} b_{kj} \left( \frac{x_1 + x_2}{2} \right). \tag{5.14}
\]

This symmetry has no effect on diagonal coefficients \( b_{jj} \) and reduces the number of independent off-diagonal coefficients \( b_{jk} \) (\( j \neq k \)) by two. Thus the most general form of the two-body density which accounts for the present symmetry reads as
\[
n^{(2)} \left( \frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2 \right) = e^{-\frac{1}{2}[x_1 - x_2]^2 + (y_1 - y_2)^2} 
\times \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^{\gamma} 
\times \left\{ \sum_{j} b_{jj} \left( \frac{x_1 + x_2}{2} \right) [x_1 - x_2]^j(y_1 - y_2)^{2j} \right. 
+ \sum_{j<k} b_{jk} \left( \frac{x_1 + x_2}{2} \right) [(x_1 - x_2)^j(y_1 - y_2)^{2k} 
\left. + (-1)^{j+k}(x_1 - x_2)^{2k}(y_1 - y_2)^{2j} \right] \right\}, \tag{5.15}
\]
where the summation indices run over integers from 0 to \( \infty \).
5.1.2 Free-fermion point

The semi-infinite 2D OCP at the free-fermion coupling $\Gamma = 2$ was solved by Jancovici [11]. In units of $\pi n = 1$, the particle density profile was obtained in the form

$$\frac{n(x)}{n} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{t} \frac{dt}{1 + \phi(t)} e^{-\left(t - x \sqrt{2}\right)^2}, \quad (5.16)$$

where

$$\phi(t) = \frac{2}{\sqrt{\pi}} \int_{0}^{t} du e^{-u^2} \quad (5.17)$$

is the error function [6]. The function

$$\frac{n(z_j, \bar{z}_k)}{n} = \frac{2}{\sqrt{\pi}} \int_{-\infty}^{\infty} \frac{dt}{1 + \phi(t)} e^{-\left(t - z_j + \bar{z}_k \sqrt{2}\right)^2} \quad (5.18)$$

which is a two-point generalization of the density function (5.16), $n(x) = n(z, \bar{z})$. The two-body density is then expressible as [11]

$$n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) = n(z_1, \bar{z}_1)n(z_2, \bar{z}_2) - e^{-|z_1 - z_2|^2} n(z_2, \bar{z}_1)n(z_1, \bar{z}_2). \quad (5.19)$$

The symmetry relation (5.9) then evidently holds. The auxiliary function (5.11) is expressible as

$$\int_{\infty}^{t} \frac{dt}{1 + \phi(t)} e^{-\left(t - s \sqrt{2}\right)^2} \quad (5.21)$$

where the summands with odd powers of $(t - s)$ disappear as a result of the $t \leftrightarrow s$ symmetry of the kernel. Using that

$$(x_1 - x_2)^{2j} - (-1)^j(y_1 - y_2)^{2j} = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{j-1} - (-1)^j(y_1 - y_2)^{2j} \quad (5.22)$$

where the summands with odd powers of $(t - s)$ disappear as a result of the $t \leftrightarrow s$ symmetry of the kernel. Using that

$$(x_1 - x_2)^{2j} - (-1)^j(y_1 - y_2)^{2j} = [(x_1 - x_2)^2 + (y_1 - y_2)^2]^{j-1} - (-1)^j(y_1 - y_2)^{2j} \quad (5.22)$$
the coefficients $b_{jk}$ of the expansion (5.13) are expressible as
\[
b_{jk}(x_1 + x_2^2) = \frac{(2n)^2}{\pi} \int_{-\pi \sigma \sqrt{2}}^{\infty} \frac{dt}{1 + \phi(t)} e^{-\left(t - \frac{x_1 + x_2^2}{2}\right)^2} \times \int_{-\pi \sigma \sqrt{2}}^{\infty} \frac{ds}{1 + \phi(s)} e^{-\left(s - \frac{x_1 + x_2^2}{2}\right)^2} c_{jk}(t - s),
\]
where
\[
c_{jk}(t - s) = \frac{(-1)^j}{2^j + k} \sum_{l=0}^{j} \sum_{m=0}^{k} \frac{(-1)^{l+m} 2^{2l+m+1} (t - s)^{2(l+m+1)}}{2!(l+m+1)!} (j-l)(k-m)!
\]
(5.24)

The consequent symmetry
\[
(-1)^k c_{jk}(t - s) = (-1)^j c_{kj}(t - s)
\]
(5.25)
implies the same symmetry relation for the coefficients $b_{jk}(x_1 + x_2^2)$ which is in agreement with the general result (5.14).

5.2 Counter-ions only

5.2.1 General theory

Let us now consider a version of the 2D OCP with zero volume background charge density $-en = 0$. As before, the particles possess the charge $e$ and therefore they are “counter-ions” to the opposite charge line charge density $-e\sigma$ at the wall surface $x = 0$. The one-body Boltzmann factor of mobile particles is $w(x) = \exp(-\Gamma \pi \sigma x)$. Introducing the generalized Boltzmann factor
\[
w(z_j, \bar{z}_k) = \exp[-\gamma \pi \sigma (z_j + \bar{z}_k)],
\]
(5.26)
the symmetry formula (2.13) implies that
\[
n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) = (-1)^\gamma n^{(2)}(z_2, \bar{z}_1, z_1, \bar{z}_2)
\]
(5.27)
or, equivalently,
\[
n^{(2)}(x_1 + x_2^2, x_1 - x_2, y_1 - y_2) = (-1)^\gamma n^{(2)}(x_2 + x_2^2, i(y_2 - y_1), i(x_1 - x_2)).
\]
(5.28)
The short-distance expansion of the two-body density is still of type (5.2):
\[
n^{(2)}(x_1 + x_2^2, x_1 - x_2, y_1 - y_2) = \left[(x_1 - x_2)^2 + (y_1 - y_2)^2\right]^\gamma 
\times \sum_{j,k=0}^{\infty} a_{jk}(x_1 + x_2^2)(x_1 - x_2)^2j(y_1 - y_2)^2k.
\]
(5.29)
Inserting this expansion into Eq. (5.28) leads to the following symmetry relation between the expansion coefficients
\[
a_{jk}(x_1 + x_2^2) = (-1)^{j+k} a_{kj}(x_2 + x_2^2).
\]
(5.30)
Consequently, the most general form of the two-body density reads as
\[ n^{(2)}(\frac{x_1 + x_2}{2}, x_1 - x_2, y_1 - y_2) = \left[ (x_1 - x_2)^2 + (y_1 - y_2)^2 \right]^\gamma \times \left\{ \sum_j a_{jj} \left( \frac{x_1 + x_2}{2} \right) \left[ (x_1 - x_2)^2(y_1 - y_2)^2 \right]^j \right. \\
+ \sum_{j < k} a_{jk} \left( \frac{x_1 + x_2}{2} \right) \left[ (x_1 - x_2)^2j(y_1 - y_2)^{2k} \right. \\
+ (-1)^{j+k}(x_1 - x_2)^{2k}(y_1 - y_2)^{2j} \left. \right\} \right\}. \]

(5.31)

5.2.2 Free-fermion point

The 2D model of the charged wall with counter-ions only was solved at the free-fermion coupling \( \Gamma = 2 \) in Ref. [12]. The particle density profile, obtained in the form
\[ n(x) = \frac{1}{4\pi} \int_0^{4\pi} ds \, s e^{-xs}, \]  
(5.32)
evidently fulfills the electroneutrality condition
\[ \int_0^\infty dx n(x) = \sigma. \]  
(5.33)

Introducing a generalization of the density function (5.32)
\[ n(z_j, \bar{z}_k) = \frac{1}{4\pi} \int_0^{4\pi} ds \, s e^{-s(z_j + z_k)}/2, \]  
(5.34)
the two-body density is expressible as [12]
\[ n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) = n(z_1, \bar{z}_1) n(z_2, \bar{z}_2) - n(z_2, \bar{z}_1) n(z_1, \bar{z}_2). \]  
(5.35)

Since according to (5.35) one has
\[ n^{(2)}(z_2, \bar{z}_1, z_1, \bar{z}_2) = n(z_2, \bar{z}_1) n(z_1, \bar{z}_2) - n(z_1, \bar{z}_1) n(z_2, \bar{z}_2), \]  
(5.36)
the symmetry relation (5.28) holds. The two-body density is expressible as
\[ n^{(2)}(z_1, \bar{z}_1, z_2, \bar{z}_2) = \frac{1}{(4\pi)^2} \int_0^{4\pi} ds \, s e^{-s(z_1 + \bar{z}_1)} \int_0^{4\pi} dt \, t e^{-t(z_1 + \bar{z}_1)} \]
\[ \times \sum_{j=0}^\infty \left( \frac{t-s}{2(2j)!} \right)^2 \left[ (x_1 - x_2)^{2j} - (-1)^j(y_1 - y_2)^{2j} \right]. \]  
(5.37)

Using the relation (5.24), the coefficients \( a_{jk} \) of the short-distance expansion (5.29) are found to be
\[ a_{jk} \left( \frac{x_1 + x_2}{2} \right) = \frac{1}{(4\pi)^2} \int_0^{4\pi} ds \, s e^{-s(z_1 + \bar{z}_1)} \int_0^{4\pi} dt \, t e^{-t(z_1 + \bar{z}_1)} b_{jk} (t-s), \]  
(5.38)
where
\[ b_{jk}(t-s) = (-1)^k \frac{(t-s)^{2(j+k+1)}}{2^{2(j+k+1)}[2(j+k+1)]!}. \] (5.39)

Due to the equality
\[ (-1)^j b_{jk}(t-s) = (-1)^k b_{kj}(t-s), \] (5.40)
the symmetry formula (5.30) automatically takes place.

6 Conclusion

The studied short-distance symmetries of two-body densities for infinite and semi-infinite 2D OCP represent a rare occasion to get exact results not only at the free-fermion coupling constant \( \Gamma = 2 \), but also at a sequence of couplings \( \Gamma = 2 \times \text{integer} \), up to the fluid-crystal phase transition.

The guiding-center structure factor in the quantum Hall effect is proportional to a specific part \( \tilde{g}(q) \) of the Fourier transform of the pair correlation function of the bulk plasma \( \tilde{h}(q) \), given by the relation (4.20). The guiding-center factors satisfies a self-dual formula between its real space (Euclidean) and Fourier components [7,8]. The first aim of this paper was to derive this self-dual formula directly in the format of the 2D OCP by using the short-distance symmetry of the pair correlation, see Eqs. (4.12) and (4.13). As a by-product of the formalism, the large-\( q \) asymptotic behavior of \( \tilde{h}(q) \) (4.5) was obtained. An infinite sequence of zero integrals over the Fourier component \( \tilde{g}(q) \) multiplied by Laguerre polynomials of argument \( q^2/2 \) was found: see Eq. (4.22) for \( \Gamma = 2 \times \text{odd integer} \) and Eq. (4.27) for \( \Gamma = 2 \times \text{even integer} \).

The second aim was to extend the short-distance symmetry of the pair correlation function to the semi-infinite 2D OCP. This was done for the jellium model in Sect. 5.1 and for its simplified version with zero background charge density in Sect. 5.2. In both cases, the coefficients of the expansion in variables \((x_1 - x_2)^2\) and \((y_1 - y_2)^2\) exhibit a symmetry with respect to the permutation of summation indices of type (5.14) and (5.30).

The short-distance symmetry is not sufficient for determining explicitly the pair correlation function, however, it restricts substantially its possible forms. This might be useful in searching for the exact solution of the 2D OCP at the coupling constants \( \Gamma = 2 \times \text{integer} \), e.g. in the spirit of the work [17].

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