A CRITERION FOR REGULARITY OF LOCAL RINGS

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Abstract. It is proved that a noetherian commutative local ring $A$ containing a field is regular if there is a complex $M$ of free $A$-modules with the following properties: $M_i = 0$ for $i \not\in [0, \dim A]$; the homology of $M$ has finite length; $H_0(M)$ contains the residue field of $A$ as a direct summand. This result is an essential component in the proofs of the McKay correspondence in dimension 3 and of the statement that threefold flops induce equivalences of derived categories.

1. Introduction

Let $(A, m, k)$ be a local ring; thus $A$ is a noetherian commutative local ring, with maximal ideal $m$ and residue field $k$. Let $M: 0 \to M_d \to \cdots \to M_0 \to 0$ be a complex of free $A$-modules with length $\dim A$ finite and non-zero. The New Intersection Theorem [10] yields $d \geq \dim A$.

In this article we prove the following result, which is akin to Serre’s theorem that a local ring is regular when its residue field has a finite free resolution.

Theorem 1.1. Assume $A$ contains a field or $\dim A \leq 3$. If $d = \dim A$ and $k$ is a direct summand of $H_0(M)$, then $H_i(M) = 0$ for $i \geq 1$, and the ring $A$ is regular.

This result is contained in Theorem 2.4. A restricted version of such a statement occurs as [3, (4.3)]; however, as is explained in Remark 2.5, the proof of loc. cit. is incorrect.

In the remainder of the introduction, the field $k$ is algebraically closed, schemes over $k$ are of finite type, and the points considered are closed points. We write $D(Y)$ for the bounded derived category of coherent sheaves on a scheme $Y$, and $k_y$ for the structure sheaf of a point $y \in Y$. Given the theorem above, arguing as in [3, \S 5], one obtains the following corollary.

Corollary 1.2. Let $Y$ be an irreducible scheme of dimension $d$ over $k$, and let $E$ be an object of $D(Y)$. Suppose there is a point $y_0 \in Y$ such that $k_{y_0}$ is a direct summand of $H_0(E)$ and

$\Hom_{D(Y)}(E, k_y[i]) = 0$ unless $y = y_0$ and $0 \leq i \leq d$.

Then $Y$ is non-singular at $y_0$ and $E \cong H_0(E)$ in $D(Y)$. □

This result enables one to show that certain moduli spaces are non-singular and give rise to derived equivalences; see [3, (6.1)]. This has proved particularly effective in dimension three, and is an essential component in the proofs in [11, 2].
2. Proof of the main theorem

This section is dedicated to a proof Theorem 2.4. The book of Bruns and Herzog [4] is our standard reference for the notions that appear here.

Let $(A, m, k)$ be a local ring, and let $C$ be an $A$-module; it need not be finitely generated. A sequence $x = x_1, \ldots, x_n$ is $C$-regular if $xC \neq C$ and $x_i$ is a non-zero divisor on $C/(x_1, \ldots, x_{i-1})C$ for each $1 \leq i \leq n$. An $A$-module $C$ is big Cohen-Macaulay if there is a system of parameters for $A$ that is $C$-regular. If every system of parameters of $A$ is $C$-regular, then $C$ is said to be balanced. Any ring that has a big Cohen-Macaulay module has also one that is balanced; see [4] (8.5.3). Big Cohen-Macaulay modules were introduced by Hochster [6], who constructed them when $A$ contains a field, and, following the recent work of Heitmann, also when $\dim A \leq 3$; see [7].

The result below is contained in the proof of [5] (1.13) by Evans and Griffith, see also [3] (3.1)], so only an outline of an argument is provided; it follows the discussion around [4] (9.1.7); see also [3] (3.4).

Lemma 2.1. Let $M$ be a complex of free $A$-modules with $M_i = 0$ for $i \notin [0, \dim A]$, the $A$-module $H_0(M)$ finitely generated, and length$_A H_i(M)$ finite for each $i \geq 1$.

If $C$ is a balanced big Cohen-Macaulay module, then $H_i(M \otimes_A C) = 0$ for $i \geq 1$.

Sketch of a proof. Replacing $M$ by a quasi-isomorphic complex we may assume each $M_i$ is finite free. Choose a basis for each $M_i$ and let $\phi_i$ be the matrix representing the differential $\partial_i: M_i \to M_{i-1}$. Set $r_i = \sum_{j=1}^{\dim A} (-1)^{j-i} \rank A M_j$, and let $I_{r_i}(\phi_i)$ be the ideal generated by the $r_i \times r_i$ minors of $\phi_i$. Fix an integer $1 \leq i \leq d$, and let $p$ be a prime ideal of $A$ with height $I_{r_i}(\phi_i) = \height p$. If $\height p = \dim A$, then height $I_{r_i}(\phi_i) = \dim A - d \geq i$, where the first inequality holds by hypotheses. If $\height p < \dim A$, then $H_j(M \otimes_A A_p) = 0$ for $j \geq 1$, by hypotheses. Therefore

$$\height I_{r_i}(\phi_i) = \height (I_{r_i}(\phi_i) \otimes_A A_p) = \height I_{r_i}(\phi_i \otimes_A A_p) \geq i$$

where the inequality is comes from the Buchsbaum-Eisenbud acyclicity criterion [4] (9.1.6)]. Thus, no matter what $\height p$ is, one has

$$\dim A - \dim A / I_{r_i}(\phi_i) \geq \height I_{r_i}(\phi_i) \geq i$$

Thus, $I_{r_i}(\phi_i)$ contains a sequence $x = x_1, \ldots, x_i$ that extends to a full system of parameters for $R$. Since $C$ is balanced big Cohen-Macaulay module, the sequence $x$ is $C$-regular, so another application of [4] (9.1.6)] yields the desired result. □

The following elementary remark is invoked twice in the arguments below.

Lemma 2.2. Let $R$ be a commutative ring and $U, V$ complexes of $R$-modules with $U_i = 0 = V_i$ for each $i < 0$. If each $R$-module $V_i$ is flat and $H_1(U \otimes_R V) = 0$, then $H_1(H_0(U) \otimes_R V) = 0$.

Proof. Let $\tilde{U} = \text{Ker}(U \to H_0(U))$; evidently, $H_i(\tilde{U}) = 0$ for $i < 1$. Since $- \otimes_R V$ preserves quasi-isomorphisms, $H_i(\tilde{U} \otimes_R V) = 0$ for $i < 1$. The long exact sequence that results from the short exact sequence of complexes $0 \to \tilde{U} \to U \to H_0(U) \to 0$ thus yields a surjective homomorphism

$$H_1(U \otimes_R V) \longrightarrow H_1(H_0(U) \otimes_R V) \longrightarrow 0$$

This justifies the claim. □
The proposition below is immediate when $C$ is finitely generated: $\text{Tor}^A_1(C, k) = 0$ implies $C$ is free. It contains \cite{MAT} (2.5), due to Schoutens, which deals with the case when $C$ is a big Cohen-Macaulay algebra.

**Proposition 2.3.** Let $(A, m, k)$ be a local ring and $C$ an $A$-module with $mC \neq C$. If $\text{Tor}^A_1(C, k) = 0$, then each $C$-regular sequence is $A$-regular.

**Proof.** First we establish that for any ideal $I$ in $A$ one has $(IC :_R C) = I$.

Indeed, consider first the case where the ideal $I$ is $m$-primary ideal.

Let $F$ be a flat resolution of $C$ as an $A$-module, and let $V$ be a flat resolution of $k$, viewed as an $A/I$-module. Therefore $H_1(F \otimes_A V) = \text{Tor}^A_1(C, k) = 0$. The $A$-action on $k$ factors through $A/I$, so $F \otimes_A V \cong (F \otimes_A A/I) \otimes_{A/I} V$. Thus, applying Lemma 2.2 with $U = C \otimes_A A/I$ one obtains

$$\text{Tor}^A_{1/1}(C/IC, k) = H_1(H_0(U) \otimes_{A/I} V) = 0$$

The ring $A/I$ is artinian and local, with residue field $k$, thus $\text{Tor}^A_{1/1}(C/IC, k) = 0$ implies that the $A/I$-module $C/IC$ is free; see, for instance, \cite{MAT} (22.3)]. Moreover, $C/IC$ is non-zero as $mC \neq C$. Thus $(IC :_R C) = I$, as desired.

For an arbitrary ideal $I$, evidently $I \subseteq (IC :_R C)$. The reverse inclusion follows from the chain:

$$(IC :_R C) \subseteq \bigcap_{n \in \mathbb{N}} ((I + m^n)C :_R C) = \bigcap_{n \in \mathbb{N}} (I + m^n) = I$$

where the first equality holds because each ideal $(I + m^n)$ is a $m$-primary, while the second one is by the Krull Intersection Theorem. This settles the claim.

Let $x_1, \ldots, x_m$ be a regular sequence on $C$. Fix an integer $1 \leq i \leq m$, and set $I = (x_1, \ldots, x_{i-1})$. For any element $r$ in $A$, the first and the second implications below are obvious:

$$rx_i \in I \Rightarrow rx_i C \subseteq IC \Rightarrow x_i(rC) \subseteq IC \Rightarrow rC \subseteq IC \Rightarrow r \in I$$

The third implication holds because $x_i$ is regular on $C/IC$, and the last one is by the claim established above. Thus, $x_i$ is a non-zero divisor on $A/I$, that is to say, on $A/(x_1, \ldots, x_{i-1})$. Since this holds for each $i$, we deduce that the sequence $x$ is regular on $A$, as desired. \qed

The result below contains the theorem stated in the introduction.

**Theorem 2.4.** Let $(A, m, k)$ be a local ring, $M$ a complex of free $A$-modules with $M_i = 0$ for $i \notin [0, \dim A]$, the $A$-module $H_0(M)$ finitely generated, and $\text{length}_A H_i(M)$ finite for $i \geq 1$. Assume $A$ has a big Cohen-Macaulay module. If $k$ is a direct summand of $H_0(M)$, then the local ring $A$ is regular.

**Proof.** Let $C$ be a big Cohen-Macaulay $A$-module; we may assume that $C$ is balanced. Let $d = \dim A$, and let $x = x_1, \ldots, x_d$ be a system of parameters for $A$ that is a regular sequence on $C$. In particular, $xC \neq C$, and hence $mC \neq C$, since $x$ is $m$-primary.

Let $V$ be a flat resolution of $C$ over $A$. The complex $M$ is finite and consists of free modules, so $H(M \otimes_A C) \cong H(M \otimes_A V)$, and hence $H_1(M \otimes_A V) = 0$, by Lemma 2.1. Now Lemma 2.2 invoked with $U = M$ implies $H_1(H_0(M) \otimes_A V) = 0$, so $\text{Tor}^A_1(H_0(M), C) = 0$. Since $k$ is a direct summand of $H_0(M)$, this implies $\text{Tor}^A_1(k, C) = 0$, equivalently, $\text{Tor}^A_1(C, k) = 0$. 

By Proposition 2.3, the sequence $x$ is $A$-regular, therefore depth $A \geq \dim A$ and $A$ is (big) Cohen-Macaulay. Consequently, Lemma 2.1 now applied with $C = A$, implies $H_i(M) = 0$ for $i \geq 1$, as claimed. In particular, $M$ is a finite free resolution of $H_0(M)$, so the projective dimension of $H_0(M)$ is finite. Therefore, the projective dimension of $k$ is finite as well, since it is a direct summand of $H_0(M)$. Thus, $A$ is regular; see Serre [11, Ch. IV, cor. 2, th. 9].

\[\square\]

Remark 2.5. As stated in the introduction, the proof of [3, (4.3)] is incorrect; in the paragraph below we adopt the notation of loc. cit. The problem with it is the claim in display (1), on [3, pg. 639], which reads:

\[
(*) \quad \text{Tor}_p^A(N, C) = 0 \quad \text{for all} \quad p > 0
\]

This cannot hold unless we assume a priori that the local ring $A$ is Cohen-Macaulay.

Indeed, suppose the displayed claim is true. Consider the standard change of rings spectral sequence sitting in the first quadrant:

\[
E_{p,q}^2 = \text{Tor}_p^A(\text{Tor}_q^R(k, A), C) \Rightarrow \text{Tor}_{p+q}^R(k, C).
\]

The edge homomorphisms in the spectral sequence give rise to the exact sequence

\[
0 = \text{Tor}_2^A(N, C) \rightarrow \text{Tor}_1^R(k, A) \otimes_A C \rightarrow \text{Tor}_1^R(k, C) = 0
\]

where the 0 on the left holds by $(*)$ and that on the right holds because $C$ is free over $R$. Thus, the middle term is 0, which implies $\text{Tor}_1^R(k, A) = 0$. Therefore $A$ is free as an $R$-module, because $A$ is finitely generated over $R$, and hence $A$ is Cohen-Macaulay.

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