Snell’s Law for Gliders

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Snell’s law, which encompasses both refraction and total internal reflection, provides the foundation for ray optics and all lens-based instruments, from microscopes to telescopes. Refraction results when light crosses the interface between media of different refractive index, the dimensionless number that captures how much a medium retards the propagation of light. In this work, we show that the motion of self-propelled particles moving across a resistance discontinuity is governed by an analogous Snell’s law, allowing for glider ray optics. We derive a variant of Snell’s law for gliders moving across regions of different frictions. Just as the ratio of refractive indexes sets the path of a light ray, the ratio of resistance coefficients is shown to determine the trajectories of gliders. We find that the magnitude of refraction depends on the glider’s shape, specifically the aspect ratio, as analogous to the wavelength of light. This enables the demixing of a polymorphic, many-shaped, beam of gliders into distinct monomorphic, single-shaped, beams through a friction prism. In turn, beams of monomorphic gliders can be focused by spherical and gradient friction lenses. Completing the analogy, we show that the shape-dependence of the total internal reflection critical angle can be used to create glider traps. Such analogies to ray optics suggest a universe of new devices for sorting, concentrating, and analyzing microscopic gliders is possible.

Introduction

The directed transport of self-propelled particles is both a fundamental process in biology [1–3] and an important engineering goal for lab-on-a-chip [4, 5] and drug delivery [6–8] systems. Relatively little is known about how life uses self-propelled particles to create complex spatial organization, or how theory might be used to control particle trajectories in artificial systems. Different modes of microscale motion, e.g. gliding or swimming, are classified based on the mechanism of a particle’s self- propulsion and the environment in which it moves [9]. Here we study gliders, rigid particles which propel themselves via interaction with a solid surface or matrix [10, 11] and provide a general theory for how to control their trajectories.

The paths of gliders have so far been modulated using topographically and chemically-patterned surfaces, where protein filaments follow paths [12] or navigate junctions [13] that explicitly confine
the filaments. However, this approach has limited utility when it comes to dynamic and autonomous routing. Two lines of recent work suggest another approach. First, more complicated active particles, such as flexible crawling cells and swimmers, refract and scatter at adhesion or viscosity discontinuities [14–18]. Second, the theory of swimmers in viscosity gradients has predicted viscotaxis, the degree to which swimmer trajectories bend toward or away from high viscosity, as a function of swimmer symmetry and swimming mechanism [19–21]. Together these findings suggest that self-propelled particles might be treated similar to light moving through a medium of varying refractive index, with a resistivity replacing refractive index. Distilling a principle for resistive refraction in these systems, however, is challenging as even the sign of viscotaxis varies based on the details of the specific system. We therefore use gliders, in which the effects of friction can be isolated from particle flexibility, propulsion mechanism, and hydrodynamics.

In contrast to swimmers, which by definition operate in a fluid environment, gliders are a form of dry active matter [22] and their motion does not, in general, depend on hydrodynamic interactions. Instead, gliders experience an adhesive molecular friction, which is a viscous-like frictional drag [23–25]. Protein filaments on motor-coated substrates [26–29] and microorganisms with arrays of surface-interacting motors [30–32] provide the simplest examples of the gliding motion. Here we focus our study to the core physics of individual gliders as they undergo a surface friction—we do not consider systems of gliders that exhibit collective dynamics at high density [33] or gliders whose use of lubricating slime layers introduce additional hydrodynamic, elastic, and capillary effects [34].

Figure 1: Sketch of discoid and rectangular gliders moving across a resistance discontinuity. a, Depiction of the relevant forces and torques on a glider as it travels across a resistance (friction) discontinuity. b, The forces and torques from (a) cause a reorientation of the glider’s trajectory around the resistance discontinuity. c, d, Forces and re-orientation of a rectangular glider depend on the aspect ratio of the glider.
Here we show that gliders indeed refract at a resistance discontinuity and derive a simple prescriptive theory for designing systems where mindless gliders are directed by local cues in the form of friction discontinuities. We derive a Snell’s law for gliders, which predicts how a glider’s trajectory is changed when traversing a friction discontinuity, and apply this Snell’s law to rationally design optics-like devices for gliders.

**Derivation**

We consider the trajectory of a glider moving across a resistance discontinuity at low Reynold’s number. Consistent with our aim to derive a simple theory that can provide intuition for the engineered routing of gliders, we ignore hydrodynamic and Brownian interactions and assume that the interface at the discontinuity does not deform as the glider crosses through. As illustrated in Fig. 1, the non-accelerating motion of the glider follows from a balance between the resistance force, \( F_{\text{resistance}} = -R_{FU} \cdot U \), and a glide force, \( F_{\text{glide}} \):

\[
0 = -R_{FU} \cdot U + F_{\text{glide}}.
\]  

In Eq. (1), \( U \) is the translational velocity of the glider, and \( R_{FU} \) is the resistance function that gives the coupling between the friction and the velocity. The resistance tensor depends on the jump in resistivity, and the geometry—the size and shape of the glider as well as its proximity and orientation relative to the discontinuity.

A glider is an active particle, propelling off of a solid substrate to generate a propulsive glide force of the form: \( F_{\text{glide}} = F_0 q \), where \( F_0 \) is the magnitude and \( q \) the direction of propulsion. To simplify the analysis, we take the resistance tensor to be isotropic and constant, \( R_{FU} = \zeta_{tt} I \), where \( \zeta_{tt} \) is the translational resistance coefficient and \( I \) is the identity tensor. Thus, the velocity is

\[
U = \frac{F_0 q}{\zeta_{tt}}.
\]  

We consider a constant magnitude glide force \( F_0 \) and thus the speed of the glider is slower in the region of greater resistivity. For example, for a discoid glider with molecular friction \( \eta \) we have a resistance coefficient \( \zeta_{tt} \sim \eta a^2 \), where \( a \) is the glider radius.

The glide direction \( q \) changes as a function of time according to

\[
\frac{dq}{dt} = \Omega \times q,
\]  

where \( \Omega \) is the angular velocity of the glider. The glide angular velocity follows from the torque balance for the force- and torque-free motion

\[
0 = -R_{L \Omega} \cdot \Omega - R_{LU} \cdot U.
\]  

In Eq. (4) \( R_{L \Omega} \) is the resistance tensor coupling the torque \( (L) \) to the angular velocity, and \( R_{LU} \) couples the torque to the translational velocity. As for the force-velocity coupling \( (FU) \), we take the torque-angular velocity coupling to be isotropic and constant: \( R_{L \Omega} = \zeta_{rr} I \); for a discoid glider undergoing rotational molecular friction \( \zeta_{rr} \sim \eta a^4 \).

The torque-translational velocity \( (LU) \) coupling arises because as the glider crosses into a region of higher resistivity that portion of the glider engaging the more resistive surface slows down and thus the glider rotates such that its direction of motion tends to align along the normal as illustrated in Fig. 1a. The opposite occurs when moving into a less resistive region. The \( LU \) coupling is a
pseudo tensor and since the glider itself is not chiral, it must be of the form $R_{LU} = \zeta_{rt} \epsilon n$, where $\epsilon$ is the unit alternating tensor, $n$ is the normal to the discontinuity, and $\zeta_{rt}$ is the resistance coefficient. For a disk, the $LU$ coupling only arises if there is a jump in resistivity, $\zeta_{rt} \sim \Delta \eta a^3$.

Combining Eqs. (1) and (4), Eq. (3) becomes

$$\frac{dq}{dt} = \frac{\zeta_{rt}}{\zeta_{rr} \zeta_{tt}} F_0 (n \times q) \times q = \frac{\zeta_{rt}}{\zeta_{rr} \zeta_{tt}} [n - q(q \cdot n)].$$

(5)

Now, $n \cdot q = \cos \theta$, where $\theta$ is the angle between the normal and the glide direction, and thus Eq. (5) gives an equation for the evolution of $\theta(t)$:

$$\frac{d\cos \theta}{\sin^2 \theta} = \frac{\zeta_{rt}}{\zeta_{rr} \zeta_{tt}} F_0 \, dt.$$

(6)

We need to integrate Eq. (6) from the time the glider first touches the discontinuity ($t = 0$) with incident angle $\theta_0$ until it fully crosses into the next region at the final time $t_f$, which will then give the out-going angle $\theta_f$. The time to cross the interface follows from the translational velocity $dx/dt = U$, and since only the normal component of the velocity is responsible for the glider crossing we have

$$\frac{d(n \cdot x)}{dt} = \frac{F_0}{\zeta_{tt}} n \cdot q = \frac{F_0}{\zeta_{tt}} \cos \theta.$$

(7)

We can use Eq. (7) to replace $dt$ in Eq. (6) to give

$$dx_\perp = \frac{\zeta_{rr} \cos \theta d\cos \theta}{\zeta_{rt}} = -\frac{\zeta_{rr}}{\zeta_{rt}} d\ln(\sin \theta),$$

(8)

where $x_\perp = n \cdot x$ is the amount of the glider that has crossed the interface. For a discoid glider, integrating from 0 to $2a$ relates the initial to the final angle and yields a Snell’s law:

$$\sin \theta_f = e^\alpha \sin \theta_0,$$

(9)

where $\alpha = -2a \zeta_{rt}/\zeta_{rr}$.

This Snell’s law for gliders provides an intuitive perspective that prescribes how a glider can be reoriented with a resistance discontinuity. The behavior is independent of the magnitude of the propulsive force $F_0$ and the translational resistance $\zeta_{tt}$. Further by dimensional arguments, the resistance coefficient for $LU$ coupling is proportional to $a^3$ and thus $\alpha$ is independent of the size of the glider. The validity of this Snell’s law and its independence on the glider size are verified by direct simulation below.

We have made a number of approximations in arriving at this Snell’s law. First, we have assumed that the resistance coefficients $\zeta_{rr}$ and $\zeta_{rt}$ are constants (Note that $\zeta_{tt}$ cancels out in Eq. (8)). Both are proportional to the local value of the resistance of the surface and thus depend on the portion of the glider in each region. We can include this effect by noting that Eq. (8) can be written as

$$\frac{\zeta_{rt}(x_\perp)}{\zeta_{rr}(x_\perp)} dx_\perp = -d\ln(\sin \theta),$$

(10)

and integration from 0 to $2a$ again recovers Snell’s law Eq. (9) where $\alpha$ is now given by

$$\alpha = -\int_0^{2a} \frac{\zeta_{rt}(x_\perp)}{\zeta_{rr}(x_\perp)} dx_\perp.$$

(11)
One could precisely compute $\alpha$ from a micromechanical simulation for $\zeta_{rr}(x_-)$ and $\zeta_{rt}(x_\perp)$. Analytically, we can substitute the resistance coefficients in terms of friction, which allows us to approximate the solution to have the form

$$\alpha = -C \frac{\Delta \eta}{\langle \eta \rangle},$$

where $\Delta \eta = \eta_f - \eta_0$, $\langle \eta \rangle = (\eta_f + \eta_0)/2$ and $C$ is an order 1 constant that is (weakly) dependent of the friction ratio $\eta_f/\eta_0$ (Fig. 1b). Later we show that Snell’s law with Eq. (12) is in excellent agreement with detailed micromechanical simulations.

Since the reorientation arises because part of the glider finds itself in a more resistive region, if the glider is very thin relative to its glide axis, then the differential resistance across the body is small and the reorientation should be reduced. An infinitely thin glider will not reorient at all. We can account for this shape effect in a simple manner by recognizing that the amount of the glider that has crossed the discontinuity $\Delta x_\perp$ depends on the body shape and the initial orientation $\theta_0$.

For a simple rectangular glider shown in Fig. 1c,d, $\Delta x_\perp = \ell \cos \theta_0 + a(1 - \cos \theta_0)$, where $\ell$ is the half major length and $a$ is the half minor length. Using this in Eq. (8) we again have Snell’s law, but now

$$\alpha \sim -\frac{a}{\ell^2} (a + (\ell - a) \cos \theta_0) \frac{\Delta \eta}{\langle \eta \rangle},$$

where we have used the geometric scaling that $\zeta_{rt}/\zeta_{rr} \sim a/\ell^2$. We have also assumed the instantaneous orientation angle of the body could be approximated with its initial angle $\theta_0$. When $\ell = a$, Eq. (14) reduces to Eq. (12). The dependence on the aspect ratio is similar to the wavelength dependence of the refractive index. We later use numerical simulations to test this prediction.

Above we assumed that any force-angular velocity coupling $R_{FQ} = R_{LU}$ was negligible. If $R_{FQ}$ is included, the force balance Eq. (1) becomes $0 = -R_{FU} \cdot U - R_{FQ} \cdot \Omega + F_{\text{glide}}$, which, when combined with the angular momentum balance Eq. (4), will give an angular velocity $\Omega = -R_{LU}^{-1} \cdot [R_{FU} - R_{FQ} R_{LU}^{-1} R_{LU}]^{-1} \cdot F_{\text{glide}}$. The additional factor $R_{FQ} R_{LU}^{-1} R_{LU}$ will add an additional $\theta$ dependence to Eq. (6) and we can only derive a Snell’s law under the condition that $\zeta_{rt}^2/(\zeta_{rr} \zeta_{tt}) \ll 1$. Furthermore, non-discoid gliders will not, in general, have isotropic resistance tensors and it may be impossible to find a Snell-like analytical expression for the refraction of nondiscoid gliders (Section S1).

**Analysis of results**

Simulations of the a discoid glider (Fig. 2) closely agree with Eqs. (9) and (12); see Section S2 for simulation details, Section S3 for curve fitting details, and Video 1 for an example simulation. One prediction from our theory is that refraction is size independent. By simulating gliders of different sizes, we find that the angle of refraction is indeed size invariant (Section S4, Fig. S3a). Further, the form of our Snell’s law for a discoid glider predicts that there should be a symmetry about the line $\theta_f = \theta_0$, which we verified by comparing data points and curves across this axis of symmetry (Section S4, Fig. S3b).

As for Snell’s law, when $\eta_f/\eta_0 < 1$, Eq. (9) is valid up to $\theta_f = \frac{\pi}{2}$. For $\theta_f = \frac{\pi}{2}$, the incident critical angle is

$$\theta_{\text{crit}} = \arcsin e^{-\alpha}. \hspace{2cm} (15)$$
Figure 2: Gliders refract and reflect in a manner analogous to Snell’s law. a, Discoid glider refraction as a function of incident angle and friction ratios. Curves represent theory, points represent simulations. b, For incident angles above the critical angle, gliders follow the law of reflection as indicated by the black line. Note that points overlap each other, as indicated in the legend. c, Refraction and reflection for rectangular gliders of varying thicknesses. Note that points on the law of reflection line overlap each other in the same order as (b). d, Comparison between the critical angles predicted by optical Snell’s law and glider Snell’s law.

For $θ_0 > θ_{\text{crit}}$ gliders obey the law of reflection,

$$θ_f = π - θ_0,$$

which we confirm with simulations (Fig. 2b). Through symmetry, the critical angle is the same as Snell’s window\[35\], which is the greatest possible refraction angle for a given $η_f/η_0 > 1$.

For the rectangular glider, we see that refraction depends on the glider’s aspect ratio in Eq. (14). As the glider’s aspect ratio becomes smaller relative to its glide axis, the effect of refraction diminishes (Fig. 2c). In the limit that the glider becomes a 1D-line segment along the glide axis, i.e. $a = 0$, the glider undergoes no refraction. In contrast, as the glider’s aspect ratio becomes wider relative to its glide axis, there is a greater refraction effect.

Due to the similarities between our theory and optical Snell’s law, we compare the two theories in more detail. The ratio of optical refractive indices, $n$, can be rewritten as a ratio of speeds $n_f/n_0 = c/v_f = v_f/v_0$, as can the ratio of frictions $η_f/η_0 = F/v_f = v_f/v_0$. Thus, we can compare optical Snell’s
law, $v_f \sin \theta_f = v_0 \sin \theta_0$, to Eqs. (9) and (14) by looking at the critical angle as a function of the speed ratio (Fig. 2d). Overall, the critical angles share a similar scaling; gliders tend to refract less than light for the same speed ratio. However, for the wide glider $a/\ell = 6/1$, we find a critical angle and refraction curves (Section S5 and Fig. S4) that closely follow the optical Snell’s law. Based on this finding, we conjecture that as $a/\ell \rightarrow \infty$, a rectangular glider will follow the optical Snell’s law. The critical angle for discoid gliders is nearly identical to that for $a/\ell = 1/1$ rectangular gliders and therefore is not plotted in Fig. 2d (compare Fig. S2a and Fig. S2b, purple dots).

Ray optics for gliders

Figure 3: Principles of ray optics can be similarly applied to organize non-interacting gliders. All results are from micromechanical simulations (see Sections S6 and S7 ). a, Prism analog disperses rectangular gliders by their shape. Top: Gliders disperse to different angles after passing through a region of higher friction ($\eta_{\text{prism}}/\eta_{\text{out}} = 2$). Bottom: A subset of gliders reflect at a discontinuity of lower friction ($\eta_{\text{prism}}/\eta_{\text{out}} = 2/3$). b, Ball lens analog has different focal lengths and (spherical aberrations) depending on glider shape. Top: $a/\ell = 1$. Bottom: $a/\ell = 2/3$. c, Gradient lens analog focuses gliders while reducing spherical aberrations. Top: $a/\ell = 1$, inset shows friction gradient of the lens. Bottom: $a/\ell = 2/3$. d, Frames of timelapse simulation for a glider trap. Time is measured in $d/v$ where $d$ is the trap diameter and $v$ is the glider speed in the trap. e, Probability of a glider to be trapped given a random initial orientation at the indicated position. f, Mean trapping efficiency increases for smaller friction ratios and decreases for thinner gliders.

The simple form of Eq. (9) suggests an intuitive set of design principles for constructing environments to direct the transport of gliders. Specifically, we consider if the principles of ray optics can also be applied to organize non-interacting gliders. We start by creating a prism, where a triangular region has a friction that differs from the bulk area (Fig. 3a, Video 2, and Video 3). A polymorphic, multi-shaped, beam of gliders is separated into monomorphic, single-shaped, beams by tuning the friction value of the triangular region. The results from our micromechanical simulations show a
finite size effect not present for light. Although refraction and reflection angles are size independent, the centroid trajectory of a glider will appear to bend before encountering the interface (when the leading edge of the glider first contacts the interface) and continue to bend beyond the interface (until the entire body of the glider is free from the discontinuity) as can be seen in Fig. S3a.

Having demonstrated that polymorphic beams of gliders can be split into monomorphic beams, we investigate how these monomorphic beams can be shaped. A ball lens that draws gliders toward a focal point can be created using a disk-shaped region with a higher friction than the bulk area (Fig. 3b). The focus of the lens is a function of the friction ratio, the glider aspect ratio, and the size of the disk region. Like a ball lens for light, the ball lens for gliders has spherical aberration, i.e. glider trajectories do not all converge at a single focal point. In optics, aberrations are corrected by using compound lenses, aspheric lenses, or lenses with index gradients. We combine refraction with a resistance gradient to create a gradient friction lens (Fig. 3c) that can focus a collimated beam of monomorphic gliders to a focal point with significantly reduced spherical aberration. Thus, beams of gliders can be sculpted with lenses in a manner similar to light in optics.

While the principles of Snell's law can be used to route gliders in free space, they can also be used to confine gliders. We trap gliders in a disk with a higher friction than the bulk area (Fig. 3d, Video 4). In this design, a glider with an incident angle greater than its critical angle will perpetually reflect off the boundary of the disk with the same incident angle. Gliders whose orientation are below or equal to the critical angle, however, will pass through the trap. Consequently, the trap reaches its steady state by \( t = d/v \), where \( d \) is the diameter of the disk and \( v \) is the glider speed inside the trap. The efficiency of the trap increases for smaller friction ratios (Fig. 3e,f). Because gliders that are thin relative to their glide axis are trapped much less effectively than thick gliders, disk-shaped traps provide another mechanism to separate gliders based on shape.

**Conclusion**

Our derivation provides a simple equation for the behavior of gliders at friction discontinuities. This result is directly applicable to experimental systems of gliders, such as gliding filaments [28] and mycoplasma [32]. Further, our formulation of behavior using a force- and torque-balance approach is completely general and pertains to any type of self-propelled particle crossing a friction discontinuity. While the addition of terms capturing particle flexibility or hydrodynamic interactions may no longer result in a Snell's law, the resistive reorientation we derive will still apply.

Unlike other derivations of Snell's law (Section S8), our derivation of Snell's law for gliders relies purely on mechanics, and comes from a transient, shape-dependent torque experienced by the glider during its short crossing of a resistance discontinuity. To us it is surprising that such a local phenomena, so highly unrelated to light, can yield a theory of refraction so similar to that for optics. Future work may expand the theory of Snell's law for gliders to include physical or geometric properties of the glider, such as flexibility or chirality (e.g. left- or right- handed stars). These properties may allow for more exotic effects, such as a negative index of refraction or an analog of circular polarization.

The framework we describe here readily lends itself to the design of environments that control the transport of gliders. Our numerical simulations have demonstrated that glider ray optics may be possible, so that friction lenses, prisms, and traps might be combined to organize gliders to perform tasks. One possible application of glider ray optics is as an alternative to microfluidic lab-on-a-chip systems, where beams of microscopic gliders could traffic molecular cargo across a chip. A fully autonomous system might be created, in which gliders change shape based on their cargo and are routed accordingly through friction prisms, lenses, and traps.
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Author contributions

T.D.R. and P.W.K.R. developed the core concept of the work. J.F.B. and T.D.R. worked on the analytical theory. D.O. built the simulation framework. T.D.R. ran and analyzed the simulations. All authors discussed results and wrote the manuscript.

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Supplementary Information

S1 Comments on the Derivation
Non-discoid gliders will not, in general, have isotropic resistance tensors and we can expect, e.g.,
the force-velocity coupling to have the form $R_{FU} = \zeta_{\parallel}qq + \zeta_{\perp}(I - qq)$, where $\zeta_{\parallel}$ and $\zeta_{\perp}$ are drag coefficients for motion parallel and perpendicular to the glider axis, which we assume be the same as the direction of propulsion $q$. There will also be similar forms for $R_{L\Omega}$ and $R_{LU}$. Clearly, this complicates the analysis and in general it may be impossible to find a Snell-like analytical expression for the refraction of nondiscoid gliders.

S2 Numerical Simulations
We verify the theory and model glider ray optics through numerical simulation. We use a position-dependent friction term in the context of a standard Langevin dynamics approach. We solve the equation for the position, $r$, of a particle in time

$$m \frac{d^2r}{dt^2} = -\gamma(r) \frac{dr}{dt} + F(r), \quad (S1)$$

where $m$ is the mass (in practice this value is low relative to the friction, setting a very short inertial timescale but one which nevertheless allows for greater numerical accuracy when iterating forward), $\gamma(r)$ is some spatially varying friction, and $F$ are the internal and glide forces.

The internal forces arise from the microscopic potentials we use in the simulation of which there are two types. Each particle has a steric repulsion, which gives an upper bound to the possible compressibility of the objects we consider, and for which we use the Weeks-Chandler-Andersen:

$$\phi_{WCA}(r) = \begin{cases} 
4\epsilon \left( \left( \frac{\sigma}{r} \right)^{12} - \left( \frac{\sigma}{r} \right)^6 \right) + \epsilon, & r \leq 2^{1/6}\sigma \\
0, & r > 2^{1/6}\sigma, \end{cases} \quad (S2)$$

where $\sigma$ is the hard sphere diameter $\sigma = 1$ and $\epsilon = 1$.

This is supplemented with the potentials necessary to give our objects structural rigidity, for which we use harmonic potentials

$$\phi_{BOND}(r) = \frac{1}{2}k(r - r_0)^2, \quad (S3)$$

where $r_0$ is some rest length and $k$ is an energy scale. For our simulations we first create an object of a given geometry, and then we add harmonic bonds between nearest neighbor particles where the rest length is taken as the original distance. In general, we set the energy scale $k$ as large as possible in order to ensure structural rigidity of our objects.

The last element of the simulation is the gliding force exerted on the object. We take this force as acting uniformly on every particle. This force is given by

$$F_{\text{glide}} = F_0(\cos \theta, \sin \theta), \quad (S4)$$

where the angle $\theta$ is with respect to the internal axis $q$ of the object and the lab frame. We define the internal axes by taking a row of particles within the glider and averaging over the displacements between neighboring particles. The glider’s orientation is then updated on each time step.
Our simulation code is written in C++. Within a swimmer, the position of each particle is defined by one row of a csv file, where the x and y positions are the values in the first and second columns, respectfully. Bonds lengths between particles are set by the inter-particle distance of the initial configuration. Disk swimmers have a diameter of 29 particles. Rectangular swimmers have a length of 30 particles as shown in Fig. S1. The spatial dependence of viscosity is determined by the function `spatial_viscosity` in `mainShape.cpp`. To implement the prism, ball lens, gradient lens and traps, the geometries and viscosities defined in Section S6 were hard-coded in `spatial_viscosity`.

![Figure S1: Particle representations of rectangular gliders.](image1)

**S3 Curve Fitting and Determination of the Critical Angle**

As discussed in the main text, the exact form of the resistance coefficients depends on how the glider straddles the discontinuity. To generate the curves in Fig. 2a,b, and Fig. S4, we use a least-squares fit of the simulation data.

![Figure S2: Fit parameters used to generate glider Snell’s law curves. a, Fit parameter for a discoid glider. b,c, Fit parameters for rectangular glider.](image2)

For the discoid glider we use

\[
\sin \theta_f = \exp \left(-2c_1(\eta_f/\eta_0)\frac{\eta_f - \eta_0}{\eta_f + \eta_0}\right) \sin \theta_0, \tag{S5}
\]

where \(c_1(\eta_f/\eta_0)\) is the free fit parameter that is determined for each friction ratio (Fig. S2a). We take the fit for \(\sin \theta_f\) to avoid the boundary issue \(\arcsin x \in \mathbb{R}, -1 \leq x \leq 1\). With the values for \(c_1(\eta_f/\eta_0)\) in hand, we calculate the critical angle

\[
\theta_{\text{crit}} = \arcsin \left(\exp \left(2c_1(\eta_f/\eta_0)\frac{\eta_f - \eta_0}{\eta_f + \eta_0}\right)\right), \tag{S6}
\]
For a rectangular glider we follow a similar procedure, but add a second fit parameter since the true $\Delta x_\perp$ is now changed by the glider’s rotation as it moves across the interface. We use

$$\sin \theta_f = \exp \left( -\frac{a}{\ell^2} \left( c_1(\eta_f/\eta_0, a/\ell)a + c_2(\eta_f/\eta_0, a/\ell)(\ell - a) \cos \theta_0 \right) \frac{\eta_f - \eta_0}{\eta_f + \eta_0} \right) \sin \theta_0, \quad (S7)$$

where $c_1(\eta_f/\eta_0, a/\ell)$ and $c_2(\eta_f/\eta_0, a/\ell)$ are fit parameters that are functions of friction ratios and the width-to-length ratio of the glider (Fig. S2b,c). We then find the rectangular glider’s critical angle by numerically solving Eq. (S7) for $\sin \theta_{\text{crit}} = 1$.

### S4 Further Validation of Snell’s Law for Gliders

Figure S3: Size independence and symmetry of refraction. a. Comparison of discoid gliders with different radii but the same incident angle ($\theta_0 = 1.0$). The friction discontinuity is indicated by the vertical dashed line. Glider radii and the $X$-axis are given in terms of the number of particle diameters, for the particles which comprise the gliders. b. Inversion of simulation data and curves for $\eta_f/\eta_0 < 1$ to demonstrate $\theta_f = \theta_0$ symmetry. Data points are simulation and curves are theory. Curves are dashed so that the overlap can be seen.

**Size independence.** Given the same incident angle ($\theta_0 = 1.0$), gliders refract to the same final angle ($\theta_f \approx 0.283$), independent of glider radius (Fig. S3a). However, the detailed trajectory of the glider centroid differs: larger gliders begin their reorientation earlier and finish their reorientation later than smaller gliders.

**Symmetry.** Eqs. (9) and (12) imply that the Snell’s law is symmetric across the line $\theta_f = \theta_0$. We test this by taking a subset of simulation data and curves shown in Fig. 2a and inverting those corresponding to $\eta_f/\eta_0 < 1$ across the line $\theta_f = \theta_0$ (Fig. S3b). The tight overlap of the curves and data points to their mirror partner confirms the predicted symmetry.
S5 Wide Rectangular Gliders

In Fig. 2c, we establish that refraction becomes less effective as $a/\ell$ becomes smaller. Conversely, as $a/\ell$ becomes larger, the effect of refraction increases. This remains true for gliders with $a/\ell > 1$ (Fig. S4). In these simulations, the glider’s width ($2a$) is 30 particles while the length ($2\ell$) is shortened to 5 particles. For a wide glider, the we must change the geometric scaling $\zeta_{rt}/\zeta_{rr} \sim \ell/a^2$, therefore changing Eq. (14) to $\alpha \sim -\frac{1}{\alpha'}(a+(\ell-a) \cos \theta_0) \frac{\Delta \eta}{\eta_0}$. This change is accounted for when fitting the curves in Fig. S4.

We compare this result to the optical Snell’s law for both the refraction curves in Fig. S4 and the critical angles in Fig. 2d using the equivalent speed ratios. We quantify the difference between refraction curves as

$$\% \text{ error} = \frac{\int \left| \theta_{f,\text{glide}}(\theta_0) - \theta_{f,\text{light}}(\theta_0) \right| d\theta_0}{\int \theta_{f,\text{light}}(\theta_0) d\theta_0}.$$  \hspace{1cm} (S8)

Based on the low errors (Table S1), we conjecture that in the limit that $a/\ell \to \infty$, the glider will converge to the optical Snell’s law.

S6 Geometries and Frictions for Ray Optics

The prism is an equilateral triangle where the length of each side is $10\ell$. For the top prism in Fig. 3a, the friction ratio is $\eta_{\text{prism}}/\eta_{\text{out}} = 2$, while the lower prism is $\eta_{\text{prism}}/\eta_{\text{out}} = 2/3$. The ball lens is created...
by making a discoid region of radius $r/\ell = 5$ with a friction ratio $\eta_{\text{lens}}/\eta_{\text{out}} = 8$. The gradient lens is a square with sides of length $\frac{40}{3} \ell$. The parabolic gradient is $\eta_{\text{g-lens}}/\eta_{\text{out}} = 2 \left( 1 - \left( \frac{3(y/\ell - 20/3)}{40} \right)^2 \right)$.

The trap is a disk of radius $r = 30\ell$ with friction $\eta_{\text{out}}/\eta_{\text{trap}} = 0.1$.

**S7 Calculation of Trapping Probabilities**

Trapping probabilities were determined by calculating the all possible incident angles that a glider could have for every point inside the trap. Trapping occurs when $\theta_0 > \theta_{\text{crit}}$ because a glider will continuously reflect to have the same incident angle across the circle. The average trapping probability is calculated by integrating the positional trapping probability and dividing it by the area of the trap.

**S8 Snell’s law and Other Mechanical analogies**

In optics, Snell’s law has been derived based on the wave properties of light, or conservation of energy and momentum [36]. Such derivations are inaccessible here, as gliders do not have wave-like properties, momentum is negligible at low Reynold’s number, and the energy of a self-propelled glider does not follow traditional energy conservation.

Liebchen and Löwen [37] consider the interesting problem of determining the optimum navigation strategy for a swimmer in complex enviroments, such as those including shear, or vortices. For idealized swimmers that can calculate such optimum strategies and steer accordingly, the authors formulate a variational Fermat’s principle that gives an optimal path to minimize travel time, energy dissipation, or fuel consumption. Given global knowledge of the environment this approach leads to geometric paths that follow Snell’s law, or more general refraction, depending on circumstances. Thus Liebchen and Löwen’s result is prescriptive, about what a “smart” swimmer should do, if it can compute and steer, to optimize various quantities. Our result, on the other hand, talks about what an unsteerable mindless glider will do as it interacts with the environment. Their picture is that forcing Snell’s law on a swimmer leads to optimal strategies. Our picture is that a Snell-like law emerges from the local mechanics; it gives no opportunity to optimize arbitrary quantities. Their picture connects swimmers to Snell’s law and ray optics by mathematical analogy—a control algorithm within the swimmer is required to force the mathematics to align, and no environmental optical elements can be constructed. Our picture enables the construction of actual optical elements by physical analogy, with which an unguided glider can interact—these elements, rather than the swimmer’s algorithm, prescribe the path. Thus both the problems and the solutions to our respective work, are only superficially and coincidentally similar due to their connections between Snell’s law and active particles.

We note that there are a variety of other analogies between Snell’s law and other mechanical systems. For example, the movement of a particle on a rigid, two-ramp track under gravity yields a mechanical analog of Snell’s law [38] when least time is considered. This result is similar to that above for swimmer’s, in that it is prescriptive. If a track is so constructed that the two ramps of the track have appropriate slopes, then the particle will follow a least time path from beginning to the end of the track. Again, this analogy is forced by the mathematics and construction of a particular situation, instead of naturally arising from the physics under any circumstance. Analogies between Snell’s law and various mechanical situations, like a car axle moving between carpet and paper, have been used for decades as pedagogical tools [39], but to our knowledge our work is the first to
take such a physical analogy seriously in working out the implications and envisioning a full-blown mechanical analog of ray optics.