ON A DUALITY BETWEEN CERTAIN FINSLER 2-SPHERES AND WEYL ORBIFOLDS

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ABSTRACT. We show that there is a one-to-one correspondence between Finsler structures on the 2-sphere with constant Finsler–Gauss curvature 1 and all geodesics closed on the one hand, and Weyl connections on certain spindle orbifolds whose symmetric Ricci curvature is positive definite and all of whose geodesics closed on the other hand.

1. INTRODUCTION

Roughly speaking, an oriented path geometry on an oriented surface $M$ prescribes an oriented path $\gamma \subset M$ for every oriented direction in $TM$. This notion can be made precise by considering the projective circle bundle $S(TM) := (TM \setminus \{0_M\})/\mathbb{R}^+$ of $M$. Recall that $S(TM)$ carries a tautological co-orientable contact distribution $\tau = \{ w \in T_{[v]}S(TM) : \pi'(w) \wedge v = 0 \}$ where $\pi'$ denotes the differential of the basepoint projection $\pi : S(TM) \to M$. An oriented path geometry is a one-dimensional distribution $P \to S(TM)$ so that $P$ together with the vertical distribution $L = \ker \pi'$ span $\tau$. Note that $P$ is naturally oriented. We may call a non-zero vector $w \in P_{[v]}$ positive if $\pi'(w)$ is a positive multiple of $v$. The orientation of $P$ equips the leaves of the foliation $P$ defined by $P$ with an orientation as well and the oriented paths of $P$ are the projections to $M$ of the oriented leaves of $P$.

For every choice of a 1-form $\Upsilon$ on $S(TM)$ whose kernel is $\tau$ we obtain a volume form $\Omega = \Upsilon \wedge d\Upsilon$ which depends on $\Upsilon$ only up to multiplication with a positive non-vanishing function. We may call a non-zero vertical vector $w \in L$ positive if the $\pi$-pullback of an orientation compatible volume form on $M$ is a positive multiple of the interior product $-w \lrcorner \Omega$. Therefore, the vertical distribution $L$ is naturally oriented as well.

Following [7], a 3-manifold $N$ equipped with a pair of oriented one-dimensional distributions $(P, L)$ spanning a contact distribution is called an oriented generalized path geometry. In this setup the surface $M$ is replaced with the leaf space of the foliation $L$ defined by $L$ and the leaf space of the foliation $P$ defined by $P$ maybe be thought of as the space of oriented paths of the oriented generalized path geometry $(P, L)$. We may reverse the role

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of $P$ and $L$ and thus consider the dual $(-L, -P)$ of the oriented generalized path geometry $(P, L)$, where here the minus sign indicates reversing the orientation.

A natural source of oriented (generalized) path geometries are Finsler metrics $F$ on oriented surfaces. Cartan [10] has shown how to equip the unit circle bundle $\Sigma \subset TM$ of a Finsler metric $F$ on an oriented surface $M$ with a canonical coframing $(\xi, \eta, \nu)$ and, in particular, with the structure of an oriented generalized path geometry $(P, L)$ whose oriented paths are the oriented Finsler geodesics. The same construction associates a generalized path geometry to an oriented Riemannian 2-orbifold. The purpose of this note is to investigate the notion of duality in the case where all geodesics are closed. In this case the dual of the path geometry arising from a Finsler metric on the 2-sphere with constant Finsler–Gauss curvature turns out to arise from a certain generalization of a Besse 2-orbifold [14] with positive curvature. Namely, using a recent result [5] by Bryant et al. about such Finsler metrics (see Theorem 1 below), we show that the space of oriented geodesics is a spindle-orbifold $O$ (see Section 2.1) which comes equipped with a positive Weyl-Besse structure. By this we mean an affine torsion-free connection $\nabla$ on $O$ which preserves some conformal structure – a so-called Weyl connection – and which has the property that the image of every maximal geodesic of $\nabla$ is an immersed circle. Moreover, the symmetric part of the Ricci curvature of $\nabla$ is positive definite. Conversely, having such a positive Weyl-Besse structure on a spindle orbifold, we show that the dual path geometry yields a Finsler metric on $S^2$ with constant Finsler–Gauss curvature all of whose geodesics are closed. More precisely, we prove the following duality result which generalizes [8, Theorem 3] and [9, Proposition 6, Corollary 2] by Bryant:

**Theorem A.** There is a one-to-one correspondence between Finsler structures on $S^2$ with constant Finsler–Gauss curvature 1 and all geodesics closed on the one hand, and positive Weyl-Besse structures on spindle orbifolds $S^2(a_1, a_2)$ with $\kappa := \gcd(a_1, a_2) \in \{1, 2\}$, $a_1 \geq a_2$, $2|(a_1 + a_2)$ and $\kappa^3|a_1a_2$ on the other hand. More precisely,

1. such a Finsler metric with shortest closed geodesic of length $2\pi \mu \in (\pi, 2\pi)$, $\mu = p/q \in (\frac{1}{2}, 1]$, $\gcd(p, q) = 1$, gives rise to a positive Weyl-Besse structure on $S^2(a_1, a_2)$ with $a_1 = q$ and $a_2 = 2p - q$, and
2. a positive Weyl-Besse structure on such a $S^2(a_1, a_2)$ gives rise to such a Finsler metric on $S^2$ with shortest closed geodesic of length $2\pi \frac{a_1 + a_2}{2a_1} \in (\pi, 2\pi]$,

and these assignments are inverse to each other. Moreover, two such Finsler metrics are isometric if and only if the corresponding Weyl–Besse structures coincide up to a diffeomorphism.

Examples of positive Weyl–Besse structures on spindle orbifolds arise in the following ways. The construction of rotationally symmetric Zoll metrics on $S^2$ can be generalized to give an infinite-dimensional family of rotationally
symmetric Riemannian metrics on spindle orbifolds all of whose geodesics are closed [3, 14], and, in particular, an infinite-dimensional family of rotationally symmetric (Riemannian) Weyl–Besse structures most of which are positive. Moreover, in [15, 16] LeBrun–Mason construct a Weyl connection $\nabla$ on the 2-sphere $S^2$ for every totally real embedding of $\mathbb{R}P^2$ into $\mathbb{C}P^2$ which is sufficiently close to the standard “real linear” embedding. The Weyl connection has the property that all of its maximal geodesics are embedded circles and hence defines a Weyl–Besse structure. In addition, they show that every such Weyl connection on $S^2$ is part of a complex 5-dimensional family of Weyl connections having the same unparametrised geodesics (see also [18]). In particular, the Weyl connections of LeBrun–Mason that arise from an embedding of $\mathbb{R}P^2$ that is sufficiently close to the standard embedding provide examples of positive Weyl–Besse structures.

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2. Preliminaries

2.1. Background on orbifolds. Recall that a length space is a metric space in which the distance between any pair of points can be realized as the infimum of lengths of paths connecting the two points. An $n$-dimensional Riemannian orbifold $O^n$ can be defined as a length space such that for each point $x \in O$ there exists a neighbourhood $U$ of $x$ in $O$, an $n$-dimensional Riemannian manifold $M$ and a finite group $\Gamma$ acting by isometries on $M$ such that $U$ and $M/\Gamma$ are isometric. In this case we call $M$ a manifold chart for $O$. Behind this definition lies the observation that an isometric action of a finite group on a simply connected Riemannian manifold can be recovered from the corresponding metric quotient [13, Lem. 2.1]. By this fact every Riemannian orbifold admits a canonical smooth structure. Roughly speaking, this means that there exist equivariant, smooth transition maps between manifold charts (see e.g. [4] for more details). Conversely, every smooth orbifold admits a Riemannian metric, i.e. a collection of Riemannian metrics on the manifold charts with respect to which the group actions and all locally defined transitions between different charts are isometric, whose induced length metric turns it into a Riemannian orbifold in the above sense. For a point $x$ on an orbifold the linearised isotropy group of a preimage of $x$ in a manifold chart is uniquely determined up to conjugation. Its conjugacy class is denoted as $\Gamma_x$ and is called the local group of $O$ at $x$. A point $x \in O$ is called regular if its local group is trivial and otherwise singular.

For example, the metric quotient $O_{a_1,a_2}$ of the unit sphere $S^3 \subset \mathbb{C}^2$ by the isometric action of $S^1 \subset \mathbb{C}$ defined by

$$z(z_1,z_2) = (z^{a_1}z_1, z^{a_2}z_2)$$
for coprime numbers \( a_1 \geq a_2 \) is a Riemannian orbifold which is topologically a 2-sphere, but which metrically has two isolated singular points with cyclic local groups of order \( a_1 \) and \( a_2 \). We denote the underlying smooth orbifold as \( S^2(a_1,a_2) \) and refer to it as a spindle orbifold. The quotient map \( \pi \) from \( S^3 \) to \( O_a \) is an example of an orbifold (Riemannian) submersion, in the sense that for every point \( z \) in \( S^3 \), there is a neighbourhood \( V \) of \( z \) such that \( M/\Gamma = U = \pi(V) \) is a chart, and \( \pi|_V \) factors as \( V \to M/\Gamma = U \), where \( \pi \) is a standard submersion. The standard Hopf action of \( S^1 \) on \( S^3 \) defined by \( z(z_0,z_1) = (zz_1,zz_2) \) commutes with the above \( S^1 \)-action and induces an isometric \( S^1 \)-action on \( O_a \). Let \( \Gamma_k \) be a cyclic subgroup of the Hopf \( S^1 \). The quotient \( S^3/\Gamma_k \) is a lens space of type \( L(k,1) \). By moding out the \( \Gamma_k \) action on \( O_a \) we obtain spindle orbifold \( S^2(a_1,a_2) \) with arbitrary \( a_1 \) and \( a_2 \) as quotients. These spaces fit in the following commutative diagram

\[
\begin{array}{ccc}
S^3 & \longrightarrow & O_a \cong S^2(a_1,a_2) \\
\downarrow & & \downarrow \\
S^2/\Gamma_k \cong L(k,1) \longrightarrow O_a/\Gamma_k \cong S^2(k'a_1,k'a_2)
\end{array}
\]

for some \( k'|k \). Here the left, vertical map is an example \( p : O \to O' \) of a (Riemannian) orbifold covering, i.e. each point \( x \in O' \) has a neighbourhood \( U \) isomorphic to some \( M/\Gamma \) for which each connected component \( U_i \) of \( p^{-1}(U) \) is isomorphic to \( M/\Gamma_i \) for some subgroup \( \Gamma_i < \Gamma \) such that the isomorphisms are compatible with the natural projections \( M/\Gamma_i \to M/\Gamma \) (see [13] for a metric definition). Thurston has shown that the theory of orbifold coverings works analogously to the theory of ordinary coverings [24]. In particular, there exist universal coverings and one can define the orbifold fundamental group \( \pi_1^{orb}(O) \) of a connected orbifold \( O \) as the deck transformation group of the universal covering. For instance, the orbifold fundamental group of \( S^2(a_1,a_2) \) is a cyclic group of order \( \gcd(a_1,a_2) \). Moreover, the number \( k' \) in the diagram is determined in [12, Theorem 4.10] to be

\[
k' = \frac{k}{\gcd(k,a_1-a_2)}.
\]

More generally, in his fundamental monograph [23] Seifert studies foliations of 3-manifolds by circles that are locally orbits of effective circle actions without fixed points (for a modern account see e.g. [22]). The orbit space of such a Seifert fibration naturally carries the structure of a 2-orbifold with isolated singularities. If both the 3-manifold and the orbit space are orientable, then the Seifert fibration can globally be described as a decomposition into orbits of an effective circle action without fixed points (see e.g. [14, Section 2.4] and the references stated therein). In particular, in [23, Chapter 11] Seifert shows that any Seifert fibration of the 3-sphere is given by the orbit decomposition of a weighted Hopf action. The classification of Seifert fibrations of lens spaces, their quotients and their behaviour under
coverings is described in detail in [12]. Let us record the following special statement which will be needed later.

**Lemma 1.** Let $F$ be a Seifert fibration of $\mathbb{RP}^3 \cong L(2,1)$ with orientable quotient orbifold. Then the quotient orbifold is a $S^2(a_1, a_2)$ spindle orbifold, $a_1 \geq a_2$, with $2|(a_1 + a_2)$, $\kappa := \gcd(a_1, a_2) \in \{1, 2\}$ and $\kappa^3 | a_1 a_2$.

**Proof.** Since $\mathbb{RP}^3$ and the quotient surface are orientable, the Seifert fibration is induced by an effective circle action without fixed points. It follows from the homotopy sequence, that the orbifold fundamental group of the quotient is either trivial or $\mathbb{Z}_2$ [22, Lemma 3.2]. In particular, the quotient has to be a spindle orbifold (see e.g. [22, Chapter 3] or [23, Chapter 10]). Moreover, such a Seifert fibration is covered by a Seifert fibration of $S^3$ [12, Theorem 5.1] with quotient $S^2(a_1^0, a_2^0)$ for coprime $a_1^0$ and $a_2^0$ with $a_i = aa_i^0$, and with

$$a = \frac{2}{\gcd(2, a_1^0 + a_2^0)} = \frac{2}{\gcd(2, a_1^0 - a_2^0)}$$

by [12, Theorem 4.10]. This implies $2|(a_1 + a_2)$, $\kappa := \gcd(a_1, a_2) \in \{1, 2\}$ and $\kappa^3 | a_1 a_2$ as claimed.

**Remark 1.** The only lens spaces that admit a Seifert fibration with a non-orientable quotient are $L(4,1)$ and $L(4,3)$ (which coincide up to orientation) [12, Proposition 4.2]. In fact, such fibrations occur as unit tangent bundles of $\mathbb{RP}^2$ fibred over $\mathbb{RP}^2$.

Usually notions that make sense for manifolds can also be defined for orbifolds. The general philosophy is to either define it in the manifold charts and demand it to be invariant under the action of the local groups (and transitions between charts as in the manifold case) like in the case of a Riemannian metric, or to demand certain lifting conditions. For instance, a map between orbifolds is called *smooth* if it locally lifts to smooth maps between manifolds charts. Let us also explicitly mention that the tangent bundle of an orbifold, like other bundles too, can be defined by gluing together quotients of the tangent bundles of manifold charts by the actions of the local groups [1, Proposition 1.21]. In particular, the tangent bundle of an orbifold is again an orbifold. Moreover, if the orbifold has only isolated singularities, then its unit tangent bundle (with respect to some Riemannian metric) is in fact a manifold. For instance, the unit tangent bundle of a $S^2(a_1, a_2)$ spindle orbifold is an $L(a_1 + a_2, 1)$ lens space [14, Lemma 3.1]. In many cases we will liberally use orbifold notions which follow this general philosophy without further explanation, and refer to the literature for a more detailed account, e.g. [21, 1, 4].

**2.2. Besse orbifolds.** The Riemannian spindle orbifolds $O_a \cong S^2(a_1, a_2)$ constructed in the preceding section have the additional property that all their geodesics are closed, i.e. any geodesic factors through a closed geodesic. Here an (orbifold) *geodesic* on a Riemannian orbifold is a path that can locally be lifted to a geodesic in a manifold chart, and a *closed geodesic* is
a loop that is a geodesic on each subinterval. We call a Riemannian metric on an orbifold as well as a Riemannian orbifold Besse, if all its geodesics are closed. The moduli space of Besse metrics on spindle orbifolds is in fact infinite-dimensional [14]. For more details about Besse orbifolds we refer to [14, 2].

2.3. Finsler structures. A Finsler metric on a manifold is – roughly speaking – a Banach norm on each tangent space varying smoothly from point to point. Instead of specifying the family of Banach norms, one can also specify the norm’s unit vectors in each tangent space. Here we only consider oriented Finsler surfaces and use definitions for Finsler structures from [7]:

A Finsler structure on an oriented surface $M$ is a smooth hypersurface $\Sigma \subset TM$ for which the basepoint projection $\pi : \Sigma \to M$ is a surjective submersion which has the property that for each $p \in M$ the fibre $\Sigma_p = \pi^{-1}(p) = \Sigma \cap T_p M$ is a closed, strictly convex curve enclosing the origin $0 \in T_p M$. A smooth curve $\gamma : [a, b] \to M$ is said to be a $\Sigma$-curve if its velocity $\dot{\gamma}(t)$ lies in $\Sigma$ for every time $t \in [a, b]$. For every immersed curve $\gamma : [a, b] \to M$ there exists a unique orientation preserving diffeomorphism $\Phi : [0, \ell] \to [a, b]$ such that $\phi := \gamma \circ \Phi$ is a $\Sigma$-curve. The number $\ell \in \mathbb{R}^+$ is the length of $\gamma$ and the curve $\dot{\phi} : [a, b] \to \Sigma$ is called the tangential lift of $\gamma$. Note that in general the length may depend on the orientation of the curve.

Cartan [10] has shown how to associate a coframing to a Finsler structure on an oriented surface $M$. For a modern reference for Cartan’s construction the reader may consult [6]. Let $\Sigma \subset TM$ be a Finsler structure. Then there exists a unique coframing $\omega = (\xi, \eta, \nu)$ of $\Sigma$ with dual vector fields $(X, H, V)$ which satisfies the structure equations

\[
\begin{align*}
    d\xi &= -\eta \wedge \nu, \\
    d\eta &= -\nu \wedge (\xi - I\eta), \\
    d\nu &= -(K\xi - J\nu) \wedge \eta,
\end{align*}
\]

for some smooth functions $I, J, K : \Sigma \to \mathbb{R}$. Moreover the $\pi$-pullback of any positive volume form on $M$ is a positive multiple of $\xi \wedge \eta$ and the tangential lift of any $\Sigma$-curve $\gamma$ satisfies

\[
\dot{\gamma}^* \eta = 0 \quad \text{and} \quad \dot{\gamma}^* \xi = dt.
\]

A $\Sigma$-curve $\gamma$ is a $\Sigma$-geodesic, that is, a critical point of the length functional, if and only if its tangential lift satisfies $\dot{\gamma}^* \nu = 0$. The integral curves of $X$ therefore project to $\Sigma$-geodesics on $M$ and hence the flow of $X$ is called the geodesic flow of $\Sigma$.

For a Riemannian Finsler structure the functions $I, J$ vanish identically whereas $K$ is constant on the fibers of $\pi : \Sigma \to M$ and therefore the $\pi$-pullback of a function on $M$ which is the Gauss curvature $K_g$ of $g$. Since in the Riemannian case the function $K$ is simply the Gauss curvature, it is usually called the Finsler–Gauss curvature. In general $K$ need not be constant on the fibers of $\pi : \Sigma \to M$. 

Let \( \Sigma \subset TM \) and \( \hat{\Sigma} \subset T\hat{M} \) be two Finsler structures on oriented surfaces with coframings \( \omega \) and \( \hat{\omega} \). An orientation preserving diffeomorphism \( \Phi : M \to \hat{M} \) with \( \Phi'(\Sigma) = \hat{\Sigma} \) is called a Finsler isometry. It follows that for a Finsler isometry \( (\Phi'|_\Sigma)^*\hat{\omega} = \omega \) and conversely any diffeomorphism \( \Xi : \Sigma \to \hat{\Sigma} \) which pulls-back \( \hat{\omega} \) to \( \omega \) is of the form \( \Xi = \Phi' \) for some Finsler isometry \( \Phi : M \to \hat{M} \).

Following [7, Def. 1], we use the following definition:

**Definition 1.** A coframing \((\xi, \eta, \nu)\) on a 3-manifold \( \Sigma \) satisfying the structure equations (2) for some functions \( I, J \) and \( K \) on \( \Sigma \) will be called a generalized Finsler structure.

As in the case of a Finsler structure we denote the dual vector fields of \((\xi, \eta, \nu)\) by \((X, H, V)\). Note that a generalized Finsler structure naturally defines an oriented generalized path geometry by defining \( P \) to be spanned by \( X \) while calling positive multiples of \( X \) positive and by defining \( L \) to be spanned by \( V \) while calling positive multiples of \( V \) positive.

**Example 1.** Let \((O, g)\) be an oriented Riemannian 2-orbifold. In particular, \( O \) has only isolated singularities. Then the unit tangent bundle

\[ SO := \{ v \in TO : |v|_g = 1 \} \subset TO \]

is a manifold, and like in the case of a smooth Finsler structure it can be equipped with a canonical coframing as well. However, in order to distinguish the Riemannian orbifold case from the smooth Finsler case, we will use the notation \((\alpha, \beta, \psi)\) instead of \((\xi, \eta, \nu)\) for the coframing. The construction is as follows: A manifold chart \( M/\Gamma \) of \( O \) gives rise to a manifold chart \( SM/\Gamma \) of \( SO \). In such a chart the first two coframing forms are explicitly given by

\[ \alpha_v(w) := g(\pi'_v(w), v), \quad \beta_v(w) := g(\pi'_v(w), iv), \quad w \in T_vSM. \]

Here \( \pi : SO \to O \) denotes the basepoint projection and \( i : TM \to TM \) the rotation of tangent vectors by \( \pi/2 \) in positive direction. Note that these expressions are invariant under the group action of \( \Gamma \) and hence in fact define forms on \( OM \). The third coframe form \( \psi \) is the Levi-Civita connection form of \( g \) and we have the structure equations

\[ d\alpha = -\beta \wedge \psi, \quad d\beta = -\psi \wedge \alpha, \quad d\psi = -(K_g \circ \pi)\alpha \wedge \beta, \]

where \( K_g \) denotes the Gauss curvature of \( g \). Moreover, note that \( \pi^*dA_g = \alpha \wedge \beta \) where \( dA_g \) denotes the area form of \( O \) with respect to \( g \). Denoting the vector fields dual to \((\alpha, \beta, \psi)\) by \((A, B, \Psi)\) we observe that the flow of \( \Psi \) is \( 2\pi \)-periodic. Finally, if \( O \) is a manifold, then the coframing \((\alpha, \beta, \psi)\) agrees with Cartan’s coframing \((\xi, \eta, \nu)\) on the Riemannian Finsler structure \( \Sigma = SO \).
2.4. Weyl structures and connections. A Weyl connection on an orbifold \( \mathcal{O} \) is an affine torsion-free connection on \( \mathcal{O} \) preserving some conformal structure \([g]\) on \( \mathcal{O} \) in the sense that its parallel transport maps are angle preserving with respect to \([g]\). An affine torsion-free connection \( \nabla \) is a Weyl connection with respect to the conformal structure \([g]\) on \( \mathcal{O} \) if for some (and hence any) conformal metric \( g \in [g] \) there exists a 1-form \( \theta \in \Omega^1(\mathcal{O}) \) such that

\[
\nabla g = 2\theta \otimes g.
\]

Conversely, it follows from Koszul’s identity that for every pair \((g, \theta)\) consisting of a Riemannian metric \( g \) and 1-form \( \theta \) on \( \mathcal{O} \) the connection

\[
(g, \theta)\nabla_X Y = g\nabla_X Y + g(X, Y)\theta^\sharp - \theta(X)Y - \theta(Y)X, \quad X, Y \in \Gamma(T\mathcal{O})
\]

is the unique affine torsion-free connection satisfying (3). Here \( g\nabla \) denotes the Levi-Civita connection of \( g \) and \( \theta^\sharp \) is the vector field dual to \( \theta \) with respect to \( g \). Notice that for \( u \in C^\infty(\mathcal{O}) \) we have the formula

\[
\exp(2u)g\nabla_X Y = g\nabla_X Y - g(X, Y)(du)^\sharp + du(X)Y + du(Y)X, \quad X, Y \in \Gamma(T\mathcal{O}).
\]

From which one easily computes the identity

\[
(\exp(2u)g, \theta + du)\nabla = (g, \theta)\nabla.
\]

Consequently, we define a Weyl structure to be an equivalence class \([g, \theta]\) subject to the equivalence relation

\[
(g, \theta) \sim (\hat{g}, \hat{\theta}) \iff \hat{g} = e^{2u}g \quad \operatorname{and} \quad \hat{\theta} = \theta + du, \quad u \in C^\infty(\mathcal{O}).
\]

Clearly, the mapping which assigns to a Weyl structure \([g, \theta]\) its Weyl connection \((g, \theta)\nabla\) is a one-to-one correspondence between the set of Weyl structures – and the set of Weyl connections on \( \mathcal{O} \).

The Ricci curvature of a Weyl connection \((g, \theta)\nabla\) on \( \mathcal{O} \) is

\[
\text{Ric}((g, \theta)\nabla) = (K_g - \delta_g \theta)g + 2d\theta
\]

where \( \delta_g \) denotes the co-differential with respect to \( g \).

**Definition 2.** We call a Weyl structure \([g, \theta]\) positive if the symmetric part of the Ricci curvature of its associated Weyl connection is positive definite.

In the case where \( \mathcal{O} \) is oriented we may equivalently say the Weyl structure \([g, \theta]\) is positive if the 2-form \((K_g - \delta_g \theta)dA_g\) – which only depends on the orientation and given Weyl structure – is an orientation compatible volume form on \( \mathcal{O} \). Note that by the Gauss–Bonnet theorem [21] simply connected spindle orbifolds are the only simply connected 2-orbifolds carrying positive Weyl structures.

We now obtain:

**Lemma 2.** Every positive Weyl structure contains a unique pair \((g, \theta)\) satisfying \( K_g - \delta_g \theta = 1 \).
Proof. We have the following standard identity for the change of the Gauss curvature under conformal change
\[ K_{e^{2u}g} = e^{-2u} (K_g - \Delta_g u) \]
where \( \Delta_g = -(d \delta_g + \delta_g d) \) is the negative of the Laplace–de Rham operator. Also, we have the identity
\[ \delta_{e^{2u}g} = e^{-2u} \delta_g \]
for the co-differential acting on 1-forms.

If \([g, \theta]\) is a positive Weyl structure, we may take any representative \((g, \theta)\), define \(u = \frac{1}{2} \ln(K_g - \delta_g \theta)\) and consider the representative \((\hat{g}, \hat{\theta}) = (e^{2u}g, \theta + du)\). Then we have
\[ K_{\hat{g}} - \delta_{\hat{g}} \hat{\theta} = K_g - \delta_g \theta = 1. \]

Suppose the two representative pairs \((g, \theta)\) and \((\hat{g}, \hat{\theta})\) both satisfy \(K_{\hat{g}} - \delta_{\hat{g}} \hat{\theta} = K_g - \delta_g \theta = 1\). Since they define the same Weyl connection, the expression for the Ricci curvature implies that \(\hat{g} = (K_{\hat{g}} - \delta_{\hat{g}} \hat{\theta})\hat{g} = (K_g - \delta_g \theta)g = g\) and hence also \(\hat{\theta} = \theta\), as claimed. \(\square\)

Definition 3. For a positive Weyl structure \([g, \theta]\) we call the unique representative pair \((g, \theta)\) satisfying \(K_g - \delta_g \theta = 1\) the natural gauge of \([g, \theta]\).

Lemma 3. Let \([g, \theta]\) be a positive Weyl structure on an orientable 2-orbifold \(O\) with natural gauge \((g, \theta)\) and let \(\pi : SO \to O\) denote the unit tangent bundle of \(g\) equipped with its canonical coframing \((\alpha, \beta, \psi)\). Then the coframing
\[ \xi := \pi^*(\ast_g \theta) - \psi, \quad \eta := -\beta, \quad \nu := -\alpha \]
defines a generalized Finsler structure of constant Finsler–Gauss curvature \(K = 1\) on \(SO\).

Proof. We compute that
\[ d\xi = d(\pi^*(\ast_g \theta) - \psi) = \pi^*((K_g - \delta_g \theta) dA_g) = \alpha \wedge \beta = -\eta \wedge \nu \]
and
\[ d\eta = -d\beta = \psi \wedge \alpha = (\xi - \pi^*(\ast_g \theta)) \wedge \nu = -\nu \wedge (\xi - \pi^*(\ast_g \theta)) \]
Now observe that \(\pi^*(\ast_g \theta) = -\Psi(\theta)\alpha + \theta \beta\) where on the right hand side we think of \(\theta\) as a real-valued function on \(SO\). Since \(\nu = -\alpha\), we thus have
\[ d\eta = -\nu \wedge (\xi - I\eta), \]
for \(I = -\theta\), again interpreted as a function on \(SO\). Likewise, we obtain
\[ d\nu = -d\alpha = \beta \wedge \psi = - (\xi - \pi^*(\ast_g \theta)) \wedge \eta = - (\xi - J\nu) \wedge \eta, \]
where \(J = \Psi(\theta)\). The claim follows. \(\square\)
Remark 2. We remark that correspondingly we have a natural gauge \((g, \theta)\) for a negative Weyl structure, that is, \((g, \theta)\) satisfy \(K_g - \delta_g = -1\). On a closed oriented surface (necessarily of negative Euler characteristic) the associated flow generated by the vector field \(A - \Psi(\theta)\Psi\) falls into the family of flows introduced in [19]. In particular, its dynamics is Anosov.

The geometric significance of the form \(\xi\) in Lemma 3 is described in the following statement. For a proof in the manifold case – which carries over mutatis mutandis to the orbifold case – the reader may consult [20, Lemma 3.1].

Lemma 4. Let \((g, \theta)\) be a pair on an orientable 2-orbifold \(O\) and let \(\pi: SO \to O\) denote the unit tangent bundle of \(g\) with canonical coframing \((\alpha, \beta, \psi)\). Then the leaves of the foliation defined by \(\{\beta, \psi - \pi^*(\star g \theta)\}^\perp\) project to \(O\) to become the (unparametrised) oriented geodesics of the Weyl connection defined by \([g, \theta]\).

We conclude this section with a definition:

Definition 4. An affine torsion-free connection \(\nabla\) on \(O\) is called Besse if the image of every maximal geodesic of \(\nabla\) is an immersed circle. A Weyl structure whose Weyl connection is Besse will be called a Weyl–Besse structure.

Note that the Levi-Civita connection of any (orientable) Besse orbifold \(O\) (see Section 2.2) gives rise to a Weyl–Besse structure on \(SO\).

3. A Duality Theorem

Let us cite the following result from [5].

Theorem 1 (Bryant, Foulon, Ivanov, Matveev, Ziller). Let \(\Sigma \subset TS^2\) be a Finsler structure on \(S^2\) with constant Finsler–Gauss curvature 1 and all geodesics closed. Then there exists a shortest closed geodesic of length \(2\pi\mu \in (\pi, 2\pi]\) and the following holds:

1. Either \(\mu = 1\) and all geodesics are closed and have the same length \(2\pi\),
2. or \(\mu = p/q \in (\frac{1}{2}, 1)\) with \(p, q \in \mathbb{N}\) and \(\gcd(p, q) = 1\), and in this case all unit-speed geodesics have a common period \(2\pi p\). Furthermore, there exists at most two closed geodesics with length less than \(2\pi p\). A second one exists only if \(2p - q > 1\), and its length is \(2\pi p/(2p - q) \in (2\pi, 2p\pi)\).

In particular, if all geodesics of a Finsler metric on \(S^2\) are closed, then its geodesic flow is periodic with period \(2\pi p\) for some integer \(p\).

We now have our main duality result:

Theorem A. There is a one-to-one correspondence between Finsler structures on \(S^2\) with constant Finsler–Gauss curvature 1 and all geodesics closed on the one hand, and positive Weyl–Besse structures on spindle orbifolds...
$S^2(a_1, a_2)$ with $\kappa := \gcd(a_1, a_2) \in \{1, 2\}$, $a_1 \geq a_2$, $2|(a_1 + a_2)$ and $\kappa^3 | a_1 a_2$

on the other hand. More precisely,

(1) such a Finsler metric with shortest closed geodesic of length $2\pi \mu \in (\pi, 2\pi]$, $\mu = p/q \in (\frac{1}{2}, 1]$, $\gcd(p, q) = 1$, gives rise to a positive Weyl–Besse structure on $S^2(a_1, a_2)$ with $a_1 = q$ and $a_2 = 2p - q$, and

(2) a positive Weyl–Besse structure on such a $S^2(a_1, a_2)$ gives rise to such a Finsler metric on $S^2$ with shortest closed geodesic of length $\frac{2\pi a_1 + a_2}{2a_1} \in (\pi, 2\pi]$,

and these assignments are inverse to each other. Moreover, two such Finsler metrics are isometric if and only if the corresponding Weyl–Besse structures coincide up to a diffeomorphism.

Proof. In case of $2\pi$-periodic geodesic flows the first statement is already contained in [9]. To prove the general statement let $\Sigma \subset TS^2$ be a $K = 1$ Finsler structure with $2\pi p$-periodic geodesic flow $\phi : \Sigma \times \mathbb{R} \to \Sigma$, i.e. the flow factorizes through a smooth, almost free $S^1$-action $\phi : \Sigma \times S^1 \to \Sigma$. The Cartan coframe will be denoted by $(\xi, \eta, \nu)$ and using the structure equations for the $S^1$-bundle compute the Lie derivative

$\mathcal{L}_X (\eta \otimes \eta + \nu \otimes \nu) = \nu \otimes \eta + \eta \otimes \nu - \eta \otimes \nu - \nu \otimes \eta = 0$.

Likewise, we compute $\mathcal{L}_X (\nu \wedge \eta) = 0$. Hence the symmetric 2-tensor $\eta \otimes \eta + \nu \otimes \nu$ and the 2-form $\nu \wedge \eta$ are invariant under $\phi$ and therefore there exists a unique Riemannian metric $g$ on $\mathcal{O}$ for which $\lambda^* g = \eta \otimes \eta + \nu \otimes \nu$ where $\lambda : \Sigma \to \mathcal{O}$ is the natural projection. We may orient $\mathcal{O}$ in such a way that the pullback of the area form $dA_g$ of $g$ satisfies $\lambda^* dA_g = \nu \wedge \eta$. The structure equations also imply that $\xi, \eta, \nu$ are invariant under $(\phi_{2n})^*$ (cf. [7, p. 186]). Therefore the map

$\varphi: \Sigma \to T\mathcal{O}$

$v \mapsto -\lambda^*_\nu(V(v))$

and the forms $\xi, \eta, \nu$ are invariant under the action of the cyclic subgroup $\Gamma < S^1$ of order $p$ on $\Sigma$. Hence $\varphi$ factors through a map $\tilde{\varphi}: \Sigma/\Gamma \to T\mathcal{O}$, and $\xi, \eta, \nu$ descend to $\Sigma/\Gamma$ where they define a generalized Finsler structure. The composition of $\tilde{\varphi}$ with the canonical projection onto the projective sphere bundle $ST\mathcal{O} := (T\mathcal{O} \setminus \{0\})/\mathbb{R}^+$ will be denoted by $\hat{\varphi}$. Note that $\hat{\varphi}$ is an immersion, thus a local diffeomorphism and by compactness of $\Sigma/\Gamma$ and connectedness of $ST\mathcal{O}$ a covering map. Since by [14, Lemma 3.1] both $\Sigma/\Gamma$ and $ST\mathcal{O}$ have fundamental group of order $2p$, it follows that $\hat{\varphi}$ is a
diffeomorphism. Therefore, \( \tilde{\varphi} \) is an embedding which sends \( \Sigma/\Gamma \) to the total space of the unit tangent bundle \( \pi: SO \to O \) of \( g \). Abusing notation, we also write \( \xi, \eta, \nu \in \Omega^1(SO) \) to denote the pushforward with respect to \( \tilde{\varphi} \) of the Cartan coframe on \( \Sigma/\Gamma \). Also we let \( \alpha, \beta, \psi \in \Omega^1(SO) \) denote the canonical coframe of \( SO \) with respect to the orientation induced by \( dA_g \).

More precisely, the pullback of \( g \) to \( SO \) is \( \alpha \otimes \alpha + \beta \otimes \beta \) and \( \psi \) denotes the Levi-Civita connection form. By construction, the map \( \varphi \) sends lifts of \( \Sigma \) geodesics onto the fibres of the projection \( \pi: SO \to O \). Moreover, for \( v \in \Sigma \) the projection \( (\pi \circ \varphi)^* V(v) \) to \( T_{(\pi \circ \varphi)(v)} O \), i.e. the horizontal component of \( \varphi^* V(v) \), is parallel to \( \varphi(v) \) and so the vertical vector field \( V \) on \( \Sigma \) is mapped into the contact distribution defined by the kernel of \( \beta \). Therefore, we see that \( \beta \) and \( \eta \) are linearly dependent and that \( \nu(\varphi^* V), \alpha(\varphi^* V) < 0 \).

However, since both \( (\alpha, \beta) \) and \( (\nu, \eta) \) are oriented orthonormal coframes for \( g \), it follows that \( \beta = -\eta \) and \( \alpha = -\nu \). The structure equations for the coframing \( (\alpha, \beta, \psi) \) imply

\[
0 = d\alpha + \beta \wedge \psi = -d\nu - \eta \wedge \psi = (\xi - J\nu) \wedge \eta - \eta \wedge \psi = -\eta \wedge (\xi - J\nu + \psi)
\]

and

\[
0 = d\beta + \psi \wedge \alpha = -d\eta - \psi \wedge \nu = \nu \wedge (\xi - I\eta) - \psi \wedge \nu = - (\xi - I\eta + \psi) \wedge \nu,
\]

where again we abuse notation by also writing \( I \) and \( J \) for the pushforward of the functions \( I \) and \( J \) with respect to \( \tilde{\varphi} \). It follows that the Levi-Civita connection form \( \psi \) of \( g \) satisfies

\[
\psi = I\eta + J\nu - \xi = - (J\alpha + I\beta) - \xi.
\]

Recall that \( \pi: SO \to O \) denotes the basepoint projection. Comparing with Lemma 3 we want to argue that there exists a unique 1-form \( \theta \) on \( O \) so that \( \pi^*(\star g \theta) = -(J\alpha + I\beta) \). Since \( J\alpha + I\beta \) is semibasic for the projection \( \pi \), it is sufficient to show that \( J\alpha + I\beta \) is invariant under the SO(2) right action generated by the vector field \( \Psi \), where \( (A, B, \Psi) \) denote the vector fields dual to \( (\alpha, \beta, \psi) \). Denoting by \( (X, H, V) \) the vector fields dual to \( (\xi, \eta, \nu) \) on \( SO \), the identities

\[
\begin{pmatrix}
\alpha \\
\beta \\
\psi
\end{pmatrix} =
\begin{pmatrix}
-\nu \\
-\eta \\
I\eta + J\nu - \xi
\end{pmatrix}
\]

imply \( \Psi = -X \). Now observe the Bianchi identity

\[
0 = d^2 \eta = (J\xi - dI) \wedge \eta \wedge \nu
\]

so that \( XI = J \). Likewise we obtain

\[
0 = d^2 \nu = -(dJ + I\xi) \wedge \eta \wedge \nu
\]

so that \( XJ = -I \). From this we compute

\[
L_X (I\eta + J\nu) = J\eta + I\nu - I\nu - J\eta = 0,
\]
so that $- (J\alpha + I\beta) = \pi^* (\star g \theta)$ for some unique 1-form $\theta$ on $O$ as desired. We obtain a Weyl structure defined by the pair $(g, \theta)$. Since

$$d(\pi^*(\star g \theta) - \psi) = d(\pi^*(- (J\alpha + I\beta) + (J\alpha + I\beta + \xi))) = d\xi = \nu \wedge \eta$$

we see that $K_g - \delta_g \theta = 1$. Therefore $(g, \theta)$ is the natural gauge for the positive Weyl structure $[g, \theta]$. Finally, by construction, the Weyl structure $[g, \theta]$ is Besse.

Conversely, let $O = S^2(a_1, a_2)$ be a spindle orbifold as in (2) with a positive Weyl-Besse structure $[g, \theta]$. Let $(g, \theta)$ be the natural gauge of $[g, \theta]$ and let $\pi : SO \to O$ denote the unit tangent bundle with respect to $g$. By [14, Lemma 3.1] the unit-tangent bundle $SO$ is a lens space of type $L(a_1 + a_2, 1)$. The canonical coframe on $SO$ as explained in Example 1 will be denoted by $(\alpha, \beta, \psi)$. By Lemma 3 the 1-forms $\xi, \eta, \nu$ on $SO$ given by

$$\xi := \pi^*(\star \theta) - \psi, \quad \eta := -\beta, \quad \nu := -\alpha$$

define a generalized Finsler structure on $SO$ of constant Finsler–Gauss curvature $K = 1$, i.e. they satisfy the structure equations

(5)  \[ d\xi = -\eta \wedge \nu, \quad d\eta = -\nu \wedge (\xi - I\eta), \quad d\nu = -(\xi - J\nu) \wedge \eta, \]

for some smooth functions $I, J : SO \to \mathbb{R}$. Moreover they parallelise $SO$ and have the property that the leaves of the foliation $F_g := \{\xi, \eta\}^\perp$ are tangential lifts of maximal oriented geodesics of the Weyl connection $(g, \theta) \nabla$ on $O$. Since this connection is Besse by assumption all of these leaves are circles. It follows from a theorem by Epstein [11] that the leaves are the orbits of a smooth, almost free $S^1$-action. Since $a_1 + a_2$ is odd, $SO$ admits a normal covering by a space $M \cong L(2, 1) \cong \mathbb{RP}^3$ with deck transformation group $\Gamma$ isomorphic to $\mathbb{Z}_{(a_1 + a_2)/2}$. The lifts of $\xi, \eta, \nu$ to $M$, which we denote by the same symbols, define a generalized Finsler structure on $M$ of constant Finsler–Gauss curvature 1. Moreover, the $S^1$-action on $SO$ lifts to a smooth, almost free $S^1$-action on $M$ whose orbits are again the leaves of the foliation $\{\xi, \eta\}^\perp$. The leaves of the foliation $F_I := \{\eta, \nu\}^\perp$ correspond to (the lifts of) the fibers of the projection $SO \to O$ (to $M$) and are in particular also all circles. We can cover the space $M$ further by $S^3$ and lift the $S^1$-action and the foliations $F_g$ and $F_I$ further to $S^3$. By the classification of Seifert fibrations of lens spaces quotienting out the foliations $F_I$ and $F_g$ of $SO$, $M$
and $S^3$ yields a diagram of maps as follows (cf. Section 2.1 and e.g. [12])

$$
\begin{align*}
\bar{O} \cong S^2(a_1/\kappa, a_2/\kappa) & \quad \longrightarrow \quad \bar{M} \cong S^3 & \quad \longrightarrow \quad \bar{O_g} \cong S^2(k_1, k_2) \\
\bar{O} \cong S^2(a_1/a, a_2/a) & \quad \longrightarrow \quad M \cong L(2, 1) & \quad \longrightarrow \quad \bar{O_g} \cong S^2(k'k_1, k'k_2) \\
O \cong S^2(a_1, a_2) & \quad \longrightarrow \quad SO \cong L(a_1 + a_2, 1) & \quad \longrightarrow \quad O_g \cong S^2(kk_1, kk_2)
\end{align*}
$$

with $a|\gcd(a_1, a_2) = \kappa \in \{1, 2\}$, $\gcd(k_1, k_2) = 1$, $k'|2$ and $k|\Gamma = (a_1 + a_2)/2$, where the horizontal maps are smooth orbifold submersions, and the vertical maps are coverings (of manifolds in the middle and of orbifolds on the left and the right). Moreover, the deck transformation groups in the middle descend to deck transformation groups of the orbifold coverings. We claim that $a = 1$. To prove this we can assume that $\kappa = 2$. In this case the coprime numbers $a_1/\kappa$ and $a_2/\kappa$ have different parity by our assumption that $\kappa^3|a_1a_2$. Since $a_1/a + a_2/a$ has to be even by Lemma 1 it follows that $a = 1$ as claimed.

The involution

$$
i : \quad SO \quad \rightarrow \quad SO \\
(x, v) \quad \mapsto \quad (x, -v).
$$

maps fibers of $F_g$ and $F_t$ to fibers of $F_g$ and $F_t$, respectively, and descends to a smooth orbifold involution $i$ of $O_g$. We claim that the same argument as in [14] shows that $i$ does not fix the singular points on $O_g$. Here we only sketch the ideas and refer to [14] for the details: If $a_1$ and $a_2$ are odd then $i$ acts freely on $O_g$, and in this case nothing more has to be said. On the other hand, if $a_1$ and $a_2$ are even, then any geodesic that runs into a singular point is fixed by the action of $i$ on $O_g$. In this case one first has to show that the lift $\bar{i} : \bar{M} \rightarrow \bar{M}$ of $i$ to the universal covering of $SO$ commutes with the deck transformation group $\Gamma$ of the covering $\bar{M} \rightarrow SO$. This can be shown based on the observation that a fiber of $F_t$ on $S^3$ over the singular point of $O$, together with its orientation, is preserved by both $\Gamma$ and $\bar{i}$ (see [14, Lemma 3.4] for the details). Now, if $a_1$ and $a_2$ are both even and a singular point on $O_g$ is fixed by $i$, then there also exists a fiber of $F_g$ on $S^3$ which is invariant under both $\Gamma$ and $\bar{i}$. However, in this case only $\Gamma$ preserves the orientation of this fiber whereas $\bar{i}$ reverses its orientation. This leads to a contradiction to the facts that $|\Gamma| = a_1 + a_2 > 2$ and that $\Gamma$ commutes with $\bar{i}$ (see [14, Lemma 3.5] for the details).

Since $i$ preserves the orbifold structure of $\Gamma_g$ and does not fix its singular points, it has to interchange the singular points. In particular, this implies that $kk_1 = kk_2$, and hence $k_1 = k_2 = 1$. Therefore the foliation $F$ on $S^3$ is the Hopf-fibration and we must have $k' = 1$ by Lemma 1. In other words, $O_g$ is a smooth 2-sphere without singular points and $\tau : M \rightarrow O_g = S^2$ is a
Consider the map
\[ \varphi : M \to TS^2 \]
\[ u \mapsto -\tau'_u(X(u)). \]
Then by [7, Prop. 1] \( \varphi \) immerses each \( \tau \)-fiber \( \tau^{-1}(x) \) as a curve in \( T_xS^2 \) that is strictly convex towards \( 0_x \). The number of times \( \varphi(\tau^{-1}(x)) \) winds around \( 0_x \) does not depend on \( x \). Since both \( M \) and \( S \) are diffeomorphic to \( L(2,1) \) the same argument as above proves that \( \varphi \) is one-to-one and so this number is 1. Therefore by [7, Prop. 2] \( \varphi(M) \) is a Finsler structure on \( S^2 \). Moreover \( S^2 \) can be oriented in such a way that the \( \varphi \)-pullback of the canonical coframing induced on \( \varphi(M) \) agrees with \( (\xi, \eta, \nu) \). In particular this implies that the Finsler structure satisfies \( K = 1 \) and has periodic geodesic flow. Moreover, because of \( a = 1 \) we have \( \mathcal{O} = \mathcal{O} \) and therefore the preimages of the leaves of \( \mathcal{F}_t \) under the covering \( M \to SO \) are connected. Since the covering \( M \to SO \) is \( (a_1 + a_2)/2 \)-fold, so is its restriction to the fibers of \( \mathcal{F}_t \). Therefore, \( p := (a_1 + a_2)/2 \) is the minimal number for which the geodesic flow of the Finsler structure on \( S^2 \) is \( 2\pi p \)-periodic. The structure of \( \mathcal{O} \) implies that all closed geodesics of \( S^2 \) have length \( 2\pi p \) except at most two exceptions which are \( q := a_1 \) and \( 2q - p = a_2 \) times shorter than the regular geodesics. In particular, the shortest geodesic has length \( 2\pi p/q = 2\pi \frac{a_1 + a_2}{2m} \) as claimed.

Finally, going through the proof shows that an isometry between two Finsler metrics as in the statement of Theorem induces a diffeomorphism between the corresponding spindle orbifolds that pulls back the two natural gauges onto each other, and vice versa. Hence, since such a pullback of a natural gauge is a natural gauge, the last statement of the Theorem follows from uniqueness of the natural gauge of a given Weyl–Besse structure. \( \square \)

**Remark 3.** In the case of the standard round metric on \( S^2 \) whose geodesics are the great circles, the corresponding complex 5-manifold of Weyl connections sharing the same unparametrized geodesics consists of the smooth quadrics in \( \mathbb{C}P^2 \) without real points [17]. On the Finsler side this recovers Bryant’s classification of Finsler structures of constant Finsler–Gauss curvature \( K = 1 \) on \( S^2 \) with linear geodesics [7].

**References**

[1] A. Adem, J. Leida, and Y. Ruan, *Orbifolds and stringy topology*, Cambridge Tracts in Mathematics **171**, Cambridge University Press, Cambridge, 2007. MR 2359514 5

[2] M. Amann, C. Lange, and M. Radeschi, Odd-dimensional orbifolds with all geodesics closed are covered by manifolds, 2018. arXiv:1811.10320 6

[3] A. L. Besse, *Manifolds all of whose geodesics are closed*, Ergebnisse der Mathematik und ihrer Grenzgebiete [Results in Mathematics and Related Areas] **93**, Springer-Verlag, Berlin-New York, 1978, With appendices by D. B. A. Epstein, J.-P. Bourguignon, L. Bérard-Bergery, M. Berger and J. L. Kazdan. MR 496885 3

[4] M. R. Bridson and A. Haefliger, *Metric spaces of non-positive curvature*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences] **319**, Springer-Verlag, Berlin, 1999. MR 1744486 3, 5
[5] R. L. Bryant, P. Foulon, S. Ivanov, V. S. Matveev, and W. Ziller, Geodesic behavior for Finsler metrics of constant positive flag curvature on $S^2$, 2017, J. Differential Geom., to appear. arXiv:1710.03736

[6] R. L. Bryant, Finsler surfaces with prescribed curvature conditions, Unpublished manuscript, 1995.

[7] R. L. Bryant, Projectively flat Finsler 2-spheres of constant curvature, Selecta Math. (N.S.) 3 (1997), 161–203. MR 1466165

[8] R. L. Bryant, Some remarks on Finsler manifolds with constant flag curvature, Houston J. Math. 28 (2002), 221–262. Special issue for S. S. Chern. MR 1898190

[9] R. L. Bryant, Geodesically reversible Finsler 2-spheres of constant curvature, in Inspired by S. S. Chern, Nankai Tracts Math. 11, World Sci. Publ., Hackensack, NJ, 2006, pp. 95–111. MR 2313331

[10] E. Cartan, Sur un problème d’équivalence et la théorie des espaces métriques généralisés, Mathemtica 4 (1930), 114–136 (French).

[11] D. B. A. Epstein, Periodic flows on three-manifolds, Ann. of Math. (2) 95 (1972), 66–82. MR 0288785

[12] H. Geiges and C. Lange, Seifert fibrations of lens spaces, Abh. Math. Semin. Univ. Hambg. 88 (2018), 1–22. MR 3785783

[13] C. Lange, Orbifolds from a metric viewpoint. arXiv:1801.03472

[14] C. Lange, On metrics on 2-orbifolds all of whose geodesics are closed, 2016, J. Reine Angew. Math., to appear. arXiv:1603.08455

[15] C. LeBrun and L. J. Mason, Zoll manifolds and complex surfaces, J. Differential Geom. 61 (2002), 453–535. MR 1979367

[16] C. LeBrun and L. J. Mason, Zoll metrics, branched covers, and holomorphic disks, Comm. Anal. Geom. 18 (2010), 475–502. MR 2747436

[17] T. Mettler, Weyl metrisability of two-dimensional projective structures, Math. Proc. Cambridge Philos. Soc. 156 (2014), 99–113. MR 3144212

[18] T. Mettler, Geodesic rigidity of conformal connections on surfaces, Math. Z. 281 (2015), 379–393. MR 3384876

[19] T. Mettler and G. Paternain, Holomorphic differentials, thermostats and Anosov flows, 2017. arXiv:1706.03554

[20] T. Mettler and G. P. Paternain, Convex projective surfaces with compatible Weyl connection are hyperbolic, 2018. arXiv:1804.04616

[21] I. Satake, The Gauss-Bonnet theorem for $V$-manifolds, J. Math. Soc. Japan 9 (1957), 464–492. MR 0095520

[22] P. Scott, The geometries of 3-manifolds, Bull. London Math. Soc. 15 (1983), 401–487. MR 705227

[23] H. Seifert, Topologie Dreidimensionaler Gefaserter Räume, Acta Math. 60 (1933), 147–238. MR 1555366

[24] W. P. Thurston, The geometry and topology of three-manifolds, 1979. arXiv:1801.03472

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