In the light of $\phi$–mapping method and topological current theory, the topological structure and the topological quantization of arbitrary dimensional topological defects are obtained under the condition that the Jacobian $J(\vec{\phi}) \neq 0$. When $J(\vec{\phi}) = 0$, it is shown that there exist the crucial case of branch process. Based on the implicit function theorem and the Taylor expansion, we detail the bifurcation of generalized topological current and find different directions of the bifurcation. The arbitrary dimensional topological defects are found splitting or merging at the degenerate point of field function $\vec{\phi}$ but the total charge of the topological defects is still unchanged.

1. Introduction

The world of topological defects is amazingly rich and have been the focus of much attention in many areas of contemporary physics\[1, 2, 3\]. The importance of the role of defects in understanding a variety of problems in physics is clear\[4, 5, 6, 7\]. Recently, some physicists noticed\[8, 9\] that the topological defects are closely related to the spontaneously broken of $O(m)$ symmetry group to $O(m-1)$ by $m$–component order parameter field $\vec{\phi}$ and pointed out that for $m = 1$, one has domain walls, $m = 2$, strings and $m = 3$, monopoles, for $m = 4$, there are textures. But for the lack of a powerful method, the topological properties are not very clear, the unified theory of describing the topological properties of all these defect objects is not established yet.

In this paper, in the light of $\phi$–mapping topological current theory\[10\], a useful method which plays a important role in studying the topological invariants\[11, 12\] and the topological structures of physical systems\[13, 14, 15\], we will investigate the topological quantization and the branch process
of arbitrary dimensional topological defects. We will show that the topological defects are generated from where \( \vec{\phi} = 0 \) and are topological quantized under the condition \( J(\vec{\phi}) \neq 0 \). While at the zero points of field function \( \vec{\phi} \) where the corresponding Jacobian determinant \( J(\vec{\phi}) \) vanishes, the defect topological current bifurcates and the topological defects split or merge at such point, this means that the topological defects system is unstable at these points.

This paper is organized as follows. In section 2, we investigate the topological quantization of these topological defect and point out that the topological charges of these defects are the Winding numbers which are determined by the Hopf indices and the Brouwer degrees of the \( \phi \)-mapping. In section 3, we study the branch process of the defect topological current at the limit points, bifurcation points and higher degenerated points systematically by virtue of the \( \phi \)-mapping theory and the implicit function theorem.

2. Topological quantization of topological defects

In our previous papers mentioned above, only the topological current of point-like particles was discussed. In this paper, in order to study the topological properties of arbitrary dimensional topological defects, we will extend the concept to present an arbitrary dimensional generalized topological current. We consider the \( \phi \)-mapping as a map between two manifolds, while the dimensions of the two manifolds are arbitrary. It is an important generalization of our previous work on topological current and is of great usefulness to theoretical physics and differential geometry.

In \( n \)-dimensional Riemann manifold \( G \) with the metric tensor \( g_{\mu\nu} \) and local coordinates \( x^\mu \) (\( \mu, \nu = 1, ..., n \)), a \( m \)-component vector order parameter field \( \vec{\phi}(x) \) can be looked upon as a mapping between the Riemann manifold \( G \) and a \( m \)-dimensional Euclidean space \( \mathbb{R}^m \)

\[ \phi : G \to \mathbb{R}^m, \quad \phi^a = \phi^a(x), \quad a = 1, ..., m. \]

The direction field of \( \vec{\phi}(x) \) is generally determined by

\[ n^a(x) = \frac{\phi^a(x)}{||\phi(x)||}, \quad ||\phi(x)|| = \sqrt{\phi^a(x)\phi^a(x)} \]  

(1)

with

\[ n^a(x)n^a(x) = 1. \]  

(2)

It is obviously that \( n^a(x) \) is a section of the sphere bundle \( S(G) \). If \( n^a(x) \) is a smooth unit vector field without singularities or it has singularities somewhere but at the point \( \vec{\phi}(x) \neq 0 \), from (2) we
have
\[ n^a \partial_\mu n^a = 0, \quad \mu = 1, \ldots, n, \] (3)
which can be looked upon as a system of \( n \) homogeneous linear equations of \( n^a \) \((a = 1, \ldots, m)\) with coefficient matrix \([\partial_\mu n^a]\). The necessary and sufficient condition that (3) has non-trivial solution for \( n^a(x) \) is rank \([\partial_\mu n^a] < m\), i.e. the Jacobian determinants
\[ D^{\mu_1 \cdots \mu_k}(\partial n) = \frac{1}{m!} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \partial_{\mu_{k+1}} n^{a_1} \cdots \partial_{\mu_{n}} n^{a_m} \] (4)
are equal to zero, where \( k = n - m \). While, at the point \( \vec{\phi} = 0 \), the above consequences are not held. In short, we have the following relations
\[ D^{\mu_1 \cdots \mu_k}(\partial n) \begin{cases} = 0, & \text{for } \vec{\phi} \neq 0, \vphantom{\frac{1}{2}} \\ \neq 0, & \text{for } \vec{\phi} = 0, \end{cases} \] (5)
which implies \( D^{\mu_1 \cdots \mu_k}(\partial n) \) behaves itself like a function \( \delta(\vec{\phi}) \). So we are focussed on the zeroes of \( \phi^a(x) \).

Suppose that the vector field \( \vec{\phi}(x) \) possesses \( l \) isolated zeroes, according to the implicit function theorem[16], when the zeroes are regular points of \( \vec{\phi} \)-mapping at which the rank of the Jacobian matrix \([\partial_\mu \phi^a]\) is \( m \), the solutions of \( \vec{\phi} = 0 \) can be expressed parameterizedly by
\[ x^\mu = z_i^\mu(u^1, \ldots, u^k), \quad i = 1, \ldots, l, \] (6)
where the subscript \( i \) represents the \( i \)-th solution and the parameters \( u^I \) \((I = 1, \ldots, k)\) span a \( k \)-dimensional submanifold with the metric tensor \( g_{IJ} = g_{\mu\nu} \frac{\partial x^\mu}{\partial u^I} \frac{\partial x^\nu}{\partial u^J} \) which is called the \( i \)-th singular submanifold \( N_i \) in the Riemannian manifold \( G \) corresponding to the \( \phi \)-mapping. For each singular manifold \( N_i \), we can define a normal submanifold \( M_i \) in \( G \) which is spanned by the parameters \( v^A \) with the metric tensor \( g_{AB} = g_{\mu\nu} \frac{\partial x^\mu}{\partial v^A} \frac{\partial x^\nu}{\partial v^B} \) \((A, B = 1, \ldots, m)\), and the intersection point of \( M_i \) and \( N_i \) is denoted by \( p_i \) which can be expressed parameterizedly by \( v^A = p^A_i \). In fact, in the words of differential topology, \( M_i \) is transversal to \( N_i \) at the point \( p_i \), i.e.
\[ T_{p_i}(G) = T_{p_i}(M_i) + T_{p_i}(N_i). \]
By virtue of the implicit function theorem, it should be held true that, at the regular point \( p_i \), the Jacobian matrices \( J(\frac{\phi}{v}) \) satisfies
\[ J(\frac{\phi}{v}) = \frac{D(\phi^1, \ldots, \phi^m)}{D(v^1, \ldots, v^m)} \neq 0. \] (7)
In the following, we will induce a rank–$k$ topological current through the integration of $D^{\mu_1 \cdots \mu_k}(\partial n)$ in \((\Sigma_i)\) on $M_i$. As is well known, the generalized Winding Number\(^{[17]}\) has been given by the Gauss map $n: \partial \Sigma_i \rightarrow S^{m-1}$.

\[
W_i = \frac{1}{A(S^{m-1})(m-1)!} \int_{\partial \Sigma_i} n^* (\epsilon_{a_1 \cdots a_m} n^{a_1} d^n a_2 \wedge \cdots \wedge d^n a_m)
\]

where

\[
A(S^{m-1}) = \frac{2\pi^{m/2}}{\Gamma(m/2)}
\]

is the area of $(m-1)$–dimensional unit sphere $S^{m-1}$, $n^*$ denotes the pull back of map $n$ and $\partial \Sigma_i$ the boundary of a neighborhood $\Sigma_i$ of $p_i$ on $M_i$ with $p_i \notin \partial \Sigma_i$, $\Sigma_i \cap \Sigma_j = \emptyset$. The generalized Winding Numbers $W_i$ can also be rewritten as

\[
W_i = \frac{1}{A(S^{m-1})(m-1)!} \int_{n[\partial \Sigma_i]} \epsilon_{a_1 \cdots a_m} n^{a_1} d^n a_2 \wedge \cdots \wedge d^n a_m
\]

which means that, when the point $x^\mu$ or $v^A$ covers $\partial \Sigma_i$ once, the unit vector $n^a$ will cover a region $n[\partial \Sigma_i]$ whose area is $W_i$ times of $A(S^{m-1})$, i.e. the unit vector $n^a$ will cover the unit sphere $S^{m-1}$ $W_i$ times. From the above equation, one can deduce that

\[
W_i = \frac{1}{A(S^{m-1})(m-1)!} \int_{\partial M_i} \epsilon_{a_1 \cdots a_m} n^{a_1} \partial_{\mu_{k+1}} n^{a_2} \cdots \partial_{\mu_n} n^{a_m} d x^{\mu_{k+2}} \wedge \cdots \wedge d x^{\mu_n}
\]

\[
= \frac{1}{A(S^{m-1})(m-1)!} \int_{M_i} \frac{1}{k!} \frac{1}{g_x} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \epsilon_{a_1 \cdots a_m} \partial_{\mu_{k+1}} n^{a_2} \cdots \partial_{\mu_n} n^{a_m} d x^{\mu_{k+2}} \wedge \cdots \wedge d x^{\mu_n}
\]

\[
= \frac{1}{A(S^{m-1})(m-1)!} \int_{M_i} \frac{1}{k!} \frac{m!}{g_x} D^{\mu_1 \cdots \mu_k}(\partial n) d \sigma_{\mu_1 \cdots \mu_k}, \quad (9)
\]

where $d \sigma_{\mu_1 \cdots \mu_k}$ is the invariant surface element of $M_i$ and $g_x = \det(g_{\mu \nu})$.

From the above discussions, especially the expressions \((\Sigma_i)\), \((\Sigma_j)\) and \((\Sigma_k)\), we can induce a generalized topological current $j^{\mu_1 \cdots \mu_k}$ which does not vanish only at the zeroes of order parameter field $\vec{\phi}(x)$, and is exactly corresponding to the generalized Winding Number,

\[
j^{\mu_1 \cdots \mu_k} = \frac{1}{A(S^{m-1})(m-1)!} \sqrt{g_x} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \epsilon_{a_1 \cdots a_m} \partial_{\mu_{k+1}} n^{a_2} \cdots \partial_{\mu_n} n^{a_m}. \quad (10)
\]

Obviously this tensor current is identically conserved, i.e.

\[
\nabla_{\mu_i} j^{\mu_1 \cdots \mu_k} = 0, \quad i = 1, \ldots, k.
\]

It is easy to see that $j^{\mu_1 \cdots \mu_k}$ are completely antisymmetric tensors.
By making use of the \( \phi \)-mapping theory, we will study the global property of the generalized topological current \( j^{\mu_1 \cdots \mu_k} \) on the whole manifold \( G \) and conclude that \( j^{\mu_1 \cdots \mu_k} \) behaves itself like the generalized function \( \delta(\vec{\phi}) \). From (1) we have

\[
\partial_\mu n^a = \frac{1}{||\phi||} \partial_\mu \phi^a + \phi^a \partial_\mu (\frac{1}{||\phi||}), \quad \frac{\partial}{\partial \phi^a} (\frac{1}{||\phi||}) = -\frac{\phi^a}{||\phi||^3}
\]

which should be looked upon as generalized functions [18]. Using these expressions the generalized topological current (10) can be rewritten as

\[
j^{\mu_1 \cdots \mu_k} = C_m \frac{1}{\sqrt{g_x}} \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \epsilon_{\alpha_1 \cdots \alpha_m} \\
\cdot \partial_{\mu_{k+1}} \phi^\alpha \partial_{\mu_{k+2}} \phi^{\alpha_2} \cdots \partial_{\mu_n} \phi^{\alpha_m} \frac{\partial}{\partial \phi^a} \frac{\partial}{\partial \phi^{\alpha_1}} (G_m(||\phi||)), \quad m > 2.
\]

(11)

where \( C_m \) is a constant

\[
C_m = \begin{cases} \\
\frac{1}{A(5^{m-1})/(m-2)(m-1)!} : & m > 2 \\
\frac{1}{2\pi} : & m = 2
\end{cases}
\]

and \( G_m(||\phi||) \) is a generalized function

\[
G_m(||\phi||) = \begin{cases} \\
\frac{1}{||\phi||^{m-2}} : & m > 2 \\
\frac{1}{\ln ||\phi||} : & m = 2
\end{cases}
\]

Defining general Jacobians \( J^{\mu_1 \cdots \mu_k}(\frac{\phi}{x}) \) as following

\[
\epsilon^{\alpha_1 \cdots \alpha_m} J^{\mu_1 \cdots \mu_k}(\frac{\phi}{x}) = \epsilon^{\mu_1 \cdots \mu_k \mu_{k+1} \cdots \mu_n} \partial_{\mu_{k+1}} \phi^\alpha \partial_{\mu_{k+2}} \phi^{\alpha_2} \cdots \partial_{\mu_n} \phi^{\alpha_m}
\]

and by making use of the \( m \)-dimensional Laplacian Green function relation [10]

\[
\Delta_\phi(\frac{1}{||\phi||^{m-2}}) = -\frac{4\pi^{m/2}}{\Gamma(m/2 - 1)} \delta(\vec{\phi})
\]

where \( \Delta_\phi = (\frac{\partial^2}{\partial \phi^a \partial \phi^\alpha}) \) is the \( m \)-dimensional Laplacian operator in \( \phi \)-space, we do obtain the \( \delta \)-function like topological current rigorously

\[
j^{\mu_1 \cdots \mu_k} = \frac{1}{\sqrt{g_x}} \delta(\vec{\phi}) J^{\mu_1 \cdots \mu_k}(\frac{\phi}{x}).
\]

(12)

We find that \( j^{\mu_1 \cdots \mu_k} \neq 0 \) only when \( \vec{\phi} = 0 \), which is just the singularity of \( j^{\mu_1 \cdots \mu_k} \). In detail, the Kernel of the \( \phi \)-mapping is the singularities of the topological tensor current \( j^{\mu_1 \cdots \mu_k} \) in \( G \). We think that this is the essential of the topological tensor current theory and \( \phi \)-mapping is the key to study this theory.
To investigate the topological properties of the generalized topological current, we should study the total expansion of the \( \delta \)-function \( \delta(\vec{\phi}) \). As is well known [19], the \( \delta \)-function \( \delta(N_i) \) in curved space-time on a submanifold \( N_i \) is

\[
\delta(N_i) = \int_{N_i} \frac{1}{\sqrt{g_x}} \delta^n(\vec{x} - \vec{z}_i(u^1, u^2)) \sqrt{g_u} d^k u,
\]

(13)

and, by analogy with the procedure of deducing \( \delta(f(x)) \), since

\[
\delta(\phi(x)) = \begin{cases} 
+\infty, & \text{for } \phi(x) = 0 \\
0, & \text{for } \phi(x) \neq 0
\end{cases}
\]

(14)

we can expand the \( \delta \)-function \( \delta(\vec{\phi}) \) as

\[
\delta(\vec{\phi}) = \sum_{i=1}^{t} c_i \delta(N_i),
\]

(15)

where the coefficients \( c_i \) must be positive, i.e. \( c_i = |c_i| \). From the definition of \( W_i \) in (8), the Winding number can also be rewritten in terms of the parameters \( v^A \) of \( M_i \) as

\[
W_i = \frac{1}{2\pi} \int_{\Sigma_i} \epsilon^{A_1\ldots A_m} \epsilon_{a_1\ldots a_m} \partial_{A_1} n^{a_1} \ldots \partial_{A_m} n^{a_m} d^m v,
\]

Then, by duplicating the above process, we have

\[
W_i = \int_{\Sigma_i} \delta(\vec{\phi}) J(\frac{\phi}{v}) d^m v,
\]

(16)

Substituting (13) into (16), and considering that only one \( p_i \in \Sigma_i \), we can get

\[
W_i = \int_{\Sigma_i} c_i \delta(N_i) J(\frac{\phi}{v}) d^m v = \int_{\Sigma_i} \int_{N_i} c_i \frac{1}{\sqrt{g_x} \sqrt{g_v}} \delta^n(\vec{x} - \vec{z}_i(u^1, u^2)) J(\frac{\phi}{v}) \sqrt{g_u} d^k u \sqrt{g_v} d^m v.
\]

(17)

where \( g_v = \det(g_{AB}) \). Because \( \sqrt{g_u} \sqrt{g_v} d^k u d^m v \) is the invariant volume element of the Product manifold \( M_i \times N_i \), so it can be rewritten as \( \sqrt{g_x} d^m x \). Thus, by calculating the integral and with positivity of \( c_i \), we get

\[
c_i = \frac{\beta_i \sqrt{g_v}}{|J(\frac{\phi}{v})|_{p_i}} = \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\phi}{v})_{p_i}},
\]

(18)

where \( \beta_i = |W_i| \) is a positive integer called the Hopf index [20] of \( \phi \)-mapping on \( M_i \), it means that when the point \( v \) covers the neighborhood of the zero point \( p_i \) once, the function \( \vec{\phi} \) covers the corresponding region in \( \vec{\phi} \)-space \( \beta_i \) times, and \( \eta_i = \text{sign} J(\frac{\phi}{v})_{p_i} = \pm 1 \) is the Brouwer degree of \( \phi \)-mapping [20]. Substituting this expression of \( c_i \) and (13) into (12), we gain the total expansion of the rank–\( k \) topological current

\[
j^{\mu_1\ldots\mu_k} = \frac{1}{\sqrt{g_x}} \sum_{i=1}^{t} \frac{\beta_i \eta_i \sqrt{g_v}}{J(\frac{\phi}{v})_{p_i}} \delta(N_i) J^{\mu_1\ldots\mu_k} \delta(\vec{\phi}).
\]
or in terms of parameters \( y^A' = (v^1, \ldots, v^m, u^1, \ldots, u^k) \)

\[
j^{A_1' \cdots A_k'} = \frac{1}{\sqrt{g_y}} \sum_{i=1}^l \beta_i \eta_i \sqrt{g_y} \delta(N_i) J^{A_1' \cdots A_k'}(\phi),
\]

From the above equation, we conclude that the inner structure of \( j^{\mu_1 \cdots \mu_k} \) or \( j^{A_1' \cdots A_k'} \) is labelled by the total expansion of \( \delta(\phi) \), which includes the topological information \( \beta_i \) and \( \eta_i \).

It is obvious that, in (6), when \( u^I \) and \( u^I (I = 2, \ldots, k) \) are taken to be time-like evolution parameter and space-like parameters, respectively, the inner structure of \( j^{\mu_1 \cdots \mu_k} \) or \( j^{A_1' \cdots A_k'} \) just represents \( l (k-1) \)-dimensional topological defects moving in the \( n \)-dimensional Riemann manifold \( G \). The \( k \)-dimensional singular submanifolds \( N_i (i = 1, \ldots, l) \) are their world sheets. Here we see that the defects are generated from where \( \phi = 0 \) and, the Hopf indices \( \beta_i \) and Brouwer degree \( \eta_i \) classify these defects. In detail, the Hopf indices \( \beta_i \) characterize the absolute values of the topological quantization and the Brouwer degrees \( \eta_i = +1 \) correspond to defects while \( \eta_i = -1 \) to antidefects. It must be pointed that the relationship between the zero points of the \( m \)-dimensional order parameter field \( \phi \) and the space position of these topological defects is distinct and clear and it is obtained rigorously without tie on any concrete model or hypothesis. Furthermore, for the first time we gain the topological charges of these defects which are determined by the Winding numbers of the \( \phi \)-mapping.

3. The branch processes of the topological defects

In this section, we will discuss the branch processes of these topological defects. In order to simplify our study, we select the parameter \( u^1 \) as the time-like evolution parameter \( t \), and let the space-like parameters \( u^I = \sigma^I (I = 2, \ldots, k) \) be fixed. In this case, the Jacobian matrices \( J^{A_1' \cdots A_k'}(\phi_y) \) are reduced to

\[
J^{A_1' \cdots I_{k-1}}(\phi_y) \equiv J^A(\phi_y), \quad J^{AB_1' \cdots I_{k-2}}(\phi_y) = 0, \quad J^{(m+1) \cdots n}(\phi_y) = J^{(\phi_y)},
\]

\[
A, B = 1, \ldots, (m+1), \quad I_j = m + 2, \ldots, n,
\]

(20)

for \( y^A = v^A (A \leq m), \ y^{m+1} = t, \ y^{m+I} = \sigma^I (I \geq 2) \). In the above section, we have studied the topological property of the topological defects in the case that the vector order parameter \( \phi \) only consists of regular points, i.e. (II) is hold true. However, when this condition fails, the above results will change in some way. It often happens when the zeros of \( \phi \) include some branch points, which lead to the branch process of topological current. The branch points are determined by the \( m + 1 \) equations

\[
\phi^a(v^1, \cdots, v^m, t, \sigma) = 0, \quad a = 1, \cdots, m
\]

(21)
and
\[ \phi^{m+1}(v^1, \ldots, v^m, t, \vec{\sigma}) \equiv J(\frac{\phi}{v}) = 0 \] (22)
for the fixed \( \vec{\sigma} \), and they are denoted as \((t^*, p_i)\). In \( \phi \)-mapping theory usually there are two kinds of branch points, namely the limit points and bifurcation points [21], satisfying
\[ J^1(\frac{\phi}{y})|_{(t^*, p_i)} \neq 0 \] (23)
and
\[ J^1(\frac{\phi}{y})|_{(t^*, p_i)} = 0, \] (24)
respectively. In the following, we assume that the branch points \((t^*, p_i)\) of \( \phi \)-mapping have been found.

A. Branch process at the limit point

In order to use the theorem of implicit function to study the branch process of topological defects at the limit point, we use the Jacobian \( J^1(\frac{\phi}{y}) \) instead of \( J(\frac{\phi}{v}) \) to discuss the problem. In fact, this means that we have replaced the parameter \( t \) by \( v^1 \). For clarity we rewrite the problem as
\[ \phi^a(t, v^2, \ldots, v^m, v^1, \vec{\sigma}) = 0, \quad a = 1, \ldots, m. \] (25)
Then, taking account of the condition (23) and using the implicit function theorem, we have an unique solution of the equations (25) in the neighborhood of the limit point \((t^*, p_i)\)
\[ t = t(v^1, \vec{\sigma}), \quad v^i = v^i(v^1, \vec{\sigma}), \quad i = 2, 3, \ldots, m \] (26)
with \( t^* = t(p^1_i, \vec{\sigma}) \). In order to show the behavior of the defects at the limit points, we will investigate the Taylor expansion of (26) in the neighborhood of \((t^*, p_i)\). In the present case, from (23) and (22), we get
\[ \frac{dv^1}{dt}|_{(t^*, p_i)} = \frac{J^1(\frac{\phi}{y})}{J(\frac{\phi}{y})}|_{(t^*, p_i)} = \infty, \] (27)
i.e.
\[ \frac{dt}{dv^1}|_{(t^*, p_i)} = 0. \]
Then we have the Taylor expansion of (26) at the point \((t^*, p_i)\)
\[ t = t(p_i, \vec{\sigma}) + \frac{dt}{dv^1}|_{(t^*, p_i)}(v^1 - p^1_i) + \frac{1}{2} \frac{d^2t}{(dv^1)^2}|_{(t^*, p_i)}(v^1 - p^1_i)^2 \]
\[ t^* + \frac{1}{2} \left( \frac{d^2 t}{(dv^1)^2} \right) (v^1 - p^1_i)^2. \]

Therefore

\[ t - t^* = \frac{1}{2} \frac{d^2 t}{(dv^1)^2} (v^1 - p^1_i)^2. \]  

which is a parabola in the \( v^1 - t \) plane. From (28), we can obtain the two solutions \( v^1 (1) (t, \vec{\sigma}) \) and \( v^1 (2) (t, \vec{\sigma}) \), which give the branch solutions of the system (21) at the limit point. If \( \frac{d^2 t}{(dv^1)^2} (v^1 - p^1_i) > 0 \), we have the branch solutions for \( t > t^* \), otherwise, we have the branch solutions for \( t < t^* \). The former is related to the origin of the topological defects at the limit points. Since the topological current of the topological defects is identically conserved, the topological quantum numbers of these two generated defects must be opposite at the limit point, i.e. \( \beta_1 \eta_1 + \beta_2 \eta_2 = 0 \). In fact, these two cases are just related to the generation and annihilation of defect-antidefect pair. The result (27) agrees with that obtained by Bray [22] who had a scaling argument associated with point defects final annihilation which leads to large velocity tail.

### B. Branch process at the bifurcation point

In the following, let us consider the case (24), in which the restrictions of the system (21) at the bifurcation point \((t^*, p_i)\) are

\[ J(\dot{\phi}/v)|_{(t^*, p_i)} = 0, \quad J(\ddot{\phi}/v)|_{(t^*, p_i)} = 0. \]  

These two restrictive conditions will lead to an important fact that the dependency relationship between \( t \) and \( v^1 \) is not unique in the neighborhood of the bifurcation point \((t^*, p_i)\). In fact, we have

\[ \frac{dv^1}{dt}|_{(t^*, p_i)} = J(\ddot{\phi}/v)|_{(t^*, p_i)} \]  

which under the restraint (24) directly shows that the tangential direction of the integral curve of equation (30) is indefinite at the point \((t^*, p_i)\). Hence, (30) does not satisfy the conditions of the existence and uniqueness theorem of the solution of a differential equation. This is why the very point \((t^*, z_i)\) is called the bifurcation point of the system (21).

In the following, we will find a simple way to search for the different directions of all branch curves at the bifurcation point. As assumed that the bifurcation point \((t^*, p_i)\) has been found from (21) and (22), the following calculations are all conducted at the value \((t^*, p_i)\). As we have mentioned above, at the bifurcation point \((t^*, p_i)\), the rank of the Jacobian matrix \( \frac{\partial \phi}{\partial v} \) is smaller than \( m \). In order to derive the calculating method, we consider the rank of the Jacobian matrix \( \frac{\partial \phi}{\partial v} \) is \( m - 1 \). The case of
a more smaller rank will be discussed in next subsection. Suppose that one of the \((m - 1) \times (m - 1)\) submatrix \(J_1(\frac{\partial \phi}{\partial v})\) of the Jacobian matrix \([\frac{\partial \phi}{\partial v}]\) is

\[
J_1(\frac{\partial \phi}{\partial v}) = \begin{pmatrix}
\phi_1^1 & \phi_1^2 & \cdots & \phi_1^m \\
\phi_2^1 & \phi_2^2 & \cdots & \phi_2^m \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{m-1}^1 & \phi_{m-1}^2 & \cdots & \phi_{m-1}^m
\end{pmatrix}
\]

(31)

and its determinant \(\text{det} J_1(\frac{\partial \phi}{\partial v})\) does not vanish at the point \((t^*, p_i)\) (otherwise, we have to rearrange the equations of (21)), where \(\phi_A^a\) stands for \(\frac{\partial \phi^a}{\partial v_A}\) \((a = 1, \ldots, m - 1; A = 2, \ldots, m)\). By means of the implicit function theorem we obtain one and only one functional relationship in the neighborhood of the bifurcation point \((t^*, p_i)\)

\[
v^A = f^A(v^1, t, \sigma^2, \ldots, \sigma^k), \quad A = 2, 3, \ldots, n
\]

(32)

with the partial derivatives

\[
f^A_1 = \frac{\partial v^A}{\partial v^1}, \quad f^A_t = \frac{\partial v^A}{\partial t}, \quad A = 2, 3, \ldots, n.
\]

Then, for \(a = 1, \ldots, m - 1\) we have

\[
\phi^a = \phi^a(v^1, f^1(v^1, t, \sigma), \ldots, f^m(v^1, t, \sigma), t, \sigma) \equiv 0
\]

which gives

\[
\sum_{A=2}^m \frac{\partial \phi^a}{\partial v^A} f^A_1 = -\frac{\partial \phi^a}{\partial v^1}, \quad a = 1, \ldots, m - 1
\]

(33)

\[
\sum_{A=2}^m \frac{\partial \phi^a}{\partial v^A} f^A_t = -\frac{\partial \phi^a}{\partial t}, \quad a = 1, \ldots, m - 1
\]

(34)

from which we can calculate the first order derivatives of \(f^A: f^A_1, f^A_t\). Denoting the second order partial derivatives as

\[
f^A_{11} = \frac{\partial^2 v^A}{(\partial v^1)^2}, \quad f^A_{1t} = \frac{\partial^2 v^A}{\partial v^1 \partial t}, \quad f^A_{tt} = \frac{\partial^2 v^A}{\partial t^2}
\]

and differentiating (33) with respect to \(v^1\) and \(t\) respectively, we get

\[
\sum_{A=2}^m \phi^a_A f^A_{11} = -\sum_{A=2}^m \sum_{B=2}^m \left[2 \phi^a_A f^A_1 + \sum_{B=2}^m (\phi^a_B f^B_1) f^A_1 \right] - \phi^a_{11}, \quad a = 1, 2, \ldots, m - 1
\]

(35)

\[
\sum_{A=2}^m \phi^a_A f^A_{1t} = -\sum_{A=2}^m \sum_{B=2}^m \left[\phi^a_A f^A_1 + \phi^a_A f^A_t + \sum_{B=2}^m (\phi^a_B f^B_t) f^A_1 \right] - \phi^a_{1t}, \quad a = 1, 2, \ldots, m - 1.
\]

(36)
And the differentiation of (34) with respect to \( t \) gives

\[
\sum_{A=2}^{m} \phi_a^A f_t^A = - \sum_{A=2}^{m} \left[ 2 \phi_a^A f_t^A + \sum_{B=2}^{m} (\phi_a^B f_t^B) f_t^A \right] - \phi_t^a, \quad a = 1, 2, \ldots, m - 1
\]  

(37)

where

\[
\phi_a^{AB} = \frac{\partial^2 \phi_a^A}{\partial v^A \partial v^B}, \quad \phi_t^A = \frac{\partial^2 \phi_t^A}{\partial v^A \partial t}.
\]

The differentiation of (34) with respect to \( v^1 \) gives the same expression as (36). If we use the Gaussian elimination method to the three vectors at the right hands of the formulas (35), (36) and (37), we can obtain the three partial derivatives \( f_{11}^A, f_{1t}^A \) and \( f_{tt}^A \). Notice that the three equations (35), (36) and (37) have the same coefficient matrix \( J_1(\frac{\partial}{\partial v}) \), which are assumed to be nonzero, and we should substitute the values of the partial derivatives \( f_{11}^A \) and \( f_{1t}^A \), which have been calculated out in the former, into the right hands of the three equations.

The above discussions do not matter to the last component \( \phi_m(v^1, \ldots, v^m, t, \vec{\sigma}) \). In order to find the different values of \( dv^1/dt \) at the bifurcation point, let us investigate the Taylor expansion of \( \phi_m(v^1, \ldots, v^m, t, \vec{\sigma}) \) in the neighborhood of \((t^*, p_i)\). Substituting the existing, but unknown, dependency relationship (32) into \( \phi_m(v^1, \ldots, v^m, t, \vec{\sigma}) \), we get the function of two variables \( v^1 \) and \( t \)

\[
F(t, v^1, \vec{\sigma}) = \phi_m(v^1, f^2(v^1, t, \vec{\sigma}), \ldots, f^m(v^1, t, \vec{\sigma}), t, \vec{\sigma})
\]

(38)

which according to (21) must vanish at the bifurcation point

\[
F(t^*, p_i) = 0.
\]

(39)

From (38), we can calculate the first order partial derivatives of \( F(t, v^1, \vec{\sigma}) \) with respect to \( v^1 \) and \( t \) respectively at the bifurcation point \((t^*, p_i)\)

\[
\frac{\partial F}{\partial v^1} = \phi'_1 + \sum_{A=2}^{m} \phi_A^m f_t^A, \quad \frac{\partial F}{\partial t} = \phi'_t + \sum_{A=2}^{m} \phi_A^m f_t^A.
\]

(40)

Using (33) and (34), the first equation of (29) is expressed by

\[
J(\frac{\partial}{\partial v})(t^*, p_i) = \begin{vmatrix}
- \sum_{A=2}^{m} \phi_A^1 f_t^A & \phi_2^1 & \cdots & \phi_m^1 \\
- \sum_{A=2}^{m} \phi_A^2 f_t^A & \phi_2^2 & \cdots & \phi_m^2 \\
\vdots & \vdots & \ddots & \vdots \\
- \sum_{A=2}^{m} \phi_A^{m-1} f_t^A & \phi_2^{m-1} & \cdots & \phi_m^{m-1} \\
\phi_m^m & \phi_m^m & \cdots & \phi_m^m
\end{vmatrix}_{(t^*, p_i)} = 0
\]
which, by Cramer’s rule, (31) and (40), can be rewritten as

\[
J\left(\frac{\phi}{v}\right)|_{(t^*, p_i)} = \begin{vmatrix}
0 & \phi_2^1 & \cdots & \phi_m^1 \\
0 & \phi_2^2 & \cdots & \phi_m^2 \\
\vdots & \vdots & \ddots & \vdots \\
\phi_1^m + \sum_{A=2}^{m} \phi_A^m f_A^1 & \phi_2^m & \cdots & \phi_m^m \\
\end{vmatrix}_{(t^*, p_i)}
\]

\[
= \frac{\partial F}{\partial v^1} \det J_1\left(\frac{\phi}{v}\right)|_{(t^*, p_i)} = 0.
\]

Since

\[
\det J_1\left(\frac{\phi}{v}\right)|_{(t^*, p_i)} \neq 0
\]

which is our assumption, the above equation leads to

\[
\frac{\partial F}{\partial v^1}|_{(t^*, p_i)} = 0. \tag{41}
\]

With the same reasons, we can prove that

\[
\frac{\partial F}{\partial t}|_{(t^*, p_i)} = 0. \tag{42}
\]

The second order partial derivatives of the function \(F(t, v^1, \sigma)\) are easily to find out to be

\[
\frac{\partial^2 F}{(\partial v^1)^2} = \phi_{11}^m + \sum_{A=2}^{m} [2\phi_{1A}^m f_A^1 + \phi_{A}^m f_{11}^A + \sum_{B=2}^{m} (\phi_{AB}^m f_B^1) f_A^1]
\]

\[
\frac{\partial^2 F}{\partial v^1 \partial t} = \phi_{1t}^m + \sum_{A=2}^{m} [\phi_{1A}^m f_{1t}^A + \phi_{A}^m f_{1t}^A + \sum_{B=2}^{m} (\phi_{AB}^m f_B f_{1t}^A)]
\]

\[
\frac{\partial^2 F}{\partial t^2} = \phi_{tt}^m + \sum_{A=2}^{m} [2\phi_{A}^m f_{t}^A + \phi_{A}^m f_{tt}^A + \sum_{B=2}^{m} (\phi_{AB}^m f_B f_{tt}^A)]
\]

which at \((t^*, p_i)\) are denoted by

\[
A = \frac{\partial^2 F}{(\partial v^1)^2}|_{(t^*, p_i)}, \quad B = \frac{\partial^2 F}{\partial v^1 \partial t}|_{(t^*, p_i)}, \quad C = \frac{\partial^2 F}{\partial t^2}|_{(t^*, p_i)}. \tag{43}
\]

Then, by virtue of (39), (31), (42) and (43), the Taylor expansion of \(F(t, v^1, \sigma)\) in the neighborhood of the bifurcation point \((t^*, p_i)\) can be expressed as

\[
F(t, v^1, \sigma) = \frac{1}{2} A(v^1 - p_i^1)^2 + B(v^1 - p_i^1)(t - t^*) + \frac{1}{2} C(t - t^*)^2 \tag{44}
\]

which is the expression of \(\phi^m(v_1, \cdots, v_m, t, \sigma)\) in the neighborhood of \((t^*, p_i)\). The expression (44) shows that at the bifurcation point \((t^*, p_i)\)

\[
A(v^1 - p_i^1)^2 + 2B(v^1 - p_i^1)(t - t^*) + C(t - t^*)^2 = 0. \tag{45}
\]
Dividing (15) by \((v^1 - p_i^1)^2\) or \((t - t^*)^2\), and taking the limit \(t \to t^*\) as well as \(v^1 \to p_i^1\) respectively, we get two equations

\[
C\left(\frac{dt}{dv^1}\right)^2 + 2B\frac{dt}{dv^1} + A = 0. \tag{46}
\]

and

\[
A\left(\frac{dv^1}{dt}\right)^2 + 2B\frac{dv^1}{dt} + C = 0. \tag{47}
\]

So we get the different directions of the branch curves at the bifurcation point from the solutions of (46) or (47).

In order to determine the branches directions of the remainder variables, we will use the relations simply

\[
dv^A = f_i^A dv^1 + f_t^A dt, \quad A = 2, 3, ..., n
\]

where the partial derivative coefficients \(f_i^A\) and \(f_t^A\) have given in (33) and (34). Then, respectively

\[
\frac{dv^A}{dv^1} = f_i^A + f_t^A \frac{dt}{dv^1}
\]

or

\[
\frac{dv^A}{dt} = f_i^A \frac{dv^1}{dt} + f_t^A. \tag{48}
\]

where partial derivative coefficients \(f_i^A\) and \(f_t^A\) are given by (33) and (34). From this relations we find that the values of \(dv^A/dt\) at the bifurcation point \((t^*, z_i)\) are also possibly different because (47) may give different values of \(dv^1/dt\). The above solutions reveal the evolution of the topological defects. Besides the encountering of the defects, i.e. two defects encounter and then depart at the bifurcation point along different branch curves, it also includes splitting and merging of defects. When a multicharged defects moves through the bifurcation point, it may split into several defects along these two different branch curves. On the contrary, several defects can merge into one defect at the bifurcation point. The identical conservation of the topological charge shows the sum of the topological charge of final defects must be equal to that of the initial defects at the bifurcation point, i.e.,

\[
\sum_f \beta_{j_f} \eta_{f} = \sum_i \beta_{j_i} \eta_{i} \tag{49}
\]

for fixed \(j\). Furthermore, from above studies, we see that the generation, annihilation and bifurcation of defects are not gradual changes, but start at a critical value of arguments, i.e. a sudden change.

C. Branch process at a higher degenerated point
In the following, let us discuss the branch process at a higher degenerated point. In the above subsection, we have analysed the case that the rank of the Jacobian matrix \( \partial \phi / \partial v \) of the equation (22) is \( m - 1 \). In this section, we consider the case that the rank of the Jacobian matrix is \( m - 2 \) (for the case that the rank of the matrix \( \partial \phi / \partial v \) is lower than \( m - 2 \), the discussion is in the same way). Let the \( (m - 2) \times (m - 2) \) submatrix \( J_2(\frac{\partial \phi}{\partial v}) \) of the Jacobian matrix \( \partial \phi / \partial v \) be

\[
J_2(\frac{\partial \phi}{\partial v}) = \begin{pmatrix}
\phi_1^1 & \phi_1^2 & \cdots & \phi_1^m \\
\phi_2^1 & \phi_2^2 & \cdots & \phi_2^m \\
\vdots & \vdots & \ddots & \vdots \\
\phi_{m-2}^1 & \phi_{m-2}^2 & \cdots & \phi_{m-2}^m \\
\end{pmatrix}
\]

and suppose that \( \det J_2(\frac{\partial \phi}{\partial v})|_{(v^*, p_i)} \neq 0 \). With the same reasons of obtaining (32), we can have the function relations

\[
v^A = f^A(v^1, v^2, t, \bar{\sigma}), \quad A = 3, 4, ..., m. \tag{50}
\]

For the partial derivatives \( f_1^A, f_2^A \) and \( f_i^A \), we can easily derive the system similar to the equations (33) and (34), in which the three terms at the right hand of can be figured out at the same time. In order to determine the 2–order partial derivatives \( f_{ij}^A, f_{i1}^A, f_{1i}^A, f_{2i}^A \) and \( f_{ii}^A \), we can use the equations similar to (35), (36) and (37). Substituting the relations (50) into the last two equations of the system (21), we have the following two equations with respect to the arguments \( v^1, v^2, t, \bar{\sigma} \)

\[
\begin{align*}
F_1(v^1, v^2, t, \bar{\sigma}) &= \phi^{m-1}(v^1, v^2, f^3(v^1, v^2, t, \bar{\sigma}), \cdots, f^m(v^1, v^2, t, \bar{\sigma}), t, \bar{\sigma}) = 0 \\
F_2(v^1, v^2, t, \bar{\sigma}) &= \phi^m(v^1, v^2, f^3(v^1, v^2, t, \bar{\sigma}), \cdots, f^m(v^1, v^2, t, \bar{\sigma}), t, \bar{\sigma}) = 0.
\end{align*} \tag{51}
\]

Calculating the partial derivatives of the function \( F_1 \) and \( F_2 \) with respect to \( v^1, v^2 \) and \( t \), taking notice of (50) and using six similar expressions to (11) and (12), i.e.

\[
\frac{\partial F_j}{\partial v^i} |_{(v^*, p_i)} = 0, \quad \frac{\partial F_j}{\partial v^2} |_{(v^*, p_i)} = 0, \quad \frac{\partial F_j}{\partial t} |_{(v^*, p_i)} = 0, \quad j = 1, 2, \tag{52}
\]

we have the following forms of Taylor expressions of \( F_1 \) and \( F_2 \) in the neighborhood of \( (t^*, p_i) \)

\[
F_j(v^1, v^2, t, \bar{\sigma}) \approx A_{j1}(v^1 - p_{i1})^2 + A_{j2}(v^1 - p_{i1})(v^2 - p_{i2}) + A_{j3}(v^1 - p_{i1}) \\
(t - t^*) + A_{j4}(v^2 - p_{i2})^2 + A_{j5}(v^2 - p_{i2})(t - t^*) + A_{j6}(t - t^*)^2 = 0 \\
\]

\[
j = 1, 2. \tag{53}
\]

In the case of \( A_{j1} \neq 0, A_{j4} \neq 0 \), by dividing (53) by \( (t - t^*)^2 \) and taking the limit \( t \to t^* \), we obtain two quadratic equations of \( \frac{dv^1}{dt} \) and \( \frac{dv^2}{dt} \)

\[
A_{j1}(\frac{dv^1}{dt})^2 + A_{j2}\frac{dv^1}{dt}\frac{dv^2}{dt} + A_{j3}\frac{dv^1}{dt} + A_{j4}(\frac{dv^2}{dt})^2 + A_{j5}\frac{dv^2}{dt} + A_{j6} = 0 \tag{54}
\]
$j = 1, 2$.

Eliminating the variable $dv^1/dt$, we obtain a equation of $dv^2/dt$ in the form of a determinant

\[
\begin{vmatrix}
A_{11} & A_{12}Q + A_{23} & A_{14}Q^2 + A_{15}Q + A_{16} & 0 \\
0 & A_{11} & A_{12}Q + A_{13} & A_{14}Q^2 + A_{15}Q + A_{16} \\
A_{21} & A_{22}Q + A_{23} & A_{24}Q^2 + A_{25}Q + A_{26} & 0 \\
0 & A_{21} & A_{22}Q + A_{23} & A_{24}Q^2 + A_{25}Q + A_{26}
\end{vmatrix} = 0 \quad (55)
\]

where $Q = dv^2/dt$, which is a 4th order equation of $dv^2/dt$

\[
a_0\left(\frac{dv^2}{dt}\right)^4 + a_1\left(\frac{dv^2}{dt}\right)^3 + a_2\left(\frac{dv^2}{dt}\right)^2 + a_3\left(\frac{dv^2}{dt}\right) + a_4 = 0. \quad (56)
\]

Therefore we get different directions at the bifurcation point corresponding to different branch curves. The number of different branch curves is four at most. If the degree of degeneracy of the matrix $\left[\frac{\partial \phi}{\partial v}\right]$ is more higher, i.e. the rank of the matrix $\left[\frac{\partial \phi}{\partial v}\right]$ is more lower than the present $(m-2)$ case, the procedure of deduction will be more complicate. In general supposing the rank of the matrix $\left[\frac{\partial \phi}{\partial z}\right]$ be $(m - s)$, the number of the possible different directions of the branch curves is $2^s$ at most. Comparing with the above subsection, the solutions in the subsection also reveal encountering, splitting and merging of the defects along more directions.

At the end of this section, we conclude that there exist crucial cases of branch processes in our topological defect theory. This means that a topological defect, at the bifurcation point, may split into several topological defects along $2^s$ different branch curves with different charges. Since the topological current is a conserved current, the total quantum number of the splitting topological defects must precisely equal to the topological charge of the original defect i.e.

\[
\sum_{j=1}^{2^s} \beta_{ij} \eta_j = \beta_i \eta_i
\]

for fixed $i$, $\beta_{ij} \eta_j$ stands for the total topological charge of defects along the $j$-th branch curve at the bifurcation point $p_i$. This can be looked upon as the topological reason of the defect splitting. Here we should point out that such splitting is a stochastic process, the sole restriction of this process is just the conservation of the topological charge of the topological defects during this splitting process.

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