Four Groups Related to Associators

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Dedicated to Professor Don Zagier
on the occasion of his 60th birthday

1. Associators

We recall the definition of associators \[\text{[Dr]}\] and explain our main results in \[\text{[F10a, PT1]}\] which are on the defining equations of associators.

Let us fix notations: Let \(k\) be a field of characteristic 0 and \(\bar{k}\) its algebraic closure. Denote by \(U\bar{\mathfrak{g}}_2 = k\langle\langle X_0, X_1\rangle\rangle\) a non-commutative formal power series ring, a universal enveloping algebra of the completed free Lie algebra \(\bar{\mathfrak{g}}_2\) with two variables \(X_0\) and \(X_1\). Its element \(\varphi = \varphi(X_0, X_1)\) is called group-like \[\text{[1]}\] if it satisfies

\[
\Delta(\varphi) = \varphi \otimes \varphi \quad \text{and} \quad \varphi(0,0) = 1
\]

with \(\Delta(X_0) = X_0 \otimes 1 + 1 \otimes X_0\) and \(\Delta(X_1) = X_1 \otimes 1 + 1 \otimes X_1\). For any \(k\)-algebra homomorphism \(\iota: U\bar{\mathfrak{g}}_2 \rightarrow \mathcal{S}\), the image \(\iota(\varphi) \in \mathcal{S}\) is denoted by \(\varphi(\iota(X_0), \iota(X_1))\).

**Definition 1 (\[\text{Dr}\]).** A pair \((\mu, \varphi)\) with a non-zero element \(\mu\) in \(k\) and a group-like series \(\varphi = \varphi(X_0, X_1) \in U\bar{\mathfrak{g}}_2\) is called an associator if it satisfies one pentagon equation in \(U\mathfrak{a}_4\)

\[
\varphi(t_{12}, t_{23} + t_{24}) \varphi(t_{13} + t_{23}, t_{34}) = \varphi(t_{23}, t_{34}) \varphi(t_{12} + t_{13}, t_{24} + t_{34}) \varphi(t_{12}, t_{23})
\]

and two hexagon equations in \(U\mathfrak{a}_4\)

\[
\exp\left\{\frac{\mu(t_{13} + t_{23})}{2}\right\} = \varphi(t_{13}, t_{12}) \exp\left\{\frac{\mu t_{13}}{2}\right\} \varphi(t_{13}, t_{23})^{-1} \exp\left\{\frac{\mu t_{23}}{2}\right\} \varphi(t_{12}, t_{23}),
\]

\[
\exp\left\{\frac{\mu(t_{12} + t_{13})}{2}\right\} = \varphi(t_{23}, t_{13})^{-1} \exp\left\{\frac{\mu t_{13}}{2}\right\} \varphi(t_{12}, t_{13}) \exp\left\{\frac{\mu t_{12}}{2}\right\} \varphi(t_{12}, t_{23})^{-1}.
\]

Here \(U\mathfrak{a}_3\) (resp. \(U\mathfrak{a}_4\)) means the universal enveloping algebra of the completed pure braid Lie algebra \(\mathfrak{a}_3\) (resp. \(\mathfrak{a}_4\)) over \(k\) with 3 (resp. 4) strings, generated by \(t_{ij}\) (\(1 \leq i, j \leq 3\) (resp. 4)) with defining relations

\[
t_{ii} = 0, \quad t_{ij} = t_{ji}, \quad [t_{ij}, t_{ik} + t_{jk}] = 0 \quad (i,j,k: \text{all distinct})
\]

and \([t_{ij}, t_{kl}] = 0\) (\(i,j,k,l: \text{all distinct}\)).

**Remark 2.**

(i). Drinfeld \[\text{[Dr]}\] proved that such a pair always exists for any field \(k\) of characteristic 0.

(ii). The equations \[\text{[2]} \sim \text{[4]}\] reflect the three axioms of braided monoidal categories \[\text{[JS]}\]. We note that for any \(k\)-linear infinitesimal tensor category \(\mathcal{C}\) each associator gives a structure of braided monoidal category on \(\mathcal{C}[[h]]\) (cf. \[\text{[C, Dr]}\]). Here \(\mathcal{C}[[h]]\) means the category whose set of objects is equal to that of \(\mathcal{C}\) and whose set of morphism \(\text{Mor}_{\mathcal{C}[[h]]}(X,Y)\) is \(\text{Mor}_{\mathcal{C}}(X,Y) \otimes k[[h]]\) (\(h\): a parameter).

\[\text{[1]}\] It is equivalent to \(\varphi \in \exp \bar{\mathfrak{g}}_2\).
(iii). Associators are essential for construction of quasi-triangular quasi-Hopf quantized universal enveloping algebras [Dr].

(iv). Le and Murakami [LMa] and Bar-Natan [Ba97] gave a reconstruction of universal Vassiliev knot invariant (Kontsevich invariant [K, Ba95]) in a combinatorial way by using associators.

Our first result is the implication of two hexagon equations from one pentagon equation.

**Theorem 3** ([F10a]). Let \( \varphi = \varphi(X_0, X_1) \) be a group-like element of \( U_{\mathfrak{g}} \). Suppose that \( \varphi \) satisfies the pentagon equation \( (2) \). Then there exists \( \mu \in k \) (unique up to signature) such that the pair \((\mu, \varphi)\) satisfies two hexagon equations \( (3) \) and \( (4) \).

Recently several different proofs of the above theorem were obtained (see [AT, BaD, W]).

One of the nice examples of associators is the Drinfeld associator below.

**Examples 4.** The Drinfeld associator \( \Phi_{KZ} = \Phi_{KZ}(X_0, X_1) \in C\langle\langle X_0, X_1 \rangle\rangle \) is defined to be the quotient \( \Phi_{KZ} = G_1(z)^{-1}G_0(z) \) where \( G_0 \) and \( G_1 \) are the solutions of the formal \( KZ \) (Kazhdan-Zamolodchikov) equation, the following differential equation over \( C\setminus\{0, 1\} \) with \( G(z) \) valued on \( C\langle\langle X_0, X_1 \rangle\rangle \)

\[
\frac{d}{dz} G(z) = \left( \frac{X_0}{z} + \frac{X_1}{z-1} \right) G(z),
\]

such that \( G_0(z) \approx z^{X_0} \) when \( z \to 0 \) and \( G_1(z) \approx (1 - z)^{X_1} \) when \( z \to 1 \) (cf. [Dr]). It is shown in [Dr] that the pair \((2\pi \sqrt{-1}, \Phi_{KZ})\) forms an associator for \( k = C \). Namely \( \Phi_{KZ} \) satisfies \( (1) \sim (4) \) with \( \mu = 2\pi \sqrt{-1} \).

**Remark 5.** (i). The Drinfeld associator is expressed as follows:

\[
\Phi_{KZ}(X_0, X_1) = 1 + \sum (-1)^m \zeta(k_1, \ldots, k_m) X_0^{k_m-1} X_1 \cdots X_0^{1} X_1 + \text{(regularized terms)}.
\]

Here \( \zeta(k_1, \ldots, k_m) \) is the multiple zeta value (MZV in short), the real number defined by the following power series

\[
\zeta(k_1, \ldots, k_m) := \sum_{0 < n_1 < \ldots < n_m} \frac{1}{n_1^{k_1} \cdots n_m^{k_m}}
\]

for \( m, k_1, \ldots, k_m \in \mathbb{N}(= Z_{>0}) \) with \( k_m > 1 \) (its convergent condition). All the coefficients of \( \Phi_{KZ} \) including its regularized terms are explicitly calculated in terms of MZV’s in [Fo03] proposition 3.2.3 by Le-Murakami’s method in [LMb].

(ii). Since MZV’s are coefficients of \( \Phi_{KZ} \), the equations \( (1) \sim (4) \) for \((\mu, \varphi) = (2\pi \sqrt{-1}, \Phi_{KZ})\) yield algebraic relations among them, which are called associator relations. It is expected that the associator relations might produce all algebraic relations among MZV’s.

Various relations among MZV’s have been found and studied so far. The regularised double shuffle relations which were initially introduced by Zagier and Ecallle in early 90’s might be one of the most fascinating ones. To state them let us fix notations again: Let \( \pi_x : k\langle\langle X_0, X_1 \rangle\rangle \to k\langle\langle Y_1, Y_2, \ldots \rangle\rangle \) be the \( k \)-linear map between non-commutative formal power series rings that sends all the words ending in \( X_0 \) to zero and the word \( X_0^{m-1} X_1 \ldots X_0^{n_1-1} X_1(n_1, \ldots, n_m \in \mathbb{N}) \) to \((-1)^m Y_{n_m} \cdots Y_{n_1} \).
Define the coproduct $\Delta_*$ on $k\langle\langle Y_1, Y_2, \ldots \rangle\rangle$ by

$$\Delta_* Y_n = \sum_{i=0}^{n} Y_i \otimes Y_{n-i}$$

with $Y_0 := 1$. For $\varphi = \sum_{W: \text{word}} c_W(\varphi) W \in U_{\mathfrak{F}_2} = k\langle\langle X_0, X_1 \rangle\rangle$ with $c_W(\varphi) \in k$ (a ‘word’ means a monic monomial element or $1$ in $U_{\mathfrak{F}_2}$), put

$$\varphi_* = \exp \left( \sum_{n=1}^{\infty} \frac{(-1)^n}{n} c_{X_0^{n-1} X_1}(\varphi) Y_1^n \right) \cdot \pi_Y(\varphi).$$

The regularised double shuffle relations for a group-like series $\varphi \in U_{\mathfrak{F}_2}$ mean

$$\Delta_*(\varphi_*) = \varphi_* \otimes \varphi_*.$$

**Remark 6.** The regularised double shuffle relations for MZV’s mean the algebraic relations among them obtained from (1) and (6) for $\varphi = \Phi_{KZ}$ (cf. [IKZ, R]). It is also expected that the relations might produce all algebraic relations among MZV’s.

The following is the simplest example of the relations.

**Examples 7.** For $a, b > 1$,

$$\zeta(a)\zeta(b) = \zeta(a, b) + \zeta(a + b) + \zeta(b, a)$$

$$= \sum_{i=0}^{a-1} \binom{b-1+i}{i} \zeta(a-i,b+i) + \sum_{j=0}^{b-1} \binom{a-1+j}{j} \zeta(b-j,a+j).$$

Our second result here is the implication of the regularised double shuffle relations from the pentagon equation.

**Theorem 8 ([F11]).** Let $\varphi = \varphi(X_0, X_1)$ be a group-like element of $U_{\mathfrak{F}_2}$. Suppose that $\varphi$ satisfies the pentagon equation (2). Then it also satisfies the regularised double shuffle relations (6).

This result attains the final goal of the project posed by Deligne-Terasoma [T]. Their idea is to use some convolutions of perverse sheaves, whereas our proof is to use Chen’s bar construction calculus.

**Remark 9.** Our theorem was extended cyclotomically in [F10b].

The following Zagier’s relation which is essential for Brown’s proof of theorem [14] might be also one of the most fascinating ones. The author does not know if it also follows from our pentagon equation (2).

**Theorem 10 ([Z]).**

$$\zeta(2^{\{a\}}, 3, 2^{\{b\}}) = 2 \sum_{r=1}^{a+b+1} (-1)^r (A_{a,b}^r - B_{a,b}^r) \zeta(2r+1) \zeta(2^{\{a+b+1-r\}})$$

with $A_{a,b}^r = \binom{2r}{2a+2}$ and $B_{a,b}^r = (1 - 2^{-2r}) \binom{2r}{2b+1}$.

2. Four Groups

We explain recent developments on the four pro-unipotent algebraic groups related to associators; the motivic Galois group, the Grothendieck-Teichmüller group, the double shuffle group and the Kashiwara-Vergne group.
2.1. Motivic Galois group. We review on the motivic Galois group, the tannakian dual group of the category of unramified mixed Tate motives.

We work in the triangulated category $DM(Q)_Q$ of mixed motives\(^2\) over $Q$ constructed by Hanamura, Levine and Voevodsky. Tate motives $Q(n)$ ($n \in Z$) are (Tate) objects of the category. Let $DMT(Q)_Q$ be the triangulated sub-category of $DM(Q)_Q$ generated by Tate motives $Q(n)$ ($n \in Z$). By the work of Levine a neutral tannakian $Q$-category $MT(Q) = MT(Q)_Q$ of mixed Tate motives over $Q$ is extracted by taking a heart with respect to a $t$-structure of $DMT(Q)_Q$. Deligne and Goncharov [DeG] introduced the full subcategory $MT(Z) = MT(Z)_Q$ of unramified mixed Tate motives inside. All objects there are mixed Tate motives $M$ (i.e. an object of $MT(Q)$) such that for each subquotient $E$ of $M$ which is an extension of $Q(n)$ by $Q(n+1)$ for $n \in Z$, the extension class of $E$ in

$$Ext^1_{MT(Q)}(Q(n), Q(n+1)) = Ext^1_{MT(Q)}(Q(0), Q(1)) = Q^\times \otimes Q$$

is equal to $Q^\times \otimes Q = \{0\}$.

In the category $MT(Z)$ of unramified mixed Tate motives, the followings hold:

\begin{align*}
(7) \quad & \dim Q Ext^1_{MT(Z)}(Q(0), Q(m)) = \begin{cases} 1 & (m = 3, 5, 7, \ldots ), \\ 0 & (m : \text{if otherwise}), \end{cases} \\
(8) \quad & \dim Q Ext^2_{MT(Z)}(Q(0), Q(m)) = 0.
\end{align*}

The category $MT(Z)$ forms a neutral tannakian $Q$-category with the fiber functor $\omega_{\text{can}} : MT(Z) \to \text{Vect}_Q$ (the category of $Q$-vector spaces) sending each motive $M$ to $\oplus_n \text{Hom}(Q(n), \text{Gr}_{W_m}^{\omega}(M))$.

**Definition 11.** The motivic Galois group here is defined to be the pro-$Q$-algebraic group $\text{Gal}^M(Z) := \text{Aut}^\otimes (MT(Z) : \omega_{\text{can}})$.

By tannakian category theory, $\omega_{\text{can}}$ induces an equivalence of categories

$$MT(Z) \simeq \text{RepGal}^M(Z)$$

where RHS means the category of finite dimensional $Q$-vector spaces with $\text{Gal}^M(Z)$-action.

**Remark 12.** The action of $\text{Gal}^M(Z)$ on $\omega_{\text{can}}(Q(1)) = Q$ defines a surjection $\text{Gal}^M(Z) \to G_m$ and its kernel $\text{Gal}^M(Z)_2$ is the unipotent radical of $\text{Gal}^M(Z)$. There is a canonical splitting $\tau : G_m \to \text{Gal}^M(Z)$ which gives a negative grading on its associated Lie algebra $\text{LieGal}^M(Z)_1$. From (7) and (8) it follows that the Lie algebra is the graded free Lie algebra generated by one element in each degree $-3, -5, -7, \ldots$ (consult [De] §8 for the full story).

The motivic fundamental group $\pi_1^M(P^1 \setminus \{0, 1, \infty\} : \tilde{0})$ constructed in [DeG] §4 is a (pro-)object of $MT(Z)$. By our tannakian equivalence (9), it gives a (pro-)object of RHS of (9), which induces a (graded) action

\begin{align*}
(10) \quad & \Psi : \text{Gal}^M(Z)_1 \to \text{Aut} \exp \tilde{S}_2.
\end{align*}

\(^2\) A part of idea of mixed motives is explained [De] §1. According to Wikipedia, “the (partly conjectural) theory of motives is an attempt to find a universal way to linearize algebraic varieties, i.e. motives are supposed to provide a cohomology theory which embodies all these particular cohomologies.”
Remark 13. For each $\sigma \in \text{Gal}^{\text{M}}(\mathbb{Z})_1(k)$, its action on $\exp \mathfrak{H}_2$ is described by $e^{X_0} \mapsto e^{X_0}$ and $e^{X_1} \mapsto \phi^{-1}_\sigma e^{X_1} \phi_\sigma$ for some $\phi_\sigma \in \exp \mathfrak{H}_2$.

The following has been conjectured (Deligne-Ihara conjecture) for a long time and finally proved by Brown by using Zagier’s relation (Theorem 10).

Theorem 14 ([Br]). The map $\Psi$ is injective.

It is a unipotent analogue of the so-called Belyi’s theorem. The theorem says that all unramified mixed Tate motives are associated with MZV’s.

2.2. Grothendieck-Teichmüller group. The Grothendieck-Teichmüller group was introduced by Drinfeld ([Dr]) in his study of deformations of quasi-triangular quasi-Hopf quantized universal enveloping algebras. It was defined to be the set of ‘degenerated’ associators. The construction of the group was also stimulated by the previous idea of Grothendieck, un jeu de Teichmüller-Lego, posed in his article Esquisse d’un programme ([G]).

Definition 15 ([Dr]). The Grothendieck-Teichmüller group $GRT_1$ is defined to be the pro-algebraic variety whose set of $k$-valued points consists of group-like series $\phi \in U\mathfrak{H}_2$ satisfying the defining equations (2) of associators with $\mu = 0$.

Remark 16. (i). By our theorem 3, it is reformulated to be the set of group-like series satisfying (2) without quadratic terms.

(ii). It forms a group [Dr] by the multiplication below

\begin{equation}
\phi_2 \circ \phi_1 := \phi_1(\phi_2 X_0 \phi_2^{-1}, X_1) \cdot \phi_2 = \phi_2 \cdot \phi_1(X_0, \phi_2^{-1} X_1 \phi_2).
\end{equation}

(iii). By the map sending $X_0 \mapsto X_0$ and $X_1 \mapsto \phi^{-1} X_1 \phi$, the group $GRT_1$ is regarded as a subgroup of $\text{Aut} \exp \mathfrak{H}_2$.

(iii). The cyclotomic analogues of associators and the Grothendieck-Teichmüller group were introduced by Enriquez ([E]). Some elimination results on the defining equations in special case were obtained in [EF].

Geometric interpretation (cf. [Dr]) of the equations (2) implies the following

Proposition 17. $\text{Im} \Psi \subset GRT_1$.

Actually it is expected that they are isomorphic.

Remark 18. (i). The Drinfeld associator $\Phi_{KZ}$ is an associator (cf. example 4) but is not a degenerated associator, i.e. $\Phi_{KZ} \notin GRT_1(\mathbb{C})$.

(ii). The $p$-adic Drinfeld associator $\Phi_{KZ}^p$ introduced in [F04] is not an associator but a degenerated associator, i.e. $\Phi_{KZ}^p \in GRT_1(\mathbb{Q}_p)$ (cf. [F07]).

2.3. Double shuffle group. The double shuffle group was introduced by Racinet as the set of solutions of the regularised double shuffle relations with ‘degeneration’ condition (no quadratic terms condition).

Definition 19 ([R]). The double shuffle group $DMR_0$ is the pro-algebraic variety whose set of $k$-valued points consists of the group-like series $\varphi \in U\mathfrak{H}_2$ satisfying the regularised double shuffle relations (6) without linear terms and quadratic terms.

Remark 20. (i). We note that $DMR$ stands for double mélange régularisé ([R]).

(ii). It was shown in [R] that it forms a group by the operation (11).

(iii). By the same way to remark 16 (iii), the group $DMR_0$ is regarded as a subgroup of $\text{Aut} \exp \mathfrak{H}_2$.
It is also shown that $\Im\Psi$ is contained in $DMR_0$ (cf. [F07]). Actually it is expected that they are isomorphic. Theorem 8 follows the inclusion between $GRT_1$ and $DMR_0$:

**Proposition 21.** $GRT_1 \subset DMR_0$.

It is also expected that they are isomorphic.

**Remark 22.** (i). The Drinfeld associator $\Phi_{KZ}$ satisfies the regularised double shuffle relations (cf. remark [F07]) but it is not an element of the double shuffle group, i.e. $\Phi_{KZ} \notin DMR_0(C)$, because its quadratic terms are non-zero, actually $\zeta(2)X_1X_0 - \zeta(2)X_0X_1$.

(ii). The $p$-adic Drinfeld associator $\Phi_{pKZ}$ satisfies the regularised double shuffle relations (cf. [BeF, FJ]) and it is an element of the double shuffle group, i.e. $\Phi_{pKZ} \in DMR_0(Q_p)$, which also follows from remark [F07](ii) and proposition 21.

2.4. **Kashiwara-Vergne group.** In [KV] Kashiwara and Vergne proposed a conjecture relating on Campbell-Baker-Hausdorff series which generalises Duflo’s theorem (Duflo isomorphism) to some extent. The conjecture was settled generally by Alekseev and Meinrenken [AM]. The Kashiwara-Vergne group was introduced as a ‘degeneration’ of the set of solution of the conjecture by Alekseev and Torossian in [AT], where they gave another proof of the conjecture by using associators.

The following is one of formulations of the conjecture stated in [AET].

**Generalized Kashiwara-Vergne problem:** Find a group automorphism $P : \exp \mathfrak{h}_2 \to \exp \mathfrak{h}_2$ such that $P$ belongs to $T\text{Aut} \exp \mathfrak{h}_2$ (that is, $P(e^{X_0}) = p_1e^{X_0}p_1^{-1}$ and $P(e^{X_1}) = p_2e^{X_1}p_2^{-1}$ for some $p_1, p_2 \in \exp \mathfrak{h}_2$) and $P$ satisfies

$$P(e^{X_0}e^{X_1}) = e^{(X_0 + X_1)}$$

and the coboundary Jacobian condition

$$\delta \circ J(P) = 0.$$

Here $J$ stands for the Jacobian cocycle $J : T\text{Aut} \exp \mathfrak{h}_2 \to \text{tr}_2$ and $\delta$ means the differential map $\delta : \text{tr}_n \to \text{tr}_{n+1}$ for $n = 1, 2, \ldots$ (for their precise definitions see [AT]). We note that $P$ is uniquely determined by the pair $(p_1, p_2)$.

The following is essential for the proof of the conjecture.

**Proposition 23 ([AT] [AET]).** Let $(\mu, \varphi)$ be an associator. Then the pair

$$(p_1, p_2) = \left( \varphi(X_0/\mu, X_\infty/\mu), e^{X_\infty/2}\varphi(X_1/\mu, X_\infty/\mu) \right)$$

with $X_\infty = -X_0 - X_1$ gives a solution to the above problem.

The Kashiwara-Vergne group is defined to be the set of solutions of the problem with ‘degeneration condition’ (‘the condition of $\mu = 0$’):

**Definition 24 ([AT] [AET]).** The Kashiwara-Vergne group $KRV$ is defined to be the set of $P \in T\text{Aut} \exp \mathfrak{h}_2$ which satisfies $P \in T\text{Aut} \exp \mathfrak{h}_2$,

$$P(e^{(X_0 + X_1)}) = e^{(X_0 + X_1)}$$

and the coboundary Jacobian condition $\delta \circ J(P) = 0$. 
It forms a subgroup of Aut exp $\mathfrak{S}_2$. We denote by $KRV_0$ the subgroup of $KRV$ consisting of $P$ without linear terms in both $p_1$ and $p_2$. Proposition 23 yields the inclusion below.

**Proposition 25.** $GRT_1 \subset KRV_0$.

Actually it is expected that they are isomorphic (cf. [AT]). Recent result of Schneps in [S] also leads

**Proposition 26.** $DMR_0 \subset KRV_0$.

### 2.5 Comparison

By combining theorem [4] and proposition [17] 21 25 and 26 we obtain

**Proposition 27.** $Gal^M(Z)_1 \subseteq GRT_1 \subseteq DMR_0 \subseteq KRV_0$.

Here we come to an interesting question on our four groups.

**Question 28.** Are they all equal? Namely,

$$Gal^M(Z)_1 = GRT_1 = DMR_0 = KRV_0$$

Though it might be not so good mathematically to believe such equalities without a strong conceptual support, the author thinks that it might be good at least spiritually to imagine/expect a hidden theory to relate them behind.

### References

[AET] Alekseev, A., Enriquez, B. and Torossian, C.; Drinfeld associators, braid groups and explicit solutions of the Kashiwara-Vergne equations, Publ. Math. Inst. Hautes Etudes Sci. No. 112 (2010), 143-189.

[AM] and Meinrenken, E.; On the Kashiwara-Vergne conjecture, Invent. Math. 164 (2006), no. 3, 615-634.

[AT] and Torossian, C.; The Kashiwara-Vergne conjecture and Drinfeld’s associators, to appear in Annals of Mathematics.

[Ba95] Bar-Natan, D.; On the Vassiliev knot invariants, Topology 34 (1995), no. 2, 423472.

[Ba97] Non-associative tangles, Geometric topology (Athens, GA, 1993), 139-183, AMAPIP Stud. Adv. Math., 2.1, Amer. Math. Soc., Providence, RI, 1997.

[BaD] and Dancso, Z.; Pentagon and hexagon equations following Furusho, arXiv:1010.0754, preprint (2010).

[BeF] Besser, A. and Furusho, H.; The double shuffle relations for $p$-adic multiple zeta values, AMS Contemporary Math, Vol 416, (2006), 9-29.

[Br] Brown, F.; Mixed Tate Motives over Spec($\mathbb{Z}$), arXiv:1102.1312 preprint (2011).

[C] Cartier, P.; Construction combinatoire des invariants de Vassiliev-Kontsevich des nœuds, C. R. Acad. Sci. Paris Ser. I Math. 316 (1993), no. 11, 1205-1210.

[De] Deligne, P.; Le groupe fondamental de la droite projective moins trois points, Galois groups over $\mathbb{Q}$ (Berkeley, CA, 1987), 79–297, Math. S. Res. Inst. Publ., 16, Springer, New York-Berlin, 1989.

[DeG] and Goncharov, A.; Groupes fondamentaux motiviques de Tate mixte, Ann. Sci. Ecole Norm. Sup. (4) 38 (2005), no. 1, 1-56.

[Dr] Drinfel’d, V. G.; On quasitriangular quasi-Hopf algebras and a group closely connected with Gal($\mathbb{Q}$/$\mathbb{Q}$), Leningrad Math. J. 2 (1991), no. 4, 829–860.

[E] Enriquez, B.; Quasi-reflection algebras and cyclotomic associators, Selecta Math. (N.S.) 13 (2007), no. 3, 391463.

[EF] and Furusho, H.; Mixed Pentagon, octagon and Broadhurst duality equation, arXiv:1103.1188 preprint (2011).

[F03] Furusho, H.; The multiple zeta value algebra and the stable derivation algebra, Publ. Res. Inst. Math. Sci. Vol 39, no 4. (2003). 695-720.

[F04] $p$-adic multiple zeta values I – $p$-adic multiple polylogarithms and the $p$-adic KZ equation, Inventiones Mathematicae, Volume 155, Number 2, 253-266(2004).
\[ \text{F07} \quad \text{p-adic multiple zeta values II – tannakian interpretations, Amer.J.Math, Vol 129, No 4, (2007),1105-1144.} \\
\[ \text{F10a} \quad \text{Pentagon and hexagon equations, Annals of Mathematics, Vol. 171 (2010), No. 1, 545-556.} \\
\[ \text{F10b} \quad \text{Geometric interpretation of double shuffle relation for multiple L-values, arXiv:1012.3911 preprint (2010).} \\
\[ \text{F11} \quad \text{Double shuffle relation for associators, Annals of Mathematics, Vol. 174 (2011), No. 1, 341-360.} \\
\[ \text{FJ} \quad \text{and Jafari, A.: Regularization and generalized double shuffle relations for p-adic multiple zeta values, Compositio Math. Vol 143, (2007), 1089-1107.} \\
\[ \text{G} \quad \text{Grothendieck, A.: Esquisse d’un programme, 1983, available on pp. 243–283. London Math. Soc. LNS 242, Geometric Galois actions, 1, 5–48, Cambridge Univ.} \\
\[ \text{IKZ} \quad \text{Ihara, K., Kaneko, M. and Zagier, D.; Derivation and double shuffle relations for multiple zeta values, Compos. Math. 142 (2006), no. 2, 307–338.} \\
\[ \text{JS} \quad \text{Joyal, A. and Street, R.; Braided tensor categories, Adv. Math. 102 (1993), no. 1, 20–78.} \\
\[ \text{KV} \quad \text{Kashiwara, M. and Vergne, M.; The Campbell-Hausdorff formula and invariant hyper-functions, Invent. Math. 47 (1978), no. 3, 249-272.} \\
\[ \text{K} \quad \text{Kontsevich, M.; Vassiliev’s knot invariants, I. M. Gelfand Seminar, 137150, Adv. Soviet Math., 16, Part 2, Amer. Math. Soc., Providence, RI, 1993.} \\
\[ \text{LMa} \quad \text{Le, T.Q.T. and Murakami, J.; The universal Vassiliev-Kontsevich invariant for framed oriented links, Compositio Math. 102 (1996), no. 1, 41-64.} \\
\[ \text{LMb} \quad \text{Kontsevich’s integral for the Kauffman polynomial, Nagoya Math. J. 142 (1996), 3965.} \\
\[ \text{R} \quad \text{Racinet, G.; Doubles melanges des polylogarithmes multiples aux racines de l’unite, Publ. Math. Inst. Hautes Etudes Sci. No. 95 (2002), 185–231.} \\
\[ \text{S} \quad \text{Schneps, L.; Double shuffle and Kashiwara-Vergne Lie algebra, preprint (2011).} \\
\[ \text{T} \quad \text{Terasoma, T.: Geometry of multiple zeta values, International Congress of Mathematicians. Vol. II, 627635, Eur. Math. Soc., Zürich, 2006.} \\
\[ \text{W} \quad \text{Willwacher, T.: M. Kontsevich’s graph complex and the Grothendieck-Teichmueller Lie algebra, arXiv:1009.1654 preprint (2010).} \\
\[ \text{Z} \quad \text{Zagier, D.: Evaluation of the multiple zeta values ζ(2, ..., 2, 3, 2, ..., 2), preprint (2010).} \\

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