Relative entropy, weak-strong uniqueness and conditional regularity for a compressible Oldroyd–B model

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Abstract

We consider the compressible Oldroyd–B model derived in [3] (J. W. Barrett, Y. Lu, E. Süli. Existence of large-data finite-energy global weak solutions to a compressible Oldroyd–B model. Comm. Math. Sci. 15 (2017), 1265–1323), where the existence of global-in-time finite-energy weak solutions was shown in two dimensional setting. In this paper, we first state a local well-posedness result for this compressible Oldroyd–B model. In the two dimensional setting, we give a (refined) blow-up criterion involving only the upper bound of the fluid density. We then show that, if the initial fluid density and polymer number density admit a positive lower bound, the weak solution coincides with the strong one as long as the latter exists. Moreover, if the fluid density of a weak solution issued from regular initial data admits a finite upper bound, this weak solution is indeed a strong one; this can be seen as a corollary of the refined blow-up criterion and the weak-strong uniqueness.

Keywords: Compressible Oldroyd–B model; weak-strong uniqueness; conditional regularity.

1 Introduction

The incompressible Oldroyd–B model is a macroscopic model involving only macroscopic quantities, such as the velocity, the pressure and the stress. It is known that from the incompressible Navier–Stokes–Fokker–Planck system which is a micro-macro model describing incompressible dilute polymeric fluids, one can derive, at least formally, the Oldroyd–B model, see [26].

A similar derivation can be performed in the compressible setting. Indeed, in [3], a compressible Oldroyd–B model was derived as a macroscopic closure of the compressible Navier–Stokes–Fokker–Planck equations studied in a series of papers by Barrett and Süli [7, 6, 4, 5, 8].

We represent the compressible Oldroyd–B model derived in [3]. Let Ω ⊂ Rd be a bounded open domain with a C2,β boundary (briefly, a C2,β domain), with β ∈ (0, 1), and d ∈ {2, 3}. We consider the following compressible Oldroyd–B model, posed in the time-space cylinder (0, T) × Ω:

\begin{align}
\partial_t \rho + \text{div}_x(\rho u) &= 0, \\
\partial_t (\rho u) + \text{div}_x(\rho u \otimes u) + \nabla_x p(\rho) - (\mu \Delta_x u + \nu \nabla_x \text{div}_x u) &= \text{div}_x(T - (kL\eta + \frac{3}{2}\eta^2)I) + \rho f, \\
\partial_t \eta + \text{div}_x(\eta u) &= \varepsilon \Delta_x \eta, \\
\partial_t T + \text{Div}_x(u T) - (\nabla_x u \cdot T + T \nabla_x^T u) &= \varepsilon \Delta_x T + \frac{k}{2\lambda} \eta I - \frac{1}{2\lambda} T,
\end{align}

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where the pressure $p$ and the density $\varrho$ of the solvent are supposed to be related by the typical power law relation:

\[ p(\varrho) = a \varrho^\gamma, \quad a > 0, \quad \gamma > 1. \]  

(1.5)

The term $\mu \Delta_x u + \nu \nabla_x \text{div}_x u$ corresponds to $\text{div}_x S(\nabla_x u)$ where $S(\nabla_x u)$ is the Newtonian stress tensor defined by

\[ S(\nabla_x u) = \mu^S \left( \frac{\nabla_x u + \nabla_x^T u}{2} - \frac{1}{d} (\text{div}_x u) I \right) + \mu^B (\text{div}_x u) I, \]  

(1.6)

where $\mu^S > 0$ and $\mu^B \geq 0$ are the shear and bulk viscosity coefficients, respectively. Indeed, a direct calculation gives

\[ \text{div}_x S(\nabla_x u) = \mu^S \frac{\Delta_x u}{2} + \left( \mu^B + \frac{\mu^S}{d} - \frac{\mu^S}{d} \right) \nabla_x \text{div}_x u = \mu \Delta_x u + \nu \nabla_x \text{div}_x u \]

with

\[ \mu := \frac{\mu^S}{2} > 0, \quad \nu := \mu^B + \frac{\mu^S}{2} - \frac{\mu^S}{d} \geq 0. \]

The velocity gradient matrix is defined as

\[ (\nabla_x u)_{1 \leq i, j \leq d} = (\partial_{x_j} u_i)_{1 \leq i, j \leq d}. \]

The symmetric matrix function $T = (T_{i,j})$, $1 \leq i, j \leq d$, defined on $(0, T) \times \Omega$, is the extra stress tensor and the notation $\text{Div}_x (u T)$ is defined by

\[ (\text{Div}_x (u T))_{i,j} = \text{div}_x (u T_{i,j}), \quad 1 \leq i, j \leq d. \]

The meanings of the various quantities and parameters appearing in (1.1)–(1.4) were introduced in the derivation of the model in [3]. In particular, the parameters $\varepsilon$, $k$, $\lambda$ are all positive numbers, whereas $z \geq 0$ and $L \geq 0$ with $z + L > 0$.

The polymer number density $\eta$ is a nonnegative scalar function defined as the integral of the probability density function $\psi$, which is governed by the Fokker–Planck equation, in the conformation vector, which is a microscopic variable in the modeling of dilute polymer chains. The term $q(\eta) := kL\eta + z\eta^2$ in the momentum equation (1.2) can be seen as the polymer pressure, compared to the fluid pressure $p(\varrho)$.

The equations (1.1)–(1.4) are supplemented by proper initial conditions for $\varrho$, $u$, $\eta$ and $T$, and the following boundary conditions are imposed:

\[ u = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \]  

(1.7)

\[ \partial_n \eta = 0 \quad \text{on} \quad (0, T) \times \partial \Omega, \]  

(1.8)

\[ \partial_n T = 0 \quad \text{on} \quad (0, T) \times \partial \Omega. \]  

(1.9)

Here $\partial_n := n \cdot \nabla_x$, where $n$ is the outer unit normal vector on the boundary $\partial \Omega$. The external force $f$ is assumed to be, at least, in $L^\infty((0, T) \times \Omega; \mathbb{R}^d)$.

There are stress diffusion terms $\varepsilon \Delta_x \eta$ and $\varepsilon \Delta_x T$ in our model. Such spatial stress diffusions are indeed allowed in some models of complex fluids, such as the creeping flow regime, as pointed out in [12]. Also in the modelling of the compressible Navier–Stokes–Fokker–Planck system arising in the kinetic theory of dilute polymeric fluids, where polymer chains immersed in a barotropic,
compressible, isothermal, viscous Newtonian solvent, Barrett and Süli [7] observed the presence of the centre-of-mass diffusion term $\varepsilon \Delta x \psi$, where $\psi$ is the probability density function depending on both microscopic and macroscopic variables; as a result, its macroscopic closure (the compressible Oldroyd–B model) contains such diffusion terms.

The incompressible Oldroyd–B model has been attracting continuous attention of mathematicians. The local-in-time well-posedness, as well as the global-in-time well-posedness with small data, in various spaces is known, thanks to the contributions of Renardy [30], Guillopé and Saut [20, 21] and Fernández-Cara, Guillén and Ortega [17]. Concerning the global-in-time existence of solutions with large data, in the corotational derivative setting where in the system for the extra stress tensor, the velocity gradient is replaced by its anti-symmetric part in one of the terms, Lions and Masmoudi [27] showed the global-in-time existence of weak solutions with large initial data. In the presence of stress diffusion, when a regularizing Laplacian term is present in the extra stress tensor, Barrett and Boyaval [2] showed the global-in-time existence of large data weak solutions in a two dimensional setting. Also in the presence of stress diffusion and in the two dimensional setting, Constantin and Kliegl [12] proved the global existence of strong solutions with large initial data, which can be seen as a extension of the global well-posedness theory for the two dimensional incompressible Navier–Stokes equations.

Many fundamental problems for the incompressible Oldroyd–B model are still open, such as the global-in-time existence of large data solutions, even weak ones, both in the two dimensional and the three dimensional setting without the stress diffusion. Even with stress diffusion, the global-in-time existence of large data solutions, strong or weak, is still open in the three dimensional setting. This is somehow expected, since the well-posedness of the incompressible Navier–Stokes equations is a well-known open problem.

Even less is known concerning the compressible Oldroyd–B models. Let us mention some mathematical results for compressible viscoelastic models, which have been the subject of active research in recent years. The existence and uniqueness of local strong solutions and the existence of global solutions near an equilibrium for macroscopic models of three-dimensional compressible viscoelastic fluids was considered in [29, 23, 24, 25]. In particular, Fang and Zi [13] proved the existence of a unique local-in-time strong solution to a compressible Oldroyd–B model and established a blow-up criterion for strong solutions. In [3], the existence of global-in-time weak solutions in a two dimensional setting for the compressible Oldroyd–B model (1.1)–(1.9) was shown.

We remark that many of the compressible Oldroyd–B type models considered before, as in [13, 25], are modifications of the incompressible Oldroyd–B model by replacing the incompressible Navier–Stokes equations by the compressible ones. The model considered in this paper is derived from a micro-macro models for dilute polymeric fluids and may have a stronger physical grounding. Moreover, this fully macroscopic model derived from a micro-macro model coincides with the formal derivation of the incompressible Oldroyd–B model from the incompressible Navier–Stokes–Fokker–Planck equations.

2 Main results

In this section, we state our main results. We first recall the result shown in [3] concerning the global-in-time existence of weak solutions. We state a theorem concerning the local-in-time well-posedness of strong solutions and a blow-up criterion. In the two dimensional setting, we give a refined blow-up criterion result where only the $L^\infty$ bound of the fluid density is needed. We then show a weak-strong uniqueness result by using the relative entropy method. As a corollary, this offers us a conditional regularity theorem.
2.1 Global-in-time finite energy weak solutions

We state our basic hypotheses on the initial data:

\( g(0, \cdot) = g_0(\cdot) \) with \( g_0 \geq 0 \) a.e. in \( \Omega \), \( g_0 \in L^7(\Omega) \),
\( u(0, \cdot) = u_0(\cdot) \in L^r(\Omega; \mathbb{R}^d) \) for some \( r \geq 2\gamma' \) such that \( g_0|u_0|^2 \in L^1(\Omega) \),
\( \eta(0, \cdot) = \eta_0(\cdot) \) with \( \eta_0 \geq 0 \) a.e. in \( \Omega \),
\[
\begin{align*}
\eta_0 & \in L^2(\Omega), & \text{if } \delta > 0, \\
\eta_0 \log \eta_0 & \in L^1(\Omega), & \text{if } \delta = 0,
\end{align*}
\]
\( T(0, \cdot) = T_0(\cdot) \) with \( T_0 = \frac{T_T}{T_0} \geq 0 \) a.e. in \( \Omega \), \( T_0 \in L^2(\Omega; \mathbb{R}^{d \times d}) \).

A related weak solution is defined as follows.

**Definition 2.1.** Let \( T > 0 \) and \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^{2,\beta} \) domain with \( 0 < \beta < 1 \). We say that \( (g, u, \eta, T) \) is a finite-energy weak solution in \( (0, T) \times \Omega \) to the system of equations (1.1)–(1.9), supplemented by the initial data (2.1), if:

- \( g \geq 0 \) a.e. in \( (0, T) \times \Omega \), \( g \in C_w([0, T]; L^7(\Omega)) \), \( u \in L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^d)) \),
  \( \eta \geq 0 \) a.e. in \( (0, T) \times \Omega \),
  \( \eta \log \eta \in L^\infty(0, T; L^1(\Omega)) \),
  \( \eta \in C([0, T]; L^2(\Omega)) \cap L^2(0, T; W^{1,2}(\Omega)) \), \( \text{if } \delta > 0 \),
  \( \eta \in C([0, T]; L^1(\Omega)) \), \( \eta \in C([0, T]; L^1(\Omega)) \), \( \text{if } \delta = 0 \),
  \( T = \frac{T_T}{T_0} \geq 0 \) a.e. in \( (0, T) \times \Omega \), \( T \in C([0, T]; L^2(\Omega; \mathbb{R}^{d \times d})) \cap L^2(0, T; W^{1,2}(\Omega; \mathbb{R}^{d \times d})) \).

- For any \( t \in (0, T) \) and any test function \( \phi \in C^\infty([0, T] \times \overline{\Omega}) \), one has
  \[
  \int_0^t \int_\Omega [g \partial_t \phi + g u \cdot \nabla_x \phi] \, dx \, dt = \int_\Omega g(t, \cdot) \phi(t, \cdot) \, dx - \int_\Omega g_0 \phi(0, \cdot) \, dx,
  \]
- For any \( t \in (0, T) \) and any test function \( \varphi \in C^\infty([0, T]; C_c^\infty(\Omega; \mathbb{R}^d)) \), one has
  \[
  \int_0^t \int_\Omega [\eta \partial_t \varphi + \eta u \cdot \nabla_x \varphi - \varepsilon \nabla_x \eta \cdot \nabla_x \varphi] \, dx \, dt' = \int_\Omega \eta(t, \cdot) \varphi(t, \cdot) \, dx - \int_\Omega \eta_0 \varphi(0, \cdot) \, dx.
  \]
- For any \( t \in (0, T) \) and any test function \( \mathcal{Y} \in C^\infty([0, T]; \mathbb{R}^{d \times d}) \), one has
  \[
  \int_0^t \int_\Omega \left[ T : \partial_t \mathcal{Y} + (u \cdot T) : \nabla_x \mathcal{Y} + (\nabla_x u \cdot T + T \nabla_x^T u) : \mathcal{Y} - \varepsilon \nabla_x T : \nabla_x \mathcal{Y} \right] \, dx \, dt' = \int_\Omega \left[ -\frac{k}{2\lambda} \eta \text{tr} (\mathcal{Y}) + \frac{1}{2\lambda} T : \mathcal{Y} \right] \, dx \, dt' + \int_\Omega (T(t, \cdot) : \mathcal{Y}(t, \cdot)) \, dx - \int_\Omega T_0 : \mathcal{Y}(0, \cdot) \, dx.
  \]
• The continuity equation holds in the sense of renormalized solutions: for any \( b \in C^1_b([0, \infty), (2.6) \]
\[
\partial_t b(q) + \text{div}_x(b(q)u) + (b'(q) - b(q)) \text{div}_x u = 0 \quad \text{in } \mathcal{D}'((0, T) \times \Omega).
\]

• For a.e. \( t \in (0, T) \), the following energy inequality holds:
\[
\begin{align*}
\int_{\Omega} \left[ \frac{1}{2} |q| u^2 + \frac{a}{\gamma - 1} q^\gamma + (kL(\eta \log \eta + 1) + 3 \eta^2) + \frac{1}{2} \text{tr} (\nabla) \right] (t, \cdot) \, dx \\
+ 2\varepsilon \int_0^t \int_{\Omega} 2kL |\nabla_x \eta|^2 \, dx \, dt' + \frac{1}{4\lambda} \int_0^t \int_{\Omega} \text{tr} (\nabla) \, dx \, dt' \\
+ \int_0^t \int_{\Omega} u |\nabla_x u|^2 + \nu |\text{div}_x u|^2 \, dx \, dt' \\
\leq \int_{\Omega} \left[ \frac{1}{2} |q_0| u_0|^2 + \frac{a}{\gamma - 1} q_0^\gamma + (kL(\eta_0 \log \eta_0 + 1) + 3 \eta_0^2) + \frac{1}{2} \text{tr} (\nabla) \right] \, dx \\
+ \int_0^t \int_{\Omega} f \cdot u \, dx \, dt' + \frac{kL}{4\lambda} \int_0^t \int_{\Omega} \eta \, dx \, dt'.
\end{align*}
\]

We recall the associated result concerning the existence of large data global-in-time finite-energy weak solutions, which can be obtained by summarizing Theorem 11.2 and Theorem 12.1 in [3].

**Theorem 2.2.** Let \( \gamma > 1 \) and \( \Omega \subset \mathbb{R}^2 \) be a bounded \( C^{2, \beta} \) domain with \( \beta \in (0, 1) \). Assume that the parameters \( \varepsilon, k, \lambda \) are all positive numbers and \( \gamma \geq 0, L \geq 0 \) with \( \gamma + L > 0 \). Then, for any \( T > 0 \), there exists a finite-energy weak solution \((q, u, \eta, \nabla)\) in the sense of Definition 2.1 with initial data (2.1). Moreover, the extra stress tensor \( \nabla \) satisfies the bound
\[
\begin{align*}
\int_{\Omega} |\nabla(t, \cdot)|^2 \, dx &+ \varepsilon \int_0^t \int_{\Omega} |\nabla_x \nabla|^2 \, dx \, dt' + \frac{1}{4\lambda} \int_0^t \int_{\Omega} |\nabla|_2 \, dx \, dt' \\
&\leq C(t, \| \eta_0 \|_{L^2(\Omega)}, E_0),
\end{align*}
\]
for a.e. \( t \in (0, T) \), where \( E_0 \) is given by
\[
E_0 := \int_{\Omega} \left[ \frac{1}{2} |\eta_0| u_0|^2 + \frac{a}{\gamma - 1} \eta_0^\gamma + (kL(\eta_0 \log \eta_0 + 1) + 3 \eta_0^2) + \frac{1}{2} \text{tr} (\nabla) \right] \, dx.
\]

### 2.2 Local well-posedness and blow-up criterion

We now state a result concerning the local-in-time existence of strong solutions. By strong solution, here we mean a weak solution that satisfies the equations (1.1)–(1.9) a.e. in the space-time cylinder under consideration.

**Theorem 2.3.** Let \( \gamma > 1 \) and \( \Omega \subset \mathbb{R}^d \) be a bounded \( C^{2, \beta} \) domain with \( \beta \in (0, 1) \). Assume the parameters \( \varepsilon, k, \lambda \) are all positive numbers, whereas \( \gamma \geq 0 \) and \( L \geq 0 \) with \( \gamma + L > 0 \). We assume the external force \( f \in W^{1,\frac{2}{d}}((0, \infty) \times \Omega) \). In addition to the assumption on the initial data in (2.1), we suppose that
\[
\begin{align*}
\rho_0 &\in W^{1,6}(\Omega), \quad \eta_0 \in W^{2,2}(\Omega), \quad \nabla_0 \in W^{2,2}(\Omega; \mathbb{R}^{d \times d}), \quad u_0 \in W^{1,2}_0 \cap W^{2,2}(\Omega; \mathbb{R}^d),
\end{align*}
\]
where the notation \( W^{2,2}(\Omega) := \{ f \in W^{2,2}(\Omega) : \partial_\alpha f = 0 \text{ on } \partial\Omega \} \). Suppose that there holds
\[
\begin{align*}
-(\mu \Delta_x u_0 + \nu \nabla_x \text{div}_x u_0) + \nabla_x p(\rho_0) - \text{div}_x \nabla_0 = \nabla_x (kL \eta_0 + \gamma \eta_0^2) = \sqrt{\rho_0} g
\end{align*}
\]
for some \( g \in L^2(\Omega; \mathbb{R}^d) \). Thus, there exists a unique strong solution \((\rho, \mathbf{u}, \eta, T)\) to (1.1)–(1.9) with a maximal existence time \( T_* \in (0, \infty] \) such that

\[
\begin{align*}
\rho &\geq 0, \quad \rho \in C([0, T_*), W^{1,6}(\Omega)), \\
\mathbf{u} &\in C([0, T_*), W_{0}^{1,2} \cap W^{2,2}(\Omega; \mathbb{R}^d)) \cap L_{\text{loc}}^{2}(0, T_*); W^{2,r}(\Omega; \mathbb{R}^d)), \\
\eta &\geq 0, \quad T = T^T \geq 0, \quad (\eta, T) \in C([0, T_*), W_{\text{loc}}^{2,2}(\Omega; \mathbb{R}^d)),
\end{align*}
\]

where \( r = 6 \) when \( d = 3 \) and \( r \in (1, \infty) \) is arbitrary when \( d = 2 \).

If \( T_* < \infty \), then there is a finite-time blow-up at \( T_* \) in the sense that

\[
\limsup_{T \to T_*} (\|\rho\|_{L^\infty((0,T) \times \Omega)} + \|\eta\|_{L^\infty((0,T) \times \Omega)} + \|T\|_{L^2(0,T; L^\infty(\Omega; \mathbb{R}^{d \times d}))}) = \infty.
\]

**Remark 2.4.** This local-in-time well-posedness result for strong solution is inspired by the study [11] for compressible Navier–Stokes equations. If the initial density is additionally assumed to have a positive lower bound, condition (2.9) is automatically satisfied. In such a setting, local-in-time well-posedness can be also obtained by employing the method presented in [31, 34].

**Remark 2.5.** Theorem 2.3 is given in a similar manner as Theorem 1.1 and Theorem 1.2 in [13] and can be proved similarly. In fact, there are extra diffusion terms in our model compared to the model considered in [13], and this makes the analysis even easier. Hence, we omit the proof of this theorem. The regularity assumption on the initial data may not be optimal and can be relaxed accordingly by employing the argument in [11].

In the two dimensional setting, we offer the following refined blow-up criterion.

**Theorem 2.6.** Let \( d = 2 \) and let \((\rho, \mathbf{u}, \eta, T)\) be the strong solution obtained in Theorem 2.3 to (1.1)–(1.9) with a maximal existence time \( T_* \in (0, \infty] \). If \( T_* < \infty \), there holds

\[
\limsup_{T \to T_*} \|\rho\|_{L^\infty((0,T) \times \Omega)} = \infty.
\]

**Remark 2.7.** The blow-up criterion (2.11), which is reproduced from [13], is inspired by the related study for the compressible Navier–Stokes equations in [32, 33] and for the incompressible Oldroyd–B model in [10]. Our refined criterion in Theorem 2.6 coincides with those in [32, 33] where only the upper bound of the fluid density is needed. Such a refinement crucially depends on the two dimensional setting and the presence of the diffusion terms in \( T \) and \( \eta \). This setting allows us to obtain improved estimates for \( T \) and \( \eta \) that are uniform in time (see Proposition 4.1 and Proposition 4.2). In the three dimensional setting, it is not known whether one can get such an improvement.

### 2.3 Weak-strong uniqueness and conditional regularity

Still in the two dimensional setting, we show the following weak-strong uniqueness result, provided the fluid density and polymer number density admit positive lower bounds.

**Theorem 2.8.** Let \( d = 2 \). Let \((\rho, \mathbf{u}, \eta, T)\) be a finite energy weak solution obtained in Theorem 2.2 and let \((\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\eta}, \tilde{T})\) be the strong solution obtained in Theorem 2.3 with the same initial data satisfying the assumptions stated in Theorem 2.3. If, in addition, the initial data satisfy

\[
\inf_{\Omega} \rho_0 > 0, \quad \inf_{\Omega} \eta_0 > 0,
\]

then there holds

\[
(\rho, \mathbf{u}, \eta, T) = (\tilde{\rho}, \tilde{\mathbf{u}}, \tilde{\eta}, \tilde{T}) \quad \text{in} \quad [0, T_*) \times \Omega.
\]
Finally, as a corollary of Theorem 2.6 and Theorem 2.8, we have the following conditional regularity result for finite energy weak solutions.

**Theorem 2.9.** Let \( d = 2 \). Let \( (\varrho, u, \eta, \mathcal{T}) \) be a finite energy weak solution obtained in Theorem 2.2 with initial data satisfying the assumptions stated in Theorem 2.3 and the additional lower bound constraints \((2.13)\). If, for some \( T > 0 \), there holds the upper bound
\[
(2.15) \quad \sup_{(0,T) \times \Omega} \varrho < \infty,
\]
then the weak solution \( (\varrho, u, \eta, \mathcal{T}) \) is actually a strong solution satisfying the estimates \((2.10)\) over the time interval \([0, T]\).

The rest of the paper is devoted to the proof of Theorems 2.6, 2.8 and 2.9. In Section 3, we recall some necessary lemmas. Theorems 2.6, 2.8 and 2.9 are proved in Sections 4 and 6.

Throughout the paper, \( C \) denotes some uniform constant whose value may differ from line to line. In the sequel, to avoid notation complications, we sometimes use \( L^r(0, T; X(\Omega)) \) to denote the scalar function space \( L^r(0, T; X(\Omega)) \), the vector valued function space \( L^r(0, T; X(\Omega; \mathbb{R}^d)) \) or the matrix valued function space \( L^r(0, T; X(\Omega; \mathbb{R}^{d \times d})) \), if there is no danger of confusion.

### 3 Preliminaries

In this section we recall some technical tools that will be used later. The first one concerns the Dirichlet problem for the Lamé system:

**Lemma 3.1.** Let \( \mu > 0, \nu \geq 0, \) and let \( G \subset \mathbb{R}^d \) be a bounded \( C^2 \) domain. Let \( u \) be the unique weak solution to
\[
\begin{aligned}
L u &:= -\mu \Delta_x u - \nu \nabla_x \text{div}_x u = f, & \text{in } G, \\
u &\quad = 0, & \text{on } \partial G,
\end{aligned}
\]
for some \( f \in L^r(G; \mathbb{R}^d) \) with \( 1 < r < \infty \).

(i). There holds the following estimate for some constant \( C = C(\mu, r, d, G) \), depending only on \( \mu, r, d \) and the \( C^2 \) norm of \( G \):
\[
\|u\|_{W^{1,r_0, r}(G)} \leq C(\mu, r, d, G)\|f\|_{L^r(G)}.
\]

(ii). If, moreover, \( f = \text{div}_x F \) with \( F = (F_{i,j})_{1 \leq i, j \leq d} \in L^s(G), 1 < s < \infty \), then
\[
\|u\|_{W^{1,s}_0(G)} \leq C(\mu, s, d, G)\|F\|_{L^s(G)}
\]
for some constant \( C(\mu, s, d, G) \) depending only on \( \mu, s, d \) and the \( C^1 \) norm of \( G \).

(iii). If, moreover, for any \( 1 \leq i, j \leq d, F_{i,j} = \text{div}_x H_{i,j} \) with \( H_{i,j} = (H_{i,j}^k)_{1 \leq k \leq d} \in L^t(G), 1 < t < \infty \), and \( H_{i,j} \cdot \text{n} = 0 \) on \( \partial G \), then
\[
\|u\|_{L^t(G)} \leq C(\mu, t', d, G)\|H\|_{L^{t'}(G)},
\]
where the constant \( C(\mu, t', d, G) \) is the same as in (i), where \( t' \) is such that \( 1/t + 1/t' = 1 \).
Remark 3.2. In the statement of (iii), it is meaningful to demand the normal trace $H_{i,j} \cdot \mathbf{n} = 0$ on $\partial G$ in the sense of distributions, due to the fact that $H_{i,j} = (H^{k}_{i,j})_{1 \leq k \leq d} \in L^t(G)$ and $\text{div}_{\sigma} H_{i,j} = F_{i,j} \in L^s(G)$ for some $s, t \in (1, \infty)$. We refer to [28, Section 3.2] or [15, Section 10.3] for detailed description.

Proof. The results in (i) and (ii) in Lemma 3.1 are collected from classical estimates for linear elliptic systems, see for example [1]. The result in (iii) can be proved by a duality argument by using the estimate in (i). Indeed, for any $\varphi \in L^{t'}(G; \mathbb{R}^d)$, we let $v \in W^{1,t'}_0 \cap W^{2,t'}(G; \mathbb{R}^d)$ be the unique solution to

$$
\begin{cases}
\mathcal{L}v = \varphi, & \text{in } G, \\
v = 0, & \text{on } \partial G,
\end{cases}
$$

satisfying, by the estimate in (i),

$$
\|v\|_{W^{1,t'}_0 \cap W^{2,t'}(G)} \leq C(\mu, t', d, G)\|\varphi\|_{L^{t'}(G)}.
$$

Then, by using the boundary conditions on $v$ and $H_{i,j}$, we have

$$
\int_G u \cdot \varphi \, dx = \int_G u \cdot \mathcal{L}v \, dx = \int_G \mathcal{L}u \cdot v \, dx = \int_G \text{div}_{\sigma} \text{div} H \cdot v \, dx = \int_G H : \nabla^2 v \, dx
\leq \|H\|_{L^t(G)} \|\nabla^2 v\|_{L^{t'}(G)} \leq C(\mu, t', d, G)\|H\|_{L^t(G)}\|\varphi\|_{L^{t'}(G)}.
$$

This is true for any $\varphi \in L^{t'}(G; \mathbb{R}^d)$. We then obtain the estimate in (iii).

In the sequel, we will use $\mathcal{L}^{-1}(f)$ to denote the solution $u$ to (3.1).

We then recall the following classical Gagliardo–Nirenberg inequality.

Lemma 3.3. Let $G \subset \mathbb{R}^d$ be a bounded Lipschitz domain; then, for any $r \in [2, \infty)$ if $d = 2$, and $r \in [2, 6]$ if $d = 3$, one has, for any $v \in W^{1,2}(G)$, that:

$$
\|v\|_{L^r(G)} \leq C(r, d, G)\|v\|_{L^2(G)}^{1-\theta}\|v\|_{W^{1,2}(G)}^\theta, \quad \theta := d(1 - \frac{1}{r}).
$$

Now we recall some regularity results for parabolic Neumann problems. We first introduce fractional-order Sobolev spaces. Let $G$ be the whole space $\mathbb{R}^d$ or a bounded Lipschitz domain in $\mathbb{R}^d$. For any $k \in \mathbb{N}$, $\beta \in (0, 1)$ and $s \in [1, \infty)$, we define

$$
W^{k+\beta,s}(G) := \{ v \in W^{k,s}(G) : \|v\|_{W^{k+\beta,s}(G)} < \infty \},
$$

where

$$
\|v\|_{W^{k+\beta,s}(G)} := \|v\|_{W^{k,s}(G)} + \sum_{|\alpha| = k} \left( \int_G \int_G \frac{|\partial^\alpha v(x) - \partial^\alpha v(y)|^s}{|x-y|^{d+s\beta}} \, dx \, dy \right)^{\frac{1}{s}}.
$$

The following classical results are taken from Section 7.6.1 in [28]. Consider the parabolic initial-boundary value problem:

$$
\partial_t \rho - \varepsilon \Delta_x \rho = h \text{ in } (0, T) \times G; \quad \rho(0, \cdot) = \rho_0 \text{ in } G; \quad \partial_n \rho = 0 \text{ in } (0, T) \times \partial G.
$$

Here $\varepsilon > 0$, $\rho_0$ and $h$ are known functions, and $\rho$ is the unknown solution. The first regularity result of relevance to us here is the following lemma.
Lemma 3.4. Let $0 < \beta < 1$, $1 < p, q < \infty$, and let $G \subset \mathbb{R}^d$ be a bounded $C^{2,\beta}$ domain with $\beta \in (0,1)$,

$$\rho_0 \in W^{2-\frac{2}{p},q}_{n} \cap \mathcal{H}(0, T; L^q(G)), h \in L^p(0, T; L^q(G)),$$

where $W^{2-\frac{2}{p},q}_{n}$ is the completion of the linear space $\{v \in C^\infty(\overline{G}) : \nabla_n v|_{\partial G} = 0\}$ with respect!the norm of $W^{2-\frac{2}{p},q}_{n}(G)$. Then, there exists a unique solution $\rho$ satisfying

$$\rho \in L^p(0, T; W^{2,q}(G)) \cap C([0, T]; W^{2-\frac{2}{p},q}_{n}(G)), \quad \partial_t \rho \in L^p(0, T; L^q(G))$$

solving \text{(3.5)} in $(0, T) \times G$; in addition, $\rho$ satisfies the Neumann boundary condition in \text{(3.5)} in the sense of the normal trace, which is well defined since $\Delta_x \rho \in L^p(0, T; L^q(G))$. Moreover,

$$\varepsilon^{-\frac{1}{p}} \|\rho\|_{L^\infty(0,T; W^{2-\frac{2}{p},q}(G))} + \|\partial_t \rho\|_{L^p(0, T; L^q(G))} + \varepsilon \|\rho\|_{L^p(0, T; W^{2,q}(G))}$$

$$\leq C(p, q, G) \left[\varepsilon^{-\frac{1}{p}} \|\rho_0\|_{W^{2-\frac{2}{p},q}(\overline{G})} + \|h\|_{L^p(0, T; L^q(G))}\right].$$

The second result concerns parabolic problems with a divergence-form source term $h = \text{div}_x g$.

Lemma 3.5. Let $0 < \beta < 1$, $1 < p, q < \infty$, and let $G \subset \mathbb{R}^d$ be a bounded $C^{2,\beta}$ domain with $\beta \in (0,1)$,

$$\rho_0 \in L^q(G), \quad g \in L^p(0, T; L^q(G; \mathbb{R}^d)).$$

Then, there exists a unique solution $\rho \in L^p(0, T; W^{1,q}(G)) \cap C([0, T]; L^q(G))$ satisfying

$$\rho(0, \cdot) = g_0(\cdot) \text{ a.e. in } G, \quad \frac{d}{dt} \int_G \rho \phi \, dx + \varepsilon \int_G \nabla_x \rho \cdot \nabla_x \phi \, dx = -\int_G g \cdot \nabla_x \phi \, dx \quad \text{in } \mathcal{D}'(0, T),$$

for any $\phi \in C^\infty(\overline{G})$. Moreover,

$$\varepsilon^{-\frac{1}{p}} \|\rho\|_{L^\infty(0,T; L^q(G))} + \varepsilon \|\nabla_x \rho\|_{L^p(0, T; L^q(G; \mathbb{R}^d))} \leq C(p, q, G) \left[\varepsilon^{-\frac{1}{p}} \|\rho_0\|_{L^q(G)} + \|g\|_{L^p(0, T; L^q(\mathbb{R}^d))}\right].$$

Finally, we recall the Bogovskiĭ operator, whose construction can be found in [9] and in Chapter III of Galdi’s book [18].

Lemma 3.6. Let $1 < p < \infty$ and $G \subset \mathbb{R}^d$ be a bounded Lipschitz domain. Let $L^p_0(G)$ be the space of all $L^p(G)$ functions with zero mean value. Then, there exists a linear operator $\mathcal{B}_G$ from $L^p_0(G)$ to $W^{1,p}_0(G; \mathbb{R}^d)$ such that for any $\rho \in L^p_0(G)$ one has

$$\text{div}_x \mathcal{B}_G(\rho) = \rho \quad \text{in } G; \quad \|\mathcal{B}_G(\rho)\|_{W^{1,p}_0(G; \mathbb{R}^d)} \leq C(p, d, G) \|\rho\|_{L^p(G)}.$$

If, in addition, $\rho = \text{div}_x g$ for some $g \in L^q(G; \mathbb{R}^d)$, $1 < q < \infty$, $g \cdot n = 0$ on $\partial G$, then the following inequality holds:

$$\|\mathcal{B}_G(\rho)\|_{L^q(G; \mathbb{R}^d)} \leq C(p, d, G) \|g\|_{L^q(G; \mathbb{R}^d)}.$$
4 A refined blow-up criterion

This section is devoted to the proof of Theorem 2.6. Let \( d = 2 \) and let \((\varrho, u, \eta, T)\) be the strong solution given in Theorem 2.3 with \( T_* \) the corresponding maximal existence time. By a contradiction argument, to prove Theorem 2.6, it is sufficient to show that, if

\[
T_* < \infty, \quad \limsup_{T \to T_*} \left( \|\varrho\|_{L^\infty((0,T) \times \Omega)} \right) < \infty,
\]

then the following uniform estimates in \( \eta \) and \( T \) hold:

\[
\limsup_{T \to T_*} \left( \|\eta\|_{L^\infty((0,T) \times \Omega)} + \|T\|_{L^2((0,T); L^\infty(\Omega; \mathbb{R}^2 \times \mathbb{R}^2))} \right) < \infty.
\]

Let \( T_1 \in (0, T_*) \) be close to \( T_* \) and to be determined later. By Theorem 2.3, there holds

\[
\|\varrho\|_{L^\infty((0,T_1); W^{1,6}(\Omega))} + \|(u, \eta, T)\|_{L^\infty((0,T_1); W^{2,2}(\Omega; \mathbb{R}^2 \times \mathbb{R} \times \Omega))} \leq C(T_1, T_*) < \infty.
\]

We remark that the constant \( C(T_1, T_*) \) may be unbounded as \( T_1 \to T_* \).

By Sobolev embedding we have \( \|\varrho\|_{L^\infty(\Omega)} \leq C \|\varrho\|_{W^{1,6}(\Omega)} \) and by noting (4.3), the assumption (4.1) is equivalent to the following assumption

\[
\|\varrho\|_{L^\infty((0,T_*))} < \infty.
\]

Now, starting from (4.4), we show our desired estimate (4.2) step by step in the rest of this section. Some ideas are based on the methods in [22, 32], while new technical difficulties arising from the terms in \( \eta \) and \( T \) need to be handled.

In the rest of this section, the constant \( C \) depends only on the initial data and the quantity \( \|\varrho\|_{L^\infty((0,T_*))} \), which is assumed to be bounded for contradiction.

4.1 A priori estimates

In this section and the next section (Section 4.2), we give some estimates that are uniform over the time interval \((0, T_*))\); in particular, these estimates hold true without assuming the condition (4.4).

We briefly recall the a priori energy estimates and we refer to Section 3 in [3] for the details of the derivation for a slightly modified model.

For any time \( t \in (0, T_*), \) calculating

\[
\int_0^t \left( \int_\Omega (1.2) \cdot u \, dx + \frac{1}{2} \int_\Omega \text{tr}(1.4) \, dx \right) dt'
\]

implies the energy inequality (2.7). In fact, here an energy equality is obtained due to the smoothness of the solution. Then, applying Gronwall’s inequality gives the following inclusions:

\[
\varrho \in L^\infty(0, T_*; L^\gamma(\Omega)), \quad u \in L^2(0, T_*; W^{1,2}_0(\Omega; \mathbb{R}^2)), \quad |\varrho| u^2 \in L^\infty(0, T_*; L^1(\Omega)),
\]

\[
\begin{aligned}
\eta &\in L^\infty(0, T_*; L^2(\Omega)) \cap L^2(0, T_*; W^{1,2}(\Omega)), & \text{if } \mathfrak{z} > 0, \\
\eta \log \eta &\in L^\infty(0, T_*; L^1(\Omega)), \quad \eta^\frac{1}{2} \in L^2(0, T_*; W^{1,2}(\Omega)), & \text{if } \mathfrak{z} = 0, \\
\text{tr}(T) &\in L^\infty(0, T_*; L^1(\Omega)).
\end{aligned}
\]
We then take the inner product of (1.4) with $\mathbb{T}$ and integrate over $\Omega$. Direct calculation implies

$$
\frac{1}{2} \frac{d}{dt} \int_{\Omega} |\mathbb{T}|^2 \, dx + \varepsilon \int_{\Omega} |\nabla_x \mathbb{T}|^2 \, dx + \frac{1}{2\lambda} \int_{\Omega} |\mathbb{T}|^2 \, dx
\leq - \int_{\Omega} \text{Div}_x(\mathbb{T} \mathbb{T}) : T \, dx + \int_{\Omega} \left( \nabla_x \mathbb{T} + T \nabla_x^T \mathbb{T} \right) : T \, dx + \frac{k^2}{2\lambda} \int_{\Omega} |\mathbb{T}|^2 \, dx + \int_{\Omega} \frac{\eta}{\lambda} \text{tr}(T) \, dx,
$$

(4.6)

where $\lambda > 0$. Applying the Gagliardo–Nirenberg inequality recalled in Lemma 3.3 in the case of $d = 2$, we have, over the time interval $(0, T_*)$, that

$$
\|\mathbb{T}\|_{L^2(\Omega)}^2 \leq C \|\mathbb{T}\|_{L^2(\Omega)}^2 \|\mathbb{T}\|_{W^{1,2}(\Omega)} \leq C \|\mathbb{T}\|_{L^2(\Omega)} \left( \|\mathbb{T}\|_{L^2(\Omega)} + \|\nabla_x \mathbb{T}\|_{L^2(\Omega)} \right).
$$

This implies

$$
3 \|\nabla_x \mathbb{T}\|_{L^2(\Omega)} \|\mathbb{T}\|_{L^1(\Omega)}^2 \leq C \|\nabla_x \mathbb{T}\|_{L^2(\Omega)}^2 \|\mathbb{T}\|_{L^2(\Omega)}^2 + \frac{1}{8\lambda} \|\mathbb{T}\|_{L^2(\Omega)}^2 + \frac{\varepsilon}{2} \|\nabla_x \mathbb{T}\|_{L^2(\Omega)}^2.
$$

We thus obtain for any $t \in (0, T_*):

$$
\int_{\Omega} |\mathbb{T}|^2(t, \cdot) \, dx + \varepsilon \int_0^t \int_{\Omega} |\nabla_x \mathbb{T}|^2 \, dx \, dt' + \frac{1}{4\lambda} \int_0^t \int_{\Omega} |\mathbb{T}|^2 \, dx \, dt'
\leq \int_{\Omega} |\mathbb{T}_0|^2 \, dx + C \int_0^t \|\nabla_x \mathbb{T}\|_{L^2(\Omega)}^2 \|\mathbb{T}\|_{L^2(\Omega)}^2 \, dt' + \frac{k^2}{\lambda} \int_0^t \int_{\Omega} \eta^2 \, dx \, dt'.
$$

(4.6)

If $\beta > 0$, we have from (4.5) that $\|\eta\|_{L^\infty(0, T_*, L^2(\Omega))} \leq C$. We thus deduce from (4.6) by using Gronwall’s inequality that

$$
\|\mathbb{T}(t, \cdot)\|_{L^2(\Omega)}^2 + \varepsilon \int_0^t \int_{\Omega} |\nabla_x \mathbb{T}|^2 \, dx \, dt' \leq C(E_0, \|\mathbb{T}_0\|_{L^2(\Omega)}), \quad \forall t \in (0, T_*).
$$

(4.7)

We now consider the case $\beta = 0$, $L > 0$. By (4.5), there holds

$$
\|\eta \log \eta\|_{L^\infty(0, T_*, L^1(\Omega))} + \|\nabla_x \eta^\frac{1}{2}\|_{L^2(0, T_*, L^2(\Omega))} \leq C.
$$

(4.8)

Then by (4.8), we have

$$
\int_{\Omega} |\nabla_x \eta| \, dx = \int_{\Omega} |2 \eta \frac{1}{2} \nabla_x \eta^\frac{1}{2}| \, dx \leq 2 \|\eta^\frac{1}{2}\|_{L^2(\Omega)} \|\nabla_x \eta^\frac{1}{2}\|_{L^2(\Omega)} \leq C.
$$

As $d = 2$, the Sobolev embedding of $W^{1,1}(\Omega)$ into $L^2(\Omega)$ gives

$$
\|\eta\|_{L^2(0, T_*, L^2(\Omega))} \leq C.
$$

Then, by (4.6) and Gronwall’s inequality, we obtain the same estimate as (4.7). Hence, there holds the uniform estimates

$$
\|\mathbb{T}\|_{L^\infty(0, T_*, L^2(\Omega; \mathbb{R}^{2 \times 2}))} + \|\mathbb{T}\|_{L^2(0, T_*, W^{1,2}(\Omega; \mathbb{R}^{2 \times 2}))} \leq C.
$$

(4.9)
4.2 Higher order estimates for $\eta$, $T$

Based on Sobolev embedding, the interpolation between Lebesgue spaces and the repeated application of Lemma 3.5, we show the following higher order estimates for the polymer number density:

**Proposition 4.1.** For any $r \in (1, \infty)$, there holds

\begin{equation}
\|\eta\|_{L^\infty(0,T;L^r(\Omega))} + \|\eta\|_{L^2(0,T;W^{1,r}(\Omega))} \leq C.
\end{equation}

**Proof.** The proof is in the same sprit as that of Proposition 12.2 in [3]. We have shown in Section 4.1 that $u \in L^2(0,T;W^{1,2}(\Omega,\mathbb{R}^2))$ and $\eta \in L^\infty(0,T;L^1(\Omega)) \cap L^2(0,T;L^2(\Omega))$. By Sobolev embedding $W^{1,2}(\Omega) \hookrightarrow L^{\frac{4}{3}}(\Omega)$, for any $\delta \in (0,1)$, we have that $\eta u \in L^{2-\frac{\delta}{2}}(0,T;L^1(\Omega;\mathbb{R}^2)) \cap L^1(0,T;L^{2-\frac{\delta}{2}}(\Omega;\mathbb{R}^2)) \hookrightarrow L^{1+c(\delta)}(0,T;L^{2-\delta}(\Omega;\mathbb{R}^2))$, for any $\delta \in (0,1)$ and some $c(\delta) > 0$. By observing that $\eta_0 \in W^{2,2}(\Omega) \subset L^\infty(\Omega)$, we can apply Lemma 3.5 to deduce that $\eta \in L^\infty(0,T;L^{2-\delta}(\Omega)) \cap L^{1+c(\delta)}(0,T;W^{1,2-\delta}(\Omega))$, for any $\delta \in (0,1)$ and some $c(\delta) > 0$.

This implies furthermore that $\eta u \in L^{2-\delta}(0,T;L^2(\Omega;\mathbb{R}^2)) \cap L^2(0,T;L^{2-\delta}(\Omega;\mathbb{R}^2))$, for any $\delta \in (0,1)$.

Applying Lemma 3.5 once more gives $\eta \in L^\infty(0,T;L^2(\Omega)) \cap L^{2-\delta}(0,T;W^{1,2}(\Omega)) \cap L^2(0,T;W^{1,2-\delta}(\Omega))$, for any $\delta \in (0,1)$.

This implies $\eta u \in L^2(0,T;L^{2-\delta}(\Omega;\mathbb{R}^2)) \cap L^{1+c(\delta)}(0,T;L^{\frac{4}{3}}(\Omega;\mathbb{R}^2))$, for any $\delta \in (0,1)$ and some $c(\delta) > 0$. Again by Lemma 3.5 we deduce $\eta \in L^\infty(0,T;L^r(\Omega)) \cap L^{1+c(r)}(0,T;W^{1,r}(\Omega))$, for any $r \in (1, \infty)$ and some $c(r) > 0$.

Again by the bound on $u$ and Sobolev embedding we have $\eta u \in L^2(0,T;L^r(\Omega;\mathbb{R}^2))$, for any $r \in (1, \infty)$.

Finally, one more application of Lemma 3.5 implies (4.10). \qed

Similarly, for the extra stress tensor, we have the following improved estimates:

**Proposition 4.2.** For any $r \in (1, \infty)$, there holds

\begin{equation}
\|T\|_{L^\infty(0,T;L^r(\Omega))} + \|T\|_{L^2(0,T;W^{1,r}(\Omega))} \leq C.
\end{equation}
Proof. We rewrite the equation for \( T \) as the equations for its each component \( T_{i,j} \), \( 1 \leq i, j \leq 2 \):

\[
(4.12) \quad \partial_t T_{i,j} - \varepsilon \Delta_x T_{i,j} = -\text{div}_x(u \circ T_{i,j}) + (\nabla_x u \circ T + T \circ \nabla_x^T u)_{i,j} + \frac{k}{2\lambda} \eta \delta_{i,j} - \frac{1}{2\lambda} T_{i,j}.
\]

Let

\[
h_{i,j} := (\nabla_x u \circ T + T \circ \nabla_x^T u)_{i,j} + \frac{k}{2\lambda} \eta \delta_{i,j} - \frac{1}{2\lambda} T_{i,j}.
\]

By (4.5) and (4.9), Sobolev embedding gives

\[
(\nabla_x u \circ T + T \circ \nabla_x^T u)_{i,j} \in L^2(0, T_*; L^1(\Omega)) \cap L^{1+c(\delta)}(0, T_*; L^{2-\delta}(\Omega)),
\]

for any \( \delta \in (0, 1) \) and some \( c(\delta) > 0 \).

Then, by the estimates in (4.9), (4.10) and (4.13), we have

\[
(4.14) \quad h_{i,j} \in L^2(0, T_*; L^1(\Omega)) \cap L^{1+c(\delta)}(0, T_*; L^{2-\delta}(\Omega)), \text{ for any } \delta \in (0, 1) \text{ and some } c(\delta) > 0.
\]

By applying the Bogovskiĭ operator (Lemma 3.6), there exists an \( H_{i,j} \in L^{1+c(\delta)}(0, T_*; W_0^{1,2-\delta}(\Omega; \mathbb{R}^2)) \) such that

\[
(4.15) \quad \text{div}_x H_{i,j} = h_{i,j} - \langle h_{i,j} \rangle \quad \text{with } \langle h_{i,j} \rangle(t) := \frac{1}{|\Omega|} \int_{\Omega} h_{i,j}(t, x) dx \in L^2(0, T_*).
\]

By Sobolev embedding, there holds

\[
(4.16) \quad \| H_{i,j} \|_{L^{1+c(\delta)}(0, T_*; L^{\frac{1}{2}}(\Omega))} \leq C, \text{ for any } \delta \in (0, 1) \text{ and some } c(\delta) > 0.
\]

Thus, we can rewrite (4.12) as

\[
(4.17) \quad \partial_t T_{i,j} - \varepsilon \Delta_x T_{i,j} = \text{div}_x(-u \circ T_{i,j} + H_{i,j}) + \langle h_{i,j} \rangle.
\]

Let \( \bar{T} < T_* \), where \( \bar{T} \) can be arbitrarily close to \( T_* \). Recall that there holds the estimates in (4.3) over the time interval \( (0, \bar{T}] \) so that the right-hand side of (4.17) has good integrability, for example in \( L^\infty(0, \bar{T}; L^r(\Omega)) \) for any \( r \in (1, \infty) \). Now we consider the equation (4.17) in the restricted time interval \( (0, \bar{T}) \).

Let \( T_{i,j}^{(1)} \) and \( T_{i,j}^{(2)} \) be the solutions to the equations

\[
(4.18) \quad \partial_t T_{i,j}^{(1)} - \varepsilon \Delta_x T_{i,j}^{(1)} = \text{div}_x(-u \circ T_{i,j} + H_{i,j})
\]

and

\[
(4.19) \quad \partial_t T_{i,j}^{(2)} - \varepsilon \Delta_x T_{i,j}^{(2)} = \langle h_{i,j} \rangle,
\]

respectively, subject to the Neumann boundary conditions

\[
\partial_n T_{i,j}^{(1)} = \partial_n T_{i,j}^{(2)} = 0 \quad \text{on } (0, \bar{T}) \times \partial \Omega.
\]

The initial data for \( T_{i,j}^{(1)} \) and \( T_{i,j}^{(2)} \) are taken to be in \( W_n^{2,2}(\Omega; \mathbb{R}^{d \times d}) \) satisfying

\[
T_{i,j}^{(1)}(0) + T_{i,j}^{(2)}(0) = T_{i,j}(0) \quad \text{in } \Omega,
\]

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where a simple choice is to take $T_{i,j}^{(1)}(0) = T_{i,j}(0), \ T_{i,j}^{(2)}(0) = 0$.

By the estimates in (4.3), the right-hand sides of (4.18) and (4.19) are both in $L^\infty(0, \bar{T}; L^r(\Omega))$ for any $r \in (1, \infty)$. Hence, we can apply Lemma 3.4 to show the existence, uniqueness and maximal regularity for their solutions over $(0, \bar{T})$. However, some related estimates may blow-up as $\bar{T} \to T_\ast$.

In the sequel, we will show some estimates that are uniform as $\bar{T} \to T_\ast$, which implies our desired estimates (4.11) over the time interval $(0, T_\ast)$.

A direct consequence of the uniqueness of solutions to the parabolic initial-boundary value problem (3.5) is that, for any $\bar{T} < T_\ast$,

$$T_{i,j}^{(1)} + T_{i,j}^{(2)} = T_{i,j} \quad \text{a.e. in } (0, \bar{T}) \times \Omega.$$  \hfill (4.20)

By (4.5) and (4.9), Sobolev embedding gives

$$u \in L^2(0, T_\ast; L^\frac{1}{2}(\Omega)), \quad T \in L^{2+c(\delta)}(0, T_\ast; L^\frac{1}{2}(\Omega)), \quad \text{for any } \delta \in (0, 1) \text{ and some } c(\delta) > 0.$$  \hfill (4.21)

Thus,

$$u \mathbb{T}_{i,j} \in L^{1+c(\delta)}(0, T_\ast; L^\frac{1}{2}(\Omega)), \quad \text{for any } \delta \in (0, 1) \text{ and some } c(\delta) > 0.$$  \hfill (4.22)

This allows us to apply Lemma 3.5, together with the estimates (4.16) and (4.22), and the initial regularity $T_{i,j}^{(1)}(0) \in W^{2,2}(\Omega; \mathbb{R}^{2 \times 2}) \subset L^\infty(\Omega; \mathbb{R}^{2 \times 2})$, to obtain

$$\|T_{i,j}^{(1)}\|_{L^\infty(0, \bar{T}; L^\frac{1}{2}(\Omega))} + \|T_{i,j}^{(1)}\|_{L^{1+c(\delta)}(0, \bar{T}; W^{1,\frac{1}{2}}(\Omega))} \leq C,$$  \hfill (4.23)

where the constant $C$ only depends on the stress diffusion coefficient $\varepsilon$, the $C^{2,\beta}$ norm of $\partial\Omega$, the initial datum norm $\|T_{i,j}^{(1)}(0)\|_{W^{2,2}(\Omega)}$ and the norm of the source term $\| - u \mathbb{T}_{i,j} + H_{i,j}\|_{L^{1+c(\delta)}(0, T_\ast; L^\frac{1}{2}(\Omega))}$. In particular, $C$ is independent of $\bar{T}$ and is surely uniformly bounded as $\bar{T} \to T_\ast$.

For $T_{i,j}^{(2)}$, by the fact $\langle h_{i,j} \rangle \in L^2(0, T_\ast)$ depending only on the time variable $t$ and the initial regularity $T_{i,j}^{(2)}(0) \in W^{2,2}(\Omega) \subset W^{1,r}(\Omega)$ for any $1 < r < \infty$, applying Lemma 3.4 implies

$$\|T_{i,j}^{(2)}\|_{L^\infty(0, \bar{T}; W^{1,r}(\Omega))} + \|T_{i,j}^{(2)}\|_{L^2(0, \bar{T}; W^{2,r}(\Omega))} \leq C,$$  \hfill (4.24)

where the constant $C$ is independent of $\bar{T}$ and is surely uniformly bounded as $\bar{T} \to T_\ast$.

Thus, by the uniform in time estimates in (4.23) and (4.24), by the fact in (4.20), we can pass $\bar{T} \to T_\ast$ to deduce that

$$\|T_{i,j}\|_{L^\infty(0, T_\ast; L^r(\Omega))} \leq C,$$  \hfill (4.25)

for any $1 < r < \infty$.

Again by (4.21), together with the new estimate (4.25), we deduce that

$$u \mathbb{T}_{i,j} \in L^2(0, T_\ast; L^\frac{1}{2}(\Omega)), \quad \|\nabla_x u \mathbb{T} + \mathbb{T} \nabla_x^T u\|_{i,j} \in L^2(0, T_\ast; L^{2-\delta}(\Omega)), \quad \text{for any } \delta \in (0, 1).$$

By properties of the Bogovskii operator and Sobolev embedding, we have

$$-u \mathbb{T}_{i,j} + H_{i,j} \in L^2(0, T_\ast; L^r(\Omega)), \quad \text{for any } 1 < r < \infty.$$  \hfill (4.26)

This allows us to apply Lemma 3.5 to obtain

$$\|T_{i,j}^{(1)}\|_{L^\infty(0, \bar{T}; L^r(\Omega))} + \|T_{i,j}^{(1)}\|_{L^2(0, \bar{T}; W^{1,r}(\Omega))} \leq C,$$  \hfill (4.27)

for any $1 < r < \infty$.  

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where again the constant $C$ is independent of $\hat{T}$.

Hence, by (4.20), (4.24) and (4.26), and by passing $\hat{T} \to T_*$, we deduce the estimate (4.11) and complete the proof.

\begin{remark}
From the proof, we see that Propositions 4.1 and 4.2 hold true for the case $\gamma = 0$, $L > 0$. We remark that the estimates shown in Sections 4.1 and 4.2 depend only on the initial data; more precisely, they only depend on the norms given in (2.1) and the norm $\|\eta_0, T_0\|_{W^{2,2}(\Omega)}$; particularly the estimates are independent of $\|g\|_{L^\infty(0,T_*,\Omega)}$.
\end{remark}

### 4.3 Estimates for $g|u|^\alpha$ with $\alpha > 2$

The goal of this section is to prove, by assuming (4.4), that

\begin{equation}
\|g|u|^\alpha\|_{L^\infty(0,T_*;L^1(\Omega))} \leq C < \infty, \text{ for some } \alpha > 2.
\end{equation}

As the derivation of a priori energy estimates in Section 4.1, the idea to show (4.27) is done by multiplying (1.2) by $\alpha|u|^{\alpha-2}u$ for some $\alpha > 2$ and integrate over $\Omega$. By the fact $\partial|u|^\alpha = \alpha|u|^{\alpha-2}u \cdot \partial u$ and integration by parts, one can obtain

$$
\int_\Omega \partial_t (\rho u) \cdot \alpha|u|^{\alpha-2}u \, dx + \int_\Omega \text{div}_x (\rho u \otimes u) \cdot \alpha|u|^{\alpha-2}u \, dx = \frac{d}{dt} \int_\Omega g|u|^\alpha \, dx.
$$

By repeatedly using $\partial|u|^\alpha = \alpha|u|^{\alpha-2}u \cdot \partial u$ and integration by parts, through direct calculation, we deduce that

\begin{equation}
\frac{d}{dt} \int_\Omega g|u|^\alpha \, dx + \int_\Omega \alpha|u|^{\alpha-2} \left( \mu|\nabla_x u|^2 + \nu|\text{div}_x u|^2 \right) \, dx
\end{equation}

$$
+ \alpha(\alpha - 2) \int_\Omega \left( \mu|u|^{\alpha-2}|\nabla_x u|^2 + \nu|\text{div}_x u|u|^{\alpha-3}u \cdot (\nabla_x u) \right) \, dx
\end{equation}

$$
= \alpha \int_\Omega p(\rho)|\nabla_x u|^2 \, dx + \alpha \int_\Omega (kL\eta + 3\eta^2)|\nabla_x u|^{\alpha-2}u \, dx
\end{equation}

$$
- \alpha \int_\Omega \hat{T} : \nabla_x (|u|^{\alpha-2}u) \, dx + \alpha \int_\Omega g\cdot |u|^{\alpha-2}u \, dx.
\end{equation}

Observing that $|\nabla_x u| = ||u|^{-1}u \cdot \nabla_x u| \leq |\nabla_x u|$, and taking $2 < \alpha \leq 3$ close to 2 such that $(\alpha - 2)\nu \leq \frac{1}{2}$, implies

$$
(\alpha - 2) \int_\Omega \nu(|\nabla_x u| |u|^{\alpha-3}u \cdot (\nabla_x u) \, dx \leq \frac{1}{2} \int_\Omega \mu|u|^{\alpha-2}|\nabla_x u|^2 \, dx.
$$

We thus deduce from (4.28) that, for any $t \in (0,T_*)$:

\begin{equation}
\int_\Omega g|u|^\alpha \, dx + \frac{\alpha}{2} \int_0^t \int_\Omega \mu|u|^{\alpha-2}|\nabla_x u|^2 \, dx \, dt' \leq \int_\Omega g_0|u_0|^\alpha \, dx
\end{equation}

$$
+ \alpha \int_0^t \int_\Omega p(\rho)|\nabla_x u|^2 \, dx \, dt' + \alpha \int_0^t \int_\Omega (kL\eta + 3\eta^2)|\nabla_x u|^{\alpha-2}u \, dx \, dt'
\end{equation}

$$
- \alpha \int_0^t \int_\Omega \hat{T} : \nabla_x (|u|^{\alpha-2}u) \, dx \, dt' + \alpha \int_0^t \int_\Omega g\cdot |u|^{\alpha-2}u \, dx \, dt'.
\end{equation}

\footnotetext{1}{By applying the Cauchy–Schwarz inequality and more precise analysis, this condition can be relaxed to $(\alpha - 2)\nu < 4\mu$. If there holds $8\mu - \nu > 0$, one may choose some $\alpha > 3$ in this step. We remark that the condition $8\mu - \nu > 0$ appears in the study of related 3D problems, for instance in [33].}
Note the fact that $|\text{div}_x(|u|^{\alpha-2}u)| \leq (\alpha-1)|u|^{\alpha-2}|
abla_x u|$. Then Hölder’s inequality implies that

$$
(4.30) \quad \int_0^t \int_\Omega p(\rho) \text{div}_x(|u|^{\alpha-2}u) \, dx \, dt' \leq C(\|\rho\|_{L^\infty((0,T_*) \times \Omega)}) \int_0^t \int_\Omega \rho |u|^\alpha + |\nabla_x u|^{\frac{2}{\alpha}} \, dx \, dt',
$$

where the integral related to $|\nabla_x u|^{\frac{2}{\alpha}}$ is uniformly bounded in $t \in (0,T_*)$ as long as $\alpha \leq 4$.

We then calculate

$$
(4.31) \quad \int_0^t \int_\Omega (kL\eta + \frac{3}{2} \eta^2) \text{div}_x(|u|^{\alpha-2}u) \, dx \, dt' \leq C\|\eta + \eta^2\|_{L^\infty((0,T;L^4(\Omega))}) \|\nabla_x u\|_{L^2((0,t) \times \Omega))} \|u|^{\alpha-2}\|_{L^2((0,t;L^4(\Omega))}.
$$

By (4.5), Proposition 4.1 and Sobolev embedding, the quantity on the right-hand side of (4.31) is uniformly bounded in $t \in (0,T_*)$ as long as $2(\alpha - 2) \leq 2$, which is equivalent to $\alpha \leq 3$.

By Proposition 4.2 and Sobolev embedding inequality, we have

$$
(4.32) \quad - \int_0^t \int_\Omega \nabla_x(|u|^{\alpha-2}u) \, dx \, dt' \leq C\|\nabla_x u\|_{L^2((0,t) \times \Omega)} \|\nabla_x u\|_{L^2((0,t) \times \Omega)} \|u|^{\alpha-2}\|_{L^2((0,t;L^4(\Omega))},
$$

which is uniformly bounded in $t \in (0,T_*)$ provided that $2(\alpha - 2) \leq 2 \Leftrightarrow \alpha \leq 3$. Similarly,

$$
(4.33) \quad \int_0^t \int_\Omega \rho \cdot |u|^{\alpha-2}u \, dx \, dt' \leq C\int_0^t \int_\Omega |u|^\alpha \, dx \, dt' + C.
$$

Summing up the estimates (4.29)–(4.33), Gronwall’s inequality gives our desired estimate in (4.27).

### 4.4 Improved estimates for $\nabla_x u$

We consider $v_\theta$, $v_\eta$, $v_\tau$ that solve the following Dirichlet problems for the Lamé system:

$$
\begin{cases}
- \mu \Delta_x v_\theta - \nu \nabla_x \text{div}_x v_\theta = \nabla_x p(\rho), & \text{in } \Omega, \\
v_\theta = 0, & \text{on } \partial \Omega, \\
- \mu \Delta_x v_\eta - \nu \nabla_x \text{div}_x v_\eta = \nabla_x (kL\eta + \frac{3}{2} \eta^2), & \text{in } \Omega, \\
v_\eta = 0, & \text{on } \partial \Omega, \\
- \mu \Delta_x v_\tau - \nu \nabla_x \text{div}_x v_\tau = - \nabla_x T, & \text{in } \Omega, \\
v_\tau = 0, & \text{on } \partial \Omega.
\end{cases}
$$

By the notation introduced below Lemma 3.1, we write

$$
v_\theta = \mathcal{L}^{-1}(\nabla_x p(\rho)), \quad v_\eta = \mathcal{L}^{-1}(\nabla_x (kL\eta + \frac{3}{2} \eta^2)), \quad v_\tau = \mathcal{L}^{-1}(\text{div}_x T).
$$

Since $\mathcal{L}^{-1}$ is a linear operator independent of the time variable $t$, we have, as long as the calculation makes sense, that

$$
n \partial_t \mathcal{L}^{-1}(v) = \mathcal{L}^{-1}(\partial_t v).
$$

By (4.4), Propositions 4.1 and 4.2, the application of Lemma 3.1 gives

$$
(4.35) \quad v_\theta \in L^\infty(0,T_\ast;W_0^{1,r}(\Omega;\mathbb{R}^2)), \quad v_\eta \in L^\infty(0,T_\ast;W_0^{1,r}(\Omega;\mathbb{R}^2)) \cap L^2(0,T_\ast;W^{2,r}(\Omega;\mathbb{R}^2)), \\
v_\tau \in L^\infty(0,T_\ast;W_0^{1,r}(\Omega;\mathbb{R}^2)) \cap L^2(0,T_\ast;W^{2,r}(\Omega;\mathbb{R}^2)), \quad \text{for any } r \in (1,\infty).
$$
We then introduce

\[(4.36) \quad w := u - v, \quad v := (v_\eta + v_\tau)\]

that solves over \((0, T_*) \times \Omega\) the system

\[(4.37) \quad \partial_t w - \Delta w - \nu \nabla w \cdot \nabla w = -\rho u \cdot \nabla u - \rho \partial_t v,\]

with no-slip boundary condition

\[(4.38) \quad w = 0 \quad \text{on} \quad (0, T_*) \times \partial \Omega.\]

The main goal of this section is to prove the following proposition, which is inspired by Proposition 3.2 in [32]. The proof here is more difficult and technical due to the presence of the extra terms involving \(\eta\) and \(T\).

**Proposition 4.4.** Under the assumption (4.4), we have for some \(T_1 \in (0, T_*)\) that

\[w \in L^\infty(0, T_1; W^{1,2}_0(\Omega; \mathbb{R}^2)) \cap L^2(0, T_1; W^{1,r}(\Omega; \mathbb{R}^2)) \cap L^2(T_1, T_*; W^{2,2}(\Omega; \mathbb{R}^2)), \quad \text{for any} \quad r \in (1, \infty).\]

**Proof.** By (4.3) and (4.35), there holds, for any \(T_1 \in (0, T_*)\), that

\[w \in L^\infty(0, T_1; W^{1,2}_0) \cap L^2(0, T_1; W^{1,r}(\Omega; \mathbb{R}^2)), \quad \text{for any} \quad r \in (1, \infty).\]

Thus, it is sufficient to prove for some \(T_1 \in (0, T_*)\), which shall be fixed later on close to \(T_*\), that

\[(4.39) \quad w \in L^\infty(T_1, T_*; W^{1,2}_0) \cap L^2(T_1, T_*; W^{1,r})(T_1, T_*; W^{2,2}) \quad \text{for any} \quad r \in (1, \infty).\]

From (4.37), we deduce by direct calculation that, for any \(t \in (0, T_*)\),

\[
\int_\Omega \rho|\partial_t w|^2 \, dx + \frac{1}{2} \frac{d}{dt} \int_\Omega \mu|\nabla w|^2 \, dx \leq \int_\Omega \rho|\partial_t w| |u \cdot \nabla w - \partial_t v| \, dx \\
\leq \frac{1}{2} \int_\Omega \rho|\partial_t w|^2 \, dx + \int_\Omega \rho|u \cdot \nabla w|^2 \, dx + \int_\Omega \rho|\partial_t v|^2 \, dx.
\]

This gives, for any \(T_1 \in (0, T_*)\) and any \(t \in (T_1, T_*)\), that

\[(4.40) \quad \int_\Omega \mu|\nabla w|^2(t, \cdot) \, dx + \int_{T_1}^t \int_\Omega \rho|\partial_t w|^2 \, dx \, dt' \\
\leq \int_\Omega \mu|\nabla w|^2(T_1, \cdot) \, dx + 2 \int_{T_1}^t \int_\Omega \rho|u \cdot \nabla w|^2 \, dx \, dt' + 2 \int_{T_1}^t \int_\Omega \rho|\partial_t v|^2 \, dx \, dt'.
\]

We need to estimate the last two terms in (4.40). For the penultimate one, by (4.27), Hölder's inequality, Young's inequality, and Gagliardo–Nirenberg inequality, we have, for some \(\alpha > 2\), that

\[
\|\sqrt{\rho} |u \cdot \nabla w|\|_{L^2(T_1, t; L^2(\Omega))} \leq C\|\sqrt{\rho}|u|\|_{L^\infty(T_1, t; L^\infty(\Omega))}\|\nabla w|\|_{L^2(T_1, t; L^{2\alpha}(\Omega))} \\
\leq C(1 + \|\nabla w\|_{L^2(T_1, t; L^{2\alpha}(\Omega))}^{\frac{1}{2}}) \\
\leq C(1 + \|\nabla w\|_{L^2(T_1, t; L^2(\Omega))}^\theta \|\nabla^2 w\|_{L^2(T_1, t; L^2(\Omega))}^{1-\theta}) \\
\leq C + C_\delta \|\nabla w\|_{L^2(T_1, t; L^2(\Omega))} + \delta \|\nabla^2 w\|_{L^2(T_1, t; L^2(\Omega))}
\]

for some \(\theta \in (0, 1)\) determined by \(\alpha\), for any \(\delta > 0\) and some \(C_\delta > 0\).
We now estimate the $L^2$ norm of $\partial_t v = \partial_t v_\phi + \partial_t v_\eta + \partial_t v_\tau$. For $\partial_t v_\phi$, we have
\[
\partial_t v_\phi = \partial_t \mathcal{L}^{-1}(\nabla_x p(\phi)) = \mathcal{L}^{-1}(\nabla_x (p'(\phi) \partial_t \phi)) = -\mathcal{L}^{-1}(\nabla_x (p'(\phi) \text{div}_x (\phi u))) = -\mathcal{L}^{-1}(\nabla_x \text{div}_x (p(\phi) u)) - \mathcal{L}^{-1}(\nabla_x ((p'(\phi) \phi - p(\phi)) \text{div}_x u)).
\]

By (4.4), applying Lemma 3.1 gives
\[
\|\partial_t v_\phi\|_{L^2(0,t;L^2(\Omega))} \leq C\|\nabla_x u\|_{L^2(0,t;L^2(\Omega))} \leq C.
\]

It is much more complicated to estimate $\partial_t v_\eta$. By the equation in $\eta$, we have
\[
\partial_t v_\eta = \partial_t \mathcal{L}^{-1}(\nabla_x (kL\eta + \varepsilon_\eta^2)) = k\mathcal{L}^{-1}(\nabla_x (\partial_t \eta)) + 2\varepsilon_\eta \mathcal{L}^{-1}(\nabla_x (\eta \partial_t \eta))
\]
\[
= k\mathcal{L}^{-1}(\nabla_x (-\text{div}_x (\eta u) + \varepsilon \Delta_x \eta)) + 2\varepsilon \mathcal{L}^{-1}(\nabla_x (-\eta \text{div}_x (\eta u) + \varepsilon \Delta_x \eta))
\]
\[
= k\mathcal{L}^{-1}(\nabla_x (-\text{div}_x (\eta u + \varepsilon \nabla_x \eta))) + 3\mathcal{L}^{-1}(\nabla_x (-\eta^2 \text{div}_x u)) + 3\mathcal{L}^{-1}(\nabla_x (-\eta \nabla_x \eta)).
\]

By Lemma 3.1, (4.5), Proposition 4.1 and Sobolev embedding, we have
\[
\|\mathcal{L}^{-1}(\nabla_x (-\text{div}_x (\eta u + \varepsilon \nabla_x \eta)))\|_{L^2(0,t;L^2(\Omega))} \leq C\|\eta u + \nabla_x \eta\|_{L^2(0,t;L^2(\Omega))}
\]
\[
\leq C\|\eta\|_{L^\infty(0,t;L^1(\Omega))}\|u\|_{L^2(0,t;L^4(\Omega))} + C\|\eta\|_{L^2(0,t;W^{1,2}(\Omega))} \leq C.
\]

Similarly,
\[
\|\mathcal{L}^{-1}(\nabla_x (-\eta^2 \text{div}_x u))\|_{L^2(0,t;L^2(\Omega))} \leq C\|\eta^2 u\|_{L^2(0,t;L^2(\Omega))} \leq C,
\]
\[
\|\mathcal{L}^{-1}(\nabla_x (-\eta \nabla_x \eta))\|_{L^2(0,t;L^2(\Omega))} \leq C\|\text{div}_x \eta\|_{L^2(0,t;L^2(\Omega))} \leq C\|\text{div}_x \eta\|_{L^2(0,t;L^2(\Omega))} \leq C.
\]

Then for $\partial_t v_\eta$ it is left to estimate $\mathcal{L}^{-1}(\nabla_x (\eta \Delta_x \eta))$, which is the most difficult term to estimate. Observing the fact
\[
\eta \Delta_x \eta = \frac{1}{2}(\Delta_x \eta^2 - |\nabla_x \eta|^2) = \text{div}_x (\eta \nabla_x \eta) - \frac{1}{2}|\nabla_x \eta|^2,
\]
and applying Lemma 3.1 and Proposition 4.1, gives that
\[
\|\mathcal{L}^{-1}(\nabla_x (\eta \Delta_x \eta))\|_{L^2(T_1,t;L^2(\Omega))}
\]
\[
\leq \|\mathcal{L}^{-1}(\nabla_x \text{div}_x (\eta \nabla_x \eta))\|_{L^2(T_1,t;L^2(\Omega))} + \frac{1}{2}\|\mathcal{L}^{-1}(\nabla_x (|\nabla_x \eta|^2))\|_{L^2(T_1,t;L^2(\Omega))}
\]
\[
\leq C\|\nabla_x \eta\|_{L^2(T_1,t;L^2(\Omega))} + C\|\nabla_x \eta\|_{L^2(T_1,t;L^2(\Omega))}^2
\]
\[
\leq C + C\|\nabla_x \eta\|_{L^2(T_1,t;L^2(\Omega))}^2.
\]

We benefit from the parabolic equation (1.3) in $\eta$, by Lemma 3.5, to obtain the control:
\[
\|\nabla_x \eta\|_{L^4(T_1,t;L^4(\Omega))} \leq C\|\eta u\|_{L^4(T_1,t;L^4(\Omega))} \leq C\|\eta\|_{L^\infty(T_1,t;L^{12}(\Omega))}\|u\|_{L^4(T_1,t;L^4(\Omega))}
\]
\[
\leq C\|\text{div}_x u\|_{L^2(T_1,t;L^2(\Omega))}^\frac{1}{2}\|\nabla_x u\|_{L^2(T_1,t;L^2(\Omega))}^\frac{1}{2}.
\]

where we used Gagliardo–Nirenberg inequality. Therefore,
\[
\int_t^{t'} \int_\Omega \rho |\partial_t \eta|^2 \, dx \, dt' \leq C\|\partial_t \eta\|_{L^2(T_1,t;L^2(\Omega))}^2 \leq C + C\|u\|_{L^\infty(T_1,t;L^2(\Omega))}^2\|\nabla_x u\|_{L^2(T_1,t;L^2(\Omega))}^2.
\]
We remark that there is no control for \( \|u\|_{L^\infty(T_1,t;L^2(\Omega))}^2 \) so far. We will see later on that this term can be absorbed into a positive term on the left-hand side by using the smallness of \( \|\nabla_x u\|_{L^2(T_1,t;L^2(\Omega))}^2 \) when \( T_1 \) is close to \( T_* \).

For \( \partial_t v_\tau \), direct calculation gives
\[
\partial_t v_\tau = \partial_t \mathcal{L}^{-1}(-\nabla_x T) = \mathcal{L}^{-1} \nabla_x (-\partial_t T) = -\mathcal{L}^{-1} \nabla_x \left(-\nabla_x (u \cdot T) + (\nabla_x u \cdot T + T \nabla_x^T u) + \varepsilon \Delta_x T + \frac{k}{2\lambda} \eta - \frac{1}{2\lambda} T \right).
\]

Then by (4.5), Lemma 3.1 and Propositions 4.1 and 4.2, we obtain
\[
\begin{align*}
\|\partial_t v_\tau\|_{L^2(0,t;L^2(\Omega))} & \leq C \left( \|\nabla_x u\|_{L^2(0,t;L^4(\Omega))} + \|(u|T|, \nabla_x T, \eta)\|_{L^2(0,t;L^2(\Omega))} \right) \\
& \leq C \left( \|u\|_{L^2(0,t;L^4(\Omega))} \|T\|_{L^\infty(0,t;L^4(\Omega))} + \|\nabla_x u\|_{L^2(0,t;L^2(\Omega))} \|T\|_{L^\infty(0,t;L^4(\Omega))} + 1 \right) \leq C.
\end{align*}
\]

By rewriting (4.37) as the Lamé system
\[
-\mu \Delta_x x - \nu \nabla_x x = -\varrho \partial_t w - \varrho u \cdot \nabla_x u - \varrho \partial_t x, v,
\]
supplemented with the no-slip boundary condition (4.38), we can apply Lemma 3.1 to obtain
\[
\|\nabla_x^2 w\|_{L^2(T_1,t;L^2(\Omega))} \leq C \|\partial_t w + \varrho u \cdot \nabla_x u + \varrho \partial_t x\|_{L^2(T_1,t;L^2(\Omega))}.
\]

Again by the estimates in (4.41), (4.42), (4.43) and (4.44) in (4.40) implies
\[
\begin{align*}
\int_\Omega \mu |\nabla_x w|^2(t) \, dx + \int_{T_1}^t \int_\Omega \varrho |\partial_t w|^2 \, dx \, dt' & \leq \int_\Omega \mu |\nabla_x w|^2(T_1) \, dx + \delta \|\nabla_x^2 w\|_{L^2(T_1,t;L^2(\Omega))}^2 + C \|u\|_{L^\infty(T_1,t;L^2(\Omega))}^2 \|\nabla_x u\|_{L^2(T_1,t;L^2(\Omega))}^2 + C.
\end{align*}
\]

By (3.35) and Poincaré’s inequality, we have
\[
\|u\|_{L^\infty(T_1,t;L^2(\Omega))} \leq C \|\nabla_x u\|_{L^\infty(T_1,t;L^2(\Omega))} \leq C \left( 1 + \|\nabla_x w\|_{L^\infty(T_1,t;L^2(\Omega))} \right).
\]
Together with (4.50), we obtain for any \( t \in (T_1, T_*) \) that

\[
\int_{\Omega} \mu |\nabla x w|^2 (t, \cdot) \, dx + \frac{1}{2} \int_{T_1}^{t} \int_{\Omega} g |\partial_t w|^2 \, dx \, dt' \leq \int_{\Omega} \mu |\nabla x w|^2 (T_1, \cdot) \, dx \\
+ C \int_{T_1}^{t} \int_{\Omega} |\nabla x w|^2 \, dx \, dt' + C \| \nabla w \|_{L^\infty(T_1,t;L^2(\Omega))}^2 \| \nabla x u \|_{L^2(T_1,t;L^2(\Omega))}^2 + C.
\]

This implies that, the following nonnegative quantity

\[
\xi(t) := \| \nabla x w \|_{L^\infty(T_1,t;L^2(\Omega))}^2
\]

satisfies for any \( t \in (T_1, T_*) \):

\[
(4.53) \quad \xi(t) \leq \xi(T_1) + C \int_{t}^{T_1} \xi(t') \, dt' + C \xi(t) \| \nabla x u \|_{L^2(T_1,T_*;L^2(\Omega))}^2 + C.
\]

Since \( \| \nabla x u \|_{L^2(0,T_*;L^2(\Omega))} < \infty \), there holds

\[
\| \nabla x u \|_{L^2(T_1,T_*;L^2(\Omega))} \to 0, \quad \text{as } T_1 \to T_*.
\]

Thus, by choosing \( T_1 \in (0, T_*) \) close to \( T_* \) such that \( C \| \nabla x u \|_{L^2(T_1,T_*;L^2(\Omega))} \leq 1/2 \), we deduce from (4.53) that

\[
\xi(t) \leq 2\xi(T_1) + C \int_{T_1}^{t} \xi(t') \, dt' + C.
\]

Gronwall’s inequality implies for any \( t \in (T_1, T_*) \):

\[
(4.54) \quad \xi(t) \leq e^{C(t-T_1)}(2\xi(T_1) + C) \implies \| \nabla x w \|_{L^\infty(T_1,t;L^2(\Omega))} \leq C.
\]

Combining the estimates in (4.52) and (4.54) gives for any \( t \in (T_1, T_*) \) that

\[
(4.55) \quad \| \nabla x w \|_{L^\infty(T_1,t;L^2(\Omega))} + \| \sqrt{g} \partial_t w \|_{L^2(T_1,t;L^2(\Omega))} \leq C.
\]

Together with (4.48) and (4.51), we obtain

\[
(4.56) \quad \| \nabla^2 x w \|_{L^2(T_1,t;L^2(\Omega))} \leq C, \quad \text{for any } t \in [T_1, T_*).
\]

By (4.55) and (4.56), using Sobolev embedding, we thus obtain our desired estimates in (4.39). The proof is then completed.

By the estimates in (4.3) and (4.35), a direct corollary of Proposition 4.4 is the following:

**Proposition 4.5.** Under the assumption (4.4), we have

\[
u \in L^\infty(0,T_*;W^{1,2}_0) \cap L^2(0,T_*;W^{1,r}) \quad \text{for any } r \in (1, \infty).
\]
4.5 End of the proof

We are now ready to prove the $L^\infty$ bound on $\eta$ and $T$. We rewrite (1.3) as
\[
\partial_t \eta - \varepsilon \Delta x \eta = -\nabla x \eta \cdot u - \eta \text{div}_x u.
\]

By Proposition 4.1 and Proposition 4.5, we have for any $r \in (1, \infty)$ that
\[
\| -\nabla x \eta \cdot u - \eta \text{div}_x u \|_{L^2(0,T^*;L^r(\Omega))} \leq \| \nabla x \eta \|_{L^2(0,T^*;L^2(\Omega))} \| u \|_{L^\infty(0,T^*;L^2(\Omega))} + \| \text{div}_x u \|_{L^2(0,T^*;L^2(\Omega))} \| \eta \|_{L^\infty(0,T^*;L^r(\Omega))} \leq C.
\]

This allows us to apply Lemma 3.4 with $p = 2$, $q = r$ to deduce
\[(4.57) \quad \| \eta \|_{L^\infty(0,T^*;W^{1,r}(\Omega))} + \| \partial_t \eta \|_{L^2(0,T^*;L^r(\Omega))} + \| \eta \|_{L^2(0,T^*;W^{2,r}(\Omega))} \leq C.
\]

This implies, by choosing $r > 2$ and the Sobolev embedding theorem, that
\[
\| \eta \|_{L^\infty(0,T^*;L^\infty(\Omega))} \leq C < \infty.
\]

While for $T$, a similar argument implies
\[
\| T \|_{L^\infty(0,T^*;L^\infty(\Omega))} \leq C < \infty.
\]

We have thus obtained our desired estimate (4.2) and finished the proof of Theorem 2.6.

5 Relative entropy

To prove the weak-strong uniqueness stated in Theorem 2.8, in the same spirit as the study in [19] and in [14, 16] for the compressible Navier–Stokes equations, we introduce a proper relative entropy and build a relative entropy inequality, for which a consequence is the weak-strong uniqueness through tedious analysis.

Firstly, we introduce a suitable relative entropy for our compressible Oldroyd–B model. Based on the relative entropy used in [19, 14] for the compressible Navier–Stokes equations, some modifications related to the additional terms in $\eta$, $T$ need to be done. The modifications are not a direct result by analyzing the a priori energy estimate (2.7). For example, the term $\text{tr}(T)$ on the left-hand side of the energy estimate (2.7) has a sign due to the positive definite property of $T$, but $\text{tr}(T - \bar{T})$ has no sign given two positive definite matrices $T$ and $\bar{T}$. We will see later on that we do not include $T - \bar{T}$ in our definition of the relative entropy.

For notational convenience, we define
\[(5.1) \quad H(s) := \frac{a}{\gamma - 1} s^\gamma, \quad G(s) := (kLs \log s + \frac{3}{2} s^2), \quad \forall s \in [0, \infty)
\]
satisfying
\[(5.2) \quad H'(s)s - H(s) = p(s), \quad G'(s)s - G(s) = q(s), \quad H''(s) = p'(s)/s, \quad G''(s) = q'(s)/s,
\]

where $p(s) = as^\gamma$ and $q(s) = kLs + \frac{3}{2} s^2$ denote the fluid pressure and the polymer pressure functions, respectively.
Now we introduce the relative entropy. Let \((\varrho, \bf{u}, \eta, \mathbb{T})\) be a finite energy weak solution in the sense of Definition 2.1 and obtained in Theorem 2.2. Let \(\tilde{\varrho}, \tilde{\bf{u}}, \tilde{\eta}, \tilde{\mathbb{T}}\) be the so-called relative functions which have sufficient regularity. Define the following two relative entropies:

\[
\mathcal{E}_1(t) = \mathcal{E}_1(\varrho, \bf{u}, \tilde{\varrho}, \tilde{\bf{u}})(t) := \int_\Omega \frac{1}{2} |\bf{u} - \tilde{\bf{u}}|^2 + (H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho})) (t, \cdot) \, dx, \\
\mathcal{E}_2(t) = \mathcal{E}_2(\eta, \tilde{\eta})(t) := \int_\Omega (G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta})) (t, \cdot) \, dx.
\] (5.3)

**Remark 5.1.** The relative entropy \(\mathcal{E}_1\) is the same as in [19, 14]. The new one \(\mathcal{E}_2\) is built in a similar manner. We remark that the extra stress tensor \(\mathbb{T}\) is not included in the relative entropies. One reason, explained above, is that \(\text{tr}(\mathbb{T} - \tilde{\mathbb{T}})\) has no sign. Another reason is that we do not want the remainder term \(\mathcal{R}\) in the relative entropy inequality, shown later on in Proposition 5.3, becomes too massy. This is enough to show the weak-strong uniqueness. Indeed, as we shall see later on in Section 6, based on the relative entropy inequality obtained in this section, together with an \(L^2\) type estimate for \(\mathbb{T} - \tilde{\mathbb{T}}\), we can derive the weak-strong uniqueness.

We prove some properties that we will use for the quantities appearing in the relative entropies:

**Lemma 5.2.** There exist \(\delta > 0, c > 0\) depending only on \(a\) and \(\gamma\) such that for any \(\varrho, \tilde{\varrho} \geq 0,\)

\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \geq \begin{cases} 
\frac{c}{\gamma - 1}(\varrho - \tilde{\varrho})^2, & \text{if } \delta \tilde{\varrho} \leq \varrho \leq \delta^{-1} \tilde{\varrho}, \\
c \max\{\varrho^\gamma, \tilde{\varrho}^\gamma\}, & \text{otherwise}.
\end{cases}
\] (5.4)

For any \(\eta, \tilde{\eta} \geq 0\), there holds

\[
G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta}) \geq 2\delta (\eta - \tilde{\eta}) + \begin{cases} 
\frac{kL(\eta - \tilde{\eta})^2}{2\tilde{\eta}}, & \text{if } \eta \leq 2\tilde{\eta}, \\
\frac{kL\eta}{4}, & \text{if } \eta \geq 2\tilde{\eta}.
\end{cases}
\] (5.5)

**Proof of Lemma 5.2.** We recall that \(a > 0, \gamma > 1\). We use the definition of \(H\) in (5.1) to obtain

\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) = \frac{a}{\gamma - 1} \varrho^\gamma - \frac{a\gamma}{\gamma - 1} \tilde{\varrho}^{\gamma - 1}\varrho + a\tilde{\varrho}^\gamma.
\] (5.6)

Let \(0 < \delta \leq 1/2\). We suppose that \(\varrho \leq \delta \tilde{\varrho}\). Then

\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) = a\tilde{\varrho}^\gamma + \frac{a}{\gamma - 1} (\varrho^\gamma - \gamma \tilde{\varrho}^{\gamma - 1}\varrho) \\
\geq a\tilde{\varrho}^\gamma + \frac{a}{\gamma - 1} ((\delta \tilde{\varrho})^\gamma - \gamma \tilde{\varrho}^{\gamma - 1}(\delta \tilde{\varrho})) = a\tilde{\varrho}^\gamma (1 + \frac{1}{\gamma - 1} (\delta^\gamma - \delta \gamma)),
\] (5.7)

where we have used the fact that the function \(f(\varrho) := \varrho^\gamma - \gamma \tilde{\varrho}^{\gamma - 1}\varrho\) is decreasing for \(\varrho \in [0, \tilde{\varrho}]\). The limit \(\lim_{\delta \to 0}(\delta^\gamma - \delta \gamma) = 0\) implies that there exists some \(\delta \in (0, \frac{1}{2})\) depending only on \(\gamma\) such that

\[
1 + \frac{1}{\gamma - 1} (\delta^\gamma - \delta \gamma) \geq \frac{1}{2}.
\]

Together with (5.7), we obtain

\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \geq a\tilde{\varrho}^\gamma / 2, \text{ if } \varrho \leq \delta \tilde{\varrho}.
\] (5.8)
We then consider the case $\varrho \geq \delta^{-1}\tilde{\varrho}$. By (5.6), we have
\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) = \frac{a}{\gamma - 1}\varrho^\gamma + a(\tilde{\varrho}^\gamma - \frac{\gamma}{\gamma - 1}\varrho^\gamma - \tilde{\varrho}^\gamma - \varrho^\gamma) = \frac{a}{\gamma - 1}\varrho^\gamma(1 + (\gamma - 1)\delta\gamma - \gamma\delta\gamma^{-1}),
\]
where we used the fact that the function $g(\tilde{\varrho}) := \tilde{\varrho}^\gamma - \frac{\gamma}{\gamma - 1}\varrho^\gamma - \tilde{\varrho}^\gamma - \varrho^\gamma$ is decreasing for $\tilde{\varrho} \in [0, \varrho]$. The limit $\lim_{\delta \to 0}(\gamma - 1)\delta\gamma - \gamma\delta\gamma^{-1} = 0$ implies for some $\delta > 0$ small and determined by $\gamma$ that
\[
1 + (\gamma - 1)\delta\gamma - \gamma\delta\gamma^{-1} \geq \frac{1}{2}.
\]
Thus, for such a fixed $\delta$,
\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \geq \frac{a}{2(\gamma - 1)}\varrho^\gamma, \text{ if } \varrho \geq \delta^{-1}\tilde{\varrho}. \tag{5.9}
\]

Now we consider the case $\delta\tilde{\varrho} \leq \varrho \leq \delta^{-1}\tilde{\varrho}$. By Taylor’s formula, direct calculation gives
\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) = H''(\tilde{\varrho})(\varrho - \tilde{\varrho})^2 = a\gamma\varrho^{\gamma-2}(\varrho - \tilde{\varrho})^2
\]
for some $\tilde{\varrho}$ between $\varrho$ and $\tilde{\varrho}$. Thus,
\[
H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \geq a\gamma \min\{\delta\gamma^{-2}, \delta^2\gamma^{-1}\}\varrho^{\gamma-2}(\varrho - \tilde{\varrho})^2, \text{ if } \delta\tilde{\varrho} \leq \varrho \leq \delta^{-1}\tilde{\varrho}. \tag{5.10}
\]

Summing up the estimates in (5.8), (5.9) and (5.10) gives our desired result (5.4).

By Taylor’s formula, we have
\[
G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta}) = 2\tilde{g}(\eta - \tilde{\eta})^2 + 2kL\tilde{\eta}^{-1}(\eta - \tilde{\eta})^2,
\]
for some $\tilde{\eta}$ between $\eta$ and $\tilde{\eta}$. This implies directly our desired estimate (5.5). \hfill \Box

We now state the relative entropy inequality in our setting.

**Proposition 5.3.** Let $T > 0$ and $\Omega \subset \mathbb{R}^2$ be a $C^{2,\beta}$ domain with $\beta \in (0, 1)$. Let $(\varrho, u, \eta, T)$ be a finite energy weak solution in the sense of Definition 2.1 and obtained in Theorem 2.2. Let $\tilde{\varrho}$, $\tilde{u}$, $\tilde{\eta}$ be smooth functions in $(t, x) \in [0, T] \times \Omega$ with constrains
\[
\tilde{u} = 0 \text{ on } [0, T] \times \partial \Omega, \quad \inf_{(0,T)\times\Omega} \tilde{\varrho} > 0, \quad \inf_{(0,T)\times\Omega} \tilde{\eta} > 0.
\]

Then there holds the following relative entropy inequality: for a.e. $t \in (0, T]$,
\[
\mathcal{E}_1(\varrho, u, \tilde{\varrho}, \tilde{u})(t) + \mathcal{E}_2(\eta, \tilde{\eta})(t) + \int_0^t \int_\Omega \mu |\nabla_x(u - \tilde{u})|^2 + \nu |\text{div}_x(u - \tilde{u})|^2 \, dx \, dt'
\]
\[
+ 2\varepsilon \int_0^t \int_\Omega 2kL|\nabla_x(\eta - \tilde{\eta})|^2 + \tilde{g} |\nabla_x(\eta - \tilde{\eta})|^2 \, dx \, dt'
\]
\[
\leq \mathcal{E}_1(\varrho_0, u_0, \tilde{\varrho}_0, \tilde{u}_0) + \mathcal{E}_2(\eta_0, \tilde{\eta}_0) + \int_0^t \mathcal{R}(t') \, dt', \tag{5.11}
\]
where \((\varrho_0, u_0, \bar{\varrho}_0, \bar{u}_0, \eta_0, \bar{\eta}_0)\) denotes the corresponding initial values and \(\mathcal{R}(t) = \sum_{j=1}^{5} \mathcal{R}_j(t)\) with

\[
\mathcal{R}_1(t) := \int_{\Omega} \varrho (\partial_t \bar{u} + u \cdot \nabla_x \bar{u}) \cdot (\bar{u} - u) \, dx \\
+ \int_{\Omega} \mu \nabla_x \bar{u} : \nabla_x (\bar{u} - u) + \nu \text{div}_x \bar{u} \text{div}_x (\bar{u} - u) \, dx \\
+ \int_{\Omega} (\bar{\varrho} - \varrho) \partial_t H'(\bar{\varrho}) + (\bar{\varrho} u - \varrho \bar{u}) \cdot \nabla_x H'(\bar{\varrho}) \, dx \\
+ \int_{\Omega} \text{div}_x \bar{u} (p(\bar{\varrho}) - p(\varrho)) \, dx,
\]

(5.12) \[ \mathcal{R}_2(t) := \int_{\Omega} (\bar{\eta} - \eta) \partial_t G'(\eta) + (\bar{\eta} u - \eta \bar{u}) \cdot \nabla_x G'(\eta) \, dx \\
+ \int_{\Omega} \text{div}_x \bar{u} (q(\bar{\varrho}) - q(\varrho)) \, dx, \]

\[ \mathcal{R}_3(t) := -4\varepsilon k \int_{\Omega} \nabla_x \bar{\eta}^{\frac{1}{2}} \cdot \nabla_x (\eta^{\frac{1}{2}} - \bar{\eta}^{\frac{1}{2}}) + \nabla_x \eta^{\frac{1}{2}} \cdot \nabla_x \bar{\eta}^{\frac{1}{2}} (1 - \bar{\eta}^{-\frac{1}{2}} \eta^{\frac{1}{2}}) \, dx, \]

\[ \mathcal{R}_4(t) := -2\varepsilon k \int_{\Omega} \nabla_x \bar{\eta} \cdot \nabla_x (\eta - \bar{\eta}) \, dx, \]

\[ \mathcal{R}_5(t) := \int_{\Omega} \Theta : \nabla_x (\bar{u} - u) \, dx. \]

**Proof of Proposition 5.3.** We calculate

(5.13) \[ \int_{\Omega} \frac{1}{2} \varrho |\bar{u} - \bar{u}|^2 \, dx = \int_{\Omega} \frac{1}{2} \varrho |u|^2 \, dx - \int_{\Omega} \varrho u \cdot \bar{u} \, dx + \int_{\Omega} \frac{1}{2} \varrho |\bar{u}|^2 \, dx, \]

where for the second and third terms we have, by taking \(\bar{u}\) as a test function in the weak formulation of the momentum equation (2.4) and taking \(\frac{1}{2} |\bar{u}|^2\) as a test function in the weak formulation of the continuity equation (2.2), that

(5.14) \[ \int_{\Omega} \varrho u \cdot \bar{u} \, dx = \int_{\Omega} \varrho_0 u_0 \cdot \bar{u}_0 \, dx + \int_{\Omega} \left[ \varrho u \cdot \partial_t \bar{u} + (\varrho \otimes u) : \nabla_x \bar{u} + p(\varrho) \text{div}_x \bar{u} \right. \]

\[ + q(\eta) \text{div}_x \bar{u} - (\mu \nabla_x u : \nabla_x \bar{u} + \nu \text{div}_x u \text{div}_x \bar{u}) - \Theta : \nabla_x \bar{u} + \varrho \bar{f} \cdot \bar{u} \left. \right] \, dx \, dt', \]

and

(5.15) \[ \int_{\Omega} \frac{1}{2} \varrho |\bar{u}|^2 \, dx = \int_{\Omega} \frac{1}{2} \varrho_0 |\bar{u}_0|^2 \, dx + \int_{\Omega} \int_{t_0}^{t} \varrho \bar{u} \cdot \partial_t \bar{u} + \varrho \bar{u} \cdot \nabla_x \bar{u} \cdot \bar{u} \, dx \, dt'. \]

Similarly, testing the continuity equation by \(H'(\bar{\varrho})\) gives

(5.16) \[ \int_{\Omega} \varrho H'(\bar{\varrho}) \, dx = \int_{\Omega} \varrho_0 H'(\bar{\varrho}_0) \, dx + \int_{\Omega} \int_{t_0}^{t} \varrho \partial_t H'(\bar{\varrho}) + \varrho u \cdot \nabla_x H'(\bar{\varrho}) \, dx \, dt'. \]
Plugging (5.14) and (5.15) into (5.13) and using (5.16) and the energy inequality (2.7) gives (5.17)

\[
\int_\Omega \frac{1}{2} \phi |\mathbf{u} - \mathbf{\tilde{u}}|^2 + (H(\phi) - H'(\tilde{\phi})\phi) \, dx + \int_0^t \int_\Omega \mu |\nabla_x (\mathbf{u} - \mathbf{\tilde{u}})|^2 + \nu |\text{div}_x (\mathbf{u} - \mathbf{\tilde{u}})|^2 \, dx \, dt' \\
+ \int_\Omega G(\eta) + \frac{1}{2} \text{tr}(\mathbb{T}) \, dx + 2\varepsilon \int_0^t \int_\Omega 2kL|\nabla_x \eta|^2 + 3|\nabla_x \eta|^2 \, dx \, dt' + \frac{1}{4\lambda} \int_0^t \int_\Omega \text{tr}(\mathbb{T}) \, dx \, dt' \\
\leq \int_\Omega \frac{1}{2} \phi_0 |\mathbf{u}_0 - \mathbf{\tilde{u}}_0|^2 + (H(\phi_0) - H'(\tilde{\phi}_0)\phi_0) \, dx + \int_\Omega G(\eta_0) + \frac{1}{2} \text{tr}(\mathbb{T}_0) \, dx + \frac{k}{2\lambda} \int_0^t \int_\Omega \eta \, dx \, dt' \\
+ \int_0^t \int_\Omega \psi \cdot (\mathbf{u} - \mathbf{\tilde{u}}) \, dx \, dt' + \int_0^t \int_\Omega \mu \nabla_x \mathbf{\tilde{u}} : \nabla_x (\mathbf{\tilde{u}} - \mathbf{u}) + \nu \text{div}_x \mathbf{\tilde{u}} \text{div} (\mathbf{\tilde{u}} - \mathbf{u}) \, dx \, dt' \\
- \int_0^t \int_\Omega [\phi \mathbf{u} \cdot (\partial_t \mathbf{\tilde{u}} + \mathbf{\nu} \nabla_x \mathbf{\tilde{u}}) + p(\phi) \text{div}_x \mathbf{\tilde{u}} + q(\eta) \text{div}_x \mathbf{\tilde{u}} - \mathbb{T} : \nabla_x \mathbf{\tilde{u}}] \, dx \, dt' \\
+ \int_0^t \int_\Omega \phi \partial_t H'(\tilde{\phi}) + \phi \mathbf{u} \cdot \nabla_x H'(\tilde{\phi}) \, dx \, dt'.
\]

By (5.2), we have

\[
\phi \partial_t H'(\tilde{\phi}) + \phi \mathbf{u} \cdot \nabla_x H'(\tilde{\phi}) \\
= \partial_t H'(\tilde{\phi}) + \mathbf{\tilde{u}} \cdot \nabla_x H'(\tilde{\phi}) + (\phi - \tilde{\phi})\partial_t H'(\tilde{\phi}) + (\phi \mathbf{u} - \tilde{\phi} \mathbf{u}) \cdot \nabla_x H'(\tilde{\phi}) \\
= (\phi - \tilde{\phi})\partial_t H'(\tilde{\phi}) + (\phi \mathbf{u} - \tilde{\phi} \mathbf{u}) \cdot \nabla_x H'(\tilde{\phi}) + \partial_t p(\tilde{\phi}) + \mathbf{\tilde{u}} \cdot \nabla_x p(\tilde{\phi}) \\
= (\phi - \tilde{\phi})\partial_t H'(\tilde{\phi}) + (\phi \mathbf{u} - \tilde{\phi} \mathbf{u}) \cdot \nabla_x H'(\tilde{\phi}) + \partial_t p(\tilde{\phi}) + \text{div}_x (p(\tilde{\phi}) \mathbf{u}) - p(\tilde{\phi}) \text{div} \mathbf{u}.
\]

Taking the test function \(\mathcal{Y} = \mathbb{I}/2\) in the weak formulation (2.5) implies

\[
\int_\Omega \frac{1}{2} \text{tr}(\mathbb{T}) \, dx + \frac{1}{4\lambda} \int_0^t \int_\Omega \text{tr}(\mathbb{T}) \, dx \, dt' = \int_\Omega \frac{1}{2} \text{tr}(\mathbb{T}_0) \, dx + \int_0^t \int_\Omega \frac{k}{2\lambda} \eta + \mathbb{T} : \nabla_x \mathbf{u} \, dx \, dt'.
\]

Using (5.18) and (5.19) in (5.17), together with the fact \(p(\tilde{\phi}) = H'(\tilde{\phi})\tilde{\phi} - H(\tilde{\phi})\), implies

\[
\mathcal{E}_1(t) + \int_\Omega G(\eta) \, dx + \int_0^t \int_\Omega \mu |\nabla_x (\mathbf{u} - \mathbf{\tilde{u}})|^2 + \nu |\text{div}_x (\mathbf{u} - \mathbf{\tilde{u}})|^2 \, dx \, dt' \\
+ 2\varepsilon \int_0^t \int_\Omega 2kL|\nabla_x \eta|^2 + 3|\nabla_x \eta|^2 \, dx \, dt' \\
\leq \mathcal{E}_1(0) + \int_\Omega G(\eta_0) \, dx + \int_0^t \int_\Omega \phi \mathbf{f} \cdot (\mathbf{u} - \mathbf{\tilde{u}}) \, dx \, dt' \\
+ \int_0^t \int_\Omega \mu \nabla_x \mathbf{\tilde{u}} : \nabla_x (\mathbf{\tilde{u}} - \mathbf{u}) + \nu \text{div}_x \mathbf{\tilde{u}} \text{div} (\mathbf{\tilde{u}} - \mathbf{u}) \, dx \, dt' - \int_0^t \int_\Omega q(\eta) \text{div}_x \mathbf{\tilde{u}} \, dx \, dt' \\
+ \int_0^t \int_\Omega [p(\tilde{\phi}) - p(\phi)] \text{div}_x \mathbf{\tilde{u}} \, dx \, dt' - \int_0^t \int_\Omega (\phi - \tilde{\phi})\partial_t H'(\tilde{\phi}) + (\phi \mathbf{u} - \tilde{\phi} \mathbf{u}) \cdot \nabla_x H'(\tilde{\phi}) \, dx \, dt' \\
+ \int_0^t \int_\Omega \phi (\partial_t \mathbf{\tilde{u}} + \mathbf{\nu} \nabla_x \mathbf{\tilde{u}}) \cdot (\mathbf{\tilde{u}} - \mathbf{u}) \, dx \, dt' + \int_0^t \int_\Omega \mathbb{T} : \nabla_x (\mathbf{u} - \mathbf{\tilde{u}}) \, dx \, dt'.
\]

Now we include \(\mathcal{E}_2 = \mathcal{E}_2(\eta, \tilde{\eta})\). We take \(G'(\tilde{\eta})\) as a test function in (2.3) to obtain

\[
\int_\Omega \eta G'(\tilde{\eta}) \, dx = \int_\Omega \eta G'(\tilde{\eta}_0) \, dx + \int_0^t \int_\Omega [\eta \partial_t G'(\tilde{\eta}) + \eta \mathbf{u} \cdot \nabla_x G'(\tilde{\eta}) - \varepsilon \nabla_x \eta \cdot \nabla_x G'(\tilde{\eta})] \, dx \, dt'.
\]

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We deduce from (5.2) that
\[
\eta \partial_t G' (\tilde{\eta}) + \eta u \cdot \nabla_x G' (\tilde{\eta}) \\
= \tilde{\eta} \partial_t G' (\tilde{\eta}) + \tilde{\eta} u \cdot \nabla_x G' (\tilde{\eta}) + (\eta - \tilde{\eta}) \partial_t G' (\tilde{\eta}) + (\eta u - \tilde{\eta} \tilde{u}) \cdot \nabla_x G' (\tilde{\eta}) \\
= (\eta - \tilde{\eta}) \partial_t G' (\tilde{\eta}) + (\eta u - \tilde{\eta} \tilde{u}) \cdot \nabla_x G' (\tilde{\eta}) + \partial_t q (\tilde{\eta}) + \tilde{u} \cdot \nabla_x q (\tilde{\eta}) \\
= (\eta - \tilde{\eta}) \partial_t G' (\tilde{\eta}) + (\eta u - \tilde{\eta} \tilde{u}) \cdot \nabla_x G' (\tilde{\eta}) + \partial_t q (\tilde{\eta}) + \text{div}_x (q (\tilde{\eta}) \tilde{u}) - q (\tilde{\eta}) \text{div} \tilde{u}.
\]

By (5.20), (5.21) and (5.22), and the fact \( q (\tilde{\eta}) = G' (\tilde{\eta}) \tilde{\eta} - G (\tilde{\eta}) \), we have
\[
\mathcal{E}_1 (t) + \mathcal{E}_2 (t) + \int_0^t \int_\Omega \mu |\nabla_x (u - \tilde{u})|^2 + \nu |\text{div}_x (u - \tilde{u})|^2 \, dx \, dt' \\
+ 2\varepsilon \int_0^t \int_\Omega 2kL |\nabla_x \eta|^2 + 2 |\nabla_x \eta|^2 \, dx \, dt' \\
\leq \mathcal{E}_1 (0) + \mathcal{E}_2 (0) + \int_0^t \int_\Omega q f \cdot (u - \tilde{u}) \, dx \, dt' \\
+ \int_0^t \int_\Omega \mu \nabla_x \tilde{u} : \nabla_x (\tilde{u} - u) + \nu \text{div}_x \tilde{u} \text{div} (\tilde{u} - u) \, dx \, dt' \\
+ \int_0^t \int_\Omega [p(\tilde{u}) - p(\rho)] \text{div}_x \tilde{u} \, dx \, dt' - \int_0^t \int_\Omega (\rho - \tilde{\rho}) \partial_t H' (\tilde{\rho}) + (\rho \tilde{u} - \tilde{\rho} \tilde{u}) \cdot \nabla_x H' (\tilde{\rho}) \, dx \, dt' \\
+ \int_0^t \int_\Omega [q (\tilde{\eta}) - q (\eta)] \text{div}_x \tilde{u} \, dx \, dt' - \int_0^t \int_\Omega (\eta - \tilde{\eta}) \partial_t G' (\tilde{\eta}) + (\eta u - \tilde{\eta} \tilde{u}) \cdot \nabla_x G' (\tilde{\eta}) \, dx \, dt' \\
+ \int_0^t \int_\Omega \rho (\partial_t \tilde{u} + u \cdot \nabla_x \tilde{u}) \cdot (\tilde{u} - u) \, dx \, dt' + \int_0^t \int_\Omega \mathbb{T} : \nabla_x (u - \tilde{u}) \, dx \, dt' \\
+ \int_0^t \int_\Omega \varepsilon \nabla_x \eta \cdot \nabla_x G' (\tilde{\eta}) \, dx \, dt'.
\]

By (5.1), there holds
\[
\nabla_x \eta \cdot \nabla_x G' (\tilde{\eta}) = \nabla_x \eta \cdot (kL \tilde{\eta}^{-1} + 2 \tilde{\eta}) \nabla_x \tilde{\eta} = kL \tilde{\eta}^{-1} \nabla_x \eta \cdot \nabla_x \tilde{\eta} + 2 \tilde{\eta} \nabla_x \eta \cdot \nabla_x \tilde{\eta}.
\]

We then calculate:
\[
(5.25) \quad 4 |\nabla_x \eta|^2 - \tilde{\eta}^{-1} \nabla_x \eta \cdot \nabla_x \tilde{\eta} = 4 |\nabla_x (\eta - \tilde{\eta})|^2 + 4 \nabla_x \tilde{\eta} \cdot \nabla_x (\eta - \tilde{\eta}) + 4 \nabla_x \eta \cdot \nabla_x \tilde{\eta} \cdot (1 - \eta \tilde{\eta}^{-\frac{1}{2}})
\]

and
\[
(5.26) \quad |\nabla_x \eta|^2 - \nabla_x \eta \cdot \nabla_x \tilde{\eta} = |\nabla_x (\eta - \tilde{\eta})|^2 + \nabla_x \tilde{\eta} \cdot \nabla_x (\eta - \tilde{\eta}).
\]

Finally, plugging (5.24)–(5.26) into (5.23) gives our desired inequality (5.11). The proof is completed.

\[\square\]

**Remark 5.4.** By the proof of Proposition 5.3, we see that the regularity constraints on \((\tilde{\rho}, \tilde{u}, \tilde{\eta})\) can be relaxed accordingly, as long as all the integrals in (5.11) make sense.
6 Weak-strong uniqueness

This section is devoted to proving Theorem 2.8. We shall employ the relative entropy inequality shown in the last section to achieve our goal. Let \((\varrho, u, \eta, T)\) be the finite energy weak solution obtained in Theorem 2.2 and \((\tilde{\varrho}, \tilde{u}, \tilde{\eta}, \tilde{T})\) be the strong solution obtained in Theorem 2.3 with the same initial data satisfying the assumptions in Theorem 2.3 and the lower bound constraint (2.13). Then for any \(T < T^*_s\), by the continuity equation (1.1), we have

\[
(6.1) \quad \inf_{[0,T] \times \Omega} \tilde{\varrho} \geq e^{-\int_0^T \|\text{div}_x \tilde{u}(t)\|_{L^\infty(\Omega)} dt} \inf_{\Omega} \varrho_0 > 0, \quad \inf_{[0,T] \times \Omega} \tilde{\eta} \geq e^{-\int_0^T \|\text{div}_x \tilde{u}(t)\|_{L^\infty(\Omega)} dt} \inf_{\Omega} \eta_0 > 0.
\]

Let \(T < T^*_s\) be arbitrary and fixed. In the rest of this section we restrict \(t \in [0, T]\). Thus, by (6.1), we can choose \((\tilde{\varrho}, \tilde{u}, \tilde{\eta})\) as the relative functions in the entropy inequality (5.11). We then analyze the corresponding right-hand side of (5.11) until some level that allows us to use Gronwall type inequalities to show the relative entropy is identically zero, which implies that the weak solution and the strong solution are equal. This is done in the rest of this section, step by step.

6.1 A new expression for the remainder

Since \((\tilde{\varrho}, \tilde{u}, \tilde{\eta}, \tilde{T})\) is the strong solution to (1.1)–(1.9) satisfying (6.1), we have

\[
(6.2) \quad \partial_t \tilde{u} + \tilde{u} \cdot \nabla_x \tilde{u} + \tilde{\varrho}^{-1} \nabla_x p(\tilde{\varrho}) - \tilde{\varrho}^{-1} (\mu \Delta_x \tilde{u} + \nu \nabla_x \text{div}_x \tilde{u}) = \tilde{\varrho}^{-1} \text{div}_x \tilde{T} - \tilde{\varrho}^{-1} \nabla_x q(\tilde{\eta}) + f.
\]

Plugging (6.2) into \(\mathcal{R}_1\) in (5.12), together with the fact \(\tilde{\varrho}^{-1} \nabla_x p(\tilde{\varrho}) = \nabla_x H'(\tilde{\varrho})\), gives

\[
\mathcal{R}_1 = \int_{\Omega} \varrho (u - \tilde{u}) \cdot \nabla_x \tilde{u} \cdot (\tilde{u} - u) dx + \int_{\Omega} (\mu \Delta_x \tilde{u} + \nu \nabla_x \text{div}_x \tilde{u}) (\tilde{\varrho}^{-1} \varrho - 1) \cdot (\tilde{u} - u) dx
\]

\[
- \int_{\Omega} \varrho \nabla_x H'(\tilde{\varrho}) \cdot (\tilde{u} - u) dx - \int_{\Omega} \tilde{\varrho} \nabla_x q(\tilde{\eta}) \cdot (\tilde{u} - u) dx + \int_{\Omega} \tilde{\varrho}^{-1} \text{div}_x \tilde{T} \cdot (\tilde{u} - u) dx
\]

\[
+ \int_{\Omega} (\tilde{\varrho} - \varrho) \partial_t H'(\tilde{\varrho}) + (\tilde{\varrho} - \varrho u) \cdot \nabla_x H'(\tilde{\varrho}) dx + \int_{\Omega} \text{div}_x \tilde{u} (p(\tilde{\varrho}) - p(\varrho)) dx.
\]

By the continuity equation (1.1), we have

\[
- \varrho \nabla_x H'(\tilde{\varrho}) \cdot (\tilde{u} - u) - (\tilde{\varrho} - \varrho) \partial_t H'(\tilde{\varrho}) + (\tilde{\varrho} - \varrho u) \cdot \nabla_x H'(\tilde{\varrho})
\]

\[
= (\tilde{\varrho} - \varrho) (\partial_t H'(\tilde{\varrho}) + \tilde{u} \cdot \nabla_x H'(\tilde{\varrho}))
\]

\[
= (\tilde{\varrho} - \varrho) \left[ \partial_t H'(\tilde{\varrho}) + \text{div}_x (\tilde{u} H'(\tilde{\varrho})) + (H''(\tilde{\varrho}) \tilde{\varrho} - H'(\tilde{\varrho}) \text{div}_x \tilde{u}) - (\tilde{\varrho} - \varrho) H''(\tilde{\varrho}) \text{div}_x \tilde{u} \right]
\]

\[
= -(\tilde{\varrho} - \varrho) p'(\tilde{\varrho}) \text{div}_x \tilde{u}.
\]
Then we can write \( R_1 := \sum_{j=1}^6 R_{1,j} \) with
\[
R_{1,1} := \int_\Omega \varrho \delta (\mathbf{u} - \mathbf{\hat{u}}) \cdot \nabla_x \mathbf{\hat{u}} \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx,
\]
\[
R_{1,2} := \int_\Omega (\mu \Delta_x \mathbf{\hat{u}} + \nu \nabla_x \text{div}_x \mathbf{\hat{u}}) \tilde{\varrho}^{-1} (\varrho - \tilde{\varrho}) \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx,
\]
\[
R_{1,3} := \int_\Omega \text{div}_x \mathbf{\hat{u}} (p(\tilde{\varrho}) - p(\varrho) - p'(\tilde{\varrho})(\tilde{\varrho} - \varrho)) \, dx,
\]
\[
R_{1,4} := \int_\Omega \tilde{\varrho}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx + \int_\Omega \tilde{\varrho}^{-1}(\varrho - \tilde{\varrho}) \text{div}_x \mathbf{\hat{\eta}} \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx,
\]
\[
R_{1,5} := - \int_\Omega \nabla_x q(\tilde{\eta}) \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx = - \int_\Omega \tilde{\eta} \nabla_x G'(\tilde{\eta}) \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx,
\]
\[
R_{1,6} := \int_\Omega \text{div}_x \mathbf{\hat{\eta}} \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx = - \int_\Omega \mathbf{T} : \nabla_x (\mathbf{\hat{u}} - \mathbf{u}) \, dx.
\]
By (5.12), we have
\[
R_{1,5} + R_{1,6} = \int_\Omega (\tilde{\eta} - \eta) \left( \partial_t G'(\tilde{\eta}) + \mathbf{\hat{u}} \cdot \nabla_x G'(\tilde{\eta}) \right) \, dx + \int_\Omega \text{div}_x \mathbf{\hat{u}} (q(\tilde{\eta}) - q(\eta)) \, dx,
\]

(6.4)
\[
R_{1,6} + R_5 = \int_\Omega (\mathbf{T} - \mathbf{\hat{T}}) : \nabla_x (\mathbf{\hat{u}} - \mathbf{u}) \, dx.
\]

By equation (1.3) and similar calculations as in (6.3),
\[
R_{1,5} + R_{1,2} = \int_\Omega \text{div}_x \mathbf{\hat{u}} (q(\tilde{\eta}) - q(\eta)) \, dx + \varepsilon \int_\Omega (\tilde{\eta} - \eta) \tilde{\varrho}^{-1} q'(\tilde{\eta}) \Delta x \tilde{\eta} \, dx
\]
(6.6)
\[
= R_{2,1} + \varepsilon \int_\Omega (\tilde{\eta} - \eta)(kL\tilde{\eta}^{-1} + 2\tilde{\eta}) \Delta x \tilde{\eta} \, dx,
\]

where
\[
R_{2,1} := \int_\Omega \text{div}_x \mathbf{\hat{u}} (q(\tilde{\eta}) - q(\eta)) \, dx.
\]

Thus, direct calculation gives
\[
R_{1,5} + R_{1,2} + R_3 + R_4 = R_{2,1} + \varepsilon kL \int_\Omega (\tilde{\eta} - \eta) \tilde{\varrho}^{-1} \Delta x \tilde{\eta} \, dx
\]
(6.8)
\[
- 4\varepsilon kL \int_\Omega \nabla_x \tilde{\eta}^\frac{1}{2} \cdot \nabla_x (\eta^\frac{1}{2} - \tilde{\eta}^\frac{1}{2}) + \nabla_x \eta^\frac{1}{2} \cdot \nabla_x \tilde{\eta}^\frac{1}{2} (1 - \tilde{\eta}^{-\frac{1}{2}} \eta^\frac{1}{2}) \, dx
\]
\[
= R_{2,1} + \varepsilon kL \int_\Omega (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \Delta x \tilde{\eta} \cdot \nabla_x \tilde{\eta} \cdot \nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) - \tilde{\eta}^{-1} \Delta x \tilde{\eta} (\eta^\frac{1}{2} - \tilde{\eta}^\frac{1}{2})^2 \, dx.
\]

Summarizing the calculations in (6.4)–(6.8), we obtain a new expression for the remainder \( R \):
\[
R = \int_\Omega \varrho (\mathbf{u} - \mathbf{\hat{u}}) \cdot \nabla_x \mathbf{\hat{u}} \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx + \int_\Omega (\mu \Delta_x \mathbf{\hat{u}} + \nu \nabla_x \text{div}_x \mathbf{\hat{u}}) \tilde{\varrho}^{-1} (\varrho - \tilde{\varrho}) \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx
\]
\[
+ \int_\Omega \text{div}_x \mathbf{\hat{u}} (p(\tilde{\varrho}) - p(\varrho) - p'(\tilde{\varrho})(\tilde{\varrho} - \varrho)) \, dx + \int_\Omega \text{div}_x \mathbf{\hat{u}} (q(\tilde{\eta}) - q(\eta) - q'(\tilde{\eta})(\tilde{\eta} - \eta)) \, dx
\]
(6.9)
\[
+ \int_\Omega \tilde{\varrho}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx + \int_\Omega \tilde{\varrho}^{-1}(\varrho - \tilde{\varrho}) \text{div}_x \mathbf{\hat{\eta}} \cdot (\mathbf{\hat{u}} - \mathbf{u}) \, dx
\]
\[
+ \varepsilon kL \int_\Omega (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \nabla_x \tilde{\eta} \cdot \nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) - \tilde{\eta}^{-1} \Delta x \tilde{\eta} (\eta^\frac{1}{2} - \tilde{\eta}^\frac{1}{2})^2 \, dx
\]
\[
+ \int_\Omega (\mathbf{T} - \mathbf{\hat{T}}) : \nabla_x (\mathbf{\hat{u}} - \mathbf{u}) \, dx.
\]
6.2 Estimate for the remainder

We estimate the right-hand side of (6.9) term by term. For notational convenience, let $\zeta(t)$ be a universal nonnegative integrable function in $L^1(0, T)$; its value may differ from line to line.

By (2.10) and Sobolev embedding $W^{2,6}(\Omega) \subset W^{1,\infty}(\Omega)$, we have $\nabla_x \tilde{u} \in L^2(0, T; L^\infty(\Omega))$. Thus,

\[(6.10) \quad \int_{\Omega} g(u - \tilde{u}) \cdot \nabla_x \tilde{u} \cdot (u - \tilde{u}) \, dx \leq ||\nabla_x \tilde{u}(t)||_{L^\infty(\Omega)} \int_{\Omega} g(u - \tilde{u})^2 \, dx \leq \zeta(t)E_1(t).\]

Together with the fact

\[p(\tilde{\eta}) - p(\eta) - p'(\tilde{\eta})(\tilde{\eta} - \eta) = (\gamma - 1)^{-1} [H(\tilde{\eta}) - H(\eta) - H'(\tilde{\eta})(\tilde{\eta} - \eta)],\]

\[q(\tilde{\eta}) - q(\eta) - q'(\tilde{\eta})(\tilde{\eta} - \eta) = 2\delta(\tilde{\eta} - \eta)^2 \leq G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta}),\]

we thus have

\[(6.11) \quad \int_{\Omega} \text{div}_x \tilde{u}(p(\tilde{\eta}) - p(\eta) - p'(\tilde{\eta})(\tilde{\eta} - \eta)) \, dx + \int_{\Omega} \text{div}_x \tilde{u}(q(\tilde{\eta}) - q(\eta) - q'(\tilde{\eta})(\tilde{\eta} - \eta)) \, dx \leq \zeta(t)(E_1(t) + E_2(t)).\]

By (2.10) (or by the argument of proving (4.57) using Lemma 3.4), we have, by Sobolev embedding, for any $r \in (2, \infty)$,

\[\tilde{\eta} \in L^\infty(0, T; W^{1,r}(\Omega)) \cap L^2(0, T; W^{2,r}(\Omega)) \subset L^\infty(0, T; L^\infty(\Omega)) \cap L^2(0, T; W^{1,\infty}(\Omega)).\]

Together with the lower bound for $\tilde{\eta}$ in (6.1), we have

\[(6.12) \quad \varepsilon k L \int_{\Omega} 4\tilde{\eta}^{-\frac{1}{2}}(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \nabla_x \tilde{\eta}^\frac{1}{2} \cdot \nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \, dx \leq 2\varepsilon k L ||\tilde{\eta}^{-1}(t)||_{L^\infty(\Omega)} ||\nabla_x \tilde{\eta}(t)||_{L^\infty(\Omega)} ||(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)} \||\nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}
\leq 4\varepsilon k L ||\tilde{\eta}^{-1}(t)||_{L^\infty(\Omega)} ||\nabla_x \tilde{\eta}(t)||_{L^\infty(\Omega)} ||(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}^2 + \varepsilon k L ||\nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}^2.
\]

By (5.5), we have the bound

\[(6.13) \quad (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})^2 \leq 4 \left(G(\eta) - G(\tilde{\eta}) - G'(\tilde{\eta})(\eta - \tilde{\eta})\right),\]

which is actually uniform in $\eta, \tilde{\eta} \in (0, \infty)$. Together with (6.12), we have

\[(6.14) \quad \varepsilon k L \int_{\Omega} 4\tilde{\eta}^{-\frac{1}{2}}(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \nabla_x \tilde{\eta}^\frac{1}{2} \cdot \nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \, dx \leq \zeta(t)E_2(t) + \varepsilon k L ||\nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}^2.\]

By (6.13), we have the estimate

\[(6.15) \quad \varepsilon k L \int_{\Omega} -\tilde{\eta}^{-1} \Delta_x \tilde{\eta}(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})^2 \, dx \leq \varepsilon k L \int_{\Omega} \tilde{\eta}^{-2} |\nabla_x \tilde{\eta}|^2(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})^2 \, dx - \varepsilon k L \int_{\Omega} 2\tilde{\eta}^{-1} \nabla_x \tilde{\eta} \cdot \nabla(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2}) \, dx
\leq ||\tilde{\eta}^{-1} \nabla_x \tilde{\eta}(t)||_{L^\infty(\Omega)}^2 \int_{\Omega} \varepsilon k L (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})^2 \, dx
+ 4\varepsilon k L ||\tilde{\eta}^{-1}(t)||_{L^\infty(\Omega)} ||(\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}^2 + \varepsilon k L ||\nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}^2
\leq \zeta(t)E_2(t) + \varepsilon k L ||\nabla_x (\tilde{\eta}^\frac{1}{2} - \eta^\frac{1}{2})(t)||_{L^2(\Omega)}^2.\]
We now consider
\begin{equation}
(6.16) \quad \int_{\Omega} \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\tilde{u} - u) \, dx = I_1 + I_2 + I_3,
\end{equation}
where, for $\delta$ be chosen as in Lemma 5.2,
\begin{equation}
I_1 := \int_{\delta \tilde{\varrho} \leq \varrho \leq \delta^{-1} \tilde{\varrho}} \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\tilde{u} - u) \, dx,
\end{equation}
\begin{equation}
I_2 := \int_{\delta \tilde{\varrho} \leq \varrho \leq \delta^{-1} \tilde{\varrho}} \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\tilde{u} - u) \, dx,
\end{equation}
\begin{equation}
I_3 := \int_{\varrho \leq \delta \tilde{\varrho}} \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\tilde{u} - u) \, dx.
\end{equation}

By (5.4), (2.10), the lower bound of $\tilde{\varrho}$ in (6.1) and Sobolev embedding, we have for some $\sigma > 0$ small that
\begin{equation}
(6.18) \quad I_1 \leq C(\sigma) \|\nabla_x q(\tilde{\eta})\|_{L^2(\Omega)}^2 \|\varrho - \tilde{\varrho}\|_{L^2(\delta \tilde{\varrho} \leq \varrho \leq \delta^{-1} \tilde{\varrho})}^2 + \sigma \|(\tilde{u} - u)\|_{L^3(\Omega)}^2
\leq \zeta(t) \int_{\delta \tilde{\varrho} \leq \varrho \leq \delta^{-1} \tilde{\varrho}} H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \, dx + \frac{\mu}{16} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)}^2.
\end{equation}

Similarly, for $I_2$ we have
\begin{equation}
I_2 \leq C \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} |\nabla_x q(\tilde{\eta})| |\varrho| |\tilde{u} - u| \, dx \leq C \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} \varrho |\nabla_x q(\tilde{\eta})|^2 \, dx + C \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} |\varrho| |\tilde{u} - u|^2 \, dx
\leq C \|\nabla_x q(\tilde{\eta})\|_{L^\infty(\Omega)} \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} \varrho^2 \, dx + C \int_{\Omega} |\varrho| |\tilde{u} - u|^2 \, dx
\leq \zeta(t) \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \, dx + C \int_{\Omega} |\varrho| |\tilde{u} - u|^2 \, dx.
\end{equation}

Finally for $I_3$ we have
\begin{equation}
I_3 \leq C \int_{\varrho \leq \delta \tilde{\varrho}} |\nabla_x q(\tilde{\eta})| |(\tilde{u} - u)| \, dx \leq C(\sigma) \|\nabla_x q(\tilde{\eta})\|_{L^2(\Omega)}^2 \|1\|_{L^2(\varrho \leq \delta \tilde{\varrho})}^2 + \sigma \|(\tilde{u} - u)\|_{L^3(\Omega)}^2
\leq \zeta(t) \int_{\varrho \leq \delta \tilde{\varrho}} H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \, dx + \frac{\mu}{16} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)}^2.
\end{equation}

Summing up (6.18)--(6.20), we obtain
\begin{equation}
(6.21) \quad \int_{\Omega} \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \nabla_x q(\tilde{\eta}) \cdot (\tilde{u} - u) \, dx \leq \zeta(t) E_1(t) + \frac{\mu}{8} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)}^2.
\end{equation}
A similar argument implies
\begin{equation}
(6.22) \quad \int_{\Omega} \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \text{div}_x \tilde{T} \cdot (\tilde{u} - u) \, dx \leq \zeta(t) E_1(t) + \frac{\mu}{8} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)}^2
\end{equation}
and
\begin{equation}
(6.23) \quad \int_{\Omega} (\mu \Delta_x \tilde{u} + \nu \nabla_x \text{div}_x \tilde{u}) \tilde{\omega}^{-1}(\tilde{\varrho} - \varrho) \cdot (\tilde{u} - u) \, dx \leq \zeta(t) E_1(t) + \frac{\mu}{8} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)}^2.
\end{equation}
We remark that, unlike $\nabla_x q(\tilde{\eta})$ or $\text{div}_x \tilde{T}$, we do not have control over the $L^2(0,T; L^\infty(\Omega))$ norm of $\Delta_x \tilde{u}$ in Theorem 2.3. Thus further steps need to be taken concerning the estimate (6.23); precisely, we need to modify the estimate of the following term compared to (6.19):

$$I_4 := \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} (\mu \Delta_x \tilde{u} + \nu \nabla_x \text{div}_x \tilde{u}) \tilde{\varrho}^{-1} (\varrho - \tilde{\varrho}) \cdot (\tilde{u} - u) \, dx.$$ 

Indeed, we have

$$I_4 \leq C \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} \varrho \|\mu \Delta_x \tilde{u} + \nu \nabla_x \text{div}_x \tilde{u}\| \tilde{u} - u \, dx.$$

(6.24)

For the case $\gamma \geq 2$, by the lower bound on $\tilde{\varrho}$ in (6.1), Lemma 5.2 and Sobolev embedding, we have

$$I_4 \leq C \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} \varrho^2 \|\mu \Delta_x \tilde{u} + \nu \nabla_x \text{div}_x \tilde{u}\| \tilde{u} - u \, dx$$

(6.25)

Then, by the estimate on $\nabla^2_x \tilde{u}$ in (2.10), we deduce from (6.24) and (6.25) that:

$$I_4 \leq \zeta(t) \int_{\varrho \geq \delta^{-1} \tilde{\varrho}} H(\varrho) - H(\tilde{\varrho}) - H'(\tilde{\varrho})(\varrho - \tilde{\varrho}) \, dx + \frac{\mu}{16} \|\nabla_x (\tilde{u} - u)\|_{L^2(\Omega)}^2.$$

(6.26)

Hölder’s inequality implies

$$\int_\Omega (T - \tilde{T}) : \nabla_x (\tilde{u} - u) \, dx \leq C \int_\Omega |T - \tilde{T}|^2 \, dx + \frac{\mu}{8} \|\nabla_x (\tilde{u} - u)\|_{L^2(\Omega)}^2.$$

(6.27)

Finally, summarizing the estimates in (6.10), (6.11), (6.14), (6.15), (6.21), (6.22), (6.23) and (6.27), we deduce from (6.9) that

$$\mathcal{R}(t) \leq \zeta(t)(\mathcal{E}_1 + \mathcal{E}_2)(t) + \frac{\mu}{2} \|\nabla(\tilde{u} - u)\|_{L^2(\Omega)}^2 + 2\varepsilon kL\|\nabla_x (\tilde{\eta}^2 - \eta^2)\|_{L^2(\Omega)}^2 + C \int_\Omega |T - \tilde{T}|^2 \, dx.$$

(6.28)

It is left to deal with $\int_\Omega |T - \tilde{T}|^2 \, dx$. 

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6.3 End of the proof

First of all, since the initial data are assumed to be regular enough as in Theorem 2.3, we can employ Propositions 4.1 and 4.2 to obtain better estimates for \( \eta \) and \( T \):

(6.29) \( (\eta, T) \in L^\infty(0, T_\ast; L^r(\Omega)) \cap L^2(0, T_\ast; W^{1,r}(\Omega)) \), for any \( r \in (1, \infty) \).

This allows us to take \( \bar{T} \) as a test function in the weak formulation (2.5). We can also take the strong solution \( \tilde{T} \) as a test function in (2.5). Together with the equation in \( \tilde{T} \), through tedious but rather direct calculations, we obtain

\[
\int_\Omega \frac{1}{2} |T - \bar{T}|^2 \, dx + \frac{1}{2\lambda} \int_0^t \int_\Omega |T - \bar{T}|^2 \, dx \, dt' + \varepsilon \int_0^t \int_\Omega |\nabla_x(T - \bar{T})|^2 \, dx \, dt'
= -\int_\Omega \int_0^t \left[ \text{Div}_x\left((u - \tilde{u})\bar{T}\right) + \text{Div}_x\left(\tilde{u}(T - \bar{T})\right)\right] : (T - \bar{T}) \, dx \, dt'
+ \int_0^t \int_\Omega \left[ \nabla_x(u - \tilde{u})T + \nabla_x\tilde{u}(T - \bar{T})\right] : (T - \bar{T}) \, dx \, dt' + \frac{k}{2\lambda} \int_0^t \int_\Omega (\eta - \bar{\eta}) \text{tr} (T - \bar{T}) \, dx \, dt'.
\]

We use (6.29) to get the estimate

(6.30)
\[
-\int_\Omega \text{Div}_x\left((u - \tilde{u})\bar{T}\right) : (T - \bar{T}) \, dx = -\int_\Omega (\text{Div}_x(u - \tilde{u})\bar{T} + (u - \tilde{u}) \cdot \nabla_x\bar{T}) : (T - \bar{T}) \, dx
\leq (\|\text{div}_x(u - \tilde{u})\|_{L^2(\Omega)} \|T\|_{L^\infty(\Omega)} + \|u - \tilde{u}\|_{L^6(\Omega)} \|\nabla_x\bar{T}\|_{L^2(\Omega)}) \|T - \bar{T}\|_{L^2(\Omega)}
\leq \zeta(t)\|T - \bar{T}\|_{L^2(\Omega)}^2 + \frac{\mu}{8}\|\nabla_x(u - \tilde{u})\|_{L^2(\Omega)}^2.
\]

Similarly as in (6.30), we have

\[
-\int_\Omega \text{Div}_x\left(\tilde{u}(T - \bar{T})\right) : (T - \bar{T}) \, dx = -\frac{1}{2} \int_\Omega (\text{div}_x\tilde{u}) |T - \bar{T}|^2 \, dx \leq \|\text{div}_x\tilde{u}\|_{L^\infty(\Omega)} \|T - \bar{T}\|_{L^2(\Omega)}^2.
\]

The other terms can be estimated similarly. At last, we arrive at

(6.31)
\[
\int_\Omega \frac{1}{2} |T - \bar{T}|^2 \, dx + \varepsilon \int_0^t \int_\Omega |\nabla_x(T - \bar{T})|^2 \, dx \, dt'
\leq \zeta(t)\|T - \bar{T}\|_{L^2(\Omega)} + \frac{k}{2\lambda} \|\eta - \bar{\eta}\|_{L^2(\Omega)}^2 + \frac{\mu}{4}\|\nabla_x(u - \tilde{u})\|_{L^2(\Omega)}^2.
\]

Denote

(6.32) \( \mathcal{E}(t) := \mathcal{E}_1(t) + \mathcal{E}_2(t) + \int_\Omega \frac{1}{2} |T - \bar{T}|^2(t, \cdot) \, dx. \)

Thus, by the estimates (6.28) and (6.31), by Proposition 5.3, we derive for any \( t \in (0, T) \):

\[
\mathcal{E}(t) + \int_0^t \int_\Omega \frac{\mu}{4} |\nabla_x(u - \tilde{u})|^2 + 2\varepsilon kL |\nabla_x(\bar{\eta}^\frac{1}{2} - \eta^\frac{1}{2})|^2 + 2\varepsilon \delta |\nabla_x(\bar{\eta} - \eta)|^2 + \varepsilon |\nabla_x(T - \bar{T})|^2 \, dx \, dt'
\leq \int_0^t \zeta(t')\mathcal{E}(t') \, dt', \quad \text{for some } \zeta(t) \in L^1(0, T).
\]

Gronwall’s inequality gives \( \mathcal{E}(t) \equiv 0 \) for \( t \in [0, T] \) for any \( T \in (0, T_\ast) \) which implies the weak-strong uniqueness (2.14). The proof of Theorem 2.8 is completed.
6.4 Conditional regularity

At last, we prove Theorem 2.9 concerning the conditional regularity for weak solutions. This is indeed a consequence of the refined blow-up criterion and the weak-strong uniqueness. By Theorem 2.3, we let \((\tilde{\rho}, \tilde{u}, \tilde{\eta}, \tilde{T})\) be the strong solution with \(T_*\) the maximal existence time issued from the same initial data as the weak solution \((\rho, u, \eta, T)\).

We firstly show that \(T < T_*\). For contradiction we assume that \(T \geq T_*\). Then for any \(T_1 < T_*\), since \(\tilde{\rho}\) and \(\tilde{\eta}\) has a positive lower bound over \([0, T_1] \times \Omega\) (see (6.1)), we can apply Theorem 2.8 to derive that the weak solution coincides with the strong one over \([0, T_1]\). This implies that

\[
\sup_{[0, T_1] \times \Omega} \tilde{\rho} = \sup_{[0, T_1] \times \Omega} \rho < \infty,
\]

where the upper bound is uniform as \(T_1 \to T_*\). Thus,

\[
\sup_{[0, T_*] \times \Omega} \tilde{\rho} \leq \sup_{[0, T] \times \Omega} \rho < \infty.
\]

This implies, by Theorem 2.6, that \(T_* = \infty\) which contradicts the assumption that \(T \geq T_*\).

Now we have \(T < T_*\). We can choose \(T_1\) such that \(T \leq T_1 < T_*\). Since \(\tilde{\rho}\) and \(\tilde{\eta}\) have positive lower bounds over \([0, T_1] \times \Omega\) (see (6.1)), we can apply Theorem 2.8 to deduce that the weak solution coincides with the strong solution over \([0, T_1] \supset [0, T]\). The proof of Theorem 2.9 is then completed.

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