Equivalences of LLT polynomials via lattice paths

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Abstract

The LLT polynomials $L_{\beta/\gamma}(X; t)$ are a family of symmetric polynomials indexed by a tuple of (possibly skew-)partitions $\beta/\gamma = (\beta^{(1)}/\gamma^{(1)}, \ldots, \beta^{(k)}/\gamma^{(k)})$. It has recently been shown that these polynomials can be seen as the partition function of a certain vertex model whose boundary conditions are determined by $\beta/\gamma$. In this paper we describe an algorithm which gives a bijection between the configurations of the vertex model with boundary condition $\beta/\gamma = (\beta^{(1)}/\gamma^{(1)}, \beta^{(2)}/\gamma^{(2)})$ and those with boundary condition $(\beta/\gamma)_{\text{swap}} = (\beta^{(2)}/\gamma^{(2)}, \beta^{(1)}/\gamma^{(1)})$. We prove a sufficient condition for when this bijection is weight-preserving up to an overall factor of $t$, which in turn implies that the corresponding LLT polynomials are equal up to the same overall factor. Using these techniques, we are also able to systematically determine linear relations within families of LLT polynomials.

1 Introduction

Originally defined by Lascoux, Leclerc, and Thibon [10] as the generating function of a spin statistic on ribbon tableaux, the eponymously named LLT polynomials are a family of symmetric polynomials which can be seen as a $t$-deformation of products of Schur polynomials. In [9] the LLT polynomials were reformulated as the generating function for an inversion statistic on tuples of semistandard Young tableaux, with the relationship between ribbon tableaux and tuples of SSYT given by the Stanton-White correspondence [14]. Most recently, in [3], the authors use a new formulation of the LLT polynomials in their work on the generalization of the shuffle theorem. We will use this formulation in what follows, and we will refer to them as the coinversion LLT polynomials.

We study the LLT polynomials from the perspective of vertex models. Vertex models have long been studied in relation to integrable systems and statistical mechanics (see [13] and references therein). Recently, they have been used to gain new insights on symmetric polynomials and their non-symmetric variants (for example, but by no means an exhaustive list, [6, 8, 5, 4]). It was shown in [1, 7] that the LLT polynomials could be expressed as the partition function of a certain vertex model. In [1] it was shown that, in fact, the LLT polynomial vertex model was a degeneration of a more general vertex model related to the quantized affine Lie superalgebra $U_q(\mathfrak{sl}(1|n))$. We will not need that level of generality here.

We say that two LLT polynomials are equivalent if they are equal up to an overall factor of $t$. In this paper, we use the vertex model structure to prove a sufficient condition for when swapping a pair of partitions in the indexing tuple of an LLT polynomial results in an equivalent LLT polynomial. Our main result is
Theorem. If there is a unique non-crossing matching $M$ of the sequence of beads associated to $\beta/\gamma$, then $L_{\beta/\gamma}(X_n; t)$ and $L_{(\beta/\gamma)_{\text{swap}}}(X_n; t)$ are equivalent. In particular,

$$L_{\beta/\gamma}(X_n; t) = \left( \prod_{a \in M} w(a) \right) L_{(\beta/\gamma)_{\text{swap}}}(X_n; t)$$

where the product is over all arcs $a$ in the matching and the weight of an arc is given by (7).

We then extend our techniques to construct linear relations between certain LLT polynomials. As one possible application we show

Theorem. For every $\beta$ in the family of $\binom{2n}{n}$ partitions given in (8), the LLT polynomial $L_{\beta}(X_n; t)$ can be written

$$L_{\beta}(X_n; t) = \frac{C_n}{\sum_{j=1}^{\ell_0(\beta)} g_j(X_n; t)}$$

where $C_n = \frac{1}{n+1} \binom{2n}{n}$ is the $n^{th}$ Catalan number, $n_j(\beta) \in \mathbb{Z}$ for each $j$ and $\beta$, and the $g_j$ are polynomials symmetric in the $X_i$.

The layout of this paper is as follows: In Section 2 we define the coinversion LLT polynomials. We briefly explain how the LLT polynomials can be seen as the partition function of a certain vertex model. In Section 3 we describe an algorithm which selectively swaps the color of certain paths in a configuration of the vertex model. We use this algorithm to give a bijection between the configurations of the vertex model with boundary condition $\beta/\gamma = (\beta^{(1)}/\gamma^{(1)}, \beta^{(2)}/\gamma^{(2)})$ and those with boundary condition $(\beta/\gamma)_{\text{swap}} = (\beta^{(2)}/\gamma^{(2)}, \beta^{(1)}/\gamma^{(1)})$. In Section 4 we prove a sufficient condition for when this bijection is weight preserving up to an explicit overall power of $t$, which implies that the corresponding LLT polynomials are equal up to the same overall factor. In Section 5 we show how we can use the tools developed in the previous sections to determine linear relations within families of LLT polynomials.

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2 LLT polynomials

In this section we give a brief description of the coinversion LLT polynomials. We then review their characterization as lattice paths introduced in [4, 7].

Let $\lambda = (\lambda_1 \geq \cdots \geq \lambda_m \geq 0)$ be a partition with $l(\lambda) = m$ parts. Note that we consider our partitions to have a fixed number of parts, but allow for the possibility of parts of zero. We associate to $\lambda$ its Young (or Ferrers) diagram $D(\lambda) \subseteq \mathbb{Z} \times \mathbb{Z}$, given as

$$D(\lambda) = \{(i,j) \mid 1 \leq i \leq \ell(\lambda), 1 \leq j \leq \lambda_i\}$$

We draw our diagrams in French notation, in the first quadrant, such as below

$$\lambda = (4, 2, 1), \quad D(\lambda) = \begin{array}{|c|c|c|}
\hline
& & \\
& \bullet & \\
\hline
\end{array}$$

We refer to the elements in $D(\lambda)$ as cells. The cell labelled above has coordinates (1,3).
The content of a cell $u = (i, j)$ in row $i$ and column $j$ of any Young diagram is $c(u) = j - i$. Given a tuple $\beta/\gamma = (\beta^{(1)}/\gamma^{(1)}, \ldots, \beta^{(k)}/\gamma^{(k)})$ of skew partitions, define a semistandard Young tableau $T$ of shape $\beta/\gamma$ to be a semistandard Young tableau on each $\beta^{(j)}/\gamma^{(j)}$, that is,

$$SSYT(\beta/\gamma) = SSYT(\beta^{(1)}/\gamma^{(1)}) \times \cdots \times SSYT(\beta^{(k)}/\gamma^{(k)})$$

We can picture this as placing the Young diagrams diagonally “on content lines” with the first shape in the South-West direction and the last shape in the North-East direction. See Example 2.1 below.

**Example 2.1.** Let $\beta/\gamma = ((3, 1), (2, 2)/(1, 1, 1), (1), (2, 1)/(2))$. The top row labels the contents of each line.

```
-3 -2 -1 0 1 2
3
7
6
4
2
5
9
8
```

Given a tuple $\beta/\gamma$ of skew partitions, we say that three cells $u, v, w \in \mathbb{Z} \times \mathbb{Z}$ form a triple of $\beta/\gamma$ if (i) $v \in \beta/\gamma$, (ii) they are situated as below

```
| u | w |
```

namely with $v$ and $w$ on the same content line and $w$ in a later shape, and $u$ on a content line one smaller, in the same row as $w$, and (iii) if $u, w$ are in row $r$ of $\beta^{(j)}/\gamma^{(j)}$, then $u$ and $w$ must be between the cells $(r, \gamma^{(j)}_r - 1), (r, \beta^{(j)}_r + 1)$, inclusive. It is important to note that while $v$ must be a cell in $\beta/\gamma$, we allow the cells $u$ and $w$ to not be in any of the skew shapes, in which case $u$ must be at the end of some row in $\gamma$ and $w$ must be the cell directly to the right of the end of some row in $\beta$.

**Definition 2.2.** Let $\beta/\gamma$ be a tuple of skew partitions and let $T \in SSYT(\beta/\gamma)$. Let $a, b, c$ be the entries in the cells of a triple $(u, v, w)$, where we set $a = 0$ and $c = \infty$ if the respective cell
is not in $\beta/\gamma$. Given the triple of entries

\[
\begin{bmatrix}
  a & c \\
  b
\end{bmatrix}
\]

we say this is a coinversion triple of $T$ if $a \leq b \leq c$.

**Definition 2.3.** Let $\beta/\gamma$ be a tuple of skew partitions. The coinversion LLT polynomial is the generating function

\[
\mathcal{L}_{\beta/\gamma}(X; t) = \sum_{T \in \text{SSYT}(\beta/\gamma)} t^{\text{coinv}(T)} x^T
\]

where $\text{coinv}(T)$ is the number of coinversion triples of $T$.

We will only consider the case when $X$ is a finite alphabet $X_n = \{x_1, \ldots, x_n\}$. The coinversion LLT polynomials are related to the inversion LLT polynomials in a simple way

\[
\mathcal{L}_{\beta/\gamma}(X; t) = t^m G_{\beta/\gamma}(X; t^{-1})
\]

where $G$ is the inversion LLT polynomial and $m$ is the total number of triples in $\beta/\gamma$.

In [1, 7] it was shown that there is a bijection between tuple of SSYT and a certain vertex model consisting of several colors of lattice paths. We review the construction from [7] here, and refer the readers to the original papers for details.

Consider a lattice model consisting of a up-right lattice paths of $k$ different colors where paths of the same color are not allowed to intersect. At each face of our lattice we assign a label in $\{0, 1\}^k$ to the sides of the face as follows:

\[
\begin{array}{c}
  K \\
  J \\
  I
\end{array}
\]

where the $x$ indicates a parameter that will be used in defining the weight of the face. One should interpret, for example, a 1 in the $i^{th}$ component of $I$ as indicating that a path of color $i$ crosses the bottom boundary of the face. The weights of the face are given by

\[
L_x(I, J, K, L) = \prod_{\text{colors } i \text{ exiting the vertex to the right}} \# \text{ colors larger than } i \text{ that appear in the vertex}
\]

whenever $I + J = K + L$ and there is no $i \in [k]$ such that $I_i = J_i = 1$, and $L_x(I, J; K, L) = 0$ otherwise. The condition $I + J = K + L$ ensures that any path that enters a face from the bottom or left must exit the face from the top or right, while the condition that there is no $i \in \{1, \ldots, k\}$ such that $I_i = J_i = 1$ ensures that paths of a given color are non-intersecting.

Let us define the boundary condition of our vertex model. Given a tuple of partitions $\mu = (\mu^{(1)}, \ldots, \mu^{(k)})$ and an integer $i$, let $\mu(i) \in \{0, 1\}^k$ be the vector whose $j$-th component is 1 if and only if

\[
i = \mu^{(j)}_m - m + 1
\]
for some $m \in \{1, \ldots, \ell(\mu^{(j)})\}$, for each index $j \in \{1, \ldots, k\}$. Let $\beta/\gamma = (\beta^{(1)}/\gamma^{(1)}, \ldots, \beta^{(k)}/\gamma^{(k)})$ be a tuple of skew partitions. Let

$$r = r(\beta/\gamma) = \min\{i \in \mathbb{Z} : \gamma(i) \neq 0\},$$

$$s = s(\beta/\gamma) = \max\{i \in \mathbb{Z} : \beta(i) \neq 0\}$$

where $s - r + 1$ gives the number of columns necessary in the vertex model for $\beta/\gamma$, $\beta(r + i)$ will give the top boundary condition for the $i^{th}$ column from the left, and $\gamma(r + i)$ will give the bottom boundary condition for the $i^{th}$ column from the left. With this notation, we introduce the lattice that will be of particular interest to us:

$$L_{\beta/\gamma} := \begin{array}{ccc}
\beta(r) & \cdots & \beta(s) \\
\vdots & \ddots & \vdots \\
0 & \cdots & 0 \\
\gamma(r) & \cdots & \gamma(s)
\end{array}$$

where the left is a possible semistandard filling of the tuple of partitions and the right is the corresponding lattice paths. Note that there is a simple bijection between fillings on the sequence of tableaux and lattice paths in which each row of tableaux $i$ is a path of color $i$ and the entries in the row correspond to the height of the horizontal steps of the path. The columns of the vertex model correspond to the content lines in the tableaux formulation (see the labelling in the above example). The green cells in the tableaux correspond to a coinvolution triple that corresponds to the face highlighted in green in the vertex model.

We let $Z_{\beta/\gamma}(X_n; t)$ denote the partition function of $L_{\beta/\gamma}$, that is

$$Z_{\beta/\gamma}(X_n; t) = \sum_{L \in LC_{\beta/\gamma}} \text{weight}(L).$$

where $LC_{\beta/\gamma}$ denotes the set of path configurations on $L_{\beta/\gamma}$ satisfying the boundary conditions and weight($L$) is the product of the weight of each face in $L$.

**Theorem 2.4.** Let $\beta/\gamma$ be a tuple of skew partitions. Then,

$$Z_{\beta/\gamma}(X_n; t) = \mathcal{L}_{\beta/\gamma}(X_n; t).$$

We refer the reader to [7] for a proof.

As an example of the above constructions, consider $\beta/\gamma = ((2,2)/(1,0),(1))$ with $n = 2$. We have

where the left is a possible semistandard filling of the tuple of partitions and the right is the corresponding lattice paths.
We now make some definitions and conventions that will be useful for us later on. We only consider tuples with two partitions, $\beta / \gamma = (\beta^{(1)} / \gamma^{(1)}, \beta^{(2)} / \gamma^{(2)})$. We always draw the path corresponding to the first partition in blue, and those corresponding to the second in red.

We say that path incident to a boundary at a face is a singleton if no path of other colors are also incident to the same boundary at the same face. In the above example, the rightmost blue path on the top boundary is a singleton, while all three paths on the bottom boundary are singletons.

We say that two LLT polynomials are equivalent if they are equal up to an overall power of $t$.

3 Partition Swapping Algorithm

Given a tuple of skew-partitions $\beta / \gamma = (\beta^{(1)} / \gamma^{(1)}, \beta^{(2)} / \gamma^{(2)})$, in this section we will construct an algorithm which defines a bijection between path configurations in the vertex model with boundary conditions $\beta / \gamma$ and path configurations in the vertex model with boundary conditions $(\beta / \gamma)_{\text{swap}} = (\beta^{(2)} / \gamma^{(2)}, \beta^{(1)} / \gamma^{(1)})$. We will do this by defining a procedure which starts with a configuration with boundary condition $\beta / \gamma$, selects certain segments of paths in this configuration, then swaps the colors of the selected path segments resulting in a configuration with boundary conditions $(\beta / \gamma)_{\text{swap}}$.

We say that we are traveling forward if while following a path we are travelling upward or to the right, otherwise we say that we are traveling backward. Our procedure is as follows: we choose a singleton red path (resp. blue path) at the top boundary (resp. bottom boundary). We follow the red (resp. blue) path traveling backward (resp. forward) until we hit a face containing a path of the other color. We then switch to the new segment of path according to the rules:

\[
\downarrow \rightarrow \quad \uparrow \quad \rightarrow \downarrow \quad \leftarrow \downarrow
\]

that is, for example, if we enter the face on a red path from the top we exit the face on a blue path traveling right. We then repeat this process on the new segment of path. We will show (Lemma 3.1) that this procedure ends at the boundary after a finite number of steps. For example, consider a configuration with boundary condition given by $\beta / \gamma = ((8, 7, 6), (4, 3, 2)/(2, 0, 0))$. Using this procedure we have

\[
\rightarrow
\]

where the path segments we follow after starting at the top-left red path are given in bold on the right. Note the rules (5) imply we alternate traveling backward and forward, and that we also alternate colors (which in turn implies that the rules presented above are sufficient to describe all situation that can arise using this procedure).
Running this algorithm from from a singleton path on the boundary, we call the sequence of path segments that are traversed the walk starting from that boundary path. These walks satisfy some straightforward properties.

**Lemma 3.1.** The walk cannot enter a loop. In particular, the walk must terminate at the top or bottom boundaries.

*Proof.* Note that from the rules in (5), if we are at a face where our walk changes color and we know how we exit the face, then we know along what path segment we must have entered. So if we are on a path segment in a loop, then the path segment previous in the walk must also be in the loop as it is the only way to get to the current path segment. Continuing this, we see that every previous path segment must be in the loop. But this contradicts that we started at the boundary.

**Lemma 3.2.** If we begin the walk at a red path on the top boundary, it will terminate at either a red path on the bottom boundary or a blue path on the top boundary. Similarly, if we begin at a blue path on the bottom boundary, the walk will terminate at either a red path on the bottom boundary or a blue path at the top boundary.

*Proof.* Suppose we begin at a red path at the top boundary traveling backward. At each step of this procedure we alternate direction of travel and color, so we will always be traveling forward on blue paths and backwards on red paths. By the previous Lemma, the walk must end on the boundary. If it ends at the top boundary, we must be traveling forward and thus be on a blue path. If it ends on the bottom boundary, we must be traveling backward along a red path. A similar argument works if we began on a blue path at the bottom boundary.

Now we repeat this procedure for every singleton red path at the top boundary path and every singleton blue path at the bottom boundary. In our example, we have

![Diagram](image)

where we highlight all the traversed path segments. From the rules given in (5), it is easy to see that

**Lemma 3.3.** If we start this procedure at two different points on the boundary, then the walks for each cannot cross (although they may touch at a corner).

Finally, let \( \Phi \) by the operator that take a configuration with boundary condition given by \( \beta/\gamma \), enacts the above procedure, then swaps the color of all the traversed path segments. In
our example, this would give

\[
\Phi^{-1}
\]

where we swap the color of all highlighted paths in the right configuration to get the left configuration.

**Proposition 3.4.** \(\Phi\) is a bijection between configurations with boundary condition given by \(\beta/\gamma\) and configurations with boundary condition given by \((\beta/\gamma)_{swap}\).

**Proof.** First, it is easy to see that after swapping the color of all the highlighted edges we still have a valid configuration of our vertex model, by simply checking what changes can possibly happen at a single face.

Next, we show that the boundary conditions change appropriately, that is, we need all singleton blue boundary paths to become red and vice-versa (we are not concerned with points on the boundary with both colors, as they appear for both the \(\beta/\gamma\) and \((\beta/\gamma)_{swap}\) boundary conditions). It suffices to show that all singleton boundary paths are traversed in our procedure. Suppose that this was not the case and there was, say, a single boundary path on the bottom boundary was not included in a walk. Then starting at that path and running the procedure in reverse (that is, reversing all the arrows in the set of rules (5)) we must end at either an singleton red boundary path at the top or a singleton blue boundary path at the bottom. But this path will have already been included in a walk as we use it as a starting point. So we must have traversed all singleton boundary paths.

Finally, it is easy to see that running this procedure again returns us to our original configuration, so \(\Phi\) is invertible.

This algorithm can be seen as a generalization of the procedure used in proving the fact that if each of the partitions in the indexing tuple of \(L_{\lambda^{(1)},\ldots,\lambda^{(k)}}(X_n; t)\) are single rows, then any rearrangement of the order of the partitions will result in an equivalent LLT polynomial given in [7].

### 4 Calculating the weight

While \(\Phi\) is always a bijection, we are interested in the case when the LLT polynomials \(L_{\beta/\gamma}(X_n, t)\) and \(L_{(\beta/\gamma)_{swap}}(X_n, t)\) are equivalent. That is, we wish to know for what \(\beta/\gamma \Phi\) is weight-preserving up to an overall power of \(t\).

Towards this end, we associate a sequence of colored beads to the boundary conditions of the path given by \(\beta/\gamma\) as follows: We consider two rows of beads. Scanning the columns of our lattice model from left-to-right, for every singleton path along the top boundary we add a bead of the same color on the top row, and for every singleton path along the bottom boundary we add a bead of the same color to the bottom row, ensuring that the beads keep the same ordering (from left-to-right) as the paths. We label the beads by number of paths to the right of their corresponding path.
We define a matching of this sequence of beads to be a set of arcs such that either the arc connects two beads of different colors in the same row or the arc connects two beads of the same color in different rows. A matching is non-crossing if the arcs (drawn so they always remain between the two rows) do not cross one another.

In our example, the sequence of beads associated to the boundary conditions is

\[
\begin{array}{ccccccccc}
1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\
\end{array}
\]

\[
\begin{array}{ccccccccc}
\bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet & \bullet \\
\end{array}
\]

It is clear that the procedure in Section 3 associates a non-crossing matching to every configuration of the vertex model. Each walk in the vertex model corresponds to an arc in the matching. The non-crossing matching associated to our example configuration is

This mapping from vertex model configurations to non-crossing matchings is useful since, as we will now see, the change in the power of \( t \) under \( \Phi \) depends only on the matching. First, we prove several lemmas.

**Lemma 4.1.** For a walk starting and ending at the same boundary we have

\[
\# \begin{array}{c}
\uparrow \\
\downarrow
\end{array} + \# \begin{array}{c}
\leftarrow \\
\rightarrow
\end{array} - \# \begin{array}{c}
\rightarrow \\
\downarrow
\end{array} - \# \begin{array}{c}
\leftarrow \\
\uparrow
\end{array} = 1,
\]

while for a walk starting at one boundary and ending at the other we have

\[
\# \begin{array}{c}
\uparrow \\
\downarrow
\end{array} + \# \begin{array}{c}
\leftarrow \\
\rightarrow
\end{array} - \# \begin{array}{c}
\rightarrow \\
\downarrow
\end{array} - \# \begin{array}{c}
\leftarrow \\
\uparrow
\end{array} = 0.
\]

**Proof.** The first two terms give the number of times we switch from traveling backwards to forwards, the second two terms give the number of times we switch from traveling forwards to backwards. For a walk that starts and ends on the same boundary, it must end traveling the opposite direction of how it started. So the net difference must be 1.

For a walk that crosses between the boundaries, it must end traveling the same direction that it started. So the net difference must be 0. \( \square \)
Lemma 4.2. For a walk starting at a path with $j$ paths to its right and ending at a path with $i$ paths to its right, we have
\[ \#\left(\begin{array}{c} \downarrow \rightarrow \\ \end{array}\right) - \#\left(\begin{array}{c} \uparrow \leftarrow \\ \end{array}\right) + \#\left(\begin{array}{c} \rightarrow \downarrow \\ \end{array}\right) - \#\left(\begin{array}{c} \leftarrow \uparrow \\ \end{array}\right) = j - i. \]

Proof. Steps of the form
\[ \begin{array}{c} \uparrow \rightarrow \\ \end{array} \quad \text{and} \quad \begin{array}{c} \rightarrow \downarrow \\ \end{array} \]
always put us on a new section of path that has one fewer paths to its right. Steps of the form
\[ \begin{array}{c} \uparrow \leftarrow \\ \end{array} \quad \text{and} \quad \begin{array}{c} \leftarrow \uparrow \\ \end{array} \]
always put us on a new section of path with one more path to its right. As the walk starts $j$ paths to its right and ends with $i$ paths to its right, we have the identity. \[ \square \]

Lemma 4.3. A $\begin{array}{c} \leftarrow \uparrow \\ \end{array}$ cannot be immediately followed and preceded by a $\begin{array}{c} \downarrow \rightarrow \\ \end{array}$ and vice versa. Similarly, a $\begin{array}{c} \rightarrow \downarrow \\ \end{array}$ cannot be immediately followed and preceded by a $\begin{array}{c} \leftarrow \uparrow \\ \end{array}$ and vice versa.

Proof. Consider a face in which the walk takes a step of the form $\begin{array}{c} \leftarrow \uparrow \\ \end{array}$. Suppose it was immediately followed and preceded by $\begin{array}{c} \downarrow \rightarrow \\ \end{array}$. Then we must have a configuration of the form
\[ \begin{array}{c} \leftarrow \uparrow \\ \end{array} \quad \begin{array}{c} \downarrow \rightarrow \\ \end{array} \quad \begin{array}{c} \rightarrow \downarrow \\ \end{array} \]
No matter how the two lower faces are arranged, either the dashed blue path or the dashed red path must intersect one of the solid paths. This would mean that there is a step of our walk in between those drawn, contradicting that these steps immediately follow and precede the $\begin{array}{c} \leftarrow \uparrow \\ \end{array}$. The other cases can be done similarly. \[ \square \]
Lemma 4.4. For a walk starting and ending at the top we have

\[
\# \begin{pmatrix} \downarrow \rightarrow \\ \end{pmatrix} - \# \begin{pmatrix} \uparrow \leftarrow \\ \end{pmatrix} - \# \begin{pmatrix} \rightarrow \downarrow \\ \end{pmatrix} + \# \begin{pmatrix} \leftarrow \uparrow \\ \end{pmatrix} = \pm 1
\]

where the RHS is +1 when the walk ends to the right of where it started and -1 if it ends to the left. For a walk starting and ending at the bottom we have

\[
\# \begin{pmatrix} \downarrow \rightarrow \\ \end{pmatrix} - \# \begin{pmatrix} \uparrow \leftarrow \\ \end{pmatrix} - \# \begin{pmatrix} \rightarrow \downarrow \\ \end{pmatrix} + \# \begin{pmatrix} \leftarrow \uparrow \\ \end{pmatrix} = \pm 1
\]

where the RHS is +1 when the walk ends to the left of where it started and -1 if it ends to the right. For a walk starting at one boundary and ending at the other we have

\[
\# \begin{pmatrix} \downarrow \rightarrow \\ \end{pmatrix} - \# \begin{pmatrix} \uparrow \leftarrow \\ \end{pmatrix} - \# \begin{pmatrix} \rightarrow \downarrow \\ \end{pmatrix} + \# \begin{pmatrix} \leftarrow \uparrow \\ \end{pmatrix} = 0
\]

Proof. Let us consider the case where the walk starts and ends on the top. Note that the walk is made up of straight sections of path and right angled corners. These corners can either occur on a segment of path, or at a face where the walk switches from one segment to the other.

Suppose that the walk ends to right of where it started. Then we must have that \(\#(\text{left turns}) = \#(\text{right turns}) + 2\), where left and right are defined relative to the direction of travel of the walk. We can restate the lemma as saying

\(\#(\text{left turns where the walk switches color}) - \#(\text{right turns where the walk switches colors}) = 1\).

For turns that occur on a path segment the difference between the number of left turns and the number of right turns can only be \(\pm 1\) or 0. In fact, we can say precisely how this difference depends on the corners the path segment starts and ends at:

\[
\begin{array}{c}
R \ R \\
R \ 0 \\
L \ 0 \\
L \ L \\
\end{array}
\]

where the box \(L/R\) indicates a left/right turn at a corner where the walk switches color, and the line segment \(L/R/0\) indicates the the segment of path connecting the corners has a net left/right/equal number of turns. We also need to consider what can happen for first and last path segments, there are two possibilities for each

| First:  | Last: |
|--------|-------|
| \(R \ R\) | \(R \ 0\) |
| \(0 \ L\) | \(L \ L\) |

Here we redraw the walk in example (6) and give the sequence of turns:
Lemma 4.3 can be interpreted as saying that looking only at turns occurring where the walk switches colors, we cannot have three rights or three lefts in a row. That is, we cannot have \[ R R R \] or \[ L L L \].

We see that left turns where the walk switches colors occur in pairs \[ L L \] or alone \[ L \], similarly for right turns where the walk changes color. Let \( L_1 \) (\( R_1 \)) be the number of left (right) turns where the walk changes color that appear alone and \( L_2 \) (\( R_2 \)) be those that occur in pairs. Each pair of left turns where the walk switches color contributes three lefts in total, as the path segment connecting the m also contribute a left turn, while the lefts that occur alone only contribute a single left turn as the path connecting them to the adjacent rights contribute a net zero. Taking into the account the first and last path segment, from the preceding discussion we have

\[
\begin{align*}
\text{#(left turns)} - \text{#(right turns)} &= \begin{cases} \\
L_1 + 3L_2 - R_1 - 3R_2 - 1, & \text{if (first, last)} = R R \ R L \ R 0 \\
L_1 + 3L_2 - R_1 - 3R_2, & \text{if (first, last)} = R R \ R L \ L 0 \\
L_1 + 3L_2 - R_1 - 3R_2, & \text{if (first, last)} = R L \ R L \ L 0 \\
L_1 + 3L_2 - R_1 - 3R_2 + 1, & \text{if (first, last)} = R L \ L L \ L 0 \\
\end{cases}
\end{align*}
\]
Counting the number of the lefts and rights, we must also have

\[ L_1 + L_2 - R_1 - R_2 = \begin{cases} 
-1, & \text{if (first, last)} = \begin{pmatrix} R & R & 0 \\ R & R & L \\ R & L & 0 \\ 0 & L & 0 \\ 0 & L & L \\ 0 & L & L \end{pmatrix} \\
0, & \text{if (first, last)} = \begin{pmatrix} R & R & 0 \\ R & R & L \\ R & L & 0 \\ 0 & L & 0 \\ 0 & L & L \\ 0 & L & L \end{pmatrix} \\
0, & \text{if (first, last)} = \begin{pmatrix} R & R & 0 \\ R & R & L \\ R & L & 0 \\ 0 & L & 0 \\ 0 & L & L \\ 0 & L & L \end{pmatrix} \\
1, & \text{if (first, last)} = \begin{pmatrix} R & R & 0 \\ R & R & L \\ R & L & 0 \\ 0 & L & 0 \\ 0 & L & L \\ 0 & L & L \end{pmatrix} 
\end{cases} \]

Using the above equations, a little algebra shows

\[
#(\text{left turns where walk switches color}) - #(\text{right turns where walk switches color}) = L_1 + 2L_2 - R_1 - 2R_2 = 1
\]

as desired. If the walk instead ended to the left of where it started we would have

\[
#(\text{left turns}) - #(\text{right turns}) = -2
\]

but otherwise the same equations. Computing gives

\[
#(\text{left turns where walk switches color}) - #(\text{right turns where walk switches color}) = -1.
\]

Here we worked out the case where the walk starts and ends on the top row. A similar analysis works for the other walks.

For each type of arc between the labelled beads define the weight of the arc by

\[
\begin{align*}
\alpha: & \begin{cases} 
i & j \\ j & i \\ i & j \\ j & i \end{cases} \\
w(\alpha): & \begin{cases} t^{(j-i)/2} & t^{(j-i)/2} \\ t^{(j-i)/2} & t^{(j-i)/2} \end{cases}
\end{align*}
\]

Proposition 4.5. For a single walk, the change in the power of \( t \) after swapping the color of all the segments of paths traversed in the walk depends only on its corresponding arc in the matching. In particular, the change in the power of \( t \) is equal to the weight of the arc.

Proof. The change in the power of \( t \) after swapping the colors of all the path segments of a walk is given by

\[
#\begin{pmatrix} \downarrow \uparrow \\ \uparrow \downarrow \end{pmatrix} - #\begin{pmatrix} \uparrow \downarrow \\ \downarrow \uparrow \end{pmatrix}.
\]
This can easily be computed from the previous lemmas. For example, suppose we are the case where our walk starts at the top boundary and end at the top boundary to the right of where it started. From Lemmas 4.2 and 4.3 we have

\[
\begin{align*}
\# &\left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) + \# \left(\begin{array}{c}
\rightarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\leftarrow \\
\end{array}\right) = j - i \\
\# &\left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\rightarrow \\
\end{array}\right) + \# \left(\begin{array}{c}
\leftarrow \\
\end{array}\right) = 1.
\end{align*}
\]

Adding them together gives

\[
\# \left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) = (j - i + 1)/2
\]

which agrees with

\[
w\left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) \approx t^{(j-i+1)/2}.
\]

Note that this implies that \(j - i + 1\) is even. Alternatively, we can see this since on the boundary between the starting and ending path there must be an even number of paths, either pairs of singleton boundary paths that are connected by a walk or non-singleton boundary paths.

For a walk starting at a red path at the top boundary and ending at a red path on the bottom boundary, Lemmas 4.2 and 4.3 give

\[
\begin{align*}
\# &\left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) + \# \left(\begin{array}{c}
\rightarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\leftarrow \\
\end{array}\right) = j - i \\
\# &\left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\rightarrow \\
\end{array}\right) + \# \left(\begin{array}{c}
\leftarrow \\
\end{array}\right) = 0
\end{align*}
\]

from which we have

\[
\# \left(\begin{array}{c}
\downarrow \\
\end{array}\right) - \# \left(\begin{array}{c}
\uparrow \\
\end{array}\right) = (j - i)/2
\]

which agrees with

\[
w\left(\begin{array}{c}
\downarrow \\
\end{array}\right) = t^{(j-i)/2}.
\]

Note that in this case we see that \(j - i\) is even.

The same analysis can be done for walks starting on blue paths.

Proposition 4.5 give a sufficient condition for when the the LLT polynomials \(L_{\beta/\gamma}(X_n; t)\) and \(L_{(\beta/\gamma), wup}(X_n; t)\) are equivalent.
Theorem 4.6. If there is a unique non-crossing matching $M$ of the sequence of beads associated to $\beta / \gamma$, then $L_{\beta / \gamma}(X_n; t)$ and $L_{(\beta / \gamma)_{\text{swap}}}(X_n; t)$ are equivalent. In particular,

$$L_{\beta / \gamma}(X_n; t) = \left( \prod_{a \in M} w(a) \right) L_{(\beta / \gamma)_{\text{swap}}}(X_n; t)$$

where the product is over all arcs $a$ in the matching and the weight of an arc is given by [7].

Proof. We know that the algorithm from Section 3 associates to every configuration of the vertex model a non-crossing matching which determines the change in weight of the configuration under the bijection $\Phi$. If there is a unique non-crossing matching then each configuration is associated to the same matching and the change in weight is the same for all configurations. Thus the bijection is weight preserving up to an overall power of $t$. □

In our running example, we have the tuple of partitions $\beta / \gamma = ((8, 7, 6), (4, 3, 2)/(2, 0, 0))$ for which there is a unique non-crossing matching for the sequence of beads associated to $\beta / \gamma$. It is given by

![Diagram](image)

Using Theorem 4.6 we have

$$L_{\beta / \gamma}(X_n; t) = t^5 L_{(\beta / \gamma)_{\text{swap}}}(X_n; t)$$

where $(\beta / \gamma)_{\text{swap}} = ((4, 3, 2)/(2, 0, 0), (8, 7, 6))$.

Remark 4.7. Suppose $\beta / \gamma = (\ldots, \beta^{(i)} / \gamma^{(i)}, \beta^{(i+1)} / \gamma^{(i+1)}, \ldots)$ and $(\beta / \gamma)_{\text{swap}} = (\ldots, \beta^{(i+1)} / \gamma^{(i+1)}, \beta^{(i)} / \gamma^{(i)}, \ldots)$ are two tuples of partitions which are the same except for having their $i^{th}$ and $(i + 1)^{st}$ partitions swapped. Then if there is a unique non-crossing matching $M$ of the sequence of beads associated to the tuple of partitions $(\beta / \gamma),_{i+1} = (\beta^{(i)} / \gamma^{(i)}, \beta^{(i+1)} / \gamma^{(i+1)})$, it still holds that

$$L_{\beta / \gamma}(X_n; t) = \left( \prod_{a \in M} w(a) \right) L_{(\beta / \gamma)_{\text{swap}}}(X_n; t)$$

where the product is over all arcs $a$ in the matching and the weight of an arc is given by [7].

In Appendix A, we classify which sequences of beads have unique non-crossing matchings. We find that

Proposition 4.8. For a single row, sequences of beads that have a unique non-crossing matching are of the form

![Diagram](image)

with $p + r = q$, or the same configurations as above with red and blue swapped.
Proposition 4.9. With two rows, sequence of beads that have a unique non-crossing matching are given by

or the same sequences as above with the rows or colors swapped, where the difference between the number of red and blue beads in the top row is equal to the difference in the number of red and blue beads in the bottom row.

We then pull back these conditions to conditions on the tuple $\beta/\gamma$ for which Theorem 4.6 holds.

While the procedure in Section 3 associates to every vertex model configuration a non-crossing matching of a sequence of labelled beads, it is not true in general that there is vertex model configuration associated to every non-crossing matching. For example, let $\beta/\gamma = ((5,4,4)/(2,2,0),(3,1,1))$. There are two non-crossing matchings for the sequence of beads associated to the tuple

![Diagram](image)

An example configuration which realizes the second matching is given by

![Diagram](image)

but it is impossible to realize the first matching with paths. That is, the map from configuration of the vertex model to non-crossing matchings is not generally onto.

However, if $\beta/\gamma = \beta$ is a tuple of straight shapes, each with the same number of parts, then we can indeed construct a path configuration for the vertex model corresponding to every non-crossing matching (provided that the vertex model has sufficiently many rows). We leave the construction of such a configurations to the interested reader. In what follows, we will consider either this case or specific examples in which it is clear that all matchings can be obtained.

5 Linear Relations between LLT polynomials

In this section we show how we can use the techniques we have developed to exhibit linear relations between different LLT polynomials.
Let's start with a small example. Consider the tuples of partitions \( \lambda_1 = ((3,3),(1)) \), \( \lambda_2 = ((2,2),(3)) \), and \( \lambda_3 = ((3,2),(2)) \). The vertex model boundary conditions and the non-crossing matchings associated to the tuples are given by

\[
\begin{align*}
\lambda_1 &= ((3,3),(1)) : \\
\lambda_2 &= ((2,2),(3)) : \\
\lambda_3 &= ((3,2),(2)) :
\end{align*}
\]

\[
\begin{align*}
M_1 : \\
M_2 :
\end{align*}
\]

As there is a unique non-crossing matching associated to \( \lambda_1 \), the previous section allow us to conclude that \( \mathcal{L}_{\lambda_1}(X_n;t) = t^2 \mathcal{L}_{((1),(3,3))}(X_n;t) \). However, we can do more. Given a path configuration with boundary condition given by \( \lambda_1 \), we can switch the color of all path segments in the walk starting from red path on the top boundary. For example,

\[
\begin{align*}
\xrightarrow{\text{}}
\end{align*}
\]

This results in a configuration with boundary condition given by \( \lambda_3 \), in particular, a configuration which corresponds to matching \( M_1 \). Doing this for every configuration gives a bijection \( \{\text{config. with boundary condition } \lambda_1\} \rightarrow \{\text{config. with boundary condition } \lambda_3\ \text{corresponding to } M_1\} \)

where under this mapping the change in the power of \( t \) is given by

\[
\omega \begin{pmatrix} 2 \\ 1 \end{pmatrix} = t.
\]

Similarly, for any configuration in the vertex model for \( \lambda_2 \), we can swap the color of all path segments along the walk starting at the red path on the top row. This gives a bijection \( \{\text{config. with boundary condition } \lambda_2\} \rightarrow \{\text{config. with boundary condition } \lambda_3\ \text{corresponding to } M_2\} \)
where under this mapping the change in the power of $t$ is given by

$$w\left(\begin{array}{cc}
1 & 0 \\
0 & 0
\end{array}\right) = t^{-1}.$$ 

All together this shows that

$$\mathcal{L}_{\lambda_3}(X; t) = t^{-1} \mathcal{L}_{\lambda_1}(X; t) + t \mathcal{L}_{\lambda_2}(X; t).$$

As one possible application of this type of calculation, we can reprove a relation between LLT polynomials indexed by single rows given in [15]. Note that the precise powers of $t$ differ than that of [15] as we are working with coinversion LLT polynomials.

**Lemma** (Lemma 3.17, [15]). Let $\beta_1, \gamma_1, \beta_2, \gamma_2$ be positive integers such that $\gamma_1 < \gamma_2 \leq \beta_1 < \beta_2$. Then

$$\mathcal{L}_{(\beta_1/\gamma_1, \beta_2/\gamma_2)}(X_n; t) = \mathcal{L}_{(\beta_2/\gamma_2, \beta_1/\gamma_1)}(X_n; t) + (t^{-1} - 1)\mathcal{L}_{(\beta_2/\gamma_1, \beta_1/\gamma_2)}(X_n; t).$$

**Proof.** We began by drawing the boundary conditions and matchings corresponding to the tuple of partitions of each of these LLT polynomials:

- $$(\beta/\gamma)_1 = (\beta_1/\gamma_1, \beta_2/\gamma_2)$$
- $$(\beta/\gamma)_2 = (\beta_2/\gamma_2, \beta_1/\gamma_1)$$
- $$(\beta/\gamma)_3 = (\beta_2/\gamma_1, \beta_1/\gamma_2)$$

Note that the terms corresponding to $M_2$ in $(\beta/\gamma)_1$ and $(\beta/\gamma)_2$ are equal as we can swap the colors along the arcs connecting the two rows with no cost. Similarly, terms corresponding to $M_1$ in $(\beta/\gamma)_1$ and $(\beta/\gamma)_2$ are equal to those in $(\beta/\gamma)_3$ up a power of $t$ coming from swapping the color along one of the arcs. Putting this together, we have

$$\mathcal{L}_{(\beta_1/\gamma_1, \beta_2/\gamma_2)}(X_n; t) - t^{-1} \mathcal{L}_{(\beta_2/\gamma_2, \beta_1/\gamma_1)}(X_n; t) = \mathcal{L}_{(\beta_2/\gamma_1, \beta_1/\gamma_2)}(X_n; t) - \mathcal{L}_{(\beta_2/\gamma_1, \beta_1/\gamma_2)}(X_n; t)$$

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from which the result follows.

For a more complicated example, consider the family of tuples of partitions

\[ \lambda_1 = ((4, 4, 4), (1, 1, 1)), \quad \lambda_2 = ((4, 4, 3), (2, 1, 1)) \]
\[ \lambda_3 = ((4, 4, 2), (2, 2, 1)), \quad \lambda_4 = ((4, 3, 3), (3, 1, 1)) \]
\[ \lambda_5 = ((4, 3, 2), (3, 2, 1)), \quad \lambda_6 = ((4, 4, 1), (2, 2, 2)) \]
\[ \lambda_7 = ((4, 1, 1), (3, 3, 3)), \quad \lambda_8 = ((4, 2, 2), (3, 3, 1)) \]
\[ \lambda_9 = ((4, 2, 1), (3, 3, 2)), \quad \lambda_{10} = ((4, 3, 1), (3, 2, 2)) \]

and the tuples \( \lambda_{10+i} \) which we obtain by swapping the order of the partitions in \( \lambda_i \). These are all the tuples that can be obtained by starting with the boundary conditions for \( \lambda_1 \)

and rearranging the order of the paths on the top boundary (without changing which columns have paths). These tuples of partitions are all associated to matchings on six beads. If we ignore the color of the beads, there are five possible non-crossing matchings:

We choose a specific way to color the beads of each of these matchings, so that an arc in the matching connects a red bead to a blue bead:

Each of these sequences of colored beads corresponds to an LLT polynomial in our family. In particular, our coloring of \( M_1 \) corresponds to \( L_{\lambda_1} \), \( M_2 \) corresponds to \( L_{\lambda_2} \), \( M_3 \) corresponds to \( L_{\lambda_3} \), \( M_4 \) corresponds to \( L_{\lambda_4} \), and \( M_5 \) corresponds to \( L_{\lambda_5} \). Note that, by construction, \( M_i \) is a non-crossing matching associated to \( L_{\lambda_i} \), but \( L_{\lambda_i} \) might have other non-crossing matchings associated to it as well. Define the polynomial \( g_i(X; t) \) as the terms in \( L_{\lambda_i}(X; t) \) which correspond to the matching \( M_i \) under the algorithm in Section 3.

If \( M_j \) is a non-crossing matching associated to \( L_{\lambda_j} \), let \( w_{ij}(t) \) be the difference in the power of \( t \) between the weight of our choice of coloring of \( M_j \) and the coloring of \( M_j \) given by \( L_{\lambda_j} \). For example, there are are two non-crossing matchings associated to \( L_{\lambda_2} \):

```plaintext
and
```

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The first is precisely our choice of coloring of $M_2$, so $w_{22}(t) = 1$. However, to make our choice coloring of $M_1$ match with the coloring given here we would need to swap the 3-2 arc which gives a weight of $t$. So $w_{32}(t) = t$. This allows us to write

$$\mathcal{L}_{\lambda_i}(X_n; t) = \sum_j w_{ij}(t) g_j(X_n; t)$$

for each of $\lambda_i$.

Consider the matrix $M(t)$ that in the $i^{th}$ row and $j^{th}$ column has $w_{ij}(t)$ if $M_j$ is a matching associated to $\mathcal{L}_{\lambda_i}$, and zero otherwise. Then $M(t)$ is the matrix that transforms $(g_1, \ldots, g_5)^T$ into $(\mathcal{L}_{\lambda_1}, \ldots, \mathcal{L}_{\lambda_5})^T$. For the above, this matrix is

$$M(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 \\ 0 & t & 1 & 0 & 0 \\ 0 & t^{-1} & 0 & 1 & 1 \\ t^2 & t^{-1} & t & t^{-1} & 1 \end{pmatrix}, \quad M^{-1}(t) = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -t & 1 & 0 & 0 & 0 \\ t^2 & -t & 1 & 0 & 0 \\ 1 & -t^{-1} & 0 & 1 & 0 \\ 1 - t^{-1} - t^2 - t^3 & t^{-2} - t^{-1} + t^2 & -t & -t^{-1} & 1 \end{pmatrix}$$

where we also write its inverse. The inverse tells us how to write the $g_i$ as a linear combination of the LLT polynomials. For example,

$$g_3(X_n; t) = \mathcal{L}_{\lambda_3}(X_n; t) - t\mathcal{L}_{\lambda_1}(X_n; t) + t^2\mathcal{L}_{\lambda_1}(X_n; t).$$

Though we chose a particular coloring of the beads in each of the $M_i$, any other choice of color would have resulted in the same polynomial $g_i(X_n; t)$ up to an overall factor of $t$, as we can always switch the color along an arc. Furthermore, we see that all $g_i(X_n; t)$ are all symmetric in the $X_n$ as they are linear combinations of LLT polynomials. Since every tuple of partitions in this family must be made up of configurations corresponding to these five matchings, we have

$$\mathcal{L}_{\lambda_i}(X_n; t) = \sum_{j=1}^5 n_{ij} g_j(X_n; t)$$

for some integers $n_{ij}$.

For instance, there are two non-crossing matchings associated to $\lambda_8$:

For the first matching, we can get from our choice of coloring for $M_1$ to the coloring given here by changing the color along the the arc connecting the beads labelled 4 and 1 at the cost of a $t^{-2}$, and along the arc connecting the beads labelled 3 and 2 at the cost of a $t^{-3}$. For the second matching we can get from our coloring of $M_3$ to this coloring by swapping the color along the arc connecting beads labelled 3 and 2 at the cost of a $t^{-1}$. All together we have

$$\mathcal{L}_{\lambda_8}(X_n; t) = t^{-3} g_1(X_n; t) + t^{-1} g_5(X_n; t).$$

While we chose a specific family of LLT polynomials in this example, the same result holds for any family of tuples of partitions which are associated to the same sequence of beads (allowing for changes in the labelling of the beads). The only difference will be the specific powers of $t$ that will appear. But these can easily be computed as they only depend on the matchings.
This computation can be generalized. Consider $2n$ real numbers $\beta_1 > \beta_2 > \ldots > \beta_{2n} \geq 0$. Consider the family of tuples of partitions of the form

$$\beta = ((\beta_1 - n + 1, \ldots, \beta_m - n + n), (\beta_{m+1} - n + 1, \ldots, \beta_{2n} - n + n))$$

where $\beta_1 > \ldots > \beta_{2n}$, $\beta_1 > \ldots > \beta_{m}$, and the $\beta_i$'s and $\beta_j$'s make up all of the $\beta_1, \ldots, \beta_{2n}$. There are $\binom{2n}{n}$ tuples of partitions in this family. As in the previous example, we will show that these LLT polynomials can all be written as sums of a smaller collection of symmetric polynomials associated to the possible non-crossing matchings.

**Theorem 5.1.** Consider the family of partitions given in (8). Then for every $\beta$ in this family, the LLT polynomial $\mathcal{L}_\beta(X_n; t)$ can be written

$$\mathcal{L}_\beta(X_n; t) = \sum_{j=1}^{C_n} i_{nj}(\beta) g_i(X_n; t)$$

where $C_n = \frac{1}{n+1}\binom{2n}{n}$ is the $n$th Catalan number, $i_{nj}(\beta) \in \mathbb{Z}$ for each $j$ and $\beta$, and the $g_i$ are polynomials symmetric in the $X_n$.

**Proof.** The given family of partitions corresponds to sequence of beads on only one row with $n$ red and $n$ blue beads. For example, the tuple of partitions $\beta = ((\beta_1 - n + 1, \ldots, \beta_m - n + n), (\beta_{m+1} - n + 1, \ldots, \beta_{2n} - n + n))$ corresponds to the LLT polynomial with the sequence of beads

```
2n-1
n
n
n
n
0
```

Ignoring the color of beads, there are $C_n = \frac{1}{n+1}\binom{2n}{n}$ non-crossing matchings of $2n$ beads. The proof of this theorem will follow the same lines as in the example. We will choose an ordering of the non-crossing matchings. Then choosing a coloring of the beads in the matching picks out a partition in our family, we call $\beta_i$, where $\mathcal{L}_{\beta_i}(X_n; t)$ has $M_i$ as one of its associated matchings. We define $g_i(X_n; t)$ as the terms in $\mathcal{L}_{\beta_i}(X_n; t)$ which give the matching $M_i$ under the algorithm in Section 3. We have

$$\mathcal{L}_{\beta_i}(X_n; t) = \sum_{j=1}^{C_n} w_{ij}(t) g_j(X_n; t)$$

where $w_{ij}(t)$ is defined as in the previous example. While it is clear we can write the LLT polynomials as the sum of the $g_i$, it is not obvious that the $g_i$ are symmetric.

To show this, let $M(t)$ be the matrix which takes $(g_1, \ldots, g_{C_n})^T$ to $(\mathcal{L}_{\lambda_1}, \ldots, \mathcal{L}_{\lambda_{C_n}})^T$ as before. We’ll show that by appropriately choosing the ordering of the matchings and their coloring, the matrix $M(t)$ will be lower triangular with ones on the diagonal. In particular, $M(t)$ is invertible with $M^{-1}(t)$ describing how to write $g_i(X_n; t)$ as a linear combination of the $\mathcal{L}_{\beta_i}(X_n; t)$. Then it follows that the $g_i$ are symmetric in the $X_n$. Since every partition in our family has to be made up of terms corresponding to one of $C_n$ matchings, the result follows.

We order these matchings inductively. When there are zero beads there is a unique non-crossing matching (the empty matching). Suppose know how to correctly order the matchings of $2k, k < n$, beads. When there are $2n$ beads, we can uniquely describe any matching by the triple $(a, b, c), 0 \leq a \leq n, 0 < b \leq C_a, 0 < c \leq C_{n-a-1}$, by decomposing it into the $b^{th}$ matching
of 2a beads \(M_b^{(a)}\) and the \(c^{th}\) matching of \(2(n - a - 1)\) beads \(M_{c}^{(n-a-1)}\) through

\[
\begin{array}{c}
\hline
2(n-a) - 1 \\
M_b^{(a)} \\
\hline
M_{c}^{(n-a+1)} \\
0
\end{array}
\]

We then order the matchings on \(2n\) beads by the rule \((a_1, b_1, c_1) < (a_2, b_2, c_2)\) iff \(a_1 < a_2\), or \(a_1 = a_2\) and \(b_1 < b_2\), or \(a_1 = a_2\) and \(b_1 = b_2\) and \(c_1 < c_2\).

Similarly, we choose to color each of the matchings inductively. When there are no beads, there is a only the empty matching. Now suppose we know the coloring how to correctly color a subsequence of bead between the rightmost blue bead and the \(2^k\) beads.

Let \(h\) be the number of blue beads minus the number of red beads up to the rightmost blue bead. By construction the rightmost bead in this subsequence is also blue. A simple induction argument shows that \(h_{n-a}(k) \geq 0\) for all \(k\). Noting that \(h_{n-a}(k) + 1 = h_n(k + 1)\) for all \(k \in \{0, \ldots, 2(n-a)\}\) finishes the proof.

\[\Box\]

We emphasize again that this holds for any family of partitions corresponding to the same sequence of beads (allowing for differences in labelling which only affect the specific powers of \(t\)).
6 Conclusion

In this paper we prove a sufficient condition for when $L(\beta_1/\gamma_1, \beta_2/\gamma_2)(X_n; t)$ is equivalent to $L(\beta_2/\gamma_2, \beta_1/\gamma_1)(X_n; t)$. We do so by giving a bijection between configurations of the vertex model for each of the LLT polynomials. We show that the change in weight under this map is determined by a matching of a sequence of colored beads that can be associated to the boundary conditions of the vertex model. When the sequence of beads has a unique non-crossing matching, the bijection is weight preserving up to an overall power of $t$ and it follows that the LLT polynomials are equivalent. From this general theorem, we can make statements about specific families of partitions. For example, as a corollary to the main result, we can show

**Corollary 6.1.** Let $\lambda^{(1)}$ and $\lambda^{(2)}$ be rectangular partitions such that the Young diagram of one is contained inside the Young diagram of the other. Then $L(\lambda^{(1)}, \lambda^{(2)})(X_n; t)$ and $L(\lambda^{(2)}, \lambda^{(1)})(X_n; t)$ are equivalent.

Using these techniques we are also able to construct linear relations between different LLT polynomials. Knowing these relations has been instrumental in proving results about the expansion of LLT polynomials into Schur and $k$-Schur polynomials [2, 11, 12, 15]. Our new techniques give a systematic way to determine these relations. While we focused on a few specific families of partitions, we wish to emphasize that these techniques could be applied to many different families without much additional effort. We hope that these techniques will be helpful in further understanding the Schur expansion of LLT polynomials.

7 A: Classification

In this section we classify the sequences of beads for which there is a unique non-crossing matching. Since we can associate a sequence of beads to every $B/\gamma$, this gives a sufficient condition on tuples of partitions for which the bijection $\Phi$ is weight-preserving up to an overall power of $t$.

**Lemma 7.1.** A sequence of beads has at least one non-crossing matching if and only if the difference between the number of red beads and blue beads on the top row equals the difference in the number of red beads and blue beads on the bottom row. In particular, if there is only one row then it must have an equal number of red and blue beads.

**Proof.** From our definition of a matching of the sequence of beads, it’s obvious that if we have a matching then the difference between the number of red beads and blue beads on the top row must equal the difference in the number of red beads and blue beads on the bottom row.

Now assume that the constraint on the beads is holds. As a base case, when there are two beads we can either have a red and blue bead in the same row, or beads of the same color in different row. In either case we have a matching.

Suppose there are $n$ beads. For concreteness let the left most bead on the top row be red. Order the beads from left to right on the top row, then from right to left on the bottom row. Let $f : \{1, \ldots, n\} \rightarrow \mathbb{Z}$ be the function such that $f(i)$ returns the difference between the number of red beads and blue beads on the top row minus the difference in the number of red beads and blue beads on the bottom row in the first $i$ beads in our sequence. Note that $f(n) = 0$ due to the constrain on the beads, and $f(1) = 1$ as we assume the first bead is red.

Let the $k^{th}$ bead under this ordering be the first such that $f(k) = 0$. Then $f(k-1)$ must be positive, otherwise, by the intermediate value theorem, $f$ must have been zero on an earlier bead. This implies that if the $k^{th}$ bead is in the top row, it must be blue since for beads in the top row $f(i) < f(i-1)$ if the $i^{th}$ bead is blue. Similarly, if the $k^{th}$ bead is in the bottom row, then it must be red.
Adding an arc that connects the first bead to the $k^{th}$ bead splits the sequence of beads into two pieces: those beads beads between the 1st and the $k^{th}$, and those after the $k^{th}$. For each piece the difference between the number of red beads and blue beads on the top row equals the difference in the number of red beads and blue beads on the bottom row. By induction each of the pieces has a non-crossing matching. Thus the whole sequence of beads has a non-crossing matching.

**Proposition 7.2.** For a single row, sequences of beads that have a unique non-crossing matching are of the form

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
p & q & r
\end{array}\]

with $p + r = q$, or the same configurations as above with red and blue swapped.

**Proof.** It is easy to see that such a sequence admits only one non-crossing matching. It is left to show that if we have any other type of configuration there are multiple non-crossing matchings. In particular, if we have a sequence of beads not of the above form it must contain a subsequence of the form

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

or

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

We will show that in this case there are at least two non-crossing matchings. We will do this by constructing two matchings. As a base case, when there are two blue and two red beads, we have the two matchings

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

and

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

Now suppose we have $n$ red and $n$ blue beads. Suppose our sequence of beads contains a subsequence of the form red-blue-red-blue (the case of blue-red-blue-red is handled the same way with all the colors swapped). We can always choose the subsequence so the first red and blue beads in the sequence are adjacent.

From Lemma 7.1 we know there is at least one matching. Let us consider where this first red bead matches. We mark it the red bead with a star. There are three cases:

Case 1: The red matches to the right of the blue. Then we must have a matching of the form

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

Note that any arcs in interior\(_1\) must stay in interior\(_1\), and similarly for arcs in interior\(_2\). Here we can simply exhibit a second matching

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

\[\begin{array}{ccc}
\bullet & \cdot & \bullet \\
& & \\
\end{array}\]

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where the matching of all the other beads remains the same.
Case 2: The red matches to the left. Here we have two subcases, the blue matching to the left or the blue matching to the right. The matchings take the form

where again any arcs in interior\(_1\) must stay in interior\(_1\), and similarly for arcs in interior\(_2\).

Again we can simply exhibit a second matching

where the matching of all the other beads remains the same.
Case 3: The red matches with the blue. Since we know this pair of beads is the first red-blue in a red-blue-red-blue subsequence we can instead look at the second red-blue pair. Again we can always choose them to be adjacent. Repeating the analysis from case 1 and 2 above, we are left only with the case when the the red and blue beads in this pair also match. The matching then looks like

where we label the other beads as being in the interior or exterior.

Note if the interior is empty we can exhibit a second matching as in the base case. Similarly if beads in the interior only match with other beads in the interior, we can exhibit a second matching. So suppose there are beads in the interior which match with beads in the exterior.

If a red bead in the interior matches with a blue bead in the left exterior, then this these beads and all those in between them form a sequence that contains a blue-red-blue-red subsequence which by induction has more than one matching. Similarly, if a blue bead in the interior matches with a red bead in the right exterior, by induction we have more than one matching. So we are left to consider the case in which either some blue beads in the interior match with red beads in the left exterior, some red beads in the interior match with blue beads in the right exterior, or both.

Let us look only at the rightmost blue bead in the interior that matches with the left exterior (or the original left red-blue pair if no such blue bead exists) and the left most red bead in the interior that matches with the right exterior (or the original right red-blue pair if no such red bead exists). The matching takes the form

where now beads in the interior must match only with other beads in the interior. Now we can
exhibit a second matching

We see that if the sequence of beads has a red-blue-red-blue subsequence, there are at least two non-crossing matchings. Repeating the analysis but swapping all the colors gives the same result for blue-red-blue-red subsequences. Thus the only sequences of beads that have a unique non-crossing matching are those in the statement of the proposition.

**Proposition 7.3.** With two rows, sequence of beads that have a unique non-crossing matching are given by

or the same sequences as above with the rows or colors swapped, where the difference between the number of red and blue beads in the top row is equal to the difference in the number of red and blue beads in the bottom row.

**Proof.** It easy to check that the above sequences of bead do in fact have a unique non-crossing matching. It is left to show that these are the only sequences for which this holds. We will do this by induction the difference between the number of red and blue beads in a row.

As base case, suppose there are an equal number of red and blue beads in each row. Then viewing each row individually, from Lemma 7.1 we know there exist a matching of each row on its own. So there is a matching of the two rows with no arcs connecting them. For there to be a unique non-crossing matching then each row individually must have a unique non-crossing matching. Let’s assume the left most bead on the top row is red. If there are no beads in the bottom row, we know the possible configurations of the top row from Prop. 7.2 which agrees with the case in this Proposition.

Now suppose the bottom row is non-empty and the left most bead of the bottom row is red. We know there is a matching in which these beads match with blue beads to their right in their own row. Given this matching is easy to construct a second matching by making the reds match with red, and blues match with blues, across the rows.

We see for there to be a unique non-crossing matching, the bottom row must start with a
blue bead. If the bottom row ends with a blue bead, we similarly have two matchings

\[ \begin{array}{ccc}
& \bullet & \bullet \\
& \bullet & \bullet \\
& \bullet & \bullet
\end{array} \quad \text{and} \quad \begin{array}{ccc}
& \bullet & \bullet \\
& \bullet & \bullet \\
& \bullet & \bullet
\end{array} \]

where in the first matching no arcs connect the two rows. So the bottom row must end in a red bead. Applying this argument a third time we can show that for there to be a unique non-crossing matching, the top row must end in a blue bead.

The only sequences of beads satisfying these constraints, as well as those of Prop. 7.2, are of the form

\[ \begin{array}{ccc}
& \bullet & \bullet \\
& \bullet & \bullet \\
& \bullet & \bullet
\end{array} \]

with \( q - p = r - s = 0 \), as desired.

Now suppose we are in the case where there are \( k \) more blue beads than red in each row. Any matching must have at least \( k \) arcs connecting the two rows. Given any matching, consider the left most such arc. Suppose it connects between two blue beads, the case where it connects between two red beads can be done by swapping all the colors. This divides the sequence of beads into two pieces

\[ \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \]

For the whole sequence to have a unique non-crossing matching each piece must also have a unique non-crossing matching. Let’s consider piece A. Since the difference between the number of blue and red beads in each row is zero in this piece, by induction we can see that this portion must take the form

\[ \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \quad \text{or} \quad \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \]

or the same sequences as above with the rows or colors swapped. Including the arc between the two blue beads, the only case in which we are unable to construct a second non-crossing matching is

\[ \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet
\end{array} \]
or the same sequences with the rows swapped. For example, if both rows are non-empty we have the two matchings

\[
\begin{array}{c}
\text{. . .}
\end{array}
\quad
\begin{array}{c}
\text{. . .}
\end{array}
\]

where the first is the unique non-crossing matching of the beads in piece A (we draw the arc of particular interest to us) as well as the arc connecting the two rightmost blue beads, and in the second the matching changes as shown.

Now consider piece B (including the arc between the two blue beads). The difference between the number of blue and red beads in each row is \(k - 1\) in this piece, so again by induction we can see that this portion must take the form

\[
\begin{array}{c}
\text{. . .}
\end{array}
\quad
\begin{array}{c}
\text{. . .}
\end{array}
\quad
\begin{array}{c}
\text{. . .}
\end{array}
\quad
\begin{array}{c}
\text{. . .}
\end{array}
\]

or the same sequences as above with the rows or colors swapped. Including the arc between the two blue beads, the only sequence of beads that still has a a unique non-crossing matching is

\[
\begin{array}{c}
\text{. . .}
\end{array}
\]

or the same configuration with the rows swapped.

Finally, combining the two pieces we see that the configuration of beads must be in the form given in the statement of the proposition.

We can pull this constraint on matching back to a constraint on the partition \(\beta/\gamma = (\beta^{(1)}/\gamma^{(1)}, \beta^{(2)}/\gamma^{(2)})\). Recall that the top row of beads corresponds to the singleton paths in the top boundary condition of the vertex model, which depends only on \(\beta^{(1)}\) and \(\beta^{(2)}\). Similarly, the bottom row of beads depends only on \(\gamma^{(1)}\) and \(\gamma^{(2)}\). For simplicity, we’ll assume \(l(\beta^{(1)}) = l(\beta^{(2)}) = n\). Let \(\delta_n = (n-1, n-2, \ldots, 0)\) be the staircase partition of length \(n\). Consider the strict partitions \(\tilde{\beta}^{(1)} = \beta^{(1)} + \delta_n\) and \(\tilde{\beta}^{(2)} = \beta^{(2)} + \delta_n\). Recall that each row of the partitions \(\beta^{(1)}/\gamma^{(1)}, \beta^{(2)}/\gamma^{(2)}\) correspond to a path in the vertex model. The length of the rows of \(\tilde{\beta}^{(1)}, \tilde{\beta}^{(2)}\) encode the number of horizontal steps the paths takes.

We can construct a third strict partition \(\sigma_\beta\) whose parts are the parts of \(\tilde{\beta}^{(1)}\) and \(\tilde{\beta}^{(2)}\) sorted into the decreasing order, along with the condition that anytime a part of \(\tilde{\beta}^{(1)}\) is equal to a part of \(\tilde{\beta}^{(2)}\) both are removed. The order of the parts in this partition gives the order of the singleton boundary paths from rightmost to leftmost. If we color the parts coming from \(\tilde{\beta}^{(1)}\) blue and those coming from \(\tilde{\beta}^{(2)}\) then this is precisely the ordering of the sequence of colored beads associated to the top boundary. We see that the top row of the sequence of colored beads is equivalent to the order of the parts of \(\sigma_\beta\) if we keep track of each parts color. The bottom row of the sequence of colored beads comes from same sorted partition using the \(\gamma\)'s rather than the \(\beta\)'s.
For example, when $\beta/\gamma = ((5, 5, 1)/(2, 1, 0), (4, 3, 2)/(1, 0, 0))$. Then $\tilde{\beta}^{(1)} = (7, 6, 1), \tilde{\beta}^{(2)} = (6, 5, 2), \tilde{\gamma}^{(1)} = (4, 2, 0), \text{ and } \tilde{\gamma}^{(2)} = (3, 1, 0)$. The Young diagrams for $\sigma_{\beta}$ and $\sigma_{\gamma}$ are then

\[
\sigma_{\beta} = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}, \quad \sigma_{\gamma} = \begin{array}{ccc}
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\bullet & \bullet & \bullet \\
\end{array}
\]

corresponding to the vertex model boundary condition and sequence of colored beads

With this the constraints on the sequence of beads can then be translated to a constraint on $\sigma_{\beta}$ and $\sigma_{\gamma}$.

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