RIGID ANALYTIC FUNCTIONS ON THE UNIVERSAL VECTOR EXTENSION

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Abstract. Let $K$ be a non-trivially valued complete non-Archimedean field. Given an algebraic group over $K$ such that every regular function is constant, every rigid analytic function on it is shown to be constant. In particular, an algebraic group whose analytification is Stein (in Kiehl's sense) is necessarily affine—a remarkable difference between the complex and the non-Archimedean worlds.

0. Introduction

0.1. Motivation. Let $K$ be a complete non-trivially valued non-Archimedean field. Let $X$ be an algebraic variety over $K$, that is, a finite type $K$-scheme. Let $X^{\text{an}}$ denote the $K$-analytic space attached to it. The algebraic variety $X$ is said to be $\text{Stein}$ if there is a closed embedding of $K$-analytic spaces $i: X^{\text{an}} \to \mathbb{A}^{n,\text{an}}_K$ for some $n \in \mathbb{N}$. Of course, affine varieties are Stein.

The question underlying this note is whether there exists an algebraic variety over $K$ which is Stein but not affine. One main result of this paper is that such an example cannot be constructed within algebraic groups (that is, group $K$-schemes of finite type):

Theorem A. An algebraic group over $K$ is Stein if and only if it is affine.

This result is in contrast to what happens over the complex numbers. Thus, before describing more in detail the content of this article, let me recall the situation over $\mathbb{C}$.

0.1.1. Serre’s example. As pointed out by Serre, there are non-affine complex algebraic groups that admit a closed holomorphic embedding in $\mathbb{C}^n$ (for some $n \in \mathbb{N}$).

The leading example is the universal vector extension $A^\sharp$ of a (non-trivial) complex abelian variety $A$, that is, the moduli space of rank 1 connections on the dual abelian variety $\hat{A}$. The tensor product of line bundles endows $A^\sharp$ with the structure of a complex algebraic group, which sits in the following short exact sequence:

$$0 \rightarrow \mathcal{V}(\omega_{\hat{A}}) \rightarrow A^\sharp \rightarrow A \rightarrow 0,$$

where the map $A^\sharp \to A$ forgets the connection on an algebraically trivial line bundle on $\hat{A}$, and the map $\mathcal{V}(\omega_{\hat{A}}) \to A^\sharp$ associates to a differential form $\omega$ on $\hat{A}$, the connection on the trivial line bundle $\mathcal{O}_A$ given by the sum $d + \omega$ of the canonical derivation and $\omega$. (The space $H^0(\hat{A}, \Omega^1)$ of global differential forms on $\hat{A}$ is identified with the dual $\omega_{\hat{A}}$ of the Lie algebra of $A$.)

On the one hand, $A^\sharp$ is not affine because the quotient of an affine algebraic group by a subgroup is affine—and $A$ is certainly not so. On the other, the Riemann-Hilbert correspondence yields a biholomorphism

$$A^{\sharp,\text{an}} \overset{\sim}{\longrightarrow} \text{Hom}_{\text{Groups}}(\pi_1(\hat{A}(\mathbb{C}), 0), \mathbb{C}^\times) \simeq (\mathbb{C}^\times)^{2g},$$

where $g := \dim A$, and $\pi_1(\hat{A}(\mathbb{C}), 0)$ is the topological fundamental group of $\hat{A}(\mathbb{C})$ with 0 as base-point. In particular, $A^{\sharp,\text{an}}$ admits a closed holomorphic embedding in $\mathbb{C}^{4g}$. 


The biholomorphism given by the Riemann-Hilbert correspondence admits a quite explicit description, which should hopefully ease the reading of the paper. Let \( \text{exp}: \text{Lie} A \to A^{\text{an}} \) be the exponential map. Identify \( V := \text{Lie} A \) with the universal cover of \( A(\mathbb{C}) \), and \( \Lambda := \ker \text{exp} \) with the fundamental group. The pull-back of \( A^{\text{an}} \) to \( \text{Lie} A \) splits as an affine bundle because the cohomology group \( H^1(V^{\text{an}}, \mathcal{O}_{V^{\text{an}}}) \) vanishes. More precisely, Hodge theory identifies \( \omega_A \) with the conjugated complex vector space \( V = \text{Lie} A \). Let \( \theta_A: \Lambda \to V \) be the inclusion. Then \( A^{\text{an}} \) is the quotient \((V \times V)/\Lambda \) with \( \Lambda \) embedded diagonally. By choosing a basis of \( \Lambda \), one sees that \( A^{\text{an}} \) is biholomorphic to \((\mathbb{C}/\mathbb{Z})^{2g}, \) hence to \((\mathbb{C}^*)^{2g} \).

0.1.2. Neeman’s counter-example to Hilbert’s fourteenth problem. The difference between affine and Stein varieties is quite subtle. Neeman exhibited a quasi-affine (that is, admitting an open immersion in an affine variety) complex variety which is Stein but not affine. Let me recall his construction.

Let \( A \) be the universal vector extension of an abelian variety \( A \). Let \( H \) be an ample line bundle on \( A \). Let \( P \to A \) be the total space of \( H \) deprived of its zero section. Since \( H \) is ample, \( P \) is quasi-affine. Since \( P \) is a principal \( \mathbb{G}_m \)-bundle over \( A \), it is not affine: otherwise, by GIT ([GIT, Theorem 1.1]), the quotient \( P/\mathbb{G}_m \cong A \) would be so. However \( P \) is Stein because \( A^{\text{an}} \) is so ([MM60]).

As remarked by Neeman ([Nee88, Proposition 5.5]), a quasi-affine complex variety \( X \) which is Stein but not affine forces its \( \mathbb{C} \)-algebra of global sections \( H^0(X, \mathcal{O}_X) \) not to be finitely generated. As \( P \) is such a variety, it yields a counter-example of Hilbert’s fourteenth problem, as extended by Zariski (loc. cit. Remark 8.2).

0.2. Main results.

0.2.1. Rigid analytic functions on anti-affine groups. Needless to say, in order to prove Theorem A one has to understand rigid analytic functions on algebraic groups.

Employing Brion’s nomenclature, an algebraic group \( G \) over a field \( K \) is said to be \( \text{anti-affine} \) if \( H^0(G, \mathcal{O}_G) = K \). An anti-affine group is commutative (otherwise the image of its adjoint representation would be a non-trivial affine quotient of \( G \)), reduced (otherwise \( G/G_{\text{red}} \) would be a non-trivial finite algebraic group) and connected (otherwise \( G/G^0 \) would be a non-trivial finite group).

Let \( K \) be a complete non-trivially valued non-Archimedean valued field.

**Theorem B** (infra Theorem 4.20). Let \( G \) be an anti-affine algebraic group over \( K \). Then, all \( K \)-analytic functions on \( G^{\text{an}} \) are constant.

Given an algebraic group \( G \), the \( K \)-algebra \( K[G] := H^0(G, \mathcal{O}_G) \) is of finite type. The structure of Hopf algebra on \( K[G] \) induces a structure of algebraic group on \( G^{\text{aff}} = \text{Spec} K[G] \). The canonical morphism \( \pi: G \to G^{\text{aff}} \) is a faithfully flat morphism of algebraic groups whose kernel \( G_{\text{ant}} \) is anti-affine ([DG70 III.3.8]).

**Theorem C** (infra Theorem 4.21). Let \( G \) be an algebraic group over \( K \). With the notation above, precomposing a \( K \)-analytic function on \( G^{\text{an}} \) with \( \pi^{\text{an}} \) yields an isomorphism

\[
H^0(G^{\text{an}}, \mathcal{O}_{G^{\text{an}}}) \cong H^0(G^{\text{an}}, \mathcal{O}_{G^{\text{an}}}).
\]

Theorem A is a straightforward consequence of Theorem C. Indeed, a \( K \)-analytic morphism \( f: G^{\text{an}} \to K^{n^{\text{an}}} \) factors through \( G^{\text{aff}} \), if \( f \) is a closed embedding, then \( G^{\text{ant}} \) is trivial. In turn, Theorem C is easily deduced from B.

0.2.2. Brion’s generalization of Neeman’s counter-example. Let \( G \) be a commutative, reduced and connected algebraic group over a field \( K \). Thanks to Chevalley’s theorem ([Con02]), \( G \) is an extension of an abelian variety \( A \) by a linear algebraic group \( L \):

\[
0 \to L \to G \to A \to 0.
\]
Let $H$ be an ample line bundle on $A$. Let $P \to G$ be the total space of $\pi^*H$ deprived of its zero section. Since $H$ is ample, $P$ is quasi-affine. Brion showed that if $G$ is anti-affine and $L$ is non-trivial, then $H^0(P, \mathcal{O}_P)$ is not Noetherian ([Bri09, Theorem 3.9]). This furnishes a counter-example to Zariski’s version of Hilbert’s fourteenth problem on any field of characteristic 0, and any field of characteristic $p > 0$ which is not an algebraic extension of $\mathbb{F}_p$.

**Theorem D** (infra Theorem 4.23). Let $K$ be a complete non-trivially valued non-Archimedean field. With the notation above, the algebraic variety $P$ is not Stein.

Arguing by contradiction, if $P$ were Stein, then $G$ would be Stein. By Theorem A, the algebraic group $G$ would be affine, contradicting the fact that it is anti-affine.

### 0.3. Rigid analytic functions on the universal vector extension.

The crucial case of Theorem B is that of the universal vector extension of an abelian variety. Actually, the techniques to prove it work when the abelian variety is replaced by an abeloid variety—the rigid analytic analogue of a complex torus. To state the result in full generality and outline its proof, let me recall the set up of Raynaud’s uniformization of abeloid varieties. (A thorough discussion is to be found in Section 4.2.)

#### 0.3.1. Raynaud’s uniformization.

Let $B$ be the generic fiber (in the sense of Raynaud) of a formal abelian scheme over the ring of integers of $K$. Let $\tilde{B}$ be the generic fiber of the dual formal abelian scheme. Let $T$ be (the analytification of) a split $K$-torus with group of characters $\tilde{M}$. A group homomorphism $\tilde{c}: \tilde{M} \to \tilde{B}(K)$ gives rise, as in the algebraic case, to an extension $G$ of $B$ by $T$. Let $M$ be a free abelian group of rank $\dim T$.

Let $\varepsilon: M \to G(K)$ be an injective group homomorphism with discrete image. Let $\tilde{T}$ be the split $K$-torus with group of characters $\tilde{M}$. Let $c: M \to B(K)$ be the composition of $\varepsilon$ with the projection $G \to B$. To $c$ is associated an extension $\hat{G}$ of $\hat{B}$ by $\hat{T}$. Moreover, $\varepsilon$ induces an injective group homomorphism with discrete image $\varepsilon: M \to \hat{G}(K)$.

Suppose that the quotient $A := G/\varepsilon(M)$ is a proper $K$-analytic space. Proper $K$-analytic groups arising in this way are called abeloid varieties. Abelian varieties (or, better, their analytification) become abeloid after a finite separable extension of the base field—if $K$ is discretely valued, this is a consequence of the Semi-Stable Reduction Theorem ([Ray71]); in the general case, this is proved by Bosch-Lütkebohmert ([BL85]). Whether this is true for every proper $K$-analytic group is unknown, yet widely believed. Anyhow, the $K$-analytic space $\hat{A} := \hat{G}/\varepsilon(M)$ represents the functor associating to a $K$-analytic space $S$ the set of isomorphism classes of homogeneous line bundles on $A \times_K S$. The considerable deal of notation is resumed in the following diagram:

\[
\begin{array}{ccc}
M & \xrightarrow{\varepsilon} & G \\
\downarrow & & \downarrow \\
T & \xrightarrow{c} & B \\
\downarrow & & \downarrow \\
A & & A
\end{array}
\]

\[
\begin{array}{ccc}
\tilde{M} & \xrightarrow{\tilde{c}} & \tilde{G} \\
\downarrow & & \downarrow \\
\tilde{T} & \xrightarrow{\tilde{c}} & \tilde{B} \\
\downarrow & & \downarrow \\
\tilde{A} & & \tilde{A}
\end{array}
\]

Let $A^\natural$ denote the moduli space of line bundles on $\hat{A}$ endowed with a connection.

**Theorem E.** If $K$ is of characteristic 0, then all $K$-analytic functions on the universal vector extension $A^\natural$ of $A$ are constant.
Let me sketch the main steps of the proof of Theorem 1. Also, I take profit of this introduction to step a little aside from the treatment presented in the paper, and reformulate the arguments in a more geometric, yet equivalent, way.

In order to do so, I find especially suggestive the language of Berkovich spaces because they permit to carry out truly topological arguments, in fact remarkably close to the complex intuition: within this framework, the $K$-analytic space $G$ is identified with the topological universal cover of the $K$-analytic space $A$, and $M$ with the topological fundamental group of $A$ with base-point $0$.

### 0.3.2. Totally degenerate case.

Suppose $A$ has totally degenerate reduction, that is $G = T$ (or, equivalently, $B = 0$). Here the topological argument explained in the complex case can be followed closely—yet it leads to a completely different conclusion.

The pull-back of $A^\flat$ to the topological universal cover $T$ splits as an affine bundle because the cohomology group $H^1(T, \mathcal{O}_T)$ vanishes. More precisely, let $\theta_M : M \to \omega_T$ be the map associating to a character $\omega \in \mathbb{G}_m$ the differential form $m^* dt/t$, where $t$ is the coordinate on $\mathbb{G}_m$. (Recall that $M$ is by definition the group of characters of $T$.) Then $A^\flat$ is the quotient of $T \times \mathbb{V}(\omega_T)$ by the fundamental group $M$ being embedded via $(\varepsilon, \theta_M)$; see Proposition 4.15.

Analytic functions on $A^\flat$ are those $K$-analytic functions on the topological universal cover $T \times \mathbb{V}(\omega_T)$ that are invariant under the action of the fundamental group $M$. Since the lattice $\theta_M(M)$ of $\omega_T$ accumulates in 0, such functions are seen to be constant.

Here the difference with the complex situation is quite striking. With the notation introduced above, the lattice $\Lambda$ is discrete in $V$ and there are plenty of holomorphic functions on $V$ that are $A$-invariant, while the lattice $\theta_M(M)$ accumulates in 0 and leaves no non-constant $K$-analytic function invariant.

To stress the analogy with the case of good reduction, note that $\theta_M$ is the universal vector hull of $M$: given a finite-dimensional $K$-vector space $E$ and a morphism of $K$-analytic groups $f : M \to \mathbb{V}(E)$, there is a unique $K$-linear map $\varphi : \omega_T \to E$ such that $f = \mathbb{V}(\varphi) \circ \theta_M$.

### 0.3.3. Good reduction case.

In the diametrically opposite case, $T = M = 0$ (or, equivalently, $G = B$). In this case, in order to simplify notation, let $A$ be a formal abelian scheme over the ring of integers $R$ of $K$.

Fix a topologically nilpotent element $\varpi \in R \setminus \{0\}$. For $n \in \mathbb{N}$, let $A^\flat_n$ denote the $n$-th infinitesimal thickening $A^\flat \times_R R/\varpi^n R$ of $A^\flat$. When the residue characteristic of $K$ is 0, Coleman proves that, for each $n \in \mathbb{N}$, functions on $A^\flat_n$ are constant (Corollary 2.4). Then, the result follows immediately.

From now on suppose that the residue characteristic is $p > 0$. Then one could see that each infinitesimal neighbourhood $A^\flat$ admits non-constant functions: the proof boils down to showing that such functions cannot be lifted to $A^\flat$. In the good reduction case, the topology offers no information because the Raynaud generic fiber $A_K$ of $A$ is contractible. The idea is to replace the topological universal cover by the “perfectoid” one

$$\breve{A} = \lim_{\leftarrow_p} A,$$

where the transition maps are the multiplication by $p$. (The projective limit exists as a formal $R$-scheme because of the finiteness of the transition maps.)

Let $u : \breve{A} \to A$ be the projection onto the first factor, and let $T_p A$ be the kernel of $u$. By definition, $T_p A$ is the projective limit (as a formal $R$-scheme) of the finite and flat group formal $R$-schemes $A[p^i]$ for $i \in \mathbb{N}$. When $K$ is algebraically closed,
the $R$-valued points of $T_p A$ form the usual $p$-adic Tate module

$$T_p A(R) = \projlim_{i \in \mathbb{N}} A[p^i](R).$$

The pull-back of $A^\flat$ to $\tilde{A}$ splits because the cohomology group $H^1(\tilde{A}, \mathcal{O}_{\tilde{A}})$ vanishes ([Bha19, Proposition 2.2.1]). More precisely, let $\theta_{T_p A}: T_p A \to \mathbb{V}(\omega_A)$ be the universal vector hull of $T_p A$. It is the morphism of group formal $R$-schemes with the following property: given a free $R$-module of finite rank $E$ and a morphism of group formal $R$-schemes $f: T_p A \to \mathbb{V}(E)$, there is a unique homomorphism of $R$-modules $\varphi: \omega_A \to E$ such that $f = \mathbb{V}(\varphi) \circ \theta_{T_p A}$. The universal vector extension $A^\flat$ of $A$ is the quotient of $\mathbb{V}(\omega_A) \times \tilde{A}$ by $T_p A$ embedded via $\theta_{T_p A}$; see paragraph 3.1.3 and Proposition 2.16.

Therefore formal functions on $A^\flat$ are those on $\mathbb{V}(\omega_A) \times \tilde{A}$ that are invariant under the action of $T_p A$. Since $H^0(\tilde{A}, \mathcal{O}_{\tilde{A}}) = R$ ([Bha19, Proposition 2.2.1]), one has

$$H^0(\mathbb{V}(\omega_A) \times \tilde{A}, \mathcal{O}_{\mathbb{V}(\omega_A) \times \tilde{A}}) = H^0(\mathbb{V}(\omega_A), \mathcal{O}_{\omega_A}),$$

so that formal functions are identified, by restriction to $\mathbb{V}(\omega_A)$, with those on $\mathbb{V}(\omega_A)$ that are invariant under translation by the image of $\theta$. According to Coleman ([Col84, p. 379], [Col91, §4]) and Faltings ([Fal87, Theorem 4]), the map $\theta_{T_p A}$ is the one appearing in the Hodge-Tate decomposition of $H^1_{et}(A_K, \mathbb{Q}_p)$: for algebraically closed $K$, the $K$-linear map induced by $\theta_{T_p A}$,

$$T_p A(R) \otimes_{\mathbb{Z}_p} K \longrightarrow \omega_A \otimes_R K,$$

is surjective. In particular, the image of $T_p A(R)$ is a lattice in $\omega_A \otimes_R K$. By using again the aforementioned accumulation of lattices, one concludes that all formal functions on $A^\flat$ are constant.

0.3.4. Intermediate reduction. When $A$ has intermediate reduction, that is, neither $T$ nor $B$ is trivial, the argument is a combination of the previous two. However, in order to do so, one has to understand the topological universal cover of $A^\flat$, and how the topological fundamental group acts on it. This is more or less equivalent to describe the canonical extension (in the sense of Mazur-Messing [MM74, §1.5]) of the Néron model of an abelian variety with (split) semi-stable reduction.

Let $B^\flat$ be the universal vector extension of $B$. Let $G^\flat$ denote the push-out of $B^\flat \times_B G$ along the morphism of $K$-analytic groups $\mathbb{V}(\omega_B) \to \mathbb{V}(\omega_G)$ induced by the tangent map of the surjection $G \to B$:

$$
\begin{array}{ccc}
0 & \longrightarrow & \mathbb{V}(\omega_B) \\
\downarrow & & \downarrow \\
0 & \longrightarrow & \mathbb{V}(\omega_G) \\
& & \downarrow \\
& & G \\
& & \downarrow \\
& & G \\
& & 0.
\end{array}
$$

There is an injective homomorphism of groups $\varepsilon^\flat: M \to G^\flat(K)$ whose composition with the projection $G^\flat \to G$ is $\varepsilon$ and such that the universal vector extension $A^\flat$ is the quotient of $G^\flat$ by $\varepsilon^\flat(M)$. The quotient of $G$ by $B^\flat \times_B G$ is $\mathbb{V}(\omega_T)$. The composition of $\varepsilon^\flat$ with the projection $G^\flat \to \mathbb{V}(\omega_T)$ is the universal vector hull $\theta_M: M \to \mathbb{V}(\omega_T)$ of $M$ (Propositions 4.10 and 4.17).

Analytic functions on $A^\flat$ correspond to $K$-analytic functions on $G^\flat$ that are invariant under $\varepsilon^\flat(M)$. However, $G^\flat$ is an affine extension of a toric extension of $B$. Therefore, in order to conclude, one cannot simply use the computation of $K$-analytic functions on the universal vector extension of $B$: one needs to compute $K$-analytic functions on an extension of $B$ by a product of a torus and a vector group (cf. Corollaries 3.10 and 4.17). The proof is then achieved by showing that
the only analytic functions on $G^2$ left invariant by $e^z(M)$ are the constant ones—this is done by expanding in “Fourier series” sections of certain line bundles (cf. section 1.4.3).

0.4. **Organization of the paper.** The paper has four sections. The first one passes in review the definition of the universal vector extension as moduli space of rank 1 connections on an abelian scheme, as opposed to Mazur-Messing ([MM74]) presentation of the universal vector extension as a solution of a universal problem. Hopefully, this should make it clear to the reader that the theory of the universal vector extension of an abelian can be translated immediately in the context of formal and rigid geometry.

The second section contains the computation of global functions on vector extensions of an arbitrary abelian scheme. Functions on vector extensions are naturally seen as global sections of unipotent vector bundles. Inspired by the work of Deninger-Werner on the $p$-adic Narasimhan-Seshadri theorem ([DW05a], [DW05b]), such sections are seen as invariants under the action of the Tate module.

The formal case, contained in the third section, is handled by computing formal functions as projective limits of functions on successive infinitesimal thickenings. The only thing left to do is realizing that the representation of the $p$-adic Tate module associated with the universal vector extension is the one appearing in the Hodge-Tate decomposition.

The fourth and final section deals with bad reduction. As already mentioned during the introduction, the main concern is an accurate description of the topological universal cover of the universal vector extension.

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1. **The Universal Vector Extension of an Abelian Scheme**

Let $S$ be a scheme. Let $\alpha: A \to S$ be an abelian scheme. Let $e: S \to A$ be the zero section. Let $\omega_A = e^*\Omega^1_{A/S}$.

1.1. **Connections on homogeneous line bundles.**

1.1.1. **Connections and Aitihah extension.** Let $f: X \to S$ be a separated morphism of schemes. The diagonal morphism $\Delta: X \to X \times_S X$ is a closed immersion. Let $I$ be the kernel of the restriction $\mathcal{O}_{X \times X} \to \Delta_* \mathcal{O}_X$.

Let $\Delta_1$ be the first infinitesimal neighbourhood of the diagonal, that is, the closed subscheme of $X \times_S X$ defined by the sheaf of ideals $I^2$. The sheaf $I/I^2$ (resp. $\mathcal{O}_{X \times X}/I^2$) is the push-forward along $\Delta$ of the sheaf of differentials $\Omega^1_{X \times X}$ (resp. the sheaf of principal parts $\mathcal{P}^1_{X \times X}$ of order 1) relative to $f$. For $i = 1, 2$, let $\pi_i: \Delta_i \to X$ be the morphism induced by the $i$-th projection. Each $\pi_i$ induces a homomorphism of $f^{-1}\mathcal{O}_S$-algebras $j_i: \mathcal{O}_X \to \mathcal{P}^1_{X \times X}$. The canonical derivation $d_f: \mathcal{O}_X \to \Omega^1_{X \times X}$ is the homomorphism of $f^{-1}\mathcal{O}_S$-modules $j_2 - j_1$.

**Definition 1.1.** Let $E$ be a vector bundle on $X$. A connection on $E$ is an isomorphism of $\mathcal{O}_{\Delta_1}$-modules $\nabla: p^*_1 E \to p^*_2 E$ such that $\Delta^* \nabla = \nabla |_{\Delta_1}$.

Let $E$ be a vector bundle on $X$. Let $\Omega^1_{\Delta_1}(E) = \Omega^1_1 \otimes E$ and $\mathcal{P}^1_{\Delta_1}(E) = p_1^* p_2^* E$. Consider the short exact sequence of $\mathcal{O}_X$-modules:

$$0 \to \Omega^1_{\Delta_1}(E) \to \mathcal{P}^1_{\Delta_1}(E) \to E \to 0. \quad (1.1)$$
The datum of a connection on $E$ is equivalent to the datum of a splitting of \[\Gamma(E).\] In turn, it is equivalent to the datum of a $f^{-1}\mathcal{O}_S$-linear homomorphism $\delta: E \to \Omega^1_f(E)$ satisfying the Leibniz rule $\delta(gs) = g\delta(s) + d_f(g) \otimes s$.

**Definition 1.2.** Let $At_f(E) := \text{Hom}(E, \mathcal{P}_f^1(E))$. Taking the tensor product of $E^\vee$ with $E^\vee$ yields the following short exact sequence of $\mathcal{O}_X$-modules, called the Atiyah extension of $E$ relative to $f$:

\[0 \to \Omega^1_f(\text{End} E) \to At_f(E) \to \text{End} E \to 0.\]

**Proposition 1.3.** Let $\pi: Y \to X$ be a morphism of $S$-schemes. Let $g := f \circ \pi$. The Atiyah extension of $\pi^* E$ is the push-out of the pull-back of the Atiyah extension of $E$ along $d\pi \otimes \text{id}: \pi^* \Omega^1_f(\text{End}(E)) \to \Omega^1_g(\text{End}(E))$:

\[
\begin{array}{cccccc}
0 & \to & \pi^* \Omega^1_f(\text{End}(E)) & \to & \pi^* At_f(E) & \to & \pi^* \text{End}(E) & \to & 0 \\
& & \downarrow \delta \pi \otimes \text{id} & & \downarrow & & \downarrow & & \\
0 & \to & \Omega^1_g(\text{End}(\pi^* E)) & \to & At_g(\pi^* E) & \to & \text{End}(\pi^* E) & \to & 0.
\end{array}
\]

**Proof.** Left to the reader. \hfill $\square$

1.1.2. Connections and rigidifications. Let $pr_1, pr_2, m: A \times_S A \to A$ be respectively the first projection, the second projection, and the group law on $A$.

**Notation 1.4.** Let $G$ be a commutative group $S$-scheme. Let $e_G: S \to G$ be the zero section. Let $i_G: G_1 \to G$ be the first infinitesimal neighbourhood of the zero section. Let $\pi_G: G_1 \to S$ be the structural morphism. If there is no confusion on the commutative group being considered, the subscript $G$ is dropped from the notation.

Adopt the notation above with $G = A$.

**Definition 1.5.** Let $E$ be a vector bundle on $A$. A rigidification on $E$ is an isomorphism of $\mathcal{O}_{A_1}$-modules $\rho: \pi^* e^* E \to \iota^* E$ such that $e^* \rho$ is the identity.

Let $\Delta: A \to A \times_S A$ be the diagonal. Let $\Delta_1$ be the first infinitesimal neighbourhood of the diagonal. For $i = 1, 2$ let $p_i: \Delta_1 \to A$ the morphism induced by the $i$-th projection. Let $\tau: A_1 \to \Delta_1$ be the morphism determined by $p_1 \circ \tau = e \circ \pi$ and $p_2 \circ \tau = \iota$. With this notation, the following square is commutative:

\[
\begin{array}{ccc}
A_1 & \xrightarrow{\pi} & S \\
\downarrow \tau & & \downarrow e \\
\Delta_1 & \xrightarrow{p_1} & A
\end{array}
\]

Consider the homomorphism of $\mathcal{O}_S$-modules obtained by base-change

\[
\psi: e^* \mathcal{P}_1^\alpha(E) = e^* p_1^* p_2^* E \to \pi^* \iota^* p_2^* E = \pi^* \iota^* E.
\]

Let $e_1: S \to A_1$ be the unique morphism such that $\iota \circ e_1 = e$. The kernel of the restriction map $\iota^* E \to e_1^* e^* E$ is $e_1^* (\omega_A \otimes \mathcal{O})$. Pushing forward along $\pi$ yields the following short exact sequence of $\mathcal{O}_S$-modules

\[
0 \to \omega_A \otimes E \to \pi^* \iota^* E \to e^* E \to 0. \tag{1.2}
\]

**Proposition 1.6.** With the notation above,

1. The homomorphism $\psi$ sits in the following commutative and exact diagram of $\mathcal{O}_S$-modules

\[
\begin{array}{cccccc}
0 & \to & \omega_A \otimes E & \to & e^* \mathcal{P}_1^\alpha(E) & \to & e^* E & \to & 0 \\
& & \downarrow & & \downarrow \psi & & \downarrow & & \\
0 & \to & \omega_A \otimes E & \to & \pi^* \iota^* E & \to & e^* E & \to & 0
\end{array}
\]
where the upper short exact sequence is the pull-back of (1.1) along \( e \), and the lower short exact sequence is (1.2). In particular, \( \psi \) is an isomorphism.

(2) The datum of a rigidification on \( E \) is equivalent to the datum of a splitting of (1.2).

Proof. (1) The short exact sequence (1.1) is the push-forward along \( p_1 \) of the short exact sequence of \( O_{\Delta_1} \)-modules
\[
0 \to \delta_0 \Omega_{\Delta_1}^1(E) \to p_2^*E \to \delta_1 E \to 0,
\]
where \( \delta: \Delta_1 \to \Delta \) is the closed immersion induced by the diagonal morphism.

Consider the homomorphism of \( O_{\Delta_1} \)-modules \( \psi: p_2^*E \to \delta_1 \tau^*E = \tau_1^*E \) given by adjunction. Note that, for an \( O_{\Delta_1} \)-module \( F \), one has \( \tau^* \delta_1 E = e_1^* e^* F \). By applying this to \( F = \delta_1 \Omega_{\Delta_1}^1(E) \), one sees that the homomorphism \( \psi \) sits in the following commutative and exact diagram
\[
\begin{array}{ccccccccc}
0 & \to & \delta_0 \Omega_{\Delta_1}^1(E) & \to & p_2^*E & \to & \delta_1 E & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
& & \tau_1^* e_1^*(\omega_A \otimes E) & \to & \tau_1^* \varphi E & \to & \tau_1^* e_1^* E & \to & 0 \\
\end{array}
\]
where the lower exact sequence is obtained by pulling-back (1.3) along \( \tau \), and then pushing-forward along \( \tau \) the result. (A posteriori, the lower row will be short exact. )

Pushing-forward the previous diagram along \( p_1 \) yields the following commutative and exact diagram of \( O_A \)-modules
\[
\begin{array}{ccccccccc}
0 & \to & \Omega_{\Delta_1}^1(E) & \to & p_1 p_2^*E & \to & E & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & e_*(\omega_A \otimes E) & \to & e_*(\varphi e^* E) & \to & e_*(e^* E) & \to & 0 \\
\end{array}
\]

The homomorphism \( p_1 \varphi \) is the composition of \( e_\psi \) : \( e_\psi p_1 p_2^*E \to e_\psi \varphi E \) with the restriction homomorphism \( p_1 p_2^*E \to e_\psi p_1 p_2^*E \). The result follows.

(2) Giving a \( O_{\Delta_1} \)-linear isomorphism \( \varphi^* E \to \varphi E \) such that \( \varphi \rho = \text{id} \) is equivalent to the datum of a \( O_{\Delta_1} \)-linear homomorphism \( \varphi: \pi_* \varphi^* E \to \pi_* \varphi E \) such that the following diagram is commutative
\[
\begin{array}{ccccccccc}
0 & \to & \omega_A \otimes e^* E & \to & \pi_* \varphi^* E & \to & e^* E & \to & 0 \\
& & \downarrow & & \downarrow & & \downarrow & & \\
0 & \to & \omega_A \otimes e^* E & \to & \pi_* \varphi E & \to & e^* E & \to & 0 \\
\end{array}
\]

Since \( \pi_* \varphi^* E = e^* E \otimes \omega_A \otimes e^* E \), such an isomorphism \( \varphi \) corresponds to a splitting of (1.2). \( \square \)

Definition 1.7. A homogeneous line bundle on \( A \) is the datum of a line bundle on \( A \) and of an isomorphism \( \varphi: m^* L \to p_1^* L \otimes p_2^* L \).

Let \( (L, \varphi) \) be a homogeneous line bundle on \( A \). To a connection \( \nabla: p_1^* L \to p_2^* L \), one associates the rigidification \( \tau^* \nabla: \pi_* \varphi^* E \to \varphi E \).

Conversely, given a rigidification \( \rho: \pi_* \varphi^* E \to \varphi E \), one associates a connection on \( L \) as follows. By definition of \( A_1 \), the map \( p_2 - p_1: \Delta_1 \to A_1 \) factors through \( A_1 \). Write \( p_2 - p_1 = \eta \circ \iota \) with \( \eta: \Delta_1 \to A_1 \). Consider the following commutative diagram:
\[
\begin{array}{cccccc}
\pi^* e^* L \otimes p_1^* L & \xrightarrow{\iota^* \rho \circ \text{id}} & (p_2 - p_1)^* L \otimes p_1^* L \\
\end{array}
\]
Then the wanted connection is \((p_2 - p_1, p_1)^*\varphi^{-1} \circ (\eta^*\rho \otimes \text{id}) \circ (e \circ \pi, p_1)^*\varphi\).

**Proposition 1.8.** The previous constructions are inverse to each other, and define a bijection between the set of connections on \(L\) and the set of rigidifications on \(L\).

**Proof.** Let \(\rho\) be a rigidification and let \(\nabla\) be the associated connection defined above. Since \(\eta \circ \tau = \text{id}_{A_1}\), one has \(\tau^*\nabla = \rho\). On the other hand, let \(\nabla', \nabla''\) be connections on \(L\). They differ by a global differential form \(\omega \in \Omega^1(A, \Omega^1_{A/S})\). Since \(\Omega^1_{A/S} = \alpha^*\omega_A\), the connections \(\nabla, \nabla'\) coincide if and only the associated rigidifications \(\tau^*\nabla, \tau^*\nabla'\) do. The result follows. (See the proof of (2) and (3) in the proof of [MM74, Proposition 3.2.3], p. 39-40.) \(\Box\)

**Proposition 1.9.** Let \((L, \varphi)\) be a homogeneous line bundle on \(A\).

1. If \(H^1(S, \omega_A) = 0\), then \(L\) admits a connection.
2. The sequence of \(\mathcal{O}_S\)-modules \(0 \to \alpha_*\Omega^1_\alpha \to \alpha_*\text{At}_\alpha(L) \to \alpha_*\mathcal{O}_A \to 0\) obtained by pushing forward along \(\alpha\) of the Atiyah extension is short exact, and the homomorphism of short exact sequence of \(\mathcal{O}_A\)-modules obtained by adjunction

\[
\begin{array}{cccccc}
0 & \longrightarrow & \alpha^*\alpha_*\Omega^1_\alpha & \longrightarrow & \alpha^*\alpha_*\text{At}_\alpha(L) & \longrightarrow & \alpha^*\alpha_*\mathcal{O}_A & \longrightarrow & 0 \\
\downarrow_{\lambda} & & \downarrow_{\mu} & & \downarrow_{\nu} & & \downarrow_{\nu} & & \\
0 & \longrightarrow & \Omega^1_\alpha & \longrightarrow & \text{At}_\alpha(L) & \longrightarrow & \mathcal{O}_A & \longrightarrow & 0
\end{array}
\]

is an isomorphism.

**Proof.** (1) According to Proposition 1.8 it suffices to show that \(L\) admits a rigidification. By Proposition 1.6 (2) this amounts a splitting of the short exact sequence (1.2) splits. The isomorphism class of the extension \(e^*\mathcal{P}^1(L)\) lives in the cohomology group \(H^1(S, \omega_A)\). The latter vanishes by hypothesis thus (1.2) splits.

(2) The fact that \(0 \to \alpha_*\Omega^1_\alpha \to \alpha_*\text{At}_\alpha(L) \to \alpha_*\mathcal{O}_A \to 0\) is exact because, according to (1), locally on \(S\) the line bundle \(L\) admits a connection. Now, \(\lambda\) is an isomorphism because the sheaf of relative differentials \(\Omega^1_\alpha\) is isomorphic to \(\alpha^*\omega_A\): \(\nu\) is an isomorphism because \(\alpha_*\mathcal{O}_A = \mathcal{O}_S\). It follows from the Five Lemma that \(\mu\) is an isomorphism. \(\Box\)

**Remark 1.10.** It follows from Proposition 1.9 that the evaluation at the zero section \(\alpha_*\text{At}_\alpha(L) \to e^*\text{At}_\alpha(L)\) is an isomorphism. On the other hand, Proposition 1.6 gives an isomorphism \(\psi: e^*\mathcal{P}^1_\alpha(L) \to \pi_*e^*L\). By taking the tensor product with \(e^*L^\vee\) yields to an isomorphism of \(\mathcal{O}_S\)-modules

\[
\psi \otimes \text{id}: e^*\text{At}_\alpha(L) = e^*\mathcal{P}^1_\alpha(L) \otimes e^*L^\vee \longrightarrow \pi_*e^*L \otimes e^*L^\vee.
\]

Note that \(e^*L\) is the trivial line bundle on \(S\) because \(L\) is homogeneous. It follows that \(e^*\text{At}_\alpha(L)\) is isomorphic to \(\pi_*e^*L\).

**1.2. The universal extension of the trivial line bundle.** The dual abelian scheme of \(\hat{A}\) is the \(S\)-scheme \(\hat{\alpha}: \hat{A} \to S\) representing the functor associating to a \(S\)-scheme \(S'\), the set of isomorphism classes of homogeneous line bundles on \(A \times_S S'\) ([FC98, Theorem 1.9], [Oort65, Proposition 18.4]). The tensor product of line bundles endows \(\hat{A}\) with the structure of \(S\)-group scheme. Let \(\mathcal{L}\) denote the Poincaré bundle, that is, the universal homogeneous line bundle on \(A \times_S \hat{A}\).

**1.2.1. Definition.** Let \(q: A \times_S \hat{A} \to \hat{A}\) be the projection onto the second factor. Let \(\text{At}_q(\mathcal{L})\) the Atiyah extension relative to \(q\) of the Poincaré bundle \(\mathcal{L}\). By pushing-forward along \(q_1\) one obtains the following short exact sequence of \(\mathcal{O}_A\)-modules (Proposition 1.9):

\[
0 \longrightarrow \hat{\alpha}^*\omega_A \longrightarrow q_*\text{At}_q(\mathcal{L}) \longrightarrow \mathcal{O}_A \longrightarrow 0.
\]
\textbf{Definition 1.11.} The extension $\mathcal{U}_A := q_\ast \text{At}_q(L)$ is called the \textit{universal extension} of $\mathcal{O}_A$. The pull-back of the Atiyah extension of $L$ along $(\text{id}_A, \epsilon)$ is the Atiyah extension of $\mathcal{O}_A$, therefore the derivation $d_\ast : \mathcal{O}_A \to \Omega^1_A$ induces a splitting of it, called the \textit{canonical splitting}.

For further use, let us point out the following fact. Let $\tilde{m}, \tilde{p}, \tilde{q} : A \times \tilde{A} \to \tilde{A}$ be respectively the group law, the first, and the second projection. Let $\varphi : (\text{id}, \tilde{m})^\ast L \to (\text{id}, \tilde{p})^\ast L \otimes (\text{id}, \tilde{q})^\ast L$ be the isomorphism endowing $L$ with the structure of universal homogeneous line bundle.

\textbf{Proposition 1.12.} With the notation above,

1. The $\mathcal{O}_{A \times \tilde{A}}$-module $(\pi, \text{id}_{A \times A})_\ast ((t, \tilde{p})^\ast L \otimes (t, \tilde{q})^\ast L) \otimes (\epsilon, \tilde{p})^\ast L^\vee \otimes (e, \tilde{q})^\ast L^\vee$ is isomorphic to the Baer sum $\tilde{p}^\ast \mathcal{U}_A + B \tilde{q}^\ast \mathcal{U}_A$ of the extensions $\tilde{p}^\ast \mathcal{U}_A$ and $\tilde{q}^\ast \mathcal{U}_A$;

2. via the above identification, the isomorphism $\varphi$ induces an isomorphism $(\pi, \text{id}_{A \times A})_\ast (t, \text{id}_{A \times A})^\ast \varphi : \tilde{m}^\ast \mathcal{U}_A \iso \tilde{p}^\ast \mathcal{U}_A + B \tilde{q}^\ast \mathcal{U}_A$.

In particular, in the cohomology group $H^1(\tilde{A} \times \tilde{A}, \omega_A \otimes \mathcal{O}_{A \times \tilde{A}})$, one has the equality of cohomology classes $\tilde{m}^\ast \mathcal{U}_A = \tilde{p}^\ast \mathcal{U}_A + \tilde{q}^\ast \mathcal{U}_A$. By following [Ser59, VII, 15], this permits to define a vector extension of the abelian scheme $A$—it will be more explicit in Section 1.3.

\textbf{Proof.} (1) The proof is a consequence of the following general remark.

Let $X$ be a scheme. Let $M$ be a quasi-coherent $\mathcal{O}_X$-module. Let $p : Y \to X$ be the relative spectrum of the quasi-coherent $\mathcal{O}_X$-algebra $A := \mathcal{O}_X \oplus M$, with multiplication defined by $(f, m)(f', m') = (ff', fm' + f'm)$. Let $\epsilon : X \to Y$ be the section of $p$ induced by the homomorphism $A \to \mathcal{O}_X$, $(f, m) \mapsto f$. Let $F$, $F'$ be quasi-coherent $\mathcal{O}_Y$-modules. Let $F_0 := \epsilon^\ast F$ and $F'_0 := \epsilon^\ast F'$. Then, the natural homomorphism of $\mathcal{O}_Y$-modules $\lambda : p_\ast F \otimes p_\ast F' \to p_\ast (F \otimes F')$ sits in the following commutative diagram

\[
\begin{array}{ccc}
0 & \longrightarrow & M \otimes F_0 \otimes F'_0 \oplus M \otimes F_0 \otimes F'_0 \\
\downarrow & & \downarrow \lambda \\
0 & \longrightarrow & M \otimes F_0 \otimes F'_0 \\
\end{array}
\]

The homomorphism $\lambda$ identifies $p_\ast (F \otimes F')$ with the Baer sum of the extensions $p_\ast F \otimes F'_0$ and $p_\ast F' \otimes F_0$ of $F_0 \otimes F'_0$ by $M \otimes F_0 \otimes F'_0$.

(2) Clear.

\textbf{1.2.2. Isomorphism class.} In this section, we compute the isomorphism class of the universal extension of $\mathcal{O}_A$, that is, the global section $\Psi$ of $R^1 \hat{\alpha}_\ast \hat{\alpha}^\ast \omega_A$ defined as connecting homomorphism of the long exact sequence of $\mathcal{O}_S$-modules

\[
0 \longrightarrow \omega_A \longrightarrow \hat{\alpha}_\ast \mathcal{U}_A \to \mathcal{O}_S \to R^1 \hat{\alpha}_\ast \hat{\alpha}^\ast \omega_A \longrightarrow \cdots ,
\]

obtained by pushing-forward $0 \to \hat{\alpha}^\ast \omega_A \to \mathcal{U}_A \to \mathcal{O}_A \to 0$ along $\hat{\alpha}$. Via the identification $R^1 \hat{\alpha}_\ast \hat{\alpha}^\ast \omega_A = R^1 \hat{\alpha}_\ast \mathcal{O}_A \otimes \omega_A = \text{Hom}(\text{Lie} A, R^1 \hat{\alpha}_\ast \mathcal{O}_A)$, the homomorphism $\Psi$ can be seen as a homomorphism of $\mathcal{O}_S$-modules $\text{Lie} A \to R^1 \hat{\alpha}_\ast \mathcal{O}_A$.

\textbf{Notation 1.13.} For a $S$-scheme $\xi : X \to S$, let $\mathcal{O}_X[\mathcal{E}]$ be the $\mathcal{O}_X$-algebra of dual numbers over $\mathcal{O}_X$. Let $X[\mathcal{E}] = \text{Spec}_X \mathcal{O}_X[\mathcal{E}]$. Let $\text{pr}_X : X[\mathcal{E}] \to X$ be the structural morphism and $\xi[\mathcal{E}] = \xi \circ \text{pr}_X$. Let $j_X : X \to X[\mathcal{E}]$ be the section of $\xi$ defined by $\mathcal{E} = 0$. 

An isomorphism of $\mathcal{O}_S$-modules $\Phi: R^1\hat{\alpha}_*\mathcal{O}_A \to \text{Lie } A$ is defined as follows ([BLR90, 8.4, Theorem 1]). Identify $A$ with the neutral component of the Picard scheme of $A$. The Lie algebra of $A$ is the kernel of the natural map $R^1\hat{\alpha}_*(\mathcal{O}_A^\times) \to R^1\hat{\alpha}_*\mathcal{O}_A^\times$. The truncated exponential map $\exp: f \mapsto 1 + \varepsilon f$ gives rise to the following exact sequence of sheaves of abelian groups on $S$:

$$0 \to R^1\hat{\alpha}_*\mathcal{O}_A \xrightarrow{\exp} R^1\hat{\alpha}_*(\mathcal{O}_A^\times) \to R^1\hat{\alpha}_*\mathcal{O}_A^\times \to \cdots$$

This yields an isomorphism $\Phi: R^1\hat{\alpha}_*\mathcal{O}_A \to \text{Lie } A$.

**Proposition 1.14.** With the notation above,

$$\Psi = \Phi^{-1}.$$  

The proof goes by identifying $\Phi^{-1}$ and $\Psi$ with a third map $\Psi'$ constructed as follows. By definition, a global section $v$ of $\text{Lie } A$ is a morphism of $S$-schemes $f: S[\varepsilon] \to A$ such that $f \circ j_S = e$.

Let $L = (f, \text{id}_A)^*\mathcal{L}$. Define $\Psi'(v)$ as the homomorphism $\mathcal{O}_S \to R^1\hat{\alpha}_*\mathcal{O}_A$, using the push-forward the short exact sequence $0 \to \varepsilon \mathcal{O}_A \to \text{pr}_{A,*} L \to \mathcal{O}_A \to 0$ along $\hat{\alpha}$. (Note that the kernel of the canonical homomorphism $L \to j_A, j_A^* L$ is $\varepsilon j_A^* \mathcal{O}_A$.) Performing this construction with $S$ replaced by any of its open subsets leads to a homomorphism of $\mathcal{O}_S$-modules $\Psi'$: $\text{Lie } A \to R^1\hat{\alpha}_*\mathcal{O}_A$.

**Lemma 1.15.** The homomorphisms $\Phi$ and $\Psi'$ are inverse to each other.

**Proof.** Since the definition of $\Phi$ and $\Psi'$ are compatible with base change, one reduces to the case where $S$ is affine. The proof consists in computing explicitly the exponential map $\exp: H^1(A, \mathcal{O}_A) \to H^1(A[\varepsilon], \mathcal{O}_A^\times)$ on Cech cocycles.

Let $A = \bigcup^n_{i=1} A_i$ an affine cover of $A$. A cohomology class $c \in H^1(A, \mathcal{O}_A)$ is represented by a 1-cocycle $(f_{i,j})_{i,j}$ with $f_{i,j} \in H^0(A_{ij}, \mathcal{O}_{A_{ij}})$. Let $L$ be the line bundle on $A[\varepsilon]$ obtained by glueing $\mathcal{O}_{A_i[\varepsilon]}$ along the isomorphisms $\exp(f_{i,j}) = 1 + \varepsilon f_{i,j}$. By definition the isomorphism class of $L$ is $\Phi(c)$.

The push-forward $\text{pr}_{A,*} L$ is the glueing of the $\mathcal{O}_{A_i[\varepsilon]}$ along the isomorphisms $1 + \varepsilon f_{i,j}$. Then, the isomorphism class $\Psi'(\Phi(c)) \in H^1(A, \mathcal{O}_A)$ of the extension $\text{pr}_{A,*} L$ is represented by the 1-cocycle $(f_{i,j})_{i,j}$. This concludes the proof. □

**Proof of Proposition [1.14]** Since the constructions of $\Psi$ and $\Psi'$ are compatible with base change, it suffices to show that they induce the same homomorphism on global sections of Lie $A$.

Let $v$ be a global section of Lie $A$. Let $f: S[\varepsilon] \to A$ be the corresponding morphism of $S$-schemes. Consider the line bundle $L = (f, \text{id}_A)^*\mathcal{L}$ on $A[\varepsilon]$. The morphism $f$ writes uniquely as $f = \iota \circ \tilde{f}$ with $\tilde{f}: S[\varepsilon] \to A_1$. By adjunction, the equality $(\tilde{f}, \text{id}_A)^*(\iota, \text{id}_A)^*\mathcal{L} = L$ gives a homomorphism of $\mathcal{O}_{A_1 \times A}$-modules $(\iota, \text{id}_A)^*\mathcal{L} \to (\tilde{f}, \text{id}_A)^* L$. Pushing it forward along $(\pi, \text{id}_A)$ gives a commutative and exact diagram of $\mathcal{O}_A$-modules

$$
\begin{array}{cccc}
0 & \to & \hat{\alpha}^*\omega_A & \to & \mathcal{U}_A & \to & \mathcal{O}_A & \to & 0 \\
0 & \to & \mathcal{O}_A & \to & \text{pr}_{A,*}(f, \text{id}_A)^*\mathcal{L} & \to & \mathcal{O}_A & \to & 0,
\end{array}
$$

where $(\pi, \text{id}_A), (\iota, \text{id}_A)^*\mathcal{L}$ is identified with $\mathcal{U}_A$ as in Remark [1.10]. Note that the leftmost vertical is the evaluation of a differential form along the tangent vector $v$. Pushing-forward along $\hat{\alpha}$ the previous diagram yields a commutative and exact diagram of $\mathcal{O}_S$-modules:
The commutativity of the rightmost square implies $\Psi(v) = \Psi'(v)$, that is, what we wanted to prove. \hfill \Box

1.2.3. Universal property. Keep the notation introduced in paragraph 1.2.2. Let $E$ be a vector bundle on $S$. For an extension $0 \to \tilde{\alpha}^* E \to V \to O_A \to 0$ let $\gamma_V: O_S \to R^1\tilde{\alpha}_*\tilde{\alpha}^* E$ be the homomorphism of $O_S$-modules obtained as the first connecting homomorphism in the long exact sequence of cohomology

$$0 \to E \to \tilde{\alpha}_* V \to O_S \xrightarrow{\gamma_V} R^1\tilde{\alpha}_*\tilde{\alpha}^* E \to \cdots.$$ 

Let $\Phi^*: \omega_A \to (R^1\tilde{\alpha}_* O_A)^\vee$ be the homomorphism of $O_S$-modules dual to $\Phi$. Via the identification $R^1\tilde{\alpha}_*\tilde{\alpha}^* E = R^1\tilde{\alpha}_* O_A \otimes E = \mathcal{H}om((R^1\tilde{\alpha}_* O_A)^\vee, E)$, the global section $\gamma_V$ can be seen as a homomorphism of $O_S$-modules $(R^1\tilde{\alpha}_* O_A)^\vee \to E$. Set $\varphi_V := \gamma_V \circ \Phi^*: \omega_A \to E$. Let $\delta$ be the canonical splitting of $\check{\epsilon}^* U_A$.

**Proposition 1.16.** Let $E$ be a vector bundle on $S$. Let $0 \to \tilde{\alpha}^* E \to V \to O_A \to 0$ be an extension such that $\check{\epsilon}^* E$ admits a splitting $\sigma: E \oplus O_S \to \check{\epsilon}^* V$. Then, there is a unique homomorphism of $O_A$-modules $f_{V,\sigma}: U_A \to V$ such that the following diagram is commutative and exact

$$0 \to \tilde{\alpha}^* E \to U_A \xrightarrow{\tilde{\alpha}^* \varphi_V} O_A \to 0$$

and $\check{\epsilon}^* f_{V,\sigma} \circ \delta = \sigma$.

Before showing it, let us state and prove a consequence of Proposition 1.16.

**Lemma 1.17.** Let $E$ be a vector bundle on $S$. Let $\psi: \omega_A \to E$ be a homomorphism of $O_S$-modules. Let $W$ be the push-out of the universal extension $U_A$ of $O_A$ along the homomorphism $\tilde{\alpha}^* \psi$:

$$0 \xrightarrow{\tilde{\alpha}^* \psi} U_A \xrightarrow{\tilde{\alpha}^*} O_A \xrightarrow{} 0$$

Then $\gamma_W = \psi \circ \Phi^\vee$ and $\varphi_W = \psi$.

**Proof of Lemma 1.17.** Pushing forward along $\tilde{\alpha}$ the diagram in the statement yields the following commutative and exact diagram of $O_S$-modules:

$$0 \xrightarrow{\omega_A \to \tilde{\alpha}_* U_A \to O_S \xrightarrow{\Psi} R^1\tilde{\alpha}_*\tilde{\alpha}^* \omega_A \to \cdots}$$

$$0 \xrightarrow{E \to \tilde{\alpha}_* W \to O_S \xrightarrow{\gamma_W} R^1\tilde{\alpha}_*\tilde{\alpha}^* E \to \cdots}$$

Via the equalities of $O_S$-modules

$$R^1\tilde{\alpha}_*\tilde{\alpha}^* \omega_A = \mathcal{H}om((R^1\tilde{\alpha}_* O_A)^\vee, \omega_A), \quad R^1\tilde{\alpha}_*\tilde{\alpha}^* E = \mathcal{H}om((R^1\tilde{\alpha}_* O_A)^\vee, E),$$

the homomorphism $R^1\tilde{\alpha}_*\tilde{\alpha}^* \psi$ is nothing but composing with $\psi$. With an abuse of notation, in paragraph 1.2.2 the homomorphism $\operatorname{Lie} A \to R^1\tilde{\alpha}_* O_A$ corresponding...
to the the global section $\Psi$ of $R^1\hat{\alpha}_*\hat{\alpha}^*\omega_A$ is still denoted $\Psi$. To be consistent with this, the homomorphism $(R^1\hat{\alpha}_*\mathcal{O}_A)^\vee \to \omega_A$ corresponding to $\Psi$ is its dual $\Psi^\vee$.

Keeping track of identifications and abuses of notation, $\gamma_W = \psi \circ \Psi^\vee$. Then, by definition, $\varphi_W = \psi \circ \Psi^\vee \circ \Phi^\vee = \psi$ because $\Phi$ and $\Psi$ are inverse to each other (Proposition 1.14).

Proof of Proposition 1.16 (Uniqueness.) Let $f, f' : \mathcal{U}_A \to V$ be homomorphisms of $\mathcal{O}_A$-modules with properties as in the statement. They differ by a global section $t : \mathcal{O}_A \to \hat{\alpha}^*E$. Since $\hat{\alpha}^*\hat{\alpha}_*E = E$, the section $t$ is constant. Its pull-back along $\hat{e}$ vanishes because $\hat{e}^*f = \hat{e}^*f'$. Therefore $t = 0$ and $f = f'$.

(Existence.) Because of the uniqueness, we may assume that $S$ is affine. Under this hypothesis $H^0(S, R^1\hat{\alpha}_*\hat{\alpha}^*E) = H^1(A, \hat{\alpha}^*E)$. Let $W$ be the the push-out of the universal extension of $\mathcal{O}_A$ along $\hat{\alpha}^*\varphi_V$. In order to prove that the statement, it suffices to show that the extensions $V$ and $W$ are isomorphic, that is, that their isomorphism classes in $H^1(A, \hat{\alpha}^*E)$ are equal. According to Lemma 1.17 the isomorphism class $\varphi_W$ of $W$ is $\varphi_V$. This concludes the proof.

Corollary 1.18. Let $E$ be a vector bundle on $S$. The homomorphism of $H^0(S, \mathcal{O}_S)$-modules $H^1(A, \hat{\alpha}^*E) \to H^0(S, R^1\hat{\alpha}_*\hat{\alpha}^*E)$, associating to $0 \to \hat{\alpha}^*E \to V \to \mathcal{O}_A \to 0$ the homomorphism of $\mathcal{O}_S$-modules $\gamma_V : \mathcal{O}_S \to R^1\hat{\alpha}_*\hat{\alpha}^*E$, induces an isomorphism

$$
\begin{align*}
\left\{ \text{Isomorphism classes of extensions } V \\
\text{of } \mathcal{O}_A \text{ by } \hat{\alpha}^*E \text{ such that } \hat{e}^*V \text{ splits} \right\} \\
\sim \to H^0(S, R^1\hat{\alpha}_*\hat{\alpha}^*E).
\end{align*}
$$

Proof. (Injective.) Let $V$ be a extension of $\mathcal{O}_A$ by $\hat{\alpha}^*E$ such that $\hat{e}^*V$ splits. Suppose $\gamma_V = 0$. Then $\varphi_V = 0$, and by Proposition 1.16 the extension $V$ splits.

(Surjective.) A global section of $R^1\hat{\alpha}_*\hat{\alpha}^*E$ can be seen as a homomorphism of $\mathcal{O}_S$-modules $\omega_A \to E$. Via this identification, let $\varphi : \omega_A \to E$ be a homomorphism of $\mathcal{O}_S$-modules. Let $W$ be the push-out of $\mathcal{U}_A$ along $\hat{\alpha}^*\varphi$. Let $c \in H^1(A, \hat{\alpha}^*E)$ be the isomorphism class of the extension $W$. By Lemma 1.17 the image of $c$ is $\varphi$. To conclude the argument, it suffices to remark that the canonical splitting of $\hat{e}^*\mathcal{U}_A$ induces a splitting of $\hat{e}^*W$. □

1.3. Moduli space of connections.

Definition 1.19. The $S$-scheme $\hat{A}^3 = \mathbb{P}(\mathcal{U}_A) \smallsetminus \mathbb{P}(\hat{\alpha}^*\omega_A)$ is called the universal vector extension of $\hat{A}$.

Remark 1.20. Let $X$ be a scheme. Let $0 \to W \to V \to \mathcal{O}_X \to 0$ be a short exact sequence of vector bundles on $X$. Let $\mathbb{P}(V)$ (resp. $\mathbb{P}(W)$) denote the projective bundle associated with $V$ (resp. $W$). Here, the dual of Grothendieck’s convention is adopted; namely, $\mathbb{P}(V)$ parametrizes line bundles $L \subseteq V$ such that $V/L$ is locally free.

The affine $X$-scheme $Y = (\mathbb{P}(V) \smallsetminus \mathbb{P}(W)$ represents the functor associating to a $X$-scheme $f : X' \to X$ the set of splittings of the short exact sequence of $\mathcal{O}_X$-modules $0 \to f^*W \to f^*V \to \mathcal{O}_X \to 0$.

Proposition 1.21. The universal vector extension $\hat{A}^3$ represents the functor associating to a $S$-scheme $S'$ the set of isomorphism classes of triples $(L, \varphi, \nabla)$ made of a homogeneous line bundle $(L, \varphi)$ on $A \times_S S'$ and a connection $\nabla$ on $L$.

Proof. Since $\hat{A}$ represents the functor associating to a $S$-scheme $S'$ the set of isomorphism classes of homogeneous line bundles on $A \times_S S'$, it suffices to study the fibers of the morphism $\hat{A}^3 \to \hat{A}$. Let $(L, \varphi)$ be a homogeneous line bundle on $A \times_S S'$. According to Remark 1.20 $S'$-valued points of $\hat{A}^3$ mapping to $(L, \varphi)$ are by definition splittings of the Atiyah extension of $L$, that is, connections on $L$. □
The tensor product endows $\mathbb{A}^2$ with a structure of group $S$-scheme. The natural projection $\mathbb{A}^2 \to \mathbb{A}, (L, \varphi, \nabla) \mapsto (L, \varphi)$ is a faithfully flat morphism of group $S$-schemes whose kernel is made of connections on the trivial bundle. A connection on the trivial line bundle is of the form $d + \omega$ for a global differential form on $\mathbb{A}$. Via the isomorphism $\alpha_\ast \Omega^1_\mathbb{A} \cong \omega_\mathbb{A}$, this yields to a short exact of $S$-group schemes

$$0 \to \nabla(\omega_\mathbb{A}) \to \mathbb{A}^2 \to \mathbb{A} \to 0.$$  

2. Finite vector bundles on an abelian scheme

Let $S$ be a scheme. Let $\alpha : A \to S$ be an abelian scheme. Let $e : S \to A$ be the zero section.

2.1. Definitions and basic properties.

**Definition 2.1.** A vector bundle $V$ on $A$ is said to be

1. **unipotent** if there is an increasing filtration $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d = V$ of $V$ by sub-$\mathcal{O}_A$-modules such that, for $i = 1, \ldots, d$, the $\mathcal{O}_A$-module $V_i/V_{i-1}$ is isomorphic to $\alpha^\ast E_i$ for some vector bundle $E_i$ on $S$ of positive rank. Note that, for such a filtration, the $\mathcal{O}_A$-modules $V_i$ are automatically locally free.

2. **finite** if there is a positive integer $m \geq 1$ such that $[m]^\ast V$ is isomorphic to the pull-back of a vector bundle on $S$.

**Remark 2.2.** A vector bundle $W$ on $A$ is the pull-back of a vector bundle on $S$ if and only if the following two conditions are satisfied:

1. the restriction on the zero section $\alpha_\ast W \to e^\ast W$ is an isomorphism;
2. the homomorphism of $\mathcal{O}_A$-modules $\alpha^\ast \alpha_\ast W \to W$ is an isomorphism.

The direct sum, tensor product, internal homs, symmetric and exterior powers of unipotent vector bundles (resp. finite) are unipotent (finite).

**Lemma 2.3.** Let $E, E'$ be vector bundles on $S$. Let $V$ be an extension of $\alpha^\ast E$ by $\alpha^\ast E'$. Assume there is a positive integer $m \geq 1$ which kills $S$.

Then, the vector bundle $[m]^\ast V$ is the pull-back of an extension of $E$ by $E'$. In particular, if $S$ is affine, then $[m]^\ast V$ is the trivial extension.

**Proof.** Since the conditions in Remark 2.2 are local on $S$, one may assume $S$ affine. The pull-back by $[m]$ acts on

$$H^1(A, \mathcal{H}om(\alpha^\ast E, \alpha^\ast E')) = H^0(S, \mathcal{R}^1\alpha_\ast \mathcal{H}om(\alpha^\ast E, \alpha^\ast E'))$$

as the multiplication by $m$. Since $m = 0$ on $S$, $[m]^\ast V$ is the trivial extension of $\alpha^\ast E$ by $\alpha^\ast E'$. In particular, $[m]^\ast V$ satisfies conditions (1) and (2) in Remark 2.2. □

**Lemma 2.4.** Let $V$ be a unipotent vector bundle on $A$ of rank $r$. Assume there is a positive integer $m \geq 1$ which kills $S$. Then $[m]^\ast V$ is isomorphic to the pull-back of a vector bundle on $S$.

**Proof.** Let $0 = V_0 \subseteq V_1 \subseteq \cdots \subseteq V_d = V$ be a filtration satisfying the properties in the definition of a unipotent vector bundle. Then, the vector bundles $V_i$ are themselves unipotent. By induction on the rank of $V$, one reduces to the case where $V$ is an extension of the pull-back of vector bundles on $S$, concludes by applying Lemma 2.3. □

**Proposition 2.5.** Let $0 \to V' \to V \to V'' \to 0$ be an exact sequence of vector bundles on $A$. Suppose $V'$ and $V''$ are finite. Assume there is a positive integer $m \geq 1$ which kills $S$. Then, $V$ is finite.

\footnote{Namely, the sequence of the associated (fpf) sheaves is short exact.}
Remark 2.9. Let $n \geq 1$ be a positive integer such that $[n]^* V'$ and $[n]^* V''$ are isomorphic to the pull-back of vector bundles on $S$. Then, the vector bundle $[n]^* V$ is unipotent. One concludes by applying Lemma 2.4. □

2.2. Finite vector bundles as representations.

2.2.1. Universal cover and Tate group scheme.

Definition 2.6. The universal cover of $A$ is the projective limit of the projective system $([m]: A \to A)_{m \in \mathbb{N} \setminus \{0\}}$, the partial order on $\mathbb{N} \setminus \{0\}$ being divisibility. The Tate group scheme $T_A$ is the projective limit of the finite flat $S$-group schemes $A[m]$, for $m \in \mathbb{N} \setminus \{0\}$, the transition maps $A[n] \to A[m]$ being multiplication by $\frac{m}{n}$ whenever $m$ divides $n$.

For a prime number $p$, the $p$-adic universal cover of $A$ is the projective limit of the projective system $([p^n]: A \to A)_{n \in \mathbb{N}}$ with the usual order on $\mathbb{N}$. The $p$-adic Tate group scheme $T_{pA}$ of $A$ is the limit of the projective system of $S$-group schemes $(A[p^n], [p]: A[p^{n+1}] \to A[p^n])_{n \in \mathbb{N}}$.

The existence of such projective limits is granted by the finiteness (hence affineness) of the transition maps $[m]: A \to A$; see [Stacks, Lemma 01YX]. The construction of the Tate group scheme of an abelian scheme is compatible with base change ([Stacks, Lemma 01YZ]). Moreover, the canonical projections $T_A \to T_{pA}$ induce an isomorphism of group $S$-schemes,

$$T_A = \prod_{p \text{ prime}} T_{pA}.$$  

Proposition 2.7. Let $I$ be a directed set. Let $(T_i, \tau_{ji}: T_j \to T_i)$ be an inverse system of affine $S$-schemes. Let $T = \text{proj lim } T_i$ and, for $i \in I$, $\text{pr}_i: T \to T_i$, the canonical projection.

Suppose $S$ quasi-compact. Let $X$ be a quasi-compact $S$-scheme of finite type. Let $f: T \to X$ be a morphism of $S$-schemes. Then, there is $i \in I$ and a morphism of $S$-schemes $f_i: T_i \to X$ such that $f = f_i \circ \text{pr}_i$.

Proof. See [Stacks, Proposition 01ZC]. □

Definition 2.8. Let $G$ be a group $S$-scheme. A representation of $G$ is the datum of vector bundle $E$ on $S$ and a morphism of group $S$-schemes $\rho: G \to \text{GL}(E)$.

A representation $\rho: G \to \text{GL}(E)$ is said to be unipotent if there is an increasing filtration $0 = E_0 \subseteq E_1 \subseteq \cdots \subseteq E_d = E$ by sub-$O_S$-modules of $E$ such that, for $i = 1, \ldots, d$,

- $E_i$ and $E_i/E_{i-1}$ are locally free;
- the vector bundle $E_i$ is stable under the action of $G$;
- the induced representation $\rho: G \to \text{GL}(E_i/E_{i-1})$ is trivial.

2.2.2. Representation associated with a finite vector bundle. Let $m \in \mathbb{N}$. The multiplication-by-$m$ map $[m]: A \to A$ makes $A$ a fppf principal $A[m]$-bundle over $A$. Given a quasi-coherent $O_A$-module $V$, its pull-back $[m]^* V$ is naturally $A[m]$-linearized ([FGT 05, Theorem 4.46]). Since $A$ is flat and proper over $S$, the quasi-coherent $O_S$-module $\alpha_* [m]^* V$ inherits a $A[m]$-linearization ([SGA3, Exp. I, 6.6]).

Remark 2.9. As $A[m]$ acts trivially on $S$, the $A[m]$-linearization corresponds to a linear action of $A[m]$ on $\alpha_* [m]^* V$ which can be described as follows.

Let $x \in A(S)$ be an $m$-torsion point. Let $tr_x: A \to A$ be the translation by $x$. Let $\varphi_x: [m]^* V \to tr_x, tr_x^*[m]^* V = tr_x[m]^* V$ be the homomorphism of $O_S$-modules given by adjunction: it is an isomorphism, because $tr_x$ is one. Pushing-forward $\varphi_x$ along $\alpha$ yields an isomorphism of $O_S$-modules

$$\alpha_* \varphi_x: \alpha_* [m]^* V \to \alpha_* tr_x[m]^* V = \alpha_* [m]^* V,$$
through which acts the $S$-point $x$. (By base-change, one obtains a similar description for $m$-torsion points with values on a flat $S$-scheme.)

Let $V$ be a finite vector bundle. By definition, there is a positive integer $m \geq 1$ such that $[m]^\ast V$ is the pull-back of a vector bundle on $S$. It follows that the restriction to the zero section $\alpha_\ast [m]^\ast V \to e^\ast V$ is an isomorphism. Via this isomorphism, the $A[m]$-linearization of $\alpha_\ast [m]^\ast V$ endows $e^\ast V$ with a $A[m]$-linearization, which corresponds to a representation $\rho_{V,m} : A[m] \to \text{GL}(e^\ast V)$.

**Definition 2.10.** Let $\rho_V$ denote the representation $\rho_{V,m} \circ \text{pr}_m : TA \to \text{GL}(e^\ast V)$.

The construction of $\rho_V$ is functorial on $V$, meaning that, given finite vector bundles $V, W$ on $A$ and a homomorphism of $\mathcal{O}_A$-modules $\varphi : V \to W$, the restriction of $\varphi$ to the zero section $e^\ast \varphi : e^\ast V \to e^\ast W$ is $TA$-equivariant.

**Proposition 2.11.** Suppose $S$ quasi-compact. Then, the functor $V \mapsto \rho_V$ defined above induces an equivalence between the category of finite vector bundles (whose arrows are homomorphisms of $\mathcal{O}_A$-modules) and that of representation of $TA$ (whose arrows are $TA$-equivariant homomorphisms of $\mathcal{O}_S$-modules). Furthermore,

1. the functor $\rho$ is compatible with direct sums, tensor products, internal homs, symmetric and exterior powers;
2. for a finite vector bundle $V$ and a positive integer $m \geq 1$ such that $[m]^\ast V$ is isomorphic to the pull-back of a vector bundle on $S$, the representation $\rho_V$ factors through $A[m]$;
3. for a finite vector bundle $V$ the restriction map $\alpha_\ast V \to (e^\ast V)^{TA}$ is an isomorphism;
4. for a morphism of schemes $f : S' \to S$ and a finite vector bundle $V$ on $A$, the morphism of $S'$-group schemes

\[ \rho_V \times \text{id}_{S'} : (TA) \times_S S' = T(A \times_S S') \to \text{GL}(e^\ast V) \times_S S' = \text{GL}(f^\ast e^\ast V), \]

is the representation of the Tate group scheme of $A' := A \times_S S'$ associated with the finite vector bundle $\text{pr}_A^\ast V$ on $A'$, where $\text{pr}_A : A' \to A$ is the first projection.

Suppose moreover that there is a positive integer $m \geq 1$ killing $S$. Then,

5. the functor $V \mapsto \rho_V$ induces an equivalence between the category of unipotent vector bundles (whose arrows are homomorphisms of $\mathcal{O}_A$-modules) and that of unipotent representations of $TA$ (whose arrows are $TA$-equivariant homomorphisms of $\mathcal{O}_S$-modules);
6. if $V$ is a unipotent vector bundle of rank $r$, then $\rho_V$ factors through $A[m^r]$;
7. if $V$ is an extension of $\alpha^\ast E''$ by $\alpha^\ast E'$ for some vector bundles $E', E''$ on $S$, then $\rho_V$ factors through $A[m]$.

**Proof.** Since $S$ is a quasi-compact, a representation of the Tate module comes from a representation of $A[m]$, for some $m \geq 1$ (Proposition 2.7). The statement is then a reformulation of faithfully flat descent (cf. [FGI+05, Theorem 4.46]), and Lemmas 2.3 and 2.4. \qed

2.3. Explicit computation of the representations.

2.3.1. **Reminder on Cartier duality and the universal vector hull.** Let $G$ be a finite flat commutative $S$-group scheme. The Cartier dual $G^D$ of $G$ is the $S$-scheme representing the functor associating to a $S$-scheme $S'$ the group $\text{Hom}_{S'}(G_{S'}, \mathbb{G}_m)$ of group $S'$-schemes morphisms $G_{S'} \to \mathbb{G}_m_{S'}$.

Let $m \in \mathbb{N}$. Weil’s pairing is the morphism of $S$-group schemes $\tilde{A}[m] \to A[m]^D$ defined as follows (cf. [Oda69, Section 1]). Let $S'$ be a $S$-scheme. Let $L$ be a $m$-torsion homogeneous line bundle on $A' = A \times_S S'$. Let $x \in A(S')$ be a $m$-torsion
point. Pick an isomorphism $\lambda: [m]^*L \to \mathcal{O}_{A'}$. Let $\text{tr}_x: A' \to A'$ be the translation by $x$. The automorphism of $\mathcal{O}_{A'}$, 
\[ \mathcal{O}_{A'} \xrightarrow{\lambda^*} [m]^*L = \text{tr}_x^*[m]^*L \xrightarrow{\text{tr}_x^* \lambda} \mathcal{O}_{A'} = \mathcal{O}_{A'}, \]

is given by an invertible element $u \in H^0(A', \mathcal{O}_{A'}) = H^0(S', \mathcal{O}_S)$. The unit $u$ does not depend on the chosen isomorphism $\lambda$ and one defines 
\[ \langle x, L \rangle_{A[m]} := u \in \mathbb{G}_m(S'). \]

The so-defined morphism $\hat{A}[m] \to A[m]^D$ is an isomorphism of $S$-group schemes (loc.cit., Theorem 1.1).

**Lemma 2.12.** With the notation introduced above, 
\[ \langle -, L \rangle_{A[m]} = \rho_{L,m}. \]

**Proof.** This follows from the explicit description of the representation $\rho_{L,m}$ given in Remark 2.7. \[ \square \]

Let $\chi: G \to \mathbb{G}_m$ be a morphism of group $S$-schemes. According to Notation 2.4, the morphism $\chi \circ \iota_G$ is by definition an automorphism of $\mathcal{O}_{G_1}$. Its push-forward along $\pi_G$ is an automorphism of $\pi_G \mathcal{O}_{G_1}$, still denoted $\chi \circ \iota_G$. It fits in the commutative diagram of $\mathcal{O}_S$-modules

\[
\begin{array}{ccc}
0 & \longrightarrow & \omega_G & \longrightarrow & \pi_G \mathcal{O}_{G_1} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0 \\
& & \downarrow & & \downarrow \chi \circ \iota_G & & \downarrow & & \\
0 & \longrightarrow & \omega_G & \longrightarrow & \pi_G \mathcal{O}_{G_1} & \longrightarrow & \mathcal{O}_S & \longrightarrow & 0.
\end{array}
\]

The difference $\chi \circ \iota_G - \text{id}$ is a global section $\theta_G^\chi(\chi)$ of $\omega$. Down-to-earth, $\theta_G^\chi(\chi)$ is the pull-back $\chi^* \frac{dt}{t}$ of the invariant differential $\frac{dt}{t}$ on $\mathbb{G}_m$ (cf. [CL99, 4.2]).

**Definition 2.13.** By replacing $S$ with an arbitrary $S$-scheme, the previous construction defines a morphism of $S$-group schemes $\theta_G\chi: G^D \to \mathcal{V}(\omega_G)$ called the universal vector hull of $G^D$.

The universal vector hull owes its name to the following property:

**Proposition 2.14 ([MM74, Proposition 1.4]).** Let $E$ be a vector bundle on $S$. With the notation introduced above, let $f: G^D \to \mathcal{V}(E)$ be a morphism of $S$-group schemes.

Then, there is a unique homomorphism of $\mathcal{O}_S$-modules $\varphi: \omega_G \to E$ such that $f = \mathcal{V}(\varphi) \circ \theta_G\chi$, where $\mathcal{V}(\varphi): \mathcal{V}(\omega_G) \to \mathcal{V}(E)$ is the morphism induced by $\varphi$.

**Proposition 2.15.** Let $A$ be an abelian scheme over $S$. Let $E$ be a vector bundle on $S$. Let $f: TA \to \mathcal{V}(E)$ be a morphism of group $S$-schemes. If $m$ kills $S$, then there is a unique morphism of group schemes $f_m: A[m] \to \mathcal{V}(E)$ such that $f = f_m \circ \text{pr}_m$.

**Proof.** According to Proposition 2.7, there are $n \in \mathbb{N}$ and a morphism of group $S$-schemes $f_n: A[n] \to \mathcal{V}(E)$ such that $f = f_n \circ \text{pr}_m$. By the universal property of the universal vector hull, it suffices to treat the case $f_n = \theta_A[n]$. Moreover, we may assume that $n$ is a multiple of $m$. Let $r = \frac{n}{m}$. The module of invariant differentials $\omega_A[m]$ and $\omega_A[mr]$ are equal to $\omega_A$, and $\theta_A[n] = \theta_A[m] \circ [r]$. \[ \square \]

2.3.2. Extension of the trivial bundle. Let $m \in \mathbb{N}$ be an integer killing $S$. Let $U_A$ be the universal extension of $\mathcal{O}_A$. Let $\rho_{U_A,m}: A[m] \to \text{GL}(e^m U_A)$ the associated representation. Let $\sigma: \omega_A \oplus \mathcal{O}_S \to e^m U_A$ be a splitting. Via $\sigma$, the representation $\rho_{U_A,m}$ has matrix

\[
\begin{pmatrix}
\text{id}_A & \theta_{U_A} \\
0 & 1
\end{pmatrix}.
\]
for a morphism of group $S$-schemes $\theta_{U_A}: A[m] \rightarrow \mathcal{V}(\omega_A)$. Note that $\theta_{U_A}$ does not depend on the chosen splitting $\sigma$.

**Proposition 2.16.** With the notation introduced above,

$$\theta_{U_A} = \theta_{A[m]}.$$  

**Proof.** Let $\mathcal{L}$ be the Poincaré bundle $A \times \check{A}$. According to Remark 1.10,  

$$\mathcal{U}_A = (\text{id}_A, \pi_A)_* (\text{id}_A, \tau_A)^* \mathcal{L} \otimes (\text{id}, \check{e})^* \mathcal{L}^\vee.$$  

By definition of the Poincaré bundle, $(\text{id}, \check{e})^* \mathcal{L} = \mathcal{O}_A$ and $A[m]$ acts trivially on it. Therefore, in what follows, there is no harm in identifying $\mathcal{U}_A$ with the vector bundle $(\text{id}_A, \pi_A)_* (\text{id}_A, \tau_A)^* \mathcal{L}$. Let $\mathcal{L}_m := \mathcal{L}_{A \times A[m]}$. Since $m = 0$, the closed immersion $\check{A}[m]_1 \rightarrow \check{A}_1$ is an isomorphism. Therefore, 

$$\mathcal{U}_A = (\text{id}_A, \pi_{A[m]})_* (\text{id}_A, \tau_{A[m]})^* \mathcal{L}_m.$$  

By seeing $A \times \check{A}[m]$ as an abelian scheme over $\check{A}[m]$, the line bundle $\mathcal{L}_m$ is $m$-torsion. Note that the corresponding morphism of group $\check{A}[m]$-schemes 

$$\rho_{\mathcal{U}_m}: A[m] \times \check{A}[m] \rightarrow \text{GL}(e, \text{id}_{A[m]})^* \mathcal{L}_m = \mathbb{G}_m \times \check{A}[m]$$  

composed with $\text{pr}_{\mathbb{G}_m}$ is Weil’s pairing $A[m] \times \check{A}[m] \rightarrow \mathbb{G}_m$. Similarly, the representation corresponding to the $m$-torsion line bundle $(\text{id}_A, \tau_{A[m]})^* \mathcal{L}_m$ on the abelian scheme $A \times \check{A}[m]_1 \rightarrow \check{A}[m]_1$ is $\rho_{\mathcal{U}_m} \circ (\text{id}_A, \tau_{A[m]})$. The latter can be seen as an automorphism $\varphi$ of the line bundle 

$$(\alpha_m \circ e, \tau_{A[m]})^* \mathcal{L}_m = \mathcal{O}_{A[m] \times \check{A}[m]_1},$$  

where $\alpha_m: A[m] \rightarrow S$ is the structural morphism.

Pushing forward $\varphi$ along $(\text{id}_{A[m]}, \pi_{A[m]})$ yields an automorphism $\psi$ of the $\mathcal{O}_{A[m]}$-module $(\text{id}_{A[m]}, \pi_{A[m]})_* (\alpha_m \circ e, \tau_{A[m]})^* \mathcal{L}_m$ fitting in the following commutative and exact diagram

$$
\begin{array}{cccccc}
0 & \rightarrow & \alpha_m^* \omega_A & \rightarrow & (\text{id}_{A[m]}, \pi_{A[m]})_* (\alpha_m \circ e, \tau_{A[m]})^* \mathcal{L}_m & \rightarrow & \mathcal{O}_{A[m]} & \rightarrow & 0 \\
0 & \rightarrow & \alpha_m^* \omega_A & \rightarrow & (\text{id}_{A[m]}, \pi_{A[m]})_* (\alpha_m \circ e, \tau_{A[m]})^* \mathcal{L}_m & \rightarrow & \mathcal{O}_{A[m]} & \rightarrow & 0.
\end{array}
$$

The universal vector hull of $A[m]$ is by its very definition given by the global section $\theta_{A[m]} = \psi - \text{id}$. On the other hand, via the equality 

$$(\text{id}_{A[m]}, \pi_{A[m]})_* (\alpha_m \circ e, \tau_{A[m]})^* \mathcal{L}_m = \alpha_m^* \mathcal{O}_A,$$  

one sees that $\psi$ is the $A[m]$-linearization of $\alpha_m^* \mathcal{O}_A$ defining the representation $\rho_{\mathcal{U}_A}$. This concludes the proof. \(\square\)

Let $E$ be a vector bundle on $S$. Let $\varphi: \omega_A \rightarrow E$ be a homomorphism of $\mathcal{O}_A$-modules. Let $V$ be the push-out of $\mathcal{U}_A$ along $\varphi$. Let $\rho: A[m] \rightarrow \text{GL}(e^* V)$ denote the representation associated with the unipotent vector bundle $V$. Via a splitting $E \oplus \mathcal{O}_S \simeq e^* V$, the representation $\rho$ has matrix 

$$
\begin{pmatrix}
\text{id}_E & \theta_V \\
0 & 1
\end{pmatrix},
$$

where $\theta_V = \begin{pmatrix} 0 & \partial \varphi \\ 1 & 0 \end{pmatrix}$.

\(^2\)Indeed, any other splitting $\sigma'$ is of the form $(x, \lambda) \mapsto \sigma(x, \lambda) + \lambda v$ for some global section $v: \mathcal{O}_S \rightarrow \omega_A$. By using $\sigma'$ instead of $\sigma$, one ends up with the re representation $\rho_{\mathcal{U}_A,m}$ has matrix 

$$
\begin{pmatrix}
\text{id}_A & v \\
v & 0 \\
0 & 1
\end{pmatrix}
\begin{pmatrix}
\text{id}_A & \theta_{U_A} \\
\text{id}_A & -v \\
0 & 1
\end{pmatrix}
= 
\begin{pmatrix}
\text{id}_A & \theta_{U_A} \\
0 & 1
\end{pmatrix}.$$
for a morphism of group $S$-schemes $\theta_V : A[m] \to V(E)$. Note that $\theta_V$ does not depend on the chosen splitting.

**Corollary 2.17.** With the notation introduced above,

$$\theta_V = \mathbb{V}(\varphi) \circ \theta_{A[m]}.$$  

**Proof.** One reduces to the case $E = \omega_A$ and $\varphi = \text{id}$. Then the result is Proposition 2.16. □

2.3.3. **A geometric point of view.** Proposition 2.16 admits a more geometric description that will be useful later. Let $m \geq 1$ be a positive integer killing $S$. Since multiplication by $m$ on $\omega_A$ is $0$, the morphism $A[m] : A \to A$ factors through a morphism of group $S$-schemes $\sigma : A \to A$ fitting in the following commutative and exact diagram:

$$
\begin{array}{c}
0 & \longrightarrow & A[m] & \longrightarrow & A & \longrightarrow & A[m] & \longrightarrow & 0 \\
& & \downarrow{\sigma}_{|A[m]} & & \downarrow{\sigma} & & \downarrow{} & & \\
0 & \longrightarrow & \mathbb{V}(\omega_A) & \longrightarrow & A^\natural & \longrightarrow & A & \longrightarrow & 0 \\
\end{array}
$$

where $\pi : A^\natural \to A$ is the structural morphism.

**Lemma 2.18** (**cf.** [MM74 2.6.2]). With the notation above, $\sigma_{|A[m]} = \theta_{A[m]}$.

**Proof.** A morphism of $S$-schemes $\tau : A \to A$ such that $\pi \circ \tau = [m]$ corresponds to a splitting $\alpha^* \omega_A \oplus O_A \simeq [m]^* U_A$, still denoted $\tau$. Since two such splittings differ by a global section of $\omega_A$, a splitting $\tau : \alpha^* \omega_A \oplus O_A \simeq [m]^* U_A$ is uniquely determined by its restriction to the zero section $e^* \tau$.

The morphism $\sigma$ is a group morphism, thus sends the neutral element of $A$ to that of $A^\natural$. By seeing $A^\natural$ as the moduli space of rank 1 connections on $A$, the neutral element of $A^\natural$ is the canonical derivation on the trivial line bundle on $A$. By definition, this corresponds to the canonical splitting $\delta : \omega_A \oplus O_S \to e^* U_A$ (Definition 1.11). In particular, the splitting $\alpha^* \omega_A \oplus O_A \simeq [m]^* U_A$ corresponding to the morphism $\sigma$ sends $(0,1)$ to $\delta(0,1)$, and it is unique doing so.

Identify $e^* U_A$ with $\omega_A \oplus O_S$ via $e^* \sigma$. Proposition 2.10 states that $A[m]$ acts on $\omega_A \oplus O_S$ via the matrix

$$
\begin{pmatrix}
\text{id}_{\omega_A} & \theta_{A[m]} \\
0 & 1
\end{pmatrix}.
$$

The restriction $\sigma$ to $A[m]$ is the action of $A[m]$ on the section $(0,1)$, that is $(\theta_{A[m]}, 1)$, which concludes the proof. □

2.4. **Global sections on vector extensions.** Let $S$ be a scheme. Let $A$ be an abelian scheme over $S$. Let $E$ be a vector bundle on $S$. Let $\varphi : \omega_A \to E$ be a homomorphism of $O_S$-modules. Let $V$ be the push-out of $U_A$ along $\varphi$. Then, one has the following commutative diagram of $O_A$-modules
Let $X = \mathbb{P}(V)$, $D = \mathbb{P}(\alpha^*E)$ and $Y = X \setminus D$. Let $\pi: X \to A$, $\eta: Y \to S$ be the structural morphisms. Let $s: \mathcal{O}_A \to V^\vee$ the section obtained by duality from the surjection $V \to \mathcal{O}_A$. Then $Y$ is identified with the closed subscheme of $\mathbb{V}(V)$ given by the equation $s = 1$. Let $C = \text{Coker}(\varphi)^{\vee}$. Via the isomorphism $\alpha^* \text{Coker}(\varphi) \simeq V/\mathcal{U}_A$, one obtains a morphism of $S$-schemes

$$\text{pr}_{\mathbb{V}(C)}: Y \to \mathbb{V}(V) \to \mathbb{V}(\alpha^*C) = \mathbb{V}(C) \times A \xrightarrow{pr_2} \mathbb{V}(C).$$

If $S$ is killed some $m \in \mathbb{N} \setminus \{0\}$, let $\rho: A[m] \to \text{GL}(e^*V)$ be the representation associated with the unipotent vector bundle $V$. By construction, $e^*V$ admits a splitting $\sigma: E \oplus \mathcal{O}_S \to e^*V$. Via the splitting $\sigma$, the representation $\rho$ has matrix

$$(\text{id}_E \quad \theta) \quad 0 \quad 1,$$

for a morphism of $S$-group schemes $\theta: A[m] \to \mathbb{V}(E)$. The group $S$-scheme $A[m]$ acts on $\mathbb{P}(e^*V) \setminus \mathbb{P}(E) \simeq A \times \mathbb{V}(E)$ via the representation $\rho$. The action of $A[m]$ on $\mathbb{V}(E)$ is the translation by $\theta$: for a $S$-scheme $S'$, and $S'$-valued points $a \in A[m](S')$, $x \in \mathbb{V}(E)(S')$, the action is given by $(a,x) \mapsto x + \theta(a)$.

Let $(\text{Sym } E^\vee)^A[m]$ be subsheaf of $\mathcal{O}_S$-algebras of $\text{Sym } E^\vee$ made of functions on $\mathbb{V}(E)$ invariant under $A[m]$.

**Proposition 2.19.** With the notation introduced above,

1. if $S$ is flat over $\mathbb{Z}$, then composing a function on $\mathbb{V}(C)$ with $\text{pr}_{\mathbb{V}(C)}$ induces an isomorphism

$$\text{Sym } C^\vee \xrightarrow{\sim} \eta_* \mathcal{O}_Y;$$

2. if some integer $m \geq 1$ kills $S$, then the restriction map $\eta_* \mathcal{O}_Y \to \text{Sym } E^\vee$ induces an isomorphism

$$\eta_* \mathcal{O}_Y \xrightarrow{\sim} (\text{Sym } E^\vee)^A[m];$$

3. if $L$ is a line bundle on $A$ such that $\alpha_* L = 0$, then $\eta_* (\pi^* L_{|Y}) = 0$.

**Proof.** The three statements are local on $S$, therefore we may assume $S$ affine. Let $j$ denote the open immersion of $Y$ in $X$.

For $d \in \mathbb{N}$ let $\mathcal{O}_V(d)$ (resp. $\mathcal{O}_{\alpha^*E}(d)$) be the $d$-th tensor power of the tautological bundle on $\mathbb{P}(V)$ (resp. $\mathbb{P}(\alpha^*E)$). Let $F$ be a vector bundle on $A$. Since $X$ is quasi-compact and (quasi-)separated, the natural map

$$\varprojlim_{d \in \mathbb{N}} \pi^* F \otimes \mathcal{O}_V(d) \longrightarrow j_* j^* \pi^* F,$$

is an isomorphism (the transition maps $\pi^* F \otimes \mathcal{O}_V(d) \to \pi^* F \otimes \mathcal{O}_V(d + 1)$ in the direct limit are the tensor products with the canonical section $s: \mathcal{O}_X \to \mathcal{O}_V(1)$); see [Stacks][Lemma 009F].
For each integer

\[ \text{Proof of the Claim.} \]

Argue by induction on

\[ F \]

The

\[ \text{Sym} \]

0

where the lowest rightmost arrow is

This is equivalent to say that, for each

\[ d \]

extension.

\[ \text{Sym} \]

homomorphism

\[ \alpha \]

homomorphism

\[ \alpha \]

commutative and exact:

Since the tensor product is right exact, the natural homomorphism of

\[ \text{Sym} \]

is surjective (here ‘\text{Sym}’ is abbreviated by ‘S’). Note that \( \text{Im} \) (\( \alpha \) \( S^{d-1}V \rightarrow S^{d-1}E \)) is by definition \( F_{d-1} \). By the inductive hypothesis, \( F_{d-1} = \text{Sym}^{d-1}C \) and \( F_d \) is the image of the natural map \( C^V \otimes \text{Sym}^{d-1}C \rightarrow \text{Sym}^{d}E \). Since the image of

\[ C^V \otimes \text{Sym}^{d-1}C \]

is \( \text{Sym}^{d}C \), this concludes the proof of the Claim. ✷
According to the Claim, $\operatorname{inj} \operatorname{lim}_{d \in \mathbb{N}} \alpha_* \operatorname{Sym}^d V^\vee = \operatorname{Sym} C^\vee$ which concludes the proof of (1).

(2) By applying Proposition [2.11](3), for each $d \in \mathbb{N}$,
$$\alpha_* \operatorname{Sym}^d V^\vee = (e^* \operatorname{Sym}^d V^\vee)^{A[m]}.$$  
Because of invariance of the section $s$ under $A[m]$, and of the compatibility with symmetric powers of the representation associated with a unipotent bundle,
$$\operatorname{inj} \operatorname{lim}_{d \in \mathbb{N}} (e^* \operatorname{Sym}^d V^\vee)^{A[m]} = \left( \operatorname{inj} \operatorname{lim} \operatorname{Sym}^d e^* V^\vee \right)^{A[m]}.$$  
Via $\sigma$, the $\mathcal{O}_S$-algebra $\operatorname{Sym} e^* V^\vee$ is identified with $\operatorname{Sym} E^\vee \otimes \mathcal{O}_S[s]$. This gives rise to an $A[m]$-equivariant isomorphism of $\mathcal{O}_S$-algebras
$$\operatorname{inj} \operatorname{lim}_{d \in \mathbb{N}} \operatorname{Sym}^d e^* V^\vee \simeq (\operatorname{Sym} E^\vee \otimes \mathcal{O}_S[s])/(s - 1) \simeq \operatorname{Sym} E^\vee,$$  
which concludes the proof.

(3) Let $i : P(\alpha^* E) \to P(V)$ the closed immersion. For $d \in \mathbb{N}$, consider the short exact sequence of $\mathcal{O}_X$-modules
$$0 \to \mathcal{O}_V(d) \xrightarrow{s} \mathcal{O}_V(d + 1) \xrightarrow{i_* \mathcal{O}_{\alpha^* E}(d + 1)} 0,$$  
and take its tensor product with $\pi^* L$:
$$0 \to \pi^* L \otimes \mathcal{O}_V(d) \xrightarrow{s} \pi^* L \otimes \mathcal{O}_V(d + 1) \xrightarrow{\pi^* L \otimes i_* \mathcal{O}_{\alpha^* E}(d + 1)} 0.$$  
By the projection formula, pushing forward along $\pi$ the previous short exact sequence yields the following exact sequence of $\mathcal{O}_A$-modules:
$$0 \to L \otimes \operatorname{Sym}^d V^\vee \xrightarrow{s} L \otimes \operatorname{Sym}^{d+1} V^\vee \to L \otimes a^* \operatorname{Sym}^{d+1} E^\vee,$$  
because $\pi_* i_* \mathcal{O}_{\alpha^* E}(d + 1) = a_* \operatorname{Sym}^{d+1} E^\vee$. Again by the projection formula, pushing forward $a$ gives the following short exact sequence of $\mathcal{O}_S$-modules
$$0 \to a_* (L \otimes \operatorname{Sym}^d V^\vee) \xrightarrow{s} a_* (L \otimes \operatorname{Sym}^{d+1} V^\vee) \to a_* L \otimes \operatorname{Sym}^{d+1} E^\vee.$$  
Since by assumption $a_* L$ vanishes, multiplication by $s$ is an isomorphism
$$a_* (L \otimes \operatorname{Sym}^d V^\vee) \xrightarrow{\sim} a_* (L \otimes \operatorname{Sym}^{d+1} V^\vee).$$  
Therefore,
$$\eta_* (\pi^* L|_Y) = \operatorname{inj} \operatorname{lim}_{d \in \mathbb{N}} a_* (L \otimes \operatorname{Sym}^d V^\vee) = a_* L = 0. \quad \square$$

3. The Universal Vector Extension of a Formal Abelian Scheme

The definitions and the results expounded in section [1] concerning the universal extension of an abelian scheme are adopted here for a formal abelian scheme.

3.1. Almost finite vector bundles. Let $K$ be a non-trivially valued complete non-Archimedean field. Let $R$ be the ring of integers of $K$. Let $\varpi \in \widehat{R}$ be a topologically nilpotent element. For a formal $R$-scheme $X$ and $n \in \mathbb{N}$, let $X_n$ be the closed subscheme defined by the equation $\varpi^n = 0$. Let $S$ be a formal $R$-scheme.
3.1.1. The formal Tate group scheme. Let \((X^{(i)}, f^{(i)}): X^{(i+1)} \to X^{(i)})_{i \in \mathbb{N}}\) be a projective system of formal \(S\)-schemes with affine transition maps. For \(n \in \mathbb{N}\), let \((X^{(i)}_n, f^{(i)}_n)\) be the projective system of \(S_n\)-schemes obtained by base change to \(S_n\). Since the transition maps are affine, the projective limit \(X_n := \text{proj lim}_{i \in \mathbb{N}} X^{(i)}_n\) exists and represents the functor associating to a \(S_n\)-scheme \(T\) the projective limit \(\text{proj lim}_{i \in \mathbb{N}} X^{(i)}_n(T)\); see [Stacks, Lemma 01YX]. Let \(\text{pr}_{n,i}: X_n \to X^{(i)}_n\) be the projection onto the \(i\)-th factor. For \(m \leq n\), identify \(X_n \times_{S_n} S_m\) with \(X_m\) ([Stacks, Lemma 01YZ]).

**Definition 3.1.** The \(S\)-formal scheme \(X := \text{inj lim}_{n \in \mathbb{N}} X_n\) is called the projective limit of the projective system \((X^{(i)}, f^{(i)}_{i \in \mathbb{N}})\). For a non-negative integer \(i \in \mathbb{N}\), the maps \((\text{pr}_{n,i})_{n \in \mathbb{N}}\) define a morphism of formal \(S\)-schemes \(\rho^{(i)}: X \to X^{(i)}\).

**Lemma 3.2.** With the notation above, the \(S\)-formal scheme \(X\) represents the functor associating to an \(S\)-formal scheme \(S'\) the projective limit \(\text{proj lim}_{i \in \mathbb{N}} X^{(i)}(S')\).

**Proof.** The statement follows from the universal property of projective limits of schemes ([Stacks, Lemma 01YX]). \(\square\)

Let \(A\) a formal abelian scheme over \(R\). The definition of universal cover, Tate group scheme and, for a prime number \(p\), the \(p\)-adic universal cover and \(p\)-adic Tate scheme is the formal analogous of Definition 2.6. The reader should be warned that, when the residue field of \(K\) is of positive characteristic, the Tate group scheme may very well be non reduced.

3.1.2. Representations. The notion of finite (resp. unipotent) vector bundle on \(A\) is the verbatim translation of Definition 2.3 into the context of formal geometry. Similarly, the concept of representation (resp. unipotent representation) is the obvious analogue of Definition 2.4 (resp. Definition 2.1).

**Definition 3.3.** A vector bundle \(V\) on \(A\) is almost finite if, for every \(n \in \mathbb{N}\), the restriction \(V|_{A_n}\) is finite.

Let \(V\) be an almost finite vector bundle on \(A\). By applying the construction associated with a finite vector bundle, one defines a representation \(\rho_V: TA \rightarrow GL(e^*V)\).

**Proposition 3.4.** The functor \(V \mapsto \rho_V\) defined above induces an equivalence between the category of almost finite vector bundles on \(A\) (whose arrows are \(O_A\)-modules) onto that of representations of \(TA\) (whose arrows are \(TA\)-equivariant homomorphisms of \(O_S\)-modules). Furthermore,

1. the functor \(\rho\) is compatible with direct sums, tensor products, internal homs, symmetric and exterior powers;
2. for an almost finite vector bundle \(V\) the restriction map \(\alpha_V: \omega_A \to (e^*V)^TA\) is an isomorphism.

Suppose \(K\) is a valued extension of \(\mathbb{Q}_p\). Then, the functor \(V \mapsto \rho_V\) defines an equivalence between the category of unipotent vector bundles on \(A\) (whose arrows are \(O_A\)-modules) onto that of unipotent representations of \(T_pA\) (whose arrows are \(T_pA\)-equivariant homomorphisms of \(O_S\)-modules).

**Proof.** This is obtained by applying Proposition 2.11 to \(A_n\) for all \(n \in \mathbb{N}\). \(\square\)

3.1.3. The universal vector hull and the Hodge-Tate decomposition. Suppose \(K\) is a valued extension of \(\mathbb{Q}_p\). Let \(T_pA\) be the \(p\)-adic Tate group scheme of \(A\). For \(i \in \mathbb{N}\), let \(\text{pr}_i: T_pA \to A[p^i]\) be the canonical projection. For non-negative integers \(n, i\), let \(\theta_{n,i}: A_n[p^i] \to V(\omega_{A_n[p^i]})\) be the universal vector hull of the flat group \(S_n\)-scheme \(A_n[p^i]\).
There is a unique morphism of group formal $S$-schemes $\theta_{T_pA} : T_pA \to \mathcal{V}(\omega_A)$ such that, for non-negative integers $n \leq i$, 
\[ \theta_{T_pA|T_pA^n} = \theta_{n,i} \circ \text{pr}_i. \]

**Proposition 3.5.** With the notation above, $\theta_{T_pA}$ is the universal vector hull of $T_pA$, i.e. given a vector bundle $E$ on $S$ and a morphism of group formal $S$-schemes $f : T_pA \to \mathcal{V}(E)$, there is a unique homomorphism of $\mathcal{O}_S$-modules $\varphi : \omega_A \to E$ such that $f = \mathcal{V}(\varphi) \circ \theta_{T_pA}$.

**Proof.** This follows directly from the definition of $\theta_{T_pA}$. \qed

Let $U_A$ be the universal extension of $\mathcal{O}_A$. Let $\rho : T_pA \to \text{GL}(e^*U_A)$ be the representation associated with the unipotent bundle $U_A$. Let $\sigma : \omega_A \otimes \mathcal{O}_S \simeq e^*U_A$ be a splitting. By applying Proposition 2.16 to $A_n$, for each $n \in \mathbb{N}$, one sees that via the splitting $\sigma$ the representation $\rho$ has matrix
\[
\begin{pmatrix}
3d_{\omega_A} & \theta_{T_pA} \\
0 & 1
\end{pmatrix}.
\]

Let $\tilde{A}$ denote the $p$-adic universal cover of $A$. Consider the morphism of $S$-group schemes $s : \tilde{A} \to A^\sharp$ defined as follows. Let $n \in \mathbb{N}$. Since the multiplication by $p^n$ on $\omega_{A_n}$ is 0, there is a morphism of $S_n$-group schemes $s_n : A_n \to A^\sharp_n$ fitting in the following commutative and exact diagram (Lemma 2.18):

\[
\begin{array}{cccccccc}
0 & \longrightarrow & A_n[p^n] & \longrightarrow & A_n[p^n] & \longrightarrow & A_n & \longrightarrow & 0 \\
\downarrow{\theta_{A[p^n]}} & & \downarrow{s_n} & & \downarrow{=} & & \downarrow{=} & & \downarrow{=} \\
0 & \longrightarrow & \mathcal{V}(\omega_{A_n}) & \longrightarrow & A^\sharp_n & \longrightarrow & A_n & \longrightarrow & 0
\end{array}
\]

There is a unique morphism of formal $S$-schemes $s : \tilde{A} \to A^\sharp$ whose restriction to $\tilde{A}_n$, for $n \in \mathbb{N}$, is $s_n \circ \text{pr}_n$. By construction $s$ fits in the following commutative and exact diagram:

\[
\begin{array}{cccccccc}
0 & \longrightarrow & T_pA & \longrightarrow & \tilde{A} & \longrightarrow & A & \longrightarrow & 0 \\
\downarrow{\theta_{T_pA}} & & \downarrow{=} & & \downarrow{s} & & \downarrow{=} & & \downarrow{=} \\
0 & \longrightarrow & \mathcal{V}(\omega_A) & \longrightarrow & A^\sharp & \longrightarrow & A & \longrightarrow & 0
\end{array}
\]

Let $S'$ be a formal $S$-scheme. Suppose $S'$ affine. According to the analogue of Proposition 1.14 in the context of formal schemes, given an $S'$-valued point $x$ of $A$ there is an $S'$-valued point $y$ of $A^\sharp$ whose image in $A$ is $x$.

**Lemma 3.6.** Let $S'$ be a formal $S$-scheme which is affine as a formal $R$-scheme. Let $x \in \tilde{A}(S')$. For $i \in \mathbb{N}$, let $y_i$ be an $S'$-valued point of $A^\sharp$ whose image in $A$ is $x_i := \text{pr}_i(x)$. With the notation introduced above, 
\[ s(x) = \lim_{i \to \infty} p^iy_i. \]

**Proof.** By definition, for each $n \in \mathbb{N}$ and $y \in A^\sharp_n \{S'_n\}$, $s_n(\pi_n(y)) = p^ny$, where $\pi : A^\sharp \to A$ is the structural morphism. The statement follows. \qed

**Theorem 3.7** (Hodge-Tate decomposition). Suppose $K$ algebraically closed and $S = \text{Spec}(R)$. Then, the map induced by $\theta_{T_pA}$,
\[ T_pA(R) \otimes_{\mathbb{Z}_p} K \longrightarrow \omega_A \otimes_R K \]
is surjective.

**Proof.** This is [SW13, Proposition 5.1.6]. Indeed, by the previous Lemma, the map $\alpha_G$ (with Scholze-Weinstein notation) restricted to $T_pA(R)$ is $\theta_{T_pA}$ (loc.cit. Definition 3.2.3 and Lemma 3.5.1). \qed
Let $E$ be a vector bundle on $S$. Let $\varphi: \omega_A \to E$ be a homomorphism of $\mathcal{O}_S$-modules. Let $V$ be the push-out of $\mathcal{U}_A$ along $\varphi$:

\[
\begin{array}{c}
0 \\
\downarrow^\alpha & \\
0 \\
\end{array}
\begin{array}{c}
\alpha^*\omega_A \\
\downarrow & \\
E \end{array}
\begin{array}{c}
\to \mathcal{U}_A \\
\downarrow & \\
V \\
\to \mathcal{O}_A \\
\end{array}
\begin{array}{c}
\to 0 \\
\end{array}
\]

The vector bundle $V$ is unipotent. Let $\rho_V: T_pA \to \text{GL}(e^*V)$ the associated representation of $T_pA$. Via a splitting $\tau: \omega_A \otimes \mathcal{O}_A \simeq e^*V$, the representation $\rho$ has matrix

\[
\begin{pmatrix}
\text{id}_E & \theta_V \\
0 & 1
\end{pmatrix},
\]

for a morphism of group formal $S$-schemes $\theta: T_pA \to \mathcal{V}(E)$. Note that $\theta_V$ does not depend on the chosen splitting $\tau$.

**Proposition 3.8.** With the notation above,

\[\theta_V = \mathcal{V}(\varphi) \circ \theta_{T_pA}.\]

**Proof.** This is an immediate application of Corollary 2.17. \hfill \Box

### 3.2. Formal functions on vector extensions

Let $K$ be a complete non-trivially valued non-Archimedean field of characteristic 0. Let $R$ be its ring of integers. Let $\varpi \in R$ be a topologically nilpotent element.

For $\nu \in \mathbb{N}$, let $R_\nu := R/\varpi^\nu R$. For a formal $R$-scheme $S$, an $\mathcal{O}_S$-module $F$, and $\nu \in \mathbb{N}$, let $S_\nu = S \times_R R_\nu$, and $F_\nu := F|_{S_\nu}$. For a morphism of formal $R$-schemes $f: S \to S'$ and $\nu \in \mathbb{N}$, let $f_\nu: S_\nu \to S'_\nu$ be the morphism deduced by base-change.

#### 3.2.1. Statements

Let $\alpha: A \to \text{Spf}(R)$ be a formal abelian scheme. Let $E$ a free $R$-module of finite rank. Let $\varphi: \omega_A \to E$ be a homomorphism of $R$-modules. Let $V$ be the push-out of $\mathcal{U}_A$ along $\varphi$. Let $X = \mathbb{P}(V)$, $D = \mathbb{P}(\alpha^*E)$ and $Y = X \setminus D$. Let $\pi: X \to A$ be the structural morphism. Let $C$ be the torsion-free quotient of $\text{Coker}(\varphi)$. By arguing as in 2.3 one obtains a morphism formal $R$-schemes

\[\text{pr}_{\mathcal{V}(C)}: Y \to \mathcal{V}(V) \to \mathcal{V}(\alpha^*C) = \mathcal{V}(C) \times A \xrightarrow{\text{pr}_S} \mathcal{V}(C).\]

**Theorem 3.9.** With the notation introduced above,

1. composing with $\text{pr}_{\mathcal{V}(C)}$ induces an isomorphism

   \[H^0(\mathcal{V}(C), \mathcal{O}_{\mathcal{V}(C)}) \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y);\]

2. for a line bundle $L$ on $A$ such that $H^0(A, L) = 0$,

   \[H^0(Y, \pi^*L) = 0.\]

Let $N$ be a finitely generated free abelian group. Let $S$ be the formal split torus with group of characters $N$; namely, $S$ represents the functor associating to a formal $R$-scheme $T$, the group $\text{Hom}(N, \mathcal{O}(T)^\times)$.

Let $d: N \to \hat{A}(R)$ be a group homomorphism. Let $H$ be the extension of $A$ by $S$ determined by $d$:

\[0 \to S \to H \to A \to 0.\]

Explicitly, it is constructed as follows. For $n \in N$, consider the line bundle $L_n := (\text{id}_A, d(n))^*\mathcal{L}$ on $A$. Then, $H$ is the relative spectrum\(^3\) of the $\mathcal{O}_A$-algebra of finite presentation

\[\mathcal{O}_A[H] := \bigoplus_{n \in N} L_n^\vee.\]

\(^3\)Let $X$ a formal $R$-scheme. Let $A$ be a $\mathcal{O}_X$-algebra of finite presentation. The relative spectrum of $A$ is the formal $X$-scheme representing the functor associating to a formal $X$-scheme $f: X' \to X$ the set $\text{Hom}_{\mathcal{O}_{X'}^{\vee}}(f^*A, \mathcal{O}_{X'})$. 

(Multiplication on \(O_A[H]\) is defined via the isomorphism \(L_n \otimes L_n' \simeq L_{n+n'}\) given by homogeneity of \(L\).

Let \(S_0\) be the formal split torus with group of characters \(\text{Ker}(d)\). Consider the \(O_A\)-algebra of finite presentation

\[
O_A[H]_0 := \bigoplus_{n \in \text{Ker}(d)} L_n'.
\]

For \(n \in \text{Ker}(d)\), the line bundle \(L_n\) is by definition trivial. It follows that the relative spectrum of \(O_A[H]_0\) is \(A \times S_0\). On the other hand, the inclusion of \(O_A[H]_0\) in \(O_A[H]\) induces a morphism of formal \(R\)-schemes

\[
\text{pr}_{S_0}: H \longrightarrow A \times S_0 \longrightarrow S_0.
\]

Consider the morphism of formal \(R\)-schemes

\[
\text{(pr}_{S_0}, \text{pr}_V(C)) : H \times_A Y \longrightarrow S' \times V(C).
\]

**Corollary 3.10.** With the notation above, composing a function on \(S_0 \times V(C)\) with \((\text{pr}_{S_0}, \text{pr}_V(C))\) induces an isomorphism

\[
H^p(S_0 \times V(C), O_{S_0 \times V(C)}) \sim \rightarrow H^p(H \times_A Y, O_{H \times_A Y}).
\]

**Proof of Corollary 3.10** admitting Theorem 3.9. Consider the following cartesian square:

\[
\begin{array}{ccc}
H \times_A Y & \overset{f}{\longrightarrow} & Y \\
\downarrow^{p} & & \downarrow^{p} \\
H & \overset{h}{\longrightarrow} & A
\end{array}
\]

where \(p\) is the composition of \(\pi: X \rightarrow A\) with the open immersion \(Y \rightarrow X\). Since the morphism \(p_\nu\) is flat for each \(\nu \in \mathbb{N}\), by flat base-change,

\[
f_\nu O_{H \times A Y} = \bigoplus_{\nu} p_\nu L_{n, \nu}'.
\]

By definition, the \(A_\nu\)-scheme \(Y_\nu\) is the relative spectrum of the \(O_{A_\nu}\)-algebra of finite presentation \(\bigoplus_{n \in \mathbb{N}} L_{n, \nu}'.\) In particular,

\[
p_\nu q_\nu O_{H_\nu} = \bigoplus_{n \in \mathbb{N}} p_\nu L_{n, \nu}'.
\]

By taking the projective limit, one obtains

\[
f_* O_{H \times A Y} = \bigoplus_{n \in \mathbb{N}} p^* L_n'.
\]

Since topological direct sums commute with taking global sections, one concludes by applying Theorem 3.9. \(\square\)

**Proof of Theorem 3.9** For a free \(R\)-module of finite rank \(F\) let

\[
\widetilde{\text{Sym}} F := \text{proj lim}_{\nu \in \mathbb{N}} \text{Sym} F \otimes_R R_\nu,
\]

so that \(H^0(V(F), O_{V(F)}) = \widetilde{\text{Sym}} F^\vee\).

To begin with, remark that it suffices to prove the statement for algebraically closed \(K\).

1. If the residue field of \(R\) is of characteristic 0, Proposition 2.19 (2) implies, for \(\nu \in \mathbb{N}\), \(H^0(Y_\nu, O_{Y_\nu}) = H^0(V(C_\nu), O_{V(C_\nu)})\). Therefore, by taking the projective limit,

\[
H^0(Y, O_Y) = H^0(V(C), O_{V(C)}).
\]

Suppose that the residue field of \(R\) is of characteristic \(p > 0\), hence \(K\) is a complete valued extension of \(\mathbb{Q}_p\). In this case, take \(\varpi = p\). Let \(\rho: T_p A \rightarrow \text{GL}(e^* V)\)}
be the representation associated with the unipotent vector bundle $V$ by Proposition 3.4. Via a splitting $\tau: E \otimes R \simeq E^*V$, the representation $\rho$ has matrix
\[
\begin{pmatrix}
\text{id}_E & \theta_V \\
0 & 1
\end{pmatrix},
\]
for a morphism of formal $S$-group schemes $\theta: T_pA \to \mathcal{V}(E)$. According to Corollary 3.8,
\[
\theta_V = \mathcal{V}(\varphi) \circ \theta_{T_pA}.
\]

Via the representation $\rho$, the $p$-adic Tate formal group scheme $T_pA$ acts on the lattice at the zero section $Y_\circ$ of $Y$. By identifying $Y_\circ$ with $\mathcal{V}(E)$ via the splitting $\tau$, the action of $T_pA$ on $\mathcal{V}(E)$ is the translation by $\theta$: for a formal $R$-scheme $S$, and $S$-valued points $t \in T_pA(S)$, $x \in \mathcal{V}(E)(S)$, the action is given by $(t, x) \mapsto x + \theta(t)$.

Let $H^0(\mathcal{V}(E), \mathcal{O}_{\mathcal{V}(E)})^{T_pA}$ be the $R$-subalgebra of $H^0(\mathcal{V}(E), \mathcal{O}_{\mathcal{V}(E)})$ of $T_pA$-invariants. Then, Proposition 2.19 (2) implies, for $\nu \in \mathbb{N}$,
\[
H^0(Y_\circ, \mathcal{O}_{Y_\circ}) = H^0(\mathcal{V}(E_\nu), \mathcal{O}_{\mathcal{V}(E_\nu)})^{T_pA_\nu}.
\]

Therefore, passing to the projective limit,
\[
H^0(Y, \mathcal{O}_Y) = H^0(\mathcal{V}(E), \mathcal{O}_{\mathcal{V}(E)})^{T_pA}.
\]

In order to conclude, one has to prove
\[
H^0(\mathcal{V}(E), \mathcal{O}_{\mathcal{V}(E)})^{T_pA} = H^0(\mathcal{V}(C), \mathcal{O}_{\mathcal{V}(C)}).
\]

The image $T_pA$ in $\mathcal{V}(C)$ is 0, thus the functions coming from $H^0(\mathcal{V}(C), \mathcal{O}_{\mathcal{V}(C)})$ are invariant under the action of $T_pA$. On the other hand, according to the Hodge-Tate decomposition (Theorem 5.7), the $K$-linear homomorphism
\[
\varphi \circ \theta_{T_pA}: T_pA(R) \otimes_{\mathbb{Z}_p} K \longrightarrow E \otimes_R K
\]
surjects onto the image of $\varphi \otimes \text{id}_K$.

Here comes the key point. Unlike the complex case, in a non-Archimedean vector space a lattice accumulates in 0, and functions invariant under translation by such a lattice are necessarily constant. To be more precise:

**Lemma 3.11.** Suppose $K$ is a complete valued extension of $\mathbb{Q}_p$. Let $B$ be a $p$-adically complete flat $R$-algebra. Let $\Lambda$ a free $\mathbb{Z}_p$-module of finite rank. Let $F := \Lambda \otimes_{\mathbb{Z}_p} R$. Let $f \in B \otimes_R \hat{\text{Sym}} F^\vee$ be such that, for all $\lambda \in \Lambda$,
\[
f(x + \lambda) = f(x).
\]

Then $f$ belongs to $B$.

The proof of the Lemma is postponed to the end of the proof of the Theorem. Let $I = \text{Ker}(E \to (E/\text{Im} \varphi) \otimes_R K)$ be the saturation in $E$ of the image of $\varphi$. Pick an isomorphism $\psi: E \to I \otimes C$. The injection $\text{Im} \varphi \to I$ induces an injection
\[
\hat{\text{Sym}} I^\vee \longrightarrow \hat{\text{Sym}} \text{Im}(\varphi)^\vee.
\]

Via $\psi$ one identifies $\hat{\text{Sym}} E^\vee$ with $\hat{\text{Sym}} C^\vee \hat{\text{Sym}} I^\vee$, which injects into
\[
\hat{\text{Sym}} C^\vee \hat{\text{Sym}} \text{Im}(\varphi)^\vee.
\]

One concludes by applying the Lemma with $B = \hat{\text{Sym}} C^\vee$ and $\Lambda = \text{Im}(\varphi)$.

(2) Since the line bundle $L$ is non-trivial, there is $v_0 \in \mathbb{N}$ such that, for $\nu \geq v_0$, the line bundle $L_\nu$ is non-trivial. It is tempting to apply directly Proposition 2.19 (3) to $L_\nu$, by arguing that non-trivial homogeneous line bundles do not have nonzero global sections. Alas, this is not true over non-reduced bases such as Spec $R_\nu$, and a more involved argument is needed.
Let \( j : Y \to X \) be the open immersion. Let \( s : \mathcal{O}_A \to V^\vee \) be the section obtained by duality from the surjection \( V \to \mathcal{O}_A \). For \( d \in \mathbb{N} \), let \( \mathcal{O}_V(d) \) be \( d \)-th power of the tautological bundle on \( X \), and \( \pi^* L(d) := \pi^* L \otimes \mathcal{O}_V(d) \). Now,
\[
j_\ast j^* \pi^* L = \proj \lim_{\nu \in \mathbb{N}} j_{\nu \ast} j_{\nu}^* \pi^*_\nu \mathcal{L}_\nu = \proj \lim_{\nu \in \mathbb{N}} \inj \lim_{d \in \mathbb{N}} \pi^*_\nu \mathcal{L}_\nu(d),
\]
where the transition map \( \pi^*_\nu \mathcal{L}_\nu(d) \to \pi^*_\nu \mathcal{L}_\nu(d + 1) \) is the multiplication by \( s \). By pushing forward along \( \pi \),
\[
\pi_\ast j_\ast j^* \pi^* L = \proj \lim_{\nu \in \mathbb{N}} \inj \lim_{d \in \mathbb{N}} \pi^*_\nu \pi^*_\nu \mathcal{L}_\nu(d) = \proj \lim \inj \lim \mathcal{L}_\nu \otimes \text{Sym}^d V^\vee.
\]
(Note that \( \pi \nu \) is quasi-compact and (quasi-)separated, so injective limits commute with pushing forward along \( \pi \nu \); see \cite[Lemma 009F]{stacks}.)

Let \( \nu \leq \nu' \) be integers. Let \( F \) be a vector bundle on \( A \). Because of flatness of \( F_\nu \), the restriction map \( H^0(A_\nu, F) \to H^0(A_\nu', F) \) has kernel \( \pi^\ast H^0(A_\nu, F) \) and gives rise to a short exact sequence:
\[
0 \to H^0(A_\nu, F)/\pi^\ast H^0(A_\nu, F) \to H^0(A_\nu, F) \to H^0(A_\nu', F)/H^0(A_\nu, F) \to 0.
\]
Let \( F, F' \) be vector bundles on \( A \). Let \( \psi : F \to F' \) be a homomorphism of \( \mathcal{O}_A \)-modules. Applying the above to \( F \) and \( F' \), the homomorphism \( \psi \) yields a commutative and exact diagram of \( R \)-modules:
\[
\begin{array}{cccccc}
0 & \to & H^0(A_\nu, F) & \to & H^0(A_\nu, F) & \to & 0 \\
\pi^\ast H^0(A_\nu, F) & \to & H^0(A_\nu, F) & \to & H^0(A_\nu, F) & \to & 0 \\
0 & \to & H^0(A_\nu', F') & \to & H^0(A_\nu', F') & \to & 0.
\end{array}
\]

Let \( d, i \in \mathbb{N} \). Apply this with \( F = L \otimes \text{Sym}^d V^\vee, F' = L \otimes \text{Sym}^{d+i} V^\vee \), and \( \psi(f) = f \otimes s^i \). Then, one obtains a homomorphism of \( R \)-modules
\[
\Phi_{d, i, \nu, \nu'} : H^0(A_\nu, L \otimes \text{Sym}^d V^\vee) \to H^0(A_\nu', L \otimes \text{Sym}^{d+i} V^\vee).
\]

Claim 3.12. With the notation above, the homomorphism of \( R \)-module \( \Phi_{d, i, \nu, \nu'} \) is injective.

Proof of the Claim. The short exact sequence of \( \mathcal{O}_A \)-modules
\[
0 \to \text{Sym}^d V^\vee \xrightarrow{s^i} \text{Sym}^{d+i} V^\vee \to \text{Sym}^{d+i} E^\vee \to 0,
\]
yields the following commutative diagram:
\[
\begin{array}{cccccc}
\text{Sym}^d V^\vee & \xrightarrow{s^i} & \text{Sym}^{d+i} V^\vee & \to & \text{Sym}^{d+i} E^\vee & \to & 0 \\
H^0(A_\nu, L \otimes \text{Sym}^d V^\vee) & \xrightarrow{s^i} & H^0(A_\nu, L \otimes \text{Sym}^{d+i} V^\vee) & \to & H^0(A_\nu, L \otimes \text{Sym}^{d+i} E^\vee) & \to & 0 \\
\pi^\ast H^0(A_\nu, L \otimes \text{Sym}^d V^\vee) & \to & \pi^\ast H^0(A_\nu, L \otimes \text{Sym}^{d+i} V^\vee) & \to & \pi^\ast H^0(A_\nu, L \otimes \text{Sym}^{d+i} E^\vee) & \to & 0 \\
H^0(A_\nu, L \otimes \text{Sym}^d V^\vee) & \xrightarrow{s^i} & H^0(A_\nu, L \otimes \text{Sym}^{d+i} V^\vee) & \to & H^0(A_\nu, L \otimes \text{Sym}^{d+i} E^\vee) & \to & 0.
\end{array}
\]
where we abbreviated \( \text{Sym} \) with \( S \). By definition, \( \Phi_{d, i, \nu, \nu'} \) is the homomorphism induced on the cokernels of the two leftmost vertical arrows.

Since \( s^i : H^0(A_\nu, L \otimes \text{Sym}^d V^\vee) \to H^0(A_\nu, L \otimes \text{Sym}^{d+i} V^\vee) \) is injective, the Snake Lemma implies the result. \( \square \)
For convenience, set $\text{Sym}^{-1} V^\nu = 0$. Let $f \in H^0(Y, \pi^* L)$. For each $\nu \in \mathbb{N}$ let $d_\nu \in \mathbb{Z}$ be the smallest integer $\geq -1$ such that the image $\tilde{f}_\nu$ of $f$ in $H^0(Y, \pi^* L)$ comes from a section $\tilde{f}_\nu \in H^0(A_\nu, L \otimes \text{Sym}^{d_\nu} V^\nu)$.

**Claim 3.13.** $\tilde{f}_\nu = 0$.

**Proof of the Claim.** By contradiction, suppose $\tilde{f}_\nu$ nonzero. Then $d_\nu \geq 0$. Let $\nu' \geq \nu$. By definition, the image of $s^{d_\nu, \nu'} \tilde{f}_\nu$ in

$$
\frac{H^0(A_\nu, L \otimes \text{Sym}^{d_\nu} V^\nu)}{H^0(A_{\nu'}, L \otimes \text{Sym}^{d_{\nu'}} V^{\nu'})}
$$

is 0, because $\tilde{f}_\nu$ maps to $s^{d_\nu, \nu'} \tilde{f}_\nu$. Therefore, according to the previous Claim, the image of $\tilde{f}_\nu$ in

$$
\frac{H^0(A_\nu, L \otimes \text{Sym}^{d_\nu} V^\nu)}{H^0(A_{\nu'}, L \otimes \text{Sym}^{d_{\nu'}} V^{\nu'})}
$$

is 0. This means that there is $g_{\nu'} \in H^0(A_{\nu'}, L \otimes \text{Sym}^{d_{\nu'}} V^{\nu'})$ whose restriction to $A_\nu$ is $f_\nu$.

Now, the projective system $(H^0(A_\nu, L))_{\nu \in \mathbb{N}}$ satisfies the Mittag-Leffler condition. Since $H^0(A, L) = 0$ by hypothesis, there is $\nu' \geq \nu$ such that the image of $H^0(A_{\nu'}, L)$ in $H^0(A_\nu, L)$ is 0. Because of the commutative diagram

$$
\begin{CD}
H^0(A_{\nu'}, L \otimes \text{Sym}^{d_{\nu'}} V^{\nu'}) @>>> H^0(A_{\nu'}, L) \otimes \text{Sym}^{d_{\nu'}} E^{\nu'} \\
@VVV @VVV \\
H^0(A_{\nu'}, L \otimes \text{Sym}^{d_{\nu'}} V^{\nu'}) @>>> H^0(A_{\nu'}, L) \otimes \text{Sym}^{d_{\nu'}} E^{\nu'}
\end{CD}
$$

the image of $\tilde{f}_\nu$ in $H^0(A_{\nu'}, L) \otimes \text{Sym}^{d_{\nu'}} E^{\nu'}$ coincides with the image of $g_{\nu'}$ in it, which is 0. Therefore there is $h \in H^0(A_\nu, L \otimes \text{Sym}^{d_{\nu'-1}} V^{\nu'})$ such that $\tilde{f}_\nu = sh$, contradicting the minimality of $d_\nu$.

The Claim implies $f_\nu = 0$ for all $\nu \in \mathbb{N}$, thus $f = 0$. □

**Proof of Lemma 3.11.** Arguing by induction on the rank of $\Lambda$ permits to reduce to the case of rank 1. By picking a generator of $\Lambda$, identify $\Lambda$ with $\mathbb{Z}_p$ and $B \hat{\otimes} \text{Sym}(\Lambda \otimes_{\mathbb{Z}_p} R)^\vee$ with

$$
B\{x\} := \text{proj lim}_{n \in \mathbb{N}} B/p^n B[x].
$$

Expand $f$ in powers series $f(x) = \sum_{k=0}^{\infty} b_k x^k$, with $b_k \in \mathbb{Z}_p$ such that $b_k \to 0$ as $k \to \infty$. Let $t \in R$ be non-zero. Then, the Taylor expansion of $f(x + t)$ is

$$
\sum_{k=0}^{\infty} b_k (x + t)^k = \sum_{k=0}^{\infty} \sum_{i=0}^{k} \binom{k}{i} b_k x^i t^{k-i} = \sum_{i=0}^{\infty} \left( \sum_{k \geq i} \binom{k}{i} b_k t^{k-i} \right) x^i.
$$

By comparing it with the Taylor expansion of $f(x)$, one obtains the following equality, for all $i \in \mathbb{N}$,

$$
b_i = \sum_{k \geq i} \binom{k}{i} b_k t^{k-i} = b_i + \sum_{k > i+1} \binom{k}{i} b_k t^{k-i}.
$$


4If the field $K$ were discretely valued, this would be [EGA III, Corollaire 4.1.7] or [FGA*05, 8.2.7, p. 191]. Since $K$ is supposed algebraically closed (thus densely valued), the result is to be found in [Abb10, Corollaire 2.11.7] or [FK13, Proposition 11.3.3].
After canceling \( b_i \) on both sides of the previous equality, and dividing by \( t \) the result (which is licit because \( B \) has no \( R \)-torsion), one has the following relation for \( i \geq 1 \):

\[
ib_i = - \sum_{k \geq i+1} \left( \frac{k}{i-1} \right) b_k t^{k-i}.
\]

Now, the right-hand side of the previous equality tends to 0 as soon as \( t \) does. Since we are in characteristic 0, this implies \( b_i = 0 \) for all \( i \geq 1 \), that is what we wanted to prove. \( \square \)

4. The universal vector extension of an abeloid variety

Let \( K \) be complete non-Archimedean valued field. Let \( R \) be its ring of integers. In this paper \( K \)-analytic space are considered in the sense of Berkovich ([Ber93]). By an abuse of notation, given a \( K \)-analytic space \( X \), an \( O_X \)-module here is an \( O_X \)-module with the notation of loc.cit.. Note that, when the space \( X \) is good (i.e., every point admits an affinoid neighbourhood), the two notions coincide; see loc.cit. Proposition 1.3.4.

4.1. Fourier expansion. Let \( X \) be a \( K \)-analytic space. Let \( \rho \in \mathbb{N} \). Let \( L_1, \ldots, L_\rho \) be line bundles on \( X \). Let \( p: P \to X \) be the relative spectrum\(^5\) of the \( O_X \)-algebra of finite presentation

\[
O_X[P] := \bigoplus_{m \in \mathbb{Z}^\rho} L_m^\rho,
\]

where, for \( m = (m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho \), \( L_m = L_1^{\otimes m_1} \cdots \otimes L_\rho^{\otimes m_\rho} \).

**Definition 4.1.** Let \( F \) be a coherent sheaf of \( O_X \)-modules. Consider the subset

\[
\bigoplus_{m \in \mathbb{Z}^\rho} H^0(X, F \otimes L_m^\rho) \subseteq \prod_{m \in \mathbb{Z}^\rho} H^0(X, F \otimes L_m^\rho)
\]

made of sequences \((f_m)_{m \in \mathbb{Z}^\rho}\) satisfying the following condition. Given

- an affinoid domain \( X' \subseteq X \) such that, for \( i = 1, \ldots, \rho \), the line bundle \( L_{i|X'} \) is trivial,
- for \( i = 1, \ldots, \rho \), a trivialization \( s_i: O_{X'} \to L_{i|X'}^\rho \),
- a norm \( \| \cdot \| \) on \( H^0(X', F) \) defining the topology of the finite \( H^0(X', O_{X'}) \)-module \( H^0(X', F) \)
- real numbers \( r \geq 1, \varepsilon > 0 \),

there is a finite subset \( M \subset \mathbb{Z}^\rho \) such that, for all \( m \in \mathbb{Z}^\rho \setminus M \),

\[
\| \tilde{f}_m \| r^{m_1 + \cdots + m_\rho} < \varepsilon,
\]

where, for \( m = (m_1, \ldots, m_\rho) \in \mathbb{Z}^\rho \), \( s^m \) is the trivialization \( s_1^{\otimes m_1} \otimes \cdots \otimes s_\rho^{\otimes m_\rho} \) of \( L_m^\rho \), and \( \tilde{f}_m \in H^0(X', F) \) is the unique section such that \( f_m|_{X'} = \tilde{f}_m \otimes s^m \).

**Proposition 4.2.** With the notation introduced above,

\[
H^0(P, p^* F) = \bigoplus_{m \in \mathbb{Z}^\rho} H^0(X, F \otimes L_m^\rho).
\]

\(^5\)Let \( X \) be a \( K \)-analytic space. Let \( A \) be an \( O_X \)-algebra of finite presentation. The relative spectrum of \( A \) is the \( X \)-analytic space representing the functor associating to a \( X \)-analytic space \( f: X' \to X \) the set \( \text{Hom}_{O_X \text{-alg}}(f^* A, O_{X'}) \).

\(^6\)Such a norm is given for instance by the quotient norm induced by a \( H^0(X', O_{X'}) \)-linear surjection \( H^0(X', O_{X'})^N \to H^0(X', F) \), for some \( N \in \mathbb{N} \).
Proof. It suffices to prove the statement when $X$ is an affinoid domain such that, for $i = 1, \ldots, \rho$, the line bundle $L_i$ is trivial.

For $i = 1, \ldots, \rho$, let $s_i : \mathcal{O}_X \to L_i^\vee$ be a trivialization. The section $s_i$ can be seen as $K$-analytic morphism $P \to \mathbb{G}_m$, still denoted $s_i$. The $K$-analytic morphism $s = (s_1, \ldots, s_{\rho}) : P \to \mathbb{G}_m^\rho \times X$ is an isomorphism. Let $\| \cdot \|$ be a norm defining the topology of $H^0(X, F)$. For $r \geq 1$, let

$$C_{p,r} := \{ t \in \mathbb{G}_m^\rho : r^{-1} \leq \| p_i(t) \| \leq r, i = 1, \ldots, \rho \},$$

$$P_{s,r} := s^{-1}(C_{p,r} \times X),$$

where $p_i : \mathbb{G}_m^\rho \to \mathbb{G}_m$ is the projection onto the $i$-th factor.

For $1 \leq r < r'$, the affinoid domain $P_{s,r}$ is contained in the topological interior of $P_{s,r'}$. Moreover, $(P_{s,r})_{r \in \mathbb{N}, r \geq 1}$ is an affinoid $G$-cover of $P$. Consider the subset

$$\bigoplus_{m \in \mathbb{Z}^\rho} H^0(X, F) \cdot s^m \leq \prod_{m \in \mathbb{Z}^\rho} H^0(X, F) \cdot s^m$$

made of sequences $(f_m \otimes s^m)_{m \in \mathbb{Z}^\rho}$ such that, for every $\varepsilon > 0$, there is a finite subset $M \subseteq \mathbb{Z}^\rho$ such that, for all $m \in \mathbb{Z}^\rho \setminus M$, $\| f_m \|_r^{m_1+\cdots+m_\rho} < \varepsilon$.

Then,

$$H^0(P_{s,r}, p^* F) = \bigoplus_{m \in \mathbb{Z}^\rho} H^0(X, F) \cdot s^m.$$

One concludes because $H^0(P, p^* F) = \text{proj} \lim_{r \in \mathbb{N}, r \geq 1} H^0(P_{s,r}, p^* F).$ \hfill $\Box$

Let $m \in \mathbb{Z}^\rho$. Consider the homomorphisms of $\mathcal{O}_X$-modules $i_m : F \otimes L_m^\vee \to p_* p^* F$ and $p_{r,m} : p_* p^* F \to F \otimes L_m^\vee$ defined as follows. Let $X'$ be an analytic domain of $X$ and identify $H^0(X', p^* F)$ with

$$\bigoplus_{m \in \mathbb{Z}^\rho} H^0(X', F \otimes L_m^\vee)$$

thanks to Proposition 4.2. Then $i_m$ is the canonical injection of the factor $H^0(X', F \otimes L_m^\vee)$ in $H^0(X', p^* F)$, and $p_{r,m}$ the projection onto the factor $H^0(X', F \otimes L_m^\vee)$.

**Proposition 4.3.** With the notation above,

1. for an integer $q \geq 1$, the higher direct image $R^q p_* p^* F$ vanishes;
2. for $m \in \mathbb{Z}^\rho$ and $q \in \mathbb{N}$, the $K$-linear map $H^q(X, F \otimes L_m^\vee) \to H^q(P, p^* F)$ induced by $i_m$ is injective.

**Proof.** (1) Let $(X_\lambda)_{\lambda \in \Lambda}$ be a $G$-cover of $X$ where, for each $\lambda \in \Lambda$, $X_\lambda$ is an affinoid domain such that $L_{i|X_\lambda}$ is trivial for $i = 1, \ldots, \rho$. For $\lambda \in \Lambda$, the $K$-analytic space $p^{-1}(X_\lambda)$ is isomorphic to $\mathbb{G}_m^\rho \times X_\lambda$. In particular, for all integer $q \geq 1$,

$$H^q(p^{-1}(X_\lambda), p^* F) = 0.$$

Therefore $(p^{-1}(X_\lambda))_{\lambda \in \Lambda}$ is an acyclic $G$-cover for $p^* F$, whence the statement.

(2) For $m \in \mathbb{Z}^\rho$, the endomorphism $p_{r,m} \circ i_m$ of the $\mathcal{O}_X$-module $F \otimes L_m^\vee$ is the identity. Since $H^q(F, p^* F) = H^q(X, p_* p^* F)$ thanks to (1), one concludes. \hfill $\Box$

**4.2. Reminder on Tate-Raynaud uniformization.** Let $S$ be an admissible formal $R$-scheme. Let $\mathcal{B}$ be a formal abelian scheme over $S$ and $\hat{\mathcal{B}}$ its dual one. Let $S$ (resp. $\hat{B}$, resp. $\hat{B}$) be generic fiber of $S$ (resp. $\hat{B}$, resp. $\hat{B}$). Let $\beta : B \to S$ and $\hat{\beta} : \hat{\mathcal{B}} \to S$ be the structural morphisms. Let $\mathcal{L}_B$ be the generic fibre of the Poincaré bundle on $\mathcal{B} \times \hat{\mathcal{B}}$.

Let $\tilde{M}$ be a free abelian group of finite rank. Let $\tilde{M}$ the constant $S$-analytic group $\tilde{M} \times_R S$. 


4.2.1. Datum of a toric extension. Let \( \tilde{c} : \tilde{M}_S \to \tilde{B} \) be a morphism of \( S \)-analytic groups. Let \( T \) be the split \( S \)-torus with group of characters \( \tilde{M} \), that is, the relative spectrum of \( \mathcal{O}_S \)-algebra of finite presentation

\[
\mathcal{O}_S[T] = \bigoplus_{\tilde{m} \in \tilde{M}} \mathcal{O}_S \chi_{\tilde{m}},
\]

with multiplication defined, for \( \tilde{m}, \tilde{m}' \in \tilde{M} \), by \( \chi_{\tilde{m}} \chi_{\tilde{m}'} = \chi_{\tilde{m} + \tilde{m}'} \). Consider the extension \( \gamma : G \to S \) of \( B \) by \( T \) determined by \( \gamma \):

\[
0 \longrightarrow T \longrightarrow G \overset{p}{\longrightarrow} B \longrightarrow 0.
\]

Concretely, it is constructed as follows. An element \( \tilde{m} \in \tilde{M} \) can be seen as a morphism \( S \to \tilde{M}_S \) still denoted \( \tilde{m} \). Let \( \tilde{c}(\tilde{m}) \) denote the composite morphism \( S \to \tilde{B} \). Given \( \tilde{m} \in \tilde{M} \), the pull-back of the Poincaré bundle \( L_B \) along the morphism \((\text{id}_B, \tilde{c}(\tilde{m})) : B \to B \times_S \tilde{B} \) is denoted \((\text{id}_B, \tilde{c}(\tilde{m}))^* L_B \). Then, \( G \) is the relative spectrum of the \( \mathcal{O}_B \)-algebra of finite presentation

\[
\mathcal{O}_B[G] := \bigoplus_{\tilde{m} \in \tilde{M}} (\text{id}_B, \tilde{c}(\tilde{m}))^* \mathcal{L}_B^\vee.
\]

In the previous formula, The multiplication on \( \mathcal{O}_B[G] \) is defined via the isomorphism, for \( \tilde{m}, \tilde{m}' \in \tilde{M} \),

\[
(\text{id}_B, \tilde{c}(\tilde{m}))^* \mathcal{L}_B^\vee \otimes (\text{id}_B, \tilde{c}(\tilde{m}'))^* \mathcal{L}_B^\vee \xrightarrow{\sim} (\text{id}_B, \tilde{c}(\tilde{m} + \tilde{m}'))^* \mathcal{L}_B^\vee
\]

given by homogeneity of \( \mathcal{L}_B \). By definition of the relative spectrum, for an \( S \)-analytic space \( S' \), an \( S' \)-valued point \( g \) of \( G \) corresponds to the datum of a \( S' \)-valued point \( b := p(g) \) of \( B \) and a homomorphism of \( \mathcal{O}_S \)-algebras \( \psi : b^* \mathcal{O}_B[G] \to \mathcal{O}_{S'} \).

In turn, this corresponds to the datum, for each \( \tilde{m} \in \tilde{M} \), of a global section \( (g, \tilde{m}) \in \mathcal{O} \) of the pull-back \((b, \tilde{c}(\tilde{m}))^* L_B \) of the Poincaré bundle \( L_B \) along the morphism \((b, \tilde{c}(\tilde{m})) : S \to B \times_S \tilde{B} \) with the following property: for \( \tilde{m}, \tilde{m}' \in \tilde{M} \),

\[
(g, \tilde{m}) \otimes (g, \tilde{m}') = (g, \tilde{m} + \tilde{m}')^G.
\]

(The equality has to be understood via the isomorphism \([4.1]\).) It follows that the section \( (g, \tilde{m}) \in \mathcal{O} \) is a trivialization of \((b, \tilde{c}(\tilde{m}))^* L_B \).

The section \( (g, \tilde{m}) \in \mathcal{O} \) can also be interpreted in the following manner. Consider the \( \mathcal{O}_B \)-algebra of finite presentation

\[
\mathcal{O}_B[G_{\tilde{m}}] := \bigoplus_{i \in \mathbb{Z}} (\text{id}_B, \tilde{c}(i\tilde{m}))^* \mathcal{L}_B^\vee.
\]

Its relative spectrum \( G_{\tilde{m}} \) is the total space \( \mathcal{V}(\text{id}_B, \tilde{c}(\tilde{m}))^* L_B \) of the line bundle \((\text{id}_B, \tilde{c}(\tilde{m}))^* L_B \) deprived of its zero section. The inclusion \( \mathcal{O}_B[G_{\tilde{m}}] \to \mathcal{O}_B[G] \) induces a morphism between relative spectra \( \text{pr}_{G, \tilde{m}} : G \to G_{\tilde{m}} \) fitting in the following commutative and exact diagram of \( S \)-analytic groups:

\[
\begin{array}{ccc}
0 & \longrightarrow & T & \longrightarrow & G & \longrightarrow & B & \longrightarrow & 0 \\
& & & & \downarrow & & & \downarrow & \\
0 & \longrightarrow & G_{\tilde{m}} & \longrightarrow & G & \longrightarrow & T & \longrightarrow & 0.
\end{array}
\]

Then,

\[
(g, \tilde{m})^G = \text{pr}_{G, \tilde{m}} \circ g.
\]

(Given an \( S \)-analytic space \( S' \), a morphism \( f : S' \to G_{\tilde{m}} \) corresponds to the datum of a morphism of \( S \)-analytic spaces \( b : S' \to B \) and a trivialization of the line bundle \((b, \tilde{c}(\tilde{m}))^* L_B \).)
4.2.2. **Dual datum.** Let $M$ be a free abelian group of rank $\dim T = \text{rk}\ M$. Let $M_S$ be the constant $S$-analytic group $M \times_K S$. Let $\varepsilon : M_S \rightarrow G$ be an injective morphism of $S$-analytic groups. Let $\hat{T}$ the split $S$-torus with group of characters $M$. As before, the group homomorphism $c = p \circ \varepsilon : M_S \rightarrow B$, defines an extension $\hat{G}$ of $\hat{B}$ by $\hat{T}$:

$$0 \rightarrow \hat{T} \rightarrow \hat{G} \xrightarrow{\hat{\varepsilon}} \hat{B} \rightarrow 0.$$ 

Let $\hat{\varepsilon} : \hat{G} \rightarrow S$ be the structural morphism. Given a homogeneous line bundle $\hat{L}$ on $\hat{G}$, $\hat{L}$ is fiberwise discrete in $S$, and $\hat{L}$ is proper over $S$.

4.2.3. **Quotient.** Suppose $M_S$ is fiberwise discrete in $G$, and let $\alpha : A \rightarrow S$ be the quotient $G/M_S$. Suppose $A$ proper over $S$. Then, $M_S$ is fiberwise discrete in $G$ and the quotient $\hat{A} := \hat{G}/M_S$ is proper over $S$ [BL91] Proposition 3.4]. Let $\hat{\alpha} : \hat{A} \rightarrow S$ be the structural morphism. Let $u : \hat{G} \rightarrow A$ and $\hat{u} : \hat{G} \rightarrow \hat{A}$ be the quotient maps.

The situation is resumed in the following diagrams:

\[
\begin{array}{ccc}
T & \rightarrow & G \\
\downarrow{\alpha} & & \downarrow{p} \\
A & \rightarrow & B
\end{array}
\quad \begin{array}{ccc}
\hat{T} & \rightarrow & \hat{G} \\
\downarrow{\hat{\alpha}} & & \downarrow{\hat{p}} \\
\hat{A} & \rightarrow & \hat{B}
\end{array}
\]

4.2.4. **Homogeneous line bundles.** Homogeneous line bundles on $A$ are defined in the evident way. Given a homogeneous line bundle $L$ on $A$, there are

- a homogeneous line bundle $F$ on $B$,
- an isomorphism $\rho : u^*L \rightarrow p^*F$,
- a trivialization $r : \mathcal{O}_{M_S} \rightarrow c^*F$ such that, for $m, m' \in M$,

$$r(m) \otimes r(m') = r(m + m),$$

where the equality has to be understood via the isomorphism of line bundles $c(m)^*F \otimes c(m')^*F \simeq c(m + m')^*F$ given by homogeneity of $F$,

with the following property: the $M$-linearization on $p^*F$ induced by the natural $M$-linearization of $u^*L$ via the isomorphism $\rho$ is given, for $m \in M$, by the isomorphism

$$p^*F \rightarrow p^*F \otimes \gamma^*c(m)^*F,$$

$$s \mapsto s \otimes r(m).$$

Conversely, given a homogeneous line bundle $F$ and trivialization $r$ of $c^*F$ satisfying (1.2), the above formula defines an $M$-linearization on $p^*F$. By descent, the
couple \((F, r)\) defines a homogeneous line bundle \(L\) on \(A\). For a proof, see [BL91, Theorem 6.7].

For \(i = 1, 2\), let \(F_i\) be a homogeneous line bundle on \(B\), \(r_i\) a trivialization of \(c^*F_i\), satisfying (4.2), and \(L_i\) the homogeneous line bundle on \(A\) induced by \((F_i, r_i)\). Then the line bundles \(L_1, L_2\) are isomorphic if and only if there is \(\hat{m} \in \hat{M}\) such that \(F_2\) is isomorphic to \(F_1 \otimes (\text{id}_B, \hat{c}(m))^*L_B\) and, via this isomorphism, for all \(m \in M\),

\[
\hat{r}_2(m) = r_1(m) \otimes (m, \hat{m}).
\]

See [BL91, Corollary 4.10]. For further reference, a direct consequence of this is the following:

**Lemma 4.4.** Let \(\hat{m} \in \hat{M}\). Let \(L_{\hat{m}}\) be the homogeneous line on \(A\) associated with the homogeneous line bundle \(F_{\hat{m}} := (\text{id}_B, \hat{c}(m))^*L_B\) on \(B\) and the trivialization \(m \mapsto (m, \hat{m})\). Then, the line bundle \(L_{\hat{m}}\) is trivial.

4.2.5. **Duality.** The proper \(S\)-analytic group \(\hat{A}\) represents the functor associating to a \(S\)-analytic space \(S'\) the group of homogeneous line bundles on \(A \times S'\) (up to isomorphism); see [BL91, Theorem 6.8].

Let \(L_A\) be the Poincaré bundle on \(A \times \hat{A}\). For \(m \in M\), \(\hat{m} \in \hat{M}\), let \(\text{tr}_m : G \to G\), \(\text{tr}_{\hat{m}} : \hat{G} \to \hat{G}\) be respectively the translation by \(m\), \(\hat{m}\). The \(M_S \times M_S\)-linearized line bundle \((u, \hat{u})^*L_A\) is isomorphic to \((p, \hat{p})^*L_B\) together with the linearization given, for \(m \in M\) and \(\hat{m} \in \hat{M}\), by the isomorphism \(\lambda_{m, \hat{m}} : (p, \hat{p})^*L_B \to (p \circ \text{tr}_m, \hat{p} \circ \text{tr}_{\hat{m}})^*L_B\) described as follows: for a \(S\)-analytic space \(S'\), \(g \in G(S')\), \(\hat{g} \in \hat{G}(S')\), and a global section \(s\) of \((p(g), \hat{p}(\hat{g}))^*L_B\),

\[
(g, \hat{g})^*\lambda_{m, \hat{m}}(s) = ((g, \hat{m})_G \otimes (m, \hat{m})_G) \otimes ((m, \hat{g})_G \otimes s).
\]

In the formula above, for \(b, b' \in B(S')\) (resp. \(\hat{b}, \hat{b}' \in \hat{B}(S')\)), the symbol \(\otimes\) (resp. \(\hat{\otimes}\)) stands for the isomorphism

\[
(b, \text{id}_B)^*L_B \otimes (b, \text{id}_B)^*L_B \xrightarrow{\sim} (b + b', \text{id}_B)^*L_B
\]

(resp. \((\text{id}_B, \hat{b})^*L_B \otimes (\text{id}_B, \hat{b'})^*L_B \xrightarrow{\sim} (\text{id}_B, \hat{b} + \hat{b'})^*L_B\))

given by homogeneity of \(L_B\).

4.2.6. **The affine quotient of \(G\).** Suppose \(S = \text{Spf}(R)\). Let \(T_0\) be the split \(K\)-torus with group of characters \(\text{Ker}(\hat{c})\): it is the relative spectrum of the \(K\)-algebra of finite presentation:

\[
K[\text{Ker}(\hat{c})] = \bigoplus_{\hat{m} \in \text{Ker}(\hat{c})} K_{\chi_{\hat{m}}}.
\]

By definition, for \(\hat{m} \in \text{Ker}(\hat{c})\), the line bundle \((\text{id}_B, \hat{c}(\hat{m}))^*L_B\) is trivial. Therefore, the relative spectrum of the \(\mathcal{O}_B\)-algebra of finite presentation

\[
\mathcal{O}_B[G]_0 := \bigoplus_{\hat{m} \in \text{Ker}(\hat{c})} (\text{id}_B, \hat{c}(\hat{m}))^*L_B
\]

is \(T_0 \times B\). The inclusion \(\mathcal{O}_B[G]_0 \subseteq \mathcal{O}_B[G]\) therefore induces a morphism of \(K\)-analytic groups

\[
\text{pr}_{T_0} : G \longrightarrow T_0 \times B \xrightarrow{\text{pr}_1} T_0.
\]

**Proposition 4.5.** With the notation above, composing a function with \(\text{pr}_{T_0}\) induces an isomorphism

\[
H^0(T_0, \mathcal{O}_{T_0}) \longrightarrow H^0(G, \mathcal{O}_G).
\]

**Proof.** According with Proposition [4.2] and the notation therein introduced,

\[
H^0(G, \mathcal{O}_G) = \bigoplus_{\hat{m} \in \hat{M}} H^0(B, (\text{id}_B, \hat{c}(\hat{m}))^*L_B^\vee).
\]
One concludes because a non-trivial homogeneous line bundle on $B$ has no nonzero sections.  

4.3. Uniformization of the universal vector extension.

4.3.1. The universal extension on the universal cover. The definitions and the results expounded in section concerning the universal extension of an abelian scheme are adopted here for the abeloid variety $A$.

Let $U_B$ be generic fiber of the universal extension $U_B$ of $O_B$. Let $U_G$ be the push-out of $U_B$ along $dp$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & \gamma^*\omega_B & \longrightarrow & \tilde{p}^*U_B & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0 \\
& & \downarrow{dp} & & \downarrow{\gamma} & & \downarrow & & \\
0 & \longrightarrow & \gamma^*\omega_G & \longrightarrow & U_G & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0.
\end{array}
$$

Adopt the rigid-analytic analogue of Notation Consider $\mathcal{L}_G := (p, \tilde{p})^*\mathcal{L}_B$, $\iota = (\iota_G, \text{id}_G)$, $\pi = (\pi_G, \text{id}_G)$ and $e = (e_G, \text{id}_G)$.

**Proposition 4.6.** With the notation above,

1. the homomorphism of $O_G$-modules $du: u^*\Omega^1_A \to \Omega^1_G$ is an isomorphism;
2. the canonical isomorphism $\Phi: (p, \tilde{p})^*\mathcal{L}_B \to (u, \tilde{u})^*\mathcal{L}_A$ induces an isomorphism $\Psi: \tilde{u}^*\mathcal{U}_A \to \mathcal{U}_G$ making the following diagram commutative:

\[
\begin{array}{cccccc}
0 & \longrightarrow & \gamma^*\omega_A & \longrightarrow & \tilde{u}^*\mathcal{U}_A & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0 \\
& & \downarrow{du} & & \downarrow{\gamma} & & \downarrow & & \\
0 & \longrightarrow & \gamma^*\omega_G & \longrightarrow & \mathcal{U}_G & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0.
\end{array}
\]

3. the isomorphism of $O_A$-modules constructed in Remark induces a homomorphism of $O_G$-modules $U_G \xrightarrow{\sim} \pi_*\iota^*\mathcal{L}_G \otimes e^*\mathcal{L}_G$.

**Proof.** (1) This is because the morphism $u: G \to A$ is étale [Ber93, 3.3]).

(2) Let $q_A: A \times A \to A$, $q_B: B \times B \to B$ and $q_G: G \times G \to G$ be the projections onto the second factor. By the analogue of Proposition 1.3 in the analytic framework, the Atiyah extension of $(u, \tilde{u})^*\mathcal{L}_A$ relative to $q_G$ is the push-out of $(u, \tilde{u})^*\mathit{At}_{q_A}(\mathcal{L}_A)$ along $du$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & (u, \tilde{u})^*\Omega^1_A & \longrightarrow & (u, \tilde{u})^*\mathit{At}_{q_A}(\mathcal{L}_A) & \longrightarrow & \mathcal{O}_{G \times G} & \longrightarrow & 0 \\
& & \downarrow{du} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^1_{q_G} & \longrightarrow & \mathit{At}_{q_G}((u, \tilde{u})^*\mathcal{L}_A) & \longrightarrow & \mathcal{O}_{G \times G} & \longrightarrow & 0.
\end{array}
$$

Let $q_B: B \times B \to B$ be the second projection. Similarly, the Atiyah extension of $(p, \tilde{p})^*\mathcal{L}_B$ relative to $q_G$ is the push-out of $(p, \tilde{p})^*\mathit{At}_{q_B}(\mathcal{L}_B)$ along $dp$:

$$
\begin{array}{cccccc}
0 & \longrightarrow & (p, \tilde{p})^*\Omega^3_B & \longrightarrow & (p, \tilde{p})^*\mathit{At}_{q_B}(\mathcal{L}_B) & \longrightarrow & \mathcal{O}_{G \times G} & \longrightarrow & 0 \\
& & \downarrow{dp} & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & \Omega^3_{q_B} & \longrightarrow & \mathit{At}_{q_B}((p, \tilde{p})^*\mathcal{L}_B) & \longrightarrow & \mathcal{O}_{G \times G} & \longrightarrow & 0.
\end{array}
$$

The isomorphism $\Phi: (u, \tilde{u})^*\mathcal{L}_A \to (p, \tilde{p})^*\mathcal{L}_B$ induces an isomorphism between Atiyah extensions $\mathit{At}_{q_A}: \mathit{At}_{q_A}((u, \tilde{u})^*\mathcal{L}_A) \to \mathit{At}_{q_A}((p, \tilde{p})^*\mathcal{L}_B)$. The wanted isomorphism is $\Psi := (e_G, \text{id}_G)^*\mathit{At}_{q_G} \Phi$.

(3) Clear.
The isomorphism $\Psi: \tilde{u}^*U_\tilde{A} \to U_\tilde{G}$ endows $U_\tilde{G}$ with a $\tilde{M}_S$-linearization given, for $\tilde{m} \in \tilde{M}$, by an isomorphism $\lambda_{U_{\tilde{G}}, \tilde{m}}: U_\tilde{G} \to \text{tr}_m^*U_\tilde{G}$. Let $\tilde{m} \in \tilde{M}$. Let $\chi_{\tilde{m}}: T \to \mathbb{G}_m$ be the corresponding character of $T$. By definition, $\chi_{\tilde{m}}$ is a basis of the rank 1 free $O_S$-module $(e_G, \tilde{m})^*L_\tilde{G}$. Moreover,

$$\chi_{\tilde{m}}((e_G, \tilde{m})) = 1,$$

so that $\chi_{\tilde{m}}$ is the basis dual to $\langle e_G, \tilde{m} \rangle$. For $\tilde{m} \in \tilde{M}$, consider the section

$$\langle \chi_{\tilde{m}} \rangle \otimes \chi_{\tilde{m}}: O_S \to \tilde{m}^*U_\tilde{G} = \pi_G, (e_G, \tilde{m})^*L_\tilde{G} \otimes (e_G, \tilde{m})^*L_\tilde{G}.$$

Let $\mu_1, \mu_2: \tilde{A} \to \tilde{A}$ respectively the group law, the first, and the second projection.

**Proposition 4.7.** Let $\tilde{m} \in \tilde{M}$. The isomorphism

$$(\text{id}_A, \mu)^*L_A \sim (\text{id}_A, \mu_1)^*L_A \otimes (\text{id}_A, \mu_2)^*L_A$$

giving $L_A$ the structure of the universal homogeneous line bundle, induces an isomorphism $\varphi$ of $\text{tr}_m^*U_\tilde{G}$ with the Baer sum $U_\tilde{G} + B \gamma^*m^*U_\tilde{G}$ of the extensions $U_\tilde{G}$ and $\gamma^*m^*U_\tilde{G}$. The isomorphism $\varphi$ fits in the following commutative diagram of $O_{\tilde{G}}$-modules

$$
\begin{array}{cccccc}
0 & \rightarrow & \gamma^*\omega_{\tilde{G}} & \rightarrow & \text{tr}_m^*U_\tilde{G} & \rightarrow & O_{\tilde{G}} & \rightarrow & 0 \\
0 & \rightarrow & \gamma^*\omega_{\tilde{G}} & \rightarrow & \tilde{U}_{\tilde{G}} + B \gamma^*m^*\tilde{U}_{\tilde{G}} & \rightarrow & O_{\tilde{G}} & \rightarrow & 0
\end{array}
$$

and is such that $\varphi \circ \lambda_{U_{\tilde{G}}, \tilde{m}}: U_{\tilde{G}} \to U_{\tilde{G}} + B \gamma^*m^*U_{\tilde{G}}$ is the homomorphism induced by the section $\langle (e_G, \tilde{m}) \rangle \otimes \chi_{\tilde{m}}$.

**Proof.** For the existence of the isomorphism $\varphi$, see Proposition [11.12]. Now, the isomorphism $\Phi: (u, \tilde{u})^*L_A \rightarrow (p, \tilde{p})^*L_B$ induces isomorphisms of $O_{\tilde{G}}$-modules

$$\tilde{u}^*(\pi_A, \text{id}_A) \circ (\tau_A, \text{id}_A)^*L_A \sim (\pi_{\tilde{G}}, \text{id}_{\tilde{G}}) \circ (e_G, \text{id}_G)^*L_B,$$

$$\tilde{u}^*(e_A, \text{id}_A)^*L_A \sim (e_G, \text{id}_G)^*(p, \tilde{p})^*L_B.$$ 

Via these isomorphisms, the $O_{\tilde{G}}$-modules $\pi_{\tau^*L_G}$ and $e^*L_G$ acquire natural $\tilde{M}_S$-linearizations. Moreover, the isomorphism of $O_{\tilde{G}}$-modules

$$U_\tilde{G} \sim \pi_{\tau^*L_G} \otimes e^*L_G$$

is $\tilde{M}_S$-equivariant. Because of the formula [11.3], the $\tilde{M}_S$-linearization of $e^*L_G$ (resp. $\tau^*L_G$) is given, for $\tilde{m} \in \tilde{M}$, by the isomorphism

$$e^*\lambda_{0, \tilde{m}}: e^*L_G \rightarrow e^*L_G \otimes (e_G, \tilde{m})^*L_G,$$

$$(\text{resp.} \tau^*\lambda_{0, \tilde{m}}: \tau^*L_G \rightarrow \tau^*L_G \otimes (e_G, \tilde{m})^*L_G,$$

which concludes the proof. \qed
Consider the following commutative and exact diagram of $\mathcal{O}_G$-modules:

$$
\begin{array}{cccc}
0 & 0 & 0 & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \gamma^*\omega_B & \bar{\rho}^*\mathcal{U}_B & \mathcal{O}_G & 0 \\
\downarrow_{dp} & \downarrow & \downarrow & \downarrow \\
0 & \gamma^*\omega_G & \mathcal{U}_G & \mathcal{O}_G & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & \gamma^*\omega_T & \sim \mathcal{U}_G/\bar{\rho}^*\mathcal{U}_B & 0 & 0 \\
\end{array}
$$

(4.5)

The exactness of the lower line is obtained by applying the Snake Lemma. Via the isomorphism $\gamma^*\omega_T \simeq \mathcal{U}_G/\bar{\rho}^*\mathcal{U}_B$ one defines a homomorphism of $\mathcal{O}_S$-modules $pr_{\omega_T}: \mathcal{U}_G \rightarrow \gamma^*\omega_T$.

**Definition 4.8.** Let $\theta_M: \tilde{M}_S \rightarrow \mathcal{V}(\text{Lie}T)$ be the morphism of $S$-analytic groups defined as follows.

Let $t$ be the coordinate function on $\mathbb{G}_m$. Let $\frac{dt}{T}$ be the invariant differential on $\mathbb{G}_m$. For $m \in \tilde{M}$ let $\chi_m: T \rightarrow \mathbb{G}_m$ be the corresponding character. Then $\theta_M(m)$ is the differential form $\gamma^*\frac{dt}{T}$ on $T$.

**Lemma 4.9.** Let $E$ be a vector bundle on $S$. With the notation above, the homomorphism of $H^0(S, \mathcal{O}_S)$-modules

$$
\text{Hom}_{\mathcal{O}_S}-\text{mod}(\omega_T, E) \rightarrow \text{Hom}_S(\tilde{M}_S, \mathcal{V}(E))
$$

$$
\varphi \mapsto \mathcal{V}(\varphi) \circ \theta_M
$$

is an isomorphism.

The group $\tilde{M}_S$ is by definition the Cartier dual $\text{Hom}(T, \mathbb{G}_m)$ of $T$. The previous statement says that the morphism of $S$-analytic groups $\theta_M: \tilde{M}_S \rightarrow \mathcal{V}(\text{Lie}T)$ is the universal vector hull of the $S$-analytic group $M_S$.

**Proof.** This follows from the fact that the $\mathcal{O}_S$-module generated by $\text{Im} \theta_M$ is $\omega_T$. 

**Proposition 4.10.** With the notation above,

$$
pr_{\omega_T}(\langle e_G, \tilde{m} \rangle \otimes \chi_{\tilde{m}}) = \theta_M(\tilde{m}).
$$

**Proof.** Consider the homomorphism $\eta: \pi_{G, 1}(e_G, \tilde{m})^*\mathcal{L}_G \rightarrow \omega_T \otimes (e_G, \tilde{m})^*\mathcal{L}_G$ obtained from the homomorphism $\tilde{m}^*\rho_{\omega_T}: \mathcal{U}_G \rightarrow \tilde{m}^*\mathcal{U}_G \equiv \omega_T$ by taking the tensor product with $(e_G, \tilde{m})^*\mathcal{L}_G$. Then, the statement amounts to the identity

$$
\eta(\langle e_G, \tilde{m} \rangle) = \theta_M(\tilde{m}) \otimes (e_G, \tilde{m}).
$$

Let $p_1: G_1 \rightarrow B_1$ the morphism induced by $p$. Let $e_{B_1}: S \rightarrow B_1$ the closed embedding of the zero section in $B_1$. Consider the following commutative diagram of $\mathcal{O}_{B_1}$-modules:

$$
\begin{array}{cccc}
0 & e_{B_1}^*\omega_B & \mathcal{O}_{B_1} & e_{B_1}^*\mathcal{O}_S & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & e_{B_1}^*\omega_B & p_{1*}\mathcal{O}_{G_1} & p_{1*}e_{G_1}^*\mathcal{O}_S & 0 \\
\downarrow & \downarrow & \downarrow & \downarrow \\
0 & e_{B_1}^*\omega_T & \sim p_{1*}\mathcal{O}_{G_1}/\mathcal{O}_{B_1} & 0.
\end{array}
$$
By taking the tensor of the previous commutative diagram with \((i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B\), one sees that the cokernel of the natural homomorphism
\[
(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B \rightarrow p_1*p_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B = p_1^*(\iota_G, \bar{m})^*\mathcal{L}_G
\]
is isomorphic to \(e_{B_1,*}(\omega_T \otimes (e_G, \bar{m})^*\mathcal{L}_G)\). Moreover, \(\eta\) is the push-forward along \(\pi_B: B_1 \rightarrow S\) of the surjection
\[
\varphi: p_1*p_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B \rightarrow e_{B_1,*}(\omega_T \otimes (e_G, \bar{m})^*\mathcal{L}_G).
\]
Therefore one has to prove the equality
\[
\varphi(\iota_G, \bar{m}) = \theta_{\mathfrak{M}}(\bar{m}) \otimes \{e_G, \bar{m}\}.
\]

Let \(H = G_{\bar{m}}\) be total space of the line bundle \((\mathrm{id}_B, \bar{c}(\bar{m}))^*\mathcal{L}_B\) deprived of its zero section. Let \(f = \mathrm{pr}_{G, \bar{m}}: G \rightarrow H\) be the projection onto \(H\). By definition, the section \(\langle \iota_G, \bar{m}\rangle\) is the morphism \(f \circ \iota_G\). The morphism \(f \circ \iota_G\) writes as \(\iota_H \circ f_1\) for a morphism \(f_1: G_1 \rightarrow H_1\).

Let \(q: H \rightarrow B\) be the structural morphism. Let \(q_1: H_1 \rightarrow B_1\) the morphism induced by \(q\). Let \(s: O_{B_1} \rightarrow q_1*q_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B\) the section corresponding to the closed immersion \(\iota_H: H_1 \rightarrow H\). Then, the following diagram of \(\mathcal{O}_{B_1}\)-modules

\[
\begin{array}{ccc}
\mathcal{O}_{B_1} & \overset{\langle \iota_G, \bar{m}\rangle}{\longrightarrow} & p_1*p_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B \\
\downarrow \uparrow & & \uparrow \\
\mathcal{O}_{B_1} & \overset{s}{\longrightarrow} & q_1*q_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B
\end{array}
\]
is commutative. Arguing similarly to what done above, one sees that the cokernel of the natural homomorphism \((i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B \rightarrow q_1*q_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B\) is isomorphic to \(e_{B_1,*}(\omega_{\mathfrak{g}_m} \otimes (e_G, \bar{m})^*\mathcal{L}_G)\).

Let \(\psi: q_1*q_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B \rightarrow e_{B_1,*}(\omega_{\mathfrak{g}_m} \otimes (e_G, \bar{m})^*\mathcal{L}_G)\) be the projection onto the cokernel. Let \(d\chi_m: \omega_{\mathfrak{g}_m} \rightarrow \omega_T\) be the differential of the character \(\chi_m\). Then, the following diagram of \(\mathcal{O}_{B_1}\)-modules

\[
\begin{array}{ccc}
(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B & \overset{\varphi}{\longrightarrow} & e_{B_1,*}(\omega_T \otimes (e_G, \bar{m})^*\mathcal{L}_G) \\
\downarrow & & \downarrow \psi \\
(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B & \overset{d\chi_m \otimes \mathrm{id}}{\longrightarrow} & e_{B_1,*}(\omega_{\mathfrak{g}_m} \otimes (e_G, \bar{m})^*\mathcal{L}_G)
\end{array}
\]
is commutative.

By combining the commutative diagrams \((4.6)\) and \((4.7)\), one obtains the equality
\[
\varphi(\iota_G, \bar{m}) = (d\chi_m \otimes \mathrm{id})(\psi(s)).
\]

In order to compute the right-hand side of the previous identity, remark first that the natural homomorphism induced by base change \(e_{B_1,*}q_1*q_1^*(i_B, \bar{c}(\bar{m}))^*\mathcal{L}_B \rightarrow (e_G, \bar{m})^*\mathcal{L}_G\) is an isomorphism. Second, note that the section \(s\) fits in the following commutative diagram of \(\mathcal{O}_{B_1}\)-modules

\[
\begin{array}{ccc}
0 & \longrightarrow & e_{B_1,*}\omega_B \\
\downarrow & & \downarrow \psi \otimes (e_G, \bar{m}) \\
e_{B_1,*}(\omega_H \otimes (e_G, \bar{m})^*\mathcal{L}_G) & \longrightarrow & e_{B_1,*}(e_G, \bar{m})^*\mathcal{L}_G
\end{array}
\]

Third, the homomorphism of \(\mathcal{O}_{B_1}\)-modules \(\psi\) factors through the homomorphism of \(\mathcal{O}_S\)-modules \((e_G, \bar{m})^*\mathcal{L}_G \rightarrow \omega_{\mathfrak{g}_m} \otimes (e_G, \bar{m})^*\mathcal{L}_G\),
\[
(e_G, \bar{m}) \mapsto \mathcal{L}_{\mathfrak{g}_m} \otimes (e_G, \bar{m}).
\]
Altogether, this gives
\[ \varphi((\xi_1, \xi_2)) = d\chi_2(\xi_1) \otimes \langle e_G, \xi_2 \rangle = \theta_{x_1}(\xi_2) \otimes \langle e_G, \xi_2 \rangle, \]
that is what we wanted to prove.

4.4. Unipotent bundles on abeloid varieties.

4.4.1. Extensions of the trivial line bundle. Let \( f: \tilde{M}_S \to \mathbb{V}(E) \) be a morphism of \( S \)-analytic groups. Consider the \( \tilde{M}_S \)-linearization of the vector bundle \( W := \tilde{\gamma}^* E \oplus \mathcal{O}_E \) defined, for \( \tilde{m} \in \tilde{M} \), by the isomorphism \( W \to W \),
\[ (x, \lambda) \mapsto (x + \lambda f(\tilde{m}), \lambda). \]

The projection \( W \to \mathcal{O}_A \), \( (x, \lambda) \mapsto \lambda \) and the injection \( \tilde{\gamma}^* E \to W \), \( x \mapsto (x, 0) \) are \( \tilde{M}_S \)-equivariant with respect to the trivial action on \( \mathcal{O}_A \) and \( \tilde{\gamma}^* E \). This gives rise to a \( \tilde{M}_S \)-equivariant short exact sequence of \( \mathcal{O}_E \)-modules
\[ 0 \to \tilde{\gamma}^* E \to W \to \mathcal{O}_E \to 0. \]

Let \( V \) be the vector bundle on \( \tilde{A} \) deduced from \( W \) (endowed with the linearization induced by \( f \)). The previous short exact sequence descends to a short exact sequence of \( \mathcal{O}_A \)-modules
\[ 0 \to \tilde{\alpha}^* E \to V \to \mathcal{O}_A \to 0. \]

Let \( \eta(f) \in H^1(\tilde{A}, \tilde{\alpha}^* E) \) its isomorphism class. Since the extension \( \tilde{\alpha}^* V \) splits, the cohomology class \( \eta(f) \) lies in \( H^0(S, R^1 \tilde{\alpha}^* \tilde{\alpha}^* E) \).

**Lemma 4.11.** The so-defined map \( \eta: \text{Hom}_S(\tilde{M}_S, \mathbb{V}(E)) \to H^0(S, R^1 \tilde{\alpha}^* \tilde{\alpha}^* E) \) is an injective homomorphism of \( \mathcal{O}(S) \)-modules.

**Proof.** With the notation above, saying that \( \eta(f) \) vanishes means that there is an isomorphism of \( \mathcal{O}_E \)-modules \( \varphi: \tilde{\gamma}^* E \oplus \mathcal{O}_E \to W \) fitting in the following commutative and exact diagram
\[
\begin{array}{cccccc}
0 & \to & \tilde{\gamma}^* E & \to & \tilde{\gamma}^* E \oplus \mathcal{O}_E & \to & \mathcal{O}_E & \to & 0 \\
0 & \to & \tilde{\gamma}^* E & \to & W & \to & \mathcal{O}_E & \to & 0,
\end{array}
\]
equivariant with respect to the trivial action on \( \tilde{\gamma}^* E \oplus \mathcal{O}_E \) and the linearization on \( W \) given by \( f \). Such an isomorphism \( \varphi \) is of the form \( (x, \lambda) \mapsto (x + \lambda \sigma, \lambda) \) for some global section \( \sigma: \mathcal{O}_E \to \tilde{\gamma}^* E \). Equivariance then translates into the equality, for each \( \tilde{m} \in \tilde{M} \),
\[ (x + \lambda \sigma, \lambda) = (x + \lambda(\sigma + f(\tilde{m})), \lambda), \]
which in turn implies \( f(\tilde{m}) = 0 \).

Let \( \text{Hom}_S(\tilde{M}_S, \mathbb{V}(E)) \) the sheaf associating to an analytic domain \( S' \subseteq S \), the set \( \text{Hom}_S(\tilde{M}_S, \mathbb{V}(E|_{S'})) \) of morphisms of \( S' \)-analytic groups \( \tilde{M}_S \to \mathbb{V}(E|_{S'}) \). The structure of \( H^0(S', \mathcal{O}_{S'}) \)-module on \( \mathbb{V}(E|_{S'}) \) defines a structure of an \( \mathcal{O}_{S'} \)-module on \( \text{Hom}_S(\tilde{M}_S, \mathbb{V}(E)) \). By applying the definition of the homomorphism \( \eta \) to each analytic domain \( S' \), one obtains a homomorphism of \( \mathcal{O}_{S'} \)-modules
\[ \eta: \text{Hom}_S(\tilde{M}_S, \mathbb{V}(E)) \to R^1 \tilde{\alpha}^* \tilde{\alpha}^* E. \]

According to Lemma 4.11, the map \( \text{Hom}_S(\omega_T, E) \to \text{Hom}_S(\tilde{M}_S, \mathbb{V}(E)) \), defined by \( \varphi \mapsto \mathbb{V}(\varphi) \circ \theta_{x_T} \), is an isomorphism. Let \( \xi \) be its inverse.
Let $\Phi_A : \mathbf{R}^1 \tilde{\alpha}_* \mathcal{O}_A \to \text{Lie } A$ the rigid-analytic analogue of the isomorphism $\Phi$ constructed in section 1.2.2 and $\Phi_B : \mathbf{R}^1 \tilde{\beta}_* \mathcal{O}_B \to \text{Lie } B$ the generic fiber of the isomorphism $\Phi_B : \mathbf{R}^1 \tilde{\beta}_* \mathcal{O}_B \to \text{Lie } B$ defined analogously for the formal abelian scheme $B$. For a vector bundle $E$ on $S$, consider the commutative diagram of $\mathcal{O}_S$-modules

$$
\begin{array}{c}
\mathbf{R}^1 \tilde{\alpha}_* \tilde{\alpha}^* E & \longrightarrow & \mathbf{R}^1 \tilde{\beta}_* \tilde{\beta}^* E \\
\Phi_A \otimes \text{id}_E & \downarrow \Phi_B \otimes \text{id}_E \\
\text{Hom}(\omega_A, E) & \longrightarrow & \text{Hom}(\omega_B, E),
\end{array}
$$

where the lower horizontal arrow is the composition with $(d\bar{u})^{-1} \circ \bar{p} : \omega_B \longrightarrow \omega_G \longrightarrow \omega_A$, and the upper horizontal arrow is the unique making such a square commutative.

**Proposition 4.12.** Let $E$ be a vector bundle on $S$. Then, the following diagram of $\mathcal{O}_S$-modules is commutative and exact:

$$
\begin{array}{c}
0 & \longrightarrow & \text{Hom}(\mathcal{M}_S, \mathcal{V}(E)) & \longrightarrow & \mathbf{R}^1 \tilde{\alpha}_* \tilde{\alpha}^* E & \longrightarrow & \mathbf{R}^1 \tilde{\beta}_* \tilde{\beta}^* E & \longrightarrow & 0 \\
0 & \longrightarrow & \text{Hom}(\omega_T, E) & \longrightarrow & \text{Hom}(\omega_A, E) & \longrightarrow & \text{Hom}(\omega_B, E) & \longrightarrow & 0.
\end{array}
$$

**Proof.** Since the vertical arrows are isomorphisms, and the right-most square is commutative by definition, it suffices to show that the diagram of $\mathcal{O}_S$-modules

$$
\begin{array}{c}
\text{Hom}(\mathcal{M}_S, \mathcal{V}(E)) & \longrightarrow & \mathbf{R}^1 \tilde{\alpha}_* \tilde{\alpha}^* E \\
\downarrow \xi & & \downarrow \Phi_A \otimes \text{id}_E \\
\text{Hom}(\omega_T, E) & \longrightarrow & \text{Hom}(\omega_A, E)
\end{array}
$$

is commutative. For, up to replacing $S$ by any of its open subsets, it suffices to show that the diagram induced on global sections is commutative.

Let $f : \mathcal{M}_S \to \mathcal{V}(E)$ be a morphism of $S$-analytic groups. Let $\varphi : \omega_T \to E$ be the unique homomorphism of $\mathcal{O}_S$-modules such that $f = \mathcal{V}(\varphi) \circ \theta_A$. Let $\tilde{\varphi}$ be the composition of $\varphi$ with the projection $\omega_G \to \omega_T$. Let $V$ be the push-out of $\mathcal{U}_A$ along $\tilde{\alpha}^* \tilde{\varphi} \circ d\bar{u}$:

$$
\begin{array}{c}
0 & \longrightarrow & \tilde{\alpha}^* \omega_A & \longrightarrow & \mathcal{U}_A & \longrightarrow & \mathcal{O}_A & \longrightarrow & 0 \\
\downarrow \tilde{\alpha}^* \tilde{\varphi} \circ d\bar{u} & & & & \| & & & & \\
0 & \longrightarrow & \tilde{\alpha}^* E & \longrightarrow & V & \longrightarrow & \mathcal{O}_A & \longrightarrow & 0
\end{array}
$$

The statement amounts to $\eta(f)$ being the isomorphism class of $V$.

Consider the $\mathcal{M}_S$-linearization of the vector bundle $W := \tilde{\gamma}^* E \oplus \mathcal{O}_G$ defined, for $\bar{m} \in \mathcal{M}$, by the isomorphism $W \to W$, $(x, \lambda) \mapsto (x + \lambda f(\bar{m}), \lambda)$. Via the isomorphism $\mathcal{U}_G / \mathcal{p} \mathcal{U}_G \cong \tilde{\gamma}^* \omega_G$, one obtains a homomorphism of $\mathcal{O}_G$-modules $\psi : \mathcal{U}_G \to W$ fitting in the following commutative diagram of $\mathcal{O}_G$-modules:

$$
\begin{array}{c}
0 & \longrightarrow & \tilde{\gamma}^* \omega_G & \longrightarrow & \mathcal{U}_G & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0 \\
\downarrow \tilde{\gamma}^* \tilde{\varphi} & & & & \| & & & & \\
0 & \longrightarrow & \tilde{\gamma}^* E & \longrightarrow & W & \longrightarrow & \mathcal{O}_G & \longrightarrow & 0
\end{array}
$$

By definition of $\varphi$, for $\bar{m} \in \mathcal{M}$, one has $f(\bar{m}) = \varphi(\theta_\mathcal{M}(\bar{m}))$. Proposition 4.10 implies that $\psi$ is $\mathcal{M}_S$-equivariant, hence that the extension deduced from $W$ by descent is
the push-out of \( U_A \) along \( \hat{\alpha}^* \tilde{\varphi} \circ \tilde{\mu} \). As the latter is \( V \) by definition, this concludes the proof.

**Corollary 4.13.** Let \( E \) be a vector bundle on \( S \).

1. Let \( V \) be an extension of \( \mathcal{O}_A \) by \( \hat{\alpha}^* E \) such that \( \hat{\epsilon}^* V \) splits. Then, there are an extension \( W \) of \( \mathcal{O}_B \) by \( \hat{\beta}^* E \) and an isomorphism \( \hat{\alpha}^* V \cong \hat{\beta}^* W \).

2. Suppose \( H^1(S, \text{Hom}(\omega_T, E)) = 0 \). Let \( W' \) be an extension of \( \mathcal{O}_B \) by \( \hat{\beta}^* E \).

Then, there are an extension \( V' \) of \( \mathcal{O}_A \) by \( \hat{\alpha}^* E \) and an isomorphism of extensions \( \hat{\alpha}^* V' \cong \hat{\beta}^* W' \).

**Proof.** This is a consequence of Proposition 4.12. (1) The extension \( W \) is the image of the isomorphism class of \( V \) via the homomorphism \( R^1 \hat{\alpha}_* \hat{\alpha}^* E \rightarrow R^1 \hat{\beta}_* \hat{\beta}^* E \).

(2) The homomorphism \( H^1(S, \text{Hom}(\omega_A, E)) \rightarrow H^1(S, \text{Hom}(\omega_B, E)) \) is surjective because of the hypothesis, and the statement follows. \( \square \)

4.5. **Universal cover of the universal vector extension.** Consider the affine bundle \( \hat{G}^x := \mathbb{P}(U_G) \setminus \mathbb{P}(\hat{\gamma}^* \omega_G) \). By definition, the \( \mathcal{O}_G \)-module \( U_G \) is the push-out of \( \hat{\alpha}^* U_B \) along \( dp \). The homomorphism of \( \mathcal{O}_G \)-modules \( \hat{\alpha}^* U_B \rightarrow U_G \) induces a morphism of \( G \)-analytic spaces \( i: \hat{B}^x \times_B \hat{G} \rightarrow \hat{G}^x \).

The isomorphism \( \Psi: \hat{\alpha}^* U_A \rightarrow U_G \) induces an isomorphism of \( S \)-analytic groups \( \hat{A}^x \times_A G \rightarrow \hat{G}^x \). The morphism \( \hat{\alpha}^2: \hat{G}^x \rightarrow \hat{A}^x \) deduced from this identification fits in the following commutative diagram of \( S \)-analytic groups:

\[
\begin{array}{ccc}
0 & \longrightarrow & \mathcal{V}(\omega_G) & \longrightarrow & \hat{G}^x & \longrightarrow & \hat{G} & \longrightarrow & 0 \\
& & \downarrow & \downarrow & \hat{\alpha}^2 & & \hat{\alpha} & & \\
0 & \longrightarrow & \mathcal{V}(\omega_A) & \longrightarrow & \hat{A}^x & \longrightarrow & \hat{A} & \longrightarrow & 0.
\end{array}
\]

For \( \hat{m} \in \hat{M} \), consider the section \( \langle \epsilon_G, \hat{m} \rangle \otimes \chi_{\hat{m}}: \mathcal{O}_S \rightarrow \hat{m}^* U_G \). Via the identification \( U_G \) with \( \pi_* \epsilon^* \mathcal{L}_G \otimes \epsilon^* \mathcal{L}_G^\vee \), the projection \( U_G \rightarrow \mathcal{O}_G \) is the trace map \( \pi_* \epsilon^* \mathcal{L}_G \otimes \epsilon^* \mathcal{L}_G^\vee \rightarrow \epsilon^* \mathcal{L}_G \otimes \epsilon^* \mathcal{L}_G^\vee \). Therefore, because of the equality (1.4), the section \( \langle \epsilon_G, \hat{m} \rangle \otimes \chi_{\hat{m}} \) defines a splitting of the short exact sequence of \( \mathcal{O}_S \)-modules

\[
0 \rightarrow \omega_G \longrightarrow \hat{m}^* U_G \longrightarrow \mathcal{O}_S \rightarrow 0.
\]

Let \( \hat{\varepsilon}^2(\hat{m}) \) denote the so-defined \( S \)-valued point of \( \hat{G}^x \).

**Proposition 4.14.** With the notation above, the map \( \hat{\varepsilon}^2: \hat{M}_S \rightarrow \hat{G}^x \) enjoys the following properties:

1. the composition \( \hat{\varepsilon}^2 \) with \( \hat{G}^x \rightarrow \hat{G} \) is the embedding \( \hat{\varepsilon} \): \( \hat{M}_S \rightarrow \hat{G} \);

2. it is injective;

3. it has a discrete image;

4. it is a group homomorphism;

5. \( \text{Ker} \hat{\mu}^2 = \text{Im} \hat{\varepsilon}^2 \);

6. \( \hat{\mu}^2 \) induces an isomorphism \( \hat{G}^2 / \hat{\varepsilon}^2(\hat{M}) \cong \hat{A}^x \).

**Proof.** (1) holds by construction. (2) follows from (1) because \( \hat{\varepsilon} \) is injective. (3) also follows from (1) because \( \hat{\varepsilon} \) has discrete image. (4) follows from the formulas, for \( \hat{m}, \hat{m}' \in \hat{M} \),

\[
\langle \epsilon_G, \hat{m} \rangle \otimes \langle \epsilon_G, \hat{m}' \rangle = \langle \epsilon_G, \hat{m} + \hat{m}' \rangle,
\]

\[
\langle \mu_G, \hat{m} \rangle \otimes \langle \mu_G, \hat{m}' \rangle = \langle \mu_G, \hat{m} + \hat{m}' \rangle.
\]

(5) The projection \( \hat{G}^x \rightarrow \hat{G} \) induces an isomorphism \( \text{Ker}(\hat{\mu}^2) \cong \text{Ker}(\hat{\mu}) \). Since \( \text{Ker}(\hat{\mu}) = \text{Im} \hat{\varepsilon} \), the statement follows from (1). (6) follows from (5). \( \square \)

Because of the commutative diagram (1.4), by arguing as in 2.4 one obtains a morphism of \( K \)-analytic spaces \( \text{pr}_{\mathcal{V}(\omega_T)}: \hat{G}^x \rightarrow \mathcal{V}(\omega_T) \).
Proposition 4.15. With the notation above,
(1) \( \text{Ker} \, pr_{V(\omega_T)} = \mathcal{B}^* \times_B \mathcal{G} \);
(2) for \( \tilde{m} \in \tilde{M} \), \( \text{pr}_{V(\omega_T)}((\tilde{\varepsilon}^*(\tilde{m}))) = \theta_{\tilde{M}}(\tilde{m}) \).

Proof. (1) follows from the commutative diagram \([\text{1.5}]\). (2) is Proposition \([\text{4.10}]\). □

4.6. Analytic functions on vector extensions. Suppose \( S = \text{Spf}(R) \).

4.6.1. Statements. Let \( E \) a finite-dimensional \( K \)-vector space. Let \( \varphi : \omega_A \to E \) be a \( K \)-linear homomorphism. Let \( V \) be the push-out of \( \mathcal{U}_A \) along \( \varphi \). Let \( X = \mathbb{P}(V) \), \( D = \mathbb{P}(\alpha^*E) \) and \( Y = X \setminus D \). Let \( \pi : X \to A \) be the structural morphism. Let \( C = \text{Coker}(\varphi) \). By arguing as in \([\text{2.4}]\) one obtains a morphism of \( K \)-analytic spaces

\[ \text{pr}_{V(C)} : Y \longrightarrow \mathbb{V}(V) \longrightarrow \mathbb{V}(\alpha^*C) = \mathbb{V}(C) \times A \overset{pr_{\mathcal{V}}}{\longrightarrow} \mathbb{V}(C). \]

Theorem 4.16. Suppose \( \text{char} \, K = 0 \). With the notation introduced above,
(1) composing with \( \text{pr}_{V(C)} \) induces an isomorphism

\[ H^0(V(C), \mathcal{O}_{V(C)}) \xrightarrow{\sim} H^0(Y, \mathcal{O}_Y); \]
(2) for a non-trivial homogeneous line bundle \( L \) on \( A \),

\[ H^0(Y, \pi^*L) = 0. \]

Let \( N \) be a finitely generated free abelian group. Let \( S \) be the split \( K \)-torus with group of characters \( N \). Let \( d : N \to A(R) \) be a group homomorphism. Let \( H \) be the extension of \( A \) by \( S \) determined by \( d \). Let \( S_0 \) be the split \( K \)-torus with group of characters \( \text{Ker}(d) \). By arguing as in \([\text{3.2.1}]\) one obtains a morphism of \( K \)-analytic spaces \( \text{pr}_{S_0} : H \to A \times S_0 \to S_0 \). Consider the morphism of \( K \)-analytic spaces

\[ (\text{pr}_{S_0}, \text{pr}_{V(C)}) : H \times_A U \longrightarrow S_0 \times \mathbb{V}(C). \]

Corollary 4.17. With the notation above,
(1) composing a function on \( S_0 \) with \( \text{pr}_{S_0} \) induces an isomorphism

\[ H^0(S_0, \mathcal{O}_{S_0}) \xrightarrow{\sim} H^0(H, \mathcal{O}_H); \]
(2) if \( \text{char} \, K = 0 \), then composing a function on \( S_0 \times \mathbb{V}(C) \) with \( (\text{pr}_{S_0}, \text{pr}_{V(C)}) \)

induces an isomorphism

\[ H^0(S_0 \times \mathbb{V}(C), \mathcal{O}_{S_0 \times \mathbb{V}(C)}) \xrightarrow{\sim} H^0(H \times_A Y, \mathcal{O}_{H \times_A Y}). \]

Proof of Corollary 4.17. admits Theorem 4.16 (1) is a direct consequence of Proposition 4.2. (2) According to Proposition 4.2 and the notation therein introduced,

\[ H^0(H \times_A Y, \mathcal{O}_{H \times_A Y}) = \bigoplus_{n \in N} H^0(Y, \pi^*L_n^\vee). \]

One concludes thanks to Theorem 4.16. □

4.6.2. Proof of Theorem 4.16 in the case of good reduction. Suppose first \( A \) with good reduction, that is, \( \tilde{M} = 0 \) or equivalently \( A = B \). Under this hypothesis, set \( A := B \).

Let \( \mathcal{E} \subseteq E \) be a free \( R \)-module of finite rank containing \( \varphi(\omega_A) \) and such that \( \mathcal{E} \otimes_R K = E \). Let \( \mathcal{U}_A \) be the universal extension of \( \mathcal{O}_A \), and \( \mathcal{V} \) be the push-out of \( \mathcal{U}_A \) along \( \varphi : \omega_A \to E \). Consider the admissible formal \( R \)-schemes \( \mathcal{X} = \mathbb{P}(\mathcal{V}) \), \( \mathcal{D} = \mathbb{P}(\mathcal{E}) \times A \) and \( \mathcal{Y} = \mathcal{X} \setminus \mathcal{D} \). The generic fiber \( \mathcal{Y}_K \) of \( \mathcal{Y} \) is a compact analytic domain of \( Y \) such that the projection map \( \mathcal{Y}_K \to A \) is surjective (because \( A \) is proper over \( R \)), and

\[ Y_K \cap \mathbb{V}(E) = \mathbb{V}(E)_K. \]
Let $\mathcal{C}$ be the torsion-free quotient of $\varphi: \omega_A \to \mathcal{E}$. For $\varpi \in K^\times$, multiplication by $\varpi$ on $E$ induces an isomorphism $f_{\varpi}: Y \to Y$ sitting in the following commutative diagram:

$$
\begin{array}{c}
0 \longrightarrow \mathbb{V}(E) \longrightarrow Y \longrightarrow A \longrightarrow 0 \\
\varpi \downarrow \quad \quad \quad \downarrow f_{\varpi} \\
0 \longrightarrow \mathbb{V}(E) \longrightarrow Y \longrightarrow A \longrightarrow 0
\end{array}
$$

Note that $f_{\varpi}^{-1}(Y_K) \cap \mathbb{V}(E) = \mathbb{V}(\frac{1}{\varpi}\mathcal{E})_K$. Fix a topologically nilpotent element $\varpi \in R \setminus \{0\}$. For $\nu \in \mathbb{N}$, consider the compact analytic domain

$$Y_\nu := f_{\varpi}^{-1}(Y_K).$$

The analytic domain $Y_\nu$ is contained in the topological interior of $Y_{\nu+1}$, and $\{Y_\nu\}_{\nu \in \mathbb{N}}$ is a $G$-cover of $Y$.

(1) Theorem 3.9 (1) states $H^0(Y, O_Y) = H^0(\mathcal{V}(C), O_{\mathcal{V}(C)})$. Since the $R$-formal scheme $\mathcal{Y}$ is quasi-compact,

$$H^0(Y_K, O_{Y_K}) = H^0(\mathcal{V}(C)_K, O_{\mathcal{V}(C)}).$$

For $\nu \in \mathbb{N}$, via the isomorphism $f_{\varpi}^\nu: Y_\nu \to Y_K$, one deduces

$$H^0(Y_\nu, O_{Y_\nu}) = H^0(\mathcal{V}(\frac{1}{\varpi}\mathcal{C})_K, O_{\mathcal{V}(C)}).$$

By taking the projective limit for $\nu \in \mathbb{N}$,

$$H^0(Y, O_Y) = \lim_{\nu \in \mathbb{N}} H^0(Y_\nu, O_{Y_\nu}) = \lim_{\nu \in \mathbb{N}} H^0(\mathcal{V}(\frac{1}{\varpi}\mathcal{C})_K, O_{\mathcal{V}(C)}) = H^0(\mathcal{V}(\mathcal{C}), O_{\mathcal{V}(C)}).$$

(2) Let $\mathcal{A}$ be the dual formal abelian scheme of $A$. Because of properness of $\mathcal{A}$, the homogeneous line bundle $L$ is the generic fiber of a homogeneous line bundle $\mathcal{L}$ on $\mathcal{A}$. Since the line bundle $\mathcal{L}$ is homogeneous and non-trivial,

$$H^0(\mathcal{A}, \mathcal{L}) = 0.$$  

Then, Theorem 3.9 (2) implies $H^0(Y_K, \pi^*L) = 0$, thus $H^0(Y, \pi^*L) = 0$. 

4.6.3. Proof of Theorem 4.16 in the general case. Identify $\omega_A$ with $\omega_G$ via $d\tilde{a}$. Let $V'$ be the push-out of $U_B$ along $\varphi_{\omega_B}: \omega_B \to E$:

$$
\begin{array}{c}
0 \longrightarrow \beta^* \omega_B \longrightarrow U_B \longrightarrow O_B \longrightarrow 0 \\
\beta^* \varphi_{\omega_B} \downarrow \quad \quad \quad \downarrow \\
0 \longrightarrow \beta^* E \longrightarrow V' \longrightarrow O_B \longrightarrow 0
\end{array}
$$

Let $X' = \mathbb{P}(V')$, $D' = \mathbb{P}(\beta^* E)$ and $Y' = X' \setminus D'$. Let $\pi': X' \to B$ be the structural morphism. According to the description of the universal vector extension of $A$ given in 4.5 one sees that $G \times_A Y$ is isomorphic to $G \times_B Y'$.

For $\tilde{m} \in \tilde{M}$, consider the line bundle $L_{\tilde{m}} = (\text{id}, \tilde{c}(\tilde{m}))^* \mathcal{L}_B$ on $B$ together with the $M$-linearization of $p^* L_{\tilde{m}}$ given, for $m \in M$, by the isomorphism

$$(\text{id}_G, \tilde{m})^* \lambda_{0,m}: p^* L_{\tilde{m}} \longrightarrow tr^*_{\tilde{m}} p^* L_{\tilde{m}} = p^* L_{\tilde{m}} \otimes \gamma^* c(m)^* L_{\tilde{m}}$$

$$s \mapsto s \otimes (m, \tilde{m}).$$

According to Lemma 4.4 the $M$-linearized line bundle $L_{\tilde{m}}$ induces the trivial line bundle on $A$.

The morphism of $K$-analytic groups $A^2 \to Y$ given by construction induces a morphism of $K$-analytic groups $\mu: G^2 \to G \times_A Y$. The group $M$ is embedded in $G \times_A Y$ via $\mu \circ \tilde{c}$, and by Proposition 4.14

$$Y = (G \times_A Y)/\mu(\tilde{c}^2(M)).$$
The image of $M$ in $\mathcal{V}(C')$ is described as follows. The linear map $\varphi: \omega_M \to E$ induces a $K$-linear map $\tilde{\varphi}: \omega_G \to C'$. Because of Proposition 4.15, the image of $M$ in $\mathcal{V}(C')$ is the image of the group homomorphism $\theta := \varphi \circ \theta_M: M \to \omega_K \to C'$.

Let $L$ be a homogeneous line bundle on $A$. According to the description of homogeneous line bundles given in 4.2.4, the datum of $L$ corresponds to the datum of homogeneous line bundle $F$ on $B$ together with a trivialization $r: M \to c^*F$ such that, for $m, m' \in M$,

$$r(m + m') = r(m) \otimes r(m').$$

(As usual, equality has to be understood via $c(m)^*F \otimes c(m')^*F \simeq c(m + m')^*F$.)

By Proposition 4.2, with the notation therein introduced,

$$H^0(G \times_B Y', h^*L) = \bigoplus_{m \in \hat{M}} H^0(Y', \pi'^*F \otimes \pi'^*L_{\hat{m}}^\vee),$$

where $h: G \times_B Y' \to B$ is the structural morphism. By taking $M$-invariants,

$$H^0(Y, \pi'^*L) = \bigoplus_{\hat{m} \in \hat{M}} H^0(Y', \pi'^*F \otimes \pi'^*L_{\hat{m}}^\vee)^M.$$

Let $[F] \in \hat{B}(K)$ be the isomorphism class of $F$. Let

$$\hat{M}_F := \{ \hat{m} \in \hat{M} : c(\hat{m}) = [F] \}.$$

For $\hat{m} \in \hat{M} \setminus \hat{M}_F$, the homogeneous line bundle is $F \otimes (id, c(\hat{m}))^*L_B^\vee$ on $B$ is not trivial. Therefore, Theorem 4.16 \((2)\) in the good reduction case, implies

$$H^0(Y', \pi'^*F \otimes \pi'^*L_{\hat{m}}^\vee) = 0.$$

Let $\hat{m} \in \hat{M}_F$. Then, the line bundle $F \otimes L_{\hat{m}}^\vee$ is trivial. According to Theorem 4.16 \((2)\) in the good reduction case, global sections of $\mathcal{O}_{Y'}$ come from global sections on $\mathcal{V}(C')$.

In order to conclude the proof, one has to compute the $M$-invariant global sections of $\pi'^*F \otimes \pi'^*L_{\hat{m}}^\vee$. To do so, one has to distinguish when $L$ is trivial, and when it is not.

**Case 1: $L$ trivial.** Suppose $L = \mathcal{O}_A$. In this case, $L_{\hat{m}} = \mathcal{O}_B$. Through this identification, the $M$-linearization of $\mathcal{O}_B$ is given by the character $m \mapsto \langle m, \hat{m} \rangle$.

Therefore, $M$-invariant sections of $\pi'^*L_{\hat{m}}^\vee$ are identified with analytic functions $f \in H^0(\mathcal{V}(C'), \mathcal{O}_{\mathcal{V}(C')})$ such that, for all $m \in M$, all $K$-analytic space $S$, and all $x \in \mathcal{V}(C')(S)$,

$$f(x + \theta(m)) = \langle m, \hat{m} \rangle^{-1} f(x).$$

If $\hat{m} \neq 0$, then the character $m \mapsto \langle m, \hat{m} \rangle$ is non trivial, and such an $M$-invariant function $f$ is necessarily 0.

Suppose $\hat{m} = 0$. In this case, $M$-invariant sections of $\pi'^*L_{\hat{m}}^\vee$ are analytic functions $f \in H^0(\mathcal{V}(C), \mathcal{O}_{\mathcal{V}(C)})$ that are invariant under translation by $\theta: M \to C'$.

The afore-mentioned phenomenon of accumulation of abelian groups in a non-Archimedean vector space here occurs again, and implies the constancy of such invariant functions (cf. Lemma 3.11).

**Lemma 4.18.** Let $Z$ be a $K$-analytic space. Let $\Lambda$ be a free abelian group of finite rank. Let $F := \Lambda \otimes Z K$. Let $f \in H^0(Z \times \mathcal{V}(F), \mathcal{O}_{Z \times \mathcal{V}(F)})$ be such that, for each $\lambda \in \Lambda$, each $K$-analytic space $S$, each $x \in \mathcal{V}(C')(S)$, and each $z \in Z(S)$,

$$f(z, x + \lambda) = f(z, x).$$

Then $f$ belongs to $H^0(Z, \mathcal{O}_Z)$.

**Proof of Lemma 4.18** The proof is analogous to that of Lemma 3.11 and for this reason left to the reader. \(\square\)
Let $E' = \text{Im} \varphi$. By choosing a section of the projection $C' \to C = C'/E'$, identify $C$ with a $K$-vector subspace of $C'$ and write $C' = E' \oplus C$. One concludes by applying the preceding Lemma with $\Lambda = \text{Im} \theta$, $F = E'$ and $Z = \mathcal{V}(C)$.

Case 2: $L$ non-trivial. Suppose $L$ is non-trivial. Even though the line bundle $F \otimes L_0^*$ is trivial, the $M$-linearization of $\rho^*(F \otimes L_0^*)$ is not: otherwise $L$ would be trivial, contradicting the hypothesis. In practice this means that there is $m_0 \in M$ such that $r(m_0) \neq \langle m_0, \hat{m} \rangle$.

Let $r(m)/\langle m, \hat{m} \rangle$ denote the unique $t \in K^\times$ such that $r(m) = t\langle m, \hat{m} \rangle$. It follows that $M$-invariant global sections of $F \otimes L_0^*$ on $Y'$ are analytic functions $f \in H^0(Y', \mathcal{O}_{Y'})$ such that, for all $m \in M$,

$$\text{tr}_m^* f = \frac{r(m)}{\langle m, \hat{m} \rangle} f,$$

where $\text{tr}_m : Y' \to Y'$ is the translation by the image of $m$ in $Y'$.

Therefore $M$-invariant global sections of $F \otimes L_0^*$ correspond to analytic functions $f \in H^0(\mathcal{V}(C'), \mathcal{O}_{\mathcal{V}(C')})$ such that, for all $m \in M$,

$$f(x + \theta(m)) = \frac{r(m)}{\langle m, \hat{m} \rangle} f(x).$$

Since the character $m \mapsto r(m)/\langle m, \hat{m} \rangle$ is non-trivial, such a function is necessarily 0.

4.6.4. Application to algebraic groups. Let me recall Brion’s classification of anti-affine groups. By Chevalley’s theorem ([Cor12], a connected algebraic group $G$ is the extension of an abelian variety $A$ by a linear group $L$. If $G$ is moreover commutative, then the linear part $L$ of $G$ is of the form $T \times U$ for a $K$-torus $T$ and a unipotent group $U$.

Let $K^\circ$ denote a separable closure of $K$. For a $K$-scheme $X$, denote $X^\circ$ the $K^\circ$-scheme deduced from $X$ by extending scalars to $K^\circ$. The $K^\circ$-torus $T^\circ$ is split. Let $\Lambda$ be its group of characters. The semi-abelian variety $G^\circ/U^\circ$ is determined by a homomorphism of groups $\lambda : \Lambda \to \hat{A}(K^\circ)$, where $\hat{A}$ is the dual abelian variety.

When the unipotent group $U$ is of the form $\mathcal{V}(E)$ for a finite-dimensional $K$-vector space $E$ (for instance, if $K$ is of characteristic 0, cf. [Ser59] §2.7, p. 172), the vector extension $G/T$ is the push-out of the universal vector extension $A^3$ along a $K$-linear map $\varphi : \omega_{\hat{A}} \to E$.

**Theorem 4.19** ([Brion99] Theorem 2.7]). Let $G$ be a commutative, connected, reduced algebraic group. With the notation above,

- if $\text{char } K = 0$, then $G$ is anti-affine if and only if $c$ is injective and $\varphi$ surjective;
- if $\text{char } K > 0$, then $G$ is anti-affine if and only if $c$ is injective and $U = 0$.

**Theorem 4.20.** Let $K$ be a non-trivially valued complete non-Archimedean field. Let $G$ be an anti-affine algebraic group. Then all analytic functions on $G^{\text{an}}$ are constant.

**Proof.** Up to extending scalars, the field $K$ may be supposed algebraically closed. In this case, the toric part of $G$ is split and the analytification of the abelian variety $A$ is abeloid.

If $\text{char } K = 0$, then $\varphi$ is surjective and $c$ is injective. If $\text{char } K = p > 0$, then $U = 0$ and $c$ is injective. In both cases, Corollary 4.17 implies that every analytic function on $G^{\text{an}}$ is constant. □
For an algebraic group $G$ over $K$, the $K$-algebra $H^0(G,\mathcal{O}_G)$ is of finite type. Let $G_{\text{aff}} := \text{Spec} H^0(G,\mathcal{O}_G)$. The canonical morphism $\pi: G \to G_{\text{aff}}$ is faithfully flat.

**Theorem 4.21.** Let $K$ be a non-trivially valued complete non-Archimedean field. Let $G$ be an algebraic group. With the notation above, composing with $\pi^\text{an}$ induces an isomorphism

$$H^0(G_{\text{aff}},\mathcal{O}_{G_{\text{aff}}}) \xrightarrow{\sim} H^0(G^\text{an},\mathcal{O}_{G^\text{an}}).$$

**Proof.** Let $G_{\text{ant}}$ be the kernel of $\pi$. Consider the following cartesian square

$$\begin{array}{ccc}
X := G \times_{G_{\text{aff}}} G & \xrightarrow{\text{pr}_2} & G \\
\downarrow{\text{pr}_1} & & \downarrow{\pi} \\
G & \xrightarrow{\pi} & G_{\text{aff}}
\end{array}$$

Let $i = 1, 2$. The fibered product $\text{pr}_i: X \to G$ is a principal $G_{\text{ant}}$-bundle over $G$. The diagonal morphism $\Delta: G \to X$ is a section of $\text{pr}_i$, thus $X$ is isomorphic to $G \times G_{\text{ant}}$. Since every $K$-analytic function on $G_{\text{ant}}$ is constant by Theorem 4.20, composing a $K$-analytic function on $G$ with $\text{pr}_i$ induces an isomorphism

$$H^0(G^\text{an},\mathcal{O}_G) \xrightarrow{\sim} H^0(X^\text{an},\mathcal{O}_X).$$

That is, for $f \in H^0(X^\text{an},\mathcal{O}_X)$, one has the equality

$$f = f \circ \Delta \circ \text{pr}_i.$$

By “faithfully flat descent” (Lemma 4.22 below), analytic functions on $G_{\text{aff}}$ are the ones among analytic functions $g$ on $G$ such that $g \circ \text{pr}_1 = g \circ \text{pr}_2$. Therefore, given $f \in H^0(X^\text{an},\mathcal{O}_X)$, the $K$-analytic function $f \circ \Delta$ is of the form $f \circ \pi$ for some $K$-analytic function $f$ on $G_{\text{aff}}$. □

**Lemma 4.22.** Let $K$ be a complete non-trivially valued non-Archimedean field. Let $Y, X$ be $K$-schemes of finite type. Let $p: Y \to X$ be a faithfully flat morphism of $K$-schemes. For $i = 1, 2$, let $\text{pr}_i: Y \times_X Y \to Y$ be the projection onto the $i$-th factor. Let $F$ be a coherent $\mathcal{O}_X$-module. Consider the subsheaf $\mathcal{F}$ of $p^\text{an}_* p^\text{an}! F^\text{an}$ associating to an open subset $U \subseteq X^\text{an}$, the $\mathcal{O}_X^\text{an}(U)$-module

$$\mathcal{F}(U) := \{ f \in H^0(p^\text{an}^{-1}(U), p^\text{an}! F^\text{an}) : f \circ \text{pr}_1 = f \circ \text{pr}_2 \}.$$

The homomorphism $F^\text{an} \to p^\text{an}_* p^\text{an}! F^\text{an}$ given by adjunction factors through a homomorphism of $\mathcal{O}_X$-modules $\varphi: F^\text{an} \to \mathcal{F}$. Then, $\varphi$ is an isomorphism.

**Proof.** The statement amounts to the homomorphism $\varphi_2: F^\text{an}_{x,Y} \to \mathcal{F}_x$ being bijective for all $x \in X^\text{an}$. Since the $p$ is surjective by hypothesis, the morphism $p^\text{an}$ is surjective. Let $x \in X^\text{an}$ and let $y \in Y^\text{an}$ be a pre-image of $x$.

Set $A := \mathcal{O}_{X,x}^\text{an}, B := \mathcal{O}_{Y,y}^\text{an}, M := F^\text{an}_x$ and $N := (p^\text{an}_* F^\text{an})_y$ so that $N = M \otimes_A B$. Let $f \in N$. The image of $f \circ \text{pr}_1$ in $N \otimes_B (B \otimes_A B)$ is $f \otimes 1$, while the image of $f \circ \text{pr}_2$ in $(B \otimes_A B) \otimes_B N$ is $1 \otimes f$. Via the canonical isomorphism $N \otimes_B (B \otimes_A B) \simeq (B \otimes_A B) \otimes_B N$, one sees that $\mathcal{F}_x$ is the kernel of the map $N \to N \otimes_B (B \otimes_A B)$, $f \mapsto f \otimes 1 - 1 \otimes f$.

Now, according to [Duc18, Proposition 4.2.4], the homomorphism $\mathcal{O}_{X,x}^\text{an} \to \mathcal{O}_{Y,y}^\text{an}$ is faithfully flat (a flat local ring homomorphism of local rings is always faithfully flat [Stacks, Lemma 00HR]). Therefore, by the usual faithfully flat descent in algebraic geometry, $\varphi_2$ is an isomorphism. □

Let $G$ be a commutative, reduced, connected algebraic group. By Chevalley’s Theorem, the algebraic group $G$ is extension of an abelian variety $A$ by a linear algebraic group $L$. Let $\pi: G \to A$ the projection onto $A$. Let $H$ be an ample line bundle on $A$. Let $P$ be the total space of $\pi^* H$ deprived of its zero section.
**Theorem 4.23.** Let $K$ be a non-trivially valued complete non-Archimedean field. With the notation above, if $G$ is anti-affine and $L$ is non-trivial, then $P$ is not Stein.

In order to prove Theorem 4.23 I need the following:

**Lemma 4.24.** Let $K$ be a non-trivially valued complete non-Archimedean field. Let $X$ be a proper $K$-analytic space. Let $M$ be a line bundle on $X$. Let $Y$ be a $K$-analytic space together with a morphism of $K$-analytic spaces $f: Y \to X$. Suppose

1. $H^0(Y, \mathcal{O}_Y) = K$,
2. there is $x \in X(K)$ such that $H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) \neq K$.

Let $g: Z \to Y$ be the total space of $f^*M$ deprived of its zero section. Let $I$ be the kernel of the restriction map $\rho: \mathcal{O}_Y \to \mathcal{O}_{f^{-1}(x)}$. Then,

$$H^1(Z, g^*I) \neq 0.$$

**Proof of Lemma 4.24.** The natural map $H^1(Y, I) \to H^1(Z, g^*I)$ induced by $g$ is injective (Proposition 4.3), therefore it suffices to show that $H^1(Y, I)$ does not vanish. Taking global sections of the short exact sequence of $\mathcal{O}_Y$-modules

$$0 \to I \to \mathcal{O}_Y \to \mathcal{O}_{f^{-1}(x)} \to 0$$

yields a long exact sequence of cohomology groups

$$\cdots \to H^0(Y, \mathcal{O}_Y) \to H^0(f^{-1}(x), \mathcal{O}_{f^{-1}(x)}) \to H^1(Y, I) \to \cdots.$$

By assumption the restriction map $\rho$ is not surjective, hence $H^1(Y, I) \neq 0$. □

**Proof of Theorem 4.23.** According to Theorem 4.20 all analytic functions on $G^a$ are constant. Apply Lemma 4.24 with $X = A^n, Y = G^a, Z = P^n, f = \pi$, and $x = 0$. Note that $\pi^{-1}(0) = L$ is affine and non-trivial, thus $H^0(L^a, \mathcal{O}_{L^a}) \neq K$.

Assume by contradiction that $P$ is Stein, that is, there is a closed analytic embedding $i: P^n \to A^n_K$. Higher coherent cohomology vanishes on $A^n_K$ (thus on $P^n$ (MP Proposition 2.8)). This contradicts the conclusion of Lemma 4.24. □

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