Regret Analysis of Dyadic Search
(Preliminary work)*

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Abstract
We analyze the cumulative regret of the Dyadic Search algorithm of [Bachoc et al. [2022]].

1 Setting
In this section, we introduce the formal setting for our budget convex optimization problem.

Given a bounded interval $I \subset \mathbb{R}$, our goal is to minimize an unknown convex function $f: I \to \mathbb{R}$ picked by a possibly adversarial and adaptive environment by only requesting fuzzy evaluations of $f$. At every interaction $t$, the optimizer is given a certain budget $b_t$ that can be invested in a query point $X_t$ of their choosing to reduce the fuzziness of the value of $f(X_t)$, modeled by an interval $J_t \ni f(X_t)$.

The interactions between the optimizer and the environment are described in Optimization Protocol 1.

Optimization Protocol 1
\begin{verbatim}
input: A non-empty bounded interval $I \subset \mathbb{R}$ (the domain of the unknown objective $f$)
1: for $t = 1, 2, \ldots$ do
2: The environment picks and reveals a budget $b_t > 0$
3: The optimizer selects a query point $X_t \in I$ where to invest the budget $b_t$
4: The environment picks and reveals an interval $J_t \subset \mathbb{R}$ such that $f(X_t) \in J_t$
\end{verbatim}

We stress that the environment is adaptive. Indeed, the intervals $J_t$ that are given as answers to the queries $X_t$ can be chosen by the environment as an arbitrary function of the past history, as long as they represent fuzzy evaluations of the convex function $f$, i.e., $f(X_t) \in J_t$.

Note that optimization would be impossible without further restrictions on the behavior of the environment, since an adversarial environment could return $J_t = \mathbb{R}$ for all $t \in \mathbb{N}$, making it impossible to gather any meaningful information. We limit the power of the environment by relating the amount of budget invested in a query point $X_t$ with the length of the corresponding fuzzy representation $J_t$ of $f(X_t)$. The idea is that the more budget is invested, the more accurate approximation of the objective $f$ can be determined, in a quantifiable way. This is made formal by the following assumption.

*This is a preliminary (and unpolished) version of our regret analysis of Dyadic Search. Stay tuned for the final polished paper.
Assumption 1. There exist $c \geq 0$ and $\alpha > 0$ such that, for any $t \in \mathbb{N}$, if the optimizer invested the budgets $b_1, \ldots, b_t$ in the query points $X_1, \ldots, X_t$, then

$$|J_t| \leq \frac{c}{2B_t^\alpha},$$

where $|J_t|$ denotes the length of $J_t$ and $B_t := \sum_{s=1}^t b_s I\{X_s = X_t\}$ is the total budget invested in $X_t$ up to time $t$.

The performance after $T$ interactions of an algorithm that received budgets $b_1, \ldots, b_T$ is evaluated with the cumulative regret. More precisely, we want to control the difference

$$R_T := \sum_{t=1}^T f(X_t)b_t - \inf_{x \in \bigcup_{t=1}^T X_t} \sum_{\ell=1}^T f(x)b_t$$

for any choice of the convex function $f$ and the fuzzy evaluations $J_1, \ldots, J_T$.

## 2 Dyadic Search

In this section, we present our Dyadic Search algorithm for budget convex optimization (Algorithm 2).

Before presenting its pseudo-code, we introduce some notation. For any positive integer $n \in \mathbb{N}$ we denote by $[n]$ the set $\{1, \ldots, n\}$ of the first $n$ integers. Let $\mathcal{P} = \{\bullet\bullet\bullet\bullet, \bullet\bullet\bullet, \bullet\bullet, \bullet, \bullet\}$, the blackened parts of the elements of $\mathcal{P}$ represent which portions of the active interval maintained by Dyadic Search the algorithm will delete. Additionally, we will consider the element $\bullet\bullet\bullet\bullet$ representing the case where no parts of the active interval will be deleted. Let $\mathcal{I}$ be the set of all intervals, and $\mathcal{I} \subset \mathcal{J}$ that of all bounded intervals. Furthermore, for any interval $J \in \mathcal{J}$, let

$$J^- := \inf(J) \quad \text{and} \quad J^+ := \sup(J).$$

Dyadic Search relies on four auxiliary functions: the delete function, the uniform partition function $u$, the non-uniform partition function $v$, and the update function. The delete function

$$\text{delete} : \mathcal{J}^3 \to \mathcal{P} \cup \{\bullet\bullet\bullet\bullet\}$$

is defined, for all $(J_l, J_c, J_r) \in \mathcal{J}^3$, by

\[
\begin{align*}
\text{if } J^-_l & \geq J^-_r, \text{ then } & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet \\
\text{if } J^-_c & \geq J^+_r, \text{ then } & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet \\
\text{if } J^-_l & \geq \min(J^+_c, J^-_r) \text{ and } J^-_c & \geq \min(J^+_l, J^-_r), \text{ then } & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet \\
\text{if } J^-_l & \geq \min(J^+_c, J^-_r), \text{ else } & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet \\
\text{if } J^-_c & \geq \min(J^+_l, J^-_r), \text{ else } & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet \\
\text{if } J^-_r & \geq \min(J^+_l, J^-_c), \text{ else } & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet \\
\text{else } & & \text{delete}(J_l, J_c, J_r) & = \bullet\bullet\bullet\bullet 
\end{align*}
\]

In words, the intervals $J_l, J_c, J_r$ will represent the fuzzy evaluations of three points $l < c < r$ in the domain of the unknown objective (left, center, and right). Since we are assuming that the objective is convex (hence unimodal\footnote{By unimodal, we mean that there exists a point $x$ belonging to the closure of the domain of $f$ such that $f$ is nonincreasing before $x$ and nondecreasing after $x$. More precisely, either $f$ is nonincreasing on the domain intersected with $(-\infty, x]$ and nondecreasing on the domain intersected with $(x, \infty)$ or is nonincreasing on the domain intersected with $(-\infty, x]$ and nondecreasing on the domain intersected with $[x, \infty)$.}), note that whenever an upper bound on the value of the objective at a point $x$ is lower than the lower bound at another point $y$ that is left (resp., right) of $x$, then, all points that are left (resp., right) of $y$ ($y$ included) are no better than $x$. Therefore, the function delete returns which part of an interval containing three distinct points $l < c < r$ should be deleted given the fuzzy evaluations $J_l, J_c, J_r$. (E.g., $\bullet\bullet\bullet\bullet$ represents the deletion of all points of the active interval left of $c$, $\bullet\bullet\bullet\bullet$ represents the deletion of all points of the active interval right of $r$, $\bullet\bullet\bullet\bullet$ is returned when the fuzzy evaluations are not sufficient to delete anything, etc.)
The uniform and non-uniform partition functions are defined, respectively, by
\[ u : I \rightarrow \mathbb{R}^3, \quad I \mapsto \left( \frac{1}{2}I^+, \frac{1}{2}I^- + \frac{1}{2}I^+ \right), \]
\[ \mu : I \rightarrow \mathbb{R}^3, \quad I \mapsto \left( \frac{1}{2}I^+, \frac{1}{2}I^- + \frac{1}{2}I^+ \right). \]

In words, when applied to an interval \( I \), the uniform partition function \( u \) returns the three points that are at \( \frac{1}{4}, \frac{1}{2}, \) and \( \frac{3}{4} \) of the interval, while the non-uniform partition function \( \mu \) returns the three points that are at \( \frac{1}{8}, \frac{1}{4}, \) and \( \frac{7}{8} \) of the interval (see Figure 1).

The update function
\[ \text{update} : \mathcal{I} \times \{u, \mu\} \times \mathcal{P} \rightarrow \mathcal{I} \times \{u, \mu\} \]
is defined, for all \((I, \vartheta, \text{del}) \in \mathcal{I} \times \{u, \mu\} \times \mathcal{P}\), by the following table:

| \( u \) | \( \mu \) |
|---|---|
| \( \left( \frac{1}{2}I^+ + \frac{1}{2}I^+, I^+ \right) \) | \( \left( \frac{1}{2}I^+ + \frac{1}{2}I^+, I^+ \right) \) |
| \( \left( I^+, \frac{1}{2}I^- + \frac{1}{2}I^+ \right) \) | \( \left( I^+, \frac{1}{2}I^- + \frac{1}{2}I^+ \right) \) |
| \( \left( \frac{1}{2}I^+ + \frac{1}{2}I^+, I^- \right) \) | \( \left( \frac{1}{2}I^+ + \frac{1}{2}I^+, I^- \right) \) |
| \( \left( I^-, \frac{1}{2}I^- + \frac{1}{2}I^+ \right) \) | \( \left( I^-, \frac{1}{2}I^- + \frac{1}{2}I^+ \right) \) |

In words, when applied to an interval \( I \), a type of partition \( \vartheta \), and the subset of \( I \) to be deleted modeled by \( \text{del} \), the update function returns as the first component the interval \( I \) pruned by the subset of \( I \) specified by \( \vartheta \) and \( \text{del} \), and, as the second component, how the new interval will be partitioned. It can be seen that the types of partitions returned by update are chosen so that our Dyadic Search algorithms will only query points on a (rescaled) dyadic mesh. (E.G., if \( I = [0, 1] \), Dyadic Search will only query points of the form \( k/2^n \), for \( k, n \in \mathbb{N} \).)

For all \( t \in \mathbb{N} \), if the sequence of budgets picked by the environment up to time \( t \) is \( b_1, \ldots, b_t \) and the sequence of query points selected by the optimizer is \( X_1, \ldots, X_t \), for each \( x \in \mathbb{R} \), we define the quantities
\[ \mathcal{B}_{x,t} := \sum_{s=1}^{t} b_s \mathbb{I}(X_s = x) \quad \text{and} \quad J_{x,t} := \bigcap_{s \in [t], X_s = x} J_s \]
with the understanding that \( J_{x,t} = \mathbb{R} \) whenever \( X_s \neq x \) for all \( s \in [t] \). Furthermore, define \( \mathcal{B}_{x,0} = 0 \) for all \( x \in \mathbb{R} \). In words, \( \mathcal{B}_{x,t} \) is the total budget that has been invested in \( x \) by the optimizer up to and including time \( t \), while \( J_{x,t} \) is the best fuzzy evaluation of the unknown objective at \( x \) that is available at the end of time \( t \).

The pseudocode of Dyadic Search is provided in Algorithm 2:

We note that the assignments in brackets in the initialization and Lines 5 and 7 are not needed to run the algorithm. We only added them for notational convenience of the analysis. As noted above, by definition of the update function, Dyadic Search only queries points in the rescaled dyadic mesh \( \{I^- + k \cdot 2^{-h} : |I| : h \in \mathbb{N}, k \in [2^h - 1] \} \). Moreover, we stress that Dyadic Search is any-time (it does not need to know the time horizon \( T \) \textit{a priori}), any-budget (it does not need to know the total budget \( B := \sum_{t=1}^{T} b_t \)) and does not require the unknown objective to be Lipschitz.

## 3 Cumulative Regret Analysis

**Theorem 1.** For any compact interval \( I \subset \mathbb{R} \), if the optimizer is running Dyadic Search (Algorithm 2) with input \( I \) in an environment satisfying Assumption 1 for some \( c \geq 0 \) and \( \alpha > 0 \), then, there exist \( c_1, c_2 > 0 \) such that, for any time $t$,
Algorithm 2 Dyadic Search

input: A non-empty bounded interval $I \subset \mathbb{R}$ (the domain of the unknown objective)

initialization: $I_1 := [I^-, I^+]$, $\vartheta_1 := u$, $(l_1, c_1, r_1) := \vartheta_1(I_1)$, $t_0 := 0$ [and $B_0 := 0$, $B_{1,0} := 0$]

1: for epochs $\tau = 1, 2, \ldots$ do
2: for $t = t_{\tau-1} + 1, t_{\tau-1} + 2, \ldots$ do
3: Query $X_t \in \arg\min_{x \in (l_\tau, c_\tau, r_\tau)} \mathbb{B}_{x,t-1}$
4: Let $\text{del}_t := \text{delete}(J_{l_\tau,t}, J_{c_\tau,t}, J_{r_\tau,t})$
5: [Let $B_{\tau,t} := B_{\tau,t-1} + b_t$ and $\tau_t := \tau$]
6: if $\text{del}_t \neq \emptyset$ then
7: [Let $t_* := t$, $B_* := B_{\tau,t}$, and $B_{\tau+1,t} := 0$]
8: Let $(I_{\tau+1}, \vartheta_{\tau+1}) := \text{update}(I_{\tau}, \vartheta_{\tau}, \text{del}_t)$
9: Let $(l_{\tau+1}, c_{\tau+1}, r_{\tau+1}) := \vartheta_{\tau+1}(I_{\tau+1})$
10: break

$T \in \mathbb{N}$ and every convex continuous function $f : I \to \mathbb{R}$, if budgets $b_t$ are equal to 1 for all $t \in \mathbb{N}$, the regret $R_T$ satisfies

$$R_T \leq c_1 \cdot T^{1-\alpha} \left( c \ln(MT) + 1 \right) + c_2 \cdot M,$$

where $M := \max(f) - \min(f)$.

Proof. Fix a compact interval $I$, a time horizon $T$, and a convex continuous function $f : I \to \mathbb{R}$. Up to translating and rescaling, we can (and do!) assume without loss of generality that $\min(f) = 0$ and $I = [0, 1]$. We also assume that $f$ admits a unique minimizer $x^* \in (0, 1)$ (the other cases are simpler). Redefine $t_{\tau} := T$ and $B_{\tau} := B_{\tau:T}$.

Claim 1. For $\tau \in [\tau_T]$, if $B_\tau \geq 4$ (i.e., if epoch $\tau$ lasts at least 4 rounds), then

$$\max_{x \in \{l_\tau, c_\tau, r_\tau\}} f(x) \leq \frac{4e3^\alpha}{(B_\tau - 3)^\alpha}$$

Proof of Claim. Fix any epoch $\tau \in [\tau_T]$ and assume that $B_\tau \geq 4$. Remember that, by Bachoc et al. [2022], Eq. (9), we have

$$\min_{x \in \{l_\tau, c_\tau, r_\tau\}} \sum_{s=1}^{t_\tau-1} \mathbb{I}_{\{X_s = x\}} \geq \frac{B_\tau - 3}{3}.$$

Assume that $x^* > r_\tau$ (all other cases can be treated similarly), which in turn implies that $\max_{x \in \{l_\tau, c_\tau, r_\tau\}} f(x) = f(l_\tau)$. Then, recalling that at time $t_\tau - 1$, it holds that $J_{l_\tau,t_{\tau-1}} \cap J_{c_\tau,t_{\tau-1}} \cap J_{r_\tau,t_{\tau-1}} \neq \emptyset$ (implying in particular that $J^+_{l_\tau,t_{\tau-1}} - J^-_{l_\tau,t_{\tau-1}} \geq 0$), we get

$$\max_{x \in \{l_\tau, c_\tau, r_\tau\}} f(x) = f(l_\tau) = f(l_\tau) - f(x^*) = f(l_\tau) - f(r_\tau) + \frac{f(r_\tau) - f(x^*)}{r_\tau - x^*} (r_\tau - x^*)$$

$$\leq f(l_\tau) - f(r_\tau) + \frac{f(l_\tau) - f(r_\tau)}{r_\tau - l_\tau} (r_\tau - x^*) = \frac{x^* - l_\tau}{r_\tau - l_\tau} (f(l_\tau) - f(r_\tau)) \leq 2(f(l_\tau) - f(r_\tau))$$

$$\leq 2(J^+_{l_\tau,t_{\tau-1}} - J^-_{l_\tau,t_{\tau-1}}) \leq 2(J^+_{l_\tau,t_{\tau-1}} - J^-_{l_\tau,t_{\tau-1}}) + J^+_{l_\tau,t_{\tau-1}} - J^-_{l_\tau,t_{\tau-1}}$$

$$\leq 4 \max \{|J_{l_\tau,t_{\tau-1}}|, |J_{l_\tau,t_{\tau-1}}|, |J_{r_\tau,t_{\tau-1}}|\}$$

$$\leq 4 \max \left\{ \frac{C}{(\sum_{s=1}^{t_{\tau-1}} \mathbb{I}\{X_s = l_\tau\})^\alpha}, \frac{C}{(\sum_{s=1}^{t_{\tau-1}} \mathbb{I}\{X_s = c_\tau\})^\alpha}, \frac{C}{(\sum_{s=1}^{t_{\tau-1}} \mathbb{I}\{X_s = r_\tau\})^\alpha} \right\}$$

$$= \frac{4}{(\min_{x \in \{l_\tau, c_\tau, r_\tau\}} \sum_{s=1}^{t_{\tau-1}} \mathbb{I}\{X_s = x\})^\alpha} \leq \frac{4e3^\alpha}{(B_\tau - 3)^\alpha}.$$

$\square$
Let $\tau^* \in [\tau_\tau]$ be the first epoch from which $x^* \in [l_\tau, r_\tau]$.

**Claim 2.** If $\tau^* \geq 2$, then, for each $\tau \in \{2, \ldots, \tau^* - 1\}$,

$$\max_{x \in \{l_\tau, c_\tau, r_\tau\}} f(x) \leq \frac{3}{4} \max_{x \in \{l_{\tau - 1}, c_{\tau - 1}, r_{\tau - 1}\}} f(x).$$

**Proof of Claim 2.** Assume that $\tau^* \geq 2$. Then, either for all $\tau \in [\tau^* - 1]$, it holds that $r_\tau < x^*$, or for all $\tau \in [\tau^* - 1]$, we have $l_\tau > x^*$. In the first case, for all $\tau \in \{2, \ldots, \tau^* - 1\}$,

$$\max_{x \in \{l_\tau, c_\tau, r_\tau\}} f(x) = f(l_\tau) - f(x^*) = \frac{f(l_\tau) - f(x^*)}{l_\tau - x^*} \leq \frac{f(l_{\tau - 1}) - f(x^*)}{l_{\tau - 1} - x^*} (l_\tau - x^*)$$

$$= \frac{3}{4} (f(l_{\tau - 1}) - f(x^*)) = \frac{3}{4} f(l_{\tau - 1}) = \frac{3}{4} \max_{x \in \{l_{\tau - 1}, c_{\tau - 1}, r_{\tau - 1}\}} f(x).$$

The other case can be worked out similarly.

For each $m \in \mathbb{N}$, let $A_m := \{x \in (0, 1) : 3k \in [2^m - 1], x = k/2^m\}$ be the dyadic mesh in $(0, 1)$ of index $m$. For any epoch $\tau \in \mathbb{N}$, let $m_\tau := -\log_2(c_\tau - l_\tau)$ be the index of the dyadic mesh in $(0, 1)$ at epoch $\tau$ of Dyadic Search (note that $m_\tau \geq 2$ for all $\tau \in \mathbb{N}$ because Dyadic Search begins with a step-size of $1/4$).

Note that:

- If the epoch $\tau^*$ is non-uniform, then, previous epoch has to be non-uniform as well and as soon as we change the dyadic mesh (in at most two epochs) we have 4 dyadic points in $(0, 1)$ to both sides of $x^*$.

- If the epoch $\tau^*$ is uniform, then, previous epoch can be either uniform or non-uniform.

  - If the previous epoch is non-uniform, then as soon as we change the dyadic mesh twice (in at most three epochs) we have 4 dyadic points in $(0, 1)$ to both sides of $x^*$.

  - If the previous epoch is uniform, then as soon as we change the dyadic mesh twice (in at most three epochs) we have 4 dyadic points in $(0, 1)$ to both sides of $x^*$.

Let $m^* := \min \{m \in \mathbb{N} : |A_m \cap (0, x^*]| \geq 4 \text{ and } |A_m \cap [x^*, 1]| \geq 4\}$ be the smallest index of the dyadic mesh in $(0, 1)$ such that there are at least 4 points of the dyadic mesh in $(0, 1)$ to the right and to the left of $x^*$. For each $m \geq m^*$ let $x_1^m < x_2^m < x_3^m < x_4^m \leq x^*$ be the four points of $A_m \cap (0, x^*]$ closest to $x^*$ and $x^* < x_5^m < x_6^m < x_7^m < x_8^m$ be the four points of $A_m \cap [x^*, 1)$ closest to $x^*$. The crucial observation is that, for all epochs $\tau \geq \tau^* + 3$, we have that $l_\tau, c_\tau, r_\tau \in \{x_1^m, \ldots, x_8^m\}$.

**Claim 3.** For each $m \geq m^* + 1$, we have

$$\max_{x \in \{x_1^m, \ldots, x_8^m\}} f(x) \leq \frac{4}{7} \max_{x \in \{x_1^{m - 1}, \ldots, x_8^{m - 1}\}} f(x).$$

**Proof of Claim 3.** Assume that $m \geq m^* + 1$. Then, either $\max_{x \in \{x_1^m, \ldots, x_8^m\}} f(x) = f(x_1^m)$ or $\max_{x \in \{x_1^m, \ldots, x_8^m\}} f(x) = f(x_8^m)$. In the first case, we have

$$\max_{x \in \{x_1^m, \ldots, x_8^m\}} f(x) = f(x_1^m) - f(x^*) = \frac{f(x_1^m) - f(x^*)}{x_1^m - x^*} (x_1^m - x^*) \leq \frac{f(x_1^{m - 1}) - f(x^*)}{x_1^{m - 1} - x^*} (x_1^{m - 1} - x^*)$$

$$= \frac{4}{7} (f(x_1^{m - 1}) - f(x^*)) = \frac{4}{7} f(x_1^{m - 1}) \leq \frac{4}{7} \max_{x \in \{x_1^{m - 1}, \ldots, x_8^{m - 1}\}} f(x).$$

The other case can be worked out similarly.
Define \( \tau^\# = \left[ 4 + 2 \log_{4\alpha}(MT^\alpha) \right] \) so that

\[
M \left( \frac{3}{4} \right)^{\frac{\tau^\#}{4}} = M \left( \frac{3}{4} \right)^{\left\lfloor \frac{4 + 2 \log_{4\alpha}(MT^\alpha)}{2} \right\rfloor - 1} \leq M \left( \frac{3}{4} \right)^{\log_{4\alpha}(MT^\alpha)} = M \frac{1}{MT^\alpha} = \frac{1}{T^\alpha}.
\]

Assume that \( \tau^\# < \tau^* \) and \( \tau^* + 2 + \tau^\# < \tau_T \) (the other cases can be treated analogously, omitting terms which are not there anymore). Then:

\[
\sum_{t=1}^{T} f(X_t) = \sum_{\tau=1}^{\tau^\#} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) + \sum_{\tau=\tau^*+1}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) + \sum_{\tau=\tau^*+3}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t)
\]

We analyze these five terms individually. For the first one, we further split the sum into two terms, depending on whether or not \( B_t \geq 6 \). By Claim[1] we have that

\[
\sum_{\tau=1}^{\tau^\#} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) \leq \sum_{\tau=1}^{\tau^\#} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} \frac{4c3^\alpha}{(B_{\tau} - 3)^\alpha} \leq \sum_{\tau=1}^{\tau^\#} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} \frac{4c3^\alpha}{(B_{\tau} - 2)^2} = \sum_{\tau=1}^{\tau^\#} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} \frac{4c6^\alpha}{B_{\tau}^2}
\]

By Claim[2] we have that

\[
\sum_{\tau=1}^{\tau^\#} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) \leq 5M \sum_{\tau=0}^{\infty} \left( \frac{3\alpha}{4} \right)^T = 20M
\]

Thus, the first term is upper bounded by \( \tau^\# \cdot 4c6^\alpha T^{1-\alpha} + 20M \).

For the second term, we leverage Claim[2] and the definition of \( \tau^\# \) to obtain

\[
\sum_{\tau=\tau^*+1}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) \leq M \sum_{\tau=\tau^*+1}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} \left( \frac{3\alpha}{4} \right)^{T-1} \leq M \left( \frac{3\alpha}{4} \right)^{\tau^*-1} \sum_{\tau=\tau^*+1}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} 1 \leq M \left( \frac{3\alpha}{4} \right)^{\tau^*-1} \sum_{\tau=\tau^*+1}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} 1 \leq T^{1-\alpha}
\]

For the third term, we further split the sum into two terms, depending on whether or not \( B_t \geq 6 \). Proceeding exactly as for the first term, we obtain

\[
\sum_{\tau=\tau^*+2}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) \leq 3 \cdot 4c6^\alpha T^{1-\alpha} + 15M
\]

For the fourth term, we split again the sum into two terms, depending on whether or not \( B_t \geq 6 \). If \( B_t \geq 6 \), proceeding exactly as for the corresponding part of the first term, we obtain

\[
\sum_{\tau=\tau^*+3}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) \leq \tau^\# \cdot 4c6^\alpha T^{1-\alpha}
\]

Instead, if \( B_t \leq 5 \), by Claim[3] we get

\[
\sum_{\tau=\tau^*+3}^{\tau_T} \sum_{l=t_{\tau-1}+1}^{l_{\tau}} f(X_t) \leq 5 \sum_{\tau=\tau^*+3}^{\tau_T} \max_{x \in \{1, \ldots, r_T\}} f(x) \leq 5 \sum_{\tau=\tau^*+3}^{\tau_T} \max_{x \in \{1, \ldots, r_T\}} f(x) \leq 10M \sum_{\tau=0}^{\infty} \left( \frac{3\alpha}{4} \right)^T \leq \frac{70}{3} M.
\]
For the last term, by Claim 3, we get
\[
\sum_{\tau = \tau^* + 3 \tau \#}^{T} \sum_{t = t_{\tau - 1} + 1}^{T} f(X_t) \leq \sum_{\tau = \tau^* + 3 \tau \#}^{T} \sum_{t = t_{\tau - 1} + 1}^{T} \max_{x^m_t, \ldots, x^m_n} f(x) \leq \sum_{\tau = \tau^* + 3 \tau \#}^{T} \sum_{t = t_{\tau - 1} + 1}^{T} M \left( \frac{1}{\tau - (\tau^* + 3 \tau \#)} - 1 \right)
\]
\[
\leq M \left( \frac{1}{\tau^* + 3 \tau \#} \right) \sum_{\tau = \tau^* + 3 \tau \#}^{T} \sum_{t = t_{\tau - 1} + 1}^{T} 1 \leq T^{1-\alpha}.
\]

Putting everything together, we conclude that
\[
R_T \leq \left( \tau^* \cdot 4c6^\alpha T^{1-\alpha} + 20M \right) + T^{1-\alpha} + \left( \tau^* \cdot 4c6^\alpha T^{1-\alpha} + 15M \right) + \frac{70}{3} M + T^{1-\alpha}
\]
\[
\leq \left( 4 + 2 \log_{4/3}(MT^\alpha) \right) \cdot 8c6^\alpha + 2 \right) T^{1-\alpha} + 60M.
\]

\[\square\]

References
François Bachoc, Tommaso Cesari, Roberto Colomboni, and Andrea Paudice. A near-optimal algorithm for univariate zeroth-order budget convex optimization, 2022. URL https://arxiv.org/abs/2208.06720

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