Research Article
Spectral Solutions for Fractional Black–Scholes Equations

M. A. Abdelkawy 1,2 and António M. Lopes 3

1Department of Mathematics and Statistics, College of Science, Imam Mohammad Ibn Saud Islamic University (IMSIU), Riyadh, Saudi Arabia
2Department of Mathematics, Faculty of Science, Beni-Suef University, Beni-Suef, Egypt
3LAETA/INEGI, Faculty of Engineering, University of Porto, Porto, Portugal

Correspondence should be addressed to António M. Lopes; aml@fe.up.pt

Received 19 April 2022; Revised 7 June 2022; Accepted 21 June 2022; Published 20 July 2022

1. Introduction

Fractional calculus [1–3] is the branch of calculus that generalizes the derivative and integral operators to non-integer orders. During the last few decades, the fractional calculus gained increasing importance in several fields, such as mechanics [4, 5], biology [6, 7], physics [8], chemistry [9, 10], viscoelasticity [11], engineering [12], and finance/economics [13, 14]. Fractional differential equations can be used to represent accurately a variety of phenomena. However, many equations cannot be solved analytically, and, thus, numerical approaches are required to find approximate solutions. The finite difference [15, 16] and finite element [17–20] methods have been widely used, as well as other numerical schemes, such as the Galerkin [21], Petrov-Galerkin [22, 23], pseudospectral [24–26], and fast predictor-corrector [27] methods.

The Black–Scholes model (BSM) is a mathematical description of pricing evolution [28, 29]. Specifically, the model estimates financial instruments via time variation. These instruments (such as stocks or futures) track a log-normal distribution of prices. Based on this hypothesis and taking into account other variables, the BSM derives the price of call options. Despite there being several models that describe pricing evolution, the BSM has been one of the most significant and prevalent in the last decades, and its generalization to fractional order was proposed [30, 31].

In [32], the Laplace homotopy analysis approach was employed to solve the fractional BSM (FBSM) in the sense of the Caputo–Fabrizio derivative. In [33], the authors employed the Sumudu and Laplace transforms to address various economic models with different fractional operators. In [34, 35], the Adomian and fractional Adomian decomposition were adopted to solve the FBSM. In [36], the time-fractional BSM was solved using the generalized differential transform technique. Moreover, in [37], the existence and uniqueness solution of the European-type option pricing model was discussed, while in [38], the multivariate Padé approximation was utilized to solve the Caputo fractional European vanilla call option pricing problem.
In most circumstances, it is impossible to provide explicit analytical solutions to space and/or time-fractional differential equations. Hence, developing effective numerical techniques is a critical necessity. Many high-accuracy numerical approaches have been devised to tackle diverse issues in various applications. Because of their high-order precision, fractional differential and integral equations [39] have seen tremendous development in recent years. But, comparing with the effort put into evaluating finite difference and element schemes, little research has been dedicated to designing and assessing global spectral schemes [40]. In cases with periodic boundary conditions, spectral techniques based on the Fourier expansion have been used. When compared to other schemes, spectral techniques take the main place due to their robustness and exponential rates of convergence [41–44]. Despite some drawbacks, such as the inability to represent physical processes in spectral space and the difficulty in parallelizing on distributed memory computers, they reveal excellent accuracy, high speed of convergence, and simplicity in solving many types of differential equations [45]. The spectral collocation method is particularly valuable because it can estimate the solution of a wide range of equations. Moreover, its exponential rate of convergence is extremely useful in delivering very exact solutions. The collocation approach has grown in prominence in recent decades for dealing with specific difficulties posed by fractional derivatives.

In this paper, a new spectral collocation approach is proposed to solve the FBSM [31, 46, 47], given by

\[
\mathcal{D}^\mu \mathcal{L}(x, t) + \frac{1}{2} \alpha^2 x^2 \frac{\partial^2}{\partial x^2} \mathcal{L}(x, t) + \mathcal{X}(x, t) \mathcal{X}(x, t)
\]

\[- r \mathcal{X}(x, t) + \mathcal{X}(x, t) = 0, (x, t) \in \Omega \times \bar{\Omega},
\]

with \(\mathcal{D}^\mu \mathcal{L}(x, t)\) denoting the right Riemann–Liouville fractional derivative:

\[
\mathcal{D}^\mu \mathcal{L}(x, t) = \frac{1}{\Gamma(1-\mu)} \frac{\partial}{\partial t} \int_x^t \mathcal{L}(s, t) - \mathcal{L}(x, t) \mathcal{L}(s, t) - (s - t)^\mu ds,
\]

where \(0 < \mu < 1\).

Herein, the shifted fractional Jacobi–Gauss–Radau collocation (SFJ-GR-C) and shifted fractional Jacobi–Gauss–Lobatto collocation (SFJ-GL-C) techniques are utilized to solve the FBSM. We interpolate the independent variables using the shifted fractional-order Jacobi nodes, and we approximate the solution of the model by a sequence of shifted fractional-order Jacobi orthogonal functions. After that, the residuals at the shifted fractional-order Jacobi quadrature locations are estimated. This yields an algebraic system of equations that can be solved using a suitable approach. The accuracy of the new method is demonstrated using numerical examples.

The paper is organized into four sections: Section 2 introduces the new spectral collocation technique. Section 3 applies the method to solve numerical examples. Section 4 summarizes the main conclusions.

2. Fully Spectral Collocation Technique

We start by mapping the variable \(\tau\) as \(\tau = t - \mathcal{T}\). Additionally, the Riemann–Liouville is transformed to the Caputo fractional derivative \(\mathcal{D}^\mu \mathcal{Y}(x, \tau)\) [see (31, 46)]:

\[
\frac{\partial}{\partial \tau} \mathcal{Y}(x, \tau) - \frac{\partial^2}{\partial x^2} \mathcal{Y}(x, \tau) = - r x \frac{\partial}{\partial x} \mathcal{Y}(x, \tau) + \mathcal{X}(x, \tau) \mathcal{X}(x, \tau)
\]

(3)

\[+ \mathcal{Y}(x, \tau) = 0, \quad (x, \tau) \in \Omega \times \bar{\Omega},\]

where \(\mathcal{X} \equiv [0, \mathcal{L}]\) and \(\mathcal{Y} \equiv [0, \mathcal{T}]\). Conditions are

\[
\mathcal{Y}(x, 0) = \mathcal{Y}_0(x), \quad x \in \Omega,
\]

\[
\mathcal{Y}(0, t) = \mathcal{Y}_1(t), \quad t \in \bar{\Omega},
\]

and \(\mathcal{D}^\mu \mathcal{Y}(x, \tau)\) is a Caputo fractional derivative.

We express the truncated solution as

\[
\mathcal{Y}(x, \tau) = \sum_{r_1, \ldots, r_N} \mathcal{c}_{r_1, \ldots, r_N} \mathcal{G}_{T_{r_1}, r_1}(x) \mathcal{G}_{T_{r_2}, r_2}(t).
\]

(5)

where \(\mathcal{G}_{T_{r_1}, r_1}(x)\) denotes the shifted fractional Jacobi functions on \([0, \mathcal{L}]\) [see (48, 49) for extra information].

\[
\frac{\partial}{\partial \tau} \mathcal{Y}(x, \tau) = \sum_{r_1, \ldots, r_N} \mathcal{c}_{r_1, \ldots, r_N} \mathcal{G}_{T_{r_1}, r_1}(x) \mathcal{G}_{T_{r_2}, r_2}(t).
\]

Similarly, we have [48, 49]

\[
\frac{\partial}{\partial \tau} \mathcal{Y}(x, \tau) = \sum_{r_1, \ldots, r_N} \mathcal{c}_{r_1, \ldots, r_N} \mathcal{G}_{T_{r_1}, r_1}(x) \mathcal{G}_{T_{r_2}, r_2}(t).
\]

The time Caputo fractional derivative, on the contrary, is obtained as [48, 49]

\[
\frac{\partial}{\partial \tau} \mathcal{Y}(x, \tau) = \sum_{r_1, \ldots, r_N} \mathcal{c}_{r_1, \ldots, r_N} \mathcal{G}_{T_{r_1}, r_1}(x) \mathcal{G}_{T_{r_2}, r_2}(t).
\]

(8)

Previous computations are indicated at certain nodes as
\[
\left( \frac{\partial^2}{\partial x^2} (Y(x,t)) \right)_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} = \sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M},
\]

\[
\left( \frac{\partial}{\partial x} (Y_{x_1, \ldots, x_M} (x,t)) \right)_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} = \sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M},
\]

\[
\left( \frac{\partial}{\partial x} (Y_{x_1, \ldots, x_M} (x,t)) \right)_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} = \sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M},
\]

Alternatively, the initial boundary can be obtained by

\[
\sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} (0) = \Theta_1 (x),
\]

\[
\sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} (t) = \Theta_2 (t),
\]

\[
\sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} (t) = \Theta_3 (t),
\]

Eq. (3) is constrained to be zero at \((\mathcal{N} - 1) \times (\mathcal{M})\) points:

\[
\Delta \left( x_{x_1, \ldots, x_M}^{a_1, b_1} y_{x_1, \ldots, x_M}^{a_2, b_2} \right) = \Theta_4 \left( x_{x_1, \ldots, x_M}^{a_1, b_1} y_{x_1, \ldots, x_M}^{a_2, b_2} \right),
\]

where

\[
\Delta \left( x_{x_1, \ldots, x_M}^{a_1, b_1} y_{x_1, \ldots, x_M}^{a_2, b_2} \right) = \sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} (t),
\]

\[
\frac{1}{2} \zeta^2 \left( x_{x_1, \ldots, x_M}^{a_1, b_1} \right)^2 \sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} (t),
\]

\[
-r \left( x_{x_1, \ldots, x_M}^{a_1, b_1} \right) \sum_{r_1=0,\ldots,\mathcal{N}} \sum_{r_2=0,\ldots,\mathcal{M}} \zeta_{r_1} \zeta_{r_2} \left[ \frac{\partial}{\partial x} \left( y_{x_1, \ldots, x_M} \right) \right]_{x=x_1, \ldots, x_M, t=t_1, \ldots, t_M} (t),
\]

where \(n = 1, \ldots, \mathcal{N} - 1\), and \(m = 1, \ldots, \mathcal{M}\), and
and \( \mu = 0.6 \). We verify that the new strategy is superior and that good estimates are obtained with a small number of collocation points. Figures 1 and 2 present the 3D charts of the numerical solution and the absolute error, respectively, when \( \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0, \lambda_1 = 1, \lambda_2 = 1, \) and \( \mathcal{N} = M = 20 \). Figure 3 depicts the corresponding exact and numerical solutions along \( x \). Figures 4 and 5 represent the absolute errors along \( x \) and \( t \), respectively, while Figure 6 shows the maximum absolute error \( (M_E) \) convergence for the following cases:

Case I: \( \alpha_1 = 1/2, \alpha_1 = 1/3, \alpha_2 = 2/3, \beta_2 = 1/2, \lambda_1 = 0.1, \lambda_2 = 1 \).

Case II: \( \alpha_1 = \alpha_1 = \alpha_2 = \beta_2 = 1/2, \lambda_1 = 0.5, \lambda_2 = 1 \).

Case III: \( \alpha_1 = \alpha_1 = -\alpha_2 = -\beta_2 = 1/2, \lambda_1 = 0.9, \lambda_2 = 1, \lambda_3 = 1 \).

Example 2. We solve the FBSM [46].

\[
\frac{\partial^\mu}{\partial t^\mu} (\mathcal{Y}(x, t)) - \frac{1}{2} \alpha^2 x^2 \frac{\partial^2}{\partial x^2} (\mathcal{Y}(x, t)) - r x \frac{\partial}{\partial x} (\mathcal{Y}(x, t)) + r (\mathcal{Y}(x, t)) = \mathbb{H}(x, t),
\]

with

\[
\mathcal{Y}(x, 0) = e^x + x + 1, x \in [0, 1],
\]

\[
\mathcal{Y}(0, t) = t^t + 1,
\]

\[
\mathcal{Y}(1, t) = t^t + e + 2, t \in [0, 1],
\]

choosing \( \mathbb{H}(x, t) \) such that \( \mathcal{Y}(x, t) = t^t + e^x + x + 1 \).

Table 2 summarizes the \( L_{\infty} \) errors between the exact and approximate solutions achieved by our method and those reported in [46], taking \( \alpha = 0.1, r = 0.06, \) and \( 0 < \mu < 1 \). We verify that the new strategy is superior and that good estimates are obtained with a small number of collocation points. Figures 7 and 8 present the 3D charts of the numerical solution and the absolute error, respectively, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1, \) and \( \mathcal{N} = M = 14 \). Figure 9 depicts the corresponding exact and numerical solutions along \( x \). Figures 10 and 11 represent the absolute errors along \( x \) and \( t \), respectively, while Figure 12 shows the \( M_E \) convergence for the following cases:

Case I: \( \alpha_1 = \beta_1 = \alpha_2 = \beta_2 = 0, \mu = 0.6, \lambda_1 = 1, \lambda_2 = 1 \).
Case II: $\alpha_1 = \beta_1 = 0, \alpha_2 = \beta_2 = 0, \mu = 0.6, \lambda_1 = 1, \lambda_2 = 1$.

Case III: $\alpha_1 = \beta_1 = -\alpha_2 = -\beta_2 = 1/2, \mu = 0.6, \lambda_1 = 1, \lambda_2 = 1$.

Figures 8, 10, and 11 show that the results are fairly precise, as the absolute error approaches zero. We can also see in Figures 7 and 9 that the approximate solution matches precisely the exact solution. In addition, Figure 12 clearly reveals that the method yields exponential convergence of the error.

To sum up, we emphasize that the majority of numerical techniques for the problem at hand are based on orthogonal polynomials. The use of classical polynomials in problems with nonsmooth solutions leads to low
accuracy or even failure to converge. Using the fractional, rather than the classical, Jacobi functions mitigates the problem. With the proposed method, all numerical computations could be completed with good precision and a low number of degrees of freedom. Moreover, our technique outperformed other existing approaches. Finally, we can conclude that the fully spectral collocation approach is a useful, efficient, and acceptable strategy for dealing with problems that have singular solutions.
Figure 7: Numerical solution of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1 \), and \( N = M = 14 \).

Figure 8: Absolute error of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1 \), and \( N = M = 14 \).

Figure 9: Exact and numerical solutions along \( x \) of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1 \), and \( N = M = 14 \).

Figure 10: Absolute error along \( t \) of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1 \), and \( N = M = 14 \).

Figure 11: Absolute error along \( x \) of Example 2, when \( \alpha_1 = -\alpha_2 = -0.5, \beta_1 = \beta_2 = 0, \lambda_1 = 0.9, \lambda_2 = 1 \), and \( N = M = 14 \).

Figure 12: The \( M_E \) convergence of Example 2 for the three cases.
4. Conclusion

The FBSM was treated using a fully spectral collocation technique for the two independent variables x and t. It was verified that the new technique is superior in terms of accuracy and efficiency to other methods, for both smooth and nonsmooth solutions. To deal with the FBSM, we devised an approach that yields an algebraic system from which an approximated solution can be computed. The simulation results revealed that the proposed approach is effective for the goal at hand. Furthermore, because of its ease of use, our technique is relevant to a wide range of fractional problems. In the future, we can concentrate on the usage of the spectral Galerkin and the tau approaches for solving more complicated pricing models, such as the tempered FBSM. Finally, we should mention that the maximum absolute error for a given boundary value problem with a smooth solution is exponentially convergent. For nonsmoothness in time (or in space), the method’s order of convergence degrades. This, however, can be mitigated by employing the fractional-order Jacobi functions described herein.

Data Availability

No data were used to support this study.

Conflicts of Interest

The authors declare that they have no conflicts of interest.

References

[1] R. Polonio and M. Popalizio, “On the use of matrix functions for fractional partial differential equations,” Mathematics and Computers in Simulation, vol. 81, no. 5, pp. 1045–1056, 2011.
[2] J. W. Kirchner, X. Feng, and C. Neal, “Fractal stream chemistry and its implications for contaminant transport in catchments,” Nature, vol. 403, no. 6769, pp. 524–527, 2000.
[3] D. Baleanu, S. Zibaei, N. Namjoo, and A. Jajarmi, “A non-standard finite difference scheme for the modeling and nonidentical synchronization of a novel fractional chaotic system,” Advances in Difference Equations, vol. 2021, no. 1, p. 308, 2021.
[4] D. Baleanu, S. S. Sajjadi, A. Jajarmi, and O. Defterli, “On a nonlinear dynamical system with both chaotic and non-chaotic behaviors: a new fractional analysis and control,” Advances in Difference Equations, vol. 2021, no. 1, p. 234, 2021.
[5] B. Dumitr, S. Sadat Sajjadi, J. Amin, O. Defterli, H. Jihad, and E. Cesarano, “The fractional dynamics of a linear trionic molecule,” Romanian Reports in Physics, vol. 73, no. 1, p. 105, 2021.
[6] I. Ahmad, H. Ahmad, P. Thounthong, Yu-M. Chu, and C. Cesarano, “Solution of multi-term time-fractional pde models arising in mathematical biology and physics by local meshless method,” Symmetry, vol. 12, no. 7, p. 1195, 2020.
[7] D. Baleanu, S. S. Sajjadi, J. H. Asad, A. Jajarmi, and O. Defterli, “Hyperchaotic behaviors, optimal control, and synchronization of a nonautonomous cardiac conduction system,” Advances in Difference Equations, vol. 2021, no. 1, p. 157, 2021.
[8] R. Hilfer, Applications of Fractional Calculus in Physics, World Scientific, Singapore, 2000.
[9] M. H. Srivastava, H. Ahmad, I. Ahmad, P. Thounthong, and N. M. Khan, “Numerical simulation of three-dimensional fractional-order convection-diffusion PDEs by a local meshless method,” Thermal Science, vol. 25, p. 210, 2020.
[10] S. M. Abo-Dahab, E. Ahmed, and H. Ahmad, “Fractional heat conduction model with phase lags for a half-space with thermal conductivity and temperature dependent,” Mathematical Methods in the Applied Sciences, 2020.
[11] C. Milici, G. Drăgănescu, and J. T. Machado, Introduction to Fractional Differential Equations. Nonlinear Systems and Complexity, Springer International Publishing, Berlin, Germany, 2018.
[12] H. G. Sun, Y. Zhang, D. Baleanu, W. Chen, and Y. Q. Chen, “A new collection of real world applications of fractional calculus in science and engineering,” Communications in Nonlinear Science and Numerical Simulation, vol. 64, pp. 213–231, 2018.
[13] F. Mainardi, “On the advent of fractional calculus in econophysics via continuous-time random walk,” Mathematics, vol. 8, no. 4, p. 641, 2020.
[14] V. V. Tarasova and V. E. Tarasov, “Concept of dynamic memory in economics,” Communications in Nonlinear Science and Numerical Simulation, vol. 55, pp. 127–145, 2018.
[15] C. Duman and M. Duman, “Finite element method for a symmetric tempered fractional diffusion equation,” Applied Numerical Mathematics, vol. 120, pp. 270–286, 2017.
[16] Z. Ren and H. Ren, “A reduced-order extrapolated finite difference iterative method for the Riemann-Liouville tempered fractional derivative equation,” Applied Numerical Mathematics, vol. 157, pp. 307–314, 2020.
[17] A. Kumar and B. V. R. Kumar, “Finite element method for drifted space fractional tempered diffusion equation,” Journal of Applied Mathematics and Computing, vol. 61, no. 1-2, pp. 117–135, 2019.
[18] M. Abbaszadeh and M. Abbaszadeh, “A finite difference/finite element technique with error estimate for space fractional tempered diffusion-wave equation,” Computers & Mathematics with Applications, vol. 75, no. 8, pp. 2903–2914, 2018.
[19] H. Ding, “A high-order numerical algorithm for two-dimensional time-space tempered fractional diffusion-wave equation,” Applied Numerical Mathematics, vol. 135, pp. 30–46, 2019.
[20] M. Deng and W. Deng, “A second-order accurate numerical method for the space-time tempered fractional diffusion-wave equation,” Applied Mathematics Letters, vol. 68, pp. 87–93, 2017.
[21] M. Hammad, R. M. Hafez, Y. H. Yousri, and E. H. Doha, “Exponential Jacobi-Galerkin method and its applications to multidimensional problems in unbounded domains,” Applied Numerical Mathematics, vol. 157, pp. 88–109, 2020.
[22] W. Zhang and Z. Zhang, “Variational formulation and efficient implementation for solving the tempered fractional problems,” Numerical Methods for Partial Differential Equations, vol. 34, no. 4, pp. 1224–1257, 2018.
[23] Z. Zhang, W. Deng, and G. E. Karniadakis, “A riesz basis Galerkin method for the tempered fractional Laplacian,” SIAM Journal on Numerical Analysis, vol. 56, no. 5, pp. 3010–3039, 2018.
[24] M. A. Zak, “Existence, uniqueness and numerical analysis of solutions of tempered fractional boundary value problems,” Applied Numerical Mathematics, vol. 145, pp. 429–457, 2019.
[25] E. Piret and C. Piret, “A Chebyshev pseudospectral method to solve the space-time tempered fractional diffusion equation,” SIAM Journal on Scientific Computing, vol. 36, no. 4, pp. A1797–A1812, 2014.
[26] T. Cui, S. Chen, and Y. Jiao, “Efficient hermite spectral methods for space tempered fractional diffusion equations,” East Asian Journal on Applied Mathematics, vol. 11, no. 1, pp. 43–62, 2021.
[27] J. Deng, L. Zhao, and Y. Wu, “Fast predictor-corrector approach for the tempered fractional differential equations,” Numerical Algorithms, vol. 74, no. 3, pp. 717–754, 2017.
[28] F. Black and M. Scholes, “The pricing of options and corporate liabilities,” in World Scientific Reference on Contingent Claims Analysis in Corporate Finance vol. 1, pp. 3–21, 1999.
[29] R. C. Merton, “Theory of rational option pricing,” Bell Journal of Economics and Management Science, vol. 4, no. 1, p. 141, 1973.
[30] A. Aghili, “Fractional black-scholes equation,” International Journal of Financial Engineering, vol. 4, no. 1, Article ID 1750004, 2017.
[31] W. Chen, X. Xu, and S.-P. Zhu, “Analytically pricing double barrier options based on a time-fractional Black-scholes equation,” Computers & Mathematics with Applications, vol. 69, no. 12, pp. 1407–1419, 2015.
[32] M. Özdemir and N. Özdemir, “European vanilla option pricing model of fractional order without singular kernel,” Fractal and Fractional, vol. 2, no. 1, p. 3, 2018.
[33] E. K. Akgül, A. Akgül, and M. Yavuz, “New illustrative applications of integral transforms to financial models with different fractional derivatives,” Chaos, Solitons & Fractals, vol. 146, Article ID 110877, 2021.
[34] M. Yavuz and N. Özdemir, “A quantitative approach to fractional option pricing problems with decomposition series,” Konuralp Journal of Mathematics (KJM), vol. 6, no. 1, pp. 102–109, 2018.
[35] M. Özdemir and N. Özdemir, “A different approach to the European option pricing model with new fractional operator,” Mathematical Modelling of Natural Phenomena, vol. 13, no. 1, p. 12, 2018.
[36] M. Yavuz, N. Özdemir, and Y. O. Yeliz, “Generalized differential transform method for fractional partial differential equation from finance,” in Proceedings of the International Conference on Fractional Differentiation and its Applications, pp. 778–785, Serbia, August 2016.
[37] M. Yavuz, “European Option Pricing Models Described by Fractional Operators with Classical and Generalized Mittag-Leffler Kernels,” Numerical Methods for Partial Differential Equations, vol. 38, 2020.
[38] N. Yavuz and M. Yavuz, “Numerical solution of fractional black-scholes equation by using the multivariate Padé approximation,” Acta Physica Polonica, A, vol. 132, no. 3, pp. 1050–1053, 2017.
[39] A. Ma, M. A. Zaky, H. Ali, and B. Dumitru, “Numerical simulation of time variable fractional order mobile-immobile advection-dispersion model,” Romanian Reports in Physics, vol. 67, no. 3, pp. 773–791, 2015.
[40] W. M. Abd-Elbameed and Y. H. Youssri, “Spectral solutions for fifth-order boundary value problems using generalized Jacobi operational matrix of derivatives,” International Journal of Algorithms, Computing and Mathematics, vol. 3, no. S1, pp. 883–901, 2017.
[41] Y. S. Sun, J. Ma, and B. W. Li, “Spectral collocation method for convective-radiative transfer of a moving rod with variable thermal conductivity,” International Journal of Thermal Sciences, vol. 90, pp. 187–196, 2015.
[42] X. Tang, “Efficient Chebyshev collocation methods for solving optimal control problems governed by Volterra integral equations,” Applied Mathematics and Computation, vol. 269, pp. 118–128, 2015.
[43] A. H. Zaky and M. A. Zaky, “Highly accurate numerical schemes for multi-dimensional space variable-order fractional Schrödinger equations,” Computers & Mathematics with Applications, vol. 73, no. 6, pp. 1100–1117, 2017.
[44] A. H. Zaky and M. A. Zaky, “An improved collocation method for multi-dimensional space-time variable-order fractional Schrödinger equations,” Applied Numerical Mathematics, vol. 111, pp. 197–218, 2017.
[45] M. Abdeljaww and S. A. Alyami, “Legendre-Chebyshev spectral collocation method for two-dimensional nonlinear reaction-diffusion equation with riesz space-fractional,” Chaos, Solitons & Fractals, vol. 151, Article ID 111279, 2021.
[46] Z. Cen, J. Huang, A. Xu, and A. Le, “Numerical approximation of a time-fractional Black-Scholes equation,” Computers & Mathematics with Applications, vol. 75, no. 8, pp. 2874–2887, 2018.
[47] X. An, F. Liu, M. Zheng, V. V. Anh, and I. W. Turner, “A space-time spectral method for time-fractional Black-Scholes equation,” Applied Numerical Mathematics, vol. 165, pp. 152–166, 2021.
[48] M. A. Abdeljaww, “A collocation method based on Jacobi and fractional order Jacobi basis functions for multi-dimensional distributed-order diffusion equations,” International Journal of Nonlinear Sciences and Numerical Simulation, vol. 19, no. 7–8, pp. 781–792, 2018.
[49] M. A. Abdeljaww, A. M. Lopes, and M. A. Zaky, “Shifted fractional Jacobi spectral algorithm for solving distributed order time-fractional reaction-diffusion equations,” Computational and Applied Mathematics, vol. 38, no. 2, p. 81, 2019.
[50] Y. Yang, Y. Chen, and Y. Huang, “Convergence analysis of the Jacobi spectral-collocation method for fractional integro-differential equations,” Acta Mathematica Scientia, vol. 34, no. 3, pp. 673–690, 2014.
[51] X. Huang and C. Huang, “Spectral collocation method for linear fractional integro-differential equations,” Applied Mathematical Modelling, vol. 38, no. 4, pp. 1434–1448, 2014.
[52] M. Zaky, E. H. Doha, and J. Tenreiro Machado, “A spectral framework for fractional variational problems based on fractional Jacobi functions,” Applied Numerical Mathematics, vol. 132, pp. 51–72, 2018.