LSZ reduction formula in many-dimensional theory with space-space noncommutativity

K. V. Antipin, M. N. Mnatsakanova, and Yu. S. Vernov

1 Department of Physics, Moscow State University, Moscow 119991, Russia. E-mail: kv.antipin@physics.msu.ru
2 Skobeltsyn Institute of Nuclear Physics, Moscow State University, Moscow 119992, Russia.
3 Institute for Nuclear Research, Russian Academy of Sciences, Moscow 117312, Russia.

An analogue of the Lehmann—Symanzik—Zimmermann reduction formula is obtained for the case of noncommutative space-space theory. Some consequences of the reduction formula and Haag’s theorem are discussed.

Keywords: LSZ reduction formula; noncommutative theory; axiomatic quantum field theory.

PACS numbers: 11.10.-z, 11.10.Cd

1 Introduction

It is well known that in conventional quantum field theory the LSZ reduction formula allows effective calculation of the scattering amplitudes from the Green functions [1]. In the present paper we analyze the applicability of this formula in the framework of noncommutative quantum field theory (NC-QFT). We will consider the case of a neutral scalar field in many-dimensional theory with space-space noncommutativity, so that the temporal variable commutes with the spatial ones.
Let us consider the general case of $SO(1, d)$-invariant theory with $d + 1$ commutative coordinates (including time) and an arbitrary even number $l$ of noncommutative ones. The commutation relations between $l$ noncommutative coordinates have the form

$$[\hat{x}^i, \hat{x}^j] = i\theta^{ij}, \quad i, j = 1, \ldots, l,$$

where $\theta^{ij}$ — real antisymmetric $l \times l$ matrix. As we said, the rest $(d + 1)$ variables commute with each other and all $\hat{x}^j$ from (1).

In order to formulate the theory in commutative space-time, we use the Weyl ordered symbol $\varphi(x)$ of the noncommutative field operator $\Phi(\hat{x})$:

$$\varphi(x) = \frac{1}{(2\pi)^d} \int d^d k \int \text{Tr} e^{ik(x-\hat{x})} \Phi(\hat{x}),$$

and the corresponding multiplication law $\varphi_1 \star \varphi_2$ between the two symbols in the Weyl–Moyal–Groenewold form:

$$(\varphi_1 \star \varphi_2)(x) = \left[ e^{\frac{i}{2} \theta^{\mu\nu} \frac{\partial}{\partial x^\mu} \partial'_{\nu}} \varphi_1(x') \varphi_2(x'') \right]_{x' = x'' = x}.$$  

Relation (3) admits further generalization: for the symbols (fields) taken at different points one can define twisted tensor product $\varphi(x_1) \star \ldots \star \varphi(x_n)$:

$$\varphi(x_1) \star \ldots \star \varphi(x_n) = \prod_{a < b} \exp \left( i \frac{\theta^{\mu\nu}}{2} \frac{\partial}{\partial x_a^\mu} \frac{\partial}{\partial x_b^\nu} \right) \varphi(x_1) \ldots \varphi(x_n),$$

$$a, b = 1, 2, \ldots n.$$ 

Thus the algebra of field operators is deformed, and it is not clear whether one can apply the standard LSZ formula for the noncommutative fields or not.
2 Commutation Relations for Creation and Annihilation Operators in NCQFT

As in conventional field theory, a free real scalar field in NCQFT admits a normal mode expansion:

$$\phi(x) = \phi^+(x) + \phi^-(x),$$

$$\phi^\pm(x) = \frac{1}{(2\pi)^{(d+l)/2}} \int \frac{d\vec{k}}{\sqrt{2\omega(\vec{k})}} e^{\pm ikx} a^\pm(\vec{k}) |_{k^0 = \omega(\vec{k})}.$$ (5)

where $$\omega(\vec{k}) = \sqrt{\vec{k}^2 + m^2}$$, $$\vec{k}^2 = \vec{k}_c^2 + \vec{k}_{nc}^2$$, $$\vec{k}_c$$— commutative part of the $$(d+l)$$-dimensional vector $$\vec{k}$$, $$\vec{k}_{nc}$$— noncommutative part of the same vector.

Let us obtain commutation relations for the creation and annihilation operators $$a^\pm$$ directly from the assumption that the canonical quantization of a real scalar field in NCQFT is defined by the relations

$$[\phi(x) \star \partial_0 \phi(y)]|_{x^0 = y^0} = i\delta(\vec{x} - \vec{y}),$$

$$[\phi(x) \star \phi(y)]|_{x^0 = y^0} = 0, \quad [\partial_0 \phi(x) \star \partial_0 \phi(y)]|_{x^0 = y^0} = 0.$$ (6)

Performing an inverse Fourier transform one can get the expression for $$a^\pm$$ from (5):

$$a^\pm(\vec{k}) = \frac{1}{(2\pi)^{(d+l)/2}} \int d\vec{x} e^{\mp ikx} \left[ \sqrt{\frac{k_0}{2}} \phi(x) \mp \frac{i}{\sqrt{2k_0}} \partial_0 \phi(x) \right] |_{k_0 = \omega(\vec{k})}.$$ (7)

Let us take the operator product $$a^-(\vec{k})a^+(\vec{q})$$ and multiply it by $$e^{\frac{i}{2} \theta^\mu_\nu k_\mu q_\nu} = e^{\frac{i}{2} \theta^\mu_\nu k_\mu (-i) q_\nu}$$. Expanding the phase factor in a series, we get

$$e^{\frac{i}{2} \theta^\mu_\nu k_\mu (-i) q_\nu} a^-(\vec{k})a^+(\vec{q}) = \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!} (\theta^\mu_\nu k_\mu (-i) q_\nu)^n a^-(\vec{k})a^+(\vec{q}) =$$

$$= \frac{1}{(2\pi)^{(d+l)/2}} \int d\vec{x} d\vec{y} \sum_{n=0}^{\infty} \frac{(i/2)^n}{n!} (\theta^\mu_\nu k_\mu (-i) q_\nu)^n e^{ikx} e^{-iqy} \times$$

$$\times \left( \sqrt{\frac{k_0}{2}} \phi(x) \mp \frac{i}{\sqrt{2k_0}} \partial_0 \phi(x) \right) \left( \sqrt{\frac{q_0}{2}} \phi(y) - \frac{i}{\sqrt{2q_0}} \partial_0 \phi(y) \right).$$ (8)
Next, let us replace momenta $k$ and $q$ with the derivatives, using the relation

$$(ik_\mu(-i)q_\nu)^n e^{ikx} e^{-iqy} = (\partial_\mu \partial_\nu)^n e^{ikx} e^{-iqy},$$

and perform integration by parts, so that the derivatives act on the field $\varphi$ in each term of the series. Thus we obtain $\ast$-product of the field operators:

$$e^{\pm \theta^{\mu\nu} k_\mu q_\nu} a^-(\vec{k}) a^+(\vec{q}) = \frac{1}{(2\pi)^{(d+1)}} \int \int d\vec{x} d\vec{y} e^{ikx} e^{-iqy} \times$$

$$\times \left( \sqrt{\frac{k_0}{2}} \varphi(x) + \frac{i}{\sqrt{2k_0}} \partial_0 \varphi(x) \right) \ast \left( \sqrt{\frac{q_0}{2}} \varphi(y) - \frac{i}{\sqrt{2q_0}} \partial_0 \varphi(y) \right) \bigg|_{k_0=\omega(\vec{k}), q_0=\omega(\vec{q})}.$$  \hspace{1cm} (10)

Similarly,

$$e^{\pm \theta^{\mu\nu} k_\mu q_\nu} a^+(\vec{q}) a^-(\vec{k}) = \frac{1}{(2\pi)^{(d+1)}} \int \int d\vec{x} d\vec{y} e^{ikx} e^{-iqy} \times$$

$$\times \left( \sqrt{\frac{q_0}{2}} \varphi(y) - \frac{i}{\sqrt{2q_0}} \partial_0 \varphi(y) \right) \ast \left( \sqrt{\frac{k_0}{2}} \varphi(x) + \frac{i}{\sqrt{2k_0}} \partial_0 \varphi(x) \right) \bigg|_{k_0=\omega(\vec{k}), q_0=\omega(\vec{q})}.$$  \hspace{1cm} (11)

Now, taking the fields $\varphi(x)$ and $\varphi(y)$ at equal moments of time $x^0 = y^0$ and subtracting (11) from (10), with the use of (6) we obtain:

$$e^{\pm \theta^{\mu\nu} k_\mu q_\nu} a^-(\vec{k}) a^+(\vec{q}) - e^{\pm \theta^{\mu\nu} k_\mu q_\nu} a^+(\vec{q}) a^-(\vec{k}) = \delta(\vec{k} - \vec{q}),$$

or, in a more convenient form:

$$a^-(\vec{k}) a^+(\vec{q}) = e^{-i\theta^{\mu\nu} k_\mu q_\nu} a^+(\vec{q}) a^-(\vec{k}) + e^{-i\theta^{\mu\nu} k_\mu q_\nu} \delta(\vec{k} - \vec{q}).$$  \hspace{1cm} (12)

In the same way we get

$$a^\pm(\vec{k}) a^\pm(\vec{q}) = e^{i\theta^{\mu\nu} k_\mu q_\nu} a^\pm(\vec{q}) a^\pm(\vec{k}).$$  \hspace{1cm} (13)

Commutation relations (12) and (13) are equivalent to the ones obtained in [5] from general group-theoretical considerations involving the twisted Poincaré symmetry.
3 Analogue of the LSZ reduction formula for space-space NCQFT

In [4] it was proposed that the expression for the noncommutative Wightman functions has the following form:

\[ W_\star(x_1, \ldots, x_n) = \langle 0 | \varphi(x_1) \star \cdots \star \varphi(x_n) | 0 \rangle, \tag{15} \]

where \( \star \)-product of fields taken at independent points is given by (4).

In accordance with (15) we suppose that the noncommutative Green functions are

\[ G_\star(x_1, \ldots, x_n) = \langle 0 | T(\varphi(x_1) \star \cdots \star \varphi(x_n)) | 0 \rangle, \tag{16} \]

where we defined time-ordered \( \star \)-product of fields as straightforward generalization of the usual \( T \)-product:

\[ T(\varphi_1(x_1) \star \cdots \star \varphi_n(x_n)) = \varphi_{\sigma_1}(x_{\sigma_1}) \star \cdots \star \varphi_{\sigma_n}(x_{\sigma_n}), \]

\[ x_{\sigma_1}^0 > x_{\sigma_2}^0 > \ldots > x_{\sigma_n}^0. \tag{17} \]

Below we extend the classical proof of the LSZ formula [1, 6, 7] to the case of space-space NCQFT.

Let us single out the variable \( x_1 \) and consider the expression

\[ \lim_{\bar{p}_1 \to -\omega(\bar{p}_1)} (p_1^2 - m^2) \int dx_1^0 d\vec{x}_1 e^{-ip_1 x_1} \langle 0 | T(\varphi(x_1) \star \cdots \star \varphi(x_n)) | 0 \rangle. \tag{18} \]

Dividing integration over \( dx_1^0 \) into three parts

\[ \int (\ldots) dx_1^0 = \int_{-\infty}^{-\tau} (\ldots) dx_1^0 + \int_{-\tau}^{\tau} (\ldots) dx_1^0 + \int_{\tau}^{+\infty} (\ldots) dx_1^0, \tag{19} \]

we denote the summands as \( I_1(\tau), I_2(\tau), \) and \( I_3(\tau) \) respectively.

Using expression \( (p_1^2 - m^2)e^{-ip_1 x_1} = (\Box_1 - m^2)e^{-ip_1 x_1} \) and performing integration by parts, we get:

\[ I_1(\tau) = \int d\vec{x}_1 e^{i\bar{p}_1 \cdot \vec{x}_1 + i\omega(\bar{p}_1)\tau} \langle 0 | T(\varphi(x_2) \star \cdots \star \varphi(x_n)) \star \]

\[ \star (i\omega(\bar{p}_1) - \partial_{\tau}) \varphi(-\tau, \vec{x}_1)|0\rangle - \int_{-\infty}^{-\tau} dx_1^0 \int d\vec{x}_1 e^{i\bar{p}_1 \cdot \vec{x}_1 - i\omega(\bar{p}_1)x_1^0} \times \]

\[ \times \langle 0 | T(\varphi(x_2) \star \cdots \star \varphi(x_n)) \star (\Box_1 - m^2) \varphi(x_1)|0\rangle. \tag{20} \]
Here $\Box_1 \equiv \frac{\partial^2}{\partial (x_1^1)^2} + \frac{\partial^2}{\partial (x_1^2)^2} + \frac{\partial^2}{\partial (x_1^3)^2} - \frac{\partial^2}{\partial (x_1^0)^2}$, and $\tau$ is taken sufficiently large so that the permutation of $\varphi(x_1)$ to the last position on the right is possible.

Next, we use the Fourier-expression for $\varphi(-\tau, \vec{x}_1)$:

$$\varphi(-\tau, \vec{x}_1) = \frac{1}{(2\pi)^{(d+l)/2}} \int dk_0 d\vec{k} e^{-i k_0 \tau} e^{-i \vec{k} \vec{x}_1} \hat{\varphi}(k). \quad (21)$$

All derivatives in the $\star$-product will act on the factor $e^{-i \vec{k} \vec{x}_1}$ in the Fourier expansion of $\varphi(x_1)$. Therefore, additional factor $N(k_{nc})$ will appear. Note that $N(k_{nc})$ depends only on the noncommutative part of $\vec{k}$.

Let us also take into account the asymptotic representation for the field $\varphi$:

$$\lim_{t \to -\infty} \int dk_0 e^{it(k_0 - \omega(p_1))} \hat{\varphi}(k) = \frac{1}{\sqrt{2\omega(p_1)}} a^+_{in}(\vec{k}). \quad (22)$$

Taking the limit $\tau \to \infty$, we obtain:

$$I_1 = \lim_{\tau \to \infty} I_1(\tau) = i(2\pi)^{(d+l)/2} \int dk_0 (k_0 + \omega(p_1)) \delta(k_0 - \omega(p_1)) \times$$

$$\times \langle 0| T(\varphi(x_2) \star \ldots \star \varphi(x_n)) N(p_{1, nc}) \frac{a^+_{in}(\vec{p}_1)}{\sqrt{2\omega(p_1)}} |0\rangle =$$

$$= i(2\pi)^{(d+l)/2} \sqrt{2\omega(p_1)} N(p_{1, nc}) \langle 0| T(\varphi(x_2) \star \ldots \star \varphi(x_n)) a^+_{in}(\vec{p}_1) |0\rangle. \quad (23)$$

In this limit the second term of the expression (20) is equal to null.

Similar calculations for $I_3(\tau)$ will give:

$$I_3 = i(2\pi)^{(d+l)/2} \sqrt{2\omega(p_1)} N(p_{1, nc}) \langle 0| a^+_{out}(\vec{p}_1) T(\varphi(x_2) \star \ldots \star \varphi(x_n)) |0\rangle = 0. \quad (24)$$

As to the second summand in (19), $I_2(\tau)$ can be presented as

$$\int_{-\infty}^{\infty} dx_1^0 e^{-ip_0^0 x_1^0} \chi(x_1^0, \tau) F(x_1),$$

$$\chi(x_1^0, \tau) = \begin{cases} 1, & |x_1^0| \leq \tau; \\ 0, & |x_1^0| > \tau. \end{cases} \quad (25)$$

The integrand contains a generalized function with compact support, so its Fourier-transform is a smooth function and doesn’t have a pole. For this
reason

\[
\lim_{p_1^0 \to \omega(p_1)} \left( p_1^2 - m^2 \right) \int \, dx_1^0 \int \, dx_1^0 e^{-ip_1 x_1} \langle 0 | T(\varphi(x_1) \ast \cdots \ast \varphi(x_n)) | 0 \rangle = 0.
\]

(26)

Following similar limiting procedure over \( x_1 \) and \( x_2 \) consecutively, we obtain

\[
\lim_{p_2^0 \to -\omega(p_2)} \lim_{p_1^0 \to \omega(p_1)} \left( p_2^2 - m^2 \right) \left( p_1^2 - m^2 \right) \int \, dx_1 \int \, dx_2 e^{-ip_2 x_2 - ip_1 x_1} \times
\]

\[
\times \langle 0 | T(\varphi(x_1) \ast \cdots \ast \varphi(x_n)) | 0 \rangle = \left[ i(2\pi)^{(d+1)/2} \right]^2 \sqrt{2\omega(p_1)} \sqrt{2\omega(p_2)} \times
\]

\[
\times N(p_{2,nc})N(p_{1,nc}) \langle 0 | T(\varphi(x_3) \ast \cdots \ast \varphi(x_n)) a_{in}^+(\tilde{p}_2) a_{in}^+(\tilde{p}_1) | 0 \rangle.
\]

(27)

Now let us replace the second procedure (over \( x_2 \)) with the one corresponding to transition to the bottom sheet of the mass hyperboloid, that is

\[
\lim_{p_2^0 \to -\omega(p_2)} \left( p_2^2 - m^2 \right). \text{ Making use of the asymptotic representation}
\]

\[
\lim_{t \to \pm \infty} \int \, dk_0 e^{i(k_0+\omega(\tilde{k}))} \tilde{\varphi}(k) = \frac{1}{\sqrt{2\omega(\tilde{k})}} a_{in(out)}^-(\tilde{k}),
\]

(28)

we obtain:

\[
\lim_{p_2^0 \to -\omega(p_2)} \lim_{p_1^0 \to \omega(p_1)} \left( p_2^2 - m^2 \right) \left( p_1^2 - m^2 \right) \int \, dx_1 \int \, dx_2 e^{-ip_2 x_2 - ip_1 x_1} \times
\]

\[
\times \langle 0 | T(\varphi(x_1) \ast \cdots \ast \varphi(x_n)) | 0 \rangle = \left[ i(2\pi)^{(d+1)/2} \right]^2 \sqrt{2\omega(p_1)} \sqrt{2\omega(p_2)} \times
\]

\[
\times [N(p_{2,nc})N(p_{1,nc}) | 0 | a^{-}_{out}(-\tilde{p}_2) T(\varphi(x_3) \ast \cdots \ast \varphi(x_n)) a_{in}^{+}(\tilde{p}_1) | 0 \rangle -
\]

\[
-\tilde{N}(p_{2,nc}) N(p_{1,nc}) | 0 | T(\varphi(x_3) \ast \cdots \ast \varphi(x_n)) a_{in}^{-}(-\tilde{p}_2) a_{in}^{+}(\tilde{p}_1) | 0 \rangle].
\]

(29)

The first term here is the contribution of \( I_3(\tau) \), which is not equal to zero in this limit. Let us make the substitution \( \tilde{p}_2 \rightarrow -\tilde{p}_2 \). We consider the scattering processes in which \( \tilde{p}_1 \) — incoming momentum, \( \tilde{p}_2 \) — outgoing momentum, and \( \tilde{p}_1 \neq \tilde{p}_2 \). In accordance with \( (13) \) we can commute \( a_{in}^{-}(\tilde{p}_2) \) and \( a_{in}^{+}(\tilde{p}_1) \) in the second term of \( (29) \) so that \( a_{in}^{+}(\tilde{p}_2) \) can act on the vacuum state and give null.

We can repeat the above-mentioned procedure \( n \) times — until nothing is left under the time-ordered \( \ast \)-product. Now the additional factor \( N(p_{1,nc}) \times \)
\( \ldots \times N(p_{n,nc}) \) can be expressed in explicit form: each derivative \( \partial_{\mu} \) in (4) should be replaced with \( ip_{\mu} \), and we have:

\[
N(p_1,nc) \times \ldots \times N(p_{n,nc}) = \exp \left[ -\frac{i}{2} \theta_{\mu\nu} \sum_{a<b} p_a^\mu p_b^\nu \right] \bigg|_{-p_{o\text{ut}}} ,
\]

\( a, b = 1, \ldots, n, \)

where \( \bigg|_{-p_{o\text{ut}}} \) means that outgoing momenta should be taken with the minus sign (as the result of the substitution \( \vec{p} \rightarrow -\vec{p} \) we made earlier).

The final expression for the scattering amplitude:

\[
\langle 0| a_{1\text{out}}(\vec{p}_1) \ldots a_{out}(\vec{p}_k) a_{in}^+(\vec{p}_{k+1}) \ldots a_{1\text{in}}^+(\vec{p}_n)|0 \rangle = \left[ \frac{1}{i(2\pi)^{(d+1)/2}} \right]^n \times
\]

\[
\exp \left[ \frac{i}{2} \theta_{\mu\nu} \sum_{a<b} p_a^\mu p_b^\nu \right] \prod_{j=1}^n \frac{p_j^2 - m^2}{\sqrt{2\omega(\vec{p}_j)}} G_\star(-p_1, \ldots, -p_k, p_n, \ldots, p_{k+1}),
\]

\( G_\star(p_1, \ldots, p_n) \) — Fourier transform of the noncommutative Green function:

\[
G_\star(p_1, \ldots, p_n) = \int dx_1 \ldots dx_n \exp \left[ -i \sum_{j=1}^n p_j x_j \right] G_\star(x_1, \ldots, x_n).
\]

Relation (31) is a noncommutative analogue of the LSZ reduction formula. This result corresponds to the one obtained in [8], authors of which didn’t use the \( \star \)-product between the fields taken at different points and considered the Green functions with the usual time-ordered product of noncommutative fields. The difference between the two results is the additional phase-factor (30) due to the chosen form of the Green function (16).

4 Consequences

Now we can extend to NCQFT the considerations that were originally proposed in [9] for the case of commutative theory.
Suppose that we have two noncommutative $SO(1, d)$-invariant theories on Hilbert spaces $\mathcal{H}_1$ and $\mathcal{H}_2$ respectively, related by a unitary transformation. Let $\phi_1$ and $\phi_2$ be two irreducible sets of field operators defined in $\mathcal{H}_1$ and $\mathcal{H}_2$. Let $< p'_1, \ldots, p'_n | p_1, \ldots, p_m >_i$, $i = 1, 2$ be inelastic scattering amplitudes of the process $m \rightarrow n$ for the fields $\phi_1$ and $\phi_2$ respectively. In accordance with the reduction formula (31)

$$< p'_1, \ldots, p'_n | p_1, \ldots, p_m >_i \sim \int dx_1 \ldots dx_{n+m} \exp \{i(-p_1 x_1 - \ldots - p_m x_m + p'_1 x_{m+1} + \ldots + p'_n x_{n+m})\} \times \prod_{j=1}^{n+m} (\Box_j - m^2) \langle 0 | T(\varphi_i (x_1) \ast \ldots \ast \varphi_i (x_{n+m})) | 0 \rangle,$$

$$i = 1, 2.$$ (33)

Let us also take into account the results obtained for the generalized Haag’s theorem in the context of noncommutative theory [10, 11]. Namely, it was shown that in two $SO(1, d)$-invariant theories, related by a unitary transformation, the two-, three, $\ldots$, $d + 1$-point Wightman functions coincide:

$$\langle 0 | \varphi_1 (x_1) \ast \ldots \ast \varphi_1 (x_s) | 0 \rangle = \langle 0 | \varphi_2 (x_1) \ast \ldots \ast \varphi_2 (x_s) | 0 \rangle,$$

$$2 \leq s \leq d + 1.$$ (34)

From (33) and (34) it follows that the amplitudes $< p'_1, \ldots, p'_n | p_1, \ldots, p_m >_1$ and $< p'_1, \ldots, p'_n | p_1, \ldots, p_m >_2$ coincide in the two theories if

$$m + n \leq d + 1.$$ (35)

References

[1] H. Lehmann, K. Symanzik, W. Zimmermann, Nuovo Cimento 1, 205 (1955).
[2] M. R. Douglas, N. A. Nekrasov, *Rev. Mod. Phys.* **73**, 977 (2001), [arXiv:hep-th/0106048].

[3] R. J. Szabo, *Phys. Rep.* **378**, 207 (2003), [arXiv:hep-th/0109162].

[4] M. Chaichian, M. N. Mnatsakanova, K. Nishijima, A. Tureanu, and Yu. S. Vernov, *J. Math. Phys.* **52**, 032303 (2011), [arXiv:hep-th/0402212].

[5] A.P. Balachandran, T.R. Govindarajan, G. Mangano, A. Pinzul, B.A. Qureshi, and S. Vaidya, *Phys. Rev. D* **75**, 045009 (2007), [arXiv: hep-th/0608179].

[6] J. D. Bjorken and S. D. Drell, *Relativistic Quantum Mechanics Vol. 2* (McGraw-Hill Inc., 1964).

[7] C. Itzykson, J.-B. Zuber, *Quantum Field Theory Vol. 1* (McGraw-Hill Inc., 1980).

[8] A.P. Balachandran, P. Padmanabhan, A. R. de Queiroz *Phys. Rev. D* **84**, 065020 (2011), [arXiv: hep-th/1104.1629].

[9] M. Chaichian, M. Mnatsakanova, A. Tureanu, Yu. Vernov, *Classical Theorems in Noncommutative Quantum Field Theory*, preprint [arXiv: hep-th/0612112].

[10] K. V. Antipin, M. N. Mnatsakanova, Yu. S. Vernov, *Moscow Univ. Phys. Bull.* **66**, 349 (2011), [arXiv: hep-th/1102.1195].

[11] K. V. Antipin, M. N. Mnatsakanova, Yu. S. Vernov, *Phys. of Atom. Nucl.* **76**, 965 (2013), [arXiv: hep-th/1202.0995].