On smooth moduli space of Riemann surfaces

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Abstract

In this paper we study the smooth moduli space of closed Riemann surfaces. This smooth moduli is an infinite cover of the usual moduli space $\mathcal{M}_g$ of closed Riemann surfaces, and is identified with the Schottky space of rank $g$. The main theorem of the paper is: Closed Riemann surfaces are uniformizable by Schottky groups of Hausdorff dimension less than one. This work seem to be the only paper in literature to study question of Riemann surface uniformization and its Hausdorff dimension. We develop new techniques of rational norm of homological marking of Riemann surface and, decomposition of probability measures to prove our result. As an application of our theorem we have existence of period matrix of Riemann surface in coordinates of smooth moduli space.

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1 Introduction and Main Theorem

The main theorem of this paper is:

**Theorem 1.1.** *Every closed Riemann surface can be uniformized by a Schottky group of Hausdorff dimension $< 1$.*

Throughout this paper $R_g$ denotes closed Riemann surface of genus $g$. A Kleinian group $\Gamma$ is, finitely generated and discrete subgroup of $\text{PSL}(2, \mathbb{C})$. Denote by $\Lambda_\Gamma$, the limit set of $\Gamma$, which is minimal closed $\Gamma$-invariant, nowhere dense, non-discrete, perfect subset of $\mathbb{C}$. We say $\Gamma$ is of Hausdorff dimension $\mathcal{D}_\Gamma$ if $\Lambda_\Gamma$ have Hausdorff dimension of $\mathcal{D}_\Gamma$. The region of discontinuity of $\Gamma$ is $\Omega_\Gamma = \mathbb{C} - \Lambda_\Gamma$. By Ahlfors theorem, $\Omega_\Gamma/\Gamma$ is of finite many components and each is of analytically finite type Riemann surface.

Rank-$g$ Schottky group $\Gamma$ is, $g$-generated free, purely loxodromic Kleinian group. Equivalently, $\Gamma$ is Schottky group if it is convex-cocompact representation of rank-$g$ free group $\mathbb{F}_g$ into $\text{PSL}(2, \mathbb{C})$. In particular, $\mathbb{H}^3/\Gamma$ is a handle-body and $\Omega_\Gamma/\Gamma$ is a Riemann surface.

Uniformization of $R_g$ by Schottky group $\Gamma$, see next section for details is, $R_g = \Omega_\Gamma/\Gamma$. Theorem 1.1 states that: there exists a Schottky group $\Gamma$ such that: $R_g = \Omega_\Gamma/\Gamma$ and $\mathcal{D}_\Gamma < 1$.

The main difficulty of Theorem 1.1 is the lack of relation of uniformization $\Gamma$ and its Hausdorff dimension. In fact, the question of what possible Hausdorff dimensions of Kleinian groups can uniformize $R_g$ has not been studied at all before.

There are some interesting consequences of Theorem 1.1 which we will address separately in subsequent works. In this paper, we will state at the end an immediate simple consequence of Theorem 1.1 which implies existence of period matrix in smooth moduli coordinates.
Strategy of the proof of Theorem 1.1

First we derive a criteria for handle-body to be uniformizable by a classical Schottky group. To do so, we establish an new relation between Hausdorff dimension of limit set of $\Gamma$ with it’s primitive elements. This is done by a generalization of paradoxically decomposing [9, 17] the probability measure on the limit set to its generators and integrate over all primitive bases. From this decomposition we derive an new family of probability measures supported locally about each generators. By using relations among this family of probability measures, we derive an lower bound on the growth of the mean primitive displacement of generating elements of $\Gamma$. We show that the mean primitive displacement growth implies upper bounds on Hausdorff dimension. The measure decomposition is done abstractly on Cayley graph of free group in Section 3. The growth estimate is done for handle-body and is given in Section 4.

Secondly, that given a Riemann surface $R_g$, it is identified as the conformal boundary of infinity $\partial_\infty M$ of hyperbolic 3-manifold $M = (\mathbb{H}^3 \cup \Omega_\Gamma)/\Gamma$. This conformal identification is through markings on $R_g$. More precisely, It is classical known fact that [3, 5], we have a morphism $\phi$ from canonical basis with orientation of $H_1(R_g, \mathbb{Z})$ into Schottky space. This is done by representing half basis of a given basis by conformal maps of $\mathbb{C}$. Every half basis of a given canonical basis generates a subgroup of $H_1(R_g, \mathbb{Z})$. The morphism establishes a injective corresponds of these subgroups into the Schottky space. We call a given half basis of a canonical basis a homological marking. However, it is completely unknown when a given point of Schottky space can actually cover a point of moduli space $M_g$.

We study homological markings by defining a real function $Q$ on markings of $R_g$. We call it rational norm of a marking. This is defined through ratio of geodesic representative curves with the dual geodesic to the marking geodesic as the unique minimal curve. We also define a conformal invariant of $R_g$, the $Q$-spectrum of $R_g$ as the, collection of $Q$ values under variation of all markings of $R_g$.

We then establish an inequality Lemma 5.5 between translation length of primitive elements of the image Schottky group to $Q$. This is done through the use of theory of extremal length. Where we use interpolation of hyperbolic metrics on planar domains.

Next we show that there exists some marking such that $Q$ is bounded below by $\frac{2\lambda_g}{\pi} g \log(2g)$ for some $\lambda_g > 2$. We call a marking such that $Q$ satis-
fies this lower bound is positive. This is done by study geodesic length ratios through elementary arcs, which are part of the marking curve on pants decompositions of $R_g$. The existence proof of Lemma 6.1 is by contradiction. Under assumption, we show by computation, one can always choose elementary arcs to construct curves for $[c]$ so that $Q([c])$ achieve value greater than $\frac{2\lambda_g g \log(2g)}{\pi}$. These computations are done on pair of pants.

Next we show that there exists some marking such that $Q$ satisfies the lower bound inequality as in Lemma 5.5 under action of $\text{stab}(\phi) \subset \text{Sp}(2g, \mathbb{Z})$, stabilizer subgroup of $\phi$. This is Proposition 6.4. The idea is that, by taken a marking provided by Lemma 6.1 we can show either this marking stays positive under elementary matrices $E_{lm}$ of $\text{stab}(\phi)$ or, there must exists another marking that have larger value of $Q$ which will be positive under $E_{lm}$. In fact, the ratio of lengths will increase under Dehn twist.

Proposition 6.4 implies that there exists a marking such that all primitive elements of the Schottky group will have the desired mean displacement on the handle-body. This is in Proposition 6.6.

Finally we use these marking estimates and result on handle-body Hausdorff dimension estimates to finish proof of the main theorem. As an immediate application of Theorem 1.1 we can explicitly express period matrix of points of $\mathcal{M}_g$ in coordinates of Schottky space.

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This paper is dedicate to: Ying Zhou and her mom Junlan Wang.
2 Covering of Moduli space $\mathcal{M}_g$

We give here a very brief and basic introduction of $\mathcal{J}_g$ as smooth moduli of a closed Riemann surface $R_g$. See [3], [15], [13], [25], [24] for backgrounds materials and some of the details.

Schottky group $\Gamma$ of rank $g$ is defined as convex-cocompact discrete faithful representation of $\mathbb{F}_g$ in $\text{PSL}(2, \mathbb{C})$. It follows that $\Gamma$ is freely generated by purely loxodromics $\{\gamma_i\}_{1}^{g}$. This implies we can find collection of disjointed closed topological disks $D_i, D'_i, 1 \leq i \leq g$ in the Riemann sphere $\partial \mathbb{H}^3 = \mathbb{C}$ with boundary curves $\partial D_i = \Delta_i, \partial D'_i = \Delta'_i$. By definition $\Delta_i$ are closed Jordan curves in Riemann sphere $\partial \mathbb{H}^3$, such that $\gamma_i(\Delta_i) = \Delta'_i$ and $\gamma_i(D_i) \cap D'_i = \emptyset$. Whenever there exists a set $\{\gamma_i, ..., \gamma_g\}$ of generators with all $\Delta_i, \Delta'_i$ as circles, then it is called a classical Schottky group with $\{\gamma_1, ..., \gamma_g\}$ classical generators.

Schottky space $\mathcal{J}_g$ is defined as space of all rank $g$ Schottky groups up to conjugacy by $\text{PSL}(2, \mathbb{C})$. By normalization, we can chart $\mathcal{J}_g$ by $3g - 3$ complex parameters. Hence $\mathcal{J}_g$ is $3g - 3$ dimensional complex manifold. The biholomorphic $\text{Auto}(\mathcal{J}_g)$ group is $\text{Out}(\mathbb{F}_g)$, which is isomorphic to subgroup of the handle-body group. On the other hand, $\mathcal{J}_{g,o}$ is not submanifold. In fact, it is nontrivial result due to Marden that $\mathcal{J}_{g,o}$ is non-empty and non-dense set of $\mathcal{J}$. However, it follows from a theorem of Hou [16], $\mathcal{J}_{g,o}^\lambda$ is $3g - 3$ dimensional complex submanifold. Here $\mathcal{J}_{g,o}^\lambda$ denotes space of classical Schottky groups of Hausdorff dimension $< \lambda$.

Let $\mathcal{T}_g$ and $\text{Mod}_g$ be the Teichmüller space of $R_g$ and it’s mapping class group respectively. $\mathcal{T}_g$ is the universal cover of $\mathcal{J}_g$. In fact, there exists subgroup $\text{Mod}_g^\phi \subset \text{Mod}_g$ which depends on a given symplectic-morphism $\phi : \pi_g \rightarrow \mathbb{F}_g$. The dependence of $\phi$ is only up to conjugacy within $\text{Mod}_g$. It follows $\text{Mod}_g^\phi$ is infinite index, torsion-free subgroup of $\text{Mod}_g$. Since $\mathcal{T}_g/\text{Mod}_g = \mathcal{M}_g$, we have the following commutative diagram of holomorphic covers:

$$
\begin{array}{ccc}
\mathcal{T}_g & \xrightarrow{\pi_U} & \mathcal{J}_g \\
\pi_F & \downarrow & \pi_S \\
\mathcal{M}_g & \xrightarrow{\pi_S} & \mathcal{J}_g
\end{array}
$$

In particular, $\mathcal{J}_g$ is infinite cover of $\mathcal{M}_g$, hence can be considered smooth moduli of $R_g$. We end this section by restating Theorem 1.1:

**Theorem 2.1.** Let $[R_g] \in \mathcal{M}_g$, there exist $\Gamma \in \pi_S^{-1}([R_g])$ of $\mathcal{D}_\Gamma < 1$.  


3 Probability Measures and $\|W_\Gamma\|_x$ on Schottky Space

Let $\Gamma$ be a finitely generated free group with generating set $\omega$. We denote the collection of all generating sets by $W_\Gamma$. For Schottky group with chosen point $x$ of hyperbolic space $H^3$, we also denote the minimal generator of $\omega$ with respect to $x$ by $w_x(\omega) = \sum_{\alpha \in \omega} \frac{\text{dist}(\alpha x, x)}{g}$.

We also denote the Cayley graph of $\Gamma$ with symmetric generating set $S_\omega = \omega \cup \omega^{-1}$ by $C_\Gamma S_\omega$.

Definition 3.1. For $\Gamma$ Schottky group, we define:

- $W_\Gamma$ the set of collection of all free basis $\omega$ of $\Gamma$, $\|W_\Gamma\|_x$ of $\Gamma$ at point $x$ as $\|W_\Gamma\|_x = \inf_{\omega \in W_\Gamma} w_x(\omega)$.

We call $\|W_\Gamma\|_x$, the mean norm of $W_\Gamma$.

Theorem 3.2. Let $\Gamma$ be a Schottky group of rank $g$ and Hausdorff dimension $D_\Gamma$. There exist nonatomic Borel measure $\sigma_x$ on $\Lambda_\Gamma \times W_\Gamma$ of total mass $< 1$ such that,

$$D_\Gamma \leq \log \left( \frac{\sigma_x(\Lambda_\Gamma \times W_\Gamma)}{\|W_\Gamma\|_x} \right).$$

Let $\Gamma$ be Schottky group. Suppose $\omega_m = \langle \gamma_1, ..., \gamma_g \rangle$ is the basis of minimal translational length $T_{\gamma_j}$. Then we can estimate $\sigma_x(\Lambda_\Gamma \times W_\Gamma)$ based on relations among elements of this minimal basis.

Corollary 3.3. Let $\Gamma$ be a Schottky group. Suppose there exists $\lambda > 0$ such that $T_{\gamma_j} \leq \lambda T_{\gamma_i}$ for $\gamma_i, \gamma_j \in \omega_m$. Then there exists $x \in H^3$ such that,

$$D_\Gamma \leq \frac{(\lambda - 1) \log(2) + (\lambda + 1) \log(g)}{\|W_\Gamma\|_x}.$$ 

Theorem 3.2 is proved by constructing families of Borel probability measures on the limit set $\Lambda(\Gamma)$. These probability measures are averages over $W_\Gamma$ of measures associated to elements of $W_\Gamma$.

We first state a decomposition lemma on $(C\Gamma S, W')$, with respect to some chosen symmetric generating set $S_\omega$ and word metric $W'$. Note that $(C\Gamma S, W')$ is Gromov hyperbolic, or $\delta$-hyperbolic.
Let $\mu_o$ be quasi-conformal measure, which is the Patterson-Sullivan measure \cite{8}, on $(C\Gamma S, W)$. It is given by:

$$c_\Gamma^{-1}\Psi^D_\Gamma(\gamma^{-1}o, \zeta) \leq \frac{d\gamma^*\mu_o}{d\mu_o}(\zeta) \leq c_\Gamma\Psi^D_\Gamma(\gamma^{-1}o, \zeta), \quad c_\Gamma > 0.$$ 

Where $\Psi^D_\Gamma(\gamma^{-1}o, \zeta) = e^{B_\Gamma o(\gamma^{-1}o, \zeta)}$, with $B_\Gamma o(\gamma^{-1}o, \zeta)$ the Buseman function. We normalize so that $\mu_o$ is always taken to be probability measure on $\partial C\Gamma$. In addition, when we choose a indexing of $W\Gamma$, we denote it by $\omega_j$.

Lemma 3.4. There exists Borel measures $\sigma^+_o, \sigma^-_o$ on $\partial C\Gamma \times \mathbb{W}_\Gamma$ with $\sigma^+_o$ a probability measure and, for each $\omega \in \mathbb{W}_\Gamma$ we have a family of Borel measures $\{\nu_o,\gamma\}_{\gamma \in \omega}$ and $\{\nu_o,\gamma^{-1}\}_{\gamma \in \omega}$ on $\partial C\Gamma$ and $C_\omega \in (0, c_\Gamma), c_\omega \in (c_\Gamma^{-1}, c_\Gamma)$ of the following properties:

- $\eta^+_o,\omega = \sum_{\gamma \in \omega} \nu_o,\gamma, \eta^-_o,\omega = \sum_{\gamma \in \omega} \nu_o,\gamma^{-1}$,

where $\eta^+_o,\omega$ is probability measures on $\partial C\Gamma$ and $\eta^-_o,\omega(\partial C\Gamma) < 1$, for each $\omega \in \mathbb{W}_\Gamma$.

- $\frac{d\sigma^+_o}{d\sigma^-_o}(\zeta, \omega) = \frac{d\eta^+_o,\omega}{d\eta^-_o,\omega}(\zeta)$.

- $\frac{1}{(g-1) + C_\omega} \int_{\partial C\Gamma} c_\omega \Psi^D_\Gamma(\gamma^{-1}o, \zeta) d\nu_o,\gamma^{-1} = \int_{\partial C\Gamma} d\nu_o,\gamma; \text{ for } \gamma \in \omega.$

The proof of next lemma will be based on a generalization of Culler-Shalen paradoxical measure decompositions for isometry groups of $\mathbb{H}^3$ \cite{17}. These paradoxical decomposition were first done by Culler-Shalen \cite{9} for the case of 2-generated cocompact Kleinian group, and later generalized in \cite{17} to all free finitely generated subgroups of $\text{PSL}(2, \mathbb{C})$. They were interested in the embedded Margulis tube volume bounds.

Lemma 3.5. There exists collection of Borel measures $\{\nu_\gamma\}_{\gamma \in \omega}, \{\nu_o,\gamma^{-1}\}_{\gamma \in \omega}$ on $\partial C\Gamma$ such that, $\sum_{\gamma \in \omega} \nu_o,\gamma = \eta^+_o,\omega$ is probability measure and, $\sum_{\gamma \in \omega} \nu_o,\gamma^{-1} =$
η_{\omega} is measures of total mass < 1. In addition, there exists $c_{\omega} \in (c_T^{-1}, c_T)$ such that,

$$\sum_{\gamma \in \omega} \int_{\partial C_T} c_{\omega} \Psi_o^{D_{\Gamma}}(\gamma^{-1}o, \zeta) d\nu_{o, \gamma^{-1}} = (g - 1) + C_{\omega}.$$  

Where $C_{\omega} \in (0, c_T)$. In addition we have,$$
\frac{1}{(g - 1) + C_{\omega}} \int_{\partial C_T} c_{\omega} \Psi_o^{D_{\Gamma}}(\gamma^{-1}o, \zeta) d\nu_{o, \gamma^{-1}} = \int_{\partial C_T} d\nu_{o, \gamma}.$$

Proof. Let $S = \omega \cup \omega^{-1}$. Let us write every element $\gamma \in \Gamma$ as a reduced word $w_1 \cdots w_n$ with $\{w_j\} \subset S$. Then we have the decomposition of $\Gamma$ as $\Gamma = \{1\} \bigsqcup \bigsqcup_{\gamma \in S} I_{\gamma}$, where $I_{\gamma}$ is the set of nontrivial elements in $\Gamma$ with initial letter $\gamma$. By the fact that $\Gamma$ act freely on $C_{\Gamma}$ we have $C_{\Gamma} = \Gamma o = \{o\} \bigsqcup \bigsqcup_{\gamma \in S} V_{\gamma}$ where $V_{\gamma} = \{wo : w \in I_{\gamma}\}$. Let $V$ denote the collection consisting of all sets of the form $\bigsqcup_{\gamma \in S'} V_{\gamma}$ or $\{o\} \bigsqcup \bigsqcup_{\gamma \in S'} V_{\gamma}$ for $S' \subset S$.

By Poincare series delta-mass construction,

$$\lim_{s \to D} \frac{\sum_{v \in V_{\gamma}} \exp(-s \text{ dist}(o, v)) \delta_v}{\sum_{v \in V_{\gamma}} \exp(-s \text{ dist}(o, v))},$$

with $CT$ and $V_{\gamma}$, Proposition 2.5, we get a family of Borel measures $(\mu_{y, V_{\gamma}})_{y \in CT}$ for each $\gamma \in S$, and $\text{supp}(\mu_{y, V_{\gamma}}) = V_{\gamma} \cap \partial CT$. In general, we need to prove that $\mu_{o, CT}$ is a probability measure on $\partial CT$. In fact, $\mu_{o, CT}$ is the normalized Patterson-Sullivan measure centered at point $o$. Define $\rho_{\gamma} := \mu_{o, V_{\gamma}}$ for each $\gamma \in S$. By the above decomposition of $CT$, we have $\mu_{o, CT} = \mu_{o, o} + \sum_{\gamma \in S} \rho_{\gamma}$. But $\mu_{o, o} = 0$.

Since $CT = V_{\gamma^{-1}} \bigsqcup V_{\gamma}$ we have $\gamma^{-1} V_{\gamma} \in V$. Then by quasiconformal transformation property of Patterson-Sullivan measure, we get

$$\mu_{\gamma o, V_{\gamma}} = \gamma^{-1*} \mu_{\gamma o, CT - V_{\gamma^{-1}}} = \gamma^{-1*} (\mu_{o} - \rho_{\gamma^{-1}}).$$

This implies we have,

$$d\mu_{\gamma o, V_{\gamma}} = c_{\gamma} \Psi_o^{D_{\Gamma}}(\gamma^{-1}o, \xi) d\mu_{o, V_{\gamma}}, \text{ for some } c_{\gamma} \in (c_T^{-1}, c_T).$$

From this, we get

$$\int_{\partial C_T} c_{\gamma} \Psi_o^{D_{\Gamma}}(\gamma^{-1}o, \xi) d\rho_{\gamma^{-1}} = \int_{\partial C_T} d(\gamma^{-1*}(\mu_{o, CT} - \rho_{\gamma})) = 1 - \int_{\partial C_T} d\rho_{\gamma}.$$
Since $\mu_{o,CT} = \sum_{\gamma \in \omega} \rho_{\gamma} + \sum_{\gamma \in \omega} \rho_{\gamma^{-1}}$, we have one of $\sum_{\gamma \in \omega} \rho_{\gamma}, \sum_{\gamma \in \omega} \rho_{\gamma^{-1}}$ must have total weight over $\partial CT$ of $\geq \frac{1}{2}$. This is one of the property two that we exploit later in next corollary to show lower bounds of $\sigma_o^-$ based on indexing choice of $\mathbb{W}_\Gamma$.

Set $C_\omega = \sum_{\gamma \in \omega} \int_{\partial CT} d\rho_{o,\gamma}$. There exists $c_\omega \in (c_{-1}^{-1}, c_1)$ such that,

$$\nu_{o,\gamma} - 1 = \rho_{o,\gamma} - 1,$$

satisfies our conditions.

Proof. Proposition 3.4:

For $\omega \in \mathbb{W}_\Gamma$, define measures by $\eta_{o,\omega}^+ = \sum_{\gamma \in \omega} \nu_{o,\gamma}$, and $\eta_{o,\omega}^- = \sum_{\gamma \in \omega} \nu_{o,\gamma} - 1$. By Lemma 3.5, $\eta_{o,\omega}^+$ is a probability measures on $\partial CT$.

Index $\mathbb{W}_\Gamma = \cup_{i \geq 1} \omega_i$. Set $\sigma_{o,N}^+ = \frac{1}{N} \sum_{i=1}^N \eta_{o,\omega_i}^+ \otimes \delta_{\omega_i}$, $\sigma_{o,N}^- = \frac{1}{N} \sum_{i=1}^N \eta_{o,\omega_i}^- \otimes \delta_{\omega_i}$. We extend $\sigma_{o,N}, \sigma_{o,N}^-$ to $\partial CT \times \mathbb{W}_\Gamma$ trivially by setting it to be zero on $\mathbb{W}_\Gamma - \cup_{j \leq N} \omega_j$. Note that since $\eta_{o,\omega_i}^+$ are probability measures, we have for all $N$,

$$\int_{\partial CT \times \mathbb{W}_\Gamma} d\sigma_{o,N}^+ = 1.$$

Since $\eta_{o,\omega_j}^-(\partial CT) < 1$ for all $j$ we have,

$$\int_{\partial CT} d\sigma_{o,N}^- < 1.$$

We have the weak-limit of, $\sigma_n^+ \rightarrow \sigma_o^+$ and $\sigma_n \rightarrow \sigma_o$. By construction we have $\sigma_o^\pm$ satisfies our conditions.

Proof. Since $\eta_{o,n}^+$ for all $n$ are probability measures, we can easily see by our construction in the proof of Proposition 3.4 that $\sigma_{o,n}^+ \rightarrow \sigma_o$ is indexing independent of $\mathbb{W}_\Gamma$. 

Corollary 3.6. There exists a sequence of Borel measures $\sigma_{o,n}^\pm$ on $\partial CT \times \mathbb{W}_\Gamma$ such that, support of $\sigma_{o,n}^\pm$ is $\partial CT \times \cup_{j \leq n} \omega_j$ for a given index of $\mathbb{W}_\Gamma$ and, $\sigma_{o,n}^\pm \rightarrow \sigma_o^\pm$ weakly. In addition, $\sigma_o^+$ is a probability measure, so it’s total weight is independent on index of $\mathbb{W}_\Gamma$, and there exists a indexing of $\mathbb{W}_\Gamma$ such that, $\sigma_o^-(\partial CT \times \mathbb{W}_\Gamma) \geq \frac{1}{2}$.

Proof. Since $\eta_{o,n}^+$ for all $n$ are probability measures, we can easily see by our construction in the proof of Proposition 3.4 that $\sigma_{o,n}^+ \rightarrow \sigma_o$ is indexing independent of $\mathbb{W}_\Gamma$. 

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We can choose a indexing of $\mathbb{W}_\Gamma$ such that given $\omega_i$ we have either $\omega_{i+1}, \omega_{i-1}$ is the inverse of $\omega_i$. Since $\mu_\omega = \eta_{o,\omega}^+ + \eta_{o,\omega}^-$ we have by using this indexing, $\sigma_{o,n}(\partial\mathcal{C} \times \mathcal{W}_\Gamma) \to \frac{\mu_\omega(\partial\mathcal{C})}{2}$. Alternatively, if given a indexing such that $\sigma_{o,n}(\partial\mathcal{C} \times \mathcal{W}_\Gamma) < \frac{1}{2}$ then we can replace all $\omega_i$ with $\omega_i^{-1}$. This would imply $\frac{1}{N} \sum_{i=1}^N \eta_{o,\omega_i}^-(\partial\mathcal{C} \times \mathcal{W}_\Gamma) \geq \frac{1}{2}$ for large $N$. Hence we have required result.

\section{Hausdorff Dimension of Handlebody}

Take $\Gamma$ to Schottky group. Then the parameter in Lemma 3.5 is $c = 1$.

Let $\Lambda_\Gamma$ be the limit set of $\Gamma$, which is the minimal invariant uniformly perfect closed no-where dense subset of $\mathbb{C}$. The open invariant set $\Omega_\Gamma = \mathbb{C} - \Lambda_\Gamma$ is the region of discontinuity of $\Gamma$. Then $\mathcal{D}_\Gamma$ from previous section is the Hausdorff dimension of $\Lambda_\Gamma$, which we denote by $\mathcal{D}_\Gamma$. We also have $\partial\mathcal{C}$ is identified with $\Lambda_\Gamma$. Since $\Gamma$ is convex-cocompact, $\mathcal{C}_\Gamma$ is quasi-isometric to $\mathcal{H}_\Gamma$, convex hull of $\Gamma$. In addition, the density $\Psi_x(\gamma^{-1}x, \xi)$ is the Poisson kernel on $\mathbb{H}^3 \cup \mathbb{C}$. Explicitly we have $[17]$,

$$
\Psi_x(\gamma^{-1}x, \xi)^{\mathcal{D}_\Gamma} = \left(\frac{1}{\cosh(\text{dist}(x, \gamma x)) - \sinh(\text{dist}(x, \gamma x)) \cos \angle \gamma^{-1}xx\xi}\right)^{\mathcal{D}_\Gamma}.
$$

Here $\text{dist}(x, y)$ is the hyperbolic distance for $x, y \in \mathbb{H}^3$. Hence we can proceed explicit computation of measure over $\Lambda_\Gamma$.

We summary this in the following:

\begin{corollary}
Let $\Gamma$ be Schottky group. Lemma 3.5 holds for $c = 1$ and $\Psi_x^{\mathcal{D}_\Gamma}(\gamma o, \xi)$ given by the Poisson Kernel.
\end{corollary}

We denote $\Psi_x(\gamma^{-1}x, 0)$ for $\Psi_x(\gamma^{-1}x, \xi)$ when $\angle \gamma^{-1}xx\xi = 0$. This is simply $\exp(\text{dist}(x, \gamma x))$.

Given $\alpha \in \omega$ we define, $\omega(\alpha)$ is the generating set given by $\alpha$ and $\{\gamma \alpha\}_{\gamma \in \omega - \alpha}$. This is the generating set given by shifting the original $\omega$ by $\alpha$.

Next we will prove a lemma that will bound the mean displacement of any given $\omega$ at a point to that of the derivative of $\eta_{x, \omega(\alpha)}$ with respect to some transformed measure. This will provide connection between the mean displacement of $\omega$ with measures $\sigma_\pm$.

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Lemma 4.2 (Shifting lemma). Define $d\eta^\ast_{x,\omega}(\zeta) = \sum_{\gamma \in \omega} \Psi^G_x(\gamma^{-1}x, \zeta) d\nu_{x,\gamma^{-1}}$. Then there exists $\alpha \in \omega$ such that,

$$
\frac{d\eta^\ast_{x,\omega}}{d\eta^\ast_{x,\omega(\alpha)}}(\zeta) \geq \sum_{\gamma \in \omega} \Psi^G_x(\gamma^{-1}x, 0) \cdot g.
$$

Remark 4.3. Note that $\Psi^G_x(\gamma^{-1}x, 0)$ is notation for $\exp(\mathcal{D}_\Gamma \text{dist}((\gamma^{-1}x, x)))$.

Recall that Patterson-Sullivan measures provide exact quasiconformal distortions under group transformation. The idea of this lemma is that, we want to somehow gauge the distortion created when we change the measure $\nu_{x,\gamma}$ under transformations. More precisely, we want to estimate the distortion under Nielsen transformation. The point of shifting lemma is to estimate the lower bound of this distortion, which states that the distortion is at least the average of overall distortion.

Proof. Let $\delta > 0$ small. Define subset $E_\delta(x, \gamma) = \{ \zeta \in \Lambda \mid \angle x, x, \zeta < \delta \}$. Let $\alpha_\delta \in \omega$ such that

$$
\nu_{x,\alpha_\delta^{-1}}(E_\delta(x, \alpha_\delta)) \leq \nu_{x,\gamma^{-1}}(E_\delta(x, \gamma)), \text{ for all } \gamma \in \omega.
$$

Note that $\text{supp}(\nu_{x,\gamma^{-1}}) = \hat{V}_{\gamma^{-1}} \cap \Lambda$. For $\beta \in \omega(\alpha_\delta) - \alpha_\delta$ we have,

$$
\text{supp}(\nu_{x,\beta^{-1}}) = \text{supp}(\nu_{x,\alpha_\delta^{-1} \gamma^{-1}}) \subset \text{supp}(\nu_{x,\alpha_\delta^{-1}}),
$$

$$
\nu_{x,\beta^{-1}} \leq \nu_{x,\alpha_\delta^{-1}}.
$$

This implies that for $\gamma' \in I_{\alpha_\delta^{-1}}$ (word start with $\alpha_\delta$) we have,

$$
\nu_{x,\alpha_\delta^{-1}}(E_\delta(x, \gamma')) \geq \nu_{x,\beta^{-1}}(E_\delta(x, \gamma')), \text{ for all } \beta \in \omega(\alpha_\delta).
$$

Hence we have,
\[
\lim_{\delta \to 0} \frac{\sum_{\gamma \in \omega} \int_{E_\delta(x, \gamma)} \Psi^\Gamma_x (\gamma^{-1}x, \zeta) d\nu_{x, \gamma^{-1}}}{\sum_{\gamma \in \omega(\alpha_\delta)} \int_{E_\delta(x, \gamma')} d\nu_{x, \gamma^{-1}}}
\geq \lim_{\delta \to 0} \frac{\sum_{\gamma \in \omega} \inf_{\xi \in E_\delta(x, \gamma')} \Psi^\Gamma_x (\gamma^{-1}x, \zeta) \int_{E_\delta(x, \gamma')} d\nu_{x, \gamma^{-1}}}{\sum_{\gamma \in \omega(\alpha_\delta)} \int_{E_\delta(x, \gamma')} d\nu_{x, \gamma^{-1}}}
\geq \lim_{\delta \to 0} \frac{\sum_{\gamma \in \omega} \inf_{\xi \in E_\delta(x, \gamma')} \Psi^\Gamma_x (\gamma^{-1}x, \zeta) \nu_{x, \alpha_\delta^{-1}}(E_\delta(x, \gamma'))}{g \nu_{x, \alpha_\delta^{-1}}(E_\delta(x, \gamma'))}
\geq \sum_{\gamma \in \omega} \frac{\Psi^\Gamma_x (\gamma^{-1}x, 0)}{g}
\]

**Proof.** Theorem 3.2:

Let \( N > 0 \) be a large integer. By Lemma 3.4 and Corollary 3.6, there exists \( \epsilon_N > 0 \) such that,

\[
\int_{\Lambda^\Gamma \times \Omega^\Gamma} d\sigma_x^+ \geq \sum_{1 \leq j \leq N} \frac{1}{N} \int_{\Lambda^\Gamma} d\eta_x^+_{\omega_j}
\geq \sum_{1 \leq j \leq N} \sum_{\gamma \in \omega_j} \frac{1}{N} \int_{\Lambda^\Gamma} d\nu_{x, \gamma^+}
\geq \sum_{1 \leq j \leq N} \sum_{\gamma \in \omega_j} \frac{1}{N(g-1+C_{\omega_j})} \int_{\Lambda^\Gamma} \Psi^\Gamma_x (\gamma^{-1}x, \zeta) d\nu_{x, \gamma^{-1}}
\]

Since \( c_\Gamma = 1 \) and \( C_{\omega_j} \in (0,1) \) for all \( j \) we have,

\[
\geq \frac{\sum_{1 \leq j \leq N} \sum_{\gamma \in \omega_j} \frac{1}{N} \int_{\Lambda^\Gamma} \Psi^\Gamma_x (\gamma^{-1}x, \zeta) d\nu_{x, \gamma^{-1}}}{g}
\]

By shifting Lemma 4.2, for every \( \omega_j \) there exists \( \alpha_j \in \omega_j \) such that,

\[
\frac{\sum_{\gamma \in \omega_j} \Psi^\Gamma_x (\gamma^{-1}x, \zeta) d\nu_{x, \gamma^{-1}}}{\sum_{\gamma \in \omega_j(\alpha_j)} d\nu_{x, \gamma^{-1}}} \geq \sum_{\gamma \in \omega_j} \frac{\Psi^\Gamma_x (\gamma^{-1}x, 0)}{g}
\]
Hence we have,

\[ \int_{\Lambda \times W} d\sigma_x^+ \geq \frac{1 - \epsilon_N}{g} \inf_{\omega \in \Omega} \sum_{\gamma \in \omega} \Psi_x^\Theta (\gamma^{-1} x, 0) \sum_{1 \leq j \leq N} \sum_{\gamma \in \omega_j (\alpha_j)} \frac{1}{N} \int_{\Lambda} d\nu_{x, \gamma^{-1}}. \]

Note the above inequality holds for any chosen indexing of \( W \), and by Corollary \ref{corollary_3.6} we can choose some indexing such that \( \sigma_x^- (\Lambda \times (W)_\Gamma) \geq \frac{1}{2} \). However, our shifting is indexing dependent hence we can’t in general simply just bound by \( \frac{1}{2} \), unless we have some bounds on generators translation length.

\[ \int_{\Lambda \times W} d\sigma_x^+ \geq \frac{1 - \epsilon_N}{g} \inf_{\omega \in \Omega} \sum_{\gamma \in \omega} \Psi_x^\Theta (\gamma^{-1} x, 0) \sum_{1 \leq j \leq N} \sum_{\gamma \in \omega_j (\alpha_j)} \frac{1}{N} \int_{\Lambda} d\nu_{x, \gamma^{-1}}. \]

Since \( \Psi_x^\Theta (\gamma^{-1}, 0) = \exp(\Theta \text{dist}(x, \gamma^{-1} x)) \),

\[ \geq \frac{1 - \epsilon_N'}{g} \inf_{\omega \in \Omega} \sum_{\gamma \in \omega} e^{\Theta \text{dist}(x, \gamma^{-1} x)} \int_{\Lambda \times W} d\sigma_x^- \].

By \( \sigma_x^+ (\Lambda \times W) = 1 \) and \( \sigma_x^- (\Lambda \times W) \) we have,

\[ 1 \geq \frac{(1 - \epsilon') \sigma_x^- (\Lambda \times W)}{g} \inf_{\omega \in \Omega} \sum_{\gamma \in \omega} e^{\Theta \text{dist}(x, \gamma^{-1} x)} \frac{g}{g}. \]

This implies that,

\[ \inf_{\omega \in \Omega} \sum_{\gamma \in \omega} e^{\Theta \text{dist}(x, \gamma^{-1} x)} \leq \frac{g}{(1 - \epsilon') \sigma_x^- (\Lambda \times W)}. \]

From the inequality,

\[ \inf_{\omega \in \Omega} \sum_{\gamma \in \omega} \Theta \text{dist}(x, \gamma^{-1} x) \leq \inf_{\omega \in \Omega} \log \left( \sum_{\gamma \in \omega} \frac{e^{\Theta \text{dist}(x, \gamma^{-1} x)}}{g} \right), \]

and since \( \epsilon_N' \) is arbitrarily small, hence we have,

\[ \Theta \leq \frac{\log \left( \frac{g}{\sigma_x^- (\Lambda \times W)} \right)}{\| W \|_x}. \]
To give best possible estimate of \( \sigma_\Sigma(x, W) \) we need some control of the shifting of \( W \). This shifting information is provided by the relationship among the generators. The idea is to pick the best possible indexing of \( W \) such that the shifting will remain sufficiently bounded below along the indexing.

**Corollary 4.4.** Let \( \Gamma \) be a Schottky group such that there exists \( x \in \mathbb{H}^3 \) and indexing of \( W = \cup_j \{\omega_j\} \) such that all shifting \( \omega_j(\alpha_j) \) have \( \nu_{x,\gamma_i} \geq \frac{1}{2g} \) for \( \omega_j(\alpha_j) \) then,

\[
D_\Gamma \leq \frac{2 \log(g)}{\|W\|_x}.
\]

We can obtain an similar estimate with average bounds on \( \nu_{x,\gamma_i} \) as follows.

**Corollary 4.5.** Let \( \Gamma \) be a Schottky group such that there exists \( x \in \mathbb{H}^3 \) and indexing of \( W = \cup_j \{\omega_j\} \) with \( \nu_{x,\gamma_i}(\Lambda_\Gamma) \geq \frac{1}{g+1} \sum_{j \in \omega_j - \gamma} \nu_{x,\gamma}(\Lambda_\Gamma) \) for every \( \gamma \in \omega_j \). Then

\[
D_\Gamma \leq \frac{2 \log g}{\|W\|_x}.
\]

Let \( \Gamma \) be Schottky group. Suppose \( \omega_m = \langle \gamma_1, \ldots, \gamma_g \rangle \) is the basis of minimal translational length \( T_{\gamma_j} \). Then we can estimate \( \nu_{x,\omega_j} \) based on relations among elements of this minimal basis.

**Corollary 4.6.** Let \( \Gamma \) be a Schottky group. Suppose there exists \( \lambda \geq 1 \) such that \( T_{\gamma_i} \leq \lambda T_{\gamma_j} \) for \( \gamma_i, \gamma_j \in \omega_m \). Then there exists \( x \in \mathbb{H}^3 \) such that

\[
D_\Gamma \leq \frac{(\lambda - 1) \log(2) + (\lambda + 1) \log(g)}{\|W\|_x}.
\]

**Proof.** For convex cocompact \( \Gamma \), Patterson-Sullivan measure is the unique \( D_\Gamma \)-Hausdorff measure. This implies \( \nu_{x,\gamma_i} \), \( \gamma \in \omega \) is absolutely continuous to the Hausdorff measure of support \( \tilde{V}_{g-1} \cap \Lambda_\Gamma \). By \( T_{\gamma_i} \leq \lambda T_{\gamma_j} \), we can choose \( x \in \mathbb{H}^3 \) such that \( \nu_{x,\gamma_i}(\Lambda_\Gamma) \geq \frac{1}{2g} \) for some \( \gamma_i \in \omega_m \), and the total mass of \( \nu_{x,\gamma_j} \) is bounded by \( \nu_{x,\gamma_j}(\Lambda_\Gamma) \geq \nu_{x,\gamma_i}(\Lambda_\Gamma) \). Let \( \omega_m(\gamma_j) = \{\gamma_j, \gamma_i\} \neq j \) be the shifted basis. Since \( \sum_{\gamma \in \omega_m(\gamma_j)} \nu_{x,\gamma}(\Lambda_\Gamma) = \nu_{x,\gamma_j}(\Lambda_\Gamma) \) we have,

\[
\sum_{\gamma \in \omega_m(\gamma_j)} \nu_{x,\gamma}(\Lambda_\Gamma) \geq \nu_{x,\gamma}(\Lambda_\Gamma) + \sum_{\gamma \in \omega_m(\gamma_j)} \nu_{x,\gamma}(\Lambda_\Gamma) \\
\geq \nu_{x,\gamma_i}(\Lambda_\Gamma) + \nu_{x,\gamma_j}(\Lambda_\Gamma) \geq \frac{2}{(2g)^{\lambda}}.
\]
Using the notations in the proof of Theorem 3.2, since $\omega_m$ is the minimal length generator basis, we can choose an indexing of $W_\Gamma$ such that,

$$\sum_{1 \leq i \leq N} \frac{1}{N} \sum_{\gamma \in \omega_{j_i}(\alpha_{j_i})} \nu_{x,\gamma^{-1}}(\Lambda \Gamma) \geq \frac{2}{(2g)\lambda}.$$  

This implies that, $\sigma_x(\Lambda \Gamma \times W_\Gamma) \geq \frac{2}{(2g)\lambda}$ which give the result. \hfill \Box

Rather than uniform bound of $\lambda$, we can relax the condition of Corollary 4.6 between $T_i, T_j$. Arrange $T_{j_i} \leq T_{j_j}$, for $i \leq j$ as follows: Suppose $T_j \leq \lambda_{j_i} T_i$ for some collection of $\lambda_{j_i} \geq 1$, $i \leq j$. We take convention that $\lambda_{ij} = 1$ for $i \leq j$. If $\gamma_j$ is the shifting generator for $\omega_m(\gamma_j)$, then by replacing generator $\gamma_k$ for, $k = j - 1$ if $j > 1$ and $k = 2$ when $j = 1$, with its inverse in $\omega_m$ if necessary, we can assume that $\nu_{x,\gamma_k^{-1}}(\Lambda \Gamma) \geq \frac{1}{(2g)\lambda}$. This implies that, $\sum_{\gamma \in \omega_m(\gamma_j)} \nu_{x,\gamma_j^{-1}}(\Lambda \Gamma) \geq \frac{2}{(2g)^{\lambda_{jk}}}$. Set $\bar{\lambda} = \sup_{jk} \lambda_{jk}$ we have the following version of above corollary:

**Corollary 4.7.** Let $\Gamma$ be a Schottky group. Suppose there exists $\lambda_{ij} \geq 1$ such that $T_{j_i} \leq \lambda_{j_i} T_{j_i}, i \leq j$ for $\gamma_i, \gamma_j \in \omega_m$. Then there exists $x \in \mathbb{H}^3$ such that,

$$D_\Gamma \leq \left( \bar{\lambda} - 1 \right) \log(2) + (\bar{\lambda} + 1) \log(g) \left\| W_\Gamma \right\|_x.$$  

Finally, we mention couple of interesting consequences of Theorem 3.2 related to injectivity radius and classical Schottky groups.

Define $H_c = \sup \{ \lambda \}$ such that all Schottky group of $D_\Gamma < \lambda$ is classical. $H_c$ is the maximal parameter such that if $\Gamma$ have Hausdorff dimension $< H_c$ then $\Gamma$ is classical Schottky group. It follows from theorem of Hou [16], such that $H_c$ exists. We have next obvious corollary:

**Corollary 4.8.** There exists $\tau_c > 0$ such that any Schottky group $\Gamma$ of rank $g$ with $\left\| W_\Gamma \right\|_x > \tau_c \log \left( \frac{g}{\sigma_x(\Lambda \Gamma \times W_\Gamma)} \right)$ for all $x \in \mathbb{H}^3$ is classical Schottky group.

**Proof.** By a theorem of Hou [16], there exists maximal $H_c > 0$ such that any finitely generated free Kleinian $\Gamma$ with Hausdorff dimension $< H_c$ is classical Schottky group. Set $\tau = \frac{1}{H_c}$ completes the proof. \hfill \Box

**Corollary 4.9.** Let $H = \mathbb{H}^3/\Gamma$ be hyperbolic handlebody of rank $g > 1$ such that $\sigma_x^{-1}(\Lambda \Gamma \times W_\Gamma) \geq 1/2$. There exists universal $\tau_c > 0$ such that, if the injectivity radius $i_H$ of $H$ satisfies $i_H > \tau_c \log(2g)$, then $H$ is uniformized by classical Schottky group $\Gamma$. 

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For a hyperbolic 3-manifold $M$, if the $\pi_1(M)$ is not free then there is imbedded surface which would put an upper bounds on the injectivity radius. Hence for sufficiently large injectivity of a given hyperbolic 3–manifold, the fundamental group must be free. So we have the following corollary:

**Corollary 4.10.** Let $M$ be a hyperbolic 3-manifold with fundamental group of rank $g$. If $i_M > c\log \left( \frac{g}{\sigma_x(\Lambda_1 \times \Lambda_1)} \right)$, then $M$ is uniformized by a classical Schottky group.

## 5 Rational Norm $Q$ of Homological Markings on $R_g$ and extremal length

Let $H_1(R_g, \mathbb{Z})$ be the first homology group and denote by $B_1$ the set of canonical basis of $H_1(R_g, \mathbb{Z})$. This is given by class: $[\alpha], [\beta]$ satisfies $< [\alpha], [\alpha] > = 0$ and $< [\alpha], [\beta] > = 0$.

Let $\pi : B_1 \to BH_1$ be the projection to the collection of first set of $g$ cycles $\{\alpha_i\}_i$. Denote $B_\alpha$ the subgroup of $H_1(R_g, \mathbb{Z})$ generated by $[\alpha] \subset BH_1$. We define $B_{1/2}$ to be the collection of all such subgroups $B_\alpha$.

There exists a $\phi : BH_1 \to \mathfrak{h}_g$ maps into the Schottky space. The map $\phi$ is a morphism such that maps all $\phi(\alpha_j) = \gamma_j$, $1 \leq j \leq g$. The curves $\{\alpha_1, ..., \alpha_g\}$ are called cut system. Denote $\Gamma_{[\alpha]} =< \gamma_1, ..., \gamma_g >$ the image Schottky group of $\alpha$. Each $B_\alpha$, subgroup of $H_1(R_g, \mathbb{Z})$ generated by $\{\alpha_1, ..., \alpha_g\}$ uniquely determines $\Gamma_{[\alpha]}$. Different set of $\alpha'$ which generates same subgroup of $H_1(R_g, \mathbb{Z})$ gives same Schottky group under $\phi$, of different set of generators, which corresponds to $\alpha'$ cut system. We have injective map of $B_{1/2}$ into $\mathfrak{h}_g$. The fundamental domain of $\Gamma_{[\alpha]}$ is conformally equivalent to the planar (genus zero) domain, $R_g - \bigcup_{1\leq i\leq g} \alpha_i$. For details, see [3, 5].

We denote the region of discontinuity of the image Schottky group $\Gamma_{[\alpha]}$ by $\Omega_{[\alpha]}$. The domain $\Omega_{[\alpha]}$ is hyperbolic planar domain with hyperbolic metric $\rho_{[\alpha]}$, which is the $\Gamma_{[\alpha]}$-invariant Poincare metric of hyperbolic disk. If $\rho_R$ is the Poincare hyperbolic metric of $R_g$ then, we have holomorphic covering map, $\pi_s : (\Omega_{[\alpha]}, \rho_{[\alpha]}) \to (R_g, \rho_R)$.

$B_1$ is invariant under the symplectic group $Sp(2g, \mathbb{Z})$. Let $\text{stab}(\phi)$ denote the normal subgroup of stabilizer of $B_\alpha$. The subgroup $\text{stab}(\phi)$ is generated by elements which corresponds to Nielsen transformations of generators of $\Gamma_{[\alpha]}$. 

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Let $\theta \in Sp(2g, \mathbb{Z})$, we define the action $\theta[c]$ for $[c] \in BH_1$ as: $\pi \theta^{-1}[c]$. We also set $BH_1 = BH_1/stab(\phi)$. Denote elements of $BH_1$ by $[c]$.

Given $[c] = \{\alpha_i\}_i^g \in BH_1$, the collection of $g$ cycles $\{\sigma_j\}_j^g \in \pi^{-1}[c]$ are called dual cycles of $\alpha_i$ cycles. Denote this collection by $D(\alpha_i)$. We also denote by $D(\alpha_i)$ to be the collection of all dual cycles to $\alpha_i$ in $D(\alpha)$, i.e. simple closed curves in $D(\alpha)$ of intersection 1 with $\alpha_i$.

For any given curve $\sigma \subset (R_g, \rho_R)$, we denote the hyperbolic length of $\sigma$ by $\ell(\sigma)$. This is also the hyperbolic length in $\pi_s^{-1}(\sigma) \subset (\Omega_{\Gamma[c]}, \rho_{\Gamma[c]})$.

For $[c] \in BH_1$, Denote $[c]_i$ as $\alpha_i$, the $i$-th cut system cycles. $\ell([c]_i)$ is the hyperbolic length of the geodesic representative curve of $\alpha_i$. Since $R_g$ is compact, there exists a unique $\beta_i^* \in D(\alpha_i)$, geodesic representative curve such that, $\ell(\beta_i^*) = \inf_{\beta \in D(\alpha_i)} \ell(\beta)$. We also define the following notations:

$$
\|\|c\|\| = \ell^2([c]_i), \quad \|\|c\|\|_D = \ell^2(\beta_i^*).
$$

**Remark 5.1.** We make convention that for a $[c] \in BH_1$ represented by geodesics $\{\alpha_i\}_i^g$, we identify $[c]$ to a basis in $H_1(R_g, \mathbb{Z})$ by adjoining the $\beta_i^*$, the unique geodesic cycles as: $\alpha_i, \beta_i^*$. So we speak of $[c]$ as basis and element of $BH_1$ interchangeably through this identification.

**Definition 5.2.** Let $[c] \in BH_1$. We also define the Rational Norms $Q([c]_i)$ of $[c]$ as the collection of all $Q([c])$:

$$Q([c]_i) = \frac{\|\|c\|\|_D}{\|\|c\|\|}, \quad Q([c]) = \sum_{i=1}^g Q([c]_i).
$$

**Definition 5.3.** We define $Q_\lambda$ the $Q$-spectrum of $R_g$ as:

$$Q_\lambda(R_g) = \{Q([c]) \mid [c] \in BH_1\}.
$$

Note $Q_\lambda(R_g)$ is non-discrete countable set. Next proposition is obvious:

**Proposition 5.4.** $Q_\lambda(R_g)$ is conformal invariant and defines a set-valued function on $\mathcal{M}_g$.

Let $\{T_{[c],j}\}_j^g$ denote the collection of translation length of elements $\{\gamma_i\} \in \Gamma_{[c]}$. Set $\mathcal{T}_{[c]} = \frac{1}{g} \sum_{1 \leq i \leq g} T_{[c],i}$. We call $\mathcal{T}_{[c]}$ the Schottky length of $[c]$.

Given $R_g$ a Riemann surface or domain of $\mathbb{C}$. Denote $\text{conf}(R_g)$ space of all conformal metric on $R_g$. Locally, $\psi \in \text{conf}(R_g)$ is given by quadratic differential $\psi(z)dz^2$.
Let \( \Phi \) be a collection of curves in \( \mathbb{R}^g \). Recall the extremal length \( \mathcal{E}_{\mathbb{R}^g}(\Phi) \) is given by:
\[
\mathcal{E}_{\mathbb{R}^g}(\Phi) = \sup_{\psi \in \text{conf}(\mathbb{R}^g)} \inf_{\sigma \in \Phi} \left( \int_\sigma |\psi|^2 \right)^{1/2} \int_\mathbb{R}^g |\psi|^2
\]
Note that it is a simple fact that \( \mathcal{E}_{\mathbb{R}^g}(\Phi) \) is conformal invariant.

Next we will use extremal length to establish lower bounds of \( T_{[c]} \) by \( Q([c]) \).

**Lemma 5.5.**
\[
T_{[c]} \geq \frac{\pi}{2} Q([c]).
\]

**Proof.** Let \( C_i, C_i' \subset \mathbb{C} \) be the lift of \([c]\), which are Jordan curves that bounds disjoint closed disks \( D_i, D_i' \), such that: \( \gamma_i(C_i) = C_i', \gamma_i(D_i^c) \cap D_i^c \). Here \( \gamma_i \) is generators of \( \Gamma_{[c]} \) given by \( \phi([c]) \). As before, \( \rho_{[c]} \) is the hyperbolic metric on \( \Omega_{[c]} \).

Let \( \Phi_i \) be the collection of all paths connecting \( C_i \) to \( C_i' \) in \( \mathbb{C} \). Let \( R_i = \mathbb{C} - (D_i \cup D_i') \). Also set \( R = \mathbb{C} - \bigcup_i (D_i \cup D_i') \), and \( D_i^c = \bigcup_{j \neq i} (D_j \cup D_j') \).

Note that since \( R \subset \Omega_{[c]} \), \( \rho_{[c]} \) defines a hyperbolic metric on \( \Omega_{[c]} \).

Let \( U \) collection of all curves in \( R_i \) connecting \( D_i, D_i' \). Choose a conformal metric \( h_w \) on \( R_i \) such that, \( \inf_{\sigma \in U} \ell_{h_w}(u) > \ell(w) \). Let \( \epsilon > 0 \) such that, the \( D_{\epsilon, \epsilon} \)-neighborhood of \( D_i^c \): \( D_{\epsilon, \epsilon} \cap (D_i \cup D_i') = \emptyset \). Choose a \( \sigma_\epsilon(z) \) smooth function of \( \mathbb{C} \), which is approximate characteristic cut-off function such that:
\[
\sigma_\epsilon(z) = \begin{cases} 
1 & \text{if } z \in R \\
0 & \text{if } x \in D_{\epsilon, \epsilon}
\end{cases}
\]
and, \( d\rho_{\epsilon, w}^2 = (d\rho_{[c]}^2)^{\sigma_\epsilon} (dh_w^2)^{1-\sigma_\epsilon} \) is of negatively pinched curvature. If we denote metric density by the same notation and write hyperbolic metric in conformal factors \( e^{\rho_{[c]}|dz|} \) and \( e^{h_w|dz|} \) we have, \( \rho_{\epsilon, w} = \sigma_\epsilon \rho_{[c]} + (1 - \sigma_\epsilon) h_w \) is \( \epsilon \)-family of pinched negatively curved metric on \( \mathcal{R}_i \).

First we establish bounds of \( Q([c]) \) by extremal length \( \mathcal{E}_{\mathcal{R}_i}(\Phi) \).

It follows from the isoperimetric inequality for negatively pinched manifold \([2, 27]\) we have,
\[
\int_{\mathcal{R}_i} |\rho_{\epsilon, w}|^2 \leq \left( \int_{C_i \cup C_i'} |\rho_{\epsilon, w}| \right)^2 .
\]
\[ \mathcal{E}_{\mathcal{A}_i}(\Phi_i) = \sup_{\psi \in \text{conf}(\mathcal{A}_i)} \frac{\inf_{\sigma \in \Phi_i} \left( \int_\sigma |\psi|^2 \right)^2}{\int_{\mathcal{A}_i} |\psi|^2} \geq \frac{\inf_{\sigma \in \Phi_i} \left( \int_\sigma |\rho_{\epsilon,w}|^2 \right)^2}{\left( \int_{C_i \cup C'_i} |\rho_{\epsilon,w}| \right)^2}. \]

By isoperimetric inequality we have,

\[ \geq \frac{\inf_{\sigma \in \Phi_i} \left( \int_\sigma |\rho_{\epsilon,w}| \right)^2}{\left( \int_{C_i \cup C'_i} |\rho_{\epsilon,w}| + \delta_{\epsilon} \right)^2}. \]

Since assuming \( \epsilon \) is sufficiently small we have, \( \int_{C_i \cup C'_i} |\rho_{\epsilon,w}| < \int_{C_i \cup C'_i} |\rho_{c}| + \delta_{\epsilon} \), with \( \delta_{\epsilon} \to 0 \). We have,

\[ > \frac{\inf_{\sigma \in \Phi_i} \left( \int_\sigma |\rho_{\epsilon,w}| \right)^2}{\left( \int_{C_i \cup C'_i} |\rho_{c}| + \delta_{\epsilon} \right)^2}. \]

By \( \inf_{u \in U} \ell_{h_w}(u) > \ell(w) \), implies all curves \( u \) in \( \mathcal{A}_i \) connecting \( D_i, D'_i \) that intersects \( D'_i \) must have \( \ell_{\rho_{\epsilon,w}}(u) > \ell(w) \). Let \( V \) denote curves in \( \mathcal{A} \) connecting \( D_i, D'_i \). Since \( w \) is a curve in \( \mathcal{A} \), hence we have,

\[ \inf_{\sigma \in \Phi_i} \int_\sigma |\rho_{\epsilon,w}| \geq \inf_{v \in V_i} \int_v |\rho_{\epsilon,w}| \]

Since for \( v \subset \mathcal{A} \) we have \( \int_v |\rho_{\epsilon,w}| < \int_v |\rho_{c}| - \delta'_{\epsilon} \) for sufficiently small \( \epsilon \). This implies,

\[ \mathcal{E}_{\mathcal{A}_i}(\Phi_i) > \frac{\inf_{v \in V_i} \left( \int_v |\rho_{c}| - \delta'_{\epsilon} \right)^2}{\left( \int_{C_i \cup C'_i} |\rho_{c}| + \delta_{\epsilon} \right)^2} \geq \frac{\inf_{v \in V_i} \left( \int_v |\rho_{c}| - \delta'_{\epsilon} \right)^2}{\left( 2 \int_{C_i} |\rho_{c}| + \delta_{\epsilon} \right)^2} > \frac{\inf_{\beta \in D(\alpha_i)} \ell^2(\beta) - \delta''_{\epsilon}}{4 \ell^2([c]_i) + \delta''''_{\epsilon}}. \]

Since \( \delta'', \delta''' \) can be made arbitrarily small by choose \( \epsilon \) sufficiently small, the last inequality implies,

\[ \mathcal{E}_{\mathcal{A}_i}(\Phi_i) \geq \frac{1}{4} Q([c]_i). \]

Let \( g \) be the Mobius transformation so that \( g\gamma_i g^{-1} \) of fixed points \( 0, \infty \). We have \( g(\mathcal{A}_i) = A_i \) is annulus centered at origin of radii \( r_2 > r_1 \). Since,

\[ g^* \mathcal{E}_{\mathcal{A}_i}(\Phi_i) = \mathcal{E}_{A_i}(g(\Phi_i)) = \frac{1}{2\pi} \log \left( \frac{r_2}{r_1} \right). \]
Also note that the translation length of $g_{\gamma_i}g$ is $\log(\frac{r_2}{r_1})$, i.e: $T_{[c_i]} = \log(\frac{r_2}{r_1})$. Hence by conformal invariance of $\mathcal{E}_{\Phi_i}(\Phi_i)$ we have, $T_{[c_i]} \geq \frac{\pi}{2} Q([c_i])$.

\section{Pants decomposition and bound of $Q([c])$}

Next we show the existence of homological basis which gives some lower bounds for the Rational Norm $Q([c])$.

\begin{lemma}
There exists $[c] \in BH_1$ such that

$$Q([c]) > \frac{2\lambda_g}{\pi} g \log(g),$$

for some $\lambda_g > 2$, when $g = 2$ and $\lambda_g > 3$ if $g > 2$.
\end{lemma}

\begin{proof}
We will choose $[c]$ which is of relatively short length to it’s $\beta^*_i$ by compare arcs on pair of pants. The case $g = 1$ is trivial. Assume $g = 2$. For $c \in [c]$, denote by $c = \{\alpha_1, \alpha_2\}$ with $\alpha_i$ the non-separating curves and it’s dual curves (intersection $< \alpha_i, \beta_j >= \delta_{ij}$) by $\beta_i$.

Since $g = 2$, take $\{\alpha_1, \alpha_2, \beta_1, \beta_2\}$ then, $\frac{\ell(\alpha_i)}{\ell(\beta_i)} < 1$ or inverse is $< 1$ for $1 \leq i \leq 2$. So we can always choose $c$ so that $\frac{\ell(\alpha_i)}{\ell(\beta_i)} > 1$ for $1 \leq i \leq 2$. Hence we have $\sum Q([c_i]) > 2$ In addition, we have some $\lambda_g > 2$.

For $g > 2$, we need to decompose $R_g$ into $2g-2$ pair of pants and estimate $\ell(\beta)/\ell(\alpha)$ on pant components.

Given $[c] \in BH_1$ we complete $[c]$ with separating curves and let $P = \{P^k\}, 1 \leq k \leq 2g-2$ denote the associated pants decomposition. For each $P^k \in P$ we cut into two hexagonal pieces and we mark it by border geodesic arcs by $B_k = \{g_k, b_k, c_k, c'_k, e_k, f_k, g'_k\}$. The arcs are: $a_k$ right geodesic arc; $b_k$ top connector geodesic arc; $c_k$ top left geodesic arc, $d_k$ top right left geodesic arc; $e_k$ middle connector geodesic arc; $f_k$ bottom geodesic arc; $g_k$ left geodesic arc. Conversely, given any $2g-2$ pair of pants decomposition of $R_g$ there exists homological basis $\{\alpha_i, \beta_i\}, 1 \leq i \leq g$ which are not separating curves. To compare homological length we use idea of elementary arcs in [7]. A elementary arc $e$ is a arc on $P^k$ with end points lie on the boundary of $P^k$ such that it intersect border geodesic arcs $B_k$ at most two points in the interior of $P^k$. By definition all border geodesic arcs in $B_k$ are elementary arcs.
The homological curve that we will be looking for must have minimal, although not necessarily zero Dehn twist, since by triangle inequality one can always shorten a curve by reduce its twist. We want to able to show that there exists \( c \) such that \( \alpha_i \), the cut curves can be made successively relatively short compare to its, dual \( \beta_i \) curves.

The idea is that, if there are no such \( c \) exists on \( R_g \) then we will have an contradiction with the hyperbolicity of \( R_g \). This contradiction is reached through length computations of arcs on \( P^k \). To do so, we will compute relative \( \ell(e) \), \( \ell(e') \) for different pair of \( e, e' \) and show that it under conditions have sufficient bounds on them. These bounds will allow us to construct curves \( \alpha_i \) which must satisfies Lemma 6.1. Of course to bound \( Q \), one must keep in mind that our curve have minimal Dehn twist, otherwise this relative length can be made arbitrarily large or small but won’t provide any meaningful bounds of \( Q \).

Denote \( a_k \in \{g_k, g'_k\}, d_k \in \{c_k, c'_k\} \) for \( 1 \leq k \leq 2g - 2 \). We make the convention that given \( a_k \) and \( d_k \) we set \( a'_k, d'_k \) to be the other arc in the pair collection.

Assume that the lemma is false. There are several cases that we need to consider. We first show that there must exists a \( P \) of \( R_g \) such that there exists \( P^k \in P \) with \( \frac{\ell(d_k)}{\ell(b_k)} > \log^{4/5}(2g) \):

**Lemma 6.2.** Assume Lemma 6.1 is not true. For every \( P \) there must exists some \( P^{k} \in P \) such that \( \frac{\ell(d_k)}{\ell(b_k)} \geq \log^{4/5}(2g) \).

Proof. We prove by contradiction. So we assume all \( P \) pants decompositions of \( R_g \) have a \( P^k \) such that \( \frac{\ell(d_k)}{\ell(b_k)} < \log^{4/5}(2g) \).

Let \( \kappa > 1 \). Consider the following two cases:

- (A) : \( \ell(a_k)\ell(b_k) \geq \kappa \log(2g) \), for all \( k \).
- (B) : There exists some \( k \) such that \( \ell(a_k)\ell(b_k) < \kappa \log(2g) \).

Case (A) : Cut \( P_k \) into hexagon and it follows from hyperbolic hexagonal \([10]\) formulas we have, \( \sinh(\ell(e_k))\sinh(\ell(b_k)/2) = \cosh(\ell(a_k)) \). This gives:

\[
\ell(e_k) \leq \ell(a_k) + \sinh^{-1}(\frac{3}{\ell(b_k)})
\]

which implies,

\[
\frac{\ell(a_k)}{\ell(e_k)} \geq \frac{\ell(a_k)}{\ell(a_k) + \sinh^{-1}(\frac{1}{\ell(b_k)})}.
\]
Hence we have
\[ \ell(a_k) \geq \kappa \log(2g), \]
and \( g \geq 3 \) we have from the above inequality,
\[ \frac{\ell(a_k)}{\ell(e_k)} \geq \frac{\ell(a_k)}{\ell(a_k) + \sinh^{-1}\left(\frac{1}{\ell(e_k)}\right)} > f(\kappa). \]

Note \( f(\kappa) \) is increasing function of \( \kappa \) and \( f(\kappa) < 1 \).

Let \( e_K \) be such that \( \ell(e_K) = \min_{1 \leq k \leq 2g - 2} \{\ell(e_k)\} \). If \( \ell \leq g - 1 \) then we set \( \alpha_1 = d_1 \cup d_2 \). Here \( d_2 \) is the cut image curve of \( d_k \), so \( \ell(d_k) = \ell(d_K) \). On the other hand, if \( \ell > g - 1 \) then, we set \( \alpha_1 = d_2g \cup d_{2g-2} \). In either case we set, \( \alpha_i = 2e_{i-1} \cup 2e_i \) for \( i \geq 2 \).

It follows from our choice of \( \alpha_i \) we have, the curve homotopic to \( \beta_i \) must have arcs homotopic to \( a_i \) arcs. Now if \( \ell \leq g - 1 \) then set \( \beta_i = \bigcup_{2g-2 \geq i \geq K} (a_i \cup a_i) \), and if \( \ell > g - 1 \) then we set \( \beta_i = \bigcup_{1 \leq i \leq K} (a_i \cup a_i) \). And for \( i \geq 2 \) we also set \( \beta_i = \bigcup_{2g-2 \geq i} (a_i \cup a_i) \). Then it follows that we must have,
\[ \frac{\ell(\beta_i)}{\ell(\alpha_i)} \geq \frac{\ell(\beta_i)}{\ell(\alpha_i)} \geq \sum_{1 \leq i \leq g-1} f(\kappa) = f(\kappa)(g-1). \]

Hence we have,
\[ \sum_{1 \leq i \leq 2g-2} \frac{\ell(\beta_i)}{\ell(\alpha_i)} > (f(\kappa)(g-1))^2, \quad \text{for} \quad g \geq 3. \]

Since \( \lim_{\kappa \to \infty} f(\kappa) = 1 \) is increasing function so, there exists \( \kappa_o \) such that for \( \kappa \geq \kappa_o \) we have for \( g = 3 \), \( (f(\kappa)(2))^2 > \frac{3g}{\pi} \log(6) \). Now by the fact that,
\[ \frac{\pi (f(\kappa)(g-1))^2}{2g \log(2g)}, \quad \text{is increasing function of} \quad g. \]

Hence we have \( Q(|c|) > \frac{2}{\pi} \log(2g) \) for \( |c| \) consists of the chosen curves, which is a contradiction.

Next we consider case (B) : \( \frac{\kappa \log(2g)}{\ell(a_{k_m})} > \ell(b_{k_m}) > \frac{\ell(d_{k_m})}{\log^{4/5}(2g)} \) for some \( k_m \).

Now if \( \ell(a_{k_m}) \geq \kappa \log(2g) \) for all the \( k_m \) then,
\[ \frac{\ell(a_{k_m})}{\ell(d_{k_m})} > \frac{\ell(a_{k_m})}{\log^{4/5}(2g)} > 1. \]

Let \( k_m \) such that \( \ell(d_{k_m}) = \min_{k_m} \ell(d_{k_m}) \). Here we choose as following:
Let $|\{k_m\}|$ denote number of elements of the collection. If $|\{k_m\}| \geq g - 1$ then we choose, 

$$\alpha_1 = d_{k_m} \cup d_{\tilde{k_m}}, \quad \beta_1 = \cup_{j \leq 2g - 2} a_j \cup a_j',$$

and $\alpha_j = 2e_j \cup 2e_{j+1}$ for $j \neq k$.

From our choice of $\alpha_i$, the curve $\beta_1$ must have arcs homotopic to $a_j$ curves. Hence $\ell(\beta_1) \geq \ell(\beta_1)$ and we have,

$$\frac{\ell(\beta_1)}{\ell(\alpha_1)} > \sum_{1 \leq j \leq 2g-2} 1 = 2g - 2.$$ 

Hence for this basis we have, $\sum_{1 \leq i \leq g} Q([c], i) > (2g - 2)^2$. Since 

$$\frac{(2g - 2)^2}{g \log(2g)} > 1, \quad g \geq 3,$$

we have contradiction.

On the other hand if $|\{k_m\}| < g - 1$ then we choose in combination with case (A): Let $\{d_k\}$ be elements of $\{d_{k_m}\}$ such that among all curves $\beta_i$ defined in (A) consist of $a_k \cup a_k'$ which do not intersect $d_k$, gives $\frac{\ell(\beta_k)}{\ell(\alpha_k)}$ maximal value. Let $d_k^*$ be the minimal length curve of $d_k$. Then we have,

$$\sum_{1 \leq i \leq g} \frac{\ell^2(\beta_i)}{\ell^2(\alpha_i)} \geq \frac{\ell^2(\beta_k)}{\ell^2(\alpha_k)} + \frac{\ell^2(\beta_{k^*})}{\ell^2(\alpha_{k^*})} > \left( \frac{f(\kappa)(g - \frac{|\{k_m\}|}{2} - 1)}{2} \right)^2 + \left( \frac{|\{k_m\}|}{2} - 1 \right)^2$$

Hence by previous estimates we have $Q([c])$ satisfies the inequality, which gives contradiction.

Now suppose that $\ell(a_{k_m}) < \kappa \log(2g)$ for some of the $m$. We consider this as the Case (C). Here we breakdown the case (C) into two subcases:

- (C1) : $\frac{\ell(a_k)}{\ell(b_k)} \geq 1$, for all $k$
- (C2) : $\frac{\ell(a_k)}{\ell(b_k)} < 1$, some $k_n$. 

Consider (C_1) : In this case we have by our global condition \( \ell(b_k) > \frac{\ell(d_k)}{\log^{1/3}(2g)} \)
implies, \( \frac{\ell(a_k)}{\ell(d_k)} > \frac{1}{\log^{1/3}(2g)} \).

We set \( \bar{\beta}_1 = \cup_{2g-2j} a_j \cup a'_j \), \( \alpha_1 = d_k \cup d_k \), where \( \ell(d_k) = \min_k \{ \ell(d_k) \} \).
And set \( \alpha_i = 2e_{i-1} \cup 2e_i \) for \( i \geq 2 \). Choose \( \beta_{i \geq 2} \) so that \( \{ \alpha_i, \beta_i \}_{i \leq g} \) form a basis. Similarly as before we have,
\[
\frac{\ell(\beta_1)}{\ell(\alpha_1)} \geq \frac{\ell(\bar{\beta}_1)}{\ell(\alpha_1)} > \frac{2g - 2}{\log^{4/5}(2g)}.
\]

Since,
\[
\frac{\pi(2g - 2)^2}{2g \log^{14/5}(2g)} > 1, \quad \text{for } g \geq 2
\]
and it is increasing function of \( g \), it follows that we have \( Q([c]) \) satisfies the inequality which give us a contradiction.

Consider (C_2). In this case we have, \( \ell(a_{k_n}) < \ell(b_{k_n}) \).

By the hyperbolic identity \( \sinh(\ell(e_k)) \sinh(\ell(b_k)) = \cosh(\ell(a_k)) \) we have,
\[
\frac{\ell(a_{k_n})}{\ell(e_{k_n})} > \frac{\ell(a_{k_n})}{\sinh^{-1}\left( \frac{\cosh(\ell(a_{k_n}))}{\sinh(\ell(b_{k_n}))} \right)}.
\]

Here we further subdivide into cases as:

- (C'_2) : \( \ell(a_{k_n}) \geq \frac{\rho}{\sqrt{2g}} \).
- (C''_2) : \( \ell(a_{k_n}) < \frac{\rho}{\sqrt{2g}} \).

For (C'_2), we have from the above inequality after some simple computations we have, \( \frac{\ell(a_{k_n})}{\ell(e_{k_n})} > \left( \frac{\rho}{\sqrt{2g}} \right)^{14/30} \). Note that this inequality holds for \( \ell(a_{k_n}) \geq \frac{\rho}{\sqrt{2g}} \) and \( g \geq 3 \).

Set \( \alpha_1 = d_k \cup d_k, \bar{\beta}_1 = \cup_{1 \leq j \leq 2g-2} a_j \cup a'_j \), and \( \alpha_1 = e_{2i-1} \cup e_{2i}, \bar{\beta}_1 = \cup_{i \leq j \leq 2g-2} a_j \cup a'_j \) and \( \bar{\beta}_{i-1} = \cup_{j \leq i-1} a_j \cup a'_j \), for \( i \neq k \). Then by similar computations as above then show that this basis \([c]\) satisfies,
\[
\sum_{1 \leq i \leq g} \frac{\ell^2(\bar{\beta}_i)}{\ell^2(\alpha_i)} \geq \frac{4(g - |n| - 1)^2}{\log^2(2g)} + \frac{\rho^{24} |n|^2}{(2g)^{11/30}}.
\]
This inequality follows from that the maximal number of $a_{kn}$ arcs that don’t pass through $d_k$ is $|n|/2$. Here $|n|$ is the number of $k_n$.

This implies that, if $|n| > \frac{g}{4}$ then, we can always choose curves $\alpha_i$ as above so that, $\sum_{1 \leq i \leq g} \frac{\ell^2(\beta_i)}{\ell^2(\alpha_i)} > \frac{\rho^{49/30}}{2^{1/30}}$. By setting $\rho = \frac{6}{10}$ we have,

$$(\pi \rho^{49/30}/2^{71/30})/(2g \log(2g)) > 1, \quad \text{for} \quad g \geq 3.$$ 

It follows that $Q(\lceil \rho \rceil) > g$ gives our contradiction.

Case $(C'\delta)$. Here we have, $\ell(a_{kn}) < \frac{\rho}{\sqrt{2g}}$. Note by the global condition $\ell(d_k) < \ell(b_k) \log^{4/5}(2g)$ we also have, $\ell(b_{kn}) < \frac{\kappa \log(2g)}{\ell(a_{kn})}$.

we have by hexagonal hyperbolic formula,

$$\cosh(\ell(a_{kn})) = \sinh(\ell(a_{kn})) \sinh(b_{kn}) \cosh(\ell(d_{kn})) - \cosh(\ell(a_{kn})) \cosh(\ell(b_{kn}))$$

gives

$$\ell(d_{kn}) = \cosh^{-1} \left( \frac{\cosh(\ell(a_{kn}))(1 + \cosh(\ell(b_{kn})))}{\sinh(\ell(a_{kn})) \sinh(\ell(b_{kn}))} \right).$$

by our conditions this implies,

$$\ell(d_{kn}) \geq \cosh^{-1} \left( \frac{\cosh(\ell(a_{kn}))(1 + \cosh(\kappa \log(2g)/\ell(a_{kn})))}{\sinh(\ell(a_{kn})) \sinh(\kappa \log(2g)/\ell(a_{kn}))} \right)$$

$$\geq \cosh^{-1} \left( \frac{\cosh(\frac{6}{\sqrt{2g}})(1 + \cosh(\kappa \log(2g)/\ell(a_{kn})))}{\sinh(\frac{6}{\sqrt{2g}}) \sinh(\kappa \log(2g)/\ell(a_{kn}))} \right)$$

$$> \frac{17}{10} \log(2g)$$

By $\ell(d_k) < \ell(b_k) \log^{4/5}(2g)$ implies that, $\ell(b_{kn}) > \frac{17}{10} \log^{1/5}(2g)$. We have,

$$\frac{\ell(b_{kn})}{\ell(e_{kn})} > \frac{(2) \frac{17}{10} \log^{1/5}(2g)}{\sinh^{-1} \left( \frac{\cosh(\frac{6}{\sqrt{2g}})}{\sinh(\frac{6}{\sqrt{10} \log^{1/5}(2g)})} \right)}$$

With some computations one shows that, $\frac{\ell(b_{kn})}{\ell(e_{kn})} > 3 \log(2g)$. 

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Here we set $\alpha_1 = \cup_{1 \leq j \leq 2g-2} a_j$ and $\alpha_i = 2e_{2i-1} \cup 2e_{2i}$ for all the $i$ that within the collection of $P_{k_n}$. We select the remaining curves $\alpha_j$ which are not consist of $e_j$. By construction it follows that, $\beta_i$ the geodesic curve must have arcs homotopic to $a_i$ and, $\sum_{i \leq j \leq g} \frac{\ell(\beta_i)}{\ell(a_i)} > \frac{9|n|}{4} \log^2(2g)$. Now if $|n| \geq \frac{g}{2}$ then we have, $Q([c]) > \frac{2}{\pi} g \log(2g)$. On the other hand, if $|n| < \frac{g}{2}$ we can then choose curves given by previous construction. Hence we always curves that gives our contradiction, this completes our prove of the result. 

Now we establish the induction process of showing:

**Corollary 6.3.** Assume Lemma [6.7] is false. Let $1 < i \leq g$. Suppose there exists $P^{k_j}$ for every $j < i$ such that, $\ell(d_{k_i}) \geq \log^{4/5}(2g - 2j + 2)$. Then there must exists $P^{k_i} \in P - \cup_{j<i} P^{k_j}$ such that $\ell(d_{k_i}) \geq \log^{4/5}(2g - 2i + 2)$.

**Proof.** The pants $P^{k_i} \in P - \cup_{j<i} P^{k_j}$ consists of decomposition of surface of genus $g - i + 1$. It follows from Lemma [6.2] there exists $d_{k_i}$ such that $\ell(d_{k_i}) \geq \log^{4/5}(2g - i + 1))$. \hfill \Box

Now we can finish the proof of Lemma [6.1] If the Lemma is false then, we choose the collection of geodesic curves on surface $R_g$ represented by $\alpha_i = b_i \cup b'_i$ for $1 \leq i \leq g$. Since the shortest geodesic $\beta_i$ intersecting $\alpha_i$ is given by either $c_i \cup c'_i$ or $c'_i \cup c''_i$, i.e. $\beta_i = d_i \cup d'_i$. Hence it follows from Corollary [6.3] we have that,

$$\frac{\ell(\beta_i)}{\ell(\alpha_i)} \geq \log^{4/5}(2g - 2i + 2), \quad \text{for} \quad 1 \leq i \leq g.$$  

After simple computation one shows that, $\sum_{i=1}^g \log^{8/5}(2g - 2i + 2) > \frac{2}{\pi} g \log(2g)$. Hence $Q([c]) > \frac{4}{\pi} g \log(g)$, a contradiction. \hfill \Box

**Proposition 6.4.** There exists a $[c] \in BH_1$ such that,

$$Q(\theta, [c]) > \frac{2\lambda g}{\pi} g \log(g), \quad \text{for all} \quad \theta \in \text{stab}(\phi).$$

**Remark 6.5.** Note that under the action of $\theta$ we could have $i$ permuted, hence only the summation of the above inequality is preserved. We call any such $[c]$ satisfies the inequality, positive. Call $[c]$ invariately positive if having property of Lemma [6.4].

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Proof. We need to examine \( \theta \) action on elementary curves on pants decomposition. Note that it is enough to show it is true for generators of \( \text{stab}(\phi) \). We can written elements of \( SP(2g,\mathbb{Z}) \) as composition of several types of elementary symplectic matrices. For elements of \( \text{stab}(\phi) \), which is subgroup generated by elementary matrices which do not intertwines the \( \alpha_i \) and \( \beta_j \) basis. These elementary matrix correspond to a Nielsen transformation on the generators of \( \phi([c]) \).

The idea is that, take \([c]\) positive which exists by Lemma 6.1, by similar computations as in the proof of Lemma 6.1 using elementary arcs, if \( \theta,[c]\) is not positive then, we can appropriately modify the original curve of \([c]\) to a new \([c']\) so that this, \([c']\) will only increase \( Q \) under \( \theta \). This is achieved by compare different elementary arc length as we have done previously.

Let \( E_{ij} \) denote a elementary matrix such that \( E_{ij} \) map \([c]\) into basis with \( \alpha_i \) replaced by \( \alpha_i + \alpha_j \) and \( \beta_j \) replaced by \( \beta_j - \beta_i \), and rest unchanged. Example \( E_{12} \) for \( g = 3 \) of \( SP(6,\mathbb{Z}) \):

\[
E_{12} = \begin{pmatrix}
1 & 1 & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 0 \\
0 & 0 & 0 & -1 & 1 & 0 \\
0 & 0 & 0 & 0 & 0 & 1
\end{pmatrix}.
\]

We need to consider cases on pair pants. Assume the statement is false and there exists \( l,m \) such that \( \sum_j Q[E_{lm},[c]] < \frac{2}{g} \log(2g) \) for all positive \([c]\). It follows from the proof of Lemma 6.1 it is suffice to consider \( \alpha_l, \alpha_m \) are curves consist of elementary arcs. Let \( h_{lm} \) denote the geodesic curve representation of homological class \([\alpha_l + \alpha_m]\). Let \([f_{lm}]\) be a class of curves which is canonical dual to \([h_{lm}]\) and non-intersecting to rest of basis. We denote the \( f_{lm} \) to be the geodesic representative of \([f_{lm}]\).

Take a \([c]\) provided by Lemma 6.1. We will show that either this \([c]\) is invariantly positive or it implies another \([c]\) is invariantly positive. To do so we will examine all possible cases of \([c]\). We continue to use our notations and definitions given in the proof of Lemma 6.1. We need to examine all possible curves given by the \( E_{lm}.[c]\).

As in the Lemma 6.1 first assume that we have \( \frac{\ell(d_v)}{\ell(b_v)} < \log^{4/5}(2g) \) for all \( P^k \). Note that we have some positive \([c]\). Since \( h \) is geodesic closed curve homotopically connects \( b_l, b_m \), the length is bounded above by sum of these
curves and twice the connecting geodesic curve. As in the proof of Lemma 6.1, if \( \ell(a_k)\ell(b_k) \geq \log(2g) \) for all \( k \), then \( \ell(e_i) + \ell(e_m) > \ell(d_i) + \ell(d_m) \). So if we set \( \delta = \ell(h_{lm}) - (\ell(b_l) + \ell(b_m)) \), then \( \ell(f_{lm}) \geq \ell(e_i) + \ell(e_m) + \delta \). Hence we have, \( \frac{\ell(f_{lm})}{\ell(h_{lm})} \geq \frac{\log(2g)}{\log(2g)} \), and we have \( E_{lm}[\hat{c}] \) positive. On the other hand if, \( \ell(a_k)\ell(b_k) < \log(2g) \) for some \( P_k \) contains \( b_l, b_m \) then, we have a curve \( \alpha \) which consists of \( a_k \) as elementary arcs and \( \ell(d_j) + \ell(\alpha) > \log(2g) \), \( j \in \{l, m\} \). We replace one of the original \( \alpha_l \) with \( \alpha \) to form basis \( \hat{c} \). It is obvious that \( \hat{c} \) is positive.

For simplicity we use same notation for the geodesic representation curves for \( \hat{c} \). Then \( h \) have \( \alpha_l \) a Dehn twist around \( b_m \) which trace off from \( a_k \) arcs. The geodesic \( f_{lm} \) consists of arcs homotopic to arcs of \( \beta_m, \beta_l, b_l \). Hence \( \ell(f_{lm}) \geq \frac{1}{2}(\ell(\beta_m) + \ell(\beta_l)) + \ell(b_l) \). This implies we have lower bound of

\[
\frac{\ell(f_{lm})}{\ell(b_l)} \geq \frac{\ell(\beta_m)}{\ell(b_l)} + 1.
\]

Hence we have, \( E_{lm}[\hat{c}] \) positive. By repeat this replacement for rest of curves of \( \hat{c} \) if necessary we can claim that \( E_{lm}[\hat{c}]; 1 \leq l, m \leq g. \) satisfies same inequality. Note that \( E_{lm}.E_{lm'}[\hat{c}] \) will increase Dehn twist which will increase the lower bound by twist number. i.e. denote \( f_{l_1m_1...l_jm_j} \) be the geodesic representative of \( [f] \) corresponds to \( E_{l_1m_1}...E_{l_jm_j}[\hat{c}] \) we have,

\[
\frac{\ell(f_{l_1m_1...l_jm_j})}{\ell(b_l)} \geq \frac{\ell(\beta_m)}{\ell(b_l)} + j.
\]

Hence we have, \( \hat{c} \) is invariantly positive.

Now it follows from Corollary 6.3 and Lemma 6.2 that, every pants decompositions have cut curves \( \alpha_i \) such that, \( \beta_i \) have the property of \( \frac{\ell(\beta_i)}{\ell(\alpha_i)} \geq \log^{4/5}(2g) \), \( 1 \leq i \leq g \). This implies that there exists a invariantly positive \( c \).

From Lemma 6.1 and Lemma 6.4 we can prove the following inequality of the Schottky length,

**Proposition 6.6.** There exists \( [c] \in BH_1 \) such that,

\[
R[c] > \lambda y \log(g).
\]
Proof. Let \([c]\) be given by Lemma 6.1. By Lemma 5.5 we have,
\[ T_{[c],i} \geq \frac{\pi}{2} Q([c],i). \]
Since \(\sum_i Q([c],i) > 2\lambda g \log(g)\), this implies that \(\sum_{i=1}^g T_{[c],i} > \lambda g \log(g)\). Since \(\mathcal{R}_c = \frac{1}{g} \sum_{i=1}^g T_{[c],i}\), hence the above strict inequality implies the result. 

7 Proof of Theorem 1.1

Let \([c] \in \hat{BH}_1\). For \([c]\) \(\in [c]\), \(\omega_{[c]}\) is a generators set of \(\Gamma_{[c]}\). Since \(\Gamma_{[c]} = \phi([c])\), \(\forall [c] \in [c]\) and \(\bigcup_{[c] \in [c]} \omega_{[c]} = \mathcal{W}_\Gamma\) we have our next corollary,

**Proposition 7.1.** Let \([c]\) be given by Proposition 6.6. Then \(\|\mathcal{W}_\Gamma\|_x > \lambda g \log(g)\).

**Proof.** It follows from Proposition 6.6 and \(w_x(\omega_{[c]}) = \mathcal{R}_c\), we have:
\[ w_x(\omega_{[c]}) > \lambda g \log(g). \]
Since \(\bigcup_{[c] \in [c]} \omega_{[c]} = \mathcal{W}_\Gamma\), by the invariant positivity result of Lemma 6.4 we have:
\[ w_x(\omega_{[c]}) > \lambda g \log(g), \quad \forall [c] \in [c]. \]
Hence we have:
\[ \|\mathcal{W}_\Gamma\|_x = \inf_{\omega_{[c]} \in \mathcal{W}_\Gamma} w_x(\omega_{[c]}) > \lambda g \log(g). \]

**Proof.** Theorem 1.1

For a \(R_g\), it follows from Proposition 7.1, there exists homological marking \([c] \in BH_1\) of \(R_g\) such that, \(\Gamma_{[c]} = \phi([c])\), the covering Schottky group satisfies, \(\|\mathcal{W}_\Gamma\|_x > \lambda g \log(g)\) for all \(\mathbb{H}^3\). Hence by Corollary 4.5 and Corollary 4.6 and note that for such \([c]\) we have \(\lambda < 2\), which implies \(\mathcal{D}_{\Gamma_{[c]}} < 1\).

Finally, we give two obvious applications of our theorem. The first application Corollary 7.2 address a folklore question that was originally due to Bers.

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Corollary 7.2. If $H_c \geq 1$ then, for all $[R_g] \in \mathcal{M}_g$, $\pi_S^{-1}([R_g])$, have classical fiber.

Proof. Assume that $H_c \geq 1$, so $\tau_c \leq 1$. Since $\tau_c \leq 1$, Proposition 3.2 and Theorem 2.1 implies that $\Gamma$ is classical Schottky group. Hence for $[R_g] \in \mathcal{M}_g$, $\pi_S^{-1}([R_g]) \cap \mathfrak{J}_{g,o} \neq \emptyset$. Therefore, $\pi_S|_{\mathfrak{J}_{g,o}}$ is surjective.

The second application is presentation of period matrix of $R_g$ in Schottky coordinates. It’s well known theorem of Torelli that there exists an injective map from $\mathcal{M}_g$ into Siegel’s space of symmetric $g \times g$ matrix over $\mathbb{C}$. This is given by the $P_{mn}$ period matrix of $R_g$. Many beautiful theorems has been proved on $P_{mn}$, such as Buser-Sarnak’s theorem [6] states that the locus of Jacobians lie in very small neighborhood of the boundary of space of principally polarized abelian varieties for large genus $R_g$.

It’s well known fact [4], that $P_{mn}$ can be represented in local coordinates of $\mathfrak{J}_g$. Recall that local coordinates of $\mathfrak{J}_g$, which is $3g-3$-dimensional complex manifold, are given by variables $\lambda_i, z_{-,i}, z_{+,i}, 1 \leq i \leq g-1$ multiplier and two fixed points respectively. Recall, given $z_1, z_2, z_3, z_4 \in \mathbb{C}$, the cross ratio is:

$$\left[ z_1, z_2, z_3, z_4 \right] = \frac{(z_1 - z_3)(z_2 - z_4)}{(z_1 - z_4)(z_2 - z_3)}.$$

Let $\Gamma \in \mathfrak{J}_g$ be generated by $< \gamma_1, ..., \gamma_g >$, we denote by $\Gamma_l$ the subgroup of $\Gamma$ generated by $\gamma_l$. Then the following formal presentation of $P_{mn}$ is well known and go back to Schottky himself [4, 21]:

$$P_{mn} = \sum_{\gamma \in \Gamma \cap \Gamma_l} \log[z_{-,n}, \gamma z_{-,m}, z_{+,n}, \gamma z_{+,m}] + \delta_{mn} \log \lambda_n.$$

However, even though $P_{mn}$ has been formally known for a very long time but, in general as $P_{mn}$ is infinite sum, so it is not always convergent for an arbitrary $\Gamma \in \mathfrak{J}_g$, and it is not known in general. In this respect, we have our second simple application of Theorem 1.1:

Corollary 7.3. Let $[R_g] \in \mathcal{M}_g$. There exists $\Gamma \in \mathfrak{J}_g$ such that the period matrix of $P_{mn}$ of $[R_g]$ is given by the above presentation.

Proof. The only thing needs to be verified is that $P_{mn}$ is convergent for some $\Gamma \in \pi_S^{-1}([R_g])$. By Theorem 2.1, we have a $\hat{\Gamma} \in \pi_S^{-1}([R_g])$ such that $\mathcal{D}_{\hat{\Gamma}} < 1$.

Note that $P_{mn}$ is obtained as integral around canonical basis cycles of $H_1(R_g, \mathbb{Z})$ of $\hat{\Gamma}$-invariant holomorphic cocycles on $\Omega_{\hat{\Gamma}}$:

$$\omega_n(z) = \sum_{\gamma \in \Gamma \cap \hat{\Gamma}} d \log \left( \frac{z - \gamma z_{+,n}}{z - \gamma z_{-,n}} \right).$$
It follows from the residue formula we have $P_{mn}$.

By change of variable we have $\omega_n$ is convergent if the Poincare series: \[
\sum_{\gamma \in \hat{\Gamma} \backslash \Gamma} |\gamma'(z)|
\]
is convergent. Since $D_\Gamma < 1$, we have \[
\sum_{\gamma \in \hat{\Gamma} \backslash \Gamma} |\gamma'(z)|
\]
is convergent. Hence $P_{mn}$ exists for $\hat{\Gamma}$. \qed

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