Modeling Spacing Distribution of Queuing Vehicles in Front of a Signalized Junction Using Random-Matrix Theory

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Abstract

Modeling of headway/spacing between two consecutive vehicles has many applications in traffic flow theory and transport practice. Most known approaches only study the vehicles running on freeways. In this paper, we propose a model to explain the spacing distribution of queuing vehicles in front of a signalized junction based on random-matrix theory. We show that the recently measured spacing distribution data well fit the spacing distribution of a Gaussian symplectic ensemble (GSE). These results are also compared with the spacing distribution observed for car parking problem. Why vehicle-stationary-queuing and vehicle-parking have different spacing distributions (GSE vs GUE) seems to lie in the difference of driving patterns.

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I. INTRODUCTION

The spacing is usually defined as the distance between two successive vehicles measured from the same common feature of the vehicles (e.g. rear axle, front bumper). Because the distribution of spacing reflects the unmeasurable interaction forces or potentials between vehicles that governs their motions, increasing investigation are put into this field to reveal the complex dynamics of vehicle traffic flow and explain some important phenomena, i.e. phrase transition [1], [2], [3], [4].

One interesting topic is to discuss spacing distributions observed during the formation of and transitions between different vehicle queues: static queues (vehicles parked on a line in the parking lot, or vehicle queues fully-stopped in front of signalized intersections), moving queues which may contain diversified inter-arrival and inter-departure queuing interactions [4], [6], [5]. In this short paper, we will focus on the not so popular static queues.

The vehicle parking problem was first introduced by Renyi in [7] as: how many randomly parking motorists can be accommodated on a line street of a given length on average. The most famous solution to this question is based on Random Sequential Adsorption (RSA), which looks it as an irreversible process in which particles are adsorbed sequentially and without overlap onto randomly chosen positions on a surface. This 1D RSA problem can be solved analytically when all the vehicles with the same equal length, see [8], [9], [10]. However, it is hard to directly apply this method to other vehicle queues by definition.

Differently in [11], the random-matrix theory is used to study the car-parking problem, where the nature of interaction between the particles in a Dyson’s Coulomb gas model is assumed to be consistent with the tendency of the drivers to park their cars near to each other and in the same time keep a distance sufficient for departure. It was shown that the recently measured gap-size distribution of parked cars in a number of roads in central London can be well represented by the spacing distribution of a Gaussian unitary ensemble.

For a similar purpose, in this paper, the Dyson’s Coulomb gas model is adopted to explain the formation of vehicle queues fully-stopped in front of signalized intersections. We found a good agreement between the empirical data and the spacing distribution for Gaussian symplectic ensemble of random matrices.
II. COULOMB GAS MODEL AND QUEUING VEHICLES

Considers a gas of $N$ charges whose positions are denoted by $x_1, x_2, ..., x_N$ and these charges are free to move on the line $0 < x < +\infty$, see Fig. 1. Suppose the potential energy of this Coulomb gas is given as

$$ V = \frac{1}{2} \sum_i x_i^2 - \sum_{i<j} \ln |x_i - x_j| $$

where the first term in (1) represents a harmonic potential that attracts each charge independently towards the coordinate origin; and the second term represents an electrostatic repulsion between each pair of these charges.

![Fig. 1: Queuing vehicles in front of a signalized junction analogue to the Dyson Gas.](image)

Let $P(s)$ denote the nearest-neighbor spacing distribution of these charges. The accurate solution of $P(s)$ is not easy to find. However, for such systems, the probability density function for the position of the charges can be approximately calculated by using the so-called Wigner surmise [13], [14], [15].

Suppose the gas is in thermodynamical equilibrium at temperature $T = 1/(K\beta)$, where $K$ is the Boltzmann constant. The probability density function can be given as below by the Boltzmann factor obtained from the Gibbs-Boltzmann canonical distribution by integration over the momenta of the particles.

$$ P(x_1, x_2, ..., x_N) = Ce^{-\beta V} $$

(2)
Combining (1) and (2), the Wigner surmise solutions for $P(s)$ can be gotten by taking the additional assumption of $\beta$. The role of this inverse temperature $\beta$ denotes the level-repulsion power of the matrices eigenvalues. Particularly, they are suggested by Wigner as below respectively. For $\beta = 0$, we get the well-known Poisson Ensembles (PE)

$$P_{PE}(s) = e^{-s}$$ (3)

for $\beta = 1$, we get Gaussian Orthogonal Ensembles (GOE)

$$P_{GOE}(s) = \frac{1}{2\pi s} e^{-\frac{s}{4} s^2}$$ (4)

for $\beta = 2$, we get Gaussian Unitary Ensemble (GUE)

$$P_{GUE}(s) = \frac{32}{\pi^3} s^2 e^{-\frac{4}{3} s^2}$$ (5)

and for $\beta = 4$, we get Gaussian Orthogonal Ensembles (GSE)

$$P_{GSE}(s) = \frac{2^{18}}{3^6 \pi^3} s^4 e^{-\frac{56}{9} s^2}$$ (6)

Similar as discussed in [11], we can abstract the movements of vehicles of different size into point particles, because we are only interested in the spacing distribution here. A natural guess for the queuing dynamics of vehicles in front of a signalized junction is that the system can also be approximately formulated into this Coulomb model.

The single-particle term in (1) can be viewed to reflect the tendency of driving closer and the repulsive two-body term in (1) indicate the tendency of maintaining the safe distance. Analogously, the basic instinct of a driver is to maintain a small and safe gap between him/her and his/her leading vehicle, especially when he/she is queuing. The superposition of these two potentials, which appears an overall repulsion for small spacings and attraction for the large ones, expresses the fact that it is unlikely to see too small or too large spacings between queuing vehicles. However, no one can always keep an ideal headway due to disturbances (unexpected acceleration/deceleration of the leading vehicle, occasional absence of mind, etc.). Thus, these vehicles (particles) are perturbed by environment simultaneously.

Noticing the above analogue and inspiriting by the report that the spacing distribution of vehicle parking is in agreement with GUE type distribution, we conjecture that the empirical
The spacing distribution of a queuing vehicle systems might fit one kind of Wigner surmises \((3)-(6)\), too.

### III. COMPARISON WITH EMPIRICAL RESULTS

To test this conjecture, we collected 700 sample spacings of queuing vehicles in front of several different signalized junctions in Beijing, China. Some details about data collection can be found in [16].

The average spacing size observed is 1.43m here. Fig. 2 shows the probability distribution function \(P(s)\) in a form of normalized histogram, where the values of the \(x\)-axis is defined as the ratio of the spacing to the mean value. This modification is introduced to make the data comparable with the nearest-neighbor spacing distribution for system (1).

![Normalized spacing distribution](image)

**FIG. 2:** The *normalized* spacing distribution of queuing vehicles in front of several signalized junctions in Beijing compared with the theoretical spacing distribution for Poisson, GOE, GUE, GSE.

In the Coulomb gas model, the inverse temperature \(\beta\) of the gas characterizes the degree
of repulsion. Fig. 2 shows the theoretical spacing distribution curves for $P_{PE}(s)$, $P_{GOE}(s)$, $P_{GUE}(s)$ and $P_{GSE}(s)$ as well as the empirical spacing distribution histogram. We can see that different from the vehicle-parking scenarios, the vehicle-queuing scenarios well fit the GSE type model instead of GUE type model, although in these two scenarios, drivers all aim to driving close enough but not too close. This suggests that the vehicles queuing process at a signalized junction can be added to the long list of the systems with RMT-like fluctuations.

An interesting question is why the spacing distributions of vehicle-parking and vehicle-queuing are different. A reasonable explanation is this difference comes from the dissimilar driving patterns. In vehicle-parking scenarios, drivers would like to try several times and move back-and-forth to adjust the gaps so as to park to an “ideal” position; while in vehicle-queuing scenarios, driver won’t be able to drive back. This phenomenon can be interpreted as: in the vehicle-parking scenarios, the repelling force from the neighboring vehicles is relatively “loose”; however, in the vehicle-queuing scenarios, the repel force from the neighboring vehicles is a kind of “rigid”. As pointed out in [7], such a difference will result in the different values of inverse temperature $\beta$. At low temperatures ($\beta$ is bigger), the charges tend to be regularly spaced in a crystalline lattice arrangement and the randomness of the positions of the charges are small. At higher temperatures ($\beta$ is smaller), the fluctuations of the charges become tenser. Thus, for vehicle-parking scenarios, we get $\beta = 2$ and for vehicle-queuing, we get $\beta = 4$.

We also guess that in different cities, the spacing of vehicle queues may still hold GSE type distribution but may have different mean value. Further experiments will be carried out to test this guess soon. Any vehicles queuing data collected in cities other than Beijing are welcome and highly appreciated.

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