A resolution of the puzzle of low $V_{us}$ values from inclusive flavor-breaking sum rule analyses of hadronic $\tau$ decay data

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Abstract: Continuum and lattice methods are used to investigate systematic issues in the sum rule determination of $V_{us}$ using inclusive hadronic $\tau$ decay data. Results for $V_{us}$ employing assumptions for $D > 4$ OPE contributions used in previous conventional implementations of this approach are shown to display unphysical dependence on the sum rule weight, $w$, and choice of upper limit, $s_0$, of the relevant experimental spectral integrals. Continuum and lattice results suggest a new implementation of the sum rule approach with not just $|V_{us}|$, but also $D > 4$ effective condensates, fit to data. Lattice results are also shown to provide a quantitative assessment of truncation uncertainties for the slowly converging $D = 2$ OPE series. The new sum rule implementation yields $|V_{us}|$ results free of unphysical $s_0$- and $w$-dependences and $\sim 0.0020$ higher than that obtained using the conventional implementation. With preliminary new experimental results for the $K\tau$ branching fraction, the resulting $|V_{us}|$ is in excellent agreement with that based on $K\tau$, and compatible within errors with expectations from three-family unitarity.

Key words: $V_{us}$, lattice, sum rules

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1 Introduction

The conventional $\tau$ decay determination of $|V_{us}|$ employs finite-energy sum rules (FESRs) and flavor-breaking (FB) combinations of inclusive hadronic $\tau$ decay data.[1] With $\Pi^{(i)}_{V/A,ij}(s)$ the $i = 0, 1$ components of flavor $ij = ud, us$ vector (V) or axial vector (A) current 2-point functions, $\rho^{\tau}_{V/A,ij}(s)$ the corresponding spectral functions, and $\Delta \Pi_{\tau} \equiv \left[\Pi^{(0+1)}_{V/A,ud} - \Pi^{(0+1)}_{V/A,us}\right]$, the FESR relation

$$\int_0^{s_0} w(s) \Delta \rho_{\tau}(s) ds = -\frac{1}{2\pi i} \int_{|s|=s_0} w(s) \Delta \Pi_{\tau}(s) ds , \quad (1)$$

is valid for any $s_0$ and any analytic $w(s)$. The spectral function of $\Delta \Pi_{\tau}$, $\Delta \rho_{\tau}$, is experimentally accessible in terms of the differential distribution, $dR_{V/A,ij}/ds$, of $R_{V/A,ij} \equiv \Gamma[\tau^- \to \nu_{\tau} hadrons_{V/A,ij}(\gamma)]/\Gamma[\tau^- \to \nu_\tau e^\pm \bar{\nu}_\tau(\gamma)]$. Explicitly [2]

$$\frac{dR_{V/A,ij}}{ds} = c_{\tau}^{\text{EW}} |V_{ij}|^2 \int_0^{s_0} w_s(s) \rho^{(0+1)}_{V/A,ij}(y_s) \left[ \frac{-w_L(y_s) \rho^{(0)}_{V/A,ij}(s)}{y_s} \right] , \quad (2)$$

with $y_s = s/m_{\tau}^2$, $w_s(y) = (1 - y)^2(1 + 2y)$, $w_L(y) = 2y(1 - y)^2$, $c_{\tau}^{\text{EW}}$ a known constant, and $V_{ij}$ the flavor $ij$ CKM matrix element. The RHS of Eq. (1) is treated using the OPE.

One uses the $J = 0 + 1$ FESR Eq. (1), rather than the analogue involving the spectral function combination in Eq. (2), because of the very bad behavior of the integrated $J = 0$, $D = 2$ OPE series [3]. $\rho^{(0+1)}_{V/A,ud,us}(s)$ is obtained after subtracting phenomenologically determined $J = 0$ contributions from $dR_{V/A,ud,us}/ds$. This subtraction is dominated by the accurately known, non-chirally-suppressed $\pi$ and $K$ pole terms. Continuum $\rho^{(0)}_{V/A,ud}$ contributions are $\propto (m_d \pm m_u)^2$ and numerically negligible. Small, but not totally negligible, $(m_d \pm m_u)^2$-suppressed continuum $\rho^{(0)}_{V/A,us}$ contributions are determined using highly constrained dispersive and sum rule methods [4,5]. With $|V_{ud}|$ fixed [6], $\Delta \rho_{\tau}(s)$ is determined by experimental data and $|V_{us}|$. $|V_{us}|$ is then obtained using the OPE on the RHS and data on the LHS of Eq. (1).

From the distribution $dR^{(0+1)}_{V/A,ud,us}/ds$, obtained after subtracting $J = 0$ contributions, re-weighted $J = 0 + 1$ versions, $R^{(0+1)}_{V/A,ij}(s_0) \equiv \int_0^{s_0} ds w(s) \frac{dR^{(0+1)}_{V/A,ij}(s)}{ds}$, of $R_{V/A,ud,us}$, may be constructed for any $w$ and $s_0 \leq m_{\tau}^2$. With $\delta R^{w,\text{OPE}}_{V/A,ij}(s_0)$ the OPE representation of

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Two obvious theoretical systematic issues exist which might account for the observed $w_\tau$- and $s_0$-instabilities. The first concerns the treatment of $D = 6$, 8 OPE contributions. Both the conventional implementation and generalized versions just mentioned \cite{8}, estimate $D = 6$ contributions using the vacuum saturation approximation (VSA) and neglect $D = 8$ contributions. The VSA $D = 6$ estimate is very small due to significant cancellations, both in the individual $ud$ and $us$ V+A sums and in the subsequent FB difference of these sums. With sizeable channel-dependent VSA breaking observed in the flavor $ud$ V and A channels \cite{8}, such strong cancellations make the VSA estimate potentially quite unreliable. The second possibility concerns the slow convergence of the $D = 2$ OPE series for $\Delta \Pi_{\tau}$. With $\bar{a} = \alpha_s(Q^2)/\pi$, and $m_s(Q^2)$, $\alpha_s(Q^2)$ the running strange quark mass and coupling in the $\overline{MS}$ scheme, one has, to four loops \cite{10} (neglecting $O(m^4_{u,d}/m_s^2)$ corrections)

$$\left| \Delta \Pi_{\tau}(Q^2) \right|_{D=2}^{OPE} = \frac{3}{2\pi^2} \frac{m_s(Q^2)}{Q^2} \left[ 1 + \frac{7}{3} \bar{a} + 19.93 \bar{a}^2 + 208.75 \bar{a}^3 + \cdots \right].$$

(4)

Since $\bar{a}(m_s^2) \simeq 0.1$, convergence at the spacelike point on $|s| = s_0$ is marginal at best, raising questions concerning the choice of truncation order and truncation error estimates for the corresponding integrated series. The $D = 2$ convergence/truncation issue is also evident in the significant difference (increasing from $\sim 0.0010$ to $\sim 0.0020$ between 3- and 5-loop truncation order) in $|V_{us}|$ results obtained using alternate (fixed-order (FOPT) and contour-improved (CIPT)) prescriptions (prescriptions differing

\begin{align*}
\delta R_{V+A}(s_0) &\equiv \frac{R_{V+A,\text{incl}}(s_0)}{|V_{ud}|^2} - \frac{R_{V+A,\text{incl}}(s_0)}{|V_{us}|^2}, \\
|V_{us}| &\equiv \sqrt{R_{V+A,\text{incl}}(s_0)/|V_{ud}|^2 - \delta R_{V+A,\text{OPE}}(s_0)}.
\end{align*}

(3)

The resulting $|V_{us}|$ should be independent of $s_0$ and the choice of weight, $w$, provided all experimental data, and any assumptions employed in evaluating $\delta R_{V+A,\text{OPE}}(s_0)$, are reliable. Since integrated $D = 2k + 2$ OPE contributions scale as $1/s_0^k$, problems with assumptions about higher $D$ non-perturbative contributions, e.g., will produce an unphysical $s_0$-dependence in $|V_{us}|$.

The conventional implementation of Eq. (3) \cite{1} employs $w = w_\tau$ and $s_0 = m^2_\tau$. With this choice, the spectral integrals $R_{V+A,\text{incl,us}}(m^2_\tau)$ are determinable from inclusive non-strange and strange hadronic $\tau$ branching fractions, but assumptions about higher dimension $D = 6,8$ OPE contributions, in principle present for a degree 3 weight like $w_\tau$, are unavoidable. Using a single $w$ and single $s_0$ precludes subjecting these assumptions to $w$- and $s_0$-independence tests. It is a long-standing puzzle that this implementation produces inclusive $\tau$- $|V_{us}|$ determinations $> 3\sigma$ below 3-family-unitarity expectations (the most recent version, $|V_{us}| = 0.2176(21)$ \cite{7}, e.g., lies 3.6$\sigma$ below the current unitarity expectation, $|V_{us}| = 0.2258(9)$ \cite{6}). Tests of the conventional implementation, however, show sizeable $s_0$- and $w$-dependence \cite{8} (see also, e.g., the left panel, and solid lines in the right panel, of Fig. 4, indicating the existence of systematic problems in the conventional implementation. The dashed lines in the right panel show the results of the alternate implementation discussed below.

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only by terms beyond the common truncation order) for the truncated integrated $D = 2$ series \[8\].

In what follows, we first investigate the treatment of the $D = 2$ OPE series using lattice data for $\Delta \Pi_\tau$, then test the $D = 6, 8$ assumptions of the conventional implementation by comparing FESR results for a judiciously chosen pair of weights, $w_s(y)$ and $\hat{w}(y) = (1-y)^2$, $y = s/s_n$. Results obtained employing an alternate implementation of the FB FESR approach suggested by these investigations are then presented.

## 2 Lattice and continuum investigations of the OPE representation of $\Delta \Pi_\tau$

Data for $\Delta \Pi_\tau(Q^2)$ can be generated over a wide range of Euclidean $Q^2$ using the lattice. The (tight) cylinder cut which must be applied to avoid lattice artifacts at higher $Q^2$ has been determined, for the ensemble employed here, in a recent analysis aimed at using lattice current-current two-point function data to determine $\alpha_s[11]$. We first consider data at high enough $Q^2$ that $[\Delta \Pi_\tau]_{OPE}$ will be safely dominated by the leading $D = 2$ and 4 contributions. The latter are determined by light and strange quark masses and condensates and hence known. We use FLAG results for physical quark masses [12] and GMOR for the light condensate. \(\langle \bar{s}s \rangle\) then follows from \(\langle \bar{s}s \rangle/(\bar{u}u)\) (the HPQCD physical-$m_s$ version of this ratio [13] is easily translated to the $m_s$ of the ensemble employed using NLO ChPT [14]). We then consider various combinations of truncation order and log-resummation schemes for the $D = 2$ OPE series, investigating whether a choice exists which produces a good match between the resulting $D = 2 + 4$ OPE sum and the lattice data in the high-$Q^2$ region.

For this high-$Q^2$ study, we employ the RBC/UKQCD $n_f = 2 + 1, 32^4 \times 64$, $1/a = 2.38 \text{ GeV}$, $m_{\pi} \sim 300 \text{ MeV}$ domain wall fermion ensemble [15]. We find that 3-loop $D = 2$ truncation with fixed-scale choice (the analogue of the FOPT FESR prescription) provides an excellent OPE-lattice match over a wide range of $Q^2$, extending from near $\sim 10 \text{ GeV}^2$ down to just above $\sim 4 \text{ GeV}^2$. The comparison is shown, for fixed scale choice $\mu^2 = 4 \text{ GeV}^2$, in the left panel of Fig. 2 where, for ease of display, we plot results for the product $Q^2 \Delta \Pi_\tau(Q^2)$ (this removes a factor of $1/Q^2$ present in the $D = 2$ OPE contributions). The right panel of Fig. 2 shows the analogous comparison for the alternate local-scale ($\mu^2 = Q^2$) choice (analogous to the CIPT FESR prescription). It is clear that the $Q^2$ dependence of the lattice data prefers the fixed-scale treatment of the $D = 2$ series.

![Comparison of lattice data and OPE $D = 2 + 4$ expectations for $Q^2 \Delta \Pi_\tau(Q^2)$, for various truncation orders and either the fixed-scale treatment (left panel) or local-scale treatment (right panel) of the $D = 2$ series.](image)

The lattice data also provides us with the possibility of investigating the reliability of conventional methods for estimating the theoretical error to be associated with the truncated OPE. This is of particular relevance given the very slow convergence of the $D = 2$ series, which might raise doubts about the suitability of such conventional estimates in the case of the $D = 2$ truncation uncertainty. Fig. 3 shows the $D = 2 + 4$ OPE error band obtained using the 3-loop-truncated, fixed-scale $D = 2$ OPE treatment and such conventional OPE error estimates, taking into account uncertainties in the input OPE parameters and using the magnitude of the last term kept to estimate the $D = 2$ series truncation uncertainty. One sees that, despite the very slow convergence of the $D = 2$ series, the resulting conventionally determined error turns out to be extremely conservative.
We now switch our attention to lower $Q^2$, the goal being to use lattice data to test the assumptions about $D > 4$ contributions underlying the standard implementation, namely that $D > 6$ contributions are safely negligible and $D = 6$ contributions can be reasonably well approximated using the VSA. Fig. 4 shows the comparison of lattice data for $\Delta \Pi_\tau(Q^2)$ in the region below $\sim 4$ GeV$^2$ with two versions of the truncated OPE. The dashed line represents the 3-loop-truncated, fixed-scale $D = 2 + 4$ OPE sum discussed above, which provides an excellent match to the lattice data at higher scales, while the solid line shows the result of supplementing the $D = 2 + 4$ sum with the VSA estimate for the $D = 6$ contribution. The results show clear evidence for the onset of $D > 4$ contributions below $Q^2 \sim 4$ GeV$^2$ significantly larger than those obtained using the VSA estimate for $D = 6$ and neglecting $D > 6$ contributions. The VSA $D = 6$ estimate is not only too small in magnitude to bring the low-$Q^2$ truncated OPE into agreement with the lattice data but, in fact, moves the truncated OPE sum slightly in the wrong direction. Unfortunately, the Euclidean lattice data provides no means of selectively isolating contributions of different $D > 4$ in this lower $Q^2$ region. Further investigation of the higher $D$ question thus requires continuum FESR methods.

Our continuum FESR studies employ the $D = 2, 4$ OPE treatment favored by lattice data, detailed above. Spectral integral input is as follows: $\pi\pi$, $K_{\ell 2}$ and Standard Model expectations for the $\pi$ and $K$ pole contributions, recent ALEPH data for the continuum $ud$ V+A distribution, BaBar $\pi^0\pi^0$ and $K^0\pi^+$ distributions, BaBar results for the $K^+\pi^0\pi^-$ distribution, Belle results for the $K^0\pi^+\pi^-$ distribution and 1999 ALEPH results for the combined distribution of those strange modes not remeasured by the B-factory experiments.

The BaBar and Belle exclusive mode distributions are unit-normalized and must have their overall scales fixed using experimental branching fractions. In the results quoted below, HFAG strange exclusive mode branching fractions have been used, with the exception of the $K^-\pi^0$ mode, for which the updated version from the recent BaBar Adametz thesis has been employed. A corresponding (very) small rescaling is applied to the continuum $ud$ V+A distribution to restore unitarity.

Neglecting $\alpha_s$-suppressed logarithmic corrections, $D > 4$ OPE contributions to $\Delta \Pi_\tau(Q^2)$ can be written

$$\sum_{D > 4} C_D/Q^D$$

with $C_D$ an effective dimension $D$ condensate. The degree 3 weights $w_r(y) = 1 - 3y^2 + 2y^3$ and $\tilde{w}(y) = 1 - 3y + 3y^2 - y^3$ generate integrated OPE contributions up to $D = 8$ only. The integrated $D = 6, 8$ results,

$$-\frac{3C_6}{s_0} - \frac{2C_8}{s_0} \text{ for } w_r \text{ and } \frac{3C_6}{s_0} + \frac{C_8}{s_0} \text{ for } \tilde{w},$$

have $D = 6$ contributions identical in magnitude but opposite in sign, and a $\tilde{w}$ $D = 8$ contribution similarly opposite in sign but half in magnitude that of $w_r$. Were the assumptions of the conventional implementation to be correct, with $D = 6, 8$ contributions numerically negligible in the $w_r$ FESR, this will necessarily also be the case for the $\tilde{w}$ FESR. The two FESRs should then produce results for $|V_{us}|$ which not only agree, but are both $s_0$-independent. In contrast, if the $D = 6$ and/or $8$ contributions to the $w_r$ FESR are not, in fact, negligible, the results for $|V_{us}|$ from the two FESRs, obtained assuming they are, should show $s_0$-instabilities of opposite sign, decreasing in magnitude with increasing $s_0$ for both. The left panel of Fig. 1 shows the latter scenario to be the one
actually realized. The sizeable $s_0$- and weight-choice de-
pendences demonstrate unambiguously the breakdown of the
assumptions underlying the conventional implementa-
tion. The $\delta \sigma$ low $|V_{us}|$ results obtained employing them
are thus afflicted with significant previously unquantified
systematic uncertainties.

3 An alternate implementation of the
FB FESR approach

With previously employed approaches to estimating $D > 4$ effective OPE condensates shown to be untenable, this
only option is to fit these condensates to data. This
can be done only by exploiting the fact that integrated OPE contributions of different $D$ scale differently with
$s_0$, and hence requires working with FESRs involving a
range of values of the variable $s_0$. This precludes deter-
mining the required spectral integrals solely in terms of
hadronic $\tau$ decay branching fractions.

Considering FESRs based on different weights pro-
vides further tests of possible theoretical systematics. To suppress duality violating contributions, we restrict
our attention to weights having at least a double zero at
$s = s_0$. The weights $w_N(y) = 1 - \frac{N-1}{N} y + \frac{1}{N} y^N, N \geq 2$ [23] are particularly convenient in this regard since the corre-
sponding integrated OPE involves a single $D > 4$ contribu-
tion (with $D = 2N+2$). With $D = 2+4$ OPE contribu-
tions under control (as discussed above) this leaves $|V_{us}|$
and the effective condensate $C_{2N+2}$ as the only para-
ters to be determined. These are obtained through a fit
to the set of $w_N$-weighted spectral integrals in the chosen
$s_0$ fit window. Further tests of the analysis are provided by verifying (i) that the $|V_{us}|$ obtained from the different
$w_N$ FESRs are in good agreement and (ii) that the fitted
$C_D$ are physically plausible, in the sense of showing FB
cancellation relative to the results of Ref. [9] for the cor-
responding effective flavor $ud$ channel condensates. We have analyzed the $w_N$ FESRs for $N = 2, 3, 4$ and verified
that the results pass these self-consistency tests.

In the right panel of Fig. [1] we display, as dashed/dotted lines, the results which follow from taking as input the central value for the condensate, $C_{2N+2}$, ob-
tained from the $w_N$ FESR analysis, and solving Eq. (3)
for $|V_{us}|$ as a function of $s_0$. The results are displayed for each of the $w_2$, $w_3$ and $w_4$ FESR cases. The results
make clear (i) the underlying excellent match between the fitted OPE and spectral integral sets, (ii) the excel-

| Table 1. Error budgets for the $w_2$, $w_3$ and $w_4$ determinations of $|V_{us}|$, using the 3-loop-truncated, fixed-scale treatment of the $D = 2$ OPE series |

| Error source | $\delta|V_{us}|$ (w2 FESR) | $\delta|V_{us}|$ (w3 FESR) | $\delta|V_{us}|$ (w4 FESR) |
|--------------|--------------------------|--------------------------|--------------------------|
| $\delta s_0$  | 0.00027                  | 0.00028                  | 0.00028                  |
| $\delta m_{s_s}$ (2 GeV) | 0.00017                  | 0.00019                  | 0.00019                  |
| $\delta (m_s ks)$ | 0.000035                | 0.000035                | 0.000035                |
| $\delta (long corr)$ | 0.00009                | 0.00009                | 0.00009                |

In Table 1 we give the error budgets for the $|V_{us}|$
determinations based on the $w_2$, $w_3$ and $w_4$ FESRs, using the 3-loop, fixed-scale treatment of the $D = 2$ OPE series favored by lattice data. The errors in the first half are those associated with input uncertainties on the theory side, with “long corr” labelling those associated with the sum rule/dispersive determinations of the small, doubly-chirally-suppressed continuum $us$ V and A channel $J = 0$ subtractions. Combining these errors in quadrature yields a total theory error of 0.0004 for each of the three cases. The errors listed in the second half of the table are those induced by the errors and covariances

The excellent $s_0$-stability and agreement between the results from the different $w_N$ FESRs allows us to arrive at a final result for $|V_{us}|$ obtained by performing a combined fit to the $w_2$, $w_3$ and $w_4$ FESRs. We find

$$|V_{us}| = 0.2228(23)_{exp}(5)_{th}.$$  (6)

This is in excellent agreement with the results, 0.2235(4)_{exp}(9)_{th} and 0.2231(4)_{exp}(7)_{th}, obtained using the 2014 FlaviaNet experimental $K_{l3}$ update [25] and
of the most recent $n_f = 2 + 1$ [26] and $n_f = 2 + 1 + 1$ [27] lattice results for $f_+(0)$. It is also compatible within errors with (i) the results, 0.2251(3)exp(9)$_{th}$ and 0.02250(3)exp(T)$_{th}$ obtained using the 2014 update for the experimental ratios $\Gamma[K_{\mu2}]/\Gamma[\pi_{\mu2}]$ [29] and the most recent $n_f = 2 + 1$ [28] and $n_f = 2 + 1 + 1$ [29] lattice determinations of $f_K/f_{\pi}$ and (ii) the expectations of 3-family unitarity [31].

It is worth noting that, among the methods mentioned above, the one having the smallest theoretical error is, in fact, the FB FESR determination. This error is, moreover, as we have seen, a very conservative one. At present the experimental error on the FB FESR determination (resulting almost entirely from uncertainties in the $u\bar{s}$ exclusive mode distributions) is larger than those of the competing methods. We note, however, that the $u\bar{s}$ spectral integral error is currently dominated by the uncertainty on the branching fraction normalizations for the exclusive strange modes, and hence systematically improvable through improvements in these branching fraction values in the near future. In the longer term, it will be important to complete the analysis of the exclusive mode $u\bar{s}$ distributions not yet remeasured by the B-factory experiments and to finalize the analyses of the covariances of those unit normalized exclusive distributions which are not yet complete at present.

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