Theta constants associated with the cyclic triple coverings of the complex projective line branching at six points *

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Abstract

Let $\psi$ be the period map for a family of the cyclic triple coverings of the complex projective line branching at six points. The symmetric group $S_6$ acts on this family and on its image under $\psi$. In this paper, we give an $S_6$-equivariant expression of $\psi^{-1}$ in terms of fifteen theta constants.

1 Introduction

Let $C(\lambda)$ be the cyclic triple covering of the complex projective line $\mathbb{P}^1$ branching at six points $\lambda_1, \ldots, \lambda_6$:

$$C(\lambda) : w^3 = \prod_{i=1}^{6} (z - \lambda_i).$$

The moduli space of such curves with a homology marking can be regarded as the configuration space $\Lambda$ of ordered six distinct points on $\mathbb{P}^1$, which is defined by

$$GL_2(\mathbb{C}) \{ \lambda = (\lambda_{ij}) \in M(2, 6) \mid \lambda \langle i j \rangle = \begin{vmatrix} \lambda_{1i} & \lambda_{1j} \\ \lambda_{2i} & \lambda_{2j} \end{vmatrix} \neq 0 \}/(\mathbb{C}^*)^6.$$
Note that the symmetric group $S_6$ naturally acts on $\Lambda$. It is shown in [Yo2] that the map

$$\iota : \Lambda \ni \lambda \mapsto \left[\ldots, \lambda(ij)\lambda(kl)\lambda(mn)\ldots\right] \in \mathbb{P}^{14}$$

is an $S_6$-equivariant embedding and that its image is an open subset of $Y$ defined by linear and cubic equations.

The normalized period matrix $\Omega$ of $C(\lambda)$ with a homology marking belongs to the Siegel upper half space $S^4$ of degree 4. By our assignment of a homology marking, $\Omega$ can be identified with an element of 3-dimensional complex ball $B^3 = \{x \in \mathbb{P}^3 | \langle x, Hx \rangle < 0\}$, where $H = \text{diag}(1, 1, 1, -1)$. In this way, we get a multi-valued map $\psi : \Lambda \to B^3 \subset S^4$, which is called the period map. Results in [DM] and [Ter] imply that the image of $\psi$ is an open dense subset of $B^3$, the monodromy group of $\psi$ is the principal congruence subgroup $\Gamma(1 - \omega)$ of level $(1 - \omega)$ of $\Gamma = \{g \in GL_4(\mathbb{Z}[\omega]) | \langle gHg, H \rangle = H\}$, and that the inverse of $\psi$ is single valued.

In this paper, we express the inverse of the period map $\psi$ in terms of fifteen theta constants. More precisely, for the two isomorphisms $\psi : \Lambda \to \psi(\Lambda)/\Gamma(1 - \omega)$ and $\iota : \Lambda \to \iota(\Lambda) \subset Y \subset \mathbb{P}^{14}$, we present an isomorphism $\Theta : \psi(\Lambda)/\Gamma(1 - \omega) \to \iota(\Lambda)$ such that the following diagram commutes:

$$\begin{array}{ccc}
\Lambda & \xrightarrow{\psi} & \psi(\Lambda)/\Gamma(1 - \omega) \\
\downarrow \iota & & \uparrow \Theta \\
\iota(\Lambda) & \subset & Y \subset \mathbb{P}^{14}
\end{array}$$

(1)

The map $\Theta$ is given by the ratio of the cubes of the fifteen theta constants on $S^4$ which are invariant under the action of $\Gamma(1 - \omega)$ embedded in $Sp(8, \mathbb{Z})$. In particular, linear and cubic relations among the cubes of fifteen theta constants coincide with the defining equations of $Y \subset \mathbb{P}^{14}$.

It is known that $\Gamma/\langle \Gamma(1 - \omega), -I_4 \rangle$ is isomorphic to $S_6$, which naturally acts on $\psi(\Lambda)/\Gamma(1 - \omega)$. The period map $\psi$ is $S_6$-equivariant. By considering the action $S_6 \simeq \Gamma/\langle \Gamma(1 - \omega), -I_4 \rangle$ on the fifteen theta characteristics, we label fifteen theta constants as $(ij; kl; mn)$, where $\{i, j, k, l, m, n\} = \{1, \ldots, 6\}$. Then it turns out that the diagram (1) is $S_6$-equivariant.

An explicit expression of $\psi^{-1}$ is given in [Gon]. We want to know the combinatorial structure of $\psi^{-1}$ in order to study the inverse of the period map.
map from a family of smooth cubic surfaces to the 4-dimensional complex ball $\mathbb{B}^4$ in [ACT].

For a 2-dimensional subfamily of ours defined by $\lambda_5 = \lambda_6$, the period map and its inverse are studied in [Pic] and [Shi].

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2 Configuration space $\Lambda$ of six points on $\mathbb{P}^1$

Let $M(m, n)$ be the set of complex $(m \times n)$ matrices. We define the configuration space $\Lambda$ of ordered six distinct points on the complex projective line $\mathbb{P}^1$ as

$$\Lambda = GL_2(\mathbb{C}) \backslash M'(2, 6)/(\mathbb{C}^*)^6,$$

where

$$M'(2, 6) = \{ \lambda = (\lambda_{ij}) \in M(2, 6) \mid \lambda(kl) = \begin{vmatrix} \lambda_{1k} & \lambda_{1l} \\ \lambda_{2k} & \lambda_{2l} \end{vmatrix} \neq 0 \ (1 \leq k \neq l \leq 6) \},$$

and $GL_2(\mathbb{C})$ and $(\mathbb{C}^*)^6$ (regarding as the group of $(6 \times 6)$ diagonal matrices) act naturally on $M'(2, 6)$ from the left and right, respectively. Note that we regard the column vectors of $\lambda \in M'(2, 6)$ as the homogeneous coordinates of six points on $\mathbb{P}^1$ and the action of $GL_2(\mathbb{C})$ as the projective transformation. Six distinct points $\lambda_1, \ldots, \lambda_6$ on $\mathbb{C}$ are expressed by an element of $\Lambda$ by $(2 \times 6)$ matrix

$$\lambda = \begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 \\ \lambda_1 & \lambda_2 & \lambda_3 & \lambda_4 & \lambda_5 & \lambda_6 \end{pmatrix}.$$

By normalizing $(\lambda_1, \lambda_2, \lambda_3)$ as $(\infty, 0, 1)$, matrices of the form

$$\begin{pmatrix} 0 & 1 & 1 & 1 & 1 & 1 \\ 1 & 0 & 1 & \ell_1 & \ell_2 & \ell_3 \end{pmatrix}, \quad \ell_i \neq 0, 1, \ell_j \ (1 \leq i < j \leq 3)$$

represent $\Lambda$.

We define a map $\iota$ from $\Lambda$ to the 14-dimensional projective space $\mathbb{P}^{14}$ by

$$\iota: \Lambda \ni \lambda \mapsto [\ldots, y_{ij;kl;mn}, \ldots] = [\ldots, \lambda(ij) \lambda(kl) \lambda(mn), \ldots] \in \mathbb{P}^{14},$$

where $\lambda$ is a $(2 \times 6)$ matrix represent of an element of $\Lambda$ and projective coordinates of $\mathbb{P}^{14}$ are labeled by $I = \langle ij; kl; mn \rangle$ ($\{i, j, k, l, m, n\} = \{1, \ldots, 6\}$, $i <$
Since the image $\iota(\lambda)$ is invariant under the actions $\text{GL}_2(\mathbb{C})$ and $(\mathbb{C}^*)^6$, this map is well defined. We use the following convention

$$y_{(ij;kl;mn)} = y_{(kl;ij;mn)} = y_{(ij;mn;kl)} = -y_{(ji;kl;mn)}.$$ \(\text{The image } \iota(A) \text{ is studied in [Yo2], it is described as the following.}

**Fact 2.1** The closure $Y = \overline{\iota(A)}$ of $\iota(A)$ is a subvariety of $\mathbb{P}^{14}$ defined by the linear and cubic equations

$$y_{(ij;kl;mn)} - y_{(ij;km;ln)} + y_{(ij;kn;lm)} = 0$$

$$y_{(ij;kl;mn)}y_{(ik;jn;lm)}y_{(im;jl;kn)} = y_{(ij;kn;lm)}y_{(ik;jl;mn)}y_{(im;jn;kl)}.$$

We define $\hat{A}$ as the compactification of $A$ isomorphic to $Y$.

## 3 Period matrix of $C$

Let $C = C(\lambda)$ be the triple covering of $\mathbb{P}^1$ branching at six distinct points $\lambda_i$s:

$$C(\lambda) : w^3 = \prod_{i=1}^6(z - \lambda_i);$$

this curve is of genus 4. Let $\rho$ be the automorphism of $C$ defined by

$$\rho : C \ni (z, w) \mapsto (z, \omega w) \in C,$$

where $\omega = \frac{-1 + \sqrt{-3}}{2}$. We give a basis of the vector space of holomorphic 1-forms on $C$ as follows

$$\varphi_1 = \frac{dz}{w}, \quad \varphi_2 = \frac{dz}{w^2}, \quad \varphi_3 = \frac{zdz}{w^2}, \quad \varphi_4 = \frac{z^2dz}{w^2},$$ \hfill (2)

For a fixed $\lambda$ such that $\lambda_i \in \mathbb{R}$, $\lambda_1 < \ldots < \lambda_6$, we take a symplectic basis $\{A_1, \ldots, A_4, B_1, \ldots, B_4\}$ of $H_1(C, \mathbb{Z})$ (i.e., $A_i \cdot A_j = B_i \cdot B_j = 0$, $B_i \cdot A_j = \delta_{ij}$) such that

$$\rho(B_i) = A_i \quad (i = 1, 2, 3), \quad \rho(B_4) = -A_4,$$ \hfill (3)

see Figure 1.
Put
\[ \left( \int_{A_1} \varphi_j \right)_{i,j} = \left( \Omega_A \right) \left( \Omega_B \right). \]

Let \( \varphi \) be the normalized basis of vector space of holomorphic 1-forms so that \( \Omega_B \) becomes \( I_4 \). Note that the normalized period \( \Omega = \Omega_A \Omega_B^{-1} \) belongs to the Siegel upper half space \( S^4 \) of degree 4. The next proposition shows that \( \Omega \) can be expressed in terms of
\[
x = t(x_1, \ldots, x_4) = t \left( \int_{A_1} \varphi_1, \ldots, \int_{A_4} \varphi_1 \right).
\]

**Proposition 3.1** We have
\[
\Omega = \omega [I_4 - (1 - \omega)(x^t H x)/ (x^t x H x)] H = \omega [H - (1 - \omega)(x^t x)/ (x^t x H x)]
\]
\[
= \begin{pmatrix}
\omega & \omega & \omega & \omega \\
\omega & \omega & \omega & \omega \\
\omega & \omega & \omega & \omega \\
\omega & \omega & \omega & \omega 
\end{pmatrix} \frac{-\sqrt{-3}}{x_1^2 + x_2^2 + x_3^2 - x_4^2} \begin{pmatrix}
x_1 x_1 & x_1 x_2 & x_1 x_3 & x_1 x_4 \\
x_2 x_1 & x_2 x_2 & x_2 x_3 & x_2 x_4 \\
x_3 x_1 & x_3 x_2 & x_3 x_3 & x_3 x_4 \\
x_4 x_1 & x_4 x_2 & x_4 x_3 & x_4 x_4
\end{pmatrix},
\]
where \( H = \text{diag}(1, 1, 1, -1) \) and \( x^t H x < 0 \).

**Proof.** Put \( \Omega_A = (x, b, c, d) \); by (3) and (4), \( \Omega_B \) can be expressed as
\[
\Omega_B = (H x, \omega^2 H b, \omega^2 H c, \omega^2 H d) = \omega^2 H \Omega_A + (\omega - \omega^2) H (x, O).
\]
We have
\[
\Omega^{-1} = \Omega_B \Omega_A^{-1} = \omega^2 H + (\omega - \omega^2) H (x, O) \Omega_A^{-1}.
\]
Put
\[
\Omega_A^{-1} = \begin{pmatrix}
\xi \\
\ast
\end{pmatrix}, \quad \xi = (\xi_1, \ldots, \xi_4);
\]
noting that
\[
\xi x = \sum_{i=1}^{4} \xi_i x_i = 1.
\]
We have
\[
H (x, O) \Omega_A^{-1} = H x \xi = \frac{1}{\xi x} \begin{pmatrix}
x_1 \xi_1 & x_1 \xi_2 & x_1 \xi_3 & x_1 \xi_4 \\
x_2 \xi_1 & x_2 \xi_2 & x_2 \xi_3 & x_2 \xi_4 \\
x_3 \xi_1 & x_3 \xi_2 & x_3 \xi_3 & x_3 \xi_4 \\
x_4 \xi_1 & x_4 \xi_2 & x_4 \xi_3 & x_4 \xi_4 \\
-x_4 \xi_1 & -x_4 \xi_2 & -x_4 \xi_3 & -x_4 \xi_4
\end{pmatrix}, \quad (4)
\]
which must be symmetric. Thus we have
\[ x_i \xi_j = x_j \xi_i \quad (1 \leq i < j \leq 3), \quad x_i \xi_4 = -x_4 \xi_i \quad (i = 1, 2, 3). \]
By eliminating \( \xi_i \) in (4), we have
\[ H(x, O) \Omega^{-1}_A = (Hx^t xH)/(t xHx). \]
Then
\[ \Omega^{-1} = \omega^2 H[I_4 - (1 - \omega^2)(x^t xH)/(t xHx)]. \]
It is easy to see that
\[ [I_4 - (1 - \omega^2)(x^t xH)/(t xHx)]^{-1} = I_4 - (1 - \omega)(x^t xH)/(t xHx), \]
we have
\[ \Omega = \omega[I_4 - (1 - \omega)(x^t xH)/(t xHx)]H. \]
The imaginary part of \( \Omega \) is \( \sqrt{n/2} \) times
\[ H - x^t x/(t xHx) - \bar{x}^t \bar{x}/(t \bar{x}H\bar{x}), \quad (5) \]
which must be positive definite. If \( x_4 = 0 \) then the \( (4, 4) \) component of (3) is \(-1\), which implies that (3) can not be positive definite. Thus we have \( x_4 \neq 0 \).

Put
\[ \eta = \begin{pmatrix} x_4 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_4 \\ x_1 & x_2 & x_3 \end{pmatrix}; \]
note that \( (\eta, x) \in GL_4(\mathbb{C}) \) and that \( t xH\eta = (0, 0, 0) \). We have
\[ t(\eta, x)H \begin{pmatrix} x_4 & 0 & 0 \\ 0 & x_4 & 0 \\ 0 & 0 & x_4 \\ x_1 & x_2 & x_3 \end{pmatrix} = \begin{pmatrix} t \bar{x}H\eta & 0 \\ 0 & -t \bar{x}H\bar{x} \end{pmatrix}. \]
If
\[ -\bar{x}Hx = -|x_1|^2 - |x_2|^2 - |x_3|^2 - |x_4|^2 > 0 \]
then the \( 3 \times 3 \) matrix
\[ t \bar{x}H\eta = |x_4|^2 I_3 - \begin{pmatrix} \bar{x}_1 \\ \bar{x}_2 \\ \bar{x}_3 \end{pmatrix} (x_1, x_2, x_3) \]
is positive definite. Hence the matrix (3) is positive definite if and only if
\[ t \bar{x}Hx = |x_1|^2 + |x_2|^2 + |x_3|^2 - |x_4|^2 < 0. \]

We embedded the domain \( \mathbb{B}^3 = \{ x \in \mathbb{P}^3 \mid t \bar{x}Hx < 0 \} \) in \( \mathbb{S}^4 \) by the map
\[ j : \mathbb{B}^3 \ni x \mapsto \Omega = \omega[I_4 - (1 - \omega)(x^t xH)/(t xHx)]H \in \mathbb{S}^4. \]
4 Monodromy

Let \((\lambda_1, \ldots, \lambda_6)\) vary as an element in \(\Lambda\), we have two multi-valued map

\[
\psi : \Lambda \rightarrow \mathbb{B}^3, \quad \lambda \mapsto x = t\left(\int_{A_1} \psi_1, \ldots, \int_{A_4} \psi_1\right),
\]

\[
\tilde{\psi} = j \circ \psi : \Lambda \rightarrow S^4, \quad \lambda \mapsto \Omega = j(\psi(\lambda)).
\]

We call them period maps. The map \(\psi\) and its monodromy group were studied in \([\text{DM}], [\text{Ter}], [\text{Yo1}]\) and \([\text{Yo2}]\), the results are as follows.

**Fact 4.1** The image of \(\psi\) is open dense in \(\mathbb{B}^3\). The monodromy group of \(\psi\) is conjugate to the congruence subgroup

\[
\Gamma(1 - \omega) = \{g \in \Gamma \mid g \equiv I_4 \mod (1 - \omega)\}
\]

of the modular group

\[
\Gamma = \{g \in GL_4(\mathbb{Z}[\omega]) \mid {}^t g H g = H\}.
\]

The Satake compactification \(\hat{\mathbb{B}^3}/\Gamma(1 - \omega)\) of \(\mathbb{B}^3/\Gamma(1 - \omega)\) is isomorphic to \(Y\).

For a column vector \(v \in \mathbb{C}^4\) such that \({}^t \bar{v} Hv \neq 0\), we define reflections \(R^\omega(v)\) and \(R^\zeta(v)\) with root \(v\) and exponent \(\omega\) and \(\zeta = -\omega^2\), respectively, as

\[
R^\omega(v) = I_4 - (1 - \omega)v(\bar{v} Hv)^{-1} {}^t \bar{v} H, \quad R^\zeta(v) = I_4 - (1 - \zeta)v(\bar{v} Hv)^{-1} {}^t \bar{v} H.
\]

It is shown in \([\text{All}]\) that \(\Gamma(1 - \omega)\) can be generated by fifteen reflections \(R^\omega_{ij} = R^\omega(v_{ij})\) \((1 \leq i < j \leq 6)\) and that \(\Gamma\) by \(-I_4\) and five reflections \(R^\zeta_{i,i+1} = R^\zeta(v_{i,i+1})\) \((1 \leq i \leq 5)\), where

\[
\begin{align*}
&v_{12} = t(1, 0, 0, 0), \quad v_{13} = t(-1, 1, 0, 1), \quad v_{14} = t(-1, -\omega^2, 0, 1), \\
&v_{15} = t(\omega^2, 0, -\omega^2, 1), \quad v_{16} = t(\omega^2, 0, \omega, 1), \quad v_{23} = t(\omega^2, 1, 0, 1), \\
&v_{24} = t(\omega^2, -\omega^2, 0, 1), \quad v_{25} = t(-\omega, 0, -\omega^2, 1), \quad v_{26} = t(-\omega, 0, \omega, 1), \\
&v_{34} = t(0, 1, 0, 0), \quad v_{35} = t(0, -\omega, \omega, 1), \quad v_{36} = t(0, -\omega, -1, 1), \\
&v_{45} = t(0, 1, \omega, 1), \quad v_{46} = t(0, 1, -1, 1), \quad v_{56} = t(0, 0, 1, 0).
\end{align*}
\]

The reflections correspond to the following movements of \(\lambda_i\)'s. When \(\lambda_i\) goes near to \(\lambda_j\) in the upper half space and turns around \(\lambda_j\) and returns, \(x\) becomes
When $\lambda_i$ and $\lambda_j$ are exchanged in the upper half space, $x$ becomes $R^c_{ij}x$. Since $R^c_{i,i+1}$’s are representations of braids, they satisfy

$$R^c_{i-1,i}R^c_{i,i+1}R^c_{i-1,i} = R^c_{i,i+1}R^c_{i-1,i}R^c_{i,i+1} \quad (2 \leq i \leq 5).$$

The embedding $j$ induces the following homomorphism from $U(3, 1; \mathbb{C})$ to

$$Sp(8, \mathbb{R}) = \left\{ g \in GL_8(\mathbb{R}) \mid {}^t gJg = J = \begin{pmatrix} O & -I_4 \\ I_4 & O \end{pmatrix} \right\} :$$

$$\tilde{j} : U(3, 1; \mathbb{C}) \ni P + \omega Q \mapsto \begin{pmatrix} P \\ -HQ \\ QH \\ H(P - Q)H \end{pmatrix} \in Sp(8, \mathbb{R}),$$

where $P$ and $Q$ are real $4 \times 4$ matrices. Note that

$$\tilde{j}^{-1} : Sp(8, \mathbb{R}) \supset \tilde{j}(U(3, 1; \mathbb{C})) \ni \begin{pmatrix} A \\ B \\ C \\ D \end{pmatrix} \mapsto A + \omega BH = (-HC + HDH) - \omega HC \in U(3, 1; \mathbb{C}).$$

Let us express the images of $R^w(v)$ and $R^c(v)$ under the map $\tilde{j}$. The image of $\omega I_4$ under $\tilde{j}$ is given by

$$W = \begin{pmatrix} O & H \\ -H & -I_4 \end{pmatrix} \in Sp(8, \mathbb{Z}).$$

For a column vector $v = a + \omega b$ ($a, b \in \mathbb{R}^4$), define column vectors $v_1 = \begin{pmatrix} a \\ -Hb \end{pmatrix}$ and $v_2 = Wv_1$ and form a $(8 \times 2)$ matrix $V = (v_1, v_2)$. Straightforward calculation shows the following.

**Proposition 4.1** If $\,^t\bar{v}Hv \neq 0$, then $\tilde{j}(R^w(v)) = \tilde{R}^w(v)$ and $\tilde{j}(R^c(v)) = \tilde{R}^c(v)$ are given by

$$I_8 - (I_8 - W)V(\,^tVJV)^{-1}\,^tVJ, \quad I_8 - (I_8 + W^2)V(\,^tVJV)^{-1}\,^tVJ,$$

respectively.

Systems of generators of $\tilde{\Gamma}(1 - \omega) = \tilde{j}(\Gamma(1 - \omega))$ and $\tilde{\Gamma} = \tilde{j}(\Gamma)$ are given by $\tilde{R}^w_{ij}$’s and $\tilde{R}^c_{i,i+1}$’s.
5 Riemann theta constants

The Riemann theta function
\[ \vartheta(z, \tau) = \sum_{n=(n_1, \ldots, n_r) \in \mathbb{Z}^r} \exp[\pi \sqrt{-1}(n^t \tau n + 2n^t z)] \]

is holomorphic on \( \mathbb{C}^r \times S^r \) and satisfies
\[ \vartheta(z + p, \tau) = \vartheta(z, \tau), \quad \vartheta(z + p \tau, \tau) = \exp[-\pi \sqrt{-1}(p \tau^t p + 2z^t p)] \vartheta(z, \tau), \]
where \( S^r \) is the Siegel upper half space of degree \( r \) and \( p \in \mathbb{Z}^r \). It is well known that for \((z, \tau) \in \mathbb{C} \times \mathbb{H}\), \( \vartheta(z, \tau) = 0 \) if and only if \( z = \frac{1 + i \tau}{2} + p + q \tau \) \((p, q \in \mathbb{Z})\).

The theta function \( \vartheta_{a,b}(z, \tau) \) with characteristics \( a, b \) is defined by
\[ \vartheta_{a,b}(z, \tau) = \sum_{n \in \mathbb{Z}^n} \exp[\pi \sqrt{-1}((n + a)^t \tau (n + a) + 2(n + a)^t (z + b))], \]
where \( a, b \in \mathbb{Q}^r \). Note that
\[ \vartheta_{-a,-b}(z, \tau) = \vartheta_{a,b}(-z, \tau), \quad \vartheta_{a+p,b+q}(z, \tau) = \exp(2\pi \sqrt{-1}a^t q)\vartheta_{a,b}(z, \tau). \]

The function \( \vartheta_{a,b}(\tau) = \vartheta_{a,b}(0, \tau) \) of \( \tau \) is called the theta constant with characteristics \( a, b \). If \( \tau \) is diagonal, then this function becomes the product of Jacobi’s theta constants:
\[ \vartheta_{a,b}(\tau) = \prod_{i=1}^r \vartheta_{a_i,b_i}(\tau_i), \]
where
\[ a = (a_1, \ldots, a_r), \quad b = (b_1, \ldots, b_r), \quad \tau = \text{diag}(\tau_1, \ldots, \tau_r). \]

The following transformation formula can be found in [Igu] p.176.

**Fact 5.1** For any \( g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in Sp(2r, \mathbb{Z}) \) and \((a, b) \in \mathbb{Q}^{2r}\), we put
\[ g \cdot (a, b) = (a, b) g^{-1} + \frac{1}{2}(dv(C^t D), dv(A^t B)) \]
\[ \phi(a,b)(g) = \frac{1}{2}(a^t D B^t a - 2a^t B C^t b + b^t C A^t b) + \frac{1}{2}(a^t D - b^t C)^t (dv(A^t B)), \]
where $dv(A)$ is the row vector consisting of the diagonal components of $A$. Then for every $g \in Sp(2r, \mathbb{Z})$, we have

$$
\vartheta_{g(a,b)}((A\tau + B)(C\tau + D)^{-1}) = \kappa(g) \exp(2\pi\sqrt{1} \phi_{(a,b)}(g)) \det(C\tau + D)^{\frac{1}{2}} \vartheta_{(a,b)}(\tau),
$$
in which $\kappa(g)^2$ is a 4-th root of 1 depending only on $g$.

**Proposition 5.1** There are $81 = 3^4$ theta characteristics

$$(a, b) = (a_1, \ldots, a_4, b_1, \ldots, b_4)$$

such that

$$g \cdot (a, b) \equiv (a, b) \mod \mathbb{Z}^8$$

for any $g \in \tilde{\Gamma}(1 - \omega) \subset Sp(8, \mathbb{Z})$; they are given by

$$b = -aH, \ a_i \in \{\frac{1}{6}, \frac{3}{6}, \frac{5}{6}\} \ (i = 1, \ldots, 4). \quad (8)$$

**Proof.** Since

$$W \cdot (a, b) = (-a + bH, -aH) + \frac{1}{2}(1, 1, 1, -1, 0, 0, 0, 0),$$

we have

$$-aH \equiv b, \ -2a + \frac{1}{2}(1, 1, 1, -1) \equiv a \mod \mathbb{Z}^4.$$

Thus we have the condition (8). It is easy to check such theta characteristics are invariant under the action on 15 reflections $\tilde{R}_{ij}^\omega$. []

We label the 81 characteristics $a$’s by combinatorics of six letters; they are classified to 4 classes. The list of the correspondence between the label of $a$ and $6a$ is as follows:

$$(12;34:56) \leftrightarrow \pm(3, 3, 3, -1) \quad (12;35:46) \leftrightarrow \pm(3, 1, 1, -3) \quad (12;36:45) \leftrightarrow \pm(3, 1, -1, -3)$$

$$(13;24:56) \leftrightarrow \pm(1, 1, 3, -3) \quad (13;25:46) \leftrightarrow \pm(1, -1, 1, -1) \quad (13;26:45) \leftrightarrow \pm(-1, 1, 1, 1)$$

$$(14;23:56) \leftrightarrow \pm(1, -1, 3, -3) \quad (14;25:36) \leftrightarrow \pm(1, 1, 1, -1) \quad (14;26:35) \leftrightarrow \pm(1, 1, -1, -1)$$

$$(15;23:46) \leftrightarrow \pm(1, 1, -1, 1) \quad (15;24:36) \leftrightarrow \pm(1, -1, -1, 1) \quad (15;26:34) \leftrightarrow \pm(1, 3, 1, -3)$$

$$(16;23:45) \leftrightarrow \pm(1, 1, 1, 1) \quad (16;24:35) \leftrightarrow \pm(-1, 1, 1, 1) \quad (16;25:34) \leftrightarrow \pm(1, 3, -1, -3)$$
(1^2) \leftrightarrow (1, 3, 3, 3) \quad (1^2) \leftrightarrow (5, 1, 3, 5) \quad (1^2) \leftrightarrow (5, 5, 3, 5)

(1^4) \leftrightarrow (5, 3, 1, 1) \quad (1^6) \leftrightarrow (5, 3, 5, 1) \quad (2^3) \leftrightarrow (1, 1, 3, 5)

(2^4) \leftrightarrow (1, 5, 3, 5) \quad (2^5) \leftrightarrow (1, 3, 1, 1) \quad (2^6) \leftrightarrow (1, 3, 5, 1)

(3^4) \leftrightarrow (3, 1, 3, 3) \quad (3^5) \leftrightarrow (3, 5, 1, 5) \quad (3^6) \leftrightarrow (3, 5, 5, 5)

(4^5) \leftrightarrow (3, 1, 1, 5) \quad (4^6) \leftrightarrow (3, 1, 5, 5) \quad (5^6) \leftrightarrow (3, 3, 1, 3)

\(ij^2 \leftrightarrow -a \) for \(i^2j\), \(1 \leq i < j \leq 6\)

(123) \leftrightarrow (3, 1, 3, 5) \quad (124) \leftrightarrow (3, 5, 3, 5) \quad (125) \leftrightarrow (3, 3, 1, 1)

(126) \leftrightarrow (3, 3, 5, 1) \quad (134) \leftrightarrow (1, 3, 3, 1) \quad (135) \leftrightarrow (1, 1, 1, 3)

(136) \leftrightarrow (1, 1, 5, 3) \quad (145) \leftrightarrow (1, 5, 1, 3) \quad (146) \leftrightarrow (1, 5, 5, 3)

(156) \leftrightarrow (1, 3, 3, 5) \quad (l mn) \leftrightarrow -a \) for \((ijk) \quad \{i, j, k, l, m, n\} = \{1, \ldots, 6\}

(123456) \leftrightarrow (3, 3, 3, 3).

The first class is characterized by \((6a)H \equiv (6a) \equiv 2 \mod 24\) and the characteristics \((a, -aH)\) with label \((ij; kl; mn)\) is invariant under the actions \(\tilde{R}_{ij}\), \(\tilde{R}_{kl}\) and \(\tilde{R}_{mn}\); the second class is characterized by \((6a)H \equiv (6a) \equiv 10 \mod 24\) and the characteristics \((a, -aH)\) with label \((i^2j)\) is invariant under the actions \(\tilde{R}_{ij}^c\) \(((i, j) \cap \{k, l\} = \emptyset)\) and \(\tilde{R}_{ij} \cdot (a, -aH) = (-a, aH)\) with label \((ij^2)\); the third class is characterized by \((6a)H \equiv (6a) \equiv 18 \mod 24\) and the characteristics \((a, -aH)\) with label \((ijk)\) is invariant under the actions \(\tilde{R}_{im}^c\) \(((i, j, k) \cap \{l, m\} = \emptyset \text{ or } \{l, m\}\). We denote \(\vartheta_{a,-aH}(\Omega)\) by \(\vartheta_{(6a)}(\Omega)\) or \(\vartheta(ij; kl; mn)\), \(\vartheta(i^2j)\), \(\vartheta(ijk)\) and \(\vartheta(123456)\) for corresponding characteristics \(a\). Note that for \(p, q \in \mathbb{Z}^4\),

\[
\vartheta(a(\Omega-H) + p\Omega + q, \Omega) = \exp[-\pi \sqrt{-1}(p\Omega \cdot p + 2p(\Omega-H) \cdot a)]\vartheta(a(\Omega-H), \Omega) = \exp[-\pi \sqrt{-1}(p\Omega \cdot p + 2p(\Omega-H) \cdot a + a\Omega \cdot a - 2aH \cdot a)]\vartheta_{a,-aH}(\Omega) = \exp[2\pi \sqrt{-1}(a + p)H \cdot (a + p)]\exp[-\pi \sqrt{-1}(a + p)\Omega \cdot (a + p)]\vartheta_{(a)}(\Omega).
\]
Proposition 5.2 The theta constants \( \vartheta(i^2j), \vartheta(ijk) \) and \( \vartheta(123456) \) are identically zero on \( j(\mathbb{B}^3\rangle \). The theta constants \( \vartheta(ij; kl; mn) \) are not identically zero on \( j(\mathbb{B}^3\rangle \).

Proof. We apply Fact 3.1 for 
\[
\tau = \Omega = j(x), \quad g = W = \begin{pmatrix} 0 & H \\ -H & -I_4 \end{pmatrix}, \quad (a, b) = (a, -aH).
\]
Note that 
\[
W \cdot \Omega = \Omega, \quad W \cdot (a, -aH) = \left( a - (3a - \frac{1}{2} \text{diag}(H)), -aH \right)
\]
and that 
\[
\phi(a, -aH)(W) = \frac{3}{2} a H^t a = \frac{1}{24} (6a) H^t (6a), \quad \det(C\Omega + D) = \omega.
\]
Since \( \kappa(W) \) is an 8-th root of 1, the sufficient condition for 
\[
\kappa(W) \exp(2\pi \sqrt{-1} \phi(a, b)(W)) \det(C\Omega + D)^{\frac{1}{2}} = 1 \tag{9}
\]
is \( (6a) H^t (6a) \equiv 2 \mod 24 \). If \( (6a) H^t (6a) \not\equiv 2 \mod 24 \), then \( \vartheta_{a, -aH}(\Omega) \) vanishes. Thus the theta constants \( \vartheta(i^2j), \vartheta(ijk) \) and \( \vartheta(123456) \) are identically zero on \( j(\mathbb{B}^3\rangle \).

For \( a = (\frac{1}{6}, \ldots, \frac{1}{6}) \) and \( x = (0, 0, 0, 1) \), \( \vartheta_{a, -aH}(\Omega) \) reduces to 
\[
\vartheta_{(\frac{1}{6}, \ldots, \frac{1}{6})}(\vartheta)(\frac{1}{6}, \ldots, \frac{1}{6})(-\omega^2),
\]
which does not vanish. Hence \( \vartheta(ij; kl; mn) \)'s survive. Note that \( \kappa(W)^2 = -1 \) by (9). []

Proposition 5.3 We have 
\[
\vartheta(i, i + 1; kl; mn)(\tilde{R}^\kappa_{i,i+1} \cdot j(x))^3 = -\chi(\tilde{R}^\kappa_{i,i+1}) \vartheta(i, i + 1; kl; mn)(j(x))^3,
\]
\[
\vartheta(ik; i + 1, l; mn)(\tilde{R}^\kappa_{i,i+1} \cdot j(x))^3 = \chi(\tilde{R}^\kappa_{i,i+1}) \vartheta(il; i + 1, k; mn)(j(x))^3,
\]
where 
\[
\chi(\tilde{R}^\kappa_{i,i+1}) = \left( \frac{t(R^\kappa_{i,i+1}x)H(R^\kappa_{i,i+1}x)}{txHx} \right)^{3/2},
\]
which takes 1 on the mirror of \( R^\kappa_{i,i+1} \).
Proof. For $\tilde{R}_{i,i+1}^\zeta = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$, straightforward calculation shows

$$\det(CJ(x) + D) = \frac{t'(R_{i,i+1}^\zeta x)H(R_{i,i+1}^\zeta x)}{\det(R_{i,i+1}^\zeta)} \cdot \frac{t'(R_{i,i+1}^\zeta x)H(R_{i,i+1}^\zeta x)}{\det(R_{i,i+1}^\zeta)} = \frac{t'(R_{i,i+1}^\zeta x)H(R_{i,i+1}^\zeta x)}{-\omega^2 t'xHx}.$$

By computing $\phi_{a,b}(\tilde{R}_{i,i+1}^\zeta)$ in Fact 5.1 and using (7), we have

$$\vartheta(i,i+1; kl; mn)(\tilde{R}_{i,i+1}^\zeta) \cdot (J(x))^3 = -c\chi(\tilde{R}_{i,i+1}^\zeta)\vartheta(i,i+1; kl; mn)(J(x))^3,$$

$$\vartheta(il; i+1, k; mn)(\tilde{R}_{i,i+1}^\omega) \cdot (J(x))^3 = c\chi(\tilde{R}_{i,i+1}^\zeta)\vartheta(il; i+1, k; mn)(J(x))^3,$$

where $c$ is a certain constant depending only on $\tilde{R}_{i,i+1}^\zeta$. If we restrict $J(x)$ on the mirror of $\tilde{R}_{i,i+1}^\zeta$, we have

$$\tilde{R}_{i,i+1}^\zeta \cdot (J(x)) = J(x), \quad \chi(\tilde{R}_{i,i+1}^\zeta) = \left(\frac{t'(R_{i,i+1}^\zeta x)H(R_{i,i+1}^\zeta x)}{t'xHx}\right)^{3/2} = 1.$$

Since $\vartheta(i,k; i+1, l; mn) = \vartheta(i,l; i+1, k; mn)$ on the mirror of $R_{i,i+1}^\zeta$ and it does not vanish, the constant $c$ must be 1.

Since $\tilde{R}_{pq}^\zeta$ can be expressed in terms of $\tilde{R}_{i,i+1}^\zeta$ and $\tilde{R}_{pq}^\omega = (\tilde{R}_{pq}^\zeta)^2$, we have the following two propositions.

**Proposition 5.4** We have

$$\vartheta(ij; kl; mn)(\tilde{R}_{pq}^\omega \cdot J(x))^3 = \chi(\tilde{R}_{pq}^\omega)\vartheta(ij; kl; mn)(J(x))^3,$$

where

$$\chi(\tilde{R}_{pq}^\omega) = \left(\frac{t'(R_{pq}^\omega x)H(R_{pq}^\omega x)}{t'xHx}\right)^{3/2},$$

which takes 1 on the mirror of $R_{pq}^\omega$.

**Proposition 5.5** The function $\vartheta(ij; kl; mn)(J(x))$ vanishes on the $\Gamma(1-\omega)$ orbits of the mirrors of $R_{ij}^\omega$, $R_{kl}^\omega$, and $R_{mn}^\omega$.

**Proof.** By Proposition 5.3, when we restrict $J(x)$ on the mirrors of $R_{12}^\omega$, $R_{34}^\omega$, and $R_{56}^\omega$, we have

$$\vartheta(12; 34; 56)(J(x))^3 = -\vartheta(12; 34; 56)(J(x))^3 = 0.$$

For the $\Gamma(1-\omega)$ orbits, use the previous proposition. In order to show for general $\vartheta(ij; kl; mn)(J(x))$'s, use Proposition 5.3.
6 The inverse of the period map

Proposition 6.1 Let $\Omega$ be the period matrix of

$$C(\lambda) : w^3 = z(z-1)(z-\ell_1)(z-\ell_2)(z-\ell_3)$$

given in Proposition 3.1. We have

$$\ell_1 = \frac{\vartheta^3(13; 24; 56)(\Omega)}{\vartheta^3(14; 23; 56)(\Omega)},$$  
$$\ell_2 = \frac{\vartheta^3(13; 25; 46)(\Omega)}{\vartheta^3(15; 23; 46)(\Omega)},$$  
$$\ell_3 = \frac{\vartheta^3(13; 26; 45)(\Omega)}{\vartheta^3(16; 23; 45)(\Omega)}.$$  

Proposition 6.2 For the period matrix $\Omega$ of $C(\lambda)$, linear and cubic relations among $\vartheta^3(ij; kl; mn)(\Omega)$ coincide with the defining equations of $Y \subset \mathbb{P}^{14}$:

$$\vartheta^3(ij; kl; mn)(\Omega) - \vartheta^3(ik; jl; mn)(\Omega) + \vartheta^3(il; jk; mn)(\Omega) = 0,$$

$$\vartheta^3(ij; kl; mn)(\Omega)\vartheta^3(ik; jn; lm)(\Omega)\vartheta^3(im; jl; kn)(\Omega) = \vartheta^3(ij; kn; lm)(\Omega)\vartheta^3(ik; jl; mn)(\Omega)\vartheta^3(im; jn; kl)(\Omega).$$

Propositions 6.1 and 6.2 imply the following.

Theorem 6.1 Let $\Theta$ be the map from $\mathbb{B}^3/\Gamma(1-\omega)$ to $Y$ defined by

$$x \mapsto [\ldots, y_{(ij;kl;mn)}, \ldots] = [\ldots, \vartheta^3(ij; kl; mn)(\varrho(x)), \ldots].$$

We have the following $S_6$-equivariant commutative diagram:

$$\Lambda \xrightarrow{\psi} \mathbb{B}^3/\Gamma(1-\omega)$$

$$\downarrow \Theta$$

$$Y \subset \mathbb{P}^{14}.$$  

In order to prove Propositions 6.1, 6.2 we state two facts in [Mum]: the one is Riemann’s theorem and the other is Abel’s theorem.
**Fact 6.1** We suppose $z$ is a fix point on the Jacobi variety $\text{Jac}(R)$ of a Riemann surface $R$ of genus $r$. The multi-valued function $\vartheta(z + \int_{P_0}^{P} \varphi, \tau)$ of $P$ on $X$ has $r$ zeros $P_1, \ldots, P_r$ provided not to be constantly zero, where $\varphi = (\varphi_1, \ldots, \varphi_r)$ is the normalized basis of the vector space of holomorphic 1-forms on $R$ such that $(\int_{B_i} \varphi_j)_{ij} = I_r$ for a symplectic basis $\{A_1, \ldots, A_r, B_1, \ldots, B_r\}$ of $H_1(R, \mathbb{Z})$, and $\tau = (\int_{A_i} \varphi_j)_{ij}$. Moreover, there exists a point $\Delta$ on $\text{Jac}(R)$ called Riemann’s constant such that

$$z = \Delta - \sum_{i=1}^{r} \int_{P_i}^{P_0} \varphi.$$ 

**Fact 6.2** Let $R$ be a Riemann surface of genus $r$ with an initial point $P_0$. Suppose $\sum_{i=1}^{d} P_i$ and $\sum_{i=1}^{d} Q_i$ be effective divisors of degree $d$ satisfying

$$\sum_{i=1}^{d} \int_{P_0}^{P_i} \varphi = \sum_{i=1}^{d} \int_{P_0}^{Q_i} \varphi,$$ 

where $\varphi$ is the normalized basis of vector space of holomorphic 1-forms on $R$. Then there exists a meromorphic function $f$ on $R$ such that

$$(f) = \sum_{i=1}^{d} Q_i - \sum_{i=1}^{d} P_i;$$

$f$ can be expressed as

$$f(P) = c \prod_{i=1}^{d} \vartheta(e + \int_{Q_i}^{P} \varphi, \tau) \prod_{i=1}^{d} \vartheta(e + \int_{P_i}^{P} \varphi, \tau),$$

where $c$ is a constant, $\tau$ is the period matrix of $R$, $e$ satisfies $\vartheta(e) = 0$,

$$\vartheta(e + \int_{P_i}^{P} \varphi, \tau) \neq 0, \quad \vartheta(e + \int_{Q_i}^{P} \varphi, \tau) \neq 0,$$

as multi-valued functions of $P$ on $R$, and paths from $P_i$ and $Q_i$ to $P$ are the inverse of the paths in (15) followed by a common path from $P_0$ to $P$. 

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Proof of Proposition 6.1. We take $R$ as $C : w^3 = z(z - 1)(z - \ell_1)(z - \ell_2)(z - \ell_3)$ with the initial point $P_0 = (0, 0)$ and put $P_\infty = (\infty, \infty)$, $P_1 = (1, 0)$, $P_i = (\ell_i, 0)$ $(i = 1, 2, 3)$. Let us define a meromorphic function $f$ on $C$ by $(f) = 3P_0 - 3P_\infty$. We construct a meromorphic function on $C$ with poles $3P_\infty$ and zeros $3P_0$ by following the recipe given in Fact 6.2. Let $\gamma_i(z_1, z_2)$ $(i = 1, 2, 3)$ be a path in $C$ from $(z_1, w_1)$ to $(z_2, w_2)$ in the $i$-th sheet. Since $\omega^2 + \omega + 1 = 0$, we have $3 \sum_{i=1}^3 \int_{\gamma_i(0, \infty)} \varphi = (0, 0, 0, 0) = 3 \int_{P_0} \varphi$ for three paths $\gamma_i(0, \infty)$ from $P_0$ to $P_\infty$. We give the following table:

| Path | Expression |
|------|------------|
| $\int_{\gamma_1(\infty, 0)} \varphi$ | $\frac{1}{3} \int A_1 - B_1 \varphi$ |
| $\int_{\gamma_2(\infty, 0)} \varphi$ | $\frac{1}{3} \int A_1 + 2B_1 \varphi$ |
| $\int_{\gamma_3(\infty, 0)} \varphi$ | $\frac{1}{3} \int A_1 + A_2 - A_4 + B_1 - 2B_2 + 2B_4 \varphi$ |
| $\int_{\gamma_1(0, 1)} \varphi$ | $\frac{1}{3} \int A_1 + A_2 - A_4 + 2B_1 - B_2 - B_4 \varphi$ |
| $\int_{\gamma_2(0, 1)} \varphi$ | $\frac{1}{3} \int A_1 + 2A_2 + A_4 - B_1 - B_2 - B_4 \varphi$ |
| $\int_{\gamma_3(0, 1)} \varphi$ | $\frac{1}{3} \int A_1 - 2A_2 + 2A_4 - B_1 - B_2 - B_4 \varphi$ |
| $\int_{\gamma_1(\ell_1, \ell_2)} \varphi$ | $\frac{1}{3} \int -2A_2 + A_3 + 2A_4 - B_2 + 2B_3 - B_4 \varphi$ |
| $\int_{\gamma_2(\ell_1, \ell_2)} \varphi$ | $\frac{1}{3} \int A_2 - B_2 \varphi$ |
| $\int_{\gamma_3(\ell_1, \ell_2)} \varphi$ | $\frac{1}{3} \int A_3 - B_3 \varphi$ |

Put $e = \frac{1}{6} \int 3A_1 + A_2 + 3A_3 + 5A_4 - 3B_1 - B_2 - 3B_3 + 5B_4 \varphi$, corresponding to the characteristic $\frac{1}{6}(3, 1, 3, 5)$ with label $(123)$, and define a meromorphic function $F$ of $P = (z, w)$ on $C$ as

$$F(P) = \frac{\vartheta \left( e + \int_{\gamma_1(0, z)} \varphi, \Omega \right)^3}{\prod_{i=1}^3 \vartheta \left( e + \int_{\gamma_i(\infty, 0)} + \gamma_i(0, z) \varphi, \Omega \right)}, \quad (16)$$
where $\Omega$ is the period matrix of $C$. Since $\vartheta(123)$ vanishes, we have $\vartheta(e) = 0$. We check that neither the denominator nor the numerator of $F$ identically vanishes. We put $P = P_1, P_2, P_3$ and use (6) and (7), then we have

\[
F(P_1) = cf(P_1) = c\ell_1 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11} + 1)\right] \frac{\vartheta^3_{[-1,-1,1,1]}(\Omega)}{\vartheta^3_{[1,1,1,1]}(\Omega)},
\]

\[
F(P_2) = cf(P_2) = c\ell_2 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11} + 1)\right] \frac{\vartheta^3_{[-1,-1,1,1]}(\Omega)}{\vartheta^3_{[1,1,1,1]}(\Omega)},
\]

\[
F(P_3) = cf(P_3) = c\ell_3 = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11} + 1)\right] \frac{\vartheta^3_{[-1,1,1,1]}(\Omega)}{\vartheta^3_{[1,1,1,1]}(\Omega)},
\]

where $c$ is a constant depending on $\Omega$. By Proposition 5.2, neither the denominator nor the numerator of $F$ identically vanishes.

We put $P = P_\infty, P_0, P_1$ the denominator and the numerator of $F$ vanish at these points by Proposition 5.2. Since $(F) = 3P_0 - 3P_\infty$, $P_\infty$ and $P_0$ are zeros of higher order of the denominator and numerator of $F$, respectively. The number of zeros of the denominator and numerator of $F$ are 4 by Fact refRiemann, thus $P_1$ is a simple zero. We consider $\lim_{P \to P_1} F(P)$. Let $t$ be a local coordinate for $P$ around $P_1$ and $z(t)$ be $\int_{P_1}^P \varphi$. We have

\[
F(P) = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11} - 2)\right] \frac{\vartheta^3_{[-1,-3,3,1]}(z(t), \Omega)}{\vartheta^3_{[1,3,3,3]}(z(t), \Omega)}.
\]

When $P \to P_1$, we have $t \to 0$ and $z(t) \to (0, 0, 0, 0)$. Since $t = 0$ is simple zero, we have

\[
\lim_{t \to 0} \frac{\vartheta^3_{[-1,3,3,3]}(z(t), \Omega)}{\vartheta^3_{[1,3,3,3]}(z(t), \Omega)} = \lim_{t \to 0} \frac{\vartheta^3_{[1,3,3,3]}(-z(t), \Omega)}{\vartheta^3_{[1,3,3,3]}(z(t), \Omega)} = -1,
\]

which implies $c = \exp\left[\frac{\pi\sqrt{-1}}{3}(2\Omega_{11} + 1)\right]$. Hence we have the expressions (11), (11) and (12).

**Proof of Proposition 6.2.** In order to obtain a cubic relation among $\vartheta^3(ij; kl; mn)$'s, put

\[
e = \frac{1}{6} \int 3A_1 + 5A_2 + 3A_3 + 5A_4 - 3B_1 - 5B_2 - 3B_3 + 5B_4 \varphi,
\]

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corresponding to the characteristic $\frac{1}{6}(3, 5, 3, 5)$ with label (124), then $\vartheta(e) = 0$; and define a meromorphic function $F$ by (16). We have

$$F(P_1) = cf(P_1) = c = \exp\left[\frac{\pi \sqrt{-1}}{3} (2\Omega_{11} + 1)\right] \frac{\vartheta^3_{[-1, 3, -3]}(\Omega)}{\vartheta^3_{[1, 3, 3]}(\Omega)},$$

$$F(P_2) = cf(P_2) = c\ell_2 = \exp\left[\frac{\pi \sqrt{-1}}{3} (2\Omega_{11} + 1)\right] \frac{\vartheta^3_{[-1, -1, -1]}(\Omega)}{\vartheta^3_{[1, -1, -1]}(\Omega)},$$

$$F(P_3) = cf(P_3) = c\ell_3 = \exp\left[\frac{\pi \sqrt{-1}}{3} (2\Omega_{11} + 1)\right] \frac{\vartheta^3_{[-1, 1, -1]}(\Omega)}{\vartheta^3_{[1, -1, -1]}(\Omega)},$$

and

$$c\ell_1 = cf(P_{\ell_1}) = \lim_{P \to P_{\ell_1}} F(P) = \exp\left[\frac{\pi \sqrt{-1}}{3} (2\Omega_{11} - 2)\right] \frac{\vartheta^3_{[-1, -3, -3]}(\int_{P_{\ell_1}}^P \varphi, \Omega)}{\vartheta^3_{[1, 3, 3]}(\int_{P_{\ell_1}}^P \varphi, \Omega)} = \exp\left[\frac{\pi \sqrt{-1}}{3} (2\Omega_{11} + 1)\right].$$

These imply

$$\ell_1 = \frac{cf(P_{\ell_1})}{cf(P_1)} = \frac{\vartheta^3(13; 24; 56)(\Omega)}{\vartheta^3(14; 23; 56)(\Omega)},$$

$$\ell_2 = \frac{cf(P_{\ell_2})}{cf(P_1)} = \frac{\vartheta^3(14; 25; 36)(\Omega)\vartheta^3(13; 24; 56)(\Omega)}{\vartheta^3(15; 24; 36)(\Omega)\vartheta^3(14; 23; 56)(\Omega)},$$

$$\ell_3 = \frac{cf(P_{\ell_3})}{cf(P_1)} = \frac{\vartheta^3(14; 26; 35)(\Omega)\vartheta^3(13; 24; 56)(\Omega)}{\vartheta^3(16; 24; 35)(\Omega)\vartheta^3(14; 23; 56)(\Omega)}.$$

Compare with the above expression of $\ell_2$ and (11), we have a cubic relation among the $\vartheta^3(ij; kl; mn)$’s. By letting $\tilde{\Gamma}$ act on theta constants, we have more cubic relations among $\vartheta^3(ij; kl; mn)$’s.

Let us lead a linear relation among the $\vartheta^3(ij; kl; mn)$’s. We start with the meromorphic function $f' : (z, w) \mapsto z - 1$; note that $(f') = 3P_1 - 3P_\infty$. Put

$$e = \frac{1}{6} \int_{3A_1 + A_2 + 3A_3 + 5A_4 - 3B_1 - B_2 - 3B_3 + 5B_4} \varphi,$$

corresponding to the characteristic $\frac{1}{6}(3, 1, 3, 5)$ with label (123), and define a
meromorphic function $F'$ of $P = (z, w)$ on $C$ as

$$F'(P) = \frac{\prod_{i=1}^3 \vartheta \left( e + \int_{\gamma_i(1,0)+\gamma_i(0,z)} \varphi, \Omega \right)}{\prod_{i=1}^3 \vartheta \left( e + \int_{\gamma_i(\infty,0)+\gamma_i(0,z)} \varphi, \bar{\Omega} \right)}.$$ 

Since $\vartheta(123)$ vanishes, we have $\vartheta(e) = 0$. We consider $\lim_{P \to P_0} F'(P)$ and put $P = P_{\ell_1}$ then we have

$$F'(P_0) = c f'(P_0) = -c = \lim_{P \to P_0} K \exp\left[\frac{4\pi \sqrt{-1}}{3} \vartheta_{[-5,-1,-3,-5]}(\int_{P_0}^{P} \varphi, \Omega) \right] \vartheta_{[5,1,3,5]}(\int_{P_0}^{P} \varphi, \Omega),$$

$$F'(P_{\ell_1}) = c f'(P_{\ell_1}) = c(\ell_1 - 1) = K \exp\left[\frac{4\pi \sqrt{-1}}{3} \vartheta_{[3,3,3,-1]}(\Omega) \right] \vartheta_{[1,-1,3,3]}(\Omega),$$

where

$$K = \exp\left[-\frac{2\pi \sqrt{-1}}{3} e' \Omega \, ^t(e' - e_1) + \frac{4\pi \sqrt{-1}}{3} e' H \, ^t(e' - e_1) \right]$$

and $e' = (1, -1, 0, 1)$. Now we have the expression

$$\ell_1 - 1 = \frac{\vartheta^3(12; 34; 56)(\Omega)}{\vartheta^3(14; 23; 56)(\Omega)}.$$

Since we had in (10)

$$\ell_1 = \frac{\vartheta^3(13; 24; 56)(\Omega)}{\vartheta^3(14; 23; 56)(\Omega)},$$

we get a relation

$$\frac{\vartheta^3(12; 34; 56)(\Omega)}{\vartheta^3(14; 23; 56)(\Omega)} - \frac{\vartheta^3(13; 24; 56)(\Omega)}{\vartheta^3(14; 23; 56)(\Omega)} + 1 = 0,$$

which is equivalent to

$$\vartheta^3(12; 34; 56)(\Omega) - \vartheta^3(13; 24; 56)(\Omega) + \vartheta^3(14; 23; 56)(\Omega) = 0.$$

Action of $\bar{\Gamma}$ produces other the linear relations among the $\vartheta^3(ij; kl; mn)$'s.
7 Appendix

In this section, we give a geometrical meaning of the label of a’s. In order to do this, we determine Riemann’s constant $\Delta$.

**Fact 7.1**  Riemann’s constant $\Delta$ is given by

$$
\Delta = \sum_{i=1}^{m+r-1} \int_{P_0}^{P_i} \varphi - \sum_{j=1}^{m} \int_{P_0}^{Q_j} \varphi
$$

(17)

for a certain divisor $D_0 = \sum_{i=1}^{m+r-1} P_i - \sum_{j=1}^{m} Q_j$ such that $2D_0$ is linearly equivalent to the canonical divisor of $R$. It is easy to see that Riemann’s constant $\Delta$ is a half period on $\text{Jac}(R)$ if and only if $(2r-2)P_0$ is a canonical divisor.

For our case, Riemann’s constant $\Delta$ is a half period on $\text{Jac}(C(\lambda))$ since we have $6P_0 = (\varphi_4)$ for any $C(\lambda)$.

**Proposition 7.1**  Riemann’s constant $\Delta$ is invariant under the action of the monodromy group $\tilde{\Gamma}(1-\omega)$. Hence we have

$$
\Delta = (\frac{1}{2}, \ldots, \frac{1}{2}).
$$

*Proof.* Let $\gamma$ be a closed path in $\Lambda$ and $g \in \tilde{\Gamma}(1-\omega)$ be its representation. Since $\Delta$ is a half period point of $\text{Jac}(C(\lambda))$, it is expressed by $c = (c_1, \ldots, c_8)$ $(c_i \in \{0, 1/2\})$. When $\lambda$ moves a little, this vector is invariant and presents $\Delta$. By the continuation along $\gamma$, $\Delta$ is presented by the vector $c$ with respect to the transformed homology basis by $g$; i.e., it is presented by $g \cdot c$ with respect to the initial homology basis.

On the other hand, $\Delta$ is invariant as a point of $\text{Jac}(C(\lambda))$ under the continuation along $\gamma$ with respect to the initial basis by the expression [L7]. Thus we have $g \cdot c = c$. There is only one half characteristic $(\frac{1}{2}, \ldots, \frac{1}{2})$ invariant under $\tilde{\Gamma}(1-\omega)$. [\[\]

By straightforward calculation, we have the following proposition giving a geometrical meaning of the label of a’s.
Proposition 7.2 The points \((a, -aH)\) of \(Jac(C)\) for \(a\) with label \((ijk)\) and \((i^2j)\) are expressed as

\[
\Delta - \int_{P_0}^{P_{\lambda_1}} \varphi - \int_{P_0}^{P_{\lambda_3}} \varphi - \int_{P_0}^{P_{\lambda_4}} \varphi,
\]

\[
\Delta - 2 \int_{P_0}^{P_{\lambda_1}} \varphi - \int_{P_0}^{P_{\lambda_3}} \varphi,
\]

respectively.

We have the necessary and sufficient condition for \(\vartheta(z, \tau) = 0\).

Fact 7.2 For a period matrix \(\tau\) of Riemann’s surface \(R\) of genus \(r\), \(\vartheta(z, \tau) = 0\) if and only if there exists an effective divisor \(\sum_{i=1}^{r-1} P_i\) such that

\[
z = \Delta - \sum_{i=1}^{r-1} \int_{P_0}^{P_i} \varphi.
\]

Proposition 7.3 The theta constant \(\vartheta(ij; kl; mn)(j(x))\) vanishes only on the \(\Gamma(1 - \omega)\) orbit of the mirrors of \(R_{ij}^\omega\), \(R_{kl}^\omega\) and \(R_{mn}^\omega\).

Proof. The function \(\vartheta(13; 24; 56)(j(x))\) is a non-zero constant times

\[
\vartheta(\Delta - \int_{P_0}^{P_{\infty}} \varphi - \int_{P_0}^{P_0} \varphi - \int_{P_0}^{P_1} \varphi + \int_{P_0}^{P_2} \varphi, j(x)).
\]

By the previous fact, \(\vartheta(13; 24; 56)(j(x)) = 0\) if and only if there exists an effective divisor \(Q_1 + Q_2 + Q_3\) such that

\[
Q_1 + Q_2 + Q_3 \equiv P_{\infty} + 2P_0 + P_1 - P_{\ell_1} = E.
\]

By the Riemann-Roch theorem, the dimension of vector space of meromorphic functions \(f\) such that \((f) + E \geq 0\) is equal to that of meromorphic 1-forms \(\phi\) such that

\[
(\phi) - E \geq 0.
\]

Since we have

\[
(\varphi_1) = P_{\infty} + P_0 + P_1 + P_{\ell_1} + P_{\ell_2} + P_{\ell_3}, \quad (\varphi_2) = 6P_{\infty},
\]

\[
(\varphi_3) = 3P_{\infty} + 3P_0, \quad (\varphi_4) = 6P_0,
\]

there does not exist a meromorphic 1-form satisfying (18). Thus if \(\lambda \in \Lambda\) then no effective divisor \(Q_1 + Q_2 + Q_3\) such that \(Q_1 + Q_2 + Q_3 \equiv E\).

The zeros of theta constants on mirrors are studied in [Shi], which yields this proposition. □
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Figure 1. Our basis of $H_1(C,\mathbb{Z})$