Possible textures of neutrino and charged lepton mass matrices

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Abstract

We study a variety of different texture patterns in the basis in which the charged lepton mass matrix or neutrino mass matrix or neither are diagonal. The experimental results on the neutrinos provide sufficient restrictions to allow only a small number of simple patterns. We discuss particularly the texture zeroes which provide important relations among the mass eigenvalues and the mixing matrix.
1 Introduction

Recent results from superKamiokande on the atmospheric neutrino experiments \[1\] has provided definite evidence for neutrino mass. It has also been established that the solution of the atmospheric neutrino anomaly requires a small mass squared difference between $\nu_\mu$ and $\nu_\tau$ with almost maximal mixing. Oscillations to sterile neutrinos is ruled out at 90% confidence level \[2\]. Another independent source of information on the mass of the neutrinos are the solar neutrino oscillations \[3\]. The latest results from superKamiokande (mainly the day-night spectrum, which rules out the small angle MSW solution and the vacuum oscillation solutions) \[4\] and the analysis of the solar neutrino data \[5\] have narrowed down the allowed region of the parameter space to only one small region. Although the mixing angle is very large, maximal mixing is not allowed. The oscillation of $\nu_e$ into a sterile neutrino is also disfavoured.

Table 1: Present experimental constraints on neutrino masses and mixing

| Neutrino Type          | Mass Squared Difference | Mixing Angle Constraints |
|------------------------|-------------------------|--------------------------|
| Solar Neutrino         | $\Delta m^2 \sim (2.5 - 15) \times 10^{-5} eV^2$ | $0.25 < \sin^2 2\theta < 0.65 \implies 0.26 < \sin \theta < 0.45$ |
| Atmospheric Neutrino   | $\Delta m^2_{\mu\tau} \sim (1.5 - 5) \times 10^{-3} eV^2$ | $\sin^2 2\theta_{\mu\tau} > 0.88 \implies \sin \theta_{\mu\tau} > 0.57$ |
| Neutrinoless Double Beta Decay | $m_{\nu_e} < 0.2 eV$ | |
| CHOOZ                  | $\Delta m^2_{eX} < 7 \times 10^{-4} eV^2$ for $\sin \theta_{eX} \sim 1$ | `or $\sin^2 2\theta_{eX} < 0.1 \implies \sin \theta_{eX} < 0.16$` |

In addition to these results, there are also some bounds from the laboratory experiments. The long baseline reactor experiment, CHOOZ, has set stringent bounds on the disappearances of the $\nu_e$ \[6\]. In the situation under consideration with no sterile neutrinos, a small mass difference between $\nu_e$ and one of the combinations of $\nu_\mu$ and $\nu_\tau$ can explain the solar neutrino anomaly. The other combination will then have a mass squared difference of the order of
the $\nu_\mu$ and $\nu_\tau$ mass squared difference. For a mass squared difference of this amount, the CHOOZ result gives a strong bound on the mixing angle. We also assume that the neutrinos are Majorana particles, so that the lepton number violation at a large scale can also explain the baryon asymmetry of the universe [7]. This then implies that there is constraint on the $\nu_e$ mass from the non-observation of the neutrinoless double beta decay [8]. All these constraints are summarized in table 1.

There have been several attempts to obtain textures of neutrino mass matrices and to relate them with the quark sector [11], However, most of these studies have been based on specific models that tried to get maximal mixing for the solar neutrino solution.

We will study the structures of the neutrino and charged lepton mass matrices in a model-independent way paying attention to any symmetry or regularity, particularly, the presence of any texture zeroes. We will also take into account the recent result that there is only one solution to the solar neutrino problem, the one which does not allow maximal mixing. As it turns out there are then very few choices that are left for us.

The atmospheric neutrino anomaly is explained by a maximal mixing of $\nu_\mu$ and $\nu_\tau$. We thus assume that the mass eigenvalues $m_2$ and $m_3$ correspond to the states $\nu_2$ and $\nu_3$, which are admixtures of the states $\nu_\mu$ and $\nu_\tau$. The mass squared difference required by the atmospheric neutrino anomaly then can be written as,

$$\Delta_{atm} = \Delta m^2_{\mu\tau} = \Delta m^2_{23} = m^2_3 - m^2_2. \tag{1}$$

The corresponding mixing angle is written as, $\sin \theta_{\mu\tau}$, which is almost maximal. The solar neutrino problem has a MSW solution [9], in which $\nu_1$ oscillates into $\nu_2$ with a mass squared difference,

$$\Delta_{sol} = \Delta m^2_{e2} = \Delta m^2_{12} = m^2_2 - m^2_1. \tag{2}$$

The corresponding mixing angle is $s \sim \sin \theta_{e2}$. The third mass squared difference is then determined in terms of these two mass differences. The corresponding mixing angle $U_{e3}$ is then constrained by the CHOOZ data.

We shall consider the mixing angle for the atmospheric neutrino anomaly to be maximal, $\sin \theta_{\mu\tau} \sim 1/\sqrt{2}$. However, we will also consider $\sin \theta_{\mu\tau} < 0.57$ which is experimentally allowed. Similarly, if we assume, for simplicity, $U_{e3} = 0$, the neutrino mixing matrix becomes,

$$U_\nu = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ s & 0 & 1 \end{pmatrix} \tag{3}$$
where \( c = \sqrt{1 - s^2} \). With this restrictive mixing matrix we find that there is no neutrino mass matrix with any texture zeroes (in the basis where the charged lepton is diagonal). For exact bi-maximal solutions, \( i.e. \), with \( s = c = 1/\sqrt{2} \) there are solutions with texture zeroes, but when we consider the present bound on \( s \), there are no solutions. So, to generalise our analysis we would also allow \( U_{e3} \) to be non-vanishing, but constrained to be small.

In our present analysis we shall not include CP violations. The mass matrices and the mixing matrices would then be real. For mass matrices, \( M_e \) and \( M_\nu \) and the corresponding charged current interaction, the Lagrangian is given by,

\[
L = \frac{g}{\sqrt{2}} \ell_m L \gamma^\mu \nu_{\mu L} W^-_{\mu} - \bar{\ell}_m L M'_{emn} \ell_n R - \nu_{mL} M_{\nu mn} \nu_{nL}.
\] (4)

The charged lepton mass matrix is not symmetric, but can be diagonalised by a bi-unitary transformation

\[
V_{emR}^{\dagger} M_{emn} V_{enL} = M_{\alpha e} \delta_{\alpha \beta},
\]

where \( \alpha, \beta = e, \mu, \tau \) are the physical states. Since the right handed mixing matrix \( V_{emaR} \) does not enter into the charged current interactions, we can write \( V_{emaR} = K_{emn} V_{maL} \) and without loss of generality symmetrize the charged lepton mass matrix. The new symmetric charged lepton mass matrix, \( M_{emn} = M_{eml}' K_{eln} \) can now be diagonalised by the unitary matrix,

\[
V_{emL}^{\dagger} M_{emn} V_{enL} = M_{\alpha e} \delta_{\alpha \beta}.
\]

The Majorana mass matrix of the neutrinos is always symmetric and hence can be diagonalised by a unitary transformation,

\[
V_{\nu im}^{\dagger} M_{\nu mn} V_{\nu nj} = M_{\nu i} \delta_{ij}.
\] (5)

Since we are working with real matrices, all unitary matrices can be replaced by orthogonal matrices and we have not introduced the Majorana phase matrix.

In this flavour basis, when the charged lepton mass matrix is diagonal, the charged current interaction contains the mixing matrix \( U_{\nu ai} \) which is given in equation (3). This mixing matrix \( U_{\nu ai} \), which is measured from experiment can be written in terms of the unitary matrices which diagonalise the mass matrices \( V_{ema} \) and \( V_{vim} \) as,

\[
U_{\nu ai} = V_{em}^{\dagger} V_{\nu mi}.
\] (6)

This enters in the charged current interactions of the physical charged lepton states with the neutrino mass eigenstates. This is analogous to the Cabibbo-Kobayashi-Maskawa matrix.
of the quark sector. In the basis in which the charged lepton mass matrix is diagonal, the mixing matrix is simply \( U_\nu = V_\nu \), while in the basis in which the neutrino mass matrix is diagonal, the mixing matrix is given by, \( U_\nu = V_\nu^T \).

\section{Diagonal Charged Lepton Mass Matrix}

We shall first consider the neutrino mass matrices in the basis in which the charged lepton mass matrix is diagonal. A general form of the mass matrix, which gives the mixing matrix of equation (3), may be obtained from a diagonal neutrino mass, \( M_\nu^{\text{diag}} = \text{Diag.}\{m_1, m_2, m_3\} \) as,

\[ M_e = M_\nu^{\text{diag}} \]  

and

\[ M_\nu = U_{\nu i} M_\nu^{\text{diag}} U_{\nu i}^T = \begin{pmatrix} m + \Delta \cos 2\theta & \frac{1}{\sqrt{2}} \Delta \sin 2\theta & \frac{1}{\sqrt{2}} \Delta \sin 2\theta \\ \frac{1}{\sqrt{2}} \Delta \sin 2\theta & \frac{1}{2} (M - \Delta \cos 2\theta) & \frac{1}{2} (\Delta_3 - \Delta \cos 2\theta) \\ \frac{1}{\sqrt{2}} \Delta \sin 2\theta & \frac{1}{2} (\Delta_3 - \Delta \cos 2\theta) & \frac{1}{2} (M - \Delta \cos 2\theta) \end{pmatrix}, \]  

where, \( m = (m_1 + m_2)/2, \Delta = (m_1 - m_2)/2, M = m + m_3, \Delta_3 = m - m_3 \) and \( \cos 2\theta = c^2 - s^2 \).

Although this matrix does not allow any texture zeroes, it is instructive in the sense that it gives \( M_{12} = M_{13} \) and \( M_{22} = M_{33} \). Moreover, if we require an inverted hierarchy with a very small \( m_3 \) then we also have \( C = D \). Using such symmetries, several forms of the neutrino mass matrix can be obtained. We list a few possible mass matrices in table 2 along with the mixing matrix and eigenvalues for a representative set of parameters.

The mass matrix

\[ \begin{pmatrix} 0 & a & a \\ a & 0 & b \\ a & b & 0 \end{pmatrix} \]

has been studied [10], that has texture zeroes along the diagonal. This is pattern (A) in table 2. For the choice, \( m_1 \approx -m_2 \), we have \( \Delta \neq 0 \) and \( m \approx 0 \) and, therefore, for the (11) element to vanish we must have \( \cos 2\theta \approx 0 \). This leads to bi-maximal mixing matrix. Although this has maximum number of texture zeroes, this mass matrix cannot give non-maximal mixing angle indicated by the solar neutrino problem.
Table 2: Neutrino mass matrices and the corresponding mixing matrices for some representative choice of parameters with $M_{
u}$ diagonal. In all these cases, the mass differences lie in the range $\Delta m_{12}^2 \sim (4.5 - 8) \times 10^{-5}$ eV$^2$ and $\Delta m_{23}^2 \sim (3.2 - 3.6) \times 10^{-3}$ eV$^2$.

| Mass matrices | Parameters | Masses in eV | Mixing matrices |
|---------------|------------|--------------|-----------------|
| $\begin{pmatrix} 0 & a & a \\ a & 0 & b \\ a & b & 0 \end{pmatrix}$ | $a = 0.04$ | $m_1 = 0.05682$ | $\begin{pmatrix} 0.71 & 0.71 & 0 \end{pmatrix}$ |
| A             | $b = 0.00025$ | $m_2 = -0.05632$ | $\begin{pmatrix} 0.50 & 0.50 & 0.71 \end{pmatrix}$ |
| $\begin{pmatrix} a & b & b \\ b & b & -b \\ b & -b & b \end{pmatrix}$ | $a = 0.00025$ | $m_1 = 0.05682$ | $\begin{pmatrix} 0.71 & 0.71 & 0 \end{pmatrix}$ |
| B             | $b = 0.04$ | $m_2 = -0.05632$ | $\begin{pmatrix} 0.50 & 0.50 & 0.71 \end{pmatrix}$ |
| $\begin{pmatrix} a & a & a \\ a & b & -b \\ a & -b & b \end{pmatrix}$ | $a = 0.005$ | $m_1 = 0.01$ | $\begin{pmatrix} 0.82 & 0.58 & 0 \end{pmatrix}$ |
| C             | $b = 0.03$ | $m_2 = -0.005$ | $\begin{pmatrix} 0.41 & 0.58 & 0.71 \end{pmatrix}$ |
| $\begin{pmatrix} 0 & a & a \\ a & b & -b \\ a & -b & c \end{pmatrix}$ | $a = 0.0014$ | $m_1 = -0.0011$ | $\begin{pmatrix} 0.93 & 0.36 & 0.007 \end{pmatrix}$ |
| D             | $b = 0.025$ | $m_2 = 0.007$ | $\begin{pmatrix} 0.28 & 0.74 & -0.60 \end{pmatrix}$ |
| $\begin{pmatrix} 0 & a & 0 \\ a & b & b \\ 0 & b & c \end{pmatrix}$ | $a = 0.003$ | $m_1 = 0.0009$ | $\begin{pmatrix} 0.92 & 0.38 & 0.03 \end{pmatrix}$ |
| E             | $b = 0.03$ | $m_2 = 0.0055$ | $\begin{pmatrix} 0.30 & 0.70 & 0.65 \end{pmatrix}$ |
| $\begin{pmatrix} 2s^4 & \sqrt{2}s^3 & \sqrt{2}s^3 \\ \sqrt{2}s^3 & 1 + s^2 & s^2 - 1 \\ \sqrt{2}s^3 & s^2 - 1 & 1 + s^2 \end{pmatrix}$ | $m = 0.03$ | $m_1 = 0$ | $\begin{pmatrix} -0.93 & 0.35 & 0 \end{pmatrix}$ |
| F             | $m = 0.03$ | $m_2 = 0.009$ | $\begin{pmatrix} 0.25 & 0.66 & -0.71 \end{pmatrix}$ |
For another simple pattern given in (B) in table 2,
\[
\begin{pmatrix}
  a & b & b \\
  b & b & -b \\
  b & -b & b \\
\end{pmatrix},
\]
the choice of \( m_1 \approx -m_2 \) (therefore \( \Delta \neq 0 \)) and \( m_3 \gg m_1 \) leads to \( m \approx 0 \), \( M \approx m_3 \), and \( \Delta \approx -m_3 \). Therefore, for (22) and (23) matrix elements to have opposite signs as above, we once again have \( \cos 2\theta \approx 0 \), which leads to bi-maximal mixing. A judicious choice of the parameters \( m_1, m_2 \) and \( m_3 \) can then be made to have the (12) and (22) elements the same.

Pattern (C), because of the choice of \( m_1, m_2, m_3 \), is an example which does not allow bi-maximal mixing. The simplest of the mass matrices with one and two texture zeroes are given, respectively, by
\[
\begin{pmatrix}
  0 & a & a \\
  a & b & -b \\
  a & -b & c \\
\end{pmatrix},
\]
which is pattern (D), and
\[
\begin{pmatrix}
  0 & a & 0 \\
  a & b & b \\
  0 & b & c \\
\end{pmatrix},
\]
which is pattern (E). We note that in order to fit the neutrino data (particularly, \( \theta_{\mu\tau} \approx 45^\circ \)), the structure of both the matrices must be such that \( c \neq b \). That is, the (33) element is no longer equal to the (22) element as implied by (8). As a consequence, one finds that neither is \( U_{e3} = 0 \). If \( U_{e3} \) were to vanish then there is no solution with (non-diagonal) texture zeroes.

### 3 Diagonal neutrino mass matrix

We next consider the case where the charged lepton mass matrix is not diagonal. The simplest situation would be where the neutrino mass matrix is diagonal and the entire mixing matrix comes from the diagonalisation of the charged lepton mass matrix. This is what happens in the case of democratic mass matrix [12], in which case the charged lepton mass matrix is democratic with rank one. However, in that case the mixing angle for the solar neutrino case comes out to be maximal, which is now ruled out at the 95% confidence level. Since
the neutrino mass matrix is diagonal, the mixing matrix is the transpose of the matrix diagonalising the charged lepton mass matrix,

$$V_e = U_\nu^T = \begin{pmatrix} c & s & s \sqrt{2} \\ -s & c & -s \sqrt{2} \\ 0 & 0 & 1 \end{pmatrix}. \quad (9)$$

Using this mixing matrix, we can now write down the most general charged lepton mass matrix (pattern 1 in table 3) in terms of the mass eigenvalues as,

$$M_\nu = M_\nu^{\text{diag}} \quad \text{and} \quad M_e = V_e M_\nu^{\text{diag}} V_e^T = \begin{pmatrix} m_+ s^2 + m_e c^2 & (m_+ - m_e) sc & m_- s \\ (m_+ - m_e) sc & m_+ c^2 + m_e s^2 & m_- c \\ m_- s & m_- c & m_+ \end{pmatrix} \quad (10)$$

where $M_\nu^{\text{diag}} = \text{Diag}\{m_e, m_\mu, m_\tau\}$, $m_+ = (m_\mu + m_\tau)/2$ and $m_- = (m_\mu - m_\tau)/2$.

From the above expressions a few points become clear. None of the elements in $M_e$ could vanish and be consistent with the above mixing matrix. The smallness of the electron mass $m_e$ implies that the various elements of the mass matrix have to be given very precisely in order to get, simultaneously, the mass hierarchy between the charged lepton masses and the required mixing matrix. So, this form of the mass matrix is unlikely to predict any texture zeroes. In other words, it is not possible for texture zeroes to exist without invoking new parameters, which have to be fine tuned to get the required mixing matrix. A simple vanishing of any elements of the mass matrix will, therefore, not allow us to get the required mixing matrix.

### 4 General textures

Let us consider the case where one of the mixing angles comes from the charged lepton mass matrix and the other from the diagonalisation of the neutrino mass matrix \cite{ref13}. If we now require that the mixing angle of the atmospheric neutrino anomaly comes from the neutrino sector and the (almost) maximal mixing angle for the solar neutrino comes from the charged lepton mixing matrix, then again we need fine tuning. For any of the elements of the charged lepton mass matrix to vanish, it is not possible to get a solution, since the maximal mixing in the neutrino sector would then make the $U_{e3}$ element large. The only possibility, therefore, is to get maximal mixing between $\nu_\mu$ and $\nu_\tau$ from the charged lepton sector. In this case, in the above charged lepton mass matrix we can put $s = 0$ allowing the mixing in the $\nu_e$ to $\nu_\mu$.
Table 3: Patterns for $M_\nu$ and $M_e$ for different textures which are consistent with the experimental values of the mixing matrix $U_\nu$. We defined $m_\pm = (m_\mu \pm m_\tau)/2$, $m_{e\tau} = \sqrt{m_e m_\tau}$, $m_{e\mu} = \sqrt{m_e m_{\mu}}$, $\tilde{m} = \sqrt{2} s_{12}$ and $\tilde{m}_3 = \frac{\Delta}{2} s_{12}$ and considered $s \approx 0.35$.

|   | $M_\nu$ | $M_e$ | $U_\nu$ |
|---|---------|-------|---------|
| 1* | $\begin{pmatrix} m_1 & 0 & 0 \\ 0 & m_2 & 0 \\ 0 & 0 & m_3 \end{pmatrix}$ | $\begin{pmatrix} a & b & m_+ s \\ b & d & m_- c \\ m_- s & m_- c & m_+ \end{pmatrix}$ | same as (3) |
| 2 | $\frac{m_2}{2} \begin{pmatrix} s^2 & sc & 0 \\ sc & c^2 & 0 \\ 0 & 0 & \frac{2m_3}{m_2} \end{pmatrix}$ | $\begin{pmatrix} m_e & 0 & 0 \\ 0 & m_+ & m_- \\ 0 & m_- & m_+ \end{pmatrix}$ | same as (3) |
| 3† | $\frac{m_2}{2} \begin{pmatrix} s^2 & sc & 0 \\ sc & c^2 & 0 \\ 0 & 0 & \frac{2m_3}{m_2} \end{pmatrix}$ | $m_0 \begin{pmatrix} 0 & a & a \\ a & b & b \\ 0 & b & d \end{pmatrix}$ | $\begin{pmatrix} -0.93 & 0.37 & 0 \\ -0.28 & -0.70 & 0.66 \\ 0.24 & 0.61 & 0.75 \end{pmatrix}$ |
| 4† | $\frac{m_2}{2} \begin{pmatrix} s^2 & sc & 0 \\ sc & c^2 & 0 \\ 0 & 0 & \frac{2m_3}{m_2} \end{pmatrix}$ | $m_0 \begin{pmatrix} 0 & a & 0 \\ a & b & b \\ 0 & b & d \end{pmatrix}$ | $\begin{pmatrix} -0.95 & 0.30 & 0.05 \\ 0.19 & 0.72 & -0.67 \\ -0.24 & -0.63 & -0.74 \end{pmatrix}$ |
| 5 | $\begin{pmatrix} m - \tilde{m} & \frac{\Delta}{\sqrt{2}} + \tilde{m}_3 & \frac{\Delta}{\sqrt{2}} - \tilde{m}_3 \\ \frac{\Delta}{\sqrt{2}} + \tilde{m}_3 & \frac{\Delta}{2} + \frac{\tilde{m}}{2} & \frac{\Delta}{2} - \frac{\tilde{m}}{2} \\ \frac{\Delta}{\sqrt{2}} - \tilde{m}_3 & \frac{\Delta}{2} + \frac{\tilde{m}}{2} & M_2 \end{pmatrix}$ | $\begin{pmatrix} 0 & m_{e\mu} & 0 \\ m_{e\mu} & m_\mu & m_{e\tau} \\ 0 & m_{e\tau} & m_\tau \end{pmatrix}$ | same as (3) |
| 6 | $\begin{pmatrix} m + \frac{\Delta}{\Delta_3} & 0 & \frac{3\Delta}{2\sqrt{2}} \\ 0 & \frac{M_2 - \Delta_1}{\Delta_2} & \frac{\Delta_2 - \Delta_3}{2\Delta_3} \\ \frac{3\Delta}{2\sqrt{2}} & \frac{\Delta_2}{\Delta_3} - \frac{\Delta_2}{2\Delta_3} & \frac{M_2}{2} \end{pmatrix}$ | $\begin{pmatrix} 0 & m_{e\mu} & 0 \\ m_{e\mu} & m_\mu & m_{e\tau} \\ 0 & m_{e\tau} & m_\tau \end{pmatrix}$ | same as (3) |

*a = m_+ s^2 + m_e c^2, b = (m_+ - m_e) s c, and d = m_+ c^2 + m_e s^2.*

† $m_0 = 0.82$ GeV, $a = 0.025$, $b = 1$ and $d = 1.285$.

‡ $m_0 = 1$ GeV, $a = 0.01$, $b = 0.9$ and $d = 1.1$.  

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sector to come from the neutrino sector. The mass matrices in this case are given by,

$$M_e = \begin{pmatrix} m_e & 0 & 0 \\ 0 & m_+ & m_- \\ 0 & m_- & m_+ \end{pmatrix} \quad \text{and} \quad M_\nu = \frac{m_2}{2} \begin{pmatrix} s^2 & sc & 0 \\ sc & c^2 & 0 \\ 0 & 0 & \frac{2m_3}{m_2} \end{pmatrix}.$$  \hspace{1cm} (11)

This is pattern 2 in table 3. The corresponding unitary matrices, which diagonalise these matrices are,

$$V_e = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ 0 & -\frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix} \quad \text{and} \quad V_\nu = \begin{pmatrix} c & -s & 0 \\ s & c & 0 \\ 0 & 0 & 1 \end{pmatrix} \hspace{1cm} (12)$$

which gives the required neutrino mixing matrix of equation (3) $$U_\nu = V_e^TV_\nu$$.

We shall now check which of the charged lepton mass matrices resemble the textures observed in the quark sector where two prominent patterns have been observed \cite{14, 15}

$$\begin{pmatrix} 0 & * & * \\ * & * & * \\ * & * & * \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} 0 & * & 0 \\ * & * & * \\ 0 & * & * \end{pmatrix}.$$  \hspace{1cm} (13)

We shall not impose any condition on the neutrino mass matrices, since the origin of the neutrino mass matrix appears quite different from the up quark sector. So, we assume the same neutrino mass matrix as above and obtain the form of the charged lepton mass matrix which can give us the required mixing matrix and the mass eigenvalues. There exists two possible charged lepton mass matrices corresponding to these two textures, which are,

$$M_e^A = m_o \begin{pmatrix} 0 & a & a \\ a & b & b \\ a & b & c \end{pmatrix} \quad \text{or} \quad M_e^B = m_o \begin{pmatrix} 0 & a & 0 \\ a & b & b \\ 0 & b & c \end{pmatrix}.$$  \hspace{1cm} (14)

We point out that the (33) elements above do no follow the pattern given in (8) which, therefore, allows the texture zeroes to develop in (14). These two patterns (patterns 3 and 4) are given in table 3 for some typical parameters.

5 Comparison with the Quark Texture

We will now take a closer look at the quark sector by comparing the individual matrix elements. From the analysis of the textures of the $u$– and $d$–quark mass matrices it is
known that an excellent representation for the $d$–quark is given by [14, 15],

$$D = \begin{pmatrix}
0 & \sqrt{m_1 m_2} & 0 \\
\sqrt{m_1 m_2} & m_2 & \sqrt{m_1 m_3} \\
0 & \sqrt{m_1 m_3} & m_3
\end{pmatrix}$$ (15)

where the masses are in hierarchical order with $m_1 = m_d$, $m_2 = m_s$ and $m_3 = m_b$. This matrix correctly reproduces $V_{us}$ and $V_{cb}$ of the CKM matrix as,

$$V_{us} \approx \sqrt{\frac{m_d}{m_s}}, \quad V_{cb} \approx \sqrt{\frac{m_d}{m_b}}$$ (16)

If we assume that the charged-lepton mass matrix has an identical form to $D$, then we can write an analogous relation

$$M_e = \begin{pmatrix}
0 & \sqrt{m_e m_\mu} & 0 \\
\sqrt{m_e m_\mu} & m_\mu & \sqrt{m_e m_\tau} \\
0 & \sqrt{m_e m_\tau} & m_\tau
\end{pmatrix}$$ (17)

Unlike the $D$, several different representations are possible for $U$. Furthermore, it is not at all clear, because of neutrino’s Majorana character and the possible presence of a see-saw mechanism, that a similarity exists between the neutrino and the $u$–sector. Therefore we write, with (17) as the basis for $M_e$, the most general matrix for $M_\nu$

$$M_\nu = \begin{pmatrix}
a & b & c \\
b & d & e \\
c & e & f
\end{pmatrix}$$ (18)

We will obtain $M_\nu$ above by comparing it to (8) after we diagonalize $M_e$ in (17) since (8) is actually in the basis with $M_e$ diagonal.

We write

$$M_{e}^{\text{diag}} = T^\dagger M_e T$$ (19)

where $M_e$ is given by (17) and $T$ is given, in the small angle approximation as

$$T = \begin{pmatrix}
1 & s_{12} & 0 \\
-s_{12} & 1 & s_{23} \\
0 & -s_{23} & 1
\end{pmatrix}$$ (20)

where

$$s_{12} \approx \sqrt{\frac{m_e}{m_\mu}}, \quad s_{23} \approx \sqrt{\frac{m_e}{m_\tau}}$$ (21)
Thus in the representation where $M_e$ is diagonal

$$M_e = M_e^\text{diag} \quad (22)$$

one can express $M_\nu$ as

$$M_\nu = T \begin{pmatrix} a & b & c \\ b & d & e \\ c & e & f \end{pmatrix} T^\dagger \quad (23)$$

and obtain the elements $a, b, \ldots$ etc, by comparing the above expression with (8). Since $s_{23} \ll s_{12}$ from (21), we will keep only $s_{12}$ to obtain (with $\theta = 45^\circ$, for simplicity)

$$a = m - \sqrt{2} \Delta s_{12}$$

$$b = \frac{\Delta}{\sqrt{2}} + \frac{\Delta_3}{2} s_{12}$$

$$c = \frac{\Delta}{\sqrt{2}} - \frac{\Delta_3}{2} s_{12}$$

$$d = \frac{M}{2} + \sqrt{2} \Delta s_{12}$$

$$e = \frac{\Delta_3}{2} + \frac{\Delta}{\sqrt{2}} s_{12}$$

$$f = \frac{M}{2} \quad (24)$$

This is pattern 5 in table 3. Once again there are no texture zeroes in $M_\nu$ unlike the $u$–sector where, as we mentioned previously, several representations with two or three texture zeroes are found [13] for the same pattern in the $d$- and, therefore, in the $M_\nu$ structures.

What if we demand a texture zero in $M_\nu$? The only possible spots that would physically make sense are the (12) (and (21)), or (13) (and (31)) locations in the mass matrix (23) in which case we find

$$\Delta = \pm \frac{\Delta_3}{\sqrt{2}} s_{12} \quad (25)$$

From (8) and (21) one can express this relation as

$$\Delta_{12}^2 = \left( \frac{2\sqrt{2}m}{m + m_3} \right) \sqrt{\frac{m_e}{m_\mu}} \Delta_{23}^2 \quad (26)$$

where $m$ is the average mass of $m_1$ and $m_2$ and $\Delta_{12}^2$ and $\Delta_{23}^2$ are defined in (1) and (2).

If we take a hierarchical structure for the neutrino masses and take $m_1 = 0$ then from (1), (2) and Table 1 we find that the above relation is, indeed, quite well satisfied. If we choose the (12) (and (21)) element to vanish then the matrix is as given by pattern 6 in table 3.
These are the only simple forms of the charged lepton masses matrices allowed which can give the mixing angles for the solar neutrino problem, when the atmospheric neutrino mixing comes from the diagonalisation of the neutrino mass matrix. This completes the possible forms of the mass matrices which are consistent with the present experiments.

In summary, we have presented different possible forms of the neutrino and charged lepton mass matrices which have simple patterns and which can give us the required mixing matrix and the mass eigenvalues. There are very few texture zero forms of the mass matrices which are still consistent. In all the cases some of the elements are required to be equal to each other. If we demand that the charged lepton mass matrix be same as the down quark mass matrix in texture, then some interesting results emerge, particularly if we also demand texture zeroes in the neutrino mass matrix. In the basis in which the charged lepton mass matrix is diagonal there are very few solutions with textures zeroes.

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