THE PROBABILISTIC VS THE QUANTIZATION APPROACH TO KÄHLER-EINSTEIN GEOMETRY

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Abstract. In the probabilistic construction of Kähler-Einstein metrics on a complex projective algebraic manifold $X$ - involving random point processes on $X$ - a key role is played by the partition function. In this work a new quantitative bound on the partition function is obtained. It yields, in particular, a new direct analytic proof that $X$ admits a Kähler-Einstein metric if it is uniformly Gibbs stable. The proof makes contact with the quantization approach to Kähler-Einstein geometry.

1. Introduction

A complex projective algebraic manifold $X$ admits a Kähler-Einstein metric with positive Ricci curvature if and only if $X$ is a Fano manifold satisfying an algebro-geometric condition called K-stability; this is the content of the solution of the Yau-Tian-Donaldson (YTD) conjecture for Fano manifolds [20]. The proof in [20] is based on a variant of Aubin’s method of continuity [1], extended to Aubin’s original method in [22]. It involves the following equations for a Kähler metric $\omega_t$, parameterized by “time” $t$:

$$\text{Ric} \omega_t = t \omega_t + (1 - t) \text{Ric} \ dV,$$

where $dV$ is a fixed a volume form $dV$ on $X$, which may be taken to have positive Ricci curvature $\text{Ric} \ dV$ (since $X$ is Fano). The supremum over all $t \in [0, 1]$ for which a solution $\omega_t$ exists defines an invariant of $X$, denoted by $R(X)$, which is strictly positive [29]. As $t$ is increased towards $R(X)$ either $\omega_t$ blows-up or it converges towards a Kähler-Einstein metric (in which case $R(X) = 1$). The first alternative is precisely what it is shown to be excluded by the condition of K-stability [22]. While it is usually assumed that $t \in [0, 1]$ it will in the present work be important to allow $t$ to be any real number.

A probabilistic construction of Kähler-Einstein metrics with negative Ricci curvature was introduced in [4], where the Kähler-Einstein metric emerges from a random point process on $X$ with $N$ points as $N$ tends to infinity (see also [30] for a different probabilistic framework involving random $N \times N$ Hermitian matrices, also inspired by the YTD conjecture). A conjectural extension to Kähler-Einstein metrics with positive Ricci curvature was proposed in [5] and conditional convergence results were given in [6, 8]. In this probabilistic approach the role of K-stability is played by a new type of stability, dubbed Gibbs stability, which amounts to the finiteness of the corresponding partition functions. In the survey [7] connections to the variational proof of the uniform YTD conjecture [13] (involving uniform K-stability) are explained, including non-Archimedean aspects. In the present paper a new quantitative lower bound on the partition functions is obtained, which yields a new direct analytic proof that uniform Gibbs stability implies the existence of a unique Kähler-Einstein metric on $X$. The proof makes contact with the quantization approach to Kähler geometry and, in particular, with K. Zhang’s new remarkably direct proof of the (uniform) YTD conjecture [46].
1.1. Background on the probabilistic approach. Let $X$ be a Fano manifold. Given a positive integer $k$ we denote by $N$ the dimension of the space of all holomorphic sections of the $k$th tensor power of the anti-canonical line bundle $-K_X$ (i.e. the top exterior power of the tangent bundle of $X$):

$$N := \dim H^0(X, -kK_X)$$

(loading notation for tensor products of line bundles). The Fano assumption on $X$ ensures, in particular, that $N \to \infty$, as $k \to \infty$ (more precisely, $N \sim k^{\dim X}$). Given a basis $s_1^{(k)}, \ldots, s_N^{(k)}$ in $H^0(X, -kK_X)$ denote by $\det S^{(k)}$ the corresponding holomorphic section of the line bundle $-(kK_X)^N \to X^N$ defined as the Slater determinant

$$\det S^{(k)}(x_1, x_2, \ldots, x_N) := \det \left( s_i^{(k)}(x_j) \right).$$

Given a volume form $dV$ on $X$ and a parameter $\beta > 0$, the $N$-fold product $X^N$ is endowed with the following probability measure:

$$\mu^{(N)}_\beta := \frac{\| \det S^{(k)} \|^{2\beta/k} dV^{\otimes N}}{Z_N(\beta)}, \quad Z_N(\beta) := \int_{X^N} \| \det S^{(k)} \|^{2\beta/k} dV^{\otimes N}$$

where $\| \cdot \|$ denotes the metric on $-K_X$ (and its tensor powers) induced by $dV$. In statistical mechanical terms this probability measure represents the equilibrium distribution of $N$ interacting particles on $X$ at inverse temperature $\beta$ and $Z_N(\beta)$ is the corresponding partition function. The probability measure $\mu^{(N)}_\beta$ is, in fact, independent of the choice of bases. It will be convenient to fix a reference volume form $dV_X$ on $X$ with positive Ricci curvature and a basis $(s_i^{(k)})$ in $H^0(X, -kK_X)$ which is orthonormal with respect to Hermitian product on $H^0(X, -kK_X)$ induced by $dV_X$.

The probability measure $\mu^{(N)}_\beta$ is symmetric (since the determinant is anti-symmetric) and thus defines a random point process on $X$ with $N$ points $x_1, \ldots, x_N$. By [4, Thm 5.7] the corresponding empirical measure $\delta_N$, i.e. the discrete measure on $X$ defined by

$$\delta_N := \frac{1}{N} \sum_{i=1}^N \delta_{x_i},$$

converges in probability, as $N \to \infty$, towards a normalized volume form $dV_\beta$ on $X$ with the property that the Kähler form

$$\omega_\beta := \frac{1}{\beta} \frac{i}{2\pi} \partial \bar{\partial} \log dV_\beta$$

is the unique solution to Aubin’s continuity equation [12] with $t := -\beta$. The convergence of $\delta_N$ towards $dV_\beta$ also implies that the following convergence holds in the weak topology of currents on $X$:

$$\omega_{k, \beta} := \frac{1}{\beta} \frac{i}{2\pi} \partial \bar{\partial} \left( \log \int_{X^{kN-1}} \| \det S^{(k)} \|^{2\beta/k} dV^{\otimes N-1} \right) \to \omega_\beta, \quad k \to \infty,$$

where $\omega_{k, \beta}$ is a Kähler form, for $k$ sufficiently large (to ensure that $-kK_X$ is very ample).

In fact, the convergence of $\delta_N$ towards $dV_\beta$ was shown to hold at an exponential speed in the sense of Large deviation theory [23]. More precisely, a Large Deviation Principle (LDP) was established, which may be symbolically expressed as

$$\langle \delta_N \rangle \cdot \left( \| \det S^{(k)} \|^{2\beta/k} dV^{\otimes N} \right) \sim e^{-NF_\beta(\mu)}, \quad N \to \infty,$$
where the left hand side defines a measure on the space of all probability measures \( \mathcal{P}(X) \) on \( X \) and \( F_\beta(\mu) \) is a free energy type functional on \( \mathcal{P}(X) \) (see formula 2.3). Expressing \( \mu \) as the normalized volume form of a Kähler metric \( \omega \) in the space \( \mathcal{H} \) of all Kähler metrics representing the first Chern class of \( X \), the free energy functional \( F_\beta(\mu) \) gets identified with the twisted Mabuchi functional on \( \mathcal{H} \) (which is minimized precisely by the unique Kähler metrics \( \omega_\beta \) solving Aubin’s equation 1.1 with \( t = -\beta \)):

\[
F_\beta(\frac{\omega^n}{V}) = \mathcal{M}_\beta(\omega)
\]

(cf. formula 2.2). The LDP 1.5 thus implies that

\[
\lim_{N \to \infty} -\frac{1}{N} \log Z_N(\beta) = \inf_{\mathcal{H}} \mathcal{M}_\beta
\]

1.1.1. The case \( \beta < 0 \). In the case when \( \beta \) is negative the probability measure \( \mu_\beta^{(N)} \) is well-defined for \( \beta \) sufficiently close to 0. The main case of interest is when \( \beta = -1 \). In this case the measure \( \mu_\beta^{(N)} \) is canonically attached to \( X \), i.e. it is independent of the choice of volume form \( dV \) (since the contributions from the metric \( \| \cdot \| \) on \( -K_X \) and the volume form \( dV \) on \( X \) cancel).

Hence, if \( \mu_\beta^{(N)} \) is well-defined when \( N \) is sufficiently large, i.e. if \( Z_N(-1) < \infty \) - in which case \( X \) is called Gibbs stable - one obtains canonical random point processes on \( X \) with \( N \) points. It was conjectured in [5] that the corresponding empirical measures \( \delta_N \) converge towards a unique Kähler-Einstein metric on \( X \) [5], as \( N \to \infty \). A conjectural extension of the LDP for positive \( \beta \) in formula 1.5 to any negative \( \beta \) was also put forth in [5]. In a weaker form this conjecture may be formulated as follows:

**Conjecture 1.1.** Let \( X \) be a Fano manifold endowed with a volume form \( dV \). Given a negative number \( \beta_0 \) the following is equivalent:

1. For any given \( \beta > \beta_0 \) the partition function \( Z_N(\beta) \) is finite, when \( N \) is sufficiently large.
2. For any given \( \beta > \beta_0 \) the twisted Mabuchi functional \( \mathcal{M}_\beta \) admits a minimizer in \( \mathcal{H} \).

Moreover, if \( \beta_0 \) satisfies the first condition, then for any given \( \beta > \beta_0 \) the empirical measure \( \delta_N \) of the ensemble \( (X^N, \mu_\beta^{(N)}) \) converges in probability as \( N \to \infty \) - after perhaps passing to a subsequence - towards a volume form \( dV_\beta \) such that the corresponding Kähler metric \( \omega_\beta \) (formula 1.4) minimizes \( \mathcal{M}_\beta \) on \( \mathcal{H} \).

The reason that one has to pass to a subsequence in the conjectural convergence above is that a minimizer of \( \mathcal{M}_\beta \) need not be uniquely determined, unless \( dV \) is assumed to have positive Ricci curvature and \( \beta > -1 \). The integrability condition \( Z_N(\beta) < \infty \) is, however, independent of choice of volume form \( dV \). Accordingly, one obtains invariants of the Fano manifold \( X \) by setting

\[
\gamma_k(X) = \sup_{\gamma > 0} \{ \gamma : Z_{N_k}(-\gamma) < \infty \} \quad \gamma(X) := \lim_{k \to \infty} \inf \gamma_k(X).
\]

and \( X \) is called uniformly Gibbs stable if \( \gamma > 1 \). This is, a priori, a stronger condition than Gibbs stability (which amounts to the condition that \( \gamma_k(X) > 1 \) for any sufficiently large \( k \)).

The validity of the equivalence "1 \iff 2" in the previous conjecture would, in particular, imply that a Fano manifold \( X \) is uniformly Gibbs stable iff \( X \) admits a unique Kähler-Einstein metric (in analogy to the uniform version of YTD [13], which, in fact, is equivalent to the ordinary formulation of the YTD [34]). The general equivalence "1 \iff 2" may be reformulated as the
following identity:

\[ \gamma(X) = \sup_{\gamma > 0} \left\{ \gamma : \inf_{\mathcal{H}} \mathcal{M}_{-\gamma} > -\infty \right\}, \]

as follows from [2, Thm 1.2] (when restricted to \( \beta \geq -1 \) the supremum in the right hand side above coincides with the maximal existence time \( R(X) \) for Aubin’s equations [1,1]). Moreover, as shown in [3, Section 7] and [8, Thm 2.3], in order to prove the conjectured convergence towards a minimizer of \( \mathcal{M}_\beta \) it is enough to extend the asymptotics [1,6] to \( \beta < 0 \).

1.2. Main results. For \( \beta < 0 \) the limsup upper bound in formula [1,6] was established in [5, Thm 6.7] (by combining Gibbs variational principle in statistical mechanics with the asymptotics for transfinite diameters in [9, Thm 6.7]). The main new result in the present work is the following quantitative upper bound that holds for any fixed \( k \), shown using a completely different argument. Henceforth, we set \( \gamma := -\beta \).

**Theorem 1.2.** There exists a constant \( C > 0 \) (depending only on the reference volume for \( dV_X \)) such that for any \( \gamma > 0 \) and positive integer \( k \)

\[ -\frac{1}{N} \log Z_N(-\gamma) \leq \frac{k + \gamma}{k + 1} \inf_{\mathcal{H}} \mathcal{M}_{-\gamma c_k} + k^{-1} \gamma \left( C + (|1 - \gamma| + C) \log \left\| \frac{dV}{dV_X} \right\|_{L^\infty(X)} \right), \]

where \( c_k := (1 - Ck^{-1})(k+1)/(k+\gamma) \).

For \( \gamma \leq 1 \) the first term in the right hand side of the previous inequality may be replaced by the infimum of \( \mathcal{M}_{-\gamma(1-Ck^{-1})} \) (see Section 2.4).

The previous theorem immediately implies one direction of the conjectured equality (1.8).

**Corollary 1.3.** The following inequality holds

\[ \gamma(X) \leq \sup_{\gamma > 0} \left\{ \gamma : \inf_{\mathcal{H}} \mathcal{M}_{-\gamma} > -\infty \right\}. \]

In other words “1 \( \implies \) 2” in Conjecture (1.1). In particular, if \( X \) is uniformly Gibbs stable, then \( X \) admits a unique Kähler-Einstein metric.

As next explained this corollary also follows from combining the algebro-geometric results in [32, Thm 6.7] with the solution of the (uniform) YTD-conjecture in [20, Thm 6.7] (or [13, Thm 6.7]) and its very recent generalization in [40, Thm 6.7] (which applies to general \( \beta \)). More precisely, exploiting that \( \gamma_k(X) \) may be realized as the log canonical threshold (lct) of an anti-canonical divisor on \( X \) of \( k - \)basis type, it is shown in [32, Thm 2.5] that \( \gamma_k(X) \) is bounded from above by the invariant \( \delta_k(X) \) introduced in [32]:

\[ \gamma_k(X) \leq \delta_k(X) := \inf_{\Delta_k} \text{lct} \left( \Delta_k \right), \]

where the infimum is taken over all anti-canonical \( \mathbb{Q} \)-divisors \( \Delta_k \) on \( X \) of \( k \)-basis type, i.e. \( \Delta_k \) is the normalized sum of the \( N \) zero-divisors on \( X \) defined by the members of a given basis in \( H^0(X, -kK_X) \). In particular,

\[ \gamma(X) \leq \delta(X) := \limsup_{k \to \infty} \delta_k(X), \]

where the invariant \( \delta(X) \) characterizes uniform K-stability; \( \delta(X) > 1 \) iff \( X \) is uniformly K-stable [33]. Recently, it was shown in [39] that \( \delta_k(X) \) coincides with the coercivity threshold of the quantized Ding functional on the symmetric space \( GL(N, \mathbb{C})/U(N) \). Combining this result with the quantized maximum principle in [15], it was then shown in [40] that \( \delta(X) \) coincides with the
coercivity threshold of the Ding functional (as further discussed in Section 1.3). Finally, Cor 1.3 follows from [2, Thm 3.4], which implies that the coercivity thresholds of the Ding and the Mabuchi functionals coincide.

1.2.1. Outline of the proof of Theorem 1.2. The proof of Theorem 1.2 is surprisingly simple. The key new observation is an inequality which, in its simplest form, $\beta = -1$ (i.e. $\gamma = 1$), may be formulated as follows:

\[ -\log Z_N \leq (1 + k^{-1}) \inf_{\mathcal{H}_k} D_k + \frac{1}{kN} \log N \]

where the infimum runs over the space $\mathcal{H}$ of all metrics on $-K_X$ with positive curvature and $D_k$ is a certain (scale invariant) functional on $\mathcal{H}$, approximating the twisted Ding functional $D$ (in the sense that $D_k$ converges towards $D$ as $k \to \infty$); see formula (2.5). Next, by an inequality established in [12] (leveraging the positivity of direct image bundles in [14]) there exists a constant $C$ such that $D_k \leq D - C k^{-1} \mathcal{E}$ on $\mathcal{H}_0$, where $\mathcal{H}_0$ denotes the subspace of all sup-normalized metrics on $\mathcal{H}$ and $\mathcal{E}$ denotes the standard functional on $\mathcal{H}$ defined as the primitive of the Monge-Ampère operator (which is non-positive on $\mathcal{H}_0$). Finally, using the well-known fact that $D$ is bounded from above by the Mabuchi functional $M$ this proves Theorem 1.2 when $\gamma = 1$ (by absorbing the error term $-C k^{-1} \mathcal{E}$ in the subscript $\gamma$ of the twisted Mabuchi functional $M_\gamma$). A slight twist of this argument yields the inequality in Theorem 1.2 for a general $\gamma$, using the thermodynamical formalism in [2].

1.3. Comparison with the quantization approach. In the quantization approach to Kähler geometry, which goes back to [44, 41, 26, 27], the space $\mathcal{H}(L)$ of all Hermitian metrics on a holomorphic line bundle $L$ over a complex manifold $X$ is approximated by the finite dimensional space $\mathcal{H}_k(L)$ of all Hermitian metrics on the $N$–dimensional complex vector space $H^0(X, kL)$. The space $\mathcal{H}_k(L)$ may be identified with the symmetric space $GL(N, \mathbb{C})/U(N)$. When $X$ is Fano and $L = -K_X$ a quantization of the Ding functional $D$ on $\mathcal{H}$ was introduced in [10], building on [29], which defines a functional on $\mathcal{H}_k$ that we shall denote by $D_k$ (formula 3.2). Here it is observed (Prop 3.1) that

\[ \inf_{\mathcal{H}_k} D_k = (1 + k^{-1}) \inf_{\mathcal{H}} D_k, \]

where $D_k$ is the approximation on $\mathcal{H}$ of the Ding functional $D$ which appeared in the inequality (1.10). As a consequence,

\[ -\log Z_N \leq \inf_{\mathcal{H}_k} D_k + \frac{1}{kN} \log N. \]

A similar inequality holds for a general $\gamma$ (see Theorem 3.3) which yields a new proof of the inequality (1.9).

This line of reasoning is inspired by K.Zhang’s very recent new proof of the uniform YTD conjecture for Fano manifolds [46]. In fact, the author discovered the equality (1.11) while trying

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1In physical terms $\mathcal{H}_k(L)$ can be viewed as the quantization of $\mathcal{H}$ with $k^{-1}$ playing the role of Planck’s constant in quantum mechanics [26].
to find a conceptual replacement for an inequality used in the proof of [46, Thm 5.1] (involving Tian’s $\alpha$–invariant [40]). One virtue of the present approach is that it directly yields a quantitative estimate on the infimum of $D_{k,\beta}$ over $\mathcal{H}_k$:

$$\inf_{\mathcal{H}_k} D_{k,-\gamma} \leq \frac{k + \gamma}{k + 1} \inf_{\mathcal{H}} \mathcal{M}_{-\gamma c_k} + k^{-1} \gamma \left( C + (|1 - \gamma| + C) \log \left\| \frac{dV}{dV_X} \right\|_{L^\infty(X)} \right)$$

by combining formula 1.11 (extended to general $\gamma$) with the inequality 1.10 (extended to general $\gamma$). As in [46, Thm 5.1] this shows that uniform K-stability of $X$ implies that $X$ admits a unique Kähler-Einstein metric. Indeed, as shown in [36], building on [32, 33], uniform K-stability is equivalent to the existence of some $\epsilon > 0$ such that the infimum of $D_{k,-1-\epsilon}$ on $\mathcal{H}_k$ is finite for $k$ sufficiently large. By the inequality 1.13 this implies that $\mathcal{M}_{-1-\epsilon}$ is bounded from below (or equivalently, that $\mathcal{M}_{-1}$ is coercive) which, in turn, implies that $X$ admits a unique Kähler-Einstein metric (as first shown in [43] using Aubin’s method of continuity and then using a direct variational approach in [2, 11], which applies to any $\gamma$).

### 1.4. Outlook on converse bounds and exceptional Fano orbifolds

The converse of the inequality 1.13 also holds, as follows from [10, Lemma 7.7]. As a consequence,

$$\delta(X) = \sup_{\gamma > 0} \left\{ \gamma : \inf_{\mathcal{H}} \mathcal{M}_{-\gamma} > -\infty \right\}.$$  

This identity is equivalent to the result [46, Thm 5.1] (which is formulated in terms of the infimum of $D_{k,\beta}$, but, by [2, Thm 1.1], this infimum coincides with the infimum of $\mathcal{M}_{\beta}$). It remains, however, to establish a similar lower bound on $-\log Z_{N,-\gamma}$ or, at least, the missing lower bound on $\gamma(X)$ in the conjectured formula 1.8. By formula 1.14 this amounts to upgrading the inequality between $\gamma(X)$ and $\delta(X)$ in Cor 1.3 to an equality. In contrast, it should be stressed that the inequality 1.9 between $\gamma_k(X)$ and $\delta_k(X)$ is not an equality, in general. For example, when $X$ is the Riemann sphere, i.e. the complex projective line $\mathbb{P}^1$,

$$\gamma_k(X) = 1 - \frac{1}{2k + 1}, \quad \delta_k(X) = 1.$$  

[31, 36]. This discrepancy becomes even more pronounced in the more general setting of Fano orbifolds $X$, where the role of $K_X$ is played by the orbifold canonical line bundle $K_{X_{orb}}$. All the results in the present paper readily extend to the orbifold setting. For example, any Fano orbifold curve is of the form

$$X = \mathbb{P}^1/G$$

where $G$ is the finite group acting on $\mathbb{P}^1$ induced by the action on $\mathbb{C}^2$ of a finite subgroup of $SU(2)$. By the “ADE-trichotomy” such groups fall into the three classes, corresponding to the classification of simply laced Dynkin diagrams; two infinite series $A_n$ and $D_n$ and three exceptional cases $E_6$, $E_7$ and $E_8$. As it turns out, the ADE-trichotomy is detected by the corresponding partition functions at the canonical value $\gamma = 1$ (as follows from [3, Thm 3.5]):

- (A) $Z_N(-1) = \infty$ for all $N$ (i.e. $\gamma_k(X) < 1$ for all $k$)
- (D) $Z_N(-1) < \infty$ for $N \gg 1$, but not all $N$ (i.e. $\gamma_k(X) > 1$ for $k$ sufficiently large)
- (E) $Z_N(-1) < \infty$ for all $N$ (i.e. $\gamma_k(X) > 1$ for all $k$).

\footnote{using, in particular, the uniform asymptotics for Bergman measures on orbifolds in [21, Thm 1.4] as a replacement for the inequality 2.9}
Moreover, $\gamma_k(X)$ is strictly increasing wrt $k$. On the hand it can be shown that

$$\delta_k(X) = \delta(X)$$

and thus $\delta(X) = \gamma(X)$, while, $\gamma_k(X) < \delta_k(X)$.

The notion of exceptionality has been extended to general Fano orbifolds [17], motivated by the Minimal Model Program in birational algebraic geometry [37]. A Fano orbifold $X$ is said to be exceptional if $\alpha(X) > 1$, where $\alpha(X)$ denotes Tian’s alpha-invariant [40] (which, in algebro-geometric terms, coincides with the global log canonical threshold of $X$ [24]). For example, in [17 Cor 1.1] a finite list of exceptional Fano orbifold surfaces $X$ is given, realized as hypersurfaces in weighted three-dimensional complex projective space. In general, it follows readily from the definitions that

$$\alpha(X) \leq \gamma_k(X)$$

(cf. [6, Lemma 7.1]). As a consequence, if $X$ is exceptional, then $Z_N$ is finite for any $N$. Does the converse also hold? For Fano orbifold curves this is, indeed, the case, according to the ADE-list above.

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2. Proof of Theorem

2.1. Setup. We will use additive notation for line bundles and metrics. Accordingly, the $k$ the tensor power of a holomorphic line bundle $L$ over an $n$–dimensional complex manifold $X$ will be denoted by $kL$ and if $\phi$ is a metric on $L$ then $k\phi$ denotes the induced metric on $kL$. Accordingly, if $s$ is a holomorphic section of $L$, i.e. $s \in H^0(X, L)$, the point-wise norm of $s$ with respect to a metric $\phi$ on $L$ is denoted by $|s|_\phi$. Given a local trivializing section of $L$ we may identify $s$ with a local holomorphic function on $X$ and $\phi$ with a local smooth function so that

$$|s|_\phi^2 := |s|^2 e^{-\phi}$$

and the normalized curvature of the metric $\phi$ may be expressed as

$$ddc \phi := \frac{i}{2\pi} \partial \bar{\partial} \phi.$$  

A smooth metric $\phi$ on $L$ is said to have positive curvature if $ddc \phi > 0$ and semi-positive curvature of $ddc \phi \geq 0$ (when identified with an $n \times n$ Hermitian matrix). Equivalently, this means that, locally, $\phi$ is plurisubharmonic (psh) and strictly psh, respectively. Given a metric $\phi$ with semi-positive curvature we denote by $MA(\phi)$ the corresponding Monge–Ampère measure, normalized to have unit total mass:

$$MA(\phi) := \frac{1}{V} (ddc \phi)^n.$$  

2.1.1. The anti-canonical setup. Henceforth, the line bundle $L$ will be taken to be the anti-canonical line bundle $-K_X$ of $X$, i.e. top exterior power of the tangent bundle of $X$. Then any smooth metric $\phi$ on $-K_X$ induces a volume form on $X$ that we shall, abusing notation slightly, denote by $e^{-\phi}$. This notation is intended to reflect the fact that if $z_1, ..., z_n$ are local holomorphic coordinates on $X$ and $\phi$ is locally represented by a function with respect to the local trivialization $\partial/\partial z_1 \wedge \cdots \wedge \partial/\partial z_n$ of $-K_X$, then the volume form in question has density $e^{-\phi}$ with respect to the local Euclidean volume form corresponding to $z_1, ..., z_n$.  

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Given a metric $\phi$ on $-K_X$ and a volume form $\mu$ on $X$ we shall denote by $H^{(k)}(\phi, \mu)$ the corresponding Hermitian metric on the $N$-dimensional complex vector space $H^0(X, -kK_X)$, defined by

$$H^{(k)}(\phi, \mu)(s, s) := \int_X |s|_{k\phi}\mu.$$ 

The space of all metrics on $\phi$ on $-K_X$ with positive curvature will be denoted by $\mathcal{H}$. We will fix once and for all a reference metric $\psi_0$ in $\mathcal{H}$ and a basis $\phi_1^{(k)}, \ldots, \phi_N^{(k)}$ in $H^0(X, -kK_X)$ which is orthonormal with respect to the corresponding Hermitian norm $H^{(k)}(\psi_0, \psi_0)$ (in the notation used in Section 1 $e^{-\psi_0} = dV_X$). Accordingly, we can identify a Hermitian metric $H$ on $H^0(X, -kK_X)$ with the corresponding $N \times N$ positive definite Hermitian matrix $H(\phi_1^{(k)}, \phi_j^{(k)})$.

2.1.2. Energies. Following (essentially) the notation in [9] we denote by $E$ the functional on $\mathcal{H}$ uniquely determined by the following conditions:

$$dE_{\phi} = MA(\phi), \quad E(\psi_0) = 0$$

Alternatively, $E(\phi)$ may be explicitly defined by

$$E(\phi) := \frac{1}{V(n+1)} \int_X \sum_{j=0}^{n} (\phi - \psi_0)(dd^c \phi)^{n-j} \wedge (dd^c \psi_0)^j$$

Dually, following [10], the pluricomplex energy of a probability measure $\mu$ on $X$ (wrt the reference metric $\psi_0$) is defined by

$$E(\mu) = \sup_{\phi \in \mathcal{H}} \left( E(\phi) - \int_X (\phi - \psi_0)\mu \right)$$

(however in [10] the functional $E(\phi)$ is denoted by $E(\phi)$ and the pluricomplex energy is denoted by $E^*$). We will use the following basic

**Lemma 2.1.** There exists a positive constant $c_X$ only depending on $X$ such that

$$(2.1) \quad -E(\phi) + \sup_X (\phi - \psi_0) \leq nE(\phi) + c_X.$$ 

**Proof.** This is essentially well-known but for completeness a short proof is provided. First observe that there exists a constant $c_X$ such that $\sup_X (\phi - \psi_0) - c_X$ is bounded from above by the integral of $(\phi - \psi_0)$ against $MA(\psi_0)$. Indeed, this follows directly follows from the submean property of plurisubharmonic functions and the compactness of $X$. Hence, the proof is concluded by invoking the following basic inequality (see [2, Lemma 2.13]):

$$J(\phi) := -E(\phi) + \int_X (\phi - \psi_0)MA(\psi_0) \leq nE(\phi)$$

$\square$

2.1.3. The twisted Ding and Mabuchi functional associated to $(\phi_0, \gamma)$. Given a volume form $dV$ on $X$ we will denote by $\phi_0$ the corresponding metric on $-K_X$ (i.e. $dV = e^{-\phi_0}$). To the pair $(\phi_0, \gamma)$ we attach the twisted Ding functional on $\mathcal{H}$ defined by

$$D_{-\gamma}(\phi) := -E(\phi) - \frac{1}{\gamma} \log \int_X e^{-\gamma(\phi + (1-\gamma)\phi_0)}.$$ 

(coinciding with the ordinary Ding functional when $\gamma = -1$). The definition is made so that $D_{-\gamma}$ is scale invariant, i.e. invariant under $\phi \mapsto \phi + c$ for any $c \in \mathbb{R}$. The corresponding (twisted) Mabuchi functional is usually defined, modulo an additive constant, by demanding that its first
variation is proportional to the (twisted) scalar curvature \[35\], \[38\], but here it will be convenient to use the thermodynamical formalism introduced in [2, Prop 4.1]:

\[ (2.2) \quad M_{-\gamma}(\phi) := F_{-\gamma}(\mu), \quad \mu = MA(\phi), \]
where \( F_{-\gamma}(\mu) \) is the free energy of a probability measure \( \mu \) on \( X \) defined by

\[ (2.3) \quad F_{-\gamma}(\mu) := -\gamma \left( E(\mu) + \int_X (\phi_0 - \psi_0) \mu \right) + \text{Ent} \left( \mu | e^{-(-\gamma \psi_0 + (1-\gamma)\phi_0)} \right), \]

where \( \text{Ent}(\mu | \nu) \) denotes the entropy of a measure \( \mu \) on \( X \) relative to the measure \( \nu \) on \( X \) (using the sign convention that renders \( \text{Ent}(\mu | \nu) \) non-negative when \( \mu \) and \( \nu \) are both probability measures). By [2, Prop 3.5]

\[ (2.4) \quad D_{-\gamma}(\phi) \leq \gamma^{-1} M_{-\gamma}(\phi) \]

(moreover, the two functionals \( D_{-\gamma} \) and \( M_{-\gamma} \) have the same infimum over \( \mathcal{H} \), but his fact will not be needed here).

**Remark 2.2.** In the notation of [2], \( D_{-\gamma} = -G_{-\gamma} \) and the definition of the free energy \( F_{-\gamma} \) used here is \( -\gamma \) times the definition employed in [2]. When \( \gamma = 1 \) formula (2.2) is equivalent to the Tian-Chen formula for the Mabuchi functional [42], [19] and the case \( \gamma \neq 1 \) is closely related to the generalized Mabuchi functional introduced in [38], Def 6.1.

### 2.2. Two inequalities.

The key new observation in the proof of Theorem 1.2 is the following proposition which yields a bound, from below, on the partition function

\[ Z_{N,-\gamma} = \int_X \| \det S^{(k)} \|_{k\phi_0}^{-2\gamma/k} (e^{-\phi_0})^N, \]

in terms of the infimum over the space of all metrics \( \phi \) on \( -K_X \) of the functional \( D_{k,-\gamma} \) defined by

\[ (2.5) \quad D_{k,-\gamma}(\phi) := -L_k(\phi) - \frac{1}{\gamma} \log \int_X e^{-(\gamma \phi + (1-\gamma)\phi_0)}, \]

where

\[ L_k(\phi) := -\frac{1}{N(k+\gamma)} \log \det H^{(k)} \left( \phi, e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right). \]

The normalization have been chosen to ensure that

\[ L_k(\phi + c) = L_k(\phi) + c, \quad \forall c \in \mathbb{R}. \]

As a consequence, since \( L_k(\phi) \) is increasing wrt \( \phi \), its differential \( dL_{k|\phi} \) may be represented by a probability measure on \( X \). For future reference we note that the probability measure in question coincides with the Bergman measure associated to the Hermitian metric \( H^{(k)} \left( \phi, e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \):

\[ (2.6) \quad dL_{k|\phi} = B_{k\phi} := \rho_{k\phi} e^{-(\gamma \phi + (1-\gamma)\phi_0)}, \quad \rho_{k\phi} := \frac{1}{N} \sum_{i=1}^{N_k} |S_i|^{2k}, \]

where \( S_i \) denotes any bases in \( H^0(X, -kK_X) \) which is orthonormal wrt \( H^{(k)} \left( \phi, e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \) (as follows from [2], Lemma 2.1]).
Proposition 2.3. Given \((\phi_0, \gamma)\) the following inequality holds for any \(k:\)

\[-\frac{1}{\gamma N_k} \log Z_{N_k}(-\gamma) \leq (1 + \gamma k^{-1}) \inf_{\phi} D_{k,-\gamma}(\phi) + \frac{1}{kN} \log N\]

where the infimum runs over all smooth metrics \(\phi\) on \(-K_X\).

**Proof.** Let \(\phi\) be a metric on \(-K_X\). Then we can rewrite

\[Z_{N,-\gamma} := \int_{X^N} \left\| \det S^{(k)} \right\|_{k,\phi}^{-2\gamma/k} (e^{-\phi_0}) \otimes N = \int_{X^N} \left\| \det S^{(k)} \right\|_{k,\phi}^{-2\gamma/k} \left( e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \otimes N.\]

Indeed, locally on each factor of \(X^N\) this simply amounts to rewriting

\[(e^{-k\phi_0})^{-\gamma/k} e^{-\phi_0} = (e^{-k\phi_0})^{-\gamma/k} e^{-\phi_0} e^{-(\gamma \phi + (1-\gamma)\phi_0)}\]

Now assume that \(\phi\) has the property that \(e^{-(\gamma \phi + (1-\gamma)\phi_0)}\) is a probability measure. Then, applying Hölder’s inequality with negative exponent \(-\gamma/k\) (or Jensen’s inequality applied to the convex function \(t \mapsto t^{-\gamma/k}\) on \([-\infty, \infty]\)) yields

\[Z_{N,-\gamma} \geq \left( \int_{X^N} \left\| \det S^{(k)} \right\|^2_{k,\phi} \left( e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \otimes N \right)^{-\gamma/k}.\]

Taking logarithms this means that

\[-\frac{1}{\gamma N} \log Z_{N,-\gamma} \leq \frac{1}{kN} \log \int_{X^N} \left\| \det S^{(k)} \right\|^2_{k,\phi} \left( e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \otimes N.\]

Now, for any metric \(\phi\) on \(-K_X\) we may apply the previous inequality to \(\phi + \log \int_X e^{-(\gamma \phi + (1-\gamma)\phi_0)}\)

and deduce that \[-\frac{1}{\gamma N} \log Z_{N,-\gamma}\]
is bounded from above by

\[(1 + \gamma k^{-1}) \left( \frac{1}{(k + \gamma)N} \log \int_{X^N} \left\| \det S^{(k)} \right\|^2_{k,\phi} \left( e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \otimes N - \frac{1}{\gamma} \log \int_X e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right).\]

The proof is thus concluded by invoking the following formula [9] Lemma 5.3, which holds for any volume form \(\mu\) on \(X\):

\[\int_{X^N} \left\| \det S^{(k)} \right\|^2_{k,\phi} \mu \otimes N = N! \det \left( H^{(k)}_\mu (\phi, \mu) \right)\]

\(\square\)

We will also use the following slight generalization of the inequality in [12] formula 3.4 (to the case \(\gamma \neq 0\)):

**Lemma 2.4.** There exists a constant \(C_0\) depending only on \(\psi_0\) such that the following inequality holds for \(\phi^{(\epsilon)} := \phi(1 - \epsilon) + \epsilon \psi_0\) with \(\epsilon := \frac{1}{k+1}:

\[-\frac{1}{(1 - \epsilon)} L_k \left( \phi^{(\epsilon)} \right) \leq -E(\phi) + C_0 k^{-1} \left( -E(\phi) + \sup_X (\phi - \psi_0) \right) + \frac{\gamma - 1}{k + 1} \| \phi_0 - \psi_0 \|_{L^\infty(X)} \]

**Proof.** This is shown in essentially the same way as in the proof of [12] formula 3.4, but to pinpoint the exact dependence on the constant we recall the argument. Let \(\psi_t\) be a weak geodesic connecting \(\phi\) (at \(t = 1\)) with \(\psi_0\) (at \(t = 0\)) [13]. In particular, this means that \(t \mapsto \psi_t\) is a psh path (aka a subgeodesic) in the following sense: extending \(\psi_t\) to \(X \times ([0,1] \times iR)\), so that \(\psi_t\) is independent of the imaginary part of \(t\), the corresponding local function \((z,t) \mapsto \psi_t(z)\) is psh locally on \(X \times ([0,1] \times iR)\). Moreover, it will be convenient to use the following regularity properties [13]: \(dd^c \psi_t \in L^\infty_{loc}\) for any fixed \(t\) and \(t \mapsto \psi_t\) is \(C^1\)—differentiable up to the boundary.
of \([0,1]\) (but, as explained in [12], for the proof it is enough to use that \(\psi_t\) is in \(L^\infty_{loc}\) for any fixed \(t\)). Now,

\[
(2.7) \quad (i) \ t \mapsto \mathcal{E}(\psi_t) \text{ is affine,} \quad (ii) \ t \mapsto \mathcal{L}_k \left( \psi_t^{(\epsilon)} \right) \text{ is concave}
\]

if \(\epsilon\) is sufficiently small. In fact, the first statement characterizes the geodesic \(\phi_t\) among all psh paths \(\phi_t\) as above [13] Thm 1.7 and the second one follows from [14], only using that \(\psi_t^{(\epsilon)}\) is a psh path. To see this rewrite \(-kK_X = (k+1)L + K_X\) for \(L = -K_X\). Then, locally, rewriting \(e^{-k\phi}e^{-(\gamma\phi + (1-\gamma)\phi_0)} = e^{-(k\phi + \gamma\phi + (1-\gamma)\phi_0)}\) the Hermitian metric \(H^{(k)}\left(\phi_t, e^{-(\gamma\phi + (1-\gamma)\phi_0)}\right)\) coincides with the \(L^2\)-metric on \(H^0(X, (k+1)L + K_X)\) induced by the metric \(k\phi + \gamma\phi + (1-\gamma)\phi_0\) on \((k+1) + L_X\). Accordingly, \(\mathcal{L}_k (\phi)\) may be identified with the \(L^2\)-metric on the determinant line of \(H^0(X, (k+1)L + K_X)\). Now replace \(\phi\) with \(\psi_t^{(\epsilon)}\) and decompose the corresponding metric on \((k+1)L + K_X\) as

\[
(2.8) \quad k\psi_t^{(\epsilon)} + \gamma\psi_t^{(\epsilon)} + (1-\gamma)\phi_0 = (k+\gamma)(1-\epsilon)\psi_t + ((k+\gamma)\epsilon\psi_0 + (1-\gamma)\phi_0).
\]

The second term above has non-negative curvature on \(X\) if \(\epsilon\) is sufficiently large. For simplicity, we will first consider the special case that \(\phi_0 = \psi_0\). Then the non-negativity in question holds if \(\epsilon \geq \frac{1}{k+\gamma}\). Henceforth will assumed that \(\epsilon = \frac{1}{k+\gamma}\) (then \((1-\epsilon) = (k+\gamma)\)). Since \(\psi_t\) is locally psh on \(X \times (0,1) \times \mathbb{R}\) the whole expression in formula (2.8) is thus locally psh. Hence, the convexity of \(t \mapsto \mathcal{L}_k \left( \psi_t^{(\epsilon)} \right)\) follows from the positivity of direct image bundles in [14], applied the to trivial fibration \(X \times (0,1) \times \mathbb{R} \rightarrow \{0,1\} \times \mathbb{R}\). This concludes the proof of the properties in formula (2.7). As a consequence, the function \(t \mapsto -\frac{1}{(1-\epsilon)} \mathcal{L}_k \left( \psi_t^{(\epsilon)} \right) + \mathcal{E}(\psi_t)\) is concave, giving,

\[
-\frac{1}{(1-\epsilon)} \mathcal{L}_k \left( \phi^{(\epsilon)} \right) + \mathcal{E}(\phi) \leq -\mathcal{L}_k \left( \psi_0^{(\epsilon)} \right) + \mathcal{E}(\psi_0) + \int_X \left( -\frac{d\psi_t^{(\epsilon)}}{dt} \bigg|_{t=0} \right) \left( \frac{1}{1-\epsilon} \mathcal{L}_k \left( \phi^{(\epsilon)} \right) - \mathcal{E} \right) \bigg|_{\psi_0}.
\]

Now assume first that \(\phi\) is sup-normalized, i.e. that \(\sup_X (\phi - \psi_0) = 0\). Then it follows from the convexity of \(t \mapsto \phi_t\) that \(\frac{d\psi_t^{(\epsilon)}}{dt} \bigg|_{t=0} \leq 0\). Next, since we are considering the special case \(\phi_0 = \psi_0\) and \(\psi_0^{(\epsilon)} = \psi_0\) the term \(\mathcal{L}_k \left( \psi_0^{(\epsilon)} \right)\) vanishes and so does \(\mathcal{E}(\psi_0)\) (by definition). Moreover, since the differential of the functional

\[
\phi \mapsto \frac{1}{(1-\epsilon)} \mathcal{L}_k \left( \phi^{(\epsilon)} \right)
\]

is given by the Bergman measure \(B_k\) associated to the Hermitian metric \(H^{(k)}(\psi_0, e^{-\psi_0})\) (by formula (2.6) it follows from Bergman kernel asymptotics [12] that there exists a constant \(C_0\) (depending only on \(\psi_0\)) such that

\[
(2.9) \quad \left( \frac{1}{1-\epsilon} \mathcal{L}_k \left( \phi^{(\epsilon)} \right) - \mathcal{E} \right) \bigg|_{\psi_0} \leq -C_0 k^{-1} \mathcal{E}(\psi_0).
\]

(in fact only an upper bound on \(B_k\) is needed for which there is an elementary proof [12] Prop 2.4]). Hence,

\[
-\frac{1}{(1-\epsilon)} \mathcal{L}_k \left( \phi^{(\epsilon)} \right) + \mathcal{E}(\phi) \leq C_0 k^{-1} \int_{11} \left( -\frac{d\psi_t^{(\epsilon)}}{dt} \bigg|_{t=0} \right) (\mathcal{E}) \bigg|_{\psi_0} = -C_0 k^{-1} \mathcal{E}(\phi),
\]
Next, we rewrite the first two terms in the right hand side above as

\[ M \text{ where the second factor above is bounded from above by} \]

using, in the last equality (i) in formula 2.7. Replacing a general \( \phi \in \mathcal{H} \) with its sup-normalized version \( \phi - \sup_{x} (\phi - \psi_{0}) \) we deduce that

\[ - \frac{1}{1-\epsilon} \mathcal{L}_{k} \left( \phi^{(\epsilon)} \right) + \mathcal{E}(\phi) \leq C_{0} k^{-1} \left( -\mathcal{E}(\phi) + \sup_{x} (\phi - \psi_{0}) \right). \]

This concludes the proof when \( \phi_{0} = \psi_{0} \). Finally, to handle the case of a general case note that replacing \( \phi_{0} \) with \( \psi_{0} \) in the definition of \( \mathcal{L}_{k} \left( \phi^{(\epsilon)} \right) \) just gives rise to an extra term which, after multiplication by \( \frac{1}{1-\epsilon}, \) may be obtained from above by

\[ \frac{1}{1-\epsilon} \frac{1}{k+\gamma} \log e^{(\gamma-1)\sup_{x} (\phi_{0} - \psi_{0})} \leq \sup_{x} |\phi_{0} - \psi_{0}| \frac{1}{1-\epsilon} \frac{1}{(k+\gamma)} |\gamma - 1| = \sup_{x} |\phi_{0} - \psi_{0}| \frac{1}{k+1} |\gamma - 1|. \]

2.3. Conclusion of the proof of Theorem 1.2. By Prop 2.3, the following inequality holds for any metric \( \phi \) on \( -K_{X} \) and number satisfying \( (1-\epsilon) \geq 0 \):

\[ - \frac{1}{\gamma N} \log \mathcal{Z}_{N}(-\gamma) \leq \left( 1 + \gamma k^{-1} \right) \frac{1}{(1-\epsilon)} \left( \frac{1}{N(k+\gamma)(1-\epsilon)} \mathcal{L}_{k} (\phi) - \frac{1}{\gamma (1-\epsilon)} \log \int_{X} e^{-(\gamma \phi + (1-\gamma)\phi_{0})} \right) \]

Taking \( \epsilon = \frac{2}{\gamma k} \) and replacing \( \phi \) with \( \phi^{(\epsilon)} \) (defined as in the previous lemma) and setting \( \gamma^{(\epsilon)} := (1-\epsilon) \gamma \) thus yields (using that \( (1 + \gamma k^{-1}) (1-\epsilon) = 1 + k^{-1} \))

\[ - \frac{1}{\gamma N} \log \mathcal{Z}_{N}(-\gamma) \leq (1 + k^{-1}) \left( \frac{1}{N(k+\gamma)(1-\epsilon)} \log \det H^{(k)} (\phi^{(\epsilon)}, \gamma) - \frac{1}{\gamma (1-\epsilon)} \log \int_{X} e^{-(\gamma^{(\epsilon)} \phi + (1-\gamma^{(\epsilon)})\phi_{0})} \right) \]

Next, in order to fix ideas, we first consider the special case when \( \phi_{0} = \psi_{0} \). Then, by the previous lemma, the right hand side above is bounded from above by

\[ (2.10) \quad (1 + k^{-1}) \left( \mathcal{D}_{\gamma^{(\epsilon)}(\phi)} + C_{0} k^{-1} \left( -\mathcal{E}(\phi) + \sup_{x} (\phi - \psi_{0}) \right) \right) \]

Since (trivially) \( \mathcal{D}_{\gamma^{(\epsilon)}(\phi)} \leq -\mathcal{E}(\phi) + \sup_{x} (\phi - \psi_{0}) \) it follows that

\[ - \frac{1}{\gamma N} \log \mathcal{Z}_{N}(-\gamma) \leq \gamma \mathcal{D}_{\gamma^{(\epsilon)}(\phi)} + C_{0} k^{-1} \mathcal{E}(MA(\phi)) \]

Invoking the inequality 2.1 thus reveals that there exists a constant \( C \) only depending on \( \psi_{0} \) such that

\[ (2.11) \quad - \frac{1}{N} \log \mathcal{Z}_{N}(-\gamma) \leq \gamma \mathcal{D}_{\gamma^{(\epsilon)}(\phi)} + C \gamma k^{-1} \mathcal{E}(MA(\phi)) + C \gamma k^{-1} \]

Next, we rewrite the first two terms in the right hand side above as

\[ \gamma \mathcal{D}_{\gamma^{(\epsilon)}(\phi)} + C \gamma k^{-1} \mathcal{E}(MA(\phi)) = (1-\epsilon)^{-1} \left( \gamma^{(\epsilon)} \mathcal{D}_{\gamma^{(\epsilon)}(\phi)} + C \gamma^{(\epsilon)} k^{-1} \mathcal{E}(MA(\phi)) \right), \]

where the second factor above is bounded from above by \( MA_{\gamma^{(\epsilon)}(1-Ck^{-1})} \), as follows from the inequality 2.24 and the free energy formula 2.25. Hence,\n
\[ - \frac{1}{N} \log \mathcal{Z}_{N}(-\gamma) \leq (1-\epsilon)^{-1} \mathcal{M}_{\gamma^{(\epsilon)}(1-Ck^{-1})} + C \gamma k^{-1}, \]

proving the theorem in the case when \( \phi_{0} = \psi_{0} \). The general case is shown in essentially the same way, by first including the error term involving \( \phi_{0} \) from Lemma 2.4 in formula 2.10 and then, in formula 2.11, estimating

\[ \mathcal{E}(MA(\phi)) \leq \left( \mathcal{E}(MA(\phi)) + \int (\phi_{0} - \psi_{0}) MA(\phi) \right) + ||\phi_{0} - \psi_{0}||_{L^{\infty}(X)}, \]
so that the first term in the right hand side above can be absorbed into the twisted Mabuchi functional, as before.

2.4. The case $\gamma \leq 1$. For $\gamma \leq 1$ the estimate in Theorem 1.2 implies that (after perhaps increasing the constant $C$):

$$-\frac{1}{N} \log Z_N(-\gamma) \leq \inf_{H} \mathcal{M}_{-\gamma(1-Ck^{-1})} + Ck^{-1} + k^{-1}\gamma (|1-\gamma| + C) \log \left\| \frac{dV}{\|dV\|_{L^\infty(X)}} \right\|.$$  

Indeed, by a simple scaling argument (applied to Lemma 2.4) it is enough to consider the case when $e^{-\gamma\psi_o+(1-\gamma)\phi_0}$ is a probability measure. This implies that the entropy term in the free energy $F_\gamma$ is non-negative. It then follows readily from the definition that the function

$$T \mapsto \inf_{T} TF_{T^{-1}}$$

is increasing in $T$ (where the infimum is taken over a given subset of $\mathcal{P}(X)$). In particular, applied to the present setup at $T_0 = (k + \gamma)/(k + 1)$ and $T_1 = 1$ this monotonicity yields (since $T_0 \leq T_1$ when $\gamma \leq 1$)

$$\frac{k + \gamma}{k + 1} \inf_{H} \mathcal{M}_{-\gamma(1-Ck^{-1})(k+1)/(k+\gamma)} \leq \inf_{H} \mathcal{M}_{-\gamma(1-Ck^{-1})},$$

as desired.

3. Comparison with the quantization approach

Given a holomorphic line bundle $L$ over a compact complex manifold $X$ and a positive integer $k$ denote by $\mathcal{H}_k(L)$ the space of all Hermitian metrics on the $N$–complex vector space $H^0(X, kL)$. The “Fubini-Study map” $FS$ maps $\mathcal{H}_k(L)$ into the space $\mathcal{H}(L)$ of all metrics on $L$ with positive curvature:

$$FS : \mathcal{H}_k(L) \rightarrow \mathcal{H}(L), \quad FS(H) := k^{-1} \log \left( \frac{1}{N} \sum_{i=1}^{N} |s_i^H|^2 \right)$$

where $(s_i^H)$ is any basis in $H^0(X, kL)$ which is orthonormal wrt $H$. The normalization by $N$ used here is non-standard, but it will simplify some of the formulas below.

Henceforth, we shall specialize to the anti-canonical setting in Section 2.1.1. Thus $X$ is a Fano manifold and $L = -K_X$. We will abbreviate $\mathcal{H}_k(-kK_X) = \mathcal{H}_k$ and $\mathcal{H}(-K_X) = \mathcal{H}$. Consider now the functional $D_{k,-\gamma}$ on $\mathcal{H}_k$ defined by

$$D_{k,-\gamma}(H_k) := \frac{1}{kN_k} \log \det H^{(k)} - \frac{1}{\gamma} \log \int_X e^{-\gamma FS(H_k)+(1-\gamma)\phi_0},$$

which is invariant under scaling by positive numbers:

$$D_{k,-\gamma}(e^c H_k) = D_{k,-\gamma}(H_k) \ \forall c \in \mathbb{R}.$$  

As is well-known the functional $D_{k,-\gamma}$ on $\mathcal{H}_k$ can be viewed as a quantization of the functional $D_{-\gamma}$ on $\mathcal{H}$ [10][36]. The following proposition relates the functional $D_{k,-\gamma}$ on $\mathcal{H}_k$ to the functional $D_{k,-\gamma}$ on $\mathcal{H}$ defined in formula 2.5.

**Proposition 3.1.** For any metric $\phi$ on $-K_X$

$$D_{k,-\gamma} \left( H^{(k)} \left( \phi, e^{-\gamma\phi+(1-\gamma)\phi_0} \right) \right) \leq (1 + \gamma k^{-1}) D_{k,-\gamma}(\phi)$$

and for any $H \in \mathcal{H}_k$

$$(1 + \gamma k^{-1}) D_{k,-\gamma}(FS(H)) \leq D_{k,-\gamma}(H)$$
In particular,

\[ \inf_{\mathcal{H}_k} D_{k,-\gamma} = (1 + \gamma k^{-1}) \inf_{\mathcal{H}} D_{k,-\gamma} \]

Proof. To prove the first inequality let \( \phi \) be a given metric on \( -K_X \) and set \( \psi_k := FS \left( H^{(k)} \left( \phi, e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \right) \). Then

\[
\int_X e^{-(\gamma \psi_k + (1-\gamma)\phi_0)} \geq \left( \int_X e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right)^{(1+\gamma/k)}.
\]

Indeed, rewriting \( e^{-(\gamma \psi_k + (1-\gamma)\phi_0)} = e^{-\gamma(\psi_k-\phi)}e^{-(\gamma \phi + (1-\gamma)\phi_0)} \) and using that

\[ e^{(\psi_k-\phi)} = \rho_{k\phi} \]

(as follows directly from the definition of \( \rho_{k\phi} \) in formula 2.40) gives

\[
\int_X e^{-(\gamma \psi_k + (1-\gamma)\phi_0)} = \int_X (\rho_{k\phi})^{-\gamma/k} e^{-(\gamma \phi + (1-\gamma)\phi_0)} \geq \left( \int_X e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right)^{(1+\gamma/k)}
\]

using Hölder’s inequality with negative exponent \(-\gamma/k\) (or Jensen’s inequality applied to the convex function \( t \mapsto t^{-\gamma/k} \) on \([-\infty, \infty]\)). The integral appearing in the first factor in the right hand side above is precisely the integral of the Bergman measure \( B_{k\phi} \) (defined in formula 2.40) and thus equal to one, which proves the inequality 3.3. Hence, using \((1 + \gamma k^{-1})/(k + \gamma) = 1/k,

\[
(1 + \gamma/k)D_{k,-\gamma} \left( H^{(k)} \left( \phi, e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) \right) \leq \frac{1}{kN} \log \det H^{(k)} \left( \phi, e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right) - \gamma^{-1} \log \left( \int_X e^{-(\gamma \phi + (1-\gamma)\phi_0)} \right),
\]

which proves the first inequality stated in the proposition. To prove the second one first observe that for any \( H \) and and volume form \( \mu \) on \( X \)

\[
\det \left( H^{(k)} (FS(H), \mu) \right) \leq \det H \cdot \left( \int_X \mu \right)^N
\]

Indeed, for any given \( \phi \) in \( \mathcal{H} \) and \( H \in \mathcal{H}_k \), taking a basis \((s_i^H)\) in \( H^0(X, -kK_X) \) which is orthonormal wrt \( H \), we can factorize

\[
\det \left( H^{(k)} (\phi, \mu) \right) = \det H \cdot \det \left( H^{(k)} (\phi, \mu) (s_i^H, s_j^H) \right), \]

where the second factor arises as the determinant of the change of bases matrix between the reference basis \((s_i^{(k)})\) in \( H^0(X, -kK_X) \) and \((s_i^H)\). Next, by the arithmetic/geometric means inequality

\[
\left( \det \left( H^{(k)} (\phi, \mu) (s_i^H, s_j^H) \right) \right)^{1/N} \leq \frac{1}{N} \sum_{i=1}^N H^{(k)} (\phi, \mu) (s_i^H, s_i^H).
\]

Now assume that \( \phi = FS(H) \). Then

\[
H^{(k)} (\phi, \mu) (s_i^H, s_i^H) := \int_X \frac{|s_i^H|^2}{N-1 \sum_{j=1}^N |s_j^H|^2} \mu.
\]

Hence, the second factor in the right hand side in formula 3.3 is bounded from above by the \( N \)th power of the integral of \( \mu \), proving the inequality 3.4. Thus, if \( H \) is a given element in
\[ \mathcal{H}_k \] which is normalized so that \( \int e^{-\gamma F_S(H) + (1-\gamma) \phi_0} = 1 \), then applying the inequality \( \delta_0 \) to \( \mu = e^{-\gamma F_S(H) + (1-\gamma) \phi_0} \) proves the second inequality for any normalized \( H \) in \( \mathcal{H}_k \). Finally, since both sides of the inequality in question are invariant under scaling, \( H \rightarrow e^c H \), this concludes the proof for a general \( H \) in \( \mathcal{H}_k \).

\[ \square \]

Remark 3.2. The previous proposition refines a monotonicity result [3, Lemma 2.6], concerning Donaldson’s iteration in the anti-canonical setting of [29]. Indeed, applying the first inequality to \( \phi = F_S(H) \) and then the second inequality reveals that \( D_{k,-\gamma}(H) \) is decreasing under Donaldson’s map on \( \mathcal{H}_k \), defined as the composition of the maps \( F \rightarrow F_S(H) \) and \( \phi \rightarrow H^{(k)}(\phi, e^{-(\gamma \phi + (1-\gamma) \phi_0)}) \). As in [3] one gets equality in the first equality in the proposition when \( \phi = F_S(H) \) iff \( H \) is a balanced metric in \( \mathcal{H}_k \) in the anti-canonical sense of [29, 10, 36].

Combining Prop 2.3 and the equality for the infima in Prop 3.1 (only the upper bound is needed) we thus arrive at the following result:

**Theorem 3.3.** Let \( X \) be a Fano manifold. Given \( (\phi_0, \gamma) \) the following inequality holds for any \( k \):

\[ -\frac{1}{\gamma N} \log Z_{N_k}(-\gamma) \leq \inf_{\mathcal{H}_k} D_{k,-\gamma} + \frac{1}{k N} \log N. \]

As a consequence, if \( Z_{N_k}(-\gamma) \) is finite, then the infimum of \( D_{k,-\gamma} \) over \( \mathcal{H}_k \) is finite. In other words, the invariant \( \gamma_k(X) \) defined by formula \( \gamma_k \) is smaller than or equal to the coercivity threshold of the functional \( D_k \) on \( \mathcal{H}_k \) which, by [36], coincides with the invariant \( \delta_k(X) \) introduced in [32] (appearing in formula \( 1.9 \)). We thus arrive at a new proof of the following inequality first shown in [33] (see \( 1.9 \) for a reformulation of the proof in terms of non-Archimedean pluripotential theory).

**Corollary 3.4.** [32] Thm 2.5. For a Fano manifold \( X \) the following inequality holds:

\[ \gamma_k(X) \leq \delta_k(X) \]

Combining the equality for the infima in Prop 3.1 with the argument employed in Section 2.3 also yields the following analog of Theorem 1.2

**Theorem 3.5.** There exists a constant \( C > 0 \) (depending only on the reference volume for \( dV_X \)) such that for any \( \gamma > 0 \) and positive integer \( k \)

\[ \inf_{\mathcal{H}_k} D_{k,-\gamma} \leq \frac{1}{N} \log Z_N(-\gamma) \leq \frac{k + \gamma}{k + 1} \inf_{\mathcal{H}_k} M_{\gamma c_k + k^{-1} \gamma} \left( C + (|1 - \gamma| + C) \log \left\| \frac{dV}{dV_X} \right\|_{L^\infty(X)} \right), \]

where \( c_k := (1 - C k^{-1})(k + 1)/(k + \gamma) \).

As explained in Section 1.3 this inequality is closely related to results in [10].

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