LOWER BOUND OF COARSE RICCI CURVATURE ON METRIC MEASURE SPACES AND EIGENVALUES OF LAPLACIAN

YU KITABEPPU

Abstract. In this paper, we investigate the coarse Ricci curvature on metric spaces. We prove that a Bishop-Gromov inequality gives a lower bound of coarse Ricci curvature. The lower bound does not coincide with the constant corresponding to curvature in Bishop-Gromov inequality. As a corollary, we obtain a lower bound of coarse Ricci curvature for a metric space satisfying the curvature-dimension condition. Moreover we give an important example, Heisenberg group, which does not satisfy the curvature-dimension condition for any constant but has a lower bound of coarse Ricci curvature. We also have an estimate of the eigenvalues of the Laplacian by a lower bound of coarse Ricci curvature.

1. Introduction

In this paper, we investigate coarse Ricci curvature on metric spaces. Ollivier [12] defined a notion of coarse Ricci curvature. He obtained a relation between the coarse Ricci curvature and the ordinary Ricci curvature on a Riemannian manifold (see [12]). On the other hand, Lott, Villani and Sturm introduced the curvature-dimension condition, which is a generalization of a Ricci curvature bounded by below for geodesic metric spaces [10,14,15]. However, no relation of these two definitions have been known. One of our motivation is to find such a relation for general metric measure spaces. In this paper we prove that a Bishop-Gromov inequality (see Definition 1.2) gives a lower bound of the coarse Ricci curvature. Note that the curvature-dimension condition implies the Bishop-Gromov inequality [10,15]. We also prove that the Heisenberg group with the sub-Riemannian metric is an example which does not satisfy the curvature-dimension condition but has a lower bound of coarse Ricci curvature. As another result, we establish an eigenvalue estimate for the Laplacian by a lower bound of the coarse Ricci curvature. For graphs, more precise estimates of a lower bound of the coarse Ricci curvature and the eigenvalue of Laplacian have already been given by Lin-Yau, Jost-Bauer-Liu, Jost-Liu [3,8,9]. Our eigenvalue estimate is for general metric spaces and for not only positive lower bound but also lower bound of any real number.

Let \((X,d)\) be a complete separable metric space. We denote by \(\mathcal{P}(X)\) the set of all Borel probability measures on \(X\). We call a family \(\{m_x\}_{x \in X}\) of measures in \(\mathcal{P}(X)\) a random walk (see Section 2 for the precise definition). The following is defined by Ollivier [12].

Definition 1.1 (Coarse Ricci curvature). For any two points \(x, y \in X\), the coarse Ricci curvature \(\kappa(x,y)\) associated with \(\{m_x\}_{x \in X}\) along \(xy\) is defined by

\[
\kappa(x,y) := 1 - \frac{W_1(m_x,m_y)}{d(x,y)},
\]

where \(W_1\) is the \(L^1\)-Wasserstein metric.

Partly supported by the Grant-in-Aid for JSPS Fellows, The Ministry of Education, Culture, Sports, Science and Technology, Japan.
It is clear that $\kappa(x, y) \leq 1$ for any $x, y \in X$.
In this paper we call a complete separable metric space with a positive Radon measure a metric measure space. We consider the following random walk.

- The $r$-step random walk:
  \[
  m^r_x = \frac{\chi_{B_r(x)}}{\nu(B_r(x))} \nu,
  \]
on a metric measure space $(X, d, \nu)$, where $\chi_{B_r(x)}$ denotes the characteristic function of $B_r(x)$. In this paper, we denote the open ball of radius $r > 0$ and centered at $x \in X$ by $B_r(x)$.

We define an important notion.

**Definition 1.2.** For two real numbers $K$ and $N > 1$, we define a function $s_{K, N} : [0, \infty) \to \mathbb{R}$, by

\[
s_{K, N}(t) := \begin{cases} 
\sqrt{(N - 1)/K \sin(t\sqrt{K/(N - 1)})} & \text{if } K > 0, \\
t & \text{if } K = 0, \\
\sqrt{(N - 1)/K \sinh(t\sqrt{K/(N - 1)})} & \text{if } K < 0.
\end{cases}
\]

A metric measure space $(X, d, \nu)$ satisfies the Bishop-Gromov inequality $[BG_{K, N}]$ if

\[
\nu(B_R(x)) \leq \int_0^R s_{K, N}(t)^{N-1} dt
\]

holds for any $x \in X$ and for any $0 < r < R \leq \pi \sqrt{(N - 1)/\max\{K, 0\}}$ with the convention $1/0 = \infty$.

A metric space $(X, d)$ is a geodesic metric space if for any $x, y \in X$, there exists a curve $\gamma : [0, 1] \to X$ joining $x$ to $y$ such that $d(\gamma(s), \gamma(t)) = |s - t|d(x, y)$ for any $s, t \in [0, 1]$. The following theorem is one of the main results in this paper.

**Theorem 1.3.** Let $(X, d, \nu)$ be a geodesic metric measure space satisfying $[BG_{K, N}]$ for two real numbers $K$ and $N > 1$. Then, for any $0 < r < \pi \sqrt{(N - 1)/\max\{K, 0\}}$, the coarse Ricci curvature associated with the $r$-step random walk satisfies

\[
\inf_{x, y \in X} \kappa(x, y) \geq 1 - 2\frac{r}{\int_0^R s_{K, N}(t)^{N-1}} dt.
\]

**Remark 1.4.** Ollivier [12] obtained a relation between the coarse Ricci curvature associated with the $r$-step random walk and the ordinal Ricci curvature in the Riemannian case. However he had only an asymptotic estimates as $r$ tends to zero.

In particular, his estimate gives no lower bound of the coarse Ricci curvature if the manifold is noncompact. Theorem 1.3 gives a priori estimate for each $r > 0$.

**Remark 1.5.** For any $r > 0$, calculating the right-hand side of (1.3), we have $\inf_{x, y \in X} \kappa(x, y) \geq 1 - 2N$ on a metric measure space $X$ which satisfies $[BG_{0, N}]$.

In the case of $[BG_{K, N}]$, we obtain

\[
\lim_{r \to 0} \inf_{x, y \in X} \kappa(x, y) \geq 1 - 2N.
\]

According to [7], the $n$-dimensional Heisenberg group $\mathbb{H}^n$ with left invariant sub-Riemannian metric does not satisfy $[CD_{K, N}]$ for any $K$ and $N > 1$. Nevertheless $\mathbb{H}^n$ satisfies $[BG_{0, 2n+3}]$ (see [7, 11]), which together with Theorem 1.3 implies the following.

**Corollary 1.6.** The coarse Ricci curvature associated with the $r$-step random walk on the $n$-dimensional Heisenberg group $\mathbb{H}^n$ satisfies

\[
\inf_{x, y \in \mathbb{H}^n} \kappa(x, y) \geq 1 - 2(2n + 3).
\]
Remark 1.7. Corollary 1.9 says that even if the coarse Ricci curvature is bounded by below, the curvature-dimension condition does not hold in general.

To state another result, we define a version of Laplacian. Let \((X,d,\{m_x\}_{x \in X})\) be a complete separable metric space with a random walk.

**Definition 1.8 (Laplacian).** We define the Laplacian of a function \(f : X \to \mathbb{R}\) by
\[
\Delta f(x) := f(x) - \int_X f(y) \, dm_x(y)
\]
for \(x \in X\), whenever the right-hand side is defined.

We consider the eigenvalue problem for the Laplacian on \((X,d,\{m_x\}_{x \in X})\). The following result holds even for discrete spaces and even for a general random walk.

**Theorem 1.9.** Assume that \(\kappa(x,y) \geq \kappa\) for any \(x,y \in X\) and for a constant \(\kappa\). Then we have the following.

1. If there exists a nonconstant Lipschitz function \(f\) on \(X\) such that \(\Delta f = \lambda f\) for a real number \(\lambda\), then it turns out that \(\kappa \leq \lambda \leq 2 - \kappa\).

2. If \(\kappa\) is positive and if a Lipschitz function \(f\) on \(X\) satisfies \(\Delta f = 0\), then \(f\) is a constant function.

Theorem 1.9 (2) is a Liouville type theorem. Moreover, if \(X\) is compact, we have the following.

**Corollary 1.10.** Let \((X,d,\nu)\) be a compact geodesic metric measure space with a lower bound, say \(\kappa\), of the coarse Ricci curvature associated with the \(r\)-step random walk. Assume that \(X\) satisfies a Bishop-Gromov inequality, then any eigenvalue \(\lambda\) of the Laplacian satisfies
\[
\kappa \leq \lambda \leq 2 - \kappa.
\]

**Remark 1.11.** Ollivier proved a similar claim in his paper [12], for which he assumed a positivity of coarse Ricci curvature and a reversibility of invariant distribution.

2. **Definition and Basic Properties of Coarse Ricci Curvature**

Let \((X,d)\) be a separable metric space and \(\mathcal{P}(X)\) the set of all Borel probability measures on \(X\). For \(\mu, \nu \in \mathcal{P}(X)\), a measure \(\xi \in \mathcal{P}(X \times X)\) is called a coupling between \(\mu\) and \(\nu\) if \(\xi(A \times X) = \mu(A)\) and \(\xi(X \times B) = \nu(B)\) for any Borel sets \(A,B \subset X\). \(\Pi(\mu, \nu)\) denotes the set of all couplings between \(\mu\) and \(\nu\). Let \(\mathcal{P}_p(X)\) for \(p \geq 1\) be the set of all Borel probability measures which have finite \(p\)-th moment, where the \(p\)-th moment of \(\mu \in \mathcal{P}(X)\) is defined to be \(\int_X d(x,o)^p \, d\mu(x)\) for a point \(o\) in \(X\). It is easy to check that \(\mathcal{P}_p(X) \subset \mathcal{P}_q(X)\) for \(p > q\). We define the \(L^p\)-Wasserstein metric \(W_p\) on \(\mathcal{P}_p(X)\) by
\[
W_p(\mu, \nu) := \inf_{\xi \in \Pi(\mu, \nu)} \left\{ \int_{X \times X} d(x,y)^p \, d\xi(x,y) \right\}^{1/p}
\]
for \(\mu, \nu \in \mathcal{P}_p(X)\). The metric space \((\mathcal{P}_p(X), W_p)\) is called the \(L^p\)-Wasserstein space. The following is a well-known result (see [10][17]).

**Lemma 2.1 (Kantorovich-Rubinstein duality).** For any \(\mu, \nu \in \mathcal{P}_1(X)\) we have
\[
W_1(\mu, \nu) = \sup_{f \in \text{Lip}_1(X)} \left( \int_X f \, d\mu - \int_X f \, d\nu \right),
\]
where \(\text{Lip}_1(X)\) is the set of all 1-Lipschitz functions on \(X\).
We define a random walk on $X$ as a family $\{m_x\}_{x \in X}$ of Borel probability measures on $X$ such that the map $x \mapsto m_x$ is a Borel measurable map from $(X, \mathcal{B})$ to $(\mathcal{P}_1(X), \mathcal{W}_1)$. The following proposition is needed for the proof of Theorem 1.3.

**Proposition 2.2 (12)**. Let $(X, d)$ be a geodesic metric space, $\{m_x\}_{x \in X}$ a random walk and $K \in \mathbb{R}$. Assume that there exists $\epsilon > 0$ such that $\kappa(x, y) \geq K$ holds for any $x, y \in X$ with $d(x, y) < \epsilon$. Then we have $\inf_{x, y \in X} \kappa(x, y) \geq K$.

**Remark 2.3**. If there exists a limit $\lim_{n \to \infty} \kappa(x, y)$ and if the limit is bounded below by $\kappa$ for any $x \in X$, then we have $\inf_{x, y \in X} \kappa(x, y) \geq \kappa$ by Proposition 2.2.

### 3. Proof of Main Theorems

We define a function $F : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ by $F(r) := \int_0^r s_{K,N}(t)^{N-1} dt$ where $\mathbb{R}_{\geq 0}$ denotes the set of nonnegative real numbers. We need the following lemma to prove Theorem 1.3.

**Lemma 3.1**. Let $(X, d, \nu)$ be a geodesic metric measure space satisfying $[BG_{K,N}]$. For given $r > 0$ we take two points $x, y \in X$ such that $d(x, y) < r$. We have

\[
\nu(B_r(x) \setminus B_r(y)) \leq \nu(B_r(x)) \leq 1 - \frac{F(r - d(x, y)/2)}{F(r + d(x, y)/2)} = \frac{F(r - d(x, y)/2)}{F(r + d(x, y)/2)}d(x, y) + o(d(x, y))
\]

as $d(x, y) \to 0$.

**Proof.** We prove (3.1). Since $X$ is a geodesic metric space, there exists $z \in X$ such that $d(x, y)/2 = d(x, z) = d(y, z)$. Setting $d := d(x, y)$, we have $B_{r-d/2}(z) \subset B_r(x) \cap B_r(y) \subset B_r(x) \subset B_{r+d/2}(z)$. By the condition $[BG_{K,N}]$, we get

\[
\nu(B_r(x)) \leq \nu(B_{r+d/2}(z)) \leq \frac{F(r + d/2)}{F(r - d/2)}\nu(B_{r-d/2}(z)).
\]

Hence we have

\[
\nu(B_r(x) \setminus B_r(y)) = \nu(B_r(x)) - \nu(B_r(x) \cap B_r(y)) \\
\leq \nu(B_r(x)) - \nu(B_{r-d/2}(z)) \\
\leq \left(1 - \frac{F(r - d/2)}{F(r + d/2)}\right)\nu(B_r(x))
\]

which implies the inequality in (3.1). Since $F$ is a $C^1$ function, we have

\[
F\left( r + \frac{d}{2} \right) = F\left( r - \frac{d}{2} \right) + F'\left( r - \frac{d}{2} \right) \frac{d}{2} + o(d)
\]

as $d \to 0$. This completes the proof. \qed

**Proof of Theorem 1.3**. Take any two points $x, y \in X$ with $d(x, y) < r$. Without loss of generality, we may assume $\nu(B_r(x)) \geq \nu(B_r(y))$. By the assumption, $\nu(B_r(x) \cap B_r(y))/\nu(B_r(y)) \leq \nu(B_r(x) \cap B_r(y))/\nu(B_r(y))$. By Lemma 3.1 we have

\[
m_x(B_r(x) \setminus B_r(y)) \leq \frac{F'(r - d/2)}{F'(r + d/2)}d + o(d),
\]

where $d := d(x, y)$.

We consider the following transportation plan. We transport $m_x$ to $m_y$. Since the mass in $B_r(x) \cap B_r(y)$ measured by $m_x$ does not have to move, we simply
transport the mass in $B_r(x) \setminus B_r(y)$ to $B_r(y) \setminus B_r(x)$. In general, this plan is far from an optimal one. We have

$$W_1(m_x, m_y) \leq m_x (B_r(x) \setminus B_r(y)) \cdot (d + 2r) \leq (d + 2r) \left\{ \frac{F'(r - d/2)}{F(r + d/2)} d + o(d) \right\}$$

as $d \to 0$. Then, by the definition of the coarse Ricci curvature and by the continuity of $F$,

$$\kappa(x, y) = 1 - \frac{W_1(m_x, m_y)}{d(x, y)} \geq 1 - (d + 2r) \left\{ \frac{F'(r - d/2)}{F(r + d/2)} + \frac{o(d)}{d} \right\}$$

$$\to 1 - 2r \frac{F'(r)}{F(r)} \text{ as } d \to 0.$$ 

By Proposition 2.2 we have (3.3) for all $x, y \in X$. This completes the proof of Theorem 1.3. $lacksquare$

**Remark 3.2 (Heat kernel on Riemannian manifold).** Let $p_t(x, y), t > 0, x, y \in M$, be the heat kernel on a Riemannian manifold $M$. $p_t(x, \cdot)$ is a fundamental solution of the heat equation $\partial_t u = \Delta u$, where $\Delta$ denotes the Laplace-Beltrami operator. A complete Riemannian manifold with Ricci curvature bounded below always has the heat kernel [18]. We set $\nu_t = \nu^* p_t = (\int_M p_t(x, \cdot) dv(x)) vol$, where $vol$ denotes the Riemannian volume measure. We agree that $\nu_0 = \nu$. We call $\{\nu_t\}_{t \geq 0}$ the heat distribution or heat flow emanating from $\nu$. The following is taught us by Nicola Gigli.

**Proposition 3.3 ([1] p.44).** Let $(M, g)$ be a complete Riemannian manifold with Ricci curvature bounded below by $K$. Let $m^t_x$ be the heat distribution emanating from the Dirac measure $\delta_x$ for $x \in M$. Then the coarse Ricci curvature associated with $\{m^t_x\}_{x \in X}$ for any $t > 0$ satisfies

$$\inf_{x, y \in M} \kappa(x, y) \geq 1 - e^{-Kt}.$$ 

In [1], they proved more general spaces, called Riemann Ricci curvature bounded from below, have a lower bound of coarse Ricci curvature.

**Proof of Theorem 1.9.** As (2) follows from (1), it suffices to show (1).

By the definition of the coarse Ricci curvature, a lower bound $\kappa$ should satisfy $\kappa \leq 1$. Then we have $\kappa \leq 1 \leq 2 - \kappa$. Accordingly we assume $\lambda \neq 1$. Let $f$ be
a non-constant Lipschitz function on $X$ such that $\Delta f = \lambda f$ and $k$ the smallest Lipschitz constant of $f$. Replacing $f$ by $(1/k)f$ if necessary, we assume that

$$\sup_{x \neq y} \frac{|f(x) - f(y)|}{d(x,y)} = 1. \tag{3.4}$$

By using (2.2), we obtain that

$$d(x, y)(1 - \kappa) \geq W_1(m_x, m_y) \geq \int f dm_x - \int f dm_y = f(x) - f(y) + \Delta f(y) - \Delta f(x) = (1 - \lambda)(f(x) - f(y)) \tag{3.5}$$

for any $x, y \in X$. Since (3.5) is symmetric for $x$ and $y$, we have

$$|1 - \lambda| \cdot |f(x) - f(y)| \leq d(x, y)(1 - \kappa). \tag{3.5}$$

By (3.5) and (3.4) it turns out that

$$\frac{1 - \kappa}{|\lambda - 1|} \geq 1.$$

Thus we get $\kappa \leq \lambda \leq 2 - \kappa$. This completes the proof of Theorem 1.9.

To prove Corollary 1.10, it suffices to show the following Lemma.

**Lemma 3.4.** Let $(X, d, \nu, \{m_x\})$ be as in Corollary 1.10 and $\nu$ an eigenfunction of the Laplacian for the eigenvalue $\lambda \neq 1$. Then $\nu$ is a Lipschitz function.

**Proof.** Setting $u := (1 - \lambda)\nu$, we have

$$u(x) = \int u dm_x = \frac{1}{\nu(B_r(x))} \int_{B_r(x)} u d\nu.$$

Then it suffices to prove that $u$ is a Lipschitz function. It is easy to see that $u$ is continuous. In fact, since $X$ is geodesic space,

$$|u(x) - u(y)| = \left| \frac{1}{\nu(B_r(x))} \int_{B_r(x)} u d\nu - \frac{1}{\nu(B_r(y))} \int_{B_r(y)} u d\nu \right|$$

$$\leq \frac{1}{\nu(B_r(x))} \int_{B_r(x) \Delta B_r(y)} |u| d\nu + \frac{\nu(B_r(x) \Delta B_r(y))}{\nu(B_r(x)) \nu(B_r(y))} \int_{B_r(y)} |u| d\nu$$

$$\to 0 \quad \text{as} \quad y \to x$$

where $A \Delta B := (A \setminus B) \cup (B \setminus A)$. Since $B_r(x), B_r(y) \subset B_{2r}(x)$ for $d(x, y) < r$ and $X$ satisfies a Bishop-Gromov inequality, we have

$$|u(x) - u(y)| \leq \frac{1}{\nu(B_r(x))} \int_{B_r(x) \Delta B_r(y)} |u| d\nu + \frac{\nu(B_r(x) \Delta B_r(y))}{\nu(B_r(x)) \nu(B_r(y))} \int_{B_r(y)} |u| d\nu$$

$$\leq 2 \sup_{B_{2r}(x)} |u| \cdot m_x(B_r(x) \Delta B_r(y))$$

$$= 2 \sup_{B_{2r}(x)} |u| \cdot m_x(B_r(x) \setminus B_r(y))$$

$$\leq 2 \sup_{B_{2r}(x)} |u| \left( \frac{F(r - d(x, y)/2)}{F(r + d(x, y)/2)} d(x, y) + o(d(x, y)) \right)$$

which leads us to

$$|u(x) - u(y)| \leq C d(x, y)$$
for any $y$ sufficiently close to $x$, where $C = 2(F'(r)/F(r) + 1)\sup_{B_2(x)} |u|$. This means that $u$ is a local Lipschitz function. Since $X$ is a compact metric space, $u$ is a Lipschitz function. □

Proof of Corollary 1.10. By Lemma 3.4, any eigenfunction for the eigenvalue $\lambda \neq 1$ is Lipschitz. Then we have $\kappa \leq \lambda \leq 2 - \kappa$ for any $\lambda$ by Theorem 1.9. This completes the proof. □

Corollary 3.5. Let $(X,d,\nu)$ be a compact geodesic metric measure space satisfying $[BG_{K,N}]$ for two real numbers $K$ and $N > 1$. Then any eigenvalue $\lambda$ of the Laplacian associated with the $r$-step random walk satisfies

$$-2r \frac{s_{K,N}(r)^{N-1}}{\int_0^r s_{K,N}(t)^{N-1} \, dt} \leq \lambda \leq 2 + 2r \frac{s_{K,N}(r)^{N-1}}{\int_0^r s_{K,N}(t)^{N-1} \, dt}.$$ 

Proof. The corollary follows from Theorem 1.9 and Corollary 1.10. □

Corollary 3.6. Let $X$ be a finite state space and $\{m_x\}_{x \in X}$ a random walk. Assume that $\kappa(x,y) \geq \kappa$ for any $x, y \in X$ and for a constant $\kappa$. Then any eigenvalue $\lambda$ of the Laplacian satisfies $\kappa \leq \lambda \leq 2 - \kappa$.

Proof. Any function on a finite set is Lipschitz. We apply Theorem 1.9. □

Remark 3.7. Ollivier proved Corollary 3.6 if $\kappa > 0$ and if an unique invariant distribution is reversible. We do not need these assumption.

Acknowledgement

The author is grateful to Professor Nicola Gigli for pointing out Proposition 3.3, Professor Shin-ichi Ohta for helpful comments and Professor Takashi Shioya for reading this paper and giving useful advices.

References

[1] L. Ambrosio, N. Gigli, and G. Savaré, Metric measure spaces with Riemannian Ricci curvature bounded from below, arXiv:1109.0222.
[2] , Calculus and heat flow in metric measure spaces and applications to spaces with Ricci bounds from below, arXiv:1106.2090.
[3] J. J. Frank Bauer and Shoping Liu, Ollivier-Ricci curvature and the spectrum of the normalized graph Laplace operator, arXiv:1105.3803v1.
[4] M. Erbar, The heat equation on manifolds as a gradient flow in the Wasserstein space, Ann. Inst. Henri Poincaré Probab. Stat. 46 (2010), no. 1, 1–23 (English, with English and French summaries).
[5] N. Gigli, K. Kuwada, and S.-i. Ohta, Heat flow on Alexandrov spaces, arXiv:1008.1319.
[6] R. Jordan, D. Kinderlehrer, and F. Otto, The variational formulation of the Fokker-Planck equation, SIAM J. Math. Anal. 29 (1998), no. 1, 1–17.
[7] N. Juillet, Geometric inequalities and generalized Ricci bounds in the Heisenberg group, Int. Math. Res. Not. IMRN 13 (2009), 2347–2373.
[8] Jürgen Jost and Shiping Liu, Ollivier’s Ricci curvature, local clustering and curvature dimension inequalities on graphs, arXiv:1103.4037v2.
[9] Y. Lin and S.-T. Yau, Ricci curvature and eigenvalue estimate on locally finite graphs, Math. Res. Lett. 17 (2010), no. 2, 343–356.
[10] J. Lott and C. Villani, Ricci curvature for metric-measure spaces via optimal transport, Ann. of Math. (2) 169 (2009), no. 3, 903–991.
[11] S.-i. Ohta, On the measure contraction property of metric measure spaces, Comment. Math. Helv. 82 (2007), no. 4, 805–828.
[12] Y. Ollivier, Ricci curvature of Markov chains on metric spaces, J. Funct. Anal. 256 (2009), no. 3, 810–864.
[13] G. Savaré, Gradient flows and diffusion semigroups in metric spaces under lower curvature bounds, C. R. Math. Acad. Sci. Paris 345 (2007), no. 3, 151–154 (English, with English and French summaries).
[14] K.-T. Sturm, *On the geometry of metric measure spaces. I*, Acta Math. **196** (2006), no. 1, 65–131.

[15] ———, *On the geometry of metric measure spaces. II*, Acta Math. **196** (2006), no. 1, 133–177.

[16] C. Villani, *Topics in optimal transportation*, Graduate Studies in Mathematics, vol. 58, American Mathematical Society, Providence, RI, 2003.

[17] ———, *Optimal transport*, Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], vol. 338, Springer-Verlag, Berlin, 2009. Old and new.

[18] M.-K. von Renesse and K.-T. Sturm, *Transport inequalities, gradient estimates, entropy, and Ricci curvature*, Comm. Pure Appl. Math. **58** (2005), no. 7, 923–949.