The Balian-Brézin Method in Relativistic Quantum Mechanics

H.-C. Jean\textsuperscript{1}, G. L. Payne\textsuperscript{2} and W. N. Polyzou\textsuperscript{2,*}

\textsuperscript{1}Department of Physics, Florida State University
Tallahassee, FL 32306, USA
\textsuperscript{2}Department of Physics and Astronomy, The University of Iowa
Iowa City, IA 52242, USA

July 13, 2021

Abstract

The method suggested by Balian and Brézin for treating angular momentum reduction in the Faddeev equations is shown to be applicable to the relativistic three-body problem.

1 Introduction

The Faddeev equations\textsuperscript{[1]} provide a formulation of the quantum mechanical three-body problem as a compact kernel integral equation. Compact kernel integral equations are well suited to numerical solution because the kernel of the equation can be uniformly approximated by a finite dimensional matrix, reducing the dynamical equations to a system of linear algebraic equations. Numerical solutions of the Faddeev equations require that the abstract operator equations be expressed as integral equations formulated in a chosen basis. The equations can be simplified by using bases that exploit the rotational and translational symmetries of the problem. The method suggested by Balian and Brézin\textsuperscript{[2]} has proved useful in applications. This paper illustrates how the Balian-Brézin method generalizes to the relativistic case.

*Research Supported by the U.S. Department of Energy, Contract DE-FG02-86ER40286.
2 The Non-Relativistic Case

The Hamiltonian for a system of three particles interacting with two-body interactions has the form

$$H = H_0 + \sum_{i<j} V_{ij},$$

where $H_0$ is the kinetic energy operator for three particles and $V_{ij} = \hat{V}_{ij} \otimes I_k$ is the two-body interaction between particle $i$ and $j$ imbedded in the three particle Hilbert space. In what follows the interaction $V_{12}$ is denoted by $V_3$, and likewise for cyclic permutations of 1,2, and 3.

The three-body 2-cluster to 2-cluster transition operators are

$$T^{ij} \equiv V^j + V^i R(z) V^j, \quad V^i = \sum_{j \neq i} V_j,$$

where $R(z) := (z - H)^{-1}$ with $z = E \pm i0^+$. The second resolvent identity implies that the transition operators satisfy the coupled integral equations:

$$T^{ij}(z) = V^j + \sum_{k \neq i} V_k R_k(z) T^{kj}(z),$$

where $R_k(z) := (z - H_0 - V_k)^{-1}$ is the resolvent of the Hamiltonian for the interacting pair plus spectator.

Equations (3) have the same kernel as the Alt-Grassberger-Sandhas equations [3], which are used instead of the Faddeev equations in most numerical applications.

They are also representative of the type of equation to which the Balian Brézin method can be applied. The analysis that follows is valid for any type of connected kernel three-body equation.

In applications Eq. (3) is replaced by a linear system of algebraic equations for the matrix elements of $T^{ij}(z)$ in a chosen basis. The complexity of the algebraic equations is reduced by choosing a suitable basis.

The Hilbert space for the three-body system is the tensor product of three single-particle Hilbert spaces. Let $p_i$, $m_i$, $s_i$ and $\mu_i$ denote the momentum, mass, spin, and magnetic quantum number of the $i$-th particle. Plane wave basis vectors for the three-body system are

$$|p_1\mu_1p_2\mu_2p_3\mu_3\rangle$$

with normalization

$$\langle p_1\mu_1p_2\mu_2p_3\mu_3|p_1'\mu_1'p_2'\mu_2'p_3'\mu_3'\rangle = \prod_{i=1}^{3} \delta(p_i - p_i') \delta_{\mu_i \mu_i'}.$$

The kernel of Eq. (3)

$$K(i) := V_i R_i(z)$$

commutes with the total linear momentum operator and the linear momentum operator of the spectator particle, $i$. It is separately invariant under rotations of the spectator and the interacting pair.
The number of non-zero matrix elements of the kernel can be reduced by evaluating it in a basis that exploits these symmetries. One basis commonly used is the basis of angular momentum eigenstates defined by the following linear combination of the basis vectors (4),

$$|P_q k_i; \mu_j L S j l s\rangle = \int d\hat{q}_i d\hat{k}_i |P_q k_i; \mu_j \mu_j \mu_k\rangle Y_{\mu_L \mu_L}(\hat{q}_i) Y_{\mu_i}(\hat{k}_i) \langle s_j \mu_j s_k \mu_k | s \mu_s \rangle \times \langle l \mu_1 s_\mu_s | j \nu_j \rangle \langle s_i \mu_i j \nu_j | S \mu_S \rangle \langle L \mu_L S \mu_S | J \mu_J \rangle ,$$

where

$$|P_q k_i; \mu_1 \mu_2 \mu_3\rangle = \int d\hat{p}_1 d\hat{p}_2 d\hat{p}_3 |P_q \mu_1 \mu_2 \mu_2 \mu_3\rangle \delta(\hat{p}_1 - \hat{p}_1(P, q_i, k_i)) \times \delta(\hat{p}_2 - \hat{p}_2(P, q_i, k_i)) \delta(\hat{p}_3 - \hat{p}_3(P, q_i, k_i))$$

and the momenta $P$, $q_i$, and $k_i$ are related to the single particle momenta by

$$P = p_1 + p_2 + p_3 ,$$

$$k_i = \frac{m_k}{m_j} p_j - \frac{m_i}{m_j} p_k ,$$

and

$$q_i = \frac{m_{jk}}{M} p_i - \frac{m_i}{M} p_{jk} ,$$

where $p_{jk} := p_j + p_k$, $M = m_1 + m_2 + m_3$, $m_{jk} = m_j + m_k$, and $i, j, k$ are cyclic permutations of 1, 2, 3. With this choice the Jacobian of the variable change $\{P, q_i, k_i\} \rightarrow \{p_1, p_2, p_3\}$ is unity.

The matrix elements of the kernel of Eq. (3) in this basis are

$$\langle P_q k_i; J \mu_j L S j l s | V_i R_i(z) | P' q_{i'} k_{i'}; J' \mu_j' L' S' j' l' s' \rangle = \delta(P - P') \frac{\delta(q_i - q_{i'})}{q_{i'}^2} \delta_{j j'} \delta_{\mu_j \mu_j'} \delta_{l l'} \delta_{S S'} \times \langle k_{i} j l s | \tilde{V}_i \tilde{R}_i \left( z - \frac{P^2}{2M} - q_{i'}^2 \frac{M}{2m_i m_{jk}} \right) | k_{i'} j' l' s' \rangle ,$$

where

$$\tilde{V}_i \tilde{R}_i \left( z - \frac{P^2}{2M} - q_{i'}^2 \frac{M}{2m_i m_{jk}} \right)$$

is

$$\tilde{V}_i \left[ z - \frac{P^2}{2M} - q_{i'}^2 \frac{M}{2m_i m_{jk}} - k_{i'}^2 \frac{m_{jk}}{2m_j m_k} - \hat{V}_i \right]^{-1}$$

and $\tilde{V}_i$ is the two-body interaction.

The basis (7) exploits the translational and rotational invariance of the system and the spectator particle. The coupling implicit in Eq. (3) breaks the translational and rotational invariance associated with the original spectator particle. This can be illustrated by considering the iterated kernel. Let $K(i)$ denote the kernel with particle $i$ as spectator and let $|i\rangle$ denote the basis vector (7) corresponding to particle $i$ being the spectator. The matrix elements of the iterated kernel are

$$\sum_{j \neq i} \langle i | K(i) K(j) | j \rangle = \sum_{j \neq i} \langle i | K(i) | i \rangle \langle i | j \rangle \langle j | K(j) | j \rangle ,$$

(14)
where in order to express the kernel in a basis where it has the form \( \langle 1 | 2 \rangle \) it is necessary to introduce the change of basis \( \langle i | j \rangle \). Since \( j \neq i \) this changes the spectator and breaks the invariance associated with the spectator particle.

The change of basis is computed from the definitions by transforming the bras and kets in Eq. \( \langle 12 \rangle \) to the tensor product of one-body basis vectors and evaluating the overlap. The general result is determined by taking cyclic permutations of

\[
\langle 1 | 2 \rangle = \delta(P - P') \delta(E - E') \frac{\delta_{J,J'} \delta_{\mu,J} \delta_{\mu',J'}}{2J + 1} \\
\times \frac{8\pi^2 m_{23} m_{13}}{m_1 m_2 m_3 k_1 k_2 q_1 q_2} Y_{\mu L}^* (\hat{q}_1) Y_{\nu L'}^* (\hat{k}_1) Y_{\mu' L} (\hat{q}_2) Y_{\nu' L'} (\hat{k}_2) \\
\times \langle J J' | L \mu L S | j \mu j \rangle \langle j \nu | l \mu s \mu s \rangle \langle s \mu s | s_2 \mu_2 s_3 \mu_3 \rangle \\
\times \langle s_3 \mu_3 s_1 s_2 | s' \mu' s' \rangle \langle l' \mu' s' | j' \nu' \rangle \langle s_2 \mu_2 j' \nu' | S' \mu' S \rangle \langle L' \mu' L' S' | J' \rangle \langle J' \rangle , \tag{15}
\]

where the sums over all repeated magnetic quantum numbers are implicit. This expression includes four two-dimensional integrals over the angles associated with each relative momenta. Two of the integrals can be done in terms of the angular parts of the delta functions. This still leaves two two-dimensional integrals over the angles and two delta functions that fix the magnitude of the relative momenta. In this form the change of basis is a complicated transformation to implement numerically.

Balian and Brézin exploit the rotational invariance of the overlap matrix elements \( \langle 12 \rangle \) to facilitate the computation. Rotational invariance implies that the matrix element \( \langle 12 \rangle \) is equal to kronecker deltas in \( \delta_{J,J'} \delta_{\mu,J} \delta_{\mu',J'} \) multiplied by an expression independent of \( \mu, J \). It follows that \( \langle 12 \rangle \) is equal to its average over \( \mu, J \). The averaging makes the integrand in \( \langle 12 \rangle \) invariant under simultaneous rotations of the vectors \( q_1, k_1, q_2, \) and \( k_2 \) which implies that the integrand depends only on the independent invariants \( q^2_1, q^2_2 \) and \( q_1 \cdot k_1 \). The quantity \( q_1 \cdot k_1 \) is fixed in terms of \( q^2_1, q^2_2, \) and \( k^2_1, q^2_2, \) and \( k^2_2 \). The result is that, after the delta function fixes the integral over the cosine of the angle between \( \hat{q}_1 \) and \( \hat{k}_1 \), the integral over the remaining angles is equal to a phase-space factor times the invariant integrand. The integrand can be evaluated by choosing the vectors \( q_1, k_1, q_2, \) and \( k_2 \) to have any convenient values consistent with the kinematics:

\[
\langle 1 | 2 \rangle = \delta(P - P') \delta(E - E') \frac{\delta_{J,J'} \delta_{\mu,J} \delta_{\mu',J'}}{2J + 1} \\
\times \frac{8\pi^2 m_{23} m_{13}}{m_1 m_2 m_3 k_1 k_2 q_1 q_2} Y_{\mu L}^* (\hat{q}_1) Y_{\nu L'}^* (\hat{k}_1) Y_{\mu' L} (\hat{q}_2) Y_{\nu' L'} (\hat{k}_2) \\
\times \langle J J' | L \mu L S | j \mu j \rangle \langle j \nu | l \mu s \mu s \rangle \langle s \mu s | s_2 \mu_2 s_3 \mu_3 \rangle \\
\times \langle s_3 \mu_3 s_1 s_2 | s' \mu' s' \rangle \langle l' \mu' s' | j' \nu' \rangle \langle s_2 \mu_2 j' \nu' | S' \mu' S \rangle \langle L' \mu' L' S' | J' \rangle \langle J' \rangle , \tag{16}
\]

where \( E \) is the kinetic energy and the unit vectors are fixed by \( k_i \) and \( q_i \), which can be chosen arbitrarily subject to the constraints that fix \( q^2_i, k^2_i \), and \( k_i \cdot q_i \). Note that the above expression includes a sum over the overall magnetic quantum number, \( \nu \).

Balian and Brézin suggest three choices of fixing the direction of \( k_i \) and \( q_i \), subject to the constraint on the scalar product, which facilitate the computation of these expressions. These three choices lead to additional simplifications. This method for treating the change of basis matrix elements is employed in many existing numerical treatments of the three-body problem. The same considerations apply to configuration space computations.
The form of the recoupling coefficient in Eq. (16) uses one of the delta functions in the two momentum variables to do the integral over \( \hat{k}_i \cdot \hat{q}_i \) while the other gives the energy-conserving delta functions. In principle these delta functions can be used to perform integrals over any pair of variables. In applications it is convenient to use the delta functions in the relative momenta to do the two integrals over the relative momenta. In this case the one-dimensional integral over \( \hat{k}_i \cdot \hat{q}_i \) must be calculated numerically.

3 The Relativistic Case

The Balian-Brézin method is also applicable to large class of relativistic formulations of the three-body problem, which includes all generalized Bakamjian-Thomas formulations, including specific Bakamjian-Thomas formulations in any of Dirac’s forms of the dynamics.

Relativistic invariance of a quantum model requires that all probabilities have values independent of the choice of inertial coordinate system, where by definition any two inertial coordinate systems are related by Poincaré transformations continuously connected to the identity. Wigner’s theorem states that this condition implies the existence of a unitary ray representation of the Poincaré group on the three-body Hilbert space.

The problem of constructing the unitary representation of the Poincaré group plays the same role in relativistic quantum mechanics as constructing the unitary representation of the one-parameter time evolution group in non-relativistic quantum mechanics. In the non-relativistic case it is sufficient to solve the eigenvalue problem for the Hamiltonian. The one-parameter time evolution group is expressed in terms of the eigenfunctions and eigenvalues of the Hamiltonian as

\[
U(t) = \sum_E |E\rangle e^{-iEt} \langle E| .
\]

In the relativistic case it is sufficient to solve the eigenvalue problem for the Casimir operators of the Poincaré group. For system of massive particles this is equivalent to finding the simultaneous eigenstates of the mass and spin operators. Representation theory for the Poincaré group is used to construct the unitary representation, \( U(\Lambda, a) \), of the Poincaré group in terms of the eigenvalues and eigenstates of the mass and spin similar to the manner that the eigenvalues and eigenstates of \( H \) are used to construct \( U(t) \) in Eq. (17).

Bakamjian-Thomas models are models with interactions that commute with and are independent of three independent functions of the non-interacting four momentum and all components of a chosen non-interacting spin. The analysis that follows is limited to Bakamjian-Thomas type models. The dynamical problem in a Bakamjian-Thomas dynamics is to find simultaneous eigenstates of the mass and non-interacting spin operator in a suitable basis. The Balian-Brézin method can be applied to Bakamjian-Thomas models because the mass operator commutes with the non-interacting spin vector, however, the explicit justification in the relativistic case is more complicated because the quantities that replace \( q_i \) and \( k_i \) in the relativistic case do not generally transform as vectors under rotations.

The mass operator in the relativistic case has the same form as the Hamiltonian in the non-relativistic case, consisting of a mass operator \( M_0 \) for three non-interacting particles plus pairwise interactions

\[
M = M_0 + \sum_{i<j} V_{ij} .
\]
implies the components of the transition operators satisfy the coupled equations

\[ T^{ij} := V^j + V^i R(z) V^j, \quad V^i = \sum_{j \neq i} V_j, \quad (19) \]

where \( R(z) = (z - M)^{-1} \) is the resolvent of the mass operator. The second resolvent identity implies the components of the transition operators satisfy the coupled equations

\[ T^{ij}(z) = V^j + \sum_{k \neq i} V_k R_k(z) T^{kj}(z), \quad (20) \]

where for the relativistic equation \( R_k(z) := (z - M_0 - V_k)^{-1} \) is the resolvent of the mass operator for the interacting pair plus spectator. Eq. (20) has the same form as the non-relativistic equation. Differences appear in the relation of the two-body interactions in the three-particle Hilbert space and the two-body interactions in the two-body Hilbert space. The structure of the interactions is given in Eq. (13) below.

The Hilbert space for a single particle in relativistic quantum mechanics is an irreducible representation space for the Lorentz group corresponding to the mass and spin of the particle. Vectors in this space can be expanded as linear combinations of simultaneous eigenstates of the linear momentum and \( z \)-component of spin. In relativistic quantum mechanics there are many different spin operators that satisfy \( SU(2) \) commutation relations whose square is the total spin. Although the spectrum of the magnetic quantum numbers is fixed by the \( SU(2) \) commutation relations, the physical interpretation of the spin operator is determined by the transformation properties of the single particle states under the Lorentz group. In general the single particle vectors \( |p\mu\rangle \) transform as mass-\( m \) spin-\( s \) irreducible representations of the Lorentz group

\[ U(\Lambda, a)|p\mu\rangle = e^{i\Lambda p_m \cdot a} |p\Lambda\mu'\rangle \left| \frac{\omega_m(p\Lambda)}{\omega_m(p)} \right|^{1/2} D_{\mu\mu'}^s (B^{-1}(\Lambda p_m) \Lambda B(p_m)), \quad (21) \]

where \( \omega_m(p) := \sqrt{p^2 + m^2} \) is the energy, \( p_m = (\omega_m(p), p) \) is the four momentum of a particle with mass \( m \) and momentum \( p \), \( p\Lambda := \Lambda p_m \) is the Lorentz transform of the four momentum \( p_m \), and \( B(p_m) \) is a Lorentz-boost-valued function of \( p_m \) with the property

\[ B(p_m)p_0 = p_m, \quad (22) \]

where \( p_0 := (m, 0, 0, 0) \) is the 0-momentum four-vector. The quantity

\[ \frac{\left| \frac{\omega_m(p\Lambda)}{\omega_m(p)} \right|^{1/2}}{\left| \frac{\partial p\Lambda}{\partial p} \right|^{1/2}} \quad (23) \]

fixes the normalization of the transformed state to ensure the unitarity of \( U(\Lambda, a) \).

The combination \( R_w(\Lambda, p_m) := B^{-1}(\Lambda p_m) \Lambda B(p_m) \) is a rotation for any \( \Lambda \), called the Wigner rotation associated with the boost \( B(p_m) \). The interpretation of the magnetic quantum number is determined by the choice of \( B(p_m) \). A boost \( B(p_m) \) is defined up to a rotation valued function \( R(p_m) \) of \( p_m \)

\[ B(p_m) \rightarrow B'(p_m) = B(p_m)R(p_m). \quad (24) \]
The canonical boost $B_c(p_m)$ is the unique boost with the properties

$$B_c(p_0) = I$$

(25)

and

$$R_w(R, p_m) = B^{-1}_c(R p_m) R B_c(p_m) = R$$

(26)

for any rotation $R$. Eq. (26) states that for canonical boosts the Wigner rotation of a rotation is the rotation itself. This is not true for other types of boosts.

Any other boost is related to a canonical boost by a $p$-dependent rotation as in Eq. (24). This rotation is called a generalized Melosh rotation\textsuperscript{[7]}. The helicity spin and front-form spin are examples of spins corresponding to non-canonical boosts\textsuperscript{[7]}. These two choices are distinguished by other special properties.

A basis for the three-particle Hilbert space is the tensor product of three single particle bases. Basis vectors have the form

$$|p_1 \mu_1 p_2 \mu_2 p_3 \mu_3\rangle$$

(27)

with the same normalization convention as the non-relativistic expression (4).

A basis for the three-body system that simplifies the matrix elements of the kernel $K(i) = V_i R_i(z)$ of the relativistic three-body equations (4) is constructed by finding the coefficients of the linear transformation that take the product of three irreducible representations of the Poincaré group to a direct integral of irreducible representations. This is equivalent to the problem of constructing Clebsch-Gordan coefficients of the Poincaré group. This replaces the successive pairwise coupling of the irreducible representation spaces of the Euclidean group (rotations and translations) used in the non-relativistic reduction.

The construction of the Clebsch-Gordan coefficients of the Poincaré group is most easily understood by successive coupling of pairs of irreducible representations. To treat the general case first consider the case of canonical spin. Kinematic variables for the two particle system are

$$P = p_1 + p_2 , \quad m_{12} = \sqrt{-P \cdot P} , \quad k_i = B^{-1}_c(P) p_i ,$$

(28)

corresponding to the total four momentum of the non-interacting pair, the invariant mass of the non-interacting pair, and the relative momentum of the non-interacting pair, respectively. The relative momentum defined in Eq. (28) is not a true four-vector. It undergoes Wigner rotations when the system is Lorentz-transformed.

The two-body basis vectors that transform irreducibly under the action of the tensor product of one-body representations of the Poincaré group are

$$|P \mu(kjls)\rangle := U_1[B_c(P)] \otimes U_2[B_c(P)] \int d\hat{k} |k, \mu_1, -k, \mu_2\rangle Y_{l\mu_l}(\hat{k})$$

\begin{align*}
&\times \langle s_{1 \mu_1 s_{2 \mu_2}}|s_{\mu s}\rangle \langle \mu | s_{\mu s}\rangle j \mu \\
&= \int d\hat{k} \langle p_1(P, k), \mu_{1 l}, p_2(P, k), \mu_{2 l}\rangle D^3_{\mu_{1 l} \mu_{2 l}}(R_{wc}(B_c(P), k_1)) D^2_{\mu_{2 l} \mu_2}(R_{wc}(B_c(P), k_2)) Y_{l\mu_l}(\hat{k}_1) \\
&\times \langle s_{1 \mu_1 s_{2 \mu_2}}|s_{\mu s}\rangle \langle \mu | s_{\mu s}\rangle j \mu ,
\end{align*}

(29)
where \( \mathbf{k} = \mathbf{k}_1 = -\mathbf{k}_2 \). The factor \( m_{12}/\omega_{m_{12}}(\mathbf{P}) \) fixes the normalization to be

\[
\langle \mathbf{P}_\mu(kjls)|\mathbf{P}'_{\mu'}(k'j'l's') \rangle = \delta(\mathbf{P} - \mathbf{P'}) \frac{\delta(k - k')}{k^2} \delta_{jj'} \delta_{\mu\mu'} \delta_{ss'} .
\]

The single particle and combined spins can be transformed from canonical to any other type of spin (i.e., a spin associated with an arbitrary boost \( B_x(p_m) \)) with the unitary transformation\[7\]

\[
|\mathbf{P}_\mu \rangle_x = |\mathbf{p}_\nu \rangle_c D^{i}_{\nu\mu}(\mathbf{R}_{x_c}(\mathbf{p})) , \quad \mathbf{R}_{x_c}(\mathbf{p}) := B^{-1}_c(\mathbf{p})B_x(\mathbf{p}) ,
\]

where \( \mathbf{R}_{x_c}(\mathbf{p}) \) is a generalized Melosh rotation. A direct calculation using the relations

\[
Y_{\mu\nu}(\mathbf{k}_c) D^{i}_{\mu\nu}(\mathbf{R}_{x_c}(\mathbf{p})) = Y_{\mu\nu}(\mathbf{R}_{x_c}(\mathbf{p})\mathbf{k}_c) = Y_{\mu\nu}(\mathbf{k}_x)
\]

and

\[
\mathbf{R}_{x_c}(\mathbf{p}_1)\mathbf{R}_{wc}(B_c(p), k_{1c})\mathbf{R}_{x_c}(\mathbf{p}) = \mathbf{R}_{wx}(B_x(p), k_{1x})\mathbf{R}_{x_c}(\mathbf{k}_{1x}) \quad (1 \leftrightarrow 2)
\]

shows

\[
|\mathbf{P}_\mu(kjls) \rangle_x := \int d\mathbf{k} |\mathbf{p}_1(\mathbf{P}, \mathbf{k}), \mu_1, \mathbf{p}_2(\mathbf{P}, \mathbf{k}), \mu_2 \rangle_x \frac{\omega_{m_1}(\mathbf{P}_1)}{\omega_{m_1}(\mathbf{k}_1)} \frac{\omega_{m_2}(\mathbf{P}_2)}{\omega_{m_2}(\mathbf{k}_2)} \frac{m_{12}}{\omega_{m_1}(\mathbf{P})}^{1/2} \times D^{i}_{\mu_1}\mu_1[R_{wx}(\mathbf{B}_x(\mathbf{P}), k_{1x})] D^{i}_{\mu_2}\mu_2[R_{wx}(\mathbf{B}_x(\mathbf{P}), k_{2x})] \times Y_{\mu\nu}(\mathbf{k}_{1x}) \langle s_1 \mu_1 s_2 \mu_2 | s \mu \rangle \langle l \mu_1 s \mu_1 | l \mu \rangle .
\]

In expression (34) the argument of the \( D \)-functions is the product of a generalized Melosh rotation followed by the Wigner rotation associated with the \( x \)-spin. The quantity \( \mathbf{k}_x \) is related to the canonical \( \mathbf{k}_c \) by a generalized Melosh rotation

\[
\mathbf{k}_x = B^{-1}_x(\mathbf{p})B_c(\mathbf{p})\mathbf{k}_c = \mathbf{R}_{x_c}(\mathbf{p})\mathbf{k}_c .
\]

As a result of this construction the state \( |\mathbf{P}_\mu(kjls) \rangle_x \) transforms like a particle with mass \( m_{ij} = \sqrt{m_i^2 + k^2} + \sqrt{m_j^2 + k^2} \) and spin \( j \) where the magnetic quantum number transforms as an \( x \)-spin:

\[
U_i(\Lambda, a) \otimes U_j(\Lambda, a) |\mathbf{P}_\mu(kjls) \rangle_x = e^{i \Lambda \cdot p} a |\mathbf{P}_\mu(kjls) \rangle_x \frac{\omega_{m_1}(\mathbf{P}_\Lambda)}{\omega_{m_1}(\mathbf{P})}^{1/2} D^{j}_{\mu}\mu[R_{wx}(\Lambda, P)] .
\]

The transformation properties of this two-particle state are identical to those of a single particle.

To construct the three particle basis the tensor product for a pair plus spectator

\[
|\mathbf{p}_{ijk \mu \nu}(kjls) \rangle_x \otimes |\mathbf{p}_i \mu \rangle_x
\]

is decomposed into a direct integral of irreducible representations by repeating the above analysis replacing one of the single particle states by the two-body state (34). The new kinematic quantities are

\[
P = p_i + p_{jk} , \quad M_0 = \sqrt{-\mathbf{P} \cdot \mathbf{P}} , \quad q_i = B^{-1}_c(\mathbf{P})p_i , \quad q_{jk} = B^{-1}_c(\mathbf{P})p_{jk} .
\]
where $\mathbf{q} := \mathbf{q}_i = -\mathbf{q}_{jk}$. In addition, the two-body kinematic variables are
\[
k_j = B_c^{-1}(p_{jk})p_j, \quad k_k = B_c^{-1}(p_{jk})p_k, \quad \mathbf{k} := \mathbf{k}_j = -\mathbf{k}_k.
\]
The resulting basis is related to the single particle bases by
\[
|\mathbf{P}; qJLSkjl\rangle := \int d\mathbf{q}d\mathbf{k}|\mathbf{p}_i\mu'_i|\mathbf{p}_j\mu'_j|\mathbf{p}_k\mu'_k\rangle \\
\times \left|\frac{\omega_{m_1}(\mathbf{p}_i)}{\omega_{m_1}(\mathbf{p}_j)}\right| \left|\frac{\omega_{m_j}(\mathbf{p}_j)}{\omega_{m_j}(\mathbf{p}_k)}\right| \left|\frac{\omega_{m_k}(\mathbf{p}_k)}{\omega_{m_k}(\mathbf{q}_i)}\right| \left|\frac{\omega_{m_j}(\mathbf{q}_i)}{\omega_{m_j}(\mathbf{q}_k)}\right| \sqrt{M_0(M_0)}^{1/2} \\
\times D^{\mu'_i\mu_j}_{\mu_k\mu_j}[R_{wx}(B_x(p_{jk}), k_{jx})R_{xc}(k_{jx})]D^{\mu_j\mu_k}_{\mu_k\mu_k}[R_{wx}(B_x(p_{jk}), k_{kk})R_{xc}(k_{kk})] \\
\times Y_{1\mu_i}(\mathbf{k}_{ix})\langle s_j\mu_j s_k\mu_k | s_j s_k | j'_{jx} \rangle \\
\times D^{\nu'_j\nu_j}_{\mu_j\mu_j}[R_{wx}(B_x(P), q_{jx})R_{xc}(q_{jx})]D^{\nu_j\nu_k}_{\mu_k\mu_k}[R_{wx}(B_x(P), q_{kx})R_{xc}(q_{kx})] \\
\times Y_{L\mu_L}(\mathbf{q}_{ix})\langle j'_{jx} s_j s_k | s_j s_k | J'_{jx} \rangle.
\]
This basis also transforms irreducibly under the tensor product of three one-body representation of the Poincaré group,
\[
U_1(\Lambda, a) \otimes U_2(\Lambda, a) \otimes U_3(\Lambda, a)|\mathbf{P}; qJLSkjl\rangle
\]
are related to two-body interactions $v_i$ with matrix elements $\langle jkls|v_i|jk'l's'\rangle$ by
\[
\hat{V}_i(q^2)\hat{R}_i(z; q^2) = \left[\sqrt{q^2 + (m_{jk} + v_i)^2} - \sqrt{q^2 + m_{jk}^2}\right] \\
\times \left[z - \sqrt{q^2 + m_i^2} - \sqrt{q^2 + (m_{jk} + v_i)^2}\right]^{-1}.
\]
in the components of the linear momentum must be replaced by delta functions in the
appropriate functions of the non-interacting four-momentum. For instance, in Dirac’s point-
form dynamics the three components of the four-velocity replace the three components
of the momentum. In the case of Dirac’s front-form dynamics the delta functions in the
components of the three momentum are replaced by delta functions in the three components
of the four momentum that generate translations tangent to the light front, \(x^3 + t = 0\). The
front-form case also requires a special choice of spin operator.

As in the non-relativistic case, when Eq. (20) is iterated, a new interaction is introduced,
varying the symmetries associated with the initial spectator particle. Thus if \(\langle i|K(i)|i\rangle\) denotes the expression in Eq. (22) the matrix elements of the iterated kernel are

\[
\sum_{j \neq i} \langle i|K(i)K(j)|j \rangle = \sum_{j \neq i} \langle i|K(i)|i'\rangle \langle i'|j\rangle \langle j'|K(j)|i \rangle.
\]  

(44)

In order to take advantage of the simple form of the kernel in Eq. (22) it is necessary to compute the overlap \(\langle i|j\rangle\) where

\[
\langle 1|2\rangle := \langle P_\mu; q_1 J L_1 S_1 k_{1x} j_{1z} l_{1x} s_1|P'_\mu'; q'_2 J' L'_2 S'_2 k'_{2x} j'_{2z} l'_{2x} s'_2 \rangle,
\]  

(45)

and cyclic permutations. By direct calculation the matrix element is

\[
\langle 1|2\rangle = \int d\mathbf{q}_1 d\mathbf{k}_1 \int d\mathbf{q}'_2 d\mathbf{k}'_2 \prod_{i=1}^{3} \delta[p_i (P, q_1, k_1) - p_i (P', q'_2, k'_2)] \\
\times \left| \omega_{m_2} (p_2) \omega_{m_1} (p_1) m_{23} \omega_{m_1} (q_1) m_{12} \omega_{m_2} (q_2) \right| \left| M_0 \right|^{1/2} \\
\times \langle J_\mu |L_{\mu z} S_{\mu 5} |S_{\mu 5} \rangle \langle j_\mu |s_1 \mu_1 \rangle Y^{*}_{L_{\mu 5}} (q_{1x}) \\
\times \langle R_{\mu} (q_{2z}) R_{xz} (B_x^{-1} (P), p_{23}) \rangle \langle S_{\mu 3} (q_{1z}) R_{xz} (B_x^{-1} (P), p_{1}) \rangle \\
\times \langle j_\nu |s_2 \mu_2 s_3 \mu_3 \rangle Y^{*}_{L_{\nu 3}} (k_{1x}) \\
\times \langle R_{\nu} (k_{2z}) R_{xz} (B_x^{-1} (P), p_{23}) \rangle \langle R_{\nu} (k_{3z}) R_{xz} (B_x^{-1} (P), p_{23}) \rangle \\
\times \langle R_{\nu} (k_{2z}) R_{xz} (B_x (p_{31}), k'_{3x}) \rangle \langle R_{\nu} (B_x (p_{31}), k'_{1z}) \rangle \\
\times \langle j_\mu |s_3 \mu_3 \rangle \langle j_\nu |s_3 \mu_3 \rangle \langle j' \mu |s_3 \mu_3 \rangle \langle j' \nu |s_3 \mu_3 \rangle \\
\times \langle R_{\nu} (B_x (P), q_{31x}) R_{xx} (q_{31x}) \rangle \langle R_{\nu} (B_x (P), q_{23x}) R_{xx} (q_{23x}) \rangle \\
\times \langle R_{\nu} (B_x (P), q_{31x}) R_{xx} (q_{31x}) \rangle \langle R_{\nu} (B_x (P), q_{23x}) R_{xx} (q_{23x}) \rangle. 
\]  

(46)

Following Balian and Brézin, symmetry principles are used to evaluate this matrix element. To facilitate the evaluation of the matrix element \(\langle 1|2\rangle\) note that

\[
\langle P_\mu; q_1 J L_1 S_1 k_{1x} j_{1z} l_{1x} s_1|P'_\mu'; q'_2 J' L'_2 S'_2 k'_{2x} j'_{2z} l'_{2x} s'_2 \rangle = \langle P_\mu; q_1 J L_1 S_1 k_{1x} j_{1z} l_{1x} s_1|U(\Lambda, a)U(\Lambda, a)|P'_\mu'; q'_2 J' L'_2 S'_2 k'_{2x} j'_{2z} l'_{2x} s'_2 \rangle
\]  

(47)

for any \(\Lambda\) and \(a\). The kinematic quantities are

\[
k_1 = k_{2x} = -k_{3x}, \quad k'_2 = k'_{3x} = -k'_{1x}, \quad q_{1x} = -q_{23x}, \quad q'_{2x} = -q'_{31x}
\]  

(48)
with
\[ k_{2x} = B_x^{-1}(p_{23})p_2, \quad k_{3x} = B_x^{-1}(p_{23})p_3, \quad k'_{3x} = B_x^{-1}(p'_{31})p'_3, \quad k'_{1x} = B_x^{-1}(p'_{31})p'_1 \]
and
\[ q_{1x} = B_x^{-1}(P)p_1, \quad q_{23x} = B_x^{-1}(P)p_{23}, \quad q'_{2x} = B_x^{-1}(P)p'_2, \quad q'_{31x} = B_x^{-1}(P)p'_{31}. \]

Evaluating the right hand side of Eq. (47) gives
\[ \langle 1|2' \rangle = e^{i\lambda(P' - P) \cdot a} \left| \frac{\omega_{M_0}(P') \omega_{M_0}'(P_A)}{\omega_{M_0}(P) \omega_{M_0}'(P')} \right|^{1/2} D^{J\mu}_{\nu\mu}(R_{wx}(\Lambda', P)) \left[ D^{J\nu}_{\nu\mu'}(R_{wx}(\Lambda, P')) \right] \times \langle P_A \nu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0 0 \rangle \langle q'_2 J' L'_2 S'_2 k'_2 j'_2 l'_2 s'_2 \rangle. \]  

(51)

To deduce the implications of Eq. (51) consider different choices of \( \Lambda \) and \( a \). For \( \Lambda = I \) and arbitrary \( a \) the relations (51) cannot be satisfied unless \( P = P' \). For \( \Lambda = B_x(P) \), \( P = P' = P_0 = (M_0, 0, 0, 0) \) and \( a = 0 \) it follows that
\[ \langle P_0 \mu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0' \mu' \rangle = \left| \frac{M_0}{\omega_{M_0}(P) \omega_{M_0}'(P')} \right|^{1/2} \langle P_0 \mu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0' \mu' \rangle \times D^{J\nu}_{\nu\mu}(R_{wx}(R, P_0)) \]  

which shows that Eq. (51) can be expressed as \( \delta(P - P') \) multiplied by a quantity independent of \( P \). For the case that \( \Lambda = R \) is a rotation, \( P = P_0 = R P_0 \), and \( a = 0 \) it follows that
\[ \langle P_0 \mu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0' \mu' \rangle = \langle P_0 \nu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0' \mu' \rangle \times D^{J\nu}_{\nu\mu}(R_{wx}(R, P_0)) \]  

(52)

The left hand side of Eq. (53) is independent of \( R \). Since the rest boosts \( B_x(P_0) \) are rotations (normally chosen to be the identity), \( R_{wx}(R, P_0) \) is a representation of \( SU(2) \) which can be parameterized by elements of \( SU(2) \) and integrated over the group. Since the Haar measure is normalized to unity the integral is still equal to the left side of Eq. (53). The integral over the \( D \) functions can be done explicitly using
\[ \int dR D^{J\nu}_{\nu\mu}(R_{wx}(R, P_0)) D^{J\nu'}_{\nu\mu'}(R_{wx}(R, P_0)) = \frac{1}{2J + 1} \delta_{J, J'} \delta_{\nu, \nu'} \delta_{\mu, \mu'}, \]  

(54)

with the result that
\[ \langle P_0 \mu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0' \mu' \rangle = \delta_{J, J'} \delta_{\mu, \mu'} \frac{1}{2J + 1} \langle P_0 \nu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P_0' \mu' \rangle \times D^{J\nu}_{\nu\mu}(R_{wx}(R, P_0)) \]  

(55)

The general form of the matrix element is obtained by incorporating all of the consequences of Eq. (51):
\[ \langle P \mu; q_1 J L_1 S_1 k_1 j_1 l_1 s_1 | P' \mu' \rangle = \delta_{J, J'} \delta_{\mu, \mu'} \delta(P - P') \delta[M_0(q_1, k_1) - M_0(q'_2, k'_2)] \times A_1(q_1 J L_1 S_1 k_1 j_1 l_1 s_1; q'_2 J' L'_2 S'_2 k'_2 j'_2 l'_2 s'_2) \]  

(56)
where the amplitude \( A_J(1; 2) \) is invariant under the non-interacting representation of the Poincaré group. From the definition (53) the amplitude \( A_J(1; 2) \) can be computed from the matrix element \( \langle 1|2' \rangle \) in Eq. (49),

\[
\delta(M - M')A_J(1; 2') = \frac{1}{2J + 1} \int d^3P \langle 0 \mu; q_1^JL_1S_1k_1j_1l_1s_1|P_0\mu; q_2^JL_2S_2k_2j_2l_2s_2 \rangle .
\]

(57)

Since \( A_J(1; 2') \) is invariant, the computation of this quantity is facilitated by evaluating this expression for \( P' = P_0 \).

The last step used by Balian and Brézin is to exploit the rotational invariance. To do this in the relativistic case let the total four momentum be \( P_0 = RP_0 \) and evaluate

\[
\langle P_0\mu; q_1^JL_1S_1k_1j_1l_1s_1|P_0'\mu; q_2^JL_2S_2k_2j_2l_2s_2 \rangle
\]

\[
= \int \langle P_0\mu; q_1^JL_1S_1k_1j_1l_1s_1|U(R)U(R)|P_1\mu; q_2^JL_2S_2k_2j_2l_2s_2 \rangle \times \langle 0\mu; q_2^JL_2S_2k_2j_2l_2s_2|P_0'\mu; q_2^JL_2S_2k_2j_2l_2s_2 \rangle \times \delta(R)
\]

\[
= \int D_{\nu}(R) \langle P_0\nu; q_1^JL_1S_1k_1j_1l_1s_1|RP_1\mu; RP_2\mu\nu|P_0'\nu; q_2^JL_2S_2k_2j_2l_2s_2 \rangle \times \delta(R)
\]

\[
= \int \langle P_0\nu; q_1^JL_1S_1k_1j_1l_1s_1|RP_1\mu; RP_2\mu\nu|P_0'\nu; q_2^JL_2S_2k_2j_2l_2s_2 \rangle \times \delta(R)
\]

(58)

This equation means that the matrix element \( \langle 1|2' \rangle \) averaged over the magnetic quantum numbers is invariant under simultaneous rotations of all of the single particle momenta that appear as intermediate states. The rotated \( q_i \)'s and \( k_i \)'s obtained by rotating the single particle momenta are (for \( P = P_0 \))

\[
q_{Ri} = R_{wx}(R, P_0)q_i , \quad k_{Ri} = R_{wx}(R, q_{jk})k_i .
\]

(59)

To apply the above result to the computation of Eq. (49), note that there are nine delta functions and eight variables of integration. Three of the delta functions lead to the overall momentum conserving delta function, leaving six delta functions. Since the \( k \) do not transform simply under rotations, it is practical to use four of the remaining six delta functions to perform the integrals over \( \hat{k}_1 \) and \( \hat{k}_2 \). Two delta functions remain. One factors out of the expression, giving the conservation of the invariant mass. The other can be used to fix the angle between \( \hat{q}_1 \) and \( \hat{q}_2 \). Three integrals over the two unit vectors \( \hat{q}_1 \) and \( \hat{q}_2 \) with fixed \( \hat{q}_1 \cdot \hat{q}_2 \) remain. If the matrix element is averaged over the magnetic quantum numbers then the resulting integrand is necessarily independent of the remaining three variables of integration. The integral is \( 8\pi^2 \) multiplied by the integrand. The result, after computing all of the Jacobians needed to convert the delta functions to the desired form, is:

\[
\langle 1|2' \rangle = \delta(P - P')\delta(M - M') \frac{8\pi^2}{2J + 1} \frac{1}{|k_{1x}| |k_{2x}| |q_{1x}| |q_{2x}|}
\]

\[
\times \frac{m^3_{23} m^3_{31}}{\omega_{m_2}(k_{1x}) \omega_{m_3}(k_{1x}) \omega_{m_1}(q_{1x}) \omega_{m_{23}}(q_{1x}) \omega_{m_2}(k_{2x}) \omega_{m_1}(k_{2x}) \omega_{m_2}(q_{2x}) \omega_{m_{31}}(q_{2x})}^{1/2}
\]

12
\[ \times (J \mu | L_{\mu_1} S_{\mu_2} | j_{\mu_j} s_{\mu_j} \mu_1) Y^*_{\mu_{L'}} (q_{1x}) \]

\[ \times D_{j_{\mu_j}}^i [R_{xx}(q_{23x}) R_{wx}(B^{-1}_x (P_0), q_{23x})] D^{s_1}_{\mu_{L'}} [R_{xx}(q_{1x}) R_{wx}(B^{-1}_x (P_0), q_{1x})] \]

\[ \times (j_{\mu_j} | s_{\mu_s} l_{\mu_1} | s_{\mu_s} s_{\mu_2} s_{\mu_3}) Y^*_{\mu_{L'}} (k_{lx}) \]

\[ \times D_{s_2}^s [R_{xx}(k_{2x}) R_{wx}(B^{-1}_x (q_{23x}), k_{2x})] D^{s_3}_{\mu_{L'}} [R_{xx}(k_{3x}) R_{wx}(B^{-1}_x (q_{23x}), k_{3x})] \]

\[ \times D^{s_3}_{s_2} [R_{wx}(B_x (q_{31x}) l_{k_{lx}')}, k_{3x}') R_{xx}(k_{2x})] D^{s_1}_{\mu_{L'}} [R_{wx}(B_x (q_{31x}), k_{1x}') R_{xx}(k_{1x})] \]

\[ \times \gamma_{l_{\mu_1} s_{\mu_1} \nu_1} (k_{2x}) (s_{3} \nu_1 | s'_{\mu_s} | l_{\mu_1} s_{\mu_s}) (j_{\mu_j}') \]

\[ \times D_{j_{\mu_j}'} [R_{wx}(B_x (P'_{0x}), q_{31x}) R_{x}(q_{31x})] D^{s_2}_{s_2} [R_{wx}(B_x (P'_{0x}), q_{22x}) R_{x}(q_{22x})] \]

\[ \times \gamma_{l_{\mu_1} s_{\mu_1} \nu_1} (q_{2x}) (j_{\mu_j}') s_{\mu_2} | S'_{\mu_s} | (L_{\mu_1} s_{\mu_2} | J_{\mu_1}) . \]  

(60)

The invariant part of Eq. (61) has been evaluated with \( P = P_0 \), which implies the replacement of all of the \( p_i \)'s by the corresponding \( q_i \)'s. In the case of the front-form or point form the spin must be chosen accordingly (front-from spin for front-form interactions, canonical spin for the point-form interactions) and the delta functions in Eq. (60) are replaced by

\[ \delta(P - P') \delta(M - M') \rightarrow \delta(P^+ - P'^+) \delta(P_1 - P_1') \delta(P_2 - P_2') \delta(M - M') \]  

(61)

in the front form, and by

\[ \delta(P - P') \delta(M - M') \rightarrow \delta(V - V') \frac{\delta(M - M')}{M} \]  

(62)

in the point form, where \( \{P^+, P_1, P_2\} \) are the front-form components of the four-momentum and \( V = P / M \) are the independent components of the four-velocity.

Note that if the magnitude of all of the \( k \)'s and \( q \)'s appearing in both the rotation matrices and the Jacobian factor in Eq. (61) are set equal to zero this expression has the non-relativistic expression (16) as a limit provided the rest boosts are chosen to be the identity. Specifically the \( D \) functions all become the identity and the factor inside the \(| \cdots |^{1/2} \) becomes \( m_{13} m_{23} / m_{1} m_{2} m_{3} \). The resulting expression is identical with Eq. (16).

The expression (61) explicitly involves the four unit vectors \( \hat{q}_{1x}, \hat{q}_{2x}, \hat{k}_{lx}, \) and \( \hat{k}_{2x} \). The \( \hat{q}'_{1x} \) can be evaluated in any geometry, subject to the constraint that the angle between the two unit vectors is fixed by kinematic considerations. These choices fix the quantities \( k_{lx} \). Different choices of the geometry used to evaluate the \( \hat{q}'_{1x} \) can lead to additional simplifications in the evaluation of Eq. (61), although in the relativistic case the choice of best geometry depends to some extent on the choice of spin. If the boosts satisfy \( B_x (P_0) = I \) then the rest Wigner rotations \( R_{wx}[B^{-1}_x(P_0), q] \) in Eq. (61) can be replaced by the identity. For canonical spin there are no generalized Melosh rotations, while for the front-from spin the Wigner rotations of front-form boosts are the identity, \( R_{af}(B_f(p), q) = I \). All of these properties lead to further simplifications of Eq. (61).

Each of the choices of spin and continuous variables in a Bakamjian-Thomas model implies a choice of representation. Although it is tempting to formulate a model using a choice that leads to the simplest Racah coefficient, any choice has implications for the structure of the representation of other operators, such as electromagnetic and weak current operators. The interacting system has interactions in different Poincaré generators for different choices of the Racah coefficients. For instance, the Racah coefficients leading to an instantaneous dynamics imply that the infinitesimal generators of rotationless Lorentz transformation
contain interactions, while Racah coefficients appropriate for a point-form dynamics imply interactions in the momentum operators. For more general choices of Racah coefficient there can be interactions in any number of Poincaré generators. For an operator, such as an electromagnetic current operator, that is well approximated in an impulse approximation in one representation may require large two-body contributions in another representation. Thus, other considerations may be important in choosing a representation.

As in the non-relativistic case, it is useful to replace the delta function in the kinematic mass by a delta function that expresses the invariant mass explicitly in terms of the relative momenta,

$$\delta(M - M') \rightarrow \delta(M(k_1, q_1) - M(k'_2, q'_2)),$$

where in computing this delta function it is important to note that only three of the variable can be considered independent. This replacement allow one to do the integral over one of the relative momenta.

The analysis above shows that the methods suggested by Balian and Brézin is applicable to relativistic three-body equations of the Bakamjian-Thomas type. The relativistic expressions for the recoupling coefficients were shown reduced to the non-relativistic ones in the non-relativistic limit.

References

[1] L. D. Faddeev, *Mathematical Aspects of the Three-Body Problem in Quantum Scattering Theory*, Israel Program for Scientific Translation, Jerusalem, 1965.

[2] B. Balian and E. Brézin, Il Nuovo Cim. **69**, 403 (1969).

[3] E. Alt, P. Grassberger, W. Sandhas, Nucl. Phys. **B2**, 167 (1967).

[4] B. Bakamjian and L. H. Thomas, Phys. Rev. **92**, 1300 (1953).

[5] P. A. M. Dirac, Rev. Mod. Phys. **21**, 392 (1949).

[6] E. P. Wigner, Ann. Math. **40**, 149 (1939).

[7] B. D. Keister and W. N. Polyzou, in *Advances in Nuclear Physics*, Ed. J.W.Negele and E. Vogt, Vol. 20, Plenum Press, New York, 1991.

[8] L. S. Pontryagin, *Topological Groups*, 3rd Ed. (Classics of Soviet Mathematics, VII), Gordon and Breach, New York, 1988.

[9] K. Gottfried, *Quantum Mechanics, Vol. I: Fundamentals*, p. 290, Benjamin-Cummings, New York, 1966.