The pro-unipotent radical of the pro-algebraic fundamental group of a compact Kähler manifold

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Introduction

For $X$ a compact connected Kähler manifold, consider the real pro-algebraic completion $\varpi_1(X,x)$ of the fundamental group $\pi_1(X,x)$. In [Sim92], Simpson defined a Hodge structure on the complex pro-algebraic fundamental group $\varpi_1(X,x)_\mathbb{C}$, in the form of a discrete $\mathbb{C}^*$-action. The Levi decomposition for pro-algebraic groups allows us to write

$$\varpi_1(X,x) \cong R_u(\varpi_1(X,x)) \times \varpi_1^{\text{red}}(X,x),$$

where $R_u(\varpi_1(X,x))$ is the pro-unipotent radical of $\varpi_1(X,x)$ and $\varpi_1^{\text{red}}(X,x)$ is the reductive quotient of $\varpi_1(X,x)$. This decomposition is unique up to conjugation by $R_u(\varpi_1(X,x))$. By studying the Hodge structure on $\varpi_1^{\text{red}}(X,x)$, Simpson established restrictions on its possible group structures.

The purpose of this paper is to use Hodge theory to show that $R_u(\varpi_1(X,x))$ is quadratically presented as a pro-unipotent group, in the sense that its Lie algebra

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can be defined by equations of bracket length two. This generalises both Goldman and Millson’s result on deforming reductive representations of the fundamental group \([GM88]\), and Deligne et al.’s result on the de Rham fundamental group \(\pi_1(X, x) \otimes \mathbb{R}\) \([DGMS75]\). The idea behind this paper is that in both of these cases, we are taking a reductive representation

\[ \rho_0 : \pi_1(X, x) \to G, \]

and considering deformations

\[ \rho : \pi_1(X, x) \to U \ltimes G \]

of \(\rho_0\), for \(U\) unipotent.

Effectively, \([GM88]\) considers only \(U = \exp(\text{Lie}(G) \otimes m_A)\), for \(m_A\) a maximal ideal of an Artinian local \(\mathbb{R}\)-algebra, while \([DGMS75]\) considers only the case when \(G = 1\). Since taking \(U = R_u(\varpi_1(X, x))\) pro-represents this functor when \(G = \varpi_1^\text{red}(X, x)\), the quadratic presentation for \(U\) gives quadratic presentations both for the hull of \([GM88]\) and for the Lie algebra of \([DGMS75]\). This also generalises Hain’s results \([Hai98]\) on relative Malcev completions of variations of Hodge structure, since here we are taking relative Malcev completions of arbitrary reductive representations.

Section 1 summarises standard definitions and properties of pro-algebraic groups which are used throughout the rest of the paper.

Section 2 develops a theory of deformations over nilpotent Lie algebras with \(G\)-actions. This can be thought of as a generalisation of the theory introduced in \([Pri04]\), which corresponds to the case \(G = 1\). The essential philosophy is that all the concepts for deformations over Artinian rings, developed by Schlessinger in \([Sch68]\) and later authors, can be translated to this context.

Section 3 introduces the notion of twisted differential graded algebras, which are analogous to the DGAs used in \([DGMS75]\) to characterise the real homotopy type. They are equivalent to the \(G\)-equivariant DGAs used in \([KPT05]\) to study the schematic homotopy type.

Section 4 contains various technical lemmas about pro-algebraic groups.

In Section 5 the twisted DGA arising from \(C^\infty\)-sections is defined. It is shown that this can be used to recover \(R_u(\varpi_1(X, x))\).

Section 6 uses Hodge theory to prove that, for a compact Kähler manifold, this twisted DGA is formal, i.e. quasi-isomorphic to its cohomology DGA. This can be thought of as equivalent to formality of the real schematic homotopy type. In consequence, \(R_u(\varpi_1(X, x))\) is quadratically presented — a direct analogue of the demonstration in \([DGMS75]\) that formality of the real homotopy type forces the de Rham fundamental group to be quadratically presented. This implies that if \(\pi_1(X, x)\) is of the form \(\Delta \rtimes \Lambda\), with \(\Lambda\) acting reductively on \(\Delta \otimes \mathbb{R}\), then \(\Delta \otimes \mathbb{R}\) must be quadratically presented.

I would like to thank Bertrand Toën for suggesting the connection between this work and the schematic homotopy type.
1 Review of pro-algebraic groups

In this section, we recall some definitions and standard properties of pro-algebraic groups, most of which can be found in [DMOSS82] II§2 and [Sim92]. Fix a field $k$ of characteristic zero.

**Definition 1.1.** An algebraic group over $k$ is defined to be an affine group scheme $G$ of finite type over $k$. These all arise as Zariski-closed subgroups of general linear groups $\text{GL}_n(k)$. A pro-algebraic group is a filtered inverse limit of algebraic groups, or equivalently an arbitrary affine group scheme over $k$.

**Definition 1.2.** Given a pro-algebraic group $G$, let $O(G)$ denote global sections of the structure sheaf of $G$, so that $G = \text{Spec} O(G)$. This is a sum of finite-dimensional $G \times G$-representations, the actions corresponding to right and left translation. The group structure on $G$ corresponds to a comultiplication $\Delta : O(G) \to O(G) \otimes O(G)$, coidentity $\varepsilon : O(G) \to k$, and coinverse $S : O(G) \to O(G)$, satisfying coassociativity, coidentity and coinverse axioms.

**Lemma 1.3.** If $G$ is a pro-algebraic group, and we regard $O(G)$ as a sum of finite-dimensional $G$-representations via the left action, then for any finite-dimensional $G$-representation $V$,

$$V \cong (V \otimes O(G))^G := \{ a \in V \otimes O(G) \mid (g \otimes \text{id})a = (\text{id} \otimes g)a, \forall g \in G \},$$

with the $G$-action on the latter coming from the right action on $O(G)$.

**Proof.** This follows immediately from [DMOSS82] II Proposition 2.2, which states that $G$-representations correspond to $O(G)$-comodules. Under this correspondence, $v \in V$ is associated to the function $g \mapsto g \cdot v$. $\square$

**Definition 1.4.** An algebraic group $G$ is said to be unipotent if the coproduct $\Delta : O(G) \to O(G) \otimes O(G)$ is counipotent. Unipotent algebraic groups all arise as Zariski-closed subgroups of the groups of upper triangular matrices:

$$\begin{pmatrix}
1 & * & * \\
0 & \ddots & * \\
0 & 0 & 1
\end{pmatrix}.$$

A pro-algebraic group is said to be pro-unipotent if it is an inverse limit of unipotent algebraic groups. This is equivalent to saying that $\Delta$ is ind-counipotent.

**Lemma 1.5.** There is a one-to-one correspondence between unipotent algebraic groups over $k$, and finite-dimensional nilpotent Lie algebras over $k$.

**Proof.** The Lie algebra $\mathfrak{u}$ of any unipotent algebraic group $U$ is necessarily finite-dimensional and nilpotent. Conversely, if $\mathfrak{u}$ is any finite-dimensional nilpotent Lie algebra, we define a unipotent algebraic group $\exp(\mathfrak{u})$ by

$$\exp(\mathfrak{u})(A) := \exp(\mathfrak{u} \otimes A).$$
Here, for any Lie algebra $\mathfrak{g}$, the group $\exp(\mathfrak{g})$ has underlying set $\mathfrak{g}$ and multiplication given by the Campbell-Baker-Hausdorff formula

$$g \cdot h := g + h + \frac{1}{2}[g,h] + \ldots,$$

which in this case is a finite sum, by nilpotence, so the group $\exp(\mathfrak{u})$ is indeed algebraic.

To see that these functors are inverse is most easily done by considering groups of upper triangular matrices.

**Definition 1.6.** The pro-unipotent radical $R_u(G)$ of a pro-algebraic group $G$ is defined to be the maximal pro-unipotent normal closed subgroup of $G$. A pro-algebraic group $G$ is said to be reductive if $R_u(G) = 1$, and for an arbitrary pro-algebraic group $G$, the reductive quotient of $G$ is defined by $G^{\text{red}} := G/R_u(G)$.

**Theorem 1.7 (Levi decomposition).** For any pro-algebraic group $G$, there is a decomposition

$$G \cong R_u(G) \rtimes G^{\text{red}},$$

unique up to conjugation by the pro-unipotent radical $R_u(G)$.

**Proof.** This is the Levi decomposition for pro-algebraic groups in characteristic zero, proved in [HM69], which states that for every pro-algebraic group $G$, the surjection $G \to G^{\text{red}}$ has a section, unique up to conjugation by $R_u(G)$, inducing an isomorphism $G \cong R_u(G) \rtimes G^{\text{red}}$.

**Theorem 1.8 (Tannakian duality).** A pro-algebraic group $G$ over $k$ can be recovered from its (tensor) category of finite-dimensional $k$-representations. Representations of $G^{\text{red}}$ correspond to the subcategory of semisimple representations.

**Proof.** The first part is [DMOS82] Theorem II.2.11. The second part is just the observation that, in characteristic zero, “reductive” and “linearly reductive” are equivalent.

**Definition 1.9.** Given a discrete group $\Gamma$, define the pro-algebraic completion $\Gamma^{\text{alg}}$ of $\Gamma$ to be the pro-algebraic group $G$ universal among group homomorphisms $\Gamma \to G(k)$. In other words $\Gamma^{\text{alg}}$ pro-represents the functor which sends an algebraic group $G$ to the set of representations $\Gamma \to G(k)$. Under Tannakian duality, the finite-dimensional linear $k$-representations of $\Gamma^{\text{alg}}$ are just the finite-dimensional linear $k$-representations of $\Gamma$.

The reductive quotient of $\Gamma^{\text{alg}}$ is denoted $\Gamma^{\text{red}}$, and is universal among Zariski-dense group homomorphisms $\Gamma \to G(k)$, with $G$ reductive. In other words $\Gamma^{\text{red}}$ pro-represents the functor which sends an algebraic group $G$ to the set of reductive representations $\Gamma \to G(k)$. Under Tannakian duality, the finite-dimensional linear $k$-representations of $\Gamma^{\text{red}}$ are just the semisimple finite-dimensional linear $k$-representations of $\Gamma$.

The pro-unipotent (or Malcev) completion $\Gamma \otimes k$ of $\Gamma$ is universal among group homomorphisms $\Gamma \to G(k)$, with $G$ pro-unipotent. In other words $\Gamma \otimes k$ pro-represents the functor which sends an algebraic group $G$ to the set of unipotent representations $\Gamma \to G(k)$. Under Tannakian duality, the finite-dimensional linear $k$-representations of $\Gamma \otimes k$ are just the unipotent finite-dimensional linear $k$-representations of $\Gamma$. 

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Definition 1.10. Given a representation $\rho : \Gamma \to G(k)$, for some pro-algebraic group $G$, the Malcev completion of $\Gamma$ relative to $\rho$ is defined in [Hai98] to represent the functor which sends a pro-unipotent extension $H \to G$ to the set of representations $\Gamma \to H(k)$ lifting $\rho$. Thus $\Gamma \otimes k$ is the Malcev completion of $\Gamma$ relative to the trivial representation, and $\Gamma^{\text{alg}}$ is the Malcev completion of $\Gamma$ relative to the canonical representation $\Gamma \to \Gamma^{\text{red}}$.

2 Functors on nilpotent Lie algebras with $G$-actions

This section extends the ideas of [Sch68] to a slightly different context.

Fix a field $k$ of characteristic zero. Take a pro-algebraic group $G$ over $k$, and let $\text{Rep}(G)$ be the category of finite-dimensional representations of $G$ over $k$. If $G$ is reductive, then every such representation will be decomposable into irreducibles, so $\text{Hom}$ will be an exact functor on this category. Consider the category $\hat{\text{Rep}}(G) := \text{pro}(\text{Rep}(G))$, whose objects are filtered inverse systems $\{V_\alpha\}_{\alpha \in I}$, with morphisms given by

$$\text{Hom}_{\text{pro}(\text{Rep}(G))}(\{V_\alpha\}, \{W_\beta\}) = \lim_{\leftarrow} \lim_{\rightarrow} \text{Hom}_{\text{Rep}(G)}(V_\alpha, W_\beta).$$

Given a set $\{V_i\}_{i \in I}$, with $V_i \in \text{Rep}(G)$, we make the vector space $\prod_{i \in I} V_i$ an object of $\hat{\text{Rep}}(G)$ via the formula

$$\prod_{i \in I} V_i = \lim_{J \subset I \text{ finite}} \prod_{j \in J} V_j.$$

Lemma 2.1. If $G$ is reductive, then every object of $\hat{\text{Rep}}(G)$ can be expressed as a product of irreducible finite-dimensional $G$-representations.

Proof. Since $\text{Rep}(G)$ is an Artinian category, i.e. it satisfies the descending chain condition for sub-objects, we may use [Gro95] to observe that $\hat{\text{Rep}}(G)$ is isomorphic to the category of left-exact set-valued functors on $\text{Rep}(G)$.

Take $W \in \hat{\text{Rep}}(G)$, and let $\{V_s : s \in S\}$ be a set of representatives for isomorphism classes of irreducible representations in $\text{Rep}(G)$. Now, $\text{Hom}_{\hat{\text{Rep}}(G)}(W, V_s)$ has the natural structure of a vector space over $k$. Choose a basis $t_i : i \in I_s$ for this vector space, and let

$$U := \prod_{s \in S} V_s^{I_s}.$$

There is then a natural isomorphism between $\text{Hom}_{\hat{\text{Rep}}(G)}(W, V_s)$ and $\text{Hom}_{\hat{\text{Rep}}(G)}(U, V_s)$ for all $s \in S$, so the left-exact functors defined on $\text{Rep}(G)$ by $U$ and $W$ must be isomorphic, and therefore $U \cong W$.

Definition 2.2. For any pro-algebraic group $G$, define $\mathcal{N}(G)$ to be the category whose objects are pairs $(u, \rho)$, where $u$ is a finite-dimensional nilpotent Lie algebras over $k$, and $\rho : G \to \text{Aut}(u)$ is a representation to the group of Lie algebra automorphisms of $u$. A morphism $\theta$ from $(u, \rho)$ to $(u', \rho')$ is a morphism $\theta : u \to u'$ of Lie algebras such that $\theta \circ \rho = \rho'$. Observe that $\mathcal{N}(G)$ is an Artinian category, and write $\hat{\mathcal{N}}(G)$ for the category $\text{pro}(\mathcal{N}(G))$.  

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Given \( \mathcal{L} \in \hat{\mathcal{N}}(G) \), let \( \mathcal{N}(G)_\mathcal{L} \) be the category of pairs \( (N \in \mathcal{N}(G), \mathcal{L} \to^\phi N) \), and \( \hat{\mathcal{N}}(G)_\mathcal{L} \) the category of pairs \( (N \in \hat{\mathcal{N}}(G), \mathcal{L} \to^\phi N) \). We will almost always consider the case \( \mathcal{L} = 0 \) (note that \( \hat{\mathcal{N}}(G)_0 = \mathcal{N}(G) \)), although a few technical lemmas (Propositions 2.30 and 2.31 and Corollary 2.32) require the full generality of \( \mathcal{N}(G)_\mathcal{L} \). In \( \hat{\mathcal{N}}(G)_\mathcal{L} \), \( \mathcal{L} \) is the initial object, and 0 the final object.

**Definition 2.3.** For \( N = \lim_{\alpha} N_\alpha \in \hat{\mathcal{N}}(G)_\mathcal{L} \), define the tangent space functor

\[
t_{N/\mathcal{L}} : \text{Rep}(G) \to \text{Vect}
V \mapsto \text{Hom}_{\hat{\mathcal{N}}(G)_\mathcal{L}}(N, V_\epsilon),
\]

where \( V_\epsilon \in \mathcal{N}(G) \) is the representation \( V \) regarded as an abelian Lie algebra, i.e. \([V_\epsilon, V_\epsilon] = 0\), with structure morphism \( \mathcal{L} \to^0 V_\epsilon \). This is clearly a vector space over \( k \), and

\[
t_{N/\mathcal{L}}(V) = \lim_{\alpha} \text{Hom}_{\text{Rep}(G)}(N_\alpha/\langle [N_\alpha, N_\alpha] + \mathcal{L} \rangle, V).
\]

We define the cotangent space

\[
t^*_N := N/\langle [N, N] + \mathcal{L} \rangle = \lim_{\alpha} N_\alpha/\langle [N_\alpha, N_\alpha] + \mathcal{L} \rangle \in \hat{\text{Rep}}(G).
\]

**Definition 2.4.** Given \( V \) in \( \hat{\text{Rep}}(G) \), denote the free pro-nilpotent Lie algebra on generators \( V \) by \( L(V) \). This has a natural continuous \( G \)-action, so is in \( \hat{\mathcal{N}}(G) \). Equivalently, we may use \([\text{Gro}95]\) to define \( L(V) \) as the object of \( \hat{\mathcal{N}}(G) \) pro-representing the functor \( N : \text{Hom}_{\hat{\text{Rep}}(G)}(V, N) \).

**Definition 2.5.** Given a Lie algebra \( L \), we define the lower central series of ideals \( \Gamma_n(L) \) inductively by

\[
\Gamma_1(L) = L, \quad \Gamma_{n+1}(L) = [L, \Gamma_n(L)],
\]

and define the associated graded algebra \( \text{gr}_L \) by \( \text{gr}_n L = \Gamma_n L / \Gamma_{n+1} L \)

For \( L \in \mathcal{N}(G)_\mathcal{L} \), define the ideals \( \Gamma_n^\mathcal{L}(L) \) by

\[
\Gamma_1(L) = L, \quad \Gamma_2(L) = [L, L] + \langle \mathcal{L} \rangle, \quad \Gamma_{n+1}(L) = [L, \Gamma_n(L)],
\]

and define the associated graded algebra \( \text{gr}_L^\mathcal{L} \) by \( \text{gr}_n^\mathcal{L} L = \Gamma_n^\mathcal{L} L / \Gamma_{n+1}^\mathcal{L} L \)

**Definition 2.6.** Given Lie algebras \( N, M \in \hat{\mathcal{N}}(G) \), let \( N \ast M \) be the completed free Lie algebra product, i.e. the completion with respect to the commutator filtration of the free product of the pro-Lie algebras \( N \) and \( M \). Note that \( \ast \) is sum in \( \hat{\mathcal{N}}(G) \) — the analogue in the category \( \text{pro}(\mathcal{A}) \) of pro-Artinian \( \mathcal{A} \)-algebras is \( \otimes \).

Given Lie algebras \( N, M \in \hat{\mathcal{N}}(G)_\mathcal{L} \), define the free fibre product \( N \ast_\mathcal{L} M \) similarly. Equivalently, we can use \([\text{Gro}95]\) to define this element of \( \hat{\mathcal{N}}(G) \), since all finite colimits must exist in a pro-Artinian category.

**Definition 2.7.** Define the free Lie algebra \( L_\mathcal{L}(V) := \mathcal{L} \ast L(V) \)

We will consider only those functors \( F \) on \( \mathcal{N}(G)_\mathcal{L} \) which satisfy
(H0) $F(0) = \bullet$, the one-point set.

We adapt the following definitions and results from [Sch68] (with identical proofs):  

**Definition 2.8.** For $p : N \to M$ in $\hat{N}(G)_{\mathcal{L}}$ surjective, $p$ is a semi-small extension if $[N, \ker p] = (0)$. If the pro-$G$-representation $\ker p$ is an absolutely irreducible $G$-representation, then we say that $p$ is a small extension. Note that any surjection in $\hat{N}(G)_{\mathcal{L}}$ can be be factorised as a composition of small extensions.

For $F : N(G)_{\mathcal{L}} \to \mathbf{Set}$, define $\hat{F} : \hat{N}(G)_{\mathcal{L}} \to \mathbf{Set}$ by

$$\hat{F}(\lim_{\alpha} L_{\alpha}) = \lim_{\alpha} F(L_{\alpha}).$$

Note that $\hat{F}(L) \sim \Hom(h_L, F)$, where

$$h_L : N(G)_{\mathcal{L}} \to \mathbf{Set}; \quad N \mapsto \Hom(L, N).$$

**Definition 2.9.** We will say a functor $F : N(G)_{\mathcal{L}} \to \mathbf{Set}$ is pro-representable if it is isomorphic to $h_L$, for some $L \in \hat{N}(G)_{\mathcal{L}}$. By the above remark, this isomorphism is determined by an element $\xi \in \hat{F}(L)$. We say the pro-couple $(L, \xi)$ pro-represents $F$.

**Remark 2.10.** This definition is not the strict analogue of that appearing in [Sch68], which had additional hypotheses on the finiteness of tangent spaces. This terminology coincides with that of [Gro95].

**Definition 2.11.** A natural transformation $\phi : F \to E$ in $[N(G)_{\mathcal{L}}, \mathbf{Set}]$ is called:

1. unramified if $\phi : F(V_0) \to E(V_0)$ is injective for all irreducible $G$-representations $V$.
2. smooth if for every surjection $M \to N$ in $N(G)_{\mathcal{L}}$, the canonical map $F(M) \to E(M) \times_{E(N)} F(N)$ is surjective.
3. étale if it is smooth and unramified.

**Definition 2.12.** $F : N(G)_{\mathcal{L}} \to \mathbf{Set}$ is smooth if and only if $F \to \bullet$ is smooth.

**Lemma 2.13.** A morphism $f : M \to N$ in $\hat{N}(G)_{\mathcal{L}}$ is a surjection or an isomorphism if and only if the associated graded morphism $\text{gr}^\mathcal{L} M \to \text{gr}^\mathcal{L} N$ is so. Therefore $f : M \to N$ in $\hat{N}(G)_{\mathcal{L}}$ is surjective if and only if the induced map $t^*_{M/\mathcal{L}} \to t^*_{N/\mathcal{L}}$ is surjective, and an endomorphism $f : M \to M$ in $\hat{N}(G)_{\mathcal{L}}$ is an automorphism whenever the induced map $t^*_{M/\mathcal{L}} \to t^*_{M/\mathcal{L}}$ is the identity.

**Definition 2.14.** A map $p : M \to N$ in $\hat{N}(G)_{\mathcal{L}}$ is essential if for all morphisms $q : M' \to M$, $q$ is surjective whenever $pq$ is.

From now on, we will assume that $G$ is reductive.

From the above lemma, we deduce:
Lemma 2.15. Let \( p : M \to N \) be a surjection in \( \hat{\mathcal{N}}(G)_L \). Then

1. \( p \) is essential if and only if the induced map \( t^*_M/L \to t^*_N/L \) is an isomorphism.

2. If \( p \) is a small extension, then \( p \) is not essential if and only if \( p \) has a section.

Proof. 1. Assume that \( p \) is essential. Since \( t^*_M/L \to t^*_N/L \) is surjective, so must \( M \to t^*_N/L \) be. Using the exactness of \( \text{Rep}(G) \) (since \( G \) is reductive), this map has a section in \( \hat{\text{Rep}}(G) \). Let its image be \( V \), and let \( q : L_L(V) \to M \) be the map determined by the inclusion \( V \hookrightarrow M \). Now, \( pq \) is surjective, since it induces an isomorphism on cotangent spaces. Therefore \( q \) is surjective, so \( V \to t^*_M/L \) is surjective, as required. The converse is immediate.

2. If \( p \) is not essential, construct the pro-Lie algebra \( L_L(V) \) as in the previous part, and let \( N' \hookrightarrow M \) be its image. Then, by comparing tangent spaces, we see that \( N' \to N \) is a surjection. Since \( p \) is not essential, \( t^*_M/L \to t^*_N/L \) is not injective, so \( \ker p \to t^*_M/L \) is non-zero, hence an embedding, since \( \ker p \) is absolutely irreducible. The image of \( N' \) in \( t^*_M/L \) has zero intersection with the image of \( \ker p \). Therefore \( \ker p \) and \( N' \) have zero intersection so \( N' \times \ker p \) is a sub-pro-Lie algebra of \( M \) with the same cotangent space, so \( M = N' \times \ker p \), and \( N' \cong N \), which gives the section of \( p \).

\[ \square \]

Definition 2.16. A pro-couple \((L, \xi)\) is a hull for \( F \) if the induced map \( h_L \to F \) is \'{e}tale.

Lemma 2.17. Suppose \( F \) is a functor such that

\[ F(V\epsilon \oplus W\epsilon) \xrightarrow{\sim} F(V\epsilon) \times F(W\epsilon) \]

for \( V, W \in \text{Rep}(G) \). Then \( F(V\epsilon) \) has a canonical vector space structure, and the tangent space functor

\[ t_F : \text{Rep}(G) \to \text{Vect} \]

\[ V \mapsto F(V\epsilon) \]

is additive.

Definition 2.18. Given \( F : \mathcal{N}(G)_L \to \text{Set} \), let \( N' \to N \) and \( N'' \to N \) be morphisms in \( \mathcal{N}(G)_L \), and consider the map:

\[ (\dagger) \quad F(N' \times_N N'') \to F(N') \times_{F(N)} F(N'') \]

We make the following definitions for properties of \( F \):

\( (H1) \) \((\dagger)\) is a surjection whenever \( N'' \to N \) is a small extension.

\( (H2) \) \((\dagger)\) is a bijection whenever \( N = 0 \) and \( N'' = V\epsilon \), for an irreducible \( G \)-representation \( V \).
(H4) (*) is a bijection whenever \( N' = N'' \) and \( N' \to N \) is a small extension.

Remark 2.19. These conditions are so named for historical reasons, following [Sch68]. The missing condition (H3) concerned finite-dimensionality of tangent spaces, which is irrelevant to our (weaker) notion of pro-representability.

Lemma 2.20. Let \( F : \mathcal{N}(G) \to \text{Set} \) satisfy (H1) and (H2), and \( M \to N \) be a semi-small extension in \( \mathcal{N}(G) \). Given a surjection \( M \to M_\alpha \), with \( M_\alpha \in \mathcal{N}(G) \), let \( N_\alpha = M_\alpha *_MN \). If \( \xi \in \hat{F}(N) \) has the property that for all such surjections, the image \( \xi_\alpha \in F(N_\alpha) \) of \( \xi \) lifts to \( F(M_\alpha) \), then \( \xi \) lifts to \( \hat{F}(M) \).

Proof. Since \( M \to N \) is semi-small, \( M_\alpha \to N_\alpha \) is also. Let \( I := \ker(M \to N) \) and \( I_\alpha := \ker(M_\alpha \to N_\alpha) \). Observe that we have canonical isomorphisms \( M_\alpha \times I_\alpha \cong M_\alpha \times_{N_\alpha} M_\alpha \), and let \( T_\alpha \) be the fibre of \( F(M_\alpha) \to F(N_\alpha) \) over \( \xi_\alpha \). Since \( F \) satisfies (H2), we have a map

\[
F(M_\alpha) \times t_F(I_\alpha) \to F(M_\alpha) \times_{F(N_\alpha)} F(M_\alpha),
\]

so \( t_F(I_\alpha) \) acts on \( T_\alpha \). From (H1) it follows that this action is transitive (and if \( F \) also satisfied (H4) then \( T_\alpha \) would be a principal homogeneous space under this action). Let \( K_\alpha \subseteq t_F(I_\alpha) \) be the stabiliser of \( T_\alpha \).

We wish to construct a compatible system \( \eta_\alpha \in T_\alpha \). Let \( \eta_\alpha \in T_\alpha \) be any element, and assume that we have constructed a compatible system \( \eta_\beta \in T_\beta \), for all strict epimorphisms \( M_\alpha \to M_\beta \). Then \( \eta_\beta = v_\beta(\eta_\alpha) \), for a unique \( v_\beta \in t_F(I_\beta)/K_\beta \). Observe that

\[
t_F(I_\alpha)/K_\alpha \to \varprojlim{\beta} t_F(I_\beta)/K_\beta
\]

is surjective, and that the \( v_\beta \) form an element of the right-hand side. Lift to \( v \in t_F(I_\alpha)/K_\alpha \), and let \( \eta_\alpha := v(\eta'_\alpha) \). The construction proceeds inductively (since every poset can be enriched to form a totally ordered set, and we have satisfied the hypotheses for transfinite induction).

Proposition 2.21. Let \((L, \xi), (L', \xi')\) be hulls of \( F \). Then there exists an isomorphism \( u : L \to L' \) such that \( F(u)(\xi) = \xi' \).

Proof. We wish to lift \( \xi \) and \( \xi' \) to \( u \in h_{L'}(L) \) and \( u' \in h_L(L') \). We may apply Lemma 2.20 to the the functor \( h_{L'} \) and the successive semi-small extensions \( L_{n+1} \to L_n \), where \( L_n = L/\langle \Gamma_{n}L \rangle \). By smoothness of \( h_{L'} \to F \), we obtain successive lifts of \( \xi_\alpha \in F(L_n) \) to \( u_\alpha \in h_{L'}(L_n) \). Let \( u = \varprojlim{\alpha} u_\alpha \), and construct \( u' \) similarly. We therefore obtain \( u : (L, \xi) \to (L', \xi) \) and \( u' : (L, \xi) \to (L', \xi) \), inducing identity on cotangent spaces. Therefore \( uu' \) induces the identity on \( t^*_L/L \), so is an automorphism, by Lemma 2.18.

Proposition 2.22. 1. Let \( M \to N \) be a morphism in \( \widehat{\mathcal{N}(G)}_L \). Then \( h_N \to h_M \) is smooth if and only if \( N \cong L_M(V) \), for some \( V \in \text{Rep}(G) \).

2. If \( F \to E \) and \( E \to H \) are smooth morphisms of functors, then the composition \( F \to H \) is smooth.
3. If \( u : F \to E \) and \( v : E \to H \) are morphisms of functors such that \( u \) is surjective and \( vu \) is smooth, then \( v \) is smooth.

Proof. If \( N = M \ast L(V) \), then \( h_N(g) = h_M(g) \times \text{Hom}(V,g) \), which is smooth over \( h_M \), since \( \text{Rep}(G) \) is an exact category. Conversely, assume that \( h_N \to h_M \) is smooth, and let \( V = t^*N/M \). Let \( N' = L_M(V) \), and observe that, by choosing a lift of \( V \) to \( N \), we obtain a morphism \( f : N' \to N \), inducing an isomorphism on relative cotangent spaces.

Let \( N'_n = N'/(\Gamma_{n+1}N') \), and observe that \( N'_n \to N'_{n-1} \) is a semi-small extension, as is \( N'_1 \to t^*_N/M \). We have a canonical map \( N \to t^*_{N'/M} \) arising from the isomorphism \( t^*_{N'/M} \cong t^*_N/M \). Since \( h_N \) satisfies (H1) and (H2), we may now apply Lemma 2.20 to construct a map \( g : N \to N' \) lifting this. Therefore the compositions \( fg \) and \( gf \) induce the identity on cotangent spaces, so are isomorphisms, and \( N' \cong N \), as required.

The remaining statements follow by formal arguments. \( \square \)

Remark 2.23. For the proposition above, it is essential that \( G \) be reductive, since we need the exactness of \( \text{Hom} \) on \( \text{Rep}(G) \) to ensure that \( L(V) \) is smooth.

Theorem 2.24. 1. \( F \) has a hull if and only if \( F \) has properties (H1) and (H2).

2. \( F \) is moreover pro-representable if and only if \( F \) has the additional property (H4).

Proof. This is essentially \text{Schö6}, Theorem 2.11, \textit{mutatis mutandis}. Since \( t_F \) defines a left-exact functor on \( \text{Rep}(G) \), by (H2), let it be pro-represented by \( W \in \hat{\text{Rep}}(G) \), and let \( h = L_C(W) \). The hull \( g \) will be a quotient of \( h \), which will be constructed as an inverse limit of semi-small extensions of \( W \). Let \( g_2 = W \), and \( \xi_2 \in \hat{F}(W) \) the canonical element corresponding to the pro-representation of \( t_F \). Assume that we have constructed \( (g_q, \xi_q) \). We wish to find a semi-small extension \( g_{q+1} \to g_q \), maximal among those quotients of \( h \) which admit a lift of \( \xi_q \).

Given quotients \( M, N \in \mathcal{N}(G) \) of \( h \), we write

\[
M \wedge N := M \ast_h N, \quad \text{and} \quad M \vee N := M \times_{M \wedge N} N.
\]

Note that \( M \wedge N \) is then maximal among those Lie algebras which are dominated by both \( M \) and \( N \), while \( M \vee N \) is minimal among those quotients of \( h \) which dominate both \( M \) and \( N \). Next, observe that a set \( \{ M_\alpha \in \mathcal{N}(G) \}_{\alpha \in I} \) of quotients of \( h \) corresponds to a quotient \( h \to \varprojlim_{\alpha} M_\alpha \) of \( h \) if and only if the following two conditions hold:

(Q1) If \( M_\alpha \to N \) is a surjection, for any \( \alpha \in I \) and any \( N \in \mathcal{N}(G) \), then \( N \cong M_\beta \), for some \( \beta \in I \).

(Q2) Given \( \alpha, \beta \in I \), \( M_\alpha \vee M_\beta = M_\gamma \), for some \( \gamma \in I \).

We will now form such a set of quotients by considering those \( h \to M \) satisfying:

1. \( M \to M \ast_h g_q \) is a semi-small extension.

2. The image of \( \xi_q \) in \( F(M \ast_h g_q) \) lifts to \( F(M) \).
It is immediate that this set satisfies (Q1). To see that it satisfies (Q2), take quotients $M, N$ satisfying these conditions. It is clear that $M \vee N \to (M \vee N) \ast_h \mathfrak{g}_q$ is a semi-small extension, since $M \vee N$ is a sub-Lie algebra of $M \times N$. To see that $\xi_q$ lifts to $F(M \vee N)$, let $x \in F(M), y \in F(N)$ be lifts of $\xi_q$. Now, as in the proof of Lemma 2.20, the fibre of $F(N)$ over $\xi_q$ surjects onto the fibre of $F(M \wedge N)$ over $\xi_q$. Therefore, we may assume that $x$ and $y$ have the same image in $F(M \vee N)$. Now (H1) provides the required lift:

$$F(M \vee N) \to F(M) \times_{F(M \wedge N)} F(N).$$

Let $\mathfrak{g}_{q+1} \in \hat{\mathcal{N}}(G)$ be the quotient defined by this collection. By Lemma 2.20, it follows that $\xi_q$ lifts to $\hat{F}(\mathfrak{g}_{q+1})$. Let $\mathfrak{g} := \lim g_q$, with $\xi := \lim \xi_q$.

It remains to show that this is indeed a hull for $F$. By construction, $h_\mathfrak{g} \to F$ is unramified, so we must show it is smooth. Let $p : (N', \eta') \to (N, \eta)$ be a morphism of couples in $\mathcal{N}(G)$, with $p$ a small extension, $N = N'/I$, and assume we are given $u : (\mathfrak{g}, \xi) \to (N, \eta)$. We must lift $u$ to a morphism $(\mathfrak{g}, \xi) \to (N'\eta')$. It will suffice to find a morphism $u' : \mathfrak{g} \to A'$ such that $pu' = u$, since we may then use transitivity of the action of $t_F(I)$ on $F(p)^{-1}(\eta)$.

For some $q$, $u$ factors as $(\mathfrak{g}, \xi) \to (\mathfrak{g}_q, \xi_q) \to (N, \eta)$. By smoothness of $h$, we may choose a morphism $w$ making the following diagram commute. We wish to construct the morphism $v$:

$$\begin{array}{ccc}
\mathfrak{g}_q & \xrightarrow{w} & \mathfrak{g}_q \times_N N' \\
\downarrow & & \downarrow v \\
\mathfrak{g}_{q+1} & \xrightarrow{v} & \mathfrak{g}_q.
\end{array}$$

If the small extension $pr_1$ has a section, then $v$ obviously exists. Otherwise, by Lemma 2.15, $pr_1$ is essential, so $w$ is a surjection. (H1) then provides a lift of $\xi_q$ to $F(\mathfrak{g}_q \times_N N')$, so by the construction of $\mathfrak{g}_{q+1}$, $w$ factors through $\mathfrak{g}_{q+1}$, and so $v$ must exist. This completes the proof that $h_\mathfrak{g} \to F$ is a hull.

If $F$ also satisfies (H4), then we may use induction on the length of $N$ to show that $h_\mathfrak{g}(N) \cong F(N)$, using the observation in the proof of Lemma 2.20 that all non-empty fibres over small extensions $I \to N' \to N$ are principal homogeneous $t_F(I)$-spaces.

Necessity of the conditions follows by a formal argument. \hfill \square

**Definition 2.25.** $F : \mathcal{N}(G)_L \to \text{Set}$ is homogeneous if

$$\eta : F(N' \times_N N'') \to F(N') \times_{F(N)} F(N'')$$

is an isomorphism for every $N' \to N$.

Note that a homogeneous functor satisfies conditions (H1), (H2) and (H4).

**Definition 2.26.** $F : \mathcal{N}(G)_L \to \text{Set}$ is a deformation functor if:

1. $\eta$ is surjective whenever $N' \to N$.
2. $\eta$ is an isomorphism whenever $N = 0$.

Note that a deformation functor satisfies conditions (H1) and (H2).

The following results are adapted from [Man99]:
**Definition 2.27.** Given $F : N_{L,k} \to \text{Set}$, an obstruction theory $(O,o_e)$ for $F$ consists of an additive functor $O : \text{Rep}(G) \to \text{Vect}$, the obstruction space, together with obstruction maps $o_e : F(N) \to O(I)$ for each small extension $$e : 0 \to I \to L \to N \to 0,$$ such that:

1. If $\xi \in F(N)$ can be lifted to $F(L)$ then $o_e(\xi) = 0$.
2. For every morphism $\alpha : e \to e'$ of small extensions, we have $o_e'(\alpha(\xi)) = O(\alpha)(v_e(\xi))$, for all $\xi \in F(N)$.

An obstruction theory $(O,o_e)$ is called complete if $\xi \in F(N)$ can be lifted to $F(L)$ whenever $o_e(\xi) = 0$.

**Proposition 2.28.** *(Standard Smoothness Criterion)* Given $\phi : F \to E$, with $(O,o_e) \xrightarrow{\phi'} (P,p_e)$ a compatible morphism of obstruction theories, if $(O,o_e)$ is complete, $O \xrightarrow{\phi'} P$ injective, and $t_F \to t_E$ surjective, then $\phi$ is smooth.

*Proof.* [Man99], Proposition 2.17. \hfill \Box

For functors $F : N(G)_L \to \text{Set}$ and $E : N(G)_L \to \text{Grp}$, we say that $E$ acts on $F$ if we have a functorial group action $E(N) \times F(N) \xrightarrow{\times} F(N)$, for each $N$ in $N(G)$. The quotient functor $F/E$ is defined by $(F/E)(N) = F(N)/E(N)$.

**Proposition 2.29.** If $F : N(G)_L \to \text{Set}$, a deformation functor, and $E : N(G)_L \to \text{Grp}$ a smooth deformation functor, with $E$ acting on $F$, then $D := F/E$ is a deformation functor, and if $\nu : t_E \to t_F$ denotes $h \mapsto h * 0$, then $t_D = \text{coker} \nu$, and the universal obstruction theories of $D$ and $F$ are isomorphic.

*Proof.* [Man99], Lemma 2.20. \hfill \Box

**Proposition 2.30.** For $F : N(G)_L \to \text{Set}$ homogeneous, and $E : N(G)_L \to \text{Grp}$ a deformation functor, given $a,b \in F(L)$, define $\text{Iso}(a,b) : N_{L,k} \to \text{Set}$ by

$$\text{Iso}(a,b)(L \xrightarrow{f} N) = \{g \in E(N) | g * f(a) = f(b)\}.$$ 

Then $\text{Iso}(a,b)$ is a deformation functor, with tangent space $\text{ker} \nu$ and, if $E$ is moreover smooth, complete obstruction space $\text{coker} \nu = t_D$.

*Proof.* [Man99], Proposition 2.21. \hfill \Box

**Proposition 2.31.** If $E,E'$ are smooth deformation functors, acting on $F,F'$ respectively, with $F,F'$ homogeneous, $\text{ker} \nu \to \text{ker} \nu'$ surjective, and $\text{coker} \nu \to \text{coker} \nu'$ injective, then $F/E \to F'/E'$ is injective.

*Proof.* [Man99], Corollary 2.22. \hfill \Box

This final result does have an analogue in [Man99], but proves extremely useful:
Corollary 2.32. If $F : \mathcal{N}(G)_{\mathcal{L}} \to \text{Set}$ and $E : \mathcal{N}(G)_{\mathcal{L}} \to \text{Grp}$ are deformation functors, with $E$ acting on $F$, let $D := F/E$, then:

1. If $E$ is smooth, then $\eta_D$ is surjective for every $M \to N$ (i.e. $D$ is a deformation functor).

2. If $F$ is homogeneous and $\ker \nu = 0$, then $\eta_D$ is injective for every $M \to N$.

Thus, in particular, $F/E$ will be homogeneous if $F$ is homogeneous, $E$ is a smooth deformation functor and $\ker \nu = 0$.

To summarise the results concerning the pro-representability of the quotient $D = F/E$, we have:

1. If $F$ is a deformation functor and $G$ a smooth deformation functor, then $D$ has a hull.

2. If $F$ is homogeneous and $E$ a smooth deformation functor, with $\ker \nu = 0$, then $D$ is pro-representable.

3 Twisted differential graded algebras

Throughout this section, we will adopt the conventions of [DMOS82] concerning tensor categories. In particular, the associativity isomorphisms will be denoted

$$\phi_{UW} : U \otimes (V \otimes W) \to (U \otimes V) \otimes W,$$

and the commutativity isomorphisms

$$\psi_{UV} : U \otimes V \to V \otimes U.$$

Definition 3.1. The category $\text{DGVect}$ of graded real vector spaces $\bigoplus_{i \geq 0} V^i$ is a tensor category, with the obvious tensor product

$$(U \otimes V)^n = \bigoplus_{i+j=n} U^i \otimes V^j,$$

with differential $d|_{U^i \otimes V^j} = d_{U^i} \otimes \text{id} + (-1)^i \text{id} \otimes d_{V^j}$. The associativity map is the obvious one, while the commutativity map is

$$\psi_{UV} : U \otimes V \to V \otimes U,$$

$$u \otimes v \mapsto (-1)^{|u|} v \otimes u,$$

for $u \in U^i, v \in V^j$.

Definition 3.2. A (real) DGA over a tensor category $\mathcal{C}$ is defined to be an additive functor $A : \mathcal{C} \to \text{DGVect}$, equipped with a multiplication

$$\mu_{UV} : A(U) \otimes A(V) \to A(U \otimes V),$$

functorial in $U$ and $V$, such that
1. Associativity. The following diagram commutes:

\[
\begin{array}{ccc}
AU \otimes (AV \otimes AW) & \overset{id \otimes \mu}{\longrightarrow} & AU \otimes A(V \otimes W) \\
\phi \downarrow & & \mu \downarrow \phi \\
(AU \otimes AV) \otimes AW & \overset{\mu \otimes id}{\longrightarrow} & A(U \otimes V) \otimes AW \\
& & \mu \longrightarrow A((U \otimes V) \otimes W).
\end{array}
\]

2. Commutativity. The following diagram commutes:

\[
\begin{array}{ccc}
AU \otimes AV & \overset{\mu}{\longrightarrow} & A(U \otimes V) \\
\psi \downarrow & & \psi \downarrow A \psi \\
AV \otimes AU & \overset{\mu}{\longrightarrow} & A(V \otimes U).
\end{array}
\]

Remark 3.3. Note that, if we take \(C\) to be the category of finite-dimensional complex vector spaces, then giving a DGA \(A\) over \(C\) is equivalent to giving the differential graded algebra \(A(\mathbb{R})\), which motivates the terminology.

Definition 3.4. We say that a DGA \(A\) over \(C\) is flat if for every exact sequence \(0 \to U \to V \to W \to 0\) in \(C\), the sequence \(0 \to AU \to AV \to AW \to 0\) is exact.

Lemma 3.5. Given a finite-dimensional Lie algebra \(L\) with a \(G\)-action, the graded vector space \(A(L)\) has the natural structure of a differential graded Lie algebra.

Proof. It suffices to define the Lie bracket. Let it be the composition

\[
A(L) \otimes A(L) \overset{\mu}{\longrightarrow} A(L \otimes L) \overset{A([,])}{\longrightarrow} A(L).
\]

the associativity, commutativity and compatibility axioms are easily verified. \(\square\)

Definition 3.6. Given a flat DGA \(A\) over \(\text{Rep}(G)\), the Maurer-Cartan functor \(MC_A : \mathcal{N}(G) \to \text{Set}\) is defined by

\[
MC_A(N) = \{ x \in A(N)^1 | dx + \frac{1}{2} [x, x] = 0 \}.
\]

Observe that for \(\omega \in A(N)^1\),

\[
d\omega + \frac{1}{2} [\omega, \omega] = 0 \Rightarrow (d + \text{ad}_\omega) \circ (d + \text{ad}_\omega) = 0,
\]

so \((A(N), [,], d + \text{ad}_\omega)\) is a DGLA.

Definition 3.7. Define the gauge functor \(G_A : \mathcal{N}(G) \to \text{Grp}\) by

\[
G_A(N) = \exp(A(N)^0),
\]

noting that nilpotence of \(N\) implies nilpotence of \(A(N)^0\).
We may now define the DGLA \((A(N))_d\) as in [Man99]:

\[
(A(N))_i^d = \begin{cases} 
(A(N))_1^d \oplus \mathbb{R}d & i = 1 \\
(A(N))_i^d & i \neq 1,
\end{cases}
\]

with

\[d_d(d) = 0, \quad [d, d] = 0, \quad [d, a]_d = da, \quad \forall a \in (A(N)).\]

**Lemma 3.8.** \(\exp(A(N)^0)\) commutes with \([,]\) when acting on \((A(N))_d\) via the adjoint action.

**Corollary 3.9.** Since \(\exp(A(N)^0)\) preserves \((A(N)^1) + d \subset (A(N))_d\) under the adjoint action, and

\[x \in MC_A(N) \iff [x + d, x + d] = 0,
\]

the adjoint action of \(\exp(A(N)^0)\) on \((A(N)^1 + d)\) induces an action of \(G_A(N)\) on \(MC_A(N)\), which we will call the gauge action.

**Definition 3.10.** \(Def_A = MC_A/G_A\), the quotient being given by the gauge action \(\alpha(x) = \text{ad}_a(x + d) - d\). Observe that \(G_A\) and \(MC_A\) are homogeneous. Define the deformation groupoid \(\mathcal{D}ef_A\) to have objects \(MC_A\), and morphisms given by \(G_A\).

Now, \(t_{G_A}(V) = A^0(V)\), and \(t_{MC_A}(V) = Z^1(A(V))\), with action

\[t_{G_A} \times t_{MC_A} \to t_{MC_A};
\]

\[(b, x) \mapsto x + db, \text{ so}
\]

\[t_{Def_A}(V) = H^1(A(V)).\]

**Lemma 3.11.** \(H^2(A)\) is a complete obstruction space for \(MC_A\).

**Proof.** Given a small extension

\[e : 0 \to I \to N \to M \to 0,
\]

and \(x \in MC_A(M)\), lift \(x\) to \(\tilde{x} \in A^1(N)\), and let

\[h = d\tilde{x} + \frac{1}{2}[	ilde{x}, \tilde{x}] \in A^2(N).
\]

In fact, \(h \in A^2(I)\), as \(dx + \frac{1}{2}[x, x] = 0\).

Now,

\[dh = d^2\tilde{x} + [d\tilde{x}, \tilde{x}] = [h - \frac{1}{2}[	ilde{x}, \tilde{x}], \tilde{x}] = [h, \tilde{x}] = 0,
\]

since \([\tilde{x}, \tilde{x}], \tilde{x}] = 0\) and \([I, N] = 0\). Let

\[o_e(x) = [h] \in H^2(A(I)).\]

This is well-defined: if \(y = \tilde{x} + z\), for \(z \in A^1(K)\), then

\[dy + \frac{1}{2}[y, y] = d\tilde{x} + dz + \frac{1}{2}[	ilde{x}, \tilde{x}] + \frac{1}{2}[z, z] + [\tilde{x}, z] = h + dz,
\]

as \([I, N] = 0\).

This construction is clearly functorial, so it follows that \((H^2(A), o_e)\) is a complete obstruction theory for \(MC_A\).
Now Proposition 2.29 implies the following:

**Theorem 3.12.** Def$_A$ is a deformation functor, $t_{\text{Def}_A} \cong \text{H}^1(A)$, and $\text{H}^2(A)$ is a complete obstruction theory for Def$_A$.

The other propositions of Section 2 can be used to prove:

**Theorem 3.13.** If $\phi : A \to B$ is a morphism of DGAs over Rep(G), and

$$\text{H}^i(\phi) : \text{H}^i(A) \to \text{H}^i(B)$$

are the induced maps on cohomology, then:

1. If $\text{H}^1(\phi)$ is bijective, and $\text{H}^2(\phi)$ injective, then $\text{Def}_A \to \text{Def}_B$ is étale.
2. If also $\text{H}^0(\phi)$ is surjective, then $\text{Def}_A \to \text{Def}_B$ is an isomorphism.
3. Provided condition 1 holds, $\text{Def}_A \to \text{Def}_B$ is an equivalence of functors of groupoids if and only if $\text{H}^0(\phi)$ is an isomorphism.

**Proof.** [Man99], Theorem 3.1, mutatis mutandis. $\square$

**Theorem 3.14.** If $\text{H}^0(A) = 0$, then Def$_A$ is homogeneous.

**Proof.** Proposition 2.32 $\square$

Thus, in particular, a quasi-isomorphism of DGAs gives an isomorphism of deformation functors and of deformation groupoids.

**Remark 3.15.** The category of DGAs over Rep(G) is, in fact, equivalent to the category of $G$-equivariant differential graded algebras. Given a DGA $A$ over Rep(G), we consider the structure sheaf $O(G)$ of $G$, regarded as a $G$-representation via the left action. Then $O(G) \in \text{ind}(\text{Rep}(G))$, and we therefore set $B = A(O(G))$, which has a DGA structure arising from the algebra structure on $O(G)$, and a $G$-action given by the right action on $O(G)$. Conversely, given a $G$-equivariant DGA $B$, we define $A(V) := B \otimes^G V$, the subspace of $G$-invariants of $B \otimes V$.

By Lemma 1.3, the vector space $O(G) \otimes^G V$ is isomorphic to $V$, with the $G$-action on $O(G)$ coming from the right action of $G$. This implies that the functors above define an equivalence. This equivalence will mean that the twisted DGA considered in Section 6 is a model for the schematic homotopy type considered in [KPT05].

### 4 Relative Malcev completions

**Definition 4.1.** Given a group $\Gamma$ with a representation $\rho_0 : \Gamma \to G$ to a reductive real pro-algebraic group, define the functor

$$\mathcal{R}_{\rho_0} : \mathcal{N}(G) \to \text{Grpd}$$

of deformations of $\rho_0$ so that the objects of $\mathcal{R}_{\rho_0}(u)$ are representations

$$\rho : \Gamma \to \exp(u) \rtimes G$$

lifting $\rho_0$, and isomorphisms are given by the conjugation action of the unipotent group $\exp(u)$ on $\exp(u) \rtimes G$. Explicitly, $u \in \exp(u)$ maps $\rho$ to $\rho u \rho^{-1}$. 
Lemma 4.2.  1. The functor $R_{\rho_0}$ of objects of $\mathcal{R}_{\rho_0}$ is a deformation functor, with tangent space $V \mapsto H^1(\Gamma, \rho_0^\sharp V)$ and obstruction space $V \mapsto H^2(\Gamma, \rho_0^\sharp V)$.

2. Given $\omega, \omega' \in \mathcal{R}_{\rho_0}(\mathfrak{g})$, the functor on $N(G)_\mathfrak{g}$ given by

$$u \mapsto \text{Iso}_{\mathcal{R}_{\rho_0}(u)}(\omega, \omega')$$

is homogeneous, with tangent space

$$V \mapsto H^0(\Gamma, \rho_0^\sharp V)$$

and obstruction space

$$V \mapsto H^1(\Gamma, \rho_0^\sharp V).$$

Proposition 4.3. Let $\Gamma_{\text{alg}}$ be the pro-algebraic completion of $\Gamma$, and $\rho_0 : \Gamma \rightarrow \Gamma_{\text{red}}$ its reductive quotient. Then the Lie algebra $L(R_{u}(\Gamma_{\text{alg}}))$ of the pro-unipotent radical of $\Gamma_{\text{alg}}$, equipped with its $\Gamma_{\text{red}}$-action as in Theorem 1.4, is a hull for the functor $R_{\rho_0}$.

Proof. By definition,

$$R_{\rho_0}(U) = \text{Hom}(\Gamma, U \rtimes \Gamma_{\text{red}})_{\rho_0}/U = \text{Hom}(\Gamma_{\text{alg}}, U \rtimes \Gamma_{\text{red}})_{\rho_0}/U.$$ 

If we now fix a Levi decomposition $\Gamma_{\text{alg}} \cong R_{u}(\Gamma_{\text{alg}}) \rtimes \Gamma_{\text{red}}$, we may rewrite this as

$$\text{Hom}(R_{u}(\Gamma_{\text{alg}}) \rtimes \Gamma_{\text{red}}, U \rtimes \Gamma_{\text{red}})_{\rho_0}/U.$$ 

There is a natural map

$$f : \text{Hom}_{X(\Gamma_{\text{red}})}(R_{u}(\Gamma_{\text{alg}}), U) \rightarrow \text{Hom}(R_{u}(\Gamma_{\text{alg}}) \rtimes \Gamma_{\text{red}}, U \rtimes \Gamma_{\text{red}})_{\rho_0}/U,$$

and we need to show that this map is surjective, and an isomorphism on tangent spaces. For surjectivity, take

$$\rho : R_{u}(\Gamma_{\text{alg}}) \rtimes \Gamma_{\text{red}} \rightarrow U \rtimes \Gamma_{\text{red}}$$

lifting $\rho_0$. Since $\Gamma_{\text{red}}$ is reductive, $\rho(\Gamma_{\text{red}}) \leq U \rtimes \Gamma_{\text{red}}$ must be reductive. But the composition

$$\rho(\Gamma_{\text{red}}) \hookrightarrow U \rtimes \Gamma_{\text{red}} \rightarrow \Gamma_{\text{red}}$$

is a surjection, and $\Gamma_{\text{red}}$ is also the reductive quotient of $U \rtimes \Gamma_{\text{red}}$, so $\rho(\Gamma_{\text{red}})$ is a maximal reductive subgroup. By the Levi decomposition theorem, maximal reductive subgroups are conjugate under the action of $U$, so there exists $u \in U$ such that $\text{ad}_u \rho(\Gamma_{\text{red}}) = \Gamma_{\text{red}}$. Now we may replace $\rho$ by $\text{ad}_u \rho$, since they define the same element of $R_{\rho_0}(U)$. As $\text{ad}_u \rho$ preserves $\Gamma_{\text{red}}$, its restriction $R_{u}(\Gamma_{\text{alg}}) \rightarrow U$ is $\Gamma_{\text{red}}$-equivariant, so $\text{ad}_u \rho$ lies in the image of $f$.

To see that $f$ induces an isomorphism on tangent spaces, we need to show that it is injective whenever $U$ is abelian. This is immediate, since the conjugation action of $U$ on $U$ is then trivial. □
Remark 4.4. In the terminology of [Hai98], $\Gamma_{\text{alg}} \to \Gamma_{\text{red}}$ is the relative Malcev completion of the representation $\Gamma \to \Gamma_{\text{red}}$, so we can regard this section as studying Malcev completions of arbitrary Zariski-dense reductive representations.

Definition 4.5. Given a homomorphism $\theta : G \to H$ of algebraic groups, with $H$ reductive, define $\theta^\#: \hat{N}(G) \to \hat{N}(H)$ to be left adjoint to the restriction map $\theta^\#: \hat{N}(H) \to \hat{N}(G)$, so that
\[
\text{Hom}_{\hat{N}(G)}(\theta^\# L, N) \cong \text{Hom}_{\hat{N}(H)}(L, \theta^\# N).
\]
This left adjoint must exist, since the functor on the right satisfies Schlessinger’s conditions.

Lemma 4.6. If $\Gamma = \Delta \times \Lambda$, such that the adjoint action of $\Lambda$ on the pro-unipotent completion $\Delta \otimes \mathbb{R}$ is reductive, then
\[
\bar{\rho}^\# R_u(\Gamma_{\text{alg}}) \cong (\Delta \otimes \mathbb{R}) \times R_u(\Lambda_{\text{alg}}) \in \exp(\hat{N}(\Lambda_{\text{red}})),
\]
where we write $\rho$ for the composition $\Gamma \to \Lambda_{\text{red}}$, and $\bar{\rho}$ for the quotient representation $\Gamma_{\text{red}} \to \Lambda_{\text{red}}$.

Proof. We use the fact that $\bar{\rho}^\# R_u(\Gamma_{\text{alg}})$ pro-represents the functor $U \mapsto \text{Hom}(\Gamma, U \times \Lambda_{\text{red}})_{\rho}$, for $U \in \exp(\hat{N}(\Lambda_{\text{red}}))$.

A homomorphism $\Gamma \to U \times \Lambda_{\text{red}}$ lifting $\rho$ gives rise to a map $\Delta \otimes \mathbb{R} \to U$, since $U$ is unipotent. It is then clear that
\[
\bar{\rho}^\# R_u(\Gamma_{\text{alg}}) \cong (\Delta \otimes \mathbb{R}) \times R_u(\Lambda_{\text{alg}}).
\]
Finally, observe that $R_u(\Lambda_{\text{alg}})$ acts trivially on $\Delta \otimes \mathbb{R}$, since the action is reductive. \qed

5 Principal homogeneous spaces

Fix a connected differentiable manifold $X$. Let $\varpi^\text{red}_1(X, x)$ be the reductive quotient of the pro-algebraic real completion of $\pi_1(X, x)$, so that $\text{Rep}(\varpi^\text{red}_1(X, x))$ can be regarded as the category of real semisimple $\pi_1(X, x)$-representations. Given such a representation $V$, let $\mathcal{V}$ denote the corresponding semisimple local system.

Definition 5.1. We may then define a DGA over $\text{Rep}(\varpi^\text{red}_1(X, x))$ by
\[
A(V) := \Gamma(X, \mathcal{V} \otimes \mathcal{A}^\bullet),
\]
where $\mathcal{A}^\bullet$ is the sheaf of real $C^\infty$ forms on $X$. The multiplication is given by
\[
A(V) \otimes A(W) \cong \Gamma(X \times X, p_1^*(\mathcal{V} \otimes \mathcal{A}^\bullet) \otimes p_2^*(\mathcal{W} \otimes \mathcal{A}^\bullet)) \xrightarrow{\Delta^\#} \Gamma(X, (\mathcal{V} \otimes \mathcal{A}^\bullet) \otimes (\mathcal{W} \otimes \mathcal{A}^\bullet)) \xrightarrow{} \Gamma(X, (\mathcal{V} \otimes \mathcal{W}) \otimes \mathcal{A}^\bullet),
\]
the first map being the Künneth isomorphism, where $\Delta : X \to X \times X$ is the diagonal map, and $p_1, p_2 : X \times X \to X$ the projection maps. The final isomorphism is the composition of the multiplication on $\mathcal{A}^\bullet$ with the relevant associativity and commutativity isomorphisms.
The aim of this section is to prove that the groupoids $\text{Def}_A(g)$ are functorially equivalent to the groupoid of $\exp(g)$-torsors, where $g$ is the sheaf of Lie algebras associated to $g$.

**Definition 5.2.** Given a locally constant sheaf $G$ of groups on $X$, define $\mathcal{B}(G)$, the category of $G$-torsors (or principal homogeneous $G$-spaces) to consist of sheaves of sets $\mathcal{B}$ on $X$, together with a multiplication $G \times \mathcal{B} \to \mathcal{B}$ such that $g \cdot (h \cdot b) = (gh) \cdot b$, and the stalks $\mathcal{B}_x$ are isomorphic (as $G_x$-spaces) to $G_x$.

**Lemma 5.3.** There is a canonical morphism $\mathcal{B} : \text{Def}_A(g) \to \mathcal{B}(\exp(g))$, functorial in $g \in N(\mathcal{W}^{\text{red}}_1(X,x))$.

**Proof.** Given $\omega \in \text{MC}_A$, let $\mathcal{B}_\omega := D^{-1}(\omega)$, where

$$D : \exp(g \otimes \mathcal{A}^0) \to g \otimes \mathcal{A}^1$$

$$\alpha \mapsto d\alpha \cdot \alpha^{-1}.$$ 

Then $\mathcal{B}_\omega$ is a principal $\exp(g)$-sheaf on $X$. □

**Lemma 5.4.** 1. The functor $g \mapsto B(\exp(g))$, the set of isomorphism classes of $\mathcal{B}(\exp(g))$, is a deformation functor with tangent space

$$V \mapsto H^1(X,V),$$

and obstruction space

$$V \mapsto H^2(X,V).$$

2. Given $\omega, \omega' \in \mathcal{B}(\exp(g))$, the functor on $N(\mathcal{W}^{\text{red}}_1(X,x))_g$ given by

$$h \mapsto \text{Iso}_{\mathcal{B}(\exp(h))}(\omega, \omega')$$

is homogeneous, with tangent space

$$V \mapsto H^0(X,V)$$

and obstruction space

$$V \mapsto H^1(X,V).$$

**Proof.** Take an cover $\{U_i\}$ of $X$ by open discs. Then a $V$-torsor $\mathcal{B}$ is determined by fixing isomorphisms $V|_{U_i} \cong \mathcal{B}|_{U_i}$ and specifying transition maps in $V_{U_i \cap U_j}$ satisfying the cocycle condition. The result follows by considering isomorphism classes of these data. □

**Theorem 5.5.** The functor $\mathcal{B}$ is an equivalence of groupoids.

**Proof.** We begin by proving essential surjectivity. The morphism $\text{Def}_A(g) \to B(\exp(g))$ induces an isomorphism on tangent and obstruction spaces $H^1(X,V)$, so is étale (by Proposition 2.28). Now, $\text{Iso}(\omega, \omega') \to \text{Iso}(\mathcal{B}_\omega, \mathcal{B}_{\omega'})$ on $N(G)_g$ is similarly étale, so must be an isomorphism, both functors being pro-representable. □
We will look at an algebraic interpretation of the groupoids we have been considering.

**Lemma 5.6.** If $X$ is a connected differentiable manifold and $\Gamma = \pi_1(X, x)$ is its fundamental group, then there is a canonical equivalence of groupoids

$$\mathbb{B} : \mathcal{R}_{\rho_0}(g) \to \mathcal{B}(\exp(g)),$$

for $\rho_0 : \pi_1(X, x) \to \mathfrak{w}_1^\text{red}(X, x)$, and $g \in \mathcal{N}(\mathfrak{w}_1^\text{red}(X, x))$.

**Proof.** Let $\tilde{X} \xrightarrow{\pi} X$ be the universal covering space of $X$, on which $\Gamma$ acts. Then, associated to any representation $\rho : \Gamma \to H$, we have the $H$-torsor

$$\mathbb{B}_{\rho, H} := \mathcal{P}(\pi^*H, \rho).$$

Associated to any $\rho : \Gamma \to \exp(g) \rtimes G$ lifting $\rho_0$, we have a representation $\rho : \Gamma \to H$, where $H = \exp(g) \rtimes \Gamma$. This gives rise to the $H$-torsor $\mathbb{B}_{\rho, H}$. Let $\mathbb{B}_\rho := \mathbb{B}_{\rho, H}/\Gamma$ be the quotient sheaf under the $\Gamma$-action (using $\Gamma \leq H$). It follows that this is an $\exp(g)$-torsor.

Finally to see that $\mathbb{B}$ defines an equivalence, observe that the maps on tangent and obstruction spaces are

$$H^i(\Gamma, V) \to H^i(X, \mathcal{V}),$$

which are isomorphisms for $i = 0, 1$, and injective for $i = 2$. The equivalence then follows from Proposition 2.28. \[ \square \]

## 6 Hodge theory

Let $X$ be a compact connected Kähler manifold, and let $\mathfrak{w}_1^\text{red}(X, x)$ be the reductive pro-algebraic completion of $\pi_1(X, x)$. Recall that the DGA $A$ is defined over $\text{Rep}(\mathfrak{w}_1^\text{red}(X, x))$ by

$$A(V) := \Gamma(X, \mathcal{V} \otimes \omega^*).$$

Since $\text{Rep}(\mathfrak{w}_1^\text{red}(X, x))$ is an exact category, note that all DGAs over $\text{Rep}(\mathfrak{w}_1^\text{red}(X, x))$ are flat.

**Theorem 6.1.** The DGA $A$ is formal, i.e. weakly equivalent to its cohomology DGA.

**Proof.** We have an operator $d^c = J^{-1}dJ$ on $A$, where $J$ is the complex structure. This satisfies $dd^c + d^c d = 0$. We then have the following morphisms of DGAs:

$$H_{d^c}(A)(V) \leftarrow Z_{d^c}(A)(V) \to (A(V)),$$

where $Z_{d^c}(A)(V)^n = \ker(d^c : A(V)^n \to A(V)^{n+1})$, with differential $d$, and $H_{d^c}(A)(V)$ also has differential $d$. Since $d^c(a \cup b) = (d^c a) \cup b + (-1)^{\deg a} a \cup (db)$, these are indeed both DGAs. It follows from \[ \text{Sim92} \] Lemmas 2.1 and 2.2, using the $dd^c$ lemma instead of the $\partial \bar{\partial}$ lemma, that these morphisms are quasi-isomorphisms, and that $d = 0$ on $H_{d^c}(A)(V))$. \[ \square \]

**Remark 6.2.** It follows from Remark 3.15 that this is equivalent to \[ \text{KPT05} \] Theorem 3.2.3, which states that the complex schematic homotopy type is formal.
Corollary 6.3. The Lie algebra $\mathcal{L}(R_u(\varpi_1(X,x)))$ associated to $R_u(\varpi_1(X,x))$ is quadratically presented (i.e. defined by equations of bracket length 2) as an element of $\mathcal{N}(\varpi_1^\text{red}(X,x))$, and has a weight decomposition (as a pro-vector space), unique up to inner automorphism.

Proof. The functor $MC_{\mathcal{H}(A)}$ is homogeneous, hence pro-representable, and $MC_{\mathcal{H}(A)} \to \text{Def}_{\mathcal{H}(A)}$ is étale, so $MC_{\mathcal{H}(A)}$ is pro-represented by a hull for $\text{Def}_{\mathcal{H}(A)}$. By Theorem 6.1, $\text{Def}_{\mathcal{H}(A)}$ is isomorphic to $\text{Def}_A$, which by Theorem 5.5 and Lemma 5.6 is isomorphic to $R_{\text{rl}}$. By Lemma 4.3 this has hull $R_u(\varpi_1(X,x))$. Therefore $R_u(\varpi_1(X,x))$ pro-represents $MC_{\mathcal{H}(A)}$, by the uniqueness of hulls.

Now,

$$MC_{\mathcal{H}(A)}(g) = \{ \omega \in H^1(X,g) \mid [\omega, \omega] = 0 \in H^2(X,g) \}$$

As in Remark 3.15 we may replace $A$ by the $\varpi_1^\text{red}(X,x)$-equivariant DGA $B := A(O(\varpi_1^\text{red}(X,x)))$. Letting $\mathcal{O}$ denote the ind-local system on $X$ associated to the representation $O(\varpi_1^\text{red}(X,x))$, $H(A)$ then corresponds to the $\varpi_1^\text{red}(X,x)$-equivariant DGA

$$H(B)^n := H^n(X, \mathcal{O}).$$

The ind-local system $\mathcal{O}$ was defined using the left $\varpi_1^\text{red}(X,x)$-action on $O(\varpi_1^\text{red}(X,x))$, and the $\varpi_1^\text{red}(X,x)$-action on $B$ is then defined using the right action.

Therefore $H(A)^n(V) = H^n(X, \mathcal{O}) \otimes_{\varpi_1^\text{red}(X,x)} V$, so

$$MC_{\mathcal{H}(A)}(g) = \{ \omega \in H^1(X, \mathcal{O}) \otimes_{\varpi_1^\text{red}(X,x)} g \mid [\omega, \omega] = 0 \in H^2(X, \mathcal{O}) \otimes_{\varpi_1^\text{red}(X,x)} g \}.$$

Let

$$H_i := H^i(X, \mathcal{O})^\vee \in \text{Rep}(\varpi_1^\text{red}(X,x)).$$

There are canonical isomorphisms

$$\text{Hom}_{\mathcal{N}(\varpi_1^\text{red}(X,x))}(L(H_1), g) \cong \text{Hom}_{\text{Rep}(\varpi_1^\text{red}(X,x))}(H_1, g) \cong H^1(X,g),$$

where $L$ denotes the free pro-nilpotent Lie algebra functor.

Now, the cup product

$$H^1(X, \mathcal{O}) \otimes H^1(X, \mathcal{O}) \xrightarrow{\cup} H^1(X, \mathcal{O})$$

gives a coproduct

$$\Delta : H_2 \to H_1 \otimes H_1 \to \bigwedge^2 H_1 \subset L(H_1).$$

Finally, observe that $MC_{\mathcal{H}(A)}(g)$ is isomorphic to the set

$$\{ \omega \in \text{Hom}_{\mathcal{N}(\varpi_1^\text{red}(X,x))}(L(H_1), g) \mid \omega \circ \Delta(H_2) = 0 \},$$

so

$$\mathcal{L}(R_u(\varpi_1(X,x))) \cong L(H_1)/\Delta(H_2)$$

is a quadratic presentation.

If we set $H_1$ to have weight $-1$, and $H_2$ to have weight $-2$, then $\mathcal{L}(R_u(\varpi_1(X,x)))$ has a canonical weight decomposition arising from those on $H_1$ and $H_2$, since $\Delta$ preserves
the weights. Note that a weight decomposition on a pro-finite-dimensional vector space is an infinite product, rather than an infinite direct sum. This decomposition is only unique up to inner automorphism, since the hull morphism is; this is equivalent to saying that we have not made a canonical choice of Levi decomposition.

**Corollary 6.4.** Let \( G \) be an arbitrary reductive real algebraic group, acting on a real unipotent algebraic group \( U \) defined by homogeneous equations, i.e. \( u \cong \text{gru} \) as Lie algebras with \( G \)-actions. If

\[
\rho_2 : \pi_1(X, x) \to (U/[U, [U, U]]) \rtimes G
\]

is a Zariski-dense representation, then

\[
\rho_1 : \pi_1(X, x) \to (U/[U, U]) \rtimes G
\]

lifts to a representation

\[
\rho : \pi_1(X, x) \to U \rtimes G.
\]

**Proof.** Observe that these representations correspond to \( \pi_1(X, x) \)-equivariant homomorphisms \( R_u(\pi_1(X, x)) \) to \( U \). Let \( g \) be the Lie algebra associated to \( R_u(\pi_1(X, x)) \). We must show that the surjective map \( \rho_1 : g \to u/[u, u] \) lifts to \( u \). Let \( g = L(V)/\langle W \rangle \), for \( W \subset \wedge^2 V \). Since \( V \cong \text{gr}_1 g = g/[g, g] \), the morphism \( \rho_1 \) gives us a map \( \theta : L(V) \to \text{gr}_u \), so it will suffice to show that \( \theta \) annihilates \( W \). But \( \wedge^2 g \to \text{gr}_2 u \) must send \( u \wedge v \) to \( [\rho_1(u), \rho_1(v)] = [\rho_2(u), \rho_2(v)] \), which annihilates \( W \), as required.

**Corollary 6.5.** For semisimple \( \pi_1(X, x) \)-representations \( V_1, \ldots V_n \), with \( n \geq 3 \), the Massey products

\[
H^1(\pi_1(X, x), V_1) \otimes H^1(\pi_1(X, x), V_2) \otimes \ldots \otimes H^1(V_n) \to H^2(\pi_1(X, x), V_1 \otimes V_2 \otimes \ldots \otimes V_n)
\]

are all zero.

**Proof.** This follows from the observation that these maps all arise as quotients of higher obstruction maps for quotients of \( L(V_1 \oplus V_2 \oplus \ldots \oplus V_n) \). Alternatively, it can be deduced directly from Theorem 6.1, which implies that all the higher Massey products are zero on the cohomology of \( X \) with semisimple coefficients.

**Remarks 6.6.** Note that Corollary 6.3 implies the results on the fundamental group of [DGMS75], of [GMSS] and of [Hai98]. The pro-unipotent completion \( \varpi_1(X, x) \otimes \mathbb{R} \) studied in [DGMS75] is just the maximal quotient of \( R_u(\varpi_1(X, x)) \) on which \( \pi_1(X, x) \) acts trivially.

The problem considered in [GMSS] (and generalised in [Sim92]) is to fix a reductive representation \( \rho_0 : \pi_1(X, x) \to G(\mathbb{R}) \), and consider lifts \( \rho : \pi_1(X, x) \to G(A) \), for Artinian rings \( A \). The hull of this functor is the functor

\[
A \mapsto \text{Hom}_{\pi_1(X, x)}(R_u(\varpi_1(X, x)), \exp(g \otimes m_A)),
\]

22
where $\mathfrak{g}$ is the Lie algebra of $G$, regarded as the adjoint representation. It follows that this hull then has generators $\text{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_1)$, and relations

$$\text{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_2) \rightarrow S^2\text{Hom}_{\pi_1(X,x)}(\mathfrak{g}, H_1)$$

given by composing the coproduct and the Lie bracket.

The statement of Corollary 6.3 is equivalent to saying that the Malcev completion of any Zariski-dense representation $\rho : \pi_1(X, x) \rightarrow G(\mathbb{R})$, for $G$ reductive, is quadratically presented. In [Hai98] Theorem 13.14, this is proved only for those $\rho$ which are polarised variations of Hodge structure, and no consequences are given.

**Proposition 6.7.** If $\pi_1(X, x) = \Delta \rtimes \Lambda$, with $\Lambda$ acting reductively on the pro-unipotent completion $\Delta \otimes \mathbb{R}$, then $\Delta \otimes \mathbb{R}$ is quadratically presented.

**Proof.** By Lemma 4.6, we know that

$$\rho^*_R(\varpi_1(X, x)) \cong (\Delta \otimes \mathbb{R}) \times R_u(\Lambda_{\text{alg}}) \in \exp(\hat{\Lambda}_{\text{red}}),$$

for $\rho : \varpi_1(X, x) \rightarrow \Lambda_{\text{red}}$. From Theorem 6.3, we know that $R_u(\varpi_1(X, x))$ is quadratically presented, hence so is $\rho^*_R(\varpi_1(X, x))$.

Now, to give a quadratic presentation for a Lie algebra $\mathfrak{g}$ is equivalent to giving a homomorphism $\theta : \text{gr}(\mathfrak{g}) \rightarrow \mathfrak{g}$ from its associated graded Lie algebra such that $\text{gr}(\theta) : \text{gr}(\mathfrak{g}) \rightarrow \text{gr}(\mathfrak{g})$ is the identity, provided that $\text{gr}(\mathfrak{g})$ is quadratic as a graded Lie algebra. Now, if $\mathfrak{g} \oplus \mathfrak{h}$ is quadratically presented, then so is $\text{gr}(\mathfrak{g} \oplus \mathfrak{h}) = \text{gr}(\mathfrak{g}) \oplus \text{gr}(\mathfrak{h})$, hence so is $\text{gr}(\mathfrak{g})$. Taking the composition

$$\text{gr}(\mathfrak{g}) \rightarrow \text{gr}(\mathfrak{g}) \oplus \text{gr}(\mathfrak{h}) \xrightarrow{\theta} \mathfrak{g} \oplus \mathfrak{h} \rightarrow \mathfrak{g}$$

then gives a quadratic presentation for $\mathfrak{g}$.

Combining these results, we see that $\Delta \otimes \mathbb{R}$ must be quadratically presented. \hfill \square

**Remark 6.8.** Since [Sim92] Lemma 4.5 states that properly rigid reductive representations underlie variations of Hodge structure, Proposition 6.7 can be deduced directly from [Hai98] whenever the composition $\pi_1(X, x) \rightarrow L \rightarrow \text{Aut}(\Delta \otimes \mathbb{R})$ is properly rigid and reductive.

**Example 6.9.** Let $\mathfrak{h} = \mathbb{R}^2 \oplus \mathbb{R}$, with Lie bracket $[\mathfrak{h}, \mathbb{R}] = 0$ and $[u, v] = u \wedge v \in \mathbb{R}$ for $u, v \in \mathbb{R}^2$, so $\exp(\mathfrak{h})$ is isomorphic to the real three-dimensional Heisenberg group. The Campbell-Baker-Hausdorff formula enables us to regard $\exp(\mathfrak{h})$ as the group with underlying set $\mathfrak{h}$ and product $a \cdot b = a + b + \frac{1}{2}[a, b]$, since all higher brackets vanish. It then follows that the lattice

$$H := \exp(\mathbb{Z}^2 \oplus \frac{1}{2}\mathbb{Z}),$$

is closed under this multiplication, so forms a discrete group, with $H \otimes \mathbb{R} = \exp(\mathfrak{h})$.

Now, $\text{SL}_2(\mathbb{Z})$ acts on $H$ by the formula:

$$A(v, w) := (Av, (\det A)w) = (Av, w),$$

for $v \in \mathbb{Z}^2$ and $w \in \frac{1}{2}\mathbb{Z}$. 23
Let \( L := \mathbb{Z}^2 = \mathbb{Z}a \oplus \mathbb{Z}b \) act on \( H \) via the homomorphism \( \vartheta : L \to \text{SL}_2(\mathbb{Z}) \) given by

\[
\vartheta(a) = \vartheta(b) = M := \begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix}.
\]

Then the Zariski closure of this representation is isomorphic to \( \mathbb{G}_m(\mathbb{R}) = \mathbb{R}^* \), being the torus in \( \text{SL}_2(\mathbb{R}) \) containing \( M \), so the action of \( L \) on \( \mathfrak{h} \) is reductive. Since \( \mathfrak{h} \) is not quadratically presented, the group \( \Gamma := H \rtimes \text{SL}_2(\mathbb{Z}) \) cannot be the fundamental group of any compact Kähler manifold.

Note that [DGMS75] cannot be used to exclude this group: since the commutator \([a, (v, 0)] = (Mv - v, 0)\) and \( M - I \) is non-singular, \([\Gamma, \Delta]\) is of finite index in \( \Delta \), so \( \Gamma \otimes \mathbb{R} = L \otimes \mathbb{R} \), which is quadratically presented.

Furthermore, this result cannot be obtained by substituting Remark 6.8 and [Hai98] Theorem 13.14 for Corollary 6.3, since \( \vartheta \) is not rigid.

Alternatively, we could use Corollary 6.4 to prove that \( \Gamma \) is not a Kähler group. Let \( G = \mathbb{G}_m(\mathbb{R}), u = L(\mathbb{R}^2) \) and \( U = \exp(u) \). Observe that \( \mathfrak{h} \cong u/[u, [u, u]] \), and let

\[
\rho_2 : \Gamma \times L \to \exp(\mathfrak{h}) \rtimes \mathbb{G}_m(\mathbb{R}),
\]

be given by combining the standard embedding with \( \vartheta \).

Since all triple commutators vanish in \( H \), this does not lift to a representation

\[
\rho : H \times L \to U \rtimes \mathbb{G}_m(\mathbb{R})
\].

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