On interpretations and constructions of classical dynamical $r$-matrices

L. Fehér and A. Gábor

Department of Theoretical Physics, University of Szeged
Tisza Lajos krt 84-86, H-6720 Szeged, Hungary
E-mail: lfeher@sol.cc.u-szeged.hu

Abstract

In this note we complement recent results on the exchange $r$-matrices appearing in the chiral WZNW model by providing a direct, purely finite-dimensional description of the relationship between the monodromy dependent 2-form that enters the chiral WZNW symplectic form and the exchange $r$-matrix that governs the corresponding Poisson brackets. We also develop the special case in which the exchange $r$-matrix becomes the ‘canonical’ solution of the classical dynamical Yang-Baxter equation on an arbitrary self-dual Lie algebra.

---

1Based on a talk given by L.F. at the QTS2 symposium, 18-21 July 2001, Kraków, Poland.
1 Introduction

Let $G$ be a connected (real or complex) Lie group whose Lie algebra $\mathfrak{g}$ is self-dual in the sense of admitting a nondegenerate invariant scalar product $\langle \cdot , \cdot \rangle$. The corresponding chiral WZNW phase space consists of $G$-valued quasiperiodic fields on the real line, $g(x + 2\pi) = g(x)M$ $(\forall x \in \mathbb{R})$, where $M \in G$ is the ‘monodromy matrix’. After restricting $M$ to some open submanifold $\tilde{G} \subset G$ (see below), this phase space is equipped \cite{1, 2} with a Poisson structure that can be symbolically described as follows:

$$\kappa \{g(x) \otimes g(y)\} = (g(x) \otimes g(y)) \left(\hat{r}(M) + \frac{1}{2} \hat{I} \text{sign}(y - x)\right), \quad 0 < x, y < 2\pi,$$

(1.1)

where $\hat{r} : \tilde{G} \to \mathfrak{g} \otimes \mathfrak{g}$, that is $\hat{r}(M) = r^{\alpha\beta}(M)T_\alpha \otimes T_\beta$, is an antisymmetric ‘exchange $r$-matrix’, and $\hat{I} = T_\alpha \otimes T_\alpha$ using dual bases $\{T_\alpha\}$, $\{T_\beta\}$ of $\mathfrak{g}$; $\kappa$ is some constant. The Jacobi identity of the Poisson bracket (1.1) is equivalent to a certain dynamical generalization of the (modified) classical Yang-Baxter equation on the exchange $r$-matrix, which we call the ‘$G$-CDYBE’. If for any function $\psi$ on $G$ we introduce

$$(\mathcal{L}_\alpha \psi)(M) := \left. \frac{d}{dt} \psi(e^{tT_\alpha} M) \right|_{t=0}, \quad (\mathcal{R}_\alpha \psi)(M) := \left. \frac{d}{dt} \psi(Me^{tT_\alpha}) \right|_{t=0},$$

(1.2)

then the $G$-CDYBE \cite{2} is given as follows:

$$[\hat{r}_{12}, \hat{r}_{23}] + T_1^\alpha \left(\frac{1}{2} \mathcal{D}_\alpha^+ + r^{\beta\gamma}_\alpha \mathcal{D}_\beta^- \right)\hat{r}_{23} + \text{cycl. perm.} = -\frac{1}{4} \hat{f}.$$  

(1.3)

Here $\mathcal{D}_\alpha^\pm := \mathcal{R}_\alpha \pm \mathcal{L}_\alpha$, the cyclic permutations act on the three tensorial factors as usual, and $\hat{f} = f^{\alpha\beta\gamma}T_\alpha \otimes T_\beta \otimes T_\gamma$ with the structure constants of $\mathfrak{g}$. The Poisson structure (1.1) arises by inverting \cite{2} a (weak) symplectic form on the chiral WZNW phase space, which is defined \cite{1} with the aid of a 2-form $\rho$ on $G$ whose exterior derivative coincides with the restriction of the canonical 3-form on $G$. In this context, the restriction to some $\tilde{G} \subset G$ is necessary since the canonical 3-form on $G$ is closed but not exact. In the next section we provide a self-contained finite-dimensional characterization of the relationship between the 2-form $\rho$ and the exchange $r$-matrix $\hat{r}$. This relationship has previously been established \cite{2} by indirect arguments relying on the properties of the infinite-dimensional chiral WZNW phase space.

2 The 2-form $\rho$ and the exchange $r$-matrix

For any (real or complex) manifold $Q$, denote by $\mathcal{F}(G, Q)$ the set of (smooth or holomorphic) maps from $G$ to $Q$, and let $\mathcal{F}(G)$ be the set of (smooth or holomorphic) scalar functions on $G$. Let $M \in \mathcal{F}(G, \operatorname{End}(G))$ be the map

$$\bar{M} : M \mapsto \operatorname{Ad} M \quad \forall M \in G.$$

(2.1)
With any $X \in \mathcal{F}(G, \mathcal{G})$ associate the vector field $\mathcal{R}_X$ on $G$ by the definition

$$(\mathcal{R}_X \psi)(M) := \frac{d}{dt} \psi(M e^{tX(M)}) \bigg|_{t=0}, \quad \forall M \in G, \quad \psi \in \mathcal{F}(G).$$

(2.2)

Let $\tilde{G}$ be an open domain in $G$ and choose a function $r \in \mathcal{F}(\tilde{G}, \text{End}(\mathcal{G}))$. Define $r_\pm := r \pm \frac{1}{2} I$ and introduce $A \in \mathcal{F}(\tilde{G}, \text{End}(\mathcal{G}))$ as

$$A := (\bar{M}^{-1} \circ r_+ - r_-).$$

(2.3)

With these notations, the $G$-CDYBE (1.3) can be rewritten as follows:

$$\langle [r_\xi, r_\eta], \zeta \rangle - \langle \mathcal{R}_A(\xi) r_\eta, \zeta \rangle + \text{c.p.} = -\frac{1}{4} \langle [\xi, \eta], \zeta \rangle \quad \forall \xi, \eta, \zeta \in \mathcal{G},$$

(2.4)

where the cyclic permutations act on $\xi, \eta, \zeta$.

Consider the canonical 3-form $\Phi$ on $G$,

$$\Phi(\mathcal{R}_X, \mathcal{R}_Y, \mathcal{R}_Z) := \frac{1}{2} \langle [X, Y], Z \rangle,$$

(2.5)

where $[X, Y](M) := [X(M), Y(M)]$. Let $q \in \mathcal{F}(\tilde{G}, \text{End}(\mathcal{G}))$ be a map for which $q(M)$ is antisymmetric with respect to $\langle , \rangle$ for any $M \in \tilde{G}$. With such a map $q$ associate a 2-form $\rho$ on $\tilde{G}$ by

$$\rho(\mathcal{R}_X, \mathcal{R}_Y) = \langle X, q Y \rangle \quad \forall X, Y \in \mathcal{F}(\tilde{G}, \mathcal{G}).$$

(2.6)

**Proposition 1.** Suppose that $d \rho = \Phi$ on an open submanifold $\tilde{G} \subset G$ and

$$\det \left( q_+(M) - q_-(M) \circ \text{Ad}(M^{-1}) \right) \neq 0 \quad \forall M \in \tilde{G},$$

(2.7)

where $q_\pm := q \pm \frac{1}{2} I$. Then

$$r := \frac{1}{2} \left( q_+ - q_- \circ \tilde{M}^{-1} \right)^{-1} \circ \left( q_+ + q_- \circ \tilde{M}^{-1} \right)$$

(2.8)

solves the $G$-CDYBE (2.4) on $\tilde{G}$.

**Remark.** In fact, under the assumption (2.7) formula (2.8) gives the unique solution of the factorization equation

$$q_+ \circ r_- = q_- \circ \tilde{M}^{-1} \circ r_+$$

(2.9)

for $r$, and one also has the equivalent formula

$$r_- = -q_- \circ \left( q_- - \tilde{M} \circ q_+ \right)^{-1}.$$

(2.10)

**Proof of proposition 1.** We start by noting that, as a special case of a standard identity between the Lie and the exterior derivatives, any 2-form $\rho$ on $G$ (or $\tilde{G}$) satisfies

$$(d \rho)(\mathcal{R}_X, \mathcal{R}_Y, \mathcal{R}_Z) = \mathcal{R}_X(\rho(\mathcal{R}_Y, \mathcal{R}_Z)) - \rho([\mathcal{R}_X, \mathcal{R}_Y], \mathcal{R}_Z) + \text{c.p.},$$

(2.11)

3
where the cyclic permutations act on \(X, Y, Z\). The Lie bracket of the vector fields \(\mathcal{R}_X, \mathcal{R}_Y\) is given by

\[
[\mathcal{R}_X, \mathcal{R}_Y] = \mathcal{R}_{\mathcal{B}(X,Y)} \quad \text{with} \quad \mathcal{B}(X,Y) = [X,Y] + \mathcal{R}_XY - \mathcal{R}_YX.
\]

(2.12)

By using the Leibniz rule to evaluate the first term of (2.11) and inserting (2.12) into (2.11), (2.6) implies that

\[
(dp)(\mathcal{R}_X, \mathcal{R}_Y, \mathcal{R}_Z) = \langle q[X,Y], Z \rangle - \langle (\mathcal{R}_Xq)Y, Z \rangle + c.p.
\]

(2.13)

Now taking into account the assumption \(dp = \Phi\), we obtain the identity

\[
-\langle (\mathcal{R}_Xq)Y, Z \rangle + c.p. = \langle [X,Y], qZ \rangle + \frac{1}{6}\langle [X,Y], Z \rangle + c.p.
\]

(2.14)

\(\forall X, Y, Z \in \mathcal{F}(\mathcal{G}, \mathcal{G})\). Then a further important identity is

\[
\langle (\mathcal{R}_{\mathcal{A}(\xi)}r_\eta, \zeta \rangle = \langle (\mathcal{R}_{\mathcal{A}(\xi)}q)\mathcal{A}(\eta), \mathcal{A}(\zeta) \rangle - \langle (\tilde{M} \circ r_- - r_+)\xi, [r_+\eta, r_+\zeta] \rangle
\]

(2.15)

\(\forall \xi, \eta, \zeta \in \mathcal{G}\). One can derive this by calculating the derivative \(\mathcal{R}_{\mathcal{A}(\xi)}r\) from the formula (2.10), and using many times (2.9), the invariance of \(\langle , \rangle\), and the antisymmetry of \(r\) and \(q\). The identities (2.14), (2.15) allow us to express the derivative term in the \(G\)-CDYBE (2.4) in terms of non-derivative terms containing \(r\). More precisely, we need one more identity to do this,

\[
\langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], q\mathcal{A}(\zeta) \rangle = \frac{1}{2}\langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], \mathcal{A}(\zeta) \rangle + \langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], r_-\zeta \rangle,
\]

(2.16)

which follows from the definition of \(\mathcal{A}\) (2.3) and (2.10). With the last three identities in hand, the \(G\)-CDYBE (2.4) can be verified straightforwardly as

\[
\langle [r_\xi, r_\eta], r_\zeta \rangle - \langle (\mathcal{R}_{\mathcal{A}(\xi)}r_\eta, \zeta \rangle + c.p. = \langle [r_\xi, r_\eta], \zeta \rangle - \langle (\mathcal{R}_{\mathcal{A}(\xi)}q)\mathcal{A}(\eta), \mathcal{A}(\zeta) \rangle
\]

\[
\quad + \langle (\tilde{M} \circ r_- - r_+)\xi, [r_+\eta, r_+\zeta] \rangle + c.p.
\]

\[
= \langle [r_\xi, r_\eta], \zeta \rangle + \langle (\tilde{M} \circ r_- - r_+)\xi, [r_+\eta, r_+\zeta] \rangle
\]

\[
\quad + \langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], q\mathcal{A}(\zeta) \rangle + \frac{1}{6}\langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], \mathcal{A}(\zeta) \rangle + c.p.
\]

\[
= \langle [r_\xi, r_\eta], \zeta \rangle + \langle [\tilde{M}^{-1} \circ r_+\xi, \tilde{M}^{-1} \circ r_+\eta], r_-\zeta \rangle
\]

\[
\quad - \langle [\tilde{M}^{-1} \circ r_+\xi, \tilde{M}^{-1} \circ r_+\eta], \tilde{M}^{-1} \circ r_+\zeta \rangle
\]

\[
\quad + \langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], r_-\zeta \rangle + \frac{2}{3}\langle [\mathcal{A}(\xi), \mathcal{A}(\eta)], \mathcal{A}(\zeta) \rangle + c.p.
\]

\[
= \langle [r_\xi, r_\eta], \zeta \rangle - \frac{1}{3}\langle [\tilde{M}^{-1} \circ r_+\xi, \tilde{M}^{-1} \circ r_+\eta], \tilde{M}^{-1} \circ r_+\zeta \rangle
\]

\[
\quad + \frac{1}{3}\langle [r_-\xi, r_-\eta], r_-\zeta \rangle + c.p. = -\frac{1}{4}\langle [\xi, \eta], \zeta \rangle,
\]

(2.17)

which proves the proposition. \textit{Q.E.D.}

\textbf{Proposition 2.} Suppose that \(r \in \mathcal{F}(\mathcal{G}, \text{End}(\mathcal{G}))\) solves the \(G\)-CDYBE (2.4) on an open submanifold \(\mathcal{G} \subset G\) and

\[
\det \left( \text{Ad}(\tilde{M}^{-1} \circ r_+(M) - r_-(M)) \right) \neq 0 \quad \forall M \in \mathcal{G}.
\]

(2.18)
Then the 2-form $\rho$ defined by (2.6) with
\[ q := \frac{1}{2} \big( \bar{M}^{-1} \circ r_+ + r_- \big) \circ \big( \bar{M}^{-1} \circ r_+ - r_- \big)^{-1} \tag{2.19} \]
satisfies $d\rho = \Phi$ on $\hat{G}$.

**Proof.** Consider the equivalent formula of the $q$
\[ q_- = r_- \circ \big( \bar{M}^{-1} \circ r_+ - r_- \big)^{-1} = r_- \circ A^{-1}, \tag{2.20} \]
where $A$ (2.3) is invertible due to (2.18). By taking the derivative of this equation one can show that relation (2.15) is valid in this case as well. By combining (2.15) with (2.4) we obtain that
\begin{align*}
\langle (R_A(\xi), A(\eta), A(\zeta)) + c.p. =
\langle [r_\xi, r_\eta], \zeta \rangle + & \frac{1}{12} \langle [\xi, \eta], \zeta \rangle - \langle [r_+ - \bar{M} \circ r_-] \xi, [r_+ \eta, r_+ \zeta] \rangle + c.p.
\langle q[A(\xi), A(\eta)], A(\zeta) \rangle - & \frac{1}{6} \langle [A(\xi), A(\eta)], A(\zeta) \rangle + c.p. \tag{2.21} \end{align*}
\forall \xi, \eta, \zeta \in G$. The second equality here is derived by using several times (2.20) and the corresponding equation for $q_+$, i.e., $q_+ = \bar{M}^{-1} \circ r_+ \circ A^{-1}$. By (2.13) and (2.5), this allows us to conclude that
\[ (d\rho)(R_A(\xi), R_A(\eta), R_A(\zeta)) = \Phi(R_A(\xi), R_A(\eta), R_A(\zeta)). \tag{2.22} \]
Since the correspondence $\xi \mapsto A(\xi)$ is invertible due to (2.18), this implies that $d\rho = \Phi$ holds.

Q.E.D.

### 3 Recovering a canonical solution of the CDYBE

Let us now suppose that $\hat{G}$ is diffeomorphic to a domain $\hat{G} \subset G$ by the exponential parametrization, whereby we write $\hat{G} \ni M = e^\omega$ with $\omega \in \hat{G}$. Then choose [2] the 2-form $\rho$ on $\hat{G} \simeq \hat{G}$ to be
\[ \rho(\omega) := -\frac{1}{2} \int_0^1 dx \langle d\omega \uparrow d\bar{\omega} e^{-x\omega} \rangle. \tag{3.1} \]
It is not difficult to check that $d\rho = \Phi$ holds and one can also calculate the explicit form of the operator $q(\omega)$ that corresponds to $\rho$ in (3.1) by means of (2.6). Let $Q$ and $R$ denote the complex meromorphic functions
\[ Q(z) = \frac{2z + e^{-z} - e^z}{2(e^z - 1)(1 - e^{-z})}, \quad R(z) := \frac{1}{2} \coth \frac{z}{2} - \frac{1}{z}, \tag{3.2} \]
which are regular around $z = 0$. In fact, (3.1) implies the formula
\[ q(\omega) = Q(\text{ad} \omega), \tag{3.3} \]
where the right hand side is defined by means of the power series expansion of the function $\mathcal{Q}$ around $z = 0$ if $\omega$ is near to the origin. By inserting this into equation (2.9) that determines $r$ in terms of $q$, one finds that the exchange $r$-matrix associated with the 2-form $\rho$ in (3.1) is given by

$$r(\omega) = \mathcal{R}(\text{ad } \omega).$$  \hfill (3.4)

Consider the holomorphic complex function

$$h : z \mapsto e^z - 1,$$  \hfill (3.5)

and recall that for a curve $t \mapsto A(t)$ of finite-dimensional linear operators

$$\frac{de^{\pm A(t)}}{dt} = \pm e^{\pm A(t)} h(\mp \text{ad}_{A(t)})(\dot{A}(t)), \quad \dot{A}(t) := \frac{dA(t)}{dt}. \hfill (3.6)$$

The right hand side of this equation is defined by means of the Taylor expansion of $h(z)$ around $z = 0$, and $\text{ad}_{A(t)}(\dot{A}(t)) = [A(t), \dot{A}(t)]$. By using relation (3.6) it is not difficult to derive (3.3). Relation (3.6) also implies that the $r$-matrix in (3.4) satisfies the following identity:

$$\left(\frac{1}{2} \mathcal{D}_\alpha^+ + r_\alpha^\beta \mathcal{D}_\beta^-\right) = \frac{\partial}{\partial \omega^\alpha}, \hfill (3.7)$$

where we use that $M = e^\omega$ and $\mathcal{D}_\alpha^\pm = (\mathcal{R}_\alpha \pm \mathcal{L}_\alpha)$ are given by (1.2). Incidentally, requiring (3.7) for the ansatz $r(\omega) = \mathcal{R}(\text{ad } \omega)$ with some odd function $\mathcal{R}(z)$ that is holomorphic around the origin leads uniquely to the function $\mathcal{R}(z)$ in (3.2).

As a consequence of proposition 1, we know that $r(\omega) = \mathcal{R}(\text{ad } \omega)$ with (3.2) satisfies the $G$-CDYBE (1.3). We now notice from (3.7) that for this particular $r$-matrix (1.3) becomes the standard CDYBE [3] on the Lie algebra $\mathcal{G}$:

$$[\hat{r}_{12}, \hat{r}_{23}] + T_1^\alpha \frac{\partial}{\partial \omega^\alpha} \hat{r}_{23} + \text{cycl. perm.} = -\frac{1}{4} \hat{f}. \hfill (3.8)$$

We call the $r$-matrix in (3.4) the ‘canonical’ solution of the CDYBE (3.8) on $\mathcal{G}$ since many solutions of the CDYBE with respect to subalgebras $\mathcal{H} \subset \mathcal{G}$ can be derived from it by applying Dirac reduction [4] to the dynamical variable $\omega$. The above proof of the statement that the canonical $r$-matrix solves (3.8) was extracted from the study of the WZNW model, where it arose in a natural manner [2]. In this context, (3.7) is in fact equivalent to having the Poisson brackets $\kappa\{g(x), \omega_a\} = g(x)T_a$, which means that the logarithm of the monodromy matrix serves as the infinitesimal generator (momentum map) for a classical $\mathcal{G}$-symmetry on the chiral WZNW phase space. For other applications of the canonical $r$-matrix and its reductions, and for different verifications of the fact that it solves the CDYBE, we refer to the literature [3, 5, 6].

**Acknowledgements.** This investigation was supported in part by the Hungarian Scientific Research Fund (OTKA) under T034170 and M028418.
A Some calculations

For convenience, we here sketch the derivation of some of the statements mentioned in the text.

First, let us show that \( d\rho = \Phi \) holds for \( \rho \) in (3.1). By using the exponential parametrization, we introduce the \( \mathcal{G} \)-valued 1-form \( \theta_x := (de^{x\omega})e^{-x\omega} \) for any \( 0 \leq x \leq 1 \). Plainly, \( \Phi \) (2.5) can be written as

\[
\Phi = \frac{1}{2} \langle [\theta_1, [\theta_1, \theta_1]] \rangle = \frac{1}{2} \int_0^1 dx \frac{d}{dx} \langle \theta_x, [\theta_x, \theta_x] \rangle \quad \text{on} \quad \mathcal{G} = \exp(\mathcal{G}). \tag{A.1}
\]

By noting that \( \frac{d\theta}{dx} = [\omega, \theta_x] + d\omega \), we obtain

\[
\frac{d}{dx} \langle \theta_x, [\theta_x, \theta_x] \rangle = \langle \langle [\omega, \theta_x], [\theta_x, \theta_x] \rangle + \langle \theta_x, [[\omega, \theta_x], \theta_x] \rangle + \langle \theta_x, [\theta_x, [\omega, \theta_x]] \rangle 
+ \langle d\omega, [\theta_x, \theta_x] \rangle + \langle \theta_x, [d\omega, \theta_x] \rangle + \langle \theta_x, [\theta_x, d\omega] \rangle = \langle d\omega \wedge [\theta_x, \theta_x] \rangle. \tag{A.2}
\]

In the second equality we used the invariance of \( \langle , \rangle \) and the usual rules of calculation for (Lie algebra valued) differential forms. It follows from (3.1) that

\[
d\rho = \frac{1}{2} \int_0^1 dx \langle d\omega \wedge [\theta_x, \theta_x] \rangle, \tag{A.3}
\]

and therefore \( d\rho = \Phi \) holds indeed on \( \exp(\mathcal{G}) \).

Next, let us describe how (3.1) leads to formula (3.3) of the operator \( q(\omega) \) by means of (2.6). For this, we need the complex functions

\[
h_1(z) = h(z) = \frac{e^z - 1}{z}, \quad h_2(z) = h(-z), \quad h_i^{-1}(z) = \frac{1}{h_i(z)}. \tag{A.4}
\]

The well known relation (3.6) implies that

\[
\alpha(\omega) := -\frac{1}{2} \int_0^1 dx \langle de^{x\omega}e^{-x\omega} \rangle = -\frac{1}{2} \int_0^1 dx h(x \text{ad} \omega)(x d\omega) = f(\text{ad} \omega)(d\omega) \tag{A.5}
\]

with the holomorphic function

\[
f(z) = \frac{1}{2z} + \frac{1 - e^z}{2z^2}. \tag{A.6}
\]

The third equality in (A.5) follows by termwise integration of the series of \( xh(x \text{ad} \omega) \). Thus \( \rho(\omega) \) in (3.1) can be written as

\[
\rho(\omega) = \langle d\omega \wedge \alpha(\omega) \rangle = \langle d\omega \wedge f(\text{ad} \omega)(d\omega) \rangle = \langle d\omega, F(\text{ad} \omega)(d\omega) \rangle \tag{A.7}
\]

with

\[
F(z) = f(z) - f(-z). \tag{A.8}
\]

\(^2\)For example, if \( \beta, \gamma \) are \( \mathcal{G} \)-valued 1-forms and \( A, B \) are vector fields, then \([\beta, \gamma] \) is the \( \mathcal{G} \)-valued 2-form \([\beta, \gamma](A, B) = [\beta(A), \gamma(B)]\); \([\beta \wedge \gamma](A, B) = \langle \beta(A), \gamma(B) \rangle - \langle \beta(B), \gamma(A) \rangle\); and analogously for \( k \)-forms.
As another consequence of (3.6), we have

\[ d\omega = h_2^{-1}(\text{ad} \, \omega)(M^{-1}dM). \]  

(A.9)

By substituting this into (A.7) and using the invariance of \( \langle , \rangle \), we obtain

\[ \rho(\omega) = \langle M^{-1}dM, Q(\text{ad} \, \omega)(M^{-1}dM) \rangle, \]

(A.10)

where

\[ Q(z) = \frac{1}{h_1(z)}F(z) \frac{1}{h_2(z)} = \frac{2z + e^{-z} - e^z}{2(e^z - 1)(1 - e^{-z})}. \]

(A.11)

By means of (2.6), this is clearly equivalent to the equality claimed in (3.3). It is readily verified that the functions \( Q(z) \) and \( R(z) \) in (3.2) enjoy the identity

\[ \left( Q(z) + \frac{1}{2} \right) \left( R(z) - \frac{1}{2} \right) = \left( Q(z) - \frac{1}{2} \right) e^{-z} \left( R(z) + \frac{1}{2} \right), \]

which implies the factorization equation (2.9) for the corresponding operators \( q(\omega) = Q(\text{ad} \, \omega) \) and \( r(\omega) = R(\text{ad} \, \omega) \). Let us also remark that (3.7) is guaranteed by the following identity satisfied by the function \( R(z) \):

\[ \frac{1}{2}(h_1^{-1} + h_2^{-1}) + R(h_1^{-1} - h_2^{-1}) = 1. \]

(A.13)

To see that this implies (3.7), one needs to express \( L_\alpha, R_\alpha \) with the aid of the relation (3.6) as

\[ L_\alpha = \left( h_2^{-1}(\text{ad} \, \omega) \right)_\alpha^\beta \frac{\partial}{\partial \omega^\beta}, \quad R_\alpha = \left( h_1^{-1}(\text{ad} \, \omega) \right)_\alpha^\beta \frac{\partial}{\partial \omega^\beta}. \]

(A.14)

Here the functions \( h_1, h_2 \) are given in (A.4), and in particular \( h_1^{-1}(z) - h_2^{-1}(z) = -z \). It is thus clear that (A.13) determines \( R(z) \) uniquely, and hence (3.7) leads uniquely to the canonical \( r \)-matrix upon the ansatz \( r(\omega) = R(\text{ad} \, \omega) \), as mentioned in section 3.

References

[1] F. Falceto and K. Gawędzki, J. Geom. Phys. 11, 251 (1993).
[2] J. Balog, L. Fehér and L. Palla, Nucl. Phys. B 568, 503 (2000).
[3] P. Etingof and A. Varchenko, Commun. Math. Phys. 192, 77 (1998).
[4] L. Fehér, A. Gábor and B.G. Pusztai, J. Phys. A 34, 7235 (2001).
[5] A. Alekseev and E. Meinrenken, Invent. Math. 139, 135 (2000).
[6] B.G. Pusztai and L. Fehér, J. Phys. A 34, 10949 (2001).