Dynamic Adaptive Streaming using Index-Based Learning Algorithms

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Abstract—We provide a unified framework using which we design scalable dynamic adaptive video streaming algorithms based on index based policies (dubbed DAS-IP Fig. 2) to maximize the Quality of Experience (QoE) provided to clients using video streaming services. Due to the distributed nature of our algorithm DAS-IP, it can be easily implemented in lieu of popular existing Dynamic Adaptive Streaming over HTTP (DASH) algorithm which is used by various Cloud based video streaming services, Content Delivery Networks (CDNs), Cache networks, wireless networks, vehicular networks etc.

We begin by considering the simplest set-up of a one-hop wireless network in which an Access Point (AP) transmits video packets to multiple clients over a shared unreliable channel. The video file meant for each client has been fragmented into several packets, and the server maintains multiple copies (each of different quality) of the same video file. Clients maintain individual packet buffers in order to mitigate the effect of uncertainty on video interruption. Streaming experience, or the Quality of Experience (QoE) of a client depends on several factors: i) starvation/outage probability, i.e., average time duration for which the client does not play video because the buffer is empty; ii) average video quality; iii) average number of starvation periods; iv) temporal variations in video quality etc.

We pose the problem of making dynamic streaming decisions in order to maximize the total QoE as a Constrained Markov Decision Process (CMDP). A consideration of the associated dual MDP suggests us that the problem is vastly simplified if the AP is allowed to charge a price per unit bandwidth usage from the clients. More concretely, a “client-by-client” QoE optimization leads to the networkwide QoE maximization, and thus provides us a decentralized streaming algorithm.

This enables the clients to themselves decide the optimal streaming choices in each time-slot, and yields us a much desired client-level adaptation algorithm. The optimal policy has an appealing simple threshold structure, in which the decision to choose the video-quality and power-level of transmission depends solely on the buffer-level. In case the clients are unaware of their (possibly random and time-varying) system parameters, we develop algorithms that learn the indices while utilizing the structure of the optimal decentralized policy. The decentralized nature of optimal policy implies that the DAS-IP has a much “smaller” policy space to explore from, and hence converges fast.

I. INTRODUCTION

Mobile video traffic accounted for 55% of total mobile data traffic in 2015, and the share is expected to increase to 75% by 2020. Unlike traditional QoS metrics such as throughput or delay, the user experience for video streaming applications depends on several complex factors, and is in itself an active area of research [7], [1], [2]. In order to meet the stringent Quality of Experience (QoE) requirements imposed by video streaming applications, service providers have switched to advanced platforms such as Cloud based services [3], Content Delivery Networks (CDNs) [4] etc. Moreover, they also use adaptive bitrate streaming algorithms such as DASH [5], HTTP Live Streaming (HLS) in order to continually monitor and improve the streaming experience.

Take the example of popular cloud services such as Microsoft Azure, IBM Cloud, Google Cloud, Amazon CloudFront, Apple’s iCloud that provide live-streaming,
on-demand video, online gaming services etc (Fig. 1). A party subscribing to live video streaming service with such a cloud will generate video file and upload it to the cloud in real-time. The cloud transcodes this data into multiple bit-rates, and the audience of this particular stream are served the video file using DASH. DASH enables a viewer to switch to low quality video in case his connection bandwidth is low, thus avoiding video interruptions. Since a major chunk of video data is demanded by mobile devices (that typically have bandwidth fluctuations), this enables the streaming service to reach a wider range of audiences. Since the CPU and storage-intensive video tasks get shifted to the cloud, this also enables computing-limited devices such as smartphones, access to high-quality, high-definition video in a wide range of locations. Figure 1 depicts such a cloud service.

However, the state-of-the-art adaptive streaming algorithms are unable to provide a satisfactory Quality of Experience (QoE) video streaming. As an example, the popular DASH algorithm is either too slow to respond to changes in congestion levels, or it is overly sensitive to short-term network bandwidth variations [6]. Similarly, for clients served over wireless networks [7], rate adaptation needs to take complex factors such as channel fading into account. Experimental studies into studying the rate adaptation techniques employed by popular DASH clients such as Microsoft Smooth Streaming [8], Adobe OSMF [9], Netflix have demonstrated that these algorithms perform poorly.

Thus, the focus of this paper is to develop adaptive streaming algorithms that optimize the user experience by taking various factors into account while making streaming decisions. As will be shown later, our algorithms are decentralized, easily implementable and can be computed in a distributed fashion. This enables them to be embedded into existing techniques such as DASH.

As a side-product of our analysis, we also touch upon a somewhat related problem of pricing the service resources so as to maximize the operator’s revenue.

A. Past Works

Previous works on video streaming have analyzed various relevant trade-offs. Trade-off between outage probability and number of initially buffered packets (initial delay time) is analyzed in [10], [11], [12], [13], while [14] studies the effect of variations in the temporal quality of videos on the global video quality. [15] studies the impact of flow level dynamics (flows entering and leaving the system) on the streaming Quality of Experience (QoE). [6] considers the problem of controlling the rate at which a single client requests data from the server in order that the requested rate closely flows the TCP throughput available to it. However, the model assumes that only a single client is present in the network, ignores inherent system randomness and proposes a heuristic scheme. [16] provides an extensive survey on the QoE related works from human computer interaction and networking domains.

References [17], [18], [19], [20], [21], [?] develop a framework to design policies which provide services to clients in a regular fashion, though not in the context of video streaming QoE.

B. Challenges

Several metrics such as starvation probability (average time spent without video streaming), start-up delay, time spent in rebuffering, average video quality [1], temporal quality variations [14] etc. collectively decide the end user streaming experience. Thus, an optimized streaming necessarily involves achieving optimal trade-offs between these competing metrics. As an example, a higher time spent rebuffering (and hence increased delay before video begins) leads to a lesser playback interruptions [22]. A lower video quality increases transmission rate, and hence reduces the starvation probability. Though dynamically switching between different qualities definitely improves upon a combination of packet starvation-video quality, it also introduces temporal variations in video quality, which are known to affect streaming experience.

Other than these trade-offs, the network dynamics also need to be taken into account [2]. For example, the algorithm should switch to a low quality video upon detecting a reduction in the bandwidth of a client. Similarly, it needs to re-allocate streaming resources upon an entry or exit of clients with time.

C. Our Contribution

Contrary to the vast existing literature which provides no theoretical guarantees on the QoE properties of the proposed schemes, our algorithms maximize the combined QoE of all the users in the network. As an example, [6] devises policy to minimize interruptions for a single client. However, a networkwide deployment of such a policy at each client need not maximize the combined QoE, i.e., a client-by-client optimization need not maximize the overall QoE experienced by the combined set of all clients [23].

Moreover, previous works address at most one particular aspect of the optimal streaming problem, in contrast we devise algorithms that optimize the overall experience by taking into account all the factors that decide final end user QoE.

We begin by considering the simple scenario of single last-hop case for ease of exposition, and pose the problem of maximizing the combined streaming experience (QoE) of N clients as a Constrained MDP (CMDP). We then consider the associated dual problem, and show that it yields a simple, decentralized solution. This allows the clients to themselves decide upon the optimal streaming decisions.

In case the clients do not know their system parameters, e.g. wireless channel’s reliability or fading
model, then we develop a simple online Reinforcement Learning [24] algorithm using which they can “infer” the optimal streaming decisions. Since we are able to show that the optimal policy lies in the space of decentralized policies, we are able to cut down drastically (from exponential in $N$ to linear) on the policy space that needs to be explored. Consequently, the learning occurs at a client level, and converges fast.

II. SYSTEM DESCRIPTION

Video File Description The server has multiple copies (files) of the same video, where each file has a different video quality or bitrate. Each such video file has been broken into a sequence of small HTTP-based file segments. While the complete video file is typically several hours in duration (e.g. movie or a live broadcast), each segment’s playback time is of the order of few seconds [6]. Such an assumption is common in popular adaptive bitrate streaming techniques such as DASH [5].

Network Description In this section, we exclusively focus on the case of a single hop wireless network serving $N$ clients. We choose to present this simple case first in order to simplify the exposition and introduce the key concepts without resorting to complex notations. Later Sections will deal with more complex scenarios. Moreover, we ignore the interaction of our proposed algorithms with the TCP. We assume that the “bottleneck link” is the last single hop link connecting the client to access point (AP).

A single wireless channel is shared by $N$ clients. Time is discretized, and the AP can attempt a packet transmission in a single time-slot. The buffer size of client $n$ is $B_n$ packets, and it plays a single packet for a duration of $T_n$ time-slots, see Fig. 3. After finishing streaming of a packet, it starts streaming the packet at the head of its buffer. However, if it finds that the buffer is empty, then the streaming is interrupted, thus causing an “outage”. This event is also called “starvation”, i.e., a client is “starved” of packets for sufficiently long duration, thereby causing video interruptions.

The clients are connected through unreliable wireless channels, so that the packet transmissions are assumed to be random. For simplicity, we will exclusively deal with the case of i.i.d. channel states. The scheduler can pick the packet quality from $\{1, 2, \ldots, Q_n\}$. A lower index of quality class is associated with a higher streaming experience. Thus, the “cost” or disutility incurred by client $n$ for streaming a packet of quality $q$ is equal to $\lambda_{q,n}$.

The scheduler has the option to choose the power/energy level used for transmitting client $n$’s packets from the set $\{E_1, E_2, \ldots, E_M\}$. Throughout $E_1 = 0$ and refers to the case of no transmission. The link reliabilities depend upon the transmission power, and the packet file size (in bits), so that a client $n$ packet of quality $q$, which is transmitted at power $E$, is delivered with a probability $P_n(q,E)$.

Streaming Experience Cost The streaming quality experienced by a single client is assumed to depend upon

1) The average number of outages.
2) How “often” the video gets interrupted, i.e., the number of outage-periods which is equal to the number of time-slots characterized by a “non-outage” to outage transition.
3) Temporal variations, i.e., the number of times the video-quality is changed.
4) Average video quality which in turn is measured by the average disutility associated with the different video quality types.

III. PROBLEM FORMULATION

We begin with some notation. For simplicity, we will restrict ourselves to optimizing the factors 1)-3) listed in the streaming cost. Objective 4) will be dealt with separately in a later section.

Let $O_n(s)$ be the random variable that is 1 if the $n$-th client faces an outage at time $s$, and 0 otherwise, and thus is the indicator random variable of the outage event. Let $E_n(s)$ be the transmission power of client $n$ at time-slot $s$ and let $I_{q,n}(s)$ be the indicator random variable of the event that a packet of quality $q$ is delivered to client $n$ in time-slot $s$. For simplicity, we will begin with a consideration of the scheduling problem under an average power constraint on AP, which models the case when the AP is operated by a battery that needs periodical charging. Streaming under constraints on the peak power, and on the number of orthogonal/independent channels which can be utilized for concurrent packet transmissions will be subject of later sections.

The streaming problem is to the following Constrained Markov Decision Process (CMDP) [25] for the optimal

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1Fading channels will be covered in a later Section XII.
The optimal policy can be implemented in a decentralized fashion and computed using distributed updates [26]. Moreover, an analysis of the single client MDP shows that the optimal policy has a simple threshold structure, and hence is easy to implement.

IV. THE DUAL MDP

We begin with some notation. We will use $\bar{C}_n$ to denote the time-average cost incurred by client $n$ under a stationary policy $\pi$, thereby suppressing the dependence on policy. Similarly $\bar{E}_n$ will be the average power consumption of client $n$.

Letting $\lambda_E$ be the Lagrangian multiplier associated with the average power constraint $\sum_n \bar{E}_n \leq E$, the Lagrangian for the problem (Primal MDP) is given by

$$\mathcal{L}(\pi, \lambda_E) = \sum_n \bar{C}_n + \lambda_E (\bar{E}_n(s) - \bar{E}),$$

while the dual function is given by

$$D(\lambda_E) = \min_{\pi} \mathcal{L}(\pi, \lambda_E),$$

and the dual problem is stated as

$$\max_{\lambda_E \geq 0} D(\lambda_E).$$

We realize that the Lagrangian [3] decomposes into the sum of individual costs $\bar{C}_n + \lambda_E \bar{E}_n$ incurred by each client $n$, and hence in order to compute the dual function $D(\lambda_E)$, each of these individual client costs could be optimized separately by designing $\pi_n$, the policy for client $n$. This decomposition is key to the attractive properties of our proposed policies that we mentioned in the previous section. Thus,

**Lemma 1:** $D(\lambda_E) = \sum_n V_n(\lambda_E) - \lambda_E \bar{E}$.

Since the policies $\pi_n$ can be combined in straightforward manner (by implementing $\pi_n$ for each client $n$) in order to obtain a policy $\pi = \otimes \pi_n$ for the overall system, we shift our focus to solving the optimal policy for a single client that minimizes its cost $\bar{C}_n + \lambda_E \bar{E}_n$.

V. SINGLE CLIENT PROBLEM

In this entire section, we will omit the sub-script $n$ with the understanding that all the quantities being referred to belong to client $n$.

Client has a buffer of capacity $B$ time-slots of playtime video in which stores video packets. Note that this assumption is equivalent to assuming a buffer of size $B$ packets because a packet gets played for $T$ time-slots. In each time-slot $t$, the AP has to choose the following two control quantities for the client: i)

1) The video quality $q(t) \in \{1, 2, \ldots, Q\}$ of the packet transmitted to the client, and,
2) the power $E(t) \in \{E_1, E_2, \ldots, E_n\}$ utilized for the transmission.

Let $l(t)$ be the play-time duration of the packets present in the buffer at time $t$, and hence it denotes the state of the client at time $t$. The wireless channel connecting
be the state values resulting from a successful and failed
while it assumes the value 

\((l(t) - 1)^+ + T\) with a probability \(P(q, E)\),
while it assumes the value \((l(t) - 1)^+ \) with a probability
\(1 - P(q, E)\). But if \(l(t) > B - T + 1\), then because accepting
a new packet in the buffer will cause it to overflow, the
system state at time \(t + 1\) is equal to \(l(t) - 1\) with a
probability 1.

Let
\[
S(x) := \begin{cases} 
(x - 1)^+ + T, & \text{if } x \leq B - T + 1, \\
1, & \text{if } B - T + 1 < x \leq B,
\end{cases}
\]
\[
\mathcal{F}(x) := (x - 1)^+,
\]
be the state values resulting from a successful and failed
packet transmission respectively when the system state is \(x\).
Let \(u(t) := (q(t), E(t))\) be the control action chosen
by AP at time \(t\), where \(q(t), E(t)\) are the video quality
and transmission power level chosen at time \(t\).

**Single Client Unit Step Cost**:
The client is charged a cost of \(\lambda E \times E\) for transmitting a packet at power level \(E\).
It faces a penalty of 1 units if there is an outage at time
\(t\), and similarly a penalty of \(\lambda_o\) units if a new outage-period begins at time \(t\).
Delivery of a packet of quality \(q\) incurs a cost of \(\lambda_q\).

Because the probability distribution of \(l(t + 1)\) is
completely determined by the value of \(l(t)\), the action
\(u(t) = (q(t), E(t))\) chosen at time \(t\), and the unit step
cost can be expressed solely as a function of \(l(t)\), the
system state can be taken to be \(l(t)\). The problem is thus
a Markov Decision Process (MDP) involving only a finite
number of actions and states, and the cost is optimized
by a stationary Markov policy [27].

We have to solve,
\[
\min_{\pi} \bar{C} + \lambda E \bar{E}.
\]
Denote by \(\pi^*_n(\lambda_E)\), the optimal policy which solves the
single client problem. We also let
\[
V_n(\lambda_E) = \min_{\pi_n} \{\bar{C} + \lambda E \bar{E}\},
\]
be the optimal cost, and \(V_n(\lambda_E, \pi)\) be the cost associated
with a policy \(\pi\) when the power usage is priced at \(\lambda_E\).

**VI. Threshold Structure of the Optimal Policy
for the Single Client Problem**

We next show that the optimal policy for the sin-
gle client problem has a certain “threshold structure”.
Precise definition will be given shortly. Roughly this
means that the transmission power should increase, and
quality of transmitted packet decrease as the buffer
level of video playtime decreases. Such a property is
quite appealing because the decision process is simple
to describe. Moreover, it has an added computational
advantages since it reduces the size of the policy search
space.

In the below, we omit the subscript \(n\). Our approach
to showing the threshold structure of the optimal pol-
cy will be to analyze the corresponding \(\beta\)-discounted
optimization problem, and show that the solution to it
has threshold structure. The result for the undiscounted
problem then follows straightforward by using results in [28].

We begin with a discussion of the \(\beta \in (0, 1)\) discounted
infinite horizon cost problem for the single client. Let
\[
V_{\beta}(x) = \min_{\pi} \lim_{t \to \infty} \mathbb{E}_x \left[ \sum_{t=0}^{\infty} \beta^t \left( C(l(t), u(t)) + \lambda_E E E(t) \right) \right],
\]
be the minimum \(\beta\)-discounted infinite horizon cost for
the system starting in state \(x\) at time 0, where \(x\) can
assume values in the set \(\{0, 1, \ldots, B\}\), and where \(C(l, u)\)
is the one-step cost associated with choosing a scheduling
action \(u\) when the buffer level is \(l\).

Similarly let \(V^*_\beta(x)\) denote the minimum discounted
cost incurred in \(s\) time-slots when the starting value
of the state is \(x\),
\[
V^*_\beta(x) = \min_{\pi^*} \mathbb{E}_x \left[ \sum_{t=0}^{s} \beta^t \left( C(l(t), u(t)) + \lambda_E E E(t) \right) \right],
\]
where \(\pi^*\) is a policy for the \(s\) horizon \(\beta\)-discounted
problem. The functions \(V_{\beta}(x), V^*_\beta(x)\) are not to be confused
with their undiscounted counterparts \(V_n(\lambda_E)\) that were
defined in the previous section.

Th one-step Dynamic programming backward induct-
ion can be written as,
\[
V^*_\beta(x) = \min_{(q, E)} 1(x = 0) + \lambda_E E
+ P(q, E) \left[ \lambda_q + \beta V^*_\beta(S(x)) \right]
+ (1 - P(q, E)) \left[ 1(x = 1)\lambda_O + \beta V^*_\beta(F(x)) \right]
= 1(x = 0) + 1(x = 1)\lambda_O + \left[ 1 + \beta \right] V^*_\beta(F(x))
+ \min_u \left\{ C(u) - P(u) D^*_\beta(x) \right\},
\]
where
\[
D^*_\beta(x) := 1(x = 1)\lambda_O + \beta \left\{ V^*_\beta(F(x)) - V^*_\beta(S(x)) \right\},
\]
\(s = 1, 2, \ldots, \)
\[
\text{while } \hat{C}(u) := \lambda_E E + P(q, E)\lambda_q,
\]
is defined to be the one-step cost associated with choos-
ing the action \(u = (q, E)\), which is further composed of
two terms i) cost for using power of amount \(E\), and ii)
the disutility associated with delivering quality \(q\) video
packet.

We will assume that the packet transmission success
probability \(P(q, E)\) is
1) increasing in \( q \) for a fixed value of power \( E \), i.e., for a fixed value of transmission power, a reduction in video quality increases the data transmission rate, or equivalently increases the chances of successful delivery of packets.

2) increasing in \( E \) for a fixed value of video quality \( q \).

**Definition 1 (Threshold Policy):** We say a policy is of threshold-type if it satisfies the following for each stage \( s \):

- Fix any \( E \in \{E_1, E_2, \ldots, E_n\} \). If the policy chooses the action \( (q, E) \) in state \( x \), then it does not choose the actions \( \{(\tilde{q}, E) : \tilde{q} < q\} \) for any state \( 1 \leq y \leq x \).
- Put differently, for a fixed choice of transmission power, a threshold policy does not switch to a higher video quality if the buffer level is reduced.
- Fix any \( q \in \{Q_1, Q_2, \ldots, Q_n\} \). If the policy chooses the action \( (q, E) \) in state \( x \), then it does not choose the actions \( \{(q, \tilde{E}) : \tilde{E} < E\} \) for any state \( 1 \leq y \leq x \).
- For a fixed choice of video quality, the policy does not switch to a lower transmission power if the buffer level is decreased.

The following fact follows from the definition of a threshold policy.

**Lemma 2:** Let \( x, y \in \{1, 2, \ldots, B\} \) be two values of buffer levels such that \( x > y \). Let \( \pi \) be a threshold policy, and denote by \( u_x, u_y \) the actions that \( \pi \) chooses for the state values \( x \) and \( y \). Then the transmission success probabilities satisfy \( P(u_x) < P(u_y) \).

In the following, \((u, \pi)\) is the policy that chooses the action \( u \) in the first slot (irrespective of the system state \( l \)), and thereafter implements the policy \( \pi \). Let \( V_{\beta}^{\pi}(x) \) be the discounted cost incurred by the system starting in state \( x \) and operating for \( s \) time-slots under the application of policy \( \pi \). We have,

**Lemma 3:** Let \( u_1, u_2 \) be two actions satisfying \( P(u_2) > P(u_1) \). Then,

\[
V_{\beta}^{\pi(u_2, \pi^*)}(F(x)) - V_{\beta}^{\pi(u_1, \pi^*)}(S(x)) = \\
P(u_1) \left\{ \beta V_{\beta}^{s-1}(S(F(x))) - V_{\beta}^{s-1}(S(S(x))) \right\} \\
+ (1 - P(u_2)) \left\{ 1(\mathcal{F}(x) = 1)\lambda_0 + \beta V_{\beta}^{s-1}(\mathcal{F}(F(x))) \right\} \\
- V_{\beta}^{s-1}(\mathcal{F}(S(x))) + \hat{C}(u_2) - \hat{C}(u_1) \\
= P(u_1) \left\{ \beta V_{\beta}^{s-1}(F(S(x))) - V_{\beta}^{s-1}(S(S(x))) \right\} \\
+ (1 - P(u_2)) \left\{ 1(\mathcal{F}(x) = 1)\lambda_0 + \beta V_{\beta}^{s-1}(\mathcal{F}(F(x))) \right\} \\
- V_{\beta}^{s-1}(\mathcal{F}(S(x))) + \hat{C}(u_2) - \hat{C}(u_1).
\]

The following result is crucial to show the threshold nature of policy.

**Lemma 4:** The functions \( D_{\beta}^s(x), x = 1, 2, \ldots, B-T+1 \) are decreasing in \( x \) for each \( s = 1, 2, \ldots \).

**Proof:** Within this proof, let \( \pi_s^* \) be the optimal policy for the \( \beta \)-discounted \( s \) time-slots problem, and let \((u, \pi_{s-1}^*)\) be the policy for \( s \) time-slots which takes the action \( u \) at the first time-slot, and then follows the policy \( \pi_{s-1}^* \). In order to prove the claim, we will use induction on \( s \), the number of time-slots.

Let us assume that the statement is true for the functions \( D_{\beta}^s(x) \), for all \( z \leq s \). In particular the function,\n
\[
1(x = 1)\lambda_0 + \beta \left\{ V_{\beta}^{s-1}(\mathcal{F}(x)) - V_{\beta}^{s-1}(S(x)) \right\},
\]

is decreasing for \( x \in \{1, 2, \ldots, B - T + 1\} \).

First we will prove the decreasing property for \( x \in \{2, 3, \ldots, B - T + 1\} \). Now the assumption \((14)\) made above, and \((11)\), together imply that \( \pi_s^* \) is of threshold-type.

Fix an \( x \in \{1, 2, \ldots, B - T\} \) and denote by \( u_1, u_2, u_3, u_4 \), the optimal actions at stage \( s \) for the states \( S(x), \mathcal{F}(x), S(x + 1), \mathcal{F}(x + 1) \) respectively. Note that the threshold nature of \( \pi_s^* \) implies that,

\[
P(u_1) < P(u_2), P(u_3) < P(u_4),
\]

This is true because as the value of state decreases in the interval \( \{1, 2, \ldots, B\} \), a threshold policy switches to an action that has a higher transmission success probability. So it follows from Lemma \(5\) that

\[
V_{\beta}^{\pi_s^*}(\mathcal{F}(x + 1)) - V_{\beta}^{\pi_s^*}(S(x + 1)) \\
\leq V_{\beta}^{\pi_s^*(u_2, \pi_{s-1}^*)}(\mathcal{F}(x + 1)) - V_{\beta}^{\pi_s^*(S(x + 1))} \\
\leq \hat{C}(u_2) - \hat{C}(u_3) \\
+ P_c(u_3) \times \beta \left[ V_{\beta}^{s-1}(\mathcal{F}(S(x + 1))) - V_{\beta}^{s-1}(S(S(x + 1))) \right] \\
+ (1 - P_c(u_2)) \times \left\{ 1(\mathcal{F}(x + 1) = 1) + \beta V_{\beta}^{s-1}(\mathcal{F}(\mathcal{F}(x + 1))) \right\} \\
- V_{\beta}^{s-1}(S(S(x + 1))) \\
\leq \hat{C}(u_2) - \hat{C}(u_3) \\
+ P_c(u_3) \times \beta \left[ V_{\beta}^{s-1}(\mathcal{F}(S(x))) - V_{\beta}^{s-1}(S(S(x))) \right] \\
+ (1 - P_c(u_2)) \times \left[ 1(\mathcal{F}(x) = 1) + \beta V_{\beta}^{s-1}(\mathcal{F}(\mathcal{F}(x))) - V_{\beta}^{s-1}(S(S(x))) \right] \\
\leq V_{\beta}^{\pi_s^*}(\mathcal{F}(x)) - V_{\beta}^{\pi_s^*}(S(x)),
\]

where the first inequality follows since a sub-optimal action in the state \( \mathcal{F}(x + 1) \) increases the cost-to-go for \( s \) time-slots, the second inequality is a consequence of the assumption that the functions \( V_{\beta}^{s-1}(\mathcal{F}(x)) - V_{\beta}^{s-1}(S(x)) \) are decreasing in \( x \), while the last inequality follows from the fact that a sub-optimal action in the state \( S(x) \) will increase the cost-to-go for \( s \) time-slots. Thus we have proved the decreasing property of \( D_{\beta}^s(\cdot) \) for \( x \in \{2, 3, \ldots, B - T + 1\} \), and it remains to show that \( D_{\beta}^{s+1}(1) > D_{\beta}^{s+1}(2) \).

Once again, let \( u_1, u_2, u_3, u_4 \) be the optimal actions at stage \( s \) for the states \( T, T + 1 \) respectively. Using the same argument as above (i.e., assuming that the actions taken in stage \( s \) at states \( T, T + 1 \) are the same, and the
actions taken in the states 0, 1 are the same), it follows that
\[
D_{s+1}(1) - D_{s+1}(2) \geq (1 + \lambda_O - \beta \lambda_O) - (V^*_\beta(T) - V^*_\beta(T + 1)).
\]
However, then \(V^*_\beta(T) - V^*_\beta(T + 1) \leq 1 + \lambda_O - \beta \lambda_O\) (for \(s\) stages, apply the same actions for the system starting in state \(T\), as that for a system starting in state \(T + 1\), and note that the two systems couple at a stage \(t - 1\), when the latter system hits the state 1 at any stage \(t\); the hitting stage is of course random). This gives us,
\[
D_{s+1}(1) - D_{s+1}(2) \geq 0,
\]
and thus we conclude that the function \(D_{s+1}(x)\) is decreasing for \(x \in \{1, 2, \ldots, B\}\). In order to complete the proof, we notice that for \(s = 1\), we have,
\[
D^*_1(x) = 1(x = 1)\lambda_O,
\]
and thus the assertion of Lemma is true for \(s = 1\).

**Theorem 1:** For the single client scheduling problem \([8]\) there is a threshold policy that is Blackwell optimal \([29]\), i.e., it is optimal for the discounted cost problem \([10]\) for all values of discount parameter \(\beta \in (\beta, 1)\) for some \(\beta \in (0, 1)\). Moreover this policy is also optimal for the average cost problem \([8]\). Thus \(\pi^*_n(\lambda_E)\) is of threshold-type.

**Proof:** Fix a \(q\) and let \(E_i, E_j, i > j\) be two power levels. Without loss of generality, let \(u_1 = (q, E_i), u_2 = (q, E_j)\). Clearly \(\hat{C}(u_1) > \hat{C}(u_2)\) \([13]\). In the Bellman equation \([11]\), consider the term depending on \(u\), i.e., the term \(\hat{C}(u) - P(u)D^3_\beta(x)\). For \(x, y \in \{1, 2, \ldots, B - T + 1\}, x > y\), we have,
\[
\hat{C}(u_1) - P(u_1) D^3_\beta(x) - \left( \hat{C}(u_2) - P(u_2) D^3_\beta(x) \right)
- \left\{ \hat{C}(u_1) - P(u_1) D^3_\beta(y) - \left( \hat{C}(u_2) - P(u_2) D^3_\beta(y) \right) \right\}
= (P(u_1) - P(u_2)) \left( D^3_\beta(y) - D^3_\beta(x) \right)
\geq 0,
\]
where the last inequality follows from Lemma\([4]\). Thus it follows that if action \(u_1\) is preferred over action \(u_2\) for any state \(x\), then \(u_1\) will also be preferred over action \(u_2\) for any state \(y < x, y \in \{1, 2, \ldots, B - T + 1\}\). Finally note that it follows from the Bellman equation \([11]\) and \([6]\), that the optimal action for states \(x > B - T + 1\) is to let \(E = 0\) (since any packet that is received will be lost due to buffer overflow). The proof for variations in power levels is similar. Thus it follows from the definition of a threshold policy that the optimal policy is of threshold type.

Finally note that the statement regarding Blackwell optimality follows from the result in the above paragraph, and because the state-space is finite.

**VII. Solution of Primal MDP**

We now utilize the solution of the dual MDP and present the solution of the Primal Problem which involves minimizing the net operation cost \([1]\) subject to average power constraints \((\text{Primal MDP})\).

**Theorem 2:** Consider the Primal MDP \((\text{Primal MDP})\) and its associated dual problem defined in \([4]\). There exists a price \(\lambda^*_E\) such that \((\pi^*(\lambda^*_E), \lambda^*_E)\) is an optimal primal-dual pair and thus the policy \(\pi^*(\lambda^*_E)\) solves the Primal MDP.

**Proof:** We observe that there is a one-to-one correspondence between any stationary randomized policy, and the measure it induces on the state-action space, and thus the Primal MDP can be posed as a linear program \([30], [31]\). Thus it follows from Slater’s condition \([32]\) that for the Primal MDP, strong duality holds if there exists a policy \(\pi\) that satisfies the constraints \(\sum_n E_n < E\). However the policy which never schedules any packets incurs a net power expenditure of 0, and thus Slater’s condition is true for the Primal MDP if \(E > 0\). The claim of the Theorem then follows from Lemma\([4]\). \(\pi^*(\lambda^*_E)\) is Decentralized: Since the optimal decision \(u_n(t) = (q_n(t), E_n(t))\) for client \(n\) at time \(t\) can be obtained by solving the single client MDP \([5]\), the decision \(u_n(t)\) can be taken by client \(n\) itself. The system behaves as if there are \(N\) clients operating in parallel without any interaction amongst themselves. They are coupled only through the price \(\lambda_E\). This eliminates the need for a centralized controller at the AP.

We still need to address a couple of issues in order to complete the discussion on obtaining optimal policy. In order to implement the optimal policy \(\pi^*(\lambda^*_E)\), the clients need to know \(\lambda^*_E\), i.e., the optimal price of power. \(\lambda^*(E)\) can be solved for if the entire system parameters are known at the AP. However, we seek a decentralized solution, in which the clients do not share their parameters with the AP. This can be attained by iterating on the price \(\lambda(E)\) via the sub-gradient descent method \([33]\) as explained below.
to solve for the optimal policy \( \pi \) and optimal policy \( \lambda \) for a time-varying system parameters such as channel reliabilities \( P(q,E) \), buffer sizes etc., they do not want to solve for the optimal policy \( \pi^*(\lambda_E) \), we would like to bypass the policy calculation step of iterations (15), (16) (see Fig. 5). Our problem thus lies in the realms of reinforcement learning [24], and we can use online learning schemes such as Q-learning etc. [24]. These schemes adaptively “learn” the optimal policy by simultaneously “exploring” the policy space, and “exploiting” the past experience in the form of collected rewards (video streaming experienced so far). It involves them to keep track of the Q-factors \( Q(l,u) \), which represent the “positive effect” of choosing the action \( u \) when the buffer level is \( l \). The Q factors should not be confused with the video quality. The Q values are updated as,

\[
Q^t(l(t),u(t))(1 - \beta_t) + \beta_t \left\{ C(t) + \min_u Q^t(l(t+1), u) \right\}
\]

\[
\leftarrow Q^{t+1}(l(t),u(t)),
\]

where \( \beta_t \) is the learning rate, and the action chosen at time \( t \) is the one which greedily minimizes the cost \( Q^t(l(t),u) \) of taking action \( u \) when system state is \( l(t) \), i.e.,

\[
u(t) \in \arg\min_u Q(l(t), u).
\]

The Q-learning iterations stated above can be combined with the “price learning iterations” (15) by performing the latter on a slower time-scale, i.e., letting \( \alpha_t = o(\beta_t) \). The resulting algorithm, which simultaneously learns the optimal price and optimal policies is guaranteed to converge to the optimal values [34], [35].

VIII. STREAMING UNDER HARD CONSTRAINTS

We now consider an important extension of the video streaming problem which involves making decisions under “hard constraints” imposed on the scheduling actions chosen at each time \( t \) such as i) the number of orthogonal channels that can be utilized for transmission at each time \( t \), and ii) the peak power that can be utilized for packet transmissions. We will discuss derive algorithms only for i), since the algorithms for ii) can be derived analogously.

Depending upon the complexity of the underlying decision process, we classify the problem instances that involve hard constraints into two distinct categories.

1) Prioritizing Clients: In this case, if a client is chosen for scheduling, then there is only a single option for power-level and video quality. Thus, the problem of the scheduler is to choose \( M \) clients in each time slot for packet transmission, or in other words, prioritize the clients for packet transmissions. Here \( u_n(t) = 1 \) or \( 0 \) accordingly whether client \( n \) was chosen or not.

2) Quality-Power Adaptation: The scheduler also has to make dynamic quality-power adaptations on top of prioritizing the clients for packet transmissions as in i). This is the more general set-up in which the action for client \( n \) is given as \( u_n(t) = (q_n(t), E_n(t)) \). In either of the above set-up, we will propose easily implementable and simple to compute index based policies.

IX. PRIORITIZING CLIENTS

The scheduling has to be performed under the constraint on the number of orthogonal channels available for transmission. The set-up is thus equivalent to
the restless multiarmed bandit problem (RMABP) [36], [37]. It is well known that the Whittle’s Index policy [36], [38] is asymptotically optimal for the RMABP in the limit the number of client $N$ and the number of orthogonal channels $M$ are scaled to $\infty$, while keeping their ratio $M/N$ a constant [39].

We now briefly describe the notion of indexability and introduce Whittle’s Index Policy. Consider the following single-client MDP parameterized by the “transmission power price” $\lambda$,

$$\min_n \bar{C}_n + \lambda \bar{U}_n.$$  \hspace{1cm} (17)

In this modified problem, a client is charged a price of $\lambda$ units if it utilizes one unit of power, and the system evolution of the client proceeds exactly as described in earlier sections. The transmission price roughly corresponds to the minimum price that should be charged in order that the net utilization of the bandwidth matches the corresponding index policies. We will assume that the value function $V(\cdot)$ corresponding to the scheduling problem is separable, i.e., there exist functions $g_n(\cdot), n = 1, 2, \ldots, N$ such that,

$$V(l_1, l_2, \ldots, l_N) = \sum_n g_n(l_n),$$

for all possible values of the state vector $(l_1, l_2, \ldots, l_N)$. Under the separable value function assumption, the optimal policy chooses $M$ clients in the decreasing order of the indices,

$$\max_u C_n(l_n, u) + P_n(u) V_n(S(l_n)) + (1 - P_n(u)) V_n(F(l_n)),$$

providing us a convenient index based policy.

Since we would like to use online data from system operation in order to compute the above indices, we propose a Q-learning type algorithm.

**Algorithm 1** Q-learning based Index Policy

At each time $t$, for each client $n$ maintain $Q$-values $Q_n^t(l, u)$.

1) **Scheduling** : Implement the action $(n, u)$ with a probability

$$\frac{\exp(\tau Q_n^t(l, u))}{\sum_{m, t, u} \exp(\tau Q_n^t(l, u))}.$$  

2) **Q-Update**: Let client $n$ be served in time $t$. Then the update occurs as,

$$Q_n^{t+1}(l_n(t), u_n(t)) = Q_n^t(l_n(t), u_n(t)) (1 - \alpha_t)$$  

$$\quad + \alpha_t \left\{ C_n(t) + \max_u Q_n^t(l_n(t + 1), u) \right\}$$

XI. EXTENSIONS

Several useful extensions can be considered. We briefly describe the set-up and mention the approach to designing algorithm.

**Temporal Variations** in video quality are also an important factor that affect the user engagement [14]. Infact [14] shows that it might play an even more role than the video quality. In order to optimize over temporal variations, we simply need to augment client $n$’s state $l_n(t)$ by including $g_n(\cdot)$, the video quality of the latest packet delivered to client $n$ up till time $t$. Index policies can then be derived using the augmented state variable.

**Multi-Hop Networks**: Video packets have to traverse a path comprising of multiple nodes before they reach the end user. The network is described by graph $G = (V, E)$, 

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**Multi-Hop Networks**: Video packets have to traverse a path comprising of multiple nodes before they reach the end user. The network is described by graph $G = (V, E)$,
where \( V \) is the set of nodes, and \( E \) is the set of directed links that can be used for packet transmissions. The link capacities \( C_\ell \) are stochastic processes. Each node \( i \in V \) maintains a packet buffer for each client whose packets are routed through node \( i \). The state of client \( n \), i.e., \( l_n(t) \), is now described by the \( E \) dimensional vector \( \{ l_n,i(t) \}_{i \in V} \), where \( l_n,i(t) \) is the buffer occupancy of client \( n \) at node \( i \) at time \( t \). In case a client does not use a node \( i \) for routing packets, one can simply set the corresponding buffer to be zero.

Within this set-up, two important class of policies can be considered.

1) Client-Level Policies: Network operator charges price at the rate \( \lambda_\ell \) for using bandwidth at link \( \ell \). The vector \( \lambda \geq 0 \) comprising the prices \( \lambda_\ell \) are known to the clients. Client \( n \) then employs the policy \( \pi_n(\lambda) \) that optimizes its individual operating cost \( C_n + \sum_\ell \lambda_\ell E_n,\ell \), where \( E_n,\ell \) represents the average bandwidth utilized by it on link \( \ell \). Note that \( \pi_n \) maps the vector \( l_n(t) \) the state of flow \( f \), so that the policy needs to know the network wide state of client \( n \). An index policy similar to that provided in Sections [18] and [19].

2) Node-Level Policies: Client-Level Policies require obtaining the “global” state of client \( n \), which might be too much of a communication overhead. We can restrict ourselves to the class of decentralized policies, so that the scheduling decisions at a node \( i \) depend only on \( l_n,i(t) \), i.e., the \( i \)-th component of client state. The policies can be modified in a straightforward fashion to yield the optimal policy within this class.

We also mention that various function approximation techniques can be utilized [40], [41] in order to combat the resulting state space explosion.

We would also like to mention that we can consider the extension where the clients join and leave the system dynamically.

XII. FADING CHANNELS

The results in the previous sections can be extended in a straight forward manner to the case of fading channels. Let the channel conditions for client \( n \) be described by a Markov process evolving on finitely many states \( \{1, 2, \ldots, C_n\} \) having a transition matrix \( \Pi_n \). The state of client \( n \) is described by the vector \( x_n(t) := (l_n(t), c_n(t)) \), where \( l_n(t) \) is the time-duration of the packets present in the buffer at time \( t \), and \( c_n(t) \) is the channel condition at time \( t \). If the client \( n \) is scheduled a packet transmission of quality \( q \) at an power \( E \) at time \( t \), then the system state at time \( t + 1 \) is \( (S(l(t)), \bar{c}) \) with a probability \( P_{n,c_n}(q,E)\Pi(c_n(t),\bar{c}) \), while it is \( (F(l(t)), \bar{c}) \) with a probability \( P_{n,c_n}(q,E)\Pi(c_n(t),\bar{c}) \).

However now the cost associated to an action \( u \) also depends on the channel condition, i.e.,

\[
C_n(u) := \lambda E + P_c(l,E)\lambda_q,
\]

and a threshold policy will have a threshold structure for each value of channel condition (as defined in Section [V]).

REFERENCES

[1] F. Dobrian, V. Sekar, A. Awan, I. Stoica, D. Joseph, A. Ganjam, J. Zhan, and H. Zhang, “Understanding the impact of video quality on user engagement,” in Proceedings of the ACM SIGCOMM 2011 Conference, ser. SIGCOMM ’11. New York, NY, USA: ACM, 2011, pp. 362–373.

[2] M. Z. Shafiq, J. Erman, L. Li, A. X. Liu, J. Pang, and J. Wang, “Understanding the impact of network dynamics on mobile video user engagement,” SIGMETRICS Perform. Eval. Rev., vol. 42, no. 1, pp. 367–379, Jun. 2014.

[3] W. Zhu, C. Luo, J. Wang, and S. Li, “Multimedia cloud computing,” IEEE Signal Processing Magazine, vol. 28, no. 3, pp. 59–69, May 2011.

[4] G. Pallis and A. Vakali, “Insight and perspectives for content delivery networks,” Commun. ACM, vol. 49, no. 1, pp. 101–106, Jan 2006.

[5] ISO/IEC, “Dynamic adaptive streaming over http (dash),” International Standard DIS 23009-1-2, 2012.

[6] G. Tian and Y. Liu, “Towards agile and smooth video adaptation in dynamic http streaming,” in Proceedings of the 8th International Conference on Emerging Networking Experiments and Technologies, ser. CoNEXT ’12, 2012, pp. 109–120.

[7] J. De Vriendt, D. De Vleeschauwer, and D. Robinson, “Model for estimating qoe of video delivered using http adaptive streaming,” in IP/IIEEE International Symposium on Integrated Network Management (IM 2013), 2013, May 2013, pp. 1288–1293.

[8] Microsoft, “Iis smooth streaming.”

[9] Adobe, “Open source media framework.”

[10] A. ParandehGheibi, M. Mdard, A. E. Ozdaglar, and S. Shakkottai, “Avoiding Interruptions - A QoE Reliability Function for Streaming Media Applications.” IEEE Journal on Selected Areas in Communications, vol. 29, no. 5, pp. 1064–1074, 2011.

[11] G. Liang and G. Liang, “Effect of delay and buffering on jitter-free streaming over random vbr channels,” Multimedia, IEEE Transactions on, vol. 10, no. 6, pp. 1128–1141, Oct 2008.

[12] T. Hossfeld, S. Egger, R. Schatz, M. Fiedler, K. Masuch, and C. Lorentzen, “Initial delay vs. interruptions: Between the devil and the deep blue sea,” in Quality of Multimedia Experience (QoMEX), 2012 Fourth International Workshop on, July 2012, pp. 1–6.

[13] Y. Xu, E. Altman, R. El-Azouzi, M. Haddad, S. Elayoubi, and T. Jimenez, “Analysis of buffer starvation with application to objective qoe of streaming services,” Multimedia, IEEE Transactions on, vol. 16, no. 3, pp. 813–827, April 2014.

[14] C. Yin and A. C. Bovik, “Evaluation of temporal variation of video quality in packet loss networks,” Image Commun., vol. 26, no. 1, pp. 24–38, Jan. 2011.

[15] Y. Xu, S. Elayoubi, E. Altman, and R. El-Azouzi, “Impact of flow-level dynamics on qoe of video streaming in wireless networks,” in INFOCOM, 2013 Proceedings IEEE, April 2013, pp. 2715–2723.

[16] M. Seufert, S. Egger, M. Slanina, T. Zinner, T. Hobfeld, and P. Tran-Gia, “A survey on quality of experience of http adaptive streaming,” IEEE Communications Surveys & Tutorials, vol. 17, no. 1, pp. 499–502, Firstquarter 2015.

[17] R. Singh, M. Biswas, P. Kumar, and R. El-Azouzi, “Pathwise performance of stochastic network calculus for wireless networks with delay constraints,” in IEEE INFOCOM, 2014 Proceedings, April 2014, pp. 2396–2400.

[18] Rahul Singh, I-Hong Hou and P.P. Kumar, “Pathwise performance of data based policies for wireless networks with hard delay constraints,” in Decision and Control (CDC), 2013 IEEE 52nd Annual Conference on, Dec 2013, pp. 7838–7843.

[19] R. Singh, X. Guo, and P. P. Kumar, “Index policies for optimal mean-variance trade-off of inter-delivery times in real-time sensor networks,” in INFOCOM 2015, IEEE, (to appear) 2015.

[20] X. Guo, R. Singh, P. Kumar, and Z. Niu, “A high reliability asymptotic approach for packet inter-delivery time optimization in cyber-physical systems,” in Proceedings of the 16th ACM International Symposium on Mobile Ad Hoc Networking and Computing, ser. MobiHoc ’15. New York, NY, USA: ACM, 2015,
[21] R. Singh and A. Stolyar, “Maxweight scheduling: Asymptotic behavior of unscaled queue-differentials in heavy traffic,” in Proceedings of the 2015 ACM SIGMETRICS International Conference on Measurement and Modeling of Computer Systems, ser. SIGMETRICS ’15. New York, NY, USA: ACM, 2015, pp. 431–432.

[22] Y. Xu, S. E. Elayoubi, E. Altman, and R. El-Azouzi, “Impact of flow-level dynamics on qoe of video streaming in wireless networks,” in IEEE INFOCOM, 2013 Proceedings, April 2013, pp. 2715–2723.

[23] Y.-C. Ho et al., “Team decision theory and information structures in optimal control problems–part i,” IEEE transactions on automatic control, vol. 17, no. 1, pp. 15–22, 1972.

[24] R. S. Sutton and A. G. Barto, Reinforcement Learning: An Introduction. MIT Press, 1998. [Online]. Available: http://www.cs.ualberta.ca/~sutton/book/ebook/the-book.html

[25] E. Altman, Constrained Markov Decision Processes. Chapman and Hall/CRC, March 1999.

[26] J. Tsitsiklis and D. Bertsekas, “Distributed asynchronous optimal routing in data networks,” IEEE Transactions on Automatic Control, vol. 31, no. 4, pp. 325–332, 1986.

[27] M. L. Puterman, Markov Decision Processes: Discrete Stochastic Dynamic Programming, 1st ed. New York, NY, USA: John Wiley & Sons, Inc., 1994.

[28] D. Blackwell, “Discrete Dynamic Programming.” Annals of Mathematical Statistics, vol. 33, pp. 719–726, 1962.

[29] D. Blackwell and M. Girshick, Theory of games and statistical decisions. New York: John Wiley and Sons, 1954, republished by Dover in 1979. MR:0070134. Zbl:0056.36303.

[30] Alan S. Manne, “Linear programming and sequential decisions,” Management Science, vol. 6, pp. 259 –267, 1960.

[31] V. S. Borkar, “Control of Markov Chains with Long-Run Average Cost Criterion,” in The IMA Volumes in Mathematics and Its Applications, W. Fleming and P. Lions, Eds. Springer, 1988, pp. 57–77.

[32] D. P. Bertsekas, Nonlinear Programming, ser. Athena scientific. Athena Scientific, 1999.

[33] K. C. K. N.Z. Shor and A. Ruszcynski, Minimization Methods for Non-differentiable Functions. New York, NY, USA: Springer-Verlag New York, Inc., 1985.

[34] V. S. Borkar, “Stochastic approximation with two time scales,” Systems & Control Letters, vol. 29, no. 5, pp. 291 – 294, 1997.

[35] ——, Stochastic Approximation : A Dynamical Systems Viewpoint. Cambridge: Cambridge University Press New Delhi, 2008.

[36] P. Whittle, “Restless bandits: Activity allocation in a changing world,” Journal of Applied Probability, vol. 25, pp. 287–298, 1988.

[37] A. Mahajan and D. Teneketzis, “Foundations and applications of sensor management,” G. De Micheli, R. Ernst, and W. Wolf, Eds. Springer-Verlag, 2008, ch. Multi-armed bandit problems, pp. 121–151.

[38] J.C. Gittins, K. Glazebrook and R. Weber, Multi-armed Bandit Allocation Indices. John Wiley & Sons, 2011.

[39] R. R. Weber and G. Weiss, “On an Index Policy for Restless Bandits,” Journal of Applied Probability, vol. 27, no. 3, pp. 637–648, 1990.

[40] V. R. Konda and J. N. Tsitsiklis, “Actor-critic algorithms.” in NIPS, vol. 13, 1999, pp. 1008–1014.

[41] D. P. Bertsekas and J. N. Tsitsiklis, “Neuro-dynamic programming: an overview,” in Decision and Control, 1995., Proceedings of the 34th IEEE Conference on, vol. 1. IEEE, 1995, pp. 560–564.