Abstract

Inequalities play an important role in pure and applied mathematics. In particular, Jensen’s inequality, one of the most famous inequalities, plays a main role in the study of the existence and uniqueness of initial and boundary value problems for differential equations. In this work we prove some new Jensen-type inequalities for $m$-convex functions, and we apply them to generalized Riemann-Liouville-type integral operators. It is remarkable that, if we consider $m = 1$, we obtain new inequalities for convex functions.

AMS Subject Classification (2010): 26A33, 26A51, 26D15

Key words and phrases: Jensen-type inequalities, convex functions, $m$-convex functions, fractional derivatives and integrals, fractional integral inequalities.
1 Introduction

Integral inequalities are used in countless mathematical problems such as approximation theory and spectral analysis, statistical analysis and the theory of distributions. Studies involving integral inequalities play an important role in several areas of science and engineering.

In recent years there has been a growing interest in the study of many classical inequalities applied to integral operators associated with different types of fractional derivatives, since integral inequalities and their applications play a vital role in the theory of differential equations and applied mathematics. Some of the inequalities studied are Gronwall, Chebyshev, Hermite-Hadamard-type, Ostrowski-type, Grüss-type, Hardy-type, Gagliardo-Nirenberg-type, reverse Minkowski and reverse Hölder inequalities (see, e.g., [2, 5, 11, 12, 14–18]).

In this work we obtain new Jensen-type inequalities for convex and $m$-convex functions, and we apply them to the generalized Riemann-Liouville-type integral operators defined in [1], which include most of known Riemann-Liouville-type integral operators.

2 Preliminaries

One of the first operators that can be called fractional is the Riemann-Liouville fractional derivative of order $\alpha \in \mathbb{C}$, with $\text{Re}(\alpha) > 0$, defined as follows (see [4]).

**Definition 1** Let $a < b$ and $f \in L^1((a, b); \mathbb{R})$. The right and left side Riemann-Liouville fractional integrals of order $\alpha$, with $\text{Re}(\alpha) > 0$, are defined, respectively, by

$$RLJ_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t (t-s)^{\alpha-1} f(s) \, ds,$$

and

$$RLJ_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b (s-t)^{\alpha-1} f(s) \, ds,$$

with $t \in (a, b)$.

When $\alpha \in (0, 1)$, their corresponding Riemann-Liouville fractional derivatives are given by

$$\left( RL D_{a^+}^\alpha f \right)(t) = \frac{d}{dt} \left( RL J_{a^+}^{1-\alpha} f(t) \right) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \frac{f(s)}{(t-s)^\alpha} \, ds,$$

and

$$\left( RL D_{b^-}^\alpha f \right)(t) = -\frac{d}{dt} \left( RL J_{b^-}^{1-\alpha} f(t) \right) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \frac{f(s)}{(s-t)^\alpha} \, ds.$$

Other definitions of fractional operators are the following ones.
Definition 2 Let $a < b$ and $f \in L^1((a,b); \mathbb{R})$. The right and left side Hadamard fractional integrals of order $\alpha$, with $\text{Re}(\alpha) > 0$, are defined, respectively, by

$$H_{a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \left( \log \frac{t}{s} \right)^{\alpha-1} \frac{f(s)}{s} ds,$$

and

$$H_{b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \left( \log \frac{s}{t} \right)^{\alpha-1} \frac{f(s)}{s} ds,$$

with $t \in (a,b)$.

When $\alpha \in (0,1)$, Hadamard fractional derivatives are given by the following expressions:

$$(^H D_{a^+}^\alpha f)(t) = t \frac{d}{dt} (H_{a^+}^{1-\alpha} f(t)) = \frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_a^t \left( \log \frac{t}{s} \right)^{-\alpha} \frac{f(s)}{s} ds,$$

$$(^H D_{b^-}^\alpha f)(t) = -t \frac{d}{dt} (H_{b^-}^{1-\alpha} f(t)) = -\frac{1}{\Gamma(1-\alpha)} \frac{d}{dt} \int_t^b \left( \log \frac{s}{t} \right)^{-\alpha} \frac{f(s)}{s} ds,$$

with $t \in (a,b)$.

Definition 3 Let $0 < a < b$, $g : [a,b] \to \mathbb{R}$ an increasing positive function on $(a,b)$ with continuous derivative on $(a,b)$, $f : [a,b] \to \mathbb{R}$ an integrable function, and $\alpha \in (0,1)$ a fixed real number. The right and left side fractional integrals in $[7]$ of order $\alpha$ of $f$ with respect to $g$ are defined, respectively, by

$$I_{g,a^+}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_a^t \frac{g'(s) f(s)}{(g(t) - g(s))^{1-\alpha}} ds,$$

and

$$I_{g,b^-}^\alpha f(t) = \frac{1}{\Gamma(\alpha)} \int_t^b \frac{g'(s) f(s)}{(g(s) - g(t))^{1-\alpha}} ds,$$

with $t \in (a,b)$.

There are other definitions of integral operators in the global case, but they are slight modifications of the previous ones.

3 General fractional integral of Riemann-Liouville type

Now, we give the definition of a general fractional integral in $[1]$. 

**Definition 4** Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b]$ with continuous positive derivative on $(a, b)$, and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. Let us define the function $T : [a, b] \times [a, b] \times (0, \infty) \to \mathbb{R}$ by

$$
T(t, s, \alpha) = \frac{G(g(t) - g(s)), \alpha}{g'(s)}.
$$

The right and left integral operators, denoted respectively by $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$, are defined for each measurable function $f$ on $[a, b]$ as

$$
J_{T,a}^\alpha f(t) = \int_a^t \frac{f(s)}{T(t, s, \alpha)} \, ds,
$$

$$
J_{T,b}^\alpha f(t) = \int_t^b \frac{f(s)}{T(t, s, \alpha)} \, ds,
$$

with $t \in [a, b]$.

We say that $f \in L_1^1[a, b]$ if $J_{T,a}^\alpha |f|(t), J_{T,b}^\alpha |f|(t) < \infty$ for every $t \in [a, b]$.

Note that these operators generalize the integral operators in Definitions 1, 2 and 3.

(A) If we choose

$$
g(t) = t, \quad G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha) |t-s|^{1-\alpha},
$$

then $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$ are the right and left Riemann-Liouville fractional integrals $RLJ_{a}^\alpha$ and $RLJ_{b}^\alpha$, respectively. Its corresponding right and left Riemann-Liouville fractional derivatives are

$$
\left( (RLD_{a}^\alpha f) \right)(t) = \frac{d}{dt} \left( (RLJ_{a}^{1-\alpha} f) \right)(t), \quad \left( (RLD_{b}^\alpha f) \right)(t) = -\frac{d}{dt} \left( (RLJ_{b}^{1-\alpha} f) \right)(t).
$$

(B) If we choose

$$
g(t) = \log t, \quad G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha) \log \left| \frac{t}{s} \right|^{1-\alpha},
$$

then $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$ are the right and left Hadamard fractional integrals $H_{a}^\alpha$ and $H_{b}^\alpha$, respectively. Its corresponding right and left Hadamard fractional derivatives are

$$
\left( (HD_{a}^\alpha f) \right)(t) = t \frac{d}{dt} \left( (HD_{a}^{1-\alpha} f) \right)(t), \quad \left( (HD_{b}^\alpha f) \right)(t) = -t \frac{d}{dt} \left( (HD_{b}^{1-\alpha} f) \right)(t).
$$

(C) If we choose a function $g$ with the properties in Definition 4 and

$$
G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha) \frac{|g(t) - g(s)|^{1-\alpha}}{g'(s)},
$$

then $J_{T,a}^\alpha$ and $J_{T,b}^\alpha$ are the right and left fractional integrals $I_{g,a}^\alpha$ and $I_{g,b}^\alpha$, respectively.
Definition 5 Let $a < b$ and $\alpha \in \mathbb{R}^+$. Let $g : [a, b] \to \mathbb{R}$ be a positive function on $(a, b)$ with continuous positive derivative on $(a, b)$, and $G : [0, g(b) - g(a)] \times (0, \infty) \to \mathbb{R}$ a continuous function which is positive on $(0, g(b) - g(a)] \times (0, \infty)$. For each function $f \in \mathcal{L}^1_{T}[a, b]$, its right and left generalized derivative of order $\alpha$ are defined, respectively, by

\[
D^\alpha_{T,a+}f(t) = \frac{1}{g'(t)} \frac{d}{dt} \left( J^1_{T,a+}f(t) \right),
\]

\[
D^\alpha_{T,b-}f(t) = -\frac{1}{g'(t)} \frac{d}{dt} \left( J^1_{T,b-}f(t) \right).
\]

(9)

for each $t \in (a, b)$.

Note that if we choose

\[ g(t) = t, \quad G(x, \alpha) = \Gamma(\alpha) x^{1-\alpha}, \quad T(t, s, \alpha) = \Gamma(\alpha) |t - s|^{1-\alpha}, \]

then $D^\alpha_{T,a+}f(t) = RL^D_{a+}f(t)$ and $D^\alpha_{T,b-}f(t) = RL^D_{b-}f(t)$. Also, we can obtain Hadamard and others fractional derivatives as particular cases of this generalized derivative.

4 Jensen-type inequalities for $m$-convex functions

The property of $m$-convexity for functions on $[0, b], b > 0$ was introduced in [10] as an intermediate property between the usual convexity and starshaped property. Since then many properties, especially inequalities, have been obtained for them (cf. [11, 12, 13]). One of the classical integral inequalities frequently studied in this setting is Jensen’s inequality, which relates the value of a convex function of an integral to the integral of the convex function. It was proved in 1906 [14], and it can be stated as follows:

Let $\mu$ be a probability measure on the space $X$. If $f : X \to (a, b)$ is $\mu$-integrable and $\varphi$ is a convex function on $(a, b)$, then

\[ \varphi \left( \int_X f \, d\mu \right) \leq \int_X \varphi \circ f \, d\mu. \]

Definition 6 Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $m \in (0, 1]$. A function $\varphi : I \to \mathbb{R}$ is said to be $m$-convex if the inequality

\[ \varphi(tx + m(1-t)y) \leq t\varphi(x) + (1-t)\varphi(y), \]

(10)

holds for every pair of points $x, y \in I$ and every coefficient $t \in [0, 1]$.

If $m \in (0, 1)$, then the hypothesis $0 \in I$ guarantees that $tx + m(1-t)y \in I$.

It is clear that taking $m = 1$ in Definition 6 we recover the concept of classical convex functions on $I$. Note that in this case it is not necessary the hypothesis $0 \in I$, since $tx + (1-t)y \in I$ for every $x, y \in I$. 

5
Note that if we choose the coefficient $t = 0$ in (10), we get the inequality $\varphi(mx) \leq m\varphi(y)$.

Also, Definition 6 is equivalent to

$$\varphi(mtx + (1-t)y) \leq mt\varphi(x) + (1-t)\varphi(y),$$

for all $x, y \in I$ and $t \in [0,1]$.

The following discrete Jensen-type inequality for $m$-convex functions was established in [13, Theorem 3.2]:

**Theorem 7** Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $\sum_{k=1}^{n} w_k x_k$ be a convex combination of points $x_k \in I$ with coefficients $w_k \in [0,1]$. If $\varphi$ is an $m$-convex function on $I$, with $m \in (0,1]$, then

$$\varphi\left(m \sum_{k=1}^{n} w_k x_k\right) \leq m \sum_{k=1}^{n} w_k \varphi(x_k).$$

(12)

This inequality is a discrete version of the following one for continuous $m$-convex functions [13, Corollary 4.2]:

**Theorem 8** Let $\mu$ be a probability measure on the space $X$. If $I \subseteq \mathbb{R}$ is an interval containing the zero, $f : X \to I$ is $\mu$-integrable and $\varphi$ is a continuous $m$-convex function on $I$, with $m \in (0,1]$, then

$$\varphi\left(m \int_{X} f \, d\mu\right) \leq m \int_{X} \varphi \circ f \, d\mu.$$  

(13)

The following discrete Jensen-type inequality for convex functions appears in [10, Theorem 1.2]:

**Theorem 9** Let $x_1 \leq x_2 \leq \cdots \leq x_n$ and let $\{w_k\}_{k=1}^{n}$ be positive weights whose sum is 1. If $\varphi$ is a convex function on $[x_1, x_n]$, then

$$\varphi\left(x_1 + x_n - \sum_{k=1}^{n} w_k x_k\right) \leq \varphi(x_1) + \varphi(x_n) - \sum_{k=1}^{n} w_k \varphi(x_k).$$

Our purpose is to prove continuous versions of the above discrete inequality in the setting of $m$-convexity (see Theorems 14 and 15). Before stating such a result, we require some properties of the $m$-convex functions.

**Lemma 10** Let $I \subseteq \mathbb{R}$ be an interval containing the zero, and let $\varphi$ be an $m$-convex function on $I$ with $m \in (0,1]$. For $\{x_k\}_{k=1}^{n} \subset I$ such that $x_1 \leq x_2 \leq \cdots \leq x_n$, the following inequalities hold:

$$\varphi(x_1 + mx_n - mx_k) \leq \varphi(x_1) + m\varphi(x_n) - \varphi(mx_k), \quad 1 \leq k \leq n.$$  

(14)
Proof. Let us consider \( y_k = x_1 + mx_n - mx_k \). Then \( x_1 + mx_n = y_k + mx_k \) and so, the pairs \( x_1, mx_n \) and \( y_k, mx_k \) have the same mid-point. Since \( x_1 \leq y_k \) and \( mx_k \leq mx_n \), we have \( x_1 \leq y_k, mx_k \leq mx_n \) and there exists \( \lambda \in [0,1] \) such that

\[
mx_k = \lambda x_1 + m(1 - \lambda)x_n,
\]
\[
y_k = (1 - \lambda)x_1 + m\lambda x_n,
\]
for \( 1 \leq k \leq n \). From Definition 4 and its equivalent form (11) we get

\[
\varphi(y_k) \leq m\lambda\varphi(x_n) + (1 - \lambda)\varphi(x_1)
\]
\[
= \varphi(x_1) + m\varphi(x_n) - [\lambda\varphi(x_1) + m(1 - \lambda)\varphi(x_n)]
\]
\[
\leq \varphi(x_1) + m\varphi(x_n) - \varphi(\lambda x_1 + m(1 - \lambda)x_n)
\]
\[
= \varphi(x_1) + m\varphi(x_n) - \varphi(mx_k),
\]
and (14) follows.

Note that since \( m \in (0,1] \), the hypothesis \( 0 \in I \) guarantees that \( mx_k \in I \).

The following two results generalize Theorem 9 in the setting of \( m \)-convexity.

**Theorem 11** Let \( I \subseteq \mathbb{R} \) be an interval containing the zero, let \( \{x_k\}_{k=1}^n \subset I \) with \( x_1 \leq x_2 \leq \cdots \leq x_n \) and let \( \{w_k\}_{k=1}^n \) be positive weights whose sum is 1. If \( \varphi \) is an \( m \)-convex function on \( I \), with \( m \in (0,1] \), then

\[
\varphi(mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k) \leq m\varphi(x_1) + m^2\varphi(x_n) - m\sum_{k=1}^n w_k\varphi(mx_k). \quad (15)
\]

**Remark 12** Theorem 9 gives that if \( m = 1 \), the inequality in Theorem 11 also holds if we remove the hypothesis \( 0 \in I \).

Proof. First, note that

\[
x_1 + mx_n - m\sum_{k=1}^n w_kx_k = \sum_{k=1}^n w_k(x_1 + mx_n - mx_k),
\]
and thus

\[
mx_1 + m^2x_n - m^2\sum_{k=1}^n w_kx_k = m\sum_{k=1}^n w_k(x_1 + mx_n - mx_k).
\]
Then it follows from (12) and (14) that

\[ \varphi(mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_k) = \varphi\left(m \sum_{k=1}^{n} w_k (x_1 + mx_n - mx_k)\right) \]

\[ \leq m \sum_{k=1}^{n} w_k \varphi(x_1 + mx_n - mx_k) \]

\[ \leq m \sum_{k=1}^{n} w_k (\varphi(x_1) + m\varphi(x_n) - \varphi(mx_k)) \]

\[ = m\varphi(x_1) + m^2\varphi(x_n) - m \sum_{k=1}^{n} w_k \varphi(mx_k), \]

and this concludes the proof of the inequality.

Let us check that the hypothesis 0 \( \in I \) guarantees that \( mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_k \in I \):

Assume that \( I = [a, b] \). Then

\[ mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_k \geq mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_n \]

\[ = mx_1 \geq \min\{0, x_1\} \geq a. \]

Also,

\[ mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_k \leq mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_1 \]

\[ = mx_1 - m^2x_1 + m^2x_n. \]

If \( x_1 \leq 0 \), then \( mx_1 - m^2x_1 \leq 0 \) and so,

\[ mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_k \leq mx_1 - m^2x_1 + m^2x_n \leq m^2x_n \leq \max\{0, x_n\} \leq b. \]

Assume now that \( x_1 > 0 \). Let us consider the function \( v(t) = tx_1 - t^2x_1 + t^2x_n \).

Since \( v'(t) = x_1 + 2t(x_n - x_1) > 0 \) and \( m \in (0, 1] \), we have

\[ v(m) \leq v(1) = x_n \leq b, \]

\[ mx_1 + m^2x_n - m^2 \sum_{k=1}^{n} w_k x_k \leq mx_1 - m^2x_1 + m^2x_n = v(m) \leq b. \]

If \( I \) is not a closed interval \([a, b]\), a similar argument gives the result. ■

If \( \varphi \) is a continuous \( m \)-convex function, we can obtain the following improvement of Theorem 11.
Theorem 13 Let $a \leq 0 \leq b$, let $\{y_k\}_{k=1}^n \subset [a, b]$ and let $\{w_k\}_{k=1}^n$ be positive weights whose sum is 1. If $\varphi$ is a continuous $m$-convex function on $[a, b]$, with $m \in (0, 1]$, then

$$\varphi \left( ma + m^2b - m^2 \sum_{k=1}^n w_k y_k \right) \leq m\varphi(a) + m^2\varphi(b) - m \sum_{k=1}^n w_k \varphi(my_k).$$ \hspace{1cm} (16)

Proof. If we consider $0 < \varepsilon < 1$, $y_0 = a$, $y_{n+1} = b$, $w'_k = (1 - \varepsilon)w_k$ ($1 \leq k \leq n$), $w'_0 = \varepsilon/2$ and $w'_{n+1} = \varepsilon/2$, then $\sum_{k=0}^{n+1} w'_k = 1$ and Theorem 11 gives

$$\varphi \left( ma + m^2b - m^2 \sum_{k=1}^n (1 - \varepsilon)w_k y_k \right) \leq m\varphi(a) + m^2\varphi(b) - m \sum_{k=1}^n (1 - \varepsilon)w_k \varphi(my_k).$$

Since $\varphi$ is a continuous function on $[a, b]$, if we take $\varepsilon \to 0^+$, we obtain (16). $\blacksquare$

Next, we present a continuous version of the above discrete inequality.

Theorem 14 Let $\mu$ be a probability measure on the space $X$ and $a \leq 0 \leq b$ real constants. If $f : X \to [a, b]$ is a measurable function and $\varphi$ is a continuous $m$-convex function on $[a, b]$, with $m \in (0, 1]$, then $f$ and $\varphi(mf)$ are $\mu$-integrable functions and

$$\varphi \left( ma + m^2b - m^2 \int_X f \, d\mu \right) \leq m\varphi(a) + m^2\varphi(b) - m \int_X \varphi(mf) \, d\mu.$$ \hspace{1cm} (17)

If $m = 1$, this inequality also holds if we remove the hypothesis $0 \in [a, b]$.

Proof. Since $a \leq f \leq b$ and $\varphi$ is a continuous function on $[a, b]$, we have that $f$ and $\varphi(mf)$ are bounded measurable functions on $X$. And using that $\mu$ is a probability measure on $X$, we conclude that $f$ and $\varphi(mf)$ are $\mu$-integrable functions.

For each $n \geq 1$ and $0 \leq k \leq 2^n$, let us consider the sets

$$E_{n,k} = \{ x \in X : a + k2^{-n}(b - a) \leq f(x) < a + (k + 1)2^{-n}(b - a) \}.$$

Since $f$ is a measurable function satisfying $a \leq f \leq b$, we have that $\{E_{n,k}\}_{k=0}^{2^n}$ are pairwise disjoint measurable sets and $X = \bigcup_{k=0}^{2^n} E_{n,k}$ for each $n$. Thus,

$$\sum_{k=0}^{2^n} \mu(E_{n,k}) = 1$$

for each $n$. 

9
Since \( f \) is a measurable function satisfying \( a \leq f \leq b \) and \( \{E_{n,k}\}_{k=0}^{2^n} \) is a partition of \( X \), the sequence of simple functions

\[
f_n = \sum_{k=0}^{2^n} (a + k2^{-n}(b - a)) \chi_{E_{n,k}}
\]

satisfies \( a \leq f_n \leq b \) and \( f - 2^{-n}(b - a) < f_n \leq f \) for every \( n \) and so,

\[
\lim_{n \to \infty} f_n = f.
\]

Note that

\[
\int_X f_n \, d\mu = \sum_{k=0}^{2^n} (a + k2^{-n}(b - a)) \mu(E_{n,k}).
\]

Since \( \{E_{n,k}\}_{k=0}^{2^n} \) is a partition of \( X \), we have

\[
\varphi(mf) = \sum_{k=0}^{2^n} \varphi(ma + mk2^{-n}(b - a)) \chi_{E_{n,k}},
\]

\[
\int_X \varphi(mf) \, d\mu = \sum_{k=0}^{2^n} \varphi(ma + mk2^{-n}(b - a)) \mu(E_{n,k}).
\]

Hence, Theorem 13 gives

\[
\varphi\left(ma + m^2b - m^2 \int_X f_n \, d\mu\right) \leq m\varphi(a) + m^2\varphi(b) - m \int_X \varphi(mf) \, d\mu. \tag{18}
\]

If \( m = 1 \), Theorem 9 gives the above inequality without the hypothesis \( 0 \in [a, b] \).

Since \( a \leq f_n \leq b \) for every \( n \), \( \mu \) is a finite measure and \( \lim_{n \to \infty} f_n = f \), dominated convergence theorem gives

\[
\lim_{n \to \infty} \int_X f_n \, d\mu = \int_X f \, d\mu.
\]

If \( m \in (0, 1) \), then the hypothesis \( 0 \in [a, b] \) guarantees that \( ma + m^2b - m^2 \int_X f_n \, d\mu \in [a, b] \):

Since \( a \leq 0 \leq b \), we have

\[
ma + m^2b - m^2 \int_X f_n \, d\mu \leq ma + m^2b - m^2a = m(1 - m)a + m^2b \leq m^2b \leq b,
\]

\[
ma + m^2b - m^2 \int_X f_n \, d\mu \geq ma + m^2b - m^2b = ma \geq a.
\]

If \( m = 1 \), then

\[
a + b - \int_X f_n \, d\mu \leq a + b - a = b,
\]

\[
a + b - \int_X f_n \, d\mu \geq a + b - b = a.
\]
and so, we do not need the hypothesis $0 \in [a, b]$.

Since
\[ a \leq ma + m^2b - m^2 \int_X f_n \, d\mu \leq b \]
for every $n$ and $m \in (0, 1]$, and $\varphi$ is a continuous function on $[a, b]$,
\[ \lim_{n \to \infty} \varphi \left( ma + m^2b - m^2 \int_X f_n \, d\mu \right) = \varphi \left( ma + m^2b - m^2 \int_X f \, d\mu \right). \]

Since $a \leq mf_n \leq b$ for every $n$, $\lim_{n \to \infty} f_n = f$ and $\varphi$ is a continuous function on $[a, b]$, $\lim_{n \to \infty} \varphi(mf_n) = \varphi(mf)$.

Again, from the continuity of $\varphi$ on $[a, b]$, there exists a positive constant $K$ with $|\varphi| \leq K$ on $[a, b]$ and so, $|\varphi(mf_n)| \leq K$ for every $n$.

In view of the finiteness of $\mu$, dominated convergence theorem guarantees that
\[ \lim_{n \to \infty} \int_X \varphi(mf_n) \, d\mu = \int_X \varphi(mf) \, d\mu. \]

Combining the foregoing facts with (18), we obtain (17).

If $m = 1$, it is possible to improve Theorem 14, by removing the hypothesis of continuity.

**Theorem 15** Let $\mu$ be a probability measure on the space $X$ and $a \leq b$ real constants. If $f : X \to [a, b]$ is a measurable function and $\varphi$ is a convex function on $[a, b]$, then $f$ and $\varphi \circ f$ are $\mu$-integrable functions and
\[ \varphi \left( a + b - \int_X f \, d\mu \right) \leq \varphi(a) + \varphi(b) - \int_X \varphi \circ f \, d\mu. \]

**Proof.** Since $\varphi$ is a convex function on $[a, b]$, $\varphi$ is continuous on $(a, b)$ and there exist the limits
\[ \lim_{s \to a^+} \varphi(s), \quad \lim_{s \to b^-} \varphi(s). \]

Define $\varphi^*$ as follows
\[ \varphi^*(t) = \begin{cases} \varphi(t) & \text{if } t \in (a, b), \\ \lim_{s \to a^+} \varphi(s) & \text{if } t = a, \\ \lim_{s \to b^-} \varphi(s) & \text{if } t = b. \end{cases} \]

Hence, $\varphi^*$ is a continuous convex function on $[a, b]$, and by Theorem 14, with $m = 1$, we have
\[ \varphi^* \left( a + b - \int_X f \, d\mu \right) \leq \varphi^*(a) + \varphi^*(b) - \int_X \varphi^* \circ f \, d\mu. \]
Assume that \( f = a \mu \text{-a.e.} \) or \( f = b \mu \text{-a.e.} \); in the first case,

\[
\varphi(a + b - \int_X f \, d\mu) = \varphi(a + b - a) = \varphi(b)
\]

\[
= \varphi(a) + \varphi(b) - \varphi(a) = \varphi(a) + \varphi(b) - \int_X \varphi \circ f \, d\mu;
\]

in the second case,

\[
\varphi(a + b - \int_X f \, d\mu) = \varphi(a + b - b) = \varphi(a)
\]

\[
= \varphi(a) + \varphi(b) - \varphi(b) = \varphi(a) + \varphi(b) - \int_X \varphi \circ f \, d\mu.
\]

Otherwise, \( a < \int_X f \, d\mu < b \) and \( a < a + b - \int_X f \, d\mu < b \). Consequently,

\[
\varphi(a + b - \int_X f \, d\mu) = \varphi^*(a + b - \int_X f \, d\mu).
\]

If we define

\[
\Delta_a = \varphi(a) - \varphi^*(a) \geq 0, \quad \Delta_b = \varphi(b) - \varphi^*(b) \geq 0,
\]

then

\[
\varphi = \varphi^* + \Delta_a \chi_{\{a\}} + \Delta_b \chi_{\{b\}},
\]

\[
\varphi \circ f = \varphi^* \circ f + \Delta_a \chi_{\{f = a\}} + \Delta_b \chi_{\{f = b\}},
\]

where \( \chi_A \) is the function with value 1 on the set \( A \) and 0 otherwise (i.e., the characteristic function of \( A \)). Hence,

\[
\int_X \varphi \circ f \, d\mu = \int_X \varphi^* \circ f \, d\mu + \Delta_a \mu(\{f = a\}) + \Delta_b \mu(\{f = b\}),
\]

and we have

\[
\varphi(a + b - \int_X f \, d\mu) = \varphi^*(a + b - \int_X f \, d\mu)
\]

\[
\leq \varphi^*(a) + \varphi^*(b) - \int_X \varphi^* \circ f \, d\mu
\]

\[
= \varphi(a) - \Delta_a + \varphi(b) - \Delta_b - \int_X \varphi \circ f \, d\mu + \Delta_a \mu(\{f = a\}) + \Delta_b \mu(\{f = b\})
\]

\[
= \varphi(a) + \varphi(b) - \int_X \varphi \circ f \, d\mu - \Delta_a \mu(\{f = a\}) - \Delta_b \mu(\{f = b\})
\]

\[
\leq \varphi(a) + \varphi(b) - \int_X \varphi \circ f \, d\mu.
\]

\( \square \)

Theorem 15 has the following direct consequence.
Corollary 16 Let \( a \leq b \), let \( \{y_k\}_{k=1}^n \subset [a, b] \) and let \( \{w_k\}_{k=1}^n \) be positive weights whose sum is 1. If \( \phi \) is a convex function on \([a, b] \), then

\[
\phi\left(a + b - \sum_{k=1}^n w_k y_k \right) \leq \phi(a) + \phi(b) - \sum_{k=1}^n w_k \phi(y_k).
\]

Note that Theorem 15 provides a kind of converse of the classical Jensen’s inequality for convex functions.

Proposition 17 Let \( \mu \) be a probability measure on the space \( X \) and \( a \leq b \) real constants. If \( f : X \to [a, b] \) is a measurable function and \( \phi \) is a convex function on \([a, b] \), then \( f \) and \( \phi \circ f \) are \( \mu \)-integrable functions and

\[
\phi\left( \int_X f \, d\mu \right) \leq \int_X \phi \circ f \, d\mu \leq \phi(a) + \phi(b) - \phi\left(a + b - \int_X f \, d\mu \right).
\]

Theorems 15 and 14 have, respectively, the following direct consequences for general fractional integrals of Riemann-Liouville type.

Proposition 18 Let \( c < d \) and \( a \leq b \) be real constants. If \( f : [c, d] \to [a, b] \) is a measurable function, \( \phi \) is a convex function on \([a, b] \), and

\[
\psi \left( \alpha \right) = \int_c^d \frac{1}{T(d, s, \alpha)} \, ds = \int_0^g \frac{dx}{G(x, \alpha)} < \infty,
\]

then \( f(s) / T(d, s, \alpha), \phi(f(s)) / T(d, s, \alpha) \in L^1[c, d] \) and

\[
\phi\left(a + b - \frac{1}{\psi(\alpha)} \int_c^d f(s) \, ds \right) \leq \phi(a) + \phi(b) - \frac{1}{\psi(\alpha)} \int_c^d \phi(f(s)) \, ds.
\]

Proposition 19 Let \( c < d \) and \( a \leq 0 \leq b \) be real constants. If \( f : [c, d] \to [a, b] \) is a measurable function, \( \phi \) is a continuous \( m \)-convex function on \([a, b] \), with \( m \in (0, 1] \), and

\[
\psi \left( \alpha \right) = \int_c^d \frac{1}{T(d, s, \alpha)} \, ds = \int_0^g \frac{dx}{G(x, \alpha)} < \infty,
\]

then \( f(s) / T(d, s, \alpha), \phi(mf(s)) / T(d, s, \alpha) \in L^1[c, d] \) and

\[
\phi\left(ma + m^2 b - \frac{m^2}{\psi(\alpha)} \int_c^d f(s) \, ds \right) \leq m\phi(a) + m^2\phi(b) - \frac{m}{\psi(\alpha)} \int_c^d \phi(mf(s)) \, ds.
\]
Acknowledgments

The research of Yamilet Quintana, José M. Rodríguez and José M. Sigarreta is supported by a grant from Agencia Estatal de Investigación (PID2019-106433GB-I00 / AEI / 10.13039/501100011033), Spain. The research of José M. Rodríguez and Yamilet Quintana is supported by the Madrid Government (Comunidad de Madrid-Spain) under the Multiannual Agreement with UC3M in the line of Excellence of University Professors (EPUC3M23), and in the context of the V PRICIT (Regional Programme of Research and Technological Innovation).

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