Non-Abelian Monopole and Dyon Solutions in a Modified Einstein-Yang-Mills-Higgs System.

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Abstract

We have studied a modified Yang-Mills-Higgs system coupled to Einstein gravity. The modification of the Einstein-Hilbert action involves a direct coupling of the Higgs field to the scalar curvature. In this modified system we are able to write a Bogomol’nyi type condition in curved space and demonstrate that the positive static energy functional is bounded from below. We then investigate non-Abelian spherically symmetric static solutions in a similar fashion to the ‘t Hooft-Polyakov monopole. After reviewing previously studied monopole solutions of this type, we extend the formalism to included electric charge and we present dyon solutions.

1 Introduction

In a classic paper \cite{1}, Dirac proposed the possible existence of a magnetic monopole, the analogue of an isolated electrically charged particle. Motivated principally to restore the symmetries between electric and magnetic forces, Dirac found that the existence of a monopole provided a natural explanation for the quantization of electric charge. A description of such a monopole consistent with quantum mechanics would lead to the famous Dirac charge quantization condition,

$$\frac{e^2}{\hbar c} \equiv \frac{1}{2n}$$

where $e^2/\hbar c$ is the fine structure constant, $g$ is the monopole charge, and $n$ an integer. Dirac’s theory required a $U(1)$ valued gauge potential which was singular along a line (Dirac string) originating from the monopole and extending to infinity. Later Dirac’s theory was reformulated by Wu and Yang \cite{2} within the framework of fiber bundles. The singular line is avoided at the expense of introducing coordinate patches on a sphere surrounding the monopole. However, the transition functions between the coordinate patches that are elements of the $U(1)$ gauge group are singular. Thus, there is a complete equivalence between the two descriptions \cite{3} and the monopole emerges on a sound footing like any other particle in nature. Consequently, during the past decades, extensive effort has gone into experimental search for monopoles, but unfortunately has as yet had no success.

In spite of the lack of experimental success, the monopole continues to thrive in the theoretical laboratory. With the pioneering work of ‘t Hooft and Polyakov \cite{4}, the monopole was reinvented in a new form as a finite...
energy, particle-like soliton in non-Abelian gauge theories with spontaneous symmetry breaking. Moreover, such objects are generic in any spontaneously broken non-Abelian gauge theory which has an unbroken U(1) gauge symmetry. Such monopoles are expected to be produced in abundance in phase transitions of grand unified theories, which has implications for early universe cosmology. New searches for such relic monopoles are under way [5, 6].

More recently, a great deal of activity has centered around monopoles in curved space-time in order to study the effects of gravity. New insights have emerged from a study of a coupled Einstein-Yang-Mills-Higgs [EYMH] system with solutions describing black holes with magnetic charge, black holes within magnetic monopoles and magnetic monopoles within black holes [7–11]. Consequently, gravity cannot be dispensed with, arguing that the strength of its interaction is weak. The coupled set of equations for the EYMH system lead to non-trivial consequences and provide a fertile ground for the study of the interplay between gravitation and other interactions.

The present paper is an extension of the work by Nguyen and Wali [12] to include electric charge and to study dyons coupled to gravity. The starting point is a modified EYMH system with a specific coupling of the Higgs field to the Einstein term in the action. In the static case, this enables us, with the help of a Bogomol’nyi-type [9] condition, to reduce the energy functional to a form that resembles the energy functional in flat space-time and derive a lower bound on the energy and hence the mass of the dyon in terms of its electric and magnetic charges.

In the next section, we begin with a review of the general formalism and the field equations for the coupled EYMH system. In Section 3, we derive the field equations and the energy functional in the static case. Through the Higgs field equation, we express the Einstein term in the action in terms of metric fields, find a positive definite expression for the energy functional, and derive a lower bound on the energy. We also discuss in this section, the relation between the mass and the charge of the dyon. Section 4 is devoted to the derivation of the basic equations in the context of a spherically symmetric static metric and the spherically symmetric ’t Hooft-Polyakov ansatz for the gauge and Higgs fields. In Section 5, we specialize to the Higgs vacuum and find monopole and dyon solutions. The final section is devoted to some concluding remarks.

2 General Framework; Field Equations

We begin by defining the action,

\[ S = \int d^4x \sqrt{-g} (\mathcal{L}_E + \mathcal{L}_M), \]

where

\[ \mathcal{L}_E = -\frac{R - 2\Lambda}{16\pi G v^2} \Phi^2, \]

with \( R \), the Ricci scalar, \( \Lambda \), the cosmological constant and \( \Phi \), the Higgs scalar field. Our metric \( g_{\mu\nu} \) is chosen to have signature (+−−−) with indicies \( \mu, \nu, \ldots \) running from 0 to 3 and indicies \( i, j, \ldots \) from 1 to 3, also \( g = \det |g_{\mu\nu}|. \) The matter content is given by

\[ \mathcal{L}_M = -\frac{1}{4}g^{\mu\rho}g^{\nu\lambda}F_{\mu\nu}F_{\rho\lambda} + \frac{1}{2}g^{\mu\nu}(D_{\mu}\Phi)(D_{\nu}\Phi) - \frac{\lambda}{4}(\Phi^2 - v^2)^2, \]

where

\[ F_{\mu\nu} = -\frac{1}{i\alpha}[D_\mu, D_\nu], \]

with

\[ D_\mu = \nabla_\mu - i\alpha[A_\mu, \cdot]. \]

Thus \( F_{\mu\nu} \) is the field strength associated with the gauge field \( A_\mu \), \( \alpha \) being the strength of the gauge coupling. \( \nabla_\mu \) is the covariant derivative with the metric compatible, torsion free connection coefficients.
More explicitly,
\[ F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu - i\alpha [A_\mu, A_\nu], \] (2.6)

and in the component form,
\[ F_{\mu\nu}^a = \partial_\mu A_\nu^a - \partial_\nu A_\mu^a + \alpha f^{abc} A_\mu^b A_\nu^c, \] (2.7)

where \( f^{abc} \) are the structure constants of a gauge group \( G \), which for the most part for our purposes will be \( SU(2) \). The scalar field \( \Phi \) belongs to the adjoint representation of \( G \). Its covariant derivative is given by
\[ D_\mu \Phi = \nabla_\mu \Phi - i\alpha [A_\mu, \Phi] \] (2.8)

and in component form
\[ D_\mu \Phi^a = \partial_\mu \Phi^a + \alpha f^{abc} A_\mu^b \Phi^c. \] (2.9)

We note that in the broken phase of the gauge symmetry, when the Higgs field \( \Phi \) assumes its vacuum expectation value, \( \Phi^2 = v^2 \), \( L_E \) in (2.2) is the conventional Einstein-Hilbert Lagrangian. \( L_M \) in (2.3) represents the standard Yang-Mills-Higgs Lagrangian in curved space-time.

By varying the action \( S \) with respect to \( A_\mu \), \( \Phi \) and \( g_{\mu\nu} \), we obtain the coupled Yang-Mills, Higgs and Einstein equations of motion:
\[ \frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} F^\mu) = i\alpha [\Phi, D^\nu \Phi], \] (2.10)
\[ \frac{1}{\sqrt{-g}} D_\mu (\sqrt{-g} D^\mu \Phi) \equiv \left( \frac{R - 2\Lambda}{8\pi Gv^2} + \lambda (\Phi^2 - v^2) \right) \Phi, \] (2.11)

and
\[ G_{\mu\nu} = R_{\mu\nu} - \frac{R - 2\Lambda}{2} g_{\mu\nu} = \frac{8\pi Gv^2}{\Phi^2} T_{\mu\nu}, \] (2.12)

where the energy-momentum tensor \( T_{\mu\nu} \) is given by
\[ T_{\mu\nu} = - \left( L_M + \frac{1}{2} \Box \Phi^2 \right) g_{\mu\nu} - F_{\mu\rho} \cdot F_{\rho\nu} + D_\mu \Phi \cdot D_\nu \Phi + \nabla_\mu \nabla_\nu \Phi^2. \] (2.13)

The terms involving \( \Phi \) in (2.13) arise because of its presence in the modified Einstein-Hilbert action [13] in eqn. (2.2). We further note that although the field equations are written in terms of the covariant derivative \( \nabla_\mu \) in \( D_\mu \), we can easily show that they can all be reduced to ordinary partial derivatives since
\[ \nabla_\sigma g_{\mu\nu} = 0; \quad \nabla_\mu \sqrt{-g} = \partial_\mu \sqrt{-g} - \sqrt{-g} \Gamma^\nu_{\mu\nu} = 0, \] (2.14)

for torsion-free, metric compatible connection coefficients \( \Gamma^\lambda_{\mu\nu} = \Gamma^\lambda_{\nu\mu} \). Therefore, henceforth, in our equations,
\[ D_\mu = \partial_\mu - i\alpha [A_\mu, \cdot]. \] (2.15)

3 Static Field Equations; Bogomol’nyi Bound

We are interested in static solutions to equations (2.10) - (2.12). Setting the time derivatives of all fields equal to zero, we find equation (2.10) reduces to two equations,
\[ \frac{1}{\sqrt{-g}} D_i (\sqrt{-g} F^{0i}) = i\alpha [\Phi, D^i \Phi], \] (3.1)
\[ D_0 F^{0j} + \frac{1}{\sqrt{-g}} D_i (\sqrt{-g} F^{ij}) = i\alpha [\Phi, D^i \Phi]. \] (3.2)
The Higgs field equation (2.11) becomes
\[
\frac{1}{\sqrt{-g}} [D_0 (\sqrt{-g} D^0 \Phi) + D_i (\sqrt{-g} D^i \Phi)] = - \left( \frac{R - 2\Lambda}{8\pi G v^2} + \lambda (\Phi^2 - v^2) \right) \Phi.
\] (3.3)

In order to find a Bogomol’nyi-type first-order equation [14] to our problem, we make the ansatz [15],
\[
F_{ij} = \sqrt{-g} \epsilon_{ijk} (D^k + u^k) \Phi,
\] (3.4)
where \( u^k = \partial_k f \) is an arbitrary time-independent function, and \( \tilde{g} = \det g_{ij} \).

With the above ansatz, we find
\[
D_i F^{ij} = i\alpha [\Phi, D^j \Phi] + \left( \frac{\partial_i \sqrt{g_{00}}}{\sqrt{g_{00}}} - \partial_i f \right) F^{ij}.
\] (3.5)

Substituting (3.5) in (3.2), we have
\[
[D_0, F^{0j}] + \left( \frac{\partial_i \sqrt{g_{00}}}{\sqrt{g_{00}}} - \partial_i f \right) F^{ij} = 0.
\] (3.6)

For future reference, we note that the Yang-Mills equation (3.1) implies
\[
D_i (\sqrt{-g} F^{ij}) \cdot \Phi = 0.
\] (3.7)

Using (3.4) together with (3.5), after some algebra and the use of the Bianchi identity, the Higgs field equation (3.3) reduces to
\[
D_0 D^0 \Phi + \left( \frac{\partial_i \sqrt{g_{00}}}{\sqrt{g_{00}}} - \partial_i f \right) D^i \Phi - \frac{1}{\sqrt{-g}} \partial_i \left( \sqrt{-g} \tilde{g} f \right) \Phi = - \left( \frac{R - 2\Lambda}{8\pi G v^2} + \lambda (\Phi^2 - v^2) \right) \Phi.
\] (3.8)

In deriving the above equations, with static spherically symmetric solutions in mind, we have assumed
\[
g_{00} > 0 \quad g_{0i} = 0, \quad \text{and} \quad -\tilde{g} > 0.
\] (3.9)

The static energy functional, \( \mathcal{E} \), that follows from (2.1) is given by
\[
\mathcal{E} = \int d^3x \frac{\sqrt{-g}}{2} \left[ \frac{R - 2\Lambda}{8\pi G v^2} \Phi^2 - F^{0i} \cdot F_{0i} + \frac{1}{2} F^{ij} \cdot F_{ij} - D^i \Phi \cdot D_i \Phi + \frac{\lambda}{2} (\Phi^2 - v^2)^2 \right],
\] (3.10)

Next we define electric and magnetic fields, \( E_i, B_i \) and corresponding \( E^i, B^i \) to be given by
\[
E_i = \frac{\sqrt{-g}}{\sqrt{g_{00}}} F_{0i} \quad ; \quad E^i = \sqrt{-g} \sqrt{g_{00}} F^{0i},
\] (3.11)
\[
B_i = \frac{1}{2} \sqrt{-g} \sqrt{-g} \epsilon_{ijk} F^{jk} \quad ; \quad B^i = \frac{1}{2} \sqrt{-g} \sqrt{-g} \epsilon^{ijk} F_{jk},
\] (3.12)
and
\[
\chi = \sqrt{g_{00}} \Phi,
\] (3.13)
where all the fields are functions of spacial coordinates only.
With these definitions and after substituting in (3.10) for \((R - 2\Lambda)/8\pi G v^2\) from (3.8), we obtain

\[
\mathcal{E} = \frac{1}{2} \int d^3x \left\{ - \left[ E^i \cdot E_i + B^i \cdot B_i + \left( \sqrt{-g} \left( D_i \chi \right) \right) \left( \sqrt{-g} D_i \chi \right) - 2 \sqrt{-g} D_0 \chi \cdot D^0 \chi \right] \\
+ \frac{1}{2} \sqrt{-g} \left( \sqrt{-g} \frac{\partial^i \sqrt{g}}{\sqrt{g_0}} - \partial^i f \right) \frac{\partial_i \chi}{2} - \frac{\lambda}{2} \sqrt{-g} \chi^2 (\chi^4 - g_{00} v^4) + \partial_i \left( \sqrt{-g} \frac{\chi^2}{g_0} \partial^i f \right) \right\}. \tag{3.14}
\]

In the spherically symmetric case in which we are interested, with the assumed signature for the metric, we note that

\[g_{ij} = 0, i \neq j \quad \text{and} \quad -g_{ii} \geq 0\] (3.15)

Consequently, each of the terms in the first parenthesis in (3.14) is positive. They are exact analogues of the corresponding terms in the case of flat space-time. The next two terms vanish in the Higgs vacuum, \(\chi^2 = g_{00} v^2\). Finally, the last term is a total divergence, giving rise to a finite surface term. Ignoring the terms that vanish when \(\chi^2 = g_{00} v^2\), we have a reduced positive-definite energy functional, \(\mathcal{E}\), given by

\[
\mathcal{E} = \frac{1}{2} \int d^3x \left\{ E^i \cdot E_i + B^i \cdot B_i + \left( \sqrt{-g} \left( D_i \chi \right) \right) \left( \sqrt{-g} D_i \chi \right) \\
+ 2 \sqrt{-g} D_0 \chi \cdot D^0 \chi + \partial_i \left( \sqrt{-g} \chi^2 \partial^i f \right) \right\}. \tag{3.16}
\]

As in the case of flat space-time (3.16), we can write

\[
\mathcal{E} = \frac{1}{2} \int d^3x \{ \left[ E^i - \sin \theta \sqrt{-g} \sqrt{g_0} D_i \chi \right] \left( E_i - \sin \theta \sqrt{-g} \sqrt{g_0} D_i \chi \right) + \left[ B^i - \cos \theta \sqrt{-g} \sqrt{g_0} D_i \chi \right] \left( B_i - \cos \theta \sqrt{-g} \sqrt{g_0} D_i \chi \right) + 2 \left[ \sin \theta \sqrt{-g} \sqrt{g_0} E^i \cdot D_i \chi \right. \\
\left. + \cos \theta \sqrt{-g} \sqrt{g_0} B^i \cdot D_i \chi + \sqrt{-g} \sqrt{g_0} D_0 \chi \cdot D^0 \chi + \frac{1}{2} \partial_i \left( \sqrt{-g} \chi^2 \partial^i f \right) \right\}. \tag{3.17}
\]

Hence, \(\mathcal{E}\), has a lower bound,

\[
\mathcal{E} \geq \int d^3x \left[ \sin \theta \sqrt{-g} \sqrt{g_0} E_i \cdot D_i \chi + \cos \theta \sqrt{-g} \sqrt{g_0} B^i \cdot D_i \chi + \sqrt{-g} \sqrt{g_0} D_0 \chi \cdot D^0 \chi + \frac{1}{2} \partial_i \left( \sqrt{-g} \chi^2 \partial^i f \right) \right], \tag{3.19}
\]

reaching the lower bound when the Bogomol'nyi-type equations are satisfied, that is when

\[
E_i - \sin \theta \sqrt{-g} \sqrt{g_0} D_i \chi = B_i - \cos \theta \sqrt{-g} \sqrt{g_0} D_i \chi = 0. \tag{3.20}
\]

Now from our definition for \(B_i\) in (3.12),

\[
\sqrt{-g} \sqrt{g_0} B^i = \frac{1}{2} e^{ijk} F_{jk},
\]

and from the Bianchi identity, we have the conservation law,

\[
D_i \left( \sqrt{-g} \sqrt{g_0} B^i \right) = 0. \tag{3.21}
\]
Consequently,

$$\int d^3x \sqrt{-g} \sqrt{g_{00}} B^i \cdot D_i \chi = \int d^3x D_i \left( \frac{\sqrt{-g}}{\sqrt{g_{00}}} B^i \cdot \chi \right).$$

(3.22)

Similarly,

$$\sqrt{-g} E^i = F^{i0},$$

(3.23)

and

$$D_i \left( \frac{\sqrt{-g}}{\sqrt{g_{00}}} E^i \cdot \chi \right) = \chi \cdot D_i F^{i0} + E^i \cdot \frac{\sqrt{-g}}{\sqrt{g_{00}}} D_i \chi$$

$$= E^i \cdot \frac{\sqrt{-g}}{\sqrt{g_{00}}} D_i \chi,$$

(3.24)

on account of (3.7).

Hence,

$$\int d^3x \left( E^i \cdot \sqrt{-g} \sqrt{g_{00}} D_i \chi \right) = \int d^3x D_i \left( \frac{\sqrt{-g}}{\sqrt{g_{00}}} E^i \cdot \chi \right).$$

(3.25)

When the Bogomol’nyi equation (3.20) is satisfied, the lower bound on the energy functional is satisfied and we have,

$$\mathcal{E} = \sin \theta \int d^3x D_i \left( \frac{\sqrt{-g}}{\sqrt{g_{00}}} E^i \cdot \chi \right) + \cos \theta \int d^3x D_i \left( \frac{\sqrt{-g}}{\sqrt{g_{00}}} B^i \cdot \chi \right)$$

$$+ \int d^3x \left[ \sqrt{-g} \sqrt{g_{00}} D_0 \chi \cdot D^0 \chi + \frac{1}{2} \partial_0 \left( \frac{\sqrt{-g}}{\sqrt{g_{00}}} \chi^2 \partial^i f \right) \right].$$

(3.26)

Now, if we have finite-energy configurations with finite extension and asymptotically flat space-time,

$$D_\mu \Phi = 0, \quad \Phi^2 = v^2,$$

(3.27)

leading to the condition that the gauge potential $A_\mu$ is given by

$$A_\mu = \frac{i}{\alpha v^2} [\Phi, \partial_\mu \Phi] + \frac{1}{v} \Phi W_\mu,$$

(3.28)

where $W_\mu$ is an arbitrary Abelian field. The field strength $F_{\mu\nu}$ corresponding to the above gauge potential is

$$F_{\mu\nu} = \frac{1}{v} \Phi F_{\mu\nu},$$

where

$$F_{\mu\nu} = \frac{i}{\alpha v^3} [\partial_\mu \Phi, \partial_\nu \Phi] \cdot \Phi + \partial_\mu W_\nu - \partial_\nu W_\mu.$$

(3.29)

In the static case, asymptotically,

$$A_0 = \frac{\Phi W_0}{v}, \quad A_i = \frac{i}{\alpha v^2} [\Phi, \partial_i \Phi],$$
and

\[
F_{i0} = \Phi \frac{1}{v} \partial_i W_0 = -E_i,
\]
\[
F_{ij} = \frac{1}{2v} \Phi \left\{ \frac{i}{\alpha v} (\partial_i \Phi, \partial_j \Phi) \cdot \Phi \right\},
\]
where we have assumed \( W_i = 0, \ i = 1, 2, 3 \).

The above field configurations imply that outside a finite region, the non-Abelian gauge field is purely in the direction \( \Phi \), the direction of the unbroken \( U(1) \) electromagnetic field \( F_{\mu\nu} \). Using (3.30), we can convert the divergence integrals of \( E_i \) and \( B_i \) into flux integrals of electric and magnetic fields over a surface at infinity and obtain

\[
\mathcal{E} = \sin \theta Q_E + \cos \theta Q_M + \frac{1}{2} \int d^3x \partial_i \left( \sqrt{-g} g^{00} \chi^2 \partial^i f \right),
\]
where

\[
Q_E = \lim_{R \to \infty} \int_{S_R} d\sigma R \frac{1}{v} (\partial_i W_0) \chi^2,
\]
and

\[
Q_M = \lim_{R \to \infty} \frac{1}{2\alpha v^3} \int_{S_R} d\sigma R \epsilon^{ijk} \chi \cdot [\partial_j \chi, \partial_k \chi].
\]

Which leads to

\[
Q_M = \frac{4\pi}{\alpha} n,
\]
where \( n \), an integer, is the winding number of the mapping \( \chi : S^2_\infty \to S^2 \),

\[
n = \frac{1}{8\pi v^3} \int_{S^2_\infty} d\sigma \epsilon^{ijk} \chi \cdot [\partial_j \chi, \partial_k \chi].
\]

\( Q_E \) has no such interpretation, but its finiteness leads to a condition on \( W_0 \).

If we define

\[
\sin \theta = \frac{Q_E}{\sqrt{Q_E^2 + Q_M^2}}, \quad \cos \theta = \frac{Q_M}{\sqrt{Q_E^2 + Q_M^2}},
\]
then

\[
\mathcal{E} \geq \sqrt{Q_E^2 + Q_M^2} + \frac{1}{2} \int d^3x \partial_i \left( \sqrt{-g} g^{00} \chi^2 \partial^i f \right),
\]
the equality holding when equation (3.20) is satisfied.

4 Basic Equations with Spherically Symmetric Ansatz

The considerations in the previous section are valid for any compact group \( G \). In this section, we shall specialize to \( SU(2) \). We define a spherically symmetric static metric

\[
ds^2 = A^2(r) dt^2 - B^2(r) dx^2 - r^2 (d\theta^2 + \sin^2 \theta d\varphi^2),
\]

(4.1)
and assume the spherically symmetric 't Hooft-Polyakov ansatz [4] for the gauge and Higgs fields,

\[ A_0^a = \frac{\dot{x}^a}{\alpha} J(r), \quad (4.2) \]

\[ \eta_{ab} A_i^b = \epsilon_{aij} \dot{x}^j (1 - W(r)) / \alpha r, \quad \eta_{ab} = (-1, -1, -1), \quad (4.3) \]

and

\[ \Phi^a = \dot{x}^a vH(r). \quad (4.4) \]

In spherical polar coordinates,

\[ A_0^a = \frac{J(r)}{\alpha} \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad A_\theta^a = 0, \]

\[ A_\varphi^a = \frac{1 - W(r)}{\alpha} \begin{bmatrix} \sin \varphi \\ \cos \varphi \\ 0 \end{bmatrix}, \quad A_r^a = \frac{1 - W(r)}{\alpha} \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix}, \quad (4.5) \]

leading to the field strengths

\[ F_{0r}^a = -\frac{J'(r)}{\alpha} \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad F_{0\theta}^a = -\frac{J(r)W'(r)}{\alpha} \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix}, \]

\[ F_{0\varphi}^a = -\frac{J(r)W(r)}{\alpha} \begin{bmatrix} -\sin \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad F_{r\theta}^a = -\frac{W'(r)}{\alpha} \begin{bmatrix} \sin \varphi \\ -\cos \varphi \\ 0 \end{bmatrix}, \]

\[ F_{r\varphi}^a = -\frac{W(r)}{\alpha} \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix} \sin \theta, \quad F_{\varphi r}^a = \frac{W^2(r) - 1}{\alpha} \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix} \sin \theta, \quad (4.6) \]

and

\[ D_r \Phi^a = vH'(r) \begin{bmatrix} \sin \theta \cos \varphi \\ \sin \theta \sin \varphi \\ \cos \theta \end{bmatrix}, \quad D_\theta \Phi^a = vW(r)H(r) \begin{bmatrix} \cos \theta \cos \varphi \\ \cos \theta \sin \varphi \\ -\sin \theta \end{bmatrix}, \]

\[ D_\varphi \Phi^a = -vW(r)H(r) \begin{bmatrix} \sin \varphi \\ -\cos \varphi \\ 0 \end{bmatrix} \sin \theta, \quad (4.7) \]

where 'prime' denotes derivatives with respect to \( r \).

Expressed in spherical polar coordinates, our ansatz (3.4) that leads to the Bogomol'nyi-type equation takes the form

\[ D_r \chi + \left( f'(r) - \frac{A'(r)}{A(r)} \right) \chi = -\frac{A(r)B(r)}{r^2 \sin \theta} F_{\theta \varphi}, \quad (4.8) \]

\[ D_\theta \chi = -\frac{A(r)}{B(r)} \frac{1}{\sin \theta} F_{\varphi r}, \quad (4.9) \]

\[ D_\varphi \chi = -\frac{A(r)}{B(r)} \sin \theta F_{r\theta}. \quad (4.10) \]
Substituting the spherically symmetric ansatz for $\chi$ and $F$, we obtain two independent equations,

$$H'(r) + f'(r)H(r) = \frac{B(r)(1 - W^2(r))}{\alpha r^2},$$

$$W'(r) = -\alpha e W(r)H(r).$$

The Yang-Mills field equations (4.13) and (4.14), and the Higgs field equations (4.8) are transformed into

$$J''(r) + \left( \frac{2}{r} - \frac{A'(r)}{A(r)} - \frac{B'(r)}{B(r)} \right) J'(r) - \frac{2B^2(r)}{r^2} J(r)W(r)^2 = 0,$$

$$J^2(r)W(r) - \left( f'(r) - \frac{A'(r)}{A(r)} \right) A^2(r)W'(r) = 0,$$

$$\left( -f'(r) + \frac{A'(r)}{A(r)} \right) \frac{H'(r)}{H(r)B^2(r)} + \frac{1}{B^2(r)} \left( \frac{B'(r)}{B(r)} - \frac{2}{r} \right) f'(r) - \frac{1}{B^2(r)} f''(r) = \frac{R - 2\Lambda}{8\pi Gv^2} + \lambda e^2 (H^2(r) - 1).$$

We would like to remark that in the case of pure monopole in the $A_0 = 0$ gauge then $J(r) = 0$. Thus if $f'(r) = \frac{A'(r)}{A(r)}$, the Yang-Mills equations (4.13) and (4.14) are automatically satisfied and the Higgs field equation (4.11) becomes

$$\frac{1}{B^2(r)} \left[ \left( \frac{B'(r)}{B(r)} - \frac{2}{r} \right) \frac{A'(r)}{A(r)} - \frac{A'(r)}{A(r)} \right] = \frac{R - 2\Lambda}{8\pi Gv^2} + \lambda e^2 (H^2(r) - 1).$$

From the metric (4.1), we calculate the components of the Einstein tensor, $G_{\mu\nu}$, and the Ricci scalar, $R$, and find them to be

$$G_{00} = \frac{A^2}{B^2} \left[ \frac{1}{r^2} (B^2 - 1) + \frac{2B'}{r B} \right],$$

$$G_{rr} = \frac{1}{r^2} (1 - B^2) + \frac{2A'}{r A},$$

$$G_{\theta\theta} = \frac{r^2}{B^2} \left( \frac{A''}{A} + \frac{A'}{r A} - \frac{A' B'}{A B} - \frac{1}{r B} \right),$$

$$G_{\varphi\varphi} = \sin^2 \theta G_{\theta\theta},$$

and

$$R = \frac{2}{r^2} \left( \frac{1}{B^2} - 1 \right) + \frac{2}{B^2} \left( \frac{A''}{A} + \frac{2A'}{r A} - \frac{A' B'}{A B} - \frac{2B'}{r B} \right).$$

The rest of the components are zero. Likewise, we express the components of the energy-momentum tensor using (4.8)-(4.10), and find

$$T_{00} = \frac{1}{\alpha^2} \left[ \frac{1}{2B^2} J'^2 + \frac{1}{r^2} J^2 W^2 + \frac{A^2}{B^2 r^2} W^2 - \frac{1}{2} \frac{A^2}{r^2} (W^2 - 1)^2 \right] + v^2 \left[ \frac{A^2}{2B^2} H'^2 + \frac{1}{2} \frac{A^2}{r^2} W^2 H^2 + \lambda \frac{3}{4} A^2 v^2 (H^2 - 1)^2 \right],$$

$$T_{rr} = \frac{1}{\alpha^2} \left[ -\frac{1}{2A^2} J'^2 + \frac{B^2}{A^2} J^2 W^2 + \frac{1}{r^2} W^2 - \frac{B^2}{2r^4} (W^2 - 1)^4 \right].$$
\[ +v^2 \left[ \frac{1}{2} H^2 - \frac{B^2}{r^2} W^2 H^2 - \frac{\lambda}{4} B^2 v^2 (H^2 - 1)^2 \right], \]  
\[ T_{\theta \theta} = \frac{1}{2\alpha^2} \left[ \frac{1}{A^2 B^2} r^2 J'^2 + \frac{1}{r^2} (W'^2 - 1)^2 \right] - \frac{v^2}{2} \left[ \frac{r^2}{B^2} H'^2 + \frac{\lambda}{2} r^2 v^2 (H^2 - 1)^2 \right], \]  
\[ T_{\varphi \varphi} = \sin^2 \theta T_{\theta \theta}. \]  

Using (4.17) and (4.19), it is straightforward to write Einstein field equations
\[ G_{\mu \nu} + \Lambda g_{\mu \nu} = 8\pi G v^2 \Phi T_{\mu \nu}. \]

In what follows, we eliminate from them their dependence on \( W'(r) \) and \( W(r) \) by using equations (4.11) and (4.12) and take linear combinations of the resulting equations. These are
\[ \frac{1}{r} \left( \frac{A'}{A} + \frac{B'}{B} \right) = 8\pi G \left( \frac{v^2}{2} \left( \frac{H'}{H} \right)^2 + B^2 \left( 1 - v^2 r^2 \left( \frac{H' + f' H}{B} \right) \right) \left( \frac{J^2}{A^2 H^2} + \frac{v^2 B^2}{r^2} \right) \right), \]
\[ \frac{A''}{A} - \frac{A'}{AB} + \frac{B^2 - 1}{r^2} = 8\pi G \left[ \frac{1}{\alpha^2} \frac{J'^2}{A^2 H^2} + \frac{1}{2} v^2 \left( \frac{H'}{H} \right)^2 + \frac{v^2 B^2}{r^2} \right. \]
\[ + f' v^2 \left( \frac{2 H'}{H} + f' \right) - v^3 B \alpha \left( H' + f' H \right), \]
\[ \frac{A''}{A} + \frac{1}{r} \left( \frac{A'}{A} - \frac{B'}{B} \right) - \frac{A'}{AB} + \Lambda B^2 = 4\pi G \left[ \frac{1}{\alpha^2} \frac{J'^2}{A^2 H^2} + f' v^2 \left( \frac{2 H'}{H} + f' \right) - \frac{1}{2} B^2 v^4 \left( \frac{H^2 - 1}{H^2} \right) \right]. \]

5 Higgs Vacuum; Monopole and Dyon Solutions

We seek solutions to the set of coupled non-linear equations (4.11)-(4.15) and (4.23)-(4.25) in the Higgs vacuum, that is, the Higgs field frozen to a constant, \( H = 1 \). The equations simplify considerably. To this end, we introduce the dimensionless co-ordinate \( x \) and define dimensionless \( J \),
\[ x = \alpha r, \quad \frac{J}{\alpha v} \rightarrow J. \]

We shall also work in units of the dimensionless coupling \( 4\pi G v^2 \) set equal to unity and henceforth all the field variables are functions of \( x \) and ‘prime’ will denote derivatives with respect to \( x \). We shall suppress their dependence on \( x \) for simplicity.

From eqns (4.12) and (4.14), it follows that
\[ f' = \frac{A'}{A} - \frac{J^2 B}{A^2}, \]  
and hence the Bogomol’nyi conditions (4.11) and (4.12) take the form
\[ \frac{A'}{AB} = \frac{J^2}{A^2} + \frac{B(1 - y)}{x^2}, \]  
\[ B = -\frac{1}{2} \frac{y}{y}, \]
where we have defined \( y = W^2 \).

Then the remaining Yang-Mills, Higgs and Einstein field equations (4.13), (4.15), (4.23)-(4.25) are given by

\[
J'' + \left( \frac{2}{x} - \frac{A'}{A} - \frac{B'}{B} \right) J' - \frac{2}{x^2} B^2 J y = 0, \tag{5.5}
\]

\[
B^2 \Lambda + \frac{B^2 - 1}{x^2} - 2 \frac{A''}{A} + 2 \frac{B'}{x B} + \frac{A'}{A} \left( \frac{A'}{A} + 2 \frac{B'}{B} - \frac{4}{x} \right) = 2 \frac{J B}{A^2} \left( J \left( \frac{A'}{A} - \frac{1}{x} \right) - J' \right), \tag{5.6}
\]

\[
\frac{1}{x} \left( \frac{A'}{A} + \frac{B'}{B} \right) - \frac{2B^2}{x^2} \left( 1 - x^2 \frac{A'}{AB} \right) = 2J^2 \frac{B^2}{A^2} \left( 1 + \frac{1}{x^2} - \frac{A'}{AB} + \frac{J^2}{A^2} \right), \tag{5.7}
\]

\[
\frac{A''}{A} - \frac{A'}{A} \left( \frac{B'}{B} + 2 \frac{A'}{A} - 2 B \right) - \frac{B^2 + 1}{x^2} = 2 \frac{J^2}{A^2} + 2J^2 \frac{B^2}{A^2} \left( 1 - \frac{2 A'}{AB} + \frac{J^2}{A^2} \right), \tag{5.8}
\]

\[
\frac{A''}{A} + \frac{1}{x} \left( \frac{A'}{A} - \frac{B'}{B} \right) - \frac{A'}{A} \left( \frac{A'}{A} + \frac{B'}{B} \right) + AB^2 = \frac{J^2}{A^2} - 2B \frac{J^2 A'}{A^2} + B^2 \frac{J^4}{A^4}. \tag{5.9}
\]

5.1 Monopole solutions

Equations for the monopole are obtained by setting \( J = 0 \) in the above set of equations. At the outset, we note that eqn. (5.5) is automatically satisfied. By taking suitable linear combinations of (5.6), (5.7), (5.8) and (5.9), we find

\[
\Lambda \left( 3B^2 + 1 \right) = 0, \tag{5.10}
\]

which implies \( \Lambda = 0 \) for \( B \) to be real. Thus, there is no monopole solution for non-vanishing cosmological constant in our model. With \( \Lambda = 0 \), we are left with three independent equations,

\[
\frac{A'}{A} = \frac{1}{x^2} B \left( 1 - y \right), \quad B = -\frac{1}{2} \frac{y'}{y}, \quad B = 1 + x \frac{A'}{A}, \tag{5.11}
\]

or equivalently,

\[
y' = -\frac{2xy}{x + y - 1}, \tag{5.12}
\]

\[
B = \frac{x}{x + y - 1}, \quad \frac{A'}{A} = \frac{1 - y}{x (x + y - 1)}. \tag{5.13}
\]

Equation (5.13) is the well known Abel’s differential equation of the second type. The two equations in (5.13) are determined in terms of the solutions of Abel’s equation. Abel’s equation has no known analytical solution other than \( y = 0 \), in which case, we have an Abelian magnetic monopole with metric functions given by

\[
A^2 = \left( 1 - \frac{1}{x} \right)^2, \quad B^2 = \frac{1}{A^2} = \left( 1 - \frac{1}{x} \right)^{-2}. \tag{5.14}
\]
This Abelian monopole solution is also an extreme Reissner-Nordstrom black hole with mass $M$ and charge $Q$ given by

$$M = \frac{4\pi G v}{\alpha}, \quad Q = \frac{M}{\alpha v}. \tag{5.15}$$

In the general case, with non-vanishing $y$, we can solve equation (5.12) numerically. Results are shown in Figs 1, 2 and 3.

Figure 1: Plot of family of solutions to Abel’s equation, i.e. $y(x)$, for a range of initial conditions $y(1) = 0, 0.2, 0.4, 0.6, 0.8, 1.0$.

Figure 2: Plot of $A(x)$, $B(x)$ and $y(x)$ against $x$ for an extremal non-Abelian monopole ($y(1) = 0$). This solution is a black hole since we see the event horizon at $x = 1$.

We observe that we have a family of solutions that have exponentially vanishing non-Abelian components $y = W^2$. They are determined by the initial condition $y(1) = y_{init}$ (see Fig. 3). These solutions are characterized by a non-Abelian core, outside of which only an Abelian component remains. The solutions to the
metric components $A$ and $B$ in general have a minimum and maximum respectively in the vicinity of $x = 1$. From eqn. (5.13) the metric component $A$ is dependent on an initial condition, however, once one fixes its asymptotic behavior ($A \to 1$ as $x \to \infty$) it is uniquely determined. The metric component $B$ automatically has the correct asymptotic behavior. We see that as the initial value of $y$ at $x = 1$ approaches zero, the minimum of $A$ also approaches zero at $x = 1$ (and the maximum $B$ at $x = 1$ gets larger). The solution for $y(1) = 0$ (see Fig. 2) represents the extremal case where an event horizon has formed. This extremal solution is a black hole with a non-Abelian magnetic monopole.

For the extremal solution, the non-Abelian magnetic field is confined within the black hole horizon. Thus it is only natural that the metric coefficients outside the horizon are identical to that of a Reissner-Nördstrom black hole. In Fig. 3 we see that the metric coefficient for $A$ vanishes inside the horizon, $x \leq 1$. This is not a problem since $A$ has been normalized at $x = \infty$. An observer at infinity can not observe the interior of the black hole since for him an object takes an infinite amount of time to reach the horizon. If one choses to normalize $A$ at the origin one would have a perfectly well defined metric inside the horizon which is infinite outside. This does not affect the determination of the non-Abelian magnetic field $y$ which is independent of the normalization of $A$. This behaviour is also observed in the monopole solutions of Lue and Weinberg [9].

In Fig. 3 we see a non-extremal monopole solution that is not a black hole.

5.2 Dyonic solutions

The set of seven coupled non-linear equations (5.3)-(5.9) involve four functions, $y$, $J$, $A$ and $B$. We do not expect to find analytic solutions to this set of equations. Thus we are forced to look for numerical solutions.

First, however we can analyse some features of these equations. Equation (5.7) is a quadratic equation in $J^2/A^2$ which has the solutions

$$2 \frac{J^2}{A^2} = - \left( 1 + \frac{1}{x^2} - \frac{A'}{AB} \right) \pm \sqrt{ \left( 1 - \frac{1}{x^2} + \frac{A'}{AB} \right)^2 + \frac{2}{x^2} \frac{A'}{A} + \frac{B'}{B} }.$$  \hspace{1cm} (5.16)
Combining equations (5.3) and (5.16) for $J^2/A^2$, we obtain

$$
\left(1 - \frac{1}{x^2} + \frac{A'}{AB}\right) + \frac{y}{x^2} = \pm \sqrt{\left(1 - \frac{1}{x^2} + \frac{A'}{AB}\right)^2 + \frac{2}{xB^2} \left(\frac{A'}{A} + \frac{B'}{B}\right)}.
$$

(5.17)

By inspection, we see that if $y = 0$, the case of an Abelian dyon, then the above equation implies that

$$\frac{A'}{A} + \frac{B'}{B} = 0,$$

(5.18)

assuming $A, B \neq 0$. This in turn implies that the product $AB$ is constant.

Thus the Yang-Mills field equation (5.5) reduces to

$$J'' + 2J' = 0,$$

(5.19)

which has the analytic solution

$$J(x) = c_1 + c_2/x,$$

(5.20)

where $c_1, c_2$ are arbitrary constants.

Furthermore, if the charge distribution is assumed to vanish asymptotically, then $c_1 = 0$. Together with the asymptotic conditions on the metric functions $A, B$, we can show that the cosmological constant $\Lambda$ must vanish. However, inspection of equation (5.9) leads to a contradiction which proves that there are no Abelian dyonic solutions to our equations (in contrast to the monopole case).

For the general case of non-Abelian dyon solutions with $y \neq 0$, examination of the asymptotic behaviour of equation (5.9) reveals that for consistent solutions the cosmological constant must vanish. Proceeding, we find two coupled equations in $y$ and $J$, whose solutions yield consistent solutions to all the equations. These are

$$y''x^3 - \frac{1}{2}y'^3 \left(x^2 - 2y\right) - \frac{1}{2}y^2 \left(1 - 7y + 2x^3\right) + 2y'yx^2 + 2yx^2 \left(x + y'\right) \left(y' + \frac{yJ'}{J}\right) \frac{y'}{J} = 0,$$

(5.21)

$$
\left(\frac{J'}{J}\right)^4 + \frac{y'}{y} \left(\frac{J'}{J}\right)^3 - \left(\frac{1}{x} + \frac{y'}{2x^2}\right) \left(\frac{y'}{y}\right)^2 \left(\frac{J'}{J}\right)
+ \frac{1}{x^2} \left(\frac{y'}{y}\right)^2 \left[\left(\frac{y'}{y}\right)^2 \left(x^2 - 5y^2 + 2y - 1\right) + \frac{y' \left(1 - 5y\right)}{4x} - \frac{5}{4}\right] = 0.
$$

(5.22)

The solutions to $y$ and $J$ yield the metric functions $A$ and $B$ via the following equations:

$$\frac{A'}{A} = -\frac{1}{2} \frac{y'}{y} \left(1 + \frac{1 - 3y}{x^2}\right) + 2 \frac{y'}{y} \left(\frac{J'}{J} + \frac{1}{2} \frac{y'}{y}\right)^2 + \frac{2}{x},$$

(5.23)

$$B = \frac{-\frac{1}{2} \frac{y'}{y}}{y}.$$

(5.24)

Equation (5.22) is a quartic polynomial in $J'/J$ in terms of $y$ and $y'$. Since $J$ is positive and the requirement that $J$ vanish asymptotically implies that $J'/J$ should be negative everywhere (as long as $J$ is mononic) we select the negative real root of the quartic polynomial. Equation (5.23) is a non-linear second order differential equation in $y$ which can be readily integrated numerically using a 4th order Runge-Kutta with initial conditions set at $x = 1$. 

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Inspection of equations (5.21) and (5.22) reveals that in order for $y$ to vanish, it can only occur at $x = 1$. One can perform a similar inspection of Abel’s equation in the monopole case where it is also evident that $y$ can also only vanish at $x = 1$. Thus, like in the monopole case one expects that the extremal black hole solution will be reach as the initial condition for $y(x = 1) \to 0$. However, unlike the monopole case there is an additional parameter, the initial value of $y'$ at $x = 1$.

Figures 4 and 5 contain plots of the solutions for $y$, $J$, $A$ and $B$ for initial conditions $y(1) = 1$, $y'(1) = -1$, $A(1) = 2$ and $J(1) = 1$. The initial value of $A$ is set in order to obtain it’s correct asymptotic behavior. For these initial conditions the metric function $A$ does not have a minimum and $B$ does not have a maximum at finite $x$. However, as one adjusts the initial conditions closer to the critical value, $y(1) = 0$, a minima and maxima begin to appear. This is evident in figures 6 and 7 where the initial conditions are $y(1) = 0$, $y'(1) = -1$, $A(1) = 0$, $J(1) = 1$. While inspecting the parameter space of initial conditions for $y'(1)$ one finds that $y'(1) \in (-2, 0)$ in order for solutions to exist. For a value of $y'(1)$ in this range one finds that $y(1)$ can not be made arbitrarily close to zero if one requires the numerical solution to be continuously defined for all $x$. The smallest value of $y(1)$ for a well defined solution depends on the value of $y'(1)$. Solutions for values of $y(1)$ closer to zero exhibit numerical singularities in the region $x < 1$. The probable explanation for this is that unlike an extremal monopole which only has only one horizon, a dyonic black hole naturally has two horizons. Thus, as the solution approaches the black hole limit the appearance of one of the horizons might occur first (meaning the metric coefficient $A$ becomes small and $B$ becomes large at some value of $x$). Also, with the existence of two horizons it is clear that our metric (equation (4.1)), which has positive coefficients $A^2$ and $B^2$, can not properly describe the region between two horizons where the space-like and time-like co-ordinates flip. Thus, it is not unreasonable that as our dyonic solution becomes a black hole (by tuning the initial conditions closer to their critical values), that the co-ordinates we used are not appropriate for all values of $x$.

6 Conclusions

We have shown in Section 3 that, with an ansatz, eqn. (3.4), we are led to Bogomol’nyi type equations and an energy functional that is bounded from below in the case of a dyon. In flat space-time, solutions that saturate the Bogomol’nyi bound are known to exist. Demonstrating this in the case of a non-Abelian
dyon in curved space-time is a new result. These results follow from the form of the action in eqn. (2.1), where
the Higgs scalar is directly coupled to the Ricci scalar. In the monopole case, this unconventional coupling
led to a first order classic Abel’s equation, the solution of which yielded solutions to all the equations in the
problem [12]. In the dyon case, the problem turns out to be more complicated; nonetheless, contains similar
simplifications in contrast to the more standard approaches in the literature.

In Section 5, we specialize to Higgs vacuum solutions to the general equations in Section 4. For the
sake of completeness, we begin with a review of the non-Abelian monopole solutions [12]. We expand on
the discussions therein. These monopole solutions depend upon initial conditions; for a specific choice the
solution represents a non-Abelian extremal black hole. In addition we show that in our model there are no
solutions with a non-vanishing cosmological constant in contrast to the conventional model in [10]. We also
present numerical evidence for non-Abelian dyonic solutions. For a range of initial conditions the defining
equations yield stable numerical results which are well-defined to the center of the dyon. However, as the initial
conditions are tuned closer to critical values, the numerical results have singularities that likely correspond to
the formation of horizons. Also as in the monopole case, physically interesting solutions require a vanishing
cosmological constant, and further, unlike in the case of the monopole, there is no Abelian dyon solution in
our model.

Finally, to the best of our knowledge, a direct proof of Dirac charge quantization in the case of the EYMH
system does not exist. However, there are compelling reasons to think, that the topological considerations
in the case of flat space-time can be extended to curved space-time. If so, it has some extremely interesting
consequences that have been pointed out by Ignatev, Joshi and Wali [17]. Among them, are the consequences
that magnetically or electrically charged black holes obey a lower bound on their mass and the consequence
that is not widely appreciated, namely, the spectrum of magnetically (or electrically) charged extremal black
holes are evenly spaced in mass due to charge quantization.

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Figure 6: Plot of $y(x)$ and $J(x)$ against $x$ for a non-Abelian dyon with initial conditions $y(1) = 0.5$, $y'(1) = -1.3$, $A(1) = 0.4$, $J(1) = 1$.

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Figure 7: Plot of $A(x)$ and $B(x)$ against $x$ for a non-Abelian dyon with initial conditions $y(1) = 0.5$, $y'(1) = -1.3$, $A(1) = 0.4$, $J(1) = 1$.

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