THE SUP-NORM PROBLEM IN THE LEVEL ASPECT FOR COCOMPACT
SUBGROUPS FROM DIVISION ALGEBRAS

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Abstract. Let \( D \) be a central division algebra of prime degree \( p \) over \( \mathbb{Q} \). We obtain subconvex bounds in the level aspect for the sup-norm of Hecke-Maass forms on the cocompact quotients of \( \text{SL}_p(\mathbb{R})/\text{SO}(p) \) by unit groups of orders in \( D \). We do so by generalising arguments of Saha and Templier in degree 2. The exponents in the bounds are explicit and polynomial in \( p \).

1. Introduction

1.1. Motivation and historical context. The sup-norm problem arises as a natural question in analysis and quantum physics and has received considerable attention in the number theory community. It is the problem of bounding the \( L^\infty \)-norm of eigenfunctions on Riemannian manifolds in terms of their \( L^2 \)-norm. To make this a reasonable endeavour, one chooses some parameters for the eigenfunctions and estimates the quotient of the two norms while these parameters vary.

For example, let \( X \) be a compact Riemannian manifold of dimension \( n \) and let \( \phi \) be an \( L^2 \)-normalised eigenfunction of the Laplacian \( \Delta_X \) with eigenvalue \( \lambda > 0 \). Motivated by semi-classical analysis, one would like to bound \( \| \phi \|_\infty \) in terms of \( \lambda \) when \( \lambda \to \infty \). In general, local analysis gives the sharp bound

\[
\| \phi \|_\infty \ll \lambda^{(n-1)/4+\varepsilon},
\]

for large enough \( \lambda \). The bound is attained on the round \( n \)-spheres.

If \( \phi \) is assumed to be an eigenfunction for a larger algebra of operators, then we can expect better bounds. Indeed, if \( X = \Gamma \setminus S \) is a locally symmetric space of rank \( r \) and \( G(S) \) is the groups of isometries of the symmetric space \( S \), then one can consider the algebra of \( G(S) \)-invariant differential operators. This algebra is generated by \( r \) operators, including the Laplacian. If \( \phi \) is an \( L^2 \)-normalised joint eigenfunction of these operators, then

\[
(1.1) \quad \| \phi \|_\infty \ll \lambda^{(n-r)/4+\varepsilon}.
\]

For more details on the above paragraphs, see [Sar04].

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\(^1\)Here and in the rest of this article, the implied constants are allowed to depend on \( \varepsilon \).
In special cases when $X$ has arithmetic structure, we expect to obtain better, so-called subconvex bounds, i.e. an exponent $(n - r)/4 - \delta$ in (1.1) with $\delta > 0$. In these cases, there is an additional algebra of commutative normal Hecke operators, which commute with the differential operators above. In this arithmetic setting, the sup-norm problem is to find a subconvex bound for $\|\phi\|_{\infty}$, where $\phi$ is a joint eigenfunction of the invariant differential operators and the Hecke algebra. The prototype of such a result is due to Iwaniec and Sarnak [IS95] in the case of $X = \Gamma \backslash \mathfrak{h}^2$, where $\mathfrak{h}^2$ is the hyperbolic plane and $\Gamma \leq \text{SL}_2(\mathbb{R})$ is a cocompact arithmetic subgroup. They showed that $\|\phi\|_{\infty} \ll \lambda^{1/4 - 1/24 + \varepsilon}$ for an $L^2$-normalised Hecke-Maaß form $\phi$. Another parameter that one could choose is the volume of $X$. This is particularly interesting in the arithmetic case and is reminiscent of the level aspect in the subconvexity problem of $L$-functions (indeed, the two problems are very much related in methodology and numerology). One thoroughly studied example is the family of non-compact spaces $X_0(N) := \Gamma_0(N) \backslash \mathfrak{h}^2$, where $\Gamma_0(N)$ is the Hecke congruence subgroup of level $N$. Note that $\Gamma_0(N) \backslash \mathfrak{h}^2$ has volume $N^{1 + o(1)}$. To isolate the level aspect, we assume that the Laplace eigenvalue $\lambda$ is bounded by some fixed number. If $\phi$ is a Hecke-Maaß newform of level $N$ with eigenvalue $\lambda$ such that

$$\int_{X_0(N)} |\phi|^2 d\mu = 1,$$

where $\mu$ is the invariant measure on $\mathfrak{h}^2$, then a “convexity” bound (borrowing terminology from $L$-functions) would be $\|\phi\|_{\infty} \ll N^\varepsilon$. This was shown for instance in [AU95] for squarefree $N$, where it was used to compare the Arakelov and the Poincaré metrics on $X_0(N)$. A great amount of work was dedicated to achieving subconvex bounds in more and more general settings, for example in [BH10], [Tem10], [TH13], [Sah17], [Ass17]. See the introduction of [HS20] for a more complete set of references with the corresponding bounds.

In general, the level aspect seems to factorise into the case of squarefree level and the case of powerful level, in particular prime powers. The latter is called the depth aspect and is amenable to techniques from $p$-adic analysis that are not available in the squarefree case. To describe an example, let $N_1$ denote the smallest positive integer such that $N_1^2 | N$. Note that $N_1 = N$ if $N$ is squarefree, yet $N_1 \approx \sqrt{N}$ if $N$ is a high prime power. As an example of a subconvex bound in the case of $X_0(N)$ and the interplay between squarefree and powerful levels, it was shown in [Sah17] that

$$\|\phi\|_{\infty} \ll N^{-2/6+\varepsilon} N_1^{1/6}.$$

Closer to the topic of these notes is the case of cocompact surfaces, where the arithmetic subgroup is given by the norm 1 units $O^1$ of an order $O$ in a division quaternion algebra $D$ over $\mathbb{Q}$. To define the level, we choose a fixed maximal order $O_m$ containing $O$ and define $N = [O_m : O]$, which is approximately equal to the volume of $O^1 \backslash \mathfrak{h}^2$ (see Section 2.4). For Eichler orders, the local bound $\|\phi\|_{\infty} \ll N_{1/2+\varepsilon}^{-1/2}$ was shown by Marshall [Mar14]. For squarefree $N$ this corresponds to the bound $N^\varepsilon$, which was first improved by Templier [Tem10], who obtained the bound $N^{-1/24+\varepsilon}$ for general $N$. For general orders, Saha [Sah20] combines these two bounds to $N^{-11/24+\varepsilon}N_1^{5/12}$.

\footnote{In particular, the measure on $O^1 \backslash \mathfrak{h}^2$ is defined analogously to $X_0(N)$.}
The depth aspect was recently improved in [HS20]. If the level is a prime power $p^n$, then Hu and Saha obtain the bound $p^{n(5/24+\epsilon)}$.

The sup-norm problem has also been pushed in the last decade to the case of higher-rank groups, such as $GL(n)$ for $n > 2$. For example, Blomer and Maga [BM16] prove that if $\phi$ is a Hecke-Maaß cusp form for the Hecke congruence group $\Gamma_0(N) \leq SL_n(\mathbb{Z})$, then

$$\|\phi\|_\infty \ll N^\epsilon \lambda^{n-\epsilon/8}(1-\delta_n),$$

for some fixed compact set $\Omega \subset h^n = SL_n(\mathbb{R})/SO(n)$ and effectively computable $\delta_n > 0$.

In higher-rank, the only other bound related to the level aspect (that is available to the author) is a bound in the depth aspect given by Hu [Hu18]. The result is stated only for automorphic forms corresponding to minimal vectors, which seem to be more suitable for the $p$-adic analysis of the depth aspect.

1.2. The main theorem and methods. The purpose of this article is to establish the first non-trivial sup-norm bounds for Hecke-Maaß forms in the level aspect in unbounded rank. More precisely, we show the following.

**Theorem 1.** Let $p$ be a prime and $D$ a central division algebra of degree $p$ over $\mathbb{Q}$. Let $\mathcal{O}_m$ be a fixed maximal order in $D$ and $\mathcal{O} \subset \mathcal{O}_m$ an order of level $[\mathcal{O}_m : \mathcal{O}] = N$. If $\phi$ is an $L^2$-normalised Hecke-Maaß form on $\mathcal{O}^1 \backslash h^n$, then

$$\|\phi\|_\infty \ll N^{-\frac{1}{4p-1}(p^2-1)+\epsilon},$$

where the implied constant depends on $p$, $\mathcal{O}_m$, the spectral parameters $\mu$ of $\phi$, and $\epsilon$.

Note that Theorem 1 recovers the result of Templier for $p = 2$. By looking more carefully at the Hecke algebra, we allow arbitrary, not necessarily Eichler orders in the theorem (see Remark 6).

The reason we only work over $\mathbb{Q}$ is explained in Remark 2. It is the only field relevant to our problem in higher rank, i.e. for $p > 2$.

The argument has the same structure as in Section 6 of [Tem10] and Section 2.4 of [Sah20], which was inspired by the former. In turn, the proof in [Tem10] was inspired by the work of Silberman and Venkatesh [SV16] on quantum unique ergodicity, which has many similarities to the sup-norm problem. As [SV16] treats more generally division algebras of prime degree, it seems only natural that the sup-norm problem argument should extend to this setting, and this is achieved in these notes. In [Sah20], Saha also provides a flexible argument for improving the bound for powerful levels, which should generalise well (see Section 1.6, ibid.), yet this is not the purpose of this article.

As usual in the treatment of the sup-norm problem, the argument starts with an amplified pretrace formula. We embed the form $\phi$ into a basis of Hecke-Maaß forms $(\phi_j)$ for $L^2(\mathcal{O}^1 \backslash h^n)$ and we spectrally expand an automorphic kernel with respect to this basis. This leads to an equality between a weighted sum (the spectral side) of the form

$$\sum_j A_j |\phi_j(z)|^2$$
and a sum over elements in sets constructed from the group $O^1$ (the geometric side).

To choose an amplifier essentially means to find suitable non-negative weights $A_j$ so that the contribution of $\phi$ is large and that of the other forms is little, hopefully negligible. For a chance at obtaining subconvex bounds, the amplifier has to make use of the spectral properties of the forms in the basis, as well as the Hecke operators. Adelicly, this corresponds to choosing appropriate test functions at every place for the pretrace formula. This is a problem in analysis and combinatorics (or real and $p$-adic analysis), and was solved for example in [BM14] for the groups $\text{PGL}_n(\mathbb{R})$. Restricting to unramified places, we are also able to use the amplifier of Blomer and Maga.

After choosing an amplifier, we can then drop all but one terms and obtain a bound for $|\phi_j(z)|^2$ in terms of a sum over certain elements in $O$, determined by the amplifier, which turns into a counting problem. This is where techniques in number theory enter the picture. We count elements $\gamma \in O$ of norms up to some parameter $L$, such that the distance between $z$ and $\gamma z$ is small. This is usually done quite explicitly in the non-compact case of congruence subgroups of $\text{SL}_2(\mathbb{Z})$. In our case, we rely on the very rigid structure of division algebras of prime degree.

More precisely, we may assume that the elements we are counting lie in a proper subalgebra of $D$ at the cost of upper bounds for the parameter $L$. Here we make crucial use of the degree being prime (and nowhere else in an essential way). In this case, the only proper subalgebras of $D$ are fields, where we have better techniques available. In particular, it suffices to count ideals and units with certain conditions in the ring of integers, and the resulting number of elements is extremely small, almost best-possible. Thus, the bound in the theorem is dictated by how large we can take $L$ to be. We note that the scarceness of subalgebras in the prime degree case is also the reason why Silberman and Venkatesh prove their results in this setting (see Section 1.3 in [SV16]).

As already mentioned, the structure of the argument is essentially present in [SV16], [Tem10], [Sah20]. The main difficulty in generalising it is the counting argument for units. In this article, we handle this by bounding the possibilities for the characteristic polynomial of such units. Since we are counting in a (commutative) field, this automatically bounds the number of units. Here again we make use of the fact that proper subalgebras of $D$ must be fields, which is not true any more in composite degree.

We also revisit the case of degree 2 so as to clarify some details in the literature. For the counting argument for units, [Sah20] redirects to [Tem10], where some details seem to be missing. Over $\mathbb{Q}$, this is covered by our general argument mentioned above. Over number fields, we add an application of Dirichlet’s unit theorem for a watertight argument.

Finally, we remark that it is not obvious how the linear index $N = [O_m : O]$ is related to actual parameter of the level aspect, that is the volume of $O^1 \setminus \mathbb{H}$. This is indeed readily available for quaternion algebras, for which there is extensive literature available. We generalise the argument to show that $[O_m^1 : O^1] = N^{1+o(1)}$ in Lemma 3. The author was not able to find a direct reference for this fact, but the calculations are certainly implicitly present in other works on zeta functions of division algebras. Since this fact might be useful in particular in the theory of automorphic forms, we provide the details here to serve as reference.
Notation. We recall the Vinogradov notation $f(x) \ll g(x)$ for two functions $f, g$, meaning that $f(x) \leq C \cdot g(x)$, at least for large enough $x$, for some $C > 0$ called the implied constant. Also, we sometimes work with more general degrees $n \in \mathbb{N}$ for the division algebra $D$, and restrict where necessary to prime degrees $p$.

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2. Division algebras and arithmetic subgroups

Let $D$ be a central division algebra of degree $n$ over $\mathbb{Q}$. Let $O \subset D$ be an order, i.e. a subring with 1 that is a full $\mathbb{Z}$-lattice. Suppose that $D$ splits over $\mathbb{R}$, meaning that there is an embedding $D \hookrightarrow M_n(\mathbb{R})$. For an element $x \in D$, the reduced norm $\text{nr}(x)$ and the reduced trace $\text{tr}(x)$ are given by the determinant and the trace, respectively, of its image under this embedding. The group $O^1 = \{ \gamma \in O : \text{nr}(\gamma) = 1 \}$ now embeds into $\text{SL}_n(\mathbb{R})$ as a cocompact arithmetic lattice (see [Mor15], Proposition 6.8.9). In particular, denoting the symmetric space of $\text{SL}_n(\mathbb{R})$ by $\mathbb{H}^n = \text{SL}_n(\mathbb{R})/\text{SO}(n)$, then the quotient $O^1 \setminus \mathbb{H}^n$ is compact.

Note that for $n$ odd, $D$ splits automatically over $\mathbb{R}$. Indeed, by the Albert-Brauer-Hasse-Noether theorem, all central simple algebras of finite degree over a number field are cyclic, meaning that they contain a strictly maximal subfield that is a Galois extension of $\mathbb{Q}$ of degree $n$ (for background on these statements and the following, see [Pie82], Theorem 18.6 and sections 13.1 through 13.3). The strictly maximal subfield of $D$ splits $D$ and is Galois of odd degree over $\mathbb{Q}$, so it must be totally real, in particular contained in $\mathbb{R}$.

In the special case $n = 2$, $D$ is called a quaternion algebra and we may replace the ground field $\mathbb{Q}$ by any totally real number field $F$. Let $[F : \mathbb{Q}] = n$ and denote by $\mathbb{O}_F$ the ring of integers of $F$. For $D$ to be split over $\mathbb{R}$, we assume that there is an embedding $\sigma_0 \in \text{hom}(F, \mathbb{R})$ such that $D \otimes_{\mathbb{O}_F} \mathbb{R} \cong M_2(\mathbb{R})$. For all other embeddings $\sigma_0 \neq \sigma \in \text{hom}(F, \mathbb{R})$ assume that $D \otimes_{\mathbb{O}_F} \mathbb{R} \cong \mathcal{H}(\mathbb{R})$, where $\mathcal{H}(\mathbb{R})$ is the Hamilton quaternion algebra. We may view $D$ as embedded (diagonally) into $D_{\infty} \cong M_2(\mathbb{R}) \times \mathcal{H}(\mathbb{R})^{n-1}$, and similarly for the norm 1 elements,

$$D^1 \hookrightarrow \text{SL}_2(\mathbb{R}) \times \text{SO}(3)^{n-1}.$$ 

We use $\phi_0$ to denote the projection onto the first component $M_2(\mathbb{R})$ and $\phi_i, i = 1, \ldots, n-1$, to denote the projections onto the Hamiltonian components.

Generalising our setting, let $O$ be an $\mathbb{O}_F$-order. By restriction of scalars, the projection $\phi_0(O^1) \subset \text{SL}_2(\mathbb{R})$ of the group of units of reduced norm 1 onto the split component gives a cocompact arithmetic lattice.

Remark 2. For $\phi_0(O^1)$ to be a cocompact arithmetic lattice in the split component $\text{SL}_2(\mathbb{R})$, it is important that the other components, in this case all isomorphic to $\text{SO}(3)$, are compact (see the definition of an arithmetic group in [Mor15], Definition 5.1.19). If
$D$ is a central division algebra over a number field $F \neq \mathbb{Q}$ and $\deg(D) = n > 2$, this is not possible any more.

Indeed, the process of restriction of scalars requires us to embed $D^1$ into the product of its completions at all infinite places. Now a central simple algebra over $\mathbb{R}$ is isomorphic to a matrix algebra over a division algebra by Wedderburn’s theorem. By a theorem of Frobenius (see [Pie82], Corollary 13.1 c) these are either matrix algebras over $\mathbb{R}$ or over the Hamiltonians $\mathcal{H}$. Since $n > 2$, the group of norm 1 units in these algebras cannot be compact any more. Thus, the number field case is simply not relevant to us in higher degree.

It will be useful later to note that the tower rule holds for division algebras (also called skew fields). More precisely, the notion of vector space over a division algebra and its dimension is the same as for commutative fields. If $D' \subset D$ is a subalgebra, then $D$ may be viewed as a vector space over $D'$, where $D'$ acts by multiplication from the left (or from the right, according to taste). We denote $\dim_{D'} D$ by $[D : D']$.

(2.1) $[D_3 : D_1] = [D_3 : D_2] \cdot [D_2 : D_1]$ holds and is proven as in the commutative case. Thus, if $D$ is a finite dimensional division algebra over $\mathbb{Q}$, then the dimension over $\mathbb{Q}$ of any subalgebra of $D$ must divide $\dim_{\mathbb{Q}} D$. Moreover, if $D$ is central, then any subfield of $D$ must have dimension over $\mathbb{Q}$ dividing the degree of $D$ (see [Pie82], Corollary 13.1 a).

2.1. The volume approximation. Let $O_m$ be a maximal order in $D$ containing $O$. Because of their lattice structure, it is useful to work with the index $[O_m : O]$, which we call the level of $O$ in $O_m$. Yet the volume of $O^1 \setminus \mathfrak{h}^n$, the relevant parameter in our sup-norm problem, is given by the volume of $O_m^1 \setminus \mathfrak{h}^n$ and the multiplicative index $[O_m^1 : O^1]$. Fortunately these two indices are related in an explicit way. For our purposes (and because the exact formulae would involve too many cases in general), it suffices to prove that they are approximately equal. The proper equalities obtained in the proof can be used together with the machinery of zeta functions and Tamagawa numbers to produce a formula for the volume of $O^1 \setminus \mathfrak{h}^n$ (as in [Voi21], 39.2.8), but this is beyond the scope of these notes.

Lemma 3. Let $A$ be a central simple algebra over $\mathbb{Q}$, but not a definite quaternion algebra. Let $O \subset O_m$ be two orders in $A$. If $[O_m : O] = N$, then $[O_m^1 : O^1] = N^{1+o(1)}$.

This lemma is a (approximate) generalisation of the well-know fact that the index of the Hecke congruence group $\Gamma_0(N)$ in $\text{SL}_2(\mathbb{Z})$ is

$$N \prod_{p \mid N} (1 + 1/p).$$

In this case, $A$ is the matrix algebra $M_2(\mathbb{Q})$, the maximal order is $M_2(\mathbb{Z})$ and $O$ is the suborder of level $N$ of integral matrices with lower left entry divisible by $N$.

The proof of Lemma 3 generalises the argument for quaternion algebras in [Voi21], Lemma 26.6.7, which in turn follows an argument of Körner. We provide here full details for the sake of completeness.
The first ingredient is the strong approximation theorem (see Kneser’s article in [BM66]), which allows us to reduce the statement to a local one. We denote by $A_p = A \otimes \mathbb{Q}_p$ and $O_p = O \otimes \mathbb{Z}_p$ the completions at a prime $p$. For all but finitely many primes $p$, the completion $D_p$ is split, i.e. $A_p \cong M_n(O_p)$ (see Proposition 18.5 coupled with Corollary 17.10.a in [Pie82]). Additionally, for all but finitely many primes $p$, the completion $O_p$ is a maximal ideal of $A_p$ (see Lemma 10.4.4 in [Voi21]). In particular, at these primes we have $O_{m,p} = O_p$. The primes where equality does not hold will be referred to as ramified.

We embed $O$ diagonally into $\hat{O} = \prod_p O_p$ and, similarly, $O^1$ into $\hat{O}^1 = \prod_p O^1_p$, where $p$ runs over all prime numbers. Then strong approximation implies that $O^1$ is dense in $\hat{O}^1$ (see [Voi21], Corollary 18.5.14, and more generally [Kle00], Theorem 4.4). Explicitly, if $S$ a set of finite places, $a_p \in O_p$ and $t_p$ for each $p \in S$, then we can find $x \in O^1$ such that

$$x \equiv a_p \pmod{p^t_p O_p} \quad (p \in S).$$

**Lemma 4.** For two orders $O \subset O_m$ as above, the level and the index of the unit groups can be computed locally, that is,

$$[O_m : O] = \prod_p [O_{m,p} : O_p] \quad \text{and} \quad [O^1_m : O^1] = \prod_p [O^1_{m,p} : O^1_p].$$

**Proof.** Note first that the products contain only finitely many factors not equal to 1, as in the remarks above. Next, we start the proof for the unit groups. The claim follows by showing that the map

$$O^1_m/O^1 = \prod_p O^1_{m,p}/O^1_p$$

is bijective.

Injectivity follows by noting that $\bigcap_p O_p = O$. Surjectivity follows by strong approximation. Indeed, let $(a_p) \in \prod_p O^1_{m,p}$. Choose an integer $N$ such that $NO_{m,p} \subset pO_p$ for all ramified primes $p$. Strong approximation supplies us with an element $b \in O^1_m$ such that $b = a_p + NO_{m,p}$. Thus $b = a_p \cdot u_p$, where $u_p \in 1 + NO_{m,p}$, so that $u_p \in O^1_p$.

The proof for the factorisation of the level is similar, where the corresponding strong approximation theorem is the Chinese Remainder Theorem.

In the following we work with the localised orders at a prime $p$, which we suppress in notation for simplicity. We now remove the condition on the norm to work with the full group of units. We have the short exact sequence

$$0 \to O^1 \to O^\times \to \text{nr}(O^\times) \to 0,$$

and similarly for $O_m$. By defining a non-canonical bijection

$$O^1_m/O^1 \times \text{nr}(O^\times_m)/\text{nr}(O^\times) \to O^\times_m/O^\times,$$

we obtain that

$$[O^1_m/O^1] \cdot [\text{nr}(O^\times_m)/\text{nr}(O^\times)] = [O^\times_m/O^\times].$$

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3Note that the groups in question are not abelian and not necessarily normal, so that we cannot apply the snake lemma directly.
Lemma 5. For $\mathbb{Z}_p$ orders $\mathcal{O} \subset \mathcal{O}_m$, we have

$$[\mathcal{O}_m^\times : \mathcal{O}^\times] = [\mathcal{O}_m : \mathcal{O}] \cdot p^{o(1)}.$$  

Proof. The proof starts as in Lemma 26.6.7 in [Voi21]. Let $n$ be such that $p^n \mathcal{O}_m \subset p \mathcal{O}$. Note that $1 + p \mathcal{O} \subset \mathcal{O}^\times$ by the convergence of the geometric series. We now have

$$[\mathcal{O}_m^\times : \mathcal{O}^\times] = \frac{[\mathcal{O}_m^\times : 1 + p \mathcal{O}_m] \cdot [1 + p \mathcal{O}_m : 1 + p^n \mathcal{O}_m]}{[\mathcal{O}^\times : 1 + p \mathcal{O}] \cdot [1 + p \mathcal{O} : 1 + p^n \mathcal{O}]}.$$  

For $\alpha, \beta \in 1 + p \mathcal{O}$, we have $\alpha \beta^{-1} \in 1 + p^n \mathcal{O}_m$ if and only if $\alpha - \beta \in p^n \mathcal{O}_m$, so that

$$[1 + p \mathcal{O} : 1 + p^n \mathcal{O}_m] = [p \mathcal{O} : p^n \mathcal{O}_m] = [\mathcal{O} : p^{n-1} \mathcal{O}_m],$$  

and similarly for $\mathcal{O}_m$. If follows that

$$\frac{[1 + p \mathcal{O}_m : 1 + p^n \mathcal{O}_m]}{[1 + p \mathcal{O} : 1 + p^n \mathcal{O}_m]} = \frac{[\mathcal{O}_m : p^{n-1} \mathcal{O}_m]}{[\mathcal{O} : p^{n-1} \mathcal{O}_m]} = [\mathcal{O}_m : \mathcal{O}].$$  

To further compute the factors $[\mathcal{O}^\times : 1 + p \mathcal{O}]$, we employ the strategy in [Voi21], Lemma 24.3.12, of introducing the Jacobson radical $\text{rad} \mathcal{O} =: J$. We have $p \mathcal{O} \subset J$ and there is an integer $r$ such that $J^r \subset p \mathcal{O}$ (see [Rei03], Theorem 6.13), which we assume to be minimal. Thus $1 + J \subset \mathcal{O}^\times$ and we obtain a filtration

$$\mathcal{O}^\times \supset 1 + J \supset 1 + J^2 \supset \ldots \supset 1 + p \mathcal{O} \supset 1 + J^r.$$  

Being kernels, all subgroups are normal inside their parent groups. It follows that

$$[\mathcal{O}^\times : 1 + p \mathcal{O}] = [\mathcal{O}^\times / 1 + J : 1 + J / 1 + J^2 \ldots / 1 + J^{r-1} / 1 + p \mathcal{O}].$$  

On the additive side, we also have a filtration $\mathcal{O} \supset J \supset \ldots \supset p \mathcal{O}$ and the quotients $\mathcal{O} / J, J / J^2, \ldots, J^{r-1} / p \mathcal{O}$ are $\mathbb{F}_p$-algebras. If $R$ is the rank of $\mathcal{O}$, then

$$p^R = [\mathcal{O} : p \mathcal{O}] = [\mathcal{O} : J][J : J^2] \ldots [J^{r-1} : p \mathcal{O}].$$  

We now reduce the multiplicative indices to the additive ones. Indeed $1 + J / 1 + J^2 \cong J / J^2$, and similarly for all powers of $J$, and $1 + J^{r-1} / 1 + p \mathcal{O} \cong J^{r-1} / p \mathcal{O}$, since $J^{2^{(r-1)}} \subset p \mathcal{O}$ (at least for $r > 1$; the case $r = 1$ is simpler and can be done directly). Therefore,

$$[1 + J / 1 + J^2] = [J / J^2], \ldots, [1 + J^{r-1} / 1 + p \mathcal{O}] = [J^{r-1} / p \mathcal{O}].$$  

Thus,

$$[\mathcal{O}^\times : 1 + p \mathcal{O}] = p^R [\mathcal{O}^\times / 1 + J].$$  

Now one can easily see that $\mathcal{O}^\times / 1 + J \cong (\mathcal{O} / J)^\times$. The reason for working with the Jacobson radical is that $\mathcal{O} / J$ is a semisimple $\mathbb{F}_p$-algebra, meaning that

$$\mathcal{O} / J \cong M_{d_1}(D_1) \times \cdots \times M_{d_t}(D_t),$$  

for some finite division algebras $D_i$ over $\mathbb{F}_p$. Since finite division algebras are fields by Wedderburn’s theorem, one can check by counting that $|GL_{d_i}(D_i)| = |M_{d_i}(D_i)|^{1 - o(1)}$ (this can be made precise, but the approximation is sufficient for our purposes).

Since $\mathcal{O}$ and $\mathcal{O}_m$ have the same rank, it follows that

$$\frac{[\mathcal{O}_m^\times : 1 + p \mathcal{O}_m]}{[\mathcal{O}^\times : 1 + p \mathcal{O}] = p^{o(1)}.}$$
This finishes the proof. □

For the groups of norms, we note that $\mathbb{Z}_p \times \mathbb{Z}_p \subset O \times \mathbb{Z}_p$, and so $(\mathbb{Z}_p \times \mathbb{Z}_p)^n \leq n r(O \times \mathbb{Z}_p) \leq \mathbb{Z}_p$. Now $[\mathbb{Z}_p : \mathbb{Z}_p]^n \ll n$ by Korollar 5.8 in [Neu92]. This contributes to the global index by $n^{\omega(N)} \ll n d(N) \ll N^\varepsilon$, where $\omega(N)$ is the number of different primes dividing the level $N$, $d(N)$ is the number of divisors of $N$, and $\varepsilon$ is any positive real number. This completes the proof of Lemma 3.

3. The amplified pretrace formula

The space of automorphic forms $L^2(O^1 \backslash G)$ has a discrete decomposition, admitting a basis of Hecke-Maaß forms $(\phi_j)_{j \in \mathbb{N}}$, that is, eigenfunctions of the algebra of invariant differential operators and of the Hecke algebra (described below). Denote the spectral parameters of each form $\phi_j$ by $\mu_j$.

For a function $f \in C^\infty_c(K \backslash G/K)$, the pretrace formula states that

$$\sum_{j \in \mathbb{N}} \hat{f}(\mu_j) \phi_j(z) \phi_j(z') = \sum_{\gamma \in O^1} f(z^{-1} \gamma z'),$$

for $z, z' \in G$, where $\hat{f}$ is the spherical transform of $f$.

Suppose $\phi$ is a form in our basis with spectral parameter $\mu$. Blomer and Maga (e.g. see [BM16], Section 3) show that we can find $f_\mu \in C^\infty_c(K \backslash G/K)$ such that the spherical transform $\hat{f}_\mu$ satisfies $\hat{f}_\mu(\lambda) \geq 0$ for all possible spectral parameters $\lambda$ and $\hat{f}_\mu(\mu) \geq 1$. In fact, we can also assume certain decay properties of $f_\mu$, but since in this paper we isolate the level aspect, it suffices to note that $f_\mu \ll 1$, where the implied constant depends continuously on $\mu$.

To amplify the contribution of $\phi$ in the pretrace formula, Blomer and Maga also constructed a general amplifier using Hecke operators (see [BM14], Section 6) for $SL_n(\mathbb{Z})$. This amplifier applies in our situation as well, as long as we only use unramified places. To be precise, we recall some facts about the Hecke algebra, for which we assume that our ground field is $\mathbb{Q}$.

First, we define the group $U_O$ as

$$U_O = GL_n^+(\mathbb{R}) \times \prod_p O_p^\times$$

Note that

$$O^1 = U_O \cap D^\times.$$

Next, let $d_O$ be the product of all primes $p$ at which $D$ ramifies or $O_p$ is not maximal. Thus, $d_O$ is a product of a number depending only on $D$, and primes dividing $N$, as in Lemma 4. We define the semigroup $S_O$ inside the adelisation $A_\mathbb{A}^n$ by

$$S_O = \left( GL_n^+(\mathbb{R}) \times \prod_p S_p \right) \cap A_\mathbb{A}^n,$$

where $S_p = \{ a \in O_p : nr(a) \neq 0 \}$ for $p \mid d_O$ and $S_p = O_p^\times$ for $p \nmid d_O$. This distinction means that we only consider the unramified Hecke algebra. Finally, let

$$\Delta_O = D_O \cap O.$$
We can now define the classical Hecke algebra \( R(O_1, \Delta_O) \), which is generated by double cosets of the form \( O_1 \xi O_1 \), where \( \xi \in \Delta_O \), and similarly the adelic Hecke algebra \( R(U_O, S_O) \). For more details, see [Miy89], Sections 2.7 and 5.3.

The adelic point of view is advantageous since we automatically obtain a factorisation of \( R(U_O, S_O) \) as the tensor product \( \bigotimes_p R(O_p^\times, S_p) \) of the local Hecke algebras. Fortunately in our case, there is essentially nothing lost in translation between the classical and the adelic Hecke algebra. Indeed, they are isomorphic under the simple correspondence \( O_1 \xi O_1 \mapsto U_O \xi U_O \). This can be seen by carefully applying the argument in the proof of Theorem 5.3.5 in [Miy89]. The proof makes crucial use of approximation theorems.

Remark 6. When generalising the argument in [Miy89], it is crucial that we only consider the unramified Hecke algebra, i.e. the local Hecke algebras \( R(O_p^\times, S_p) \) are trivial for \( p \mid d_O \). At unramified primes \( p \), the local order \( O_p \) is maximal. This condition could be relaxed to only asking the norm of \( O_p \) to be surjective onto \( \mathbb{Z}_p \). If this is true at all primes, then \( O \) is called locally norm maximal. Examples of such orders are Eichler orders. This property implies that the idelic quotient defined by \( O \) has only one connected component. In particular, the dictionary between classical automorphic forms and adelic forms is simpler.

These are, perhaps, a few “cosmetic” reasons why several works in the literature only consider orders of class number one (see for instance the use of Eichler orders in [Tem10], and [SV16], Remark 6.3.1). From the point of view of the Hecke algebra (in our case, the only place we make use of the adelic language), treating all orders on equal footing is just a matter of staying away from the ramified primes.

Now at unramified primes \( p \), the local Hecke algebra \( R(O_p^\times, S_p) \) is isomorphic to the Hecke algebra of \( GL_n(\mathbb{Z}_p) \). Therefore, we can use the same Hecke operators as Blomer and Maga. Note that the determinantal divisors in [BM14] do not translate into our setting, yet the norm does, as one can easily check using the explicit isomorphism between the classical and adelic Hecke algebras above.

For \( m \in \mathbb{Z} \), let

\[
O(m) := \{ \gamma \in O \mid \text{nr}(\gamma) = m \}.
\]

If \( L > 5 \) is a parameter, let \( P \) be the set of primes in \([L, 2L]\) that are unramified. We have the pretrace inequality (see [BM16], (2.5))

\[
(3.1) \quad |P|^2 \cdot |\phi(z)|^2 \ll_{\mu} |P| + \sum_{\nu=1}^{n} \sum_{l_1, l_2 \in P} \frac{1}{L^{(n-1)\nu}} \sum_{\gamma \in O(l_1^{(n-1)\nu})} |f_{\mu}(z^{-1}\gamma z)|,
\]

where \( \tilde{\gamma} = \gamma / \text{nr}(\gamma)^{1/n} \in SL_n(\mathbb{R}) \).

Since \( f_{\mu} \) has compact support and \( f_{\mu} \ll_\mu 1 \), to obtain an explicit bound from the pretrace inequality we must count the number of elements \( \gamma \in O(m) \) such that \( d(z, \tilde{\gamma}z) \ll 1 \), where \( d \) is the invariant distance function on \( \mathbb{H}^n \). Because we are interested in the level aspect, we need to count uniformly in the level \( N \). To deduce anything about the sup-norm of \( \phi \), the counting must also be done uniformly in \( z \), at least in a fundamental domain for \( O^1 \). Though compact, this fundamental domain grows with the level. As in [Sah20], we resolve this uniformity problem as follows.
Choose a maximal order $\mathcal{O}_m$ containing $\mathcal{O}$, and fix a fundamental domain $\mathcal{J} \subset \mathfrak{h}^n$ for the action of $\mathcal{O}_m$. If $(g_i)$ is a system of coset representatives for $\mathcal{O}_m^1/\mathcal{O}^1$, then the set $\bigcup g_i^{-1} \mathcal{J}$ is a fundamental domain for $\mathcal{O}^1$. Let $z_0 \in \mathcal{J}$ and note that
\[
\{ \gamma \in \mathcal{O}(m) \mid d(g_i^{-1} z_0, \gamma g_i^{-1} z_0) < \delta \} = \{ \gamma \in \mathcal{O}(m) \mid d(z_0, \gamma g_i^{-1} z_0) < \delta \}
\]
where $\mathcal{O}' = g_i \mathcal{O} g_i^{-1} \subset \mathcal{O}_m$.

Thus, we may always assume that $z \in \mathcal{J}$ at the cost of conjugating the order. Crucially, conjugation preserves the level and the counting argument only depends on $\mathcal{O}_m$ and the level, as we shall see in the next section.

For the following counting result assume that $n = p$, a prime.

**Proposition 7.** Let $m \ll N^{2(p-1)(p-1)}$ with implicit constant as in Lemma 9 and let $z \in \mathcal{J}$. Then $\# \mathcal{O}(m,z,\delta) \ll_p m^r$, where the implicit constant depends only on $\varepsilon$, $\delta$, $z$, and $p$.

We postpone the proof to the next section and apply the proposition to the pretrace inequality. Assume that $L p^2 \ll N^{2(p-1)(p-1)}$ and that $z \in \mathcal{J}$ as above. Using Proposition 7, the inequality (3.1) reduces to
\[
|\mathcal{P}| \cdot |\phi(z)|^2 \ll_{\mu, \varepsilon} |\mathcal{P}| \cdot L^\varepsilon.
\]

By the prime number theorem, $L^{1-\varepsilon} \ll |\mathcal{P}|$, at least for $L$ large enough, so that the ramified primes we leave out are negligible (there are only $\ll N^\varepsilon$ such primes). It follows that $|\phi(z)|^2 \ll L^{1+\varepsilon}$. Taking $L$ as large as possible, that is, $L \gg N^{2(p-1)(p-1)}$, and putting together the remarks on uniformity in $z$ and the implied constants explained in the next section, we arrive at Theorem 1.

4. **The counting argument**

4.1. **The counting argument over the rational numbers.** From now on, assume that $\deg(D) = p \geq 2$, a prime. Recall that $\mathcal{O} \subset \mathcal{O}_m$ are orders in $D$, the latter a maximal one, and we fix a compact fundamental domain $\mathcal{J}$ for the action of $\mathcal{O}_m^1$ on $\mathfrak{h}^n$. Let $\delta$ be a positive real number, which will later be related to a bound for the radius of the compact support of $f_\mu$.

We are interested in bounding the cardinality of
\[
\mathcal{O}(m; z, \delta) = \{ \gamma \in \mathcal{O} : \text{nr}(\gamma) = m, d(z, \gamma z) = O(\delta) \}.
\]
As in Section 2.4 of [Sah20], we can obtain slightly better bounds by working with the submodule of traceless elements of $\mathcal{O}$, that is
\[
\mathcal{O}_0 = \{ \gamma \in \mathcal{O} : \text{tr} \gamma = 0 \}.
\]
We have that $\mathbb{Z} \oplus \mathcal{O}_0 \subset \mathcal{O}$ and $[\mathcal{O} : \mathcal{O}_0 \oplus \mathbb{Z}] \leq p$. To see the latter, note that for any $\gamma \in \mathcal{O}$ we have $\text{tr} \gamma \in \mathbb{Z}$, and it is clear that $\gamma \in \mathbb{Z} \oplus \mathcal{O}_0$ is equivalent to $\text{tr} \gamma \in p\mathbb{Z}$. Therefore, $\mathcal{O}/\mathcal{O}_0 \oplus \mathbb{Z}$ is a subgroup of $\mathbb{Z}/p\mathbb{Z}$.

Since $\mathcal{O}_0$ is a $\mathbb{Z}$-submodule of $\mathcal{O}$, by the above it follows that $\mathcal{O}_0$ has rank $p^2 - 1 =: P$. We can apply the same reasoning to the maximal order, so that $\mathcal{O}_{m,0}$ is also $\mathbb{Z}$-module
of rank $P$. Since $O_0 \subset O_{m,0}$, by elementary divisor theory we can find a basis $(b_j)$, of $P$ elements of $O_{m,0}$, and positive integers $M_1 | M_2 | \ldots | M_P$, such that
\begin{equation}
O_{m,0} = \mathbb{Z}b_1 \oplus \cdots \oplus \mathbb{Z}b_P \quad \text{and} \quad O_0 = M_1 \mathbb{Z}b_1 \oplus \cdots \oplus M_P \mathbb{Z}b_P.
\end{equation}
Recall that we call $N = [O_m : O]$ the level of $O$. By the previous discussion, we have that $M_1 \cdots M_P \asymp_p N$.

To keep bounds uniform in the choice of $O$ (so implicitly uniform in the level), we choose a fixed basis for $O_{m,0}$, say $(i_1, \ldots, i_P)$. Thus
\begin{equation}
O_{m,0} = Z i_1 \oplus \cdots \oplus Z i_p.
\end{equation}

To compare the two bases, we can find a matrix $\nu \in GL_p(\mathbb{Z})$ for the change of basis from $(i_1, \ldots, i_p)$ to $(b_1, \ldots, b_P)$.

To motivate the following lemmata, we recall the tower rule \((2.1)\) for division algebras. Especially for prime degree $p$, this severely restricts the possible dimensions of subalgebras in $D$. If one can show that the subalgebra generated by the elements we are counting is proper, then the tower rule drastically reduces the dimension of the counting problem, automatically. In our case, the subalgebra will actually be commutative, which is crucial in our argument. To show properness in the first place, we use a version of the determinant method, for which we need good control over a basis of a vector space.

**Lemma 8.** The $\mathbb{Q}$-algebra generated by $\bigcup_{1 \leq m \leq L} O(m; z, \delta)$ is contained in the $\mathbb{Q}$-vector space spanned by $\bigcup_{1 \leq m \leq L^{2p-2}} O(m; z, (2p - 2)\delta)$.

**Proof.** By the tower rule, a subalgebra of $D$ is of the form $\mathbb{Q}$, $\mathbb{Q}(x)$, or $\mathbb{Q}(x, y)$, where $x, y \in D$. The algebra $\mathbb{Q}(x, y)$ is generated as a vector space by monomials of degree at most $2p - 2$.

Now if $a_j \in \bigcup_{1 \leq m \leq L} O(m; z, \delta)$ for $j = 1, \ldots, 2p - 2$, then the reduced norm of $\prod a_j$ is at most $L^{2p-2}$ and, by the triangle inequality, $d(z, \prod a_j \cdot z) \ll (2p - 2)\delta$. The order structure ensures that $\prod a_j$ lies in $O$. \qed

**Lemma 9.** The $\mathbb{Q}$-vector space spanned by $\bigcup_{1 \leq m \leq L^{2p-2}} O(m; z, (2p - 2)\delta)$ is proper, i.e. not equal to $D$, if $L \ll N^{\frac{2p-1}{2p-2}}$, where the implicit constant depends only on $p$, $J$, $\delta$, and the maximal order $O^\max$.

**Proof.** Let $\alpha_1, \ldots, \alpha_p \in \bigcup_{1 \leq m \leq L^{2p-2}} O(m; z, (2p - 2)\delta)$. It suffices to show that $1, \alpha_1, \ldots, \alpha_p$ are linearly dependent over $\mathbb{Q}$. Multiplying these elements by $p$ if necessary, we may assume $\alpha_1, \ldots, \alpha_p \in \bigcup_{1 \leq m \leq p^2 L^{2p-2}} O'(m; z, p(2p - 2)\delta)$, where $O' = \mathbb{Z} \oplus O_0$. By our choice of basis above, we can write
\[ \alpha_j = a_0^{(j)} + a_1^{(j)} i_1 + \cdots + a_p^{(j)} i_p, \]
with $a_i^{(j)} \in \mathbb{Z}$. The conclusion follows by proving that $\det(A) = 0$ for $A = (a_i^{(j)})_{i,j=1,\ldots,p}$. Changing basis to $(b_1, \ldots, b_P)$ and recalling \((4.1)\), we deduce that the $j$-th row of $V \cdot A$ is divisible by $M_j$. Since $\det(V) = \pm 1$, it follows that $M_1 \cdots M_P | \det(A)$ and since $M_1 \cdots M_P \asymp N$, the level of $O$, we have
\[ N \ll \det(A). \]
On the other hand, the set
\[ \Omega_{J,\delta} = \{ \gamma \in \text{SL}_p(\mathbb{R}) : d(z, \gamma z) \leq \delta \text{ for all } z \in J \} \]
is compact, since \( J \) is compact and the stabilisers of all points are compact (as conjugates of \( \text{SO}(p) \)). If \( \alpha = a_0 + a_1 \tilde{z}_1 + \cdots + a_p \tilde{z}_p \) has norm \( m \) and satisfies \( d(z, \tilde{\alpha} z) \leq \delta \), then \( \alpha/m^{1/p} \in \Omega_{J,\delta} \). By compactness, there is an implicit constant only depending on the basis \((\tilde{z}_1, \ldots, \tilde{z}_p)\) and on \( \delta \) and \( J \) such that \( a_j \ll m^{1/p} \) for all \( j \).

Therefore, \( a_i^{(j)} \ll L^{(2p-2)/p} \) and so
\[
\det(A) \ll L^{(2p-2)/p}.
\]

This finishes the proof. \( \square \)

Thus, if \( L \) is small enough, we can assume that we are counting matrices in a proper subalgebra of \( D \), which must be \( \mathbb{Q} \) or a field extension \( E/\mathbb{Q} \) of degree \( p \). This is where the use of \( p \) being a prime is crucial.

We are now counting certain elements in \( \mathcal{O}_E \), the ring of integers of \( E \), which certainly includes \( \mathcal{O} \cap E \). We do so by counting ideals and units. Since the units in \( \mathbb{Z} \) are only \( \pm 1 \), we can concentrate on the non-trivial extensions, which must have an infinite group of units, at least if \( p > 2 \). It is important to note that the reduced norm and the reduced trace in \( D \) of an element in a subfield \( E \subset D \) such that \( [E : \mathbb{Q}] = p \) are the same as the number field norm, resp. trace of \( E/\mathbb{Q} \) (see [Pie82, Sect. 16.2]).

**Lemma 10.** Let \( E/\mathbb{Q} \) be a cyclic extension of degree \( p \) that is a subfield of \( D \) and let \( \mathcal{O} \) be an order of \( D \). The number of units \( \xi \in \mathcal{O}^\times \cap E \) such that \( d(z, \xi z) \leq \delta \) for a fixed \( z \in \mathbb{B}^p \) is \( \ll p^p (1 + \delta)^{p-1} \).

**Proof.** Let \( \xi \in \mathcal{O}^\times \cap E \). Then \( \xi \in \mathcal{O}_E^\times \), where \( \mathcal{O}_E \) is the ring of integers of \( L \), by integrality over \( \mathbb{Z} \). Thus, \( \text{nr}(\xi) = N_{E/\mathbb{Q}}(\xi) = \pm 1 \).

Next, the condition \( d(z, \xi z) \leq \delta \) is equivalent to \( \xi \in z B(\delta) z^{-1} \), where \( B(\delta) \) is a union of \( \delta \)-balls around all elements of \( \text{SO}(p) \). Applying the trace, we find that \( \text{tr}(\xi) \ll p(1 + \delta) \).

Since \( \text{tr}(\xi) \in \mathbb{Z} \) by integrality, we see that there are \( \ll p(1 + \delta) \) possibilities for the value of \( \text{tr}(\xi) \).

We may apply the same reasoning to \( \xi^j \) and derive that there are \( \ll p(1 + j\delta) \) possibilities for the value of \( \text{tr}(\xi^j) \). Indeed, \( d(z, \xi^j z) \leq d(z, \xi z) + d(\xi z, \xi^j z) = d(z, \xi z) + d(z, \xi^{j-1} z) \leq j \delta \), inductively. Note also that \( \xi^j \in \mathcal{O}^\times \cap E \) since \( \mathcal{O} \) is closed under multiplication.

Now the characteristic polynomial of \( \xi \) is
\[
X^p - \text{tr}(\xi) X^{p-1} + \frac{1}{2} [\text{tr}(\xi)^2 + \text{tr}(\xi^2)] X^{p-2} + \ldots \pm \text{det}(\xi).
\]

By Newton’s identities, each coefficient is determined by the values of \( \text{tr}(\xi^j) \) for certain \( j \). By the bounds above, there are only \( \ll \prod_{j=1,\ldots,p-1} p(1 + j\delta) \ll [p(1 + \delta)]^{p-1} \) polynomials that are satisfied by a unit \( \xi \) as in the statement of the lemma. Since each polynomial can have at most \( p \) different roots, the proof is finished. \( \square \)
Lemma 11. Let $E \subset D$ be a field of degree $p$ over $\mathbb{Q}$. Then for a fixed $z \in b^p$ and any positive integer $m$ we have
\[ O_L(m; z, \delta) \ll_p \tau(m)^{p-1} \cdot (1 + \delta)^{p-1}. \]

Proof. Let $\gamma \in O_E$ with $\text{nr}(\gamma) = N_{E/Q}(\gamma) = m$. Up to units, there are only $\tau(m)^{p-1}$ elements of $O_E$ with norm $m$. Indeed, a principal ideal is determined by its generator up to units and the norm of the ideal is equal to the norm of the generator. Since ideals factorise uniquely into prime factors, we only need to count prime ideals.

Above each rational prime, there are at most $p$ prime ideals of $O_E$. Therefore, if $q^\nu \mid m$ for a prime $q$, then we need to choose at most $p$ numbers $a_1, \ldots, a_p \in \mathbb{Z}_{\geq 0}$ such that $a_1 + \ldots + a_p = v$ to determine an ideal of norm $q^v$. The number of such tuples is $v + p - 1$ choose $p - 1$, that is $\ll v^{p-1}$. Thus, there are at most
\[ \ll \prod_{q' \mid m} v^{p-1} = \tau(m)^{p-1} \]
ideals of norm $m$.

Now if $\gamma \in O_E(m; z, \delta)$ and $\xi \gamma \in O_E(m; z, \delta)$ for some unit $\xi \in O_E^\times$, then
\[ d(z, \xi z) \leq d(z, \xi \gamma z) + d(\xi \gamma z, \xi z) = d(z, \xi \gamma z) + u(z, \gamma z) \leq 2\delta. \]
Thus, we finish the proof by counting such units using Lemma 11. \qed

4.2. Revisiting the counting argument for degree 2 over number fields. The case of quaternion algebras has been dealt with in [Sah20] over $\mathbb{Q}$. Yet, in this case, the full classification of cocompact arithmetic subgroups requires us to consider quaternion algebras over number fields. Templier [Tem10] treats the counting problem in this more general setting (though only for Eichler orders), yet some details seem to be missing. This section is meant to clarify the argument, using the same techniques as for degree $p > 2$. We first recall the theoretical background.

Let $F$ be a totally real number field of degree $n$. We denote by $O_F$ its ring of integers. Let $D$ be a division quaternion algebra over $F$ and assume that there is an embedding $\sigma_0 \in \text{hom}(F, \mathbb{R})$ such that $D \otimes_{\sigma_0} \mathbb{R} \cong M_2(\mathbb{R})$. For all other embeddings $\sigma_0 \neq \sigma \in \text{hom}(F, \mathbb{R})$ assume that $D \otimes_{\sigma} \mathbb{R} \cong \mathcal{H}(\mathbb{R})$, where $\mathcal{H}(\mathbb{R})$ is the Hamilton quaternion algebra. Now let $O \subset O_{\text{max}}$ be an $O_F$-order in a fixed maximal order.

We may view $D$ as embedded in $D_{\text{max}} \cong M_2(\mathbb{R}) \times \mathcal{H}(\mathbb{R})^{n-1}$. We use $\phi_0$ to denote the projection onto the first component $M_2(\mathbb{R})$ and $\phi_i$, $i = 1, \ldots, n - 1$, to denote the projections onto the Hamiltonian components.

Note that $\sigma_i(\text{tr}_D(\gamma)) = \text{tr}(\phi_i(\gamma))$ for $\phi_i$, the projection onto the $\sigma_i$ component. The trace on the right hand side refers to the matrix trace, respectively to the quaternion trace. Similarly, $\sigma_i(\text{nr}_D(\gamma)) = \text{nr}(\phi_i(\gamma))$.

Hecke operators are associated to principal ideals of $O_F$ and are explained in Section 5.5 of [Tem10]. We may now follow Templier’s argument, which is essentially the same as the degree $p > 2$ argument up to Section 6.3, ibid. Recall that we can reduce the counting problem to counting $\gamma \in O \cap E$, where $F \subset E$ is a quadratic extension of $F$, in particular a field.
Now Lemma 11 goes through with the same proof if we can control the action of units. This cannot be done directly as in Lemma 10 since \( N_{E/F}(\xi) \in O_E^\times \) for any unit \( \xi \in O_E^\times \), and the group \( O_F^\times \) is infinite for \( F \neq \mathbb{Q} \). We can balance this out by noting that we only need to count \( \xi \) up to units in \( O_F^\times \), since these act trivially on our locally symmetric space.

**Lemma 12 (Lemma 10 revisited).** Let \( E/F \) be an extension of degree 2 that is a subfield of \( D \) and let \( O \) be an order of \( D \). The number of units \( \xi \in O_F^\times \cap E \) up to multiplication by units in \( O_F^\times \), such that \( u(z, \phi_0(\xi)z) \leq \delta \) for a fixed \( z \in \mathbb{H} \), is \( \ll_{z,F} (1 + \delta)^2 \).

**Proof.** We begin by investigating the quantity \( (tr_{E/F}(\xi))^2/ N_{E/F}(\xi) \) and proving that it can only take finitely many values. For any embedding \( \sigma \neq \sigma_0 \), we have

\[
\sigma \left( \frac{(tr_{E/F}(\xi))^2}{N_{E/F}(\xi)} \right) \in [0,4].
\]

Indeed, for the corresponding projections \( \phi_i, i \neq 0, \phi_i(\xi) \) is an element of the real Hamilton quaternion algebra. For an arbitrary such element \( a+ib+jc+kd \) in the usual notation with \( a,b,c,d \in \mathbb{R} \), its trace is equal to \( 2a \) and its norm is \( a^2+b^2+c^2+d^2 \), whence the inequality.

Next, the condition \( u(z, \phi_0(\xi)z) \leq \delta \) is equivalent to

\[
\frac{\phi_0(\xi)}{N_D(\xi)^{1/2}} \in zB(\delta)z^{-1},
\]

as in the proof of Lemma 10 noting that \( N(\xi)^{1/2} \) may not be trivial in our case. This now implies that

\[
\sigma_0 \left( \frac{(tr_{E/F}(\xi))^2}{N_{E/F}(\xi)} \right) \ll_{z,F} (1 + \delta)^2.
\]

Since \( \xi \in O_E^\times \), the maximal order, the quantity \( (tr_{E/F}(\xi))^2/ N_{E/F}(\xi) \) must lie in \( O_F \). Recall that the image of \( O_F \) inside \( \mathbb{R}^n \) under all embeddings is a discrete lattice. Since the image of \( (tr_{E/F}(\xi))^2/ N_{E/F}(\xi) \) is bounded, it follows that the number of possibilities for the value of this quantity is bounded by \( (1 + \delta)^2 \), up to a constant depending on \( z \) and \( F \).

For the last step, recall Dirichlet’s unit theorem, stating that \( O_F^\times \) is a finitely generated group. This implies that \( O_F^\times/(O_F^\times)^2 \) is finite. Now if \( \kappa \in O_F^\times \), then \( N(\kappa \xi) = \kappa^2 N(\xi) \). Thus, if we are only counting \( \xi \in O_F^\times \cap O_F^\times \), then the value of \( N(\xi) \) can only lie in \( O_F^\times/(O_F^\times)^2 \).

Nevertheless, we have

\[
\frac{(tr_{E/F}(\kappa \xi))^2}{N_{E/F}(\kappa \xi)} = \frac{(tr_{E/F}(\xi))^2}{N_{E/F}(\xi)}.
\]

Since there are only finitely many possibilities for this quantity and finitely many for \( N_{E/F}(\xi) \), if follows that there are only finitely many possibilities for \( tr_{E/F}(\xi) \). Finally, \( \xi \) is determined up to the action of \( \text{Gal}(E/F) \) (which has order 2) by its minimal polynomial. This polynomial is determined by the trace and norm of \( \xi \). The lemma now follows by bookkeeping.

With this lemma, we can complete the proof of Proposition 6.5 in [Tem10].
Remark 13. We remark that Lemma 6.4, part (i), of [Tem10] might not hold in general. Indeed, in the proof, the condition $u(z, \phi_0(\xi)z) \leq \delta$ is said to be equivalent to
\[
\frac{\phi_0(\xi)}{\sigma_0(N_D(\xi))^{1/2}} \in zB(\delta)z^{-1},
\]
for a single $\delta$-ball $B(\delta)$ around the identity. This cannot be true in general. For instance if $z = i$, corresponding to the identity matrix, and $\xi = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}$, then $u(z, \phi_0(\xi)z) = 0$, yet $\xi$ lies far from the identity.

Our proof uses the idea of Lemma 6.3 in [Tem10] and couples it with an application of Dirichlet’s unit theorem, so as to apply the argument using characteristic polynomials, as in the higher degree case.

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