On multiplicative renormalizability of Yang-Mills theory with the background field method in the BV-formalism

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Abstract

In the study of gauge-dependencies and multiplicative renormalizability of 4D pure Yang-Mills theory using the background field method and the BV-formalism we derive a classical master-equation by introducing antifield partners to the background fields and parameters. The classical master-equation is homogeneous with respect to the antibracket. The constructed model can be renormalized via the standard method of introducing counterterms. This model does not obey (exact) multiplicative renormalizability but it obeys this property in the physical sector (quasi-multiplicative renormalizability).

Keywords: background-field method, Yang-Mills theory, renormalizability, gauge dependence, BV-formalism.

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1 Introduction

This paper analyzes multiplicative renormalization of 4D pure Yang-Mills (YM) theory \[1\] and the issue of gauge-dependencies by combining the powerful tools of the background field method \[2, 3, 4\] and the BV formalism \[5, 6\].

It is well-known that BRST symmetry \[7, 8, 9\] is instrumental in proving renormalizability of Yang-Mills theories.

The traditional main benefit of using the background field method (which introduces a background field \(B_\mu\) to the Yang-Mills gauge field \(A_\mu\)) is that a background gauge symmetry is preserved after gauge-fixing, thereby reducing the list of possible gauge-invariant counterterms when discussing renormalization \[10, 11, 12\].

The main idea of the present paper is to try (possibly selectively) to extend antisymplectic phase space by auxiliary sectors. We managed to realize this idea in the model under consideration and this procedure can be generalized to YM theories matter. Of course, we cannot yet say that there is a systematic procedure that works for an arbitrary gauge theory.

Although there are many papers devoted to various aspects of renormalizability of Yang-Mills theories, the gauge dependence of the renormalization constants has been studied explicitly only in the gauge field sector \[13\].

In 1975 Kluberg-Stern & Zuber \[13\] introduced a Grassmann-odd background field \(L_\mu(x)\) (which we will call \(\theta_\mu(x)\)) as a shift-symmetry for the BRST transformation of \(A_\mu\), and a Grassmann-odd \(x\)-independent parameter \(L\) (which we will call \(\chi\)). The latter is associated with another kind of shift symmetry for the BRST transformation. In 1985 Piguet & Sibold \[14\] included a BRST transformation for the \(\xi\) parameter (known from the \(R_\xi\) gauges). In our previous 2 papers \[11, 12\] on the topic (written by 3 of the 4 authors) in the presence of fermionic and scalar matter, the BV-formalism was also not applied and a generalization of action from the paper \[13\] has been used. However the corresponding master equations for the classical and effective actions were not on the closed BV antibracket form because of the presence of inhomogeneous terms. This complicates the cohomological analysis concerning existence & uniqueness of solutions.

In this paper we bring the master-equation on homogeneous form by introducing antifields to \(\theta_\mu, \chi\) and \(\xi\).

The paper is organized as follows. In Section 2 we find a solution in the minimal sector to the master-equation and compare it with previous solutions in the Literature. In Sections 3-4 we derive the general solution of the classical master-equation in full antisymplectic phase space.
Notation

Condensed DeWitt notation [15] is used through the paper. A left (functional) derivative wrt. a field $\Phi$ is denoted $\partial_\Phi$ while the corresponding right (functional) derivative is denoted $\overset{\leftarrow}{\partial}_\Phi$. Space-time indices are kept as lower indices and denoted by letters from the middle of the Greek alphabet. Lie algebra/color indices are kept as upper indices and denoted by letters from the beginning of the Greek alphabet. Einstein summation convention is used with the slight modification that an (inverse) metric tensor is implicitly implied for each pair of repeated indices.

2 BV-action

We start with the pure 4D Yang-Mills action with $SU(2)$ Lie group
\begin{align*}
S_{YM}(A) &= \int dx \left(-\frac{1}{4}G^\alpha_{\mu\nu}(A)G^\alpha_{\mu\nu}(A)\right), \\
G^\alpha_{\mu\nu}(A) &= \partial_\mu A^\alpha_\nu - \partial_\nu A^\alpha_\mu + g\varepsilon^{\alpha\beta\gamma}A^\beta_\mu A^\gamma_\nu.
\end{align*}

The Lie group is here assumed to be $SU(2)$ but the construction works with almost no changes for any simple compact Lie group. The action (2.1) is invariant $\delta_\omega S_{YM}(A) = 0$ under the gauge transformation
\begin{align*}
\delta_\omega A^\alpha_\mu &= D^{\alpha\beta}_\mu(A)\omega^\beta, \\
D^{\alpha\beta}_\mu(A) &= \partial_\mu \delta^{\alpha\beta} + g\varepsilon^{\alpha\beta\gamma}A^\gamma_\mu,
\end{align*}
with gauge parameters $\omega^\alpha = \omega^\alpha(x)$. In the background field method we replace $A^\alpha_\mu \rightarrow A^\alpha_\mu + B^\alpha_\mu$,
\begin{align*}
S_{YM}(A) \rightarrow S_{YM}(A + B),
\end{align*}
where $B$ is the background field. This action is invariant under the gauge transformation of the form
\begin{align*}
\delta_\lambda A^\alpha_\mu = \delta A^\alpha_\mu \lambda, \\
\delta A^\alpha_\mu = D^{\alpha\beta}_\mu(V)C^\beta, \\
V = A + B.
\end{align*}

We will follow the BV-formalism. The minimal sector consists of the following set of fields and antifields,
\begin{align*}
\Phi &= \{A^\alpha_\mu(x), C^\alpha(x), B^\alpha_\mu(x), \theta^\alpha_\mu(x)\}, \\
\Phi^* &= \{A^{\ast\alpha}_\mu(x), C^{\ast\alpha}(x), B^{\ast\alpha}_\mu(x), \theta^{\ast\alpha}_\mu(x)\}.
\end{align*}

Next we give the table of ”quantum” numbers of fields, antifields, auxiliary fields and constant parameters used in construction of the action, cf. Table II. Recall that
\begin{align*}
\text{gh}(\Phi^*) = -1 - \text{gh}(\Phi), \\
\dim(\Phi^*) = 3 - \dim(\Phi).
\end{align*}

Using Table II of quantum numbers it is easy to establish the quantum numbers of any quantities met in the text. We will denote as $S^{(1)} = S^{(1)}(\Phi, \Phi^*)$ an action to be constructed. We will assume that the action has all the standard properties:
\begin{align*}
\varepsilon(S^{(1)}) = \text{gh}(S^{(1)}) = \dim(S^{(1)}) = 0,
\end{align*}
\[3\]
and \( S^{(1)} \) satisfies the master-equation,
\[
S^{(1)} \int dx \left[ \overleftarrow{\partial} \Phi \partial \Phi^{\ast} \right] S^{(1)} = 0, \quad S^{(1)} \bigg|_{C, B, \theta, \Phi^{\ast} = 0} = S_{YM}(A). \quad (2.10)
\]

We suppose also that \( S^{(1)} \) has a background gauge symmetry
\[
S^{(1)} \overleftarrow{h}^{\alpha} \omega^{\alpha} = 0, \quad (2.11)
\]
where
\[
\overleftarrow{h}^{\alpha} \omega^{\alpha} = \int dx \left\{ \left[ \frac{\partial}{\partial B_{\mu}^{b}} D_{\mu}^{b} (B) \right] + g \varepsilon^{\beta \gamma \alpha} \left( \frac{\partial}{\partial A_{\mu}^{a}} A_{\mu}^{\gamma} + \frac{\partial}{\partial C_{\beta}^{a}} C_{\beta}^{\gamma} + \frac{\partial}{\partial \theta_{\mu}^{a}} \theta_{\mu}^{\gamma} \right) \right\} \omega^{\alpha}. \quad (2.12)
\]

The equation (2.11) means that \( S^{(1)} \) is required to be invariant under the background field gauge transformation.

Represent the action in the form
\[
S^{(1)} = S_{A}^{(1)} (A, B) + S_{\Phi}^{(1)} (\Phi, \Phi^{\ast}), \quad S_{\Phi^{\ast}}^{(1)} \bigg|_{\Phi^{\ast} = 0} = 0. \quad (2.13)
\]

It is easy to see that \( S_{\Phi^{\ast}}^{(1)} \) is linear in \( \Phi^{\ast} \), i.e.,
\[
S_{\Phi^{\ast}}^{(1)} = S_{A^{\ast}}^{(1)} + S_{C^{\ast}}^{(1)} + S_{B^{\ast}}^{(1)} + S_{\theta^{\ast}}^{(1)}, \quad (2.14)
\]
where
\[
S_{A^{\ast}}^{(1)} = \int dx [A^{\ast} D(V) C] + \int dx [A^{\ast} \theta]. \quad (2.15)
\]

In deriving (2.15) we have used the equality (2.11). In what follows we will systematically use this method. So for the rest of actions in (2.14) we obtain
\[
S_{C^{\ast}}^{(1)} = Z_{2} \int dx \left[ C^{\ast \alpha} \frac{g}{2} \varepsilon^{\alpha \beta \gamma} C^{\beta} C^{\gamma} \right], \quad (2.16)
\]
Let $A^*$ be the left-hand side of eq. (2.10). Each action derivative is labelled with an identification number $n$. In the following we write down separately various anticanonical sectors of the vertical line in eqs. (2.25)-(2.28).

Thus, $\tilde{S}_{A^*} = \tilde{S}_{A^*} + \tilde{S}_{B^*}$. Then

$$S_{A^*}^{(1)} + S_{B^*}^{(1)} = S_{A^*}^{(1)} + S_{B^*}^{(1)},$$

$$S_{A^*}^{(1)} = S_{A^*}^{(1)} + S_{B^*}^{(1)}, \quad S_{A^*}^{(1)} = \int dx \left[ \tilde{A}^* D(V) C \right],$$

$$S_{B^*}^{(1)} = -Z_4 \int dx \left[ B^* \theta \right].$$

Thus, $S_{\Phi^*}$ has the form

$$S_{\Phi^*}^{(1)} = S_{A^*}^{(1)} + S_{C^*}^{(1)} + S_{B^*}^{(1)} + S_{B^*}^{(1)},$$

$$S_{A^*}^{(1)} = (2.21), \quad S_{C^*}^{(1)} = (2.16), \quad S_{B^*}^{(1)} = (2.22), \quad S_{B^*}^{(1)} = (2.19).$$

### 2.1 Master-equation

We will omit the tildes in (2.23) from now on. We next turn to the consequences of the master-equation (2.10). In the following we write down separately various anticanonical sectors of the left-hand side of eq. (2.10). Each action derivative is labelled with an identification number $n$ as a convenient shorthand. The identification number $n$ is typed as a subscript behind a vertical line in eqs. (2.25)-(2.28).

$$\bar{\partial}_A \partial_A^* :$$

$$\left[ S_{A^*}^{(1)} \bar{\partial}_{A^*}^A \right]_1 + S_{A^*}^{(1)} \bar{\partial}_{A^*}^A \right]_2 + S_{B^*}^{(1)} \bar{\partial}_{B^*}^A \right]_6 + S_{B^*}^{(1)} \bar{\partial}_{B^*}^A \right]_7 \left[ \partial_{A^*} S_{A^*}^{(1)} \right]_3 + \partial_{A^*} S_{A^*}^{(1)} \right]_4;$$

$$\bar{\partial}_C \partial_C^* :$$

$$\left[ S_{A^*}^{(1)} \bar{\partial}_{A^*}^C \right]_5 + S_{C^*}^{(1)} \bar{\partial}_{C^*}^C \right]_6 + S_{B^*}^{(1)} \bar{\partial}_{B^*}^C \right]_7 + S_{B^*}^{(1)} \bar{\partial}_{B^*}^C \right]_8 + S_{B^*}^{(1)} \bar{\partial}_{B^*}^C \right]_9 + S_{B^*}^{(1)} \bar{\partial}_{B^*}^C \right]_10 \partial_{C^*} S_{C^*}^{(1)} \right]_11;$$

$$\bar{\partial}_B \partial_B^* :$$

$$\left[ S_{B^*}^{(1)} \bar{\partial}_{B^*}^B \right]_9 + S_{A^*}^{(1)} \partial_{A^*}^B \right]_10 + S_{B^*}^{(1)} \partial_{B^*}^B \right]_11 \partial_{B^*} S_{B^*}^{(1)} = -Z_4 \theta^\alpha \right]_12;$$

$$\bar{\partial}_\theta \partial_\theta^* :$$

$$\left[ S_{A^*}^{(1)} \bar{\partial}_{A^*}^\theta \right]_13 + S_{B^*}^{(1)} \partial_{B^*}^\theta \right]_14 \partial_{B^*} S_{B^*}^{(1)} \right]_15 + \partial_{B^*} S_{B^*}^{(1)} \right]_21 + \partial_{B^*} S_{B^*}^{(1)} \right]_22.$$
Below, the notation \( n_1 \cdot n_2 \) is a shorthand for the antibracket constructed from the action derivatives with identification numbers \( n_1 \) and \( n_2 \). The abbreviation “SPT” means “summands proportional to”.

SPT \( B^* \theta C \) :

\[
13 \cdot 15 = S_{B^*}^{(1)} \frac{\leftrightarrow}{\theta^\alpha \theta^\gamma} S_{B^*}^{(1)} = -Z_4Z_5g \int dx \left[ B^* \varepsilon^{\alpha \beta \gamma} \theta^\beta C^\gamma \right] = 0 \\
\implies Z_4Z_5 = 0 \implies Z_4 = 0 \text{ or } Z_5 = 0. \tag{2.29}
\]

Let \( Z_4 = 0 \). Consider SPT \( A^k B^\theta \) in eq. \((2.10)\) :

\[
0 = 1 \cdot 4 = \int dx \left[ S_{A^\alpha}^{(1)} \frac{\leftrightarrow}{A^\alpha} \theta^\alpha \right] \implies S_{A^1}^{(1)} \frac{\leftrightarrow}{A^\alpha} = 0 \\
\implies Z_4 \neq 0 \implies Z_5 = 0. \tag{2.30}
\]

Now, consider SPT \( B^* \) in eq. \((2.10)\). It follows from eq. \((2.10)\)

\[
0 = 13 \cdot (21 + 22) = \int dx \left\{ S_{B^*}^{(1)} \frac{\leftrightarrow}{\theta^\alpha \theta^\gamma} \left[ S_{B^*}^{(1)} + S_{B^*}^{(1)} \right] \right\} \\
= \int dx \left\{ Z_4B^* [Z_6 \theta^\varepsilon C(D(V)C) + Z_7 g \varepsilon A(\varepsilon CC)] \right\} \\
\implies Z_6 = Z_7 = 0 \implies S_{B^*}^{(1)} = 0. \tag{2.31}
\]

The surviving sectors of the left-hand side of eq. \((2.10)\) reduce to the following:

\[
\frac{\leftrightarrow}{A^\alpha} : \\
\left[ S_{A^1}^{(1)} \frac{\leftrightarrow}{A^\alpha} \right] + S_{A^1}^{(1)} \frac{\leftrightarrow}{A^\alpha} \left[ \partial_{A^\alpha} S_{A^1}^{(1)} + \partial_{A^\alpha} S_{A^1}^{(1)} \right] ; \tag{2.32}
\]

\[
\frac{\leftrightarrow}{C^\alpha} : \\
\left[ S_{C^1}^{(1)} \frac{\leftrightarrow}{C^\alpha} + S_{C^1}^{(1)} \frac{\leftrightarrow}{C^\alpha} \left[ \partial_{C^\alpha} S_{C^1}^{(1)} \right] \right] ; \tag{2.33}
\]

\[
\frac{\leftrightarrow}{B^\alpha} : \\
\left[ S_{B^1}^{(1)} \frac{\leftrightarrow}{B^\alpha} + S_{B^1}^{(1)} \frac{\leftrightarrow}{B^\alpha} \right] \left[ \partial_{B^\alpha} S_{B^1}^{(1)} \right] . \tag{2.34}
\]

We obtain:

SPT \( A^* C \theta \) :

\[
2 \cdot 4 + 10 \cdot 11 = 0 \implies Z_4 = 1; \tag{2.35}
\]

SPT \( A^* CD(V)C \) :

\[
2 \cdot 3 + 5 \cdot 8 = (Z_2 - 1)\frac{g}{2} \int dx \left[ A^* D^{\alpha} (V) \varepsilon^{\alpha \beta \gamma} C^\beta C^\gamma \right] \implies Z_2 = 1, \tag{2.36}
\]

where we used the identity

\[
\varepsilon^{\alpha \beta \gamma} \left[ D^{\sigma \beta} (V) C^\beta \right] C^\gamma = \frac{1}{2} D^{\alpha \sigma} (V) \left( \varepsilon^{\sigma \beta \gamma} C^\beta C^\gamma \right); \tag{2.37}
\]
SPT $S_1^{(1)} \stackrel{\partial}{\rightarrow} A^x, \theta^a$ and $S_1^{(1)} \stackrel{\partial}{\rightarrow} B^x, \theta^a$ :

$$1 \cdot 4 + 9 \cdot 11 = 0 = S_1^{(1)} \int dx \left[ \left( \partial \theta^a - \partial \theta^a \right) \theta^a \right] \implies S_1^{(1)}(A, B) = S_1^{(1)}(V); \quad (2.38)$$

SPT $C^*CCC$ :

$$6 \cdot 8 = g^2 \int dx \left( C^* \varepsilon^\beta \gamma^\alpha C^\gamma \varepsilon^{\alpha \sigma \delta} C^\sigma C^\delta \right) \equiv 0; \quad (2.39)$$

$$0 = 1 \cdot 3 = S_1^{(1)}(V) \int dx \left[ \partial V \cdot D(V) C \right] \implies S_1^{(1)}(V) = S_{YM}(V), \quad (2.40)$$

which is in agreement with the boundary condition of the master-equation $(2.10)$. So, we obtain the following action as a solution to the master-equation with given boundary condition

$$S_1^{(1)} = S_{YM}(V) + \int dx \left[ A^* D(V) C \right] + \int dx \left[ C^* \frac{g^2}{2} \varepsilon^{\alpha \beta \gamma} C^\beta C^\gamma \right] + \int dx \left[ A^* \theta \right] - \int dx \left[ B^* \theta \right]. \quad (2.41)$$

The first 3 terms in eq. $(2.41)$ constitute the standard minimal action. The last 2 terms are expected because the combination $A^* - B^*$ antibracket-commutes with $V \equiv A + B$.

### 2.2 Anticanonical transformation

By shifting the action as

$$S^{(1)} \rightarrow S^{(2)} = S^{(1)} + \int dx \overline{C}^x B + \xi^* \chi, \quad (2.42)$$

we impose then the gauge fixing via the anticanonical transformation

$$\Phi^* = Y(\Phi, \Phi^{*\prime}) \partial \Phi, \quad \Phi^\prime = \partial \Phi^* Y(\Phi, \Phi^{*\prime}) \quad (2.43)$$

$$Y(\Phi, \Phi^{*\prime}) = \int dx \Phi^{*\prime} \Phi + \Lambda(\Phi) \implies \quad (2.44)$$

$$\Phi^\prime = \Phi, \quad \Phi^* = \Phi^{*\prime} + \Lambda(\Phi) \partial \Phi. \quad (2.45)$$

$$\Phi = \{ A, C, B, \theta, \overline{C}, B, \xi, \chi \}, \quad \Phi^* = \{ A^*, C^*, \theta^*, \overline{C}^x, B^*, \xi^*, \chi^* \}. \quad (2.46)$$

We choose $\Lambda(\Phi)$ in the form

$$\Lambda(\Phi) = \int dx \overline{C} \left( D(B) A + \frac{\xi}{2} B \right) \implies \quad (2.47)$$

$$A^* = A^{*\prime} - D(B) \overline{C}, \quad C^* = C^{*\prime}, \quad \overline{C}^* = \overline{C}^{*\prime} + D(B) A + \frac{\xi}{2} B, \quad (2.48)$$

$$B^* = B^{*\prime} + g (\varepsilon A \overline{C}), \quad \xi^* = \xi^{*\prime} + \frac{1}{2} \int dx \overline{C} B, \quad (2.49)$$
\[ S_{\text{ext}} = S_{YM}(V) + \int dx [A^* D(V) C] + \int dx [A^* \theta] + \int dx [\theta D(V) \bar{C}] + \int dx [\bar{C} D(B) D(V) C + BD(B) A + (\xi/2) BB] - \frac{1}{2} \chi \int dx (\bar{C} B) + \int dx \left[ C^{* \alpha} \frac{g}{2} e^{\alpha \beta \gamma} C^\beta C^\gamma \right] + \int dx \left[ \bar{C}^{* \alpha} B^\alpha \right] - \int dx [B^* \theta] + \xi^* \chi. \] (2.50)

With the exception of the last three terms, this action is analogously with the actions considered in the papers by Kluberg-Stern & Zuber [13] and by Piguet & Sibold [14].

### 3 Action \( S_{\text{ext}} \) as solution of a set of equations

In what follows we will often omit the symbol of the integral to have more compact presentation of formulas. The action \( S_{\text{ext}} \) satisfies the following set of equations: There are (i) equations linear in functional and partial derivatives with respect to fields and antifields and constant parameters respectively,

\[ \partial_{\bar{\theta}} S_{\text{ext}} = \partial_{B^*} S_{\text{ext}} = \partial_{\chi^*} S_{\text{ext}} = 0, \] (3.1)

\[ \partial_{\bar{B}} S_{\text{ext}} = -\theta, \quad \partial_{\bar{\chi}} S_{\text{ext}} = \chi, \quad \partial_{C} S_{\text{ext}} = B, \] (3.2)

\[ \partial_{B} S_{\text{ext}} = D(B) A + \xi B - \frac{1}{2} \chi C + \bar{C}^\epsilon, \] (3.3)

\[ [D^\alpha(B) \partial_{A^{* \beta}} - \partial_{C^\alpha}] S_{\text{ext}} = -g e^{\alpha \beta \gamma} A^\beta \theta^\gamma - \frac{1}{2} \chi B^\alpha, \] (3.4)

(ii) master-equation

\[ S_{\text{ext}} \dot{\Phi} \partial_{\Phi^*} S_{\text{ext}} = 0 \implies S_{\text{ext}} \dot{A} \partial_{A^*} S_{\text{ext}} + S_{\text{ext}} \dot{C} \partial_{C^*} S_{\text{ext}} + [\chi \partial_\xi - \theta \partial_B - B \partial_C] S_{\text{ext}} = 0, \] (3.5)

and (iii) background gauge invariance

\[ S_{\text{ext}} \dot{H}^\alpha \omega = 0, \] (3.6)

where

\[
\dot{H}^\alpha \omega = \int dx \left\{ \left[ \frac{\delta}{\delta B^\alpha_{\mu}} D_{\mu}^{\beta \alpha}(B) + g e^{\beta \gamma \alpha} \left( \frac{\delta}{\delta A_{\mu}^{\beta}} A_{\mu}^\gamma + \frac{\delta}{\delta C_{\mu}^\beta} C_{\mu}^\gamma + \frac{\delta}{\delta \theta_{\mu}^\beta} \theta_{\mu}^\gamma + \frac{\delta}{\delta C^\beta} C_{\mu}^\gamma + \frac{\delta}{\delta B_{\beta}} B_{\beta}^\gamma \right) \right] \omega^\alpha \right\}. \] (3.7)
4 General solution of equation system (3.1) - (3.7)

We will find a functional $P = P(\Phi, \Phi^*)$ which has all quantum numbers of the functional $S_{\text{ext}}$ and satisfies the equation system (3.1) - (3.7) (with substitution $S_{\text{ext}} \to P$)

\[
\begin{aligned}
\partial_{\theta^*} P &= \partial_{C^*} P = \partial_{\chi^*} P = 0, \\
\partial_{\Phi} P &= -\theta, \\
\partial_{C} P &= B, \\
\partial_{\xi} P &= \chi,
\end{aligned}
\]

(4.1)

\[
\begin{aligned}
\partial_{B^*} P &= D(B)A + \xi B - \frac{1}{2} \chi C + C^*, \\
[D^\alpha\beta(B)\partial_{A^\alpha\beta} - \partial_{C^*}] P &= -g\varepsilon^{\alpha\beta\gamma} A^\beta \theta^\gamma - \frac{1}{2} \chi B^\alpha,
\end{aligned}
\]

(4.2)

master-equation

\[
\begin{aligned}
P \partial_{\Phi} \partial_{\Phi^*} P &= 0 \quad \implies P \partial_{A^\alpha} P - \partial_{C^\alpha} \partial_{C^\alpha} P + [\chi \partial_{\xi} - \theta \partial_{B^*} - B \partial_{C^*}] P = 0,
\end{aligned}
\]

(4.3)

(4.4)

background gauge invariance

\[
P \hat{H}^\alpha \omega^\alpha = 0,
\]

(4.5)

where the operator $\hat{H}^\alpha \omega^\alpha$ is given by eq. (3.7).

It follows from eqs. (4.1) that the functional $P(\Phi, \Phi^*)$ does not dependent of the antifields $\theta^*, B^*$ and fermion parameter $\chi^*$. Because $\theta^*, B^*$ and $\chi^*$ be not appear in what follows, we will mean the set $\Phi^*$ as set $\Phi^* = \{A^*, C^*, B^*, C^*, \xi^*\}$. The operator $\hat{H}^\alpha \omega^\alpha$ is reduced to the form

\[
\hat{H}^\alpha \omega^\alpha = \int dx \left\{ \left[ \frac{\delta}{\delta B^{\alpha}} D^{\alpha\beta}(B) + g\varepsilon^{\beta\gamma\alpha} \left( \frac{\delta}{\delta A^{\mu\beta}} A^\gamma + \frac{\delta}{\delta C^\beta} C^\gamma + \frac{\delta}{\delta \theta_\mu} \theta^\gamma + \frac{\delta}{\delta B^\beta} B^\gamma \right) \right] \omega^\alpha \right\},
\]

(4.6)

(4.7)

Represent $P$ in the form

\[
P = P_{00} + \tilde{P}, \quad \tilde{P} = P^{(1)} + \chi P^{(2)},
\]

(4.8)

\[
P_{00} = \int dx [C B^* \theta + BD(B)A + (\xi/2)BB] + g[C \varepsilon A \theta] - \frac{1}{2} \chi \int dx [C B] + \xi^* \chi.
\]

(4.9)

Note that $P_{00}$ satisfies the equation of the type (4.6),

\[
P_{00} \hat{H}^\alpha \omega^\alpha = 0.
\]

(4.10)

It follows from eqs. (4.2)-(4.3)

\[
\begin{aligned}
\partial_{B^*} P^{(k)} &= \partial_{C^*} P^{(k)} = \partial_{\xi} P^{(k)} = \partial_B P^{(k)} = 0, \quad k = 1, 2,
\end{aligned}
\]

(4.11)
such that
\[
P^{(k)} = P^{(k)}(\tilde{\phi}, \Phi^{(1)*}), \quad \tilde{\phi} = \{\phi^{(1)}, \phi^{(2)}\},
\]
\[
\phi^{(1)} = \{A, C\}, \quad \Phi^{(2)} = \{\mathcal{B}, \theta, \overline{C}, \xi, \chi\}, \quad \phi^{(1)*} = \{A^*, C^*\}.
\]
(4.12)

It follows from eqs. (4.4)
\[
[D^{\alpha\beta}(\mathcal{B})\partial^\alpha_{A^*} - \partial^\alpha_{C^*}]P^{(k)} = 0, \quad k = 1, 2.
\]
(4.14)

So, the functionals $P^{(k)}$ satisfy eq. (4.14), the equations
\[
P^{(k)}\int h^\alpha \omega^\alpha = 0, \quad k = 1, 2,
\]
(4.15)

\[
\int h^\alpha \omega^\alpha = \int dx \left\{ \left[ \frac{\delta}{\delta B^\mu_\alpha} D^{\beta\mu}(\mathcal{B}) + g\varepsilon^{\beta\gamma\alpha} \left( \frac{\delta}{\delta A^\mu_\alpha} A^\gamma + \frac{\delta}{\delta C^\mu_\alpha} C^\gamma + \frac{\delta}{\delta \theta^\mu_\alpha} \theta^\gamma + \frac{\delta}{\delta C^\mu_\alpha} C^\gamma \right) \right] \omega^\alpha \right\},
\]
(4.16)

and the equations which follow from eq. (4.5).

Represent the functionals $P^{(k)}$ in the form
\[
P^{(k)}(A^*, \overline{C}, \Psi) = \tilde{P}^{(k)}(A^*, \overline{C}, \Psi),
\]
(4.17)

where $\Psi$ is the set of all additional to $A^*, \overline{C}$ variables of the functionals $P^{(k)}$,
\[
\Psi = \{A, C, \mathcal{B}, \theta, \xi, C^*\},
\]
(4.18)

and we introduce a notation $A^*$,
\[
A^* = A^* - D(\mathcal{B})\overline{C}.
\]
(4.19)

It follows from eq. (4.14) that
\[
\partial^\alpha_{A^*} \tilde{P}^{(k)}(A^*, \overline{C}, \Psi) \bigg|_{A^*, \Psi} = 0 \quad \implies \quad P^{(k)}(A^*, \overline{C}, \Psi) = \tilde{P}^{(k)}(A^*, \Psi), \quad k = 1, 2,
\]
(4.20)

where $A^*$ is given by eq. (4.19).

### 4.1 A shift

Make a transformation of variables $A^\mu_\alpha$, $\mathcal{B}^\alpha_\mu$ and $\overline{C}^\alpha$,
\[
A^* = A^{\prime} + D(\mathcal{B})\overline{C}, \quad \mathcal{B} = \mathcal{B}', \quad \overline{C} = \overline{C}'.
\]
(4.21)
\( \partial_A^* F = \partial_A^* \tilde{F}, \quad \partial_{B^*}^* F = [\partial_{B^*} + D(B') \partial_{A^*}] \tilde{F}, \)  \( \tag{4.22} \)

\( \partial_{B^*} \tilde{F} = [\partial_{B^*} + g \varepsilon C \partial_{A^*}] \tilde{F}, \)  \( \tag{4.23} \)

\( \partial_A^* \tilde{F} = \partial_A^* F, \quad \partial_{B^*}^* \tilde{F} = [\partial_{B^*} - D(B) \partial_{A^*}] F, \)  \( \tag{4.24} \)

\( \partial_{B^*} \tilde{F} = \partial_{B^*} F, \quad \partial_{C^*} \tilde{F} = \left[ \partial_{C^*} + D(B) \partial_{A^*} \right] F, \)  \( \tag{4.25} \)

\( P^{(k)} \int dx \left[ -\frac{\xi}{\delta B} D(B) + \frac{\xi}{\delta A^*} A^* \varepsilon - \frac{\xi}{\delta C} C \varepsilon \right] \omega = \tilde{P}^{(k)} \int dx \left[ -\frac{\xi}{\delta B} D(B') + \frac{\xi}{\delta A^*} A^* \varepsilon \right] \omega, \)  \( \tag{4.26} \)

where \( A^* \) be considered as an independent variable at the action on the functional \( \tilde{F} \), \( A^* \equiv A' \), and

\( F = F(A^*, B, C, \ldots) = \tilde{F} = \tilde{F}(A^*, B', C', \ldots), \)  \( \tag{4.27} \)

where ellipsis “…” means all the remaining invariable arguments.

Eqs. (4.15) - (4.16) are reduced to (omitting primes)

\( \tilde{P}^{(k)} \frac{\xi}{\delta A} \omega^a = 0, \quad k = 1, 2, \)  \( \tag{4.28} \)

\( \frac{\xi}{\delta A} \omega^a = \int dx \left\{ \left[ \frac{\xi}{\delta B} D_{\mu} A^\mu(B) + g \varepsilon^{\beta \gamma} \left( \frac{\xi}{\delta A^\mu} A^\gamma + \frac{\xi}{\delta C^\beta} C^\gamma + \frac{\xi}{\delta C} \right) \right] \omega^a \right\}. \)  \( \tag{4.29} \)

From eq. (4.15) follows the equations for \( \tilde{P}^{(k)} \) (omitting primes),

\( \tilde{P}^{(k)} \frac{\xi}{\delta A} \partial_A \tilde{P}^{(k)} + \tilde{P}^{(k)} \frac{\xi}{\delta C} \partial_C \tilde{P}^{(k)} - \theta \partial_B \tilde{P}^{(k)} = 0, \)  \( \tag{4.30} \)

\( \partial_k \tilde{P}^{(k)} = \tilde{P}^{(k)} \frac{\xi}{\delta A} \partial_A \tilde{P}^{(k)} - \tilde{P}^{(k)} \frac{\xi}{\delta A} \partial_A \tilde{P}^{(k)} \)

\( + \tilde{P}^{(k)} \frac{\xi}{\delta C} \partial_C \tilde{P}^{(k)} - \tilde{P}^{(k)} \frac{\xi}{\delta C} \partial_C \tilde{P}^{(k)} - \theta \partial_B \tilde{P}^{(k)}, \quad A \equiv A. \)  \( \tag{4.31} \)

### 4.2 Solving of eq. (4.30)

Eq. (4.30) for \( \tilde{P}^{(1)} \) is already solved in Ref. [11], section 3.1, eq. (3.51). The result is (setting spinor arguments in Ref. [11] equal to zero),

\( \tilde{P}^{(1)} = \int dx \left[ -\frac{1}{4} Z_{14} G(U) G(U) - Z_6 \theta A^* + Z_6 A^* D(U) C + \frac{Z_6}{Z_5} g C^* \varepsilon C \right], \)  \( \tag{4.32} \)

\( U = B + \frac{1}{Z_5} A, \quad Z_5, Z_6, Z_{14} \neq 0. \)  \( \tag{4.33} \)
4.3 Solving of eq. (4.31)

First, we find an explicit form of the functional $\tilde{P}^{(2)}$. The set of quantum numbers of $\tilde{P}^{(2)}$ and eqs. (4.28) give

$$\tilde{P}^{(2)} = \int dx \left[ Z_1 A^* A + Z_2 C^* C \right].$$

(4.34)

Then eq. (4.31) is reduced to

$$\partial_t \tilde{P}^{(1)} - \mathcal{L} \tilde{P}^{(1)} = 0,$$

(4.35)

$$\mathcal{L} = \int dx \left[ Z_1 (A \partial A - A^* \partial A^*) + Z_2 (C \partial C - C^* \partial C^*) \right].$$

(4.36)

Represent functional $\partial_t \tilde{P}^{(1)} - \mathcal{L} \tilde{P}^{(1)}$ in the form of linear combination of independent monomials $V_k(A, C, A^*, C^*, \theta)$ with coefficients (differential operators) $M_k(\mathcal{B})$,

$$\partial_t \tilde{P}^{(1)} - \mathcal{L} \tilde{P}^{(1)} = \sum_k M_k(\mathcal{B}) V_k(A, C, A^*, C^*, \theta),$$

(4.37)

$$M_1 V_1 = m_1 \int dx [A^* \theta], \quad m_1 = \dot{Z}_5 - Z_1 Z_5,$$

(4.38)

$$M_2 V_2 = m_2 \int dx \left[ \frac{g}{2} C^* \varepsilon CC \right], \quad m_2 = \left( \frac{Z_6}{Z_5} \right) - Z_2 \frac{Z_6}{Z_5},$$

(4.39)

$$M_3 V_3 = m_3 \int dx \left[ - \frac{1}{4} G(\mathcal{B}) G(\mathcal{B}) \right], \quad m_3 = \dot{Z}_{14},$$

(4.40)

$$M_k |_{m_1=m_2=m_3=0} = 0, \quad k \geq 4.$$  

(4.41)

It follows from eq. (4.35) that

$$m_1 = 0 \implies Z_1 = \frac{\dot{Z}_5}{Z_5}, \quad m_2 = 0 \implies Z_2 = \frac{\dot{Z}_6}{Z_6} - \frac{\dot{Z}_5}{Z_5},$$

(4.42)

$$m_3 = 0 \implies \dot{Z}_{14} = 0.$$

(4.43)

If $Z_\ell$ are the coefficients in counterterms, then $Z_\ell = Z_\ell(\eta)$ can be represented in the form of the Taylor series,

$$Z_\ell = \sum_{n=0}^{\infty} \eta^n z_{\ell,n},$$

(4.44)

where $z_{\ell,n}$ are formed from $n$-loop diagrams. Then, we find in the tree approximation

$$Z_{5,0} = 1, \quad \dot{Z}_{5,0} = 0, \quad Z_{6,0} = 1, \quad \dot{Z}_{6,0} = 0,$$

$$Z_{1,0} = \dot{Z}_{5,0} = 0, \quad Z_{2,0} = \dot{Z}_{6,0} - \dot{Z}_{5,0} = 0,$$

(4.45)

that is, the vertices $\chi \int dx [A^* A]$ and $\chi \int dx [C^* C]$ are absent in the tree approximation.
5 Summary

We have found that the construction of renormalized action in the BV-formalism with the background field method leads to a breakdown of (exact) multiplicativity. However, it does not lead to any difficulties if we are only interested in the physical sector. In the model under consideration we have obtained a theory with renormalized action $S_{\text{ext}} = P$ and can make the theory finite by using the standard scheme of counterterms. If we put $A^* = C^* = 0$ then we obtain a sector in which the renormalization is already multiplicative and contains the full physical sector. In particular, the property that the renormalization constant $Z_{14}$ does not depend on gauge, is preserved, cf. eq. (4.43). We call theories that possess this feature for quasi-multiplicative renormalizable.

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