Gravity on $AdS_3$ and flat connections in the boundary CFT

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We consider the CFT that arises from the D1-D5 system in the presence of a constant background gauge potential which couples to the R-charge of the theory: this potential effectively changes the periodicities of the fermions. By the AdS/CFT correspondence, the effect of this connection should be obtained by finding a smooth solution to the vector field in AdS space which couples to the constant mode of the R-current in the CFT. We investigate such solutions for small values of the connection, and contrast these with spacetimes which have ‘Wilson lines for the bulk ’ – spacetimes that are locally AdS times a sphere but have a global deformation. The latter class are in general singular spacetimes. We comment on some aspects of the recently found geometries corresponding to the D1-D5 state with angular momentum, observing relations between scales in the microscopic theory and scales in the geometry.

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1. Introduction

The D1-D5 system has been crucial in the study of black holes in string theory, since it provides a microscopic description of black hole entropy and Hawking radiation\cite{1}. According to the conjecture by Maldacena \cite{2}, The D1-D5 CFT is dual to IIB string theory compactified on $AdS_3 \times S^3 \times M_4$ where the 4-manifold $M_4$ can be $T^4$ or $K3$. When this duality is examined in more detail, one encounters the question of how the boundary CFT is related to the bulk spacetime \cite{3} \cite{4}. For the Euclidean theory the CFT is placed at the boundary of a large ball shaped region of $AdS$ space. For Lorentzian signature, the ‘global $AdS$’ space is dual to a CFT on a cylindrical boundary with the state on the cylinder being the Neveu-Schwarz (NS) vacuum. The near horizon geometry of branes, on the other hand, gives the Poincare patch of $AdS$ space.

The question that naturally arises is whether one can find a representation of the Ramond (R) vacuum of the CFT. The zero mass BTZ black hole solution of the gravity theory is expected to be a Ramond sector ground state \cite{5} \cite{6} \cite{7}. The Ramond sector has periodic fermions, while the NS sector has anti-periodic fermions. One would therefore like to ask if there are supergravity backgrounds with all other possible intermediate boundary conditions on the fermions. In particular, we can consider the CFT in the presence of a constant background gauge potential coupling to the R-charge of the CFT. This R charge is carried by the fermions of the theory, and such a gauge potential effectively modifies the boundary conditions on the fermions. The family of theories with different values of the gauge potential can be related by a ‘spectral flow’. Thus we can start with the NS vacuum (at zero gauge potential) and look at the states that are obtained as we continuously increase the potential.

The R-symmetry of the CFT is $su(2) \oplus su(2)$, with the first $su(2)$ being carried by the left movers and the second $su(2)$ being carried by the right movers. Let the background gauge field be called $A_{\text{CFT}}^{\text{ext}}$; it will have the parts $A_{L}^{\text{ext},a}$ and $A_{R}^{\text{ext},a}$ ($a = 1, 2, 3$) for the left and right movers. Let the currents of the CFT be $J$; these will have the left and right parts $J_{L}^{a}$ and $J_{R}^{a}$. At least to leading order in the potential the CFT will have an action

\[
S = S_0 + A_{\text{CFT}}^{\text{ext}}J
\]

(1.1)

where $S_0$ is the action in the absence of any background potential. This minimal coupling is what we would get for example by looking at the orbifold CFT that we get at the orbifold
point of the D1-D5 system, and we expect that it will be valid more generally when the current $J$ has been appropriately defined in the presence of the gauge field.

The external gauge fields $A_{\text{CFT}}^{\text{ext}}$ get related to the fields $A_{\text{SUGRA}}$ of the supergravity theory which are dual to the CFT currents $J$, as follows. If $\phi_i$ are fields of the supergravity theory dual to operators $O_i$ in the CFT, then we compute correlators of the $O_i$ from the relation

$$\int DX e^{-S_{\text{CFT}} - O_i \phi^b_i} = e^{-S_{\text{SUGRA}}(\phi^b_i)}$$

(1.2)

where the supergravity action is computed on the field configuration obtained by extremising with the $\phi_i$ fixed to have the values $\phi^b_i$ at the boundary of $AdS$. If we want to turn on an external gauge field in the CFT, then we would get

$$\int DX e^{-S_{\text{CFT}} - O_i \phi^b_i - A_{\text{CFT}}^{\text{ext}} J} = \int DX e^{-S_{\text{CFT}} - O_i \phi^b_i \left[ 1 - A_{\text{CFT}}^{\text{ext}} J + \ldots \right]}$$

$$= e^{-S_{\text{SUGRA}}(\phi^b_i)} + A_{\text{CFT}}^{\text{ext}} \left[ \frac{\delta}{\delta A_{\text{SUGRA}}^b} e^{-S_{\text{SUGRA}}(\phi^b_i + A_{\text{SUGRA}}^b) \left| A_{\text{SUGRA}}^b = 0 + \ldots \right.} \right]$$

(1.3)

at least to leading order in the gauge field where the issue of contact interactions between multiple $J$ insertions does not arise; we expect that with a suitable definition of the field $A_{\text{SUGRA}}$ any magnitude of $A_{\text{CFT}}^{\text{ext}}$ will be identified with a value of $A_{\text{SUGRA}}^b$.

Thus it would appear that to take into account a constant external gauge potential in the boundary CFT we must look for smooth solutions to the supergravity equations that take a specified value at the boundary for the gauge field $A_{\text{SUGRA}}$ that is dual to the current operator $J$ in the CFT; we then evaluate all other correlation functions in the presence of this supergravity field.

In an interesting paper [8] it was argued (extending earlier results of [9]) that a family of $AdS_3$ geometries with varying conical defect angles would be the gravity descriptions of the family of CFT states connected by spectral flow between the NS and R sectors. These spacetimes had in addition to the conical defect a Wilson line of a gauge field from the supergravity multiplet: this flat connection would change the boundary condition for the fields charged under the internal $su(2) \oplus su(2)$ symmetry group of the CFT.

In [8] the supergravity theory was extended supergravity in three dimensions, and the gauge fields thus arose as superpartners of the gravity fields. But if the $AdS_3$ spacetime is obtained by compactification of 10-d IIB string theory, then it must be possible to identify this gauge field among the fields of the 10-d supergravity multiplet, and look at the 10-d
spacetime corresponding to the spacetimes of interest. We carry out such an identification, and find that the IIB spacetimes that represent the $AdS$ construction of \cite{8} appear to be spacetimes that are locally $AdS_3 \times S^3$ (we will suppress the compact manifold $M_4$ in most of what follows) but which have identifications that make them not globally a simple product of $AdS_3$ and $S^3$. But these spacetimes are generally \textit{singular} at $r = 0$, and thus would not arise as smooth solutions of the supergravity fields with given values at the boundary. We comment on this issue again at the end of this paper.

In this paper we do the following:

(a) We look at the fields in supergravity that carry the quantum numbers of the operators $J$ in the CFT. We solve their coupled equations, and observe the ‘chiral’ nature of their solutions: The behavior of the field on $S^3$ must correlate with the behavior in $AdS^3$. This is the case because the former determines which $su(2)$ out of the $su(2) \oplus su(2)$ R symmetry the field carries, while the behavior of the field near the $AdS_3$ boundary determines whether it couples to left or right movers in the CFT.

(b) We examine the spacetimes that are locally of the form $AdS_3 \times S^3$ but are not globally a direct product. We look at three different kinds of deformations of $AdS_3 \times S^3$ in this category. We may term these deformations ‘Wilson lines’ of the bulk supergravity theory. (By what was said above these may not represent Wilson lines in the boundary CFT.) These deformed spacetimes are in general singular spacetimes (except when the ‘twist’ equals a multiple of $2\pi$; then it can be removed by a coordinate transformation). We look at the wavefunctions and energy levels for localized states of a scalar field in the backgrounds that have ‘twists’ and conical defects (which are two of the above mentioned three kinds of deformations). A subclass of these wavefunctions generalise the wavefunctions that were constructed in \cite{3} as duals to primary fields of the CFT when the spacetime was globally $AdS_3 \times S^3$. However, the significance of these geometries (and these localised wavefunctions) in the context of the AdS/CFT correspondence is not quite clear.

(c) In some recent papers \cite{10,11} a set of geometries was given that corresponded to the D1-D5 system with no momentum charge but with nonzero angular momentum. We comment on several interesting aspects of these geometries. In particular we note that from the microscopic viewpoint these systems are in a different ‘phase’ from the D1-D5-momentum black holes. We observe several interesting correspondences between scales of the gravity theory and scales of the microscopic dual.
The plan of this paper is the following. In section 2 we look at the linearized equations of the fields that carry the quantum numbers of the current operators in the CFT, and analyze some of their solutions. In section 3 we look at ‘pure gauge’ solutions that are spacetimes that are locally but not globally $AdS_3 \times S^3$; these are finite deformations of the spacetime that appear at the linearised level as pure gauge deformations of the fields. In section 4 we look at solutions to the scalar wave equation around some of the deformed backgrounds studied in section 3. In section 5 we examine some interesting relations between the solutions presented in [10][11] and the microscopic theory of the D1-D5 system. Section 6 is a summary and discussion.

Note added: Several of the computations described in this paper were performed at various points over the past year, as part of a general study of the AdS/CFT correspondence for the D1-D5 system. These notes have been brought to the form of the present paper partly due to the interest in this issue generated by recent studies of the spacetimes corresponding to states in the Ramond sector of the CFT [10][11]. We discuss the relation of our computations to these latter studies towards the end of this paper, and comment on some interesting aspects of the ideas emerging from the above references.

2. The linearised theory

2.1. Field equations

Let us consider the compactification of type IIB string theory on a K3 or $T^4$. We will get a set of low energy supergravity fields. We will focus on the graviton and the 2-forms arising in the 6 noncompact directions. The lagrangian and linearised equations of these fields can be read off from [12] where a general $D = 6, N = 4b$ supergravity theory was studied. (If the compact space is a K3 then there will be no 1-forms from in 6-d; for $T^4$ there will be 1-forms but they are not the ones of interest to us below.) In subsection 2.3 below we will obtain these equations directly from IIB supergravity in 10-D, reduced to 6-D on $T^4$.

The 2-form fields can be split into self-dual and anti-self-dual sets, and one of the self-dual fields acquires a vacuum expectation value to give a compactification the 6-d space to a geometry $AdS_3 \times S^3$. We will use indices $A, B, \ldots$ to represent 6-d indices, $\mu, \nu, \ldots$ to represent $AdS_3$ indices, and $a, b, \ldots$ to represent $S^3$ indices. We call the $AdS$ coordinates $x$ and the $S^3$ coordinates $y$. We will also ignore the scalars in the discussion below, and
so will not distinguish between the elementary field strengths $G = dB$ and the modified field strengths $H$. Letting the radii of the two factors be unity each, we get the curvature tensor and self dual field strengths

$$R_{\mu\nu\rho\sigma} = -(g_{\mu\rho} g_{\nu\sigma} - g_{\nu\rho} g_{\mu\sigma}), \quad R_{abcd} = g_{ac} g_{bd} - g_{ad} g_{bc} \quad (2.1)$$

$$H_{\mu\nu\rho} = \partial_\mu B_{\nu\rho} + \partial_\nu B_{\rho\mu} + \partial_\rho B_{\mu\nu}, \quad H_{abc} = \partial_a B_{bc} + \partial_b B_{ca} + \partial_c B_{ab} \quad (2.2)$$

$$H_{\mu\nu\rho} = \epsilon_{\mu\nu\rho}, \quad H_{abc} = \epsilon_{abc} \quad (2.3)$$

The sphere $S^3$ has the symmetry group $SO(4) = (SU(2) \times SU(2))/Z_2$, and thus the algebra of diffeomorphisms is $su(2) \oplus su(2)$. We identify these $su(2)$ factors with the left and right current algebras of the boundary CFT. To find the fields dual to the currents $J^i_L, J^i_R$ of the boundary theory we must look at supergravity fields that are vectors in the $AdS_3$ and vectors on the sphere $S^3$. The components of the graviton $h^a_\mu$ are such an object; we also have $B^a_\mu$ from the 2-form field which acquires an expectation value. The other 2-form fields in the theory are related by symmetry: the self dual fields had an $SO(5)$ symmetry which is broken to $SO(4)$ when we single out one field to give it an expectation value, and the anti-self-dual fields have a $SO(n)$ symmetry, where $n$ is the number of tensor multiplets in the $D = 6, N = 4b$ theory. Thus we do not expect fields from these categories to be dual to the currents of the boundary theory (the currents do not carry these $SO(4)$ or $SO(n)$ symmetries.)

In appendix A we describe explicitly the vector fields that correspond to the symmetries of the sphere, and indicate their separation into the two $su(2)$ factors. But in the boundary CFT we must find the left movers to have one such $su(2)$ and the right movers to have the other one. Thus $h^a_\mu$ or $B^a_\mu$ must yield a correlation between the behavior of the index of the sphere and the index in the $AdS$. Such a correlation arises because of Chern-Simmons like couplings between the $h^a_\mu$ and the $B^a_\mu$ fields, and in this section we would like to analyse the dynamics of these coupled fields at the linearized level.

Before we write the explicit equations, let us recall why the two fields $h^a_\mu$, which is the perturbation of the graviton, and $b_{a\mu}$, which is a perturbation of the 2-form field $B$, are coupled at the linearized level. The action has a term $H_{ABC} H^{ABC}$. Suppose we have an excitation $b_{a\mu}$. If we have derivatives in the directions $\mu, \nu$ then this gives a component $\delta H_{\nu a \mu} = \partial_\nu b_{a\mu} - \partial_\mu b_{a\nu}$. This field strength couples to the background field strength $H_{\mu\nu\lambda}$ if there is a nonzero value for $h_{\mu a}$, thus giving a quadratic order contribution to $H^2$ involving.
one \( B_{a\mu} \) and one \( h_{\mu a} \) perturbation. This mechanism is similar to that which mixes the two 2-form fields in the presence of a background 5-form field strength in \( AdS_5 \times S^5 \).

Following \[12\] we write
\[
h_{\mu a} = K_{\mu a}, \quad b_{\mu a} = Z_{\mu a}
\]
\[
K_{\mu a} = K_{\mu}^{1,\pm 1}(x)Y_a^{1,\pm 1}(y), \quad Z_{\mu a} = Z_{\mu}^{1,\pm 1}(x)Y_a^{1,\pm 1}(y)
\]

The equations relating \( K \) and \( Z \) are \[12\]
\[
K_{\mu;\nu}^{1,\pm 1} - 2K_{\nu}^{1,\pm 1} + 4\epsilon_{\mu\nu\lambda} \partial_\nu Z_\lambda^{1,\pm 1} \mp 8Z_{\mu}^{1,\pm 1} = 0
\]
\[
\epsilon_{\mu\nu\lambda} \partial_\nu Z_\lambda^{1,\pm 1} \pm 2Z_{\mu}^{1,\pm 1} + K_{\mu}^{1,\pm 1} = 0
\]

Let us work with the upper sign. Writing
\[
A_\mu \equiv K_{\mu}^{1,1}, \quad B_\mu = 2Z_{\mu}^{1,1}
\]
\[
F_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu, \quad G_{\mu\nu} = \partial_\mu A_\nu - \partial_\nu A_\mu
\]
we get the equations
\[
F_{\nu\mu}^{\mu} + 2\epsilon^{\mu\nu\rho} G_{\nu\rho} = 0
\]
\[
\epsilon^{\mu\nu\rho} G_{\nu\rho} + 4(A_\mu + B_\mu) = 0
\]

Let us write
\[
F_{\mu\nu} = (dA)_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu}, \quad (*F)_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} F^{\nu\rho}, \quad (d\ast F)_{\mu\nu} = \partial_\mu (*F)_\nu - \partial_\nu (*F)_\mu, \quad (*d\ast F)_\mu = \frac{1}{2} \epsilon_{\mu\nu\rho} (d\ast F)^{\nu\rho}
\]

Then the equation (2.10) becomes
\[
(*d\ast F) + 4(*dB)_\mu = 0
\]
This gives
\[
d\ast F + 4B = 0, \quad B = -\frac{1}{4} F + d\Lambda
\]
The equation (2.11) becomes
\[
2\ast G + 4(A + B) = 0
\]
Thus once we find a solution to (2.14) with $\Lambda = 0$, adding in the term $d\Lambda$ to $B$ does not change $G = dB$, and from (2.15) requires $A \rightarrow A - d\Lambda$. This freedom

$$B \rightarrow B + d\Lambda, \quad A \rightarrow A - d\Lambda \quad (2.16)$$

gives the pure gauge freedom of the equations; we will set $\Lambda = 0$ in the calculation below and return to the issue of flat connections later. Then $*G = -2A + \frac{1}{2} * F$, and (2.13) becomes

$$*d * dA - 8A + 2 * dA = 0 \quad (2.17)$$

We can write this as

$$(*d + 4)(*d - 2)A = 0 \quad (2.18)$$

which can be solved in two ways

$$(*d + 4)A = 0 \quad \text{or} \quad (*d - 2)A = 0 \quad (2.19)$$

If on the other hand we had chosen the lower sign in passing from (2.6), (2.7) to (2.10), (2.11) then we would have a replacement $* \rightarrow -*$, and the equations would be different. Thus the different 1-forms on the sphere $S^3$ lead to different solutions in the $AdS_3$ spacetime.

2.2. A special solution

We write in explicit form the equation (2.19) in Appendix D, and reduce it to an equation of hypergeometric form. Here we extract only the solutions relevant to the constant connection in the boundary CFT. Let us solve for the condition $(*d - 2)A = 0$. In Appendix D we have considered solutions starting with the form

$$A_u = e^{-i\alpha u} f(r), \quad A_r = e^{-i\alpha u} h(r), \quad A_v = e^{-i\alpha u} q(r)$$

where $u = t + \phi, v = t - \phi$. To get a constant connection at the boundary we set $\alpha = 0$. Then using the relations in Appendix D we get $h = 0$. Further, we find

$$q_r = \frac{C}{r(1 + r^2)} \quad (2.20)$$

(Note that in the notation of Appendix D we have $Q = 2$ for the equation that we are solving.) For a solution regular at $r = 0$ we take $q_r = 0$, and thus set $q = 1$. Then we get
\[ f = (1 + 2r^2). \] Note that near \( r = 0 \) we have \( A_\phi = A_u - A_v = f - q = O(r^2) \), so that the connection is regular at \( r = 0 \). Near \( r = \infty \), \( A_u \) dominates over \( A_v \), and we thus get a connection at the boundary that is a left moving 1-form on the boundary.

If we had taken instead the condition \((*d + 4)A = 0\) from (2.18) then we would find that the solution that is large at infinity is not chiral. For this reason it appears reasonable to identify the solution of \((*d - 2)A = 0\) as the function dual to the constant mode of the current operator in the boundary.

Having solved for the field \( A_\mu \) we can recover the field \( B_\mu \) from the relation (2.14). Thus we observe that we can obtain solutions for the fields \( A_\mu, B_\mu \) that go over to a constant value of the connection at the boundary, and give solutions that are smooth inside the \( AdS_3 \) spacetime. These fields lift to a smooth geometry of \( AdS_3 \times S^3 \) when we recall the relation between the gauge group index and the vector fields on \( S^3 \), listed in Appendix A. We have also found along the way excitations that are pure gauge, in eqn (2.14); we take \( \Lambda \neq 0 \) but \( F = G = 0 \). We will look at such solutions (at the nonperturbative level) in the next section, but note here that such locally pure gauge connections are typically singular at \( r = 0 \), in contrast to the smooth solutions listed in this subsection.

### 2.3. Obtaining the field equations from 10-D supergravity

The equations (2.6)-(2.7) were obtained in [12] for a general \( N = 4 \) supergravity theory in \( D=6 \). Let us see how we get such a set of equations from dimensional reduction of Type IIB supergravity in \( D = 10 \), compactified on \( AdS_3 \times S^3 \times M_4 \). The relevant part of the supergravity action is

\[
S = \int d^{10} x \sqrt{-g} [R - \frac{1}{2} \partial \phi \partial \phi - \frac{1}{3} e^\phi H^2] \tag{2.21}
\]

where we are using the Einstein metric, \( \phi \) is the dilaton, and \( H = dB \). (We have chosen the normalization of \( B \) to agree with the notation of [12].) The field equation for \( \phi \) is

\[
\Delta \phi = \frac{1}{3} e^\phi H^2 \tag{2.22}
\]

Thus if \( H^2 = 0 \) then we can consistently set \( \phi = 0 \). The background (2.1)-(2.3) satisfies

\[
H = *H, \quad H^2 = H \wedge *H = H \wedge H = 0 \tag{2.23}
\]

where the \( * \) is taken in \( AdS_3 \times S^3 \).
Let us look for deformations $B \rightarrow B + \delta B$ which continue to satisfy $H = \ast H$, so that $H^2 = 0$ and the dilaton can continue to be set to zero. The Einstein equations give

$$R_{a\mu} = H_{aAB} H_{\mu}^{\ AB}$$  \hspace{1cm} (2.24)

Let the nonzero components of the perturbation be of the form

$$h_{\mu a} = K_{\mu a} (x) Y_a^{1,1} (y), \quad B_{a\mu} = Z_{\mu a} (x) Y_a^{1,1} (y)$$

$$\epsilon_a^{\ bc} \partial_b h_{\mu c} = 2h_{\mu a}, \quad \epsilon_a^{\ bc} \partial_b B_{c\mu} = 2h_{a\mu}$$  \hspace{1cm} (2.25)

Then we get to lowest order in the perturbation

$$R_{a\mu} = -\frac{1}{2} [h_{\mu a;A}^{\ ;A} + h_{A;\mu a}^{\ ;A} - h_{A;\mu a}^{\ ;A} - h_{aA;\mu}^{\ ;A}]$$

$$= -\frac{1}{2} [K_{\mu;\nu} - K_{\nu;\mu} - 2K_{\mu}]$$  \hspace{1cm} (2.27)

On the other hand

$$H_{\mu AB} H_{\ a}^{\ AB} = \epsilon_\mu^{\ \nu\lambda} [B_{a\nu,\lambda} - B_{a\lambda,\nu}] + [B_{b\mu,\nu} - B_{b\nu,\mu}] \epsilon_a^{\ bc} = 2\epsilon_\mu^{\ \nu\lambda} Z_{\nu,\lambda} - 4Z_\mu$$  \hspace{1cm} (2.28)

where we have expanded to lowest order in the perturbation, and used the background values of $H$ and (2.26). Thus we obtain from (2.24) the equation

$$\triangle K_{\mu} - K_{\nu;\mu}^{\ ;\nu} - 2K_{\mu} + 4\epsilon_\mu^{\ \nu\lambda} Z_{\nu,\lambda} - 8Z_\mu = 0$$  \hspace{1cm} (2.29)

Now let us look at the condition $H = \ast H$. The relevant component of this equation is

$$H_{\nu\lambda c} = (\ast H)_{\nu\lambda c}$$  \hspace{1cm} (2.30)

But

$$H_{\nu\lambda c} = \partial_\nu B_{\lambda c} - \partial_\lambda B_{\nu c}$$

$$\ast H)_{\nu\lambda c} = \frac{1}{6} \epsilon_{\nu\lambda c;ABC} H_{A'B'C'} g_{AA'}^{BB'} g_{CC'} = \epsilon_{\nu\lambda \mu} h_{c}^{\mu} + 2\epsilon_{\nu\lambda \mu} B_{c\mu}$$  \hspace{1cm} (2.31)

Contracting (2.31) with $\epsilon_\mu^{\ \nu\lambda}$ (and using (2.26)) we get

$$\epsilon_\mu^{\ \nu\lambda} \partial_\nu Z_\lambda + 2Z_\lambda + K_\mu = 0$$  \hspace{1cm} (2.32)
Thus we get the equations in \[12\] if we had used the perturbations \(K^{1,-1}, Z^{1,-1}\) instead then we would get the lower signs in \((2.6), (2.7)\).

To evaluate correlation functions using the AdS/CFT correspondence we must evaluate the value of the action of the solution of the supergravity equations. The action to quadratic order is

\[
S = \int d^3x \sqrt{-g} \left[ -\frac{1}{4} F_{\mu\nu} F^{\mu\nu} - 2A_\mu A^\mu + 4(-2B_\mu B^\mu - \frac{1}{4} G_{\mu\nu} G^{\mu\nu} \pm 2A_\mu B^\mu - \frac{1}{2} \epsilon^{\mu\nu\lambda} A_\mu G_{\nu\lambda}) \right]
\]

(2.34)

where \(F = dA, G = dB\). The first two terms come from the dimensional reduction of \(\sqrt{-g}R\) in 6-D, and the others come from reduction of \(-\frac{1}{3}H^2\). The upper and lower signs correspond to the modes \(Y^{1,\pm 1}\) in the harmonic expansions on \(S^3\). We are assuming here that we excite fields in a way such that \(H\) remains self-dual, so that the dilaton is not excited.

The above action should also yield the anomalies of the CFT, following the method used for the case \(AdS_5 \times S^5\) in \([4]\).

3. ‘Locally pure gauge’ deformations

3.1. Different types of deformations

In our analysis of section 2.1 we saw that there were locally pure gauge excitations given in \((2.16)\) where the field strengths satisfied \(G = F = 0\). These solutions can be nevertheless nontrivial if we have a nonzero value for the integral of the connection around a cycle. There are no nonvanishing cycles in \(AdS_3\), but if we allow \(r = 0\) to be a singular point, then we can find nontrivial flat connections. Such flat connections can be locally gauged away, so locally the spacetime must look like \(AdS_3 \times S^3\) though globally it need not reduce to this product form.

Thus we wish to look at the possible ways that one can have a space that looks locally the same as \(AdS_3 \times S^3\); with this analysis we will then not be constrained to the linear order of ‘locally pure gauge’ deformations. For simplicity of presentation let us first look at

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\[1\] In \([10]\) it was argued that to get a consistent dimensional reduction from 6-D to 3-D we must use both self-dual and anti-self dual parts of the 2-form gauge field. This does not contradict the fact that we can obtain a class of solutions using only the self-dual part of the field and obtain for these the equations in \([12]\); we are investigating these special solutions to obtain the dual of a constant background potential in the CFT.
a ‘toy model’ where the space is locally $AdS_3 \times S^1$. Let us therefore start with a ‘product metric’

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2 + d\chi^2$$

(3.1)

where $0 \leq \chi < 2\pi$ is a circle of unit length. To get a space that looks locally the same we can do three things:

(a) ‘Twists’ We can write the metric as

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2 + (d\chi - \alpha d\phi)^2$$

(3.2)

with $\alpha$ a constant. We then say that the variables $\phi, \chi$ are still the ones that are well defined in the sense that $\phi \rightarrow \phi + 2\pi$ and $\chi \rightarrow \chi + 2\pi$ give the identifications of the spacetime. (Thus even though with the definition $\chi' = \chi - \alpha \phi$ the metric (3.2) looks locally like (3.1), the two metrics are really different.) In a ‘Kaluza-Klein’ ansatz where the coordinate $\chi$ denotes an internal direction, we would write the metric as

$$\hat{g}_{AB} = \begin{pmatrix} g_{\chi \chi} & A_\phi^\chi g_{\chi \chi} \\ A_\mu^\chi g_{\chi \chi} & g_{\mu \nu} + A_\mu^\chi A_\nu^\chi g_{\chi \chi} \end{pmatrix}$$

(3.3)

(see for example [14]). Here $A, B$ are coordinates of the 4-d space in (3.3), $\hat{g}_{AB}$ is the metric (3.2) and $\mu, \nu$ are the coordinates $t, r, \phi$. The 1-form gauge connection is $A^\chi_\mu$, with $F_{\mu \nu} = \partial_\mu A^\chi_\nu - \partial_\nu A^\chi_\mu$. We find $A^\chi_\phi = \alpha$, so the potential is locally pure gauge. The effective 3-d metric is then

$$g_{\mu \nu} dx^\mu dx^\nu = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2$$

(3.4)

Note that $g_{\chi \chi} = 1$, so the scalar corresponding to the length of the compact circle is not excited. The 4d metric (3.2) is not in general regular at $r = 0$. (It is regular if $\alpha$ is an integer $n$ in which case the deformation can be undone by a coordinate transformation.)

We will call this type of deformations ‘twists’.

(b) ‘Magnetic solutions’ We can write a metric

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 (d\phi + \beta d\chi)^2 + d\chi^2$$

(3.5)

where again the coordinates $\phi, \chi$ are periodic with period $2\pi$. Now $g_{\chi \chi} = 1 + r^2 \beta^2$, so the scalar corresponding to the size of the compact circle is excited, and in fact grows monotonically with increasing $r$. The gauge potential is $A^\chi_\phi = (g_{\chi \chi})^{-1} g_{\chi \phi} = \beta \frac{r^2}{1 + r^2 \beta^2}$. 

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There is a nonvanishing magnetic field strength $F_{r\phi}$. The effective metric $g_{\mu\nu}$ in the Kaluza-Klein ansatz is

$$g_{\mu\nu}dx^\mu dx^\nu = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + \frac{r^2}{1 + r^2 \beta^2}d\phi^2$$  \hspace{1cm} (3.6)

(This is the ‘string metric’ which has a gravity action $\int \sqrt{-g}e^{-2\phi}R + \ldots$ where $\phi = -\frac{1}{4} \log g_{\chi\chi}$.) Note that at $r \approx 0$ the metric (3.6) approaches (3.1), and so is regular at $r = 0$. If we set the $t, r$ part of the metric to equal $-dt^2 + dr^2$ then this metric would be equivalent to the one appearing in the Melvin solution\[15\].

(c) We can take the metric (3.1) but include a ‘conical defect’; i.e., let the range of $\phi$ be $0 \leq \phi < 2\pi \gamma$. We can define a new coordinate $\phi' = \phi/\gamma$ that still runs over $0 \leq \phi' < 2\pi$ by writing

$$ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 \gamma^2 d\phi'^2 + d\chi^2$$

$$= -\left(\gamma^2 + (r\gamma)^2\right)d(\gamma t/\gamma)^2 + \frac{d(r\gamma)^2}{\gamma^2 + (r\gamma)^2} + (r\gamma)^2 d\phi'^2 + d\chi^2$$

$$= -\left(\gamma^2 + r'^2\right)dt'^2 + \frac{dr'^2}{\gamma^2 + r'^2} + r'^2 d\phi'^2 + d\chi^2$$  \hspace{1cm} (3.7)

where we have defined $r' = r\gamma, t' = t/\gamma$ in the last step.\[16\]

It can be seen that we can ‘mix’ the deformations of types (a), (b), (c) to make other spacetimes. Thus we can take an arbitrary linear combinations of $\phi, \chi$ (with constant coefficients) to multiply the terms in (3.1) with coefficients $r^2$ and unity, and for any such construction we can limit the range of the coordinate $\phi$ as in (c).

---

1 I thank A. Tseytlin for pointing out these references.

2 Writing $\gamma^2 = \delta$, we can continue (3.7) to $\delta = -A$. Defining $\tilde{\phi} = it, \tilde{t} = i\phi, \tilde{r}^2 = r^2 - A$, it is interesting that we get

$$ds^2 = -(r^2 - A)dt^2 + \frac{dr^2}{r^2 - A} + r^2 d\phi^2 = -((\tilde{r}^2 + A)dt^2 + \frac{d\tilde{r}^2}{\tilde{r}^2 + A} + \tilde{r}^2 d\tilde{\phi}^2$$

Further, after allowing rotations in $t, \phi$ it would be interesting to investigate the possible relation of such metrics to the ‘kinks’ observed in thermal strings in [16].
3.2. ‘Twists’ for AdS$_3 \times S^3$

The metric of the undeformed AdS$_3 \times S^3$ is

$$ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 + d\Omega_3^2$$

(3.8)

where we have taken unit radius for the AdS and sphere factors. The vector field $A_\mu$ in the above section arose from $h^a_\mu$, and a corresponding perturbation of the 2-form field $B$. A nonzero value for $h^a_\mu$ indicates that the AdS space is not orthogonal to the sphere, though since we are trying to make a flat connection, locally it should be possible to choose coordinates where no effect of the gauge field is seen at all.

Let the sphere be given by

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1$$

(3.9)

Let

$$x_1 = q \cos \chi$$
$$x_3 = q \sin \chi$$
$$x_2 = \sqrt{1 - q^2} \cos \psi$$
$$x_4 = -\sqrt{1 - q^2} \sin \psi$$

(3.10)

In Appendix A we list the vector fields on $S^3$ that correspond to the decomposition $so(4) = su(2) \oplus su(2)$. Let $M_{ij}$ be the generators of the rotations of the Cartesian coordinates $x_i$ of the sphere (see Appendix A). Let us make the analogue of the deformation of type (a) in the above subsection. We wish to choose a fixed generator of rotations of this $S^3$, and rotate the $S^3$ at a constant rate as we increase the AdS coordinate $\phi$. We can take any linear combination of the generators $M_{ij}$, or equivalently, of the left and right $su(2)$ generators $J^a, K^a$ ($a = 1, 2, 3$). For presentational purposes the simplest case is where we have an equal magnitude of rotation in the two $su(2)$ factors. We can choose coordinates on the sphere so that these rotations are in the directions $J^3, K^3$, and then for equal rotations in the two $su(2)$s the generator of rotations is proportional to

$$J_3 + K_3 = \frac{1}{2} [M_{13} + M_{42}] + \frac{1}{2} [M_{13} - M_{42}] = M_{13} \rightarrow \frac{\partial}{\partial \chi}$$

(3.11)

Thus we get a rotation of the circle $\chi$ as we increase the AdS coordinate $\phi$, so this is similar to the situation in (3.2). But note that the radius of the $\chi$ circle $q$ is not a constant but varies in the range $0 \leq q \leq 1$. 

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We can find coordinates $\phi', \chi'$ in the $\phi, \chi$ space where the metric in this subspace is

$$g' = \begin{pmatrix} g'_{\chi' \chi'} & 0 \\ 0 & g'_{\phi' \phi'} \end{pmatrix} = \begin{pmatrix} q^2 & 0 \\ 0 & r^2 \end{pmatrix}$$  \hspace{1cm} (3.12)$$

(The other coordinates are orthogonal to these two and their metrics are left unchanged.)

The coordinate $\phi'$ still labels the different spheres over different points on a fixed $\rho$, fixed $t$ circle in the $AdS$ space, but the coordinate $\chi'$ is not a periodic one in the spacetime. Let the periodic coordinate be instead

$$\chi = \chi' - \alpha \phi', \quad \alpha = \text{constant} \hspace{1cm} (3.13)$$

Writing

$$\phi = \phi' \hspace{1cm} (3.14)$$

we get the metric in the $\phi, \chi$ subspace in the unprimed coordinates

$$g \equiv \begin{pmatrix} g_{\chi' \chi'} & g_{\chi' \phi'} \\ g_{\phi' \chi'} & g_{\phi' \phi'} \end{pmatrix} = \begin{pmatrix} g'_{\chi' \chi'} & \alpha g'_{\chi' \phi'} \\ \alpha^2 g'_{\chi' \chi'} & g'_{\phi' \phi'} + \alpha^2 g'_{\chi' \chi'} \end{pmatrix}$$ \hspace{1cm} (3.15)$$

The determinant is unchanged

$$\text{det}g = \text{det}g' = g'_{\phi' \phi'} g'_{\chi' \chi'}$$ \hspace{1cm} (3.16)$$

The inverse metric in this subspace is

$$\left(g\right)^{-1} = \begin{pmatrix} -\frac{1}{g'_{\chi' \chi'}} - \frac{\alpha^2}{g'_{\phi' \phi'}} & \frac{\alpha}{g'_{\phi' \phi'}} \\ \frac{\alpha}{g'_{\phi' \phi'}} & \frac{1}{g'_{\phi' \phi'}} \end{pmatrix}$$ \hspace{1cm} (3.17)$$

3.3. Analogues of the other deformations

We can also make deformations of $AdS_3 \times S^3$ which are analogous to the deformations of type (b) (‘magnetic deformations’) and type (c) (‘conical defects’). The conical defects are made in the same way as for $AdS_3 \times S^1$, since the space is a direct product of an $AdS_3$ space with conical defect (eq. (3.7)) with the internal space which can now be taken to be $S^3$ instead of $S^1$. To make the analogue of the ‘magnetic deformations’ we take the circle parametrized by $\chi$ on $S^3$ (eq. (3.9)) and use this to replace the $S^1$ in the construction of type (b) above. Note however that at $q = 0$ the size of this circle goes to zero, and thus the spacetime is singular in general.
3.4. Changing to a chiral gauge

The above construction gives us a connection

\[ A^J_\phi = A^K_\phi = \alpha \]  \hspace{1cm} (3.18)

In the boundary CFT the left movers can carry the charge \( J^a \) and the right movers can carry the charge \( K^a \). We can change gauge to let the \( J^a \) component of \( A_\mu \) be of the form \( A_\phi + t \) and the \( K^a \) component be of the form \( A_{\phi - t} \). To do this we must make a gauge rotation linear in \( t \), such that we get

\[ A^J_t = -A^K_t = \alpha \]  \hspace{1cm} (3.19)

But

\[ J_3 - K_3 = \frac{1}{2}[M_{13} + M_{42}] - \frac{1}{2}[M_{13} - M_{42}] = M_{42} \rightarrow \frac{\partial}{\partial \psi} \]  \hspace{1cm} (3.20)

so that the above gauge transformation corresponds to

\[ \psi = \psi' - \alpha t', \quad t = t' \]  \hspace{1cm} (3.21)

analogous to (3.13), (3.14). The metric in the \( \psi, t \) subspace is

\[ g' = \begin{pmatrix} 1-q^2 & 0 \\ 0 & -(1+r^2) \end{pmatrix} \]  \hspace{1cm} (3.22)

\[ g = \begin{pmatrix} g'_{\psi^\prime \psi^\prime} & g'_{\psi^\prime t^\prime} \\ g'_{t^\prime \psi^\prime} & g'_{t^\prime t^\prime} + \alpha^2 g'_{\psi^\prime \psi^\prime} \end{pmatrix} \]  \hspace{1cm} (3.23)

Note however that the only geometric invariants of these flat connections are the periods of the \( J \) and \( K \) subgroups around the compact circle \( \phi \). The \( t \) direction is noncompact, and so we can adjust the generator \( \hat{A}_t \) to be any constant we want by a gauge transformation.

3.5. Unequal values of the connection in the two subgroups

We list for completeness the metric that arises from the case where we have a connection with unequal magnitudes in the left and right \( su(2) \) factors. We can choose coordinates to again put the connections along \( J^3, K^3 \). Let the connection in the \( \phi \) direction \( \hat{A}_\phi \) have the form

\[ \hat{A}_\phi = \alpha M_{13} + \beta M_{42} = \alpha(J^3 + K^3) + \beta(J^3 - K^3) = (\alpha + \beta)J^3 + (\alpha - \beta)K^3 \]  \hspace{1cm} (3.24)
If we wish to make the $J$ part of the connection to be of the form $A_\phi + t$ and the $K$ part of the connection of the form $A_\phi - t$, then we let

$$
\hat{A}_t = (\alpha + \beta) J^3 - (\alpha - \beta) K^3 = \alpha (J^3 - K^3) + \beta (J^3 + K^3) = \alpha M_{42} + \beta M_{13}
$$

(3.25)

Recall that $M_{13} \to \frac{\partial}{\partial \chi}$, $M_{42} \to \frac{\partial}{\partial \psi}$. We then get the metric

$$
ds^2 = - (r^2 + 1 - \beta^2 \cos^2 \theta - \alpha^2 \sin^2 \theta) dt^2 + (r^2 + \alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\phi^2 + \frac{dr^2}{r^2 + 1}
+ d\theta^2 + 2(\alpha d\phi + \beta dt) \cos^2 \theta d\chi + 2(\alpha dt + \beta d\phi) \sin^2 \theta d\psi + \cos^2 \theta d\chi^2 + \sin^2 \theta d\psi^2
$$

(3.26)

If we start instead with a metric having a conical defect (which had been put in the form given by the primed coordinates in (3.7)) and perform the twists as above, then we would get the more general metric

$$
ds^2 = - (r^2 + \gamma^2 - \beta^2 \cos^2 \theta - \alpha^2 \sin^2 \theta) dt^2 + (r^2 + \alpha^2 \cos^2 \theta + \beta^2 \sin^2 \theta) d\phi^2 + \frac{dr^2}{r^2 + \gamma^2}
+ d\theta^2 + 2(\alpha d\phi + \beta dt) \cos^2 \theta d\chi + 2(\alpha dt + \beta d\phi) \sin^2 \theta d\psi + \cos^2 \theta d\chi^2 + \sin^2 \theta d\psi^2
$$

(3.27)

4. Localised modes for the ‘twisted geometries’

In [17] it was shown that if we consider a scalar of mass $m$ in $AdS_3$ then we obtain normalisable solutions localised near $\rho = 0$ (over a scale of the order of the $AdS$ radius) which are dual to primary operators in the boundary CFT of dimension $h = \bar{h} = 1/2(1 + \sqrt{m^2 + 1})$. It is not clear what the significance of the general ‘twisted geometry’ is, so it is not clear how to interpret solutions to the scalar wave equation for general twists. Nevertheless the spectrum of these modes appears to be interesting and solutions parallel in some ways the wavefunctions in [17] that were dual to chiral primary operators, so we carry out this computation below.

In our case the 6-d space does not factorize globally into an $AdS_3$ part and a sphere, so we look at scalars in the 6-d theory. We let the scalar be massless and minimally coupled at the linear order (there are several such scalars in the theory). For a given $R$-charge the lowest dimension CFT operators typically arise from supergravity fields that have two indices along the $S^3$. Since we are starting with a scalar, we are considering supersymmetry descendents of chiral primaries (this is analogous to the operator $tr F^2$ dual to the dilaton in $AdS_5 \times S^5$).
Consider the metric that corresponds to the connection $A^J_\phi = A^K_\phi = \alpha$:

$$ds^2 = -(1+r^2)dt^2 + \frac{dr^2}{r^2+1} + d\theta^2 + 2\alpha \cos^2 \theta d\phi d\chi + (r^2 + \alpha^2 \cos^2 \theta)d\phi^2 + \cos^2 \theta d\chi^2 + \sin^2 \theta d\psi^2$$

(4.1)

The scalar $f$ in 6-d satisfies the equation

$$\frac{1}{\sqrt{-g}} [f_{,M} g^{M N} \sqrt{-g}]_{,N} = 0$$

(4.2)

Consider the ansatz

$$f = e^{-i\omega t} e^{i m \phi} Y^{l_1,0}(y)$$

(4.3)

where $Y^{l_1,0}$ is a spherical harmonic for the scalar (the left and right spins must be equal for a scalar function: $j = k = l_1/2$, $-j \leq j_3 \leq j$, $-k \leq k_3 \leq k$), and $y$ denotes the coordinates on the sphere. With this ansatz we find the wave equation

$$\frac{1}{r} [f_{,r}(1+r^2)]_{,r} + \omega^2 (1+r^2)^{-1} f - \left[ l_1(l_1+2) + (\alpha(j_3+k_3) - m)^2 r^{-2} \right] f = 0$$

(4.4)

In Appendix E we bring this equation to a hypergeometric form that parallels the equation for the 1-form fields studied in section 2.

In [17] the wavefunctions dual to the chiral primaries had the form

$$f = e^{-i\omega t} (cosh \rho)^{-\mu} Y^{l_1,0}(y)$$

(4.5)

where $r = sinh \rho$. To get the generalisation of these special solutions to the twisted geometries consider the ansatz

$$f = e^{-i\omega t} (cosh \rho)^{-\mu} (sinh \rho)^{-\nu} Y^{l_1,0}(y)$$

(4.6)

(We have added a parameter $\nu$ since as we will see we will encounter a term singular at $\rho = 0$ in our case.)

Substituting in the wave equation we find that we need to have vanish the expression

$$\omega^2 (cosh \rho)^{-2-\mu} (sinh \rho)^{-\nu} + \mu^2 (cosh \rho)^{-\mu-2} (sinh \rho)^{-\nu+2}$$

$$+ (\mu(\nu - 2) + \nu(\mu - 2))(cosh \rho)^{-\mu} (sinh \rho)^{-\nu} + \nu^2 (cosh \rho)^{-\mu+2} (sinh \rho)^{-\nu-2}$$

$$- [2j(j+1) + 2k(k+1)](cosh \rho)^{-\mu} (sinh \rho)^{-\nu} - \alpha^2 (j_3 + k_3)^2 (cosh \rho)^{-\mu} (sinh \rho)^{-\nu-2}$$

(4.7)
The most singular terms at the origin must cancel, so we set
\[ \nu^2 = \alpha^2 (j_3 + k_3)^2 \] (4.8)

Let us take \( \alpha > 0 \). We get two choices
\[ \nu = \pm \alpha (j_3 + k_3) \] (4.9)

Equating the coefficients of the other independent expressions to zero we get
\[ -\mu^2 + \omega^2 = 0, \quad \omega = \pm \mu \] (4.10)

(We will take \( \omega > 0 \)). and
\[ \mu^2 + \nu^2 + \mu(\nu - 2) + \nu(\mu - 2) - l_1(l_1 + 2) = 0 \] (4.11)

Then (4.11) gives
\[ (\mu + \nu)(\mu + \nu - 2) - l_1(l_1 + 2) = 0 \] (4.12)

Thus we get two choices
\[ \mu + \nu = l_1 + 2, \quad \mu + \nu = -l_1 \] (4.13)

For a convergent solution at infinity we need \( \mu + \nu > 0 \). So we discard the second choice. Then we have
\[ \omega = \pm \mu = \pm [(l_1 + 2) \mp \alpha (j_3 + k_3)] \] (4.14)

Note that states with negative \( \nu \) are singular at \( \rho = 0 \) while those with positive \( \nu \) are regular. It is interesting that we get solutions that are regular both at \( r = 0 \) and at \( r = \infty \).

If we go to a gauge where the connections for the two gauge groups are purely left moving or purely right moving forms then we get the metric (3.26) (with \( \beta = 0 \)). The solutions to the wave equation are of course the same as above since we have just made a coordinate transformation. However in this new frame the frequencies can be read off from the dependence
\[ e^{-i\omega t} e^{i(j_3 - k_3)\psi} = e^{-i\omega t'} e^{i(j_3 - k_3)(\psi' + \alpha t')} = e^{-i(\omega - \alpha (j_3 - k_3))t'} e^{i(j_3 - k_3)\psi'} \] (4.15)

so that
\[ \omega' = \omega - \alpha (j_3 - k_3) \] (4.16)

and in particular for the modes (4.14)
\[ \omega' = \pm (l_1 + 2) - 2\alpha j_3, \quad \omega' = \pm (l_1 + 2) + 2\alpha k_3 \] (4.17)

for the two signs respectively in (4.14).
4.1. Including a ‘conical defect’

Let us take the metric (4.1) but let the range of $\phi$ be $0 \leq \phi \leq 2\pi \gamma$. In the ansatz (4.3) we need just take $m = m'/\gamma$ where $m'$ is an integer, and the rest of the computation for the frequencies is unaffected. But it is also interesting to see the effect of the conical defect on the specific modes (4.6). There is no explicit $\phi$ dependence, so the frequencies (4.14) in the coordinate $t$ are unaffected. But $\phi, t$ are not asymptotically AdS coordinates now, and to change to asymptotically AdS coordinates we must write $\phi' = \phi/\gamma, t' = t/\gamma$ (see eq. (3.7)). Thus in these coordinates $\phi', t'$ the frequencies are given through

$$e^{-i\omega t} = e^{-i\omega' t'/\gamma}, \quad \omega' = \omega \gamma$$

(4.18)

Thus the frequencies in the asymptotically AdS coordinates are not in general integral multiples of the natural AdS frequency.

5. Metrics from the rotating string

In [10], [11] a set of metrics was constructed that corresponded to the geometry of a D1-D5 system carrying angular momentum. The string in 6-D (after compactifying spacetime on a $T^4$) is wrapped on the circle $\phi$ and the fermions have periodic boundary conditions on this circle. Thus the D1-D5 system has Ramond type boundary conditions. The metric as given in [11] is (after adapting the notation to the one we use)

$$ds^2 \sqrt{k} = \frac{1}{h} [-dt^2 + d\phi^2] + hf[ d\theta^2 + \frac{r^2 dr^2}{(r^2 + \gamma_1^2)(r^2 + \gamma_2^2)}]$$

$$+ \frac{2}{h f} [ (\gamma_2 dt + \gamma_1 d\phi) \cos^2 \theta d\chi + (\gamma_1 dt + \gamma_2 d\phi) \sin^2 \theta d\psi]$$

$$+ [h(r^2 + \gamma_2^2) + (\gamma_1^2 - \gamma_2^2) \cos^2 \theta] \cos^2 \theta d\chi^2 + [h(r^2 + \gamma_1^2) - (\gamma_1^2 - \gamma_2^2) \sin^2 \theta] \sin^2 \theta d\psi^2,$$

$$f = r^2 + \gamma_1^2 \cos^2 \theta + \gamma_2^2 \sin^2 \theta,$$

$$h = \sqrt{k} \frac{R_5^2 Q_1}{R_y} (1 + \frac{R_5^2 Q_5}{k f})^{1/2} (1 + \frac{R_5^2 Q_5}{k f})^{1/2} \quad (5.1)$$

Here $k = N_1 N_5$, where $N_1, N_5$ are the numbers of D1 and D5 branes.

For equal rotations in the left and right $su(2)$ factors the near horizon geometry is

$$ds^2 \sqrt{k} = -(r^2 + \gamma_1^2 \cos^2 \theta)[dt^2 - d\phi^2] + \frac{dr^2}{r^2 + \gamma_1^2}$$

$$+ 2\gamma_1 d\phi d\chi \cos^2 \theta + 2\gamma_1 dt d\psi \sin^2 \theta + \cos^2 \theta d\chi^2 + \sin^2 \theta d\psi^2 \quad (5.2)$$
The coefficient of $dt^2$ is
\[-(r^2 + \gamma_1^2 \cos^2 \theta) = -(r^2 + \gamma_1^2 - \gamma_1^2 \sin^2 \theta) \quad (5.3)\]
Thus the metric (5.2) is of the form (3.27) if we have a conical defect with $\gamma = \gamma_1$ and perform ‘twists’ with $\alpha = \gamma_1$, $\beta = 0$. (The case $\gamma_1 \neq \gamma_2$ of [11] is not of the form (3.26) since the metric differs by a shift in the coefficient of $d\phi^2$ of the form $r^2 \rightarrow r^2 + C$.)

5.1. Microscopics of the $D1$-$D5$ system

To comment on some interesting aspects of the solution in [10], [11] we recall the microscopics of the $D1$-$D5$ system. We can think of the $D1$-$D5$ bound state as a set of $N_1$ $D1$-branes living inside $N_5$ $D5$-branes. Each $D$-string in the $5$-branes can be regarded as being ‘fractionated’ into $N_5$ substrings, giving $N_1N_5$ substrings in all. Each substring can be placed in a state which is spin $(0,0)$ under $su(2) \oplus su(2)$, in a state which is $(1/2, 1/2)$, or into fermionic states that are $(1/2, 0) + (0, 1/2)$. Two or more substrings can be joined into a ‘long string’, but this long string can carry only the spins that a single substring could have carried. Thus in particular if we join all the substrings into one long string then we can only get a spin of order unity, which will not be visible at the classical level. On the other hand if we do not join any of the fractionated strings to each other, and orient the spins of each to all lie in the same direction, then we can get a state with left and right angular momenta $(j_L, j_R) = (k/2, k/2)$, where $k = N_1N_5$.

(One way to see the above properties is to perform a sequence of $S$ and $T$ dualities (along the compact $T^4$ and $\phi$ directions) such that the $N_5$ $D5$-branes and $N_1$ $D$-strings transform to an $N_5$ times wound elementary string carrying $N_1$ units of momentum. The $N_1$ units of momentum can be carried by oscillators (acting on the elementary string winding state) which each carry a momentum in multiples of $1/N_5$, and a vector index of the space transverse to the string; this index can be chosen to give spin 1 in the $so(4) = su(2) \oplus su(2)$ which must be used to describe the angular momenta of the system.)

(a) First we note that the $D1$-$D5$ system that has been given an angular momentum $(j_L, j_R) = (k/2, k/2)$ (with no momentum charge) is in a different ‘phase’ from the $D1$-$D5$ system which gives the $D1$-$D5$-momentum black holes. In the 3-charge black hole, the microscopic state that is most favoured is one where the substrings of the $D1$-$D5$ system all join up into one long string [19] (thus reducing their own angular momentum to essentially zero) and then any angular momentum that the system may have is carried
by momentum modes that run up and down the string (the left moving momentum modes carry \( j_L \), the right moving ones carry \( j_R \)). The 3-charge extremal hole has only left moving momentum modes, and so we only have \( j_L \). In the limit that we reduce this momentum to zero, we get no momentum modes to carry any angular momentum, and so we are forced to \( j_L = 0, j_R = 0 \). This fact is mirrored in the fact that the left and right temperatures of the hole (which combine to give the Hawking temperature) are complex for the classical configuration that has only D1 and D5 charges, unless \( j_L = j_R = 0 \). (The general expressions for these temperatures can be found in [20].)

But from the microscopic picture we can see that starting from the 3-charge system we can reduce the momentum (and the angular momentum) to zero, but then break up the joined substrings into separate substrings (which can now each carry a spin \((1/2, 1/2)\) under \( su(2) \oplus su(2) \)); thus the system acquires angular momentum by a different mechanism from the three charge hole. Closely related to this phenomenon are the ‘phase transitions’ where the substrings join or split to maximise entropy [21] (see also [22]).

(b) Now we look at some interesting energy and angular momentum scales that appear from the geometry of [11], and find corresponding scales in the microscopic D1-D5 theory. This analysis is rough, and it is not possible to claim from this that there is in fact an identification between the physical quantities being compared. But it is nevertheless interesting to see related scales appearing in the two pictures. We will look at the case where the rotation parameters of [11] are \( \gamma_1 = 1, \gamma_2 = 0 \), which gives a nonsingular spacetime.

If we take \( R_y \) large then we find that we have a large region that is approximately \( AdS_3 \times S^3 \), and can relate the analysis around \( r = 0 \) in the full metric of the spinning string to quantities computed in the \( AdS \) geometry. In what follows we assume that such a limit has been taken.

The near horizon geometry of the D1-D5 system is governed by the number \( N_5 \). Thus a giant graviton made of a D-string has an angular momentum \( N_5 \) units [23]. This is the same as the maximum spin that can be achieved by a D1 brane when it dissolves in the D5 branes. There is a ‘twist’ between the coordinates natural to the AdS region and natural at infinity. Thus while the angular momentum of the giant graviton in the AdS region can have any orientation, these differently oriented states have different angular momenta as seen from infinity, depending on the relation between the direction of rotation and the direction of the ‘twist’ in the geometry. In particular consider the configuration where the D string is wound on the circle \( \phi \), and is extracted away from the AdS region to infinity. Here
we can have a static configuration that will have no angular momentum. This appears to be a reflection of the fact that when we extract a D1 brane from the D5 branes then it can no longer have a spin greater than unity.

(c) Note that between $r = 0$ and $r = \infty$ there is a redshift

$$\frac{g_{tt}^{1/2}(r = \infty)}{g_{tt}^{1/2}(r = 0)} = \frac{h^{1/2}(r = 0)}{h^{1/2}(r = \infty)} \approx \frac{R_y}{k^{1/4}} \quad (5.4)$$

where at $r \approx 0$ we have written $r^2 + \cos^2 \theta \approx 1$. In section 4 we had seen that there are localised solutions to the scalar wave equation in the ‘twisted’ AdS spaces. If the full geometry has a large $AdS$ type region then there can be long lived modes localised near $r = 0$. Such a mode has a wavelength of order the $AdS$ radius $\sim k^{1/4}$, and thus a frequency in the $t$ coordinate $\omega \sim k^{-1/4}$. After taking into account the redshift, this gives an energy as seen from infinity

$$E \sim k^{-1/4} \left( \frac{R_y}{k^{1/4}} \right)^{-1} \sim \frac{1}{R_y} \quad (5.5)$$

But now note that from the microscopic view of the D1-D5 system at weak coupling, the energy of excitation of the substrings is also $\sim 1/R_y$; since in the state with $(j_L, j_R) = (k/2, k/2)$ all the substrings are separately wound on a circle of length $R_y$ and not joined together into longer strings.

The scalar wavefunctions localised in the $AdS$ region have frequencies in the coordinate $t$ that satisfy $\omega \geq 1$, so that they become travelling waves at $r = \infty$. Thus any wavepacket localised near $r = 0$ will escape to infinity. But as we increase $R_y$, we will have a wavefunction that decays for a longer distance in the $AdS$ part of the geometry, thus having a smaller overlap with the travelling wave at infinity, and thus a longer lifetime. This agrees qualitatively with the microscopic intuition that excitations on a longer string take longer to ‘collide’ and escape as radiation.

(d) Now consider a D-string that is dissolved in the $N_5$ D5 branes, and let its $N_5$ substrings join up into one long string. Then we expect the microscopic picture to yield an energy of excitation

$$E \sim \frac{1}{R_y N_5} \quad (5.6)$$

Let us see if we can find such an energy scale anywhere in the classical geometry. The D-string under consideration is not joined to other substrings in the system, and can be thus a giant graviton wrapped in the $AdS$ region. The mass of this giant graviton is $N_5$ times the natural energy scale $R_{AdS}^{-1}$ of the $AdS$ space. Since $R_{AdS} \sim k^{1/4}$, we get for this
mass \( M = N_5 k^{-1/4} \). For \( AdS_3 \) the giant graviton does not have a radius that is fixed by its angular momentum; rather the angular momentum is fixed and there is no potential for the variable that gives the radius of the D-string. Thus we can have a slow variation of this radius, getting the physics of a massive particle that is moving without potential. Let us set the momentum of this radial motion to be \( P \sim 1/R_{AdS} \sim k^{-1/4} \). Then the energy of the motion is \( P^2/(2M) \). Taking into account the redshift (5.4) we get the energy as seen from infinity

\[
E \sim \left( \frac{R_y}{k^{1/4}} \right)^{-1} \frac{P^2}{M} \sim \left( \frac{R_y}{k^{1/4}} \right)^{-1} \frac{k^{-1/2}}{N_5 k^{-1/4}} \sim \frac{1}{R_y N_5} \quad (5.7)
\]

which agrees with (5.6).

6. Discussion

We have examined the equations of the fields in supergravity that have the correct quantum numbers to couple to the R-currents of the boundary CFT, and argued that these fields must be used to get the supergravity description corresponding to adding a constant external gauge potential to the boundary CFT. Such an external gauge field effectively changes the boundary condition on the fermions, and thus can be used to generate a 1-parameter family of CFTs starting with the CFT in the NS sector, which is dual to \( AdS_3 \times S^3 \). We have not followed the smooth solutions beyond the lowest order in the fields, but based on a comparison of quantum numbers we expect that for boundary values of the supergravity fields that represent flat connections in the CFT we can find solutions at the nonlinear level using only the fields that we have considered here.

But in this process we argued that one must look at smooth solutions of these supergravity fields in the interior of \( AdS \) space. (As we follow the family of configurations to larger values of the connection, at some point the solution may of course become singular as a solution of supergravity.) We can on the other hand look at ‘Wilson lines in the bulk’ which are deformations of the \( AdS_3 \times S^3 \) such that locally the space is still \( AdS_3 \times S^3 \). We looked at three different deformations of this kind, but these are spacetimes which are generically singular. While singular solutions can have a nonzero value of \( A_\phi \) as \( r \to 0 \) (which can at special values be gauged away to zero) the family of regular solutions has \( A_\phi \to 0 \) as \( r \to 0 \).

The proposal of [8] was to represent spectral flow in the CFT (which corresponds to changing boundary conditions of the fermions) by putting a Wilson line of the gauge field in
the bulk which couples to the R-current, while also including a conical defect. The singular
spacetimes could represent the supergravity solutions required to spectral flow the theory if
the action on these configurations was lower than the action on the smooth configurations;
checking this however would require resolving the singularity in string theory and then
computing the resulting action. More generally, we have to distinguish states of the CFT
from deformations of the Hamiltonian of the CFT; the former are localised near the center
of AdS while the latter correspond to solutions that grow towards the boundary [24]. The
addition of a background gauge potential to the CFT is a deformation of the Hamiltonian.
The issue of singular versus regular configurations is certainly one deserving further study.

We examined the wavefunctions of a scalar field that were localised near the center of
the AdS spaces with twists and conical defects. If we included only a twist, the shift in the
spectrum (4.14) was formally similar to the effect of spectral flow. But after we change to a
gauge where the left connection carries a left moving index and the right connection carries
only a right moving index, the form the the frequencies becomes different. (It is also not
clear how to get the momentum of the state in the AdS to correspond to the momentum
in the CFT.) If we include conical defects then the frequencies of these wavefunctions
(expressed in a coordinates that are asymptotically AdS) are not integer multiples of the
natural AdS frequency.

In [11] the solution for $\gamma_1 = 1, \gamma_2 = 0$ was shown to be nonsingular. At $r = \infty$ the
fermions of the theory are periodic around the circle $\phi$, which becomes a compact circle of
fixed radius. In this region the gauge connection has gone to zero, so we can say that we
have Ramond sector fermions. As we move to smaller $r$, the fermion wavefunction remains
periodic by continuity, but we develop a gauge connection. When we reach the region that
can be approximated by $AdS_3 \times S^3$ then the value of this connection is $\int A_\phi d\phi = 2\pi$. We
can do a coordinate transformation to get $A = 0$, but in the process the fermions in the
new coordinate system would be antiperiodic around the $\phi$ circle. This means that they
can be smoothly continued to the origin at $r = 0$. Thus in the region where the spacetime
is approximately $AdS_3 \times S^3$ we are in the NS sector in the locally correct coordinates; an
R fermion at infinity gives an NS fermion in the AdS region. Thus this case differs from

\footnote{In [9],[10] a similar construction of Wilson lines and conical defects was used, but in these
references the goal was to construct supersymmetric conical defects. In [10] the conical defect
and ‘Wilson lines in the bulk’ were argued to be related to spectral flow, though in a manner
somewhat different from [8]; the singular spacetimes were argued to be ensembles of states in the
R sector.}
the spacetime which we would obtain if we try to deform $AdS_3 \times S^3$ to get the R sector of
the CFT.

In [25] a method was found to compute the correlation functions of twist operators in
the bosonic CFT which arises from a sigma model with target space $M^N/S_N$. Here $S_N$ is
the symmetric group, so the target space is an orbifold with this symmetry group. In [26]
this computation is extended to the supersymmetric CFT which is expected to correspond
to the D1-D5 system at the orbifold point [27]. The latter computation also allows us
to compute correlation functions in the Ramond sector of the CFT, thus giving a ‘mini
black hole S-matrix’. By the observation of the preceding paragraph, for the maximally
spinning D1-D5 system, if we wish to replace the physics of the AdS region with an effective
boundary CFT then this CFT will be in the NS ground state. It would be interesting to
compare correlation functions in the CFT with calculations in the dual string theory: it is
not clear which quantities will be protected against change when we move from the orbifold
point to the point in moduli space where the supergravity description is good.

The general issue of whether we can get nonsingular metrics from branes is very
interesting. The 6-brane of type IIA theory lifts to a nonsingular brane in 11-D (the
geometry shares some features with the metric (5.1) at $\gamma_1 = 1, \gamma_2 = 0$). Also, we can
dualize any brane to a zero brane, which can be regarded as just a gravitational wave
in M-theory. The geometry formed by D3-branes is nonsingular, but the Poincare
patch appears to have a natural extension to a second region behind the horizon, which effectively
doubles up the spacetime [28]. In this case perhaps one should orbifold across the horizon,
and thus keep only one copy of asymptotic infinity; we would get additional states at the
orbifold plane which could be excited in the process of absorption by the branes. The
metrics of [10][11] provide new interesting examples where the ‘matter’ representing the
branes has effectively been completely represented by the metric and gauge fields produced
by the branes.

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Appendix A. Vector fields on the sphere

Consider the sphere $S^3$. We will need to study vector fields on this sphere, since supergravity fields like $h_{a\mu, b}$ have one index on this sphere, and so give rise to vector fields on this sphere. The lowest modes of these vector fields will give rise to the modes of interest to us, so we construct them explicitly below.

The symmetry group of this sphere is $SO(4)$, which has the algebra

$$[M_{ab}, M_{cd}] = \delta_{bc}M_{ad} + \delta_{ad}M_{bc} - \delta_{ac}M_{bd} - \delta_{bd}M_{ac}$$

The combinations

$$J_1 = \frac{1}{2}[M_{12} + M_{34}], \quad J_2 = \frac{1}{2}[M_{23} + M_{14}], \quad J_3 = \frac{1}{2}[M_{13} + M_{42}] \quad (A.1)$$

form an $SU(2)$, as do the combinations

$$K_1 = \frac{1}{2}[M_{12} - M_{34}], \quad K_2 = \frac{1}{2}[M_{23} - M_{14}], \quad K_3 = \frac{1}{2}[M_{13} - M_{42}] \quad (A.2)$$

Thus we observe the decomposition $SO(4) = SU(2) \times SU(2)$:

$$[J_i, J_j] = \epsilon_{ijk}J_k, \quad [K_i, K_j] = \epsilon_{ijk}K_k, \quad [J_i, K_j] = 0 \quad (A.3)$$

To find a representation of this algebra on vector fields on $S^3$, consider the unit sphere

$$x_1^2 + x_2^2 + x_3^2 + x_4^2 = 1 \quad (A.4)$$

We can find vector fields on the sphere that form a representation of the generators $M_{ab}$, by writing $M_{ab}$ as matrices:

$$(M_{ab})_{a'b'} = \delta_{aa'}\delta_{bb'} - \delta_{ab'}\delta_{ba'} \quad (A.5)$$

Under a generator $\epsilon M_{ab}$ we get the differmorphism of $S^3$

$$\delta X_{a'} = \epsilon (M_{ab})_{a'b'}X_{b'} \equiv \epsilon \frac{1}{2}V_{a'} \quad (A.6)$$

(Thus these vector fields are the tangent vectors arising from the nonlinear action of the rotation group on the sphere. They are Killing vector fields for $S^3$.)
The following are three vector fields arising from the combinations $J_i$: we list their components at a point $(x_1, x_2, x_3, x_4)$:

\[
V^{(1)} = \{x_2, -x_1, x_4, -x_3\}, \quad V^{(2)} = \{x_4, x_3, -x_2, -x_1\}, \quad V^{(3)} = \{x_3, -x_4, -x_1, x_2\}
\]

(A.7)

These fields have unit norm everywhere, and are orthogonal to each other at each point of the sphere.

The fields arising from the combinations $K_i$ are

\[
W^{(1)} = \{x_2, -x_1, -x_4, x_3\}, \quad W^{(2)} = \{-x_4, x_3, -x_2, x_1\}, \quad W^{(3)} = \{x_3, x_4, -x_1, -x_2\}
\]

(A.8)

The fields $V^{(i)}$ give a representation $(3, 1)$ and the fields $W^{(i)}$ give a representation $(1, 3)$ under the $SU(2) \times SU(2)$. Each set of fields give a representation 3 under the diagonal $SU(2)$ subgroup $J_i + K_i$.

Appendix B. Symmetries of anti-de-Sitter space

We can obtain the anti-de-Sitter space $AdS_3$ with unit radius as the surface

\[
-x_1^2 - x_2^2 + x_3^2 + x_4^2 = -1
\]

(B.1)

The symmetry group is $SO(2, 2) = SU(1, 1) \times SU(1, 1)$. Let

\[
\eta_{ab} = \eta^{ab} = \text{diag}\{1, 1, -1, -1\}
\]

(B.2)

be a metric that will be used to raise and lower indices. The generators of the algebra can be written as the matrices

\[
(\tilde{M}_{ab})^{a'}_{b'} = \delta^{a'}_a \eta_{bb'} - \eta_{ab'} \delta^a_b
\]

(B.3)

which satisfy the algebra

\[
[\tilde{M}_{ab}, \tilde{M}_{cd}] = \eta_{bc}\tilde{M}_{ad} + \eta_{ad}\tilde{M}_{bc} - \eta_{ac}\tilde{M}_{bd} - \eta_{bd}\tilde{M}_{ac}
\]

(B.4)

The generator $\epsilon \tilde{M}_{ab}$ acts on $AdS_3$ as

\[
\delta X^{a'} = \epsilon (\tilde{M}_{ab})^{a'}_{b'} X^{b'} \equiv \frac{1}{2} V^{a'}
\]

(B.5)
The combinations
\[ \tilde{J}_1 = \frac{1}{2} [\tilde{M}_{12} - \tilde{M}_{34}], \quad \tilde{J}_2 = \frac{1}{2} [\tilde{M}_{23} + \tilde{M}_{14}], \quad \tilde{J}_3 = \frac{1}{2} [\tilde{M}_{13} + \tilde{M}_{42}] \] (B.6)
give
\[ [\tilde{J}_1, \tilde{J}_2] = \tilde{J}_3, \quad [\tilde{J}_2, \tilde{J}_3] = -\tilde{J}_1, \quad [\tilde{J}_3, \tilde{J}_1] = \tilde{J}_2 \] (B.7)
and the combinations
\[ \tilde{K}_1 = \frac{1}{2} [\tilde{M}_{12} + \tilde{M}_{34}], \quad \tilde{K}_2 = \frac{1}{2} [\tilde{M}_{23} - \tilde{M}_{14}], \quad \tilde{K}_3 = \frac{1}{2} [\tilde{M}_{13} - \tilde{M}_{42}] \] (B.8)
also give the same algebra
\[ [\tilde{K}_1, \tilde{K}_2] = \tilde{K}_3, \quad [\tilde{K}_2, \tilde{K}_3] = -\tilde{K}_1, \quad [\tilde{K}_3, \tilde{K}_1] = \tilde{K}_2 \] (B.9)

Now we note that the combinations
\[ L_0 = i\tilde{J}_1, \quad L_1 = i\tilde{J}_2 + \tilde{J}_3, \quad L_{-1} = i\tilde{J}_2 - \tilde{J}_3 \] (B.10)
form the anomaly free part of the Virasoro algebra
\[ [L_0, L_1] = -L_1, \quad [L_0, L_{-1}] = L_{-1}, \quad [L_1, L_{-1}] = 2L_0 \] (B.11)
(We have \( L_i = L^i_{-1} \) since the \( J_i \) are anti-Hermitian.) We get a similar copy of the Virasoro algebra from the generators \( \tilde{K}_i \).

Let us now choose convenient coordinates on anti-de-Sitter space. In the relation (B.1) write
\[ x_3^2 + x_4^2 = r^2, \quad x_3 = r \cos \phi, \quad x_4 = r \sin \phi \]
\[ x_1^2 + x_2^2 = 1 + r^2, \quad x_1 = \sqrt{1 + r^2} \cos t, \quad x_2 = \sqrt{1 + r^2} \sin t \] (B.12)
Then we get on anti-de-Sitter space the metric
\[ ds^2 = -(1 + r^2)dt^2 + \frac{dr^2}{1 + r^2} + r^2 d\phi^2 \] (B.13)
We take the covering space of this space, so that \(-\infty < t < \infty\) (we still have \(0 \leq \phi < 2\pi\)).

If we write \( r = \sinh \rho \) we get
\[ ds^2 = -\cosh^2 \rho dt^2 + d\rho^2 + \sinh^2 \rho d\phi^2 \] (B.14)

Writing the action of the Virasoro generators (B.11) in terms of their action of functions (an action \( X^i \to X^i + \epsilon V^i \) gives rise to an operation \( -\epsilon V^i \partial_i \)) we get
\[ L_0 = i\partial_u \] (B.15)
\[ L_1 = ie^{iu}(\text{Coth}(2\rho)\partial_u - \frac{1}{\text{Sinh}(2\rho)}\partial_v - \frac{i}{2}\partial_\rho) \] (B.16)
\[ L_{-1} = ie^{-iu}(\text{Coth}(2\rho)\partial_u - \frac{1}{\text{Sinh}(2\rho)}\partial_v + \frac{i}{2}\partial_\rho) \] (B.17)
where \( u = t + \phi, v = t - \phi \).
Appendix C. The $N = 4$ algebra and spectral flow

The theory has a left moving and a right moving chiral algebra, which are each of the form

\[ [L_m, L_n] = (m - n)L_{m+n} + \frac{c}{12}m(m^2 - 1)\delta_{m+n,0} \quad (C.1) \]

\[ [J^i_m, J^j_n] = i\epsilon^{ijk}L_{m+n} + \frac{c}{12}m\delta_{m+n,0}, \quad [L_m, J^i_n] = -nJ^i_{m+n} \quad (C.2) \]

\[ \{G^a_r, G^b_s\} = 2\delta^a_b L_{r+s} - 2(r-s)(\sigma^i)^a_b J^i_{r+s} + \frac{c}{12}(4r^2 - 1)\delta^a_b\delta_{r+s,0} \quad (C.3) \]

\[ \{G^a_r, G^b_s\} = 0, \quad \{G^a_r, \bar{G}^b_s\} = 0 \quad (C.4) \]

\[ [L_m, G^a_r] = \left(\frac{m}{2} - r\right)G^a_{m+r}, \quad [L_m, \bar{G}^a_{r}] = \left(\frac{m}{2} - r\right)\bar{G}^a_{m+r} \quad (C.5) \]

\[ [J^i_m, G^a_r] = -\frac{1}{2}(\sigma^i)^a_b G^b_{m+r}, \quad [J^i_m, \bar{G}^a_r] = \frac{1}{2}\bar{G}^b_{m+r}(\sigma^i)^b_a \quad (C.6) \]

Under a spectral flow by parameter $\alpha$ we get

\[ h' = h - \alpha j_3 + \frac{\alpha^2}{24} \quad (C.7) \]

\[ j'_3 = j_3 - \frac{\alpha}{12} \quad (C.8) \]

Thus the NS vacuum with $h = 0, j_3 = 0$ flows for $\alpha = 1$ to $h' = \frac{c}{24}, j'_3 = -\frac{c}{12}$. This value of $h'$ corresponds to a R sector ground state.

While the spectral flow is formally just an automorphism of the algebra, we can look at the following physical situation. We increase the value of the flat connection starting from zero, and follow any chosen state of the system. Focusing for simplicity on a U(1) in the gauge group, we note that $\int Adx = 4\pi$ the theory returns to what it was at $A = 0$ (at $\int Adx = 2\pi$ the fermions are antiperiodic even though the bosons have returned to their original Lagrangian). But the state that we were following does not return to the same state, and thus we get a map from states of the system to states of the same system. This map causes the changes in quantum numbers listed above under the spectral flow.

The D1-D5 system has $c = 6N = 6Q_1Q_5$. The length scale given by the radius of the space $AdS_3$ and the sphere $S^3$ is $l = \sqrt{\alpha'}g_6^{-1}N^4$. Here $g_6$ is the 6-dimensional string coupling constant, and is given by $g_6 = g/\sqrt{v}$, where $g$ is the 10-d string coupling constant and the volume of the compact 4-d space (K3 or $T^4$) is $(2\pi)^4\alpha'^2v$. 

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Appendix D. Solving the first order equations for a 1-form

In this appendix we look at the equations (2.19) that arise from the coupled equations for the graviton and the 2-form field. Let us look at a more general set of equations of the form

\[ *dW + QW = 0 \]

where \( Q \) is a constant.

Let

\[ u = t + \phi, \quad v = t - \phi, \quad t = \frac{1}{2}(u + v), \quad \phi = \frac{1}{2}(u - v) \]

Then

\[ \epsilon_{uvr} = -\frac{1}{2} \epsilon_{\hat{t}\hat{\phi}\hat{r}} \]

We let \( \epsilon_{\hat{t}\hat{\phi}\hat{r}} = 1 \).

Since the background has translational invariance in \( t, \phi \) we can separate harmonics in these variables. We look at an ansatz of the form

\[ W_u = e^{-i\alpha u} f(r) \]
\[ W_r = e^{-i\alpha u} h(r) \]
\[ W_v = e^{-i\alpha u} q(r) \]

(The parameter \( \alpha \) here does not have any relation to the \( \alpha \) used earlier in the paper to denote the ‘twist’ of the spacetime; we are at present looking at perturbations around \( AdS_3 \times S^3 \times M_4 \) without any twist.) Then we have

\[ F_{ur} = W_{r,u} - W_{u,r} = [-i\alpha h - f']e^{-i\alpha u} \]
\[ F_{vr} = W_{r,v} - W_{v,r} = [-q']e^{-i\alpha u} \]
\[ F_{uv} = W_{v,u} - W_{u,v} = [-i\alpha q]e^{-i\alpha u} \]

This gives

\[ (*F)_u = -\frac{1}{2r}[(1 + 2r^2)(i\alpha h + f') - q']e^{-i\alpha u} \quad (D.1) \]
\[ (*F)_v = -\frac{1}{2r}[(i\alpha h + f') - (1 + 2r^2)q']e^{-i\alpha u} \quad (D.2) \]
\[ (*F)_r = -\frac{2i\alpha}{r(1 + r^2)}qe^{-i\alpha u} \quad (D.3) \]
From (D.3) we get
\[ h = \frac{2i\alpha}{Qr(1 + r^2)}q \] (D.4)

From (D.4), (D.2) we get
\[ f = \frac{2(r + r^3)q'}{Q} + (1 + 2r^2)q \] (D.5)

Putting this in (D.2) we get
\[ r(1 + r^2)[r(1 + r^2)q']' + [Q(2 - Q)r^2(1 + r^2) - \alpha^2]q = 0 \] (D.6)

Let us write
\[ Q(2 - Q) = P, \quad r^2 = x, \quad q = \left(\frac{r}{\sqrt{1 + r^2}}\right)^\alpha z \]

We then get the equation of hypergeometric type
\[ x(1 + x)z_{xx} + (\alpha + 1 + 2x)z_x + \frac{P}{4}z = 0 \] (D.7)

For \( Q = 2 \) (which is one of the cases in (2.19)) we get \( P = 0 \). Then we get the solution
\[ q = x^{\alpha/2}(1 + x)^{-\alpha/2}[\frac{1}{\alpha}(1 + x)^\alpha + Cx + D] \]

**Appendix E. The wave equation for scalars in the ‘twisted’ geometries**

Consider a massless scalar field in the ‘twisted’ geometry (4.1). For an ansatz
\[ f = e^{-i\omega t}f(r)Y^{l_1,0}(y) \]
we get the equation for the radial function
\[ \frac{1}{r}[f, r(1 + r^2)]_r + \omega^2(1 + r^2)^{-1}f - [l_1(l_1 + 2) + \alpha^2(j_3 + k_3)^2r^{-2}]f = 0 \] (E.1)

Writing
\[ r^2 = x, \quad f = r^{-\alpha(j_3 + k_3)}(1 + r^2)^{-\frac{i}{2}\omega z} \]
we get the equation of hypergeometric form
\[ x(1 + x)z_{xx} + z_x[(1 - \alpha(j_3 + k_3))(1 + x) + (1 - \omega)x] + \frac{P}{4}z = 0 \]
where
\[ P = (\omega + \alpha(j_3 + k_3))(\omega + \alpha(j_3 + k_3) - 2) - l_1(l_1 + 2) \]
The solutions in section 4 with \( \omega = l_1 + 2 - \alpha(j_3 + k_3) \) have \( P = 0, \quad z = 1 \).

If we include a dependence \( e^{im\phi} \) on the angular coordinate of AdS, then the equation (E.1) becomes
\[ \frac{1}{r}[f, r(1 + r^2)]_r + \omega^2(1 + r^2)^{-1}f - [l_1(l_1 + 2) + (\alpha(j_3 + k_3) - m)^2r^{-2}]f = 0 \] (E.2)

Thus we just get a replacement of \( \alpha(j_3 + k_3) \) by \( \alpha(j_3 + k_3) - m \) in the above discussion.
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