GLOBAL BRANCHING LAWS BY GLOBAL OKOUNKOV BODIES

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ABSTRACT. Let $G'$ be a complex semisimple group, and let $G \subseteq G'$ be a semisimple subgroup. We show that the branching cone of the pair $(G, G')$, which (asymptotically) parametrizes all pairs $(W, V)$ of irreducible finite-dimensional $G$-representations $W$ which occur as sub-representations of a finite-dimensional irreducible $G'$-representation $V$, can be identified with the pseudo-effective cone, $\overline{\text{Eff}}(Y)$, of some GIT quotient $Y$ of the flag variety of the group $G \times G'$. Moreover, we prove that the quotient $Y$ is a Mori dream space.

As a consequence, the global Okounkov body $\Delta(Y)$, with respect to some admissible flag of subvarieties of $Y$, is fibred over the branching cone of $(G, G')$, and the fibre $\Delta(Y)_{(W, V)}$ over a point $(W, V)$ carries information about (the asymptotics of) the multiplicity of $W$ in $V$. Using the global Okounkov body $\Delta(Y)$, we easily derive a multidimensional generalization of Okounkov’s result about the log-concavity of asymptotic multiplicities.

1. INTRODUCTION

Let $G'$ be a complex semisimple algebraic group, and let $G \subseteq G$ be a complex semisimple subgroup. We are interested in decomposing finite dimensional irreducible representations $V$ of $G'$ into irreducible $G$-representations;

$$V = \bigoplus_j m_j W_j,$$

where $W_j$ is an irreducible $G$-representation and $m_j = \dim(\text{Hom}_G(W_j, V))$ is its multiplicity in $V$.

By the Borel-Weil Theorem, we can reformulate the problem into a question concerning sections of line bundles as follows. Each finite-dimensional irreducible $G'$-representation can be realized geometrically as the space $H^0(G'/B', L_\lambda)$ of all sections of a line bundle $L_\lambda$ over the flag variety $G'/B'$, where $B'$ is a Borel subgroup of $G'$, and the parameter $\lambda$ is a dominant weight with respect to a maximal torus $T' \subseteq B'$ and a given choice of positive roots. Likewise, the irreducible $G$-representations are realized as spaces of sections of line bundles over $G$-flag varieties, so that the representation-theoretic problem of determining subspaces of $G$-invariants in the tensor products $W_j^* \otimes V$ amounts to the geometric problem of determining the $G$-invariant sections in the spaces $H^0(G/B \otimes G'/B', L_\mu \otimes L_\lambda)$, where $B \subseteq G$ is a Borel subgroup of $G$, and $\mu$ is a dominant weight with respect to a maximal
torus $T \subseteq G$ and a choice of positive roots for the root system of $G$. In other
words, we are interested in $G$-invariant sections of line bundles $L$ over the
flag variety $X := (G \times G')/(B \times B')$.

If $L$ is ample we can form the GIT quotient $X^{ss}(L)//G$. Then there
exists a $q \in \mathbb{N}$ and a line bundle $L_0$ over $X^{ss}(L)//G$, such that we have
isomorphisms

$$H^0(X, L^{kq})^G \cong H^0(X^{ss}(L)//G, L^k_0)$$

for all $k \in \mathbb{N}$. In other words, we have translated the problem of finding
$G$-invariant sections of powers of $L$ into the problem of studying the full
linear series of a line bundle $L_0$ on the GIT-quotient $X^{ss}(L)//G$.

Note that the quotient $X^{ss}(L)//G$ depends on the line bundle $L$. It would
be desirable to be able to simultaneously study the $G$-invariant sections of
all line bundles over $X$ in terms of understanding all sections of all line
bundles over some variety $Y$.

In this note we use VGIT to construct a “universal quotient”, $Y$, of $X$ by
$G$, yielding a geometric formulation of the problem of finding all branching
laws for the pair $(G, G')$ in the following sense: For every line bundle $L$ on $X$,
there exists a $q \in \mathbb{N}$ and a line bundle $L_0$ on $Y$, such that

$$H^0(X, L^{kq})^G \cong H^0(Y, L^k_0), \quad k \in \mathbb{N}.$$ 

In order to explain our main results in more detail, we recall that the
branching cone $\Gamma(G, G')$ of the pair $(G, G')$ is the closed convex cone (in
the direct sum of two given Cartan subalgebras of the Lie algebras of $G$
and $G'$, respectively) generated by all pairs $(\mu, \lambda)$, where $\lambda$ is a dominant
weight for $G'$, and $\mu$ is a dominant weight for $G$, such that the correspond-
ing irreducible $G$-representation occurs in the irreducible $G'$-representation
of highest weight $\lambda$. If a rational point $(\mu, \lambda)$ lies in $\Gamma(G, G')$ we thus
know that for some $k \in \mathbb{N}$ the irreducible $G$-representation $W_\mu$ occurs as
a $G$-subrepresentation in the irreducible $G'$-representation $V_\lambda$. However,
$\Gamma(G, G')$ gives no information about the multiplicity of $W_\mu$ in $V_\lambda$.

In this paper, we show how to construct a cone which is fibred over
$\Gamma(G, G')$, and whose fibres above points in $\Gamma(G, G')$ describe asymptotic
multiplicities. We collect the main results of the paper in the following the-
orem.

**Theorem 1.1.** There exists a cone $\Delta = \Delta_Y$, depending on the universal
quotient $Y$, and a surjective linear map $p : \Delta \rightarrow \Gamma(G, G')$, such that

(i) For each rational point $(\mu, \lambda)$ in the interior of $\Gamma(G, G')$,

$$\lim_{k \rightarrow \infty} \frac{\dim \text{Hom}_G(W_\mu, V_\lambda)}{k^n} = \text{Vol}_n(p^{-1}(\mu, \lambda)), \quad n \in \mathbb{N}$$

where $n$ is the dimension of the quotient $Y$, and the right hand side denotes
the $n$-dimensional volume of the fibre $p^{-1}(\mu, \lambda) \subseteq \Delta_Y$.

(ii) The branching cone $\Gamma(G, G')$ is linearly isomorphic to the presudo-
effective cone $\overline{Eff}(Y)$ of $Y$, and the cone $\Delta_Y$ is the global Okounkov body of
$Y$ with respect to some admissible flag of subvarieties of $Y$.  


(iii) The quotient $Y$ is a Mori dream space.

We would like to end this introduction with a brief history of the branching cone. It was proven by Brion and Knop (cf. [E92]) that the semigroup of all pairs $(\mu, \lambda)$ of dominant weights which generate the branching cone is a finitely generated semigroup of the group of all integral weights of the respective Cartan subalgebra of the Lie algebra $\mathfrak{g} \oplus \mathfrak{g}'$ of $G \times G'$. Berenstein and Sjamaar ([BS00]) obtained a list of inequalities determining the branching cone, which, however, is redundant. Later, Ressayre ([R10]) obtained a minimal list of inequalities defining this cone, i.e., a list where none of the inequalities is redundant. We would like to remark that Ressayre also used the identification of the branching cone with the $G$-ample cone of the flag variety $X = G/B \times G'/B'$. For a more thorough history of the branching cone, involving special cases, we refer to [R10].

As a final remark on branching cones, we point out that in general $G'$ and $G$ are only assumed to be reductive, and not semisimple, whereas we have chosen the more restrictive semisimple setting. The reason is not merely technical, as in the use of the (essential) uniqueness of the moment map in Section 3, but also since we do not expect the result on the identification of the branching cone with a pseudo-effective cone to hold in the more general setting (cf. Remark 4.4).

Finally, we would like to mention a couple of other approaches to the branching problem. First of all, instead of studying $G$-invariant sections of line bundles over the product $X$, one could study sections of line bundles over $G'/B'$, or indeed any projective $G$-variety, which are invariant under a maximal unipotent subgroup $U$ of $G$, and consider the Okounkov bodies defined using only $U$-invariants. This was the original approach in Okounkov’s famous paper ([O96]), and it was later generalized by Kaveh and Khovanskii ([KK10]). (Our main reason for passing to quotients is that we want to get rid of all conditions of invariance under any subgroup in order to be able to use generic arguments, e.g., Bertini-type arguments (cf. Remark 5.3).)

Secondly, C. Manon ([M11], [M14]) has also addressed the problem of constructing cones above the branching cones using Okounkov bodies. However, these Okounkov bodies are of a more combinatorial nature than in our approach.

The paper is organized as follows. In Section 2 we introduce the setting and recall some results about the variation of GIT quotients; VGIT. In Section 3 we study the descent of line bundles over the flag variety $X$ to various GIT quotients of $X$. In Section 4 we construct a GIT quotient $Y$ which is a Mori dream space, and we identify the pseudo-effective cone, $\text{Eff}(Y)$, of $Y$ with the $G$-ample cone $C^G(X)$, i.e., with the branching cone for the pair $(G, G')$. Finally, in Section 5 we use global Okounkov bodies of $Y$ to study asymptotics of the function $k \mapsto \dim \text{Hom}_G(W_{k\mu}, V_{k\lambda})$, $k \in \mathbb{N}$, for a fixed pair $(\mu, \lambda)$ in the branching cone of $(G, G')$. Using the global Okounkov body of $Y$, we easily derive a multi-dimensional generalization of a log-concavity result for multiplicities due to Okounkov ([O96]).

Acknowledgement: I would like to thank M. Brion for interesting discussions about GIT during the preparation of this paper, as well as for
helpful remarks on a preliminary version. I am also grateful to V. Tsanov for comments.

2. Preliminaries

Let \( T \subseteq G, T' \subseteq G' \), and \( B \subseteq G \) and \( B' \subseteq G' \) be maximal tori, and Borel subgroups of \( G' \) and \( G \), respectively, such that \( T \subseteq B \) and \( T' \subseteq B' \). Let \( g, g', b, b', t, t' \) be the Lie algebras of the groups \( G, G', B, B', T, \) and \( T' \). Fix a choice of roots \( \mathcal{R}^+ \) for the root system on \( g \oplus g' \) with respect to the Cartan subalgebra \( t \oplus t' \) such that the Borel subalgebra \( b \oplus b' \) is the direct sum of \( t \oplus t' \) and the sum of the negative root spaces,

\[
b \oplus b' = t \oplus t' \oplus \bigoplus_{\alpha \in \mathcal{R}^+} (g \oplus g')_\alpha.
\]

Let \( X := G/B \times G'/B' \) be the product of the corresponding flag varieties. Then \( G \) acts on \( X \) by the diagonal action

\[
(f, (gB, hB')) := (fgB, fhB'), \quad f, g \in G, \ h \in G'.
\]

Now, let \( U \subseteq G \) and \( U' \subseteq G' \) be maximal compact subgroups, such that the complex maximal tori \( T \) and \( T' \) are complexifications of maximal tori \( T_\mathbb{R} \subseteq U \) and \( T'_\mathbb{R} \subseteq U' \), respectively. Then the flag variety \( X \) is naturally isomorphic as a complex manifold to the quotient \((U \times U')/(T_\mathbb{R} \times T'_\mathbb{R})\), and can thus also be realized as a coadjoint \((U \times U')\)-orbit in the dual of Lie algebra \( u \oplus u' \) of \( U \times U' \).

For a line bundle \( L \to X \), and a section \( s \in H^0(X, L) \), let

\[
X_s := \{ x \in X \mid s(x) \neq 0 \}.
\]

The set of semi-stable points of \( X \) with respect to \( L \) is the set

\[
X^{ss}(L) := \{ x \in X \mid \exists m \in \mathbb{N} \exists s \in H^0(X, L^m) \text{ such that } s(x) \neq 0 \}.
\]

The set of stable points of \( X \) with respect to \( L \) is the set

\[
X^s(L) = \{ x \in X^{ss}(L) \mid \text{the orbit } Gx \text{ is closed in } X^{ss}(L) \text{ and } Gx \text{ is finite} \}
\]

Let \( N^1(X) \cong \mathbb{Z}^{\dim(T \times T')} \) be the Néron-Severi group of \( X \), and let \( N^1(X)_\mathbb{R} := N^1(X) \otimes_{\mathbb{Z}} \mathbb{R} \). We recall the following definitions (cf. [DH98]).

**Definition 2.1.** (i) A line bundle \( L \to X \) is \( G \)-effective, if \( X^{ss}(L) \neq \emptyset \) and \( X_s \subseteq X \) is affine.

(ii) The \( G \)-ample cone \( C^G(X) \) is the closed convex cone in \( N^1(X)_\mathbb{R} \) generated by all \( G \)-effective ample line bundles.

3. Descent of Line Bundles to Quotients

In this section we study the descent of line bundles to GIT quotients of \( X \), in particular its dependence on the location of the line bundle used to define semi-stability in the \( G \)-ample cone; more precisely, in terms of the GIT chambers in \( C^G(X) \). In doing so, although we have otherwise taken the algebro-geometric approach, we will use transcendental methods, notably the moment maps of the manifold \( X \) with respect to various symplectic forms and the action of a maximal compact subgroup of \( G \). In the following sections we will, however, only be interested in quotients defined
by a chamber in $C^G(X)$, as in Theorem 3.6 ii), and for this purpose the
algebro-geometric approach is sufficient.

Let $t$ and $t'$ be the Lie algebras of $T$ and $T'$, respectively, and let $P^+(t \oplus t')$
denote the set of dominant integral weights of the Lie algebra $t \oplus t'$ with
respect to the positive system $\Phi^*$.

The Cox ring $\text{Cox}(X)$, or total coordinate ring, of $X$ is then the $P^+(t \oplus t)$-graded ring

$$\text{Cox}(X) = \bigoplus_{\nu \in P^+(t \oplus t)} H^0(X, L_\nu),$$

where $L_\nu := (G \times G') \times_\nu \mathbb{C}$ is the line bundle on $X$ induced from the character
of $B \times B'$ defined by $\nu \in (t \oplus t)^*$. Since the multiplication map

$$H^0(X, L_\nu) \otimes H^0(X, L_{\nu'}) \to H^0(X, L_{\nu + \nu'})$$

is surjective for all pairs of dominant weights $\nu$ and $\nu'$, the Cox ring of $X$
is finitely generated, namely by the $H^0(X, L_\nu)$, where $\nu$ is a fundamental weight. Hence, the subring of invariants

$$\text{Cox}(X)^G := \bigoplus_{\nu \in P^+(t \oplus t)} H^0(X, L_\nu)^G,$$

where $H^0(X, L_\nu)^G$ denotes the subspace of $G$-invariant sections of $H^0(X, L_\nu)$,
is also finitely generated. Moreover, the generators can be chosen to be ho-

gogeneous with respect to the $P^+(t \oplus t)$-grading. Thus, let

$$s_i \in H^0(X, L_{\nu_i})^G, \ i = 1, \ldots, r$$

be a set of homogeneous generators of $\text{Cox}(X)^G$. For each $i \in \{1, \ldots, r\}$, let
$D_i$ be a divisor with $\mathcal{O}_X(D_i) \cong L_{\nu_i}$.

**Proposition 3.1.** (i) The cone $C^G(X)$ is the convex cone in $N^1(X)_{\mathbb{Z}}$ generated by the divisors $D_1, \ldots, D_r$.

(ii) The cone $C^G(X)$ is of full dimension in $t^* \oplus (t')^*$ if no non-zero ideal of the Lie algebra $g$ is an ideal in $g'$.

**Proof.** We first claim that $C^G(X)$ is nonempty. Indeed, we can choose
Cartan subalgebras $t_0 \subseteq g, t'_0 \subseteq g'$ and Borel subalgebras $b_0 \subseteq g, b'_0 \subseteq g'$,
containing the respective Cartan subalgebras, such that $t_0 \subseteq t'_0$ and $b_0 \subseteq b'_0$.
The restriction of linear functionals then defines a surjective linear map
$p : (t_0)^* \to (t'_0)^*$ such that the dominant chamber in $(t'_0)^*$, with respect to
the positive system defined by the choice of the Borel subalgebra $b'_0$,
is mapped into the dominant chamber of $(t_0)^*$ defined by $b_0$. Since $p$ is
surjective, some regular dominant weight $\lambda_0 \in (t'_0)^*$ is mapped to a regular
dominant weight $\mu_0 \in (t_0)^*$. If $V_{\lambda_0}$ is the highest weight representation of
$G'$ - with respect to the Borel subalgebra $b'_0$ - with highest weight $\lambda_0$, and
$v_{\lambda_0} \in V_{\lambda_0}$ is a highest weight vector, then the $G$-submodule generated by
$v_{\lambda_0}$ is a highest weight representation of $G$, with $v_{\lambda_0}$ being a highest weight
vector of highest weight $p(\lambda_0) = \mu_0$. Hence, the pair $(\lambda_0, \mu_0)$ defines a pair of regular weights in the branching cone of $(G, G')$ (defined by this new
choice of Borel subalgebras). The corresponding line bundle over $X$ is thus
ample and admits $G$-invariant sections. This shows that the cone $C^G(X)$ is nonempty.

Since $X$ is a homogeneous variety, so that every effective divisor is nef (cf. [L04, Example 1.4.7]), (i) now follows by a straightforward approximation of nef divisors by ample divisors, using the fact that the ample $\mathbb{R}$-divisors in $C^G(X)$ form a dense subset of $C^G(X)$.

For (ii), we refer to [R10]. □

Let $V^G(X) \subseteq \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R} \cong (t^* \oplus (t')^*)_{\mathbb{R}}$, where the subscript indicates that the right hand side is viewed as a vector space over $\mathbb{R}$, be the real vector space generated by the cone $C^G(X)$.

We recall ([DH98]) that the equivalence relation $\sim$ on the set of equivalence classes of line bundles over $X$ defined by $L_\nu \sim L_{\nu'}$ if and only if $X^{ss}(L_\nu) = X^{ss}(L_{\nu'})$ extends to an equivalence relation on the $G$-ample cone $C^G(X)$. The equivalence classes are called GIT-equivalence classes. A special case of GIT-equivalence classes is given by the chambers.

**Definition 3.2.** A GIT-equivalence class $C \subseteq C^G(X)$ is called a chamber, if $X^{ss}(\ell) = X^{s}(\ell)$, for every $\ell \in C$.

Recall here that the notions of semi-stability and stability extend to $\mathbb{R}$-divisors (cf. [DH98]). By VGIT (cf. [DH98, R00]) there are only finitely many GIT-equivalence classes. In particular, there exists a GIT-equivalence class $C \subseteq C^G(X)$ with non-empty interior. In fact, chambers exist in our setting (cf. [DH98, Cor. 4.1.9]).

Let $D \in C^G(X)$ be an ample integral divisor, $L = \mathcal{O}_X(D)$ the corresponding line bundle, and let $Y := X^{ss}(L)/G$, with projection morphism $\pi : X^{ss}(L) \to Y$, be the GIT quotient defined by $L$. Recall that a line bundle $L_\mu$ is said to descend to a line bundle on $Y$ if the sheaf on $Y$ defined by

$$\mathcal{F}(U) := H^0(\pi^{-1}(U), L_\mu)^G,$$

for $U \subseteq Y$ open, is an invertible sheaf of $\mathcal{O}_Y$-modules, i.e., a line bundle over $Y$, and that the isomorphism of $G$-line bundles

$$\pi^*(\mathcal{F}) \cong L_\mu$$

holds on the open subset $X^{ss}(L)$ of $X$. In particular, there exists a $q \in \mathbb{N}$, such that $L^q$ descends to a ample line bundle on $Y$ (cf. [S95]).

In order to study descent of line bundles from $X$ to the quotient $Y$, we recall some facts about the structure of $X$ as a symplectic manifold.

If $\nu \in P^*(t \oplus t')$ is any regular dominant weight, there is a natural identification of $X$ with the $(U \times U')$-coadjoint orbit $\mathcal{O}_\nu \subseteq (u \oplus u')^*$, and this inclusion map is the moment map for the $(U \times U')$-action with respect to the Kirillov-Kostant-Souriau symplectic form $\omega_{\nu}$ on $\mathcal{O}_\nu$. The moment map $\Phi_\nu : \mathcal{O}_\nu \to u^*$ for the action of the subgroup $U$ on $\mathcal{O}_\nu$ is simply given be
restriction of linear functionals to $u$, i.e., we have

$$(\Phi_\nu(\lambda))(x) = \lambda(x), \quad \lambda \in \mathcal{O}_\nu \subseteq (u \oplus u')^*, \quad x \in u.$$ 

If $L_\nu$ is the ample line bundle defined by $\nu$, and

$$f_{L_\nu} : X \to \mathbb{P}(H^0(X, L_\nu)^*)$$

is the associated projective embedding, the moment map $\Phi_\nu$ equals the pull-back, $\Phi_{L_\nu}$, by $f_{L_\nu}$ of the moment map defined by the Fubini-Study symplectic form on the projective space $\mathbb{P}(H^0(X, L_\nu)^*)$, up to a complex scalar factor. Since there zero sets of moment maps will be our main interest, we will therefore now assume that $\Phi_\nu = \Phi_{L_\nu}$. If $L_\nu$ and $L_{\nu'}$ are ample line bundles with moment maps $\Phi_{L_\nu}$ and $\Phi_{L_{\nu'}}$, it follows from the above description of these moment maps in terms of the respective coadjoint orbits that $\Phi_\nu + \Phi_{\nu'}$ is the moment map for the tensor product line bundle $L_\nu \otimes L_{\nu'} = L_{\nu + \nu'}$. That is, the identity

$$\Phi_{L_1 \otimes L_2} = \Phi_{L_1} + \Phi_{L_2}$$

holds for all ample line bundles $L_1$ and $L_2$ on $X$.

Now, if $L_\nu$ is not ample, i.e., the weight $\nu$ is not regular, there exists a unique parabolic subgroup $P_\nu \subseteq G \times G'$ and an ample line bundle $L_\nu \to (G \times G')/P_\nu$ such that $L_{\nu} = q^* L_{\nu'}$, where $q : (G \times G')/(B \times B') \to (G \times G')/P_\nu$ is the natural quotient map. We then define $\Phi_{L_\nu} := q^* \Phi_{\nu'}$, where $\Phi_{\nu'}$ is the moment map for the action of $U$ on $(G \times G')/P_\nu$ defined by the projective embedding

$$f_{\nu'} : (G \times G')/P_\nu \to \mathbb{P}(H^0((G \times G')/P_\nu, L_{\nu'})^*).$$

In this way we have now defined a map $\Phi_{L_\nu} : X \to u^*$ for each effective line bundle $L_\nu$ on $X$.

**Lemma 3.3.** The identity

$$\Phi_{L_1 \otimes L_2} = \Phi_{L_1} + \Phi_{L_2}$$

holds for all effective line bundles $L_1$ and $L_2$ on $X$.

**Proof.** Let $\lambda_1, \ldots, \lambda_r \in (t \oplus t')^*$ be the fundamental weights of the root system of $(g \oplus g', t \oplus t')$. It then suffices to prove that

$$\Phi_{L_\nu} = m_1 \Phi_{L_{\lambda_1}} + \ldots + m_r \Phi_{L_{\lambda_r}},$$

for each dominant weight $\nu \in \mathcal{P}^+(t \oplus t')$ which has the representation

$$(1) \quad \nu = m_1 \lambda_1 + \ldots + m_r \lambda_r,$$

with $m_1, \ldots, m_r \geq 0$.

Recall that, for each dominant weight $\nu \in \mathcal{P}^+(t \oplus t')$, the complex manifold $(G \times G')/P_\nu$ is naturally identified with the coadjoint orbit $\mathcal{O}_\nu \subseteq (u \oplus u')^*$. The Kirillov-Kostant-Souriau symplectic form $\omega_\nu \in \Gamma(\mathcal{O}_\nu, \Lambda^2 T^*(\mathcal{O}_\nu))$ is induced from the map

$$T_\nu : \mathcal{O}_\nu \to \Lambda^2(u \oplus u')^*,$$

$$T_\nu(\lambda)(x, y) := \lambda([x, y]), \quad x, y \in u \oplus u'.$$
Now, (1) yields
\[ T_\nu(\text{Ad}^*(u)(\nu))(x,y) = (m_1\text{Ad}^*(u)(\nu_1)([x,y]) + \cdots + (m_r\text{Ad}^*(u)(\nu_r)([x,y]), \quad u \in U. \]

If \( P_\lambda \) is the maximal parabolic subgroup of \( G \times G' \) defined by the fundamental weight \( \lambda_i \) with \( m_i > 0 \), let \( q_i : (G \times G')/P_\nu \to X_i := (G \times G')/P_{\lambda_i} \) be the associated quotient map. It follows from (2) that
\[ \omega_\nu = \sum_{i=1}^{r} q_i^* \omega_{\lambda_i}, \]
where \( \omega_{\lambda_i} \) is the Kirillov-Kostant-Souriau symplectic form on \((G \times G')/P_{\lambda_i}\) defined by \( \lambda_i \).

Now, each \( q_i \) is \((G \times G')\)-equivariant, and in particular \( U \)-equivariant. Thus, if \( \xi, \xi_i \in \mathfrak{u} \) induces the vector field \( \xi^X \) on \( X \) and the vector field \( \xi^{X_i} \) on \( X_i \), we have
\[ (dq_i)(x)(\xi^X(x)) = \xi^{X_i}(q_i(x)), \quad x \in X, \]
so that
\[ d\Phi_{L_{\lambda_i}}(x)(v,\xi) = q_i^*(d\Phi_{L_{\lambda_i}})(x)(v,\xi) = d\Phi_{L_{\lambda_i}}(q_i(x))(dq_i(x)v,\xi^{X_i}(x)) = \omega_{\lambda_i}(dq_i(x)v,\xi^{X_i}(x)) = q_i^*\omega_{\lambda_i}(v,\xi^X(x)), \quad x \in X, v \in T_x(X), \xi \in \mathfrak{u}. \]

Hence, using (3), it follows that
\[ d\Phi_{L_\nu}(x)(v,\xi^X(x)) = \omega_\nu(v,\xi^X(x)) = m_1d\Phi_{L_{\lambda_1}}(x)(v,\xi^X(x)) + \cdots + m_r d\Phi_{L_{\lambda_r}}(x)(v,\xi^X(x)), \quad x \in X, v \in T_x(X), \xi \in \mathfrak{u}. \]

Since \( X \) is connected, and \( U \) is semi-simple, we thus get
\[ \Phi_{L_\nu} = m_1 \Phi_{L_{\lambda_1}} + \cdots + m_r \Phi_{L_{\lambda_r}}. \]

This finishes the proof. \( \square \)

We recall the following descent criterion by Kempf.

**Proposition 3.4.** [KKV89 Prop. 4.2.] The line bundle \( L_\nu \) on \( X \) descends to a line bundle on \( Y \) if and only if for every \( x \in X^{ss}(L) \) whose \( G \)-orbit \( G.x \) is relatively Zariski-closed in \( X^{ss}(L) \), the stabilizer \( G_x \) acts trivially on the fibre \((L_\nu)_x\).

We now turn to an infinitesimal version of this condition.

If the orbit \( G.x \), for \( x \in X^{ss}(L) \), is relatively closed in \( X^{ss}(L) \), the stabilizer \( G_x \) is the complexification of the stabilizer \( U_x \), i.e., \( G_x = (U_x)^C \) (cf. [Sj95 Prop. 1.6, Prop. 2.4]).

Let \( D_{L_\nu} \) be the Chern connection of \( L_\nu \) with respect to the unique Hermitian metric \( h_{L_\nu} \) defined by the 2-form \( \omega_\nu \). (Note that this is defined by pulling back the natural Hermitian metric from an ample line bundle on the flag variety \((G \times G')/P_\nu\)
For a tangent vector $v \in T_\eta(L_\nu)$, $\eta \in L_\nu$, let $v_h$ be the horizontal component of $v$ as defined by the splitting of the tangent bundle $T(L_\nu)$ induced by the connection $D_{L_\nu}$. Since the group $U$ acts on the line bundle $L_\nu$ as automorphisms preserving the Hermitian metric $h_{L_\nu}$, for any $\xi \in u$, the vector field $\xi^{L_\nu}$ on $L_\nu$ defined by $\xi$ is given by

$$\xi^{L_\nu}(\eta) = (\xi^{L_\nu}(\eta))_h + \Phi_{L_\nu}(x)(\xi)\zeta(\eta), \quad \eta \in (L_\nu)_x,$$

where $\zeta$ is the vector field on $L_\nu$ generating the $S^1$-action on $L_\nu$ defined by fibrewise multiplication (cf. [K70, Theorem 3.3.1.]).

In particular, the infinitesimal action of the Lie algebra $u_x$ of the stabilizer $U_x$ of $x$ on the fibre $(L_\nu)_x$ is given by

$$\xi^{L_\nu}(\eta) = \Phi_{L_\nu}(x)(\xi)\zeta(\eta), \quad \eta \in (L_\nu)_x.$$

A necessary condition for the descent of the line bundle $L_\nu$ to a line bundle on $Y$ is thus that the condition

\[(4) \quad \Phi_{L_\nu}(x)(\xi) = 0\]

hold for every $x \in X^{ss}(L)$ for which the orbit $G.x$ is relatively closed in $X^{ss}(L)$, and $\xi \in u_x$. On the other hand, if the condition \[(i)\] holds, the identity component $(G_x)^0$ acts trivially on $(L_\nu)_x$ for every $x \in X^{ss}(L)$ with relatively closed $G$-orbit. Therefore the action of $G_x$ on $(L_\nu)_x$ factorizes through an action of the finite group $G_x/(G_x)^0$ of connected components of $G_x$. If $q = q(x)$ is the order of this group, the action of $G_x$ on the fibre $(L_\nu)_x$ at $x$ of the $q$-th tensor power of $L_\nu$ is thus trivial. Since there are only finitely many conjugacy classes of stabilizers of points $x \in X^{ss}(L)$ with relatively closed $G$-orbit $G.x \subseteq X^{ss}(L)$, the condition \[(i)\] implies the existence of a uniform $q \in \mathbb{N}$ such that $G_x$ acts trivially on $(L_\nu)_x$, for all $x \in X^{ss}(L)$ with relatively closed $G$-orbit. Hence, the line bundle $L_\nu^q$ descends to $Y$. We have thus proved the following proposition.

**Proposition 3.5.** For a line bundle $L_\nu$ on $X$, the following are equivalent:

\[(i)\] There exists a natural number $q$, such that $L_\nu^q$ descends to a line bundle on $Y$;

\[(ii)\] For every point $x \in X^{ss}(L)$ for which the $G$-orbit $G.x$ is relatively Zariski-closed in $X^{ss}(L)$, and $\xi \in u_x$, the condition

$$\Phi_{L_\nu}(x)(\xi) = 0$$

holds.

We are now ready to state the main theorem of this section.

**Theorem 3.6.** (i) Assume that the subgroup $G$ is semi-simple. Let $F$ be a cell in $C^G(X)$, and let $E_1, \ldots, E_m \in C^G(X)$ be divisors such that for some nonempty open subset $V \subseteq (\mathbb{R}^+)^m$, the divisors $t_1E_1+\cdots+t_mE_m, (t_1, \ldots, t_m) \in V$, lie in the interior of $F$. Then, for every divisor $E$ in the subgroup of $Pic(X)$ generated by the divisors $E_1, \ldots, E_m$, there exists an $m \in \mathbb{N}$ such that the line bundle $O_X(mE)$ descends to a line bundle on $Y = X^{ss}(F)/G$. In particular, this holds if the closure $\overline{F}$ is a rational polyhedral cone generated
by the divisors $E_1, \ldots, E_m \in C^G(X)$

(ii) If $F = C$ is a chamber, and $G \subseteq G'$ is a reductive subgroup, then each integral divisor $E$ on $X$ admits a multiple $mE$, for some $m \in \mathbb{N}$, such that $\mathcal{O}_X(mE)$ descends to a line bundle on $Y$

Proof. For the first part it suffices to prove that the claim holds for all the generators $E_i, i = 1, \ldots, m$.

Let $D$ be a divisor in the interior of $F$ which can be written as a linear combination $D = \sum_{j=1}^m t_j E_j$, for some $(t_1, \ldots, t_m) \in V$. Then, for any $i \in \{1, \ldots, m\}$, the divisor $D$ can be expressed as a linear combination

$$D = \sum_{j=1}^m t_j E_j, \quad t_1, \ldots, t_m \geq 0,$$

where $t_i > 0$. By renumbering the divisors $E_i$, if necessary, we may assume that $i = 1$, and that the linear combination is of the form

$$D = \sum_{j=1}^\ell t_j E_j, \quad t_1, \ldots, t_\ell > 0,$$

where $2 \leq \ell \leq r$. Let $\Phi_j : X \to \mathfrak{u}^*$ be the moment map for the action of $U$ on $X$ with respect to the line bundle $\mathcal{O}_X(E_j), j = 1, \ldots, m$, and let $\Phi_D : X \to \mathfrak{u}^*$ be the moment map for the $U$-action with respect to $L = \mathcal{O}_X(D)$. Then, by Lemma 3.3,

$$\Phi_D = \sum_{j=1}^\ell t_j \Phi_j.$$

Now, let $x \in X^s(L)$ be a point for which the orbit $G.x \subseteq X^s(L)$ is closed in the relative Zariski topology, and let $\xi \in \mathfrak{u}_x$. Since the line bundle $\mathcal{O}_X(mD)$ descends, we have

$$\Phi_D(x)(\xi) = 0.$$

We now prove that $\Phi_1(x)(\xi) = 0$ also holds. Indeed, for any tuple $(\tau_1, \ldots, \tau_\ell) \in \mathbb{R}^\ell$ of positive rational numbers for which $D_\tau := \tau_1 E_1 + \cdots + \tau_\ell E_\ell \in F$, the $\mathbb{Q}$-divisor $D_\tau$ admits some integral multiple $pD_\tau$ such that $X^s(\mathcal{O}_X(pD_\tau)) = X^s(L)$. Therefore we have $\tau_1 \Phi_1(x)(\xi) + \cdots + \tau_\ell \Phi_\ell(x)(\xi) = 0$, by Proposition 3.5 since some positive power of the line bundle $\mathcal{O}_X(pD_\tau)$ descends to a line bundle on $Y$. Hence, the set

$$\{(\tau_1, \ldots, \tau_\ell) \in \mathbb{R}^\ell \mid \tau_1 \Phi_1(x)(\xi) + \cdots + \tau_\ell \Phi_\ell(x)(\xi) = 0\}$$

contains the open neighbourhood $V$ of the point $(t_1, \ldots, t_\ell)$. It follows that the map

$$f : \mathbb{R}^\ell \to \mathbb{R}, \quad f(\tau_1, \ldots, \tau_\ell) := \tau_1 \Phi_1(x)(\xi) + \cdots + \Phi_\ell(x)(\xi)$$

is the zero map. In particular, $f(1,0,\ldots,0) = \Phi_1(x)(\xi) = 0$. The Lie algebra $\mathfrak{u}_x$ of the stabilizer $U_x$ thus acts trivially on the fibre of $\mathcal{O}_X(E_1)$ at $x$. By Proposition 3.5 there thus exists a $q \in \mathbb{N}$ such that the line bundle $\mathcal{O}_X(qE_1)$ descends to a line bundle on $Y$. This proves the first claim.

If $G$ is semisimple, then, for divisors $E \in C^G(X)$ (ii) follows from (i) using

$$(E_1, \ldots, E_m) = (D_1, \ldots, D_r).$$
Hence, every integral divisor in \( V^G(X) \) admits an integral multiple which descends.

On the other hand, for a general reductive \( G \subseteq G' \), every stabilizer \( G_x \) of a point \( x \in X^{ss}(C) \) is finite since each \( C \)-semi-stable point is \( C \)-stable. Hence, if \( E \) is an arbitrary integral divisor on \( X \), and \( m \) is the smallest common multiple of the orders of the finite groups \( G_x \) above, the line bundle \( \mathcal{O}_X(mE) \) descends to \( Y \), for each. This proves the second claim. \( \square \)

From now on we will focus on GIT quotients \( Y = X^{ss}(D)/\!\!/G = X^{ss}(C)/\!\!/G \), where \( D \in C^G(X) \) is an ample divisor in a chamber \( C \).

Since the cone \( C^G(X) \), where divisors are naturally defined with integral weights, is of full dimension in the vector space \( V^G(X) \), the subset \( \{D_1, \ldots, D_r\} \) contains an \( \mathbb{R} \)-basis for \( V^G(X) \). Without loss of generality, assume that \( \{D_1, \ldots, D_r\} \) is such a basis. For each \( D_i, i = 1, \ldots, s \), let \( m_i \) be the smallest natural number such that \( \mathcal{O}_X(m_iD_i) \) descends to a line bundle \( \mathcal{L}_i \mid Y \). Put \( F_i := m_iD_i, i = 1, \ldots, r \), and define the \( \mathbb{R} \)-linear map

\[
\sigma : V^G(X) \to \text{Pic}(Y) \otimes \mathbb{R},
\]

\[
\sigma(x_1F_1 + \cdots + x_sF_s) := x_1\mathcal{L}_1 + \cdots + x_s\mathcal{L}_s, \quad x_1, \ldots, x_s \in \mathbb{R}.
\]

4. A Mori dream space quotient

In this section we refine the choice of the quotient \( Y \) from the previous section.

**Proposition 4.1.** There exists an ample divisor \( D \in C^G(X) \) such that the set of unstable points \( X \setminus X^{ss}(D) \) is of codimension at least two. Moreover, \( D \) can be chosen to lie in a chamber.

**Proof.** Let \( D' \in C^G(X) \) be an arbitrary divisor, and let \( Y = X^{ss}(D')/\!\!/G \) be the associated quotient. By replacing \( L' = \mathcal{O}_X(D') \) with some power, if necessary, we may assume that \( L' \) descends to a line bundle \( L'_0 \) on \( Y \).

Since \( L'_0 \) is semi-ample, the multiplication maps

\[
H^0(Y, (L'_0)^k) \otimes H^0(Y, (L'_0)^\ell) \to H^0(Y, (L'_0)^{k+\ell})
\]

are surjective for \( k \) and \( \ell \) sufficiently big ([L04, Example 2.1.29]). In particular, for \( k \) big enough, the section ring \( \bigoplus_{m=1}^{\infty} H^0(Y, (L'_0)^{km})^G \) is generated in degree one. Hence, the ring of invariants \( \bigoplus_{m=1}^{\infty} H^0(X, (L^m)^G) \) is generated in degree one (cf. Remark 4.3). By again replacing \( L' \) by a sufficiently high power, we may assume that \( k = 1 \), so that the linear series \( \bigoplus_{m=1}^{\infty} H^0(X, (L^m)^G) \) is generated in degree one. The set of unstable points \( X \setminus X^{ss}(D') \) thus equals the common zero set of all sections in \( H^0(X, L^m)^G \). Let \( E_1, \ldots, E_r \) be the one-dimensional irreducible components of \( X \setminus X^{ss}(D') \), and, for each \( i \), let \( m_i \in \mathbb{N} \) be the smallest order to which some section in \( H^0(X, L'_0)^G \) vanishes along \( E_i \). Since \( X \setminus X^{ss}(D) \) is \( G \)-invariant, each \( E_i \) is also \( G \)-invariant. Hence, each \( E_i \) is the zero set of a section \( \xi_i \in H^0(X, \mathcal{O}_X(E_i))^G \). Now, for each \( g \in G \), the section \( g, \xi_i \in H^0(X, \mathcal{O}_X(E_i))^G \) vanishes on the zero set, \( E_i \), of \( \xi_i \), so that, by the normality of \( X \), \( g, \xi_i \mid \xi_i \) defines a global regular function on \( X \), i.e., \( g, \xi_i = c(g)\xi_i \), for some \( c(g) \in \mathbb{C} \). The function \( g \mapsto c(g) \) then defines a character of \( G \), so
it has to be constant by the semisimplicity of \( G \); that is, the section \( \xi_i \) is \( G \)-invariant.

Since each section \( s \in H^0(X, L')^G \) is divisible by the section

\[
\xi := \xi_1^{m_1} \cdots \xi_{\ell}^{m_\ell} \in H^0(X, \mathcal{O}_X(E))^G,
\]

where \( E := m_1 E_1 + \cdots + m_{\ell} E_{\ell} \), the map

\[
\varphi : \bigoplus_{k=1}^{\infty} H^0(X, \mathcal{O}_X(k(D' - E)))^G \to \bigoplus_{k=1}^{\infty} H^0(X, \mathcal{O}_X(kD'))^G
\]

\[
\varphi(s) := s \cdot \xi^k, \quad s \in H^0(X, \mathcal{O}_X(k(D' - E)))^G
\]
defines an isomorphism of graded rings. Put \( D'' := D' - E \) and \( L'' := \mathcal{O}_X(D'') \).

By construction, the unstable locus of \( \mathcal{O}_X(D'') \) is a subset of the union of the irreducible components of \( X \setminus X^{ss}(D') \) which are of codimension at least two. We now claim that the divisor \( D'' \) can be chosen to lie in the interior of \( C^G(X) \). Indeed, if we start with a divisor \( D_1' \), and the resulting divisor from the above construction, \( D_2'' \), would happen to lie in some face \( F_1 \) (of maximal dimension) of the boundary of \( C^G(X) \), we proceed as follows. Let \( F_2 \) be an extremal ray of \( C^G(X) \), i.e., a face of minimal dimension, such that \( F_2 \cap F_1 = \emptyset \) (we may assume here that \( \dim C^G(X) > 1 \), since \( C^G(X) \) will otherwise lie in the ample cone of \( X \)), and let \( D_2'' \in F_2 \) be an integral divisor in this face. By applying the above construction to \( D_2'' \), we get a decomposition of \( D_2'' \) as the sum \( D_2'' = D_2' + E_2 \), where \( D_2' \) has an unstable locus of codimension at least two. Since \( D_2' \) lies in the face \( F_2 \), we must also have \( D_2' \in F_2 \) and \( E_2 \in F_2 \). Then, since \( F_1 \) and \( F_2 \) are disjoint faces of \( C^G(X) \), the divisor \( D'' := D'' + D_2'' \) lies in the interior of \( C^G(X) \). In particular, \( D'' \) is ample.

In order to see that \( D'' \) also has an unstable locus of codimension at least two, assume that \( X \setminus X^{ss}(D'') \) contains some divisor \( Z \). Then, by replacing \( D_1'' \) by some multiple \( mD_1'' \), \( m \in \mathbb{N} \), we can, using the fact that the unstable locus of \( D_1'' \) contains no divisor, assume that the exists a section \( s_1 \in H^0(X, \mathcal{O}_X(D_1''))^G \) which does not vanish on \( Z \). However, for every \( m \in \mathbb{N} \), and every section \( t \in H^0(X, \mathcal{O}_X(mD_2''))^G \), the product section \( s_1^m \cdot t \in H^0(X, \mathcal{O}_X(mD''))^G \) vanishes on \( Z \). It follows that the divisor \( Z \) lies in the stable base locus of \( D_2'' \); a contradiction. This shows that \( X \setminus X^{ss}(D'') \) is of codimension at least two.

If \( D'' \) happens to lie in a chamber of \( C^G(X) \), we are done. Otherwise, \( D'' \) lies in a cell \( C \). Now choose an effective divisor \( E \) and a natural number \( k \) so that the divisor \( kD'' - E \) lies in some chamber \( C \). We now claim that the set of unstable points of \( C \), i.e., of \( kD'' - E \), is of codimension at least two. Indeed, the section ring

\[
R(kD'' - E, E) = \bigoplus_{m_1, m_2 \in \mathbb{Z}} H^0(X, \mathcal{O}_X(m_1(kD'' - E) + m_2 E))
\]
is finitely generated since \( kD'' - E \) and \( E \) are semi-ample divisors (\cite[Lemma 2.8]{HK00}). Hence, the invariant ring

\[
R(kD'' - E, E)^G = \bigoplus_{m_1, m_2 \in \mathbb{Z}} H^0(X, \mathcal{O}_X(m_1(kD'' - E) + m_2 E))^G
\]
is also finitely generated. Thus, there exists an $\ell \in \mathbb{N}$ such that, for $m \geq \ell$, any invariant section $s \in H^0(X, \mathcal{O}_X(mkD''))^G$ can be written as

$$s = \sum_{j=1}^{p} s_j t_j,$$

for some $s_j \in H^0(X, \mathcal{O}_X(m(kD'' - E)))^G$, $t_j \in H^0(X, \mathcal{O}_X(mE))^G$, $j = 1, \ldots, p$. Now, if all invariant sections $t \in H^0(X, \mathcal{O}_X(m(kD'' - E)))^G$, $m \in \mathbb{N}$, would vanish on some divisor $Z$ of $X$, then every $s \in H^0(X, \mathcal{O}_X(mkD''))^G$, for $m$ sufficiently big, would also vanish on the divisor $Z$ - in contradiction to the fact that the unstable locus of $D''$ contains no divisors. Hence, the set of unstable points of $D := kD'' - E$ is of codimension at least two. \qed

Now, let $D \in C^G(X)$ be an ample divisor which satisfies the conditions in the above proposition, and put $L := \mathcal{O}_X(D)$.

**Theorem 4.2.** (i) For any integral divisor $F \in C^G(X)$ such that $\mathcal{O}_X(F)$ descends to a line bundle $\mathcal{O}_Y(F_Y)$ on $Y$,

$$H^0(Y, \mathcal{O}_Y(F_Y)) \cong H^0(X^{ss}(L), \mathcal{O}_X(F)|_{X^{ss}(L)})^G \cong H^0(X, \mathcal{O}_X(F))^G.$$  

(ii) The Picard group $\text{Pic}(Y)$ is finitely generated, and the restriction of the map \([5]\) defines an isomorphism of cones

$$\sigma \mid_{C^G(X)} : C^G(X) \xrightarrow{\cong} \overline{\text{Eff}}(Y) \subseteq \text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}.$$

**Proof.** First of all, the natural restriction map $\text{Pic}(X) \to \text{Pic}(X^{ss}(L))$ defines an isomorphism of groups, since $X \smallsetminus X^{ss}(L)$ does not contain any divisors.

For (i), we first note that the first isomorphism holds by the definition of the sheaf $\mathcal{O}_Y(F_Y)$. Moreover, since $X$ is normal, and the unstable locus of $L$ is of codimension at least two, any section $s \in H^0(X^{ss}(L), \mathcal{O}_X(F)|_{X^{ss}(L)})$ extends uniquely to a section $S$ of $\mathcal{O}_X(F)$ over $X$. If $s$ is $G$-invariant, then so is $S$, since the identity $g.S = S$ holds on $X$ if it holds on $X^{ss}(L)$. This shows that the second isomorphism holds. This proves (i).

The natural restriction map

$$(6) \quad \text{Pic}(X) \to \text{Pic}(X^{ss}(L))$$

defines an isomorphism of groups, since $X \smallsetminus X^{ss}(L)$ does not contain any divisors.

If $E$ is a divisor on $Y$, the divisor $\pi^*(E)$ on $X^{ss}(L)$ extends to divisor on $X$, which we will also denote by $\pi^*(E)$, by taking its closure in $X$. The map $\pi^*$ extends to an injective linear map $\pi^* : \text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R} \to \text{Pic}(X) \otimes_{\mathbb{Z}} \mathbb{R}$ of real vector spaces. Since \([3]\) is an isomorphism of groups, the map

$$\sigma : V^G(X) \to \text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$$

is an inverse to $\pi^*$, so that we have an isomorphism $V^G(X) \xrightarrow{\cong} \text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$ of real vector spaces. The map $\pi^*$ maps $\text{Pic}(Y)$ injectively into the finitely generated abelian group $\text{Pic}(X)$, so that $\text{Pic}(Y)$ is also finitely generated.
Hence, $N^1(Y)_{\mathbb{R}} \cong \text{Pic}(Y) \otimes_{\mathbb{Z}} \mathbb{R}$, so that $\sigma$ indeed defines an isomorphism between $V^G(X)$ and $N^1(Y)_{\mathbb{R}}$. By (i), we also have $\sigma(C^G(X)) = \overline{\text{Eff}}(Y)$. \hfill \Box

**Remark 4.3.** The second isomorphism in part (i) of the above theorem actually also holds when the unstable locus $X \setminus X^{ss}(L)$ is not of codimension at least two. Indeed, if $Y$ and $S$ is finitely generated.

In the general setting of a pair $(G, G')$, where $G$ and $G'$ are merely reductive, the second part of the above theorem can not hold in this generality. Indeed, if both $G'$ and $G$ have nontrivial centres, and $x \in (t')^*$ is an integral weight which exponentiates to a nontrivial character of $G'$, i.e., of the centre of $G'$, and $\lambda \in (t)^*$ is the restriction of $\lambda$, and thus exponentiates to a character of $G$, then both $(\lambda, \lambda')$ and $(-\lambda, -\lambda')$ lie in the branching cone of $(G, G')$, i.e., in $C^G(X)$. If the restricted character of $G$ is nontrivial, the points $(\lambda, \lambda')$ and $(-\lambda, -\lambda')$ in $C^G(X)$ are distinct; they both represent the trivial line bundle over $X$, but equipped with two distinct structures of a $G$-line bundle. In particular, the cone $C^G(X)$ then contains the real line through the point $(\lambda, \lambda')$. However, the pseudo-effective cone of a projective variety can not contain any lines ([LM09, Lemma 4.6]).

**Theorem 4.5.** The quotient $Y := X^{ss}(D)/G$ is a Mori dream space.

**Proof.** The variety $Y$, being the GIT quotient of a normal variety, is normal. Moreover, since $Y$ is a geometric quotient of the $\mathbb{Q}$-factorial variety $X^{ss}(D)$, it is $\mathbb{Q}$-factorial ([HK00, Lemma 2.1]). By Theorem 1.2 the group $\text{Pic}(Y)$ is finitely generated. In order to prove the claim, it thus suffices to prove the Cox ring of $Y$ is finitely generated.

For this, we first choose a rational polyhedral cone $C \subseteq \text{Pic}(Y) \otimes \mathbb{R} = N^1(Y)_{\mathbb{R}}$ such that $C$ is generated by an integral basis $\{Z_1, \ldots, Z_s\}$ for the $\mathbb{R}$-vector space $N^1(Y)_{\mathbb{R}}$, and $\overline{\text{Eff}}(Y)$ lies in the interior of $C$. (This is possible, since the pseudo-effective cone of a projective variety does not contain any lines.) Then, the divisors $Z_i$ pull back to divisors $\pi^*Z_i$ on $X$ with

$$H^0(X, \mathcal{O}_X(\pi^*(m_1Z_1 + \cdots + m_sZ_s)))^G \cong H^0(Y, \mathcal{O}_Y(m_1Z_1 + \cdots + m_sZ_s)), \quad m_1, \ldots, m_s \in \mathbb{Z}.$$ 

Now, $H^0(X, \mathcal{O}_X(\pi^*(m_1Z_1 + \cdots + m_sZ_s)))^G \neq \{0\}$ if and only if

$$\pi^*(m_1Z_1 + \cdots + m_sZ_s) \in C^G(X).$$

Let $\Gamma \subseteq V^G(X)$ be the integral lattice generated by the divisors $\pi^*Z_i$, $i = 1, \ldots, s$, and put $S(\Gamma) := \Gamma \cap C^G(X)$. By the fact that each integral divisor in
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\(V^G(X)\) admits a multiple which descends to \(Y\), the cone \(C^G(X)\) is the closed convex cone generated by the semigroup \(S(\Gamma)\). Since \(C^G(X)\) is a rational polyhedral cone, Gordan’s lemma shows that the semigroup \(S(\Gamma)\) is finitely generated, say, by divisors \(E_1, \ldots, E_\ell\). Moreover, since every effective divisor on \(X\) is semi-ample, the section ring

\[
R(E_1, \ldots, E_\ell) := \bigoplus_{m_1, \ldots, m_\ell \in \mathbb{N}_0} H^0(X, \mathcal{O}_X(m_1 E_1 + \cdots + m_\ell E_\ell))
\]

is finitely generated, so that

\[
\bigoplus_{m_1, \ldots, m_\ell \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\pi^*(m_1 Z_1 + \cdots + m_\ell Z_\ell))) \cong R(E_1, \ldots, E_\ell)
\]

is a finitely generated ring. By taking invariants, it follows that

\[
\text{Cox}(Y) = \bigoplus_{m_1, \ldots, m_\ell \in \mathbb{Z}} H^0(X, \mathcal{O}_X(\pi^*(m_1 Z_1 + \cdots + m_\ell Z_\ell))^G
\]

is a finitely generated ring. □

5. Global branching laws

In this section we use the identification of the branching cone for the pair \((G, G')\) with the pseudo-effective cone \(\overline{\text{Eff}}(Y)\) to study the global branching laws.

We first briefly recall the construction of Okounkov bodies. For a thorough treatment, we refer to the seminal papers [O96], [KK09], and [LM09]. Let \(n := \dim Y\). An admissible flag of subvarieties of \(Y\) is a flag

\[
\{p\} = Y_n \subseteq Y_{n-1} \subseteq \cdots \subseteq Y_1 \subseteq Y_0 := Y
\]

of normal irreducible subvarieties, where \(Y_i\) has codimension \(i\), \(i = 0, \ldots, n\), and where the point \(p\) is a non-singular point of each \(Y_i\). Let

\[
v : \mathbb{C}(Y)^* \to \mathbb{Z}^n
\]

be the valuation on the ring of rational functions defined by the flag \(Y_\bullet\), where \(\mathbb{Z}^n\) is equipped with the lexicographic order.

If \(E\) is an effective divisor on \(Y\), by identifying the section ring

\[
R(E) = \bigoplus_{k=0}^\infty H^0(Y, \mathcal{O}_Y(kE))
\]

of \(E\) with a subring of \(\mathbb{C}(Y)\), the valuation \(v\) yields a valuation-like function

\[
\bigcup_{k \geq 0} H^0(Y, \mathcal{O}_Y(kE)) \setminus \{0\} \to \mathbb{N}_0^n,
\]

i.e., a function having the ring-theoretic properties of a valuation, although it is only defined on nonzero homogeneous elements.

Now, let \(\Sigma \subseteq \overline{\text{Eff}}(Y)\) be the semigroup generated by all effective divisors. Using values of all effective divisors in \(\Sigma\), we define the semigroup

\[
S_{Y_\bullet}(Y) := \{(v(s), E) \in \mathbb{N}_0^n \times \Sigma \mid s \in H^0(Y, \mathcal{O}_Y(E)) \setminus \{0\}\}.
\]

Finally, we define the global Okounkov body for \(Y\), with respect to the flag \(Y_\bullet\), to be the closed convex cone

\[
\Delta_{Y_\bullet}(Y) \in \mathbb{R}^n \times \overline{\text{Eff}}(Y)
\]

generated by the semigroup \(S_{Y_\bullet}(Y)\).
For a fixed effective divisor $E$, we define the semigroup
\[ S_Y(E) := \{(v(s), k) \in \mathbb{N}_0^n \times \mathbb{N}_0 \mid s \in H^0(Y, \mathcal{O}_Y(kE)) \setminus \{0\}\}, \]
and the Okounkov body for $E$, with respect to the flag $Y_\bullet$, as the closed convex hull
\[ \Delta_{Y_\bullet} := \text{conv} \left\{ \frac{v(s)}{k} \mid (v(s), k) \in S_Y(E) \right\} \subseteq \mathbb{R}^n. \]

If $p_2: \mathbb{R}^n \times \overline{\text{Eff}(Y)} \rightarrow \overline{\text{Eff}(Y)}$ denotes the projection onto the second factor, the identity
\[ p_2^{-1}(E) \cap \Delta_{Y_\bullet}(Y) = \Delta_{Y_\bullet}(E) \tag{7} \]
then holds for every effective divisor $E \in \Sigma$ (cf. [LM09]). It therefore makes sense to define Okounkov bodies for $R$-divisors by
\[ \Delta_{Y_\bullet}(\xi) := p_2^{-1}(\xi) \cap \Delta_{Y_\bullet}(Y), \quad \xi \in \overline{\text{Eff}(Y)}. \tag{8} \]

For each $E \in \Sigma$, let $d(E) \in \{0, \ldots, n\}$ be the dimension of $\Delta_{Y_\bullet}(E)$, i.e., the dimension of the smallest subspace of $\mathbb{R}^n$ containing the convex compact set $\Delta_{Y_\bullet}(E)$, and let $\text{Vol}_{d(E)}(\Delta_{Y_\bullet}(E))$ denote the volume of $\Delta_{Y_\bullet}(E)$ with respect to the Lebesgue measure on $\mathbb{R}^{d(E)}$. In particular, if $E$ is big, that is, in the interior of $\overline{\text{Eff}(Y)}$, then $d(E) = n$. The volume of the Okounkov body $\Delta_{Y_\bullet}(E)$, for a big divisor $E \in \Sigma$, encodes the asymptotics of the spaces of sections $H^0(Y, \mathcal{O}_Y(kE)), k \in \mathbb{N}$, by the identity
\[ \lim_{k \rightarrow \infty} \frac{\dim H^0(Y, \mathcal{O}_Y(kE))}{k^n} = \text{Vol}_n(\Delta_{Y_\bullet}(E)) \tag{9} \]
(cf. [LM09]). Hence, we have the following theorem.

**Theorem 5.1.** If $F \in C^G(X)$ is an ample divisor in the interior of the ample cone $C^G(X)$ which descends to a divisor $F_Y$ on $Y$, then
\[ \lim_{k \rightarrow \infty} \frac{\dim H^0(X, \mathcal{O}_X(kF))}{k^n} = \text{Vol}_n(p_2^{-1}(F_Y) \cap \Delta_{Y_\bullet}(Y)). \]

**Proof.** The claim follows immediately from the identities (7), (9), and the isomorphism
\[ H^0(X, \mathcal{O}_Y(kF))^G \cong H^0(Y, \mathcal{O}_Y(kF_Y)), \quad k \in \mathbb{N}. \]

As an immediate corollary of the log concavity of the volume function in the interior of the pseudo-effective cone ([L04, p. 157], [LM09, Cor. 4.12]), we obtain the following generalization of Okounkov’s result ([O96]) on log concavity of asymptotic multiplicities.

**Theorem 5.2.** The function
\[ f : \text{Int}(\overline{\text{Eff}(Y)}) \rightarrow \mathbb{R}, \quad f(\xi) := \log(\text{Vol}_n(\Delta_{Y_\bullet}(\xi))) \]
(cf. (8)) is concave.
Remark 5.3. So far in this section we have been working with an arbitrary admissible flag $Y$, and the shape of the global Okounkov body $\Delta_Y(Y)$ of course depends on this flag. An interesting question is now whether $Y$ admits an admissible flag $Y$ for which $\Delta_Y(Y)$ is a rational polyhedral cone. In particular, each single Okounkov body $\Delta_Y(E)$, being a fibre of $\Delta_Y(Y)$, would then be a rational polyhedral polytope, and the asymptotics of the corresponding branching law would be (approximately at least) given by counting integral points of this polytope.

Note that we can at least make one slice of $\Delta_Y(Y)$ rational polyhedral by a suitable choice of $Y$. Indeed, if we choose the $Y_i$, for $i = 1, \ldots, n-1$, to be complete intersections of generic divisors in the linear system of the line bundle $L_0^q$ on $Y$, and the point $Y_n \in Y_{n-1}$ generically, then the Okounkov body of the divisor of $L_0^q$ is, up to scaling, a standard simplex (cf. [Se12]). An interesting question is now for which other divisors on $Y$ a flag $Y$ of this type gives a rational polyhedral Okounkov body. A natural first attempt would be to study the Okounkov bodies $\Delta_Y(E)$ of divisors $E$ coming from the interior of the GIT-equivalence class $C$ of $L = O_X(D)$.

In this context, it may be very useful to know that $Y$ is a Mori dream space, since the pseudo-effective cone $\overline{\text{Eff}}(Y)$ exhibits a chamber structure similar to that of a smooth projective surface. It is therefore conceivable that the results in ([SS14]), notably the construction of a Minkowski base for an Okounkov body, could be generalized to our setting, or even to general Mori dream spaces.

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