THE INTERIOR GRADIENT ESTIMATE FOR SOME NONLINEAR CURVATURE EQUATIONS

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Abstract. In this paper, we obtain the interior gradient estimate of some nonlinear equations which arise naturally from prescribed curvature problem of graphs in hyperbolic space. The method depends on the maximum principle.

1. Introduction. In this paper, we study the interior gradient estimate of the equation

$$\sigma_k(\lambda\{A_{ij}\}) = H(x,u,\nu)(1 + |\nabla u|^2)^{\frac{k}{2}},$$

where

$$A_{ij} := \delta_{ij} + \left( \frac{u_{ij}}{w(w+1)} \right) - \sum_{l=1}^{n} \frac{u_{il}u_{lj}}{w(w+1)} + \sum_{m,l=1}^{n} \frac{u_{im}u_{ml}u_{lj}}{w^2(w+1)^2},$$

with \( w := \sqrt{1 + |\nabla u|^2} \) and \( \lambda\{A_{ij}\} \) denote eigenvalues of the matrix \( \{A_{ij}\} \), and \( \sigma_k(\lambda) \) denotes the \( k \)-th elementary symmetric function of \( \lambda \). (see [2])

This equation arises naturally from prescribed curvature problem of graphs in hyperbolic space, see for example [10, 11] and the references therein. To study Weingarten hypersurfaces of prescibed curvature in hyperbolic space \( H^{n+1} \), we seek a complete hypersurface \( \Sigma \subset H^{n+1} \) satisfying

$$f(\kappa[\Sigma]) = H(x,u,\nu),$$

where \( f \) is a smooth symmetric function of \( n \) variables and \( \kappa[\Sigma] := (\kappa_1, \ldots, \kappa_n) \) denotes the induced hyperbolic principal curvatures of \( \Sigma \).

For convenience, we use the half-space model,

$$H^{n+1} = \{(x,x_{n+1}) \in \mathbb{R}^{n+1} : x_{n+1} > 0\},$$

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equipped with the hyperbolic metric
\[ ds^2 = \sum_{i=1}^{n+1} \frac{dx_i^2}{x_i^{n+1}}. \]

Suppose \( \Sigma \) locally can be represented as the graph of some function \( u \in C^2(\Omega) \), \( u > 0 \), where \( \Omega \subset \mathbb{R}^{n+1} \):
\[ \Sigma = \{(x,u(x)) \in \mathbb{R}^{n+1} : x \in \Omega\}, \]
taking \( \nu \) be the upward unit normal vector field to \( \Sigma \):
\[ \nu = \left( -\nabla u, \frac{1}{w} \right), \quad w = \sqrt{1 + |\nabla u|^2}. \]

The induced Euclidean metric and second fundamental form of \( \Sigma \) are given by
\[ g_{ij} = \delta_{ij} + u_i u_j, \quad h_{ij} = \frac{u_{ij}}{w}, \]
and the inverse of \( g_{ij} \) and its square root can be represented as
\[ (g_{ij})^{-1} = \delta_{ij} - \frac{u_i u_j}{w(w+1)}. \]

The Euclidean principal curvatures \( \kappa^0[\Sigma] \) are the eigenvalues of the symmetric matrix
\[ b_{ij} := (g^{ik})^{1/2}(h_{kl})(g^{lj})^{1/2}, \]
using the relation between the hyperbolic and Euclidean principal curvatures (see Section 2 in \[10\] or \[11\])
\[ \kappa_i = \frac{1}{w} + u \kappa_i^0, \quad i = 1, \ldots, n. \]

That is, the hyperbolic principal curvatures \( \kappa[\Sigma] \) are the eigenvalues of the matrix
\[ \widehat{h}_{ij} := \frac{1}{w} \left[ \delta_{ij} + uw(g^{ik})^{1/2}(h_{kl})(g^{lj})^{1/2} \right], \]
so the equation (1.2) becomes
\[ f(\lambda\{\widehat{h}_{ij}\}) = H(x,u,\nu). \]

For convenience, we rewrite the above equation as
\[ F(A_{ij}) := f(\lambda\{A_{ij}\}) = H(x,u,\nu)(1 + |\nabla u|^2)^{1/2} := \tilde{f}(x,u,\nu), \]
where \( f(\lambda) = \sigma_k(\lambda) \), denote \( A_{ij} := \delta_{ij} + u a_{ij} \) with \( a_{ij} := u_{ij} - \sum_{l=1}^{n} \frac{u_i u_l u_j u_l}{w(w+1)} - \sum_{l=1}^{n} \frac{u_i u_k u_i u_l u_j u_l}{w(w+1)^2} \).

The interior gradient estimate for high order prescribed curvature equation in Euclidean space has been obtained by Korevaar \[12\] for Weingarten equations, Li \[15\] and Trudinger \[19\] for general nonlinear elliptic equations of prescribed curvatures. Later, Wang gave a new proof in \[20\] using the maximum principle, where he constructed a celebrated auxiliary function. In this paper, we adapt the method used in \[5\] and \[20\] to obtain the interior gradient estimate for equation (1.1), but here more terms are involved and should be handled with care.

The related parabolic type interior gradient estimate for curvature equation has been obtained in \[9, 7\] for mean curvature flow of graphs and \[6\] for the anisotropic
mean curvature flow of a graph-like hypersurface. Later, the interior gradient estimate has been obtained in [1] for quasilinear parabolic equations having coefficients depending on the gradient, which include both isotropic and anisotropic flows, and [16] for some modified mean curvature flow (MMCF) in hyperbolic space.

We recall the notion of admissible solution.

**Definition 1.1** (See [4]). For $1 \leq k \leq n$, define

$$
\Gamma_k := \{ \lambda \in \mathbb{R}^n : \sigma_1(\lambda) > 0, \ldots, \sigma_k(\lambda) > 0 \}.
$$

A function $u \in C^2(\Omega)$ is called admissible if $\kappa := (\kappa_1, \kappa_2, \ldots, \kappa_n) \in \Gamma_k$, where $\kappa_1, \kappa_2, \ldots, \kappa_n$ are the principal curvatures of the graph of the function $u$.

Now we are ready to establish our main theorem as follows,

**Theorem 1.2.** For any $0 < \varepsilon < 1$, let $u \in C^3(B_R(0))$ be an admissible solution of the $k$-th curvature equation (1.5), suppose $u \geq \varepsilon$, $H(x,u,\nu) \in C^1(B_R(0) \times \mathbb{R} \times \mathbb{R}^{n+1})$ with the structure condition

$$
\frac{\partial}{\partial u} (u^{-k}H(x,u,\nu)) \geq 0, \tag{1.6}
$$

and $|H^\frac{k}{n}|_{C^1} \leq C_0$. Then

$$
|\nabla u(0)| \leq \exp[\left(\frac{M}{R} + \frac{M}{\sqrt{\varepsilon}} + 1\right)^2], \tag{1.7}
$$

where $M := \sup_{B_R(0)} u$, and $C$ is a positive constant depending only on $n, k$ and $C_0$.

In particular, when $k = 1$, equation (1.5) is just the hyperbolic mean curvature equation:

$$
\frac{n}{w} + \frac{u}{w} a^{ij} u_{ij} = H(x,u,\nu), \tag{1.8}
$$

where $a^{ij} := \delta_{ij} - \frac{u_i u_j}{1+|\nabla u|^2}$. Then immediately we have the following corollary.

**Corollary 1.3.** For any $0 < \varepsilon < 1$, let $u \in C^3(B_R(0))$ be the solution of mean curvature equation (1.8), suppose $u \geq \varepsilon$, $H(x,u,\nu) \in C^1(B_R(0) \times \mathbb{R} \times \mathbb{R}^{n+1})$ with condition $uH_u \geq H \geq 0$ and $|H|_{C^1} \leq C_0$. Then

$$
|\nabla u(0)| \leq \exp[\left(\frac{M}{R} + \frac{M}{\sqrt{\varepsilon}} + 1\right)^2], \tag{1.9}
$$

where $M := \sup_{B_R(0)} u$, and $C$ is a positive constant depending only on $n, k$ and $C_0$.

**Remark 1.4.** The assumption $u \geq \varepsilon$ is necessary in the sense that equation (1.4) is degenerate at $u = 0$. The same idea has been used in [11] and [16], in which they consider the corresponding approximate problem for a fixed sufficiently small $\varepsilon > 0$.

**Remark 1.5.** The gradient estimate was obtained in [18] for minimal surface equation (i.e. $H \equiv 0$ in (1.8)), which is key for them to obtain the existence of solution for Dirichlet boundary value problem in bounded planar convex domain. In the paper [16] (Section 9), they obtained the similar interior gradient estimate results for MMCF, which can be seen as the parabolic version of Corollary 1.3. For further interesting related results, please see papers [16, 18] and references therein.

The rest of the paper is organized as follows. In section 2, we collect some basic properties of $\sigma_k$ which shall be used to prove the interior gradient estimate. In section 3, we adapt the idea used in [5] and [20] to prove Theorem 1.2.
2. Preliminary. In this section, we collect some basic facts about $k$-th elementary symmetric functions which shall be used to prove Theorem 1.2.

We denote $\sigma_i(\lambda|i)$ be the symmetric function $\sigma_i(\lambda)$ with $\lambda_i = 0$, and $\sigma_i(\lambda|ij)$ be the symmetric function $\sigma_i(\lambda)$ with $\lambda_i = \lambda_j = 0$. Also denote $\sigma_i(A|i)$ be the symmetric function with $A$ deleting the $i$-row and $i$-column, and $\sigma_i(A|ij)$ be the symmetric function with $A$ deleting the $i, j$-rows and $i, j$-columns, for all $1 \leq i, j, l \leq n$.

**Proposition 2.1.** For $\lambda = (\lambda_1, \ldots, \lambda_n) \in \mathbb{R}^n$ and $k = 1, 2, \ldots, n$, then

$$
\sigma_k(\lambda) = \sigma_k(\lambda|i) + \lambda_i \sigma_{k-1}(\lambda|i), \quad \forall 1 \leq i \leq n;
$$

$$
\sum_{i=1}^{n} \lambda_i \sigma_{k-1}(\lambda|i) = k \sigma_k(\lambda);
$$

$$
\sum_{i=1}^{n} \sigma_k(\lambda|i) = (n-k) \sigma_k(\lambda).
$$

**Proof.** See [14].

**Proposition 2.2.** If $\lambda \in \Gamma_k$, then

$$
\sigma_i(\lambda|i) > 0, \quad \forall 1 \leq l \leq k-1, \quad 1 \leq i \leq n;
$$

and

$$
\sum_{i=1}^{n} \frac{\partial \sigma_k^i(\lambda)}{\partial \lambda_i} \geq \left( C_n^k \right)^k.
$$

**Proof.** See [14].

**Proposition 2.3.** If $\lambda \in \Gamma_k$, suppose that $\lambda_1 \geq \cdots \geq \lambda_n$. Then

$$
\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_k > 0;
$$

$$
\sigma_k(\lambda) \leq C_n^k \lambda_1 \lambda_2 \cdots \lambda_k;
$$

$$
\sigma_{k-1}(\lambda) \geq \lambda_1 \lambda_2 \cdots \lambda_{k-1}, \quad 1 \leq k-1.
$$

**Proof.** The first and second inequalities can be seen in [15]. And the last inequality follows from

$$
\sigma_{k-1}(\lambda) = \sigma_{k-1}(\lambda|1) + \lambda_1 \sigma_{k-2}(\lambda|1) = \sigma_{k-1}(\lambda|1) + \lambda_1 \sigma_{k-2}(\lambda|12) + \lambda_1 \lambda_2 \sigma_{k-3}(\lambda|12)
$$

$$
= \sigma_{k-1}(\lambda|1) + \lambda_1 \sigma_{k-2}(\lambda|12) + \cdots + \lambda_1 \lambda_2 \cdots \lambda_{k-1}
$$

$$
\geq \lambda_1 \lambda_2 \cdots \lambda_{k-1}.
$$

**Proposition 2.4.** If $v \in C^2(\mathbb{R}^n)$ satisfies $\lambda(\nabla^2 v) \in \Gamma_k$, $v_{11} < 0$, $(v_{ij})_{i,j \geq 2}$ is diagonal and $v_{22} \geq \cdots \geq v_{nn}$. Then

$$
\{(v_{ij})_{2 \leq i,j \leq n}\} \in \Gamma_k;
$$

and

$$
F^{11} \geq \frac{\mathcal{F}}{n-k+1}, \quad F^{ii} = -v_{1i} \sigma_{k-2}(\nabla^2 v|1i), \quad \forall i \geq 2;
$$

and

$$
\mathcal{F} \geq v_{22} v_{33} \cdots v_{kk},
$$

where $F(\nabla^2 v) := \sigma_k(\nabla^2 v) = \sigma_k(\lambda(\nabla^2 v))$, $F^{ij} := \frac{\partial \sigma_k(v_{ij})}{\partial v_{ij}}$ and $\mathcal{F} := \sum_{i=1}^{n} F^{ii}$. 

Proof. Since for $0 \leq t \leq n - 1$,

$$
\sigma_{i+1}(\nabla^2 v) = \sigma_{i+1}(\nabla^2 v | 1) + v_{11} \sigma_i(\nabla^2 v | 1) - \sum_{i=2}^{n} v_{1i} v_{jj} \sigma_{i-1}(\nabla^2 v | 1i)
$$

$$
= + \sum_{2 \leq i \neq j \leq n} v_{1i} v_{ij} v_{jj} \sigma_{i-2}(\nabla^2 v | 1ij) + \mathcal{R},
$$

where $\mathcal{R}$ be terms which contain at least two $v_{ij} (2 \leq i \neq j \leq n)$. So when $(v_{ij})_{i,j \geq 2}$ is diagonal, we have $\mathcal{R} = 0$.

Due to $\frac{\sigma_k}{n-k+1} = \sigma_{k-1}(\nabla^2 v)$ and $F^{11} = \sigma_{k-1}(\nabla^2 v | 1)$, and combining with the Proposition 2.2, Proposition 2.3, we have proved the Proposition 2.4.

\[ \square \]

3. Proof of Main Theorem 1.2. In this section, we give the proof of main Theorem 1.2.

Proof. Consider the auxiliary function:

$$
\phi(x) := g(x) \varphi(u) \log |\nabla u|(x), \quad x \in B_R(0), \tag{3.1}
$$

where $g(x) := R^2 - |x|^2$, $\varphi(u) := 1 + \frac{u}{M}$. Note that, $1 \leq \varphi(u) \leq 2$, for $0 < u \leq M$.

Suppose $\phi$ takes the maximum at point $x_0 \in B_R(0)$. By rotating the coordinate $(x_1, x_2, \ldots, x_n)$, satisfy that

$$
|\nabla u|(x_0) = u_1(x_0) > 0, \{u_{ij}(x_0)\}_{2 \leq i,j \leq n} \text{ is diagonal}. \tag{3.2}
$$

Then the function

$$
G(x) := \log g_1(x) + \log \varphi(u) + \log g(x), \quad x \in B_R(0), \tag{3.3}
$$

attains local maximum at the point $x_0 \in B_R(0)$.

All the calculations are done at $x_0$ below. And in the following, we always assume that $u_1(x_0) \geq 4$, otherwise the proof is done.

So at $x_0$, the first derivative of the auxiliary function has,

$$
0 = G_i = \frac{g_1}{g} + \frac{\varphi_i}{\varphi} + \frac{u_{1i}}{u_1 \log u_1}, \tag{3.4}
$$

As $u_j(x_0) = 0, \forall j \geq 2$, hence we have

$$
\frac{u_{1j}}{\log u_1} = -u_1 \frac{g_{1j}}{g} (j \geq 2), \quad \frac{u_{11}}{\log u_1} = -u_1 \frac{g_1}{g} + \frac{\varphi'_{u_1}}{\varphi_{u_1}}. \tag{3.5}
$$

Taking the second derivative to the auxiliary function gives us

$$
G_{ij} = \frac{g_{ij}}{g} - \frac{g_{ij} g_1}{g^2} + \frac{\varphi_{ij}}{\varphi} - \frac{\varphi_i \varphi'_j}{\varphi^2} + \frac{u_{1ij}}{u_1 \log u_1} - \left( 1 + \frac{1}{\log u_1} \right) \frac{u_{11} u_{1j}}{u_1 \log u_1}. \tag{3.6}
$$

Using equality (3.4), we obtain

$$
\frac{g_{ij} g_1}{g^2} + \frac{\varphi_{ij} \varphi'_j}{\varphi^2} = \frac{u_{11} u_{1j}}{u_1 \log^2 u_1} - \left( \frac{\varphi_i g_j + \varphi_j g_i}{\varphi g} \right). \tag{3.7}
$$
If $u$ is admissible, so $F$ is elliptic, and by substituting the above equality into (3.6), we have

$$0 \geq \sum_{i,j=1}^{n} \tilde{F}^{ij} G_{ij}$$

$$= \sum_{i,j=1}^{n} \tilde{F}^{ij} \left[ \frac{g_{ij}}{g} - \frac{g_i g_j}{g^2} \frac{\varphi}{\varphi_i} \frac{\varphi_j}{\varphi_j} + \frac{u_{1ij}}{u_1 \log u_1} - \left(1 + \frac{1}{\log u_1}\right) \frac{u_{11} u_{1j}}{u_1^2 \log u_1} \right]$$

$$= \sum_{i,j=1}^{n} \tilde{F}^{ij} \left( \frac{g_{ij}}{g} + \frac{\varphi_{ij}}{\varphi_i} + \frac{2\varphi_i g_j}{\varphi g} \right)$$

$$+ \sum_{i,j=1}^{n} \tilde{F}^{ij} \left[ \frac{u_{1ij}}{u_1 \log u_1} - \left(1 + \frac{2}{\log u_1}\right) \frac{u_{11} u_{1j}}{u_1^2 \log u_1} \right]$$

$$:= I + II,$$

where the linearized operator coefficient

$$\tilde{F}^{ij} := F^{ij} - \sum_{l=1}^{n} F^{ij} u_{1l} \frac{u_{1l} g_{ij}}{w(1+w)} - \sum_{l=1}^{n} F^{il} u_{1j} \frac{u_{1j} g_{ij}}{w(1+w)} + \sum_{m,l=1}^{n} F^{ml} u_{1m} u_{1j} u_{1l} \frac{w_{1j} u_{1j} g_{ij}}{w^2(1+w)^2}. \quad (3.9)$$

At $x = x_0$, we have

$$\tilde{F}^{ij} = \begin{cases} \frac{F^{ij}_{11}}{w}, & i = j = 1, \\ \frac{F^{ij}_{1i}}{w}, & i \geq 2, j = 1, \\ \frac{F^{ij}_{i1}}{w}, & i, j \geq 2. \end{cases} \quad (3.10)$$

And we denote $\tilde{F} := \sum_{i=1}^{n} \tilde{F}^{ii}$ below.

First we deal with the term $I$, observe that

$$I := \sum_{i,j=1}^{n} \tilde{F}^{ij} \left[ \frac{g_{ij}}{g} + \frac{\varphi_{ij}}{\varphi_i} + \frac{2\varphi_i g_j}{\varphi g} \right]$$

$$= -\frac{2 \tilde{F}}{g} + \frac{\varphi'}{\varphi} \sum_{i,j=1}^{n} \tilde{F}^{ij} u_{1j} + \frac{2\varphi'}{\varphi g} \sum_{j=1}^{n} \tilde{F}^{1j} u_{1j} g_j. \quad (3.11)$$

By using Proposition 2.1,

$$\sum_{i,j=1}^{n} \tilde{F}^{ij} u_{1j} = \sum_{i,j=1}^{n} F^{ij} u_{1j} = \frac{1}{u} \sum_{i,j=1}^{n} F^{ij} (A_{ij} - \delta_{ij}) = \frac{1}{u} \left( k\tilde{f}(x,u,\nu) - \mathcal{F} \right), \quad (3.12)$$

and

$$\sum_{j=1}^{n} \tilde{F}^{1j} u_{1j} g_j = \tilde{F}_{11} u_{11} g_1 + \sum_{i=2}^{n} \tilde{F}^{1i} u_{1i} g_i$$

$$= \frac{u_{11} g_1}{w} \tilde{F}_{11} + \frac{u_{1i} g_i}{w} \sum_{i=2}^{n} \tilde{F}^{1i} u_{1i} g_i$$

$$= \frac{u_{11} g_1}{w} \tilde{F}_{11} + \frac{u_1}{w^2} \sum_{i=2}^{n} \left[ -u u_{1i} \sigma_{k-2} (A|1_i) g_i \right]$$

$$= \frac{u_{11} g_1}{w} \tilde{F}_{11} + \frac{u_1^2}{w^2} \sum_{i=2}^{n} \left[ u \log u_1 \sigma_{k-2} (A|1_i) g_i^2 \right],$$
where we have used (3.5) in the last equality above. Then
\[
I = -\frac{\tilde{F}}{g} + \frac{\varphi'}{\varphi u} (k \bar{f}(x, u, \nu) - \varphi) + \frac{2\varphi'}{\varphi g} u_1 g_1 F^{11} \\
= + \frac{2\varphi' u_1^2 u \log u_1}{\varphi g^2 u^2} \sum_{i=2}^{n} (\sigma_{k-2} A[1i] g_i^2)
\]
\[
\geq -\frac{2\tilde{F}}{g} + \frac{\varphi' (kH w^k)}{u} - \frac{\varphi}{\varepsilon M} + 2 \frac{\varphi' u_1 g_1 F^{11}}{w^2} \\
\geq -\frac{2\tilde{F}}{g} + \frac{\varphi' kH w^k u}{u} - \frac{\varphi}{\varepsilon M} + 4 \frac{\varphi' R F^{11}}{\varphi g w} F^{11}.
\]
Next we handle the term II in (3.8),
\[
II := \sum_{i,j=1}^{n} \tilde{F}^{ij} \frac{u_{ij}}{u_1 \log u_1} - \sum_{i,j=1}^{n} \tilde{F}^{ij} (1 + \frac{2}{\log u_1}) \frac{u_{ij} u_{j1}}{u_1^2 \log u_1} (3.14)
\]
\[
:= 1 + 2.
\]
Taking the first derivative to equation (1.5), we have
\[
D_1 \vec{f} = \sum_{i,j=1}^{n} F^{ij} A_{i,j,1} = \sum_{i,j=1}^{n} F^{ij} (u a_{ij}),1 = u_1 \sum_{i,j=1}^{n} F^{ij} a_{ij} + u \sum_{i,j=1}^{n} F^{ij} a_{ij,1}, (3.15)
\]
that is,
\[
\sum_{i,j=1}^{n} F^{ij} (A_{i,j} - \delta_{ij}) \frac{u_1}{u} + u \left( F^{11} a_{11,1} + 2 \sum_{i=2}^{n} F^{1i} a_{i1,1} + \sum_{i,j=2}^{n} F^{ij} a_{ij,1} \right) = D_1 \vec{f}. (3.16)
\]
By direct computation, we get
\[
a_{ij,k} = u_{ijk} - \frac{u_{ik} u_{ij}}{w(1+w)} + \frac{u_{ij} u_{ik}}{w(1+w)} + \frac{u_{ij} u_{ijk}}{w^2(1+w)} \frac{u_1}{w} + \frac{u_{ij} u_{ij} u_{mk}}{w^2(1+w)^2} \\
- \frac{u_{ik} u_{ij}}{w(1+w)} - \frac{u_{ij} u_{ik}}{w(1+w)} - \frac{u_{ij} u_{ijk}}{w(1+w)} + \frac{u_{ij} u_{ij} u_{mk}}{w^2(1+w)^2} \frac{u_1}{w} + \frac{u_{ij} u_{ij} u_{mk} u_{mk}}{w^2(1+w)^2} \\
+ \frac{u_{ik} u_{ij} u_{mn} u_{mk}}{w^2(1+w)^2} + \frac{u_{ik} u_{ij} u_{ij} u_{mn}}{w^2(1+w)^2} \frac{u_1}{w} + \frac{u_{ij} u_{ij} u_{mn} u_{mk} u_{mk}}{w^2(1+w)^2} + \frac{u_{ij} u_{ij} u_{ij} u_{mn} u_{mn}}{w^2(1+w)^2}. \]
Then at \( x = x_0 \),
\[
a_{11,1} = \frac{u_{111}}{w} - \frac{2u_1}{w^4} u_{11}^2 - \frac{2u_1}{w^2(1+w)} \sum_{k=2}^{n} u_{ik}^2. (3.17)
\]
For \( i \geq 2 \)
\[
a_{i1,1} = \frac{u_{i11}}{w} - \frac{u_1}{w(w+1)} u_{i1} u_{ii} - \frac{u_1(2w+1)}{w^3(1+w)} u_{11} u_{ii}. (3.18)
\]
For \( i, j \geq 2 \)
\[
a_{ij,1} = u_{ij1} - \frac{2u_1}{w(w+1)} u_{ij} u_{jj}. (3.19)
\]
By adding 1

Therefore,

\[
\sum_{i,j=1}^{n} F^{ij} a_{ij,1} = \sum_{i,j=1}^{n} \tilde{F}^{ij} u_{ij1} - \frac{2u_1}{w^4} F^{11} u_{11}^2 - \frac{2u_1}{w^2(w+1)} F^{11} \sum_{k=2}^{n} u_{1k}^2
\]

\[-2 \sum_{i=2}^{n} F^{11i} \left[ \frac{u_1}{w(w+1)} u_{i1} u_{ii} + \frac{u_1(2w+1)}{w^3(w+1)} u_{11} u_{i1} \right] - \frac{2u_1}{w(w+1)} \sum_{i,j=2}^{n} F^{ij} u_{ij1} u_{ij1}.
\]

So

\[
u \sum_{i,j=1}^{n} \tilde{F}^{ij} u_{ij1} = D_1 \tilde{f} - u_1 \sum_{i,j=1}^{n} F^{ij} a_{ij} + u \left\{ \frac{2u_1}{w^4} F^{11} u_{11}^2 + \frac{2u_1}{w^2(w+1)} F^{11} \sum_{k=2}^{n} u_{1k}^2
\right. \]

\[+ 2 \sum_{i=2}^{n} F^{11i} \left[ \frac{u_1}{w(w+1)} u_{i1} u_{ii} + \frac{u_1(2w+1)}{w^3(w+1)} u_{11} u_{i1} \right] + \frac{2u_1}{w(w+1)} \sum_{i,j=2}^{n} F^{ij} u_{ij1} u_{ij1} \}.
\]

Therefore,

(1) \[:= \sum_{i,j=1}^{n} \tilde{F}^{ij} \frac{u_{1ij}}{u_1 \log u_1}
\]

\[= \frac{1}{u_1 \log u_1} \left( D_1 \tilde{f} - u_1 \sum_{i,j=1}^{n} F^{ij} a_{ij} \right)
\]

\[+ \frac{1}{u_1 \log u_1} \left\{ \frac{2u_1}{w^4} F^{11} u_{11}^2 + \frac{2u_1}{w^2(w+1)} F^{11} \sum_{k=2}^{n} u_{1k}^2 \right. \]

\[= 2 \sum_{i=2}^{n} F^{11i} \left[ \frac{u_1}{w(w+1)} u_{i1} u_{ii} + \frac{u_1(2w+1)}{w^3(w+1)} u_{11} u_{i1} \right] + \frac{2u_1}{w(w+1)} \sum_{i,j=2}^{n} F^{ij} u_{ij1} u_{ij1} \}.
\]

(2) \[:= - \sum_{i,j=1}^{n} \tilde{F}^{ij} \left( 1 + \frac{2}{\log u_1} \right) \frac{u_{1ij1}}{u_1^2 \log u_1}
\]

\[= -\frac{1}{u_1^2 \log u_1} \left( 1 + \frac{2}{\log u_1} \right) \frac{1}{w^2} F^{11} u_{11}^2 + \frac{2}{w} \sum_{k=2}^{n} F^{1k} u_{11} u_{1k} + \sum_{i,j=2}^{n} F^{ij} u_{ij1} u_{ij1} \}.
\]

By adding (1) and (2) together, we obtain

\[
\Pi = (1) + (2) = \frac{D_1 H w^k}{u_1 \log u_1} + \frac{k H w^{k-2} u_{11}}{u \log u_1} - \frac{k H w^k - \bar{F}}{u^2 \log u_1} 
\]

\[+ \frac{1}{\log u_1} \left( \frac{2}{w} - \frac{1}{u_1^2 \log u_1} \right) F^{11} u_{11}^2
\]

\[+ \frac{2}{w^2(w+1)} F^{11} \sum_{k=2}^{n} u_{1k}^2 + \frac{2}{w(w+1)} \sum_{i,j=2}^{n} F^{ij} u_{ij1} u_{ij1}.
\]
\[2 \left[ \frac{2w + 1}{w^w(w + 1)} - \frac{1}{u_1^w w} + \frac{2}{\log u_1} \right] \sum_{i=2}^{n} F_{1i} u_{1i} + \left[ \frac{2}{w(w + 1)} - \frac{1}{u_1^w (1 + \frac{2}{\log u_1})} \right] \sum_{i,j=2}^{n} F_{ij} u_{1i} u_{1j} \].

Assume that \( \frac{\varphi'}{2 \varphi} u_1 \geq |g_1| \), otherwise the proof is done, since if
\[
\frac{\varphi'}{2 \varphi} u_1 < |g_1|,
\]
we get
\[
u_1 g \leq \frac{2 \varphi}{\varphi'} |g_1| \leq 8MR,
\]
then
\[
\log |\nabla u(0)| \leq \frac{g(x_0) \varphi(u(x_0))}{g(0) \varphi(u(0))} \log u_1(x_0) \leq \frac{2g(x_0)}{R^2} \log 8MR \leq \frac{16M}{R}.
\]

By above assumption, we have
\[
\frac{u_{11}}{u_1 \log u_1} = - \frac{g_1}{g} - \frac{\varphi'}{\varphi} u_1 \leq - \frac{\varphi'}{2 \varphi} u_1 < 0.
\]

Then
\[
A_{11} = 1 + \frac{u_1}{w} u_{11} - \frac{\varphi'}{\varphi} u_1^2 \leq 1 - \frac{\epsilon}{4 \varphi'} \log u_1 < 0.
\]

If the last inequality above does not holds, that is
\[
\log u_1(x_0) \leq \frac{4 \varphi}{\epsilon \varphi'} \leq \frac{8M}{\epsilon}.
\]

Then
\[
\log |\nabla u(0)| \leq \frac{g(x_0) \varphi(u(x_0))}{g(0) \varphi(u(0))} \log u_1(x_0) \leq \frac{16M}{\epsilon},
\]
and the proof is complete.

Since at \( x = x_0, \{u_{ij}(x_0)\}_{2 \leq i,j \leq n} \) is diagonal, we assume that \( u_{22} \geq \cdots \geq u_{nn} \), so \( A_{ij} = \delta_{ij} + u_{ij} \), for \( 2 \leq i \neq j \leq n \), and \( A_{22} \geq \cdots \geq A_{nn} \).

Since \( A_{11} < 0 \), using Proposition 2.4, we have
\[
F^{11} \geq \frac{\mathcal{F}}{n - k + 1},
\]
and
\[
\mathcal{F} \geq A_{22} \cdots A_{kk}.
\]

For index \( j \in \Lambda := \{ 2 \leq i \leq n : A_{ii} \geq 0 \} \) and using equation (3.5), Proposition 2.3 and Proposition 2.4, we have
\[
\sum_{i=2}^{n} F_{1i} u_{1i} u_{1i} = \sum_{i=2}^{n} - A_{1i} \sigma_{k-2}(A|1i) u_{1i} u_{1i} = - \frac{1}{w} \sum_{i=2}^{n} (A_{ii} - 1) \sigma_{k-2}(A|1i) u_{1i}^2 \\
\geq - \frac{1}{w} \sum_{i=2}^{n} A_{ii} \sigma_{k-2}(A|1i) u_{1i}^2 \geq - \frac{C_{n-2} \log^2 u_1}{w} \sum_{i=1}^{n} u_{1i}^2 A_{22} \cdots A_{kk} = \sum_{i=2}^{n} \frac{g_i^2}{g_{ii}^2} A_{22} \cdots A_{kk}
\]
By structure condition (1.6), we have
\[\geq -(n - k + 1)C_{n-2}^k \frac{u_1^2 \log^2 u_1}{w} \sum_{i \in \Lambda} \frac{g_i^2}{g^2} F_{i1}.\]
Thus the following term in II,
\[\frac{2}{w(w + 1) \log u_1} \sum_{i = 2}^n F_{i1} u_i u_{i1} \geq -\frac{C}{w(w + 1) \log u_1} \frac{u_1^2 \log^2 u_1}{w} \sum_{i \in \Lambda} \frac{g_i^2}{g^2} F_{i1}\]
\[\geq -\frac{C|\nabla g|^2 \log u_1}{g^2 w} F_{i1}.\] (3.29)
Again using equation (3.5), we have
\[\frac{D_1 H w^k}{u_1 u \log u_1} = \frac{H_1 w^k}{u_1 u \log u_1} + \frac{H_u w^k}{u \log u_1} + \frac{H_{\nu,1} w^{k-3}}{u} \left( \frac{g_1}{g} + u_1 \frac{\varphi'}{\varphi} \right)\]
\[= + \sum_{i = 2}^n \left( H_{\nu,1} w^{k-1} \frac{g_i}{g u} + H_{\nu,1} w^{k-3} \frac{u_1 u_{i1} w^{k-3}}{u} \left( \frac{g_1}{g} + u_1 \frac{\varphi'}{\varphi} \right)\right)\] (3.30)
\[\geq \frac{H_u w^k}{u \log u_1} - C_n |DH| w^{k-1} \left( \frac{1}{\varepsilon} + \frac{R}{\varepsilon g} + \frac{1}{\varepsilon M} \right).\]
Observe that
\[\sum_{i = 2}^n F_{i1} u_i = \sum_{i = 2}^n -A_{i1} \sigma_{k-2}(A|1i) u_{i1} = -\frac{u}{w} \sum_{i = 2}^n \sigma_{k-2}(A|1i) u_{i1}^2 \leq 0.\] (3.31)
Therefore, providing that \(u_1(x_0)\) is suitably large, we get
\[\Pi \geq \frac{H_u w^k}{u \log u_1} - C_n |DH| w^{k-1} \left( \frac{1}{\varepsilon} + \frac{R}{\varepsilon g} + \frac{1}{\varepsilon M} \right) + \frac{k H w^{k-2} u_{11}}{u \log u_1}\]
\[\geq - \frac{1}{2w^4 \log u_1} - \frac{1}{2w^2 \log u_1} F_{n1} \frac{u_{11}^2}{g^2 w} = \frac{C|\nabla g|^2 \log u_1}{g^2 w} F_{i1}.\] (3.32)
By structure condition (1.6), we have
\[\frac{H_u w^k}{u \log u_1} - \frac{k H w^k}{u^2 \log u_1} = (H_u - \frac{k H w}{u \log u_1}) \frac{w^k}{u \log u_1} \geq 0.\] (3.33)
Using equation (3.23), we have
\[\frac{u_{11}^2}{2w^4 \log u_1} \geq \frac{C|\nabla g|^2 \log u_1}{g^2 w} \geq \frac{\varphi'^2 u_1^4 \log u_1}{8 \varphi'^2 w^4} - \frac{C|\nabla g|^2 \log u_1}{g^2 w}\]
\[\geq \left( \frac{\varphi'^2}{32 \varphi^2} - \frac{C|\nabla g|^2}{g^2 w} \right) \log u_1\] (3.34)
\[\geq \frac{\varphi'^2}{64 \varphi^2} \log u_1,\]
where the last inequality above holds under the assumption that \(\frac{\varphi'^2}{64 \varphi^2} \geq \frac{C|\nabla g|^2}{g^2 w}\), otherwise if
\[\frac{\varphi'^2}{64 \varphi^2} < \frac{C|\nabla g|^2}{g^2 w},\] (3.35)
we get
\[ g^2 \log^2 u_1(x_0) \leq g^2 u_1(x_0) \leq g^2 w < \frac{64C|\nabla g|^2 \varphi^2}{\varphi^2} \leq C' R^2 M^2, \] (3.36)
then
\[ \log |\nabla u|(0) \leq \frac{\varphi(u(x_0))}{g(0) \varphi(u(0))} g(x_0) \log u_1(x_0) \leq \frac{2}{R^2} C' R M = \frac{2C' M}{R}. \] (3.37)
the proof is complete.

From Proposition 2.2, we get
\[ (C_n^k)^{\frac{1}{k}} \leq \sum_{i=1}^{n} \frac{\sigma_i^2}{\partial A_{ii}} = \frac{1}{k} F^{\frac{1}{k} - 1} \sum_{i=1}^{n} F^{ii} = \frac{1}{k} F^{\frac{1}{k} - 1} \frac{\varphi'}{\varphi} = \frac{1}{k} (H w^k)^{\frac{1}{k} - 1} \frac{\varphi'}{\varphi}, \] (3.38)
then
\[ \varphi \geq k (C_n^k)^{\frac{1}{k}} H^{1 - \frac{1}{k}} w^{k - 1}. \] (3.39)
By assumption, \(|DH|^k| \leq C_0\), we obtain \( \varphi = \sum_i F^{ii} \geq C |DH| w^{k - 1} \). Using equation (3.5), we have
\[ \frac{k H w^{k-2} u_{11}}{u \log u_1} + \frac{\varphi' k H w^k}{\varphi} \frac{u_{11} g_1}{u} + \frac{\varphi' k H w^{k-2}}{\varphi} \frac{u_{11} g_1}{u} \geq -C_k |H| w^{k-1} \left( \frac{R}{\varepsilon g} + \frac{1}{\varepsilon M} \right). \] (3.40)
Adding above (3.13), (3.32), (3.33), (3.34) and (3.40) together, we get
\[ 0 \geq I + II \geq -C_n \varphi \left( \frac{1}{g} + \frac{1}{\varepsilon M} \right) - C_k |H| w^{k-1} \left( \frac{R}{\varepsilon g} + \frac{1}{\varepsilon M} \right) \]
\[ - C_n |DH| w^{k-1} \left( \frac{1}{\varepsilon} + \frac{R}{\varepsilon g} + \frac{1}{\varepsilon M} \right) + \frac{\varphi^2}{64 \varphi^2} \log u_1 F^{11} - \frac{4 \varphi' R}{\varphi g w} F^{11}. \]
Therefore we obtain
\[ g \log u_1(x_0) \leq C \left( M^2 + MR + \frac{MR^2}{\varepsilon} + \frac{M^2 R^2}{\varepsilon} + \frac{M^2 R^2}{\varepsilon} \right). \] (3.41)
So
\[ \log |\nabla u|(0) \leq \frac{\varphi(u(x_0))}{g(0) \varphi(u(0))} g(x_0) \log u_1(x_0) \]
\[ \leq C \left( \frac{M^2}{R^2} + \frac{M}{R} \frac{M}{\varepsilon R} + M \frac{M}{\varepsilon} + \frac{M^2}{\varepsilon} \right). \]
Finally, our interior gradient bound has the form
\[ |\nabla u(0)| \leq \exp \left[ C \left( \frac{M}{R} + \frac{M}{\sqrt{\varepsilon}} + 1 \right)^2 \right]. \]
Then we complete the proof.

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