Operator-algebraic renormalization and wavelets

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We report on a rigorous operator-algebraic renormalization group scheme and construct the free field with a continuous action of translations as the scaling limit of Hamiltonian lattice systems using wavelet theory. A renormalization group step is determined by the scaling equation identifying lattice observables with the continuum field smeared by compactly supported wavelets. Causality follows from Lieb-Robinson bounds for harmonic lattice systems. The scheme is related with the multi-scale entanglement renormalization ansatz and augments the semi-continuum limit of quantum systems.

INTRODUCTION

Lattice regularization is a standard procedure to define continuum quantum field theories [1] which has led to extraordinary results in the ab-initio determination of the Hadron mass spectrum [2] and may serve as a starting point for the quantum simulation of quantum field theories [3]. While interacting models have been rigorously constructed in the classical works of Glimm-Jaffe and others [4], the lattice and continuum theories are often related indirectly in terms of correlation functions.

A recent attempt to build a continuum conformal field theory (CFT) by embedding a quantum spin chain from coarser to finer lattices, coined the semi-continuum limit and inspired by block-spin renormalization, resulted in a discontinuous action of symmetries, even the translatinal symmetries [5–8]. Here, we explain how this deficiency can be remedied by utilizing an observable-based, i.e. operator-algebraic, approach to the Wilson-Kadanoff renormalization group (RG) [9–11] for lattice field theories [12, 13]. As an important, instructive example [14, 15], we construct the massive continuum free field with its continuous action of spacetime translations via the scaling limit of lattice systems in their ground states approaching the unstable, massless fixed point ([16] for details and proofs). More recently, the presented method has been extended to CFTs based on free fermions [17] invoking the Koo-Saleur formula [18].

Our RG is defined in terms of compactly supported, regular wavelets [19] allowing for simultaneous control of locality properties in real and momentum space. We take inspiration from renormalization in classical systems [20] and use a scaling function and its multiresolution analysis to define a RG step: While block-spin renormalization would correspond to a step function, we use a Daubechies scaling function (see Figure 1), cf. [21, 22]. Thereby we avoid the obstacles encountered in [5, 7, 8] to implement continuous symmetries in the scaling limit, cf. [23]. Mapping observables from coarser to finer lattices results in a real-space RG dual to coarse graining the Hamiltonian or density matrices, e.g. the density matrix renormalization group (DMRG) [15, 24, 25]. Our method applies in all dimensions as we explicitly demonstrate for scalar lattice fields. Moreover, our approach yields a rigorous proof that spacetime locality (in the sense of the Haag-Kastler axioms [26]) in the continuum follows from Lieb-Robinson bounds [27–31].

As real-space RG schemes have received rapidly growing interest in recent years, especially in the context of tensor networks [32] and the multi-scale entanglement renormalization ansatz (MERA) [33–35], we show as an important application that our approach yields a rigorous analytic MERA in any dimension d which is not restricted to critical (massless) models [36, 37]. The discrete dimension of the d + 1-dimensional tensor network of the MERA is identified with the sequence of scales the given quantum system is observed at.

The letter is organized as follows. First, we outline our general renormalization scheme. Then, we apply it to lattice scalar fields by constructing explicit renormalization maps in terms of compactly supported wavelets, and we discuss the connection with the MERA. Finally, in the example of the free scalar field, we show that imposing a suitable renormalization condition on lattice ground states at different scales, we fully recover the continuum massive...
field in the scaling limit including the action of spacetime translations. The letter closes with an outlook on possible future developments.

OPERATOR-ALGEBRAIC RENORMALIZATION

As discussed in [13], the RG approach to the lattice approximation of continuum theories can be rephrased in terms of observables, that is operator algebras, as follows. We fix a family of lattices \( \Lambda_N \) in \( \mathbb{R}^d \) with lattice constant \( \varepsilon_N = 2^{-N} \varepsilon \), and consider a sequence of Hamiltonian quantum systems \( \{ \mathfrak{A}_N, \mathcal{H}_N, \Pi^{(N)}_0 \} \) indexed by the scale \( N \). At each scale \( N \), we have an algebra of observables \( \mathfrak{A}_N \) generated by (bounded functions of) basic time-zero lattice fields \( \Phi_N(x) \), their momenta \( \Pi_N(x) \), and a Hamiltonian \( H^{(N)}_0 \) both acting on the Hilbert space \( \mathcal{H}_N \). The quantum state at each scale is initially given by a density matrix \( \rho_0^{(N)} \), e.g. in terms of a Hamiltonian: \( \rho_0^{(N)} = (Z_0^{(N)})^{-1} e^{-H_0^{(N)}} \). The RG connects systems at different scales via (coarse graining) quantum operations, mapping density matrices on the finer system to the coarser system

\[
E^{N+M}_N(\rho^{(N+M)}_0) = E^{(N)}_M(\rho^{(N+M)}_0), \quad E^{N+1}_N \circ E^{N+2}_N = E^{N+2}_N,
\]

where \( \rho^{(N)}_M \) corresponds to the \( \langle M \rangle \) times renormalized Hamiltonian \( H^{(N)}_0 \) at scale \( N \). Because quantum states \( \rho \) are positive, linear maps \( \omega : \mathfrak{A}_N \to \mathbb{C} \), by \( \omega(A) = \operatorname{tr}(\rho A) \), and the field correlation functions are given by \( \langle \Phi_N(x) \ldots \Pi_N(y) \rangle^{(N)} := \omega^{(N)}(\Phi_N(x) \ldots \Pi(y)) \), we can state (1) as:

\[
E^{N+M}_N(\omega^{(N+M)}_0) = \omega^{(N+M)}_0 \circ \alpha^{N+M}_N = \omega^{(N)}_M,
\]

where \( \alpha^{N+M}_N : \mathfrak{A}_N \to \mathfrak{A}_{N+M} \) is the dual of \( E^{N+M}_N \) (the ascending superoperators [34]). \( \omega^{(N)}_0 \) and \( \omega^{(N)}_M \) characterize the initial and renormalized states on \( \mathfrak{A}_N \) corresponding to \( \rho^{(N)}_0 \) and \( \rho^{(N)}_M \). We call the collection \( \alpha^{N+M}_N \) the scaling maps or renormalization group. The structure is neatly summarized by an adaptation of Wilson’s triangle of renormalization [10, p. 790] in Figure 2. If the limit \( \omega_N^{(N)} := \lim_{M \to \infty} \omega^{(M)}_N \) exists (in a suitable sense), the sequence \( \omega^{(N)}_N \), called the scaling limit of the initial states \( \omega^{(N)}_0 \), is stable under coarse graining:

\[
E^{N+M}_N(\omega^{(N+M)}_N) = \omega^{(N)}_N, \quad N < N'.
\]

Employing operator-algebraic techniques (see [16] for details), we obtain a Hilbert space \( \mathcal{H}_N \) and an algebra \( \mathfrak{A}_N \) generated by continuum fields \( \Phi, \Pi \), acting on it. Following [5, 6, 12, 13, 38] we call \( \mathfrak{A}_N \) the semi-continuum limit, see also [39, 40]. Moreover, we have isometries \( V_N : \mathcal{H}_N \to \mathcal{H}_N \) and a state \( \Omega \in \mathcal{H}_N \) realizing the correlations of the scaling limit \( \omega = (\Omega, \Omega) \). The finite-scale fields \( \Phi_N, \Pi_N \) are embedded in the continuum fields \( \Phi, \Pi \) through \( \alpha^{N}_N : \mathfrak{A}_N \to \mathfrak{A}_\infty \):

\[
\alpha^{N}_N(\Phi_N(x))V_N = V_N \Phi_N(x), \quad \omega_N^{(N)} = \omega \circ \alpha^{N}_N.
\]

FIG. 2: Wilson’s triangle of renormalization: Vertical lines represent renormalization steps, either by coarse graining states (\( \varepsilon \)’s) or by refining fields (\( \alpha \)’s). Horizontal lines represent sequences of renormalized states considered on the algebra generated by fields and momenta at a fixed scale (right column).

WAVELETS AND THE SCALAR FIELD

We now apply the above framework to lattice scalar fields, setting up a specific renormalization scheme involving compactly supported wavelets [19, 41]. To avoid infrared divergence at finite scale, we take lattices \( \Lambda_N = \varepsilon_N \{-L, \ldots, L_N - 1\}^d \) representing a discretization of the torus \( [-L, L]^d = T^d \) (periodic boundary conditions, \( L_N \equiv -L_N \equiv L_N \) fixed). We denote by \( \Gamma_N = \frac{2}{L}(-L, \ldots, L_N - 1)^d \) the dual momentum space lattices. The kinematical setup of the lattice scalar field systems is given by the Fock space \( \mathcal{H}_N \), built from the action of momentum-space creation and annihilation operators \( a_N(k), a^\dagger_N(k) \) on the vacuum vector \( \Omega_N \) subject to the canonical commutation relations (CCR), \( [a_N(k), a^\dagger_N(l)] = (2L_N)^d \delta_{k,l} \), and by the algebra \( \mathfrak{A}_N \) generated by the local (dimensionless) canonical lattice field for \( x \in \Lambda_N \):

\[
\Phi_N(x) = \frac{1}{\sqrt{2L_N^d}} \sum_{k \in \Gamma_N} [a^\dagger_N(k)e^{-ikx} + a_N(k)e^{ikx}],
\]

and its momentum (with a similar formula) satisfying:

\[
\{\Phi_N(x), \Pi_N(y)\} = i \delta_{x,y}. \quad \text{The scaling maps } \alpha^{N'}_N : \mathfrak{A}_N \to \mathfrak{A}_{N'} \text{ are the most important input in our framework determining the existence and structure of the continuum limit. Our choice using wavelets is motivated by the block-spin case and its locality properties in real space corresponding to the smearing of continuum fields with the simplest member of the Daubechies’ wavelet, the } \text{Haar wavelet } \chi_{0,1} \text{ (see Figure 1). But, as the approximation of momenta requires higher regularity, the latter does not suffice as explained below.}

Scaling maps from a scaling function. We consider an orthonormal scaling function \( s \) that satisfies the scaling equation [19, 42, 43]:

\[
s(x) = \sum_{n \in \mathbb{Z}^d} h_n 2^\frac{d}{2} s(2x - n),
\]

such that its integer translates \( s(\cdot - n) \) are orthonormal. To build local operators, we further take \( s \) compactly sup-
ported an orthonormal, compactly supported wavelet basis in $L^2(\mathbb{R}^d)$, and the sum (5) is necessarily finite ($h_\ast$ is a finite low-pass filter [19]). We denote by $s(x) = \varepsilon^{-d} s(\varepsilon^{-1}(x))$ the scaling function localized near $x \in \mathbb{Z}^d$ at length scale $\varepsilon$, periodized on the torus $\mathbb{Z}^d_L$. With the scaling relation (5) in mind, we define $\alpha_{N+1}$ using the low-pass filter $h_\ast$:

$$\alpha_{N+1}(x) = 2^{-\frac{d}{2}} \sum_{n \in \mathbb{Z}^d} h_n \Phi_{N+1}(x + n \varepsilon_{N+1}), \quad (6)$$

and similarly for $\Pi_N$. Now, the associated semi-continuum limit algebra $\mathfrak{A}_\infty$ can be identified with the algebra generated by continuum fields smeared with the functions $s(x)$ over all scales $N$: The map,

$$\Phi(x) \mapsto \alpha_\infty(\Phi(x)) = \varepsilon^{-\frac{d}{2}} \int dy \, \Phi(y) s(x) (y), \quad (7)$$

identifies the lattice fields at scale $N$ with the continuum fields smeared with $s(x)$ (and analogously for $\Pi_N(x)$). The RG elements $\alpha_N$, defined by (6) have two intriguing properties: First, the lattice field $\Phi_N(x)$ at one scale is decomposed into a linear combination of the fields at the successive scale. Second, the embedding (7) into the continuum field theory is compatible with this decomposition, $\alpha_{N+1} \circ \alpha_N = \alpha_\infty$, realizing the correct CCR:

$$[\alpha_\infty(\Phi_N(x)), \alpha_\infty(\Pi_N(y))] = [\Phi(s(x)), \Pi(s(y))] = id_{x,y}. \quad (8)$$

Furthermore, we have $\Phi(s(x)) = \sum_{n \in \mathbb{Z}^d} h_n \Phi(s(x+n \varepsilon_{N+1}))$ (linearity and (5)) with an analogous formula for $\Pi$. This means that the lattice fields and their realization in terms of the continuum field have the same algebraic structure.

Concrete choice of a scaling function. The simplest scaling function, $\chi_{[0,1]}$, corresponds to the block-spin renormalization (6) (see Figure 1). By taking a more regular scaling function, e.g., $K \cdot s$ with $K \geq 2$ of Daubechies’ D2K wavelet family, we achieve that the smeared continuum momentum $\Pi_K(s(x))$ is a well-defined operator (technically $s$ needs to be in the Sobolev space $H^\frac{d}{2}$). In addition, the compact support of $K \cdot s$ leads to locality in real space, i.e., the lattice fields $\Phi_N(x), \Pi_N(x)$ can be used to approximate local operators in the continuum because $\Phi(s(x)), \Pi(s(x))$ are spatially localized in compact regions. In comparison with the block-spin renormalization, we trade some locality (the support of the Daubechies scaling function $K \cdot s$ is larger than the support of $\chi_{[0,1]}$) for higher regularity improving approximations. With this price, we gain the continuum realization of $\Pi_N(x)$, and we recover the correlation functions and space-time symmetries (translations) in the scaling limit (see below).

Connection with multi-scale entanglement renormalization. Considering the embedding $I_N^1(\Phi_N(x)) = 2^{-\frac{d}{2}} \Phi_{N+1}(x)$ resulting from identifying $\Lambda_N$ as a sublattice of $\Lambda_{N+1}$, and the Bogoliubov unitary,

$$U_{N+1} \Phi_{N+1}(x) = \sum_{n \in \mathbb{Z}^d} h_n \Phi_{N+1}(x + n \varepsilon_{N+1}) U_{N+1}, \quad (8)$$

implementing the redistribution of field values according to the low-pass filter $h_\ast$, the scaling map $\alpha_N$ decomposes into MERA form [13, 33–35, 38]:

$$\alpha_N(\cdot) = U_{N+1}(\cdot \otimes I_{N+1}) U_{N+1}^*, \quad (9)$$

Here, $\cdot \otimes I_{N+1}$ is the tensor product with the identity on the ancillary Fock space, $\mathcal{H}_{N+1} = \mathcal{H}_N \otimes \mathcal{H}_{N+1}^{(a)}$, and the dual quantum channel $\mathcal{E}_{N+1} = \text{Tr}_{H_{N+1}}(U_{N+1}^* \cdot \otimes \mathcal{U}_N^*)$ is given by a twisted partial trace on the ancillary. From (9), we find that $U_{N+1}$ serves as MERA disentangler recovered from the isometries, $V_N^N : \mathcal{H}_N \rightarrow \mathcal{H}_{N+1}$, between Fock spaces resulting from coarse-graining stability (3):

$$\mathcal{O}_\infty = \frac{V_N^N}{\mathcal{O}_{N+1}}, \quad (10)$$

where $\mathcal{O}_\infty$ is the vector implementing the scaling limit $\omega_\infty$ at scale $N$. The embedding into the continuum Hilbert space $\mathcal{H}_\infty$ can be explicitly computed from (4).

Summarizing, we observe that one layer of MERA isometries and disentanglers is recovered from $\alpha_N$ and the scaling limit $\omega_\infty$. This structure is further elucidated by the action of the isometries $V_N^N$ on coherent or Glauber states, $\mathcal{E}_{N+1}^\infty \Psi_N(f, g) = e^{i(\Phi_N(f) + \Pi_N(g))} \mathcal{O}_\infty$, using the identification (7) (see Figure 3). In this sense, our operator-

![FIG. 3: Illustration of the analytic MERA in $d=1$ induced by the wavelet scaling maps. From bottom to top: the first layer represents the isometric embedding $I_N^1$ and the second layer represents the action of the (dis)entangler $U_{N+1}$ at scale $N+1$.](image)

algebraic RG scheme produces an analytic MERA. Specifically, the scaling limits of free lattice ground states, which we construct below, exhibit a structure similar to an analytic MERA in arbitrary dimensions and off criticality [36, 44–46].

**SCALING LIMITS OF HARMONIC LATTICE SYSTEMS**

We are now in a position to apply the RG $\alpha_N$, defined by (6) to find the ground-state scaling limits of the free lattice Hamiltonian on $\mathcal{H}_N$:

$$H_0^{(N)} = \varepsilon_N^{-\frac{d}{2}} \sum_{x \in \Lambda_N} \sum_{y \in \Lambda_N} \left[ \mu_N \Phi_N(x) \Phi_N(y) + \sum_{(x,y) \in \Lambda_N} \Phi_N(x) \Phi_N(y) \right], \quad (11)$$

where $\mu_N \geq 2d$ is a “mass” parameter. The ground state $\mathcal{O}_0^{(N)}$ of $H_0^{(N)}$ can be encoded into the expectation $\omega_0^{(N)}$ on $\mathcal{H}_N$ determined by the two-point functions:

$$\omega_0^{(N)}(\Phi_N(x) \Phi_N(y)) = \frac{1}{(2\pi N)^d} \sum_{k \in \Gamma_N} \frac{1}{\sin\pi N(k)} e^{i(kx-ky)}, \quad (12)$$
with the dispersion relation \( \gamma_{N}^{2}(k) = \varepsilon_{N}^{2}(\mu_{N}^{2} - 2d) + 2\varepsilon_{N}^{2} \sum_{j=1}^{d}(1 - \cos(\varepsilon_{N}k_{j})) \), and analogous formulae for \( \omega_{0}(\Pi_{N}(x)\Pi_{N}(y)) \) and \( \omega_{0}(\Pi_{N}(x)\Pi_{N}(y)) \), the latter being most singular.

**Scaling limit of the ground states.** We choose (12) as our initial states to generate a sequence of renormalized states \( \omega_{m}^{(N)} \) at each scale \( N \) (Figure 2). To avoid the RG-fixed points \( \mu_{N}^{2} = 2d \) (massless, unstable) and \( \mu_{N}^{2} = \infty \) (ultralocal, stable) and hit the unstable manifold of the relevant \( \Phi^{2} \)-operator, we impose the renormalization condition

\[
\lim_{N \to \infty} \varepsilon_{N}^{2}(\mu_{N}^{2} - 2d) = m^{2},
\]

for some \( m > 0 \). This leads to the massive continuum dispersion, \( \lim_{\varepsilon_{N} \to \infty} \gamma_{m}^{(N)}(k)^{2} = m^{2} + k^{2} = \gamma_{m}(k)^{2} \), and the scaling limit (using (6) & (12), and similar for \( \Pi_{N} \)):

\[
\lim_{N \to \infty} \gamma_{m,n}^{(N)}(x)\Phi_{N}(y) = \frac{1}{(2\pi)^{d}}\sum_{k \in \mathbf{Z}^{d}} \frac{|\varepsilon_{N}(k)|^{2}}{2\pi \gamma_{m}(k)} e^{ik(x-y)},
\]

where \( \Gamma_{n} = \frac{\varepsilon_{n}}{2}Z^{d} \) is the momentum space of the torus \( \mathbf{T}_{n}^{d} \). Since the two-point function of the momentum \( \Pi_{N} \) is the most singular, the limit states are well defined for \( T_{m} \geq 1 \). Because of localization and compact support, and similar for \( \Pi_{N} \):

\[
\omega_{m,n}^{(N)}(x)\Phi_{N}(y) = \frac{1}{(2\pi)^{d}}\sum_{k \in \mathbf{Z}^{d}} \frac{|\varepsilon_{N}(k)|^{2}}{2\pi \gamma_{m}(k)} e^{ik(x-y)},
\]

and similarly for \( \Pi_{N} \), uniformly on bounded intervals of \( t \in \mathbb{R} \), with the free continuum Hamiltonian \( H \) on the torus \( \mathbf{T}_{n}^{d} \). Since \( \gamma_{m} \) is the free, massive relativistic dispersion relation, we know that the dynamics generated by \( H \) has propagation speed \( c = 1 \) and, thus, the scaling limit theory satisfies Einstein causality, i.e., \( e^{itH}e_{N}^{1}(\Phi_{N}(x)) e^{-itH} \) and \( e^{itH}e_{N}^{2}(\Phi_{N}(x)) e^{-itH} \) commute if the support of \( s_{N,\xi}(x) \) at time \( t \) and the support of \( s_{N,\xi}(x) \) at time \( s \) are spacelike separated on the torus. A more lattice-intrinsic and model-independent way to conclude recovery of causality in the scaling limit is via Lieb-Robinson bounds [28, 29]. Considering the extension of the finite-scale time translations \( \sigma_{t}^{(N)} = e^{itH_{0}^{(N)}}(\cdot)e^{-itH_{0}^{(N)}} \) to \( \mathfrak{A}_{N} \) by (9), said bounds for harmonic lattice systems [30] imply:

\[
\lim_{N \to \infty} \left[ \sigma_{t}^{(N)}(A), B \right] = 0,
\]

exponentially fast and uniformly for \( |t| \leq T \) with (bounded) \( A, B \in \mathfrak{A}_{N} \) localized in sets \( S_{A}, S_{B} \subset \mathbf{T}_{n}^{d} \) such that \( \text{dist}(x, S_{A}) \geq cT \) for all \( x \in S_{B} \), for some \( c' > 1 \). Because \( c' > 1 \), the causality implied by (15) is not strict likely due to a non-optimal bound on the Lieb-Robinson velocity [28]. Another important feature of our approximation of dynamics (or symmetries in general) is the possibility for uniform error bounds in time and within a fixed range of field and momentum amplitudes at a given scale \( N \); For the free continuum time evolution \( \sigma_{t} = e^{itH}(\cdot)e^{-itH} \) we have [16]:

\[
\| (\sigma_{t}^{(N)} - \sigma_{t}^{(N)})(A) \psi \| \leq C \sup_{k \in \Gamma_{n}} \left( \frac{\gamma_{m}(k)}{(1 + \varepsilon_{N}|k|)^{2}} \right),
\]

for exponentials \( A = \alpha_{N}^{1}(e^{i(\Phi_{N}(x) + \Pi_{N}(y))} \cdot) \) of fields and momenta on coherent states \( \psi = c(\varepsilon_{N}^{-\frac{1}{2}} s_{N,\xi}(x), \varepsilon_{N}^{-\frac{1}{2}} s_{N,\xi}(y)) \) at scale \( N, C \) only depends on \( N, \varepsilon_{N}, m, T \) for \( |t| \leq T \), and \( s \). While the specific form of these bounds reflects the free-field situation, our general method to obtain such uniform bounds at fixed approximation scale \( N \) is not restricted to this situation (cf. conclusion).

**CONCLUSIONS AND OUTLOOK**

Our results show that the existence and properties of continuum limits depend decisively on the choice of a renormalization scheme. Correctly choosing the initial states allows us to reconstruct the continuum field theory from the lattice approximation through the semi-continuum limit. For the free massive scalar field, our renormalization scheme, given by compactly supported wavelets, yields continuous spacetime translations, avoiding the apparent no-go results stated in [5, 7]. Obtaining a similar convergence statement for Lorentz transformations or even conformal transformations requires further work [17]. Apart from the question of approximation of symmetries, our method proves ((14) and (16)) that time-dependent and
spatially translated correlation functions of the continuum field theory for any insertions of fields and momenta, \( A_N = \Phi_N(x_1) \ldots \Pi_N(x_n) \) and \( B_N = \Phi_N(x_{n+1}) \ldots \Pi_N(x_{n+m}) \), at any scale \( N \) are approximated by the correlation functions of the lattice models (suppressing scaling maps \( \alpha_N^N, \alpha_\infty^N \)):

\[
|\omega_0^{(N')}\langle A_N \sigma_{(t,x)}^{(N')}(B_N) \rangle - \omega\langle A_N \sigma_{(t,x)}(B_N) \rangle|^{N' \to \infty} \to 0, \quad (17)
\]

where \( \sigma_{(t,x)} \) and \( \sigma_{(t,x)}^{(N')} \) are the continuum respectively discrete spacetime translations for \((t, x) \in \mathbb{R} \times \Lambda_N\). We point out that the convergence in (17) only mildly depends on the choice of scaling function \( s \) (requiring sufficient regularity). This presents a significant conceptual and presumably computational difference in comparison with a related construction using wavelet theory [46] focusing on locality in one-particle space and relying on a continuous adaptation of the choice of scaling function to achieve a given accuracy goal for the approximation of equal-time correlation function similar to (17). An application of the wavelet method to (free) lattice fermions lead to similar results as those presented here [17, 47]. Our general framework can also include interacting lattice systems, e.g. \( \Phi^4 \)-models, although we will need approximations by analytical and numerical expansion or perturbative methods [25, 48, 49]. Moreover, Lieb-Robinson bounds for anharmonic lattice systems [31] offer a possibility to obtain spacetime locality directly from the lattice [28, 29]. In view of the classical results by Glimm-Jaffe and others [4] on \( P(\Phi) \)-models in \( d = 1 \), our method is directly applicable to those using a low-pass filter implementing momentum-space cutoffs [16] thereby providing the same regularized continuum fields as in [50], and we expect a possible extension to the wavelet setting. Therefore, it would be interesting whether the convergence to the scaling limit can be shown exploiting the results in [51] supplemented by explicit error bounds similar to (16).

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