Abstract. We prove a growth lemma for integrodifferential operators and derive regularity estimates for solutions to integrodifferential equations. Our emphasis is on kernels with a critically low singularity which does not allow for standard scaling.

In recent years, regularity results for linear and nonlinear integrodifferential operators have been addressed by many research articles. Scaling properties are crucially used in these approaches. Here, we consider limit cases where standard scaling properties do not hold anymore. We consider linear operators of the form

$$Au(x) = \int_{\mathbb{R}^d} (u(x + h) - u(x) - \langle \nabla u(x), h \rangle 1_{B_1}(h)) K(x, h) \, dh,$$

which, provided certain assumptions on $K(x, h)$ are satisfied, are well defined for smooth and bounded functions $u : \mathbb{R}^d \to \mathbb{R}$. In many applications where the linear operator $A$ is related to a stochastic process, $K(x, \lambda h)$ is comparable to $|\lambda|^{-d-\alpha} k(h)$ for a given function $k : \mathbb{R}^d \to [0, \infty)$, a given exponent $\alpha \in (0, 2)$, all $x, h \in \mathbb{R}^d$ and all reals $\lambda \neq 0$. This property reflects that the increments of the process that is used as the main building block in the construction of the process related to the operator $A$ have stable distributions. Looking at the operator $A$ as an integrodifferential operator, this property is important because it allows to use scaling techniques. These techniques themselves are crucial when studying regularity properties of functions $u : \mathbb{R}^d \to \mathbb{R}$ satisfying an equation $Au = f$ in a domain $\Omega \subset \mathbb{R}^d$ for some function $f : \Omega \to \mathbb{R}$. In this work we study such properties with the emphasis on two features. We do not assume any regularity except for boundedness of the kernel function $K$ with respect to the first variable. Moreover, and this is the main new contribution, we systematically study classes of kernels which do not possess the aforementioned scaling property.

Our results include a growth lemma (expansion of positivity) and Hölder regularity estimates. Recall that, in case of an elliptic operator of second order $A$ like $Au = \sum_{i,j} a_{ij}(\cdot) \partial_{x_i} \partial_{x_j} u$, the standard growth lemma reads as follows:

**Lemma.** There is a constant $\theta \in (0, 1)$ such that, if $R > 0$ and $u : \mathbb{R}^d \to \mathbb{R}$ with

$$- Au \leq 0 \text{ in } B_{2R}, \quad u \leq 1 \text{ in } B_{2R}, \quad |(B_{2R} \setminus B_R) \cap \{u \leq 0\}| \geq \frac{1}{2} |B_{2R} \setminus B_R|,$$

then $u \leq 1 - \theta$ in $B_R$. 

Date: December 23, 2014.

2010 Mathematics Subject Classification. Primary 35B65; Secondary: 35R09, 47G20, 60J75, 31B05.

Key words and phrases. integrodifferential operators, regularity, jump processes, intrinsic scaling.

Research supported by German Science Foundation (DFG) via SFB 701. Research supported by MZOS grant 037-0372790-2801.
In this work we prove a similar growth lemma for integrodifferential operators, see \textbf{Theorem 1} below. The model case of an operator $A$ that we have in mind is

$$Au(x) = \int_{\mathbb{R}^d} \left[u(x+h) - u(x)\right] a(x, h)|h|^{-d} \mathbb{1}_{B_1}(h) \, dh,$$

for some measurable function $a : \mathbb{R}^d \times \mathbb{R}^d \to [1, 2]$. Note that, in the last years, similar results have been studied for kernels of the form $a(x, h)|h|^{-d-\alpha}$ for $\alpha \in (0, 2)$ and we refer the reader to the discussion in [KM13]. The case $\alpha = 0$ is of particular interest because in this case results like the growth lemma are known to fail, even if one prescribes some bounds on $u$ outside of $B_{2R}$. We provide a general class of kernels that can be used in (1) instead of $\mathbb{1}_{B_1}(h)a(x, h)|h|^{-d}$.

Although in our results we assume the solutions $u$ to be twice differentiable, the assertions do not depend on the regularity of the functions $u$. Thus, the techniques and assertions presented here can be applied to nonlinear problems. Our method is based on a purely analytic technique introduced in [Sil06] where extensions to nonlinear problems are discussed.

This note extends and enhances the results of [KM13]. Let us comment on the differences between [KM13] and the present article:

1. Here, we are able to allow for source terms $f \neq 0$ which is not trivial in our setup since standard scaling procedures seem not to be suitable for such an extension.
2. We do not use stochastic analysis or related Markov processes in this note. We interpret the equation $Au = f$ in the classical sense. Our proofs are entirely different from those in [KM13].
3. We prove a growth lemma (\textbf{Theorem 1}) which itself is an important result for later applications.
4. Our assumptions on the singularity of the kernels are weaker than the ones in [KM13].

After [KM13] had appeared, several articles have made use of the ideas therein. In [Bae14] nonlocal problems are studied where the kernels are supposed to satisfy certain upper and lower scaling conditions. These assumptions do not include limit cases like (1) since some sort of comparability with respect to kernels like $|h|^{-d-\alpha}$ for $\alpha \in (0, 2)$ is still assumed. In [KKL14] the authors study fully nonlinear problems with similar assumptions on the kernels as in [Bae14]. In [CZ14] the authors extend the regularity estimates of [KM13] to time-dependent equations with drifts. The article [JW14] is not directly related to [KM13] but mentions the problem of considering $f \neq 0$. We solve this problem.

\textbf{Acknowledgement}: The authors thank Tomasz Grzywny and Jongchun Bae for discussions about the assumptions on the kernel function $K$.

1. \textbf{Assumptions and main results}

Let us state our main assumptions. Let $K : \mathbb{R}^d \times (\mathbb{R}^d \setminus \{0\}) \to [0, \infty)$ be a measurable function such that $K(x, h) = K(x, -h)$ for all $x, h$ and

$$\kappa^{-1}|h|^{-d}\ell(|h|) \leq K(x, h) \leq \kappa|h|^{-d}\ell(|h|) \quad \text{for} \quad 0 < |h| < R_0 \quad \text{(A1)}$$
$$K(x, h) \leq c_0|h|^{-d}\ell(|h|) \quad \text{for} \quad |h| \geq R_0, \quad \text{(A2)}$$
where $\kappa, c_0 \geq 1$, $R_0 \in (0, \infty]$ and $\ell: (0, \infty) \to (0, \infty)$ is a function satisfying, for some $c_L \in (0, 1)$, $c_U \geq 1$, and $\gamma \in (0, 2)$ the following:

$$\int_r^\infty \frac{\ell(s)}{s} \, ds < \infty \quad \text{for every } r > 0, \quad \ell(r, \lambda) \geq c_L \lambda^{-\gamma} \quad \text{for all } r > 0 \text{ and } \lambda \geq 1,$$

$$\ell(r, \lambda) \leq c_U \lambda^d \quad \text{for all } r > 0 \text{ and } \lambda \geq 1 \text{ such that } r\lambda \leq R_0.$$

The above conditions imply

$$\sup_{x \in \mathbb{R}^d} \int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^2) K(x, h) \, dh < \infty,$$

which is condition $(K_1)$ in [KM13]. This estimate follows, because for $x \in \mathbb{R}^d$

$$\int_{\mathbb{R}^d \setminus \{0\}} (1 \wedge |h|^2) K(x, h) \, dh \leq c \int_0^1 s^{2-1} \ell(s) \, ds + c \int_1^\infty s^{-1} \ell(s) \, ds \leq c \ell(1) c_L^{-1} \int_0^1 s^{1-\gamma} \, ds + c L(1) = c \ell(1) c_L^{-1} (2 - \gamma)^{-1} + c L(1).$$

Note that conditions $(\ell_2)$ and $(\ell_3)$ are satisfied if the function $\ell$ is regularly varying of non-positive order $-\alpha \in (-2, 0]$. The behavior of $\ell$ at 0 can be very different from the behavior at $+\infty$. One important example is $\ell(s) = \mathbb{1}_{(0,1)}(s) + s^{-\alpha} \mathbb{1}_{[1,\infty)}(s)$. Another standard example satisfying the above conditions would be $\ell(s) = s^{-\alpha} g(s)$ for $\alpha \in (0, 2)$ and $g$ a function which varies slowly at 0 and at $+\infty$. Let us provide some more sophisticated examples. For $\alpha \in [0, 2)$, $\varepsilon > 0$ we could consider

$$\ell(s) = \begin{cases} 0 & \text{if } 0 < s \leq 2, \\ s^{-\alpha} \ln(s)^{-1-\varepsilon} & \text{if } s > 2, \end{cases} \quad \text{or} \quad \ell(s) = \begin{cases} \ln(4/s) & \text{if } 0 < s \leq 2, \\ \ln(s)^{-1-\varepsilon} & \text{if } s > 2. \end{cases}$$

**Remark:** It is important to note that, unlike for $\ell(|h|)$, the function $h \mapsto K(x, h)$ might be zero for large values of $|h|$ because we do not assume any lower bound on $K(x, h)$ in this range.

We define an auxiliary function $L: (0, \infty) \to (0, \infty)$ by $L(r) = \int_r^\infty \frac{\ell(s)}{s} \, ds$, which turns out to be strictly decreasing. Furthermore, we define a measure $\mu$ by

$$\mu(dy) = \frac{\ell(|y|)}{L(|y|)} \frac{dy}{|y|^d}$$

and, for $a > 1$, a function $\varphi_a = \varphi: (0, \infty) \to (0, \infty)$ by $\varphi(r) = L^{-1}(a^{-1} L(r))$.

Define an operator $A: C_b^2(\mathbb{R}^d) \to C(\mathbb{R}^d)$ by

$$Au(x) = \int_{\mathbb{R}^d \setminus \{0\}} \left( u(x + h) - u(x) - (\nabla u(x), h) \mathbb{1}_{B_1(h)} \right) K(x, h) \, dh \quad (2)$$

where $K: \mathbb{R}^d \times \mathbb{R}^d \setminus \{0\} \to [0, \infty)$ satisfies $(A_1)$ and $(A_2)$.
Now we can formulate our first main result, a growth lemma for nonlocal operators. We formulate the result for functions which, together with their first and second derivatives, are continuous and bounded. The strength of the result is that the regularity does not show up in the estimate. The result is tailored for later applications to viscosity solutions of fully nonlinear partial differential equations.

**Theorem 1.** Let \( \eta, \delta \in (0,1) \) and \( c > 0 \). There exist constants \( a > 2 \) and \( \theta \in (0,1) \) such that for every \( r > 0 \) such that \( \frac{5}{2} \varphi(r) < R_0 \) and every function \( v \in C^2_0(\mathbb{R}^d) \) satisfying

\[
-Av(x) \leq L(\varphi(r)) \quad (= a^{-1}L(r)) \quad \text{for } x \in B_{\varphi(r)},
\]

\[
v(x) \leq 1 \quad \text{for } x \in B_{\varphi(r)},
\]

\[
v(x) \leq c \left( \frac{L(\varphi(r))}{L(|x|)} \right)^{\eta} \quad \text{for } x \in \mathbb{R}^d \setminus B_{\varphi(r)},
\]

\[
\mu(B_{\varphi(r)} \setminus B_r) \cap \{v \leq 0\} \geq \delta \mu(B_{\varphi(r)} \setminus B_r),
\]

the following is true:

\[
v(x) \leq 1 - \theta \quad \text{for all } x \in B_r.
\]

**Remark:** As the proof shows, the value of \( \theta \) is a multiple of \( a^{-1} \).

Note that, in the by now well-known case where \( \ell(r) = r^{-\alpha} \) and \( L(r) \propto r^{-\alpha} \) for \( \alpha \in (0,2) \), this result reduces to a growth lemma which is very similar to those given in [Sil06] and [CS09]. Let us now formulate the second main result. The following theorem is an analog to Theorem 1.4 in [KM13]. Note that we treat \( f \neq 0 \) here, too.

**Theorem 2.** There exist constants \( c > 0 \) and \( \beta \in (0,1) \) such that for any \( r \in (0,R_0/2) \), \( f \in L^\infty(B_r) \) and function \( u \in C^2_0(\mathbb{R}^d) \) satisfying \( Au = f \) in \( B_r \) the following holds

\[
\sup_{x,y \in B_{r}/4} \frac{|u(x) - u(y)|}{L(|x-y|)} \leq cL(r)^{\beta} \|u\|_\infty + cL(r)^{\beta - 1} \|f\|_{L^\infty(B_r)}.
\]

In the case \( \ell(r) = r^{-\alpha} \), \( \nu = \alpha \beta \), estimate (4) reduces to

\[
\sup_{x,y \in B_{r}/4} \frac{|u(x) - u(y)|}{|x-y|^\nu} \leq cr^{-\nu} \|u\|_\infty + cr^{\alpha - \nu} \|f\|_{L^\infty(B_r)},
\]

which one would expect from standard scaling behavior.

In the case \( R_0 = \infty \) we obtain a Liouville theorem.

**Corollary 3.** Assume that \( R_0 = \infty \). Any bounded function \( u: \mathbb{R}^d \to \mathbb{R} \) satisfying \( Au = 0 \) on \( \mathbb{R}^d \) is a constant function.

**Proof.** Since \( u \) is harmonic in any ball \( B_r \) we can let \( r \to \infty \) in Theorem 2 and use that \( \lim_{r \to \infty} L(r) = 0 \) to get that \( u \) must be a constant function. \( \square \)

Before we proceed to the proofs, let us provide two auxiliary statements which explain the link of the scale function \( \varphi \) with the kernels \( K \).

**Lemma 4.** Assume \( r > 0 \) and \( \lambda \geq 1 \). Then

\[
L(r) \geq \gamma^{-1}c_L \ell(r) \quad \text{and} \quad \frac{L(\lambda r)}{L(r)} \geq c_L \lambda^{-\gamma}.
\]
Proof. From (\ell_2) we deduce the following

\[ L(r) = \ell(r) \int_r^\infty \frac{\ell(s) ds}{s^\gamma} \geq c_L \ell(r) r^\gamma \int_r^\infty s^{-1-\gamma} ds = c_L r^{-\gamma} \ell(r). \]

and

\[ L(r\lambda) = \int_r^\infty \ell(s) ds = \int_r^{r\lambda} \ell(s) ds = \int_r^{r\lambda} \frac{\ell(s\lambda)}{\ell(s)} \ell(s) ds \geq c_L \lambda^{-\gamma} \int_r^\infty \ell(s) ds = c_L \lambda^{-\gamma} L(r). \]

\[ \square \]

**Lemma 5.** Assume \( a > 1 \). Then \( \mu(B_{\varphi_a(r)} \setminus B_r) = |\partial B_1| \ln a \), where \( |\partial B_1| \) denotes the surface area of the unit sphere in \( \mathbb{R}^d \).

Proof. The proofs follows by introducing polar coordinates:

\[ \mu(B_{\varphi_a(r)} \setminus B_r) = |\partial B_1| \int_r^{\varphi_a(r)} \frac{1}{L(r)} \ell(s) ds = |\partial B_1| \ln \frac{L(r)}{L(\varphi_a(r))} = |\partial B_1| \ln a. \]

\[ \square \]

**Lemma 6.** Set \( j(s) = s^{-d} \ell(s) \) for \( s > 0 \) where \( \ell \) satisfies (\ell_2) and (\ell_3). Let \( M \geq 1 \). Then

\[ s \leq Mt \quad \text{and} \quad t \leq R_0 \quad \text{imply} \quad j(t) \leq cj(s), \]

with \( c = \max\{c_U, M^{\gamma+d} c_L^{-1}\} \).

Proof. Assume \( s, t > 0 \) with \( s \leq Mt \) for some \( M \geq 1 \). We consider two cases. If \( s < t \), then

\[ j(t) = t^{-d} \ell(t) = (s/t)^d s^{-d} \ell(s/t/s) \leq c_U s^{-d} \ell(s), \]

where we have applied (\ell_3). If \( t \leq s \leq Mt \), then

\[ j(t) = t^{-d} \ell(t) \leq (s/t)^d c_L^{-1} (M^{-1} s)^{-d} \ell(s) \leq M^{\gamma+d} c_L^{-1} j(s), \]

where we have applied (\ell_2). The proof is complete. \[ \square \]

2. Proof of Theorem 1

We provide the proof of Theorem 1.

Proof. Let \( \beta: [0, \infty) \to [0, \infty) \) be a smooth and strictly decreasing function satisfying \( \beta(0) = 1 \) and define

\[ b(x) := \beta(|x|) \quad \text{and} \quad b_r(x) := \beta_r(x) := \beta(r^{-1} |x|) \quad \text{for} \quad x \in \mathbb{R}^d, \ r > 0. \]
Let \( r > 0 \) be such that \( \frac{1}{2} \varphi(r) < R_0 \). First we estimate \(-Ab_r\). By (\( \ell_1 \)), (\( \ell_2 \)) and Lemma 4 it follows that

\[
-Ab_r(x) \leq c_1 \int_{\mathbb{R}^d} \left( b(\frac{x}{r}) - b(\frac{x+y}{r}) - \nabla b(\frac{x}{r}) \cdot \frac{y}{|y|^d} \right) \frac{\ell(|y|)}{|y|^d} \, dy
\]

\[
\leq c_2 \int_{\mathbb{R}^d} \left( \left( \frac{|y|}{r} \right)^2 1_{B_r}(y) + 1_{B_r^c}(y) \right) \frac{\ell(|y|)}{|y|^d} \, dy
\]

\[
= c_3 \left( r^{-2} \int_0^r s \ell(s) \, ds + \int_r^\infty \ell(s) \frac{ds}{s} \right)
\]

\[
= c_3 \left( r^{-2} \ell(r) \int_0^r s \frac{s}{r(s)} \, ds + L(r) \right)
\]

\[
\leq c_4 \left( r^{-2} \ell(r) \int_0^r s \left( \frac{r}{s} \right)^{\gamma} \, ds + L(r) \right)
\]

\[
\leq c_5 (\ell(r) + L(r)) \leq c_6 L(r).
\]

Hence,

\[
\sup_{x \in \mathbb{R}^d} -Ab_r(x) \leq c_6 L(r). \tag{5}
\]

Set \( \theta := \frac{1}{a}(\beta(1) - \beta(\frac{3}{2})) = \frac{1}{a}(\beta_r(r) - \beta_r(\frac{3r}{4})) \), where \( a > 2 \) will be chosen later independently of \( v \) and \( r \).

We claim that one can choose \( a > 2 \) so large that \( v(x) \leq 1 - \theta \) for \( x \in B_r \). Assume that this is not true. Then for any \( a > 2 \) there is \( x_0 \in B_r \) satisfying

\[
v(x_0) \geq 1 - \theta = 1 - a^{-1} \beta_r(r) + a^{-1} \beta_r(\frac{3r}{4}).
\]

In particular, since \( |x_0| < r \),

\[
v(x_0) + a^{-1} b_r(x_0) \geq 1 + a^{-1} \beta_r(|x_0|) - a^{-1} \beta_r(r) + a^{-1} \beta_r(\frac{3r}{4})
\]

\[
> 1 + a^{-1} \beta_r(\frac{3r}{4})
\]

\[
\geq v(y) + a^{-1} \beta_r(|y|) \quad \text{for all} \quad y \in B_{\varphi(r)} \setminus B_{\frac{3r}{4}}, \tag{6}
\]

where the last inequality follows from the assumption \( v(x) \leq 1 \) for \( x \in B_{\varphi(r)} \) and \( \beta_r(|y|) \leq \beta_r(\frac{3r}{4}) \). By choosing \( a \) sufficiently large, we will make sure that \( \varphi(r) > \frac{3r}{4} \). It follows from (6) that \( v + a^{-1} b_r \) attains its maximum at \( x_1 \in B_{\frac{3r}{4}} \) and \( (v + a^{-1} b_r)(x_1) > 1 \).

Figure 1. \( B_r \subset B_{\frac{3r}{4}} \subset B_{\varphi(r)} \)
The idea now is to establish a contradiction by evaluating \(-A(v + a^{-1}b_r)(x_1)\) in two different ways. First, by (5)

\[-A(v + a^{-1}b_r)(x_1) \leq a^{-1}L(r) + c_6a^{-1}L(r) = (1 + c_6)a^{-1}L(r).\]

On the other hand, since \(v + a^{-1}b_r\) attains maximum at \(x_1\), \(\nabla(v + a^{-1}b_r)(x_1) = 0\) and hence

\[-A(v + a^{-1}b_r)(x_1) = \int_{\mathbb{R}^d \setminus \{0\}} ((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)) K(x, y) dy
\]

\[= \int_{\{y \in \mathbb{R}^d \setminus \{0\}: x_1 + y \in B_{\varphi(r)}\}} ((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)) K(x, y) dy
\]

\[+ \int_{\{y \in \mathbb{R}^d \setminus \{0\}: x_1 + y \not\in B_{\varphi(r)}\}} ((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)) K(x, y) dy
\]

\[=: I_1 + I_2.\]

Since \((v + a^{-1}b_r)(x_1) > 1\) is maximum of \(v + a^{-1}b_r\) on \(B_{\varphi(r)}\) and \(|y| \leq |x_1| + |x_1 + y| \leq \frac{3r}{2} + \varphi(r) \leq \frac{3}{2} \varphi(r) < R_0\) for \(x_1 + y \in B_{\varphi(r)}\), by (A1) we obtain

\[I_1 \geq \int_{\{y \in \mathbb{R}^d \setminus \{0\}: x_1 + y \in B_{\varphi(r)}\} \setminus B_r} ((v + a^{-1}b_r)(x_1) - (v + a^{-1}b_r)(x_1 + y)) K(x, y) dy
\]

\[\geq c_7(1 - a^{-1}\|b\|_\infty) \int_{\{y \in \mathbb{R}^d \setminus \{0\}: x_1 + y \in B_{\varphi(r)}\} \setminus B_r, v(x_1 + y) \leq 0} j(|y|) dy,
\]

with \(j(s) := s^{-d} \ell(s)\).

Using \(|y| \leq |x_1 + y| + |x_1| \leq |x_1 + y| + \frac{3r}{2} \leq \frac{5}{2} |x_1 + y|\) for \(x_1 + y \in B_{\varphi(r)} \setminus B_r\), we deduce from (\(L_2\)) that \(j(|y|) \geq c_8 j(|x_1 + y|)\). Here we have applied Lemma 6. Together with the assumptions of the lemma we obtain

\[I_1 \geq c_9(1 - a^{-1}\|b\|_\infty) \int_{\{y \in \mathbb{R}^d \setminus \{0\}: x_1 + y \in B_{\varphi(r)}\} \setminus B_r, v(x_1 + y) \leq 0} j(|x_1 + y|) dy
\]

\[= c_9(1 - a^{-1}) \int_{\{y \in \mathbb{R}^d \setminus \{0\}: y \in B_{\varphi(r)} \setminus B_r, v(y) \leq 0} j(|y|) dy
\]

\[= c_9(1 - a^{-1}) \int_{\{y \in \mathbb{R}^d \setminus \{0\}: y \in B_{\varphi(r)} \setminus B_r, v(y) \leq 0} L(|y|) \mu(dy)
\]

\[\geq c_9(1 - a^{-1}) L(\varphi(r)) \mu((B_{\varphi(r)} \setminus B_r) \cap \{v \leq 0\})
\]

\[\geq c_9(1 - a^{-1}) a^{-1} L(r) \delta \mu(B_{\varphi(r)} \setminus B_r)
\]

\[\geq c_{10}(1 - a^{-1}) a^{-1} L(r) \frac{a}{\alpha},
\]

where in the last inequality we have used Lemma 5.

If we consider \(a > c_L^{-1}(5/2)^\gamma\), then Lemma 4 implies \(\frac{L(r)}{L(\frac{5r}{2})} \leq c_L^{-1}(5/2)^\gamma\). Hence

\[L(\varphi(r)) = a^{-1} \frac{L(r)}{L(\frac{5r}{2})} L(\frac{5r}{2}) \leq L(\frac{5r}{2}).\]
and, since $L$ is decreasing, we obtain that $\varphi(r) \geq \frac{5r}{2}$. To estimate $I_2$ we note that for $x_1 + y \notin B_{\varphi(r)}$ it follows from (6) that

$$(v + a^{-1}b_r)(x_1) - a^{-1}b_r(x_1 + y) \geq 1 + a^{-1}\beta_r(\frac{5r}{2}) - a^{-1}\beta_r(\varphi(r)) \geq 1.$$ 

Hence, this together with the assumption on growth of $v$ yields

$$I_2 \geq -c_{11} \int_{\{y \in \mathbb{R}^d \setminus \{0\} : x_1 + y \notin B_{\varphi(r)}\}} \left(\frac{L(\varphi(r))}{L(x_1 + y)}\right)^{\eta} j(|y|) \, dy.$$ 

Note that $x_1 + y \notin B_{\varphi(r)}$ implies $|y| \geq |x_1 + y| - |x_1| \geq \frac{5r}{2} - \frac{3r}{2} = r$ and

$$|x_1 + y| \leq \frac{3r}{2} + |y| \leq \frac{3}{2}|y| + |y| = \frac{5}{2}|y| \text{ and } y \notin B_{\frac{5}{2}\varphi(r)}.$$ 

In this case $L(|x_1 + y|) \geq L(\frac{5}{2}|y|)$ and

$$\{y \in \mathbb{R}^d \setminus \{0\} : x_1 + y \notin B_{\varphi(r)}\} \subset \{y \in \mathbb{R}^d \setminus \{0\} : |y| \geq \frac{5}{2}\varphi(r)\}.$$ 

Thus we obtain

$$I_2 \geq -c_{12} \int_{\{y \in \mathbb{R}^d \setminus \{0\} : x_1 + y \notin B_{\varphi(r)}\}} \left(\frac{L(\varphi(r))}{L(y)}\right)^{\eta} j(|y|) \, dy \geq -c_{13} \int_{\frac{5}{2}\varphi(r)}^{\infty} \left(\frac{L(\varphi(r))}{L(s)}\right)^{\eta} \frac{j(s)}{s} \, ds \geq -c_{14} \int_{\frac{5}{2}\varphi(r)}^{\infty} \left(\frac{L(\varphi(r))}{L(s)}\right)^{\eta} (-L'(s)) \, ds = -c_{14}L(\varphi(r))^{\eta}(1 - \eta)^{-1}L(\frac{5}{2}\varphi(r))^{1-\eta} \geq -c_{15}L(\varphi(r)) = -c_{15}L(\frac{r}{a}),$$ 

where in the third inequality Lemma 4 and in the last inequality monotonicity of $L$ and Lemma 4 again have been used.

Finally, we obtain

$$(1 + c_6)\frac{L(r)}{a} \geq c_{10}(1 - a^{-1})\delta L(r)\frac{\ln a}{a} - c_{15}\frac{L(r)}{a},$$ 

or written in another way $(1 + c_6 + c_{15}) \geq c_{10}(1 - a^{-1})\delta \ln a$. Choosing $a > 2$ large enough leads to a contradiction.

This means that we have proved that there exists $a > 2$ such that

$$v(x) \leq 1 - a^{-1}(\beta(1) - \beta(\frac{5}{2})) = 1 - \theta \quad \text{for all} \quad x \in B_r.$$ 

Note that our choice of $a$ does not depend on $r$; hence the assertion of the theorem holds for any $r > 0$ satisfying $\frac{5}{2}\varphi(r) < R_0$ with the same choice of $a$ and $\theta$. 

\[ \square \]

3. Proof of Theorem 2

\textit{Proof.} Let $r \in (0, \frac{R_0}{2})$ where $R_0$ is as in (A1). Assume $u \in C^{2}_{b}(\mathbb{R}^{d})$ satisfies $Au = f$ in $B_r$ where $f$ is essentially bounded. We assume $u \neq 0$. We prove assertion (4) in the simplified
case \(\|u\|_\infty \leq 1/2\) and \(\|f\|_{L^\infty(B_r)} \leq \frac{1}{2}L(r/2)\). We briefly explain why this is sufficient. In the general case we would set
\[
\tilde{u} = \frac{u}{2\|u\|_\infty + 2L(r/2)^{-1}\|f\|_{L^\infty(B_r)}},
\]
If \(u\) solved \(Au = f\) in \(B_r\), then \(\tilde{u}\) would solve \(A\tilde{u} = \tilde{f}\) in \(B_r\) with \(\|\tilde{u}\|_\infty \leq 1/2\) and \(\|\tilde{f}\|_{L^\infty(B_r)} \leq \frac{1}{2}L(r/2)\). Thus we could apply the result in the simplified case and obtain:
\[
\sup_{x,y \in B_{r/4}} \frac{|u(x) - u(y)|}{L(|x - y|)} \leq \left( \frac{\gamma}{2}L(r)^\beta + \frac{\gamma}{2}L(r/2)L(r)^{\beta - 1} \right) (2\|u\|_\infty + 2L(r/2)^{-1}\|f\|_{L^\infty(B_r)}) \tag{7}
\]
\[
\leq \tilde{c}L(r)^\beta \|u\|_\infty + \tilde{c}L(r)^{\beta - 1}\|f\|_{L^\infty(B_r)}, \tag{8}
\]
where \(\tilde{c}\) is another constant, depending on \(c_L\) and \(\gamma\) because of Lemma 4.

Hence we can restrict ourselves to \(\|u\|_\infty \leq 1/2\) and \(\|f\|_{L^\infty(B_r)} \leq \frac{1}{2}L(r/2)\). Let \(x_0 \in B_{r/4}\).

![Diagram](image.png)

**Figure 2. Reduction of oscillation at \(x_0\)**

It is sufficient to show that
\[
|u(x) - u(x_0)| \leq c\frac{L(|x - x_0|)^{-\beta}}{L(r)^{-\beta}} \quad \text{for any } x \in B_r.
\]

Define \(r_n = L^{-1}(a^{n-1}L(r/2))\) for \(n \in \mathbb{N}\), where \(a > 2\) will be chosen in the course of the proof independently of \(r, u\) and \(f\). We will construct a nondecreasing sequence \((c_n)_{n \in \mathbb{N}}\) and non-increasing sequence \((d_n)_{n \in \mathbb{N}}\) of positive numbers so that
\[
c_n \leq u(x) \leq d_n \quad \text{for all } x \in B_{r_n}(x_0) \quad \text{and} \quad d_n - c_n \leq b^{-n+1},
\tag{9}
\]
where \(b = \frac{2}{\sqrt{2\theta}} \in (1, 2)\) and \(\theta \in (0, 1)\) will be chosen later independently of \(r, u\) and \(f\). This will be enough, since for \(r_{n+1} \leq |x - x_0| < r_n\) we will then have
\[
|u(x) - u(x_0)| \leq b^{-n+1} = b \left( \frac{b}{a^{n-1}} \right)^{\frac{\ln b}{\ln a}} = b \left( \frac{L(r/2)}{a^{n-1}L(r/2)} \right)^{\frac{\ln b}{\ln a}} < b \left( \frac{L(r/2)}{L(|x - x_0|)} \right)^{\frac{\ln b}{\ln a}},
\]
where in the last inequality we have used Lemma 4.

We prove (9) and construct sequences \((c_n)\) and \((d_n)\) inductively. We set
\[
c_1 := \inf_{\mathbb{R}^d} u \quad \text{and} \quad d_1 := c_1 + 1.
\]
Let $n \in \mathbb{N}$, $n \geq 2$. Assume that $c_k$ and $d_k$ have been constructed for $k \leq n$ and that (9) holds for $k \in \mathbb{N}, k \leq n$. We are now going to construct $c_{n+1}$ and $d_{n+1}$.

Set $m := \frac{c_n + d_n}{2}$. By (9) it follows for $x \in B_{r_n}$
$$u(x) - m \leq \frac{1}{2}(d_n - c_n) \leq \frac{1}{2}b^{-n+1}. $$
Define a function $v : \mathbb{R}^d \to \mathbb{R}$ by $v(x) := 2b^{n-1}(u(x_0 + x) - m)$. Then $v(x) \leq 1$ for $x \in B_{r_n}$ and $Av = 2b^{n-1}f$ in $B_{r_n}$.

Assume that $\mu(\{x \in B_{r_n} \setminus B_{r_{n+1}} : v(x) \leq 0\}) \geq \frac{1}{2}\mu(B_{r_n} \setminus B_{r_{n+1}})$. We recall that the ball $B_{r/2}$ has center 0 and the balls $B_{r_n}$ have center $x_0$. For $x \in B_{r/2} \setminus B_{r_n}$ there exists $k \in \mathbb{N}, k \leq n - 1$ such that $r_{k+1} \leq |x| < r_k$. Then by (9) we have
$$v(x) = 2b^{n-1}(u(x_0 + x) - m) \leq 2b^{n-1}(d_k - m) \leq 2b^{n-1}(d_k - c_k)$$
$$= 2b^{n-k} = 2b \left( \frac{a^{n-1}L(r/2)}{a^nL(r/2)} \right)^{\frac{ln b}{ln a}} \leq 2b \left( \frac{L(r_n)}{L(r/2)} \right)^{\frac{ln b}{ln a}}.$$
If $x \in B_{r/2}$, then $L(|x|) \leq L(r/2)$ and $u(x_0 + x) - m \leq d_1 - c_1 = 1$; hence
$$v(x) \leq 2b^{n-1} = 2 \left( \frac{a^{n-1}L(r/2)}{L(r/2)} \right)^{\frac{ln b}{ln a}} \leq 2 \left( \frac{L(r_n)}{L(r/2)} \right)^{\frac{ln b}{ln a}}.$$

We want to apply Theorem 1 with $r = r_{n+1}$. Note that $r_n = \varphi_n(r_{n+1})$ and $\tilde{\varphi}(r_{n+1}) \leq \frac{5}{2}\varphi(r_2) = \frac{5}{2}r_1 = \frac{5}{2}R_0$. In order to apply Theorem 1 we need to verify that $2b^{n-1}|f| \leq L(\varphi(r_{n+1})) = a^{n-1}L(r/2)$. But this holds true because $|f| \leq \frac{1}{2}L(r/2)$ and $(b/a)^{n-1} \leq b/a \leq 2a^{-1} \leq 1$. Thus we obtain that for some $a > 2$ and $\theta \in (0, 1)$, not depending on $v$ and $r$,

$v(x) \leq 1 - \theta$ on $B_{r_{n+1}}$. Going back to $u$, we deduce
$$u(x) \leq \frac{1 - \theta}{2}b^{1-n} + \frac{c_n + d_n}{2} \quad \text{for} \quad x \in B_{r_{n+1}}(x_0).$$

Hence we may take $c_{n+1} = c_n$ and $d_{n+1} = \min\{d_n, \frac{1 - \theta}{2}b^{1-n} + \frac{c_n + d_n}{2}\}$. This choice implies $d_{n+1} - c_{n+1} \leq b^{-n}$.

In the case $\mu(\{x \in B_{r_n} \setminus B_{r_{n+1}} : v(x) \leq 0\}) < \frac{1}{2}\mu(B_{r_n} \setminus B_{r_{n+1}})$ we repeat the previous argument with $-v$ instead of $v$ and deduce
$$u(x) \geq -\frac{1 - \theta}{2}b^{1-n} + \frac{c_n + d_n}{2} \quad \text{for} \quad x \in B_{r_{n+1}}(x_0).$$

This time we choose $c_{n+1} = \max\{c_n, -\frac{1 - \theta}{2}b^{1-n} + \frac{c_n + d_n}{2}\}$ and $d_{n+1} = d_n$. Finally, we set $\beta := \frac{\ln b}{\ln a}$.

References

[Bac14] J. Bae: Regularity for fully nonlinear equations driven by spatial-inhomogeneous nonlocal operators (2014), http://arxiv.org/pdf/1405.1824v2

[CS09] L. Caffarelli, L. Silvestre: Regularity theory for fully nonlinear integro-differential equations, Comm. Pure Appl. Math., 62(5):597-638 (2009)

[CZ14] Z.-Q. Chen, X. Zhang: Hölder estimates for nonlocal-diffusion equations with drifts (2014), http://arxiv.org/pdf/1410.8243v1
[JW14] S. Jarohs, T. Weth: Symmetry via antisymmetric maximum principles in nonlocal problems of variable order (2014), http://arxiv.org/pdf/1406.6181v1, to appear in Annali Matematica Pura ed Applicata

[KKL14] S. Kim, Y.-C. Kim, K.-A. Lee: Regularity for fully nonlinear integro-differential operators with regularly varying kernels (2014), http://arxiv.org/pdf/1405.4970v2

[KM13] M. Kassmann, A. Mimica: Intrinsic scaling properties for nonlocal operators (2013), http://arxiv.org/pdf/1310.5371v2

[Sil06] L. Silvestre: Hölder estimates for solutions of integro-differential equations like the fractional Laplace, Indiana Univ. Math. J., 55(3):1155–1174 (2006)

Fakultät für Mathematik, Universität Bielefeld, Postfach 100131, D-33501 Bielefeld

E-mail address: moritz.kassmann@uni-bielefeld.de

Department of Mathematics, University of Zagreb, Bijenička c. 30, 10000 Zagreb, Croatia

E-mail address: amimica@math.hr