The reduced phase space of spherically symmetric Einstein-Maxwell theory including a cosmological constant

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Abstract
We extend here the canonical treatment of spherically symmetric (quantum) gravity to the most simple matter coupling, namely spherically symmetric Maxwell theory with or without a cosmological constant. The quantization is based on the reduced phase space which is coordinatized by the mass and the electric charge as well as their canonically conjugate momenta, whose geometrical interpretation is explored.

The dimension of the reduced phase space depends on the topology chosen, quite similar to the case of pure (2+1) gravity.

We investigate several conceptual and technical details that might be of interest for full (3+1) gravity. We use the new canonical variables introduced by Ashtekar, which simplifies the analysis tremendously.

1 Introduction
The introduction of a new set of canonical variables due to Ashtekar [4] brings the initial value constraints of general relativity into polynomial form. This tremendous simplification of the algebraic form of the constraint functionals has far reaching consequences: since the first step in the canonical quantization programme is the solution of the constraints (meaning that an appropriate operator version of the constraint functionals vanishes on physical states) there is justified hope that with these new canonical variables the roadblock that one meets when using the old (ADM) [14] variables (namely that the constraints are not even analytical in the basic variables so that it is not even clear how to define the constraint operators) can actually be overcome.

It is suggested to verify this assumption by first trying to quantize simplified models of pure and matter-coupled gravity. As expected, the complete quantization of model

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systems in the new variables (e.g. [1, 2, 5, 15, 21]) proves to be more feasible and actually led to new, surprising results. Moreover, it gives some confidence in attacking the much more complicated problem of quantizing full (3+1) general relativity via canonical techniques.

Up to now, no model for gravity with matter has been quantized in the new variables. Especially interesting would be a gauge field as the matter content because in this case the scalar constraint turns out to be of fourth, rather than second, order in the momenta which is a quite unusual situation that has been barely dealt with in the literature so far, in fact, the author is not aware of any such case.

In this paper we are going to rigorously quantize a spherically symmetric charged \( (\text{Reissner-Nordstrøm [14]} \) black hole. This is the most simple gauge field that one can imagine to couple to gravity, simple enough in order that all the steps of the canonical quantization programme can be carried out. Only in that way one can hope to gain some insight in the problems that will occur in the full theory and how to solve them. Although this model only has a finite number of degrees of freedom after solving the constraints, it is a true field theory to start with and therefore ”more similar” to full general relativity than, say, cosmological models.

We can also include a cosmological constant into the analysis if we choose a closed (rather than asymptotically flat) topology for the initial value hypersurface.

This paper is the natural extension of [1] (hereafter referred to as I). These two papers treat the vacuum case of a spherically symmetric black hole (similar results were obtained also in [22]). Therefore, this paper, although in principle self-contained, uses many of the results (that is, computational results) of I in order to avoid repetition.

It turns out that the application of the operator constraint method (Dirac method [18]) fails due to the fourth order (in the momenta) of the scalar constraint when using the Ashtekar variables. More precisely, the author was unable to find consistent orderings (without regularization) of the constraint operators such that they form a closed algebra (point splitting regularizations are available but the associated manipulations are ill-defined [4]). However, we succeed in using the framework of symplectic reduction (degenerate Hamilton-Jacobi method [10]).

The computational part of this paper is quite large, however, as it is well known, one actually loses physical degrees of freedom if one is not very careful with the functional analysis, at least for asymptotically flat topologies. It is therefore for physical reasons that we display in detail the fine tuned interplay between the role of boundary conditions (fall-off behaviour of the various fields), the difference between symmetries and gauge, the finiteness of the functionals on the reduced phase space and their (functional) differentiability. These items show up only in the symplectic reduction of true field theories, models that are finite-dimensional even before reduction like cosmologies (see ref. [3]) or those that deal only with closed topologies like (2+1) gravity (see ref. [2]) do not share these problems and so it seems to be justified to dwell a little bit on the mathematical techniques involved.

The organization of the paper is as follows:

Section 2 introduces the model in terms of Ashtekar’s variables. This serves mainly in order to fix our notation.

Section 3 discusses the various available topologies for the initial data hypersurface.
After a topology is fixed (once and for all - this is one of the disadvantages of the canonical quantization scheme), the associated fall-off behaviour of the various fields can be derived. It will turn out that the dimension of the reduced phase space depends on the topology chosen - quite similar to the case of the (2+1) gravity model. The role of the genus of the Riemann surface which is the initial value surface for (2+1) gravity is played by the number of the asymptotic ends for spherically symmetric gravity (irrespective of whether with or without sources).

The fall-off behaviour of the fields is chosen more general than the one used by Ashtekar [4].

Section 4 then comprises the symplectic reduction of the present model. Quite surprisingly, all constraints can be strictly solved for the momenta which consist of the Einstein-Maxwell 'electric' fields in the Ashtekar polarization of the phase space. However, the equations for the electric fields become 4th order algebraic equations and hence we were not able to apply the theorem proved in I in order to solve the associated Hamilton-Jacobi equations due to the complicated appearance of the solution formulas for the electric fields. The basic idea is to change the polarization in order to simplify the Hamilton-Jacobi equations and to return to the Ashtekar-polarization after the solution has been found.

Section 5 discusses the topology and reality structure of the reduced phase space obtained in section 4. The topology turns out to be quite complicated due to the appearance of several "sectors" (the solution of an n-th order scalar constraint in terms of its momenta has n roots) especially for non-zero cosmological constant. Furthermore we prove that with the function spaces derived in section 3, the observables (i.e. coordinates on the reduced phase space) are finite and functionally differentiable.

Having obtained this result, we are able to prove in section 6 that the observables are gauge invariant but transform non-trivially under symmetries of the asymptotic structure. This in turn allows to give a definite interpretation of the observables conjugate to the mass and the electric charge respectively, at least for open topologies: they are the eigentime of an asymptotic observer and the variable conjugate to the electric charge that also appears in canonical (1+1) Maxwell theory. The latter can be interpreted as a "formal magnetic flux" (which has nothing to do with the monopole that we might or might not introduce) and we will refer to it as the magnetic flux in the sequel. We want to stress that these observables are genuine volume integrals and no surface terms as has been expected by some authors which is important because otherwise the reduced phase space for closed topologies would only carry a presymplectic, rather than a symplectic structure. They do not seem to have been discovered before, probably because the Birkhoff theorem ([14]) excludes the existence of these variables. We recall from I why this is no contradiction. Moreover, as already stressed in the second paper of [1], the reduced system adopts the form of an integrable model whereby the role of the action variables is played by the mass and the charge whereas the angle variables are their canonically conjugate momenta.

Section 7 is dedicated to quantum theory. After the quantum theory via the group theoretical approach is derived ([8]) one is able to study the solutions of the Schroedinger-equation. The eigenvectors of the Hamilton-operator are peaked on the classical eigentime and classical magnetic flux.

Another subsection deals with the operator constraint or Dirac algorithm for quantizing field theories with constraints. As for all Yang-Mills theories, the scalar constraint functional becomes non-analytic in the electric fields so that this method (in the Ashtekar-polarization) only makes sense after multiplying the scalar constraint functional with an
appropriate power of the electric fields. The unregulated constraint operator does not close the quantum algebra whereas the regulated one is ill-defined as already said above. If one works, however, with the polarization suggested already by the classical theory, then one recovers the same quantum theory as obtained via the reduced phase space method.

The paper concludes with some remarks on what has been learnt by studying this model.

In an appendix we carry out a tedious computation which proves the statements given in section 6.

2 Introduction of the model

We use the same notation as in I. The spherically symmetric reduction of the Einstein sector of the model is defined identically as in I. Furthermore, only gravitational fields contribute to the (ADM)-energy-momentum. Finally, the reality conditions on the Ashtekar variables are unchanged when coupling bosonic matter only, hence the formulas of I except for the boundary conditions can be taken over without change directly to the present case. Suffice it to recall that after the spherically symmetric reduction the Ashtekar variables \( E_i^a = \det(e_i^a) e_i^a \) and \( A_i^a = \Gamma_i^a + i K_{ab} e_i^b \), where \( e_i^a \) is the spatial triad and \( K_{ab} \) is the extrinsic curvature of the initial data hypersurface, become

\[
(E_x^x, E_x^y, E_x^z) = (\sin(\theta) E^1 n_t^x, \frac{\sin(\theta)}{\sqrt{2}} (E^2 n_t^x + E^3 n_t^z), \frac{1}{\sqrt{2}} (E^2 n_t^x - E^3 n_t^z))
\]

\[
(A_x^x, A_x^y, A_x^z) = (A_1 n_t^x, \frac{1}{\sqrt{2}} (A_2 n_t^y + (A_3 - \sqrt{2}) n_t^x), \frac{\sin(\theta)}{\sqrt{2}} (A_2 n_t^y - (A_3 - \sqrt{2}) n_t^x))
\]

Here \( \theta, \phi \) are the usual polar coordinates on \( S^2 \) and the internal vectors \( n_x, n_\theta, n_\phi \) are the standard orthonormal vectors \([1]\). The functions \( E^I, A_I, I = 1, 2, 3 \) depend on the spatial variable \( x \) and the time variable \( t \) only. \( E^I \) is real while \( A_I - \Gamma_I \) is imaginary where

\[
(\Gamma_1, \Gamma_2, \Gamma_3) = \left( -\frac{(E^3)^{\prime} E^2 - (E^2)^{\prime} E^3}{(E^2)^2 + (E^3)^2}, -\frac{(E^1)^{\prime} E^3}{(E^2)^2 + (E^3)^2}, \frac{(E^1)^{\prime} E^2}{(E^2)^2 + (E^3)^2} \right)
\]

We can now proceed to the source terms.

We require that the Maxwell electric \((\epsilon^e)\) and magnetic fields \((\mu^a)\) are spherically symmetric, i.e. they are Lie annihilated by the generators of the SO(3) Killing group. The unique solution of this definition are radially symmetric fields

\[
(\epsilon^x, \epsilon^\theta, \epsilon^\phi) := (\epsilon(x, t), 0, 0),
\]

\[
(\mu^x, \mu^\theta, \mu^\phi) := (\mu(x, t), 0, 0).
\]

(2.1)

Now we exploit that the magnetic field comes from a spatial potential \( \omega_a \) i.e. \( \mu^a = 1/2 \epsilon^{abc} \partial_b \omega_c \). Then it follows (locally) from the Bianchi identity

\[
2 \partial_x \mu^x = - (\partial_\theta \mu^\theta + \partial_\phi \mu^\phi) = 0,
\]

(2.2)

i.e. \( \mu = \mu(t) \) is a spatial constant, the magnetic charge. The Maxwell potential is thus given by

\[
(\omega_x, \omega_\theta, \omega_\phi) = (\omega(x, t), 0, 0) + (\Omega_a(x, t, \theta, \phi)),
\]

(2.3)
where $\Omega_a$ is a monopole solution with charge $\mu : \star d \wedge \Omega = \mu$ (it has no radial part).
The cosmological constant will be labelled by the (real) parameter $\lambda$. Then we are already in the position to complete
the reduction to spherical symmetry by plugging the formulas (2.1) and (2.3) into the canonical Yang-Mills action (given below for arbitrary (semisimple) gauge group $G$)

$$ Y^M S = \int_R dt \int_{\Sigma^3} d^3 x \text{tr}\{ \dot{\omega}^a \epsilon^a - [-U D_a \epsilon^a + N^a \epsilon_{abc} \mu^c \epsilon^b + N^1 1 \frac{1}{2} q_{ab}(\epsilon^a \epsilon^b + \mu^a \mu^b)] \} \quad (2.4) $$

which for spherical symmetry and $G = U1$ becomes after integration over the sphere (using $\Sigma^3 = S^2 \times \Sigma$ where $\Sigma$ is 1-dimensional)

$$ M S = 4\pi \int_R dt \int_{\Sigma} dx \{ \dot{\omega} \epsilon - [ -U \epsilon' + N^1 1 \frac{1}{2} (E^2)^2 + (E^3)^2 ] \} \quad (2.5) $$

Note that the monopole potential was projected out since it has no radial component and that for the same reason there is no Maxwell-contribution to the vector constraint. Furthermore we do not care about boundary terms at this stage. We define $p^2 := \epsilon^2 + \mu^2$, rescale the Lagrange-multiplier $U$ and the radial part of the Maxwell-connection by $p / \sqrt{p^2 - \mu^2}$ and we arrive at the same action without magnetic charge. Formally we have carried out a ‘duality rotation’ which leaves the energy-momentum tensor of the Maxwell-field unchanged.

Finally, we have for the cosmological constant term

$$ C S = \int_R dt \int_{\Sigma} d^3 x N \lambda \sqrt{q} = 4\pi \int_R dt \int_{\Sigma} dx \lambda N^1 1 \frac{1}{2} ((E^2)^2 + (E^3)^2) E^1 \quad (2.6) $$

The model has thus 4 canonical pairs $(\omega, p ; \omega, A_f, E_f)$ and is subject to the 4 constraints, defined by the following 4 constraint functionals:

Maxwell Gauss constraint :

$$ M G = p' \quad (2.7) $$

Einstein Gauss constraint :

$$ E G = (E^1)' + A_2 E^3 - A_3 E^2 \quad (2.8) $$

Vector constraint :

$$ V = B^2 E^3 - B^3 E^2 \quad (2.9) $$

scalar constraint :

$$ C = (B^2 E^2 + B^3 E^3) E^1 + \frac{1}{2} ((E^2)^2 + (E^3)^2) (B^1 + \kappa \frac{p^2}{2E^1} + \kappa \lambda E^1) \quad (2.10) $$

which implies that there will be only a finite number of degrees of freedom left on the reduced phase space.

Additionally to the ADM energy-momentum (I) there is the electric charge boundary term (this interpretation follows from inserting a Reissner-Nordstrøm solution)

$$ + \int_{\partial \Sigma} U p \quad (2.11) $$
3 Topologies and function spaces

3.1 Possible topologies

For spherically symmetric systems, the topology of the 3 manifold is necessarily of the form \( \Sigma^{(3)} = S^2 \times \Sigma \) where \( \Sigma \) is a 1-dimensional manifold.

We will deal with 2 kinds of topologies:

a) compact without boundary
In this case the only possible choice is \( \Sigma = S^1 \), i.e. we have the topology of a compactified wormhole.

b) open with boundary
As was motivated already in I, we choose now

\[ \Sigma = \Sigma_n, \quad \Sigma_n \cong K \cup \bigcup_{A=1}^{n} \Sigma_A, \]

i.e. the hypersurface is the union of a compact set \( K \) (diffeomorphic to a compact interval) and a collection of ends (each of which is diffeomorphic to the positive real line without the origin) i.e. asymptotic regions with outward orientation and all of them are joined to \( K \). This means, we have \( n \) positive real lines, including the origin, but one end of each line is common to all of them, i.e. these parts are identified. Since the identity map is smooth, this is still a \( C^\infty \) (Hausdorff) manifold except in a neighbourhood of the end point of the common line. This kind of topology is illustrated in the figure below.

We want to point out here three items:

1) The boundary of the compactum \( K \) is to be understood topologically, for a given value of the mass \( m \) it is not fixed a priori at the coordinate value \( r = m \) because otherwise the topology would depend on a dynamical object, the mass of the system, while in the canonical framework the choice of the topology is a kinematical ingredient, it is fixed once and for all right from the beginning before solving the dynamics of the system.

2) Boundary conditions should not only be imposed in the asymptotic regions but also at the origin inside the compactum \( K \). Note that this is never done in the literature since one is usually only interested in issues like the positivity of the gravitational energy ([19]) or the asymptotic Poincaré group ([11]).

3) The compactum \( K \) could be replaced by another topological object so as to obtain the Reissner-Nordstrøm topology ([14]) that avoids closed timelike curves. The formalism does not force us to do that, that is, the Reissner-Nordstrøm topology does not follow from
Einstein’s equations because the then necessary extension of K lies outside the domain of
dependence of the initial data hypersurface. This is also why we do not have to distinguish
between the cases where the electric charge squared is less or greater than or equal to the
mass squared of the black hole. We refrain from analyzing the most general situation in
the present paper and prefer to deal with a geodesically incomplete manifold.

3.2 Derivation of the function spaces

a) Asymptotically flat topologies

Up to now we did not modify the Einstein sector of the theory at all compared to I.
In contrast to I we will derive a new set of function spaces (fall-off behaviour of the fields)
based on the following 2 minimal requirements, following [9]:

1) finiteness of the symplectic structure,
2) finiteness and functional differentiability of the constraint functionals. Requirement 2)
further depends on the set of asymptotic symmetries that one is willing to allow.
In [9] (which is based on the old (ADM) variables) these requirements 1) and 2) including
asymptotical Poincare transformations can be satisfied as follows:

\[ q_{ab} \rightarrow ^0q_{ab} + \frac{f_{ab}(x^c/r, t)}{r} + O(1/r^{1+\epsilon}) \]

\[ p^{ab} \rightarrow ^0p^{ab} + \frac{k^{ab}(x^c/r, t)}{r^2} + O(1/r^{2+\epsilon}) \]  (3.1)

as \( r \to \infty \), \( r := ^0q_{ab}x^ax^b \) whereby \( ^0q_{ab} \) is a fixed (nondynamical) flat metric of Euclidean
signature and \( x^a \) are cartesian coordinates with respect to it. Furthermore, it must be
required that the functions \( f_{ab} \) and \( k^{ab} \) respectively are even and odd respectively under
reflections of the asymptotically flat frame.

It is clear that for spherical symmetry we are not able to impose the above parity con-
ditions because the reduction to spherical symmetry excludes all modes of the fields
(considered as expanded into spherical harmonics) which have angular momentum different
from zero. Hence we have to modify the strategy slightly.

Comparing the spherically symmetric metric

\[ q_{ab} = \frac{(E^2)^2 + (E^3)^2}{2E^1}x_ax_b + E^1h_{ab} \]  (3.2)

with the Euclidean metric in spherical coordinates (we choose the coordinate \( x \) to coincide
asymptotically with the radial coordinate of the asymptotical Euclidean frame)

\[ ^0q_{ab} = x_ax_b + x^2h_{ab} \],  (3.3)

where \( h_{ab} \) is the standard metric on the sphere, we conclude the following fall-off
properties :

\[ (E^1, E^2, E^3) \rightarrow (x^2[1 + \frac{f^1(t)}{x} + O(1/x^2)], \]

\[ \sqrt{2}x[E^2 + \frac{f^2(t)}{x} + O(1/r^2)], \sqrt{2}x[E^3 + \frac{f^2(t)}{x} + O(1/x^2)] \]  (3.4)
whereby $(\bar{E}^2)^2 + (\bar{E}^3)^2 = 1$. As well as the motivation for the fall-off behaviour of the metric is that it should approach asymptotically a Schwarzschild solution, the motivation for the fall-off of the Maxwell electric field is that it should approach asymptotically a Coulomb solution. The Coulomb solution in Minkowski space in spherical coordinates is just $e^a = e x_a$ where $e_{,x} = 0$, hence we conclude

$$p \to e(t) + O(1/x).$$

(3.5)

The Einstein-Maxwell connection behaves as a one-form under diffeomorphisms. That means that compared to cartesian coordinates in spherical coordinates the $\theta, \phi$ components adopt an additional power of $x$. Recalling the definition of the Ashtekar connection from [4] as well as the definition of the Maxwell electric field $\epsilon^a = q^{ab}(F_{tb} + N^c F_{bc})/\bar{N}$ which reduces here to $p = q^{xx}/\bar{N} (\dot{\omega} - U')$ we conclude that

$$(A_1, A_2, A_3 - \sqrt{2}) \rightarrow \left( \frac{a_1(t)}{x^2} + O(1/x^3), \frac{a_2(t)}{x} + O(1/x^2), \frac{a_3(t)}{x} + O(1/x^2) \right)$$

$$\omega \rightarrow \frac{b(t)}{x^2} + O(1/x^3).$$

(3.6)

Since, as we noted before, there is no parity freedom left, the requirements 1) and 2) discussed above will not be satisfied yet. Let us explore what further restrictions are there to be imposed.

The symplectic structure on the large phase space can be read off from the action. The non-vanishing brackets are

$$\{ A_I(x), E^J(y) \} = i\kappa \delta^J_I \delta(x,y), \{ \omega(x), p(y) \} = \delta(x,y) \text{ for all } x,y \text{ in } \Sigma.$$ (3.7)

Written as a 2-form on the space of the variations of the fields:

$$\Omega = \int_\Sigma dx [-i/\kappa dE^I \wedge dA_I + dp \wedge d\omega]$$

$$= \int_\Sigma dx \left( \frac{i}{\kappa x} [da_1 \wedge df^1 + \sqrt{2} (da_2 \wedge df^2 + da_3 \wedge df^3)] + O(1/x^2) \right).$$

(3.8)

Hence we can satisfy requirement 1) by restricting the variations to be such that

$$da_1 \wedge df^1 + \sqrt{2} (da_2 \wedge df^2 + da_3 \wedge df^3) = 0.$$ (3.9)

As for requirement 2) we first have to agree on the set of allowed symmetries at infinity. We want to incorporate only asymptotic translations as well as asymptotic U(1)-transformations of the Maxwell-field, and do also allow for asymptotic O(2) transformations of the Einstein fields. Why do we not consider asymptotic boosts of the 2-dimensional flat structure (rotations do not exist in 1 dimension anyway)? In the literature, one looks at Schwarzschild-solutions in arbitrarily boosted frames (see ref. [9], for example). However, these boosts are really boosts with respect to the 4-dimensional spacetime which violate spherical symmetry of the initial data. The ’boosts’ that we were able to discuss here must be meant with respect to the effective 2-dimensional spacetime coordinatized by the variables $x$ and $t$ in order not to violate spherical symmetry, they are thus not physical anyway. But since we do not have this parity freedom at our disposal
our 'boost' generator diverges. So we would have to impose much more restrictive fall-off conditions than in (3.4)-(3.6) which, in particular, would exclude Reissner-Nordstrøm configurations and for that reason we refrain from doing so. An option would be to impose some 'reflection conditions' in different ends of \( \Sigma \) to make the boost generator converge, however then masses and charges in different ends would not evolve independently of each other although they are spacelike separated and this seems to contradict causality. Hence we neither impose such a condition.

The same is actually true for asymptotic translations: only radial translations preserve the spherical symmetry of the fields, that is, translations of the form \( x^a \rightarrow x^a + cx^a/r \) where \( c \) is a constant but these are then position-dependent (on the sphere) and do not correspond to the translation subgroup of the Poincaré group, rather they are odd supertranslations ([9]). They correspond to a translation of the radial coordinate \( r \) by \( c \).

Recalling that \( N^x = N^a \partial x / \partial x^a \) we have in this case (i.e. for \( N^a = cx^a/r \)) really \( N^x = c \).

Obviously, we have then for symmetry transformations the following fall-off behaviour of the Lagrange multipliers:

\[
(\Lambda, N^x, N, U) \rightarrow \left( \frac{\text{const.}}{x^2} + O(1/x^3), \frac{\text{const.}}{x^2} + O(1/x^3), \frac{\text{const.}}{x^2} + O(1/x^3) \right)
\]

while for gauge transformations we require, for simplicity, that the Lagrange multipliers are of compact support.

We now compute the leading order behaviour of the integrands of the constraint functionals: the Maxwell Gauss constraint functional is already finite, it becomes functionally differentiable when adding the electric charge counterterm. For the rest of the constraints we have

\[
E \mathcal{G} \rightarrow 2x(1 - \bar{E}^2) + f^1 + \sqrt{2}(a_2 \bar{E}^3 - \sqrt{2} f^2 - \bar{E}^3 a_3) + O(1/x)
\]

which becomes a finite functional when imposing \( \bar{E}^2 = 1 \) i.e. \( \bar{E}^3 = 0 \). Functional differentiability can be achieved without adding the \( \text{O}(2) \) charge given in I.

We want here to draw attention to the following subtlety: the constraints follow from setting the variation of the action with respect to the Lagrange multipliers equal to zero. If now the variation of a Lagrange multiplier happens to occur outside its support, its variation also vanishes. Hence the constraints hold only in the support of the Lagrange multipliers. What support is valid: that for symmetries or for gauge? Since the constraint equations are field equations the variations are set equal to zero at spacelike and timelike infinity upon deriving the Euler-Lagrange equations, hence it is consistent to impose the constraints only off the boundaries although for simplicity one usually imposes them everywhere.

Note that weakly (i.e. on the constraint surface) we have, requiring the Gauss constraint to hold even at infinity,

\[
f^1 - 2f^2 - \sqrt{2}a_3 = 0 .
\]

It is convenient first to compute the asymptotic form of the Einstein magnetic fields

\[
B^1 \rightarrow -\frac{\sqrt{2}a_3}{x} + O(1/x^2),
B^2 \rightarrow \frac{a_3}{x^2} + O(1/x^3),
B^3 \rightarrow \frac{a_2 + \sqrt{2}a_1}{x^2} + O(1/x^3)
\]
to conclude for the vector constraint

\[ V \rightarrow -\sqrt{2}a_2 + \sqrt{2}a_1 + O(1/x^2) . \]  

Hence we have to impose

\[ a_2 + \sqrt{2}a_1 = 0 \]  

in order to make this functional finite and differentiability can be achieved by adding the ADM-momentum as in I. Finally, it is easy to see that with this restriction the scalar constraint functional is already finite and functionally differentiable when adding the ADM-energy, provided we set the cosmological constant equal to zero.

Now it is possible to make the restriction that comes from requirement 1) more concrete. We have

\[ 0 = da_1 \wedge df^1 + \sqrt{2}(da_2 \wedge df^2 + da_3 \wedge df^3) \]
\[ = -\frac{1}{\sqrt{2}}da_2 \wedge d(f^1 - 2f^2) + \sqrt{2}da_3 \wedge df^3 \]
\[ = -\frac{1}{\sqrt{2}}da_2 \wedge d(f^1 - 2f^2 - \sqrt{2}a_3) + da_3 \wedge d(\sqrt{2}f^3 + a_2) . \]  

Note that the bracket of the 1st wedge product in the last line of (3.16) vanishes weakly according to (3.12). Hence it is consistent with the constraint equations to impose

\[ \sqrt{2}f^3 + a_2 = 0 . \]  

This completes the boundary conditions at the ends of the hypersurface. What about the interior, the compactum K? For gauge transformations one requires that the asymptotic structure is untouched. Since the compactum K is also such a kinematical ingredient of the formalism, we also require that for gauge transformations the Lagrange multipliers have compact support \emph{outside and inside} the compactum K while for symmetries they shall be smooth functions on all of \( \Sigma \). Hence there is a transition region between the asymptotic ends and the compactum K. As for the fields, it is motivated to adapt their behaviour in K in such a way that \emph{observables} are well-defined. We therefore have to postpone this item at this stage and come back to it after the formal expressions for the observables have been found.

Note that in the literature one usually assumes that ‘there exists a regular initial data set on the hypersurface’ ([19]). Since initial data are in one to one correspondence with the Dirac observables, what we do here in choosing the boundary conditions in the interior is nothing else than a realization of this assumption in a concrete example. Accordingly, the definition of the fall-off behaviour of the fields becomes (partly) a dynamical ingredient of the formalism.

We have by now succeeded to give a definition of the phase space which relies on minimal requirements and which is general enough to allow for non-trivial dynamics on the reduced phase space (compare also the 2nd paper in I).

b) Compact topologies

Here it is sufficient to require the fields and Lagrange multipliers to be smooth and finite everywhere. The cosmological constant may take any finite value. Obviously, the case of compact topologies is much more easier to handle from a technical point of view.
4 Symplectic reduction of the model

We will use some basic facts from the theory of symplectic reduction which can be looked up in the second paper of I and in much greater detail in [8], [10]. We can apply that theory here because, as was shown in I, the present model is a field theory with first class constraints. According to that theory we are thus first of all interested in the solutions of constraint equations.

Recall from I that the following set of 'cylinder' coordinates was suggested from the transformation properties of the gravitational variables under the gravitational Gauss constraint

\[(A_2, A_3) = \sqrt{A}(\cos(\alpha), \sin(\alpha)), \quad (E^2, E^3) = \sqrt{E}(\cos(\beta), \sin(\beta))\].

(4.1)

Now, recall the following result from I: Lemma: The reduced symplectic potential with respect to the Gauss constraint is given by

\[i\kappa \Theta[\partial_t] = \int_\Sigma dx(\dot{\gamma}\pi_\gamma + \dot{B}^1\pi_1 + \dot{\omega}p)\],

(4.2)

where

\[\gamma := A_1 + \alpha', \quad \pi_\gamma := E^1, \quad B^1 := \frac{1}{2}(A - 2) \quad \text{and} \quad \pi_1 := \sqrt{E/A}\cos(\alpha - \beta).

(4.3)

Proof: compare I, second paper

In the following p will already be taken as a constant. Also we will deal with an arbitrary cosmological constant for the sake of generality. We take then the following linear combinations of the vector and the scalar constraint functional

\[E^1E^2V + E^3C = E(E^1B^3 + \frac{1}{2}E^3(\frac{\kappa p^2}{2E^1}\kappa + \kappa\lambda E^1 + B^1))\]

\[-E^1E^3V + E^2C = E(E^1B^2 + \frac{1}{2}E^2(\frac{\kappa p^2}{2E^1}\kappa + \kappa\lambda E^1 + B^1))\],

(4.4)

where \(E = (E^2)^2 + (E^3)^2\).

Setting these expressions strongly zero we obtain, exactly as in I, 2 possible solutions:

Case I: \(E = 0\) (degenerate case)

Looking at the formula for the metric (3.2) we see that there is no radial distance now. From the reality of the triads we conclude further that \(E^2 = E^3 = 0\) whence we conclude \(E^1 = E^1(t)\) via setting the Gauss constraint equal to zero. Obviously this solution of the constraint equations is not valid in the asymptotic ends since it violates the asymptotic conditions on the fields. It can therefore only hold in the compactum K. For compact topologies it is a global solution of the constraints. We can thus apply the above framework of symplectic reduction only for that part of the symplectic potential which corresponds to K or \(S^1\) and obtain the reduced symplectic potential

\[\dot{\Theta}[\partial_t] = E^1\frac{d}{dt}(\frac{-i}{\kappa} \int_M d x A_1) + p\frac{d}{dt} \int_M d x \omega =: m\dot{T} + p\dot{\Phi}\]

(4.5)
where \( M \) means \( K \) or \( S^1 \).

Case II : \( E \neq 0 \) (nondegenerate case)

We now conclude

\[
0 = E^1 B^3 + \frac{1}{2} E^3 (\frac{\kappa p^2}{2E^1} + \kappa \lambda E^1 + B^1),
\]

\[
0 = E^1 B^2 + \frac{1}{2} E^2 (\frac{\kappa p^2}{2E^1} + \kappa \lambda E^1 + B^1)
\]  \quad (4.6)

and can further distinguish between a) \( f = 0 \) and b) \( f \neq 0 \) where \( f = \frac{\kappa p^2}{2E^1} + \kappa \lambda E^1 + B^1 \).

We assume the generic case \( p \neq 0 \) to hold in the following. Then in order that the scalar constraint makes sense at all, we must have \( E^1 \neq 0 \).

Subcase a)

Here it follows from (4.6) that \( B^2 = B^3 = 0 \). Hence, from the Bianchi-identity \( A_3 B^2 - A_2 B^3 = (B^1)' = 0 \), we infer \( (B^1)' = 0 \) i.e. \( B^1 \) is a spatial constant whence from \( f = 0 \), \( E^1 \) is also a spatial constant. Therefore, this solution of the constraint equations can only refer to the compactum \( K \) or to the compact case. Finally, we have \( 0 = A_2 B^2 + A_3 B^3 = A\gamma = 2(1 + B^1)\gamma \Rightarrow \gamma = 0 \) or \( B^1 = -1 \). Using the above lemma we can finally carry out the pull-back on \( M \in \{ K, S^1 \} \):

\[
\hat{\Theta} [\partial_t] = \hat{B}^1 (\frac{-i}{\kappa} \int_M dx \pi_1) + p \left( \frac{d}{dt} \int_M dx \omega \right) =: m \hat{T} + p \hat{\Phi}. \]  \quad (4.7)

or

\[
\hat{\Theta} [\partial_t] = \pi_1 \frac{d}{dt} (\frac{-i}{\kappa} \int_M dx \gamma) + p \left( \frac{d}{dt} \int_M dx \omega \right) =: m \hat{T} + p \hat{\Phi}. \]  \quad (4.8)

We do not consider the trivial case \( A = \gamma = 0 \) which is equivalent to 2-dimensional pure Maxwell theory without dynamics.

Subcase b)

We can, by virtue of \( f \neq 0 \), divide by \( f \) to solve eqs. (4.6) for the momenta \( E^2 \) and \( E^3 \)

\[
E^2 = -\frac{2(E^1)^2}{\kappa(p^2/2 + \lambda(E^1)^2 + B^1E^1)} B^2,
\]

\[
E^3 = -\frac{2(E^1)^2}{\kappa(p^2/2 + \lambda(E^1)^2 + B^1E^1)} B^3
\]  \quad (4.9)

and insert this into into the Gauss constraint :

\[
0 = (\kappa(p^2/2 + \lambda(E^1)^2) + B^1E^1)(E^1)' + 2(E^1)^2(B^1)' \]  \quad (4.10)

Eqn. (4.10) can be written as the derivative of a constant function of \( B^1 \) and \( E^1 \) with respect to \( x \) after multiplying it with the integrating multiplicator \( (E^1)^{-3/2} \) :

\[
\left( \frac{1}{\sqrt{E^1}} [\kappa(-p^2/2 + \lambda(E^1)^2/3) + B^1E^1] \right)' = 0 \]  \quad (4.11)

i.e. \( [\kappa(-p^2/2 + \lambda(E^1)^2/3) + B^1E^1]^2 = PE^1 \). \quad (4.12)

The integration constant, \( P \), is real because, as proved in I, the magnetic fields are (weakly) real. Note further that the rhs. of (4.15) is non-negative whence \( PE^1 \geq 0 \). In the asymptotical ends \( E^1 \) is positive such that \( P \) is non-negative and thus can be
written $P = m^2$ where $m$ is real and it is easy to see the relation of $m$ with the gravitational mass $m_G$ : using (3.4), (3.13) and (4.16) and expanding in powers of $r$ one finds $m^2 = (-\sqrt{2}a_3)^2$ together with $a_3 = -\sqrt{2}m_G$ and we will fix the sign ambiguity by requiring that $m = +\sqrt{2}a_3$.

If one uses the positive censorship conjecture that there are no naked singularities, then the positive energy theorem (see ref. [11]) tells us that $m$ is not positive because, although the energy density of matter $E/(2E^1)p^2$ is manifestly non-negative, in the positive $m$ case the singularity at the origin $x = 0$ is timelike and therefore no spacelike hypersurface with everywhere regular initial data exists such that the positive energy theorem does not apply. We will, however, not make such an assumption (that case is treated in the 2nd paper of I).

Note that (4.12) is a purely algebraic equation of fourth order for $E^1$ in terms of $B^1$. Although algebraic eqs. of fourth order can be solved analytically, the corresponding formulas are too complicated as that the theorem in I could be applied (solving the Hamilton-Jacobi equations by quadrature techniques) in the general case. An exception is the special case $P = p = 0$ : in that case the electric fields are proportional to the magnetic ones

$$E^I = -\frac{3}{\kappa \lambda} B^I, I = 1, 2, 3$$

and the unique solution of the Hamilton-Jacobi equations $\delta S/\delta A_I = -3/(\kappa \lambda) B^I$ is precisely the reduction to spherical symmetry of the SO(3)-Chern-Simons functional

$$S = -\frac{6}{4\pi \kappa \lambda} \int_{\Sigma^3} d^3x tr[A \wedge (F - \frac{1}{3} A \wedge A)] = -\frac{3}{\kappa \lambda} \int_{\Sigma} dx \gamma B^1$$

which was to be expected from the corresponding result for the full theory (see ref. [12]). However, since the solution of the Hamilton-Jacobi equation does not depend on any free parameter, it follows that the reduced symplectic potential vanishes in that case. This solution has thus only the status of a total differential that can be added to the action and gives rise to a $\theta$-term as in Yang-Mills theory.

Note also that formula (4.12) can be obtained by the method of CDJ (see ref. [13]) restricted to spherical symmetry (see I), however its derivation is not simplified by that method, so we do not display it here. That the CDJ-method only applies for case II.b follows from the fact that only then ‘electric and magnetic metric’ are non-degenerate.

In order to actually reduce the theory in the general case, one can proceed as follows : the basic observation is that formula (4.12) can be solved easily for the magnetic field $B^1$ which, according to the above lemma, can be chosen as a canonical coordinate. Thus, if one simply changes the polarization so that $B^1$ becomes a momentum, the chance that one can complete the reduction becomes significantly larger. Accordingly, let us write formula (4.12) as

$$B^1 = \frac{\kappa p^2}{2\pi \gamma} - \frac{\lambda \kappa}{3(\pi \gamma)^2} + \frac{m}{\sqrt{\pi \gamma}}, \quad (4.15)$$

where we have confined ourselves to an asymptotic region in order to have $P$ positive or zero, the compact case can be treated analogously. From the transformation properties of the fields under diffeomorphisms (see I) we know that the Gauss-reduced vector constraint
must be
\[ V = -\gamma \pi_1' + (B_1')\pi_1 \]  
(4.16)
which can also be checked explicitly, of course. Substituting for \( B_1 \) from (4.15) we can solve (4.16) for \( \gamma \):
\[ \gamma = \pi_1 \left( -\frac{\kappa \pi_1^2}{2(\pi_1)^2} + \frac{2\lambda \kappa}{3(\pi_1)^3} - \frac{m}{2(\sqrt{\pi_1})^3} \right) . \]  
(4.17)
Since for the asymptotically flat topologies we must have \( \lambda = 0 \) we conclude from (4.15) that asymptotically \( E_1 \propto (B_1)^{-2} \propto x^2 \), i.e. we have finally found a solution of the constraints that fits into the fall-off requirements valid for the asymptotic regions. Furthermore \( E_1 > 0 \) asymptotically, so \( P := m^2 \) is positive while \( m \) is real and thus \( E_1 \geq 0 \) in the whole asymptotic region.

The last step is then to pull back the symplectic potential. We comprise total differentials that appear during the reduction process in a functional \( S \) after having displayed them. Then we have
\[
(i^*\Theta)[\partial_t] = -i^*[\int_{\Sigma} d\pi \( \hat{p}\omega - i/\kappa(\hat{\pi}_1 + \hat{\pi}_1 B_1) \)]
+ \frac{d}{dt} \int_{\Sigma} d\pi \( p\omega - i/\kappa(\pi_1 + \pi_1 B_1) \))
= -\int_{\Sigma} d\pi \( \hat{p}\omega - i/\kappa\hat{\pi}_1 \pi_1(-\frac{\kappa \pi_1^2}{2(\pi_1)^2} + \frac{2\lambda \kappa}{3(\pi_1)^3} - \frac{m}{2(\sqrt{\pi_1})^3} + \lambda \pi_1)} + \hat{S}
= -\int_{\Sigma} d\pi \omega + i/\kappa \frac{\pi_1}{\pi_1^3} \int_{\Sigma} d\pi \( \frac{2\pi_1}{\pi_1^3} - \frac{\pi_1}{\pi_1^2} \) + \hat{S}
= \frac{d}{dt} \int_{\Sigma} d\pi \( \frac{\pi_1}{\sqrt{\pi_1}} - \frac{\lambda \kappa}{3} \int_{\Sigma} d\pi \( \frac{\pi_1}{\pi_1^2} \) \) + \hat{S}
= \hat{p}\Phi + \hat{m}T + \hat{S} \]  
(4.18)
where we have assumed that the cosmological constant is time-independent (otherwise we introduce a new time variable \( \tau \) according to \( d\tau(t)/dt = \lambda^{-1} \) and absorb a factor of \( 1/\lambda \) into the variables \( T \) and \( \Phi \) which is, of course, only possible if \( \lambda(t) \) is nowhere vanishing).

5 The classical \(*\)-algebra
5.1 Finiteness and functional differentiability

Let us first check whether the formal integral expressions for $T$ and $\Phi$ derived in the last section, which serve as candidates for the coordinates of the reduced phase space, are actually finite and functionally differentiable with respect to the coordinates of the large phase space.

Case I:

Finiteness is trivial since the integrals involved are over a compact set and we require the fields $A_1$ and $\omega$ to be smooth and finite there. Functional differentiability is also trivial since no spatial derivatives appear in the integrands of $T$ and $\Phi$.

Case II.a:

$\gamma = 0$: since we have $A = 2(B^1 + 1) = \text{const.} \neq 0$ in general, the factor $\sqrt{E/A}$ is smooth and finite in general. Furthermore, it follows from the Gauss constraint

$$0 = (E^1)' = \sin(\alpha - \beta)\sqrt{AE},$$

(5.1)

hence $\cos(\alpha - \beta) = 1$. Since we again integrate over a compactum, $T$ is again finite in general if we impose the fields $A$ and $E$ to be finite and smooth there. Since no spatial derivatives are involved, functional differentiability is trivial.

$A = 0$: we require $\gamma$ to be smooth and finite, so finiteness of $T$ is no problem because we integrate it over a compactum. For $T$ to be functionally differentiable, we need it to vanish at the boundaries of $K$.

Case II.b:

First of all, let us express the observables $m$ and $p$ as functionals on the phase space. To that end, let $D$ be a scalar density of weight one with respect to the natural metric on $\Sigma$ (derived from its distance function, available because $\Sigma$ is a normed space) normalized to 1, i.e.

$$\int_M dx D(x) = 1$$

(5.2)

(one could also choose $D$ to depend on the fields, e.g. $D = (A/2)'$). Then one can express $m$ and $p$ as

$$m := \int_M dx D B^1 E^1 - \kappa(p^2/2 - \lambda/3(E^1)^2)/\sqrt{E^1},$$

$$p = \int_M dx Dp.$$  

(5.3)

These functionals are obviously finite and functionally differentiable on the whole phase space because $D$ is at least $O(1/x^2)$, the rest of the integrands is $O(1)$ and there are no spatial derivatives involved.

Because of the subtlety related to the support of the Lagrange multipliers, $m$ and $p$ may take different constant values in the disconnected parts of the support of the Lagrange multipliers on $\Sigma$ corresponding to gauge transformations. Between these regions of constancy, in the transition region between $K$ and the asymptotic ends, $m$ and $p$ should change smoothly. Now it would be satisfactory if these independent, different possible values of $m$ in the different regions of $\Sigma$ would correspond to different canonical variables because the various asymptotic regions should correspond to different asymptotic observers. This is a physical motivation and no mathematical prediction of the formalism! Readers that feel uneasy may skip the following paragraph in which we introduce some
further structure alluded to at the end of section 3 in order to achieve this aim, and may view \( m \) and \( p \) as constant over all of \( \Sigma \) in the sequel. We will simply require in addition to the restrictions mentioned in section 3 the support of the integrands of \( T \) and \( \Phi \) to avoid the transition region and the origin in \( K \) (the same for the Lagrange multipliers, even for symmetries). The part of \( \Sigma \) which is neither origin nor a transition region will be referred to as support region. Since there are 8 basic fields but only 4 reduced coordinates per region, the smoothness and support requirements on the integrands of the latter can easily be satisfied.

Then the reduced symplectic potential splits into a sum over ends and the compactum

\[
\dot{\Theta} = \dot{p}_K \Phi_K + \dot{m}_K T_K + \sum_{A=1}^{n} (\dot{p}_A \Phi_A + \dot{m}_A T_A)
\]  

(5.4)

because on the support of the integrands \( i_T \) and \( i_\Phi \) of \( T \) and \( \Phi \), \( m \) and \( p \) are constrained to be spatially constant and can therefore be dragged out of the integral. This furnishes our aim. Due to our support conditions, the degrees of freedom of the fields in the transition regions are so to say frozen: the support conditions are never changed under evolution because the right hand side of the equations of motion (6.1) vanish trivially if \( x \) happens to fall into the transition region since by definition the Lagrange multipliers do not have their support there, so even all the field variables do never alter their value there. Of course, this means that our family of hypersurfaces actually does not form a foliation because those parts of \( \Sigma \) where the lapse has no support will never move. This, eventually, will also restrict the spacetime manifold if the hypersurfaces are to stay spacelike.

Note also that because of the just introduced support conditions on the integrands of \( T \) and \( \Phi \) there will never arise boundary terms corresponding to \( K \) when varying \( T \) and \( \Phi \). Finiteness and functional differentiability are then already assured for the the variables corresponding to \( K \).

All calculations in the rest of this paragraph will therefore deal with respect to one asymptotic end only in order to avoid the labelling of the various ends i.e. we set \( M := \Sigma_A \) for some \( A \in 1..n \) in the sequel.

Since \( \omega = O(1/x^2) \) we can focus at the other expressions in the asymptotic regions and we have \( \lambda = 0 \) (in the case of \( \Sigma = S^1 \) we are already done). Hence we need to determine the fall-off behaviour of \( \pi_1 \). Let

\[
g := -2(E^1)^2/(\kappa|p^2/2 + \lambda(E^1)^2] + B^1E^1),
\]  

(5.5)

then we have \( E^2 = gB^2, \ E^3 = gB^3, \) and \( (E^1)' = g(B^1)' \) on the constraint surface. Computing \( E \) we are interested in

\[
(B^2)^2 + (B^3)^2 = \frac{1}{A}([B^1]'^2 + (A\gamma)^2),
\]  

(5.6)

a formula which was obtained by writing the magnetic fields in terms of 'cylinder coordinates'. On the other hand we have

\[
\alpha - \beta = \arctan(A_3/A_2) - \arctan(B^3/B^2)
= \arctan(A_3B^2 - A_2B^3) = \frac{(B^1)'}{A\gamma}.
\]  

(5.7)
We now express the cosine function in terms of the tangens function,
\[ \cos(x) = \pm \frac{A \gamma}{\sqrt{(B^1)^2 + (A \gamma)^2}}. \] (5.8)
So we end up finally with
\[ \pi_1 = \sqrt{\frac{E}{A}} \cos(\alpha - \beta) = \pm \gamma g. \] (5.9)
However, only the upper sign is appropriate since also \( \pi_1 = (A_2 E^2 + A_3 E^3) = \gamma g \).
Looking at the expression for \( g \) in terms of \( B^1 \) and \( E^1 \) we find that \( g = O(x^3) \) for \( \lambda = 0 \). Furthermore
\[ \gamma = \frac{A_2 B^2 + A_3 B^3}{2(1 + B^1)} = O(1/x^3) \] (5.10)
when recalling formula (3.13). Thus \( \pi_1 = O(1) \) and the integral involving \( \pi_1 \) in the expression for \( \Phi \) is already finite due to \( E^1 = \pi_\gamma = O(x^2) \) whereas the integral \( T \) seems to be logarithmically divergent. However, it is possible to prove that the divergent part vanishes on the constraint surface. This is not unexpected since an observable on the full phase space is in any case only determined up to the addition of a linear combination of the constraint functionals. In order to obtain a manifestly finite expression for \( T \) on the full phase space we have therefore to add a term which is proportional to the constraints and which also diverges off the constraint surface.
A first hint how this expression should look like gives the observation that if one replaced \( A_3 \) by \( A_3 - \sqrt{2} \) in \( A \gamma = A_2 B^2 + A_3 B^3 \) then the part of the integrand of \( T \propto (A_3 - \sqrt{2}) B^3 \) would already be \( O(1/x^2) \) and therefore finite. Accordingly, it is motivated to look for a linear combination of constraints that precisely accomplishes for that subtraction of \( \sqrt{2} \) from \( A_3 \). The idea is thus to subtract from \( T \) the expression
\[ (-i/\kappa) \int \Sigma \, dx \frac{g \sqrt{2}}{\sqrt{E^1 A}} (B^3 - g^{-1} E^3) \] (5.11)
since the bracket term is constrained to vanish. Recalling from section 3 how this bracket term could be obtained in terms of the constraints, formula (4.4), we propose the improved expression for \( T \)
\[ T_{finite} := T + \frac{i}{\kappa} \int_M \, dx \frac{g \sqrt{2}}{(\sqrt{E^1})^3 E A} (E^1 E^2 V + E^3 C) \]
\[ = (-i/\kappa) \int_M \, dx \frac{g}{\sqrt{E^1 A}} [(A_3 - \sqrt{2}) B^3 + (A_2 B^2 - \sqrt{2} g^{-1} E^3)]. \] (5.12)
The prefactor of the square bracket in the last line of (5.12) is \( O(x^2) \), the first term in the bracket is \( O(1/x^4) \) by construction and finally we have for the leading order part of the second term in the bracket
\[ A_2 B^2 - g^{-1} \sqrt{2} E^3 \to (-a_2 a_3 x^3 + O(1/x^4)) - (\frac{\sqrt{2} a_3 f^3}{x^3} + O(x^4)) \]
\[ = -\frac{a_3 (a_2 + \sqrt{2} f^3)}{x^3} + O(1/x^4) = O(1/x^4) \] (5.13)
i.e. $T_{finite}$ is already a finite functional on $\Gamma$ due to the important eqn. (3.17). It is interesting to see that the finiteness of the symplectic structure enforces the finiteness of the observable $T_{finite}$ and how fine tuned this finiteness comes about to hold! Next we come to discuss functional differentiability:

The only terms that could spoil differentiability are those that appear with spatial derivatives in the integrand of $T_{finite}$ or $\Phi$ because they give rise to a boundary term in a variation of these functionals. Let us study these boundary terms (note that $g \to (x^3\sqrt{2})/a_3 + O(x^2)$):

\[
i\kappa(\delta \Phi)|_{\text{boundary term}} = \kappa p \int_{\partial M} \frac{g}{E^1 A} (A_2 \delta A_3 - A_3 \delta A_2) = -\frac{p}{a_3} \int_{\partial M} \delta a_2
\]

\[
i\kappa(\delta T_{finite})|_{\text{boundary term}} = \int_{\partial M} \frac{g}{\sqrt{E^1 A}} (A_2 \delta A_3 - (A_3 - \sqrt{2}) \delta A_2)
\]

\[= \int_{\partial M} \frac{1}{\sqrt{2a_3}} (a_2 \delta a_3 - a_3 \delta a_2). \quad (5.14)\]

Hence both observables fail to be functionally differentiable. Even more serious: the boundary term of the variation is not exact and it seems that one is not able to add a counterterm in order to restore differentiability. What saves the day is that it follows from the transformation law of the fields to require $\delta a_2 = \delta a_3 = 0$ which has the consequence that $\Phi$ and $T_{finite}$ already become differentiable:

looking at the variation of $A_2, A_3$ as derived from computing the Poisson brackets with the constraint functionals (which are already functionally differentiable by construction of the phase space in section 3.2), equation (6.1), and inserting the asymptotic behaviour of the various fields, we conclude $\delta a_2 = \delta a_3 = 0$ even for the symmetry transformations that we allow. The result $\delta a_2 = \delta a_3 = 0$ means that the dynamical part (due to $\sqrt{2a_1+a_2}=0$) of $A_I$ rests in the higher order terms in their asymptotic expansions.

### 5.2 Reality conditions

In all cases the set of coordinates $E^I, \omega, p$ is real. Therefore, in cases I and II.a, $\Phi$ is already known to be real. The range of $E^I$ is not always the whole real axis (see below).

Case I:

The reality condition for $A_1$ is given by $\dot{A}_1 + A_1 = 2\Gamma_1$ where $\Gamma_1$ is given by $-\beta'$ (see I). Obviously, there is a problem because $\beta = \arctan(E^3/E^2)$ but $E^2 = E^3 = 0$. Let for example $E^1 = h + k, E^2 = he^2, E^3 = he^3$ where $e^2, e^3, h$ are arbitrary real smooth functions and $k$ is a spatial constant. Then the constraint surface is defined by $h = 0$ however $\beta$ is ill-defined. This shows that Ashtekar’s formalism is not, although frequently said so (ref. [4]), really an extension of Einstein’s theory in the sense that it allows for degenerate metrics, because the degeneracy makes the reality conditions ill-defined (this is also true for the full theory because the spin connection is a homogeneous function of degree zero in terms of the densitized triads). In order to have a real reduced theory we may motivate to set $\beta = const.$ so that $T$ becomes real. In that case we would have a cotangent bundle over $R^2$ as the reduced phase space for the compactum K. There is no reality condition on $A_2, A_3$ because we have no motivation to give the spin-connections $\Gamma_2, \Gamma_3$ a definite real value.
Case II.a:

The solution of $f = 0$ is given by

$$E^1 = -B^1/(2\kappa \lambda) \pm \sqrt{(B^1/(2\kappa \lambda))^2 - p^2/(2\lambda)}$$

for $\lambda \neq 0$ (5.15)

whereas for $\lambda = 0$ we have $E^1 = -p^2/(2B^1)$. Hence the range of $E^1$ depends here on the value of the cosmological constant and there is also the constraint ($B^1 \leq -1$ for $\gamma = 0$, see below)

$$B^1 \leq \min\{\{-1,-\sqrt{2\lambda} |p| \kappa\}\} \text{ for } \lambda > 0 \text{ (and only } B^1 \leq -1 \text{ for } \lambda < 0)$$

(5.16)

in order to guarantee the reality of $E^1$. Hence, since in the case $B^1 = -1$ of subcase II.a the reality of $E^1$ cannot be guaranteed for $\lambda \neq 0$ we have to stick with $\gamma = 0$ in that case. For $\lambda = 0$ both cases $\gamma = 0$ and $B^1 = -1$ are possible.

More precisely the ranges of $E^1, B^1, p$ are linked as follows for the various values of the cosmological constant:

i) $\lambda < 0$ : $E^1$ is monotonously decreasing (increasing) with decreasing $B^1$ for the lower (upper) sign and therefore its upper (lower) bound, which is negative (positive), is given by inserting the upper bound of $B^1$ into formula (5.18). There is thus a gap in the range of $E^1$ of width $2\sqrt{(B^1/(2\kappa \lambda))^2 - p^2/(2\lambda)}$ symmetrically around zero.

ii) $\lambda = 0$ : $E^1$ is positive but bounded from above by $p^2/2$.

iii) $\lambda > 0$ : $E^1$ is monotonously increasing with decreasing $B^1$ and therefore its lower bound, which is positive, is given by inserting the upper bound of $B^1$ into formula (5.18) with the negative sign.

Now let us discuss the two subcases of II.a:

$\gamma = 0$:

Since $E \neq 0$ but $(E^1)' = 0$ it follows that $\Gamma_2 := -(E^1)'E^3/E = \Gamma_3 := (E^1)'E^2/E = 0$ whence $A_2, A_3$ are imaginary. Thus, $\sqrt{E/A} = \pi_1$ is imaginary, i.e. $T$ is again real while $m = B^1 = A/2 - 1$ is bounded from above by $-1$ for $\lambda \leq 0$ and by $\min(-1, -\sqrt{2\lambda} |p| \kappa)$ for $\lambda > 0$. Thus, we obtain the cotangent bundle over the half-plane $R \times R \leq -1$ for $\lambda = 0$ or over the 'cut wedge' $\{(p, B^1); p \in R, B^1 \leq \min(-1, -\sqrt{2\lambda} |p|)\}$ as the reduced phase space respectively.

$B^1 = -1$:

Since $A = 0$ we have $A_2 = \pm i A_3$ whence $\alpha' = 0$ i.e. $\gamma = A_1$. Furthermore $B^3 = \mp i B^2 = (A_3)' + i A_1 A_3 = 0$ whence $A_1 = \pm i [\ln(A_3)]'$ i.e. $\gamma$ is imaginary because $A_3$ is imaginary and thus we conclude $\Gamma_1 = -\beta' = 0$. Hence $T$ is real with range over the whole real axis while $m = E^1 < p^2/2$ because we have $\lambda = 0$.

The reduced phase space is therefore the cotangent bundle over $\{(p, m); p \in R, m < p^2/2\}$.

Case II.b:

In I it was proved that the function $\gamma$ is (weakly) imaginary, hence $\pi_1$ is imaginary while $\pi_\gamma = E^1$ is real. Accordingly, the integrands of $T$ and $\Phi$ are both real.

The reduced phase space can thus be described as follows : in every asymptotic end $A$ we have a cotangent bundle over $R^2$ as well as in $K$.

Note that we also could glue together case II.b in the asymptotic regions with one of
the cases I and II.a in the compactum for open topologies while we have to make a choice for compact topologies between the 3 cases. Of course, one could also imagine to have arbitrarily many regions in which one of the 3 cases holds (for both types of topologies) but in order to make the associated observables again differentiable one would have to impose additional structure (support conditions for these additional regions) which we do find unnatural.

5.3 Geometry of the constraint surface

We have obtained, over each of the regions of $\Sigma$, up to three apparently unrelated constraint surfaces and reduced phase spaces.

The question arises whether various observables on the 3 different constraint surfaces should be treated as independent of each other or not. This is an important question because, as already pointed out in I, it affects the dimension of the reduced phase space. From the way we found the constraint surface it is clear how this split into apparently three different leaves came about: by taking linear combinations of the original constraint generators we find first equivalent constraint generators but in such a way that each of them can be written as a product. A product vanishes if any of the factors vanishes. Hence we obtain new inequivalent constraint generators depending on which set of factors we chose to vanish. These new sets of constraints form again, as one can show, a 1st class algebra and the symplectic reduction works for every set separately. However, the flow of the constraint generator corresponding to one of these sets lies only tangential to the constraint surface defined by this set and thus never penetrates the other parts of the complete constraint surface (corresponding to the other set) except for possible intersection subsets between the different leaves of the constraint surface. Now, points of the full constraint surface which lie on the same flow line of a Hamiltonian vector field corresponding a constraint generator are to be identified. Accordingly, intersection points will lead to an identification of some points in the two reduced phase spaces.

Once one has obtained a (partial) identification of points of the various leaves of the reduced phase space, one can glue these together along these points and one obtains one big reduced phase space which consists of leaves which communicate through the identification process.

Altogether, one gets a topology of the complete reduced phase space which is similar to a Riemann surface with various leaves and cuts in it.

A detailed analysis of that problem, including the general theory of how to treat a factorizing constraint, is given in [20], but we will not need these results here and just restrict ourselves to a short description:

One would need to compute the intersection domain between the leaves of the constraint surface for our model. It is possible to show that all leaves have non-empty mutual intersection, that the intersection domain is only presymplectic and that the resulting reduced phase space has a quite complicated topology.

How does one do quantum theory on these glued leaves? There are two obvious strategies available:

1) Probably the only constructive way to deal with the problem is to ’hide’ the complicated topology of the reduced phase in a set of relations between the variables of an overcomplete set and to proceed along the lines of the algebraic quantization programme.
(compare third reference of [4] and references therein).
2) One excludes the intersection domain by hand (this is motivated by its typically presymplectic nature anyway, in particular this is true for our model). This disconnects the two reduced phase spaces by brute force and they can be treated just as phase spaces of two unrelated theories, that is, separately.

We will in quantum theory choose the second strategy.

6 Proof of conjugacy and gauge invariance and derivation of evolution

In the derivation of the observables in section 5 we neglected several boundary terms and total differentials. It is therefore not unnecessary to check whether the improved quantities of section 6 are really conjugate variables, if they are really gauge invariant and what their evolution equations are. Of course, in the case of compact topology without boundary the following analysis is unnecessary.

By construction the symmetry generators are functionally differentiable. The variation of the coordinates of the full phase space is given by ($G := G[\Lambda, L, M, U] := \int_{\Sigma} dx [(\Lambda - LA_1)G + LV + MC + U^M G] - \int_{\partial \Sigma} [(\Lambda - LA_1)E^1 + L((A_3 - \sqrt{2})E^3 + A_2 E^2) + M((A_3 - \sqrt{2})E^2 - A_2 E^3) E^1 + Up]$ is the full symmetry generator and we absorb the small variation parameter $\epsilon$ into the Lagrange multipliers $\Lambda, L, M, U$)

\[
\frac{1}{i\kappa} \delta A_1 := \frac{1}{i\kappa} \{ A_1, G \} = -\Lambda' + (LA_1)' + M(B^2 E^2 + B^3 E^3 + E\kappa/2(-p^2/(2E^1)^2) + \lambda))
\]
\[
\frac{1}{i\kappa} \delta A_2 := \frac{1}{i\kappa} \{ A_2, G \} = -\Lambda A_3 + LA_2' + M(E^1 B^2 + E^2(B^1 + \kappa(p^2/(2E^1) + \lambda E^1))
\]
\[
\frac{1}{i\kappa} \delta A_3 := \frac{1}{i\kappa} \{ A_3, G \} = \Lambda A_2 + LA_3' + M(E^1 B^3 + E^3B^1 + \kappa(p^2/(2E^1) + \lambda E^1))
\]
\[
\frac{1}{i\kappa} \delta E^1 := \frac{1}{i\kappa} \{ E^1, G \} = (E^1)' - ME^1(A_2 E^2 + A_3 E^3)
\]
\[
\frac{1}{i\kappa} \delta E^2 := \frac{1}{i\kappa} \{ E^2, G \} = -\Lambda E^3 + (LE^2)' - ME^3 E^1)' - ME^1(A_1 + 1/2A_2 E)
\]
\[
\frac{1}{i\kappa} \delta E^3 := \frac{1}{i\kappa} \{ E^3, G \} = \Lambda E^2 + (LE^3)' + ME^2 E^1)' - ME^3 E^1(A_1 + 1/2A_3 E)
\]
\[
\delta \omega := \{ \omega, G \} = -U' + ME/(2E^1)p
\]
\[
\delta p := \{ p, G \} = 0 .
\]

The symmetry algebra is given by (on $\bar{\Gamma}$, compare I)

\[
\frac{1}{i\kappa} \{ G[\Lambda_1, L_1, M_1, U_1], G[\Lambda_2, L_2, M_2, U_2] \} \bar{\Gamma}
\]
\[
= \int_{\partial \Sigma} [(\Lambda_1 L_2 - A_2 L_1)(A_2 E^3 - A_3 E^2)
+ (\Lambda_1 M_2 - A_2 M_1)(A_2 E^2 + A_3 E^3)E^1 - (L_1 M_2 - L_2 M_1)
+ (A_1 E^1(A_2 E^2 + A_3 E^3) + 1/2 E(B^1 + \kappa(p^2/(2E^1) + \lambda E^1))]
\]

i.e. it is abelian (recall the boundary conditions to show that the rhs of (6.2) vanishes identically) as it should be since we are dealing only with the translation subgroup of the
Poincare group at spatial infinity.

We can now proceed to vary the expressions for the observables. By construction they are functionally differentiable so we do not need to worry about boundary terms, the Poisson bracket with \( G \) is a volume integral again. We are interested in the restriction of the Poisson bracket to the constraint surface only, hence when varying the part of \( T \) which is proportional to the constraint generators, we do not need to care about the variations of the prefactors of these generators. Rather, they can be treated as multipliers so that we can apply formula (6.1) when computing the action of the symmetry generators on the observables. One might object that the part of the observables which is proportional to a constraint generator is by itself a divergent expression so that it is doubtful whether formula (6.2) can be applied, however, since we compute the variation of 2 divergent expressions whose associated divergent parts cancel each other, our argument is indeed accurate (recall that we first integrate over a finite range of \( x \) and then take the limit).

Then we obtain on the constraint surface

\[
\delta m = \int_{\Sigma} dx D \left[ \frac{B^1 E^1 + \kappa(p^2/2 + \lambda(E^1)^2}{2(\sqrt{E^1})^3} \delta E^1 + \sqrt{E^1}(A_2 \delta A_2 + A_3 \delta A_3) + \frac{kp}{2\sqrt{E^1}} \delta p \right]
\]

\[
\delta p = \int_{\Sigma} dx D \delta p
\]

\[
\delta T = (-i/\kappa) \int_{\Sigma} dx \left[ \frac{g}{\sqrt{E^1}} \delta A_1 + \left( \frac{g^2 \gamma}{2E^1} A_2 + \left( \frac{g}{\sqrt{E^1}} \right) \frac{A_3}{2} \right) \delta A_2 + \left( \frac{g^2 \gamma}{2E^1} A_3 - \left( \frac{g}{\sqrt{E^1}} \right) \frac{A_2}{2} \right) \delta A_3 + \frac{\gamma}{\sqrt{E^1}} (3/2g + \frac{g^2}{2E^1}(B^1 + 2/3\lambda\kappa E^1)) \delta E^1 \right]
\]

\[
\delta \Phi = (-i/\kappa) p \int_{\Sigma} dx \left[ \frac{\gamma g}{pE^1} \delta p - \frac{1}{p} \delta \omega + \frac{g}{E^1} \delta A_1 + \left( \frac{g^2 \gamma}{2E^1} A_2 + \left( \frac{g}{E^1} \right) \frac{A_3}{2} \right) \delta A_2 + \left( \frac{g^2 \gamma}{2E^1} A_3 - \left( \frac{g}{E^1} \right) \frac{A_2}{2} \right) \delta A_3 + \frac{\gamma}{(E^1)^2} (g + \frac{g^2}{2E^1}(B^1 + 2/3\lambda\kappa E^1)) \delta E^1 \right],
\]  

(6.3)

where in the last line of the equation for \( \delta T \), \( \delta G \) has to be replaced by \( \int_{\Sigma} dx \{ A_I, G \} \delta E^I - \{ E^I, G \} \delta A_I + \{ \omega, G \} \delta p - \{ p, G \} \delta \omega \) according to eqn. (6.1) with the 'Lagrange-multipliers'

\[
\Lambda = A_1 L \text{ and } L = \frac{g\sqrt{2}E^1 E^2}{(\sqrt{E^1})^3 E A},
\]

\[
M = \frac{g\sqrt{2}E^3}{(\sqrt{E^1})^3 E A},
\]  

(6.4)

We have now all necessary formulas to compute the Poisson brackets between physically relevant quantities. The actual computation is rather tedious and the reader is referred to the appendix. However, these computations again show the fine tuning and interrelation between the well-definedness of the various objects that one is dealing with in general relativity.

Computing Poisson brackets among the observables and between observables and symmetry generators reaffirms the canonical structure that has been formally derived in the last
section and that the observables are really gauge invariant when choosing the Lagrange-multippliers of compact support. For symmetry transformations on the other hand we obtain

\[
\begin{align*}
\{m_A, G[\Lambda, N^x, N, U]\} &= 0, \\
\{T_A, G[\Lambda, N^x, N, U]\} &= N_A, \\
\{p_A, G[\Lambda, N^x, N, U]\} &= 0, \\
\{\Phi_A, G[\Lambda, N^x, N, U]\} &= U_A
\end{align*}
\] (6.5)

while \(m_K, T_K, p_K\) and \(\Phi_K\) are all constant and we have defined

\[
N_A(t) := N(x = \partial \Sigma_A, t), \quad U_A(t) := U(x = \partial \Sigma_A, t)
\] (6.6)

where \(N := \det(q)^{1/2}N\) is the lapse function. Hence the observables are invariant under radial translations and O(2)-rotations at spatial infinity while they react nontrivially under time-translations and phase transformations at spatial infinity.

It is expected but for field theories not completely obvious that the equations of motion (6.5) coincide with the equations of motion that follow from the reduced Hamiltonian

\[
H_{red}[m, T, p, \Phi] := G[\Lambda, N^x, M, U]|_{\bar{\Gamma}} = \sum_{A=1}^{n} m_AN_A + p_AU_A
\] (6.7)

(for systems with a finite number of degrees of freedom and Hamiltonian \(H\) it is easy to prove that for any functional \(O\) on the full phase space holds

\[
\{O, H\}|_{\bar{\Gamma}} = \{O_{\bar{\Gamma}}, H_{red}\}
\]

if and only if \(O\) is an observable) provided that we neglect the O(2) charge \(\int_{\partial \Sigma} \Lambda E^1\) which does not spoil the differentiability of the Einstein-Gauss constraint for symmetries because the variation of the charge vanishes anyway. We will do this in the sequel.

Note that if we had not made the observables manifestly finite and functionally differentiable but had computed the various brackets in a naive way, not caring about boundary terms occurring in variations, then we would not have obtained the contributions from the constraint part and the evolution equations would change significantly. This shows how subtle the treatment of the asymptotically flat case is (compare the appendix to see this technically).

It is easy to solve the equations of motion (6.5) : introduce functions \(\tau_A(t)\) and \(\phi_A(t)\) defined by

\[
\frac{d}{dt}\tau_A = N_A \quad \text{and} \quad \frac{d}{dt}\phi_A = U_A
\] (6.8)

Then the solution can be written

\[
\begin{align*}
m_A(t) &= \text{const.}, \\
T_A(t) &= \text{const.} + \tau_A(t), \\
p_A(t) &= \text{const.}, \\
\Phi_A(t) &= \text{const.} + \phi_A(t) \quad A = 1..n
\end{align*}
\] (6.9)
i.e. the reduced system adopts the form of an integrable system whereby the role of the action variables is played by the masses and the charges whereas their conjugate variables take the role of the angle variables.

What now is the interpretation of this second set of conjugate variables (compare also the second reference in [1])?

The interpretation of m and p follows simply from the fact that they can be derived from the reduced Hamiltonian, i.e. they are the well-known surface integrals ADM-energy and Maxwell-charge. However, their conjugate partners are genuine volume integrals and we are not able to write them as known surface integrals. Nevertheless it is possible to give an interpretation: Recall that for vanishing shift vector, the \( g_{tt} \) component of the spacetime metric is just given by \( g_{tt} = -N^2 \) and that one defines the eigentime of a local observer by \( d\tau = \pm \sqrt{-g_{tt}} dt \) where the upper (lower) sign is valid when \( dt \) is future (past) directed (note that \( dt \) is proportional to the normal to \( \Sigma \) which is assumed to be future directed. Hence, for vanishing shift, \( d\tau = \pm N dt \), whence it follows from the solution (6.9) that 'on shell' the variable \( T_A \) is nothing else than the eigentime of an asymptotic observer at spatial infinity of the end \( \Sigma_A \). That \( T \) must be a time variable follows also from a dimensional analysis since m and T are conjugate and m has the dimension of an energy. The eigentime of an observer is intuitively something 'observable' such that this interpretation sounds quite satisfactory.

Next recall that \( U = \omega_t \) is the t-component of the Maxwell connection, i.e. the scalar potential of Minkowski space at infinity, i.e. the vacuum potential. Now the equation of motion \(-d/dt(-\Phi) = U\) looks like the induction law of electrodynamics! Hence, it seems that \(-\Phi\) (as it should by a dimensional analysis) should represent something like a magnetic flux. Can this formal consideration be given a physical meaning? Only approximately: Looking at the part

\[ + \int_{\Sigma_A} dx \omega \]

(6.10)

of \( \Phi \) and comparing it with the formula for a magnetic Maxwell flux through a surface \( S \), \( \int_S d \wedge \omega = \int_{\partial S} \omega \) we see that our interpretation is formally correct although the line integral over \( \Sigma_A \) is not closed and it is not possible to find a closed line in \( \Sigma^3 \) including \( \Sigma_A \) such that the integral of \( \omega_a dx^a \) reduces to our expression. Note however, that the expression (6.10) is also the variable conjugate to the electric field in (1+1) Maxwell theory on a Minkowski background and that the remainder of \( \Phi \) vanishes for Minkowski space (Minkowski space corresponds to vanishing connection \( A_1, A_2, A_3 - \sqrt{2} \to 0, \ E^1 \to x^2 \) i.e. \( \gamma = 0, g/E^1 = -4x^2/(\kappa p^2) \)). So, \(-\Phi\) can also be interpreted as the curved analogue of this observable because the spherically symmetric reduction leads us to an effectively 2-dimensional spacetime. However, the dynamics of \( \Phi \) is completely different from (1+1) Maxwell theory: there \( \dot{\Phi} \propto p \) whereas here there is no dependence of \( \dot{\Phi} \) on the charge!

This finishes the issue of interpretation of the theory at least for asymptotically flat topologies.

In the compact case, we have no Hamiltonian and the observables found are constants of motion. They are, nevertheless, nonvanishing in general. How can we interpret the theory in that case? Here one can apply the theory of deparametrization (see ref. [5]). In the terminology of that paper, \( m, T, p \) and \( \Phi \) are 'time-independent' Dirac observables and we have applied the so-called "frozen formalism" so far.

The application of that theory is beyond the scope of the present paper.
Let us now mention two objections that were raised in discussions about the results in the first paper of I and which were resolved in the second paper of I (compare also [20]):

1) For a Reissner Nordstrøm foliation, the observable $T$ indeed vanishes. This foliation follows from setting the shift equal to zero everywhere (so-called static foliation) and by choosing $E^1 = x^2$ since if we check whether the gauge choice $E^1 = r^2$ is preserved under evolution we find on the constraint surface (recall formula (6.1))

$$
\delta E^1 = \delta x^2 - 0 = -4x^6 M(1 + B^1) \frac{\gamma}{i\kappa[x^2 B^1 + \kappa(p^2/2 + \lambda x^4)]}
$$

from which follows $\gamma = 0$ ($B^1$ cannot vanish due to the constraint (4.18)) i.e. $\pi_1 = 0$ and hence $T=0$. This means that nonvanishing $T$ indicates a deviation from the usual Reissner Nordstrøm-foliation. Now there seems to appear a problem: according to the equations of motion (6.5) a vanishing $T$ is not stable under evolution! However, in the literature one chooses a static foliation at each instant of time (e.g. ref. [14]) while here this seems to be dynamically impossible. The contradiction is resolved in the same way as in the second paper of I, namely by choosing the lapse at the ends of $\Sigma$ appropriately.

2) The (extended) Birkhoff theorem (ref. [14]) states that the 4-diffeomorphism inequivalent solutions of Einstein-Maxwell-theory reduced to spherical symmetry are only labelled by mass and charge. The space of solutions of gauge-inequivalent solutions of a field theory on the other hand are in 1-1 correspondence with the reduced phase space. We, however, find besides the mass and the charge the eigentime and the flux as additional observables.

The apparent contradiction can be concisely resolved as follows (compare [13] for a related phenomenon for Bianchi cosmologies):

obviously, following Birkhoff’s theorem, the observables $T$ and $\Phi$ are considered as pure gauge, i.e. they can be set to zero for all times (looking at the proof given by Birkhoff, one finds that it is purely geometrical in nature, that is, it does not care about fall-off properties of diffeomorphisms and fields etc.). From the Hamiltonian point of view, this is impossible because we showed in section 5 that these 2 observables are 

\textit{definitely} gauge invariant. Accordingly, Birkhoff’s theorem is an "overkill" in the sense that not due care has been taken of the functional analysis involved (compare I and [20] to make this statement more precise).

Birkhoff’s theorem refers to the asymptotically flat case only. In the compact case we neither are able to gauge $T$ and $\Phi$ to zero (note that for compact topologies there is no difference between gauge and symmetries) because the observables are manifestly gauge invariant for any kind of gauge transformations since surface terms never arise.

7 Quantum theory

7.1 Group theoretical quantization

We finally come to the quantization of the system. We follow the group theoretical quantization scheme (see ref. [8]) for each leaf of the constraint surface separately as discussed at the end of section 5.
The configuration space in case II.a has a quite complicated topology and the discussion of the explicit solution for the corresponding measure and Hilbert space (except for the case $\gamma = 0, \lambda \leq 0$ to which we therefore restrict in the sequel) would by far exceed the scope of the present paper (the interested reader is referred to [20]).

We then have, referring to [8], as Hilbert spaces in case

1) $L_2(R^2, dx \wedge dy)$,
2) $L_2(R^+ \times R, dx/x \wedge dy)$ ,
3) $L_2(R^2, dx \wedge dy)$.

The operators associated to the configuration variables act simply by multiplication whereas those corresponding to momentum operators are represented by $\hat{p} = -\hbar \partial/\partial x$ if the underlying configuration space is the whole real line and by $\hat{p} = -\hbar x \partial/\partial x$ if it is only the positive part of it. They are obviously self-adjoint with respect to the associated inner products.

Over $K$ and for the compact case we so construct 3 different Hilbert spaces and sets of elementary operators. We take the direct sum of these Hilbert spaces (thus creating 'sectors'). The various elementary operators have then only diagonal elements. In formulae we have

$$\Psi = \begin{pmatrix} \Psi_I \\ \Psi_{II,a} \\ \Psi_{II,b} \end{pmatrix}$$

(7.1)

where we labelled the states that we defined to belong to different sectors by the associated subscripts and for any operator $\hat{O}$ we have similarly

$$\hat{O} = \begin{pmatrix} \hat{O}_I \\ \hat{O}_{II,a} \\ \hat{O}_{II,b} \end{pmatrix}.$$  

(7.2)

Over each asymptotic region, on the other hand, only the Hilbert space corresponding to case II.b above is appropriate.

### 7.2 The Schroedinger equation

Let us restrict to the asymptotically flat case and for simplicity that mass and charge vanish in the interior compactum $K$. We thus treat only the last entry in the decompositions (7.1) and (7.2).

The set of elementary observables is then given by the masses and charges in the various ends and their conjugate momenta, in other words the Hilbert space is just $L_2(R^{2n}, d^{2n}x)$. We will choose the representation for which the eigentime and the flux act by multiplication and the mass and the charge by differentiation. The substitution of classical observables by their quantum analogues is then unambiguous for the ADM-Hamiltonian and leads to the following Schroedinger equation

$$i\hbar \frac{\partial}{\partial t} \Psi(t; \{T_A\}, \{\Phi_A\}) = (-i\hbar \sum_{A=1}^n [N_A(t) \frac{\partial}{\partial T_A} + U_A(t) \frac{\partial}{\partial \Phi_A}]) \Psi(t; \{T_A\}, \{\Phi_A\}).$$

(7.3)

It can be solved trivially by separation:

$$\Psi(t; \{m_A\}, \{\Phi_A\}) := \prod_{A=1}^n \psi_A(t, m_A, \Phi_A)$$

(7.4)
and by introducing the functions defined by integrating
\[ \dot{r}_A(t) := N_A(t) , \quad \dot{\phi}_A(t) := U_A(t). \] (7.5)

We then find as the general solution
\[ \psi_A(t, T_A, \phi_A) = C_A \exp(i \bar{h} \left[ T_A - \tau_A(t) \right]) \times \exp(i \bar{h} \left[ \Phi_A - \phi_A(t) \right]) \] (7.6)

where \( C_A \) is a complex number, whereas \( k_A, l_A \) must be real because the spectrum of the momenta, which are self-adjoint, is real. This set of \( 2n \) real numbers labels the state which in particular is an eigenstate of all momentum operators. The states are normalizable (to delta distributions) and thus lie in the completion of the Hilbert-space. These solutions of the time-dependent Schrödinger equation are obviously peaked at an instant of 'time' \( t \) around the classical solutions (see (6.9)) in the sense that they are strongly oscillating off the classical trajectory. The fact that these states look more like momentum eigenstates than energy eigenstates (for the time independent Schrödinger equation) is due to the fact that the energy is linear in all momenta.

The quantum-mechanical 'time' happens to coincide with the hypersurface label \( t \). It is not a Dirac observable, but there are enough candidates for an intrinsic time in the model, e.g. \( T_1 \), which is an observable. Thus again the so-called Schrödinger time \( t \) is not quantized, quantized is the intrinsic time \( T_1 \).

For compact topologies the states are independent of the Schrödinger time \( t \). This can be interpreted (see ref. [16]) as the definition of the Heisenberg picture of quantum theory. Hence, what evolves are not the states but the observables. We constructed the 'time'-independent Dirac observables. 'Time'-dependent Dirac observables are available along the lines of the theory of deparametrization ([5]).

### 7.3 Comparison with the operator constraint method

As for any Yang-Mills theory, the scalar constraint is not any longer polynomial in the Einstein electric fields (see ref. [17]). One option how to make sense out of this constraint functional as an operator when applying the operator constraint method (Dirac method) is to multiply this constraint by a sufficient power of the electric fields although this leads to new solutions of the scalar constraint if one does not restrict to non-degenerate metrics.

Let us pursue this recipe for our model. A glance at formula (2.10) reveals that it is sufficient to multiply the scalar constraint with a factor of \( E^1 \) so that in the ordering in which all momenta (for the Ashtekar polarization) stand to the right, the scalar constraint becomes

\[
[(B^2 \frac{\delta}{\delta A^2} + B^3 \frac{\delta}{\delta A^3}) \frac{\delta^2}{\delta^2 A^2} + \frac{1}{2} (B^1 \frac{\delta}{\delta A^1} + \kappa(-\frac{1}{2} \frac{\delta^2}{\delta \omega^2} + \lambda \frac{\delta^2}{\delta A^2})) \times
\]

\[
(\frac{\delta^2}{\delta A^2} + \frac{\delta^2}{\delta A^3}) \Psi[A_I, \omega] = 0
\] (7.7)

which is a 4th order functional differential equation. This equation, of course, does not make any sense the way it stands. First of all, it involves products of operator-valued distributions and thus should actually be smeared. Moreover, these distributions are
evaluated at the same point \( x \in \Sigma \) and are thus meaningless, in general, unless one regularizes them. Finally, consistency of the Dirac method (see ref. [18]) requires that all constraints form a closed operator subalgebra. A formal calculation of the commutators of the constraints in which all momenta are to the right reveals that they do not close in the sense that the commutator is proportional to a constraint operator, however the constant of proportionality depends on the fields and stands to the right. A trial and error procedure in order to find a correct ordering did not succeed. However, the way the constraints were solved classically suggests how to solve the quantum constraints: by a suitable choice of polarization. In the following we will only treat the case II.b for one asymptotical end or for the compact case and we denote this region by \( M \).

We simply choose the following set of canonical pairs (compare (4.3))

\[
(p, -\omega; \pi_\gamma, -\gamma; \pi_1, -B_1^1; \alpha, \mathcal{G})
\]  

(7.8)

and choose the representation in which state functionals depend only on \( p, \pi_\gamma, \pi_1 \) and \( \alpha \).

The leaf of the constraint surface corresponding to case II.b is defined (in these coordinates) by

\[
0 = \mathcal{G} 
\]  

(7.9)

\[
0 = p' 
\]  

(7.10)

\[
0 = \alpha' \mathcal{G} + \pi_\gamma'(-\gamma) - (-\pi_1)(B_1^1)' 
\]  

(7.11)

\[
0 = \pi_\gamma'((\pi_\gamma B_1^1 + \kappa(p^2/2 + \lambda \pi_1^2)) - (\pi_1 B_1^1 + \kappa(p^2/2 + \lambda \pi_1^2))\mathcal{G} + 2\pi_\gamma^2(B_1^1)' 
\]  

(7.12)

where one obtains (7.11) and (7.12) most conveniently as follows: one writes formulas (4.6) in the above coordinates and obtains 2 equations

\[
C_1 = \sqrt{E}(\pi_\gamma B_1^1 + \kappa(p^2/2 + \lambda \pi_1^2))\cos(\beta) + 2\pi_\gamma^2(\gamma \sqrt{A} \cos(\alpha) + \frac{(B_1^1)'}{\sqrt{A}} \sin(\alpha))
\]

\[
C_2 = \sqrt{E}(\pi_\gamma B_1^1 + \kappa(p^2/2 + \lambda \pi_1^2))\sin(\beta) + 2\pi_\gamma^2(\gamma \sqrt{A} \sin(\alpha) - \frac{(B_1^1)'}{\sqrt{A}} \cos(\alpha))
\]

(7.13)

of which one takes the following combinations

\[
\frac{1}{\sqrt{A}}(\cos(\alpha)C_1 + \sin(\alpha)C_2) \text{ and } \sqrt{A}(\sin(\alpha)C_1 - \cos(\alpha)C_2)
\]

(7.14)

which lead directly to (7.11) and (7.12).

Now in the polarization chosen, the constraints are linear in momenta except for the term \( \propto \mathcal{G} \) (which could also be dropped alternatively without losing a constraint) in (7.12). However, the coordinate \( \alpha \) does nowhere appear in our constraints such that we can simply treat these momenta as C numbers when computing commutators. Since, as proved in section 5.3, the constraints on the leaf also close and since they are linear in the momenta \( \gamma \) and \( B_1^1 \) it follows that they close also as operators irrespective of the ordering chosen. For quantum theory, the ordering that they stand to the right is the most useful one and thus we take (7.9)-(7.12) as the ordering for quantum theory with the substitutions

\[
\mathcal{G} \to \kappa \frac{\delta}{\delta \alpha}, \quad \omega \to i \frac{\delta}{\delta p}, \quad B_1^1 \to -\kappa \frac{\delta}{\delta \pi_1}, \quad \gamma \to -\kappa \frac{\delta}{\delta \pi_1}, \quad \gamma \to -\kappa \frac{\delta}{\delta \pi_1}.
\]

(7.15)
Let us then solve the quantum constraints. (7.9) and (7.10) applied to a functional of the configuration variables and set equal to zero reduces the dependence of a physical state on these variables to the form

\[ \tilde{\Psi}[p, \alpha, \pi_1, \pi_\gamma] = \delta[p']\Psi(p; \pi_1, \pi_\gamma) \] (7.16)

where the notation means that \( \Psi \) depends on the functions \( \pi_1, \pi_\gamma \) but on the parameter \( p \).

The dependence of a physical state on the variable \( \pi_1 \) is determined by (7.12): multiplying this constraint by \( \pi_1^{3/2} \) from the left one obtains

\[
\left[ \frac{-\kappa}{\sqrt{\pi_\gamma}} (-\pi_\gamma \frac{\delta \Psi}{\delta \pi_1} + \frac{\lambda}{3} \pi_\gamma^2 \Psi - \frac{1}{2} p^2 \Psi) \right]' = 0 .
\] (7.17)

This equation says that the bracket term is an arbitrary spatial constant \( P \). It equation has the general solution

\[
\Psi = \exp(\int_M dx \pi_1 (\frac{\lambda}{3} \pi_\gamma - \frac{p^2}{2 \pi_\gamma})) \times \psi(p, \int_M dx \pi_1 / \sqrt{\pi_\gamma}; \pi_\gamma) =: \exp(\theta(p))\psi(p, T; \pi_\gamma)
\] (7.18)

where we recognize the expression for the eigentime \( T \) in the first argument of \( \psi_\pm \) and there is still a functional dependence on \( \pi_\gamma \). Finally, we obtain from the remaining constraint (7.17)

\[
0 = \left[ \pi_1 \left( \frac{\delta}{\delta \pi_1} \right)' - \pi_\gamma \left( \frac{\delta}{\delta \pi_\gamma} \right) \right] \Psi \\
= -\pi_\gamma' \exp(\theta) \frac{\delta \psi}{\delta \pi_\gamma}
\] (7.19)

whence \( \psi \) is an ordinary function of \( T \) only, i.e.

\[
\psi := \psi(p, T) .
\] (7.20)

Thus, the nontrivial information about the quantum state \( \tilde{\Psi} \) is contained in the function \( \psi(p, T) \) and therefore we choose the measure for the inner product to be

\[
d\mu := \exp(-\theta)dp \land dT
\]

which turns the conjugate momenta \( -\Phi, m \) into self-adjoint operators. We therefore have established that the quantum theories as obtained via either the reduced phase space approach or the operator constraint approach can be made equivalent in this case by going to the appropriate quantum representation.

8 Conclusions

Let us summarize the new results of the present paper:

- We showed that the reduced phase space method can be applied to spherically symmetric Einstein-Maxwell theory to complete the full quantization programme with full
mathematical rigour.
• The analysis was carried out in the Ashtekar variables rather than in the geometro-
dynamical (ADM) variables. This is the first model for (3+1) gravity coupled to gauge
fields that has been quantized completely (in the Ashtekar variables).
• The reduced phase space is coordinatized completely, in every asymptotical end, by the
gravitational mass, the eigentime, as well as the electric charge and the (formal) magnetic
flux as measured by an asymptotic observer in that end of the time slice of the underlying
4-manifold. These interpretations of the Dirac observables hold ”on shell” of the reduced
dynamical phase space, in the asymptotically flat case. These observables are never men-
tioned in the textbook treatments of the Reissner-Nordstrøm solution [14] since no due
care is taken of the distinction between gauge and symmetry.
• In the compact case we were able to treat also the case of a nonvanishing cosmological
constant.
• There are several ”sectors” of the theory due to the fact that the scalar constraint is
nonlinear in the momenta. These sectors are carefully treated using group theoretical and
algebraic quantization techniques.
• Classically we were able to compute the Hamilton-Jacobi functional, in quantum theory
we succeeded in solving the Schroedinger equation.
• Besides performing the quantization via symplectic techniques, we were also able to
complete the quantization using the operator constraint (Dirac) method. The resulting
quantum theories obtained turn out to be equivalent.
• Perhaps the most interesting technical result of the present paper is the successful quan-
tization of a model for (3+1) gravity coupled to an abelian gauge field whose constraints
are fourth order in the momenta.

The conclusions that we may draw are as follows:

• We found it convenient to work out the symplectic reduction of the model not in
the Ashtekar polarization but in a polarization which mixes both the triad and the con-
nection coefficients although, as usual, it turned out that it is convenient to start with
the Ashtekar polarization since it simplifies the analysis tremendously. Of course, after
the difficult part of the work is performed, i.e. to find the observables, one can return
to the Ashtekar polarization : one has simply to solve equation (4.12) for $E^1$ in terms
of $B^1$, $p$ and $\lambda$ and plug this into the expressions for the observables. For nonvanishing
cosmological constant, eq. (4.12) and the corresponding formulas are somewhat lengthy
and one has 4 roots which have to be reconciled with the reality of $E^1$. Let us restrict
therefore to the more feasible and physically more relevant case $\lambda = 0$. Now, one only
has to solve a quadratic equation. The 2 roots are

$$E^1 = \frac{1}{2(B^1)^2} \left( [p^2 \kappa B^1 + m^2] \pm \sqrt{[p^2 \kappa B^1 + m^2]^2 - [p^2 \kappa B^1]^2} \right), \quad (8.1)$$

but only the one with the positive sign is physical since in the case of vanishing charge we
have to recover the result in I (the negative sign leads to $E^1 = 0$). Note that the reality
of $E^1$ requires that $B^1 \leq -m^2/(2p^2 \kappa)$, however, this imposes no further constraints on
the range of $T$, $\Phi$ because this yields for the integrand of the variables $T$ and $\Phi + \int_\Sigma dx \omega$
respectively

\[
-2A_1 + [\arctan\left(\frac{A_2}{A_3}\right)]' \frac{[p^2\kappa + \frac{m^2}{B^1}] + \sqrt{[p^2\kappa + \frac{m^2}{B^1}]^2 - [p^2\kappa]^2}]^{2-n}}{(2B^1)^{2-n}} p^2\kappa + 1/2[\frac{m^2}{B^1} + \sqrt{[p^2\kappa + \frac{m^2}{B^1}]^2 - [p^2\kappa]^2}]^{2-n} \]

(8.2)

where \( n = 1/2 \) and \( n = 1 \) respectively and \( B^1 = 1/2((A_2)^2 + (A_3)^2 - 2) \). This expression is much more complicated than the one in (4.21) in terms of \( \pi_1 \) and \( \pi_\gamma \), and it is thus suggested that in general the polarization that one starts with will not turn out to be the natural one for the problem at hand.

- Formula (8.2) also enables one to straightforwardly rewrite all the results given in the present paper in terms of the geometrodynamical variables along the lines of the procedure given in the first reference of [1]. After all, the Ashtekar variables differ from the ADM variables merely by a canonical transformation. However, as the reader may check by himself, the computations are rather tedious and lengthy and the formulas become less feasible than when using the Ashtekar variables.

- It should be stressed that the observables found have genuine volume integral representations and therefore they do not vanish even in the compact case. Furthermore, the matter-coupling suggests that the T-variable should be considered on equal footing with the \( \Phi \)-variable. Since the latter also occurs in (1+1) canonical Maxwell-theory in a Minkowski background geometry, the very existence of T, against which various arguments were raised in the past, e.g. that it always should be possible to gauge it to zero, that spherically symmetric gravitational fields have no proper reduced phase space etc., should appear less unnatural.

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### A Computations related to section 6

We give here the explicit calculations necessary to prove the gauge invariance of the observables, their evolution laws and that they satisfy a canonical Poisson algebra. In order to display the fine tuning referred to earlier, let us compute the Poisson bracket between the contribution to the observables, that vanishes or does not vanish on the constraint surface respectively, with the symmetry generators, separately.

In the following calculations we will make frequent use of the constraint equations \( (E^1)' = g(B^1)', \ E^2 = gB^2 \) and \( E^3 = gB^3 \) where \( g := -2(E^1)^2/(B^1E^1 + \kappa(\lambda(E^1)^2 + p^2/2)) \), (see (4.12)), which is allowed since we are working on the constraint surface. What we have to do is simply to insert the variations of the basic variables (6.1) in the expressions for the variation of the observables (6.3).

Let us start with the variation of \( m \) under a general symmetry transformation (recall the
The equation for $T$ is much more complicated. We begin with the variation of the non-constraint part of $\delta T$:

$$\frac{1}{i\kappa} \delta m = \frac{1}{i\kappa} \int \Sigma dx D\sqrt{E^1} \left[ -g^{-1}\delta E^1 + (A_2\delta A_2 + A_3\delta A_3 + \frac{\kappa p}{2(E^1)^2}\delta p) \right]$$

$$= \int \Sigma dx D\sqrt{E^1} \left[ -g^{-1}(L(E^1)' - ME^1(A_2E^2 + A_3E^3)) \right.$$ 
$$+ (A_2(-\Lambda A_3 + LA_2' + M(E^1B^2 - E^2(B^1 + \kappa(p^2/(2E^1) + \lambda E^1))))$$
$$+ A_3(\Lambda A_2 + LA_3' + M(E^1B^3 + E^3(B^1 + \kappa(p^2/(2E^1) + \lambda E^1))) + \frac{\kappa p}{2(E^1)^2}\delta p) \right]$$

$$= \int \Sigma dx D\sqrt{E^1} \left[ -g^{-1}(L(E^1)' + ME^1(A_2E^2 + A_3E^3)) + L(B^1)' + ME^1\gamma A \right]$$

$$= \int \Sigma dx D\sqrt{E^1} L[-g^{-1}(E^1)' + (B^1)'] = 0 \quad \text{(A.1)}$$

Hence, $m$ is a constant of motion, not only gauge invariant.

The equation for $p$ is trivial, since it is simultaneously a basic variable:

$$\delta p = 0 \quad \text{(A.2)}$$

The equation for $T$ is much more complicated. We begin with the variation of the non-constraint part of $\delta T$:

$$i\kappa \int \Sigma dx \left[ \frac{g}{\sqrt{E^1}}\delta A_1 + \left( \frac{g^2\gamma}{2\sqrt{E^1}} \right) A_2 + \left( \frac{g}{\sqrt{E^1}} \right) A_3 \right] \delta A_2$$

$$+ \left( \frac{g^2\gamma}{2\sqrt{E^1}} \right) A_3 - \left( \frac{g}{\sqrt{E^1}} \right) \gamma A_2 + \left( \frac{\gamma}{\sqrt{E^1}} \right) \gamma A_3 + \frac{\kappa p}{2(E^1)^2}(B^1 + 2/3\lambda\kappa E^1))\delta E^1]$$

$$= \int \Sigma dx \left[ \frac{g}{\sqrt{E^1}}(-\Lambda' + (LA_1)' + M(B^2E^2 + B^3E^3) + E\kappa/2(-p^2/(2E^1)^2)$$

$$+ \lambda))$$

$$+ \left( \frac{g^2\gamma}{2\sqrt{E^1}} \right) A_2 + \left( \frac{g}{\sqrt{E^1}} \right) A_3 \right] \gamma A_2 + L^2E_3')$$

$$+ \left( \frac{g^2\gamma}{2\sqrt{E^1}} \right) A_2 - \left( \frac{g}{\sqrt{E^1}} \right) \gamma A_2 + \left( \frac{\gamma}{\sqrt{E^1}} \right) \gamma A_3 + \frac{\kappa p}{2(E^1)^2}(B^1 + \kappa(p^2/(2E^1)$$

$$+ \lambda E^1)))$$

$$+ \gamma \frac{\partial}{\partial E^1} \left( \frac{g}{\sqrt{E^1}}(L(E^1)' - ME^1(A_2E^2 + A_3E^3)) \right]$$

$$= \int \Sigma dx \left[ \frac{g}{\sqrt{E^1}}(-\Lambda' + (LA_1)' + ME(1/g + \kappa/2(-p^2/(2E^1)^2) + \lambda))$$

$$+ \left( \frac{g^2\gamma}{2\sqrt{E^1}} \right)(L(B^1)' - ME^1\gamma A) + \left( \frac{g}{\sqrt{E^1}} \right) \frac{1}{A}(-\Lambda A - L\alpha' A$$

$$+ M(E^1(B^1)' - 1/g(A_3E^2 + A_2E^3)2E^1)$$

$$+ \gamma \frac{\partial}{\partial E^1} \left( \frac{g}{\sqrt{E^1}}(L(E^1)' - MgE^1A^2) \right] \quad \text{A.3}$$

where we used the identity $A_2B^2 + A_3B^3 = A\gamma$. We observe that the 2 first terms in the first line of the last equality can be combined with 2 companions in the 2nd bracket of the
2nd line to obtain a total spatial derivative and we use the Gauss constraint in the same bracket to simplify this bracket further. We thus obtain when using $\partial g/\partial B^1 = g^2/(2E^1)$ for the non-constraint part of $\delta T$

$$\int d\Sigma[(\frac{g}{\sqrt{E^1}}(LA_1 - \Lambda))' + \frac{g}{\sqrt{E^1}}ME(1/g + \kappa/2(-p^2/(2(E^1)^2) + \lambda))$$

$$+(\frac{g^2\gamma}{2\sqrt{E^1}}(L(B^1)') - ME^1\gamma A) - (\frac{g}{\sqrt{E^1}})'(L\gamma A + ME^1(B^1)')$$

$$+\gamma\frac{\partial}{\partial E^1}(\frac{g}{\sqrt{E^1}})(L(E^1)' - MgE^1A)\gamma]$$

$$= \int d\Sigma[(\frac{g}{\sqrt{E^1}}(LA_1 - \Lambda))' + \frac{g}{\sqrt{E^1}}ME(1/g + \kappa/2(-p^2/(2(E^1)^2) + \lambda))$$

$$+(\frac{g^2\gamma}{2\sqrt{E^1}}(L(B^1)') - ME^1\gamma A) - (\frac{\partial}{\partial E^1}(\frac{g}{\sqrt{E^1}})(E^1)') + \frac{\partial}{\partial B^1}(\frac{g}{\sqrt{E^1}})(B^1)')(\frac{M}{A}E^1(B^1)')$$

$$-\gamma\frac{\partial}{\partial E^1}(\frac{g}{\sqrt{E^1}})MgE^1A\gamma]$$

$$= \int d\Sigma[(\frac{g}{\sqrt{E^1}}(LA_1 - \Lambda))' + \frac{g}{\sqrt{E^1}}ME(1/g + \kappa/2(-p^2/(2(E^1)^2) + \lambda))$$

$$-A\gamma^2 + \frac{1}{A}((B^1)')^2)(\frac{g^2}{2\sqrt{E^1}}ME^1 + \frac{\partial}{\partial E^1}(\frac{g}{\sqrt{E^1}})MgE^1)] .$$

We now use the identity $A\gamma^2 + ((B^1)')^2/A = (B^2)^2 + (B^3)^2 = E/(g^2)$ and can collect all non-boundary terms with a common prefactor $E$ in the last equation:

$$\int d\Sigma[(\frac{g}{\sqrt{E^1}}(LA_1 - \Lambda))' + ME(\frac{g}{\sqrt{E^1}}(1/g + \kappa/2(-p^2/(2(E^1)^2) + \lambda))$$

$$-\frac{1}{2\sqrt{E^1}} + E^1\frac{\partial}{g\partial E^1}(\frac{g}{\sqrt{E^1}}))].$$

(A.5)

Now, one has only to use $\partial/\partial E^1(\frac{\sqrt{E^1}}{g}) = g^2/(4(E^1)^{7/2})(- (B^1 E^1 + \kappa p^2) + \kappa(\lambda(E^1)^2 - p^2/2))$ in order to show that the last equation becomes

$$\int_{\partial\Sigma} \frac{g}{\sqrt{E^1}}(LA_1 - \Lambda).$$

(A.6)

As for the constraint part of $\delta T$, we can, as already derived in section 6, refer to formula 6.2 and obtain on the constraint surface

$$-\delta G[A_1g\sqrt{2}E^1E^2/(\sqrt{E^1})^3EA, g\sqrt{2}E^1E^2/(\sqrt{E^1})^3EA, g\sqrt{2}E^3/(\sqrt{E^1})^3EA, 0]$$

$$= -\int_{\partial\Sigma} [(LA_1 - \Lambda)g\sqrt{2}E^1E^2/(\sqrt{E^1})^3EA](A_2E^3 - A_3E^2)$$

33
\begin{align*}
+ (MA_1 \frac{g \sqrt{2} E_1 E^2}{(\sqrt{E_1})^3 EA} - \Lambda \frac{g \sqrt{2} E^3}{(\sqrt{E_1})^3 EA}) (A_2 E^2 + A_3 E^3) E^1 \\
+ (L \frac{g \sqrt{2} E^3}{(\sqrt{E_1})^3 EA} - M \frac{g \sqrt{2} E^1 E^2}{(\sqrt{E_1})^3 EA} \times \nonumber
(A_1 E^1 (A_2 E^2 + A_3 E^3) + \frac{1}{2} E (B^1 + \kappa (p^2/(2E^1) + \lambda E^1)))
\end{align*}
\hspace{8cm} (A.7)

We use the boundary conditions for symmetry transformations to simplify this expression of which most terms vanish at infinity. The second term is already \(O(1/x)\). The first summand in the first bracket in the last term is \(O(1/x^2)\) the second is \(O(1/x)\), while in the second bracket of the last term the first summand is \(O(1)\) and the second has a \(O(x)\) term in leading order (since we are dealing with an asymptotic region we have to set the cosmological constant equal to zero). Finally, using the Gauss constraint in the first term we see that it is \(O(1)\) altogether so that we end up with the following contribution of the constraint part of \(\delta T\):

\[- \int_{\partial \Sigma} [(L A_1 - \Lambda) \frac{g \sqrt{2} E_1 E^2}{(\sqrt{E_1})^3 EA} (E^1)'] + M \frac{g \sqrt{2} E^1 E^2}{(\sqrt{E_1})^3 EA^2} EB^1 \] \hspace{1cm} (A.8)

Now, since \(L A_1 - \Lambda = O(1/x^2)\), \(A \to 2\), \(E^1 \to x^2\), \((E^1)’ \to 2x\), \(g = O(x^3)\), \(\sqrt{2} E^2 \to 2x\), \(E \to 2x^2\) (plus higher orders respectively), we can replace the first term in the last formula by

\[- \int_{\partial \Sigma} (L A_1 - \Lambda) \frac{g}{\sqrt{E_1}} \] \hspace{1cm} (A.9)

which exactly cancels the contribution (A.6) from the part of \(\delta T\) which does not vanish on the constraint surface.

Finally, since \(g \to \frac{-2x^3}{(\sqrt{2} a_3)} = 2^3/m\), \(M x^2 = O(1)\), \(B^1 \to -m/x\), we can conclude that the complete variation of \(T\) is indeed given by

\[\delta T = \lim_{x \to +\infty} \int_{\partial \Sigma} (M x^2) \] \hspace{1cm} (A.10)

which is just the lapse at infinity for a symmetry transformation while \(T\) is indeed gauge invariant for a gauge transformation.

The variation of \(\Phi\) is now fairly easy to derive : it is nearly the same as that for \(T\) except that one has to divide all expressions by one more power of \(\sqrt{E_1}\). The total differentials that arise therefore all vanish except for that coming from the variation of \(\omega\):

\[\delta - \int_{\Sigma} dx \omega = + \int_{\partial \Sigma} U - \frac{\kappa p}{2} \int_{\Sigma} ME/E^1 \] \hspace{1cm} (A.11)

The second term in the last equation is due to the following identity

\[\sqrt{E_1} \left( \frac{E^1}{g} \frac{\partial}{\partial E^1} (g E^1) + \frac{1}{2E^1} \right) = \frac{E^1}{g} \frac{\partial}{\partial E^1} (g \sqrt{E_1}) \] \hspace{1cm} (A.12)

which shows that the changes in the term \(\propto E\) due to the power change of \(\sqrt{E_1}\) and the appearance of the \(\omega\) in \(\Phi\) as compared to \(T\) exactly cancel each other so that in conclusion

\[\delta \Phi = + \lim_{x \to +\infty} \int_{\partial \Sigma} U \] \hspace{1cm} (A.13)

\[34\]
We have thus arrived at the expected transformation laws for our reduced variables as claimed in section 6.
It should be clear by now how to prove that they form a closed canonical Poisson algebra in which \((T,m)\) and \((\Phi,p)\) are canonical pairs. The computations are rather tedious to perform and do not give further insight into the ideas involved, so we refrain from displaying them here.

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