In this lecture we argue that the fluctuations of Dirac eigenvalues on the finest scale, i.e. on the scale of the average level spacing do not depend on the underlying dynamics and can be obtained from a chiral random matrix theory with the same low energy effective theory. We present three pieces of evidence supporting that such microscopic correlations of lattice QCD Dirac spectra are given by chiral random matrix theory. First, we find that the spectral correlations of eigenvalues in the bulk of the spectrum obey the Dyson-Mehta-Wigner statistics. Second, we show that the valence quark mass dependence for sufficiently small quark masses, as calculated by the Columbia group, can be obtained from the microscopic spectral density of chiral random matrix theory. Third, in the framework of chiral random matrix models, we present results showing that the microscopic spectral density is strongly universal, i.e. is insensitive to the details of the probability distribution.

1 Introduction

Although not a physical observable itself, the spectrum of the Euclidean QCD Dirac operator is an essential ingredient for the calculation of hadronic correlation functions. It also serves as the order parameter for the chiral phase transition. This becomes clear by expressing the quark propagator in terms of the eigenfunctions and the eigenvalues of the Dirac operator

\[ S(x, y) = -\sum_k \frac{\phi_k(x)\phi_k^*(y)}{\lambda_k + im}, \]

where

\[ i\gamma D\phi_k = \lambda_k \phi_k, \]

and \( i\gamma D \) is the Euclidean Dirac operator. Because \( \{\gamma_5, i\gamma D\} = 0 \), all nonzero eigenvalues occur in pairs \( \pm \lambda_k \). The eigenvalues near zero are of great importance for the propagation of light quarks. This is the reason that instanton field configurations, with one zero eigenvalue per instanton, determine the main characteristics of hadronic correlation functions.
With regards to chiral symmetry breaking, the Dirac operator is directly related to the chiral condensate

\[
\langle \bar{q}q \rangle = -\lim_{V} \frac{1}{V} \int d^4x S(x, x),
\]

\[
= \lim_{V} \frac{1}{V} \int \frac{2m i \rho(\lambda) d\lambda}{m^2 + \lambda^2},
\]

\[
= \frac{\pi i \rho(0)}{V}, \quad (3)
\]

which is the celebrated Banks-Casher formula. Here, the spectral density is defined as

\[
\rho(\lambda) = \sum_k \langle \delta(\lambda - \lambda_k) \rangle, \quad (4)
\]

and the average is over gauge field configurations weighted according to the Euclidean action. The space-time volume is denoted by \(V\). The limit is defined such that the thermodynamic limit is taken before the chiral limit.

In this lecture, I wish to argue that some properties of the Dirac spectrum are completely determined by the global symmetries of the QCD partition function and are not sensitive to the dynamics of the theory. In general such universal properties are given by fluctuations of the eigenvalues on the finest scale, i.e. on the scale of individual level spacings. Because the spectrum is symmetric about zero, we have to distinguish spectral correlations near zero virtuality from correlations in the bulk of the spectrum. Although the overall spectral density is certainly not universal, we wish to convince the reader that the spectral density near zero, on the scale of the average level spacing, is universal. It is called the microscopic spectral density. In the bulk of the spectrum, the spectral correlations will be shown to obey universal level statistics. From the study of quantum chaos, we know that the latter correlations are strongly universal. In the framework of a random matrix model we will show that the microscopic spectral density has strong universality properties as well. Finally, we make contact with lattice QCD calculations. In this context the Dirac spectrum was investigated via the valence quark mass dependence of chiral condensate, and, for a relatively small number of configurations, by a full diagonalization of the Dirac operator. Both the valence quark mass dependence and the eigenvalue correlations in the bulk of the spectrum are shown to be in complete agreement with random matrix theory (RMT) within its range of applicability.
2 Universality

Leutwyler and Smilga have argued that in the range

\[ \frac{1}{\Lambda} \ll L \ll \frac{1}{\sqrt{m\Lambda}}, \]

(5)

where \( L \) is the linear size of the Euclidean box, \( m \) the quark mass and \( \Lambda \) a typical hadronic mass scale, the mass dependence of the QCD partition function is given by the effective partition function

\[ Z_{\text{eff}}(M, V) = \int_{U \in G/H} dU \exp(\text{Re} V \Sigma \text{tr} MU^2 e^{i\theta/N_f}). \]

(6)

Here, \( M \) is the mass matrix, \( \theta \) is the vacuum angle and \( \Sigma = |\langle \bar{q}q \rangle| \). The integration is over the Goldstone manifold \( G/H \) which is determined by the scheme of chiral symmetry breaking. In QCD with three colors and \( N_f \) fundamental fermions, \( G/H = SU(N_f) \).

However, QCD is not the only theory that can be reduced to the effective partition function (6). In particular, chiral random matrix theories can be reduced to this partition function. This leads us to the notion of universality: a property will be called universal if it is the same for theories with the same low energy effective theory. This allows us to calculate such quantities for the simplest possible theory, for example, for a chiral random matrix theory. In particular, properties that are determined by the effective partition function are trivially universal. As an example, I mention the Leutwyler-Smilga sum rules. Let us for the moment consider the sector of zero topological charge. Then, by expanding both sides of the equality (valid in the range (5))

\[ \frac{Z_{\text{QCD}}^{\nu=0}(m, V)}{Z_{\text{QCD}}^{\nu=0}(m = 0, V)} = \frac{Z_{\text{eff}}^{\nu=0}(m, V)}{Z_{\text{eff}}^{\nu=0}(m = 0, V)} \]

(7)

in powers of \( m \), and performing the group integrals we obtain an infinite family of sum rules. The simplest sum rule is given by

\[ \frac{1}{V^2} \sum_{\lambda_k > 0} \left\langle \frac{1}{\lambda_k^2} \right\rangle_{\nu=0} = \frac{\Sigma^2}{4N_f}. \]

(8)

The generalization of this formula to a sector with topological charge \( \nu \) and different coset spaces (\( G/H = SU(2N_f)/Sp(2N_f) \) for \( SU(2) \) color with fundamental fermions, \( G/H = SU(N_f)/SO(N_f) \) for adjoint fermions with gauge
The sum in (9) can be replaced by an integral over the spectral density
\[
\frac{1}{V^2} \sum_{\lambda_k > 0} \left\langle \frac{1}{\lambda_k^2} \right\rangle = \frac{\Sigma^2}{4 (|\nu| + (\text{dim(coset)} + 1)/N_f)}.
\]

(9)

The sum in (9) can be replaced by an integral over the spectral density
\[
\frac{1}{V^2} \sum_{\lambda_k > 0} \left\langle \frac{1}{\lambda_k^2} \right\rangle = \Sigma^2 \int_0^\infty \frac{du}{u^2} \frac{1}{V \Sigma} \rho\left( \frac{u}{V \Sigma} \right),
\]

(10)

where we have introduced the microscopic variable \( u = \lambda V \Sigma \). The combination
\[
\rho_S(u) = \frac{1}{V \Sigma} \rho\left( \frac{u}{V \Sigma} \right),
\]

(11)

will be called the microscopic spectral density. It is constrained by the effective partition function but not determined by it. Our conjecture is that, in the thermodynamic limit, it is universal function that can be calculated with the help of chiral random matrix theory.

A more intuitive interpretation of \( \rho_S(u) \) is obtained if we start from the Banks-Casher relation and the fact that chiral symmetry is broken. The eigenvalues near zero virtuality are roughly equally spaced, with spacing given by
\[
\Delta \lambda = \frac{1}{\rho(0)} = \frac{\pi}{\Sigma V}.
\]

(12)

and smallest nonzero eigenvalue of the Dirac operator is equal to
\[
\lambda_{\text{min}} \sim \frac{1}{V \Sigma}.
\]

(13)

The microscopic variable \( u \) parameterizes the spectral density on the scale of the average level spacing, and the microscopic spectral density is obtained by magnifying the region around \( \lambda = 0 \) by a factor \( V \).

An observable directly related to the microscopic spectral density is the valence quark mass dependence of the chiral condensate \[\Sigma(m)\] which can be expressed as
\[
\frac{\Sigma(m)}{\Sigma} = \int_0^\infty \frac{2mV \Sigma}{u^2 + m^2 V^2 \Sigma^2} \rho_S(u) du.
\]

(14)

It can be obtained from random matrix theory in the range (5). This range (5) can also be written as
\[
m \ll \sqrt{\lambda_{\text{min}} \Lambda}, \quad \lambda_{\text{min}} \ll \Lambda,
\]

(15)
which might be called the mesoscopic window of QCD.

To summarize this section, we conclude that the eigenvalues of the QCD Dirac operator are strongly correlated for the following reasons:

- The Leutwyler-Smilga sum-rules.
- The eigenvalue spacing near zero $\Delta \lambda \sim 1/V$ instead of $\Delta \lambda \sim 1/V^{1/4}$ for a noninteracting system.
- If the eigenvalues were uncorrelated, then, in the chiral limit $\rho(\lambda) \sim \lambda^{2N_f}$, and chiral symmetry would not be broken.

3 Universal level correlations of quantum spectra

Quantum spectra of complex systems have been investigated in great detail both experimentally and theoretically. Already, in the fifties it was observed that the spectral correlations of nuclear resonances do not depend on the details of the system and can be obtained from random matrix theory. More recently, it has been shown that the essential ingredient to obtain random matrix spectra is that the corresponding classical system is chaotic. Even systems with only two degrees of freedom show spectral correlations that are in perfect agreement with the random matrix prediction. It was already noted early on that the average spectral density is not given by random matrix theory. Typically, for a physical system, it is a monotonously increasing function whereas for random matrix theory it is a semicircle. To deal with this problem, a first important hypothesis was introduced, namely, that we have a separation of scales with regards to the average spectral density and the microscopic spectral fluctuations. The second hypothesis is that the microscopic fluctuations are given by the random matrix ensemble with same symmetries as the underlying theory. In particular, anti-unitary symmetries play an important role in identifying the correct RMT.

In practice, the average spectral density is folded out by transforming the original spectrum $\{\lambda_k\}$ to a new spectrum with average spectral density equal to one. Below we will use the following statistics to measure the spectral correlations of the unfolded level sequence: the nearest neighbor spacing distribution, $P(S)$, the number statistics and the $\Delta_3$-statistic. The number statistics are obtained by counting the number of eigenvalues $n_k$ in consecutive intervals of length $n$. The number variance is defined as $\Sigma_2(n) = var\{n_k\}$, and the $\Delta_3$-statistic is obtained from a convolution of $\Sigma_2(n)$ with a 'smoothening'-kernel.

The above statistics have been obtained analytically (see) for the invariant random matrix ensembles defined by a matrix of independently distributed
gaussian matrix elements, consistent with the hermiticity of the matrix. Depending on the anti-unitary symmetry, the matrix elements are real (Gaussian Orthogonal Ensemble (GOE), \( \beta = 1 \)), complex (Gaussian Unitary Ensemble (GUE), \( \beta = 2 \)) or quaternion real (Gaussian Symplectic Ensemble (GSE), \( \beta = 4 \)). The most notable property is the stiffness of RMT spectra: the number variance behaves asymptotically as \( \Sigma_2(n) \sim \frac{2}{\pi^2} \log(n) \) instead of \( \Sigma_2(n) \sim n \) for independently distributed eigenvalues.

4 Chiral random matrix theory

Chiral random matrix theories are theories with the global symmetries of the QCD Dirac operator, but otherwise independently Gaussian distributed random matrix elements. The chiral random matrix model that obeys these conditions is defined by

\[
Z_\nu^\beta = \int DW \prod_{j=1}^{N_f} \det(\mathcal{D} + m_f) \exp\left(-\frac{N\Sigma_2^\beta}{4} \text{Tr}W^\dagger W\right),
\]

where

\[
\mathcal{D} = \begin{pmatrix}
0 & iW + i(\Omega_T - \text{arg}(P)) + \mu \\
iW^\dagger + i(\Omega_T - \text{arg}(P)) + \mu & 0
\end{pmatrix},
\]

and \( W \) is a rectangular \( n \times m \) matrix with \( \mu = n - m \) and \( N = n + m \). An early version of this model can be found in \cite{18}. The temperature dependence introduced in \cite{19, 21} is included in the form of Matsubara frequencies in the diagonal matrix \( \Omega_T \). The chemical potential, \( \mu \), introduced in \cite{22} results in a non-Hermitean Dirac operator with eigenvalues that are scattered in the complex plane. The term \( \text{arg}(P) \), the argument of the Polyakov loop was introduced by Stephanov \cite{21} in order to explain the dependence of the critical temperature on the expectation value of the \( Z_N \)-phase \cite{11}. In this paper we only discuss models with \( \mu = 0 \) and temperature dependence with only the lowest Matsubara frequencies included. This partition function reproduces the following symmetries of QCD:

- The \( U_A(1) \) symmetry. All nonzero eigenvalues of the random matrix Dirac operator occur in pairs \( \pm \lambda \).
- The topological structure of the QCD partition function. The matrix \( \mathcal{D} \) has exactly \( |\nu| \equiv |n - m| \) zero eigenvalues. This identifies \( \nu \) as the topological sector of the model.
• The flavor symmetry is the same as in QCD. For $\beta = 2$ it is $SU(N_f) \times SU(N_f)$. For $\beta = 1$ it is $SU(2N_f)$ and for $\beta = 4$ it is $SU(N_f)$.

• The chiral symmetry is broken spontaneously for two or more flavors according to the pattern $SU(N_f) \times SU(N_f)/SU(N_f)$, $SU(2N_f)/Sp(N_f)$ and $SU(N_f)/O(N_f)$ for $\beta = 2$, 1 and 4, respectively, the same as in QCD. The chiral condensate satisfies the Banks-Casher relation

$$\Sigma = \partial_{m_f} \log Z = \lim_{N \to \infty} \frac{\pi \rho(0)}{N},$$

(18)

Therefore, the parameter $\Sigma$ in the random matrix model is identified as the chiral condensate and $N$ as the (dimensionless) volume of space time.

• The anti-unitary symmetries. For three and more colors with fundamental fermions the Dirac operator has no anti-unitary symmetries, and, in general, the matrix elements of the Dirac operator are complex. The matrix elements of $W$ of the corresponding random matrix ensemble are chosen arbitrary complex as well ($\beta = 2$). For $N_c = 2$ and fundamental fermions the Dirac operator satisfies

$$[C\tau_2 K, i\gamma D] = 0,$$

(19)

where $C$ is the charge conjugation matrix and $K$ is the complex conjugation operator. Because, $(C\tau_2 K)^2 = 1$, the matrix elements of the Dirac operator can always be chosen real, and the corresponding random matrix ensemble is defined with real matrix elements ($\beta = 1$). For two or more colors with fermions in the adjoint representation $i\gamma D$ has the symmetry

$$[C K, i\gamma D] = 0,$$

(20)

but now $(C K)^2 = -1$, which allows us to rearrange the matrix elements of the Dirac operator into real quaternions. The matrix elements of $W$ of the corresponding random matrix ensemble are chosen quaternion real as well ($\beta = 4$).

5 Universal spectral correlations

As already stated before, we have to distinguish spectral correlations near zero and spectral correlations in the bulk of the spectrum. The latter have the advantage that, under the assumption of spectral ergodicity, they can be calculated by a spectral average rather than an ensemble average. This requires no more than a few statistically independent gauge field configurations, whereas
the calculation of the microscopic spectral density requires a prohibitively large number of independent gauge field configurations.

We will present three pieces of evidence that spectra of the lattice QCD Dirac operator show universal correlations on the finest scale. First, we will show that spectral correlations in the bulk of the spectrum are given by the invariant random matrix ensembles. Second, we will show that the valence quark mass dependence in the mesoscopic range of QCD can be obtained from the microscopic spectral density. Third, we will present random matrix results showing that the microscopic spectral density is insensitive to the probability distribution of the matrix elements and to the temperature as introduced in (16).

5.1 Correlations in the bulk of lattice spectra

Recently, complete Dirac spectra for reasonably large lattices were obtained by Kalkreuter both for Kogut-Susskind fermions (with four flavors) and Wilson fermions (with two flavors) with $SU(2)$-color. It is important that all eigenvalues were obtained. Because the sum of the squares of the eigenvalues satisfy rigorous sum-rules, this allows a very stringent test on the accuracy of the calculation.

In the case of the $SU(2)$ color group, the anti-unitary symmetries of Kogut-Susskind fermions are different from the continuum theory. We have have

$$[\tau_2 K, D^{KS}] = 0 \quad \text{with} \quad (\tau_2 K)^2 = -1,$$

($K$ is the complex conjugation operator) whereas for Wilson fermions in the form considered by Kalkreuter we have

$$[\gamma_5 C K \tau_2, \gamma_5 \gamma_5 D_{\text{Wilson}}] = 0 \quad \text{with} \quad (\gamma_5 C K \tau_2)^2 = 1.$$

According to random matrix theory, if the square of the anti-unitary operator is $-1$ the spectral correlations are given by the GSE, whereas, if the square is 1, the correlations are given by the GOE.

We have analyzed the spectra of a score of lattice QCD configurations. In Fig. 1 we show the results for the correlations of the Dirac eigenvalues for Kogut Susskind fermions (right) on a $12^4$ lattice and Wilson fermions (left) on a $8^3 \times 12$ lattice for parameters as indicated in labels of the figure. We show the number variance, $\Sigma_2(n)$, the $\Delta_3$ statistic and the nearest neighbor spacing distribution $P(S)$. The points represent the lattice-data and the random matrix results are given by the curves. The corresponding RMT is given in the label of the figure. We find perfect agreement with the above predictions. In addition, we have also confirmed that the correlations are stationary over
Figure 1: Spectral correlations of Dirac eigenvalues for Wilson fermions (left) and Kogut-Susskind fermions (right). The number variance, $\Sigma_2(n)$, the $\Delta_3$-statistic and the nearest neighbor spacing distribution are shown in the upper, middle and lower figure, respectively.
Figure 2: The valence quark mass dependence of the chiral condensate $\Sigma^V(m)$ plotted as $\Sigma^V(m)/\Sigma$ versus $mN\Sigma$. The dots and squares represent lattice results by the Columbia group [11] for values of $\beta$ as indicated in the label of the figure.

5.2 The valence quark mass dependence of the chiral condensate.

Using standard methods of random matrix theory it is possible to obtain analytically the microscopic spectral density. The result for $N_f$ flavors in the sector of topological charge $\nu$ is

$$\rho_S(u) = \frac{u}{2} \left( J^2_{N_f+\nu}(u) - J_{N_f+\nu+1}(u)J_{N_f+\nu-1}(u) \right).$$

(23)

Using (14) we obtain the valence quark mass dependence by an elementary integration

$$\frac{\Sigma(x)}{\Sigma} = x(I_{N_f+\nu}(x)K_{N_f+\nu}(x) + I_{N_f+\nu+1}(x)K_{N_f+\nu-1}(x)).$$

(24)

Asymptotically, for large $x \equiv mV\Sigma$, $\Sigma(x) \sim \Sigma(1 - (N_f + \nu)/x)$ and for $x \to 0$, $\Sigma(x) \sim x\Sigma/2(N_f + \nu)$ for $N_f + \nu \neq 0$ and $\Sigma(x) \sim -x\Sigma\log(x)$ for $N_f + \nu = 0$. The valence quark mass was calculated by the Columbia group [11] for unquenched Kogut-Susskind fermions on a $16^3 \times 4$ lattice with $m_{\text{sea}}a = 0.01$, two flavors and three colors. In Fig. 2 we plot their results in terms of the dimensionless quantities suggested by the valence quark mass dependence [14], namely as $\Sigma(x)/\Sigma$ versus $x$. We observe that the lattice data fall on a
universal curve given by (24) for \( N_f + \nu = 0 \). The reason that \( N_f = 0 \) is that
\[ \lambda_{\text{min}} a \sim 1.0 e^{-4} \ll m_{\text{sea}} a, \]
so that the calculation is quenched on the scale of the smallest eigenvalues. It is also not surprising that \( \nu = 0 \) because the fermionic zero modes are completely mixed with the rest of the states. For larger \( x \) the lattice data deviate from the universal curve. This was to be expected because in terms of \( x \) the domain of validity is given by
\[ x \ll \sqrt{\frac{\Lambda}{\lambda_{\text{min}}}}. \]  

5.3 Universality in random matrix theory

The microscopic spectral density was calculated for the simplest possible random matrix ensemble. Recently, the microscopic spectral density for \( N_f = \nu = 0 \) was calculated for ensembles with a non-Gaussian probability distribution. The authors found that the addition of a polynomial in \( \text{tr}(W^\dagger W) \) to the exponent of the probability distribution had no effect on the microscopic spectral density. We studied the random matrix model (17) with only the lowest Matsubara frequencies included. It can be shown that it is sufficient to include only the positive frequency. This model has a critical temperature, \( T_c \), above which \( \rho(0) = 0 \). In spite of the fact that the average spectral density changes drastically between \( T = 0 \) and \( T = T_c \) (it is given by the solution of a cubic equation), we find that the microscopic spectral density remains unaffected.

Numerically, we studied a variety other ensembles. In particular, I mention the chiral Cauchy ensemble, with a with a diverging second moment of the matrix elements. Nevertheless, we find the same universal microscopic spectral density. Finally, we wish to mention that according to general universality arguments in random matrix theory, the correlations in the bulk of the spectrum of the chiral random matrix ensembles are given by the invariant random matrix ensembles. Therefore, we find that the microscopic spectral density and the correlations in the bulk of the spectrum are mutually inclusive and both are strongly universal.

6 Conclusions

We have argued that the spectral fluctuations on the finest scale are determined by the global symmetries of the QCD partition function, and can be obtained from chiral random matrix theories. In particular, the microscopic spectral density determines the valence quark mass dependence of the condensate in the mesoscopic range of QCD. In this range we find perfect agree-
ment with recent lattice QCD data. We have also observed that correlations in the bulk of the spectrum are given by the invariant random matrix ensembles. This implies that the spectral fluctuations are strongly suppressed \[ \Sigma_2(n) \sim \log(n)/\pi^2, \] instead of \( n \) for uncorrelated eigenvalues. To some extent, QCD is self-quenching.

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