DIXMIER TRACES AND EXTRAPOLATION DESCRIPTION OF NONCOMMUTATIVE LORENTZ SPACES

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Abstract. We study the relationships between Dixmier traces, \( \zeta \)-functions and traces of heat semi-groups beyond the dual of the Macaev ideal and in the general context of semifinite von Neumann algebras. We show that the correct framework for this investigation is that of operator Lorentz spaces possessing an extrapolation description. We demonstrate the applicability of our results to Hörmander-Weyl pseudo-differential calculus on \( \mathbb{R}^n \). In that context, we prove that the Dixmier trace of a pseudo-differential operator coincide with the ‘Dixmier integral’ of its symbol.

Keywords: Dixmier traces, Singular traces, Lorentz (Marcinkiewicz) spaces, \( \zeta \)-functions, Heat kernels, Extrapolation, Hörmander-Weyl pseudo-differential operators

Introduction

This article is devoted to the study of singular traces beyond the dual of the Macaev ideal. The theory of singular traces has been intensively developed recently, especially after Alain Connes’ application of that theory to noncommutative geometry and physics [10]. In particular, singular traces now have a broad application to several fields of mathematics and theoretical physics, including noncommutative symmetric spaces, geometry of Banach spaces, pseudo-differential operators, index theory and geometric analysis, fractal geometry, noncommutative integration theory, renormalization and quantum field theory. So far, known results and applications concentrate around singular traces arising from the logarithmic divergence of a normal trace. But the importance of singular traces is also seen from the fundamental fact that an arbitrary tracial weight on a semifinite von Neumann algebra is uniquely decomposed into the sum of a normal trace and a singular trace. Hence the understanding of the structure of singular traces contributes to the understanding of the structure of general non-normal traces. Also, we strongly believe that singular traces associated with more general divergences should have natural applications in all the above mentioned areas. In this paper, we demonstrate such application to the theory of Hörmander-Weyl pseudo-differential calculus on \( \mathbb{R}^n \). More specifically, the main motivation of the present article is to characterize the class of noncommutative Lorentz spaces of \( \tau \)-compact operators within a semifinite von Neumann algebra, for which one can generalize the ‘classical’ relationships between:

1) Dixmier traces, 2) residues of \( \zeta \)-functions and 3) asymptotics of traces of heat semigroups.

Our main discovery is the existence of a strong connection between singular traces and extrapolation theory. At the present stage, extrapolation only appears as a technical device for the study of singular traces. We leave a more profound investigation of their interconnections for a future work.
Let us first review the ‘classical’ results we wish to extend to more general Lorentz spaces. To fix some conventions, set \( \mathbb{N} := \{1, 2, 3, \ldots \} \), \( \mathbb{N}_0 := \{0, 1, 2, \ldots \} \), \( \mathbb{R}_+ = [0, \infty) \) and \( \mathbb{R}_+^* = (0, \infty) \). Let \( \mathcal{M}^{1,\infty}(\mathcal{H}) \) be the dual of the Macaev ideal of compact operators on a separable Hilbert space \( \mathcal{H} \):

\[
\mathcal{M}^{1,\infty}(\mathcal{H}) := \left\{ T \in \mathcal{K}(\mathcal{H}) : \|T\|_{1,\infty} := \sup_{n \in \mathbb{N}} \frac{1}{\log(1+n)} \sum_{k=1}^{n} \mu(T,k) < \infty \right\}.
\]

Here \( \mathcal{K}(\mathcal{H}) \) denotes the ideal of compact operators and \( \{\mu(T,n)\}_{n \in \mathbb{N}} \) is the sequence of singular values of a compact operator \( T \). We will always embed \( \ell^\infty(\mathbb{N}) \) onto \( L^\infty(\mathbb{R}_+^*) \) via the map

\[
\{x_n\}_{n \in \mathbb{N}} \mapsto \sum_{n \in \mathbb{N}} x_n \chi_{[n,n+1)},
\]

and, according to this embedding, a state on \( L^\infty(\mathbb{R}_+^*) \) defines a state on \( \ell^\infty(\mathbb{N}) \) by restriction. In his seminal paper [11], Jacques Dixmier constructed singular traces (i.e. traces vanishing on finite rank operators), which for \( \mathcal{M}^{1,\infty}(\mathcal{H}) \) are given by the linear extension of the unitarily invariant weights

\[
\text{Tr}_\omega : \mathcal{B}(\mathcal{H})^+ \to [0, \infty], \quad T \mapsto \omega \left( \left\{ \frac{1}{\log(1+n)} \sum_{k=1}^{n} \mu(T,k) \right\}_{n \in \mathbb{N}} \right),
\]

where \( \omega \) is an arbitrary dilation invariant (see [12]) state on \( L^\infty(\mathbb{R}_+^*) \). We stress that the computation of a Dixmier trace can be an highly non-trivial task in concrete examples. This is not really stipulated by the presence of an extended limit \( \omega \) in the formula [1]. To our knowledge and within the framework of spectral triples in noncommutative geometry, all the examples of computations of Dixmier traces involve ‘measurable operators’, that is operators in \( \mathcal{B}(\mathcal{H}) \) which all the Dixmier traces take the same value. For instance, this is the case for classical pseudo-differential operators on a closed manifold (a very short and simple proof is displayed in [19 Theorem 3]). Of course, concrete examples of non-measurable operators are also known. A construction of such operators in the context of pseudo-differential operators on \( \mathbb{R}^n \) with non-homogeneous symbol, is given in [19 Theorem 4]. The real problem for the computation of a Dixmier trace is that, typically, one has no access to the sequence of singular values of an operator. This explains why in concrete applications of Dixmier traces, the defining formula [1] is of little utility. A remedy to this problem exists and consists in expressing a Dixmier traces \( \text{Tr}_\omega \) in term of the ordinary trace \( \text{Tr} \), which is a computable object. The following theorem (for which we refer to [10], see also [17,8,32,33]) is an example to such a remedy and a main sample of ‘classical’ result which we set to generalize.

**Theorem 0.1.** Let \( T \in \mathcal{M}^{1,\infty}(\mathcal{H})^+ \), \( B \in \mathcal{B}(\mathcal{H}) \) and \( \omega \) be an exponentiation invariant (see [12]) state on \( L^\infty(\mathbb{R}_+^*) \). Then we have

\[
\text{Tr}_\omega(BT) = \omega \left( \left[ r \mapsto \frac{\zeta_B(T,1+\log(r)^{-1})}{\log(1+r)} \right] \right) = \omega \left( \left[ \lambda \mapsto \frac{1}{\log(1+\lambda)} \int_0^{\lambda} \xi_B(T,t^{-1}) \frac{dt}{t^2} \right] \right).
\]

Here, \( \zeta_B(T,.) \) and \( \xi_B(T,.) \) are the generalized \( \zeta \)-function and heat-trace function, respectively defined by

\[
\zeta_B(T,z) := \text{Tr}(BT^z), \quad z \in \mathbb{C}, \quad \Re(z) > 1 \quad \text{ and } \quad \xi_B(T,t) := \text{Tr}(Be^{-tT^{-1}}), \quad t > 0.
\]

The first version of this result (for \( T \) measurable, \( B = 1 \) and with different assumptions on \( \omega \)) is due to Alain Connes and appeared already in his book [10]. It has then been followed by an important list of improvements and generalizations of many kinds [1,7,8,33]. In the form we phrase it here, this
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result first appeared in [32]. We do not want to do a complete review of the different steps leading to this ‘final’ answer. The reader interested in the history of this result may consult the book [23].

We should also mention two other approaches to obtain alternative expressions for a Dixmier trace. The first one is linked to a spectral formula, which is somewhat analogous to the Lidskii’s formula for the ordinary trace [30]:

\[
\text{Tr}_\nu(T) = \nu\left(\left\{ \frac{1}{\log(1+n)} \sum_{|\lambda| \geq \log(n)/n} \lambda \right\}_{n \in \mathbb{N}} \right),
\]

where \( T \) is arbitrary in \( M_{1,\infty}(\mathcal{H}) \). One of the main feature of this formula is that it extends to more general operator ideals than \( M_{1,\infty}(\mathcal{H}) \), in particular to the Lorentz ideal \( M_\psi(\mathcal{H}) \) (with \( \psi \in \Omega \) satisfying condition (8) – see below). The second alternative approach relies on the results of [19], and allows to compute Dixmier traces from expectation values [23, Corollary 8.2.4]:

\[
\text{Tr}_\nu(T) = \nu\left(\left\{ \frac{1}{\log(1+n)} \sum_{k=1}^{n} \langle T e_k, e_k \rangle \right\}_{n \in \mathbb{N}} \right).
\]

Here \( T \) is required to belong to \( L_{1,\infty}(\mathcal{H}) \) (the non-closed subspace of \( M_{1,\infty}(\mathcal{H}) \) whose elements have their sequence of singular values dominated by the harmonic sequence), and is such that there exists \( 0 \leq V \in L_{1,\infty}(\mathcal{H}) \) with \( \sup_{t>0} t^{1/2} \|T(1+tV)^{-1}\|_2 < \infty \) and \( \{e_n\}_{n \in \mathbb{N}} \) is an eigenbasis for \( V \), ordered in such a way that the corresponding sequence of eigenvalues is decreasing. This formula has a great impact on the theory of pseudo-differential operators [19], and especially on Connes’ Trace Theorem, which connects Dixmier traces with Wodzicki residue.

We now explain to what extent we generalize Theorem 0.1. Fix \((\mathcal{N}, \tau)\) a semifinite von Neumann algebra and a normal faithful semifinite trace. In that context, there is an analogous notion of singular numbers, the generalized s-numbers (see for instance [14]). The latter object is a (right-continuous and decreasing) function on the positive half-line, \( \mu(T) \), attached to every operator \( T \) in \( \mathcal{N} \). The generalized s-numbers extend both singular numbers of compact operators and decreasing rearrangement of measurable functions on \( \sigma \)-finite measure spaces. Let then \( \psi : [0, \infty) \to [0, \infty) \) be a concave and increasing function, vanishing at zero and diverging at infinity. We denote by \( \Omega \) the set of such functions. Then, one can generalize the ideal \( M_{1,\infty}(\mathcal{H}) \) by setting:

\[
\mathcal{M}_\psi(\mathcal{N}, \tau) := \left\{ T \in \mathcal{N} : \|T\|_{\mathcal{M}_\psi} := \sup_{t>0} \frac{1}{\psi(t)} \int_0^t \mu(T, s) \, ds < \infty \right\}.
\]

Of course, when \( \mathcal{N} = \mathcal{B}(\mathcal{H}) \) and \( \tau = \text{Tr} \), \( \mu(T) \) is a step function and if moreover \( \psi(t) = \log(1+t) \), we have \( \mathcal{M}_\psi(\mathcal{N}, \tau) = M_{1,\infty}(\mathcal{H}) \) isometrically.

A comment on terminology is in order. In many sources, including most of our own papers on this subject, these ideals are called (operator) Marcinkiewicz spaces. However, in a commutative setting, these spaces were first introduced by G. Lorentz in 1950 [25]. In this paper, we refer to these spaces as (operator) Lorentz spaces, which is consistent with the terminology employed in the book of Bennett and Sharpley [5]. Again, we refer to the book [23], where historical details of the development of the theory of Lorentz spaces are given.
With these definitions at hand, the extension of Dixmier’s construction of singular traces\(^1\) to more general Lorentz spaces is rather direct. Namely, given \(\psi \in \Omega\) and \(\omega\) an arbitrary dilation invariant state on \(L^\infty(\mathbb{R}_+^\ast)\), one considers the unitarily invariant functional:

\[
\tau_{\psi,\omega} : \mathcal{M}_\psi(N,\tau)^+ \to [0,\infty), \quad T \mapsto \omega\left( \left[ t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T,s) \, ds \right] \right).
\]

It is then shown in \(\text{[20]}\) that the latter is positively additive (i.e. it defines a weight on \(\mathcal{N}\)) if and only if \(\omega\) is compatible with \(\psi\), in the sense that

\[
\omega\left( \left[ t \mapsto \frac{\psi(at)}{\psi(t)} \right] \right) = 1, \quad \forall a > 1.
\]

In this case, the linear extensions of the functionals \((3)\) define traces on \(\mathcal{M}_\psi(N,\tau)\), by which we mean unitarily invariant linear functionals. Moreover, these traces are singular, that is they vanish on projections of finite trace \(\tau\). The main difference with the classical case of \(\mathcal{B}(\mathcal{H})\), is that in the general semifinite setting, these singular traces may be supported either at zero or at infinity (or at both). By this we mean that they may either come from the lack of integrability of the generalized \(s\)-numbers function \(\mu(T)\) at infinity, or of the unboundness of \(\mu(T)\) at zero (or both). (Equivalently, a singular trace supported at infinity vanishes on \(L^1(N,\tau)\) and a singular trace supported at zero vanishes on \(\mathcal{N}\).) Here however, we are mainly concerned with singular traces supported at infinity.

Again, having in general no practical access to the generalized \(s\)-numbers of an operator in a semifinite von Neumann algebra, the computation of these singular traces remains an elusive problem. This is mostly why we aim to generalize Theorem \(\text{[1]}\). The hope is that Theorem \(\text{[1]}\) holds for more general Lorentz spaces, with \(\tau\) instead of \(\text{Tr}\) (including in the definition of the \(\zeta\)-function and heat-kernel) and with \(\psi\) instead of the logarithm in the denominators of the double equality \((2)\). (This is known to be true when \(\psi\) behaves like a logarithm at infinity and for any semifinite von Neumann algebra and related ideas can be found in \([8,28,29]\).) More precisely, the question we are addressing in this paper is the following:

**What is the class of functions \(\psi \in \Omega\) for which we have**

\[
(4) \quad \tau_{\psi,\omega}(BT) = C_\zeta(\psi) \omega\left( \left[ r \mapsto \frac{\zeta_B(T,1 + \log(r)^{-1})}{\psi(r)} \right] \right) = C_\xi(\psi) \omega\left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_0^\lambda \zeta_B(T,t^{-1}) \, dt \right] \right),
\]

**for all semifinite von Neumann algebras \(N\) with a fixed normal semifinite faithful trace \(\tau\), all positive elements \(T\) in the Lorentz space \(\mathcal{M}_\psi(N,\tau)\), all \(B\) in \(\mathcal{N}\), all exponentiation invariant (see \([12]\)) states \(\omega\) on \(L^\infty(\mathbb{R}_+^\ast)\) and for certain positive constants \(C_\zeta(\psi)\) and \(C_\xi(\psi)\) which depend on \(\psi\) only?**

Here the functions \(\zeta_B\) and \(\xi_B\) are defined exactly in the same fashion as their ‘classical’ counterparts in \([2]\) via replacing \(\text{Tr}\) with \(\tau\).

We do not fully answer this question as we have only found sufficient conditions on \(\psi \in \Omega\) for \((4)\) to hold true. However, these conditions are wide enough to cover new applications. For instance, they apply to \(\psi(t) = \log(1 + t^{1/\beta})^\beta\), \(\beta > 0\), and to \(\psi(t) = \log(1 + \log(1 + t)\ldots)\). The first condition that we have found, and this is the main novelty of our approach, relies on an extrapolation description of the Lorentz space \(\mathcal{M}_\psi(N,\tau)\). The extrapolated space we need to consider first appeared in \([1,2]\).

\(^1\)For type \(I_\infty\) factors and \(\psi \in \Omega\) satisfying \(\lim_{t \to \infty} \psi(2t)/\psi(t) = 1\), this construction already appeared in \([11]\).
and is of the following nature. Let $\| \cdot \|_p$, $1 \leq p < \infty$, be the $L^p$-norm associated to $\tau$. Then, we let $\mathfrak{L}_\psi(N, \tau)$ be the Banach space of operators in $N$ such that

$$\| T \|_{\mathfrak{L}_\psi} := \sup_{p>1} \frac{\| T \|_p}{\psi(\epsilon^{(p-1)} - 1)} < \infty.$$ 

It can be proven that $\mathfrak{L}_\psi(N, \tau)$ always embeds in $\mathcal{M}_\psi(N, \tau)$ and our first requirement is

(5) $\mathfrak{L}_\psi(N, \tau) = \mathcal{M}_\psi(N, \tau).$

The second condition we need to impose, can be interpreted as the existence of an exponentiation index for $\psi$:

(6) for all $\alpha > 1$, the limit $\lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)}$ exists.

We can now state our answer to the question above, which is the combination of Theorem 3.4 and Theorem 4.7, the main results of this paper:

**Theorem 0.2.** Let $(N, \tau)$ be a semifinite von Neumann algebra with a normal semifinite faithful trace and let $\psi$ be an element of $\Omega$ satisfying conditions (5) and (6). Then, the double equality (4) holds for all $T \in \mathcal{M}_\psi(N, \tau)^+$, all $B \in N$ and all exponentiation invariant (see (12)) states $\omega$ on $L^\infty(\mathbb{R}_1^+)$ with

$$C_\zeta(\psi) = \Gamma \left(1 + \log \left( \lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} \right) \right)^{-1} \quad \text{and} \quad C_\xi(\psi) = 1.$$

Some comments are in order. Observe first that the condition (5) is necessary for the middle term in (4) to be well defined for all $T \in \mathcal{M}_\psi(N, \tau)^+$, and that condition (6) warrants the existence of the constant $C_\zeta(\psi)$. Hence, we believe that conditions (5) and (6) are necessary as well, for the first equality in (4) to hold true. However, we do not believe that conditions (5) and (6) are necessary for the equality between the term on the left and the term on the right in (4). The reason for this is that the term on the right in (4) is well defined under a strictly weaker condition than (6).

We stress that Theorem 0.2 is new even for type $I_\infty$ factors. To illustrate it, we give in Section 5 an application to Weyl pseudo-differential operators on $\mathbb{R}^n$:

$$\text{OP}_W : S'(\mathbb{R}^{2n}) \to \mathcal{L}(S(\mathbb{R}^n), S'(\mathbb{R}^n)),$$

with symbols in the general Hörmander classes $S(m, g)$. To formulate the results obtained in this section, we set $\mathcal{M}_\psi(\mathbb{R}^{2n})$ for the commutative Lorentz space associated with $L^\infty(\mathbb{R}^{2n})$ and Lebesgue integral and $\mathcal{M}_\psi(L^2(\mathbb{R}^n))$ for the type $I_\infty$ Lorentz space associated with $\mathcal{B}(L^2(\mathbb{R}^n))$ and operator trace. We also set $\int_{\psi, \omega}$ for a Dixmier trace on $\mathcal{M}_\psi(\mathbb{R}^{2n})$, that we may call a ‘Dixmier integral’. In this context, we are able to deduce the following consequence of the relation between the Dixmier trace and the trace of the heat semigroup, as stated in Theorem 0.2:

**Theorem 0.3.** Let $\psi \in \Omega$ satisfy condition conditions (5) and (6) and let also $(g, m)$ be an Hörmander pair (see Definition 5.1) such that $m \in \mathcal{M}_\psi(\mathbb{R}^{2n})$. Then,

$$\text{OP}_W : S(m, g) \to \mathcal{M}_\psi(L^2(\mathbb{R}^n)),$$
continuously. Moreover, for any symbol \( f \in S(m,g) \) and any exponentiation invariant state \( \omega \) on \( L^\infty(\mathbb{R}^*_+) \), we have

\[
\text{Tr}_{\psi,\omega}(\text{OP}_W(f)) = \int_{\psi,\omega} f.
\]

This result is proven in the text in Theorems 5.4 and 5.6. Observe that it uses the full strength of the theory of singular traces for semifinite von Neumann algebras. It also nicely complements the classical relation between the operator trace of a Weyl pseudo-differential operator and the integral of its symbol:

\[
\text{Tr}(\text{OP}_W(f)) = \int f,
\]

whenever \( \text{OP}_W(f) \) is trace class\(^2\). In particular, the relation between the Dixmier trace and \( \zeta \)-function of Theorem 0.2 applied to \( L^\infty(\mathbb{R}^{2n}) \), gives a very elementary way to compute the Dixmier trace of an Hörmander-Weyl pseudo-differential operator:

**Theorem 0.4.** Let \( \psi \in \Omega \) satisfy conditions (5) and (6) and let also \((g,m)\) be an Hörmander pair such that \( m \in \mathcal{M}_\psi(\mathbb{R}^{2n}) \). Then, for any symbol \( f \in S(m,g) \) and any exponentiation invariant state \( \omega \) on \( L^\infty(\mathbb{R}^*_+) \), we have:

\[
\text{Tr}_{\psi,\omega}(\text{OP}_W(f)) = C_{\zeta}(\psi) \omega\left( r \mapsto \frac{1}{\psi(r)} \int_{\mathbb{R}^{2n}} f(x,\xi) \frac{|f(x,\xi)|^{1/\log(r)}}{\log(r)} \, dx \, d\xi \right),
\]

where \( C_{\zeta}(\psi) \) is the constant associated to \( \psi \), given in Theorem 0.2.

We should also mention that Theorem 0.4 gives very simple alternative proofs of [15, Proposition 4.17] and [26, Theorem 4.7].

**Acknowledgments.** It is a pleasure to thank Sergei Astashkin, Bruno Iochum, Steven Lord, Evgeni Semenov, Alexandr Usachev and Dmitriy Zanin for various discussions and suggestions during the development of this project.

1. **Preliminary results**

In this section, we first recall the basis of the theory of operator Lorentz spaces in the semifinite setting. We introduce then a subclass of Lorentz spaces on which we focus. This subclass is characterized by a single condition, which is displayed in (7). In Proposition 1.6, we compare this condition with those which are most commonly used in the literature (those being conditions (8) and (9) in the text). Next, we recall the fundamentals of singular traces on Lorentz spaces. The main role is played by the notion of \( \psi \)-compatible state. In Proposition 1.11 we construct singular traces in the context of exponentiation invariant \( \psi \)-compatible states. We also exhibit relationships between dilation invariant \( \psi \)-compatible states and Boyd indices in Proposition 1.16. We conclude this section with a spectral formula for singular traces, again in the context of exponentiation invariant states. This is done in Proposition 1.17.

\(^2\)This occurs for instance when \( f \) is a Schwartz function.
1.1. **Notations.** Set $\mathbb{N} := \{1, 2, 3, \ldots\}$, $\mathbb{N}_0 := \{0, 1, 2, \ldots\}$, $\mathbb{R}_+ = [0, \infty)$ and $\mathbb{R}_+^* = (0, \infty)$. Let $\mathcal{N}$ be a semifinite von Neumann algebra endowed with $\tau$, a faithful semifinite normal trace. We denote by $\mathcal{L}_0(\mathcal{N}, \tau) \equiv \mathcal{L}_0$ the $\ast$-algebra of $\tau$-measurable operators affiliated with $\mathcal{N}$. Recall [14] that a densely defined closed operator $T$ affiliated with $\mathcal{N}$ is $\tau$-measurable if

$$
\lim_{\lambda \to \infty} \tau(1 - E_{|T|}(\lambda)) < \infty,
$$

where $E_{|T|}$ denotes the spectral family of $|T|$. For $p \in [1, \infty)$, we set $L^p(\mathcal{N}, \tau) \equiv \mathcal{L}^p$ for the noncommutative $L^p$-space:

$$
\mathcal{L}^p := \{T \in \mathcal{L}_0 : \|T\|_p := \tau(|T|^p)^{1/p} < \infty\}.
$$

We also consider $\mathcal{F}(\mathcal{N}, \tau) \equiv \mathcal{F}$, the subset of $\mathcal{N}$ consisting of finite linear combinations of projections of finite trace $\tau$ and we denote by $\mathcal{K}(\mathcal{N}, \tau) \equiv \mathcal{K}$ the norm closure of $\mathcal{F}$ in $\mathcal{N}$. The latter is termed the algebra of $\tau$-compact operators. For $T \in \mathcal{L}_0$, a $\tau$-measurable operator, we let $\mu(T) = [t \mapsto \mu(T, t)]$ be the generalized $s$-numbers function. Recall that the latter is defined by

$$
\mu(T, t) := \inf \{\|PT\| : \tau(1 - P) \leq t\}, \quad t > 0,
$$

where the infimum runs over the lattice of projections in $\mathcal{N}$. We also define the distribution function, $n(T) = [t \mapsto n(T, t)]$, of a $\tau$-measurable operator $T$ by:

$$
n(T, t) := \tau(\chi_{(t, \infty)}(T)), \quad t > 0,
$$

where $\chi_{(t, \infty)}$ is the indicator function of the interval $(t, \infty)$. It can be shown that

$$
\mu(T, s) = \inf \{t > 0 : n(T, t) \leq s\} \quad \text{and} \quad n(T, t) \leq s \Leftrightarrow \mu(T, s) \leq t.
$$

If $T$ and $S$ belong to $\mathcal{L}_0$, we say that $T$ majorizes $S$ in the sense of Hardy-Littlewood-Pólya, denoted $S \preceq T$, if and only if

$$
\int_0^t \mu(S, s) \, ds \leq \int_0^t \mu(T, s) \, ds, \quad \forall t > 0.
$$

Let $\mathcal{E}(\mathcal{N}, \tau) \equiv \mathcal{E}$ be a Banach subspace of $L^1(\mathcal{N})$, with norm $\|\cdot\|_\mathcal{E}$. Then $\mathcal{E}$ is called a symmetric operator space if for all $A \in \mathcal{E}^+$ and $B \in (L^1 + \mathcal{N})^+$ with $B \leq A$, we have $B \in \mathcal{E}$ with $\|B\|_\mathcal{E} \leq \|A\|_\mathcal{E}$ and if for all $A \in \mathcal{E}$ and $B \in L^1 + \mathcal{N}$ with $\mu(B) = \mu(A)$, we have $B \in \mathcal{E}$ with $\|B\|_\mathcal{E} = \|A\|_\mathcal{E}$. Observe that $\mathcal{E}$ is a bimodule over $\mathcal{N}$ (or an ideal in $\mathcal{N}$ if $\mathcal{E} \subset \mathcal{N}$) since the condition above implies that for all $A \in \mathcal{E}$ and all $B, C \in \mathcal{N}$, we have $BAC \in \mathcal{E}$ with $\|BAC\|_\mathcal{E} \leq \|B\|_\mathcal{E} \|A\|_\mathcal{E} \|C\|$. If moreover for all $A \in \mathcal{E}$ and $B \in L^1 + \mathcal{N}$ with $B \preceq A$, we have $B \in \mathcal{E}$ with $\|B\|_\mathcal{E} \leq \|A\|_\mathcal{E}$, then $\mathcal{E}$ is called a fully symmetric operator space. Symmetric (resp. fully symmetric) operator spaces are the noncommutative counterparts of symmetric (resp. fully symmetric) Banach spaces of (classes of) measurable functions on $\sigma$-finite measure spaces. Note that given $(\mathcal{E}, \|\cdot\|_\mathcal{E})$, a Banach space of Lebesgue measurable functions on the interval $[0, \infty)$, we can construct an operator subspace of $\mathcal{L}_0$, by setting

$$
E(\mathcal{N}, \tau) := \{T \in \mathcal{L}_0 : \mu(T) \in \mathcal{E}\},
$$

normed with $T \mapsto \|\mu(T)\|_\mathcal{E}$. The latter is symmetric (resp. fully symmetric) when $E$ is symmetric (resp. fully symmetric). If moreover $\mathcal{N}$ is either non-atomic or a type $I_\infty$ factor, then every symmetric (resp. fully symmetric) operator space arises this way [9,12,21,31]. When a symmetric operator space $\mathcal{E}$ is of the form $E(\mathcal{N}, \tau)$, for some Banach space $E$ of functions on $[0, \infty)$, we define its fundamental function $\varphi_\mathcal{E}$ as the corresponding fundamental function of $E$:

$$
\varphi_\mathcal{E}(t) := \|\chi_{(0,t)}\|_E.
$$
In this situation, we also define the lower and upper Boyd indices of $E$, as the ordinary Boyd indices of $E$. Hence, with $D_a$, $a > 0$, the dilation operator acting on functions on $[0, \infty)$ via $D_a f(t) := f(at)$, we set
\[ p_E := \lim_{a \downarrow 0} \frac{\log a^{-1}}{\log \| D_a \|_{E \to E}}, \quad q_E := \lim_{a \to \infty} \frac{\log a^{-1}}{\log \| D_a \|_{E \to E}}. \]
Recall that we always have $1 \leq p_E \leq q_E \leq \infty$.
Whenever $f$ and $g$ are real valued function on a set $X$, we write $f \asymp g$ when there exists $0 < c \leq C$ such that for all $x \in X$, we have $c f(x) \leq g(x) \leq C f(x)$.

### 1.2. Lorentz spaces.

An important class of fully symmetric operator spaces is given by the Lorentz spaces. To introduce this class of operator spaces, we consider the following sets of concave functions

\[ \text{Definition 1.1. Let } \Omega \text{ be the set of all increasing concave functions } \psi : [0, \infty) \to [0, \infty) \text{ such that } \psi(0) = 0 \quad \text{and} \quad \lim_{t \to \infty} \psi(t) = \infty, \]

and let $\Omega_b$ be the subset of $\Omega$ whose elements satisfy further:
\[ \psi(t) = O(t), \quad t \to 0. \]

Associated with an element $\psi \in \Omega$, one defines the operator Lorentz space $\mathcal{M}_\psi(\mathcal{N}, \tau) \equiv \mathcal{M}_\psi$ to be the collection of all $\tau$-measurable operators, such that
\[ \| T \|_{\mathcal{M}_\psi} := \sup_{t > 0} \frac{1}{\psi(t)} \int_0^t \mu(T, s) \, ds < \infty. \]

We denote by $\mathcal{M}_\psi^0$, the closure of $\mathcal{F}$ in $\mathcal{M}_\psi$ and by $\mathcal{M}_\psi^+$, the positive cone of $\mathcal{M}_\psi$. We also use the notation $\mathcal{M}_\psi(\mathbb{R}_+)$ for the commutative Lorentz space of (classes of) functions on the positive half-line, i.e. for $\mathcal{N} = L^\infty(\mathbb{R}_+)$ and $\tau = \int_0^\infty \cdot \, dx$. Note that $t/\psi(t)$ is the fundamental function of $\mathcal{M}_\psi$ and that $\mathcal{M}_\psi$ is continuously embedded in $\mathcal{N}$ if and only if $\psi \in \Omega_b$ or equivalently if $\psi' \in L^\infty(\mathbb{R}_+)$. Indeed, as $\psi' \in \mathcal{M}_\psi(\mathbb{R}_+)$ and $\| \psi' \|_\infty = \lim_{t \downarrow 0} \psi'(t)$, we deduce that
\[ \mathcal{M}_\psi \subset \mathcal{N} \quad \Rightarrow \quad \psi' \in L^\infty(\mathbb{R}_+) \quad \Rightarrow \quad \frac{\psi(t)}{\psi'(0)} \leq t. \]

Conversely, if $\psi(t) \leq Ct$ for $t \in (0, 1)$, we obtain
\[ \| T \|_{\mathcal{M}_\psi} \geq \int_0^1 \frac{\mu(T, s)}{\psi(t)} \, ds \geq \frac{\mu(T, t)}{C}, \quad \forall t \in (0, 1), \quad \forall T \in \mathcal{M}_\psi, \]
which entails that $T \in \mathcal{N}$ as
\[ \| T \| = \lim_{t \downarrow 0} \mu(T, t). \]

This justifies our notation for $\Omega_b$. Observe that when $\psi \in \Omega_b$, then the Lorentz space $\mathcal{M}_\psi$ consists of bounded operators and therefore the latter is an ideal in $\mathcal{N}$. The distinction between $\Omega$ and $\Omega_b$, plays a minor role in this paper. In fact, this distinction will only appear in Section 2 (which is also of independent interest) while in Sections 3 and 4 even for $\psi \in \Omega \setminus \Omega_b$, we are forced to consider the elements of $\mathcal{M}_\psi \cap \mathcal{N}$ only.

\[ \text{We warn the reader that in many sources, the dilation operator } D_a \text{ is defined with } a^{-1} \text{ instead of } a. \]
In this article, we are mostly interested in Lorentz spaces associated with the subset of $\Omega$ whose elements are characterized by the additional condition:

$$\forall \alpha > 1, \quad \text{the limit} \quad A_\psi(\alpha) := \lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} \quad \text{exists.} \quad (7)$$

We will see in a moment that condition (7) is strictly stronger than the condition used in [30]:

$$\lim_{t \to \infty} \frac{\psi(t^\alpha \psi(t))}{\psi(t)} = 1, \quad (8)$$

and that the latter is strictly stronger than the standard condition (used in particular in [11] and [8]):

$$\forall a > 1, \quad \lim_{t \to \infty} \frac{\psi(at)}{\psi(t)} = 1. \quad (9)$$

We first observe that Theorem 6 of [20], proved there for extended limits, can be adapted to the ordinary limit:

**Lemma 1.2.** Let $\psi \in \Omega$. Then condition (9) is equivalent to any of the following two conditions:

$$\exists a > 1 : \lim_{t \to \infty} \frac{\psi(at)}{\psi(t)} = 1, \quad \lim_{t \to \infty} \frac{t\psi'(t)}{\psi(t)} = 0. \quad (9')$$

**Proof.** Call (9) the condition $\exists a > 1 : \lim_{t \to \infty} \frac{\psi(at)}{\psi(t)} = 1$ and (9') the condition $\lim_{t \to \infty} \frac{t\psi'(t)}{\psi(t)} = 0$.

Observe first that the concavity and the monotony of $\psi$, imply that for any $a > 1$ there exists $C > 0$ such that:

$$0 \leq \frac{at\psi'(at)}{\psi(at)} \leq C \frac{\psi(at) - \psi(t)}{\psi(t)},$$

and it also imply that for any $a > 1$:

$$1 \leq \frac{\psi(at)}{\psi(t)} \leq 1 + (a - 1)\frac{t\psi'(t)}{\psi(t)}.$$

Hence we deduce that (9) $\Rightarrow$ (9) $\Rightarrow$ (9') $\Rightarrow$ (9). \hfill $\square$

We next observe a simple but important consequence of condition (7):

**Lemma 1.3.** Let $\psi \in \Omega$ satisfy condition (7). Then we have $A_\psi(\alpha) = a^{\log A_\psi(e)}$ and in particular

$$\lim_{\alpha \to 1} A_\psi(\alpha) = 1.$$

**Proof.** Observe first that if $A_\psi(\alpha)$ exists for all $\alpha > 1$, then it exists for all $\alpha > 0$. Indeed, we clearly have $A_\psi(1/\alpha) = 1/A_\psi(\alpha)$. Analogously, we also observe that $A_\psi(\alpha \beta) = A_\psi(\alpha)A_\psi(\beta)$ for all $\alpha, \beta > 0$. These two observations entail that $A_\psi(\alpha) = a^{\log A_\psi(e)}$ for all $\alpha \in \exp\{\mathbb{Q}\}$. As the function $[\alpha \mapsto A_\psi(\alpha)]$ is increasing, we conclude from what precedes that it is also continuous and thus $A_\psi(\alpha) = a^{\log A_\psi(e)}$ for all $\alpha > 0$. \hfill $\square$

**Definition 1.4.** In the sequel, for $\psi \in \Omega$ satisfying condition (7), we set

$$k_\psi := \log (A_\psi(e)),$$

and we observe that we always have $A_\psi(e) \geq 1$ and thus $k_\psi \geq 0$. 

Remark 1.5. Note also the equivalence of the following two conditions:

\[ \forall \alpha > 1, \quad \lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} = 1 \quad \iff \quad \exists \alpha > 1 : \lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} = 1. \]

Indeed, assume that there exists \( \alpha_0 > 1 \) such that \( \lim_{t \to \infty} \frac{\psi(t^{\alpha_0})}{\psi(t)} = 1 \). Then for \( \alpha \in (1, \alpha_0] \) and \( t \geq 1 \), we deduce

\[ 1 \leq \frac{\psi(t^\alpha)}{\psi(t)} \leq \frac{\psi(t^{\alpha_0})}{\psi(t)}, \]

which implies that \( \lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} = 1 \). For \( \alpha \in (\alpha_0, \infty) \), fix \( \beta \in (1, \alpha_0] \) and \( n \in \mathbb{N} \) such that \( \beta^n \geq \alpha \). Then we get

\[ 1 \leq \frac{\psi(t^\alpha)}{\psi(t)} \leq \frac{\psi(t^{\beta^n})}{\psi(t)} = \frac{\psi(t^{\beta^{n-1}})}{\psi(t^{\beta^{n-2}})} \cdot \ldots \cdot \frac{\psi(t^{1})}{\psi(t)}, \]

and the assertion follows from the preceding argument.

Example 1. Consider the functions

\[ \psi_{n,\beta}(t) := \left( \log(1 + \log(1 + \cdots \log(1 + t^{1/\beta}) \ldots)) \right)^\beta, \quad \beta > 0, \quad n \text{ iterations}, \]

and for a suitable constant \( C > 0 \)

\[ \tilde{\psi}_{n,\beta}(t) := \begin{cases} C^{-1} t, & t \in [0, C] \\ \exp \left\{ \log(\cdots \log(t) \cdots) \right\}^\beta, & t > C \end{cases}, \quad \beta \in (0, 1), \quad n \text{ iterations}. \]

It is clear that all these functions belong to \( \Omega_b \). Moreover, \( \psi_{n,\beta} \) satisfies condition (7), with \( k_{\psi} = \beta \) when \( n = 1 \) and \( k_{\psi} = 0 \) when \( n > 1 \). Also, \( \tilde{\psi}_{n,\beta} \) satisfies condition (7) for \( n > 1 \) only, with \( k_{\psi} = 0 \).

Proposition 1.6. Condition (7) is strictly stronger than condition (8) and the latter is strictly stronger than condition (9).

Proof. First, we prove that condition (7) implies condition (9). So, fix \( a > 1 \). Since \( \psi \) is increasing, we have

\[ 1 \leq \liminf_{t \to \infty} \frac{\psi(at)}{\psi(t)}. \]

Then, choose \( \alpha > 1 \) arbitrary. Since \( at \leq t^\alpha \) for \( t \geq a^{1/(\alpha-1)} \), we deduce by Lemma 1.3

\[ \limsup_{t \to \infty} \frac{\psi(at)}{\psi(t)} \leq \limsup_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} = \lim_{t \to \infty} \frac{\psi(t^\alpha)}{\psi(t)} = \alpha^{k_{\psi}}. \]

As \( \alpha > 1 \) is arbitrary, we get

\[ \limsup_{t \to \infty} \frac{\psi(at)}{\psi(t)} \leq 1, \]

and the claim follows. Next, observe that

\[ 1 \leq \liminf_{t \to \infty} \frac{\psi(t \psi(t))}{\psi(t)}. \]

This example generalizes [30, Example 6].
Then, choose $\varepsilon > 0$ arbitrary. By [4, Lemma 4.1], condition (9) implies that $\psi(t) \leq C_\varepsilon t^\varepsilon$. Thus from Lemma [13] and what we have already proven, we get:

$$\limsup_{t \to \infty} \frac{\psi(t^2 \psi(t))}{\psi(t)} \leq \limsup_{t \to \infty} \frac{\psi(C_\varepsilon t^{1+\varepsilon})}{\psi(t)} = \lim_{t \to \infty} \frac{\psi(C_\varepsilon t^{1+\varepsilon}) \psi(t^{1+\varepsilon})}{\psi(t)} = (1 + \varepsilon)^k_\psi.$$ 

As $\varepsilon > 0$ is arbitrary, we deduce that

$$\limsup_{t \to \infty} \frac{\psi(t^2 \psi(t))}{\psi(t)} \leq 1,$$

and thus condition (7) implies condition (8). That condition (8) implies condition (9) is obvious. To see that all these implications are strict, we consider the family of elements $\psi_\beta := \hat{\psi}_{1, \beta} \in \Omega_\beta$, $\beta \in (0, 1)$, where $\hat{\psi}_{n, \beta}$ is defined in (11). Condition (9) is satisfied for all $\beta \in (0, 1)$. Indeed, for $t > 1$ and $a > 1$, we have

$$\frac{\psi_\beta(at)}{\psi(t)} = \exp \left\{ \log(t)^\beta \left( \left( 1 + \frac{\log(a)}{\log(t)} \right)^\beta - 1 \right) \right\} \sim \exp \left\{ \beta \log(a) \log(t)^{\beta-1} \right\} \to 1, \quad t \to \infty.$$ 

Condition (8) is satisfied for $\beta \in (0, 1/2)$ but not for $\beta \in [1/2, 1)$:

$$\frac{\psi_\beta(t^2 \psi(t))}{\psi_\beta(t)} = \exp \left\{ \log(t)^\beta \left( \left( 1 + \log(t)^{\beta-1} \right)^\beta - 1 \right) \right\} \sim \exp \left\{ \beta \log(t)^{2\beta-1} \right\} \to \begin{cases} 
1, & \beta \in (0, \frac{1}{2}) \\
\sqrt{e}, & \beta = \frac{1}{2}, t \to \infty. \\
\infty, & \beta \in \left(\frac{1}{2}, 1\right) 
\end{cases}$$

Condition (7) is never satisfied. Indeed, for $\alpha > 1$, we have:

$$\frac{\psi_\beta(t^\alpha)}{\psi_\beta(t)} = \exp \left\{ (\alpha^\beta - 1) \log(t)^\beta \right\} \to \infty, \quad t \to \infty.$$ 

This concludes the proof. □

1.3. Dixmier traces and $\psi$-compatible states. We say that a positive continuous linear functional $\varphi : \mathcal{M}_\psi \to \mathbb{C}$ is singular if its restriction to $\mathcal{M}_\psi^0$ is zero, symmetric if $\varphi(A) = \varphi(B)$ for all $A, B \in \mathcal{M}_\psi^+$ such that $\mu(A) = \mu(B)$ and fully symmetric if moreover $\varphi(A) \leq \varphi(B)$ for all $A, B \in \mathcal{M}_\psi^+$ such that $A \prec \prec B$. Note that every fully symmetric functional is symmetric and bounded and if $\mathcal{M}_\psi \not\subset \mathcal{L}^1$, then every symmetric functional is singular.

For $a > 0$, we define the translation $T_a$, dilation $D_a$ and exponentiation $E_a$ operators on $L^\infty(\mathbb{R}^*_+)$ by

$$T_a f(t) := f(t + a), \quad D_a f(t) := f(at), \quad E_a f(t) := f(t^a), \quad f \in L^\infty(\mathbb{R}^*_+).$$

A continuous linear functional $\omega \in (L^\infty(\mathbb{R}^*_+))^*$ is said to be translation invarient, dilation invariant or exponentiation invariant if for all $f \in L^\infty(\mathbb{R}^*_+)$ and all $a > 0$, we have

(12) \hspace{1cm} \omega(T_a f) = \omega(f), \quad \omega(D_a f) = \omega(f) \quad \text{or} \quad \omega(E_a f) = \omega(f).$

Definition 1.7. A state $0 \leq \omega \in (L^\infty(\mathbb{R}^*_+))^*$ is called singular if it vanishes on functions converging to zero at infinity.

We observe that translation, dilation and exponentiation invariant states are singular. Singular states may also be called extended limits, as they are extensions (via the Hahn-Banach Theorem) of the ordinary limit functional on the linear subset of $L^\infty(\mathbb{R}^*_+)$ consisting in functions converging at infinity.
It is proven in [13, Theorem 3.4], that a given Lorentz space \( \mathcal{M}_\psi \) possesses a nontrivial fully symmetric singular trace supported at infinity (i.e. which vanishes on \( \mathcal{M}_\psi \cap L^1 \)) if and only if
\[
\exists a > 1 : \liminf_{t \to \infty} \frac{\psi(at)}{\psi(t)} = 1.
\]
Thus, Proposition 1.6 shows that the Lorentz spaces associated with the elements of \( \Omega \) satisfying condition (13) do possess nontrivial singular traces.

**Remark 1.8.** As in Lemma 1.2, the arguments of [20, Theorem 6] can be generalized to show that condition (13) is equivalent to any of the following two conditions:
\[
\forall a > 1, \liminf_{t \to \infty} \frac{\psi(at)}{\psi(t)} = 1, \quad \liminf_{t \to \infty} \frac{t\psi'(t)}{\psi(t)} = 0.
\]
The most important class of fully symmetric singular traces is certainly the class of Dixmier traces. The most general way to construct Dixmier traces comes from [20], where we borrow the following notion:

**Definition 1.9.** A state \( 0 \leq \omega \in (L^\infty(\mathbb{R}^*_+))^* \) is said to be \( \psi \)-compatible if there exists \( a > 1 \) such that
\[
\omega \left( \left[ t \mapsto \frac{\psi(at)}{\psi(t)} \right] \right) = 1.
\]
It is proven in [20, Proposition 10] that given \( \omega \), a \( \psi \)-compatible and dilation invariant state on \( L^\infty(\mathbb{R}^*_+) \), the (unitarily invariant and singular) functional
\[
\tau_{\psi,\omega} : \mathcal{M}_\psi^+ \to \mathbb{R}_+, \quad T \mapsto \omega \left( \left[ t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T,s) \, ds \right] \right),
\]
is positively additive (and positively homogeneous too), and its extension by linearity on \( \mathcal{M}_\psi \) is called a Dixmier trace. As proven in [20, Proposition 10], it is a fully symmetric normalized tracial functional on \( \mathcal{M}_\psi \). Surprisingly enough, the converse also holds true (see [20, Theorem 11]): every fully symmetric normalized tracial linear functional on \( \mathcal{M}_\psi \) is a Dixmier trace.

**Remark 1.10.** Observe that when \( \psi \in \Omega \) satisfies condition (9), then every singular state is \( \psi \)-compatible. Hence, Proposition 1.6 shows that it is still the case when condition (8) or condition (7) is satisfied.

The next result is strongly inspired from [32, Proposition 5]. It relates an analogous notion of \( \psi \)-compatibility in the context of exponentiation invariant states, to the existence of Dixmier traces:

**Proposition 1.11.** Let \( \omega \) be an exponentiation invariant state of \( L^\infty(\mathbb{R}^*_+) \) and \( \psi \in \Omega \), such that
\[
\lim_{\alpha \downarrow 1} \omega \left( \left[ t \mapsto \frac{\psi(t^\alpha)}{\psi(t)} \right] \right) = 1.
\]
Then the functional
\[
\tau_{\psi,\omega} : \mathcal{M}_\psi^+ \to [0, \infty), \quad T \mapsto \omega \left( \left[ t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T,s) \, ds \right] \right),
\]

\( ^5 \)A functional \( \varphi \) on \( \mathcal{M}_\psi \) is normalized if \( \varphi(T) = 1 \) for all positive \( T \in \mathcal{M}_\psi \) such that \( \mu(T) = \psi' \).
is positively additive. Since its extension by linearity is fully symmetric and normalized, there exists a $\psi$-compatible and dilation invariant state $\omega'$, such that this functional coincides with the Dixmier trace $\tau_{\psi,\omega'}$.

**Proof.** We just have to prove additivity on $\mathcal{M}^+_\psi$. So let $T, S \in \mathcal{M}^+_\psi$. By [14, Theorem 4.4 (ii)], we have

$$
\int_0^{t/2} (\mu(T, s) + \mu(S, s)) \, ds \leq \int_0^t \mu(T + S, s) \, ds \leq \int_0^t (\mu(T, s) + \mu(S, s)) \, ds,
$$

which yields by positivity of $\omega$:

$$
\omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t (\mu(T, s) + \mu(S, s)) \, ds \right]\right) \leq \omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T + S, s) \, ds \right]\right),
$$

and

$$
\omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t (\mu(T, s) + \mu(S, s)) \, ds \right]\right) \leq \omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^{2t} \mu(T + S, s) \, ds \right]\right).
$$

Since for every $\varepsilon > 0$ and $t$ sufficiently large we have $2t \leq t^{1+\varepsilon}$, we deduce

$$
\omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t (\mu(T, s) + \mu(S, s)) \, ds \right]\right) \leq \omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^{t^{1+\varepsilon}} \mu(T + S, s) \, ds \right]\right).
$$

On the other hand, for an arbitrary $0 \leq g \in L^\infty(\mathbb{R}_+^*)$, as $\psi(t^{1+\varepsilon})/\psi(t) \geq 1$ for $t \geq 1$ and $\varepsilon > 0$, we deduce from our hypotheses on $\omega$:

$$
0 \leq \omega\left(\left[ t \mapsto \left(\frac{\psi(t^{1+\varepsilon})}{\psi(t)} - 1\right) g(t) \right]\right) \leq \|g\|_\infty \omega\left(\left[ t \mapsto \frac{\psi(t^{1+\varepsilon})}{\psi(t)} - 1\right]\right) \to 0, \quad \varepsilon \to 0.
$$

Since,

$$
\sup_{t > 0} \frac{1}{\psi(t^{1+\varepsilon})} \int_0^{t^{1+\varepsilon}} \mu(T + S, s) \, ds = \|T + S\|_{\mathcal{M}_\psi}, \quad \forall \varepsilon > 0,
$$

we deduce

$$
\omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t (\mu(T, s) + \mu(S, s)) \, ds \right]\right) \leq \|T + S\|_{\mathcal{M}_\psi} \omega\left(\left[ t \mapsto \frac{\psi(t^{1+\varepsilon})}{\psi(t)} - 1\right]\right)
$$

$$
+ \omega\left(\left[ t \mapsto \frac{1}{\psi(t^{1+\varepsilon})} \int_0^{t^{1+\varepsilon}} \mu(T + S, s) \, ds \right]\right).
$$

By exponentiation invariance of $\omega$, we obtain

$$
\omega\left(\left[ t \mapsto \frac{1}{\psi(t^{1+\varepsilon})} \int_0^{t^{1+\varepsilon}} \mu(T + S, s) \, ds \right]\right) = \omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T + S, s) \, ds \right]\right)
$$

Thus, taking the limit $\varepsilon \to 0$, we get

$$
\omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t (\mu(T, s) + \mu(S, s)) \, ds \right]\right) \leq \omega\left(\left[ t \mapsto \frac{1}{\psi(t)} \int_0^t \mu(T + S, s) \, ds \right]\right).
$$

The last statement follows from [20, Theorem 11].

**Corollary 1.12.** Let $\psi \in \Omega$ satisfying condition (7). Then for any exponentiation invariant state $\omega$, the linear extension of the map [16] defines a Dixmier trace on $\mathcal{M}_\psi$. 

\[\square\]
Proof. This follows from Lemma 1.3.

Remark 1.13. Observe that condition (15) is not satisfied for the elements \( \tilde{\psi}_{1,\beta} \), \( \beta \in (0, 1) \) given in (11). Indeed, we have already seen that in this case \( \lim_{t \to \infty} \psi(t^a)/\psi(t) = \infty \) for all \( \alpha > 1 \).

The goal of the next result is to clarify the relationship between the notion of \( \psi \)-compatibility and conditions (13) and (9). The first part of this result is implicitly contained in [20] and largely discussed in [23]. For convenience of the reader, we supply the detailed argument.

Proposition 1.14. Let \( \psi \in \Omega_b \).
(i) Condition (13) is satisfied if and only if there exists a \( \psi \)-compatible dilation invariant state.
(ii) Condition (9) is satisfied if and only if every translation invariant state is \( \psi \)-compatible.

Proof. By Remark 1.8 we may assume that \( a = 2 \) in condition (13). If a \( \psi \)-compatible dilation invariant state \( \omega \) exists, then by [20, Proposition 10] the map \( \tau_{\psi,\omega} \) is a fully symmetric singular trace on \( \mathcal{M}_\psi \), which by [13, Theorem 3.4] implies (since \( \psi \in \Omega_b \) is equivalent to \( \mathcal{M}_\psi \subset \mathcal{N} \) and thus nontrivial singular traces are necessarily supported at infinity) that \( \liminf_{t \to \infty} \psi(2t)/\psi(t) = 1 \). (See also the discussion prior to Definition 9 in [20] for a direct proof of this fact.) Now assume that condition (13) holds. Then, by [23, Lemma 6.3.4], there exists a dilation invariant state \( \omega \in (L^\infty(\mathbb{R}_+^*))^* \), such that \( \omega([t \mapsto t\psi'(t)/\psi(t)]) = 0 \). Arguing as in Lemma 1.2, we see that \( \omega \) is \( \psi \)-compatible as well. This proves the first equivalence. We next prove the second equivalence. Clearly, if the condition (9) holds, then every singular state is \( \psi \)-compatible. In particular every translation invariant state is \( \psi \)-compatible. Next assume that every translation invariant state is \( \psi \)-compatible. By [20, Theorem 6], the condition of \( \psi \)-compatibility of Definition 1.9 is equivalent to \( \omega([t \mapsto \psi(at)/\psi(t)]) = 1 \), for all \( a > 1 \). By Lemma 1.2 we can therefore prove condition (9) with \( a = 2 \). This means that the map \( [t \mapsto \psi(2t)/\psi(t)] \) is almost convergent (in the sense of Lorentz [24]). By [22, Theorem 3.3], this implies that

\[
\lim_{u \to \infty} \frac{1}{u} \int_0^u \frac{\psi(2t)}{\psi(t)} \, dt = 1. \tag{17}
\]

Last, by [16, Section 6.8] if there exists \( C > 0 \) such that

\[
t \frac{d}{dt} \left( \frac{\psi(2t)}{\psi(t)} \right) \geq -C, \tag{18}
\]

then (17) is equivalent to (9). Hence, we just need to prove the inequality (18) to conclude the proof. For this, we remark that

\[
t \frac{d}{dt} \left( \frac{\psi(2t)}{\psi(t)} \right) = \frac{t^2 \psi'(2t)\psi(t) - t \psi(2t)\psi'(t)}{\psi(t)^2} \geq - \frac{t^2 \psi(2t)\psi'(t)}{\psi(t)^2} = - \frac{t \psi'(t) \psi(2t)}{\psi(t) \psi(t)},
\]

and observing that by the concavity of \( \psi \), we have:

\[
0 \leq \frac{t \psi'(t)}{\psi(t)} \leq 1, \quad 1 \leq \frac{\psi(2t)}{\psi(t)} \leq 1 + \frac{t \psi'(t)}{\psi(t)} \leq 2,
\]

the result follows with \( C = 2 \).

As the property (9) also implies that every dilation invariant state of \( L^\infty(\mathbb{R}_+^*) \) is \( \psi \)-compatible, we deduce the following:
Corollary 1.15. If every translation invariant state of $L^\infty(\mathbb{R}^*_+)$ is $\psi$-compatible, then every dilation invariant state of $L^\infty(\mathbb{R}^*_+)$ is $\psi$-compatible.

We next show that the existence of dilation invariant and $\psi$-compatible states is also related to the value of the Boyd indices of the associated Lorentz spaces.

Proposition 1.16. Let $\psi \in \Omega_b$.
(i) There exists a dilation invariant and $\psi$-compatible state if and only if $p_{M\psi} = 1$.
(ii) Every dilation invariant state is $\psi$-compatible if and only if $p_{M\psi} = q_{M\psi} = 1$.

Proof. For the first part, we observe by [3, Proposition 2.3], that condition (13) is equivalent to $p_{M\psi} = 1$. One then concludes with Proposition 1.14 (i).

For the second assertion, we start by [20, Theorem 8], which shows that every dilation-invariant state is $\psi$-compatible if and only if
\[
\forall \varepsilon > 0, \exists C > 0 : \forall t > 0, \forall s > 1, \psi(st) \leq C s^\varepsilon \psi(t).
\]
(19)

On the other hand, we have by definition of the upper Boyd index of $M\psi$:
\[
q_{M\psi} = \lim_{a \to \infty} \frac{\log a^{-1}}{\log \| D_a \|_{M\psi \to M\psi}}.
\]
Thus $q_{M\psi} = 1$ if and only if $\| D_a \|_{M\psi \to M\psi} = a^{-1+o(1)}$, $a \to \infty$. Next, we observe that
\[
\| D_a \|_{M\psi \to M\psi} = \sup_{t > 0} \frac{\psi(at)}{a \psi(t)}.
\]
(20)

Indeed, as $\psi'$ belongs to the unit sphere of $M\psi(\mathbb{R}^*_+)$ and as $\psi'$ is decreasing, we first deduce:
\[
\| D_a \|_{M\psi \to M\psi} \geq \| D_a \psi' \|_{M\psi}.
\]

Also, since for an arbitrary $f \in M\psi(\mathbb{R}^*_+)$ we have $\mu(f) \ll \| f \|_{M\psi} \psi'$ and $\mu(D_a f) = D_a \mu(f)$, we obtain
\[
\frac{\| D_a f \|_{M\psi}}{\| f \|_{M\psi}} \leq \| D_a \psi' \|_{M\psi},
\]
and (20) follows from the computation:
\[
\| D_a \psi' \|_{M\psi} = \sup_{t > 0} \frac{1}{\psi(t)} \int_0^t \psi'(as) ds = \sup_{t > 0} \frac{\psi(at)}{a \psi(t)}.
\]

Hence $q_{M\psi} = 1$ if and only if
\[
\sup_{t > 0} \frac{\psi(at)}{\psi(t)} = a^{o(1)}, \quad a \to \infty,
\]
which is equivalent to
\[
(21) \quad \forall \varepsilon > 0, \exists a > 1 : \forall t > 0, \psi(at) \leq a^\varepsilon \psi(t).
\]

All that remains to do is to show that the conditions (19) and (21) are equivalent. We first assume that condition (19) is satisfied. Then fix $\varepsilon > 0$ and let $C(\varepsilon/2)$ be the constant given in condition (19) for $\varepsilon/2$ (we may assume this constant to be strictly greater than one, since in the opposite case the conclusion is obvious). Define then
\[
a(\varepsilon) := C(\varepsilon/2)^{2/\varepsilon} > 1.
\]
We then get that for all \( t > 0 \)
\[
\psi(a(\varepsilon)t) \leq C(\varepsilon/2)a(\varepsilon)^{\varepsilon/2}\psi(t) = C(\varepsilon/2)^2\psi(t) = a(\varepsilon)^\varepsilon\psi(t),
\]
which is condition (21). Next, assume that condition (21) holds. Fix \( \varepsilon > 0 \) and let \( a(\varepsilon) > 1 \) as given by condition (21). For an arbitrary \( s > 1 \), let \( n \in \mathbb{N} \) be such that \( s \in [a(\varepsilon)^n, a(\varepsilon)^{n+1}) \). By monotony of \( \psi \) and using repeatedly condition (21), we get for arbitrary \( t > 0 \):
\[
\psi(st) \leq a(\varepsilon)^{n+1}\psi(t) \leq a(\varepsilon)^\varepsilon a(\varepsilon)^n\psi(t) \leq \cdots \leq a(\varepsilon)^{(n+1)\varepsilon}\psi(t) \leq a(\varepsilon)^\varepsilon s^\varepsilon\psi(t),
\]
which is condition (19) with constant \( C(\varepsilon) = a(\varepsilon)^\varepsilon \). \( \square \)

In the last result of this section, we generalize [8, Proposition 4.3]:

**Proposition 1.17.** Let \( \omega \) be an exponentiation invariant state of \( L^\infty(\mathbb{R}^n_+) \) and \( \psi \in \Omega \). If condition (15) holds and if
\[
\forall \varepsilon \in (0, 1), \exists C > 0 : \forall t > 1, \psi(t) < Ct^\varepsilon,
\]
then for every \( T \in \mathcal{M}_\psi^+ \), we have
\[
\tau_{\psi, \omega}(T) = \omega\left( \left[ t \mapsto \frac{\tau(T\chi_{[1, \infty)})(T)}{\psi(t)} \right] \right) = \omega\left( \left[ t \mapsto \frac{\tau(T\chi_{[1, 1]})(T)}{\psi(t)} \right] \right).
\]

**Proof.** Under the condition (22) and by exactly the same arguments than those of [8, Proposition 4.3], we get that for any \( \varepsilon > 0 \) and \( t > 0 \) sufficiently large:
\[
\int_0^t \mu(T,s)dt \leq \tau(T\chi_{[1, \infty)})(T) + C \leq \int_0^{t^{1+\varepsilon}} \mu(T,s)dt + C,
\]
for an absolute constant \( C > 0 \). Dividing these inequalities by \( \psi(t) \), applying \( \omega \) and using the same argument as in Proposition 1.11, we get the result. The argument for the second equality is identical to those of [8, Corollary 4.4]. \( \square \)

**Remark 1.18.** Gathering Remark 1.13, Proposition 1.6 and [8, Lemma 4.1], we deduce that the condition (22) is satisfied for \( \psi \in \Omega \) verifying condition (7).

2. An extrapolation description of Lorentz spaces close to \( L^1 \)

This section contains our chief innovation to study singular traces on generic Lorentz spaces. It relies on a characterization of Lorentz spaces which possess a description as an extrapolated space. Our results (partly) extend [1,2] in the noncommutative setting and for Lorentz spaces which are close to \( L^1 \). (The authors of [1,2] study the commutative and dual situation of Lorentz spaces which are close to \( L^\infty \).) We first introduce the category of operator spaces close to \( L^1 \). Applied to Lorentz spaces, we show in Corollary 2.3 that this notion is closely related to the values of the Boyd indices. We then construct an extrapolation functor, \( \mathcal{L} \), from the category of Banach lattice ideals of measurable functions on the interval \( (1, \infty) \), to the category of fully symmetric operator spaces on any semifinite von Neumann algebra. This leads us to introduce the subcategory \( \mathcal{X}_1 \) of extrapolated operator spaces, given by the intersection of the image of our extrapolation functor with the category of operator spaces close to \( L^1 \). We show in Proposition 2.8 that the objects in \( \mathcal{X}_1 \) have trivial Boyd indices. We apply our extrapolation functor to a pair \( (F_\psi, F^\psi) \) of Banach spaces of functions on \( (1, \infty) \), associated to any element \( \psi \in \Omega \) and call \((\mathcal{L}_\psi, \mathcal{L}^\psi)\) the resulting pair of fully symmetric operator spaces. The first important result of
this section is Proposition 2.13, where we prove that one always has \( \mathcal{L}_p(\psi) \subset \mathcal{M}_\psi \subset \mathcal{L}_p(\psi) \) with continuous inclusions. We then give different characterizations of when \( \mathcal{M}_\psi = \mathcal{L}_p(\psi) \) and \( \mathcal{L}_p(\psi) = \mathcal{M}_\psi \). This is done in Proposition 2.14 and Proposition 2.15 respectively. We conclude by exhibiting a convenient sufficient condition to have the equality \( \mathcal{L}_p(\psi) = \mathcal{M}_\psi \), which in turn, is one of the the main step in the analysis we perform in section 3. This is done in Proposition 2.18.

2.1. Lorentz spaces close to \( \mathcal{L}^1 \).

**Definition 2.1.** A symmetric operator space is said to be close to \( \mathcal{L}^1 \), if it is continuously embedded in \( \mathcal{L}^p \) for all \( p > 1 \), and if it is not contained in \( \mathcal{L}^1 \).

In the context of Lorentz spaces, we have an easy characterization of those which are close to \( \mathcal{L}^1 \):

**Lemma 2.2.** Let \( \psi \in \Omega \). Then \( \mathcal{M}_\psi \) is close to \( \mathcal{L}^1 \) if and only if \( \psi' \in L^p(\mathbb{R}^+) \) for all \( p > 1 \) and \( \psi' \notin L^1(\mathbb{R}^+) \).

**Proof.** Note first that for \( T \in \mathcal{M}_\psi \) and since \( \psi(0) = 0 \):

\[
\int_0^t \mu(T, s) ds \leq \|T\|_{\mathcal{M}_\psi} \psi(t) = \|T\|_{\mathcal{M}_\psi} \int_0^t \psi(s) ds, \quad \forall t > 0.
\]

Thus \( \mu(T) \ll \|T\|_{\mathcal{M}_\psi} \psi' \), which implies that for all \( p \geq 1 \), \( \mu(T)^p \ll \|T\|_{\mathcal{M}_\psi}^p \psi^p \), that is

\[
\int_0^t \mu(|T|^p, s) ds = \int_0^t \mu(T, s)^p ds \leq \|T\|_{\mathcal{M}_\psi}^p \int_0^t \psi(s)^p ds, \quad \forall t > 0.
\]

So taking the limit \( t \to \infty \) for \( p > 1 \), we deduce:

\[
(23) \quad \|T\|_p \leq \|T\|_{\mathcal{M}_\psi} \|\psi'\|_p, \quad \forall T \in \mathcal{M}_\psi.
\]

Assume first that \( \mathcal{M}_\psi \) is close to \( \mathcal{L}^1 \). We then observe that \( \psi' \in L^p(\mathbb{R}^+) \) for all \( p > 1 \) as \( \psi' \in \mathcal{M}_\psi(\mathbb{R}^+) \).

Now, as \( \mathcal{M}_\psi \not\subset \mathcal{L}^1 \), there exists \( T_0 \in \mathcal{M}_\psi \) such that \( \|T_0\|_1 = \infty \). Taking the limit \( p \to 1 \) in (23), gives

\[
\lim_{p \to 1} \|\psi'\|_p \geq \lim_{p \to 1} \frac{\|T_0\|_p}{\|T_0\|_{\mathcal{M}_\psi}} = \infty,
\]

and the first implication is proven. Next, assume the second condition. Then the inequality \( 23 \) shows that \( \mathcal{M}_\psi \subset \mathcal{L}^p \) for all \( p > 1 \). But \( \mathcal{M}_\psi \not\subset \mathcal{L}^1 \) as \( \psi' \in \mathcal{M}_\psi(\mathbb{R}^+) \) and \( \psi' \notin L^1(\mathbb{R}^+) \). This proves the second implication. \( \square \)

**Remark 2.3.** In the light of the previous result, it is clear that the Lorentz spaces associated with the elements of \( \Omega_0 \) given in (10) and (11), are close to \( \mathcal{L}^1 \).

**Lemma 2.4.** Let \( \psi \in \Omega \) be such that \( 22 \) holds. Then \( \mathcal{M}_\psi \) is close to \( \mathcal{L}^1 \).

**Proof.** Observe that since \( \psi(0) = 0 \), we have \( \|\psi'\|_1 = \lim_{t \to \infty} \psi(t) = \infty \) and thus \( \psi' \notin L^1(\mathbb{R}^+) \). By the concavity of \( \psi \), we deduce that \( t \psi'(t) \leq \psi(t) \) and thus for \( t > 1 \), we get \( \psi'(t) \leq C(\varepsilon) t^{-1+\varepsilon} \) for all \( \varepsilon \in (0, 1) \). Hence \( \psi' \in L^p(\mathbb{R}^+) \) for all \( p > (1 - \varepsilon)^{-1} \). As \( \varepsilon \in (0, 1) \) is arbitrary, \( \psi' \in L^p(\mathbb{R}^+) \) for all \( p > 1 \). The proof then follows by Lemma 2.2. \( \square \)

Since the property \( 22 \) is implied by \( 13 \), and that \( 19 \) is equivalent to the \( \psi \)-compatibility of every dilation invariant state of \( L^\infty(\mathbb{R}^+) \), we deduce by Proposition 1.16 (ii):
Corollary 2.5. Let $\psi \in \Omega_b$ be such that $p_{\mathcal{M}_\psi} = q_{\mathcal{M}_\psi} = 1$, then $\mathcal{M}_\psi$ is close to $\mathcal{L}^1$. In particular, this happens when condition (I) holds.

2.2. An extrapolation functor. Consider $(F, \|\cdot\|_F)$, a Banach ideal lattice of (class of) measurable functions on the open interval $(1, \infty)$. Given $T \in \mathcal{L}_0$, we define $\eta_T$ to be the $[0, \infty]$-valued measurable function on $(1, \infty)$ given by

$$\eta_T := [p \in (1, \infty) \mapsto \|T\|_p].$$

We then define an extrapolation functor, from the category of Banach lattice ideals of measurable functions on the interval $(1, \infty)$, to the category of fully symmetric operator spaces on any semifinite von Neumann algebra, by

$$\mathcal{E} : F \mapsto \mathcal{L}_F := \{ T \in \mathcal{L}_0 : \eta_T \in F \}.$$ 

It is obvious that $\mathcal{L}_F$ is a bi-module over $\mathcal{N}$ and that it carries most of the properties of $F$. More specifically, we have:

Lemma 2.6. Let $F$ be a Banach ideal lattice on the interval $(1, \infty)$. Normed with

$$\|T\|_{\mathcal{L}_F} := \|\eta_T\|_F,$$

the operator space $\mathcal{L}_F$, becomes a fully symmetric operator space and it has the Fatou property if and only if $F$ does. Moreover, if the evaluation maps

$$F \to \mathbb{C}, \quad f \mapsto f(p), \quad p \in (1, \infty),$$

are continuous, then $\mathcal{L}_F$ embeds continuously in $\mathcal{L}^p$, for all $p \in (1, \infty)$.

Proof. First note that the $\mathcal{L}^p$-spaces are symmetric and have the Fatou property. Next observe that if $T \in \mathcal{L}_0$ and $S \in \mathcal{L}_F$ are such that $T \ll S$, then for $p \geq 1$, $|T|^p \ll |S|^p$ and thus for $p > 1$, $\eta_T(p) = \|T\|_p / \|S\|_p = \eta_S(p)$, which entails that $T \in \mathcal{L}_F$, with $\|T\|_{\mathcal{L}_F} \leq \|S\|_{\mathcal{L}_F}$, since $F$ is Banach lattice. This proves the first claim. Last, if for all $p > 1$, there exists a constant $C_p > 0$ such that for all $f \in F$ we have $|f(p)| \leq C_p \|f\|_F$, we deduce that for all $T \in \mathcal{L}_F$, we have $\|T\|_p = \eta_T(p) \leq C_p \|\eta_T\|_F = C_p \|T\|_{\mathcal{L}_F}$, proving the second claim.

Next, we turn to the notion of extrapolated operator spaces close to $\mathcal{L}^1$.

Definition 2.7. Let $\mathcal{E}$ be a fully symmetric operator space. Then $\mathcal{E}$ is called an extrapolated operator space if $\mathcal{E} = \mathcal{L}_F$ with equivalent norms, where $F$ is a Banach ideal lattice of measurable functions on the interval $(1, \infty)$, which contains the constant unit function and such that the evaluation maps

$$F \to \mathbb{C}, \quad f \mapsto f(p), \quad p > 1,$$

are continuous. We let $\mathcal{X}_1$ be the class of all extrapolated operator spaces close to $\mathcal{L}^1$.

The class $\mathcal{X}_1$ contains the standard Lorentz space $\mathcal{M}^{1, \infty}$ and, more generally, the Lorentz spaces associated with the elements of $\Omega_b$ given in (II). We observe that an extrapolated operator space close to $\mathcal{L}^1$ has always trivial Boyd indices:

Proposition 2.8. Let $\mathcal{E}$ be a fully symmetric operator space such that $\mathcal{E} \in \mathcal{X}_1$ and such that $\mathcal{E} \subset \mathcal{N}$ continuously. Then $p_\mathcal{E} = q_\mathcal{E} = 1$. 

Proof. By assumption, there exists a Banach lattice $F$ on the interval $(1, \infty)$ such that $\mathcal{E} = \mathcal{L}_F$ with equivalent norms. Thus, the Boyd indices of $\mathcal{E}$ and of $\mathcal{L}_F$ coincide. Moreover, by assumption too, $\mathcal{L}_F \subseteq \mathcal{N}$ continuously. This implies that $\mathcal{L}_F = \hat{\mathcal{L}}_F$ with equivalent norms, where

$$\hat{\mathcal{L}}_F := \{ T \in \mathcal{N} : \bar{\eta}_T \in F \} \quad \text{and} \quad \bar{\eta}_T(p) := \eta_T(p) + \|T\|.$$ 

Indeed since $\eta_T \leq \bar{\eta}_T$ and since $F$ is a Banach lattice which contains the unit function, for $T \in \hat{\mathcal{L}}_F$ we have

$$\|T\|_{\hat{\mathcal{L}}_F} = \|\eta_T\|_F \leq \|\bar{\eta}_T\|_F = \|T\|_{\hat{\mathcal{L}}_F}.$$ 

On the other hand, we know by assumption that there exists $C > 0$ such that for all $T \in \mathcal{L}_F$, we have $\|T\| \leq C \|T\|_{\hat{\mathcal{L}}_F}$. This entails that

$$\|T\|_{\hat{\mathcal{L}}_F} = \|\bar{\eta}_T\|_F \leq \|\eta_T\|_F + \|T\| \leq (1 + C \|1\|_F) \|T\|_{\hat{\mathcal{L}}_F},$$

which is enough to conclude as the constant unit function belongs to $F$ by assumption (see Definition 2.7). Thus, the Boyd indices of $\mathcal{E}$ and of $\hat{\mathcal{L}}_F$ coincide as well. Now, fix $q > 1$. We consider the closed subspace $F_q$ of $F$ defined by

$$F_q := \{ f \chi_{(1,q)} : f \in F \},$$

where $\chi_{(1,q)}$ is the indicator function of the interval $(1, q]$. Equipped with the norm

$$\|f\|_{F_q} := \|f\chi_{(1,q)}\|_F,$$

$F_q$ becomes a Banach lattice as well. Moreover, $\hat{\mathcal{L}}_F = \hat{\mathcal{L}}_{F_q}$ with equivalent norms. Indeed as $F$ is a Banach lattice, for $T \in \hat{\mathcal{L}}_F$, we have

$$\|T\|_{\hat{\mathcal{L}}_{F_q}} = \|\bar{\eta}_T\|_{F_q} = \|\bar{\eta}_T\chi_{(1,q)}\|_F \leq \|\bar{\eta}_T\|_F = \|T\|_{\hat{\mathcal{L}}_F},$$

while for $T \in \hat{\mathcal{L}}_{F_q}$, we deduce

$$\|T\|_{\hat{\mathcal{L}}_F} = \|\bar{\eta}_T\|_F \leq \|\bar{\eta}_T\chi_{(1,q)}\|_F + \|\bar{\eta}_T\chi_{(q,\infty)}\|_F \leq \|T\|_{\hat{\mathcal{L}}_{F_q}} + \|\chi_{(q,\infty)}\|_F \sup \bar{\eta}_T(p).$$

Next, we observe that for $p > q$, the Young inequality yields:

$$\bar{\eta}_T(p) = \|T\| + \|T\|_p \leq \|T\| + \|T\|^{-q/p}\|T\|^{q/p} \leq 2\|T\| + \|T\|_q \leq 2\bar{\eta}_T(q).$$

Observe also that as the evaluation map is continuous on $F$ (by assumption) and as $F_q \subseteq F$, the evaluation map is also continuous on $F_q$ for its own topology. Indeed, for $f \in F_q$ and $p \in (1,q]$ we have

$$f(p) = f(p)\chi_{(1,q]}(p) \leq \|f\chi_{(1,q]}\|_F = \|f\|_{F_q}.$$ 

Consequently, we obtain

$$\|T\|_{\hat{\mathcal{L}}_{F_q}} \leq \left(1 + 2\|\chi_{(q,\infty)}\|_F \sup_{p>q} \bar{\eta}_T(p)\right) \|T\|_{\hat{\mathcal{L}}_{F_q}},$$ 

and the equivalence of the norms of $\hat{\mathcal{L}}_F$ and $\hat{\mathcal{L}}_{F_q}$ is proven, since by assumption $F$ contains the constant unit function and thus contains $\chi_{(q,\infty)}$.

All these preliminary considerations show that the Boyd indices of $\mathcal{E}$ coincide with those of $\hat{\mathcal{L}}_{F_q}$, with $q > 1$ arbitrary. So, let now $f$ be a measurable function on $(1, \infty)$ with finite $\hat{\mathcal{L}}_{F_q}$-norm. Then, for $a > 1$, we have

$$\|D_af\|_{\hat{\mathcal{L}}_{F_q}} = \|[p \mapsto \|D_af\|_p]\|_{F_q} = \|[p \mapsto a^{-1/p}\|f\|_p]\|_{\chi_{(1,q]}\} F_q} \geq a^{-1/q}\|f\|_{\hat{\mathcal{L}}_{F_q}}.$$
This entails \( q \ell \leq q \), which gives the result since \( q \) can be chosen arbitrarily close to 1.

In the context of extrapolated Lorentz spaces contained in \( \mathcal{N} \), we can relate the property of being close to \( \mathcal{L}^1 \) to the values of the Boyd indices.

**Corollary 2.9.** Let \( \psi \in \Omega_b \) be such that \( \mathcal{M}_\psi \) is an extrapolated operator space. Then \( \mathcal{M}_\psi \) is close to \( \mathcal{L}^1 \) (and thus \( \mathcal{M}_\psi \in \mathcal{X}_1 \)) if and only if \( p_{\mathcal{M}_\psi} = q_{\mathcal{M}_\psi} = 1 \). In this case, \( \mathcal{M}_\psi \) admits nontrivial singular traces supported at infinity, since then every dilation invariant state is \( \psi \)-compatible.

**Proof.** This a combination of Corollary 2.5 and of Proposition 2.8 \( \square \)

### 2.3. The lower and upper approximations

Our next task is to construct lower and upper approximations of a Lorentz space by extrapolated spaces.

**Definition 2.10.** Let \( \psi \in \Omega \) be such that \( \mathcal{M}_\psi \) is close to \( \mathcal{L}^1 \). We then let

\[
F_\psi := \{ f : (1, \infty) \to \mathbb{C}, \text{ measurable} : \| f \|_{F_\psi} := \text{ess sup}_{p>1} \frac{|f(p)|}{\psi(p)} < \infty \},
\]

\[
F_\psi := \{ f : (1, \infty) \to \mathbb{C}, \text{ measurable} : \| f \|_{F_\psi} := \text{ess sup}_{p>1} \frac{|f(p)|}{\psi(p)} < \infty \}.
\]

Accordingly, we let \( \mathcal{L}_\psi \equiv \mathcal{L}_{F_\psi} \) and \( \mathcal{L}_\psi \equiv \mathcal{L}_{F_\psi} \) be the associated extrapolated operator spaces. Recall that their norms are given by

\[
\| T \|_{\mathcal{L}_\psi} := \| \eta_T \|_{F_\psi} = \text{ess sup}_{p>1} \frac{\| T \|_p}{\psi(p)}, \quad \| T \|_{\mathcal{L}_\psi} := \| \eta_T \|_{F_\psi} = \text{ess sup}_{p>1} \frac{\| T \|_p}{\psi(p)}.
\]

We also use the notations \( \mathcal{L}_\psi(\mathbb{R}_+) \) and \( \mathcal{L}_\psi(\mathbb{R}_+) \) to denote the extrapolated spaces described above for the commutative von Neumann algebra \( \mathcal{L}^\infty(\mathbb{R}_+) \) with trace given by the Lebesgue integral.

**Remark 2.11.** Observe that the fundamental functions of \( \mathcal{L}_\psi \) and \( \mathcal{L}_\psi \) are respectively given by:

\[
\varphi_\psi \equiv \varphi_{\mathcal{L}_\psi}(t) = \text{sup}_{p>1} \frac{t^{1/p}}{\psi(p)}, \quad \varphi_\psi \equiv \varphi_{\mathcal{L}_\psi}(t) = \text{sup}_{p>1} \frac{t^{1/p}}{\psi(p)}.
\]

Since the evaluation maps are continuous on \( F_\psi \) and \( F_\psi \):

\[
|f(p)| \leq \psi(e^{p-1}1) \| f \|_{F_\psi}, \quad |f(p)| \leq \| f \|_{F_\psi},
\]

we deduce from Lemma 2.6 that \( \mathcal{L}_\psi \) and \( \mathcal{L}_\psi \) embed continuously in \( \mathcal{L}^p \) for all \( p > 1 \). We also observe that \( \mathcal{L}_\psi \subset \mathcal{N} \) for all \( \psi \in \Omega \). Indeed, suppose that there exists an element \( T \in \mathcal{L}_\psi \) such that \( T \notin \mathcal{N} \). Then \( \| T \|_p \to \infty, p \to \infty \), while \( \| T \|_p \leq \| T \|_{\mathcal{L}_\psi} \psi(e^{p-1}1) \to \| T \|_{\mathcal{L}_\psi} \psi(1), p \to \infty \), a contradiction. In particular, we get for \( T \in \mathcal{L}_\psi \):

\[
\| T \| \leq \psi(1) \| T \|_{\mathcal{L}_\psi}.
\]

When \( \psi \in \Omega_b \), by the same reasoning we have further that \( \mathcal{L}_\psi \subset \mathcal{N} \), as then \( \psi' \in \mathcal{L}^\infty(\mathbb{R}_+) \). Last, as the constant unit function belongs to both \( F_\psi \) and \( F_\psi \), we see that \( \mathcal{L}_\psi \) and \( \mathcal{L}_\psi \) belong to the class \( \mathcal{X}_1 \). Hence, Proposition 2.8 applies, and shows that \( \mathcal{L}_\psi \) has trivial Boyd indices. This is also true for \( \mathcal{L}_\psi \) when \( \psi \in \Omega_b \).
We make an important observation. Since $\psi(0) = 0$, we get from the Hölder inequality:

$$\psi(e^r) = \int_0^{e^r} \psi'(t) \, dt \leq \|\chi_{[0,e^r]}\|_{1+r} \|\psi'\|_{1+1/r} = e^{r/(1+r)} \|\psi'\|_{1+1/r} \leq e \|\psi'\|_{1+1/r}, \quad r > 0.$$  

Specifying this to the case $r = (p - 1)^{-1}, p > 1$, leads us to the following embedding result:

**Lemma 2.12.** Let $\psi \in \Omega$ be such that $M_\psi$ is close to $L^1$. Then $F_\psi \subset F^\psi$ with $\|\cdot\|_{F^\psi} \leq e \|\cdot\|_{F_\psi}$. Consequently, $L_\psi \subset L^\psi$ with $\|\cdot\|_{L^\psi} \leq e \|\cdot\|_{L_\psi}$.

Our approximation result for a Lorentz space close to $L^1$ is that one always have

(28) 

$$L_\psi \subset M_\psi \subset L^\psi,$$

with continuously inclusions:

**Proposition 2.13.** Let $\psi \in \Omega$ be such that $M_\psi$ is close to $L^1$. Then $L_\psi \subset M_\psi \subset L^\psi$ with

$$\|\cdot\|_{L^\psi} \leq \|\cdot\|_{M_\psi} \leq \max \left\{ e, \frac{\psi(1)}{\psi'(1)} \right\} \|\cdot\|_{L_\psi}.$$  

**Proof.** In the proof of Lemma 2.2, we have obtained for $T \in M_\psi$ and for $p > 1$ the inequality:

$$\|T\|_p \leq \|T\|_{M_\psi} \|\psi'\|_p.$$  

This implies the first inequality. Take next $T \in L_\psi$. As noticed before, we have $L_\psi \subset N$ with $\|T\| \leq \psi(1) \|T\|_{L_\psi}$. Thus for $t \in (0,1]$, we can use the estimates

$$\int_0^t \mu(T,s) \, ds \leq \|T\|_t \leq \psi(1) \|T\|_{L_\psi} t$$  

and $\psi(t) = \int_0^t \psi'(s) \, ds \geq \psi'(1) t$, to get

$$\frac{1}{\psi(t)} \int_0^t \mu(T,s) \, ds \leq \frac{\psi(1)}{\psi'(1)} \|T\|_{L_\psi}, \quad t \in (0,1].$$  

For $t > 1$, the Hölder inequality gives for any $\varepsilon \in (0,1)$:

$$\int_0^t \mu(T,s) \, ds \leq \left( \int_0^t \mu(|T|^{1+\varepsilon},s) \, ds \right)^{1/(1+\varepsilon)} \left( \int_0^t \, ds \right)^{\varepsilon/(1+\varepsilon)} \leq \|T\|_{1+\varepsilon} t^{\varepsilon/(1+\varepsilon)}$$

$$\leq \|T\|_{L_\psi} t^{\varepsilon/(1+\varepsilon)} \psi(e^{\varepsilon-1}).$$

Choosing $\varepsilon = 1/\log(t)$, one gets

$$\frac{1}{\psi(t)} \int_0^t \mu(T,s) \, ds \leq e \|T\|_{L_\psi}, \quad t > 1,$$

proving the second inequality. \ Qt

Our next task is to understand when $M_\psi$ coincides with the extrapolated space $L_\psi$. This is a very important feature since the main results of this article, Theorem 3.4 and Theorem 4.7, are valid on the positive cone of $L_\psi$, only. But it is also important to understand when $M_\psi = L^\psi$, since this equality of spaces is a necessary condition to have $M_\psi = L_\psi$: 
Proposition 2.14. Let $\mathcal{M}_\psi$ be a Lorentz space close to $L^1$. Then, the three following conditions are equivalent:

(i) $\mathcal{M}_\psi \in \mathcal{X}_1$,

(ii) $\mathcal{M}_\psi = \mathcal{L}^\psi$ with equivalent norms,

(iii) We have

$$\psi(t) \asymp \left( \sup_{p>1} t^{1/p-1} \|\psi^t\|_p \right)^{-1}.$$ 

Proof. Assume that (ii) holds true. Then, the fundamental functions of $\mathcal{M}_\psi$ and $\mathcal{L}^\psi$ are equivalent, which implies (iii) using the first equality in (27) and the fact that the fundamental function of $\mathcal{M}_\psi$ is $t/\psi(t)$. Next, assume that (iii) holds true. Note that by Proposition 2.13 \( \|\cdot\|_{\mathcal{L}^\psi} \leq \|\cdot\|_{\mathcal{M}_\psi} \), so that the fundamental function of $\mathcal{L}^\psi$ is always smaller than that of $\mathcal{M}_\psi$. Since (iii) also contains the converse inequality, (iii) implies that the fundamental functions of $\mathcal{L}^\psi$ and $\mathcal{M}_\psi$ are equivalent. As $\mathcal{M}_\psi$ is the largest symmetric operator space with a given fundamental function, we deduce that $\mathcal{M}_\psi \supset \mathcal{L}^\psi$, which implies (ii) by Proposition 2.13. Thus (ii) $\Leftrightarrow$ (iii).

That (i) $\Leftrightarrow$ (ii) is trivial. Thus, we just need prove that (i) $\Rightarrow$ (ii). So assume that the exists a Banach lattice $F$ on the interval $[1, \infty)$ such that $\mathcal{M}_\psi = \mathcal{L}^F$. Take $f \in F^\psi$. Note that $|f(p)| \leq \|f\|_{F^\psi} \|\psi^t\|_p$, $p > 1$. Since moreover $\psi^t \in \mathcal{M}_\psi(\mathbb{R}_+)$, we deduce that

$$\left[ p \in (1, \infty) \Rightarrow \|\psi^t\|_p \right] \in F.$$ 

As $F$ is a Banach lattice, we deduce that $f \in F$. Hence $F^\psi \subset F$ and consequently $\mathcal{L}^\psi \subset \mathcal{L}^F = \mathcal{M}_\psi$. As the reversed inclusion $\mathcal{L}^\psi \supset \mathcal{M}_\psi$ is always true by Proposition 2.13 we get (i) $\Leftrightarrow$ (ii). \( \square \)

Proposition 2.14 tells us that we cannot have $\mathcal{L}_\psi = \mathcal{M}_\psi \subset \mathcal{L}^\psi$ (because $\mathcal{L}_\psi = \mathcal{M}_\psi$ implies that $\mathcal{M}_\psi \in \mathcal{X}_1$ and $\mathcal{M}_\psi \subset \mathcal{L}^\psi$ implies that $\mathcal{M}_\psi \notin \mathcal{X}_1$), but the three other possibilities $\mathcal{L}_\psi = \mathcal{M}_\psi = \mathcal{L}^\psi$, $\mathcal{L}_\psi \subset \mathcal{M}_\psi = \mathcal{L}^\psi$ and $\mathcal{L}_\psi \subset \mathcal{M}_\psi \subset \mathcal{L}^\psi$, are a priori possible. In particular, $\mathcal{L}_\psi = \mathcal{M}_\psi$ is equivalent to $\mathcal{L}_\psi = \mathcal{M}_\psi = \mathcal{L}^\psi$. Observe also that $\mathcal{L}^\psi \subset \mathcal{N}$ if and only if $\psi \in \Omega_\psi$ and $\psi \in \Omega_\psi$ if and only if $\psi \in \Omega_\psi$. Conversely, if $\psi \in \Omega_\psi$, then $\psi^t \in L^\infty(\mathbb{R}_+)$ which as observed in Remark 2.11 entails that $\mathcal{L}^\psi \subset \mathcal{N}$. Hence, if $\psi \in \Omega \setminus \Omega_\psi$, we get $\mathcal{L}_\psi \supset \mathcal{M}_\psi$. Last we observe that $\psi^t$ is always in the unit ball of $\mathcal{L}^\psi$ but it needs not to belong to $\mathcal{L}_\psi$. In fact, $\psi^t \in \mathcal{L}_\psi$ is equivalent to $\mathcal{L}_\psi = \mathcal{M}_\psi = \mathcal{L}^\psi$.

Proposition 2.15. Let $\mathcal{M}_\psi$ be a Lorentz space close to $L^1$. Then, the three conditions below are equivalent:

(i) $\mathcal{L}_\psi = \mathcal{M}_\psi$, with equivalent norms,

(ii) $\psi^t \in \mathcal{L}_\psi(\mathbb{R}_+)$,

(iii) There exists $C > 0$ such that for all $p > 1$ we have $\|\psi^t\|_p \leq C \psi(e^{(p-1)^{-1}})$.

Moreover, any of these conditions implies that $\mathcal{M}_\psi = \mathcal{L}^\psi$ and also that $\psi \in \Omega_\psi$.

Proof. That (i) $\Rightarrow$ (ii), follows from the fact that $\psi^t \in \mathcal{M}_\psi(\mathbb{R}_+)$. Next, by definition of the norm of $\mathcal{L}_\psi$, we have

$$\|\psi^t\|_p \leq \|\psi^t\|_{\mathcal{L}_\psi} \psi(e^{(p-1)^{-1}}), \quad \forall p > 1,$$
hence (ii) ⇒ (iii). For the last part, note that in the proof of Proposition 2.13, we have obtained
\[ \|T\|_p \leq \|T\|_{L, \psi'} \|\psi'\|_p, \forall T \in M, \forall p > 1. \] Thus, if (iii) holds we get for all \( T \in M \):
\[ \frac{\|T\|_p}{\psi(e^{(p-1)}-1)} \leq C \|T\|_{M, \psi}, \]
and thus \( \|T\|_{\mathcal{L}, \psi} \leq C \|T\|_{M, \psi} \), which entails that \( M \subset \mathcal{L}_{\psi} \). As the converse inclusion has been proven in Proposition 2.13, we deduce that (iii) ⇒ (i), completing the proof. □

From Proposition 2.14 we also deduce a criterion for the equality \( \mathcal{L}_{\psi} = M \), when \( M = \mathcal{L}^\psi \).

**Proposition 2.16.** Assume that \( M \in X_1 \). Then, a sufficient condition to have \( \mathcal{L}_{\psi} = M \), is that
\[ (29) \sup \left\{ \frac{\|\psi'\|_p}{\psi(t)} : p, q \geq 1, \right\} < \infty. \]

**Proof.** Under the assumption that \( M \in X_1 \), we have by Proposition 2.14 (iii):
\[ \frac{1}{\psi(t)} \leq C \sup_{p > 1} \|\psi'\|_p. \]
Hence
\[ \|\psi'\|_{\mathcal{L}_{\psi}} = \sup_{p > 1} \frac{\|\psi'\|_p}{\psi(e^{(p-1)}-1)} \leq C \sup_{p > 1} \|\psi'\|_p(e^{(p-1)}-1)^{1/q-1} \leq C \sup_{p, q > 1} \|\psi'\| e^{\frac{1}{q}(p-1)} \].
Thus if \( \psi' \in \mathcal{L}_{\psi}(\mathbb{R}_+) \), so Proposition 2.15 gives \( \mathcal{L}_{\psi} = M \) as needed. □

In order to produce examples of element \( \psi \in \Omega_b \) such that \( \mathcal{L}_{\psi} = M = \mathcal{L}^\psi \), we make the following easy observation:

**Lemma 2.17.** Let \( \psi \in \Omega_b \) be such that \( M \) is close to \( L^1 \). Then \( \mathcal{L}_{\psi} = M = \mathcal{L}^\psi \) if and only if there exists \( C > 0 \) such that for all \( \varepsilon > 0 \), we have
\[ \int_{e^{\varepsilon-1}}^\infty \psi'(t)^{1+\varepsilon} dt \leq C \psi(e^{-\varepsilon-1}), \]

**Proof.** For \( \varepsilon > 0 \) small enough, \( \|\psi'\|_{1+\varepsilon} \geq 1 \) and thus since \( \psi' \) is decreasing,
\[ \|\psi'\|_{1+\varepsilon} \leq \|\psi'\|_{1+\varepsilon} \leq \psi'(0)^{\varepsilon}\int_0^{e^{\varepsilon-1}} \psi'(t) dt + \int_{e^{\varepsilon-1}}^\infty \psi'(t)^{1+\varepsilon} dt = \psi'(0)^{\varepsilon} \psi(e^{-\varepsilon-1}) + \int_{e^{\varepsilon-1}}^\infty \psi'(t)^{1+\varepsilon} dt. \]
Hence, Proposition 2.15 (iii) shows that \( \mathcal{L}_{\psi} = M \). The converse is obvious by Proposition 2.15 (iii) again:
\[ \int_{e^{\varepsilon-1}}^\infty \psi'(t)^{1+\varepsilon} dt \leq \|\psi'\|_{1+\varepsilon}^{1+\varepsilon} \leq C \psi(e^{-\varepsilon-1})^{1+\varepsilon} \leq C' \psi(e^{-\varepsilon-1}), \]
since \( \psi(t) \leq \psi'(0)t \) and \( \psi'(0) = \|\psi'\|_\infty < \infty \) as \( \psi \in \Omega_b \). □

We are now able to give an handy criterion for the equality \( \mathcal{L}_{\psi} = M = \mathcal{L}^\psi \):

\[ \int_{e^{\varepsilon-1}}^\infty \psi'(t)^{1+\varepsilon} dt \leq \|\psi'\|_{1+\varepsilon}^{1+\varepsilon} \leq C \psi(e^{-\varepsilon-1})^{1+\varepsilon} \leq C' \psi(e^{-\varepsilon-1}), \]
Proposition 2.18. Let $\psi \in \Omega_b$. Assume that for all $\delta > 0$, the map $[t \mapsto t^{-\delta} \psi(t)]$ is decreasing and that there exists $\rho > 0$ such that the map $[t \mapsto \psi(\exp(t^\rho))]$ is still concave. Then $\mathcal{L}_\psi = \mathcal{M}_\psi = \mathcal{L}^\psi$.

Proof. Let $\varphi(t) := \psi(e^t)$. By concavity of $\varphi$ we get
\[ \rho t^\rho \, e^t \varphi'(e^t) = t \varphi'(t) \leq \varphi(t) = \psi(e^t). \]
Performing the change of variable $t \mapsto \log(t)^{1/\rho}$ gives for $t > 1$:
\[ \psi'(t) \leq \frac{\psi(t)}{\rho t \log(t)}. \]
This clearly implies that $\mathcal{M}_\psi$ is close to $\mathcal{L}^1$ as $\psi(t) \leq C t^\delta$ for all $\delta > 0$ by assumption, and moreover
\[ \int_{e^{-1}}^\infty \psi'(t)^{1+\varepsilon} \, dt \leq \rho^{-1-\varepsilon} \int_{e^{-1}}^\infty \frac{\psi(t)^{1+\varepsilon}}{\log(t)^{1+\varepsilon}} \, dt \]
\[ \leq \rho^{-1-\varepsilon} \sup_{s \geq e^{-1}} \left\{ \frac{\psi(s)^{1+\varepsilon}}{s^{1/2} \log(s)^{1+\varepsilon}} \right\} \int_{e^{-1}}^\infty t^{-1-\varepsilon/2} \, dt = \frac{2}{e} \rho^{1+\varepsilon} \psi(e^{-1})^{1+\varepsilon} \varepsilon. \]
The conclusion follows from Lemma 2.17. $\square$

Remark 2.19. The concavity of $[t \mapsto \psi(\exp(t^\rho))]$ implies that $\lim_{t \to 0} t\psi'(t)/\psi(t) = 0$, which by Lemma 2.17 implies that condition (9) is satisfied.

Remark 2.20. The elements $\psi_{n,\beta}$ given in (10), satisfies the assumptions of Proposition 2.18 for $n \geq 1$ and $\beta > 0$. Also, the elements $\tilde{\psi}_{n,\beta}$ given in (11), satisfies the assumptions of Proposition 2.18 for $n > 1$ and $\beta \in (0, 1)$. For $n = 1$ and $\beta \in (0, 1)$, there is no $\rho > 0$ such that $[t \mapsto \tilde{\psi}_{1,\beta}(\exp(t^\rho))]$ is concave. Moreover, $\mathcal{L}_{\tilde{\psi}_{1,\beta}} \not\subset \mathcal{M}_{\tilde{\psi}_{1,\beta}}$. For example, take $\beta = 1/2$. We then have $\tilde{\psi}_{1,1/2}'(t) = (2t \log t)^{-1}e^{\sqrt{t}/\log t}$, so performing the change of variable $t \mapsto \log(t)^{1/2}$, we get:
\[ \int_1^\infty \tilde{\psi}'_{1,1/2}(t)^{1+\varepsilon} \, dt = \int_0^\infty e^{-t^2} t^{-\varepsilon} e^{(1+\varepsilon)t} \, dt \]
\[ \geq \int_0^\infty e^{-t^2} e^t \, dt = \frac{\sqrt{\pi}}{2\sqrt{\varepsilon}} e^{1/4\varepsilon} \left( 1 + \operatorname{Erf} \left( \frac{1}{2\sqrt{\varepsilon}} \right) \right) \sim_{\varepsilon \to 0} \frac{\sqrt{\pi}}{\varepsilon} e^{1/4\varepsilon}, \]
while $\psi_{1,1/2}'(e^{1/\varepsilon}) = e^{1/\sqrt{\varepsilon}}$, and we conclude using Lemma 2.17. However, we don’t know in these cases if $\mathcal{L}_{\tilde{\psi}_{1,\beta}} \not\subset \mathcal{L}_{\tilde{\psi}_{1,\beta}}$ or $\mathcal{M}_{\tilde{\psi}_{1,\beta}} = \mathcal{L}_{\tilde{\psi}_{1,\beta}}$.

Last, we give stability properties for the class of elements $\psi$ in $\Omega_b$ such that $\mathcal{L}_\psi = \mathcal{M}_\psi = \mathcal{L}^\psi$.

Proposition 2.21. (i) Let $\psi \in \Omega_b$ be such that $\mathcal{L}_\psi = \mathcal{M}_\psi$ and such that for all $\delta > 0$, the map $[t \mapsto t^{-\delta} \psi(t)]$ is decreasing. Then for any $\alpha > 0$, we have $\mathcal{L}_{\psi^\alpha} = \mathcal{M}_{\psi^\alpha}$.
(ii) Let $\psi_1, \psi_2 \in \Omega_b$ be such that $\mathcal{L}_{\psi_1} = \mathcal{M}_{\psi_1}$ and such that for all $\delta > 0$, the map $[t \mapsto t^{-\delta} \psi_1(t)]$ is decreasing. Then we have $\mathcal{L}_{\psi_1 \psi_2} = \mathcal{M}_{\psi_1 \psi_2}$.
(iii) Let $\psi_1, \psi_2 \in \Omega_b$ be such that for all $\delta > 0$, the map $[t \mapsto t^{-\delta} \psi_2(t)]$ is decreasing and such that there exists $\rho > 0$ such that the map $[t \mapsto \psi_2(\exp(t^\rho))]$ is still concave, then $\mathcal{L}_{\psi_1 \psi_2} = \mathcal{M}_{\psi_1 \psi_2}$. 

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Proof. To prove (i), fix \( \delta > 0 \), and observe that by concavity of \( \psi \), we have
\[
\psi'(t)^{\varepsilon \delta} \psi(t)^{(\alpha - 1)(1 + \varepsilon)} \leq \psi(t)^{\varepsilon \delta + (\alpha - 1)(1 + \varepsilon) t^{-\varepsilon \delta}}, \quad \forall \varepsilon > 0.
\]
Thus, we obtain:
\[
\int_{e^{-\varepsilon}}^{\infty} \psi'(t)^{1+\varepsilon} \psi(t)^{(\alpha - 1)(1 + \varepsilon)} \, dt \leq \sup_{s \geq e^{-\varepsilon}} \left\{ s^{-\varepsilon \delta} \psi(s)^{\varepsilon \delta + (\alpha - 1)(1 + \varepsilon)} \right\} \| \psi' \|_{1+\varepsilon(1-\delta)}^{1+\varepsilon(1-\delta)} \psi\left( e^{-1} \right)^{(\alpha - 1)(1 + \varepsilon)} \| \psi' \|_{1+\varepsilon(1-\delta)}^{1+\varepsilon(1-\delta)},
\]
since the map \( t \mapsto \psi(t)^{\varepsilon \delta + (\alpha - 1)(1 + \varepsilon) t^{-\varepsilon \delta}} \) is decreasing. Letting \( \delta \) going to zero, we finally deduce:
\[
\int_{e^{-\varepsilon}}^{\infty} \psi'(t)^{1+\varepsilon} \psi(t)^{(\alpha - 1)(1 + \varepsilon)} \, dt \leq \psi\left( e^{-1} \right)^{(\alpha - 1)(1 + \varepsilon)} \| \psi' \|_{1+\varepsilon}^{1+\varepsilon} \leq C \psi\left( e^{-1} \right)^{(\alpha - 1)} \| \psi' \|_{1+\varepsilon},
\]
where we used \( \psi(t) \leq \psi(0) t^{\alpha} \) and \( \psi'(0) < \infty \) as \( \psi \in \Omega_b \), when \( \alpha \geq 1 \). One concludes using Proposition 2.15 (iii) and Lemma 2.17. The proof of (ii) is similar. On starts from:
\[
\| (\psi_1 \psi_2)' \|_{1+\varepsilon} \leq \| \psi_1 \psi_2' \|_{1+\varepsilon} + \| \psi_1' \psi_2 \|_{1+\varepsilon},
\]
and one uses the fact that \( \psi_1 \) is increasing to get:
\[
\int_0^{e^{-\varepsilon}} \psi_1(t)^{1+\varepsilon} \psi_2'(t)^{1+\varepsilon} \, dt \leq \psi_1\left( e^{-1} \right)^{1+\varepsilon} \int_0^{e^{-\varepsilon}} \psi_2'(t)^{1+\varepsilon} \, dt \leq C \psi_1\left( e^{-1} \right) \psi_2\left( e^{-1} \right).
\]
Then one uses the concavity of \( \psi_2 \), to estimate with \( \delta > 0 \) arbitrary:
\[
\psi_1(t)^{1+\varepsilon} \psi_2'(t)^{\varepsilon \delta} \leq \psi_1(t)^{1+\varepsilon} \psi_2(t)^{\varepsilon \delta} t^{-\varepsilon \delta},
\]
which from the same reasoning as in the proof of (i) entails that:
\[
\int_{e^{-\varepsilon}}^{\infty} \psi_1(t)^{1+\varepsilon} \psi_2'(t)^{1+\varepsilon} \, dt \leq \psi_1\left( e^{-1} \right)^{1+\varepsilon} \| \psi_2' \|_{1+\varepsilon}^{1+\varepsilon} \leq C \psi_1\left( e^{-1} \right) \psi_2\left( e^{-1} \right),
\]
and one concludes with Proposition 2.15 (iii). The case of \( \psi_1' \psi_2 \) is identical, proving (ii). Last, from \( \psi_1(t) \leq \psi_1(0) t \) we observe that the conditions of Proposition 2.18 are satisfied for \( \psi_1 \circ \psi_2 \) and (iii) follows. This concludes the proof. \( \square \)

3. Dixmier traces and \( \zeta \)-functions

In this section, we will formulate the first main result of this article, which is Theorem 3.4. It answers positively the first part of the question raised in the introduction, for all the Lorentz spaces \( \mathcal{M}_\psi \) which are associated with elements \( \psi \in \Omega_b \) satisfying condition (7) and which are such that \( \mathfrak{L}_\psi = \mathcal{M}_\psi \). Note that this result connects section 1 and section 2. The proof of Theorem 3.4 relies on Proposition 1.17 and Proposition 3.2 and can be thought as a variant of the weak*-Karamata Theorem, generalizing [7, Theorem 2.2]. Its proof essentially relies on the following Lemma:

Lemma 3.1. Let \( \beta : [0, \infty) \rightarrow [0, \infty) \) be a non-negative, increasing and right continuous function and let \( \varphi : [0, \infty) \rightarrow [0, \infty) \) be a measurable function satisfying
\[
\lim_{r \to \infty} \frac{\varphi(r/n)}{\varphi(r)} = n^{-k}, \quad \text{for all } n \in \mathbb{N} \text{ and for some } k \geq 0.
\]
Suppose also that
\[ \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} d\beta(t) \right] \in L^\infty(\mathbb{R}_+^*), \]
and let \( \omega \) be a dilation invariant state of \( L^\infty(\mathbb{R}_+^*) \). Then, for every \( f \in C[0,1] \), we have
\[ \omega \left( \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty f(e^{-t/r}) e^{-t/r} d\beta(t) \right] \right) = C(f) \omega \left( \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} d\beta(t) \right] \right), \]
where
\[ C(f) := \begin{cases} \frac{1}{\Gamma(k)} \int_0^1 f(t) |\log(t)|^{k-1} dt, & \text{for } k > 0, \\ f(1), & \text{for } k = 0. \end{cases} \]

Proof. First of all, observe that since \( f \) is bounded, both sides of the equality (32) are well defined. Indeed, for the left hand side we have
\[ \sup_{r > 0} \frac{1}{\varphi(r)} \left| \int_0^\infty f(e^{-t/r}) e^{-t/r} d\beta(t) \right| \leq \| f \|_\infty \sup_{r > 0} \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} d\beta(t), \]
which is finite by assumption, and for the right hand side we clearly have \( |C(f)| \leq \| f \|_\infty \) in both cases \( k > 0 \) and \( k = 0 \). In particular, both sides of the equality (32) are continuous in \( f \in C[0,1] \) for the uniform topology. By density of polynomials in \( C[0,1] \) and by linearity, it is sufficient to verify (32) for \( f_n(t) := t^n \), \( n \in \mathbb{N}_0 \). For such a function, we have \( C(f_n) = 1 \) when \( k = 0 \) and when \( k > 0 \):
\[ \frac{1}{\Gamma(k)} \int_0^1 f_n(t) |\log(t)|^{k-1} dt = \frac{1}{\Gamma(k)} \int_0^1 t^n |\log(t)|^{k-1} dt = \frac{1}{\Gamma(k)} \int_0^\infty e^{-u(n+1)} u^{k-1} du = (n+1)^{-k}. \]

Now, for the left hand side of (32), we find for \( k \geq 0 \):
\[ \omega \left( \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-(n+1)t/r} d\beta(t) \right] \right) = \omega \left( \left[r \mapsto \frac{\varphi(r/(n+1))}{\varphi(r)} \frac{1}{\varphi(r/(n+1))} \int_0^\infty e^{-(n+1)t/r} d\beta(t) \right] \right) \\
= (n+1)^{-k} \omega \left( \left[r \mapsto \frac{1}{\varphi(r/(n+1))} \int_0^\infty e^{-(n+1)t/r} d\beta(t) \right] \right) \\
= (n+1)^{-k} \omega \left( \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} d\beta(t) \right] \right), \]
where the second equality follows from assumption (31) and the third one follows from the dilation-invariance of \( \omega \).

\[ \square \]

Proposition 3.2. Under the assumptions on the functions \( \varphi, \beta \) and on the extended limit \( \omega \) of Lemma 3.1 we have
\[ \omega \left( \left[r \mapsto \frac{\beta(r)}{\varphi(r)} \right] \right) = \frac{1}{\Gamma(1+k)} \omega \left( \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} d\beta(t) \right] \right). \]

Proof. Consider \( L \), the continuous linear functional given by the left hand side of (32), namely:
\[ L : C[0,1] \to \mathbb{C}, \ f \mapsto \omega \left( \left[r \mapsto \frac{1}{\varphi(r)} \int_0^\infty f(e^{-t/r}) e^{-t/r} d\beta(t) \right] \right). \]
We then obtain
\[
L(g) = \omega \left( \left[ r \mapsto \frac{1}{\varphi(r)} \int_{0}^{\infty} \chi_{[e^{-1},1]}(e^{-t/r}) \, d\beta(t) \right] \right) = \omega \left( \left[ r \mapsto \frac{1}{\varphi(r)} \int_{0}^{r} d\beta(t) \right] \right) = \omega \left( \left[ r \mapsto \frac{\beta(r)}{\varphi(r)} \right] \right).
\]
Thus, all that remains is to show that the right hand side of (33) coincides with \(L(g)\) too. To this aim, we choose two sequences \(\{g_{n}^{\pm}\}_{n \in \mathbb{N}}\) in \(C[0,1]\), which converge point-wise to \(g\) and such that there exist \(a < b \in \mathbb{R}\) with \(a \leq g_{n}^{+} \leq g \leq g_{n}^{-} \leq b, \ \forall n \in \mathbb{N}\). By positivity, we have \(L(g_{n}^{+}) \leq L(g) \leq L(g_{n}^{-})\) and thus it suffices to prove that the numerical sequences \(\{L(g_{n}^{\pm})\}_{n \in \mathbb{N}}\) converge to the right hand side of (33). We first deduce from Lemma 3.1 that
\[
L(g_{n}^{\pm}) = C(g_{n}^{\pm}) \omega \left( \left[ r \mapsto \frac{1}{\varphi(r)} \int_{0}^{\infty} e^{-t/r} \, d\beta(t) \right] \right),
\]
with
\[
C(g_{n}^{\pm}) = \begin{cases} 
\frac{1}{\Gamma(k)} \int_{0}^{1} g_{n}^{\pm}(t) |\log(t)|^{k-1} \, dt, & \text{for } k > 0, \\
g_{n}^{\pm}(1), & \text{for } k = 0.
\end{cases}
\]
Accordingly, it suffices to prove that the numerical sequences \(\{C(g_{n}^{\pm})\}_{n \in \mathbb{N}}\) converge to \(\Gamma(1 + k)^{-1}\). For \(k = 0\) this is obvious since \(C(g_{n}^{\pm}) = g_{n}^{\pm}(1)\) converges to \(g(1) = 1\). For \(k > 0\), we use dominated convergence to get that \(C(g_{n}^{\pm})\) converges to \(C(g)\). Last, we compute
\[
C(g) = \frac{1}{\Gamma(k)} \int_{0}^{1} g(t) |\log(t)|^{k-1} \, dt = \frac{1}{\Gamma(k)} \int_{e^{-1}}^{1} |\log(t)|^{k-1} \frac{dt}{t} = \frac{1}{\Gamma(k)} \int_{0}^{1} u^{k-1} \, du = \frac{1}{k \Gamma(k)} = \frac{1}{\Gamma(1 + k)},
\]
which completes the proof. \(\square\)

**Theorem 3.3.** Let \(\psi \in \Omega\) satisfying condition (7). Let also \(k_{\psi} \geq 0\) be the constant associated to \(\psi\) given as in Definition 1.1.1 and let finally \(\omega\) be an exponentiation invariant state of \(L^{\infty}(\mathbb{R}_{+}^{*})\). Then, for all \(T \in \mathfrak{L}^{+}_{\psi}\) (the subcone of \(\mathcal{M}_{\psi}^{+}\) given as in Definition 2.1.10), we have
\[
\omega \circ \log \left( \left[ r \mapsto \frac{\tau(T^{1+1/r})}{\psi(e^{r})} \right] \right),
\]
where \(\omega \circ \log\) is the dilation invariant state of \(L^{\infty}(\mathbb{R}_{+}^{*})\) defined by
\[
\omega \circ \log(f) = \omega(f \circ \log).
\]

**Proof.** That \(\tau_{\psi,\omega}\) defines a Dixmier trace for an exponentiation invariant state \(\omega\) and \(\psi\) satisfying condition (7) follows from Proposition 1.1.11 and the fact that condition (7) implies condition (15). Indeed, we have
\[
\omega \left( \left[ t \mapsto \frac{\psi(t^{\alpha})}{\psi(t)} \right] \right) = A_{\psi}(\alpha), \ \ \alpha > 1,
\]
thus from Lemma 1.3 we deduce
\[
\lim_{\alpha \downarrow 1} A_{\psi}(\alpha) = 1.
\]
Fix $T \in \mathfrak{L}_\psi^+$ and set $E_T$ for the spectral family of $T$ and define
\[ \beta(t) := \tau(T E_T(e^{-t})). \]

Note that $\beta$ is finite (since $T$ is $\tau$-compact), real valued, increasing and right continuous. Define also $\varphi := \psi \circ \exp$. Now, observe that for $n \in \mathbb{N}$, we get by Lemma 1.3
\[ \lim_{r \to \infty} \frac{\varphi(r/n)}{\varphi(r)} = \lim_{r \to \infty} \frac{\psi(e^{r/n})}{\psi(e^r)} = \lim_{t \to \infty} \frac{\psi(t^{1/n})}{\psi(t)} = A_\psi(1/n) = n^{-k_\psi}. \]

Next, we observe that by spectral theory
\[ \int_0^\infty e^{-t/r} \, d\beta(t) = \int_0^\infty e^{-t/r} \, d\tau(T E_T(e^{-t})) = \tau(T \int_0^\infty e^{-t/r} \, dE_T(e^{-t})) = \tau(T^{1+1/r}). \]

By assumption we have $T \in \mathfrak{L}_\psi^+$ and so
\[ \left[ r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} \, d\beta(t) \right] \in L^\infty(\mathbb{R}_+^*). \]

Next, by Proposition 1.17 and Remark 1.18 we have:
\[ \tau_{\psi,\omega}(T) = \omega \left( \left[ t \mapsto \frac{1}{\psi(t)} \tau(T \chi_{(1/t,1)}(T)) \right] \right) = \omega \left( \left[ t \mapsto \frac{\beta(\log(t))}{\psi(t)} \right] \right) = \omega \circ \log \left( \left[ t \mapsto \frac{\beta(t)}{\varphi(t)} \right] \right). \]

Hence, the previous computations and Proposition 3.2 applied to $\beta$ and $\varphi$ as defined above, and to the dilation invariant state $\omega \circ \log$, give:
\[ \tau_{\psi,\omega}(T) = \omega \circ \log \left( \left[ t \mapsto \frac{\beta(t)}{\varphi(t)} \right] \right) = \frac{1}{\Gamma(1+k_\psi)} \omega \circ \log \left( \left[ r \mapsto \frac{1}{\varphi(r)} \int_0^\infty e^{-t/r} \, d\beta(t) \right] \right) \]
\[ = \frac{1}{\Gamma(1+k_\psi)} \omega \circ \log \left( \left[ r \mapsto \tau(T^{1+1/r}) \right] \right), \]

This completes the proof. \hfill \Box

The next result is a generalization of [7, Proposition 3.6]:

**Theorem 3.4.** Let $\psi \in \Omega$ satisfying condition 7. Let also $\omega$ be an exponentiation invariant state of $L^\infty(\mathbb{R}_+^*)$, $T \in \mathfrak{L}_\psi^+$ and $B \in \mathcal{N}$. Then, with the notations of Theorem 3.3 we have:
\[ \tau_{\psi,\omega}(BT) = \frac{1}{\Gamma(1+k_\psi)} \omega \circ \log \left( \left[ r \mapsto \frac{\tau(BT^{1+1/r})}{\psi(e^r)} \right] \right). \]

**Proof.** First observe that both sides of the equality (35) are linear in $B$. We can therefore assume that $B \geq 0$. Next, assume for a moment that the result holds when $B \geq \text{Id}_{\mathcal{N}}$. To simplify the notations, set $R_T(B)$ for the right hand-side of (35). So for arbitrary $B \geq 0$, we deduce by Theorem 3.3 and by our (momentary) extra assumption:
\[ \tau_{\psi,\omega}(BT) = \tau_{\psi,\omega}((B + \text{Id}_{\mathcal{N}})T) - \tau_{\psi,\omega}(T) = R_T(B + \text{Id}_{\mathcal{N}}) - \tau_{\psi,\omega}(T) \]
\[ = R_T(B) + R_T(\text{Id}_{\mathcal{N}}) - \tau_{\psi,\omega}(T) = R_T(B). \]
Hence, we may assume without loss of generality that $B \geq \text{Id}_\mathcal{N}$. Then, by traciality and Theorem 8.3 it suffices to prove that

$$
\omega \circ \log \left( \left[ r \mapsto \tau \left( \frac{(B^{1/2}TB^{1/2})^{1+1/r}}{\psi(e^r)} \right) \right] \right) = \omega \circ \log \left( \left[ r \mapsto \tau \left( \frac{B^{1/2}T^{1+1/r}B^{1/2}}{\psi(e^r)} \right) \right] \right).
$$

But when $B \geq \text{Id}_\mathcal{N}$, since $\mathcal{L}_\psi \subset \mathcal{N}$, we have by [7, Lemma 3.3] the operator inequality:

$$
0 \leq (B^{1/2}TB^{1/2})^{1+1/r} - B^{1/2}T^{1+1/r}B^{1/2} \leq (\|B\|^{1/r} - 1)B^{1/2}T^{1+1/r}B^{1/2}, \quad \forall r \geq 1,
$$

which gives the result as $\|B\|^{1/r} - 1 \to 0$, $r \to \infty$. \qed

We conclude this section by applying our results to the elements of $\Omega_b$ given in Example 1. First, for $\psi(t) = \log(1 + t^{1/\beta})$, $\beta > 0$, condition (7) is satisfied with $k_\psi = \beta$ and also we have $\mathcal{L}_\psi = \mathcal{M}_\psi$ by Remark 2.20. Therefore Theorem 3.4 shows that for any $T \in \mathcal{M}_\psi^+$, $B \in \mathcal{N}$ and exponentiation invariant state $\omega$, we have:

$$
\tau_{\psi,\omega}(BT) = \frac{\beta^\beta}{\Gamma(1 + \beta)} \omega \circ \log \left( \left[ r \mapsto \frac{\tau(B^{1+1/r})}{\psi(e^r)} \right] \right).
$$

In particular, we observe that for the Lorentz space associated to the function $\psi(t) = \log(1 + t^{1/\beta})^n$ with $n \in \mathbb{N}$, when meromorphic, the $\zeta$-function $\zeta(T,z) = \tau(T^z)$ of $T \in \mathcal{M}_\psi^+$ has a pole of order $n$ at $z = 1$. This generalizes the well known fact that, when meromorphic, the $\zeta$-function of a positive operator in $\mathcal{M}_1^{1,\infty}$ has a simple pole at $z = 1$.

Next, take for example $\psi(t) = \log(1 + \log(1 + t))$. Recall that we have then $k_\psi = 0$ and $\mathcal{L}_\psi = \mathcal{M}_\psi$. Hence, for any $T \in \mathcal{M}_\psi^+$, $B \in \mathcal{N}$ and an exponentiation invariant state $\omega$, we have

$$
\tau_{\psi,\omega}(BT) = \omega \circ \log \left( \left[ r \mapsto \frac{\tau(B^{1+1/r})}{\log(r)} \right] \right).
$$

4. Dixmier traces and heat kernels

In this section, we answer positively the second part of the question raised in the introduction. This is done in Theorem 4.7. As for Theorem 3.4 the answer is limited to the Lorentz spaces $\mathcal{M}_\psi$ which are associated with elements $\psi \in \Omega_b$ satisfying condition (7) and which are such that $\mathcal{L}_\psi = \mathcal{M}_\psi$. We start by introducing the subset $\mathcal{C}_\psi$ of $\tau$-measurable operators in $\mathcal{N}$ which are such that the right hand side of the equality (2) is well defined. We then show in Proposition 4.3 and Proposition 4.4 that we have $\mathcal{L}_\psi \subset \mathcal{C}_\psi \subset \mathcal{M}_\psi$ for all $\psi \in \Omega$ and with continuous inclusions. Moreover, we show in Corollary 4.6 that condition (8) is sufficient to ensure that $\mathcal{C}_\psi = \mathcal{M}_\psi$. This is why we strongly believe that Theorem 4.7 should hold with condition (8) instead of the stronger condition (7) that we use.

In what follows, for $T \in \mathcal{N}^+$, we define the operator $e^{-T^{-1}}$ to be zero on $\ker(T)$ and on $\ker(T)^\perp$ in the usual way by the functional calculus.

**Definition 4.1.** For $\psi \in \Omega$, we let $\mathcal{C}_\psi(\mathcal{N},\tau) \equiv \mathcal{C}_\psi$ to be the subset of $\mathcal{L}_0$ such that

$$
\sup_{\lambda > 0} \frac{1}{\psi(\lambda)} \int_0^\lambda \tau \left( e^{-u^{-1}|T|^{-1}} \right) \frac{du}{u^2} < \infty.
$$
Hence, we see that a sufficient condition to have 

$$\int_0^\lambda \tau(e^{-u^{-1}|T|^{-1}}) \frac{du}{u^2} = \tau(|T|e^{-\lambda^{-1}|T|^{-1}}),$$

so that condition (37) is equivalent to

$$\left[ \lambda \mapsto \frac{\tau(|T|e^{-\lambda^{-1}|T|^{-1}})}{\psi(\lambda)} \right] \in L^\infty(\mathbb{R}^*_+).$$

But this observation is of little help for one of our motivations, that is to derive formulas for Dixmier traces from short-time expansion of the heat-kernel type functions $\xi(T,t) = \tau(e^{-t|T|^{-1}})$. With this regard, the correct way to understand condition (37) goes as follows. First write

$$\frac{1}{\psi(\lambda)} \int_0^\lambda \tau(e^{-u^{-1}|T|^{-1}}) \frac{du}{u^2} = \frac{1}{\psi(\lambda)} \int_0^\lambda \frac{\tau(e^{-u^{-1}|T|^{-1}})}{u^2 \psi'(u)} \psi'(u) du.$$  

Then, set $M$ for the Cesàro mean of the additive group $(\mathbb{R}, +)$:

$$M : L^\infty(\mathbb{R}^*_+) \to L^\infty(\mathbb{R}^*_+), \quad f \mapsto \left[ \lambda \mapsto \frac{1}{\lambda} \int_0^\lambda f(u) du \right].$$

and observe that the map

$$M_\psi : L^\infty(\mathbb{R}^*_+) \to L^\infty(\mathbb{R}^*_+), \quad f \mapsto \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_0^\lambda f(u) \psi'(u) du \right],$$

coincides with the Cesàro mean twisted by $\psi$:

$$M_\psi(f) = \left( M_+(f \circ \psi^{-1}) \circ \psi \right), \quad \forall f \in L^\infty(\mathbb{R}^*_+).$$

Hence, we see that a sufficient condition to have $T \in C_\psi$ is that

$$\left[ \lambda \mapsto \frac{\tau(e^{-\lambda^{-1}|T|^{-1}})}{\lambda^2 \psi'(\lambda)} \right] \in L^\infty(\mathbb{R}^*_+).$$

For instance, the bound

$$\tau(e^{-t|T|^{-1}}) \leq Ct^{-1} |\log t|^\beta, \quad \beta \geq -1,$$

implies that $T \in C_\psi$ for $\psi(t) = \log(1 + t^{1/(1+\beta)})^{1+\beta}$, when $\beta > -1$. For $\beta = -1$ it implies that $T \in C_\psi$ for $\psi(t) = \log(1 + \log(1 + t))$.

For latter use, we first observe the following evident property:

**Lemma 4.2.** Let $\psi \in \Omega$, $T \in C^+_\psi$ and $S \in \mathcal{L}_q$ for some $q \in (1, \infty)$. Then, we have

$$\lim_{\lambda \to \infty} \frac{1}{\psi(\lambda)} \int_1^\lambda \tau(S e^{-t^{-1}T^{-1}}) \frac{dt}{t^2} = 0.$$  

**Proof.** Let $p = (1 - q^{-1})^{-1}$. The Hölder estimate on the von Neumann algebra $L^\infty([1, \lambda]) \otimes \mathcal{N}$ for the trace \( \int_1^\lambda \frac{dt}{t^2} \otimes \tau \) gives:

$$\left| \int_1^\lambda \tau(S e^{-t^{-1}T^{-1}}) \frac{dt}{t^2} \right| \leq \|S\|_q \left( \int_1^\lambda t^{-2} dt \right)^{1/q} \left( \int_1^\lambda \tau(e^{-p^{1/q}t^{-1}T^{-1}}) t^{-2} dt \right)^{1/p}.$$
Since $T \in C^+_\psi$, we have
\[ \sup_{\lambda \geq 1} \frac{1}{\psi(\lambda) \lambda^{1/p}} \left( \int_1^\lambda \tau(e^{-pt^{-1}T^{-1}}) t^{-2} dt \right)^{1/p} \leq \sup_{\lambda \geq 1} \frac{1}{\psi(\lambda) \lambda^{1/p}} \left( \int_0^\lambda \tau(e^{-pt^{-1}T^{-1}}) t^{-2} dt \right)^{1/p} < \infty. \]

Last, we observe that
\[ \lim_{\lambda \to \infty} \frac{1}{\psi(\lambda)^{1/q}} \left( \int_1^\lambda t^{-2} dt \right)^{1/q} = 0, \]
which concludes the proof. \( \square \)

The two following results show that we have $L^1_{\psi} \subset C^1_{\psi} \subset M^1_{\psi}$, with almost no condition on $\psi \in \Omega$. The first one is a direct generalization of [6, Proposition 3.8].

**Proposition 4.3.** Let $T \in N$. Then we have
\[ \int_0^t \mu(T, s) ds \leq \|T\| t \chi_{(0,1)}(t) + \left( \int_0^t \tau(e^{-\lambda^{-1}|T|^{-1}}) \frac{d\lambda}{\lambda^2} + 1 \right) \chi_{[1,\infty)}(t). \]
Consequently, for $\psi \in \Omega$, we have $C^1_{\psi} \cap N \subset M^1_{\psi}$ with:
\[ \|T\|_{M^1_{\psi}} \leq \|T\| \sup_{t \in (0,1)} \frac{t}{\psi(t)} + \sup_{t \geq 1} \frac{1}{\psi(t)} \left( \int_0^t \tau(e^{-\lambda^{-1}|T|^{-1}}) \frac{d\lambda}{\lambda^2} + 1 \right), \quad \forall T \in C^1_{\psi} \cap N. \]

**Proof.** Assume $T \geq 0$. For $t \in (0,1)$, we can use the estimates $\int_0^t \mu(T, s) ds \leq \|T\| t$. For $t \geq 1$, let
\[ A_t := \int_0^{1/t} e^{-sT^{-1}} ds \quad \text{and} \quad B_t := \int_{1/t}^\infty e^{-sT^{-1}} ds, \]
so that by Laplace transform, we have $T = A_t + B_t$ for all $t \geq 1$. Note first that
\[ \|B_t\|_1 = \tau(B_t) = \int_1^\infty \tau(e^{-\lambda^{-1}t^{-1}}) d\lambda = \int_0^t \tau(e^{-\lambda^{-1}T^{-1}}) \frac{d\lambda}{\lambda^2}. \]

On the other hand, we have
\[ \|A_t\| \leq \int_0^{1/t} \|e^{-sT^{-1}}\| ds \leq 1/t. \]

Thus for $t > 0$, we have
\[ \int_0^t \mu(T, s) ds = \inf \{ \|B_1 + tA\| : T = B + A \in L^1 + N \} \leq \|B_t\|_1 + t\|A_t\| \leq \int_0^t \tau(e^{-\lambda^{-1}T^{-1}}) \frac{d\lambda}{\lambda^2} + 1. \]

This concludes the proof. \( \square \)

**Proposition 4.4.** Let $\psi \in \Omega$ be such that $M^1_{\psi}$ is close to $L^1$. Then $L^1_{\psi} \subset C^1_{\psi}$ continuously.
Proof. Assume $T \geq 0$. Recall that $L_\psi \subset N$. We have for $\varepsilon > 0$:

$$
\frac{1}{\psi(e^{-\varepsilon})} \int_0^{e^{-\varepsilon}} \tau(e^{-\lambda}T^{-1}) \frac{d\lambda}{\lambda^2} = \frac{1}{\psi(e^{-\varepsilon})} \int_{-\varepsilon}^{\infty} \tau(e^{-t}T^{-1}) dt = \frac{1}{\psi(e^{-\varepsilon})} \frac{\tau(T e^{-e^{-\varepsilon}} T^{-1})}{\psi(e^{-\varepsilon})} \leq \|T\|^{1+\varepsilon} \|T^{-\varepsilon} e^{-\varepsilon} T^{-1}\| \leq \|T\|^{1+\varepsilon} \|\Sigma_\psi^{-\varepsilon} \psi(e^{-\varepsilon})\| \|T^{-\varepsilon} e^{-\varepsilon} T^{-1}\|.
$$

By functional calculus, we easily deduce

$$
\|T^{-\varepsilon} e^{-\varepsilon} T^{-1}\| = e^\varepsilon e^{-\varepsilon} e \leq \varepsilon, \quad \varepsilon \in (0, 1].
$$

This is enough to conclude since $\psi(t) \leq Ct$ for $t$ large enough. \hfill \Box

In conclusion, we observe that for a Lorentz space close to $L^1$, one always has

$$
L_\psi \subset C_\psi \subset M_\psi \subset L^\psi,
$$

with continuous inclusions. Our next goal is to show that under condition $[3]$ (hence under the stronger condition $[2]$ too), we have the equality $C_\psi = M_\psi$. We will prove a more general result, similar to $[35]$ Theorem 40:

**Proposition 4.5.** Let $\psi \in \Omega$ satisfy condition $[3]$. Then, for all $f \in L^\infty(R^*_+) \cap C^2(R^*_+)$ such that $f'(0) = f(0) = 0$, for all $T \in M^*_\psi$ and all $B \in N$, we have

$$
[t \mapsto \frac{1}{\psi(t)} \int_0^t \tau(B f(uT)) \frac{du}{u^2}] \in L^\infty(R^*_+).
$$

Proof. By assumptions on the function $f$, we have $|f(t)| \leq \min\{1, t^2\} = t^2 \chi_{[0,1)}(t) + \chi_{[1,\infty)}(t)$. Hence,

$$
(39) \quad |\tau(B f(uT))| \leq \|B\| \left( u^2 \tau(T^2 \chi_{[0,1/u)}(T)) + \tau(\chi_{[1/u,\infty)}(T)) \right).
$$

For the second term in the inequality above, we get with $E_T(\lambda)$ the spectral projection of $T$:

$$
\int_0^t \tau(\chi_{[1/u,\infty)}(T)) \frac{du}{u^2} = \tau \left( \int_0^t \left( \int_0^{\infty} \chi_{[1/u,\infty)}(\lambda) dE_T(\lambda) \right) \frac{du}{u^2} \right).
$$

But

$$
\int_0^t \chi_{[1/u,\infty)}(\lambda) \frac{du}{u^2} = \int_0^t \chi_{[1/\lambda,\infty)}(u) \frac{du}{u^2} = \int_0^{1/\lambda} \chi_{[1/t,\infty)}(\lambda) \leq \chi_{[1/t,\infty)}(\lambda),
$$

and thus by Fubini’s Theorem, we get

$$
\int_0^t \tau(\chi_{[1/u,\infty)}(T)) \frac{du}{u^2} \leq \tau \left( \int_0^{\infty} \lambda \chi_{[1/t,\infty)}(\lambda) dE_T(\lambda) \right) = \tau(T \chi_{[1/t,\infty)}(T)) = \int_0^{\mu(T,s)} \mu(T,s) ds,
$$

where $n(T)$ is the distribution function of $T$. Now, by $[30]$ Lemma 7 (this is where condition $[3]$ is used), for all $\varepsilon > 0$, there exists $t_0 > 0$ such that for all $t \geq t_0$, we have $n(T, 1/t) \leq (1 + \varepsilon)\|T\|_{M_\psi} t_0 \psi(t)$. Hence, with $\varepsilon > 0$ fixed, we get for $t$ large enough

$$
\frac{1}{\psi(t)} \int_0^t \tau(\chi_{[1/u,\infty)}(T)) \frac{du}{u^2} \leq \frac{1}{\psi(t)} \int_0^{(1+\varepsilon)\|T\|_{M_\psi} t \psi(t)} \mu(T,s) ds \leq \frac{\psi((1 + \varepsilon)\|T\|_{M_\psi} t \psi(t))}{\psi(t)} \|T\|_{M_\psi}.
$$
But the fraction on right most of the inequality above is bounded since condition (8) implies condition (9), and thus
\[
\frac{\psi((1 + \varepsilon)\|T\|_{\mathcal{M}_\psi} t \psi(t))}{\psi(t)} = \frac{\psi((1 + \varepsilon)\|T\|_{\mathcal{M}_\psi} t \psi(t)) \psi(t \psi(t))}{\psi(t)} \to 1, \quad t \to \infty.
\]

For the contribution corresponding to the first term in the right hand side of (39), we get
\[
\int_0^t \tau(T^2 \chi_{[1/u, \infty)}(T)) \, du = \tau\left( \int_0^\infty \lambda^2 \left( \int_0^t \chi_{[0,1/u)}(\lambda) \, du \right) dE_T(\lambda) \right).
\]
Since
\[
\int_0^t \chi_{[0,1/u)}(\lambda) \, du = \min\{t, 1/\lambda\} = t \chi_{[0,1/t)}(\lambda) + \lambda^{-1} \chi_{[1/t, \infty)}(\lambda),
\]
we deduce
\[
\int_0^t \tau(T^2 \chi_{[1/u, \infty)}(T)) \, du = t \tau(T^2 \chi_{[0,1/t)}(T)) + \tau(T \chi_{[1/t, \infty)}(T)).
\]
The conclusion for the contribution coming from the second term above is reached using the same argument as above. Thus, all what remains to show is that
\[
\tau(T^2 \chi_{[0,1/t)}(T)) = O(\psi(t)/t).
\]
To this aim, we write:
\[
\tau(T^2 \chi_{[0,1/t)}(T)) = \int_0^\infty \mu(T, s)^2 \chi_{\{ \mu(T) \leq 1/t \}}(s) \, ds
\]
\[
= \int_0^{\|T\|_{\mathcal{M}_\psi} t \psi(t)} \mu(T, s)^2 \chi_{\{ \mu(T) \leq 1/t \}}(s) \, ds + \int_0^\infty \mu(T, s)^2 \chi_{\{ \mu(T) \leq 1/t \}}(s) \, ds
\]
\[
\leq \|T\|_{\mathcal{M}_\psi} \frac{\psi(t)}{t} + \int_0^\|T\|_{\mathcal{M}_\psi} \psi(t) \, ds.
\]
Next, we use
\[
\|T\|_{\mathcal{M}_\psi} \geq \frac{1}{\psi(t)} \int_0^t \mu(T, s) \, ds \geq \frac{t \mu(T, t)}{\psi(t)},
\]
to get for all \( \delta \in (0, 1) \):
\[
\int_0^\infty \frac{\mu(T, s)^2 \, ds}{\|T\|_{\mathcal{M}_\psi} t \psi(t)} \leq \|T\|_{\mathcal{M}_\psi}^2 \int_0^\infty s^{-2} \psi(s)^2 \, ds
\]
\[
\leq \|T\|_{\mathcal{M}_\psi}^2 \psi(\|T\|_{\mathcal{M}_\psi} t \psi(t))^2 \left( \|T\|_{\mathcal{M}_\psi} t \psi(t) \right)^{-\delta} \int_0^\infty s^{-2+\delta} \, ds = \frac{\|T\|_{\mathcal{M}_\psi} \psi(\|T\|_{\mathcal{M}_\psi} t \psi(t))^2}{1 - \delta} \frac{1}{t \psi(t)}.
\]
Letting \( \delta \to 0 \), we eventually get:
\[
\tau(T^2 \chi_{[0,1/t)}(T)) \leq \|T\|_{\mathcal{M}_\psi} \frac{\psi(t)}{t} \left( 1 + \frac{\psi(\|T\|_{\mathcal{M}_\psi} t \psi(t))^2}{\psi(t)^2} \right),
\]
which concludes the proof since the right most fraction above is bounded as already shown. \( \square \)

**Corollary 4.6.** Let \( \psi \in \Omega \) satisfy condition (8). Then we have \( \mathcal{C}_\psi = \mathcal{M}_\psi \).
Proof. By Proposition 4.3 we only need to prove that if $T \in \mathcal{M}_\psi^+$ then
\[
\sup_{\lambda > 0} \frac{1}{\psi(\lambda)} \int_0^\lambda \tau(e^{-t^{-1}T^{-1}}) \frac{dt}{t^2} < \infty.
\]
But the result follows from Proposition 4.5 applied to the function $f(x) = e^{-x}$. \qed

We can now state the main result of this section, which relates Dixmier traces and heat kernels, when the strongest condition (7) holds.

**Theorem 4.7.** Let $\psi \in \Omega$ satisfy condition (7). Then, for any exponentiation invariant state $\omega$ of $L^\infty(\mathbb{R}_+^*)$, any $T \in \mathcal{L}_\psi^+$ (a subcone of $\mathcal{C}_\psi^+$ which coincides with $\mathcal{M}_\psi^+$ in this case) and any $B \in \mathcal{N}$, we have
\[
\tau_{\psi, \omega}(BT) = \omega\left(\left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \tau(Be^{-t^{-1}T^{-1}}) \frac{dt}{t^2} \right]\right).
\]

Proof. By linearity, we may assume $B \in \mathcal{N}^+$ and by traciality and Theorem 3.4 we are left to prove that:
\[
\omega\left(\left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_0^\lambda \tau(B^{1/2}e^{-t^{-1}T^{-1}}B^{1/2}) \frac{dt}{t^2} \right]\right) = \frac{1}{\Gamma(1 + k_\psi)} \omega \circ \log \left(\left[ r \mapsto \frac{1}{\psi(e^r)} \tau(B^{1/2}T^{1+1/r}B^{1/2}) \right]\right),
\]
with $k_\psi$ the constant associated to $\psi \in \Omega$, given as in Definition 4.4. Using the integral representation $T^{1+1/r} = \Gamma(1 + 1/r)^{-1} \int_0^\infty t^{1/r}e^{-tT^{-1}}dt$ and since $\Gamma(1 + 1/r) \to 1$, $r \to \infty$, we deduce
\[
\omega \circ \log \left(\left[ r \mapsto \frac{1}{\psi(e^r)} \tau(B^{1/2}T^{1+1/r}B^{1/2}) \right]\right) = \omega \circ \log \left(\left[ r \mapsto \frac{1}{\psi(e^r)} \int_0^\infty t^{1/r}e^{-tT^{-1}}B^{1/2} \frac{dt}{t^2} \right]\right).
\]

Next, observe that for $t > 1$, we have since $\mathcal{L}_\psi \subset \mathcal{N}$:
\[
B^{1/2}e^{-tT^{-1}}B^{1/2} \leq e^{-(t-1)/\|T\|} B^{1/2}e^{-T^{-1}}B^{1/2},
\]
which implies that
\[
\int_1^\infty t^{1/r} \tau(B^{1/2}e^{-t^{-1}T^{-1}}B^{1/2}) \frac{dt}{t^2} \leq \tau(B^{1/2}e^{-T^{-1}B^{1/2}}) \int_1^\infty t e^{-(t-1)/\|T\|} \frac{dt}{t^2},
\]
for $r \geq 1$. Hence,
\[
\lim_{r \to \infty} \frac{1}{\psi(e^r)} \int_1^\infty t^{1/r} \tau(B^{1/2}e^{-t^{-1}T^{-1}}B^{1/2}) \frac{dt}{t^2} = 0,
\]
and thus
\[
\omega \circ \log \left(\left[ r \mapsto \frac{1}{\psi(e^r)} \tau(B^{1/2}T^{1+1/r}B^{1/2}) \right]\right) = \omega \circ \log \left(\left[ r \mapsto \frac{1}{\psi(e^r)} \int_0^1 t^{1/r} \tau(B^{1/2}e^{-t^{-1}T^{-1}}B^{1/2}) \frac{dt}{t^2} \right]\right).\]

Set
\[
\beta(u) := \int_0^u e^{-\nu} \tau(B^{1/2}e^{-e^{-\nu}T^{-1}B^{1/2}}) \frac{du}{\nu}.
\]
Then, $\beta$ is real valued, increasing and right continuous on $\mathbb{R}_+^*$ with $\beta(0) = 0$ and a little computation shows that
\[
\int_0^\infty e^{-\mu/r} \frac{d\beta(\mu)}{\mu} = \int_0^1 t^{1/r} \tau(B^{1/2}e^{-t^{-1}T^{-1}B^{1/2}}) \frac{dt}{t^2}.
\]
Therefore, we get
\[ \omega \circ \log \left( \left[ r \mapsto \frac{1}{\psi(e^r)} \tau (B^{1/2}T^{1+1/r}B^{1/2}) \right] \right) = \omega \circ \log \left( \left[ r \mapsto \frac{1}{\psi(e^r)} \int_0^\infty e^{-\mu/r} \, d\beta(\mu) \right] \right). \]

Since \( T \in \mathfrak{L}_{\psi}^+ \), we also get from what precedes:
\[ \left[ r \mapsto \frac{1}{\psi(e^r)} \int_0^\infty e^{-\mu/r} \, d\beta(\mu) \right] \in L^\infty (\mathbb{R}_+^*). \]

Thus, we can apply Proposition 3.2 to deduce:
\[ \omega \circ \log \left( \left[ r \mapsto \frac{1}{\psi(e^r)} \int_0^\infty e^{-\mu/r} \, d\beta(\mu) \right] \right) = \Gamma(1 + k_\psi) \omega \circ \log \left( \left[ r \mapsto \frac{\beta(r)}{\psi(e^r)} \right] \right) = \Gamma(1 + k_\psi) \omega \left( \left[ r \mapsto \frac{\beta \log(r)}{\psi(r)} \right] \right). \]

But
\[ \frac{\beta \log(r)}{\psi(r)} = \frac{1}{\psi(r)} \int_0^{\log(r)} e^{-\nu} \tau (B^{1/2}e^{-e^{-\nu}T^{-1}}B^{1/2}) \, d\nu = \frac{1}{\psi(r)} \int_1^r \tau (B^{1/2}e^{-\lambda^{-1}T^{-1}}B^{1/2}) \frac{d\lambda}{\lambda^2}, \]

and finally,
\[ \frac{1}{\Gamma(1 + k_\psi)} \omega \circ \log \left( \left[ r \mapsto \frac{\tau (B^{1/2}T^{1+1/r}B^{1/2})}{\psi(e^r)} \right] \right) = \omega \left( \left[ r \mapsto \frac{1}{\psi(r)} \int_1^r \tau (B^{1/2}e^{-\lambda^{-1}T^{-1}}B^{1/2}) \frac{d\lambda}{\lambda^2} \right] \right), \]

which completes the proof. \( \square \)

Last, we apply the previous result in the case of a short-time asymptotic expansion of the heat trace function with logarithms in the leading term.

**Corollary 4.8.** Let \( T \in \mathcal{N}^+ \) and \( \beta \in [-1, \infty) \) be such that there exists \( C(T) > 0 \) with
\[ (40) \quad \xi(T, t) := \tau (e^{-tT^{-1}}) \sim C(T) t^{-1/2} |\log t|^{\beta}, \quad t \downarrow 0. \]

Then \( T \in \mathcal{M}_\psi \) for \( \psi(t) = \log(1 + t^{1/(1+\beta)})^{1+\beta} \) when \( \beta > -1 \) and for \( \psi(t) = \log(1 + \log(1 + t)) \) when \( \beta = -1 \). Moreover, for any exponentiation invariant state \( \omega \) on \( L^\infty (\mathbb{R}_+^*) \), we have
\[ \tau_\psi \omega (T) = C(T). \]

**Proof.** Note first that since \( T \in \mathcal{N}^+ \), the map \( [t \in \mathbb{R}_+^* \mapsto \tau (e^{-tT^{-1}})] \) is decreasing and thus the behavior \( \xi(T, t) \) gives rise to a bound \( \tau (e^{-tT^{-1}}) \leq C t^{-1/2} |\log t|^{\beta} \) for all \( t > 0 \) and some constant \( C > 0 \). Then, the discussion right after Definition 4.4 combined with Proposition 4.5 shows that \( T \in \mathcal{M}_\psi \), for the elements \( \psi \in \Omega_b \) given as above. Then Theorem 4.7 gives the result for \( \mathfrak{L}_{\psi}^+ \), the subcone of \( \mathcal{N}^+ \). But Proposition 2.18 implies that in these cases, we have \( \mathfrak{L}_{\psi}^+ = \mathcal{M}_\psi \), concluding the proof. \( \square \)
5. Applications to pseudo-differential operators on $\mathbb{R}^n$

In this last section, we apply our results to the setting of Hörmander-Weyl pseudo-differential operators. The framework that we will introduce in a moment, is strongly related to the work of Nicola and Rodino in [27]. In fact, the first main result of this section, Theorem 5.4, generalizes their main result (which is stated as Theorem 1.1 there) in two directions. First, it works for more general Lorentz spaces $\mathcal{M}_p$ than the dual to the Macaev ideal. Second, it is not restricted to the non-closed subspace characterized by $\mu(T) = O(\psi')$. But we go further: we are able to express the Dixmier trace of an Hörmander-Weyl pseudo-differential operator in term of its symbol only. This is done in Theorem 5.6 and Corollary 5.7.

Recall that the Weyl quantization map, $\text{OP}_W$, is a continuous linear map from the space of tempered distributions on $\mathbb{R}^{2n}$ (with the strong dual topology) to the space of continuous linear maps acting from the space of Schwartz functions on $\mathbb{R}^n$ (with its standard Fréchet topology) to the space of tempered distributions on $\mathbb{R}^n$:

$$\text{OP}_W \in \mathcal{L}\left(\mathcal{S}'(\mathbb{R}^{2n}), \mathcal{L}(\mathcal{S}(\mathbb{R}^n), \mathcal{S}'(\mathbb{R}^n))\right),$$

given, with a little abuse of notations, by

$$\left(\text{OP}_W(T)\phi\right)(x) = (2\pi)^{-n} \int_{\mathbb{R}^{2n}} e^{i\xi x} T\left(\frac{x+y}{2}, \xi\right) \phi(y) \, dy \, d\xi, \quad T \in \mathcal{S}'(\mathbb{R}^{2n}), \ \phi \in \mathcal{S}(\mathbb{R}^n).$$

For $T \in \mathcal{S}'(\mathbb{R}^{2n})$, the linear operator $\text{OP}_W(T)$ from $\mathcal{S}(\mathbb{R}^n)$ to $\mathcal{S}'(\mathbb{R}^n)$ is called the Weyl pseudo-differential operator with symbol $T$. The most important properties of this quantization scheme, not shared by other variants of pseudo-differential calculus on $\mathbb{R}^n$, is that it maps real symbols to self-adjoint operators:

$$\text{OP}_W(T^*) = \text{OP}_W(T)^*,$$

and defines a unitary operator from the Hilbert space of $L^2$-function on $\mathbb{R}^{2n}$, to the Hilbert space of Hilbert-Schmidt operators acting on $L^2$-functions on $\mathbb{R}^n$:

$$\text{OP}_W \in \mathcal{U}\left(L^2(\mathbb{R}^{2n}), L^2(L^2(\mathbb{R}^n))\right),$$

that is to say

$$\text{Tr}\left(\text{OP}_W(f_1)^*\text{OP}_W(f_2)\right) = \int_{\mathbb{R}^{2n}} \overline{f_1(x,\xi)} f_2(x,\xi) \, d^n x \, d^n \xi, \quad \forall f_1, f_2 \in L^2(\mathbb{R}^{2n}).$$

It is then easy to see that the relation (11) still holds for a pair of symbols $(f_1, f_2) \in L^\infty(\mathbb{R}^{2n}) \times L^1(\mathbb{R}^{2n})$ such that $\text{OP}_W(f_1)$ is bounded and $\text{OP}_W(f_2)$ is trace class (and vice versa).

We consider here symbols in $\mathcal{S}(m,g)$, the space of Hörmander symbols. This class of symbols is associated to $g$, a slowly varying and $\sigma$-temperate metric in $\mathbb{R}^{2n}$, satisfying the uncertainty principle and to $m : \mathbb{R}^{2n} \to \mathbb{R}_+^*$, a $g$-continuous and $(\sigma, g)$-temperate weight function, see [15, Definition 18.5.1]. The most important feature of $\mathcal{S}(m,g)$ is that it is a Fréchet space, for the topology associated to the seminorms:

$$\|f\|_{k; m, g} := \sup_{(x,\xi) \in \mathbb{R}^{2n}} \sup_{X_1, \ldots, X_k \in T_{x,\xi} \mathbb{R}^{2n}} \frac{|f^{(k)}((x,\xi); X_1, \ldots, X_k)|}{g_{x,\xi}(X_1)^{1/2} \cdots g_{x,\xi}(X_k)^{1/2}} < \infty, \quad k \in \mathbb{N}_0.$$
Here, $f^{(k)}((x, \xi); \ldots)$ denotes the $k$-multilinear form on $T_{x,\xi}^j \mathbb{R}^{2n}$, the tangent space of $\mathbb{R}^{2n}$ at $(x, \xi)$, given by the differential of order $k \in \mathbb{N}_0$ of the function $f$ at the point $(x, \xi) \in \mathbb{R}^{2n}$. Accordingly, we let $\text{OPS}(m, g)$ to be the class of Weyl pseudo-differential operators with symbols in the Hörmander class $S(m, g)$. We call the elements of $\text{OPS}(m, g)$ the Hörmander-Weyl pseudo-differential operators. We also define $g^*$, the symplectic dual metric of $g$, by

$$g^*_{x,\xi}(t, \tau) := \sup \{ \sigma(t, \tau; y, \eta)^2 : g_{x,\xi}(y, \eta) = 1 \}, \quad (x, \xi), (t, \tau) \in \mathbb{R}^{2n},$$

where $\sigma$ is the standard symplectic form of $\mathbb{R}^{2n} \cong T^*\mathbb{R}^n$. We also let $h_g$ be the so called Planck function. It is defined by:

$$h^2_g(x, \xi) := \sup_{(t, \tau) \in \mathbb{R}^{2n}} g^*_{x,\xi}(t, \tau)^{\frac{1}{n}}.$$

The uncertainty principle mentioned above, correspond to the condition that $h_g \leq 1$. Here, we make the further assumption that there exists $g \in (1, \infty)$ such that $h_g \in L^g(\mathbb{R}^{2n})$. Observe however that this condition is weaker than the one used in [27], which is $h_g(x, \xi) \leq C(1 + |x| + |\xi|)^{-\delta}$, for some $C, \delta > 0$.

We refer to [18, Chapter XVIII] for precise definitions and details as here we just need to know the following facts. First, $\text{OP}_W$ sends $S(m, g)$ to $\mathcal{B}(L^2(\mathbb{R}^n))$ continuously, if and only if $m$ is bounded [18, Theorem 18.6.3]. Second, $\text{OP}_W$ sends $S(m, g)$ to $\mathcal{K}(L^2(\mathbb{R}^n))$ continuously, if and only if $m \to 0$ at $\infty$ [18, Theorem 18.6.6]. If moreover $m \in S(m, g)$ then $\text{OP}_W(m)$ is self-adjoint and its spectrum is bounded from below and if $m \to \infty$ at $\infty$, then the spectrum is discrete [17, Theorem 3.4]. Following the discussion of [18, page 143], given a $g$-continuous and $(\sigma, g)$-temperate weight function $m$, we can always find another $g$-continuous and $(\sigma, g)$-temperate weight function $m'$ such that $m' \in S(m', g)$ and $S(m', g) = S(m, g)$. Hence, the assumption $m \in S(m, g)$ (that we will make in what follows) is, in fact, irrelevant. We will also use the following trace norm estimate, proved in [17, Theorem 3.9]: for all $k \in \mathbb{N}$, there exists $C_k > 0$ such that for all $f \in S(m, g)$, we have

$$||\text{OP}_W(f)||_1 \leq C_k ||f||_1 + ||h^k_g m||_1 ||f||_{k_m, g}.$$  

Set $\star$ for he composition product associated to the Weyl calculus, i.e. defined by the relation

$$\text{OP}_W(f_1)\text{OP}_W(f_2) = \text{OP}_W(f_1 \star f_2).$$

The last property we need to know, is that the operation $\star$ defines a continuous bilinear map:

$$\star : S(m_1, g) \times S(m_2, g) \to S(m_1 m_2, g),$$

and if $f_j \in S(m_j, g), j = 1, 2$, we have

$$f_1 \star f_2 - f_1 f_2 \in S(m_1 m_2 h, g).$$

This is proven in [18, Theorem 18.5.4]. To simplify the discussion, it is useful to introduce the following terminology:

**Definition 5.1.** An Hörmander pair $(g, m)$ consists of a slowly varying and $\sigma$-temperate metric $g$ in $\mathbb{R}^{2n}$ such that the Planck function $h_g$, given in [13], belongs to $L^q(\mathbb{R}^{2n}) \cap L^\infty(\mathbb{R}^{2n})$ for some $q \in (1, \infty)$ and a weight function $m : \mathbb{R}^{2n} \to \mathbb{R}^+_+$ which is $g$-continuous, $(\sigma, g)$-temperate and which satisfies $m \in S(m, g)$. 


Finally, we need to record some results of [27].

**Proposition 5.2.** Let \((g, m)\) an Hörmander pair such that \(m \to 0\) at \(\infty\).

(i) There exists \(c > 0\), such that setting \(\tilde{m} := (c + m^{-1})^{-1}\), the operator \(\text{OP}_W(\tilde{m}^{-1})\) is positive, boundedly invertible and with compact inverse in \(\text{OPS}(m, g)\).

(ii) For all \(t \geq 0\), we have

\[
e^{-t \text{OP}_W(\tilde{m}^{-1})} = \text{OP}_W(b_t) + S_t,
\]

where \(\{b_t\}_{t>0}\) is a bounded family of symbols in \(S(1, g)\) and \(\{S_t\}_{t>0}\) is a family of trace class operators such that we have \(\|S_t\|_1 \leq Ct\), for some \(C > 0\) and all \(t > 0\).

(iii) For fixed \(t > 0\), \(b_t \in \cap_{\epsilon \in \mathbb{N}} S(m^\epsilon, g)\)

(iv) The symbol \(b_t\) can be written as

\[b_t = e^{-t \tilde{m}^{-1}} + \sum_{j=1}^{N} b_{t,j}\]

with \(|b_{t,j}| \leq C_j e^{-t \tilde{m}/2} h_g^j\),

for some constants \(C_j > 0\) independent of \(t > 0\). In particular, we have for some \(C > 0\):

\[|b_t| \leq C e^{-t \tilde{m}/2}\]

**Proof.** (i) is a restatement of [27] Lemma 3.2. (ii) is [27] Theorem 3.3. (iii) and the first part of (iv) follow from the intermediate estimates used in the proof of [27] Theorem 3.3 and the last part of (iv) follows from \(h_g \in L^\infty(\mathbb{R}^{2n})\), by assumption. \(\square\)

The following result is mostly a consequence of Proposition 5.2. The Lorentz spaces we consider there are associated with the type \(I_\infty\) factor of all bounded operators on \(L^2(\mathbb{R}^n)\) and with the standard trace. To make the notations explicit, we denote this Lorentz space by \(\mathcal{M}_\psi(L^2(\mathbb{R}^n))\). We stress that in this type \(I_\infty\) factor setting, there is no need to distinguish \(\Omega\) and \(\Omega_b\).

**Proposition 5.3.** Let \(\psi \in \Omega_b\) and let \((g, m)\) be an Hörmander pair. If furthermore

\[
(46) \quad \sup_{\lambda > 0} \frac{1}{\psi(\lambda)} \int_{\mathbb{R}^{2n}} e^{-t \tilde{m}^{-1}(x, \xi)} d^n x d^n \xi \frac{dt}{t^2} < \infty,
\]

then \(\text{OPS}(m, g) \subset \mathcal{M}_\psi(L^2(\mathbb{R}^n))\) continuously.

**Proof.** We start by observing that the condition (46) implies that \(m \to 0\) at \(\infty\). Indeed assume it does not. Then, in the proof of [27] Lemma 3.1 and under the assumption that \(m\) is \(g\)-admissible, the authors have constructed a Borel set \(B \subset \mathbb{R}^{2n}\) and of infinite Lebesgue measure (\(B = \bigcup_{j=1}^{\infty} B_j\) within the notations of [27] Lemma 3.1) and such that \(B \subset \{m > C\}\), for a constant \(C > 0\). This entails that

\[
\int_{\mathbb{R}^{2n}} e^{-t \tilde{m}^{-1}(x, \xi)} d^n x d^n \xi \geq \int_{B} e^{-t \tilde{m}^{-1}(x, \xi)} d^n x d^n \xi \geq \int_{B} e^{-tC^{-1}} d^n x d^n \xi = e^{-tC^{-1}} \text{meas}(B) = \infty,
\]

a contradiction with the assumption (46). Thus, we can apply Proposition 5.2. We adopt the following notations:

\[
\tilde{m}_\delta := (c + \delta + m^{-1})^{-1}, \quad \text{for} \quad \delta > 0, \quad \text{and} \quad \tilde{m} := \tilde{m}_0.
\]
Next, we show that (46) implies that $OP_W(\tilde{m}_\delta)^{-1}$ belongs to $\mathcal{M}_\psi(L^2(\mathbb{R}^n))$ for all $\delta > 0$. Indeed, since $OP_W(\tilde{m}_\delta)^{-1}$ is bounded and positive by Proposition 5.2 (i), we get from the first estimate of Proposition 5.3 and for $t > 0$:

$$\int_0^t \mu(\mathcal{OP}_W(\tilde{m}_\delta)^{-1}, s) \, ds \leq \|\mathcal{OP}_W(\tilde{m}_\delta)^{-1}\|_t t \chi_{(0,1)}(t) + \left( \int_0^t \text{Tr}(e^{-u^{-1}\mathcal{OP}_W(\tilde{m}_\delta)^{-1}}) \frac{du}{u^2} + 1 \right) \chi_{[1,\infty)}(t).$$

(48)

Writing, $\mathcal{OP}_W(\tilde{m}_\delta)^{-1} = \mathcal{OP}_W(\tilde{m}_\delta) + \delta$, we deduce by Proposition 5.2 (ii):

$$\int_0^\lambda \text{Tr}(e^{-u^{-1}\mathcal{OP}_W(\tilde{m}_\delta)^{-1}}) \frac{du}{u^2} \leq \int_0^\lambda \left( \|\mathcal{OP}_W(b_{u^{-1}})\|_1 + S_{u^{-1}} \|_{11} \right) e^{-u^{-1}\delta} \frac{du}{u^2} \leq \int_0^\lambda \left( \|\mathcal{OP}_W(b_{u^{-1}})\|_1 + \frac{C}{u} \right) e^{-u^{-1}\delta} \frac{du}{u^2}.$$

The second term is bounded uniformly in $\lambda > 0$ and since $\psi(t) = O(t)$, $t \to 0$ (by the assumption that $\psi \in \Omega_b$), the corresponding contribution to (48) is bounded. For the first term, we use [17 Theorem 3.9] to get that for every $N \in \mathbb{N}$, there exists $k \in \mathbb{N}$ such that we have

$$\|\mathcal{OP}_W(b_{u^{-1}})\|_1 \leq C \left( \|b_{u^{-1}}\|_1 + \|b_g^N\|_1 \|b_{u^{-1}}\|_{k;1;g} \right).$$

By assumption, we have $\|h_N\|_1 < \infty$ for $N \geq q$ and we know by Proposition 5.2 (ii) that $[u \mapsto \|b_{u^{-1}}\|_{k;1;g}]$ belongs to $L^\infty(\mathbb{R}_+^*)$ for each $k \in \mathbb{N}$. Thus, the corresponding contribution in (48) is finite. For the contribution coming from $\|b_{u^{-1}}\|_1$, Proposition 5.2 (iv) shows that

$$\|b_{u^{-1}}\|_1 \leq C \int_{\mathbb{R}^{2n}} e^{-u^{-1}m^{-1}(x,\xi)/2} \, d^n x \, d^n \xi,$$

and we conclude using our condition (46). Now, take any symbol $f \in S(m,g)$ then we write

$$\mathcal{OP}_W(f) = \mathcal{OP}_W(f) \mathcal{OP}_W(\tilde{m}_\delta)^{-1} \mathcal{OP}_W(\tilde{m}_\delta)^{-1}.$$
the Lebesgue integral of its symbol\(^6\). This result relies on the properties of the Hörmander-Weyl calculus and on the following simple lemma.

**Lemma 5.5.** Let \( \psi \in \Omega_0 \) satisfy condition \((\mathcal{S})\) and \((g,m)\) an Hörmander pair. Let also \( 0 < f \in \mathcal{M}_\psi(\mathbb{R}^{2n}) \). Then, we have

\[
\lim_{\lambda \to \infty} \frac{1}{\psi(\lambda)} \int_1^\lambda \left( \int_{\mathbb{R}^{2n}} h_g(x,\xi) e^{-t^{-1}f^{-1}(x,\xi)} d^n x \, d^n \xi \right) dt = 0.
\]

**Proof.** Since \( \psi \) satisfies condition \((\mathcal{S})\), we deduce by Corollary 4.6 that \( \mathcal{C}_\psi(\mathbb{R}^{2n}) = \mathcal{M}_\psi(\mathbb{R}^{2n}) \). By assumption that \((g,m)\) is an Hörmander pair, we have \( h_g \in L^q(\mathbb{R}^{2n}) \) for some \( q \in (1, \infty) \). Thus, the proof follows directly from Lemma 4.2. \(\square\)

We can now formulate the second main result of this section. Here, \( \int_{\psi,\omega} \) denotes a Dixmier trace for the commutative Lorentz space \( \mathcal{M}_\psi(\mathbb{R}^{2n}) \) associated with \( L^\infty(\mathbb{R}^{2n}) \) and with the Lebesgue integral. For this result to hold, condition \((\mathcal{S})\) is not enough and we need to consider the stronger condition \((\mathcal{O})\).

**Theorem 5.6.** Let \( \psi \in \Omega \) satisfy condition \((\mathcal{O})\) and assume further that \( \mathcal{L}_\psi = \mathcal{M}_\psi \). Let also \((g,m)\) be an Hörmander pair such that \( m \in \mathcal{M}_\psi(\mathbb{R}^{2n}) \). Then, for any symbol \( f \in S(m,g) \) and any exponential invariant state \( \omega \) of \( L^\infty(\mathbb{R}^n_+) \), we have

\[
\text{Tr}_{\psi,\omega}(\text{OP}_W(f)) = \int_{\psi,\omega} f.
\]

**Proof.** First, fix \( \delta > 0 \) and write as in Proposition 5.3 and with the notations given in 17:

\[
(49) \quad \text{OP}_W(f) = \text{OP}_W(f) \text{OP}_W(\tilde{m}_\delta^{-1}) \text{OP}_W(\tilde{m}_\delta^{-1})^{-1} = \text{OP}_W(f * \tilde{m}_\delta^{-1}) \text{OP}_W(\tilde{m}_\delta^{-1})^{-1}.
\]

We have shown in the proof of Proposition 5.3 that

\[
\text{OP}_W(\tilde{m}_\delta^{-1})^{-1} \in \mathcal{M}_\psi(L^2(\mathbb{R}^n))^+ = \mathcal{L}_\psi(L^2(\mathbb{R}^n))^+.
\]

Since \( f \in S(m,g) \) and \( \tilde{m}_\delta^{-1} \in S(m^{-1},g) \), we deduce by 18 Theorem 18.5.4 that \( f * \tilde{m}_\delta^{-1} \in S(1,g) \) and by 18 Theorem 18.6.3 that

\[
\text{OP}_W(f * \tilde{m}_\delta^{-1}) \in \mathcal{B}(L^2(\mathbb{R}^n)),
\]

hence, using 19, we get \( \text{OP}_W(f) \in \mathcal{M}_\psi(L^2(\mathbb{R}^n)) \). Thus, we can apply Theorem 1.7 to get:

\[
\text{Tr}_{\psi,\omega}(\text{OP}_W(f)) = \omega\left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \text{Tr} \left( \text{OP}_W(f * \tilde{m}_\delta^{-1}) e^{-t^{-1} \text{OP}_W(\tilde{m}_\delta^{-1})} \frac{dt}{t^2} \right) \right] \right).
\]

Next, by Proposition 5.2 (ii) we have

\[
e^{-t^{-1} \text{OP}_W(\tilde{m}_\delta^{-1})} = e^{-t^{-1} \text{OP}_W(\tilde{m}_\delta^{-1})} e^{-\delta/t} = (\text{OP}_W(b_{t^{-1}}) + S_{t^{-1}}) e^{-\delta/t} \quad \text{with} \quad \|S_{t^{-1}}\|_1 \leq Ct^{-1}.
\]

Hence, as

\[
|\text{Tr}(\text{OP}_W(f * \tilde{m}_\delta^{-1}) S_{t^{-1}})| \leq \|\text{OP}_W(f * \tilde{m}_\delta^{-1})\| \|S_{t^{-1}}\|_1,
\]

\(\text{OP}_W(\tilde{m}_\delta^{-1})\). This follows from the fact that, for a kernel operator of trace class, its trace coincides with the integral of its kernel on the diagonal.
we get
\[ \frac{1}{\psi(\lambda)} \int_1^\lambda \frac{\sqrt{\lambda}}{t^2} \left( \int \frac{dt}{t^2} \right) \leq C \left\| \text{OP}_W(f * \tilde{m}_\delta^{-1}) \right\| \frac{1}{\psi(\lambda)} \int_1^\lambda \frac{dt}{t^2} \to 0, \quad \lambda \to \infty. \]
Accordingly, we get
\[ \text{Tr}_{\psi, \omega}(\text{OP}_W(f)) = \omega \left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \frac{\sqrt{\lambda}}{t^2} \left( \int \frac{dt}{t^2} \right) \right] \right). \]
Since \( f * \tilde{m}_\delta^{-1} \in L^\infty(\mathbb{R}^{2n}), \) \( b_{t-1} \in L^1(\mathbb{R}^{2n}), \) \( \text{OP}_W(f * \tilde{m}_\delta^{-1}) \) is bounded and \( \text{OP}_W(b_{t-1}) \) is trace class, we can employ the relation \( (41) \), to get
\[ \text{Tr}(\text{OP}_W(f * \tilde{m}_\delta^{-1})\text{OP}_W(b_{t-1})) = \int_{\mathbb{R}^{2n}} f * \tilde{m}_\delta^{-1}(x, \xi) b_{t-1}(x, \xi) \, dx \, d\xi. \]
Consequently
\[ \text{Tr}_{\psi, \omega}(\text{OP}_W(f)) = \omega \left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \int_{\mathbb{R}^{2n}} f * \tilde{m}_\delta^{-1}(x, \xi) b_{t-1}(x, \xi) \, dx \, d\xi \right] \right). \]
But by \cite[Theorem 18.5.4]{15}, we have \( f * \tilde{m}_\delta^{-1} - f \tilde{m}_\delta^{-1} \in S(h_g, g), \) which, in particular, gives
\[ |f * \tilde{m}_\delta^{-1} - f \tilde{m}_\delta^{-1}| \leq Ch_g. \]
Hence, since \( |b_{t-1}| \leq Ce^{-t^{-1}\tilde{m}^{-1}/2} \) by Proposition \ref{prop:5.2} (iv), since \( \tilde{m} \in \mathcal{M}_\psi(\mathbb{R}^{2n}), \) we can use Lemma \ref{lem:5.3} to deduce that
\[ \omega \left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \int_{\mathbb{R}^{2n}} (f * \tilde{m}_\delta^{-1}(x, \xi) - f(x, \xi)\tilde{m}_\delta^{-1}(x, \xi)) b_{t-1}(x, \xi) \, dx \, d\xi \right] \right) = 0. \]
Accordingly, we get
\[ \text{Tr}_{\psi, \omega}(\text{OP}_W(f)) = \omega \left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \int_{\mathbb{R}^{2n}} f(x, \xi) \tilde{m}_\delta^{-1}(x, \xi) b_{t-1}(x, \xi) \, dx \, d\xi \right] \right). \]
Similarly, we have by Proposition \ref{prop:5.2} (iv) that
\[ |b_{t-1} - e^{-t^{-1}\tilde{m}^{-1}}| \leq C h_g e^{-t^{-1}\tilde{m}^{-1}/2}. \]
Hence, from Lemma \ref{lem:5.3} again, we deduce that
\[ \text{Tr}_{\psi, \omega}(\text{OP}_W(f)) = \omega \left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \int_{\mathbb{R}^{2n}} f(x, \xi) \tilde{m}_\delta^{-1}(x, \xi) e^{-t^{-1}\tilde{m}_\delta^{-1}(x, \xi)} \, dx \, d\xi \right] \right). \]
However, since \( f \tilde{m}_\delta^{-1} \in S(1, g) \subset L^\infty(\mathbb{R}^{2n})^+ \) and since \( \tilde{m}_\delta \in \mathcal{M}_\psi(\mathbb{R}^{2n})^+ = \mathcal{L}_\psi(\mathbb{R}^{2n})^+ \), we can use Theorem \ref{thm:4.7} for the von Neumann algebra \( L^\infty(\mathbb{R}^{2n}) \) with Lebesgue integral, to deduce
\[ \omega \left( \left[ \lambda \mapsto \frac{1}{\psi(\lambda)} \int_1^\lambda \int_{\mathbb{R}^{2n}} f(x, \xi) \tilde{m}_\delta^{-1}(x, \xi) e^{-t\tilde{m}_\delta^{-1}(x, \xi)} \, dx \, d\xi \right] \right) = \int_{\psi, \omega} (f \tilde{m}_\delta^{-1} \tilde{m}_\delta) = \int_{\psi, \omega} f. \]
This completes the proof. \hfill \Box
Our last result combines Theorem \ref{thm:3.3} and Theorem \ref{thm:5.6} (with \( B = \text{Sign}(f) \) and \( T = |f| \)). It complements Theorem \ref{thm:5.6} as it gives a very simple way to compute the Dixmier trace of an Hörmander-Weyl pseudo-differential operator.
Corollary 5.7. Let $\psi \in \Omega_b$ satisfy condition (7) and assume further that $\mathcal{L}_\psi = \mathcal{M}_\psi$. Let also $(g,m)$ be an Hörmander pair such that $m \in M_\psi(\mathbb{R}^n)$. Then, for any symbol $f \in S(m,g)$ and any exponentiation invariant state $\omega$ of $L^\infty(\mathbb{R}_+^*)$, we have:
\[
\text{Tr}_{\psi,\omega}(\text{OP}_W(f)) = \frac{1}{\Gamma(1 + k_\psi)} \omega \left( \left[ r \mapsto \frac{1}{\psi(r)} \int_{\mathbb{R}^n} f(x,\xi) |f(x,\xi)|^{1/\log(r)} \, dx \, d^n\xi \right] \right),
\]
where $k_\psi$ is the constant associated to $\psi$ given in Definition 1.4.

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