A Remark on Fourier Transform

Guangwu Xu∗

Abstract

In this note, we describe an interpretation of the (continuous) Fourier transform from the perspective of the Chinese Remainder Theorem. Some related issues, including a new derivation of Poisson summation formula, are discussed.

Keywords: Chinese Remainder Theorem; Fourier Transform.

1 Introduction

The usual Chinese Remainder Theorem (CRT) concerns reconstructing an integer given its remainders with respect to a set of coprime moduli. For a set of pairwise coprime positive integers \( \{m_0, m_1, \ldots, m_{\mu-1}\} \), we let \( \Gamma = \prod_{j=0}^{\mu-1} m_j \). Then there is an isomorphism

\[
\mathbb{Z}/(\Gamma) \rightarrow \prod_{j=0}^{\mu-1} \mathbb{Z}/(m_j) \tag{1}
\]

\[
n \mapsto (n \pmod{m_0}, n \pmod{m_1}, \cdots, n \pmod{m_{\mu-1}}).
\]

The crucial part is to verify the inverse (of the isomorphism), which is given by the CRT: finding (the unique) solution \( n < \Gamma \) of the system of congruence

∗SCST, Shandong University, Qingdao, Shandong, China; e-mail: gxu4sdq@sdu.edu.cn.
equations
\[
\begin{align*}
&x \equiv r_0 \pmod{m_0} \\
&x \equiv r_1 \pmod{m_1} \\
&\quad \cdots \\
&x \equiv r_{\mu-1} \pmod{m_{\mu-1}}
\end{align*}
\]

CRT provides a formula (reported by Jiushao Qin (aka Chin Chiu-shao) in 1247 [4]) for the solution:
\[
n = \sum_{j=0}^{\mu-1} r_j u_j \frac{\Gamma}{m_j} \pmod{\Gamma}
\]

where \( u_j = (\frac{\Gamma}{m_j})^{-1} \pmod{m_j} \). It is remarked that in [4], Qin explained a beautiful algorithm for computing (the positive value) \( u_j \) (under the name of ‘Dayan deriving one’). As discussed in [6, 7], Qin’s algorithm is somehow more concise and efficient than the modern Extended Euclidean algorithm.

We would like to point out that Qin also noted the following relation
\[
\sum_{j=0}^{\mu-1} u_j \frac{\Gamma}{m_j} = 1 + \ell \Gamma,
\]

and gave it the names ‘positive use’ (for \( \ell = 1 \)) and ‘universal use’ (for \( \ell > 1 \)). Using this relation to get (3) is more convenient as illustrated in [2].

Modeling a computational task to the Chinese remainder representation is of great significance. Each component of the representation is restricted in a ring (e.g., \( \mathbb{Z}/(m_j) \)) of smaller size and arithmetic operations on those smaller rings are closed and independent, so computation task of large scale can be decomposed into that of smaller scales and performed in parallel, see [1, 2]. Other applications of CRT in fast computation can be found in public key cryptography.

It is well-known that the Chinese remainder theorem has a general formulation to decompose a certain ring to be a product of ‘smaller’ quotient rings. The correctness of the decomposition in essence is (3). As an example, replacing \( \mathbb{Z}/(\Gamma) \) with \( \mathbb{R}[x]/(x^n - 1) \) and \( \mathbb{Z}/(m_j) \) with \( \mathbb{R}[x]/(x - e^{\frac{2\pi i}{n_j}}) \), we get the (finite) discrete Fourier transform: for a polynomial \( f \in \mathbb{R}[x] \) with
degree less than \( n \),

\[
f \mapsto \left( f(1), f(e^{-2\pi i/n}), f(e^{-4\pi i/n}), \cdots, f(e^{-2(n-1)\pi i/n}) \right).
\]

The inverse of the transform is exactly in the same format like (3) and it is now commonly called the Lagrange interpolation formula. For the computational significance of the discrete Fourier transform, besides being parallelizable, the symmetry of the roots of \( x^n - 1 = 0 \) enables the use of divide and conquer strategy, so we have the famous Cooley-Tukey algorithm — the fast Fourier transform (FFT).

In this note, we would like to remark that the continuous version of Fourier transform can also be interpreted as some (approximation) form of the Chinese remainder representation. Such algebraic treatment enables different and maybe elementary approach to some related issues.

## 2 Continuous Fourier Transform and Chinese Remainder Theorem

Now we consider a discretization of the continuous Fourier transform under the framework of Chinese remainder theorem. We shall assume functions involved are all good enough (e.g., in Schwartz space [5]) so the convergence will not be an issue.

Let \( f : \mathbb{R} \to \mathbb{C} \) be a rapidly decreasing smooth function, its Fourier transform is given by

\[
\hat{f}(y) = \int_{\mathbb{R}} f(x) e^{-2\pi i xy} dx \tag{5}
\]

The inverse of the transform is proved to be

\[
f(x) = \int_{\mathbb{R}} \hat{f}(y) e^{2\pi i xy} dy. \tag{6}
\]

We will use carefully formulated finite sums to approximate these two absolutely convergent integrals. Let \( N, M \) be large even integers (depend on \( f \)).
For the spatial domain $\mathbb{R}$, we choose interval $[-\frac{N}{2}, \frac{N}{2}]$ and partition it with each unit interval being divided into $M$ equal subintervals. Take the following partition points $x_j = -\frac{N}{2} + \frac{j}{M}$ for $j = 0, 1, \cdots, MN - 1$.

For the frequency domain $\mathbb{R}$, we choose interval $[-\frac{M}{2}, \frac{M}{2}]$ and partition it with each unit interval being divided into $N$ equal subintervals. Take the following partition points $y_k = -\frac{M}{2} + \frac{k}{N}$ for $k = 0, 1, \cdots, MN - 1$.

To approximately evaluate the transform at partition points $y_k$, we see that

$$\hat{f}(y_k) \approx \int_{-\frac{N}{2}}^{\frac{N}{2}} f(x) e^{-2\pi i x y_k} \, dx \approx \sum_{j=0}^{MN-1} f(x_j) e^{-2\pi i x_j y_k} \frac{1}{M}$$

$$= \frac{1}{M} \sum_{j=0}^{MN-1} f(x_j) e^{-2\pi i (Mx_j)(Ny_k)/MN}.$$  \hspace{1cm} (7)

Similarly, we can get

$$f(x_j) \approx \frac{1}{N} \sum_{k=0}^{MN-1} \hat{f}(y_k) e^{2\pi i (Ny_k)(Mx_j)/MN}$$ \hspace{1cm} (8)

We should note that $Ny_k$ (resp. $Mx_j$) runs over all integers from $-\frac{MN}{2}$ to $\frac{MN}{2} - 1$ for $k = 0, 1, \cdots, MN - 1$ (for $j = 0, 1, \cdots, MN - 1$), with the same order. Write $w = \frac{NM}{2}$ and and $\omega = e^{-2\pi i \frac{1}{MN}}$. Denote

$$P(X) = \frac{1}{M} \left(f(x_w) + f(x_{w+1})X + \cdots + f(x_{2w-1})X^{w-1} + f(x_0)X^w + f(x_1)X^{w+1} + \cdots + f(x_{w-1})X^{2w-1} \right).$$

Then (7) says that

$$P(X) \pmod{X - \omega^{Ny_k}} \approx \hat{f}(y_k) = \hat{f}(y_{Ny_k + \frac{M}{2w}}),$$

and hence $\hat{f}$ can be approximated by the Chinese remainder representation of $P(X)$ (treated as an element of $\mathbb{C}[X]/(X^{MN} - 1)$).

With this interpretation, we can illustrate a way of recovering $f$ given the values of $\hat{f}(y_k)$, $k = 0, 1, \cdots, NM - 1$ by using (3). Note that in this case $\Gamma = X^{MN} - 1$ and $m_j = X - \omega^j$. We also note that $\left(\frac{\Gamma}{m_j}(\omega^j)\right)^{-1}$ is $\left(\frac{\Gamma}{m_j}\right)^{-1}$
We note that \( \frac{1}{M} f(x_j) \) is the coefficient of \( X^q \) in \( P(X) \) with \( q = (j + \frac{MN}{2}) \) (mod \( MN \)), in order to get its expression, we take \( q \)th derivatives of both sides and evaluate them at \( X = 0 \):

\[
f(x_j) \approx \frac{1}{N} \sum_{k=0}^{MN-1} \hat{f}(y_k) e^{2\pi i \frac{(Ny_k)(Mx_j)}{MN}}.
\]

As mentioned above, this derivation of the approximation formula for inverse Fourier transform we just illustrated is rather heuristic. A more rigorous treatment inside this framework can be done by using the Dirichlet kernel which is defined as

\[
D(x) = \begin{cases} 
\frac{\sin(\pi Mx + \frac{x}{N})}{\sin(\frac{x}{N})} & \text{if } \frac{x}{N} \notin \mathbb{Z} \\
MN + 1 & \text{if } \frac{x}{N} \in \mathbb{Z}
\end{cases}
\]

We shall simply explain this for the case of \( x_j = 0 \), i.e., \( j = \frac{MN}{2} \).

\[
\frac{1}{N} \sum_{k=0}^{MN-1} \hat{f}(y_k) = \frac{1}{N} \sum_{k=0}^{MN-1} \int_{\mathbb{R}} f(x) e^{-2\pi i xy_k} dx = \frac{1}{N} \int_{\mathbb{R}} f(x) \sum_{k=0}^{MN-1} e^{-2\pi i xy_k} dx
\]

\[
= \frac{1}{N} \int_{\mathbb{R}} f(x) \sum_{k=0}^{MN} e^{-2\pi i \frac{x}{N} \left( -\frac{MN}{2} + k \right)} dx - \frac{1}{N} \int_{\mathbb{R}} f(x) e^{-2\pi i Mx} dx
\]

\[
= \frac{1}{N} \int_{\mathbb{R}} f(x) D(x) dx + \frac{O(1)}{N} \approx f(0).
\]

We also want to remark that parallelization and FFT are thus possible for numerical evaluations of the transform.

\footnote{It is interesting to note that we can omit the (mod ) operation here since the degree of both sides are less than \( NM \). We also note that \( \sum_{j=0}^{N-1} u_j \frac{r_j}{m_j} = \sum_{j=0}^{N-1} \frac{1}{\omega_j} = 1. \)
2.1 Poisson summation formula

The classical 1-dimensional Poisson summation formula states that for a rapidly decreasing smooth function $f$, one has

$$\sum_{n \in \mathbb{Z}} f(n) = \sum_{n \in \mathbb{Z}} \hat{f}(n). \quad (9)$$

It is noted that from our previous formula (5),

$$\hat{f}(-\frac{M}{2}) + \hat{f}(\frac{M}{2}) + \cdots + \hat{f}(\frac{M}{2} - 1)$$

$$= \hat{f}(y_0) + \hat{f}(y_N) + \hat{f}(y_{2N}) + \cdots + \hat{f}(y_{(M-1)N})$$

$$\approx \frac{1}{M} \sum_{\ell=0}^{M-1} \sum_{j=0}^{MN-1} f(x_j) e^{-2\pi i \frac{(MN + (N)Mx_j)}{MN}} = \frac{1}{M} \sum_{j=0}^{MN-1} f(x_j) \sum_{\ell=0}^{M-1} \left( e^{-2\pi i \frac{j}{MN}} \right)^\ell$$

$$= f(-\frac{N}{2}) + f(-\frac{N}{2} + 1) + \cdots + f(\frac{N}{2} - 1)$$

Taking limits, we get the Poisson summation formula (9). This argument is essentially of the form of the finite group version of the Poisson summation formula. This can be done because our partitions for approximating integrals have a natural subgroup structure.

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