Influence of a road on a population in an ecological niche facing climate change

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Abstract
We introduce a model designed to account for the influence of a line with fast diffusion—such as a road or another transport network—on the dynamics of a population in an ecological niche. This model consists of a system of coupled reaction-diffusion equations set on domains with different dimensions (line / plane). We first show that, in a stationary climate, the presence of the line is always deleterious and can even lead the population to extinction. Next, we consider the case where the niche is subject to a displacement, representing the effect of a climate change. We find that in such case the line with fast diffusion can help the population to persist. We also study several qualitative properties of this system. The analysis is based on a notion of generalized principal eigenvalue developed and studied by the authors (2019).

Keywords KPP equations · Reaction-diffusion system · Line with fast diffusion · Generalized principal eigenvalue · Moving environment · Climate change · Forced speed · Ecological niche

Mathematics Subject Classification 35K57 · 92D25 · 35B40 · 35K40 · 35B53

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1 Setting of the problem and main results

1.1 Introduction

It has long been known that the spreading of invasive species can be enhanced by human transportations. This was the case for instance for the “tiger mosquito”, *Aedes Albopictus*. Originating from south-east Asia, eggs of this mosquito were introduced in several places around the world, mostly via shipments of used tires (Hawley et al. 1987). Recently, there has been evidence of passive dispersal of adult tiger mosquitoes by cars, leading to the colonizations of new territories along road networks (Millner and Loaiza 2015; Eritja et al. 2017).

Other kind of networks with fast transportation appear to help the dispersal of biological entities. For instance, rivers can accelerate the spreading of plant pathogens (Jung and Blaschke 2004). It has also been observed by Mckenzie et al. (2012) and Hillen and Painter (2013) that populations of wolves in the Western Canadian Forest move and concentrate along seismic lines (paths traced in forests by oil companies for testing of oil reservoirs). In a different register, we mention that the road network is known to have a driving effect on the spreading of epidemics. The “black death” plague, for instance, spread first along the silk road and then spread along the main commercial roads in Europe (Siegfried 1960).

All these facts suggest that networks with fast diffusion (roads, rivers, seismic lines...) are important factors to take into account in the study of the spreading of species. A mathematical formulation of a model accounting for this phenomenon was introduced by Berestycki et al. (2013a). The width of the lines with fast diffusion being much smaller than the natural scale of the problem, this model consists in a system of coupled reaction-diffusion equations set on domains of different dimensions, namely a line and the plane or half-plane. An important feature is that it is homogeneous, in the sense that the environment does not change from one place to another.

This “homogeneity” hypothesis does not hold in several situations. For instance, many observations suggest that the spreading of invasive species can happen only when the environment is “favorable enough”. Considering again the tiger mosquito, the climate is known to limit its range of expansion. In America, the tiger mosquito has reached its northernmost boundary in New Jersey, southern New York and Pennsylvania. It is believed that cold temperatures are responsible for stopping its northward progression. This means that the ecological niche of the tiger mosquito is limited by the climate conditions. In this paper, we call an ecological niche a portion of the space where a population can reproduce, surrounded by an unfavorable domain, lethal for the population. From a biological perspective, the niche can be characterized by a suitable temperature range, or by a localization of resources, for instance.

An important feature of an ecological niche is that it can move as time goes by. For instance, the temperature to the north of the territory currently occupied by the tiger mosquito in north america is currently increasing, as a consequence of global warming. This leads to a displacement of the ecological niche of the tiger-mosquito, and it should entail the further spreading of the mosquito into places that were unaccessible before Rochlin et al. (2013). The displacement of the ecological niche could also result from
seasonal variation of resources. Potapov and Lewis (2004) and the first author of this paper together with Diekmann, Nagelkerke and Zegeling (2009) have introduced a model designed to describe the evolution of a population facing a shifting climate. We review some of their results in the next section.

In the present paper, we introduce and study a model of population dynamics which takes into account both these phenomena: it combines a line with fast diffusion and an ecological niche, possibly moving in time, as a consequence for instance of a climate change. Consistently with the existing literature on the topic, we will refer in the sequel to the line with fast diffusion as the “road” and to the rest of the environment as “the field”. The two phenomena we consider are in some sense in competition: the road enhances the diffusion of the species, while the ecological niche confines its spreading. Two questions naturally arise.

**Question 1** Does the presence of the road help or, on the contrary, inhibit the persistence of the species living in an ecological niche?

**Question 2** What is the effect of a moving niche?

### 1.2 The model

The goal of this paper is to investigate these questions. We consider a two dimensional model, where the road is the one-dimensional line \( \mathbb{R} \times \{0\} \) and the field is the upper half-plane \( \mathbb{R} \times \mathbb{R}^+ \). Let us mention that we can consider as well a field given by the whole plane. This does not change the results presented here, as we explain in Sect. 2.2 below. However, the notations become somewhat cumbersome. We refer to the paper by Berestycki et al. (2019), where road-field systems on the whole plane are considered.

As in the paper by Berestycki et al. (2013a), we use two distinct functions to represent the densities of the population on the road and in the field respectively: \( u(t, x) \) is the density on the road at time \( t \) and point \( (x, 0) \), while \( v(t, x, y) \) is the density of population in the field at time \( t \) and point \( (x, y) \in \mathbb{R} \times \mathbb{R}^+ \).

In the field, we assume that the population is subject to diffusion, and also to reaction, accounting for reproduction and mortality. The presence of the ecological niche is reflected by an heterogeneous reaction term which is negative outside a bounded set. For part of our study, we also allow the niche to move with constant speed \( c \in \mathbb{R} \). On the road, the population is only subject to diffusion. The diffusions in the field and on the road are constant but *a priori* different. Moreover, there are exchanges between the road and the field: the population can leave the road to go into the field and can enter the road from the field with some (a priori different) probabilities.

Combining these definitions and effects, our system writes:

\[
\begin{align*}
\partial_t u - D \partial_{xx} u &= v v|_{y=0} - \mu u, & t > 0, \ x \in \mathbb{R}, \\
\partial_t v - d \Delta v &= f(x - ct, y, v), & t > 0, \ (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d \partial_y v|_{y=0} &= \mu u - vv|_{y=0}, & t > 0, \ x \in \mathbb{R}. 
\end{align*}
\]
The first equation accounts for the dynamic on the road, the second for the dynamic in the field, and the third for the exchanges between the field and the road. Note that the term \( v v|_{y=0} - \mu u \) represents the balance of the exchange between the road and the field (gained by the road and lost by the field). Unless otherwise specified, we consider classical solutions when dealing with the parabolic problem (1). In the system (1), \( D, d, \mu, \nu \) are strictly positive constants and \( c \) is a real number. Without loss of generality, we consider only the case \( c \geq 0 \), that is, the niche is moving to the right. The nonlinear term \( f \) depends on the variable \( x - ct \). This implies that the spatial heterogeneities are shifted with speed \( c \) in the direction of the road. We will first consider the case \( c = 0 \) that is, when there is no shift.

Throughout the paper, besides some regularity hypotheses (see Sect. 1.4) we assume that \( f(x, y, v) \) vanishes at \( v = 0 \) (neither reproduction nor deaths, nor interactions occurs if there are no individuals) and that the environment has a maximal carrying capacity:

\[ \exists S > 0 \text{ such that } f(x, y, v) < 0 \text{ for all } v \geq S, (x, y) \in \mathbb{R} \times (0, +\infty). \]  

We then assume that the per capita net growth rate is a decreasing function of the size of the population, that is,

\[ v \mapsto \frac{f(x, y, v)}{v} \text{ is strictly decreasing for } v \geq 0, (x, y) \in \mathbb{R} \times [0, +\infty). \]  

In particular, \( f \) satisfies the Fisher-KPP hypothesis: \( f(x, y, v) \leq f_{v}(x, y, 0)v \), for \( v \geq 0 \).

Last, we assume that the ecological niche is bounded:

\[ \lim \sup_{|(x, y)| \to +\infty} f_{v}(x, y, 0) < 0. \]  

An example of a nonlinearity satisfying the above assumptions is \( f(x, y, v) = \xi(x, y)v - v^{2} \), with \( \lim \sup_{|(x, y)| \to +\infty} \xi(x, y) < 0 \).

To address Questions 1 and 2, we will compare the situation “with the road” with the situation “without the road”. When there is no road, the individuals in the field who reach the boundary \( \mathbb{R} \times \{0\} \) bounce back in the field instead of entering the road. In other terms, removing the road from system (1) leads to the Neumann boundary problem

\[ \begin{align*}
\partial_{t} v - d \Delta v &= f(x - ct, y, v), \quad t > 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^{+}, \\
\partial_{y} v|_{y=0} &= 0, \\
\end{align*} \]

By a simple reflection argument, it is readily seen that the dynamical properties of this system are the same as for the problem in the whole plane, as explained in Sect. 2.2 below.

In the sequel, we call (1) the system “with the road”, while (5) is called the system “without the road”. Problem (5) describes the evolution of a population subject to a climate change only. In the next section, we recall the basic facts on this model.
Questions 1 and 2 translate in terms of the comparative dynamics of (1) and (5): we will see that, depending on the parameters, the solutions of these systems either asymptotically in time stabilize to a positive steady state, in which case the population persists, or vanish, meaning that the population goes extinct. Therefore, we will compare here the conditions under which one or the other of these scenarios occurs, for systems (1) and (5).

1.3 Related models and previous results

We present in this section some background about reaction-diffusion equations as well as the system from which (1) is originated. These results will be used in the sequel.

Consider first the classical reaction-diffusion equation introduced by Fisher (1937) and Kolmogorov et al. (1937):

\[ \partial_t v - d \Delta v = f(v), \quad t > 0, \quad x \in \mathbb{R}^N, \]  

(6)

with \( d > 0 \), \( f(0) = f(1) = 0 \), \( f(v) > 0 \) for \( v \in (0, 1) \) and \( f(v) \leq f'(0)v \) for \( v \in [0, 1] \). The archetypical example is the logistic nonlinearity \( f(v) = v(1 - v) \).

Under these conditions, we refer to (6) as the Fisher-KPP equation. It is shown by Aronson and Weinberger (1978) that invasion occurs for any nonnegative and not identically equal to zero initial datum. That is, any solution \( v \) arising from such an initial datum converges to 1 as \( t \) goes to \( +\infty \), locally uniformly in space. Moreover, if the initial datum has compact support, one can quantify this phenomenon by defining the speed of invasion as a value \( c_{KPP} > 0 \) such that:

\[ \forall c > c_{KPP}, \quad \sup_{|x| \geq ct} v(t, x) \longrightarrow 0, \]

and

\[ \forall c < c_{KPP}, \quad \sup_{|x| \leq ct} |v(t, x) - 1| \longrightarrow 0. \]

The speed of invasion can be explicitly computed in this case: \( c_{KPP} = 2\sqrt{df'(0)} \).

Building on equation (6), Potapov and Lewis (2004) and Berestycki et al. (2009) proposed a model describing the effect of a climate change on a population in dimension 1. Berestycki and Rossi (2008) have further studied this model in higher dimensions and under more general hypotheses. It consists in the following reaction-diffusion equation

\[ \partial_t v - d \Delta v = f(x - ct, y, v), \quad t > 0, \quad (x, y) \in \mathbb{R}^2, \]

(7)

with \( f \) satisfying the same hypotheses (3) and (4) extended to the whole plane. The favorable zone moves with constant speed \( c \) in the \( x \)-direction. Let us mention that, if the nonlinearity \( f \) is even with respect to the vertical variable, i.e., if \( f(\cdot, y, \cdot) = f(\cdot, -y, \cdot) \) for every \( y \in \mathbb{R} \), then equation (7) is equivalent to the problem (5) “without the road”, at least for solutions which are even in the variable \( y \). It turns out that the
results of (Berestycki and Rossi 2008) hold true for such problem, as we explain in
details in Sect. 2.2 below.

In the frame moving with the favorable zone, (7) rewrites
\[ \partial_t v - d \Delta v - c \partial_x v = f(x, y, v), \quad t > 0, \quad (x, y) \in \mathbb{R}^2. \]  
(8)

The dependance of the nonlinear term in \( t \) disappears and is replaced by a drift-term. 
From a modeling point of view, a drift term can also describe a stream or a wind, 
or any such transport. Intuitively, the faster the wind, the harder it would be for the 
population to stay in the favorable zone (that does not move in this frame). Hence, 
the faster the favorable zone moves, the harder it should be for the population to keep track with it. This intuition is made rigorous by Berestycki and Rossi (2008), who proved the following:

**Proposition 1** (Berestycki and Rossi (2008)). There exists \( c_N \geq 0 \) such that

(i) If \( 0 \leq c < c_N \), there is a unique bounded positive stationary solution of (8), and any solution arising from a non-negative, not identically equal to zero, bounded initial datum converges to this stationary solution as \( t \) goes to \( +\infty \).

(ii) If \( c \geq c_N \), there is no bounded positive stationary solution of (8) and any solution arising from a non-negative, not identically equal to zero initial datum converges to zero uniformly as \( t \) goes to \( +\infty \).

In this proposition and in the sequel, we shall use the subscript \( N \), as in \( c_N \), to refer to quantities relative to models where there is no road.

Our system (1) is also inspired by the *road-field model*, introduced by Berestycki et al. (2013a). They studied the influence of a line with fast diffusion on a population in an environment governed by a homogeneous Fisher-KPP equation. Their model reads

\[
\begin{aligned}
\partial_t u - D \partial_{xx} u &= vv|_{y=0} - \mu u, \quad t > 0, \quad x \in \mathbb{R}, \\
\partial_t v - d \Delta v &= f(v), \quad t > 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d \partial_y v|_{y=0} &= \mu u - vv|_{y=0}, \quad t > 0, \quad x \in \mathbb{R}.
\end{aligned}
\]  
(9)

The novelty in our system (1) with respect to system (9) is that we allow the nonlinearity to depend on space and time variables. The main result on this model by Berestycki et al. (2013a) can be summarized as follows.

**Proposition 2** (Theorem 1.1 of (Berestycki et al. 2013a)). Invasion occurs in the direction of the road for system (9) with a speed \( c_H \). That is, for any solution \((u, v)\) of (9) arising from a compactly supported non-negative not identically equal to zero initial datum, there holds

\[
\forall h > 0, \forall c < c_H, \quad \sup_{|x| \leq ct, |y| \leq h} |v(t, x, y) - 1| \to 0, \quad \sup_{|x| \leq ct} \left| u(t, x) - \frac{v}{\mu} \right| \to 0, \quad t \to +\infty.
\]
Influence of a road on a population…

and

$$\forall c > c_H, \quad \sup_{|x| \geq ct, y \geq 0} v(t, x, y) \longrightarrow 0, \quad \sup_{|x| \geq ct} u(t, x) \longrightarrow 0.$$ 

Moreover, $c_H \geq c_{KPP}$ and

$$c_H > c_{KPP} \quad \text{if and only if} \quad D > 2d.$$ 

Recall that $c_{KPP} = 2\sqrt{df'(0)}$ is the speed of invasion for (6), that is, in the absence of a road. Hence, this result means that the speed of invasion in the direction of the road is enhanced, provided the diffusion on the road $D$ is large enough compared to the diffusion in the field $d$.

Several works have subsequently extended model (9) in several ways. Berestycki et al. (2013b) study the influence of drift terms and mortality on the road. In a further paper, Berestycki et al. (2016) compute the spreading speed in all directions of the field. Giletti et al. (2015) treats the case where the exchanges coefficients $\mu, \nu$ are not constant but periodic in $x$. Pauthier (2015, 2016) studies non-local exchanges and Dietrich (2015) considers a combustion nonlinearity instead of the KPP one together with other aspects of the problem. Berestycki et al. (2014, 2015) study the effect of non-local diffusion. Different geometric situations are considered by Rossi et al. (2017) and Ducasse (2018), namely the case where the field is a cylinder with its boundary playing the role of the road, and the case where the road is curved, respectively. Tellini (2016) considers the case of a strip with a line with fast diffusion on one side of the boundary and Dirichlet boundary conditions on the other side considered.

We mention that the climate-change model (7) is extended to more general nonlinearities than the KPP one considered here by Bouhours and Nadin (2015).

1.4 Main results

We assume in the whole paper that the nonlinearity $f$ is globally Lipschitz-continuous and that $v \mapsto f(x, y, v)$ is of class $C^1$ in a neighborhood of 0, uniformly in $(x, y)$. The hypotheses (2), (3) and (4) will also be understood to hold throughout the whole paper without further mention. For notational simplicity, we define

$$m(x, y) := f_v(x, y, 0).$$

This is a bounded function on $\mathbb{R} \times \mathbb{R}^+$.

As in the case of the climate change model (7), it is natural to work in the frame moving along with the forced shift. There, the system “with the road” (1) rewrites

$$\begin{cases}
\partial_t u - D\partial_{xx} u - c\partial_x u = \nu v|_{y=0} - \mu u, & t > 0, \quad x \in \mathbb{R}, \\
\partial_t v - d\Delta v - c\partial_x v = f(x, y, v), & t > 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d\partial_y v|_{y=0} = \mu u - \nu v|_{y=0}, & t > 0, \quad x \in \mathbb{R}.
\end{cases}$$
Likewise, in the moving frame the system “without the road” (5) takes the form:

$$\begin{align*}
\partial_t v - d\Delta v - c\partial_x v &= f(x, y, v), \quad t > 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-\partial_y v|_{y=0} &= 0, \quad t > 0, \quad x \in \mathbb{R}.
\end{align*}$$

(11)

In this paper we investigate the long-time behavior of solutions of (10) in comparison with (11). We will derive a dichotomy concerning two opposite scenarios: extinction or persistence.

**Definition 1** For the systems (10) or (11), we say that

(i) **extinction** occurs if every solution arising from a non-negative compactly supported initial datum converges uniformly to zero as $t$ goes to $+\infty$;

(ii) **persistence** occurs if every solution arising from a non-negative not identically equal to zero compactly supported initial datum converges locally uniformly to a positive stationary solution as $t$ goes to $+\infty$.

We will show that the positive stationary solution, when it exists, takes the form of a traveling pulse. In the original frame, it decays to zero as the spatial variable goes to infinity, due to the assumption (4).

In the Fisher-KPP setting considered in this paper, it is natural to expect the phenomena of extinction and persistence to be characterized by the stability of the null state $(0, 0)$, i.e., by the sign of the smallest eigenvalue $\lambda$ of the linearization of the system (10) around $(u, v) = (0, 0)$:

$$\begin{align*}
-D\partial_{xx}\phi - c\partial_x \phi - [v\psi|_{y=0} - \mu\phi] &= \lambda\phi, \quad x \in \mathbb{R}, \\
-d\Delta \psi - c\partial_x \psi - m(x, y)\psi &= \lambda\psi, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d\partial_y \psi|_{y=0} &= \mu\phi - v\psi|_{y=0}, \quad x \in \mathbb{R}.
\end{align*}$$

(12)

The smallest eigenvalue of an operator, when associated with a positive eigenfunction, is called the principal eigenvalue. When dealing with operators that satisfy some compactness and monotonicity properties, the existence of the principal eigenvalue can be deduced from the Krein-Rutman theorem, see (Krein and Rutman 1950). However, the system (12) is set on an unbounded domain. Hence, the Krein-Rutman theorem does not directly apply. Therefore, we make use in this paper of a notion of generalized principal eigenvalue, in the spirit of the one introduced by Berestycki et al. (1994) to deal with elliptic operators on non-smooth bounded domains under Dirichlet conditions. The properties of this notion have been later extended by Berestycki and Rossi (2015) to unbounded domains. The authors of the present paper introduced a notion of generalized principal eigenvalue adapted for road-field systems in (Berestycki et al. 2019), that we will use here.

In the sequel, $\lambda_1 \in \mathbb{R}$ denotes the generalized principal eigenvalue of (12). Its precise definition is given in (16) below and we also recall in Sect. 2 its relevant properties. Our first result states that indeed the sign of $\lambda_1$ characterizes the long-time behavior of (10). Namely, there is a dichotomy between persistence and extinction given in Definition 1, that is completely determined by the sign of $\lambda_1$.

**Theorem 3** Let $\lambda_1$ be the generalized principal eigenvalue of system (12).
Influence of a road on a population... 1067

(i) If $\lambda_1 < 0$, system (10) admits a unique positive bounded stationary solution and there is persistence.

(ii) If $\lambda_1 \geq 0$, system (10) does not admit any positive stationary solution and there is extinction.

A result analogous to Theorem 3 holds true for the system without the road (11), with $\lambda_1$ replaced by the corresponding generalized principal eigenvalue, see Proposition 15 below. This is a consequence of the results of (Berestycki and Rossi 2008), owing to the equivalence of behaviors in (11) and (8), that we explain in Sect. 2.2 below.

Questions 1 and 2 are then tantamount to understanding the relation between the generalized principal eigenvalues associated with models (10) and (11), and to analyze their dependance with respect to the parameters.

To this end, it will sometimes be useful to quantify the “size” of the favorable zone by considering terms $f$ given by

$$f_L(x, y, v) := \chi(|(x, y)| - L)v - v^2.$$  (13)

Here, $L \in \mathbb{R}$ represents the scale of the favorable region and $\chi$ is a smooth, decreasing function satisfying

$$\chi(r) \to 1 \text{ as } r \to -\infty \quad \text{and} \quad \chi(r) \to -1 \text{ as } r \to +\infty.$$

The nonlinearity $f^L$ satisfies both the Fisher-KPP condition (3) and the “bounded favorable zone” hypothesis (4). Consistently with our previous notations, we define

$$m^L(x, y) := f^L_v(x, y, 0) = \chi(|(x, y)| - L).$$  (14)

In this case, the favorable zone is the ball of radius $L + \chi^{-1}(0)$ (intersected with the upper half-plane), which is empty for $L \leq -\chi^{-1}(0)$. The fact that the favorable zone is a half-ball does not play any role in the sequel, and one could envision more general conditions.

We will first consider the case where $c = 0$, that is, when the niche is not moving (there is no climate change).

**Theorem 4** Assume that $c = 0$.

(i) Whatever the values of the parameters $D, \mu, v$ are, if extinction occurs for the system “without the road” (11), then extinction also occurs for the system “with the road” (10).

(ii) When $f = f^L$, there exist some values of the parameters $d, D, L, \mu, v$ for which persistence occurs for the system “without the road” (11) while extinction occurs for the system “with the road” (10).

Theorem 4 answers Question 1. Indeed, statement (i) means that the presence of the road can never entail the persistence of a population which would be doomed to extinction without the road. In other words, the road never improves the chances of survival of a population living in an ecological niche. Observe that this result was not obvious a priori: first, there is no death term on the road, so the road is not a lethal environment.
Second, if the favorable niche were made of, say, two connected components, one might have thought that a road “connecting” them might have improved the chances of persistence. Statement (i) shows that this intuition is not correct.

Statement (ii) asserts that the road can actually make things worse: there are situations where the population would persist in an ecological niche, but the introduction of a road drives it to extinction. This is due to an effect of “leakage” of the population due to the road.

In the context of this result, we can discuss the roles of the diffusion parameters $d$ and $D$, that represent the amplitudes of the random motion of individuals in the field and on the road.

**Theorem 5** Consider the system “with the road” (10), with $c = 0$. Then, there exists $d^* \geq 0$ depending on $D, \mu, \nu$ such that persistence occurs if and only if $0 < d < d^*$. In particular, extinction occurs for every $d > 0$ when $d^* = 0$.

This result is analogous to the one discussed for the model without road in the one dimensional case by Potapov and Lewis (2004) and Berestycki et al. (2009). The interpretation is that the larger $d$, the farther the population will scatter away from the favorable zone, with a negative effect for persistence. Observe that when $d^* = 0$, then persistence never occurs (the set $(0, d^*)$ is empty). This is the case if there is no favorable niche at all. However, $d^* > 0$ as soon as $f > 0$ somewhere. It is natural to wonder if a result analogous to Theorem 5 holds true when, instead of the diffusion $d$ in the field, one varies the diffusion on the road $D$, keeping $d > 0$ fixed. We show that this is not the case because there are situations where persistence occurs for all values of $D > 0$, see Proposition 19 below.

Next, we turn to Question 2, that is, we consider the case $c > 0$, corresponding to a moving niche. This movement can be caused e.g. by a climate change. We start with analyzing the influence of $c$ on the survival of the species for the system “with the road” (10). Owing to Theorem 3, this amounts to studying the generalized principal eigenvalue $\lambda_1$ as a function of $c$. For $x \in \mathbb{R}$, we write $[x]^+ := \max\{x, 0\}$.

**Theorem 6** There exist $0 \leq c_* \leq c^* \leq 2\sqrt{\max\{d, D\} \sup m}$, such that the following holds for the system (10):

(i) Persistence occurs if $0 \leq c < c_*$.

(ii) Extinction occurs if $c \geq c^*$.

Moreover, if persistence occurs for $c = 0$ then $c_* > 0$.

The quantities $c_*$ and $c^*$ are called the lower and upper critical speeds respectively for (10). Theorem 6 has a natural interpretation: on the one hand, if $c$ is large, the population cannot keep pace with the moving favorable zone, and extinction occurs. On the other hand, if persistence occurs in the absence of climate change, it will also be the case with a climate change with moderate speed.

We do not know if the lower and upper critical speeds actually always coincide, that is, if $c_* = c^*$. We prove that this is the case when $d = D$, but we leave the general case as an open question.

Finally, we investigate the consequences of the presence of a road for a population facing a climate change. To this end, we focus on the case where $f = f^L$, given
by (13). Observe that formally, as \( L \) goes to \(+\infty\), (10) reduces to the system (9), in the same moving frame. This suggests that the critical speeds for (10) should converge to the spreading speed \( c_H \) of Proposition 2. The next result makes this intuition rigorous.

**Theorem 7** Assume that \( f = f^L \) in (10). Then, the lower and upper critical speed \( c_\star, c^\star \) satisfy:

\[
c_\star, c^\star \nearrow c_H \quad \text{as} \quad L \nearrow +\infty,
\]

where \( c_H \) is given by Proposition 2.

The above theorem has the following important consequence.

**Corollary 8** Assume that \( D > 2d \). There are \( L > 0 \) and \( 0 < c_1 < c_2 \) such that, when \( f = f^L \),

(i) If \( c \in [0, c_1) \), persistence occurs for the model “with the road” (10) as well as for the model “without the road” (11).

(ii) If \( c \in (c_1, c_2) \), persistence occurs for the model “with the road” (10) whereas extinction occurs for the model “without the road” (11).

This result answers Question 2. Indeed, it means that, in some cases, the road can help the population to survive faster climate change than it would if there were no road. The threshold \( D > 2d \) in the theorem is the same threshold obtained by Berestycki et al. (2013a) for the road to induce an enhancement of the asymptotic speed of spreading.

The paper is organized as follows. In Sect. 2, we recall some results from (Berestycki et al. 2019), concerning the generalized principal eigenvalue for system (12). We explain in Sect. 2.2 why the systems on the half plane are equivalent to the systems on the whole plane. We prove Theorem 3 in Sect. 2.3. In Sect. 3, we study the effect of a road on an ecological niche, i.e., we consider (10) with \( c = 0 \). We prove Theorems 4 and 5 in Sects. 3.1 and 3.2 respectively. Section 4 deals with the effect of a road on a population facing climate change, i.e., system (10) with \( c > 0 \). We prove Theorem 6 in Sect. 4.1 and Theorem 7 in Sect. 4.2. Section 5 contains a brief discussion of the case where the favorable niche is not bounded, that is, when (4) does not hold. This situation is still by and large open.

## 2 The generalized principal eigenvalue and the long-time behavior

### 2.1 Definition and properties of the generalized principal eigenvalue

In this section, we recall some technical results from (Berestycki et al. 2019) concerning \( \lambda_1 \), the generalized principal eigenvalue for system (12). To simplify notations, we define the following linear operators:

\[
\begin{align*}
\mathcal{L}_1(\phi, \psi) & := D\partial_{xx}\phi + c\partial_x\phi + v\psi|_{y=0} - \mu\phi, \\
\mathcal{L}_2(\psi) & := d\Delta\psi + c\partial_x\psi + m(x, y)\psi, \\
\mathcal{B}(\phi, \psi) & := d\partial_y\psi|_{y=0} + \mu\phi - v\psi|_{y=0}.
\end{align*}
\]  

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These operators are understood to act on functions \((\phi, \psi) \in W^{2,p}_{loc}(\mathbb{R}) \times W^{2,p}_{loc}(\mathbb{R} \times [0, +\infty))\). We restrict to \(p > 2\), in order to have the imbedding in \(C^1_{loc}(\mathbb{R}) \times C^1_{loc}(\mathbb{R} \times [0, +\infty))\).

The generalized principal eigenvalue of (12) is defined by

\[
\lambda_1 := \sup \left\{ \lambda \in \mathbb{R} : \exists (\phi, \psi) \geq (0, 0), (\phi, \psi) \not\equiv (0, 0) \text{ such that} \right. \\
\left. L_1(\phi, \psi) + \lambda \phi \leq 0 \text{ on } \mathbb{R}, \quad L_2(\psi) + \lambda \psi \leq 0 \text{ on } \mathbb{R} \times \mathbb{R}^+, \right. \\
\text{and } B(\phi, \psi) \leq 0 \text{ on } \mathbb{R}\}. 
\]

(16)

Above and in the sequel, unless otherwise stated, the differential equalities and inequalities are understood to hold almost everywhere.

Owing to Theorem 3, proved below in this section, the sign of \(\lambda_1\) completely characterizes the long-time behavior for system (10). Therefore, to answer Questions 1 and 2, and the other questions addressed in this paper, we will study the dependence of \(\lambda_1\) with respect to the coefficients \(d, D, c\) as well as with respect to the parameter \(L\) in (14). The formula (16) is not always easy to handle, but there are two other characterizations of \(\lambda_1\) which turn out to be handy. First, \(\lambda_1\) is the limit of principal eigenvalues of the same problem restricted to bounded domains that converge to the half-plane. More precisely, calling \(B_R\) the (open) ball of radius \(R\) and of center \((0, 0)\) in \(\mathbb{R}^2\), we consider the increasing sequences of (non-smooth) domains \((\Omega_R)_{R>0}\) and \((I_R)_{R>0}\) given by

\[
\Omega_R := B_R \cap (\mathbb{R} \times \mathbb{R}^+) \quad \text{and} \quad I_R = (-R, R).
\]

We introduce the following eigenproblem:

\[
\begin{cases}
-\mathcal{L}_1(\phi, \psi) = \lambda \phi \quad \text{in } I_R, \\
-\mathcal{L}_2(\psi) = \lambda \psi \quad \text{in } \Omega_R, \\
B(\phi, \psi) = 0 \quad \text{in } I_R, \\
\psi = 0 \quad \text{on } (\partial \Omega_R) \setminus (I_R \times \{0\}), \\
\phi(-R) = \phi(R) = 0.
\end{cases}
\]

(17)

Here, the unknowns are \(\lambda \in \mathbb{R}, \phi \in W^{2,p}(I_R)\) and \(\psi \in W^{2,p}(\Omega_R)\). The existence of a principal eigenvalue and its connection with the generalized principal eigenvalue are given by the next result.

**Proposition 9** (Theorem 2.2 in (Berestycki et al. 2019)). For \(R > 0\), there is a unique \(\lambda_R \in \mathbb{R}\) and a unique (up to multiplication by a positive scalar) positive pair \((\phi_R, \psi_R) \in W^{2,p}(I_R) \times W^{2,p}(\Omega_R)\) that satisfy (17).

Moreover, the following limit holds true

\[
\lambda_1^R \xrightarrow{R \to +\infty} \lambda_1.
\]
Finally, there is a generalized principal eigenfunction associated with $\lambda_1$, that is, a pair $\varphi, \psi \in W^{2, p}_{\text{loc}}(\mathbb{R}) \times W^{2, p}_{\text{loc}}(\mathbb{R} \times \{0, +\infty\})$, $(\varphi, \psi) \geq 0$, $(\varphi, \psi) \neq (0, 0)$ satisfying $L_1(\varphi, \psi) = \lambda_1 \varphi$, $L_2(\psi) = \lambda_1 \psi$ and $B(\varphi, \psi) = 0$.

We refer the reader to (Berestycki et al. 2019) for the details. The real number $\lambda^R_1$ and the pair $(\varphi^R, \psi^R)$ are called respectively the principal eigenvalue and eigenfunction of (17).

The next characterization of $\lambda_1$ is obtained in the case when $c = 0$. It is in the spirit of the classical Rayleigh-Ritz formula. We introduce the following Sobolev space:

$$
\tilde{H}^1_0(\Omega_R) := \{ u \in H^1(\Omega_R) : u = 0 \text{ on } (\partial B_R) \cap (\mathbb{R} \times \mathbb{R}^+) \text{ in the sense of the trace} \}.
$$

**Proposition 10** (Proposition 4.5 in (Berestycki et al. 2019)). Assume that $c = 0$. The principal eigenvalue $\lambda^R_1$ of (17) satisfies

$$
\lambda^R_1 = \inf_{(\varphi, \psi) \in \mathcal{H}_R} \frac{\mu \int_{I_R} D|\varphi'|^2 + \nu \int_{\Omega_R} (d|\nabla \psi|^2 - m \psi^2) + \int_{I_R} (\mu \varphi - \nu \psi|_{y=0})^2}{\mu \int_{I_R} \varphi^2 + \nu \int_{\Omega_R} \psi^2},
$$

where we recall that $m = f_v(\cdot, \cdot, 0)$, and

$$
\mathcal{H}_R := H^1_0(I_R) \times \tilde{H}^1_0(\Omega_R).
$$

Let us also recall the following result concerning the continuity and monotonicity of $\lambda_1$. We use the notation $\lambda_1(c, L, d, D)$ to indicate the generalized principal eigenvalue of (12), with coefficients $c, d, D$ and with $m = m^L$ given by (14). Then, we treat $\lambda_1$ as a function from $(\mathbb{R})^2 \times (\mathbb{R}^+)^2$ to $\mathbb{R}$. Analogous notations will be used several times in the sequel. The following proposition gathers Propositions 2.4 and 2.5 of (Berestycki et al. 2019).

**Proposition 11** Let $\lambda_1(c, L, d, D)$ be the generalized principal eigenvalue of system (12) with $m = m^L$ defined by (14). Then,

- $\lambda_1(c, L, d, D)$ is a locally Lipschitz-continuous function on $(\mathbb{R})^2 \times (\mathbb{R}^+)^2$.
- If $c = 0$, then $\lambda_1(c, L, d, D)$ is non-increasing with respect to $L$ and non-decreasing with respect to $d$ and $D$.
- If $c = 0$ and $\lambda_1(c, L, d, D) \leq 0$, then $\lambda_1(c, L, d, D)$ is strictly decreasing with respect to $L$ and strictly increasing with respect to $d$ and $D$.

Next, we consider the generalized principal eigenvalue for the model “without the road” (11), that we require to answer Questions 1 and 2. The eigenproblem associated with the linearization around $v = 0$ of the stationary system associated with (11) reads

$$
\begin{align*}
-L_2(\psi) &= \lambda \psi, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-\partial_y \psi|_{y=0} &= 0, \quad x \in \mathbb{R},
\end{align*}
$$

(19)
where \( L_2 \) is defined in (15). The generalized principal eigenvalue of (19) is given by

\[
\lambda_N := \sup \left\{ \lambda \in \mathbb{R} : \exists \psi \geq 0, \psi \neq 0 \text{ such that } (L_2(\psi) + \lambda \psi) \leq 0 \text{ on } \mathbb{R} \times \mathbb{R}^+, \partial_y \psi |_{y=0} \leq 0 \text{ on } \mathbb{R} \right\}.
\]

(20)

The subscript \( N \) refers to the Neumann boundary condition, and also to the fact that there is no road in (11). Again, the test functions \( \psi \) in (20) are assumed to be in \( W^{2,p}_{loc}(\mathbb{R} \times [0, +\infty)) \). We also consider the principal eigenvalue on the truncated domains \( \Omega_R \), which is the unique quantity \( \lambda_{N}^R \) such that the problem

\[
\begin{cases}
-\mathcal{L}_2(\psi) = \lambda_{N}^R \psi, & (x, y) \in \Omega_R, \\
-\partial_y \psi |_{y=0} = 0, & x \in I_R, \\
\psi(x, y) = 0, & (x, y) \in (\partial \Omega_R) \setminus (I_R \times \{0\}),
\end{cases}
\]

(21)

admits a positive solution \( \psi \in W^{2,p}(\Omega_R) \). The results concerning \( \lambda_1 \) hold true for \( \lambda_N \). We gather them in the following proposition.

**Proposition 12** Let \( \lambda_N \) be the generalized principal eigenvalue of the model “without the road” (19), and let \( \lambda_{N}^R \) be the principal eigenvalue of (21). Then

- \( \lambda_N \) is the decreasing limit of \( \lambda_{N}^R \), i.e.,

\[
\lim_{R \to +\infty} \lambda_{N}^R = \lambda_N.
\]

(22)

- If \( c = 0 \), then

\[
\lambda_{N}^R = \inf_{\psi \in \tilde{H}_0^1(\Omega_R), \psi \neq 0} \frac{\int_{\Omega_R} (d|\nabla \psi|^2 - m\psi^2)}{\int_{\Omega_R} \psi^2}.
\]

(23)

- If the zero-th order term in (19) is given by \( m = m^L \), defined in (14), then \( L \mapsto \lambda_N(L) \) is a continuous and non-increasing function.

The two first points are readily derived from (Berestycki and Rossi 2008): indeed, \( \lambda_N \) coincides with the generalized principal eigenvalue of the problem in the whole space (8) with \( f \) extended by symmetry, as explained in the next Sect. 2.2. The third point concerning the monotonicity and the continuity comes from the paper by Berestycki and Rossi (2015).

### 2.2 The case of the whole plane

Systems (10) and (11) are set on half-planes. Let us explain here how these models are actually equivalent to the same systems set on the whole plane under a symmetry hypothesis on the nonlinearity. When writing a road-field system where the road is not
the boundary of a half-plane but a line in the middle of a plane, one needs to consider 3 equations: one equation for each portion separated by the road and an equation on the road, completed with two exchange conditions between the road and each side of the field. We assume that the exchanges are the same between the road and the two sides of the field. Moreover, we assume that the environmental conditions are symmetric with respect to the road, that is, the nonlinearity $f$ on the field is even with respect to the $y$ variable, i.e., $f(x, y, v) = f(x, -y, v)$ for every $(x, y) \in \mathbb{R}^2$ and $v \geq 0$. The system then writes (in the moving frame that follows the climate change):

\[
\begin{aligned}
\partial_t u - D \partial_{xx} u - c \partial_x u &= v(v|_{y=0^+} + v|_{y=0^-}) - \mu u, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_t v - d \Delta v - c \partial_x v &= f(x, y, v), \quad t > 0, \ (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
\pm d \partial_y v|_{y=0^=} &= \frac{\mu}{2} u - \nu v|_{y=0^+}, \quad t > 0, \ x \in \mathbb{R}.
\end{aligned}
\]

(24)

We point out that the set in the second equation has two connected components and thus it can be treated as two distinct equations. The last line in (24) are also two equations with the proportion $\mu$ of $u$ leaving the road evenly split among the two sides.

Under these hypotheses of symmetry, the dynamical properties of the system (24) are the same as those of the system on the half-plane (10). This is clear if one restricts to a symmetric initial datum $(u_0, v_0)$, i.e., such that $v_0(x, y) = v_0(x, -y)$ for every $(x, y) \in \mathbb{R}^2$. Indeed, the corresponding solution $(u, v)$ of (24) also satisfies $v(t, x, y) = v(t, x, -y)$ for every $t > 0, (x, y) \in \mathbb{R}^2$, hence

\[
\begin{aligned}
\partial_t u - D \partial_{xx} u - c \partial_x u &= 2v v|_{y=0^+} - \mu u, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_t v - d \Delta v - c \partial_x v &= f(x, y, v), \quad t > 0, \ (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
- d \partial_y v|_{y=0^=} &= \frac{\mu}{2} u - \nu v|_{y=0^+}, \quad t > 0, \ x \in \mathbb{R}.
\end{aligned}
\]

It follows that $(\tilde{u}, v) := (\frac{1}{2} u, v)$ is a solution of the system with the road in the half-plane (10). For non-symmetric solutions of (24), the long-time behavior also turns out to be governed by (10). Indeed, any solution of (24) arising from a non zero initial datum is strictly positive at time $t = 1$, and can then be nested between two symmetric solutions, both converging to the same stationary solution.

Let us mention that, in the paper (Berestycki et al. 2019), where we define and study the notion of generalized principal eigenvalues for road-field systems, we also consider the case of non-symmetric fields.

By the same arguments as above, the problem without the road in the half-plane (11) is also seen to share the same dynamical properties as the equation in the whole plane (8). Actually, a stronger statement holds true concerning the linearized stationary equations

\[ - \mathcal{L}_2(\psi) = 0, \quad (x, y) \in \mathbb{R}^2, \]

without any specific assumptions on $m$ (besides regularity).

**Lemma 1** Assume that the nonlinearity $f(x, y, v)$ in (8) is even with respect to the variable $y$. Then extinction (resp. persistence) occurs for (8) if and only if it occurs for (11).
Moreover, (25) admits a positive supersolution (resp. subsolution) if and only if (19) with λ = 0 does.

**Proof** We have explained before that the long-time behavior for (8) can be reduced to the one for (11), that is, the first statement of the lemma holds.

For the second statement, consider a positive supersolution ω of (25). We have $L_2(ω) ≤ 0$ on $\mathbb{R}^2$ and this inequality also holds true for $\tilde{ω}(x, y) := ω(x, -y)$. Hence, the function $ψ := ω + \tilde{ω}$ satisfies $L_2(ψ) ≤ 0$ on $\mathbb{R} \times \mathbb{R}^+$ and $∂_y ψ|_{y=0} = 0$, i.e., $ψ$ is a positive supersolution for (19).

Take now a positive supersolution $ψ$ of (19), that is, $L_2(ψ) ≤ 0$ on $\mathbb{R} \times \mathbb{R}^+$ and $∂_y ψ|_{y=0} ≤ 0$ on $\mathbb{R}$. One would like to use the function $ψ(t, x, |y|)$ as a supersolution for (25), however this function is not in $W^{2,p}(\mathbb{R}^2)$. To overcome this difficulty, define $\tilde{ψ}(t, x, y)$ to be the solution of the parabolic problem

$$∂_t \tilde{ψ} = L_2(\tilde{ψ}), \quad t > 0, \quad (x, y) \in \mathbb{R}^2,$$

with initial datum $ψ(x, |y|)$. The function $\tilde{ψ}(t, x, y)$ is positive (this is a consequence of the parabolic comparison principle) and is in $W^{2,p}(\mathbb{R}^2)$. Moreover the parabolic comparison principle yields $∂_y \tilde{ψ} ≤ 0$, hence the function $ω(x, y) := \tilde{ψ}(1, x, y)$ satisfies $L_2(ω) ≤ 0$ on $\mathbb{R}^2$, i.e., it is a positive supersolution to (8).

The second part of Lemma 1 applied to $L_2 + λ$ implies that the operator $L_2$ set on $\mathbb{R} \times \mathbb{R}^+$ with Neumann boundary conditions and the operator $L_2$ set on $\mathbb{R}^2$ share the same generalized principal eigenvalue, that is:

$$λ_N = \sup \left\{ λ \in \mathbb{R} : \exists ψ ≥ 0, \ ψ \neq 0 \text{ such that } (L_2(ψ) + λψ) ≤ 0 \text{ on } \mathbb{R}^2 \right\}.$$

### 2.3 The long-time behavior for the system with the road

This section is dedicated to proving Theorem 3. We first derive in Sect. 2.3.1 a Liouville-type result, namely we show that there is at most one non-negative, not identically equal to zero, bounded stationary solution of the semilinear system (10). This will be used in Sect. 2.3.2 to characterize the asymptotic behavior of solutions of the evolution problem (10) in terms of the generalized principal eigenvalue $λ_1$.

For future convenience, let us state the parabolic strong comparison principle for the road-field system (10). It was obtained by Berestycki et al. (2013a) with $f$ independent of $x, y$, but the proof does not change if one adds this dependence.

We say that a pair $(u, v)$ is a supersolution (resp. subsolution) of (10) if it solves (10) with all the signs $=$ replaced by $≥$ (resp. $≤$).

**Proposition 13** Let $(u_1, v_1)$ and $(u_2, v_2)$ be respectively a bounded sub and supersolution of (10) such that $(u_1, v_1) ≤ (u_2, v_2)$ at time $t = 0$. Then $(u_1, v_1) ≤ (u_2, v_2)$ for all $t > 0$, and the inequality is strict unless they coincide until some $t > 0$. 
Remark 1 The previous comparison principle applies in particular to stationary sub and supersolutions, providing the strong comparison principle for the elliptic system associated to (10). Namely, if a stationary subsolution touches from below a stationary supersolution then they must coincide everywhere.

2.3.1 A Liouville-type result

We derive here the uniqueness of stationary solutions for (10).

Proposition 14 There is at most one non-null bounded positive stationary solution of (10).

Before turning to the proof of Proposition 14, we state a technical lemma.

Lemma 2 Let \((u, v)\) be a solution of the evolution problem (10) arising from a bounded non-negative initial datum. Then

\[
\sup_{|x| \geq R} u(t, x) \rightarrow 0 \text{ and } \sup_{|x, y| \geq R} v(t, x, y) \rightarrow 0.
\]

Proof We first show that the conclusion of the lemma holds for the component \(v\). Assume by contradiction that there are \(\varepsilon > 0\) and two diverging sequences, \((t_n)_{n \in \mathbb{N}}\) in \((0, +\infty)\) and \(((x_n, y_n))_{n \in \mathbb{N}}\) in \(\mathbb{R} \times (0, +\infty)\), such that

\[
\liminf_{n \to +\infty} v(t_n, x_n, y_n) \geq \varepsilon.
\]

The idea is to consider the equations satisfied by some translations of \(u, v\). We divide the discussion into two different cases.

First case: \((y_n)_{n \in \mathbb{N}}\) is unbounded.

Up to extraction of a subsequence, we assume that \(y_n\) goes to \(+\infty\) as \(n\) goes to \(+\infty\). We define the translated functions

\[
u_n := u(\cdot + t_n, \cdot + x_n) \text{ and } v_n := v(\cdot + t_n, \cdot + x_n, \cdot + y_n).
\]

Because \(((x_n, y_n))_{n \in \mathbb{N}}\) diverges, the zone where \(f\) is positive disappears in the limit. More precisely, (4) and (3) yield that there is \(K > 0\) such that

\[
\limsup_{n \to +\infty} f(x + x_n, y + y_n, z) \leq -Kz \text{ for } (x, y) \in \mathbb{R}^2, z \geq 0. \tag{26}
\]

The parabolic estimates (see, for instance, Theorems 5.2 and 5.3 in the book by Ladyzenskaja et al. (1967)) and (26) yield that, up to another extraction, \(v_n\) converges as \(n\) goes to \(+\infty\) locally uniformly to some \(v_\infty\), entire (i.e., defined for all \(t \in \mathbb{R}\)) subsolution of

\[
\partial_t v - d \Delta v - c \partial_x v + K v = 0, \quad t \in \mathbb{R}, \ (x, y) \in \mathbb{R}^2. \tag{27}
\]
Moreover,
\[ v_\infty(0, 0, 0) \geq \varepsilon. \]
Observe that, for any \( A > 0 \), the space-independent function \( w(t) := Ae^{-Kt} \) is a solution of (27). Because \( v_\infty \) is bounded, we can choose \( A \) large enough so that, for every \( \tau \geq 0 \),
\[ v_\infty(-\tau, \cdot, \cdot) \leq w(0). \]
The parabolic comparison principle for (27) yields
\[ v_\infty(t - \tau, \cdot, \cdot) \leq Ae^{-Kt} \text{ for } t \geq 0, \tau \in \mathbb{R}. \]
Choosing \( t = \tau \), we get
\[ \varepsilon \leq v_\infty(0, 0, 0) \leq Ae^{-K\tau}. \]
Taking the limit \( \tau \to +\infty \) yields a contradiction.

**Second case:** \((y_n)_{n\in\mathbb{N}}\) is bounded.
Up to a subsequence, we assume that \( y_n \) converges to some \( y_\infty \geq 0 \) as \( n \) goes to \( +\infty \).
We now define the translated functions with respect to \((t_n)_{n\in\mathbb{N}}, (x_n)_{n\in\mathbb{N}}\) only:
\[ u_n := u(\cdot + t_n, \cdot + x_n) \quad \text{and} \quad v_n := v(\cdot + t_n, \cdot + x_n, \cdot). \]
Arguing as in the first case, we find that \(((u_n, v_n))_{n\in\mathbb{N}}\) converges locally uniformly (up to a subsequence) as \( n \) goes to \( +\infty \) to \((u_\infty, v_\infty)\), entire subsolution of
\[
\begin{aligned}
\partial_t u - D \partial_{xx} u - c \partial_x u &= \nu v|_{y=0} - \mu u, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}, \\
\partial_t v - d \Delta v - c \partial_x v + M v &= 0, \quad t \in \mathbb{R}, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d \partial_y v|_{y=0} &= \mu u - \nu v|_{y=0}, \quad t \in \mathbb{R}, \quad x \in \mathbb{R}.
\end{aligned}
\] (28)
Moreover, we have
\[ v_\infty(0, 0, y_\infty) \geq \varepsilon. \]
Unlike the previous case, one does not have suitable space-independent solutions of (28). Instead, we look for a supersolution of the form
\[ A(e^{-\eta t}, \gamma e^{-\eta t} (e^{-\beta y} + 1)). \]
An easy computation shows that for this to be a supersolution of (28) for any \( A > 0 \), it is sufficient to choose the parameters \( \eta, \gamma, \beta > 0 \) so that
\[
\begin{aligned}
-\eta(e^{-\beta y} + 1) - d \beta^2 e^{-\beta y} + M(e^{-\beta y} + 1) &\geq 0, \quad \text{for all } y \geq 0, \\
-d \beta \gamma &\geq \mu - 2\nu \gamma.
\end{aligned}
\]
For \( \eta < \mu \), we take \( \gamma = \frac{\mu - \eta}{2\nu} > 0 \), \( \beta = \frac{\eta}{d \gamma} \), so that the first and third inequalities are automatically satisfied. For the second inequality to hold true for \( y \geq 0 \), it is sufficient to have \( -\eta - d \beta^2 + M \geq 0 \). We can take \( \eta \) sufficiently small so that the latter is fulfilled.
We now choose \( A \) large enough so that \( A \geq \sup u_\infty, \gamma A \geq \sup v_\infty \). A contradiction is reached by arguing as in the first case, with the difference that now we need to use the parabolic comparison principle for the full road-field system, Proposition 13.

We have shown that

\[
\sup_{|t-x|\geq R} v(t,x,y) \to 0 \quad \text{as} \quad R \to +\infty.
\]

Let us now derive the result for \( u \). Assume by contradiction that there are \( \varepsilon > 0 \) and two diverging sequences \( (t_n)_{n \in \mathbb{N}} \) and \( (x_n)_{n \in \mathbb{N}} \) such that

\[
\liminf_{n \to +\infty} u(t_n,x_n) > \varepsilon.
\]

Then, because of (29), for \( n \) large enough the third equation in (10) gives us that

\[
\partial_y v(t_n,x_n,0) \leq -\frac{\mu}{2d} \varepsilon.
\]

The parabolic estimates then provide a constant \( C > 0 \) such that for, say, \( y \in (0,1) \), there holds that

\[
v(t_n,x_n,y) \leq v_n(t_n,x_n,0) - \frac{\mu}{2d} \varepsilon y + Cy^2.
\]

From this, taking \( y > 0 \) small enough and then \( n \) large enough, and using again (29), we deduce that \( v(t_n,x_n,y) < 0 \), which is impossible, hence the contradiction. \( \square \)

We now turn to the proof of Proposition 14.

**Proof of Proposition 14** Let \((u,v)\) and \((\tilde{u},\tilde{v})\) be two non-null non-negative bounded stationary solutions of the system (10). We will prove that they coincide. We define, for \( \varepsilon > 0 \),

\[
(u_\varepsilon,v_\varepsilon) := \left(u + \varepsilon, v + \varepsilon\right),
\]

and

\[
\theta_\varepsilon := \max\{\theta > 0 : (u_\varepsilon,v_\varepsilon) \geq \theta(\tilde{u},\tilde{v})\},
\]

which is positive. Let us show that one of the following occurs:

- Either \( \exists x_\varepsilon' \in \mathbb{R}, u_\varepsilon(x_\varepsilon') = \theta_\varepsilon \tilde{u}(x_\varepsilon) \)
- or \( \exists (x_\varepsilon,y_\varepsilon) \in \mathbb{R} \times [0, +\infty), v_\varepsilon(x_\varepsilon,y_\varepsilon) = \theta_\varepsilon \tilde{v}(x_\varepsilon,y_\varepsilon) \).

By definition of \( \theta_\varepsilon \), for every \( n \in \mathbb{N} \), we can find either \( x_n' \) such that \( u_n(x_n') < (\theta_\varepsilon + \frac{1}{n}) \tilde{u}(x_n') \), or \( (x_n,y_n) \) such that \( v_n(x_n,y_n) < (\theta_\varepsilon + \frac{1}{n}) \tilde{v}(x_n,y_n) \). For every \( \varepsilon > 0 \), the norm of these points is bounded independently of \( n \in \mathbb{N} \), because \( \tilde{u} \) and \( \tilde{v} \) converge to zero at infinity, by Lemma 2. Therefore, because either \( (x_n')_n \) or \( ((x_n,y_n))_n \) has an infinite number of elements, we can define \( x'_{\varepsilon} \) or \( (x_\varepsilon,y_\varepsilon) \) as the limit of a subsequence of either \( (x_n')_n \) or \( ((x_n,y_n))_n \) respectively.
By definition, the positive reals \((\theta_\varepsilon)_{\varepsilon \geq 0}\) are increasing with respect to \(\varepsilon\). We define 
\[
\theta_0 := \lim_{\varepsilon \to 0^+} \theta_\varepsilon.
\]
The rest of the proof is dedicated to show that \(\theta_0 \geq 1\). This will yield that \(u \geq \tilde{u}\) and \(v \geq \tilde{v}\). Exchanging the roles of \((u, v)\) and \((\tilde{u}, \tilde{v})\) in what precedes, we then get \(u = \tilde{u}\) and \(v = \tilde{v}\), hence uniqueness.

We argue by contradiction, assuming that \(\theta_0 < 1\). From now on, we assume that \(\varepsilon\) is small enough so that \(\theta_\varepsilon < 1\). We proceed in two steps: we first derive some estimates for the contact points \(x_\varepsilon'\) or \((x_\varepsilon, y_\varepsilon)\), then we use them to get a contraction.

**Step 1. Boundedness of the contact points.**

This step is dedicated to show that there is \(R > 0\) such that
\[
\forall \varepsilon > 0, \quad |x_\varepsilon'| \leq R \text{ or } |(x_\varepsilon, y_\varepsilon)| \leq R,
\]
i.e., \(|x_\varepsilon'|\) or \(|(x_\varepsilon, y_\varepsilon)|\) are bounded independently of \(\varepsilon\). From the uniform regularity of \(f\) together with (4), we infer that there is \(\eta > 0\) so that \(v \mapsto f(x, y, v)\) is non-increasing in \([0, \eta]\) if \(|(x, y)| \geq R\). Because \((u, v)\) is a stationary solution of (10), Lemma 2 implies that \(v(x, y)\) goes to zero as \(|(x, y)|\) goes to \(+\infty\). Hence, up to decreasing \(\varepsilon\) so that \(\varepsilon < \eta\) and up to increasing \(R\), we assume that
\[
v(x, y) \leq \eta - \varepsilon \quad \text{for } |(x, y)| \geq R.
\]

Hence, if \(|(x, y)| \geq R\), we see that
\[-d \Delta v_\varepsilon - c \partial_x v_\varepsilon - f(x, y, v_\varepsilon) \geq f(x, y, v) - f(x, y, v_\varepsilon) \geq 0.
\]

Therefore,
\[
\begin{cases}
-d \partial_x u_\varepsilon - c \partial_x u_\varepsilon = \nu v_\varepsilon|_{y=0} - \mu u_\varepsilon, & \text{for } |x| > R, \\
-d \Delta v_\varepsilon - c \partial_x v_\varepsilon \geq f(x, y, v_\varepsilon), & \text{for } |(x, y)| > R, \\
-d \partial_y v_\varepsilon|_{y=0} = \mu u_\varepsilon - \nu v_\varepsilon|_{y=0}, & \text{for } |x| > R.
\end{cases}
\]

Moreover, because we assumed that \(\theta_\varepsilon < 1\), we have \(\theta_\varepsilon f(x, y, \tilde{v}) \leq f(x, y, \theta_\varepsilon \tilde{v})\). Hence, \((\theta_\varepsilon \tilde{u}, \theta_\varepsilon \tilde{v})\) is a stationary subsolution of (10). Because \((u_\varepsilon, v_\varepsilon) \geq (\theta_\varepsilon \tilde{u}, \theta_\varepsilon \tilde{v})\), the elliptic strong comparison principle (see Remark 1) yields that, if the point at which we have either \(u_\varepsilon(x_\varepsilon') = \theta_\varepsilon \tilde{u}(x_\varepsilon')\) or \(v_\varepsilon(x_\varepsilon, y_\varepsilon) = \theta_\varepsilon \tilde{v}(x_\varepsilon, y_\varepsilon)\) satisfied
\[
|x_\varepsilon'| > R \text{ or } |(x_\varepsilon, y_\varepsilon)| > R,
\]
then we would have
\[
(u_\varepsilon, v_\varepsilon) \equiv \theta_\varepsilon (\tilde{u}, \tilde{v}).
\]

This is impossible because \(u(x) \to \varepsilon \frac{\nu}{\mu} \tilde{u}\) and \(\tilde{u}(x) \to 0\) as \(|x|\) goes to \(+\infty\). We have reached a contradiction, showing that (30) holds true.

**Step 2. Taking the limit \(\varepsilon \to 0\).**

\[\square \text{ Springer}\]
The estimate (30) implies that, up to extraction of a suitable subsequence, either \( x'_\varepsilon \) or \((x_\varepsilon, y_\varepsilon)\) converge as \( \varepsilon \) goes to zero to some limit \( x_0 \in \mathbb{R} \) or \((x_0, y_0) \in \mathbb{R} \times [0, +\infty)\). Hence

\[
\begin{align*}
  u &\geq \theta_0 \tilde{u}, \\
v &\geq \theta_0 \tilde{v},
\end{align*}
\]

and either \( u(x_0) = \theta_0 \tilde{u}(x_0) \) or \( v(x_0, y_0) = \theta_0 \tilde{v}(x_0, y_0) \). Because \((u, v) > (0, 0)\), owing to the elliptic strong comparison principle (cf. Remark 1), this yields \( \theta_0 > 0 \). As before, we can use the elliptic strong comparison principle for (10) (see Remark 1) with the solution \((u, v)\) and the subsolution \(\theta_0(\tilde{u}, \tilde{v})\) to find that these couples coincide everywhere, namely

\[
\begin{align*}
u_0 \geq \theta_0 \tilde{u}, \quad v &\equiv \theta_0 \tilde{v}.
\end{align*}
\]

Plotting the latter in (10) we obtain

\[
\theta_0 f(x, y, \tilde{v}) = f(x, y, \theta_0 \tilde{v}) \quad \text{for} \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+.
\]

Recalling that we assumed that \( \theta_0 < 1 \), we find a contradiction with the hypothesis (3). This shows that \( \theta_0 \geq 1 \), which concludes the proof.

2.3.2 The persistence/extinction dichotomy

We proved in the previous section that there is at most one non-trivial bounded positive stationary solution of the semilinear problem (10). Building on that, we prove now Theorem 3.

Proof of Theorem 3 In the whole proof, \((u, v)\) denotes the solution of the parabolic problem (10) arising from a non-negative not identically equal to zero compactly supported initial datum \((u_0, v_0)\). We prove separately the two statements of the Theorem.

Statement (i).

Assume that \( \lambda_1 < 0 \). Owing to Proposition 9, we can take \( R > 0 \) large enough so that \( \lambda_1^R < 0 \). Let \((\phi_R, \psi_R)\) be the corresponding principal eigenfunction provided by Proposition 9. Using the fact the \( u(1, \cdot) > 0 \) and \( v(1, \cdot, \cdot) > 0 \), as a consequence of the parabolic comparison principle Proposition 13, and that \((\phi_R, \psi_R)\), extended by \((0, 0)\) outside of its support, is compactly supported, we can find \( \varepsilon > 0 \) such that

\[
\varepsilon(\phi_R, \psi_R) \leq (u(1, \cdot), v(1, \cdot, \cdot)).
\]

Up to decreasing \( \varepsilon \), the regularity hypotheses on \( f \) combined with the fact that \( \lambda_1^R < 0 \) implies that \( \varepsilon(\phi_R, \psi_R) \) (extended by \((0, 0)\) outside its support) is a generalized stationary subsolution of (10). On the other hand, for \( M > 0 \) sufficiently large, the pair \((\frac{\nu}{\mu} M, M)\) is a stationary supersolution of (10), due to hypothesis (2). Up to increasing \( M \), we can assume that \((\frac{\nu}{\mu} M, M) > (u(1, \cdot), v(1, \cdot, \cdot))\).

As a standard application of the parabolic comparison principle, Proposition 13, one sees that the solution of (10) arising from \( \varepsilon(\phi_R, \psi_R) \) (respectively from \((\frac{\nu}{\mu} M, M)\)) is time-increasing (respectively time-decreasing), and converges locally uniformly to
a stationary solution, thanks to the parabolic estimates. Owing to the elliptic strong comparison principle (see Remark 1), this solution is positive. Proposition 14 implies that these limiting solutions are actually equal, and by comparison \((u, v)\) also converges to this positive stationary solution. This proves the statement \((i)\) of the theorem.

**Statement (ii).**
Assume now that \(\lambda_1 \geq 0\). Let \((U, V)\) be a bounded non-negative stationary solution of (10). We start to show that \((U, V) \equiv (0, 0)\). We argue by contradiction, assuming that this stationary solution is not identically equal to zero. Let \((\phi, \psi)\) be a positive generalized principal eigenfunction associated with \(\lambda_1\), provided by Proposition 9. The fact that \(\lambda_1 \geq 0\), combined with the Fisher-KPP hypothesis (3), implies that \((\phi, \psi)\) is a supersolution of (10). For \(\varepsilon > 0\), we define

\[
\theta_\varepsilon := \max \left\{ \theta > 0 : (\phi, \psi) + \left( \frac{\psi}{\mu}, \varepsilon \right) \geq \theta (U, V) \right\}.
\]

Hence, for \(\varepsilon > 0\), there is either \(x'_\varepsilon \in \mathbb{R}\) or \((x_\varepsilon, y_\varepsilon) \in \mathbb{R} \times [0, +\infty)\) such that

\[
\phi(x'_\varepsilon) + \varepsilon \frac{\psi}{\mu} = \theta_\varepsilon U(x'_\varepsilon) \quad \text{or} \quad \psi(x_\varepsilon, y_\varepsilon) + \varepsilon = \theta_\varepsilon V(x_\varepsilon, y_\varepsilon).
\]

Arguing as in the proof of Proposition 14, Step 1, we find that the norm of the contact points \(x'_\varepsilon\) or \((x_\varepsilon, y_\varepsilon)\) is bounded independently of \(\varepsilon\). Because \(\theta_\varepsilon\) is increasing with respect to \(\varepsilon\), it converges to a limit \(\theta_0 \geq 0\) as \(\varepsilon\) goes to 0. Up to extraction, we have that either \(x'_0\) or \((x_0, y_0)\) converges to some \(x'_0 \in \mathbb{R}\) or \((x_0, y_0) \in \mathbb{R} \times [0, +\infty)\) as \(\varepsilon\) goes to zero. Taking the limit \(\varepsilon \to 0\) then yields

\[
(\phi, \psi) \geq \theta_0 (U, V),
\]

and either \(\phi(x'_0) = \theta_0 U(x'_0)\) or \(\psi(x_0, y_0) = \theta_0 V(x_0, y_0)\). In both cases, the elliptic strong comparison principle, cf. Remark 1, implies that

\[
(\phi, \psi) \equiv \theta_0 (U, V),
\]

Owing to the hypothesis (3) on \(f\), this is possible only if \(\theta_0 = 0\), but this would contradict the strict positivity of \((\phi, \psi)\). We have reached a contradiction: there are no non-negative non-null bounded stationary solutions of (10) when \(\lambda_1 \geq 0\).

We can now deduce that extinction occurs: for a given compactly supported initial datum \((u_0, v_0)\), we choose \(M > 0\) large enough so that the couple \(\left( \frac{\psi}{\mu} M, M \right)\) is a stationary supersolution of (10) and, in addition,

\[
\left( \frac{\psi}{\mu} M, M \right) \geq (u_0, v_0).
\]

We let \((\bar{u}, \bar{v})\) denote the solution of (10) arising from the initial datum \((\frac{\psi}{\mu} M, M)\). It is time-decreasing and converges locally uniformly to a stationary solution. The only one being \((0, 0)\), owing to Proposition 14, we infer that \((\bar{u}, \bar{v})\) converges to zero.
locally uniformly as \( t \) goes to \(+\infty\). Lemma 2 implies that the convergence is actually uniform. The same holds for \((u, v)\), thanks to the parabolic comparison principle, Proposition 13. 

We conclude this section by stating the dichotomy analogous to Theorem 3 in the case of the system “without the road”, (11).

**Proposition 15** Let \( \lambda_N \) be the generalized principal eigenvalue of the system (19).

(i) If \( \lambda_N < 0 \), the system (11) admits a unique positive bounded stationary solution and persistence occurs.

(ii) If \( \lambda_N \geq 0 \), the system (11) does not admit any positive stationary solution, and extinction occurs.

This result can be proved similarly to Theorem 3, or, alternatively, one can recall the results of Berestycki and Rossi (2008). In that paper, the authors consider the problem (8) set on the whole plane, but their results adapt to the problem (11) thanks to Lemma 1.

### 3 Influence of a road on an ecological niche

In this section, we study the effect of a road on an ecological niche. In terms of our models, this means that we compare the system “with the road” (10) with the system “without the road” (11), when \( c = 0 \), i.e., when the niche does not move.

#### 3.1 Deleterious effect of the road on a population in an ecological niche

This section is dedicated to the proof of Theorem 4, which answers Question 1. We start with a technical result.

**Proposition 16** Assume that \( c = 0 \). Let \( \lambda_1 \) and \( \lambda_N \) be the generalized principal eigenvalues of the model “with the road” (12) and of the model “without the road” (19) respectively. Then,

\[
\lambda_N \geq 0 \implies \lambda_1 \geq 0.
\]

This proposition readily yields the statement \((i)\) of Theorem 4. Indeed, suppose that extinction occurs for the system “without the road” (11). Then Proposition 15 implies that \( \lambda_N \geq 0 \) and thus Proposition 16 gives us that \( \lambda_1 \geq 0 \). Theorem 3 then entails that extinction occurs for (10).

**Remark 2** In view of Proposition 16, it might be tempting to think that \( \lambda_1 \geq \lambda_N \). However, this is not always the case. Indeed, taking \( \psi = 0 \) in (18), we find that

\[
\lambda_1^R \leq \frac{\pi^2}{4DR^2} + \mu,
\]

and therefore

\[
\lambda_1 = \lim_{R \to +\infty} \lambda_1^R \leq \mu.
\]
Now, $\lambda_N$ does not depend on $\mu$, and, for fixed $\mu$, it can be made arbitrarily large. This can be achieved for instance by choosing $m = \rho m^L$, where $m^L$ is from (14) with $L$ sufficiently negative and $\rho$ sufficiently large.

**Proof of Proposition 16** Assume that $\lambda_N \geq 0$. Then $\lambda_N^R \geq 0$ for any $R > 0$, thanks to (22). Because $c = 0$, on the one hand, the variational formula (23) for $\lambda_N^R$ gives us, for all $R > 0$

$$\forall \psi \in \tilde{H}_0^1(\Omega_R), \quad \int_{\Omega_R} \left( d|\nabla \psi|^2 - m\psi^2 \right) \geq 0.$$ 

On the other hand, (18) implies that

$$\lambda_1^R \geq \inf_{(\phi, \psi) \in \mathcal{H}_R \setminus \{(0,0)\}} \frac{\int_{\Omega_R} (d|\nabla \psi|^2 - m\psi^2)}{\mu \int_{I_R} \phi^2 + \nu \int_{\Omega_R} \psi^2}.$$ 

Gathering these inequalities, we get $\lambda_1^R \geq 0$. Because this is true for every $R > 0$, taking the limit $R \to +\infty$ proves the result. $\square$

The proof of Theorem 4 (ii) is more involved. The key tool is the following.

**Proposition 17** Assume that $c = 0$ and that the parameters $d, \mu, \nu$ are fixed. For $L \in \mathbb{R}$ and $D > 0$, let $\lambda_N(L)$ and $\lambda_1(L, D)$ denote the generalized principal eigenvalues of (19) and (12) respectively, with zero-th order term $m = m^L$ given by (14). Then, for every $D > 0$, there exists $L^* \in \mathbb{R}$ such that

$$\lambda_N(L^*) < 0 < \lambda_1(D, L^*).$$

**Proof** Step 1. Finding $L$ that yields $\lambda_N = 0$. 
Let us first observe that

$$\lim_{L \to -\infty} \lambda_N(L) > 0 > \lim_{L \to +\infty} \lambda_N(L).$$

Indeed, owing to formula (23), $L \mapsto \lambda_N(L)$ is non-increasing on $\mathbb{R}$, then it admits limits as $L$ goes to $\pm \infty$. Moreover, Proposition 12 yields, for every $R > 0$,

$$\lim_{L \to +\infty} \lambda_N(L) \leq \inf_{\psi \in \tilde{H}_0^1(\Omega_R) \setminus \{0\}} \frac{d \int_{\Omega_R} |\nabla \psi|^2}{\int_{\Omega_R} \psi^2} - 1.$$ 

Then, taking the limit as $R \to +\infty$ and using the well-known fact that the quantity

$$\inf_{\psi \in \tilde{H}_0^1(\Omega_R) \setminus \{0\}} \frac{\int_{\Omega_R} |\nabla \psi|^2}{\int_{\Omega_R} \psi^2}$$

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coinsides with the principal eigenvalue of the Laplace operator on $B_R \subset \mathbb{R}^2$ under Dirichlet boundary condition, which converges to 0 as $R$ goes to $+\infty$, we find
\[
\lim_{L \to +\infty} \lambda_N(L) < 0.
\]
Now, by definition of $m^L$, if $-L$ is large enough, we have that $m^L < -\frac{1}{2}$. Hence, (23) implies that, for every such $L$ and for $R > 0$,
\[
\lambda_N^R(L) = \inf_{\psi \in H^1_0(\Omega_R) \setminus \{0\}} \frac{\int_{\Omega_R} (d|\nabla \psi|^2 - m^L \psi^2)}{\int_{\Omega_R} \psi^2} \geq \frac{1}{2},
\]
and then $\lim_{L \to -\infty} \lambda_N(L) > 0$. Owing to Proposition 12, $L \mapsto \lambda_N(L)$ is a continuous function on $\mathbb{R}$, and we can then define
\[
\overline{L} := \max\{L \in \mathbb{R} : \lambda_N(L) \geq 0\}.
\]
It follows that $\lambda_N(\overline{L}) = 0$ and $\lambda_N(L) < 0$ for $L > \overline{L}$.

**Step 2.** For every $D > 0$, there holds $\lambda_1(\overline{L}, D) > 0$.
Assume by contradiction that there is $D > 0$ such that $\lambda_1(\overline{L}, D) \leq 0$. Owing to the last statement of Proposition 11, for any $D' \in (0, D)$, we have
\[
0 \geq \lambda_1(\overline{L}, D) > \lambda_1(\overline{L}, D').
\]
This means that $\lambda_1(\overline{L}, D') < 0 = \lambda_N(\overline{L})$, contradicting Proposition 16.

**Step 3. Conclusion.**
Let $D > 0$ be given. As stated in Proposition 11, $L \mapsto \lambda_1(L, D)$ is a continuous function and thus, by Step 2, $\lambda_1(L^*, D) > 0$ if $L^* > \overline{L}$, with $L^*$ sufficiently close to $\overline{L}$. On the other hand, recalling the definition of $\overline{L}$, we have $\lambda_N(L^*) < 0$. This concludes the proof. \(\square\)

Combining Proposition 17 with Theorem 3 and Proposition 15 we derive the statement $(ii)$ of Theorem 4.

### 3.2 Influence of the diffusions $D$ and $d$

#### 3.2.1 Extinction occurs when $d$ is large

This section is dedicated to the proof of Theorem 5. A similar result is derived for the model “without the road” in the paper by Berestycki and Rossi (2008). Throughout this section, we let $\lambda_1(d)$ be the generalized principal eigenvalue of (12) with $c = 0$, $D$, $\mu$, $\nu$, $f$ fixed and $d$ variable. We start with a technical proposition.

**Proposition 18** Let $\lambda_1(d)$ be the generalized principal eigenvalue of (12) with $c = 0$ and diffusion in the field $d > 0$. Then,
\[
\liminf_{d \to +\infty} \lambda_1(d) \geq \min \left\{ \mu, - \limsup_{|(x, y)| \to +\infty} m(x, y) \right\} > 0.
\]
Proof The proof relies on the construction of suitable test functions for the formula (16). It is divided into five steps.

Step 1. Defining the test function.
We take \( \lambda \geq 0 \) such that
\[
\lambda < \min\{\mu, -\limsup_{|x, y| \rightarrow +\infty} m(x, y)\}.
\]
By hypothesis (4), we can find \( M \) large enough so that
\[
K := -\sup_{|x, y| \geq M} m(x, y) > 0
\]
whence \( \lambda \in (0, \min\{\mu, K\}) \). We take \( A \in (\lambda, \mu) \) and we define
\[
\begin{align*}
\Psi(x, y) &:= \phi(x) + \psi(y), \\
\Phi(x) &:= \frac{\nu}{\mu - A} \Psi(x, 0),
\end{align*}
\]
with \( \beta \in \left(\frac{1}{2}, 1\right) \) and \( \alpha_d \) being the unique real number such that
\[
\alpha_d \in \left(0, \frac{\pi}{2M}\right) \quad \text{and} \quad \tan(\alpha_d M) = \frac{1}{d^\beta \alpha_d}.
\]
Then, the functions \( \Phi \) and \( \Psi \) given by (31) are non-negative and belong to \( W^{2,\infty}(\mathbb{R}) \) and \( W^{2,\infty}(\mathbb{R} \times [0, +\infty)) \) respectively. They are suitable test functions for formula (16).

The next steps are dedicated to proving the following inequalities
\[
\begin{align*}
D\Phi'' - \mu \Phi + \nu \Psi |_{y=0} + \lambda \Phi &\leq 0, \\
d\Delta \Psi + m \Psi + \lambda \Psi &\leq 0, \\
d \partial_y \Psi |_{y=0} - \nu \Psi |_{y=0} + \mu \Phi &\leq 0.
\end{align*}
\]
Observe that \( \Psi \leq 2, \Phi \leq \frac{2\nu}{\mu - A} \). These inequalities will be used several times in the following computations.

Step 2. The boundary condition.
Let us first check that \( (\Phi, \Psi) \) satisfies the third inequality of (33). We have
\[
\begin{align*}
d \partial_y \Psi |_{y=0} - \nu \Psi |_{y=0} + \mu \Phi &= d \psi'(0) + A \Phi \\
&\leq -d^{1-\beta} + \frac{2A\nu}{\mu - A}.
\end{align*}
\]
Because $1 - \beta > 0$, this is negative is $d$ if large enough.

**Step 3. Equation for $\Phi$.**

Let us check that the first inequality of (33) holds true almost everywhere. First, assume that $|x| < M$. Then

\[
D\Phi'' - \mu \Phi + \nu \Psi |_{y=0} + \lambda \Phi = D\Phi'' - A\Phi + \lambda \Phi
\]

\[
= \frac{\nu}{\mu - A} (-D\alpha_d^2 \cos(\alpha_d x) + (\lambda - A)(\cos(\alpha_d x) + 1)).
\]

Because $\lambda < A$, this is negative. Now, if $|x| > M$, we have

\[
D\Phi'' - \mu \Phi + \nu \Psi |_{y=0} + \lambda \Phi = D\Phi'' - A\Phi + \lambda \Phi
\]

\[
= \frac{\nu}{\mu - A} \left( \left( \frac{D}{d^{2\beta}} + \lambda - A \right) \cos(\alpha_d M)e^{\frac{1}{d^{\beta}}(M-|x|)} + \lambda - A \right).
\]

Because $\lambda < A$, this is negative if $d$ is large enough.

**Step 4. Equation for $\Psi$.**

Finally, let us check that the second inequality of (33) holds almost everywhere. First, when $|x| < M$, we have (a.e.)

\[
d\Delta \Psi + m\Psi + \lambda \Psi = d(\phi'' + \psi''') + (\lambda + m)(\phi + \psi)
\]

\[
\leq -d\alpha_d^2 \cos(\alpha_d M) + d^{1-2\beta} + 2(m + \lambda).
\]

Observe that

\[
\alpha_d^2 \sim \frac{1}{Md^{\beta}} \quad \text{as} \quad d \quad \text{goes to} \quad +\infty.
\]

Therefore,

\[-d\alpha_d^2 \cos(\alpha_d M) \sim -\frac{1}{M}d^{1-\beta} \quad \text{as} \quad d \quad \text{goes to} \quad +\infty.
\]

Because $\beta < 1$, this goes to $-\infty$ as $d$ goes to $+\infty$. On the other hand, $d^{1-2\beta}$ goes to zero as $d$ goes to $+\infty$, because $1 - 2\beta < 0$. Then, for $d$ large enough,

\[-d\alpha_d^2 \cos(\alpha_d M) + d^{1-2\beta} + 2(m + \lambda)
\]

is negative.

If $|x| > M$, we have

\[
d\Delta \Psi + m\Psi + \lambda \Psi = d(\phi'' + \psi''') + (\lambda + m)(\phi + \psi)
\]

\[
\leq d \left( \frac{1}{d^{2\beta}} \cos(\alpha_d M)e^{\frac{1}{d^{\beta}}(M-|x|)} + d^{-2\beta} e^{-\frac{1}{d^{\beta}}} \right)
\]

\[
+ (\lambda + m)(\phi + \psi)
\]

\[
\leq (d^{1-2\beta} + \lambda - K)\phi + (d^{1-2\beta} + \lambda - K)\psi.
\]

Because $\lambda < K$ and $\beta > \frac{1}{2}$, this is negative for $d$ large enough.
Step 5. Conclusion.
Gathering all that precedes, we have shown that, for \( d \) large enough, (33) is verified. Owing to the formula (16) defining \( \lambda_1(d) \), this implies that \( \lambda_1(d) \geq \lambda \), for \( d \) large enough. The fact that we choose \( \lambda \) arbitrarily yields the result. \( \square \)

We are now in a position to prove Theorem 5.

Proof of Theorem 5 Owing to Proposition 18, we see that there is \( \bar{d} > 0 \) such that
\[
\forall d > \bar{d}, \quad \lambda_1(d) \geq 0.
\]
It is readily seen from the variational formula (18), combined with Proposition 9, that the function \( d \in \mathbb{R}^+ \mapsto \lambda_1(d) \) is non-decreasing. We can define
\[
d^* := \min\{d \geq 0 : \lambda_1(d) \geq 0\}.
\]
Theorem 3 then yields the result. \( \square \)

We have proved that extinction occurs when the diffusion in the field is above a certain threshold. It is natural to wonder whether the same result holds in what concerns the diffusion on the road: is there \( D^\star \) such that extinction occurs for (10) with \( c = 0 \) when \( D \geq D^\star \)? Without further assumptions on the coefficients, the answer is no in general, as shown in the following section.

3.2.2 Influence of the diffusion on the road
This section is dedicated to proving the following statement.

Proposition 19 Consider the system (10), with \( c = 0 \) and \( f = f^L \) given by (13). For \( L \) large enough (depending on \( d \)), persistence occurs for every \( D, \mu, \nu > 0 \).

This result is the counterpart of Theorem 5, which asserts that increasing the diffusion \( d \) in the field, the system is inevitably led to extinction (under assumption (4)). Proposition 19 shows that this is not always the case if, instead of \( d \), one increases the diffusion on the road. It is also interesting to compare it with Theorem 4. While the latter states that the road always has a deleterious influence on the population, Proposition 19 means that this effect is nevertheless limited. Indeed, if the favorable zone is sufficiently large, then, no matter what happens on the road, that is, regardless of the coefficients \( D, \mu, \nu \), there will always be persistence.

Proof of Proposition 19 For \( R > 0 \), let \( \lambda_R \) and \( \phi_R \) denote the principal eigenvalue and (positive) eigenfunction of \(-\Delta\) on \( B_R \subset \mathbb{R}^2 \), under Dirichlet boundary condition. We take \( R \) large enough so that \( \lambda_R < \frac{1}{d} \) (it is well known that \( \lambda_R \searrow 0 \) as \( R \) goes to \( +\infty \)). Then define \( \psi(x, y) := \phi_R(x, y - 2R) \) for \( (x, y) \in \overline{B}_R(0, 2R) \). The definition of \( f^L \) and the fact that \( d\lambda_R < 1 \) allows us to find \( L \) sufficiently large so that
\[
\min_{(x, y) \in \overline{B}_{3R}} m^L(x, y) > d\lambda_R.
\]
As a consequence,

\[-d \Delta v = d \lambda_R v < m^L(x, y)v \quad \text{in } \overline{B}_R(0, 2R).\]

Owing to the regularity of \(f\), we can take \(\varepsilon > 0\) small enough so that \(\varepsilon v\) satisfies

\[-d \Delta (\varepsilon v) < f^L(x, y, \varepsilon v) \text{ in } \overline{B}_R(0, 2R).\]

The parabolic comparison principle in the ball \(B_R(0, 2R)\) implies that the solution of (10) with \(c = 0\), arising from the initial datum \((0, \varepsilon v)\) with \(\varepsilon v\) extended by 0 outside \(\overline{B}_R(0, 2R)\), is larger than or equal to \((0, \varepsilon v)\) for all positive times. In particular, extinction does not occur and hence, by Theorem 3, we necessarily have persistence. Because this is true independently of the values of \(D, \mu, \nu\), the proof is complete. \(\square\)

It is worthwhile to note a few observations about this result. To prove Proposition 19, we compared system (10) with the single equation in a ball, under Dirichlet boundary condition, for which we are able to show that persistence occurs provided \(L\) is sufficiently large. The population dynamics intuition behind this argument is clear: the Dirichlet condition means that the individuals touching the boundary are “killed”; it is therefore harder for the population to persist. One could have compared our system with the system with Dirichlet condition on the road instead, by showing that the generalized principal eigenvalue of the latter is always larger than \(\lambda_1\). As a matter of fact, it is also possible to compare system (10) with the system with Robin boundary condition:

\[
\begin{align*}
\frac{\partial}{\partial t} v - d \Delta v &= f(x, y, v), \quad t > 0, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d \frac{\partial y}{\partial v} |_{y=0} + \nu v |_{y=0} &= 0 \quad \text{for } t > 0, \quad x \in \mathbb{R} \times \{0\}.
\end{align*}
\]

This system describes the situation where the individuals can enter the road, but cannot leave it. It can actually be shown that the generalized principal eigenvalue of the linearization of this system, which we denote by \(\lambda_{\text{Robin}}\), is larger than \(\lambda_1\). We conjecture that

\[
\lambda_1 \xrightarrow{D \to +\infty} \min\{\lambda_{\text{Robin}}, \mu\}.
\]

This is based on the intuition that, as \(D\) becomes large, the population on the road diffuses “very fast” and then is sent “very far” into the unfavorable zone, where it dies.

### 4 Influence of a road on a population facing a climate change

#### 4.1 Influence of the speed \(c\)

This section is dedicated to proving Theorem 6. In this whole section, we assume that the coefficients \(D, d, \mu, \nu\) are fixed in (12), and we let \(\lambda_1(c)\) denote the generalized principal eigenvalue of (12) and \(\lambda_1^R(c)\) the principal eigenvalue of (17), as functions of the parameter \(c \geq 0\). We start with proving two preliminary results.
Proposition 20 Let $\lambda_1(c)$ be the generalized principal eigenvalue of (12). Then

$$\lambda_1(c) \geq \frac{1}{4} \min \left\{ \frac{1}{d}, \frac{1}{D} \right\} c^2 - [\sup m]^+. $$

Proof Let $c \geq 0$ be chosen. For $R > 0$, let $(\phi_R, \psi_R)$ denote the principal eigenfunction of (17) and $\lambda^R_1$ the associated principal eigenvalue. Take $\kappa \in \mathbb{R}$. The idea is to multiply the system (17) by the weight $x \mapsto e^{\kappa x}$, and to integrate by parts. At the end, optimizing over $\kappa$ will yield the result. We define

$$I_\psi := \int_{\Omega_R} \psi_R(x, y) e^{\kappa x} dy \quad \text{and} \quad I_\phi := \int_{I_R} \phi_R(x) e^{\kappa x} dx.$$

We multiply the equation for $\psi_R$ in (17) by $e^{\kappa x}$ and integrate over $\Omega_R$ to get

$$-d \int_{\Omega_R} (\Delta \psi_R) e^{\kappa x} - c \int_{\Omega_R} (\partial_x \psi_R) e^{\kappa x} = \int_{\Omega_R} m(x, y) \psi_R e^{\kappa x} + \lambda^R_1 I_\psi. \quad (34)$$

We let $e_x$ denote the unit vector in the direction of the road, i.e., $e_x := (1, 0)$, and $\nu$ the exterior normal vector to $\Omega_R$. We have

$$\int_{\Omega_R} (\partial_x \psi_R) e^{\kappa x} = -\kappa \int_{\Omega_R} \psi_R e^{\kappa x} + \int_{\Omega_R} \partial_x (\psi_R e^{\kappa x}).$$

Because $\psi_R = 0$ on $\partial \Omega_R \setminus I_R$, the Fubini theorem implies that $\int_{\Omega_R} \partial_x (\psi_R e^{\kappa x}) = 0$. Hence

$$\int_{\Omega_R} (\partial_x \psi_R) e^{\kappa x} = -\kappa I_\psi.$$

Using the divergence theorem, as well as the above equivalence, we find that

$$-d \int_{\Omega_R} (\Delta \psi_R) e^{\kappa x} = -d \int_{\partial \Omega_R} (\partial_v \psi_R) e^{\kappa x} + d\kappa \int_{\Omega_R} (\partial_x \psi_R) e^{\kappa x}$$

$$= -d\kappa^2 I_\psi - d \int_{\partial \Omega_R \setminus I_R} (\partial_v \psi_R) e^{\kappa x} + d \int_{I_R} (\partial_y \psi_R) e^{\kappa x}$$

$$\geq -d\kappa^2 I_\psi + d \int_{I_R} (\partial_y \psi_R) e^{\kappa x},$$

where the last inequality comes from the fact that we have $\psi_R = 0$ on $\partial \Omega_R \setminus I_R$ and $\psi_R \geq 0$ elsewhere, hence $\partial_v \psi_R \leq 0$ on $\partial \Omega_R \setminus I_R$. Then, (34) yields

$$0 \leq (\sup m)^+ + \lambda^R_1 - \kappa c + d\kappa^2 I_\psi + \int_{I_R} (-d\partial_y \psi_R) e^{\kappa x}. \quad (35)$$
Now, the boundary condition in (17) combined with the equation satisfied by \( \phi_R \) gives us
\[
\int_{I_R} (-d \partial_y \psi_R) e^k x = \int_{I_R} (D \partial_{xx} \phi_R + c \partial_x \phi_R + \lambda_1^R \phi_R) e^k x
\]
\[
= \int_{I_R} (D \partial_{xx} \phi_R + c \partial_x \phi_R) e^k x + \lambda_1^R I_\phi.
\]
Integrating by parts and arguing as before, we obtain
\[
\int_{I_R} (-d \partial_y \psi_R) e^k x \leq \left( D \kappa^2 - c \kappa + \lambda_1^R \right) I_\phi.
\]
Then, (35) implies that
\[
0 \leq \left( d \kappa^2 - c \kappa + [\sup m]^+ + \lambda_1^R \right) I_\psi + \left( D \kappa^2 - c \kappa + \lambda_1^R \right) I_\phi.
\]
We write \( \kappa := \alpha c \), using \( \alpha \in \mathbb{R} \) as the new optimization parameter. Because \( I_\phi \) and \( I_\psi \) are positive, we deduce that one of the following inequalities necessarily holds:
\[
(d \alpha^2 - \alpha) c^2 + [\sup m]^+ + \lambda_1^R \geq 0, \quad (D \alpha^2 - \alpha) c^2 + \lambda_1^R \geq 0.
\]
Namely, we derive
\[
\lambda_1^R (c) \geq \sup_{\alpha \in \mathbb{R}} \left( \min\{-(d \alpha^2 - \alpha) c^2 - [\sup m]^+, -(D \alpha^2 - \alpha) c^2\} \right)
\]
\[
\geq \sup_{\alpha \in \mathbb{R}} \left( \min\{\alpha - d \alpha^2, \alpha - D \alpha^2\} c^2 - [\sup m]^+ \right)
\]
\[
\geq \frac{1}{4} \min \left\{ \frac{1}{d}, \frac{1}{D} \right\} c^2 - [\sup m]^+.
\]
Letting \( R \) go to \( +\infty \), we get the result. \( \square \)

Proposition 20 implies that \( \lambda_1 (c) \geq 0 \) if
\[
c \geq 2 \sqrt{\max[d, D] \sup m}^+.
\]
Owing to the continuity of \( c \mapsto \lambda_1 (c) \) (recalled in Proposition 11 above), this allows us to define
\[
c_* := \min\{c \geq 0 : \lambda_1 (c) \geq 0\}, \quad c^* := \sup\{c \geq 0 : \lambda_1 (c) < 0\},
\]
with the convention that \( c^* = 0 \) if the set in its definition is empty. Moreover, \( c_* > 0 \) if and only if \( \lambda_1 (0) < 0 \). Thanks to Theorem 3, we have thereby proved Theorem 6.

As we mentioned in the introduction, Sect. 1.4, we actually conjecture that \( c_* = c^* \). We prove that this is true when \( d = D \).
Proposition 21 Assume that \( d = D \) in (10). Then

\[
c_* = C = 2\sqrt{d[-\lambda_1(0)]^+}.
\]

This proposition is readily derived using the change of functions

\[
\tilde{\phi} := \phi e^{\frac{c}{2}x}, \quad \tilde{\psi} := \psi e^{\frac{c}{2}x}
\]

in (16) to get

\[
\lambda_1(c) = \frac{c^2}{4d} + \lambda_1(0).
\]

We conclude this section by showing that \( c \mapsto \lambda_1(c) \) attains its minimum at \( c = 0 \). This has a natural interpretation: it means that a population is more likely to persist if the favorable zone is not moving; in other words, the climate change always has a deleterious effect on the population, at least for what concerns survival.

Proposition 22 Let \( c \geq 0 \). Then

\[
\lambda_1(c) \geq \lambda_1(0).
\]

Proof Take \( R \geq 0 \) and let \((\phi_R, \psi_R)\) be the principal eigenfunction of (17) and \( \lambda_1^R \) be the associated eigenvalue. We multiply the equation for \( \psi_R \) in (17) by \( \psi_R \) and integrate over \( \Omega_R \) to get

\[
\int_{\Omega_R} d|\nabla \psi_R|^2 - d \int_{\partial \Omega_R} \psi_R \partial_x \psi_R - \int_{\Omega_R} m \psi_R^2 = \lambda_1^R(c) \int_{\Omega_R} \psi_R^2,
\]

where we have used the fact that

\[
\int_{\Omega_R} c \psi_R \partial_x \psi_R = \frac{c}{2} \int_{\Omega_R} \partial_x (\psi_R)^2 = 0.
\]

Then, the boundary condition yields

\[
\int_{\Omega_R} d|\nabla \psi_R|^2 - \int_{I_R} \psi_R|_{y=0}(\mu \phi_R - \nu \psi_R|_{y=0}) - \int_{\Omega_R} m \psi_R^2 = \lambda_1^R(c) \int_{\Omega_R} \psi_R^2.
\]

Likewise, multiplying the equation on the road by \( \phi_R \) and integrating, we have

\[
\int_{I_R} D|\phi_R'|^2 + \int_{I_R} \phi_R(\mu \phi_R - \nu \psi_R|_{y=0}) = \lambda_1^R(c) \int_{I_R} \phi_R^2.
\]
Multiplying (36) by \( v \) and (37) by \( \mu \) and summing the two resulting equations yields

\[ \lambda^R_1(c) = \frac{\mu \int_{I^R} D|\phi'_R|^2 + v \int_{\Omega^R} (d|\nabla \psi_R|^2 - m \psi^2_R) + \int_{I^R} (\mu \phi_R - v \psi_R|_{y=0})^2}{\mu \int_{I^R} \Phi^2_R + v \int_{\Omega^R} \psi^2_R}. \]

Owing to Proposition 10, this is greater than \( \lambda^R_1(0) \), hence the result. \( \square \)

### 4.2 Positive effect of the road in keeping pace with a climate change

In this section, we prove Theorem 7, whose Corollary 8 answers Question 2. The key observation is that, when \( L \) goes to +\( \infty \), the nonlinearity \( f^L \) converges to \( f^\infty(\nu) :=\nu(1-\nu) \), the favorable zone then being the whole space, and the system (10) becomes, at least formally

\[
\begin{aligned}
\partial_t u - D \partial_{xx} u - c \partial_x u &= v v|_{y=0} - \mu u, \quad t > 0, \ x \in \mathbb{R}, \\
\partial_t v - d \Delta v - c \partial_x v &= \psi(1-\nu), \quad t > 0, \ (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d \partial_y \psi|_{y=0} &= \psi u - \psi v|_{y=0}, \quad t > 0, \ x \in \mathbb{R}.
\end{aligned}
\]

This system is the road-field model (9), recalled in Sect. 1.3, rewritten in the frame moving in the direction of the road with speed \( c \in \mathbb{R} \). The results of Berestycki et al. (2013a), summarized here in Proposition 2, are obtained by constructing explicit supersolutions and subsolutions, and do not use principal eigenvalues. We need to rephrase Proposition 2 in terms of the generalized principal eigenvalue. Namely, we define

\[
\lambda_H := \sup \left\{ \lambda \in \mathbb{R} : \exists (\phi, \psi) \geq (0, 0), \ (\phi, \psi) \neq (0, 0) \text{ such that} \right. \\
\begin{aligned}
D \phi'' + c \phi' - \mu \phi + v \psi|_{y=0} + \lambda \phi &\leq 0 \text{ on } \mathbb{R}, \\
d \Delta \psi + c \partial_x \psi + \psi + \lambda \psi &\leq 0 \text{ on } \mathbb{R} \times \mathbb{R}^+, \\
\text{and } d \partial_y \psi - \nu \psi|_{y=0} + \mu \phi &\leq 0 \text{ on } \mathbb{R}.
\end{aligned}
\]

Then, \( \lambda_H \) is the generalized principal eigenvalue of

\[
\begin{aligned}
-D \partial_{xx} \phi - c \partial_x \phi &= v \psi|_{y=0} - \mu \phi + \lambda \phi, \ x \in \mathbb{R}, \\
-d \partial_y \psi|_{y=0} &= \mu \phi - \nu \psi|_{y=0}, \quad (x, y) \in \mathbb{R} \times \mathbb{R}^+, \\
-d \partial_y \psi|_{y=0} &= \mu \phi - \nu \psi|_{y=0}, \quad x \in \mathbb{R}.
\end{aligned}
\]

This is the eigenproblem associated with the linearization at \( (u, v) = (0, 0) \) of the stationary system associated with the road-field model (38), in the frame moving in the direction of the road with speed \( c \). For \( R > 0 \), we let \( \lambda^R_H \) denote the principal eigenvalue of (40) on the truncated domains \( \Omega^R \) in the field and \( I^R \) on the road. We know from the paper (Berestycki et al. 2019) that \( \lambda^R_H \rightarrow \lambda_H \) as \( R \) goes to +\( \infty \) and that there is a positive generalized principal eigenfunction associated with \( \lambda_H \).

Consistently with our previous notations, we let \( \lambda_H(c) \) denote the generalized principal eigenvalue of (40) with \( D, d, \mu, \nu > 0 \) fixed and with \( c \in \mathbb{R} \) variable.
Lemma 3  Let \( \lambda_H(c) \) denote the generalized principal eigenvalue of (40). Then

\[ \lambda_H(c) < 0 \text{ for } c \in [0, c_H) \text{ and } \lambda_H(c) \geq 0 \text{ for } c \geq c_H. \]

**Proof** We argue by contradiction. Assume that there is \( c \in [0, c_H) \) such that \( \lambda_H(c) \geq 0 \). Let \((\phi, \psi) \geq (0, 0)\) be a generalized principal eigenfunction associated with \( \lambda_H(c) \). Then \((\phi, \psi)\) is a stationary supersolution of (38), owing to the Fisher-KPP property, and because \( \lambda_H(c) \geq 0 \). We normalize it so that \( \psi(0, 0) = \frac{1}{2} \).

Now, let \((u_0, v_0)\) be a non-negative, not identically equal to zero compactly supported initial datum such that

\[ (u_0, v_0) \leq (\phi, \psi). \]

Let \((u, v)\) be the solution of (38) arising from \((u_0, v_0)\). The parabolic comparison principle Proposition 13 implies that

\[ (u, v) \leq (\phi, \psi). \]

However, because \( 0 \leq c < c_H \), the main result of the paper (Berestycki et al. 2013a), Proposition 2 above, when translated in the moving frame of (38), yields

\[ (u(t, x), v(t, x, y)) \rightarrow_{t \rightarrow +\infty} \left( \frac{v}{\mu}, 1 \right), \]

locally uniformly in \( x \) and \((x, y)\). This contradicts (41) because \( \psi(0, 0) = \frac{1}{2} \). Therefore, \( \lambda_H(c) < 0 \) when \( c \in [0, c_H) \).

Now, take \( c > c_H \). If we had \( \lambda_H(c) < 0 \), we could argue as in the proof of Theorem 3 to show that persistence occurs. However, in view of Proposition 2, this cannot be the case. Hence, \( \lambda_H(c) \geq 0 \). Because \( \lambda_H \) is a continuous function of \( c \) (see Proposition 2.4 in the paper by Berestycki et al. (2019)), it must be the case that \( \lambda_H(c_H) = 0 \). This concludes the proof. \( \square \)

The next proposition states that, in some sense, the system (10) converges to the homogeneous system (38) as \( L \) goes to \( +\infty \). In agreement with our previous notations, we let \( \lambda_1(c, L) \) denote the generalized principal eigenvalue of (12) with parameters \( d, D > 0 \) fixed and with \( c \in \mathbb{R} \) and zero-th order term \( m = m^L \) given by (14), \( c \) and \( L \) being variable.

**Proposition 23** Let \( \lambda_1(c, L) \) be the generalized principal eigenvalue of (12) with zero-th order term \( m = m^L \) defined in (14). Then

\[ \lambda_1(c, L) \rightarrow_{L \rightarrow +\infty} \lambda_H(c) \text{ locally uniformly in } c. \]

**Proof** First, because \( m^L \leq 1 \), formulae (16) and (39) yield

\[ \forall L > 0, \ c \geq 0, \ \lambda_H(c) \leq \lambda_1(c, L), \]
Influence of a road on a population...

hence

$$\lambda_H(c) \leq \liminf_{L \to +\infty} \lambda_1(c, L).$$

Let $\varepsilon \in (0, 1)$ be fixed. In view of the definition of $m^L$ (14), for any $R > 0$, we can take $L_R > 0$ such that, for $L \geq L_R$, $m^L(x, y) \geq 1 - \varepsilon$ on $\Omega_R$. Therefore,

$$\forall R > 0, \ L \geq L_R, \ \lambda^R_1(c, L) \leq \lambda^R_H(c) + \varepsilon. \quad (42)$$

Because $\lambda^R_1(c, L) \geq \lambda_1(c, L)$ and by arbitrariness of $\varepsilon$ in (42), we find that

$$\limsup_{L \to +\infty} \lambda_1(c, L) \leq \lambda_H(c).$$

Hence,

$$\lambda_1(c, L) \xrightarrow{L \to +\infty} \lambda_H(c).$$

This convergence is locally uniform with respect to $c \geq 0$. Indeed, the continuity of the functions $c \mapsto \lambda_1(c, L)$ and $c \mapsto \lambda_H(c)$ combined with the fact that the family $(\lambda_1(c, L))_{L \geq 0}$ is decreasing and converges pointwise to $\lambda_H(c)$ as $L$ goes to $+\infty$ allows us to apply Dini’s theorem.

We are now in a position to prove Theorem 7.

**Proof of Theorem 7** As explained in the proof of Proposition 23 above, we have

$$\forall L \in \mathbb{R}, \ c \geq 0, \ \lambda_H(c) \leq \lambda_1(c, L).$$

Owing to Lemma 3, we have $\lambda_H(c) \geq 0$ for every $c \geq c_H$. By definition of $c^*$, we find that

$$\forall L \in \mathbb{R}, \ c^* \leq c_H. \quad (43)$$

Take $\eta \in (0, c_H)$. Lemma 3 yields $\lambda_H(c) < 0$ for every $c \in [0, c_H - \eta]$. Because $\lambda_1(c, L)$ converges locally uniformly to $\lambda_H(c)$ as $L$ goes to $+\infty$, by Proposition 23, we find that there is $L^*$ such that

$$\forall L \geq L^*, \ c \in [0, c_H - \eta], \ \lambda_1(c, L) < 0.$$

Theorem 3 then implies that persistence occurs in (10) if $c \in [0, c_H - \eta]$ and $L \geq L^*$. It follows from the definition of $c^*$ that

$$\forall L \geq L^*, \ c^* \geq c_H - \eta. \quad (44)$$

We can take $\eta$ arbitrarily close to zero, up to increasing $L^*$ if need be. Combining (43) and (44) yields that $c_*$ and $c^*$ converge to $c_H$ as $L$ goes to $+\infty$. This convergence is monotonically non-decreasing because $L \mapsto \lambda_1(c, L)$ in non-increasing for every $c > 0$. 

\[ Springer \]
We can now deduce Corollary 8 from Theorem 7.

**Proof of Corollary 8** Assume that $D > 2d$. Consider the system (10) with nonlinearity $f^L$ given by (13). Proposition 2 implies that $c_H > c_{KPP} = 2\sqrt{d}$, and then, in view of Theorem 7, we can choose $L$ sufficiently large to have $c_* > c_{KPP}$.

Now, taking $\psi = e^{-c^2d x}$ in the formula (20) and using $m^L \leq 1$, shows that $\lambda_N(c) \geq \frac{c^2}{4d} - 1$. It follows that $\lambda_N(c) \geq 0$ when $c \geq c_{KPP}$. From Propositions 1 and 15, we infer that $c_N \leq c_{KPP}$. We eventually conclude that $c_N \leq c_{KPP} < c_*$. That is, the two statements of the corollary hold with $c_1 := c_N$ and $c_2 := c_*$. ☐

5 Extension to more general reaction terms

Throughout the whole paper, up to now, we assumed (4), that is, that the favorable zone is bounded. It is natural to wonder whether this condition can be weakened. This question turns out to be rather delicate and it is still by and large open.

Nevertheless, we point out that the notion of generalized principal eigenvalue introduced and studied by Berestycki et al. (2019), and the basic technical facts recalled in Sect. 2.1, do not require hypothesis (4). However, when it comes to studying the long-time behavior and the qualitative properties of road-field models of the type (10), the boundedness of the ecological niche is crucial.

For instance, without (4), the uniqueness of stationary solutions is not guaranteed anymore. In addition, it is not clear that the solutions of road-field models (10) would converge to stationary solutions as $t$ goes to $+\infty$. As a consequence, the keystone of our analysis, Theorem 3 (which states that the sign of $\lambda_1$ completely characterizes the long-time behavior of solutions) is not known in this context.

Therefore, in this framework, the notions of persistence and extinction, as stated in Definition 1, do not make much sense. For this reason, we introduce the following modified notions:

**Definition 2** For the system (10), that is, the system in the moving frame, we say that

(i) **local extinction in the moving frame** occurs if every solution arising from a non-negative compactly supported initial datum converges locally uniformly to zero as $t$ goes to $+\infty$;
(ii) **local persistence in the moving frame** occurs if every solution arising from a non-negative not identically equal to zero compactly supported initial datum satisfies, for every $R > 0$,

$$\liminf_{t \to +\infty} \left( \inf_{x \in I_R} u(t, x) \right) > 0, \quad \liminf_{t \to +\infty} \left( \inf_{(x, y) \in \Omega_R} v(t, x, y) \right) > 0.$$

The fact that we are working in the moving frame is important. Indeed, in a system exhibiting a climate change ($c \neq 0$), if the favorable zone is bounded, i.e., (4) is
verified, then every solution of the road-field model (1) in the original frame goes to zero locally uniformly as \( t \) goes to \( +\infty \) (even when we have persistence in the moving frame).

We have the following weak version of Theorem 3:

**Theorem 24** Let \( \lambda_1 \) be the generalized principal eigenvalue of system (12), with \( m \) not necessarily satisfying (4).

(i) If \( \lambda_1 < 0 \), local persistence in the moving frame occurs for (10).
(ii) If \( \lambda_1 > 0 \), local extinction in the moving frame occurs for (10).

The proof of this result is similar to that of Theorem 3. We briefly sketch it here for completeness.

**Proof** Let \((u, v)\) be a solution of (10) arising from a compactly supported non-null initial datum.

Statement (i).

As in the proof of statement (i) of Theorem 3, we can take \( R > 0 \) and \( \varepsilon > 0 \) such that \( \varepsilon(\phi_R, \psi_R) \) is a stationary subsolution of (10) and such that \((u(1, \cdot), v(1, \cdot, \cdot)) \geq \varepsilon(\phi_R, \psi_R)\). Then, the parabolic comparison principle yields that the solution of (10) arising from the initial datum \( \varepsilon(\phi_R, \psi_R) \) is time-increasing and converges to a positive stationary solution, hence statement (i) of Theorem 24 follows.

In the proof of statement (i) of Theorem 3, we also used \((\frac{\nu}{\mu} M, M)\), with \( M > 0 \) large enough, as an initial datum in (10) to bound from above \((u, v)\). We could do the same here. However, for want of an uniqueness result, the solution of (10) arising from \((\frac{\nu}{\mu} M, M)\) may converge to a stationary solution different from the one obtained by starting with the initial datum \( \varepsilon(\phi_R, \psi_R) \).

Statement (ii).

Observe that, for \( M > 0 \), the pair \( M e^{-\lambda_1 t} (\phi, \psi) \) is supersolution of (10). Taking \( M \) large enough so that \((u_0, v_0) \leq M(\phi, \psi)\), the parabolic comparison principle yields that \((u, v)\) goes to zero, locally uniformly as \( t \) goes to \( +\infty \). The converge to zero is only locally uniform because the generalized principal eigenfunction \((\phi, \psi)\) may not be bounded.

Is is natural to wonder whether there are situations where local extinction occurs but not extinction. In such situation, the population would move inside favorable zones but would neither settle definitively there nor go extinct. We leave this as an open question.

6 Conclusion

We have introduced a model that aims at describing the effect of a line with fast diffusion (a road) on the dynamics of an ecological niche. We incorporate in the model the possibility of a climate change. We have found that this model exhibits two contrasting influences of the road. The first one is that, when the niche is not moving (there is no climate change), the presence of the line with fast diffusion can lead to the extinction of a population that would otherwise persist: the effect of the line is
deleterious. On the other hand, if the ecological niche is moving, because of a climate change, then there are situations where a population that would otherwise be doomed to extinction manages to survive thanks to the presence of the road.

The first result is not a priori intuitive: in our model, the line with fast diffusion is not lethal, in the sense that there is no death term there. The second result, that is, the fact that the line can “help” the population, is also surprising, because there is no reproduction on the line either.

These results are derived through a careful analysis of the properties of a notion of generalized principal eigenvalue for elliptic systems set in different spatial dimensions, introduced in our previous work (Berestycki et al. 2019).

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