Bessel functions and local converse conjecture of Jacquet

Jingsong Chai

Abstract

In this paper, we prove the local converse conjecture of Jacquet over \( p \)-adic fields for \( \text{GL}_n \) using Bessel functions.

Keywords. Bessel functions, Howe vectors, local converse conjecture of Jacquet

1 Introduction

Let \( F \) be a \( p \)-adic field, and \( \psi \) be a nontrivial additive character of \( F \) with conductor precisely \( \mathcal{O} \), the ring of integers of \( F \). Let \( \pi \) be an irreducible admissible generic representation of \( \text{GL}_n(F) \). If \( \rho \) is an irreducible admissible generic representation of \( \text{GL}_r(F) \), one can attach an important invariant local gamma factor \( \gamma(s, \pi \times \rho, \psi) \) via the theory of local Rankin-Selberg integrals by Jacquet, Piatetski-Shapiro and Shalika (JPSS83). This invariant can also be defined by Langlands-Shahidi method (Sh84).

These invariants can be used to determine the representation \( \pi \) up to isomorphism. In [He], Henniart proved that, for irreducible admissible generic representations \( \pi_1 \) and \( \pi_2 \) of \( \text{GL}_n(F) \), if the family of invariants \( \gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi) \), for all \( r \) with \( 1 \leq r \leq n-1 \), and for all irreducible admissible generic representations \( \rho \) of \( \text{GL}_r(F) \), then \( \pi_1 \cong \pi_2 \). This is then strengthened by J.Chen (Ch06) and Cogdell and Piatetski-Shapiro (CPS99) by decreasing \( r \) from \( n-1 \) to \( n-2 \). Such type of results together with global version first appeared in [ILa] for \( \text{GL}(2) \) and [JPSS79] for \( \text{GL}(3) \). A general conjecture, which is due to Jacquet can be formulated as follows.

**Conjecture 1 (Jacquet).** Assume \( n \geq 2 \). Let \( \pi_1 \) and \( \pi_2 \) be irreducible generic smooth representations of \( \text{GL}_n(F) \). Suppose for any integer \( r \), with \( 1 \leq r \leq \lfloor \frac{n}{2} \rfloor \), and any irreducible generic smooth representation \( \rho \) of \( G_r \), we have

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\[
\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi),
\]
then \(\pi_1\) and \(\pi_2\) are isomorphic.

In the present paper, we will prove **Conjecture 1** using Bessel functions. The author was recently informed that this conjecture has also been proved by H. Jacquet and Baiying Liu independently using a different method, see [JL].

By the work of Dihua Jiang, Chufeng Nien and Shaun Stevens in section 2.4, [JNS], this conjecture has been reduced to the following conjecture when both \(\pi_1, \pi_2\) are unitarizable irreducible supercuspidal representations.

**Conjecture 2.** Assume \(n \geq 2\). Let \(\pi_1\) and \(\pi_2\) be irreducible unitarizable and supercuspidal smooth representations of \(GL_n(F)\). Suppose for any integer \(r\), with \(1 \leq r \leq \left\lfloor \frac{n}{2} \right\rfloor\), and any irreducible generic smooth representation \(\rho\) of \(G_r\), we have

\[
\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi),
\]
then \(\pi_1\) and \(\pi_2\) are isomorphic.

It is this conjecture that we will prove in this paper. The main result can be stated as follows.

**Theorem 1.1.** **Conjecture 2** is true, and so is **Conjecture 1**.

A crucial ingredient in the proof is Bessel function, which has its own interests. Given an irreducible admissible generic representation \(\pi\) of \(GL_n(F)\), one can attach a Bessel function \(j_\pi\) to \(\pi\). Such functions were first defined over \(p\)-adic fields by D. Soudry in [S] for \(GL_2(F)\), and then were generalized to \(GL_n(F)\) by E. M. Baruch in [B05], to other split groups by E. Lapid and Zhengyu Mao in [LM13], respectively. A general philosophy is that the local gamma factors \(\gamma(s, \pi \times \rho, \psi)\) are intimately related to the Bessel functions of \(\pi, \rho\). In many cases, we know that local gamma factors can be expressed as certain Mellin transform of Bessel functions, see for example [CPS98, Sh02, S]. This expression is the starting point of proving stability of local gamma factors, which is crucial in applying converse theorem to Langlands functoriality problems. For the case at hand, such Mellin transform is also expected but has not yet been proved according to the author’s knowledge. However, it is still possible to derive an equality of Bessel functions from equalities of local gamma factors via local Rankin-Selberg integrals. Then the above conjecture follows as the Bessel function \(j_\pi\) determines the representation \(\pi\) up to isomorphism by the weak kernel formula (Theorem 4.2 in [Chai15]). This is the basic idea of our proof.

There are much progress made towards this conjecture in recent years. In particular, Chufeng Nien in [N] proved an analogue of this conjecture in finite field case. Dihua Jiang, Chufeng Nien and Shaun Stevens in [JNS] formulated an approach using constructions of supercuspidal representations to attack this conjecture in \(p\)-adic case, and proved it in
many cases, including the cases when $\pi_1, \pi_2$ are supercuspidal representations of depth zero. Later on based on this approach, Moshe Adrian, Baiying Liu, Shaun Stenvens and Peng Xu in [ALSX] proved the conjecture for $GL_n(F)$ when $n$ is prime. There are also some work on similar problems for other groups. See [B95, B97, JS, Zh1, Zh2] for examples. For a more comprehensive survey on local converse problems and related results, see relevant sections in [Jiang, JN].

An important ingredient of the approach suggested in [JNS] is to reduce the conjecture to show the existence of certain Whittaker functions (called special pair of Whittaker functions) for a pair of unitarizable supercuspidal representations $\pi_1, \pi_2$ of $GL_n(F)$. In [JNS] and [ALSX], such Whittaker functions were found in many cases using the constructions of supercuspidal representations.

We will explain our proof in more details, and the above ingredient is also important. To prove Conjecture 2, as explained above it suffices to show that, under the assumptions of the conjecture, the unitarizable supercuspidal representations $\pi_1, \pi_2$ have the same Bessel functions. By Proposition 5.3 in [Chai15], it is reduced to show that

$$\tilde{W}_{\omega_n,v_m}^1\left(\begin{pmatrix} g & x & I_r \\ 1 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} \omega_{2r} \\ 1 \end{pmatrix} \alpha^{r+1} a \right) W'(g)|\det(g)|^{s-\frac{r+1}{2}} dg = \tilde{W}_{\omega_n,v_m}^2\left(\begin{pmatrix} g & x & I_r \\ 1 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} \omega_{2r} \\ 1 \end{pmatrix} \alpha^{r+1} a \right) W'(g)|\det(g)|^{1-s-\frac{r+1}{2}} dg,$$

where $\rho$ is any generic irreducible smooth representation of $GL_r(F)$.

By Lemma 2.3 below, it suffices to show

$$\tilde{W}_{\omega_n,v_m}^1\left(\begin{pmatrix} g & x & I_r \\ 1 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} \omega_{2r} \\ 1 \end{pmatrix} \alpha^{r+1} a \right) = \tilde{W}_{\omega_n,v_m}^2\left(\begin{pmatrix} g & x & I_r \\ 1 & 1 & 1 \end{pmatrix} \left(\begin{pmatrix} \omega_{2r} \\ 1 \end{pmatrix} \alpha^{r+1} a \right)$$

on certain open dense subset of the domain in the integrals. Inspired by the work of [JNS], we will first in section 3 show that, the normalized Howe vectors satisfy certain properties similar to special pair of Whittaker functions in a slightly weak form on certain Bruhat cells. Then combining with the work of Jeff Chen in [Ch06], these properties will imply the above identities, which finishes the proof in the odd case. The even case can then be deduced from the odd case.
We finally remark that Nien used Bessel functions in her’s proof of finite field analogue ([N]), and in this paper we give the first Bessel function proof of Conjecture 1 over p-adic fields.

The paper is organized as follows. In section 2, we recall some backgrounds and preparations on Rankin-Selberg integrals and Bessel functions. We then study Howe vectors in detail in section 3. In the last section, we prove the theorem.

2 Preparations

Use $G_n$ to denote $GL_n(F)$, and embed $G_{n-1}$ into $G_n$ on the left upper corner. Let $N_n$ be subgroup of the upper triangular unipotent matrices. $A_n$ the subgroup of diagonal matrices. Use $P_n$ to denote the mirabolic subgroup consisting of matrices with the last row $(0, ..., 0, 1)$. We extend the additive character $\psi$ to $N_n$, still denoted as $\psi$, by setting

$$\psi(u) = \psi\left(\sum_{i=1}^{n-1} u_{i,i+1}\right) u = (u_{ij}) \in N_n.$$

If $\pi$ is an irreducible admissible generic representation of $G_n$, use $W(\pi, \psi)$ to denote the Whittaker model of $\pi$ with respect to $\psi$. If $W \in W(\pi, \psi)$, define

$$\tilde{W}(g) := W(\omega_n, {}^t g^{-1})$$

where $\omega_n = \begin{pmatrix} \ddots & 1 \\ \vdots & \ddots \\ 1 & \ddots \\ 1 & \cdots & 1 \end{pmatrix}$, then the space of functions

$$\{\tilde{W}(g) : W \in W(\pi, \psi)\}$$

is the Whittaker model of $\pi^*$ with respect to $\psi^{-1}$, where $\pi^*$ denotes the contragredient of $\pi$.

Suppose $\pi$ and $\pi'$ are irreducible admissible generic representations of $G_n$ and $G_r$ respectively, with associated Whittaker models $W(\pi, \psi)$ and $W(\pi', \psi^{-1})$. For our purpose, we will assume $r < n$. For any $W \in W(\pi, \psi)$, $W' \in W(\pi', \psi^{-1})$, $s \in \mathbb{C}$ a complex number, and any integer $j$ with $n - r - 1 \geq j \geq 0$, set $k = n - r - 1 - j$, and let

$$I(s, W, W', j) = \int_{N_r \backslash G_r} \int_{M(j \times r)} W(g \begin{pmatrix} x & 0 & 0 \\ 0 & I_j & 0 \\ 0 & 0 & I_{k+1} \end{pmatrix}) W'(g | \text{det} g|^{s-(n-r)/2} dxdg,$$

where $M(j \times r)$ denotes the space of matrices of size $j \times r$.

We have the following basic result in the theory of local Rankin-Selberg integrals.
Theorem 2.1. ([JPSS83]) For any pair of generic irreducible admissible representations
\( \pi \) and \( \pi' \) on \( G_n \) and \( G_r \), we have:

1. The integrals \( I(s, W, W', j) \) converge absolutely for \( \text{Re}(s) \) large;

2. The integrals \( I(s, W, W', j) \) span a fractional ideal in \( \mathbb{C}[q^s, q^{-s}] \) with a unique
   generator \( L(s, \pi \times \pi') \) such that \( L(s, \pi \times \pi') \) has the form \( P(q^{-s})^{-1} \) for some polynomial
   \( P \in \mathbb{C}[x] \) with \( P(0) = 1 \);

3. There exists a meromorphic function \( \gamma(s, \pi \times \pi', \psi) \), independent of the choices
   of \( W, W' \), such that

\[
I(1-s, \pi^*(\omega_{n,r})\overline{W}, \overline{W'}, k) = \omega_{\pi'}(-1)^{n-1-1}\gamma(s, \pi \times \pi', \psi)I(s, W, W', j),
\]

where \( n - r - 1 \geq j \geq 0, k = n - r - 1 - j, \omega_{n,r} = \begin{pmatrix} \omega_r & \omega_{n-r} \\ \omega_{n-r} & \omega_r \end{pmatrix} \) and \( \omega_{\pi'} \) is the central
character of \( \pi' \).

Remark. The meromorphic function \( \gamma(s, \pi \times \pi', \psi) \) is called the local \( \gamma \)-factor of \( \pi \) and
\( \pi' \).

Now let \( \pi \) be a generic irreducible unitarizable representation of \( G_n \). Consider the
space of functions

\[
\overline{W} := \{ \overline{W}(g) : W(g) \in \mathcal{W}(\pi, \psi) \}
\]

where \( \overline{\cdot} \) denotes the complex conjugate.

Then with the right translation by \( G_n \), \( \overline{W} \) is an irreducible representation of \( G_n \), with
\( \overline{W}(ug) = \psi^{-1}(u)\overline{W}(g) \) for \( u \in N_n, g \in G_n \). Thus \( \overline{W} \) is the Whittaker model with respect
to \( \psi^{-1} \) of some generic irreducible unitarizable representation \( \tau \) of \( G_n \).

Since \( \pi \) is unitarizable, we have a \( G_n \)-invariant inner product \( \langle W_1, W_2 \rangle \) on some complex
Hilbert space. Then for \( \overline{W} \in \overline{W} \), view it as a smooth linear functional on \( \mathcal{W}(\pi, \psi) \)
via the above form

\[
l_{\overline{\tau}} : W_v(g) \to \langle W_v, \overline{W} \rangle.
\]

Since \( (*) \) is \( G_n \)-invariant, this gives an isomorphism between the representation \( \tau \) on \( \overline{W} \)
and \( \pi^* \). We record it as a proposition.

Proposition 2.2. If the representation \( \pi \) is generic irreducible unitarizable with Whittaker
model \( \mathcal{W}(\pi, \psi) \), then the representation \( (\tau, \overline{W}) \) is a Whittaker model of \( \pi^* \) with respect to
\( \psi^{-1} \).

We also need the following lemma, which is Corollary 2.1 in [Ch06].
Lemma 2.3. Let $H$ be a complex smooth function on $G_r$ satisfying

$$H(ug) = \psi(u)H(g)$$

for any $g \in G_r$, $u \in N_r$.

If for any irreducible generic smooth representation $\rho$ of $G_r$, and for any Whittaker function $W \in \mathcal{W}(\rho, \psi^{-1})$, the function defined by the following integral vanishes for $\text{Re}(s)$ large

$$\int_{N_r \backslash G_r} H(g)W(g)|\det(g)|^{s-k}dg = 0,$$

where $k$ is some fixed constant, then $H \equiv 0$.

Remark. We don’t need to require the function $H$ to be in the Whittaker model of some generic representation as in [Ch06]. The proof there works for general Whittaker functions.

Next we introduce the Bessel functions briefly. For more details see [B05, Chai15]. Let $(\pi, V)$ be a generic irreducible representation of $G_n$. Take $W_v \in \mathcal{W}(\pi, \psi)$. Consider the integral for $g \in N_nA_n\omega_n N_n$

$$\int_{N_n} W_v(gu)\psi^{-1}(u)du.$$ 

This integral stabilizes along any compact open filtration of $N_n$, and the map

$$v \to \int_{N_n}^* W_v(gu)\psi^{-1}(u)du,$$

where $\int^*$ denotes the stabilized integral, defines a Whittaker functional on $V$. By the uniqueness of Whittaker functional, there exists a scalar $j_{\pi,\psi}(g)$ such that

$$\int_{N_n}^* W_v(gu)\psi^{-1}(u)du = j_{\pi,\psi}(g)W_v(I).$$

Definition 2.4. The assignment $g \to j_\pi(g) = j_{\pi,\psi}$ defines a function on $N_nA_n\omega_n N_n$, which is called the Bessel function of $\pi$ attached to $\omega_n$.

We extend $j_\pi$ to $G_n$ by putting $j_\pi(g) = 0$ if $g \notin N_nA_n\omega_n N_n$, and still use $j_\pi$ to denote it and call it the Bessel function of $\pi$. It is easy to check that $j_\pi$ is locally constant on $N_nA_n\omega_n N_n$(See Theorem 1.7 and remarks above it in [B05]) and $j_\pi(u_1gu_2) = \psi(u_1)\psi(u_2)j_\pi(g)$ for any $u_1, u_2 \in N_n, g \in G_n$.

A property of Bessel function which is important to us is the following weak kernel formula, see Theorem 4.2 in [Chai15].
Theorem 2.5. (Weak Kernel Formula) For any \( b_\omega_n, b = \text{diag}(b_1, \ldots, b_n) \in A_n \), and any \( W \in \mathcal{W} \), we have

\[
W(b_\omega_n) = \int j_\pi \left( \begin{array}{c}
\begin{pmatrix}
 a_1 & a_2 & \cdots & a_n \\
x_{21} & x_{n-1,1} & \cdots & x_{n-1,n-2} \\
 & 1
\end{pmatrix}^{-1}
\end{array}
\right) W \left( \begin{array}{c}
\begin{pmatrix}
 a_1 & a_2 & \cdots & a_n \\
x_{21} & x_{n-1,1} & \cdots & x_{n-1,n-2} \\
 & 1
\end{pmatrix}
\end{array}
\right) |a_1|^{-(n-1)} da_1 |a_2|^{-(n-2)} dx_{21} da_2 \cdots |a_{n-1}|^{-1} dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1},
\]

where the right side is an iterated integral, \( a_i \) is integrated over \( F^\times \subset F \) for \( i = 1, \ldots, n-1 \), \( x_{ij} \) is integrated over \( F \) for all relevant \( i, j \), and all measures are additive self-dual Haar measures on \( F \).

3 Howe vectors

In this section, we will introduce and study Howe vectors. Following [B05], for a positive integer \( m \), let \( K_n^m = I_n + M_n(p^m) \), here \( p \) is the maximal ideal of \( \mathcal{O} \) and \( \mathcal{O} \) is the ring of integers of \( F \). Use \( \varpi \) to denote an uniformizer of \( F \). Let

\[
d = \begin{pmatrix}
1 \\
\varpi^2 \\
\varpi^4 \\
\cdots \\
\varpi^{2n-2}
\end{pmatrix}
\]

Put \( J_{n,m} = d^m K_n^m d^{-m}, N_{n,m} = N_n \cap J_{n,m}, \bar{N}_{n,m} = \bar{N}_n \cap J_{n,m}, \bar{B}_{n,m} = \bar{B}_n \cap J_{n,m} \) and \( A_{n,m} = A_n \cap J_m \), then

\[
J_{n,m} = \bar{N}_{n,m} A_{n,m} N_{n,m} = \bar{B}_{n,m} N_{n,m} = N_{n,m} \bar{B}_{n,m}.
\]

For \( j \in J_{n,m} \), write \( j = \bar{b}_j n_j \) with respect to the above decomposition, as in [B05], define a character \( \psi_m \) on \( J_{n,m} \) by

\[
\psi_m(j) = \psi(n_j).
\]

Remark. We will write \( J_m \) for \( J_{n,m} \), and \( \psi \) for \( \psi_m \) when there is no confusion.

In this section we assume \( \pi \) is a generic irreducible unitarizable representation of \( G_n \).
Definition 3.1. \( W \in \mathcal{W}(\pi, \psi) \) is called a **Howe vector** of \( \pi \) of level \( m \) with respect to \( \psi \) if

\[
W(gj) = \psi_m(j)W(g)
\]

for all \( g \in G_n, j \in J_m = J_{n,m}. \)

**Remark.** It follows that the level \( m \) must be greater than or equal to the conductor of the central character \( \omega_{\pi} \) of \( \pi \) if the Howe vector exists.

For each \( W \in \mathcal{W}(\pi, \psi) \), let \( M \) be a positive constant such that \( R(K^M_n)W = W \) where \( R \) denotes the action of right multiplication. For any \( m > 3M \), put

\[
W_m(g) = \int_{N_{n,m}} W(gu)\psi^{-1}(u)du,
\]

then by Lemma 7.1 in [BOS], we have

\[
W_m(gj) = \psi_m(j)W_m(g), \quad \forall j \in J_m, \quad \forall g \in G_n.
\]

This gives the existence of Howe vectors when \( m \) is large enough. The following lemma establishes its uniqueness in Kirillov model, see Theorem 5.2 in [Chai15] for the proof.

**Lemma 3.2.** Assume \( W \in \mathcal{W}(\pi, \psi) \) satisfying (3.1). Let \( h \in G_{n-1} \), if

\[
W\begin{pmatrix} h \\ 1 \end{pmatrix} \neq 0,
\]

then \( h \in N_{n-1}B_{n-1,m}. \) Moreover

\[
W\begin{pmatrix} h \\ 1 \end{pmatrix} = \psi(u)W(I)
\]

if \( h = ub \), with \( u \in N_{n-1}, \ b \in B_{n-1,m}. \)

**Remark.** Thus, given \( \pi \), if \( m \) is large enough, there exists a unique vector \( W_{vm} \in \mathcal{W}(\pi, \psi) \) satisfying (3.1) and \( W_{vm}(I) = 1. \) We will call this vector as the normalized Howe vector of level \( m \) with respect to \( \psi \).

**Remark.** By the constructions above, Howe vectors exist if their levels \( m \) are sufficiently large. So when we talk about Howe vectors, we implicitly mean the levels are large enough so that these vectors exist.

**Proposition 3.3.** If \( W_{vm} \) is the normalized Howe vector of \( \pi \) of level \( m \) with respect to \( \psi \), then \( \overline{W}_{\omega,vm} \) is the normalized Howe vector of level \( m \) with respect to \( \psi^{-1} \) for \((\pi^*, \overline{\mathcal{W}})\), and

\[
\overline{W}_{vm}(g) = W_{vm}(\omega_n^t g^{-1} \omega_n) \quad \forall g \in G_n.
\]
Proof. Consider the Whittaker function $\tilde{W}_{\omega_n,v_m}$. For $j \in J_m$, one can check $\omega_n^t j^{-1} \omega_n \in J_m$, and

$$\psi_m(\omega_n^t j^{-1} \omega_n) = \psi_m^{-1}(j).$$

Thus for any $g \in G_n, j \in J_m$,

$$\tilde{W}_{\omega_n,v_m}(gj) = \tilde{W}_{j^{-1} \omega_n,v_m}(g) = \tilde{W}_{j^{-1} \omega_n,v_m}(g) = \psi_m^{-1}(j)\tilde{W}_{\omega_n,v_m}(g).$$

This shows that $\tilde{W}_{\omega_n,v_m}$ is the normalized Howe vector of level $m$ with respect to $\psi_m^{-1}$ for $(\pi^*, \tilde{W})$.

On the other hand, by Proposition 2.2, $W$ is a Whittaker model of $\pi^*$ with respect to $\psi^{-1}$. Take $W_{v_m} \in \mathcal{W}$, for any $g \in G_n, j \in J_m$,

$$W_{v_m}(gj) = W_{v_m}(g)\psi_m^{-1}(j)W_{v_m}(g),$$

which implies that $W_{v_m}$ is the normalized Howe vector of level $m$ with respect to $\psi_m^{-1}$. Thus by the uniqueness of Howe vectors, we have

$$\tilde{W}_{\omega_n,v_m}(g) = W_{v_m}(g) \quad \forall g \in G_n \quad \cdots \cdots (2),$$

which proves the proposition. □

**Proposition 3.4.** Assume $n = 2r + 1$. Let $g = \begin{pmatrix} 0 & a & u_r & 0 & u_r \omega_r a' u \\ b \omega_r & 0 & 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix}$, with $a = \text{diag}(a_1, \ldots, a_{r+1}) \in A_{r+1}, \ a' = \text{diag}(a'_1, \ldots, a'_{r}) \in A_{r}, \ b = \text{diag}(b_1, \ldots, b_r) \in A_r, \ u_r \in N_r, u = (u_{ij}) \in N_r$. $W_{v_m}$ is the normalized Howe vector. If $W_{v_m}(g) \neq 0$, then

$$\frac{a_i}{a_{i+1}} \in 1 + p^m \text{ for } i = 1, 2, \ldots, r, \quad \begin{pmatrix} I_r & 0 & \omega_r a' u \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \in J_m,$$

and

$$W_{v_m}(g) = W_{v_m} \begin{pmatrix} 0 & a & u_r & 0 & 0 \\ b \omega_r & 0 & 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix},$$

Proof. Take $j_1 = \begin{pmatrix} I_r & 0 & 0 \\ 0 & 1 & x_1 \\ 0 & 0 & I_r \end{pmatrix} \in J_m$ with $x_1 = (x_{11}, \ldots, x_{1r})$. As $\pi(j_1).v_m = \psi_m(j_1)v_m$, we find
\[\psi(x_{11})W_{v_m}(g) = W_{v_m}(gj_1)\]
\[= W_{v_m}\left(\begin{pmatrix}0 & a \\ b \omega_r & 0\end{pmatrix}\begin{pmatrix}u_r & 0 \\ 0 & u_r \omega_r a' u\end{pmatrix}j_1\right)\]
\[= W_{v_m}\left(\begin{pmatrix}0 & a \\ b \omega_r & 0\end{pmatrix}j_1\begin{pmatrix}u_r & 0 \\ 0 & u_r \omega_r a' u\end{pmatrix}\right)\]
\[= W_{v_m}\left(\begin{pmatrix}1 & x'_1 \\ 0 & I_r \end{pmatrix}\begin{pmatrix}0 & a \\ b \omega_r & 0\end{pmatrix}\begin{pmatrix}u_r & 0 \\ 0 & u_r \omega_r a' u\end{pmatrix}\right)\]
\[= \psi(a_1 a_2^{-1} x_{11})W_{v_m}(g),\]

where \(x'_1 = (a_1 a_2^{-1} x_{11}, ..., a_1 a_r^{-1} x_{1r})\).

Since \(W_{v_m}(g) \neq 0\), we have \(\psi(x_{11}(1 - a_1 a_2^{-1})) = 1\). As \(x_{11} \in \mathfrak{p}^{-m}\) is arbitrary, we get \(1 - a_1 a_2^{-1} \in \mathfrak{p}^m\), which means \(a_1 a_2^{-1} \in 1 + \mathfrak{p}^m\).

Let \(j'_1 = \begin{pmatrix}I_r & 0 & 0 \\ y_1 & 1 & 0 \\ 0 & 0 & I_r\end{pmatrix}\in J_m\) with \(y_1 = (y_{11}, ..., y_{1r})\). Then we have

\[W_{v_m}(g) = W_{v_m}(gj'_1)\]
\[= W_{v_m}\left(\begin{pmatrix}0 & a \\ b \omega_r & 0\end{pmatrix}\begin{pmatrix}I_r & 0 \\ y_1 u_1^{-1} & 1 - y_1 \omega_1 a' u\end{pmatrix}\begin{pmatrix}u_r & 0 \\ 0 & u_r \omega_r a' u\end{pmatrix}\right)\]
\[= W_{v_m}\left(\begin{pmatrix}1 & y'_1 \\ 0 & I_r \end{pmatrix}\begin{pmatrix}0 & a \\ b \omega_r & 0\end{pmatrix}\begin{pmatrix}u_r & 0 \\ 0 & u_r \omega_r a' u\end{pmatrix}\right)\]
\[= \psi(a_1 a_2^{-1} y_{1r} a'_1)W_{v_m}(g),\]

where \(y'_1 = (a_1 a_2^{-1} y_{1r} a'_1, ...).\)

Since \(W_{v_m}(g) \neq 0\), we have \(\psi(a_1 a_2^{-1} y_{1r} a'_1) = 1\). As \(a_1 a_2^{-1} \in 1 + \mathfrak{p}^m\), and \(y_{1r} \in \mathfrak{p}^{3m}\) is arbitrary, we find \(a'_1 \in \mathfrak{p}^{-3m}\).
Write

\[
g = \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & \omega_r a' u \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_{r-1} & 0 & 0 & 0 \\ 0 & 1 & 0 & a'_1 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & I_{r-1} \end{pmatrix},
\]

where the matrix \((0, u'_1) = \omega_r a' u - \begin{pmatrix} 0 \\ a'_1 \\ 0 \end{pmatrix}\).

Note that the last matrix belongs to \(J_m\), thus if we set

\[
g_1 = \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix},
\]

we have

\[
W_{vm}(g) = W_{vm} \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} = W_{vm}(g_1).
\]
Take \( j_2 = \begin{pmatrix} I_{r+1} & 0 & 0 \\ 0 & 1 & x_2 \\ 0 & 0 & I_{r-1} \end{pmatrix} \) \( \in J_m \) with \( x_2 = (x_{21}, \ldots, x_{2,r-1}) \). Then we have

\[
\psi(x_{21})W_{vm}(g) = \psi(x_{21})W_{vm}(g_1) = \psi(x_{21})W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) \]

\[
= W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) j_2 
\]

\[
= W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) j_2 
\]

\[
= W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) j_2 
\]

\[
= W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) j_2 
\]

\[
= \psi(a_2 a_3^{-1} x_{21})W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) 
\]

\[
= \psi(a_2 a_3^{-1} x_{21})W_{vm}(g_1) = \psi(a_2 a_3^{-1} x_{21})W_{vm}(g), 
\]

where \( x'_2 = (a_2 a_3^{-1} x_{21}, \ldots, a_2 a_3^{-1} x_{2,r-1}) \).

As \( W_{vm}(g) \neq 0 \) and \( x_{21} \in \mathfrak{p}^{-m} \) is arbitrary, we must have \( 1 - a_2 a_3^{-1} \in \mathfrak{p}^m \), that is, \( a_2 a_3^{-1} \in 1 + \mathfrak{p}^m \).

Now take \( j'_2 = \begin{pmatrix} I_r & 0 & 0 \\ 0 & 1 & 0 \\ y_2 & 0 & 1 \\ 0 & 0 & I_{r-1} \end{pmatrix} \) \( \in J_m \) with \( y_2 = (y_{21}, \ldots, y_{2r}) \). Similarly as above, we have

\[
W_{vm}(g) = W_{vm}(g_1) = W_{vm}(g_1 j'_2) 
\]

\[
= W_{vm} \left( \begin{pmatrix} 0 & a \\ b \omega_r & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{pmatrix} \begin{pmatrix} I_r & 0 & u'_1 \\ 0 & I_2 & 0 \\ 0 & 0 & I_{r-1} \end{pmatrix} \right) j'_2 
\]
if we set
\[
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\]
\[W_{v_m} \left( \begin{pmatrix}
0 & a \\
\omega_r & 0
\end{pmatrix}
\right) \left(
\begin{pmatrix}
u_r & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_r & 0 & 0 & 0 \\
0 & 1 & 0 & 0 \\
y_2 & 0 & 1 & -y_2 u_1' \\
0 & 0 & 0 & I_{r-1}
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_r & 0 & u_1' \\
0 & I_2 & 0 \\
0 & 0 & I_{r-1}
\end{pmatrix}
\right)
\]
\[= W_{v_m} \left( \begin{pmatrix}
1 & 0 & 0 & 0 \\
0 & 1 & y_2' & 0 \\
0 & 0 & I_{r-1} & 0 \\
0 & 0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
0 & a \\
\omega_r & 0 \\
0 & I_{r-1} & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
u_r & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_r & 0 & u_1' \\
0 & I_2 & 0 \\
0 & 0 & I_{r-1}
\end{pmatrix}
\right)
\]
\[= \psi(-a_2 a_3^{-1}(y_2 a_1(u_{12} + y_2 r_{r-1} a_2')) W_{v_m}(g_1) = \psi(-a_2 a_3^{-1}(y_2 a_1(u_{12} + y_2 r_{r-1} a_2')) W_{v_m}(g),
\]
where \(y_2' = (-a_2 a_3^{-1}(y_2 a_1(u_{12} + y_2 r_{r-1} a_2)), \ldots).\)

Again, as \(W_{v_m}(g) \neq 0, a_2 a_3^{-1} \in 1 + p^m \) and \(y_2 r_{r-1} \in p^{7m}, y_2 r \in p^{5m} \) are arbitrary, we get \(a_2' \in p^{-7m}, a_1' u_{12} \in p^{-5m} \), thus it follows that
\[
g_1 = \left( \begin{pmatrix}
0 & a \\
\omega_r & 0 \\
0 & I_{r-1} & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
u_r & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_r & 0 & u_1' \\
0 & I_2 & 0 \\
0 & 0 & I_{r-1}
\end{pmatrix}
\right)
\]
\[= \left( \begin{pmatrix}
0 & a \\
\omega_r & 0 \\
0 & I_{r-1} & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
u_r & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_r & 0 & u_1' \\
0 & I_3 & 0 \\
0 & 0 & I_{r-2}
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_{r-2} & 0 & 0 & 0 & 0 \\
0 & 1 & 0 & 0 & a_2' \\
0 & 0 & 1 & 0 & a_1' u_{12} \\
0 & 0 & 0 & I_2 & 0 \\
0 & 0 & 0 & 0 & 1
\end{pmatrix}
\right),
\]
where \(u_1' = (0 \ u_2') + \left( \begin{pmatrix}
a_2' \\ a_1' u_{12}
\end{pmatrix}
\right)\).

Note that the last matrix belongs to \(J_m\). So we find
\[
W_{v_m}(g) = W_{v_m}(g_1) = W_{v_m}(g_2)
\]
if we set \(g_2 = \left( \begin{pmatrix}
0 & a \\
\omega_r & 0 \\
0 & I_{r-1} & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
u_r & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \left(
\begin{pmatrix}
I_r & 0 & u_2' \\
0 & I_3 & 0 \\
0 & 0 & I_{r-2}
\end{pmatrix}
\right).
\]

Now take \(j_3 = \left( \begin{pmatrix}
I_{r+2} & 0 & 0 \\
0 & 1 & x_3 \\
0 & 0 & I_{n-2}
\end{pmatrix}
\right), j_3' = \left( \begin{pmatrix}
1 \\
1 \\
y_3 & 1
\end{pmatrix}
\right) \ldots \in J_m, \) and then argue

as above inductively, eventually, we will find \(a_3 a_4^{-1}, \ldots, a_r a_{r+1}^{-1} \in 1+p^m, \left( \begin{pmatrix}
I_r & 0 & \omega_r a' u \\
0 & 1 & 0 \\
0 & 0 & I_r
\end{pmatrix}
\right) \in \)
\[ J_m, \text{ and} \]
\[ W_{v_m}(g) = W_{v_m} \left( \begin{pmatrix} 0 & a \\ b_{\omega_r} & 0 \end{pmatrix} \begin{pmatrix} u_r & 0 \\ 0 & I_r \end{pmatrix} \right), \]

which finishes the proof. \( \square \)

**Proposition 3.5.** Assume \( n = 2r + 1 \). Let \( g = \left( \begin{array}{ccc} 0 & a \\ b_{\omega_r} & 0 \end{array} \right) \begin{pmatrix} u_r & 0 \\ 0 & I_r \end{pmatrix}, \) with \( a = \text{diag}(a_1, \ldots, a_{r+1}) \in A_{r+1}, a' = \text{diag}(a'_1, \ldots, a'_{r}) \in A_r, b = \text{diag}(b_1, \ldots, b_r) \in A_{r}, u_r \in N_r, u = (u_{ij}) \in N_r, W_{v_m} \text{ is the normalized Howe vector. Let} \ u_1 = \left( \begin{array}{ccc} u_r & 0 \\ 0 & I_r \end{array} \right), \ u_2 = \omega_n^t u_1 \omega_n. \text{ If} W_{v_m}(g) \neq 0, \text{ then} \]
\[ \overline{W}_{v_m}(u_2g) = W_{v_m}(g^{-1}u_2^{-1}). \]

**Proof.** By Proposition 3.4, if \( W_{v_m}(g) \neq 0, \text{ then} \ a_i a_{i+1}^{-1} \in 1+p^m, i = 1, 2, \ldots, r, \ \left( \begin{array}{ccc} I_r & 0 & \omega_r' a' u \\ 0 & 1 & 0 \\ 0 & 0 & I_r \end{array} \right) \in J_m \text{ and} \]
\[ W_{v_m}(g) = W_{v_m}(g'), \]
where \( g' = \left( \begin{array}{ccc} 0 & a \\ b_{\omega_r} & 0 \end{array} \right) u_1 \). Note that \( W_{v_m}(g^{-1}u_2^{-1}) = W_{v_m}(g'^{-1}u_2^{-1}). \) Hence it suffices to prove the proposition for \( g' \), that is,
\[ \overline{W}_{v_m}(u_2g') = W_{v_m}(g'^{-1}u_2^{-1}). \]

By Proposition 3.3, we have
\[ \overline{W}_{v_m}(u_2g') = W_{v_m}(\omega_n^t u_2^{-1}g'^{-1}u_2^{-1}) \]
\[ = W_{v_m}(u_1^{-1}\omega_n \left( \begin{array}{ccc} 0 & a^{-1} \\ b^{-1}_{\omega_r} & 0 \end{array} \right) u_1^{-1} \omega_n) \]
\[ = W_{v_m}(u_1^{-1} \left( \begin{array}{ccc} 0 & \omega_r b^{-1} \\ \omega_r a^{-1}_{r+1} \omega_{r+1} & 0 \end{array} \right) u_2^{-1}) \]
\[ = \psi(u_1^{-1})W_{v_m}(\left( \begin{array}{ccc} I_{r+1} & \omega_r b^{-1} a_{r+1} \\ \omega_r a^{-1}_{r+1} \omega_{r+1} & I_{r+1} \end{array} \right) \left( \begin{array}{ccc} I_{r+1} & 0 \\ 0 & \omega_r^t u_2^{-1} \omega_r \end{array} \right)) \]
\[ = \psi(u_1^{-1}) \omega_n (a_{r+1}^{-1}) W_{v_m}(\left( \begin{array}{ccc} I_{r+1} & \omega_r b^{-1} a_{r+1} \\ \omega_r a^{-1}_{r+1} \omega_{r+1} & I_{r+1} \end{array} \right) \left( \begin{array}{ccc} I_{r+1} & 0 \\ 0 & \omega_r^t u_2^{-1} \omega_r \end{array} \right)) \]
\[ = \psi(u_1^{-1}) \omega_n (a_{r+1}^{-1}) \omega_r^t(u_2^{-1} a''), \]

where \( a'' = \text{diag}(1, a_{r+1} a_{r+1}^{-1}, \ldots, a_{r+1} a_r^{-1}, 1, \ldots, 1). \)
Since $a_i^{-1}a_{i+1}^{-1} \in 1+p^m$ for $i = 1, 2, \ldots, r$, we then have $a_{r+1}^{-1}a_{r+2}^{-1} \in 1+p^m$ for $i = 1, \ldots, r$.

It follows that $a'' \in J_m$, and we find

$$
\overline{W}_{VM}(u_2 g') = \psi(u_1^{-1})\omega(\omega(a_{r+1}^{-1})) W_{VM} \left( \left( I_{r+1} \begin{array}{cc} \omega r b^{-1} a_{r+1} & \omega r t u_{r-1}^{-1} \omega r \\ a_{r+1}^{-1} & \omega r t u_{r-1}^{-1} \omega r \end{array} \right) \right) \cdots (3).
$$

On the other hand,

$$
W_{VM}(g^{-1} u_2^{-1}) = W_{VM} \left( \begin{array}{cc} a_1^{-1} & 0 \\ 0 & a_{r+1}^{-1} \omega r b^{-1} \end{array} \right) u_2^{-1})
\psi(u_1^{-1})W_{VM} \left( \left( I_{r+1} \begin{array}{cc} \omega r b^{-1} a_{r+1} & \omega r t u_{r-1}^{-1} \omega r \\ a_{r+1}^{-1} & \omega r t u_{r-1}^{-1} \omega r \end{array} \right) \right) \cdots (3).
$$

where $a'' = \text{diag}(a_{r+1}^{-1}a_1^{-1}, \ldots, a_{r+1}^{-1}a_1^{-1}, 1, 1, \ldots, 1)$.

Similarly, $a'' \in J_m$, and we get

$$
W_{VM}(g^{-1} u_2^{-1}) = \psi(u_1^{-1})\omega(\omega(a_{r+1}^{-1})) W_{VM} \left( \left( I_{r+1} \begin{array}{cc} \omega r b^{-1} a_{r+1} & \omega r t u_{r-1}^{-1} \omega r \\ a_{r+1}^{-1} & \omega r t u_{r-1}^{-1} \omega r \end{array} \right) \right) \cdots (4).
$$

Compare (3) and (4), we find

$$
\overline{W}_{VM}(u_2 g') = W_{VM}(g^{-1} u_2^{-1})
$$

and the proposition follows.

We will record the following analog property of $W_{VM}$ on the big Bruhat cell though we don’t need it in the present paper.

**Proposition 3.6.** For $g = a_1 a_2 \omega a_2 \in N_n \omega A_n N_n$, let $u = \omega t u_2 \omega \omega n u_1^{-1}$, then

$$
\overline{W}_{VM}(u g) = W_{VM}(g^{-1} u^{-1}).
$$

**Proof.** Use Proposition 3.3.

\[\square\]

## 4 Proof of the Main Result

In this section, we will prove **Conjecture 2** which will imply the local converse conjecture of Jacquet by the results in [JNS]. We first recall the conjecture as follows.

**Conjecture 2.** Assume $n \geq 2$. Let $\pi_1$ and $\pi_2$ be irreducible unitarizable and supercuspidal smooth representations of $GL_n(F)$. Suppose for any integer $r$, with $1 \leq r \leq \lfloor \frac{n}{2} \rfloor$, and any irreducible generic smooth representation $\rho$ of $G_r$, we have
\[ \gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi), \]

then \( \pi_1 \) and \( \pi_2 \) are isomorphic.

From section 3.1, [Ch06], we have the following disjoint decomposition

\[ G_n = \bigsqcup_{i=0}^{n-1} N_n \alpha^i P_n, \]

where \( \alpha = \begin{pmatrix} 1 & I_{n-1} \\ 0 & 1 \end{pmatrix} \). We note that for \( 1 \leq i \leq n-1 \),

\[ t(\alpha^i)^{-1} = \alpha^i. \]

The following is Proposition 3.1 in [Ch06].

**Proposition 4.1.** Let \( \pi_1, \pi_2 \) be two generic irreducible representations of \( G_n \) with the same central character, and let \( W_1, W_2 \) be two Whittaker functions for \( \pi_1, \pi_2 \) respectively, which agree on \( P_n \). If the local gamma factors \( \gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi) \) for all irreducible generic smooth representation \( \rho \) of \( G_i \), then \( W_1, W_2 \) agree on \( N_n \alpha^i P_n \).

**Theorem 4.2.** Assume \( n = 2r + 1 \). Let \( \pi_1 \) and \( \pi_2 \) be generic irreducible unitarizable representations of \( G_n \). Suppose for any integer \( l \), with \( 1 \leq l \leq \left\lfloor \frac{n}{2} \right\rfloor = r \), and any irreducible generic smooth representation \( \rho \) of \( G_l \), we have

\[ \gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi). \]

Let \( W^i_{\omega_n} \) be normalized Howe vectors of \( \pi_i, i = 1, 2 \). Then for any \( a \in A_n \), we have

\[ W^1_{\omega_n}(a\omega_n) = W^2_{\omega_n}(a\omega_n). \]

**Proof.** By Proposition 3.3, \( \tilde{W}^i_{\omega_n} \) is the normalized Howe vector of \( \pi_i^*, i = 1, 2 \). For any \( a \in A_n \), consider the following Rankin-Selberg integrals

\[ \gamma(s, \pi_i^* \times \rho, \psi^{-1}) \omega_{\rho}(-1)^{2r} \int_{N_r \backslash G_r} \int_{M_{r \times r}} \tilde{W}^i_{\omega_n, \omega_m} \left( \begin{pmatrix} g & x \\ I_r & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \omega_{2r} & \alpha^{r+1} a \\ 1 & 1 \end{pmatrix} \right) W'(g). \]

\[ |det(g)|^{\frac{s}{2} - \frac{r+1}{4}} dxdg = \int \tilde{W}^i_{\omega_n, \omega_m} \left( \begin{pmatrix} g & \omega_{n,r} \\ I_{r+1} & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \omega_{2r} & \alpha^{r+1} a^{-1} \omega_{n,r} \alpha^{r+1} a^{-1} \omega_{n,r} \alpha^{r+1} a^{-1} \omega_{n,r} \end{pmatrix} \right) \tilde{W}'(g) |det(g)|^{1-s} \frac{1}{4} dg \cdots (5), \]

where \( \rho \) is any generic irreducible smooth representation of \( G_r \).

We first look at left hand side of (5), it equals (we will write \( \gamma(s, \pi_i^* \times \rho, \psi^{-1}) \) simply as \( \gamma \) to save space)

\[ \gamma(s, \pi_i^* \times \rho, \psi^{-1}) \omega_{\rho}(-1)^{2r} \int_{N_r \backslash G_r} \int_{M_{r \times r}} \tilde{W}^i_{\omega_n, \omega_m} \left( \begin{pmatrix} g & x \\ I_r & 1 \end{pmatrix} \right) \left( \begin{pmatrix} \omega_{2r} & \alpha^{r+1} a \\ 1 & 1 \end{pmatrix} \right) W'(g) |det(g)|^{\frac{s}{2} - \frac{r+1}{4}} dxdg \]
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Then the above integral equals to

\[ \gamma \int_{N_r \setminus G_r} \int_{M_{r \times r}} W_i^j \begin{pmatrix} \omega_n \begin{pmatrix} t g^{-1} & -t g^{-1} x \\ I_r & 1 \end{pmatrix} & \omega_r \\ \omega_{r+1} & \omega_n \end{pmatrix} W'(g) |\det(g)|^{s - \frac{r+1}{2}} dxdg \]

where \( d\bar{x} \) denotes the Haar measure on \( G_r \). We continue to get the above equal to

\[ \gamma \int_{N_r \setminus G_r} \int_{G_r} W_i^j \begin{pmatrix} \omega_n \begin{pmatrix} t g^{-1} & -t \bar{x} \\ I_r & 1 \end{pmatrix} & \omega_r \\ \omega_{r+1} & \omega_n \end{pmatrix} W'(g) |\det(g)|^{s + \frac{r-1}{2}} |\det(\bar{x})|^r d\bar{x}dg \]

if we write \( a^{-1} = \begin{pmatrix} a' \\ a'_{r+1} \\ a'' \end{pmatrix} \) with \( a' = \text{diag}(a_1^{-1}, ..., a_r^{-1}) \), \( a'' = \text{diag}(a_1^{-1}, ..., a_2^{-1}_r) \) \( \in A_r \).

\[ \gamma \int_{N_r \setminus G_r} \int_{G_r} W_i^j \begin{pmatrix} 0 & a_{r+1}^{-1} & 0 \\ a'' \omega_r & 0 & 0 \\ -\omega_r t g^{-1} \omega_r a' \omega_r & 0 & 0 \end{pmatrix} W'(g) |\det(g)|^{s + \frac{r-1}{2}} |\det(\bar{x})|^r d\bar{x}dg \]

Write \( \bar{x} = -\omega_r v_r \omega_r c u_r \omega_r^{-1} \omega_r \) uniquely, where \( v_r, u_r \in N_r, c \in A_r \). Note that \( \bar{x} \) runs through an open dense subset of \( G_r \), as \( u_r, v_r \) run through \( N_r \) and \( c \) runs through \( A_r \).

Then the above integral equals to

\[ \gamma \int_{N_r \setminus G_r} \int_{N_r \times A_r \times N_r} W_i^j \begin{pmatrix} 0 & a_{r+1}^{-1} & 0 \\ a'' \omega_r & 0 & 0 \\ v_r \omega_r \omega_r^{-1} s \end{pmatrix} W'(g) |\det(g)|^{s + \frac{r-3}{2}} |\det(-ca''^{-1})|^r \delta(v_r, c, u_r, a'') dgdv_r d'ed\omega_r \]

where \( \delta(v_r, c, u_r, a'') \) is certain Jacobian as a function of the indicated variables.

Let \( \Omega_r = N_r \omega_r A_r N_r \). Let \( \Omega_{r}' = \{ u_r \in N_r : \omega_r u_r \omega_r \in \Omega_r \} \).

Claim: \( \Omega_{r}' \) is open dense in \( N_r \).

proof of the claim: First observe that the claim is equivalent to the following

\[ \overline{\Omega}_r \cap \Omega_r \text{ is open dense in } \overline{\Omega}_r \]

as \( \omega_r N_r \omega_r = \overline{\Omega}_r \cap \Omega_r \).

To prove (7), in general, if \( g = (g_{ij}) \in G_r \), then by Proposition 10.3.6 in [Go](Proposition 10.3.6 is over \( \mathbb{R} \), but the proof works equally well over p-adic fields), \( g \in \Omega_r \) if and
only if all the bottom left minors are nonzero, that is, \( g_{r1} \neq 0 \), \( det \begin{pmatrix} g_{r-1,1} & g_{r-1,2} \\ g_{r1} & g_{r2} \end{pmatrix} \neq 0 \), \( ... \), \( det(g) \neq 0 \).

Thus the complements of \( N_r \cap \Omega_r \) in \( \overline{N_r} \) is a union of finitely many closed subvarieties with strictly smaller dimensions than \( \overline{N_r} \), thus we obtain (7) and the claim follows.

Let's continue the proof of the theorem. For any \( c \in A_r \), the set

\[ N_r c u_r \omega_r A_r N_r = \Omega_r \]

is open dense in \( G_r \) if \( u_r \in N'_{r} \), which implies the set

\[ \omega_r N_r c u_r \omega_r A_r N_r = \omega_r \Omega_r \]

is also open dense in \( G_r \) if \( u_r \in N'_{r} \). Hence if \( u_r \in N'_{r} \), the subset of cosets

\[ \Omega'_r := \overline{N_r} \cdot c u_r \omega_r A_r N_r = \omega_r \Omega_r \]

is open dense in \( \overline{N_r} \setminus G_r \).

Since \( g \in N_r \setminus G_r \) if and only if \( t^r g^{-1} \in \overline{N_r} \setminus G_r \), and \( \Omega'_r \) is open dense in the latter space, combining with the claim, we can rewrite the integral (6) as

\[
\gamma \int_{N_r \setminus G_r} \int_{N_r \times A_r \times N_{r}'} W^i_{vm} \begin{pmatrix} 0 & a_{r+1}^{-1} & 0 \\ a'' \omega_r & 0 & 0 \\ v_r \omega_r c u_r & 0 & \omega_r t g^{-1} \omega_r a' \omega_r \end{pmatrix} W'(g) |det(g)|^{s + \frac{r - 1}{2}} |det(-c a^{r-1} g^{-1})|^r \delta(v_r, c, u_r, a') dgdv_r d' cdur_r
\]

\[
= \gamma \int_{\Omega'_r} \int_{N_r \times A_r \times N_{r}'} W^i_{vm} \begin{pmatrix} 0 & a_{r+1}^{-1} & 0 \\ a'' \omega_r & 0 & 0 \\ v_r \omega_r c u_r & 0 & -v_r \omega_r c u_r a''^{-1} v_r b \end{pmatrix} W'(g) |det(g)|^{s + \frac{r - 1}{2}} |det(-c a^{r-1} g^{-1})|^r \delta(v_r, c, u_r, a') dgdv_r d' cdur_r \cdots (8)
\]

if we write \( t^r g^{-1} = -\omega_r v_r \omega_r c u_r \omega_r a''^{-1} v_r b \omega_r a''^{-1} \omega_r \) where \( b \in A_r, v_r' \in N_r \).

Now we are going to show that the integrals in (8) are equal to each other for \( i = 1, 2 \) based on results in section 3 and Proposition 4.1 which will imply the left sides of (5) are also equal.

Let

\[
h = \begin{pmatrix} 0 & a_{r+1}^{-1} & 0 \\ a'' \omega_r & 0 & 0 \\ -\omega_r^i t \omega_r a'' \omega_r & 0 & \omega_r^i t g^{-1} \omega_r a' \omega_r \end{pmatrix}.
\]
Whenever they are nonzero, which implies $t^*p^{-1}d\alpha^r u\omega_\pi, i = 1, 2$, agrees on $N_{2r+1}\alpha^r P_n$, and it is here we need to require the number of twists is at least $[\frac{n}{2}] = r$ as we will see.

By Corollary 2.7, \cite{JNS}, $\pi_1, \pi_2$ have the same central characters. By Lemma 3.2, $\tilde{W}^{i}_{\omega_n, v_m}$ agree on $P_n, i = 1, 2$. As $\pi_1, \pi_2$ have the same local gamma factors twisted by irreducible generic representations of $G_l$ with $1 \leq l \leq [\frac{n}{2}] = r$, by Proposition 3.4, $\tilde{W}^{i}_{\omega_n, v_m}, i = 1, 2$ also agree on $N_{n}\alpha^r P_n$. As $\alpha^r p^{-1}\omega_n t u\omega_n$ is an element in $N_{n}\alpha^r P_n$, by the above computation we find

\[
\tilde{W}^{1}_{v_m}(uh) = \omega_\pi^{-1}(d)\tilde{W}^{1}_{\omega_n, v_m}(\alpha^r p^{-1}\omega_n t u\omega_n) = \omega_\pi^{-1}(d)\tilde{W}^{2}_{\omega_n, v_m}(\alpha^r p^{-1}\omega_n t u\omega_n) = \tilde{W}^{2}_{v_m}(uh)
\]

whenever they are nonzero, which implies $W^{1}_{v_m}(h) = W^{2}_{v_m}(h)$. Hence the integrals in (8) are equal to each other for $i = 1, 2$. It is clear that we need to require the number of twists is at least $[\frac{n}{2}] = r$ as $\alpha^r p^{-1}\omega_n t u\omega_n \in N_{n}\alpha^r P_n$.

Since $\gamma(s, \pi^*_i \times \rho, \psi)(1 - s, \pi^*_i \times \rho^*, \psi^{-1}) = 1$ by the statements after Lemma 3.1 in \cite{He}, by the assumptions on local gamma factors, we get $\gamma(s, \pi^*_2 \times \rho, \psi) = \gamma(s, \pi^*_2 \times \rho, \psi)$. Then we can conclude that the left hand sides of (5) are equal for $i = 1, 2$, which means the right hand sides are equal to each other.
Now apply Lemma 2.3 we conclude that
\[
\widetilde{W}^1_{\omega_{n,v_m}} \left( \begin{pmatrix} g & \omega_{2r} \\ I_{r+1} & 1 \end{pmatrix} \omega_{n,r} \left( \begin{pmatrix} \omega_{2r} & \\ 1 \end{pmatrix} \alpha^{r+1}a^{-1} \right) \right) = \widetilde{W}^2_{\omega_{n,v_m}} \left( \begin{pmatrix} g & \omega_{2r} \\ I_{r+1} & 1 \end{pmatrix} \omega_{n,r} \left( \begin{pmatrix} \omega_{2r} & \\ 1 \end{pmatrix} \alpha^{r+1}a^{-1} \right) \right) \cdots (9).
\]

Let \( g = \omega_r \) in the above identity, we finally proved the theorem.

\[\square\]

**Theorem 4.3.** *Conjecture 2 is true when \( n = 2r + 1 \) is odd.*

**Proof.** Let \( \pi_1, \pi_2 \) be irreducible unitarizable supercuspidal representations of \( G_n \) satisfying the assumptions in *Conjecture 2*. By Theorem 4.2 their normalized Howe vectors \( W^i_{v_m} \), \( i = 1, 2 \) satisfy
\[
W^1_{v_m}(a\omega_n) = W^2_{v_m}(a\omega_n).
\]
As this is true for all levels of Howe vectors, then by Proposition 5.3 in [Chai15], we find
\[
j_{\pi_1}(a\omega_n) = j_{\pi_2}(a\omega_n)
\]
for all \( a \in A_n \). Thus \( j_{\pi_1}(g) = j_{\pi_2}(g) \) for all \( g \in \Omega_n \).

Consider \( g = u_1\omega_nau_2 \in \Omega_n \), by the weak kernel formula Theorem 2.5 we have
\[
W^i_{v_m}(g) = W^i_{v_m}(u_1\omega_nau_2) = \psi(u_1)W^i_{u_2,v_m}(\omega_n a)
\]
\[
= \int j_{\pi_i} \left( \begin{pmatrix} a_1 & a_2 & \cdots \\ x_{21} & x_{n-1,1} & \cdots \\ & \ddots & \ddots & \ddots \\ & & x_{n-1,n-2} & x_{n-1,n-1} \\ & & & x_{n-1,1} \\ & & & 1 \end{pmatrix} \right)^{-1} \begin{pmatrix} a_1 \\ x_{21} \\ \vdots \\ x_{n-1,1} \\ 1 \end{pmatrix} W^i_{u_2,v_m} \left( \begin{pmatrix} a_1 \\ x_{21} \\ \vdots \\ x_{n-1,1} \\ 1 \end{pmatrix} \right) \left( \begin{pmatrix} a_1 & a_2 & \cdots \\ x_{21} & x_{n-1,1} & \cdots \\ & \ddots & \ddots & \ddots \\ & & x_{n-1,n-2} & x_{n-1,n-1} \\ & & & x_{n-1,1} \\ & & & 1 \end{pmatrix} \right) u_2
\]
\[
\left| a_1 \right|^{-(n-1)} \left| a_2 \right|^{-(n-2)} dx_{21} da_2 \cdots \left| a_{n-1} \right|^{-1} dx_{n-1,1} \cdots dx_{n-1,n-2} da_{n-1}.
\]

We note that the element
\[
\begin{pmatrix} a_1 & a_2 & \cdots \\ x_{21} & x_{n-1,1} & \cdots \\ & \ddots & \ddots & \ddots \\ & & x_{n-1,n-2} & x_{n-1,n-1} \\ & & & x_{n-1,1} \\ & & & 1 \end{pmatrix}^{-1}
\]

satisfy the assumptions in the above identity.
is in $\Omega_n$, so

$$j_{\pi_1} \left( \begin{pmatrix} a_1 & a_2 \\ x_{21} & 1 \\ & \ddots & \ddots \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} \end{pmatrix} \right)^{-1} b\omega_n \left( \begin{pmatrix} a_1 & a_2 \\ x_{21} & 1 \\ & \ddots & \ddots \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} \end{pmatrix} \right)^{-1} = j_{\pi_2} \left( \begin{pmatrix} a_1 & a_2 \\ x_{21} & 1 \\ & \ddots & \ddots \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} \end{pmatrix} \right).$$

The element

$$\left( \begin{pmatrix} a_1 \\ x_{21} \\ \ddots \\ x_{n-1,1} & \cdots & x_{n-1,n-2} & a_{n-1} \end{pmatrix} u_2 \right)$$

is in $P_n$. By Lemma 5.2, $W^{1}_{v_m}(p) = W^{2}_{v_m}(p), \forall p \in P_n$. Hence the last integrals are equal to each other for $i = 1, 2$, which implies $W^{1}_{v_m}(g) = W^{1}_{v_m}(g)$ for all $g \in \Omega_n$. As $\Omega_n$ is open dense in $G_n$, we eventually get that $W^{1}_{v_m}(g) = W^{2}_{v_m}(g)$ for all $g \in G_n$ which finishes the proof by the multiplicity one theorem on Whittaker models.

**Theorem 4.4.** Conjecture 2 is true when $n = 2r$ is even.

**Proof.** Suppose $\pi_1, \pi_2$ are irreducible unitarizable supercuspidal representations of $G_{2r}$ satisfying the assumptions in Conjecture 2. Take a unitary character $\chi$ of $G_1$, and form the normalized induced representations $\tau_1 = Ind(\pi_1 \otimes \chi), \tau_2 = Ind(\pi_2 \otimes \chi)$. By Theorem 4.2 in [BZ], both $\tau_1, \tau_2$ are irreducible unitarizable supercuspidal representations of $G_{2r+1}$. For any $l$ with $1 \leq l \leq r$, and any irreducible generic smooth representation $\rho$ of $G_l$, we have

$$\gamma(s, \pi_1 \times \rho, \psi) = \gamma(s, \pi_2 \times \rho, \psi).$$

By the multiplicativity of local gamma factors, we get

$$\gamma(s, \tau_1 \times \rho, \psi) = \gamma(s, \tau_2 \times \rho, \psi).$$

Then by Theorem 4.2, for all normalized Howe vector $W^i_{v_m}$ of $\tau_i, i = 1, 2$, we have

$$W^{1}_{v_m}(a\omega_n) = W^{2}_{v_m}(a\omega_n) \cdots (10).$$

In the following, we will present three different approaches to finish the proof. The first is based on well expected property of Bessel functions: local integrability. The other two approaches are based on well established results: Derivatives of smooth representations of $G_n$ and Shahidi’s formula expressing local coefficients as Mellin transforms of partial Bessel functions. All three approaches have its own interests and they are quite independent to one another. They all illustrate the power of Bessel functions.
The first approach. The first way is based on the well expected property: local integrability of Bessel functions. As (10) is true for all levels of Howe vectors, by Proposition 5.3 in [Chai15], we find
\[ j_{\pi_1}(a\omega_n) = j_{\pi_2}(a\omega_n) \]
for all \( a \in A_n \). Thus \( j_{\pi_1}(g) = j_{\pi_2}(g) \) for all \( g \in \Omega_n \). It then follows from Corollary 7.2 [Chai16] that \( \pi_1 \cong \pi_2 \). This finishes the first approach.

The second approach. The second is based on the theory of derivatives of smooth representations on \( G_n \). We first recall a result of Cogdell and Piatetski-Shapiro about derivatives. Let \( \pi \) be an irreducible generic representations of \( G_n \). Take a Whittaker function \( W \in W(\pi, \psi) \), and a Schwartz function \( \Phi_0 \in S(F^{n-1}) \) which is supported in a sufficiently small neighborhood of 0, if the first derivative \( \pi^{(1)} \) of \( \pi \) is irreducible, then there is a Whittaker function \( W' \in W(\pi^{(1)}, \psi) \), such that for all \( g \in G_{n-1} \), we have
\[
W \left( \begin{pmatrix} g & 1 \\ 1 & 1 \end{pmatrix} \right) \Phi_0(\epsilon_{n-1}g) = |\det(g)|^{1/2}W'(g)\Phi_0(\epsilon_{n-1}g),
\]
where \( \epsilon_{n-1} = (0, \ldots, 0, 1) \). This is a special case of the second half of corollary to Proposition 1.7 in [CPS].

Let \( W^i_{\alpha,\nu_m} \) be the normalized Howe vector of level \( m \) in \( \tau_i, i = 1, 2 \). Recall \( \alpha = \begin{pmatrix} I_{n-1} \\ 1 \end{pmatrix} \). Take \( W = W^i_{\alpha,\nu_m} \), \( \Phi_0 \) to be the characteristic function of a sufficiently small neighborhood of 0. Note that \( \tau_i^{(1)} \cong \pi_i \) by Lemma 4.5 in [BZ] and it is irreducible. Apply the above result of Cogdell and Piatetski-Shapiro, we conclude that there exists some \( W'_i \in W(\tau_i^{(1)}, \psi) \), such that for all \( g \in G_{n-1} \), we have
\[
W^i_{\alpha,\nu_m} \left( \begin{pmatrix} g & 1 \\ 1 & 1 \end{pmatrix} \right) \Phi_0(\epsilon_{n-1}g) = |\det(g)|^{1/2}W'_i(g)\Phi_0(\epsilon_{n-1}g) \cdots (11).
\]

Given \( g \in G_{n-1}, j \in J_{n-1,m} \), where \( J_{n-1,m} \) is the same as in the Definition 3.1. Choose \( z \) in the center of \( G_{n-1} \) so that \( \Phi_0(\epsilon_{n-1}gz) = 1 \) and \( \Phi_0(\epsilon_{n-1}gzj) = 1 \). By (11), we have
\[
W^i_{\alpha,\nu_m} \left( \begin{pmatrix} gzj & 1 \\ 1 & 1 \end{pmatrix} \right) = |\det(gzj)|^{1/2}W'_i(gzj) \cdots (12)
\]
and
\[
W^i_{\alpha,\nu_m} \left( \begin{pmatrix} gz & 1 \\ 1 & 1 \end{pmatrix} \right) = |\det(gz)|^{1/2}W'_i(gz) \cdots (13).
\]
On the other hand, note that \( \begin{pmatrix} j & 1 \\ 1 & 1 \end{pmatrix} \alpha = \alpha \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \in J_{n,m} \) and \( \psi \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} = \).
Then the left hand side of (12) is

\[ W_{i,vm}^i \left( g^j \right) = W_{i,vm}^i \left( g^z, \alpha \left( j \right) \right) = \psi(j) W_{i,vm}^i \left( g^z \right) \]

(by (13)).

This equals the right hand side of (12), hence

\[ \psi(j) |\det(g)|^{1/2} W_i'(g) = |\det(gj)|^{1/2} W_i'(gj), \]

which implies that

\[ \psi(j) W_i'(g) = W_i'(gj). \]

As \( \tau_i^{(1)} \cong \pi_i \) and it has a central character. It follows that

\[ \psi(j) W_i'(g) = W_i'(gj), \]

which proves that \( W_i' \) is the Howe vector of level \( m \) for \( \pi_i, i = 1, 2 \).

Now suppose \( a \in G_{n-1} \) is diagonal, choose \( z \) in the center of \( G_{n-1} \) which is sufficiently close to 0 so that \( \Phi_0(e_{n-1} \omega_{n-1} za) = 1 \). Apply (11) to \( g = \omega_{n-1} za \), we have

\[ W_{i,vm}^i \left( \omega_{n-1} za \right) = |\det(\omega_{n-1} za)|^{1/2} W_i'(\omega_{n-1} za). \]

As \( W_{i,vm}^i \left( \omega_{n-1} za \right) = W_{i,vm}^i \left( 1 \right) \), it follows from (10) that

\[ W_1'(\omega_{n-1} za) = W_2'(\omega_{n-1} za), \]

which implies

\[ W_1'(\omega_{n-1} a) = W_2'(\omega_{n-1} a). \]

As this is true for all Howe vectors \( W_i' \) and all diagonal \( a \), we conclude that the Bessel functions of \( \pi_1, \pi_2 \) are equal to each other by Proposition 5.3 in [Chai15]. Since \( \pi_1, \pi_2 \) are supercuspidal representations, as in the proof of Theorem 4.3, it follows from the weak kernel formula (Theorem 2.5) that they are in fact isomorphic. This ends the proof of the second approach.
The third approach. We first need to recall Shahidi’s formula (Theorem 6.2 in \[Sh02\]) expressing local coefficients as Mellin transform of partial Bessel functions in our case. Let $P = MU$ be the standard parabolic subgroup of $G_{2r}$ with Levi $M = G_r \times G_r$. $U$ is the unipotent part, with opposite $\bar{U}$. Put $\omega_0 = \begin{pmatrix} I_r & \emptyset \\ I_r & \emptyset \end{pmatrix}$ and $\omega_M = \begin{pmatrix} \omega_r & \omega_r \\ \omega_r & \omega_r \end{pmatrix}$.

Let $N_M = N_{2r} \cap M$. Use $Z, Z_M$ to denote the centers of $M$ and $G_{2r}$ respectively. Let $Z_0^M = Z \setminus Z_M$.

As in \[Sh02\], we start with the following decomposition

$$\omega_0^{-1} n = mn' \bar{n} \quad \cdots (\ast 1)$$

valid for almost all $n \in U$, where $m \in M, n' \in U, \bar{n} \in \bar{U}$. The Bruhat decomposition of $m$ is

$$m = u_1 t \omega u_2 \quad \cdots (\ast 2),$$

where $u_1, u_2 \in N_m, t \in A_{2r}$ and $\omega$ is certain Weyl group element of $M$. As in section 3 of \[CPSS08\], if we set $u' = \omega_0 u_1^{-1} \omega_0^{-1}$ and $n_1 = u' n (u')^{-1}$, then the map $n \to n_1$ gives a bijection from the set of all $n$ satisfying ($\ast 1$) onto the Bruhat double coset $\bar{B}_{2r} \omega_0 \omega \bar{U} N_M$ of $G_{2r}$.

Shahidi’s formula involves certain unipotent integral defining partial Bessel functions (see ($\ast 3$) below). For this integral to be nonzero, $m \in M$ appearing in the integration must support a Bessel function in the sense of \[CPSS05\], at least for some full measure subset. Note that the cell $\bar{B}_{2r} \omega_2 \bar{U} N_M$ is the unique Bruhat double coset of $G_{2r}$ intersecting $U$ in an open dense subset. By Proposition 3.2 in \[CPSS08\], we have $\omega_2r = \omega_0 \omega$ which implies that $\omega = \omega_M$, and this Weyl element does support a Bessel function.

$Z^0_M N_M$ acts on $U$ by conjugation, we will use $Z^0_M N_M \setminus U$ to denote $Z^0_M N_M$ orbits in $N$, $dn$ is certain measure on this set of orbits. We first consider $N_M$ orbits in $U$. For $\begin{pmatrix} u_1 & \\ u_2 \end{pmatrix} \in N_M, \begin{pmatrix} I_r & X \\ I_r & \end{pmatrix} \in U$, we have

$$\begin{pmatrix} u_1 & \\ u_2 \end{pmatrix} \begin{pmatrix} I_r & X \\ I_r & \end{pmatrix} \begin{pmatrix} u_1 & \\ u_2 \end{pmatrix}^{-1} = \begin{pmatrix} I_r & u_1 X u_2^{-1} \\ I_r & \end{pmatrix}.$$ 

Hence the matrices like

$$\begin{pmatrix} I_r & \omega_r t \\ \omega_r t & I_r \end{pmatrix}$$

with $t \in A_r$, form a set of representatives of an open dense subset of $N_M \setminus U$. Direct computation shows that the decomposition ($\ast 1$) for such matrices is

$$\begin{pmatrix} I_r & \omega_r t \\ \omega_r t & I_r \end{pmatrix} = \begin{pmatrix} - (\omega_r t)^{-1} & \\ \omega_r t & \end{pmatrix} \begin{pmatrix} I_r & - \omega_r t \\ \omega_r t & (\omega_r t)^{-1} \end{pmatrix}.$$
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It then follows that we can find a set of representatives of a full measure subset $\Omega$ of $Z_M^0 N_M \setminus U$, and satisfy decomposition $\omega_0^{-1} n = mn' \bar{n}$, where $n \in \Omega$ and $m$ has the form $\omega_M a$ for certain diagonal matrices $a \in A_{2r}$. This is a weak version of Proposition 4.2.3 in [TS].

Let $\pi, \rho$ be generic irreducible representations of $G_r$, denote by $\sigma = \pi \otimes \rho$ which is a generic irreducible representations of $M$. Then the central characters $\omega_\sigma = \omega_\pi \otimes \omega_\rho$. We also define for $t \in F^*$, define characters of $F^*$ by $\omega_\sigma(t) = \omega_\sigma(\alpha^\vee(t))$ and $(\omega_0.\omega_\sigma)(t) = \omega_\sigma(\omega_0^{-1} \alpha^\vee(t) \omega_0)$, where $\alpha^\vee(t) = \begin{pmatrix} tI_r & \cr & I_r \end{pmatrix}$.

Now if $W_\bar{v}$ is a Whittaker function in $\sigma$ with $W_\bar{v}(I_{2r}) = 1$. Let $\hat{U}_0$ be a sufficiently large open compact subgroup of $\hat{U}$ and $\phi$ its characteristic function. For $n \in \Omega, \omega_0^{-1} n = mn' \bar{n}, y \in F^*$, we define the partial Bessel function

$$j_{\bar{v}, \hat{U}_0}(m, y) = \int_{N_M, n \setminus N_M} W_\bar{v}(mu^{-1}) \phi(zu \bar{u}^{-1}z^{-1}) \psi(u) du \cdots (3),$$

where $N_{M,n} = \{ u \in N_M : unu^{-1} = n \}$ and $z = \alpha^\vee(y^{-1} \bar{\chi}_0)$ is certain element in $Z_M^0$. If the character $\omega_\sigma(\omega_0.\omega_\sigma^{-1})$ is ramified, Theorem 6.2 in [Sh02] for the local coefficient $C(s, \sigma)$, applied to our case, can be stated as follows.

$$C(s, \sigma)^{-1} = \gamma(2 < \bar{\alpha}, \alpha^\vee > s, \omega_\sigma(\omega_0.\omega_\sigma^{-1}), \psi)^{-1} \times \int_{Z_M^0 N_M \setminus U} j_{\bar{v}, \hat{U}_0}(\hat{m}, y_0) \omega_0^{-1}(\bar{\chi}_0)(\omega_0.\omega_\sigma)(\bar{\chi}_0) q^{(s\bar{\alpha} + \rho, H_M(\bar{m}))} d\bar{n} \cdots (4).$$

In this formula, $y_0 \in F^*$ is an element with $ord(y_0) = -d - f$, where $d, f$ are conductors of $\psi$ and $\omega_\sigma^{-1}(\omega_0.\omega_\sigma)$, respectively. The choice of $y_0$ is irrelevant. The above integral is independent of the choice of $\bar{v}$ and $\hat{U}_0$ as long as $W_\bar{v}(I_{2r}) = 1$ and $\hat{U}_0$ is a sufficiently large compact open subgroup of $\hat{U}$. $\hat{m}, \hat{n}$ are related by (*1). Moreover by choosing representatives $\hat{n}$ in $\Omega, \hat{m}$ have the form $\omega_M a$ for certain diagonal matrices $a \in A_{2r}$ as we discussed above. We refer to [Sh02] for the unexplained terms in the formula.

We also note that, by Lemma 3.11 in [CPSS08], the domain of integration in the definition of $j_{\bar{v}, \hat{U}_0}(m, y_0)$ is independent of $m$, and depends only on $y_0$ and $\hat{U}_0$.

Now we begin the third proof. Let $\rho$ be an irreducible generic representation of $G_{2r}$, choose a character $\chi'$ of $G_1$, so that the normalized induced representation $\sigma = Ind(\rho \otimes \chi')$ is generic and irreducible. Consider $\tau_\chi \otimes \sigma$, which is an irreducible generic representation of $M = G_{2r+1} \times G_{2r+1}$. The central character of $\tau_\chi \otimes \sigma$ is $\omega_{\tau_\chi} \otimes (\omega_\rho \chi')$. Recall that for $t \in F^*$, $\omega_\sigma(t) = \omega_\sigma(\alpha^\vee(t))$ and $(\omega_0.\omega_\sigma)(t) = \omega_\sigma(\omega_0^{-1} \alpha^\vee(t) \omega_0)$, where
\[ \alpha^\vee(t) = \begin{pmatrix} tI_r \\ I_r \end{pmatrix}. \] Hence \( \omega_{\tau_i \otimes \sigma}(\omega_0, \omega_{\tau_i \otimes \sigma}^{-1}) = \omega_{\tau_i} \cdot (\omega_{\rho, \chi'})^{-1}. \) So we can choose \( \chi' \) to require further that the characters \( \omega_{\tau_i \otimes \sigma}(\omega_0, \omega_{\tau_i \otimes \sigma}^{-1}), i = 1, 2 \) are ramified.

Now we want to apply Shahidi’s formula \(^{(4)}\) to \( \tau_i \otimes \sigma, i = 1, 2 \), and to show that \( C(s, \tau_1 \otimes \sigma) = C(s, \tau_2 \otimes \sigma) \). For this purpose, we first choose \( \tilde{U}_0 \) large enough satisfying \(^{(4)}\) for both \( \tau_i \otimes \sigma, i = 1, 2 \) and fix \( y_0 \). Then take positive integer \( l \) sufficiently large so that \( N_{2r+1,l} \times N_{2r+1,l} \) contains the domain of integration in \( j_{\tilde{v}_i, \tilde{U}_0}(m, y_0) \), where \( N_{2r+1,l} = N_{2r+1} \cap J_{2r+1,l} \) as in section 3. Now choose \( W_{\tilde{v}_i} \left( \begin{array}{cc} g_1 \\ g_2 \end{array} \right) = W_{\tilde{v}_i}(g_1)W'(g_2) \), where \( W' \) is a Whittaker function of \( \sigma \) with \( W'(I_{2r+1}) = 1 \).

So with this \( W_{\tilde{v}_i} \), and plug the integral defining \( j_{\tilde{v}_i, \tilde{U}_0} \) into the formula \(^{(4)}\) for \( C(s, \tau_i \otimes \sigma) \). We get formula
\[
\int_{Z_{M,1} \setminus U} \int_{N_{M,a} \setminus N_M} W_{\tilde{v}_i}(mu^{-1}) \phi(zu^{-1}z^{-1}) \psi(u) du \omega_{\tau_i \otimes \sigma}(m, y) \chi_\alpha(s, \tau_i \otimes \sigma, \gamma \chi_\alpha)(\omega_0, \omega_{\tau_i \otimes \sigma}) \chi_\alpha(q^{(s\alpha + \rho, H_M(\tilde{m}))} d\tilde{m}.
\]
Note that \( \tilde{m} \) has particular form
\[
\begin{pmatrix} \omega_{2r+1} \\ \omega_{2r+1} \end{pmatrix}
\]
with diagonal matrices \( a, b \), and the integration with \( u \) is over \( N_{2r+1,l} \times N_{2r+1,l} \). By \( (10) \) and the definition of Howe vectors \( W_{\tilde{v}_i} \), we have
\[
W_{\tilde{v}_i}(\omega_{2r+1}au) = W_{\tilde{v}_i}(\omega_{2r+1}au)
\]
for all diagonal matrices \( a \) and \( u \in N_{2r+1,l} \). This then implies that
\[
W_{\tilde{v}_1}(\tilde{m}u^{-1}) = W_{\tilde{v}_2}(\tilde{m}u^{-1}),
\]
which means \( C(s, \tau_1 \otimes \sigma) = C(s, \tau_2 \otimes \sigma) \).

By the relation between local coefficient \( C(s, \tau_i \otimes \sigma) \) and gamma factors \( \gamma(s, \tau_i \otimes \sigma, \psi) \) and their multiplicativities, it follows that
\[
\gamma(s, \tau_1 \otimes \rho, \psi) = \gamma(s, \tau_2 \otimes \rho, \psi).
\]
By the \((2r + 1, 2r)\)-local converse theorem in \([H\delta]\), we then conclude that \( \tau_1 \cong \tau_2 \). Now apply Bernstein-Zelevinsky’s classification of irreducible admissible representations of \( G_n \) in terms of segments, for example Theorem 6.1 in \([Z]\), we can conclude that \( \pi_1 \cong \pi_2 \), which finishes the proof.

\[ \square \]
Theorem 4.5. Conjecture 1 is true.

Proof. This follows from Theorem 4.3, 4.4 and the work [JNS].

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