FORWARD ATTRACTION OF PULLBACK ATTRACTORS AND SYNCHRONIZING BEHAVIOR OF GRADIENT-LIKE SYSTEMS WITH NONAUTONOMOUS PERTURBATIONS

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ABSTRACT. We consider the nonautonomous perturbation $x_t + Ax = f(x) + \varepsilon h(t)$ of a gradient-like system $x_t + Ax = f(x)$ in a Banach space $X$, where $A$ is a sectorial operator with compact resolvent. Assume the non-perturbed system $x_t + Ax = f(x)$ has an attractor $\mathcal{A}$. Then it can be shown that the perturbed one has a pullback attractor $\mathcal{A}_\varepsilon$ near $\mathcal{A}$. If all the equilibria of the non-perturbed system in $\mathcal{A}$ are hyperbolic, we also infer from [4, 6] that $\mathcal{A}_\varepsilon$ inherits the natural Morse structure of $\mathcal{A}$. In this present work, we introduce the notion of nonautonomous equilibria and give a more precise description on the Morse structure of $\mathcal{A}_\varepsilon$ and the asymptotically synchronizing behavior of the perturbed system. Based on the above result we further prove that the sections of $\mathcal{A}_\varepsilon$ depend on time symbol continuously in the sense of Hausdorff distance. Consequently, one concludes that $\mathcal{A}_\varepsilon$ is a forward attractor of the perturbed nonautonomous system. It will also be shown that the perturbed system exhibits completely a global forward synchronizing behavior with the external force.

1. Introduction. The theory of attractors plays a fundamental role in the study of dynamical systems. This is because that if a system has an attractor, then all its long-term dynamics near the attractor will be captured by the attractor. The theory of attractors for autonomous dynamical systems has been fully developed in the past decades, in both finite and infinite dimensional cases; see e.g. [2, 13, 15, 20, 25, 26, 28], etc. In contrast to autonomous systems, the understanding of the dynamics of nonautonomous systems seems to be far more difficult, and even some fundamental notions such as attractors are still undergoing investigations.

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For nonautonomous dynamical systems, usually one can define two different types of nonautonomous attractors: pullback attractor and forward attractor, corresponding to pullback attraction and forward attraction, respectively. The existence of the former one is well known and can be obtained under quite general hypotheses similar to those to guarantee the existence of attractors for autonomous systems; see e.g. [5, 12, 14, 29]. This aroused great interest in a systematic study on pullback behavior of nonautonomous systems arising in applications in the past two decades. On the other hand, in most cases we are naturally more concerned with the forward behavior of a system. However, a pullback attractor may provide little information on this aspect. This can be seen from the following easy example (see also the example in [6, (4.8)]).

**Example 1.1.** Consider the scalar system on $\mathbb{R}$:

$$x'(t) = (2h(t) - 1)x,$$

where $h$ is a continuous monotone function on $\mathbb{R}$ with

$$\lim_{t \to -\infty} h(t) = 0, \quad \lim_{t \to \infty} h(t) = 1.$$  

It is easy to verify that this system has a global pullback attractor $A = \{A(t)\}_{t \in \mathbb{R}}$ with $A(t) = \{0\}$ for all $t \in \mathbb{R}$. On the other hand, any solution $x(t)$ of the system goes to infinity as $t \to +\infty$ except the trivial one (hence system (1.1) has neither a uniform attractor nor a forward one).

Theoretically, forward attractors sound to be quite suitable to describe the forward behavior of nonautonomous systems. Unfortunately, by far the existence of this type of attractors remains an open problem except in some particular cases such as the asymptotically autonomous and the periodic ones, and is still under investigations; see e.g. Cheban, Kloeden and Schmalfuß [9], Wang, Li and Kloeden [31], Kloeden [18] and Carvalho, Langa, Robinson and Surez [6, pp. 595]. It is well known that for a cocycle system $\Phi(t,p)x$ on a complete metric space $X$ with compact base space $P$, if a pullback attractor $\mathcal{A} = \{A(t)\}_{t \in \mathbb{R}}$ with $A(t) = \{0\}$ for all $t \in \mathbb{R}$ then $A$ is a forward attracting. Based on this fact it was shown in [9] that if the section $A(p)$ is lower semicontinuous in $p$ then $\mathcal{A}$ is a forward attractor (see also [31]). But in general the verification of the lower semicontinuity of $A(p)$ in $p$ seems to be a difficult task.

In this paper we consider the nonautonomous perturbation

$$x_t + Ax = f(x) + \varepsilon h(t)$$

of a gradient-like system

$$x_t + Ax = f(x)$$

in a Banach space $X$, where $A$ is a sectorial operator in $X$ with compact resolvent, and $h$ is translation compact. Suppose that all the equilibria of the non-perturbed system (1.3) are hyperbolic and that the system has an (local) attractor $\mathcal{A}$. We show that the perturbed system has a pullback attractor $\mathcal{A}_\varepsilon$ near $\mathcal{A}$ provided $\varepsilon$ is sufficiently small. We then introduce the notion of nonautonomous equilibria and give a more precise description on the Morse structure of the pullback attractor $\mathcal{A}_\varepsilon$. Based this result we finally prove that the section of $\mathcal{A}_\varepsilon$ depends on the time symbol continuously in the sense of Hausdorff distance. It then follows that $\mathcal{A}_\varepsilon$ is a forward attractor. Our results demonstrate simultaneously that the perturbed system exhibits completely a forward synchronizing behavior with the external perturbation $h$ in both point-wise and global sense.
Now let us give a more detailed description of our work. As usual we embed the system (1.2) into the following one:

\[ x_t + Ax = f(x) + \varepsilon p(t), \quad p \in \mathcal{H}, \]

where \( \mathcal{H} := \mathcal{H}[h] \) is the hull of \( h \) (in some appropriate function space). Suppose \( f \) is a locally Lipschitz map from \( X^\alpha \) to \( X \) for some \( \alpha \in [0, 1) \). Then (1.4) generates a cocycle system \( \varphi_\varepsilon(x) = \varphi_\varepsilon(t,p)x \) on \( X^\alpha \) with base space \( \mathcal{H} \) and driving system \( \theta_t \), where \( \theta_t \) is the shift operator on \( \mathcal{H} \) for each \( t \in \mathbb{R} \).

First, we introduce the notion of nonautonomous equilibria. Specifically, a nonautonomous equilibrium of \( \varphi_\varepsilon \) is a continuous map \( \Gamma \in C(\mathcal{H}, X^\alpha) \) such that \( \gamma(t) = \Gamma(\theta_t h) \) is a full solution of the system for each \( p \in \mathcal{H} \). We show that if \( E \) is a hyperbolic equilibrium of the non-perturbed system \( \varphi_0 \), then for \( \varepsilon > 0 \) sufficiently small the perturbed system \( \varphi_\varepsilon \) has a unique nonautonomous equilibrium \( \Gamma \) in a small neighborhood \( \mathcal{U} \) of \( E \). Moreover, the local \( p \)-section unstable manifold \( W^u_p(\Gamma, p) \) (see Section 2 for definition) of \( \Gamma \) depends on \( p \) continuously. It is worth mentioning that \( \gamma(t) = \Gamma(\theta_t h) \) gives a full solution of (1.2) that is completely synchronizing with the external force \( h \). Indeed, if \( h \) is periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic), then so is \( \sigma(t) := \theta_t h \). Thus the continuity of \( \Gamma(p) \) in \( p \) implies that \( \gamma(t) \) is periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic) as well. Quite similar results can be found in the nice work of Kloeden and Rodrigues [19] in a more general setting. Roughly speaking, the authors introduced a class of functions defined on the real line extending periodic and almost periodic functions, and prove that if the forcing term of a perturbed nonautonomous system belongs to this class, then the system has a full solution which belongs to the same class as well. In contrast, in our case the description on the synchronizing behavior of the system given here seems to be a little more explicit.

Second, we discuss Morse structures of pullback attractors of the system \( \varphi_\varepsilon \). This problem was addressed as early as in Carvalho et al [6]. Suppose the non-perturbed system has an attractor \( \mathcal{A} \) which contains only a finite number of hyperbolic equilibria \( E_i \) \((1 \leq i \leq n)\). Then the perturbed nonautonomous system has a pullback attractor \( \mathcal{A}_\varepsilon \) near \( \mathcal{A} \) (see Section 4). It was proved in [6] that \( \mathcal{A}_\varepsilon \) contains the same number of full solutions which forms a Morse decomposition of \( \mathcal{A}_\varepsilon \). This work was further developed and extended to more general cases in Carvalho and Langa [4] and Vishik, Zelik and Chepyzhov [30], and to discrete dynamical systems in Cheban, Mammana and Michetti [10]. One can even find some interesting results concerning randomly perturbed case in Caraballo, Langa and Liu [8].

Based on the existence of nonautonomous equilibria of \( \varphi_\varepsilon \), here we give a more precise description on the Morse structure of \( \mathcal{A}_\varepsilon \) and the point-wise asymptotic behavior of the system. We show that if \( \varepsilon \) is sufficiently small then the pullback attractor \( \mathcal{A}_\varepsilon \) contains the same number of nonautonomous equilibria \( \Gamma_i \) \((1 \leq i \leq n)\), which completely characterize the Morse structure of \( \mathcal{A}_\varepsilon \). Furthermore, the section \( \mathcal{A}_\varepsilon(p) \) of \( \mathcal{A}_\varepsilon \) consists of precisely all the \( p \)-section unstable manifolds \( W^u(\Gamma_i, p) \) \((1 \leq i \leq n)\). We also show that there is a neighborhood \( U \) of \( \mathcal{A}_\varepsilon \) such that for any \( x \in U \) and \( p \in \mathcal{H} \), one has

\[ \lim_{t \to +\infty} d(\varphi_\varepsilon(t,p)x, \Gamma_i(\theta_t p)) = 0 \quad (1.5) \]
for some $i$. Since $\Gamma(\theta t h)$ is synchronizing with $h$, (1.5) indicates that the nonautonomous system (1.2) exhibits completely a forward asymptotically synchronizing behavior with the external force.

Third, making use of the above results, we prove that the section $A_x(p)$ is continuous in $p$ in the sense of Hausdorff distance. Consequently one concludes that $A_x$ is uniformly forward attracting, and hence is a (uniform) forward attractor. More precisely, there is a neighborhood $B$ of $A_u$ such that

$$
\lim_{t \to +\infty} H(\varphi_x(t)pB, A_x(\theta t p)) = 0, \quad \forall p \in \mathcal{H},
$$

where $H(\cdot, \cdot)$ denotes the Hausdorff semidistance in the state space.

As we have mentioned earlier, if the function $h$ in (1.2) is periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic), then so is $\theta_t p$ for any fixed $p \in \mathcal{H}$. Further by the continuity of $A_x(p)$ in $p$, we deduce that $A(t) = A_x(\theta t h)$ is a periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic) function taking values in $\mathcal{U}(X^\alpha)$, where $\mathcal{U}(X^\alpha)$ denotes the metric space consisting of all the nonempty compact subsets of $X^\alpha$ equipped with the Hausdorff metric. This along with (1.6) then asserts that system (1.2) also exhibits a global forward synchronizing behavior with the external force.

This paper is organized as follows. In Section 2 we present some basic definitions and results, and in Section 3 we study nonautonomous perturbations of hyperbolic equilibria of (1.3). In Section 4 we discuss Morse structures of pullback attractors and the point-wise asymptotically synchronizing behavior of the perturbed system (1.4). Finally in Section 5, we investigate the forward attraction of pullback attractors of (1.4) and demonstrate the global synchronizing behavior of the system.

2. Preliminaries. Let $X$ be a complete metric space with metric $d(\cdot, \cdot)$. Given $M \subset X$, we denote $\overline{M}$, $\text{int} M$ and $\partial M$ the closure, interior and boundary of $M$ of $X$, respectively. A set $U \subset X$ is called a neighborhood of $M \subset X$, if $\overline{M} \subset \text{int} U$. For any $\rho > 0$, denote by

$$
B_X(M, \rho) := \{x \in X : d(x, M) < \rho\}
$$

the $\rho$-neighborhood of $M$ in $X$, where $d(x, M) = \inf_{y \in M} d(x, y)$.

The Hausdorff semidistance and the Hausdorff distance in $X$ are defined, respectively, as

$$
H_X(M, N) = \sup_{x \in M} d(x, N), \quad \forall M, N \subset X,
$$

$$
\delta_X(M, N) = \max\{H_X(M, N), H_X(N, M)\}, \quad \forall M, N \subset X.
$$

2.1. Semiflow. Let $\mathbb{R}^+ = [0, \infty)$. A continuous map $S: \mathbb{R}^+ \times X \to X$ is called a semiflow on $X$, if it satisfies

i) $S(0, x) = x$ for all $x \in X$; and

ii) $S(t + s, x) = S(t, S(s, x))$ for all $x \in X$ and $t, s \in \mathbb{R}^+$.

Remark 1. If $\mathbb{R}^+$ is replaced by $\mathbb{R}$ in the above definition, then $S$ is called a flow on $X$.

Let $S$ be a given semiflow on $X$. As usual, we will rewrite $S(t, x)$ as $S(t)x$. Henceforth we always assume $S$ is asymptotically compact, namely, (AC) each bounded subset $B$ of $X$ is admissible (that is, for any sequences $x_n \in B$ and $t_n \to +\infty$ with $S([0, t_n])x_n \subset B$ for all $n$, the sequence $S(t_n)x_n$ has a convergent subsequence).
Let \( J \subset \mathbb{R} \) be an interval. A solution (trajectory) \( \gamma \) of \( S \) on \( J \) is a map \( \gamma : J \to X \) satisfying \( \gamma(t) = S(t-s)\gamma(s) \) for any \( s,t \in J \) with \( s \leq t \). In case \( J = \mathbb{R} \), we will simply call \( \gamma \) a full solution.

A set \( B \subset X \) is called invariant under \( S \) if \( S(t)B = B \) for all \( t \geq 0 \).

Let \( B \) and \( M \) be subsets of \( X \). We say that \( M \) attracts \( B \) (under \( S \)), if
\[
\lim_{t \to \infty} H_X(S(t)B, M) = 0.
\]

**Definition 2.1.** A compact subset \( A \subset X \) is called an attractor for \( S \), if it is invariant under \( S \) and attracts one of neighborhood of itself.

The attraction basin of \( A \) is defined as
\[
\Omega(A) = \{ x \in X : \lim_{t \to \infty} d(S(t)x, A) = 0 \}.
\]

Let \( A \) be an attractor of \( S \). Since \( A \) is invariant, the restriction \( S_A(t) \) of \( S \) on \( A \) is also a semiflow.

A compact set \( A \) is said to be an attractor of \( S \) in \( A \), if \( A \) is an attractor of the restricted system \( S_A \) in \( A \).

**Lemma 2.2.** [17] An attractor \( A \) of \( S \) in \( A \) is also an attractor in \( X \).

**Definition 2.3.** Let \( A \) be an attractor, and let \( M = \{ M_1, \ldots, M_l \} \) be an ordered collection of pairwise disjoint compact invariant subsets of \( A \). We call \( M \) a Morse decomposition of \( A \), if for every \( x \in A \) and every full solution \( \gamma \) with \( \gamma(0) = x \), one of the following alternatives holds:

1. There is a \( M_i \) such that \( \gamma(t) \in M_i \) for all \( t \in \mathbb{R} \).
2. There are indices \( i < j \) such that \( \alpha(\gamma) \subset M_j \) and \( \omega(\gamma) \subset M_i \), where \( \alpha(\gamma) \) and \( \omega(\gamma) \) denote the \( \alpha \)-limit set and \( \omega \)-limit set of \( \gamma \), respectively.

**Remark 2.** Let \( M = \{ M_1, \ldots, M_l \} \) be a Morse decomposition of \( A \). Set
\[
A_k = \bigcup_{1 \leq i \leq k} W^u(M_i), \quad 1 \leq k \leq l,
\]
where
\[
W^u(M_i) = \{ \gamma(0) \mid \gamma \text{ is a full solution in } A \text{ with } \alpha(\gamma) \subset M_i \}
\]
is the unstable manifold of \( M_i \). Then each \( A_k \) is an attractor of \( S \) in \( A \); see [23], Section 2.

Let \( A \) be an attractor of \( S \) with the basin of attraction \( \Omega = \Omega(A) \). \( S \) is called gradient-like on \( \Omega \), if there exists a continuous function \( L : \Omega \to \mathbb{R} \) which decreases along each non-equilibrium solution in \( \Omega \).

**Remark 3.** It is known that if \( S \) is a gradient-like system on \( \Omega \) with only a finite number of equilibria in \( \Omega \), then \( A \) has a natural Morse decomposition consisting of precisely the equilibria of \( S \) in \( \Omega \).

2.2. Cocycle semiflows. A nonautonomous system consists of a “base flow” and a “cocycle semiflow” that is in some sense driven by the base flow.

A base flow \( \theta = \theta(t) \) (\( t \in \mathbb{R} \)) is a flow on a metric space \( \Sigma \) such that \( \theta(t)\Sigma = \Sigma \) for all \( t \in \mathbb{R} \). Let \( \theta \) be a base flow on \( \Sigma \). For notational simplicity, from now on we will rewrite \( \theta(t) \) as \( \theta_t \).

**Definition 2.4.** A cocycle semiflow \( \varphi \) on the phase space \( X \) over \( \theta \) is a continuous map \( \varphi : \mathbb{R}^+ \times \Sigma \times X \to X \) satisfying

- \( \varphi(0, \sigma, x) = x \) for all \( (\sigma, x) \in \Sigma \times X \), and
• $\varphi(t+s, \sigma, x) = \varphi(t, \theta_s \sigma, \varphi(s, \sigma, x))$ for all $(\sigma, x) \in \Sigma \times X$ and $t, s \geq 0$ (cocycle property).

**Remark 4.** The flow $\theta$ and the space $\Sigma$ in the above definition are usually called the driving system and base space of $\varphi$, respectively.

Let $\varphi$ be a given cocycle semiflow on $X$ with driving system $\theta$ and base space $\Sigma$. As usual, we will write $\varphi(t, \sigma, x) = \varphi(t, \sigma)x$.

For convenience, a family of subsets $B = \{B_\sigma\}_{\sigma \in \Sigma}$ of $X$ is called a nonautonomous set. Let $B = \{B_\sigma\}_{\sigma \in \Sigma}$ be a nonautonomous set. We usually rewrite $B_\sigma$ as $B_\sigma(\sigma)$, called the $\sigma$-section of $B$. We also denote $P(B) = \bigcup_{\sigma \in \Sigma} B_\sigma(\sigma) \times \{\sigma\}$.

Note that $P(B)$ is a subset of $X \times \Sigma$.

A nonautonomous set $B$ is said to be closed (resp. open, compact), if $P(B)$ is closed (resp. open, compact) in $X \times \Sigma$.

Let $B$ and $M$ be two nonautonomous subsets of $X$. We say that $M$ pullback (resp. forward) attracts $B$ under $\varphi$, if for any $\sigma \in \Sigma$,

$$
\begin{align*}
\lim_{t \to +\infty} H_X(\varphi(t, \theta_t \sigma)B(\theta_t \sigma), M(\sigma)) &= 0 \\
(\lim_{t \to +\infty} H_X(\varphi(t, \sigma)B(\sigma), M(\sigma))) &= 0.
\end{align*}
$$

**Definition 2.5.** A nonautonomous set $A$ is called a pullback (resp. forward) attractor, if it is invariant and pullback (forward) attracts a neighborhood of itself.

**Proposition 1.** Assume that the base space $\Sigma$ is compact. Let $A$ be a pullback (resp. forward) attractor of $\varphi$. If the section $A(\sigma)$ is continuous in $\sigma$ (in the sense of Hausdorff distance), then $A$ is a forward attractor.

**Proof.** The argument is a slight modification of that of [9, Theorem 3.3] (see also [9, Theorem 4.3]), and is thus omitted. \hfill $\Box$

2.3. **Non-autonomous equilibria.** Let $\varphi$ be a given cocycle semiflow on $X$ with driving system $\theta$ and base space $\Sigma$, and $J \subset \mathbb{R}$ be an interval.

A map $\gamma : J \to X$ is called a solution of $\varphi$ on $J$, if there exists $\sigma \in \Sigma$ such that

$$
\gamma(t) = \varphi(t-s, \theta_s \sigma)\gamma(s), \quad \forall \; t, s \in J, \quad t \geq s.
$$

A solution on $J = \mathbb{R}$ is called a full solution.

**Remark 5.** We will also call a solution $\gamma$ defined as above a $\sigma$-solution of $\varphi$ to emphasize the relation between $\gamma$ and $\sigma$.

**Definition 2.6.** A nonautonomous equilibrium of $\varphi$ is a continuous map $\Gamma \in C(\Sigma, X)$ such that $\gamma(t) = \Gamma(\theta_t \sigma)$ is a full solution of the system for each $\sigma \in \Sigma$. 


Definition 2.7. Let Γ be a nonautonomous equilibrium of ϕ, and U a neighborhood of Γ. Let σ ∈ Σ. The σ-section unstable manifold and local σ-section unstable manifold of Γ in U of Γ are defined, respectively, to be the sets

\[ W^u(Γ, σ) = \left\{ x ∈ X \mid \begin{array}{l}
\text{there is a } σ\text{-solution } γ(t) \text{ on } (−∞, 0] \text{ with } γ(0) = x \\
\text{such that } \lim_{t \to −∞} d(γ(t), Γ(θ_t σ)) = 0
\end{array} \right\}, \]

\[ W^u_U(Γ, σ) = \left\{ x ∈ X \mid \begin{array}{l}
\text{there is a } σ\text{-solution } γ(t) \text{ on } (−∞, 0] \text{ with } γ(0) = x \\
\text{and } γ(t) ∈ U(θ_t σ) \text{ for all } t ≤ 0 \text{ such that } \\
\lim_{t \to −∞} d(γ(t), Γ(θ_t σ)) = 0
\end{array} \right\}. \]

3. Nonautonomous perturbation of hyperbolic equilibria of (1.3). In this section, we first discuss nonautonomous perturbation of hyperbolic equilibrium points of the autonomous system (1.3). Specifically, we show that the perturbed system (1.4) has a unique nonautonomous equilibrium Γε = Γε(p) near each hyperbolic equilibrium of the non-perturbed system (1.3). Furthermore, there is a nonautonomous neighborhood U of Γε such that the local section unstable manifold W^u_U(Γε, p) depends on p continuously. These results will play an important role in discussing persistence of the topological structures of attractors under nonautonomous perturbations. However, they may be of independent interest in their own right.

3.1. Mathematical setting. Let X be the Banach space appeared in the introduction. Denote C_b(ℝ, X) the set of bounded continuous functions from ℝ to X. Equip C_b(ℝ, X) with either the uniform convergence topology generated by the metric

\[ r(h_1, h_2) = \sup_{t ∈ ℝ} \|h_1(t) − h_2(t)\|, \]

or the compact-open topology generated by the metric

\[ r(h_1, h_2) = \sum_{n=1}^{∞} \frac{1}{2^n} \cdot \frac{\max_{t ∈ [−n, n]} \|h_1(t) − h_2(t)\|}{1 + \max_{t ∈ [−n, n]} \|h_1(t) − h_2(t)\|}. \]

Then C_b(ℝ, X) is a complete metric space.

Let h ∈ C_b(ℝ, X) be the function in (1.2). Define the hull of h as follows

\[ H := H[h] = \{h(τ + ·) : τ ∈ ℝ\} \subset C_b(ℝ, X). \]

As we have mentioned in the introduction, we assume that H is compact. Note that this covers all the cases where p is a periodic function, quasiperiodic function, almost periodic function, local almost periodic function [13, 22] or uniformly almost automorphic function [27]. One can also easily verify that if h is uniformly continuous on ℝ, then H is compact with respect to the compact-open topology.

The translation group θ on H is defined as

\[ θ_t p = p(t + ·), \quad ∀ t ∈ ℝ, \ p ∈ H. \]

Let A be the sectorial operator in (1.2). Pick a number a > 0 sufficiently large so that

\[ \min \{\Re z : z ∈ σ(A)\} > 0, \]

where Λ = A + aI. Then one can define fractional powers Λ^α of Λ for α ≥ 0; see Henry [16] for details. For each α ≥ 0, define X^α = D(Λ^α). As usual, X^α is quipped with the norm \( \| x \|_{α} \) defined by

\[ \| x \|_{α} = \| Λ^α x \|, \quad x ∈ X^α. \]
It is well known that $X^\alpha$ is a Banach space, whose definition is independent of the choice of the number $\alpha$.

Now we turn our attention to the cocycle system (1.4).

From now on we always assume that $f$ is a locally Lipschitz map from $X^\alpha$ to $X$ for some $\alpha \in [0, 1)$. We infer from Henry [16] that the initial value problem of (1.4) is well-posed. More precisely, we have

**Proposition 2.** [16] Given $x_0 \in X^\alpha$, $p \in \mathcal{H}$ and $t_0 \in \mathbb{R}$, there exists $T > t_0$ such that (1.4) has a unique (mild) solution $x(t) = x_\varepsilon(t, t_0; x_0, p)$ on $[t_0, T)$ with $x(t_0) = x_0$, namely, there is a unique function $x \in C([t_0, T), X)$ satisfying

$$x(t) = e^{-\Lambda(t-t_0)}x_0 + \int_{t_0}^{t} e^{-\Lambda(t-s)}[f(x(s)) + \varepsilon p(s)]ds, \quad t \in [t_0, T). \quad (3.1)$$

For convenience, we assume that the solution $x_\varepsilon(t, t_0; x_0, p)$ given in the above proposition is globally defined for all $x_0 \in X^\alpha$, $p \in \mathcal{H}$ and $t_0 \in \mathbb{R}$. Set

$$\phi_\varepsilon(t, p)x := x_\varepsilon(t, 0; x, p), \quad x \in X^\alpha, \quad p \in \mathcal{H}.$$ 

Then $\phi_\varepsilon$ is a **cocycle semiflow** on $X^\alpha$ driven by the translation group $\theta$ on $\mathcal{H}$.

**Remark 6.** When $\varepsilon = 0$, the equation (1.4) reduces to the autonomous one. Consequently $\phi_0$ coincides with the semiflow $S$ on $X^\alpha$ defined by the solutions of the initial value problem of the equation (1.3).

We now assume, without loss of generality, that $0$ is a hyperbolic equilibrium of the non-perturbed system (1.3). Then the systems (1.4) and (1.3) can be rephrased, respectively, as follows:

$$x_1 + Lx = g(x) + \varepsilon p(t), \quad p \in \mathcal{H} := \mathcal{H}[h], \quad \text{for } \alpha \in [0, 1]$$

where

$$L = A - f'(0), \quad g = o(\|x\|_\alpha) \text{ as } \|x\|_\alpha \to 0.$$ 

Denote $k(p)$ the Lipschitz constant of $g$ in $B_{X^\alpha}(p)$, where $B_{X^\alpha}(r)$ denotes the ball in $X^\alpha$ centered at 0 with radius $r$. Then $\lim_{p \to 0} k(p) = 0$. Note that

$$\|g(x_1) - g(x_2)\| \leq k(p)\|x_1 - x_2\|_\alpha, \quad \forall x_1, x_2 \in B_{X^\alpha}(p). \quad (3.4)$$

The spectrum $\sigma(L)$ of $L$ has a decomposition $\sigma(L) = \sigma_1 \cup \sigma_2$, where

$$\sigma_1 = \sigma(L) \cap \{\Re \lambda < 0\}, \quad \sigma_2 = \sigma(L) \cap \{\Re \lambda > 0\}.$$ 

Accordingly, the space $X$ has a direct sum decomposition: $X = X_1 \oplus X_2$. Let

$$\Pi_i : X \to X_i, \quad i = 1, 2$$

be the projection from $X$ to $X_i$. Set

$$X_i^\alpha := X_i \cap X^\alpha, \quad i = 1, 2.$$ 

Then $X^\alpha = X_1^\alpha \oplus X_2^\alpha$. Note that $X_i^\alpha$ coincides with $X_i$ since $\dim(X_i) < \infty$.

Denote $L_1 = L|_{X_1}$ and $L_2 = L|_{X_2}$. By the basic knowledge on sectorial operators (see Henry [16]), we know that there exist $M \geq 1$, $\beta > 0$

$$\|\Lambda^\alpha e^{-L_1 t}\| \leq M e^{\beta t}, \quad \|e^{-L_1 t}\| \leq M e^{\beta t}, \quad t \leq 0, \quad (3.5)$$

$$\|\Lambda^\alpha e^{-L_2 t}\Pi_2 \Lambda^{-\alpha}\| \leq M e^{-\beta t}, \quad \|\Lambda^\alpha e^{-L_2 t}\| \leq Mt^{-\alpha} e^{-\beta t}, \quad t > 0, \quad (3.6)$$

where $\Lambda = A + aI$. 


3.2. A basic lemma. This subsection consists of a fundamental lemma which will play a crucial role in our discussion.

**Lemma 3.1.** The following assertions hold:

(a) Let $x : (-\infty, 0) \to X^\alpha$ be continuous and bounded. Then $x$ is the solution of (1.4) if and only if it solves the following integral equation

$$x(t) = e^{-L_1 t} \Pi_1 x(0) + \int_0^t e^{-L_1(t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds$$

$$+ \int_{-\infty}^t e^{-L_2(t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds. \quad (3.7)$$

(b) Let $x : \mathbb{R} \to X^\alpha$ be continuous and bounded. Then $x$ is the solution of (1.4) if and only if it solves the integral equation

$$x(t) = \int_{-\infty}^t e^{-L_2(t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds$$

$$- \int_t^\infty e^{-L_1(t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds. \quad (3.8)$$

**Proof.** (a) Let $x : (-\infty, 0) \to X^\alpha$ be a bounded solution of (1.4). We write $x(t) = x_1(t) + x_2(t)$, where $x_i(t) = \Pi_i x(t) \ (i = 1, 2)$. Then

$$x_1(t) = e^{-L_1 t} x_1(0) + \int_0^t e^{-L_1(t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds, \quad (3.9)$$

$$x_2(t) = e^{-L_2(t-t_0)} x_2(t_0) + \int_{t_0}^t e^{-L_2(t-s)} \Pi_2 [g(x(s)) + p(s)] ds, \quad (3.10)$$

for $t \geq t_0$. By the boundedness of $x$ in $X^\alpha$, we have

$$\|e^{-A_2(t-t_0)} x_2(t_0)\|_\alpha \leq Me^{-\beta(t-t_0)} \sup_{t \leq 0} \|x(t)\|_\alpha \to 0, \quad \text{as} \ t_0 \to -\infty.$$ 

It follows that

$$x_2(t) = \int_{-\infty}^t e^{-L_2(t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds.$$ 

Therefore

$$x(t) = x_1(t) + x_2(t)$$

$$= e^{-L_1 t} x_1(0) + \int_0^t e^{-L_1(t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds$$

$$+ \int_{-\infty}^t e^{-L_2(t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds.$$ 

This is precisely what we desired in (3.7).

If $x$ satisfies (3.7), then it clearly solves (1.4) on $(-\infty, 0]$. The proof of (a) is complete.

(b) Let $x : \mathbb{R} \to X^\alpha$ be a bounded solution of (1.4). Let $t_0 \in \mathbb{R}$. Similar to (a), $x(t) \ (t \leq t_0)$ solves the following integral equation:

$$x(t) = e^{-L_1(t-t_0)} \Pi_1 x(t_0) + \int_{t_0}^t e^{-L_1(t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds$$

$$+ \int_{-\infty}^t e^{-L_2(t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds.$$
By the boundedness of $x$, we have $e^{-L_1(t-t_0)}\Pi_1 x(t_0) \rightarrow 0$, as $t_0 \rightarrow \infty$. It follows that $x$ solves (3.8). If $x$ satisfies (3.8), then it clearly solves (1.4) on $\mathbb{R}$. This completes the proof of (b).

3.3. Main results. The main results in this section are summarized in the following theorem.

**Theorem 3.2.** Assume that $0$ is a hyperbolic equilibrium of (1.3). Then there exist $\rho, \epsilon' > 0$ such that the following assertions hold:

(a) $\varphi_\epsilon$ has a nonautonomous equilibrium $\Gamma := \Gamma_\epsilon \in C(\mathcal{H}, \overline{B}_{X^\alpha}(\rho))$ for $\epsilon \leq \epsilon'$.

(b) There exists $\vartheta > 0$ (independent of $\epsilon \in [0, \epsilon']$) and a family of Lipschitz continuous maps

$$\xi_p : \overline{B}_{X^\alpha}(\vartheta) \rightarrow X^\alpha_2,$$

such that

$$W^u_\mathcal{U}(\Gamma, p) = \{ y + \xi_p(y) : y \in \overline{B}_{X^\alpha}(\vartheta) \}.$$  

(3.11)

Here $\mathcal{U}$ is a nonautonomous neighborhood of $\Gamma$ defined by

$$\mathcal{U}(p) = \{ x \in X^\alpha : \|\Pi_1 x\|_\alpha \leq \vartheta \} \cap \overline{B}_{X^\alpha}(\rho), \quad p \in \mathcal{H}.$$

(c) The map $\Xi : \mathcal{H} \rightarrow C(\overline{B}_{X^\alpha}(\vartheta), X^\alpha_2)$ defined by

$$\Xi(p) = \xi_p, \quad p \in \mathcal{H}$$

is continuous.

**Remark 7.** The continuous dependence of $\xi_p$ on $p$ implies that

$$\lim_{p \rightarrow q} \delta_{X^\alpha}(W^u_\mathcal{U}(\Gamma, p), W^u_\mathcal{U}(\Gamma, q)) = 0, \quad \forall q \in \mathcal{H}. \quad (3.12)$$

**Remark 8.** As we have mentioned in the introduction, the non-autonomous perturbation problem has already been investigated comprehensively in the work of Kloeden and Rodrigues [19] in a more general setting. But comparing with the results in [19], our description on the synchronizing behavior of system (3.2) seems to be a little more explicit.

**Proof of Theorem 3.2.** (a). Set

$$M_\beta := M \int_0^\infty (1 + s^{-\alpha})e^{-\beta s} ds,$$

where $M, \beta$ are the constants in (3.5)-(3.6). Fix a number $\rho > 0$ with

$$M_\beta k(\rho) < 1,$$

and choose a number $\epsilon' > 0$ sufficiently small so that

$$M_\beta (k(\rho) + \epsilon'\| h \|_{L^\infty(\mathbb{R}, X^\alpha)} / \rho) < 1. \quad (3.14)$$

For each $p \in \mathcal{H}$, one can use the righthand side of equation (3.8) to define a map $\mathcal{T} = \mathcal{T}_p$ on the space

$$\mathcal{X}^\ast := \{ x \in C(\mathbb{R}, X^\alpha) : \| x \|_{L^\infty(\mathbb{R}, X^\alpha)} \leq \rho \}$$

as follows:

$$\mathcal{T} x(t) = \int_{-\infty}^t e^{-L_2(t-s)}\Pi_2 [g(x(s)) + \epsilon p(s)] ds$$

$$- \int_t^\infty e^{-L_1(t-s)}\Pi_1 [g(x(s)) + \epsilon p(s)] ds, \quad \forall x \in \mathcal{X}^\ast.$$
We show that \( T^r \mathcal{X} \subset \mathcal{X} \) and is a contraction map.

For every \( t \in \mathbb{R} \), simple computations show that
\[
\|Tx(t)\|_\alpha \leq M \int_t^\infty e^{\beta(t-s)} (k(\rho)\|x(s)\|_\alpha + \varepsilon\|p(s)\|) \, ds
\]
\[
+ M \int_{-\infty}^t (t-s)^{-\alpha} e^{-\beta(t-s)} (k(\rho)\|x(s)\|_\alpha + \varepsilon\|p(s)\|) \, ds
\]
\[
\leq M \left( \int_0^\infty e^{-\beta s} ds \right) \left( k(\rho)\rho + \varepsilon\|p\|_{L^\infty(\mathbb{R}, X)} \right)
\]
\[
+ M \left( \int_0^\infty s^{-\alpha} e^{-\beta s} ds \right) \left( k(\rho)\rho + \varepsilon\|p\|_{L^\infty(\mathbb{R}, X)} \right)
\]
\[
\leq M_\beta \left( k(\rho)\rho + \varepsilon\|h\|_{L^\infty(\mathbb{R}, X)} \right) \leq (by \ (3.14)) \leq \rho.
\]  

Here we have used the simple fact that
\[
\|p\|_{L^\infty(\mathbb{R}, X)} \leq \|h\|_{L^\infty(\mathbb{R}, X)} < \infty, \quad \forall \ p \in \mathcal{H}.
\]

Since \( t \) is arbitrary, we see that \( \|T(x)\|_{L^\infty(\mathbb{R}, X^\alpha)} \leq \rho \). Hence \( T(x) \in \mathcal{X} \).

Now let \( x, z \in \mathcal{X} \). We have
\[
\|Tx(t) - Tz(t)\|_\alpha \leq M k(\rho) \int_t^\infty e^{\beta(t-s)} \|x(s) - z(s)\|_\alpha \, ds
\]
\[
+ M k(\rho) \int_{-\infty}^t (t-s)^{-\alpha} e^{-\beta(t-s)} \|x(s) - z(s)\|_\alpha \, ds
\]
\[
\leq M_\beta k(\rho) \|x - z\|_{L^\infty(\mathbb{R}, X^\alpha)}, \quad t \in \mathbb{R}.
\]

Hence
\[
\|T x - T z\|_{L^\infty(\mathbb{R}, X^\alpha)} \leq M_\beta k(\rho) \|x - z\|_{L^\infty(\mathbb{R}, X^\alpha)}.
\]

Since \( M_\beta k(\rho) < 1 \), we conclude that \( T \) is a contraction map on \( \mathcal{X} \).

Thanks to the classical Banach fixed-point theorem, \( T = T_p \) has a unique fixed point \( x_p \in \mathcal{X} \). It is easy to see that \( x_p \) is precisely a full solution of (1.4) satisfying (3.8). Define \( \Gamma := \Gamma_\varepsilon : \mathcal{H} \to X^\alpha \) as
\[
\Gamma(p) = x_p(0), \quad p \in \mathcal{H}.
\]

One easily verifies that \( x_p(t) = \Gamma(\theta_t p), \ t \in \mathbb{R} \). To prove that \( \Gamma \) is a nonautonomous equilibrium of \( \varphi_\varepsilon \), there remains to check the continuity of \( \Gamma \).

For notational convenience, given \( \mu, \beta \geq 0 \), we denote
\[
\|\|x\|\|_{\mu, \beta} = \sup_{t \in \mathbb{R}^\pm} \left( e^{-\mu|t|} \|x(t)\|_\beta \right), \quad \forall \ x \in C(\mathbb{R}^\pm, X^\beta)
\]
and
\[
\|\|x\|\|_{\mu, \beta} = \sup_{t \in \mathbb{R}} \left( e^{-\mu|t|} \|x(t)\|_\beta \right), \quad \forall \ x \in C(\mathbb{R}, X^\beta).
\]

By (3.13), we can pick a \( \beta' \in (0, \beta) \) so that \( M_{\beta'} k(\rho) < 1 \), where
\[
M_{\beta'} := M \int_0^\infty (1 + s^{-\alpha}) e^{-\beta's} ds.
\]

Set \( \mu = \beta - \beta' \). One observes that
\[
e^{-\mu|t|} = e^{-\mu|s+(t-s)|} \leq e^{-\mu(|s|-(t-s))} = e^{-\mu|s| e^{\mu|t-s|}}, \quad \forall \ t, s \in \mathbb{R}. \tag{3.16}
\]
Recall that $x_p(t) = \Gamma(t)p$ solves (3.8). Let $p, q \in \mathcal{H}$. Then for $t \in \mathbb{R}$, by (3.8) and (3.16) we deduce that

$$e^{-\mu|t|} \|x_p(t) - x_q(t)\|_{\alpha}$$

$$\leq Mk(\rho) \left( \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} e^{\mu|t-s|} (e^{-\mu|s|} \|x_p(s) - x_q(s)\|_{\alpha}) ds \right)$$

$$+ Mk(\rho) \left( \int_{t}^{\infty} e^{-\beta(s-t)} e^{\mu|t-s|} (e^{-\mu|s|} \|x_p(s) - x_q(s)\|_{\alpha}) ds \right)$$

$$+ M\varepsilon \left( \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} e^{\mu|t-s|} \left( e^{-\mu|s|} \|p(s) - q(s)\| \right) ds \right)$$

$$+ M\varepsilon \left( \int_{t}^{\infty} e^{-\beta(s-t)} e^{\mu|t-s|} \left( e^{-\mu|s|} \|p(s) - q(s)\| \right) ds \right)$$

(3.17)

Hence

$$e^{-\mu|t|} \|x_p(t) - x_q(t)\|_{\alpha}$$

$$\leq Mk(\rho) \left( \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} ds + \int_{t}^{\infty} e^{-\beta(s-t)} ds \right) \|x_p - x_q\|_{\mu,\alpha}$$

$$+ M\varepsilon \left( \int_{-\infty}^{t} (t-s)^{-\alpha} e^{-\beta(t-s)} ds + \int_{t}^{\infty} e^{-\beta(s-t)} ds \right) \|p - q\|_{\mu,0}$$

$$\leq M_\beta k(\rho) \|x_p - x_q\|_{\mu,\alpha} + M_\beta \|p - q\|_{\mu,0}, \quad \forall t \in \mathbb{R}.$$ (3.18)

Thus

$$\|x_p - x_q\|_{\mu,\alpha} \leq \frac{M_\beta}{1 - M_\beta k(\rho)} \|p - q\|_{\mu,0}.$$ (3.19)

Therefore

$$\|\Gamma(p) - \Gamma(q)\|_{\alpha} = \|x_p(0) - x_q(0)\|_{\alpha} \leq \frac{M_\beta}{1 - M_\beta k(\rho)} \|p - q\|_{\mu,0}.$$ (3.19)

Since $p \to q$ (in $\mathcal{H}$) implies that $\|p - q\|_{\mu,0} \to 0$, we conclude from (3.19) that

$$\|\Gamma(p) - \Gamma(q)\|_{\alpha} \to 0, \quad \text{as } p \to q,$$

which completes the proof of the continuity of $\Gamma$.

(b) and (c). Take a $\varrho > 0$ small enough so that

$$M_\varrho \rho + M_\beta (k(\rho) + \varepsilon' \|h\|_{L^\infty(\mathbb{R},X)}) < 1,$$ (3.20)

where $\varepsilon'$ is the number in (3.14). Let

$$\mathcal{Y} := \{x \in C(\mathbb{R}^-,X^\alpha) : \|x\|_{L^\infty(\mathbb{R}^-,X^\alpha)} \leq \rho\}.$$
For each \( p \in \mathcal{H} \) and \( y \in \overline{B}_{X_1}(\varrho) \), using the right hand side of equation (3.7) we can define a map \( \hat{T} = \hat{T}_{p,y} \) on \( \mathcal{W} \) as follows:

\[
\hat{T}(t) = e^{-L_1 t} y + \int_0^t e^{-L_1 (t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds \\
+ \int_{-\infty}^t e^{-L_2 (t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds.
\]

We first show that \( \hat{T} \) maps \( \mathcal{W} \) to itself. Indeed, let \( x \in \mathcal{W} \). Then for any \( t \leq 0 \),

\[
\| \hat{T}(t) \|_{\alpha} \leq M e^{\beta t} \| y \|_{\alpha} + M \int_0^t e^{\beta (t-s)} (k(\rho) \| x(s) \|_{\alpha} + \varepsilon \| p(s) \|) ds \\
+ M \int_{-\infty}^t (t-s)^{-\alpha} e^{-\beta (t-s)} (k(\rho) \| x(s) \|_{\alpha} + \varepsilon \| p(s) \|) ds
\]

\[
\leq M \| y \|_{\alpha} + M \left( \int_0^t e^{-\beta s} ds + \int_0^\infty s^{-\alpha} e^{-\beta s} ds \right) (k(\rho) \rho + \varepsilon \| h \|_{L^\infty(\mathbb{R},X)})
\]

\[
\leq M \| y \|_{\alpha} + M \beta (k(\rho) \rho + \varepsilon \| h \|_{L^\infty(\mathbb{R},X)}) \leq (by 3.20) \leq \rho,
\]

from which it immediately follows that \( \hat{T}(x) \in \mathcal{W} \).

Now we check that \( \hat{T} \) is contracting on \( \mathcal{W} \). For \( x, z \in \mathcal{W} \), we have

\[
\| \hat{T}(t) - \hat{T}(z) \|_{\alpha} \leq M k(\rho) \int_0^t e^{\beta (t-s)} \| x(s) - z(s) \|_{\alpha} ds \\
+ M k(\rho) \int_{-\infty}^t (t-s)^{-\alpha} e^{-\beta (t-s)} \| x(s) - z(s) \|_{\alpha} ds
\]

\[
\leq M k(\rho) \left( \int_0^t e^{-\beta s} ds + \int_0^\infty s^{-\alpha} e^{-\beta s} ds \right) \| x - z \|_{L^\infty(\mathbb{R},X^\alpha)}
\]

\[
= (M \beta k(\rho)) \| x - z \|_{L^\infty(\mathbb{R},X^\alpha)}, \quad t \leq 0.
\]

Since \( M \beta k(\rho) < 1 \), we see from the above estimate that \( \hat{T} \) is indeed a contraction map.

By the Banach fixed-point theorem, \( \hat{T} \) has a unique fixed point \( x_{p,y} \) in \( \mathcal{W} \). \( x(t) = x_{p,y}(t) \) is precisely a bounded solution of (1.4) on \( \mathbb{R}^- \) with \( \Pi_1 x(0) = y \), which equivalently solves the integral equation

\[
x(t) = e^{-L_1 t} y + \int_0^t e^{-L_1 (t-s)} \Pi_1 [g(x(s)) + \varepsilon p(s)] ds \\
+ \int_{-\infty}^t e^{-L_2 (t-s)} \Pi_2 [g(x(s)) + \varepsilon p(s)] ds, \quad t \in \mathbb{R}^-.
\]

We claim that \( x_{p,y}(0) \) is Lipschitz continuous in \( y \) uniformly with respect to \( p \in \mathcal{H} \). Indeed, let \( y, z \in \overline{B}_{X_2}(\varrho) \). Then for \( t \leq 0 \), simple computations show that

\[
\| x_{p,y}(t) - x_{p,z}(t) \|_{\alpha} \leq M e^{\beta t} \| y - z \|_{\alpha} + M k(\rho) \int_0^t e^{\beta (t-s)} \| x_{p,y}(s) - x_{p,z}(s) \|_{\alpha} ds \\
+ M k(\rho) \int_{-\infty}^t (t-s)^{-\alpha} e^{-\beta (t-s)} \| x_{p,y}(s) - x_{p,z}(s) \|_{\alpha} ds
\]
\[ \leq M \|y - z\|_\alpha + M\kappa(\rho) \left( \int_0^t e^{-\beta s} ds + \int_\infty^t s^{-\alpha} e^{-\beta s} ds \right) \|x_{p,y} - x_{p,z}\|_{L^\infty(\mathbb{R}^-,X^\alpha)} \]
\[ \leq M \|y - z\|_\alpha + M\beta\kappa(\rho) \|x_{p,y} - x_{p,z}\|_{L^\infty(\mathbb{R}^-,X^\alpha)}. \]

Hence
\[ \|x_{p,y} - x_{p,z}\|_{L^\infty(\mathbb{R}^-,X^\alpha)} \leq M \|y - z\|_\alpha + M\beta\kappa(\rho) \|x_{p,y} - x_{p,z}\|_{L^\infty(\mathbb{R}^-,X^\alpha)}. \]

Therefore
\[ \|x_{p,y}(0) - x_{p,z}(0)\|_\alpha \leq \|x_{p,y} - x_{p,z}\|_{L^\infty(\mathbb{R}^-,X^\alpha)} \leq \frac{M}{1 - M\beta\kappa(\rho)} \|y - z\|_\alpha, \]

which justifies our claim.

Define
\[ \xi_p(y) := \int_{-\infty}^0 e^{Lz \Pi_2[g(x_{p,y}(s))] + \varepsilon p(s)]} ds, \quad y \in \overline{B}_{X^\alpha_1}(\varrho). \] (3.22)

Setting \( t = 0 \) in (3.21) leads to
\[ x_{p,y}(0) = y + \xi_p(y), \quad y \in \overline{B}_{X^\alpha_1}(\varrho). \] (3.23)

The Lipschitz continuity of \( x_{p,y}(0) \) in \( y \) then implies \( \xi_p : \overline{B}_{X^\alpha_1}(\varrho) \to X^\alpha_1 \) is a Lipschitz continuous map.

Set
\[ \mathcal{M}_\varrho^u(p) = \{ y + \xi_p(y) : y \in \overline{B}_{X^\alpha_1}(\varrho) \}. \]

Clearly \( \mathcal{M}_\varrho^u(p) \) is homeomorphic to \( \overline{B}_{X^\alpha_1}(\varrho) \). Let \( \mathcal{U} = \{ U_p \}_{p \in \mathcal{H}} \), where
\[ U_p = \{ x \in X^\alpha : \|\Pi_1 x\|_\alpha \leq \varrho \} \cap \overline{B}_{X^\alpha}(\varrho) := U, \quad p \in \mathcal{H}. \] (3.24)

Then \( \mathcal{U} \) is a (nonautonomous) neighborhood of \( \Gamma \). We show that
\[ W_{\mathcal{U}}^u(\Gamma, p) = \mathcal{M}_\varrho^u(p). \] (3.25)

Note that \( W_{\mathcal{U}}^u(\Gamma, p) \subset \mathcal{M}_\varrho^u(p) \). Thus we only need to check that \( \mathcal{M}_\varrho^u(p) \subset W_{\mathcal{U}}^u(\Gamma, p) \). For this purpose, it suffices to verify that for each \( y \in \overline{B}_{X^\alpha_1}(\varrho) \),
\[ \lim_{t \to -\infty} \|x_{p,y}(t) - \Gamma(\theta_t p)\|_\alpha = 0. \] (3.26)

(It is clear that \( x_{p,y} \) is a \( p \)-solution of \( \varphi_\varepsilon \) on \( \mathbb{R}^- \).)

We argue by contradiction and suppose that (3.26) was false. Then there would exist \( \delta > 0 \) and a sequence \( 0 \geq t_n \to -\infty \) such that
\[ \|x_{p,y}(t_n) - \Gamma(\theta_{t_n} p)\|_\alpha \geq \delta. \]

Let
\[ x_n(t) = x_{p,y}(t + t_n), \quad t \in [t_n, -t_n], \]

and
\[ y_n(t) = \Gamma(\theta_{t+t_n} p), \quad t \in [t_n, -t_n]. \]

Then both \( x_n(t) \) and \( y_n(t) \) are solutions of the equation
\[ x_t + Ax = f(x) + \varepsilon p_n(t) \] (3.27)
on \([t_n, -t_n]\), where \( p_n = \theta_{t_n} p \). Note that
\[ \|x_n(0) - y_n(0)\|_\alpha \geq \delta > 0 \] (3.28)

for all \( n \). By the compactness of \( \mathcal{H} \) we may assume that \( \theta_{t_n} p \to p_0 \in \mathcal{H} \). Thus there exists a subsequence \( n_k \) of \( n \) such that \( x_{n_k}(t) \) and \( y_{n_k}(t) \) converge uniformly
on every compact interval, respectively, to bounded full solutions \(x(t)\) and \(y(t)\) of the equation:

\[
x_t + Ax = f(x) + \varepsilon p_0(t).
\]

(3.29)

It is obvious that

\[
\|x(t)\|_\alpha, \|y(t)\|_\alpha \leq \rho
\]

for all \(t \in \mathbb{R}\). By (3.28) we also have \(\|x(0) - y(0)\|_\alpha \geq \delta\). Moreover, we infer from the proof of (a) that for fixed \(p_0\) the bounded full solutions of (3.29) satisfying (3.30) are unique. Hence \(x(t) \equiv y(t)\) on \(\mathbb{R}\), which leads to a contradiction and completes the proof of (3.25).

Finally, we verify that \(\xi_p\) is continuous in \(p\), which finishes the proof of (e).

Set \(\mu = \beta - \beta'\). Let \(y \in \overline{B_{X^\gamma}(\varrho)}\), and \(p, q \in \mathcal{H}\). Then similar computations as in (3.17) and (3.18) apply to show that

\[
\|x_{p,y} - x_{q,y}\|_{\mu,0} \leq M_{\beta'} k(\varrho) \|x_{p,y} - x_{q,y}\|_{\mu,0} + M_{\beta'} \|p - q\|_{\mu,0}.
\]

Therefore

\[
\|x_{p,y}(0) - x_{q,y}(0)\|_\alpha \leq \|x_{p,y} - x_{q,y}\|_{\mu,0} \leq \frac{M_{\beta'}}{1 - M_{\beta'} k(\varrho)} \|p - q\|_{\mu,0}.
\]

It then follows by (3.23) that

\[
\|\xi_p(y) - \xi_q(y)\|_\alpha \leq \frac{M_{\beta'}}{1 - M_{\beta'} k(\varrho)} \|p - q\|_{\mu,0}.
\]

(3.31)

It is trivial to check that if \(p \to q\) in \(\mathcal{H}\) then \(\|p - q\|_{\mu,0} \to 0\). By (3.31) one immediately concludes that

\[
\lim_{p \to q, y \in \Pi_{X^\gamma} (\varrho)} \sup_{y \in \Pi_{X^\gamma} (\varrho)} \|\xi_p(y) - \xi_q(y)\|_\alpha = 0.
\]

\(\square\)

**Remark 9.** Let \(U \subset X^\alpha\) be the set given in (3.24). We infer from the proof of Theorem 3.2 that if \(\gamma(t)\) \((t \leq 0)\) is a backward \(p\)-solution of \(\varphi_\varepsilon\) satisfying \(\gamma(t) \subset U\) \((t \leq -t_0)\) for some \(t_0 > 0\), then

\[
\lim_{t \to -\infty} \|\gamma(t) - \Gamma(\theta_t p)\|_\alpha = 0.
\]

Using a similar argument as in verifying (3.26), it can be shown that if \(\gamma(t) = \varphi_\varepsilon(t, p)x\) is a forward \(p\)-solution satisfying \(\gamma(t) \subset U\) \((t \geq t_0)\) for some \(t_0 > 0\), then

\[
\lim_{t \to \infty} \|\gamma(t) - \Gamma(\theta_t p)\|_\alpha = 0.
\]

**4. Morse structures of pullback attractors of the perturbed system.** It is known that for an autonomous gradient-like system with only a finite number of equilibria, an attractor can be completely characterized via Morse decomposition, and the internal dynamics in the attractor is explicit: each full solution in the attractor goes forward (backward) to a single equilibrium and there is no homoclinic structure associated to the equilibria. If all the equilibria are hyperbolic, this structure maintains stable under both autonomous and nonautonomous perturbations \([4, 6, 11, 15]\).

In this present work, based on the results in the previous section we give a more precise description on the stability of Morse structures of attractors for gradient-like systems under nonautonomous perturbations.
4.1. Stability of attractors under nonautonomous perturbations. Let us first present a very basic result concerning the stability of attractors under nonautonomous perturbations. Specifically, suppose the non-perturbed system $S = \varphi_0$ has an attractor $\mathcal{A}$. We show that the perturbed system $\varphi_\varepsilon$ possesses a pullback attractor $\mathcal{A}_\varepsilon$ with

$$\lim_{\varepsilon \to 0} H_{X^\alpha}(\mathcal{A}_\varepsilon(p), \mathcal{A}) = 0$$

uniformly with respect to $p \in \mathcal{H}$.

First, we have the following fundamental result.

**Lemma 4.1.** For every $T > 0$ and bounded subset $B$ of $X^\alpha$, we have

$$\lim_{\varepsilon \to 0} \| \varphi_\varepsilon(t,p)x - \varphi_\varepsilon(t,p)x \| = 0, \quad (4.1)$$

uniformly with respect to $(t,x) \in [0,T] \times B$ and $p \in \mathcal{H}$.

**Proof.** Since $A$ is sectorial, there exist $M \geq 1$ and $b \in \mathbb{R}$ such that

$$\| A^a e^{-At} \| \leq Mt^{-a}e^{-bt}.$$

Denote $x_\varepsilon(t) = \varphi_\varepsilon(t,p)x$. Then

$$\| x_\varepsilon(t) - x_\varepsilon(t) \| \leq |\varepsilon - \varepsilon'|M \int_0^t (t-s)^{-a}e^{-b(t-s)}\|p(s)\|ds + Mk(\rho) \int_0^t (t-s)^{-a}e^{-b(t-s)}\|x_\varepsilon(s) - x_\varepsilon(s)\|_\alpha ds. \quad (4.2)$$

Let

$$a(t) = M\|h\|_{L^\infty(R,\mathcal{X})} \int_0^t (t-s)^{-a}e^{-b(t-s)}ds.$$

Applying the Gronwall inequality (see e.g. [16, Lemma 7.1.1]) to (4.2) it yields

$$\| x_\varepsilon(t) - x_\varepsilon(t) \| \leq |\varepsilon - \varepsilon'| \left( a(t) + \eta \int_0^t Z_1^{-\alpha}(\eta(t-s))e^{-b(t-s)}a(ds) \right), \quad (4.3)$$

where $\eta = [Mk(\rho)\Gamma(1-\alpha)]^{1-\alpha}$, and

$$\Gamma(1-\alpha) = \int_0^\infty s^{-\alpha}e^{-s}ds, \quad Z_1^{-\alpha}(s) = \sum_{n=0}^\infty s^{n(1-\alpha)} / \Gamma[n(1-\alpha) + 1].$$

The conclusion then immediately follows. \qed

**Lemma 4.2.** Assume that $\mathcal{A} \subset X^\alpha$ is an attractor of $S$. Let $N$ be a neighborhood of $\mathcal{A}$ which is such that $\mathcal{A}$ attracts $N$. Then for every $\delta > 0$, there exist $0 < \varepsilon_0 < 1$ and $T > 0$ such that if $\varepsilon \in [0,\varepsilon_0]$ then

$$\varphi_\varepsilon(t,p)N \subset B_{X^\alpha}(\mathcal{A},\delta), \quad t \geq T, \ p \in \mathcal{H}. \quad (4.4)$$

**Proof.** The proof can be obtained by modifying some argument in the proof of [21, Theorem 4.1]. We omit the details. \qed

Now we state and prove the following stability result concerning attractors under nonautonomous perturbations.

**Theorem 4.3.** Assume that $\mathcal{A} \subset X^\alpha$ is an attractor of $S$. Then there exists $\varepsilon_0 > 0$ such that if $\varepsilon \leq \varepsilon_0$, then $\varphi_\varepsilon$ has a pullback attractor $\mathcal{A}_\varepsilon$; furthermore,

$$\lim_{\varepsilon \to 0} d_H(\mathcal{A}_\varepsilon(p), \mathcal{A}) = 0 \quad (4.5)$$

uniformly with respect to $p \in \mathcal{H}$. 
Proof. Take a \( \delta > 0 \) sufficiently small so that \( B_{X^\omega}(A, 2\delta) \subset N \), where \( N \) is the neighborhood of \( A \) given in Lemma 4.2. Then by Lemma 4.2 there exist \( \varepsilon_0 > 0 \) and \( T > 0 \) such that (4.4) holds for all \( \varepsilon \in [0, \varepsilon_0] \). Set

\[
\mathcal{A}_\varepsilon(p) = \bigcap_{\tau > 0} \bigcup_{t \geq \tau} \varphi_{\varepsilon}(t, \theta_{-t}p)N.
\]

Then \( \mathcal{A}_\varepsilon = \{ \mathcal{A}_\varepsilon(p) \}_{p \in \mathcal{H}} \) is a pullback attractor of \( \varphi_{\varepsilon} \). Clearly

\[
\mathcal{A}_\varepsilon(p) \subset \overline{B}_{X^\omega}(A, \delta), \quad \forall p \in \mathcal{H}.
\]

The conclusion in (4.5) is a simple consequence of the fact that \( \delta \) can be chosen arbitrarily small.

4.2. Morse structures of perturbed pullback attractors. Let \( S \) be the autonomous system generated by (1.3), and let \( A \) be an attractor of \( S \) with the attraction basin \( \Omega = \Omega(A) \). Assume that \( S \) is a gradient-like system on the basin of attraction \( \Omega \) of an attractor \( A \), with a finite number of hyperbolic equilibria \( \mathcal{E} = \{ E_i \}_{i=1}^n \). Then \( \mathcal{E} \) forms a natural Morse decomposition of \( A \). We have

**Theorem 4.4.** There exists \( \varepsilon' > 0 \) such that if \( \varepsilon < \varepsilon' \), the perturbed system \( \varphi_{\varepsilon} \) has a pullback attractor \( \mathcal{A}_\varepsilon = \{ \mathcal{A}_\varepsilon(p) \}_{p \in \mathcal{H}} \) near \( A \); furthermore, the following assertions hold.

(1) \( \varphi_{\varepsilon} \) has a finite number of nonautonomous equilibria \( \Gamma_{\varepsilon}^i = \Gamma_{\varepsilon}^i(p) \) \( (i = 1, 2, \ldots, n) \) with each \( \Gamma_{\varepsilon}^i \) being close to \( E_i \).

(2) For each \( p \in \mathcal{H} \), \( \mathcal{A}_\varepsilon(p) \) is the union of the section unstable manifolds of \( \Gamma_{\varepsilon}^i \) \( (i = 1, 2, \ldots, n) \) at \( p \), namely,

\[
\mathcal{A}_\varepsilon(p) = \bigcup_{i=1}^n W^u(\Gamma_{\varepsilon}^i, p).
\]

(3) There is a neighborhood \( N \) of \( A \) (independent of \( \varepsilon \)) such that for each \( x \in N \), there is a \( \Gamma_{\varepsilon}' \) such that

\[
\lim_{t \to -\infty} d \left( \varphi_{\varepsilon}(t, p)x, \Gamma_{\varepsilon}'(\theta_{t}p) \right) = 0.
\]

**Proof.** The existence part of the pullback attractor \( \mathcal{A}_\varepsilon \) is contained in Theorem 4.3, and the conclusion in (1) is contained in Theorem 3.2.

(2) The argument below is adapted from the proof of [6, Theorem 2.11].

Take \( \varepsilon' > 0 \) sufficiently small so that (1) holds for \( \varepsilon \leq \varepsilon' \); furthermore, \( \varphi_{\varepsilon} \) possesses a pullback attractor \( \mathcal{A}_\varepsilon \) closed to \( A \). We infer from the proof of Theorem 3.2 and Remark 9 that for each \( 1 \leq i \leq n \), there is a neighborhood \( U_i \subset X^\omega \) of \( E_i \) such that for any solution \( \gamma \) of \( \varphi_{\varepsilon} \) on \( \mathbb{R}^- \) in \( U_i \), it holds that

\[
\lim_{t \to -\infty} \| \gamma(t) - \Gamma_{\varepsilon}'(\theta_{t}p) \|_\alpha = 0
\]

for some \( p \in \mathcal{H} \). It can be assumed that \( U_i \cap U_j = \emptyset \) for \( i \neq j \). To prove the second conclusion (2), it now suffices to check if \( \varepsilon > 0 \) is sufficiently small, then for any full solution \( \gamma \) in \( \mathcal{A}_\varepsilon \), there exist \( 1 \leq i \leq n \) and \( t_0 > 0 \) such that

\[
\gamma(t) \in U_i, \quad t \leq -t_0.
\]

Since \( \mathcal{M} = \{ E_1, \ldots, E_n \} \) is a Morse decomposition of the attractor \( A \) of the non-perturbed system \( S = \varphi_0 \), for every full solution \( \gamma \) of \( S \) in \( A \), we have either
\( \gamma(\mathbb{R}) \equiv E_i \) for some \( i \), or else there are indices \( i < j \) such that
\[
\lim_{t \to -\infty} \gamma(t) = E_j, \quad \lim_{t \to \infty} \gamma(t) = E_i.
\] (4.8)

Furthermore, \( A_k = \bigcup_{1 \leq i \leq k} W^u(E_i) \) is an attractor of \( S \) in \( \mathcal{A} \) for each \( 1 \leq k \leq n \).

By [17, Lemma 2.5] we also deduce that each \( A_k \) is an attractor of \( S \) in \( X^\alpha \). Note that
\[ \{E_1\} = A_1 \subset A_2 \subset \cdots \subset A_n = \mathcal{A} \]

By Lemma 4.2 there exists \( \delta, \delta > 0 \) such that \( N_k = \overline{B}_{X^\alpha}(A_k, \delta) \) uniformly attracts itself under the perturbed system \( \varphi_x \) for all \( \varepsilon \leq \delta \) and \( 1 \leq k \leq n \). More precisely, there exists \( T > 0 \) such that if \( \varepsilon \leq \delta \), then
\[
\varphi_x(t, p)N_k \subset N_k, \quad t > T, \ p \in \mathcal{H}
\] (4.9)
for all \( \varepsilon \leq \delta \) and \( 1 \leq k \leq n \). For convenience, we assign \( N_0 := \emptyset \). We may assume that \( \delta \) is sufficiently small so that
\[ U_{k+1} \cap N_k = \emptyset \]
for all \( k \) (recall that \( U_{k+1} \) is a neighborhood of \( E_{k+1} \) and \( E_{k+1} \notin A_k \)).

We first claim that for each full solution \( \gamma \) of \( \varphi_x \) in \( \mathcal{A}_\varepsilon (\varepsilon \leq \delta) \), there is a \( 1 \leq k \leq n \) such that
\[ \gamma(t) \in N_k \setminus N_{k-1} \] (4.10)
for all \( t \) sufficiently large. Indeed, by (4.9) we see that \( \gamma(\mathbb{R}) \subset N_n \). Let \( k \) be the smallest number such that \( \gamma(\mathbb{R}) \subset N_k \). We show that (4.10) holds true, which completes our claim. Suppose the contrary. There would exist a sequence \( t_m \to -\infty \) such that \( \gamma(t_m) \in N_{k-1} \). For any \( t \in \mathbb{R} \), pick a \( t_m \) such that \( t - t_m > T \) (\( T \) is the number in (4.9)). Then for some \( p \in \mathcal{H} \),
\[ \gamma(t) = \varphi_x(t - t_m, p)\gamma(t_m) \in N_{k-1} \]
Hence \( \gamma(\mathbb{R}) \subset N_{k-1} \). This leads to a contradiction.

In what follows, we again argue by contradiction. Suppose that (4.7) failed to be true. Then there would exist \( \varepsilon \geq \varepsilon_m \to 0 \) such that for each \( \varepsilon_m \), \( \varphi_{\varepsilon_m} \) has a full solution \( \gamma_m \) in \( \mathcal{A}_{\varepsilon_m} \) and a sequence \( 0 \geq t_l^m \to -\infty \) (as \( l \to \infty \)) such that
\[
\gamma_m(t_l^m) \notin \bigcup_{j=1}^n U_j, \quad l \geq 1.
\] (4.11)
Since there are only a finite number of \( N_k \)'s, by the claim above we deduce that there is a \( k \in \{1, 2, \ldots, n\} \) and a subsequence of \( \gamma_m \) (\( m = 1, 2, \ldots \)), still denoted by \( \gamma_m \), such that for each \( m \geq 1 \),
\[ \gamma_m(t) \in N_k \setminus N_{k-1}, \quad \forall t \leq T_m \]
for some \( T_m < 0 \). For each \( m \), pick two numbers \( t_l^m \) and \( t_l^m \) with
\[ t_l^m < T_m, \quad t_l^m - t_l^m > -m. \]
Write \( \tau_m = t_l^m - t_l^m \). Then \( \tau_m \to +\infty \). Define
\[ x_m(t) = \gamma_m(t + t_l^m), \quad t \in J_m = [-\tau_m, \tau_m] \]
It is clear that
\[ x_m(0) = \gamma_m(t_l^m) \notin \bigcup_{j=1}^n U_j, \quad x_m(J_m) \subset N_k \setminus N_{k-1} \] (4.12)
for all $m$. Now by very standard argument, it can be shown that the sequence $x_m(t)$ has a subsequence, still denoted by $x_m(t)$, such that $x_m(t)$ converges uniformly on any compact interval of $\mathbb{R}$ to a bounded full solution $x(t)$ of $S$ in $A$ (recall that $\varepsilon_m \to 0$ as $m \to \infty$). By (4.12) we conclude that $x(0) \notin \bigcup_{j=1}^{n} U_j$ and

$$x(\mathbb{R}) \subset N_k \setminus N_{k-1}.$$  

It follows by $x(0) \notin \bigcup_{j=1}^{n} U_j$ that $x(t)$ is not an equilibrium solution. Thus by (4.8) there exist two distinct equilibria $E_i, E_j$ such that

$$\lim_{t \to -\infty} x(t) = E_j, \quad \text{and} \quad \lim_{t \to \infty} x(t) = E_i.$$  

This leads to a contradiction, as there is only one equilibrium in $N_k \setminus N_{k-1}$.

(3) Let $N_k (1 \leq k \leq n)$ and $\varepsilon > 0$ be the same as above, and let $N = N_n$. Then for each $x \in N$,

$$\varphi_{\varepsilon}(t,p)x \in N_n, \quad t \geq T.$$  

Let $k$ be the smallest number such that $\varphi_{\varepsilon}(t,p)x \in N_k$ for all $t > 0$ sufficiently large. Then as in (4.10) it can be shown that

$$\varphi_{\varepsilon}(t,p)x \in N_k \setminus N_{k-1}, \quad \forall t \geq T'.$$

for some $T' > 0$. Based on this result, one can prove by a fully analogous argument as in the verification of (4.7) that if $\varepsilon > 0$ is sufficiently small, then for any $x \in N$ and $p \in \mathcal{H}$, there exists $1 \leq i \leq n$ and $t_0 > 0$ such that

$$\varphi_{\varepsilon}(t,p)x \in U_i, \quad t \geq -t_0.$$  

(4.13)

It is clear that $x$ is attracted by an equilibrium $E_i$ under $S$ for some $i$. Similar to the proof of Lemma 4.2, there is a $T > 0$ such that for sufficiently small $\varepsilon$ and $p \in \mathcal{H}$,

$$\varphi_{\varepsilon}(t,p)x \in U_i, \quad t \geq T.$$  

It then follows by Remark 9 that

$$\lim_{t \to \infty} d(\varphi_{\varepsilon}(t,p)x, \Gamma^i_{\varepsilon}(\theta_t p)) = 0.$$  

This completes the proof. \hfill \Box

**Remark 10.** The interested reader is referred to [1, 7, 24] etc. for some general theories on Morse decompositions of nonautonomous dynamical systems.

5. **Forward attraction of the perturbed attractors and synchronizing behavior of the perturbed nonautonomous system.** As we have mentioned in the introduction, in general the existence of forward attractors of nonautonomous systems remains an open problem. In this section we show that the pullback attractor $\mathcal{A}_{\varepsilon}$ of the perturbed system $\varphi_{\varepsilon}$ given in Theorem 4.4 is a forward attractor, using the Morse structures of pullback attractors.

For the sake of clarity, in this section we drop the subscript "$\varepsilon$" in $\mathcal{A}_{\varepsilon}$ and rewrite $\mathcal{A}_{\varepsilon} = \mathcal{A}$.

**Theorem 5.1.** Assume the hypotheses in Theorem 4.4, and let $\mathcal{A} = \{\mathcal{A}(p)\}_{p \in \mathcal{H}}$ be the pullback attractor of the perturbed system $\varphi_{\varepsilon}$ given in Theorem 4.4. Then

(1) $\mathcal{A}(p)$ is continuous in $p$ (in the sense of Hausdorff distance); and

(2) $\mathcal{A}$ is a forward attractor of $\varphi_{\varepsilon}$. 
Remark 11. The forward attraction of $\mathcal{A}$ means that there is a neighborhood $B$ of $\mathcal{A}_u := \bigcup_{p \in \mathcal{H}} \mathcal{A}(p)$ such that
\begin{equation}
\lim_{t \to \infty} H_{X^{\alpha}}(\varphi_{\varepsilon}(t, p)B, \mathcal{A}(\theta t p)) = 0.
\end{equation}

Remark 12. Recall that if the external force $h$ in (1.2) is periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic), then $\theta t p$ is periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic) for any fixed $p \in \mathcal{H} := \mathcal{H}[h]$. Thus under the assumptions of Theorem 5.1, $\mathcal{A}(p)$ is continuous in $p$ and $\mathcal{A}(\theta t p)$ is periodic (resp. quasiperiodic, almost periodic, uniformly almost automorphic). The convergence (5.1) then indicates that the system exhibits a global synchronizing behavior with the external force $h$.

Proof of Theorem 5.1. (1) The verification of the upper semicontinuity of $\mathcal{A}(p)$ can be performed easily. Since the argument is quite standard, we omit the details. In what follows we only check the lower semicontinuity of $\mathcal{A}(p)$ in $p$. For this purpose, by virtue of [5, Lemma 3.2, pp. 75] it suffices to verify that for any $p_0 \in \mathcal{H}$, $x_0 \in \mathcal{A}(p_0)$ and sequence $p_k \to p_0$, there is a sequence $x_k$ with $x_k \in \mathcal{A}(p_k)$ (for each $k$) such that $x_k \to x_0$ as $k \to \infty$.

We may assume that $\varepsilon > 0$ is sufficiently small so that Theorem 4.4 holds. Since $x_0 \in \mathcal{A}(p_0) = \bigcup_{i=1}^{n} W^u_i(\Gamma_i, p_0)$, where $\Gamma_i := \Gamma_\varepsilon (1 \leq i \leq n)$ are the non-autonomous equilibria of $\varphi_{\varepsilon}$ given by Theorem 4.4, there is a $j \in \{1, \ldots, n\}$ such that $x_0 \in W^u_j(\Gamma_j, p_0)$. Thus we can find a full solution $\gamma$ with $\gamma(0) = x_0$ such that
\begin{equation}
d(\gamma(t), \Gamma_j(\theta t p_0)) \to 0 \quad \text{as } t \to -\infty.
\end{equation}

Let $U_j$ be the neighborhood of the equilibrium point $E_j$ of the autonomous system
\begin{equation}
S = \varphi_0 \text{ given in the proof of Theorem 4.4, and let } U_j = U_j \times \mathcal{H}.
\end{equation}

Choose a $\tau > 0$ such that $z = \gamma(-\tau) \in W^u_{U_j}(\Gamma_j, q_0)$, where $q_0 = \theta_{-\tau} p_0$.

Let $q_k = \theta_{-\tau} p_k$. Clearly $q_k \to q_0$. Thus by the continuity of $W^u_{U_j}(\Gamma_j, p)$ in $p$ (see Remark 7) we can choose a sequence $\{z_k\}_{n \in \mathbb{N}}$ with
\begin{equation}
z_k \in W^u_{U_j}(\Gamma_{j, q_k}) \subset \mathcal{A}(q_k)
\end{equation}

(for each $k$) such that $z_k \to z$ as $k \to \infty$. Note that by the continuity property of $\varphi_{\varepsilon}$, we have
\begin{equation}
\lim_{n \to \infty} \|\varphi_{\varepsilon}(\tau, q_k)z_k - \varphi_{\varepsilon}(\tau, q_0)z\|_\alpha = \lim_{n \to \infty} \|\varphi_{\varepsilon}(\tau, q_k)z_k - x_0\|_\alpha = 0.
\end{equation}

Let $x_k := \varphi_{\varepsilon}(\tau, q_k)z_k$. Then by the invariance property of $\mathcal{A}$ we have $x_k = \varphi_{\varepsilon}(\tau, q_k)z_k \in \varphi_{\varepsilon}(\tau, q_k)\mathcal{A}(q_k) = \mathcal{A}(\theta t q_k) = \mathcal{A}(p_k)$.

Then (5.2) asserts that $x_k \to x_0$ as $k \to \infty$. This completes the proof.

(2) The forward attraction property of $\mathcal{A}$ follows from the continuity of $\mathcal{A}(p)$ in $p$ and Proposition 1.

Finally, let us give an example to illustrate our results.

Example 5.1. Consider the nonautonomous system:
\begin{equation}
\begin{cases}
 u_t - \Delta u = f(u) + \varepsilon h(x, t), & t > 0, x \in \Omega; \\
 u = 0, & t > 0, x \in \partial\Omega,
\end{cases}
\end{equation}
where $\Omega$ is a bounded domain in $\mathbb{R}^3$ with smooth boundary, $\varepsilon > 0$, and

$$f(s) = -b_3 s^3 + b_2 s^2 + b_1 s + b_0, \quad b_3 > 0.$$  

Denote by $A$ the operator $-\Delta$ associated with the homogeneous Dirichlet boundary condition. Then $A$ is a sectorial operator on $X = L^2(\Omega)$ with compact resolvent, and $\mathcal{D}(A) = H^2(\Omega) \cap H_0^1(\Omega)$.

The system (5.3) can be written into an abstract equation on $X$:

$$u_t + Au = f(u) + \varepsilon h(t),$$

where $h(t) = h(\cdot, t)$. The non-perturbed system reads

$$u_t + Au = f(u), \quad (5.4)$$

which is a gradient-like system, with energy functional

$$L(u) = \frac{1}{2} \int_{\Omega} \left( |\nabla u|^2 - F(u) \right) dx,$$

where $F$ is a primitive of $f$. It is well known that (5.4) has a global attractor $A$ (see e.g. [28]). We also infer from [3] (see Sect. 3) that all equilibria of (5.4) are hyperbolic generically for $(b_0, \cdots, b_3) \in \mathbb{R}^4$. Thus we can reasonably assume that (5.4) has only a finite number of equilibria $E_1, \cdots, E_n$.

Consider the cocycle system:

$$u_t + Au = f(u) + \varepsilon p(t), \quad p \in \mathcal{H}. \quad (5.5)$$

Denote $\varphi_\varepsilon := \varphi_\varepsilon(t, p)u$ the cocycle semiflow on $H_0^1(\Omega)$ driven by the base flow (translation group) $\theta$ on $\mathcal{H}$. Assume that $h \in C(\mathbb{R}, X)$ is translation compact with hull $\mathcal{H} := \mathcal{H}[h]$. Then the system is dissipative and hence has a global pullback attractor $\mathcal{A} = \{ \mathcal{A}(p) \}_{p \in \mathcal{H}}$ such that $\mathcal{A}(p)$ approaches $A$ as $\varepsilon \to 0$ for each $p \in \mathcal{H}$.

Since the hypotheses in the main theorems are fulfilled, we obtain some interesting results concerning the dynamics of the perturbed system as follows.

**Theorem 5.2.** If $\varepsilon$ is sufficiently small then the following assertions about the pullback attractor $\mathcal{A} = \{ \mathcal{A}(p) \}_{p \in \mathcal{H}}$ of the perturbed system hold true.

1. $\varphi_\varepsilon$ has a finite number of non-autonomous equilibria $\Gamma_i = \Gamma_i(p) (i = 1, 2, \cdots, n)$ such that for each $p \in \mathcal{H}$,

$$\mathcal{A}(p) = \bigcup_{i=1}^{n} W^u(\Gamma_i, p).$$

2. For each $u \in H_0^1(\Omega)$, there is a $\Gamma_j$ such that

$$\lim_{t \to \infty} d(\varphi_\varepsilon(t, p)u, \Gamma_j(\theta_t p)) = 0.$$  

3. $\mathcal{A}$ is a uniform forward attractor.

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