Complete phase diagrams of the coevolving spreading dynamics in complex networks

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Epidemic spreading processes in the real world can interact with each other in a cooperative, competitive or asymmetric way, requiring a description based on coevolution dynamics. Rich phenomena such as discontinuous outbreak transitions and hystereses can arise, but a full picture of these behaviors in the parameter space is lacking. We develop a theory for coevolving spreading dynamics on complex networks through spectral dimension reduction. In particular, we derive from the microscopic quenched mean-field equations a two-dimensional system in terms of the macroscopic variables, which enables a full Phase diagram to be determined analytically. The diagram not only generates critical phenomena that have been known, but also predicts those that were previously unknown such as the interplay between discontinuous transition and hysteresis as well as the emergence and role of tri-critical points.

I. INTRODUCTION

Spreading dynamics of diseases, behaviors and information in nature and human society are rarely independent processes but interact with each other in a complex way. Weakened immunity to other viruses due to HIV infection and suppression of spreading due to disease-related information exchange on the social media are known examples. To better understand, predict, and control spreading on networks, coevolution of epidemics must be taken into account. In network science, recent years have seen increasing efforts in developing coevolving epidemic models, which can generate surprising behaviors that cannot be predicted by any single-virus epidemic model. For example, spreading of one epidemic can facilitate that of another, leading to a first-order or explosive transition in the outbreak with significant real-world implications. Many factors can affect the critical behaviors of coevolving spreading dynamics, such as self-evolution of each epidemic, interaction between two epidemics, and network structure.

By now, spreading dynamics of a single epidemic on complex networks have been well studied. For coevolving spreading dynamics, the special case of well-mixed populations has been treated. In addition, theories based on percolation and annealed mean field have been articulated to study the effect of network structure on coevolving spreading, leading to a qualitative understanding of critical phenomena. There are difficulties with these theories. For example, the annealed mean-field theory takes into account only the nodal degrees and is not applicable to quenched networks (especially networks with a high clustering coefficient and modularity). For such networks, quenched mean-field theories such as those based on Markov chains and the N-intertwined method is needed. A deficiency of such a theory is that it uses a large number of nonlinear differential equations, with two difficulties: (a) high computational overload for large networks and (b) lack of any analytic insights. Such a theory, due to its heavy reliance on numerics, can lead to inconsistent or even contradicting predictions. To our knowledge, a general analytic theory capable of providing a more complete understanding of coevolving spreading dynamics is lacking.

In this paper, we develop an analytic theory for coevolving spreading dynamics on complex networks through the approach of dimension reduction for complex networks. From the quenched mean-field equations, we derive a two-dimensional system that is capable of analytically yielding the full Phase diagrams underlying coevolving spreading dynamics on any complex network, from which the conditions for various phase transitions can be derived. The analytic model predicts not only phase transitions that were previously known but also previously unknown phenomena such as the interplay between discontinuous outbreak transitions and hystereses as well as the emergence of tri-critical points, providing a solid theoretical foundation for understanding coevolving spreading dynamics and articulating optimal control strategies.

II. MODEL DESCRIPTION

We consider the susceptible-infected-susceptible (SIS) model of coevolving spreading dynamics on simple complex networks. In the classic SIS model, a single epidemic spreads in the network and a node can be either in the susceptible or infected state. Susceptible nodes are infected by their infected neighbors at rate λ and infected nodes recover at rate γ. For coevolving SIS dynamics, two epidemics, say 1 and 2, spread simultaneously and interact with each other. Each node infected by a ∈ {1, 2} transmits the infection to neighbors...
that are susceptible for both epidemics with probability $\lambda_a$. If a neighbor is susceptible for \( a \) but infected by the other epidemic, the infection will be transmitted with rate \( \lambda_{a,i} \). All the nodes infected by \( a \) recover to being susceptible with rate \( \gamma_a \). Without loss of generality, we set \( \gamma_a = 1 \) for both \( a \in \{1, 2\} \). In general, the nature of the coevolving SIS dynamics depends on the interplay between the rates \( \lambda_a \) and \( \lambda_{a,i} \). In particular, for \( \lambda_{a,i} > \lambda_a \), the two epidemics tend to facilitate each other, leading to cooperative SIS dynamics, whereas if \( \lambda_{a,i} < \lambda_a \), infection with one epidemic will suppress infection with the other, giving rise to competitive SIS dynamics. For \( \lambda_{a,i} > \lambda_a \) but \( \lambda_{a,i} < \lambda_a \) for \( a, b \in \{1, 2\} \) with \( b \neq a \), the interactions are asymmetric.

### III. SPECTRAL DIMENSION REDUCTION

The coevolving SIS model represents a paradigm to study the rich dynamical behaviors such as first-order outbreak transitions and hysteresis [31]. In this paper, we study the coevolving SIS model by applying the quenched mean field theory (QMF) [32]. QMF ignores the dynamical correlations among neighbors [33], yet predicts the phase transitions in good accuracy. Let \( p_{a,i} \) be the probability that node \( i \in \{1, \cdots, N\} \) is infected by \( a \in \{1, 2\} \) at time \( t \). In first-order mean-field analysis [33], the evolution of \( p_{a,i} \) on a simple network with adjacency matrix \( G \) is governed by

$$
\frac{dp_{a,i}}{dt} = -p_{a,i} + \lambda_{a,i}^\dagger (1 - p_{a,i}) p_{b,i} \sum_j G_{ij} p_{a,j} + \lambda_a (1 - p_{a,i}) (1 - p_{b,i}) \sum_j G_{ij} p_{a,j}
$$

(1)

for \( a \in \{1, 2\} \) and \( i \in \{1, \cdots, N\} \). The first term on the right side of Eqs. (1) is the rate of recovery from epidemic \( a \) for node \( i \), while the second (third) term corresponds to the rate of infection for epidemic \( a \) with (without) \( i \) already infected by \( b \neq a \). For a network of size \( N \), the number of equations in (1) is \( 2N \). To derive an analytic model, we exploit the technique of spectral dimension reduction (SDR) [29] to arrive at an equivalent description of the original system in terms of two macroscopic observables - one for each epidemic. In particular, let \( \alpha \) be a vector with nonnegative entries and normalized as \( \sum_i \alpha_i = 1 \). The entries of \( \alpha \) represent the nodal weights. We define linear observables as \( \psi_a = \alpha^T p_a \) for \( a \in \{1, 2\} \). Since the entries of \( \alpha \) are summed to unity, \( \psi_a \) is a weighted average. The evolution of \( \psi_a \) is determined by the equation

$$
\frac{d\psi_a}{dt} = \sum_{i=1}^N \alpha_i \frac{dp_{a,i}}{dt},
$$

(2)

which can be written in terms of the macroscopic observables \( \psi_a \), which can be decomposed as

$$
\begin{align*}
\rho_{a,i} &= \rho_a \psi_a + \delta \rho_{a,i} \\
\rho_{b,i} &= \mu_a \psi_b + \delta \rho_{b,i} \\
\rho_{a,j} &= \nu_a \psi_a + \delta \rho_{a,j}
\end{align*}
$$

(3)

where \( \rho_a, \mu_a, \) and \( \nu_a \) are parameters to be determined, and \( \delta \rho_{a,i}, \delta \rho_{b,i}, \) and \( \delta \rho_{a,j} \) are correction terms. Substituting Eqs. (1) and Eqs. (3) into Eqs. (2) gives

$$
\frac{d\psi_a}{dt} = -\psi_a + \lambda_{a,i}^\dagger \alpha \mu_a \nu_a (1 - \rho_a \psi_a) \psi_b \psi_a + \mu_a \nu_a \delta \rho_a (1 - \mu_a \psi_b) \psi_a + R_a,
$$

(4)

where \( \alpha = \sum_{i,j} \alpha_i G_{ij} \) and \( R_a \) is the remainder term that can be decomposed as

$$
R_a = R_{a,1} + R_{a,2} + R_{a,3},
$$

(5)

with \( R_{a,1}, R_{a,2} \) and \( R_{a,3} \) containing the first-, second- and third-order terms in the corrections \( \{\delta \rho_{a,i}\} \), respectively. Let \( K \) be the diagonal matrix with \( K_{ii} \) being the degree of node \( i \). The first-order correction \( R_{a,1} \) is given by

$$
R_{a,1} = \left[ (\lambda_a - \lambda_{a,i}^\dagger) \mu_a \nu_a \psi_a - \lambda_{a,i} \mu_a \psi_a \right] \alpha^T K \delta \rho_a + \left[ (\lambda_a - \lambda_{a,i}^\dagger) \nu_a \psi_a \right] \psi_a \alpha^T K \delta \rho_b + \mu_a (1 - \rho_a \psi_a) \psi_a \alpha^T G \delta \rho_a,
$$

(6)

where \( \delta \rho_a \) is a vector with \( \delta \rho_{a,i} \) in the \( i \)th entry and \( \delta \rho_b \) is defined analogously. The second-order remainder term is

$$
R_{a,2} = (\lambda_a - \lambda_{a,i}^\dagger) (1 - \rho_a \psi_a) \sum_{i=1}^N \sum_{j=1}^N G_{ij} \alpha_i \delta \rho_b, \delta \rho_{a,j}
$$

$$
+ \left[ (\lambda_a - \lambda_{a,i}^\dagger) \mu_a \psi_a - \lambda_{a,i} \mu_a \psi_a \right] \sum_{i=1}^N \sum_{j=1}^N G_{ij} \alpha_i \delta \rho_{a,i} \delta \rho_{a,j}
$$

$$
+ (\lambda_a - \lambda_{a,i}^\dagger) \nu_a \psi_a \sum_{i=1}^N \sum_{j=1}^N G_{ij} \alpha_i \delta \rho_{a,i} \delta \rho_{b,i}
$$

(7)

and the third-order remainder term is

$$
R_{a,3} = (\lambda_a - \lambda_{a,i}^\dagger) \sum_{i=1}^N \sum_{j=1}^N \sum_{k=1}^N G_{ijk} \alpha_i \delta \rho_{a,i} \delta \rho_{b,i} \delta \rho_{a,j}.
$$

(8)

From Eq. (6), the dominant remainder term \( R_{a,1} \) vanishes if the following equations hold

$$
\alpha^T K \rho_a = \hat{\alpha} \rho_a \psi_a,
$$

(9)

$$
\alpha^T K \rho_b = \hat{\alpha} \mu_a \psi_b,
$$

$$
\alpha^T G \rho_a = \hat{\alpha} \nu_a \psi_a,
$$

where \( \rho_a \) is a vector with \( \rho_{a,i} \) in the \( i \)th entry and \( \rho_b \) is defined similarly. In general, the equations cannot be satisfied
simultaneously. In Ref. [29], \( \alpha \) is chosen to be the eigenvalue associated with the leading eigenvalue \( \omega \) of \( G \). In this paper we consider connected undirected networks, therefore, the eigenvector associated with the leading eigenvalue of \( G \) has positive entries. The third equation in Eqs. (9) implies \( \omega \psi_a = \alpha \nu a \psi_a \) and, hence, \( \omega = \nu a \alpha \). Using the definition \( \hat{\alpha} = 1^T G \alpha = \omega \), we have \( \nu a = 1 \). The remaining two parameters, \( \rho a \) and \( \mu b \), are chosen such that the first two equations in Eqs. (9) are satisfied. The quantities \( \rho a \) and \( \mu b \) can be chosen by minimizing the following squared vector norm

\[
\rho a^* = \mu a^* = \arg \min_x \| K \alpha - x \hat{\alpha} \|^2, \tag{10}
\]

which yields

\[
\mu : = \frac{1}{\omega} \frac{\alpha^T K \alpha}{\alpha^T \alpha} = \rho a^* = \mu a^*. \tag{10}
\]

A justification of the parameter choices was given in Ref. [29].

With the parameters chosen, \( R_{a,1} \) can be made as small as possible and can be neglected, so can the higher order terms \( R_{a,2} \) and \( R_{a,3} \). Substituting the values of \( \rho a \), \( \mu b \) and \( \nu a \) into Eqs. (4) and ignoring the remainder term, we get

\[
\frac{d\psi a}{dt} = - \psi a + \lambda a^\dagger \omega (1 - \mu \psi a) \psi b \psi a + \\
+ \lambda a (1 - \mu \psi a) (1 - \mu \psi b) \psi a + R_a. \tag{11}
\]

The first term on the right side of Eqs. (11) accounts for the rate of recovery and the second (third) term represents the rate of infection for epidemic \( a \) with (without) being infected by \( b \neq a \). The quantity \( R_a \) in Eqs. (11) characterizes the fluctuations of the microscopic observables \( p_{a,i} \) about the macroscopic observables \( \psi a \), which is small in comparison to other terms on the right side of Eqs. (11) because \( \alpha \) is the leading eigenvector. Now we neglect \( R_a \) in Eqs. (11) and study the mean-field equations only depend on the macroscopic observables \( \psi a \). The resulting errors from neglecting the remainder terms near the phase boundaries are insignificant, as will be verified numerically in Sec. IV.

In Eqs. (11), the order parameters are \( \langle p_{a,i} \rangle \) for \( a \in \{1, 2\} \), where \( \langle \cdot \rangle \) is the unweighted average over the nodes. In Eqs. (11), the order parameters can be chosen to be \( \psi a = \alpha^T p a \), a weighted average over the nodes. Since \( \alpha \) is the eigenvector associated with the leading eigenvalue of \( G \), its entries are strictly positive. As a result, \( \psi a = 0 \) (\( \psi a > 0 \)) implies \( \langle p_{a,i} \rangle = 0 \) (\( \langle p_{a,i} \rangle > 0 \)). When crossing a phase boundary, at least one of \( \psi a \) for \( a \in \{1, 2\} \) becomes either zero or nonzero, guaranteeing that the corresponding \( \langle p_{a,i} \rangle \) becomes either zero or nonzero, respectively. We have that \( \langle p_{a,i} \rangle \) and \( \psi a \) give the same phase diagram, which can be obtained analytically through the 2D mean-field system.

IV. PHASE DIAGRAM OF THE REDUCED SYSTEM

The reduced mean-field equations are amenable to analytic treatment. In this section, we first calculate the equilibrium points of the reduced system and analyze their stability, which enables us to obtain the full phase diagram of the reduced system and the equations for the phase boundaries. We then discuss the types of phase transitions crossing the various boundaries and study the interplay between the transitions and the phenomenon of hysteresis. Finally, we derive the conditions under which a hysteresis can arise.

For convenience, for the rest of the paper, we use the convention that, if variables indexed by \( a, b \in \{1, 2\} \) (e.g., \( \psi a \) and \( \psi b \)) appear together in an equation or an inequality, the assumption is \( a \neq b \).

A. Sketch of the results

The analyse of the 2D mean-field system are performed in the following steps. First, for each point in the parameter space \( (\lambda a, \lambda a^\dagger) \), we determine the equilibrium points of Eqs. (11) (Sec. IV.B) and their stability (Sec. IV.C). The equilibrium points have to further satisfy the probability constraint to be physical meaningful. A detailed analysis of the stability and probability constraints of the equilibrium points leads to the following functions of \( \lambda a \) and \( \lambda a^\dagger \):

\[
s_{a,0} = \lambda b + \lambda a^\dagger - \lambda a - \omega \lambda b \lambda a^\dagger, \quad s_{a,1} = \lambda a^\dagger - \lambda b - \lambda a + 2 \omega \lambda b \lambda a^\dagger - \omega \lambda b \lambda a^\dagger, \quad s_{a,2} = \lambda b - \lambda b, \quad s_{a,3} = \lambda a - \omega^{-1}, \quad \text{and} \quad s_{a\Delta} = (\lambda a^\dagger - \lambda b + \lambda a - \omega \lambda b \lambda a^\dagger)^2 - 4 (\lambda a^\dagger - \lambda a) (\lambda^2 - \lambda b),
\]

for \( a, b \in \{1, 2\} \) and \( a \neq b \). If an equilibrium point is physical or stable is determined by the signs of these functions. Then, based on the numbers of stable and unstable equilibrium points as well as their relations, we can divide the parameter space into distinct regions, where a boundary-crossing between two neighboring regions gives rise to a phase transition (Sec. IV.E). A region either can have a unique stable equilibrium point or can have two stable equilibrium points with one unstable point in between, where crossing the latter will result in a hysteresis (IV.E).

B. Equilibrium points of the reduced system

The equilibrium points are obtained by setting the right side of Eqs. (11) to zero:

\[
- \psi a + \lambda a^\dagger \omega (1 - \mu \psi a) \psi b \psi a + \\
+ \lambda a (1 - \mu \psi a) (1 - \mu \psi b) \psi a + R_a = 0
\]

(12)

for \( a, b \in \{1, 2\} \) and \( a \neq b \). Further, the physical solutions have to satisfy the probability constraints \( 0 \leq \psi a \leq 1 \).

Because of the appearance of terms such as \( (1 - \mu \psi a) \) in Eqs. (11), it is necessary to impose the physical condition \( (1 - \mu \psi a) \leq 1 \). To prepare for a discussion of the equilibrium points, we first prove that \( \mu \) given by Eq. (10) satisfies \( \mu \geq 1 \). It can also be verified that any point with \( \psi_1 = \mu^{-1} \) or \( \psi_2 = \mu^{-1} \) cannot be an equilibrium point. These, together the probability constraints, imply that all the equilibrium points must satisfy the inequality \( 0 \leq \psi a < \mu^{-1} \) for \( a \in \{1, 2\} \).

To prove \( \mu \geq 1 \), we rewrite Eq. (10) as

\[
\mu^{-1} = \frac{\alpha^T G \alpha}{\alpha^T K \alpha}.
\]

(13)
Since \( K \) is positive definite, it can be decomposed as \( K = K^{1/2}K^{1/2} \), where \( K^{1/2} \) is a diagonal matrix whose entries are the square root of the degrees. Let \( y = K^{1/2}\alpha \). We have \( \alpha = K^{-1/2}y \). Substituting this back to \( \mu^{-1} \) gives

\[
\mu^{-1} = \frac{yK^{-1/2}GK^{-1/2}y}{y^T y},
\]

which is the Rayleigh quotient of matrix \( K^{-1/2}G K^{-1/2} \) and, hence, we have \( \mu^{-1} \leq \delta_1 \), where \( \delta_1 \) is the largest eigenvalue of the matrix \( K^{-1/2}G K^{-1/2} \). Recall that the symmetric normalized Laplacian matrix of \( G \) is defined as

\[
L^{\text{sym}} = I - K^{-1/2}G K^{-1/2},
\]

which has a smallest eigenvalue \( \zeta_n = 0 \). As a result, we have \( \delta_1 = 1 - \zeta_n = 1 \), which gives \( \mu \geq 1 \).

We are now in a position to discuss the types of equilibrium points of the reduced mean-field equations.

(i) Epidemic free. The trivial solution \((\psi_1, \psi_2) = (0, 0)\) is always an equilibrium point.

(ii) Partial infection of epidemic 1. For \( \psi_1 \neq 0 \) and \( \psi_2 = 0 \), Eqs. (12) become

\[
-1 + \lambda_1 \omega (1 - \mu \psi_1) = 0,
\]

which gives

\[
\psi_1 = \frac{\lambda_1 \omega - 1}{\mu \lambda_1 \omega}.
\]

The solution further has to satisfy the probability constraint \( 0 < \psi_1 \leq \mu^{-1} \). We have that \( \psi_1 > 0 \) is guaranteed for \( \lambda_1 \omega > 1 \) while \( \psi_1 \leq \mu^{-1} \) always holds.

(iii) Partial infection of epidemic 2. Similar to case (ii), we have the equilibrium point:

\[
(\psi_1, \psi_2) = \left(0, \frac{\lambda_2 \omega - 1}{\mu \lambda_2 \omega}\right).
\]

(iv) Coinfection. If \( \psi_a \neq 0 \) for both \( a \in \{1, 2\} \), Eqs. (12) become

\[
\begin{align*}
-1 + \lambda_1^a \omega \mu (1 - \mu \psi_1) \psi_2 + \lambda_1 \omega (1 - \mu \psi_1) (1 - \mu \psi_2) &= 0, \\
-1 + \lambda_1^a \omega \mu (1 - \mu \psi_2) \psi_1 + \lambda_2 \omega (1 - \mu \psi_2) (1 - \mu \psi_1) &= 0.
\end{align*}
\]

Rearranging the second equation, we get

\[
\psi_2 = \frac{1}{\frac{\lambda_1^a \omega \mu}{\lambda_2 - \lambda_2^a} \mu^2 \omega \psi_2 + \lambda_2 \mu \omega}.
\]

Substituting this relation into Eq. (19a), we get an equation that depends on \( \psi_1 \) only. Similarly we can obtain the equation that determines \( \psi_2 \). The two equations for \( \psi_1 \) and \( \psi_2 \) have the following symmetric form:

\[
g_{a,2} \psi_a^2 + g_{a,1} \psi_a + g_{a,0} = 0
\]

for \( a \in \{1, 2\} \), where

\[
\begin{align*}
g_{a,2} &= \omega^2 \mu^3 \lambda_a^4 \left( \lambda_a^b - \lambda_b \right) \\
g_{a,1} &= \omega \mu^2 \left( \lambda_a^b - \lambda_a^b - \lambda_a + 2 \omega \lambda_b \lambda_a^b - \omega \lambda_a^b \lambda_b^a \right) \\
g_{a,0} &= \omega \mu \left( \lambda_a^b + \lambda_a - \lambda_a - \omega \lambda_b \lambda_a^b \right)
\end{align*}
\]

for \( b \in \{1, 2\} \) and \( b \neq a \).

If \( \lambda_a^b \neq \lambda_a \) holds for \( a \in \{1, 2\} \), \( g_{a,2} \neq 0 \) and Eqs. (21) will be quadratic, leading to two solutions

\[
\psi_a^{\pm} = \frac{-g_{a,1} \pm \sqrt{g_{a,1}^2 - 4g_{a,2}g_{a,0}}}{2g_{a,2}}.
\]
The solutions for $a \in \{1, 2\}$ are paired as
\[ (\psi_1, \psi_2) = \left( \psi_1^+, \psi_2^+ \right), \quad (\psi_1, \psi_2) = \left( \psi_1^-, \psi_2^- \right). \tag{24} \]

If $\lambda_a = \lambda_a$ for one or both $a \in \{1, 2\}$, we have
\[ \psi_a = -\frac{g_{a,0}}{g_{a,1}}, \tag{25} \]

To discuss the probability constraints of the equilibrium points, we consider the following cases.

(i) If $g_{a,2} = 0$ for one of $a \in \{1, 2\}$, i.e., $\lambda_b = \lambda_b$, then there is a unique solution given by
\[ \psi_a = \mu^{-1} - \frac{\lambda_b}{\mu (\lambda_a - \lambda_b + \omega \lambda_b \lambda_b^1)}, \quad \psi_b = \mu^{-1} - \frac{1}{\mu \omega \lambda_b}. \tag{26} \]

The probability constraints imply
\[ \lambda_a - \lambda_b - \lambda_b^1 + \omega \lambda_b \lambda_b^1 > 0, \quad \omega \lambda_b > 1. \tag{27} \]

Further, if we have $\lambda_b^1 = \lambda_b$, the two epidemics will become independent of each other with the solution
\[ \psi_a = \mu^{-1} - \frac{1}{\omega \mu \lambda_a}, \quad \psi_b = \mu^{-1} - \frac{1}{\omega \mu \lambda_b}. \tag{28} \]

(ii) Suppose $\lambda_b^1 > \lambda_b$ for both $a \in \{1, 2\}$. In this case there are two solutions, as shown in Eqs. (24).

Consider the solution
\[ (\psi_1, \psi_2) = \left( \psi_1^+, \psi_2^+ \right). \]

Firstly, it is necessary to have $g_{a,1}^2 - 4g_{a,2}g_{a,0} \geq 0$ for $a \in \{1, 2\}$ to make the solutions real. Because of the condition $g_{a,2} > 0$, the probability constraints $0 < \psi_a < \mu^{-1}$ imply
\[ g_{a,1} < \sqrt{g_{a,1}^2 - 4g_{a,2}g_{a,0}} < 2\mu^{-1}g_{a,2} + g_{a,1}, \tag{29} \]
for $a \in \{1, 2\}$. The second inequality can be written as
\[ \mu^{-2}g_{a,2} + \mu^{-1}g_{a,1} + g_{a,0} > 0. \tag{30} \]

Substituting these into Eqs. (22), we have
\[ g_{a,0} + \mu^{-1}g_{a,1} + \mu^{-2}g_{a,2} = \mu \omega \lambda_b^1 > 0, \tag{31} \]
indicating that the second inequality always holds.

It remains to consider the first inequality in Eqs. (29). Suppose $g_{a,0} < 0$, then both the first and the inequality $g_{a,1}^2 - 4g_{a,2}g_{a,0} > 0$ hold. Otherwise, suppose $g_{a,0} > 0$, it is necessary to have $g_{a,1} < 0$ and $g_{a,1}^2 - 4g_{a,2}g_{a,0} \geq 0$.

Combining the discussions above, we have that $(\psi_1^+, \psi_2^+)$ is a physical solution either for $g_{a,0} < 0$ or for $g_{a,0} > 0$, $g_{a,1} < 0$, $g_{a,1}^2 - 4g_{a,2}g_{a,0} > 0$ for both $a \in \{1, 2\}$.

We now consider the solution $(\psi_1, \psi_2) = (\psi_1^-, \psi_2^-)$. In order for the solution to be meaningful, it has to be guaranteed that $g_{a,1}^2 - 4g_{a,2}g_{a,0} \geq 0$ for $a \in \{1, 2\}$. The probability constraints give
\[ g_{a,1} < -\sqrt{g_{a,1}^2 - 4g_{a,2}g_{a,0}} < 2\mu^{-1}g_{a,2} + g_{a,1}. \tag{32} \]

The first inequality implies $g_{a,1} < 0$ and $g_{a,0} > 0$. Consider the second inequality. If $2\mu^{-1}g_{a,2} + g_{a,1} > 0$, then the second inequality will be satisfied. Else if $2\mu^{-1}g_{a,2} + g_{a,1} \leq 0$, the second inequality can be written as
\[ \mu^{-2}g_{a,2} + \mu^{-1}g_{a,1} + g_{a,0} \leq 0. \tag{33} \]

Substituting these into Eqs. (22), we have
\[ g_{a,0} + \mu^{-1}g_{a,1} + \mu^{-2}g_{a,2} = \mu \omega \lambda_b^1 > 0, \tag{34} \]
which leads to a contradiction. It is thus necessary to have $2\mu^{-1}g_{a,2} + g_{a,1} \geq 0$ for $a \in \{1, 2\}$. In fact, the inequalities $g_{a,1} < 0$ and $g_{a,0} > 0$ are sufficient to guarantee the condition $2\mu^{-1}g_{a,2} + g_{a,1} \geq 0$. For $g_{1,1} + g_{2,1} < 0$, we have
\[ \lambda_1^1 + \lambda_2^1 - \omega \lambda_1^1 \lambda_2^1 < 0. \tag{35} \]

We then have
\[ 2\mu^{-1}g_{1,2} + g_{1,1} = \omega \mu^2 \left( \lambda_2^1 - \lambda_2 - \lambda_1^1 + \lambda_1 + \omega \lambda_1^1 \lambda_2^1 \right) \]
\[ > \omega \mu^2 \left( 2\lambda_2^1 - \lambda_2 - \lambda_1 \right) > 0. \tag{36} \]

Similarly, we obtain $2\mu^{-1}g_{2,2} + g_{2,1} > 0$.

Combining the conditions discussed above, we have that the solution $(\psi_1, \psi_2) = (\psi_1^+, \psi_2^+)$ is physical for $g_{a,2} > 0$, $g_{a,1} < 0$, $g_{a,0} > 0$ and $g_{a,1}^2 - 4g_{a,2}g_{a,0} > 0$ for $a \in \{1, 2\}$. Comparing with the conditions for $(\psi_1, \psi_2) = (\psi_1^-, \psi_2^-)$, we see that, for $g_{a,0} < 0$, only one physical solution is possible.

(iii) Suppose $\lambda_b^1 < \lambda_a$ for $a \in \{1, 2\}$ and $\lambda_b^1 > \lambda_b$ for $b \in \{1, 2\}$ and $b \neq a$. We have $g_{a,2} > 0$ and $g_{b,2} < 0$. Consider the solution $(\psi_1, \psi_2) = (\psi_1^+, \psi_2^+)$. The probability constraints imply
\[ g_{a,1} < \sqrt{g_{a,1}^2 - 4g_{a,2}g_{a,0}} < 2\mu^{-1}g_{a,2} + g_{a,1}, \tag{37a} \]
\[ g_{b,1} > \sqrt{g_{b,1}^2 - 4g_{b,2}g_{b,0}} > 2\mu^{-1}g_{b,2} + g_{b,1}. \tag{37b} \]

From Eq. (37a) we must have either $g_{a,0} < 0$ or $g_{a,0} > 0$, $g_{a,1} < 0$, $g_{a,1}^2 - 4g_{a,2}g_{a,0} > 0$. The first inequality in Eq. (37b) implies $g_{b,1} > 0$ and $g_{b,0} < 0$. Now consider the second inequality in Eq. (37b). For $2\mu^{-1}g_{b,2} + g_{b,1} \leq 0$, the second inequality in Eq. (37b) holds. Otherwise if $2\mu^{-1}g_{b,2} + g_{b,1} > 0$, the second inequality implies
\[ c_{b,0} + \mu^{-1}c_{b,1} + \mu^{-2}c_{b,2} > 0 \tag{38} \]
which always holds since the left side of the above inequality equals $\mu \omega \lambda_b^1$.

Recall that, from Eq. (37a), we can have either $g_{a,0} < 0$ or $g_{a,0} > 0$, $g_{a,1} < 0$, $g_{a,1}^2 - 4g_{a,2}g_{a,0} > 0$. We can show that the latter case contradicts with the conditions $g_{b,1} > 0$ and $g_{b,0} < 0$. In particular, from
\[ g_{b,0} = \omega \mu \left( \lambda_a \left( 1 - \omega \lambda_b^1 \right) + \lambda_b \right) < 0, \tag{39} \]
we have \(1 - \omega \lambda^1_a < 0\) and similarly
\[
g_{a,0} = \omega \mu (\lambda_b (1 - \omega \lambda^1_b) + \lambda^\dagger_a - \lambda_a) > 0, \tag{40}
\]

implying \(1 - \omega \lambda^1_a > 0\). Since
\[
g_{a,1} = \omega \mu^2 \left( \lambda^1_b - \lambda_b - \lambda^1_a + \lambda_a + 2 \omega \lambda_b \lambda^1_a - \omega \lambda^\dagger_a \lambda^\dagger_b \right)
\]
\[
= \omega \mu^2 \left( \left( \lambda^\dagger_b - \lambda_b \right) \left( 1 - \omega \lambda^1_a \right) + \lambda_a - \lambda^\dagger_a \left( 1 - \omega \lambda_b \right) \right),
\tag{41}
\]

then \(g_{a,1} < 0\) implies
\[
\lambda_a - \lambda^\dagger_a \left( 1 - \omega \lambda_b \right) < 0.
\tag{42}
\]

As a result, we have \((1 - \omega \lambda_b) > \lambda_a/\lambda^1_b > 1\) and \(\omega \lambda_b < 0\), leading to a contradiction.

Summarizing the above discussions about the equilibrium points, we have that \((\psi_1, \psi_2) = (\psi^+_1, \psi^+_2)\) is physical for \(g_{a,0} < 0, g_{b,0} < 0\) and \(g_{b,1} > 0\). Note that \(g_{b,1} > 0\) is implied by the other two. Since
\[
g_{b,1} + \mu g_{b,0} = \omega \mu^2 \left( \lambda^1_a + \omega \lambda^1_b \left( \lambda_a - \lambda^\dagger_a \right) \right) > 0,
\tag{43}
\]

we have that \(g_{b,1} > 0\) always holds given \(g_{b,0} < 0\). Together, it is sufficient to have \(g_{a,0} < 0\) and \(g_{b,0} < 0\).

We consider the solution \((\psi_1, \psi_2) = (\psi^-_1, \psi^-_2)\). The probability constraints imply
\[
g_{a,1} < -\sqrt{g^2_{a,1} - 4g_{a,2}g_{a,0}} < 2\mu^{-1}g_{a,2} + g_{a,1}, \tag{44a}
\]
\[
g_{b,1} < -\sqrt{g^2_{b,1} - 4g_{b,2}g_{b,0}} > 2\mu^{-1}g_{b,2} + g_{b,1}. \tag{44b}
\]

From Eq. (44b) we have \(2\mu^{-1}g_{b,2} + g_{b,1} < 0\), giving
\[
g_{b,0} + \mu^{-1}g_{b,1} + \mu^{-2}g_{b,2} < 0. \tag{45}
\]

which cannot hold since its left side equals \(\mu \omega \lambda^1_2\). Thus, in this region, no physical solution of \((\psi_1, \psi_2) = (\psi^+_1, \psi^+_2)\) exists.

(iv.4) Suppose \(\lambda^\dagger_a < \lambda_a\) for \(a \in \{1, 2\}\), then \(c_{a,2} < 0\). For the solution \((\psi_1, \psi_2) = (\psi^-_1, \psi^-_2)\), we must have \(g_{a,1} > 0\) and \(g_{a,0} < 0\) for \(a \in \{1, 2\}\). Similar to the discussions in the case (iv.3), we have that the sufficient condition for an equilibrium point is \(g_{a,0} < 0\) for \(a \in \{1, 2\}\). The solution \((\psi_1, \psi_2) = (\psi^-_1, \psi^-_2)\) is nonphysical - see the discussion in (iv.3).

C. Stability analysis

The starting point to study the stability of the equilibrium points is the Jacobian matrix \(J\) of the 2D mean-field system, whose entries are
\[
J_{11} = -1 + \lambda^1_b \omega (1 - 2x_1) \psi_2 + \lambda_1 \omega (1 - 2x_1) \psi_1 (1 - \mu_2),
\]
\[
J_{12} = \omega \mu \left( \lambda^1_1 - \lambda_1 \right) (1 - \mu_1) \psi_1, \tag{46}
\]
\[
J_{21} = \omega \mu \left( \lambda^1_2 - \lambda_2 \right) (1 - \mu_2) \psi_2, \]
\[
J_{22} = -1 + \lambda^1_b \omega (1 - 2x_2) \psi_1 + \lambda_2 \omega (1 - 2x_2) (1 - \mu_1).
\]

We analyze the stability of the different classes of equilibrium points as discussed in Sec. IV.B.

(i) Epidemic free. For \((\psi_1, \psi_2) = (0, 0)\), the Jacobian matrix is
\[
J = \begin{pmatrix}
-1 + \lambda_1 \omega & 0 \\
0 & -1 + \lambda_2 \omega
\end{pmatrix}.
\tag{47}
\]

The equilibrium point is stable for \(\lambda_a < \omega^{-1}\) for \(a \in \{1, 2\}\).

(ii) Partial infection of epidemic 1. In this case, we have
\[
(\psi_1, \psi_2) = \left( \frac{\lambda_1 \omega - 1}{\mu \lambda_1 \omega}, 0 \right),
\tag{48}
\]

and \(J_{21} = 0\), so the Jacobian is upper triangular, whose eigenvalues are simply the diagonal entries:
\[
J_{1,1} = 1 - \lambda_1 \omega,
\]
\[
J_{2,2} = -1 + \frac{\lambda_2}{\lambda_1} + \frac{\lambda_1 \omega - 1}{\lambda_1} \tag{49}
\]

The equilibrium point is stable for \(J_{1,1} < 0\) and \(J_{2,2} < 0\), i.e.,
\[
\lambda_1 > \omega^{-1},
\]
\[
\lambda_2 - \frac{\lambda_2}{\lambda_1} - \lambda_1 + \omega \lambda_1 \lambda_2 < 0.
\tag{50}\]

(iii) Partial infection of epidemic 2. For this type of equilibrium point, we have
\[
(\psi_1, \psi_2) = \left( 0, \frac{\lambda_2 \omega - 1}{\mu \lambda_2 \omega} \right). \tag{51}
\]

It is stable under the following conditions:
\[
\lambda_2 > \omega^{-1},
\]
\[
\lambda_1 - \frac{\lambda_2}{\lambda_1} - \lambda_2 + \omega \lambda_2 \lambda_1 < 0.
\tag{52}\]

(iv) Coinfection. Suppose we have \(\psi_a \neq 0\) for \(a \in \{1, 2\}\). Substituting Eqs. (19) into Eqs. (46), the Jacobian matrix has entries
\[
J_{11} = \mu \psi_1 / (\mu \psi_1 - 1), \tag{53}
\]
\[
J_{12} = \omega \mu \left( \lambda^1_1 - \lambda_1 \right) (1 - \mu \psi_1) \psi_1,
\]
\[
J_{21} = \omega \mu \left( \lambda^1_2 - \lambda_2 \right) (1 - \mu \psi_2) \psi_2,
\]
\[
J_{22} = \mu \psi_2 / (\mu \psi_2 - 1).
\]
A necessary and sufficient condition for a two-dimensional matrix to have two negative eigenvalues is to have a negative trace \((\text{tr}(J) < 0)\) but a positive determinant \((\det(J) > 0)\). Since \(\mu \psi_a - 1 < 0\), the negativity of the trace always holds. The stability of an equilibrium point in this class is fully determined by the determinant. It is stable when \(\det(J) > 0\) and unstable when \(\det(J) > 0\). The stable condition from the determinant is

\[
\det(J) = \frac{\mu^2 \psi_a \psi_b}{(1 - \mu \psi_a)(1 - \mu \psi_b)} + \omega^2 \mu^2 (\lambda_1^a - \lambda_a) (\lambda_1^b - \lambda_b) \times (1 - \mu \psi_a)(1 - \mu \psi_b) \psi_a \psi_b > 0.
\]

Let \(z = (1 - \mu \psi_a)(1 - \mu \psi_b)\), the inequality can be written as

\[
\frac{1}{z} > \omega^2 (\lambda_1^a - \lambda_a) (\lambda_1^b - \lambda_b) z.
\]

Equations (52) can be rewritten as

\[
\lambda_1^a \omega (1 - \mu \psi_1) = 1 + (\lambda_1^a - \lambda_1) \omega z, \quad (56a)
\]

\[
\lambda_2^a \omega (1 - \mu \psi_2) = 1 + (\lambda_2^a - \lambda_2) \omega z. \quad (56b)
\]

Multiplying the above two equations, we get

\[
d_2 z^{-2} + d_1 z^{-1} + d_0 = 0, \quad (57)
\]

where

\[
d_2 = 1, \\
d_1 = \omega \left(\lambda_1^a - \lambda_1 + \lambda_2^a - \lambda_2 - \omega \lambda_1^a \lambda_2^a\right), \\
d_0 = \omega^2 \left(\lambda_1^a - \lambda_1\right) \left(\lambda_2^a - \lambda_2\right).
\]

Multiplying both sides of Eq. (57) by \(z\) and substituting the result into Eq. (55), we obtain

\[
\frac{1}{z} > -\frac{d_1}{2}.
\]

That is, an equilibrium point is stable if and only if Eq. (59) holds and is unstable otherwise. It remains to find the solutions of Eq. (57) to verify whether Eq. (59) is satisfied.

If the condition \(\lambda_1^b = \lambda_b\) holds for one or both values of \(b \in \{1, 2\}\), then \(d_0 = 0\). In this case, we have \(z^{-1} = -d_1/d_2\) and Eq. (55) implies the solution is stable for \(d_1 < 0\).

For \(\lambda_1^b \neq \lambda_b\) for any \(a \in \{1, 2\}\), from Eq. (57), we see that \(1/z\) has two solutions

\[
\left(\frac{1}{z}\right)^\pm = \frac{-d_1 \pm \sqrt{d_1^2 - 4d_2d_0}}{2d_2}.
\]

Since we have a pair of solutions for \((\psi_1, \psi_2)\) as in Eq. (24), the following hold:

\[
\left(\frac{1}{z}\right)^+ = \frac{1}{(1 - \mu \psi_1^a)(1 - \mu \psi_1^b)}, \\
\left(\frac{1}{z}\right)^- = \frac{1}{(1 - \mu \psi_2^a)(1 - \mu \psi_2^b)}.
\]

Substituting Eq. (60) into Eq. (59), we have

\[
\pm \sqrt{d_1^2 - 4d_2d_0} \geq 0.
\]

We see that, given \(d_1^2 - 4d_2d_0 > 0\), the solution \((\psi_1^+, \psi_2^+)\) is always stable, while \((\psi_1^-, \psi_2^-)\) is always unstable. It remains to check the validity of the inequality \(d_1^2 - 4d_2d_0 > 0\). After some algebra, we have

\[
d_1^2 - 4d_2d_0 = g_{1,1}^2 - 4g_{1,2}g_{1,0} = g_{2,1}^2 - 4g_{2,2}g_{2,0} = \omega \mu^2 \left(\lambda_1^a - \lambda_1 + \lambda_2^b - \lambda_2 - \omega \lambda_1^a \lambda_2^b\right)^2 + 4\omega \mu^2 \left(\lambda_1^b - \lambda_1\right) \left(\lambda_2^b - \lambda_2\right) \geq 0.
\]

Thus the inequalities \(d_1^2 - 4d_2d_0 > 0\) and \(g_{a,1}^2 - 4g_{a,2}g_{a,0} > 0\) are equivalent to each other for \(a \in \{1, 2\}\).

D. Phase diagrams

With full knowledge about the equilibrium points and their stability, we can obtain the phase diagram of the reduced mean-field system. Define the following set of functions:

\[
s_{a,0} = \lambda_b + \lambda_1^a - \lambda_a - \omega \lambda_1^a \lambda_2^b, \\
s_{a,1} = \lambda_1^b - \lambda_b + \lambda_1^a + \lambda_a - 2\omega \lambda_b \lambda_2^a - \omega \lambda_1^a \lambda_2^b, \\
s_{a,2} = \lambda_2^b - \lambda_b, \\
s_{a,3} = \lambda_a - \omega - 1, \\
s_{\Delta} = \left(\lambda_1^a - \lambda_1 + \lambda_2^b - \lambda_2 - \omega \lambda_1^a \lambda_2^b\right)^2 - 4 \left(\lambda_1^b - \lambda_1\right) \left(\lambda_2^b - \lambda_2\right) \geq 0,
\]

for \(a, b \in \{1, 2\}\) and \(a \neq b\). The distinct phase regions can be defined via various inequalities among these functions.

(i) Epidemic free. The solution \((\psi_1, \psi_2) = (0, 0)\) is stable for \(s_{a,3} < 0\) for both \(a \in \{1, 2\}\).

(ii) Partial infection of epidemic 1. The phase has a stable equilibrium point

\[
(\psi_1, \psi_2) = \left(\frac{\lambda_1 \omega - 1}{\mu \lambda_1 \omega}, 0\right).
\]

Combining the probability constraints and the stability analysis, we obtain the phase region as

\[
s_{2,0} > 0, \ s_{1,3} > 0.
\]

(iii) Partial infection of epidemic 2. The phase is characterized by

\[
(\psi_1, \psi_2) = \left(0, \frac{\lambda_2 \omega - 1}{\mu \lambda_2 \omega}\right).
\]

The phase region is given by

\[
s_{1,0} > 0, \ s_{2,3} > 0.
\]
(iv) Coinfection. In this region, there is an equilibrium point with both \( \psi_1 \) and \( \psi_2 \) nonzero, corresponding to the case of double epidemic outbreaks. For cooperative coevolution, i.e., \( \lambda_a^+ > \lambda_a \) for \( a \in \{1, 2\} \), a point in the parameter space belongs to this phase if

\[
s_{a,0} > 0, \ s_{a,1} < 0, \ s_{a,2} > 0, \ s_\Delta > 0,
\]

or

\[
s_{a,0} < 0, \ s_{a,2} > 0
\]

for both \( a \in \{1, 2\} \). When coevolution is not cooperative, the coinfection region is given by

\[
s_{a,0} < 0
\]

for both \( a \in \{1, 2\} \). We have verified that the case of \( \lambda_a = \lambda_a^+ \) for one or both \( a \in \{1, 2\} \) is well covered by this inequality.

(i \& iv). Hysteresis region 1. A hysteresis region appears when there are two stable equilibrium points and one unstable equilibrium point in between. The stability analysis indicates that the solution \((\psi_1^+, \psi_2^+)\) is always stable while \((\psi_1^-, \psi_2^-)\) is unstable. In addition to these two equilibrium points, a third stable solution is necessary for a hysteresis to arise. This is only possible when region (iv) overlaps with regions (i), (ii) and (iii). Checking the equilibrium points and their stability, we find that a hysteresis region exists only when the inequality \( \lambda_a^+ > \lambda_a \) holds for \( a \in \{1, 2\} \). The region where (i) and (iv) overlap is

\[
s_{a,0} > 0, \ s_{a,1} < 0, \ s_{a,2} > 0, \ s_{a,3} < 0, \ s_\Delta > 0, \tag{72}
\]

where the first inequality \( s_{a,0} > 0 \) can in fact be implied by the other inequalities. Since \( g_{1,1} + g_{2,1} < 0 \), we have

\[
\lambda_1^+ + \lambda_2^+ - \omega \lambda_1^+ \lambda_2^+ < 0, \tag{73}
\]

which further implies \( \omega \lambda_1^+ > 1 \) and \( \omega \lambda_2^+ > 1 \). Since \( s_{a,3} < 0 \), we have

\[
s_{a,0} = \lambda_a \left(1 - \omega \lambda_1^+ \right) + \lambda_a^+ - \lambda_a > \omega^{-1} - \lambda_a > 0. \tag{74}
\]

Altogether, the region is given by

\[
s_{a,1} < 0, \ s_{a,2} > 0, \ s_{a,3} < 0, \ s_\Delta > 0 \tag{75}
\]

for \( a \in \{1, 2\} \).

(ii \& iv). Hysteresis region 2. This region is where (ii) and (iv) overlap, which is bounded by the inequalities

\[
s_{a,0} > 0, \ s_{a,1} < 0, \ s_{a,2} > 0, \ s_{1,3} > 0, \ s_{2,3} < 0, \ s_\Delta > 0 \tag{76}
\]

for \( a \in \{1, 2\} \).

(iii \& iv). Hysteresis region 3. Similarly, the region where (iii) and (iv) overlap is bounded by

\[
s_{a,0} > 0, \ s_{a,1} < 0, \ s_{a,2} > 0, \ s_{1,3} < 0, \ s_{2,3} > 0, \ s_\Delta > 0 \tag{77}
\]

for \( a \in \{1, 2\} \).

E. Types of phase transition

A phase transition occurs when a point in the parameter space crosses a boundary between two neighboring phase regions. Depending on different combinations of phase-region pairs, the resulting phase transitions can be characteristically distinct. To be concrete, we focus on the phase transitions in the \( \lambda_1 - \lambda_2 \) plane with fixed values of \( \lambda_1^+ \) and \( \lambda_2^+ \). Both continuous and discontinuous phase transitions can arise, as we will show below.

(i) \( \iff \) (ii): We have that the equations \( s_{1,0} = 0 \) and \( s_{2,0} = 0 \) intersect at the point \( (\lambda_1, \lambda_2) = (\omega^{-1}, \omega^{-1}) \), so the two phases are separated by the line \( s_{1,3} = 0 \) in the \( \lambda_1 - \lambda_2 \) plane. When approaching the line \( s_{1,3} = 0 \) from phase (ii), the equilibrium point

\[
(\psi_1, \psi_1) = \left(\frac{\lambda_1 \omega - 1}{\mu \lambda_1 \omega}, 0\right) \tag{78}
\]

approaches \((\psi_1, \psi_1) = (0, 0)\). As a result, a continuous phase transition arises.

(ii) \( \iff \) (iii): Similar to the preceding case, the phase transition is continuous.

(ii) \( \iff \) (iv)\&(ii \& iv): The two phases are separated by the line \( s_{2,0} = 0 \). When the stable equilibrium point \((\psi_1^+, \psi_2^+)\) in Eqs. 25 approaches the line, for \( s_{2,1} > 0 \) we have

\[
(\psi_1^+, \psi_2^+) \rightarrow \left(-\frac{g_{1,2} + g_{2,1} - g_{2,2}}{g_{1,1}}, \frac{g_{2,2}}{g_{2,1}}\right), \tag{80}
\]

so the phase transition is discontinuous. It remains to discuss the sign of \( s_{2,1} \). Substituting \( s_{2,0} = 0 \) into \( s_{2,1} \), we get

\[
s_{2,1} = \lambda_1^+ + \omega \lambda_1 \lambda_2^+ - \omega \lambda_1^+ \lambda_2^+ \tag{81}
\]

First consider the case where the coevolution dynamics are not cooperative, i.e., \( \lambda_a \geq \lambda_a^+ \) for at least one of \( a \in \{1, 2\} \). In this case, the region (ii \& iv) is empty. Suppose \( \lambda_1 \geq \lambda_1^+ \), it can be immediately seen that \( s_{2,1} > 0 \). For \( \lambda_2 \geq \lambda_2^+ \), we have \( s_{2,0} > 0 \), implying \( \omega \lambda_2^+ \leq 1 \) and consequently \( s_{2,1} > 0 \).

Now consider the case of cooperative coevolution dynamics, where a point in the region (ii \& iv) satisfies \( s_{2,1} < 0 \). Further, we can prove that, if a point is in the region (iv)\&(ii \& iv), then \( s_{2,1} > 0 \). This is accomplished by showing that if a point has \( s_{2,1} < 0 \) then it must be in the region (ii \& iv). Notice that the equations \( s_{2,1} = 0 \), \( s_{2,0} = 0 \) and \( s_\Delta = 0 \) intersect at the point

\[
(\lambda_1, \lambda_2) = \left(\lambda_1^+ - \lambda_2^+, \frac{2 \lambda_1^+ + \lambda_2^+ - \omega \lambda_1^+ \lambda_2^+ - \lambda_1^+}{\omega \lambda_2^+}\right) \tag{82}
\]

in the \( \lambda_1 - \lambda_2 \) plane. Since \( s_{2,1} \) is an increasing function of \( \lambda_1 \) along \( s_{2,0} = 0 \), as can be seen from Eq. 81, we have that,
if a point in the line defined by \( s_{2,0} = 0 \) in the \( \lambda_1 - \lambda_2 \) plane has \( \lambda_1 < \lambda_1^t - \lambda_2^t/\omega \lambda_2^t \), it will satisfy \( s_{2,1} < 0 \). Furthermore, since \( \lambda_1 > \omega^{-1} \), the inequality \( \lambda_1 < \lambda_1^t - \lambda_1/\omega \lambda_2^t \) implies

\[
\lambda_1^t + 2 \lambda_2^t < \omega \lambda_1^t \lambda_2^t. \tag{83}
\]

Along the line \( s_{2,0} = 0 \), \( s_{1,0} \) can be written as

\[
s_{1,0} = (\omega \lambda_1^t \lambda_2 - \lambda_2^t - \lambda_2^t) \lambda_1 + \lambda_1^t + \lambda_2^t - \omega \lambda_1^t \lambda_2^t. \tag{84}
\]

which is an increasing function of \( \lambda \). Since the curves \( s_{1,0} = 0 \) and \( s_{2,0} = 0 \) intersect at the point \( (\lambda_1, \lambda_2) = (\omega^{-1}, \omega^{-1}) \), we have \( s_{1,0} = 0 \). We thus have \( s_{1,0} > 0 \) for \( \lambda_1 > \omega^{-1} \). Similarly, along the line \( s_{2,0} = 0 \), we have

\[
s_{1,1} = -\lambda_1^t + \omega \lambda_1 \lambda_2^t + 2 \omega \lambda_2 \lambda_1^t - \omega \lambda_1^t \lambda_2^t < 0.
\]

The first inequality is the result of \( \lambda_1 < \lambda_1^t - \lambda_1/\omega \lambda_2^t \) and the second inequality is due to the fact \( \lambda_2 < \omega^{-1} \) along the line \( s_{2,0} = 0 \) for \( \lambda_1 > \omega^{-1} \). Lastly, a point in region (ii) can always make \( s_{\Delta} > 0 \) if it is sufficiently close to the line \( s_{2,0} = 0 \).

To summarize, if a point in region (ii) has \( s_{2,1} < 0 \) near the phase boundary \( s_{2,0} = 0 \), then all the conditions under which the point is in region (ii \( \cap \) iv) hold. Thus, if the point is in the region (iv) \( \cap \) (ii \( \cap \) iv), we have \( s_{2,1} > 0 \), which makes the phase transition continuous.

(iii) \( \cap \) (iv), \( \cap \) (iv). Following a similar treatment to the preceding case, we have that the phase transition is continuous.

With discussions similar to those in the (ii \( \cap \) iv) \( \rightarrow \) (iv) case, we find that all transitions as a result of entering or leaving the hysteresis region are of the discontinuous type, due to the fact that, in the hysteresis region, the inequality \( s_{a,1} < 0 \) holds. The discontinuous transitions include (ii \( \cap \) iv) \( \rightarrow \) (iv), (ii \( \cap \) iv) \( \rightarrow \) (iv), (ii \( \cap \) iv) \( \rightarrow \) (ii), (iii \( \cap \) iv) \( \rightarrow \) (iii), (ii \( \cap \) iv) \( \rightarrow \) (ii) \( \cap \) (ii \( \cap \) iv) \( \rightarrow \) (ii) \( \cap \) (iii \( \cap \) iv) \( \rightarrow \) (i) \( \cap \) (i) \( \cap \) (i). The tri-critical points that separate the continuous from the discontinuous transition lie in the boundaries of the hysteresis region where \( q_{a,1} = 0 \) holds for either \( a \in \{1, 2\} \). One such point is given by Eq. (82). The second tri-critical point can be obtained similarly as

\[
(\lambda_1, \lambda_2) = \left( 2 \lambda_1^t + \lambda_1^t - \omega \lambda_2^t / \omega \lambda_1^t \lambda_2^t - \lambda_2^t / \omega \lambda_1^t, \lambda_2^t - \lambda_2^t / \omega \lambda_1^t \right).
\]

\[
(\lambda_1, \lambda_2) = \left( 2 \lambda_1^t + \lambda_1^t - \omega \lambda_2^t / \omega \lambda_1^t \lambda_2^t - \lambda_2^t / \omega \lambda_1^t, \lambda_2^t - \lambda_2^t / \omega \lambda_1^t \right).
\]

F. Conditions on \( \lambda_2^t \) for hysteresis

For fixed values of \( \lambda_1^t \) and \( \lambda_2^t \), a hysteresis can arise in the \( \lambda_1 - \lambda_2 \) plane. To determine these values, we first note that a hysteresis is possible only when the coevolution dynamics are cooperative, i.e., \( \lambda_1^a > \lambda_2^a \) for both \( a \in \{1, 2\} \). A point in the hysteresis region must satisfy the inequality \( g_{1,1} + g_{2,1} < 0 \). Consequently, we have

\[
\lambda_1^t + \lambda_2^t - \omega \lambda_1^t \lambda_2^t < 0, \tag{87}
\]

which provides a necessary condition for a hysteresis to arise. We can show that this is also sufficient to guarantee the occurrence of a hysteresis. In particular, suppose inequality Eq. (87) holds. Since \( s_{1,0} = 0 \) and \( s_{2,0} = 0 \) intersect at \( (\lambda_1, \lambda_2) = (\omega^{-1}, \omega^{-1}) \) in the \( \lambda_2 - \lambda_0 \) plane, there is a neighborhood near \( (\omega^{-1}, \omega^{-1}) \) in which the inequalities \( s_{1,0} > 0 \) and \( s_{2,0} > 0 \) hold. At the point \( (\lambda_a, \lambda_b) = (\omega^{-1}, \omega^{-1}) \), we have

\[
s_{a,1} = s_{a,1} = \lambda_1^t + \lambda_2^t - \omega \lambda_1^t \lambda_2^t < 0. \tag{88}
\]

It remains to check whether the inequality \( s_{\Delta} > 0 \) holds. Let \( \omega \lambda_2^t \lambda_1^t = \lambda_2^t + \lambda_2^t + \epsilon \), where \( \epsilon > 0 \) is a constant. Then at the point \( (\lambda_a, \lambda_b) = (\omega^{-1}, \omega^{-1}) \), we have

\[
s_{\Delta} = (2 \omega^{-1} + \epsilon)^2 - 4 (\epsilon \omega^{-1} + \omega^{-2}) = \epsilon > 0. \tag{89}
\]

We thus have that a hysteresis region exists in the \( \lambda_1 - \lambda_2 \) plane if and only if Eq. (87) holds.

G. Summary of the phase diagrams

Concluding the above calculations, the structure of our analytically predicted phase diagram is as follows.

(i) Epidemic free region. In this region, Eqs. (11) has the equilibrium point \( (\psi_1, \psi_2) = (0,0) \), indicating extinction of both epidemics. The solution is stable for \( s_{a,3} < 0 \) (\( a \in \{1, 2\} \)). The phase boundary \( \lambda_a = \omega^{-1} \) is also the outbreak threshold for the classic SIS model with a single epidemic.

(ii) Partial infection of epidemic 1. In this phase, epidemic 1 outbreaks and epidemic 2 becomes extinct. The equilibrium point is given by \( (\psi_1, \psi_2) = (\lambda_1 \omega - 1 / (\mu \lambda_1 \omega), 0) \), which is stable for \( s_{2,0} > 0 \) and \( s_{1,3} > 0 \), where the latter gives \( \lambda_1 > \omega^{-1} \), indicating that epidemic 1 can outbreak independently. Similarly, \( s_{2,0} > 0 \) implies \( \lambda_2 < \lambda_1 \) and \( \lambda_2^t < (\lambda_1 - \lambda_2) / (\omega \lambda_1 - 1) \), stipulating that \( \lambda_2^t \) cannot be too large to make epidemic 2 outbreak as a result of the outbreak of 1.

(iii) Partial infection of epidemic 2. Analogous to (ii), this phase is defined by \( (\psi_1, \psi_2) = (0, (\lambda_2 \omega - 1) / (\mu \lambda_2 \omega)) \), which is stable for \( s_{1,0} > 0 \) and \( s_{2,3} > 0 \).

(iv) Coinfection. In this region the reduced system has a stable equilibrium point with both \( \psi_1 \) and \( \psi_2 \) nonzero, leading to a simultaneous outbreak of two epidemics. The stable equilibrium point \( (\psi_1, \psi_2) = \psi_1 \equiv (-s_{1,1} + \sqrt{s_{1,2}}) / (2 \mu \omega \lambda_1 \lambda_2) \) and \( \psi_2 \equiv (-s_{2,1} + \sqrt{s_{2,2}}) / (2 \mu \omega \lambda_2 \lambda_2) \). For cooperative coevolution, i.e., \( \lambda_1^a > \lambda_2^a \) for \( a \in \{1, 2\} \), we have \( s_{a,2} > 0 \). A point in the parameter space belongs to this phase if it further satisfies either \( s_{a,0} > 0 \), \( s_{a,1} < 0 \), \( s_{\Delta} > 0 \), or \( s_{a,0} < 0 \) for
\(a \in \{1, 2\}\). The former case is where region (iv) overlaps with regions (i), (ii) and (iii). As a result, hystereses can arise.

For competitive or asymmetric coevolution, the coinfection region is given by \(s_{a,0} < 0\) for \(a \in \{1, 2\}\). The boundaries of this region are relatively complex. To get an intuitive picture, we consider the degenerate case of \(\lambda_1 = \lambda_2\) and \(\lambda_1^* = \lambda_2^*\).

For \(s_{a,0} < 0\), we have \(\omega \lambda_1 = \omega \lambda_2 > 1\) so that both epidemics are able to outbreak by themselves. On the contrary, for \(s_{a,0} > 0\) so that neither epidemic can outbreak by itself, we have \(s_{a,1} < 0\) and \(\lambda_1^* > 2\lambda_1\). The condition \(s_{a,2} > 0\) requires \((\omega \lambda_1^*)^2 > 4(\omega \lambda_1^2 - \omega \lambda_1^*\lambda_2^*),\) and \(\lambda_1^* > 2\lambda_2\), leading to the necessary condition \(\omega \lambda_1^* > 2\). In this case, the interaction transmission rate must at least double the threshold value of classic SIS outbreak to have coinfection.

\((i \cap iv)\) Hysteresis region 1. A hysteresis arises when there are two stable equilibrium points and one unstable equilibrium point in between, which occurs when region (iv) overlaps with regions (i), (ii) and (iii). Our analysis reveals that a hysteresis region emerges only for cooperative coevolution. The region where (i) and (iv) overlap is bounded by the inequalities \(s_{a,1} < 0, s_{a,2} > 0, s_{a,3} < 0,\) and \(s_{a} > 0\) for \(a \in \{1, 2\}\).

\((ii \cap iv)\) Hysteresis region 2. This is where (ii) and (iv) overlap. Besides the cooperative condition \(s_{a,2} > 0\), the region is nonempty if the inequalities \(s_{a,0} > 0, s_{a,1} < 0,\) \(s_{a,3} > 0, s_{a,3} < 0,\) and \(s_{a,0} > 0\) hold for \(a \in \{1, 2\}\).

\((iii \cap iv)\) Hysteresis region 3. Similarly, for cooperative coevolution with \(s_{a,2} > 0\), the region where (iii) and (iv) overlap is bounded by \(s_{a,0} > 0, s_{a,1} < 0, s_{a,3} < 0, s_{a,2} > 0,\) and \(s_{a,0} > 0\) for \(a \in \{1, 2\}\).

The types of phase transitions that occur when crossing a phase boundary are determined by checking if the stable solution varies continuously (see Sec. V). We find all possible phase transitions as a result of crossing a hysteresis region are discontinuous, whereas other transitions are continuous. A striking result revealed by our analysis of the Phase diagrams, which has not been established previously, is that the precursor of a discontinuous transition with an abrupt outbreak of at least one epidemic is a hysteresis. Continuous and discontinuous phase transitions are separated by two tri-critical points in the \(\lambda_1 - \lambda_2\) plane: \((\lambda_1, \lambda_2) = (\lambda_1^* - \lambda_1^*/(\omega \lambda_2^*), 2\lambda_1^* + \lambda_2^* - \omega \lambda_1^* \lambda_2^* - \lambda_1^*/(\omega \lambda_1^*)\) and \((\lambda_1, \lambda_2) = (2\lambda_2^* + \lambda_1^* - \omega \lambda_1^* \lambda_2^* - \lambda_1^*/(\omega \lambda_1^*), \lambda_2^* - \lambda_1^*/(\omega \lambda_1^*)\). The phase diagram also makes it possible to obtain the conditions in the interaction strengths \(\lambda_1^*\) and \(\lambda_2^*\) for a hysteresis to occur. In Sec. V we obtain the necessary and sufficient condition

\[
\lambda_1^* + \lambda_2^* - \omega \lambda_1^* \lambda_2^* < 0, \tag{90}
\]

where there is a hysteresis region with \(\lambda_1 < \lambda_1^*\) and \(\lambda_2 < \lambda_2^*\).

V. NUMERICAL RESULTS

We provide a numerical illustration of the analytic prediction on the interplay between discontinuous transitions and hystereses with an Erdős-Rényi graph of size \(N = 100\) and average degree \((k) = 4\). The inequality \((87)\) divides the \(\lambda_1 - \lambda_2\) plane into two regions, as shown in Fig. (1a). Above the curve defined by \(\lambda_1^* + \lambda_2^* - \omega \lambda_1^* \lambda_2^* = 0\), a hysteresis region appears while it is absent below. In the limit \(\lambda_0 \to \infty\), the curve approaches \(\lambda_b^* = \omega^{-1}\), as shown by the orange dashed lines in Fig. (1a). Note that the curve avoids the dashed lines for finite \(\lambda_1^*\). Since \(\omega^{-1}\) is also the outbreak threshold of the classic SIS model for a single epidemic, a necessary condition for a hysteresis is that \(\lambda_1^*\) must be larger than the classic threshold. A special case is \(\lambda_1^* = \lambda_2^*\), where \((87)\) implies that, for a hysteresis to arise, the inequality \(\lambda_1^* = \lambda_2^* > 2\omega^{-1}\) must hold.

That is, the interactive transmission rate must at least twice the classic SIS threshold for a hysteresis to arise, suggesting that networks with a larger leading eigenvalue are more prone to hystereses. Two representative phase diagrams in the \(\lambda_1 - \lambda_2\) plane with fixed values of \(\lambda_1^*\) and \(\lambda_2^*\) are shown in Figs. (1b) and (1c), corresponding to the points \(b\) and \(c\) in Fig. (1a), respectively. For point \(b\), no hysteresis can arise for any values of \((\lambda_1, \lambda_2)\) and the phase transitions between different neighboring phase regions are continuous, as indicated by the orange solid lines in Fig. (1b). For point \(c\) that is slightly above the hysteresis boundary, region (iv) overlaps with regions (i), (ii) and (iii), where a hysteresis can arise. Crossing into region (iv) from any one of the phase regions (i) \(\cap\) (iv), (ii) \(\cap\) (iv) and (iii) \(\cap\) (iv), a discontinuous outbreak transition occurs with some \(\psi_0\) changing abruptly from zero to a nonzero value. Along the path \((i \cap iv) \to (i)\), \((ii \cap iv) \to (ii)\) and \((iii \cap iv) \to (iii)\), the system displays a discontinuous transition to extinction at which at least one epidemic changes abruptly from a nonzero value to zero. All the phase boundaries with discontinuous transitions are indicated by the orange dashed lines in Fig. (1c), where the two tri-critical points separating continuous from discontinuous transitions are marked (white circles).

Are the phase diagrams obtained from the reduced mean field equations accurate in comparison with those from the original mean field equations? In the presence of the fluctuation terms \(R_0\), Eqs. (11) are exactly equivalent to Eqs. (1). Consider a system of dimension \(2N + 2\), which consists of Eqs. (1) and Eqs. (11). A stable equilibrium point \((p_{1,i}, p_{2,i})\) of the subsystem Eqs. (1) is also one for the \(2N + 2\) system with \(\psi_0 = \alpha^2 p_0\). Consider a stable equilibrium point with which neither epidemic has an outbreak. Substituting \(p_{a,1} = \cdots = p_{a,N} = 0\) and \(\psi_0 = 0\) into the remainder term (with full expression in Sec. III), we have \(R_0 = 0\) for \(a \in \{1, 2\}\). In this case the remainder terms can be ignored. Since a zero stable equilibrium point of Eqs. (1) implies the existence of exactly such a point of Eqs. (11) (with no remainder terms) and vice versa, any outbreak transition threshold from phase (i) is expected to be exact for Eqs. (1).

There are two cases where the remainder terms do not vanish and can lead to inaccuracies of the analytic prediction. The first case is when Eqs. (1) exhibit a stable equilibrium point at which there is an outbreak for epidemic 1 but extinction for epidemic 2: \(R_1 = 0\) and \(R_2 \neq 0\). The second case is when Eqs. (1) have a stable equilibrium point with an outbreak for both epidemics: \(R_0 \neq 0\) for \(a \in \{1, 2\}\). Since the remainder terms are small by construction, they lead to corrections that can be neglected, which have been verified numerically.
Especially, for the Erdős-Rényi network in Fig. 1 we solve Eqs. (1) numerically and compare the solutions with the analytic phase diagram obtained from Eqs. (11). The values of $\psi_u$ obtained from Eqs. (11) in the $\lambda_1-\lambda_2$ plane are shown in Fig. 2 for $\lambda_1^+ = 3.5\omega^{-1}$ and $\lambda_2^+ = 2.5\omega^{-1}$ [so that (57) is satisfied, guaranteeing a hysteresis]. Since in the hysteresis region there are two stable equilibrium points for each $\psi_u$, we plot separately the two solutions for $\psi_1$ in Figs. 2(a) and 2(b), and those for $\psi_2$ in Figs. 2(c) and 2(d), respectively. The phase diagram from original Eqs. (11) is also shown in Fig. 2 by the orange solid and dashed lines for continuous and discontinuous transitions, respectively. Our analytical phase diagram predicts accurately all outbreak transitions: $(i) \rightarrow (ii)$, $(i) \rightarrow (iii)$, $(i \cap iv) \rightarrow (iv)$, $(ii \cap iv) \rightarrow (iv)$ and $(iii \cap iv) \rightarrow (iv)$. However, quantitatively, the predicted extinction transitions $(i \cap iv) \rightarrow (i)$, $(ii \cap iv) \rightarrow (ii)$ and $(iii \cap iv) \rightarrow (iii)$ are less accurate, due to the nonzero remainders $R_1$ and $R_2$ as a result of the loss of stability of an equilibrium point with an outbreak for both epidemics. The values of the remainders $R_1$ and $R_2$ at equilibrium are shown in Fig. 3. Nonetheless, the predictions are qualitatively correct.

Next we consider tests and validation of our analytic prediction from Eqs. (11) for a variety of networks, including synthetic networks with strong and weak degree heterogeneity, and real-world networks. For synthetic networks, we have already shown the results for (1) ER network above. In addition, we consider networks that generated from the uncorrelated configuration model (UCM) with a power-law degree distribution $p(k) \sim k^{-\beta}$. Specifically, we consider three networks with different degree exponents: (2) PL-2.3 with $\beta = 2.3$, (3) PL-3 with $\beta = 3$ and (4) PL-4 with $\beta = 4$. For empirical networks, we have (5) Dolphins [34], a social network of bottle-nose dolphins; (6) HIV [35], a network of sexual contacts between people involved in the early spread of the human immunodeficiency virus (HIV); (7) Highschool [36], a friendship network between boys in a small high school, and (8) Jazz [37], a collaboration network between Jazz musicians. The networks are downloaded from Ref. [38].

Basic statistic of the networks considered are shown in Table I. Note that Highschool is a directed and weighted network. Here we simply take it as undirected, by assuming that there is an undirected edge between node $i$ and $j$ if there is at least a directed edge between the two nodes in either direction. Also the edges weighted are ignored.

For all the networks, we set $\lambda_1^+ = 3.5\omega^{-1}$ and $\lambda_2^+ = 2.5\omega^{-1}$ to guarantee the emergence of a hysteresis region. To have an idea of the size of the correction terms $R_a$, we also show the values of $R_a$ at equilibrium. The results of (1) ER, (2) PL-2.3 (3) PL-3, (4) PL-4, (5) Dolphins, (6) HIV, (7) Highschool and (8) Jazz are shown in Figs. 3[3] 3[4] 3[5] 3[6] 3[7] 3[8] 3[9] and 3[10] respectively. In each of the figures, subfigures (a) and (b) correspond to the values of $\psi_1$, while (e) and (f) are the corresponding values of $R_1$. Similarly, (c) and (d) correspond to the values of $\psi_2$, while (e) and (f) are the corresponding values of $R_2$.

We see that, for all the networks tested, the analytic phase diagram predicts quantitatively and accurately the outbreak transitions, while the predicted extinction transitions are qualitatively correct. The values of correction terms $R_a$ are near zero for the outbreak transitions, while have relatively larger magnitudes near extinction transitions.

VI. CONCLUSION

To summarize, we have analytically predicted the phase diagram of coevolving SIS spreading dynamics using the technique of spectral dimension reduction and provided numerical validation. The analytic phase diagram not only reveals previously known phase transitions but also elucidates the interplay between discontinuous transitions and hystereses as well as...
the emergence of tri-critical points. This method can also be applied to study other coevolving epidemic models. For general epidemic models, a one-dimensional description of each epidemics could not be sufficient, as shown in [29]. Determining the number of macroscopic observables that required for general epidemic models needs further exploration.

Our work gives a full picture that how the phase transitions depend on network topology and spreading parameters. Therefore, it lays a foundation for the intervening of coevolving spreading processes. For instance, controlling the type of phase transition by network structure perturbations, or controlling one spreading process by another coevolving process. How to apply our suggested theory to study other irreversible epidemic models (e.g., susceptible-infected-recovered, SIR model) still requires more rigorous studies.

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FIG. 3. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the Erdős-Rényi network for $\lambda_1^* = 3.5\omega^{-1}$ and $\lambda_2^* = 2.5\omega^{-1}$.

FIG. 4. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the power-law network with degree exponent $\beta = 2.3$ for $\lambda_1^* = 3.5\omega^{-1}$ and $\lambda_2^* = 2.5\omega^{-1}$.
FIG. 5. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the power-law network with degree exponent $\beta = 3$ for $\lambda_1^\ast = 3.5\omega^{-1}$ and $\lambda_2^\ast = 2.5\omega^{-1}$.

FIG. 6. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the power-law network with degree exponent $\beta = 4$ for $\lambda_1^\ast = 3.5\omega^{-1}$ and $\lambda_2^\ast = 2.5\omega^{-1}$.
FIG. 7. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the Dolphins network for $\lambda_1^* = 3.5\omega^{-1}$ and $\lambda_2^* = 2.5\omega^{-1}$.

FIG. 8. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the HIV network for $\lambda_1^* = 3.5\omega^{-1}$ and $\lambda_2^* = 2.5\omega^{-1}$.
FIG. 9. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the Highschool network for $\lambda_1^* = 3.5\omega^{-1}$ and $\lambda_2^* = 2.5\omega^{-1}$.

FIG. 10. Comparison of analytic phase diagram from the reduced mean-field system with the numerical diagram of the original mean-field system for the Jazz network for $\lambda_1^* = 3.5\omega^{-1}$ and $\lambda_2^* = 2.5\omega^{-1}$.