Ground State Entropy of the Potts Antiferromagnet on Strips of the Square Lattice

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Abstract

We present exact solutions for the zero-temperature partition function (chromatic polynomial $P$) and the ground state degeneracy per site $W$ (= exponent of the ground-state entropy) for the $q$-state Potts antiferromagnet on strips of the square lattice of width $L_y$ vertices and arbitrarily great length $L_x$ vertices. The specific solutions are for (a) $L_y = 4$, ($FBC_y, PBC_x$) (cyclic); (b) $L_y = 4$, ($FBC_y, TPBC_x$) (M"obius); (c) $L_y = 5, 6$, ($PBC_y, FBC_x$) (cylindrical); and (d) $L_y = 5$, ($FBC_y, FBC_x$) (open), where $FBC$, $PBC$, and $TPBC$ denote free, periodic, and twisted periodic boundary conditions, respectively. In the $L_x \to \infty$ limit of each strip we discuss the analytic structure of $W$ in the complex $q$ plane. The respective $W$ functions are evaluated numerically for various values of $q$. Several inferences are presented for the chromatic polynomials and analytic structure of $W$ for lattice strips with arbitrarily great $L_y$. The absence of a nonpathological $L_x \to \infty$ limit for real nonintegral $q$ in the interval $0 < q < 3$ ($0 < q < 4$) for strips of the square (triangular) lattice is discussed.

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1 Introduction

The $q$-state Potts antiferromagnet (AF) \cite{1,2} exhibits nonzero ground state entropy, $S_0 > 0$ (without frustration) for sufficiently large $q$ on a given lattice $\Lambda$ or, more generally, on a graph $G$. This is equivalent to a ground state degeneracy per site $W > 1$, since $S_0 = k_B \ln W$. Such nonzero ground state entropy is important as an exception to the third law of thermodynamics \cite{3}. There is a close connection with graph theory here, since the zero-temperature partition function of the above-mentioned $q$-state Potts antiferromagnet on a graph $G$ satisfies

$$Z(G, q, T = 0)_{PAF} = P(G, q)$$

(1.1)

where $P(G, q)$ is the chromatic polynomial expressing the number of ways of coloring the vertices of the graph $G$ with $q$ colors such that no two adjacent vertices have the same color (for reviews, see \cite{4-6}). The minimum number of colors necessary for such a coloring of $G$ is called the chromatic number, $\chi(G)$. Thus

$$W(\{G\}, q) = \lim_{n \to \infty} P(G, q)^{1/n}$$

(1.2)

where $n = v(G)$ is the number of vertices of $G$ and $\{G\} = \lim_{n \to \infty} G$. At certain special points $q_s$ (typically $q_s = 0, 1, \ldots, \chi(G)$), one has the noncommutativity of limits

$$\lim_{q \to q_s} \lim_{n \to \infty} P(G, q)^{1/n} \neq \lim_{n \to \infty} \lim_{q \to q_s} P(G, q)^{1/n}$$

(1.3)

and hence it is necessary to specify the order of the limits in the definition of $W(\{G\}, q_s)$ \cite{7}. Denoting $W_{qn}$ and $W_{nq}$ as the functions defined by the different order of limits on the left and right-hand sides of (1.3), we take $W \equiv W_{qn}$ here; this has the advantage of removing certain isolated discontinuities that are present in $W_{nq}$.

Using the expression for $P(G, q)$, one can generalize $q$ from $\mathbb{Z}_+$ to $\mathbb{C}$. The zeros of $P(G, q)$ in the complex $q$ plane are called chromatic zeros; a subset of these may form an accumulation set in the $n \to \infty$ limit, denoted $\mathcal{B}$, which is the continuous locus of points where $W(\{G\}, q)$ is nonanalytic. \cite{1} The maximal region in the complex $q$ plane to which one can analytically continue the function $W(\{G\}, q)$ from physical values where there is nonzero ground state entropy is denoted $R_1$. The maximal value of $q$ where $\mathcal{B}$ intersects the (positive) real axis is labelled $q_c(\{G\})$. This point is important since it separates the interval $q > q_c(\{G\})$ on the positive real $q$ axis where the Potts model (with $q$ extended from $\mathbb{Z}_+$ to $\mathbb{R}$) exhibits nonzero ground state entropy (which increases with $q$, asymptotically approaching $S_0 = k_B \ln q$ for

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\footnote{For some families of graphs $\mathcal{B}$ may be null, and $W$ may also be nonanalytic at certain discrete points.}
large $q$, and which for a regular lattice $\Lambda$ can be calculated approximately via large–$q$ series expansions) from the interval $0 \leq q \leq q_c(\{G\})$ in which $S_0$ has a different analytic form.

In the present work we report exact solutions for chromatic polynomials $P(G, q)$ for strips of the square lattice with arbitrarily great length $L_x$ vertices of the following types: (a) width $L_y = 4$ vertices and $(FBC_y, PBC_x) = $ cyclic; (b) $L_y = 4$ and $(FBC_y, TPBC_x) = $ Môbius, (c) $L_y = 5$ and $(PBC_y, FBC_x) = $ cylindrical; and (d) $L_y = 5$ and $(FBC_y, FBC_x) = $ open, where $FBC$, $PBC$, and $TPBC$ denote free, periodic, and twisted periodic (i.e. periodic with reversed orientation) boundary conditions, respectively. For each of these, taking the infinite-length limit, we calculate the degeneracy per site, $W(\{G\}, q)$, and the continuous nonanalytic locus $B$. A comparative discussion is given of these results together with previous exact solutions for strips of smaller widths. These strips of regular lattices are examples of recursive families of graphs, where the latter are constructed by successive additions of subgraph units to an initial subgraph.

There are several motivations for this work. We have mentioned the basic importance of nonzero ground state entropy in statistical mechanics [3]. Physical examples are provided by ice [8]-[10] and certain other hydrogen-bonded molecular crystals [11]. From the point of view of rigorous statistical mechanics, exact solutions are always valuable for the insight that they give into the behavior of the given system under study. Although infinite-length finite-width strips are quasi-one-dimensional systems and hence (for finite-range spin-spin interactions) do not have finite-temperature phase transitions, their zero-temperature critical points are of interest. Indeed, the presence of a zero-temperature critical point for the Ising antiferromagnet on infinite-length, finite-width strips of the square lattice has an interesting connection with the behavior of the singular locus $B$ for the strips that we have studied with global circuits: in these cases, this singular locus passes through the point $q = 2$ in the complex $q$ plane. Our exact solutions for $P$ and, in the $L_x \to \infty$ limit, $W$, thus show quantitatively the relation between critical behavior as a function of temperature (at $T = 0$) in the free energy and singularities as a function of $q$ in the per-site ground state degeneracy $W$. The present results also show many interesting connections with mathematical graph theory, as is clear from the identity (1.1), and algebraic geometry, as follows from the fact that for these strips, $B$ is an algebraic curve. Besides the works already cited, some related

\footnote{We recall that on bipartite graphs such as cyclic strips of the square lattice with even $L_x$, an elementary mapping shows the Ising ferromagnet and antiferromagnet to be equivalent; since the $L_x \to \infty$ limit can be taken with even $L_x$, this implies that the critical behavior of the Ising ferromagnet is equivalent to that of the Ising antiferromagnet on infinite-length limits cyclic strips of the square lattice. By similar elementary reasoning, one can show this equivalence for the infinite-length limit of Môbius strips of the square lattice.}

\footnote{A global circuit is a route following a lattice direction which has the topology of the circle, $S^1$, and a length $\ell_{g.c.}$ that goes to infinity as $n \to \infty$. For strip graphs, global circuits are equivalent to periodic or twisted periodic boundary conditions.}
work on chromatic polynomials for recursive graphs includes [12]–[37]; further discussion of background and references may be found in [33].

A generic form for chromatic polynomials for a strip graph of type $G_s$, width $L_y$, and length $L_x$ is

$$P(G_s, L_y \times L_x, BC_y, BC_x, q) = \sum_{j=1}^{N_\lambda} c_j(q)(\lambda_j(q))^{L_x}$$

where $c_j(q)$ and the $N_\lambda$ terms $\lambda_j(q)$ depend on the type of strip graph $G_s$ including the boundary conditions but are independent of $L_x$.

2 $L_y = 4$ Square-Lattice Strips with $(FBC_y, (T)PBC_x)$

In this section we give our solutions for the chromatic polynomials of the $L_y \times L_x$ strips of the square lattice with $(FBC_y, PBC_x)$ and $(FBC_y, TPBC_x)$, i.e. cyclic and Möbius, boundary conditions, respectively. For both the cyclic and Möbius strips, for $L_x \geq 3$ to avoid certain degenerate cases, the square lattice strips of width $L_y$ have $n = L_xL_y$ vertices and $e = L_x(2L_y - 1)$ edges. The cyclic square strips have $\chi = 2$ for $L_x$ even and $\chi = 3$ for $L_x$ odd, independent of $L_y$. For Möbius square strips, $\chi = 2$ for $(L_x, L_y) = (e, o)$ or $(o, e)$, and $\chi = 3$ for $(L_x, L_y) = (e, e)$ or $(o, o)$, where $e$ and $o$ denote even and odd.

We calculate the chromatic polynomials by iterated use of the deletion-contraction theorem [5], together with coloring matrix methods [14, 30]. The calculation is considerably more involved than that for the $L_y = 3$ cyclic strip given in [28], as is indicated by the number of $\lambda_j$ terms in eq. (1.4), namely, $N_\lambda = 26$, as contrasted with the value $N_\lambda = 10$ for the $L_y = 3$ cyclic strip. Elsewhere we have given a general determination of $N_\lambda$ as a function of $L_y$ [36]. As $L_y$ increases, the number of terms $N_\lambda$ in (1.4) grows rapidly; it is 70, 192, and 534 for $L_y = 5$, 6, and 7. We obtain the exact solutions of the form (1.4)

$$P(sq(4 \times m, FBC_y, PBC_x), q) = \sum_{j=1}^{26} c_{sq4,j}(\lambda_{sq4,j})^m$$

and

$$P(sq(4 \times m, FBC_y, TPBC_x), q) = \sum_{j=1}^{26} c_{sq4Mb,j}(\lambda_{sq4,j})^m$$

where $L_x = m$. The fact that the $\lambda_j$’s for a Möbius strip must be the same as those for the cyclic strip of the same width and lattice type was proved in [29]; this also proves, a fortiori, that (i) the total number, $N_\lambda$, of $\lambda_j$’s, and (ii) the continuous nonanalytic locus $B$, including the point $q_c$, are the same for the cyclic and Möbius strips of a given type. For $B$ and $q_c$,
we shall often indicate this by the notation \((FBC_y, (T)PBC_x)\). The explicit \(\lambda_{sq,j}\)'s that we calculate are as follows. The first six are

\[
\begin{align*}
\lambda_{sq,1} &= 1 \\
\lambda_{sq,2} &= 3 - q \\
\lambda_{sq,3} &= 1 - q \\
\lambda_{sq,4,5} &= (3 \pm \sqrt{2}) - q \\
\lambda_{sq,6} &= (q - 1)(q - 3)
\end{align*}
\]

and

\[
\lambda_{sq,6} = (q - 1)(q - 3).
\]

Three of the remaining \(\lambda_{sq,j}\)'s, labelled \(j = 7, 8, 9\), including the one that is dominant in region \(R_1\), are identical to the three that enter into the chromatic polynomial for the \(L_y = 4\) strip with \((FBC_y, FBC_x)\) boundary conditions, which was previously calculated in [23]. This identity was shown in [32]. The remaining \(\lambda_{sq,j}\)'s for \(10 \leq j \leq 26\) are roots of another cubic equation, for \(j = 10, 11, 12\); a quartic equation for \(13 \leq j \leq 16\); and two 5th degree equations, for \(17 \leq j \leq 26\). Since the equations defining these \(\lambda_j\)'s are somewhat lengthy, we give them in the appendix. In Table 1 we list various properties of our calculation and compare them with the properties that we have found for other related strips of the square (and triangular) lattice. The results for the \(L_y \to \infty\) limit for the triangular lattice with \((PBC_y, FBC_x)\) are from [16]. Comparisons for other lattices such as honeycomb and kagomé were given in [21, 23, 24, 28].

In particular, the fact that for this width, \(L_y = 4\) for the cyclic strip of the square lattice, we encounter equations of degree 5 for the \(\lambda_j\)'s means that it is not possible to solve for the corresponding \(\lambda_j\)'s as algebraic roots. Our experience with lattice strips of a given width \(L_y\) (and arbitrary length) and a given set of boundary conditions is that the maximal degrees of the factors in the general equation for the \(\lambda_j\)'s are non-decreasing functions of \(L_y\). Thus, assuming that this property of non-decreasing degrees of algebraic factors in the equation for the \(\lambda_j\)'s continues for higher \(L_y\), our present results indicate that the exact solutions in [28] for the \(\lambda_j\)'s for the width \(L_y = 3\) strip of the square lattice have completed the program of obtaining exact algebraic expressions for these terms for this type of lattice strip.

Although no closed-form algebraic expression can be obtained for the \(\lambda_j\)'s, a theorem on symmetric polynomials of roots of algebraic equations, discussed in [29], enables one to calculate the chromatic polynomials exactly to arbitrary order. The key to this is the property that since the chromatic polynomial for a cyclic strip is a symmetric polynomial in the various roots, it can be expressed in terms of the coefficients of the algebraic equations...
Table 1: Properties of $P$, $W$, and $B$ for strip graphs $G_s$ of the square (sq) and triangular (tri) lattices. New results in this work are marked with an asterisk in the first column. The properties apply for a given strip of type $G_s$ of size $L_y \times L_x$; some apply for arbitrary $L_x$, such as $N_\lambda$, while others apply for the infinite-length limit, such as the properties of the locus $B$. For the boundary conditions in the $y$ and $x$ directions ($BC_y$, $BC_x$), $F$, $P$, and $T$ denote free, periodic, and orientation-reversed (twisted) periodic, and the notation (T)P means that the results apply for either periodic or orientation-reversed periodic. The column denoted eqs. describes the numbers and degrees of the algebraic equations giving the $\lambda_{G_s,j}$: for example, $\{3(1),2(2),1(3)\}$ indicates that there are 3 linear equations, 2 quadratic equations and one cubic equation. The column denoted BCR lists the points at which $B$ crosses the real $q$ axis; the largest of these is $q_c$ for the given family $G_s$. The notation “none” in this column indicates that $B$ does not cross the real $q$ axis. The notation “int; $q_1$; $q_c$” refers to cases where $B$ contains a real interval, there is a crossing at $q_1$, and the right-hand endpoint of the interval is $q_c$. Column labelled “SN” refers to whether $B$ has support for negative $Re(q)$, indicated as yes (y) or no (n).

| $G_s$ | $L_y$ | $BC_y$ | $BC_x$ | $N_\lambda$ | eqs. | BCR | SN |
|-------|-------|--------|--------|-------------|------|-----|----|
| sq    | 1     | F      | F      | 1           | \{1(1)\} | none | n  |
| sq    | 2     | F      | F      | 1           | \{1(1)\} | none | n  |
| sq    | 3     | F      | F      | 2           | \{1(2)\} | 2    | n  |
| sq    | 4     | F      | F      | 3           | \{1(3)\} | int; 2.265; 2.30 | n |
| **sq**| 5     | F      | F      | 7           | \{1(7)\} | 2.43 | n  |
| sq    | 1     | F      | P      | 2           | \{2(1)\} | 2, 0 | n  |
| sq    | 2     | F      | (T)P   | 4           | \{4(1)\} | 2, 0 | n  |
| sq    | 3     | F      | (T)P   | 10          | \{5(1),1(2),1(3)\} | 2.34, 2, 0 | y |
| **sq**| 4     | F      | (T)P   | 26          | \{4(1),1(2),2(3),1(4),2(5)\} | 2.49, 2, 0 | y |
| sq    | 3     | P      | F      | 1           | \{1(1)\} | none | n  |
| sq    | 4     | P      | F      | 2           | \{1(2)\} | int; 2.30; 2.35 | n |
| **sq**| 5     | P      | F      | 2           | \{1(2)\} | 2.69 | n  |
| **sq**| 6     | P      | F      | 5           | \{1(5)\} | 2.61 | y  |
| sq    | 3     | P      | F      | 8           | \{8(1)\} | 3, 2, 0 | n |
| sq    | 3     | P      | TP     | 5           | \{5(1)\} | 3, 2, 0 | n  |
| tri   | 2     | F      | F      | 1           | \{1(1)\} | none | –  |
| tri   | 3     | F      | F      | 2           | \{1(2)\} | 2.57 | n  |
| tri   | 4     | F      | F      | 4           | \{1(4)\} | none | n  |
| tri   | 5     | F      | F      | 9           | \{1(9)\} | 3    | n  |
| tri   | 2     | F      | (T)P   | 4           | \{2(1),1(2)\} | 3, 2, 0 | n |
| tri   | 3     | F      | (T)P   | 10          | \{3(1),2(2),1(3)\} | 3, 2, 0 | n |
| tri   | 4     | F      | P      | 26          | \{1(1),2(4),1(8),1(9)\} | 3.23, 3, 2, 0 | y |
| tri   | 3     | P      | F      | 1           | \{1(1)\} | none | n  |
| tri   | 4     | P      | F      | 2           | \{1(2)\} | 4, 3.48 | n |
| tri   | 5     | P      | F      | 2           | \{1(2)\} | 3.28, 3.21 | n |
| tri   | ∞     | P      | F      | –           | –         | 4, 3.82, 0 | y |
| tri   | 3     | P      | P      | 11          | \{5(1),3(2)\} | 3.72, 2, 0 | n |
| tri   | 3     | P      | TP     | 5           | \{5(1)\} | 3.72, 2, 0 | n |
that determine these $\lambda_j$’s. However, the fact that it is no longer possible to calculate the $\lambda_j$’s as algebraic roots when the width of the cyclic square strip is 4 means that the determination of the nonanalytic locus $B$ must be done in a somewhat more cumbersome manner than in our previous work where we had exact algebraic expressions for these $\lambda_j$’s.

The coefficients $c_j$ that enter into the expressions for the chromatic polynomial (1.4) for the cyclic and Möbius strip of the square lattice of width $L_y$ are certain polynomials that we denote $c^{(d)}$, given by

$$c^{(d)} = \prod_{k=1}^{d} (q - q_{d,k})$$

(2.8)

where

$$q_{d,k} = 2 + 2 \cos \left( \frac{2\pi k}{2d+1} \right) \text{ for } k = 1, 2, ... d$$

(2.9)

with $0 \leq d \leq L_y$. We list below the specific $c^{(d)}$’s that appear in our results for the $L_y = 4$ square lattice strip:

$$c^{(0)} = 1 \ , \quad c^{(1)} = q - 1 \ , \quad c^{(2)} = q^2 - 3q + 1 \ ,$$

(2.10)

$$c^{(3)} = q^3 - 5q^2 + 6q - 1 \ ,$$

(2.11)

and

$$c^{(4)} = (q - 1)(q^3 - 6q^2 + 9q - 1) \ .$$

(2.12)

In ascending order of degrees of $c^{(d)}$, we calculate

$$c_{sq4,j} = c^{(0)} \text{ for } 6 \leq j \leq 9$$

(2.13)

$$c_{sq4,j} = c^{(1)} \text{ for } 13 \leq j \leq 21$$

(2.14)

$$c_{sq4,j} = c^{(2)} \text{ for } 10 \leq j \leq 12 \text{ and } 22 \leq j \leq 26$$

(2.15)

$$c_{sq4,j} = c^{(3)} \text{ for } 2 \leq j \leq 5$$

(2.16)

and

$$c_{sq4,1} = c^{(4)} \ .$$

(2.17)

In [32] it was shown that the coefficient for the $\lambda_j$ that is leading in region $R_1$ must be 1. We define

$$C(G) = \sum_{j=1}^{N_{\lambda_G}} c_{G,j} \ .$$

(2.18)
where the \( G \)-dependence in the coefficients is indicated explicitly. Note that for recursive graphs like the strip graphs considered here, the \( c_{G,j} \) depend on \( L_y \) and the boundary conditions, but not on \( L_x \). Our results above give

\[
C(G) = q(q - 1)^3 \quad \text{for} \quad G = sq(L_y = 4, FBC_y, PBC_x).
\]

in accord with the generalization \[28, 29\]

\[
C(G_s(L_y \times L_x, FBC_y, PBC_x), q) = P(T_{L_y}, q) = q(q - 1)^{L_y-1}
\]

for \( G_s \) a strip of the square (or triangular) lattice, where \( P(T_{n}, q) \) is the chromatic polynomial for the tree graph \( T_n \). This is in accord with the coloring matrix approach \[31, 30\].

For the \( L_y = 4 \) Möbius strip of the square lattice, we find

\[
c_{sq4Mb,j} = c^{(0)} \quad \text{for} \quad 7 \leq j \leq 12
\]

\[
c_{sq4Mb,j} = -c^{(0)} \quad \text{for} \quad j = 6 \quad \text{and} \quad 22 \leq j \leq 26
\]

\[
c_{sq4Mb,j} = c^{(1)} \quad \text{for} \quad 17 \leq j \leq 21
\]

\[
c_{sq4Mb,j} = -c^{(1)} \quad \text{for} \quad j = 1 \quad \text{and} \quad 13 \leq j \leq 16
\]

\[
c_{sq4Mb,j} = c^{(2)} \quad \text{for} \quad j = 4, 5
\]

and

\[
c_{sq4Mb,j} = -c^{(2)} \quad \text{for} \quad j = 2, 3.
\]

Hence, the sum of the coefficients is

\[
C(G) = 0 \quad \text{for} \quad G = sq(L_y = 4, FBC_y, TPBC_x)
\]

in accord with the general result for the Möbius strip of the square (and triangular) lattice \[36\]

\[
\sum_{j=1}^{N_{xG}} c_{G(L_y,Mb),j} = \begin{cases} 
P(T_{L_y+1}, q) & \text{for odd } L_y \\
0 & \text{for even } L_y 
\end{cases}
\]

Chromatic zeros for the cyclic strip of the square lattice with \( L_y = 4, L_x = m = 20 \) and hence \( n = 80 \) are shown in Fig. \ref{fig:chromatic_zeros}; with this value of \( m \), the complex chromatic zeros lie close to the boundary \( B \) and give an approximate indication of its position. Note that there is a zero very close to \( q = 2 \), but \( P(sq(L_y \times L_x, FBC_y, PBC_x), q) \) is nonzero for \( q = 2 \) for the case shown, where \( L_x = m \) is even, as is clear from the fact that \( \chi = 2 \) in this case.
Figure 1: Chromatic zeros for the $L_y = 4$, $L_x = m = 20$ (i.e., $n = 80$) cyclic strip of the square lattice.

The maximal point at which $B$ crosses the real axis, $q_c$, is determined as a solution of the equation of degeneracy of leading terms $|\lambda_{eq7-9,\text{max}}| = |\lambda_{eq22-26,\text{max}}|$, where $\lambda_{eq7-9,\text{max}}$ and $\lambda_{eq22-26,\text{max}}$ are the roots of eqs. (8.1.1) and (8.1.5) with the largest magnitudes, respectively. Since only two $\lambda_j$’s are degenerate in magnitude at this point, it is a regular point on the algebraic curve $B$ in the terminology of algebraic geometry. This is also the case for the $L_y = 3$ (and $L_y = 1$) strip of the square lattice [28, 29], whereas, in contrast, $q_c$ is a multiple point on $B$ for $L_y = 2$. We find

$$q_c \approx 2.4928456 \quad \text{for} \quad sq(4 \times \infty, FBC_y, (T)PBC_x)$$

(2.29)

This may be compared with the values $q_c = 2$ for the $L_y \times \infty$ strip of the square lattice with $L_y = 1, 2$ and the same $(FBC_y, (T)PBC_x)$ boundary conditions [4], and the value $q_c \approx 2.33654$ for $L_y = 3$ [28]. We calculate that $W(sq, 4 \times \infty, FBC_y, BC_x) = 1.2697336$ at the value $q = q_c$ in eq. (2.29).

The locus $B$ also crosses the real $q$ axis at $q = 2$ and at $q = 0$. In addition to region $R_1$ which extends outward from the envelope of $B$ and includes the real axis for $q > q_c$ and $q < 0$, there are two other regions that contain segments of the real axis: $R_2$, including the interval $2 < q < q_c$ and $R_3$, including the interval $0 < q < 2$. In region $R_1$, the dominant $\lambda_j$...
is the root of the cubic equation (8.1.1) with the largest magnitude, which we label $\lambda_{7-9,\text{max}}$. In region $R_2$, the dominant $\lambda_j$ is the root of the fifth-degree equation (8.1.3) with the largest magnitude, which we label $\lambda_{22-26,\text{max}}$. In region $R_3$, the dominant $\lambda_j$ is the root of the fifth-degree equation (8.1.4) with the largest magnitude, which we label $\lambda_{17-21,\text{max}}$. We have

$$W = (\lambda_{7-9,\text{max}})^{1/4}, \quad \text{for} \quad q \in R_1$$

$$|W| = |\lambda_{22-26,\text{max}}|^{1/4}, \quad \text{for} \quad q \in R_2$$

$$|W| = |\lambda_{17-21,\text{max}}|^{1/4}, \quad \text{for} \quad q \in R_3$$

(In regions other than $R_1$, only the magnitude $|W|$ can be determined unambiguously [4].)

The locus $B$ has support for $\Re(q) < 0$ as well as $\Re(q) \geq 0$. It separates the $q$ plane into several regions, including the three described above and two complex-conjugate ones which we denote $R_4$ and $R_4^*$, centered approximately at $q \simeq 2.6 \pm 0.8i$. In the regions $R_4$ and $R_4^*$, we have

$$|W| = |\lambda_{13-16,\text{max}}|^{1/4}, \quad \text{for} \quad q \in R_4, R_4^*$$

Just as complex-conjugate pairs of tiny sliver regions were found for the cyclic $L_y = 3$ square [28] and triangular [35] strips, so also these may be present here; we have not carried out a search for such regions (but have ruled out the possibility of tiny regions on the real axis).

### 3 $L_y = 5, 6$ Square-Lattice Strips with $(PBC_y, FBC_x)$

Here we report our exact solutions for the chromatic polynomials for the width $L_y = 5, 6$ strips of the square lattice of arbitrary length and with $(PBC_y, FBC_x)$, i.e., cylindrical, boundary conditions. Results for the cases $L_y = 3$ and $L_y = 4$ were given previously in [24, 26, 32]. We recall that $N_\lambda = 1$ for $L_y = 3$ and $N_\lambda = 2$ for $L_y = 4$. For $L_y = 5$ and $L_y = 6$ we calculate $N_\lambda = 2$ and $N_\lambda = 5$, respectively. As before, it is convenient to present the results in terms of a generating function, denoted $\Gamma(G_s, q, x)$. The chromatic polynomial $P((G_s)_m, q)$ is determined as the coefficient in a Taylor series expansion of this generating function in an auxiliary variable $x$ about $x = 0$:

$$\Gamma(G_s, q, x) = \sum_{m=0}^{\infty} P((G_s)_m, q)x^m.$$  

The generating function $\Gamma(G_s, q, x)$ is a rational function of the form

$$\Gamma(G_s, q, x) = \frac{N(G_s, q, x)}{D(G_s, q, x)}$$  

9
with

$$\mathcal{N}(G_s, q, x) = \sum_{j=0}^{d_N} A_{G_s,j}(q)x^j$$

(3.3)

and

$$\mathcal{D}(G_s, q, x) = 1 + \sum_{j=1}^{d_D} b_{G_s,j}(q)x^j$$

(3.4)

where the $A_{G_s,i}$ and $b_{G_s,i}$ are polynomials in $q$, and $d_N \equiv \text{deg}_x(\mathcal{N})$, $d_D \equiv \text{deg}_x(\mathcal{D})$. In factorized form

$$\mathcal{D}(G_s, q, x) = \prod_{j=1}^{d_D} (1 - \lambda_{G_s,j}(q)x) .$$

(3.5)

Equivalently, the $\lambda_{G_s,j}$ are roots of the equation

$$\xi^{d_D} \mathcal{D}(G_s, q, 1/\xi) = \xi^{d_D} + \sum_{j=1}^{d_D} b_{G_s,j}\xi^{d_D-j}.$$  

(3.6)

The general formula expressing $P(G_m, q)$ in terms of these quantities is [23]

$$P(G_m, q) = \sum_{j=1}^{d_D} \left[ \sum_{s=0}^{d_N} A_s \lambda_j^{d_D-s-1} \right] \left[ \prod_{1 \leq i \leq d_D; i \neq j} \frac{1}{(\lambda_j - \lambda_i)} \right] \lambda_j^m .$$

(3.7)

For $L_y = 5$ we find

$$\lambda_{sq5PF,j} = \frac{1}{2} \left[ T_{sq5PF} \pm \sqrt{R_{sq5PF}} \right], \quad j = 1, 2$$

(3.8)

where

$$T_{sq5PF} = q^5 - 10q^4 + 46q^3 - 124q^2 + 198q - 148$$

(3.9)

and

$$R_{sq5PF} = q^{10} - 20q^9 + 188q^8 - 1092q^7 + 4356q^6 - 12596q^5 + 27196q^4 - 44212q^3 + 52708q^2 - 41760q + 16456 .$$

(3.10)

The coefficients $c_{sq5PF,j}$ can be computed using eq. (3.7) in terms of the generating function, which is given in the appendix.

In the $L_x \to \infty$ limit, the locus $\mathcal{B}$ includes five arcs, consisting of two complex-conjugate pairs and a fifth, self-conjugate, arc. The endpoints of these arcs are located at the five
Figure 2: Locus $B$ for the width $L_y = 5$ strip (tube) of the square lattice with $(PBC_y, FBC_x)$ boundary conditions. Thus, the cross sections of the tube form pentagons. For comparison, chromatic zeros calculated for the strip length $L_x = m + 2 = 16$ (i.e., $n = 80$ vertices) are shown.
Figure 3: Locus $\mathcal{B}$ for the width $L_y = 6$ strip (tube) of the square lattice with $(PBC_y, FBC_x)$ boundary conditions. Thus, the cross sections of the tube form hexagons. For comparison, chromatic zeros calculated for the strip length $L_x = m + 2 = 16$ (i.e., $n = 96$ vertices) are shown.
complex-conjugate pairs of roots of $R_{sq5PF}$. The self-conjugate arc crosses the real axis at the real zero of $T_{sq5PF}$, namely at

$$q_c \simeq 2.691684 \quad \text{for} \quad sq(5 \times \infty, PBC_y, FBC_x)$$  \hspace{1cm} (3.11)

In Fig. 3 we show a plot of chromatic zeros for the $L_y = 6$ strip of the square lattice with $(PBC_y, FBC_x)$ and length $L_x = m + 2 = 16$ vertices, so that the strip has $n = 96$ vertices in all. With this large a value of $m$, the complex chromatic zeros lie close to the boundary $B$ and give an approximate indication of its position. (We have not searched for very minute features in $B$.) From our exact analytic results, we calculate indication of its position. From our exact analytic results, we calculate

$$q_c \simeq 2.6089 \quad \text{for} \quad sq(6 \times \infty, PBC_y, FBC_x)$$  \hspace{1cm} (3.12)

The morphology of chromatic zeros for this long $6 \times 16$ cylindrical strip is similar to that found for a $8 \times 8$ patch of the square lattice, again with cylindrical boundary conditions, in [16]. In both cases, the chromatic zeros have support for $Re(q) < 0$ and prongs extending to the right; further, our exact calculation shows that in the limit $L_x \to \infty$ with $L_y = 6$, the locus $B$ has support for $Re(q) < 0$. In Fig. 3 one of the chromatic zeros is very close to $q = 2$, but for $q = 2$ exactly, the chromatic polynomial is nonzero, equal to 2, in accord with the fact that this strip is bipartite for any value of $L_x$.

For the $L_x \to \infty$ limit of these respective strips we have

$$W(sq(5 \times \infty, PBC_y, FBC_x), q) = (\lambda_{sq5PF,j,\text{max}})^{1/5}$$  \hspace{1cm} (3.13)

and

$$W(sq(6 \times \infty, PBC_y, FBC_x), q) = (\lambda_{sq6PF,j,\text{max}})^{1/6}$$  \hspace{1cm} (3.14)

where $\lambda_{sq5PF,j,\text{max}}$ and $\lambda_{sq6PF,j,\text{max}}$ denote the solutions to the respective equations (3.6) with maximal magnitude in region $R_1$.

It is of interest to use this exact result to study further the approach of $W$ to the limit for the full infinite 2D square lattice. This extends our previous study in [26]. In Table 2 we list various values of $W(sq(L_y \times \infty, PBC_y, BC_x), q)$ (which, for this range of $q$, are independent of $BC_x$), denoted as $W(sq(L_y), P, q)$, together with Monte Carlo measurements of $W$ for the full 2D square lattice, $W(sq, q)$ from [21] and the $q = 3$ value $W(sq, 3) = (4/3)^{3/2}$ from [9]. We also list the ratio

$$R_W(\Lambda(L_y), BC_y, q) = \frac{W(\Lambda(L_y), BC_y, q)}{W(\Lambda, q)}$$  \hspace{1cm} (3.15)

for the present square lattice $\Lambda = sq$. One sees that for $L_y = 5$ and moderate values of $q$, say 5 or 6, the agreement of $W(sq(L_y), q)$ for the infinite-length, finite-width strips with
Table 2: Values of $W(sq(L_y \times \infty, PBC_y, BC_x, q)$ (for any $BC_x$), denoted $W(sq(L_y), P, q)$ for short, with $W(sq, q) = W(sq(\infty \times \infty), q)$ for $3 \leq q \leq 10$. For each value of $q$, the quantities in the upper line are identified at the top and the quantities in the lower line are the values of $R_W(sq(L_y), PBC_y, q)$.

| $q$ | $W(sq(3), P, q)$ | $W(sq(4), P, q)$ | $W(sq(5), P, q)$ | $W(sq(6), P, q)$ | $W(sq, q)$ |
|-----|------------------|------------------|------------------|------------------|-------------|
| 3   | 1.25992          | 1.58882          | 1.43097          | 1.56168          | 1.53960(\ldots) |
|     | 0.8183           | 1.032            | 0.9294           | 1.014            | 1            |
| 4   | 2.22398          | 2.37276          | 2.31865          | 2.34339          | 2.3370(7)    |
|     | 0.9516           | 1.015            | 0.9921           | 1.0027           | 1            |
| 5   | 3.17480          | 3.26878          | 3.24518          | 3.25196          | 3.2510(10)   |
|     | 0.9766           | 1.0055           | 0.9982           | 1.0003           | 1            |
| 6   | 4.14082          | 4.21082          | 4.19790          | 4.20058          | 4.2003(12)   |
|     | 0.9858           | 1.002505         | 0.9994           | 1.0001           | 1            |
| 7   | 5.11723          | 5.17377          | 5.16557          | 5.16689          | 5.1669(15)   |
|     | 0.9904           | 1.0013           | 0.9997           | 1.0000           | 1            |
| 8   | 6.10017          | 6.14792          | 6.14221          | 6.14296          | 6.1431(20)   |
|     | 0.9930           | 1.0008           | 0.9999           | 1.0000           | 1            |
| 9   | 7.08734          | 7.12881          | 7.12458          | 7.12506          | 7.1254(22)   |
|     | 0.9947           | 1.0005           | 0.9999           | 1.0000           | 1            |
| 10  | 8.07737          | 8.11409          | 8.11083          | 8.1111           | 8.1122(25)   |
|     | 0.9957           | 1.0002           | 0.9998           | 0.9999           | 1            |

the respective values $W(sq, q)$ for the infinite square lattice is excellent; the differences are of order $10^{-3}$ to $10^{-4}$. As noted before [26], for $PBC_y$ (and any $BC_x$) this approach is not monotonic.

4 $L_y = 5$ Square-Lattice Strips with $(FBC_y, FBC_x)$

We have also gone beyond the previous studies in [23, 24] to calculate the chromatic polynomial for the strip of the square lattice with width $L_y = 5$ and $(FBC_y, FBC_x)$, i.e., open, boundary conditions. A related study on wide strips is in [37]. In [23], a given strip $(G_s)_m$ was constructed by $m$ successive additions of a subgraph $H$ to an endgraph $I$; here, $I = H$, so that, following the notation of [23], the total length of the strip graph $(G_s)_m$ is $L_x = m + 2$ vertices, or equivalently, $m + 1$ edges in the longitudinal direction. The results are conveniently expressed in terms of the coefficient functions in the generating function, as discussed above.

For the width $L_y = 5$ strip of the square lattice we find $deg_x(D) = N_\lambda = 7$. The coefficient
functions \( b_{sq5FF,j} \) in eq. (3.4) that determine the \( \lambda_{sq5FF,j} \)'s via eq. (3.6) are listed in the appendix. Because the \( A_{sq5FF,j} \)'s (cf. eq. (3.3)) are quite lengthy, we do not give them here\(^4\). In Table 1 this result is compared with the findings from the previous calculations in [23, 24] for narrower open strips of the square lattice, and with strips of the triangular lattice [28, 35]. One observes that the equation (3.6) defining the \( \lambda_j \)'s increases in degree as \( L_y \) increases for the open strips. In particular, because we now encounter an equation (3.6) of degree higher than 4 (specifically, degree 7), it is not possible to solve for the \( \lambda_j \)'s as algebraic roots. Furthermore, assuming that this increase in degree of (3.6) continues for greater widths \( L_y \) of open strips, our present results show that the previous calculations of the \( \lambda_j \)'s in [23] up to \( L_y = 4 \) have completed the program of calculating these terms exactly as algebraic roots for open strips of the square lattice. As noted above, because of the theorem on symmetric polynomial functions of roots an algebraic equations [29], one can still calculate the chromatic polynomial in terms of the coefficients of the algebraic equation for the \( \lambda_j \)'s.

![Figure 4: Chromatic zeros for the \( L_y = 5 \) open strip of the square lattice of length \( L_x = m + 2 = 16 \) vertices (i.e. total number of vertices \( n = 80 \).)](image)

In Fig. 4 we show a plot of chromatic zeros for the open strip of the square lattice with

\(^4\)The \( A_{sq5FF,j} \) are listed in the copy of this paper in the cond-mat archive.
$L_y = 5$ and length $L_x = m + 2 = 16$ vertices, so that the strip has $n = 80$ vertices in all. With this large a value of $m$, the complex chromatic zeros lie close to the boundary $\mathcal{B}$ and give an approximate indication of its position. From an analysis of the degeneracy of leading $\lambda_j$’s, we find that (in the $L_x \to \infty$ limit where $\mathcal{B}$ is defined)

$$q_c \simeq 2.42843 \quad \text{for} \quad sq(5 \times \infty, FBC_y, FBC_x) \quad (4.1)$$

This is in agreement with the chromatic zeros shown in Fig. 4. Comparing Fig. 4 with the corresponding plots for $L_y = 2$ and $L_y = 3$ (Fig. 3(a,b) of [23]), we see that the arcs forming $\mathcal{B}$ are elongating and that the arc endpoints nearest to the origin are approaching more closely to the origin. This agrees with the behavior that we had observed earlier from narrower strips and with the conclusions that were drawn from that behavior [23, 24, 25, 32], in particular, the statement that these results are consistent with, and provide further support for, the hypothesis that in the limit as $L_y \to \infty$, the locus $\mathcal{B}$ will extend all the way through the origin of the $q$ plane and will separate this plane into different regions containing the real axis.

For $q > q_c$, we have, for the physical ground state degeneracy per site of the $q$-state Potts antiferromagnet,

$$W(sq(5 \times \infty, FBC_y, FBC_x), q) = (\lambda_{sq5FF,j,max})^{1/5} \quad (4.2)$$

where $\lambda_{sq5FF,j,max}$ denotes the solution of eq. (3.6) with the coefficients (8.3.1)-(8.3.7) that has the maximal magnitude in region $R_1$.

As with the cylindrical strips, we can use our new exact solution for the $L_y = 5$ open square strip to study the approach of $W$ to the limit for the infinite 2D square lattice, extending [26]. In Table 3 we list various values of $W(sq(L_y \times \infty, FBC_y, BC_x), q)$ (which, for this range of $q$, are independent of $BC_x$), denoted as $W(sq(L_y), F, q)$, together with Monte Carlo measurements of $W$ for the full 2D square lattice, $W(sq, q)$ from [21] and the $q = 3$ value $W(sq, 3) = (4/3)^{3/2}$ from [9]. We also list the ratio $R_W(sq(L_y), FBC_y, q)$ defined in (3.15). In [26] it was proved that for $FBC_y$ the approach of $W$ to the $L_y = \infty$ limit is monotonic. One sees from Table 3 that for $L_y = 5$ and moderate values of $q$, say 5 or 6, the agreement of $W(sq(L_y), q)$ for the infinite-length, finite-width strips with the respective values $W(sq, q)$ for the infinite square lattice is very good, accurate to a few per cent, although the approach is somewhat slower for open strips than for cylindrical strips. This is understandable since the condition of periodic boundary conditions in the transverse direction minimizes finite-size effects in this direction.
Table 3: Values of $W(sq(L_y \times \infty, FBC_y, BC_x, q)$ (for any $BC_x$), denoted $W(sq(L_y), F, q)$ for short, with $W(sq, q) = W(sq(\infty \times \infty), q)$ for $3 \leq q \leq 10$. For each value of $q$, the quantities in the upper line are identified at the top and the quantities in the lower line are the values of $RW(sq(L_y), FBC_y, q)$.

| $q$ | $W(sq(1), F, q)$ | $W(sq(2), F, q)$ | $W(sq(3), F, q)$ | $W(sq(4), F, q)$ | $W(sq(5), F, q)$ | $W(sq, q)$ |
|-----|-----------------|-----------------|-----------------|-----------------|-----------------|------------|
| 3   | 2.000           | 1.73205         | 1.65846         | 1.624945        | 1.60597         | 1.53960..  |
|     | 1.299           | 1.125           | 1.077           | 1.055           | 1.043           |            |
| 4   | 3.000           | 2.64575         | 2.53800         | 2.48590         | 2.45517         | 2.3370(7)  |
|     | 1.284           | 1.132           | 1.086           | 1.064           | 1.051           |            |
| 5   | 4.000           | 3.60555         | 3.48304         | 3.42336         | 3.38805         | 3.2510(10) |
|     | 1.230           | 1.109           | 1.071           | 1.053           | 1.042           |            |
| 6   | 5.000           | 4.58258         | 4.45136         | 4.38717         | 4.34910         | 4.2003(12) |
|     | 1.190           | 1.091           | 1.060           | 1.0445          | 1.035           |            |
| 7   | 6.000           | 5.56776         | 5.43073         | 5.36348         | 5.32353         | 5.1669(15) |
|     | 1.161           | 1.078           | 1.051           | 1.038           | 1.030           |            |
| 8   | 7.000           | 6.55744         | 6.41623         | 6.34677         | 6.30545         | 6.1431(20) |
|     | 1.1395          | 1.067           | 1.0445          | 1.033           | 1.026           |            |
| 9   | 8.000           | 7.54983         | 7.40548         | 7.33434         | 7.29199         | 7.1254(22) |
|     | 1.123           | 1.060           | 1.039           | 1.029           | 1.023           |            |
| 10  | 9.000           | 8.54400         | 8.39720         | 8.324745        | 8.28157         | 8.1122(25) |
|     | 1.109           | 1.053           | 1.035           | 1.026           | 1.021           |            |
5 Comparative Discussion on $P$ and $B$

In this section we give a general discussion of some properties of (i) the chromatic polynomials for cyclic lattice strips with both arbitrarily great length and arbitrarily great width, and (ii) the loci $B$ for the infinite-length limit of strips of the square and triangular lattice with various boundary conditions. This discussion incorporates the exact solutions given in the present work and also in our previous papers.

1. From our exact solutions for cyclic and Möbius strips of the square and triangular lattices, we draw the following inference: for these lattice strips, with arbitrary $L_y$ (independent of $L_x$), the $\lambda_{G,s,j}$ in eq. (1.4) with the highest-degree $c^{(d)}$, namely $c^{(L_y)}$ (see eq. (2.8)), is

$$\lambda_{G,s,1} = (-1)^L_y$$

(5.1)

2. A second inference concerns the set of terms $\lambda_{G,s,j}$ for the cyclic strips of the square and triangular lattice with coefficients $c_{G,s,j} = c^{(L_y-1)}$. Let $\bar{\lambda}_{G,s,L_y,j} = (-1)^{L_y}\lambda_{G,s,L_y,j}$ for $G_s = sq, tri$. Then the $\bar{\lambda}_{sq,L_y,j}$'s and hence the $\lambda_{sq,L_y,j}$'s with coefficients $c_{sq,L_y,j} = c^{(L_y-1)}$ can be calculated as follows. Denote the equation whose solution is $\bar{\lambda}_{sq,L_y,j}$ as $f(sq, L_y, \xi)$. Thus, $f(sq, 1, \xi) = \xi + (q - 1)$ and

$$f(sq, 2, \xi) = f(sq, 1, \xi)(\xi + q - 3)\ .$$

(5.2)

The $\lambda_{sq,L_y,j}$'s for higher values of $L_y$ are then given by

$$f(sq, L_y, \xi) = f(sq, L_y - 1, \xi)(\xi + q - 3) - f(sq, L_y - 2, \xi)\quad\text{for } L_y \geq 3\ .$$

(5.3)

We find that in the chromatic polynomial for the cyclic strip of the square lattice, of the $\lambda_{sq,j}$'s with coefficient $c_{sq,j} = c^{(L_y-1)}$, (i) one is $\lambda_{sq,j} = (-1)^{L_y}(1 - q)$; (ii) if $L_y$ is even, then another is $3 - q$; (iii) if $L_y = 0 \mod 3$, two others are $(-1)^{L_y}(2 - q)$ and $(-1)^{L_y}(4 - q)$; (iv) if $L_y = 0 \mod 6$, then two others are $3 \pm \sqrt{3} - q$. (This is not an exhaustive list of special factors.)

For cyclic strips of the triangular lattice, denote the equation whose solution is $\bar{\lambda}_{tri,L_y,j}$ as $f(tri, L_y, \xi)$. Thus, $f(tri, 1, \xi) = \xi + (q - 1)$ and

$$f(tri, 2, \xi) = f(tri, 1, \xi)(\xi + q - 3) - (\xi - 1)$$

(5.4)

The $\lambda_{tri,L_y,j}$'s for higher values of $L_y$ are then given by

$$f(tri, L_y, \xi) = f(tri, L_y - 1, \xi)(\xi + q - 3) - \xi f(tri, L_y - 2, \xi)\quad\text{for } L_y \geq 3\ .$$

(5.5)
The equations defining the $\lambda_{\text{tri},L_y,j}$'s involve progressively higher degrees in $\xi$.

It was shown in [29] that the $\lambda_{G_s,j}$'s are the same for the cyclic and Möbius strips of a given lattice with width $L_y$. Therefore, for each width $L_y$, the $\lambda_{G_s,j}$'s identified above also occur in the respective Möbius strips of the square and triangular lattices, although they do not, in general, have the same coefficients $c_{G_s,j}$.

These inferences are important because they show how one can reduce the problem of calculating the $\lambda_{G_s,j}$'s for larger-width strips graphs of cyclic and Möbius type from those for lower widths without recourse to the usual iterative application of the deletion-contraction or coloring matrix methods. That is, after having used these latter methods to obtain the chromatic polynomials for the first few values of $L_y$, the rest can be obtained purely algebraically, without further direct analysis of the graphs involved. Work on constructing the recursive formulas for the other $\lambda_{G_s,j}$'s is currently in progress.

3. For the infinite-length limit of a given strip graph $G_s$, the dominant $\lambda_{G_s,j}$ in region $R_1$ is independent of the longitudinal boundary condition, and its coefficient is $c^{(0)} = 1$ [32]. In particular, this $\lambda_{G_s,j}$ is the same for $(FBC_y,T)PBC_x$ and $(FBC_y,FBC_x)$ boundary conditions. When this $\lambda_{G_s,j}$ is the root of an algebraic equation of degree higher than linear, then, for the theorem on symmetric functions of roots of algebraic equations to apply and guarantee the polynomial nature of $P(G_s, q)$, it is necessary and sufficient that all of the other roots of this equation enter with the same coefficient [29]. Hence, the analysis given in [36] for cyclic strips of type $G_s$ that determines the number of $\lambda_{G_s,j}$'s with a specified $c_{G_s,j} = c^{(d)}$ places, for $d = 0$, an upper bound on the number $N_\lambda$ of $\lambda_{G_s,j}$'s that occur in the strip of type $G_s$ with $(FBC_x, FBC_y)$ boundary conditions. In particular, for $L_y$ from 1 through 8, this number $n_P(L_y, 0)$ takes the values 1, 1, 2, 4, 9, 21, 51, 127. For the square lattice, one finds, for $L_y$ from 1 through the current results presented here for $L_y = 5$, the values $N_\lambda = 1, 1, 2, 3, 7$, as listed in Table 1. For the strips of the triangular lattice, this upper bound is realized as an equality: for $L_y$ from 1 to 5, the open strips have $N_\lambda = 1, 1, 2, 4, 9$. The reason that the inequality is realized as an equality for the strips of the triangular lattice is a consequence of the different behavior of the coefficients of the square and triangular lattice strips in the Möbius case [36].

4. For all of the strips of the square lattice containing global circuits that we have studied, the locus $B$ encloses regions of the $q$ plane including certain intervals on the real axis and passes through $q = 0$ and $q = 2$ as well as other possible points, depending on the family. Note that the presence of global circuits is a sufficient, but not necessary,
condition for $B$ to enclose regions, as was shown in [24] (see Fig. 4 of that work). Our present results for the square lattice are in accord with, and strengthen the evidence for, the inference (conjecture) [32, 33] that

$$B \supset \left\{ q = 0, 2 \right\}$$

for $sq(L_y, FBC_y, (T)PBC_x) \quad \forall L_y \geq 1$

and $sq(L_y, PBC_y, (T)PBC_x) \quad \forall L_y \geq 3$.  \hspace{1cm} (5.6)

(For the upper line of this equation, note that the $L_y = 1$ graphs with $(FBC_y, TPBC_x)$ and $(FBC_y, PBC_x)$ boundary conditions are identical.)

5. The crossing of $B$ at the point $q = 2$ for the (infinite-length limit of) strips with global circuits nicely signals the existence of a zero-temperature critical point in the Ising antiferromagnet (equivalent to the Ising ferromagnet on bipartite graphs). This has been discussed in [33] in the context of exact solutions for finite-temperature Potts model partition functions on the $L_y = 2$ cyclic and M"obius strips of the square lattice. In contrast, this connection is not, in general, present for strips with free longitudinal boundary conditions since $B$ does not, in general, pass through $q = 2$. (Of the strips with $FBC_x$ that we have studied so far, such a crossing at $q = 2$ was only found for the $L_y = 3$ ($FBC_y, FBC_x$) case, as one can see from Table [1].) Furthermore, for the strips without global circuits, there is no indication of any motion of the respective loci $B$ toward $q = 2$ as $L_y$ increases.

6. Our exact solutions show that in the limit as $L_x \to \infty$, the respective loci $B$ for the $W$ functions for the infinite strips of the square lattice with the fixed values of $L_y$ considered and with (i) periodic or twisted periodic longitudinal boundary conditions and (ii) free longitudinal boundary conditions differ; in particular, the loci $B$ for cases with (i) pass through $q = 2$, whereas the loci for cases with (ii) do not. This dependence of $B$ on the boundary conditions means that an $n \to \infty$ limit does not exist in a manner independent of these boundary conditions. If one fixes $q = 2$ at the outset, i.e. considers the Ising antiferromagnet on the square-lattice strips (or if one fixes $q$ to the trivial value $q = 1$ at the outset) and then calculates $W$, there are no pathologies; these arise when one considers nonintegral real $q$ in the range $0 < q < 3$. This was already discussed in the more general context of the full temperature-dependent free energy for the Potts antiferromagnet in [33], together with other pathologies such as a negative partition function (lack of Gibbs measure), noted earlier in [20], and negative specific heat. In general, the the conclusion is that a nonpathological $n \to \infty$ limit of the antiferromagnetic Potts model fails to exist at sufficiently low temperature and
sufficiently small real nonintegral positive \( q \) on strips of the square lattice. Since these strips are of fixed width, the \( L_x \to \infty \) limit may be considered to be effectively quasi-one-dimensional; in contrast, a true two-dimensional thermodynamic limit would be \( L_x \to \infty, \ L_y \to \infty \), with the ratio \( L_y/L_x \) a finite nonzero constant in this limit. However, as is clear from the random cluster representation of the Potts model, the problem of a negative partition function (lack of Gibbs measure) for sufficiently small positive real nonintegral \( q \) is present for both infinite-length, finite width strips and for the above two- or higher-dimensional infinite volume limit \([20, 33]\). Our exact results for infinite-length strips of various widths and our inference above that in the \( L_y \to \infty \) limit, the loci \( B \) and \( W \) functions obtained with periodic (or twisted periodic) versus free longitudinal boundary conditions would differ is in connected with the other pathologies noted above. From the analysis in \([35]\), we also conclude that a nonpathological \( L_x \to \infty \) limit for the antiferromagnetic Potts model fails to exist at sufficiently low temperature and sufficiently small positive nonintegral \( q \) on strips of the triangular lattice and a nonpathological thermodynamic limit fails to exist at sufficiently low temperature for nonintegral \( 0 < q < 4 \) for the full triangular lattice. One could infer a generalization of this for other lattices also: a thermodynamic limit would fail to exist for the Potts antiferromagnet at sufficiently low temperature for positive nonintegral \( q \) in the range from 0 to \( q_c \) for the given 2D lattice, e.g., \( q_c = 3 \) for the kagomé lattice. Our exact solutions are consistent with the understanding that the point \( q_c \) for the infinite 2D (or higher-dimensional) lattice is independent of the boundary conditions used to define this infinite lattice.

7. For cyclic strips, we note a correlation between the coefficient \( c_{G_s,j} \) of the respective dominant \( \lambda_{G_s,j} \)'s in regions that include intervals of the real axis. Before, it was shown \([22]\) that the \( c_{G_s,j} \) of the dominant \( \lambda_{G_s,j} \) in region \( R_1 \) including the real intervals \( q > q_c(\{G\}) \) and \( q < 0 \) is \( c^{(0)} = 1 \), where the \( c^{(d)} \) were given in eqs. \((2.8), (2.9)\). We observe further that the \( c_{G_s,j} \) that multiplies the dominant \( \lambda_{G_s,j} \) in the region containing the intervals \( 0 < q < 2 \) is \( c^{(1)} \). For the cyclic \( L_y = 3 \) and \( L_y = 4 \) strips, there is also another region containing an interval \( 2 \leq q \leq q_c \) on the real axis, where \( q_c \approx 2.34 \) and 2.49 for \( L_y = 3, 4 \); in this region, we find that the \( c_{G_s,j} \) multiplying the dominant \( \lambda_{G_s,j} \) is \( c^{(2)} \).

8. Our new results on cylindrical and open strips with \( L_y = 5 \) confirm and extend various features that had been discussed earlier \([23, 24, 25]\); for these values of \( L_y \), \( B \) forms arcs, and as \( L_y \) increases, these arcs elongate and move closer together, with the arc endpoints nearest to the origin moving toward this point. This is consistent with the
inference that in the $L_y \to \infty$ limit, the arcs would close to form a closed boundary that contained $q = 0$ and $q = q_c(sq) = 3$. One sees this general trend in the $L_y = 6$ cylindrical strip (Fig. 3). However, in contrast with the strip graphs containing global circuits, for which the loci $B$ contained a region-enclosing boundary passing through $q = 0$ for any $L_y$, this feature is evidently only approached in the limit as $L_y \to \infty$ for the strips that do not contain global circuits. The earlier calculations of cylindrical strips of the triangular lattice showed an example of a strip, namely the $L_y = 4$ case, where $B$ contains arcs and an self-conjugate oval on the real axis \cite{24}, but for the cylindrical strips of the square lattice that we have investigated so far, we have not yet encountered such an oval.

9. For the $L_x \to \infty$ limit of all of the strips of the square lattice containing global circuits, a $q_c$ is defined, and our results for the cyclic and Möbius strips with widths from $L_y = 1$ through $L_y = 4$ indicate that $q_c$ is a non-decreasing function of $L_y$ in these cases. The same behavior was found for the strips of the triangular lattice with $L_y = 2$ \cite{28} (and subsequently also $L_y = 3, 4$ \cite{35}). This motivated the inference (conjecture) that $q_c$ is a non-decreasing function of $L_y$ for strips of regular lattices with $(FBC_y, (T)PBC_x)$ boundary conditions \cite{32}, and our present results strengthen the support for this inference. Given that, as $L_y \to \infty$, $q_c$ reaches a limit, which is the $q_c$ for the 2D lattice of the specified type (square, triangular, etc.), this inference leads to the following inequality:

$$q_c(\Lambda, L_y \times \infty, BC_y, (T)PBC_x) \leq q_c(\Lambda). \quad (5.7)$$

Our exact solutions show that this inequality can be saturated. For example, $q_c = 3$ for the $L_y = 3$ torus and Klein bottle strip of the square lattice \cite{31}, which is equal to the $q_c$ value for the infinite 2D square lattice \cite{1}. In contrast, for (the $L_x \to \infty$ limit of) strips without global circuits, the locus $B$ does not necessarily cross the real axis, and hence there is not necessarily any $q_c$ defined, as was shown in \cite{23}. Furthermore, in these cases, even if $B$ does cross the real axis, so that a $q_c$ is defined, the value of $q_c$ is not a non-decreasing function of $L_y$. This is shown by our calculations of $B$ for the $L_y = 4, L_y = 5$, and $L_y = 6$ strips of the triangular lattice with cylindrical boundary conditions in \cite{24}; for these we get $q_c = 4$ for $L_y = 4$ but $q_c = 3.28$ for $L_y = 5$ and $q_c = 3.25$ for $L_y = 6$. Similarly, for the $L_y = 5$ and $L_y = 6$ cylindrical strips of the square lattice we get $q_c = 2.69$ and $q = 2.61$, respectively.

10. A generalized conjecture would be to consider a slab of a $d$-dimensional lattice $\Lambda$ of size $L_1 \times L_2 \times ... \times L_d$, and let $d - 1$ of the lengths of this slab go to infinity, holding
one length, which can be chosen without loss of generality to be $L_d$, fixed and finite, and to define $W$ via (5.2) as

$$W(\Lambda, L_d \times \infty^{d-1}, BC_1, ..., BC_d, q) = \lim_{L_j \to \infty, \ j=1,...,d-1} P(\Lambda, L_1 \times ... \times L_d, BC_1, ..., BC_d, q)^{1/n} \tag{5.8}$$

For each of these $W$ functions, one would consider the corresponding continuous singular locus $B$ and its $q_c$, for choices of the $BC_j$ and $L_d$ where this point exists. We display the dependence of $q_c$ on these inputs by writing it as $q_c(\Lambda_d, L_d \times \infty^{d-1}, BC_1, ..., BC_d)$. Next, we define a $W$ function for the $d$-dimensional lattice as

$$W(\Lambda_d, BC_1, ..., BC_d, q) = \lim_{L_j \to \infty, \ j=1,...,d} P(\Lambda, L_1 \times ... \times L_d, BC_1, ..., BC_d, q)^{1/n} \tag{5.9}$$

and a corresponding singular locus and $q_c(\Lambda_d)$. As indicated in the notation, one expects that this $q_c$ would be independent of the $BC_j$, $j = 1, ..., d$ just as is the case for the exactly known $q_c$ values for certain 2D lattices. Then we conjecture the inequality

$$q_c(\Lambda_d, L_d \times \infty^{d-1}, BC_1, ..., BC_d) \leq q_c(\Lambda_d) \tag{5.10}$$

Similarly, a generalization of our inference that $q_c$ is a non-decreasing function of $L_y$ for the strips with $(FBC_y, (T)PBC_x)$ would be the conjecture that $q_c(\Lambda_d, L_d \times \infty^{d-1}, (T)PBC_1, ..., (T)PBC_{d-1}, FBC_d)$ is a non-decreasing function of $L_d$. Our exact solutions for strips with $(PBC_y, FBC_x)$ boundary conditions show that if one uses periodic rather than free boundary conditions in the direction in which the slab is finite, then the resultant $q_c$ is not, in general, a non-decreasing function of $L_d$.

11. Our exact solutions for the $L_y = 4$ cyclic and Möbius strips of the square lattice yield a singular locus $B$ that has support for $Re(q) < 0$. In comparison (see Table 3), this was also true for the same type of strip with $L_y = 3$, while for $L_y = 1, 2$, $B$ only had support for $Re(q) \geq 0$, and the only point on $B$ with $Re(q) = 0$ was $q = 0$ itself. This shows that for a given type of strip, increasing $L_y$ can shift the left-most chromatic zeros and, in the $L_x \to \infty$ limit, the left-most portion of the locus $B$ into the $Re(q) < 0$ half-plane. The same type of behavior was found for the cyclic and Möbius strips of the triangular lattice; for $L_y = 2$ and $L_y = 3$, $B$ and chromatic zeros had support only for $Re(q) \geq 0$, while for $L_y = 4$, this support extended into the $Re(q) < 0$ region. In [23] it was conjectured that global circuits were a necessary condition for lattice strips to have chromatic zeros and, in the limit $L_x \to \infty$, a locus $B$ with support for $Re(q) < 0$. However, this conjecture was ruled out by our exact solutions for
chromatic polynomials, $W$, and $\mathcal{B}$ for homeomorphic expansions\footnote{We recall two definitions from graph theory: (i) a homeomorphic expansion of a graph is obtained by inserting one or more degree-2 vertices on edge(s) of the graph; (ii) the girth of a graph is the number of edges or vertices in a minimum-distance circuit.} of lattice strips with $(FBC_y, FBC_x)$ boundary conditions in \cite{25}, as also by the results for lattice strips with $(PBC_y, FBC_x)$ in \cite{16}. The homeomorphic expansions in \cite{25} have the effect of increasing the girth of these strip graphs, and it was found that for a given type of open strip graph, increasing the degree of homeomorphic expansion and hence the girth shifts the left-most chromatic zeros and, in the limit $L_x \to \infty$, the left-most portion of $\mathcal{B}$, farther to the left. This is thus a different way of getting chromatic zeros and part of $\mathcal{B}$ to have support for $Re(q) < 0$ than in the present case of cyclic strips, where this result is obtained as a consequence of increasing the width of the strip while the girth remains constant. We remark that for all of these families of graphs, the magnitudes of the chromatic zeros and points $q$ on $\mathcal{B}$ are bounded. Yet another way to get chromatic zeros and $\mathcal{B}$ with negative real parts involves families with unbounded chromatic zeros and loci $\mathcal{B}$; indeed, in \cite{38} we constructed families where these zeros and loci $\mathcal{B}$ had arbitrarily large negative $Re(q)$.

There have been a number of theorems proved concerning real chromatic zeros. An elementary result is that no chromatic zeros can lie on the negative real axis $q < 0$, since a chromatic polynomial has alternating coefficients. It has also been proved that there are no chromatic zeros in the intervals $0 < q < 1$, and $1 < q < 32/27$ \cite{40}. The bound of $32/27$ in \cite{40} has been shown to be sharp; i.e., for any $\epsilon > 0$, there exists a graph with a chromatic zero at $q = 32/27 + \epsilon$ \cite{11}. Based on our studies of strips of the square (and triangular) lattices with all of the various boundary conditions considered, we make the following observation: for such strips, we have not found any chromatic zeros, except for the zero at $q = 1$, in the interior of the disk $|q - 1| = 1$. This motivates the conjecture that for these strips, there are no chromatic zeros with $|q - 1| < 1$ except for the zero at $q = 1$. Assuming that this conjecture is valid, the bound would be a sharp bound, since the circuit graph with $n$ vertices, $C_n$, has chromatic zeros lying precisely on the circle $|q - 1| = 1$ and at $q = 1$ \cite{7}.

## 6 Values of $W$ for Low Integral Values of $q$

In previous works \cite{7, 26, 35} and sections of the present paper we have discussed values of $W$ for various infinite-length, finite-width lattice strips. For infinite-length limits of strips with global circuits, where the region(s) of the positive real axis in the interval $0 < q < q_c$
are not analytically connected with the region $R_1$ including $q > q_c$ (and $q < 0$), the ground state degeneracy per site, $W$, has a qualitatively different behavior than for integer or real $q \geq q_c$. A comparative discussion of this was given in [7] with the results available at that time, and it is worthwhile to use our new exact solutions to study this behavior further. In particular, it is of interest to inquire what the values of the $W$ functions are for the infinite-length limits of various strips of the square lattice at the points $q = 0, 1, \text{ and } 2$. Our exact analytic expressions yield the numerical values listed in Table 4. The notation follows that in Table 1. As was noted in [7], in general, for regions other than $R_1$, it is only possible to determine $|W|$ unambiguously. Hence, for uniformity, we list $|W|$ for all of the strips, including those with only a region $R_1$. For comparison, we also include values of $|W|$ at $q = 0, 1, \text{ and } 2$ for infinite-length strips of the triangular lattice in Table 5. In addition, for families where, in the $L_x \to \infty$ limit, there exists a $q_c$, we include the respective values of $W$ at $q_c$. For the smallest widths, the $|W|$ values are relatively simple analytic expressions, e.g., for the square strips, $|W| = 3^{3/2}$ for $(FBC_y, BC_x)$, $L_y = 2, q = 0$; $|W| = 13^{3/3}$ for $(PBC_y, BC_x), L_y = 3, q = 0$, and so forth. In the case of the triangular lattice, $L_y = \infty$, $(PBC_y, FBC_x)$, the values of $|W|$ for $q = 0$ and $q = 4$ are from [14]; the exact value for $q = 3$ is our analytic evaluation, and the numerical values for $q = 1, 2$ are our numerical evaluations, of an integral representation in [16]. Although we list the values in the tables only to three significant figures, we note that the $q = 1$ value, $|W(\text{tri})| \simeq 3.1716$, is different from the $q = 0$ value $|W(\text{tri}, 3 \times \infty, PBC_y, BC_x)| \simeq 3.1748$.

For these values of $q$, the noncommutativity of eq. (1.3) occurs [7]. Thus, for any connected graph $G$, and in particular, the lattice strips considered here, the chromatic polynomial $P(G, q)$ vanishes at $q = 0$ and $q = 1$ and hence also the function $W_{nq}$ defined via the order of limits on the right-hand side of eq. (1.3) vanishes. In contrast, in general, $W_{qn}$ defined by the limits on the left-hand side of (1.3) is nonzero. For cyclic strips of the square lattice of length $L_x$, at $q = 2$, the chromatic polynomial $P$ is equal to 2 if $L_x$ is even but 0 if $L_x$ is odd, so that at $q = 2$, no $W_{nq}$ is defined, since the limit on the right-hand side of (1.3) does not exist; however, $W_{qn}$ is well-defined and, in general, nonzero. Analogous comments apply for strips of the triangular lattice: at $q = 2$, the chromatic polynomial $P$ vanishes identically, so $W_{nq} = 0$, but $W_{qn}$ is, in general, nonzero. For cyclic strips of the triangular lattice, at $q = 3$, then $P = 3!$ if $L_x = 0 \text{ mod } 3$, and $P = 0$ if $L_x = 1$ or $2 \text{ mod } 3$; hence, no $W_{nq}$ is defined, since the limit on the right-hand side of (1.3) does not exist, but $W_{qn}$ is well-defined and, in general, nonzero. As with the other results given in this paper, the values of $W$ given in Tables 4 and 5 follow the definition $W \equiv W_{qn}$.

Some general comments follow:
Table 4: Values of $W(sq, L_y \times \infty, BC_y, BC_x, q)$ for low integral $q$ and for respective $q_c$, if such a point exists, where $BC_y$ and $BC_x$ denote the transverse and longitudinal boundary conditions.

| $L_y$ | $BC_y$ | $BC_x$ | $|W_{q=0}|$ | $|W_{q=1}|$ | $|W_{q=2}|$ | $q_c$ | $W_{q=q_c}$ |
|-------|--------|--------|-------------|-------------|-------------|-------|-------------|
| 1     | F      | F      | 1           | 0           | 1           | n     | –           |
| 2     | F      | F      | 1.73        | 1           | 1           | n     | –           |
| 3     | F      | F      | 2.06        | 1.44        | 1           | 2     | 1           |
| 4     | F      | F      | 2.24        | 1.68        | 1.25        | 2.30  | 1.14        |
| 5     | F      | F      | 2.36        | 1.82        | 1.39        | 2.43  | 1.22        |
| 1     | F      | P      | 1           | 1           | 1           | 2     | 1           |
| 2     | F      | (T)P   | 1.73        | 1.41        | 1           | 2     | 1           |
| 3     | F      | (T)P   | 2.06        | 1.66        | 1.26        | 2.34  | 1.18        |
| 4     | F      | (T)P   | 2.24        | 1.81        | 1.41        | 2.49  | 1.27        |
| 3     | P      | F      | 2.35        | 1.59        | 1           | n     | –           |
| 4     | P      | F      | 2.58        | 1.89        | 1.32        | 2.35  | 1.16        |
| 5     | P      | F      | 2.68        | 2.05        | 1.51        | 2.69  | 1.15        |
| 6     | P      | F      | 2.73        | 2.14        | 1.62        | 2.61  | 1.39        |
| 3     | P      | (T)P   | 2.35        | 1.91        | 1.44        | 3     | 1.26        |

Table 5: Values of $W(tri, L_y \times \infty, BC_y, BC_x, q)$ for low integral $q$ and for respective $q_c$, if such a point exists, where $BC_y$ and $BC_x$ denote the transverse and longitudinal boundary conditions.

| $L_y$ | $BC_y$ | $BC_x$ | $|W_{q=0}|$ | $|W_{q=1}|$ | $|W_{q=2}|$ | $|W_{q=3}|$ | $q_c$ | $W_{q=q_c}$ |
|-------|--------|--------|-------------|-------------|-------------|-------------|-------|-------------|
| 2     | F      | F      | 2           | 1           | 0           | 1           | n     | –           |
| 3     | F      | F      | 2.49        | 1.66        | 1           | 2           | 2.57  | 0.656       |
| 4     | F      | F      | 2.77        | 1.41        | 1           | n           | –     | –           |
| 5     | F      | F      | 2.95        | 2.25        | 1.66        | 1           | 3     | 1           |
| 2     | F      | (T)P   | 2           | 1.62        | 1           | 1           | 3     | 1           |
| 3     | F      | (T)P   | 2.49        | 1.64        | 1           | 1           | 3     | 1           |
| 4     | F      | (T)P   | 2.77        | 1.68        | 1.21        | 3.23        | 1.13  | –           |
| 3     | P      | F      | 3.17        | 2.22        | 1.26        | 1           | n     | –           |
| 4     | P      | F      | 3.44        | 2.26        | 1.78        | 0           | 4     | 1.19        |
| 5     | P      | F      | 3.56        | 2.29        | 2.05        | 1.15        | 3.28  | 0.772       |
| 6     | P      | F      | 3.63        | 2.90        | 2.21        | 1.41        | 3.25  | 1.11        |
| ∞     | P      | F      | 3.77        | 3.17        | 2.60        | 2           | 4     | 1.46        |
| 3     | P      | (T)P   | 3.17        | 2.62        | 2           | 1.71        | 3.72  | 1.41        |
1. As is evident in Tables 4 and 5 for values of \( q \) that are positive but sufficiently small, for a given lattice, boundary conditions, and value of \( L_y \) studied, \(|W|\) is a non-increasing function of \( q \). In contrast, for sufficiently large \( q \), \(|W|\) increases with \( q \). For families of graphs that involve global circuits, these two different types of behavior occur, respectively, for \( 0 < q < q_c \) and \( q > q_c \). The latter behavior is the one expected for the \( q \)-state Potts antiferromagnet, since increasing \( q \) increases the physical ground state entropy. As examples, for the \((L_x \rightarrow \infty \) limit of\) circuit graph, \( W \) is constant for \( 0 \leq q \leq 2 \), while for the cyclic or Möbius strip of the square lattice \( L_y = 2 \), it decreases as \(|W| = |3 - q|\) in this interval; in both of these cases, \( q_c = 2 \) and \( W \) is real and increasing for \( q > q_c \). For the cyclic or Möbius \( L_y = 3 \) and \( L_y = 4 \) strips of the square lattice, \(|W|\) has a different analytic form in the interval \( 0 \leq q \leq 2 \) and \( 2 \leq q \leq q_c \) but is everywhere decreasing for \( 0 < q < q_c \), for the respective values of \( q_c \). As an example of a strip with no \( q_c \), for the open line, \( L_y = 1 \), \(|W|\) decreases from 1 to 0 as \( q \) increases from 0 to 1 and increases for larger \( q \). As another example of a strip with no \( q_c \), for the \( L_y = 3 \) strip of the square lattice with \((PBC_y, FBC_x)\) boundary conditions, \(|W|\) decreases monotonically as \( q \) increases from 0 and vanishes at \( q \approx 2.453 \); for larger values of \( q \), \( W \) is real and positive and increases with \( q \).

2. For the \( L_x \rightarrow \infty \) limit of strips with free transverse boundary conditions, \( FBC_y \) and any longitudinal boundary conditions \( BC_x \), it was proved that for a fixed physical \( q \geq q_c \), \( W \) is a monotonically decreasing function of \( L_y \) \([23]\). However, as is evident from Tables 4 and 5 for sufficiently small positive values of \( q \) (smaller than \( q_c \) for strips with a \( q_c \)), \(|W|\) is a non-decreasing function of \( L_y \).

3. For the strips that we have studied whose \( L_x \rightarrow \infty \) limit yields a locus \( B \) with a \( q_c \), \(|W(q)|\) for fixed \( q \in [0, q_c] \) is a non-decreasing function of \( L_y \).

4. It has been shown that for physical values of \( q \) in the \( q \)-state Potts antiferromagnet, in the \( L_x \rightarrow \infty \) limit of a strip of a given type of lattice \( \Lambda \), \( W(\Lambda, L_y \times \infty, BC_y, BC_x, q) \) is independent of the longitudinal boundary condition \( BC_x \) \([32]\). However, for small positive values of \( q \), \(|W|\) does depend on both \( BC_y \) and \( BC_x \), as is evident from Tables 4 and 5. One observes that for the small integral values of \( q \) shown in these tables, \(|W(\Lambda, L_y \times \infty, FBC_y, FBC_x, q)| \leq |W(\Lambda, L_y \times \infty, FBC_y, (T)PBC_x, q)| \) and \(|W(\Lambda, L_y \times \infty, PBC_y, FBC_x, q)| \leq |W(\Lambda, L_y \times \infty, PBC_y, (T)PBC_x, q)|\) .

5. It was observed in \([7]\) and proved in (section 7 of) \([22]\) that for integer, and, by analytic continuation, real, values of \( q > max(q_c) \) for the square and triangular lattice strips, i.e., \( q \geq 4 \), \( W(tri, q) < W(sq, q) \). Most of the values of \(|W|\) shown in Tables 4 and 5...
show the opposite inequality. Together with various other properties noted above, this shows that \(|W|\) behaves qualitatively differently for sufficiently small positive values of \(q\) than for larger values.

7 Conclusions

In conclusion, we have presented exact solutions of the zero-temperature partition function (chromatic polynomial \(P\)) and the ground state degeneracy per site \(W\) (= exponent of the ground-state entropy) for the \(q\)-state Potts antiferromagnet on strips of the square lattice of width \(L_y\) vertices and arbitrarily great length \(L_x\) vertices. The specific solutions were for (a) \(L_y = 4\), \((FBC_y, PBC_x)\) (cyclic); (b) \(L_y = 4\), \((FBC_y, TPBC_x)\) (Möbius); (c) \(L_y = 5, 6\), \((PBC_y, FBC_x)\) (cylindrical); and (d) \(L_y = 5\), \((FBC_y, FBC_x)\) (open), where \(FBC, PBC,\) and \(TPBC\) denote free, periodic, and twisted periodic boundary conditions, respectively.

Some inferences were given for certain terms \(\lambda_{G_z,j}\) for cyclic and Möbius strip graphs of the square and triangular lattice that allow one to calculate them for arbitrarily wide strips (of any length). These are important because they show how one can reduce the problem of calculating the \(\lambda_{G_z,j}\)'s for these strips of arbitrarily large width from those for lower widths without recourse to the usual iterative application of the deletion-contraction or coloring matrix methods. A comparative discussion was given of the continuous nonanalytic locus \(B\) for these strips and numerical results of \(W\) were given for a range of values of \(q\). In general, our exact solutions give further insight into the properties of the Potts antiferromagnet in the setting of infinite-length, finite width systems.

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8 Appendix

8.1 Terms \(\lambda_j\) for the Cyclic \(4 \times m\) Strip of the Square Lattice

In this appendix we give the equations for the terms \(\lambda_j\) for \(7 \leq j \leq 26\). The \(\lambda_{sq4,j}; j = 7, 8, 9,\) are roots of the cubic equation

\[
\xi^3 + (-q^4 + 7q^3 - 23q^2 + 41q - 33)\xi^2
\]

\(^6\)Accordingly, the U.S. government retains a non-exclusive royalty-free license to publish or reproduce the published form of this contribution or to allow others to do so for U.S. government purposes.
\[+(2q^6 - 23q^5 + 116q^4 - 329q^3 + 553q^2 - 517q + 207)\xi\]
\[+(-q^8 + 16q^7 - 112q^6 + 449q^5 - 1130q^4 + 1829q^3 - 1858q^2 + 1084q - 279) = 0 .\]

(8.1.1)

The \(\lambda_j\) for \(j = 10, 11, 12\) are the roots of another cubic equation
\[\xi^3 - 2(q - 2)(q - 3)\xi^2 + (q^4 - 11q^3 + 42q^2 - 68q + 38)\xi\]
\[-(q - 1)(q - 3)(-q^3 + 7q^2 - 15q + 11) = 0 .\]

(8.1.2)

The \(\lambda_j\) for \(13 \leq j \leq 16\) are the roots of the quartic equation
\[\xi^4 + (2q^3 - 12q^2 + 28q - 23)\xi^3\]
\[+(q^6 - 13q^5 + 73q^4 - 224q^3 + 396q^2 - 381q + 152)\xi^2\]
\[-(q - 1)(q - 3)(q^6 - 12q^5 + 62q^4 - 179q^3 + 304q^2 - 288q + 119)\xi\]
\[-q^9 + 17q^8 - 127q^7 + 549q^6 - 1518q^5 + 2790q^4 - 3411q^3 + 2673q^2 - 1215q + 243 = 0 .\]

(8.1.3)

Finally, there are two sets of roots of two degree-5 equations. The set \(\lambda_j\) for \(17 \leq j \leq 21\) are the roots of the equation
\[\xi^5 + (q - 3)(2q^2 - 9q + 14)\xi^4\]
\[+(q^6 - 17q^5 + 119q^4 - 446q^3 + 947q^2 - 1080q + 515)\xi^3\]
\[-(q - 3)(2q^7 - 32q^6 + 224q^5 - 883q^4 + 2106q^3 - 3028q^2 + 2428q - 835)\xi^2\]
\[+(q - 2)(q^9 - 22q^8 + 213q^7 - 1193q^6 + 4267q^5 - 10120q^4 + 15922q^3 - 16013q^2 + 9329q - 2394)\xi\]
\[+(q - 1)(q^8 - 17q^7 + 125q^6 - 520q^5 + 1342q^4 - 2206q^3 + 2261q^2 - 1325q + 341)(q - 3)^2 = 0 .\]
The set $\lambda_j$ for $22 \leq j \leq 26$ are the roots of the equation
\[
\xi^5 + (-4q^2 + 19q - 26)\xi^4 + (6q^4 - 58q^3 + 214q^2 - 354q + 219)\xi^3 \\
+(-4q^6 + 60q^5 - 370q^4 + 1198q^3 - 2144q^2 + 2013q - 773)\xi^2 \\
+(q - 2)(q^7 - 20q^6 + 162q^5 - 693q^4 + 1697q^3 - 2391q^2 + 1805q - 565)\xi \\
+(q - 1)(q - 3)(q^7 - 16q^6 + 106q^5 - 378q^4 + 788q^3 - 967q^2 + 653q - 189) = 0 .
\]

8.2 Generating Functions for the $L_y = 5, 6$ Strips of the Square Lattice with $(PBC_y, FBC_x)$

For the $L_y = 5$ strip we calculate a generating function of the form (3.2) with $d_D = 2$, $d_N = 1$ and, in the notation of eqs. (3.4) and (3.3), we find
\[
b_{sq5PF,1} = -q^5 + 10q^4 - 46q^3 + 124q^2 - 198q + 148	ag{8.2.1}
\]
\[
b_{sq5PF,2} = q^8 - 19q^7 + 159q^6 - 767q^5 + 2339q^4 - 4627q^3 + 5800q^2 - 4212q + 1362	ag{8.2.2}
\]
\[
A_{sq5PF,0} = q(q - 1)(q - 2)(q^7 - 12q^6 + 67q^5 - 225q^4 + 494q^3 - 719q^2 + 650q - 282)	ag{8.2.3}
\]
\[
A_{sq5PF,1} = -q(q - 1)(q - 2)(q^2 - 2q + 2)(q^8 - 19q^7 + 159q^6 - 767q^5 \\
+2339q^4 - 4627q^3 + 5800q^2 - 4212q + 1362)	ag{8.2.4}
\]

For the $L_y = 6$ strip we calculate a generating function of the form (3.2) with $d_D = 5$, $d_N = 4$, with
\[
b_{sq6PF,1} = -q^6 + 12q^5 - 68q^4 + 234q^3 - 524q^2 + 727q - 483	ag{8.2.5}
\]
\[ b_{sq6PF,2} = 2q^{10} - 44q^9 + 456q^8 - 2917q^7 + 12710q^6 - 39322q^5 + 87323q^4 - 137193q^3 + 145624q^2 - 94100q + 28114 \] (8.2.6)

\[ b_{sq6PF,3} = -q^{14} + 33q^{13} - 509q^{12} + 4872q^{11} - 32374q^{10} + 158152q^9 - 586234q^8 + 1676100q^7 - 3715937q^6 + 6358772q^5 - 8268225q^4 + 16703951q^3 - 429510 \] (8.2.7)

\[ b_{sq6PF,4} = -q^{17} + 38q^{16} - 681q^{15} + 7649q^{14} - 60357q^{13} + 355400q^{12} - 1618550q^{11} + 5828269q^{10} - 16812727q^9 + 39098146q^8 - 73327191q^7 + 110295876q^6 - 131415610q^5 + 121386275q^4 - 83893487q^3 + 40850378q^2 - 12502528q + 1809361 \] (8.2.8)

\[ b_{sq6PF,5} = q^{19} - 41q^{18} + 794q^{17} - 9658q^{16} + 82760q^{15} - 531052q^{14} + 2647330q^{13} - 10495556q^{12} + 33592560q^{11} - 87588439q^{10} + 186851845q^9 - 326185418q^8 + 464098186q^7 - 533530852q^6 + 488389118q^5 - 347889815q^4 + 185960167q^3 - 70211630q^2 + 16703951q - 1884267 \] (8.2.9)

\[ A_{sq6PF,0} = q(q - 1)(q^{10} - 17q^9 + 136q^8 - 674q^7 + 2296q^6 - 5640q^5 + 10183q^4 - 13457q^3 + 12563q^2 - 7517q + 2183) \] (8.2.10)

\[ A_{sq6PF,1} = -q(q - 1)(2q^{14} - 54q^{13} + 695q^{12} - 5631q^{11} + 31999q^{10} - 134668q^9 + \ldots \)
\[ +432404q^8 - 1075802q^7 + 2085064q^6 - 3137110q^5 + 3615627q^4 - 3106751q^3 + 1890461q^2 \\
-733250q + 137516 \]  

\[ (8.2.11) \]

\[ A_{sq6PF,2} = q(q - 1)(q^{18} - 38q^{17} + 684q^{16} - 7756q^{15} + 62128q^{14} - 373554q^{13} \\
+1748131q^{12} - 6513823q^{11} + 19602672q^{10} - 48032023q^9 + 96128905q^8 - 15692032q^7 \\
+207640116q^6 - 220043849q^5 + 183010634q^4 - 115543495q^3 + 52290297q^2 \\
-15182726q + 2135038) \]  

\[ (8.2.12) \]

\[ A_{sq6PF,3} = q(q - 1)(q^{21} - 43q^{20} + 881q^{19} - 11444q^{18} + 105796q^{17} - 740641q^{16} \\
+4078480q^{15} - 18111664q^{14} + 65961019q^{13} - 199240735q^{12} + 502713558q^{11} \\
-1063474616q^{10} + 1887470282q^9 - 2803761470q^8 + 3465937164q^7 - 3530769703q^6 + 2919336052q^5 \\
-1914246633q^4 + 960052617q^3 - 346744827q^2 + 80479446q - 9033772) \]  

\[ (8.2.13) \]

\[ A_{sq6PF,4} = -q(q - 1)(q^4 - 5q^3 + 10q^2 - 10q + 5)(q^{19} - 41q^{18} + 794q^{17} - 9658q^{16} \\
+82760q^{15} - 531052q^{14} + 2647330q^{13} - 10495556q^{12} + 33592560q^{11} - 87588439q^{10} \\
+186851845q^9 - 326185418q^8 + 464098186q^7 - 533530852q^6 + 488389118q^5 \\
-347889815q^4 + 185960167q^3 - 70211630q^2 + 16703951q - 1848467) \]  

\[ (8.2.14) \]
8.3 Generating Function for the $L_y = 5$ Open Strip of the Square Lattice

For this strip we calculate a generating function of the form \( (3.2) \) with \( d_D = 7 \) and \( d_N = 6 \). In the notation of eq. \( (3.4) \) we find

\[
b_{sq(5),1} = -q^5 + 9q^4 - 40q^3 + 107q^2 - 167q + 118 \quad (8.3.1)
\]

\[
b_{sq5FF,2} = 4q^8 - 63q^7 + 458q^6 - 2011q^5 + 5840q^4 - 11477q^3 + 14844q^2 - 11466q + 4003 \quad (8.3.2)
\]

\[
b_{sq5FF,3} = -6q^{11} + 136q^{10} - 1432q^9 + 9250q^8 - 40749q^7 + 128594q^6
\]
\[-296624q^5 + 499762q^4 - 601803q^3 + 492117q^2 - 245164q + 56113 \quad (8.3.3)
\]

\[
b_{sq5FF,4} = 4q^{14} - 120q^{13} + 1685q^{12} - 14681q^{11} + 88695q^{10} - 393187q^9 + 1319323q^8
\]
\[-3404712q^7 + 6790667q^6 - 10414582q^5 + 12084263q^4 - 10278730q^3
\]
\[+6051725q^2 - 2204111q + 373840 \quad (8.3.4)
\]

\[
b_{sq5FF,5} = -(q - 1)(q^{16} - 38q^{15} + 674q^{14} - 7419q^{13} + 56807q^{12} - 321258q^{11}
\]
\[+1389731q^{10} - 4696189q^9 + 12540817q^8 - 26576855q^7 + 44582788q^6 - 58613690q^5
\]
\[+59234653q^4 - 4497390q^3 + 23436736q^2 - 7733009q + 1204091) \quad (8.3.5)
\]

\[
b_{sq5FF,6} = -(q - 1)^2(q^{17} - 38q^{16} + 682q^{15} - 7680q^{14} + 60795q^{13} - 359135q^{12} + 1639962q^{11}
\]
\[-5915021 q^{10} + 17065698 q^9 - 39623309 q^8 + 74056302 q^7 - 110813572 q^6 \]
\[+131155616 q^5 - 120231650 q^4 + 82455281 q^3 - 39872376 q^2 \]
\[+12141916 q - 1753922 \] (8.3.6)

\[b_{sq5FF,7} = (q - 1)^3 (q - 2)^2 (q^{15} - 34 q^{14} + 538 q^{13} - 5259 q^{12} + 35541 q^{11} - 176036 q^{10} \]
\[+ 660682 q^9 - 1914798 q^8 + 4324155 q^7 - 7615130 q^6 + 10381339 q^5 - 10768339 q^4 \]
\[+ 8235159 q^3 - 4388527 q^2 + 1459163 q - 228580) . \] (8.3.7)

Since the \(A_{sq5FF,j}\) are rather lengthy, they are given in the copy of this paper in the cond-mat archive. With the definition \(A_{sq5FF,j} = q(q - 1)A_{sq5FF,j}\), we have
\[
\bar{A}_{sq5FF,0} = (D_4)^4 \] (8.3.8)

where \(D_4 = q^2 - 3q + 3; \)

\[
\bar{A}_{sq5FF,1} = -4q^{11} + 75q^{10} - 653q^9 + 3478q^8 - 12572q^7 + 32346q^6 \]
\[-60381 q^5 + 81687 q^4 - 78370 q^3 + 50664 q^2 - 19788 q + 3517 \] (8.3.9)

\[
\bar{A}_{sq5FF,2} = 6q^{14} - 154q^{13} + 1854q^{12} - 13864q^{11} + 71883q^{10} \]
\[-273164 q^9 + 784036 q^8 - 1725384 q^7 + 2923023 q^6 - 3789945 q^5 + 3697547 q^4 \]
\[-2627998 q^3 + 1283656 q^2 - 384667 q + 53170 \] (8.3.10)

\[
\bar{A}_{sq5FF,3} = -4q^{17} + 132q^{16} - 2056q^{15} + 20067q^{14} - 137414q^{13} \]
\[+ 700413 q^{12} - 2750993 q^{11} + 8502143 q^{10} - 20926266 q^9 + 41238149 q^8 \]

34
\[-65036748q^7 + 81574624q^6 - 80321984q^5 + 60731068q^4 - 34014621q^3 \\
+13280602q^2 - 3222200q + 365089 \quad (8.3.11)\]

\[\tilde{A}_{sq5_{FF,4}} = (q - 1)^2(q^{18} - 40q^{17} + 751q^{16} - 8804q^{15} + 72287q^{14} - 441816q^{13} \\
+2084720q^{12} - 7769759q^{11} + 23199424q^{10} - 55934061q^9 + 109187879q^8 \\
-172199452q^7 + 217795442q^6 - 217905554q^5 + 168626444q^4 - 97343549q^3 \\
+39444397q^2 - 1000178q + 1191823) \quad (8.3.12)\]

\[\tilde{A}_{sq5_{FF,5}} = (q - 1)^3(q^{19} - 40q^{18} + 759q^{17} - 9082q^{16} + 76836q^{15} - 488373q^{14} \\
+2418556q^{13} - 9549855q^{12} + 30509903q^{11} - 79556288q^{10} + 169998158q^9 \\
-297636195q^8 + 425129474q^7 - 490934225q^6 + 451509111q^5 - 323033449q^4 \\
+173271795q^3 - 65533774q^2 + 15574281q - 1747588) \quad (8.3.13)\]

\[\tilde{A}_{sq5_{FF,6}} = -(q - 1)^6(q - 2)^2(q^{15} - 34q^{14} + 538q^{13} - 5259q^{12} + 35541q^{11} \\
-176036q^{10} + 660682q^9 - 1914798q^8 + 4324155q^7 - 7615130q^6 + 10381339q^5 \\
-10768339q^4 + 8235159q^3 - 4388527q^2 + 1459163q - 228580) \quad (8.3.14)\]

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