$\sigma^2/N$ fraction of real history players

memory

fraction of real history players
\[ H = \frac{1}{P} \sum_{\mu} H(\rho^\mu) \]
1 Figure captions

Fig. 1. Normalized volatility averaged over 40 runs (generated by standard MG for $N = 101$, during $t = 10000$ iterations after stationary regime has been attained $t_0 = 10000$) as a function of memory size and fraction of players using real history information.

Fig. 2. Normalized complexity as a function of memory size and fraction of players using real history information. Data were obtained under the same conditions as in Fig.1. The inset shows projection of complexity on the memory plane for real (below) and invented history (above) players.

Fig. 3. Normalized complexity of the binary string of outcomes of length 1000 bits, averaged over 40 runs. For each value of $N$, memory $m$ was varied in $\alpha = 2^m/N$.

Fig. 4. Entropy $H_\mu$ for standard MG as a function of $\alpha = P/N$. Probability $p^\mu$ was evaluated by series expansion of erfc function. For each fixed memory size parameter $\alpha$ was varied by changing the number of players. Following a maximum value of 1, $H_\mu$ starts decreasing at $\alpha \sim 0.34$, the value at phase transition.
Quantifying Complexity in the Minority Game

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Abstract

A Lempel-Ziv complexity measure is introduced into the theory of a Minority Game (MG) in order to capture some features that volatility, one of the central quantities in this model of interacting agents, is not able to. Extracted solely from the binary string of outcomes of the game complexity offers new and valuable information on collective behavior of players. Also, we show that an expression for volatility may be included in the analytical expression for complexity.

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Key words: Complexity measure; Minority Game; Economy; Phase transition

Introduction

The Minority Game has been a topic of considerable recent attention as a paradigm of a complex adaptive system where individual actions of each participant (agent) give rise to cooperative effects reflected in the collective behavior of the whole population of players. The model is also a valuable tool for studying financial markets as particular cases of an interacting agent systems. The central issue of this work is the complexity of a binary string which represents outcome of the game, and its role in the understanding of collective behavior of players. The Lempel-Ziv complexity is used as a measure of complexity and it turns out that it captures many features otherwise eluding usual analysis. The paper is organized such that first a brief presentation of this complexity measure is given followed by an introduction to the setup of the Minority Game (MG). The relationship between phase transitions, symmetry breaking and complexity is also presented in this section followed by a comparison with a similar work on physical complexity in MG. Finally we discuss results and their relevance in explaining collective behavior of players.
1 The Complexity Measure of Lempel and Ziv

According to the widely accepted notions of complexity \[1, 2\] the complexity of a string of zeros and ones (i.e. a binary string) is given by the number of bits of the shortest computer program which can generate the string. From the pool of all possible programs Lempel and Ziv \[3\] have chosen the ones that involve only two operations: copying and inserting, and instead of calculating the length of the programs the calculation is focused on a single number \(c(n)\) which is a useful measure of this length. Very briefly the algorithm consists in scanning a given \(n\)-digit sequence from left to right and at each point of the sequence a test is performed to check whether the rest of the sequence may be reconstructed by simple copying or whether one has to insert new digits. The number of production steps reflected in the number of newly inserted digits represents a complexity of a binary string. Details of the algorithm may be found in the original paper of Lempel and Ziv \[3\]. In order to shed more light on the results we shall consider some analytic features of \(c(n)\). The complexity of a binary string of length \(n\) in the limit of large number of uncorrelated random digits tends to

\[
\lim_{n \to \infty} c(n) = c_R(n) \equiv n / \log_2 n.
\]

\(c_R(n)\) hence represents the asymptotic value of complexity of a binary random string\(^1\). Naturally in the case when probabilities of finding each of two digits in a binary representation, say 1 and 0, are different from 0.5 we would expect the complexity to be smaller than in the case when probabilities of finding both digits are the same. In this case \(c(n)\) tends to \[3, 4\]

\[
\lim_{n \to \infty} c(n) = H(p) \frac{n}{\log_2 n} = -[p \log_2 p + (1 - p) \log_2 (1 - p)] \frac{n}{\log_2 n}, \tag{1}
\]

where \(H(p)\) is known as the source entropy. Clearly this quantity is maximal for \(p = 0.5\) and once \(p\) and \(H(p)\) are determined it can be tested whether deviation of \(\lim_{n \to \infty} [c(n)/c_R(n)]\) from 1 is due to the fact that \(H(p)\) differs from 1 (and the string is still as random as possible) or, as is the case when \(\lim_{n \to \infty} [c(n)/c_R(n)] < H(p)\), due to the presence of correlations in the string.

2 Minority Game

Minority game \[5\] consists of \(N\) players trying at each time step to be in minority taking an action \(a_i(t) = \pm 1\), where, for example, 1 may be interpreted

\(^1\) We normalize all our numerically obtained complexity values by the maximal complexity of a random string evaluated by averaging over 100 000 random samples whose length corresponds to the length used in simulations.
as buying and −1 as selling an asset. The choice is made based on the information stored as a binary time series of the last m actions taken by the minority

\( \mu_t = (\chi_{t-1}, ..., \chi_{t-m}) \).

Each player has at his disposal the same number s (but in general different) of strategies among the S possible strategies, denoted by

\( s_{\pm} \), which give a prediction for the next outcome of the game based on the history of the last m outcomes. Since there are \( 2^m \) possible histories there are \( S = 2^{2m} \) possible strategies. Each strategy is dynamically assigned a virtual payoff based on its performance and at each time step every player uses his most successful strategy in terms of the highest payoff (highest \(-a_i(t)A(t)\) where the difference in the population of agents choosing the + and the − sign at time t is

\( A(t) = \sum_i^N a_i(t) \)). Therefore, this interaction rewards the minority of agents (those who took the action \( a_i(t) = \text{sign} \ A(t) \)), and hence the name minority game. If a player i follows his \( s_{\pm} \)-th strategy and the history information is \( \mu \) his action is denoted by

\( a_{\mu,s,i}^\alpha \).

The variance \( \sigma \) of \( A(t) \) (volatility in financial context) represents one of the most interesting observables since the smallest value attained by \( \sigma \) represents the maximal payoff distribution to agents and minimal global waste of resources by the community of agents.

\[ \text{3 Complexity in the Minority game} \]

\[ \text{3.1 Phase transition, symmetry breaking and complexity} \]

The MG undergoes a second order phase transition with symmetry breaking as the control parameter \( \alpha = 2^m / N \) is varied [6, 10]: the system is in the symmetric phase (temporal average of \( A(t) \) conditional to \( \mu(t) = \mu \) equals 0, i.e. \( \langle A^\mu \rangle = 0 \)) for \( \alpha < \alpha_c \) and it is in the asymmetric phase for \( \alpha > \alpha_c \). In other words in the symmetric phase both actions are taken by the minority with the same frequency while in the asymmetric phase the minority prefers one action over the other. Moreover, for \( \alpha < \alpha_c \) all players use all their available strategies while for \( \alpha > \alpha_c \) a finite population of players uses only one strategy (analogous to spontaneous magnetization in the spin formalism). It has been suggested that behavior of \( \sigma \) does not depend on the real history of the game [7], however it was shown that this assertion is true only in the symmetric phase [8]. Additionally, volatility was criticized for its insensitivity to the real history of the game and another measure based on the algorithmic complexity was suggested [11]. Actually, in reference [11] the physical complexity (non normalized Shannon entropy) of substrings of the string of outcomes is evaluated which is equal to zero for random sequences, and the effect of memory size on the degree of randomness was in the focus of the study. On the other hand the Lempel-Ziv complexity is maximal for white noise and the focus of our work lies in proving that this complexity measure may be used to detect phase transitions and other information about players and the game in general.
that can not be obtained by using volatility alone. From our point of view also, volatility is inferior in many aspects to the complexity measure of Lempel and Ziv, or variations of it, however due to its important role in theory of spin glass systems which are analogous to the setup of MG, it would be useful if it appears in the analytic expression for complexity. In order to clearly illustrate the advantage of using the complexity measure of Lempel and Ziv, we numerically simulate a game with two types of players: those who use real history in selecting an optimal strategy and those who use assigned invented (random) history at each time step. The behavior of $\sigma^2$ is shown in Fig. 1. Note that in this case too there is a second order phase transition with symmetry breaking and two phases of the game can be clearly distinguished. However, when the game is played by only one group of players the graph of $\sigma^2$ vs. memory is qualitatively the same while the graph of complexity, presented in Fig. 2 clearly shows the difference. Specifically, when the game is played solely by players using invented history no phase transition is detected and complexity is practically constant in the complete range of memory values (inset of Fig. 2). When only players using real history take part in the game there is a phase transition at $m = 5$ indicated by the maximum value of complexity. Comparing the minimum value of volatility with respect to memory size with the corresponding maximum value of complexity we may expect that maximal collaboration among the players (and minimal global waste) is achieved very close to the maximum value of complexity. Maximal collaboration is an inherently complicated occurrence considering the fact that none of the players are aware of actions taken by other participants in the game. Based on these findings we may add that history is relevant! An important information conveyed by complexity is its ability to distinguish types of players involved in the game based on the information contained only in the binary string of outcomes. In the economic context this would mean that such inference may be obtained from the direction of the sign of the order imbalance, i.e. from returns. In addition, this finding supports the intermediate version of the efficient market hypothesis, namely that a number of important informations, including the ones that are not available to the players (agents) playing the game (taking part in the financial market activity), are reflected in the outcome (price index) at each moment of time. One more important feature is that the maximum value of complexity (an indication of phase transition) is independent of memory as indicated in Fig. 3, where complexity is presented as a function of the usual order parameter $\alpha = P/N \equiv 2^m/N$, with all players relying on the true history of the game. Note that for each maximal value of complexity the order parameter value is $\sim 0.34$, a universally accepted value at phase transition.
3.2 Complexity and Entropy

Inspired by the expression (1) for source entropy, we introduce the following entropy function

\[ H_\mu = -\frac{1}{P} \sum_\mu [p_\mu \log p_\mu + (1 - p_\mu) \log(1 - p_\mu)], \]  

(2)

where \( P \) is the state space of all histories of size \( m \), i.e. \( P = 2^m \), and the summation is over the histories. We also assume that the probabilities figuring in the above expression depend on the history of the outcomes of the game. Namely, the probability \( p_\mu \) represents the probability of finding a minority sign \( a_t = -\text{sign} \ A_t \) provided a specific history of outcomes occurred. Hence, it is a conditional probability

\[ p_\mu = \text{Prob} \{ A(t) > 0 \mid \mu(t) = \mu \}. \]

In the limit of large \( N \), \( A_\mu \) is a gaussian variable with average \( \langle A_\mu \rangle \) and variance \( \langle (A_\mu)^2 \rangle - \langle A_\mu \rangle^2 \). A calculation yields

\[ p_\mu = \frac{1}{2} \text{erf} \left( \frac{\langle A_\mu \rangle}{\sqrt{2 \langle (A_\mu)^2 \rangle - \langle A_\mu \rangle^2}} \right) \approx \frac{1}{2} - \frac{1}{\sqrt{2\pi}} \frac{1}{\sqrt{\sigma^2 \ h}} - 1, \]

(3)

where \( \sigma^2 = \langle (A_\mu)^2 \rangle \), and \( \langle A_\mu \rangle^2 = h \), and where we have expanded the erf function to linear order under the assumption that \( \langle A_\mu \rangle \ll \text{Var}(A_\mu) \). Consequently, this entropy expression contains both \( \sigma^2 \) and \( h \) keeping the analogy with the spin glass systems (the dynamics of the Minority Game is similar to spin dynamics with hamiltonian \( \sigma^2 \)) [10]. In Fig. 4 we show the entropy given by expression (2) for standard MG as a function of \( \alpha = P/N \). A striking feature of this graph immediately draws attention, namely for an initial range of the order parameter \( \alpha \) the entropy is maximal (\( \approx 1 \)) and then it starts decreasing at \( \alpha \approx 0.34 \), the value of the control parameter at phase transition. In the symmetric phase of the game, when both actions are taken by the minority with same frequency the entropy is maximal and close to 1, and this result may be easily predicted using expressions (2) and (3). However when minority starts preferring one action over the other the entropy starts decreasing and a phase transition occurs. Since the Lempel Ziv algorithm is optimal in the sense that at least for ergodic and infinite sequences complexity approaches the Shannon entropy [9], it is of interest to compare the Shannon entropy and the entropy given in expression (2) in order to shed more light on the difference between the complexity shown in Fig. 3 and the entropy presented in Fig. 4. Numerically, the Shannon entropy from a finite binary sequence of length \( N \) is usually estimated by calculating all block (subsequence or ”word”) probabilities \( p(s_1, ..., s_n) \) by the standard likelihood estimate,

\[ \hat{p}_n = n_{s_1...s_n}/N \]  

(4)
where \( n_{s_1...s_n} \) is the number of occurrences of the block \( s_1, ..., s_n \), and \( s_i \in \{0, 1\} \), and where the expression for the Shannon entropy is

\[
H_S = \lim_{N \to \infty} -\frac{1}{N} \sum_{\{s_i\}} p_n(s_1, ..., s_n) \log p_n(s_1, ..., s_n). \tag{5}
\]

Hence, the probabilities (3) and (4) figuring in the entropy expressions (2) and (5) respectively, are inherently different. As a final remark we may add that in the context of MG complexity may be linked to Nash equilibria, collaborative behavior among players and bounded rationality [12].

4 Summary

A complexity measure based on the algorithm of Lempel and Ziv offers new evidence that history is relevant in MG, and that maximal complexity corresponds to the maximal collaboration among players. It reaches its maximum at phase transition value of the order parameter regardless of the memory size. Also, types of players involved in the game may be recognized using the complexity measure and overall findings support the hypothesis that important information is contained in the outcomes. Inspired by an analytic expression for asymptotic complexity we introduce an entropy expression which contains volatility, a central quantity in the theory of the Minority Game, enabling links to the exact solution and analytical results [13].

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References

[1] A. N. Kolmogorov, Prob. Inf. Transmission 1 (1965) 1.
[2] G. J. Chaitin, J. Assoc. Comput. Machinery 13 (1966) 547.
[3] A. Lempel, J. Ziv, IEEE Trans. Inf. Theory 22 (1976) 75.
[4] F. Kaspar, H. G. Schuster, Phys. Rev. A 36 (1987) 842.
[5] D. Challet, Modelling Market Dynamics: Minority Games and Beyond, PhD Thesis, Univ. de Fribourg (2000).
[6] R. Savit, R. Manuca, R. Riolo, Phys. Rev. Lett. 82 (1999) 2203.

[7] A. Cavagna, Phys. Rev. E 59 (1999) R3783.

[8] D. Challet, M. Marsili, Phys. Rev. E 62 (2000) 1862.

[9] J. Ziv, IEEE Trans. Information Theory 24 (1978) 405.

[10] D. Challet, M. Marsili, Phys. Rev. E 60 (1999) 6271.

[11] R. Mansilla, Phys. Rev. E 62 (2000) 4553.

[12] M. Rajković, Z. Mihailović, in preparation.

[13] M. Marsili, D. Challet, R. Zecchina, Physica A 280 (2000) 522.