Abstract

We prove that for a holomorphic submersion of reduced complex spaces, the basic Oka property implies the parametric Oka property. It follows that a stratified subelliptic submersion, or a stratified fiber bundle whose fibers are Oka manifolds, enjoys the parametric Oka property.

Résumé

Lés Applications d’Oka. Nous prouvons que, pour une submersion holomorphe des espaces complexes réduits, la propriété d’Oka simple implique la propriété d’Oka paramétrique. En particulier, toute submersion sous-elliptique stratifié possède la propriété d’Oka paramétrique.

1. Oka properties of holomorphic maps

Let $E$ and $B$ be reduced complex spaces. A holomorphic map $\pi: E \to B$ is said to enjoy the Basic Oka Property (BOP) if, given a holomorphic map $f: X \to B$ from a reduced Stein space $X$ and a continuous map $F_0: X \to E$ satisfying $\pi \circ F_0 = f$ (a lifting of $f$) such that $F_0$ is holomorphic on a closed complex subvariety $X'$ of $X$ and in a neighborhood of a compact $\mathcal{O}(X)$-convex subset $K$ of $X$, there is a homotopy of liftings $F_t: X \to E$ ($t \in [0, 1]$) of $f$ to a holomorphic lifting $F_1$ such that for every $t \in [0, 1]$, $F_t$ is holomorphic in a neighborhood of $K$ (independent of $t$), $\sup_{x \in K} \text{dist}(F_t(x), F_0(x)) < \epsilon$, and $F_t|_{X'} = F_0|_{X'}$ (the homotopy is fixed on $X'$).

By definition, a complex manifold $Y$ enjoys BOP if and only if the trivial map $Y \to \text{point}$ does. This is equivalent to several other properties, from the simplest Convex Approximation Property (CAP) to the Parametric Oka Property (POP) concerning compact families of maps from reduced Stein spaces to $Y$. A complex manifold enjoying these equivalent properties...
is called an Oka manifold \[2, 11\]; these are precisely the fibrant complex manifolds in Lárusson’s model category \[9\]. Here we prove that BOP \(\implies\) POP also holds for holomorphic submersions. (The submersion condition corresponds to requiring smoothness as part of the definition of a variety being Oka. The singular case is rather problematic.)

**Theorem 1.1.** For every holomorphic submersion \(\pi: E \to B\) of reduced complex spaces, the basic Oka property implies the parametric Oka property.

Recall \[9\] that a holomorphic map \(\pi: E \to B\) enjoys the Parametric Oka Property (POP) if for any triple \((X, X', K)\) as above and for any pair \(P_0 \subset P\) of compact subsets in a Euclidean space \(\mathbb{R}^m\) the following holds. Given a continuous map \(f: P \times X \to B\) that is \(X\)-holomorphic (that is, \(f(p, \cdot): X \to B\) is holomorphic for every \(p \in P\)) and a continuous map \(F_0: P \times X \to E\) such that (a) \(\pi \circ F_0 = f\), (b) \(F_0(p, \cdot)\) is holomorphic on \(X\) for all \(p \in P_0\) and is holomorphic on \(K \cup X'\) for all \(p \in P\), there exists for every \(\epsilon > 0\) a homotopy of continuous liftings \(F_t: P \times X \to E\) of \(f\) to an \(X\)-holomorphic lifting \(F_1\) such that the following hold for all \(t \in [0, 1]\):

(i) \(F_t = F_0\) on \((P_0 \times X) \cup (P \times X')\), and

(ii) \(F_t\) is \(X\)-holomorphic on \(K\) and \(\sup_{p \in P, x \in K} \text{dist}(F_t(p, x), F_0(p, x)) < \epsilon\).

A stratified subelliptic holomorphic submersion, or a stratified fiber bundle with Oka fibers, enjoys BOP \[3, 4\]. Hence Theorem \[1.1\] implies:

**Corollary 1.2.** (i) Every stratified subelliptic submersion enjoys POP. 
(ii) Every stratified holomorphic fiber bundle with Oka fibers enjoys POP.

If \(\pi: E \to B\) enjoys the Oka property then by considering liftings of constant maps \(X \to b \in B\) we see that every fiber \(E_b = \pi^{-1}(b)\) is an Oka manifold. For stratified fiber bundles the converse holds by Corollary \[1.2\].

**Question 1.3.** Does every holomorphic submersion with Oka fibers enjoy the Oka property?

A holomorphic map is said to be an Oka map if it is a topological (Serre) fibration and it enjoys POP. Such maps are intermediate fibrations in Lárusson’s model category \[3, 10\]. Corollary \[1.2\] implies:
Corollary 1.4. (i) Every holomorphic fiber bundle projection with Oka fiber is an Oka map.

(ii) A stratified subelliptic submersion, or a stratified holomorphic fiber bundle with Oka fibers, is an Oka map if and only if it is a Serre fibration.

Corollary 1.2 (i) and the proof by Ivarsson and Kutzschebauch [8] gives the following solution of the parametric Gromov-Vaserstein problem [7, 12].

Theorem 1.5. Assume that $X$ is a finite dimensional reduced Stein space, $P$ is a compact subset of $\mathbb{R}^m$, and $f: P \times X \to \text{SL}_n(\mathbb{C})$ is a null-homotopic $X$-holomorphic mapping. Then there exist a natural number $N$ and $X$-holomorphic mappings $G_1, \ldots, G_N: P \times X \to \mathbb{C}^{n(n-1)/2}$ such that

$$f(p, x) = \begin{pmatrix} 1 & 0 & & & \phantom{1} \\ G_1(p, x) & 1 & & & \\ & 0 & & & \\ \vdots & & & & \phantom{1} \\ & & & & \phantom{1} \\ G_N(p, x) & & & & 1 \end{pmatrix}$$

is a product of upper and lower diagonal unipotent matrices.

2. Reduction of Theorem 1.1 to an approximation property

Assume that $\pi: E \to B$ enjoys BOP and that $(X, X', K, P, P_0, f, F_0)$ are as in the definition of POP, with $P_0 \subset P \subset \mathbb{R}^m \subset \mathbb{C}^m$. Set

$$Z = \mathbb{C}^m \times X \times E, \quad Z_0 = \mathbb{C}^m \times X \times B, \quad \psi = (\text{id}_{\mathbb{C}^m \times X}) \times \pi: Z \to Z_0. \quad (1)$$

Observe that $\psi$ enjoys BOP (resp. POP) if and only if $\pi$ does. To the map $f: P \times X \to B$ we associate the $X$-holomorphic section

$$g: P \times X \to Z_0, \quad g(p, x) = (p, x, f(p, x)) \quad (p \in P, \ x \in X), \quad (2)$$

and to the $\pi$-lifting $F_0: P \times X \to E$ of $f$ we associate the section

$$G_0: P \times X \to Z, \quad G_0(p, x) = (p, x, F_0(p, x)) \quad (p \in P, \ x \in X). \quad (3)$$

Then $\psi \circ G_0 = g$, $G_0$ is $X$-holomorphic over $K \cup X'$, and $G_0|_{P_0 \times X}$ is $X$-holomorphic. We must find a homotopy $G_t: P \times X \to Z$ $(t \in [0, 1])$ such that $\psi \circ G_t = g$ for all $t \in [0, 1]$, $G_1$ is $X$-holomorphic, and for all $t \in [0, 1]$ the map $G_t$ has the same properties as $G_0$, $G_t$ is uniformly close to $G_0$ on $K \times P$, and $G_t = G_0$ on $(P_0 \times X) \cup (P \times X')$. Set

$$Q = [0, 1] \times P, \quad Q_0 = ([0] \times P) \cup ([0, 1] \times P_0).$$

The following result is the key to the proof of Theorem 1.1.
Proposition 2.1. If the submersion \( \psi: Z \to Z_0 \) enjoys the basic Oka property, then it also enjoys the following

Parametric Homotopy Approximation Property (PHAP): Let \( K \subset L \) be compact \( O(X) \)-convex subsets and let \( U \supset K, V \supset L \) be open neighborhoods in \( X \). Assume that \( g: P \times V \to Z_0 \) is an \( X \)-holomorphic section of the form (2) and \( G_t: P \times V \to Z \) (\( t \in [0, 1] \)) is a homotopy of sections (3) satisfying

(a) \( \psi \circ G_t = g \) for all \( t \in [0, 1] \),
(b) \( G_t(p, \cdot) \) is holomorphic on \( U \) for \( (t, p) \in Q \), and
(c) \( G_t(p, \cdot) = G_0(p, \cdot) \) for \( (t, p) \in Q_0 \), and these are holomorphic on \( V \).

Let \( \epsilon > 0 \). After shrinking the neighborhoods \( U \supset K \) and \( V \supset L \), there exists a homotopy \( \tilde{G}_t: P \times V \to Z \) (\( t \in [0, 1] \)) of the form (3) such that

(i) \( \psi \circ \tilde{G}_t = g \) for all \( t \in [0, 1] \),
(ii) for each \( (t, p) \in Q \) the map \( \tilde{G}_t(p, \cdot): V \to Z \) is holomorphic and it satisfies \( \sup_{x \in K} \text{dist}(\tilde{G}_t(p, x), G_t(p, x)) < \epsilon \), and
(iii) \( \tilde{G}_t(p, \cdot) = G_t(p, \cdot) \) for each \( (t, p) \in Q_0 \).

Furthermore, there is a homotopy from \( \{G_t\} \) to \( \{\tilde{G}_t\} \) consisting of homotopies with the same properties as \( \{G_t\} \).

For families of sections of a holomorphic submersion \( \pi: Z \to X \) over a Stein space \( X \), PHAP holds if \( Z \to X \) admits a fiber-dominating spray over a neighborhood of \( L \) (7, 5), or a finite fiber-dominating family of sprays (1). Submersions admitting such sprays over small open subsets of \( X \) are called elliptic, resp. subelliptic. If PHAP holds over small open subsets of \( X \) then sections \( X \to Z \) satisfy the parametric Oka property (Gromov [7, Theorem 4.5]; the details can be found in [3, 5]). The same proof applies in our situation (see [4, Theorem 4.2]) and shows that PHAP implies Theorem 1.1.

3. Proof of Proposition 2.1

Let \( h: E \to Z \) denote the holomorphic vector bundle whose fiber over a point \( z \in Z \) equals the tangent space at \( z \) to the (smooth) fiber of \( \psi \). The restriction \( E|_{\Omega} \) to any open Stein domain \( \Omega \subset Z \) is a reduced Stein space. By standard techniques we obtain for every such \( \Omega \) an open Stein neighborhood

1. [1]
2. [2]
3. [3]
4. [4]
5. [5]
6. [6]
7. [7]
In the sequel the set \( V \) with such that \( S \) is restricted bundle \( E|_W \) convex, and it follows that \( \xi \) metric version of the Oka-Weil theorem we can approximate \( (\text{See Fig. 1.}) \) For a given collection \( (\Omega_j, W_j, s_j) \) the existence of homotopies \( \xi_t \) is stable under sufficiently small perturbations of the homotopy \( G_t \).

Consider the homotopy of sections \( \{\xi_t\}_{t \in [0,t_1]} \) of \( E|_{G_0(P \times U')} \). By the parametric version of the Oka-Weil theorem we can approximate \( \xi_t \) uniformly on \( P \times K \) by \( X \)-holomorphic sections \( \tilde{\xi}_t \) of \( E|_{G_0(P \times V')} \) for an open set \( V' \subset X \) with \( L \subset V' \subset V \). Further, we may choose \( \tilde{\xi}_t = \xi_t \) for \( t = 0 \) and on \( P_0 \times V' \). In the sequel the set \( V' \) may shrink around \( L \).

By [6] Corollary 2.2 there is an open Stein neighborhood \( \Omega \) of \( S_0 \) in \( Z \) such that \( S_0 \) is \( \mathcal{O}(\Omega) \)-convex. Hence \( \Sigma_0 = G_0(P \times K) \subset S_0 \) is also \( \mathcal{O}(\Omega) \)-convex, and it follows that \( W_0 \cap \Sigma_0 \) is exhausted by \( \mathcal{O}(\Sigma_0) \)-convex compact sets. Since \( \mathcal{E}|_{\Omega} \) is a reduced Stein space and \( s_0 \) extends continuously to \( \mathcal{E}|_{\Omega} \) preserving the property \( \psi \circ s_0 = \psi \circ h \), the assumed BOP of \( \psi \) implies that \( s_0 \) can be approximated on the range of the homotopy \( \{\xi_t : t \in [0,t_1]\} \) (which
Figure 1: Lifting sections $G_t$ to the spray bundle $\mathcal{E}|_{G_0(P \times V)}$ is contained in $W_0 \cap \mathcal{E}|_{\Sigma_0}$ by a holomorphic map $\tilde{s}_0: \mathcal{E}|_{\Omega} \to Z$ which equals the identity on the zero section and satisfies $\psi \circ \tilde{s}_0 = \psi \circ h$. The homotopy

$$\tilde{G}_t = \tilde{s}_0 \circ \tilde{\xi}_t \circ G_0: P \times V' \to Z \quad (t \in [0,t_1])$$

is fixed over $P_0$, $X$-holomorphic on $V'$, $\tilde{G}_0 = G_0$, and $\tilde{G}_t$ approximates $G_t$ uniformly on $P \times K$ (also uniformly with respect to $t \in [0,t_1]$). If the approximation is sufficiently close, we obtain a new homotopy $\{G_t\}_{t \in [0,1]}$ that agrees with $\tilde{G}_t$ for $t \in [0,t_1]$ (hence is $X$-holomorphic on $L$), and that agrees with the initial homotopy for $t \in [t'_1,1]$ for some $t'_1 > t_1$ close to $t_1$.

We now repeat the same argument with the parameter interval $[t_1,t_2]$ using $G_{t_1}$ as the initial reference map. This produces a new homotopy that is $X$-holomorphic on $L$ for all values $t \in [0,t_2]$. After $N$ steps of this kind we obtain a homotopy satisfying the conclusion of Proposition 2.1.

Acknowledgments. I wish to thank Finnur Lárusson for his helpful remarks on a preliminary version of the paper. Research was supported in part by grants P1-0291 and J1-2043-0101 from ARRS, Republic of Slovenia.

References

[1] Forstnerič, F.: The Oka principle for sections of subelliptic submersions. Math. Z. 241 (2002) 527–551

[2] Forstnerič, F.: Oka Manifolds. C. R. Acad. Sci. Paris, Ser. I 347 (2009) 1017-1020
[3] Forstnerič, F.: The Oka principle for sections of stratified fiber bundles. Pure and Appl. Math. Quarterly, 6 (2010), no. 3, 843–874

[4] Forstnerič, F.: Invariance of the parametric Oka property. P. Ebenfelt, N. Hungerbuehler, J. J. Kohn, N. Mok, E. J. Straube, eds., Complex Analysis. Trends in Mathematics, Birkhäuser (2010)

[5] Forstnerič, F., Prezelj, J.: Oka’s principle for holomorphic submersions with sprays. Math. Ann. 322 (2002) 633–666

[6] Forstnerič, F., Wold, E. F.: Fibrations and Stein Neighborhoods. Proc. Amer. Math. Soc., to appear. arXiv: 0906.2424

[7] Gromov, M.: Oka’s principle for holomorphic sections of elliptic bundles. J. Amer. Math. Soc. 2 (1989) 851-897

[8] Ivarsson, B., Kutzschebauch, F.: A solution of Gromov’s Vaserstein problem. C. R. Acad. Sci. Paris, Ser. I 346 (2008) 1239–1243

[9] Lárusson, F.: Model structures and the Oka principle. J. Pure Appl. Algebra 192 (2004) 203–223

[10] Lárusson, F.: Mapping cylinders and the Oka principle. Indiana Univ. Math. J. 54 (2005) 1145–1159

[11] Lárusson, F.: What is an Oka manifold? Notices Amer. Math. Soc. 57 (2010), no. 1, 50–52. http://www.ams.org/notices/201001/

[12] Vaserstein, L.: Reduction of a matrix depending on parameters to a diagonal form by addition operations. Proc. Amer. Math. Soc. 103 (1988) 741–746