On Zweier paranorm I-convergent double sequence spaces

Vakeel A. Khan*, Nazneen Khan and Yasmeen Khan

Abstract: In this article, we introduce the Zweier Paranorm I-convergent double sequence spaces \(z^I_0(q)\), \(z^I_{00}(q)\) and \(z^I_{0}(q)\) for \(q = (q_{ij})\), a sequence of positive real numbers. We study some algebraic and topological properties on these spaces.

Keywords: Engineering Technology; Mathematics Statistics; Science; Technology

1. Introduction

Let \(\mathbb{N}, \mathbb{R}\) and \(\mathbb{C}\) be the sets of all natural, real and complex numbers, respectively. We write

\[\omega = \{ x = (x_k) : x_k \in \mathbb{R} \text{ or } \mathbb{C} \},\]

the space of all real or complex sequences.

Let \(l_\infty, c\) and \(c_0\) denote the Banach spaces of bounded, convergent and null sequences, respectively, normed by \(\|x\|_\infty = \sup_k |x_k|\).

The following subspaces of \(\omega\) were first introduced and discussed by Maddox (1969).

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PUBLIC INTEREST STATEMENT

The term sequence has a great role in analysis. Sequence spaces play an important role in various fields of real analysis, complex analysis, functional analysis and Topology. They are very useful tools in demonstrating abstract concepts through constructing examples and counter examples. Convergence of sequences has always remained a subject of interest to the researchers. Later on, the idea of statistical convergence came into existence which is the generalization of usual convergence. Statistical convergence has several applications in different fields of Mathematics like Number Theory, Trigonometric Series, Summability Theory, Probability Theory, Measure Theory, Optimization and Approximation Theory. The notion of Ideal convergence (I-convergence) is a generalization of the statistical convergence and equally considered by the researchers for their research purposes since its inception.
where \( p = (p_k) \) is a sequence of strictly positive real numbers.

After then Lascarides (1971, 1983) defined the following sequence spaces

\[
l_1(p) := \{ x \in \omega : \sum_k |x_k|^p_k < \infty \},
\]

\[
l_\infty(p) := \{ x \in \omega : \sup_k |x_k|^p_k < \infty \},
\]

\[
c(p) := \{ x \in \omega : \lim_k |x_k|^l = 0, \text{ for some } l \in \mathbb{C} \},
\]

\[
c_0(p) := \{ x \in \omega : \lim_k |x_k|^l = 0 \},
\]

where \( p = (p_k) \) is a sequence of strictly positive real numbers.

A double sequence of complex numbers is defined as a function \( x : \mathbb{N} \times \mathbb{N} \rightarrow \mathbb{C} \). We denote a double sequence as \( (x_{ij}) \) where the two subscripts run through the sequence of natural numbers independent of each other. A number \( a \in \mathbb{C} \) is called a double limit of a double sequence \( (x_{ij}) \) if for every \( \epsilon > 0 \) there exists some \( N = N(\epsilon) \in \mathbb{N} \) such that (Khan & Sabiha, 2011)

\[
| x_{ij} - a | < \epsilon, \quad \forall i, j \geq N
\]

Therefore we have,

\[
\ell_2 = \{ x = (x_{ij}) \in \mathbb{R} \text{ or } \mathbb{C} \},
\]

the space of all real or complex double sequences.

Each linear subspace of \( \omega \), for example, \( \lambda, \mu \subset \omega \) is called a sequence space.

The notion of I-convergence is a generalization of the statistical convergence. At the initial stage it was studied by Kostyrko, Šalát, and Wilczynski (2000). Later on it was studied by Šalát, Tripathy, and Zíman (2004), Tripathy and Hazarika (2009) and Demirci (2001).

2. Preliminaries and definitions

Here, we give some preliminaries about the notion of I-convergence and Zweier sequence spaces. For more details one refer to Das, Kostyrko, Malik, and Wilczyński (2008), Gurdal and Ahmet (2008), Khan and Khan (2014a, 2014b), Mursaleen and Mohiuddine (2010, 2012), Esi and Sapsizoğlu (2012), Fadile Karababa and Esi (2012), Khan et al. (2013b).

**Definition 2.1** If \( (X, \rho) \) is a metric space, a set \( A \subset X \) is said to be nowhere dense if its closure \( \bar{A} \) contains no sphere, or equivalently if \( A \) has no interior points.

**Definition 2.2** Let \( X \) be a non-empty set. Then a family of sets \( I \subset 2^X \) (denoting the power set of \( X \)) is said to be an ideal in \( X \) if

(i) \( \emptyset \in I \)

(ii) \( I \) is finitely additive i.e. \( A, B \in I \Rightarrow A \cup B \in I \).

(iii) \( I \) is hereditary i.e. \( A \in I, B \subset A \Rightarrow B \in I \).
An ideal $I \subseteq 2^X$ is called non-trivial if $I \neq 2^X$. A non-trivial ideal $I \subseteq 2^X$ is called admissible if \( \{ \{ X \} : X \in X \} \subseteq I \).

A non-trivial ideal $I$ is maximal if there cannot exist any non-trivial ideal $J \neq I$ containing $I$ as a subset.

**Definition 2.3** A double sequence $(x_{ij})$ is said to be

(i) $I$-convergent to a number $L$ if for every $\varepsilon > 0$,
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L| \geq \varepsilon \} \subseteq I.
\]
In this case we write $I - \lim x_{ij} = L$.

(ii) A double sequence $(x_{ij})$ is said to be $I$-null if $L = 0$. In this case, we write
\[
I - \lim x_{ij} = 0.
\]

(iii) A double sequence $(x_{ij})$ is said to be $I$-cauchy if for every $\varepsilon > 0$ there exist numbers $m = m(\varepsilon), n = n(\varepsilon)$ such that
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - x_{mn}| \geq \varepsilon \} \subseteq I.
\]

(iv) A double sequence $(x_{ij})$ is said to be $I$-bounded if there exists $M > 0$ such that
\[
\{(i, j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}| < M \}.
\]

**Definition 2.4** A double sequence space $E$ is said to be solid or normal if $(x_{ij}) \in E$ implies $(\alpha_{ij}x_{ij}) \in E$ for all sequence of scalars $(\alpha_{ij})$ with $|\alpha_{ij}| < 1$ for all $(i, j) \in \mathbb{N} \times \mathbb{N}$.

**Definition 2.5** Let $X$ be a linear space. A function $g : X \rightarrow \mathbb{R}$ is called a paranorm, if for all $x, y, z \in X$,

(i) $g(x) = 0$ if $x = \theta$,

(ii) $g(-x) = g(x)$,

(iii) $g(x + y) \leq g(x) + g(y)$,

(iv) $\lim \frac{f(\lambda_n)}{\lambda_n} = \lambda (n \rightarrow \infty)$ and $x_n, a \in X$ with $x_n \rightarrow a (n \rightarrow \infty)$, in the sense that $g(x_n - a) \rightarrow 0 (n \rightarrow \infty)$, in the sense that $g(\lambda_n x_n - \lambda a) \rightarrow 0 (n \rightarrow \infty)$. The concept of paranorm is closely related to linear metric spaces. It is a generalization of that of absolute value (see Lascarides, 1971; Tripathy & Hazarika, 2009).

A sequence space $\mathcal{J}$ with linear topology is called a K-space provided each of maps $p_i : \mathcal{J} \rightarrow \mathcal{I}$ defined by $p_i(x) = x_i$ is continuous for all $i \in \mathbb{N}$.

A K-space $\mathcal{J}$ is called an FK-space provided $\mathcal{J}$ is a complete linear metric space.

An FK-space whose topology is normable is called a BK-space.

Let $\mathcal{J}$ and $\mu$ be two sequence spaces and $A = (a_{nk})$ is an infinite matrix of real or complex numbers $a_{nk}$, where $n, k \in \mathbb{N}$. Then we say that $A$ defines a matrix mapping from $\mathcal{J}$ to $\mu$, and we denote it by writing $A : \mathcal{J} \rightarrow \mu$.

If for every sequence $x = (x_n) \in \mathcal{J}$ the sequence $Ax = ((Ax)_n)$, the $A$ transform of $x$ is in $\mu$, where
\[
(Ax)_n = \sum_k a_{nk}x_k, \quad (n \in \mathbb{N})
\]
By \((\lambda : \mu)\), we denote the class of matrices \(A : \lambda \rightarrow \mu\).

Thus, \(A \in (\lambda : \mu)\) if and only if series on the right side of (1) converges for each \(n \in \mathbb{N}\) and every \(x \in \lambda\).

The approach of constructing new sequence spaces by means of the matrix domain of a particular limitation method have been recently employed by Altay, Başar, and Mursaleen (2006), Başar and Altay (2003), Malkowsky (1997), Ng and Lee (1978) and Wang (1978).

Şengönül (2007) defined the sequence \(y = (y_i)\) which is frequently used as the \(Z^p\) transform of the sequence \(x = (x_i)\) i.e.

\[ y_i = px_i + (1 - p)x_{i-1} \]

where \(x_{-1} = 0, 1 < p < \infty\) and \(Z^p\) denotes the matrix \(Z^p = (z_{ik})\) defined by

\[ z_{ik} = \begin{cases} p, & (i = k) \\ 1 - p, & (i - 1 = k); (i, k \in \mathbb{N}), 0, & \text{otherwise.} \end{cases} \]

Following Basar and Altay (2003), Şengönül (2007), introduced the Zweier sequence spaces \(Z\) and \(Z_0\) as follows

\[ Z = \{ x = (x_k) \in \omega : Z^p x \in c \} \]
\[ Z_0 = \{ x = (x_k) \in \omega : Z^p x \in c_0 \} \]

Here, we quote below some of the results due to Şengönül (2007) which we will need in order to establish the results of this article.

**Theorem 2.1** The sets \(Z\) and \(Z_0\) are the linear spaces with the co-ordinate wise addition and scalar multiplication which are the BK-spaces with the norm

\[ ||x||_Z = ||x||_{Z_0} = ||Z^p x||_c. \]

**Theorem 2.2** The sequence spaces \(Z\) and \(Z_0\) are linearly isomorphic to the spaces \(c\) and \(c_0\) respectively, i.e. \(Z \cong c\) and \(Z_0 \cong c_0\).

**Theorem 2.3** The inclusions \(Z_0 \subset Z\) strictly hold for \(p \neq 1\).

The following Lemma and the inequality has been used for establishing some results of this article.

**Lemma 2.4** If \(I \subset Z^p\) and \(M \subset N\). If \(M \notin I\) then \(M \cap N \notin I\) (Şengönül, 2007).

Let \(p = (p_k)\) be the bounded sequence of positive real numbers. For any complex \(\lambda\), whenever \(H = \sup p_k < \infty\), we have \(|\lambda|^{p_k} \leq \max(1, |\lambda|^{p_k})\). Also, whenever \(H = \sup p_k\) we have \(|\alpha + b_j|^{p_k} \leq C(|\alpha|^{p_k} + |b_j|^{p_k})\) where \(C = \max(1, 2^{k-1})\). (Maddox, 1969) cf. (Khan Ebadullah, Ayhan Esi, Khan, & Shafiq, 2013a; Khan & Khan, 2014b; Khan & Sabiha, 2011; Malkowsky, 1997; Ng & Lee, 1978).

Recently Khan Ebadullah, Ayhan Esi, Khan, and Shafiq (2013a) introduced various Zweier sequence spaces the following sequence spaces.

\[ Z^1_I = \{ x = (x_k) \in \omega : \{ k \in \mathbb{N} : I - \lim Z^p x = L, \text{ for some } L \} \in I \} \]
\[ Z^1_0 = \{ x = (x_k) \in \omega : \{ k \in \mathbb{N} : I - \lim Z^p x = 0 \} \in I \} \]
\[ Z^{\infty}_I = \{ x = (x_k) \in \omega : \{ k \in \mathbb{N} : \sup |Z^p x| < \infty \} \in I \} \]

We also denote by

\[ m^I_x = Z^{\infty}_x \cap Z^I \quad \text{and} \quad m^I_{x_0} = Z^{\infty}_{x_0} \cap Z^I. \]
In this article, we introduce the following sequence spaces.

For any \( \varepsilon > 0 \), we have
\[
\ell_2(q) = \{ x = (x_{ij}) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L_{ij}|^q \geq \varepsilon \} \in I_2, \text{ for some } L \in \mathbb{C} ;
\]
\[
\ell_2^{L_0}(q) = \{ x = (x_{ij}) \in \mathbb{N} \times \mathbb{N} : |x_{ij}|^q \geq \varepsilon \} \in I_2; \]
\[
\ell_2^{\infty}(q) = \{ x = (x_{ij}) \in \mathbb{N} \times \mathbb{N} : \sup_{ij} |x_{ij}|^q < \infty \}.
\]

We also denote by
\[
\ell_2^{M_1}(q) = \ell_2^{\infty}(q) \cap \ell_2(q)
\]
and
\[
\ell_2^{M_0}(q) = \ell_2^{\infty}(q) \cap \ell_2^{L_0}(q)
\]
where \( q = (q_{ij}) \) is a double sequence of positive real numbers.

Throughout the article, for the sake of convenience now we will denote by \( Z^p x = x' \) for all \( x \in \mathbb{L} \).

### 3. Main results

**Theorem 3.1** The sequence spaces \( \ell_2^{M_1}(q) \), \( \ell_2^{M_2}(q) \), \( \ell_2^{M_0}(q) \) are linear spaces.

**Proof** We shall prove the result for the space \( \ell_2^{M_1}(q) \).

The proof for the other spaces will follow similarly.

Let \((x_{ij}), (y_{ij}) \in \ell_2^{M_1}(q)\) and let \( \alpha, \beta \) be scalars. Then for a given \( \varepsilon > 0 \), we have
\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L_{ij}|^q \geq \frac{\varepsilon}{2M_1} \} \in I_2
\]
and
\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L_{ij}|^q \geq \frac{\varepsilon}{2M_2} \} \in I_2
\]
where
\[
M_1 = D.\max \{1, \sup_{ij} |\alpha|^q\}
\]
\[
M_2 = D.\max \{1, \sup_{ij} |\beta|^q\}
\]
and
\[
D = \max \{1, 2^{H-1} \} \quad \text{where } H = \sup_{ij} q_{ij} \geq 0.
\]

Let
\[
A_1 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij} - L_{ij}|^q < \frac{\varepsilon}{2M_1} \} \in I_2
\]
\[
A_2 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |y_{ij} - L_{ij}|^q < \frac{\varepsilon}{2M_2} \} \in I_2
\]
be such that \( A_1, A_2 \in I \). Then
\[
A_3 = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |(\alpha x_{ij} + \beta y_{ij}) - (\alpha L_1 + \beta L_2)|^q < \varepsilon \}
\]
\[
\sup \{(i,j) \in \mathbb{N} \times \mathbb{N} : |\alpha|^q |x_{ij} - L_{ij}|^q < \frac{\varepsilon}{2M_1} |\alpha|^q \}
\]
\[
\cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : |\beta|^q |y_{ij} - L_{ij}|^q < \frac{\varepsilon}{2M_2} |\beta|^q \}
\]
Thus $A_1^c \subseteq A_1^c \cup A_2^c \in I$. Hence $(\alpha x'_j + \beta y'_j) \in \| z \|_q(\mathcal{Z})$. Therefore $\| z \|_q(\mathcal{Z})$ is a linear space. Proof of $\| z \|_q(\mathcal{Z})$ follows since it is a special case of $\| z \|_q(\mathcal{Z})$.

**Remark** The sequence spaces $\| m \|_q^2_m(q), \| m \|_q^2_m(q)$, are linear spaces since each is an intersection of two of the linear spaces in Theorem 3.1.

**Theorem 3.2** Let $(q_m) \in J^\infty$. Then $\| m \|_q^2_m(q)$ and $\| m \|_q^2_m(q)$ are paranormed spaces, paranormed by $g(x') = \sup \mid x'_j \mid^2$ where $M = \max(1, \sup _{i,j} q_{ij})$.

**Proof** Let $x' = (x'_j), y' = (y'_j) \in \| m \|_q^2_m(q)$.

(1) Clearly, $g(x') = 0$ if and only if $x' = 0$.

(2) $g(x') = g(-x')$ is obvious.

(3) Since $\frac{q_{ij}}{M} \leq 1$ and $M > 1$, using Minkowski’s inequality, we have

\[
g(x' + y') = g(x'_j + y'_j) = \sup _{j} \mid x'_j + y'_j \mid^2 \leq \sup _{j} \mid x'_j \mid^2 + \sup _{j} \mid y'_j \mid^2 = g(x'_j) + g(y'_j)
\]

Therefore, $g(x' + y') \leq g(x') + g(y')$ for all $x', y' \in \| m \|_q^2_m(q)$.

(4) Let $(\lambda_j)$ be a double sequence of scalars with $(\lambda_j) \to \lambda, (i, j \to \infty)$ and $x' = (x'_j), x_0 = (x_{0,ij}) \in \| m \|_q^2_m(q)$ with $g(x'_j) \to g(x_{0,ij}) (i, j \to \infty)$.

Note that $g(\lambda x'_j) \leq \max(1, |\lambda|)g(x'_j)$. Then since the inequality $g(x'_j) \leq g(x'_j - x'_j) + g(x'_j)$ holds by subadditivity of $g$, the sequence $\{ g(x'_j) \}$ is bounded. Therefore, $| g(\lambda x'_j) - g(\lambda x'_0) | = | g(\lambda x'_j) - g(\lambda x'_0) | + | g(\lambda x'_0) - g(\lambda x'_0) | < \epsilon$, and $| x'_j - x'_j | \to 0$ as $(i, j \to \infty)$. That is to say that the scalar multiplication is continuous. Hence $\| m \|_q^2_m(q)$ is a paranormed space.

**Theorem 3.3** $\| m \|_q^2_m(q)$ is a closed subspace of $J^\infty(q)$.

**Proof** Let $(x_{ij}^{(mn)})$ be a Cauchy sequence in $\| m \|_q^2_m(q)$ such that $x_{ij}^{(mn)} \to x'$. We show that $x' \in \| m \|_q^2_m(q)$.

Since $(x_{ij}^{(mn)}) \in \| m \|_q^2_m(q)$, then there exists $(a_{mn})$ such that

\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}^{(mn)} - a_{mn}| \geq \epsilon \} \subseteq I
\]

We need to show that

(1) $(a_{mn})$ converges to $a$.

(2) If $U = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}^{(n)} - a| < \epsilon \}$ then $U^c \in I$.

Since $(x_{ij}^{(mn)})$ is a Cauchy sequence in $\| m \|_q^2_m(q)$ then for a given $\epsilon > 0$, there exists $(i_0, j_0) \in \mathbb{N} \times \mathbb{N}$ such that

\[
\sup _{ij} \mid x_{ij}^{(mn)} - x_{ij}^{(np)} \mid < \frac{\epsilon}{3}, \text{ for all } (m,n),(p,q) \geq (i_0,j_0)
\]

For a given $\epsilon > 0$, we have

$B_{mnpq} = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}^{(mn)} - x_{ij}^{(np)}| < \frac{\epsilon}{3} \}$

$B_{mnp} = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}^{(np)} - a_{pq}| < \frac{\epsilon}{3} \}$

$B_{mn} = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x_{ij}^{(mn)} - a_{mn}| < \frac{\epsilon}{3} \}$
Then \( B'_{mn,pq} \cap B'_{pq} \in I \).

Let \( B' = B'_{mn,pq} \cap B'_{pq} \cap B'_{mnpq} \) where
\[
B = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |a_{pq} - a_{mn}| < \epsilon \}.
\]

Then \( B' \in I \).

We choose \((i_0,j_0) \in B' \), then for each \((m,n),(p,q) \geq (i_0,j_0)\), we have
\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : |a_{pq} - a_{mn}| < \epsilon \} \supseteq \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(pq)}_y - a_{pq}| < \epsilon/3 \}
\]
\[
\cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(mn)}_y - x^{(pq)}_y| < \epsilon/3 \}
\]
\[
\cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(mn)}_y - a_{mn}| < \epsilon/3 \}
\]

Then \((a_{mn})\) is a Cauchy sequence of scalars in \( \mathbb{C} \), so there exists a scalar \( a \in \mathbb{C} \) such that \( a_{mn} \to a \), as \((m,n) \to \infty \).

For the next part let \( 0 < \delta < 1 \) be given. Then we show that if \( U = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x'_y - a|^3 < \delta \} \), then \( U \in I \).

Since \( x^{(mn)} \to x' \), then there exists \((p_0,q_0) \in \mathbb{N} \times \mathbb{N} \) such that
\[
P = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(pq)}_y - x'_y| < \left(\frac{\delta}{3D}\right)^2\}
\]

which implies that \( P' \in I \). The number \((p_0,q_0)\) can be so chosen that together with (1), we have
\[
Q = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |a_{pq} - a|^3 < \left(\frac{\delta}{3D}\right)^2\}
\]

such that \( Q' \in I \).

Since \( \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(pq)}_y - a_{pq}| \geq \delta \} \in I \). Then we have a subset \( S \) of \( \mathbb{N} \times \mathbb{N} \) such that \( \mathbb{N} \times \mathbb{N} \), where
\[
S = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(pq)}_y - a_{pq}|^3 < \left(\frac{\delta}{3D}\right)^2\}.
\]

Let \( U' = P' \cap Q' \cap S \), where \( U = \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x'_y - a|^3 < \delta \} \).

Therefore for each \((i,j) \in U' \), we have
\[
\{(i,j) \in \mathbb{N} \times \mathbb{N} : |x'_y - a|^3 < \delta \} \supseteq \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(pq)}_y - x'_y| < \left(\frac{\delta}{3D}\right)^2\}
\]
\[
\cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : |x^{(pq)}_y - a_{pq}|^3 < \left(\frac{\delta}{3D}\right)^2\}
\]
\[
\cap \{(i,j) \in \mathbb{N} \times \mathbb{N} : |a_{pq} - a|^3 < \left(\frac{\delta}{3D}\right)^2\}
\]

Then the result follows.

**Theorem 3.4** The spaces \( \ell_2(q) \) and \( \ell_2(q) \) are nowhere dense subsets of \( \ell_1(q) \).

**Proof** Since the inclusions \( \ell_2(q) \subset \ell_1(q) \) and \( \ell_2(q) \subset \ell_1(q) \) are strict so in view of Theorem 3.3 we have the following result.

**Theorem 3.5** The spaces \( \ell_2(q) \) and \( \ell_2(q) \) are not separable.
Proof. We shall prove the result for the space \( z_{M}^{2}(q) \). The proof for the other spaces will follow similarly.

Let \( M \) be an infinite subset of \([\mathbb{N} \times \mathbb{N}]\) of such that \( M \in I \). Let

\[
q_{ij} = \begin{cases} 
1, & \text{if } (i,j) \in M, \\
2, & \text{otherwise}.
\end{cases}
\]

Since \( B = \{B(x', \frac{1}{2}) : x' \in P_{0}\} \).

Then for all sufficiently large \( \ell \), we have

\[
B_\ell = \{(i,j) \in [\mathbb{N} \times \mathbb{N}] : |x'_j|^\gamma_i \geq \epsilon \} \in I
\]

Let \( G_0 = K^c \cup B_0 \). Then \( G_0 \in I \).

Then for all sufficiently large \( \ell \),

\[
\{(i,j) \in [\mathbb{N} \times \mathbb{N}] : |x'_j|^\gamma_i \geq \epsilon \} \subseteq \{(i,j) \in [\mathbb{N} \times \mathbb{N}] : |x'_j|^\gamma_i \geq \epsilon \} \in I.
\]

Therefore \( x'_j \in z_{M}^{2}(q) \).

The converse part of the result follows obviously.

The other inclusion follows by symmetry of the two inequalities.

4. Conclusion

The notion of Ideal convergence (I-convergence) is a generalization of the statical convergence and equally considered by the researchers for their research purposes since its inception. Along with this the very new concept of double sequences has also found its place in the field of analysis. It is also being further discovered by mathematicians all over the world. In this article, we introduce para-norm ideal convergent double sequence spaces using Zweier transform. We study some topological and algebraic properties. Further we prove some inclusion relations related to these new spaces.
Acknowledgements
The authors would like to record their gratitude to the reviewer for his careful reading and making some useful corrections which improved the presentation of the paper.

Funding
The authors received no direct funding for this research.

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Citation information
Cite this article as: On Zweier paranorm I-convergent sequence spaces.

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