A SOLUTION OF THE SCHOTTKY-TYPE PROBLEM
FOR CURVES WITH AUTOMORPHISMS

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Abstract. In this paper, an explicit hierarchy of differential equations for the \( \tau \)-functions defining the moduli space of curves with automorphisms as a subscheme of the Sato Grassmannian is obtained. The Schottky problem for Riemann surfaces with automorphisms consists of characterizing those p.p.a.v. that are Jacobian varieties of a curve with a non-trivial automorphism. A characterization in terms of hierarchies of p.d.e. for theta functions is also given.

1. Introduction

The objective of the Schottky problem is to characterize the principally polarized abelian varieties (p.p.a.v.) which are Jacobians of smooth algebraic curves. This problem was solved, in the framework of the theory of KP equations, by Shiota (Sh). The analogous problem for Prym varieties was studied and partially solved by Shiota (Sh2) in terms of the BKP hierarchy.

The Schottky problem for Pryms is also related to the characterization of Jacobians of algebraic curves which admit a non-trivial involution. The moduli space of curves with non-trivial automorphisms was studied in the previous paper ([GMP]), and the points of the Sato Grassmannian defined by those curves were characterized. Moreover, an explicit set of algebraic equations defining the moduli space of curves

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with automorphisms as a subscheme of the Sato Grassmannian were obtained.

In §3 of the present paper, these equations are given as an explicit hierarchy of differential equations for the $\tau$-functions (Theorems 3.14 and 3.15). The first non-trivial equations are studied in detail and related to the KP and KdV hierarchies.

These results do not solve the Schottky problem for Riemann surfaces with automorphisms. For solving this problem one must characterize the conditions that a theta function of a p.p.a.v. should satisfy in order to be a jacobian theta function of a curve with a non-trivial automorphism. Such a characterization is obtained in this paper, as follows.

Let $X_\Omega$ be an irreducible p.p.a.v. of dimension $g$. For a natural number $r$, the $r$-th Baker-Akhiezer functions are defined from the theta function of $X_\Omega$ and some auxiliary data; namely, an $r$-tuple of $g \times \infty$-matrices $(A^{(1)}, \ldots, A^{(r)})$ of rank $g$ and $r$ symmetric quadratic forms $Q^{(1)}, \ldots, Q^{(r)}$ (see subsection 4.B for precise definitions).

**Theorem** (Characterization). Let $X_\Omega$ be an irreducible p.p.a.v. of dimension $g > 1$.

Then the following conditions are equivalent.

1. There exists a projective irreducible smooth genus $g$ curve $C$ with a non-trivial automorphism $\sigma : C \to C$ such that $X_\Omega$ is isomorphic as a p.p.a.v. to the Jacobian of $C$.
2. There exist a prime number $p$, $p$ matrices $A^{(1)}, \ldots, A^{(p)}$ ($A^{(j)}$ being a $g \times \infty$-matrix of rank $g$) and $p$ symmetric quadratic forms $Q^{(1)}, \ldots, Q^{(p)}$, such that
   a) for some $\xi_0 \in \mathbb{C}^g$ the corresponding BA-functions satisfy the $(1, p, 1)$-KP hierarchy
   
   \[
   \text{Res} \left( \sum_{j=1}^{p} z^{-\delta_j u - \delta_j v} \psi^{(j)}_{u, \xi_0}(z, t)\psi^{*(j)}_{v, \xi_0}(z, s) \right) \, dz = 0, \quad \text{and}
   \]

   b) there exists $\xi_1 \in \mathbb{C}^g$ (depending on $\xi_0$) such that
   
   \[
   \text{Res} \left( \sum_{j=1}^{p} z^{-\delta_j u - \delta_j v} \psi^{(j+1)}_{v+1, \xi_0}(z, \sigma^*(t))\psi^{*(j)}_{u, \xi_1}(z, s) \right) \, dz = 0
   \]

   where $\sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \ldots, t^{(p-1)})$.

This theorem can be understood as a translation into equations of the characterization theorems of §5, where they are stated in terms of orbits of finite dimension as in the approach of Mulase ([M]) and Shiota ([Sh]). The idea of our proof is similar to the first part of the paper of...
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Shiota; in fact, our main ingredient is a generalization of Theorem 6 of Shiota to the case of the \((1,\ldots,1)\)-KP hierarchy (see Theorem 6.2).

Standard arguments allow us to reinterpret the above identities as a hierarchy of PDE for the associated tau functions. The final step of our program should consist of proving that the “first” non-trivial equations of the hierarchy suffice to characterize the theta functions of Riemann surfaces with non-trivial automorphisms. This problem has an analytical counterpart which is the analogue of Shiota’s proof of Novikov conjecture. We hope to study it in a future paper.

2. Preliminary results

2.A. Formal group schemes. We establish the notation and recall some of the results proved in the papers [MP2] and [GMP] that will be used in subsequent sections.

In what follows, \(V\) will denote a finite \(\mathbb{C}((z))\)-algebra endowed with an action of the group \(\mathbb{Z}/p\mathbb{Z}\), where \(p\) is a prime number, such that its fixed subset \(V^{\mathbb{Z}/p\mathbb{Z}}\) is equal to \(\mathbb{C}((z))\). Let \(\sigma\) denote a (fixed) generator of this subgroup of \(\text{Aut}_{\mathbb{C}((z))}V\).

Since \(V\) is a finite \(\mathbb{C}((z))\)-algebra, there are canonical maps given by the trace and the norm

\[
\text{Tr}: V \to \mathbb{C}((z)) \\
\text{Nm}: V \to \mathbb{C}((z))
\]

which map an element \(g\) in \(V\) to the trace (respectively norm) of the homothety of \(V\) defined by \(g\) (as a \(\mathbb{C}((z))\)-vector space).

Throughout the paper, we will consider only the following two cases.

(a) Ramified case: \(V = V_{\mathbb{R}} = \mathbb{C}((z_1))\), where the \(\mathbb{C}((z))\)-algebra structure is given by mapping \(z\) to \(z^p_1\) and \(\sigma\) is given by \(\sigma(z_1) = \omega z_1\), where \(\omega\) is a primitive \(p\)-th root of 1 in \(\mathbb{C}\). In this case we set \(V^+_\mathbb{R} = \mathbb{C}[[z_1]]\) and \(V^-_\mathbb{R} = z_1^{-1}\mathbb{C}[z_1^{-1}]\).

(b) Non-ramified case: \(V = V_{\mathbb{R}} = \mathbb{C}((z_1)) \times \cdots \times \mathbb{C}((z_p))\), where the \(\mathbb{C}((z))\)-algebra structure is given by mapping \(z\) to \((z_1,\ldots,z_p)\) and \(\sigma\) is given by \(\sigma(z_i) = z_{i+1}\) (for \(i < p\)) and \(\sigma(z_p) = z_1\). In this case we set \(V^+_{\mathbb{R}} = \mathbb{C}[[z_1]] \times \cdots \times \mathbb{C}[[z_p]]\) and \(V^-_{\mathbb{R}} = z_1^{-1}\mathbb{C}[z_1^{-1}] \times \cdots \times z_p^{-1}\mathbb{C}[z_p^{-1}]\).

Example 2.1. Let \(C\) be a projective irreducible smooth curve with an order \(p\) automorphism, denoted by \(\sigma_C : C \to C\).

If \(\pi : C \to \overline{C} := C/\langle \sigma_C \rangle\) denotes the quotient map and \(p \in \overline{C}\) is a smooth point, then the \(\widehat{\mathcal{O}}_{\overline{C},p}\)-algebra \(\widehat{\mathcal{O}}_{C,\pi^{-1}(p)}\) is isomorphic to one of the above types.
The formal base curve is \( \hat{C} := \text{Spf } \mathbb{C}[[z]] \) and the formal spectral cover is \( \hat{C}_V := \text{Spf } V^+ \). Let \( \Gamma_V \) be the formal group scheme representing the functor

\[
\{ \text{category of formal } \mathbb{C}\text{-schemes} \} \sim \{ \text{category of groups} \} \\
S \sim (V \hat{\otimes}_\mathbb{C} H^0(S, \mathcal{O}_S))^*\] 

where the subscript 0 denotes the connected component of the identity and the superscript * denotes the invertible elements. Replacing \( V \) by \( V^+ \) (respectively by \( 1 + V^- \)) we define the subgroup \( \Gamma^+_V \) (respectively \( \Gamma^-_V \)) and thus obtain the decomposition

\[
\Gamma_V = \Gamma^-_V \times \Gamma^+_V .
\]

The formal Jacobian of the formal spectral cover is the formal group scheme \( J(\hat{C}_V) := \Gamma^-_V \). A straightforward calculation shows that \( J(\hat{C}_V) \) is the formal spectrum of the ring

\[
\mathcal{O}(J(\hat{C}_V)) = \mathbb{C}\{\{t_1^{(1)}, t_2^{(1)}, \ldots\}\} \hat{\otimes} \ldots \hat{\otimes} \mathbb{C}\{\{t_1^{(r)}, t_2^{(r)}, \ldots\}\}
\]

where \( t_i^{(j)} \) are indeterminates, \( \mathbb{C}\{\{t_1, t_2, \ldots\}\} \) denotes the inverse limit \( \lim_{\leftarrow} \mathbb{C}[[t_1, \ldots, t_n]] \), and \( r = 1 \) for the ramified case (in which case the superscript (1) is dropped) and \( r = p \) for the non-ramified one.

Replacing \( V, V^+ \) and \( V^- \) by \( \mathbb{C}((z)), \mathbb{C}[[z]] \) and \( z^{-1}\mathbb{C}[z^{-1}] \) respectively in the previous constructions, one obtains formal schemes \( \Gamma, \Gamma^+ \) and \( J(\hat{C}) := \Gamma^- \). It is straightforward that the canonical morphism \( \mathbb{C}((z)) \hookrightarrow V \) gives rise to \( \Gamma \hookrightarrow \Gamma_V \) and that the trace and the norm yield corresponding morphisms \( \Gamma_V \rightarrow \Gamma \).

Recall that the Abel map \( \phi_V : \hat{C}_V \longrightarrow J(\hat{C}_V) \) is the morphism corresponding to the \( \hat{C}_V \)-valued point of \( \Gamma_V \) associated to the \( r \)-tuple of series

\[
(2.2) \quad \left( \left(1 - \frac{\bar{z}_1}{z_1}\right)^{-1}, \ldots, \left(1 - \frac{\bar{z}_r}{z_r}\right)^{-1} \right)
\]

where \( \hat{C}_V \simeq \text{Spf } (\mathbb{C}[[\bar{z}_1]] \times \cdots \times \mathbb{C}[[\bar{z}_r]]) \), with \( r = 1 \) in the ramified case and \( r = p \) in the non-ramified one.

**Proposition 2.3.** The Albanese variety of \( \hat{C}_V \) is the pair \((J(\hat{C}_V), \phi_V)\).

*Proof.* This statement is proved in [MP] for the case \( V = \mathbb{C}((z)) \). The present case follows from that result and from the fact that the Albanese variety of a disjoint union is the product of the corresponding Albanese varieties. \( \square \)
The natural morphism \( \hat{C}_V \to \hat{C} \) induces two group homomorphisms; namely, the pull-back

\[
\pi^* : J(\hat{C}) \to J(\hat{C}_V)
\]

and the Albanese

\[
J(\hat{C}_V) \to J(\hat{C})
\]

The Albanese map coincides with the restriction of the \( Nm \) to \( J(\hat{C}_V) \); it will also be denoted by \( Nm \).

Observe that the automorphism \( \sigma \) of \( V \) gives rise to automorphisms of \( \Gamma V \) and \( J(\hat{C}_V) \) (also denoted by \( \sigma \)) such that \( \Gamma V^\sigma = \mathbb{C}(z) \) and \( J(\hat{C}_V)^\sigma = J(\hat{C}) \). In particular, it follows that \( \text{Tr} \) (respectively \( Nm \)) maps an element \( g \in \Gamma V \) to \( \sum_{i=0}^{p-1} \sigma^i(g) \) (respectively \( \prod_{i=0}^{p-1} \sigma^i(g) \)).

2.B. Infinite Grassmannians. Recall that the infinite Grassmaninan \( \text{Gr}(V) \) of the pair \( (V, V^+) \) is a \( \mathbb{C} \)-scheme, which is not of finite type, whose set of rational points is

\[
\bigg\{ \text{subspaces } U \subset V \text{ such that } U \to V/V^+ \text{ has finite dimensional kernel and cokernel} \bigg\}.
\]

This scheme is equipped with the determinant bundle, \( \text{Det}_V \), which is the determinant of the complex of \( \mathcal{O}_{\text{Gr}(V)} \)-modules

\[
\mathcal{L} \to V/V^+ \otimes_{\mathbb{C}} \mathcal{O}_{\text{Gr}(V)},
\]

where \( \mathcal{L} \) is the universal submodule of \( \text{Gr}(V) \) and the morphism is the natural projection. The connected components of the Grassmannian are indexed by the Euler–Poincaré characteristic of the complex. The connected component of index \( m \) will be denoted by \( \text{Gr}^m(V) \).

The group \( \Gamma V \) acts by homotheties on \( V \), and this action gives rise to a natural action on \( \text{Gr}(V) \)

\[
\Gamma V \times \text{Gr}(V) \to \text{Gr}(V).
\]

Furthermore, this action preserves the determinant bundle.

These facts allow us to introduce \( \tau \)-functions and Baker-Akhiezer functions of points of \( \text{Gr}(V) \). Let us recall the definition and some properties of these functions (\cite{MP2}, §3).

The determinant of the morphism \( \mathcal{L} \to V/V^+ \otimes_{\mathbb{C}} \mathcal{O}_{\text{Gr}(V)} \) gives rise to a canonical global section

\[
\Omega_+ \in H^0(\text{Gr}^0(V), \text{Det}_V^*).
\]

In order to extend this section to \( \text{Gr}(V) \) (in a non-trivial way), we fix elements \( v_m \in V^+ \) for \( m > 0 \) such that \( \dim V^+/v_mV^+ = m \). Setting \( v_{-m} := v_m^{-1} \) for \( m > 0 \), we define \( \Omega_+(U) := \Omega_+(v_m^{-1}U) \) for \( U \in \text{Gr}^m(V) \).
Now, the $\tau$-function and BA functions will be introduced following [MP2]. The $\tau$-function of $U$, $\tau_U(t)$, is a function on $\mathcal{J}(\mathcal{C}_V)$ and is introduced as a suitable trivialization of the function $g \mapsto \Omega_+(gU)$ for $g \in \mathcal{J}(\mathcal{C}_V)$,

$$\tau_U(g) = \frac{\Omega_+(gU)}{g\delta_U}$$

where $\delta_U$ is a non-zero element in the fibre of $\text{Det}_U^*$ over $U$.

Let $t$ be the set of variables $(t^{(1)}, \ldots, t^{(r)})$ (where $t^{(j)} = (t^{(j)}_1, t^{(j)}_2, t^{(j)}_3, \ldots)$) and $z_j$ denote $(z_1, \ldots, z_r)$. For $1 \leq u, v \leq r$, let $[z_u] := (z_u, \frac{z_u}{2}, \frac{z_u}{3}, \ldots)$, $t + [z_u] := (t^{(1)} + [z_u], \ldots, t^{(r)} + [z_u])$, $U_{uu} = U$ and, if $u \neq v$, $U_{uv} := (1, \ldots, z_u, \ldots, z_v^{-1}, \ldots, 1) \cdot U$.

The $u$-th Baker-Akhiezer function of a point $U \in \text{Gr}(V)$ is the $V$-valued function

$$\psi_{u,U}(z, t) := \left( \exp \left( - \sum_{i \geq 1} \frac{t^{(1)}_i}{z^i_1} \right), \frac{\tau_{u,t}(t + [z_1])}{\tau_U(t)}, \ldots, \frac{\tau_{u,t}(t + [z_r])}{\tau_U(t)} \right).$$

From the decomposition $V = \prod_{i=1}^r \mathbb{C}((z_i))$ we can write

$$\psi_{u,U}(z, t) = (\psi_{u,U}^{(1)}(z_1, t), \ldots, \psi_{u,U}^{(r)}(z_r, t)) \in \prod_{i=1}^r \left( \mathbb{C}((z_i)) \otimes \mathcal{O}(\mathcal{J}(\mathcal{C}_V)) \right).$$

Let $p_{it}(t)$ be the Schur polynomials and $\tilde{\delta}_{t^{(j)}} = (\partial_{t^{(j)}_1}, \frac{1}{2}\partial_{t^{(j)}_2}, \frac{1}{3}\partial_{t^{(j)}_3}, \ldots)$. Using the identity $\exp \left( \sum_{i \geq 1} z^i_j \tilde{\delta}_{t^{(j)}} \right) \tau_{u,t}(t) = \tau_{u,t}(t + [z_j])$, we obtain

$$\psi_{u,U}^{(j)}(z_j, t) = \exp \left( - \sum_{i \geq 1} \frac{t^{(j)}_i}{z^i_j} \right) \tau_{u,t}(t + [z_j]) =$$

$$\left( \sum_{i \geq 0} p_{it}(t^{(j)}) \frac{\tau_{u,t}(t + [z_j])}{\tau_U(t)} \right).$$

The main property of these Baker-Akhiezer functions is that they can be understood as generating functions for $U$ as a subspace of $V$, as we recall next.

**Theorem 2.5 ([MP2])**. Let $U \in \text{Gr}^m(V)$. Then

$$\psi_{u,U}(z, t) = v^{-1}_m(1, \ldots, z_u, \ldots, 1) \sum_{i \geq 0} \left( \psi_{u,U}^{(i,1)}(z_1), \ldots, \psi_{u,U}^{(i,r)}(z_r) \right) p_{ui,U}(t)$$

where

$$\{(\psi_{u,U}^{(i,1)}(z_1), \ldots, \psi_{u,U}^{(i,r)}(z_r)) \mid i > 0, 1 \leq u \leq r\}$$
is a basis of $U$ and $p_{ui,U}(t)$ are functions in $t$.

Consider the following pairing

$$V \times V \to \mathbb{C}$$

$$(a, b) \mapsto \text{Res}_{z=0} \text{Tr}(a, b) \, dz.$$  

Since it is non-degenerate, there is an involution of $\text{Gr}(V)$ which maps any point $U$ to its orthogonal complement $U^\perp$. This involution sends the connected component $\text{Gr}^m(V)$ to $\text{Gr}^{1-m-p}(V)$ in the ramified case and to $\text{Gr}^{-m}(V)$ in the non-ramified one.

Finally, the adjoint Baker-Akhiezer functions of $U$ are defined by

$$\psi_{u,U}^*(z, t) := \psi_{u,U^\perp}(z, -t),$$

whose components are given by

$$\psi_{u,U}^{\sigma(j)}(z_j, s) = \left( \sum_{i \geq 0} p_i(s^{(j)}) \frac{\left( \sum_{i \geq 0} p_i(-\tilde{\partial}_s^{(j)}) \frac{z_j^i}{\tau_U(s)} \right) \tau_{U^\sigma}(s)}{\tau_U(s)} \right).$$

3. Moduli of curves with an order $p$ automorphism

The aim of this section is to give explicit differential equations that characterize the tau functions coming from curves with an order $p$ automorphism among the tau functions coming from curves.

Recall from §5 of [GMP] that $\mathcal{M}^\infty(p, R)$ (respectively $\mathcal{M}^\infty(p, NR)$) is the subscheme of $\text{Gr}(V_R)$ (respectively $\text{Gr}(V_{NR})$) parametrizing $(C, \sigma_C, x, t_x)$ where $C$ is a curve, $\sigma_C$ is an order $p$ automorphism of $C$, $x$ is a smooth point of $C$ fixed under $\sigma_C$ (respectively an orbit consisting of $p$ pairwise distinct points) and $t_x$ is a formal parameter at $x$.

More precisely, if $\mathcal{M}^\infty(1)$ (respectively $\mathcal{M}^\infty(p)$) is the moduli space parameterizing $(C, x, t_x)$ where $C$ is a curve, $x$ is a smooth point (respectively $p$ pairwise distinct smooth points) and a formal parameter at $x$, then the Krichever map induces an immersion

$$\text{Kr}: \mathcal{M}^\infty(1) \hookrightarrow \text{Gr}(V_R)$$

(respectively $\text{Kr}: \mathcal{M}^\infty(p) \hookrightarrow \text{Gr}(V_{NR})$) such that

$$\mathcal{M}^\infty(p, R) = \mathcal{M}^\infty(1) \cap \text{Gr}(V_R)^\sigma$$

(respectively $\mathcal{M}^\infty(p, NR) = \mathcal{M}^\infty(p) \cap \text{Gr}(V_{NR})^\sigma$) where $\text{Gr}(V)^\sigma$ denotes the set of points in $\text{Gr}(V)$ fixed under the action of $\sigma$.

Therefore, our task consists of writing down the hierarchies corresponding to these subschemes.
3.A. Hierarchies for invariant subspaces. The automorphism of \( \text{Gr}(V) \) induced by \( \sigma : V \to V \) will also be denoted by \( \sigma \); it preserves the determinant bundle. If we denote by \( \text{Gr}(V)^{\sigma} \) the set of points in \( \text{Gr}(V) \) fixed under the action of \( \sigma \), then it is known that \( \text{Gr}(V)^{\sigma} \) is a closed subscheme.

The action of \( \sigma \) on \( \text{Gr}(V) \) can be easily described in terms of the Baker-Akhiezer and the tau functions, as we show next.

Let us begin by studying the ramified case. We denote
\[
\lambda t = (\lambda t_1, \lambda^2 t_2, \ldots, \lambda^j t_j, \ldots)
\]
for any \( \lambda \in \mathbb{C} \) and
\[
\sigma^*(t) := \omega^{-1} t.
\]

Then
\[
\tau_{\sigma(U)}(t) = \tau_U(\omega^{-1} t)
\]
(up to a constant) and
\[
(3.2) \quad \psi_{\sigma(U)}(z_1, t) = \psi_U(\omega^{-1} z_1, \omega^{-1} t) = \sigma^{-1}(\psi_U(z_1, \sigma^*(t)))
\]
where the expression on the r.h.s. corresponds to the action of \( \sigma \) on \( V \); that is, since \( \psi_U(z_1, t) \) is \( V \)-valued, \( \sigma \) acts on \( z_1 \) and acts trivially on \( t \).

It is also known that a point \( U \in \text{Gr}(V) \) lies in \( \text{Gr}(V)^{\sigma} \) if and only if its BA functions satisfy the following identities ([MP2], Theorem 3.15).

\[
(3.3) \quad \text{Res}_{z=0} \text{Tr} \left( \frac{1}{z_1} \psi_{\sigma(U)}(z_1, t) \psi_U^*(z_1, s) \right) \frac{dz}{z} = 0
\]

Then, one has the following

**Theorem 3.4** (ramified case). Let \( V = V_R \) and let \( U \) be a closed point of \( \text{Gr}(V) \).

Then \( U \) is a point of \( \text{Gr}(V)^{\sigma} \) if and only its \( \tau \)-function \( \tau_U \) satisfies the following differential equations
\[
(3.5) \quad \sum_{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = 1 \atop \alpha_1 - \beta_1 \equiv j \pmod{p}} \left( D_{\lambda_1, \alpha_1}(-\tilde{\partial}_x) p_{\beta_1}(\tilde{\partial}_s) D_{\lambda_2, \alpha_2}(\tilde{\partial}_s) p_{\beta_2}(-\tilde{\partial}_x) \right) \tau_U(x) \tau_U(s) = 0
\]
for all Young diagrams \( \lambda_1, \lambda_2 \) and all integers \( j \in \{0, \ldots, p-1\} \), where the definition of \( D_{\lambda, \alpha} \) is
\[
D_{\lambda, \alpha}(\tilde{\partial}_y)(f(y)) := \sum_{\lambda - \mu = (\alpha)} \chi_{\mu}(\tilde{\partial}_y)(f(y))|_{y=0}.
\]
Proof. Let $U$ be a closed point of $\text{Gr}(V)$. The residue condition given by (3.3) is equivalent to the vanishing of the constant term of

$$
\sum_{j=1}^{p} \frac{1}{\omega^j z_1} \psi_{\sigma(U)}(\omega^j z_1, t) \psi^*_U(\omega^j z_1, s) = \sum_{j=1}^{p} \frac{1}{\omega^j z_1} \psi_U(\omega^j-1 z_1, \omega^{-1} t) \psi^*_U(\omega^j z_1, s)
$$

Using (2.3) and (2.6) this coefficient turns out to be

$$
\sum_{j=1}^{p} \sum_{\substack{\beta_1 + \beta_2 - \alpha - 1 = 1 \\
\alpha_1 - \beta_1 \equiv \beta_1 (\mod p)}} \omega^{j(\beta_1 + \beta_2 - \alpha_2)} \cdot p_{\alpha_1}(-\omega^{-1} t) p_{\beta_1}(-\tilde{\partial}_t \tau_U(\omega^{-1} t)) \cdot p_{\alpha_2}(s) p_{\beta_2}(-\tilde{\partial}_s \tau_U(s)) .
$$

Since this coefficient must vanish for each $p$-th root of $\omega^j$ of $1$ (note that the case $\omega^0 = 1$ is precisely the KP-hierarchy, see [MP]), we obtain the equivalent conditions

$$
\sum_{\substack{\beta_1 + \beta_2 - \alpha - 1 = 1 \\
\alpha_1 - \beta_1 \equiv \beta_1 (\mod p)}} p_{\alpha_1}(-\omega^{-1} t) p_{\beta_1}(-\tilde{\partial}_t \tau_U(\omega^{-1} t)) p_{\alpha_2}(s) p_{\beta_2}(-\tilde{\partial}_s \tau_U(s)) = 0
$$

for all $j$ in $\{0, \ldots, p-1\}$ and all $t$ and $s$.

Equivalently (substituting $\omega^{-1} t$ by $x$), we obtain

$$
(3.6) \sum_{\substack{\beta_1 + \beta_2 - \alpha - 1 = 1 \\
\alpha_1 - \beta_1 \equiv \beta_1 (\mod p)}} p_{\alpha_1}(-x) p_{\beta_1}(\tilde{\partial}_x \tau_U(x)) p_{\alpha_2}(s) p_{\beta_2}(-\tilde{\partial}_s \tau_U(s)) = 0
$$

for all values of $x$ and $s$, and for all $j$ in $\{0, \ldots, p-1\}$.

Since the vanishing of a function $f(x, s)$ (such as the left hand side of (3.6)) for all values of $x$ and $s$ is equivalent to the vanishing of $\chi_{\lambda_1}(\tilde{\partial}_x) \chi_{\lambda_2}(\tilde{\partial}_s) f(x, s)|_{x=0, s=0}$ for all Young diagrams $\lambda_1$ and $\lambda_2$, a calculation shows that (3.6) is equivalent to (3.3), thus proving the theorem. \(\square\)

Remark 3.7. We now compute the first equations in the previous statement and relate them to the KP equations. Observe that if $E_j$ denotes the l.h.s. of equation (3.3) for $j \in \{0, \ldots, p-1\}$, then the KP hierarchy is $E_0 + \cdots + E_{p-1} = 0$. Let us make this explicit.

The first non trivial equation in the KP hierarchy, which corresponds to the Young diagrams $\lambda_1 = (1, 1, 1)$ and $\lambda_2 = 0$, is the celebrated KP equation

$$
\tau_U(0) \chi_{(2,2)}(\tilde{\partial}_s) \tau_U(s)|_{s=0} - p_1(\tilde{\partial}_x) \tau_U(x)|_{x=0} \cdot \chi_{(2,1)}(\tilde{\partial}_s) \tau_U(s)|_{s=0}
$$

$$
+ p_2(\tilde{\partial}_x) \tau_U(x)|_{x=0} \chi_{(1,1)}(\tilde{\partial}_s) \tau_U(s)|_{s=0} = 0 .
$$
On the other hand, equations (3.5) for the same Young diagrams are
\[
\begin{align*}
p_1(\partial_x)\tau_U(x)_{|x=0} \cdot \chi_{(2,1)}(\partial_s)\tau_U(s)_{|s=0} &= 0 \\
\tau_U(0) \chi_{(2,2)}(\partial_s)\tau_U(s)_{|s=0} + p_2(\partial_x)\tau_U(x)_{|x=0} \chi_{(1,1)}(\partial_s)\tau_U(s)_{|s=0} &= 0,
\end{align*}
\]
for \( p = 2 \) and
\[
\begin{align*}
p_1(\partial_x)\tau_U(x)_{|x=0} \cdot \chi_{(2,1)}(\partial_s)\tau_U(s)_{|s=0} &= 0 \\
\tau_U(0) \chi_{(2,2)}(\partial_s)\tau_U(s)_{|s=0} &= 0 \\
p_2(\partial_x)\tau_U(x)_{|x=0} \chi_{(1,1)}(\partial_s)\tau_U(s)_{|s=0} &= 0.
\end{align*}
\]
for \( p \neq 2 \)

**Example 3.8.** As an example we give some more equations from (3.5) for other pairs of Young diagrams. Note that the corresponding equations in the KP hierarchy are all trivial.

1. Consider the Young diagrams \( \lambda_1 = (1,0) \) and \( \lambda_2 = (1) \) and the corresponding \( p \) equations given by (3.5).

   If \( p = 2 \), both equations are trivial.

   However, if \( p \neq 2 \), two of the equations (3.5) (the cases \( \alpha_1 - \beta_1 \equiv 0 \) (mod \( p \)) and \( \alpha_1 - \beta_1 \equiv -2 \) (mod \( p \)) are equivalent to the following

   \[\tau_U(0) \cdot p_2(\partial_x)\tau_U(x)_{|x=0} = 0.\]

   Similarly, the consideration of \( \lambda_1 = (1,0) \) and \( \lambda_2 = 0 \) in (3.5) yields trivial equations for \( p = 2 \), whereas for \( p \neq 2 \) we obtain

   \[\tau_U(0) \cdot p_2(-\partial_x)\tau_U(x)_{|x=0} = 0.\]

2. More generally, for \( n \geq 2 \) and not divisible by \( p \) we obtain

   \[\tau_U(0) \cdot p_n(\partial_t)\tau_U(t)_{|t=0} = 0,\]

   \[\tau_U(0) \cdot p_n(-\partial_t)\tau_U(t)_{|t=0} = 0,\]

   when considering (3.5) with \( \lambda_i = (n-1,1) \) and \( \lambda_j = 0 \).

3. When \( n \geq 2 \) is not divisible by \( p \), we obtain

   \[p_1(\partial_t)\tau_U(t)_{|t=0} \cdot p_{n+1}(\partial_s)\tau_U(s)_{|s=0} = 0,\]

   \[p_1(\partial_t)\tau_U(t)_{|t=0} \cdot p_{n+1}(-\partial_s)\tau_U(s)_{|s=0} = 0\]

   by considering \( \lambda_i = (n-1,1) \) and \( \lambda_j = 0 \).

4. When \( n \geq 1 \) is not divisible by \( p \), the equations become

   \[p_2(\partial_t)\tau_U(t)_{|t=0} \cdot p_{n+2}(\partial_s)\tau_U(s)_{|s=0} = 0,\]

   \[p_2(-\partial_t)\tau_U(t)_{|t=0} \cdot p_{n+2}(-\partial_s)\tau_U(s)_{|s=0} = 0,\]

   for \( \lambda_i = (n+1,2) \) and \( \lambda_j = 0 \).
Remark 3.9. Let us study the relation with the KdV hierarchy. Consider an invariant point $U \in \text{Gr}(V)^{\sigma}$ (in the ramified case). If we impose the condition that $\mathbb{C}[z^{-1}] \cdot U = U$, then we obtain the $p$-KdV hierarchy. To see this, recall that the $\tau$-function of $U$, $\tau_U$, is (up to a constant) the pullback of the global section $\Omega_{+}$ to $\Gamma_V$ by

$$\Gamma_V \times \{U\} \longrightarrow \text{Gr}(V).$$

Since the condition means that $\Gamma^{-} \cdot U = U$, we obtain the following diagram

$$\xymatrix{ \mathcal{P} \ar[r] & \Gamma_V^- \ar[r] & \text{Gr}(V) \ar[ld] \ar[rd] \ar[r] & \Gamma_V / \Gamma^- \ar[l] \ar[d] \ar[r] \ar[l] & }$$

where

$$\mathcal{P} := \{g \in \mathcal{J}(\hat{C}_V) \mid \text{Nm}(g) = 1\} = \left\{\exp\left(\sum t_i z_i^{-1}\right) \in \Gamma_V \mid t_i = 0 \text{ for } i = p\right\}.$$}

Therefore, for the ramified case and for $p = 2$, it makes sense to write $\tau_U = \tau_U(t_1, t_3, \ldots)$ as an element of $\mathcal{O}(\mathcal{P}) = \mathbb{C}\{t_1, t_3, \ldots\}$ so that the $\tau$-function only depends on $t_i$ with $i$ odd. The resulting hierarchy is the classical KdV hierarchy (see also [SW], Proposition 5.11).

Now we focus in the non-ramified case. Denote

$$\sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \ldots, t^{(p-1)});$$

then the corresponding relations are as follows.

$$\tau_{\sigma(u)}(t) = \tau_{\sigma(u)}((t^{(1)}, \ldots, t^{(p)})) = \tau_U((t^{(p)}, t^{(1)}, t^{(2)}, \ldots, t^{(p-1)})) = \tau_U(\sigma^*(t))$$

(up to a constant) and

$$\psi_{u, \sigma(u)}(z, t) = (\psi^{(2)}_{u+1, U}(z_1, \sigma^*(t)), \psi^{(3)}_{u+1, U}(z_2, \sigma^*(t)), \ldots, \psi^{(1)}_{u+1, U}(z_p, \sigma^*(t))) = \sigma^{-1}(\psi^{(p)}_{u+1, U}(z, \sigma^*(t)))$$

where the action of $\sigma$ on the r.h.s. is the action on $V$-valued functions.

It is also known that a point $U \in \text{Gr}(V)$ lies in $\text{Gr}(V)^{\sigma}$ if and only if its BA functions satisfy the following identities (NP2)

$$\text{Res}_{z=0} \text{Tr} \left( \frac{\psi_{u, \sigma(u)}(z, t)}{(1, \ldots, z, \ldots, 1)} \cdot \frac{\psi_{v, U}^*(z, s)}{(1, \ldots, z, \ldots, 1)} \right) \, dz = 0$$

for all $u, v \in \{1, \ldots, p\}$ where $(1, \ldots, z_u, \ldots, 1)$ denotes the element of $V$ with entries equal to 1 except the $u$-th, which is $z_u$. 
In a similar manner to the ramified case but this time using (3.11), we obtain the following result.

**Theorem 3.12 (non-ramified).** Let $V = V_{NR}$ and let $U$ be a closed point of $\text{Gr}(V)$.

Then $U$ is a point of $\text{Gr}(V)^*$ if and only the $\tau$-functions of $U$, $\tau_{U_{uv}}$, satisfy the following differential equations.

$$
\sum_{j=1}^{p} \sum_{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = \delta_{j_u} + \delta_{j_v} - 1} (D_{\lambda_j, \alpha_1}(-\hat{\partial}_{x(j+1)})p_{\beta_1}(\hat{\partial}_{x(j+1)})D_{\lambda_j}^{+1}(\hat{\partial}_{x}) \tau_{U_{u+1,j+1}}(x) \cdot \\
\cdot D_{\mu_j, \alpha_2}(\hat{\partial}_{s(j)})p_{\beta_2}(\hat{\partial}_{s(j)})D_{\mu_j}^{+1}(\hat{\partial}_{s}) \tau_{U_{j_v}}(s) = 0
$$

for all Young diagrams $\lambda_1, \mu_1, \ldots, \lambda_p, \mu_p$, where the differential operator $D_{\mu_j}^{+1}$ is given by

$$
D_{\mu_j}^{+1}(\hat{\partial}_{s}(f(s))) = \prod_{\ell \neq k} \chi_{\mu_k}(\hat{\partial}_{s\ell})(f(y))|_{y=0}.
$$

**Proof.** Let $U$ be a closed point of $\text{Gr}(V)$. The residue condition given by (3.11) is equivalent to the vanishing of

$$
\text{Res} \left( \sum_{j=1}^{p} x^{-\delta_{j_u} - \delta_{j_v}} \psi_{u,\sigma(U)}^{(j)}(z,t)\psi_{v,U}^{*(j)}(z,s) \right) \text{ dz} = \\
= \text{Res} \left( \sum_{j=1}^{p} x^{-\delta_{j_u} - \delta_{j_v}} \psi_{u+1,\sigma(U)}^{(j+1)}(z,\sigma^*(t))\psi_{v,U}^{*(j)}(z,s) \right) \text{ dz}
$$

Using (2.6) and (2.7) and denoting $x = \sigma^*(t)$, the coefficient of $z^{-1}$ in the latter sum turns out to be

$$
(3.13) \sum_{j=1}^{p} \sum_{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = \delta_{j_u} + \delta_{j_v} - 1} p_{\alpha_1}(-x^{(j+1)})p_{\beta_1}(\hat{\partial}_{x(j+1)}) \tau_{U_{u+1,j+1}}(x) \cdot \\
\cdot p_{\alpha_2}(s^{(j)})p_{\beta_2}(\hat{\partial}_{s(j)}) \tau_{U_{j_v}}(s).
$$

Since the vanishing of the function $f(x, s)$ one given by (3.13) for all values of $x$ and $s$ is equivalent to the vanishing of

$$
\prod_{1 \leq a, b \leq p} \chi_{\lambda_a}(-\hat{\partial}_{\alpha(a)})\chi_{\mu_b}(\hat{\partial}_{s(b)})f(x, s)|_{x=0, s=0}
$$

for all Young diagrams $\lambda_1, \mu_1, \ldots, \mu_p$, a calculation shows that the vanishing of (3.13) is equivalent to (3.11), thus proving the theorem. \(\square\)
3.B. Equations of the moduli space. Recalling the relation \[3.14\] and the Theorem \[3.4\], we can now write down the characterization of \(\mathcal{M}^\infty(p, R)\) in terms of differential equations for the \(\tau\)-functions.

**Theorem 3.14** (ramified case). Let \(V = V_R\) and let \(U\) be a closed point of \(\text{Gr}^m(V)\).

Then \(U\) is a point of \(\mathcal{M}^\infty(p, R)\) if and only if its \(\tau\)-function \(\tau_U(t)\) satisfies the following set of differential equations, for all Young diagrams \(\lambda_1, \lambda_2, \lambda_3, \lambda\), and all integers \(j\) in \(\{0, 1, \ldots, p - 1\}\):

1. The equations \([3.5]\):
   \[
   \sum_{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = 1 \atop \alpha_1 - \beta_1 \equiv j \pmod{p}} \left( D_{\lambda_1, \alpha_1} (-\tilde{\partial}_x) p_{\beta_1} (\tilde{\partial}_x) D_{\lambda_2, \alpha_2} (\tilde{\partial}_s) p_{-\beta_2} (-\tilde{\partial}_s) \right) \tau_U(x) \tau_U(s) = 0
   \]

2. \[
   \sum_{\beta - \alpha = m} \left( D_{\lambda, \alpha} (\tilde{\partial}_t) p_{\beta} (-\tilde{\partial}_t) \right) \tau_U(t) = 0
   \]

3. \[
   \sum_{\beta_1 + \beta_2 + \beta_3 - \alpha_1 - \alpha_2 - \alpha_3 = 2 - m} \left( D_{\lambda_1, \alpha_1} (-\tilde{\partial}_x) p_{\beta_1} (\tilde{\partial}_x) D_{\lambda_2, \alpha_2} (-\tilde{\partial}_s) p_{\beta_2} (\tilde{\partial}_s) \right) \cdot
   \left( D_{\lambda_3, \alpha_3} (\tilde{\partial}_u) p_{\beta_3} (-\tilde{\partial}_u) \right) \tau_U(x) \tau_U(s) \tau_U(t) = 0.
   \]

Similarly, we can write down differential equations for \(\mathcal{M}^\infty(p, NR)\) for the non-ramified case. In this case the non-trivial curves with automorphisms appear when \(m \leq 0\); if \(-m = qp + f\) with \(0 \leq f < p\), then \(v_m = z_1^{q+1} \cdot \ldots \cdot z_f^{q+1} \cdot z_{f+1}^{q} \cdot \ldots \cdot z_p^{q}\), and \(v_{-m} = v_m^{-1}\).

**Theorem 3.15** (non ramified case). Let \(V = V_{NR}\) and let \(U\) be a closed point of \(\text{Gr}^m(V)\).

Then \(U\) is a point of \(\mathcal{M}^\infty(p, NR)\) if and only if the following set of differential equations is satisfied by its \(\tau\)-functions \(\tau_{uvw}(t)\), for all Young diagrams \(\lambda = \{\lambda_1, \ldots, \lambda_p\}\), \(\mu = \{\mu_1, \ldots, \mu_p\}\), \(v = \{v_1, \ldots, v_p\}\), and all integers \(u, v, w\) in \(\{1, 2, \ldots, p\}\).

1. The equations of Theorem \[3.12\]
   \[
   \sum_{j=1}^p \sum_{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = \delta_{ju} + \delta_{jv} - 1} \left( D_{\lambda_j, \alpha_1} (-\tilde{\partial}_{x(j+1)}) p_{\beta_1} (\tilde{\partial}_{x(j+1)}) D^{-1}_{\lambda_j, \delta_{ju} + \delta_{jv} - 1} (\tilde{\partial}_x) \right) \tau_{uvw+1,j+1}(x) \cdot
   \left( D_{\mu_j, \alpha_2} (\tilde{\partial}_{s(j)}) p_{\beta_2} (-\tilde{\partial}_{s(j)}) D^{-1}_{\mu_j, \alpha_2} (\tilde{\partial}_s) \right) \tau_{uvw+1,j+1}(s) = 0
   \]
\[
\sum_{j=1}^{f} \sum_{\beta-\alpha = \delta j u - 2 - q} \left( D_{\lambda j, \alpha}(\tilde{\partial}_{t(j)}) p_{\beta}(\tilde{\partial}_{t(j)}) \tilde{D}^j_{\lambda}(\tilde{\partial}_t) \right) \tau_{\nu_{ju}}(t) + \\
\sum_{j=f+1}^{p} \sum_{\beta-\alpha = \delta j u - 1 - q} \left( D_{\lambda j, \alpha}(\tilde{\partial}_{t(j)}) p_{\beta}(\tilde{\partial}_{t(j)}) \tilde{D}^j_{\lambda}(\tilde{\partial}_t) \right) \tau_{\nu_{ju}}(t) = 0
\]

\[
\sum_{j=1}^{f} \sum_{\beta_1+\beta_2+\beta_3-\alpha_1-\alpha_2-\alpha_3 = q+\delta j u + \delta j u + \delta j w} \left( D_{\lambda j, \alpha_1}(\tilde{\partial}_{t(j)}) p_{\beta_1}(\tilde{\partial}_{t(j)}) \tilde{D}^j_{\lambda}(\tilde{\partial}_t) \tilde{D}^j_{\lambda}(\tilde{\partial}_t) \right) \tau_{\nu_{ju}}(t) \tau_{\nu_{vj}}(s) \tau_{\nu_{juw}}(x) + \\
\sum_{j=f+1}^{p} \sum_{\beta_1+\beta_2+\beta_3-\alpha_1-\alpha_2-\alpha_3 = q+\delta j u + \delta j w - 1} \left( D_{\lambda j, \alpha_1}(\tilde{\partial}_{t(j)}) p_{\beta_1}(\tilde{\partial}_{t(j)}) \tilde{D}^j_{\lambda}(\tilde{\partial}_t) \tilde{D}^j_{\lambda}(\tilde{\partial}_t) \right) \tau_{\nu_{ju}}(t) \tau_{\nu_{vj}}(s) \tau_{\nu_{juw}}(x) = 0 .
\]

4. \textit{\tau\text{-functions and theta functions of Jacobians}

4.A. Geometrical meaning of “formal” objects. This section aims at motivating the relation between \(\tau\)-functions and theta functions of Jacobians. In particular, \(\tau\)-functions attached to a Riemann surface with marked points will be defined following the works of Fay, Krichever, Shiota and Adler–Shiota–van Moerbecke (\cite{FKShASvM}).

Throughout this section, \(C\) will be an integral complete curve over \(\mathbb{C}\) of genus \(g\). Let \(J_{g-1}(C)\) denote the scheme parametrizing invertible sheaves of degree \(g - 1\). For the sake of clarity, we will assume \(C\) to be smooth although most of the results established here hold in greater generality.

We let \(r = 1\) in the ramified case and \(r = p\) in the non-ramified case.

Let us fix data \((C, \tilde{x}, t_{\tilde{x}})\), where \(\tilde{x} = \{x_1, \ldots, x_r\}\) are \(r\) pairwise distinct points of \(C\) and \(t_{\tilde{x}}\) is a collection of formal parameters \(\{t_{x_1}, \ldots, t_{x_r}\}\) giving corresponding isomorphisms \(t_{x_j} : \tilde{O}_{C,x_j} \sim \mathbb{C}[z_j]\).
Proposition 4.1. For each invertible sheaf $L \in J_{g-1}(C)$, there is a canonical morphism $\gamma : J(\hat{C}_V) \to J_{g-1}(C)$ such that the diagram

$$
\begin{array}{ccc}
\hat{C}_V & \xrightarrow{\alpha} & C \\
\downarrow \phi_V & & \downarrow \phi_L \\
J(\hat{C}_V) & \xrightarrow{\gamma} & J_{g-1}(C)
\end{array}
$$

is commutative. Here $\phi_V$ is the Abel map and $\phi_L$ sends a point $x' \in C$ to $L(r \cdot x' - \bar{x})$.

Proof. The data $(C, \bar{x}, t_{\bar{x}})$ gives rise to a canonical morphism:

$$
O_C \hookrightarrow \hat{O}_{C, \bar{x}} \xrightarrow{\sim} V^+.
$$

Then we define $\alpha$ to be the induced morphism between the corresponding schemes.

Because of the Albanese property given in Proposition 2.3, the composition $\hat{C}_V \to J_{g-1}(C)$ factors through the Abel map $\phi_V : \hat{C}_V \to J(\hat{C}_V)$, thus defining $\gamma$. □

Let $J_{g-1}^\infty(C, \bar{x})$ be the scheme parametrizing pairs $(L, \phi)$ where $L \in J_{g-1}(C)$ and $\phi : \hat{L}_{\bar{x}} \sim \hat{O}_{C, \bar{x}}$. It carries a canonical action of $\hat{\Gamma}_V^+$ since this group acts by homotheties on the trivialization of $L$. Summing up, there is an exact sequence of group schemes

$$
0 \to \hat{\Gamma}_V^+ \to J_{g-1}^\infty(C, \bar{x}) \to J_{g-1}(C) \to 0.
$$

This sequence has also a formal counterpart

$$
0 \to \hat{\Gamma}_V^+ \to \Gamma_V \to J(\hat{C}_V) \to 0.
$$

Since this last sequence splits, $\Gamma_V \simeq J(\hat{C}_V) \times \hat{\Gamma}_V^+$ and every element $g \in \Gamma_V$ can be written as $(g_-, g_+)$ with $g_- \in J(\hat{C}_V)$, $g_+ \in \hat{\Gamma}_V^+$ and $g = g_- \cdot g_+$. 

Proposition 4.2. Let $(L, \phi)$ be a point in $J_{g-1}^\infty(C, \bar{x})$. Then the Abel maps $\phi_V$ and $\phi_L$ have canonical lifts to $\Gamma_V$ and $J_{g-1}^\infty(C, \bar{x})$ respectively, and the map $\gamma$ has a lift

$$
\begin{array}{ccc}
\Gamma_V & \xrightarrow{\hat{\gamma}} & J_{g-1}^\infty(C, \bar{x}) \\
\downarrow & & \downarrow \\
J(\hat{C}_V) & \xrightarrow{\gamma} & J_{g-1}(C)
\end{array}
$$

compatible with those of $\phi_V$ and $\phi_L$. 

Proof. Let $E(u, v)$ denote the prime form of $C$ as a meromorphic function on $C \times C$. Then the meromorphic function $\frac{E(x, x_i)}{E(x, x')}$ has a zero at $x = x_i$ and a pole at $x = x'$. Moreover, if $z_i$ is a coordinate at $x_i$ such that $z_i(x_i) = 0$ and $\bar{z}_i := z_i(x')$, then the expansion of this function at $x_i$ is the following series.

$$t_{x_i} \left( \frac{E(x, x_i)}{E(x, x')} \right) \in \left( 1 - \frac{\bar{z}_i}{z_i} \right)^{-1} \cdot (1 + z_i \mathbb{C}[[z_i]]).$$

Therefore, the $r$-tuple consisting of the expansions

$$t \left( \frac{E(x, x_1)}{E(x, x')}, \ldots, \frac{E(x, x_r)}{E(x, x')} \right)$$

corresponds to a morphism $\hat{C}_V \to \Gamma_V$ which lifts the Abel morphism $\phi_V$ defined by the $r$-tuple (2.2).

To define the lift of $\phi_L$, it is enough to observe that if the line bundle $L$ carries a formal trivialization $\phi : \hat{L} \sim \hat{O}_{C, \bar{x}}$, then $L(r \cdot x' - \bar{x})$ is canonically endowed with the trivialization given by

$$\phi \cdot \prod_{i=1}^{r} \frac{E(x, x_i)}{E(x, x')}.$$

Finally, the lift of $\gamma$ is defined by

$$\tilde{\gamma} : \Gamma_V \longrightarrow J_{g-1}^\infty(C, \bar{x})$$

$$g \longmapsto (L \otimes L_{g-}, \phi \cdot g)$$

where $L_{g-}$ is given as follows: let $D_i$ be a small disk around $x_i$ such that $z_i$ defines a coordinate in $D_i$, then $L_{g-}$ consists of gluing the trivial bundles on $C - \bar{x}$ and on $\bar{D}_1, \ldots, \bar{D}_r$ by the transition functions $(g^{-})_1, \ldots, (g^{-})_r$ (see [SW], Remark 6.8 and [Sh], Lemma 4). □

Recall that the Krichever map associated to $(C, \bar{x}, t_{\bar{x}})$ is the morphism

$$J_{g-1}^\infty(C, \bar{x}) \to \text{Gr}(V)$$

$$(L, \phi) \mapsto (t_{\bar{x}} \circ \phi)(H^0(C - \bar{x}, L)).$$

Then, we obtain the following

**Theorem 4.3.** Let $\text{Kr}$ denote the Krichever map associated to $(C, \bar{x}, t_{\bar{x}})$. For $(L, \phi) \in J_{g-1}^\infty(C, \bar{x})$, let $U = \text{Kr}(L, \phi) \in \text{Gr}(V)$.

Then the composition

$$\Gamma_V \xrightarrow{\tilde{\gamma}} J_{g-1}^\infty(C, \bar{x}) \xrightarrow{\text{Kr}} \text{Gr}(V)$$

coincides with the morphism $\mu_U : \Gamma_V \cong \Gamma_V \times \{U\} \to \text{Gr}(V)$ mapping $g$ to $g \cdot U$. 
Proof. We have to check that \((Kr \circ \hat{\gamma})(g) = g \cdot U\), but this is a straightforward consequence of the definition of \(\hat{\gamma}\). \(\square\)

In particular, we have obtained morphisms

\[
\begin{array}{ccc}
\Gamma_V & \xrightarrow{\hat{\gamma}} & J^\infty_{g-1}(C, \bar{\pi}) \\
& \searrow & \downarrow \pi \\
& & J_{g-1}(C)
\end{array}
\]

As a straightforward consequence of the determinantal construction of the theta divisor in \(J_{g-1}(C)\) and of the determinant bundle on \(\text{Gr}(V)\), it follows that

\[
Kr^* \text{Det}^*_V \simeq \pi^* \mathcal{O}_{J_{g-1}(\Theta)}
\]

Remark 4.4. It is easy to check that there is a canonical isomorphism of formal schemes

\[
\Gamma_V(U)/\hat{\Gamma}_V^+ \simeq J_{g-1}(C)_L^-,
\]

where \(J_{g-1}(C)_L^-\) denotes the formal completion of \(J_{g-1}(C)\) at the point \(L\). If \(A_U = \{v \in V : v \cdot U \subseteq U\}\) is the stabilizer of \(U\), then the above isomorphism yields, at the level of tangent spaces,

\[
T_1 \left( \Gamma_V(U)/\hat{\Gamma}_V^+ \right) = V/(A_U + V^+) \simeq H^1(C, \mathcal{O}_C)
\]

since \(A_U = t_{\bar{x}}(H^0(C - \bar{x}, \mathcal{O}_C))\). This is related to Mulase’s characterization of Jacobian varieties \([\mathbb{M}]\).

4.B. \(\tau\)-function attached to a Jacobian. The relations just described between the determinant bundle and the theta line bundle induce an explicit relation between \(\tau\)-functions and theta functions, which we now study.

Let \(J(C)\) denote the Jacobian variety of \(C\). Choose a symplectic basis \(\{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}\) for \(H_1(C, \mathbb{Z})\) and let \(\{\omega_1, \ldots, \omega_g\}\) denote the corresponding canonical basis of holomorphic 1-forms on \(C\); that is,

\[
\int_{\alpha_i} \omega_j = \delta_{ij} \quad \text{and} \quad \int_{\beta_i} \omega_j = \Omega_{ij}
\]

where \(\Omega = (\Omega_{ij})\) is the period matrix of \(C\). Then, as a complex torus, we have

\[
J(C) = \mathbb{C}^g/\mathbb{Z}^g + \Omega \mathbb{Z}^g.
\]

We also recall that the Riemann theta function associated to \(\Omega\) is the quasi-periodic function on \(\mathbb{C}^g\) (the universal cover of \(J(C)\)) given
By the Riemann-Kempf Theorem, there is an identification of $J_t$ variables ($J$ and the point $\Theta \subset \Theta$) which maps each $\omega \in \Theta$ to $z$ phic coordinates.

Since we have chosen bases on both sides, for each $\omega \in \Theta$ we have a $g \times \infty$ matrix over $C$, $A^{(j)}$, associated to the map $H^0(C, \Omega_C) \to \mathbb{C}[[z_j]]dz_j$. Moreover, rank $A^{(j)} = g$ for each $j \in \{1, \ldots, r\}$.

**Remark 4.5.** The transpose of the above map is related to the map induced by Proposition 4.1 at the level of tangent spaces.

We will now define the $\tau$-function of $(C, x_1, \ldots, x_r, z_1, \ldots, z_r, \xi)$, where $\{x_1, \ldots, x_r\}$ are $r$ different points in $C$, with respective local holomorphic coordinates $z_1, \ldots, z_r$, and $\xi \in \mathbb{C}^g$ is a point in the universal cover of $\Omega(C)$.

For each $j \in \{1, \ldots, r\}$ and each natural number $n$, we let $\eta^{(j)}_n$ denote the normalized meromorphic 1-form on $C$ with a unique pole of order $n$ at $x_j$ of the form $d(z_j^{-n+1}) + O(1)$ and such that

$$\int x \eta^{(j)}_n = z_j^{-n} + O(z_j) \quad \text{at } x = x_j.$$

We also consider the complex numbers $q^{(j)}_{nm}$ defined by the following identities.

$$\int x \eta^{(j)}_n = z_j^{-n} - 2 \sum_{m=1}^{\infty} q^{(j)}_{nm} \frac{z_j^m}{m} \quad \text{at } x = x_j.$$

If we let $t$ be the $r$-tuple $(t^{(1)}, \ldots, t^{(r)})$ where each $t^{(j)}$ is a family of variables $(t_1^{(j)}, t_2^{(j)}, \ldots)$, then we define the following quadratic form

$$Q(t) = \sum_{n,m \geq 1} q^{(1)}_{nm} t^{(1)}_n t^{(1)}_m + \ldots + \sum_{n,m \geq 1} q^{(r)}_{nm} t^{(r)}_n t^{(r)}_m$$

and the point $A(t)$ of $J(C)$ with values in $C\{\{t^{(1)}, \ldots, t^{(r)}\}\}$ given by

$$A(t) = A^{(1)} t^{(1)} + \ldots + A^{(r)} t^{(r)}.$$
Finally, the \( \tau \)-function of \((C, x_1, \ldots, x_r, z_1, \ldots, z_r, \xi)\) is defined as follows

\[
\tau(\xi, t) := \exp(Q(t)) \theta(A(t) + \xi)
\]

for \( t = (t^{(1)}, \ldots, t^{(r)}) \).

It is worth pointing out that standard calculations shows that the \( \tau \)-function \( \tau(\xi, t) \) coincides with the \( \tau \)-function \( \tau_U(t) \) of the point \( U \) defined through the Krichever map \( [SW, ASvM] \).

Now, let us introduce the associated BA function for the ramified case \((r = 1)\) by

\[
\psi_{\xi}(z, t) := \exp\left(-\sum_{i \geq 1} \frac{t_i}{z^i} \right) \cdot \frac{\tau(\xi, t + [z])}{\tau(\xi, t)}
\]

and the adjoint BA function by

\[
\psi^*_{\xi}(z, t) := \exp\left(\sum_{i \geq 1} \frac{t_i}{z^i} \right) \cdot \frac{\tau(\xi, t - [z])}{\tau(\xi, t)}
\]

where \( t = (t_1, t_2, \ldots) \) and \( t + [z] = (t_1 + z, t_2 + z^2/2, \ldots, t_n + z^n/n, \ldots) \).

Then it follows from \([DJKM, P] \) that this BA function coincides with the BA function of \( U \) (the corresponding point under the Krichever map) and that the following equality holds

\[
\text{Res}_{z=0} \psi_{\xi}(z, t) \psi^*_{\xi}(z, s) \frac{dz}{z^2} = 0
\]

for all \( t \) and \( s \).

In the non-ramified case \((r = p)\), BA functions will be introduced as follows. Let \( u, v \) be two integers in \( \{1, \ldots, p\} \) and let \( T_{uv} \) denote the homothety of \( V \) given by the element \((1, \ldots, z_u, \ldots, z_{v-1}, \ldots, 1)\) if \( u \neq v \), and the identity if \( u = v \). Then it is easy to check that \( T_{uv} \) induces an automorphism of each connected component of \( \text{Gr}(V) \) and that these automorphisms lift canonically to automorphisms of the determinant bundle, which will also be denoted by \( T_{uv} \). In particular, we have automorphisms of \( H^0(\text{Gr}(V), \text{Det}^*) \).

Recalling the bosonization isomorphism and the fact that \( \mathcal{O}(\Gamma_V^*) \tribe \mathcal{O}(\Gamma_V^-) \), we obtain isomorphisms

\[
T_{uv}^* : \mathcal{O}(\Gamma_V^-) \xrightarrow{\sim} \mathcal{O}(\Gamma_V^-)
\]

and we define

\[
\tau_{uv}(\xi, t) := T_{uv}^*(\tau(\xi, t))
\]

where \( \tau(\xi, t) \) is given by (4.6). Note that \( \tau_{U_{uv}} \) coincides with \( T_{uv}^*(\tau_U) \) for any \( U \in \text{Gr}(V) \).
We will also consider the corresponding formal $u-$BA functions as follows

\begin{equation}
\psi_{u,\xi}(z, t) := \left( \psi_{u,\xi}^{(1)}(z_1, t), \ldots, \psi_{u,\xi}^{(r)}(z_r, t) \right)
\end{equation}

where

\begin{equation}
\psi_{u,\xi}^{(j)}(z_j, t) := \exp \left( -\sum_{i \geq 1} \frac{t_{i}^{(j)}}{z_{j}^{i}} \frac{\tau_{uj}(\xi, t + [z_j])}{\tau(\xi, t)} \right),
\end{equation}

and the formal adjoint $u-$BA functions by

\begin{equation}
\psi_{u,\xi}^{*}(z, t) := \left( \psi_{u,\xi}^{*(1)}(z_1, t), \ldots, \psi_{u,\xi}^{*(r)}(z_r, t) \right)
\end{equation}

where

\begin{equation}
\psi_{u,\xi}^{*(j)}(z_j, t) := \exp \left( \sum_{i \geq 1} \frac{t_{i}^{(j)}}{z_{j}^{i}} \frac{\tau_{ju}(\xi, t - [z_j])}{\tau(\xi, t)} \right).
\end{equation}

Then, from [MP2], we obtain that the BA functions satisfy the $(1, p, 1)$-KP-hierarchy

$$\text{Res}_{z=0} \text{Tr} \left( \frac{\psi_{u,u}(z, t)}{(1, \ldots, z_u, \ldots, 1)} \cdot \frac{\psi_{v,v}(z, s)}{(1, \ldots, z_v, \ldots, 1)} \right) dz = 0$$

for all $u, v \in \{1, \ldots, p\}$. Equivalently, in terms of $\tau$-functions we have

\begin{equation}
\sum_{j=1}^{p} \sum_{\beta_1 + \beta_2 - \alpha_1 - \alpha_2 = \delta_j + \delta_{ju} - 1} p_{\alpha_1}(-t^{(j)} p_{\beta_1} \tilde{\partial}_{(j)} \tau_{U_{u,j}}(t) \cdot p_{\alpha_2}(s^{(j)}) p_{\beta_2}(-\tilde{\partial}_{(s)} \tau_{U_{ju}}(s) = 0.
\end{equation}

Moreover, it follows from [MP2] that these hierarchies characterize when the BA-functions (respectively $\tau$-function) are the BA-functions (respectively $\tau$-function) of a point of $\text{Gr}(V)$.

5. Geometric characterization of Jacobian varieties with automorphisms in terms of Sato Grassmannian

In this section we characterize the points of the Sato Grassmannian that arise from geometric data over an algebraic curve with automorphisms via the Krichever construction. We prove that these points are those whose orbit under the action of $\Gamma_V$ is finite dimensional (up to the action of $\bar{\Gamma}_V^+$). A related result has been established in [GMP]. This type of characterizations dates back to the approach of Mulase ([M]).

Let us denote by $\bar{\text{Pic}}(C)$ the moduli space of rank 1 torsion free sheaves on $C$. 

We say that \((C, \bar{x}, L)\) is maximal for a curve \(C\), a divisor \(\bar{x}\) composed by \(r\) pairwise distinct points \(x_1, \ldots, x_r\) in \(C\), and \(L \in \text{Pic}(C)\) when one of the following equivalent condition holds (see page 38 of [SW]):

- let \((C', \bar{x}', L')\) be another such triple, and suppose there exists \(\psi : C' \to C\) a birational morphism such that \(\psi(\bar{x}') = \bar{x}\) and \(\psi_*L' \sim L\); then \(\psi\) is an isomorphism.
- The canonical map \(\mathcal{O}_C \to \text{End}(L)\) is an isomorphism.

We begin with a detailed study of the ramified case and then generalize the results to the non-ramified case.

5.A. Ramified case. For the sake of notation, in this subsection the subscript \(R\) in \(V_R, V_R^+, \Gamma_{VR}, \ldots\) will be omitted.

**Definition 5.1.** Let \(\text{Pic}^\infty(p, R)\) be the contravariant functor from the category of \(C\)-schemes to the category of sets defined by

\[ S \mapsto \{(C, \sigma_C, x, t_x, L, \phi_x)\} \]

where

1. \(p_C : C \to S\) is a proper and flat morphism whose fibres are geometrically integral curves.
2. \(\sigma_C : C \to C\) is an order \(p\) automorphism (over \(S\)).
3. \(x : S \to C\) is a smooth section of \(p_C\), fixed under \(\sigma_C\), such that the Cartier divisor \(x(s)\) is a smooth point of \(C_s := p_C^{-1}(s)\) for all closed points \(s \in S\).
4. \(t_x\) is an equivariant formal parameter along \(x(S)\); that is, an equivariant isomorphism of \(\mathcal{O}_S\)-modules \(t_x : \hat{O}_{C, x(S)} \cong \hat{V}_S^+\).
5. \(L \in \text{Pic}(C)\) satisfies that \((C_s, x(s), L|_{C_s})\) is maximal for all closed point \(s \in S\).
6. \(\phi_x\) is a formal trivialization of \(L\) along \(x(S)\); that is, an isomorphism \(\phi_x : \hat{L}_{x(S)} \cong \hat{O}_{C, x(S)}\).
7. \((C, \sigma_C, x, t_x, L, \phi_x)\) and \((C', \sigma_C', x', t_{x'}, L', \phi_{x'})\) are said to be equivalent when there is an isomorphism of \(S\)-schemes \(C \cong C'\) compatible with all the data.

The Krichever morphism for the functor \(\text{Pic}^\infty(p, R)\) is the morphism of functors

\[ \text{Kr} : \text{Pic}^\infty(p, R) \to \text{Gr}(V) \]

which sends the \(S\)-valued point \((C, \sigma_C, x, t_x, L, \phi_x)\) to the following submodule of \(\hat{V}_S := V \otimes \mathcal{O}_S\)

\[ (t_x \circ \phi_x) \left( \lim_{m \to \infty} (p_C)_* L(m \cdot x) \right) \subset \hat{V}_S. \]
Theorem 5.2. The functor $\text{Pic}^\infty(p, R)$ is representable by a subscheme $\text{Pic}^\infty(p, R)$ of $\text{Gr}(V)$.

Proof. Consider the morphism from $\text{Pic}^\infty(p, R)$ to $\text{Gr}(V) \times \text{Gr}(V)$ which sends the $S$-valued point $(C, \sigma_C, x, t_x, L, \phi_x)$ to the following pair of submodules

$$
\left( t_x \left( \lim_{m \to \infty} (p_C)_* \mathcal{O}_C(m \cdot x) \right), (t_x \circ \phi_x) \left( \lim_{m \to \infty} (p_C)_* L(m \cdot x) \right) \right) \in \text{Gr}(V) \times \text{Gr}(V)
$$

where $p_C : C \times S \to C$ is the projection.

From the inverse construction of the Krichever map ([K, SW]) one has that this map is injective and that the image is contained in the set $Z$ of those pairs $(\mathcal{A}, \mathcal{L})$ in $\text{Gr}(V) \times \text{Gr}(V)$ such that

$$
\mathcal{O}_S \subset \mathcal{A}, \quad \mathcal{A} \cdot \mathcal{A} \subseteq \mathcal{A}, \quad \mathcal{A} \cdot \mathcal{L} \subseteq \mathcal{L}, \quad \sigma(\mathcal{A}) = \mathcal{A}.
$$

Let us examine the maximality condition. For $(\mathcal{A}, \mathcal{L})$ satisfying the above conditions, let $A_{\mathcal{L}}$ denote the stablizer of $\mathcal{L}$

$$
A_{\mathcal{L}} := \{ v \in \hat{V}_S \text{ such that } v \cdot \mathcal{L} \subseteq \mathcal{L} \},
$$

and let $(C', \sigma_{C'}, x', t_{x'}, L', \phi_{x'})$ be the geometric data defined by the pair $(A_{\mathcal{L}}, \mathcal{L})$. Then the inclusion $\mathcal{A} \subseteq A_{\mathcal{L}}$ gives rise to an equivariant morphism of $S$-schemes $\psi : C' \to C$ such that $\psi(x') = x$ and $\psi_*(L') \cong L$.

The maximality condition says that $\psi_s$ is an isomorphism for every closed point $s \in S$. That is, $A_{\mathcal{L}}$ is a finite $\mathcal{A}$-module such that $A_s = (A_{\mathcal{L}})_s$ for all $s$. Therefore we have that $\mathcal{A} = A_{\mathcal{L}}$. Summing up, we are interested on the subset $Z_0$ of $Z$ consisting of those pairs $(\mathcal{A}, \mathcal{L})$ such that $\mathcal{A} = A_{\mathcal{L}}$.

From the proof of Theorem 6.5 of [MP] we know that the condition $A_{\mathcal{L}} \subseteq \mathcal{A}$, and hence $Z_0$, is closed. The Krichever construction implies that $Z_0$ represents $\text{Pic}^\infty(p, R)$. Finally, $p_2|_{Z_0} : Z_0 \to \text{Gr}(V)$ is a closed immersion (where $p_2$ is the projection onto the second factor), and the theorem is proved.  \[\square\]

Let us consider the following action of $\Gamma_V$ on $\text{Gr}(V)^p := \text{Gr}(V) \times \ldots \times \text{Gr}(V)$

$$
\mu^p : \Gamma_V \times \text{Gr}(V)^p \to \text{Gr}(V)^p
$$

$$
(g, (U_1, \ldots, U_p)) \mapsto (gU_1, \ldots, gU_p).
$$

Then the morphism

$$
\text{Gr}(V) \hookrightarrow \text{Gr}(V)^p
$$

$$
U \mapsto U_\sigma := (U, \sigma(U), \ldots, \sigma^{p-1}(U))
$$

is a $\Gamma_V$-equivariant closed immersion.
The orbit of $U_\sigma \in \text{Gr}(V)^p$ under the action of $\Gamma_V$ is the schematic image of $\mu^p_{U_\sigma} := \mu^p|_{\Gamma_V \times \{U_\sigma\}}$; it will be denoted by $\Gamma_V(U_\sigma)$. Section 4 of [GMP] implies that $\Gamma_V(U_\sigma)/\bar{\Gamma}_V^+$ is a formal scheme whose tangent space is

$$T_{U_\sigma} \left( \Gamma_V(U_\sigma)/\bar{\Gamma}_V^+ \right) \simeq T_1\Gamma_V/(\text{Ker} \, d\mu^p_{U_\sigma} + T_1\bar{\Gamma}_V^+) ,$$

where

$$d\mu^p_{U_\sigma} : T_1\Gamma_V \longrightarrow T_{U_\sigma}\text{Gr}(V)^p$$

is the map induced by $\mu^p_{U_\sigma}$ on the respective tangent spaces.

**Theorem 5.3** (ramified). Let $U$ be a closed point of $\text{Gr}(V)$. Then the following conditions are equivalent.

1. $\dim_C T_{U_\sigma} \left( \Gamma_V(U_\sigma)/\bar{\Gamma}_V^+ \right) < \infty$ where $V = V_R$, and
2. there exists $(C, \sigma_C, x, t_x, L, \phi_x) \in \overline{\text{Pic}}^\infty(p, R)$ such that its image by the Krichever morphism is $U$.

**Proof.** The result follows straightforwardly from similar arguments to those of Theorems 4.13 and 4.15 of [GMP].

**Remark 5.4.** If the conditions of the theorem hold, it follows that $\text{Ker} \, d\mu^p_{U_\sigma} = \cap_{i=0}^{p-1} \sigma_i^* A_U$, with $A_U$ the stabilizer of $U$. It is then straightforward to show that a similar characterization exists in terms of two copies of $\text{Gr}(V)$ instead of $p$ copies.

**Theorem 5.5.** Let $\overline{\text{Pic}}^\infty(p, R)$ denote the subfunctor of $\overline{\text{Pic}}^\infty(p, R)$ consisting of those data $(C, \sigma_C, x, t_x, L, \phi_x)$ such that the fibres $C_s$ are smooth curves for all closed points $s \in S$.

Then $\overline{\text{Pic}}^\infty(p, R)$ is representable by an open subscheme $\text{Pic}^\infty(p, R)$ of $\overline{\text{Pic}}^\infty(p, R)$.

**Proof.** Let $C \rightarrow \overline{\text{Pic}}^\infty(p, R)$ be the universal curve. Then $\overline{\text{Pic}}^\infty(p, R)$ is given by the open subscheme of $\overline{\text{Pic}}^\infty(p, R)$ consisting of the points $s$ such that $C_s$ is a smooth curve.

**Remark 5.6.** Observe that we have a forgetful map

$$\overline{\text{Pic}}^\infty(p, R) \longrightarrow \mathcal{M}^\infty(p, R)$$

which sends $(C, \sigma_C, x, t_x, L, \phi_x)$ to $(C, \sigma_C, x, t_x)$. The fibre of $(C, \sigma_C, x, t_x)$ when $C$ is a smooth curve is essentially the scheme studied in Section 4.A.
5.B. Non-ramified case. Similarly to the ramified case we now consider the functor $\text{Pic}^\infty(p, \text{NR})$. To define it, it suffices to replace conditions (1) and (3) in Definition 5.1 respectively by

$$(1')\ p_C: C \to S$$
is a proper and flat morphism whose fibres are geometrically reduced curves.

$$(3')\ x_i: S \to C$$
are disjoint smooth sections of $p_S$ ($i = 1, \ldots, p$) such that $\sigma_C(x_i) = x_{i+1}$ for $i < p$ and $\sigma_C(x_p) = x_1$. Also, for every closed point $s \in S$ and each irreducible component of $C_s$, there is at least one $i$ such that $x_i(s)$ lies on that component.

We also take $V, V^+, \Gamma_V, \ldots$ to be $V_{\text{NR}}, V^+_{\text{NR}}, \Gamma_{V_{\text{NR}}}, \ldots$ in this case.

With arguments similar to those in the previous section, we can prove the following results.

**Theorem 5.7.** The functor $\text{Pic}^\infty(p, \text{NR})$ is representable by a subscheme $\text{Pic}^\infty(p, \text{NR})$ of $\text{Gr}(V)$.

The subfunctor $\text{Pic}^\infty(p, \text{NR})$ of $\text{Pic}^\infty(p, \text{NR})$ consisting of those data $(C, \sigma_C, \bar{x}, t_x, L, \phi_x)$ such that the fibres $C_s$ are smooth curves for all closed points $s \in S$ is representable by an open subscheme $\text{Pic}^\infty(p, \text{NR})$ of $\text{Pic}^\infty(p, \text{NR})$.

**Theorem 5.8** (non-ramified). Let $U$ be a closed point of $\text{Gr}(V)$. Then the following conditions are equivalent.

1. $\dim_C T_{U_\sigma}(\Gamma_V(U_\sigma) / \bar{\Gamma}_V^+ < \infty$ where $V = V_{\text{NR}},$

2. there exists $(C, \sigma_C, \bar{x}, t_x, L, \phi_x) \in \text{Pic}^\infty(p, \text{NR})$ such that its image by the Krichever morphism is $U$.

**Remark 5.9.** Analogously to the ramified case, it follows that there is a similar statement in terms of two copies of $\text{Gr}(V)$ instead of $p$ copies.

6. Characterization of Jacobian theta functions of Riemann surfaces with non-trivial automorphisms

In this section we give the conditions that a theta function of a p.p.a.v. should satisfy in order to be the theta function of the Jacobian of a smooth irreducible projective curve.

We begin with the proof of a generalization of the known theorems of Mulase (M) and Shiota (SH) in terms of the Sato Grassmannian.

We will use the following notation.

Let $\Omega \in \mathbb{C}^g$ be a point in the Siegel upper half space such that the principally polarized abelian variety $X_{\Omega} := \mathbb{C}^g / (\mathbb{Z}^g + \Omega \mathbb{Z}^g)$ is irreducible. Let $\theta(z) = \theta(z, \Omega)$ denote the Riemann theta function of $X_{\Omega}$. 

Let \( r \) be a natural number, let \( A^{(j)} = (a^{(j)}_1, a^{(j)}_2, \ldots) \in (\mathbb{C}^g)^\infty \) be a \( g \times \infty \)-matrix of rank \( g \) for each \( j \in \{1, \ldots, r\} \), let \( Q^{(j)}(t^{(j)}) = \sum_{i,k=1}^\infty q^{(j)}_{ik} t^{(j)}_i t^{(j)}_k \), with \( q^{(j)}_{ik} \in \mathbb{C} \), be a quadratic form for each \( j \) in \( \{1, \ldots, r\} \), and let \( \xi \) be in \( \mathbb{C}^g \).

**Definition 6.1.** The \( \tau \)-function and BA-functions associated to the data \((X_\Omega, \Omega, \{A^{(j)}, \ldots, A^{(r)}\}, \{Q^{(1)}, \ldots, Q^{(r)}\}, \xi)\) are given formally by formulae (4.6) and (4.10)-(4.13).

The \( \tau \)-function will be denoted by \( \tau(\xi,t) \) while the BA-function (respectively adjoint BA-function) will be denoted by \( \psi_{u,\xi}(z,t) \) (respectively \( \psi^*_{u,\xi}(z,t) \)).

**Theorem 6.2.** Let \( X_\Omega \) be an irreducible p.p.a.v. of dimension \( g \).

Then the following conditions are equivalent.

1. There exists a triple \((C, \bar{x}, t_\bar{x})\), where \( C \) is a projective irreducible smooth curve of genus \( g \), \( \bar{x} = (x_1, \ldots, x_r) \) is an \( r \)-tuple of distinct points in \( C \) and \( t_\bar{x} = (t_{x_1}, \ldots, t_{x_r}) \) is an \( r \)-tuple of local parameters at the corresponding \( x_j \), such that \( X_\Omega \) is isomorphic as a p.p.a.v. to the Jacobian of \( C \).

2. For each \( j \in \{1, \ldots, r\} \) there exist a \( g \times \infty \)-matrix of rank \( g \), \( A^{(j)} = (a^{(j)}_1, a^{(j)}_2, \ldots) \), with \( a^{(j)}_i \in \mathbb{C}^g \), and a quadratic form \( Q^{(j)}(t^{(j)}) = \sum_{i,k=1}^\infty q^{(j)}_{ik} t^{(j)}_i t^{(j)}_k \), with \( q^{(j)}_{ik} \in \mathbb{C} \), such that for every \( \xi \in \mathbb{C}^g \), the corresponding \( \tau \)-function \( \tau(\xi,t) \) is a \( \tau \)-function of the \((1, \ldots, 1)\)-KP-hierarchy [4.14].

Moreover, if one of the conditions is fulfilled, the matrices \( A^{(j)} \) and the quadratic forms \( Q^{(j)} \) are equal to the data associated to the triple \((C, \bar{x}, t_\bar{x})\) in Section 4.6.

**Proof.**

1. \( \Rightarrow \ 2. \) It follows from Section 4.

2. \( \Rightarrow \ 1. \) We denote

\[
A(t) = \sum_{j=1}^r A^{(j)} t^{(j)} \quad \text{and} \quad Q(t) = \sum_{j=1}^r \sum_{i,k \geq 1} q^{(j)}_{ik} t^{(j)}_i t^{(j)}_k .
\]

Since \( \tau(\xi,t) \) is a \( \tau \)-function of the \((1, \ldots, 1)\)-KP-hierarchy for every \( \xi \in \mathbb{C}^g \), it follows that \( \tau(\xi,t) \) defines a point \( U_\xi \in \text{Gr}(V) \) (with \( V = \mathbb{C}((z)) \times \ldots \times \mathbb{C}((z)) \)) such that \( \tau(\xi,t) = \tau_{U_\xi}(t) \) (up to a constant).

From Theorem 3.12 in [MP2], we have that

\[
U_\xi = \left\langle \left( p_i(\bar{\partial}_t) \psi_{u,\xi}^{(1)}(z_1, t)_{|z_1=0}, \ldots, p_i(\bar{\partial}_t) \psi_{u,\xi}^{(r)}(z_r, t)_{|z_r=0} \right) , i \geq 0, 1 \leq u \leq r \right\rangle .
\]

Therefore, we have obtained a morphism

\[
\varphi : \mathbb{C}^g \longrightarrow \text{Gr}(V) \\
\xi \longmapsto U_\xi .
\]
We claim that this morphism induces an injection
\[ X \hookrightarrow \text{Gr}(V). \]
Indeed, given \( \xi_1 \) and \( \xi_2 \) in \( C^g \), the condition \( U_{\xi_1} = U_{\xi_2} \) is equivalent to \( \tau(\xi_1, t) = \tau(\xi_2, t) \) for all \( t \) (up to a constant), which is in turn equivalent to \( \theta(A(t) + \xi_1) = \theta(A(t) + \xi_2) \) for all \( t \) (up to a constant), and therefore equivalent to \( \xi_1 - \xi_2 \in \mathbb{Z}^g + \Omega \mathbb{Z}^g \), since \( \Theta \) is a principal polarization on \( X \).

Now the function \( A \) can be interpreted as a surjective linear map
\[ C^\infty \times \ldots \times C^\infty \rightarrow C^g \]
and, with the identifications \( C^\infty \times \ldots \times C^\infty = T_1 \Gamma_V^- \) and \( T_\xi X = C^g \), \( A \) corresponds to a surjective morphism of formal group schemes
\[ \Gamma_V^- \xrightarrow{A_\xi} \hat{X}_\xi. \]

We claim now that the surjective morphism
\[ \mu_\xi : \Gamma_V^- \twoheadrightarrow \Gamma(U_\xi)/\Gamma_V^+ \]
\[ g \mapsto g \cdot U_\xi \]
factorizes by \( A_\xi : \Gamma_V^- \rightarrow \hat{X}_\xi \). Observe that if \( s = (s^{(1)}, \ldots, s^{(r)}) \in \text{Ker} A \) then
\[ \tau_{U_\xi}(t + s) = \tau(\xi, t + s) = q_\xi(t, s) \exp(Q(t)) \theta(A(t) + \xi) = q_\xi(t, s) \tau(\xi, t) \]
where \( q_\xi(t, s) \) is an exponential of a linear function in \( t \). Generalizing Lemma 3.8 of [SW] to the case of \( \Gamma_V \), there exists \( g \in \Gamma_V^+ \) (which depends on \( s \)) such that
\[ \tau_{U_\xi}(t + s) = q_\xi(t, s) \tau(\xi, t) = \tau_g U_\xi(t). \]

Hence there is a factorization
\[ \begin{array}{ccc}
\Gamma_V^- & \xrightarrow{\mu_\xi} & \Gamma(U_\xi)/\Gamma_V^+ \\
& \searrow & \nearrow \\searrow
& A_\xi
\end{array} \]
\[ \hat{X}_\xi \]

In particular, it follows that \( \dim T_{U_\xi} \Gamma(U_\xi)/\Gamma_V^+ \) is finite, and applying the results of [M] one has that there exists data \((C_\xi, \bar{x}_\xi, t_\xi, L_\xi, \phi_\xi)\) associated to \( U_\xi \) under the Krichever map.

Let us check that the piece of data \((C_\xi, \bar{x}_\xi, t_\xi)\) does not depend on \( \xi \). Indeed, for \( \xi, \xi' \in X \) let \( s \in C^\infty \times \ldots \times C^\infty \) be such that \( A(s) = \xi' - \xi \). Then
\[ \tau_\xi(t + s) = \exp(Q(t + s)) \theta(A(t + s) + \xi) = q_{\xi'}(t) \tau_{\xi'}(t) \]
\[ = q_{\xi'}(t) \exp(Q(t)) \theta(A(t) + \xi') = q_{\xi'}(t) \tau_{\xi'}(t) \]
The generalization of Lemma 3.8 of [SW] implies that there exists $g_\xi(z) \in V$ such that $U_{\xi'} = g_\xi(z)U_{\xi}$. From this fact it follows that $U_{\xi'}$ and $U_{\xi}$ have the same stabilizer and that, therefore, $(C_\xi, \bar{x}_\xi, t_\xi)$ does not depend on $\xi$. It will be denoted by $(C, \bar{x}, t_{\bar{x}})$.

The latter fact does have further consequences. It implies that the map (6.3) takes values into $\text{Pic}_\infty(C, \bar{x})$ (the subscheme of $\text{Gr}(V)$ parameterizing torsion free sheaves of rank 1 on $C$ with a formal trivialization along $\bar{x}$). Furthermore, it says that the composite map

$$
\begin{array}{cccc}
X & \longrightarrow & \text{Pic}_\infty(C, \bar{x}) & \longrightarrow & \text{Pic}(C) \\
\xi' & \longmapsto U_{\xi'} & \longmapsto L_{\xi'}
\end{array}
$$

takes values in $\text{Pic}^0(C) \cdot L_\xi$, the orbit of $L_\xi \in \text{Pic}(C)$ under the action of $\text{Pic}^0(C)$. Using the surjectivity of $A$ we can show that the induced map

$$
X \longrightarrow \text{Pic}^0(C) \cdot L_\xi
$$

is surjective. Since $(C, \bar{x}, t_{\bar{x}}, L_\xi, \phi_\xi)$ is maximal (see §5), the action of $\text{Pic}^0(C)$ on $\overline{\text{Pic}}(C)$ is free. So $\text{Pic}^0(C)$ is a quotient of an abelian variety and, therefore, $C$ is a smooth complete curve of genus at most $g$.

To finish the implication $2. \Rightarrow 1.$, one has only to show that $X \to \text{Pic}^0(C)$ is an isomorphism of p.p.a.v.'s: Given $(X, \xi)$ and $(J(C), \xi)$ we consider the tau-functions $\tau_X = \tau(\xi, t)$ associated to $X$ as in (4.6) and $\tau_J = \tau(\xi, t)$ associated to $J(C)$ as in 4.6. By the construction of the data $(C_\xi, \bar{x}_\xi, t_\xi, L_\xi, \phi_\xi)$, it follows that $\tau_X = \tau_J$ (up to a constant) and hence

$$
\theta_X(A(t) + \xi) = \exp(q(t)) \theta_J(A_J(t) + \xi) \quad \text{(up to a constant)},
$$

where $q(t)$ is a quadratic function.

Therefore

$$
\Theta_X = \Theta_J \quad \text{(up to translation)}.
$$

In particular, the curve $C$ is irreducible and of genus $g$. \hfill \Box

**Remark 6.4.** Considering $r = 1$ in the previous theorem we obtain the characterization of Jacobian varieties given by Shiota ([Sh], Theorem 6).

We will now apply this result to give a sufficient and necessary condition for a theta function of a p.p.a.v. to be the theta function of a curve with an automorphism of prime order $p$ with a fixed point.

**Theorem 6.5** (ramified case). Let $X_\Omega$ be an irreducible p.p.a.v. of dimension $g$.

Then the following conditions are equivalent.
There exists a quadruple \((C, \sigma_C, x, t_x)\), where \(C\) is a projective irreducible smooth curve of genus \(g\), \(\sigma_C\) is an automorphism of order \(p\) of \(C\), \(x\) is a fixed point of \(\sigma_C\) in \(C\), and \(t_x\) is a local parameter at \(x\), such that \(X_\Omega\) is isomorphic as a p.p.a.v. to the Jacobian of \(C\).

(2) There exist a \(g \times \infty\)-matrix \(A\) of rank \(g\) and a symmetric quadratic form \(Q(t)\) such that for each \(\xi_0\) in \(\mathbb{C}^g\) there exists \(\xi_1\) in \(\mathbb{C}^g\) so that the corresponding BA-functions satisfy

\[ \text{Res}_{z=0} \psi_{\xi_0}(z, t) \psi_{\xi_0}^*(z, s) \frac{dz}{z^2} = 0, \] and

\[ \text{Res}_{z=0} \psi_{\xi_0}(\omega^{-1} z, \omega^{-1} t) \psi_{\xi_1}^*(z, s) \frac{dz}{z^2} = 0 \]

for all \(t\) and \(s\), where \(\omega\) is a primitive \(p\)-th root of 1.

**Proof.** 1. \(\Rightarrow\) 2.

From the results given in Section 4.3 with \(r = 1\), we know that given \((C, \sigma_C, x, t_x)\) as in condition 1., there exist \(A\) and \(Q\) as in condition 2) such that its associated BA-functions \(\psi_{\xi_0}(z, t)\) and \(\psi_{\xi_0}^*(z, s)\), defined by (4.7) and (4.8) respectively, satisfy (4.9) for each \(\xi_0\) in \(\mathbb{C}^g\). That is, there exists a point \(U_{\xi_0} \in \text{Gr}(V_\mathbb{R})\) such that \(\psi_{\xi_0} = \psi_{U_{\xi_0}}\) and 2.a) follows.

Furthermore, the induced embedding \(J(C) \hookrightarrow \text{Gr}(V_\mathbb{R})\) (given by (6.3)) is compatible with the actions of \(\sigma_C\) in \(J(C)\) and of \(\sigma\) on \(\text{Gr}(V_\mathbb{R})\); that is, \(\sigma(U_{\xi}) = U_{\sigma_{\mathbb{C}}^*\xi}\). Having this in mind and taking into account the relation between the BA-function of \(U_{\xi_0}\) and \(U_{\xi_1} = \sigma(U_{\xi_0})\) given by (3.2), 2.b) follows.

2. \(\Rightarrow\) 1.

We know by Theorem 6.2 (with \(r = 1\)) that the first identity implies that there exists a triple \((C, x, t_x)\) such that \(X_\Omega \cong J(C)\) as p.p.a.v.’s.

The second identity implies that \(U_{\xi_1} = \sigma(U_{\xi_0})\). In particular, we have that \(A_{\xi_1} = \sigma(A_{\xi_0})\) where \(A_{\xi}\) denotes the stabilizer of \(U_{\xi}\). Then the orbit \(\Gamma_{V_\mathbb{R}}(U_{\xi_0}, \sigma(U_{\xi_0})/\Gamma_{V_\mathbb{R}}^+\) is finite dimensional and, by Theorem 6.3 (see Remark 5.4), we obtain that \(\sigma\) induces an automorphism of \(A_{\xi_0}\). Since \(A_{\xi_0} = t_3(H^0(C - \overline{x}, \mathcal{O}_C))\), because of the Krichever construction, the result follows. \(\square\)

**Remark 6.6.** Under the hypotheses of the above Theorem, suppose there are \(\xi_0\) and \(\xi_1\) in \(\mathbb{C}^g\) such that \(\xi_0 - \xi_1 \in \mathbb{Z}^g + \Omega \mathbb{Z}^g\) and the equations of the theorem are satisfied. Then there exists a line bundle \(L\) on \(C\) such that \(\sigma^*(L) \simeq L\).
Theorem 6.7 (non-ramified). Let $X_\Omega$ be an irreducible p.p.a.v. of dimension $g$.

Then the following conditions are equivalent.

1. There exists a quadruple $(C, \sigma, \bar{x}, t_{\bar{x}})$, where $C$ is a projective irreducible smooth curve of genus $g$, $\sigma$ is an automorphism of order $p$ of $C$, $\bar{x} = \{x_1, \ldots, x_p\}$ is an orbit of $\sigma$ consisting of $p$ different points in $C$, and $t_{\bar{x}} = \{t_{x_1}, \ldots, t_{x_p}\}$ is a collection of local parameters $t_{x_j}$ at each respective $x_j$, such that $X_\Omega$ is isomorphic as a p.p.a.v. to the Jacobian of $C$.

2. There exist $p$ matrices $A^{(1)}, \ldots, A^{(p)}$, where $A^{(j)}$ is a $g \times \infty$-matrix of rank $g$, and $p$ symmetric quadratic forms $Q^{(1)}, \ldots, Q^{(p)}$, such that for each $\xi_0 \in C^g$ there exists $\xi_1 \in C^g$ so that their BA-functions satisfy

   a) the $(1, \ldots, 1)$-KP hierarchy
   
   \[
   \text{Res} \left( \sum_{j=1}^{p} z^{-\delta_{j\nu}-\delta_{j\nu}} \psi_{u,\xi_0}^{(j)}(z, t) \psi_{v,\xi_0}^{*(j)}(z, s) \right) \; dz = 0 , \text{ and} \]

   b) the identity
   
   \[
   \text{Res} \left( \sum_{j=1}^{p} z^{-\delta_{j\nu}-\delta_{j\nu}} \psi_{v+1,\xi_0}^{(j+1)}(z, \sigma^*(t)) \psi_{u,\xi_1}^{*(j)}(z, s) \right) \; dz = 0
   \]

   where $\sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \ldots, t^{(p-1)})$.

Proof. 1. $\Rightarrow$ 2.

Given $(C, \sigma, \bar{x}, t_{\bar{x}})$ satisfying condition 1., we construct $A(t)$ and $Q(t)$ as in Section 4.B. By Theorem 6.2, for each $\xi \in C^g$ the corresponding BA-functions satisfy 2.a).

Similarly to the ramified case, the actions of $\sigma$ in $J(C)$ and of $\sigma$ on $\text{Gr}(V_{NR})$ are compatible; that is, $\sigma(U_\xi) = U_{\sigma \xi}$. Taking into account the relation (3.10) between $\psi_{u,\xi_0}$ and $\psi_{v,\xi_1}$ for $L_{\xi_1} = \sigma^* L_{\xi_0}$, the second part of 2. follows.

2. $\Rightarrow$ 1.

From 2.a) and Theorem 6.2, we know that there exists a triple $(C, \bar{x}, t_{\bar{x}})$ such that $X_\Omega$ is isomorphic as a p.p.a.v. to the Jacobian of $C$.

Now 2.b) shows that $U_{\xi_1} = \sigma(U_{\xi_0})$, so the orbit $\Gamma_V(U_{\xi_0}, U_{\xi_1})/\bar{T}_V$ is finite dimensional. Theorem 5.8 (see also Remark 5.9) implies that $\sigma$ induces an automorphism of $C$ satisfying the conditions of 1.

Remark 6.8. Observe that if the condition 2. of Theorem 6.2 holds for one $\xi \in C^g$, then it holds for every $\xi$ (see [S], Theorem 6). Therefore,
if the condition 2. of Theorem 6.5 or 6.7 holds for a given \( \xi_0 \in \mathbb{C}^g \), then it holds for every \( \xi_0 \).

Finally, we obtain a solution of the Schottky problem for curves with automorphisms.

**Theorem 6.9** (Characterization). Let \( X_\Omega \) be an irreducible p.p.a.v. of dimension \( g > 1 \).

Then the following conditions are equivalent.

1. There exists a projective irreducible smooth curve \( C \) of genus \( g \) with a non-trivial automorphism \( \sigma_C : C \to C \) such that \( X_\Omega \) is isomorphic as a p.p.a.v. to the Jacobian of \( C \).

2. There exist a prime number \( p \), \( p \) matrices \( A^{(1)}, \ldots, A^{(p)} \) \( (A^{(j)} \) being a \( g \times \infty \)-matrix of rank \( g \)) and \( p \) symmetric quadratic forms \( Q^{(1)}, \ldots, Q^{(p)} \), such that
   a) for some \( \xi_0 \in \mathbb{C}^g \), the corresponding BA-functions satisfy the \( (1, p, \ldots, 1) \)-KP hierarchy
   \[
   \text{Res} \left( \sum_{j=1}^{p} z^{-\delta_{j\mu}-\delta_{j\nu}} \psi^{(j)}_{u,\xi_0}(z,t) \psi^{(j)}_{v,\xi_0}(z,s) \right) dz = 0
   \]
   b) there exist \( \xi_1 \in \mathbb{C}^g \) (depending on \( \xi_0 \)) such that
   \[
   \text{Res} \left( \sum_{j=1}^{p} z^{-\delta_{j\mu}-\delta_{j\nu}} \psi^{(j+1)}_{v+1,\xi_0}(z,\sigma^*(t)) \psi^{(j)}_{u,\xi_1}(z,s) \right) dz = 0
   \]
   where \( \sigma^*(t) := (t^{(p)}, t^{(1)}, t^{(2)}, \ldots, t^{(p-1)}) \).

**Proof.** Observe that any curve with non-trivial automorphism group admits an automorphism of prime order \( p \) with an orbit consisting of \( p \) pairwise distinct points. By the previous theorem, we conclude.

\[\square\]

**Remark 6.10.** Recall that standard arguments allow us to express the above equations as an infinite system of partial differential equations for the \( \tau \)-function.

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