ASYMPTOTICS FOR THE ELECTRIC FIELD WHEN M-CONVEX INCLUSIONS ARE CLOSE TO THE MATRIX BOUNDARY

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Abstract. In the perfect conductivity problem of composites, the electric field may become arbitrarily large as $\varepsilon$, the distance between the inclusions and the matrix boundary, tends to zero. The main contribution of this paper lies in developing a clear and concise procedure to establish a boundary asymptotic formula of the concentration for perfect conductors with arbitrary shape in all dimensions, which explicitly exhibits the singularities of the blow-up factor $Q[\varphi]$ introduced in [29] by picking the boundary data $\varphi$ of $k$-order growth. In particular, the smoothness of inclusions required for at least $C^{3,1}$ in [27] is weakened to $C^{2,\alpha}$, $0 < \alpha < 1$ here.

1. Introduction and main results

It is well known that field concentrations appear widely in nature and industrial applications. These fields include extreme electric, heat fluxes and mechanical loads. Motivated by the issue of material failure initiation, in this paper we are devoted to the investigation of blow-up phenomena arising from high-contrast fiber-reinforced composites with the densely packed fibers. The key feature of the concentrated fields is that the blow-up comes from the narrow regions between fibers and the thin gaps between fibers and the matrix boundary. It is worth emphasizing that the latter is more interesting due to the interaction from the boundary data. Although there has made great progress in the engineering and mathematical literature since Babuška et al’s famous work [6] over the past two decades, accurate numerical computation of the concentrated field are still very hard for lack of fine characterization to develop an efficient numerical scheme. So, it is significantly important from a practical point of view to precisely describe the singular behavior of such high concentration.

In the context of electrostatics, the field is the gradient of a solution to the Laplace equation and the blow-up rate of the gradient were captured accurately. Denote the distance between two inclusions or between inclusions and the matrix boundary by $\varepsilon$. It has been proved that for the perfect conductivity problem, the blow-up rate of the gradient is $\varepsilon^{-1/2}$ in two dimensions [3,4,7,12,20,34,35], while it is $|\varepsilon \ln \varepsilon|^{-1}$ in three dimensions [2,13,26,30].

Besides these foregoing estimates of the singularities for the field, there is another direction of investigation to establish the asymptotic formula of $\nabla u$ in the thin gap of electric field concentration. In two dimensions, consider the following...
of \( D \) and Yun \cite{Lim2008} derived an asymptotic formula for the perfect conductivity problem (1.1) to the case when inclusions are spherical and \( 0 < \alpha < 1 \) (see \cite{Kang2009}). Recently, Ammari, Ciraolo, Kang, Lee, Yun \cite{Ammari2010} extended the characterization of the singularities of \( \nabla u \) with \( D_1 \) and \( D_2 \) being disks as follows

\[
\nabla u(x) = \frac{2r_1r_2}{r_1 + r_2} (n \cdot \nabla H)(p) \nabla h(x) + \nabla g(x),
\]

where \( h(x) = \frac{1}{2r} |\ln| x-p_1 || - |\ln| x-p_2 || \) with \( p_1 \in D_1 \) and \( p_2 \in D_2 \) being the fixed point of \( R_1 R_2 \) and \( R_2 R_1 \) respectively, \( R_j \) is the reflection with respect to \( \partial D_j \), \( n \) is the unit vector in the direction of \( p_2 - p_1 \), \( p \) is the middle point of the shortest line segment connecting \( \partial D_1 \) and \( \partial D_2 \), and \( |\nabla g| \) is bounded independently of \( \varepsilon \) on any bounded subset of \( \mathbb{R}^2 \backslash \overline{D_1 \cup D_2} \). Obviously \( \nabla h \) characterizes the singular behavior of \( \nabla u \) explicitly. Ammari, Ciraolo, Kang, Lee, Yun \cite{Ammari2010} extended the characterization (1.2) to the case when inclusions \( D_1 \) and \( D_2 \) are strictly convex domains in \( \mathbb{R}^2 \) by utilizing disks osculating to convex domains. In three dimensions, Kang, Lim and Yun \cite{Kang2009} derived an asymptotic formula for \( \nabla u \) for two spherical perfect conductors with the same radii. The asymptotics for perfectly conducting particles with different radii can be seen in \cite{Kang2009}. More related work can be seen in \cite{Ammari2010}.

However, to the best of our knowledge, previous investigations on the asymptotics of the field concentration only focused on the narrow region between inclusions. This paper, by contrast, aims at deriving a completely asymptotic characterization for the perfect conductivity problem \cite{Ammari2010} with \( m \)-convex inclusions close to the matrix boundary and the boundary data of \( k \)-order growth in all dimensions. The asymptotic results in this paper also provide an efficient way to compute the electrical field numerically.

To state our main results in a precise manner, we first describe our domain and notations. Let \( D \subset \mathbb{R}^n (n \geq 2) \) be a bounded domain with \( C^{2,\alpha} \) \( (0 < \alpha < 1) \) boundary, which has a \( C^{2,\alpha} \)-subdomain \( D_1^* \) touching matrix boundary \( \partial D \) only at one point. That is, by a translation and rotation of the coordinates, if necessary, \( \partial D_1^* \cap \partial D = \{ \theta' \} \subset \mathbb{R}^{n-1} \).
Throughout the paper, we use superscript prime to denote \((n-1)\)-dimensional domains and variables, such as \(\Sigma'\) and \(x'\). After a translation, we set

\[
D^*_1 := D^*_1 + (0', \varepsilon),
\]

where \(\varepsilon > 0\) is a sufficiently small constant. For the sake of simplicity, denote

\[
D_1 := D^*_1, \quad \text{and} \quad \Omega := D \setminus \overline{D}_1.
\]

The conductivity problem with inclusions close to touching matrix boundary can be modeled by the following scalar equation with piecewise constant coefficients

\[
\begin{cases}
\text{div}(a_k(x)\nabla u) = 0, & \text{in } D, \\
u = \varphi, & \text{on } \partial D,
\end{cases}
\]

where

\[
a_k(x) = \begin{cases}
k \in [0,1) \cup (1,\infty], & \text{in } D_1, \\
1, & \text{on } D \setminus D_1.
\end{cases}
\]

Actually, equation (1.3) can also be used to describe more physical phenomenon, such as dielectrics, magnetism, thermal conduction, chemical diffusion and flow in porous media.

When the conductivity of \(D_1\) degenerates to be infinity, problem (1.3) turns into the perfect conductivity problem as follows

\[
\begin{cases}
\Delta u = 0, & \text{in } D \setminus D_1, \\
u = C_1, & \text{in } \overline{D}_1, \\
\int_{\partial D_1} \frac{\partial u}{\partial \nu} + = 0, & \text{on } \partial D, \\
u = \varphi, & \text{on } \partial D,
\end{cases}
\]

where the free constant \(C_1\) is determined later by the third line of (1.4). There has established the existence, uniqueness and regularity of weak solutions to (1.4) in [12] with a minor modification. We further assume that there exists a small constant \(R > 0\) independent of \(\varepsilon\), such that the portions of \(\partial D\) and \(\partial D_1\) near the origin can be written as

\[
x_n = \varepsilon + h_1(x') \quad \text{and} \quad x_n = h(x'), \quad x' \in B^n_{2R},
\]

where \(h_1\) and \(h\) satisfy that for \(m \geq 2\),

\[
\begin{align*}
(H1) \quad & h_1(x') - h(x') = \lambda |x'|^m + O(|x'|^{m+1}), \\
(H2) \quad & |\nabla_i h_1(x')|, |\nabla_i h(x')| \leq \kappa_1 |x'|^{m-i}, \quad i = 1, 2, \\
(H3) \quad & \|h_1\|_{C^{2,\alpha}(B^n_{2R})} + \|h\|_{C^{2,\alpha}(B^n_{2R})} \leq \kappa_2,
\end{align*}
\]

where \(\lambda\) and \(\kappa_j, j = 1, 2\), are three positive constants independent of \(\varepsilon\).

To explicitly uncover the effect of boundary data \(\varphi\) on the singularities of the field, we classify \(\varphi \in C^2(\partial D)\) according to its parity as follows. Denote the bottom boundary of \(\Omega_R\) by \(\Gamma^-_R = \{x \in \mathbb{R}^n \mid x_n = h(x'), |x'| < R\}\). Suppose that for \(x \in \Gamma^-_R\),

\[
(S1) \quad \varphi \text{ satisfies the k-order growth condition, that is,} \\
\varphi(x) = \eta |x'|^k;
\]

\[
(S2) \quad \varphi \text{ is odd with respect to some } x_{i_0}, \quad i_0 \in \{1, \cdots, n-1\},
\]
where $\eta > 0$ and $k > 1$ is a positive integer.

For $z' \in B_R^1$, $0 < t \leq 2R$, denote
\[
\Omega_t(z') := \{ x \in \mathbb{R}^n \mid h(x') < x_n < \varepsilon + h_1(x'), \ |x' - z'| < t \}.
\]
We will use the abbreviated notation $\Omega_t$ for the domain $\Omega_t(0')$. Before stating our main results, we first introduce two scalar auxiliary functions $\bar{u} \in C^2(\mathbb{R}^n)$ and $\bar{u}_0 \in C^2(\mathbb{R}^n)$ such that $\bar{u} = 1$ on $\partial D_1$, $\bar{u} = 0$ on $\partial D$ and
\[
\bar{u}(x) = \frac{x_n - h(x')}{\varepsilon + h_1(x') - h(x')}, \quad \text{in} \ \Omega_{2R}, \quad \|\bar{u}\|_{C^2(\Omega;\mathbb{R})} \leq C, \quad (1.5)
\]
and $\bar{u}_0 = 0$ on $\partial D_1$, $\bar{u}_0 = \varphi(x)$ on $\partial D$, and
\[
\bar{u}_0 = \varphi(x', h(x'))(1 - \bar{u}), \quad \text{in} \ \Omega_{2R}, \quad \|\bar{u}_0\|_{C^2(\Omega;\mathbb{R})} \leq C. \quad (1.6)
\]
To simplify notations used in the following, for $i = 0$, and $i = k, k$ is the order of growth defined in $(S1)$, we denote
\[
\rho_i(n, m; \varepsilon) = \begin{cases}
\varepsilon^{\frac{n+i-1}{m}}, & m > n + i - 1, \\
|\ln \varepsilon|, & m = n + i - 1, \\
1, & m < n + i - 1,
\end{cases}
\]
and
\[
\Gamma_{n+i}^m = \begin{cases}
\Gamma \left(1 - \frac{n+i-1}{m}\right) \Gamma \left(\frac{n+i-1}{m}\right), & m > n + i - 1, \\
1, & m = n + i - 1,
\end{cases}
\]
where $\Gamma(s) = \int_0^{+\infty} t^{s-1} e^{-t} dt$, $s > 0$ is the Gamma function. Denote by $\omega_{n-1}$ the area of the surface of unit sphere in $(n-1)$-dimension. For $(z', z_n) \in \Omega_{2R}$, denote
\[
\delta(z') := \varepsilon + h_1(z') - h(z'). \quad (1.7)
\]
Let $\Omega^* := D \setminus \overline{D_1}$. We define a linear functional with respect to $\varphi$,
\[
Q^*[\varphi] := \int_{\partial D_1} \frac{\partial v_0^*}{\partial \nu}, \quad (1.8)
\]
where $v_0^*$ is a solution of the following problem:
\[
\begin{cases}
\Delta v_0^* = 0, & \text{in} \ \Omega^*, \\
v_0^* = 0, & \text{on} \ \partial D_1^*, \\
v_0^* = \varphi(x), & \text{on} \ \partial D.
\end{cases} \quad (1.9)
\]
Note that the definition of $Q^*[\varphi]$ is valid under case $(S2)$ but only valid for $m < n + k - 1$ under case $(S1)$. For $m < n - 1$, define
\[
a_{11}^* := \int_{\Omega^*} |\nabla v_1^*|^2, \quad (1.10)
\]
where $v_1^*$ satisfies
\[
\begin{cases}
\Delta v_1^* = 0, & \text{in} \ \Omega^*, \\
v_1^* = 1, & \text{on} \ \partial D_1^* \setminus \{0\}, \\
v_1^* = 0, & \text{on} \ \partial D.
\end{cases} \quad (1.11)
\]
Unless otherwise stated, in what following $C$ represents a constant, whose values may vary from line to line, depending only on $\lambda, \kappa_1, \kappa_2, R$ and an upper bound of the $C^{2,\alpha}$ norms of $\partial D_1$ and $\partial D$, but not on $\varepsilon$. We also call a constant having such dependence a \textit{universal constant}. Without loss of generality, we set $\varphi(0) = 0$. 


Otherwise, we substitute \( u - \varphi(0) \) for \( u \) throughout this paper. For simplicity of discussions, we assume that convexity index \( m \geq 2 \) and growth order index \( k > 1 \) are all positive integers in the following.

**Theorem 1.1.** Assume that \( D_1 \subset D \subseteq \mathbb{R}^n (n \geq 2) \) are defined as above, conditions (H1)–(H3) and (S1) hold. Let \( u \in H^1(D; \mathbb{R}^n) \cap C^1(\Omega; \mathbb{R}^n) \) be the solution of (1.4). Then for a sufficiently small \( \varepsilon > 0 \) and \( x \in \Omega_R \),

(i) for \( m \geq n + k - 1 \),

\[
\nabla u = \frac{\eta \Gamma_m (1 + O(\varepsilon))}{\lambda n_m} \rho_0(n, m; \varepsilon) \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{1 - \frac{m}{k}} \| \varphi \|_{C^2(\partial D)};
\]

(ii) for \( n - 1 \leq m < n + k - 1 \), if \( Q^*[\varphi] \neq 0 \),

\[
\nabla u = m \lambda^{\frac{n-1}{m}} Q^*[\varphi] \frac{1 + O(\varepsilon)}{(n-1)\omega_{n-1}} \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{1 - \frac{m}{k}} \| \varphi \|_{C^2(\partial D)};
\]

(iii) for \( m < n - 1 \), if \( Q^*[\varphi] \neq 0 \),

\[
\nabla u = \frac{Q^*[\varphi]}{a_{11}} (1 + O(\varepsilon)) \nabla \bar{u} + \nabla \bar{u}_0 + O(1)\delta^{1 - \frac{m}{k}} \| \varphi \|_{C^2(\partial D)},
\]

where \( \rho_0(n, m; \varepsilon) = \rho_k(n, m; \varepsilon) / \rho_0(n, m; \varepsilon) \), \( \bar{u} \) and \( \bar{u}_0 \) are defined by (1.3) and (1.0), respectively, \( \delta \) is defined by (1.4), \( a_{11} \) is defined by (1.10), and

\[
r_\varepsilon = \begin{cases}
\frac{\varepsilon^{\frac{1}{k} + k - 1}}{m_n}, & m > n + k, \\
\frac{\varepsilon^{\frac{1}{k} + k - 1 - n}}{m_n}, & m = n + k - 1, \\
\frac{\varepsilon^{\frac{1}{k} + k - 1 - n}}{m_n}, & m = n - 1, \\
\max\{\frac{\varepsilon^{\frac{1}{k} + k - 1 - n}}{m_n}, \varepsilon^{\frac{1}{k}}\}, & m < n - 1.
\end{cases}
\]

**Remark 1.2.** There is a great difference between interior asymptotics and boundary asymptotics. Specifically, the blow-up factor \( Q_\varepsilon[\varphi] \) defined in [27] is bounded for any boundary data \( \varphi \), while \( Q[\varphi] \) here can increase the singularities of the field by \( \varepsilon^{\frac{1}{k} + k - 1} \) if \( m > n + k - 1 \) or \( |\ln \varepsilon| \) if \( m = n + k - 1 \) for the boundary data \( \varphi \) with \( k \)-order growth. In addition, when \( m > 2 \), the remainder of order \( O(\varepsilon^{1 - 2/m}) \) in the shortest line segment between the conductors and the matrix boundary provides a more precise characterization on the asymptotic behavior of the concentration than that of \( m = 2 \). Finally, the concisely main terms \( \nabla \bar{u} \) and \( \nabla \bar{u}_0 \) together with their coefficients can completely describe the singular effect of the geometry, which will greatly reduce the complexity of numerical computation for \( \nabla u \).

**Remark 1.3.** The asymptotics of \( \nabla u \) in Theorem 1.1 indicate that

1. if \( m \leq n + k - 1 \), then its maximum achieves only at \( \{x' = 0\} \cap \Omega \);
2. if \( m > n + k - 1 \), then the maximum achieves at both \( \{x' = 0\} \cap \Omega \) and \( \{|x'| = \varepsilon^{\frac{1}{k}}\} \cap \Omega \).

**Remark 1.4.** In order to further reveal the effect of principal curvatures of the geometry, we take \( n = 3 \) relevant to physical dimension for example. Consider

\[
\varphi = \eta_1 |x_1|^k + \eta_2 |x_2|^k, \quad x \in \{\lambda_1 |x_1|^m + \lambda_2 |x_2|^m < R, x_3 = h(x')\},
\]
and
\[ h_1(x') - h(x') = \lambda_1|x_1|^m + \lambda_2|x_2|^m, \quad x' \in \{ \lambda_1|x_1|^m + \lambda_2|x_2|^m < R \}, \tag{1.13} \]
where \( \lambda_i, \eta_i, \ i = 1, 2, \) are four positive constant independent of \( \varepsilon. \) Then by the same method as in Theorem 1.1 we find that the coefficient of the main term \( \nabla \bar{u} \) has an explicit dependence on \( \lambda_1 \) and \( \eta_1 \) in the form of that \( \eta_1 \lambda_1^{-\frac{1}{m}} + \eta_2 \lambda_2^{-\frac{1}{m}} \) for \( m \geq k + 2 \) and \( \sqrt{\lambda_1 \lambda_2} \) for \( m < k + 2. \)

**Theorem 1.5.** Assume that \( D \subset D \subseteq \mathbb{R}^n (n \geq 2) \) are defined as above, conditions \((H1)\)–\((H3)\) and \((S2)\) hold, \( Q^*[\varphi] \neq 0. \) Let \( u \in H^1(D; \mathbb{R}^n) \cap C^1(\overline{D}; \mathbb{R}^n) \) be the solution of (1.4). Then for a sufficiently small \( \varepsilon > 0, \)

(i) for \( m \geq n - 1, \)
\[
\nabla u = \frac{m \lambda_1^{\frac{1}{m}} Q^*[\varphi]}{(n-1) \omega_{n-1} \rho_0(n, m; \varepsilon)} \frac{1 + O(\tilde{r}_\varepsilon)}{\nabla \bar{u} + \nabla u_0} \frac{\nabla \bar{u} + O(1)\delta^{1-\frac{2}{m}}}{\| \varphi \|_{C^2(\partial D)}},
\]

(ii) for \( m < n - 1, \)
\[
\nabla u = \frac{Q^*[\varphi]}{a_{11}} (1 + O(\tilde{r}_\varepsilon)) \nabla \bar{u} + \nabla u_0 + O(1)\delta^{1-\frac{2}{m}} \| \varphi \|_{C^2(\partial D)},
\]
where \( \bar{u} \) and \( u_0 \) are defined by (1.5) and (1.6), respectively, \( \delta \) is defined by (1.7), \( a_{11} \) is defined by (1.10), and \( \tilde{r}_\varepsilon \) is defined by (1.11), and
\[
\tilde{r}_\varepsilon = \begin{cases} \varepsilon^{\frac{m+n-2}{(m+1)(2m+n-2)}}, & m > n - 1, \\ \varepsilon^{-\frac{1}{2}}, & m = n - 1, \\ \max\{\varepsilon^{\frac{m+n-2}{(m+1)(2m+n-2)}}, \varepsilon^\frac{1}{m}\}, & m < n - 1. \end{cases}
\]

**Remark 1.6.** The asymptotics of \( \nabla u \) in Theorem 1.5 imply that

(1) if \( m < n, \) then its maximum attains only at \( \{ x' = 0 \} \cap \Omega; \)

(2) if \( m = n, \) then the maximum attains at \( \{ x' = 0 \} \cap \Omega \) and \( \{|x'| = \varepsilon^{\frac{1}{m}}\} \cap \Omega \) simultaneously;

(3) if \( m > n, \) then the maximum attains at \( \{|x'| = \varepsilon^{\frac{1}{m}}\} \cap \Omega. \)

**Remark 1.7.** If (1.16) holds in Theorem 1.5, we can obtain that the coefficient of the main term \( \nabla \bar{u} \) has an explicit dependence of \( \sqrt{\lambda_1 \lambda_2}. \)

The organization of this paper is as follows. In section 2, we carry out a linear decomposition of the solution \( u \) to problem (1.1) as \( v_0 \) and \( v_1, \) defined by (2.2) and (2.3) below, and we prove the correspondingly main terms \( \bar{u}_0 \) and \( \bar{u} \) constructed by (1.5) and (1.6), respectively, in Lemma 2.2 and Theorem 2.1. Based on the results obtained in section 2, we give the proofs of Theorem 1.1 and Theorem 1.5 consisting of the asymptotics of blow-up factor \( Q[\varphi] \) and \( a_{11} \) in section 3.

2. Preliminary

2.1. Solution split. As in [29], we decompose the solution \( u \) of (1.4) as follows
\[
u(x) = C_1 v_1(x) + v_0(x), \quad \text{in } D \setminus \overline{D}_1, \tag{2.1} \]
where \( v_i, i = 0, 1 \), verify
\[
\begin{align*}
\Delta v_0 &= 0, & \text{in } \Omega, \\
v_0 &= 0, & \text{on } \partial D_1, \\
v_0 &= \varphi(x), & \text{on } \partial D,
\end{align*}
\] (2.2)
and
\[
\begin{align*}
\Delta v_1 &= 0, & \text{in } \Omega, \\
v_1 &= 1, & \text{on } \partial D_1, \\
v_1 &= 0, & \text{on } \partial D,
\end{align*}
\] (2.3)
respectively. Similarly as (1.8) and (1.9), we define a linear functional of \( \varphi \) as follows
\[
Q[\varphi] = \int_{\partial D_1} \frac{\partial v_0}{\partial \nu},
\] (2.4)
where \( v_0 \) is defined by (2.2). Denote
\[
a_{11} := \int_{\Omega} |\nabla v_1|^2 \, dx.
\]
Then, it follows from the third line of (1.4) and the decomposition (2.1) that
\[
C_1 \int_{\partial D_1} \frac{\partial v_1}{\partial \nu} + \int_{\partial D_1} \frac{\partial v_0}{\partial \nu} = 0.
\]
Recalling the definition of \( v_1 \) and making use of integration by parts, we have
\[
\nabla u = \frac{Q[\varphi]}{a_{11}} \nabla v_1 + \nabla v_0.
\] (2.5)

2.2. **A general boundary value problem.** To obtain the asymptotic expansion for \( \nabla u \), we first consider the following general boundary value problem:
\[
\begin{align*}
\Delta v &= 0, & \text{in } \Omega, \\
v &= \psi, & \text{on } \partial D_1, \\
v &= 0, & \text{on } \partial D,
\end{align*}
\] (2.6)
where \( \psi \in C^2(\partial D_1) \) is a given scalar function. Note that if \( \psi = 1 \) on \( \partial D_1 \), then \( v_1 = v \). Extend \( \psi \in C^2(\partial D_1) \) to \( \psi \in C^2(\Omega) \) such that \( \|\psi\|_{C^2(\Omega)} \leq C \|\psi\|_{C^2(\partial D_1)} \).

Construct a cutoff function \( \rho \in C^2(\Omega) \) satisfying \( 0 \leq \rho \leq 1, |\nabla \rho| \leq C \) on \( \Omega \), and
\[
\rho = 1 \text{ on } \Omega_{2R}, \quad \rho = 0 \text{ on } \Omega \setminus \Omega_{2R}.
\] (2.7)

For \( x \in \Omega \), we define
\[
\tilde{v}(x) = [\rho(x)\psi(x', \varepsilon + h_1(x')) + (1 - \rho(x))\psi(x)]\bar{u}(x),
\]
where \( \bar{u} \) is defined by (1.5). Specially,
\[
\tilde{v}(x) = \psi(x', \varepsilon + h_1(x'))\bar{u}(x), \quad \text{in } \Omega_R.
\]
Due to (1.5), we have
\[
\|\tilde{v}\|_{C^2(\Omega \setminus \Omega_R)} \leq C \|\psi\|_{C^2(\partial D_1)}.
\] (2.8)

Similarly as in [29], we can obtain an asymptotic expansion of the gradient for problem (2.3).
Theorem 2.1. Assume as above. Let \( v \in H^1(\Omega) \) be a weak solution of (2.6). Then, for a sufficiently small \( \varepsilon > 0 \),

\[
|\nabla (v - \bar{v})(x)| \leq C \delta^{1 - \frac{m}{1 - m}} \left( |\varphi(x', \varepsilon + h_1(x'))| + \delta^{\frac{m}{1 - m}} \|\varphi\|_{C^2(\partial D_1)} \right), \quad \text{in } \Omega_R. \tag{2.9}
\]

Consequently, (2.9), together with choosing \( \psi = 1 \) on \( \partial D_1 \), yields that

\[
\nabla v_1 = \nabla \bar{u} + O(1) \delta^{1 - \frac{m}{1 - m}}, \quad \text{in } \Omega_R, \tag{2.10}
\]

and

\[
\|\nabla v\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C \|\psi\|_{C^2(\partial D_1)}.
\]

where \( v_1 \in H^1(\Omega) \) is a weak solution of (2.3).

Note that when \( m > 2 \), the remainder of order \( O(1) \) in (2.9) is improved to that of order \( O(\varepsilon^{1 - 2/m}) \) for \( x \in \{x' = 0'\} \cap \Omega_R \) here. For readers’ convenience, the detailed proof of Theorem 2.1 is left in the Appendix. Similarly, by applying Theorem 2.1, we can find that the leading term of \( \nabla v_0 \) is \( \nabla \bar{u}_0 \) in the following.

Lemma 2.2. Assume as above. Let \( v_0 \) be the weak solution of (2.2). Then, for a sufficiently small \( \varepsilon > 0 \),

\[
\nabla v_0 = \nabla \bar{u}_0 + O(1) \delta^{1 - \frac{m}{1 - m}} (|\varphi(x', h(x'))| + \delta^{\frac{m}{1 - m}} \|\varphi\|_{C^2(\partial D)}), \quad \text{in } \Omega_R, \tag{2.11}
\]

and

\[
\|\nabla v_0\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C \|\varphi\|_{C^2(\partial D)}, \quad \|\nabla v_0\|_{L^\infty(\Omega \setminus \Omega_R)} \leq C \|\varphi\|_{C^2(\partial D)} \tag{2.12}
\]

where \( \bar{u}_0 \) is defined by (1.6).

Therefore, recalling the decomposition (2.5) and in view of (2.10) and (2.11), for the purpose of deriving the asymptotic of \( \nabla u \), it suffices to establish the following two aspects of expansions:

(i) Expansion of \( Q[\varphi] \);

(ii) Expansion of \( a_{11} \).

3. Proofs of Theorem 1.1 and Theorem 1.5

3.1. Expansion of \( Q[\varphi] \). Before proving Theorem 1.1 and Theorem 1.5, we first give an expansion of \( Q[\varphi] \) with respect to \( \varepsilon \).

Lemma 3.1. Assume as above. Then, for a sufficiently small \( \varepsilon > 0 \),

(a) if (S1) holds for \( m \geq n + k - 1 \) in Theorem 1.1

\[
Q[\varphi] = \frac{(n - 1)\omega n_{n-1} \gamma^m}{m \Lambda^\frac{k}{m}} \rho_k(n, m; \varepsilon) \begin{cases} 1 + O(1)\varepsilon^{\frac{m}{1 - m}}, & m > n + k, \\ 1 + O(1)\varepsilon^{\frac{m}{1 - m}} \ln \varepsilon, & m = n + k, \\ 1 + O(1)\ln \varepsilon, & m = n + k - 1; \end{cases}
\]

(b) if (S1) holds for \( m < n + k - 1 \) in Theorem 1.1

\[
Q[\varphi] = Q^*[\varphi] + O(1)\varepsilon^{\frac{n-k-1-m}{m(n+k-1(n+k)}}.
\]

(c) if (S2) holds in Theorem 1.5

\[
Q[\varphi] = Q^*[\varphi] + O(1)\varepsilon^{\frac{m+n-2}{m+n}}.
\]
follows from integration by parts that

\[
\nu = \frac{(\nabla x' h_1(x'), -1)}{\sqrt{1 + |\nabla x' h_1(x')|^2}} \quad \text{in } \Omega_R.
\]

In light of \((H2)\), we obtain that for \(i = 1, \cdots, n - 1,\)

\[
|\nu_i| \leq C|x'|^{m-1}, \quad |\nu_n| \leq 1, \quad \text{in } \Omega_R.
\]

Recalling the definition of \(Q[\varphi]\), it follows from \((2.11)\)–\((2.12)\) and \((3.1)\) that

\[
Q[\varphi] = \int_{\partial D_1} \partial_n v_0 \nu_0 + \int_{\partial D_1} \sum_{i=1}^{n-1} \partial_i v_0 \nu_i
\]

\[
= \int_{|x'| < R} \frac{\eta |x'|^k}{|x'|^m} + O(1) \int_{|x'| < R} \frac{\eta |x'|^{k+1}}{|x'|^m} + O(1) \|\varphi\|_{C^2(\partial D)}
\]

\[
= \frac{(n-1)\omega_{n-1}}{m\lambda^{\frac{n-1}{2}}} \int_0^1 \int_0^{\infty} \frac{\epsilon^{\frac{n-1+k}{m}}}{|\varphi|} + O(1) \frac{m^{\frac{n-1}{m}}}{|\varphi|} + O(1) \ln \epsilon \|\varphi\|_{C^2(\partial D)},
\]

\[
\text{Step 1.} \quad \text{Proofs of } (b) \text{ and } (c). \quad \text{In view of the definitions of } Q[\varphi] \text{ and } Q^*[\varphi], \text{ it follows from integration by parts that}
\]

\[
Q[\varphi] = \int_{\partial D} \frac{\partial v_1}{\partial \nu} \varphi(x), \quad Q^*[\varphi] = \int_{\partial D} \frac{\partial v^*_1}{\partial \nu} \varphi(x),
\]

where \(v_1\) and \(v^*_1\) are defined by \((2.13)\) and \((1.11)\). Thus,

\[
Q[\varphi] - Q^*[\varphi] = \int_{\partial D} \frac{\partial (v_1 - v^*_1)}{\partial \nu} \cdot \varphi(x).
\]

To estimate \(v_1 - v^*_1\), we first introduce a scar auxiliary functions \(\bar{u}^*\) satisfying \(\bar{u}^* = 1 \text{ on } \partial D^*_1 \setminus \{0\}, \bar{u}^* = 0 \text{ on } \partial D, \) and

\[
\bar{u}^* = \frac{x_n - h(x')}{h_1(x') - h(x')}, \quad \text{in } \Omega^*_R, \quad \|\bar{u}^*\|_{C^2(\Omega^*_R)} \leq C,
\]

where \(\Omega^*_R := \Omega^* \cap \{|x'| < r\}, 0 < r \leq 2R. \) In view of \((H2)\), we obtain that for \(x \in \Omega^*_R,\)

\[
|\nabla x' (\bar{u} - \bar{u}^*)| \leq \frac{C}{|x'|},
\]

and

\[
|\partial_n (\bar{u} - \bar{u}^*)| \leq \frac{C\epsilon}{|x'|^{m}(\epsilon + |x'|^m)}.
\]

Applying Theorem \((2.1)\) to \((1.11)\), it follows that for \(x \in \Omega^*_R,\)

\[
|\nabla (v^*_1 - \bar{u}^*)| \leq C|x'|^{m-2},
\]

and

\[
|\nabla x' v^*_1| \leq \frac{C}{|x'|}, \quad |\partial_n v^*_1| \leq \frac{C}{|x'|^m}.
\]

For 0 < \(r < R,\) denote

\[
C_r := \left\{ x \in \mathbb{R}^n \mid |x'| < r, \frac{1}{2} \min_{|x'| \leq r} h(x') \leq x_n \leq \epsilon + 2 \max_{|x'| \leq r} h_1(x') \right\}.
\]
We now divide into two steps to estimate $|Q[\varphi] - Q^*[\varphi]|$.

**Step 2.1.** Note that $v_1 - v_1^*$ solves

$$
\begin{aligned}
\Delta (v_1 - v_1^*) &= 0, &\text{in } D \setminus (D_1 \cup D_1^*), \\
v_1 - v_1^* &= 1 - u^*, &\text{on } \partial D_1 \setminus D_1^*, \\
v_1 - v_1^* &= v_1 - 1, &\text{on } \partial D_1^* \setminus (D_1 \cup \{0\}), \\
v_1 - v_1^* &= 0, &\text{on } \partial D.
\end{aligned}
$$

We first estimate $|v_1 - v_1^*|$ on $\partial(D_1 \cup D_1^*) \setminus C_\gamma$, where $0 < \gamma < 1/2$ to be determined later. In light of the definition of $v_1^*$, we derive that

$$
|\partial_n v_1^*| \leq C, \quad \text{in } \Omega^* \setminus \Omega_R^*.
$$

Therefore,

$$
|v_1 - v_1^*| \leq C\varepsilon, \quad \text{for } x \in \partial D_1 \setminus D_1^*. \quad (3.7)
$$

It follows from (2.10) that

$$
|v_1 - v_1^*| \leq C\varepsilon^{1-m\gamma}, \quad \text{on } \partial D_1^* \setminus (D_1 \cup C_\gamma). \quad (3.8)
$$

Combining Theorem 2.1 and (3.3)–(3.4), we obtain that for $x \in \Omega_R^* \cap \{|x'| = \varepsilon\gamma\}$,

$$
|\partial_n (v_1 - v_1^*)| \leq |\partial_n (v_1 - \bar{\varphi})| + |\partial_n (\bar{\varphi} - \bar{\varphi}^*)| + |\partial_n (v_1^* - \bar{\varphi}^*)|
\leq C \left( \frac{1}{\varepsilon^{2m\gamma-1}} + \varepsilon^{(m-2)\gamma} \right),
$$

which together with $v_1 - v_1^* = 0$ on $\partial D$ yields that

$$
|(v_1 - v_1^*)(x', x_n)| = |(v_1 - v_1^*)(x', x_n) - (v_1 - v_1^*)(x', h(x'))| \leq C(\varepsilon^{1-m\gamma} + \varepsilon^{2(m-1)\gamma}). \quad (3.9)
$$

Take $\gamma = \frac{1}{m+1}$. Then, it follows from (3.7)–(3.9) that

$$
|v_1 - v_1^*| \leq C\varepsilon^{\frac{1}{m+1}}, \quad \text{on } \partial(D \setminus (D_1 \cup D_1^* \cup C_\varepsilon)).
$$

Making use of the maximum principle, we obtain

$$
|v_1 - v_1^*| \leq C\varepsilon^{\frac{-1}{m+1}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup C_\varepsilon). \quad \text{in } (3.7)
$$

This, together with the standard interior and boundary estimates, leads to that, for any $\frac{m-1}{m(m+1)} < \bar{\gamma} < \frac{1}{m+1}$,

$$
|\nabla (v_1 - v_1^*)| \leq C\varepsilon^{m\bar{\gamma} - \frac{m+1}{m+1}}, \quad \text{in } D \setminus (D_1 \cup D_1^* \cup C_\varepsilon),
$$

which implies that

$$
|A_{\text{out}}| := \left| \int_{\partial D \setminus C_\varepsilon} \frac{\partial (v_1 - v_1^*)}{\partial \nu} \cdot \varphi(x) \right| \leq C \|\varphi\|_{L^\infty(\partial D)} \varepsilon^{m\bar{\gamma} - \frac{m+1}{m+1}}, \quad (3.10)
$$

where $\frac{m-1}{m(m+1)} < \bar{\gamma} < \frac{1}{m+1}$ to be determined later.
Step 2.2. We further estimate

\[ A^\ast := \int_{\partial D \cap C, \frac{1}{m+\epsilon} - \gamma} \frac{\partial(v_1 - v_1^\ast)}{\partial \nu} \cdot \varphi(x) \]

\[ = \int_{\partial D \cap C, \frac{1}{m+\epsilon} - \gamma} \frac{\partial(w_1 - w_1^\ast)}{\partial \nu} \cdot \varphi(x) + \int_{\partial D \cap C, \frac{1}{m+\epsilon} - \gamma} \frac{\partial(\bar{u} - \bar{u}^\ast)}{\partial \nu} \cdot \varphi(x) \]

\[ =: A_w + A_u, \]

where \( w_1 = v_1 - \bar{u} \) and \( w_1^\ast = v_1^\ast - \bar{u}^\ast \). To begin with, applying Theorem 2.1, we obtain that

\[ |A_w| \leq C\eta \int_{\partial D \cap C, \frac{1}{m+\epsilon} - \gamma} |\varphi'|^{m+k-2} \leq C\eta \varepsilon^{\left(\frac{m}{m+\epsilon} - \gamma\right)(m+n+k-3)}. \quad (3.11) \]

To estimate \( A_u \), we split it into two parts as follows.

\[ A_u = \int_{\partial D \cap C, \frac{1}{m+\epsilon} - \gamma} \sum_{i=1}^{n-1} \partial_i(\bar{u} - \bar{u}^\ast) \nu_i \varphi(x) + \int_{\partial D \cap C, \frac{1}{m+\epsilon} - \gamma} \partial_n(\bar{u} - \bar{u}^\ast) \nu_n \varphi(x) \]

\[ =: A_u^1 + A_u^2. \]

Case 1. If (S1) holds for \( m < n + k - 1 \) in Theorem 1.1, owing to (3.1) and (3.2)–(3.3), we obtain that

\[ |A_u^1| \leq C\eta \varepsilon^{\left(\frac{m}{m+\epsilon} - \gamma\right)(n+k+m-3)}, \quad |A_u^2| \leq C\eta \varepsilon^{\left(\frac{m}{m+\epsilon} - \gamma\right)(n+k-m-1)}. \]

Then

\[ |A_u| \leq C\eta \varepsilon^{\left(\frac{m}{m+\epsilon} - \gamma\right)(n+k-1)}. \]

This, together with (3.10)–(3.11) and picking \( \gamma = \frac{n+k-2}{(n+k-1)(m+1)} \), yields that

\[ |Q[\varphi] - Q^\ast[\varphi]| \leq C(\eta + \|\varphi\|_{L^\infty(\partial D)}) \varepsilon^{\frac{n+k-1}{(n+k-1)(m+1)}}. \]

Case 2. If (S2) holds in Theorem 1.1, based on the fact that the integrating domain is symmetric with respect to \( x_i, i = 1, \cdots, n - 1 \), we have

\[ |A_u^1| \leq C\eta \varepsilon^{\left(\frac{m}{m+\epsilon} - \gamma\right)(n+m-2)}, \quad A_u^2 = 0. \]

Hence,

\[ |A_u| \leq C\eta \varepsilon^{\left(\frac{m}{m+\epsilon} - \gamma\right)(n+m-2)}. \]

This, together with (3.10)–(3.11) and taking \( \gamma = \frac{2n-3}{(2m+n-2)(m+1)} \), leads to that

\[ |Q[\varphi] - Q^\ast[\varphi]| \leq C(\eta + \|\varphi\|_{L^\infty(\partial D)}) \varepsilon^{\frac{m+n-2}{(m+n-2)(m+1)}}. \]

Consequently, it follows from Step 1 and Step 2 that Lemma 3.1 holds. \( \square \)
3.2. Expansion of $a_{11}$. Before stating the asymptotic of $a_{11}$ with respect to $\varepsilon$, we first introduce a notation used in the following. Denote

$$
A := \int_{\Omega \setminus \Omega_R^*} |\nabla v_1|^2 + 2 \int_{\Omega_R^*} \nabla \bar{u}^* \cdot \nabla (v_1^* - \bar{u}^*)
$$

$$
+ \int_{\partial \Omega_R^*} (|\nabla (v_1^* - \bar{u}^*)|^2 + |\partial x \cdot \bar{u}^*|^2).
$$

(3.12)

Lemma 3.2. Assume as in Theorem 1.1 and Theorem 1.5. Then, for a sufficiently small $\varepsilon > 0$,

(i) for $m \geq n - 1$,

$$
a_{11} = \frac{(n - 1)\omega_{n-1} n^m}{m \lambda^{m-1}} \rho_0(n, m; \varepsilon) \begin{cases} 
1 + O(1)\varepsilon^{\frac{m}{2}}, & m > n, \\
1 + O(1)\varepsilon^{\frac{m-1}{2}} |\ln \varepsilon|, & m = n, \\
1 + O(1)|\ln \varepsilon|^{-1}, & m = n - 1;
\end{cases}
$$

(ii) for $m < n - 1$,

$$
a_{11} = a_{11}^* + O(1)\varepsilon^{\frac{1}{2}},
$$

where $a_{11}^*$ is defined by (3.10).

Proof. Fix $\bar{\gamma} = \frac{1}{6m}$. We first split $a_{11}$ into three parts as follows.

$$
a_{11} = \int_{\Omega, \gamma} |\nabla v_1|^2 + \int_{\Omega \setminus \Omega_R^*} |\nabla v_1|^2 + \int_{\Omega_R^*} |\nabla v_1|^2 =: I + II + III.
$$

Step 1. As for $I$, recalling the definition of $\bar{u}$ and using Theorem 2.1 we obtain that

$$
I = \int_{\Omega, \gamma} |\partial_n \bar{u}|^2 + \int_{\Omega, \gamma} |\partial \gamma \bar{u}|^2 + 2 \int_{\Omega, \gamma} \nabla \bar{u} \cdot \nabla (v_1 - \bar{u}) + \int_{\Omega, \gamma} |\nabla (v_1 - \bar{u})|^2
$$

$$
= \int_{|x'| < \varepsilon} |\bar{u}| dx' + h_1(x') - h(x') + O(1)\varepsilon^{\frac{m-1}{6m}}.
$$

(3.13)

For the second term $II$, we further decompose it into three parts as follows

$$
\begin{align*}
II_1 &= \int_{(\Omega_R \setminus \Omega^* \setminus \Omega_R^*) \setminus (\Omega_R^* \setminus \Omega^*)} |\nabla v_1|^2, \\
II_2 &= \int_{\Omega_R^* \setminus \Omega^*} |\nabla (v_1 - v_1^*)|^2 + 2 \int_{\Omega_R^* \setminus \Omega^*} \nabla v_1^* \cdot \nabla (v_1 - v_1^*), \\
II_3 &= \int_{\Omega_R^* \setminus \Omega^*} |\nabla v_1^*|^2.
\end{align*}
$$

Due to the fact that the thickness of $(\Omega_R \setminus \Omega^*) \setminus (\Omega_R^* \setminus \Omega^*)$ is $\varepsilon$, it follows from (2.10) that

$$
II_1 \leq C\varepsilon \int_{\varepsilon < |x'| < R} dx' \leq C \begin{cases} 
\varepsilon^{\frac{4m+n-1}{6m}}, & m > \frac{n-1}{2}, \\
\varepsilon |\ln \varepsilon|, & m = \frac{n-1}{2}, \\
\varepsilon, & m < \frac{n-1}{2}.
\end{cases}
$$

(3.14)

By picking $\gamma = \frac{1}{2m}$ in Step 2.1 of the proof of Lemma 3.1 it follows from (3.7–3.9) and the maximum principle that

$$
|v_1 - v_1^*| \leq C\varepsilon^{\frac{1}{2}}, \quad \text{in} \ D \setminus (D_1 \cup D_1^* \cup \varepsilon^{\frac{1}{2m}}).
$$
Similarly as before, utilizing the standard interior and boundary estimates, we derive that
\[ |\nabla (v_1 - v_1^*)| \leq C \varepsilon^{\frac{1}{2}}, \quad \text{in} \ D \setminus \left( D_1 \cup D_1^* \cup C_{\varepsilon^{-m}} \right). \quad (3.15) \]

Then combining (3.5) and (3.15), we obtain that
\[ |II_1| \leq C \varepsilon^{\frac{1}{2}}. \tag{3.16} \]

As for II_3, it follows from (3.4) and (3.5) that
\[ II_3 = \int_{\Omega_R \setminus \Omega_R^*} |\nabla v_1|^2 + 2 \int_{\Omega_R \setminus \Omega_R^*} \nabla \bar{v}_3^* \cdot \nabla (v_1^* - \bar{v}_3^*) + \int_{\Omega_R \setminus \Omega_R^*} |\nabla (v_1^* - \bar{v}_3^*)|^2 \]
\[ = \int_{\varepsilon^2 < |x'| < R} \frac{dx'}{h_1(x') - h(x')} + A - \int_{\Omega_R^* \setminus \Omega_R^*} |\nabla v_1|^2 + O(1) \varepsilon^{(n+3-2)\gamma}, \]
where A is defined by (3.12). This, together with (3.14) and (3.16), leads to that
\[ II = \int_{\varepsilon^2 < |x'| < R} \frac{dx'}{h_1(x') - h(x')} + A - \int_{\Omega_R^* \setminus \Omega_R^*} |\nabla v_1|^2 \]
\[ + O(1) \begin{cases} \varepsilon^{\frac{n-1}{m}}, & n = 2, \\ \varepsilon^{\frac{1}{2}}, & n \geq 3. \end{cases} \tag{3.17} \]

For the last term III, due to the fact that $|\nabla v_1|$ is bounded in $D_1^* \setminus (D_1 \cup \Omega_R)$ and $D_1 \setminus D_1^*$ and the fact that the volume of $D_1^* \setminus (D_1 \cup \Omega_R)$ and $D_1 \setminus D_1^*$ is of order $O(\varepsilon)$, it follows from (3.14) that
\[ III = \int_{D_1 \setminus (D_1 \cup \Omega_R)} |\nabla v_1|^2 + O(1) \varepsilon \]
\[ = \int_{D_1 \setminus (D_1 \cup \Omega_R)} |\nabla v_1^*|^2 + 2 \int_{D_1 \setminus (D_1 \cup \Omega_R)} \nabla v_1^* \cdot \nabla (v_1 - v_1^*) \]
\[ + \int_{D_1 \setminus (D_1 \cup \Omega_R)} |\nabla (v_1 - v_1^*)|^2 + O(1) \varepsilon \]
\[ = \int_{\Omega_R \setminus \Omega_R^*} |\nabla v_1^*|^2 + O(1) \varepsilon^{\frac{1}{2}}. \]

This, together with (3.13) and (3.17), yields that
\[ a_{11} = \int_{\varepsilon^2 < |x'| < R} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < \varepsilon^2} \frac{dx'}{\varepsilon + h_1(x') - h(x')} \]
\[ + A + O(1) \begin{cases} \varepsilon^{\frac{n-1}{m}}, & n = 2, \\ \varepsilon^{\frac{1}{2}}, & n \geq 3. \end{cases} \]

**Step 2.** Denote
\[ \text{Main} := \int_{\varepsilon^2 < |x'| < R} \frac{dx'}{h_1(x') - h(x')} + \int_{|x'| < \varepsilon^2} \frac{dx'}{\varepsilon + h_1(x') - h(x')} \]
(i) For $m \geq n - 1$,
\[
\text{Main} = \int_{|x'| < R} \frac{dx'}{\varepsilon + h_1 - h} + \int_{\varepsilon < |x'| < R} \frac{\varepsilon dx'}{(h_1 - h)(\varepsilon + h_1 - h)}
\]
\[
= \int_{|x'| < R} \frac{1}{\varepsilon + \lambda|x'|^m} + \int_{|x'| < R} \left( \frac{1}{\varepsilon + h_1 - h} - \frac{1}{\varepsilon + \lambda|x'|^m} \right) + O(1)\varepsilon^{\frac{4m+n-1}{m}}
\]
\[
= (n-1)\omega_{n-1} \int_0^R \frac{s^{n-2}}{\varepsilon + \lambda s^m} + O(1) \int_0^R \frac{s^{n-1}}{\varepsilon + \lambda s^m}
\]
\[
= \frac{n-1}{m\lambda} \int_0^R \frac{s^{n-1}}{\varepsilon + \lambda s^m} + O(1)\varepsilon^{\frac{m+n-1}{m}},
\]
\[
\text{for } m > n,
\]
\[
\text{Main} = \int_{|x'| < R} \frac{\varepsilon dx'}{(h_1 - h)(\varepsilon + h_1 - h)}
\]
\[
= \int_{\Omega_R^n} |\partial_n \bar{u}^*|^2 + O(1)\varepsilon^{\frac{4m+n-1}{m}}.
\]

Therefore, it follows from Step 1 and Step 2 that Lemma 3.2 holds.

3.3. **Proof of Theorem 1.1** Recalling decomposition (2.5) and combining the results derived in Theorem 2.1, Lemma 2.2, Lemma 3.1 and Lemma 3.2, we complete the proofs of Theorem 1.1 and Theorem 1.5.

\[\Box\]

4. **Appendix: The proof of Theorem 2.1**

In light of assumptions (H1) and (H2), it follows from a direct calculation that for $i = 1, \ldots, n-1$, $x \in \Omega_{2R}$,
\[
|\partial_i \bar{v}| \leq \frac{C|\psi(x', \varepsilon + h_1(x'))|}{\sqrt{\varepsilon + |x'|^m}} + C\|\nabla \psi\|_{L^\infty(\partial D_1)},
\]
\[
(4.1)
\]
\[
|\partial_n \bar{v}| = \frac{|\psi(x', \varepsilon + h_1(x'))|}{\delta(x')}, \quad \partial_n \bar{v} = 0,
\]
\[
(4.2)
\]
\[
|\Delta \bar{v}| \leq \frac{|\psi(x', \varepsilon + h_1(x'))|}{(\varepsilon + |x'|^m)^{\frac{2}{m}}} + \frac{\|\nabla \psi\|_{L^\infty(\partial D_1)}}{\sqrt{\varepsilon + |x'|^m}} + \|\nabla^2 \psi\|_{L^\infty(\partial D_1)}.
\]
\[
(4.3)
\]

Here and throughout this section, for simplicity of notations, we use $\|\nabla \psi\|_{L^\infty}$, $\|\nabla^2 \psi\|_{L^\infty}$ and $\|\psi\|_{C^2}$ to denote $\|\nabla \psi\|_{L^\infty(\partial D_1)}$, $\|\nabla^2 \psi\|_{L^\infty(\partial D_1)}$ and $\|\psi\|_{C^2(\partial D_1)}$, respectively.

Define
\[
w := v - \bar{v}.
\]
\[
(4.4)
\]

**STEP 1.** Let $v \in H^1(\Omega)$ be a weak solution of (2.6). Then
\[
\int_{\Omega} |\nabla w|^2 dx \leq C\|\psi\|_{C^2(\partial D_1)}^2.
\]
\[
(4.5)
\]
Invoking (1.4), \( w \) satisfies
\[
\begin{align*}
\Delta w &= -\Delta \bar{v}, & \text{in } \Omega, \\
w &= 0, & \text{on } \partial \Omega.
\end{align*}
\]

Multiplying the equation in (4.6) and integrating by parts, it follows from the Poincaré inequality, Sobolev trace embedding theorem, (2.8) and (4.11)-(4.12) that
\[
\int_{\Omega} |\nabla w|^2 = \int_{\Omega_R} \omega \Delta \bar{v} + \int_{\Omega \setminus \Omega_R} \omega \Delta \bar{v}
\leq \sum_{i=1}^{n-1} \left( \int_{\Omega_R} \omega \partial_i \bar{v} \right) + C\|\psi\|_{C^2} \int_{\Omega \setminus \Omega_R} |w|
\leq C\|\nabla w\|_{L^2(\Omega_R)} \|\nabla \bar{v}\|_{L^2(\Omega_R)} + C\|\psi\|_{C^2(\partial D_1)} \|\nabla w\|_{L^2(\Omega \setminus \Omega_R)}
\leq C\|\psi\|_{C^2(\partial D_1)} \|\nabla w\|_{L^2(\Omega)}.
\]

Then (4.5) is proved.

**Step 2.** Proof of
\[
\int_{\Omega_s(z')} |\nabla w|^2 dx \leq C\delta^{n+2-\frac{n}{m}} \left( |\psi(z', \varepsilon + h_1(z'))|^2 + \delta \frac{\varepsilon}{\varepsilon^{\frac{n}{m}} - 1}\|\nabla \bar{v}\|^2_{C^2(\partial D_1)} \right),
\]
where \( \delta \) is defined by (1.17). As seen in (20), we have the iteration formula as follows:
\[
\int_{\Omega_s(z')} |\nabla w|^2 dx \leq \frac{C}{(s-t)^2} \int_{\Omega_s(z')} |w|^2 dx + C(s-t)^2 \int_{\Omega_s(z')} |\Delta \bar{v}|^2 dx.
\]

We next divide into two cases to prove (4.7).

**Case 1.** If \( |z'| < \frac{\varepsilon}{C}, \) \( 0 < s < \varepsilon \frac{C}{C}, \) we have \( \varepsilon \leq \delta(x') \leq C\varepsilon \) in \( \Omega \cap \varepsilon(z'). \) In light of (1.3), we derive
\[
\int_{\Omega_s(z')} |\Delta \bar{v}|^2 \leq C|\psi(z', \varepsilon + h_1(z'))|^2 \frac{s^{n-1}}{\varepsilon^{\frac{n}{m} - 1}} + C\varepsilon^{n-1} \frac{\varepsilon^{\frac{n}{m} - 1}}{\varepsilon^{\frac{n}{m}}} \|\nabla \bar{v}\|^2_{C^2},
\]
while, due to the fact that \( w = 0 \) on \( \Gamma_1 := \{ x \in \mathbb{R}^n | x_n = h(x'), |x'| < R \}, \)
\[
\int_{\Omega_s(z')} |w|^2 \leq C\varepsilon^2 \int_{\Omega_s(z')} |\nabla w|^2.
\]
Denote
\[
F(t) := \int_{\Omega_t(z')} |\nabla w|^2.
\]
It follows from (1.3) and (4.9) that for \( 0 < t < s < \varepsilon \frac{C}{C}, \)
\[
F(t) \leq \left( \frac{c_1 \varepsilon}{s-t} \right)^2 F(s) + C(s-t)^2 \frac{s^{n-1}}{\varepsilon^{\frac{n}{m} - 1}} \frac{|\psi(z', \varepsilon + h_1(z'))|^2}{\varepsilon^{\frac{n}{m}}} + C \|\nabla \bar{v}\|^2_{C^2} + \frac{\|\nabla \bar{v}\|^2_{C^2}}{\varepsilon^{\frac{n}{m}}},
\]
where \( c_1 \) and \( C \) are universal constants.

Pick \( k = \left[ \frac{1}{4c_1 \varepsilon \frac{C}{C}} \right] + 1 \) and \( t_i = \delta + 2c_1 \varepsilon, \) \( i = 0, 1, 2, \cdots, k. \) Then, (4.10), together with \( s = t_i+1 \) and \( t = t_i, \) leads to
\[
F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{n-1} \frac{\varepsilon^{n+2-\frac{n}{m}}}{\varepsilon^{\frac{n}{m}}} \left[ |\psi(z', \varepsilon + h_1(z'))|^2 + \varepsilon^{\frac{n}{m}} \|\nabla \bar{v}\|^2_{C^2} \right].
\]
It follows from \( k \) iterations and (1.3) that for a sufficiently small \( \varepsilon > 0, \)
\[
F(t_0) \leq C\varepsilon^{n+2-\frac{n}{m}} \left( |\psi(z', \varepsilon + h_1(z'))|^2 + \varepsilon^{\frac{n}{m}} \|\nabla \bar{v}\|^2_{C^2} \right).
\]
Consequently, for $h \leq |z'| \leq R$ and $0 < s < \frac{2|z'|}{3}$, we have $\frac{|z'|^m}{C} \leq \delta(x') \leq C|z'|^m$ in $\Omega_{2|z'|}(z')$. Similar to (4.8) and (4.9), we obtain

$$\int_{\Omega_{s}(z')} |\Delta \tilde{v}|^2 \leq C|\psi(z', \varepsilon + h_1(z'))|^2 \frac{s^{n-1}}{|z'|^{4-m}} + Cs^{n-1}|z'|^m-2\|\psi\|_{C^2}^2,$$

and

$$\int_{\Omega_{s}(z')} |w|^2 \leq C|z'|^{2m}\int_{\Omega_{s}(z')} |\nabla w|^2.$$

Moreover, for $0 < t < s < \frac{2|z'|}{3}$, estimate (4.10) becomes

$$F(t) \leq \left( \frac{c_2|z'|^m}{s-t} \right)^2 F(s) + C(s-t)^2 s^{n-1} \left( \frac{|\psi(z', \varepsilon + h_1(z'))|^2}{|z'|^{4-m}} + |z'|^m-2\|\psi\|_{C^2}^2 \right).$$

Similarly as above, pick $k = \left\lceil \frac{1}{2c_2|z'|} \right\rceil + 1$, $t_i = \delta + 2c_2i|z'|^m$, $i = 0, 1, 2, \ldots, k$ and take $s = t_{i+1}$, $t = t_i$, then we obtain

$$F(t_i) \leq \frac{1}{4} F(t_{i+1}) + C(i+1)^{n-1}|z'|^{m(n+2)-4} \left( |\psi(z', \varepsilon + h_1(z'))|^2 + |z'|^2\|\psi\|_{C^2}^2 \right).$$

Likewise, by using $k$ iterations, we have

$$F(t_0) \leq C|z'|^{m(n+2)-4} \left( |\psi(z', \varepsilon + h_1(z'))|^2 + |z'|^2\|\psi\|_{C^2}^2 \right). \quad (4.12)$$

Consequently, (4.12), together with (4.11), yields that (4.7) holds.

**STEP 3.** Proof of

$$|\nabla w(x)| \leq C \delta^{-\frac{m}{2}} (|\psi(x', \varepsilon + h_1(x'))| + \delta^\frac{m}{2}\|\psi\|_{C^2(\partial D_1)}) \quad \text{in } \Omega_R. \quad (4.13)$$

As in (2.9), combining the rescaling argument, Sobolev embedding theorem, $W^{2,p}$ estimate and bootstrap argument, we obtain

$$||\nabla w||_{L^\infty(\Omega_{\delta z'}(z'))} \leq \frac{C}{\delta} \left( \delta^{-\frac{m}{2}} ||\nabla w||_{L^2(\Omega_{\delta z'}(z'))} + \delta^2 ||\Delta \tilde{v}||_{L^\infty(\Omega_{\delta z'}(z'))} \right).$$

In view of (4.8) and (4.7), we obtain that for $|z'| \leq R$,

$$\delta ||\Delta \tilde{v}||_{L^\infty(\Omega_{\delta z'}(z'))} \leq C \delta^{-\frac{m}{2}} (|\psi(z', \varepsilon + h_1(z'))| + \delta^\frac{m}{2}\|\psi\|_{C^2}),$$

and

$$\delta^{-\frac{m}{2}} ||\nabla w||_{L^2(\Omega_{\delta z'}(z'))} \leq C \delta^{-\frac{m}{2}} (|\psi(z', \varepsilon + h_1(z'))| + \delta^\frac{m}{2}\|\psi\|_{C^2}).$$

Consequently, for $h(z') < z_n < \varepsilon + h_1(z')$,

$$|\nabla w(z', z_n)| \leq C \delta^{-\frac{m}{2}} (|\psi(z', \varepsilon + h_1(z'))| + \delta^\frac{m}{2}\|\psi\|_{C^2}).$$

Estimate (2.9) is established. On the other hand, it follows from the standard interior estimates and boundary estimates for the Laplace equation that

$$||\nabla v||_{L^\infty(\Omega \setminus \Omega_R)} \leq C ||\psi||_{C^2(\partial D_1)}.$$ 

Thus, Theorem 2.1 is proved.

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