Conformal invariance and the conformal–traceless decomposition
of the gravitational field

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Einstein’s theory of general relativity is written in terms of the variables obtained from a conformal–traceless decomposition of the spatial metric and extrinsic curvature. The determinant of the conformal metric is not restricted, so the action functional and equations of motion are invariant under conformal transformations. With this approach the conformal–traceless variables remain free of density weights. The conformal invariance of the equations of motion can be broken by imposing an evolution equation for the determinant of the conformal metric \( g \). Two conditions are considered, one in which \( g \) is constant in time and one in which \( g \) is constant along the unit normal to the spacelike hypersurfaces. This approach is used to write the Baumgarte–Shapiro–Shibata–Nakamura system of evolution equations in conformally invariant form. The presentation includes a discussion of the conformal thin sandwich construction of gravitational initial data, and the conformal flatness condition as an approximation to the evolution equations.

I. INTRODUCTION

The conformal–traceless decomposition of the gravitational field was originally introduced by Lichnerowicz \(^1\) and York \(^2\)\(^3\) in their work on the initial value problem. Since that time the same decomposition has appeared in various guises in mathematical and numerical relativity. The BSSN \(^4\), \(^5\) (Baumgarte–Shapiro–Shibata–Nakamura) system of evolution equations for general relativity is currently in widespread use in the numerical relativity community. The BSSN equations are based on a conformal–traceless splitting. The recently discovered conformal thin sandwich construction \(^6\), \(^7\), like the earlier techniques for solving the initial data problem, uses the conformal–traceless decomposition. Unlike its predecessors, the conformal thin sandwich scheme mixes a subset of the evolution equations with the constraints to facilitate the analysis. The conformal flatness condition (CFC) is an approximation to the Einstein equations that uses the conformal–traceless decomposition. It has been applied to numerical simulations of binary neutron star systems \(^8\) and supernovae \(^9\).

The conformal–traceless (CT) decomposition is

\[
 h_{ab} = \varphi^4 g_{ab}, \quad K_{ab} = \varphi^{-2} A_{ab} + \frac{1}{3} \varphi^4 g_{ab} \tau,
\]

(1)

where \( h_{ab} \) is the physical spatial metric and \( K_{ab} \) is the physical extrinsic curvature. The definitions (1) are redundant, in the sense that multiple sets of fields \( g_{ab}, \varphi, A_{ab}, \tau \) yield the same physical metric and extrinsic curvature. To be precise, these definitions are invariant under the conformal transformation

\[
 g_{ab} \rightarrow \bar{g}_{ab} = \xi^4 g_{ab}, \quad \varphi \rightarrow \bar{\varphi} = \xi^{-1} \varphi,
 A_{ab} \rightarrow \bar{A}_{ab} = \xi^{-2} A_{ab}, \quad \tau \rightarrow \bar{\tau} = \tau,
\]

(2a)

(2b)

(2c)

(2d)

for any field \( \xi \). In many recent applications involving the conformal–traceless splitting the conformal invariance \(^2\) is broken, and the redundancy in the CT variables removed, by the condition that \( g_{ab} \) should have unit determinant. There are various insights to be gained by leaving the determinant of \( g_{ab} \) unspecified at this point in the analysis. Thus, I will not impose \( g = \det(g_{ab}) = 1 \).

In most of the original works on the initial value problem and on gravitational degrees of freedom \(^2\)\(^11\), the determinant \( g \) is not restricted. Rather, the field \( g_{ab} \) is described as representing a conformal equivalence class of metrics that defines a conformal geometry. The approach taken here is equivalent. However, instead of describing the results in terms of conformal geometry, I treat \( g_{ab} \) as an ordinary spatial metric (symmetric type \(^0\)^2 tensor) and emphasize the conformal invariance \(^2\) of the action and equations of motion.

\(^1\) The arrow notation means the following: Replace the fields \( g_{ab}, \varphi, A_{ab}, \) and \( \tau \) with their “barred” counterparts, which differ from the unbarred fields by certain factors of \( \xi \). In Eq. (1) the factors of \( \xi \) completely cancel in each term so that \( h_{ab} \) and \( K_{ab} \) are independent of \( \xi \). Thus, \( h_{ab} \) and \( K_{ab} \) are conformally invariant.
The physical metric and extrinsic curvature are also invariant under the “trace transformation” defined by $\tau \rightarrow \tau + \zeta$ and $A_{ab} \rightarrow A_{ab} = A_{ab} - \zeta \bar{\phi} g_{ab}/3$, with $g_{ab}$ and $\phi$ unchanged, for any field $\zeta$. This redundancy in the CT variables is usually removed by the condition that $A_{ab}$ should have zero trace. It turns out that there is no particular advantage to be gained by leaving the trace of $A_{ab}$ unspecified. Thus, I will impose $A = A_{ab} g^{ab} = 0$. Then the CT variable $\tau$ is the trace of the extrinsic curvature, $\tau = K_{ab} h^{ab}$.

At first sight it might appear surprising that I have chosen to keep the conformal invariance but eliminate the trace invariance. The reason is that the trace transformation is essentially trivial. It just corresponds to a change in the splitting of the trace of $K_{ab}$ into the two terms $\tau$ and $A \phi^{-6}$. With the choice $A = 0$, the trace of the extrinsic curvature is placed entirely in $\tau$. In a similar fashion, the conformal transformation can be viewed as a change in the splitting of the determinant of $h_{ab}$ into the two factors $\phi^{12}$ and $g$. However, this is a nontrivial change precisely because the field $\xi$ can carry a nonzero density weight. Throughout this paper I define the “unbarred” CT variables that appear in the definitions as having no density weight. Thus, $g_{ab}$ and $A_{ab}$ are spatial tensors of type $(0,2)$, and $\phi$ and $\tau$ are spatial scalars. If $\xi$ carries a density weight, then the conformally transformed variables, the “barred” variables in Eqs. (9), acquire density weights.

By not specifying $g = 1$ from the outset, the equations of motion we obtain are conformally invariant. We can then consider breaking the conformal invariance by fixing $g$. In principle $g$ can be chosen as any $t$-dependent scalar density of weight 2. In practice I expect that the most natural way to break conformal invariance is to choose $g$ arbitrarily on the initial time slice and then evolve $g$ according to some simple prescription. Two natural prescriptions are considered, one in which $g$ remains fixed along the time flow vector field (the “Lagrangian condition”) and one in which $g$ remains fixed along the normal to the spacelike hypersurfaces (the “Eulerian condition”). The equations of motion differ between these two cases in the way that the shift terms appear. These differences can be interpreted as changes in the density weights of the CT variables.

In this paper I carefully examine the full Einstein theory, constraints and evolution equations, in terms of the conformal–traceless variables. One of my motivations for this analysis is to help clarify various subtle issues that arise along the way. Most of the subtleties concern density weights. Consider what happens when $\zeta = 1$. In principle $\zeta$ can be chosen as any $t$-dependent scalar density of weight 2. In practice I expect that the most natural way to break conformal invariance is to choose $g$ arbitrarily on the initial time slice and then evolve $g$ according to some simple prescription. Two natural prescriptions are considered, one in which $g$ remains fixed along the time flow vector field (the “Lagrangian condition”) and one in which $g$ remains fixed along the normal to the spacelike hypersurfaces (the “Eulerian condition”). The equations of motion differ between these two cases in the way that the shift terms appear. These differences can be interpreted as changes in the density weights of the CT variables.

In this paper I use the lapse anti–density $\alpha$ as the undetermined multiplier for the Hamiltonian constraint. The lapse anti–density carries density weight $-1$ and the Hamiltonian constraint carries density weight 2. York and collaborators [8, 11] have pointed out a number of reasons why this choice is preferred over the traditional scalar lapse function. It has also been shown [10] that the BSSN equations are equivalent to a strongly hyperbolic system with physical characteristic speeds when the lapse has density weight $-1$.

In Sec. II I begin by writing down the action and equations of motion in Hamiltonian form. In Sec. III the action is expressed in terms of the CT variables. Because the action is conformally invariant, it can also be expressed in terms of a set of conformally invariant variables that includes the density weight $-2/3$ metric $g_{ab} = g^{-1/3} g_{ab}$. The equations of motion for the CT variables are derived in Sec. IV. There, it is pointed out that the conformal invariance can be broken with either the Lagrangian or Eulerian condition on the determinant of $g_{ab}$. This leads to two sets of evolution equations that differ in the way that the shift terms enter. In Sec. V I examine the BSSN system. It is written first in conformally invariant form, then with the Eulerian and Lagrangian conditions applied. The conformal thin sandwich equations are derived in Sec. VI. They follow directly from the Hamiltonian and momentum constraints and the evolution equations of Sec. IV. The CFC approximation is discussed in Sec. VII. There it is pointed out that the CFC equations are identical to the conformal thin sandwich equations with the restrictions that the conformal metric $g_{ab}$ is flat and the time slicing is maximal, $\tau = 0$. The main results are summarized briefly in Sec. VIII.
II. ADM ACTION AND EQUATIONS OF MOTION

The Arnowitt–Deser–Misner (ADM) gravitational action is

$$S^g[h_{ab}, \pi^a, \alpha, \beta^a] = \int dt d^3 x \left[ P^{ab} h_{ab} - \alpha H^g - \beta^a M^g_{\alpha} \right],$$  \hspace{1cm} (3)$$

which is a functional of the physical 3–metric $h_{ab}$, its conjugate $P^{ab}$, the lapse anti–density $\alpha$, and the shift vector $\beta^a$. The dot denotes a time derivative, $h_{ab} \equiv \partial h_{ab}/\partial t$. The gravitational momentum is related to the extrinsic curvature by

$$P^{ab} = \frac{1}{2} \sqrt{g} (h^{ab} h_{cd} - h^{ac} h^{bd}) K_{cd}.$$  \hspace{1cm} (4)$$

The gravitational Hamiltonian and momentum densities are defined by

$$\mathcal{H}^g = 2 P^{ab} P_{ab} - P^2 - h R / 2,$$
$$M^g_{\alpha} = -2 \nabla_a P^b_a,$$

where $P = P^{ab} h_{ab}$ is the trace of the gravitational momentum and $h$ is the determinant of $h_{ab}$. Also, $\nabla_a$ and $R \equiv h_{ab} R_{ab}$ are the covariant derivative and scalar curvature for the spatial metric $h_{ab}$. If matter fields are present the complete action $S = S^g + S^m$ includes a functional $S^m$ of the matter fields in addition to the gravitational action. I will assume that the matter is “minimally coupled” to gravity, so the matter action does not depend on derivatives of the spatial metric or on $P^{ab}$. Throughout this paper I use units in which $8 \pi G = 1$, where $G$ is Newton’s constant.

Variations of the ADM action plus matter action $S^m$ with respect to the lapse anti–density and shift vector yield the constraints

$$\delta_{\alpha} S^m = -h \rho, \quad M^g_{\alpha} = \sqrt{h} j_a,$$

where $\rho$ and $j_a$ are the energy and momentum densities for the matter fields. They are defined, along with the spatial stress tensor $s^{ab}$, in terms of the functional derivatives of the matter action by

$$\rho = -\frac{1}{h} \frac{\delta S^m}{\delta \alpha}, \quad j_a = \frac{1}{\sqrt{h}} \frac{\delta S^m}{\delta \beta^a}, \quad s^{ab} = \frac{2}{\alpha h} \frac{\delta S^m}{\delta h_{ab}} - \frac{h_{ab}}{h} \frac{\delta S^m}{\delta \alpha}.$$  \hspace{1cm} (7)$$

Variations of the action with respect to $h_{ab}$ and $P^{ab}$ yield the well–known ADM evolution equations

$$\partial_\perp h_{ab} = \alpha \left( 4 P_{ab} - 2 P h_{ab} \right),$$  \hspace{1cm} (8a)$$
$$\partial_\perp P^{ab} = \alpha \left( -4 P^{ac} P_c^b + 2 P_{ab} + \frac{1}{2} h R h_{ab} - \frac{1}{2} h R^{ab} \right) + \frac{1}{2} \nabla^a \nabla^b \alpha - \frac{1}{2} h h^{ab} \nabla^2 \alpha + \frac{1}{2} \alpha h (s^{ab} - \rho h_{ab}).$$  \hspace{1cm} (8b)$$

The derivative operator on the left–hand sides of these equations is defined by

$$\partial_\perp \equiv \partial / \partial t - L_{\beta}$$  \hspace{1cm} (9)$$

where $L_{\beta}$ is a Lie derivative along the shift vector field $\beta^a$. Thus, $(\alpha \sqrt{h})^{-1} \partial_\perp$ is the derivative with respect to proper time along the unit normal to the spacelike hypersurfaces. Specifically, the Lie derivatives are defined by

$$L_{\beta} h_{ab} \equiv \beta^c \partial_c h_{ab} + h_{ac} \partial_b \beta^c + h_{cb} \partial_a \beta^c = 2 \nabla_{(a} \beta_{b)},$$
$$L_{\beta} P^{ab} \equiv \partial_c (\beta^c P^{ab}) - P^{ac} \partial_c \beta^b - P^{bc} \partial_c \beta^a = \nabla_{(a} (\beta^c P^{b)}) - 2 P^{[a} \nabla_{c} \beta^{b]},$$  \hspace{1cm} (10a)$$

where $\partial_a = \partial / \partial x^a$ and $(a \cdots b)$ denotes symmetrization on the indices $a$ and $b$. Note that $h_{ab}$ is a symmetric type $(0,2)$ tensor on the spatial manifold, and $P^{ab}$ is a symmetric type $(2,0)$ tensor density of weight 1 on the spatial manifold. (The terminology of density weights is fixed by noting that $\sqrt{h}$ is a scalar density of weight 1.)

The time derivatives in the action can be written in terms of the normal derivative $\partial_\perp$ with the result

$$S^g[h_{ab}, P^{ab}, \alpha, \beta^a] = \int dt d^3 x \left[ P^{ab} \partial_\perp h_{ab} - \alpha H^g \right],$$  \hspace{1cm} (11)$$

The difference between the actions $S^g$ and $(11)$ is a total derivative, $2 \int dt d^3 x \nabla_a (P^{ab} \beta_b)$. This term vanishes for compact spatial manifolds, and integrates to a boundary term for non–compact spatial manifolds. The presence of boundary terms in the action affect the admissible boundary conditions to be imposed in the variational principle, but they do not affect the resulting equations of motion. I will ignore such boundary terms in the present work.
III. ACTION IN CONFORMAL–TRACELESS VARIABLES

With the physical spatial metric and extrinsic curvature written as in Eq. (1), the gravitational momentum becomes

\[ P^{ab} = -\frac{1}{2} \sqrt{g} \varphi^{-4} A^{ab} + \frac{1}{3} \sqrt{g} \varphi^{2} \tau g^{ab}. \]  

(12)

Recall that \( A_{ab} \) is trace free, \( A = 0 \). Also note that indices on \( A_{ab} \) are raised with the inverse conformal metric \( g^{ab} \).

We can express the ADM action in terms of the conformal–traceless (CT) variables by substituting the expressions for \( h_{ab} \) and \( P^{ab} \) into Eq. (13). The result is

\[ S^{(g)}[g_{ab}, A_{ab}, \varphi, \tau, \alpha, \beta^{a}] = \int dt d^{3}x \left[ -\frac{1}{2} g^{5/6} A^{ab} \partial_{\perp} \left( g^{-1/3} g_{ab} \right) + \frac{2}{3} \tau \partial_{\perp} \left( g^{1/2} \varphi^{6} \right) - \alpha \mathcal{H}^{(g)} \right], \]  

(13)

where the gravitational contribution to the Hamiltonian constraint is

\[ \mathcal{H}^{(g)} = \frac{1}{2} g A^{ab} A_{ab} - \frac{1}{3} g \varphi^{12} \tau^{2} - \frac{1}{2} g \varphi^{8} R + 4 g \varphi^{7} D^{2} \varphi. \]  

(14)

Here, \( R \) is the scalar curvature of \( g_{ab} \), and \( D_{a} \) is the covariant derivative compatible with \( g_{ab} \). Note that the “velocity” term \( \partial_{\perp} \left( g^{-1/3} g_{ab} \right) \) in the action is trace free. It can be written as \( \partial_{\perp} \left( g^{-1/3} g_{ab} \right) = g^{-1/3} \left( \partial_{\perp} g_{ab} \right)^{\text{TF}} \), where TF stands for the trace–free part of the expression enclosed in parentheses.

Observe that the traceless property of \( A_{ab} \) must be preserved when the action is varied. This condition can be enforced with a Lagrange multiplier. The result is equivalent to simply demanding that the functional derivative should be symmetric.

A key piece of the calculation for the action (13) is the expression for the scalar curvature of the physical metric \( h_{ab} \) in terms of the conformal metric \( g_{ab} \) and conformal factor \( \varphi \):

\[ R = \varphi^{-4} R - 8 \varphi^{-5} D^{2} \varphi. \]  

(15)

This result can be derived in a straightforward way by inserting the decomposition \( h_{ab} = \varphi^{4} g_{ab} \) into the definition for the Ricci scalar, \( R = 2 g^{ab} \left( \partial_{\perp} \Gamma_{ba}^{c} + \Gamma_{dc}^{a} \Gamma_{bca}^{d} \right) \), where \( \Gamma_{bc}^{a} = k_{a}^{cd} \left( \partial_{b} h_{dc} + \partial_{c} h_{bd} - \partial_{d} h_{bc} \right) / 2 \) are the Christoffel symbols.

The gravitational field contributions to the momentum constraint are “hidden” in the \( \partial_{\perp} \) terms in the action, since these terms include Lie derivatives with respect to the shift vector \( \beta^{a} \). The Lie derivatives are defined by the tensor character of the variables on which they act. For example, \( \mathcal{L}_{\beta} g_{ab} \) and \( \mathcal{L}_{\beta} A^{ab} \) are given by expressions (10) with \( h_{ab} \) replaced by \( g_{ab} \), \( P^{ab} \) replaced by \( A^{ab} \), and \( \nabla_{a} \) replaced by \( D_{a} \). (Indices on \( \beta^{a} \) and \( D_{a} \), like \( A_{ab} \), are raised and lowered with the conformal metric \( g_{ab} \) and its inverse \( g^{ab} \).) In the calculations here and below, it is convenient to use the formula

\[ \mathcal{L}_{\beta} T = g^{w/2} \mathcal{L}_{\beta} \left( T g^{-w/2} \right) + w T D_{c} \beta^{c}. \]  

(16)

for the Lie derivative, where \( T \) is a tensor density of weight \( w \). (The indices have been suppressed on \( T \).) Then the shift terms in the action (13) are

\[ S^{(g)} \big|_{\text{shift terms}} = \int dt d^{3}x \sqrt{g} \left[ A^{ab} \left( D_{a} \beta_{b} \right)^{\text{TF}} - \frac{2}{3} \tau D_{a} \left( \beta^{a} \varphi^{6} \right) \right], \]  

(17)

and the gravitational contribution to the momentum constraint is

\[ \mathcal{M}_{a}^{(g)} \equiv \frac{\delta S^{(g)}}{\delta \beta^{a}} = \sqrt{g} D_{b} A^{b} - \frac{2}{3} \sqrt{g} \varphi^{6} D_{a} \tau. \]  

(18)

The full Hamiltonian and momentum constraints are \( 0 = \mathcal{H}^{(g)} + g \varphi^{12} \rho \) and \( 0 = \mathcal{M}_{a}^{(g)} - \sqrt{g} \varphi^{6} j_{a} \), respectively. They are invariant under the conformal transformation (1).
The action as written in Eq. (13) is a functional of the symmetric type \( (0, 0) \) tensor \( g_{ab} \), the symmetric type \( (0, 2) \) tensor \( A_{ab} \), the scalars \( \varphi \) and \( \tau \), and the lapse anti-density and shift vector. Because the CT variables are redundant the equations of motion obtained by varying the action are not independent. To be precise, the action is conformally invariant so its variation vanishes when the CT fields are varied by an infinitesimal conformal transformation \( \varphi \). This leads to the relation

\[
0 = 4g_{ab} \frac{\delta S^{(g)}}{\delta g_{ab}} - \varphi \frac{\delta S^{(g)}}{\delta \varphi} - 2A_{ab} \frac{\delta S^{(g)}}{\delta A_{ab}}
\]

for all field configurations, not just those that satisfy the classical equations of motion.

The conformal invariance of the theory defined by the action \( S^{(g)} \) can be displayed in an elegant form by treating it as a constrained Hamiltonian system \( \mathcal{H} \). In that case the action becomes

\[
S^{(g)} = \int dt d^3x \left[ \mathcal{P}_{\text{metric}} \partial_1 \mathcal{Q}_{\text{metric}}^{\text{metric}} + \mathcal{P}_{\text{phi}} \partial_1 \mathcal{Q}_{\text{phi}} + \mathcal{P}_{\text{root}} \partial_1 \mathcal{Q}_{\text{root}}^{\text{root}} - \alpha \mathcal{H}^{(g)} - \epsilon \mathcal{C} \right],
\]

where \( \mathcal{P}_{\text{metric}}, \mathcal{P}_{\text{phi}}, \) and \( \mathcal{P}_{\text{root}} \) are the momenta conjugate to the canonical coordinates \( \mathcal{Q}_{\text{metric}}^{\text{metric}} = g^{-1/3} g_{ab}, \mathcal{Q}_{\text{phi}} = \varphi^6, \) and \( \mathcal{Q}_{\text{root}}^{\text{root}} = \sqrt{g} \), respectively. These variables are restricted by the first class constraint \( \mathcal{C} = \mathcal{Q}_{\text{phi}} P_{\text{phi}} - \mathcal{Q}_{\text{root}} P_{\text{root}} \). The constraint \( \mathcal{C} \) appears in the action with an undetermined multiplier \( \epsilon \). The momenta are related to the original CT variables by \( \mathcal{Q}_{\text{phi}} P_{\text{phi}} = \mathcal{Q}_{\text{root}} P_{\text{root}} = (2/3) \sqrt{g} \varphi^6 \tau \) and \( \mathcal{P}_{\text{metric}} = -(1/2) g^{5/6} A^{ab} \). These relations can be used to rewrite the Hamiltonian constraint \( \mathcal{C} \) in terms of the canonical variables. The “smeared” constraint \( \int d^3x \epsilon \mathcal{C} \) generates an infinitesimal conformal transformation \( \tilde{\mathcal{C}} \) through the Poisson brackets with \( \xi = 1 - \epsilon/6 \).

Let us return to the action as expressed in Eq. (13). The equations of motion obtained from this action are redundant, as shown by Eq. (19), because the CT variables are not unique. Said another way, the action \( S^{(g)} \) does not depend on \( g_{ab}, \varphi, A_{ab}, \) and \( \tau \) separately but only on the combinations

\[
\begin{align*}
\tilde{g}_{ab} &\equiv g^{-1/3} g_{ab} = h^{-1/3} h_{ab}, \\
\tilde{\varphi} &\equiv g^{1/12} \varphi = h^{1/12}, \\
\tilde{A}^{ab} &\equiv g^{5/6} A^{ab} = -2h^{1/3} \left( P^{ab} - P h^{ab}/3 \right), \\
\tilde{\tau} &\equiv \tau = P/\sqrt{h}.
\end{align*}
\]

These quantities are invariant under the conformal transformation \( \varphi \). I will refer to \( \tilde{g}_{ab}, \tilde{\varphi}^6, \tilde{A}^{ab} \) and \( \tilde{\tau} \) as the “invariant CT variables.” Note that \( \tilde{g}_{ab} \) is the unit determinant metric. It is a type \( (0, 2) \) tensor density with weight \(-2/3\). The variable \( \tilde{A}^{ab} \) is a traceless type \( (2, 2) \) tensor density with weight \(5/3\). It is proportional to the trace–free part of the extrinsic curvature. The variable \( \tilde{\varphi} \) is a scalar density with weight \(1/6\), and \( \tilde{\tau} \) is the trace of the extrinsic curvature with no density weight. Note that indices on \( \tilde{A}^{ab} \) and other “tilde” quantities are raised and lowered with \( \tilde{g}_{ab} \) and its inverse \( \tilde{g}^{ab} \).

The invariant CT variables (the variables with tilde’s) are combinations of the CT variables (the variables without tilde’s) that are invariant under the conformal transformation \( \varphi \). But they can also be viewed as the CT variables transformed by Eqs. (2) with \( \xi = g^{-1/12} \). This implies that the action, which is conformally invariant, has the same expression in terms of the invariant CT variables as it does in terms of the original CT variables. Thus, we can rewrite Eqs. (13), (14), and (15) by placing tilde’s on each of the CT variables:

\[
S^{(g)}[\tilde{g}_{ab}, \tilde{A}_{ab}, \tilde{\varphi}, \tilde{\tau}, \alpha, \beta^a] = \int dt d^3x \left[ -\frac{1}{2} \tilde{A}^{ab} \partial_1 \tilde{g}_{ab} + \frac{2}{3} \tilde{\tau} \partial_1 \tilde{\varphi}^6 - \alpha \mathcal{H}^{(g)} \right],
\]

where

\[
\begin{align*}
\mathcal{H}^{(g)} &= \frac{1}{2} \tilde{A}^{ab} \tilde{A}_{ab} - \frac{1}{3} \tilde{\varphi}^{12} \tilde{\tau}^2 - \frac{1}{2} \tilde{\varphi}^8 \tilde{R} + 4 \tilde{\varphi}^7 \tilde{D}^2 \tilde{\varphi}, \\
\mathcal{M}_a^{(g)} &= \tilde{D}_b \tilde{A}_a^b - \frac{2}{3} \tilde{\varphi}^6 \tilde{D}_b \tilde{\tau}.
\end{align*}
\]

Note that the determinant of \( \tilde{g}_{ab} \) has been set to one. The full Hamiltonian and momentum constraints are \( \mathcal{H} = \mathcal{H}^{(g)} + \tilde{\varphi}^{12} \rho \) and \( \mathcal{M}_a = \mathcal{M}_a^{(g)} - \tilde{\varphi}^6 j_a \), respectively.

In the Hamiltonian density \( \mathcal{H}^{(g)} \), the term \( \tilde{R} \) is constructed from \( \tilde{g}_{ab} \) according to the usual formula \( \tilde{R} = 2\tilde{g}^{ab} \left( \partial_c \tilde{\Gamma}_a_{cb} + \tilde{\Gamma}_a_{ab} \tilde{\Gamma}_c_{d} \right) \) where \( \tilde{\Gamma}_a_{bc} = \tilde{g}^{ad} \left( \partial_b \tilde{g}_{dc} + \partial_c \tilde{g}_{db} - \partial_d \tilde{g}_{bc} \right)/2 \). Likewise, the Laplacian operator acting on \( \tilde{\varphi} \) is defined by \( \tilde{D}^2 \tilde{\varphi} = \tilde{g}^{ab} \left( \partial_a \partial_b \tilde{\varphi} - \tilde{\Gamma}_a_{ab} \partial_c \tilde{\varphi} \right) \). Because the metric \( \tilde{g}_{ab} \) itself carries a density weight it transforms under
a change of spatial coordinates with an extra factor of $J^{-2/3}$ as compared with a weight–zero metric. Here, $J$ is the Jacobian of the transformation. As a consequence of the density weighting on $\bar{g}_{ab}$, the terms $\bar{R}$ and $\bar{D}^2\bar{\varphi}$ are not separately scalars under spatial coordinate transformations. However, the combination $\bar{\varphi}^{-4} \bar{R} - 8 \bar{\varphi}^{-6} \bar{D}^2 \bar{\varphi}$, which equals the physical scalar curvature $\mathcal{R}$, is a scalar. Then together the terms $-\bar{\varphi}^8 \bar{R}/2 + 4 \bar{\varphi}^7 \bar{D}^2 \bar{\varphi}$ that appear in the Hamiltonian density (23a) transform as a scalar density of weight 2. Recall that the lapse anti–density $\alpha$ carries weight $-1$, so $\alpha \mathcal{H}^{(g)}$ is a weight 1 density as it should be.

Also observe that the Laplacian $\bar{D}^2$ acts on $\bar{\varphi}$ in Eq. (26a) as if $\bar{\varphi}$ were a scalar rather than a scalar density of weight $1/6$. This is because the difference between $\bar{D}_a$ acting on a scalar and acting on a scalar density is a term proportional to $\partial_a \bar{g}$, which vanishes because $\bar{g}_{ab}$ has unit determinant. Likewise, in the momentum density (23b), the covariant derivative $\bar{D}_b$ acts on $\bar{A}_a^b$ as if $\bar{A}_a^b$ were a type $(1,1)$ tensor with no density weight.

IV. EQUATIONS OF MOTION FOR THE CONFORMAL–TRACELESS VARIABLES

Let us derive the dynamical equations of motion by varying the action (22) with respect to the invariant CT variables. Note that the variations $\delta \bar{A}^{ab}$ and $\delta \bar{g}_{ab}$ are traceless. As discussed in the previous section, the functional derivatives with respect to $\bar{A}^{ab}$ and $\bar{g}_{ab}$ are trace free:

$$\frac{\delta S^{(g)}}{\delta \bar{g}_{ab}} = \frac{1}{2} \left( \partial_{\perp} \bar{A}^{ab} \right)^{\text{TF}} - \left( \frac{\delta H^{(g)}}{\delta \bar{g}_{ab}} \right)^{\text{TF}},$$  \hspace{1cm} (24a)$$

$$\frac{\delta S^{(g)}}{\delta \bar{A}^{ab}} = -\frac{1}{2} \left( \partial_{\perp} \bar{g}_{ab} \right)^{\text{TF}} - \left( \frac{\delta H^{(g)}}{\delta \bar{A}^{ab}} \right)^{\text{TF}}.$$  \hspace{1cm} (24b)

Here, $H^{(g)} \equiv \int d^3 x \alpha \mathcal{H}^{(g)}$ is the gravitational field contribution to the Hamiltonian where $\mathcal{H}^{(g)}$ is given by Eq. (23a). The terms $\partial_{\perp} \bar{g}_{ab}$ and $\delta H^{(g)}/\delta \bar{A}^{ab} = \alpha \bar{A}_{ab}$ in Eq. (24b) are already traceless, so the equation of motion obtained by varying the action with respect to $\bar{A}^{ab}$ is $\partial_{\perp} \bar{g}_{ab} = -2\alpha \bar{A}_{ab}$. The term $\left( \partial_{\perp} \bar{A}^{ab} \right)^{\text{TF}}$ can be written as

$$\left( \partial_{\perp} \bar{A}^{ab} \right)^{\text{TF}} = \partial_{\perp} \bar{A}^{ab} - \frac{1}{3} \bar{g}^{ab} \bar{g}_{cd} \partial_{\perp} \bar{A}^{cd}$$
$$= \partial_{\perp} \bar{A}^{ab} + \frac{1}{3} \bar{g}^{ab} \bar{A}^{cd} \partial_{\perp} \bar{g}_{cd}$$
$$= \partial_{\perp} \bar{A}^{ab} - \frac{2}{3} \alpha \bar{g}^{ab} \bar{A}^{cd} \partial_{\perp} \bar{A}_{cd},$$  \hspace{1cm} (25)

where the equation $\partial_{\perp} \bar{g}_{ab} = -2\alpha \bar{A}_{ab}$ has been used. The evolution equation for $\bar{A}^{ab}$ is found by setting $\delta S^{(g)}/\delta \bar{g}_{ab}$ equal to $-\delta S^{(m)}/\delta \bar{g}_{ab} = -(\alpha \bar{\varphi}^{16}/2) (s^{ab})^{\text{TF}}$ in Eq. (24b) and using the result from the calculation (24a). The remaining equations of motion $\delta S/\delta \bar{\varphi} = 0$ and $\delta S/\delta \bar{\tau} = 0$ are straightforward to derive. The complete set is

$$\partial_{\perp} \bar{g}_{ab} = -2\alpha \bar{A}_{ab},$$  \hspace{1cm} (26a)$$

$$\partial_{\perp} \bar{A}^{ab} = 2\alpha \bar{A}^{ac} \bar{A}_c^b - 8 \bar{\varphi}^7 \left[ \alpha \bar{D}^{a} \bar{D}^b \bar{\varphi} + \bar{D}^{(a} \bar{\alpha} \bar{D}^b) \bar{\varphi} + \frac{1}{8} \bar{\varphi} \bar{D}^{a} \bar{D}^b \alpha - \frac{1}{8} \alpha \bar{\varphi} \bar{D}^{a} \bar{D}^b \alpha + \frac{1}{8} \alpha \bar{\varphi}^9 s^{ab} \right]^{\text{TF}},$$  \hspace{1cm} (26b)$$

$$\partial_{\perp} \bar{\varphi} = -\frac{1}{6} \alpha \bar{\varphi}^7 \bar{\tau},$$  \hspace{1cm} (26c)$$

$$\partial_{\perp} \bar{\tau} = \alpha \bar{\varphi}^6 \bar{\varphi}^2 + \alpha \bar{\varphi}^2 \bar{R} - 14 \alpha \bar{\varphi} \bar{D}^2 \bar{\varphi} - \bar{\varphi}^2 \bar{D}^2 \alpha - 14 \bar{\varphi} \bar{D}^a \bar{\varphi} \bar{D}_a \alpha - 42 \alpha \bar{D}^a \bar{\varphi} \bar{D}_a \bar{\varphi} + \frac{1}{2} \alpha \bar{\varphi}^5 (s - 3\rho).$$  \hspace{1cm} (26d)

Note that the matter variables have not been conformally scaled, and the indices on the spatial stress tensor are lowered with the physical metric; in particular, $s = s^{ab}p_{ab}$.

The equations of motion for the original set of CT variables, $\bar{g}_{ab}$, $\bar{A}_{ab}$, $\bar{\varphi}$, and $\bar{\tau}$, can be found by extremizing the action (13). These equations are redundant, as implied by the relation in Eq. (19). Alternatively, we can obtain the
independent equations of motion by inserting the definitions \( g_{ab} \) into Eqs. (26):

\[
\begin{align*}
\partial_\perp g_{ab} &= \frac{1}{3} g_{ab} \partial_\perp \ln g - 2\alpha \sqrt{g} A_{ab} , \\
\partial_\perp A_{ab} &= -\frac{1}{6} A_{ab} \partial_\perp \ln g - 2\alpha \sqrt{g} A_{ac} A_c^b , \\
\partial_\perp g &= -\frac{8\sqrt{g} \varphi}{\delta F/\delta g} \left[ \alpha D_a D_b \varphi + D_{(a} \alpha D_{b)} \varphi + \frac{1}{8} \varphi D_a D_b \alpha - \frac{1}{8} \alpha \varphi R_{ab} + \frac{1}{8} \alpha \varphi s_{ab} \right]_{\text{TF}} , \\
\partial_\perp \alpha &= -\frac{1}{12} \varphi \partial_\perp \ln g - \frac{1}{6} \alpha \sqrt{g} \varphi^7 \tau , \\
\partial_\perp \tau &= \sqrt{g} \left[ \alpha \varphi^6 \varphi^2 + \alpha \varphi^2 R - 14 \alpha \varphi \varphi D^2 \varphi - \varphi^2 D^2 \alpha - 14 \varphi^7 D^a \varphi D_a \alpha - 42 \alpha \varphi^8 \varphi D_a \varphi + \frac{1}{2} \alpha \varphi^6 (s - 3\rho) \right] ,
\end{align*}
\] (27a)

These equations are essentially identical to Eqs. (26), apart from various factors of \( g \) and the absence of tilde’s. Note that the indices are up in Eq. (26b), and down in Eq. (27b). This difference gives rise to the sign difference between the first term on the right–hand side of Eq. (26b) and the second term on the right–hand side of Eq. (27b).

It might not be obvious that the terms involving the Ricci tensor \( R_{ab} \) and covariant derivative \( D_a \) on the right–hand sides of Eqs. (26) and (27) will simplify to the corresponding terms involving \( R_{ab} \) and \( D_a \) on the right–hand sides of Eqs. (26b) and (27b). However, the following argument shows that this must be the case. The terms involving \( R_{ab} \) and \( D_a \) on the right–hand sides of Eqs. (26) can be obtained from the functional derivatives of

\[
F = -\frac{1}{2} \int d^3x \, \alpha h R = -\frac{1}{2} \int d^3x \, \alpha g \left( \varphi^8 R - 8 \varphi^7 D^2 \varphi \right) .
\] (28)

Since \( F \) is conformally invariant, we can view it either as a functional of \( g_{ab} \) and \( \varphi \) or as a functional of the conformally invariant variables \( g^{1/3} g_{ab} \) and \( g^{1/12} \varphi \). The functional derivatives with respect to \( g_{ab} \) and \( \varphi \) are defined by

\[
\delta F = \int d^3x \left[ \frac{\delta F}{\delta g_{ab}} \delta g_{ab} + \frac{\delta F}{\delta \varphi} \delta \varphi \right] .
\] (29)

By splitting \( \delta F/\delta g_{ab} \) into its trace and trace–free parts, we can rewrite this expression as

\[
\delta F = \int d^3x \left[ -g^{-1/3} g_{ac} g_{bd} \left( \frac{\delta F}{\delta g_{cd}} \right)_{\text{TF}} \delta \left( g^{1/3} g_{ab} \right) + g^{-1/12} \frac{\delta F}{\delta \varphi} \delta \left( g^{1/12} \varphi \right) \right] .
\] (30)

Then by explicit calculation, we find

\[
\frac{\delta F[g, \varphi]}{\delta (g^{1/3} g_{ab})} = -g^{-1/3} g_{ac} g_{bd} \left( \frac{\delta F}{\delta g_{cd}} \right)_{\text{TF}} \frac{1}{2} g^{2/3} \varphi^7 \left[ \alpha \varphi R_{ab} - \varphi D_a D_b \alpha - 8 D_{(a} \alpha D_{b)} \varphi - 8 \alpha D_a D_b \alpha \right]_{\text{TF}} ,
\]

\[
\frac{\delta F[g, \varphi]}{\delta (g^{1/12} \varphi)} = g^{-1/12} \frac{\delta F}{\delta \varphi} = 4 g^{1/12} \varphi^5 \left[ -\alpha \varphi^2 R + 14 \alpha \varphi \varphi D^2 \varphi + 42 \alpha \varphi^8 \varphi D_a \varphi + 14 \varphi^7 D^a \varphi D_a \alpha + \varphi^2 D^2 \alpha \right] .
\] (31a)

Since \( F, g^{1/3} g_{ab} \), and \( g^{1/12} \varphi \) are conformally invariant, the expressions on the right–hand sides of Eqs. (31a) and (31b) must be conformally invariant. Therefore these expressions are unchanged if we conformally transform the CT variables to the invariant CT variables. We do this by applying the transformation (2) with \( \xi = g^{-1/12} \). This argument shows that we can place tilde’s on the right–hand sides of Eqs. (31) without changing the values of these expressions. Armed with this result, it is straightforward to show that Eqs. (27) follow from Eqs. (26).

The CT equations (27) extremize the action and are equivalent to the ADM equations (8). They are invariant under the conformal transformation (2). Thus, these equations do not determine the evolution of \( g \), the determinant of the conformal metric \( g_{ab} \). For practical (numerical) calculations, it can be useful to fix \( g \). As discussed in the introduction, a common choice is to specify \( g = 1 \). However, we are free to choose \( g \) to be any \( t \)–dependent spatial scalar density of weight 2. There are two natural cases to consider for the evolution of \( g \). The first case is \( \partial_\perp g = 0 \); that is, \( g \) can be chosen to be constant along the normal to the spacelike hypersurfaces in spacetime. I will refer to this as the Eulerian condition, since \( g \) is constant for the observers who are at rest in the spacelike hypersurfaces. The
second case is $\partial g/\partial t = 0$; that is, $g$ is chosen to be constant along the time flow vector field in spacetime. I will refer to this as the Lagrangian condition, since $g$ is constant for the observers who move along the “flow lines” defined by the spatial coordinates. Note that the choice $g = 1$ is a special case of the Lagrangian condition.

For the Eulerian case the terms $\partial_\perp \ln g$ vanish in the equations of motion. Thus, we have

\begin{equation}
\frac{\partial g_{ab}}{\partial t} = 2D_{(a}\beta_{b)} - 2\alpha \sqrt{g} A_{ab} ,
\end{equation}

\begin{equation}
\frac{\partial A_{ab}}{\partial t} = \beta^c D_c A_{ab} + 2 A_{c(a} D_{b)} \beta^c - 2 \alpha \sqrt{g} A_{ac} A_b^c ,
\end{equation}

\begin{equation}
\frac{\partial \varphi}{\partial t} = \beta^c D_c \varphi - \frac{1}{6} \alpha \sqrt{g} \varphi^2 \tau ,
\end{equation}

\begin{equation}
\frac{\partial \tau}{\partial t} = \beta^c D_c \tau + \sqrt{7} \left[ \alpha \varphi \tau^2 + \alpha \varphi^2 R - 14 \alpha \varphi D^2 \varphi - \varphi^2 D^2 \alpha - 14 \varphi D \varphi D_{a} \alpha \right. \\
\left. - 42 \alpha D \varphi D_{a} \varphi + \frac{1}{2} \alpha \varphi^6 (s - 3 \rho) \right] ,
\end{equation}

when the condition $\partial_\perp g = 0$ holds. For the Lagrangian case $\partial g/\partial t = 0$ we have

\begin{equation}
\frac{\partial g_{ab}}{\partial t} = \left\{ \text{right–hand side of Eq. (32a)} \right\} - \frac{2}{3} g_{ab} D_c \beta^c ,
\end{equation}

\begin{equation}
\frac{\partial A_{ab}}{\partial t} = \left\{ \text{right–hand side of Eq. (32b)} \right\} + \frac{1}{3} A_{ab} D_c \beta^c ,
\end{equation}

\begin{equation}
\frac{\partial \varphi}{\partial t} = \left\{ \text{right–hand side of Eq. (32c)} \right\} + \frac{1}{6} \varphi D_c \beta^c ,
\end{equation}

\begin{equation}
\frac{\partial \tau}{\partial t} = \left\{ \text{right–hand side of Eq. (32d)} \right\} .
\end{equation}

By contracting with $g^{ab}$ we find that Eq. (32b) preserves the Eulerian condition. Likewise, Eq. (32c) preserves the Lagrangian condition.

The terms $2D_{(a} \beta_{b)}$ in Eqs. (32a) and (32b) come from the Lie derivatives of the type $\left( \begin{array}{l} 0 \\ 0 \end{array} \right)$ tensor $g_{ab}$. The extra term $-(2/3)g_{ab} D_c \beta^c$ that appears in the Lagrangian case combines with $2D_{(a} \beta_{b)}$ to give the same result one would obtain by computing the Lie derivative as if $g_{ab}$ were a type $\left( \begin{array}{l} 0 \\ 0 \end{array} \right)$ tensor density with weight $-2/3$. Similarly, the extra terms that appear in the equations of motion for $A_{ab}$ and $\varphi$ in the Lagrangian case combine with the Lie derivatives $\mathcal{L}_\beta A_{ab}$ and $\mathcal{L}_\beta \varphi$ to give the same results one would obtain by computing those Lie derivatives as if $A_{ab}$ were a type $\left( \begin{array}{l} 0 \\ 0 \end{array} \right)$ tensor density of weight $1/3$ and $\varphi$ were a scalar density of weight $1/6$. As discussed in the introduction, there is no conceptual advantage in leaving the trace of $A_{ab}$ unspecified. If $A = 0$ were not imposed from the beginning, the independent equations of motion would be identical to Eqs. (32a) with the replacements $\tau \rightarrow \tau + \varphi^{-6} A$ and $A_{ab} \rightarrow (A_{ab})^{\text{TF}}$. The resulting equations would not determine the evolution of $\tau$ and $A$ separately, but only the “trace invariant” combination $\tau + \varphi^{-6} A$. We could choose $A$ to be any $t$-dependent spatial scalar. In practice it might be convenient to determine $A$ by removing the trace–free (TF) symbol from the terms in square brackets on the right–hand side of Eq. (32a).

V. BSSN EQUATIONS

The conformally invariant BSSN system of evolution equations \cite{24, 25} is based on the CT equations \cite{27}. It uses new fields, the “conformal connection functions”, defined by

\begin{equation}
\Gamma^c \equiv g^{ab} \Gamma^c_{ab} = - \frac{1}{\sqrt{g}} \partial_a (\sqrt{g} g^{ac}) .
\end{equation}

Here, $\Gamma^c_{ab}$ are the Christoffel symbols built from the conformal metric $g_{ab}$ and $\partial_a$ stands for the partial derivative with respect to the spatial coordinate $x^a$. In the original treatment of the BSSN system, the condition $g = 1$ was imposed from the outset. As in the previous sections, I do not impose $g = 1$.

The key idea of the BSSN system is to replace certain derivatives of the metric that appear in the Ricci tensor on the right–hand side of Eq. (27) by the conformal connection functions $\Gamma^a$. By explicit calculation, we have

\begin{equation}
R_{ab} = - \frac{1}{2} g^{cd} \partial_c \partial_d g_{ab} + g_{c(a} \partial_b \Gamma^c + \Gamma^c \Gamma_{(ab)c} + g^{cd} \left( 2 \Gamma^e_{c(a} \Gamma_{b)ed} + \Gamma^e_{ac} \Gamma_{ebd} \right) ,
\end{equation}

where $\Gamma^c_{ab}$ are the Christoffel symbols built from the conformal metric $g_{ab}$ and $\partial_a$ stands for the partial derivative with respect to the spatial coordinate $x^a$.
where $\Gamma_{abc} \equiv g_{ad} \Gamma^d_{bc}$. The CT system \(^{(27)}\) must be extended to include an evolution equation for $\Gamma^a$. This is found by computing $\partial_a \Gamma^a$ from its definition \(^{(34)}\). For this calculation, it is helpful to introduce the invariant conformal connection functions, $\Gamma^c \equiv \bar{\gamma}^{ab} \Gamma^c_{ab} = -\partial_m \bar{g}^{ac}$. These variables are invariant under the conformal transformation \(^{(2)}\) since they are built from $\bar{g}_{ab} = g^{-1/3} g_{ab}$. They are related to $\Gamma^a$ by

$$\tilde{\Gamma}^a = g^{1/3} \Gamma^a + (1/6) g^{-2/3} g^{ab} \partial_b g .$$

The evolution equation for $\tilde{\Gamma}^a$ is straightforward to compute once we recognize that $\partial_\perp$ and $\partial_\parallel$ commute.\(^2\) The result can be expressed as

$$\partial_\perp \left( g^{1/3} \Gamma^a \right) = -\frac{1}{6} \partial_\perp \left( g^{1/3} g^{ab} \partial_b \ln g \right) - 2 \partial_b \left( \alpha g^{5/6} A^{ab} \right) ,$$

where $\partial_b \ln g = g^{-1} \partial_b g$.

The equations \(^{(27)}\), \(^{(26)}\) appear different from the familiar expression of the BSSN equations largely due to differences in notation. Let me define a set of "BSSN variables", denoted by carets:

$$\phi \equiv e^{\bar{\phi}} ,$$
$$ A_{ab} \equiv e^{\bar{\phi}} \bar{A}_{ab} ,$$
$$ \alpha \equiv e^{-6\bar{\phi}} \bar{\alpha} / \sqrt{\bar{g}} .$$

The definitions for the metric $g_{ab}$, trace of extrinsic curvature $\tau$, and shift vector $\beta^a$ are unchanged. Under a conformal transformation, the BSSN variables change by

$$g_{ab} \rightarrow \xi^4 g_{ab} ,$$
$$ \bar{\phi} \rightarrow \bar{\phi} - \ln \xi ,$$
$$ \bar{A}_{ab} \rightarrow \xi^4 \bar{A}_{ab} ,$$
$$ \tau \rightarrow \tau ,$$
$$ \Gamma^a \rightarrow \xi^{-4} \Gamma^a - 2 \xi^{-5} g^{ab} \partial_b \xi .$$

The scalar lapse function $\bar{\alpha}$ and the shift vector $\beta^a$ are conformally invariant.

With the change of notation \(^{(28)}\), the BSSN system \(^{(27)}\), \(^{(26)}\) becomes

$$\partial_\perp g_{ab} = -\frac{1}{6} \partial_\perp \ln g - 2 \bar{\alpha} \bar{A}_{ab} ,$$
$$ \partial_\perp \bar{A}_{ab} = -\frac{1}{3} \bar{A}_{ab} \partial_\perp \ln g - 2 \bar{\alpha} \bar{A}_{ac} \bar{A}^c_{ab} + \bar{\alpha} \bar{A}_{ab} \tau + e^{-2\bar{\phi}} \left[ -2 \bar{\alpha} D_a D_b \bar{\phi} + 4 \bar{\alpha} D_a \bar{\phi} D_b \bar{\phi} + 4 D_a (\bar{\alpha} D_b \bar{\phi}) - D_a D_b \bar{\phi} + \bar{\alpha} R_{ab} - \bar{\alpha} S_{ab} \right]_{TF} ,$$
$$ \partial_\perp \bar{\phi} = -\frac{1}{12} \partial_\perp \ln g - \frac{1}{6} \bar{\alpha} \tau ,$$
$$ \partial_\perp \tau = \bar{\alpha} \tau^2 + e^{-2\bar{\phi}} \left( \bar{\alpha} R - 8 \bar{\alpha} D^2 \bar{\phi} + 8 \bar{\alpha} D^a \bar{\phi} D_a \bar{\phi} + D^2 \bar{\alpha} - 2 D^a \bar{\alpha} D_a \bar{\phi} \right) + \frac{1}{2} \bar{\alpha} (s - 3 \rho) ,$$
$$ \partial_\perp \Gamma^a = -\frac{1}{3} \bar{\Gamma}^a \partial_\perp \ln g - \frac{1}{6} g^{ab} \partial_b \partial_\perp \ln g - \frac{2}{\sqrt{\bar{g}}} \partial_b \left( \bar{\alpha} \sqrt{\bar{g}} \bar{\phi} \bar{A}^{ab} \right) .$$

We can further modify these equations by making use of the Hamiltonian and momentum constraints $H^{(g)} = -g \phi^{12} \rho$ and $M^{(g)}_{\alpha} = \sqrt{\bar{g}} \phi^{\beta} j_{\alpha}$, where $H^{(g)}$ and $M^{(g)}_{\alpha}$ are given by Eqs. \(^{(13)}\) and \(^{(15)}\), respectively. First, we rewrite the momentum constraint as

$$\partial_b \left( \sqrt{\bar{g}} e^{6\bar{\phi}} \bar{A}^{ab} \right) = \sqrt{\bar{g}} e^{6\bar{\phi}} \left( -\Gamma^c_{bc} \bar{A}^{ab} + \frac{2}{3} g^{ab} \partial_b \tau + g^{ab} j_b \right) ,$$

\(^2\) That is, $L_\beta$ and $\partial_\parallel$ commute. To show this, recall that the components of, say, a contravariant vector in a “barred” coordinate system $\bar{x}^a$ are related to the components of that vector in an unbarred coordinate system by $\bar{x}^a = \bar{v}^a(x) \partial_\parallel x^a$. The coordinate derivatives of the vector components are related by $\partial_\parallel \bar{v}^a(x) = (\partial_\parallel \bar{v}^a) \partial_\parallel (v^a(x) \partial_\parallel x^a)$. Similar expressions hold for the components, and derivatives of components, of other types of tensors. Define the Lie derivative by $L_\beta T \equiv T(x) - T(x)$ where $\bar{x}^a = x^a - \beta^a$ and $\beta$ is infinitesimal. (Indices have been suppressed on $T$.) One then finds that $L_\beta (\partial_\parallel T) = \partial_\parallel (L_\beta T)$.
and use this result to replace the spatial derivatives of $\hat{A}^{ab}$ on the right-hand side of Eq. (10a). Next, we rewrite the Hamiltonian constraint as

$$R = e^{4\hat{\varphi}} \left( \hat{A}_{ab} \hat{A}^{ab} - \frac{2}{3} \tau^2 + 2\rho \right) + 8 \left( D^2 \hat{\varphi} + D^a \hat{\varphi} D_a \hat{\varphi} \right), \quad (42)$$

and use this result to replace the Ricci scalar $R$ on the right-hand side of Eq. (10a). The end result of these changes is

$$\partial_\perp g_{ab} = \frac{1}{3} g_{ab} \partial_\perp \ln g - 2\hat{\alpha} \hat{A}_{ab}, \quad (43a)$$
$$\partial_\perp \hat{A}_{ab} = \frac{1}{3} \hat{A}_{ab} \partial_\perp \ln g - 2\hat{\alpha} \hat{A}_{ac\hat{A}}_{b} + \hat{\alpha} \hat{A}_{ab\tau} + e^{-4\hat{\varphi}} \left[ -2\hat{\alpha} D_a \hat{D}_b \hat{\varphi} + 4\hat{\alpha} D_a \hat{D}_b \hat{\varphi} + 4D_{(a} \hat{\alpha} D_{b)} \hat{\varphi} - D_a D_b \hat{\alpha} + \hat{\alpha} R_{ab} - \hat{\alpha} s_{ab} \right]_{TF}, \quad (43b)$$
$$\partial_\perp \hat{\varphi} = -\frac{1}{12} \partial_\perp \ln g - \frac{1}{6} \hat{\alpha} \tau, \quad (43c)$$
$$\partial_\perp \tau = \frac{1}{3} \hat{\alpha} \tau^2 + \hat{\alpha} \hat{A}_{ab} \hat{A}^{ab} - e^{-4\hat{\varphi}} \left( D^2 \hat{\alpha} + 2D^a \hat{\alpha} D_a \hat{\varphi} \right) + \frac{1}{2} \hat{\alpha} (s + \rho), \quad (43d)$$
$$\partial_\perp \Gamma^a = -\frac{1}{3} \Gamma^a \partial_\perp \ln g - \frac{1}{6} g^{ab} \partial_b \partial_\perp \ln g - 2\hat{\alpha} \hat{B}_{ab} \partial_b \hat{\alpha} + 2\hat{\alpha} \left[ 6\hat{A}^{ab} \partial_b \hat{\varphi} + \Gamma^a_{bc} \hat{\varphi}^{bc} - \frac{2}{3} g^{ab} \partial_b \tau - g^{ab} j_b \right]. \quad (43e)$$

Equations (43) with the Ricci tensor $R_{ab}$ given by Eq. (8) define the conformally invariant BSSN system.

We can rewrite the BSSN equations in a more compact form by using the identities

$$\nabla^2 \hat{\alpha} = e^{-4\hat{\varphi}} \left( D^2 \hat{\alpha} + 2D^a \hat{\alpha} D_a \hat{\varphi} \right) \quad (44)$$

in Eq. (43d) and

$$[\hat{\alpha} R_{ab} - \nabla_a \nabla_b \hat{\alpha}]_{TF} = \left[ -2\hat{\alpha} D_a D_b \hat{\varphi} + 4\hat{\alpha} D_a \hat{D}_b \hat{\varphi} + 4D_{(a} \hat{\alpha} D_{b)} \hat{\varphi} - D_a D_b \hat{\alpha} + \hat{\alpha} R_{ab} - \hat{\alpha} s_{ab} \right]_{TF} \quad (45)$$

in Eq. (43a). Recall that $R_{ab}$ and $\nabla_a$ are the Ricci tensor and covariant derivative constructed from the physical metric $h_{ab} = e^{4\hat{\varphi}} g_{ab}$.

The BSSN equations (43) are invariant under the conformal transformation (39). As with the CT equations (27), these equations do not determine the evolution of $g$. We must specify $\partial_\perp g$, which appears in several places on the right-hand sides of Eqs. (43), as a separate condition. Alternatively, we could write the BSSN system in terms of conformally invariant variables, including the invariant conformal connection functions $\tilde{\Gamma}^a$. This would remove the terms proportional to $\partial_\perp \ln g$ in the BSSN equations but add certain density weights to each of the variables (except $\tau$). As defined here, $g_{ab}$ and $\hat{A}_{ab}$ are type $(0,2)$ tensors with no density weight, and $\hat{\varphi}$ and $\tau$ are scalars with no density weight. The coordinate transformation rule for the conformal connection functions $\tilde{\Gamma}^a$ follow from their definition (44) and the familiar inhomogeneous transformation rule for the Christoffel symbols. In particular, $\tilde{\Gamma}^a$ carries no density weight.

Two natural choices for the evolution of $g$ are the Eulerian condition $\partial_\perp g = 0$ and the Lagrangian condition $\partial g / \partial t = 0$. With the Eulerian condition, we have

$$\partial g_{ab} / \partial t = 2D_{(a} \beta_b) - 2\hat{\alpha} \hat{A}_{ab}, \quad (46a)$$
$$\partial \hat{A}_{ab} / \partial t = \beta^c D_c \hat{A}_{ab} + 2 \hat{A}_{c(a} \beta_{b)} - 2\hat{\alpha} \hat{A}_{ac\hat{A}}_{b} + \hat{\alpha} \hat{A}_{ab\tau} + e^{-4\hat{\varphi}} \left[ \hat{\alpha} R_{ab} - \nabla_a \nabla_b \hat{\alpha} - \hat{\alpha} s_{ab} \right]_{TF}, \quad (46b)$$
$$\partial \hat{\varphi} / \partial t = \beta^c D_c \hat{\varphi} - \frac{1}{6} \hat{\alpha} \tau, \quad (46c)$$
$$\partial \tau / \partial t = \beta^c D_c \tau + \frac{1}{3} \hat{\alpha} \tau^2 + \hat{\alpha} \hat{A}_{ab} \hat{A}^{ab} - \nabla^2 \hat{\alpha} + \frac{1}{2} \hat{\alpha} (s + \rho), \quad (46d)$$
$$\partial \Gamma^a / \partial t = \beta^c \partial_c \Gamma^a - \Gamma^c \partial_c \beta^a + g^{bc} \partial_b \partial_c \beta^a - 2\hat{A}^{ab} \partial_b \hat{\alpha} + 2\hat{\alpha} \left[ 6\hat{A}^{ab} \partial_b \hat{\varphi} + \Gamma^a_{bc} \hat{\varphi}^{bc} - \frac{2}{3} g^{ab} \partial_b \tau - g^{ab} j_b \right]. \quad (46e)$$

The identities (44) and (45) have been used to express these equations in compact form. For the Lagrangian condition,
we have
\[
\partial g_{ab}/\partial t = \left\{ \text{right–hand side of Eq. \ref{eq:47a}} \right\} - \frac{2}{3} g_{ab} D_c \beta^c, \tag{47a}
\]
\[
\partial \hat{A}_{ab}/\partial t = \left\{ \text{right–hand side of Eq. \ref{eq:47b}} \right\} - \frac{2}{3} \hat{A}_{ab} D_c \beta^c, \tag{47b}
\]
\[
\partial \dot{g}/\partial t = \left\{ \text{right–hand side of Eq. \ref{eq:47c}} \right\} + \frac{1}{6} D_c \beta^c, \tag{47c}
\]
\[
\partial \tau/\partial t = \left\{ \text{right–hand side of Eq. \ref{eq:47d}} \right\}, \tag{47d}
\]
\[
\partial \Gamma^a/\partial t = \left\{ \text{right–hand side of Eq. \ref{eq:47e}} \right\} + \frac{2}{3} \Gamma^a D_c \beta^c + \frac{1}{3} D^a D_c \beta^c. \tag{47e}
\]

The Eulerian and Lagrangian conditions are preserved by equations (46a) and (46b), respectively. Note that the restriction \( g = 1 \) is a special case of the Lagrangian condition. Also note that with \( g = 1 \), the extra terms in Eqs. (47) simplify since in that case \( D_c \beta^c = \partial_c \beta^c \).

It has been suggested that the “Gamma freezing” condition, \( \partial \Gamma^a/\partial t = 0 \) might be useful as a means of specifying the shift vector for numerical evolutions based on the BSSN system. The related “Gamma driver” conditions, which are less time–consuming to solve numerically than the Gamma freezing condition, have been used with some success. Here we note that these conditions depend specifically on the way that the shift terms enter the evolution equation for \( \Gamma^a \). If we choose the Eulerian condition \( \partial_L g = 0 \) to break the conformal invariance, then the Gamma freezing shift equation is
\[
g^{bc} \partial_b \partial_c \beta^a = \text{terms containing at most first derivatives of } \beta^a. \tag{48}
\]

If we choose the Lagrangian condition \( \partial g/\partial t = 0 \) to break conformal invariance, the Gamma freezing condition becomes
\[
g^{bc} \partial_b \partial_c \beta^a + \frac{1}{3} g^{ab} \partial_b \partial_c \beta^c = \text{terms containing at most first derivatives of } \beta^a. \tag{49}
\]

It might be interesting to explore the difference between these two shift conditions.

\[\text{VI. CONFORMAL THIN SANDWICH EQUATIONS}\]

Initial data for general relativity must satisfy the constraint equations. In the original York–Lichnerowicz conformal decomposition \ref{eq:18}, the gravitational field parts of the constraints are written as in Eqs. \ref{eq:14} and \ref{eq:18}. The Hamiltonian constraint \( \mathcal{H}(\varphi) = -\varphi^{12} g \rho \) can be solved for the Laplacian of the conformal factor \( \varphi \),
\[
D^2 \varphi = -\frac{1}{8} \varphi^{-7} A_{ab} A_{ab} + \frac{1}{12} \varphi^5 \tau^2 + \frac{1}{8} \varphi R - \frac{1}{4} \varphi^5 \rho.
\]

The trace free extrinsic curvature \( A_{ab} \) is split into a transverse part and a longitudinal part. With the so–called “conformal transverse–traceless decomposition”, the longitudinal part of \( A_{ab} \) is expressed in terms of derivatives of a vector \( X^a \). Then the principal part of the momentum constraint \( \mathcal{M}_a = \varphi^6 \sqrt{g} j_a \) is proportional to
\[
\Delta_L X_a \equiv D_b (ILX)^b_a,
\]
where
\[
(ILX)^a_{ab} = 2 D_{(a} X_{b)} - (2/3) g_{ab} D_c X^c.
\]

\( \Delta_L \) is an elliptic operator. In this way, the Hamiltonian and momentum constraints are expressed as a system of elliptic equations for \( \varphi \) and \( X^a \). The freely specifiable parts of the gravitational field are the conformal metric \( g_{ab} \), the trace of the extrinsic curvature \( \tau \), and the transverse part of \( A_{ab} \).

More recently, York \ref{eq:18} has recognized that the momentum constraint can be expressed in terms of the elliptic operator \( \Delta_L \) acting on the shift vector. Observe that the trace–free part of the conformal metric velocity is \( (\partial g_{ab}/\partial t)^{TF} = \partial g_{ab}/\partial t - (g_{ab}/3) \partial (\ln g)/\partial t \). Using the “dot” notation for time derivatives, we find that Eq. \ref{eq:47} becomes
\[
A_{ab} = -\frac{1}{2\alpha \sqrt{g}} \left( \dot{g}_{ab}^{TF} - (IL\beta)_{ab} \right).
\]
where \((\mathbf{L} \beta)_{ab}\) is defined in Eq. (52). Inserting this result into the momentum constraint, we find

\[
\Delta_L \beta_a = D^b g_{ab}^{\text{TF}} + 2\sqrt{g} A_{ab} D^b \alpha + \frac{4}{3} \alpha \sqrt{g} \varphi^6 D_a \tau + 2 \alpha \sqrt{g} \varphi^6 j_a,
\]

(54)

where \(\Delta_L\) is defined by Eq. (51). As described in Ref. [1], one can specify freely the gravitational quantities \(g_{ab}, \dot{g}_{ab}^{\text{TF}}, \tau,\) and \(\alpha\) then solve Eqs. (56) and (57) for \(\varphi\) and \(\beta^a\). Wherever \(A^{ab}\) appears, it is written in terms of the conformal metric and its derivatives via Eq. (55).

The conformal thin sandwich construction [1] is an extension of this analysis to include the lapse anti–density as one of the unknowns. By using the Hamiltonian constraint (40) to eliminate the Laplacian of \(\varphi\), we find that Eq. (27d) can be written as

\[
D^2 \alpha = -\frac{1}{\sqrt{g}} \varphi^{-2} \partial_\perp \tau - \frac{1}{6} \alpha \varphi^4 \tau^2 - \frac{3}{4} \alpha R + \frac{7}{4} \alpha \varphi^{-8} A^{ab} A_{ab} - 14 \varphi^{-1} D^a \varphi D_\alpha \alpha - 42 \alpha \varphi^{-2} D^a \varphi D_\alpha \varphi + \frac{1}{2} \alpha \varphi^4 (s + 4 \rho).
\]

(55)

The principle part of this equation is the conformal Laplacian operator acting on \(\alpha\). For the conformal thin sandwich initial data construction the freely specified quantities are the conformal metric \(g_{ab}\), the trace–free part of the metric velocity \(\dot{g}_{ab}^{\text{TF}}\), the trace of the extrinsic curvature \(\tau\), and its time derivative \(\dot{\tau} = \partial r / \partial \bar{t}\). Equations (51), (54), and (55), along with the expression for \(A^{ab}\) given in Eq. (53), constitute an elliptic system of equations to be solved for the conformal factor \(\varphi\), the shift vector \(\beta^a\), and the lapse anti–density \(\alpha\). Once these equations are solved, the initial data in terms of the physical metric \(h_{ab}\) and extrinsic curvature \(K_{ab}\) can be obtained from Eqs. (1).

The conformal thin sandwich equations (50), (53), (54), and (55) are derived from the CT equations and the Hamiltonian and momentum constraints, all of which are conformally invariant. It follows that the conformal thin sandwich equations are invariant under the conformal transformation [2]. To be precise, let \(\varphi, \beta^a,\) and \(\alpha\) denote the solution of the conformal thin sandwich equations for a given set of input data \(g_{ab}, \dot{g}_{ab}^{\text{TF}}, \tau,\) and \(\dot{\tau}\). Then the solution for input data \(\xi^a g_{ab}, \xi^a \dot{g}_{ab}^{\text{TF}}, \tau,\) and \(\dot{\tau}\) will be \(\varphi / \xi, \beta^a,\) and \(\alpha\). The physical metric \(h_{ab}\) and extrinsic curvature \(K_{ab}\) are the same in either case.

In evolving the conformal thin sandwich data one can choose the lapse anti–density and shift vector \(\alpha\) and \(\beta^a\) freely, without regard to the values obtained from the initial data construction. However, if the initial values of the lapse and shift are chosen to coincide with the values of \(\alpha\) and \(\beta^a\) that were computed from the thin sandwich equations, then initially the trace–free part of the conformal metric velocity will coincide with the chosen value of \(\dot{g}_{ab}^{\text{TF}}\) and the initial time derivative of the extrinsic curvature’s trace will be given by the chosen value of \(\dot{\tau}\). This will be the case whether the data is evolved via the ADM equations (8), the CT equations (27), or the BSSN equations (43). In the later two cases, we are free to choose the Eulerian or Lagrangian condition, or any other condition for the evolution of \(g\).

VII. CFC EQUATIONS

The approximate evolution equations obtained from the CFC have been used in numerical studies of binary neutron star coalescence [9, 13] and supernovae [9, 20]. This approximation to general relativity appears to be quite good [21], at least for systems that are not too far from spherical symmetry.

It turns out that the CFC equations are precisely the conformal thin sandwich equations, (27a), (27d), and (27l). The CFC approximation can be described in a generalized way as follows. Let the conformal metric \(g_{ab}\) and the trace of the extrinsic curvature \(\tau\) be specified freely for all times. Then \(g_{ab}, \dot{g}_{ab}^{\text{TF}}, \tau,\) and \(\dot{\tau}\) are known for all times and the thin sandwich equations can be solved for the gravitational data \(\alpha, \beta^a, \varphi,\) and \(A_{ab}\) as soon as the source data \(\rho, j_a,\) and \(s^{ab}\) is available. The idea, then, is to solve the conformal thin sandwich equations at the initial time using the initial values for the sources. We then evolve the sources forward in time to the first timestep using the matter equations of motion. These equations depend on the physical metric \(h_{ab} = \varphi^4 g_{ab}\), lapse anti–density, and shift. From the source values at the first timestep, the conformal thin sandwich equations are solved to complete the gravitational data at the first timestep. The process repeats.

For the CFC approximation, one sets the conformal metric equal to a flat metric for all times. In particular \(\dot{g}_{ab}^{\text{TF}} \equiv 0\) in Eq. (57). Also, maximal slicing is assumed so that \(\tau = 0\) for all times. Recall that the conformal thin sandwich equations are equivalent to the Hamiltonian constraint, the momentum constraint, and two of the four CT evolution equations, namely, Eqs. (27a) and (27d). Thus, the approximation in the CFC formalism consists in ignoring the evolution equations for \(A_{ab}\) and \(\varphi,\) namely, Eqs. (27b) and (27c).
VIII. SUMMARY

The conformal–traceless decomposition of the gravitational field appears in a number of contexts in general relativity, most notably in the analysis of the initial value problem and in the construction of the BSSN system of evolution equations. In this paper I have written the action functional and equations of motion in terms of conformal–traceless variables. I do not invoke the frequently imposed condition $g = 1$ on the determinant of the conformal metric. As a consequence, the action and equations of motion are conformally invariant. I have presented two possibilities for breaking conformal invariance, namely, the Eulerian condition $\partial_\perp g = 0$ and Lagrangian condition $\partial g / \partial t = 0$ on the evolution of $g$. I also extended the equations of motion to obtain a conformally invariant version of the BSSN system. For this system as well, the invariance can be broken by specifying the Eulerian or Lagrangian condition. I showed that the conformal thin sandwich equations for gravitational initial data are obtained from the Hamiltonian and momentum constraints along with a subset of the evolution equations written in conformal–traceless variables. Finally, I have pointed out that the CFC approximation to general relativity consists in solving the conformal thin sandwich equations at each time step, assuming the conformal metric is flat and the time slicing is maximal.

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