ON THE COMPACTNESS OF BERGMAN-TYPE INTEGRAL OPERATORS

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Abstract. Recently, the author and his collaborator [8] completely characterized the \( L^p - L^q \) boundedness of Bergman-type integral operators \( K_\alpha, K_\alpha^+ \) and the \( L^p - L^q \) compactness of \( K_\alpha \) on the unit ball. The key method used for compactness relies on the fact that \( K_\alpha \) maps \( L^p \) space into Bergman space or Bloch space. However, \( K_\alpha^+ \) has no such analytic property, thus the compactness characterization of \( K_\alpha^+ \) remains. In this paper, we will use a substantially new method to completely characterize the \( L^p - L^q \) compactness of \( K_\alpha^+ \), but also prove that the \( L^p - L^q \) compactness of operators \( K_\alpha, K_\alpha^+ \) is in fact equivalent. Moreover, we completely characterize Schatten class Bergman-type integral operator \( K_\alpha \) on \( L^2 \) space and Bergman space via an inequality related to \( \alpha \) and the dimension of the ball.

1. Introduction

This paper is a systematic research on the \( L^p - L^q \) compactness of Bergman-type integral operators on the unit ball. While there exist abundant works on the boundedness of Bergman-type integral operators, the investigation of compactness aspect was started only a few years.

Let us start with recalling some notations and terminologies. Let \( \mathbb{B}^d \) be the open unit ball in the \( d \)-dimensional complex Euclidian space \( \mathbb{C}^d \) with the normalized Lebesgue measure \( dv \), which means the measure of \( \mathbb{B}^d \) is one. For any \( \alpha \in \mathbb{R} \), the Bergman-type integral operator \( K_\alpha \) on \( L^1(\mathbb{B}^d, dv) \) is defined by

\[
K_\alpha f(z) = \int_{\mathbb{B}^d} \frac{f(w)}{(1 - \langle z, w \rangle)^\alpha} dv(w),
\]

where \( \langle z, w \rangle = z_1 \bar{w}_1 + \cdots + z_d \bar{w}_d \) is the standard Hermitian inner product on \( \mathbb{C}^d \). In particular, when \( \alpha = d + 1 \), \( K_{d+1} \) is the standard Bergman projection on \( \mathbb{B}^d \), since the function \( K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{d+1}} \) is the Bergman kernel of \( \mathbb{B}^d \) with the measure \( dv \). However, the Bergman-type integral operator \( K_\alpha^+ \) on \( L^1(\mathbb{B}^d, dv) \), mainly concerned in this paper, is given by

\[
K_\alpha^+ f(z) = \int_{\mathbb{B}^d} \frac{f(w)}{|1 - \langle z, w \rangle|^\alpha} dv(w).
\]

For any \( \alpha > 0 \), if restrict \( K_\alpha \) to Bergman spaces, then every \( K_\alpha \) is a special fractional radial differential operator [23 24 25] which is a kind of useful operators in the Bergman space theory on the unit ball, see Lemma 3.6 below. On the other hand, the operator \( K_\alpha^+ \) in the complex analysis category, to some extent, is analogous to the Riesz potential operators in the real analysis. The classical Riesz potential operator
$R_{\alpha}$ is defined on the real Euclidian space $\mathbb{R}^d$, whose basic result concerning mapping properties is the Hardy-Littlewood-Sobolev theorem, which characterizes the boundedness of Riesz potential operator $R_{\alpha} : L^p(\mathbb{R}^d) \rightarrow L^q(\mathbb{R}^d)$ and has been applied to the PDE theory for a long time; we refer the reader to \[14, 19\].

For convenience, we replace $L^p(\mathbb{R}^d, dv)$ by $L^p(\mathbb{B}^d)$ or $L^p$ for any $1 \leq p \leq \infty$ without confusion arises. Bergman-type operators $K_\alpha, K_\alpha^+$ are called $L^p-L^q$ bounded (or compact) if $K_\alpha, K_\alpha^+ : L^p(\mathbb{B}^d) \rightarrow L^q(\mathbb{B}^d)$ are bounded (or compact), where $1 \leq p, q \leq \infty$. Researches on $L^p-L^q$ problems of Bergman operators $K_\alpha, K_\alpha^+$ go back to the boundedness of Bergman projection on bounded domains. F. Forelli and W. Rudin \[10\] proved that the Bergman projection on the unit ball is $L^p-L^p$ bounded if and only if $1 < p < \infty$, in their method an asymptotic estimate of integral for Bergman kernel plays an important role; indeed, following the same method, one can characterize the $L^p-L^p$ boundedness of more general Bergman-type operators, see \[24, 25\]. Around the same time, H. Phong and E. Stein \[18\] proved more general results on a class of bounded strongly pseudoconvex domains; see \[13, 15, 16\] for more results along this line.

Recently, G. Cheng and X. Fang et al \[4\] completely characterized the $L^p-L^q$ boundedness of $K_\alpha$ on the unit disk $\mathbb{D}$, i.e. the case $d = 1$, and they conjectured that there should exist similar results in the high dimensional case. Their proofs depend on techniques of harmonic analysis and coefficient multiplier theory of Bergman space on the unit disk. Unfortunately, this method can not be directly applied to the case of unit ball. But, for the spacial case $\alpha = 1$, G. Cheng, C. Liu et al \[5\] solved the $L^p-L^q$ boundedness problem of $K_1$ on $\mathbb{B}^d$. The author and his collaborator \[8\] completely characterized the $L^p-L^q$ boundedness of $K_\alpha, K_\alpha^+$ and established some Hardy-Littlewood-Sobolev type inequalities on $\mathbb{B}^d$, motivated by an observation in \[25\] we further investigated the $L^p-L^q$ compactness of $K_\alpha$. Techniques of complex and harmonic analysis are synthetically utilized in \[8\], such as the Carleson measure theory is applied to the $L^p-L^q$ compactness problem.

The first purpose of the present paper is to continue to characterize the $L^p-L^q$ compactness of $K_\alpha^+$. The following two theorems are our main results on the $L^p-L^q$ compactness.

**Theorem 1.** Suppose $d + 1 < \alpha < d + 2$, then the following statements are equivalent.

1. $K_\alpha : L^p \rightarrow L^q$ is bounded.
2. $K_\alpha^+ : L^p \rightarrow L^q$ is bounded.
3. $K_\alpha : L^p \rightarrow L^q$ is compact.
4. $K_\alpha^+ : L^p \rightarrow L^q$ is compact.
5. $p, q$ satisfy one of the following inequalities:
   \[ a) \frac{1}{d+2-\alpha} < p < \infty, \frac{1}{q} > \left| \frac{1}{p} + \alpha - (d+1) \right|; \]
   \[ b) p = \infty, q < \frac{1}{\alpha-(d+1)}. \]

The equivalences of (1),(2),(3) and (5) in Theorem 1 have been proved in \[8\]. From Theorem 1 we know the $L^p-L^q$ boundedness and compactness of operators $K_\alpha, K_\alpha^+$ are equivalent when $d + 1 < \alpha < d + 2$.

**Theorem 2.** Suppose $0 < \alpha \leq d + 1$, then the following statements are equivalent:

1. $K_\alpha : L^p \rightarrow L^q$ is compact.
(2) \( K_\alpha^+ : L^p \to L^q \) is compact.

(3) \( p, q \) satisfy one of the following inequalities:

(a) \( p = 1, q < \frac{d+1}{d+1-\alpha} \);
(b) \( 1 < p < \frac{d+1}{d+1-\alpha}, \frac{1}{q} > \frac{1}{p} + \frac{\alpha}{d+1} - 1 \);
(c) \( p = \frac{d+1}{d+1-\alpha}, q < \infty \);
(d) \( \frac{d+1}{d+1-\alpha} < p \leq \infty \).

The equivalence of (1) and (3) in Theorem 2 has been proved in [8], combining Theorem 2 with [8, Theorem 2.3], we know the \( L^p \)-\( L^q \) boundedness and compactness of operators \( K_\alpha, K_\alpha^+ \) are different when \( 0 < \alpha \leq d+1 \). When \( \alpha \leq 0 \), Proposition 3.11 below shows that \( K_\alpha, K_\alpha^+ \) are all \( L^p \)-\( L^q \) compact for any \( 1 \leq p, q \leq \infty \); moreover, \( K_\alpha \) is a finite rank operator if \( \alpha \) is a nonpositive integer and \( K_\alpha^+ \) is a finite rank operator if \( \alpha \) is a nonpositive even integer. In contrast, when \( \alpha \geq d+2 \), there exist no \( 1 \leq p, q \leq \infty \) such that \( K_\alpha^+ : L^p \to L^q \) is compact, see Corollary 3.13 below.

Theorem 2 and Proposition 3.11 imply Bergman-type operator \( K_\alpha \) are compact on the Hilbert space \( L^2 \) only when \( \alpha < d+1 \).

As we all known, Schatten classes are more refined classification of compact operators on Hilbert spaces, but also involve global estimates of the spectrum of compact operators. The next theorem deals with the problem of Schatten class Bergman-type operators. Let \( H \) be a separable Hilbert space, denote the Schatten \( p \)-class (or ideal) on \( H \) by \( S^p(H) \), where \( 0 < p \leq \infty \); we refer the reader to [17, 21, 25, 26] for more details about Schatten class operators. Let us denote the set \( \mathcal{S}_d \subset \mathbb{R} \) by

\[
\mathcal{S}_d = \{ \alpha < d+1 : -\alpha \notin \mathbb{N} \},
\]

where \( \mathbb{N} \) is the set of nonnegative integers. For \( 1 \leq p \leq \infty \), let \( A^p = H(\mathbb{B}^d) \cap L^p \) be the \( p \)-integrable Bergman space on \( \mathbb{B}^d \). In particular, \( A^2 \) is a separable Hilbert space.

**Theorem 3.** Suppose \( \alpha \in \mathcal{S}_d \) and \( 0 < p < \infty \), then the following statements are equivalent.

1. \( K_\alpha \in S^p(L^2) \).
2. \( K_\alpha \in S^p(A^2) \).
3. \( \widetilde{K}_\alpha \in L^p(d\lambda) \).
4. \( p > \frac{d}{d+1-\alpha} \).

Where \( \widetilde{K}_\alpha \) is the Berezin transform of \( K_\alpha \) on the Bergman space \( A^2 \) and \( d\lambda \) is the Möbius invariant measure on \( \mathbb{B}^d \), the exact definition will be given in the following Section 4. We remark that the condition \( \alpha \in \mathcal{S}_d \) in Theorem 3 is necessary and sharp, namely \( \mathcal{S}_d \) is the maximal set ensures the theorem holds.

It is worth mentioning that our results are totally new even in the one-dimensional case. Moreover, the results of this paper can be generalized to the weighted Lebesgue integrable spaces and more general kernel operators on the unit ball. On the other hand, we know that the unit ball is a spacial bounded strongly pseudoconvex domain and every bounded strongly pseudoconvex domain with noncompact automorphism group is biholomorphic to the unit ball [22], so we guess that there exist some similar results on bounded strongly pseudoconvex domains.

The paper is organized as follows. In Section 2, we give the proof of Theorem 1 by using a criteria of precompactness in \( L^p \) space and the interpolation of compact
operators. Section 3 is devoted to prove Theorem 2, which is based on the hypergeometric function theory and the fractional radial differential operator theory. We also describe the phenomenon of Bergman-type operators for \( \alpha \leq 0 \). In Section 4, we will characterize Schatten class Bergman-type operator \( K_\alpha \) by estimates of the spectrum and techniques of operator theory.

2. The case of \( d + 1 < \alpha < d + 2 \)

In this section, we will prove the main Theorem 1. We need some lemmas. The following lemma gives the regularity of the image of \( K_\alpha, K_\alpha^+ \).

**Lemma 2.1.** For any \( \alpha \in \mathbb{R} \) and \( f \in L^p(\mathbb{B}^d), 1 \leq p \leq \infty \), then the followings hold.

1. \( K_\alpha f \) is holomorphic on \( \mathbb{B}^d \).
2. \( K_\alpha^+ f \) is smooth on \( \mathbb{B}^d \).

**Proof.** (1) It is sufficient to show that \( K_\alpha f, f \in L^1(\mathbb{B}^d) \) is holomorphic on every point of \( \mathbb{B}^d \). Suppose \( z \) is an arbitrary point in \( \mathbb{B}^d \). Denote \( B_r = \{ w \in \mathbb{C}^d : |w| < r \}, r > 0 \). Choose \( |z| < r_0 < r_0' < 1 \), and then choose \( 1 < r_1 < \frac{1}{r_0} \). It is easy to see that \( \{ z \} \times \mathbb{B}^d \subset B_{r_0} \times \mathbb{B}^d \subset B_{r_0'} \times B_{r_1} \) and \( 0 < r_0 < r_0 r_1 < 1 \). It follows that the function \( K_\alpha(z, w) = \frac{1}{1 - \langle z, w \rangle^\alpha} \) is holomorphic on the domain \( B_{r_0'} \times B_{r_1} \). Then [20, Proposition 1.2.6] implies that the binary function function \( K_\alpha(z, w) = \frac{1}{(1 - \langle z, w \rangle^\alpha)} \) has a global power series expansion on \( B_{r_0'} \times B_{r_1} \). Therefor, we can suppose that

\[
K_\alpha(z, w) = \sum_{n,m \geq 0} a_{n,m} z^n w^m, \tag{2.1}
\]
on \( B_{r_0'} \times B_{r_1} \). Here \( n, m \) are multi-indexes, \( n = (n_1, \ldots, n_d) \geq 0 \) means \( n_j \geq 0 \) for any \( j \), and \( z^n = \prod_{j=1}^d z_j^{n_j} \). Note that \( \overline{B_{r_0}} \times \overline{B^d} \subset B_{r_0'} \times B_{r_1} \), we get that the power series (2.1) is uniformly converge on compact set \( \overline{B_{r_0}} \times \overline{B^d} \). Then the dominated convergence theorem and the maximum modulus principle of analytic function imply that

\[
K_\alpha f(z) = \sum_{n,m \geq 0} a_{n,m} \left( \int_{\mathbb{B}^d} f \bar{w}^m \, dw \right) z^n,
\]
on \( B_{r_0} \ni z \). This implies that \( K_\alpha f \) is holomorphic at \( z \in \mathbb{B}^d \).

(2) Similarly, it suffices to consider the case \( f \in L^1 \). Denote the holomorphic function \( G_\alpha(z, w, u, v) \) on \( B_{r_0'} \times B_{r_1} \times B_{r_1} \times B_{r_0'} \) by

\[
G_\alpha(z, w, u, v) = \frac{1}{(1 - \langle z, w \rangle^\alpha) (1 - \langle u, v \rangle^\alpha)^{\frac{\alpha}{2}}}.
\]
By [20, Proposition 1.2.6] again, we know that \( G_\alpha(z, w, u, v) \) has a global power series expansion, thus we can suppose that

\[
G_\alpha(z, w, u, v) = \sum_{n,m,k,l \geq 0} a_{n,m,k,l} z^n w^m u^k v^l
\]
on \( B_{r_0'} \times B_{r_1} \times B_{r_1} \times B_{r_0'} \). Since

\[
\overline{B_{r_0}} \times \overline{B^d} \times \overline{B^d} \times \overline{B_{r_0}} \subset B_{r_0'} \times B_{r_1} \times B_{r_1} \times B_{r_0'},
\]
the function \( G_\alpha(z, w, u, v) \) is holomorphic on \( \mathbb{B}^d \times \mathbb{B}^d \times \mathbb{B}^d \times \mathbb{B}_{r_0} \).
it follows from the dominated convergence theorem and the maximum modulus principle of analytic function that

\[ F_\alpha(u, v) = \int_{\mathbb{B}^d} f(w)G_\alpha(u, \bar{w}, w, v)dv(w) \]

\[ = \sum_{n,m,k,l \geq 0} a_{n,m,k,l} \left( \int_{\mathbb{B}^d} f^{\alpha} w^m dv \right) u^n v^l, \]

is holomorphic on \( B_{r_0} \times B_{r_0} \). Then

\[ K_\alpha^+ f(z) = \int_{\mathbb{B}^d} \frac{f(w)dv(w)}{|1 - \langle z, w \rangle|^\alpha} = \int_{\mathbb{B}^d} f(w)G_\alpha(z, \bar{w}, w, \bar{z})dv(w) = F_\alpha(z, \bar{z}) \]  \hspace{1cm} (2.2)

on \( B_{r_0} \ni z \). It shows that \( K_\alpha^+ f \) is smooth at \( z \in \mathbb{B}^d \). \( \Box \)

**Remark 2.2.** The formula (2.2) in fact shows that, for any \( f \in L^1 \), the function \( K_\alpha^+ f \) admits a global power series expansion in the form

\[ K_\alpha^+ f(z) = \sum_{n,m \geq 0} a_{n,m} z^n \bar{z}^m, \]

and \( K_\alpha^+ f \) uniquely determines a holomorphic function on \( \mathbb{B}^d \times \mathbb{B}^d \).

From Lemma 2.1, we know that the image of \( K_\alpha^+ \) is continuous. The next lemma provides a result on the equicontinuity.

**Lemma 2.3.** Suppose \( \alpha \in \mathbb{R}, 1 \leq p \leq \infty \) and \( K \subset \mathbb{B}^d \) is a compact subset, then for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \) such that

\[ |K_\alpha^+ f(z_1) - K_\alpha^+ f(z_2)| \leq \|f\|_p \cdot \varepsilon, \quad \forall f \in L^p(\mathbb{B}^d), \]

whenever \( z_1, z_2 \in K \) and \( |z_1 - z_2| < \delta \).

**Proof.** Let \( B_r \) be the ball with radius \( r \) defined in Lemma 2.1. Since \( K \subset \mathbb{B}^d \) is a compact subset, there exists a \( 0 < r' < 1 \) such that \( K \subset B_{r'} \subset \overline{B_r} \subset \mathbb{B}^d \). Since the function \( \frac{1}{|1 - \langle z, w \rangle|^\alpha} \) is uniformly continuous on the compact set \( \overline{B_r} \times \mathbb{B}^d \), it yields that, for any \( \varepsilon > 0 \), there exists a \( \delta > 0 \), such that

\[ \left| \frac{1}{|1 - \langle z_1, w \rangle|^\alpha} - \frac{1}{|1 - \langle z_2, w \rangle|^\alpha} \right| \leq \varepsilon, \]

whenever \( z_1, z_2 \in K \) and \( |z_1 - z_2| < \delta \). Thus, for any \( f \in L^p(\mathbb{B}^d) \), we have

\[ |K_\alpha^+ f(z_1) - K_\alpha^+ f(z_2)| \leq \int_{\mathbb{B}^d} |f| \left| \frac{1}{|1 - \langle z_1, w \rangle|^\alpha} - \frac{1}{|1 - \langle z_2, w \rangle|^\alpha} \right| dv(w) \leq \|f\|_p \cdot \varepsilon \]

whenever \( z_1, z_2 \in K \) and \( |z_1 - z_2| < \delta \). It completes the proof. \( \Box \)

A subset in a Banach space is called precompact if the closure of the subset in the norm topology is compact. Obviously, an operator between two Banach spaces is compact if and only if the operator maps every bounded set to precompact one. Suppose \( \Omega \subset \mathbb{C}^d \) is an arbitrary bounded domain. The following lemma provides a criteria of precompactness in \( L^p(\Omega) \) with \( 1 \leq p < \infty \), see [1, Theorem 2.33] for more details.
Lemma 2.4. Let \( 1 \leq p < \infty \) and \( K \subset L^p(\Omega) \). Suppose there exists a sequence \( \{\Omega_j\} \) of subdomains of \( \Omega \) having the following properties:

1. \( \Omega_j \subset \Omega_{j+1} \).
2. The set of restrictions to \( \Omega_j \) of the functions in \( K \) is precompact in \( L^p(\Omega_j) \) for each \( j \).
3. For every \( \varepsilon > 0 \) there exists a \( j \) such that
   \[
   \int_{\Omega - \Omega_j} |f|^p dv < \varepsilon, \quad \forall f \in K.
   \]

Then \( K \) is precompact in \( L^p(\Omega) \).

The Forelli-Rudin asymptotic estimate of integral for Bergman kernel on the unit ball \( \mathbb{B}_d \) is a fundamental result in function spaces and operator theory, see [20, Proposition 1.4.10] or [25, Theorem 1.12] for more details. In the rest of this article, we will frequently use this estimate, for the sake of convenience, we state as follows.

Lemma 2.5. (Forelli-Rudin) Suppose \( c \) is real and \( t > -1 \). Then the integral

\[
J_{c,t}(z) = \int_{\mathbb{B}^d} \frac{(1 - |w|^2)^t dv(w)}{|1 - \langle z, w \rangle|^{c+t}}, \quad z \in \mathbb{B}^d,
\]

has the following asymptotic properties.

1. If \( c < d + 1 \), then \( J_{c,t} \) is bounded on \( \mathbb{B}^d \).
2. If \( c = d + 1 \), then
   \[
   J_{c,t}(z) \sim -\log(1 - |z|^2), \quad |z| \to 1^-.
   \]
3. If \( c > d + 1 \), then
   \[
   J_{c,t}(z) \sim (1 - |z|^2)^{d+1-c}, \quad |z| \to 1^-.
   \]

The notation \( A(z) \sim B(z) \) means that the ratio \( \frac{A(z)}{B(z)} \) has a positive finite limit as \( |z| \to 1^- \).

To complete the proof of Theorem 1, we also need a result on the interpolation of compact operators. The following lemma is first proved by A. Krasnoselskii [11], see also [6] or [12, Theorem 3.10].

Lemma 2.6. Suppose that \( 1 \leq p_1, p_2, q_1, q_2 \leq \infty \) and \( q_2 \neq \infty \). If a linear operator \( T \) such that \( T : L^{p_1} \to L^{q_1} \) is bounded and \( T : L^{p_2} \to L^{q_2} \) is compact, then \( T : L^p \to L^q \) is compact, if there exists a \( \theta \in (0, 1) \) such that

\[
\frac{1}{p} = \frac{\theta}{p_1} + \frac{1-\theta}{p_2}, \quad \frac{1}{q} = \frac{\theta}{q_1} + \frac{1-\theta}{q_2}.
\]

Proof of Theorem 1. The equivalence of (1) \( \leftrightarrow \) (2) \( \leftrightarrow \) (3) \( \leftrightarrow \) (5) is the main result of [8, Theorem 1], so it is enough to prove (4) \( \leftrightarrow \) (5). Note that the equivalence of (3) and (5), it implies that (4) \( \Rightarrow \) (5), since compact operators are all bounded between two Banach spaces. Now we prove (5) \( \Rightarrow \) (4), which means we need to show \( K_\alpha^* \) is \( L^p-I^q \) compact if \( K_\alpha^* \) is \( L^p-I^q \) bounded. It is easy to see that \( 1 \leq q < p \leq \infty \) under the assumption in (5). By Lemma 2.6, the proof will be completed once we prove the following two conclusions:
(a) If $p = \infty, q < \frac{1}{\alpha - (d + 1)}$, then $K^+_\alpha : L^p \rightarrow L^q$ is compact.
(b) If $p > \frac{1}{(d + 2) - \alpha}, q = 1$, then $K^+_\alpha : L^p \rightarrow L^q$ is compact.

Observe the operator $K^+_\alpha$ is adjoint by Fubini’s theorem, combing this with the well known fact that an operator between two Banach spaces is compact if and only if its adjoint is compact, we conclude that conclusions (a) and (b) are in fact equivalent. Consequently, the proof is completed if conclusion (a) is proved. Now we turn to prove the conclusion (a). Suppose $\{f_j\}$ is an arbitrary bounded sequence in $L^\infty$, without loss of generality, we can suppose that

$$
\|f_j\|_\infty \leq 1, \quad j = 1, 2, \cdots \tag{2.3}
$$

Denote the bounded domain $B'_j$ by $B'_j = \{z \in \mathbb{C}^d : |z| < 1 - \frac{1}{j}\}, j = 1, 2, \cdots$. Clearly, $B'_j$ is compactly contained in $\mathbb{B}^d$ and $B'_j \subset B'_{j+1} \subset \mathbb{B}^d$, for every $j$. We first prove that the set of restrictions to $B'_j$ of the functions in $\{K^+_\alpha f_n\}$ is precompact in $L^p(B'_j)$ for each $j$. In view of Lemma 2.1 we know the functions in $\{K^+_\alpha f_n\}$ are all continuous on $\mathbb{B}^d$ and uniformly continuous on every $\overline{B'_j}$. Combing with the fact that the embedding $C(\overline{B'_j}) \subset L^p(\overline{B'_j})$ is continuous for every $j$, it is enough to prove $\{(K^+_\alpha f_n)|_{B'_j}\}$ is precompact in $C(\overline{B'_j})$ for every $j$. Note that (2.3) and the definition of $B'_j$, we have that

$$
\|(K^+_\alpha f_n)|_{B'_j}\|_\infty = \sup_{z \in \overline{B'_j}} |K^+_\alpha f_n| = \sup_{z \in \overline{B'_j}} \left| \int_{\mathbb{B}^d} f_n(w)dv(w) \right| |(1 - \langle z, w \rangle)^\alpha|
$$

$$
\leq \|f_n\|_\infty \sup_{z \in \overline{B'_j}} \int_{\mathbb{B}^d} \frac{dv(w)}{|1 - \langle z, w \rangle|^\alpha} \tag{2.4}
$$

$$
\leq j^\alpha \|f_n\|_\infty
$$

$$
\leq j^\alpha,
$$

for every $j$. The estimate (2.4) implies that $\{(K^+_\alpha f_n)|_{B'_j}\}$ are bounded in $C(\overline{B'_j})$ for every $j$. It follows from Lemma 2.3 that $\{(K^+_\alpha f_n)|_{B'_j}\}$ is equicontinuous on $\overline{B'_j}$ for every $j$. Then Arzelà-Ascoli theorem implies that $\{(K^+_\alpha f_n)|_{B'_j}\}$ is precompact in $C(\overline{B'_j})$ for every $j$. By the well known fact that, for $t \in \mathbb{R}$, $(1 - |z|^2)^t \in L^1(\mathbb{B}^d)$ if and only if $t > -1$, we obtain that, for any fixed $t > -1$ and for any $\varepsilon > 0$, there exists a $J > 0$ satisfying

$$
\int_{\mathbb{B}^d - B'_j} (1 - |z|^2)^t dv < \varepsilon, \quad \forall j > J, \tag{2.5}
$$

since the absolute continuity of the integral and $\lim_{j \rightarrow \infty} v(\mathbb{B}^d - B'_j) = 0$. From the assumption in (a) and $d + 1 < \alpha < d + 2$, we get that $0 < q(\alpha - (d + 1)) < 1$, then by (2.5) we obtain that for any $\varepsilon > 0$, there exists a $J > 0$ such that

$$
\int_{\mathbb{B}^d - B'_j} (1 - |z|^2)^{-q(\alpha - (d + 1))} dv < \varepsilon, \quad \forall j > J. \tag{2.6}
$$
Combing (2.6) with Lemma 2.5 there exists a positive constant \( C \) such that, for any \( \varepsilon > 0 \), there exists a \( J > 0 \) satisfying
\[
\int_{B_d - B'_j} |K_\alpha^+ f_n(z)|^q dv(z) = \int_{B_d - B'_j} \left| \int_{B_d} \frac{f_n(w) dv(w)}{|1 - \langle z, w \rangle|^\alpha} \right|^q dv(z) \\
\leq \|f_n\|_\infty^q \int_{B_d - B'_j} \left| \int_{B_d} \frac{dv(w)}{|1 - \langle z, w \rangle|^\alpha} \right|^q dv(z) \\
\leq C \|f_n\|_\infty^q \int_{B_d - B'_j} (1 - |z|^2)^{-q(\alpha - (d+1))} dv \\
\leq C \varepsilon,
\]
for any \( j > J \) and \( n = 1, 2, \cdots \). Thus \( \{K_\alpha^+ f_n\} \) is precompact in \( L^q \) by Lemma 2.4, it completes the proof. \( \square \)

Remark 2.7. Theorem 1 indicates that Bergman operators \( K_\alpha, K_\alpha^+ \) from \( L^p \) into \( L^q \) are compact if and only if they are bounded when \( 1 \leq q < p \leq \infty \). In fact, this is quite expected, which can be seen from Ando’s theorem for a bounded integral operator from \( L^p \) into \( L^q \) with \( 1 \leq q < p \leq \infty \), see [3] or [12, Theorem 5.5, 5.13]. Nevertheless, we give a direct proof here.

3. The case of \( \alpha \leq d + 1 \)

This section is mainly devoted to prove Theorem 2, which characterizes the \( L^p-L^q \) compactness of \( K_\alpha^+ \) for \( 0 < \alpha \leq d + 1 \). We first recall some results on hypergeometric function theory. For complex numbers \( \alpha, \beta, \gamma \) and complex variable \( z \), we use the classical notation \( _2F_1(\alpha, \beta; \gamma; z) \) to denote
\[
_2F_1(\alpha, \beta; \gamma; z) = \sum_{j=0}^{\infty} \frac{(\alpha)_j(\beta)_j}{j!(\gamma)_j} z^j,
\]
with \( \gamma \neq 0, -1, -2, \ldots \), where \( (\alpha)_j = \prod_{k=0}^{j-1} (\alpha + k) \) is the Pochhammer for any complex number \( \alpha \). The following lemma calculates the exact value of the hypergeometric function at the point \( z = 1 \).

Lemma 3.1. [9] Section 2.8 If \( \text{Re}(\gamma - \alpha - \beta) > 0 \), then
\[
_2F_1(\alpha, \beta; \gamma; 1) = \frac{\Gamma(\gamma) \Gamma(\gamma - \alpha - \beta)}{\Gamma(\gamma - \alpha) \Gamma(\gamma - \beta)},
\]
where \( \Gamma \) is the usual Euler Gamma function.

Recall the definition of the function \( J_{\beta, \gamma} \) in Lemma 2.5. The following lemma is not only a restatement of Lemma 2.5 but also establishes a connection between the integration induced by Bergman kernel and the hypergeometric function. A function is called finite if it has finite value at every point of its domain.

Lemma 3.2. (1) [20] Suppose \( \beta \in \mathbb{R} \) and \( \gamma > -1 \), then
\[
J_{\beta-\gamma, \gamma}(z) = \frac{\Gamma(1+d) \Gamma(1+\gamma)}{\Gamma(1+d+\gamma)} _2F_1 \left( \frac{\beta}{2}, \frac{\beta}{2}; 1 + d + \gamma; |z|^2 \right),
\]
(3.1)
for $z \in \mathbb{B}^d$.

(2) $J_{\beta-\gamma,\gamma}$ is finite on the closed ball $\overline{\mathbb{B}}^d$ if and only if $\beta < d + 1 + \gamma$. Moreover, in this case, the identity (3.1) actually holds on the closed ball $\overline{\mathbb{B}}^d$.

Proof. (1) In this case, the identity (3.1) is a restatement of Lemma 2.5, we refer the reader to [20, Proposition 1.4.10] for more details.

(2) It suffices to prove $J_{\beta-\gamma,\gamma}$ is finite on the unit sphere $S^d = \{z \in \mathbb{C}^d : |z| = 1\}$ if and only if $\beta < d + 1 + \gamma$. Due to the unitary invariance of the Lebesgue measure, we know that

$$J_{\beta-\gamma,\gamma}(\eta) = \int_{\mathbb{B}^d} (1 - |w|^2)^{\gamma} dv(w) = \int_{\mathbb{B}^d} (1 - |w|^{\gamma})^{\beta} dv(w),$$

for any $\eta \in S^d$. Then [24, Lemma 1.8,1.9] imply that

$$J_{\beta-\gamma,\gamma}(\eta) = 2d \int_0^1 r^{2d-1}(1 - r^2)^{\gamma} dr \int_{\mathbb{S}^d} \frac{1}{1 - r\xi_1^{\beta}} d\sigma(\xi)$$

$$= 2d \int_0^1 r^{2d-1}(1 - r^2)^{\gamma} (d-1) \int_{\mathbb{B}^1} \frac{1}{1 - rw^{\beta}} dv_1(w) dr$$

$$= 2d \int_0^1 r^{2d-1}(1 - r^2)^{\gamma} \frac{\beta}{2} F_1 \left( \frac{\beta}{2}; d; r^2 \right) dr$$

$$= 2d \sum_{j=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)_j}{j!} \int_0^1 r^{2j+2d-1}(1 - r^2)^{\gamma} dr$$

for any $\eta \in S^d$, where $d\sigma$ and $dv_1$ are normalized Lebesgue measures on $S^d$ and $\mathbb{B}^1$, respectively. Note that $\Gamma(s + j) = (s)_j \Gamma(s)$ for all $s \in \mathbb{C}$ except the nonpositive integers. Then Stirling’s formula and (3.2) yield that $J_{\beta-\gamma,\gamma}$ is finite on $S^d$ if and only if $\beta < d + 1 + \gamma$. From Lemma 3.1 and (3.2), we know that, if $\beta < d + 1 + \gamma$, then

$$J_{\beta-\gamma,\gamma}(\eta) = \frac{\Gamma(d+1)\Gamma(\gamma+1)}{\Gamma(d+\gamma+1)} \sum_{j=0}^{\infty} \frac{\left(\frac{\beta}{2}\right)_j}{j!(d+\gamma+1)_j}$$

$$= \frac{\Gamma(d+1)\Gamma(\gamma+1)}{\Gamma(d+\gamma+1)} 2F_1 \left( \frac{\beta}{2}; \frac{\beta}{2}; d+\gamma+1; 1 \right)$$

$$= \frac{\Gamma(d+1)\Gamma(\gamma+1)\Gamma(d+\gamma+1-\beta)}{\Gamma^2(d+\gamma+1-\frac{\beta}{2})}$$

for any $\eta \in S^d$, which means that the identity (3.1) also holds on the closed ball $\mathbb{B}^d$. It completes the proof. \hfill \Box

From Lemma 3.2, we know that $J_{\beta-\gamma,\gamma}$ is a radial function on $\mathbb{B}^d$. Moreover, when $\beta < d + 1 + \gamma$, $J_{\beta-\gamma,\gamma}$ is increasing on $\mathbb{B}^d$ in the following sense,

$$J_{\beta-\gamma,\gamma}(z_1) < J_{\beta-\gamma,\gamma}(z_2),$$

(3.4)
whenever $|z_1| < |z_2| \leq 1$, since all Taylor coefficients of the hypergeometric function in (3.1) are positive. Now, we introduce the following auxiliary function $I_\alpha$ for every $\alpha < d + 1$. The function $I_\alpha(r, z)$ on $[0, 1) \times \mathbb{B}^d$ is denoted by

$$I_\alpha(r, z) = \int_{r \leq |w| < 1} \frac{1}{|1 - \langle z, w \rangle|^\alpha} dv(w).$$

Since $\alpha < d + 1$, it follows by (3.3) and (3.4) that

$$I_\alpha(r, z) \leq J_{\alpha,0}(z) \leq \sum_{j=0}^{\infty} \frac{\Gamma(j + \frac{\alpha}{2})}{\Gamma(j + 1)} \frac{\Gamma(d + 1)}{\Gamma(j + d + 1)} = \frac{\Gamma(d + 1)\Gamma(d + 1 - \alpha)}{\Gamma^2(d + 1 - \frac{\alpha}{2})},$$

(3.5)

for any $(r, z) \in [0, 1) \times \mathbb{B}^d$, which means that $I_\alpha$ is finite on $[0, 1) \times \mathbb{B}^d$. Moreover, $I_\alpha$ is increasing on $[0, 1) \times \mathbb{B}^d$ in the following sense.

Lemma 3.3. Suppose $r \in [0, 1)$, then

$$I_\alpha(r, z_1) \leq I_\alpha(r, z_2),$$

whenever $z_1, z_2 \in \mathbb{B}^d$ and $|z_1| \leq |z_2|$.

Proof. From (3.5), we know that $I_\alpha$ is finite on $[0, 1) \times \mathbb{B}^d$. Now we calculate its exact value. It follows from [24, Lemma 1.8,1.11] and the unitary invariance of the Lebesgue measure that

$$I_\alpha(r, z) = \sum_{j=0}^{\infty} \frac{\Gamma(j + \frac{\alpha}{2})}{\Gamma(j + 1)} \frac{\Gamma(d + 1)}{\Gamma(j + d + 1)} |z|^{2j} \int_{\mathbb{S}^{d-1}} |\xi_j|^2 d\sigma(\xi),$$

(3.6)

for any $z \in \mathbb{B}^d$. It leads to the desired result since all coefficients of the power series expansion about $|z|$ in (3.6) are positive. □

Proposition 3.4. If $0 < \alpha < d+1$, then $K^{+}_\alpha : L^\infty \to L^q$ is compact for any $1 \leq q \leq \infty$.

Proof. It suffices to prove that $K_\alpha : L^\infty \to L^\infty$ is compact. It is clear that $K_\alpha f$ is continuous on the open ball $\mathbb{B}^d$ for any $f \in L^\infty$ by Lemma 2.1. From Lemma 3.2, (3.3) and (3.4), we obtain that $K^{+}_\alpha f(\eta)$ exists and

$$|K^{+}_\alpha f(\eta)| \leq \|f\|_\infty \frac{\Gamma(d + 1)\Gamma(d + 1 - \alpha)}{\Gamma^2(d + 1 - \frac{\alpha}{2})}.$$
for any \( \eta \in \mathbb{S}^d \). We now turn to prove that \( K_\alpha^+ f \) is continuous on \( \overline{\mathbb{B}^d} \). It suffices to prove that \( K_\alpha^+ f \) on \( \mathbb{S}^d \). Thus we need only to prove that, for any \( \eta \in \mathbb{S}^d \) and for any point sequence \( \{z_n\} \) in \( \mathbb{B}^d \) satisfying \( z_n \to \eta \), then \( K_\alpha^+ f(z_n) \to K_\alpha^+ f(\eta) \) as \( n \to \infty \). By Lemma 3.2 and Lemma 3.4, we have

\[
|K_\alpha^+ f(z)| \leq \|f\|_\infty J_{\alpha,0}(z) \leq \|f\|_\infty J_{\alpha,0}(\eta) = \|f\|_\infty \frac{\Gamma(d + 1)\Gamma(d + 1 - \alpha)}{\Gamma^2(d + 1 - \frac{\alpha}{2})},
\]

(3.7)

for any \( z \in \mathbb{B}^d \). The absolute continuity of the integral implies that, for any \( \varepsilon > 0 \), there exists a \( 0 < \delta < 1 \), satisfying

\[
\int_F \frac{dv(w)}{|1 - \langle\eta, w\rangle|^\alpha} \leq \frac{\varepsilon}{4}, \tag{3.8}
\]

whenever \( v(F) < \delta \). Denote \( F_\delta = \{z \in \mathbb{B}^d : \sqrt{d} \frac{\delta}{2} < |z| < 1\} \). Note that \( v(F_\delta) = \frac{\delta}{2} < \delta \) and

\[
\frac{1}{|1 - \langle z_n, w\rangle|^\alpha} \to \frac{1}{|1 - \langle \eta, w\rangle|^\alpha}
\]

uniformly on \( \mathbb{B}^d \setminus F_\delta \), as \( n \to \infty \).

Then there exists a \( N > 0 \) such that, for any \( n > N \),

\[
\int_{\mathbb{B}^d \setminus F_\delta} \left| \frac{1}{|1 - \langle z_n, w\rangle|^\alpha} - \frac{1}{|1 - \langle \eta, w\rangle|^\alpha} \right| dv(w) \leq \frac{\varepsilon}{2}.
\]

Combing this with Lemma 3.3 and (3.8), we conclude that, for any \( n > N \),

\[
|K_\alpha^+ f(z_n) - K_\alpha^+ f(\eta)| \leq \|f\|_\infty \int_{\mathbb{B}^d \setminus F_\delta} \left| \frac{1}{|1 - \langle z_n, w\rangle|^\alpha} - \frac{1}{|1 - \langle \eta, w\rangle|^\alpha} \right| dv(w)
\]

\[
+ \|f\|_\infty \int_{F_\delta} \left| \frac{1}{|1 - \langle z_n, w\rangle|^\alpha} - \frac{1}{|1 - \langle \eta, w\rangle|^\alpha} \right| dv(w)
\]

\[
\leq \|f\|_\infty \int_{\mathbb{B}^d \setminus F_\delta} \left| \frac{1}{|1 - \langle z_n, w\rangle|^\alpha} - \frac{1}{|1 - \langle \eta, w\rangle|^\alpha} \right| dv(w)
\]

\[
+ 2\|f\|_\infty \int_{F_\delta} \frac{1}{|1 - \langle \eta, w\rangle|^\alpha} dv(w)
\]

\[
\leq \|f\|_\infty \frac{\varepsilon}{2} + 2\|f\|_\infty \frac{\varepsilon}{4}
\]

\[
= \varepsilon/2 + \frac{\varepsilon}{4}
\]

It completes the proof that \( K_\alpha^+ f \) is continuous on the closed ball \( \overline{\mathbb{B}^d} \) for any \( f \in L^\infty \).

Now we prove that, for any bounded sequence in \( L^\infty \), there exists a subsequence satisfying that its image under \( K_\alpha^+ \) is convergent in \( L^\infty \). Suppose that \( \{f_n\} \) is a bounded sequence in \( L^\infty \), then we have that \( \{K_\alpha^+ f_n\} \) is in \( C(\overline{\mathbb{B}^d}) \) and \( \{K_\alpha^+ f_n\} \) is uniformly bounded by (3.7). Now we prove that \( \{K_\alpha^+ f_n\} \) is also equicontinuous. From (3.9), we know that

\[
\lim_{\mathbb{B}^d \ni z \to \eta} \int_{\mathbb{B}^d} \left| \frac{1}{|1 - \langle z, w\rangle|^\alpha} - \frac{1}{|1 - \langle \eta, w\rangle|^\alpha} \right| dv(w) = 0,
\]

(3.10)

for arbitrary fixed \( \eta \in \mathbb{S}^d \). Combing (3.10) with the unitary invariance of Lebesgue measure and the symmetry of the unit ball, we have that, for any \( \varepsilon > 0 \), there exists
a $0 < \delta' < 1$, satisfying that
\[
\int_{B_d} \left| \frac{1}{|1 - \langle z, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle \eta, w \rangle|^{\alpha}} \right| dv(w) \leq \frac{\epsilon}{2}
\] (3.11)
whenever $z \in B_d, \eta \in S^d$ and $|z - \eta| < \delta'$. Denote $B''_{\frac{\delta'}{2}} = \{z \in C^d : |z| \leq 1 - \frac{\delta'}{2}\}$ and $C_{\frac{\delta'}{2}} = \{z \in C^d : 1 - \frac{\delta'}{2} < |z| \leq 1\}$. Then the closed ball $\overline{B}_d$ has the following decomposition,
\[
\overline{B}_d = B''_{\frac{\delta'}{2}} \cup C_{\frac{\delta'}{2}} \text{ and } B''_{\frac{\delta'}{2}} \cap C_{\frac{\delta'}{2}} = \emptyset.
\]
Since the function $\frac{1}{|1 - \langle z, w \rangle|^{\alpha}}$ is uniformly continuous on the compact set $B''_{\frac{\delta'}{2}} \times \overline{B}_d$, there exists a $0 < \delta'' < 1$ such that
\[
\left| \frac{1}{|1 - \langle z_1, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle z_2, w \rangle|^{\alpha}} \right| \leq \epsilon,
\] (3.12)
whenever $(z_1, w), (z_2, w) \in B''_{\frac{\delta'}{2}} \times \overline{B}_d$ and $|z_1 - z_2| < \delta''$. Take $\delta''' = \min\{\delta'', \delta''\}$. Now we prove that, for any $z_1, z_2 \in \overline{B}_d$ such that $|z_1 - z_2| < \delta''$, then
\[
\int_{\overline{B}_d} \left| \frac{1}{|1 - \langle z_1, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle z_2, w \rangle|^{\alpha}} \right| dv(w) \leq \epsilon.
\] (3.13)
In fact, there are two cases need to be considered. The first case is $z_1 \in C_{\frac{\delta'}{2}}$ or $z_2 \in C_{\frac{\delta'}{2}}$. Without loss of generality, we can assume that $z_1 \in C_{\frac{\delta'}{2}}$, then there exists an $\eta \in S^d$ satisfying that $|z_1 - \eta| < \delta'' \leq \frac{\delta'}{2}$. Thus, obviously, the triangle inequality implies that $|z_2 - \eta| \leq |z_2 - z_1| + |z_1 - \eta| < \delta'$. This together with (3.11) implies that
\[
\int_{\overline{B}_d} \left| \frac{1}{|1 - \langle z_1, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle z_2, w \rangle|^{\alpha}} \right| dv(w)
\leq \int_{\overline{B}_d} \left| \frac{1}{|1 - \langle z_1, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle \eta, w \rangle|^{\alpha}} \right| dv(w)
+ \int_{\overline{B}_d} \left| \frac{1}{|1 - \langle \eta, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle z_2, w \rangle|^{\alpha}} \right| dv(w)
\leq \epsilon
\]
The second case is $z_1, z_2 \in B''_{\frac{\delta'}{2}}$. From (3.12), it implies that
\[
\int_{\overline{B}_d} \left| \frac{1}{|1 - \langle z_1, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle z_2, w \rangle|^{\alpha}} \right| dv(w) \leq \epsilon \int_{\overline{B}_d} dv = \epsilon.
\]
It proves (3.13). Combing (3.13) with
\[
|K_{\alpha}^+ f_n(z_1) - K_{\alpha}^+ f_n(z_2)| \leq \|f_n\|_{\infty} \int_{B_d} \left| \frac{1}{|1 - \langle z_1, w \rangle|^{\alpha}} - \frac{1}{|1 - \langle z_2, w \rangle|^{\alpha}} \right| dv(w)
\]
follows that $\{K_{\alpha}^+ f_n\}$ is equicontinuous. Then Arzelà-Ascoli theorem implies that $\{K_{\alpha}^+ f_n\}$ has a convergency subsequence in the supremum norm (or is precompact). That finishes the proof. \(\square\)

**Corollary 3.5.** If $0 < \alpha < d + 1$, then $K_{\alpha}^+ : L^p \rightarrow L^1$ is compact for any $1 \leq p \leq \infty$. 
Proof. It follows from Lemma 3.4 and the fact that $K^+_\alpha$ is adjoint.

Now, recall the definition of the fractional radial differential operator $R^{s,t}$ on $H(\mathbb{B}^d)$. For any two real parameters $s$ and $t$ with the property that neither $d+s$ nor $d+s+t$ is a negative integer, the invertible operator $R^{s,t}$ is given by

$$R^{s,t}f(z) = \sum_{n=0}^{\infty} \frac{\Gamma(d+1+s)\Gamma(d+1+n+s+t)}{\Gamma(d+1+s+t)\Gamma(d+1+n+s)} f_n(z),$$

for any $f = \sum_{n=0}^{\infty} f_n \in H(\mathbb{B}^d)$ with homogeneous expansion. From [20 Proposition 1.2.6], we know every holomorphic function $f \in H(\mathbb{B}^d)$ has a global power series expansion, thus the definition is well-defined on $H(\mathbb{B}^d)$. In fact, it can be checked by the direct calculation that the invertible operator of $R^{s,t}$ is just $R^{s+t,-t}$. Be careful that the invertible operator here is unnecessarily continuous. Recall that $A^p = H(\mathbb{B}^d) \cap L^p$ is the $p$-integrable Bergman space on $\mathbb{B}^d$ for $1 \leq p \leq \infty$.

**Lemma 3.6.** Suppose $\alpha \in \mathbb{R}$ satisfying $\alpha$ is not a nonpositive integer and $1 \leq p \leq \infty$, then the following holds on $A^p$

$$K_\alpha = R^{0,\alpha-d-1}.$$

Proof. Since every holomorphic function $f \in H(\mathbb{B}^d)$ has a global power series expansion, we can suppose $f = \sum_{n=0}^{\infty} f_n \in A^p$ with the homogeneous expansion. Then the dominated convergence theorem and formula (1.21) in [24] imply that

$$K_\alpha f(z) = \int_{\mathbb{B}^d} \frac{f(w)}{(1-\langle z, w \rangle)^\alpha} dv(w)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \int_{\mathbb{B}^d} f(w)\langle z, w \rangle^n dv(w)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} 2d \int_0^{1} r^{2d-1} dr \sum_{k=0}^{\infty} r^{k+n} \int_{\mathbb{S}^d} f_k(\xi)\langle z, \xi \rangle^n d\sigma(\xi)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} 2d \int_0^{1} r^{2d-1} dr \int_{\mathbb{S}^d} f_n(r\xi)\langle z, r\xi \rangle^n d\sigma(\xi)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(n+\alpha)}{\Gamma(n+1)\Gamma(\alpha)} \int f_n(w)\langle z, w \rangle^n dv(w)$$

$$= \sum_{n=0}^{\infty} \frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)} f_n(z),$$

for any $z \in \mathbb{B}^d$. It leads to the desired result.

**Lemma 3.7.** [23 Proposition 5] Suppose $s, t$ are real parameters such that neither $n+s$ nor $n+s+t$ is a negative integer. Then, for any nonnegative integer $N$,

$$R^{s,t} \frac{1}{(1-\langle z, w \rangle)^{d+1+s+N}} = \frac{h(\langle z, w \rangle)}{(1-\langle z, w \rangle)^{d+1+s+N+t}},$$

where $h$ is a certain one-variable polynomial of degree $N$. In particular, $h \equiv 1$ if $N = 0$. 

Proposition 3.8. Suppose $0 < \alpha \leq d + 1$ and $1 < p, q < \infty$, if $K_\alpha^+ : L^p \to L^q$ is compact, then $\frac{1}{q} > \frac{1}{p} + \frac{\alpha}{d+1} - 1$.

Proof. For every $z \in \mathbb{B}^d$, denote the holomorphic function $H_z$ on $\mathbb{B}^d$ by
\[
H_z(w) = \frac{(1 - |z|^2)^{\frac{\alpha}{q}}}{(1 - \langle w, z \rangle)^{\frac{\alpha}{q} + \frac{d+1}{p} + \alpha - 1}}, \quad w \in \mathbb{B}^d,
\] (3.14)
where $s$ is a positive parameter satisfying $\frac{s}{q} - (1 - \frac{1}{p})(d + 1)$ is a large enough positive integer. Clearly $H_z$ is bounded holomorphic on $\mathbb{B}^d$ for each $z \in \mathbb{B}^d$. We first prove $|R^{\alpha - d - 1, d + 1 - \alpha} H_z| \to 0$ weakly in $L^p$, as $|z| \to 1^-$. By [2, Section 8.3.3, Theorem 2], it suffices to prove that
\begin{enumerate}
\item[(1)] $\sup_{z \in \mathbb{B}^d} \|R^{\alpha - d - 1, d + 1 - \alpha} H_z\|_p < \infty$;
\item[(2)] $\lim_{|z| \to 1^-} \int_{\mathbb{B}^d} |R^{\alpha - d - 1, d + 1 - \alpha} H_z| dv = 0$.
\end{enumerate}
It is immediate from Lemma 3.7 that
\[
|R^{\alpha - d - 1, d + 1 - \alpha} H_z(w)| = h_s(\langle w, z \rangle)(1 - |z|^2)^{\frac{\alpha}{q}}(1 - \langle w, z \rangle)^{\frac{\alpha}{q} + \frac{d+1}{p} + \alpha - 1}, \quad w \in \mathbb{B}^d,
\] (3.15)
where $h_s$ is a certain one-variable polynomial of degree $\frac{s}{q} - (1 - \frac{1}{p})(d + 1)$. From Lemma 2.5, there exists a positive constant $C$ such that
\[
\int_{\mathbb{B}^d} \frac{|h_s(\langle w, z \rangle)| dv(w)}{|1 - \langle w, z \rangle|^{\frac{\alpha}{q} + \frac{d+1}{p} + \alpha - 1}} \leq C(1 - |z|^2)^{(1 - \frac{1}{p})(d+1) - \frac{s}{q}}, \quad z \in \mathbb{B}^d.
\]
Then by (3.15), the following estimate holds,
\[
\int_{\mathbb{B}^d} |R^{\alpha - d - 1, d + 1 - \alpha} H_z| dv \leq \int_{\mathbb{B}^d} \frac{|h_s(\langle w, z \rangle)| (1 - |z|^2)^{\frac{\alpha}{q}} dv(w)}{|1 - \langle w, z \rangle|^{\frac{\alpha}{q} + \frac{d+1}{p} + \alpha - 1}} \leq C(1 - |z|^2)^{(1 - \frac{1}{p})(d+1)}.
\]
Thus, $\lim_{|z| \to 1^-} \int_{\mathbb{B}^d} |R^{\alpha - d - 1, d + 1 - \alpha} H_z| dv = 0$, namely, condition (2) holds. Similarly, we can verify the condition (1) by Lemma 2.5. Thus we prove that $|R^{\alpha - d - 1, d + 1 - \alpha} H_z| \to 0$ weakly in $L^p$, as $|z| \to 1^-$. Now, combing with the well-known fact that a compact operator maps a weakly convergent sequence into a strongly convergent one, we obtain
\[
\lim_{|z| \to 1^-} \|K_\alpha^+ (|R^{\alpha - d - 1, d + 1 - \alpha} H_z|)\|_q = 0.
\]
Note that $K_\alpha^+ (|f|) \geq |K_\alpha (f)|$ for any $f \in L^1$, then
\[
\lim_{|z| \to 1^-} \|K_\alpha (R^{\alpha - d - 1, d + 1 - \alpha} H_z)\|_q = 0.
\]
Since $R^{0, \alpha - d - 1} R^{\alpha - d - 1, d + 1 - \alpha} f = f$ for any bounded holomorphic function $f$ on $\mathbb{B}^d$, it follows from Lemma 3.6 that
\[
\lim_{|z| \to 1^-} \|H_z\|_q = \lim_{|z| \to 1^-} \|R^{0, \alpha - d - 1} R^{\alpha - d - 1, d + 1 - \alpha} H_z\|_q = \lim_{|z| \to 1^-} \|K_\alpha R^{\alpha - d - 1, d + 1 - \alpha} H_z\|_q = 0,
\]
and then
\[
\lim_{|z| \to 1^-} \int_{\mathbb{B}^d} \frac{(1 - |z|^2)^{\alpha} dv(w)}{|1 - \langle w, z \rangle|^{\frac{\alpha}{q} + \frac{d+1}{p} + \alpha - 1}} = \lim_{|z| \to 1^-} \|H_z\|_q^2 = 0.
\]
Since $s > 0$ is chosen large enough, by Lemma 2.5 again, we obtain that $\frac{1}{q} > \frac{1}{p} + \frac{\alpha}{d+1} - 1$.

Remark 3.9. We have proved $|R^{\alpha-d-1,d+1-\alpha}H_z| \to 0$ weakly in $L^p$, as $|z| \to 1^-$. Note that bounded operators map a weakly convergent sequence into a weakly convergent one. Analogously, we can prove that, if $K_+^\alpha : L^p \to L^q$ is bounded as the same condition in Proposition 3.8, then

$$\sup_{z \in B^d} \int_{B^d} \frac{(1 - |z|^2)^s dv(w)}{|1 - \langle w, z \rangle|^{q+\frac{(d+1)}{p}+q(\alpha-d-1)}} < \infty.$$ 

Then Lemma 2.5 implies that $\frac{1}{q} \geq \frac{1}{p} + \frac{\alpha}{d+1} - 1$. In fact, this gives a new proof of necessity part of [8, Lemma 4.3], which is a key step to characterize the $L^p-L^q$ boundedness of $K_\alpha, K_+^\alpha$.

Now we turn to the proof of Theorem 2, we first describe the $L^p-L^q$ compactness of $K_{d+1}^+$. It is as analogous to the proof of Theorem 1. Thus we give a sketchy proof here.

**Proposition 3.10.** $K_{d+1}^+ : L^p \to L^q$ is compact if and only if $1 \leq q < p \leq \infty$.

**Proof.** By Lemma 2.6 and the adjointness of $K_+^\alpha$, it suffices to prove $K_{d+1}^+ : L^p \to L^1$ is compact for any $1 < p \leq \infty$. Note that [8, Theorem 3] shows $K_{d+1}^+ : L^\infty \to L^1$ is bounded, combing with Lemma 2.6 again, it suffices to prove $K_{d+1}^+ : L^p \to L^1$ is compact for any $d + 1 < p < \infty$. Let $(B_j')$ be as in Proof of Theorem 1, and $(f_j)$ be a bounded sequence in $L^p$. Then we can prove $\| (K_{d+1}^\alpha f_n) |_{B_j'} \| \leq j^\alpha \| f_n \|_p$, combing with Lemma 2.1 and Arzelà-Ascoli theorem implies that $\{ (K_{d+1}^\alpha f_n) |_{B_j'} \}$ is precompact in $L^p(B_j')$. By Lemma 2.5 and Hölder’s inequality, it implies, there exists a constant $C > 0$ such that

$$\int_{B^d-B_j'} |K_{d+1}^+ f_n(z)| dv(z) \leq \| f_n \|_p \int_{B^d-B_j'} \left( \int_{B^d} \frac{dv(w)}{|1 - \langle z, w \rangle|^{\frac{p(d+1)}{q}+q(\alpha-d-1)}} \right)^{1-\frac{1}{p}} dv(z)$$

$$\leq C \| f_n \|_p \int_{B^d-B_j'} (1 - |z|^2)^{-\frac{d+1}{p}} dv(z).$$

Thus we obtain that, if $d + 1 < p < \infty$, then for any $\varepsilon > 0$, there exists a $J > 0$, such that

$$\int_{B^d-B_j'} |K_{d+1}^+ f_n| dv \leq C \varepsilon,$$

whenever $j > J$. Therefore, $\{K_{d+1}^\alpha f_n\}$ is precompact in $L^1(B^d)$ by Lemma 2.4.

**Proof of Theorem 2.** Since the equivalence of $(1) \Leftrightarrow (3)$ is the main result of [8, Theorem 3], it suffices to prove $(2) \Leftrightarrow (3)$. There are two cases $\alpha = d + 1$ and $0 < \alpha < d + 1$ to be considered. Indeed, the case $\alpha = d + 1$ has been proved in Proposition 3.10. Thus, it remains to deal with the case $0 < \alpha < d + 1$. By Lemma 2.6 and Corollary 3.5, we conclude that $K_{d+1}^+ : L^1 \to L^q$ is compact if and only if $q \leq \frac{d+1}{\alpha}$, and then $K_{d+1}^+ : L^p \to L^\infty$ is compact if and only if $p > \frac{d+1}{\alpha}$. Then apply Lemma
Proposition 3.3 and Proposition 3.8 to obtain the equivalence of (2) and (3) when \(0 < \alpha < d + 1\). It completes the proof.

We have completely characterized that \(L^p-L^q\) compactness of the Bergman-type operators \(K_\alpha, K_\alpha^+\) when \(0 < \alpha < d + 2\) so far. However, when \(\alpha \geq d + 2\), [8, Theorem 4] shows that there exist no \(1 \leq p, q \leq \infty\) such that \(K_\alpha^+ : L^p \to L^q\) is bounded.

To end this section, we describe the phenomenon for \(\alpha \leq 0\). Recall that a bounded operator between two Banach spaces is called a finite rank operator if the range of the operator has finite dimension. Obviously, finite rank operators must be compact and the finite rank operators on a Hilbert space belong to every Schatten \(p\)-class with \(0 < p \leq \infty\).

**Proposition 3.11.** Suppose \(\alpha \leq 0\), then the followings hold.

1. \(K_\alpha, K_\alpha^+ : L^p \to L^q\) are compact for any \(1 \leq p, q \leq \infty\).
2. If \(\alpha\) is a nonpositive integer, then \(K_\alpha : L^p \to L^q\) is a finite rank operator for any \(1 \leq p, q \leq \infty\).
3. If \(\alpha\) is a nonpositive even integer, then \(K_\alpha^+ : L^p \to L^q\) is a finite rank operator for any \(1 \leq p, q \leq \infty\).

**Proof.** (1) It suffices to prove that \(K_\alpha, K_\alpha^+ : L^1 \to L^\infty\) are compact when \(\alpha \leq 0\). Suppose \(\{f_j\}\) is an arbitrary bounded sequence in \(L^\infty\), without loss of generality, we can suppose that

\[
\|f_j\|_\infty \leq 1, \quad j = 1, 2, \ldots
\]

Then Lemma 2.1 implies that \(\{K_\alpha f_j\}\) is continuous function sequences on \(\mathbb{B}^d\). Note that the kernel function \(\frac{1}{(1-\langle z, w \rangle)^\alpha}\) is uniformly continuous on the compact set \(\mathbb{B}^d \times \mathbb{B}^d\) when \(\alpha \leq 0\), then we obtain that \(\{K_\alpha f_j\}\) is in fact continuous function sequence on \(\mathbb{B}^d\). Since \(\alpha \leq 0\), it follows that

\[
\|K_\alpha f_n\|_\infty = \sup_{z \in \mathbb{B}^d} \left| \int_{\mathbb{B}^d} f_n(w)dv(w) \right| (1-\langle z, w \rangle)^\alpha \\
\leq 2^{-\alpha} \|f_n\|_1 \\
\leq 2^{-\alpha},
\]

Thus \(\{K_\alpha f_j\}\) is a bounded subset in \(C(\overline{\mathbb{B}^d})\). The uniform continuity of the function \(\frac{1}{(1-\langle z, w \rangle)^\alpha}\) also yields that, for any \(\varepsilon > 0\), there exists a \(\delta > 0\), such that

\[
\left| \frac{1}{(1-\langle z_1, w \rangle)^\alpha} - \frac{1}{(1-\langle z_2, w \rangle)^\alpha} \right| \leq \varepsilon,
\]

whenever \(z_1, z_2 \in \mathbb{B}^d\) and \(|z_1 - z_2| < \delta\). Then we have

\[
|K_\alpha f_j(z_1) - K_\alpha f_j(z_2)| \leq \int_{\mathbb{B}^d} \left| f_j \right| \left| \frac{1}{(1-\langle z_1, w \rangle)^\alpha} - \frac{1}{(1-\langle z_2, w \rangle)^\alpha} \right| dv(w) \\
\leq \|f_j\|_1 \cdot \varepsilon \\
\leq \varepsilon
\]
whenever \( z_1, z_2 \in \mathbb{B}^d \) and \( |z_1 - z_2| < \delta \). That means \( \{K_\alpha f_j\} \) is equicontinuous. Then Arzelà-Ascoli theorem implies that \( \{K_\alpha f_j\} \) has a convergency subsequence in the supremum norm (or is precompact). That proves \( K_\alpha : L^1 \rightarrow L^\infty \) is compact when \( \alpha \leq 0 \). Similarly, \( K_\alpha^+ : L^1 \rightarrow L^\infty \) is compact when \( \alpha \leq 0 \).

(2) It suffices to prove that \( K_\alpha : L^1 \rightarrow L^\infty \) is a finite rank operator when \( \alpha \) a nonpositive integer. Suppose that \( \alpha = -N \), where \( N \) is a nonnegative integer. Then the binomial theorem implies that
\[
(1 - \langle z, w \rangle)^N = \sum_{|s| \leq N} a_{N,s} z^s \bar{w}^s,
\]
where \( s \geq 0 \) is multi-index and \( a_{N,s} \) is nonzero constant. Thus, for any \( f \in L^1 \), we have
\[
K_{-N} f(z) = \int_{\mathbb{B}^d} f(w)(1 - \langle z, w \rangle)^N dv = \sum_{|s| \leq N} \left( a_{N,s} \int_{\mathbb{B}^d} f \bar{w}^s dv \right) z^s.
\]
It implies that \( K_{-N} \) is a finite rank operator.

(3) It suffices to prove that \( K_\alpha^+ : L^1 \rightarrow L^\infty \) is a finite rank operator when \( \alpha \) a nonpositive even integer. Suppose that \( \alpha = -2N \), where \( N \) is a nonnegative integer. Then
\[
|1 - \langle z, w \rangle|^{2N} = (1 - \langle z, w \rangle)^N(1 - \langle w, z \rangle)^N = \sum_{|s|,|l| \leq N} a_{N,s}a_{N,l} z^s \bar{z}^l w^l \bar{w}^s.
\]
Thus, for any \( f \in L^1 \), we have
\[
K_{-2N} f(z) = \int_{\mathbb{B}^d} f |1 - \langle z, w \rangle|^{2N} dv = \sum_{|s|,|l| \leq N} \left( a_{N,s}a_{N,l} \int_{\mathbb{B}^d} f w^l \bar{w}^s dv \right) z^s \bar{z}^l.
\]
It implies that \( K_{-2N} \) is a finite rank operator. \( \square \)

**Remark 3.12.** \( K_\alpha^+ \) will not be a finite rank operator when \( \alpha \) is a nonpositive odd integer.

Thus we have the following corollary.

**Corollary 3.13.** For \( \alpha \in \mathbb{R} \), then the following statements are equivalent:

1. \( \alpha < d + 2 \);
2. there exist \( 1 \leq p, q \leq \infty \) such that \( K_\alpha : L^p \rightarrow L^q \) is bounded;
3. there exist \( 1 \leq p, q \leq \infty \) such that \( K_\alpha^+ : L^p \rightarrow L^q \) is bounded;
4. there exist \( 1 \leq p, q \leq \infty \) such that \( \tilde{K}_\alpha : L^p \rightarrow L^q \) is compact;
5. there exist \( 1 \leq p, q \leq \infty \) such that \( K_\alpha^+ : L^p \rightarrow L^q \) is compact.

4. **Schatten class Bergman-type integral operators**

In the present section, we complete the proof of Theorem 3, which gives the necessary and sufficient conditions that ensure the Bergman-type operator \( K_\alpha \) belongs to Schatten classes. The proof will be respectively given in cases \( 0 < \alpha < d + 1 \) and \( \alpha \leq 0 \). Although the theorem can be uniformly proved by the method of estimates of the spectrum, we prefer to prove theorem in the case \( 0 < \alpha < d + 1 \) by using the
method of operator theory which is inspired by the characterization of Schatten class Toeplitz operators and Hankel operators in [17, 25, 26].

Denote the point spectrum (the collections of eigenvalues) of $K_\alpha$ on the Bergman space $A^2$ by $\sigma_p(K_\alpha, A^2)$.

**Lemma 4.1.** Suppose $\alpha < d + 1$, then the followings hold.

1. If $\alpha \in S_d$, then $\sigma_p(K_\alpha, A^2) = \{\frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)} : n \in \mathbb{N}\}$.
2. If $-\alpha \in \mathbb{N}$, then $\sigma_p(K_\alpha, A^2) = \{0\} \cup \{\frac{(-1)^n\Gamma(1-\alpha)\Gamma(d+1)}{\Gamma(1-\alpha-n)\Gamma(n+d+1)} : 0 \leq n \leq -\alpha\}$.

**Proof.** (1) Denote $\mu_n = \frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)}$, $n \in \mathbb{N}$. Due to Lemma 3.6, it suffices to show that $\sigma_p(K_\alpha, A^2) \subset \{\mu_n : n \in \mathbb{N}\}$. Suppose $\mu \in \sigma_p(K_\alpha, A^2)$, then there exists a nonzero $f \in A^2$ such that

$$K_\alpha f = \mu f. \quad (4.1)$$

It is easy to see that $\cup_{n \in \mathbb{N}} P_n$ is an orthonormal basis of $A^2$, where $P_n$ is the set of homogeneous polynomials defined by

$$P_n = \left\{ c_k z^k : \sum_{j=1}^{d} k_j = n, k_j \in \mathbb{N} \right\}, \quad n \in \mathbb{N},$$

where $c_k$ is the normalized positive constant such that $\|c_k z^k\|_2 = 1$ for each $k$. Thus $f$ has the following representation

$$f = \sum_{n=0}^{\infty} \sum_{e_{n,k} \in P_n} \langle f, e_{n,k} \rangle e_{n,k},$$

where $\langle \cdot, \cdot \rangle$ is the standard Hermitian inner product on $A^2$. Combing this with Lemma 3.6 and (4.1), we conclude that

$$\sum_{n=0}^{\infty} (\mu - \mu_n) \sum_{e_{n,k} \in P_n} \langle f, e_{n,k} \rangle e_{n,k} = 0. \quad (4.2)$$

Since $f$ is nonzero, there exists a $e_{n_0,k_0}$ such that $\langle f, e_{n_0,k_0} \rangle \neq 0$. Then (4.2) implies that $f$ has the form

$$f = \sum_{e_{n,k} \in P_{n_0}} \langle f, e_{n,k} \rangle e_{n,k}, \quad (4.3)$$

and $\mu = \mu_{n_0}$. It completes the proof of $\sigma_p(K_\alpha, A^2) \subset \{\mu_n : n \in \mathbb{N}\}$. 

(2) Suppose \( f = \sum f_n \in A^2 \) with the homogeneous expansion. Since \(-\alpha\) is a natural number, it follows that

\[
K_{\alpha}f(z) = \int_{\mathbb{B}^d} \sum_{n=0}^{\infty} f_n(w)(1 - \langle z, w \rangle)^{-\alpha} dv(w)
\]

\[
= \int_{\mathbb{B}^d} \sum_{n=0}^{\infty} f_n \sum_{k=0}^{\infty} \frac{(-\alpha)!(-1)^k}{(-\alpha - k)!k!} \langle z, w \rangle^k dv(w)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-\alpha)!(-1)^n}{(-\alpha - n)!n!} \int f_n(w) \langle z, w \rangle^n \, dv(w)
\]

\[
= \sum_{n=0}^{\infty} \frac{(-1)^n \Gamma(1 - \alpha) \Gamma(d + 1)}{\Gamma(1 - \alpha - n) \Gamma(n + d + 1)} f_n(z)
\]

for any \( z \in \mathbb{B}^d \). Then, the same argument as above shows that

\[
\sigma_p(K_{\alpha}, A^2) = \{0\} \cup \left\{ \frac{(-1)^n \Gamma(1 - \alpha) \Gamma(d + 1)}{\Gamma(1 - \alpha - n) \Gamma(n + d + 1)} : 0 \leq n \leq -\alpha \right\}.
\]

\[\square\]

Theorem 2, Lemma 2.1, Lemma 3.6 and the boundedness of embedding \( A^2 \rightarrow L^2 \), imply that \( K_{\alpha} \) is a positive compact operator on the Bergman space \( A^2 \), thus we can apply the functional calculation to \( K_{\alpha} \). Set function \( F_p(x) = x^p \) on \( \mathbb{R}_{\geq 0} \) for any \( p > 0 \). Note that \( F_p : \mathbb{R}_{\geq 0} \rightarrow \mathbb{R}_{\geq 0} \) is bijective. We define \( K_{\alpha}^p = F_p(K_{\alpha}) \), which is the functional calculation of \( K_{\alpha} \in B(A^2) \) with respect to the function \( F_p(x) = x^p \), where \( B(A^2) \) is the collections of bounded operators on the Bergman space \( A^2 \).

**Lemma 4.2.** Suppose \( 0 < \alpha < d + 1 \). Then for any \( 0 < p < \frac{d+1}{d+1-\alpha} \), there exists a positive constant \( C_p \) such that the following operator inequality

\[
\frac{1}{C_p} K_{p\alpha-(p-1)(d+1)} \leq K_{\alpha}^p \leq C_p K_{p\alpha-(p-1)(d+1)}
\]

holds on the Bergman space \( A^2 \).

**Proof.** Since \( K_{\alpha} \) is positive and compact on \( A^2 \), it follows from [21, Theorem 1.9.2] that \( K_{\alpha} \) admits the following canonical decomposition

\[
K_{\alpha}f = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n \rangle e_n,
\]

where \( \{\lambda_n\} \) is the sequence of nonzero eigenvalues (counting multiplicities) with decreasing order, \( \{e_n\} \) is the corresponding orthonormal sequence of eigenvectors and \( \langle \cdot, \cdot \rangle \) is the standard Hermitian inner product on \( A^2 \). Combing with Lemma 4.1, we can further suppose

\[
K_{\alpha}f = \sum_{n=0}^{\infty} \frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)} \sum_{\epsilon_{n,k} \in P_n} \langle f, e_{n,k} \rangle e_{n,k}
\]
is the canonical decomposition of $K_\alpha$. It implies from the functional calculation and (4.4) that

$$K^p_\alpha f = \sum_{n=0}^{\infty} \left(\frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)}\right)^p \sum_{e_{n,k} \in P_n} \langle f, e_{n,k} \rangle e_{n,k},$$

is the canonical decomposition of $K^p_\alpha$. Note that the condition $0 < p < \frac{d+1}{d+1-\alpha}$ ensures $p\alpha - (p-1)(d+1)>0$. By Stirling’s formula, we conclude that

$$\left(\frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)}\right)^p \sim n^{p(\alpha-(d+1))} \sim \frac{\Gamma(d+1)\Gamma(p\alpha - (p-1)(d+1)+n)}{\Gamma(p\alpha - (p-1)(d+1))\Gamma(d+1+n)} n \to \infty.$$ Together this with Lemma 3.6 shows that there exists a positive constant $C_p$ satisfying

$$\langle \frac{1}{C_p} K_{p\alpha-(p-1)(d+1)} f, f \rangle \leq \langle K^p_\alpha f, f \rangle \leq \langle C_p K_{p\alpha-(p-1)(d+1)} f, f \rangle,$$

for any $f \in A^2$. This finishes the proof.

Now we recall the Berezin transform on the unit ball $B^d$. The Bergman kernel of $B^d$ is given by

$$K_w(z) = K(z, w) = \frac{1}{(1 - \langle z, w \rangle)^{d+1}}, \quad z, w \in B^d,$$

which is also called the reproducing kernel of $A^2$, since

$$f(z) = \langle f, K_z \rangle, \quad z \in B^d \quad (4.5)$$

for any $f \in A^2$. The normalized reproducing kernel of $A^2$ is

$$k_w(z) = \frac{K(z, w)}{\sqrt{K(w, w)}} = \frac{(1 - |w|^2)^{\frac{d+1}{2}}}{(1 - \langle z, w \rangle)^{d+1}}, \quad z, w \in B^d. \quad (4.6)$$

For a bounded operator $T \in B(A^2)$, the Berezin transform $\widetilde{T}$ of $T$ is given by

$$\widetilde{T}(z) = \langle T k_z, k_z \rangle, \quad z \in B^d.$$ The M"obius invariant measure $d\lambda$ on $B^d$ is defined by

$$d\lambda(z) = \frac{dv(z)}{(1 - |z|^2)^{d+1}}.$$ The Berezin transform is an important tool in the operator theory on the holomorphic function space, see [24, 25, 26] for more details. In what follows, we calculate the Berezin transform of $K_\alpha$.

**Lemma 4.3.** $\widetilde{K}_\alpha(z) = (1 - |z|^2)^{d+1-\alpha}$.

**Proof.** For every $w \in B^d$, denote the holomorphic function $K_{\alpha,w}(z)$ on $B^d$ by

$$K_{\alpha,w}(z) = \frac{1}{(1 - \langle z, w \rangle)^\alpha}.$$ Obviously, $K_{\alpha,w} \in A^2$ for every $w \in B^d$. By (4.5), we have

$$K_\alpha K_z(w) = \langle K_{\alpha,w}, K_z \rangle = \frac{1}{(1 - \langle w, z \rangle)^\alpha}.$$
Combining this with (4.5) and (4.6), we get that
\[
\tilde{K}_\alpha(z) = \langle K_\alpha k_z, k_z \rangle = (1 - |z|^2)^{d+1} \langle K_\alpha k_z, K_z \rangle = (1 - |z|^2)^{d+1} \langle K_{\alpha,z}, K_z \rangle = (1 - |z|^2)^{d+1} - \alpha.
\]
This completes the proof. \(\square\)

The following lemma establishes a connection between the Berezin transform and Schatten \(p\)-class on the Bergman space \(A^2\), see \([17, \text{Lemma C}]\) and \([25, \text{Lemma 7.10}]\) for more details.

**Lemma 4.4.** \([17, 25]\) If \(T \in B(A^2)\) is a positive operator, then the followings hold.

1. \(T \in S_1(A^2)\) if and only if \(\tilde{T} \in L^1(d\lambda)\). Moreover, the following trace formula holds,
   \[
   \text{Tr}(T) = \int_{\mathbb{B}^d} \tilde{T} d\lambda. \quad (4.7)
   \]

2. For \(1 < p < \infty\), then \(\tilde{T} \in L^p(d\lambda)\) if \(T \in S_p(A^2)\).

Now we can prove Theorem 3. The proof will be given in cases \(0 < \alpha < d + 1\) and \(\alpha \geq 0\), respectively.

**Proof of the case** \(0 < \alpha < d + 1\). Note that the compact operator \(K_\alpha\) is adjoint on \(L^2\) by Fubini’s theorem, it follows from \([21, \text{Theorem 1.9.2}]\) that \(K_\alpha\) on \(L^2\) admits the canonical decomposition
\[
K_\alpha f = \sum_{n=0}^{\infty} \lambda_n \langle f, e_n \rangle e_n \quad (4.8)
\]
whenever \(\alpha < d + 1\), where \(\{\lambda_n\}\) is the sequence of nonzero eigenvalues (counting multiplicities) and \(\{e_n\}\) is the corresponding orthonormal sequence of eigenvectors. It follows from (4.8) that
\[
K_\alpha e_n = \lambda_n e_n, n = 0, 1, \ldots.
\]

Since \(K_\alpha e_n\) is holomorphic by Lemma 2.1, we obtain that \(e_n\) is holomorphic for any integer \(n \geq 0\). Thus \(K_\alpha\) on \(A^2\) and \(L^2\) own the same canonical decomposition (4.8). This completes the proof that (2) is always equivalent to (1).

Now we turn to prove that (2) implies (3). Namely, we need to prove \(\tilde{K}_\alpha \in L^p(d\lambda)\) if \(K_\alpha \in S_p(A^2)\). Suppose that \(K_\alpha \in S_p(A^2)\). Note that \(\frac{d+1}{d+1-\alpha} > 1\), then Lemma 4.4 shows that \(\tilde{K}_\alpha \in L^p(d\lambda)\) if \(p \geq \frac{d+1}{d+1-\alpha}\). Thus, it suffices to consider the case \(0 < p < \frac{d+1}{d+1-\alpha}\). Observe that \(0 < p < \frac{d+1}{d+1-\alpha}\) ensures \(\alpha p - (p - 1)(d + 1) > 0\). Then, by Lemma 4.2, there exists a positive constant \(C'_p\) such that
\[
\frac{1}{C'_p} K_{\alpha(p\alpha-(p-1)(d+1)}} \leq K_\alpha^p. \quad (4.9)
\]
Note that $K^p_\alpha \in S_1(A^2)$, together with (4.9) shows that $K_{p\alpha-(p-1)(d+1)} \in S_1(A^2)$. Then Lemma 4.3 and Lemma 4.4 imply

$$\int_{\mathbb{B}^d} |\tilde{K}_\alpha|^p d\lambda = \int_{\mathbb{B}^d} (1 - |z|^2)^{p(d+1-\alpha)} d\lambda$$

$$= \int_{\mathbb{B}^d} (1 - |z|^2)^{d+1-(p\alpha-(p-1)(d+1))} d\lambda$$

$$= \int_{\mathbb{B}^d} \tilde{K}_{p\alpha-(p-1)(d+1)} d\lambda$$

$$= Tr(K_{p\alpha-(p-1)(d+1)})$$

$$< \infty.$$ 

Then $\tilde{K}_\alpha \in L^p(d\lambda)$. This shows that (2) implies (3).

Suppose $\tilde{K}_\alpha \in L^p(d\lambda)$, we go to prove that $p > \frac{d}{d+1-\alpha}$. By Lemma 4.3, we have

$$\int_{\mathbb{B}^d} |\tilde{K}_\alpha|^p d\lambda = \int_{\mathbb{B}^d} (1 - |z|^2)^{p(d+1-\alpha)} d\lambda$$

$$= \int_{\mathbb{B}^d} (1 - |z|^2)^{p(d+1-(p\alpha-1)(d+1))} d\lambda$$

Together this with the condition $\tilde{K}_\alpha \in L^p(d\lambda)$ shows that $(1 - |z|^2)^{(p-1)(d+1)-p\alpha} \in L^1(\mathbb{B}^d)$. By the fact that, for $t \in \mathbb{R}$, $(1 - |z|^2)^t \in L^1(\mathbb{B}^d)$ if and only if $t > -1$, we conclude that $(p-1)(d+1)-p\alpha > -1$. Then $p > \frac{d}{d+1-\alpha}$. This shows that (3) implies (4).

Now we turn to prove that (4) implies (2), that means that we need to prove $K_\alpha \in S_\alpha(A^2)$ if $p > \frac{d}{d+1-\alpha} > 1$, by the interpolation theorem of Schatten classes, see [25, Theorem 2.6]. Now Lemma 4.2 shows that, there exists a positive constant $C''_p$ such that

$$K^p_\alpha \leq C''_p K_{p\alpha-(p-1)(d+1)}$$

(4.10)

The condition $\frac{d}{d+1-\alpha} < p < \frac{d+1}{d+1-\alpha}$ means that

$$0 < p\alpha - (p-1)(d+1) < 1.$$ 

Then Lemma 4.3 shows that

$$\int_{\mathbb{B}^d} |\tilde{K}_{p\alpha-(p-1)(d+1)}| d\lambda = \int_{\mathbb{B}^d} (1 - |z|^2)^{p(d+1-\alpha)} d\lambda$$

$$= \int_{\mathbb{B}^d} (1 - |z|^2)^{p(d+1-(p\alpha-1)(d+1))} d\lambda$$

$$< \infty.$$ 

Combing with Lemma 4.4 follows that $\tilde{K}_{p\alpha-(p-1)(d+1)} \in S_1(A^2)$. Therefor, we obtain that $K_\alpha \in S_p(A^2)$ by (4.10). This shows that (4) implies (2). That completes the proof. □
Although $K_\alpha$ on $A^2$ and $L^2$ own the same canonical decomposition \[4.8\] when $0 < \alpha < d + 1$, the point spectrum of $K_\alpha$ on $A^2$ and $L^2$ differ by the element $0$. Indeed, the point spectrum of $K_\alpha$ on $L^2$ is $\{0\} \cup \{\Gamma(d+1)\Gamma(\alpha+n): n \in \mathbb{N}\}$.

**Proof of the case $\alpha \leq 0$**. We have proven that (1) and (2) is always equivalent when $\alpha \leq d + 1$. Thus, it suffice to prove (2), (3) and (4) are equivalent if $\alpha \leq 0$ and $\alpha$ is not a integer.

Suppose $\alpha \leq 0$ and $\alpha$ is not a integer. From Lemma \[4.1\] we know that $\{\mu_n : n \in \mathbb{N}\}$ is exactly the point spectrum of $K_\alpha$ on $A^2$, where $\mu_n = \Gamma(d+1)\Gamma(\alpha+n)/\Gamma(\alpha)\Gamma(d+1+n)$, $n \in \mathbb{N}$. Let $E_n$ be the eigenspace corresponding to $\mu_n$. It follows from (4.3) that

$$\dim E_n = \#P_n = \frac{(n+1)d-1}{(d-1)!}, \quad n \in \mathbb{N}.\leqno{4.11}$$

Combing this with the definition of $S_p(A^2)$, we know that $K_\alpha \in S_p(A^2)$ if and only if

$$\sum_{n=0}^{\infty} \mu_n^p \dim E_n = \sum_{n=0}^{\infty} \left(\frac{\Gamma(d+1)\Gamma(\alpha+n)}{\Gamma(\alpha)\Gamma(d+1+n)}\right)^p \frac{(n+1)d-1}{(d-1)!} < \infty.\leqno{4.11}$$

Then Stirling’s formula implies that (4.11) is equivalent to

$$\sum_{n=1}^{\infty} \frac{1}{n^p(d+1-\alpha)-(d-1)} < \infty.$$\leqno{4.11}

It follows that $K_\alpha \in S_p(A^2)$ if and only if $p > \frac{d}{d+1-\alpha}$. This implies that (2) and (4) are equivalent if $\alpha \leq 0$ and $\alpha$ is not a integer.

Suppose $\alpha \leq 0$ and $\alpha$ is not a integer. From Lemma \[4.3\] we obtain that $\tilde{K}_\alpha \in L^p(d\lambda)$ if and only if

$$\int_{\mathbb{B}^d} |\tilde{K}_\alpha|^p d\lambda = \int_{\mathbb{B}^d} (1 - |z|^2)^p(d+1-\alpha) d\lambda = \int_{\mathbb{B}^d} (1 - |z|^2)^p(d+1-\alpha)-(d+1) dv < \infty.\leqno{4.12}$$

By the well-known fact that $(1 - |z|^2)^t \in L^1(\mathbb{B}^d)$ if and only if $t > -1$, we conclude that (4.12) is equivalent to $p > \frac{d}{d+1-\alpha}$. This implies that (3) and (4) are equivalent if $\alpha \leq 0$ and $\alpha$ is not a integer. \hfill \Box

**Remark 4.5**. Proposition \[3.12\] follows that $K_\alpha$ is a finite operator whenever $\alpha$ is a nonpositive integer. In this case, $K_\alpha \in S_p(L^2)$ for any $p > 0$ rather than $p > \frac{d}{d+1-\alpha}$. This shows that the condition $\alpha \in \mathcal{F}_d$ in Theorem 3 is necessary and sharp.

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**References**

[1] A. Adama, F. Fournier, *Sobolev Spaces*, Pure and Applied Mathematics (Amsterdam), Elsevier/Academic Press, Amsterdam (2003)
[2] P. Akilov, V. Kantorovich, *Functional analysis*, Pergamon, Oxford (1982)
[3] T. Ando, *On the compactness of integral operators*, Indag. Math., 24, 235-239 (1962)
[4] G. Cheng, X. Fang, Z. Wang, J. Yu, *The hyper-singular cousin of the Bergman projection*, Trans. Amer. Math. Soc. 369, 8643-8662 (2017)
[5] G. Cheng, X. Hou, C. Liu, *The singular integral operator induced by Drury-Arveson kernel*, Complex Anal. Oper. Theory, 12, 917-929 (2018)
[6] F. Cobos, J. Peetre, Interpolation of compactness using Aronszajn-Gagliardo functors, Israel J. Math. 68, 220-240 (1989)
[7] S. Cui, Introduction to modern theory of partial differential equations (Chinese), Modern Mathematics Foundation Series, Higher Education Press, Beijing (2016)
[8] L. Ding, K. Wang, The $L^p$-$L^q$ problems of Bergman-type operators, \texttt{arXiv:2003.00479}
[9] A. Erdélyi, W. Magnus, F. Oberhettinger, F. G. Tricomi, Higher Transcendental Functions, Vol. I, McGraw-Hill, New York (1953)
[10] F. Forelli, W. Rudin, Projections on spaces of holomorphic functions in balls, Indiana Univ. Math. J. 24, 593-602 (1974)
[11] A. Krasnoselskii, On a theorem of M. Riesz, Soviet Math. Dokl. 1 229-231 (1960)
[12] A. Krasnoselskii et al., Integral operators in spaces of summable functions, Nordhoff, Groningen (1976)
[13] L. Lanzani, E. Stein, The Bergman projection in $L^p$ for domains with minimal smoothness, Illinois J. Math. 56, 127-154 (2013)
[14] H. Lieb, Sharp constants in the Hardy-Littlewood-Sobolev and related inequalities, Ann. Math. 118, 349-374 (1983)
[15] D. McNeal, The Bergman projection as a singular integral operator, J. Geom. Anal. 1, 91-103 (1994)
[16] D. McNeal, E. Stein, Mapping properties of the Bergman projection on convex domains of finite type, Duke Math. J. 73, 177-199 (1994)
[17] J. Pau, Characterization of Schatten-class Hankel operators on weighted Bergman spaces, Duke Math. J., 165, 2771-2791 (2016)
[18] H. Phong, E. Stein, Estimates for the Bergman and Szegö projections on strongly pseudoconvex domains, Duke Math. J. 44, 695-704 (1977)
[19] N. Plessis, Some theorems about the Riesz fractional integral, Trans. Amer. Math. Soc. 80, 124-134 (1955)
[20] W. Rudin, Function Theory in the Unit Ball of $\mathbb{C}^n$, Grundlehren der Math. Springer, New York (1980)
[21] J. Ringrose, Compact non-self-adjoint operators, Van Nostrand Reinhold Co., London (1971)
[22] B. Wong, Characterization of the unit ball in $\mathbb{C}^n$ by its automorphism group, Invent. Math. 41, 253-257 (1977)
[23] R. Zhao, K. Zhu, Theory of Bergman Spaces in the Unit Ball of $\mathbb{C}^n$, Mém. Soc. Math. Fr. 115 (2009)
[24] K. Zhu, Spaces of Holomorphic Functions in the Unit Ball, Graduate Texts in Mathematics, 226, Springer-Verlag, New York (2005)
[25] K. Zhu, Operator Theory in Function Spaces. Second Edition, Mathematical Surveys and Monographs, 138, American Mathematical Society, Providence (2007)
[26] K. Zhu, Schatten class Toeplitz operators on weighted Bergman spaces of the unit ball, New York J. Math., 13, 299-316 (2007)

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