Mass Gap and Confinement in (2+1)-Dimensional Yang-Mills Theory

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I. INTRODUCTION

While the perturbative aspects of non-Abelian gauge theories were understood many years ago, nonpertubative phenomena have been difficult to analyze. Detailed analyses and calculational techniques are still lacking, even though most of the qualitative features are more or less clear.

In particular, the issue of confinement and generation of mass gap in Quantum Chromodynamics (QCD) still defies a clear quantitative analysis. We have tried to understand these nonperturbative mechanisms in a somewhat simpler but still physically relevant framework, the (2+1)-dimensional Yang-Mills theory. To start with we have neglected the quark degrees of freedom, keeping only the gluons. The confinement mechanism applies to gluons as well. The lowest excited states of the purely gluonic theory are not the perturbative gluons but rather massive, color-singlet bound states of gluons, the so-called glueballs. The lowest glueball mass defines essentially the mass gap of the theory.

The study of non-Abelian gauge theories in a lower dimensional context, such as two spatial dimensions is interesting for at least two reasons. Primarily it can be a useful guide to the more realistic case of three dimensions; given the complexity of gauge theories, any further progress, even in a simpler lower dimensional context, should be interesting. Secondly, gauge theories in two spatial dimensions can be an approximation to the high temperature phase of QCD with the mass gap serving as the magnetic mass. It is the latter connection that makes our work relevant in a thermal field theory context, although there is no explicit mention of temperature. For a resume of our work and the relation to (3+1)-dimensional magnetic mass, see V.P. Nair’s contribution to these proceedings.

II. HAMILTONIAN ANALYSIS, GAUGE-INvariant PARAMETRIZATION

In an attempt to understand the origin of the mass gap and the confinement mechanism, we have analyzed SU(N) Yang-Mills theory in two spatial dimensions in a Hamiltonian framework in terms of gauge-invariant variables \([1–3]\). One of the advantages of the Hamiltonian approach (fixed time) is the fact that the theory is essentially two-dimensional and various mathematical techniques from two-dimensional conformal field theory become applicable and very useful. As is convenient for a Hamiltonian formulation, we use the \(A_0 = 0\) gauge. The gauge potentials are \(A_i = -i t^a A^a_i\), \(i = 1, 2\), where \(t^a\) are hermitian \((N \times N)\)-matrices which form a basis of the Lie algebra of \(SU(N)\) with \([t^a, t^b] = i f^{abc} t^c\), \(\text{Tr}(t^a t^b) = \frac{1}{2} \delta^{ab}\).

The Hamiltonian can be written as

\[
\mathcal{H} = T + V, \tag{1a}
\]

\[
T = \frac{e^2}{2} \int E_i^n E_i^a, \quad V = \frac{1}{4 \pi} \int B^n B^a \tag{1b}
\]

where \(e\) is the coupling constant, \(E_i^n = -i \frac{\delta}{\delta A_i^n}\) is the electric field and \(B^a = \frac{i}{2} \epsilon_{ijk} \partial_j A_k - \partial_k A_j + [A_j, A_k]^a\) is the magnetic field. In (2+1) dimensions, \(e^2\) has the dimension of mass.

Our basic strategy has been to reformulate the theory in terms of gauge-invariant variables and analyze the corresponding Schrödinger equation directly on the physical configuration space \(\mathcal{C} = A/\mathcal{G}\) (space of gauge potentials modulo gauge transformations). This is achieved by making use of a special parametrization of the gauge fields in terms of complex matrices in the following way.

We combine the spatial coordinates \(x_1, x_2\) into the complex combinations \(z = x_1 - i x_2\), \(\bar{z} = x_1 + i x_2\); correspondingly we have \(A \equiv A_z = \frac{1}{2} (A_1 + i A_2)\), \(\bar{A} \equiv A_{\bar{z}} = \frac{1}{2} (A_1 - i A_2) = -(A_z)^\dagger\). The parametrization we use is given by

\[
A_z = -\partial_z M M^{-1}, \quad A_{\bar{z}} = M^\dagger \partial_{\bar{z}} M \tag{2}
\]

Here \(M, M^\dagger\) are complex \(SL(N, \mathbb{C})\)-matrices.

From \([2]\), it is clear that the gauge transformation, \(A_i \rightarrow gA_ig^{-1} - \partial_i g g^{-1}\), is expressed in terms of \(M, M^\dagger\) by

\[
M \rightarrow gM, \quad M^\dagger \rightarrow M^\dagger g^{-1} \tag{3}
\]

for \(g(x) \in SU(N)\). In particular, if we split \(M\) into a unitary part \(U\) and a hermitian part \(\rho\) as \(M = U \rho\), then
$U$ is the ‘gauge part’ and it can be removed by a gauge transformation. The combination $H = M^\dagger M = \rho^2$ is a gauge-invariant field and it provides a parametrization for the configuration space $C$. The hermitian field $H$ belongs to $SL(N,\mathbb{C})/SU(N)$.

The parametrization (3) introduces a new symmetry in the theory, what we call “holomorphic” invariance. This arises as follows. The original gauge fields $A, \bar{A}$ remain invariant under the transformation $M \rightarrow M\bar{V}(\bar{z}), M^\dagger \rightarrow V(z)M^\dagger, \bar{H} \rightarrow V(z)H\bar{V}(\bar{z})$, where $V, \bar{V}$ depend only on the holomorphic coordinate $z$ and the antiholomorphic coordinate $\bar{z}$ respectively. Since the original theory is unaffected by this transformation, we should require that all physical states remain invariant under “holomorphic” transformations.

Having derived a particular parametrization for $C$ one is able to ask more detailed questions regarding its structure. An important question is: what is the integration measure (volume element) on $C$? One of the advantages of the particular parametrization that we introduced is that we can give an explicit answer to the above question.

The volume element in the original space of gauge potentials $A$ is given by

$$d\mu(A) = \prod_{x, a} dA^a(x)dA^a(x) = |\det(D\bar{D})|d\mu(M, M^\dagger) \tag{4}$$

where $D, \bar{D}$ are covariant derivatives in the adjoint representation; $D = \partial - \partial MM^{-1}$, $\bar{D} = \partial + M^{-1}\partial M^\dagger$. $d\mu(M, M^\dagger)$ is the standard volume element in the space of $SL(N,\mathbb{C})$-valued fields. Factorising $M$ into a unitary part $U$ and a hermitian part $\rho$, $M = U\rho$, $d\mu(M, M^\dagger)$ can be written as

$$d\mu(M, M^\dagger) = d\mu(H)d\mu(U) \tag{5}$$

where $d\mu(U)$ is the Haar measure for $SU(N)$ and $d\mu(H)$ the corresponding measure for hermitian fields. The volume element for $C$ is now obtained as

$$d\mu(C) = \frac{d\mu(A)}{d\mu(U)} = |\det(D\bar{D})|d\mu(H) \tag{6}$$

The problem is thus reduced to calculating the determinant of the two-dimensional operator $D\bar{D}$. Using a gauge-invariant regulator we find (4)

$$d\mu(C) \sim e^{2c_A S(H)} d\mu(H) \tag{7}$$

where $c_A$ is the adjoint quadratic Casimir ($c_A = N$ for $SU(N)$) and $S(H)$ is the Wess-Zumino-Witten (WZW) action for the hermitian matrix field $H$ given by

$$S(H) = \frac{1}{2\pi} \text{Tr} \int d^2z \; \partial H\partial H^{-1}$$
$$+ \frac{i}{12\pi} \text{Tr} \int d^3x e^{i\rho/\partial^2} H^{-1}\partial_\mu HH^{-1}\partial_\nu HH^{-1}\partial_\sigma H \tag{8}$$

As is typical for the WZW action, the second integral is over a three-dimensional space $M^3$ whose boundary is the physical two-dimensional space corresponding to the coordinates $z, \bar{z}$. The integrand thus requires an extension of the matrix field $H$ into the interior of $M^3$, but physical results do not depend on how this extension is done (3). Actually for the special case of hermitian matrices, the second term can also be written as an integral over spatial coordinates only (3).

The inner product for gauge-invariant physical states is given by

$$\langle 1|2 \rangle = \int d\mu(H)e^{2c_A S(H)} \Psi_1^*(H)\Psi_2(H) \tag{9}$$

Here we begin to see how conformal field theory could be useful; equation (3) shows that matrix elements of the $(2+1)$-dimensional gauge theory can be thought of as correlators of the hermitian WZW model, which is a conformal field theory. This point of view allows us to derive further constraints on the dependence of the wavefunction on the gauge-invariant variables $H$. The correlators of the hermitian WZW model have been studied before (3) and following conformal field theoretic arguments one can show that all wavefunctions of finite inner product are functions of the current

$$J_a = \frac{c_A}{\pi}(\partial HH^{-1})_a \tag{10}$$

The result is not surprising. The Wilson loop operator itself can be written in terms of the current as

$$W(C) = \text{Tr} P e^{-\oint_C (\partial z + \partial \bar{z})} = \text{Tr} P e^{(\pi/c_A) \oint_C J} \tag{11}$$

Since in principle all gauge-invariant functions of $A, \bar{A}$ can be constructed from $W(C)$, the current $J$ should suffice to generate all the gauge-invariant states of the theory. It is thus important to express the original Hamiltonian in terms of the current $J$, which can be thought of as a collective field.

The potential energy term can be written as

$$V = \frac{1}{2\varepsilon^2} \int B^2(x) = \frac{\pi}{m\varepsilon_A} \int \partial J_a(x)\partial J_a(x) \tag{12}$$

where $m = \frac{2\varepsilon_A}{\pi^2}$.

The kinetic energy term becomes

$$T = m \left[ \int_x J^a(x)\frac{\delta}{\delta J^a(x)} + \int \Omega_{ab}(x, y)\frac{\delta}{\delta J^a(x)}\frac{\delta}{\delta J^b(y)} \right]$$
$$\Omega_{ab}(x, y) = \frac{c_A}{\pi^2}(x-y)^2 - i\frac{\varepsilon_{abc}J^c(y)}{\pi(x-y)} \tag{13}$$

The first term in $T$ essentially counts the number of $J$’s and the second term replaces pairs of $J$’s by the operator product expansion for currents in the WZW-model. It is the first term which is responsible for the appearance of a mass gap. As it is obvious from (3), the lowest excited eigenstate of $T$ is the current $J$ itself:

$$TJ_a(x) = mJ_a(x) \tag{14}$$
The current $J$ can be thought of as the nonperturbative gauge-invariant gluon with dynamical mass $m$. Although $J_0$ is an eigenstate of $T$, it is not an acceptable physical state. As we mentioned earlier, physical states should be invariant under “holomorphic” transformations. The state with the minimum number of $J$’s which satisfies “holomorphic” invariance is a $2J$-state of the form $\bar{F}J_0 F J_0$. This is also a color-singlet state as expected. Such an expression after a careful regularization (“holomorphic” invariant point splitting) is a good candidate for the lowest glueball state $O^{++}$.

\[ \Phi_2 = \int f(x, y) \partial J(x)(H(x, y)H^{-1}(y, \bar{y}))\partial J(y) \] \[ (15) \]

Such a state has a mass gap of at least $2m$. The regulator term introduces interactions among currents which make the estimation of the binding energy rather difficult.

### III. VACUUM WAVEFUNCTION AND STRING TENSION

So far we have considered the spectrum of the kinetic energy term $T$. Notice that the potential energy term $V$, \[ (12) \], gives contributions of the order $k^2/e^2 \sim k^2/m$, where $k$ is a typical momentum. For momentum modes $k \ll e^2 \sim m$, the kinetic energy term is dominant and its spectrum provides an approximation for the spectrum of the whole theory. In this limit, which can be thought of as the strong coupling limit, the potential energy term $V$ can be treated in a perturbative fashion. Let us see how this works in the calculation of the vacuum wavefunction.

As far as the kinetic energy operator is concerned, $\Phi_0 = 1$ may be taken as the vacuum wavefunction. This might seem rather trivial, however it is important to notice that the corresponding norm $\langle \Phi_0 | \Phi_0 \rangle$, which is also the volume of the configuration space $C$, coincides with the partition function of the hermitian WZW-model, which upon appropriate regularization is finite \[ (8) \]. In other words $\Phi_0$ is normalizable with the inner product $\langle \Phi_0 | \Phi_0 \rangle$.

Essentially the exponential factor in \[ (8) \] provides the necessary damping to render the norm finite. (It is infinite in the Abelian case $e_A = 0$, where the exponential factor is absent.)

For low momentum modes, $k \ll m$, the inclusion of the potential energy term leads to a modified vacuum wavefunction which can be written as

\[ \Psi_0 = e^P \Phi_0 \] \[ (16) \]

where $P$ is a functional of the $J$’s which can be expanded in powers of $1/m$. The various terms in this expansion can be determined from the Schrödinger equation

\[ \mathcal{H} \Psi_0 = (T + V) \Psi_0 = 0 \] \[ (17) \]

The first few terms are given as \[ (8) \]

\[ P = -\frac{\pi}{m^2 c_A} \text{Tr} \int : \partial J \partial J : \]

\[ - \left( \frac{\pi}{m^2 c_A} \right)^2 \text{Tr} \int \left[ : \partial J(D \partial) \partial J + \frac{1}{3} \partial J[J, \partial^2 J] : \right] \]

\[ - 2 \left( \frac{\pi}{m^2 c_A} \right)^3 \text{Tr} \int \left[ : \partial J(D \partial)^2 \partial J + \frac{2}{9} [D \partial J, \partial J] \partial^2 J + \frac{8}{9} [D \partial^2 J, J] \partial^2 J - \frac{1}{6} [J, \partial J][\partial J, \partial^2 J] \right] \]

\[ - \frac{2}{9} [J, \partial J][J, \partial^3 J] \right] + O \left( \frac{1}{m^8} \right) \] \[ (18) \]

where $Dh = \frac{\pi}{m^2 c_A} \bar{h} - [J, h]$. The normal ordering of various terms in \[ (18) \] is necessary for $P$ to satisfy \[ (7) \]. The second derivative term in \[ (13) \] can give singularities when acting on composite operators. The normal ordering subtracts out precisely these singularities.

Our approach provides the first analytical, continuum calculation of the vacuum wavefunction. There are several interesting points regarding this. The leading order term for the vacuum wavefunction is

\[ \Psi_0 \approx \exp \left\{ - \frac{\pi}{m^2 c_A} \text{Tr} \int : \partial J \partial J : \right\} \]

\[ \approx \exp \left\{ - \frac{1}{2m c^2} \text{Tr} \int B^2 \right\} \] \[ (19) \]

The calculation of expectation values involves averaging with the factor $\Psi_0^\dagger \Psi_0 \approx e^{-S}$, where $S$, as seen from the above equation, is the action of a Euclidean two-dimensional Yang-Mills theory of coupling constant $g^2 = mc^2 = e^2 c_A/2\pi$. Thus, retaining only the leading term in $\Psi_0$, the expectation value of the Wilson loop in the fundamental representation is given by \[ (7) \]

\[ \langle W_F(C) \rangle = \exp \left\{ - \frac{e^4 c_A c_F}{4\pi} a_C \right\} \] \[ (20) \]

where $a_C$ is the area of the loop $C$ and $c_F$ is the quadratic Casimir of the fundamental representation. The expectation value of the Wilson loop exhibits an area law behavior, as expected for a confining theory. From \[ (20) \] we can identify the expression for the string tension $\sigma$ as

\[ \sigma = \frac{e^4 c_A c_F}{4\pi} = e^4 \left( \frac{N^2 - 1}{8\pi} \right) \] \[ (21) \]

Recent Monte Carlo calculations of the string tension give the values $\sqrt{\sigma}/e^2 = 0.335, 0.553, 0.758, 0.966$ for the gauge groups $SU(2), SU(3), SU(4)$ and $SU(5)$ respectively \[ (9) \]. The corresponding values calculated from \[ (21) \] are $0.345, 0.564, 0.772, 0.977$. We see that there is excellent agreement (upto $\sim 3\%$) between \[ (21) \] and the Monte Carlo results. It is further interesting to notice that our analytic expression for the string tension \[ (21) \] has the appropriate $N$-dependence as expected from large-$N$ calculations.

Another interesting point is the fact that we have obtained an expression \[ (11) \] local in terms of the current $J$.
but nonlocal in terms of the original magnetic field $B$. All previous attempts to derive an analytical expression of the vacuum wavefunction are based on the assumption that, in the strong coupling regime, such a wavefunction can be written as a local expansion in $B$. Our results obviously do not agree with this approach. It is thus interesting to investigate whether one could incorporate these nonlocal terms and verify our results in lattice calculations.

The approximation of the vacuum wavefunction by the first few terms in (18) makes sense only in the low momentum regime $k \ll m$. Although very cumbersome, in principle we could determine arbitrarily many terms in the series expansion (18), so one would expect that by resumming up various terms one might be able to get information on the vacuum wavefunction away from the low momentum region. In particular, what about the short distance behavior, $k \gg e^2$? Indeed this can be retrieved by converting the $1/m$ expansion into a series expansion in $J$'s. The terms in (18) can be naturally re-arranged into terms with two $J$'s, terms with three $J$'s, etc. The series of terms (infinite many) with only two $J$'s can be summed up to give

$$P = -\frac{1}{2e^2} \int_{x,y} B_a(x) \left[ \frac{1}{(m + \sqrt{m^2 - k^2})} \right] B_a(y) \tag{22}$$

In the weak coupling or high momentum regime $k \gg e^2$, the leading order term in (22) is

$$P = -\frac{1}{2e^2} \int_{x,y} B_a(x) \left[ (-\nabla^2)^{-\frac{1}{2}} \right] B_a(y) \tag{23}$$

This is the vacuum wavefunction for an Abelian theory as expected. Of course, in the low momentum regime $k \ll e^2$ the leading order term in (22) will reproduce (14).

Now what about the $3J$-terms and higher order terms? Part of the $3J$-contribution is just what is needed to covariantize the Laplacian operator $\nabla^2$ in (22), but that is not all. We can explicitly derive the full $3J$-contribution (2). What is important is the fact that comparing the $2J$ and $3J$-terms we find (2) that the $3J$-term is subdominant compared to the leading $2J$-term in $P$ for both the low and high momentum regimes. Similar arguments hold, based on dimensional analysis, for the higher $J$-terms. So the expansion of $P$ as a series expansion in $J$'s is consistent with both low and high momentum regimes.

So far we have discussed the structure of the vacuum wavefunction. One could, in principle, extend the above analysis for the excitation spectrum of the Schrödinger equation. For example, in the absence of the potential term $V$, the current $J$ is an eigenstate of the kinetic energy operator $T$, (14), with eigenvalue $m$. One could now ask what the corresponding modified eigenstate and eigenvalue should be, once the potential term is included. We find

$$(T + V)J_a e^P = \sqrt{m^2 \nabla^2} J_a e^P \tag{24}$$

where in the expression for $P$ we have kept only the leading $2J$-term contribution and have neglected the higher $J$-terms. As expected in a relativistic theory, the mass $m$ gets corrected to its relativistic expression $\sqrt{m^2 + k^2}$, which also displays the correct high and low momentum limits. In a similar fashion we can obtain expressions interpolating between short and long distance regimes for other quantities of interest.

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