A note on Penrose limits

Paul Tod*
Mathematical Institute,
Oxford University

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Abstract

As a footnote to ref [2], I show that, given a (time-like) umbilic 3-surface Σ in a 4-dimensional space-time $M$, the Penrose limit taken along any null geodesic $Γ$ which lies in $Σ$ is a diagonalisable plane-wave.

1 Introduction

Given a space-time $M$ and a null geodesic $Γ$ in $M$, Penrose [1] defined a limiting process which produces a plane-wave space-time. One can ask whether the various plane-wave limits of an $M$ are diagonalisable, which is to say whether coordinates can be found in which the metric has only diagonal entries. In 4-dimensions a simple algorithm for deciding whether a particular geodesic $Γ$ leads to a diagonalisable plane-wave was given in [3], and in [2] I considered the problem of finding all 4-dimensional space-times with the property that all of their plane-wave limits were diagonalisable. In the process of finding examples to give in [2], I realised that the following Proposition must hold:

Proposition Given a (time-like) umbilic 3-surface $Σ$ in a 4-dimensional space-time $M$, the Penrose limit taken along any null geodesic $Γ$ which lies in $Σ$ is diagonalisable.

Recall that a hypersurface $Σ$ is umbilic if its second fundamental form is proportional to its intrinsic metric. An umbilic hypersurface is totally-geodesic for null geodesics: any null geodesic from a point $p ∈ Σ$ and initially tangent to $Σ$ remains in $Σ$. A stronger condition on $Σ$ is for it to be extrinsically flat, that is the second fundamental form is zero and such a $Σ$ is totally-geodesic for any geodesic. Evidently a totally-geodesic surface is umbilic. One could obtain an umbilic surface as a surface orthogonal to a hypersurface-orthogonal conformal Killing vector, for example a dilatation in flat space, and a totally-geodesic surface as the fixed point set of an involution, for example $θ → π − θ$ in a spherically-symmetric

*email: tod@maths.ox.ac.uk
metric when the equatorial plane is totally-geodesic, or as a hypersurface orthogonal to a hypersurface-orthogonal Killing vector, for example the surfaces of constant $x, y$ or $z$ in the Kasner metric:

$$g = dt^2 - t^2p dx^2 - t^2q dy^2 - t^2r dz^2.$$  

Thus this Proposition explains two results from [2]: why all Penrose limits of the Schwarzschild metric are diagonalisable - since any geodesic in Schwarzschild may be supposed w.l.o.g. to lie in the equatorial plane - and why the Penrose limit of the Kasner metric for a $\Gamma$ orthogonal to one of the Killing vectors is diagonalisable.

To prove the Proposition I’ll first review how to take the Penrose limit, and then consider umbilic surfaces.

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2 The Penrose limit and umbilic surfaces.

The 4-dimensional plane-wave metric can be written in the Brinkman form as

$$g = 2du dv + H(u, \zeta, \bar{\zeta}) du - 2d\zeta d\bar{\zeta},$$  \hspace{1cm} (1)

with

$$H(u, \zeta, \bar{\zeta}) = \frac{1}{2} \left( \Psi(u)\zeta^2 + 2\Phi(u)\zeta \bar{\zeta} + \bar{\Psi}(u)\bar{\zeta}^2 \right).$$  \hspace{1cm} (2)

To take the Penrose limit of a space-time $M$ given a null geodesic $\Gamma$ in $M$, one first chooses a null vector $\ell^a$ tangent to $\Gamma$ and factorising into spinors as $\ell^a = \alpha^A\alpha^B$, and then rescales $\alpha^A$ so as to be parallelly-propagated along $\Gamma$ in the sense that

$$D\alpha^A := \ell^{BB'}\nabla_{BB'}\alpha^A = 0,$$

and an affine parameter $u$ along $\Gamma$ with $Du = 1$. Then one calculates the two curvature components

$$\Psi(u) = \Psi_{ABCD}\alpha^A\alpha^B\alpha^C\alpha^D, \hspace{1cm} \Phi(u) = \Phi_{ABA'B'}\alpha^A\alpha^B\alpha^A'\alpha^B',$$

where $\Psi_{ABCD}, \Phi_{ABA'B'}$ are respectively the Weyl and Ricci spinors of $M$. Finally one substitutes the $\Psi$ and $\Phi$ thus obtained into $H$ in (2) and then into the plane-wave metric (1).

It was shown in [3] that a plane-wave metric is diagonalisable if and only if the phase of $\Psi$ is constant along $\Gamma$. This can be equivalently stated as iff $\Psi(u)$ is real after a suitable constant rescaling of $\alpha^A$. 

2
Now suppose \( \Sigma \) is a hypersurface with unit normal \( N^a \) which we’ll take to be space-like (i.e. \( N^a N_a = -1 \) with our conventions) so that the metric of \( \Sigma \) is indefinite and there are null vectors tangent to \( \Sigma \) (and therefore null geodesics of the ambient space \( M \) lying entirely in \( \Sigma \)). The covariant derivative of \( N_a \) can be written

\[
\nabla_a N_b = K_{ab} - N_a A_b,
\]

where \( K_{ab} \) is the second fundamental form of \( \Sigma \) and \( A_b \) is the acceleration of the normals. Since \( \Sigma \) is umbilic,

\[
K_{ab} = \frac{1}{3} K h_{ab}
\]

where \( h_{ab} = g_{ab} + N_a N_b \) is the metric intrinsic to \( \Sigma \) and \( K = h^{ab} K_{ab} \).

Introduce the projection \( P^a_b = \delta^a_b + N_a N^b \) orthogonal to \( N^a \) and project the Ricci identity in the form

\[
(\nabla_a \nabla_b - \nabla_b \nabla_a) N_c = R_{abcd} N^d
\]

where \( R_{abcd} \) is the Riemann tensor, orthogonal to \( N \) on all indices. We obtain

\[
\frac{1}{3} P^a_b P^q_c P^r_d (K_{pq} h_{qr} - K_{qr} h_{pq}) = P^a_b P^q_c P^r_d N^d R_{pqrd},
\]

and the trace-free part of this gives

\[
P^a_b P^q_c P^r_d C_{pqrd} = 0,
\]

where \( C_{pqrd} \) is the Weyl tensor. Recall the definition of the electric and magnetic parts of the Weyl tensor at \( \Sigma \) as respectively

\[
E_{ab} = C_{acbd} N^c N^d, \quad H_{ab} = \frac{1}{2} \epsilon_{ac} P^q_c P^r_d N^d N^d,
\]

then in particular this means that \( H_{ab} \), the magnetic part of the Weyl tensor, is zero at \( \Sigma \). (For the totally-geodesic case, we could have deduced the vanishing of \( H_{ab} \) from the fact that \( \epsilon_{abcd} \) changes sign under reflection).

Next we choose a null geodesic \( \Gamma \) that lies in \( \Sigma \). Suppose its tangent is \( \ell^a = \alpha^A \ell_A \) with \( D \alpha^A = 0 \) as before. Since \( \Gamma \) lies in \( \Sigma \) we’ll have \( N_a \ell^a = 0 \) and so

\[
\alpha^A N_{AA'} = f \ell_{A'}
\]

for some (nonzero) \( f \). Taking \( D \) of this equation and using \( K_{ab} = \frac{1}{3} K h_{ab} \) we find \( D f = 0 \) so \( f \) is constant along \( \Gamma \), and then a constant phase change of \( \alpha^A \) will make \( f \) real. Now we consider the identity

\[
E_{ab} - i H_{ab} = 2 \Psi_{ACBD} \epsilon_A \epsilon_{AC'} \epsilon_{B'D'} N^{CC'} N^{DD'},
\]

3
where $\Psi_{ABCD}$ is the Weyl spinor of $M$, and impose the vanishing of $H_{ab}$ on this, then contract with $\ell^a \ell^b = \alpha^A \alpha^A' \alpha^B \alpha^B'$ to find

$$E_{ab} \ell^a \ell^b = 2 f^2 \Psi_{ABCD} \alpha^A \alpha^B \alpha^C \alpha^D = f^2 \Psi(u).$$

To prove the Proposition, we need to establish that $\Psi(u)$ is real (possibly after a constant phase change on $\alpha^A$) and that now follows since both $f$ and $E_{ab} \ell^a \ell^b$ are real. Thus any $\Gamma$ lying in $\Sigma$ has a real $\Psi(u)$ after suitable choice of spinor $\alpha^A$, and therefore has a diagonalisable Penrose limit.

\[\square\]

References

[1] R Penrose, *Any space-time has a plane-wave as a limit*, pp271-275 in *Differential geometry and relativity* eds M Cahen and M Flato (Reidel, Dordrecht, Netherlands 1976)

[2] Paul Tod, *Spacetimes with all Penrose limits diagonalisable* Class.Quant.Grav. 37 (2020), 075021 and arXiv:1909.07756

[3] Paul Tod, *On choosing coordinates to diagonalise the metric*, Class.Quant.Grav. 9, 1693-1705 (1992)