KRON REDUCTION OF
GENERALIZED ELECTRICAL NETWORKS

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ABSTRACT. Kron reduction is used to simplify the analysis of multi-machine power systems under certain steady state assumptions that underly the usage of phasors. Using ideas from behavioral system theory, we show how to perform Kron reduction for a class of electrical networks without steady state assumptions. The reduced models can thus be used to analyze the transient as well as the steady state behavior of these electrical networks.

1. Introduction

Multi-machine power networks are the interconnection of power generators and substations via three-phase transmission lines. This structure can be abstracted as a graph in which edges represent transmission lines and vertices represent buses that could be connected to generators and/or to substations abstracted as loads. Depending on the models used to describe the generators and the loads, we have a set of algebraic, differential, or algebro-differential equations per vertex. When the number of vertices increases this set of equations quickly becomes intractable. Common practice in the power systems literature is to reduce this set of equations, through a process called Kron reduction \cite[Sec. 9.3]{12}, that results in a simpler set of equations providing the same relationships between voltage and current at the generators’ terminals. One example of Kron reduction is the classical $\mathcal{Y}$ to $\mathcal{\Delta}$ conversion depicted in Figure 1. We regard the white vertices as boundary vertices to which generators are connected. The black vertex is an internal vertex that is connected to a load. The $\mathcal{Y}$ to $\mathcal{\Delta}$ conversion provides an equivalent circuit solely consisting of boundary vertices.

Despite its widespread use, Kron reduction is based on the use of phasors and it requires the current and voltage waveforms in each phase to be sinusoidal and with the same frequency. This assumption seems contradictory if we want to study the transient behavior of a power system during which the waveforms are not sinusoidal.

The contribution of this paper is to identify a class of electrical networks for which Kron reduction can be performed in the time domain, i.e., without resorting to phasors. The reduced models can then be used to study the transient as well as the steady state behavior of these electrical networks which can be used to describe, e.g., short transmission lines.

A graph-theoretic discussion of Kron reduction can be found in \cite{8}. Our results are based on the very same graph-theoretic constructions. The problem of Kron reduction can be understood as a system equivalence problem: when do two models - the original and the Kron reduced - describe the same terminal behavior? Here we

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is defined as $\overline{\mathbb{R}}$ of all strictly positive real numbers and $\mathbb{R}_+$ is denoted by $A$. This problem was first solved for purely resistive circuits in [5] and then for RLC circuits in [7] by use the term behavior in the sense of behavioral systems theory [6]. This problem was solved for purely resistive, purely inductive and purely capacitive circuits in [1, 2] and for a class of RL circuits called homogeneous RL networks in [3]. This paper generalizes the results in [3] by describing a larger class of electrical networks for which Kron can be performed in the time domain.

2. Notation and Definitions

Let $\mathbb{N}$ be the set of natural numbers, $\mathbb{R}$ be the set of real numbers, $\mathbb{R}_+$ be the set of all strictly positive real numbers and $\mathbb{R}_{+0} = \mathbb{R}_+ \cup \{0\}$. For any $n \in \mathbb{N}$, the set $\overline{n}$ is defined as $\overline{n} = \{1, \ldots, n\}$. The cardinality of the set $S$ is denoted by $|S|$. We also use $|x|$ to denote the absolute value of $x \in \mathbb{R}$. The set of all real $n \times m$ matrices is denoted by $\mathbb{R}^{n \times m}$. The element of $\mathbb{R}^n$ with all entries equal to 0 is denoted by $0_n$. We also use $0_n$ to denote the zero map $0_n : \mathbb{R} \to \mathbb{R}^n$, which maps every real number to $0_n \in \mathbb{R}^n$. For any matrix $M \in \mathbb{R}^{n \times m}$, we denote the element located in the $i$th row and the $j$th column by $M_{ij}$. For any set of vectors $S = \{v_1, \ldots, v_n\}$, $\text{span} S$ is the vector space spanned by the elements of $S$. For any vector space $V$, $\text{dim} V$ denotes the dimension of $V$. The kernel and the image of a linear map $f$ are denoted by $\ker f$ and $\text{im} f$. The set of all smooth functions with domain $A$ and codomain $B$ is denoted by $C^\infty(A, B)$.

We follow [9] for the definitions related with graphs. A graph $G = (V, E)$ is a two-tuple where $V$ is a finite set of vertices and $E \subseteq V \times V$ is a finite set of edges. In directed graphs, $(x, y) \in E$ implies that vertices $x$ and $y$ are connected via an edge with tail vertex $x$ and head vertex $y$. Let $v = |V|$ and $e = |E|$. Without loss of generality, we will assume $V = \overline{v}$ for the rest of the paper. Any graph $G$ is completely represented by a $v \times e$ matrix called incidence matrix $B$. The rows of the incidence matrix represent the vertices and the columns represent the edges. In directed graphs, every edge $e_k = (i, j)$, $k \in \overline{e}$ is encoded in the incidence matrix by setting $B_{ik} = -1$, $B_{jk} = 1$ and $B_{xk} = 0$ for all $x \in \overline{v}\{i, j\}$. The set of adjacent vertices $A$ is defined as $A = \{(x, y) \in V \times V : (x, y) \in E \text{ or } (y, x) \in E\}$. A path of length $\ell$ from vertex $i$ to vertex $j$ is the subset of vertices $\{r_0, \ldots, r_\ell\}$ such that $r_0 = i$, $r_\ell = j$, and $(r_{k-1}, r_k) \in A$ for all $k \in \ell$. The careful reader may notice that we allow traversing directed edges in both directions. A graph is called a connected graph if for every $i, j \in V$, $i \neq j$, there exists a path from $i$ to $j$. A cycle of a graph

\begin{figure}[h]
\centering
\includegraphics[width=0.3\textwidth]{fig1.png}
\caption{Equivalent Y (left) and \(\Delta\) (right) circuits. The internal vertex depicted in black is eliminated in the conversion from the Y to the \(\Delta\) circuit.}
\end{figure}
is a connected subgraph in which every vertex has exactly two neighbors. For the rest of the paper, we will assume that every graph is connected.

We will need the following lemma, which is a restatement of part of Theorem 3.1 in [2], to prove our main result. A similar result can also be found in Lemma 2.1 in [3].

**Lemma 2.1** (Theorem 3.1 in [2]). Let $\mathcal{G} = (\mathcal{V}, \mathcal{E})$ be a directed and connected graph with nontrivial subsets $\mathcal{V}_i \subset \mathcal{V}$ and $\mathcal{V}_b = \mathcal{V} \setminus \mathcal{V}_i \subset \mathcal{V}$. Let $B_b$ and $B_i$ be the matrices that are obtained by collecting the rows of $B$ that correspond to vertices in the sets $\mathcal{V}_b$ and $\mathcal{V}_i$, respectively. Then there exist a unique directed graph $\hat{\mathcal{G}} = (\mathcal{V}_b, \hat{\mathcal{E}})$ with incidence matrix $\hat{B}_b \in \{-1,0,1\}^{\mathcal{V}_b \times |\mathcal{E}|}$ and a diagonal matrix $W_r \in \mathbb{R}^{|\mathcal{E}| \times |\mathcal{E}|}$ such that

$$\hat{B}_b W_r B_b^T = B_b W B_T - B_b W B_T (B_r W B_T)^{-1} B_r W B_b^T.$$ 

Note that Lemma 2.1 implies that matrix $B_r W B_T$ is invertible for any diagonal matrix $W \in \mathbb{R}^{n \times n}$ with strictly positive diagonal elements, where $B_i$ is defined as in Lemma 2.1.

3. **Kirchoff’s Laws on Graphs**

We review Kirchoff’s Current Law (KCL) and Kirchoff’s Voltage Law (KVL) on abstract graphs, which were introduced in [1] [2]. Let $\mathbb{R}^S$ be the vector space of all functions from $S$ to the field of real numbers $\mathbb{R}$. The space of trajectories of currents entering into vertices is denoted by $\Lambda_0$, and it is defined as $\Lambda_0 = C^\infty(\mathbb{R}, \mathbb{R}^V)$. We can think of vertices as zero-dimensional objects and for this reason we use the subscript 0. We identify $\mathbb{R}^V$ with the vector space $\mathbb{R}^v$ and let $(\mathbb{R}^v)^*$ be the dual space of $\mathbb{R}^v$. The dual space of $\Lambda_0$ is denoted by $\Lambda^0$. We can interpret $\Lambda^0 = C^\infty(\mathbb{R}, (\mathbb{R}^v)^*)$ as the space of trajectories of voltages on the vertices. Similarly, the space of trajectories of currents flowing through the edges is denoted by $\Lambda_1$, and it is defined as $\Lambda_1 = C^\infty(\mathbb{R}, \mathbb{R}^E)$. Edges are regarded as one dimensional objects, hence the subscript 1. We identify $\mathbb{R}^E$ with the vector space $\mathbb{R}^e$ and interpret the dual space of $\Lambda_1$, denoted by $\Lambda^1 = C^\infty(\mathbb{R}, (\mathbb{R}^e)^*)$, as the space of trajectories of voltages across the edges. We can think of the incidence matrix $B$ of the directed graph $\mathcal{G}$ as the matrix representation of the linear map

$$B : \mathbb{R}^e \to \mathbb{R}^v.$$ 

Using the standard inner product in Euclidean spaces as the duality product, the dual map of (3.1) can be represented by the matrix $B^T$. The linear map (3.1) can be extended to the map $\hat{B} : \Lambda_1 \to \Lambda_0$ defined by $(\hat{B} \circ f_1)(t) = B \circ f_1(t)$ for every $f_1 \in \Lambda_1$ and every $t \in \mathbb{R}$. Similarly, we can define the dual map $\Lambda^0 \to \Lambda^1$ as $(B^T \circ g^0)(t) = B^T \circ g^0(t)$ for every $g^0 \in \Lambda^0$ and every $t \in \mathbb{R}$. For the graph $\mathcal{G} = (\mathcal{V}, \mathcal{E})$, let $\{\mathcal{V}_b, \mathcal{V}_i\}$ be a partition of its set of vertices $\mathcal{V}$. In other words, $\mathcal{V} = \mathcal{V}_b \cup \mathcal{V}_i$ and $\mathcal{V}_b \cap \mathcal{V}_i = \emptyset$. We call the elements of $\mathcal{V}_b$ boundary vertices and the elements of $\mathcal{V}_i$ internal vertices. A graph $\mathcal{G}$ with at least one boundary vertex is called an open graph [1]. We will use open graphs to model power networks connected to generators through its boundary vertices. The partition $\{\mathcal{V}_b, \mathcal{V}_i\}$ of $\mathcal{V}$ leads to a decomposition of the incidence matrix $B$. Explicitly, the incidence matrix can be decomposed as $B = [B_b^T B_i^T]^T$. In this decomposition, $B_b$ contains the rows of the incidence matrix that correspond to boundary vertices, and $B_i$ contains the rows of the incidence matrix that correspond to internal vertices. The
decomposition of $B$ into $B_b$ and $B_i$ induces the decomposition of the vector space $\Lambda_0$ into $\Lambda_{0b}$ and $\Lambda_{0i}$. Similarly, the dual space $\Lambda^0$ can be decomposed into $\Lambda^{0b}$ and $\Lambda^{0i}$. We can define the linear maps $\bar{B}_b : \Lambda_1 \to \Lambda_{0b}$, $\bar{B}_i : \Lambda_1 \to \Lambda_{0i}$ and their dual maps as before.

Kirchoff’s Current Law states that the sum of currents entering into a vertex is zero. We can conveniently describe the sum of the currents entering each vertex in a directed graph by $\bar{B}I_1$, where $I_1 \in \Lambda_1$ is the vector of currents flowing across edges. If a vertex is an internal vertex, the sum of the currents is zero. This can be stated as $\bar{B}_i I_1 = \mathbf{0}_{|V_i|}$. Since we can inject currents into the boundary vertices, the sum of the currents entering into a boundary vertex is equal to the injected current. This can be stated as $\bar{B}_b I_1 = I_{0b}$. Therefore, an open graph $\mathcal{G} = (V, E)$ with the set of boundary vertices $V_b$ satisfies KCL for $(I_{0b}, I_1) \in \Lambda_{0b} \times \Lambda_1$ if

$$
\begin{bmatrix}
\bar{B}_i \\
\bar{B}_b
\end{bmatrix} I_1 =
\begin{bmatrix}
\mathbf{0}_{|V_b|} \\
I_{0b}
\end{bmatrix}.
$$

Although KCL, as expressed in (3.2), requires a directed graph, the directions of the currents can be arbitrarily chosen. In other words, they are a modeling choice and not intrinsic to the physics of the problem.

Kirchoff’s Voltage Law states that the sum of voltages along a cycle is zero. Since the cycle space of a graph $\mathcal{G}$ with incidence matrix $B$ is $\ker B$, the sum of the voltages along a cycle $K \in \ker B$ can be written as $K^T V^1$ for $V^1 \in \Lambda^1$. Hence, KVL can be written as $K^T V^1 = 0$ for every $K \in \ker B$ or as $V^1 \in (\ker B)^\perp$ with $(\ker B)^\perp$ denoting the orthogonal complement of $\ker B$ with respect to the standard Euclidean inner product. By making the identification $(\ker B)^\perp \cong \text{im} B^T$, we can further express KVL as $V^1 = B^T \psi^0$ for some potential $\psi^0 \in \Lambda^0$.

In the remainder of the paper it will be convenient to split the potential $\psi^0 \in \Lambda^0$ into its internal components $\psi_i \in \Lambda^0_i$ and its boundary components $\psi_b \in \Lambda^0_b$ resulting in the following version of KVL

$$
V^1 = B^T \psi^0 = \bar{B}_b^T \psi_{0b} + \bar{B}_i^T \psi_{0i}.
$$

For the rest of the paper, we will abuse notation and denote $\bar{B}, \bar{B}_i, \bar{B}_b$ by $B, B_i, B_b$, respectively.

4. Main Result

Assume that we have a network of electrical components. Each electrical component has two terminals. The constitutive relation of an electrical component relates the current flowing through the electrical component to the voltage across its terminals. We consider electrical components with constitutive relations given by:

$$
\sum_{j=0}^\nu p_{kj} \frac{d}{d\nu} I_{1,k} = \sum_{j=0}^\nu q_{kj} \frac{d}{d\nu} V_k^1.
$$

where $p_{kj}, q_{kj} \in \mathbb{R}_{+0}$, $\nu \in \mathbb{N}$ is the highest degree of differentiation, $I_{1,k}$ is the current flowing through the electrical component $k \in \bar{\mathcal{C}}$ and $V_k^1$ is the voltage across the terminals of the electrical component $k$. The coefficient vectors of (4.1) are defined as $p_k = (p_0, \ldots, p_{\nu}) \in \mathbb{R}_{+0}^{\nu+1}$ and $q_k = (q_0, \ldots, q_{\nu}) \in \mathbb{R}_{+0}^{\nu+1}$. The coefficient matrices are defined as $P = [p_1 \mid \ldots \mid p_c]$ and $Q = [q_1 \mid \ldots \mid q_c]$. 

The constitutive relation for a linear ideal resistor is given by \( rI_{1,k} = V_k^1 \) and can be described by (4.1) if we take \( \nu = 0 \) and \( \frac{\partial \psi}{\partial \nu_0} = r \). Similarly, the constitutive relations for inductors and capacitors, \( \ell \dot{I}_{1,k} = V_k^1 \) and \( I_{1,k} = cV_k^1 \), are described by (4.1) when we set \( \nu = 1 \), \( p_k = 0 \), \( p_{k1} = \ell \), \( q_{k0} = 0 \), \( q_{k1} = 1 \), and \( \nu = 1 \), \( p_{k0} = 0 \), \( p_{k1} = 1 \), \( q_{k0} = 0 \), \( q_{k1} = c \), respectively. Hence, (4.1) generalizes the constitutive relations of RLC circuits.

The constant upper bound \( \nu \) in (4.1) is the same for every electrical component. In other words, the highest degree of differentiation in (4.1), which is a measure of the complexity of the electrical component, is independent of the electrical component. We can think of an electrical network as a directed graph in which each edge in the graph represents an electrical component. In this framework, electrical components relate the space of trajectories of currents flowing through the edges \( \Lambda_1 \) and its dual space, the space of trajectories of voltages across the edges \( \Lambda^1 \). In other words, for the directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \), \( (I_1, V^1) \in \Lambda_1 \times \Lambda^1 \) satisfies the constitutive relations (4.1) if for every edge \( e_k \in \mathcal{E} \), the relationship between \( I_{1,k} \) and \( V_k^1 \) is given by (4.1), where \( I_{1,k} \) is the \( k \)-th element of \( I_1 \) and \( V_k^1 \) is the \( k \)-th element of \( V^1 \). For every edge \( e_k \in \mathcal{E} \), the coefficient vectors of (4.1) are \( p_k \) and \( q_k \). For the rest of the paper, we will assume that \( p_k \neq 0_{\nu+1} \), i.e., no short-circuit edges and \( q_k \neq 0_{\nu+1} \), i.e., no open-circuit edge.

**Definition 4.1.** A generalized electrical network is a five-tuple \( \mathcal{N} = (\mathcal{G}, \mathcal{V}_b, \nu, P, Q) \). It consists of: an open directed graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) on which KCL (3.2), KVL (3.3), and constitutive relations (4.1) are satisfied; a set of boundary vertices \( \mathcal{V}_b \subset \mathcal{V} \); an order of differentiation \( \nu \) (the constant in (4.1)); and the coefficient matrices \( P, Q \in \mathbb{R}^{(\nu+1) \times \mathcal{E}} \).

We adapt the notion of terminal behavior, which was introduced in [5, 7], to our framework.

**Definition 4.2.** The terminal behavior \( \mathcal{B}_\mathcal{N} \subset \Lambda_0^0 \times \Lambda_0^{0i} \) of a generalized electrical network \( \mathcal{N} = (\mathcal{G}, \mathcal{V}_b, \nu, P, Q) \) is the relation defined by: \( (\psi, I_0) \in \mathcal{B}_\mathcal{N} \) iff there exists a \( \psi^0 \in \Lambda_0^{0i} \) such that \( \psi^0 = (\psi^0, \psi^0) \in \Lambda_0^0 \) and \( I_0 = (I^0, 0_{\nu+1}) \in \Lambda_0^0 \times \Lambda_0^{0i} \) satisfy KCL (3.2), KVL (3.3) and constitutive relations (4.1) on \( \mathcal{G} \).

The problem addressed in this note is:

**Problem 4.3** (Kron Reduction [8]). Given a generalized electrical network \( \mathcal{N} = (\mathcal{G}, \mathcal{V}_b, \nu, P, Q) \), when can we construct another generalized electrical network \( \hat{\mathcal{N}} = (\hat{\mathcal{G}}, \hat{\mathcal{V}}_b, \nu, \hat{P}, \hat{Q}) \) with \( \hat{\mathcal{G}} = (\mathcal{V}_b, \mathcal{E}) \) and \( \mathcal{B}_\hat{\mathcal{N}} = \mathcal{B}_\mathcal{N} \)?

Note that every vertex in the graph \( \hat{\mathcal{G}} \) is a boundary vertex. Therefore Problem 4.3 is equivalent to eliminating all the internal vertices of the generalized electrical network \( \mathcal{N} \) without changing the terminal behavior and the complexity of the constitutive relations measured by \( \nu \). We can now can state our main result.

**Theorem 4.4.** Problem 4.3 is solvable for the generalized electrical network \( \mathcal{N} = (\mathcal{G}, \mathcal{V}_b, \nu, P, Q) \) if we have

\[
\dim \text{span} \{p_1, \ldots, p_e\} = \dim \text{span} \{q_1, \ldots, q_e\} = 1,
\]

where \( p_j \), \( q_j \) are the coefficient vectors of (4.1) for edge \( e_j \in \mathcal{E} \), where \( j \in \{1, \ldots, e\} \).

**Proof.** Assume that \( \mathcal{N} = (\mathcal{G}, \mathcal{V}_b, \nu, P, Q) \) is a generalized electrical network satisfying (4.2); condition (4.2) states that the vector space \( \text{span} \{p_1, \ldots, p_e\} \) has
Hence, we can assume \( \lambda \). Thus for every \( k \in \mathcal{E} \), there exists a constant \( \lambda_k \) such that \( p_k = \lambda_k \bar{p} \). Since every element of \( p_k \) is nonnegative for all \( k \in \mathcal{E} \), we can assume without loss of generality that every element of \( \bar{p} \) is nonnegative. Hence, we can assume \( \lambda_k \geq 0 \) for all \( k \in \mathcal{E} \). Similarly, the basis for the vector space \( \text{span} \{ q_1, \ldots, q_e \} \) consists of a single vector \( \bar{q} = (\bar{q}_1, \ldots, \bar{q}_e) \neq 0 \). From the same reasoning, we can assume \( \gamma_k \geq 0 \). Replacing \( p_k \) and \( q_k \) in (4.1) with \( p_k = \lambda_k \bar{p} \) and \( q_k = \gamma_k \bar{q} \), we obtain

\[
\lambda_k \sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} I_{1,k} = \gamma_k \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} V_k^1,
\]

for every edge \( e_k \in \mathcal{E} \). By assumption, we have \( p_k \neq 0 \) and \( q_k \neq 0 \). Dividing (4.3) by \( \lambda_k \) for each \( k \in \mathcal{E} \) and writing equation (4.3) for every edge \( e_k \in \mathcal{E} \) in vector form, we obtain

\[
\sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} I_1 = \Gamma \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} V^1,
\]

where \( I_1 = (I_{1,1}, \ldots, I_{1,e}) \), \( V^1 = (V_1^1, \ldots, V_e^1) \) and \( \Gamma \) is a diagonal matrix with strictly positive diagonal elements. The matrix \( \Gamma \) is defined as \( \Gamma_{kk} = \frac{1}{\lambda_k} \) for all \( k \in \mathcal{E} \). From KVL (3.3), we have \( V^1 = B^T \psi^0 \) for \( \psi^0 \in \text{Lambda}_0 \). Replacing \( V^1 \) in (4.4), we obtain

\[
\sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} I_1 = \Gamma \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} B^T \psi^0 = \Gamma B^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0.
\]

and multiplication by \( B \) leads to

\[
B \sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} I_1 = \Gamma B^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0
\]

\[
\iff \sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} B I_1 = \Gamma B^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0.
\]

Using the previously defined partitioning of \( B \) into \( B_i \) and \( B_o \), we obtain the following set of equations from (4.6)

\[
\sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} B_o I_1 = B_o \Gamma B_i^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0 + B_o \Gamma B_b^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0.
\]

\[
\sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} B_i I_1 = B_i \Gamma B_i^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0 + B_i \Gamma B_b^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0.
\]

From KCL (3.2), we have \( B_o I_1 = I_{0b} \) and \( B_i I_1 = 0_{|\mathcal{V_i}|} \). Replacing \( B_o I_1 = I_{0b} \) in (4.7) and \( B_i I_1 = 0_{|\mathcal{V_i}|} \) in (4.8), we have

\[
\sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} I_{0b} = B_o \Gamma B_i^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0 + B_o \Gamma B_b^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0,
\]

\[
\sum_{j=0}^\nu \bar{p}_j \frac{d^j}{dt^j} I_{0b} = B_o \Gamma B_i^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0 + B_o \Gamma B_b^T \sum_{j=0}^\nu \bar{q}_j \frac{d^j}{dt^j} \psi^0.
\]

(4.10)
The matrix $B_i \Gamma B_i^T$ is invertible by Lemma 2.1 Therefore, we obtain from the previous equality:

$$\sum_{j=0}^{\nu} \tilde{q}_j \frac{d\psi^{0j}}{dt} = -(B_i \Gamma B_i^T)^{-1} B_i \Gamma \psi_i \sum_{j=0}^{\nu} \tilde{q}_j \frac{d\psi^{0j}}{dt}.$$  \hspace{1cm} (4.11)

Substituting $\sum_{j=0}^{\nu} \tilde{q}_j \frac{d\psi^{0j}}{dt}$ into (4.9), we obtain

$$\sum_{j=0}^{\nu} \tilde{p}_j \frac{d\psi^{0j}}{dt} I_{0b} = (B_b \Gamma B_b^T - B_b \Gamma B_b^T (B_i \Gamma B_i^T)^{-1} B_i \Gamma B_i^T) \sum_{j=0}^{\nu} \tilde{q}_j \psi^{0j}. \hspace{1cm} (4.12)$$

Smoothness of $\psi^{0b}$ implies that the left hand side of (4.12) and the left hand side of (4.11) are continuous functions. Since $\tilde{p} \neq 0_{\nu+1}$ for any $\psi^{0b} \in \Lambda_b^0$, there exists a unique $I_{0b} \in \Lambda_{0b}$ that satisfies (4.12) and a unique $\psi^{0b}$ that satisfies (4.11). Therefore, if $I_{0b}$ and $\psi_{0b}$ satisfy (4.12), then there exists a unique $\psi^{0b} \in \Lambda_b^0$ such that $\psi^0 = (\psi^{0b}, \psi^{0i}) \in \Lambda_b^0$ and $I_0 = (I_{0b}, 0_{\nu_v}) \in \Lambda_0$ satisfy KCL (3.2), KVL (3.3) and constitutive relations (4.1). Thus $(\psi_v, I_v) \in \mathcal{B}_N$ iff $I_{0b}$ and $\psi_{0b}$ satisfy (4.12). We now want to construct a generalized electrical network $\tilde{\mathcal{N}} = (\tilde{\mathcal{G}}, \tilde{\mathcal{V}}, \tilde{\mathcal{B}}, \tilde{\mathcal{P}}, \tilde{\mathcal{Q}})$ with $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ and $\tilde{\mathcal{B}} = \tilde{\mathcal{B}}$. From Lemma 2.4 there exists a graph $\tilde{\mathcal{G}} = (\mathcal{V}, \tilde{\mathcal{E}})$ with incidence matrix $\tilde{B}$ and a diagonal matrix $\Gamma$ with strictly positive diagonal elements such that

$$\tilde{B} \Gamma \tilde{B}^T = B_b \Gamma B_b^T - B_b \Gamma B_b^T (B_i \Gamma B_i^T)^{-1} B_i \Gamma B_i^T.$$  \hspace{1cm} (4.13)

We construct a generalized electrical network $\tilde{\mathcal{N}}$ from the directed graph $\tilde{\mathcal{G}}$ by defining the constitutive relations on $\tilde{\mathcal{G}}$ as

$$\sum_{j=0}^{\nu} \tilde{p}_j \frac{d\psi^{0j}}{dt} \tilde{I}_1 = \tilde{\Gamma} \sum_{j=0}^{\nu} \tilde{q}_j \frac{d\psi^{0j}}{dt} \tilde{I}_1. \hspace{1cm} (4.14)$$

Multiplying both sides of (4.14) by $\tilde{B}$ and using (3.2), (3.3), (4.13); we obtain (4.12) from (4.13). Therefore we can construct a generalized electrical circuit $\tilde{\mathcal{N}}$ that has the same terminal behavior as $\mathcal{N}$. \hfill \Box

We emphasize that the reduction process detailed in the proof of Theorem 4.3 is performed in the time domain and requires no steady state assumptions. We start with the generalized electrical network $\mathcal{N}$ that describes the relation between voltages and currents in the time domain and we construct the unique Kron reduced generalized electrical network $\tilde{\mathcal{N}}$ that has the same terminal behavior also described in the time domain by the constitutive relations (4.13). In the proof we assumed that no currents are injected at the internal vertices. However, the proof smoothly extends to the case where there are injected currents. We shall return to this observation, in the context of power networks, in Section 5.

The following example illustrates that Condition (4.12) in Theorem 4.3 is not necessary.

**Example 4.5.** Consider the generalized electrical network $\mathcal{N} = (\mathcal{G}, \mathcal{V}, \nu, \mathcal{P}, \mathcal{Q})$ with $\mathcal{G}$ given below:

```
1  V_1 \psi_1^0
\psi_1^0 I_{1,1} 3  V_2 \psi_3^0
\psi_3^0 \psi_2^0 I_{1,2}
```

```
The constitutive relations for the edges \( e_1 = (1,3) \) and \( e_2 = (3,2) \) are \( I_{1,1} = V_1^1 \) and \( \frac{d}{dt}I_{1,2} = V_2^1 \), respectively. Note that \( \nu = 1 \). From KCL (3.2), we have \( I_{0b,1} = I_{1,1} \), \( I_{1,1} = -I_{1,2} \) and \( I_{1,2} = -I_{0b,2} \). From KVL (3.3), we have \( V_1^1 = \psi_0^0 - \psi_1^0 \) and \( V_2^1 = \psi_0^0 - \psi_3^0 \). Replacing \( I_{1,1}, I_{1,2}, V_1^1 \) and \( V_2^1 \) in the constitutive relations, we obtain

\[
I_{0b,1} = -I_{0b,2} = \psi_0^0 - \psi_1^0, 
\]

\[
\frac{d}{dt}I_{0b,1} = -\frac{d}{dt}I_{0b,2} = \psi_2^0 - \psi_3^0. 
\]

Combining these equations, we obtain

\[
(4.15) \quad I_{0b,1} + \frac{d}{dt}I_{0b,1} = -I_{0b,2} - \frac{d}{dt}I_{0b,2} = \psi_2^0 - \psi_1^0. 
\]

There exists a \( \psi_0^0 \) such that \( (\psi_1^0, \psi_2^0, \psi_3^0) \) and \( (I_{0b,1}, I_{0b,2}) \) satisfy KCL (3.2), KVL (3.3) and the constitutive relations if and only if \( (\psi_1^0, \psi_2^0) \) and \( (I_{0b,1}, I_{0b,2}) \) satisfy (4.15). Therefore \( (\psi_1^0, \psi_2^0, I_{0b,1}, I_{0b,2}) \in \mathcal{B}_N \) if and only if \( (\psi_1^0, \psi_2^0, I_{0b,1}, I_{0b,2}) \) satisfy (4.15). We now construct a generalized electrical network \( \hat{N} = (\hat{G}, \hat{V}_b, 1, P, Q) \) such that \( \mathcal{B}_N = \mathcal{B}_{\hat{N}} \). The directed graph \( \hat{G} \) is given below.

\[
\begin{array}{ccc}
1 & \hat{V}_1 & 2 \\
\psi_1^0 & \hat{I}_{1,1} & \psi_2^0 \\
\end{array}
\]

We pick the constitutive relation of the single edge in \( \hat{N} \) as

\[
\hat{I}_{1,1} + \frac{d}{dt}\hat{I}_{1,1} = \hat{V}_1^1. 
\]

From KCL (3.2), we have \( I_{0b,1} = -I_{0b,2} = \hat{I}_{1,1} \). From KVL (3.3), we have \( \hat{V}_1^1 = \psi_2^0 - \psi_1^0 \). Replacing \( \hat{I}_{1,1} \) and \( \hat{V}_1^1 \) in the constitutive relations, we recover (4.15). Therefore \( (\psi_1^0, \psi_2^0, I_{0b,1}, I_{0b,2}) \in \mathcal{B}_N \) if and only if \( (\psi_1^0, \psi_2^0, I_{0b,1}, I_{0b,2}) \) satisfy (4.15). This implies that \( \mathcal{B}_N = \mathcal{B}_{\hat{N}} \). Hence Problem 4.3 is solvable. However \( p_1 = (1,0) \) and \( p_2 = (0,1) \). Therefore \( \dim \text{span} \{p_1, p_2\} = 2 \) and condition (4.2) does not hold.

We now provide an example for which condition (4.2) in Theorem 4.4 fails to hold and Kron reduction is impossible.

**Example 4.6.** Consider the generalized electrical network in the previous example. If we pick the constitutive relations for the edges of \( N \) as \( \frac{d}{dt}I_{1,1} = V_1^1 \) and \( I_{1,2} = \frac{d}{dt}V_2^1 \), condition (4.2) does not hold. Nevertheless, we can compute the constitutive relation for the Kron reduced network as

\[
(4.16) \quad \frac{d}{dt}(V_1^1 + V_2^1) = \frac{d}{dt}(\psi_2^0 - \psi_1^0) = \frac{d^2}{dt^2}I_{1,1} + I_{1,2}. 
\]

Observe that for the original network we have \( \nu = 1 \) while for the reduced network we have \( \nu = 2 \). Hence Problem 4.3 is not solvable in this case.

5. **Application to RLC Circuits and Power Networks**

Every RLC circuit can be modeled as a generalized electrical network by taking the electric components to be combinations of resistors, inductors, or capacitors. When all the circuit elements are resistors we speak of a purely resistive circuit.
Purely inductive and purely capacitive circuits can be defined similarly. It is shown in [2] that Problem 4.3 is solvable for purely resistive, inductive, or capacitive circuits. The same result can be obtained by the following corollary of Theorem 4.4.

**Corollary 5.1.** Problem 4.3 is solvable for the generalized electrical network \( \mathcal{N} \) if \( \mathcal{N} \) is a purely resistive, purely inductive or purely capacitive circuit.

**Proof.** We will prove the corollary for purely resistive circuits. The proofs for purely inductive and purely capacitive circuits are very similar. In a purely resistive circuit, \( p_i = (r_i, 0) \), and \( q_i = (1, 0) \), where \( r_i \in \mathbb{R}^+ \) is the resistance of the edge \( i \). Hence, condition (4.2) holds. The result follows from Theorem 4.4. □

One particular example of generalized electrical networks is homogeneous RL circuits [3]. Every edge of an homogeneous RL circuit is a series connection of a resistor and an inductor. The term homogeneous comes from the fact that for every two edges \( e_i, e_j \in \mathcal{E} \) with resistance values \( r_i, r_j \) and inductor values \( \ell_i, \ell_j \), we have \( \frac{r_i}{r_j} = \frac{\ell_i}{\ell_j} \). In order to represent RL circuits, it is enough to set \( \nu = 1 \) and \((p_{i1}, p_{i2}, q_{i1}, q_{i2}) = (r_i, \ell_i, 1, 0) \) in (4.1) for all \( i \in \bar{e} \), where \( r_i \) is the resistance value of the resistive component of edge \( i \) and \( \ell_i \) is the inductance value of the inductive component of edge \( i \). Note that homogeneity implies that there exists a constant \( c \in \mathbb{R}, c > 0 \) such that \( p_i = cp_j \) for every \( i, j \in \bar{e} \). Therefore condition (4.2) holds and we can recover Theorem 4.4 in [3] as a corollary of Theorem 4.4. The concept of homogeneity can be generalized to RLC circuits. A homogeneous RLC circuit is an electrical circuit such that every edge is a series combination of a resistor, an inductor and a capacitor with the following condition: for every two edges \( e_i, e_j \in \mathcal{E} \) with resistance values \( r_i, r_j \), inductor values \( \ell_i, \ell_j \), and capacitor values \( c_i, c_j \); we have \( \frac{r_i}{r_j} = \frac{\ell_i}{\ell_j} = \frac{c_i}{c_j} \). Homogeneous RC circuits and homogeneous LC circuits can be defined in a similar fashion. The previous discussion is summarized in the next result.

**Corollary 5.2.** Problem 4.3 is solvable for homogeneous RLC, RL, RC or LC circuits.

There are various transmission line models available in the literature [10], [11], [12]. If the transmission line is relatively short (less than 60 kilometers [10], or 50 miles [12]), the transmission line can be described by the short line approximation. In the short line approximation the line is modeled as a series connection of a resistor and an inductor [12]. Hence every network of short transmission lines can be modeled as a generalized electrical network with \( \nu = 1 \), \( p_i = (r_i, \ell_i) \) and \( q_i = (1, 0) \) in (4.1), where \( r_i \) is the resistance value of the resistive component of edge \( i \) and \( \ell_i \) is the inductance value of the inductive component of edge \( i \). Moreover, if the network is a homogeneous RL circuit, it follows from Corollary 5.2 that we can perform Kron reduction. To guarantee that the homogeneity condition only depends on the transmission lines, we model the loads as (possibly nonlinear) current injections at the internal vertices. As discussed after Theorem 4.4 in this case Kron reduction is still possible and the aggregated effect of the loads will appear as (possibly nonlinear) current injections at the boundary vertices of the reduced network.

Different types of transmission line models are suitable for the analysis of different types of transients [14]. The short line approximation is used to analyze the electromechanical transients [13], [14]. One can argue that the assumption that
allows us to use the short line approximation (sinusoidal voltages and currents) to study electromechanical transients also allows us to use phasors. However, in our framework we do not need to assume sinusoidal waveforms per se. As long as the short line approximation can accurately describe the transients in consideration, we can use the reduced model for transient analysis.

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