SHIFTED TABLEAUX AND PRODUCTS OF SCHUR’S SYMMETRIC FUNCTIONS

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Abstract. We introduce a new combinatorial object, semistandard increasing decomposition tableau and study its relation to a semistandard decomposition tableau introduced by Kraskiewicz and developed by Lam and Serrano. We also introduce generalized Littlewood–Richardson coefficients for products of Schur’s symmetric functions and give combinatorial descriptions in terms of tableau words. The insertion algorithms play central roles for proofs. A new description of shifted Littlewood–Richardson coefficients is given in terms of semistandard increasing decomposition tableaux. We show that a “big” Schur function is expressed as a sum of products of two Schur $P$-functions, and vice versa. As an application, we derive two Giambelli formulae for big Schur functions: one is a determinant and the other is a Pfaffian.

1. Introduction

The ordinary representation of the symmetric group plays important roles in symmetric functions, representations, and the combinatorics of Young tableaux (see e.g. [3, 4, 19] and references therein). The projective representations of the symmetric and alternating groups were studied by Schur in [21] and Schur $P$- and $Q$-functions were introduced. The expansion of the $Q$-function in terms of power-sum symmetric functions gives the characters of the projective representation, as the Schur functions do for the ordinary representation. After the invention of shifted tableaux by Thrall [25], there has been progress in connecting shifted tableaux and the projective representation. Shifted tableaux have combinatorial structures as the theory of ordinary tableaux: shifted analogue of the Robinson–Schensted–Knuth correspondence studied by Sagan [18] and Worley [26], and a shifted analogue of the Littlewood–Richardson coefficient by Stembridge [23]. Sagan and Worley introduced two combinatorial tools for the study of $Q$-functions: one is a shifted insertion and the other is shifted jeu-de-taquin. The shifted insertion possesses a shifted analogue of Schensted correspondence, namely a bijection between a permutation and a pair of shifted tableaux of the same shape. Haiman constructed a mixed insertion which is dual to the shifted insertion in [6]. Here, “dual” means that application of the mixed insertion to the inverse of a permutation $w$ gives the same tableaux as the shifted insertion applied to $w$. The pair of tableaux consists of the mixed insertion tableau and the mixed recording tableau. In [22], Serrano introduce the shifted Knuth (or shifted plactic) relations as a shifted analogue of the plactic relations introduced by Knuth [8] and further developed as the plactic monoid by Lascoux and Schützenberger [14]. Two words have the same mixed insertion tableau if and only if they are shifted Knuth-equivalent. Another combinatorial object is a semistandard decomposition tableau (SSDT) [11, 12, 22]. A SSDT is an output of Kraskiewicz insertion for a word. The algorithm was introduced by Kraskiewicz for the hyperoctahedral group [11] and further developed by Lam to study the $B_n$ Stanley symmetric functions [12]. A SSDT is also characterized by a shifted tableau, namely two words are shifted plactic equivalent if and only if they have the same semistandard Kraskiewicz insertion tableau [22].
The expansion coefficients of a product of two Schur functions in terms of Schur functions are known as a Littlewood–Richardson coefficient. There are many combinatorial models to describe this coefficient: Littlewood–Richardson (LR) tableaux \([15]\), lattice words based on plactic monoids (see e.g. \([13, 14]\)), and puzzles \([9, 10]\). Similarly, expansion coefficients of a product of two \(P\)-functions are known as Littlewood–Richardson–Stembridge (LRS) coefficients. Its combinatorial description based on the study of the projective representation is given by Stembridge in \([23]\). Another one in terms of SSDT’s is given by Cho in \([2]\). A description of expansion coefficients of a \(P\)-function in terms of Schur functions is also given in \([23]\). This coefficient also appears in an expansion of “big” Schur function in terms of \(Q\)-functions. Thus, the expansion coefficient of a product of Schur’s symmetric functions in terms of Schur functions or \(P\)-functions can be calculated by successive applications of the above-mentioned expansion coefficients.

The purpose of this paper is three-fold. First, we introduce shifted analogue of a Yamanouchi word (shifted Yamanouchi word for short) and a new concept called semistandard increasing decomposition tableau (SSIDT). We show the shifted and mixed insertion algorithms are characterized by Yamanouchi words and shifted Yamanouchi words, respectively. We construct a SSIDT from a SSDT. By construction, we have a bijection between a SSDT and a SSIDT, which implies that a SSDT is bijective to a shifted tableau. A word obtained from an ordinary Young tableau is a concatenation of weakly increasing sequences. Since a SSDT is a concatenation of hook words and its shape is a strict partition \(\lambda\), it can be viewed as a shifted analogue of an ordinary Young tableau. On the other hand, a SSIDT is expressed as a concatenation of weakly increasing sequences. The shape of a SSIDT is not \(\lambda\) but characterized by \(\lambda\). From these properties, a SSIDT is also viewed as another shifted analogue of an ordinary Young tableau. We have two types of SSIDT for a given strict partition \(\lambda\), which have the shapes \(\epsilon^+ (\lambda)\) and \(\epsilon^- (\lambda)\) (see Section 3 for definition). By construction, a SSIDT of shape \(\epsilon^\pm (\lambda)\) is a bijective to a SSDT of shape \(\lambda\). There is also a one-to-one correspondence between SSIDT’s of shape \(\epsilon^\pm (\lambda)\) and \(\epsilon^- (\lambda)\).

Secondly, we survey generalized Littlewood–Richardson coefficients. We have four types of symmetric functions: a Schur function \(s_\alpha\), a “big” Schur function \(\hat{S}_\beta\), a \(P\)-function \(P_\lambda\) and a \(Q\)-function \(Q_\lambda\) where \(\alpha\) and \(\beta\) are ordinary partitions and \(\lambda\) is a strict partition. A Schur \(Q\)-function \(Q_\lambda\) is related to the \(P\)-function by \(Q_\lambda = 2^{\ell(\lambda)} P_\lambda\) where \(\ell(\lambda)\) is the length of \(\lambda\). Roughly speaking, \(Q_\lambda\) contains the same information as \(P_\lambda\). We consider a product of these three functions and expand it in terms of Schur functions \(s_\alpha\) or \(P\)-functions \(P_\lambda\) (see Section 4 for definition). We call these expansion coefficients generalized Littlewood–Richardson coefficients. Generalized Littlewood–Richardson coefficients can be essentially calculated by using LR coefficients and LRS coefficients successively. However, this method is not efficient. We propose simple combinatorial descriptions of generalized Littlewood–Richardson coefficients in terms of tableau words. The proofs are elementary but combinatorial. We make use of insertion algorithms introduced in Section 2. We also give alternative combinatorial descriptions for the Littlewood–Richardson–Stembridge coefficients, one of which is in terms of SSIDT (see Theorem 4.33).

Finally, we study a relation between a “big” Schur function \(\hat{S}_\alpha\) and Schur \(P\)-functions. A Schur function \(\hat{S}_\alpha\) can be expressed as a sum of products of two Schur \(P\)-functions. This expansion has a remarkable property: the expansion coefficient is either 1 or \(-1\) except an overall factor, which means that it is multiplicity free. By inverting this relation, a product of two \(P\)-functions can be expanded in terms of \(\hat{S}\)-functions. Again, the expansion coefficient is 1 or \(-1\) except an overall factor. Similarly, a skew “big” Schur function \(\hat{S}_{\alpha/\beta}\) can be also expanded in terms of products of two skew \(P\)-functions. When two partitions \(\alpha\) and \(\beta\) are shift-symmetric, we recover the result by Józefiak and Pragacz \([7]\). We also give an expression of \(\hat{S}\)-functions in terms of perfect matchings. Recall that Schur function \(s_\alpha\) and \(P\)-function \(P_\lambda\) have Giambelli formulae: \(s_\alpha\) is determinant and
is devoted for a strict partition \( \lambda \) containing \( i \), i.e., the number of parts, \( i \) the main diagonal (see the right picture in Fig. 2.1). The integer \( n = \lambda/\mu \) shift-symmetric tableau is a set of boxes indexed \((i,i)\) such that \( \sum_{i=1}^{l} \lambda_{i} = n \). The \( \lambda_{i} \) are called the parts of the partition. The integer \( n \) is called the size of \( \lambda \) and denoted \(|\lambda|\). The length \( l(\lambda) \) of a partition \( \lambda \) is defined as the number of parts, i.e., \( l(\lambda) := l \). The Young diagram of \( \lambda \) is an array of boxes with \( \lambda_{i} \) boxes in the \( i \)-th row. The skew Young diagram \( \lambda/\mu \) is obtained by removing a Young diagram \( \mu \) from \( \lambda \) containing \( \mu \). We denote by \( \alpha^{t} \) the conjugate of a partition \( \alpha \), i.e., \( \alpha_{i}^{t} = \#\{j|\alpha_{j} \geq i\} \).

A strict partition \( \lambda = (\lambda_{1}, \ldots, \lambda_{l}) \) is a partition with all parts distinct, i.e., \( \lambda_{1} > \lambda_{2} > \ldots > \lambda_{l} \). For a strict partition \( \lambda \), the shifted diagram or shifted shape of \( \lambda \) is an array of boxes where the \( i \)-th row has \( \lambda_{i} \) boxes and is shifted \( i - 1 \) steps to the right with respect to the first row. The skew shifted diagram \( \lambda/\mu \) is obtained by removing a shifted diagram \( \mu \) from \( \lambda \) containing \( \mu \). The main diagonal of a skew shifted diagram is a set of boxes indexed \((i,i)\).

Let \( \lambda \) and \( \mu \) be a strict partition satisfying \( l(\lambda) = l(\mu) \) or \( l(\lambda) = l(\mu) + 1 \). We construct an ordinary Young diagram \( \alpha \) from \( \lambda \) and \( \mu \) by transposing \( \lambda \) and \( \mu \) along the main diagonal (see the right picture in Fig. 2.1). We denote the diagram by \( \alpha = \lambda \otimes \mu \). When \( \lambda = \mu \), i.e., \( \alpha = \lambda \otimes \lambda \), we call \( \alpha \) a shift-symmetric tableau.

![Figure 2.1](image)

A semistandard Young tableau \( T \) of shape \( \lambda \) is a filling of the shape \( \lambda \) with letters from the alphabet \( X = \{1 < 2 < 3 < \ldots \} \) such that

- each row is weakly increasing from left to right
- each column is increasing from top to bottom.

A skew semistandard Young tableau is defined analogously. The content of \( T \) is an integer sequence \( v := (v_{1}, v_{2}, \ldots) \), where \( v_{i} \) is the number of the letter \( i \) in \( T \). A semistandard tableaux is called standard when its content is \((1,1,\ldots,1)\). We denote the set of semistandard tableaux of shape \( \lambda \) by SSYT(\( \lambda \)).
A semistandard shifted Young tableau $T$ of (ordinary or shifted) shape $\lambda$ is a filling of the shifted shape $\lambda$ with letters from the marked alphabet $X' = \{1' < 1 < 2' < 2 < \ldots\}$ such that
- rows and columns of $T$ are weakly increasing
- each letter $i$ appears at most once in every column
- each letter $i'$ appears at most once in every row
- no primed letter on the main diagonal.

A skew shifted Young tableau is defined analogously. The content of $T$ is an integer sequence $v := (v_1, v_2, \ldots)$, where $v_i$ is the number of the letter $i$ and $i'$ in $T$. A semistandard shifted tableau called marked standard when its content is $(1, 1, \ldots, 1)$ and the letter is possibly primed once. A marked standard tableau is called standard when it is a marked standard tableau without primed letters. We denote the set of semistandard shifted tableau of shape $\lambda$ by $\text{SSShYT}(\lambda)$.

For an ordinary or shifted tableau $T$ with content $v = (v_1, v_2, \ldots)$, we denote the monomial by $x^T := x_1^{v_1}x_2^{v_2} \cdots$. For a strict partition $\lambda$, the Schur $P$- and $Q$-functions are defined as $P_\lambda := P_\lambda(X) = \sum_{T \in \text{SSShYT}(\lambda)} x^T,$ $Q_\lambda := Q_\lambda(X) = 2^{l(\lambda)}P_\lambda.$

The skew Schur $P$- and $Q$-functions $P_{\lambda/\mu}$ and $Q_{\lambda/\mu} := 2^{l(\lambda) - l(\mu)}P_{\lambda/\mu}$ are defined similarly.

2.2. Littlewood–Richardson coefficients. We define the Littlewood–Richardson coefficient $a^\gamma_{\alpha\beta}$ as $s_\alpha s_\beta = \sum_\gamma a^\gamma_{\alpha\beta} s_\gamma,$ that is, $a^\gamma_{\alpha\beta}$ is the multiplicity of $s_\gamma$ in the product of two Schur functions $s_\alpha$ and $s_\beta$.

The LR coefficient $a^\gamma_{\alpha\beta}$ is expressed in terms of Yamanouchi words [15]. We have $a^\gamma_{\alpha\beta} = \#\{T \in T(\alpha/\beta; \gamma) \mid \text{read}(T) \text{ is a Yamanouchi word}\}.$

The coefficients $a^\gamma_{\alpha\beta}$ also appear in the case of $\hat{S}$-functions (see e.g. [16]):

\begin{equation}
\hat{S}_\alpha \hat{S}_\beta = \sum_\gamma a^\gamma_{\alpha\beta} \hat{S}_\gamma.
\end{equation}

Let $\lambda, \mu$ and $\nu$ be strict partitions satisfying $\lambda, \mu \subseteq \nu$. Similarly, a Littlewood–Richardson–Stembridge coefficient $d^\nu_{\lambda\mu}$ is defined in terms of Schur $P$-function as $P_\lambda P_\mu = \sum_\nu d^\nu_{\lambda\mu} P_\nu.$
Theorem 2.2 (Stembridge [23]). We have
\[ d^\lambda_{\mu\nu} = \# \{ T \in T'(\lambda/\mu; \nu) \mid \text{read}(T) \text{ is an LRS word} \} \].

2.3. Skew functions. The skew Schur functions are expressed in terms of Littlewood–Richardson coefficients as
\[ s_{\alpha/\beta} = \sum_{\gamma} a_{\alpha\beta\gamma} s_\gamma, \]
\[ Q_{\lambda/\mu} = \sum_{\nu} d_{\mu\nu}^\lambda Q_\nu, \]
\[ \hat{S}_{\alpha/\beta} = \sum_{\gamma} a_{\alpha\beta\gamma} \hat{S}_\gamma. \]
The skew P-function is given by
\[ P_{\lambda/\mu} = 2^{l(\mu) - l(\lambda)} Q_{\lambda/\mu}. \]

2.4. Basic properties of Schur functions. We summarize properties of Schur s-functions and P-functions. The reader is referred to [16] for detailed definitions and proofs.

Let \( \Lambda := \bigoplus_{n \geq 0} \Lambda^n \) be the graded ring of symmetric functions in the variables \( x_1, x_2, \ldots \) with coefficients in \( \mathbb{Z} \). The power-sum symmetric function \( p_r, r > 0, \) is defined by
\[ p_r := x_1^r + x_2^r + \cdots, \]
and we denote \( p_\lambda := p_{\lambda_1} p_{\lambda_2} \cdots \) for a partition \( \lambda \). The set \( \{ p_\lambda \mid |\lambda| = n \} \) forms a basis of \( \Lambda^n \). Another basis of the ring \( \Lambda \) is Schur functions \( s_\lambda \). We have \( \{ s_\lambda \mid |\lambda| = n \} \) is a \( \mathbb{Z} \)-bases of \( \Lambda^n \). We define the bilinear form \( \langle \cdot, \cdot \rangle \) on \( \Lambda \) via the generating function
\[ \prod_{i,j} \frac{1}{1 - x_i y_j} = \sum_{\lambda} s_\lambda(x)s_\lambda(y). \]
Thus Schur functions are orthonormal bases of \( \Lambda^n \), namely
\[ \langle s_\lambda, s_\mu \rangle = \delta_{\lambda\mu}. \]

Let \( \Omega_Q = \bigoplus_{n \geq 0} \Omega^n_Q \) be the graded subalgebra of \( \Lambda_Q \) generated by \( \{ p_{2r-1} \mid r \geq 1 \} \) and let \( \Omega = \Omega_Q \cap \Lambda \) denote the \( \mathbb{Z} \)-coefficient graded subring of \( \Omega_Q \). Let \( OP_n \) denote the partitions of \( n \) with odd parts. Then, the set \( \{ p_\lambda \mid \lambda \in OP_n \} \) forms a basis of \( \Omega^n_Q \). We define an inner product \( [\cdot, \cdot] \) on \( \Omega^n_Q \) by the generating function
\[ \prod_{i,j} \frac{1 + x_i y_j}{1 - x_i y_j} = \sum_{\lambda \in OP_n} \frac{1}{z_\lambda} 2^{l(\lambda)} p_\lambda(x)p_\lambda(y) \]
\[ = \sum_{\lambda \in DP} Q_\lambda(x) P_\lambda(y), \]
where \( DP \) denote the set of distinct partitions. where \( z_\lambda = \prod_{i \geq 1} i^{m_i} m_i! \) for \( \lambda = (1^{m_1}, 2^{m_2}, \cdots) \). Thus, we have
\[ [p_\lambda, p_\mu] = z_\lambda 2^{-l(\lambda)} \delta_{\lambda\mu} \quad \text{for } \lambda, \mu \in OP_n, \]
\[ [P_\lambda, Q_\mu] = \delta_{\lambda\mu} \quad \text{for } \lambda, \mu \in DP. \]
The bilinear form \( \langle \cdot, \cdot \rangle \) and \( [\cdot, \cdot] \) are related by Proposition 9.1 in [23]:
\[ \langle s_\alpha, P_\lambda \rangle = [S_\alpha, P_\lambda], \]
where \( \alpha \) is an ordinary partition.
Define symmetric functions $q_n \in \Lambda^n$ by the generating function
\[\sum_{n \geq 0} q_n t^n = \prod_i \frac{1 + x_i t}{1 - x_i t}.\]

A $Q$-function $Q_\lambda$ with $l(\lambda) = 2$ can be expressed in terms of $q_n$, namely we have
\[Q_{(r,s)} = q_r q_s + 2 \sum_{i=1}^s (-1)^i q_{r+i} q_{s-i}.\]

The skew Schur $\hat{S}$-function has a determinant expression:
\[\hat{S}_{\alpha/\beta} = \det \left[ q_{\lambda-i-j} \right]_{1 \leq i,j \leq t},\]
where $q_{-r} = 0$ for $r > 0$ and $l := l(\alpha)$.

Let $S_{2n}$ be the symmetric group of order $2n$ and $F_n$ be the subset
\[F_n := \left\{ \rho \in S_{2n} \left| \begin{array}{l} \rho(1) < \rho(3) < \cdots < \rho(2n-1), \\ \rho(2i-1) < \rho(2i) \ (1 \leq i \leq n) \end{array} \right. \right\}.\]

For a skew symmetric matrix $A := (a_{ij})$ of order $2n$, we define the Pfaffian of $A$ as
\[\text{pf}[A] := \sum_{\rho \in F_n} \epsilon(\rho) \prod_{i=1}^n a_{\rho(2i-1)\rho(2i)},\]
where $\epsilon(\rho)$ is the sign of the permutation $\rho$.

Let $\tilde{l}(\lambda) := l(\lambda) \equiv 0 \ (\text{mod } 2)$ and $\tilde{l}(\lambda) := l(\lambda) + 1$ for $l(\lambda) \equiv 1 \ (\text{mod } 2)$. We denote $P[\lambda/\mu] := P_{\lambda/\mu}$. We define a skew-symmetric matrix $P_{i,j}(\lambda, \mu)$ as
\[P_{i,j}(\lambda, \mu) := \begin{cases} P[\lambda_i, \lambda_j], & 1 \leq i < j \leq \tilde{l}(\lambda), \\ P[\lambda_i - \mu(l(\lambda) + \tilde{l}(\mu) + j + 1)], & 1 \leq i \leq \tilde{l}(\lambda), \tilde{l}(\lambda) + 1 \leq j \leq \tilde{l}(\lambda) + \tilde{l}(\mu), \\ 0, & \text{otherwise,} \end{cases}\]
for $1 \leq i < j \leq \tilde{l}(\lambda) + \tilde{l}(\mu)$. A skew $P$-function is expressed as (see e.g. [7])
\[P_{\lambda/\mu} = \text{pf}[P_{i,j}(\lambda, \mu)].\]

Especially, a (non-skew) $P$-function has a simple Pfaffian expression:
\[P_\lambda = \text{pf} \left[ P[\lambda_i, \lambda_j] \right]_{1 \leq i,j \leq l(\lambda)},\]
where $\lambda_{l+1} = 0$ for $l$ odd.

2.5. Insertion algorithms. We introduce four insertion algorithms used in this paper. We start with the Robinson–Schensted–Knuth (RSK) correspondence.

**Definition 2.3 (RSK insertion [20]).** For a word $w = w_1 \cdots w_n$ in the alphabet $X$, we recursively define a sequence of tableaux $(T_0, U_0) := (\emptyset, \emptyset), (T_1, U_1), \ldots, (T_n, U_n) = (T, U)$. For $1 \leq i \leq n$, we insert $w_i$ into $T_{i-1}$ as follows.

We start with $z := w_i$, $S := T_{i-1}$ and $p = 1$.

(1) Insert $z$ into the $p$-th row of $S$, bumping out the smallest letter $a \in X$ which is strictly greater than $z$. Let new $S$ be the tableau where $a$ is replaced by $z$ in $S$. We define new $z := a$ and $p \mapsto p + 1$, and go to (1).
The insertion algorithm stops when a letter is placed at the end of a row.

A tableau $U_i$ is obtained from $U_{i-1}$ by adding a box with content $i$ on the same location as a new box added to $T_{i-1}$ to obtain $T_i$.

We call $T$ the insertion tableau and $U$ the recording tableau. When a word $w$ has a pair $(T, U)$, we denote $w = \text{RSK}^{-1}(T, U)$.

A shifted mixed insertion was introduced by Haiman for the correspondence between permutations and pairs of shifted Young tableaux. This mixed insertion can be viewed as a shifted analogue of the Robinson–Schensted–Knuth correspondence. A semistandard generalization of the mixed insertion of Haiman was given by Serrano. This extended insertion gives a correspondence between words in the alphabet $X$ and pairs of semistandard shifted and standard shifted Young tableaux.

**Definition 2.4** (Mixed insertion [6]). For a word $w = w_1 \cdots w_n$ in the alphabet $X$, we recursively define a sequence of tableaux, $(T_0, U_0) := (\emptyset, \emptyset), (T_1, U_1), \ldots, (T_n, U_n) = (T, U)$. For $1 \leq i \leq n$, we insert $w_i$ into $T_{i-1}$ as follows.

We start with $z := w_1$, $S := T_{i-1}$ and $(p, q) := (1, 1)$.

(★) Insert $z$ into the $p$-th row of $S$, bumping out the smallest letter $a \in X'$ which is strictly greater than $z$. Let new $S$ be the tableau where $a$ is replaced by $z$ in $S$ and $(p, q)$ be the position of $a$ in old $S$.

(1) If $a$ is not on the main diagonal, then

(a) if $a$ is unprimed, then we insert $a$ into the $p + 1$-th row. We bump out the smallest element $b$ which is strictly greater than $a$. We set $a := b$, $(p, q)$ be the position of $b$ in $S$ and go to (★).

(b) if $a$ is primed, then we insert $a$ into the $q + 1$-th column to the right. We bump out the smallest element $b$ which is strictly greater than $a$. We set $a := b$, $(p, q)$ be the position of $b$ in $S$ and go to (1).

(2) If $a$ is on the main diagonal ($a$ is unprimed), then we prime it and insert it into the $q + 1$-th column to the right. We bump out the smallest element $b$ which is strictly greater than $a$. We set $a := b$, $(p, q)$ be the position of $b$ in $S$ and go to (1).

The insertion algorithm stops when a letter is placed at the end of a row or a column.

A tableau $U_i$ is obtained by adding a box with content $i$ on the same location as a new box added to $T_{i-1}$ to obtain $T_i$.

We call $T$ the mixed insertion tableau and $U$ the mixed recording tableau and denote them $P_{\text{mix}}(w)$ and $Q_{\text{mix}}(w)$, respectively. When a word $w$ has a pair $(T, U)$ by the mixed insertion, we denote $w = \text{mixRSK}^{-1}(T, U)$.

In case of ordinary tableaux, Knuth has proved that two words $u$ and $v$ in $X$ have the same RSK insertion tableau if and only if $u$ and $v$ are equivalent modulo the plactic relations [8]. The following theorem is a shifted analogue of the plactic relations for the mixed insertion.

**Theorem 2.5** (shifted plactic relations [22]). Two words have the same mixed insertion tableau if and only if they are equivalent modulo the following shifted plactic relations:

\begin{align*}
(2.7) & \quad abdc \sim adbc \quad \text{for } a \leq b \leq c < d, \\
(2.8) & \quad acdb \sim acbd \quad \text{for } a \leq b < c \leq d, \\
(2.9) & \quad daeb \sim adeb \quad \text{for } a \leq b < c < d, \\
(2.10) & \quad bdac \sim bdac \quad \text{for } a < b \leq c < d,
\end{align*}
There is another insertion process for the correspondence between permutations and pairs of standard and marked standard shifted tableaux. We call this insertion process shifted insertion. We define the shifted insertion following [18].

**Definition 2.6** (Shifted insertion). For a word \( w = w_1 \ldots w_n \) in the alphabets \( X \), we recursively define a sequence of shifted tableaux \( (T_0, U_0) := (0, \emptyset), (T_1, U_1), \ldots, (T_n, U_n) = (T, U) \). For \( 1 \leq i \leq n \), we insert a letter \( w_i \) into \( T_{i-1} \) as follows:

We start with \( z := w_i \), \( S := T_{i-1} \), \( (p, q) := (1, 1) \) and \( C := \text{Schensted} \).

1. If \( C = \text{Schensted} \), then we insert \( z \) into the \( p \)-th row of \( S \), bumping out the smallest letter \( a \in X \) which is strictly greater than \( z \). Let (new) \( S \) be the tableau where \( a \) is replaced by \( z \) in \( S \). If \( a \) is on the main diagonal, we set \( z = a, q \mapsto q + 1 \) and \( C = \text{non-Schensted} \) and go to (2). Otherwise, we set \( z = a, p \mapsto p + 1 \) and go to (1).

2. If \( C = \text{non-Schensted} \), then we insert \( z \) into the \( q \)-th column of \( S \), bumping out the smallest letter \( a \in X \) which is greater than or equal to \( z \). Let (new) \( S \) be the tableau where \( a \) is replaced by \( z \) in \( S \). We set \( z = a, q \mapsto q + 1 \) and go to (2).

The insertion process stops when a letter is placed at the end of a row or a column. If \( C = \text{Schensted} \) (resp. \( C = \text{non-Schensted} \)) when the insertion process stops, we call the process Schensted (resp. non-Schensted) move.

A tableau \( U_i \) is obtained by adding a box with content \( i \) on the same location as a new box added to \( T_{i-1} \) to obtain \( T_i \). If the insertion is non-Schensted move, we put a prime on \( i \).

We call \( T \) the shifted insertion tableau and \( U \) the shifted recording tableau and denote them \( P_{\text{shift}}(w) \) and \( Q_{\text{shift}}(w) \).

For a permutation \( \pi \in S_n \), the mixed insertion and the shifted insertion are related by Theorem 6.10 in [6]:

\[
(P_{\text{mix}}(w), Q_{\text{mix}}(w)) = (Q_{\text{shift}}(w^{-1}), P_{\text{shift}}(w^{-1})).
\]

We introduce the notion of semistandard decomposition tableaux in the following.

A word \( w = w_1 \ldots w_l \) on \( X \) is called a hook word if there exists \( 1 \leq m \leq l \) such that

\[
w_1 > w_2 > \cdots > w_m \leq w_{m+1} \leq \cdots \leq w_l.
\]

We denote by \( (w \downarrow) \) the subword \( w_1w_2 \cdots w_m \) of \( w \) and by \( (w \uparrow) \) the subword \( w_{m+1} \cdots w_l \).

**Definition 2.7** (Semistandard Kraśkiewicz (SK) insertion). For a given hook word \( w = (w \downarrow) * (w \uparrow) \) with \( (w \downarrow) = w_1 \cdots w_m \) and \( (w \uparrow) = w_{m+1} \cdots w_l \) and a letter \( x \in X \), the insertion of \( x \) into \( w \) is the word \( wx \) if \( wx \) is a hook word, or the \( w' \) with an element \( u \) which is bumped out in the following way:

1. let \( y_j \) be the leftmost element in \( (w \uparrow) \) which is strictly greater than \( x \)
2. replace \( y_j \) by \( x \)
3. let \( y_i \) be the leftmost element in \( (w \downarrow) \) which is less than or equal to \( y_j \)
4. replace \( y_i \) by \( y_j \) and bump out \( u := y_i \). A word \( w' \) is a word obtained from \( w \) by replacing \( y_i \) and \( y_j \) in \( w \) by \( y_j \) and \( x \).
The insertion of $x$ into a SSDT $T$ with rows $w_1, \ldots, w_l$ is defined as follows. First, we insert $x$ into $w_1$ and bumps out $x_1$. Then, we insert $x_1$ into the second row $w_2$. We continue this process until an element $x_j$ is placed at the end of the row $w_l$.

The **SK insertion tableau** of a word $w = w_1 \cdots w_n$ is obtained from the empty tableau by inserting the letters $w_1, \ldots, w_n$. At each steps, we obtain a SSDT. We denote by $P_{SK}(w)$ the SK insertion tableau for a word $w$. The **SK recording tableau** is the standard shifted Young tableau obtained by adding a box with content $i$ on the same location as a new box added to $P_{SK}(w_1 \cdots w_{l-1})$ to obtain $P_{SK}(w_1 \cdots w_l)$.

### 2.6. Tableau words

Let $T$ be an ordinary or shifted tableau. The **reading word** $read(T)$ is obtained by reading the contents of $T$ from the bottom row to the top row and in each row from left to right.

For any letter $i$ in $X'$, we set $(i')' = i$ and $(i')' = i'$. Let $w = w_1 \cdots w_n$ be a reading word of $T$ in the alphabet $X'$. We write $w' = w_n' \cdots w_1'$. The **weak reading word** $read(T)$ is obtained by reading unprimed letters of the word $w'$.

For a word $w := w_1 \cdots w_n$, we define a reversed word $rev(w) := w_n \cdots w_1$.

Given two words $v = v_1 \cdots v_m$ and $w = w_1 \cdots w_n$, we define the concatenation of $v$ and $w$ as $v \ast w := v_1 \cdots v_m w_1 \cdots w_n$.

Let $w = w_1 \cdots w_n$ be a word in the alphabet $X$. A word with **lattice property** is a word satisfying $\# \{ w_q : w_q = i, 1 \leq q \leq p \} \geq \# \{ w_q : w_q = i+1, 1 \leq q \leq p \}$ for all $i$ and $1 \leq p \leq n$. In other words, the occurrence of $i$ in $w_1 \cdots w_p$ is greater than or equal to the occurrence of $i+1$ in $w_1 \cdots w_p$. A **Yamanouchi** word $w$ is a word such that its reversal word satisfies the lattice property.

We say that a word $w$ is a **weak Yamanouchi** word if weak reading word is a Yamanouchi word.

We follow [23] to define a Littlewood–Richardson–Stembridge (LRS) word. Let $w := w_1w_2 \cdots w_n$ be a word over the alphabet $X'$. We define $m_i(j)$ ($0 \leq j \leq 2n, i \geq 1$) depending on $w$ as

$$
m_i(j) := \text{number of } i \text{ in } w_{n-j+1}, \ldots, w_n \quad \text{for } 0 \leq j \leq n
$$

$$
m_i(n+j) = m_i(n) + \text{number of } i' \text{ in } w_1, \ldots, w_j \quad \text{for } 0 < j \leq n.
$$

The word $w$ is said to satisfy the **lattice property** if $m_i(j) = m_{i-1}(j)$, we have

$$
w_{n-j} \neq i, i' \quad \text{if } 0 \leq j < n,
$$

$$
w_{j-n+1} \neq i-1, i' \quad \text{if } n \leq j < 2n.
$$

A word $w$ is said to be an LRS word if it satisfies (i) $w$ satisfies the lattice property (ii) the first occurrence of a letter $i$ or $i'$ in $w$ is $i$.

The condition (i) for an LRS word can be rephrased as follows. For any letter $i$ in $X'$, we set $\hat{i} = (i+1)'$ and $\hat{\beta} = i$. Let $w = w_1 \cdots w_n$ be a word in the alphabet $X'$. We write $\hat{w} = \hat{w}_n \cdots \hat{w}_1$. The concatenated word of $\hat{w}w$ is a Yamanouchi word, that is, every letters $i$ or $i'$ is preceded by more occurrence of $i-1$ than that of $i$ in $\hat{w}w$ from right to left.

Let $w$ be a word in the alphabet $X$ and $max(w)$ be the maximum letter in $w$. We recursively define a strictly increasing sequence of length $r$, $seq(r) := (seq(r,i)) \mid 1 \leq i \leq r$, starting from $r = max(w)$ to $r = 1$ as follows. A number $seq(r,1)$ is the position (from left end) of the leftmost $r$ in $w$. A number $seq(r,i+1)$, $1 \leq i \leq r-1$, is the position (from left end) of the leftmost $r-i$ which is right to the position $seq(r, i)$. We delete the letters appearing in $seq(r)$ from $w$ and construct $seq(r-1)$ from the remaining letters in $w$ by the same procedure as above. We say a word $w$ is **shifted Yamanouchi** word if (i) $w$ is a Yamanouchi word, (ii) there exists at least one $i-1$ to the left
of the leftmost \( i \), and (iii) A sequence of positive integers \( \text{seq}(r, r) \) for \( 1 \leq r \leq \max(w) \) is strictly increasing, i.e., \( \text{seq}(r, r) < \text{seq}(r + 1, r + 1) \).

The shifted insertion and the mixed insertion are characterized by Yamanouchi words and shifted Yamanouchi words.

**Proposition 2.8.** Let \( w \) be a word of content \( \lambda \). Then, we have

\[
\begin{align*}
\text{(2.16)} & \quad \#\{w| \text{shape}(P_{\text{shift}}(w)) = \lambda\} = \#\{w| w \text{ is a Yamanouchi word}\}, \\
\text{(2.17)} & \quad \#\{w| \text{shape}(P_{\text{mix}}(w)) = \lambda\} = \#\{w| w \text{ is a shifted Yamanouchi word}\}.
\end{align*}
\]

**Proof.** We first show the right term of Eqn.\((2.16)\) implies the left term. Suppose that the word \( w \) is written as a concatenation of subwords \( w = w' * y * w'' \). Since \( w \) is a Yamanouchi word, the number of \( x \) satisfying \( x < y \) is strictly greater than that of \( y \) in \( w'' \). The insertion of \( y \) into \( P_{\text{shift}}(w') \) results in a tableau such that \( y \) is in the first row. The Yamanouchi property ensures that \( y \) is bumped out by a letter \( y - 1 \) and \( y \) is inserted into the second row, a letter \( y - 2 \) bumps out these \( y - 1 \) and \( y \) and insert them into the next rows, and finally a letter 1 bumps out letters from 2 to \( y \) which are inserted into the next rows. As a result, the letter \( i, 1 \leq i \leq l(\lambda) \), is in the \( i \)-th row of a tableau \( P_{\text{shift}}(w) \). This is nothing but \( \text{shape}(P_{\text{shift}}(w)) = \lambda \). Further, if \( w \) is not a Yamanouchi word, it is easy to see that \( \text{shape}(P_{\text{shift}}(w)) \neq \lambda \). Thus, Eqn.\((2.16)\) is true.

For Eqn.\((2.17)\), observe that \( P_{\text{mix}}(w) \) does not have a primed entry. Suppose that \( w \) is a shifted Yamanouchi word of content \( \lambda \). From the properties (ii) of a shifted Yamanouchi word, at least one \( x - 1 \) appears left to \( x \) in \( w \). The property (iii) implies that, in the word \( w \), there are at least one \( x, x < y \), in-between the \( i \)-th leftmost letter \( y \) and the \( (i + 1) \)-th leftmost letter \( y \) for \( 1 \leq i \leq \max(w) - y \). Suppose \( w = w' * y * v * v' \). We insert \( v' \) into a tableau \( T = P_{\text{mix}}(w' * y * v) \). The above constraint implies that if \( y \) is in the \( j \)-th row of \( T \), bumped out by \( x \) and inserted into the \((j + 1)\)-th row, there are at least one \( z, j \leq z \leq y - 1 \), in the \((j + 1)\)-th row. Thus, a letter \( y \) cannot be placed on the main diagonal in the \( j \)-th row with \( 1 \leq j < y \). After the mixed insertion, \( P_{\text{mix}}(w) \) does not have a primed entry and the Yamanouchi property ensures that \( \text{shape}(P_{\text{mix}}(w)) = \lambda \). If \( w \) is not a shifted Yamanouchi word, we have at least one primed entries, or the shape of \( P_{\text{mix}}(w) \) is not equal to \( \lambda \). This completes the proof. \( \square \)

Let \( \lambda \) be a strict partition and \( w \) be a word of content \( \lambda \). Then, the left hand side of Eqn. \((2.17)\) is equal to the number of \( Q_{\text{mix}}(w) \), that is the number of standard shifted tableaux of shape \( \lambda \). Let \( \text{SShiTab}(\lambda) \) denote the set of standard shifted tableaux of shape \( \lambda \).

**Proposition 2.9.** We have

\[
|\text{SShiTab}(\lambda)| = \#\{w| w \text{ is a shifted Yamanouchi word of content } \lambda\}.
\]

3. **Semistandard increasing decomposition tableau**

For a strict partition \( \lambda = (\lambda_1, \lambda_2, \ldots, \lambda_l) \), we define two skew shapes \( \epsilon^+(\lambda) \) and \( \epsilon^-(\lambda) \) as follows. The skew shape \( \epsilon^+(\lambda) \) is obtained by transposing \( \lambda \) and shifting the \( i \)-th row \( l - i \) steps to the right with respect to the \( \lambda_1 \)-th row. Let \( \lambda' \) be the ordinary partition whose parts are the same as the ones of \( \lambda \). The skew shape \( \epsilon^-(\lambda) \) is obtained from \( \lambda' \) by rotating \( \lambda' \) 90 degrees in anti-clockwise direction and shifting the \( i \)-th row \( l - i \) steps to the right with respect to the \( \lambda_1 \)-th row. The shapes \( \epsilon^\pm(\lambda) \) have \( l(\lambda) \) anti-diagonals and its \( i \)-th anti-diagonal is of length \( \lambda_i \). See Figure 3.1 for an example.

We denote by \( \epsilon^\pm(T) \) a semistandard tableau of the shape \( \epsilon^\pm(\lambda) \) and corresponding to a shifted tableau \( T \). Especially, we consider a semistandard tableau \( \epsilon^\pm(T) \) such that the shape of \( T \) is \( \lambda \) and \( P_{\text{mix}}(\text{read}(\epsilon^\pm(T))) = T \). We call a semistandard tableau \( \epsilon^\pm(T) \) a semistandard increasing decomposition tableau.
decomposition tableau (SSIDT). A SSIDT can be seen as a shifted analogue of an ordinary Young tableau. A reading word \( w \) of \( \varepsilon^\pm(T) \) is decomposed into sequences \( w = u_1u_{l-1} \cdots u_l \) where \( l := l(\lambda) \), each word \( u_i \) is weakly increasing and \( w \) is compatible with the shape \( \varepsilon^\pm(\lambda) \).

In the rest of this section, we construct a SSIDT \( \varepsilon^\pm(T) \) corresponding to a shifted tableau \( T \). We first construct \( \varepsilon^-(T) \) from a SSDT. Then, by using \( \varepsilon^-(T) \), we construct a SSIDT \( \varepsilon^+(T) \).

### 3.1. Construction of \( \varepsilon^-(T) \)

Let \( w \) be a word in the alphabet \( X \). We denote by \( \text{SK}(w) \) the SSDT obtained by semistandard Kraskiewicz insertion and by \( \lambda \) the shape of \( \text{SK}(w) \). Let \( R_i, 1 \leq i \leq l(\lambda), \) be the \( i \)-th row of \( \text{SK}(w) \). By definition, \( R_i \) is a hook word of length \( \lambda_i \). Recall that \( (R_i \downarrow) \) is the decreasing part of \( R_i \). Then, we have

**Proposition 3.2** (Theorem 4.1 in [12]). The decreasing parts \( (R_i \downarrow), 1 \leq i \leq l(\lambda), \) form a strict partition.

Below, we construct the tableau \( \varepsilon^-(\lambda) \) from a SSDT \( \text{SK}(w) \).

We denote by \( \mu \) the strict partition formed by the decreasing parts of \( \text{SK}(w) \). From the construction, the shape \( \mu \) is of length \( l(\lambda) \) and its part \( \mu_i \) is smaller than or equal to \( \lambda_i \). We consider a up-right path from the leftmost box in the \( l(\lambda) \)-th row to the rightmost box in the first row. A path is inside of the shape \( \lambda \) and therefore the length of a path is \( \lambda_1 \). Let \( (x_1, y_1), 1 \leq i \leq \lambda_1, \) be the coordinate of a box in the up-right path. Here, the coordinate system is matrix notation, that is, \( x \) increases from top to bottom and \( y \) increases from left to right. The coordinate of the starting point is \( (x_1, y_1) := (l(\lambda), l(\lambda)) \). First, we move along the shape of \( \mu \), i.e., the \( i \)-th coordinate is \( (x_i, y_i) = (x_{i-1}, y_{i-1} + 1) \) if the box on \( (x_{i-1}, y_{i-1} + 1) \) is in \( \mu \) and \( (x_i, y_i) = (x_{i-1} - 1, y_{i-1}) \) if the box on \( (x_{i-1}, y_{i-1} + 1) \) is not in \( \mu \). If we arrive at \( (1, \mu_1) \), we move only right from \( (1, \mu_1) \) to \( (1, \lambda_1) \). We denote by \( p_1 \) the obtained path.

Three boxes on \( (x_{i-1}, y_{i-1}), (x_{i-1}, y_{i-1} + 1) \) and \( (x_{i-1} + 1, y_{i-1}) \) are in the path \( p_1 \) and the box on \( (x_{i-1} + 1, y_{i-1} + 1) \) is not in \( p_1 \). We say that a path \( p_1 \) has a corner at \( (x_{i-1}, y_{i-1}) \). We define a path \( p_2 \) from \( p_1 \) by bending the path \( p_1 \) at a corner as follows. Suppose that \( p_1 \) has a corner at \( z_i := (x_i, y_i) \). Then, it is obvious that a box \( z_{i+1} := (x_i + 1, y_i + 1) \) is not in \( p_1 \). If the content of the box on \( z_i \) in the SSDT \( \text{SK}(w) \) is greater than or equal to that of the box on \( z_{i+1} \), we obtain a new path \( p_2 \) by locally changing the path \( p_1 \) such that the path \( p_2 \) pass through \( z_{i+1} \) instead of \( z_i \). We call this local change of a path as a bending. We perform a bending at a corner of \( p_2 \) until no bending occurs at corners in the path any more. We denote by \( p \) the obtained up-right path in \( \lambda \). Let word\((p)\) be a word reading the content of \( \text{SK}(w) \) along the path \( p \) starting from the box on \( (x_1, y_1) \). Recall that the length of word\((p)\) is \( \lambda_1 \). We put the letters in word\((p)\) on the first anti-diagonal of the shape \( \varepsilon^-(\lambda) \) from bottom to top.

We remove the boxes on the path \( p \) from the SSDT \( \text{SK}(w) \) and denote by \( \text{SK}'(w) \) the tableau with removed boxes. We introduce the reverse SK insertion of type I to obtain a new SSDT \( \text{SK}(w') \) whose shape is \( (\lambda_2, \lambda_3, \ldots, \lambda_l) \). We construct \( \text{SK}(w') \) from \( \text{SK}(w) \) by the reverse insertion defined below.

**Figure 3.1.** The shapes \( \varepsilon^\pm(\lambda) \) for \( \lambda = (4, 2, 1) \).
Definition 3.3 (reverse SK insertion of type I). Let $R$ be a hook word with $(R \downarrow) := s_1 \ldots s_m$ and $(R \uparrow) := s_{m+1} \ldots s_i$. We reversely insert a letter $r \in X$ into $R$ in the following way:

1. let $s_i$ be the rightmost element in $(R \downarrow)$ which is greater than or equal to $r$
2. replace $s_i$ by $r$
3. we insert $s_i$ into $(R \uparrow)$ such that the obtained sequence is a weakly increasing one.

We denote by $R \leftarrow \{r_1, r_2, \ldots, r_l\}$ the successive reverse SK insertions of $r_1, r_2, \ldots, r_l$ into $R$.

Let $L_i$ (resp. $R_i$) be a word in the $i$-th row of SK$(w')$ and left (resp. right) to the path $p$. Suppose that $R_{i+1} := r_1 \ldots r_k$. The $i$-th row of SK$(w')$ is obtained by the following reverse SK insertion: $L_i \leftarrow \{r_k, r_{k-1}, \ldots, r_1\}$. By construction of a path $p$, the word $R_i$ is weakly increasing.

We construct a path $p'$ from SK$(w')$ and put the word word$(p)$ on the second anti-diagonal of $\epsilon^-(\lambda)$ from bottom to top. We repeat above procedures $l(\lambda)$ times and obtain the tableau $\epsilon^-(T)$ from SK$(w)$. See Example 3.13 below.

Theorem 3.4. The construction of $\epsilon^-(T)$ is well-defined, i.e., a word obtained from SK$(w)$ is compatible with the shape $\epsilon^-(\lambda)$. Further, the tableau word read$(\epsilon^-(T))$ produces the same mixed insertion tableau as SK$(w)$.

Before we move to a proof of Theorem 3.4, we introduce five lemmas needed later.

Lemma 3.5. Suppose that the word $w = w_1 \ldots w_{2l}$ satisfies

1. $w_1 \ldots w_l$ and $w_{l+1} \ldots w_{2l}$ are weakly increasing,
2. $w_i > w_{i+1}$ for $2 \leq i \leq l$,
3. $w_1 \leq w_{2l}$.

Then, $w \sim w'$ where $w' := w_1 \ldots w_l w_2 w_{l+1} \ldots w_{2l-1}$.

Proof. By using shifted plactic relations, we have

\[
\begin{align*}
w_1 \ldots w_l \& w_{l+1} \ldots w_{2l-1} w_{2l} \sim w_1 w_2 w_{l+1} w_3 w_{l+2} w_4 \ldots w_{2l-2} w_l w_{2l-1} w_{2l} & \text{by (2.7) and (2.13)} \\
& \sim w_1 w_2 w_{l+1} \& w_3 w_{l+2} w_4 \ldots w_{2l-2} w_l w_{2l-1} & \text{by (2.8)} \\
& \sim w_1 w_2 w_{l+1} \& w_3 w_{l+2} w_4 \ldots w_{l-1} w_l w_{2l-2} w_{2l-1} & \text{by (2.8) and (2.13)} \\
& \sim w_1 \ldots w_l w_{2l} w_{l+1} \ldots w_{2l-1} & \text{by (2.8) and (2.13)}. \\
\end{align*}
\]

We introduce the inverse of the SK insertion.

Definition 3.6 (reverse SK insertion of type II). Let $w$ be a hook word satisfying Eqn. (2.15). Define $(w \downarrow) := w_1 \ldots w_{m-1}$ and $(w \uparrow) := w_m \ldots w_l$. We reversely insert a letter $x \in X$ into $w$ and obtain a new word $w'$ with the elements $(y, z)$ as follows:

1. let $w_i$ be the rightmost element in $(w \downarrow)$ which is greater than or equal to $x$, and set $y := w_i$,
2. replace $w_i$ by $x$,
3. let $w_j$ be the rightmost element in $(w \uparrow)$ which is smaller than $w_i$,
4. replace $w_j$ by $w_i$ and bump out $w_j$. A word $w'$ is obtained from $w$ by replacing $w_i$ and $w_j$ by $x$ and $w_i$, and $z := w_j$.

If there exists no $w_j$ in the step (3), we insert $w_i$ into $(w \uparrow)$ such that the obtained sequence is weakly increasing.
Remark 3.7. We apply the reverse SK insertion of type II only when \( x \leq w_1 \) and \((w \downarrow) \neq \emptyset\). Otherwise, \( xw \) is a hook word and the length of \( xw \) is the length of \( w \) plus one. This means that \( w \) cannot be a row of a semistandard decomposition tableau.

The reverse SK insertion of type II is an inverse of the SK insertion. For a given SK recording tableau, one can obtain a word by applying the reverse SK insertion of type II to the corresponding element in the SK insertion tableau.

Let \( w := w_1 \cdots w_1 \) be a word such that \( w_i \) is a hook word of length \( \lambda_i \) and \( w \) satisfies \( w = \text{read}(SK(w)) \). Since the word \( \text{read}(SK(w)) \) gives the same SK insertion tableau as \( SK(w) \) itself, \( SK(w) \) is of shape \( \lambda \) if and only if a partial word \( w_{i+1}w_i \) form a tableau word of shape \((\lambda_i, \lambda_{i+1})\). Further, if \( \lambda_i > \lambda_{i+1} + 1 \), we have a SSDT of shape \((\lambda_{i+1} + 1, \lambda_{i+1})\) by deleting from the \((\lambda_{i+1} + 2)\)-th to \( \lambda_i \)-th elements in \( w_i \). Therefore, it is enough to give criteria for a SSDT of shape \((n, n - 1)\) to check whether a word \( w \) gives a SK insertion tableau of shape \( \lambda \). Let \( w := w_2w_1 \) and we denote \( w_1 := u_1 \cdots u_n \) and \( w_2 := v_1 \cdots v_{n-1} \). We construct two sequences of letters \( y := y_1 \cdots y_{n-1} \) and \( z := z_1 \cdots z_{n-1} \) by using reverse SK insertion of type II as follows. We start the process below with \( i = n \) and \( w'_i = w_1 \), decrease \( i \) one-by-one. By reverse SK insertion of type II, we insert \( v_{i-1} \) into \( w'_i \) and obtain a new word \( w''_{i-1} \) with elements \((y_{i-1}, z_{i-1})\). Since a word \( w''_{i-1} \) is of length \( i - 1 \), we delete the last element in \( w''_{i-1} \) and obtain a new word \( w'_{i-1} \). We continue the process until we obtain words \( y \) and \( z \) of length \( n - 1 \). We denote by \( w'_{i,j} \) an \( j \)-th element from left in a word \( w'_{i} \).

Lemma 3.8. If a word \( w := w_2w_1 \) produces a SSDT of shape \((n, n - 1)\) by the SK insertion, then

1. \( v_{i-1} \leq w'_{i,1} \) for \( 2 \leq i \leq n \),
2. \( y_{n-1} > w'_{n,n} \) and \((w \downarrow) \neq \emptyset\).
3. \((w'_i \downarrow) \neq \emptyset\) for \( 2 \leq i \leq n \).

Proof. We first show that \( v_{n-1} \leq u_1, y_{n-1} > w'_{n,n} \) and \((w \downarrow) \neq \emptyset\). Suppose that \( v_{n-1} > u_1 \). Since \( w_1 \) is a hook word, a word \( v_{n-1}w_1 \) is also a hook word of length \( n + 1 \). By the SK insertion, it is obvious that the shape of \( SK(w_1w_2) \) is \((n, n - 1)\). Thus, we have \( v_{n-1} \leq u_1 \). Since \( v_{n-1} \leq u_1 \), a word \( w_2u_1 \) is a hook word of length \( n \). Recall that the reverse SK insertion of type II is the inverse of the SK insertion. By definitions of these insertions, \( y_{n-1} \) is an element which is bumped out by the insertion of \( u_n \) into \( SK(w_2u_1 \cdots u_{n-1}) \). From the process (1) in Definition 2.7, we have \( y_{n-1} > u_n \). Suppose that \((w \downarrow) = \emptyset\). Then, \( w_1 = (w_1 \uparrow) \). Since \( v_{n-1} \leq u_1 \), \( w_1w_2 \) is a hook word of length \( 2n - 1 \). The SK insertion tableau of \( w_2w_1 \) does not have the shape \((n, n - 1)\), which is a contradiction to the assumption. Thus, we have \((w \downarrow) \neq \emptyset\).

It is easy to check that the reverse SK insertion of type II is an inverse of the SK insertion. This implies that the word \( v_1 \cdots v_{i-1}w''_{i,z_i} \) produces the same SK insertion tableau as \( v_1 \cdots v_{i-1}w'_{i+1} \). Note that a word \( w'_i \) is obtained from \( w''_{i} \) by deleting the last element. If \( P_{SK}(w) \) is of shape \((n, n - 1)\), then \( P_{SK}(v_1 \cdots v_{i-1}w'_i) \) has to be of shape \((i, i - 1)\). Combining this with the argument above, we obtain \( v_{i-1} < w'_{i,1}, y_{n-1} > w'_{i,i} \) and \((w'_i \downarrow) \neq \emptyset\). \( \square \)

Example 3.9. For example, let \( w := w_1w_2 \) with \( w_1 = 123 \) and \( w_2 = 5433 \). The shape of \( P_{SK}(w) \) is not \((4, 3)\), since we have a sequence of tableaux by the reverse SK insertion of type II:

\[
\begin{array}{cccc}
5 & 4 & 3 & 3 \\
1 & 2 & 3 & \Rightarrow \\
5 & 3 & 3 & 2 \leftarrow 4, 3 \\
1 & 2 & \leftarrow 5, 3, 4, 3
\end{array}
\]

The third tableau corresponding to the word 123 violates the condition (3) in Lemma 3.8. Note that, for example, the SK insertion of 43 to 533 produces a word 35433, and one can check that the reverse SK insertion of type II is the inverse of the SK insertion.
Let $w := w_1 \ldots w_2 w_1$ be the reading word of a SSDT of shape $\lambda$. The word $w_1$ is a hook word of length $\lambda_1$. We denote by $w_{i,j}$ the $j$-th element of $w_i$. A strict partition $\lambda'$ is obtained from $\lambda$ by the reverse SK insertion of type II, namely, insert $w_{2,\lambda_2}$ into a word $w_1$. From Lemma 3.8, if we insert $w_{i,\lambda_i}$ into $w_{i-1}$ by the reverse SK insertion of type II, then we bump out $w_{i-1,\lambda_{i-1}}$ and successively insert $w_{i-1,\lambda_{i-1}}$ into $w_{i-2}$. Let $p$ be the path of length $\lambda_1$ constructed from the SSDT SK($w$). By construction, the path $p$ contains the element $w_{1,\lambda_1}$. If $\lambda_1 > \lambda_2 + 1$, the path $p$ contains the elements $w_{1,j}$ with $\lambda_2 + 1 \leq j \leq \lambda_1$. Thus, without loss of generality, it is enough to consider the case where $\lambda_1 = \lambda_2 + 1$. We consider two cases: (a) the element $w_{2,\lambda_2}$ is in the path $p$, and (b) $w_{2,\lambda_2}$ is not in $p$.

**Lemma 3.10.** Suppose that the element $w_{2,\lambda_2}$ is in the path $p$. Then, we have $(w_1 \uparrow) = \emptyset$.

**Proof.** Suppose that $(w_1 \uparrow) \neq \emptyset$. From Proposition 3.2, decreasing parts $(w_1 \downarrow)$ form a strict partition $\mu$. We have two cases: (i) $\mu_1 > \mu_2 + 1$, and (ii) $\mu_1 = \mu_2 + 1$.

Case (i). The SSDT $w$ is locally given by
\[
\cdots a_0 \ a_1 \ a_2 \ \cdots \ a_{m_1} \ b_1 \ \cdots \ b_{m_2} \\
\cdots \ c \ d_1 \ \cdots \ d_{m_1-1} \ d_{m_1} \ \cdots \ d_{m_1+m_2-1}
\]
where $b_1 = \min\{w_1\}$, $c = \min\{w_2\}$, $b_{m_2} = w_{1,\lambda_1}$ and $d_{m_1+m_2-1} = w_{2,\lambda_2}$. Since $w_1$ and $w_2$ are hook words, we have
\[
a_1 > a_2 > \cdots > a_{m_1} > b_1 \leq \cdots \leq b_{m_2},
\]
\[
c \leq d_1 \leq \cdots \leq d_{m_1+m_2-1}.
\]
From the assumption, $b_{m_2}$ and $d_{m_1+m_2-1}$ are in the path $p$. By construction of a path, a path before bending is $ca_1 \cdots a_{m_1}b_1 \cdots b_{m_2}$. By bending the path, we have constraints on $a_i$ and $d_i$, namely $d_i \leq a_i$ for $1 \leq i \leq m_1$. Similarly, we have constraints on $b_i$ and $d_i$, that is, $d_{m_1+i} \leq b_i$ for $1 \leq i \leq m_2$. By inserting elements from $d_{m_1+m_2-1}$ to $d_{m_1+1}$ into the first row by the reverse SK insertion of type II, the obtained SSDT looks locally like
\[
a_0' \ a_1' \ a_2' \ \cdots \ a_{m_1}' \ b_1' \ \cdots \ b_{m_2}',
\]
where $a_0' = a_{m_1}$. If $a_{m_1} > d_{m_1+1}$, we have $a_j' = d_{m_1+1}$ for some $j \leq m_1$. Note that $d_1 \leq d_2 \leq \cdots \leq d_{m_1} \leq d_{m_1+1} \leq b_1$ and we insert a letter into $(w_1 \downarrow)$ rather than $(w_1 \downarrow)$ in case of the reverse SK insertion of type II. Therefore, if we insert the elements form $d_{m_1}$ to $d_1$ into the first row by the reverse SK insertion of type II, we have a increasing sequence $d_1, d_2, \ldots, d_{m_1+1}$ left to the position of $a_{m_1}$ in $w$. Thus, the minimum of the first row appears at the position $a_0'$ or left to $a_0'$. On the other hand, $c$ is the minimum in the second row, this contradicts to the fact that decreasing parts form a strict partition in a SSDT. Thus, we have $(w_1 \uparrow) = \emptyset$.

Case (ii). The SSDT $w$ looks locally like
\[
\cdots \ a_1 \ c_1 \ \cdots \ c_m \\
\cdots \ a_2 \ d_1 \ \cdots \ d_m
\]
where $a_1 = \min\{w_1\}$ and $a_2 = \min\{w_2\}$. Since the path $p$ includes the elements $c_m$ and $d_m$, bending gives the constraints: $a_1 \geq d_1$ and $c_i \geq d_{i+1}$ for $1 \leq i \leq m - 1$. If $d_m \leq a_1$, $d_m$ is placed left to $a_1$ by the reverse insertion of type II. Then, the element $d_m$ is the minimum of the first row and left to $a_2$. This contradicts to Proposition 3.2. If $d_m > a_1$, let $j$ be the maximal integer such that $d_j \leq a_1$. The integer $j$ exists since we have $d_1 \leq a_1$. By a similar argument to above, $d_j$ is placed left to $a_1$ by the reverse SK insertion of type II. This also contradicts to Proposition 3.2. Thus, we have $(w_1 \uparrow) = \emptyset$. \qed
Lemma 3.11. Suppose that the element $w_{2,\lambda_2}$ is in the path $p$. Then, we have $w_{1,\lambda_1-1} \geq w_{2,\lambda_2}$.

Proof. From Lemma 3.10, we have $(w_1 \uparrow) = \emptyset$. Let $w_{2,i}$ be the smallest element in $w_2$. Then, we have two cases: (a) $i \neq \lambda_2$ and (b) $i = \lambda_2$.

For case (a), let $b = w_{2,i}$. The SSDT $SK(w)$ looks locally like

$$\begin{array}{cccc}
\ldots & a_1 & a_2 & \ldots & a_n \\
\ldots & b & c_1 & \ldots & c_{n-1} \\
\end{array}$$

where $a_n = w_{1,\lambda_1}$, $c_{n-1} = w_{2,\lambda_2}$, $a_i > a_{i+1}$ for $1 \leq i \leq n-1$ and $c_i \leq c_{i+1}$ for $1 \leq i \leq n-2$. A path before bending is $ba_1 \cdots a_n$. Since $c_{n-1}$ is included in the path $p$, we have to bend the path at corners $a_i$ for $1 \leq i \leq n-1$. From the definition of bending, we have $a_i \geq c_i$ for $1 \leq i \leq n-1$. Thus we have $a_{n-1} = w_{1,\lambda_1-1} \geq w_{2,\lambda_2} = c_{n-1}$.

For case (b), suppose that $w_{1,\lambda_1-1} < w_{2,\lambda_2}$. We insert $w_{1,i}$ for $1 \leq i \leq \lambda_1$ into $w_2$ by the SK insertion. Since $w_1$ and $w_2$ are strictly decreasing sequences, the length of the first row after the insertion is greater than $\lambda_1$. This is a contradiction to the shape of $SK(w)$. Thus, we have $w_{1,\lambda_1-1} \geq w_{2,\lambda_2}$. \hfill $\square$

Suppose that we have $\lambda_i = \lambda_{i+1} + 1$. From Lemma 3.8, we have $w_{i+1,\lambda_{i+1}} \leq w_{i,1}$ and $(w_1 \downarrow) \neq \emptyset$. Let $y$ be the rightmost element in $(w_1 \downarrow)$ which is greater than or equal to $w_{i+1,\lambda_{i+1}}$.

Lemma 3.12. Suppose that $w_{i+1,\lambda_{i+1}}$ is not in the path $p$. Then, the element $y$ is not included in the path $p$.

Proof. Suppose that a letter $y$ is included in the path $p$. From the definition of $y$, we have $y \geq w_{i+1,\lambda_{i+1}}$. Since $y \in (w_1 \downarrow)$, there exists $z \geq y$ such that a path has a corner at $z$. A SSDT $SK(w)$ locally looks like as follows:

$$\begin{array}{cccc}
\ldots & z & \ldots & y & \ldots \\
\ldots & c_0 & c_1 & \ldots & c_m & w_{i+1,\lambda_{i+1}} \\
\end{array}$$

The elements $c_0z \cdots y \cdots$ form the path. By definition, the elements right to the path form a weakly increasing sequence, we have $c_1 \leq c_2 \leq \ldots \leq c_m \leq w_{i+1,\lambda_{i+1}}$. Since $p$ has a corner at $z$, we have $z < c_1$. From these observations, we have $y \leq z < c_1 \leq w_{i+1,\lambda_{i+1}}$, which is a contradiction to the assumption. Therefore, $y$ is not included in $p$. \hfill $\square$

Proof of Theorem 3.4. We prove Proposition by induction. Let $v := read(SK(w)) = v_1 \cdots v_{|\lambda|}$ and $v' := v_1 \cdots v_{|\lambda|-1}$. We denote by $\lambda$ the shape of $SK(v)$. We assume that Proposition is true for $SK(v')$. If $l(\lambda) = 1$, it is obvious that $read(\epsilon^-(T)) = read(SK(w))$. This implies the word is compatible with the shape $\epsilon^-(\lambda)$ and $read(\epsilon^-(T))$ has the same insertion tableau as the word $v$.

Suppose that $\lambda_1 > \lambda_2 + 1$. We denote by $\lambda'$ a strict partition obtained from $\lambda$ by deleting the rightmost box of the first row of $\lambda$. The shape of $SK(v')$ is given by $\lambda'$. From the construction of paths from $SK(v')$ and $SK(v)$, the element $v_{|\lambda|}$ is placed at the rightmost box of the first antidiagonal in $\epsilon^-(\lambda)$. This means that the word obtained from $SK(v)$ is compatible with the shape $\epsilon^-(\lambda)$. Since a word in $\epsilon^-(\lambda')$ has the same mixed insertion tableau $P_{mix}(v')$ as $SK(v')$ by assumption and $v_{|\lambda|}$ is the last element of the word in $\epsilon^-(\lambda)$, the word in $\epsilon^-(\lambda)$ and $v$ have the same mixed insertion tableau. Thus, Proposition holds true in this case.

Below, we assume $\lambda_1 = \lambda_2 + 1$ without loss of generality. Let $k$ be the maximal integer such that $\lambda_1 = \lambda_k + k - 1$ and $\lambda'$ be a strict partition obtained from $\lambda$ by replacing $\lambda_k$ with $\lambda_k - 1$. We denote by $v := read(SK(w))$ the reading tableau word of $SK(w)$. By reversing the SK insertion, we have a word $w'$ of length $|\lambda| - 1$ such that $SK(w')$ is of shape $\lambda'$ and there exists a letter $x$ satisfying
successively, we obtain a new
that the length of the first row increases after the insertion. This contradicts
The insertion of $v$ to the first row of $\text{SK}(w')$ results in $v'x$. This implies
the second rightmost elements of the first row in $\text{SK}(w')$. Hence $\lambda_1 = \lambda'_1$. Thus, we have $v'_{|\lambda|-1} > x$. Let $R_1$ be the first row of $\text{SK}(w')$. Suppose that a hook word $R_1$ does not have an increasing part, i.e., $\langle R_1 \uparrow \rangle = \emptyset$. If we insert $x$ into $R_1$, the word $R_1x$ is a hook word of length $\lambda'_1 + 1$. This contradicts $\lambda_1 = \lambda'_1$. Thus, a hook word $R_1$ has an increasing part, i.e., $\langle R_1 \uparrow \rangle \neq \emptyset$.

Let $\epsilon^{-}(T)$ (resp. $\epsilon^{-}(T')$) be a tableau word of shape $\epsilon^{-}(\lambda)$ (resp. $\epsilon^{-}(\lambda')$) corresponding to $\text{SK}(w)$ (resp. $\text{SK}(w')$). We denote by $q_i$ the tableau word of length $\lambda_i$ obtained by reading the $i$-th anti-diagonal of $\epsilon^{-}(T)$. We define $q'_i$ for $\epsilon^{-}(T')$ in a similar way.

Let $p'$ be a path of length $\lambda'_1$ in $\text{SK}(w')$. Suppose that the path $p'$ is obtained from a path $p''$ by bending at corners. The path $p''$ is along the shape $\mu$ where $\mu$ is a strict partition formed by decreasing parts of $\text{SK}(w')$. We denote by read($p''$) := $p''_1 \cdots p''_{|\mu|}$ the reading word along the path $p''$ in $\text{SK}(w')$. Since $\langle R_1 \uparrow \rangle \neq \emptyset$, we have $p''_{|\mu| - 1} \leq p''_{|\mu|}$. By definition, we have $p''_{|\mu| - 1} = p''_{|\mu|}$. However, since a bending at a corner gives an equal or smaller letter, we have $p''_{|\mu| - 1} \leq p''_{|\mu| - 1}$. Similarly, for $1 \leq i \leq |\mu|$, the $i$-th row of $\text{SK}(w')$ has a non-empty increasing part since the decreasing parts of $\text{SK}(w')$ form a strict partition (see Proposition 3.2). We denote by $p'_i$ the path obtained from $\text{SK}_i(w')$, where $\text{SK}_i(w') := \text{SK}(w')$ and $\text{SK}_{i+1}(w')$ is a SSDT obtained from $\text{SK}_i(w')$ by removing boxes in $p'_i$ and successively applying the reverse $\text{SK}$ insertion of type I. Let $R'_1$ be the first row of $\text{SK}_1(w')$. Since the $i$-th row of $\text{SK}(w')$ has an increasing part and the reverse $\text{SK}$ insertion of type I does not decrease the length of the increasing part, we have $\langle R'_1 \downarrow \rangle \neq \emptyset$. From Lemma 3.10, the rightmost element of the second row of $\text{SK}_1(w')$ is not in the path $p'_1$. Therefore, the rightmost and the second rightmost elements of the first row in $\text{SK}_1(w')$ are both in the path $p'_1$. This implies $p'_{1,\lambda_1 - 1} \leq p'_{1,\lambda_1}$.

Let $r$ be the rightmost element in the second row of $\text{SK}(w')$. Since $r$ is not in the path $p'_1$, we apply the reverse $\text{SK}$ insertion of type I to $r$. By definition, $r$ bumps out an element $y$ greater than or equal to $r$, that is, $r \leq y$. Since we can not apply bending in $p'_1$, we have $p'_{1,\lambda_1 - 1} < r \leq y \leq p'_{2,\lambda_2}$, which implies $p'_{1,\lambda_1 - 1} \leq p'_{2,\lambda_2}$. By a similar argument, we have $p'_{1,\lambda_1 - 1} \leq p'_{i+1,\lambda_{i+1}}$.

Suppose that $\epsilon^{-}(T')$ corresponds to $\text{SK}(w')$. Let $a_1, \ldots, a_k$ be the elements forming the second rightmost column of $\epsilon^{-}(T')$ and $b_1, \ldots, b_k$ be the elements forming the rightmost column. By summing up the above observations, these elements satisfy $a_i \leq b_i$ for $1 \leq i \leq k - 1$, $a_i \leq b_{i+1}$ for $1 \leq i \leq k - 2$. If we denote $b_0 := x$, we have $b_0 \leq b_1$. Thus, we can put $b_0$ above $b_1$ in $\epsilon^{-}(T')$ which corresponds to the insertion of $x$ into $\epsilon^{-}(T')$. By using Lemma 3.5 successively, we obtain a new tableau $\epsilon^{-}(T'')$ of the shape $\epsilon^{-}(\lambda)$. A schematic procedure is as follows:

\[
\begin{array}{cccc}
\vdots & \vdots & \vdots & \vdots \\
 a_{k-2} & b_{k-1} & a_{k-1} & b_{k-1} \\
 a_k & a_k & & \\
\end{array}
\]

By construction, $\epsilon^{-}(T'')$ is compatible with the shape $\epsilon^{-}(\lambda)$ and its tableau word produces the same mixed insertion tableau as $\text{SK}(w)$. 

\[
\begin{array}{cccc}
 b_0 \\
b_1 \\
a_1 \ b_2 \\
\vdots \\
a_{k-2} \ b_{k-1} \\
a_k \\
\end{array}
\]
To prove Theorem, we have to show that $\epsilon^-(T)$ obtained from $\text{SK}(w)$ is nothing but $\epsilon^-(T'')$. Suppose that the path $p$ of length $\lambda_1$ constructed from $\text{SK}(w)$ contains $w_{i,\lambda_i}$ for $1 \leq i \leq q \leq k$. From Lemma 3.10, a word $w$, $1 \leq i \leq q$, is strictly decreasing. The element $w_{q,\lambda_q-1}$ is contained by $p$ since a path is an up-right path. We insert $w_{k,\lambda_k}$ into $w_{k-1} \cdots w_1$ by the reverse SK insertion of type II. The insertion is characterized by Lemma 3.8. From Lemma 3.8 and Lemma 3.12, the partial path from $w_{i(\lambda_i)}$ to $w_{q,\lambda_q-1}$ is not changed by the insertion. The partial path from $w_{q,\lambda_q-1}$ to $w_{1,\lambda_1}$ changes locally by the insertion as follows:

\[
\begin{array}{cccc}
  w_{1,\lambda_1} & w_{1,\lambda_1} & w_{2,\lambda_2} & w_{1,\lambda_1} \\
  \vdots & \vdots & \vdots & \vdots \\
  w_{q-1,\lambda_q-1} & w_{q-1,\lambda_q-1} & w_{q,\lambda_q} & w_{q-1,\lambda_q-1} \\
  w_{q,\lambda_q} & w_{q,\lambda_q} & w_{q,\lambda_q-1} & \alpha
\end{array}
\]

where $\alpha \geq w_{q,\lambda_q}$ and $\leftarrow w_{1,\lambda_1}$ means that this element is bumped out by the insertion. Since $w_{i,\lambda_i} \leq w_{i-1,\lambda_i-1}$ for $2 \leq i \leq q$ (by Lemma 3.11), the new path of length $\lambda_1$ is identical to the old path except the last element. Note that $w_{1,\lambda_1} < w_{1,\lambda_1-1}$. This condition corresponds to putting $w_{1,\lambda_1}$ above $w_{1,\lambda_1-1}$ in $\epsilon^-(T')$. To obtain paths from $\text{SK}(w)$, we delete the path of length $\lambda_1$ and perform the reverse SK insertion of type I on it. Let $p_i$ (resp. $p_i'$) be the $i$-th path of length $\lambda_i$ (resp. $\lambda_i'$) obtained from $\text{SK}(w)$ (resp. $\text{SK}(w')$). The difference between $p_i$ and $p_i'$ is the last element by using a similar argument above. More precisely, we have $p_{i,\lambda_i'} = p_{i+1,\lambda_{i+1}}$ for $1 \leq i \leq k-1$ and $p_i = p_i'$ for $k < i \leq l(\lambda)$. In this way, we obtain $\epsilon^-(T')$ from $\text{SK}(w')$. Combining these observations, we have shown that $\epsilon^-(T) = \epsilon^-(T'')$. This completes the proof.

\textbf{Example 3.13.} Let $\lambda = (7,4,3)$. An example of a SSDT of shape $(7,4,3)$ is the leftmost tableau in Figure 3.14. From this SSDT, we obtain a path $p_1$ of length 7 and its word is given by $\text{word}(p_1) = 3112344$. By removing the boxed elements, we obtain a SSDT of shape $(4,3)$ (see rightmost picture in the first row in Figure 3.14). By extracting a path $p_2$, $\text{word}(p_2) = 4535$, we obtain a SSDT of shape $(3)$. A path $p_3$ is equal to the third SSDT, namely $\text{word}(p_3) = 956$. By putting the elements in the paths $p_1, p_2$ and $p_3$ on the anti-diagonals, we obtain a semistandard increasing decomposition tableau $\epsilon^-(T)$ of shape $\epsilon^-(\lambda)$ (see Figure 3.15) corresponding to the SSDT of shape $(7,4,3)$. One can easily check that the words constructed from the SSDT and $\epsilon^-(\lambda)$ produce the same shifted tableau by the mixed insertion.

\textbf{3.2. Construction of $\epsilon^+(T)$.} Let $\lambda$ be a strict partition. We construct a bijection between a semistandard shifted tableau $T$ of shape $\lambda$ and a semistandard shifted tableau $\epsilon^+(T)$. We call also $\epsilon^+(T)$ a semistandard increasing decomposition tableau. There are five steps for this bijection:

1. perform the standardization on $T$ and obtain a standard tableau $\text{stand}(T)$,
2. perform a “dual” operation on $\text{stand}(T)$ with respect to primes and obtain $\overline{T}$,
Step 1: We denote by #(α) the number of a letter α ∈ X′ in a tableau T. We define a standard tableau \( \text{stand}(T) \) of \( T \) as follows. We replace 1′'s in \( T \) with the letter 1, 2, ..., #(1′) from top to bottom. Successively, replace 1's by the letter #(1′) + 1, #(1′) + 2, ..., #(1′) + #(1) from left to right. Continue with 2′'s in \( T \) which are replaced by #(1′) + #(1) + 1, #(1′) + #(1) + #(2′), and so on, until we obtain a standard tableau. We denote by \( \text{stand}(T) \) the standard shifted tableau corresponding to a tableau \( T \). Note that when we replace primed (resp. unprimed) letter with a letter, we work from top to bottom (resp. from left to right). We define a content as \( \#(\lambda) := (\#(1′) + \#(1), \#(2′) + \#(2), \ldots) \).

Step 2: A given standard shifted tableau \( \text{stand}(T) \), we consider the following “dual” operation:

(i) letters on the main diagonal are unchanged,
(ii) a letter \( \alpha \in X′ \) which is not on the main diagonal is replaced by \( \alpha' \). Here, we set \( (i)'' = i' \) and \( (i')'' = i \) for \( i \in X \).

The new semistandard shifted tableau is denoted by \( T' \). Note that this operation is an involution, i.e., \( T'' = T \).

Step 3: For our purpose, it is enough to find a word \( w \in X \) such that \( P_{\text{mix}}(w) = T \). This is easily done by reversing the mixed insertion starting from a pair of tableaux \( (T', U) \) where \( U \) is an arbitrary chosen standard shifted tableau. Once \( w \) is fixed, we can obtain a SSDT by the SK insertion and a tableau \( \epsilon^-(T) \) from the SSDT.

Step 4: We reflect \( \epsilon^-(T) \) over the diagonal line and obtain a standard tableau \( \epsilon^+(T) \).

Step 5: Let \( \alpha \) be the content of \( T \). We replace the letters from 1 to \( \alpha_1 \) in \( \epsilon^+(T) \) by 1, the letters from \( \alpha_1 + 1 \) to \( \alpha_1 + \alpha_2 \) in \( \epsilon^+(T) \) by 2, and so on. This destandardization of \( \epsilon^+(T) \) gives a semistandard increasing decomposition tableau \( \epsilon^+(T) \).

**Theorem 3.16.** Let \( T \) be a shifted tableau of shape \( \lambda \) and \( \epsilon^+(T) \) be a semistandard increasing decomposition tableau of shape \( \lambda \) constructed by the above steps. Then, \( \epsilon^+(T) \) is well-defined, i.e., the reading word \( w := \text{read}(\epsilon^+(T)) \) satisfies 1) the word \( w \) is compatible with the shape \( \epsilon^+(\lambda) \) and 2) \( P_{\text{mix}}(w) = T \).

Before proceeding to a proof of Theorem 3.16, we introduce a proposition and two lemmas needed for the proof.

**Proposition 3.17** (Proposition 8.8 in [6]). We have \( P_{\text{mix}}(w) = P_{\text{mix}}(\text{rev}(w)) \).

**Proposition 3.17.** Given a shifted tableau \( T \), we denote \( S = \text{stand}(T) \).
Lemma 3.18. Let $w$ be the reading word of $\epsilon^-(S)$ and $\overline{w}$ be the reading word of a tableau which is obtained by reflecting $\epsilon^-(S)$ over the diagonal line. Then, we have $\overline{w} \sim \text{rev}(w)$.

Proof. We first show Lemma is true when $\lambda = (n, n-1, \ldots, 1)$. In this case, we have $\epsilon^+(\lambda) = \epsilon^-(\lambda)$. We enumerate the rows of $\epsilon^-(\lambda)$ by 1, $\ldots$, $n$ from top to bottom and the boxes in the $i$-th row by 1, $\ldots$, $i$ for 1 $\leq$ $i$ $\leq$ $n$. Let $a_{i,j}$ be the content of the $j$-th box in the $i$-th row in $\epsilon^-(S)$. By definition, we have

$$w = a_{n,1} \cdots a_{n,n}a_{n-1,1} \cdots a_{n-1,n-1} \cdots a_{2,1}a_{2,2}a_{1,1}$$

$$\overline{w} = a_{1,1}a_{2,2} \cdots a_{n,n}a_{2,1} \cdots a_{n,n-1} \cdots a_{n-1,1}a_{n,2}a_{n,1}$$

Since $S$ is standard, $a_{i,j}$ satisfies

$$a_{i,1} < a_{i,2} < \ldots < a_{i,i},$$

$$a_{i,1} < a_{i+1,2} < \ldots < a_{n,n-i+1},$$

for 1 $\leq$ $i$ $\leq$ $n$. We depict $\text{rev}(w)$ and $\overline{w}$ as

$$\text{rev}(w) = \begin{array}{c}
\leftarrow \\
\downarrow
\end{array},
\overline{w} = \begin{array}{c}
\downarrow \\
\leftarrow
\end{array},$$

where $\leftarrow$ means that we read a word from right to left starting from the top row to bottom, and $\downarrow$ means that we read a word from top to bottom starting from the rightmost column to the leftmost column. We make use of induction on $n$. Suppose that we have $\overline{w} \sim \text{rev}(w)$ for some $n$. For $n + 1$, we have

$$(3.1)$$

$$\begin{array}{c}
\downarrow \\
\leftarrow
\end{array} \sim \begin{array}{c}
\downarrow \\
\leftarrow
\end{array} \sim \begin{array}{c}
\downarrow \\
\leftarrow
\end{array} \sim \begin{array}{c}
\downarrow \\
\leftarrow
\end{array} \sim \begin{array}{c}
\downarrow \\
\leftarrow
\end{array} \sim \begin{array}{c}
\downarrow \\
\leftarrow
\end{array}.$$  

In Eqn. (3.1), the second picture means we first read the rightmost column and then read the word in the triangle of size $n$, the third picture is obtained by applying the induction assumption on the second picture and by decomposing the triangle of size $n$ into a single row and a triangle of size $n - 1$. Since $a_{n,n}$ is the maximal content, by applying Lemma 3.5 successively, the last picture in Eqn. (3.1) is equal to

$$(3.1)$$

$$\begin{array}{c}
\leftarrow \\
\downarrow
\end{array} \sim \begin{array}{c}
\leftarrow \\
\downarrow
\end{array} \sim \begin{array}{c}
\leftarrow \\
\downarrow
\end{array} \sim \begin{array}{c}
\leftarrow \\
\downarrow
\end{array} \sim \begin{array}{c}
\leftarrow \\
\downarrow
\end{array} \sim \begin{array}{c}
\leftarrow \\
\downarrow
\end{array}.$$  

where we have used the induction assumption in the first and third equality.

For a general $\lambda$, recall that $\epsilon^+(\lambda)$ has $\lambda_i$ boxes in the $i$-th anti-diagonal. The shape $\epsilon^+(\lambda)$ is obtained as a union of the shapes $\epsilon^+(\mu)$’s where $\mu$ is a staircase. For example, if $\lambda = (4, 3, 1)$, $\epsilon^+(\lambda)$ is a union of $\epsilon^+(\mu_1)$ and $\epsilon^+(\mu_2)$ with $\mu_1 = (3, 2, 1)$ and $\mu_2 = (2, 1)$ (see Figure 3.19). Suppose that
Figure 3.19. An example of $\epsilon^+(\lambda)$.

$\mu_1$ and $\mu_2$ are staircases, and $\epsilon^+(\mu_1)$ and $\epsilon^+(\mu_2)$ overlap. We have

By applying the procedure above to $\epsilon^+(\lambda)$, we obtain $\overline{w} \sim \text{rev}(w)$. □

Lemma 3.20. Let $w$ and $T$ be a word and a shifted tableau such that $P_{\text{mix}}(w) = T$. Then

(1) If the unprimed letters from $i_1$ to $i_r$ form a vertical strip from top to bottom in $T$, then $i_r, \ldots, i_1$ form a decreasing sequence in $w$.

(2) Let $i_j := i_1 + j - 1$ for $1 \leq j \leq r$ be unprimed letters. If the letters $\alpha, i'_2, \ldots, i'_r$ with $\alpha = i_1$ or $i'_1$ form a horizontal strip from left to right in $T$, then $i_r, \ldots, i_1$ form a decreasing sequence in $w$.

Proof. We assume that $i_j$ and $i_{j+1}$ form an increasing sequence in $w$ for (1) and (2). Suppose that $i_j$ (resp. $i_{j+1}$) are in the $k$-th (resp. $k'$-th) row of a tableau.

For (1), since we insert the letter $i_j$ into a tableau before $i_{j+1}$, and $i_j$ and $i_{j+1}$ are unprimed, it is obvious that $k \geq k'$. The letters $i_j$ and $i_{j+1}$ form a horizontal strip. By taking contraposition, (1) follows.

For (2), if $i_j$ and $i_{j+1}$ are unprimed, we have $k \geq k'$ and $i_{j+1}$ is in a column right to $i_j$ from (1). An unprimed letter becomes primed when it is placed on the main diagonal and bumped out by a smaller letter. Suppose that $i_j$ is in the $m_0$-th column of the main diagonal and $i_{j+1}$ is in the $m_1$-th column with $m_1 > m_0$. If $m_1 = m_0 + 1$, the letters $i'_j$ is placed above $i'_{j+1}$ in the $m_1$-th column after bumping. If $m_1 \geq m_0 + 2$, $i'_j$ and $i_{j+1}$ form a horizontal strip from left to right. To have $i_{j+1}$ primed, we consider bumpings of $i'_{j+1}$ by smaller letters. In the case of $m_1 \geq m_0 + 2$, we arrive at a configuration such that $i'_j$ is above $i_{j+1}$ by bumping of $i_{j+1}$. When $i_{j+1}$ becomes primed, $i'_j$ and $i'_{j+1}$ form a vertical strip from top to bottom. By taking contraposition, (2) follows. □
Proof of Theorem 3.16. For (1), observe that \( \overline{T} \) is a standard tableau obtained from \( \text{stand}(T) \). Let \( i_1, \ldots, i_q \) (resp. \( i_{q+1}, \ldots, i_r \)) be unprimed letters in \( \text{stand}(T) \) corresponding to \( j' \) (resp. \( j' \)) in \( T \). In \( \overline{T} \), the letters \( i_1, \ldots, i_{q+1} \) form a vertical strip if there exists a letter \( j \) below all \( j' \)'s. In \( \overline{T} \), the letters \( i_1, \ldots, i_q \) form a vertical strip if there is no letter \( j \) below all \( j' \)'s. Similarly, the letters \( \alpha, i'_{q+2}, \ldots, i_r \) with \( \alpha = i_{q+1} \) or \( i'_{q+1} \) form a horizontal strip in \( T \). From Lemma 3.20, \( i_r, \ldots, i_1 \) form a decreasing sequence in \( \epsilon^-(\overline{T}) \). Since a row in \( \epsilon^-(\overline{T}) \) is an increasing sequence, there exists at most one \( i_s \), \( 1 \leq s \leq r \), in a row. The tableau \( \epsilon^+(T) \) is obtained from \( \epsilon^-(\overline{T}) \) by reflecting it over the anti-diagonal. Thus, there exists at most one \( i_s \) in a column of \( \epsilon^-(\overline{T}) \). After destandardization, the tableau \( \epsilon^+(T) \) contains at most one \( j \) in a column. This implies that \( \epsilon^+(T) \) is compatible with the shape \( \epsilon^+(\lambda) \).

For (2), let \( w \) be the reading word of \( \epsilon^-(\text{stand}(T)) \) and \( \overline{w} \) be the reading word of a tableau which is obtained by reflecting \( \epsilon^-(\text{stand}(T)) \) over the main diagonal. Then, from Proposition 3.17 and Lemma 3.18, we have

\[
P_{\text{mix}}(w) = P_{\text{mix}}(\text{rev}(w)) = P_{\text{mix}}(\overline{w}).
\]

Note that the reflection of \( \epsilon^-(\overline{T}) \) over the main diagonal corresponds to performing the “dual” operation with respect to primes on it. Thus, we have \( P_{\text{mix}}(\text{read}(\epsilon^+(T))) = T \).

\[\Box\]

Example 3.21. Let \( T \) be a shifted tableau in the left picture of Figure 3.22. After standardization

\[
T = \begin{array}{cccc}
1 & 2' & 3 & 4' \\
2 & 4 & 5 & 6 \\
6 & & & \\
\end{array}
\]

\[
\overline{T} = \begin{array}{cccc}
1 & 2 & 4' & 5 \\
3 & 6' & 7' & 8 \\
& & & \\
\end{array}
\]

\[\text{Figure 3.22. A tableau } T \text{ and } \overline{T}.\]

and taking dual with respect to primes, we have a tableau \( \overline{T} \) as the right picture of Figure 3.22. There are several words which form \( \epsilon^-(\overline{T}) \) as in Figure 3.23. By reflecting \( \epsilon^-(\overline{T}) \) over the diagonal line

\[
\epsilon^-(\overline{T}) = \begin{array}{cccc}
8 & 6 & 5 & 2 \\
7 & 1 & 3 & 4 \\
& & & \\
& & 4 & 7 & 8 \\
\end{array}
\]

\[\text{Figure 3.23. A SSDT } SK(w) \text{ and } \epsilon^-(\overline{T}).\]

and performing a destandardization, we obtain

\[
\epsilon^+(T) = \begin{array}{cccc}
1 & 7 & 3 & 6 \\
2 & 5 & 2 & 4 \\
& & & \\
\end{array}
\]

The mixed insertion of the reading word \( \text{read}(\epsilon^+(T)) = 24246153 \) is \( T \).
4. Generalized Littlewood–Richardson coefficients

4.1. Generalized Littlewood–Richardson rules. We define generalized Littlewood–Richardson (LR) coefficients $b^\beta_{\alpha\lambda}$, $c^\gamma_{\alpha\beta}$, $e^\alpha_{\lambda\mu}$, $f^\mu_{\lambda\alpha}$, $g^\gamma_{\alpha\beta}$, and $h^\lambda_{\alpha\beta}$ as follows:

\[
s_\alpha P_\lambda = \sum \beta b^\beta_{\alpha\lambda} s_\beta,
\]
\[
s_\alpha \hat{S}_\beta = \sum \gamma c^\gamma_{\alpha\beta} s_\gamma,
\]
\[
P_\lambda P_\mu = \sum \nu e^\alpha_{\lambda\mu} s_\alpha,
\]
\[
P_\lambda \hat{S}_\alpha = \sum \beta f^\mu_{\lambda\alpha} P_\mu,
\]
\[
\hat{S}_\alpha \hat{S}_\beta = \sum \gamma g^\gamma_{\alpha\beta} s_\gamma,
\]
\[
\hat{S}_\alpha \hat{S}_\beta = \sum \lambda h^\lambda_{\alpha\beta} P_\lambda.
\]

Let $b^\beta_{\lambda} := b^\beta_{\emptyset\lambda}$ and $e^\alpha_{\lambda} := e^\alpha_{\emptyset\lambda}$, i.e., $b^\beta_{\lambda}$ and $e^\alpha_{\lambda}$ are an expansion of $P_\lambda$ in terms of Schur functions $s_\alpha$. Then we have

\[
b^\beta_{\lambda} = e^\alpha_{\lambda}.
\]

Similarly, we have $c^\gamma_{\lambda\alpha} = g^\gamma_{\alpha\emptyset}$ and $f^\mu_{\lambda\emptyset} = h^\lambda_{\alpha\emptyset}$. Eqn. (2.2) implies

\[
b^\alpha_{\lambda} = 2^{-l(\lambda)} f^\lambda_{\emptyset\alpha}.
\]

4.2. Tableau words. We will give expressions for generalized LR coefficients in terms of tableau words.

Let $\alpha$, $\beta$ and $\gamma$ be an ordinary or shifted partitions satisfying $\beta \subseteq \alpha$. We denote by $T(\alpha/\beta; \gamma)$ be a set of tableaux (without primed letters) of shape $\alpha/\beta$ whose content is $\gamma$. Similarly, we denote by $T'(\alpha/\beta; \gamma)$ be a set of shifted tableaux (possibly with primed letters) of shape $\alpha/\beta$ whose content is $\gamma$. We denote by $T''(\alpha/\beta; \gamma)$ be a set of shifted tableaux (possibly with primed letters and diagonal elements can be primed) of shape $\alpha/\beta$ whose content is $\gamma$. We define $T''(\alpha; \ast) := \bigcup_{\beta} T''(\alpha; \beta)$.

**Theorem 4.1.** Let $\alpha, \beta$ be ordinary partitions and $\lambda$ be a strict partition. We have

\[
b_{\alpha\lambda} = \#\{T \in T'(\beta/\alpha; \lambda)\mid \text{read}(T) \text{ is an LRS word}\}.
\]

By setting $\alpha = \emptyset$ in Theorem 4.1, we have

**Corollary 4.2** (Stembridge [23]). Let $\alpha$ be an ordinary partition and $\lambda$ be a strict partition.

\[
b_{\lambda} = \#\{T \in T'(\alpha; \lambda)\mid \text{read}(T) \text{ is an LRS word}\}.
\]
Proof of Theorem 4.1. We have
\[
\begin{align*}
b_{\alpha\lambda}^\beta &= \langle s_\alpha P_\lambda, s_\beta \rangle \\
&= \langle s_{\beta/\alpha}, P_\lambda \rangle \\
&= \sum_\gamma a_{\alpha\gamma}^\beta \langle s_\gamma, P_\lambda \rangle \\
&= \sum_\gamma a_{\alpha\gamma}^\beta [\hat{S}_\gamma, P_\lambda] \\
&= [\hat{S}_{\beta/\alpha}, P_\lambda].
\end{align*}
\]

The Schur function $\hat{S}_{\beta/\alpha}$ is written by a skew Q-function as $\hat{S}_{\beta/\alpha} = Q_{\beta}^{\beta/\alpha} + \delta/\alpha + \delta = \sum_\nu d_{\beta/\alpha}^{\beta/\alpha} Q_\nu$, where $\delta := (l - 1, l - 2, \ldots, 0)$ with $l = l(\beta)$. By the orthogonality $[Q_\lambda, P_\mu] = \delta_{\lambda\mu}$, we obtain
\[
(4.1) \quad b_{\alpha\lambda}^\beta = d_{\alpha/\lambda}^{\beta/\alpha}.
\]

The skew shape $\beta/\alpha$ is equivalent to $\beta'/\alpha'$ where $\alpha' = \alpha + \delta$ and $\beta' = \beta + \delta$. We complete the proof by the equality (4.1) together with Theorem 2.2. \hfill \Box

Theorem 4.3. Let $\alpha, \beta$ and $\gamma$ be ordinary partitions. We have
\[
c_{\alpha\beta}^\gamma = \# \{ T \in T''(\gamma/\alpha; \beta) : \text{wread}(T) \text{ is a Yamanouchi word} \}.
\]

In [5], Gasharov has proved the Littlewood–Richardson rule by using the Bender–Knuth involution [1]. We show Theorem 4.3 by a shifted analogue of the proof proposed by Gasharov. We make use of the involutions introduced by Stembridge in [24].

We introduce two lemmas needed later for a proof of Theorem 4.3. For a partition $\beta$ of length $l$ and a permutation $\pi \in S_l$, we define
\[
\pi * \beta := (\beta\pi(i) - \pi(i) + i)_{1 \leq i \leq l}.
\]

Since $\hat{S}_\beta$ has a determinant expression (Eqn. (2.3)), we have
\[
(4.2) \quad \hat{S}_\beta = \sum_{\pi \in S_l} \text{sgn}(\pi) q_{\pi*\beta}.
\]

Then, $c_{\alpha\beta}^\gamma$ is rewritten as
\[
\begin{align*}
c_{\alpha\beta}^\gamma &= \langle s_\alpha \hat{S}_\beta, s_\gamma \rangle \\
&= \langle s_{\gamma/\alpha}, \hat{S}_\beta \rangle \\
&= \sum_{\pi \in S_l} \text{sgn}(\pi) \langle s_{\gamma/\alpha}, q_{\pi*\beta} \rangle.
\end{align*}
\]

When a strict partition $\lambda$ is a single row, i.e., $\lambda = (n)$, we have $q_n = 2P_n$. From Theorem 4.1, one can easily obtain a Pieri formula for a product of $s_\alpha$ and $q_{(n)}$. The factor two comes from the fact that the first unprimed integer in $T \in T''(\gamma/\alpha; (n))$ is free to choose primed or unprimed. Therefore, we have
\[
(4.3) \quad c_{\alpha\beta}^\gamma = \sum_{(\pi, T) \in \mathcal{L}} \text{sgn}(\pi),
\]

where
\[
\mathcal{L} := \{ (\pi, T) \mid \pi \in S_l, T \in T''(\gamma/\alpha; \pi * \beta) \}.
\]
Let $L_0$ be the subset of $L$:
$$L_0 := \{(\pi, T) \in L \mid \text{wread}(T) \text{ is a Yamanouchi word}\}.$$

**Lemma 4.4.** If $(\pi, T) \in L_0$, $\pi = \text{Id}$.

**Proof.** Since $(\pi, T) \in L_0$, the number of $i$ and $i'$ is equal to or greater than that of $i+1$ and $(i+1)'$. This implies that $(\pi * \beta)_i \geq (\pi * \beta)_{i+1}$, i.e., $\beta_{\pi(i)} - \pi(i) + i \geq \beta_{\pi(i+1)} - \pi(i+1) + i + 1$. On the other hand, we have $\beta_1 - 1 > \beta_2 - 2 > \cdots > \beta_l - l$. Thus we have $\pi(i) < \pi(i+1)$ for $1 \leq i \leq l$. This property can be satisfied by only $\pi = \text{Id}$. □

**Lemma 4.5.** There exists a bijection $\phi : L - L_0 \ni (\pi, T) \mapsto (\pi', T') \in L - L_0$ with $\text{sgn}(\pi) = -\text{sgn}(\pi')$.

For a construction of a bijection $\phi$, we make use of the involution on tableaux developed by Bender and Knuth for ordinary partitions [1] and by Stembridge for shifted partitions [24]. We briefly review these involutions before we move to a proof of Lemma 4.5.

A skew diagram $\lambda/\mu$ is said to be detached if $\lambda/\mu$ has at most one box on the main diagonal.

Let $R_{1,2}(\lambda/\mu)$ be the set of shifted tableaux whose contents are 1 and 2. Then, we have

**Lemma 4.6** (Bender–Knuth [1]). Suppose that $\lambda/\mu$ is detached. There exists a content-reversing involution $\omega_{1,2}$ on $R_{1,2}(\lambda/\mu)$.

**Proof.** Since contents of $\lambda/\mu$ are 1 and 2, there are at most two boxes in a column. Further, if there are two boxes in a column, the contents are 1 and 2 from top to bottom. We say that the content 1 (resp. 2) is free if there is no 2 (resp. 1) in the same column. Suppose that a 1 is not free in $\lambda/\mu$ and denote by $p_1$ the position of this 1. Since $\lambda/\mu$ is detached, a 1 left to $p_1$ in the same row is not free. Similarly, suppose that 2 is not free in $\lambda/\mu$ and denote by $p_2$ the position of this 2. Then, a 2 right to $p_2$ in the same row is not free. In a row of $\lambda/\mu$, we have non-free 1’s, $r_1$ free 1’s, $r_2$ free 2’s and non-free 2’s from left to right. The involution $\omega_{1,2}$ exchange $r_1$ and $r_2$, i.e., the row of $\omega_{1,2}(\lambda/\mu)$ has non-free 1’s, $r_2$ free 1’s, $r_1$ free 2’s and non-free 2’s from left to right. By construction, $\omega_{1,2}$ is content-reversing. □

Similarly, let $C_{1',2'}(\lambda/\mu)$ be the set of shifted tableaux whose contents are $1'$ and $2'$. Then, we have

**Corollary 4.7.** Suppose that $\lambda/\mu$ is detached. There exists a content-reversing involution $\omega_{1',2'}$ on $C_{1,2}(\lambda/\mu)$.

The involution $\omega_{1',2'}$ is obtained by transposing $\omega_{1,2}$, i.e., we replace a row in the definition of $\omega_{1,2}$ by a column.

Let $RC(\lambda/\mu)$ be the set of tableaux with contents 1 and $2'$. Similarly, $CR(\lambda/\mu)$ be the set of tableaux with contents $2'$ and 1, but with a nonstandard ordering $2' < 1$.

**Lemma 4.8** (Stembridge [24]). Suppose that $\lambda/\mu$ is detached. There exists a content-preserving bijection $\psi : RC(\lambda/\mu) \rightarrow CR(\lambda/\mu)$.

**Proof.** A skew shape $\lambda/\mu$ is a union of connected components. The action of $\psi$ is defined on each connected component. We assume that $\lambda/\mu$ is connected, which implies that $\lambda/\mu$ is a strip.

Let $T$ be a tableau in $RC(\lambda/\mu)$ and $\text{read}(T) := a_n \cdots a_1$. Since $T$ is a semistandard tableau, we have $a_i = 1$ if $a_i$ and $a_{i+1}$ are in the same row, and $a_i = 2'$ if $a_i$ and $a_{i+1}$ are in the same column except for $a_1$. The content $a_1$ is either 1 or $2'$. We define $\psi$ as a rotation of $\text{read}(T)$: $\psi(a_{i-1}) = a_i$
for \(2 \leq i \leq n\) and \(\psi(a_n) = a_1\). By construction, \(\psi(T) \in CR(\lambda/\mu)\) and \(\psi\) is invertible. Therefore, \(\psi\) is content-preserving and bijective. \qed

We construct a bijection \(\phi\) in Lemma 4.5 as follows. Given \((\pi, T) \in \mathcal{L} - \mathcal{L}_0\), we define \(\text{wread}(T) = w_N \cdots w_1\). From the definition of reading and weak reading words, there exists an integer \(N'\) such that \(w_i\) for \(1 \leq i \leq N'\) (resp. \(w_i\) for \(i > N'\)) corresponds to an unprimed (resp. primed) element in \(\text{read}(T)\). Let \(r := r(T)\) be an integer such that

\[
    r(T) = \min\{k \mid w_k \cdots w_1 \text{ is not a Yamanouchi word}\}.
\]

We have two cases: \(r \leq N'\) or \(r > N'\). First, we consider \(r \leq N'\). The element \(w_r\) is unprimed in \(\text{wread}(T)\). For a tableau word \(w\), we denote by \(m_i(w)\) the number of \(i\) in \(w\). By definition of \(r\), the word \(w_{r-1} \cdots w_1\) is a Yamanouchi word and the word \(w_r \cdots w_1\) is not. Therefore, \(w_r \geq 2\) and one can set \(w_r = i + 1\) or \((i + 1)\). More precisely, we have \(w_r = i + 1\) if \(r \leq N'\) and \(w_r = (i + 1)'\) if \(r > N'\). We have

\[
    m_1(w_{r-1} \cdots w_1) \geq \cdots \geq m_r(w_{r-1} \cdots w_1)
\]

and

\[
    (4.4) \quad m_{i+1}(w_r \cdots w_1) = m_i(w_r \cdots w_1) + 1.
\]

Recall that \(\psi_{i,(i+1)'}\) is a content-preserving bijection and we have a non-standard ordering \((i + 1)' < i\). Let \(S\) be a strip formed by \(\{i, (i+1)\}\) including the element \(w_r\) if \(w_r = (i+1)'\). We denote by \(s_1 \in S\) the most upper-right element in \(S\). Let \(\bar{T} := \psi(T)\), \(\text{wread}(\bar{T}) := v_N \cdots v_1\) and \(r' := r(\bar{T})\). Note that \(\phi\) is also shape-preserving.

**Lemma 4.9.** We have:

1. If \(r \leq N'\), then the position of \(v_r\) in \(\psi(T)\) is the same as the one of \(w_r\) in \(T\).
2. For \(r > N'\), we have the following:
   1. If \(s_1 = i\), then the position of \(v_r\) in \(\psi(T)\) is one step upper than \(w_r\).
   2. If \(s_1 = (i+1)\) and \(w_r\) is not in the most upper-right box in \(S\), then the position of \(v_r\) in \(\psi(T)\) is the same as the one of \(w_r\) in \(T\).
   3. If \(s_1 = (i+1)\) and \(w_r = s_1\), then the position of \(v_r\) in \(\psi(T)\) is the leftmost box in \(S\) which is in the same row as \(w_r\).

Further, we have

\[
    m_i(w_r \cdots w_1) = m_i(v_r \cdots v_1)
\]

for \(1 \leq i \leq r\).

**Proof.** For simplicity, we make use of the identification \(i \leftrightarrow 1, i+1 \leftrightarrow 2\). We denote by \(b_2\) the box corresponding to \(w_r\) in \(T\).

For (1), we denote by \(b_1\) the box just above \(b_2\) in \(T\). Suppose that the content of \(b_1\) (resp. \(b_2\)) is 1 (resp. 2) and there are \(k\) 2's right to the box \(b_2\) in \(T\). Denote by \(b_3\) the box \(k\) steps right to the box \(b_1\) and by \(w_s\) the content of \(b_3\). Since the order of letters is \(1 < 2' < 2\), we have \(k\) 1's right to the box \(b_1\) or \(k - 1\) 1's and one 2' right to the box \(b_1\), namely \(w_s = 1\) or \(w_s = 2'\). In the former case, since \(m_2(w_r \cdots w_1) = m_1(w_r \cdots w_1) + 1\) and \(m_2(w_r \cdots w_s) \leq m_1(w_r \cdots w_s)\), there exists \(s' < r\) satisfying \(m_2(w_{s'} \cdots w_1) = m_1(w_{s'} \cdots w_1) + 1\). This contradicts the minimality of \(r\).

Thus, the content \(w_s\) of the box \(b_3\) is 2' if \(b_1\) is 1. Further, if there is a box with content 1 left to the box \(b_1\), this also contradicts the minimality of \(r\). The above observations can be extended to the case where the letter 2's form a horizontal strip and the letters 1 and 2' form a strip \(S\) in \(T\). The strip \(S\) has to be above the box \(b_2\).
In general, we consider a configuration of letters 1, 2' and 2 such that 1's and 2's form a strip $S$ and 2's form a horizontal strip below $S$. Suppose that the $i$-th row from top contains $n_1$ 1's, the $(i + 1)$-th row contains $n_2$ 2's and the $i$-th row is not placed at the bottom of $S'$. We have $n_1 \geq n_2$ if the $i$-th row contains 2' and $n_1 \geq n_2 + 1$ if the $i$-th row does not contain 2'. After applying $\psi$ on $S$, the number of 1's in the $i$-th row is greater than or equal to the one of 2's in the $(i + 1)$-th row. The total numbers of 1's in $S$ and $\psi(S)$ are the same. This means that if $w_p \ldots w_1$ is Yamanouchi, then $v_p \ldots v_1$ is also Yamanouchi for all $p < r$. Note that in the case where the content of $b_1$ is not 1, the letter 2 in $b_2$ is irrelevant to the bijection $\psi$. Since the bijection $\psi$ is defined as a rotation of the strip $S$ and content-preserving, the position of $v_r'$ is the same as the one of $w_r$. See Figure 4.10 for an example. Therefore, the statement (1) is true.

For (2), we have $w_r = 2'$ and denote by $b_1$ the box one step left to the box $b_2$ in $T$. Suppose that the content of $b_1$ is 1'. For primed integers, we read the content of $T$ from left to right in a row. Primed integers appear at most once in a row of $T$. Therefore, we have that $m_1(w_r \ldots w_{r-1}) = m_2(w_r \ldots w_{r-1})$. This is equivalent to $m_2(w_{r-2} \ldots w_1) = m_1(w_{r-2} \ldots w_1) + 1$, which is a contradiction of the minimality of $r$. Thus, the content of $b_1$ is not 1'. If we apply $\psi$ on a strip with $s_1 = 1$, a letter 2' is moved upward by one step. A condition $s_1 = 1$ implies that this letter 1 appears in $w$ as $w_j$ with some $j \leq N'$ before and after the application of $\psi$. The statement (2a) directly follows from this observation. If $s_1 = 2'$ and we apply $\psi$ on the strip $S$, the lowest row of a new strip contains 2' but the top row does not. Combining the observation above with the fact that a letter 2' is moved upward by $\psi$, the statements (2b) and (2c) holds true. Figure 4.11 is examples of the action of $\psi$ on $S$.

By summarizing the above observations, it is obvious that $m_i(w_r \ldots w_1) = m_i(v_r' \ldots v_1)$ for $1 \leq i \leq r$. □
Let $T'$ be a tableau $T' := \psi(T)$ and denote by $T'(p,q)$ the $q$-th element in the $p$-th row in $T'$. We say that the element of $T'$, $T'(p,q) = i + 1$ (resp. $i$), is free when there exists no $i$ (resp. $i + 1$) in the same column of $T'(p,q)$. Similarly, we say that the element $i'$ (resp. $(i+1)'$) is free when there exists no $(i+1)'$ (resp. $i'$) in the same row.

We claim:

(S1) Suppose that $w_r = i + 1$. The integer $i + 1$ in $T'$, which is left to $w_r$ and in the same row as $w_r$, is free.

(S2) Suppose that $w_r = (i+1)'$. The alphabet $(i+1)'$ in $T'$, which is lower than $w_r$ and in the same column as $w_r$, is free.

Let $T'(p,q)$ be the element $w_r$ in $T'$. Suppose that $T'(p,q) = i + 1$ with $q' < q$ and $T'(p,q')$ is not free. Then, we have $T'(p-1,q') = i$. Since the ordering of alphabet in $T'$ is $i' < (i+1)' < i < i + 1$ and $T'$ is a semistandard tableau, the integer $i + 1$, which is right to $w_r$ and in the same row as $w_r$, is not free. We denote by $T'(p,q')$ the rightmost $i + 1$, which is right to $w_r$ and in the same row. We have $T'(p-1,q') = i$ and denote by $w_s := i$ the content of $T'(p-1,q')$. The above consideration also implies that the elements $T(p-1,m) = i$ with $q' \leq m < q''$. We have

$$m_i(w_r \cdots w_s) \geq m_{i+1}(w_r \cdots w_s).$$

Together with Eqn.(4.4), we obtain

$$m_i(w_{s-1} \cdots w_1) < m_{i+1}(w_{s-1} \cdots w_1),$$

which is a contradiction against the minimality of $r$. Thus, the statement (S1) holds true. By transposing the above argument, the statement (S2) follows.

**Proof of Lemma 4.5.** In the above notation, we construct a bijection $\phi(\pi, T) = (\pi', T')$ as follows. First, we define $\pi' := \pi \circ (i, i+1)$ where $(i, i+1)$ is a transposition in $S_i$. We perform two operations on $\tilde{T} := \psi_{i,i+1}(T)$. We consider two cases: 1) $v_r = i + 1$ and 2) $v_r = (i + 1)'$.

Case 1. We perform $\omega_{i+1}$ on $i$'s and $i + 1$'s except for $v_r$ which are in the same row as $v_r$. We also perform $\omega_{i,i+1}$ on $i$’s and $i + 1$’s which are in the lower rows than $v_r$. We call this operation $\Omega_{i,i+1}$. Successively, we perform $\omega_{i',(i+1)'}$ on all $i'$’s and $(i + 1)'$’s in $\Omega_{i,i+1}(\tilde{T})$. Then, we obtain $T' = \psi^{-1} \circ \omega_{i',(i+1)'} \circ \Omega_{i,i+1} \circ \psi(T)$.

Case 2. Since $v_r = (i + 1)'$, we do not perform any operation with respect to $i$ and $i + 1$. We perform $\omega_{i',(i+1)'}$ on $i'$’s and $(i + 1)'$’s which are strictly upper than $v_r$. We call this operation $\Omega'_{i',(i+1)'}$. A tableau $T'$ is given by $T' := \psi^{-1} \circ \Omega'_{i',(i+1)'} \circ \psi(T)$.

Let $\tilde{r} := r(T')$ and $\text{wread}(T') = w'_{\tilde{n}} \cdots w'_1$. We show that the weights of $T$ and $T'$ are related as $\text{wt}(T') = \pi \ast \text{wt}(T)$. From the construction of $T'$ and Lemma 4.9, we have $\tilde{r} = r$, $w_r \cdots w_1 = w'_r \cdots w'_1$ and

\[
\begin{align*}
m_i(w_N \cdots w_{r+1}) &= m_{i+1}(w'_N \cdots w'_{r+1}), \\
m_{i+1}(w_N \cdots w_{r+1}) &= m_i(w'_N \cdots w'_{r+1}).
\end{align*}
\]

Therefore, we have

\[
\begin{align*}
m_i(\text{wread}(T')) &= m_i(\text{wread}(T)) - 1, \\
&= \beta_{\pi(i)} - \pi(i+1) + i, \\
m_{i+1}(\text{wread}(T')) &= m_i(\text{wread}(T)) + 1, \\
&= \beta_{\pi(i)} - \pi(i) + i + 1,
\end{align*}
\]
which implies \( \text{wt}(T') = \pi \ast \text{wt}(T) \). Further, \((\pi', T') \in \mathcal{L} - \mathcal{L}_0 \) since \( w'_1 \cdots w'_l \) is not a Yamanouchi word.

We have \( \phi(\pi', T') = (\pi, T) \) by construction, which implies that \( \phi \) is a bijection.

\[
\text{Proof of Theorem 4.3.} \quad \text{From Eqn.(4.3), we have}
\]
\[
(4.5) \quad c^\gamma_{\alpha,\beta} = \sum_{(\pi, T) \in \mathcal{L}_0} \text{sgn}(\pi) + \sum_{(\pi, T) \in \mathcal{L} - \mathcal{L}_0} \text{sgn}(\pi).
\]

Form Lemma 4.4, the first term of Eqn.(4.5) becomes the total number of \( T \in T''(\gamma/\alpha; \beta) \) whose weak reading word is a Yamanouchi word. The second term of Eqn.(4.5) is zero from Lemma 4.5. Therefore, we complete the proof.

\[
\text{Theorem 4.12.} \quad \text{We have}
\]
\[
e^\alpha_{\lambda,\mu} = \sum_\beta \# \{(T, U) \in T(\beta; \lambda) \times T(\alpha/\beta; \mu) : \text{read}(T) \text{ and read}(U) \text{ are LRS words}\}.
\]

\[
\text{Proof.} \quad \text{The theorem directly follows from Theorems 4.1 and Theorem 2.2.}
\]

\[
\text{Theorem 4.13.} \quad \text{Let} \ T \in T'(\lambda; *) \text{ and } U \in T'(\mu; *) \text{ be shifted tableaux in } X' \text{ of shape } \lambda \text{ and } \mu. \text{ We denote by } w_\lambda \text{ and } w_\mu \text{ the weak reading words of } T \text{ and } U, \text{ respectively. Then, we have}
\]
\[
e^\alpha_{\lambda,\mu} = \# \{(w_\lambda, w_\mu) : w_\mu \ast w_\lambda \text{ is a Yamanouchi word of content } \alpha\}.
\]

Before we move to a proof of Theorem 4.13, we introduce lemmas needed later.

Let \( \alpha, \beta \) and \( \gamma \) be ordinary partitions and \( \lambda \) be a shifted partition. From Theorems 4.1 and 2.2, we have

\[
(4.6) \quad b^\beta_{\alpha,\lambda} = \sum_\gamma b^\gamma_{\alpha,\lambda} \cdot a^\beta_{\gamma,\alpha},
\]

where \( \gamma \) is an ordinary partition satisfying \( \gamma \subseteq \beta \) and \( |\gamma| = |\lambda| \). Let \( T \) be a tableau of shape \( \beta \) such that the reading word inside \( \gamma \subseteq \beta \) is an LRS word of content \( \lambda \) and the reading word for the shape \( \beta' / \gamma \) is a Yamanouchi word of content \( \alpha \). Note that contents inside \( \gamma \) can be primed and contents inside \( \beta' / \gamma \) do not have primes. We denote by \( \text{Tab}(\alpha, \lambda; \beta) \) the set of tableaux \( T \) with properties as above.

We have a unique semistandard tableau \( T_\alpha \) whose shape and weight are both \( \alpha \). Then, the reading word \( w_\alpha := \text{read}(T_\alpha) \) is a Yamanouchi word, i.e., \( w_\alpha \) consists of \( \alpha_1 \)'s, \( \alpha_{l-1} \)'s, \( l-1 \)'s, . . . , and \( \alpha_1 \)'s. We denote by \( \text{Tab}'(\alpha, \lambda; \beta) \) the set of tableaux \( U \) of shape \( \lambda \) such that the concatenation of the weak reading word of \( U \) and \( w_\alpha \) is a Yamanouchi word of content \( \beta \), i.e., \( \text{read}(U) \ast w_\alpha \) is Yamanouchi of content \( \beta \).

\[
\text{Lemma 4.14.} \quad \text{There exists a bijection } \chi : \text{Tab}(\alpha, \lambda; \beta) \rightarrow \text{Tab}'(\alpha, \lambda; \beta).
\]

\[
\text{Proof.} \quad \text{We construct a map } \chi : \text{Tab}(\alpha, \lambda; \beta) \rightarrow \text{Tab}'(\alpha, \lambda; \beta) \text{ in the following two steps.}
\]

Step 1. We perform a standardization on \( T \in \text{Tab}(\alpha, \lambda; \beta) \) as follows. Note that the region \( \gamma \subseteq \beta \) is formed by contents whose reading word is an LRS word. We enumerate boxes in \( \gamma \subseteq \beta \) by \( 1, 2, \ldots, |\lambda| \) (see Step 1 for the construction of a bijection from \( T \) to \( \epsilon^+(T) \) in Section 3.2). Successively, we enumerate boxes in \( \beta' / \gamma \) by \( |\lambda| + 1, \ldots, |\beta| \) according to the same rule as in the region \( \gamma \). We denote by \( Q_{\text{stand}}(\beta) \) the obtained standard tableau by the above operation. We denote by \( P_0(\beta) \) a unique semistandard tableau of shape \( \beta \), weight \( \beta \) and without primes. By reversing the RSK algorithm, we obtain a word \( w := \text{RSK}^{-1}(P_0(\beta), Q_{\text{stand}}(\beta)) \).
Step 2. We split the word $w$ into two words $w = w_0 * w_1$ such that the length $w_0$ (resp. $w_1$) is $|\lambda|$ (resp. $|\alpha|$). Then, we define a tableau $U$ of shape $\lambda$ by the mixed insertion $U := P_{\text{mix}}(w_0)$.

We claim:

(S3) The word $w$ is a Yamanouchi word.
(S4) A word $w_1$ satisfies $P_{\text{RSK}}(w_1) = T_\alpha$.
(S5) A tableau $U = P_{\text{mix}}(w_0)$ is of shape $\lambda$ and a concatenation $\text{wreath}(U) * w_\alpha$ is a Yamanouchi word.

Reversing the RSK insertion means that we insert an element $x$ into the upper row and bump out the largest element which is less than $x$. From the definition of $P_0(\beta)$, the contents of the $i$-th row of $P_0(\beta)$ are all integer $i$. Therefore, if we bump out the $m$-th $x$, $1 \leq x \leq l(\beta)$, as an output, we have already bumped out at least $m$ $y$'s for all $1 \leq y \leq x - 1$. This implies that the word $w$ is Yamanouchi, that is, the statement (S3) holds true.

Given a partition $\alpha$ of length $l$, we consider a sequence of partitions $\emptyset = \hat{\alpha}_0 \subset \hat{\alpha}_1 \subset \hat{\alpha}_2 \ldots \subset \hat{\alpha}_l = \alpha$ given by $\hat{\alpha}_i := (\alpha_{i-1} + 1, \alpha_{i-2} + 1, \ldots, \alpha_0)$. Note that the skew shape $\hat{\alpha}_i / \hat{\alpha}_{i-1}$ is a horizontal strip of length $\alpha_i$. We enumerate boxes in $\alpha$ by $|\alpha|, |\alpha| - 1, \ldots, 1$ in the following way. For $1 \leq i \leq l$, we put integers from $|\alpha| - \sum_{j=1}^{i-1} \alpha_i$ to $|\alpha| - \sum_{j=1}^{i} \alpha_i + 1$ on the region $\hat{\alpha}_i / \hat{\alpha}_{i-1}$ from left to right. We denote by $T_{\text{rev}}$ a tableau of shape $\alpha$ constructed above. The tableau $T_{\text{rev}}$ is standard with respect to the reversed order of letters. A word $u := u_1 u_2 \ldots u_{|\alpha|}$ is obtained from $T_{\text{rev}}$ by setting $u_i = j$ where $j$ is the number of row of $i$ in $T_{\text{rev}}$. For example, when $\alpha = (4, 2, 1)$, $T_{\text{rev}}$ is given by

\[
\begin{array}{cccc}
7 & 5 & 2 & 1 \\
6 & 3 & \ \\
4 & \ \\
\end{array}
\]

and $u = 1123121$. By construction, the word $u$ is Yamanouchi and $P_{\text{RSK}}(u) = T_\alpha$. Further, the recording tableau $Q_{\text{RSK}}(u)$ is a standard tableau numbered from 1 to $|\alpha|$ starting with the first row and from left to right in a row. It is enough to show that $w_1 = u$. Since the reading word for the shape $\beta / \gamma$ is Yamanouchi and the contents of the $i$-th row in $P_0(\beta)$ are $i$'s, it is obvious that we bump out the word $u$. This implies that the statement (S4) is true.

Let $(P_{\text{RSK}}(w_0), Q_{\text{RSK}}(w_0))$ be a pair of a semistandard tableau of content $\beta / \alpha$ and a standard tableau of shape $\gamma$ for the word $w_0$. Let $\gamma'$ and $\gamma''$ be a partition satisfying $\gamma', \gamma'' \subseteq \gamma$. Suppose that the shape $\gamma / \gamma'$ is a horizontal strip and we enumerate the boxes in $\gamma / \gamma'$ by $1, 2, \ldots, l$ from left to right, where $l$ is the number of boxes in the strip. Starting from the box with $l$, we perform the reversed RSK insertion on $P_{\text{RSK}}(w_0)$. Then, it is obvious that we obtain a word which is weakly increasing from left to right. Similarly, suppose that the shape $\gamma / \gamma''$ is a vertical strip and we enumerate the boxes in $\gamma / \gamma''$ by $1, 2, \ldots, l$ from top to bottom. By reversing the RSK insertion, we obtain a word which is strictly decreasing. Therefore, if we write $w_0 = \tilde{w}_1 \tilde{w}_2 \ldots \tilde{w}_{l(\lambda)}$ where the word $\tilde{w}_i$ is of length $\lambda_i$, the word $\tilde{w}_i$, $1 \leq i \leq l(\lambda)$, is a hook word. The word $\tilde{w}_i$ is written as a concatenation of two words, $\tilde{w}_i = (\tilde{w}_i \downarrow) * (\tilde{w}_i \uparrow)$. Recall that a tableau $T$ in $\text{Tab}(\alpha, \lambda; \beta)$ satisfies the lattice property, i.e., $\text{read}(T)$ inside $\gamma$ is an LRS word. The length of the word $(\tilde{w}_i \downarrow)$ (resp. $(\tilde{w}_i \uparrow)$) is the number of $i$'s (resp. $i$'s) in $T$ plus (resp. minus) one. The lattice property also implies the following: 1) the rightmost $i$ in $T$ is in the same column of or left to the rightmost $i - 1$, 2) the rightmost $i$ in $T$ is placed in a lower row than that of the rightmost $i - 1$, and 3) the number of $i$ in $T$ is greater than or equal to that of $i + 1$ in $T$. By a similar argument to the case of a vertical strip, 1) and 2) imply that the last element $\tilde{w}_{i, \lambda_i}$ of $\tilde{w}_i$ is strictly decreasing with respect to $1 \leq i \leq l(\lambda)$. Also, 3) implies that the length of $(\tilde{w}_i \uparrow)$ is weakly decreasing with respect to $1 \leq i \leq l(\lambda)$. Denote by $\ell_i$ the length of $(\tilde{w}_i \uparrow)$ for $1 \leq i \leq l(\lambda)$. We delete the rightmost $i$'s for $1 \leq i$ and obtain a new
tableau $T'$. We apply the same argument as above to $T'$. Then, the $(\tilde{l}_i - j)$-th element of $(\tilde{w}_i)$ is greater than $(\tilde{l}_{i+1} - j)$-th element of $(\tilde{w}_{i+1})$, i.e., $(\tilde{w}_i)[\tilde{l}_{i-j}] > (\tilde{w}_{i+1})[\tilde{l}_{i+1-j}]$ for $1 \leq j \leq \tilde{l}_{i+1}$. Let $j$ be the rightmost $(i+1)'$ or the leftmost $(i+1)$ if there is no $(i+1)'$ in $T$. Since the rightmost $i$ and $j$ form a vertical strip of length two, $\tilde{w}_{i,\lambda_i}$ is greater than $\tilde{w}_{i+1,1}$.

We have a stronger constraint on the words $\tilde{w}_i$ and $\tilde{w}_{i+1}$. Let $p$ be the position of the letter $(i+1)'$ in a tableau word read($T$). Then, the lattice property for an LRS word implies the following: 4) in the word read($T$), the number of $i$ right to $p$ is strictly greater than the one of $i+1$ right to $p$. Let $m_j$ be the number of elements in $(\tilde{w}_{i+1})'$ which is equal to or greater than $(\tilde{w}_i \downarrow)_j$ for $1 \leq j \leq l((\tilde{w}_{i+1} \downarrow))$. The constraint 4) is rephrased: the number of elements in $(\tilde{w}_i)$ which is strictly greater than $(\tilde{w}_{i+1} \downarrow)_j$ is equal to or greater than $m_j$ plus one. From these observations, we have that the shape of $P_{\text{mix}}(w_0)$ is $\lambda$. To show that the statement (S5) is true, it is enough to show the following statement:

\[(S5') \text{ The concatenation of two words } \text{read}(P_{\text{mix}}(x_i)) \text{ and } y_i \text{ for } 1 \leq i \leq l(\lambda) \text{ is Yamanouchi.} \]

where $x_i := \tilde{w}_1 \ldots \tilde{w}_i$ and $y_i := \tilde{w}_{i+1} \ldots \tilde{w}_{l(\lambda)} w_\alpha$.

We prove the above statement by induction. When $i = 1$, since the word $\tilde{w}_1$ is a hook word, we have read($P_{\text{mix}}(\tilde{w}_1)$) = $\tilde{w}_1$. From (S3), we have read($P_{\text{mix}}(\tilde{w}_1)$) = $\tilde{w}_1 * y_1 = \tilde{w}_1 * y_1 = w$ is Yamanouchi. Suppose that the statement (S5’) is true for some $i$. Note that the shape of $P_{\text{mix}}(x_i)$ is a shifted tableau $(\lambda_1, \ldots, \lambda_i)$. We insert the word $\tilde{w}_{i+1}$ into $P_{\text{mix}}(x_i)$ by the mixed insertion. When we insert $x$ into a shifted tableau, we first bump out an smallest element $y$ which is strictly greater than $x$. Then, we insert $y$ into the next row (resp. column) if $y$ is unprimed (resp. primed). Therefore, in the weak reading word of a new tableau, the bumped element $y$ (or $y'$) appears left to the element $x$. Suppose that a word $v$ is Yamanouchi and $v := v_2 * y * v_1 * x * v_0$. Note that $v_0$ is also Yamanouchi. Then, we have that the concatenation of two words read($P_{\text{mix}}(v_2 * y * v_1 * x)$) and read($P_{\text{RSK}}(v_0)$) stays Yamanouchi. This implies that (S5’) is true.

The inverse of $\chi$, $\chi^{-1} : \text{Tab}'(\alpha, \lambda; \beta) \rightarrow \text{Tab}(\alpha, \lambda; \beta)$, $U \mapsto T$, is given by the following two steps.

Step A. Let $Q_0(\lambda)$ (resp. $Q_0(\alpha)$) be a standard tableau of shape $\lambda$ (resp. $\alpha$) by enumerating boxes in $\lambda$ (resp. $\alpha$) by $1, 2, \ldots, |\lambda|$ (resp. $1, 2, \ldots, |\alpha|$) from left to right in a row starting from the top row to bottom. By reversing the insertions, we define $w_0 := \text{mixRSK}^{-1}(U, Q_0(\lambda))$ and $w_1 := \text{RSK}^{-1}(U_\alpha, Q_0(\alpha))$. We obtain $w$ by a concatenation of $w_0$ and $w_1$, that is, $w := w_0 * w_1$.

Step B. Given a word $w$, we obtain a standard tableau $Q_{\text{stand}}(\beta)$ of shape $\beta$ by the RSK insertion, $Q_{\text{stand}}(\beta) := Q_{\text{RSK}}(w)$. We perform a desstandardization on $Q_{\text{stand}}(\beta)$ with respect to $\lambda$ and $\alpha$. Here, desstandardization is a reversed procedure of standardization. The boxes with contents from 1 to $|\lambda|$ form a ordinary shape $\gamma \subseteq \beta$. If we replace the contents from $\lambda_{i-1} + 1$ to $\sum_{k=1}^i \lambda_k$ by $i'$ and $i$ for $1 \leq i \leq l(\lambda)$, the reading word for the shape $\gamma$ is an LRS word. Note that the desstandardization is unique since the first occurrence of a letter $i'$ or $i$ in an LRS word (see Section 2.6). Similarly, we perform desstandardization in the region $\beta/\gamma$ with respect to the content $\alpha$. Then, we obtain a tableau $T \in \text{Tab}(\alpha, \lambda; \beta)$.

By summarizing observations above, the map $\chi$ is a well-defined bijection. \square

Example 4.15. The product $s_{(3,2)}$ and $P_{(4,2,1)}$ contains $9s_{(5,4,2,1)}$, i.e., $t^{(5,4,2,1)}_{(3,2)(4,2,1)} = 9$. One of them is given by the left picture in Figure 4.16. Here, the integers 1, 2 and 3 form an LRS word of content $(4,2,1)$ and the integers 4 and 5 form a Yamanouchi word of content $(3,2)$. The right picture in Figure 4.16 is the corresponding standard tableau $Q_{\text{stand}}(\beta)$. Then, reversing the RSK
Figure 4.16. A configuration for the product of $s_{(3,2)}$ and $P_{(4,2,1)}$ and its standardization.

By exchanging roles of $\alpha$ and $\lambda$, we have another expression of Eqn. (4.6). Let $T$ be a shifted tableau of shape $\beta/\alpha$ and $U$ be a shifted tableau of shape $\lambda$ and its weak reading word $\text{wread}(U)$ satisfy that $\text{read}(U_\alpha) \ast \text{wread}(U)$ is a Yamanouchi word of content $\beta$. We construct a bijection $\chi'$ between a shifted tableau $T$ and a pair of $(U_\alpha, U)$ in the following two steps.

Step 1. Note that $\beta = \text{shape}(T) \cup \alpha$ and $\alpha \subseteq \beta$. We enumerate boxes in $\alpha \subseteq \beta$ by $1, 2, \ldots, |\alpha|$ from left to right in a row starting from the top row to bottom. Then, we perform a standardization on $T$ by $|\alpha| + 1, |\alpha| + 2, \ldots, |\beta|$. By reversing the RSK algorithm, we obtain a word $\tilde{w}$, i.e., $\tilde{w} := \text{RSK}^{-1}(P_0(\beta), Q_{\text{stand}}(\beta))$.

Step 2. We divide the word $\tilde{w}$ into a concatenation of two words $\tilde{w} = \tilde{w}_0 \ast \tilde{w}_1$ such that the length of $\tilde{w}_0$ (resp. $\tilde{w}_1$) is $|\alpha|$ (resp. $|\lambda|$). Finally, we define a shifted tableau $U$ of shape $\lambda$ by $U := P_{\text{mix}}(\tilde{w}_1)$ and a tableau $U_\alpha$ of shape $\alpha$ by $U_\alpha := P_{\text{RSK}}(\tilde{w}_0)$.

By summarizing the above discussion and a similar argument to Lemma 4.14, we have

Lemma 4.17. The map $\chi' : T \mapsto (U_\alpha, U)$ is a bijection.

Proof of Theorem 4.13. If we set $\alpha = \emptyset$ in Lemma 4.14, we obtain $e_{\emptyset\mu}^\alpha = b_{\emptyset\mu}^\alpha$, which is equivalent to $e_{\emptyset\mu}^\alpha = |\text{Tab}'(\emptyset, \mu; \alpha)|$. From the definition of $\text{Tab}'(\emptyset, \mu; \alpha)$, a shifted tableau of shape $\mu$ has the content $\alpha$ and its weak reading word is a Yamanouchi word. We have

$$e_{\lambda\mu}^\alpha = \sum_\gamma b_{\gamma\lambda}^\alpha e_{\emptyset\mu}^\gamma.$$  

By applying Lemma 4.14 to $b_{\gamma\lambda}^\alpha$, we have

$$e_{\lambda\mu}^\alpha = \sum_\gamma |\text{Tab}'(\gamma, \lambda; \alpha)| \cdot |\text{Tab}'(\emptyset, \mu; \gamma)|,$$

which implies Theorem 4.13 is true. □

Theorem 4.18. We have

$$f_{\lambda\alpha}^\mu = \# \{ T \in T''(\mu/\lambda; \alpha) \mid \text{wread}(T) \text{ is a Yamanouchi word} \}. $$
Proof. Since $\hat{S}_\alpha$ has a determinant expression Eqn.(4.2), the coefficient $f^\mu_{\lambda\alpha}$ is rewritten as

$$f^\mu_{\lambda\alpha} = \left[ P_\lambda \hat{S}_\alpha, Q_\mu \right] = \left[ \hat{S}_\alpha, Q_{\mu/\lambda} \right] = \sum_{\pi \in S_l} \text{sgn}(\pi) \left[ q_{\pi^*\alpha}, Q_{\mu/\lambda} \right].$$

When $\beta = (n)$, by applying Theorem 2.2, we have

$$[q_{(n)}, Q_{\mu/\lambda}] = \#\{T \in T''(\mu/\lambda) \mid \text{wread}(T) \text{ is a Yamanouchi word of content } (n)\}.$$  

Thus, $f^\mu_{\lambda\alpha}$ is given by

$$f^\mu_{\lambda\alpha} = \sum_{(\pi, T) \in \mathcal{M}} \text{sgn}(\pi),$$

where

$$\mathcal{M} := \{ (\pi, T) \mid \pi \in S_l, T \in T''(\mu/\lambda; \pi^* \alpha) \}.$$  

We define the subset $\mathcal{M}_0$ of $\mathcal{M}$ by

$$\mathcal{M}_0 := \{ (\pi, T) \in \mathcal{M} \mid \text{wread}(T) \text{ is a Yamanouchi word of content } \alpha \}.$$  

We want to show that $f^\mu_{\lambda\alpha} = |\mathcal{M}_0|$. By replacing $\gamma/\alpha$ and $\mathcal{L}$ in the proof of Theorem 4.3 by $\mu/\lambda$ and $\mathcal{M}$, we have a proof of Theorem. The difference between Theorem 4.3 and Theorem 4.18 is that we may have a skew tableau which is not detached. We can construct an involution for non-detached shape by the following procedure developed by Stembridge (see Section 6 in [24]). Suppose that a skew shape $T$ is a tableau formed by two letters and $T$ is not detached. Without loss of generality, a tableau $T$ has $k-1$ main diagonals of length two and the $k$-th main diagonal is of length one. The entries on the first main diagonal can be primed or unprimed. We delete the first $k-1$ diagonals and let $S$ be a new tableau and $a$ be the unique entry on the main diagonal in $S$. After deletion of the diagonals, the entry $a$ can be either primed or unprimed. Then, we choose one of the entries in the first main diagonal of $T$ and let $b$ be this entry. We set $a$ primed (resp. unprimed) if $b$ is primed (resp. unprimed). We have a two-to-one map $\vartheta : T \mapsto S$. One can perform involutions $\psi, \omega_{1,2}$ and $\omega_{1,2}'$ on $S$. We have an one-to-two map by reversing the map $\vartheta$. Thus, we have a two-to-two involution on $T$. This completes the proof. \hfill \Box

\textbf{Theorem 4.19.} Let $A \in T''(\alpha; \ast)$ and $B \in T''(\beta; \ast)$ be a shifted tableaux. We denote $w(A) := \text{wread}(A)$ and $w(B) := \text{wread}(B)$. We have

$$g_{\alpha\beta}^\gamma = \#\{ (w(A), w(B)) \mid w(B) \ast w(A) \text{ is a Yamanouchi word of content } \gamma \}.$$  

Let $\alpha, \beta$ and $\gamma$ be ordinary partitions. We define $w_\alpha$ as the reading word $\text{read}(T_\alpha)$ where $T_\alpha$ is a unique tableau whose shape and weight are $\alpha$.

\textbf{Lemma 4.20.} Let $G_{\alpha\beta}^\gamma$ be an expansion coefficient of $s_\alpha \hat{S}_\beta$ in terms of $s_\gamma$, i.e., $s_\alpha \hat{S}_\beta = \sum_\gamma G_{\alpha\beta}^\gamma s_\gamma$. Then, we have

$$G_{\alpha\beta}^\gamma = \#\{ T \in T''(\beta; \gamma/\alpha) \mid \text{wread}(T) \ast w_\alpha \text{ is a Yamanouchi word of content } \gamma \}.  \hspace{1cm} (4.7)$$
Proof. We have
\[ G_{n\beta}^\gamma = \langle s_\alpha \hat{S}_\beta, s_\gamma \rangle \]
\[ = \langle \hat{S}_\beta, s_\gamma/s_\alpha \rangle \]
\[ = \sum_\mu f_{\theta/\beta}^\mu \langle P_\mu, s_\gamma/s_\alpha \rangle \]
\[ = \sum_\mu f_{\theta/\beta}^\mu \left[ P_\mu, \hat{S}_\gamma/s_\alpha \right] \]
\[ = \left[ \hat{S}_\beta, \hat{S}_\gamma/s_\alpha \right] \]
\[ = \langle s_\beta, \hat{S}_\gamma/s_\alpha \rangle. \]

The skew \( \hat{S}_\gamma/s_\alpha \) has a determinant expression:
\[ \hat{S}_\gamma/s_\alpha = \det[q_{\gamma_i - \alpha_i + j}]. \]
By a similar argument to a proof of Theorem 4.3, we obtain Eqn.(4.7).

Proof of Theorem 4.19. We want to expand the product \( \hat{S}_\alpha \hat{S}_\beta \) in terms of Schur functions \( s_\gamma \). First, by setting \( \alpha = \emptyset \) in Lemma 4.20, we obtain \( \hat{S}_\alpha = \sum_\zeta G_{\emptyset/\alpha}^\zeta s_\zeta \). Given \( \zeta \), we obtain from Lemma 4.20 that \( s_\zeta \hat{S}_\beta = \sum_\gamma G_{\zeta/\beta}^\gamma s_\gamma \). Thus, the coefficient \( g_{\alpha\beta}^\gamma \) is given by
\[ g_{\alpha\beta}^\gamma = \sum_\zeta G_{\emptyset/\alpha}^\zeta G_{\zeta/\beta}^\gamma, \tag{4.8} \]
where the sum is taken over all ordinary partitions \( \zeta \)'s. Let \( w \) be a Yamanouchi word of content \( \gamma \). If we write \( w \) as a concatenation of two word \( w_0 \ast w_1 \), then the word \( w_1 \) is again a Yamanouchi word of content \( \zeta \) for some \( \zeta \subseteq \gamma \). Therefore, Eqn.(4.8) and Lemma 4.20 imply Theorem 4.19.

Theorem 4.21. Let \( A \in T''(\alpha; *) \) and \( B \in T''(\beta; *) \) be a shifted tableaux. We have
\[ h_\alpha^\lambda = 2^{|\lambda|} \cdot \# \{ (A, B) \mid \text{read}(A) \ast \text{read}(B) \text{ is an LRS word of content } \lambda \}. \]

Let \( \text{Tab}''(\lambda) \) be the set of semistandard shifted tableaux of shape \( \gamma \) such that alphabets from 1 to \( l(\lambda) \) form an LRS word of content \( \lambda \). Similarly, let \( \text{Tab}''(\alpha, \beta) \) be the set of semistandard shifted tableaux of shape \( \gamma \) without primed letters such that alphabets from 1 to \( l(\alpha) \) (resp. from \( l(\alpha) + 1 \) to \( l(\alpha) + l(\beta) \)) form a Yamanouchi word of content \( \alpha \) (resp. \( \beta \)). Let \( \text{Tab}''(\alpha, \beta; \gamma) \) be the set of pairs of shifted tableaux:
\[ \text{Tab}''(\alpha, \beta; \gamma) := \{ (A, B) \in T''(\alpha; *) \times T''(\beta; *) \mid \text{read}(A; B) \text{ is an LRS word of content } \lambda \}. \]

where \( \text{read}(A; B) := \text{read}(A) \ast \text{read}(B) \).

We will construct a bijection between a pair of tableaux \((T, U)\) where \( T \in \text{Tab}''(\lambda) \) and \( U \in \text{Tab}''(\alpha, \beta) \) and a pair of tableaux \((A, B)\) where the concatenation of two words \( \text{read}(A) \ast \text{read}(B) \) is an LRS word. Namely, we define a map \( \theta : \text{Tab}''(\lambda) \times \text{Tab}''(\alpha, \beta) \to \text{Tab}''(\alpha, \beta; \lambda), (T, U) \mapsto (A, B), \)

as the following two steps.

Step 1. We perform a standardization on \( T \) and \( U \), and obtain a pair of standard tableaux of shape \( \gamma \), \( (\text{stand}(T), \text{stand}(U)) \). Then, by reversing the RSK insertion, we obtain a word \( w := \text{RSK}^{-1}(\text{stand}(T), \text{stand}(U)) \).
Step 2. We split the word \( w \) into a concatenation of two words \( w = w_1 \ast w_2 \) where \( w_1 \) (resp. \( w_2 \)) is a word of length \( |\alpha| \) (resp. \( |\beta| \)). By the RSK insertion, we obtain a pair of standard tableaux of shape \( \alpha \) and \( \beta \), denote by \( (\text{stand}(A), \text{stand}(B)) = (P_{\text{RSK}}(w_1), P_{\text{RSK}}(w_2)) \). Finally, we perform a destandardization on the pair \( (\text{stand}(A), \text{stand}(B)) \), and we obtain the pair \( (A, B) \) of semistandard tableaux of shape \( \alpha \) and \( \beta \). Here, destandardization means that if the box \( b \) has a content \( j \) in \( \text{stand}(T) \) and \( b \) in \( T \) has a letter \( x = i \) or \( i' \), we put \( x \) on a box in \( A \) or \( B \) which has a content \( j \) in \( \text{stand}(A) \) or \( \text{stand}(B) \).

Lemma 4.22. The map \( \theta \) satisfies

(1) Two tableaux \( \text{stand}(A) \) and \( \text{stand}(B) \) are of shape \( \alpha \) and \( \beta \).

(2) Two tableaux \( A \) and \( B \) are well-defined. In other words, \( A \) and \( B \) are semistandard shifted tableaux with respect to the marked alphabet \( X' \).

(3) The word \( \text{read}(A) \ast \text{read}(B) \) is an LRS word of content \( \lambda \).

Proof. For (1), we will show that \( \text{stand}(A) \) is of shape \( \alpha \) since the case for \( \text{stand}(B) \) can be shown in a similar way. Note that the letters from \( \sum_{j=1}^{i-1} \alpha_j + 1 \) to \( \sum_{j=1}^{i} \alpha_j \) in a tableau form a horizontal strip. By reversing the RSK insertion for a horizontal strip, the words \( w_1 \) is written as \( w_1 = \tilde{w}_1 \ldots \tilde{w}_{l(\alpha)} \) where \( \tilde{w}_i \) is a weakly increasing sequence of length \( \alpha_i \). In \( \text{stand}(U) \), the position of the letter \( \sum_{j=1}^{i-1} \alpha_j \) is upper than that of the letter \( \sum_{j=1}^{i} \alpha_j \), since the alphabets from 1 to \( l(\alpha) \) form a Yamanouchi word of content \( \alpha \). By reversing the RSK insertion, this condition is rephrased as the following condition: \( \tilde{w}_{i-1,\alpha_i-1} > \tilde{w}_{i,\alpha_i} \) for \( 2 \leq i \leq l(\alpha) \). We delete the boxes with letters \( \sum_{j=1}^{i-1} \alpha_j \) and \( \sum_{j=1}^{i} \alpha_j \) from \( \text{stand}(U) \). This means that we delete the right most boxes with integer \( i-1 \) and \( i \) in \( U \). In the obtained tableau, letters \( i-1 \) and \( i \) form a Yamanouchi word of content \( (\alpha_i-1, \alpha_i-1-1) \). By applying the same argument, we obtain the condition \( \tilde{w}_{i-1,\alpha_i-1-1} > \tilde{w}_{i,\alpha_i-1} \) for \( 2 \leq j \leq \alpha_i \). These observations imply that \( \text{stand}(A) = P_{\text{RSK}}(w_1) \) is of shape \( \alpha \).

For (2), since a tableau \( \text{stand}(A) \) is obtained by the RSK insertion, it is obvious that unprimed letters in the tableau \( A \) satisfy the semistandard property, i.e., unprimed letters appear at most once in a column of \( A \). Boxes corresponding to a primed letter \( i' \) in \( T \) form a vertical strip, which means that contents corresponding to \( i' \) in \( \text{stand}(T) \) appear as a decreasing sequence in the word \( w \). This property holds true after the RSK insertion of \( w_1 \) or \( w_2 \). In other words, boxes for a primed letter \( i' \) form a vertical strip in both \( A \) and \( B \). These imply that the tableau \( A \) is a semistandard shifted tableau and so is \( B \).

For (3), note that the destandardization of the word \( w = w_1 \ast w_2 \) is an LRS word of content \( \lambda \). The destandardization of the partial word \( w_2 \) is also an LRS word. Thus, the tableau word \( \text{read}(B) \) is an LRS word. A tableau \( A \) is given by a destandardization of \( P_{\text{RSK}}(w_1) \). Suppose that \( w_1 \) is written as \( w_1 := v_1 \ldots v_l \) where \( l = l(\alpha) \). We show the statement (3) by induction. The word \( \text{read}(P_{\text{RSK}}(v_1)) = v_1 \) and a destandardization of the concatenation words \( v_1 \ast v_2 \ldots v_1 \ast w_2 \) is an LRS word. We insert \( v_i \) into \( P_{\text{RSK}}(v_1 \ldots v_{i-1}) \). By induction assumption, the destandardization of \( \text{read}(P_{\text{RSK}}(v_1 \ldots v_{i-1})) \ast v_i \ldots v_i \ast w_2 \) is an LRS word. We first bump out smallest \( x \) which is greater than \( v_i \), and bumped element \( x \) moves to the next row. Thus, in the word \( \text{read}(P_{\text{RSK}}(v_1 \ldots v_{i-1})) \), \( x \) is left to \( v_i \). Since a letter \( v_i \) stays in the first row of \( P_{\text{RSK}}(v_1 \ldots v_l) \) and the elements right to \( v_i \) are strictly greater than \( v_i \), the lattice property holds after the insertion and destandardization. In general, elements bumped out by the RSK insertion are inserted into the next row in \( P_{\text{RSK}}(v_1 \ldots v_l) \). After destandardization, the lattice property holds true. As a result, the concatenation of three words \( \text{read}(P_{\text{RSK}}(v_1 \ldots v_l)), v_{i+1} \ldots v_l \) and \( w_2 \) is an LRS word. By induction, we obtain that \( \text{read}(P_{\text{RSK}}(v_1)) \ast w_2 \) is an LRS word. Combining these observations with the fact that \( \text{read}(B) \) is an LRS word, we obtain that \( \text{read}(A) \ast \text{read}(B) \) is an LRS word of content \( \lambda \). \( \square \)
The inverse \( \theta^{-1} : \text{Tab}^\gamma(\alpha, \beta; \gamma) \to \text{Tab}^\gamma(\lambda) \times \text{Tab}^\gamma(\alpha, \beta), (A, B) \mapsto (T, U) \), can be given by the following two steps.

Step A. We perform a standardization on \((A, B)\) and obtain a pair of standard tableaux of shape \(\alpha\) and \(\beta\). We enumerate the boxes in tableaux \(A\) and \(B\) with the letter 1 by \(\#(1')\) where \(\#(1')\) is the total number of 1’ in tableaux \(A\) and \(B\). We start from the top row of the tableau \(B\) to the bottom row of \(B\), moving to the tableau \(A\), and continue to enumerate boxes from the top row to the bottom row of \(A\). Then, we continue to enumerate the boxes with letter 1 by \(\#(1') + 1, \ldots, \#(1') + \#(1)\) starting from the leftmost column of a tableau \(A\) to the rightmost column of \(A\), moving to the tableau \(B\), and successively enumerate from the leftmost column to the rightmost column in \(B\). We similarly enumerate boxes with \(i’\) or \(i\) as above. Note that the order of scanning boxes in tableaux \(A\) and \(B\) for \(i\) is reversed compared to the case of \(i’\). We denote by \(\text{std}(A)\) a tableau on which standardization is performed. Let \(Q_0(\alpha)\) be a standard tableau of shape \(\alpha\) obtained by enumerating boxes in \(\alpha\) by \(1, 2, \ldots, |\alpha|\) from left to right in a row starting from the top row to bottom. By reversing the RSK insertion, we obtain two words \(w_1\) and \(w_2\) as \(w_1 := \text{RSK}^{-1}(\text{std}(A), Q_0(\alpha))\) and \(w_2 := \text{RSK}^{-1}(\text{std}(B), Q_0(\beta))\). The word \(w\) is defined as a concatenation of \(w_1\) and \(w_2\), i.e., \(w := w_1 \ast w_2\).

Step B. Given the word \(w\), we obtain a pair of standard tableaux of shape \(\gamma\) by the RSK insertion. We define \(\text{std}(T) := P_{\text{RSK}}(w)\) and \(\text{std}(U) := Q_{\text{RSK}}(w)\). Then, we perform a destandardization on \(\text{std}(T)\) and \(\text{std}(U)\) such that \(T\) is formed by an LRS word of content \(\lambda\) and \(U\) is formed by two Yamanouchi words of content \(\alpha\) and \(\beta\). The pair of tableaux obtained by the above procedure is \((T, U)\).

**Lemma 4.23.** The map \(\theta^{-1}\) satisfies

1. A tableau \(T\) is well-defined, i.e., \(T\) is a semistandard shifted tableau of shape \(\gamma\) of content \(\lambda\).
2. A tableau \(U\) is well-defined, i.e., \(U\) is a semistandard tableau of shape \(\gamma\) which is formed by a tableau \(\alpha\) of content \(\alpha\) and a skew tableau \(\gamma/\alpha\) of content \(\beta\).

**Proof.** For (1), we observe that the positions of letters from \(\sum_{k=1}^{i-1} \lambda_k + 1\) to \(\sum_{k=1}^i \lambda_k\) form a hook word of length \(\lambda_i\) in the word \(w\). Let \(\text{pos}(j)\) be the position (from left end) of \(\sum_{k=1}^{i-1} \lambda_k + i\) for \(1 \leq i \leq \lambda_i\) in \(w\). Since \(\text{pos}(j)\)’s form a hook word, there exists a unique \(i_0\) such that \(\text{pos}(1) > \text{pos}(2) > \cdots > \text{pos}(i_0) < \text{pos}(i_0 + 1) < \cdots < \text{pos}(\lambda_i)\). Recall that a pair \((A, B)\) satisfies that the concatenation of two words \(\text{read}(A) \ast \text{read}(B)\) is an LRS word of content \(\lambda\). Two words \(w_1\) and \(w_2\) are given by the reversed RSK insertion, and destandardization of the word \(w\) is an LRS word of content \(\lambda\). Since \(\text{pos}(j)\) with \(1 \leq j \leq i_0\) form a decreasing sequence in \(w\), the letters from \(\sum_{k=1}^{i-1} \lambda_k + 1\) to \(\sum_{k=1}^i \lambda_k + \lambda_0\) form a vertical strip by the RSK insertion. Similarly, since \(\text{pos}(j)\) with \(\lambda_0 \leq j \leq \lambda_i\) form an increasing sequence in \(w\), the letters from \(\sum_{j=1}^{i-1} \lambda_j + \lambda_0\) to \(\sum_{k=1}^i \lambda_k\) form a horizontal strip by the RSK insertion. From these observations, a tableau \(T\) is a semistandard shifted tableau of content \(\lambda\). The shape \(\gamma\) is nothing but the shape of \(P_{\text{RSK}}(w)\).

For (2), observe that \(w = w_1 \ast w_2\) where the length of \(w_1\) (resp. \(w_2\)) is \(|\alpha|\) (resp. \(|\beta|\)). The word \(w_1\) is written as \(w_1 = u_1 \ldots u_{l(\alpha)}\) where \(u_i\) is a strictly increasing sequence of length \(\alpha_i\) for \(1 \leq i \leq l(\alpha)\). Since the word \(w_1\) is given by the reversed RSK insertion, we have \(u_{i+1, \alpha_{i+1}} > u_{i+1, \alpha_{i+1}+1}\). More generally, we have \(u_{i+1, \alpha_{i+1}+j} > u_{i+1, \alpha_{i+1}+j+1}\) for \(0 \leq j \leq \alpha_{i+1}\). From these conditions, the shape of insertion tableau \(P_{\text{RSK}}(w_1)\) is \(\alpha\). The tableau \(U\) is obtained by destandardization of the recording tableau \(Q_{\text{RSK}}(w_1)\) and its shape is the same as the one of \(P_{\text{RSK}}(w_1)\). We insert \(w_2\) into \(P_{\text{RSK}}(w_1)\). Let \(\gamma\) be the shape of \(P_{\text{RSK}}(w)\). The word \(w_2\) is also rewritten as a concatenation of increasing
sequences. By a similar argument to the case of (1), \( U \) has the region \( \gamma/\alpha \) with content \( \beta \). This completes the proof. \qed

Summarizing the above observations, we have

**Lemma 4.24.** There exists a bijection \( \theta : \text{Tab}^\gamma(\lambda) \times \text{Tab}^\gamma(\alpha, \beta) \to \text{Tab}''(\alpha, \beta; \lambda) \).

**Example 4.25.** The product \( \hat{S}_{(4,2,1)} \) and \( \hat{S}_{(3,2,1)} \) contains \( 464P_{(5,4,3,1)} \). Thus, we have \( 464/2^4 = 29 \) pairs which satisfy the lattice property. Suppose \( \gamma = (4,4,2,2,1) \). We have two pairs of \((T,U) \in \text{Tab}^\gamma(\lambda) \times \text{Tab}^\gamma(\alpha, \beta)\). One of them is the left picture of Figure 4.26. We have an LRS word of content \((5,4,3,1)\) in \( T \) and two Yamanouchi words of contents \((4,2,1)\) and \((3,2,1)\) in \( U \). The right picture of Figure 4.26 is its standardization. By reversing the RSK insertion, we have the word \( w = \text{RSK}^{-1}(\text{stand}(T), \text{stand}(U)) = (2,7,11,13,1,12,8,3,6,10,4,9,5) \), which is a concatenation of two words \( w_1 \ast w_2 = (2,7,11,13,1,12,8) \ast (3,6,10,4,9,5) \). We obtain a pair of standard tableaux \((\text{stand}(A), \text{stand}(B)) := (P_{\text{RSK}}(w_1), P_{\text{RSK}}(w_2))\). By destandardization, we have

\[
(A,B) = \theta(T,U) = \begin{pmatrix}
1' & 2 & 2 & 3 & 1 & 1 & 1 \\
1 & 3 & 4 & 5 & 1 & 2 & 3 & 4 \\
2 & 3' & 3 & 5 \\
3 & 3 & 4 & 6 \\
4 & 5 & 7 & 10 & 11 & 12 & 8 & 13 & 13 & 11
\end{pmatrix}.
\]

The tableau word \( \text{read}(A) \ast \text{read}(B) \) is an LRS word of content \((5,4,3,1)\).

**Proof of Theorem 4.21.** The coefficient \( h_{\alpha,\beta}^\lambda \) can be expressed as

\[
h_{\alpha,\beta}^\lambda = [\hat{S}_\alpha \hat{S}_\beta, Q_\lambda] = \sum_\gamma a_{\alpha,\beta}^\gamma [S_\gamma, Q_\lambda] = \sum_\gamma 2^{(\lambda)} a_{\alpha,\beta}^\gamma b_\lambda^\gamma.
\]

The number \( a_{\alpha,\beta}^\gamma b_\lambda^\gamma \) in Eqn.(4.9) is equal to \( |\text{Tab}^\gamma(\lambda)| \cdot |\text{Tab}^\gamma(\alpha, \beta)| \). From Lemma 4.24, we have a bijection between \( \text{Tab}^\gamma(\lambda) \times \text{Tab}^\gamma(\alpha, \beta) \) and \( \text{Tab}''(\alpha, \beta; \lambda) \). Thus, \( \sum_\gamma a_{\alpha,\beta}^\gamma b_\lambda^\gamma = |\text{Tab}''(\alpha, \beta; \lambda)| \), which implies Theorem is true. \qed

4.3. Littlewood–Richardson–Stembridge coefficients revisited. We will show three different expressions of coefficient \( d_{\mu,\nu}^\rho \).

Let \( \lambda, \mu \) and \( \nu \) be strict partitions. We consider a standard shifted tableau \( S \) of shape \( \nu/\mu \). Let \( b_1, \ldots, b_\lambda \) be boxes in \( S \) labeled by an integer in the interval \( [\sum_{j=1}^{\lambda-1} \lambda_j + 1; \sum_{j=1}^\lambda \lambda_j] \). If a box \( b_j \) is in the \( r \)-th row in \( \nu \), we denote \( \text{ht}(b_j) := r \).
The proposition below is a shifted analogue of the Remmel–Whitney algorithm [17]. It directly follows from Lemma 8.4 and Lemma 8.6 in [23].

**Proposition 4.27.** The number \( d_{\lambda \mu}^\nu \) is the number of standard shifted tableaux of \( S \) of shape \( \nu/\mu \) satisfying the following:

1. \( \text{ht}(b_1) < \text{ht}(b_2) < \ldots < \text{ht}(b_r) \geq \text{ht}(b_{r+1}) \geq \ldots \geq \text{ht}(b_{\lambda}) \) for \( 1 \leq i \leq l(\lambda) \).
2. The shape of \( P_{\text{shift}}(\text{read}(S)) \) is \( \lambda \).

**Theorem 4.28.** We have

\[
d_{\lambda \mu}^\nu = \sum_\alpha b_\alpha^\lambda \cdot \# \{ T \in \text{Tab}(\nu/\mu; \alpha) \mid T \text{ is a Yamanouchi word} \}.
\]

**Remark 4.29.** Since \( b_\alpha^\lambda = e_\alpha^\lambda \), we have

\[
d_{\lambda \mu}^\nu = \sum_\alpha e_\alpha^\lambda \cdot \# \{ T \in \text{Tab}(\nu/\mu; \alpha) \mid T \text{ is a Yamanouchi word} \}
\]

from Theorem 4.28. From Theorem 4.15, \( e_\alpha^\lambda \) is expressed in terms of Yamanouchi words. Thus, we have a description of \( d_{\lambda \mu}^\nu \) in terms of Yamanouchi words rather than LRS words. The coefficient \( e_\alpha^\lambda \) is expressed in terms of weak reading words. Thus, the complexity of an LRS word is replaced with a weak reading word.

For a proof of Theorem 4.28, we introduce the following lemma. Let \( \alpha \) be an ordinary partition and \( \lambda, \mu \) and \( \nu \) be strict partitions. Let \( \text{Tab}^\alpha(\lambda) \) (resp. \( \text{Tab}^\nu(\nu/\mu; \alpha) \)) be the set of shifted tableaux of shape \( \alpha \) (resp. \( \nu/\mu \)) which have an LRS word of content \( \lambda \), and \( \text{Tab}^\nu(\nu/\mu; \alpha) \) be the set of skew tableaux of shape \( \nu/\mu \) which have a Yamanouchi word of content \( \alpha \).

**Lemma 4.30.** There exists a bijection \( \zeta : \text{Tab}^\nu(\nu/\mu; \alpha) \to \text{Tab}^\alpha(\lambda) \times \text{Tab}^\nu(\nu/\mu; \alpha) \).

**Proof.** We construct a map \( \zeta : \text{Tab}^\nu(\nu/\mu; \alpha) \to \text{Tab}^\alpha(\lambda) \times \text{Tab}^\nu(\nu/\mu; \alpha), T \mapsto (A, B) \), as follows. Since \( \text{read}(T), T \in \text{Tab}^\nu(\nu/\mu; \alpha) \), is an LRS word of \( \lambda \), we first perform a standardization of \( T \) and define a word \( w := \text{read}(\text{stand}(T)) \). By the RSK insertion, we obtain a pair of standard tableaux \( (P_{\text{RSK}}(w), Q_{\text{RSK}}(w)) \) of shape \( \alpha \). A tableau \( A \in \text{Tab}^\nu(\nu/\mu; \alpha) \) is obtained by destandardization of \( P_{\text{RSK}}(w) \) with respect to the content \( \lambda \). We construct a tableau \( B \in \text{Tab}^\nu(\nu/\mu; \alpha) \) from \( Q_{\text{RSK}}(w) \) as follows. Let \( P_0(\alpha) \) be a standard tableau enumerated by \( 1, \ldots, |\alpha| \) from left to right in row starting from the top row to bottom. By reversing the RSK insertion, we obtain a word \( w' := \text{RSK}^{-1}(P_0(\alpha), Q_{\text{RSK}}(w)) \). A standard tableau \( \text{stand}(B) \) of shape \( \nu/\mu \) is given from \( w' \) such that the reading word \( \text{stand}(B) \) is \( w' \). Finally, we perform a destandardization on \( \text{stand}(B) \) with respect to the content \( \alpha \).

We claim:

1. A standard tableau \( \text{stand}(B) \) is well-defined. In other words, the word \( w' \) is compatible with the skew shape \( \nu/\mu \).
2. A semistandard tableau \( B \) is compatible with the skew shape \( \nu/\mu \) and the word \( \text{read}(B) \) is a Yamanouchi word of content \( \alpha \).

We show that \( \text{stand}(B) \) is a well-defined standard tableau. Let \( \xi \in \mathbb{Z}^{l(\nu)} \) be a sequence of non-negative integers of length \( l(\nu) \) defined by \( \xi := \nu - \mu \) where we set \( \mu_i = 0 \) for \( l(\mu) < i \leq l(\nu) \). In the skew shape \( \nu/\mu \), we have \( \xi_i \) boxes in the \( i \)-th row. Given \( T \in \text{Tab}^\nu(\nu/\mu; \lambda) \), the \( \xi_i \) letters in \( i \)-th row are strictly increasing in \( \text{stand}(T) \). By the RSK insertion of \( w \), the letters from \( \sum_{j=1}^{i-1} \xi_j + 1 \) to \( \sum_{j=1}^i \xi_i \) form a horizontal strip in the recording tableau \( Q_{\text{RSK}}(w) \). We obtain \( w' \) by reversing the RSK insertion starting from the pair \( (P_0(\alpha), Q_{\text{RSK}}(w)) \). The above property of \( Q_{\text{RSK}}(w) \) implies that
$w'$ is written as a concatenation of words $w' := w'_1 \cdots w'_r$ where $w'_i$ is an increasing sequence of length $\xi_i$ for $1 \leq i \leq l(\nu)$. This means that $\text{stand}(B)$ is standard in a row.

Suppose that the skew shape $\nu/\mu$ has $k$ boxes in the main diagonal. We enumerate rows of $\nu/\mu$ by $1, 2, \ldots, l(\nu)$ from bottom to top. The $k$ rows from bottom form a shifted tableau $\tau := (\tau_1, \tau_2, \ldots, \tau_k)$ and rows from $k$-th to $l(\nu)$-th form a ordinary skew shape $\beta/\gamma$. Note that both $\tau$ and $\beta/\gamma$ contain the $k$-th row of $\nu/\mu$. We consider a sequence of ordinary partitions obtained from a shifted partition $\tau$ by $\emptyset = \tau_0 \subseteq \tau_1 \subseteq \ldots \subseteq \tau_k = \tau$ where $\tau_i := (\tau_{k-i+1}, \tau_{k-i+2}, \ldots, \tau_k)$. The skew shape $\tau_i/\tau_{i-1}$ forms a horizontal strip of length $\tau_{k-i+1}$. Then, we put letters from $\sum_{i=1}^{k-1} \tau_{k-i+1} + 1$ to $\sum_{i=1}^{k-1} \tau_{k-i+1}$ on the boxes of $\tau_i/\tau_{i-1}$ from left to right. We obtain a standard tableau of ordinary shape $\tau$ and denote it by $Q(\tau)$. By the RSK insertion of $w$, the recording tableau $Q(\nu/\mu)$ contains the tableau $Q(\tau)$. We enumerate boxes in $\nu/\mu$ by $1, 2, \ldots, |\nu/\mu|$ from left to right in a row starting from the bottom rows to top. When the boxes labelled by letters in $[i, j]$ forms a row in $\nu/\mu$, the letters $[i, j]$ form a horizontal strip in $Q(\nu/\mu)$. Suppose that the box $b_j$ labelled as $j$ is in the same column as the box $b_i$ labelled as $i$ with $i < j$ in $\nu/\mu$. When the boxes $b_i$ and $b_j$ are in $\beta/\gamma$, the letter $j$ is below the letter $i$ in $Q(\nu/\mu)$. Similarly, when the boxes $b_i$ and $b_j$ are in $\tau$, the letter $j$ is below the letter $i$ or next to each other in the same row in $Q(\nu/\mu)$. Since $(P_0, Q_0) = (Q(\nu/\mu), P_0(\alpha))$ for a permutation $\rho$, we consider the inverse of $w'$ where $w' = \text{RSK}^{-1}(Q(\nu/\mu), P_0(\alpha))$. By reversing the RSK insertion, if letters $i_1 < i_2 < \ldots < i_r$ are in the same column in $\nu/\mu$, $i_r, i_{r-1}, \ldots, i_1$ appear in $w'$ as a decreasing sequence. By considering the inverse of $w'$, the letters in boxes labelled $i_1, i_2, \ldots, i_r$ are decreasing from bottom to top. This implies that $\text{stand}(B)$ is column standard. Since $\text{stand}(B)$ is standard in both rows and columns, the statement (S6) is true.

Let $\tilde{P}_0(\alpha)$ be a unique semistandard tableau of shape $\alpha$ and content $\alpha$. The statement (S7) is equivalent to show that the word $\tilde{w} := \text{RSK}^{-1}(\tilde{P}_0(\alpha), Q(\nu/\mu))$ fits to the skew shape $\nu/\mu$ and is Yamanouchi word of content $\alpha$. Since $\tilde{P}_0(\alpha)$ is the standardization of $P_0(\alpha)$, the word $\tilde{w}$ is compatible with the shape $\nu/\mu$ by a similar argument to the case of (S6). Since the contents of $i$-th row of $\tilde{P}_0(\alpha)$ are all $i$’s, it is obvious that $\tilde{w}$ is a Yamanouchi. These observations imply that (S7) is true.

From the construction, it is obvious that the map $\zeta$ has an inverse and well-defined by a similar argument to a proof of (S6) and (S7). Thus, $\zeta$ is a bijection. 

\[ \text{Example 4.31. Let } \lambda := (6, 4, 2), \mu := (5, 2) \text{ and } \nu := (7, 5, 4, 2, 1). \text{ The product } P_\lambda P_\mu \text{ contains } 10P_\nu. \text{ A tableau } T \text{ in } \text{Tab}^{\nu/\mu}(\lambda) \text{ is given by the left picture of Figure 4.32 and its standardization is in the right picture. The reading word of } \text{stand}(T) \text{ is } \tilde{w} = \text{read(stand}(T)) = \]

\begin{figure}[h]
\centering
\begin{tabular}{cccc}
* & * & * & * & 1' & 1 & * & * & * & * & 1' & 1 & * & * & * & * & 1 & 6 \\
* & * & 1 & 1 & 1 & * & * & 3 & 4 & 5 & 1 & 2' & 2 & 2 & 2 & 3 & 9 & 10 & 3 & 8 & 11 & 2 & 12
\end{tabular}
\caption{A tableau $T$ and its standardization}
\end{figure}
By the RSK insertion, we have

$$\begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 1 & 3 & 6 & 7 & 12 \\ 2 & 9 & 10 & 2 & 5 & 10 \\ 7 & 11 & 4 & 9 \\ 8 & 8 \\ 12 & 11 \end{pmatrix}.$$ 

By reversing the RSK insertion, we have a word

$$w' = \text{RSK}^{-1}(P_0(\alpha), Q_{\text{RSK}}(w)) = (12, 9, 11, 1, 6, 7, 10, 2, 3, 8, 4, 5).$$

A pair of standard tableaux $\text{stand}(A), \text{stand}(B)$ is given by

$$\begin{pmatrix} 1 & 3 & 4 & 5 & 6 & * & * & * & * & 4 & 5 \\ 2 & 9 & 10 & * & 2 & 3 & 8 \\ 7 & 11 & 1 & 6 & 7 & 10 \\ 8 & 9 & 11 \\ 12 & 12 \end{pmatrix},$$

which yields a pair of tableaux $(A, B)$:

$$\begin{pmatrix} 1' & 1 & 1 & 1 & 1 & * & * & * & * & 1 & 1 \\ 1 & 2 & 2 & * & * & 1 & 1 & 2 \\ 2' & 3' & * & 1 & 2 & 2 & 3 \\ 2 & 3 & 4 \\ 3 & 5 \end{pmatrix} \in \text{Tab}^\alpha(\lambda) \times \text{Tab}^{\nu/\mu}(\alpha).$$

Proof of Theorem 4.28. From Theorem 2.2, we have $d^\nu_\lambda \mu = |\text{Tab}^{\nu/\mu}(\lambda)|$. From Lemma 4.30, we have $|\text{Tab}^{\nu/\mu}(\lambda)| = \sum_\alpha |\text{Tab}^\alpha(\lambda)| \cdot |\text{Tab}^{\nu/\mu}(\alpha)|$, which implies Theorem 4.28. □

The coefficient $d^\nu_\lambda \mu$ can be expressed in terms of semistandard increasing decomposition tableaux $\epsilon^+(\lambda)$ and $\epsilon^+(\mu)$ as follows.

Let $T$ and $U$ be a semistandard increasing decomposition tableaux of shape $\epsilon^+(\lambda)$ and $\epsilon^+(\mu)$. We denote by $w_\lambda := \text{read}(\epsilon^+(T)), w_\mu := \text{read}(\epsilon^+(U))$. Let Word($\mu, \lambda$) be the set

$$\text{Word}(\mu, \lambda) := \{ (\epsilon^+(T), \epsilon^+(U)) \mid w_\mu * w_\lambda \text{ is a shifted Yamanouchi word of content } \nu \}.$$ 

Theorem 4.33. We have

$$d^\nu_\lambda \mu = |\text{Word}(\mu, \lambda)|.$$

We introduce a lemma used in a proof of Theorem 4.33. Recall that $\text{Tab}^{\nu/\mu}(\lambda)$ is the set of shifted tableaux of skew shape $\nu/\mu$ which have an LRS word of content $\lambda$.

Lemma 4.34. There exists a bijection $\kappa : \text{Tab}^{\nu/\mu}(\lambda) \rightarrow \text{Word}(\mu, \lambda), T \mapsto (w_1, w_2)$.

Proof. A map $\kappa : \text{Tab}^{\nu/\mu}(\lambda) \rightarrow \text{Word}(\mu, \lambda)$ is given by the following three steps.

Step 1. We perform a standardization on a tableau $T$. Then, we obtain a word $v_1$ as the inverse permutation of the reading word of stand($T$), i.e., $v_1 := (\text{read(stand(T))})^{-1}$. 

(12, 8, 11, 2, 7, 9, 10, 3, 4, 5, 1, 6). By the RSK insertion, we have

$$\begin{pmatrix} 1 & 3 & 4 & 5 & 6 & 1 & 3 & 6 & 7 & 12 \\ 2 & 9 & 10 & 2 & 5 & 10 \\ 7 & 11 & 4 & 9 \\ 8 & 8 \\ 12 & 11 \end{pmatrix}.$$
Step 2. Let $\epsilon_0^+(\lambda)$ be a tableau of shape $\epsilon^+(\lambda)$ such that the boxes are enumerated by $1, 2, \ldots, |\lambda|$ from left to right in row starting from the bottom row to top. We slide down the columns of $\epsilon_0^+(\lambda)$ such that the lowest box in a column is placed in the same height as the unique box in the leftmost column of $\epsilon_0^+(\lambda)$. We denote by $Q$ the obtained tableau. By turning $Q$ upside down and moving its rows to fit the shifted tableau $\lambda$, we obtain a standard tableau $Q_0(\lambda)$. For example, $Q_0(\lambda)$ for $\lambda = (6, 3, 1)$ is given by

$$
\begin{array}{ccccccc}
& & & & & & 10 \\
& & & & & 8 & 9 \\
3 & 4 & 6 & 7 & & & \\
& & & & 1 & 2 & 3 & 4 & 6 & 7 & \\
& & & 1 & & & & & & & & \\
\end{array}
$$

From $v_1$ and $Q_0(\lambda)$, we have a pair of shifted tableaux $(P_{\text{mix}}(v_1), Q_0(\lambda))$. The tableau $P_{\text{mix}}(v_1)$ may have primed letters but $Q_0(\lambda)$ does not. By reversing the mixed insertion, we obtain a word $v'_2 := \text{mixRSK}^{-1}(P_{\text{mix}}(v_1), Q_0(\lambda))$. We define $v_2$ as the inverse permutation of $v'_2$.

Step 3. We put letters in a tableau $U'$ of shape $\nu/\mu$ such that the reading word $\text{read}(U')$ is equal to $v_2$. We construct a tableau $U$ of shape $\nu$ from $U'$ by enumerating boxes in $\mu \subseteq \nu$ by $1, 2, \ldots, |\mu|$ as $Q_0(\mu)$ and boxes in $\nu/\mu$ by $|\mu| + 1, \ldots, |\nu|$ according to the letters in $U'$, i.e., a letter in the region $\nu/\mu$ in $U$ is a letter in $U'$ plus $|\mu|$. Let $P_0(\nu)$ be a unique tableau of shape $\nu$ and of content $\nu$. Then, by reversing the mixed insertion, we obtain a word $w := \text{mixRSK}^{-1}(P_0(\nu), U)$. The word is written as a concatenation of two words $w = w_1 * w_2$ where $w_1$ (resp. $w_2$) is of length $|\mu|$ (resp. $|\lambda|$). This gives a map $\kappa : T \mapsto (w_1, w_2)$.

We claim the following with respect to the map $\kappa$.

- (S8) The word $v_2$ is compatible with the shape $\nu/\mu$.
- (S9) The words $w_1$ and $w_2$ are compatible with the shapes $\epsilon^+(\mu)$ and $\epsilon^+(\lambda)$.

For (S8), let $\xi \in \mathbb{Z}^{l(\nu)}$ be a sequence of non-negative integers defined by $\xi := \nu - \mu$ where $\mu_i = 0$ for $l(\mu) < i \leq l(\nu)$. We enumerate the boxes in $\nu/\mu$ by $1, 2, \ldots, |\lambda|$ from left to right in a row starting from the bottom row to top. We denote by $b_i$ the box in $\nu/\mu$ labelled $i$. The $i$-th row (from top) of the shape $\nu/\mu$ has $\xi_i$ boxes. Since $\text{stand}(T)$ is a standard tableau, the word $v_1$ has the following properties: 1) the letters from $\sum_{j=i+1}^{l(\nu)} \xi_j + 1$ to $\sum_{j=i}^{l(\nu)} \xi_j$ for $1 \leq i \leq l(\nu)$ form an increasing sequence in $v_1$, and 2) if the boxes $b_i$ and $b_j$ ($i < j$) are in the same column in $\nu/\mu$, the letter $j$ is left to the letter $i$ in $v_1$. To show that $v_2$ is compatible with the shape $\nu/\mu$, it is enough to show that $v'_2$ satisfies the properties 1) and 2) (replace $v_1$ with $v'_2$). We first show that $v'_2$ satisfies the property 2). By definition, two words $v_1$ and $v_2$ have the same mixed insertion tableau. From Theorem 2.5, we have $v_1 \sim v_2$. Suppose that the box $b_{i+1}$ is just above the box $b_i$ in $\nu/\mu$. The letter $i + 1$ is left to $i$ in the word $v_1$. In the plactic relations, there is no relation which exchanges the letters $i$ and $i + 1$. Therefore, the letter $i + 1$ is left to the letter $i$ even in the word $v'_2$. Suppose that the boxes $b_{i+l}, b_{i+l+1}, \ldots, b_{i+l-1}$ are just above the boxes $b_1, b_{i+1}, \ldots, b_{i+l-1}$ in $\nu/\mu$. The letter $i + j + l$ is left to the letter $i + j$ for $0 \leq j \leq l - 1$ in the word $v_1$. If the letter $i + j$ is left to the letter $i + j + l$ in $v'_2$, we have to exchange $i + j + l$ and $i + j$ in $v_1$ by a plactic relation. Since a plactic relation is applied to a word of length four, it is enough to consider a word $\tilde{w}$ of length four including $i + j + l$ and $i + j$ such that $i + j$ and $i + j + l$ are next to each other in $\tilde{w}$. We have four cases:

(a) $j < l - 1$ and $\tilde{w}$ contains $i + j + 1$,
(b) $j < l - 1$ and $\tilde{w}$ does not contain $i + j + 1$, 

(c) $j = l - 1$ and $\tilde{w}$ contains $i + j + l - 1$.
(d) $j = l - 1$ and $\tilde{w}$ does not contain $i + j + l - 1$.

For case (a), since $i + j + l + 1$ is right to $i + j + l$ and left to $i + j + 1$, $\tilde{w} = (i + j + l)(i + j + l + 1)(i + j + 1)$. The word $\tilde{w}$ is locally equivalent to a word $cadb$ with $a < b < c < d$. None of the plactic relations exchanges $a$ and $c$. For case (b), we have two cases: (i) $\tilde{w}$ contains $i + j + l - 1$ and (ii) $\tilde{w}$ does not contain $i + j + l - 1$. For case (b-i), since the letter $i + j - 1$ is right to $i + j + l - 1$ and left to $i + j$, $\tilde{w} = (i + j + l - 1)(i + j - 1)(i + j + l)(i + j)$. Locally, $\tilde{w}$ is equivalent to $cadb$ with $a < b < c < d$. There is no plactic relation which exchanges $b$ and $d$. For case (b-ii), observe that the word $\tilde{w}$ neither contains $i + j + 1$ nor $i + j + l - 1$. If $\tilde{w}$ is formed by four letters $a, b, c$ and $d$ with $a < b < c < d$, the letters $i + j$ and $i + j + 1$ form a partial word $ab$, $bc$ or $cd$ in $\tilde{w}$. However, there is no plactic relation which exchanges $ab$, $bc$ or $cd$. By a similar argument, one can show that there is no plactic relation which exchanges $i + j$ and $i + j + l$ for cases (c) and (d). Summarizing above observations, the letter $i + j + 1$ is left to the letter $i + j$ in the word $v'_2$. This implies that $v'_2$ satisfies the property 2). By a similar argument, one can show that $v'_2$ satisfies the property 1). Thus, the statement (S8) is true.

For the statement (S9), observe that if two letters $i$ and $i + 1$ are in the same row in $\epsilon^+_0(\lambda)$, $i$ and $i + 1$ form a horizontal strip in $Q_0(\lambda)$. Similarly, if the letters $i_1 < i_2 < \ldots < i_r$ are in the same column in $\epsilon^+_0(\lambda)$, these letters form a vertical strip in $Q_0(\lambda)$. Since the reading word of the tableau $U'$ is $v_2$, these properties are hold by $U'$. The tableau $U$ is divided into two regions $\mu$ and $\nu / \mu$. Both regions have the same properties as $Q_0(\mu)$ and $U'$ (or equivalently $Q_0(\lambda)$). Finally, since the word $w$ is obtained by reversing the mixed insertion with the recording tableau $U$, the word $w_1$ and $w_2$ fit to the shapes $\epsilon^+(\mu)$ and $\epsilon^+(\lambda)$. Thus, the statement (S9) is true.

From the construction, the map $\kappa$ has the inverse. By a similar argument discussed above, it is easy to show that the map $\kappa^{-1}$ is well-defined. This completes the proof.

Example 4.35. Let $\lambda = (5, 3)$, $\mu = (3, 1)$ and $\nu = (6, 4, 2)$. The product of $P_\lambda P_\mu$ contains $4P_\nu$. An example of $T$ in Tab$^{q'_\nu / \mu}(\lambda)$ is given by

$$T = \begin{array}{ccc}
1' & 1 & 1 \\
1 & 2 & 2' \\
2 & 2 & 2
\end{array}.$$ 

Since the word read(stand($T$)) is 78236145, we have $v_1 = 63478512$. Thus, we have a pair of tableaux

$$P_{mix}(v_1), Q_0(\lambda) := \begin{pmatrix} 1 & 2 & 3' & 6' & 8 \\ 4 & 5 & 7' & 1 & 2 & 3 & 5 & 7 \end{pmatrix}.$$ 

By reversing the mixed insertion, we obtain words $v'_2$ and $v_2$ as $v'_2 = 63745182$ and $v_2 = 68245137$. The tableau $U'$ and $U$ is given from $v_2$ by

$$U' = \begin{array}{cccc}
* & * & * & 1 \\
1 & 2 & 3 & 7 \\
6 & 8 & 9 & 11
\end{array}, \quad U = \begin{array}{cccc}
1 & 2 & 3 & 5 \\
4 & 8 & 9 & 10 \\
10 & 12
\end{array}.$$ 

The word $w = w_1 * w_2$ is obtained from $P_0(\lambda)$ and $U$:

$$w = \text{mixRSK}^{-1}\left(\begin{pmatrix} 1 & 1 & 1 & 1 & 1 & 1 & 1 & 1 \\ 1 & 2 & 3 & 5 & 7 & 11 \\ 2 & 2 & 2 & 2 \\ 3 & 3 \end{pmatrix} \right) = 1221 * 31312121.$$
Note that \( w \) is a shifted Yamanouchi word and the word \( w_1 \) (resp. \( w_2 \)) is compatible with the shape \( \epsilon^+(\mu) \) (resp. \( \epsilon^+(\lambda) \)).

Proof of Theorem 4.33. From Theorem 2.2, we have \( d_{\alpha\mu}^\nu = |\text{Tab}^\nu/\mu(\lambda)| \). From Lemma 4.34, we have \( |\text{Tab}^\nu/\mu(\lambda)| = |\text{Word}(\lambda, \mu)| \), which implies Theorem is true. \( \square \)

5. \( \hat{S} \)-functions and products of \( P \)-functions

5.1. \( \hat{S} \)-function in terms of products of \( P \)-functions. Let \( \alpha \) be an ordinary partition and \( \lambda, \mu \) be strict partitions such that \( \alpha = \lambda \otimes \mu \). We regard a strict partition as a set of positive integers. Let \( A = (A_1, A_2, \ldots) := (\lambda \cup \mu) \setminus (\lambda \cap \mu) \) be a decreasing integer sequence. We assign a sign to an element of \( A \): \( \text{sign}(A_i) := (-1)^{i-1} \).

We define a pair of strict partitions \((\lambda', \mu')\) from \((\lambda, \mu)\) as follows. As sets of positive integers, \( \lambda' \) and \( \mu' \) satisfy \( \lambda' \subseteq \lambda, \mu' \supseteq \mu, \lambda' \cup \mu' = \lambda \cup \mu \) and \( \lambda' \cap \mu' = \lambda \cap \mu \). We denote by \( S_1(\lambda, \mu) \) the set of pairs of strict partitions \((\lambda', \mu')\). We define the sets

\[
S_1^+(\lambda, \mu) := \{ (\lambda', \mu') \in S_1(\lambda, \mu) \mid |\mu' \setminus \mu| \equiv 0 \pmod{2} \}, \\
S_1^-(\lambda, \mu) := S_1(\lambda, \mu) \setminus S_1^+(\lambda, \mu).
\]

Similarly, let \( S_2(\lambda, \mu) \) be the set of pairs of strict partitions \((\lambda'', \mu'')\) such that \( \lambda'' \supseteq \lambda, \mu'' \subseteq \mu, \lambda'' \cup \mu'' = \lambda \cup \mu \) and \( \lambda'' \cap \mu'' = \lambda \cap \mu \).

When \( (\lambda', \mu') \in S_1(\lambda, \mu) \) or \( (\lambda'', \mu'') \in S_2(\lambda, \mu) \), we define two signs \( \text{sign}_1(\lambda', \mu'; \lambda, \mu) \) and \( \text{sign}_2(\lambda'', \mu''; \lambda, \mu) \) as

\[
\text{sign}_1(\lambda', \mu'; \lambda, \mu) := \prod_{a \in A \cap (\lambda' \cup \lambda'')} \text{sign}(a), \\
\text{sign}_2(\lambda'', \mu''; \lambda, \mu) := (-1)^{|\mu' \setminus \mu''|} \prod_{a \in A \cap (\mu' \cap \mu'')} \text{sign}(a).
\]

Theorem 5.1. We have

\begin{equation}
\hat{S}_\alpha = 2^{l(\mu)} \sum_{(\lambda', \mu') \in S_1(\lambda, \mu)} \text{sign}_1(\lambda', \mu'; \lambda, \mu) P_{\lambda'} P_{\mu'},
\end{equation}

for \( l(\lambda) - l(\mu) = 0 \) or \( 1 \) and

\begin{equation}
\hat{S}_\alpha = 2^{l(\lambda)} \sum_{(\lambda', \mu') \in S_1^+(\lambda, \mu)} \text{sign}_1(\lambda', \mu'; \lambda, \mu) P_{\lambda'} P_{\mu'},
\end{equation}

\begin{equation}
= 2^{l(\lambda)} \sum_{(\lambda', \mu') \in S_1^-(\lambda, \mu)} \text{sign}_1(\lambda', \mu'; \lambda, \mu) P_{\lambda'} P_{\mu'},
\end{equation}

for \( l(\lambda) = l(\mu) + 1 \). We also have

\begin{equation}
\hat{S}_\alpha = 2^{l(\lambda)} \sum_{(\lambda'', \mu'') \in S_2(\lambda, \mu)} \text{sign}_2(\lambda'', \mu''; \lambda, \mu) P_{\lambda''} P_{\mu''},
\end{equation}

for \( l(\lambda) - l(\mu) = 0 \) or \( 1 \).
Example 5.2. Let $\lambda := (5, 4, 2)$ and $\mu := (3, 1)$. Then, we have $\alpha = \lambda \otimes \mu = (4, 3, 3, 2)$. The $\hat{S}_\alpha$ is written in terms of $P$-functions in three ways as
\[
\hat{S}_\alpha = 2^2 (P_{(5, 4, 2)} P_{(3, 1)} + P_{(5)} P_{(4, 3, 2, 1)} - P_{(4)} P_{(5, 3, 2, 1)} - P_{(2)} P_{(5, 4, 3, 1)})
\]
\[
= 2^2 (P_{(4, 2)} P_{(5, 3, 1)} - P_{(5, 2)} P_{(4, 3, 1)} - P_{(5, 4)} P_{(3, 2, 1)} + P_{(4, 5, 3, 2, 1)})
\]
\[
= 2^2 (P_{(5, 4, 2)} P_{(3, 1)} - P_{(5, 4, 3, 2)} P_{(1)} - P_{(5, 4, 2, 1)} P_{(3)} + P_{(5, 4, 3, 2, 1)})
\]

The first and second expressions are Eqn. (5.2) and they imply Eqn. (5.1). The third expression comes from Eqn. (5.3).

We will prove Theorem 5.1 by induction. Before we move to a proof of Theorem 5.1, we introduce two lemmas needed later.

Lemma 5.3. Suppose that $\alpha := \lambda \otimes \mu$ with $l(\lambda) = l(\mu) + 1$ and $\beta := \sigma \otimes \rho$ with $l(\sigma) = l(\rho)$. The function $\hat{S}_\alpha$ satisfies Eqn. (5.2) if Eqn. (5.1) is true for all $\beta$’s satisfying $|\beta| < |\alpha|$.

Proof. Let $\hat{\lambda}$ be a strict partition obtained by deleting $\lambda_i$ from $\lambda$. Recall that $\alpha^T$ is the conjugate partition of $\alpha$. A $\hat{S}$-function satisfies $\hat{S}_\alpha = \hat{S}_\alpha^T$ and $S^T_\alpha$ has a determinant expression (2.3). By expanding a determinant with respect to the first column, we have
\[
\hat{S}_\alpha = \hat{S}_\alpha^T
\]
\[
= \sum_{i=1}^{l(\lambda)} (-1)^{i-1} q_{\lambda_i} \hat{S}_{\mu \otimes \lambda_i}
\]
\[
= 2^{l(\lambda)} \sum_{i=1}^{l(\lambda)} \sum_{(\lambda', \mu') \in \mathcal{S}_1(\mu, \lambda_i)} (-1)^{i-1} \text{sign}_1(\lambda', \mu'; \mu, \hat{\lambda}_i) P_{\lambda_i} P_{\lambda_i}; P_{\mu'}
\]
Since $P_{\lambda}$ has an expression by a Pfaffian (see Eqn. (2.6)), we have
\[
\sum_{i=1}^{l(\lambda)} (-1)^{i-1} P_{\lambda_i} P_{\lambda_i} = \left\{ \begin{array}{ll}
0, & \text{for } l(\lambda) \equiv 0 \pmod{2} \\
1, & \text{for } l(\lambda) \equiv 1 \pmod{2}.
\end{array} \right.
\]
By Substituting Eqn. (5.5) to a product $P_{\lambda_i} P_{\mu'}$ in Eqn. (5.4) and rearranging the terms, we obtain Eqn. (5.2).

Lemma 5.4. Suppose that $\alpha := \lambda \otimes \mu$ with $l(\lambda) = l(\mu)$ and $\beta := \sigma \otimes \rho$ with $l(\sigma) = l(\rho)$. The function $\hat{S}_\alpha$ satisfies Eqn. (5.1) if Eqn. (5.1) is true for all $\beta$ such that $|\beta| < |\alpha|$ or $\mu_{l(\mu)} > \rho_{l(\rho)}$ for $|\alpha| = |\beta|$ with $l(\mu) = l(\rho)$.

Proof. Let $\gamma := \lambda' \otimes \mu'$ with $l(\lambda') = l(\mu') + 1$. Since Eqn. (5.1) is true for all $\beta$ satisfying $|\beta| < |\alpha|$, by Lemma 5.3, Eqn. (5.2) is true for all $\gamma$ with $|\gamma| \leq |\alpha|$. We have two cases: (a) $\mu_{l(\mu)} = 1$ and (b) $\mu_{l(\mu)} \geq 2$.

For (a), suppose that $\mu_{l(\mu)} = 1$. We denote by $\tilde{\mu}$ a strict partition obtained from $\mu$ by deleting $\mu_{l(\mu)}$ and by $\tilde{\alpha} := \lambda \otimes \tilde{\mu}$. Since $|\tilde{\alpha}| = |\alpha| - 1$ and $l(\lambda) = l(\tilde{\mu}) + 1$, the function $\hat{S}_{\tilde{\alpha}}$ satisfies Eqn. (5.2). We consider the product of $\hat{S}_{\tilde{\alpha}}$ and $\hat{S}_{(1)}$. A product of two $\hat{S}$-functions is expressed in terms of the Littlewood–Richardson rule (see Eqn. (2.1)),
\[
\hat{S}_{(1)} \hat{S}_{(1)} = \hat{S}_\alpha + \sum_{\alpha' \neq \alpha} \hat{S}_{\alpha'}
\]
where the sum is taken over all \( \alpha' \)'s such that \( \alpha'/\hat{\alpha} \) is a single box and \( \alpha' \neq \alpha \). Note that multiplicity of a \( \hat{S} \)-function is one. We have \( \hat{S}(1) = 2P(1) \). On the other hand, a \( \hat{S} \)-function is expressed as a sum of products of \( P \)-functions. Recall that a product \( P_{\lambda}P_{\mu}(1) \) is expressed by the Littlewood–Richardson–Stembridge rule (see Theorem 2.2), namely

\[
P_{\lambda}P_{\mu}(1) = \sum_{\lambda'} P_{\lambda'}
\]

where the sum is taken over all \( \lambda' \)'s such that \( \lambda' / \lambda \) is a single box. Thus, a product of \( P_{\lambda}P_{\mu} \) and \( \hat{S}(1) \) is expressed as

\[
\hat{S}(1)P_{\lambda}P_{\mu} = 2P_{(1)}P_{\lambda}P_{\mu} = (P_{(1)}P_{\lambda})P_{\mu} + P_{\lambda}(P_{(1)}P_{\mu})
\]

(5.7)

\[
= \sum_{\lambda'} P_{\lambda'}P_{\mu} + \sum_{\mu'} P_{\lambda}P_{\mu'},
\]

where the skew shapes \( \lambda' / \lambda \) and \( \mu' / \mu \) are a single box. In Eqn. (5.6), \( \hat{S}_{\alpha'} \) with \( \alpha' \neq \alpha \) satisfies Eqn. (5.1) or Eqn. (5.2) by the assumption. We multiply \( \hat{S}(1) \) by both sides of Eqn. (5.2) for \( \alpha \) and rearrange the terms for \( \hat{S}_{\alpha'} \). By a direct calculation, one can show that \( \hat{S}_{\alpha} \) satisfies Eqn. (5.1).

For (b), we have \( \mu(\mu) \geq 2 \). We denote by \( \tilde{\mu} \) a strict partition obtained from \( \mu \) by replacing \( \mu(\mu) \) with \( \mu(\mu) - 1 \) and define \( \hat{\alpha} := \lambda \otimes \tilde{\mu} \). We consider the product \( \hat{S}_{\alpha'}\hat{S}(1) \). By a similar argument to case (a), one can show that \( \hat{S}_{\alpha} \) satisfies Eqn. (5.1). \( \square \)

Given an ordinary partition \( \alpha \), we have \( \alpha = \lambda \otimes \mu \). Let \( l := l(\lambda) \) if \( l(\lambda) = l(\mu) + 1 \) and \( l := l(\lambda) + 1 \) if \( l(\lambda) = l(\mu) \). If \( l(\lambda) = l(\mu) + 1 \), we regard \( \mu \) as a strict partition of length \( l \) by adding zero to \( \lambda \). Similarly, if \( l(\lambda) = l(\mu) \), we regard \( \lambda \) and \( \mu \) as strict partitions of length \( l \) by adding zero to \( \lambda \) and \( \mu \).

We place integers in \( \lambda \cup \mu \) in a decreasing order from left to right. When an integer \( n \) is in \( \lambda \cap \mu \), we put two \( n \)'s next to each other and denote by \( n_{\lambda} \) (resp. \( n_{\mu} \)) the left (resp. right) \( n \). The index of \( n_{\lambda} \) stands for \( n \in \lambda \). We construct a perfect matching of length \( 2l \) by connecting two integers via an arc. A perfect matching satisfies the following conditions.

(1) We connect two integers \( n \) and \( m \) if and only if \( n \in \lambda \) and \( m \in \mu \).
(2) We do not connect the same integers \( n_{\lambda} \) and \( n_{\mu} \) for \( n \geq 1 \).

Note that two integers \( 0_{\lambda} \) and \( 0_{\mu} \) can be connected by an arc. Thus, a perfect matching characterizes a permutation \( \pi \in S_l \) since an arc connects two integers \( \lambda_i \) and \( \mu_{\pi(i)} \). We denote by \( c(\lambda, \mu; \pi) \) the number of crossings in a perfect matching \( \pi \). We assign a sign for an element of the sequence \( s := s_1 \ldots s_{2l} \) of integers \( \lambda \cup \mu \). Here, the multiplicity of \( n \) with \( n \in \lambda \cap \mu \) is two in the sequence \( s \). The sign \( \text{sign}(s) \) is defined as \((-1)^{c(\lambda, \mu; \pi)} \prod_{s_i \in \lambda} \text{sign}(s_i) \)

**Example 5.5.** Let \( \lambda = (5, 4, 3) \) and \( \mu = (4, 2) \). We have four perfect matchings:
The numbers of crossings are 0, 2, 1 and 1 from left to right. The first two perfect matchings have sign plus and the last two have minus.

We denote by \( P[m, n] \) the \( P \)-function \( P_{(m,n)} \) for simplicity. We define \( P[m, n] := P[n, m] \) for \( m < n \). By definition, note that \( P[n, n] = 0 \) for \( n \geq 1 \) and \( P[0,0] = 1 \).

**Theorem 5.6.** Suppose that \( \hat{S}_\alpha \) satisfies Eqn. (5.1) or Eqn. (5.2). Then, a \( \hat{S} \)-function is expressed in terms of \( P \)-functions as

\[
(5.8) \quad \hat{S}_\alpha = 2^{l(\lambda)} \sum_{\pi \in S_l} \text{sign}_3(\lambda, \mu; \pi) \prod_{i=1}^{l} P[\lambda_i, \mu_{\pi(i)}]
\]

where \( S_l \) is the symmetric group of order \( l \).

**Proof.** From equations (5.1) and (5.2), we have an expression of \( \hat{S}_\alpha \) in terms of products of two \( P \)-functions. Since a \( P \)-function is written in terms of a Pfaffian, \( \hat{S}_\alpha \) is expressed as a sum of products of \( P \)-functions \( P_\lambda \)'s where \( \lambda \) is of length two. A term in \( \hat{S}_\alpha \) is the from \( P_\lambda P_\mu \) and \( \mu' \supseteq \mu \). Note that \( \lambda = \lambda' \cup (\mu' \setminus \mu) \). We expand \( P_\mu \) in terms of \( P \)-functions of length two or one according to the length of \( \mu' \). By a direct calculation using an expansion formula for a Pfaffian, one can easily show that the coefficient of \( P[\mu_i, \mu_j] \) is zero. Thus, \( \hat{S}_\alpha \) is a sum of products of \( P \)-functions \( P[\lambda_\pi, \mu_{\pi(i)}] \) for some \( \pi \in S_l \). By an expansion of a Pfaffian, we have the coefficient of \( \prod_{i=1}^{l} P[\lambda_i, \mu_{\pi(i)}] \) is one except for the overall factor. By a direct calculation, one can show that the sign of \( \prod_{i=1}^{l} P[\lambda_i, \mu_{\pi(i)}] \) is given by \( \text{sign}_3(\lambda, \mu; \pi) \).

**Example 5.7.** Let \( \lambda = (5,4,3) \) and \( \mu = (4,2) \). Then, \( \alpha := \lambda \otimes \mu = (5,4,3,3,3) \).

\[
(5.9) \quad \hat{S}_\alpha = 2^3(P_{(5,4,3)}P_{(4,2)} - P_{(4)}P_{(5,4,3,2)}) - 2^2(P_{(3)}P_{(4,2)}P_{(3,2)} + P_{(5,4)}P_{(3)}P_{(4,3)} - P_{(5,2)}P_{(4)}P_{(3,2)}).
\]

Note that the terms in the last line in Eqn. (5.9) correspond to the perfect matchings in Example 5.5.

**Proof of Theorem 5.1.** We prove Theorem by induction. For \( \alpha = (1) \), it is obvious that \( \hat{S}_\alpha = 2P_{(1)} \) and Eqn. (5.1) is true.

Let \( \beta := \sigma \otimes \rho \). Suppose that Equations (5.1) and (5.2) are true for all \( \beta \)'s such that \( |\beta| < |\alpha| \) or \( \mu(\mu) > \rho(\rho) \) for \( |\alpha| = |\beta| \) with \( l(\mu) = l(\rho) \). From Lemmas 5.3 and 5.4, the function \( \hat{S}_\alpha \) satisfies either Eqn. (5.1) or Eqn. (5.2).

If we expand \( P \)-functions in Eqn. (5.3) by using an expansion formula for a Pfaffian, we obtain the same expression as Theorem 5.6. This implies that Eqn. (5.3) is true.

From Theorem 5.1, we have the following corollary.

**Corollary 5.8** (Józefiak and Pragacz [7]). Let \( \alpha \) be shift-symmetric, i.e., \( \alpha := \lambda \otimes \lambda \). Then, \( \hat{S}_\alpha \) is given by the square of \( P_\lambda \):

\[
\hat{S}_\alpha = 2^{l(\lambda)} P_\lambda^2.
\]
5.2. A product of $P$-functions in terms of $\hat{S}$-functions. Theorem 5.1 shows that a $\hat{S}$-function can be expressed as a sum of products of two Schur $P$-functions. By solving the relation reversely, one can show that a product of Schur $P$-function can be expressed in terms of a sum of Schur $\hat{S}$-function. This expression does not have positivity, i.e., a sign of the coefficient of a $\hat{S}$-function can be minus. However, the expression is multiplicity free except for the overall factor, i.e., the coefficient of a $\hat{S}$-function is either 1 or -1.

Fix strict partitions $\lambda$ and $\mu$. Let $\lambda_0$ and $\mu_0$ be sets of positive integers such that $\lambda_0 := \lambda \cup \mu$ and $\mu_0 := \lambda \cap \mu$. We define a set $S_3(\lambda, \mu)$ by
\[
S_3(\lambda, \mu) := \{\nu \mid \nu \subseteq \lambda_0 \setminus \mu_0 \text{ and } l(\nu) = |(|\lambda_0| - |\mu_0|)/2|\}.
\]
We define a sign of $\nu \subseteq \lambda_0 \setminus \mu_0$ by
\[
\text{sign}_4(\nu) := \prod_{m \in \nu} (-1)^{d(m)}
\]
where
\[
d(m) := \# \{l \in \mathbb{N} \mid l \in \lambda_0 \setminus \mu_0 \text{ and } l \geq m\}.
\]
We construct an ordinary partition $\alpha(\nu)$ from $\nu \in S_3(\lambda, \mu)$ by defining
\[
\alpha(\nu) := (\lambda_0 \setminus \nu) \otimes (\mu_0 \cup \nu).
\]
Suppose that $\lambda$ and $\mu$ are written by $\chi \subseteq \lambda_0 \setminus \mu_0$ as $\lambda = \lambda_0 \setminus \chi$ and $\mu = \mu_0 \cup \chi$. Finally, we define a sign with respect to $\lambda, \mu$ and $\alpha(\nu)$ by
\[
\text{sign}_5(\lambda, \mu; \alpha(\nu)) := (-1)^{l(\chi) + l(\chi \cup \nu)} \cdot \text{sign}_4(\nu) \cdot \text{sign}_4(\chi).
\]

Theorem 5.9. We have
\[
P_\lambda P_\mu = 2^{-(l(\lambda)+l(\mu)-l(\lambda \cap \mu))} \sum_{\nu \in S_3(\lambda, \mu)} \text{sign}_5(\lambda, \mu; \alpha(\nu)) \cdot \hat{S}_{\alpha(\nu)}.
\]

Proof. Suppose that $\lambda$ and $\mu$ is written as $\lambda = \lambda_0 \setminus \chi$ and $\mu = \mu_0 \cup \chi$ where $\lambda_0 = \lambda \cup \mu$ and $\mu_0 = \lambda \cap \mu$. Then, $l(\lambda) + l(\mu) - l(\lambda \cap \mu) = l(\lambda_0)$. By substituting Eqn. (5.8) into the right hand side of Eqn. (5.10), we obtain
\[
(5.11) \quad 2^{-(l(\nu))} \sum_{\nu \in S_3(\lambda_0, \mu_0)} \sum_{\pi \in S_\lambda} \text{sign}_5(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) \prod_{i=1}^l P[(\lambda_0 \setminus \nu)_i, (\mu_0 \cup \nu)_{\pi(i)}].
\]
Note that in the expression of $\text{sign}_5(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi)$ we have
\[
(5.12) \quad \prod_{\nu \subseteq \lambda_0 \setminus \nu} \text{sign}'(s_i) = \prod_{\nu \subseteq \lambda_0} \text{sign}'(s_i) \prod_{\nu \subseteq \mu_0} \text{sign}'(s_i).
\]
We denote
\[
\text{Sign}(\lambda, \mu; \nu) = \text{sign}_3(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) \cdot \text{sign}_5(\lambda, \mu; \alpha(\nu)).
\]
We will show that Eqn. (5.11) is equal to $P_\lambda P_\mu$. We first show that the coefficient of $P[\lambda_i, \mu_j]$ in Eqn. (5.11) is zero.

Suppose that $\lambda_i, \mu_j \notin \mu_0$, $\lambda_i \subseteq \lambda_0 \setminus \nu$ and $\mu_j \subseteq \mu_0 \cup \nu$. Then, we have $\lambda_i \notin \nu$, $\mu_j \in \nu$, $\lambda_i \notin \chi$ and $\mu_j \in \chi$. There exists $\nu'$ such that $l(\nu \cap \nu') = l(\nu) - 1$, $\lambda_i \in \nu'$ and $\mu_j \notin \nu'$. We have $\mu_j \subseteq (\chi \cap \nu')$, $\mu_j \not\subseteq (\chi \cap \nu')$ and $(\chi \cap \nu') \setminus \{\mu_j\} = \chi \cap \nu'$. This implies that $l(\chi \cap \nu') = l(\chi \cap \nu') + 1$. If $\text{sign}'(\lambda_i) = \text{sign}'(\mu_j)$, we have $\text{sign}_4(\nu) = \text{sign}_4(\nu')$, $\prod_{\nu \subseteq \nu} \text{sign}'(s_i) = \prod_{\nu \subseteq \nu'} \text{sign}'(s_i)$ and $\text{cr}(\lambda \setminus \nu, \mu_0 \cup \nu; \pi) = \text{cr}(\lambda \setminus \nu', \mu_0 \cup \nu'; \pi)$. Here, $\lambda_i$ and $\mu_j$ is connected by an arc in the perfect matching $\pi$. Thus we
have \( \text{Sign}(\lambda, \mu; \nu) = -\text{Sign}(\lambda, \mu; \nu') \). If \( \text{sign}'(\lambda_i) \neq \text{sign}'(\mu_j) \), we have \( \text{sign}_4(\nu) = -\text{sign}_4(\nu') \). We also have \( \prod_{s_i \in \nu} \text{sign}'(s_i) = -\prod_{s_i \in \nu'} \text{sign}'(s_i) \) and \( \text{cr}(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) = \text{cr}(\lambda_0 \setminus \nu', \mu_0 \cup \nu'; \pi) \). Thus we have \( \text{Sign}(\lambda, \mu; \nu) = -\text{Sign}(\lambda, \mu; \nu') \). The coefficient of \( P[\lambda, \mu] \) in Eqn. (5.11) is zero since the contributions from \( \nu \) and \( \nu' \) cancel each other.

The observation above implies that Eqn. (5.11) contains only the terms \( P[\lambda, \lambda_j] \) and \( P[\mu, \mu] \). We will show that there exits \( \nu' \) such that \( \prod_{i=1}^k P[(\lambda \setminus \nu_i), (\mu_0 \cup \nu_i), \pi(i)] = \prod_{i=1}^k P[(\lambda \setminus \nu_i'), (\mu_0 \cup \nu_i'), \pi(i)] \) and \( \nu \cap \nu' = \nu - 1 \) and \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \) if \( \lambda_i \) or \( \lambda_j \) is in \( \nu \). We consider the coefficient of \( P[\lambda, \lambda_j] \) since one can apply the same argument to \( P[\mu, \mu] \) by the symmetry between \( \lambda \) and \( \mu \).

Suppose that \( \lambda_i, \lambda_j \notin \mu_0, \lambda_i \in \lambda_0 \setminus \nu \) and \( \lambda_j \in \mu_0 \cup \nu \). Then, we have \( \lambda_i \notin \nu, \lambda_j \in \nu \) and \( \lambda_i, \lambda_j \notin \chi \). There exists a unique \( \nu' \) such that \( \nu \cap \nu' = \nu - 1 \), \( \lambda_i \notin \nu' \) and \( \lambda_j \notin \nu' \). Since \( \lambda_i \) and \( \lambda_j \) are connected by an arc in perfect matchings corresponding to \( \nu \) and \( \nu' \), we have \( \text{cr}(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) = \text{cr}(\lambda_0 \setminus \nu', \mu_0 \cup \nu'; \pi) \). If \( \text{sign}'(\lambda_i) = \text{sign}'(\lambda_j) \), we have \( \text{sign}(\nu) = \text{sign}(\nu') \) and \( \prod_{s_i \in \nu} \text{sign}(s_i) = \prod_{s_i \in \nu'} \text{sign}(s_i) \). Thus we have \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \). Similarly, if \( \text{sign}'(\lambda_i) \neq \text{sign}'(\lambda_j) \), we have \( \text{sign}_4(\nu) = -\text{sign}_4(\nu') \) and \( \prod_{s_i \in \nu} \text{sign}(s_i) = -\prod_{s_i \in \nu'} \text{sign}(s_i) \), which implies \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \). The sets \( \nu \) and \( \nu' \) give the same contribution in Eqn. (5.11), which gives the coefficient two as an overall factor.

Suppose that \( \lambda_i \in \mu_0 \) and \( \lambda_j \notin \mu_0 \). Then, \( \lambda_i \notin \nu \) and \( \lambda_j, \lambda_j \notin \chi \). We first consider the case where \( \lambda_i \notin \lambda_0 \setminus \nu \) and \( \lambda_j \notin \mu_0 \cup \nu \). Then, \( \lambda_j \in \nu \). We define a sequence of a pair of integers \( \alpha_a := (\alpha_{a,1}, \alpha_{a,2}) \), \( 1 \leq a \leq p \) for some positive integer \( p \), such that it satisfies the following five conditions:

1. \( \alpha_1 := (\lambda_i, \lambda_j) \),
2. \( \alpha_{a+1,2} = \alpha_{a,1} \in \mu_0 \) for \( 1 \leq a \leq p - 1 \),
3. \( \alpha_{a,2} \neq \alpha_{a,2} \) if \( b \neq a \),
4. \( \alpha_{p,1} \notin \mu_0 \),
5. \( \alpha_{a,1} \in \lambda_0 \setminus \nu \) and \( \alpha_{a,2} \in \mu_0 \cup \nu \) for \( 1 \leq a \leq p \).

From condition (2), we have \( \alpha_{a,1} \notin \nu \) for \( 1 \leq a \leq p \). From (4) and (5), we have \( \alpha_{p,1} \notin \nu \). There exists \( \nu' \) such that \( \alpha_{p,1} \notin \nu', \lambda_j \notin \nu' \) and \( \nu \cap \nu' = \nu - 1 \). This \( \nu' \) is characterized by a sequence of pairs of positive integers \( \beta_a, 1 \leq a \leq p \), where \( \beta_a = (\alpha_{a,2}, \alpha_{a,1}) \). We have two cases: (a) \( \alpha_{p,1} = \lambda_k \) and (b) \( \alpha_{p,1} = \mu_k \) for some integer \( k \). For (a), observe that \( \lambda_j, \lambda_k \notin \chi \) and \( \lambda_k \notin \mu_0 \). By a similar argument above, we have \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \). For (b), we have \( \mu_0 \notin \mu_0 \) and \( \mu_0 \notin \chi \). If \( \text{sign}'(\lambda_j) = \text{sign}'(\mu_k) \), we have \( \nu \cap \nu' = \nu - 1 \), \( \text{cr}(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) = \text{cr}(\lambda_0 \setminus \nu', \mu_0 \cup \nu'; \pi) = \nu - 1 \), and \( \text{sign}_4(\nu) = \text{sign}_4(\nu') \). These imply that \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \). If \( \text{sign}'(\lambda_j) \neq \text{sign}'(\mu_k) \), we have \( \nu \cap \nu' = \nu - 1 \), \( \text{cr}(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) = \text{cr}(\lambda_0 \setminus \nu', \mu_0 \cup \nu'; \pi) - 1 \), \( \text{sign}_4(\nu) = -\text{sign}_4(\nu') \) and \( \prod_{s_i \in \nu} \text{sign}(s_i) = -\prod_{s_i \in \nu'} \text{sign}(s_i) \). Thus we have \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \). In case of \( \lambda_j \notin \lambda_0 \setminus \nu \) and \( \lambda_j \notin \mu_0 \cup \nu \), one can similarly define a sequence \( \alpha_a \) and show that there exists \( \nu' \) such that \( \text{Sign}(\lambda, \mu; \nu) = \text{Sign}(\lambda, \mu; \nu') \) and \( \prod_{i=1}^p P[(\lambda \setminus \nu_i), (\mu_0 \cup \nu_i), \pi(i)] = \prod_{i=1}^p P[(\lambda \setminus \nu_i'), (\mu_0 \cup \nu_i'), \pi(i)] \). Therefore, the sets \( \nu \) and \( \nu' \) give the overall factor two.

Suppose that \( \lambda_i, \lambda_j \in \mu_0 \). Then, \( \lambda_i, \lambda_j \notin \nu \) and \( \lambda_i, \lambda_j \notin \chi \). We define a sequence of a pair of integers \( \alpha_a := (\alpha_{a,1}, \alpha_{a,2}) \), \( 1 \leq a \leq 2p \) for some integer \( p \), such that it satisfies the following four conditions:

1. \( \alpha_1 := (\lambda_i, \lambda_j) \),
2. \( \alpha_{a,1} \in \alpha_{a,2} \in \mu_0 \),
3. \( \alpha_{a+1,1} = \alpha_{a,2} \) for \( 1 \leq a \leq p \) with \( \alpha_{p+1,1} := \alpha_{1,1} \),
4. \( \alpha_{a,1} \neq \alpha_{b,1} \) if \( a \neq b \).
We define \( \beta_a, 1 \leq a \leq 2p \), by \( \beta_a := (\alpha_{a,2}, \alpha_{a,1}) \). A sequence \( \alpha_a, 1 \leq a \leq 2p \), corresponds to a product \( \prod_{i=1}^{2p} P[a_i, a_{i-1}] \) and so does \( \beta \). However, we have a freedom to choose \( a_{i-1} \in \lambda \) or \( \mu \). Thus we have two-to-two bijection the choice of \( \alpha \) or \( \beta \) and the choice of \( \lambda \) or \( \mu \). Note that these give the same contribution in Eqn. (5.11) but the corresponding perfect matchings are different.

By summarizing the observations above and taking care of the overall factor, Eqn. (5.11) is rewritten as

\[
\sum_{\rho \in \mathcal{F}_n, \sigma \in \mathcal{F}_m} \text{Sign}'(\rho, \sigma) \prod_{i=1}^{n} P[\lambda_{\rho(i-1)}, \lambda_{\rho(i)}] \prod_{i=1}^{m} P[\mu_{\sigma(i-1)}, \mu_{\sigma(i)}],
\]

where \( n = \lfloor (l(\lambda) + 1)/2 \rfloor \) and \( m = \lfloor (l(\mu) + 1)/2 \rfloor \). We will show that \( \text{Sign}'(\rho, \sigma) = \epsilon(\rho)\epsilon(\sigma) \) where \( \epsilon(\rho) \) (resp. \( \epsilon(\sigma) \)) is a sign for the permutation \( \rho \) (resp. \( \sigma \)). Since we have a symmetry between \( \lambda \) and \( \mu \) in \( P_{\lambda} P_{\mu} \), we set \( \lambda_1 \) is the largest integer in \( \lambda \cup \mu \) without loss of generality. Given \( \lambda \) and \( \mu \), we define \( \overline{\lambda} := \lambda \setminus \{\lambda_1, \lambda_r\} \), \( \overline{\lambda}_0 := \overline{\lambda} \cup \mu \) and \( \overline{\pi} := \nu \setminus \{\lambda_1, \lambda_r\} \) for some \( r \). This corresponds to considering the perfect matchings such that \( \lambda_1 \) and \( \lambda_r \) are connected by an arc. We define \( \overline{\lambda} \) such that \( \overline{\lambda} = \overline{\lambda}_0 \setminus \overline{\lambda} \) and \( \mu = \overline{\lambda}_0 \cup \overline{\lambda} \). Since we consider a perfect matching \( \pi \) which contains an arc connecting \( \lambda_1 \) and \( \lambda_r \), we denote \( \overline{\pi} \) by a perfect matching obtained from \( \pi \) by deleting this arc. For simplicity, we write \( \text{cr}(\nu; \pi) := \text{cr}(\lambda_0 \setminus \nu, \mu_0 \cup \nu; \pi) \). Since \( P_{\lambda} \) has an expression in terms of Pfaffian, to show \( \text{Sign}'(\rho, \sigma) = \epsilon(\rho)\epsilon(\sigma) \) is equivalent to show \( \text{Sign}(\lambda, \mu; \nu) = (-1)^{r} \text{Sign}(\overline{\lambda}, \overline{\mu}; \overline{\pi}) \).

Suppose that

\[
\lambda_1 \geq \lambda_{1,1} > \cdots > \lambda_{1,k_1} > \lambda_2 \geq \lambda_{2,1} > \cdots > \lambda_{2,k_2} > \lambda_3 \geq \cdots > \lambda_{r-1,k_{r-1}} > \lambda_r \geq \lambda_{r,1}.
\]

We assume that \( \lambda_i \)'s (\( 2 \leq i < r \)) are in \( \mu_0 \), \( b \mu_{i,j} \)'s are in \( \nu \) and \( c \lambda_k \)'s (\( 2 \leq k < r \)) are in \( \nu \). We define \( k := \sum_{i=1}^{r-1} k_i \). We have four cases: (1) \( \lambda_1, \lambda_r \notin \mu_0 \), (2) \( \lambda_1 \in \mu_0 \) and \( \lambda_r \notin \mu_0 \), (3) \( \lambda_1 \notin \mu_0 \) and \( \lambda_r \in \mu_0 \), and (4) \( \lambda_1, \lambda_r \in \mu_0 \). We prove \( \text{Sign}(\lambda, \mu; \nu) = (-1)^{r} \text{Sign}(\overline{\lambda}, \overline{\mu}; \overline{\pi}) \) for only case (1), since other cases can be proven in a similar way.

For case (1), we have two cases: (a) \( \lambda_1 \notin \nu \) and \( \lambda_r \notin \nu \) and \( \lambda_r \notin \nu \). For case (a), we have

\[
\begin{align*}
\text{sign}_4(\nu) &= (-1)^b(-1)^c(-1)^{k+r}\text{sign}_4(\overline{\pi}), \\
\text{sign}_4(\chi) &= (-1)^{k-a}\text{sign}_4(\overline{\lambda}), \\
\text{sign}(\nu; \pi) &= (-1)^{k+r-2c}(1)^{a}(-1)^{k+r-1} \prod_{s_i \in \overline{\lambda}_0} \text{sign}'(s_i), \\
\prod_{s_i \in \nu} \text{sign}'(s_i) &= (-1)^b(-1)^c(-1)^{k+r-1} \prod_{s_i \in \overline{\pi}} \text{sign}'(s_i), \\
l(\chi) &= l(\overline{\lambda}), \\
l(\nu \cap \chi) &= l(\overline{\pi} \cap \overline{\lambda}).
\end{align*}
\]

Therefore, we obtain \( \text{Sign}(\lambda, \mu; \nu) = (-1)^{r} \text{Sign}(\overline{\lambda}, \overline{\mu}; \overline{\pi}) \). For case (b), we have

\[
\begin{align*}
\text{sign}_4(\nu) &= (-1)^b(-1)^c(-1)^4\text{sign}_4(\overline{\pi}), \\
\prod_{s_i \in \nu} \text{sign}'(s_i) &= (-1)^b(-1)^c \prod_{s_i \in \overline{\pi}} \text{sign}'(s_i)
\end{align*}
\]

and the same Equations from (5.14) to (5.16), (5.18) and (5.19). Thus, we have \( \text{Sign}(\lambda, \mu; \nu) = (-1)^{r} \text{Sign}(\overline{\lambda}, \overline{\mu}; \overline{\pi}) \).
Note that we have \( l(\overline{\lambda}) = l(\chi) + 1 \) in case of (2) and (3) and \( l(\overline{\lambda}) = l(\chi) + 2 \) in case of (4). This completes the proof.

**Example 5.10.** Let \( \lambda := (4, 1) \) and \( \mu := (3, 2) \). We have

\[
P_{(4,1)}P_{(3,2)} = 2^{-4}(\hat{S}_{(3,2)}(2,1) + \hat{S}_{(4)}(2,1) + \hat{S}_{(4,1)}(3,2) + \hat{S}_{(2,1)}(3,2) + \hat{S}_{(4)}(2,1) - \hat{S}_{(2,1)}(3,2)).
\]

5.3. **Giambelli formula.** A Schur function \( s_\alpha \) has a Giambelli formula with determinant, and \( Q \)-function has a Giambelli formula in terms of a Pfaffian. We show two types of Giambelli formula for \( \hat{S} \)-function; one is a determinant and the other is a Pfaffian.

Let \( n \) be the length of \( \lambda \). When \( l(\mu) = n - 1 \), we regard \( \mu \) as a strict partition of length \( n \) by \( \mu_n = 0 \). When \( l(\mu) = n \), we regard \( \lambda, \mu \) as a strict partition of length \( n + 1 \) by \( \lambda_{n+1} = \mu_{n+1} = 0 \). We define a matrix \( \tilde{P}(\lambda, \mu) \) in terms of Schur \( P \)-functions as

\[
\tilde{P}(\lambda, \mu)_{i,j} := \begin{cases} 
P_{(\mu_i, \mu_j)}, & \lambda_i > \mu_j > 0, \\
-P_{(\mu_j, \lambda_i)}, & \mu_j > \lambda_i > 0, \\
-P_{(\lambda_i)}, & \lambda_i > 0, \mu_j = 0, \\
-P_{(\mu_j)}, & \lambda_i = 0, \mu_j > 0, \\
\delta_{\lambda_i,0}, & \lambda_i = \mu_j. 
\end{cases}
\]

**Theorem 5.11.** Let \( \alpha = \lambda \otimes \mu \). Then,

\[
\hat{S}_\alpha = 2^n \det \left[ \tilde{P}(\lambda, \mu)_{i,j} \right]_{1 \leq i,j \leq m}.
\]

where \( m = n \) for \( l(\lambda) = l(\mu) + 1 = n \) and \( m = n + 1 \) for \( l(\lambda) = l(\mu) = n \).

**Proof.** From the definition of \( \tilde{P}(\lambda) \), we have

\[
\det \left[ \tilde{P}(\lambda, \mu)_{i,j} \right] = \sum_{\pi \in S^m} \epsilon(\pi) \prod_{i=1}^{m} (-1)^{\delta(\mu_{\pi(i)} > \lambda_i)} \prod_{i=1}^{m} P[\lambda_i, \mu_{\pi(i)}].
\]

where \( \delta(S) = 1 \) if the statement \( S \) is true and zero otherwise. From Theorem 5.6, it is enough to show that

\[
\text{sign}_3(\lambda, \mu; \pi) = \epsilon(\pi) \prod_{i=1}^{m} (-1)^{\delta(\mu_{\pi(i)} > \lambda_i)}.
\]

First, we show Eqn. (5.20) is true for \( \pi = \text{id} \) by induction. It is easy to see that Eqn. (5.20) is true for \( l(\lambda) = 1 \). Let \( \lambda' = \lambda \setminus \{\lambda_1\} \) and \( \mu' = \mu \setminus \{\mu_1\} \). If \( \lambda_1 = \mu_1 \), then \( P[\lambda_1, \mu_1] = 0 \) by definition. Therefore, we have two cases: (1) \( \lambda_1 > \mu_1 \) and (2) \( \lambda_1 < \mu_1 \). For (1), suppose that \( \lambda_1 > \lambda_2 > \ldots > \lambda_{k+1} > \mu_1 \). Then,

\[
(-1)^{\text{cr}(\lambda, \mu; \text{id})} = (-1)^k \cdot (-1)^{\text{cr}(\lambda', \mu'; \text{id})} \prod_{s, \lambda} \text{sign}'(s_i) = (-1)^k \prod_{s, \lambda} \text{sign}'(s_i).
\]

Thus, we have \( \text{sign}_3(\lambda, \mu; \text{id}) = \text{sign}_3(\lambda', \mu'; \text{id}) \) and the right hand side of Eqn. (5.20) is equal to \( \prod_{i=2}^{m} (-1)^{\delta(\mu_i > \lambda_i)} \). By induction assumption, we have \( \text{sign}_3(\lambda', \mu'; \text{id}) = \prod_{i=2}^{m} (-1)^{\delta(\mu_i > \lambda_i)} \). Thus, Eqn. (5.20) is true for general \( \lambda \) and \( \mu \). For case (2), by a similar argument, we have \( \text{sign}_3(\lambda, \mu; \text{id}) = \prod_{i=2}^{m} (-1)^{\delta(\mu_i > \lambda_i)} \).
be a weakly decreasing integer sequence with repeated entries such that an integer in
by
\[ \hat{\lambda} \]

We have that
\[ \hat{\lambda} \]

transpositions to obtain $\text{sign}(\pi)$. For $\pi = id$, let
\[ \pi' = (i, i + 1) \circ \pi \]

arc connecting $\lambda_i$ with $\mu_{\pi(i)}$ and $\lambda_{i+1}$ with $\mu_{\pi(i+1)}$ in a perfect matching. Applying
the transposition on the perfect matching means that we connect $\lambda_i$ with $\mu_{\pi(i+1)}$ and $\lambda_{i+1}$ with $\mu_{\pi(i)}$.
Let $j := \min\{\pi(i), \pi(i+1)\}$ and $k := \max\{\pi(i), \pi(i+1)\}$. We have the cases where $\lambda_i$ and $\lambda_{i+1}$ are distinct and so do $\mu_j$ and $\mu_k$. We have six local configurations for $\lambda_i, \lambda_{i+1}, \mu_j$ and $\mu_k$: 1) $\lambda_i > \lambda_{i+1} \geq \mu_j > \mu_k$, 2) $\lambda_i \geq \mu_j \geq \lambda_{i+1} \geq \mu_k$, 3) $\lambda_i \geq \mu_j > \mu_k \geq \lambda_{i+1}$, 4) $\mu_j \geq \lambda_i > \lambda_{i+1} \geq \mu_k$, 5) $\mu_j \geq \lambda_i \geq \mu_k \geq \lambda_{i+1}$, and 6) $\mu_j > \mu_k \geq \lambda_i > \lambda_{i+1}$. For case 1), the reconnection of arcs is locally given by

\[ \lambda_i \lambda_{i+1} \mu_j \mu_k \leftrightarrow \lambda_i \lambda_{i+1} \mu_j \mu_k . \]

It is easy to see that the both sides of Eqn. (5.20) gives $(-1)$ by the reconnection. Similarly, we have a factor $(-1)$ for the cases 3), 4) and 6) and 1 for the cases 2) and 5). Starting from $\pi = id$, by successively applying the procedure above, we obtain that Eqn. (5.20) is true for any $\pi$. This completes the proof.

\[ \square \]

Example 5.12. Let $\alpha := (4, 3, 3, 3) = (4, 3, 2) \otimes (3, 1)$. The matrix $\bar{P} := \bar{P}((4, 3, 2), (3, 1))$ is given by

\[ \bar{P} = \begin{bmatrix} P[(4, 3)] & P[(4, 1)] & P[(4)] \\ 0 & P[(3, 1)] & P[(3)] \\ -4P[(3, 2)] & P[(2, 1)] & P[(2)] \end{bmatrix} . \]

We have $\hat{S}_\alpha = 2^3 \cdot \det[\bar{P}]$.

Let $n$ be the length of $\lambda$. When $l(\mu) = n - 1$, we regard $\mu$ as a strict partition of length $n$ by $\mu_n = 0$. When $l(\mu) = n$, we regard $\lambda, \mu$ as a strict partition of length $n + 1$ by defining $\lambda_{n+1} = \mu_{n+1} = 0$. Let $m = n$ when $l(\lambda) = l(\mu) + 1$ and $m = n + 1$ when $l(\lambda) = l(\mu)$. Let $A := \lambda \cup \mu$ be a weakly decreasing integer sequence with repeated entries such that an integer in $\lambda \cap \mu$ (resp. $\lambda \cup \mu \setminus (\lambda \cap \mu)$) appears twice (resp. once) in $A$. We write $A := (a_1 \geq a_2 \geq \ldots \geq a_{2m})$. For $i \in A$ and $i \notin \lambda \cap \mu$, we define the sign of $i$ as $\text{sign}(i) = +$ if $i \in \lambda$ and $\text{sign}(i) = -$ if $i \in \mu$. We denote the sort of $A$ by $\text{sign}(A) := (\text{sign}(a_1), \text{sign}(a_2), \ldots, \text{sign}(a_{2m}))$. Let $\text{sign}_0(A)$ be a sequence of alternating signs, i.e., $\text{sign}_0(A) := (+, -, +, -, \ldots)$. We denote by $d(A)$ the number of transpositions to obtain $\text{sign}(A)$ from $\text{sign}_0(A)$.

For $i \in A$ and $i \notin \lambda \cap \mu$, we assign $\text{sign}(i) = +$ for the first $i$ and $\text{sign}(i) = -$ for the second $i$. We define a skew symmetric matrix $P_{i,j}(A)$, $1 \leq i, j \leq 2m$, in terms of Schur $P$-functions:

\[ P_{i,j}(A) := \begin{cases} 0, & \text{sign}(a_i) = \text{sign}(a_j) \text{ or } a_i = a_j \neq 0, \\ P(a_i, a_j), & a_i > a_j > 0, \\ P(a_i), & a_i > a_j = 0, \\ 1, & a_i = a_j = 0, \end{cases} \]

for $1 \leq i \leq j \leq 2m$.

Theorem 5.13. Let $A$, $d(A)$ and $P_{i,j}(A)$ as above.

\[ \hat{S}_\alpha = (-1)^{d(A)} \cdot 2^n \cdot \text{pf}[P_{i,j}(A)]_{1 \leq i, j \leq 2m} . \]
Proof. Recall that a Pfaffian is defined by Eqn. (2.4) and a $\hat{S}$-function has an expression in terms of perfect matchings as Theorem 5.6. We place integers in $[1, 2m]$ from left to right. Then, for a permutation $\pi \in F_m$, we consider a perfect matching such that $\pi(2i - 1)$ and $\pi(2i)$ for $1 \leq i \leq m$ are connected by an arc. It is obvious that the sign $|\pi|$ is equal to $(-1)^{\text{cr}}$ where $\text{cr}$ is the number of crossings in the perfect matching. We have

$$\text{pf}[P_{i,j}(A)] = \sum_{\pi \in F_m} (-1)^{\text{cr}(\lambda, \mu; \pi)} \prod_{i=1}^{m} P[\lambda_i, \mu_{\pi(i)}],$$

where $\alpha = \lambda \otimes \mu$. It is easy to see

$$\prod_{s_i \in \lambda} \text{sign}'(s_i) = (-1)^{d(A)}.$$ 

Finally, $n = l(\lambda)$ by definition. Combining these observations with Theorem 5.6, we obtain Eqn. (5.21). This completes the proof.$\square$

Example 5.14. Let $\alpha := (4, 3, 3, 3) = (4, 3, 2) \otimes (3, 1)$. Then, $A = (4, 3, 2, 1, 0)$ and $\text{sign}(A) = (+, +, -, +, -)$. The skew symmetric matrix $P_{i,j}(A)$ is given by

$$P_{i,j}(A) = \begin{bmatrix}
0 & 0 & 0 & P[(4, 3)] & 0 & P[(4, 1)] & P[(4)] \\
0 & 0 & 0 & 0 & P[(4, 1)] & P[(4)] \\
-P[(4, 3)] & 0 & 0 & P[(3, 2)] & 0 & 0 \\
0 & 0 & -P[(3, 2)] & 0 & P[(2, 1)] & P[(2)] \\
-P[(4, 1)] & -P[(3, 1)] & 0 & -P[(2, 1)] & 0 & 0 \\
-P[(4)] & -P[(3)] & 0 & -P[(2)] & 0 & 0
\end{bmatrix}.$$ 

Then, $\hat{S}_\alpha = 2^3 \cdot \text{pf}[P_{i,j}(A)].$ 

5.4. Skew $\hat{S}$-functions. Given two strict partitions $\lambda$ and $\mu$, we define the sets of strict partitions $S_p(\lambda, \mu)$, $p = 1, 2$, and signs $\text{sign}_p(\lambda', \mu'; \lambda, \mu)$, $p = 1, 2$, as in Section 5.1. Let $\alpha$ and $\beta$ be ordinary partitions such that $\beta \subseteq \alpha$, $\alpha = \lambda \otimes \mu$ and $\beta = \nu \otimes \xi$. We define two sets $S(\lambda, \mu, \nu, \xi)$ and $\hat{S}(\lambda, \mu, \nu, \xi)$ as

$$S(\lambda, \mu, \nu, \xi) := \{(\lambda', \mu', \nu', \xi', \xi') | (\lambda', \mu') \in S_1(\lambda, \mu) \text{ and } (\nu', \xi') \in S_2(\nu, \xi)\},$$

$$\hat{S}(\lambda, \mu, \nu, \xi) := \{(\lambda', \mu', \nu', \xi', \xi') | (\lambda', \mu') \in S_2(\lambda, \mu) \text{ and } (\nu', \xi') \in S_1(\nu, \xi)\}.$$ 

We define $m := l(\nu)$ (resp. $m := l(\nu) + 1$) if $l(\nu) = l(\xi)$ + 1 (resp. $l(\nu) = l(\xi)$) and $n := l(\lambda)$ (resp. $n := l(\lambda) + 1$) if $l(\lambda) = l(\mu)$ + 1 (resp. $l(\lambda) = l(\mu)$).

Theorem 5.15. A $\hat{S}$-function can be expressed as a sum of products of Schur $Q$-functions:

$$\hat{S}_{\alpha/\beta} = 2^{l(\nu) - l(\lambda)} \sum_{(\lambda', \mu', \nu', \xi')} \text{sign}_1(\lambda', \mu'; \lambda, \mu) \text{sign}_2(\nu', \xi'; \nu, \xi) Q_{\lambda'/\mu'} Q_{\nu'/\xi'},$$

and

$$\hat{S}_{\alpha/\beta} = 2^{m-n} \sum_{(\lambda', \mu', \nu', \xi')} \text{sign}_2(\lambda', \mu'; \lambda, \mu) \text{sign}_1(\nu', \xi'; \nu, \xi) Q_{\lambda'/\mu'} Q_{\nu'/\xi'}.$$ 

Example 5.16. Let $\alpha = (4, 3, 2, 1)$ and $\beta = (2, 2)$. Then, $\alpha = (4, 2) \otimes (3, 1)$ and $\beta = (2, 1) \otimes (1)$, $m = 2$ and $n = 3$. Thus, we have

$$\hat{S}_{\alpha/\beta} = Q(4, 2)/(2, 1) Q(3, 1)/(1) = 2^{-1} \left[ Q(4, 2)/(2, 1) Q(3, 1)/(1) + Q(4, 2)/(1) Q(3, 1)/(2, 1) + Q(4, 2)/(2, 1) Q(1)/(1) + Q(4, 2, 1)/(2, 1) Q(3)/(1) \right].$$
We denote $Q[\lambda/\mu] := Q_{\lambda/\mu}$ and $\hat{S}[\alpha/\beta] := \hat{S}_{\alpha/\beta}$ for simplicity. Given a strict partition $\lambda$, let $\widehat{\lambda}_i$ be a strict partition obtained from $\lambda$ by deleting $\lambda_i$.

**Lemma 5.17.** Let $\lambda/\mu$ be a skew partition. Then, a skew Schur $Q$-function satisfies

\begin{equation}
Q[\lambda/\mu] = \sum_{i=1}^{l(\lambda)} (-1)^{i-1} Q[(\lambda_i - \mu_1)]Q[\widehat{\lambda}_i/\mu_1] \tag{5.24}
\end{equation}

and

\begin{equation}
\sum_{i=1}^{l(\lambda)} (-1)^{i-1} Q[(\lambda_i - \mu_1)]Q[\widehat{\lambda}_i/\mu] = 0. \tag{5.25}
\end{equation}

**Proof.** For Eqn. (5.24), recall that we have a Pfaffian expression of $Q[\lambda/\mu]$ (see Eqn. (2.5)). By expanding the Pfaffian with respect to the column corresponding to $\mu_1$, we obtain Eqn. (5.24).

For Eqn. (5.25), we expand $Q[\widehat{\lambda}_i/\mu]$ by using Eqn. (5.24). If we regard $\mu$ as a set of integers, we have $\mu_1 \in \mu$. Then, the coefficient of $Q[(\lambda_i - \mu_1)]Q[(\lambda_j - \mu_1)]$ is zero for any pair of $i$ and $j$, which implies Eqn. (5.25).

\qed

**Proof of Theorem 5.15.** From Theorem 5.1, Theorem 5.15 holds true for $\beta = 0$. We assume that Eqn. (5.22) is true for all $\alpha'$ with $|\alpha'| < |\alpha|$.

Recall that $\hat{S}[\alpha/\beta]$ has a determinant expression and $\hat{S}[\alpha/\beta] = \hat{S}[\alpha^T/\beta^T]$ by the conjugate partitions. Then, we have

\[ \hat{S}_{\alpha/\beta} = \hat{S}_{\alpha^T/\beta^T} \]

\[ = \sum_{\lambda} (-1)^{l(\lambda)} Q[\lambda/\mu] \cdot \hat{S}[\mu \otimes \widehat{\lambda}/\xi \otimes \widehat{\nu}]. \]

Note that if $(\mu', \lambda') \in S_1(\mu, \lambda)$, then $(\lambda', \mu') \in S_2(\lambda, \mu)$. We also have

\[ \text{sign}_1(\lambda', \mu'; \lambda, \mu) = (-1)^{|\lambda'|} \text{sign}_2(\mu', \lambda'; \mu, \lambda). \]

We have four cases for the power of 2 in Eqn. (5.22): (1) $l(\lambda) = l(\mu) + 1$ and $l(\nu) = l(\xi) + 1$, (2) $l(\lambda) = l(\mu) + 1$ and $l(\nu) = l(\xi)$, (3) $l(\lambda) = l(\mu)$ and $l(\nu) = l(\xi) + 1$, and (4) $l(\lambda) = l(\mu) + 1$ and $l(\nu) = l(\xi)$. In all cases, we have $l(\xi') = l(\mu') = n - m$, which is the same power as Eqn. (5.23).

By substituting Eqn. (5.22) for $\hat{S}[\mu \otimes \widehat{\lambda}/\xi \otimes \widehat{\nu}]$ into the equation above and successively applying Lemma 5.17, we obtain Eqn. (5.23).

If we assume Eqn. (5.23) for $\hat{S}[\alpha'/\beta]$ with $|\alpha'| < |\alpha|$, we obtain Eqn. (5.22) in the same way as above. \qed

**Corollary 5.18** (Józefiak and Pragacz [7]). Let $\alpha$ and $\beta$ be shift-symmetric, i.e., $\alpha := \lambda \otimes \lambda$ and $\beta := \mu \otimes \mu$. Then, $\hat{S}_{\alpha/\beta}$ is given by the square of $Q_{\lambda/\mu}$:

\[ S_{\alpha/\beta} = 2^{|\mu| - l(\lambda)} Q_{\lambda/\mu}^2. \]

Let $\alpha := \lambda \otimes \mu$ and $\beta := \nu \otimes \xi$ with $\beta \subseteq \alpha$. We define $n := l(\lambda) + 1$ if $l(\lambda) = l(\mu)$ and $l(\nu) = l(\mu)$, and $n := l(\lambda)$ otherwise. We also define $m := l(\nu)$. We regard $\mu$ (resp. $\xi$) as a partition of length $n$ (resp. $m$) by adding a zero to $\mu$ (resp. $\xi$) if $l(\mu) = l(\lambda) - 1$ (resp. $l(\xi) = l(\nu) - 1$). If $l(\lambda) = l(\mu)$ and $l(\nu) = l(\xi)$, we regard $\lambda$ and $\mu$ as a partition of length $n$ by adding a zero to $\lambda$ and $\mu$. Note that in the last case we do not add zeros to $\nu$ and $\xi$. 

We place the integers in $\lambda \cup \mu$ in a decreasing order from left to right and successively the integers in $\nu \cup \xi$ in a decreasing order from left to right. When an integer $p$ is in $\lambda \cap \mu$ or $\nu \cap \xi$, we place two $p$’s next to each other. We denote by $p_{\lambda}$ or $p_{\nu}$ (resp. $p_{\mu}$ or $p_{\xi}$) the left (resp. right) $p$. We construct a perfect matching of length $2(n+m)$ by connecting two integers via an arc or a dashed arc. A perfect matching satisfies:

1. We connect two integers $p$ and $q$ via an arc if and only if $p \in \lambda$ and $q \in \mu$.
2. We connect two integers $p$ and $q$ via a dashed arc if and only if $p \in \lambda$ and $q \in \nu$, or $p \in \mu$ and $q \in \xi$.
3. We do not connect the same integers $p_{\lambda}$ and $p_{\mu}$ for $p \geq 1$.
4. We do not connect two integers $p$ and $q$ if both $p$ and $q$ are in $\nu$ or $\xi$.

Let $\text{PM}(\alpha/\beta)$ be the set of perfect matchings satisfying conditions above. Let $\pi \in \text{PM}(\alpha/\beta)$. Then, we denote by $\text{cr}(\lambda, \mu, \nu, \xi; \pi)$ the number of crossings in $\pi$ by ignoring whether an arc is dashed or not. Let $s := s_1 \ldots s_{2(m+n)}$ be the sequence of integers in $\lambda \cup \mu$ and $\nu \cup \xi$ with repeated entries. We define a sign $\text{sign}(s_i)$ by $\text{sign}(s_i) = (-1)^{i-1}$ if $s_i \in \lambda \cup \mu$ and by $\text{sign}(s_i) = (-1)^i$ if $s_i \in \nu \cup \xi$. Then, we define a sign of a perfect matching as

$$\text{sign}_6(\lambda, \mu, \nu, \xi; \pi) := (-1)^{\text{cr}(\lambda, \mu, \nu, \xi; \pi)} \prod_{s_i \in \lambda} \text{sign}'(s_i) \cdot \prod_{s_i \in \nu} \text{sign}'(s_i).$$

Given a perfect matching $\pi$, we denote by $\text{Arc}(\pi)$ (resp. $\text{dArc}(\pi)$) the set of arcs (resp. dashed arcs). When $s_i$ and $s_j$, $i < j$, is connected via an arc (resp. a dashed arc), we denote $(i, j) \in \text{Arc}(\pi)$ (resp. $(i, j) \in \text{dArc}(\pi)$). Note that $s_i \geq s_j$ if $i < j$.

**Theorem 5.19.** Denote $Q[\lambda] := Q_{\lambda}$. Then,

$$S_{\alpha/\beta} := 2^{\text{deg}(\alpha/\beta)} \sum_{\pi \in \text{PM}(\alpha/\beta)} \text{sign}_6(\lambda, \mu, \nu, \xi; \pi) \prod_{(i, j) \in \text{Arc}(\pi)} Q[(s_i, s_j)] \prod_{(i, j) \in \text{dArc}(\pi)} Q[(s_i - s_j)],$$

where $\text{deg}(\alpha, \beta) := l(\nu) - l(\mu)$ if $l(\lambda) = l(\mu)$ and $\text{deg}(\alpha, \beta) := l(\xi) - l(\mu)$ if $l(\lambda) = l(\mu) + 1$.

**Example 5.20.** Let $\alpha = (5, 4, 3, 2, 1)$ and $\beta = (2, 1)$. We have $\alpha = (5, 3, 1) \otimes (4, 2)$ and $\beta = (2) \otimes (1)$. The signs are given by $\prod_{s_i \in \lambda} \text{sign}'(s_i) = +1$ and $\prod_{s_i \in \nu} \text{sign}'(s_i) = -1$. Then, we have

$$S_{\alpha/\beta} = 2^{-1} Q_{(3)}(Q_{(2,1)} - Q_{(1)}(Q_{(3)} Q_{(5)} Q_{(2,1)} + Q_{(1)}(Q_{(3)} Q_{(3)} Q_{(3,2)})
- Q_{(1)}(Q_{(3)} Q_{(4,1)} + Q_{(1)}(Q_{(3)} Q_{(4,1)} + Q_{(1)}(Q_{(3)} Q_{(4,3)})
- Q_{(1)}(Q_{(3)} Q_{(5,2)} + Q_{(1)}(Q_{(5,4)}.\tag{5.26}$$

In Figure 5.21, the left (resp. right) picture correspond to the fourth (resp. seventh) term in Eqn. (5.26).
Proof of Theorem 5.19. From Theorem 5.15, a $\hat{S}$-function can be expressed as a sum of products of two (skew) $Q$-functions. We expand $Q$-functions in terms of skew $Q$-functions of length one and two. By a direct calculation, it is easy to show that the coefficient of $Q[(\lambda_i, \lambda_j)]$ with $\lambda_i, \lambda_j \neq \lambda \cap \mu$ is zero. Similarly, the coefficient of $Q[(\lambda_i - \xi_j)]$ with $\lambda_i \neq \lambda \cap \mu$ and $\xi_j \neq \nu \cap \xi$ is zero. Thus, we have

$$\hat{S}_{\alpha/\beta} := 2^{\deg' (\alpha, \beta)} \sum_{\pi \in \text{PM}(\alpha/\beta)} \text{sign}_d(\lambda, \mu, \nu, \xi; \pi) \prod_{(i,j) \in \text{Arc}(\pi)} Q[(s_i, s_j)] \prod_{(i,j) \in \text{dArc}(\pi)} Q[(s_i - s_j)],$$

with some $\deg' (\alpha, \beta)$ and $\text{sign}_d(\lambda, \mu, \nu, \xi; \pi)$. We first show that $\deg' (\alpha, \beta) = \deg (\alpha, \beta)$. When $l(\lambda) = l(\mu)$, we have a single term corresponding to a perfect matching $\pi$. Thus, from Theorem 5.15, we have $\deg (\alpha, \beta) = l(\nu) - l(\lambda) = l(\xi) - l(\mu)$ if $l(\lambda) = l(\mu) + 1$ and $l(\nu) = l(\xi) + 1$. However, in case of $l(\lambda) = l(\mu) + 1$ and $l(\nu) = l(\xi)$, we have two terms coming from the right hand side of Eqn. (5.22), which corresponds to a perfect matching $\pi$. To see this, suppose that the right hand side of Eqn. (5.22) contains a term $Q[\lambda' / \nu']Q[\mu' / \xi']$ with $(\lambda', \mu', \nu', \xi') \in S(\lambda, \mu, \nu, \xi)$. Since $l(\lambda) = l(\mu) + 1$ and $l(\nu) = l(\xi)$, we have $l(\lambda) - l(\nu) = l(\mu) - l(\xi) + 1$. Assume that $\chi$ contains $\lambda_i \notin \lambda \cap \mu$. We consider the term containing $Q[\lambda_i]$. Let $\lambda_i' := \lambda_i \setminus \{\lambda_i\}$ and $\mu_i' := \mu_i \cup \{\lambda_i\}$ as sets. Then, an expansion of the term $Q[\lambda_i' / \nu']Q[\mu_i' / \xi']$ also contains $Q[\lambda_i]$. By taking the sign into account, we can show that these two terms contains the same sign. This implies that we have a factor 2 in the expansion. Thus, the power of 2 is $l(\nu) - l(\lambda) + 1 = l(\xi) - l(\mu)$.

We will show that $\text{sign}_d(\lambda, \mu, \nu, \xi; \pi) = \text{sign}_d(\lambda, \mu, \nu, \xi; \pi)$. Recall that we expand a $Q$-function in terms of $Q$-functions of length one and two. It is easy to see that the number of crossings in a perfect matching is compatible with the sign arising from the expansion of a $Q$-function. Thus, we have $\text{sign}_d(\lambda, \mu, \nu, \xi; \pi)$ is equal to $\text{sign}_d(\lambda, \mu, \nu, \xi; \pi)$ up to the overall factor. By comparing the signs $\prod_{s_i \in \lambda \cup \nu} \text{sign}(s_i)$ with the number of crossings, we conclude that the overall factor is one. This completes the proof.

Suppose that two ordinary partitions $\alpha$ and $\beta \subseteq \alpha$ are written as $\alpha := \lambda' \otimes \mu'$ and $\beta := \nu' \otimes \xi'$. When $l(\lambda') = l(\mu') + 1$, we define strict partitions of length $l(\lambda')$ by $\lambda = \lambda'$ and $\mu_i = \mu_i'$ for $1 \leq i \leq l(\mu')$ and $\mu_{l(\mu')+1} = 0$. When $l(\lambda') = l(\nu')$, we define strict partitions of length $l(\lambda') + 1$ by $\lambda_i = \lambda_i'$ for $1 \leq i \leq l(\lambda')$ and $\lambda_{l(\lambda') + 1} = 0$ and $\mu_i = \mu_i'$ for $1 \leq i \leq l(\mu')$ and $\mu_{l(\mu')+1} = 0$. We also define $\nu$ and $\xi$ from $\nu'$ and $\xi'$ as follows. If $l(\nu') = l(\xi')$, we define $\nu' = \nu$ and $\xi' = \xi$. If $l(\nu') = l(\xi') + 1$, we define $\nu := \nu'$ and define $\xi$ by adding a zero to $\xi'$.

We define a matrix $\tilde{Q}(\lambda, \mu; \nu, \xi) := (\tilde{Q}_{i,j})_{1 \leq i, j \leq l(\lambda) + l(\mu)}$ in terms of Schur $Q$-functions as follows. We define

$$\tilde{Q}_{i,j} := \begin{cases} Q(\lambda_i, \mu_j), & \lambda_i > \mu_j > 0, \\ -Q(\mu_i, \lambda_j), & 0 < \lambda_i < \mu_j, \\ Q(\lambda_i, \mu_j), & \lambda_i > \mu_j = 0, \\ -Q(\mu_i, \lambda_j), & 0 = \lambda_i < \mu_j, \\ 0, & \lambda_i = \mu_j \neq 0, \\ 1, & \lambda_i = \mu_j = 0, \end{cases}$$

for $1 \leq i, j \leq l(\lambda)$,

$$\tilde{Q}_{i,j} + l(\lambda) := \begin{cases} Q(\lambda_i - \nu_j), & \lambda_i > \nu_j, \\ 1, & \lambda_i = \nu_j, \\ 0, & \lambda_i < \nu_j. \end{cases}$$
for $1 \leq i \leq l(\lambda)$ and $1 \leq j \leq l(\nu)$,
\[
\tilde{Q}_{i+l(\lambda),j} := \begin{cases} 
-Q_{(\mu_j, -\xi_i)}, & \mu_j > \xi_i, \\
-1, & \mu_j = \xi_i, \\
0, & \mu_j < \xi_i.
\end{cases}
\]
for $1 \leq i \leq l(\nu)$ and $1 \leq j \leq l(\lambda)$, and $\tilde{Q}_{i+l(\lambda),j+l(\lambda)} := 0$ for $1 \leq i, j \leq l(\nu)$.

**Theorem 5.22.** In the above notation, a function $\hat{S}_{\alpha/\beta}$ can be expressed in terms of a determinant:
\[
\hat{S}_{\alpha/\beta} = 2^{l(\xi')-l(\mu')} \det \left[ \tilde{Q}(\lambda, \mu; \nu, \xi) \right].
\]

**Proof.** From Theorem 5.19, we have an expression of $\hat{S}$-function in terms of perfect matchings. Comparing the definition of the determinant $\det[\tilde{Q}]$ with a term in Theorem 5.19, we have that $\hat{S}_{\alpha/\beta}$ is proportional to the determinant. We have three cases: (1) $l(\lambda') = l(\mu') + 1$, (2) $l(\lambda') = l(\mu')$ and $l(\nu') = l(\xi')$, and (3) $l(\lambda') = l(\mu')$ and $l(\nu') = l(\xi') + 1$. In case of (1) and (2), the overall factor is $2^{l(\xi')-l(\mu')}$. In case of (3), note that the last elements in $\lambda, \mu$ and $\xi$ are zero. In Theorem 5.19, we have no double zeros in the sequence $s$. However, to connect the determinant to a perfect matching $\pi$ considered in Theorem 5.19, we introduce a perfect matching with double zeros. We denote such perfect matching by $\pi'$. We show that we have a one-to-one bijection between $\pi$ and $\pi'$. Suppose that $p_\mu$ is connected to $0_\xi$ by a dashed arc. Then, in $\pi'$, we connect $p_\mu$ and $0_\lambda$ by an arc and $0_\nu$ by a dashed arc or connect $p_\mu$ and $0_\xi$ by a dashed arc and $0_\lambda$ and $0_\nu$ by an arc. Note that two $\pi'$'s above give the same products of $Q$-functions and the sign is the same. Thus, the power of two becomes $l(\nu') - l(\lambda') - 1 = l(\xi') - l(\mu')$. This completes the proof. \(\square\)

**Example 5.23.** Let $\alpha = (4, 3, 2, 1)$ and $\beta = (1)$. Then, $\alpha = (4, 2) \otimes (3, 1)$ and $\beta = (1) \otimes \emptyset$. The matrix $\tilde{Q} := \tilde{Q}(\lambda, \mu; \nu, \xi)$ is given by
\[
\tilde{Q} = \begin{bmatrix}
Q_{(4,3)} & Q_{(4,1)} & Q_{(4)} & Q_{(3)} \\
-Q_{(3,2)} & Q_{(2,1)} & Q_{(2)} & Q_{(1)} \\
-Q_{(3)} & -Q_{(1)} & 1 & 0 \\
-Q_{(3)} & -Q_{(1)} & -1 & 0
\end{bmatrix}.
\]
Therefore, we have $\hat{S}_{\alpha/\beta} = 2^{-2} \det[\tilde{Q}]$.

A given $\alpha = \lambda \otimes \mu$, we define $A := (a_1 \geq a_2 \geq \ldots \geq a_{2n})$, $\text{sign}(A)$ and $d(A)$ as in Section 5.3. For $\beta = \nu \otimes \xi$, we define $b := (b_1 \geq b_2 \geq \ldots \geq b_{2m})$ and $d(B)$ similarly. We define a skew-symmetric matrix $Q(A, B) := (Q_{i,j})_{1 \leq i, j \leq 2(n+m)}$ as follows:
\[
Q_{i,j} := \begin{cases} 
0, & \text{sign}(a_i) = \text{sign}(a_j) \text{ or } a_i = a_j > 0, \\
Q_{(a_i, a_j)}, & a_i > a_j > 0, \\
Q_{(a_i)}, & a_i > a_j = 0, \\
1, & a_i = a_j = 0,
\end{cases}
\]
for $1 \leq i \leq j \leq 2n$. For $1 \leq i \leq 2n$ and $2n + 1 \leq j \leq 2(n + m)$, we define
\[
Q_{i,j} := \begin{cases} 
0, & \text{sign}(a_i) = \text{sign}(b_{j-2n}), \\
Q_{(a_i, b_{j-2n})}, & \text{otherwise}.
\end{cases}
\]
We define $Q_{i,j} := 0$ for $2n + 1 \leq i \leq j \leq 2(n + m)$. 
Theorem 5.24. Suppose that the skew shape \( \alpha/\beta \) exists for \( \alpha = \lambda \otimes \mu \) and \( \beta = \nu \otimes \xi \). Then, we have

\[
\hat{S}_{\alpha/\beta} = (-1)^{d(\lambda) + d(\mu) + m_2(\xi) - l(\mu)} \text{pf} \left[ Q(A, B) \right]_{1 \leq i, j \leq 2(n + m)}
\]

Proof. By comparing the Pfaffian \( \text{pf}[Q(A, B)] \) with Theorem 5.22, we can show that \( \hat{S}_{\alpha/\beta} \) is proportional to the Pfaffian times \( 2^l(\xi) - l(\mu) \). By calculating the overall sign, we obtain Theorem. \( \square \)

Example 5.25. Let \( \alpha := (4, 3, 2, 1) = (4, 2) \otimes (3, 1) \) and \( \beta := (1) = (1) \otimes \emptyset \). Then, we have \( A = (4, 3, 2, 1, 0, 0) \) and \( B = (1, 0) \). The skew-symmetric matrix \( Q(A, B) \) is given by

\[
Q(A, B) = \begin{bmatrix}
0 & Q[(4, 3)] & 0 & Q[(4, 1)] & 0 & Q[(4)] & Q[(3)] & 0 \\
-Q[(4, 3)] & 0 & Q[(3, 2)] & 0 & Q[(3)] & 0 & 0 & Q[(3)] \\
Q[(4, 1)] & -Q[(3, 2)] & 0 & Q[(2, 1)] & 0 & Q[(2)] & 0 & 0 \\
0 & Q[(3)] & 0 & -Q[(1)] & 0 & 1 & 0 & 0 \\
-Q[(4)] & 0 & -Q[(2)] & 0 & -1 & 0 & 0 & 1 \\
-Q[(3)] & 0 & -Q[(1)] & 0 & 0 & 0 & 0 & 0 \\
0 & -Q[(3)] & 0 & -Q[(1)] & 0 & -1 & 0 & 0
\end{bmatrix}
\]

Then, we have \( \hat{S}_{\alpha/\beta} = -2^{-2} \cdot \text{pf}[Q(A, B)] \).

References

[1] E. A. Bender and D. E. Knuth, Enumeration of plane partitions, J. Combin. Theory Ser. A 13 (1972), 40–54.
[2] S. Cho, A new Littlewood–Richardson rule for Schur P-functions, Trans. Amer. Math. Soc. 365 (2013), no. 2, 939–972.
[3] W. Fulton, Young Tableaux. With Applications to Representation Theory and Geometry, London Mathematical Society Student Texts, vol. 35, Cambridge University Press, 1997.
[4] W. Fulton and J. Harris, Representation Theory: A First Course, Graduate Texts in Mathematics, vol. 129, Springer, New York, 1991.
[5] V. Gasharov, A short proof of the Littlewood–Richardson rule, European J. Combin. 19 (1998), 451–453.
[6] M. D. Haiman, On mixed insertion, symmetry, and shifted Young tableaux, J. Combin. Theory Ser. A 50 (1989), 196–225.
[7] T. Józefiak and P. Pragacz, A determinantal formula for skew Q-functions, J. London Math. Soc. 43 (1991), 76–90.
[8] D. E. Knuth, Permutations, matrices and generalized Young tableaux, Pac. J. Math. 34 (1970), 709–727.
[9] A. Knutson and T. Tao, The honeycomb model of GL_n(C) tensor products. I. Proof of the saturation conjecture, J. Amer. Math. Soc. 12 (1999), no. 4, 1055–1090.
[10] A. Knutson, T. Tao, and C. Woodward, The honeycomb model of GL_n(C) tensor products. II. Puzzles determine facets of the Littlewood–Richardson cone, J. Amer. Math. Soc. 17 (2004), no. 1, 19–48.
[11] W. Krasikov, Reduced decompositions in hyperoctahedral groups, C. R. Acad. Sci. Paris Sér. I Math. 309 (1989), 903–907.
[12] T. K. Lam, B_n, Stanley symmetric functions, Discrete Math. 157 (1996), 241–270.
[13] A. Lascoux, B. Leclerc, and J.-Y. Thibon, The plactic monoid, in "M. Lothaire, Algebraic combinatorics on words", Cambridge University Press, Cambridge, 2002, (Chapter 5).
[14] A. Lascoux and M.-P. Schützenberger, Le monoïde plaxique, Quad. Ricerca Scient. 109 (1981), 129–156.
[15] D. E. Littlewood and A. R. Richardson, Group characters and algebras, Phi. Trans. Royal Soc. London A 233 (1934), 99–141.
[16] I. G. Macdonald, Symmetric functions and Hall polynomials, second ed., Oxford Mathematical Monographs, Oxford University Press, New York, 1995, With contributions by A. Zelevinsky.
[17] J. B. Remmel and R. Whitney, Multiplying Schur Functions, J. Algorithms 5 (1984), 471–487.
[18] B. E. Sagan, Shifted Tableaux, Schur Q-Functions and a Conjecture of R. Stanley, J. Combin. Theory Ser. A 45 (1987), 62–103.
[19] _____, The symmetric group. Representations, combinatorial algorithms, and symmetric functions, 2nd ed., Graduate Texts in Mathematics, vol. 203, Springer, New York, 2001.
[20] C. Schensted, *Longest increasing and decreasing subsequences*, Canad. J. Math. 13 (1961), 179–191.

[21] I. Schur, *Über die darstellung der symmetrischen und der alternierenden Gruppe durch gebrochene lineare Substitutionen*, J. Reine Angew. Math. 139 (1911), 155–250.

[22] L. Serrano, *The shifted plactic monoid*, Math. Z. 266 (2010), 363–392, arXiv:0811.2057.

[23] J. R. Stembridge, *Shifted tableaux and the projective representations of symmetric groups*, Adv. in Math. 74 (1989), 87–134.

[24] ———, *On symmetric functions and the spin characters of $S_n$*, Topics in Algebra (S. Balcerzyk et al., Eds.), Banach Center Publ., vol. 26 (Part 2), PWN-Polish Scientific Publishers, Warsaw, 1990, pp. 433–453.

[25] R. M. Thrall, *A combinatorial problem*, Michigan Math. J. 1 (1952), 81–88.

[26] D. R. Worley, *A Theory of Shifted Young Tableaux*, Ph.D. thesis, Massachusetts Institute of Technology, 1984.

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