WEIL RESTRICTION OF NONCOMMUTATIVE MOTIVES

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Abstract. The Weil restriction functor was introduced in the late fifties and extended recently by Karpenko to the category of Chow motives with integral coefficients. In this article we introduce the noncommutative analogue of the Weil restriction functor, where schemes are replaced by dg algebras, and extend it to Kontsevich’s category of noncommutative motives. As an application, we compute Karpenko’s restriction functor in several new cases.

1. Introduction

Weil restriction. Given a finite Galois field extension \( l/k \) with Galois group \( G \), Weil [47] introduced in the late fifties the \( R_{l/k} : \text{QProj}(l) \to \text{QProj}(k) \) from quasi-projective \( l \)-schemes to quasi-projective \( k \)-schemes. Conceptually, (1.1) is the right adjoint of the classical base-change functor. Among other properties, it preserves smoothness, projectiveness, and it is moreover symmetric monoidal. Consequently, it restricts to a \( \otimes \)-functor

\[
R_{l/k} : \text{SmProj}(l) \to \text{SmProj}(k)
\]

from smooth projective \( l \)-schemes to smooth projective \( k \)-schemes. At the beginning of the millennium, Karpenko [20] extended (1.2) to well-defined \( \otimes \)-functors

\[
\begin{array}{ccc}
\text{SmProj}(l) & \xrightarrow{R_{l/k}} & \text{SmProj}(k) \\
M & \downarrow{M} & \downarrow{M} \\
\text{Chow}(l)_Z & \xrightarrow{R_{l/k}} & \text{Chow}(k)_Z \\
\end{array}
\]

\[
\begin{array}{ccc}
\text{SmProj}(l) & \xrightarrow{R_{l/k}} & \text{SmProj}(k) \\
M^* & \downarrow{M^*} & \downarrow{M^*} \\
\text{Chow}^*(l)_Z & \xrightarrow{R_{l/k}^*} & \text{Chow}^*(k)_Z \\
\end{array}
\]

on the categories of Chow motives with integral coefficients; \( \text{Chow}^*(k)_Z \) uses correspondences of arbitrary degree. Although well-defined, these extensions are not additive! In particular, they are not the right adjoint of the base-change functor. This makes their computation a very difficult task.

Kontsevich’s noncommutative motives. In noncommutative algebraic geometry in the sense of Bondal, Drinfeld, Kaledin, Kapranov, Kontsevich, Orlov, Van den Bergh, and others (see [5, 6, 7, 8, 12, 18, 25, 26, 27]), schemes are replaced by differential graded (=dg) algebras. A celebrated result, due to Bondal and Van den Bergh [8], asserts that for every quasi-compact quasi-separated scheme \( Y \) over a base field \( K \) (e.g. \( Y \) quasi-projective) there exists a dg \( K \)-algebra \( A_Y \) whose derived

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category of perfect complexes $\text{perf}(A_Y)$ is equivalent to $\text{perf}(Y)$; see Definition 6.3. The dg algebra $A_Y$ is unique up to derived Morita equivalence (see §3) and reflects many of the properties of $Y$. For instance, $Y$ is smooth (resp. proper) if and only if $A_Y$ is smooth (resp. proper) in the sense of Kontsevich, i.e. if and only if $A_Y$ is compact as a bimodule over itself (resp. $\sum_i \dim_K H^i(A_Y) < \infty$).

Let us denote by $\text{Dga}(K)$ the category of dg $K$-algebras and by $\text{SpDga}(K)$ its full subcategory of smooth and proper dg algebras. Making use of $K$-theory, Kontsevich introduced in [25, 26, 27] the category $\text{NChow}(K)_\mathbb{Z}$ of noncommutative Chow motives (with integral coefficients) as well as a canonical $\otimes$-functor

$$U(-)_\mathbb{Z} : \text{SpDga}(K) \longrightarrow \text{NChow}(K)_\mathbb{Z};$$

consult §4 for details and the survey article [40] for several applications.

**Motivation:** The aforementioned constructions and results in the commutative and noncommutative world lead us naturally to the following motivating questions:

Q1: *Does the Weil restriction functor admits a noncommutative analogue $R^{nc}_{l/k}$?*

Q2: *Does $R^{nc}_{l/k}$ extends to a well-defined $\otimes$-functor on $\text{NChow}(K)_\mathbb{Z}$?*

Q3: *Can this new theory be used for the computation of Karpenko’s functors (1.3) ?*

In this article we provide complete answers to questions Q1-Q2 and to one half of question Q3. A partial answer to the remaining half of Q3 is also provided. In the process we develop new technology of independent interest.

### 2. Statement of results

Let $l/k$ be a finite Galois field extension of degree $n$ with Galois group $G$. The noncommutative analogue of (1.1) is given by the following $\otimes$-functor

$$R^{nc}_{l/k} : \text{Dga}(l) \longrightarrow \text{Dga}(k) \quad B \mapsto (\otimes_{\sigma \in G} ^{\sigma} B)^G,$$

where $^{\sigma} B$ is the $\sigma$-conjugate of $B$ and $G$ acts on $\otimes_{\sigma \in G} ^{\sigma} B$ by permutation of the tensor factors; see §5.1. The correctness of this construction is justified by our first main result, which answers affirmatively to the above question Q1.

**Theorem 2.2.** (i) The functor (2.1) preserves proper dg algebras.

(ii) Assume that the characteristic of $k$ does not divide $n$ (e.g. $\mathbb{Q} \subseteq k$). Under this assumption, the functor (2.1) also preserves smooth dg algebras.

(iii) For every quasi-projective $l$-scheme $Y$, the dg $k$-algebras $R^{nc}_{l/k}(A_Y)$ and $A_{R^{nc}_{l/k}(Y)}$ are derived Morita equivalent.

Thanks to Theorem 2.2, (2.1) restricts to a $\otimes$-functor

$$R^{nc}_{l/k} : \text{SpDga}(l) \longrightarrow \text{SpDga}(k).$$

Recall now from §9.1 that a binomial ring is an unital commutative ring $R$ equipped with binomial coefficients. Examples include $\mathbb{Z}$, its localizations, and also every $\mathbb{Q}$-algebra. Our affirmative answer to question Q2 is then the following:

**Theorem 2.4.** The functor (2.3) extends to a $\otimes$-functor

$$R^{nc}_{l/k} : \text{NChow}(l)_\mathbb{Z} \longrightarrow \text{NChow}(k)_\mathbb{Z}.$$

Given a binomial ring $R$, (2.5) extends furthermore to a $\otimes$-functor

$$R^{nc}_{l/k} : \text{NChow}(l)_R \longrightarrow \text{NChow}(k)_R.$$
As in the commutative world, the functors (2.5)-(2.6) are not additive. In order to describe their non-additivity we introduce the following $G$-action:

**Notation 2.7.** Given an integer $m \geq 2$, consider the following left $G$-action

$$G \times \{ G \to \{ 1, \ldots , m \} \} \to \{ G \to \{ 1, \ldots , m \} \} \quad (\rho, \alpha) \to \rho(\alpha) := (\sigma \mapsto \alpha(\rho^{-1}\sigma))$$

on the set of arbitrary maps from $G$ to $\{ 1, \ldots , m \}$. Let $O(G, m)$ be the associated set of orbits, $\#^m_G$ its cardinality, and $\text{stab}(\alpha) \subseteq G$ the stabilizer of $\alpha$.

**Theorem 2.8.** (i) The choice of a representative $\alpha_o$ for each orbit $o \in O(G, m)$ gives rise to an isomorphism

$$\mathcal{R}_{l/k}^{\text{nc}}(U(l)_R \oplus \cdots \oplus U(l)_R) \simeq \bigoplus_{o \in O(G, m)} U(l^{\text{stab}(\alpha_o)})_R . \quad (2.9)$$

(ii) When $G \simeq \mathbb{Z}/p\mathbb{Z}$, with $p$ a prime number, the right-hand-side of (2.9) reduces to the direct sum of $m$ copies of $U(k)_R$ and $(\#^m_G - m)$-copies of $U(l)_R$.

(iii) Every intermediate field extension $l/k_o/k$ can be written as $l^{\text{stab}(\alpha_o)}$ for some orbit $o \in O(G, m)$ and representative $\alpha_0$.

**Remark 2.10.** Different representatives $\alpha_o, \alpha'_o$ of an orbit $o \in O$ give rise to conjugated groups $\text{stab}(\alpha_o), \text{stab}(\alpha'_o)$ and hence to conjugated fields $l^{\text{stab}(\alpha_o)}, l^{\text{stab}(\alpha'_o)}$. As a consequence, (2.9) is canonical up to conjugation.

Intuitively speaking, Theorem 2.8 shows us that the non-additivity of (2.5)-(2.6) is encoded in the above $G$-action. Note that thanks to item (iii) the behavior is highly non-additive. Here is a (concrete) example:

**Example 2.11.** Let $B$ be a finite dimensional $\mathcal{C}$-algebra of finite global dimension. Note that $B$ is smooth since its projective dimension as a $B$-$B$-bimodule is finite; see [10, page 2]. Examples include all path algebras of finite quivers without oriented cycles and more generally their quotients by admissible ideals (e.g. the quiver algebras of Khovanov-Seidel [28] and the close relatives of Rouquier-Zimmerman [36]). As explained in [39, Remark 3.19], $U(B)_R$ identifies with the direct sum of $m$ copies of $U(\mathcal{C})_R$, where $m$ is the number of simple (right) $B$-modules. Using Theorem 2.8, one then obtains the following isomorphism

$$\mathcal{R}_{\mathbb{C}/\mathbb{R}}^{\text{nc}}(U(B)_R) \simeq \bigoplus_{m\text{-copies}} U(\mathbb{R})_R \oplus \cdots \oplus U(\mathbb{R})_R \oplus \bigoplus_{(\#^m_G - m)\text{-copies}} U(\mathcal{C})_R .$$

Now, let $Y$ be an irreducible smooth projective $l$-scheme of dimension $d$ such that

$$U(A_Y)_{\mathbb{Z}[1/(2nd)!]} \simeq \bigoplus_{m\text{-copies}} U(l)_{\mathbb{Z}[1/(2nd)!]} \oplus \cdots \oplus U(l)_{\mathbb{Z}[1/(2nd)!]} . \quad (2.12)$$

This motivic decomposition occurs for instance in the following four cases:

**I: Full exceptional collections.** Recall from Huybrechts [16, §4] the notion of a full exceptional collection. As proved in [35, §5], whenever the derived category $\text{perf}(Y)$ admits a full exceptional collection of length $m$, $U(A_Y)_\mathbb{Z}$ becomes isomorphic to the direct sum of $m$ copies of $U(l)_\mathbb{Z}$. As a consequence, (2.12) holds for every $n$. Thanks to the work of Beilinson, Kapranov, Kawamata, Kuznetsov, Manin-Smirnov, Orlov, and others (see [3, 19, 21, 31, 33, 37]), examples include:

(i) Projective spaces, rational surfaces, and moduli spaces of pointed stable curves of genus zero, in the case of an arbitrary base field $l$;
(ii) Quadrics, Grassmannians, flag varieties, Fano threefolds with vanishing odd cohomology, and toric varieties, in the case where $l$ is algebraically closed and of characteristic zero.

Conjecturally, full exceptional collections exist also for all homogeneous spaces of the form $G/P$ with $P$ a parabolic subgroup of a semisimple algebraic group $G$; see Kuznetsov-Polishchuk [32].

**II: Quadrics.** Let $l$ be a field of characteristic $\neq 2$, $V$ a finite dimensional $l$-vector space, $q : V \to l$ a non-degenerate quadratic form, and $Y := Q_q \subset \mathbb{P}(V)$ the associated smooth projective quadric of dimension $d := \dim(V) - 2$. In this generality, the derived category $\text{perf}(Y)$ does not admit a full exceptional collection. Nevertheless, as explained in [39, §3], the following holds:

(i) When $\dim(V)$ is odd, $U(A_Y)_{Z[1/2]}$ identifies with the direct sum of $d+1$ copies of $U(l)_{Z[1/2]}$. Hence, (2.12) (with $m = d+1$) holds for every $n$;

(ii) When the signed determinant $\delta(q) \in l^*/(l^*)^2$ of $q$ belongs to $(l^*)^2$, $U(A_Y)_{Z[1/2]}$ identifies with the direct sum of $d+2$ copies of $U(l)_{Z[1/2]}$. Hence, (2.12) (with $m = d+2$) holds for every $n$.

**III: Complex surfaces of general type.** Thanks to the work of Alexeev-Orlov [1], Böning-Graf von Bothmer-Sosna [9], Galkin-Shinder [14], and Gorchinskiy-Orlov [15, Propositions 2.3 and 4.3], the following holds:

(i) When $Y$ is the Godeaux surface, $U(A_Y)_{Z[1/3]}$ identifies with the direct sum of 11 copies of $U(\mathbb{C})_{Z[1/3]}$. Hence, (2.12) (with $m = 11$) holds for every $n \geq 2$;

(ii) When $Y$ is any Burniat surface, $U(A_Y)_{Z[1/2]}$ identifies with the direct sum of 6 copies of $U(\mathbb{C})_{Z[1/2]}$. Hence, (2.12) (with $m = 6$) holds for every $n$;

(iii) When $Y$ is the Beauville surface, $U(A_Y)_{Z[1/3]}$ identifies with the direct sum of 4 copies of $U(\mathbb{C})_{Z[1/3]}$. Hence, (2.12) (with $m = 4$) holds for every $n \geq 2$.

**IV: Severi-Brauer varieties.** Let $B$ be a central simple $l$-algebra of degree $\deg(B) := \sqrt{\dim(B)}$ and $Y := SB(B)$ the associated Severi-Brauer variety of dimension $d := \deg(B) - 1$. As proved in [39, §3], $U(A_Y)_{Z[1/\dim(B)]}$ identifies with the direct sum of $\deg(B)$ copies of $U(l)_{Z[1/\dim(B)]}$. Hence, (2.12) (with $m = \deg(B)$) holds for every $n$.

Our fourth main result, which answers to one half of question Q3, is the following:

**Theorem 2.13.** Let $Y$ be an irreducible smooth projective $l$-scheme of dimension $d$ and $R$ a commutative ring. Assume that (2.12) holds, that $1/(2d)! \in R$, and that $\text{char}(k) \not\mid n$. Under these assumptions, the choice of a representative $\alpha_o$ for each orbit $o \in \mathcal{O}(G, m)$ gives rise to an isomorphism

\[
\mathcal{R}^{*}_{l/k}(M^*(Y))_R \simeq \bigoplus_{o \in \mathcal{O}(G, m)} M^*(\text{spec}(l^{\text{stab}(\alpha_o)}))_R.
\]

Isomorphism (2.14) applied to each one of the above cases I-IV gives rise to several computations of Karpenko’s functor $\mathcal{R}^{*}_{l/k}$.

Here is a low dimensional example:

**Example 2.15.** Let $l/k$ be the quadratic extension $\mathbb{Q}(\zeta_3)/\mathbb{Q}$ and $Y$ the moduli space $\mathcal{M}_{0,5}$ of 5-pointed stable curves of genus zero. The $\mathbb{Q}(\zeta_3)$-scheme $\mathcal{M}_{0,5}$ is 2-dimensional and, as proved by Manin-Smirnov in [33, §3.3], the derived category $\text{perf}(\mathcal{M}_{0,5})$ admits a full exceptional collection of length 7. As a consequence, we conclude from (2.14) that

\[
\mathcal{R}^{*}_{\mathbb{Q}(\zeta_3)/\mathbb{Q}}(M^*(\mathcal{M}_{0,5}))_R \simeq \bigoplus_{i=1}^7 M^*(\text{spec}(\mathbb{Q}))_R \oplus \bigoplus_{i=1}^{21} M^*(\text{spec}(\mathbb{Q}(\zeta_3)))_R.
\]
for every commutative ring $R$ containing $\frac{1}{2}, \frac{1}{3}, \frac{1}{5}, \frac{1}{7}$.

Our partial answer to the remaining half of question Q3 is the following:

**Theorem 2.16.** Under the assumptions of Theorem 2.13, the following holds:

(i) There exist idempotent endomorphisms $e_i$, with $\sum_i e_i = \text{id}$, such that

$$\mathcal{R}_{l/k}(M(Y))_R \simeq \bigoplus_{i=0}^{nd} e_i \left( \bigoplus_{o \in \mathcal{O}(G,m)} M(\text{spec}(r^{\text{stab}(a_o)}))_R \right) \otimes L^\otimes i,$$

where $e_i(-)$ stands for the image of $e_i$ and $L$ for the Lefschetz motive.

(ii) When $Q \subseteq R$, the idempotents $e_i$ are mutually orthogonal.

(iii) When $R$ is field of characteristic zero and $G \simeq \mathbb{Z}/p\mathbb{Z}$, with $p$ a prime number, there exist non-negative integers $r_i, s_i \in \{0, \ldots, nd\}$ for which

$$\mathcal{R}_{l/k}(M(Y))_R \simeq \bigoplus_{i=1}^{m} M(\text{spec}(k))_R \otimes L^\otimes r_i \oplus \bigoplus_{i=1}^{\# \mathbb{Q} - m} M(\text{spec}(l))_R \otimes L^\otimes s_i.$$

Intuitively speaking, Theorem 2.16 show us that the existence of the motivic decomposition (2.12) “quasi-determines” the shape of the Chow motive $\mathcal{R}_{l/k}(M(Y))_R$.

The indeterminacy consists only on the idempotent endomorphisms $e_i$ or on the non-negative integers $r_i, s_i$. In particular, $\mathcal{R}_{l/k}(M(Y))_R$ is an Artin-Tate motive.

**Example 2.18.** Example 2.15 (with $R$ a field of characteristic zero) combined with item (iii) of Theorem 2.16, allow us to conclude that there exist non-negative integers $r_1, \ldots, r_7, s_1, \ldots, s_{21} \in \{0, \ldots, 4\}$ for which

$$\mathcal{R}_{\mathbb{Q}(\zeta_7)/\mathbb{Q}}(M(\mathbb{P}^1)_{\mathbb{Q}(\zeta_7)}) \simeq \bigoplus_{i=1}^{7} M(\text{spec}(\mathbb{Q}))_R \otimes L^\otimes r_i \oplus \bigoplus_{i=1}^{21} M(\text{spec}(\mathbb{Q}(\zeta_7)))_R \otimes L^\otimes s_i.$$

Note that the above indeterminacies cannot be refined. For instance, the $l$-schemes $\text{spec}(l) \times \text{spec}(l)$ and $\mathbb{P}^1$ have the same motivic decomposition (2.12) since their derived categories of perfect complexes admit a full exceptional collection of length 2. However, as explained by Karpenko in [20, Example 4.8], one has

$$\mathcal{R}_{C/R}(M(\mathbb{P}^1))_\mathbb{Z} \simeq M(\text{spec}(\mathbb{R}))_\mathbb{Z} \oplus M(\text{spec}(\mathbb{C}))_\mathbb{Z} \otimes L \oplus M(\text{spec}(\mathbb{R}))_\mathbb{Z} \otimes L^\otimes 2$$

which is a twist of

$$\mathcal{R}_{C/R}(M(\text{spec}(l) \times \text{spec}(l)))_\mathbb{Z} \simeq M(\text{spec}(\mathbb{R}))_\mathbb{Z} \oplus M(\text{spec}(\mathbb{C}))_\mathbb{Z} \oplus M(\text{spec}(\mathbb{R}))_\mathbb{Z}.$$

In conclusion, Theorem 2.16 furnish us the maximum amount of information about the Chow motive $\mathcal{R}_{l/k}(M(Y))_R$ that can be extracted from the existence of the above motivic decomposition (2.12).

**Notations.** Throughout the article, $l/k$ will denote a finite Galois field extension of degree $n := [l : k]$ with Galois group $G := \text{Gal}(l/k)$. We will reserve the letter $R$ for a (binomial) ring of coefficients, the letters $\sigma, \rho, \tau$ for elements of $G$, the letters $B, B', B''$ for dg algebras, the letters $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ for dg categories, the letters $M, M', M''$ for modules, the letters $\mathcal{B}, \mathcal{B}', \mathcal{B}''$ for bimodules, and the letters $Y, Y', Y''$ for (quasi-projective) schemes. Given a small category $\mathcal{C}$, we will write $\text{Iso}\mathcal{C}$ for its set of isomorphism classes of objects and $\mathcal{C}^\otimes$ for its idempotent completion.
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3. Differential graded preliminaries

Let $K$ be a field and $C(K)$ be the category of cochain complexes of $K$-vector spaces. A differential graded (=dg) category $\mathcal{B}$ is a category enriched over $C(K)$ and a dg functor $\mathcal{B} \to \mathcal{B}'$ is a functor enriched over $C(K)$; consult Keller’s ICM survey [22]. A dg algebra $B$ is a dg category with a single object and a dg morphism $B \to B'$ is a dg functor between dg categories with a single object. In what follows we will write $\text{Dgcat}(K)$ for the category of (small) dg categories and dg functors, and $\text{Dga}(K)$ for the category of dg algebras and dg homomorphisms.

Modules. Let $B$ be a dg category. The category $H^0(\mathcal{B})$ has the same objects as $\mathcal{B}$ and morphisms given by $H^0(\mathcal{B})(x, y) := H^0(\mathcal{B}(x, y))$, where $H^0$ denotes degree zero cohomology. The opposite $\mathcal{B}^{\text{op}}$ has the same objects as $\mathcal{B}$ and complexes of morphisms given by $\mathcal{B}^{\text{op}}(x, y) := \mathcal{B}(y, x)$. A right $\mathcal{B}$-module is a dg functor $M : \mathcal{B}^{\text{op}} \to C_{dg}(K)$ with values in the dg category $C_{dg}(K)$ of cochain complexes of $K$-vector spaces. Let us denote by $C(\mathcal{B})$ the category of right $\mathcal{B}$-modules. Recall from [22, §3.2] that the derived category $D(\mathcal{B})$ of $\mathcal{B}$ is the localization of $C(\mathcal{B})$ with respect to the class of objectwise quasi-isomorphisms. Its full subcategory of compact objects will be denoted by $D_\text{c}(\mathcal{B})$.

Tensor product. The tensor product $\mathcal{B} \otimes \mathcal{B}'$ of dg categories is defined by the cartesian product of the sets of objects of $\mathcal{B}$ and $\mathcal{B}'$ and by the tensor product of complexes $(\mathcal{B} \otimes \mathcal{B}')(x, x', (y, y')) := \mathcal{B}(x, y) \otimes \mathcal{B}'(x', y')$. This gives rise to a symmetric monoidal structure on $\text{Dgcat}(K)$ with $\otimes$-unit $K$; see [22, §2.3 and §4.3].

Bimodules. Let $\mathcal{B}, \mathcal{B}'$ be two dg categories. A $\mathcal{B}-\mathcal{B}'$-bimodule is a dg functor $\mathcal{B} : \mathcal{B}^{\text{op}} \otimes \mathcal{B}'^{\text{op}} \to C_{dg}(K)$. Standard examples are given by the $\mathcal{B}-\mathcal{B}'$-bimodule

$$\mathcal{B} \otimes \mathcal{B}'^{\text{op}} \to C_{dg}(K) \quad (x, y) \mapsto \mathcal{B}(y, x)$$

and by the $\mathcal{B}-\mathcal{B}'$-bimodule

$$\mathcal{B} \otimes \mathcal{B}'^{\text{op}} \to C_{dg}(K) \quad (x, x') \mapsto \mathcal{B}'(x', F(x))$$

and $F: \mathcal{B} \to \mathcal{B}'$ associated to a dg functor $F : \mathcal{B} \to \mathcal{B}'$.

Derived Morita equivalences. A dg functor $F : \mathcal{B} \to \mathcal{B}'$ is called a derived Morita equivalence if the induced restriction of scalars $D(\mathcal{B}') \to D(\mathcal{B})$ is an equivalence of (triangulated) categories; see [22, §4.6]. As proved in [41, Theorem 5.3], $D(\text{Dgcat}(K))$ carries a Quillen model structure whose weak equivalences are the derived Morita equivalences. Let $\text{Hmo}(K)$ be the homotopy category obtained. As explained in [41, Corollary 5.10], one has a bijection

$$\text{Hom}_{\text{Hmo}(K)}(\mathcal{B}, \mathcal{B}') \simeq \text{Iso rep}(\mathcal{B}, \mathcal{B}')$$

where $\text{rep}(\mathcal{B}, \mathcal{B}')$ denotes the full triangulated subcategory of $D(\mathcal{B}^{\text{op}} \otimes \mathcal{B}')$ consisting of those $\mathcal{B}-\mathcal{B}'$-bimodules $\mathcal{B}$ such that for every object $x \in \mathcal{B}$ the right $\mathcal{B}'$-module $\mathcal{B}(x, -)$ belongs to $D_\text{c}(\mathcal{B}')$. Under the above bijection (3.3), the composition law
on $\text{Hmo}(K)$ corresponds to the (derived) tensor product of bimodules. Moreover, the identity of an object $B$ is given by the class of the above $B\otimes B'$-bimodule (3.1).

Finally, two dg categories $B, B'$ are called derived Morita equivalent if they become isomorphic in $\text{Hmo}(K)$. This is equivalent to the existence of a $B\otimes B'$-bimodule $B \in \text{rep}(B, B')$ inducing an equivalence $- \otimes_B B : D(B) \xrightarrow{\sim} D(B')$ of (triangulated) categories; consult [22, §3.8] for details.

**Smoothness and properness.** Recall from Kontsevich [25, 26, 27] that a dg category $B$ is called smooth if the above $B\otimes B'$-bimodule induces a symmetric monoidal structure on $\text{Hmo}(K)$ and proper if for each ordered pair of objects $(x, y)$ one has $\sum \dim_K H^i(B(x, y)) < \infty$. Thanks to [11, Proposition 4.10] and [22, Theorem 4.22], every smooth and proper dg category is derived Morita equivalent to a smooth and proper dg algebra.

### 4. Kontsevich’s noncommutative motives

In this section we recall from [11, 40, 41] the construction of Kontsevich’s category of noncommutative Chow motives; consult also Kontsevich’s work [25, 26, 27].

Recall from §3 the description of the homotopy category $\text{Hmo}(K)$. Note that since the $B\otimes B'$-bimodules (3.2) belong to $\text{rep}(B, B')$, we have a well-defined functor

\begin{equation}
\text{Dgcat}(k) \to \text{Hmo}(K) \quad F \mapsto F_{\text{Bi}}.
\end{equation}

The tensor product of dg categories descends to $\text{Hmo}(K)$ giving thus rise to a symmetric monoidal structure on $\text{Hmo}(K)$ and making the functor (4.1) symmetric monoidal. The *additivization* of $\text{Hmo}(K)$ is the additive category $\text{Hmo}_0(K)$ with the same objects as $\text{Hmo}(K)$ and abelian groups of morphisms $\text{Hom}_0(\text{Hmo}(K))(B, B')$ given by the Grothendieck group $K_0\text{rep}(B, B')$ of the triangulated category $\text{rep}(B, B')$. The composition law is induced by the (derived) tensor product of bimodules. Note that we have a canonical functor

\begin{equation}
\text{Hmo}(K) \to \text{Hmo}_0(K) \quad B \mapsto |B|.
\end{equation}

Given a commutative ring of coefficients $R$, the $R$-linearization of $\text{Hmo}_0(K)$ is the $R$-linear additive category $\text{Hmo}_0(K)_R$ obtained by tensoring each abelian group of morphisms of $\text{Hmo}_0(K)$ with $R$. It gives rise to the functor

\begin{equation}
\text{Hmo}_0(K) \to \text{Hmo}_0(K)_R \quad [B] \mapsto [B] \otimes_R R.
\end{equation}

The symmetric monoidal structure on $\text{Hmo}(K)$ descends first to a bilinear symmetric monoidal structure on $\text{Hmo}_0(K)$ and then to a $R$-linear bilinear symmetric monoidal structure on $\text{Hmo}_0(K)_R$ making the above functors (4.2)-(4.3) symmetric monoidal. We hence obtain the following composition of $\otimes$-functors:

\begin{equation}
\text{Dgcat}(k) \xrightarrow{(4.1)} \text{Hmo}(K) \xrightarrow{(4.2)} \text{Hmo}_0(K) \xrightarrow{(4.3)} \text{Hmo}_0(K)_R.
\end{equation}

Kontsevich’s category of **noncommutative Chow motives** $\text{NChow}(K)_R$ (with $R$-coefficients) is then defined as the idempotent completion of the full subcategory of $\text{Hmo}_0(K)_R$ generated by the smooth and proper dg algebras$^1$. Thanks to [11, Theorem 5.8], $\text{NChow}(K)_R$ inherits a symmetric monoidal structure. By restricting (4.4) to smooth and proper dg algebras we hence obtain a well-defined $\otimes$-functor

$$U(-)_R : \text{SpDga}(K) \to \text{NChow}(K)_R.$$  

$^1$Or equivalently by the smooth and proper dg categories.
Finally, given smooth and proper dg algebras $B, B'$, the triangulated category $\text{rep}(B, B') \subset \mathcal{D}(B^\text{op} \otimes B')$ identifies with $\mathcal{D}_c(B^\text{op} \otimes B')$; see [11, §5]. As a consequence, we obtain the following isomorphism
\begin{equation}
\text{Hom}_{\text{Chow}(K)_{\text{R}}} (U(B)_R, U(B')_R) \simeq K_0 (B^\text{op} \otimes B')_R.
\end{equation}

5. DG Galois Descent

In this section we extend the classical Galois descent theory to the differential graded setting; see Proposition 5.8. Making use of it, we then introduce the non-commutative analogue of the Weil restriction functor; see §5.1.

Let $l/k$ be a finite Galois field extension with Galois group $G$. For every $\sigma \in G$ one has a $\otimes$-equivalence of categories
\begin{equation}
\sigma (-) : C(l) \xrightarrow{\sim} C(l) \quad V \mapsto \sigma V,
\end{equation}
where $\sigma V$ is obtained from $V$ by restriction of scalars along the automorphism $\sigma^{-1} : l \xrightarrow{\sim} l$. This gives naturally rise to the functor
\begin{equation}
C(l) \rightarrow C(l) \quad V \mapsto \otimes_{\sigma \in G} \sigma V.
\end{equation}

Definition 5.2. A $l/k$-Galois complex is a complex of $l$-vector spaces $W$ endowed with a left $G$-action $G \times W \rightarrow W, (\rho, w) \mapsto \rho(w)$, which is skew-linear in the sense that $\rho(\lambda) \cdot \rho(w) = \rho(\lambda \cdot w)$ for every $\lambda \in l$, $w \in W$ and $\rho \in G$.

Example 5.3. Given a complex of $k$-vector spaces $U$, the complex of $l$-vector spaces $W := U \otimes_k l$, endowed with the skew-linear left $G$-action
\begin{equation}
G \times (U \otimes_k l) \rightarrow U \otimes_k l \quad (\rho, u \otimes \lambda) \mapsto u \otimes \rho(\lambda),
\end{equation}
is a $l/k$-Galois complex.

Example 5.4. The above complex of $l$-vector spaces $W := \otimes_{\sigma \in G} \sigma V$, endowed with the skew-linear left $G$-action
\begin{equation}
G \times \otimes_{\sigma \in G} \sigma V \rightarrow \otimes_{\sigma \in G} \sigma V \quad (\rho, \otimes_{\sigma \in G} v_\sigma) \mapsto \otimes_{\sigma \in G} v_\rho^{-1} \sigma,
\end{equation}
is a $l/k$-Galois complex.

Let $C(l)^{\text{Gal}}$ be the category of $l/k$-Galois complexes and $G$-equivariant morphisms. By construction one has a forgetful functor $C(l)^{\text{Gal}} \rightarrow C(l)$. Moreover, $C(l)^{\text{Gal}}$ carries a canonical symmetric monoidal structure making the forgetful functor symmetric monoidal. The $\otimes$-unit is the complex $l$ (concentrated in degree zero) endowed with the canonical skew-linear left $G$-action, and given $l/k$-Galois complexes $W, W'$ the group $G$ acts diagonally on the underlying tensor product $W \otimes W'$.

Lemma 5.6. The above functor (5.1) lifts to a $\otimes$-functor
\begin{equation}
C(l) \rightarrow C(l)^{\text{Gal}} \quad V \mapsto \otimes_{\sigma \in G} \sigma V,
\end{equation}
where $\otimes_{\sigma \in G} \sigma V$ is endowed with the above skew-linear left $G$-action (5.5).

Proof. Clearly, every morphism $V \rightarrow V'$ in $C(l)$ gives rise to a $G$-equivariant morphism $\otimes_{\sigma \in G} \sigma V \rightarrow \otimes_{\sigma \in G} \sigma V'$. Hence, the functor (5.1) take values in $C(l)^{\text{Gal}}$. The naturality of the $G$-equivariant isomorphisms
\begin{equation}
l \xrightarrow{\sim} \otimes_{\sigma \in G} \sigma l \quad (\otimes_{\sigma \in G} \sigma V) \otimes (\otimes_{\sigma \in G} \sigma V') \rightarrow \otimes_{\sigma \in G} \sigma (V \otimes V')
\end{equation}
implies that (5.1) is symmetric monoidal. This achieves the proof. \qed
Proposition 5.8. (DG Galois descent) One has a $\otimes$-equivalence of categories

\[
\begin{align*}
\mathcal{C}(l)^{\text{Gal}} \\
\otimes_k l \bigl( - \bigr) & \cong (-)^G \\
\mathcal{C}(k),
\end{align*}
\]

where $(-)^G$ stands for the $G$-invariants functor.

Proof. If in the above Definition 5.2 one replaces the word “complex” by the word “module”, one recovers the classical notion of $l/k$-Galois module $W$; see [29, II.\S 5]. It consists of a $l$-vector space $W$ endowed with a skew-linear left $G$-action. Let $\text{Vect}(l)^{\text{Gal}}$ be the associated category of $l/k$-Galois modules. As proved in Knus-Ojanguren in [29, II. Theorem 5.3], one has the following $\otimes$-equivalence of categories

\[
\begin{align*}
\mathcal{C}(l)^{\text{Gal}} \\
\otimes_k l \bigl( - \bigr) & \cong \mathcal{C}(k).
\end{align*}
\]

Now, note that a $l/k$-Galois complex is precisely the same data as a complex of morphisms in $\mathcal{C}(l)^{\text{Gal}}$. Moreover, the symmetric monoidal structure on $\mathcal{C}(l)^{\text{Gal}}$ is induced by the symmetric monoidal structure on $\mathcal{C}(l)^{\text{Gal}}$. Since both functors $\otimes_k l$ and $(-)^G$ in (5.9) are defined degreewise, one then observes that (5.9) can be obtained from (5.10) by passing to the category of complexes. This implies that (5.9) is a $\otimes$-equivalence. $\square$

By combining Proposition 5.8 with Lemma 5.6 we obtain a well-defined $\otimes$-functor

\[
\mathcal{C}(l) \rightarrow \mathcal{C}(k) \quad V \mapsto (\otimes_{\sigma \in G} \sigma V)^G.
\]

Lemma 5.12. The functor (5.11) preserves quasi-isomorphisms. Moreover, one has a canonical isomorphism

\[
(\otimes_{\sigma \in G} \sigma V)^G \otimes_{(\otimes_{\sigma \in G} \sigma B)^G} (\otimes_{\sigma \in G} \sigma V')^G \xrightarrow{\sim} (\otimes_{\sigma \in G} \sigma (V \otimes_B V'))^G
\]

for every dg $l$-algebra $B$, right $B$-module $V$, and left $B$-module $V'$.

Proof. Since we are working over a field, the classical Künneth formula holds. Hence, the above functor (5.7) preserves quasi-isomorphisms. In order to show that the functor $(-)^G : \mathcal{C}(l)^{\text{Gal}} \rightarrow \mathcal{C}(k)$ also preserves quasi-isomorphisms, it suffices by equivalence (5.9) to show that $\otimes_k l : \mathcal{C}(k) \rightarrow \mathcal{C}(l)^{\text{Gal}}$ preserves quasi-isomorphisms. This is clearly the case and so our first claim is proved.

Now, note that one has a canonical isomorphism of complexes of $l$-vector spaces

\[
(\otimes_{\sigma \in G} \sigma V) \otimes_{(\otimes_{\sigma \in G} \sigma B)} (\otimes_{\sigma \in G} \sigma V') \xrightarrow{\sim} \otimes_{\sigma \in G} \sigma (V \otimes_B V').
\]

The isomorphism (5.14) is moreover $G$-equivariant with respect to the above skew-linear left $G$-action (5.5); $G$ acts diagonally on the left-hand-side. Thanks to the above $\otimes$-equivalence (5.9), the above isomorphism (5.13) can then be obtained from (5.14) by applying the functor $(-)^G$. In particular, it is an isomorphism. $\square$
5.1. Weil restriction of dg algebras. Since the above functor (5.11) is symmetric monoidal, it gives automatically rise to the following \( \otimes \)-functor
\[
\mathcal{R}_{l/k}^{nc} : \text{Dga}(l) \longrightarrow \text{Dga}(k) \quad B \mapsto (\otimes_{\sigma \in G} B)^G
\]
from dg \( l \)-algebras to dg \( k \)-algebras. Note that thanks to Lemma 5.12, (5.15) preserves quasi-isomorphisms.

6. Perfect complexes on schemes

In this section we collect some standard results and notations and show that perfect complexes satisfy Galois descent; see Proposition 6.7. In what follows, \( Y \) is a quasi-projective scheme.

Notation 6.1. Let \( \text{Mod}(Y) \) be the Grothendieck category of sheaves of \( \mathcal{O}_Y \)-modules and \( \text{Qcoh}(Y) \) the full subcategory of quasi-coherent \( \mathcal{O}_Y \)-modules. We will denote by \( \mathcal{D}(Y) := \mathcal{D}(\text{Qcoh}(Y)) \) the derived category of \( Y \) and by \( \text{perf}(Y) \) the full subcategory of perfect complexes. By construction, we have \( \text{perf}(Y) \subset \mathcal{D}(Y) \). Moreover, as proved by Bondal-Van den Bergh in [8, Theorems 3.1.1 and 3.1.3], \( \text{perf}(Y) \) identifies with the category of compact objects in \( \mathcal{D}(Y) \).

Let \( \mathcal{E} \) be an abelian (or exact) category. As explained in [22, §4.4], the derived dg category \( \mathcal{D}_{dg}(\mathcal{E}) \) of \( \mathcal{E} \) is defined as the dg quotient \( \mathcal{C}_{dg}(\mathcal{E})/\mathcal{A}_{dg}(\mathcal{E}) \) of the dg category of complexes over \( \mathcal{E} \) by its full dg subcategory of acyclic complexes. Note that every exact functor \( \mathcal{E} \rightarrow \mathcal{E}' \) (or more generally every dg functor \( \mathcal{C}_{dg}(\mathcal{E}) \rightarrow \mathcal{C}_{dg}(\mathcal{E}') \) which restricts to \( \mathcal{A}_{dg}(\mathcal{E}) \rightarrow \mathcal{A}_{dg}(\mathcal{E}') \)) gives rise to a dg functor \( \mathcal{D}_{dg}(\mathcal{E}) \rightarrow \mathcal{D}_{dg}(\mathcal{E}') \).

Notation 6.2. Let \( \mathcal{D}_{dg}(Y) \) be the dg category \( \mathcal{D}_{dg}(\mathcal{E}) \), with \( \mathcal{E} := \text{Mod}(Y) \), and \( \text{perf}_{dg}(Y) \) the full dg subcategory of perfect complexes. By construction, we have \( \text{H}^0(\text{perf}_{dg}(Y)) \simeq \text{perf}(Y) \) and \( \text{H}^0(\mathcal{D}_{dg}(Y)) \simeq \mathcal{D}(Y) \).

As proved by Bondal and Van den Bergh in [8, Theorem 3.1.1 and Corollary 3.1.2], the triangulated category \( \text{perf}(Y) \) is generated by a single object \( G \). This gives naturally rise to the following definition:

Definition 6.3. Let \( A_Y := \text{End}_{\text{perf}_{dg}(Y)}(G) \) be the dg algebra associated to \( Y \) and \( G \).

Remark 6.4. The inclusion \( A_Y \rightarrow \text{perf}_{dg}(Y) \) of dg categories is a derived Morita equivalence; see [22, Lemma 3.10]. Hence, given any other generator \( G' \in \text{perf}(Y) \) one obtains a zig-zag of derived Morita equivalences \( A_Y \rightarrow \text{perf}_{dg}(Y) \leftarrow A_Y' \). As a consequence, we conclude that \( A_Y \) and \( A_Y' \) are derived Morita equivalent.

Galois descent. Let \( l/k \) be a finite Galois field extension with Galois group \( G \). A \( l/k \)-Galois scheme is a \( l \)-scheme \( Y \) endowed with a left \( G \)-action \( G \times Y \rightarrow Y \), \( (\rho, y) \mapsto \rho(y) \), which is skew-linear in the sense that the square

\[
\begin{array}{ccc}
Y & \overset{\rho(-)}{\longrightarrow} & Y \\
\downarrow & & \downarrow \\
\text{spec}(l) & \overset{\text{spec}(\rho^{-1})}{\longrightarrow} & \text{spec}(l)
\end{array}
\]

commutes for every \( \rho \in G \). A \( l/k \)-Galois module (over \( Y \)) is a \( \mathcal{O}_Y \)-module \( \mathcal{F} \) together with structure isomorphisms \( s_\rho : (\rho(-))^* \mathcal{F} \simeq \mathcal{F} \) satisfying the cocycle condition \( s_{\tau \rho} = s_\tau \circ (\rho(-))^* (s_\rho) \).
**Notation 6.5.** Let \( \text{Mod}(Y;G) \) be the Grothendieck category of \( l/k \)-Galois modules and \( G \)-equivariant morphisms, and \( \text{Qcoh}(Y;G) \) the full subcategory those \( l/k \)-Galois modules that are quasi-coherent as \( \mathcal{O}_Y \)-modules. We will denote by \( \mathcal{D}(Y;G) := \mathcal{D}(\text{Qcoh}(Y;G)) \) the derived category of the \( l/k \)-Galois scheme \( Y \) and by \( \text{perf}(Y;G) \) the full subcategory of those complexes of \( l/k \)-Galois modules that are perfect as complexes of \( \mathcal{O}_Y \)-modules.

**Notation 6.6.** Let \( \mathcal{D}_{dg}(Y;G) \) be the dg category \( \mathcal{D}_{dg}(\mathcal{E}) \), with \( \mathcal{E} := \text{Mod}(Y;G) \), and \( \text{perf}_{dg}(Y;G) \) the full dg subcategory of those complexes of \( l/k \)-Galois modules that belong to \( \text{perf}(Y;G) \). By construction, we have \( H^0(\text{perf}_{dg}(Y;G)) \cong \text{perf}(Y;G) \) and \( H^0(\mathcal{D}_{dg}(Y;G)) \cong \mathcal{D}(Y;G) \).

As explained by Joukhovitski in [17, Proposition 3.4(i)], the geometric quotient \( Y/G \) exists and is quasi-projective. Let \( p : Y \rightarrow Y/G \) be the quotient map.

**Proposition 6.7.** One has an equivalence of categories and of dg categories:

\[
\text{perf}(Y;G) \cong _p \text{perf}_{dg}(Y;G) \\
\text{perf}(Y;G) \cong _p \text{perf}_{dg}(Y/G).
\]

**Proof.** Let \( \text{Vect}(Y/G) \) be the category of vector bundles over \( Y/G \) and \( \text{Vect}(Y;G) \) the full subcategory of \( \text{Mod}(Y;G) \) consisting of those \( l/k \)-Galois modules that are vector bundles as \( \mathcal{O}_Y \)-modules. As explained in [17, page 12], the quotient map \( p \) gives rise to the following equivalence of categories:

\[
\text{Vect}(Y;G) \cong _p \text{Vect}(Y/G).
\]

Now, recall from Thomason-Trobaugh [44, Corollary 3.9] that since every quasi-projective scheme admits an ample family of line bundles, one has the equivalences

\[
\mathcal{D}^b(\text{Vect}(Y)) \cong \text{perf}(Y) \quad \mathcal{D}^b(\text{Vect}(Y/G)) \cong \text{perf}(Y/G)
\]

As a consequence, by applying the constructions \( \mathcal{D}^b(-) \) and \( \mathcal{D}^b_{dg}(-) \) to (6.9) (and using the fact that the functors \( _p \) and \( _p(-)^G \) are exact) we obtain the above equivalences (6.8). \( \square \)

### 7. Proof of Theorem 2.2

**Proof of item (i).** Let \( B \) be a proper dg \( l \)-algebra. By definition, we have \( \sum_i \dim_l H^i(B) < \infty \). Clearly, \( \dim_l H^i(\sigma B) = \dim_l H^i(B) \). Hence, using the K"unneth formula and the fact that the cohomology of \( B \) is bounded, one concludes that

\[
\sum_i \dim_l H^i(\otimes_{\sigma \in G^\sigma} B) < \infty.
\]

Moreover, using equivalence (5.9), we obtain the following equality

\[
\dim_l H^i((\otimes_{\sigma \in G^\sigma} B))^G = \dim_l H^i(\otimes_{\sigma \in G^\sigma} B).
\]

By combining (7.1)-(7.2) we then conclude that the dg \( k \)-algebra \( \mathcal{R}_{l/k}^{nc}(B) \) is proper.

---

2A generalization of equivalence (5.10).
Proof of item (ii). Let \(B'\) be an arbitrary dg \(l\)-algebra. The above functor (5.11) is symmetric monoidal and, thanks to Lemma 5.12, it preserves quasi-isomorphisms. Hence, it gives rise to the following (non-additive) functor:

\[
\mathcal{D}(B') \longrightarrow \mathcal{D}(\mathcal{R}_{l/k}^{nc}(B')) \quad \quad M' \mapsto (\otimes_{\sigma \in G} \sigma M')^G.
\]

Lemma 7.4. The functor (7.3) preserves compact objects.

Proof. Note first that for every \(\sigma \in G\) one has an equivalence of categories

\[
\mathcal{D}_c(B') \sim \mathcal{D}_c(\sigma B') \quad \quad M' \mapsto \sigma M'.
\]

Since the triangulated categories \(\mathcal{D}_c(\sigma B')\) and \(\mathcal{D}_c(\otimes_{\sigma \in G} \sigma B')\) are generated by \(\sigma B'\) and \(\otimes_{\sigma \in G} \sigma B'\), respectively, we have also the following multi-triangulated functor

\[
\prod_{\sigma \in G} \mathcal{D}_c(\sigma B') \longrightarrow \mathcal{D}_c(\otimes_{\sigma \in G} \sigma B') \quad \quad \{\sigma M'\}_{\sigma \in G} \mapsto \otimes_{\sigma \in G} \sigma M'.
\]

Thanks to Proposition 5.8, the dg \(l\)-algebra \(\otimes_{\sigma \in G} \sigma B'\) is canonically isomorphic to \((\otimes_{\sigma \in G} \sigma B')^G \otimes \mathbb{K}\). Hence, since the field extension \(l/k\) is of degree \(n\), the choice of an isomorphism \(l \cong k^\otimes n\) of \(l\)-vector spaces gives rise to an isomorphism \(\otimes_{\sigma \in G} \sigma B' \cong \bigoplus_{\nu=1}^n (\otimes_{\sigma \in G} \sigma B')^G\) of right \((\otimes_{\sigma \in G} \sigma B')^G\)-modules. As a consequence, we obtain a well-defined restriction functor

\[
\mathcal{D}_c(\otimes_{\sigma \in G} \sigma B') \longrightarrow \mathcal{D}_c((\otimes_{\sigma \in G} \sigma B')^G)
\]

and consequently the following composition

\[
\mathcal{D}_c(B') \longrightarrow \prod_{\sigma \in G} \mathcal{D}_c(\sigma B') \stackrel{(7.5)}{\longrightarrow} \mathcal{D}_c(\otimes_{\sigma \in G} \sigma B') \stackrel{(7.6)}{\longrightarrow} \mathcal{D}_c((\otimes_{\sigma \in G} \sigma B')^G) \quad \quad M' \mapsto \otimes_{\sigma \in G} \sigma M'.
\]

For every \(M' \in \mathcal{D}_c(B')\) one has an inclusion \((\otimes_{\sigma \in G} \sigma M')^G \subseteq \otimes_{\sigma \in G} \sigma M'\). Since the category \(\mathcal{D}_c((\otimes_{\sigma \in G} \sigma B')^G)\) is \(k\)-linear and by assumption the characteristic of \(k\) does not divide \(n\), this inclusion admits the following retraction

\[
\frac{1}{n} \sum_{\rho \in G} (\rho, -) : \otimes_{\sigma \in G} \sigma M' \rightarrow (\otimes_{\sigma \in G} \sigma M')^G,
\]

where \((\rho, -)\) is the skew-linear left \(G\)-action (5.5). This implies that \((\otimes_{\sigma \in G} \sigma M')^G\) is a direct factor of \(\otimes_{\sigma \in G} \sigma M'\) and hence also a compact object. \(\square\)

Thanks to Lemma 7.4, the above functor (7.3) restricts to

\[
\mathcal{D}_c(B') \longrightarrow \mathcal{D}_c(\mathcal{R}_{l/k}^{nc}(B')) \quad \quad M' \mapsto (\otimes_{\sigma \in G} \sigma M')^G.
\]

Now, let \(B\) be a smooth dg \(l\)-algebra. By definition, the bimodule (3.1) (with \(B = B\)) belongs to \(\mathcal{D}_c(B^{op} \otimes B)\). The above functor (7.7) (with \(B' := B^{op} \otimes B\)), combined with the fact that \(\mathcal{R}_{l/k}^{nc}(\cdot)\) is symmetric monoidal, allow us then to conclude that the bimodule (3.1) (with \(B = \mathcal{R}_{l/k}^{nc}(B)\)) belongs to \(\mathcal{D}_c(\mathcal{R}_{l/k}^{nc}(B)^{op} \otimes \mathcal{R}_{l/k}^{nc}(B))\). This implies that the dg \(k\)-algebra \(\mathcal{R}_{l/k}^{nc}(B)\) is smooth.
Proof of item (iii). We start by recalling from Weil [47] the construction of $R_{l/k}(Y)$. Every element $\sigma \in G$ gives rise to a new $l$-scheme $\sigma Y$ (the $\sigma$-conjugate of $Y$) which is obtained from $Y$ by base-change along the isomorphism $\text{spec}(\sigma^{-1}) : \text{spec}(l) \xrightarrow{\sim} \text{spec}(l)$. The product $\prod_{\sigma \in G} \sigma Y$, equipped with the following skew-linear left $G$-action

\begin{equation}
G \times \prod_{\sigma \in G} \sigma Y \to \prod_{\sigma \in G} \sigma Y \quad (\rho, \{\sigma y\}_{\sigma \in G}) \mapsto \{\rho^{-1}\sigma y\}_{\sigma \in G},
\end{equation}

becomes then a $l/k$-Galois scheme. The Weil restriction $R_{l/k}(Y)$ of $Y$ is defined as the geometric quotient $(\prod_{\sigma \in G} \sigma Y)/G$. The proof is now divided into 4 steps:

**Step 1:** Recall from Definition 6.3 that $A_Y := \text{End}_{\text{perf}}(Y)(\mathcal{G})$, where $\mathcal{G}$ is a generator of $\text{perf}(X)$. Note that we have the following equivalence of dg categories

$$\text{perf}_{dg}(Y) \xrightarrow{\sim} \text{perf}_{dg}(\sigma Y) \quad \mathcal{F} \mapsto \sigma \mathcal{F}.$$ 

As a consequence, $\sigma \mathcal{G}$ is a generator of $\text{perf}(\sigma Y)$ and $A_{\sigma Y}$ is isomorphic to $\sigma A_Y$.

**Step 2:** As proved in [43, Proposition 6.2], one has the derived Morita equivalence

$$\bigotimes_{\sigma \in G} \text{perf}_{dg}(\sigma Y) \to \text{perf}_{dg}(\prod_{\sigma \in G} \sigma Y) \quad \{\mathcal{F}\}_{\sigma \in G} \mapsto \mathcal{E}_{\sigma \in G} \sigma \mathcal{F}.$$ 

As a consequence, $\mathcal{E}_{\sigma \in G} \sigma \mathcal{G}$ is a generator of $\text{perf}(\prod_{\sigma \in G} \sigma Y)$ and $A_{\prod_{\sigma \in G} \sigma Y}$ is quasi-isomorphic to $\mathcal{E}_{\sigma \in G} \sigma \mathcal{A}_Y$.

**Step 3:** The generator $\mathcal{E}_{\sigma \in G} \sigma \mathcal{G}$ is naturally a $l/k$-Galois module over the above $l/k$-Galois scheme (7.8). Let us denote by

$$p : \prod_{\sigma \in G} \sigma Y \to \prod_{\sigma \in G} \sigma Y/G =: R_{l/k}(Y)$$

the quotient map. As explained in [17, page 12], one has canonical isomorphisms

$$R_{l/k}(Y) \times_{\text{spec}(k)} \text{spec}(l) \simeq \prod_{\sigma \in G} \sigma Y$$

$$p_*(\mathcal{E}_{\sigma \in G} \sigma \mathcal{G}) \times_{\text{spec}(k)} \text{spec}(l) \simeq \mathcal{E}_{\sigma \in G} \sigma \mathcal{G}.$$ 

It follows then from Lemma 7.12 below (with $X = R_{l/k}(Y)$) that $p_*(\mathcal{E}_{\sigma \in G} \sigma \mathcal{G})$ is a generator of $\text{perf}(R_{l/k}(Y))$. Using Definition 6.3 and Proposition 6.7, we obtain

\begin{equation}
A_{R_{l/k}(Y)} = \text{End}_{\text{perf}_{dg}}(\prod_{\sigma \in G} \sigma Y)(\mathcal{E}_{\sigma \in G} \sigma \mathcal{G}).
\end{equation}

**Step 4:** Since the Galois group $G$ acts on $\mathcal{E}_{\sigma \in G} \sigma \mathcal{G}$ (by permutation of the $\mathcal{E}$-factors), it acts also by conjugation on the endomorphisms $\text{dg } l$-algebra

\begin{equation}
\text{End}_{\text{perf}_{dg}}(\prod_{\sigma \in G} \sigma Y)(\mathcal{E}_{\sigma \in G} \sigma \mathcal{G}).
\end{equation}

As explained in §6, the right-hand-side of (7.9) identifies with the $G$-equivariant endomorphisms in (7.10) or equivalently with the $G$-invariants of this $G$-action on (7.10). Since the composed quasi-isomorphism

\begin{equation}
\otimes_{\sigma \in G} \sigma A_Y \simeq \otimes_{\sigma \in G} A_{\sigma Y} \simeq A_{\prod_{\sigma \in G} \sigma Y} = \text{End}_{\text{perf}_{dg}}(\prod_{\sigma \in G} \sigma Y)(\mathcal{E}_{\sigma \in G} \sigma \mathcal{G})
\end{equation}

is $G$-equivariant and the functor $(-)^G$ preserves quasi-isomorphisms we conclude finally that $R_{l/k}^\text{nc}(A_Y) := (\otimes_{\sigma \in G} \sigma A_Y)^G$ is quasi-isomorphic to $A_{R_{l/k}(Y)}$. 

Lemma 7.12. Let $X$ be a (quasi-projective) $k$-scheme, $G$ a perfect complex of $\mathcal{O}_X$-modules, and $X_1 := X \times_{\spec(k)} \spec(l)$. Under these assumptions and notations, whenever $G_l$ is a generator of $\perf(X_l)$, $G$ is a generator of $\perf(X)$.

Proof. Let the canonical map $\iota : X_l \to X$ is finite we have a well-defined adjunction

$$
\perf(X_l) \xrightarrow{\sim} \iota^* \perf(X) \xrightarrow{\sim} \perf(X_l).
$$

Let us now show that $\iota_* \iota^*(F) \simeq F\oplus^n$ for every $F \in \perf(X)$. The classical base-change formula applied to

$$\begin{array}{ccc}
X_l & \xrightarrow{\iota} & X \\
\pi' \downarrow & & \pi \downarrow \\
\spec(l) & \xrightarrow{\iota'} & \spec(k),
\end{array}$$

gives rise to $\iota_*(\pi')^*(\mathcal{O}_{\spec(l)}) \simeq \pi^*(\iota')_* (\mathcal{O}_{\spec(l)})$. By choosing an isomorphism $(\iota')_* (\mathcal{O}_{\spec(l)}) \simeq (\mathcal{O}_{\spec(k)})\oplus^n$ we hence obtain $\iota_*(\mathcal{O}_{X_l}) \simeq (\mathcal{O}_X)^\oplus n$. Finally, making use of the projection formula, we conclude that

$$
\iota_* \iota^*(F) \simeq \mathcal{O}_{\spec(l)}(\Sigma^n \mathcal{G}, \iota^*(F)) \simeq \mathcal{F} \otimes \iota_*(\mathcal{O}_{X_l}) \simeq \mathcal{F} \otimes (\mathcal{O}_X)^\oplus \simeq F_{\oplus n}.
$$

We now have all the ingredients needed for the conclusion of the proof. One needs to show that if by hypothesis $\Hom_{\perf(X)}(\Sigma^n (G), F) = 0$ for all $m \in \mathbb{Z}$, then $\mathcal{F} = 0$. These assumptions give rise to the following equalities

$$0 = \Hom_{\perf(X)}(\Sigma^n (G), \iota_* \iota^*(F)) \overset{(7.13)}{=} \Hom_{\perf(X_l)}(\Sigma^n (\iota^*(G)), \iota^*(F)) = 0.
$$

Using the fact that $\iota^*(G)$ is a generator of $\perf(X_l)$, we conclude that $\iota^*(F) = 0$ and hence that $\iota_* \iota^*(F) = 0$. Finally, using the isomorphism $\iota_* \iota^*(F) \simeq F_{\oplus n}$, we conclude that $\mathcal{F} = 0$. This achieves the proof. \hfill \Box

8. Grothendieck group of dg algebras

Let $K$ be a field and $B$ a dg $K$-algebra. By definition, $K_0(B)$ is the Grothendieck group of the triangulated category $D(B)$. In this section we give a four-step description of $K_0(B)$, which will be used in the proof of Theorem 2.4.

(i) Let $\mathcal{C}_c(B)$ be the full subcategory of $\mathcal{C}(B)$ consisting of those right $B$-modules which become compact in $D(B)$. Note that $\mathcal{C}_c(B)$ is stable under direct sums. Let us then write $M_0(B)$ for the associated monoid.

(ii) Let $K_0^m(B)$ be the group completion of $M_0(B)$; see [46, II §1]. By construction, we have a well-defined homomorphism $\iota : M_0(B) \to K_0^m(B)$.

(iii) Recall from [22, Lemma 3.3] that $\mathcal{C}(B)$ carries an exact structure in the sense of Quillen [38]. The conflations are the short exact sequences of $B$-modules

$$
0 \to M \xrightarrow{i} M' \xrightarrow{s} \frac{M'}{p} \to M'' \to 0
$$

for which there exists a morphism $s$ of graded $B$-modules such that $p \circ s = \id_N$. Note that $\mathcal{C}_c(B)$ is stable with respect to these short exact sequences. Let us
then write $K^\otimes_0(B)$ for the quotient $K^\otimes_0(B)/\langle M' - M - M'' \rangle$, where $M, M', M''$ are as in (8.1). By construction, we have a homomorphism $K^\otimes_0(B) \to K^\otimes_0(B)$.

(iv) Let $Q$ be the set of pairs $(M, M'') \in C_c(B) \times C_c(B)$ for which there exists a zig-zag of quasi-isomorphisms $M \sim \cdot \sim \cdots \sim \cdot \sim M''$ relating them.

**Lemma 8.2.** The Grothendieck group $K_0(B)$ is isomorphic to the quotient

\[(8.3) \quad K^\otimes_0(B)/\langle M - M'' \mid (M, M'') \in Q \rangle.
\]

Proof. Note first that since $K_0(B)$ is a group, the above homomorphism $\iota$ factors through $K^\otimes_0(B)$. Since by construction every conflation becomes a distinguished triangle in $D_c(B)$, $\iota$ factors moreover through $K^\otimes_0(B)$. Quasi-isomorphic right-B-modules give rise to the same element of $K_0(B)$. Hence, $\iota$ descends furthermore to a group homomorphism

\[(8.4) \quad \iota : K^\otimes_0(B)/\langle M - M'' \mid (M, M'') \in Q \rangle \longrightarrow K_0(B).
\]

Let us now show that (8.4) is an isomorphism. Note that two right $B$-modules $M, M'' \in C_c(B)$ become isomorphic in $D_c(B)$ if and only if there exists a zig-zag of quasi-isomorphisms $M \sim \cdot \sim \cdots \sim \cdot \sim M''$ relating them. Moreover, every triangle in $D_c(B)$ can be represented (up to quasi-isomorphism) by a conflation in $C_c(B)$. These two facts imply that (8.4) is injective. Clearly, it is also surjective. □

9. **Polynomial maps and binomial rings**

Let $f : M \to G$ be a map from an abelian monoid to an abelian group such that $f(0) = 0$. Following Eilenberg-MacLane [13], we declare $\Delta^0 f := f$ and define the maps $\Delta^{k+1}f : M^{k+1} \to G, k \geq 0$, by the following recursive formula:

\[
\Delta^{k+1}f(m_0, \ldots, m_{k-1}, m_k, m_{k+1}) := \Delta^k f(m_0, \ldots, m_{k-1}, m_k) - \Delta^k f(m_0, \ldots, m_{k-1}, m_{k+1}) - \Delta f(m_0, \ldots, m_{k+1}).
\]

**Definition 9.1.** A map $f : M \to G$ is called polynomial if there exists an integer $N \geq 0$ such that $\Delta^N f = 0$. The smallest such $N$ is called the degree of $f$.

**Proposition 9.2.** (see Joukhovitski [17, Propositions 1.2, 1.6 and 1.7])

(i) Every polynomial map $f : M \to G$ extends uniquely to a polynomial map $\overline{f} : G(M) \to G$ defined on the group completion of $M$;

(ii) Let $f$ be a polynomial maps as in (i) and $\Omega \subseteq M \times M$. Assume that $f(m + m') = f(m + m')$ for every $m \in M$ and $(m', m'') \in \Omega$. Under these assumptions, $\overline{f}$ factors through the quotient group $G(M)/\langle m - m' \mid (m', m'') \in \Omega \rangle$;

(iii) Given polynomial maps $f : G \to G'$ and $f' : G' \to G''$ between abelian groups, their composition $f' \circ f : G \to G''$ is also polynomial;

(iv) Every polynomial map $M_1 \times \cdots \times M_n \to G$ is polynomial.

9.1. **Binomial rings.** Recall from Xantcha [45, §1] that a binomial ring is an unital commutative ring $R$ equipped with unitary operations $r \mapsto \binom{a}{n}, n \in \mathbb{N},$ subject to the following five axioms:

(i) \[\binom{a+b}{n} = \sum_{p+q=n} \binom{a}{p} \binom{b}{q};\]

(ii) \[\binom{a}{n} = \sum_{m=0}^n \binom{a}{m} \sum_{q_1 + \cdots + q_m = n, q_i \geq 1} \binom{b}{q_1} \cdots \binom{b}{q_m};\]

(iii) \[\binom{a}{m+n} = \sum_{k=0}^n \binom{a}{m+k} \binom{n}{k} \binom{k}{n};\]

(iv) \[\binom{1}{n} = 0 \text{ when } n \geq 2;\]

(v) \[\binom{a}{n} = 0 \text{ when } a < n.\]
the upperscripts (\(−\)) standard example of a polynomial functor between additive categories; see [45, Definition 5]. As explained in loc. cit., examples include \(\mathbb{Z}, \mathbb{Z}[1/r], \mathbb{Q}\), the \(p\)-adic numbers \(\mathbb{Z}_p\), and also every \(\mathbb{Q}\)-algebra.

Now, let \(f : G → G'\) be a polynomial map between abelian groups. As proved in [45, Theorem 10], \(f\) extends to a polynomial map \(f_R : G ⊗ \mathbb{Z} R → G' ⊗ \mathbb{Z} R\).

**Lemma 9.3.** (i) Given polynomial maps \(f : G → G'\) and \(f' : G' → G''\) between abelian groups, we have \((f' ◦ f)_R = f'_R ◦ f_R\);

(ii) Given polynomial maps \(f : G → G'\) and \(h : H → H''\) between abelian groups, we have \((f × h)_R = f_R × h_R\);

(iii) Given a bilinear map \((g, g') : G × G' → H\) between abelian groups, we have \((g, g')_R = (g_R, g'_R)\).

**Proof.** Recall from [45, Theorem 10] that a map \(f : G → G'\) between abelian groups is polynomial (= numerical in the sense of [45, Definition 5]) if and only if it extends uniquely to a natural transformation \(f ⊗ \mathbb{Z} - : G ⊗ \mathbb{Z} → G' ⊗ \mathbb{Z} -\) between functors defined on the category of binomial rings. As a consequence, by composing \(f \otimes \mathbb{Z} -\) with \(f' \otimes \mathbb{Z} -\) we conclude that \(f' \circ f\) is polynomial and that \((f' \circ f)_R = f'_R \circ f_R\). This proves item (i). By taking the product of \(f \otimes \mathbb{Z} -\) with \(h \otimes \mathbb{Z} -\) we conclude that \(f \times h\) is polynomial and that \((f \times h)_R = f_R \times h_R\). This proves item (ii). In what concerns item (iii), the induced natural transformation \((g \times g') \otimes \mathbb{Z} - : (G × G') \otimes \mathbb{Z} - → H \otimes \mathbb{Z} -\) allows us to conclude that \((g, g')_R = (g_R, g'_R)\). \(□\)

10. **Proof of Theorem 2.4**

Let \(B'\) be a \(dg\ \text{I-algebra.}\) Recall from (7.7) the construction of the functor

\[(10.1) \quad \mathcal{D}_c(B') → \mathcal{D}_c(\mathcal{R}_{l/k}^n(B')) M' → (\otimes_{σ ∈ G}^σ M')^G.\]

**Proposition 10.2.** The above (non-additive) functor (10.1) gives rise to a polynomial map \(K_0(B') → K_0(\mathcal{R}_{l/k}^n(B'))\).

**Proof.** Recall from Proposition 5.8 the \(⊗\)-equivalence \((-)^G : \mathcal{C}(l)^{Gal} \sim \mathcal{C}(k)\). Since \(\mathcal{R}_{l/k}^n(B') := \left(\otimes_{σ ∈ G}^σ B'\right)^G\), it gives rise to an induced equivalence of categories \(\mathcal{C}_c(\otimes_{σ ∈ G}^σ B')^Gal \sim \mathcal{C}_c(\mathcal{R}_{l/k}^n(B'))\) and induced isomorphisms:

\[
M_0(\otimes_{σ ∈ G}^σ B')^{Gal} ≃ M_0(\mathcal{R}_{l/k}^n(B')) \quad K_0^G(\otimes_{σ ∈ G}^σ B')^{Gal} ≃ K_0^G(\mathcal{R}_{l/k}^n(B'))
\]

\[
K_0^c(\otimes_{σ ∈ G}^σ B')^{Gal} ≃ K_0^c(\mathcal{R}_{l/k}^n(B')) \quad K_0(\otimes_{σ ∈ G}^σ B')^{Gal} ≃ K_0(\mathcal{R}_{l/k}^n(B')).
\]

The upperscripts \((-)^Gal\) emphasize that these constructions are obtained from \(\mathcal{C}(l)^{Gal}\) and not from \(\mathcal{C}(l)\). Under these notations, it suffices then to show that

\[(10.3) \quad \mathcal{C}_c(B') → \mathcal{C}_c(\otimes_{σ ∈ G}^σ B')^{Gal} M' → \otimes_{σ ∈ G}^σ M' \quad \text{gives rise to a polynomial map}\]

\(K_0(B') → K_0(\otimes_{σ ∈ G}^σ B')^{Gal}\). This functor is the standard example of a polynomial functor between additive categories; see [17, Definition 1.3]. Hence, one obtains from [17, Proposition 1.8] a polynomial map

\[(10.4) \quad K_0^c(B') → K_0^c(\otimes_{σ ∈ G}^σ B')^{Gal} M' → \otimes_{σ ∈ G}^σ M'.\]

We now show that the composition

\[(10.5) \quad K_0^c(B') → K_0^c(\otimes_{σ ∈ G}^σ B')^{Gal} M' → \otimes_{σ ∈ G}^σ M'.\]
descends to \( K_0^\sigma(B') \). Let \( \Omega \subseteq M_0(B') \times M_0(B') \) be the set of pairs \((M', M'')\) associated to the conflations \((8.1)\). Since \([M' + M''] = [M] + [M'']\) in \( K_0^\sigma(B') \), the group \( K_0^\sigma(B') \) identifies with the quotient \( K_0(B')/\{(M') - [M \oplus M'']\} \). Using Proposition 9.2(ii), one needs then to show that for every \( N \in C_c(B') \) the equality
\[
[\otimes_{\sigma \in G}(N \oplus M')] = [\otimes_{\sigma \in G}(N \oplus M' + M'')]
\]
holds in \( K_0(B')^{\sigma} \). Consider the following objects:
\[
\otimes_{\sigma \in G}(N \oplus M') = \left[ \oplus_{I \subseteq G, \sigma : I \rightarrow \sigma} \right] \otimes_{\sigma \in G} \left\{ \begin{array}{ll} \sigma(N \oplus M) & \text{if } \sigma \in I \\ \sigma(N \oplus M') & \text{if } \sigma \notin I \end{array} \right. \quad 0 \leq i \leq n .
\]

Note also that \( \otimes_{\sigma \in G}(N \oplus M + M'') \) is canonically isomorphic to the direct sum \( \bigoplus_{i=0}^n \text{Gr}_i(\otimes_{\sigma \in G}(N \oplus M + M'')) \). As a consequence, we obtain the equality
\[
[\otimes_{\sigma \in G}(N \oplus M + M'')] = \sum_{i=0}^n \text{Gr}_i(\otimes_{\sigma \in G}(N \oplus M + M'')).
\]

Note also that \((8.1)\) gives rise to the following conflation
\[
0 \quad \xrightarrow{\text{id} \oplus id} \quad N \oplus M \quad \xrightarrow{\text{id} \oplus s} \quad N \oplus M' \quad \xrightarrow{\text{id} \oplus p} \quad N \oplus M'' \quad \xrightarrow{\text{id} \oplus id} \quad 0 .
\]

Consider then the following objects:
\[
F_i(\otimes_{\sigma \in G}(N \oplus M')) := \sum_{I \subseteq G, \sigma : I \rightarrow \sigma} \otimes_{\sigma \in G} \left\{ \begin{array}{ll} \sigma(N \oplus M) & \text{if } \sigma \in I \\ \sigma(N \oplus M') & \text{if } \sigma \notin I \end{array} \right. \quad 0 \leq i \leq n .
\]

They form a decreasing filtration of \( \otimes_{\sigma \in G}(N \oplus M) \) and give rise to the conflations
\[
\begin{CD}
0 @>>> F_i(\otimes_{\sigma \in G}(N \oplus M')) \\
@. @VVV \\
F_{i-1}(\otimes_{\sigma \in G}(M \oplus M')) @>>> 1 \leq i \leq n , \\
@. @VVV \\
\text{Gr}_{i-1}(\otimes_{\sigma \in G}(N \oplus M'')) @>>> \end{CD}
\]

where the splitting is induced from the one of \((10.8)\). As a consequence, we obtain
\[
[\otimes_{\sigma \in G}(N \oplus M')] = \left[ F_0(\otimes_{\sigma \in G}(N \oplus M')) \right] + \sum_{i=1}^n [\text{Gr}_i(\otimes_{\sigma \in G}(N \oplus M''))] .
\]

Finally, since \( F_0(\otimes_{\sigma \in G}(N \oplus M')) = \text{Gr}_0(\otimes_{\sigma \in G}(N \oplus M'')) \), the searched equality \((10.6)\) follows from \((10.7)\) and \((10.9)\). We now show that the composition
\[
K_0^\sigma(B') \xrightarrow{(10.5)} K_0^\sigma(\otimes_{\sigma \in G}(B')^{\text{Gal}}) \xrightarrow{K_0(\otimes_{\sigma \in G}(B')^{\text{Gal}})}
\]
descends furthermore to \( K_0(B') \). Let \( Q \) be the set of pairs \((M, M'')\) described at (iv) of \S 8. Thanks to Lemma 8.2, \( K_0(B) \) identifies with the quotient group
\[K_{0}^{\otimes}(B)/\langle M - M'' | (M, M'') \rangle.\] Using Proposition 9.2(ii), one needs then to show that for every \(N \in C_{e}(B')\) the equality
\[
\otimes_{\sigma \in G}^{\sigma}(N \oplus M) = \otimes_{\sigma \in G}(N \oplus M')
\]
holds in \(K_{0}(\otimes_{\sigma \in G}^{\sigma}B')^{\text{Gal}}\). This follows automatically from the fact that \(N \oplus M\) and \(N \oplus M''\) are related by a zig-zag of quasi-isomorphisms and that the above functor (10.3) preserves quasi-isomorphisms. \(\square\)

Let \(B, B' \in \text{SpDga}(l)\). By applying Proposition 10.2 to \(B^{\text{op}} \otimes B'\), and using the fact that \(\mathcal{R}_{l/k}^{\text{nc}}(-)\) is symmetric monoidal, we obtain a polynomial map
\[
K_{0}(B^{\text{op}} \otimes B') \rightarrow K_{0}(\mathcal{R}_{l/k}^{\text{nc}}(B)^{\text{op}} \otimes \mathcal{R}_{l/k}^{\text{nc}}(B')).
\] (10.10)

We now have all the ingredients needed for the construction of the \(\otimes\)-functor (2.5). Note that by definition of Kontsevich’s category \(\mathcal{NChow}(-)_{Z}\) of noncommutative Chow motives, it suffices to treat the case of smooth and proper dg \(l\)-algebras. Recall from (4.5) that
\[
\text{Hom}_{\mathcal{NChow}(l)_{Z}}(U(B), U(B')) \simeq K_{0}(B^{\text{op}} \otimes B')
\]
\[
\text{Hom}_{\mathcal{NChow}(k)_{Z}}(U(\mathcal{R}_{l/k}^{\text{nc}}(B)), U(\mathcal{R}_{l/k}^{\text{nc}}(B')))) \simeq K_{0}(\mathcal{R}_{l/k}^{\text{nc}}(B)^{\text{op}} \otimes \mathcal{R}_{l/k}^{\text{nc}}(B')).
\]
Hence, we define the searched functor (2.5) by the above polynomial maps (10.10). Since (2.1) is a \(\otimes\)-functor, this construction is symmetric monoidal and preserves the identities. Let us now prove that the composition law is also preserved. Given \(B, B', B'' \in \text{SpDga}(l)\), one needs to show that the following diagram commutes
\[
K_{0}(B^{\text{op}} \otimes B') \times K_{0}(B^{\text{op}} \otimes B'') \rightarrow_{\otimes_{\mathcal{R}}} K_{0}(B^{\text{op}} \otimes B'')
\]
\[
K_{0}(\mathcal{R}(B)^{\text{op}} \otimes \mathcal{R}(B')) \times K_{0}(\mathcal{R}(B')^{\text{op}} \otimes \mathcal{R}(B'')) \rightarrow_{\otimes_{\mathcal{R}}} K_{0}(\mathcal{R}(B)^{\text{op}} \otimes \mathcal{R}(B''))\]
where we have written \(\mathcal{R}\) instead of \(\mathcal{R}_{l/k}^{\text{nc}}\) in order to simplify the exposition. Thanks to Proposition 9.2(iii)-(iv), both compositions are polynomial maps. In order to show that they are the same it suffices by Proposition 9.2(i) to show the their restriction to \(M_{0}(B^{\text{op}} \otimes B') \times M_{0}(B^{\text{op}} \otimes B'')\) is the same, i.e. that
\[
\mathcal{R}_{l/k}^{\text{nc}}(B) \otimes \mathcal{R}_{l/k}^{\text{nc}}(B') \otimes \mathcal{R}_{l/k}^{\text{nc}}(B'') \simeq \mathcal{R}_{l/k}^{\text{nc}}(B \otimes_{B'} B')
\]
for every \(B \in \mathcal{D}_{c}(B^{\text{op}} \otimes B')\) and \(B' \in \mathcal{D}_{c}(B^{\text{op}} \otimes B'')\). This follows from isomorphism (5.13) and so the construction of (2.5) is finished.

Let us now construct the \(\otimes\)-functor (2.6). As mentioned in §9.1, every polynomial map \(G \rightarrow G'\) between abelian groups extends to a polynomial map \(G \otimes_{Z} R \rightarrow G' \otimes_{Z} R\). By tensoring (10.10) with \(R\) we then obtain a polynomial map
\[
K_{0}(B^{\text{op}} \otimes B')_{R} \rightarrow K_{0}(\mathcal{R}_{l/k}^{\text{nc}}(B)^{\text{op}} \otimes \mathcal{R}_{l/k}^{\text{nc}}(B'))_{R}.
\] (10.11)

We define the searched functor (2.6) by the polynomial maps (10.11). As above, it suffices to show that the following diagram commutes
\[
K_{0}(B^{\text{op}} \otimes B')_{R} \times K_{0}(B^{\text{op}} \otimes B'')_{R} \rightarrow_{\otimes_{\mathcal{R}}} K_{0}(B^{\text{op}} \otimes B'')_{R}
\]
\[
K_{0}(\mathcal{R}(B)^{\text{op}} \otimes \mathcal{R}(B'))_{R} \times K_{0}(\mathcal{R}(B')^{\text{op}} \otimes \mathcal{R}(B''))_{R} \rightarrow_{\otimes_{\mathcal{R}}} K_{0}(\mathcal{R}(B)^{\text{op}} \otimes \mathcal{R}(B''))_{R}.
\]
Thanks to Lemma 9.3 this diagram can be obtained by tensoring the previous one with $R$. As a consequence, it is commutative.

11. Proof of Theorem 2.8

**Proof of item (i).** Thanks to Theorem 2.4 and the fact that the functor $U(-)_R$ sends products to direct sums, it suffices to establish the isomorphism

$$R^{nc}_{l/k}(l \times \cdots \times l) \cong \prod_{o \in \mathcal{O}(G,m)} \text{pstb}(\alpha_o).$$

**Lemma 11.2.** One has a canonical isomorphism

$$R^{nc}_{l/k}(l \times \cdots \times l) \cong \prod_{o \in \mathcal{O}(G,m)} (\prod_{\alpha \in o} l)^G,$$

where the left $G$-action on $\prod_{\alpha \in o} l$ is given by $(\rho, \{\lambda_\alpha\}_{\alpha \in o}) \mapsto \{\rho(\lambda_{\rho^{-1}\alpha})\}_{\alpha \in o}$.

*Proof.* Given $B_1, \ldots, B_m \in \text{SpDga}(l)$, one has a canonical isomorphism

$$R^{nc}_{l/k}(B_1 \times \cdots \times B_m) \cong \prod_{o \in \mathcal{O}(G,m)} \left(\prod_{\sigma \in \mathcal{G}} B_{\sigma(\sigma)}\right)^G.$$

Moreover, in the particular case where $B_1 = \cdots = B_m = l$, the isomorphism

$$l \xrightarrow{\sim} \otimes_{\sigma \in \mathcal{G}} l ^{\sigma^{-1}(\rho)} \otimes_{\sigma \in \mathcal{G}} t_{\sigma(\sigma)} \ (l_{\sigma(\sigma)} = l)$$

gives rise to a $G$-equivariant isomorphism $\prod_{\alpha \in o} l_{\alpha} \xrightarrow{\sim} \prod_{\alpha \in o} \otimes_{\sigma \in \mathcal{G}} t_{\sigma(\sigma)}$. By combining it with (11.4) we then obtain the above isomorphism (11.3). \qed

**Lemma 11.5.** For every representative $\alpha_o$ of the orbit $o \in \mathcal{O}(G,m)$ the projection at the $\alpha^\text{th}$-component gives rise to an isomorphism $p_{\alpha_o} : (\prod_{\alpha \in o} l)^G \xrightarrow{\sim} \text{pstb}(\alpha_o)$.

*Proof.* We start by showing that $p_{\alpha_o}$ is well-defined. Recall that

$$(\prod_{\alpha \in o} l)^G := \{\{\lambda_\alpha\}_{\alpha \in o} \in \prod_{\alpha \in o} l | \lambda_\alpha = \rho(\lambda_{\rho^{-1}\alpha}) \ \forall \rho \in G\}.$$

Hence, when $\alpha = \alpha_o$ and $\rho \in \text{stab}(\alpha_o)$, we have $\lambda_{\alpha_o} = \rho(\lambda_{\alpha_o})$. This implies that $p_{\alpha_o}(\{\lambda_\alpha\}_{\alpha \in o}) = \lambda_{\alpha_o} \in \text{pstb}(\alpha_o)$. Now, consider the following map

$$t_{\alpha_o} : \text{pstb}(\alpha_o) \longrightarrow (\prod_{\alpha \in o} l)^G \quad \lambda \mapsto \{\tau_\alpha(\lambda)\}_{\alpha \in o},$$

where $\tau_\alpha$ is any element of $G$ such that $\tau_\alpha(\alpha_o) = \alpha$. Note that if $\tau'_\alpha$ is any other element of $G$ such that $\tau'_\alpha(\alpha_o) = \alpha$, the multiplication $\tau^{-1}_\alpha \tau'_\alpha \in G$ belongs to $\text{stab}(\alpha_o)$. Since $\lambda \in \text{pstb}(\alpha_o)$, this gives rise to the equalities $(\tau^{-1}_\alpha \tau'_\alpha)(\lambda) = \lambda \Leftrightarrow \tau'_\alpha(\lambda) = \tau_\alpha(\lambda)$ and hence shows that $t_{\alpha_o}$ is well-defined.

Let us now prove that $t_{\alpha_o}$ and $p_{\alpha_o}$ are the inverse of each other. Clearly, we have $p_{\alpha_o} \circ t_{\alpha_o} = \text{id}$. In order to prove that $t_{\alpha_o} \circ p_{\alpha_o} = \text{id}$, one needs to show that $\lambda_{\alpha} = \tau_{\alpha}(\lambda_{\alpha})$ for every $\alpha \in o$. Since $\{\lambda_\alpha\}_{\alpha \in o}$ is $G$-invariant we have $\lambda_\alpha = \rho(\lambda_{\rho^{-1}\alpha})$ for every $\rho \in G$. Hence, by taking $\rho := \tau_\alpha$, we obtain $\lambda_\alpha = \tau_{\alpha}(\lambda_{\alpha^{-1}(\alpha)})$. From the equality $\tau_\alpha(\alpha_o) = \alpha$, we conclude finally that $\lambda_{\alpha} = \tau_{\alpha}(\lambda_{\alpha_o})$ for every $\alpha \in o$. This achieves the proof. \qed

The searched isomorphism (11.1) follows now from Lemmas 11.2 and 11.5.
**Proof of item (ii).** As explained in the proof of item (i), it suffices to consider the above isomorphism (11.1). The only subgroups of $\mathbb{Z}/p\mathbb{Z}$ are 0 and itself. As a consequence, we have the following equalities

$$l^{\text{stab}(\alpha_o)} = \begin{cases} l & \text{if } \text{stab}(\alpha_o) = 0 \\ k & \text{if } \text{stab}(\alpha_o) = \mathbb{Z}/p\mathbb{Z} \end{cases}.$$ 

Now, note that the set $\mathcal{O}(G, m)$ contains $m$ “constant” orbits. The stabilizer of each one of these constant maps is $\mathbb{Z}/p\mathbb{Z}$. As a consequence, we obtain $m$ copies of $k$. In the case of a non-constant map $\alpha: G \to \{1, \ldots, m\}$, there exist elements $\sigma \neq \sigma'$ such that $\alpha(\sigma) \neq \alpha(\sigma')$. This implies that the element $\sigma'\sigma^{-1} \in G$ does not belong to $\text{stab}(\alpha)$ and consequently that $\text{stab}(\alpha) = 0$. We conclude then that each one of the non-constant orbits gives rise to a copy of $l$. This achieves the proof.

**Proof of item (iii).** Since $l/k$ is a finite Galois field extension, one has a bijective correspondence $G \ni H \mapsto l^H$ between subgroups of $G$ and intermediate field extensions between $k$ and $l$. Hence, there exists a unique subgroup $H_o$ of $G$ such that $l^{H_o} = k_o$. Consider the following map

$$(11.6) \quad \alpha_o: G \to \{1, \ldots, m\} \quad \alpha_o(\sigma) := \begin{cases} 1 & \text{if } \sigma \in H_o \\ m & \text{if } \sigma \notin H_o \end{cases}.$$ 

Note that $\rho(\alpha_o) = \alpha_o$ when $\rho \in H_o$ and that $\rho(\alpha_o) \neq \alpha_o$ when $\rho \notin H_o$. This implies that $\text{stab}(\alpha_o) = H_o$ and so $k_o = l^{\text{stab}(\alpha_o)}$.

### 12. Orbit categories

Let $\mathcal{C}$ be an additive symmetric monoidal category and $T \in \mathcal{C}$ a $\otimes$-invertible object. The *orbit category* $\mathcal{C}/\cdot \otimes T$ has the same objects as $\mathcal{C}$ and morphisms

$$(12.1) \quad \text{Hom}_{\mathcal{C}/\cdot \otimes T}(a, b) := \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes T^\otimes r).$$

Given objects $a, b, c \in \mathcal{C}$ and morphisms

$$\underline{\mu} = \{\mu_r\}_{r \in \mathbb{Z}} \in \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(a, b \otimes T^\otimes r) \quad \underline{\nu} = \{\nu_r\}_{r \in \mathbb{Z}} \in \bigoplus_{r \in \mathbb{Z}} \text{Hom}_{\mathcal{C}}(b, c \otimes T^\otimes r),$$

the $i^{th}$-component of the composition $\underline{\nu} \circ \underline{\mu}$ is the finite sum $\sum_r (\nu_{i-r} \otimes T^\otimes r) \mu_r$.

The category $\mathcal{C}/\cdot \otimes T$ is additive and comes equipped with a canonical functor $\pi: \mathcal{C} \to \mathcal{C}/\cdot \otimes T$. Moreover, $\pi$ is endowed with a natural 2-isomorphism $\pi \circ (- \otimes T) \cong \pi$ and is 2-universal among all such functors; consult [42, §7] for further details.

### 13. Proof of Theorem 2.13

In contrast with Karpenko [20], we will work with *contravariant* functors $M, M^*$ from smooth projective schemes to the categories of Chow motives. As explained in [20, Remark 5.1], this is just a matter of taste. Recall also from *loc. cit.* that $\text{Chow}^*(-)_\mathbb{Z}$ is defined as the idempotent completion of the category $\mathcal{C}V(-)_\mathbb{Z}$ of (arbitrary) correspondences.
Lemma 13.1. For every commutative ring $R$ there is a canonical equivalence of categories $\Phi_R$ making the following diagram commute

\[
\begin{array}{ccc}
\text{SmProj}(k)^{op} & \longrightarrow & \text{SmProj}(k)^{op} \\
\downarrow M^* & & \downarrow M(-)_R \\
\text{Chow}^*(k)_Z & \longrightarrow & \text{Chow}(k)_R \\
\downarrow (-)_R & & \downarrow \pi \\
\text{Chow}^*(k)_R & \longrightarrow & \Phi_R \text{(Chow}(k)/\otimes R)\mathfrak{Z},
\end{array}
\]

where $R(1)$ stands for the Tate motive.

Proof. Thanks to description (12.1) of the morphisms in the orbit category, the fully-faithful image of the composed functor $\pi \circ M(-)_R$ identifies with the category $\mathcal{CV}(k)_R$ of (arbitrary) correspondences with $R$-coefficients. Since the category $(\text{Chow}(k)/\otimes R)\mathfrak{Z}$ is idempotent complete we hence obtain a well-defined fully-faithful functor $\Phi_R$ making the above diagram (13.2) commute. It remains then only to show that $\Phi_R$ is essentially surjective. Given a Chow motive $(X, e, r) \in \text{Chow}(k)_R$ (see [2, §4] [34]) one has a canonical isomorphism $\pi(X, e, r) \simeq \pi(X, e, 0)$ in the orbit category; see §12. As a consequence, the idempotent endomorphism $e$ belongs to $\mathcal{CV}(k)_R$ and so we conclude that $\pi(X, e, r)$ is in the image of $\Phi_R$. This implies that $\Phi_R$ is also essentially surjective. □

Every commutative ring $R$ containing $1/(2nd)!$ gives rise to an additive functor $\text{NChow}(k)_Z[1/(2nd)!] \rightarrow \text{NChow}(k)_R$. Hence, it suffices to establish isomorphism (2.14) in the particular case $R = Z[1/(2nd)!]$. Recall from (1.3) that we have the following diagram:

\[
\begin{array}{ccc}
\text{SmProj}(l)^{op} & \longrightarrow & \text{SmProj}(k)^{op} \\
\downarrow M^* & & \downarrow M^* \\
\text{Chow}^*(l)_Z & \longrightarrow & \text{Chow}^*(k)_Z \\
\downarrow (-)_Z[1/(2nd)!] & & \downarrow (-)_Z[1/(2nd)!] \\
\text{Chow}^*(l)_Z[1/(2nd)!] & \longrightarrow & \text{Chow}^*(k)_Z[1/(2nd)!].
\end{array}
\]

On the other hand, since $\text{char}(k) \nmid n$ and $Z[1/(2nd)!]$ is a binomial ring, Theorem 2.4 furnish us the following commutative diagram:

\[
\begin{array}{ccc}
\text{SpDga}(l) & \longrightarrow & \text{SpDga}(k) \\
\downarrow U(-)_Z & & \downarrow U(-)_Z \\
\text{NChow}(l)_Z & \longrightarrow & \text{NChow}(k)_Z \\
\downarrow (-)_Z[1/(2nd)!] & & \downarrow (-)_Z[1/(2nd)!] \\
\text{NChow}(l)_Z[1/(2nd)!] & \longrightarrow & \text{NChow}(k)_Z[1/(2nd)!].
\end{array}
\]
Consider now the following subcategories:

(i) Let \((R_{l/k}(Y)) \subset \text{SmProj}(k)^{\text{op}}\) be the full subcategory containing \(R_{l/k}(Y)\) and the (smooth projective) 0-dimensional \(k\)-schemes \(\{\text{spec}(k_o) | l/k_o/k\}\).

(ii) Let \((M^*(R_{l/k}(Y)) \subset \text{Chow}^*(k)_{\mathbb{Z}[1/(2nd)!]}\) be the smallest full additive subcategory containing \(M^*(R_{l/k}(Y))_{\mathbb{Z}[1/(2nd)!]}\) and \(\{M^*(\text{spec}(k_o))_{\mathbb{Z}[1/(2nd)!]} | l/k_o/k\}\).

(iii) Let \((A_{R_{l/k}(Y)}) \subset \text{SpDga}(k)\) be the full subcategory containing \(A_{R_{l/k}(Y)}\) and the (smooth and proper) dg \(k\)-algebras \(\{A_{\text{spec}(k_o)} | l/k_o/k\}\).

(iv) Let \((U(A_{R_{l/k}(Y)})) \subset \text{NChow}(k)_{\mathbb{Z}[1/(2nd)!]}\) be the smallest full additive subcategory containing \(U(A_{R_{l/k}(Y)}))_{\mathbb{Z}[1/(2nd)!]}\) and \(\{U(A_{\text{spec}(k_o)})_{\mathbb{Z}[1/(2nd)!]} | l/k_o/k\}\).

Since the \(l\)-scheme \(Y\) is of dimension \(d\) and the field extension \(l/k\) is of degree \(n\), the \(k\)-scheme \(R_{l/k}(Y)\) is of dimension \(nd\). Hence, thanks to Lemma 13.1 (with \(R = \mathbb{Z}[1/(2nd)!]\)) and [4, Proposition 6.6], there is an equivalence of categories \(\Psi^\oplus\) making the following diagram commute:

\[
\begin{array}{ccc}
\langle R_{l/k}(Y) \rangle & \xrightarrow{\{X \rightarrow A_X\}} & \langle A_{R_{l/k}(Y)} \rangle \\
\text{SmProj}(k)^{\text{op}} & \xrightarrow{M^*} & \text{SpDga}(k) \\
\text{Chow}^*(k)_{\mathbb{Z}} & \xrightarrow{U(-)_Z} & \text{NChow}(k)_{\mathbb{Z}} \\
& \xrightarrow{\cong} & \\
\langle M^*(R_{l/k}(Y)) \rangle & \xrightarrow{\Psi^\oplus} & \langle U(A_{R_{l/k}(Y)}) \rangle \\
\end{array}
\]

As a consequence, we obtain the following isomorphism

\[
\Psi^\oplus(M^*(R_{l/k}(Y))_{\mathbb{Z}[1/(2nd)!]} \simeq U(A_{R_{l/k}(Y)})_{\mathbb{Z}[1/(2nd)!]}.
\]

Thanks to diagrams (13.3)-(13.4), and to the fact that \(A_{R_{l/k}(Y)}\) is derived Morita equivalent to \(R_{l/k}^{nc}(Y)\) (see Theorem 2.2(iii)), (13.5) is equivalent to

\[
\Psi^\oplus(R_{l/k}^{nc}(M^*(Y))_{\mathbb{Z}[1/(2nd)!]} \simeq R_{l/k}^{nc}(U(A_Y))_{\mathbb{Z}[1/(2nd)!]}.
\]

Now, recall that by assumption the motivic decomposition (2.12) holds. Using Theorem 2.8(i), one then obtains the following isomorphism

\[
R_{l/k}^{nc}(U(A_Y)_{\mathbb{Z}[1/(2nd)!]} \simeq \bigoplus_{o \in O(G,m)} U(l^{\text{stab}(\alpha_o)})_{\mathbb{Z}[1/(2nd)!]}.
\]

Finally, by combining (13.6)-(13.7), we conclude that

\[
\Psi^\oplus(R_{l/k}^{nc}(M^*(Y))_{\mathbb{Z}[1/(2nd)!]} \simeq \Psi^\oplus\left( \bigoplus_{o \in O(G,m)} M^*(\text{spec}(l^{\text{stab}(\alpha_o)}))_{\mathbb{Z}[1/(2nd)!]} \right).
\]

Since \(\Psi^\oplus\) is an equivalence we hence obtain the searched isomorphism (2.14).

14. Proof of Theorem 2.16

In order to simplify the exposition we will write \(O\) instead of \(O(G,m)\), \((-)(r)\) instead of \(- \otimes R(1)^{\otimes r}\), and will remove the subscript \(l/k\) from \(R_{l/k}, R_{l/k}^*, \text{and} R_{l/k}^{nc}\).
Proof of item (i). Diagram (13.3), in the proof of Theorem 2.13, implies that \( R^*(M^*(Y))_R \) is isomorphic to \( M^*(\mathcal{R}(Y))_R \). Hence, we obtain from (2.14) an isomorphism between \( M^*(\mathcal{R}(Y))_R \) and the direct sum \( \bigoplus_{\alpha \in \mathcal{O}} M^*(\text{spec}(l^{\text{stab}(\alpha_o)}))_R \). Thanks to Lemma 13.1 and §12, there exist then well-defined morphisms in the category of Chow motives

\[
\mu = \{ \mu_r \}_{r \in \mathbb{Z}} \in \bigoplus_{r \in \mathbb{Z}} \text{Hom}(M(\mathcal{R}(Y))_R, (\bigoplus_{\alpha \in \mathcal{O}} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R)(r))
\]

\[
\nu = \{ \nu_r \}_{r \in \mathbb{Z}} \in \bigoplus_{r \in \mathbb{Z}} \text{Hom}(\bigoplus_{\alpha \in \mathcal{O}} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R, M(\mathcal{R}(Y))_R(r))
\]

such that \( \mu \circ \nu = \text{id} \) and \( \mu \circ \nu = \text{id} \). Since the k-schemes \( \text{spec}(l^{\text{stab}(\alpha_0)}) \) are 0-dimensional and \( \mathcal{R}(Y) \) has dimension \( nd \) we have \( \mu_r = 0 \) for \( r \neq \{-nd, \ldots, 0\} \) and \( \nu_r = 0 \) for \( r \neq \{0, \ldots, nd\} \). This implies that

\[
\sum_r \nu_{-r}(r) \circ \mu_r = \text{id} \quad \text{and} \quad \sum_r \mu_{-r}(r) \circ \nu_r = \text{id}.
\]

Now, the sets of morphisms \( \{ f_{-i} \mid 0 \leq i \leq nd \} \) and \( \{ g_i(-i) \mid 0 \leq i \leq nd \} \) give rise to well-defined morphisms in the category of Chow motives

\[
M(\mathcal{R}(Y))_R \longrightarrow \bigoplus_{i=0}^{nd} (\bigoplus_{\alpha \in \mathcal{O}} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R)(-i)
\]

\[
\bigoplus_{i=0}^{nd} (\bigoplus_{\alpha \in \mathcal{O}} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R)(-i) \longrightarrow M(\mathcal{R}(Y))_R.
\]

The composition \((14.3) \circ (14.2)\) agrees with the left-hand-side of (14.1). As a consequence, \((14.2) \circ (14.3)\) is an idempotent endomorphism. Concretely, since the k-schemes \( \text{spec}(l^{\text{stab}(\alpha_0)}) \) are 0-dimensional, we obtain the diagonal matrix

\[
(14.2) \circ (14.3) = \begin{pmatrix}
\tau_0 & 0 & \ldots & \ldots & 0 \\
0 & \ddots & \ddots & \vdots \\
\vdots & \ddots & \tau_i & \ddots & \vdots \\
\vdots & \ddots & \ddots & \ddots & 0 \\
0 & \ldots & \ldots & 0 & \tau_{nd}
\end{pmatrix}
\]

\( \tau_i := f_{-i} \circ g_i(-i) \).

As a consequence, (14.2) gives rise to the isomorphism

\[
M(\mathcal{R}(Y))_R \simeq \bigoplus_{i=0}^{nd} \tau_i ((\bigoplus_{\alpha \in \mathcal{O}} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R)(-i)).
\]

Now, let \( e_i := \tau_i(i) = \mu_{-i}(i) \circ \nu_i \). Note that thanks to the right-hand-side of (14.1) we have \( \sum_i e_i = \text{id} \). Note also that since the functor \((-)(i)\) is an auto-equivalence of \( \text{Chow}(k)_R \), the above isomorphism (14.4) identifies with

\[
M(\mathcal{R}(Y))_R \simeq \bigoplus_{i=0}^{nd} e_i ((\bigoplus_{\alpha \in \mathcal{O}} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R) \otimes R(1)^{\otimes -i}).
\]

Finally, since \( L := R(1)^{\otimes -1} \), we obtain from (14.5) the searched isomorphism (2.17).
Proof of item (ii). It follows from the proof of item (i) that the idempotent endomorphisms of $\bigoplus_{o \in O} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R$ are obtained from the corresponding endomorphisms of $\bigoplus_{o \in O} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_{\mathbb{Z}[1/(2nd)]}$. Hence, it suffices to prove the case $R = \mathbb{Q}$. Let $\Gamma := \text{Gal}(\overline{k}/k)$ be the absolute Galois group scheme of $k$. As explained in [2, Example 4.1.6.1], one has the following $\otimes$-equivalence
\begin{equation}
\text{Artin}(k)_\mathbb{Q} \sim \text{Rep}_{\mathbb{Q}}(\Gamma) \quad M(X)_\mathbb{Q} \mapsto \mathbb{Q}^X(\overline{k}).
\end{equation}
(14.6)
between the category of Artin motives with $\mathbb{Q}$-coefficients and the category of finite dimensional $\mathbb{Q}$-linear representations of $\Gamma$. Under (14.6), an idempotent endomorphism $e_i$ of the Artin motive $\bigoplus_{o \in O} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_\mathbb{Q}$ corresponds then to an idempotent endomorphism $\tilde{e}_i$ of the $\Gamma$-representation
\begin{equation}
\bigoplus_{o \in O} \mathbb{Q}^{\text{spec}(l^{\text{stab}(\alpha_o)})}(\overline{k}).
\end{equation}
(14.7)
In particular, $\tilde{e}_i$ is an idempotent endomorphism of the underlying $\mathbb{Q}$-vector space of (14.7), and thus can be written as a matrix $[\tilde{e}_i]$ with 1’s at certain spots of the diagonal and 0’s elsewhere. Since $\sum_i e_i = \text{id}$, we conclude that the support of the matrices $[\tilde{e}_i]$ is disjoint and hence that the idempotents $e_i$ are mutually orthogonal.

Proof of item (iii). Recall first from [2, Example 4.1.6.1] that the above $\otimes$-equivalence (14.6) holds also with $\mathbb{Q}$ replaced by a field $R$ of characteristic zero. As explained in the proof of item (ii) of Theorem 2.8, we have the following equalities
\begin{equation}
l^{\text{stab}(\alpha_o)} = \begin{cases} 
l & \text{if } \text{stab}(\alpha_o) = 0 \\
\mathbb{Z}/p\mathbb{Z} & \text{if } \text{stab}(\alpha_o) = \mathbb{Z}/p\mathbb{Z}
\end{cases}.
\end{equation}
(14.8)
Now, note that the claim of item (iii) (iii) is equivalent to the claim that every idempotent endomorphism $e_i$ is given by a subsume of $\bigoplus_{o \in O} M(\text{spec}(l^{\text{stab}(\alpha_o)}))_R$. It follows then from the proof of item (ii) that it suffices to show that the $\Gamma$-representations
\begin{equation}
R^{\text{spec}(k)}(\overline{k}) \quad R^{\text{spec}(l)}(\overline{k})
\end{equation}
are indecomposable. The left-hand-side of (14.8) is the $\otimes$-unit of $\text{Rep}_R(\Gamma)$ (i.e. $R$ with the trivial $\Gamma$-action) and hence indecomposable. In what concerns the right-hand-side, recall from Galois theory that one has a surjective group homomorphism $\Gamma \twoheadrightarrow G := \text{Gal}(l/k)$. As a consequence, $G$ becomes a $\Gamma$-set and $R^G$ a $\Gamma$-representation. Thanks to Galois theory again, the right-hand-side of (14.8) is isomorphic to this latter $\Gamma$-representation. Hence, it remains only to show that $R^G$ is indecomposable. In order to do so, recall from item (ii) that if $\tilde{e}_i$ is an idempotent endomorphism of $R^G$, then the associated $n \times n$ matrix $[\tilde{e}_i]$ has 1’s at certain places of the diagonal and 0’s elsewhere. Since by assumption the Galois group is cyclic, a simple matrix argument allow us to conclude that the unique possibilities for $[\tilde{e}_i]$ are the identity or the zero matrix. This implies our claim and concludes the proof.

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