Chiral Condensate at Nonzero Chemical Potential in the Microscopic Limit of QCD

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The chiral condensate in QCD at zero temperature does not depend on the quark chemical potential (up to one third the nucleon mass), whereas the spectral density of the Dirac operator shows a strong dependence on the chemical potential. The cancellations which make this possible also occur on the microscopic scale, where they can be investigated by means of a random matrix model. We show that they can be understood in terms of orthogonality properties of orthogonal polynomials. In the strong non-Hermiticity limit they are related to integrability properties of the spectral density. As a by-product we find exact analytical expressions for the partially quenched chiral condensate in the microscopic domain at nonzero chemical potential.

I. INTRODUCTION

One of the simplest questions one can ask regarding the phase diagram of QCD is: What is the dependence of the chiral condensate on the quark chemical potential at zero temperature? This simple question has an equally simple answer: At low temperature, the vacuum of QCD is dominated by the pions and since they have zero baryon charge, the free energy remains chemical potential-independent until µ surpasses a third of the nucleon mass. Despite its simplicity, even this answer is exceedingly hard to verify by direct lattice QCD computations. To see why, let us consider the relationship between (the magnitude of) the chiral condensate

\[ \Sigma(m) \equiv \frac{1}{V} \frac{d}{dm} \log Z(m) = \frac{1}{V} \left\langle \text{Tr} \left( \frac{1}{\mathcal{D} + \mu \gamma_0 + m} \right) \right\rangle \] (1)

and the spectral density \( \rho \) of the Euclidean Dirac operator, \( \mathcal{D} + \mu \gamma_0 \). At nonzero chemical potential, the Dirac operator is non-Hermitian and the spectral density spreads out from the imaginary axis. Since the chiral condensate is related to this \( \mu \)-dependent spectral density by

\[ \Sigma(m) = \frac{1}{V} \int_{\mathbb{C}} d^2 z \frac{1}{z + m} \rho(z, z^*, m; \mu) \] (2)
it is far from obvious that the chiral condensate obtained this way is independent of $\mu$ (for $\mu < m_N/3$ and zero temperature).

The quark mass enters the support of the Dirac spectrum when $\mu \geq m_\pi/2$ [1, 2], and by an electrostatic analogy [3] one could conclude that the chiral condensate becomes $\mu$-dependent and always vanishes in the chiral limit ($m \to 0$) when the chemical potential is non-zero. While this conclusion is correct for quenched QCD, it fails completely in the unquenced case where, as argued above, the unquenced chiral condensate is independent of $\mu$.

In the unquenced case, the generally accepted picture for two decades was that chiral symmetry breaking requires that the Dirac spectrum should accumulate to a non-zero density on the imaginary axis (like the Banks-Casher relation [4] at $\mu = 0$) despite the fact that the anti-Hermiticity of the Dirac operator is explicitly broken by the chemical potential (see [3] for a discussion of this point). Indeed, intricate cancellations due to dynamical fermions set in when $\mu$ becomes greater than $m_\pi/2$. The eigenvalues, however, do not accumulate on the imaginary axis. Direct input [5, 6] from the microscopic domain of QCD shows that the discontinuity in the chiral condensate arises due to strongly oscillating terms in the eigenvalue density [7]. The same mechanism is also responsible for chiral symmetry breaking in one dimensional QCD at nonzero $\mu$ [8]. Because of the oscillations, the unquenced eigenvalue density is not real and positive. This fact is a direct consequence of the sign problem: since the measure of the Euclidean QCD partition function includes a complex valued fermion determinant the expectation value of a real and positive function, such as $\sum_k \delta^2(z - z_k)$, needs not be real and positive.

The microscopic domain [9] is the region where the eigenvalues of the Dirac operator are in the domain

$$|z| \ll \frac{1}{\Lambda_{QCD}\sqrt{V}},$$

(3)

while the volume is much larger than $\Lambda_{QCD}^{-4}$. The quark masses may or may not be in this region. In this region the QCD partition function can be expressed in terms of microscopic scaling variables

$$\hat{z} = zV\Sigma, \quad \hat{m} = mV\Sigma \quad \text{and} \quad \hat{\mu} = \mu F_\pi \sqrt{V},$$

(4)

which stay fixed in the thermodynamic limit. This scaling limit is known as the microscopic limit. The oscillations in the spectral density with a period proportional to the inverse volume are resolved at this scale. The cancellations of these oscillating contributions with an amplitude that grows exponentially large with the volume results in a $\mu$-independent chiral condensate. They give us a direct insight into problems faced by lattice QCD at nonzero chemical potential.
FIG. 1: Phase diagram of the Dirac spectrum in the complex $z$ plane for one dynamical flavor. The support of the quenched spectrum is between the two vertical black lines. Unquenching introduces the oscillations within the ellipses which intersect the $x$-axis at $x = \pm m$ and $x = \pm (\frac{2}{3} F_{2} z_{0}^{2} / \Sigma - m)$. The labels N, $\pi$, K refer to the normal, pion and kaon condensed phases of the generating functional of the eigenvalue density.

In [7] the $\mu$-independence of the chiral condensate was established using an asymptotic approximation to the exact expression for the microscopic spectral density. The argument was carried through by complex contour integrations. In the present paper we extend this result to the complete mass dependence of the chiral condensate in the microscopic domain.

The extent of the oscillating region in the complex eigenvalue plane can be understood from a mean field treatment of the chiral Lagrangian for the generating functionals of the eigenvalue density (see [10]). The picture that emerges is shown in Fig. II where we present results for one dynamical flavor. Inside the green ellipses the eigenvalue density is strongly oscillating. Outside these regions the oscillating terms are exponentially suppressed and the eigenvalue density approaches the quenched spectral density. The spectral density can be evaluated by introducing valence quark masses $z$ and $z^{*}$ in addition to the dynamical quark with mass $m$. The first two masses can be thought of as the up and down quark whereas the mass $m$ can be interpreted as a strange quark mass. Varying the chemical potential then corresponds to varying the isospin chemical potential and the strangeness chemical potential explaining the appearance of regions we identify as the pion condensed phase (denoted by $\pi$), the kaon condensed phase (denoted by $K$) and the normal phase (denoted by $N$). In the latter phase the chiral condensate is nonzero but the Bose condensates are
vanishing. For more details we refer to \[10\].

The fact that the structure of the eigenvalue density is determined by the phases of the generating functional shows that this structure persists beyond the microscopic domain. In this paper, however, we will focus on the microscopic limit. In this limit the QCD partition function and the spectral density of the Dirac operator are uniquely determined by the global symmetries of QCD and hence can be derived from chiral random matrix theories with the same symmetries \[11\]. In random matrix theory, the microscopic limit can be taken by identifying the size of the matrices with the volume of space-time and defining a scaling limit as in \[1\]. A random matrix model for the microscopic domain of QCD at nonzero chemical potential amenable to an analysis by orthogonal polynomials was introduced in \[5\], and its properties were further analyzed in \[6\] and \[7\].

The purpose of the present paper is two-fold. First we will show that within the random matrix model \[5\] the delicate cancellations follow from orthogonality properties of orthogonal polynomials. The results are exact, independent of the size of the matrices. It allows us to establish the $\mu$-independence of the full microscopic chiral condensate as obtained from integrating the microscopic spectral density according to Eq. \[2\]. We also present a direct computation using complex contour integrations which establishes the relation between the exact $\mu$-dependent microscopic spectral density and the $\mu$-independent chiral condensate. As a by-product we will obtain the exact expressions for the partially quenched chiral condensate at non-zero chemical potential as well as the bosonic partition functions in an arbitrary fixed topological sector.

In the upcoming section the random matrix model is introduced along with the complex orthogonal polynomials and their relation, through the Cauchy transform and kernels, with the eigenvalue density. In section \[III\] we then use the orthogonal polynomials to establish the $\mu$-independence of the chiral condensate. A strategy to derive this result without the use of orthogonal polynomials is sketched in section \[IV\] but it could only be worked out in detail in the strong nonhermiticity limit (see Appendix C). In section \[V\] we turn to the bosonic partition function and the quenched and partially quenched condensates are discussed in section \[VI\]. In the appendices we provide alternate derivations of the results of section \[III\]. Appendix A reproduces the microscopic condensate starting from the microscopic density so that only universal quantities are used in the calculation. Appendix B uses a method involving complex contour integration while in Appendix C we first rewrite the eigenvalue density as a total derivative.
II. THE RANDOM MATRIX MODEL

We consider the microscopic limit of QCD at nonzero chemical potential. In this limit where ($\Sigma$ is the magnitude of the chiral condensate in the chiral limit and $F_\pi$ is the tree level pion decay constant)

$$m\Sigma V \quad \text{and} \quad \mu^2 F_\pi^2 V$$

are kept fixed for $V \to \infty$, the mass and chemical potential dependence of the QCD partition function is given by a random matrix model (see for example the review [12]). For $N_f$ quark flavors with mass $m$ and $n$ pairs of regular and conjugate quarks with masses $x$ and $y^*$, respectively, this partition function is defined by [5]

$$Z_{N_f,n}^{N_f,n}(m, x, y^*; \mu) \equiv \int d\Phi d\Psi \, w_G(\Phi)w_G(\Psi) \det N_f(D(\mu) + m) \times \det^n(D(\mu) + x) \det^n(D^\dagger(\mu) + y^*) ,$$

where the non-Hermitian Dirac operator is given by

$$D(\mu) = \begin{pmatrix}
0 & i\Phi/\alpha + \mu\Psi/\alpha \\
i\Phi^\dagger/\alpha + \mu\Psi^\dagger/\alpha & 0
\end{pmatrix} .$$

(7)

Here $\Phi$ and $\Psi$ are complex $(N + \nu) \times N$ matrices both distributed according to a Gaussian weight function

$$w_G(X) = \exp(- N \text{Tr} X^\dagger X) .$$

(8)

The parameters $\alpha$ and $\gamma$ are scale factors used to map the random matrix model onto the chiral Lagrangian, which are given below. The number of additional rows as compared to columns gives rise to $\nu$ zero modes of $D$, and $\nu$ is therefore referred to as the topological index. Inverse determinants are interpreted as bosonic quarks and will be denoted by negative values of $N_f$ or $n$. In cases where $N_f$ or $n$ are zero they will be left off along with the corresponding masses.

In the random matrix model, the microscopic limit is given by the limit $N \to \infty$ where

$$\hat{m} = 2N\alpha m \quad \text{and} \quad \hat{\mu}^2 = 2N\gamma^2 \mu^2$$

are kept fixed as $N \to \infty$. The identification with the QCD partition function is made according to (see the discussion in [6])

$$\hat{m} = 2N\alpha m \to m\Sigma V,$$

$$\hat{\mu}^2 = 2N\gamma^2 \mu^2 \to \mu^2 F_\pi^2 V.$$
This then determines the scale factors $\alpha = \Sigma V / 2N$ and $\gamma^2 = F_\mu^2 V / 2N$. From here on we will drop the factors of $\alpha$ and $\gamma$ except when explicitly needed.

Contrary to the Hermitian random matrix ensembles, it is quite nontrivial to express the partition function (6) as an integral over the joint probability distribution of the eigenvalues of $D(\mu)$. Remarkably, it was shown in [5] that an analytical form could be obtained with result given by

$$Z_N^{N_f,n}(m, x, y^*; \mu) \sim m^{\nu N_f} \int d^2 z_k P^{N_f,n}(\{z_i\}, \{z^*_i\}, m, x, y^*; \mu),$$

where the integration extends over the complex plane and the joint probability distribution reads

$$P^{N_f,n}(\{z_i\}, \{z^*_i\}, m, x, y^*; \mu) = \frac{1}{\mu} \Delta_N(\{z_i^2\})^{1/2} \prod_{k=1}^N w(z_k, z^*_k; \mu) \rho^{N_f,n}(m, x, y^* - z^2_k) N_f (y^2 - z^*^2_k).$$

The Vandermonde determinant is defined as

$$\Delta_N(\{z_i^2\}) \equiv \prod_{i>j=1}^N (z_i^2 - z_j^2),$$

and the weight function includes a modified Bessel function,

$$w(z_k, z^*_k; \mu) = |z_k|^{2\nu+2} K_\nu \left( \frac{N(1 + \mu^2)}{2\mu^2} |z_k|^2 \right) \exp \left( -\frac{N(1 - \mu^2)}{4\mu^2} (z_k^2 + z^*_k) \right).$$

The modified Bessel function is obtained as a result of the integration over the angular degrees of freedom. In the microscopic limit the same functional dependence is obtained starting from a chiral Lagrangian for the bosonic phase quenched partition function [13]. This is a strong argument for the universality of this factor.

The eigenvalue representation makes it possible to evaluate integrals over eigenvalues by means of the method of complex orthogonal polynomials [14, 15, 16, 17, 18]. Analytical expressions for the eigenvalue density [5], eigenvalue correlation functions [5] and partition functions [6, 19] have been obtained using this method.

### A. Orthogonal polynomials and their Cauchy transform

In this section we introduce the orthogonal polynomials and relate them to the quenched and unquenched eigenvalue density. The complex orthogonal polynomials corresponding to the weight function (14) are given in terms of the complex Laguerre polynomials by [5]

$$p_k(z; \mu) = \left( \frac{1 - \mu^2}{N} \right)^k k! L^\nu_k \left( -\frac{N z^2}{1 - \mu^2} \right).$$
They satisfy the orthogonality relations
\[ \int_{\mathbb{C}} d^2z \, w(z, z^*; \mu) \, p_k(z; \mu) \, p_l(z; \mu)^* = \delta_{kl} \, r_k^\nu, \] (16)
with the norm \( r_k^\nu \) given by
\[ r_k^\nu = \frac{\pi \mu^2 (1 + \mu^2)^{2k+\nu} k! (k+\nu)!}{N^{2k+2+\nu}}. \] (17)
Since the orthogonal polynomials \( p_k \) are related to the Laguerre polynomials they also satisfy an orthogonality relation on the imaginary axis,
\[ \int_{0}^{\infty} dx \, w_i(x) \, p_k(ix; \mu) \, p_l(ix; \mu) = \delta_{kl} \, s_k^{\nu}, \] (18)
with
\[ w_i(x) = x^{2\nu+1} e^{-Nx^2/(1-\mu^2)} \quad \text{and} \quad s_k^{\nu} = \frac{k! (k+\nu)! (1-\mu^2)^{2k+\nu+1}}{2N^{2k+\nu+1}}. \] (19)
We will make use of this relation in section III B 1. Since
\[ \Pi_{l-1} = \frac{p_l(z) - p_l(m)}{z^2 - m^2} \] (20)
is a polynomial of order \( l - 1 \) in \( z^2 \), as a direct consequence of the orthogonality relations, we can establish the identity
\[ \int_{\mathbb{C}} d^2z \, w(z, z^*; \mu) p_l(z; \mu)^* \frac{p_l(z) - p_l(m)}{z^2 - m^2} = 0. \] (21)

The Cauchy transform of the orthogonal polynomials is defined as
\[ h_k(m; \mu) = \int_{\mathbb{C}} d^2z \, \frac{1}{z^2 - m^2} w(z, z^*; \mu) p_k^*(z; \mu), \] (22)
where we recall that \( \mathbb{C} \) indicates that the integration extends over the complex plane.

The partition function for one fermion can be expressed in terms orthogonal polynomials as
\[ \frac{Z_{N_f=1}^N(z; \mu)}{Z_{N_f=0}^N(z; \mu)} = z^\nu p_N(z; \mu), \] (23)
and the partition function for one bosonic flavor is given by a Cauchy transform \([6, 18, 19]\)
\[ \frac{Z_{N_f=-1}^N(z; \mu)}{Z_{N_f=0}^N(z; \mu)} = -z^{-\nu} h_{N-1}^N(z; \mu). \] (24)

Note that at finite \( N \) both partition functions depend on the chemical potential. This dependence can be removed \([20]\) from the fermionic partition function \([23]\) by scaling the factors \( \alpha \) and \( \gamma \) in \([7]\), but we will not bother here since the \( \mu \)-dependence will factorize in the microscopic limit which is our main concern. The bosonic partition function has a nontrivial \( \mu \)-dependence that can not be removed through the constants \( \alpha \) and \( \gamma \) \([19, 21]\). Below we will drop \( \mu \) from the argument of \( p \) and \( h \) to make the notation less clumsy.
B. Kernels and the spectral densities

From the orthogonal polynomials and their Cauchy transforms we can construct two different kernels (see for example [6]) for later use
\[
K_N(x, y) = \sum_{k=0}^{N-1} \frac{p_k(x)p_k(y)}{r_k}, \tag{25}
\]
\[
H_N(x, y) = \sum_{k=0}^{N} \frac{p_k(x)h_k(y)}{r_k}. \tag{26}
\]
We will also use the auxiliary kernel
\[
\mathcal{N}_N(x, y) = \frac{1}{y^2 - x^2} + H_N(x, y), \tag{27}
\]
which enters in the computation of the partially quenched chiral condensate in section VI.

The $\mu$-dependent spectral density is related to the kernel $K$. In particular, the quenched spectral density is given by [5]
\[
\rho_N^Q(z, z^*; \mu) = 2w(z, z^*; \mu)K_N(z, z^*), \tag{28}
\]
and the unquenched $N_f = 1$ spectral density can be written as [5]
\[
\rho_N^{N_f=1}(z, z^*, m; \mu) = 2w(z, z^*; \mu) \left[ K_N(z, z^*) - \frac{p_N(z)}{p_N(m)}K_N(m, z^*) \right]. \tag{29}
\]
Note that the first term is equal to the quenched spectral density. The second term is responsible for the strong oscillations of the real and imaginary parts of the unquenched eigenvalue density (see figure 2). There is also an extra contribution $\nu\delta^2(z)$ to the eigenvalue density which arises due to the exact zero modes of the Dirac operator which has not been included in the spectral density.

III. THE $\mu$-INDEPENDENCE OF THE MICROSCOPIC CONDENSATE

In this section we will show how a $\mu$-independent chiral condensate in the microscopic limit can arise even though the eigenvalue density of Dirac operator which enters (2) has a strong $\mu$-dependence. Inserting the finite $N$ one flavor partition function (23) in (1) we immediately find that the chiral condensate is given by
\[
V\Sigma(m) = \frac{dp_N(m)/dm}{p_N(m)} + \frac{\mu}{m}. \tag{30}
\]
Making use of orthogonality relations for the polynomials we will show that the above expression also follows from a direct integration of the spectral density [20]. The microscopic result is then
FIG. 2: The eigenvalue density of the random matrix model \( N_f=0 \) with \( N=20, \mu=0.8, m=0.6 \) and \( \nu = 0 \). The quenched density (top) is real and positive. The real part of the unquenched density (bottom) shows oscillations. Note that the oscillating region starts out at \( z = \pm m \) and extends outward. The peak of the oscillations has been cut; the maximum amplitude is an order of magnitude larger than the scale displayed.

obtained by taking the microscopic limit of the finite N result, and as we shall see explicitly below, the \( \mu \)-dependence drops out. The topological term \( \nu/m \) follows trivially from the \( \delta \) function contribution due to the zero modes.

Alternatively one can start from the microscopic limit of the spectral density and show by
explicit evaluation of the integrals that the corresponding chiral condensate does not depend on the chemical potential. For completeness, this derivation is given in Appendix A. The techniques used here will also be useful when calculating the partially quenched chiral condensate in section VI.

**A. Chiral condensate at finite \( N \)**

In this subsection we will calculate the one flavor chiral condensate by integrating over the eigenvalue density (29) using a variant of (2) obtained from exploiting the evenness of \( \rho \) in \( z \)

\[
V \Sigma(m) = \int_\mathbb{C} d^2 z \frac{m}{m^2 - z^2} \rho_{N_f=1}^N(z, z^*, m; \mu). \tag{31}
\]

The spectral density for \( N_f = 1 \) given in Eq. (29) can be written out as

\[
\rho_{N_f=1}^N(z, z^*, m; \mu) = 2w(z, z^*; \mu) \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(z) - p_N(z)p_k(m)}{r_k}. \tag{32}
\]

By subtraction and addition of \( p_k(m) \) and \( p_N(m) \) to \( p_k(z) \) and \( p_N(z) \), respectively, we obtain

\[
\rho_{N_f=1}^N(z, z^*, m; \mu) = 2w(z, z^*; \mu) \left[ \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(z) - p_N(z)p_k(m)}{r_k} - \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(m)}{r_k} \frac{p_N(z) - p_N(m)}{p_N(m)} \right]. \tag{33}
\]

The first sum in the square brackets is well behaved in the thermodynamic limit whereas the second sum, with the exception of \( z \approx m \), shows oscillations with an amplitude that grows exponentially with the volume. The chiral condensate is given by the integral

\[
V \Sigma(m) = 2m \int_\mathbb{C} d^2 z w(z, z^*; \mu) \left[ \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(z) - p_N(z)p_k(m)}{r_k(m^2 - z^2)} - \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(m)}{r_k} \frac{p_N(z) - p_N(m)}{p_N(m)(m^2 - z^2)} \right]. \tag{34}
\]

The contribution of the first sum in (34) integrates to zero due to orthogonality. To obtain a result that is well behaved in the thermodynamic limit, the exponentially large terms in the second sum have to be canceled which indeed happens because the contributions to the chiral condensate are localized on \( z = m \). The reason is that, as a consequence of the orthogonality of the \( p_k \), the sum

\[
w(z, z^*; \mu) \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(m)}{r_k} \tag{35}
\]

is a reproducing kernel (or equivalent to the delta function \( \delta^2(z - m) \)) in the space of polynomials in \( z \) of order less than \( N \) (denoted by \( \Pi_n(z) \)). We thus have

\[
\int_\mathbb{C} d^2 z w(z, z^*; \mu) \sum_{k=0}^{N-1} \frac{p_k(z^*)p_k(m)}{r_k} \Pi_n(z) = \Pi_n(m) \quad \text{for} \quad n < N. \tag{36}
\]
Using the definition of derivative we finally find the following result for the chiral condensate,

\[ V \Sigma(m) = \frac{2m}{p_N(m)} \lim_{z \to m} \frac{p_N(z) - p_N(m)}{z^2 - m^2} = \frac{dp_N(m)/dm}{p_N(m)}. \]  

(37)

Clearly this is consistent with expressing the chiral condensate as the derivative with respect to \( m \) of the logarithm of the partition function \( \log Z \). Notice that since the spectral density \( \nu \delta^2(z) \) does not contain the trivial contribution of the zero modes, \( \nu \delta^2(z) \), the condensate calculated this way does not have the \( \nu/m \) term. A second remark is that at finite \( N \), the polynomial \( p_N(m) \) depends on the chemical potential. As will be shown in the next subsection, this dependence drops out of the chiral condensate in the microscopic limit.

**B. Microscopic limit of the finite \( N \) results**

In this section we discuss the microscopic limit of the argument presented in the previous section. First, let us show explicitly that, in the microscopic limit, we obtain the expected \( \mu \)-independent chiral condensate. The microscopic limit of the orthogonal polynomials is given by

\[ \frac{p_k(z; \mu)}{\sqrt{\gamma_k}} \xrightarrow{N \to \infty} (\alpha z)^{-\nu} I_{\nu}(2N\alpha t) \frac{N e^{-2N\gamma^2\mu^2 t^2}}{\gamma \sqrt{\pi}} \quad \text{with} \quad t = \sqrt{\frac{k}{N}}. \]  

(38)

Because of this factorization, the overall \( \mu \)-dependence of the orthogonal polynomials drops out of equation (37) in the microscopic limit resulting in a \( \mu \)-independent chiral condensate

\[ \hat{\Sigma}(\hat{m}) \equiv \lim_{N \to \infty} \frac{\Sigma(\hat{m}/2N\alpha)}{2N\alpha} = \frac{dI_{\nu}(\hat{m})/d\hat{m}}{I_{\nu}(\hat{m})} - \frac{\nu}{\hat{m}}. \]  

(39)

The term \( \nu/\hat{m} \) is canceled by the contribution of the zero modes. In the above argument we have manipulated the non-universal finite \( N \) expressions into a form which are suitable to take the microscopic limit. In Appendix A we will show that it is also possible to take the microscopic limit from the start and deal with the universal expressions throughout the argument.

In fact, a slightly stronger result is valid. In the microscopic limit, orthogonal polynomials of the same order but with different arguments have the same \( \mu \)-dependence (see [8]). Therefore, the ratio

\[ \frac{p_N(z) - p_N(m)}{p_N(m)(m^2 - z^2)} = \sum_{k=0}^{N-1} c_k(m) \frac{p_k(z)}{p_N(m)} \]  

(40)

does not depend on the chemical potential in the microscopic limit, and for the same reason, each term in the sum does not depend on \( \mu \). If we define \( \Sigma_k \) as

\[ \Sigma_k(m) = \frac{2mp_k(m)}{p_N(m)} c_k(m), \]  

(41)
so that
\[ \Sigma(m) = \sum_{k=0}^{N-1} \Sigma_k(m), \] (42)
then also the \( \Sigma_k \) are independent of \( \mu \) in the microscopic limit. Explicit results for the microscopic limit of \( \Sigma_k(m) \) will be worked out in the next subsection.

1. **Microscopic limit of \( \Sigma_k \)**

In this subsection we derive the microscopic limit of \( \Sigma_k(m) \) defined in (41). This quantity, denoted by \( \hat{\Sigma}(\hat{m}, t) \), also enters in derivation of the chiral condensate by complex contour integration (see Appendix C).

The coefficient \( c_k(m) \) defined in (40) can be calculated by using the orthogonality of the \( p_k \) on the positive imaginary axis
\[ c_k(m) = \int_0^\infty dx w_i(x) \frac{p_k(ix) p_N(m) - p_N(ix)}{s_k^\nu} \frac{1}{x^2 + m^2}. \] (43)
Now looking at the first part of the r.h.s. of the previous equation we define
\[ \Sigma^I_k(m) = 2m \frac{p_k(m)}{s_k^\nu} \int_0^\infty dx w_i(x) \frac{p_k(ix)}{x^2 + m^2}. \] (44)
In the microscopic limit we have
\[ \sum_{k=0}^{N-1} \to \int_0^1 2Nt dt \quad \text{with} \quad t = \sqrt{k/N} \quad \text{and} \quad w_i(x) = x^{2\nu+1}, \] (45)
so that
\[ \hat{\Sigma}^I(\hat{m}, t) = 2\hat{m} t \hat{m}^{-\nu} I_\nu(\hat{mt}) \int_0^\infty d\hat{x} \frac{\hat{x}^{\nu+1}}{\hat{x}^2 + \hat{m}^2} J_\nu(\hat{xt}) = 2\hat{mt} I_\nu(\hat{mt}) K_\nu(\hat{mt}). \] (46)
Likewise the second part of \( c_k(m) \) contributes to the chiral condensate as
\[ \Sigma^\Pi_k(m) = 2m \frac{p_k(m)}{p_N(m)s_k^\nu} \int_0^\infty dx w_i(x) \frac{p_k(ix)p_N(ix)}{x^2 + m^2}. \] (47)
In the microscopic limit this simplifies to
\[ \hat{\Sigma}^\Pi(\hat{m}, t) = 2\hat{m} t I_\nu(\hat{mt}) I_\nu(\hat{m}) \int_0^\infty d\hat{x} \frac{\hat{x}}{\hat{x}^2 + \hat{m}^2} J_\nu(\hat{xt})J_\nu(\hat{x}). \] (48)
The integral is known \[22\] resulting in
\[ \hat{\Sigma}^\Pi(\hat{m}, t) = 2\hat{mt} I_\nu(\hat{mt}) K_\nu(\hat{m}) I_\nu(\hat{mt}). \] (49)
We conclude that the microscopic limit of $\Sigma_k(m)$ is independent of the chemical potential.

Reminding ourselves of the identities

\[
\int_0^1 dt [\hat{I}_\nu(\hat{m}t)]^2 = \frac{1}{2}[I^2_\nu(\hat{m}) - I_{\nu+1}(\hat{m})I_{\nu-1}(\hat{m})],
\]

\[
\int_0^1 dt t K_\nu(\hat{m}t) I_\nu(\hat{m}t) = \frac{1}{2}[K_\nu(\hat{m}) I_\nu(\hat{m}) + K_{\nu+1}(\hat{m})I_{\nu-1}(\hat{m})] - \frac{\nu}{\hat{m}^2},
\]

we obtain after using the Wronskian identity

\[
\hat{\Sigma}(\hat{m}) = \int_0^1 dt [\hat{\Sigma}^I(\hat{m}, t) - \hat{\Sigma}^{\Pi}(\hat{m}, t)] = \frac{I'_\nu(\hat{m})}{I_\nu(\hat{m})} - \frac{\nu}{\hat{m}},
\]

which is the correct result after adding the contribution of the zero modes.

We have by now established the $\mu$-independence of the microscopic chiral condensate as expressed by an integral over the eigenvalue density. As discussed next, the properties of the spectral density that lead to the $\mu$-independence of the chiral condensate can be exposed further by writing the spectral density as a derivative of the partially quenched chiral condensate.

**IV. THE SPECTRAL DENSITY AS A TOTAL DERIVATIVE**

In section III we have seen that the chiral condensate can be written as an integral over a reproducing kernel in the space of complex orthogonal polynomials up to order $N - 1$:

\[
\Sigma(m) = -2m \int_C d^2z K_N(z^*, m) \frac{\Pi_{N-1}(z)}{p_N(m)}.
\]

In the thermodynamic limit the kernel can be written as

\[
\lim_{N \to \infty} K_N(z^*, m) \sim 2m\delta^2(z^2 - m^2) = \frac{1}{\pi}\partial_{z^*}\frac{1}{z^2 - m^2}.
\]

Using a partial integration this suggest that the condensate can be written as

\[
\Sigma(m) = \int_C d^2z \frac{2m}{z^2 - m^2}\partial_{z^*}F(z, z^*, m; \mu),
\]

or that the spectral density can be written as a total derivative with respect to $z^*$. Such a representation of the eigenvalue density in the complex plane is well known \[23, 24\]; the function $F$ is known as the partially quenched condensate

\[
F(z, z^*, m; \mu) = \frac{1}{V} \left\langle \text{Tr} \frac{1}{\slashed{\partial} + \mu \gamma_0 + z} \right\rangle.
\]

It is the chiral condensate evaluated at a complex mass $z$. A partial integration of (54) singles out $z = m$ and consistently reproduces the chiral condensate $\Sigma(m) = F(z = m, z^* = m, m; \mu)$. 

If we can express the eigenvalue density as a total derivative \( \rho = \partial z^* F(z, z^*, m; \mu) \) and show that \( F \) at \( z = z^* = m \) is independent of \( \mu \) we have an alternative way to show how a strongly \( \mu \)-dependent density can result in a chiral condensate that does not depend on the chemical potential. This alternative proof will be worked out in Appendix C in the strong non-Hermiticity limit for arbitrary \( \nu \). While the exact expressions for the microscopic eigenvalue density are known, the partially quenched condensates at nonzero chemical potential have not been obtained previously and will be derived in section VI.

V. CAUCHY TRANSFORM AND THE BOSONIC PARTITION FUNCTION

In this section we evaluate the Cauchy transform of the orthogonal polynomials for finite \( N \). As a new result we obtain the microscopic limit of the bosonic partition function with arbitrary topological charge. For zero topological charge the result agrees with the bosonic partition function obtained in [19] and thus proves a conjectured cancellation in the derivation of [19]. The result of this section can also be used to obtain an alternate expression for the microscopic limit of \( \mathcal{H}_N(m_1, m) \) at finite \( N \).

The Cauchy transform is defined by (see Eq. (22))

\[
 h_k(m) = \int \frac{d^2 z}{2\pi} w(z, z^*; \mu) \frac{1}{z^2 - m^2} p_k(z^*)
 = \int \frac{d^2 z}{2\pi} \frac{|z|^{2\nu+2}}{z^2 - m^2} K_\nu(b|z|^2) e^{-a(z^2 + z^{2*})} p_k(z^*)
\]

(56)

with \( a = N(1 - \mu^2)/4\mu^2 \) and \( b = N(1 + \mu^2)/2\mu^2 \). The integrals can be evaluated by writing

\[
 \frac{e^{-az^2}}{z^2 - m^2} = \frac{e^{-am^2}}{z^2 - m^2} + \frac{e^{-az^2} - e^{-am^2}}{z^2 - m^2}.
\]

(57)

The first term has a pole while the second is analytic. The angular integration for the first term is now easy since the integrand only depends on \( z^* \) (or \( |z| \)) and not \( z \). Simply expanding \( 1/(z^2 - m^2) \) in a geometric series in \( z^2/m^2 \) we obtain

\[
 \int \frac{d^2 z}{2\pi} \frac{|z|^{2\nu+2}}{z^2 - m^2} K_\nu(b|z|^2) e^{-a(m^2 + z^{2*})} p_k(z^*) = -\frac{2\pi}{m^2} \int_0^m dr r^{2\nu+2} K_\nu(br^2) e^{-a(m^2 + r^4/m^2)} p_k(r^2/m).
\]

For the second piece, since it is analytic, we assume it can be expanded in the form

\[
 \frac{e^{-az^2} - e^{-am^2}}{z^2 - m^2} = e^{-az^2} \sum_{k=0}^\infty d_k(m) p_k(z).
\]

(58)

Then by substituting this into (56) and using orthogonality we get

\[
 \int \frac{d^2 z}{2\pi} |z|^{2\nu+2} \frac{e^{-az^2} - e^{-am^2}}{z^2 - m^2} K_\nu(b|z|^2) e^{-a(m^2 + z^{2*})} p_k(z^*) = r^\nu_k d_k(m),
\]

(59)
where \( r^\nu_k \) is the normalization factor defined in (17). To evaluate \( d_k(m) \) we first assume that the series \( (58) \) converges for all \( z \) and \( m \). Then this is also valid along the imaginary axis \( z = ix \) where we can use the orthogonality relations of the orthogonal polynomials \( p_k(ix) \) defined in (18). The coefficients are thus given by

\[
d_k(m) = \frac{1}{s_k^n} \int_{x=0}^{\infty} dx w_i(x) p_k(ix) \frac{e^{-a(x^2+m^2)}}{x^2 + m^2}.
\] (60)

Using that

\[
\int_0^a dt \ e^{-t(x^2+m^2)} = - \frac{e^{-a(x^2+m^2)} - 1}{x^2 + m^2},
\] (61)

and the formula \( (22) \)

\[
\int_0^\infty e^{-st} t^\nu L_k^\nu(t) dt = \frac{\Gamma(\nu + k + 1)(s - 1)^k}{k! s^{k+\nu+1}}
\] (62)

this can be reduced to

\[
d_k(m) = - \frac{N^{k+1}}{k!(1 - \mu^2)^{k+1}} \int_0^{(1-\mu^2)/4\mu^2} dt \ e^{-m^2Nt/(1-\mu^2)} \frac{t^k}{(t + 1)^{k+\nu+1}}.
\] (63)

Combining the above expressions we thus find

\[
h_k(m) = \frac{-2\pi m^2}{\mu^2} \int_0^m dr r^{2\nu+3} K_\nu(br^2) e^{-a(m^2+r^4/m^2)} p_k(r^2/m)
\]

\[
\quad - \frac{r^\nu_k N^{k+1}}{k!(1 - \mu^2)^{k+1}} \int_0^{(1-\mu^2)/4\mu^2} dt \ e^{-m^2Nt/(1-\mu^2)} \frac{t^k}{(t + 1)^{k+\nu+1}}.
\] (64)

This form makes it easier to take the microscopic limit.

### A. Microscopic limit of the bosonic partition function

The bosonic partition function given by (24) can now be obtained using the expression (64) for the Cauchy transform. In the microscopic limit we find (as in (19,25) a factor \( \exp(\hat{\mu}^2/2) \) has been removed to ensure independence of \( \mu \) for \( \mu < m_{\pi}/2 \) in the limit \( \hat{\mu} \to \infty \)

\[
Z^{(\nu)}_{N_f=-1}(\hat{m}; \hat{\mu}) = \frac{e^{-2\hat{\mu}^2 - \hat{m}^2/8\hat{\mu}^2}}{4\hat{\mu}^2 \sqrt{\pi}} e^N \sqrt{2N} \int_0^{\hat{m}} du \exp[-\frac{u^2}{8\hat{\mu}^2}] K_\nu \left( \frac{u\hat{m}}{4\hat{\mu}^2} \right) I_\nu(u)
\]

\[
+ \frac{e^N \sqrt{2N}}{\sqrt{\pi}(2\hat{m})^{\nu/2}} \int_0^{1/8\hat{\mu}^2} ds \frac{1}{s^{\nu+1}} e^{-\hat{m}^2s^{-1}/4s}.
\] (65)

The microscopic limit of the \( N_f = -1 \) partition function in the sector of zero topological charge
was evaluated in \cite{19, 25} with the result,
\begin{equation}
Z_{N_f=1}^{\nu=0} = e^{-2\rho^2} \frac{1}{4\mu^2} e^{-\frac{\rho^2}{8^2}} \\
\times \left[ \int_0^m du u \exp\left[ -\frac{u^2}{8^2} \right] K_0\left( \frac{u\rho}{4^2} \right) I_0(u) + \int_0^\infty du u \exp\left[ -\frac{u^2}{8^2} \right] K_0\left( \frac{u\rho}{4^2} \right) K_0(u) \right].
\end{equation}

\begin{equation}
= K_0(m) - e^{-2\rho^2} \frac{1}{4\mu^2} e^{-\frac{\rho^2}{8^2}} \int_0^m du u \exp\left[ -\frac{u^2}{8^2} \right] K_0\left( \frac{u\rho}{4^2} \right) I_0(u).
\end{equation}

It is immediately clear that the first term agrees up to an overall normalization constant with the first term in the result \cite{25}. That also the second terms are in agreement can be seen from the substitution of
\begin{equation}
K_0(u) = \frac{1}{2} \int_0^\infty \frac{dt}{t} e^{-t u^2/4^2}
\end{equation}
in (66) which allows the u integration to be performed
\begin{equation}
\frac{1}{4\mu^2} e^{-2\rho^2 - \frac{\rho^2}{8^2}} \int_0^\infty du u \exp\left[ -\frac{u^2}{8^2} \right] I_0\left( \frac{u\rho}{4^2} \right) K_0(u) = \frac{1}{2} \int_0^{1/8 \rho^2} \frac{dt}{t} e^{-\rho^2 t - 1/4^2}.
\end{equation}

VI. THE PARTIALLY QUENCHED CHIRAL CONDENSATE

The partially quenched chiral condensate for $N_f$ flavors is defined by
\begin{equation}
\Sigma_{N_f}(m) = \frac{1}{V} \left[ \partial_f \ln \left( Z_{N_f=N_b}^{N_f=N_b=1} (m_1, \cdots, m_{N_f}, m + J|m) \right) \right]_{J=0}.
\end{equation}

In this section we evaluate the microscopic limit of this expression for $N_f = 0$ and $N_f = 1$ and derive simplified expressions valid in the strong non-Hermiticity limit.

For the quenched theory ($N_f = 0$) we need the ratio \cite{17}
\begin{equation}
\frac{Z_N^{N_f=1, N_b=1}(x|y; \mu)}{Z_N^{N_f=0}} \equiv \left\langle \left( \frac{x}{y} \right)^\nu \prod_{j=1}^N \frac{x^2 - z_j^2}{y^2 - z_j^2} \right\rangle_{N_f=0}
\end{equation}
\begin{equation}
= (y^2 - x^2) \left( \frac{x}{y} \right)^\nu \mathcal{N}_{N-1}(x, y),
\end{equation}
so that the quenched chiral condensate can be expressed as
\begin{equation}
V \Sigma_Q(m) = \frac{\nu}{m} - 2m \mathcal{H}_{N-1}(m, m),
\end{equation}
where we used the auxiliary kernel defined in \cite{27}. Below we will evaluate the microscopic limit of $\mathcal{H}_{N-1}(m, m)$. 
What enters in the chiral condensate for \( N_f = 1 \) is the ratio
\[
\frac{Z_{N=2,N_b=1}^{N_f=2,N_f=1}(x, m_1| y; \mu)}{Z_{N=1}^{N_f=1}(m_1; \mu)} = \frac{1}{Z_{N=1}^{N_f=1}(m_1; \mu)} \left\langle \left( \frac{x m_1}{y} \right) \nu \prod_{j=1}^{N} \frac{(x^2 - z_j^2)(m_1^2 - z_j^2)}{y^2 - z_j^2} \right\rangle_{N_f=0}^{N_f=1}
\]
(72)
where the second equality follows from [17]. The partially quenched condensate for \( N_f = 1 \) is the condensate as a function of \( m \) for fixed physical mass \( m_1 \). It can be expressed as
\[
V_{\Sigma PQ}(m, m_1) = \frac{\nu}{m} - 2m \mathcal{H}_N(m, m) + \frac{2m}{m_2^2 - m^2} + 2m \frac{p_N(m)}{p_N(m_1)} \left[ \frac{1}{m^2 - m_1^2} + \mathcal{H}_N(m_1, m) \right].
\]
(73)
This form makes it easy to take the \( m_1 = m \) limit. The extension to more flavors again only requires the function \( \mathcal{H}_N(m_1, m) \).

The integral appearing in the bosonic partition function (67) has an essential singularity at \( \mu = 0 \) [25]. The function \( \mathcal{H} \) and hence the quenched (71) and partially quenched condensate inherit this non-analyticity. For \( m_1 \neq m \) the partially quenched condensate (see (73)) is a function of the chemical potential. However, for \( m_1 = m \) it correctly reduces to the derivative with respect to \( \hat{m} \) of \( Z_{N_f=1} \) which is \( \hat{\mu} \)-independent in the microscopic limit.

**A. Microscopic limit of the \( \mathcal{H} \)-kernel**

Using the definition of the Cauchy transform, the \( \mathcal{H} \)-kernel can be written as
\[
\mathcal{H}_N(m_1, m) = \sum_{k=0}^{N} \frac{p_k(m_1) h_k(m)}{r_k}
= \int_{\mathbb{C}} d^2 z \, w(z, z^*; \mu) \frac{1}{z^2 - m^2} \sum_{k=0}^{N} p_k(m_1) p_k(z^*)/r_k.
\]
(74)
To evaluate \( \mathcal{H}_N(m_1, m) \) we substitute
\[
p_k(m_1) = (z^2 - m^2) [c_{k-1} p_{k-1}(z) + c_{k-2} p_{k-2}(z) + \ldots] + p_k(z m_1/m)
\]
(75)
into (74) to get
\[
\mathcal{H}_N(m_1, m) = \int_{\mathbb{C}} d^2 z \, w(z, z^*; \mu) \frac{1}{z^2 - m^2} \sum_{k=0}^{N} p_k(z m_1/m) p_k(z^*)/r_k.
\]
(76)
In the microscopic limit the $H$ kernel gets multiplied by an additional factor \(1/(2N)^2\). Using expressions for the microscopic limit of the orthogonal polynomials we then obtain

\[
\hat{H}(\hat{m}_1, \hat{m}) = \int_C d^2 \hat{z} \frac{|\hat{z}|^2 \hat{m}^\nu}{16\pi^2 \hat{m}_1^\nu} K_\nu \left( \frac{|\hat{z}|^2}{4\hat{\mu}^2} \right) \exp \left( -\frac{\hat{z}^2 + \hat{z}^*}{8\hat{\mu}^2} \right) \frac{1}{\hat{z}^2 - \hat{m}^2} \times \int_0^1 t dt \exp(-2\hat{\mu}^2 t^2) I_\nu(\hat{z}t\hat{m}_1/\hat{m})I_\nu(\hat{z}^* t) .
\]  

(77)

B. Strong non-Hermiticity limit

The expression for $H$ simplifies considerably in the strong non-Hermiticity limit, $\hat{\mu} \gg 1$. Then for $\hat{m}_1 < \hat{m}$ using (113) we obtain

\[
\hat{H}(\hat{m}_1, \hat{m}) = \int_C d^2 \hat{z} \frac{|\hat{z}|^2 \hat{m}^\nu}{16\pi^2 \hat{m}_1^\nu} K_\nu \left( \frac{|\hat{z}|^2}{4\hat{\mu}^2} \right) I_\nu \left( \frac{|\hat{z}|^2 \hat{m}_1}{4\hat{\mu}^2 \hat{m}_1^\nu} \right) \exp \left( -\frac{\hat{m}_1^2 - \hat{m}^2}{8\hat{\mu}^2 \hat{m}_1^\nu \hat{\mu}^2} \hat{z}^2 \right) \frac{1}{\hat{z}^2 - \hat{m}^2} .
\]  

(78)

This is now in a form that can be evaluated analytically. The angular integration gives

\[
\hat{H}(\hat{m}_1, \hat{m}) = \frac{\hat{m}^\nu - \hat{m}_1^\nu}{8\hat{\mu}^2 \hat{m}_1^\nu} \exp \left( \frac{\hat{m}_1^2 - \hat{m}^2}{8\hat{\mu}^2} \right) \int_0^{\infty} K_\nu \left( \frac{r^2}{4\hat{\mu}^2} \right) I_\nu \left( \frac{r^2 \hat{m}_1}{4\hat{\mu}^2 \hat{m}_1^\nu} \right) r^3 dr
\]

\[
- \frac{\hat{m}^\nu - \hat{m}_1^\nu}{8\hat{\mu}^2 \hat{m}_1^\nu} \int_0^{\infty} K_\nu \left( \frac{r^2}{4\hat{\mu}^2} \right) I_\nu \left( \frac{r^2 \hat{m}_1}{4\hat{\mu}^2 \hat{m}_1^\nu} \right) r^3 dr .
\]  

(79)

The $r$-integrals are known and the final result is

\[
\hat{H}(\hat{m}_1, \hat{m}) = \frac{-1}{\hat{m}^2 - \hat{m}_1^2} + \frac{\hat{m}^\nu + 1}{4\hat{\mu}^2 \hat{m}_1^\nu (\hat{m}^2 - \hat{m}_1^2)} \exp \left( \frac{\hat{m}_1^2 - \hat{m}^2}{8\hat{\mu}^2} \right) \left[ \hat{m} K_{\nu-1} \left( \frac{\hat{m}^2}{4\hat{\mu}^2} \right) I_\nu \left( \frac{\hat{m} \hat{m}_1}{4\hat{\mu}^2} \right) + \hat{m}_1 K_\nu \left( \frac{\hat{m}^2}{4\hat{\mu}^2} \right) I_{\nu-1} \left( \frac{\hat{m} \hat{m}_1}{4\hat{\mu}^2} \right) \right] .
\]  

(80)

Inserting this result in (71) and (73) gives respectively the quenched and the partially quenched chiral condensate in the strong non-Hermiticity limit. In both cases we need $\hat{H}$ at equal masses.

With help of the Wronskian identity this can be expressed as

\[
2\hat{m} \hat{H}(\hat{m}, \hat{m}) = \frac{\nu}{\hat{m}} - \frac{\hat{m}}{4\hat{\mu}^2} \left[ \nu \left( I_{\nu-1} \left( \frac{\hat{m}^2}{4\hat{\mu}^2} \right) K_\nu \left( \frac{\hat{m}^2}{4\hat{\mu}^2} \right) - I_\nu \left( \frac{\hat{m}_1^2}{4\hat{\mu}^2} \right) K_{\nu-1} \left( \frac{\hat{m}_1^2}{4\hat{\mu}^2} \right) \right) \right]
\]

\[
+ \frac{\hat{m}^2}{4\hat{\mu}^2} \left[ I_{\nu-1} \left( \frac{\hat{m}^2}{4\hat{\mu}^2} \right) K_{\nu-1} \left( \frac{\hat{m}^2}{4\hat{\mu}^2} \right) + I_\nu \left( \frac{\hat{m}_1^2}{4\hat{\mu}^2} \right) K_\nu \left( \frac{\hat{m}_1^2}{4\hat{\mu}^2} \right) \right] .
\]  

(81)

In the strong non-Hermiticity limit, the partially quenched chiral condensate also follows from the expression of the spectral density as a total derivative. In the quenched case we find

\[
\hat{\Sigma}_Q(\hat{m}) = \int_C d^2 \hat{z} \frac{1}{\hat{z} + \hat{m}} \frac{1}{\pi} \partial_{\hat{z}^*} F(\hat{z}, \hat{z}^*, \hat{m}) + \frac{\nu}{\hat{m}} = -F(\hat{z} = \hat{m}, \hat{z}^* = \hat{m}, \hat{m}) + \frac{\nu}{\hat{m}} .
\]  

(82)

with $F$ given by (see (136) in Appendix C)

\[
F(\hat{z}, \hat{z}^*, \hat{m}) = \frac{\partial_{\hat{m}} [F_A(\hat{z}, \hat{m}) G(\hat{m} \hat{z}, \hat{z} \hat{z}^*)]}{2\hat{z}} \bigg|_{\hat{m} = \hat{z}} - \text{Sing} \left[ \frac{G(\hat{z} \hat{m}, \hat{z} \hat{z}^*) \partial_{\hat{m}} F_A(\hat{z}, \hat{m})}{2\hat{z}} \bigg|_{\hat{m} = \hat{z}} \right] .
\]  

(83)
FIG. 3: The effect of unquenching on the partially quenched chiral condensate is shown by plotting $\Sigma(m) \equiv \hat{\Sigma}_{U,PQ}(\hat{m}_1 = 0, 2Nm)$, for one massless flavor with $\hat{m}_1 = 0$, $\nu = 0$ and $\tilde{\mu}^2 = 3.84$. The strong non-Hermiticity result is given by the solid curve. The squares denote the exact microscopic result, and the open circles show the result obtained in [26] by matrix diagonalization.

The subtraction of the singular term amounts to not differentiating $I_\nu(\hat{m})$ except for at $\hat{m} = 0$ which is compensated for by subtracting the term $\nu/\hat{m}$. To avoid convergence problems we subtract the asymptotic value of the quenched spectral density which is equal to $u$. The asymptotic contribution will be denoted by $\hat{\Sigma}_{as}(\hat{m})$. A simple calculation then results in

\[
\hat{\Sigma}_{Q}(\hat{m}) = \hat{\Sigma}_{as}(\hat{m}) + \frac{\hat{m}}{4\tilde{\mu}^2} \left[ \nu K_{\nu+1} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) I_{\nu} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) - \nu K_{\nu} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) I_{\nu+1} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) \right] \\
+ \frac{\hat{m}^2}{4\tilde{\mu}^2} \left[ K_{\nu} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) I_{\nu} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) + K_{\nu+1} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) I_{\nu+1} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) \right] - 1
\]

\[
= -\frac{\nu}{m} + \frac{\hat{m}}{4\tilde{\mu}^2} + \frac{\hat{m}^3}{16\tilde{\mu}^4} \left[ K_{\nu} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) I_{\nu} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) + K_{\nu+1} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) I_{\nu+1} \left( \frac{\hat{m}^2}{4\tilde{\mu}^2} \right) \right].
\]  

(84)

This result can also be obtained by direct integration the quenched spectral density using polar coordinates. The contribution $\hat{\Sigma}_{as}(\hat{m})$ is obtained by a direct calculation

\[
\hat{\Sigma}_{as}(\hat{m}) = \frac{1}{4\tilde{\mu}^2} \int dx dy \frac{1}{x + iy + \hat{m}} = 2 \frac{\hat{m}}{4\tilde{\mu}^2}.
\]  

(85)

where the $x$-integration has been taken over a symmetric interval about $x = 0$. As it should, the result [84] agrees with [71] after inserting [81].
Next we study the valence quark mass dependence of the partially quenched chiral condensate for \( N_f = 1 \) and compare with numerical simulations obtained in [26] for \( \nu = 0 \) and \( m_1 = 0 \). In this case the result for \( \Sigma_{PQ}(m_1 = 0, m) \) is particularly simple. The chiral condensate is decomposed according to (112)

\[
\hat{\Sigma}_{PQ}(\hat{m}_1 = 0, \hat{m}) = \hat{\Sigma}_Q(\hat{m}) - \hat{\Sigma}_{U,PQ}(\hat{m}_1 = 0, \hat{m}).
\]

The quenched contribution was evaluated above (see (84)) and is given by (introducing the notation \( u \equiv 1/4\hat{\mu}^2 \))

\[
\hat{\Sigma}_Q(\hat{m}) = u\hat{m} + u^2\hat{m}^3[K_0(u\hat{m}^2)I_0(u\hat{m}^2) + K_1(u\hat{m}^2)I_1(u\hat{m}^2)].
\]

Notice that this result is only valid when \( \hat{m} \) is inside the domain of eigenvalues. The unquenched contribution follows immediately from (126) and (127), or alternatively, can be calculated from the strong non-Hermiticity limit of \( \mathcal{H}(m, m_1) \) and (73). The result for \( \nu = 0 \) and \( \hat{m}_1 = 0 \) is given by

\[
\hat{\Sigma}_{U,PQ}(\hat{m}_1 = 0, \hat{m}) = -2u\hat{m}e^{-u\hat{m}^2/2}I_0(\hat{m})K_1(\hat{m}^2) + \frac{2}{\hat{m}}.
\]

The derivation of this result does not require that \( \hat{m} \) is inside the domain of eigenvalues as in the quenched case.

In Fig. 3 we show the result for \( \hat{\Sigma}_{U,PQ}(\hat{m}_1 = 0, \hat{m}) \) (solid curve) for \( \hat{\mu}^2 = 3.84 \). In this figure we also give numerical results for an ensemble of 96 \( \times \) 96 matrices obtained in [26] by direct diagonalization of the random matrices. The solid red squares are the results obtained by numerical integration of the exact result for the spectral density.

For complex valence masses we obtain the valence quark mass dependence for \( \nu = 0 \) and \( m_1 = 0 \)

\[
\hat{\Sigma}_{PQ}(\hat{m}_1 = 0, \hat{m}) = \hat{\Sigma}_Q(\hat{m}) - \hat{\Sigma}_{U,PQ}(\hat{m}_1 = 0, \hat{m})
\]

\[
= 2u\hat{m}e^{-u\hat{m}^2/2}I_0(\hat{m})K_1(\hat{m}^2) + 2/\hat{m}
\]

\[
+ u^2\hat{m}^*\hat{m}^* (I_0(\hat{m}\hat{m}^*)K_0(\hat{m}\hat{m}^*) + I_1(\hat{m}\hat{m}^*)K_1(\hat{m}\hat{m}^*)) + u\hat{m}.
\]

Two three dimensional plots of this result are shown in Figures 4 and 5 for \( \hat{\mu}^2 = 3.84 \) and \( \hat{\mu}^2 = 4 \times 3.84 = 15.36 \), respectively. Note that the \( x \) and \( y \) axis in the second plot have been scaled up by the same factor of 4 as the chemical potential in order that the oscillating area covers approximately the same part of the two figures. We observe that the amplitude of the oscillations increase strongly with increasing values of \( \hat{\mu} \).
VII. CONCLUSIONS

It has been shown that the exact microscopic expression for the dependence of the chiral condensate on the dynamical quark mass can be obtained by integrating the unquenched eigenvalue density over the complex plane. While the eigenvalue density depends strongly on the chemical potential, the chiral condensate does not depend on the chemical potential in the microscopic limit. Within a random matrix framework the cancellations leading to this result have been connected to the orthogonality properties of orthogonal polynomials in the complex plane and the factorization of the $\mu$-dependence of the polynomials in the microscopic limit. The simplifications occur even for finite size random matrices and the derivation is simpler in this case than in the microscopic limit. The proof of the microscopic limit was obtained by taking the microscopic limit of relations derived for finite order polynomials. For completeness, we also have given a derivation of the mass dependence of the chiral condensate using complex contour integrations.

A second effort in this paper was to get a firmer grip on the properties of the spectral density that lead to the $\mu$-independence of the chiral condensate. This was achieved in the strong non-Hermiticity limit for arbitrary $\nu$, where it was found that an integrability property of the spectral density is responsible for the desired cancellations. The interpretation of this property is that

FIG. 4: The real part of the unquenched contribution to the partially quenched condensate for one massless flavor, $\nu = 0$ and $\bar{\mu}^2 = 3.84$ as obtained form the strong non-Hermiticity approximation. As in Figure 2 the $x$ and $y$ axis are scaled by $2N = 96$. Note that the result for $\text{Im}[\hat{m}] = 0$ is the same as the full line in Fig.
FIG. 5: To illustrate how the oscillations of the spectral density develop we show (as above) the real part of the unquenched contribution to the partially quenched condensate for one massless flavor, \( \nu = 0 \), but now \( \tilde{\mu}^2 = 4 \times 3.84 = 15.36 \). The maximal amplitude is about 50 times larger than the scale included in this plot.

The spectral density has been written as a derivative of the (complex) mass dependence of the partially quenched chiral condensate. We have compared the result for the partially quenched chiral condensate with earlier work, and with exact results obtained by means of complex orthogonal polynomials.

A third result of this paper is the proof of a conjectured identity that was used in the analysis of the partition function for one boson. In addition, we have extended this result to arbitrary topological charge.

Finally, we wish to stress that discontinuity of the chiral condensate in the thermodynamic limit is due to oscillating terms in the eigenvalue density rather than an accumulation of eigenvalues on the imaginary axis: the Banks-Casher formula is not valid for QCD at nonzero chemical potential. The nature of the oscillations exemplifies the difficulties encountered by lattice QCD simulations at nonzero chemical potential where strong oscillations result in an exponentially suppressed phase factor \( [25] \) for \( \mu > m_\pi/2 \).

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Appendix A. DIRECT DERIVATION OF THE MICROSCOPIC RESULT

In section II we have manipulated the non-universal finite $N$ expressions into a form which is suitable to take the microscopic limit. In this appendix we show that it is also possible to take the microscopic limit from the start and deal with the universal expressions throughout the argument.

The continuum limit of the sum over the index of the orthogonal polynomials is given by

$$\sum_{k=0}^{N-1} F(k) \to N \int_0^1 2t dt F(Nt^2). \quad (90)$$

The microscopic limit of the orthogonality relation reads

$$\int_C d^2 \hat{z} w(\hat{z}, \hat{z}^*; \hat{\mu}) \hat{z}^{-\nu} I_{\nu}(s \hat{z}^*) \hat{z}^{-\nu} I_{\nu}(t \hat{z}) = 4\pi \hat{\mu}^2 s^{-1} e^{2\hat{\mu}^2 t^2} \delta(s - t), \quad (91)$$

where we used the microscopic limit of the orthogonal polynomials given in (38). The normalization can be verified by means of the large argument asymptotic expansion of the modified Bessel functions.

The microscopic limit of the identity (21) is given by

$$\int_C d^2 \hat{z} w(\hat{z}, \hat{z}^*; \hat{\mu}) \hat{z}^{\nu - \nu} I_{\nu}(\hat{z}^*) \hat{z}^{-\nu} I_{\nu}(\hat{m} t) = 0, \quad (92)$$

and the microscopic limit of the expansion (40) is given by

$$\frac{\hat{z}^{-\nu} \hat{I}_{\nu}(\hat{z}) - \hat{m}^{-\nu} I_{\nu}(\hat{m})}{\hat{z}^2 - \hat{m}^2} = \int_0^1 A(\hat{m}, s) \hat{z}^{-\nu} I_{\nu}(\hat{z} s) ds, \quad (93)$$

with expansion coefficients equal to

$$A(\hat{m}, s) = K_{\nu}(s \hat{m}) I_{\nu}(\hat{m}) - K_{\nu}(\hat{m}) I_{\nu}(s \hat{m}). \quad (94)$$

Using the definition of a derivative we obtain

$$\hat{m}^{-\nu}[I_{\nu}'(\hat{m}) - \frac{\nu}{\hat{m}} I_{\nu}(\hat{m})] = 2\hat{m} \int_0^1 A(\hat{m}, s) \hat{m}^{-\nu} I_{\nu}(s \hat{m}) ds. \quad (95)$$

The microscopic spectral density for $N_f = 1$ is given by (96)

$$\rho_{N_f=1}(\hat{z}, \hat{z}^*, \hat{m}; \hat{\mu}) = \frac{w(\hat{z}, \hat{z}^*; \hat{\mu})}{2\pi \hat{\mu}^2} \int_0^1 dt dt e^{-2\hat{\mu}^2 t^2} \left[ I_{\nu}(\hat{z}^* t) I_{\nu}(\hat{z} t) \frac{I_{\nu}(\hat{z}^* t)}{||z||^{2\nu}} - \frac{I_{\nu}(\hat{z}^* t) I_{\nu}(\hat{m} t)}{||z||^{2\nu}} \frac{I_{\nu}(\hat{z})}{I_{\nu}(\hat{m})} \right]. \quad (96)$$

By adding and subtracting $I_{\nu}(\hat{m} t)$ to $I_{\nu}(\hat{z} t)$ and $I_{\nu}(\hat{m})$ to $I_{\nu}(\hat{z})$ the spectral density can be rewritten as

$$\rho_{N_f=1}(\hat{z}, \hat{z}^*, \hat{m}; \hat{\mu}) = \frac{w(\hat{z}, \hat{z}^*; \hat{\mu})}{2\pi \hat{\mu}^2} \int_0^1 dt dt e^{-2\hat{\mu}^2 t^2} \frac{I_{\nu}(\hat{z}^* t)}{\hat{z}^* \nu} \times \left[ I_{\nu}(\hat{z}^* t) - \hat{m}^{-\nu} I_{\nu}(\hat{m} t) \right] - \left[ I_{\nu}(\hat{z}) - \hat{m}^{-\nu} I_{\nu}(\hat{m}) \right] I_{\nu}(\hat{m}) I_{\nu}(\hat{m}). \quad (97)$$
The chiral condensate is given by

\[
\hat{\Sigma}(\hat{m}) = \int_C d^2 \hat{z} \frac{\hat{m}}{\hat{m}^2 - \hat{z}^2} \hat{\rho}^{(\nu)}_{N_f=1}(\hat{z}, \hat{z}^*; \hat{m}, \hat{\mu}).
\]  

(98)

Because of the expansion (93) and the orthogonality relation (91), the first bracketed term in Eq. (97) does not contribute to the chiral condensate resulting in

\[
\hat{\Sigma}(\hat{m}) = \hat{m} \int_C d^2 \hat{z} w(\hat{z}, \hat{z}^*; \mu) \int_0^1 dt e^{-2\hat{\mu}^2 t^2} \frac{\hat{\rho}^{(\nu)}_{N_f=1}(\hat{z}, \hat{z}^*; \hat{m}, \hat{\mu})}{\hat{z}^2 - \hat{m}^2}. 
\]  

(99)

Finally, after inserting the expansion (93) and applying the orthogonality relation (91), the derivative relation (95) results in

\[
\hat{\Sigma}(\hat{m}) = \frac{I'_{\nu}(\hat{m})}{I_{\nu}(\hat{m})} - \frac{\nu}{\hat{m}},
\]  

(100)

which is the correct \(\mu\)-independent chiral condensate after including the contribution from the zero modes.

Appendix B. CHIRAL CONDENSATE FROM COMPLEX CONTOUR INTEGRATIONS

In this appendix we derive the chiral condensate by integration over the eigenvalue density using complex contour integrations. We only work out the case of \(\nu = 0\).

Let us start by writing the chiral condensate for \(N_f = 1\) as

\[
\hat{\Sigma}(\hat{m}) = \int_0^1 dt \hat{\Sigma}(\hat{m}, t)
\]  

(101)

where

\[
\hat{\Sigma}(\hat{m}, t) = \int_C d^2 \hat{z} \frac{1}{\hat{z}^2 + \hat{z}^*^2} \hat{\rho}^{(0)}_{N_f=1}(\hat{z}, \hat{z}^*; \hat{m}, \hat{\mu}),
\]  

(102)

and

\[
\hat{\rho}^{(0)}_{N_f=1}(\hat{z}, \hat{z}^*; \hat{m}, \hat{\mu}) = \frac{\hat{z} \hat{z}^*}{2\pi \hat{\mu}^2} e^{-(\hat{z}^2 + \hat{z}^*^2)/8\hat{\mu}^2} K_0(|\hat{z}|^2/4\hat{\mu}^2) te^{-2\hat{\mu}^2 t^2} I_0(\hat{z}^* t) I_0(\hat{m} t) \frac{I_0(\hat{m}) - I_0(\hat{\mu})}{I_0(\hat{m})},
\]  

(103)

which follows from (96) after use of (92). Since the chiral condensate is real, there is no need to consider imaginary contributions to \(\hat{\Sigma}(\hat{m}, t)\).

We write \(\hat{z} = x + iy\) and do the integral over \(y\) by a contour integration. Since we perform the integration over \(y\) by a contour integration, we have to distinguish the cases \(x < -\hat{m}\) and \(x > -\hat{m}\) and decompose \(I_0(\hat{z})\) as

\[
I_0(\hat{z}) = \frac{1}{\pi} \sqrt{\hat{z}} (K_0(\hat{z}) - K_0(-\hat{z})).
\]  

(104)
For $I_0(\hat{z}^* t)$, a similar decomposition is used. The pole is at $y = i(x + \hat{m})$ so that

$$\hat{z} \rightarrow -\hat{m} \quad \text{and} \quad \hat{z}^* \rightarrow 2x + \hat{m}. \quad (105)$$

Completing the contour integral we obtain for the pole contribution

$$\hat{\Sigma}(\hat{m}, t) = \frac{-\hat{m} t}{\pi \mu^2} \int dx (2x + \hat{m}) e^{-\frac{2x^2 + \hat{m}(2x + \hat{m})}{4\mu^2} - \frac{\hat{m}^2 t^2}{4}} K_0\left(-\frac{\hat{m}(2x + \hat{m})}{4\mu^2}\right) I_0(\hat{m}t) I_0(\hat{m})$$

$$\times \left[ \theta(x + \hat{m}) [-i \text{sgn}(2x + \hat{m})] K_0((2x + \hat{m})t) I_0(\hat{m}) + iK_0(\hat{m}) I_0((2x + \hat{m})t) \right]$$

$$- \theta(-x - \hat{m}) [i \text{sgn}(2x + \hat{m})] K_0(-(2x + \hat{m})t) I_0(\hat{m}) - iK_0(-\hat{m}) I_0((2x + \hat{m})t)] \quad (106)$$

There is also a contribution from the jump across the cuts along the imaginary axis. However, since these contributions are purely imaginary we do not further analyze them.

To disentangle the different contributions, we rewrite the hyperbolic Bessel functions as

$$K_0(x) = K_0(|x|) - \pi iI_0(|x|). \quad (107)$$

After the cancellation of two terms for $x + \hat{m} < 0$ we observe that the integrand is given by the same expression as for $x + \hat{m} > 0$. Using $y = 2x + \hat{m}$ as new integration variable the expression for $\text{Re} (\hat{\Sigma}(\hat{m}, t))$ simplifies to upon integration,

$$\text{Re} (\hat{\Sigma}(\hat{m}, t)) = \frac{\hat{m} t}{2\mu^2} \frac{I_0(\hat{m}t)}{I_0(\hat{m})} e^{-\frac{\hat{m}^2 t^2}{4\mu^2} - \frac{\hat{m}^2}{8\mu^2}} \int dy ye^{-\frac{y^2}{8\mu^2}}$$

$$\times \left[ \theta(y) I_0\left(\frac{\hat{m}|y|}{4\mu^2}\right) (I_0(\hat{m}) \text{sgn}(y) K_0(|y|) - K_0(\hat{m}) I_0(|y|)) \right.$$

$$\left. + \theta(-y) K_0\left(\frac{\hat{m}|y|}{4\mu^2}\right) I_0(\hat{m}) \text{sgn}(y) I_0(|y|) \right]. \quad (108)$$

The integrals can be simply rewritten as

$$\text{Re} (\hat{\Sigma}(\hat{m}, t)) = \frac{\hat{m} t}{2\mu^2} \frac{I_0(\hat{m}t)}{I_0(\hat{m})} e^{-\frac{\hat{m}^2 t^2}{4\mu^2} - \frac{\hat{m}^2}{8\mu^2}} \int_0^\infty dy ye^{-\frac{y^2}{8\mu^2}}$$

$$\times \left[ I_0\left(\frac{\hat{m}y}{4\mu^2}\right) K_0(yt) + K_0\left(\frac{\hat{m}y}{4\mu^2}\right) I_0(yt) \right] I_0(\hat{m}) - I_0\left(\frac{\hat{m}y}{4\mu^2}\right) I_0(yt) K_0(\hat{m}) \right]. \quad (109)$$

The integral of the terms multiplying $I_0(\hat{m})$ was evaluated in [25] whereas the integral containing the last term in the above equation can be found in [22]. We finally find

$$\text{Re} (\hat{\Sigma}(\hat{m}, t)) = 2\hat{m}t \frac{I_0(\hat{m}t)}{I_0(\hat{m})} [K_0(\hat{m}t) I_0(\hat{m}) - K_0(\hat{m}) I_0(\hat{m}t)] \quad (110)$$

in agreement with results derived in subsection III B 1.
Appendix C. SPECTRAL DENSITY AS A TOTAL DERIVATIVE

In this appendix we obtain the chiral condensate by writing the spectral density as a total derivative. This will be worked out in the strong non-hermiticity limit only, first for \( \nu = \frac{1}{2} \) and then for general \( \nu \).

A. Strong Non-Hermiticity

The microscopic limit of the spectral density (29) for \( N_f = 1 \) is

\[
\rho_{N_f=1}(\hat{z}, \hat{z}^*, \hat{m}; \hat{\mu}) = \frac{|\hat{z}|^2}{2\pi \hat{\mu}^2} K_{\nu} \left( \frac{|\hat{z}|^2}{4\hat{\mu}^2} \right) e^{-\frac{|\hat{z}|^2}{4\hat{\mu}^2}} 
\]

\[
\times \left( \int_0^1 dt \ t e^{-2\hat{\mu}^2 t^2} I_{\nu}(\hat{z}t)I_{\nu}(\hat{z}^*t) - \frac{I_{\nu}(\hat{z})}{I_{\nu}(\hat{m})} \int_0^1 dt \ t e^{-2\hat{\mu}^2 t^2} I_{\nu}(\hat{m}t)I_{\nu}(\hat{z}^*t) \right)
\]

The term with the first integral is the quenched density and the second term gives the effect of unquenching, which motivates the notation

\[
\rho^{(\nu)}_{N_f=1} = \rho^{(\nu)}_Q - \rho^{(\nu)}_U.
\]

It is easily checked that \( \rho_{N_f=1} \) vanishes at \( \hat{z} = \hat{m} \) since \( \rho_Q = \lim_{\hat{m} \to \hat{z}} \rho_U \). This trivial observation which holds since the fermion determinant vanishes when an eigenvalue is equal to the quark mass will be very useful below.

In the limit of strong non-Hermiticity \( \hat{\mu}^2 \gg 1 \) with \( \hat{x}/(2\hat{\mu}^2) < 1 \) and \( (\hat{x} + \hat{m})/(4\hat{\mu}^2) < 1 \) such that the saddle points of the \( t \)-integrations are inside the interval [0, 1], the integrals in (111) can be approximated by

\[
\int_0^1 dt \ t e^{-2\hat{\mu}^2 t^2} I_{\nu}((\hat{x} - i\hat{y})t)I_{\nu}(\hat{m}t) \approx \int_0^\infty dt \ t e^{-2\hat{\mu}^2 t^2} I_{\nu}((\hat{x} - i\hat{y})t)I_{\nu}(\hat{m}t)
\]

\[
= \frac{1}{4\hat{\mu}^2} e^{\frac{((\hat{x} - i\hat{y})^2 + \hat{m}^2)}{\hat{\mu}^2}} I_{\nu} \left( \frac{\hat{m}(\hat{x} - i\hat{y})}{4\hat{\mu}^2} \right).
\]

Within this limit the quenched spectral density for topological charge \( \nu \) reduces to

\[
\rho^{(\nu)}_Q = \frac{2\mu^2}{\pi} \hat{z} \hat{z}^* K_{\nu}(u\hat{z}\hat{z}^*)I_{\nu}(u\hat{z}\hat{z}^*),
\]

while the unquenched part reads

\[
\rho^{(\nu)}_U = \frac{2\mu^2}{\pi} \hat{z} \hat{z}^* K_{\nu}(u\hat{z}\hat{z}^*)I_{\nu}(u\hat{m}\hat{z}^*)I_{\nu}(\hat{z})I_{\nu}(\hat{m})e^{u(m^2 - z^2)/2}
\]

where we introduced the abbreviation \( u \equiv 1/(4\mu^2) \). Below we will express \( \rho \) as a total derivative and show that the resulting chiral condensate is independent of \( \hat{\mu} \). Because the chiral condensate...
is independent of $\hat{\mu}$ we recover the full microscopic chiral condensate even though we work in the limit of strong non-Hermiticity. Before we present the argument for general topological index we first work through the simpler case $\nu = 1/2$.

1. The case of $\nu = \frac{1}{2}$

The microscopic spectral density for $\nu = \frac{1}{2}$ and $N_f = 1$ is given by

$$\rho^{(\nu = \frac{1}{2})}_{N_f = 1}(\hat{z}, \hat{z}^*, \hat{m}; \hat{\mu}) = \rho^{(\nu = \frac{1}{2})}_Q(\hat{z}, \hat{z}^*; \hat{\mu}) - \rho^{(\nu = \frac{1}{2})}_U(\hat{z}, \hat{z}^*, \hat{m}; \hat{\mu})$$

with (using the notation $u \equiv 1/4\hat{\mu}^2$)

$$\rho^{(\frac{1}{2})}_U = \frac{u}{\pi} e^{-\frac{1}{2}\hat{z}^2 - u\hat{z}^* + \frac{u}{2}\hat{m}^2} \frac{e^{\hat{z}} - e^{-\hat{z}}}{e^\hat{m} - e^{-\hat{m}}}(e^{u\hat{m}\hat{z}^*} - e^{-u\hat{m}\hat{z}^*}).$$

and

$$\rho^{(\frac{1}{2})}_Q = \lim_{\hat{m} \to \hat{z}} \rho_U = \frac{u}{\pi} (1 - e^{-2u\hat{z}^*}).$$

It is straightforward to integrate $\rho_U$ with respect to $\hat{z}^*$:

$$\rho^{(\frac{1}{2})}_U = \frac{1}{\pi} \partial_{\hat{z}^*} \left[ \frac{f(\hat{z}, \hat{m})e^{u\hat{z}^*(\hat{m} - \hat{z})} - 1}{\hat{m} - \hat{z}} \right] + \frac{1}{\pi} \partial_{\hat{z}^*} \left[ \frac{f(\hat{z}, \hat{m})e^{-u\hat{z}^*(\hat{m} + \hat{z})} + 1}{\hat{z} + \hat{m}} \right]$$

with

$$f(\hat{z}, \hat{m}) = \frac{e^{\hat{z}} - e^{-\hat{z}}}{e^\hat{m} - e^{-\hat{m}}} e^{u(\hat{m}^2 - \hat{z}^2)}. (120)$$

The limit $\hat{m} \to \hat{z}$ and the derivative with respect to $\hat{z}^*$ can be interchanged for smooth functions. However, taking the limit first can generate singularities that result in spurious contributions after differentiating with respect to $\hat{z}^*$. Taking the limit $\hat{m} \to \hat{z}$ in (119) by naive differentiation leads to

$$\rho^{(\frac{1}{2})}_Q = \frac{1}{\pi} \partial_{\hat{z}^*} \left[ u\hat{z} + u\hat{z}^* - \frac{\cosh \hat{z}^{\ast} \sinh \hat{z}}{\sinh \hat{z}} + \frac{e^{-2u\hat{z}^{\ast}} + 1}{2\hat{z}^{\ast}} \right].$$

To eliminate the spurious singular terms we subtract

$$\tilde{\rho}^{(\frac{1}{2})}_Q = -\frac{1}{\pi} \partial_{\hat{z}^*} \left[ \cosh \hat{z}^{\ast} \sinh \hat{z} - \frac{1}{2\hat{z}^{\ast}} \right]$$

from the quenched spectral density. The chiral condensate can then be expressed as

$$\hat{\Sigma}(\hat{m}) = \int d\hat{x} d\hat{y} \frac{1}{\hat{m} + \hat{z}} (\rho^{(\frac{1}{2})}_Q - \rho^{(\frac{1}{2})}_U).$$
The contribution of the unsubtracted derivatives vanishes at \( \hat{z} = \hat{m} \). After partial integration we thus find
\[
\hat{\Sigma}(\hat{m}) = \frac{\cosh \hat{m}}{\sinh \hat{m}} - \frac{1}{\hat{m}} = \frac{I_{3/2}(\hat{m})}{I_{1/2}(\hat{m})}.
\] (124)

After adding the contribution from the zero modes given by \( \nu / \hat{m} \) we find
\[
\hat{\Sigma}(\hat{m}) = \frac{I_{3/2}(\hat{m})}{I_{1/2}(\hat{m})} + \frac{1}{2\hat{m}} = \frac{I'_{1/2}(\hat{m})}{I_{1/2}(\hat{m})}
\] (125)
in agreement with the result for arbitrary topological charge to be discussed next.

2. The general \( \nu \) case

We now turn to the case with arbitrary \( \nu \). Similar to the case \( \nu = \frac{1}{2} \) above, we express \( \rho^{(\nu)}_U \) as
\[
\rho^{(\nu)}_U = \frac{1}{\pi} \partial_{\hat{z}^*} F(\hat{z}, \hat{z}^*, \hat{m}) \left[ \hat{m} \hat{z}^* uK'_{\nu}(u\hat{m}\hat{z}^*)I'_{\nu}(u\hat{m}\hat{z}^*) - \hat{z}^* uI'_{\nu}(u\hat{m}\hat{z}^*)K''_{\nu}(u\hat{m}\hat{z}^*) \right] - 2\hat{z},
\] (126)
with
\[
F(\hat{z}, \hat{z}^*, \hat{m}) = 2\hat{z} e^{u(\hat{m}^2 - \hat{z}^2)/2} \frac{I_{\nu}(\hat{z})}{I_{\nu}(\hat{m})} [\hat{m} \hat{z}^* uK'_{\nu}(u\hat{m}\hat{z}^*)I'_{\nu}(u\hat{m}\hat{z}^*) - \hat{z}^* uI'_{\nu}(u\hat{m}\hat{z}^*)K''_{\nu}(u\hat{m}\hat{z}^*)] - 2\hat{z}.
\] (127)

Here,
\[
F_A(\hat{z}, \hat{m}) = 2\hat{z} \exp(u(\hat{m}^2 - \hat{z}^2)/2) I_{\nu}(\hat{z})/I_{\nu}(\hat{m}),
\] (128)
and
\[
G(\hat{m} \hat{z}^*, \hat{z}^*) = \hat{m} \hat{z}^* uK'_{\nu}(u\hat{m}\hat{z}^*)I'_{\nu}(u\hat{m}\hat{z}^*) - \hat{z}^* uI'_{\nu}(u\hat{m}\hat{z}^*)K''_{\nu}(u\hat{m}\hat{z}^*). \] (129)
The term \( 2\hat{z} \) has been subtracted to eliminate singularities at \( \hat{z} = \pm \hat{m} \). Using the Wronskian identity
\[
I'_{\nu}(x)K_{\nu}(x) - I_{\nu}(x)K'_{\nu}(x) = \frac{1}{x}
\] (130)
one easily shows that
\[
G(\hat{m} \hat{z}^*, \hat{z}^*) F_A(\hat{z}, \hat{m}) \big|_{\hat{m} = \pm \hat{z}} = 2\hat{z}.
\] (131)
Therefore,

\[ F_A(\tilde{z}, \hat{m})G(\hat{m}\tilde{z}^*, \hat{z}\tilde{z}^*) - 2\tilde{z} = (\hat{m}^2 - \tilde{z}^2)h(\hat{m}, \tilde{z}, \tilde{z}^*) \]  \hspace{1cm} (132)

so that

\[ \partial_{\hat{m}}[F_A G - 2\tilde{z}]_{\tilde{z} = \tilde{z}^* = \hat{m}} = -\partial_{\tilde{z}}[F_A G - 2\tilde{z}]_{\tilde{z} = \tilde{z}^* = \hat{m}} \]  \hspace{1cm} (133)

with \( h(\hat{m}, \tilde{z}, \tilde{z}^*) \) a function that is regular at \( \hat{m} = \tilde{z} \).

The quenched part of the spectral density is given by

\[ \rho_Q^{(\nu)}(\tilde{z}, \tilde{z}^*; \hat{m}, \tilde{m}) = \lim_{\hat{m} \to \tilde{z}} \rho_Q^{(\nu)}(\tilde{z}, \tilde{z}^*, \hat{m}; \tilde{m}). \]  \hspace{1cm} (134)

Naively interchanging this limit with \( \partial_{\tilde{z}^*} \) results in

\[ \rho_Q^{(\nu)}(\tilde{z}, \tilde{z}^*) = \frac{1}{\pi} \left( \partial_{\tilde{z}^*} \frac{F_A(\tilde{z}, \tilde{m})G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)}{2\tilde{z}} + \partial_{\tilde{z}^*} \frac{G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)\partial_{\tilde{m}}F_A(\tilde{z}, \hat{m})}{2\tilde{z}} \right). \]  \hspace{1cm} (135)

However, the second term may give rise to singular contributions that would be absent if we would have differentiated with respect to \( \tilde{z}^* \) before taking the limit. Therefore these contributions have to be subtracted so that the quenched spectral density is given by

\[ \rho_Q^{(\nu)}(\tilde{z}, \tilde{z}^*) = \frac{1}{\pi} \left( \partial_{\tilde{z}} \frac{\partial_{\tilde{m}}[F_A(\tilde{z}, \hat{m})G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)]}{2\tilde{z}} - \partial_{\tilde{z}^*} \text{Sing} \left[ \frac{G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)\partial_{\tilde{m}}F_A(\tilde{z}, \hat{m})}{2\tilde{z}} \right] \right). \]  \hspace{1cm} (136)

The chiral condensate can be written as

\[ \hat{\Sigma}(\hat{m}) = \int d^2\tilde{z} \frac{1}{\hat{m} + \tilde{z}} \frac{1}{\pi \tilde{z}^*} \text{Sing} \frac{\partial_{\tilde{m}}[F_A(\tilde{z}, \hat{m})G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)]}{2\tilde{z}} \]  \hspace{1cm} (137)

\[ -\partial_{\tilde{z}^*} \text{Sing} \left[ \frac{G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)\partial_{\tilde{m}}F_A(\tilde{z}, \hat{m})}{2\tilde{z}} - \frac{F_A(\tilde{z}, \tilde{m})G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)}{2\tilde{z}} \right]. \]

After partial integration with respect to \( \tilde{z}^* \) we find

\[ \hat{\Sigma}(\hat{m}) = -\frac{\partial_{\tilde{m}}[F_A(\tilde{z}, \tilde{m})G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)]}{2\tilde{z}} \]  \hspace{1cm} (138)

\[ -\text{Sing} \left[ \frac{G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)\partial_{\tilde{m}}F_A(\tilde{z}, \hat{m})}{2\tilde{z}} + \frac{\partial_{\tilde{z}}[F_A(\tilde{z}, \tilde{m})G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)]}{2\tilde{z}} \right] \]  \hspace{1cm} \tilde{z} = \tilde{z}^* = -\hat{m}.

After using (133) only the singular terms remain. Because of (132) there is no singularity at \( \tilde{z} = 0 \) after using (133)

\[ \hat{\Sigma}(\hat{m}) = \text{Sing} \left[ \frac{G(\hat{m}\tilde{z}, \hat{z}\tilde{z}^*)\partial_{\tilde{m}}F_A(\tilde{z}, \hat{m})}{2\tilde{z}} \right]_{\tilde{z} = \tilde{z}^* = -\hat{m}} = \frac{I'_\nu(\hat{m})}{I_\nu(\hat{m})} - \frac{\nu}{\hat{m}}, \]  \hspace{1cm} (139)

which gives the correct result after including the contribution from the zero modes.

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