Asymptotic stability of a free boundary problem for the growth of multi-layer tumours in the necrotic phase

Junde Wu

Department of Mathematics, Soochow University, Suzhou, Jiangsu 215006, People’s Republic of China
E-mail: wujund@suda.edu.cn

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Abstract
In this paper we study a free boundary problem for the growth of multi-layer tumours in the necrotic phase. The tumour region is strip-like and divided into a necrotic region and a proliferating region with two free boundaries. The upper free boundary is the tumour surface and is governed by a Stefan condition. The lower free boundary is the interface separating the necrotic region from the proliferating region, and its evolution is implicit and intrinsically governed by an obstacle problem. By the Nash–Moser implicit function theorem we show that the lower free boundary is smoothly dependent on the upper free boundary, and by using analytic semigroups theory we obtain asymptotic stability of the unique flat stationary solution.

Keywords: free boundary problem, asymptotic stability, necrotic tumour
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1. Introduction

In this paper, we study the following free boundary problem modelling multi-layer tumour growth in the necrotic phase:

\[
\begin{align*}
\Delta \sigma &= \sigma \chi \Omega^+(t) \quad \text{in } \Omega^+(t) \cup \Omega^-(t), \ t > 0, \\
\Delta p &= -\mu(\sigma - \bar{\sigma}) \chi \Omega^+(t) + \nu \chi \Omega^-(t) \quad \text{in } \Omega^+(t) \cup \Omega^-(t), \ t > 0, \\
\sigma &= \bar{\sigma}, \ p = \gamma \kappa \quad \text{on } \Gamma^+(t), \ t > 0, \\
\sigma &= \tilde{\sigma}, \ [\partial_n \sigma] = 0 \quad \text{on } \Gamma^-(t), \ t > 0, \\
\hat{[}p\hat{]} &= 0, \ \hat{[}\partial_n p\hat{]} = 0 \quad \text{on } \Gamma^-(t), \ t > 0, \\
\partial_n \sigma &= 0, \ \partial_n p = 0 \quad \text{on } \Gamma_0, \ t > 0, \\
V &= -\partial_n p \quad \text{on } \Gamma^+(t), \ t > 0, \\
\Gamma^+(0) &= \Gamma_0 \\
\end{align*}
\]

where \(\sigma = \sigma(x,y,t)\) and \(p = p(x,y,t)\) are unknown functions representing concentration of nutrients and internal pressure within the tumour, respectively, \(\Omega^+(t)\) and \(\Omega^-(t)\) are unknown domains occupied by tumour proliferating cells and necrotic cells at time \(t > 0\), respectively, and

\[
\Omega^+(t) := \{ (x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : \eta(x,t) < y < \rho(x,t) \}, \\
\Omega^-(t) := \{ (x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < y < \eta(x,t) \},
\]

where \(\eta(x,t)\) and \(\rho(x,t)\) are unknown functions satisfying \(0 < \eta(x,t) < \rho(x,t)\) for \(x \in \mathbb{R}^{n-1}\) and \(t > 0\). \(\Gamma^+(t)\) and \(\Gamma^-(t)\) are free boundaries, and

\[
\Gamma^+(t) := \text{graph}(\rho(x,t)) = \{ (x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = \rho(x,t) \},
\]

\[
\Gamma^-(t) := \text{graph}(\eta(x,t)) = \{ (x,y) \in \mathbb{R}^{n-1} \times \mathbb{R} : y = \eta(x,t) \},
\]

\(\Gamma_0 := \text{graph}(0)\) is the fixed bottom boundary, \(\kappa\) is the mean curvature and \(V\) is the outward normal velocity of the upper tumour surface \(\Gamma^+(t)\), respectively, \(\partial_n\) denotes the outward normal derivative with respect to \(\Omega^+(t)\), \(\bar{\sigma}, \tilde{\sigma}, \sigma, \mu, \nu\) and \(\gamma\) are positive constants, where \(\bar{\sigma}\) represents constant external nutrient supply, \(\sigma\) is a critical value for the balance of cell apoptosis and mitosis, \(\sigma\) is the nutrient level for tumour cell necrosis, \(\mu\) is the proliferation rate of tumour proliferating cells, \(\nu\) is the removal rate of necrotic cells, and \(\gamma\) is the cell-to-cell adhesiveness. We assume that \(0 < \tilde{\sigma} < \sigma < \bar{\sigma}\), \(\chi \Omega^+(t)\) is the characteristic function of \(\Omega^+(t)\), respectively. The notation \(\hat{[}p\hat{]}\) denotes the jump of \(p\) across \(\Gamma^+(t)\), and

\[
\hat{[}p\hat{]} := \mathcal{T} p^+ - \mathcal{T} p^- \quad \text{for } p^+ = p\big|_{\Omega^+(t)} \text{ and } p^- = p\big|_{\Omega^-(t)},
\]

where \(\mathcal{T}\) is the trace operator on \(\Gamma^-(t)\). Similarly, \(\hat{[}\partial_n p\hat{]}\) and \(\hat{[}\partial_n \sigma\hat{]}\) denote the jump of the normal derivatives of \(p\) and \(\sigma\) across \(\Gamma^-(t)\), respectively.

Problem (1.1) is originated from the mathematical model proposed by Byrne and Chaplain [4], for the growth of necrotic tumours in vitro which are cultivated on an impermeable support membrane, and where the tumour cells are multi-layered. The first equation describes the diffusion and consumption of nutrients in the tumour region; the second equation is based on Darcy’s law and the mass conservation law; the third equation means constant nutrient supply and pressure is continuous across the upper tumour surface, by taking cell-to-cell adhesiveness into account; the later three lines of the equations mean that nutrient concentration, pressure and their normal derivatives are continuous across the lower free boundary; nutrient
and tumour cells cannot pass through the bottom boundary, respectively. For more details we refer to [4].

Free boundary problems modelling tumour growth are very interesting, and rigorous analysis of these problems has attracted a lot of attention over the past two decades. Friedman and Reitich [16] first studied a non-necrotic solid tumour spheroid model, where the tumour region is sphere-like and consists of proliferating tumour cells only. They proved that a unique radially symmetric stationary solution exists, and it is globally asymptotically stable under radial perturbations. Friedman and Hu [13] proved that a positive threshold value $\mu^*$ exists, such that the radially symmetric non-necrotic stationary solution is linearly stable for $\mu < \mu^*$, and linearly unstable for $\mu > \mu^*$, by studying the linearized problem. Then by using power series expansions and the fixed point approach they improved this result to obtain nonlinear stability in [14]. By using centre manifold analysis and analytic semigroups theory, Cui and Escher [8] also proved that a positive threshold value $\gamma^*$ exists, such that the radially symmetric non-necrotic stationary solution is asymptotically stable under small non-radial perturbations for $\gamma > \gamma^*$, and unstable for $\gamma < \gamma^*$. The existence of symmetry-breaking stationary solutions was established in [3, 7] by using the Crandall–Rabinowitz theorem. Hopf bifurcation and linear stability of symmetry-breaking stationary solutions were studied in [15]. For a non-necrotic tumour model with Gibbs–Thomson relation, asymptotic stability of radially symmetric stationary solutions and existence of bifurcation stationary solutions were established in [23, 25]. Similar results of the non-necrotic case of the multi-layer tumour model (1.1) were obtained in [9, 26]. For other extended studies of similar non-necrotic tumour spheroid models, we refer to [11, 12, 20, 24] and references cited therein.

We observe that problem (1.1) has two free boundaries. The evolution of the upper free boundary $\Gamma^+(t)$ is governed by the equation $V = -\partial_\rho p$, but the evolution of the lower free boundary $\Gamma^-(t)$ is implicit. This is a remarkable feature and the analysis of problem (1.1) in high dimension is much more difficult than the corresponding non-necrotic tumour model. By the maximum principle, since $0 < \sigma < \hat{\sigma}$, we have $\sigma(x, y, t) \equiv \hat{\sigma}$ in $\Omega^-(t)$, and $\sigma(x, y, t) > \hat{\sigma}$ in $\Omega^+(t)$ at each $t > 0$. Let $\Omega(t) := \{(x, y) \in \mathbb{R}^{n-1} \times \mathbb{R} : 0 < y < \rho(x, t)\}$ for a given function $\rho(x, t)$, we can rewrite $\Omega^+(t) = \{(x, y) \in \Omega(t) : \sigma(x, y, t) > \hat{\sigma}\}$, $\Omega^-(t) = \text{int}\{(x, y) \in \Omega(t) : \sigma(x, y, t) = \hat{\sigma}\}$ and $\Gamma^-(t) = \partial\Omega^+(t) \cap \partial\Omega^-(t)$. We see that $(\sigma(x, y, t), \Gamma^-(t))$ satisfies an obstacle problem:

$$
\begin{cases}
-\Delta \sigma + \sigma \geq 0, & \sigma \geq \hat{\sigma}, \\
(\Delta \sigma + \sigma)(\sigma - \hat{\sigma}) = 0 & \text{in } \Omega(t), \\
\sigma = \hat{\sigma} & \text{on } \Gamma^+(t), \\
\partial_t \sigma = 0 & \text{on } \Gamma_0.
\end{cases}
$$

(1.2)

It is very difficult to study the regularity of the free boundary of obstacle problems in high dimension (see [5]). Even for a smooth domain $\Omega(t)$, the solution $\sigma \notin C^2(\Omega(t))$. It causes a difficulty arising in the necrotic tumour model, which is different from the non-necrotic case. For a similar necrotic tumour spheroid model, bifurcation solutions and linear stability of the radially symmetric stationary solution were formally obtained in [18], and numerical verifications were performed in [19].

In this paper, we give a rigorous analysis of problem (1.1). We first show that there is a unique flat stationary solution (see section 2). It implies that we can consider the obstacle problem (1.2) in a small neighbourhood of the flat stationary solution. Motivated by Cui [6] and Hamilton [17], by using the Nash–Moser implicit function theorem, for given $\Gamma^+(t) = \text{graph}(\rho(x, t))$ close to the flat equilibrium, we prove that the solution $(\sigma(x, y, t), \Gamma^-(t))$ smoothly depends on $\rho(x, t)$, and $\Gamma^-(t)$ is actually smooth in space variables (see lemma 3.2). Then we further solve the first six lines of the equations of problem (1.1) and get the solution $p(x, y, t)$ which also smoothly depends on $\rho(x, t)$. Finally by the second last equation $V = -\partial_\rho p$ we reduce problem (1.1) into an abstract differential equation $\partial_t \rho + \Psi(\rho) = 0$ only containing
function \( \rho(x,t) \). In suitable Banach spaces, we show that this abstract differential equation is of a parabolic type and the local well-posedness follows by the geometric theory of parabolic differential equations. By a delicate analysis and computation, we study the spectrum of the linearized operator at the flat stationary solution, and by the linearized stability principle we can get asymptotic stability of the flat stationary solution.

To give a precise statement of our main results, we introduce some notations.

In this paper, we only consider the case \( n = 2 \), and the higher-dimensional case can be treated similarly. We denote the solution of problem (1.1) by \((\sigma, p, \eta, \rho)\), with \( \Gamma^- (t) = \text{graph}(\eta(x,t)) \) and \( \Gamma^+ (t) = \text{graph}(\rho(x,t)) \). For the sake of simplicity, we impose that

\[
\sigma(x,y,t), \ p(x,y,t), \ \eta(x,t), \ \rho(x,t) \text{ are } 2\pi\text{-periodic in } x \in \mathbb{R}.
\]  

We identify \( \mathbb{S} = \mathbb{R}/2\pi\mathbb{Z} \), and identify continuous 2\( \pi \)-periodic function space \( C_{\text{per}}(\mathbb{R}) = C(\mathbb{S}) \).

Given \( s > 0 \), we denote by \( \text{BUC}^s(\mathbb{S}) \) the space of all bounded and uniformly Hölder continuous functions on \( \mathbb{S} \) of order \( s > 0 \). Let \( h^s(\mathbb{S}) \) denote the little Hölder space, a closure of \( \text{BUC}^\infty(\mathbb{S}) \) in \( \text{BUC}^s(\mathbb{S}) \). Similarly, we denote by \( h^s(\Omega) \) the closure of \( \text{BUC}^\infty(\Omega) \) in \( \text{BUC}^s(\Omega) \) for a bounded open domain \( \Omega \) in \( \mathbb{R}^2 \).

Our first main result is stated as follows:

**Theorem 1.1.** Let \( \bar{\sigma} > \tilde{\sigma} > 0 \) be given. There exists a positive constant \( \sigma_* > \bar{\sigma} \) depending only on \( \tilde{\sigma} \) and \( \bar{\sigma} \), such that the free boundary problem (1.1) has a unique flat stationary solution \( (\sigma_*, p_*, \eta_*, \rho_*) \) if and only if \( \bar{\sigma} > \sigma_* \).

We shall prove this result in section 2. Recall that in the non-necrotic case (see theorem 2 of [9]), there exists a unique non-necrotic flat stationary solution for all \( \bar{\sigma} > \tilde{\sigma} \). It is an interesting difference that the necrotic flat stationary solution does not exist for \( \bar{\sigma} < \tilde{\sigma} \leq \sigma_* \).

Our second main result is about asymptotic stability of the flat stationary solution.

**Theorem 1.2.** Assume \( \bar{\sigma} > \sigma_* \), then there exists a positive threshold value \( \gamma_* \) of the cell-to-cell adhesiveness such that

(i) For \( \gamma > \gamma_* \), the flat stationary solution \( (\sigma_*, p_*, \eta_*, \rho_*) \) is asymptotically stable in the following sense: there exists a constant \( \epsilon > 0 \) such that if \( \rho_0 \in h^{4+\alpha}(\mathbb{S}), ||\rho_0||_{h^{4+\alpha}(\mathbb{S})} < \epsilon \) and \( \Gamma^+_0 = \text{graph}(\rho_0 + \rho_0) \), then the solution \( (\sigma, p, \eta, \rho) \) of problem (1.1) exists for all \( t > 0 \) and converges to \( (\sigma_*, p_*, \eta_*, \rho_*) \) exponentially fast as \( t \to +\infty \).

(ii) For \( 0 < \gamma < \gamma_* \), the flat stationary solution \( (\sigma_*, p_*, \eta_*, \rho_*) \) is unstable.

The above result implies that the cell-to-cell adhesiveness \( \gamma \) plays an important role in the tumour’s stability. A smaller value of \( \gamma \) may make the tumour more aggressive. The threshold value \( \gamma_* \) of the cell-to-cell adhesiveness is given by (4.19) and (4.21), and \( \gamma_* \) can be regarded as a function of the removal rate \( \nu \). By \( d\gamma_*/d\nu \leq 0 \), we see that a smaller value of \( \nu \) may make the tumour less aggressive, while in the limiting case \( \nu = 0 \), the flat stationary solution is not asymptotically stable for all \( \gamma > 0 \) (see remark 5.2).

The structure of the rest of this paper is arranged as follows. In the next section, we study the existence and uniqueness of the flat stationary solution. In section 3, by using the implicit function theorem and classical theory of elliptic equations we reduce the free boundary problem (1.1) into a Cauchy problem in little Hölder spaces, and establish the local well-posedness. Section 4 is devoted to studying the linearized problem at the flat stationary solution and compute eigenvalues. In the last section we make a stability analysis and give a proof of theorem 1.2.
2. Flat stationary solution

In this section, we study the existence and uniqueness of the flat stationary solution of the free boundary problem (1.1).

We denote the flat stationary solution by \((\sigma_s(y), p_s(y), \eta_s, \rho_s)\) with \(0 < \eta_s < \rho_s\). It satisfies the following problem

\[
\begin{aligned}
\sigma_s''(y) &= \sigma_s(y), & p_s''(y) &= -\mu(\sigma_s(y) - \bar{\sigma}) & & \text{for } \eta_s < y < \rho_s, \\
\sigma_s'(y) &= 0, & p_s'(y) &= \nu & & \text{for } 0 < y < \eta_s, \\
\sigma_s(\eta_s) &= \bar{\sigma}, & p_s(\eta_s) &= 0, \\
\sigma_s(\rho_s) &= \hat{\sigma}, & p_s(\rho_s) &= 0, \\
p_s(\eta_s^+) &= p_s(\eta_s^-), & p_s'(\eta_s^+) &= p_s'(\eta_s^-), \\
\sigma_s'(0) &= 0, & p_s'(0) &= 0, & p_s'(\rho_s) &= 0.
\end{aligned}
\]  
(2.1)

We easily get that

\[
\sigma_s(y) = \begin{cases} 
\frac{\sigma \sinh(y - \eta_s) + \hat{\sigma} \sinh(\rho_s - y)}{\sinh(\rho_s - \eta_s)} & \text{for } \eta_s \leq y \leq \rho_s, \\
\hat{\sigma} & \text{for } 0 < y < \eta_s.
\end{cases}
\]  
(2.2)

\[
p_s(y) = \begin{cases} 
\frac{\hat{\sigma}}{\sigma_s^2 - \rho_s^2} (y^2 - \rho_s^2) + (\nu - \mu \sigma_s)(y - \rho_s) \eta_s + \mu (\sigma_s - \sigma_s(y)) & \text{for } \eta_s \leq y \leq \rho_s, \\
\frac{\hat{\sigma}}{\sigma_s^2 - \rho_s^2} (y^2 - \rho_s^2) + p_0 & \text{for } 0 < y < \eta_s.
\end{cases}
\]  
(2.3)

where \(p_0 = \frac{\hat{\sigma}}{\sigma_s^2 - \rho_s^2} (\eta_s^2 - \rho_s^2) + (\nu - \mu \hat{\sigma})(\eta_s - \rho_s) \eta_s + \mu (\hat{\sigma} - \sigma_s).\)

By \(\sigma_s'(\eta_s) = 0\), there holds

\[
cosh(\rho_s - \eta_s) = \frac{\hat{\sigma}}{\sigma_s}.
\]  
(2.4)

Using this formula,

\[
\sigma_s'(\rho_s) = \frac{\hat{\sigma} \cosh(\rho_s - \eta_s) - \sigma_s}{\sinh(\rho_s - \eta_s)} = \sqrt{\sigma_s^2 - \hat{\sigma}^2}.
\]  
(2.5)

By \(p_s'(\rho_s) = 0\) we have \(\mu \sigma_s \rho_s + (\nu - \mu \sigma_s) \eta_s - \mu \sigma_s'(\rho_s) = 0\). It implies that

\[
(\nu - \mu \sigma_s) \eta_s + \mu \sigma_s \rho_s = \mu \sqrt{\sigma_s^2 - \hat{\sigma}^2}.
\]  
(2.6)

Then from (2.4) and (2.6), we obtain

\[
\eta_s = \frac{\mu}{\nu} \left( \sqrt{\sigma_s^2 - \hat{\sigma}^2} - \sigma_s \ln(\sigma_s + \sqrt{\sigma_s^2 - \hat{\sigma}^2}) + \hat{\sigma} \ln \hat{\sigma} \right),
\]  
(2.7)

\[
\rho_s = \eta_s + \ln(\sigma_s + \sqrt{\sigma_s^2 - \hat{\sigma}^2}) - \ln \hat{\sigma}.
\]  
(2.8)

Clearly, \(\rho_s > \eta_s\) if and only if \(0 < \hat{\sigma} < \bar{\sigma}\).

Next, we only need to make sure \(\eta_s > 0\) for \(0 < \hat{\sigma} < \min\{\bar{\sigma}, \sigma_s\}\). Define a function

\[
f(a, r) := \sqrt{r^2 - 1} - a \ln(r + \sqrt{r^2 - 1}) \quad \text{for } r > 1, \ a > 1.
\]

Note that

\[
\partial_r f(a, r) = \frac{r - a}{\sqrt{r^2 - 1}}, \quad f(a, a) < 0 \quad \text{for } r > 1, \ a > 1.
\]
It follows that for any \( a > 1 \), there exists a positive constant \( a_* > a \) such that

\[
\begin{cases}
  < 0, & 1 < r < a_* , \\
  = 0, & r = a_* , \\
  > 0, & r > a_* .
\end{cases}
\]

By (2.7) we see \( \eta_\ell = \frac{\partial \eta_\ell}{\partial r} f(f(\frac{\rho}{\eta}, \frac{s}{\eta}), 0) \). Recall that \( 0 < \sigma < \min\{\bar{\sigma}, \tilde{\sigma}\} \). We immediately obtain that a positive constant \( \sigma_* > \bar{\sigma} \) exists depending only on \( \bar{\sigma} \) and \( \tilde{\sigma} \), such that \( \eta_\ell > 0 \) for \( \sigma > \sigma_* \) and \( \eta_\ell \leq 0 \) for \( \sigma \leq \sigma_* \).

In conclusion, we have

**Theorem 2.1.** Assume \( 0 < \tilde{\sigma} < \sigma \). There exists a positive constant \( \sigma_* \) depending only on \( \bar{\sigma} \) and \( \tilde{\sigma} \), such that for \( \sigma > \sigma_* \), problem (1.1) has a unique flat stationary solution \((\sigma_\ell, \rho_\ell, \eta_\ell, \rho_\ell)\) given by (2.2), (2.3), (2.7) and (2.8). If \( \bar{\sigma} \leq \sigma_* \), problem (1.1) has no flat stationary solution.

It is interesting to compare this result with the existence of the corresponding non-necrotic flat stationary solutions. From theorem 2 of [9], we see that there exists a unique non-necrotic flat stationary solution for \( 0 < \tilde{\sigma} < \sigma \). But in the necrotic case, we see for \( \tilde{\sigma} < \sigma \leq \sigma_* \), the necrotic flat stationary solution does not exist.

### 3. Reduction and well-posedness

In this section, we reduce the free boundary problem (1.1) into a Cauchy problem in little Hölder spaces, and study the local well-posedness.

First, we transform the free boundary problem (1.1) into an equivalent problem on a fixed domain. Later on, we always assume \( 0 < \tilde{\sigma} < \sigma < \sigma_* < \sigma \). By theorem 2.1, problem (1.1) has a unique flat stationary solution \((\sigma_\ell, \rho_\ell, \eta_\ell, \rho_\ell)\). Denote

\[
\begin{align*}
\Omega_{\ell} &= \{(x, y) \in S \times \mathbb{R} : 0 < y < \rho_{\ell}\}, \\
\mathbb{D}_{\ell} &= \{(x, y) \in S \times \mathbb{R} : 0 < y < \eta_{\ell}\}, \\
\Gamma_{\ell} &= S \times \{\rho_{\ell}\}, \\
J_{\ell} &= S \times \{\eta_{\ell}\}, \\
\Gamma_0 &= S \times \{0\}, \\
\mathbb{E}_{\ell} &= \Omega_\ell \setminus \mathbb{D}_{\ell}.
\end{align*}
\]

Let \( r_0 := (\rho_{\ell} - \eta_{\ell})/\delta \), \( \delta \in (0, r_0) \) and \( \alpha \in (0, 1) \), set

\[
\mathcal{O}_\delta := \{\rho \in C^{1+\alpha}(S) : \|\rho\|_{C^{1+\alpha}(S)} < \delta\}. 
\]

For \( \rho, \eta \in \mathcal{O}_\delta \), we denote

\[
\Omega_{\rho, \eta} := \Omega_\rho \setminus \mathbb{D}_{\eta} \quad \text{and} \quad \mathbb{E}_{\eta} := \Omega_\rho \setminus \mathbb{E}_{\eta}.
\]

Choose a function \( \varphi \in C^\infty(S) \) such that

\[
0 \leq \varphi(y) \leq 1, \quad \varphi(y) = \begin{cases} 1, & \text{for } |y| \leq \delta, \\
0, & \text{for } |y| \geq 3\delta, \\
\end{cases} \quad \sup |\varphi'(y)| < 1/\delta.
\]

Given \( \rho \in \mathcal{O}_\delta \), we introduce a mapping

\[
\Phi_\rho : \Omega_{\ell} \to \Omega_{\rho}, \quad (x, y) \to (x, y + \varphi(y - \rho)\rho(x)).
\]
Clearly, $\Phi_{\rho}(\Omega_s) = \Omega_\rho$, $\Phi_{\rho}(\Gamma_s) = \Gamma_\rho$ and $\Phi_{\rho}$ is a $h^{4+\alpha}$ diffeomorphism from $\Omega_s$ onto $\Omega_\rho$. Moreover, for any $\eta \in \mathcal{O}_\eta$, $\Phi_{\rho}$ is the identity mapping on $\mathbb{D}_\eta$. Define the induced push-forward operator $\Phi_{\rho}^\circ$, and pull-back operator $\Phi_{\rho}^{-1}$ by
\[
\Phi_{\rho}^\circ u = u \circ \Phi_{\rho}^{-1} \quad \text{for} \quad u \in C(\Omega_s), \quad \Phi_{\rho}^\circ v = v \circ \Phi_{\rho} \quad \text{for} \quad v \in C(\Omega_\rho).
\]
(3.3)

Next, we introduce the following transformed operators:
\[
\mathcal{A}(\rho) u := \Phi_{\rho}^\circ \Delta(\Phi_{\rho}^\circ u), \quad \mathcal{B}(\rho) u := \langle \nabla(\Phi_{\rho}^\circ u) | \mathbf{n}_\rho \rangle \quad \text{for} \quad u \in H^2(\Omega_\rho),
\]
(3.4)
where $\mathbf{n}_\rho = (-\rho_x, 1)$ is the outward normal on $\Gamma_\rho$, $\langle \cdot, \cdot \rangle$ denotes the Euclidean inner product, and $H^2(\Omega_\rho)$ stands for Sobolev space. By lemma 2.2 of [10], we have
\[
\begin{cases}
\mathcal{A} \in C^\infty(\mathcal{O}_\delta, L(h^{k+2+\alpha}(\Omega_s), h^{k+\alpha}(\Omega_s))), & 0 \leq k \leq 2, \\
\mathcal{B} \in C^\infty(\mathcal{O}_\delta, L(h^{k+1+\alpha}(\Omega_s), h^{k+\alpha}(S))), & 0 \leq k \leq 3.
\end{cases}
\]
(3.5)

Denote by $\mathcal{K}(\rho)$ the transformed mean curvature on $\Gamma_\rho$ and
\[
\mathcal{K}(\rho) = -(1 + \rho_x^2)^{-1} \rho_{xx}.
\]
(3.6)

For some $T > 0$, and a function $\rho \in C([0, T], \mathcal{O}_\delta) \cap C^1([0, T], h^{4+\alpha}(S))$, we identify $\rho(x, t) = \rho(t)(x)$ for $t \in [0, T)$ and $x \in \mathbb{S}$. By an elementary analysis, the outward normal $V$ of the tumour surface $V$ at $(t, x)$ can be given by
\[
V = \rho_t / \sqrt{1 + \rho_x^2}.
\]

Let $\chi_{\mathbb{D}_\eta}$ and $\chi_{\mathbb{S}_\eta}$ be the characteristic functions of $\mathbb{D}_\eta$ and $\mathbb{S}_\eta$, respectively. Rewrite
\[
\Gamma_0^+ = \text{graph}(\rho_s + \rho_0) \quad \text{for some} \quad \rho_0 \in \mathcal{O}_\delta,
\]
and
\[
u(x, y, t) = \Phi_{\rho}^\circ \sigma(x, y, t), \quad v(x, y, t) = \Phi_{\rho}^\circ p(x, y, t).
\]

One can easily check that the free boundary problem (1.1) is transformed into the following problem:
\[
\begin{align*}
\mathcal{A}(\rho) u & = u \chi_{\mathbb{D}_\eta} & \text{in} \ & \Omega_s, \ t > 0, \\
\mathcal{A}(\rho) v & = -\mu(u - \bar{\sigma}) \chi_{\mathbb{D}_\eta} + \nu \chi_{\mathbb{S}_\eta} & \text{in} \ & \Omega_s, \ t > 0, \\
\nu & = \gamma \mathcal{K}(\rho) & \text{on} \ & \Gamma_s, \ t > 0, \\
u & = 0 & \text{on} \ & J_s, \ t > 0, \\
\partial_{\nu} u & = 0 & \text{on} \ & \Gamma_0, \ t > 0, \\
\partial_{\nu} v & = 0 & \text{on} \ & \Gamma_0, \ t > 0, \\
\rho(0) & = \rho_0 & \text{on} \ & \mathbb{S}, \ t = 0.
\end{align*}
\]
(3.7)

By the above transformation, we have

**Lemma 3.1.** A quadruple $(u, v, \eta, \rho)$ is a solution of problem (3.7) if and only if the quadruple $(\sigma, \rho, \eta_0 + \eta, \rho_0 + \rho)$ is a solution of problem (1.1) in the neighbourhood of $(\sigma, \rho, \eta_0, \rho_0)$, with $\sigma = \Phi_{\rho}^\circ u$ and $p = \Phi_{\rho}^\circ v$.

Next we further reduce problem (3.7) into a Cauchy problem in little Hölder spaces for $\rho$ only. Given $\rho \in \mathcal{O}_\delta$, we consider the following problem:
\[
\begin{aligned}
\begin{cases}
A(\rho)u &= u \chi_{E_{\rho}} &\text{in } \Omega, \\
u|_{\Gamma_{\rho}} &= \bar{\sigma}, &\partial_{\nu}u|_{\Gamma_{\rho}} = 0, \\
u &= \bar{\sigma}, &[\partial_{\nu}u] = 0 &\text{on } J_{\eta},
\end{cases}
\end{aligned}
\]

(3.8)

For any \( \eta \in O_{\delta} \), by the maximum principle, \( u \equiv \bar{\sigma} \) in \( D_{\eta} \), and \( u > \bar{\sigma} \) in \( E_{\eta} \). On the other hand, since \( u = \bar{\sigma} \) on \( J_{\eta} \), we have

\[
\partial_{\nu}u = \frac{u_{x}y_{t} - u_{y}}{\sqrt{1 + \eta_{t}^{2}}} = -u_{t} \sqrt{1 + \eta_{t}^{2}} &\text{ on } J_{\eta}.
\]

It implies that \( [\partial_{\nu}u] = 0 \) is equivalent to \( \partial_{\nu}u = 0 \) on \( J_{\eta} \). Hence for problem (3.8), we only need to solve

\[
\begin{aligned}
\begin{cases}
A(\rho)u &= u &\text{in } E_{\eta}, \\
u &= \bar{\sigma} &\text{on } \Gamma_{\sigma}, \\
u &= \bar{\sigma} &\text{on } J_{\eta},
\end{cases}
\end{aligned}
\]

(3.9)

Motivated by Cui [6], based on the Nash–Moscow implicit function theorem, we have the following result:

**Lemma 3.2.** There exists a constant \( \delta_{1} \in (0, r_{0}) \), such that for any \( \rho \in O_{\delta_{r}} \), problem (3.9) has a unique solution \((u, \eta)\) satisfying \( u \in h^{4+\alpha}(E_{\eta}) \) and \( \eta \in C^{\infty}(S) \). Moreover, the mapping \( \rho \mapsto (u, \eta) \) from \( O_{\delta_{r}} \) to \( h^{4+\alpha}(E_{\eta}) \times C^{\infty}(S) \) is smooth.

**Proof.** Denote

\[
\mathbb{H} := \{(x, y) \in S \times \mathbb{R} : \frac{\rho_{s} + \eta_{t}}{2} < y < \rho_{s}\}, \quad \mathbb{K} := \{(x, y) \in S \times \mathbb{R} : \eta_{t} < y < \frac{\rho_{s} + \eta_{t}}{2}\}.
\]

Let \( r_{1} := \min\{(\rho_{s} - \eta_{t})/8, \eta_{t}/8\} \leq r_{0} \) and \( \delta \in (0, r_{1}) \). For any \( m \in \mathbb{N}, m \geq 4 \) and \( \alpha \in (0, 1) \), denote

\[
\tilde{O}_{\delta}^{m+\alpha} := \{ \eta \in h^{m+\alpha}(S) : \| \eta \|_{h^{m+\alpha}(S)} < \delta \}.
\]

For any \( \eta \in \tilde{O}_{\delta}^{m+\alpha} \), we introduce a mapping

\[
\tilde{\Phi}_{\eta} : \mathbb{R}^{2} \to \mathbb{R}^{2}, \quad (x, y) \to (x, y + \varphi(y - \eta_{t})\eta(x)),
\]

where \( \varphi \) is a smooth function given by (3.2). We have \( \tilde{\Phi}_{\eta} \) is a \( h^{m+\alpha} \) diffeomorphism from \( E_{\eta} \) onto \( E_{\eta} \) and \( \tilde{\Phi}_{\eta} \) is the identity mapping on \( \mathbb{H} \). Similarly as (3.3), we can define the push-forward operator \( \tilde{\Phi}_{\eta}^{\ast} \), and the pull-back operator \( \tilde{\Phi}_{\eta}^{\ast} \) induced by \( \tilde{\Phi}_{\eta} \). For any \( \rho \in O_{\delta} \) and \( \eta \in \tilde{O}_{\delta}^{m+\alpha} \), we define an operator

\[
\mathcal{A}(\rho, \eta)v := \tilde{\Phi}_{\eta}^{\ast}A(\rho)(\tilde{\Phi}_{\eta}^{\ast}v) &\text{ for } v \in BUC^{1}(E_{\rho}).
\]

Notice that for any \( \rho \in O_{\delta} \) and \( \eta \in \tilde{O}_{\delta}^{m+\alpha}, \mathcal{A}(\rho) \equiv \Delta \) in \( \mathbb{K} \), we see that \( \mathcal{A}(\rho, \eta) \) is independent of \( \eta \) on \( \mathbb{H} \), and independent of \( \rho \) on \( \mathbb{K} \). Moreover, \( \mathcal{A}(\rho, \eta) \) is uniformly elliptic and by lemma 2.2 in [10], we have
\[ \mathcal{A} \in C^\infty \left( O_\delta \times \tilde{O}_\delta^{m+\alpha}, L(h^{4+\alpha}(E_x)) \cap h^{m+\alpha}(K), h^{2+\alpha}(E_x) \cap h^{m-2+\alpha}(K) \right). \]

Set \( \tilde{u} = u \in \tilde{\Phi}_u \). The first three equations of (3.9) is equivalent to
\[ \mathcal{A}(\rho, \eta)\tilde{u} = \tilde{u} \quad \text{in} \quad E_\sigma, \quad \tilde{u} = \tilde{\sigma} \quad \text{on} \quad \Gamma_s, \quad \partial_j \tilde{u} = 0 \quad \text{on} \quad J_s. \] (3.10)

By well-known regularity theory of second-order elliptic differential equations, problem (3.10) has a unique solution \( \tilde{u} := \tilde{U}(\rho, \eta) \in h^{4+\alpha}(E_x) \), and by lemma 2.3 in [10], for \( m \geq 4 \),
\[ \tilde{U} \in C^\infty \left( O_\delta \times \tilde{O}_\delta^{m+\alpha}, h^{4+\alpha}(E_x) \right). \] (3.11)

Next, we further show some much more profound properties of \( \tilde{U} \). Recall from Part II.1 of Hamilton [17], Banach space \( h^{4+\alpha}(S) \) can be regarded as a tame Fréchet space, \( C^\infty(S) \) with a collection of seminorms \( \{ \| \|_{h^{\alpha}(S)}, m = 0, 1, 2, \cdots \} \) and \( BUC^\infty(K) \) with a collection of seminorms \( \{ \| \|_{BUC^\infty(K)}, m = 0, 1, 2, \cdots \} \) are both tame Fréchet spaces. Denote
\[ \tilde{O}_\delta^{\infty} = \{ \eta \in C^\infty(S) : \| \eta \|_{h^{\alpha}(S)} < \delta \}. \]

By theorem 3.3.5 in part II of [17], we have
\[ \tilde{U} \quad \text{is a smooth tame mapping from} \quad O_\delta \times \tilde{O}_\delta^{\infty} \quad \text{to} \quad h^{4+\alpha}(E_x) \cap BUC^\infty(K), \] (3.12)
which means that for all \( m \in \mathbb{N}, m \geq 4 \),
\[ \tilde{U} \in C^\infty \left( O_\delta \times \tilde{O}_\delta^{m+\alpha}, h^{4+\alpha}(E_x) \cap h^{m+\alpha}(K) \right), \] (3.13)
and
\[ \| \tilde{U}(\rho, \eta) \|_{h^{4+\alpha}(E_x)} + \| \tilde{U}(\rho, \eta) \|_{h^{m+\alpha}(K)} \leq C_m(1 + \| \rho \|_{h^{\alpha}(S)} + \| \eta \|_{h^{\alpha}(S)}). \] (3.14)

where \( C_m \) is a positive constant depending on \( m \).

We define a mapping \( F : O_\delta \times \tilde{O}_\delta^{m+\alpha} \rightarrow h^{m+\alpha}(S) \) by
\[ F(\rho, \eta) = \tilde{U}(\rho, \eta) \Big|_{J_s} - \tilde{\sigma}. \]

It is easy to see that \( F \in C^\infty(O_\delta \times \tilde{O}_\delta^{m+\alpha}, h^{m+\alpha}(S)) \). Moreover, by (3.12) we have
\[ F \quad \text{is a smooth tame mapping from} \quad O_\delta \times \tilde{O}_\delta^{\infty} \quad \text{to} \quad C^\infty(S). \] (3.15)

Clearly, \( F(0, 0) = 0 \) and problem (3.9) is equivalent to the equation \( F(\rho, \eta) = 0 \).

Next we compute the Fréchet derivative of \( F \) with respect to \( \eta \) at \( (\rho, \eta) \in O_\delta \times \tilde{O}_\delta^{\infty} \), which is denoted by \( D_\eta F(\rho, \eta) \). Let \( \tilde{U}(\rho, \eta) \) be the solution of the first three equations of problem (3.9). For any \( \zeta \in C^\infty(S) \), we easily verify that
\[ D_\eta F(\rho, \eta) \zeta = \mathcal{Z}(\rho, \eta, \zeta) \Big|_{J_s}, \]
where \( z = \mathcal{E}(\rho, \eta, \zeta) \) is the solution of the following problem
\[
\mathcal{A}(\rho)z = z \quad \text{in} \quad \Omega_{\eta}, \\
z = 0 \quad \text{on} \quad \Gamma_{\eta}, \\
\partial_n z = -\partial_n \mathcal{U}(\rho, \eta)\zeta \quad \text{on} \quad J_{\eta}.
\]  
(3.16)

Since \( \mathcal{U}(0,0) = \sigma_0|_{\Omega_{\eta}} \), we have \( \partial_n \mathcal{U}(0,0)|_{\Gamma_{\eta}} = \sigma_0''(\eta^+_{\eta}) = \hat{\sigma} > 0 \). Thus for sufficiently small \( \delta > 0 \), we have \( \partial_n \mathcal{U}(\rho, \eta)|_{\Gamma_{\eta}} > \hat{\sigma}/2 \) for \( (\rho, \eta) \in \mathcal{O}_\delta \times \widehat{\mathcal{O}}^\infty_\delta \). By (3.16), for any \( \xi \in C^\infty(S) \) we have
\[
[D_\eta F(\rho, \eta)]^{-1} \xi = -\frac{\partial_\eta \mathcal{T}(\rho, \eta, \xi)}{\partial_\eta \mathcal{U}(\rho, \eta)} |_{J_{\eta}},
\]
where \( z = \mathcal{T}(\rho, \eta, \xi) \) is the solution of the problem
\[
\mathcal{A}(\rho)z = z \quad \text{in} \quad \Omega_{\eta}, \\
z = 0 \quad \text{on} \quad \Gamma_{\eta}, \\
\partial_n z = \xi \quad \text{on} \quad J_{\eta}.
\]

Notice that \( D_\eta F(0,0) \) is an isomorphism from \( h^{m+\alpha}(\Sigma) \) onto \( h^{m+1+\alpha}(\Sigma) \) for all \( m \in \mathbb{N} \), so the classical implicit function theorem in Banach spaces is not available here. But on the other hand, similarly as in (3.15), we can show the mapping \( (\rho, \eta, \xi) \mapsto [D_\eta F(\rho, \eta)]^{-1} \xi \) is smooth tame from \( \mathcal{O}_\delta \times \widehat{\mathcal{O}}^\infty_\delta \times C^\infty(S) \) to \( C^\infty(S) \).

Thus by the Nash–Moser implicit function theorem (see theorem 3.3.1 in part III of [17]), there exist sufficiently small \( \delta_1, \delta_1' \in (0, n_0) \), and a unique smooth tame mapping \( S \) from \( \mathcal{O}_{\delta_1} \) to \( \widehat{\mathcal{O}}^\infty_{\delta_1} \) such that
\[
S(0) = 0 \quad \text{and} \quad F(\rho, S(\rho)) = 0.
\]

By letting \( u = \mathcal{U}(\rho, S(\rho)) \) and \( \eta = S(\rho) \), we see that \( (u, \eta) \) is the solution of problem (3.9), and the mapping \( \rho \mapsto (u, \eta) \) is smooth. The proof is complete. \( \square \)

By the proof of lemma 3.2, for any \( \rho \in \mathcal{O}_{\delta_1} \), problem (3.8) has a unique solution
\[
u
\]
\[
\begin{align*}
u &= \mathcal{U}(\rho, S(\rho)) \quad \text{in} \quad \Omega_{S(\rho)}, \\
\partial_\nu &= 0 \quad \text{on} \quad \Gamma_0, \\
\partial_\eta \nu &= 0 \quad \text{on} \quad J_{S(\rho)}, \\
\end{align*}
\]
Next we consider the following problem
\[
\begin{align*}
\mathcal{A}(\rho)w^+ &= -\mu(u - \hat{\sigma}) \chi_{\Omega_{\eta}} + \nu \chi_{\mathbb{D}_0} \quad \text{in} \quad \Omega_{\eta}, \\
v &= \gamma K(\rho) \quad \text{on} \quad \Gamma_{\eta}, \\
[w] &= 0, \quad [\partial_\nu v] = 0 \quad \text{on} \quad J_{\eta}, \\
\partial_\nu v &= 0 \quad \text{on} \quad \Gamma_0,
\end{align*}
\]
(3.18)

where \( u \) and \( \eta \) are given by (3.17). For the sake of simplicity, we first study
\[
\begin{align*}
\mathcal{A}(\rho)w^+ &= -\mu(\mathcal{U}(\rho, S(\rho)) - \hat{\sigma}) \quad \text{in} \quad \Omega_{S(\rho)}, \\
\mathcal{A}(\rho)w^- &= \nu \quad \text{in} \quad \mathbb{D}_{S(\rho)}, \\
w^+ &= w^- \quad \text{on} \quad \Gamma_{S(\rho)}, \\
\partial_\nu w^+ &= \partial_\nu w^- \quad \text{on} \quad J_{S(\rho)}, \\
\partial_\nu w^- &= 0 \quad \text{on} \quad \Gamma_0.
\end{align*}
\]  
(3.19)
Lemma 3.3. A positive constant $\delta_2 \in (0, \delta_1)$ exists such that for any $\rho \in \mathcal{O}_{\delta_1}$, problem (3.19) has a unique solution $(w^+, w^-) \in h^{k+\alpha}(\mathbb{E}_{\mathcal{S}(\rho)}) \times h^{k+\alpha}(\mathbb{D}_{\mathcal{S}(\rho)})$, and the mapping $\rho \mapsto (w^+, w^-)$ is smooth in $\mathcal{O}_{\delta_1}$.

Proof. For given $\rho \in \mathcal{O}_{\delta_1}$ and $\zeta \in h^{k+\alpha}(\mathbb{S})$, we consider

$$
\begin{cases}
A(\rho)w^+ = -\mu(U(\rho, \mathcal{S}(\rho)) - \tilde{\sigma}) & \text{in } \mathbb{E}_{\mathcal{S}(\rho)}, \\
w^+ = \zeta & \text{on } J_{\mathcal{S}(\rho)}, \\
w^+ = 0 & \text{on } \Gamma_s.
\end{cases}
$$

and

$$
\begin{cases}
A(\rho)w^- = \nu & \text{in } \mathbb{D}_{\mathcal{S}(\rho)}, \\
w^- = \zeta & \text{on } J_{\mathcal{S}(\rho)}, \\
\partial_\nu w^- = 0 & \text{on } \Gamma_0.
\end{cases}
$$

From lemma 3.2, we see $\mathcal{S}(\rho) \in C^\infty(\mathbb{S})$ and $U(\rho, \mathcal{S}(\rho)) \in h^{k+\alpha}(\mathbb{E}_{S(\rho)})$. By classical regularity theory of elliptic differential equations, problem (3.20) has a unique solution $(w^+, w^-)$ such that

$$
w^+ := W^+(\rho, \zeta) \in h^{k+\alpha}(\mathbb{E}_{\mathcal{S}(\rho)}) \quad \text{and} \quad w^- := W^-(\rho, \zeta) \in h^{k+\alpha}(\mathbb{D}_{\mathcal{S}(\rho)}).$$

Since the mappings $\mathcal{S}$ and $U$ are both smooth in $\mathcal{O}_{\delta_1}$, the mappings $W^+$ and $W^-$ are also smooth in $\mathcal{O}_{\delta_1} \times h^{k+\alpha}(\mathbb{S})$.

Recall that $\partial_\nu$ is the outward normal derivative on $J_{\mathcal{S}(\rho)}$ with respect to $\mathbb{E}_{\mathcal{S}(\rho)}$. Define a mapping $G : \mathcal{O}_{\delta_1} \times h^{k+\alpha}(\mathbb{S}) \to h^{k+\alpha}(\mathbb{S})$ by

$$G(\rho, \zeta) = \partial_\nu W^+(\rho, \zeta) \big|_{J_{\mathcal{S}(\rho)}} - \partial_\nu W^-(\rho, \zeta) \big|_{J_{\mathcal{S}(\rho)}} \quad \text{for } \rho \in \mathcal{O}_{\delta_1}, \ \zeta \in h^{k+\alpha}(\mathbb{S}).$$

(3.22)

It is easy to see that problem (3.19) is equivalent to the equation $G(\rho, \zeta) = 0$.

Since $W^+$ and $W^-$ are smooth, we have

$$G \in C^\infty(\mathcal{O}_{\delta_1} \times h^{k+\alpha}(\mathbb{S}), h^{k+\alpha}(\mathbb{S})).$$

(3.23)

By (2.1)–(2.3), we see $G(0, p_0) = 0$, where $p_0 = p_*(\eta_0)$. Note that

$$S(0) = 0, \quad U(0, S(0)) = \sigma_0, \quad W^+(0, p_0) = p_1|_{\mathbb{E}_s}, \quad W^-(0, p_0) = p_1|_{\mathbb{D}_s}.$$

By a direct computation, we have

$$D_\xi G(0, p_0) \xi = -\partial_\nu z^+ \big|_{J_\mathcal{S}} + \partial_\nu z^- \big|_{J_\mathcal{S}} \quad \text{for } \xi \in h^{k+\alpha}(\mathbb{S}),$$

where $z^+$ and $z^-$ are the solutions to the following two problems, respectively,

$$
\begin{cases}
\Delta z^+ = 0 & \text{in } \mathbb{E}_s, \\
z^+ = \xi & \text{on } J_\mathcal{S}, \\
\partial_\nu z^+ = 0 & \text{on } \Gamma_\mathcal{S}.
\end{cases}
$$

and

$$
\begin{cases}
\Delta z^- = 0 & \text{in } \mathbb{D}_s, \\
z^- = \xi & \text{on } J_\mathcal{S}, \\
\partial_\nu z^- = 0 & \text{on } \Gamma_\mathcal{S}.
\end{cases}
$$

(3.24)

For any $\xi \in C^\infty(\mathbb{S})$ with the expression $\xi(x) = \sum_{k \in \mathbb{Z}} \xi_k e^{ikx}$, we obtain

$$D_\xi G(0, p_0) \xi = \sum_{k \in \mathbb{Z}} \tau_k \xi_k e^{ikx},$$

(3.25)

where $\tau_0 = (\rho_* - \eta_0)^{-1}$ and $\tau_k = k(\coth k(\rho_* - \eta_0) + \tanh k\eta_0)$ for $k \neq 0, k \in \mathbb{Z}$. 
Obviously, there exist two positive constants \( C_1 \) and \( C_2 \) such that
\[
C_1 \sqrt{k^2 + 1} \leq \tau_k \leq C_2 \sqrt{k^2 + 1}.
\]
It implies that
\[
D_\xi G(0, p_0) \text{ is an isomorphism from } H^{r+1}(S) \text{ onto } H^r(S) \text{ for } r > 0,
\] (3.26)
where \( H^r(S) = \{ f \in L^2(S) : \sum_{k \in \mathbb{Z}} (k^2 + 1)^r |\hat{f}(k)|^2 < +\infty \}. \)

From (3.25), we easily obtain that for \( \xi \in C^\infty(S) \) with \( \xi(x) = \sum_{k \in \mathbb{Z}} \xi \xi e^{ikx} \),
\[
[D_\xi G(0, p_0)]^{-1} \xi = \sum_{k \in \mathbb{Z}} \tau_k^{-1} \xi \xi e^{ikx}.
\]

Define a function \( \tau(x) = x (\coth(\rho) - \eta) x + \tanh(\eta) x \) for \( |x| \geq 1 \). It is easy to verify that
\[
\tau_k = \tau(k) \quad \text{for } k \neq 0 \quad \text{and} \quad \sup_{|x| \geq 1} |\tau'(x)| + |x| |\tau''(x)| < +\infty.
\]

Using the above relations we can prove that
\[
\begin{cases}
\sup_{k \in \mathbb{Z}} |k| \left| \frac{1}{\tau_k} \right| < +\infty, \\
\sup_{k \in \mathbb{Z}} |k|^2 \left| \frac{1}{\tau_{k+1}} - \frac{1}{\tau_k} \right| < +\infty, \\
\sup_{k \in \mathbb{Z}} |k|^3 \left| \frac{1}{\tau_{k+1}} - \frac{2}{\tau_{k+2}} + \frac{1}{\tau_k} \right| < +\infty.
\end{cases}
\]

Then by theorem 4.5 of [2] (or [22]), we have
\[
[D_\xi G(0, p_0)]^{-1} \in L(C^r(S), C^{r+1}(S)) \quad \text{for } r > 0.
\] (3.27)

By the Sobolev embedding theorem, \( H^{4+r}(S) \rightarrow C^{4+\alpha}(S) \) for \( r > 3/2 \). Notice that \( h^{4+\alpha}(S) \) is the closure of \( H^{4+r}(S) \) in \( C^{4+\alpha}(S) \) for \( r > 3/2 \). By (3.26) and (3.27), we obtain that \( [D_\xi G(0, p_0)]^{-1} \in L(h^{4+\alpha}(S), h^{4+\alpha}(S)) \) and
\[
D_\xi G(0, p_0) \text{ is an isomorphism from } h^{4+\alpha}(S) \text{ onto } h^{4+\alpha}(S).
\]

Hence by the classical implicit function theorem in Banach spaces, there exist sufficiently small constants \( \delta_2, \delta'_2 \in (0, \delta_1) \), and a unique mapping \( R \in C^\infty(O_{\delta_2}, h^{4+\alpha}(S)) \) such that
\[
R(0) = p_0, \quad \|R(\rho) - p_0\|_{h^{4+\alpha}(S)} \leq \delta'_2 \quad \text{and} \quad G(\rho, R(\rho)) = 0.
\]

By letting \( (w^+, w^-) = (W^+(\rho, R(\rho)), W^-(\rho, R(\rho))) \), we see that \( (w^+, w^-) \) is the solution of problem (3.19), and the desired result follows immediately.

By the proof of lemma 3.3, we denote
\[
\mathcal{W}(\rho) = \begin{cases} W^+(\rho, R(\rho)) & \text{in } \mathcal{E}_S(\rho), \\ W^-(\rho, R(\rho)) & \text{in } \mathcal{D}_S(\rho), \end{cases} \quad \text{for } \rho \in O_{\delta}.
\] (3.28)

Consider the problem
\[
A(\rho) v_0 = 0 \quad \text{in } \Omega_\gamma, \quad v_0 = \gamma K(\rho) \quad \text{on } \Gamma_\gamma, \quad \partial_\nu v_0 = 0 \quad \text{on } \Gamma_0.
\] (3.29)
Note that by (3.3), we have
\[ K \in C^\infty(\mathcal{O}_{\delta_2}, h^{2+\alpha}(\mathcal{S})). \] (3.30)

By classical regularity theory of elliptic differential equations, problem (3.29) has a unique solution \( v_0 := \mathcal{V}(\rho) \in h^{2+\alpha}(\Omega_s) \). Moreover, by (3.5), (3.30) and lemma 2.3 in [10],
\[ \mathcal{V} \in C^\infty(\mathcal{O}_{\delta_2}, h^{2+\alpha}(\Omega_s)). \] (3.31)

From (3.28), (3.29) and lemma 3.3, for any \( \rho \in \mathcal{O}_{\delta_2} \), we see problem (3.18) has a unique solution
\[ v = \mathcal{V}(\rho) + \mathcal{W}(\rho). \] (3.32)

Later on, we always fix \( 0 < \delta \leq \delta_2 \). Define a mapping \( \Psi : \mathcal{O}_\delta \to h^{1+\alpha}(\mathcal{S}) \) by
\[ \Psi(\rho) := B(\rho)\mathcal{V}(\rho) + B(\rho)\mathcal{W}(\rho) \quad \text{for} \quad \rho \in \mathcal{O}_\delta. \] (3.33)

It follows from (3.5), (3.31) and lemma 3.3 that
\[ \Psi \in C^\infty(\mathcal{O}_\delta, h^{1+\alpha}(\mathcal{S})). \] (3.34)

With all the above reductions, we see that problem (3.7) is equivalent to the following Cauchy problem
\[ \begin{cases} 
\partial_t \rho + \Psi(\rho) = 0 & \text{on } \mathcal{S}, \ t > 0, \\
\rho(0) = \rho_0 & \text{on } \mathcal{S}.
\end{cases} \] (3.35)

More precisely, we have

**Lemma 3.4.** The function \( \rho \) is the solution of problem (3.35) if and only if \((u, v, \eta, \rho)\) is the solution of problem (3.7) with \((u, v, \eta)\) given by (3.17) and (3.32).

Next we study the local well-posedness of problem (3.35). For any \( \rho \in \mathcal{O}_\delta \), we define the Fréchet derivative of the nonlinear operator \( \Psi \) at \( \rho \) by
\[ D\Psi(\rho)\zeta := \lim_{\varepsilon \to 0} \frac{\Psi(\rho + \varepsilon \zeta) - \Psi(\rho)}{\varepsilon} \quad \text{for} \quad \zeta \in h^{4+\alpha}(\mathcal{S}). \]

Let \( E_0 \) and \( E_1 \) be two Banach spaces; \( E_1 \) is densely and continuously embedded into \( E_0 \). Denote by \( \mathcal{H}(E_1, E_0) \) the subspace of all linear operators \( A \in L(E_1, E_0) \) such that \(-A\) generates a strongly continuous analytic semigroup on \( E_0 \). We have the following result:

**Lemma 3.5.** \( D\Psi(\rho) \in \mathcal{H}(h^{4+\alpha}(\mathcal{S}), h^{1+\alpha}(\mathcal{S})) \) for \( \rho \in \mathcal{O}_\delta \).

**Proof.** Let \( \Psi_1(\rho) := B(\rho)\mathcal{V}(\rho) \) and \( \Psi_2 := B(\rho)\mathcal{W}(\rho) \), then
\[ \Psi(\rho) = \Psi_1(\rho) + \Psi_2(\rho) \quad \text{for} \quad \rho \in \mathcal{O}_\delta. \]

Notice that the following problem is the corresponding transformed periodic Hele-Shaw model with surface tension:
\[ \begin{cases} 
A(\rho)v_0 = 0 & \text{in } \Omega_s, \ t > 0, \\
\partial_\tau v_0 = 0 & \text{on } \Gamma_0, \ t > 0, \\
v_0 = \gamma K(\rho) & \text{on } \Gamma_s, \ t > 0, \\
\rho_t = -B(\rho)v_0 & \text{on } \mathcal{S}, \ t > 0.
\end{cases} \]
and similarly, it can be reduced to \( \partial_t \rho + \Psi_1(\rho) = 0 \) for \( t > 0 \). Thus by the well-known results of Hele-Shaw models (see [10]), we have \( D\Psi_1(\rho) \in H(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})) \), for any \( \rho \in \mathcal{O}_\delta \).

On the other hand, by lemma 3.3, (3.5) and (3.28), we have \( \Psi_2 \in C^\infty(\mathcal{O}_\delta, h^{4+\alpha}(\mathbb{S})) \) and \( D\Psi_2(\rho) \in L(h^{4+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S})) \). Since \( h^{4+\alpha}(\mathbb{S}) \) is compactly embedded into \( h^{1+\alpha}(\mathbb{S}) \), by the well-known perturbation result (see theorem I.1.5.1 in [1], or proposition 2.4.3 in [21]), we get the desired result.

The above result implies that problem (3.35) is of a parabolic type in \( \mathcal{O}_\delta \). Thus by using analytic semigroup theory and applications to parabolic differential problems (see [1] and [21]), we get the local well-posedness.

**Theorem 3.6.** Let \( 0 < \delta \leq \delta_0 \). For any given \( \rho_0 \in \mathcal{O}_\delta \), there exists a maximal \( T > 0 \) such that problem (3.35) has a unique solution \( \rho \in C([0, T), \mathcal{O}_\delta) \cap C^1([0, T), h^{1+\alpha}(\mathbb{S})) \).

By remark III 3.4.2 of Amann [1], the above solution obtained in little Hölder spaces has maximal continuous regularity. For less regularity, we can get similar local well-posedness in Hölder spaces and Besov spaces with some modifications.

From theorem 3.6, and combining lemmas 3.1 and 3.4, we see that the free boundary problem (1.1) is locally well-posed, and given \( \rho_0 \in \mathcal{O}_\delta \), there exists a unique solution \((\sigma, p, \eta, \rho)\) of problem (1.1).

### 4. Linearization and eigenvalues

In this section we study the linearization of problem (3.35) at the stationary solution \( \rho = 0 \), and compute all eigenvalues of \( D\Psi(0) \).

First, we study the linearization of the free boundary problem (1.1) at the flat stationary solution \((\sigma_0, p_0, \eta_0, \rho_0)\). Let

\[
\begin{align*}
\sigma &= \sigma_0 + \epsilon \phi(x, y, t), \quad p = p_0 + \epsilon \psi(x, y, t), \quad \eta = \eta_0 + \epsilon \xi(x, t), \quad \rho = \rho_0 + \epsilon \zeta(x, t),
\end{align*}
\]

(4.1)

where \( \phi, \psi, \xi \) and \( \zeta \) are unknown functions. At each time \( t > 0 \), by (3.6), the mean curvature of the curve \( y = \rho_0 + \epsilon \zeta \) can be expressed by

\[
\mathcal{K}(\epsilon \zeta) = -\epsilon \zeta_{xx} + O(\epsilon^2).
\]

(4.2)

Let \( \mathbf{n}_{\epsilon \zeta} = (-\epsilon \zeta_y, 1) \) be the outward normal direction on \( y = \rho_0 + \epsilon \zeta \). We compute

\[
\begin{align*}
\langle \nabla p, \mathbf{n}_{\epsilon \zeta} \rangle |_{y=\rho_0+\epsilon \zeta} &= (-\epsilon \rho_0 \zeta_x + p_0) |_{y=\rho_0+\epsilon \zeta} - \partial_t (\rho_0 + \epsilon \zeta) |_{y=\rho_0+\epsilon \zeta} + O(\epsilon^2) \\
&= -\epsilon \rho_0 \zeta_x + \epsilon \rho_0 \psi |_{y=\rho_0} + O(\epsilon^2) \\
&= -\epsilon \left[ \mu (\tilde{\sigma} - \tilde{\sigma}) \zeta - \partial_t \psi |_{y=\rho_0} \right] + O(\epsilon^2).
\end{align*}
\]

(4.3)

By substituting (4.1) into problem (1.1), collecting all first order \( \epsilon \)-terms and with the aid of (4.2), (4.3) and the fact that

\[
\begin{align*}
\sigma_x''(\eta_0^+) &= \tilde{\sigma}, & \sigma_x''(\eta_0^-) &= 0, & \sigma_x'(\rho_0) &= \sqrt{\tilde{\sigma}^2 - \tilde{\sigma}^2}, \\
p_x''(\eta_0^+) &= -\mu (\tilde{\sigma} - \tilde{\sigma}), & p_x''(\eta_0^-) &= \nu, & p_x'(\rho_0) &= 0,
\end{align*}
\]
we obtain that the linearization of problem (1.1) at \((\sigma_s, \rho_s, \eta_s, \rho_s)\) is given by

\[
\begin{align*}
\Delta \phi &= \phi \chi_{\Omega_s} & \text{in } \Omega_s, \ t > 0, \\
\Delta \psi &= -\mu \phi \chi_{\Omega_s} & \text{in } \Omega_s, \ t > 0, \\
\phi &= -\sqrt{\alpha^2 - \sigma^2} \zeta, \quad \psi = -\gamma \zeta & \text{on } \Gamma_s, \ t > 0, \\
\partial_t \phi &= 0, \quad \partial_t \psi &= 0 & \text{on } \Gamma_0, \ t > 0, \quad (4.4)
\end{align*}
\]

For any given \(\zeta \in H^{4+\alpha}(\mathbb{S})\), by solving problem (4.4) \(1\)–(4.4)\(6\), we get a unique solution \((\phi, \psi, \xi)\).

Since problem (1.1) is equivalent to problem (3.35), their corresponding linearizations at the flat stationary solution are also equivalent. It implies that

\[
D\Psi(0) \zeta = \partial_t \psi \big|_{y=-\rho_s} - \mu (\bar{\sigma} - \bar{\sigma}) \zeta \quad \text{for } \zeta \in H^{4+\alpha}(\mathbb{S}). \quad (4.5)
\]

Next, we give an explicit expression of \(D\Psi(0)\) and study its eigenvalues. For any given

\[
\zeta(x) = \sum_{k \in \mathbb{Z}} c_k e^{ikx} \in C^\infty(\mathbb{S}), \quad (4.6)
\]

set

\[
\phi(x, y) = \sum_{k \in \mathbb{Z}} a_k(y) e^{ikx}, \quad \psi(x, y) = \sum_{k \in \mathbb{Z}} b_k(y) e^{ikx}, \quad \xi(x) = \sum_{k \in \mathbb{Z}} d_k e^{ikx}, \quad (4.7)
\]

where \(a_k(y)\) and \(b_k(y)\) are unknown functions, \(d_k\) is an unknown coefficient for each \(k \in \mathbb{Z}\).

Substituting (4.6) and (4.7) into (4.4), we see that for each \(k \in \mathbb{Z}\), there holds

\[
\begin{align*}
a_k'' - k^2 a_k &= a_k \quad \text{for } \eta_h < y < \rho_s, \\
a_k(y) &= 0 \quad \text{for } 0 < y \leq \eta_h, \\
\end{align*}
\]

and

\[
\begin{align*}
b_k'' - k^2 b_k &= -\mu a_k \quad \text{for } \eta_h < y < \rho_s, \\
b_k'' - k^2 b_k &= 0 \quad \text{for } 0 < y < \eta, \\
b_k(\eta_h^+) &= b_k(\eta_h^-) + (\mu (\bar{\sigma} - \bar{\sigma}) + \nu) d_k, \\
b_k(\eta_h^-) &= b_k(\eta_h^-), \\
b_k(\rho) &= \gamma k^2 c_k, \\
b_k(0) &= 0. \quad (4.8)
\end{align*}
\]

By solving problem (4.8), we obtain that for each \(k \in \mathbb{Z}\),

\[
a_k(y) = \begin{cases} 
- \frac{\sinh \sqrt{k^2 + 1}(y-\eta_h)}{\sinh \sqrt{k^2 + 1}(\rho_s-\eta_h)} \sqrt{\alpha^2 - \sigma^2} c_k & \text{for } \eta_h \leq y \leq \rho_s, \\
0 & \text{for } 0 < y < \eta_h.
\end{cases} \quad (4.10)
\]
and
\[ d_k = \frac{\sqrt{k^2 + 1} \sqrt{\sigma^2 - \rho^2}}{\sigma \sinh \sqrt{k^2 + 1}(\rho - \eta_k)}. \] (4.11)

Then by solving problem (4.9), an elementary computation shows that for each \( k \neq 0, k \in \mathbb{Z}, \)
\[ b_k(y) = \begin{cases} -\mu a_k(y) + (\gamma k^2 - \mu \sqrt{\sigma^2 - \rho^2}) c_k \frac{\cosh k\rho}{\cosh k\eta_k} + e_k \frac{\sinh k(\rho - \eta_k)}{\sinh k(\rho - \eta_k)}, & \text{for } \eta_k \leq y \leq \rho_k, \\ (\gamma k^2 - \mu \sqrt{\sigma^2 - \rho^2}) c_k \frac{\cosh k\rho}{\cosh k\eta_k} + e_k \frac{\sinh k(\rho - \eta_k)}{\sinh k(\rho - \eta_k)}, & \text{for } 0 < y < \eta_k, \end{cases} \] (4.12)
where
\[ e_k = \frac{(\mu \bar{\sigma} - \nu) d_k}{k[\coth k(\rho - \eta) + \tanh k\eta_k]} \quad \text{for } k \neq 0, k \in \mathbb{Z}. \] (4.13)

By using (2.4) and (4.11), we have \( d_0 = c_0, \) then
\[ b_0(y) = \begin{cases} \mu \sqrt{\sigma^2 - \rho^2} \left( \frac{\sinh(\gamma - \eta_k)}{\sinh(\rho - \eta_k)} - 1 \right) + (\mu \bar{\sigma} - \nu)(\rho_k - y) c_0, & \text{for } \eta_k \leq y \leq \rho_k, \\ -\mu \sqrt{\sigma^2 - \rho^2} + (\mu \bar{\sigma} - \nu)(\rho_k - \eta_k) c_0, & \text{for } 0 < y < \eta_k. \end{cases} \] (4.14)

By (4.10)–(4.13), for \( k \neq 0, \) we compute
\[ b_k'(\rho_k) - \mu(\bar{\sigma} - \bar{\sigma}) c_k = -\mu a_k'(\rho_k) + (\gamma k^2 - \mu \sqrt{\sigma^2 - \rho^2}) c_k \frac{\tanh k\rho_k}{\sinh k(\rho - \eta_k)} - \mu(\bar{\sigma} - \bar{\sigma}) c_k = \lambda_k(\gamma) c_k, \] (4.15)
where
\[ \lambda_k(\gamma) = \gamma k^4 \tanh k\rho_k + \mu \sqrt{\sigma^2 - \rho^2} \left[ \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1}(\rho_k - \eta_k) - k \tanh k\rho_k \right] \] (4.16)
\[ + \frac{(-\mu \bar{\sigma} + \nu) \sqrt{k^2 + 1} \sqrt{\sigma^2 - \rho^2}}{\sigma \sinh k(\rho - \eta_k) \sinh k(\rho - \eta_k)} \left[ \coth k(\rho_k - \eta_k) + \tanh k\eta_k \right] - \mu(\bar{\sigma} - \bar{\sigma}), \]
for \( k \neq 0 \) and \( \gamma > 0. \)

Note that (2.4) implies \( \coth(\rho_k - \eta_k) = \bar{\sigma} / \sqrt{\sigma^2 - \rho^2}. \) Then from (4.14) we compute
\[ b_0'(\rho_k) - \mu(\bar{\sigma} - \bar{\sigma}) c_0 = \mu c_0 \sqrt{\sigma^2 - \rho^2} \coth(\rho_k - \eta_k) - (\mu \bar{\sigma} - \nu) c_0 - \mu(\bar{\sigma} - \bar{\sigma}) c_0 = \lambda_0(\gamma) c_0 - (\mu \bar{\sigma} - \nu) c_0 - \mu(\bar{\sigma} - \bar{\sigma}) c_0 = \nu c_0. \] (4.17)

In conclusion, by (4.5)–(4.7) and (4.15)–(4.17), we have

**Lemma 4.1.** For any \( \zeta \in C^\infty(\mathbb{S}) \) given by \( \zeta = \sum_{k \in \mathbb{Z}} c_k e^{ik}, \) there holds
\[ D\Psi(0) \zeta = \sum_{k \in \mathbb{Z}} \lambda_k(\gamma) c_k e^{ik}, \] (4.18)
where \( \lambda_k(\gamma) \) is given by (4.16) for \( k \neq 0, \) and \( \lambda_0(\gamma) \equiv \nu. \)

Obviously, for each \( k \in \mathbb{Z} \) and \( \gamma > 0, \) \( \lambda_k(\gamma) \) is an eigenvalue of the linearized operator \( D\Psi(0). \) We have the following properties:
Lemma 4.2.

(i) For any \( \gamma > 0 \), \( \lim_{k \to \infty} \lambda_k(\gamma) = +\infty \).

(ii) There exists a constant \( \gamma_* > 0 \), such that if \( \gamma > \gamma_* \), we have \( \lambda_k(\gamma) > 0 \) for all \( k \in \mathbb{Z} \); and if \( 0 < \gamma < \gamma_* \) there exists at least an integer \( k_0 \in \mathbb{Z} \) such that \( \lambda_{k_0}(\gamma) < 0 \).

Proof.

(i) By a direct analysis, we have

\[
\lim_{k \to +\infty} \tanh k \rho_s = \lim_{k \to +\infty} \coth k(\rho_s - \eta_s) = 1,
\]

\[
\lim_{k \to -\infty} \tanh k \rho_s = \lim_{k \to -\infty} \coth k(\rho_s - \eta_s) = -1,
\]

\[
\lim_{k \to \infty} \left( \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1} (\rho_s - \eta_s) - k \tanh k \rho_s \right) = 0.
\]

Hence by (4.16), we immediately obtain \( \lim_{k \to \infty} \lambda(\gamma) = +\infty \) for any \( \gamma > 0 \).

(ii) Define a sequence \( \{ \gamma_k \}_{k \neq 0} \) by

\[
\gamma_k := \frac{1}{k^3 \tanh k \rho_s} \left\{ \mu \sqrt{\sigma^2 - \bar{\sigma}^2} \left[ k \tanh k \rho_s - \sqrt{k^2 + 1} \coth \sqrt{k^2 + 1} (\rho_s - \eta_s) \right] \right. \\
+ \frac{(\mu - \nu) \sqrt{k^2 + 1} \sqrt{\sigma^2 - \bar{\sigma}^2}}{\sigma \sinh k(\rho_s - \eta_s) \sinh \sqrt{k^2 + 1} (\rho_s - \eta_s)} + \mu (\bar{\sigma} - \hat{\sigma}) \left. \right\}. \\
\text{(4.19)}
\]

Clearly, we have

\[
\lim_{k \to \infty} \gamma_k = 0 \quad \text{and} \quad \lim_{k \to \infty} k^3 \tanh k \rho_s \gamma_k = \mu (\bar{\sigma} - \hat{\sigma}) > 0. \\
\text{(4.20)}
\]

Let

\[
\gamma_* := \sup_{k \neq 0} \{ \gamma_k \}. \\
\text{(4.21)}
\]

By (4.20) we see that \( \gamma_* \) is well-defined and \( \gamma_* > 0 \).

By (4.19), we rewrite (4.16) as

\[
\lambda_k(\gamma) = k^3 \tanh k \rho_s (\gamma - \gamma_k) \quad \text{for} \quad k \neq 0, \ k \in \mathbb{Z}. \\
\text{(4.22)}
\]

Then the desired result follows from (4.20) and (4.21). \( \square \)

Corollary 4.3.

(i) If \( \gamma > \gamma_* \), there exists a constant \( \varpi > 0 \) such that

\[
\sigma(D\Psi(0)) \subset \{ \lambda \in \mathbb{C} : \Re \lambda \geq \varpi \}.
\]

(ii) If \( 0 < \gamma < \gamma_* \) then \( \sigma(D\Psi(0)) \cap \{ \lambda \in \mathbb{C} : \Re \lambda < 0 \} \neq \emptyset \).
Proof. Since $D \Psi(0) \in L(h^{1+\alpha}(\mathbb{S}), h^{1+\alpha}(\mathbb{S}))$, and $h^{4+\alpha}(\mathbb{S})$ is compactly embedded into $h^{1+\alpha}(\mathbb{S})$, we see that $\sigma(D \Psi(0))$ consists of all eigenvalues. By lemma 4.1, we easily show that all eigenvalues of the restriction of $D \Psi(0)$ in $h^{4+r}(\mathbb{S})$ are given by $\lambda_k(\gamma)$ for $k \in \mathbb{Z}$, since $h^{4+\alpha}(\mathbb{S})$ is the closure of $H^{4+r}(\mathbb{S})$ in $C^{4+\alpha}(\mathbb{S})$ for $r > 3/2$, we have $\sigma(D \Psi(0)) = \{ \lambda_k(\gamma); k \in \mathbb{Z} \}$.

Let $\gamma > \gamma_*$. By (4.21) and (4.22), we see that
$$\lambda_k(\gamma) > 0$$
for $k \neq 0$. Notice that $\lambda_0(\gamma) > 0$. Take $\omega \in (0, 0, \min\{\tanh(\gamma - \gamma_*), \nu\})$, then $\lambda(\gamma) > \omega$ for all $k \in \mathbb{Z}$. It implies that the assertion (i) holds. The assertion (ii) directly follows from lemma 4.2(ii). The proof is complete.

5. Asymptotic stability

In this section we study the asymptotic stability of the stationary solution $\rho = 0$ of problem (3.35) and give a proof of our main result theorem 1.2.

Since problem (3.35) is of a parabolic type in $h^{1+\alpha}(\mathbb{S})$, by using geometric theory of parabolic equations in Banach spaces, we have

Theorem 5.1.

(i) If $\gamma > \gamma_\ast$, then the stationary solution 0 of problem (3.35) is asymptotically stable. More precisely, there exists a positive constant $\epsilon$ such that for any given $\rho_0 \in \mathcal{O}_3$ with $\|\rho_0\|_{h^{4+\alpha}(\mathbb{S})} < \epsilon$, problem (3.35) has a unique solution $\rho(t) \in C([0, +\infty), \mathcal{O}_3) \cap C^1([0, +\infty), h^{1+\alpha}(\mathbb{S}))$, which converges exponentially fast to 0 as $t \to +\infty$

(ii) If $0 < \gamma < \gamma_\ast$ then the stationary solution 0 is unstable.

Proof.

(i) Let $\gamma > \gamma_*$. Recall that $h^{1+\alpha}(\mathbb{S})$ is densely and compactly embedded into $h^{1+\alpha}(\mathbb{S})$. Set

$$A := D \Psi(0)$$

and

$$G(\rho) : = -\Psi(\rho) + D \Psi(0) \rho \quad \text{for} \quad \rho \in \mathcal{O}_3.$$ (5.1)

Clearly, we have $G(0) = 0$ and $DG(0) = 0$. Problem (3.35) is equivalent to the following problem

$$\rho'(t) = A \rho(t) + G(\rho(t)) \quad \text{for} \quad t > 0, \quad \rho(0) = \rho_0.$$ (5.1)

By lemma 3.5, $A$ generates a strongly continuous analytic semigroup on $h^{1+\alpha}(\mathbb{S})$. By corollary 4.3(i), we have $\sup\{\Re \lambda : \lambda \in \sigma(A)\} < -\omega < 0$. Thus by theorem 9.1.2 of [21], there are positive constants $\omega, \epsilon$ and $M$ such that if the initial value $\rho_0 \in \mathcal{O}_3$ and $\|\rho_0\|_{h^{4+\alpha}(\mathbb{S})} < \epsilon$, then the solution $\rho(t)$ of problem (3.35) exists globally and

$$\|\rho(t)\|_{h^{4+\alpha}(\mathbb{S})} + \|\rho'(t)\|_{h^{4+\alpha}(\mathbb{S})} \leq M e^{-\omega t} \|\rho_0\|_{h^{4+\alpha}(\mathbb{S})} \quad \text{for} \quad t \geq 0.$$ (5.2)

(ii) If $0 < \gamma < \gamma_\ast$ by corollary 4.3(ii) we have $\sigma_+(A) = \sigma(A) \cap \{\lambda \in \mathbb{C} : \Re \lambda > 0\} \neq \emptyset$ and $\inf\{\Re \lambda : \lambda \in \sigma_+(A)\} > 0$. Thus by theorem 9.1.3 in [21], the stationary solution $\rho = 0$ is unstable. The proof is complete.
The proof of theorem 1.2. By lemmas 3.1, 3.4 and theorem 5.1(i), we see that the flat stationary solution $(\sigma_*, p_*, \eta_*, \rho_*)$ is asymptotically stable for $\gamma > \gamma_*$. More precisely, there is a constant $\epsilon > 0$ such that for any $\rho_0 \in \mathcal{O}_3$ satisfying $\|\rho_0\|_{\mathcal{H}^{\alpha+\alpha}(\mathbb{S})} < \epsilon$, problem (1.1) has a unique global solution $(\sigma(t), p(t), \eta(t), \rho(t))$ with the form of

$$
\sigma(t) = \Psi^*_\epsilon(t) u(t), \quad p(t) = \Phi^*_\epsilon(t) v(t), \quad \eta(t) = \eta_0 + S(\rho(t)), \quad \rho(t) = \rho_0 + \rho(t),
$$

where $\rho(t) = \tilde{\rho}(0) + u(t)$, and $(\sigma(t), p(t), \eta(t), \rho(t))$ converges exponentially fast to $(\sigma_*, p_*, \eta_*, \rho_*)$ in $\mathcal{H}^{4+\alpha}(\Omega_{\tilde{\rho}(0)} \mathcal{J}_{S(\tilde{\rho}(0))}) \times \mathcal{H}^{2+\alpha}(\Omega_{\tilde{\rho}(0)} \mathcal{J}_{S(\tilde{\rho}(0))}) \times \mathcal{H}^{4+\alpha}(\mathbb{S}) \times \mathcal{H}^{4+\alpha}(\mathbb{S})$, as time goes to infinity.

Similarly, by lemmas 3.1, 3.4 and theorem 5.1(ii), the flat stationary solution $(\sigma_*, p_*, \eta_*, \rho_*)$ is unstable for $0 < \gamma < \gamma_*$. The proof is complete.

Remark 5.2. From (4.19) and (4.21), we easily obtain $d\gamma_0/d\nu < 0$ for each $k \neq 0, k \in \mathbb{Z}$. Thus we have $d\gamma_0/d\nu < 0$. It implies that the smaller value of $\nu$ may make tumour less aggressive. In the limiting case $\nu = 0$, since $\lambda_0(\gamma) = \nu = 0$, we have $0 \in \sigma(D\Psi(0))$. It implies that the flat stationary solution is not asymptotically stable any more for all $\gamma > 0$.

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ORCID iDs

Junde Wu @ https://orcid.org/0000-0001-9916-7158

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