STOCHASTIC VERSION OF THE
ERDOS-RENYI LIMIT THEOREM

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Abstract
We generalize the Erdős-Rényi limit theorem on the maximum of partial sums of random variables to the case when the number of terms in these sums is randomly distributed. Relations between this limit theorem and the spectral theory of random graphs and random matrices are discussed.

Key words: Erdős-Rényi partial sums; random matrices; random graphs; spectral norm

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1 Introduction

The Erdős-Rényi limit theorem concerns the asymptotic behaviour of the random variables

$$\eta(n, k) = \max_{i=1, \ldots, n-k} S_i(k)/k, \quad S_i(k) = \xi_i + \xi_{i+1} + \cdots + \xi_{i+k}, \quad (1.1)$$

where $\Xi = \{\xi_i\}_{i=1}^\infty$ is a family of independent identically distributed (i.i.d.) random variables determined on the same probability space $\Omega$ and having zero mathematical expectation $E\xi = 0$. It is assumed that the function

$$\phi(\tau) = Ee^{\xi\tau} \quad (1.2)$$

is determined for $\tau \in I_\xi$, where $I_\xi \subseteq \mathbb{R}_+ = (0, +\infty)$.

In [7] it is proved that given $1 < C < \infty$ there exists, with probability 1, a non-random limit

$$\lim_{n \to \infty} \frac{\eta(n, [C \log n])}{\alpha} = \alpha \quad (1.3)$$

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determined by relation
\[ \inf_{\tau \in I} \phi(\tau)e^{-\alpha \tau} = e^{-1/C}. \] (1.4)

In the particular case when \( \xi_i \) are given as Bernoulli random variables
\[ \xi_i = \zeta_i = \begin{cases} 1, & \text{with probability } \frac{1}{2}, \\ -1, & \text{with probability } \frac{1}{2}, \end{cases} \] (1.5)
convergence (1.3) holds with \( C = c \log 2 \), where \( \alpha(c) \) is determined by relation
\[ \frac{1}{c} = 1 - h \left( \frac{1 + \alpha}{2} \right) \] (1.6)
with
\[ h(t) = -t \log_2 t - (1 - t) \log_2 (1 - t), \quad 0 < t < 1. \]
It is easy to see that in this case \( \alpha \) takes values between the mean value \( 0 = \alpha(+\infty) \) and the maximum \( 1 = \alpha(1) \) of random variables \( \zeta_i \). Obviously, one can also determine the limit \( \alpha(c) \) for the values \( c \in (0, 1) \); in this case it is equal to 1.

Further studies give more details about the convergence (1.3); in particular, the convergence in probability was proved in [1] and the estimates with probability 1 were derived for the difference \( \log \frac{\eta(n, k) - \alpha(C)}{k} \). The Erdös-Rényi theorem has found several applications (see e.g. [4, 11]) and various generalizations of it have been considered (random variables indexed by sets, non-i.i.d. random variables, random variables in Banach spaces and others).

One more version of this limit theorem is motivated by the studies of spectra of random matrices [8]. Namely, when regarding the weighted adjacency matrix of a random graph, one observes that the spectral norm of such a matrix is bounded from below by the maximum of the sums \( S_i(k) \) (1.1), where the number of terms \( k \) is distributed at random [8]. Then, in the limit of large dimension of such a sparse random matrix, one faces the problem that can be called the stochastic version of the Erdös-Rényi limit theorem. It is clear that in this direction one can find different generalizations of the Erdös-Rényi theorem. In present paper we give the proof of the results announced previously [9] in the form maximally close to (1.1). We discuss other related settings at the end of the paper.

Let us complete this introduction with expressions of gratitude to Profs. A. Rouault and E. Rio for the interest to this work and valuable discussions.

2 Main result and discussion

Let us consider the family of i.i.d. random variables \( \Lambda = \{\lambda_i\}_{i=1}^\infty \) determined on the same probability space as \( \Xi \), also independent from the family \( \Xi \). These \( \lambda_i \) take values in \( \mathbb{N} \) according to the law \( \Pr\{\lambda = k\} = q(k) \) such that \( \mathbb{E}\lambda = p \). We assume that the function \( \psi_p(t) = \mathbb{E}e^{\lambda t}, \ t \in I_\lambda \subseteq \mathbb{R}_+ \) exists,
\[ \psi_p(t) = e^{p \chi(t)(1 + o(1))} \text{ as } p \to \infty \] (2.1)
and $\chi(t)$ is analytic and satisfies conditions $\chi(t) \geq 0$, $\chi(0) = 0$. It is easy to deduce from (2.1) that

$$\Pr\{\lambda \geq l\} \leq \inf_{t \in I_L} \psi_p(t)e^{-t} = e^{-pf(l/p)(1+o(1))}, \quad p \to \infty, \quad (2.2)$$

where

$$f(y) = \sup_{t \in I_L} [yt - \chi(t)]. \quad (2.3)$$

We assume that $f(y)$ is the steep function, i.e. it takes value $+\infty$ when $y$ goes beyond the domain of definition of $f$. Then by the Gärtner-Ellis theorem (see e.g. [6])

$$\Pr\{\lambda \geq yp\} = e^{-pf(y)(1+o(1))}, \quad y \geq 0, \quad p \to \infty. \quad (2.4)$$

Let us note that $f(y)$ is non-negative, strictly convex monotone function. It attains its minimal value at the point $y' = (E\lambda)/p = 1$.

**Theorem 2.1** Let us consider the sums

$$S_i(\lambda_i) = \xi_i + \xi_{i+1} + \ldots + \xi_{i+\lambda_i}, \quad (2.5)$$

where $\{\xi_i\}$ are as in (1.1), and determine

$$\tilde{\eta}(n,p) = \max_{i=1,...,n} \eta_i(n,p), \quad \eta_i(n,p) = S_i(\lambda_i)/p.$$

There exists with probability 1 a non-random limit

$$\lim_{n \to \infty} \tilde{\eta}(n,C \log n) = \tilde{\alpha}, \quad (2.6)$$

determined by the following relations:

i) If

$$D(\tilde{\alpha}/y) = \max_{\tau \in I_L} \left[ \frac{\tilde{\alpha} \tau}{y} - \log \phi(\tau) \right], \quad (2.7)$$

then $\tilde{\alpha} = \tilde{\alpha}(C)$ is determined by relation

$$\inf_{y \geq 0} \left[ f(y) + yD(\tilde{\alpha}/y) \right] = \frac{1}{C} \quad (2.8)$$

that generalizes (1.4);

ii) In the case of Bernoulli random variables $\xi_i = \zeta_i$ (1.5), convergence (2.6) holds with $\tilde{\alpha} = \tilde{\alpha}(c)$ determined by relation (cf. (1.6))

$$\inf_{y \in (\tilde{\alpha}, +\infty)} \left\{ f(y) + y \left[ 1 - h \left( \frac{1}{2} + \frac{\tilde{\alpha}}{2y} \right) \right] \right\} = \frac{1}{c}, \quad (2.9)$$

where $c = C/\log e^2$.

To compare this theorem with results of [7], let us consider first the case (ii) of Bernoulli random variables. The next simplifying assumption is that $\lambda_i$ have
the Poisson distribution with parameter $p$. This makes (2.5) close to the model arising in the studies of sparse random matrices (see the end of this paper). One can easily derive that in this case $\chi(t) = e^t - 1$, $I_\lambda = R_+$ and

$$f(y) = \begin{cases} 0, & \text{if } y \in (0, 1), \\ y(\log y - 1) + 1, & \text{if } y \in [1, \infty). \end{cases}$$

The function $g_a(y) = y \left[1 - h \left(\frac{1}{2} + \frac{a}{2y}\right)\right]$ is positive and strictly decaying on $(a, +\infty)$; the maximum is attained at $a$ and $g_a(a) = a$, $g_a'(a) = -\infty$. The solution of (2.9) exists for all $c \in (0, +\infty)$ and $\lim_{c \to 0} \alpha(c) = 0$. This coincides with the value of $\alpha(+\infty) = 0$ given by (1.6).

It is not hard to show that $\tilde{\alpha}(c) > \alpha(c)$ for all finite values of $c$. Moreover, (2.9) implies that $\tilde{\alpha}(c)$ infinitely increases as $c \to 0$. This means that

$$\lim_{c \to 0} \tilde{\eta}(n, p) = +\infty,$$

while the corresponding value of $\alpha(c)$, $c \to 0$ remains equal to 1. This is an important difference between the usual and stochastic cases of the Erdős-Rényi limit theorem (see Section 4).

The reason for (2.11) is that in the limit $c \to 0$ the averaging in (2.5) is not sufficient and $\tilde{\eta}(n, p)$ really searches for the maximum of variables $\eta_i$. This is provided by those variables that have almost all $\zeta_i$ equal to 1; since one can see large deviations of the the number of terms in $S_i(\lambda_i)$ with respect to $p$, then one can obtain infinite values of $\tilde{\eta}(n, p)$ (2.11).

Thus, we conclude that the large fluctuations of $q(l)$ in the scale $p$ are responsible for (2.11). This proposition is supported by the following observation. Let us forget the Poisson distribution of $\lambda$ and assume that there exists a finite interval $Y \subset (0, \infty)$ such that $q(p_y) = o(e^{-p})$ for all $y \in \bar{Y} = R \setminus Y$. Then we determine $f(y)$ as $+\infty$ on $\bar{Y}$ and still consider (2.9). In this case $\sup_c \tilde{\alpha}(c)$ is finite. Finally, we observe that if $f(y)$ is close to the Dirac $\delta$-function $\delta(y - 1)$, then $\tilde{\alpha}(c)$ is close to the values $\alpha(c)$ given by (1.5).

Summing up these arguments, we arrive at the conclusion that $\lim_{c \to 0} \tilde{\alpha}(c) = \infty$ provided the fluctuations of $\lambda_i$ around $p$ are sufficiently large.

It the general case of finite but unbounded random variables $\xi_i$, the limit $\tilde{\alpha}(C)$ as $C \to 1$ can be infinite already in the classical case of $\lambda_i \equiv k = |C \log n|$.

3 Proof of Theorem 2.1.

As in [7], we give the proof of the item (ii) concerning the Bernoulli random variables $\zeta_i$ and then describe the changes needed to prove Theorem 2.1 in the general case.

Let us show that for any positive $\varepsilon$

$$\Pr\{\tilde{\eta}(n, c \log_2 n) \geq \tilde{\alpha} + \varepsilon\} = O(n^{-\delta}), \quad n \to \infty,$$  

(3.1)
where \( \delta > 0 \) depends only on \( \varepsilon \). We start with elementary inequality

\[
\Pr\left\{ \sup_{i=1,\ldots,n} \eta_i(n,p) \geq x \right\} \leq \sum_{i=1}^{n} \Pr\{\eta_i(n,p) \geq x\} = n \Pr\{\eta_1(n,p) \geq x\},
\]

where we used the fact that \( \eta_i \) are identically distributed. Observing that \( \{\omega : \eta_1 \geq x\} \subset \{\omega : \lambda \geq px\} \), we can write that

\[
\Pr\{\eta_1(n,p) \geq x\} = \sum_{l \geq px} q(l) \Pr\{S(l) \geq px\}. \tag{3.2}
\]

Using the Stirling formula, one can write that

\[
\frac{1}{2^l} \sum_{(l+px)/2 \leq j \leq l} \binom{l}{j} = \sum_{(l+px)/2 \leq j \leq l} \frac{2^{-j\log_2(j/l)-(l-j)\log_2(l-j/l)}}{\sqrt{2\pi j(1-j/l)-1}}(1+o(1)).
\]

Elementary computation shows that the last sum is estimated by its first term times a constant. Then we obtain inequality

\[
\Pr\{S(l) \geq px\} \leq \frac{U}{\sqrt{l}} 2^{-l[1-h(1/2 + px)]}. \tag{3.3}
\]

If \( x \geq \tilde{\alpha} + \varepsilon \), then there exists \( \delta > 0 \) that

\[
f(y) + y \left[ 1 - h\left(\frac{1}{2} + \frac{x}{2y}\right) \right] = f(y) + g_x(y) > \frac{1+\delta}{c} \tag{3.4}
\]

for all \( y \geq 0 \). It is clear that the minimal value of \( f(y) \) is \( f(1) = 0 \). Since \( f(y) \) is strictly convex and monotone, it is continuous. Let us denote by \( z \) the value such that \( f(y) \leq \delta/(2c) \) for \( 1 \leq y \leq z \). Then for all \( 0 \leq y \leq z \)

\[
cg_x(y) > 1 + \frac{\delta}{2}. \tag{3.5}
\]

Using monotonicity of \( g_x(\cdot) \), we derive from (3.3) inequality

\[
\Pr\{\eta \geq x\} \leq \left( \sum_{px \leq l \leq pz} + \sum_{pz \leq l} \right) \frac{U}{\sqrt{l}} q(l)2^{l[h(1/2 + px)-1]} \leq \frac{U}{\sqrt{l}} \sum_{pz \leq l \leq px} 2^{-pg_x(l/p)} + 2^{-pg_x(z)(1+o(1))} \sum_{l \geq pz} q(l).
\]

Taking into account that \( p = c \log_2 n \), using (3.5) and combination of (2.2) and (3.4), we obtain that

\[
\Pr\{\eta_1(n,p) \geq x\} = O(n^{-1-\delta/4})
\]

because the number of terms in the first sum is of the order \( O(\log_2 n) \). Relation (3.1) is proved.
To prove the almost sure estimate, we follow the scheme of [7]. Let us consider the sequence of random variables

\[ \tilde{\eta}_j \equiv \tilde{\eta}(e^{(j+1)/C} - 1, j). \]

Then (3.1) implies convergence of the series \( \sum_j \Pr\{\tilde{\eta}_j > \tilde{\alpha}\} \). Now, taking into account that \( \tilde{\eta}(n, C \log n) \leq \tilde{\eta}_j \) for all \( n \) such that \( e^{j/C} \leq n \leq e^{(j+1)/C} - 1 \), we obtain relation

\[ \Pr\{\limsup_{n \to \infty} \tilde{\eta}(n, C \log n) \leq \tilde{\alpha}\} = 1. \]

This completes the estimate from above of \( \lim \tilde{\eta}(n, c \log_2 n) \) for the case of Bernoulli random variables.

In the general case one can use inequality (see e.g. [1])

\[ \Pr\{S(l) \geq px\} = (2\pi lb)^{-1/2} e^{-lD(x/y)}, \]

(3.6)

where \( 0 < b \leq b_l \leq B < \infty \), instead of (3.3). The remaining part of the proof repeats the arguments presented above.

Now let us show that \( \Pr\{\max_i \eta_i < \tilde{\alpha} - \varepsilon'\} \) vanishes as \( n \to \infty \). To do this, we take an integer \( m \) and determine the subsets of \( \Omega \)

\[ B_n(m) = \{ \omega \in \Omega : \sup_{i=1,\ldots,n} \lambda_i < m \} \]

The next observation is that the events

\[ A_k(n, m) = \{ \omega : \eta_{km+1}(n, p) \leq x|B_n(m)\} \]

are jointly independent for all \( 0 \leq k \leq n(m) - 1, n(m) = [n/m] \). Thus, we can write that

\[ \Pr\{\sup \eta_i \leq x|B_n(m)\} = \prod_{k=1}^{n(m)} \Pr\{A_k(n, m)\} \]

\[ = \left( \frac{\Pr\{\eta_{1} \leq x \cap B_n(m)\}}{\Pr(B_n(m))} \right)^{n(m)}, \]

(3.7)

where we denoted \( \eta_{1} = \eta_{1}(n, p) \). Regarding elementary relations

\[ \Pr\{F \cap B_n(m)\} \leq 1 - \Pr\{\overline{F} \cap B_n(m)\} \]

and

\[ \Pr\{D \cap B_n(m)\} = \Pr\{D\} - \Pr\{D \cap B_n(m)\} \geq \Pr\{D\} - \Pr\{\overline{B_n(m)}\} \]

with \( F = \{ \omega : \eta_1 \leq x \} \) and \( D = \overline{F} \), we can write that

\[ \Pr\{\eta_1 \leq x \cap B_n(m)\} \leq 1 - \Pr\{\eta_1 > x\} + \Pr\{\overline{B_n(m)}\}. \]

(3.8)
Let us consider $\Pr\{\eta_1 > x\}$. If $x < \hat{\alpha}(c) - \varepsilon$, then there exist $\delta' > 0$ and $z' > 1$ that

$$f(y) + y \left[1 - h \left(\frac{1}{2} + \frac{x}{2y}\right)\right] \geq \frac{1 - \delta'}{c}, \quad \text{for all } y \geq z_1 \quad (3.9)$$

The Stirling formula implies the following inequality inverse to (3.3)

$$\frac{1}{2^l} \sum_{(l+pz)/2 \leq j \leq l} \binom{l}{j} \geq \frac{u}{\sqrt{l}} 2^{-l[1 - \delta'(\frac{1}{2} + \frac{z'}{x})]}.$$  \hspace{1cm} (3.10)

Using this estimate and remembering monotonicity of the function $g_x(\cdot)$ (2.10), we derive from (3.2) relation

$$\Pr\{\eta_1 > x\} \geq n^{-c_2 \left[1 - \delta'(\frac{1}{2} + \frac{z'}{x})\right]} \sum_{y \geq pz_1} q(y).$$

Now (2.4) together with (3.9) imply that

$$\Pr\{\eta_1 > x\} = O(n^{-1 + \delta'}). \quad (3.11)$$

In the general case, one can use (3.6) instead of (3.10) and obtain (3.11).

To estimate $\Pr\{B_n(m)\} \leq n \Pr\{\lambda_i \geq m\}$, we use again (2.2) and observe that if $z''$ is such that $f(z'') \geq 3/C$, then

$$\Pr\{B_n(m)\} = O(n^{-2}), \quad m = pz'', \quad n \to \infty. \quad (3.12)$$

Now we can derive from (3.7), (3.8), (3.11) and (3.12) that

$$\Pr\{\sup_i \eta_i \leq x|B_n(pz'')\} \leq \left(1 - \frac{O(n^{-1 + \delta'})}{1 - O(n^{-2})}\right)^{n/(pz'')} = O(e^{-n^{-\delta'/2}}).$$

Finally, writing inequality

$$\Pr\{\sup_i \eta_i \leq x\} \leq \Pr\{\sup_i \eta_i \leq x|B_n(m)\} \Pr(B_n(m)) + \Pr\{\overline{B_n(m)}\}$$

with $m = pz''$, we get that

$$\Pr\{\tilde{\eta}(n,p) < \hat{\alpha} - \varepsilon'\} = O(n^{-2}).$$

Therefore $\Pr\{\liminf_{n \to \infty} \tilde{\eta}(n,C \log n) \geq \hat{\alpha}\} = 1$. This completes the proof of Theorem 2.1. \hspace{1cm} \square

### 4 Applications to random graphs and random matrices

Let us consider the adjacency matrix $A$ of a simple graph $\Gamma$ with the sets of vertices and edges denoted by $V$ and $E$, respectively. If $|V| = N$ and the vertices are enumerated, then $A$ is an $N \times N$ real symmetric matrix with the entries

$$A_{ij}^{(N)} = \begin{cases} 1, & \text{if the edge } e(i,j) \in E, \\ 0, & \text{if } e(i,j) \notin E, \end{cases} \quad i,j = 1, \ldots, N. \quad (4.1)$$
Often one calls the set of eigenvalues of $A^{(N)}$ the spectrum of $\Gamma$. 

One of the models of random graphs (see e.g. [3]) is determined by the ensemble $\{A^{(N,p)}\}$ of matrices whose entries $\{a_{ij}, i \leq j\}$ are given as a family of jointly independent random variables with distribution

$$a_{ij} = \begin{cases} 1, & \text{with probability } p/N, \\ 0, & \text{with probability } 1 - p/N. \end{cases}$$

Having a random graph $\Gamma^{(N,p)}$, one can ask about the asymptotic behaviour of its spectrum when $N \to \infty$, in particular, what happens with the maximal (minimal) eigenvalue of $A^{(N,p)}$. This question was addressed in [8] in more general setting than (4.1).

Namely, the random matrix ensemble $W^{(N,p)}_{ij} = a_{ij}w_{ij}$ has been studied, where $\{w_{ij}, i \leq j\}$ are jointly independent random variables, also independent from $\{a_{ij}\}$. It is assumed that the probability distribution of $w_{ij}$ has all odd moments zero $m_{2k+1} = 0$ and $m_{2k} \leq k^{(1+\tau)k}$ with $\gamma \geq 0, k \geq 1$. Under these conditions, it was shown that the spectral norm of the matrix $\hat{W}^{(N,p)} = \frac{1}{\sqrt{p}}W^{(N,p)}$ in the limit $N,p \to \infty$ converges with probability 1 to the limits

$$\|\hat{W}^{(N,p)}\| = \begin{cases} 2v, & \text{if } p = O((\log n)^{1+\gamma}), \\ +\infty, & \text{if } p = O((\log n)^{1-\gamma}). \end{cases} \quad (4.2)$$

for any $\gamma > \tau$. Here we denoted $v = \sqrt{\mathbb{E}w_{ij}^2}$, $i,j = 1, \ldots N$.

Slightly modifying computations of [8], one can show that the same convergence (4.2) is valid for the spectral norm of $\|\frac{1}{\sqrt{p}}A^{(N,p)}\|$ with $\gamma > 0$.

To study the limit of $p = C\log N$, one has to carry out more accurate analysis than that of [8]. One of the possible results can be obtained by using the Theorem 2.1. Indeed, one can write inequality

$$\|\hat{W}^{(N,p)}\|^2 \geq \max_{i=1,\ldots,n} \|\hat{W}^{(N,p)}e(i)\|^2 \equiv \max_{i=1,\ldots,n} T_i(N,p),$$

where $e(i)_j = \delta_{ij}$. Observing that

$$T_i(N,p) = \frac{1}{p} \sum_{j=1}^{N} a_{ij}w_{ij}^2 \geq \frac{1}{p} \sum_{j \geq i} a_{ij}w_{ij}^2 \equiv \hat{T}_i(N,p),$$

one faces the same problem as described in Theorem 2.1. Indeed, $p\hat{T}_i(n,p)$ is given by the sum of independent random variables and the number of terms is given by $\hat{\lambda}_i = \sum_{j=i}^{N} a_{ij}$ that approaches the Poisson random variables $\lambda_i$ with parameters $ip/N \leq p$, respectively. Thus $p\hat{T}_i(n,p)$ resembles $S_i(\lambda_i)$ (2.5) with $\xi$ replaced by $\hat{\xi}_j = w_{ij}^2$. Let us denote

$$H(N,p) = \sup_{i=1,\ldots,N} T_i(N,p) \text{ and } \hat{H}(N,p) = \sup_{i=1,\ldots,N} \hat{T}_i(N,p). \quad (4.3)$$

So, the first difference between (2.5) and (4.3) is that

$$\mathbf{E}\hat{\xi}_j = v^2 > 0. \quad (4.4)$$
However, it is easy to check that Theorem 2.1 remains valid in the case of (4.4). Relations (2.6)-(2.8) do not change provided $\phi(1.2)$ is replaced by $\hat{\phi}(\tau) = E e^{\tau \hat{\xi}}$. In this case $\hat{\alpha}(+\infty) = v^2$.

The following proposition is true that
\[
\lim_{N \to \infty} H(N, C \log N) \leq \hat{\alpha}(C),
\]
where $\hat{\alpha}(C)$ is determined by (2.7) and (2.8) in terms of $\hat{\phi}(\tau)$. We put inequality in (4.5) because the parameters of random variables $\hat{\lambda}_i$ are of the order $p$ provided $i \sim 1$ but decrease to zero when $i$ increases up to $N$. This is another difference between $\hat{H}(N, p)$ and $\tilde{\eta}(n, p)$ (2.5).

In this connection, it would be interesting to develop the analogs of the Erdős-Rényi limit theorem for maximums of $\hat{\mathcal{T}}_i$ and of $\mathcal{T}_i$. It is natural to expect that $\lim H(N, C \log N) = \hat{\alpha}(C)$. Of special interest is the study of asymptotic behaviour of $\|\hat{\mathcal{W}}(N, C \log N)\|^2$ also because in the limit $C \to \infty$ it is four times greater than that of $\hat{\alpha}(C)$.

Regarding the adjacency matrix $A^{(N,p)}$, it is shown in [10] that its maximal eigenvalue is closely related with the maximal degree $\Delta$ of a random graph. Since the asymptotic behaviour of $\Delta$ is fairly well studied, this gives an important source of information on the spectra of random graphs. It could be interesting to find the limit of the spectral norm of $A^{(N,p)}$ in dependence on $C$, where $p = C \log N, N \to \infty$.

The behaviour of sums of the type (2.5) is interesting by itself in the following aspect. Assume that the random variables $\lambda_i$ are such that $E \lambda_i = p$ but the second moment of $\lambda_i$ does not exists. Then it is interesting to know does the border $p \sim \log n$ still remain to be the critical one for the maximums $\tilde{\eta}(n, p)$.

References

[1] R.R. Bahadur, R. Ranga Rao, On deviations of the sample mean, Ann. Mathem. Statist. 31 (1960) 1015-1027
[2] B. Bollobás, Random graphs, Acad. Press, New York (1985)
[3] D.M. Cvetković, M. Doob, H. Sachs, Spectra of Graphs: Theory and Applications, Ac. Press: New York (1980)
[4] F. Comets. Erdős-Rényi laws for Gibbs measures, Commun. Math. Phys. 162 (1994) 353-369
[5] P. Deheuvels, L. Devroye, J. Lynch. Exact convergence rate in the limit theorems of Erdős-Rényi and Shepp, Ann. Probab. 14 (1986) 209-223
[6] A.Dembo, O.Zeitouni, Large Deviations Techniques and Applications, Springer-New York (1993), 2d ed.
[7] P. Erdős and A. Rényi, On a new law of large numbers, J. Analyse Math. 23 (1970) 103-111
[8] A. Khorunzhy, Sparse random matrices: spectral edge and statistics of rooted trees, *Adv. Appl. Probab.* 33 (2001) 124-140

[9] A. Khorunzhy, On the stochastic version of the Erdős-Rényi limit theorem. Accepted for publication in: *Paul Erdős and his Mathematics.*, Research communications of the conference in memory of Paul Erdős, Budapest (2000)

[10] M. Krivelevich, B. Sudakov, The largest eigenvalue of sparse random graphs, Preprint xxx.lanl.gov: math.CO/0106064

[11] M. Ruszinkó and P. Vanroose, How an Erdős-Rényi-type search approach gives an explicit code construction of rate 1 for random access with multiplicity feedback. IEEE Trans. Inform. Theory 43 (1997), no. 1, 368–373.