Quantum Field Theory Description of Tunneling in the Integer Quantum Hall Effect

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Abstract

We study the tunneling between two quantum Hall systems, along a quasi one-dimensional interface. A detailed analysis relates microscopic parameters, characterizing the potential barrier, with the effective field theory model for the tunneling. It is shown that the phenomenon of fermion number fractionalization is expected to occur, either localized in conveniently modulated barriers or in the form of free excitations, once lattice effects are taken into account. This opens the experimental possibility of an observation of fractional charges with internal structure, close to the magnetic length scale. The coupling of the system to external gauge fields is performed, leading us to the exact quantization of the Hall conductivity at the interface. The field theory approach is well supported by a numerical diagonalization of the microscopic Hamiltonian.

PACS No: 73.20.Dx, 74.50.+r, 12.90.+b, 11.10.Ef

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I. INTRODUCTION

The relevance of the edges in the quantum Hall effect was stressed several years ago in a seminal paper by Halperin [1]. The basic idea is that the existence of gapless excitations at the edges, which follows from the general principle of gauge invariance, provides a mechanism for the universal character of the quantum Hall effect, or in other words, the presence of plateaus for the Hall conductivity, independently of factors such as the degree of disorder and the sample geometry. After this initial observation, the subject was left untouched for a while until the works by Stone [2] and Wen [3,4], which showed many interesting connections with the quantum field theory of (1+1)-dimensional models, in particular a relation to Kac-Moody algebras [5]. One of the consequences of these works is that they may be useful as a promising tool for the study of tunneling between quantum Hall systems [3]. Such a research program is completely supported by the recent advances in the fabrication of microstructures, which may give a real opportunity for testing ideas originated from (1+1)-dimensional quantum field theory.

Motivated by this possibility, in this paper we will concentrate on the integer quantum Hall effect in the simplest situation, viz, the tunneling along a quasi one-dimensional interface, taking into account that both samples have their first Landau levels completely filled. We will make contact here with the theory of fermion number fractionalization [7–9], which has found some applications in the condensed matter physics of polymers [10,11], superfluid $^3$He [12], and recently suggested to play an analogous role at the interface of quantum Hall systems [13,14]. In fact, as we will show, similarly to the case of polyacetylene [10,11,13,16], lattice effects may induce the presence of fractional charges moving along the interface. It is interesting, however, to observe that in the present situation the excitations have a peculiar internal structure, composed by two “subparticles”, each one localized around one of the edges.

This paper is organized as follows: in section II we present simple arguments which give the effective model of tunneling. This will be useful to show what is to be expected
from a more rigorous analysis, to be seen in further sections. In section III we build the effective model starting from the microscopic definition of an interface. For the sake of a more complete analysis, and to study the conductivity, we also consider the presence of additional external gauge fields. In section IV we explore some physical consequences of the model defined in the previous section, such as charge trapping in specific barriers or via an Aharonov-Bohm effect. The Hall conductivity is found to be quantized, regardless the presence of localized states or disorder at the interface. In section V we study lattice effects in the tunneling, showing that lattice distortions may occur, associated to a gap in the fermion spectrum as well as to fractionally charged excitations at the interface. Distortions are represented by a complex bosonic field, interacting with the fermion system. In section VI we perform a numerical analysis of the problem of charge trapping in a modulated barrier, obtaining a very good agreement between the microscopic system and the effective (1+1)-dimensional field theory. In section VII we conclude our discussion and point directions for future investigations. Finally, in order to make the paper as self-contained as possible, there is an appendix on fermion number fractionalization in field theory, where we show in detail the computation of the fermionic current for the model of tunneling, according to the method of Goldstone and Wilczek [8].

II. HEURISTIC MODEL

We will show how the general form of the effective model of tunneling may be found through phenomenological arguments, relying only on a few basic assumptions [13]. This will provide us with some motivation before taking the more complicated task of a complete microscopic analysis. In this section we assume the absence of two-body interactions and electromagnetic perturbations.

Our problem is to study what happens when we approximate two planar samples, through a common plane, taking into account that both of them have their first Landau levels completely filled. As one sample gets closer to the other, there will be some tunneling
through an approximately one-dimensional interface. A sufficient condition for the presence of tunneling is that we have a certain degree of disorder at the interface and that the distance between the samples be of the order of the magnetic length $\ell$.

If we consider one of the samples as the square $-L/2 \leq x \leq L/2$, $-L + \Lambda \leq y \leq \Lambda$, with $\Lambda \sim \ell$, its edge can be defined by $-\Lambda \leq y \leq \Lambda$. The system is under the influence of a magnetic field $\vec{B} = B\hat{z}$ and its edge may be physically generated via the introduction of an electric field $\vec{E} = E\hat{y}$, which avoids the presence of electrons (at zero temperature) in the region $y \geq \Lambda$. Working in the Landau gauge, $\vec{A} = (-By, 0)$, we have eigenfunctions localized only in the $y$ direction. According to Stone [4] we can define the charge density operator at the edge as

$$ j(x) = \int_{-\Lambda}^{\Lambda} \phi^+(x, y) \phi(x, y) \, dy , \quad (2.1) $$

where $\phi(x, y)$ is the field operator in second quantization, constructed as a sum of only first Landau level states (which is a good approximation to the case of strong magnetic fields). We expect the low energy excitations at the edge to be associated to deformations of the quantum Hall droplet, contained in $-\Lambda \leq y \leq \Lambda$ [2,17]. The Fourier expansion of $j(x)$, as given by (2.1), may be recovered from

$$ H = \int \left[ \psi_R^+(x)(-iv\partial_x) \psi_R(x) + \psi_L^+(x)v\partial_x \psi_L(x) + t\psi_R^+\psi_L + t^*\psi_L^+\psi_R \right] dx , \quad (2.2) $$

a Hamiltonian of “right-going” chiral fermions. In this expression, $v = eE/B$ is the drift velocity of electrons in the sample. From (2.2) it follows that if the sign of $v$ is changed, the electric current is inverted. In this case we would have a system of “left-going” chiral fermions, given by the field $\psi_L(x)$. This opposite situation is exactly what occurs at the edge of the other sample. Therefore, the transference of electrons between the samples will be given by the introduction, in the Hamiltonian, of the tunneling operator $t\psi_R^+\psi_L$ (and its hermitian conjugate), where $t$ is the amplitude for the tunneling of electrons across the interface. We claim, thus, that

$$ H = \int \left[ \psi_R^+(x)(-iv\partial_x) \psi_R + \psi_L^+(x)v\partial_x \psi_L + t\psi_R^+\psi_L + t^*\psi_L^+\psi_R \right] dx , \quad (2.3) $$
is the Hamiltonian which describes the tunneling process. Taking \( t \equiv -(t_1 - it_2) \),

\[
\psi = \begin{pmatrix} \psi_R \\ \psi_L \end{pmatrix},
\]

and using the chiral representation for the \( \gamma \) matrices,

\[
\begin{align*}
\gamma^0 &= \sigma_1 \\
\gamma^1 &= -i\sigma_2 \\
\gamma^5 &= \gamma^0\gamma^1 = \sigma_3,
\end{align*}
\]

where \( \sigma_1, \sigma_2 \) and \( \sigma_3 \) are the Pauli matrices, it is a simple matter to show that we can obtain (2.3) from the following Lagrangian

\[
\mathcal{L} = \bar{\psi}(i\gamma^0\partial_0 + iv\gamma^1\partial_1)\psi + \bar{\psi}(t_1 + it_2\gamma^5)\psi.
\]

The fact that \( t \) may depend on space and time variables, opens interesting experimental possibilities, related to the phenomenon of fermion number fractionalization. We will come back to this point later.

**III. MICROSCOPIC DERIVATION**

In our phenomenological description, given by the Lagrangian (2.6), the basic input is the tunneling amplitude \( t \). A deeper question, therefore, is to ask for a microscopic derivation of (2.3), in which the starting point of analysis is the exact Hamiltonian of the system. Only in this way we would know how to obtain the tunneling amplitude from a specific potential barrier, characterizing the relevant class of microscopic structures for the observation of interesting phenomena.

**A. COMPUTATION IN THE ABSENCE OF EXTERNAL GAUGE FIELDS**

Let us consider a system of independent electrons in the two-dimensional \((x,y)\) plane, under the influence of a magnetic field \( \vec{B} = B\hat{z} \) and confined only in the \( x \) direction by the
strip $|x| < L/2$. We will be interested in the limit $L \to \infty$. A simple model of interface is described by the Hamiltonian

$$H = \frac{1}{2m}[(P_x + eBy)^2 + P_y^2] + eV(x, y).$$  \hspace{1cm} (3.1)

The first term in the Hamiltonian represents an non-interacting two-dimensional electron gas. The second term can be decomposed as the sum of two contributions:

$$eV(x, y) = eV_1(y) + eV_2(x, y),$$  \hspace{1cm} (3.2)

where

$$eV_1(y) = -Ay^2; \quad A > 0$$  \hspace{1cm} (3.3)

$$eV_2(x, y) = g(x) \exp\left(-\frac{y^2}{b^2}\right).$$  \hspace{1cm} (3.4)

The potential $V_1(y)$ is a parabolic barrier, which means, as we will see, that for a certain range of the chemical potential $\mu$ there will be a region $|y| < y_0$ completely free of electrons. By varying $\mu$ we can have $y_0 \in (0, \infty)$. The other piece of $V(x)$, the term $V_2(x, y)$, breaks translation invariance in the $x$ direction, generating a modulated tunneling amplitude along the interface. It is clear that the Hamiltonian (3.1) is unbounded from below, but this does not present any problem in our approach: we could regularize the potential $V_1(y)$ to be well-behaved for $|y| \gg \ell$, which would not change the physics of tunneling at the interface.

When $g(x) = 0$, we have an exactly soluble model. In this case the time independent Schrödinger equation is

$$\left\{\frac{1}{2m} [(P_x + eBy)^2 + P_y^2] - Ay^2\right\} \varphi(x, y) = E\varphi(x, y).$$  \hspace{1cm} (3.5)

Imposing periodic boundary conditions in the $x$ direction, we can look for a solution of the form $\varphi(x, y) = \exp(ik_n x) \xi(y)$, where $k_n = \frac{2\pi n}{L}$, and $n$ is an integer. The equation for $\xi(y)$ is

$$\left\{\frac{1}{2m} [(k_n + eBy)^2 + P_y^2] - Ay^2\right\} \xi(y) = E\xi(y),$$  \hspace{1cm} (3.6)
that is
\[
\left[ \frac{P_y^2}{2m} + \left( \frac{(eB)^2}{2m} - A \right) \cdot \left( y + \left( \frac{(eB)^2}{2m} - A \right)^{-1} \cdot \frac{eB}{2m} k_n \right)^2 + \\
- \frac{1}{4} \frac{(eB)^2}{m^2} \cdot \left( \frac{(eB)^2}{2m} - A \right)^{-1} k_n^2 + \frac{1}{2m} k_n^2 \right] \xi(y) = E \xi(y). \tag{3.7}
\]

Therefore, equation (3.7) represents an one-dimensional harmonic oscillator, with eigenfunctions given by
\[
\xi_{n,p}(y) = \exp \left[ - \frac{1}{2\alpha^2} \left( y - \frac{\alpha}{\ell^2} k_n \right) \right] H_p \left( y - \frac{\alpha}{\ell^2} k_n \right), \tag{3.8}
\]
where
\[
\alpha = \left[ 2m \cdot \left( \frac{(eB)^2}{2m} - A \right) \right]^{-\frac{1}{4}} = \left[ 2m \cdot \left( \frac{1}{2m\ell^4} - A \right) \right]^{-\frac{1}{4}} \tag{3.9}
\]
and $H_p$ is the Hermite polynomial of order $p$. The energies are
\[
E_{n,p} = (p + \frac{1}{2}) \omega_c + \frac{k_n^2}{2m} \left[ 1 - \left( \frac{\alpha}{\ell} \right)^4 \right], \tag{3.10}
\]
where $\omega_c = \frac{1}{m\ell^2}$ is the cyclotron frequency.

If the external magnetic field $B$ is high enough we can limit the Hilbert space to the first Landau level. Therefore, from now on we will restrict the state space to the set of normalized wavefunctions
\[
\varphi_n(x, y) = \varphi_{n,0}(x, y) = \left( \frac{1}{\pi \alpha^2 L^2} \right)^{1/4} \exp \left\{ ik_n x - \frac{1}{2\alpha^2} (y - \frac{\alpha}{\ell^2} k_n)^2 \right\}. \tag{3.11}
\]

The electric current associated with a function $\varphi_n(x, y)$ can be immediately computed:
\[
\langle J_y \rangle = \frac{1}{m} \langle P_y \rangle = 0 \tag{3.12}
\]
\[
\langle J_x \rangle = \frac{1}{m} \langle P_x - \frac{1}{\ell^2} y \rangle = \frac{k_n}{m} \left[ 1 - \left( \frac{\alpha}{\ell} \right)^4 \right]. \tag{3.13}
\]

If we give a look at expression (3.11), we easily recognize an interesting relation between the $k_n$ space and the real $(x, y)$ space. The wavefunction $\varphi_n$ represents a delocalized state in the $x$ direction and a localized state in the $y$ direction. These states are centered in the $y$
direction around \( y_n = \frac{\alpha^4}{\varepsilon} k_n \). From (3.10) and (3.11), one can see that for each value of the energy, there are two states, one centered at \( +y_n \) and the other at \( -y_n \), so that the lesser the energy, the greater \(|y_n|\).

We see from (3.13) that \( n > 0 \) and \( n < 0 \) define, respectively, electrons moving to the right and to the left directions along the \( x \) axis. We consider \( \alpha \gtrsim \ell \), which is equivalent of saying that the wavefunctions are spread in the \( y \) direction within a region of order \( \ell \). This means that two wavefunctions, one with \( n > 0 \) and the other with \( n < 0 \) will have some overlap only if their orbitals are distant from the line \( y = 0 \) by a length of order \( \ell \). It is, thus, enough to consider \( b = \ell \) in the expression for \( V_2(x, y) \), if one wants to produce a modulated tunneling amplitude at the interface.

For each value of the chemical potential \( \mu \) in the first Landau level, there is a coordinate \( y_0 \) in such a way that all the states contained in the region \(|y| < y_0\) will be empty ones. The value of \( y_0 \) is a function of \( \mu \) and enters in our model as a phenomenological parameter. Since \( \alpha \gtrsim \ell \), it is adequate to take \( y_0 \sim \ell \) so that there is some overlap between wavefunctions situated at the opposite sides of the interface.

According to (3.13), the state centered at \( y_0 \) (or \(-y_0\)) defines a value of \( k_n \) (or \( k_{-n} = -k_n \)) given by \( \bar{k} = y_0 \ell^2 / \alpha^4 \). This state has the quantum number \( \bar{n} = \bar{k} L / 2 \pi \). All we need to do from now on is to find a theory for the modes near \( \bar{k} \) and \(-\bar{k} \).

Using the wavefunctions we found for the case of \( g(x) = 0 \), given by (3.11), we may write the second quantized field operator as

\[
\phi = \sum_{n<0} a_n^R \varphi_n(x, y) + \sum_{n>0} a_n^L \varphi_n(x, y) .
\]  

(3.14)

In this representation the Hamiltonian becomes

\[
H_{eff} = \int \int \phi^+ (x, y) H \phi (x, y) \ dy dx ,
\]  

(3.15)

where \( H \) is the first quantized Hamiltonian (3.1).

In order to identify the filled states of our model with a “Dirac sea” in the effective theory, it is necessary to make some redefinitions of the operators \( a_n^R \) and \( a_n^L \). Let us make the following transformation
The equation (3.14) now becomes
\[
\phi = \sum_{n<\bar{n}} a^n_R \varphi_{n-\bar{n}}(x,y) + \sum_{n>-\bar{n}} a^n_L \varphi_{n+\bar{n}}(x,y).
\] (3.18)

In the “large box” limit, the set of modes \( k_n \) becomes dense and we can define the continuum theory through
\[
\sum_n \rightarrow \frac{L}{2\pi} \int dk
\]
\[
a_k \rightarrow \left(\frac{2\pi}{L}\right)^{1/2} a_k , \quad \varphi_k \equiv \varphi_n(L)^{1/2}.
\] (3.20)

We, thus, obtain
\[
\phi = \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{k<\bar{k}} a^n_R \varphi_{k-\bar{k}} dk + \left(\frac{1}{2\pi}\right)^{\frac{1}{2}} \int_{k>-\bar{k}} a^n_L \varphi_{k+\bar{k}} dk.
\] (3.21)

In order to completely identify the filled states with the “Dirac sea”, it is necessary to redefine the energy too, setting to zero the energy of the modes \( \bar{k} \) and \(-\bar{k}\). In this way, we shift (3.10) to
\[
E_k = \frac{(k^2 - \bar{k}^2)}{2m} \left[1 - \left(\frac{\alpha}{\ell}\right)^4\right] - \frac{(k^2 - \bar{k}^2)}{2k} v,
\] (3.22)
\[
v = -\frac{\bar{k}}{m} \left[1 - \left(\frac{\alpha}{\ell}\right)^4\right] > 0.
\] (3.23)

Substituting (3.21) into (3.15), we get
\[
H_{\text{eff}} = \int_{k<\bar{k}} dk E_{k-k} a_k^n R a_k^n R + \int_{k>-\bar{k}} dk E_{k+k} a_k^n L a_k^n L +
+ \frac{1}{2\pi} \int_{k<k'} \int_{k'>-\bar{k}} dk dk' dx \left\{ \exp \left[i(-k + k' + 2\bar{k})x\right] g(x)c_1(k, k') a_k^n R a_{k'}^R + \text{H.c.} \right\} +
+ \frac{1}{2\pi} \int_{k,k'<\bar{k}} dk dk' dx \left\{ \exp \left[i(-k + k')x\right] g(x)c_2(k, k') a_k^n R a_{k'}^R + \right\} +
+ \frac{1}{2\pi} \int_{k,k'>-\bar{k}} dk dk' dx \left\{ \exp \left[i(-k + k')x\right] g(x)c_2(-k, -k') a_k^n L a_{k'}^L + \right\},
\] (3.24)
where
\[ c_1(k, k') = \left( \frac{1}{\pi \alpha^2} \right)^{\frac{1}{2}} \int dy \exp \left\{ -\frac{1}{2\alpha^2} \left[ \left( y - \frac{\alpha^4}{\ell^2} (k - \bar{k}) \right)^2 + \left( y - \frac{\alpha^4}{\ell^2} (k' + \bar{k}) \right)^2 \right] - \frac{y^2}{b^2} \right\} \]

(3.25)

and
\[ c_2(k, k') = \left( \frac{1}{\pi \alpha^2} \right)^{\frac{1}{2}} \int dy \exp \left\{ -\frac{1}{2\alpha^2} \left[ \left( y - \frac{\alpha^4}{\ell^2} (k - \bar{k}) \right)^2 + \left( y - \frac{\alpha^4}{\ell^2} (k' - \bar{k}) \right)^2 \right] - \frac{y^2}{b^2} \right\} \]

(3.26)

The above gaussian integrals can be computed exactly. We have
\[ c_1(k, k') = \left( \frac{b^2}{\alpha^2 + b^2} \right)^{\frac{1}{2}} \exp \left\{ \frac{\alpha^6}{2\ell^4} \left[ \frac{b^2}{2(\alpha^2 + b^2)} (k + k')^2 - ((k - \bar{k})^2 + (k' + \bar{k})^2) \right] \right\} \]

(3.27)

\[ c_2(k, k') = \left( \frac{b^2}{\alpha^2 + b^2} \right)^{\frac{1}{2}} \exp \left\{ \frac{\alpha^6}{2\ell^4} \left[ \frac{b^2}{2(\alpha^2 + b^2)} (k + k' - 2\bar{k})^2 - ((k - \bar{k})^2 + (k' - \bar{k})^2) \right] \right\} \]

(3.28)

As we defined before, \( b \simeq \alpha \simeq \ell \), so that we can write
\[ c_1(k, k') \simeq \left( \frac{1}{2} \right)^{\frac{1}{2}} \exp \left\{ \frac{\ell^2}{2} \left[ \frac{1}{4} (k + k')^2 - ((k - \bar{k})^2 + (k' + \bar{k})^2) \right] \right\} \]

(3.29)

\[ c_2(k, k') \simeq \left( \frac{1}{2} \right)^{\frac{1}{2}} \exp \left\{ \frac{\ell^2}{2} \left[ \frac{1}{4} (k + k' - 2\bar{k})^2 - ((k - \bar{k})^2 + (k' - \bar{k})^2) \right] \right\} \]

(3.30)

The existence of tunneling depends essentially on the function \( g(x) \). If, for example \( g(x) = \text{constant} \), the term in (3.24) representing the tunneling between electrons \( R \) and \( L \) vanishes. This occurs because in momentum space \( \tilde{g}(k) = \delta(k) \), whereas in the tunneling process each electron changes its momentum by approximately \( 2\bar{k} \) or \(-2\bar{k}\). Therefore, an interesting class of functions is given by \( g(x) = e^{-2i\bar{k}x} f(x) + \text{c.c.} \), where \( f(x) \) is dominated in momentum space by modes \( \omega << \bar{k} \). That is to say, writing \( f(x) = \int \exp(i\omega x) \tilde{f}(\omega) \, d\omega \), we are considering functions centered at \( \omega = 0 \), with support in the interval \( \Delta \omega << \bar{k} \). Substituting the proposed form for \( g(x) \) into (3.24), we obtain
relevant couplings. Therefore, we obtain from (3.31), a new effective Hamiltonian (3.31) only those degrees of freedom involved in the tunneling process, retaining the most associated with tunneling, which are given approximately by Hamiltonian (3.31) describes the complete system, and we want to study only the modes where, according to (3.29) and (3.30),

$$H_{k} = \int_{|\bar{k}|} dk d\omega [\tilde{f}(\omega) c_{1}(k, k - \omega) a_{k}^{R+} a_{k}^{L} + H.c.] +$$

$$+ \int_{\omega - k < |\bar{k}|} dkd\omega [\tilde{f}(\omega) c_{2}(k, k + 2\bar{k} - \omega) a_{k}^{R+} a_{k+2\bar{k}-\omega}^{R} + H.c.] +$$

$$+ \int_{|\bar{k}|, k < |\bar{k}|} dkd\omega [\tilde{f}(\omega) c_{2}(-k, -k - 2\bar{k} + \omega) a_{k}^{L+} a_{k+2\bar{k}-\omega}^{L} + H.c.] ,$$

where, according to (3.29) and (3.30),

$$c_{1}(k, k - \omega) = \left(\frac{1}{2}\right)^{\frac{1}{4}} \exp \left\{ -\frac{\ell^{2}}{2} \left[ \left( k - \frac{\omega}{2}\right)^{2} + 2 \left( k - \frac{\omega}{2}\right) \right] \right\}$$

(3.32)

$$c_{2}(k, k + 2\bar{k} - \omega) = c_{1}(k, k - \omega)$$

(3.33)

$$c_{2}(-k, -k - 2\bar{k} + \omega) = \left(\frac{1}{2}\right)^{\frac{1}{4}} \exp \left\{ -\frac{\ell^{2}}{2} \left[ \left( k - \frac{\omega}{2} + 2\bar{k}\right)^{2} + 2 \left( k - \frac{\omega}{2}\right) \right] \right\} .$$

(3.34)

The expression (3.32) shows that for $k \sim \omega \sim 0$ we have $c_{1}(k, k - \omega) \simeq \frac{1}{\sqrt{2}} e^{-3(k\ell)^{2}}$. The Hamiltonian (3.31) describes the complete system, and we want to study only the modes associated with tunneling, which are given approximately by $|k| < |\bar{k}|$. We have to select in (3.31) only those degrees of freedom involved in the tunneling process, retaining the most relevant couplings. Therefore, we obtain from (3.31), a new effective Hamiltonian

$$H'_{\text{eff}} = \int_{|\bar{k}|} dkd\omega [\tilde{f}(\omega) a_{k}^{R+} a_{\bar{k}+\omega}^{L} + H.c.] + c_{2}(-2\bar{k}, 0) \int_{|\bar{k}|} dkd\omega [\tilde{f}(\omega) a_{\bar{k}+2\bar{k}+\omega}^{R+} a_{k}^{R} + H.c.] +$$

$$+ c_{2}(0, -2\bar{k}) \int_{|\bar{k}|} dkd\omega [\tilde{f}(\omega) a_{k+2\bar{k}+\omega}^{R+} a_{\bar{k}+\omega}^{L} + H.c.] + H' ,$$

(3.35)

where

$$H' = \int_{|\bar{k}+2\bar{k}|} dkd\omega [\tilde{f}(\omega) a_{k}^{R+} a_{\bar{k}+\omega}^{L} + H.c.] + \int_{|\bar{k}-2\bar{k}|} dkd\omega [\tilde{f}(\omega) a_{k}^{R+} a_{\bar{k}+\omega}^{L}].$$

(3.36)

We can retain in $H'_{\text{eff}}$, only the first three terms. In fact, using (3.33) and (3.34) we have

$$c_{2}(-2\bar{k}, 0) = c_{2}(0, -2\bar{k}) \simeq \frac{1}{\sqrt{2}} e^{-3(k\ell)^{2}} .$$

(3.37)
That is, the ratio between \(c_2(−2\bar{k}, 0)\) and the factor \(\frac{1}{\sqrt{2}}e^{−(\bar{k}\ell)^2}\) in the tunneling amplitude is \(e^{−2(\bar{k}\ell)^2}\). Now, if \(\bar{k} \sim 1/\ell\), we have \(e^{−2(\bar{k}\ell)^2} \sim e^{-2} \sim 10^{-1}\), which indeed shows that we can keep only the first three terms of (3.35), to investigate the tunneling.

The expressions for \(E_{k−\bar{k}}\) and \(E_{k+\bar{k}}\), can be linearized around \(k = 0\). Using (3.22) we get

\[
E_{k−\bar{k}} = kv \\
E_{k+\bar{k}} = −kv .
\]

(3.38)

Taking into account these approximations, we obtain the effective Hamiltonian

\[
H_{eff}^I = \int_{|k|<\bar{k}} dk \ kv \ a_k^R a_k^+ - \int_{|k|<\bar{k}} dk \ kv \ a_k^L a_k^+ + \]
\[
c \int_{|k|<\bar{k}} dk d\omega \left[ \hat{f}(\omega) a_k^R a_{k-\omega}^L + \text{H.c.} \right],
\]

(3.39)

where

\[
c = \left(\frac{1}{2}\right)^{\frac{1}{4}} \exp[-(\bar{k}\ell)^2] = \left(\frac{1}{2}\right)^{\frac{1}{4}} \exp \left[-\left(\frac{y_0}{\ell}\right)^2\right].
\]

(3.40)

Defining now

\[
\psi_R(x) = \frac{1}{\sqrt{2\pi}} \int dk \exp(ikx) a_k^R
\]

(3.41)

\[
\psi_L(x) = \frac{1}{\sqrt{2\pi}} \int dk \exp(ikx) a_k^L
\]

(3.42)

we see that \(H_{eff}^I\) is, in coordinate space,

\[
H_{eff}^I = \int dx \left[ \psi_R^+(−iv\frac{\partial}{\partial x})\psi_R + \psi_L^+(iv\frac{\partial}{\partial x})\psi_L + cf(x) \psi_R^+ \psi_L + cf^*(x) \psi_L^+ \psi_R \right],
\]

(3.43)

which agrees with (2.3).

**B. INTRODUCTION OF GAUGE FIELDS**

We will obtain now the effective Hamiltonian for the interface, taking into account the presence of an external gauge field \(a_\mu\). We have to consider the more general microscopic Hamiltonian
\[ H = \frac{1}{2m} \left[ (P_x + eB_y - ea_1)^2 + (P_y - ea_2)^2 \right] + eV(x, y) + ea_0. \]  

(3.44)

We can write

\[ H = U^+ H_0 U, \]  

(3.45)

where

\[ H_0 = \frac{1}{2m} \left[ (P_x + eB_y)^2 + P_y^2 \right] + eV(x, y) + ea_0, \]  

(3.46)

and \( U \) is the unitary operator

\[ U = \exp \left( -ie \int_{0,c} \bar{a} \cdot d\vec{x}' \right). \]  

(3.47)

In this expression \( c \) represents a path in the \((x, y)\) plane, given by

\[ c : \begin{cases} 
  y = 0, & 0 < x' < x \\
  x = 0, & 0 < y' < y 
\end{cases} \]  

(3.48)

Let us suppose that \( a_\mu \) is a small static field with slow variations in the magnetic length scale. From \( H_0 \), equation (3.46), we are led, according to our previous computations, to the effective Hamiltonian

\[ H_{0,\text{eff}}^I = \int dx \left[ \psi_R^+ \left( -iv \frac{\partial}{\partial x} \right) \psi_R + \psi_L^+ \left( iv \frac{\partial}{\partial x} \right) \psi_L + cf(x) \psi_R^+ \psi_L + cf^*(x) \psi_I^+ \psi_R \right] + \int dx \ ea_0(x, y = 0) \left[ \psi_R^+ \psi_R + \psi_L^+ \psi_L \right]. \]  

(3.49)

The second quantized unitary operator \( U \) takes the form

\[ U \simeq 1 - ie \int \phi^+(x, y) \left[ \int_0^x a_1(x',0) \, dx' + \int_0^y a_2(x, y') \, dy' \right] \phi(x, y) \, dx dy. \]  

(3.50)

Using (3.21) and neglecting terms similar to \( a_k^{R+} a_{k'}^L \), we obtain

\[ U = 1 - \frac{ie}{2\pi} \int_{k,k' < k} dkdk' dx \exp[i(-k + k') \cdot x] \int_0^x a_1(x',0) \, dx' \, \bar{c}_2(k,k')[a_k^{R+} a_{k'}^R + a_k^{L+} a_{k'}^L] - \frac{ie}{2\pi} \int_{k,k' < k} dkdk' dx \exp[i(-k + k') \cdot x] c_3(x, k, k') a_k^{R+} a_{k'}^R + \]  

\[ - \frac{ie}{2\pi} \int_{k,k' < k} dkdk' dx \exp[i(-k + k') \cdot x] c_4(x, k, k') a_k^{L+} a_{k'}^L. \]  

(3.51)
where

\[ c_3(x, k, k') \equiv \left( \frac{1}{\pi \alpha^2} \right)^{\frac{3}{2}} \int dy \int_0^y a_2(x, y') dy' \exp \left\{ -\frac{1}{2\alpha^2} \left[ \left( y - \frac{\alpha^4}{\ell^2} (k - \bar{k}) \right)^2 + \left( y - \frac{\alpha^4}{\ell^2} (k' - \bar{k}) \right)^2 \right] \right\}, \]  

(3.52)

\[ c_4(x, k, k') \equiv \left( \frac{1}{\pi \alpha^2} \right)^{\frac{3}{2}} \int dy \int_0^y a_2(x, y') dy' \exp \left\{ -\frac{1}{2\alpha^2} \left[ \left( y - \frac{\alpha^4}{\ell^2} (k + \bar{k}) \right)^2 + \left( y - \frac{\alpha^4}{\ell^2} (k' + \bar{k}) \right)^2 \right] \right\}, \]  

(3.53)

and \( \bar{c}_2(k, k') \equiv \lim_{b \to \infty} c_2(k, k') \), with \( c_2(k, k') \) given by (3.28). We must retain in equation (3.51) only the modes associated with tunneling, \(|k| < \bar{k}\). Approximating \( a_2(x, y) \) by \( a_2(x, 0) \) and substituting \( k = k' = 0 \) in the expressions for \( \bar{c}_2, c_3 \) and \( c_4 \), we get

\[ U^I = 1 - \frac{ie}{2\pi} \int_{|k|,|k'|<\bar{k}} dk dk' dx \exp[i(-k + k')x] \int_0^x a_1(x', 0) dx'[a_k^R a_{k'}^R + a_k^{L+} a_{k'}^{L+}] - \frac{ie}{2\pi} \int_{|k|,|k'|<\bar{k}} dk dk' dx \exp[i(-k + k')x] \left( -y_0 a_k^R a_{k'}^R + y_0 [a_k^{L+} a_{k'}^{L+}] a_2(x, 0) \right), \]  

(3.54)

or alternatively, using the definitions (3.41-3.42),

\[ U^I = 1 - ie \int dx \left[ \psi_R^+ \psi_R \left( -y_0 a_2(x, 0) + \int_0^x a_1(x', 0) dx' \right) + \psi_L^+ \psi_L \left( y_0 a_2(x, 0) + \int_0^x a_1(x', 0) dx' \right) \right]. \]  

(3.55)

To compute \( H_{eff}^I = U^I H_{0,eff}^I U^I \), it is important to know the following operator products:

\[ i) \ U^I \psi_R^+(x) U^I = \]

\[ \psi_R^+(x) + ie \int dx' \left[ \psi_R^+(x') \psi_R(x') \left( -y_0 a_2(x', 0) + \int_0^{x'} a_1(x'', 0) dx'' \right) \psi_R^+(x) \right] \]

\[ = \psi_R^+ + ie \left( -y_0 a_2(x, 0) + \int_0^x a_1(x', 0) dx' \right) \psi_R^+(x) \]

\[ \approx \exp \left[ ie \left( -y_0 a_2(x, 0) + \int_0^x a_1(x', 0) dx' \right) \right] \psi_R^+(x) \]  

(3.56)

and, in the same way,
\[ ii) \quad U^T \psi^+_L(x) U^T = \exp \left[ ie \left( y_0 a_2(x, 0) + \int_x^x a_1(x', 0) \, dx' \right) \right] \psi^+_L(x). \]  

Therefore, we get, using the above relations,

\[ iii) \quad U^T \left[ \psi^+_R \frac{\partial}{\partial x} \psi_R - \psi^+_L \frac{\partial}{\partial x} \psi_L \right] U^T = \psi^+_R \left[ \frac{\partial}{\partial x} - ie \frac{\partial}{\partial x} \left( -y_0 a_2(x, 0) + \int_0^x a_1(x', 0) \, dx' \right) \right] \psi_R + \psi^+_L \left[ \frac{\partial}{\partial x} - ie \frac{\partial}{\partial x} \left( y_0 a_2(x, 0) + \int_0^x a_1(x', 0) \, dx' \right) \right] \psi_L = \psi^+_RD_x\psi_R - \psi^+_LD_x\psi_L + i\epsilon \frac{\partial a_2}{\partial x} y_0[\psi^+_R\psi_R + \psi^+_L\psi_L], \]  

where \( D_x \equiv \frac{\partial}{\partial x} - i e a_1(x, 0) \), and also

\[ iv) \quad U^T \psi^+_R \psi_L U^T = \exp(-2iea_2(x, 0) y_0) \psi^+_R\psi_L. \]  

From (3.58) and (3.59) we obtain

\[ H_{eff}^I = U^T H_{0,eff}^I U^T = \int dx [ -i\bar{\psi} \gamma^1 D_1 \psi - c\bar{\psi}(f_1 + if_2\gamma^5) \exp(2iea_2y_0\gamma^5)\psi + ev \frac{\partial a_2}{\partial x} y_0\bar{\psi}\gamma^1\gamma^5\psi]. \]  

The gauge invariant Lagrangian associated with this Hamiltonian is

\[ \mathcal{L} = i\bar{\psi} (\gamma^0 D_0 + \nu \gamma^1 D_1) \psi + c\bar{\psi}(f_1 + if_2\gamma^5) \exp(2iea_2y_0\gamma^5)\psi + eva_2y_0 \partial_\mu j_5^\mu. \]  

It is not difficult to understand why the above Lagrangian is the correct generalization of (2.6), which takes into account the presence of gauge fields. The first attempt, in order to implement gauge invariance, would be to replace, in (2.3), \( \gamma^\mu \partial_\mu \) (we are now considering \( \nu = 1 \)) by the covariant derivative, \( D = \gamma^0(\partial_0 + i e a_0) + \gamma^1(\partial_1 + i e a_1) \). This is, however, only a partial answer. It is necessary to consider that the operators \( \psi_R(x) \) and \( \psi_L(x) \) create hole states in different positions of the two-dimensional plane. In this way, if a gauge transformation is performed, \( \psi_R \) and \( \psi_L \) will be multiplied by different phase factors. Besides that, one additional requirement is that the effective model of tunneling be gauge invariant at a classical level. This condition is obtained from the decoupling between the modes at
the interface and the bulk degrees of freedom. Considering the gauge transformation (below \( \mu = 0, 1 \))

\[
a_\mu \to a_\mu + \partial_\mu \alpha \quad \text{and} \quad a_2 \to a_2 + \partial_2 \alpha|_{y=0} ,
\]

(3.62)

associated to

\[
\psi \to \exp \left[ i \left( \frac{\phi}{2} \gamma^5 - e\alpha \right) \right] \psi \quad \text{and} \quad \bar{\psi} \to \bar{\psi} \exp \left[ i \left( \frac{\phi}{2} \gamma^5 + e\alpha \right) \right] ,
\]

(3.63)

where \( \phi \equiv e\partial_2 \alpha|_{y=0}2y_0 \) is the phase factor implied by the physical distinction between \( \psi_R \) and \( \psi_L \), as fields defined in edges separated by \( 2y_0 \), a simple check shows us that (3.61) in fact satisfies all the above conditions.

We have not considered, in the microscopic derivation, the presence of two-body interactions. Indeed, as far as the wavefunctions at the edges are delocalized in the \( x \) direction, their low electric charge densities make the Coulomb repulsion generate logarithmic effects on the spectrum \( [18] \). The Coulomb potential may be relevant, however, in situations where we have localized states, induced by specific configurations of the tunneling amplitude.

**IV. CHARGE TRAPPING AND CONDUCTIVITY AT THE INTERFACE**

Let us explore some of the consequences of the effective model of tunneling, given by (3.61). We will consider first the case \( a_0 = a_1 = a_2 = 0 \). This model may exhibit the phenomenon of charge fractionalization \( [7] \), intrinsically related to the global behaviour of the external fields \( f_1(x) \) and \( f_2(x) \). A gradient expansion of the fermion current, \( \langle J^\mu \rangle \), computed assuming \( f_1 \) and \( f_2 \) as slowly varying fields in space-time, yields \[ (4.1) \]

\[
\langle J^\mu(x) \rangle = \frac{1}{(2\pi)^2} \varepsilon^{\mu\nu} \varepsilon^{ab} \frac{f_a \partial_\nu f_b}{|f|^2} = \frac{1}{(2\pi)^2} \varepsilon^{\mu\nu} \partial_\nu \left[ \tan^{-1} \left( \frac{f_2}{f_1} \right) \right] .
\]

(4.1)

For a configuration given by \( f_1 = \epsilon, \epsilon \to 0 \) and \( f_2 = A\tanh(\lambda x) \), we obtain, from the expression for \( \langle J^\mu \rangle \), a total charge \( +1/2 \) or \( -1/2 \), localized near \( x = 0 \), when \( \epsilon \to 0^+ \) or \( \epsilon \to 0^- \), respectively. The role of \( f_1 \) is limited to providing a regularization of (4.1). The two possibilities for the total charge are associated with a zero mode in the spectrum, which by
its turn implies a doubly degenerate state: the occupied zero mode, with charge 1/2 and the unoccupied one, with charge $-1/2$. According to the discussion of section III, this profile of $f$ corresponds to the modulating potential

$$g(x) = 2A \tanh(\lambda x) \sin(2\tilde{k}x).$$

(4.2)

This potential could, in principle, be manufactured with the techniques used in the fabrication of microstructures. A difficulty is that the spatial period of (4.2) occurs at not yet very manageable scales ($\sim 100\varphi A$), by present day technology. However, it is possible that in disordered interfaces (which must contain, in Fourier space, modes close to $2\tilde{k}$), modulated in adequate scales by a function qualitatively similar to $\tanh(\lambda x)$, we have a more practical way to observe the occurrence of fermion number fractionalization.

Let us now investigate the introduction of perturbing gauge fields in the system. There is a curious Aharonov-Bohm effect, which shows how a magnetic flux may control the amount of charge localized at the interface. In order to see it, we consider $a_0 = 0$, $f = \text{const.}$ and a magnetic flux $\Phi$, confined in the interior of a very thin imaginary solenoid crossing the plane at the point $(x, y) = (0, 0)$. In this way, the exponential factor which appears in (3.61) is $\exp(-ie\Phi\gamma^5)$, for $x = 0^-$ and $\exp(ie\Phi\gamma^5)$, for $x = 0^+$. According to (4.1), this discontinuity will induce an accumulation of charge $\Phi/\pi$ at $x = 0$.

It is important to remark that equation (4.1), while giving a good evaluation of the total amount of charge trapped by variations of the tunneling amplitude, must be replaced by a sharper computation if we want to know the size of the region where most of the charge is confined. This may be found through exact solutions of the Dirac equation, in special cases. In a “soliton” profile, as $f = iA \tanh(\lambda x)$, the zero mode may be explicitly computed \[19\] and the localization region inferred to be

$$\delta \sim \lim_{x \to \infty} \left( \frac{2xv}{\int_0^x c|f(y)|dy} \right).$$

(4.3)

We also note that in the situations we have been discussing, the charge concentrated at the interface will have an interesting internal structure: the relation between Fourier space
and coordinate space (the $y$ direction) will make any charge distribution have two equivalent and disjoint pieces, localized at the edges R and L. The numerical analysis of section VI will show this effect very clearly.

Since we know how the system is coupled to gauge fields, we can compute the conductivity at the interface, as a response to small electric fields.

The current density which crosses the interface, $j_{\perp}$, flowing from one edge to the other, may be calculated through

$$j_{\perp} = \frac{i}{2y_0} \frac{\delta \ln Z[a_2]}{\delta a_2} = \frac{i}{2y_0} \frac{\delta \ln Z[a_2 + \alpha]}{\delta \alpha} \bigg|_{\alpha=0}, \quad (4.4)$$

where $\alpha = \alpha(x,y)$ and

$$Z = \int D\bar{\psi}D\psi \exp \left[ i \int d^2x L_{eff} \right] \quad (4.5)$$

is the generating functional of model (3.61). We can now use the chiral anomaly of model (3.61) to find

$$Z[a_2 + \alpha] = Z[a_2] \cdot \exp \left[ i \int d^2x \frac{e^2 \alpha y_0}{2\pi} \epsilon_{\mu\nu} F^{\mu\nu} \right], \quad (4.6)$$

where $F^{\mu\nu} = \partial^\mu a^\nu - \partial^\nu a^\mu$. In this way, from (4.3) and (4.4), we obtain

$$j_{\perp} = -\frac{e^2}{2\pi} E_x \quad (4.7)$$

Above, $E_x$ is the $x$ component of the electric field. We may also study the influence of a small electric field, pointing in the $y$ direction. In this case, we have

$$a^0 = a^1 = 0 \quad \text{and} \quad a_2 = E_y t, \quad (4.8)$$

where $E_y$ represents the $y$ component of the electric field. Applying (4.4) to this problem, we get

$$\langle J^1 \rangle_x = \frac{1}{2\pi} e^{10} \partial_0 (-2ea_2 y_0) = \frac{e}{2\pi} E_y 2y_0. \quad (4.9)$$

Therefore, the above relations tell us that the response to external electric fields is given (in the usual units) by the following conductivity tensor
\[
\sigma = \begin{pmatrix}
0 & \sigma_{xy} \\
-\sigma_{xy} & 0
\end{pmatrix}
\quad \text{with} \quad \sigma_{xy} = \frac{e^2}{\hbar},
\]  
(4.10)

showing the quantization of the Hall conductivity at the interface. It is interesting to note that this result is strongly related to the imposition of gauge invariance and is independent on the specific configuration of the tunneling amplitude, even when it is associated to localized states. The above argument may be regarded as an exact version, worked out for a particular problem, of the fundamental works of Halperin [1] and Laughlin [21], , since here we do not have to take any averages over magnetic fluxes.

V. LATTICE EFFECTS

The quantum Hall effect is observed in approximately two-dimensional electron systems, confined at the interface of semiconductor devices as Si-SiO$_2$ or GaAs-Ga$_{1-x}$Al$_x$As. The latter has been preferred in recent years due to its high-mobility parameters, allowing for very precise measurements of conductivity and other physical quantities [22].

Let us consider for a study of lattice effects, the two-dimensional electron gas as having a certain thickness $2L_\perp$, experimentally found to be around 50\varphi A, and interacting with a three-dimensional atomic lattice of macroscopic size, in fact much larger than any relevant characteristic length in the process of tunneling. The lattice structure is relatively complex. The GaAs lattice, for instance, has a zincblende structure, consisting of two interpenetrating fcc sublattices, one of Ga and the other of As atoms. The inter-atomic distance, $s$, is close to 5\varphi A. We will study the effects of the lattice on the tunneling, through an approach analogous to that of Takayama et al. [15] for the case of polyacetylene.

The fortunate fact that the magnetic length is nearly 100\varphi A in the quantum Hall effect, means that we can, in order to estimate couplings, simplify the discussion assuming the lattice to have a simple cubic structure composed of interlinked atoms through an harmonic potential. Therefore, the lattice potential may be expressed as
\[ V(\vec{x}) = \sum_i [V_0(\vec{x} - \vec{x}_i) - \xi_j(\vec{x}_i)\partial_j V_0(\vec{x} - \vec{x}_i)] , \]  
(5.1)
a sum of single atomic potentials centered at sites \( \vec{x}_i = s(n_1\hat{x} + n_2\hat{y} + n_3\hat{z}) \), with \( n_1, n_2, n_3 \) integers, where we consider distortions as represented by \( \vec{\xi}(\vec{x}_i) \), in a linear approximation.

We will be interested in static distortions of the lattice, so that we can write the total Hamiltonian of the system (quasi one-dimensional interface + lattice) as

\[ H = H_{\text{eff}}^I + \frac{1}{4}K \sum_{<i,j>} (\vec{\xi}(\vec{x}_i) - \vec{\xi}(\vec{x}_j))^2 + \int d^3\vec{x} \phi^+(\vec{x})\phi(\vec{x})V(\vec{x}) , \]  
(5.2)
where \( H_{\text{eff}}^I \) is given by (3.43), \( K \sim 10^{-6}(\varphi A)^{-3} \) is the force constant associated to distortions, and \( \phi(\vec{x}) \) is defined from (3.21), according to \( \phi(x,y) \rightarrow \phi(x,y,z) = (2/(\pi L_\perp^2))^{1/4} \exp\left( -\left( z/L_\perp \right)^2 \right) \phi(x,y) \), which corresponds to three-dimensional wavefunctions confined in \( |z| < L_\perp \). The sum in (5.2) is carried over nearest neighbors. We are also assuming the absence of additional external gauge fields.

A convenient approximation is to replace sums over sites in (5.1) and (5.2) by integrals. We find, then

\[ V(\vec{x}) = \frac{1}{s^3} \int d^3\vec{x} V_0(\vec{x}) + \frac{1}{s^3} \int d^3\vec{x}_i \partial_j \xi_j(\vec{x}_i) V_0(\vec{x} - \vec{x}_i) \]  
(5.3)
and

\[ H = H_{\text{eff}}^I + \frac{K}{2s^3} \int d^3\vec{x} \partial_i \xi_i(\vec{x}) \partial_j \xi_j(\vec{x}) + \int d^3\vec{x} \phi^+(\vec{x})\phi(\vec{x})V(\vec{x}) . \]  
(5.4)
As an estimate, we can write \( V_0(\vec{x}) = -4\pi Ze^2 s^2 \delta^3(\vec{x}) \), where \( e^2 = 1/137 \) is the fine structure constant and \( Z \simeq 2 \) is an effective atomic number. Substituting the above quantities in \( V(\vec{x}) \), equation (5.3), we obtain

\[ V(\vec{x}) = -\frac{4\pi Ze^2}{s} - \frac{4\pi}{s} Ze^2 \partial_j \xi_j(\vec{x}) . \]  
(5.5)
The first term on the RHS of (5.3) is a constant and may be shifted to zero. As we noticed in section III, only certain Fourier components of \( V(\vec{x}) \) will be relevant in the tunneling. This amounts to considering a new set of fields, \( \vec{\sigma}(\vec{x}) \) and \( \varphi(\vec{x}) \), given by

20
\[ \vec{\xi}(\vec{x}) \simeq \vec{\sigma}(\vec{x}) \cos(2\bar{k}x + \varphi(\vec{x})) . \]  

(5.6)

From (5.6) we get, neglecting gradients of \( \vec{\sigma} \) and \( \varphi \),

\[ \partial_j \xi_j(x) = -2\bar{k}\sigma_1(x) \sin(2\bar{k}x + \varphi(x)) + \cos(2\bar{k}x + \varphi(x)) \partial_j \sigma_j(x) + \]
\[ -\sigma_j(x) \sin(2\bar{k}x + \varphi(x)) \partial_j \varphi(x) \simeq -2\bar{k}\sigma_1(x) \sin(2\bar{k}x + \varphi(x)) . \quad (5.7) \]

Therefore, we see that only \( \sigma_1 \) must be considered in our analysis. On the other side, the quadratic term in \( \vec{\xi} \) in the Hamiltonian (5.4) turns out to be (restricted to \( \sigma_1 \))

\[ \frac{K}{2s} \int d^3\vec{x} \partial_j \left( \sigma_1 \cos(2\bar{k}x + \varphi) \right) \partial_j \left( \sigma_1 \cos(2\bar{k}x + \varphi) \right) = \]
\[ = \frac{K}{2s} \int d^3\vec{x} \left[ (2\bar{k})^2 \sigma_1^2 \sin^2(2\bar{k}x + \varphi) + (\vec{\nabla} \sigma_1)^2 \cos^2(2\bar{k}x + \varphi) + \right. \]
\[ + \left. (\vec{\nabla} \varphi)^2 \sin^2(2\bar{k}x + \varphi) - \partial_j \sigma_1(2\bar{k}\delta_{ij} + \partial_j \varphi) \cos(2\bar{k}x + \varphi) \sin(2\bar{k}x + \varphi) \right] \]
\[ \simeq \frac{K}{4s} \int d^3\vec{x} \left[ (2\bar{k})^2 \sigma_1^2 + (\vec{\nabla} \sigma_1)^2 + (\vec{\nabla} \varphi)^2 \right] , \quad (5.8) \]

where, assuming that \( \sigma_1 \) and \( \varphi \) are dominated by slow modes, we have substituted the above trigonometric functions by their averaged values.

Defining the complex field \( \sigma = \sigma_1 \exp(i\varphi) \) and using the approximations (5.7) and (5.8), we find, for the total Hamiltonian of the system,

\[ H = H'_{\text{eff}} + \frac{K}{4s} \int d^3\vec{x} \left[ (2\bar{k})^2 |\sigma|^2 + (\partial_j \sigma)^+(\partial_j \sigma) + \right. \]
\[ + \left. \left[ \frac{4\pi Z e^2 \bar{k}}{s} \right] \int d^3\vec{x} \phi^+(\vec{x}) \phi(\vec{x}) \exp(2i\bar{k}x)\sigma(\vec{x}) + \text{H.c.} \right] . \quad (5.9) \]

It is useful to work in Fourier space, through

\[ \sigma(\vec{x}) = \frac{1}{(2\pi)^{3/2}} \int d^3\vec{q} \exp(i\vec{q} \cdot \vec{x}) \tilde{\sigma}(\vec{q}) . \quad (5.10) \]

We now define a variational ansatz for the Hamiltonian (5.9): let us consider a class of fields \( \tilde{\sigma}(\vec{q}) \), parametrized by \( \Delta_1, \Delta_2 \geq 0 \), according to

\[ \tilde{\sigma}(\vec{q}) = \begin{cases} \tilde{\sigma}(q_x, 0, 0) & \text{if} \quad |q_y| \leq \Delta_1/\ell , \quad |q_z| \leq \Delta_2/L \parallel \\ 0, & \text{otherwise}. \end{cases} \quad (5.11) \]
This means, roughly, that we are taking the distortion field \( \sigma(\vec{x}) \) as different from zero only in a certain neighborhood of the interface, given, in Fourier space by the two variational parameters \( \Delta_1 \) and \( \Delta_2 \). Substituting (5.11) in (5.9) we find, after straightforward computations, the Hamiltonian

\[
H = H^I_{\text{eff}} + \frac{K\Delta_1\Delta_2}{sL_{\perp}} \int dx \left[ (\partial_x \sigma(x))^+(\partial_x \sigma(x)) + \eta |\sigma(x)|^2 \right] + \\
+ \left[ icf(\Delta_1, \Delta_2) \int dx \psi_L^+(x)\psi_R(x)\sigma(x) + \text{H.c.} \right],
\]

(5.12)

where

\[
\sigma(x) = \frac{1}{(2\pi)^{1/2}} \int dq \exp(iqx)\tilde{\sigma}(q,0,0),
\]

(5.13)

\[
\eta = \frac{\Delta_1^2}{3\ell^2} + \frac{\Delta_2^2}{3L_{\perp}^2} + (2\bar{k})^2,
\]

(5.14)

\[
c = \frac{2\bar{k}Ze^2(2\pi)^{1/2}}{s} \exp\left(-\bar{k}\ell \right),
\]

(5.15)

and

\[
f(\Delta_1, \Delta_2) = \int_{-\Delta_1/\ell}^{\Delta_1/\ell} dq_1 \exp\left(-\frac{\ell^2}{4}q_1^2\right) \int_{-\Delta_2/L_{\perp}}^{\Delta_2/L_{\perp}} dq_2 \exp\left(-\frac{L_{\perp}^2}{8}q_2^2\right).
\]

(5.16)

In order to study the variational problem, we will consider the path-integral formalism, defining the generating functional

\[
Z = \int D\bar{\psi}D\psi D\sigma^+ D\sigma \exp(iS[\psi, \sigma]),
\]

(5.17)

where

\[
S[\psi, \sigma] = \int d^2x \left\{ i\bar{\psi}(\gamma^0 \partial_0 + v\gamma^1 \partial_1)\psi + \bar{\psi} \left( t_1 + it_2\gamma^5 - cf(\Delta_1, \Delta_2)|\sigma|\exp(-i\varphi^5) \right) \psi + \\
- \frac{K\Delta_1\Delta_2}{s\ell L_{\perp}} \left[ (\partial_1 \sigma)^+(\partial_1 \sigma) + \eta |\sigma|^2 \right] \right\}.
\]

(5.18)

We will simplify our discussion, assuming that \( t_1, t_2 = 0 \), or in other words, that we have a very clean interface, without any degree of disorder or modulating potentials. We can
integrate over the fermion fields in (5.17), neglecting variations of $|\sigma|$ and $\varphi$. This may be considered as the first term in a gradient expansion of the fermion determinant. We get, then,

$$Z_{\text{eff}} = \int D\sigma^+ D\sigma \exp(iS_{\text{eff}}[\sigma]) ,$$  \hspace{1cm} (5.19)

with

$$S_{\text{eff}} = -\int d^2x \left\{ \frac{K\Delta_1\Delta_2}{s\ell L_\perp} \left[ (\partial_1 \sigma)^+ (\partial_1 \sigma) + \eta |\sigma|^2 \right] + V_{\text{eff}}(|\sigma|^2) \right\} ,$$  \hspace{1cm} (5.20)

where

$$V_{\text{eff}}(|\sigma|^2) = \frac{(cf(\Delta_1, \Delta_2)|\sigma|)^2}{2\pi v} \left[ \ln \left( \frac{2cf(\Delta_1, \Delta_2)|\sigma|}{v\bar{k}} \right)^2 + 1 \right] .$$  \hspace{1cm} (5.21)

In the computation of (5.21) it is important to consider the presence of a cutoff at $\bar{k}$ in the fermion theory. The variational strategy is to find extremes of (5.20), in the space of configurations of $\sigma(x)$ and also in the space of parameters $\Delta_1$ and $\Delta_2$. Let us perform this analysis in two steps: first, we consider a fixed pair $(\Delta_1, \Delta_2)$ in order to find the (constant) field $\bar{\sigma}(\Delta_1, \Delta_2)$ which extremizes $S_{\text{eff}}$. Second, we look for extremes of $S_{\text{eff}}[\bar{\sigma}(\Delta_1, \Delta_2)]$, in the space $(\Delta_1, \Delta_2)$.

We obtain, in the first step, the gap in the fermion spectrum,

$$|cf(\Delta_1, \Delta_2)\bar{\sigma}| = v\bar{k} \exp \left[ -1 - \frac{\pi v K\Delta_1\Delta_2\eta}{c^2 s\ell L_\perp} \left( \frac{1}{f(\Delta_1, \Delta_2)} \right)^2 \right],$$  \hspace{1cm} (5.22)

and the effective action, evaluated for $\bar{\sigma}(\Delta_1, \Delta_2)$,

$$S_{\text{eff}}[\bar{\sigma}(\Delta_1, \Delta_2)] = -\frac{K\Delta_1\Delta_2\eta}{2s\ell L_\perp} |\bar{\sigma}|^2$$

$$= -\frac{E\Delta_1\Delta_2}{(f(\Delta_1, \Delta_2))^2} \left( F\Delta_1^2 + G\Delta_2^2 + H \right) \exp \left[ -2 - \frac{I\Delta_1\Delta_2}{(f(\Delta_1, \Delta_2))^2} \left( F\Delta_1^2 + G\Delta_2^2 + H \right) \right] ,$$  \hspace{1cm} (5.23)

where

$$E = \frac{K\bar{k}^2}{8s\ell L_\perp c^2} , \quad F = \frac{1}{3s^2} , \quad G = \frac{1}{3L_\perp^2} , \quad H = (2\bar{k})^2 , \quad I = \frac{2\pi v K}{c^2 s\ell L_\perp} .$$  \hspace{1cm} (5.24)

It is readily seen that the second step in the variational analysis, $\partial S_{\text{eff}}/\partial \Delta_1 = \partial S_{\text{eff}}/\partial \Delta_2 = 0$, leads to
\[ \frac{\partial}{\partial \Delta_{1,2}} \left[ \frac{\Delta_1 \Delta_2 (F \Delta_1^2 + G \Delta_2^2 + H)}{(f(\Delta_1, \Delta_2))^2} \right] = 0 , \]  

(5.25)

or

\[ \frac{I \Delta_1 \Delta_2 (F \Delta_1^2 + G \Delta_2^2 + H)}{(f(\Delta_1, \Delta_2))^2} = 1 . \]  

(5.26)

In order to study the above equations, we have to define \( \bar{k} \) and \( \alpha \) (\( v \) depends on \( \alpha \); see equation (3.23)). Let us, then, consider \( \bar{k} = 3/(2\ell) \), corresponding to an interface with the edges separated by \( \sim 3\ell \), and \( \alpha/\ell = 1.02 \). With these definitions, we get \( v \sim 5 \times 10^{-5} \), \( c \sim ((2\pi)^{1/2}/6) \times 10^{-4}(\varphi A)^{-2} \) and \( I \sim 4 \times 10^{-5} \). A quick inspection shows that both (5.25) and (5.26) have solutions and that the minimum of \( S_{eff}[\bar{\sigma}(\Delta_1, \Delta_2)] \) is obtained from (5.26). This equation has, in fact, many solutions, which determine a curve in the \((\Delta_1, \Delta_2)\) plane. An estimate yields an isotropic solution \( \Delta_1 \sim \Delta_2 \sim 10 \). If we look at (5.11), we see that the degeneracy in the solutions of (5.26) could mean a “torsion” in the lattice displacements, if the parameters \( \Delta_1 \) and \( \Delta_2 \) were non-trivially dependent on \( q_x \). We assume, however, that the isotropic solution represents a mean field for the fluctuations of \( \Delta_1 \) and \( \Delta_2 \). Anyway, the gap, as given by (5.22) does not depend on the degeneracy of \( \Delta_1 \) and \( \Delta_2 \). Using \( \eta^{-1/2} \) into (5.26), we find two possible “vacua”,

\[ cf(\Delta_1, \Delta_2) \bar{\sigma} \sim_{\pm} 10^{-7}(\varphi A)^{-1} , \]  

(5.27)

physically measurable as a gap at the interface.

We can, in the same way, find non-trivial solutions of the Euler-Lagrange equations for the complex field \( \sigma \). We will have, in general, solitons which interpolate the vacuum values of \( <\sigma> \). The transition region is given by the square root of the ratio between the kinetic and mass term coefficients in the action (5.20). This quantity is \( \eta^{-1/2} \). Using \( \Delta_1, \Delta_2 \sim 10 \), we have \( \eta^{-1/2} \sim 10\varphi A \). Since \( \sigma \) changes its sign in this specific configuration, we will have solitons carrying charges \( \pm 1/2 \) (compared to the electron charge) propagating along the interface. Using equation (4.3) we find that the charge will be spread, along the interface, in a region of length \( \sim 500\varphi A \). Here, as in the case of polyacetylene, we can also have soliton-antisoliton (polarons) excitations. One can suppose that the soliton states would be found
as midgap excitations, but since the gap is estimated to be small, it would be hard to observe any kind of soliton production threshold through variations of the sample temperature. We point, however, that there is a mechanism, related to Coulomb repulsion, which raises the soliton energy up to more clearly observable scales. The argument is as follows. As we have already noticed, the edges are associated to different wave-numbers of the fields $\psi_R$ and $\psi_L$. This means that all the charge-density profiles of these excitations will be symmetrically displaced at opposite sides of the interface. In this way, the Coulomb repulsion will raise the soliton energy by $\sim (e/4)^2/(2\ell) \sim 3 \times 10^{-6}(\varphi A)^{-1}$, which is close to the gap between Landau levels ($\sim 4 \times 10^{-6}(\varphi A)^{-1}$). We see, therefore, that the Coulomb interaction, which was playing a minor role in the theory so far, has an important participation in the soliton spectrum.

Another interesting point is the possibility of an enhanced electric conductivity at the interface, via soliton excitations. The same phenomenon was conjectured to be present in the polyacetylene, but in view of the relatively small polymer filaments, the solitons probably do not contribute directly to the conductivity [23,24]. In our case, however, the interface’s length may be constructed many orders of magnitude larger than the magnetic length, in such way that we could hope the solitons to be relevant in the conductivity process.

VI. NUMERICAL ANALYSIS

In order to test the accuracy of the approximations made in sec. III we performed a numerical investigation of the model of tunneling defined by relation (3.1).

Let us consider the matrix elements of (3.1) in the basis of functions $\varphi_n(x,y)$, as given by (3.11), and put it in the following form

$$ < n|H|m > = < n|H_0|m > + < n|H_I|m > , $$(6.1)

where

$$ H_0 = \frac{1}{2m}[(P_x + eBy)^2 + P_y^2] + eV_1(y) $$ (6.2)
The elementary calculation of $<n|H_0|m>$ leads to a diagonal matrix with its elements given by (3.10), with $p = 0$. In order to evaluate $<n|H_f|m>$ we must choose a specific modulating potential barrier. Since we want to explore the occurrence of fermion number fractionalization, we may take a function $f(x)$ with the same asymptotic behaviour as the modulating potential considered in sec IV or that of the predicted solitons of sec V. Some numerical improvement is obtained from the consideration of

$$f(x) = -\left[\epsilon - ig \operatorname{sgn}(x) \left(1 - e^{-\lambda|x|}\right)\right].$$

In (6.4) we have considered $\lambda = (5\ell)^{-1}$ and $g = 4/m\ell^2$. We have performed the numerical diagonalization of (6.1) considering $L/\ell \simeq 900$, $\bar{k} = 2/\ell$ and a “smooth” interface, determined by $(\alpha/\ell - 1) \sim 10^{-4}$. The eigenvalue distribution is shown in fig.1. In this figure we show the gap due to the presence of $V_2(x,y)$. The order of magnitude of the gap agrees with the value obtained from (3.43), $\Delta \simeq cg$. The two points inside the gap correspond to the “zero mode” and to a spurious wavefunction (associated to the periodic boundary conditions).

In order to obtain the charge distribution along the interface we must evaluate

$$|\psi_n(x,y)|^2 = \sum_{ij} (c_i^n)^* c_j^n \varphi_i(x,y)^* \varphi_j(x,y),$$

where $c_i^n$ is the $i$ component of the $n^{th}$ eigenvector of (6.1). The result for the localized state (zero mode) is shown in fig. 2a. On the other hand, a typical delocalized state in the band, far away from the gap, is depicted in Fig. 2b. Note that in the localized state we have peaks at both sides of the interface. The integral of the two peaks is close to $1/2$, so that at each side of the interface there is an accumulated charge of $1/4$, measured in units of the electron charge. In the numerical computation the total charge is not exactly $1/2$, in view of finite size effects, as was numerically verified.

It is interesting to analyse the “vacuum” structure of the theory. For this aim it is enough to integrate expression (6.5) in the $y$ direction. We define, then, a projected charge density,
\[ |\psi_n(x)|^2 \equiv \int_{-\infty}^{\infty} dy \, |\psi_n(x,y)|^2. \] In the field theory context, the vacuum charge density is evaluated by means of

\[ \rho = \langle 0 | : \psi^\dagger(x) \psi(x) : |0 \rangle = \frac{1}{2} \langle 0 | [\psi^\dagger(x), \psi(x)] |0 \rangle. \] (6.6)

Note that in (6.6) not only the lower band but also the upper band is considered in the calculation. The numerical result, shown in fig 3, agrees with the field theory expression. In this computation one observes that the states in the upper band cancel the states in the lower band, in such a way that the charge density of the “vacuum” turns out to be determined only by the zero mode. This fact suggests that the approximation of linearizing the energy near the Fermi energy is indeed a very good approximation. This result also confirms that our matrix Hamiltonian is adequate to obtain the physics at the interface, without interferences from the bulk. Therefore, the equivalence of the two calculations presented above is a clear evidence that the “Dirac sea” of the field theory model reproduces accurately the completely filled lower band.

**VII. CONCLUSION**

We studied the tunneling across a quasi one-dimensional interface in the integer quantum Hall effect. The particular form of the electron spectrum at the edges allowed a mapping from the microscopic definition of the interface to a relativistic (1 + 1)-dimensional quantum field theory. Gauge fields and lattice effects were considered in the description. Regarding the coupling to gauge fields, the Hall conductivity was found to be quantized, independently of the possible induction of localized states by a non-uniform tunneling amplitude. We also obtained a peculiar Aharonov-Bohm effect, which shows the influence of magnetic fluxes on the charge concentrated at the interface. The study of interactions between edge excitations and a three-dimensional lattice showed us a natural mechanism for the generation of fractionally charged solitons propagating along the interface. They are associated to topologically stable solutions of the Euler-Lagrange equations for a complex scalar field. We point that
these excitations may contribute strongly to the $\sigma_{xx}$ component of the conductivity tensor. A numerical test supported the field theory approximations in the case of charge trapping in a modulated barrier.

We believe that an experimental investigation of the above predicted phenomena is crucial for a further development of the theory, in the sense of a more accurate quantitative description. Anyway, there are a certain number of extensions of the present work which may motivate future studies. These are related to different microscopic definitions of the interface, a consideration of higher Landau levels, the introduction of spin in the formalism (which may be relevant to the case of magnetic fields of lower intensity) and a derivation of lattice effects, taking into account exact crystal structures (a Monte Carlo numerical analysis with pseudopotentials could provide some information on lattice distortions and solitons).

ACKNOWLEDGMENTS

L. M. would like to thank F.D.M. Haldane, M. Moriconi and D. Tsui for interesting conversations. This work was supported by CNPq (Brazil).

APPENDIX A: FRACTIONAL FERMION NUMBER

In order to make this paper self-contained, we will show in this appendix how the fermion current (4.1) may be obtained. We will compute it through the adiabatic method, as outlined by Goldstone and Wilczek [8]. A more rigorous approach may be found in ref. [9].

Let us consider the following fermionic Lagrangian in (1 + 1) dimensions

$$\mathcal{L} = \bar{\psi}i\partial_t \psi + c\bar{\psi}(f_1 + i\gamma^5 f_2)\psi, \quad (A1)$$

where $f_1$ and $f_2$ are classical external fields, and the $\gamma$ matrices are defined in (2.3). This model may exhibit the phenomenon of fermion number fractionalization, according to the topology of the fields $f_1$ and $f_2$. 

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The model is invariant under rotations in the \((f_1, f_2)\) plane. This is related to global chiral transformations. That is, considering\[\bar{\psi}(f_1 + i\gamma^5 f_2) \psi = |\vec{f}| \bar{\psi} \exp \left[ i \tan^{-1} \left( \frac{f_2}{f_1} \right) \gamma^5 \right] \psi ,\]where \(|\vec{f}| = (f_1^2 + f_2^2)^{\frac{1}{2}}\), we see that a rotation in the plane of coordinates \((f_1, f_2)\) by an angle \(\Theta\) gives \(\tan^{-1}(f_2/f_1) \rightarrow \tan^{-1}(f_2/f_1) + \Theta\), which can be absorbed by a global chiral transformation\[\psi \rightarrow \exp \left( -\frac{i}{2} \Theta \gamma^5 \right) \psi \]and\[\bar{\psi} \rightarrow \bar{\psi} \exp \left( -\frac{i}{2} \Theta \gamma^5 \right) .\]
The existence of this symmetry will be important to establish the topological nature of \(\langle J^\mu \rangle\).

Let \(x_0\) be a space-time point such that \(f_1(x_0) \neq 0\) and \(f_2(x_0) = 0\). If we consider \(f_1\) and \(f_2\) as slowly varying fields, we can calculate \(\langle J^\mu(x_0) \rangle\) using a "mass" \(-c f_1(x_0)\) and an interaction term \(L_I = ic \bar{\psi} \gamma^5 f_2 \psi\), as can be seen from (A1). We may write
\[
\langle J^\mu(x_0) \rangle = \langle 0 | \bar{\psi}(x_0) \gamma^\mu \psi(x_0) | 0 \rangle = \lim_{\epsilon \to 0} \langle 0 | T \bar{\psi}(x_0) \gamma^\mu \psi(x_0 + \epsilon) | 0 \rangle .
\]
That is, defining \(x_1 \equiv x_0\) and \(x_2 \equiv x_0 + \epsilon\),
\[
\langle J^\mu(x_0) \rangle = Tr[D^\mu(x_1, x_2)] ,
\]
where \(D^\mu_{\alpha \beta}(x_1, x_2) = \gamma^\mu \langle 0 | T \bar{\psi}(x_1) \gamma_\beta \psi(x_2) | 0 \rangle\) and we are suppressing the limit symbols to simplify the notation.

Expanding (A6) in a perturbative series, we find, up to the first order in \(c\)
\[
Tr \left[ D^\mu(x_1, x_2) = D^\mu_0(x_1, x_2) + c \int d^2 x \, D^\mu_0(x_1, x) \, f_2(x) \, \gamma^5 \gamma_\mu D^\mu_0(x, x_2) \right] ,
\]
where \(D^\mu_0(x_1, x_2) = \gamma^\mu S_0(x_1, x_2) = \gamma^\mu \langle 0 | T \bar{\psi}(x_1) \psi(x_2) | 0 \rangle |_{f_2 = 0}\). Now, since
\[
D^\mu_0(x_1, x) = \gamma^\mu (2\pi)^{-2} \int d^2 q_1 \exp[i \bar{q}_1(x_1 - x)] \, S_0(q_1) ,
\]
\[
f_2(x) = \int d^2 k \exp(ikx) \, \tilde{f}_2(k) ,
\]
where \(S_0(q) = (\not{q} - m)^{-1}\), we obtain

\[
\langle J^\mu(x_0) \rangle = \text{Tr} \left[ \frac{c\gamma^\mu}{(2\pi)^4} \int d^2q_1d^2kd^2q_2d^2x \ S_0(q_1) \ f_2(k) \ \gamma^5 S_0(q_2) \ \exp[iq_1(x_1 - x)] \right] \times \exp(i k x) \exp[iq_2(x - x_2)] .
\] (A10)

Integrating in \(x\) and \(q_1\) and substituting \(q_2\) by \(q\), we find

\[
\langle J^\mu(x_0) \rangle = \text{Tr} \left[ \frac{c\gamma^\mu}{(2\pi)^2} \int d^2qd^2k \ S_0(k + q) \ \gamma^5 \tilde{f}_2(k) \ S_0(q) \ \exp[iq(x_1 - x_2)] \times \exp(i k x) \right] .
\] (A11)

In the limit \(x_1 \to x_2 = x_0\), we have

\[
\langle J^\mu(x_0) \rangle = \text{Tr} \left[ \frac{c\gamma^\mu}{(2\pi)^2} \int d^2qd^2k \ S_0(k + q) \ \gamma^5 \tilde{f}_2(k) \ S_0(q) \ \exp(i k x) \right] .
\] (A12)

Substituting the expansion \(S_0(k + q) = S_0(q) + k^\nu \frac{\partial}{\partial q^\nu} S_0(q)\) in the above expression, we get

\[
\langle J^\mu(x_0) \rangle = \text{Tr} \left[ \frac{c\gamma^\mu}{(2\pi)^2} \int d^2q \ S_0(q) \ \gamma^5 S_0(q) \ f_2(x_0) \right] + \text{Tr} \left[ -ic\gamma^\mu \ \partial^\nu f_2(x_0) \int d^2q \ \frac{\partial}{\partial q^\nu} S_0(q) \ \gamma^5 S_0(q) \right] = \frac{ic\partial^\nu f_2(x_0)}{(2\pi)^2} \text{Tr} \left[ \gamma^\mu \int d^2q \ \frac{\gamma^\nu}{(q^2 - m^2)} - \frac{\not{q} + m}{(q^2 - m^2)} \right] \gamma^5 \frac{\not{q} + m}{(q^2 - m^2)} = \frac{ic\partial^\nu f_2(x_0)}{(2\pi)^2} \text{Tr} \gamma^\mu \gamma^\nu \gamma^5 \int d^2q \ \frac{m}{(q^2 - m^2)^2} .
\] (A13)

Since \(\text{Tr} \gamma^\mu \gamma^\nu \gamma^5 = 2\epsilon^{\mu\nu}\), we obtain

\[
\langle J^\mu(x_0) \rangle = \frac{2ie^{\mu\nu} c \ \partial^\nu f_2(x_0)}{(2\pi)^2} \int d^2q \ \frac{m}{(q^2 - m^2)^2} .
\] (A14)

Performing a Wick rotation, \(q_0 \to iq_0\), the above integral yields

\[
i \int d^2q \ \frac{m}{(q^2 + m^2)^2} = 2\pi i \int_0^\infty dq \ \frac{mq}{(q^2 + m^2)^2} = -\pi im \int_0^\infty dq \ \frac{d}{dq} \left( \frac{1}{q^2 + m^2} \right) = \pi_i \frac{1}{m} .
\] (A15)

In this way, using \(m = -cf_1(x_0)\), we find
\[ \langle J^\mu(x_0) \rangle = \frac{1}{(2\pi)} \epsilon^{\mu\nu} \frac{\partial \nu f_2(x_0)}{f_1(x_0)}. \]  
(A16)

As we mentioned, the theory has chiral invariance. This means that we must write the current \( \langle J^\mu(x_0) \rangle \) in a chiral invariant way. In other words, it must be invariant under rotations in the \((f_1, f_2)\) plane. Therefore, we are led to

\[ \langle J^\mu(x) \rangle = \frac{1}{(2\pi)} \epsilon^{\mu\nu} \epsilon^{ab} \frac{f_a}{f^2} \frac{\partial \nu f_b}{f_1} = \frac{1}{(2\pi)} \epsilon^{\mu\nu} \frac{\partial \nu}{\tan^{-1} \left( \frac{f_2}{f_1} \right)} \]. \]  
(A17)

It is interesting to note that from a perturbative calculation and taking into account the chiral symmetry of the model, it was possible to find the non-perturbative expression (A17).

The dependence of \( \langle J^\mu \rangle \) with the topology of the fields \( f_1 \) and \( f_2 \) may be clearly seen by considering, for example, the total charge

\[
Q = \int_{-\infty}^{\infty} \langle J^0(x) \rangle \, dx = \frac{1}{2\pi} \int_{-\infty}^{\infty} \partial_1 \tan^{-1} \left( \frac{f_2}{f_1} \right) \, dx = \\
= \frac{1}{2\pi} \Delta \tan^{-1} \left( \frac{f_2}{f_1} \right). \]  
(A18)
REFERENCES

[1] B. I. Halperin, Phys. Rev. B25, 2185 (1982)

[2] M. Stone, Ann. Phys. (N.Y.) 207, 38 (1991)

[3] X. G. Wen, Phys. Rev. B43, 11025 (1991)

[4] X. G. Wen, Int. J. Mod. Phys. B6, 1711 (1992)

[5] P. Goddard and D. Olive, Int. J. Mod. Phys. A1, 303 (1986)

[6] X. G. Wen, Phys. Rev. B44, 5708 (1991); C. de C. Chamon and X. G. Wen, Phys. Rev. Lett. 70, 2605 (1993)

[7] R. Jackiw and C. Rebbi, Phys. Rev. D13, 3398 (1976)

[8] J. Goldstone and F. Wilczek, Phys. Rev. Lett. 47, 986 (1981)

[9] A. J. Niemi and G. W. Semenoff, Phys. Rep. 135, 99 (1986)

[10] W. P. Su, J. R. Schrieffer and A. J. Heeger, Phys. Rev. Lett. 42, 1698 (1979); Phys. Rev. B22, 2099 (1980)

[11] R. Jackiw and J. R. Schrieffer, Nucl. Phys. B190, [FS3], 253 (1981)

[12] M. Stone in Symposyum on Anomalies, Geometry, Topology, Edited by William A. Bardeen and Alan R. White (World Scientific, Singapore, 1985)

[13] L. Moriconi, Phys. Rev. B46, 4996 (1992)

[14] D. G. Barci and L. Moriconi, Phys. Rev. B49, 14740 (1994)

[15] H. Takayama, Y. R. Lin-Liu and K. Maki, Phys. Rev. B21, 2388 (1980)

[16] W. P. Su and J. R. Schrieffer, Phys. Rev. Lett. 46, 738 (1981)

[17] R. Ray and B. Sakita, Ann. Phys. (N.Y.) 230, 131 (1994)

[18] S. Giovanazzi, L. Pitaevskii and S. Stringari, Phys. Rev. Lett. 72, 3230 (1994)
[19] R. Rajaraman and J. S. Bell, Phys. Lett. **116B**, 151 (1982)

[20] W. Dittrich and M. Reuter, *Selected Topics in Gauge Theories*, edited by H. Araki et al., Lecture Notes in Physics, Vol. 244 (Springer-Verlag, Berlin, 1985)

[21] R. B. Laughlin, Phys. Rev. **B23**, 5632 (1981)

[22] S. Girvin and R. Prange, *The Quantum Hall Effect* (Springer-Verlag, New York, 1987)

[23] S. Roth and H. Bleier, Adv. Phys. **36**, 385 (1985)

[24] J. Tsukamoto, ibid. **41**, 509 (1992)
FIGURES

FIG. 1.

The energy as a function of the quantum number $k$. The unit of energy is $10^{-4}(\varphi A)^{-1}$ while $k$ is measured in units of $2\bar{k}$. The dots inside the gap represent localized states. They were depicted there only to help in the visualization of their energies.

FIG. 2.

The charge density at the interface, $|\psi(x, y)|^2$, and their level curves. Fig. 2a) shows the profile of the charge trapped in the potential barrier. Fig. 2b) shows the charge density of a delocalized state.

FIG. 3.

The projected charge density at the interface, obtained from the integration of the charge density in the $y$ direction. The total charge is close to $-1/2$ (empty zero mode). The $x$ coordinate is measured in units of the magnetic length and all the wavefunctions considered in this computation were normalized to 1 in the interval $-200 \leq x \leq 200$. 

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