Chaotic D1-D5 Black Hole Dynamics through Networks

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This work studies dynamics controlling the transition between different microstates of two charge D1-D5 black holes by network methods, in which microstates of the system are defined as network nodes, while transitions between them are defined as edges. It is found that the eigenspectrum of this network’s Laplacian matrix, which is identified with Hamiltonians of the microstate system, has completely the same Nearest-Neighbor Spacing Distribution as that of general Gaussian Orthogonal Ensemble of Random Matrices. According to the BGS conjecture, this forms evidence for chaotic features of the D1-D5 microstate dynamics. This evidence is further strengthened by observations that inverse of the first/minimal nonzero eigenvalue of the Laplacian matrix is proportional to logarithms of the microstate number of the system. By Sekino and Susskind, this means that dynamics of the D1-D5 black hole microstates are not only chaotic, but also the fastest scrambler in nature.

Motivation and conclusion Understanding dynamics of the black hole microstate is a key step towards the final theory of quantum gravitation. Many important progresses on this direction have been achieved in string theory and Anti deSitter space/Conformal Field Theory [1-7] correspondences, AdS/CFT hereafter, loop quantum gravity [8,9], as well as other approaches such as semiclassic idea of [10-13]. For large mass black holes, the number of microstate available to the system is huge and considerable complicating relations must occur among them. Network method [14] is a natural tool to explore systems like this which involve many degrees of freedoms and complicated relations. A. Charles and D. Mayerson [15] considered random walks on the microstate network of black holes to study their evolution and see interesting late-time behaviors of the system.

Since the work of Hayden and Preskill [16], relations between the black hole microstate dynamics and its information processing efficiency and chaotic feature arouse wide interests on the market. Sekino and Susskind [17] suggest that black holes may be the fastest scrambler in nature. Maldacena, Shenker and Stanford show that black holes are maximally chaotic in the sense that a bound on the early time behavior of OTOC is saturated [18]. Investigations of [19-21] uncover many similarities between the Sachdev-Ye-Kitaev model and black holes and provide more precise and quantitative supports for chaotic features of the black hole dynamics. Basing on this model and through analytical continued partition function as well as correlation functions, Cotler et. al [22,23] numerically established that the late-time behavior of horizon fluctuations in large AdS black holes is governed by random matrix dynamics characteristic of quantum chaotic systems.

Our interest in this work is chaotic features of the D1-D5 black hole dynamics [3,4,24]. The CFT side and weak coupling limit of this issue are discussed in [25], we will focus on its AdS side and general coupling strength aspect, but in string gas approximation. The network idea [14,27,28] allows us to extract nontrivial information about the dynamic without knowing details of the interaction. In this idea the D1-D5 microstates are taken as nodes and transitions among them as edges of the network. We discard the late time behavior of any initial state’s evolution as the investigating goal [15], but take the network Laplacian matrix as the key object and identify it as the hamiltonian controlling transitions among different microstates. We observe that

(i) the Nearest-Neighbor Spacing Distribution (NNSD) of the Laplacian matrix’s eigenvalue, follows universal statistics of the Gaussian Orthogonal Ensemble (GOE) of Random Matrices;

(ii) the system’s typical relaxation time or inverse of the minimal nonzero eigenvalue of the Laplacian matrix manifests linear feature with logarithms of the system’s microstate number.

According to the BGS conjecture which states that systems whose NNSD of hamiltonian spectrum exhibits equal statistics as the GOE matrices will be chaotic in the classic limit, our observations (i) forms direct bulk space evidence for chaotic features of the D1-D5 black hole dynamics in the strong coupling limits, while (ii) provide strong supports for the fastest scrambling ability of black holes in the nature. Relations between the information processing ability of black holes and community structures of their microstate network are explored before summarizing section. Existence of communities in a network implies that microstates in one subnet are connected more tightly than those are connected among different sub-ones.

Network definition and microstate dynamics

The D1-D5 system [3,24] is the most well understood string theory model for black hole microstates. Constraining by computation resources, we focus on in the current work the two charge system for simplicity. The
same research to three charge systems involves no principle difficulties. The system lives in a $R^{4.1} \times S^1 \times T^4$ spacetime, with the $N_1$ D1-brane wraps on $S^1$ while the $N_5$ D5-brane wraps on the $S^1 \times T^4$. In the so called “string gas” picture, the whole system can be considered as variable $\{N_n\}$ strands of short string, each wound the $S^1$ dimension $\{w\}$ times, with the total winding number fixed as

$$\sum_{w=1}^{\infty} wN_w = N_1 \cdot N_5 \equiv N$$

(1)

This allows us to represent each microstate of the system with an unordered set $\{w_1, w_2, \ldots\}$, each consisting of an integer partition of $N$ and forming a node in the microstate network. These microstate could be transformed into each other through a parent string’s splitting or children string’s merging. This will be represented with edges on the network. FIG. 1 gives an example of network representation for a black hole of 5 possible microstate. It is very natural to understand that sizes of the network will grow rapidly with $N$.

![Network representation of microstates](image)

FIG. 1: The network representation of all microstates of D1-D5 black holes with $N_1 \cdot N_5 = 5$

Obviously, dynamics of the string interaction will affect the structure of the network directly, e.g. the linking strength of different edges. However, it is found [15] that although the adjacency matrices $A_{ij} = \omega(i)\Gamma(i \rightarrow j)\omega(j)$ affect the network structure heavily, where $\omega(i)$ is the string state degeneracy and $\Gamma(i \rightarrow j)$ is the transition rate between $i$-$j$ state. The concrete form of $\omega(i)$ and $\Gamma(i \rightarrow j)$ or details of the string interaction dynamics on the late time behavior of general initial configurations of the network are unimportant. In other words, it is the inter-node being linked or not, rather than their linking strength’s being robust or fragile that determines behavior of the network. Basing on this fact, we will in this work simply set the network adjacency matrices $A_{ij}$ to be 1 or 0, depending on nodes $i$-$j$ could be connected or not through one event of a parent string’s splitting or two children strings’ merging, with the winding number conservation as the only judging criteria. Obviously, this value assignment scheme for $A_{ij}$ is independent of the string coupling strength. So physics extracting from their spectrums is valid in general coupling strengths. Of course, string gas approximation is still a qualification on our works’ value.

According to [27, 28], evolutions of quantum state in the Hilbert space can be written as diffusions on the network controlled by matrix $A_{ij}$. Starting with an arbitrary initial state $\sum_i p_i|i\rangle\langle i|$, the system will evolve the following way

$$\frac{dp_i}{dt} = C \sum_j A_{ij}(p_j - p_i) = C \sum_j (A_{ij} - \delta_{ij}d_i)p_j$$

(2)

where $C$ is the diffusion constant and $d_i \equiv \sum_j A_{ij}$ is the node degree. Writing this in the standard form of diffusion or transportation equations on the network [14], we have, where $L$ is named the Laplacian matrix

$$\frac{dp}{dt} + CLp = 0$$

(3)

obviously $L = D - A$, with $D$ being the diagonal node degree matrix and $A$, the adjacency matrix. Denoting the eigenstates of $L$ as $|q_n\rangle$, the solutions of (3) can be formally written as, where $|ini\rangle$ represent an arbitrary initial state,

$$p_i(t) = |i\rangle e^{-CLt}|ini\rangle = \sum_n e^{-\lambda_nCLt}|i\rangle|q_n\rangle\langle q_n|ini\rangle.$$ 

(4)

So the eigenvalue and eigenvectors of the Laplacian matrix determine evolutions of the diffusion process. It can be proven that $L$ is non-negative and has one zero eigenvalue at least [14]. Denote the spectrum in ascending order $\lambda_0 \leq \lambda_1 \leq \ldots$, as $t \rightarrow \infty$, the only term surviving in $|ini\rangle$ is the $\lambda_0 = 0$ term. This means all diffusions on the network has a final equilibrium. While inverse of the first nonzero eigenvalue $\lambda_1^{-1}$ characterizes the relaxation time of the system.

All microstates represented by nodes in our network are of equal mass/energy. They are eigenstates of free string hamiltonian. However, the hamiltonian identified with the network Laplacian matrix contains interaction. It is this interaction that makes the string-string merging, or one to two string splitting process possible, i.e.

$$CL \equiv H = H_0 + H_{int}.$$ 

(5)

$H_0$ here is the Hamiltonian whose eigenstates consist the network nodes, while $H_{int}$ is the interaction which makes the transition between different microstates possible

$$H_0\psi_i = E\psi_i, \quad H\psi_i = CL_{ij}\psi_j.$$ 

(6)

The string interaction follows from the non-extremality introduced of the otherwise exactly BPS supersymmetric D1-D5 brane system. Relative to $H_0$, $H_{int}$ is not small quantities in the strong coupling limit. This will help to explain why spectrums of $H$ exhibit wide spreading instead of narrow localizing profile in the following FIG. 2.

But because $H_{int}$ takes effects only when two strings contacting each other, which happens very seldom in string gas approximations, we neglect its effects when classifying microstate of the system and characterize them by two-charges, i.e. the D1- and D5-charge, or NS1 winding number and Kaluza-Klein momentum number in dual theories.
log 8
3
30
and
40

s
λ
3
the eigenvalue density/distribution
variations of
9
λ
3
s
30
30
3

FIG.
2: The eigenvalue distribution of Laplacian matrices of
four D1-D5 black hole microstate network with N = 23, 27, 31
and 35. The horizontal axis is the possible eigenvalues and
the vertical axis is their relative appearance frequency.

FIG. 3: The NNSD of eigenvalues of the Laplacian matrix
of four typical D1-D5 black hole microstate network, the
red curve is the distribution of eigenvalue spacing of GOE.
The horizontal axis is the unfolded eigenvalue appearance fre-
quency and the vertical axis is their relative distribution.

**Evidence for chaotic dynamics** We display in
FIGs 2 and 3 the eigenvalue density/distribution \( \rho(\lambda) \) of
Laplacian matrices for four typical D1-D5 microstate net-
work and their Nearest Neighbor Spacing Distribution
\( P(s) \) explicitly. NNSD [30, 31] is the distribution of rel-
ative appearance time/frequency \( s \), \( s \) itself has already
the meaning of appearance time/frequency of \( \lambda \), unfolded
from the accumulated eigenvalue distribution

\[
\pi(\lambda) = \int_{-\infty}^{\lambda} d\lambda' \rho(\lambda'), s_i = \pi(\lambda_{i+1}) - \pi(\lambda_i).
\]  

So the NNSD \( P(s) \) is a kind of appearance frequency of
appearance frequency.

Very interestingly, our numeric results indicate that
the NNSD of D1-D5 microstate network, matches ex-
actly with those of the Gaussian Orthogonal Ensemble
of Random Matrices., which is known as

\[
P(s) = \frac{\pi}{2} s e^{-\frac{s^2}{2}},
\]

By the so called BGS conjecture, for time-inverse invari-
ant systems if their quantum hamiltonian has NNSD the
same as those of GOE, they will manifest chaotic features
in classic limit. This means that dynamics controlling
the transition among different microstates of our D1-D5
brane system is chaotic in nature. This is a highly non-
trivial but exciting results, because our network building
process introduces nowhere random ingredients directly.
Obviously, our results forms direct bulk space evidences
for chaotic features of the black hole dynamics, while
other works [21–23, 25] accomplish this goal almost ex-
clusively from the dual gauge theory side.

FIG. 4: Relations between the inverse of first nonzero eigen-
value \( \lambda^{-1}_1 \) of network Laplacian matrix versus logarithms of
the number of microstates log \( W \) of the D1-D5 two charge
black holes.

Chaotic features of black hole dynamics are also re-
lated with its ability of information processing. By ref-
ence [17] black hole is the fastest scrambler in the na-
ture, which means that information fallen into them will
be scrambled uniformly inside in times logarithmically
proportional with the system’s number of degrees of free-
dom. In our two charge D1-D5 microstate networks, the
network Laplacian matrix is identified with the Hamil-
tonian controlling the transition among different states.
So information scrambling time \( \tau_* \) of the system must
be corresponded with the inverse of first, i.e. minimal
nonzero eigenvalue \( \lambda^{-1}_1 \) of the Laplacian matrix. We dis-
play in FIG3 variations of \( \tau_* \) with respect to the loga-
ithmic value of the corresponding microstate number for
several black holes, with \( N_1 \cdot N_3 \) ranges from 17 to 35.
Obviously, our results forms strong support for the fast
scrambler conjecture of reference [17].

It is worth emphasizing that the two key observations
extracted from the Laplacian matrix spectrum in this
section are valid in general string coupling conditions.
Although string gas approximation represents a new con-
straint on our observations’ validity, it is a more loose one
than weak coupling limits.

**Community and information processing effi-
Ciency** Community [14, 23] or subnetwork structure is
another interesting feature of general networks. However, exactly defining and identifying community in a general network is a very difficult question mathematically. In practice, studying spectrums of the network adjacency matrix $A_{ij}$ may be the most convenient way. We display in FIG. 5 the community structure and corresponding $A_{ij}$ eigenvalue distribution for 4 typical networks representing the microstate of D1-D5 black holes with $N = N_1 \cdot N_5 = 23, 27, 31, 35$. From the figure we easily see that as $N$ increases the number of communities in the network equals more and more precisely the number of $A_{ij}$'s eigenvalues well separated from the big bulk of other eigenvalues, with the bulk itself also considered as a “well-separated” one. This coincides with the observation of remarkably well.

![Image 5: Communities of microstate network of four different size D1-D5 black holes](image)

Intuitively, when communities appear in the D1-D5 network, microstate in one community will be more frequently connected with others in the same community than with states in others. Basing on this intuition, it is reasonable to expect that community structure should have effects on information processing efficiency of the system. Recalling that in the two charge D1-D5 system, microstates are defined essentially the way a long string’s decomposition into many short ones. We display in FIG. 6 the distribution of short string numbers in the five communities or subnetworks consisting the D1-D5 microstate network and the whole network itself, the last one, with total mass/energy parameter $N = 35$. The horizontal axis is the number of short strings $N_s$, while the vertical one is the number of microstate $W$ with $N_s$ short strings.

![Image 6: The distribution of the number of short strings in the five community or subnetworks consisting the D1-D5 microstate network and the whole network itself](image)

of transition to evolve into each other, so connections among them are less tightly.

| $N$   | 10 | 11 | 12 | 13 | 14 | 15 | 16 | 17 | 18 | 19 | 20 | 21 | 22 |
|-------|----|----|----|----|----|----|----|----|----|----|----|----|----|
| n.c.  | 4  | 4  | 4  | 4  | 4  | 5  | 4  | 4  | 5  | 4  | 5  | 5  | 5  |
| $N$   | 23 | 24 | 25 | 26 | 27 | 28 | 29 | 30 | 31 | 32 | 33 | 34 | 35 |
| n.c.  | 5  | 5  | 4  | 5  | 4  | 5  | 5  | 5  | 5  | 5  | 5  | 5  | 5  |

TABLE I: Number of communities (n.c.) in the D1-D5 microstate network with the black hole mass/energy parameter $N$ ranges from 10 to 35

In practical operations, division and identification of communities in general networks contain big uncertainties. We display in TABLE I the variation of community numbers in our two charge D1-D5 black hole microstate network as the total energy of the system increases. From the table, we easily see that the number of communities in different size black holes approaches more and more closely to the uniform value 5, although fluctuation/errors appear unavoidably due to exact division and identification schemes’ lacking. Since in our networks, positive and negative eigenvalues of $A_{ij}$ always appear symmetrically, the asymptotic value of the number should be 5 instead of 4. Since black holes are conjectured the fastest scrambler in nature, this maybe the smallest number of communities for networks representing general multi-degree dynamic systems.

**Summary and discussion** We construct simple network representation for dynamics of two charge D1-D5
black hole microstates. In our representation, elements of the network adjacency matrix are assigned values either 1 or 0, depending on related microstate nodes can be transformed into each other or not under the only constraint of energy conservation. We identify Hamiltonians controlling dynamics of the D1-D5 microstate with the network Laplacian matrix, and find that NNSD of their spectrum has almost exactly the same distribution as those of GOE random matrices. According to the BGS conjecture, this forms a direct and concrete evidence for chaotic features of the D1-D5 microstate dynamics. As further evidence, we find that inverse of the first/minimal nonzero eigenvalue of the network Laplacian matrix, when identified with the typical relaxation time of the system, manifests simple linear increasing law with the logarithm of the microstate number of the D1-D5 black holes. This means that dynamics of the D1-D5 microstate is not only chaotic, but also the fastest information scrambler in nature. We analyze relations between this high efficiency of information processing and the community structure of the network preliminarily. All our findings or analysis are made under string gas approximation but general coupling strengths of the underlying string theory.

Network method is a powerful tool for fundamental physics. For instance, in string landscape studies it provides dynamical mechanisms for vacuum selection [39]. As prospects for futures, firstly we suggest applying the method of this work to the more complex three charge D1-D5 black holes [34] or black holes with nontrivial semi-classic inner structure and dynamics such as [10–13]. Secondly, more aspects or characteristic quantities in network theories corresponding to black hole microstate dynamics deserve investigations. For example, the network side concept of Mean-First-Passage-Time(MFPT) [3] and complexity concepts [37, 38] on the black hole side, share much similarities intuitively. The latter is defined as the minimal number of simple operations needed for |A⟩ evolving to |B⟩. While the physical meaning of MFPT (T_{ij}) is the mean value of the least walk steps from node i to j. As long as relations between MFPT and complexity can be established, in either three charge D1-D5 black holes or other ones, we will get more concrete and direct verification of the CV [37] or CA [38] conjecture.

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