ON THE CURVATURE GROUPS OF A CR MANIFOLD

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Abstract. We show that any contact form whose Fefferman metric admits a nonzero parallel vector field is pseudo-Einstein of constant pseudohermitian scalar curvature. As an application we compute the curvature groups $H^k(C(M), \Gamma)$ of the Fefferman space $C(M)$ of a strictly pseudoconvex real hypersurface $M \subset \mathbb{C}^{n+1}$.

1. Statement of results

Let $M$ be a strictly pseudoconvex CR manifold of CR dimension $n$ and $\theta$ a contact form on $M$ such that the Levi form $L_\theta$ is positive definite. Let $S^1 \to C(M) \to M$ be the canonical circle bundle and $F_\theta$ the Fefferman metric on $C(M)$, cf. [6]. Let $GL(2n + 2, \mathbb{R}) \to L(C(M)) \to C(M)$ be the principal bundle of linear frames tangent to $C(M)$ and $\Gamma : u \in L(C(M)) \mapsto \Gamma_u \subset T_u(L(C(M)))$ the Levi-Civita connection of $F_\theta$. Let $H^k(C(M), \Gamma)$ be the curvature groups of $(C(M), \Gamma)$, cf. [3] and our Section 2. Our main result is

Theorem 1. If $(M, \theta)$ is a pseudo-Einstein manifold of constant pseudohermitian scalar curvature $\rho$ then the curvature groups $H^k(C(M), \Gamma)$ are isomorphic to the de Rham cohomology groups of $C(M)$. Otherwise (that is if either $\theta$ is not pseudo-Einstein or $\rho$ is nonconstant) $H^k(C(M), \Gamma) = 0$, $1 \leq k \leq 2n + 2$.

The key ingredient in the proof of Theorem 1 is the explicit calculation of the infinitesimal conformal transformations of the Lorentz manifold $(C(M), F_\theta)$.

Corollary 1. Let $\Omega \subset \mathbb{C}^{n+1}$ be a smoothly bounded strictly pseudoconvex domain. There is a defining function $\varphi$ of $\Omega$ such that $\theta = \frac{i}{2} (\bar{\partial} - \partial) \varphi$ is a pseudo-Einstein contact form on $\partial \Omega$. If $(\partial \Omega, \theta)$ has constant pseudohermitian scalar curvature then

$$H^k(C(\partial \Omega), \Gamma) \approx H^k(\partial \Omega, \mathbb{R}) \oplus H^{k-1}(\partial \Omega, \mathbb{R}),$$

for any $1 \leq k \leq 2n + 2$. 

The first statement in Corollary 1 is a well known consequence of the fact that $T_{1,0}(\partial \Omega)$ is an embedded CR structure, cf. J.M. Lee, [7]. If for instance $\Omega$ is the unit ball in $\mathbb{C}^{n+1}$ and $\theta = \frac{i}{2}(\overline{\partial} - \partial)|z|^2$ then

$$H^k(C(\partial \Omega), \Gamma) = \begin{cases} \mathbb{R}, & k \in \{1, 2n + 1, 2n + 2\}, \\ 0, & \text{otherwise}. \end{cases}$$

The paper is organized as follows. In Section 2 we recall S.I. Goldberg & N.C. Petridis’ curvature groups of a torsion-free linear connection (cf. also I. Vaisman, [10]) as well as the needed material on CR manifolds, Tanaka-Webster connection and the Fefferman metric. Section 3 is devoted to the proof of Theorem 1 and corollaries.

2. THE CURVATURE GROUPS OF THE FEFFERMAN METRIC

Let $(M, T_{1,0}(M))$ be a $(2n + 1)$-dimensional connected strictly pseudoconvex CR manifold with the CR structure $T_{1,0}(M) \subset T(M) \otimes \mathbb{C}$. Let $\theta$ be a contact form on $M$ such that the Levi form $L_\theta(Z, \overline{W}) = -i(d\theta)(Z, \overline{W})$, $Z, W \in T_{1,0}(M)$, is positive definite. Let $H(M) = \text{Re}\{T_{1,0}(M) \oplus T_{0,1}(M)\}$ be the Levi distribution and

$$J : H(M) \to H(M), \quad J(Z + \overline{Z}) = i(Z - \overline{Z}), \quad Z \in T_{1,0}(M),$$

its complex structure. Let $\mathbb{C} \to K(M) \to M$ be the complex line bundle

$$K(M)_x = \{\omega \in \Lambda^{n+1}T^*_x(M) \otimes \mathbb{C} : T_{0,1}(M)_x | \omega = 0\}, \quad x \in M.$$

There is a natural action of $\mathbb{R}_+$ (the multiplicative positive reals) on $K(M) \setminus \{0\}$ such that $C(M) = (K(M) \setminus \{0\})/\mathbb{R}_+$ is a principal $S^1$-bundle $\pi : C(M) \to M$ (the canonical circle bundle over $M$, cf. [2], Chapter 2). The Fefferman metric $F_\theta$ is given by

$$F_\theta = \pi^*\tilde{G}_\theta + 2(\pi^*\theta) \circ \sigma,$$

(2)

$$\sigma = \frac{1}{n + 2}\left\{d\gamma + \pi^* \left( i \omega_\alpha^\alpha - \frac{i}{2} g^{\alpha\beta} g_{\alpha\beta} - \frac{\rho}{4(n + 1)} \theta \right) \right\}.$$

The Fefferman metric is a Lorentz metric on $C(M)$, cf. J.M. Lee, [6]. The following conventions are adopted as to the formulae (1)-(2). Let $T$ be the characteristic direction of $d\theta$ i.e. the tangent vector field on $M$ determined by $\theta(T) = 1$ and $T \not\parallel d\theta = 0$. We set

$$G_\theta(X, Y) = (d\theta)(X, JY), \quad X, Y \in H(M),$$

$$\tilde{G}_\theta(X, Y) = G_\theta(X, Y), \quad \tilde{G}_\theta(Z, T) = 0, \quad Z \in T(M).$$
There is a unique linear connection $\nabla$ on $M$ (the Tanaka-Webster connection of $(M, \theta)$, cf. [9] and [11]) such that i) the Levi distribution is parallel with respect to $\nabla$, ii) $\nabla J = 0$ and $\nabla g_\theta = 0$, iii) the torsion $T_\nabla$ of $\nabla$ is pure i.e.

$$T_\nabla(Z, W) = 0, \quad T_\nabla(Z, W) = 2i L_\theta(Z, W) T, \quad Z, W \in T_{1,0}(M), \quad \tau \circ J + J \circ \tau = 0.$$ 

Here $g_\theta$ is the Webster metric i.e. the Riemannian metric on $M$ given by

$$g_\theta(X, Y) = G_\theta(X, Y), \quad g_\theta(X, T) = 0, \quad g_\theta(T, T) = 1,$$

for any $X, Y \in H(M)$. Also $\tau(X) = T_\nabla(T, X)$, $X \in T(M)$, is the pseudohermitian torsion of $\nabla$. If $\{T_\alpha : 1 \leq \alpha \leq n\}$ is a local frame of $T_{1,0}(M)$ defined on the open set $U \subseteq M$ then $\omega^\alpha_\beta$ are the corresponding connection 1-forms of the Tanaka-Webster connection i.e. $\nabla T_\alpha = \omega^\alpha_\beta \otimes T_\beta$. Let $R_\nabla$ be the curvature of $\nabla$ and

$$R_{\alpha \beta} = \text{trace}\{X \mapsto R_\nabla(X, T_\alpha) T_\beta\}$$

the pseudohermitian Ricci tensor of $(M, \theta)$. Moreover $g_{\alpha \beta} = L_\theta(T_\alpha, T_\beta)$ and $\rho = g^{\alpha \beta} R_{\alpha \beta}$ is the pseudohermitian scalar curvature of $\nabla$. Also $\gamma : \pi^{-1}(U) \to \mathbb{R}$ is a local fibre coordinate on $C(M)$. Precisely let $\{\theta^\alpha : 1 \leq \alpha \leq n\}$ be the admissible local coframe associated to $\{T_\alpha : 1 \leq \alpha \leq n\}$ i.e.

$$\theta^\alpha(T_\beta) = \delta^\alpha_\beta, \quad \theta^\alpha(T_\beta) = 0, \quad \theta^\alpha(T) = 0.$$ 

The locally trivial structure of $S^1 \to C(M) \to M$ is described by

$$\pi^{-1}(U) \to U \times S^1, \quad [\omega] \mapsto (x, \frac{\lambda}{|\lambda|}), \quad \omega \in K(M)_x \setminus \{0\},$$

$$\omega = \lambda(\theta \wedge \theta^1 \wedge \cdots \wedge \theta^n)_x, \quad x \in U, \quad \lambda \in \mathbb{C} \setminus \{0\}.$$ 

Then $\gamma([\omega]) = \arg(\lambda/|\lambda|)$ where $\arg : S^1 \to [0, 2\pi)$. If $(U, x^1, \cdots, x^{2n+1})$ is a system of local coordinates on $M$ then $(\pi^{-1}(U), \tilde{x}^1, \cdots, \tilde{x}^m)$ are the naturally induced local coordinates on $C(M)$ i.e. $\tilde{x}^A = x^A \circ \pi$, $1 \leq A \leq 2n+1$, and $\tilde{x}^m = \gamma$ (with $m = 2n + 2$).

Let $\Pi : L(C(M)) \to C(M)$ be the projection and $\rho : \text{GL}(m, \mathbb{R}) \to \text{End}_{\mathbb{R}}(\mathbb{R}^m)$ the natural representation. We denote by $\Omega^k_{\rho(\text{GL}(m))}(C(M))$ the space of tensorial $k$-forms of type $\rho(\text{GL}(m, \mathbb{R}))$ i.e. each $\omega \in \Omega^k_{\rho(\text{GL}(m))}(C(M))$ is a $\mathbb{R}^m$-valued $k$-form on $L(C(M))$ such that

i) $\omega_u(X_1, \cdots, X_k) = 0$ if at least one $X_i \in \text{Ker}(d_u\Pi)$,

ii) $\omega_{gU}(d_u R_g) X_1, \cdots, (d_u R_g) X_k) = \rho(g^{-1}) \omega_u(X_1, \cdots, X_k)$ for any $g \in \text{GL}(m, \mathbb{R}), X_i \in T_u(L(C(M)))$ and $u \in L(C(M))$.  


Let $\Gamma$ be the Levi-Civita connection of $(C(M), F_\theta)$ thought of as a connection-distribution in $L(C(M)) \to C(M)$. If $\omega \in \Omega^k_{\rho(GL(m))}(C(M))$ then its covariant derivative with respect to $\Gamma$ is the tensorial $(k+1)$-form of type $\rho(GL(m, \mathbb{R}))$

$$(\nabla \omega)(X_0, \cdots, X_k) = \omega(hX_0, \cdots, hX_k),$$

for any $X_i \in T(L(C(M)))$, $0 \leq i \leq k$. Here $h_u : T_u(L(C(M))) \to \Gamma_u$ is the natural projection associated to the direct sum decomposition $T_u(L(C(M))) = \Gamma_u \oplus \text{Ker}(d_u \Pi)$. Let us consider the $C^\infty(C(M))$-module

$L^k = \Omega^k_{\rho(GL(m))}(C(M)) \times \Pi^* \Omega^k(C(M))$

and the submodule $\tilde{L}^k$ given by

$$\tilde{L}^k = \{ (\omega, \Pi^* \alpha) \in L^k : \nabla^2 \omega = 0 \}.$$

Let $\eta \in \Gamma^\infty(T^*(L(C(M))) \otimes \mathbb{R}^m)$ be the canonical 1-form i.e. $\eta_u = u^{-1} \circ (d_u \Pi)$ for any $u \in L(C(M))$. If we set

$$D^k : \tilde{L}^k \to \tilde{L}^{k+1}, \quad D^k(\omega, \Pi^* \alpha) = (\nabla \omega - \eta \wedge \Pi^* \alpha, \Pi^* d\alpha),$$

then $\tilde{L} = (\bigoplus_{k=0}^m \tilde{L}^k, D^k)$ is a cochain complex, cf. [3], p. 550. The curvature groups of $\Gamma$ are the cohomology groups

$$H^k(C(M), \Gamma) = H^k(\tilde{L}) = \frac{\text{Ker}(D^k)}{D^{k-1} \tilde{L}^{k-1}}, \quad 1 \leq k \leq m.$$

3. Infinitesimal conformal transformations

We shall establish the following

**Theorem 2.** Any infinitesimal conformal transformation of the Fef-ferman metric $F_\theta$ is a parallel vector field.

By a result of C.R. Graham, $\sigma$, $\tau$ is a connection 1-form in $S^1 \to C(M) \to M$. For each vector field $X \in T(M)$ let $X^\uparrow$ denote the horizontal lift of $X$ with respect to $\sigma$ i.e. $X^\uparrow_z \in \text{Ker}(\sigma_z)$ and $(d_z \pi)X^\uparrow_z = X_{\pi(z)}$ for any $z \in C(M)$. To prove Theorem 2 we need to recall the following

**Lemma 1.** (E. Barletta et al., [1])

The Levi-Civita connection $\nabla^{C(M)}$ of $(C(M), F_\theta)$ and the Tanaka-Webster connection $\nabla$ of $(M, \theta)$ are related by

$$(3) \quad \nabla^{C(M)} Y^\uparrow = (\nabla_X Y)^\uparrow - (d\theta)(X, Y)T^\uparrow - \{ A(X, Y) + (d\sigma)(X^\uparrow, Y^\uparrow) \} S,$$

$$(4) \quad \nabla^{C(M)} T^\uparrow = (\tau(X) + \phi X)^\uparrow,$$
(5) \[ \nabla^{C(M)}_{T^\uparrow} X^\uparrow = (\nabla_T X + \phi X)^\uparrow + 2(d\sigma)(X^\uparrow, T^\uparrow)S, \]

(6) \[ \nabla^{C(M)}_{X^\uparrow} S = \nabla^{C(M)}_{S^\uparrow} X^\uparrow = (JX)^\uparrow, \]

(7) \[ \nabla^{C(M)}_{T^\uparrow} T^\uparrow = V^\uparrow, \quad \nabla^{C(M)}_{S^\uparrow} S^\uparrow = 0, \]

(8) \[ \nabla^{C(M)}_{S^\uparrow} T^\uparrow = 0, \quad \nabla^{C(M)}_{T^\uparrow} S^\uparrow = 0, \]

for any \( X, Y \in H(M) \). Here \( A(X, Y) = g_\theta(\tau(X), Y) \). Also the vector field \( V \in H(M) \) and the endomorphism \( \phi : H(M) \to H(M) \) are given by

\[ G_\theta(V, Y) = 2(d\sigma)(T^\uparrow, Y^\uparrow), \quad G_\theta(\phi X, Y) = 2(d\sigma)(X^\uparrow, Y^\uparrow), \]

for any \( X, Y \in H(M) \).

A vector field \( \mathcal{X} \) on \( C(M) \) is an infinitesimal conformal transformation of \( F_\theta \) if

(9) \[ \nabla^{C(M)} \mathcal{X} = \lambda I, \]

for some \( \lambda \in C^\infty(C(M)) \) where \( I \) is the identical transformation of \( T(C(M)) \). Let \( S \) be the tangent to the \( S^1 \)-action (locally \( S = \partial/\partial \gamma \)).

Taking into account the decomposition \( T(C(M)) = H(M)^\uparrow \oplus RT^\uparrow \oplus RS \) the first order partial differential system (9) is equivalent to

(10) \[ \nabla^{C(M)}_{X^\uparrow} \mathcal{X} = \lambda X^\uparrow, \quad \nabla^{C(M)}_{T^\uparrow} \mathcal{X} = \lambda T^\uparrow, \quad \nabla^{C(M)}_{S^\uparrow} \mathcal{X} = \lambda S, \]

for any \( X \in H(M) \). Let \( \{X_a : 1 \leq a \leq 2n\} = \{X_\alpha, JX_\alpha : 1 \leq \alpha \leq n\} \) be a local frame of \( H(M) \). Then \( \mathcal{X} = \mathcal{X}^a X_a^\uparrow + fT^\uparrow + gS \) for some \( C^\infty \) functions \( \mathcal{X}^a, f, \) and \( g \). By (6)-(8) in Lemma 1 the last equation in (10) may be written

\[ S(\mathcal{X}^a X_a^\uparrow + \mathcal{X}^a(JX_a)^\uparrow + S(f)T^\uparrow + S(g)S = \lambda S \]

hence

(11) \[ \mathcal{X}^a = 0, \quad S(f) = 0, \quad S(g) = \lambda. \]

Similarly, by (4) and (6) in Lemma 1 the first equation in (10) may be written

\[ X^\uparrow(f)T^\uparrow + f(\tau(X) + \phi X)^\uparrow + X^\uparrow(g)S + g(JX)^\uparrow = \lambda X^\uparrow \]

hence

(12) \[ X^\uparrow(f) = 0, \quad X^\uparrow(g) = 0, \]
Lemma 2. With respect to a local frame \( \{ T_\alpha : 1 \leq \alpha \leq n \} \) of \( T_{1,0}(M) \) the endomorphism \( \phi : H(M) \otimes \mathbb{C} \to H(M) \otimes \mathbb{C} \) is given by \( \phi_{T_\alpha} + \phi_{T_\beta} \) with
\[
\phi_{T_\alpha} = \phi_{T_\beta} = 0,
\]
and \( \phi_{T_\alpha} + \phi_{T_\beta} = g^{\alpha \gamma} \phi_{T_\alpha} \).

Proof of Lemma 2. Taking the exterior derivative of (2) we obtain
\[
(n + 2) d\sigma = \pi^* (\text{id}_\omega - \frac{i}{2} dg \wedge g - \frac{1}{4(n + 1)} d(\rho \theta)).
\]
Note that \( \nabla g = 0 \) may be locally written as \( dg = g^{\alpha \gamma} \omega_{\alpha \gamma} \). Also \( g^{\alpha \beta} g_{\beta \gamma} = \delta_\gamma \) yields \( dg^{\alpha \beta} = -g^{\gamma \beta} g^{\alpha \gamma} d\theta \). Hence
\[
dg \wedge dg^{\alpha \beta} = \omega_{\alpha \beta} \wedge \omega^{\alpha \beta} + \omega_{\alpha \beta} \wedge \omega^{\alpha \beta} = 0.
\]
Let \( \{ \theta^\alpha : 1 \leq \alpha \leq n \} \) be the admissible local coframe associated to \( \{ T_\alpha : 1 \leq \alpha \leq n \} \). Then (by a result in [11], cf. also [2], Chapter 1)
\[
d\omega = R_{\lambda \nu} \theta^\lambda \wedge \theta^\nu + (W_{\alpha \lambda} \theta^\lambda - W_{\alpha \nu} \theta^\nu) \wedge \theta,
\]
where \( A_{\alpha \beta} = A(T_\alpha, T_\beta) \) and covariant derivatives are meant with respect to the Tanaka-Webster connection. Finally (by the very definition of \( \phi \))
\[
(n + 2) G_\theta (\phi X, Y) = i(R_{\alpha \beta} \theta^\alpha \wedge \theta^\beta)(X, Y) - \frac{\rho}{4(n + 1)} (d\theta)(X, Y)
\]
for any \( X, Y \in H(M) \otimes \mathbb{C} \). This yields (14). Lemma 2 is proved.

Proof of Theorem 2. Let us extend both members of (13) by \( \mathbb{C} \)-linearity. Then (13) holds for any \( X \in H(M) \otimes \mathbb{C} \). By a result in [9] \( \tau(T_{1,0}(M)) \subseteq T_{0,1}(M) \) hence \( \tau(T_\alpha) = A_\alpha \) for some \( C^\infty \) functions \( A_\alpha \).

Using (13) for \( X = T_\alpha \) we obtain
\[
0 = f A_\alpha = 0, \quad f \phi_\alpha + (i g - \lambda) \delta_\alpha = 0.
\]
By Lemma 2
\[
\phi_\alpha = \frac{i}{2(n + 2)} \left( R_{\alpha \beta} - \frac{\rho}{2(n + 1)} \delta_\beta \right).
\]
and a contraction leads to \( \dot{\phi}_\alpha^\alpha = i\rho/[4(n+1)] \). Next a contraction in the second of the identities (13) gives \( f \dot{\phi}_\alpha^\alpha + n(ig - \lambda) = 0 \) or \( i\rho f + 4n(n+1)ig = 4n(n+1)\lambda \) and then

\[
(16) \quad g = -\frac{\rho}{4n(n+1)} f
\]

and \( \lambda = 0 \) as \( f, g \) and \( \lambda \) are \( \mathbb{R} \)-valued. In particular \( \nabla^{C(M)}\mathcal{X} = 0 \). Theorem 2 is proved.

**Corollary 2.** Any infinitesimal conformal transformation of \( F_\theta \) is a parallel vector field of the form

\[
(17) \quad \mathcal{X} = a \left( T^\uparrow - \frac{\rho}{4n(n+1)} S \right), \quad a \in \mathbb{R}.
\]

In particular any contact form \( \theta \) whose Fefferman metric \( F_\theta \) admits a nontrivial parallel vector field is pseudo-Einstein of constant pseudohermitian scalar curvature and vanishing pseudohermitian torsion.

**Proof.** By (7)-(8) in Lemma 1 the middle equation in (10) may be written

\[
T^\uparrow(f)T^\uparrow + fV^\uparrow + T^\uparrow(g)S = \lambda T^\uparrow
\]

hence

\[
(18) \quad T^\uparrow(f) = \lambda, \quad T^\uparrow(g) = 0,
\]

(19)

\[
f(z) V_{\pi(z)} = 0, \quad z \in C(M).
\]

Yet \( \lambda = 0 \) (by Theorem 2) so that (by (11)-12) \( f = a \) and \( g = b \) for some \( a, b \in \mathbb{R} \). Let us assume now that \( F_\theta \) admits a parallel vector field \( \mathcal{X} \neq 0 \). Replacing from (16) into the second of the identities (15) leads to

\[
(20) \quad a \left( R^{\beta}_\alpha - \frac{\rho}{n} \delta^\beta_\alpha \right) = 0.
\]

Note that \( a \neq 0 \) (otherwise (16) implies \( b = 0 \) hence \( \mathcal{X} = 0 \)) so that (by (20)) \( R^{\beta}_\alpha = (\rho/n)g^{\beta\alpha} \) i.e. \( \theta \) is pseudo-Einstein (cf. [7]). Also (16) shows that \( \rho = \) constant. Finally the first identity in (15) implies \( \tau = 0 \). Corollary 2 is proved.

A remark is in order. Apparently (19) implies that \( a = 0 \) when \( \text{Sing}(V) \neq \emptyset \) (and then there would be no nonzero parallel vector fields on \( (C(M), F_\theta) \)). Yet we may show that

**Corollary 3.** Assume that \( \theta \) is pseudo-Einstein. Then \( V = 0 \) if and only if \( \rho \) is constant.
So (19) brings no further restriction. Proof of Corollary 3. Let $z \in L$ be given by

$$2(d\omega^\alpha)(T, T_\beta) = -W^\alpha_{\alpha \beta}, \quad 2(d\omega^\alpha)(T, T_\overline{\beta}) = W^\alpha_{\alpha \beta},$$

where $\rho_\beta = T_\beta(\rho)$ and $\rho_\overline{\beta} = \overline{T_\beta}$. Consequently

$$2(n + 2)(d\sigma)(T^\dagger, T^\dagger_\beta) = -iW^\alpha_{\alpha \beta} + \frac{1}{4(n + 1)} \rho_\beta.$$  

On the other hand (cf. [2], Chapter 5) if $\theta$ is pseudo-Einstein then

$$W^\alpha_{\alpha \beta} = -\frac{i}{2n} \rho_\beta, \quad W^\alpha_{\alpha \overline{\beta}} = W^\alpha_{\alpha \overline{\beta}}.$$  

Hence $V$ is given by (see Lemma 1 above)

$$G_\theta(V, T_\beta) = -\frac{1}{4n(n + 1)} \rho_\beta.$$  

Clearly if $\rho = \text{constant}$ then $V = 0$. Conversely if $V = 0$ then $\overline{\theta}_b\rho = 0$ i.e. $\rho$ is a $\mathbb{R}$-valued CR function. As $M$ is nondegenerate $\rho$ is constant. Corollary 3 is proved.

At this point we may prove Theorem 1. Let $\mathcal{L}^k$ be the sheaf associated to the module $\mathcal{L}^k$ i.e. for any open set $A \subseteq C(M)$

$$\mathcal{L}^k(A) = \{(\lambda, \Pi^k\lambda) : \lambda \in \Omega^k_{\rho(GL(m))}(\Pi^{-1}(A)), \nabla^2 \lambda = 0, \quad \alpha \in \Omega^k(A)\}.$$  

Let $D^k : \mathcal{L}^k \to \mathcal{L}^{k+1}$ be the sheaf homomorphism induced by the module homomorphism $D^k : \mathcal{L}^k \to \mathcal{L}^{k+1}$.

**Lemma 3.** Let $\mathcal{S}_\theta$ be the sheaf of parallel vector fields on $(C(M), F_\theta)$. For each open set $A \subseteq C(M)$ let $j_A : \mathcal{S}_\theta(A) \to \mathcal{L}^0(A)$ be given by

$$j_A(\mathcal{X}) = (f_\mathcal{X}, 0), \quad \mathcal{X} \in \mathcal{S}_\theta(A),$$

$$f_\mathcal{X} : \Pi^{-1}(A) \to \mathbb{R}^m, \quad f_\mathcal{X}(u) = u^{-1}(\mathcal{X}_{\Pi(u)}), \quad u \in \Pi^{-1}(A).$$  

Then

$$(21) \quad 0 \to \mathcal{S}_\theta \xrightarrow{j} \mathcal{L}^0 \xrightarrow{D^0} \mathcal{L}^1 \xrightarrow{D^1} \cdots \xrightarrow{D^{n-1}} \mathcal{L}^n \to 0$$

is a fine resolution of $\mathcal{S}_\theta$ so that the curvature groups of $\Gamma$ are isomorphic to the cohomology groups of $C(M)$ with coefficients in $\mathcal{S}_\theta$.

**Proof.** Let $(f, \lambda) \in \mathcal{L}^0$ such that $0 = D^0(f, \lambda) = (\nabla f - \lambda \eta, d\lambda)$. Let $z \in C(M)$ and $u \in \Pi^{-1}(z)$. We set by definition $\mathcal{X}_u = u(f(u))$. As $f \circ R_g = \rho(g^{-1}) \circ f$ for any $g \in \text{GL}(m, \mathbb{R})$ it follows that $\mathcal{X}_u$ is well defined. Let $(\Pi^{-1}(C(U)), x^i, X^i_j)$ be the naturally induced local coordinates on $L(C(M))$ i.e. $x^i(u) = \tilde{x}^i(\Pi(u))$ and $X^i_j(u) = a^i_j$ for any $u = (z, \{X_i : 1 \leq i \leq m\}) \in L(C(M))$ such that $X_j = a^i_j(\partial/\partial \tilde{x}^i)_z$. 


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Given a vector field \( X = \lambda^j \partial/\partial x^j + \lambda^i_j \partial/\partial X^i_j \) on \( L(C(M)) \)

\[
(\nabla f)_u X_u = \lambda^j(u) u^{-1} (\nabla^{C(M)}_{\partial/\partial u^j} \mathcal{X})_{u^j} , \quad u \in \Pi^{-1}(C(U)).
\]

Then \( \nabla f = \lambda j \) implies that \( \nabla^{C(M)} \mathcal{X} = \lambda I \) hence (by Theorem 2) \( \lambda = 0 \) i.e. \( (f, \lambda) = j(\mathcal{X}) \). Therefore the corresponding sequence of stalks \( 0 \to S_{\theta,z} \to L^0_z \to L^1_z \to \cdots \to L^m_z \to 0 \) is exact at \( L^0_z \) while the exactness at the remaining terms follows from the Poincaré lemma for \( D \) as in [3], p. 552. Lemma 2 is proved. In particular if \( \theta \) is a pseudo-Einstein contact form of constant pseudohermitian scalar curvature then (by Corollary 2) \( S_{\theta} = R \) and Lemma 2 furnishes a resolution \( 0 \to R \to \mathcal{L}^* \) of the constant sheaf \( R \) where

\[
j_A(a) = (f_A, 0) , \quad a \in \mathbb{R},
\]

and \( \mathcal{X} \) is given by (17) in Corollary 2 hence

\[
H^k(C(M), \Gamma) \approx H^k(C(M), \mathbb{R}) , \quad 1 \leq k \leq m.
\]

Otherwise (i.e. if \( \theta \) is not pseudo-Einstein or \( \rho \) is nonconstant) then \( S_{\theta} = 0 \). Theorem 2 is proved.

If \( M \subset \mathbb{C}^{n+1} \) is a strictly pseudoconvex real hypersurface then (by a result of J.M. Lee, [7]) \( M \) admits globally defined pseudo-Einstein contact forms. On the other hand the pullback to \( M \) of \( dz^0 \wedge \cdots \wedge dz^n \) is a global nonzero section in \( K(M) \). In particular \( C(M) \) is trivial. If \( \theta \) is a pseudo-Einstein contact form on \( M \) of constant pseudohermitian scalar curvature then (by Theorem 1 and the Künneth formula)

\[
H^k(C(M), \Gamma) = H^k(C(M), \mathbb{R}) = \sum_{p+q=k} H^p(M, \mathbb{R}) \otimes H^q(S^1, \mathbb{R}) = H^k(M, \mathbb{R}) \oplus H^{k-1}(M, \mathbb{R})
\]

and Corollary 1 is proved. Using M. Rumin’s criterion (cf. [8]) for the vanishing of the first Betti number of a pseudohermitian manifold we get

**Corollary 4.** Let \( M \subset \mathbb{C}^{n+1} \) be a connected strictly pseudoconvex real hypersurface and \( \theta \) a pseudo-Einstein contact form on \( M \) with \( \rho \) constant and \( \tau = 0 \). If \( n \geq 2 \) then \( H^1(C(M), \Gamma) = \mathbb{R} \).

An interesting question is whether one may improve Corollary 1 by choosing a contact form with \( \rho \) constant to start with. Indeed as \( T_{1,0}(M) \) is embedded one may choose a pseudo-Einstein contact form \( \theta \). On the other hand if the CR Yamabe invariant \( \lambda(M) \) is \( < \lambda(S^{2n+1}) \) then (by the solution to the CR Yamabe problem due to D. Jerison & J.M. Lee, [5]) there is a positive solution \( u \) to the CR Yamabe equation...
such that $u^{2/n} \theta$ has constant pseudohermitian scalar curvature. Yet (by a result in [7]) the pseudo-Einstein property is preserved if and only if $u$ is a CR-pluriharmonic function. It is an open problem whether the CR Yamabe equation admits CR-pluriharmonic solutions.

References

[1] E. Barletta & S. Dragomir & H. Urakawa, Yang-Mills fields on CR manifolds, submitted to J. Math. Phys., 2005.
[2] S. Dragomir & G. Tomassini, Differential geometry and analysis on CR manifolds, to be published in Progress in Math., Birkhäuser, Boston, 2006.
[3] S.I. Goldberg & N.C. Petridis, The curvature groups of a pseudo-Riemannian manifold, J. Differential Geometry; 9(1974), 547-555.
[4] C.R. Graham, On Sparling's characterization of Fefferman metrics, American J. Math., 109(1987), 853-874.
[5] D. Jerison & J.M. Lee, The Yamabe problem on CR manifolds, J. Diff. Geometry, 25(1987), 167-197.
[6] J.M. Lee, The Fefferman metric and pseudohermitian invariants, Trans. A.M.S., (1)296(1986), 411-429.
[7] J.M. Lee, Pseudo-Einstein structures on CR manifolds, American J. Math., 110(1988), 157-178.
[8] M. Rumin, Un complexe de formes différentielles sur les variétés de contact, C.R. Acad. Sci. Paris, 330(1990), 401-404.
[9] N. Tanaka, A differential geometric study on strongly pseudo-convex manifolds, Kinokuniya Book Store Co., Ltd., Kyoto, 1975.
[10] I. Vaisman, The curvature groups of a space form, Ann. Scuola Norm. Sup. Pisa, 22(1968), 331-341.
[11] S.M. Webster, Pseudohermitian structures on a real hypersurface, J. Diff. Geometry, 13(1978), 25-41.

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