GEOMETRICAL FORMALITY OF SOLVMANIFOLDS AND SOLVABLE LIE TYPE GEOMETRIES

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Abstract. We show that for a Lie group $G = \mathbb{R}^n \ltimes \mathbb{R}^m$ with a semisimple action $\phi$ which has a cocompact discrete subgroup $\Gamma$, the solvmanifold $G/\Gamma$ admits a canonical invariant formal (i.e. all products of harmonic forms are again harmonic) metric. We show that a compact oriented aspherical manifold of dimension less than or equal to 4 with the virtually solvable fundamental group admits a formal metric if and only if it is diffeomorphic to a torus or an infra-solvmanifold which is not a nilmanifold.

1. Introduction

Let $(M, g)$ be a compact oriented Riemannian $n$-manifold. We call $g$ formal if all products of harmonic forms are again harmonic. If a compact oriented manifold admits a formal Riemannian metric, we call it geometrically formal. If $g$ is formal, then the space of the harmonic forms is a subalgebra of the de Rham complex of $M$ and isomorphic to the real cohomology of $M$. By this, a geometrically formal manifold is a formal space (in the sense of Sullivan [22]). But the converse is not true (see [15] [16]). For very simple examples, closed surfaces with genus $\geq 2$ are formal but not geometrically formal. Thus we have one problem of geometrical formality of formal spaces. Kotschick’s nice work in [15] stimulates us to consider this problem.

In this paper we prove the following theorem by using computations of the de Rham cohomology of general solvmanifolds given in [14].

Theorem 1.1. Let $G = \mathbb{R}^n \ltimes \mathbb{R}^m$ with a semisimple action $\phi$. Suppose $G$ has a lattice $\Gamma$. Then $G/\Gamma$ admits an invariant formal metric.

We also study geometrical formality of low-dimensional aspherical manifolds with the virtually solvable fundamental groups. We consider infra-solvmanifolds which are quotient spaces of simply connected solvable Lie groups by subgroups of the groups of the affine transformations of $G$ satisfying some conditions (see Section 7 for the definition). We classify geometrically formal compact aspherical manifolds of dimension less than or equal to 4 with the virtually solvable fundamental groups.

Theorem 1.2. Let $M$ be a compact oriented aspherical manifold of dimension less than or equal to 4 with the virtually solvable fundamental group. Then $M$ is geometrically formal if and only if $M$ is diffeomorphic to a torus or an infra-solvmanifold which is not a nilmanifold.

2. Notation and conventions

Let $k$ be a subfield of $\mathbb{C}$. A group $G$ is called a $k$-algebraic group if $G$ is a Zariski-closed subgroup of $GL_n(\mathbb{C})$ which is defined by polynomials with coefficients in $k$. 
Let $G(k)$ denote the set of $k$-points of $G$ and $U(G)$ the maximal Zariski-closed unipotent normal $k$-subgroup of $G$ called the unipotent radical of $G$. Denote $U_n(k)$ the group of $k$-valued upper triangular unipotent matrices of size $n$.

3. Unipotent Hull of Solvable Lie Group

**Theorem 3.1.** ([19]) Let $G$ be a simply connected solvable Lie group. Then there exists a unique $\mathbb{R}$-algebraic group $H_G$ with an injective group homomorphism $\psi : G \to H_G(\mathbb{R})$ so that:

1. $\psi(G)$ is Zariski-dense in $H_G$.
2. The centralizer $Z_{H_G}(U(H_G))$ of $U(H_G)$ is contained in $U(H_G)$.
3. $\dim U(H_G) = \dim G$ (resp. rank $G$).

We denote $U_G = U(H_G)$.

**Theorem 3.2.** ([13]) Let $G$ be a simply connected solvable Lie group. Then $U_G$ is abelian if and only if $G = \mathbb{R}^n \ltimes \mathbb{R}^m$ such that the action $\phi : \mathbb{R}^n \to \text{Aut}(\mathbb{R}^m)$ is semisimple.

4. Hodge Theory

Let $(V, g)$ be a $\mathbb{R}$ or $\mathbb{C}$-vector space of dimension $n$ with an inner product $g$. Let $igwedge V = \bigoplus_{p=0}^n \bigwedge^p V$ be the exterior algebra of $V$. We extend $g$ to the inner product on $\bigwedge V$. Take $vol \in \bigwedge^n V$ such that $g(vol, vol) = 1$. We define the linear map $*g : \bigwedge^p V \to \bigwedge^{n-p} V$ as:

$$v \wedge *g u = g(v, u) vol$$

Let $\{\theta_1, \ldots, \theta_n\}$ be an orthonormal basis of $(V, g)$. Then we have

$$*g(\theta_{i_1} \wedge \ldots \wedge \theta_{i_p}) = (\text{sgn} \sigma_{I,J}) \theta_{j_1} \wedge \ldots \theta_{j_{n-p}}$$

where $J = \{j_1, \ldots, j_{n-p}\}$ is the complement of $I = \{i_1, \ldots, i_p\}$ in $\{1, \ldots, n\}$ and $\sigma_{I,J}$ is the permutation \(\begin{pmatrix} 1 & \ldots & p & p+1 & \ldots & n \\
1 & \ldots & i_p & j_1 & \ldots & j_{n-p}\end{pmatrix}\).

Let $(M, g)$ be a compact oriented Riemannian $n$-manifold. Let $(A^*(M), d)$ be the de Rham complex of $M$ with the exterior derivation $d$. For $x \in M$ by the inner product $g_x$ on $T_x M$ we define the linear map $*g : A^p(M) \to A^{n-p}(M)$ by

$$(*g(\omega))_x = *g_x \omega_x$$

for $\omega \in A^p(M)$. Define $\delta : A^p(M) \to A^{p-1}(M)$ by $\delta = (-1)^{np+n+1} *g d *g$. We call $\omega \in A^p(M)$ harmonic if $d \omega = 0$ and $\delta \omega = 0$. Let $\mathcal{H}^p(M)$ be the subspace of $A^p(M)$ which consists of harmonic $p$-forms. Let $\mathcal{H}(M) = \bigoplus \mathcal{H}^p(M)$. It is known that the inclusion $\mathcal{H}(M) \subset A^*(M)$ induces an isomorphism

$$\mathcal{H}^p(M) \cong H^p(M, \mathbb{R}).$$

In general a wedge product of harmonic forms is not harmonic and so $\mathcal{H}^p(M)$ is not a subalgebra of $A^*(M)$.

**Definition 4.1.** We call a Riemannian metric $g$ formal if all products of harmonic forms are again harmonic. We call an oriented compact manifold $M$ geometrical formal if $M$ admits a formal metric.
5. Invariant forms on solvmanifolds (proof of Theorem 1)

Let $G$ be a simply connected solvable Lie group and $\mathfrak{g}$ the Lie algebra which is the space of the left invariant vector fields on $G$. Consider the exterior algebra $\wedge \mathfrak{g}^*$ of the dual space of $\mathfrak{g}$. Denote $d : \wedge^1 \mathfrak{g}^* \rightarrow \wedge^2 \mathfrak{g}^*$ the dual map of the Lie bracket of $\mathfrak{g}$ and $d : \wedge^p \mathfrak{g}^* \rightarrow \wedge^{p+1} \mathfrak{g}^*$ the extension of this map. We can identify $(\wedge \mathfrak{g}^*, d)$ with the left invariant forms on $G$ with the exterior derivation. Let $\text{Ad} : G \rightarrow \text{Aut}(\mathfrak{g})$ be the adjoint representation. Representations of $G$ are trigonalizable in $\mathbb{C}$ by Lie’s theorem. We define the diagonal representation $\text{Ad}_s : G \rightarrow \text{Aut}(\mathfrak{g}_\mathbb{C})$ as the diagonal entries of a triangulation of $\text{Ad}$. Let $X_1, \ldots, X_n$ be a basis of $\mathfrak{g}_\mathbb{C}$ such that $\text{Ad}_s$ is represented by diagonal matrices. Then we have $\text{Ad}_s X_i = \alpha_i(g)X_i$ for characters $\alpha_i$ of $G$. Let $x_1, \ldots, x_n$ be the dual basis of $X_1, \ldots, X_n$. We assume that $G$ has a lattice $\Gamma$. Define the sub-DGA $A^*$ of the de Rham complex $A^*_\mathbb{C}(G/\Gamma)$ as

$$A^p = \left\langle \alpha_{i_1} \cdots \alpha_{i_p} x_{i_1} \wedge \cdots \wedge x_{i_p} \right| \begin{array}{c} 1 \leq i_1 < i_2 < \cdots < i_p \leq n, \\ \text{the restriction of } \alpha_{i_1} \cdots \alpha_{i_p} \text{ on } \Gamma \text{ is trivial} \end{array} \right\rangle$$

where $\alpha_{i_1} \cdots \alpha_{i_p} = \alpha_{i_1} \cdots \alpha_{i_p}$.

**Theorem 5.1.** ([14] v. Corollary 7.6) The inclusion

$$A^* \subset A^*_\mathbb{C}(G/\Gamma)$$

induces a cohomology isomorphism and $A^*$ can be considered as a sub-DGA of $\wedge \mathfrak{u}^*$ where $\mathfrak{u}$ is the Lie algebra of $U_G$ as in Section 3.

Define $g$ the Hermitian inner product as

$$g(X_i, X_j) = \delta_{ij}.$$ 

Since $\text{Ad}_s$ is an $\mathbb{R}$-valued representation, the restriction of $g$ on $\mathfrak{g}$ is an inner product on $\mathfrak{g}$. We consider $g$ as an invariant Riemannian metric on $G/\Gamma$.

**Theorem 5.2.** If $U_G$ is abelian, then $g$ is a formal metric on $G/\Gamma$.

**Proof.** By the assumption, the differential on $\wedge \mathfrak{u}^*$ is 0. By Theorem 5.1, the derivation on $A^*$ is 0 and we have an isomorphism

$$A^* \cong H^*(G/\Gamma).$$

Thus it is sufficient to show that all elements of $A^*$ are harmonic. Let $*_g$ be the star operator. Then for $\alpha_{i_1} \cdots \alpha_{i_p} x_{i_1} \wedge \cdots \wedge x_{i_p} \in A^p$ we have

$$*_g(\alpha_{i_1} \cdots \alpha_{i_p} x_{i_1} \wedge \cdots \wedge x_{i_p}) = (\text{sgn} \sigma_1 \cdots \sigma_p) \tilde{\alpha}_{i_1} \cdots \tilde{\alpha}_{i_p} x_{j_1} \wedge \cdots \wedge x_{j_{n-p}}.$$

Since the restriction of $\alpha_{i_1} \cdots \alpha_{i_p}$ on $\Gamma$ is trivial, the image $\alpha_{i_1} \cdots \alpha_{i_p}(G) = \alpha_{i_1} \cdots \alpha_{i_p}(G/\Gamma)$ is compact and hence $\alpha_{i_1} \cdots \alpha_{i_p}$ is unitary. Since $G$ has a lattice $\Gamma$, $G$ is unimodular (see [19] Remark 1.9) and hence we have

$$\tilde{\alpha}_{i_1} \cdots \tilde{\alpha}_{i_p} = \tilde{\alpha}_{i_1}^{-1} \cdots \tilde{\alpha}_{i_p} = \alpha_{j_1} \cdots \alpha_{j_{n-p}}.$$

Hence we have

$$\tilde{\alpha}_{i_1} \cdots \tilde{\alpha}_{i_p} x_{j_1} \wedge \cdots \wedge x_{j_{n-p}} = \alpha_{j_1} \cdots \alpha_{j_{n-p}} x_{j_1} \wedge \cdots \wedge x_{j_{n-p}} \in A^{n-p}$$

and thus we have $*_g(A^*) \subset A^*$. Since the derivation on $A^*$ is 0, we have $\delta(A^*) = 0$. Hence the theorem follows. □
By this theorem and Theorem 3.2 we have Theorem 1.1

Remark 5.1. Not every invariant metric on $G/\Gamma$ in Theorem 1.2 is formal. See the following example.

Example 1. Let $H = \mathbb{R} \ltimes \mathbb{R}^2$ such that $\phi(z)(x,y) = (e^{i}\pi, e^{i}\pi)$. Consider $G = H \times \mathbb{R}$. Then for some non-zero number $a \in \mathbb{R}$, $\phi(a)$ is conjugate to an element of $SL_2(\mathbb{Z})$, and hence $G$ has a lattice $\Gamma = a\mathbb{Z} \ltimes \mathbb{Z}$ for a lattice $\Gamma'$ of $\mathbb{R}^2$. Let $g$ be the Lie algebra of $G$ and $g'$ the dual of $g$. The cochain complex $(\bigwedge g^*, d)$ is generated by a basis $\{x, y, z, w\}$ such that

$$dx = -z \wedge x, \ dy = z \wedge y, \ dz = 0, \ dw = -x \wedge x.$$  

Consider the invariant metric $g = x^2 + y^2 + z^2 + w^2$. Then $z$ and $w - x$ are harmonic for $g$. But $z \wedge (w - x)$ is not harmonic. Thus $g$ is not formal.

Example 2. Let $G = \mathbb{C} \ltimes \mathbb{C}^2$ with $\phi(z)(x,y) = (e^{i}\pi, e^{i}\pi)$. For some $p, q \in \mathbb{R}$, $\phi(p\mathbb{Z} + \sqrt{-1}q\mathbb{Z})$ is conjugate to a subgroup of $SL_4(\mathbb{Z})$ and hence we have a lattice $\Gamma = (p\mathbb{Z} + \sqrt{-1}q\mathbb{Z}) \ltimes \Gamma'$ for a lattice $\Gamma'$ of $\mathbb{C}^2$ (see [12] and [9]). For any lattice $\Gamma$, $G/\Gamma$ is geometrically formal by Theorem 1.1.

Remark 5.2. In [2] for some lattice of $G$ in Example 2 it is proved that $G/\Gamma$ is geometrically formal. But the de Rham cohomology of $G/\Gamma$ varies according to a choice of a lattice $\Gamma$.

Example 3. Let $K$ be a finite extension field of $\mathbb{Q}$ with the degree $r$ for positive integers. We assume $K$ admits embeddings $\sigma_1, \ldots, \sigma_s, \sigma_{s+1}, \ldots, \sigma_{s+2t}$ into $\mathbb{C}$ such that $s + 2t = r$, $\sigma_1, \ldots, \sigma_s$ are real embeddings and $\sigma_{s+1}, \ldots, \sigma_{s+2t}$ are complex ones satisfying $\sigma_{s+i} = \sigma_{s+i+t}$ for $1 \leq i \leq t$. We suppose $s > 0$. Denote $O_K$ the ring of algebraic integers of $K$, $O_K'$ the group of units in $O_K$ and

$$O_K'^{\pm} = \{a \in O_K' : \sigma_i(a) > 0 \text{ for all } 1 \leq i \leq s\}.$$  

Define $\sigma : O_K \to \mathbb{R}^s \times \mathbb{C}^t$ by

$$\sigma(a) = (\sigma_1(a), \ldots, \sigma_s(a), \sigma_{s+1}(a), \ldots, \sigma_{s+t}(a))$$  

for $a \in O_K$. We denote

$$\sigma(a) \cdot \sigma(b) = (\sigma_1(a)\sigma_1(b), \ldots, \sigma_s(a)\sigma_s(b), \sigma_{s+1}(a)\sigma_{s+1}(b), \ldots, \sigma_{s+t}(a)\sigma_{s+t}(b))$$  

for $a, b \in O_K$. Then the image $\sigma(O_K)$ is a lattice in $\mathbb{R}^s \times \mathbb{C}^t$. Define $l : O_K'^{\pm} \to \mathbb{R}^{s+t}$ by

$$l(a) = (\log |\sigma_1(a)|, \ldots, \log |\sigma_s(a)|, 2\log |\sigma_{s+1}(a)|, \ldots, 2\log |\sigma_{s+t}(a)|)$$  

for $a \in O_K'^{\pm}$. Then by Dirichlet’s units theorem, the image $l(O_K'^{\pm})$ is a lattice in the vector space $L = \{x \in \mathbb{R}^{s+t} | \sum_{i=1}^{s+t} x_i = 0\}$. By this we have a geometrical representation of the semi-direct product $O_K'^{\pm} \ltimes O_K$ as $l(O_K'^{\pm}) \ltimes \sigma(O_K)$ with

$$\phi(t_1, \ldots, t_{s+t})(\sigma(a)) = \sigma(l^{-1}(t_1, \ldots, t_{s+t})) \cdot \sigma(a)$$  

for $(t_1, \ldots, t_{s+t}) \in l(O_K'^{\pm})$. Since $l(O_K'^{\pm})$ and $\sigma(O_K)$ are lattices of $L$ and $\mathbb{R}^s \times \mathbb{C}^t$ respectively, we have an extension $\bar{\phi} : L \to \text{Aut}(\mathbb{R}^s \times \mathbb{C}^t)$ of $\phi$ and $l(O_K'^{\pm}) \ltimes \sigma(O_K)$ can be seen as a lattice of $L \ltimes \bar{\phi}(\mathbb{R}^s \times \mathbb{C}^t)$. By Theorem 1.1 the solvmanifold $L \ltimes \bar{\phi}(\mathbb{R}^s \times \mathbb{C}^t)/l(O_K'^{\pm}) \ltimes \sigma(O_K)$ is geometrically formal by Theorem 2. For a subgroup $U \subset O_K'^{\pm}$, we have a Lie group $L' \ltimes \bar{\phi}(\mathbb{R}^s \times \mathbb{C}^t)$ which contains $l(U) \ltimes \sigma(O_K)$ as a lattice. The solvmanifold $L' \ltimes \bar{\phi}(\mathbb{R}^s \times \mathbb{C}^t)/l(U) \ltimes \sigma(O_K)$ is also geometrically formal by Theorem 1.1.
Example 4. Let $G = \mathbb{R} \ltimes U_3(\mathbb{R})$ such that
\[ \phi(t) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & e^t x & z \\ 0 & 1 & e^{-t} y \\ 0 & 0 & 1 \end{pmatrix}. \]
The left-invariant forms $\wedge g^*$ on $G$ is generated by $\{e^{-t} dx, e^t dy, dz - x dy, dt\}$. It is known that $G$ has a lattice $\Gamma$ (see [11, Chapter 7]). By simple computations, we have $H^1(g^*) = \langle dt \rangle$, $\dim H^2(g^*) = 0$ and $\dim H^3(g^*) = 1$. Since $G$ is completely solvable, we have $H^*(G/\Gamma, \mathbb{R}) \cong H^*(g^*)$ (see [10]) and hence $H(g) = H(G/\Gamma)$ where $H(g)$ is the set of left-invariant harmonic forms. By $d(\wedge^3 g^*) = 0$, for any invariant metric $g$ on $G$, we have:
\[ H^1(g) = \langle dt \rangle, \]
\[ H^2(g) = 0, \]
\[ H^3(g) = \langle (s g) dt \rangle. \]
Thus any invariant metric on $G/\Gamma$ is formal. Otherwise we have $U_G = U_3(\mathbb{C}) \times \mathbb{C}$ and hence this solvmanifold is different from examples of geometrically formal solvmanifold given in Theorem [11].

6. The extension of Theorem [11]

Let $G$ be a simply connected solvable Lie group and $g$ an invariant metric which we construct in Section 5. Denote $C_g$ the group of the isometrical automorphisms of $(G, g)$. Consider $C_g \ltimes G$ and the projection $p : C_g \ltimes G \to C_g$.

Corollary 6.1. Suppose $G = \mathbb{R}^n \ltimes \mathbb{R}^m$ with a semi-simple action $\phi$. Let $\Gamma \subset C_g \ltimes G$ be a torsion-free discrete subgroup such that $G/\Gamma$ is compact. Suppose $p(\Gamma)$ is finite.

Then the metric $g$ given in the last section is a formal metric on $G/\Gamma$.

Proof. Let $\Delta = \Gamma \cap G$. Since $\Gamma/\Delta \cong p(\Gamma)$, $\Delta$ is a finite index normal subgroup of $\Gamma$ and $G/\Delta$ is compact and hence $\Delta \subset G$ is a lattice. Denote $H(G/\Gamma)$ and $H(G/\Delta)$ the sets of the harmonic forms on $G/\Gamma$ and $G/\Delta$ for the metric $g$. Since we have $A^*(G/\Gamma) = A^*(G/\Delta)^{\Gamma/\Delta}$, we have
\[ H(G/\Gamma) = H(G/\Delta)^{\Gamma/\Delta}. \]
By Theorem [11] $H(G/\Delta)$ is closed under the wedge product, so is $H(G/\Delta)^{\Gamma/\Delta}$. Hence the corollary follows.  

Remark 6.1. Not all cocompact discrete subgroup $\Gamma$ satisfies the assumption of the finiteness of $p(\Gamma)$. See the following example.

Example 5. Let $G = \mathbb{R} \ltimes \mathbb{R}^3$ such that $\phi(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$. Then $G$ has no lattice (see [11 Chapter 7]). Consider the metric $g = e^{-2t} dx^2 + e^{-2t} dy^2 + e^{4t} dz^2 + dt^2$. Then we have $C_g = O(2) \times O(1)$ acting as rotations and reflections on the $(x, y)$-coordinates and reflection on the $z$-coordinate. $C_g \ltimes G$ admits a torsion-free cocompact discrete subgroup $\Gamma$. Since $G \cap \Gamma$ is not a lattice of $G$, $p(\Gamma)$ is not finite. In [11] Chapter 8 it is proved that $\Gamma \cong \mathbb{Z} \ltimes \mathbb{Z}^2$ and for $t \neq 0$ $\phi(t) \in SL_3(\mathbb{Z})$ has a pair of complex conjugate eigenvalues (see [11 Chapter 7]). Hence $\Gamma$ can be a
lattice of a Lie group $H = \mathbb{R} \ltimes_\phi \mathbb{R}^3$ with $\phi(t) = \begin{pmatrix} e^t \cos ct & -e^t \sin ct & 0 \\ e^t \sin ct & e^t \cos ct & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$,
and $G/\Gamma = H/\Gamma$ is geometrically formal by Theorem 1.1.

7. Thurston’s Geometries and Infrasolvmanifold

We say that a compact oriented manifold $M$ admits a geometry $(X,g)$ if $M = X/\Gamma$ where $X$ is a simply connected manifold with a complete Riemannian metric $g$ and $\Gamma$ is a cocompact discrete subgroup of the group $\text{Isom}_g(X)$ of isometries. If $(X,g)$ is a solvable Lie group with an invariant metric $g$, we call it a solvable Lie type geometry. We consider the following 3-dimensional solvable Lie type geometries.

(3-A) $X = E^3 = \mathbb{R}^3$, $g_{E^3} = dx^2 + dy^2 + dz^2$.

(3-B) $X = \text{Nil}^3 = U_3(\mathbb{R}) \ltimes \mathbb{R}^3$ with $\phi(z) = \begin{pmatrix} e^z & 0 & 0 \\ 0 & e^{-z} & 0 \\ 0 & 0 & e^{-2z} \end{pmatrix}$, $g_{\text{Nil}^3} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2$.

By the theory of geometry and topology of 3-dimensional manifolds we have the following theorem (see [21]).

**Theorem 7.1.** A compact aspherical 3-dimensional manifold with the virtually solvable fundamental group admits one of the geometries (3-A~C).

We also consider the following 4-dimensional solvable Lie type geometries (listed in [21]).

(4-A) $X = E^4 = \mathbb{R}^4$, $g_{E^4} = dx^2 + dy^2 + dz^2 + dt^2$.

(4-B) $X = \text{Nil}^4 \ltimes E = U_3(\mathbb{R}) \ltimes \mathbb{R}$, $g_{\text{Nil}^4 \ltimes E} = dx^2 + dy^2 + (dz - xdy)^2 + dt^2$.

(4-C) $X = \text{Nil}^4 = \left\{ \begin{pmatrix} 1 & 0 & 0 & t \\ x & y & z & t \\ 0 & 1 & 0 & \frac{t^2}{2} \\ 0 & 0 & 1 & 0 \end{pmatrix} : x, y, z, t \in \mathbb{R} \right\}$, $g_{\text{Nil}^4} = dx^2 + (dy - tdz)^2 + (dz - tdy + \frac{t^2}{2}dx)^2 + dt^2$.

(4-D) $X = \text{Sol}^3 \times E$, $g_{\text{Sol}^3 \times E} = e^{2z} dx^2 + e^{-2z} dy^2 + dz^2 + dt^2$.

(4-E) $X = \text{Sol}^4_{m,n} = \mathbb{R} \ltimes_\phi \mathbb{R}^4$ such that $\phi(t) = \begin{pmatrix} e^{at} & 0 & 0 \\ 0 & e^{bt} & 0 \\ 0 & 0 & e^{ct} \end{pmatrix}$, where $a, b, c$ are distinct roots of $X^3 - mX^2 - nX - 1$ for real numbers $a < b < c$ and integers $m < n$, $g_{\text{Sol}^4_{m,n}} = e^{-2at} dx^2 + e^{-2bt} dy^2 + e^{-2ct} dz^2 + dt^2$.

(4-F) $X = \text{Sol}^4_0 = \mathbb{R} \ltimes_\phi \mathbb{R}^3$ such that $\phi(t) = \begin{pmatrix} e^t & 0 & 0 \\ 0 & e^t & 0 \\ 0 & 0 & e^{-2t} \end{pmatrix}$, $g_{\text{Sol}^4_0} = e^{-2t} dx^2 + e^{-2t} dy^2 + e^{4t} dz^2 + dt^2$.

(4-G) $X = \text{Sol}^4_1 = \mathbb{R} \ltimes_\phi U_3(\mathbb{R})$ such that $\phi(t) = \begin{pmatrix} 1 & x & z \\ 0 & 1 & y \\ 0 & 0 & 1 \end{pmatrix}$, $g_{\text{Sol}^4_1} = e^{-2t} dx^2 + e^{2t} dy^2 + (dz - xdy)^2 + dt^2$. 

Let $G$ be a simply connected solvable Lie group and $g$ an invariant metric on $G$. We consider the affine transformation group $\text{Aut}(G) \ltimes G$ and the projection $p : \text{Aut}(G) \ltimes G \to \text{Aut}(G)$. Let $\Gamma \subset \text{Aut}(G) \ltimes G$ be a torsion-free discrete subgroup such that $p(\Gamma)$ is contained in a compact subgroup of $\text{Aut}(G)$ and the quotient $G/\Gamma$ is compact. We call $G/\Gamma$ an infra-solvmanifold. If $G$ is nilpotent, $G/\Gamma$ is called an infra-nilmanifold. Since $\Gamma \subset \text{Isom}_g(G)$ does not satisfy $\Gamma \subset \text{Aut}(G) \ltimes G$, a compact manifold with a solvable Lie type geometry is not an infra-solvmanifold.

By Nomizu’s theorem ([18]) we have a non-toral nilmanifold and hence not formal (see [8]). Suppose $\text{Isom}_g(G) \subset \text{Aut}(G) \ltimes G$. Then for an isometry transformation $(\phi, x) \in \text{Aut}(G) \ltimes G$, $\phi$ is an also isometry transformation. By this, for the group $C_g$ of the isometrical automorphisms of $G$, we have $\text{Isom}_g(G) = C_g \times G$. Thus in the assumption $\text{Isom}_g(G) \subset \text{Aut}(G) \ltimes G$, a compact manifold with a solvable Lie type geometry is an infra-solvmanifold. It is known that for the Euclidean geometry $(E^n, g_{E^n} = dx_1^2 + \cdots + dx_n^2)$ we have $\text{Isom}_{g_{E^n}} = O(n) \ltimes \mathbb{R}^n$ and the geometries (3-A∼C) satisfies $\text{Isom}_g(G) \subset \text{Aut}(G) \ltimes G$ (see [21]). In [11], Hillman studied the structures of $\text{Isom}_g(G)$ of the geometries (4-A∼H) and proved $\text{Isom}_g(G) \subset \text{Aut}(G) \ltimes G$. In [12], Hillman proved the following theorem.

**Theorem 7.2.** ([12], Theorem 8) A 4-dimensional infra-solvmanifold is diffeomorphic to a manifold which admits one of the geometries (4-A∼G).

**Remark 7.1.** By Baues’s result in [3], any compact aspherical manifold with the virtually solvable fundamental group is homotopy equivalent to an infra-solvmanifold $G/\Gamma$. But for dimension $\geq 4$, there may exist a compact aspherical manifold with virtually solvable fundamental group which is not diffeomorphic to an infra-solvmanifold.

8. Geometrical formality of 3-manifolds

**Theorem 8.1.** Let $M$ be a compact oriented aspherical 3-manifold with the virtually solvable fundamental group. If $M$ is a torus or not a nilmanifold, then $M$ is geometricaly formal.

**Proof.** By Theorem [21], it is sufficient to consider the geometries (3-A∼C). In the case (3-A), by Corollary 6.1 and the first Bieberbach theorem $g_{E^n}$ is a formal metric on $G/\Gamma$.

In the case (3-C), it is known that $C_g$ is isomorphic to the finite dihedral group $D(8)$ (see [21]) and hence by Corollary 6.1 $g_{\text{Sat}^3}$ is a formal metric on $G/\Gamma$.

Suppose $(G, g)$ is in the case (3-B). Then $C_g$ has two components and the identity component of $C_g$ is isomorphic to a circle $S^1$. Let $\Delta = \Gamma \cap G$. By Generalized Bieberbach’s theorem (see [11]), $\Delta$ is a finite index normal subgroup of $\Gamma$. Consider the projection $p : C_g \ltimes G \to C_g$. If $p(\Gamma)$ is trivial, then $\Gamma \subset G$ is a lattice and $G/\Gamma$ is a non-toral nilmanifold and hence not formal (see [3]). Suppose $p(\Gamma)$ is non-trivial. By Nomizu’s theorem ([18]) we have

$$H^*(G/\Delta, \mathbb{R}) \cong H^*(g)$$

where $g$ is the Lie algebra of $G$. By this we have

$$H^*(G/\Gamma, \mathbb{R}) \cong H^*(G/\Delta, \mathbb{R})^{\Gamma/\Delta} \cong H^*(g)^{\Gamma/\Delta}.$$

In [4] Lemma 13.1], it is shown that a non-trivial semisimple automorphism of a nilpotent Lie algebra $g$ acts non-trivially on $H^1(g)$. Since $\Gamma/\Delta \cong p(\Gamma)$ is a nontrivial finite group,

$$H^1(g)^{\Gamma/\Delta} \neq H^1(g).$$
Since \( \dim H^1(g) = 2, \dim H^1(g)^{\Gamma/\Delta} = 0 \) or 1. If \( \dim H^1(g)^{\Gamma/\Delta} = 0 \), then \( G/\Gamma \) is a rational homology sphere and any metric on \( G/\Gamma \) is formal. Suppose \( \dim H^1(g)^{\Gamma/\Delta} = 1 \). Then \( b_i = 1 \) for any \( 1 \leq i \leq 3 \). For any \( 1 \leq i \leq 3 \), we have

\[
H^*(G/\Gamma, \mathbb{R}) \cong H^*(g)^{\Gamma/\Delta} = \bigoplus_{i=1}^{3}([\alpha_i])
\]

for non-zero cohomology classes \([\alpha_i] \in H^i(g)\). We can choose invariant harmonic forms \( \alpha_i, i = 1, 2, 3 \) for the invariant metric \( g \). Then we have \( H(G/\Gamma) = \bigoplus_{i=1}^{3}(\alpha_i) \), Since all elements of \( \bigwedge^3 g^* \) are harmonic, \( \alpha_1 \wedge \alpha_2 \) is harmonic. For \( i < j \) with \( (i, j) \neq (1, 2) \), we have \( \alpha_i \wedge \alpha_j = 0 \). Thus \( g \) is a formal metric on \( G/\Gamma \). This completes the proof of the theorem. \( \square \)

**Remark 8.1.** There exists a closed 3-dimensional infra-nilmanifold which is not a nilmanifold. By this theorem such a manifold is geometrically formal.

**Example 6.** Consider \( \Gamma = \mathbb{Z} \times \mathbb{Z}^2 \) such that \( \phi(t) = \begin{pmatrix} (-1)^t & (-1)^t \\ 0 & (-1)^t \end{pmatrix} \). Then we can embed \( \Gamma \) in \( Isom_{g_{Nil^3}}(Nil^3) \) (see [13]). By the direct computation of the lower central series, \( \Gamma \) is non-nilpotent and hence \( Nil^3/\Gamma \) is not a nilmanifold.

9. ASPHERICAL MANIFOLDS WITH THE VIRTUALLY SOLVABLE FUNDAMENTAL GROUPS

**Theorem 9.1.** Let \( M \) be an oriented 4-dimensional infra-solvmanifold. If \( M \) is a torus or not a nilmanifold, then \( M \) is geometrically formal.

**Proof.** In the case (4-A), by Corollary 6.1 and the first Bieberbach theorem \( g_{E^4} \) is a formal metric on \( G/\Gamma \).

In the case (4-D) (resp (4-E)), \( G_g \) is isomorphic to the finite group \( D(8) \times (\mathbb{Z}/2\mathbb{Z}) \) (resp. \( (\mathbb{Z}/2\mathbb{Z})^3 \)) (see [11] Chapter 7) and hence \( g_{Sat^3 \times E} \) (resp. \( g_{Sat^4_{m,n}} \)) is a formal metric on \( G/\Gamma \) by Corollary 6.1.

As we showed in Example 5 in the case (4-F) \( G/\Gamma \) is geometrically formal.

In the case (4-B), the group of all the orientation preserving isomorphisms is \( Isom_{g_{Nil^3} \times E}(Nil^3 \times \mathbb{R}) \) (see [23], [24], or [25]). Thus as the proof of Theorem 5.1 for the case (3-B), if \( G/\Gamma \) is a nilmanifold then \( G/\Gamma \) is not formal, and if \( G/\Gamma \) is an infra-nilmanifold but not a nilmanifold then \( g_{Nil^3 \times \mathbb{R}} \) is formal.

In the case (4-C), the group of all the orientation preserving isomorphisms is \( Nil^4 \) itself (see [23], [24], or [25]). Thus oriented \( Nil^4 \) manifolds are only nilmanifolds and so all oriented \( Nil^4 \) manifolds are not formal.

In the case (4-G) we have \( Isom_{g_{Sat^4}}(Sol^4) \cong D(4) \ltimes Sol^4_1 \). For any cocompact discrete subgroup \( \Gamma \subset Isom_{g_{Sat^4}}(Sol^4_1) \), since for the projection \( p : Aut G \ltimes G \rightarrow Aut(G), p(\Gamma) \subset D(4) \) is finite, we have a subgroup \( \Delta \subset \Gamma \) which is a lattice of \( Sol^4_1 \) and we have \( H(Sol^4_1/\Gamma) = H(Sol^4_1/\Delta)^{\Gamma/\Delta} \). In Example 4 we showed that the metric \( g_{Sat^4} \) on the solvmanifold \( Sol^4_1/\Delta \) is formal. Thus the metric \( g_{Sat^4} \) on every \( Sol^4_1 \) manifold is formal. Hence the theorem follows. \( \square \)

Finally we prove:
Theorem 9.2. Let $M$ be a compact oriented aspherical manifold of dimension less than or equal to 4 with the virtually solvable fundamental group. Then $M$ is geometrically formal if and only if $M$ is diffeomorphic to a torus or an infra-solvmanifold which is not a nilmanifold.

Proof. It is sufficient to show that $M$ is an infra-solvmanifold if $M$ is geometrically formal. If $\dim M \leq 2$, it is obvious. If $\dim M = 3$, it follows from Theorem 7.1. We consider $\dim M = 4$. As Remark 7.1, $M$ is homotopy equivalent to an infra-solvmanifold $G/\Gamma$. It is known that the Euler characteristic $\chi(G/\Gamma)$ of an infra-solvmanifold is 0 (see [11, Chapter 8]). Since $G/\Gamma$ is an oriented 4-manifold, $\chi(G/\Gamma) = 0$ implies $b_1(G/\Gamma) \neq 0$. Thus we have $b_1(M) \neq 0$. If $M$ is geometrically formal, we have a submersion $M \to T^{b_1(M)}$ (see [15, Theorem 7]) and hence $M$ is a fiber bundle over a torus $T^{b_1(M)}$. Now we suppose that $M$ is a compact oriented aspherical manifold of dimension 4 with the virtually solvable fundamental group. By the exact sequence of homotopy groups associated by the fiber bundle, the fiber of $M \to T^{b_1(M)}$ is a compact aspherical manifold of dimension less than or equal 3 with the virtually solvable fundamental group, and hence it is an infra-solvmanifold. Thus $M$ is a fiber bundle whose fiber is an infra-solvmanifold and base space is a torus. By [15, Theorem 7], $M$ is diffeomorphic to an infra-solvmanifold with the fundamental group $\pi_1(M)$. \hfill \Box

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