ON MINIMAL EIGENVALUES OF SCHRÖDINGER OPERATORS
ON MANIFOLDS

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Abstract. We consider the problem of minimizing the eigenvalues of the Schrödinger operator \( H = -\Delta + \alpha F(\kappa) \) (\( \alpha > 0 \)) on a compact \( n \)-manifold subject to the restriction that \( \kappa \) has a given fixed average \( \kappa_0 \).

In the one-dimensional case our results imply in particular that for \( F(\kappa) = \kappa^2 \) the constant potential fails to minimize the principal eigenvalue for \( \alpha > \alpha_c = \mu_1/(4\kappa_0^2) \), where \( \mu_1 \) is the first nonzero eigenvalue of \( -\Delta \). This complements a result by Exner, Harrell and Loss, showing that the critical value where the circle stops being a minimizer for a class of Schrödinger operators penalized by curvature is given by \( \alpha_c \). Furthermore, we show that the value of \( \mu_1/4 \) remains the infimum for all \( \alpha > \alpha_c \). Using these results, we obtain a sharp lower bound for the principal eigenvalue for a general potential.

In higher dimensions we prove a (weak) local version of these results for a general class of potentials \( F(\kappa) \), and then show that globally the infimum for the first and also for higher eigenvalues is actually given by the corresponding eigenvalues of the Laplace–Beltrami operator and is never attained.

1. Introduction

In the last years there has been a great interest in the study of optimal properties of eigenvalues of Schrödinger operators of the form \( H = -\Delta + V \) defined on compact manifolds, when some restrictions are imposed on the potential \( V \). Some of these problems are related to several physical phenomena such as motion by mean curvature, electrical properties of nanoscale structures, etc (see, for instance, [A, AHS, EHL, HL, Ke] and the references therein).

In [EHL], the authors considered the case of potentials depending on the curvature \( \kappa \) and studied the problem of minimizing the first eigenvalue of the operator \( H = -d^2/ds^2 + \alpha \kappa^2 \) defined on a closed planar curve with length one. They proved that for \( 0 < \alpha < 1/4 \) the circle is the unique minimizer, while for \( \alpha > 1 \) this is no longer the case, leaving open the question of the value of \( \alpha \) where the transition takes place, and also what happens after this critical value.

More generally, one might consider an operator \( H \) defined on a compact \( n \)-manifold \((M, g)\) by \( H = -\Delta + \alpha F(\kappa) \) and with eigenvalues \( \lambda_0 < \lambda_1 \leq \ldots \), and study the problem of determining

\[ \Lambda_j(\alpha) = \inf_{\kappa \in K} \lambda_j(\kappa), \quad j = 0, 1, \ldots \]

where

\[ K = \left\{ \kappa \in \mathcal{C}(M; \mathbb{R}) : \frac{1}{|M|} \int_M \kappa dv_g = \kappa_0 \right\}. \]

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In particular, we are interested in knowing whether or not there exists a critical value of \( \alpha \), say \( \alpha_c \), where the constant potential stops being a global minimizer for the first eigenvalue. In this paper we show that in the one–dimensional case studied in [EHL] this critical value is in fact equal to \( 1/4 \), and that for \( \alpha \) larger than \( \alpha_c \) the infimum is identically equal to \( \pi^2 \) and is not attained. The first part of this result is a consequence of a more general result which provides an upper bound for \( \alpha_c \) holding in any dimension. Furthermore, we show that for potentials of the form \( \kappa = \kappa_0 + \varepsilon q \) where \( q \) has zero average, this bound is in fact precise for sufficiently small values of \( \varepsilon \), in the sense that for \( \alpha \) smaller than the bound, the constant potential \( \kappa_0 \) gives a smaller eigenvalue than \( \kappa \), while for larger values of \( \alpha \) this is not always the case.

These results could lead us to expect that results similar to those in one dimension would also hold in higher dimensions, that is, that there would exist a nontrivial interval \((0, \alpha_c)\) where the constant potential was the unique minimizer. However, it turns out that for dimensions higher than the first there exist potentials satisfying the given restrictions and which make the principal eigenvalue as close to zero as desired. Thus, we see that in this case the constant potential is never a global minimizer. It remains an open question if it is a local minimizer. A similar statement also holds for higher eigenvalues and for minimizations subject to other types of integral restrictions – see Theorem 4 and the remarks that follow it. The reason for this different behaviour in dimensions higher than the first is directly related to the fact that in this case, given a manifold \( M \) and a geodesic ball \( B_\delta \) of radius \( \delta \) centred at a point \( x_0 \) in \( M \), the Dirichlet eigenvalues of the Laplacian in \( \Omega_\delta = M \setminus B_\delta \) converge to those of the Laplacian in \( M \) as \( \delta \) approaches zero – for a more precise statement of this property see Section 4 and [CF].

Finally, we point out that the results in one dimension enable us to obtain a lower bound for the principal eigenvalue in the case of a general potential (Corollary 3) which, of course, corresponds also to the first eigenvalue of Hill’s equation. Note that one of the motivations behind the study of the minimization of eigenvalues when the potential is subject to integral restrictions was precisely to obtain lower bounds for eigenvalues – see [Ke].

### 2. Notation and general local results

Let \((M, g)\) be a compact Riemannian \( n \)--manifold with metric \( g \) and let \(-\Delta\) denote the Laplace–Beltrami operator defined on \( M \) with eigenvalues \( 0 = \mu_0 < \mu_1 \leq \ldots \) repeated according to their multiplicity. Denote the corresponding orthonormal (with respect to the \( L^2(M) \) inner product induced by the Riemannian measure \( v_g \)) system of eigenfunctions by \( \{v_j\}_{j=0}^{\infty} \). Consider now the operator defined on \( M \) by \( H = -\Delta + \alpha F(\kappa) \), where

\[
\frac{1}{|M|} \int_M \kappa v_g = \kappa_0,
\]

and \( F : \mathbb{R} \to \mathbb{R} \) is assumed to be of class \( C^3 \) in a neighbourhood of \( \kappa_0 \).

The main result in this section is then the following

**Theorem 1.** Assume that \( F'(\kappa_0) \neq 0 \) and define

\[
\alpha^* = \frac{\mu_1 F''(\kappa_0)}{2[F'(\kappa_0)]^2}.
\]
Then, if \( q \) is a continuous real valued function with zero average and not identically zero, we have that for \( \kappa = \kappa_0 + \varepsilon q \) with sufficiently small \( \varepsilon \) (depending on \( q \) and \( \alpha \)), the principal eigenvalue \( \lambda_0 \) of \( H \) satisfies

\[
\lambda_0(\kappa) > \alpha F(\kappa_0), \quad \text{if} \quad 0 < \alpha < \alpha^*,
\]

while for \( \alpha > \alpha^* \) there exist functions \( q \) as above for which

\[
\lambda_0(\kappa) < \alpha F(\kappa_0).
\]

**Proof.** Consider the Schrödinger operator defined on \( M \) by

\[
H_\varepsilon = -\Delta + \alpha F(\kappa_0 + \varepsilon q).
\]

Since \( M \) is compact, the spectrum of \( H_\varepsilon \) is discrete and its first (simple) eigenvalue and the corresponding (normalized) eigenfunction are analytic functions of the (real) parameter \( \varepsilon \in \mathbb{R} \). We thus expand \( \lambda_0 \) and the corresponding eigenfunction \( u \) as a power series of \( \varepsilon \) around zero:

\[
\lambda_0 = \ell_0 + \ell_1\varepsilon + \ell_2\varepsilon^2 + \ldots
\]

\[
u = \phi_0 + \phi_1\varepsilon + \phi_2\varepsilon^2 + \ldots.
\]

On the other hand, we also have that \( F(\kappa_0 + \varepsilon q) = f_0 + f_1\varepsilon + f_2\varepsilon^2 + o(\varepsilon^2) \), where

\[
f_0 = F(\kappa_0), \quad f_1 = F'(\kappa_0), \quad \text{and} \quad f_2 = \frac{1}{2} F''(\kappa_0).
\]

Substituting these expressions in the equation giving the eigenvalues for \( H_\varepsilon \) we obtain, equating like powers in \( \varepsilon \),

\[
\varepsilon^0: \quad -\Delta \phi_0 + \alpha f_0 \phi_0 = \ell_0 \phi_0
\]

\[
\varepsilon^1: \quad -\Delta \phi_1 + \alpha f_0 \phi_1 + \alpha f_1 q \phi_0 = \ell_0 \phi_1 + \ell_1 \phi_0
\]

\[
\varepsilon^2: \quad -\Delta \phi_2 + \alpha f_0 \phi_2 + \alpha f_1 q \phi_1 + \alpha f_2 q^2 \phi_0 = \ell_0 \phi_2 + \ell_1 \phi_1 + \ell_2 \phi_0.
\]

From the first equation it follows that \( \ell_0 = \alpha f_0 \) and that \( \phi_0 \) is constant, which we take to be one. Substituting this in the equation for \( \varepsilon^1 \) and integrating over \( M \) gives that \( \ell_1 \) vanishes and \( \phi_1 \) satisfies

\[
(2.1) \quad -\Delta \phi_1 = -\alpha f_1 q.
\]

Substituting now this in the last equation gives that \( \phi_2 \) satisfies

\[
-\Delta \phi_2 = -\alpha f_2 q^2 - \alpha f_1 q \phi_1 + \ell_2.
\]

Again integrating over \( M \) gives

\[
(2.2) \quad \ell_2 = \frac{\alpha f_2}{|M|} \int_M q^2 dv_g + \frac{\alpha f_1}{|M|} \int_M q \phi_1 dv_g.
\]

Taking squares on both sides of \((2.1)\) we get \( |\Delta (\phi_1)|^2 = \alpha^2 f_1^2 q^2 \). On the other hand, multiplying the same equation by \( \phi_1 \) and integrating over \( M \) gives that

\[
\alpha f_1 \int_M q \phi_1 dv_g = -\int_M |\nabla \phi_1|^2 dv_g.
\]

Substituting these two expressions into \((2.2)\) we finally obtain

\[
\ell_2 = \frac{f_2}{\alpha f_1 |M|} \int_M (\Delta \phi_1)^2 dv_g - \frac{1}{|M|} \int_M |\nabla \phi_1|^2 dv_g,
\]

and it follows from Lemma \((2.1)\) below that \( \ell_2 \) is always positive for \( \alpha < \alpha^* \).
To give an example of a function \( q \) for which \( \ell_2 \) becomes negative when \( \alpha > \alpha^* \) it is sufficient to take \( q = v_1 \). We obtain from (2.1) that in this case
\[
\phi_1 = c - \frac{\alpha}{\mu_1} f_1 v_1,
\]
where \( c \) is an arbitrary constant. Substituting this into the expression for \( \lambda_2 \) yields
\[
\ell_2 = \frac{\alpha}{|M|} \left( f_2 - \frac{\alpha}{\mu_1} f_1^2 \right),
\]
which is negative for \( \alpha > \alpha^* \).

An obvious consequence of this result is that for all \( F \) of the form above there exists a value of \( \alpha \), say \( \alpha^{**} \) such that for \( \alpha > \alpha^{**} \) the constant potential is not a minimizer of the first eigenvalue.

In the case where \( F' \) is allowed to vanish, it is also clear that if \( \kappa_0 = \kappa_0^* \) is a (local) minimizer (resp. maximizer) of \( F \), it follows that, for positive values of \( \alpha \), \( \kappa(x) \equiv \kappa_0^* \) will be a (local) minimizer (resp. maximizer). This is the case, for instance, when \( F(\kappa) = \kappa^2 \) and \( \kappa_0 = 0 \), where obviously \( \kappa = 0 \) is a global minimizer for all \( \alpha \).

The result needed to prove that \( \ell_2 > 0 \) for \( \alpha < \alpha^* \) is neither new nor difficult, but a specific reference could not be found in the literature and so, for the sake of completeness, we provide a proof here.

**Lemma 2.1.** The functional
\[
I_\alpha(u) = \int_M \alpha(\Delta u)^2 - |\nabla u|^2 dv_g
\]
is nonnegative for \( \alpha \geq 1/\mu_1 \).

**Proof.** The spectral problem corresponding to \( I_\alpha \) is
\[
\alpha \Delta^2 u + \Delta u = \gamma u,
\]
which has discrete spectrum \( \gamma_0 \leq \gamma_1 \leq \ldots \). We will prove that if \( \alpha > 1/\mu_1 \) then \( \gamma_j \geq 0 \) for all \( j = 0, 1, \ldots \). To this end rewrite (2.2) as
\[
\Delta(\alpha \Delta u + u) = \gamma u.
\]
It is not difficult to see that if \( u \) is an eigenfunction if and only if
\[
\alpha \Delta u + u = \beta v_j
\]
for some real number \( \beta \) different from zero. For \( \alpha > 1/\mu_1 \) the operator \( \alpha \Delta + I \) is invertible and thus this last equation has one and only one solution given by
\[
u = \beta v_j / (1 - \alpha \mu_j).
\]
Substituting this into (2.3) gives \( \gamma = (\alpha \mu_j - 1) \mu_j \) from which the result follows.

3. The one–dimensional case

In this section we consider the particular case studied in [EHI] with \( F(\kappa) = \kappa^2 \), and for which
\[
\alpha^* = \frac{\mu_1}{4 \kappa_0^2}.
\]
As a consequence of Theorem \ref{thm1} and the results in [EHI] we have the following

**Theorem 2.** In the one dimensional case and for \( F \) as above, \( \alpha_c = \alpha^* \). Furthermore, for \( \alpha > \alpha_c \), \( \Lambda_0(\alpha) \equiv \mu_1/4 \).
Proof. It only remains to show the result for $\alpha$ larger than $\alpha_c$. Clearly in this case $\Lambda_0(\alpha) \geq \mu_1/4$. Consider now the family of potentials given by

$$\kappa_\delta(s) = \begin{cases} \kappa_0/\delta, & 0 < s < \delta, \\ 0, & \delta < s < \ell. \end{cases}$$

Note that although $\kappa_\delta$ is not continuous on the circle, it can be approximated by continuous functions without affecting our results. For this family of potentials we obtain the functional

$$J_\delta(u) = \int_0^\ell [u']^2 \, ds + \frac{\alpha \kappa_0^2}{\delta^2} \int_0^\delta u^2 \, ds,$$

where $u$ is normalized. We now take $u(s) = \sqrt{2} \sin(\pi s/\ell)$ to obtain

$$J_\delta(u) = \frac{\mu_1}{4} + \frac{2 \alpha \kappa_0^2}{\delta^2} \int_0^\delta \sin^2\left(\frac{\pi s}{\ell}\right) ds,$$

and since

$$\lim_{\delta \to 0^+} \int_0^\delta \frac{\sin^2(\pi s) ds}{\delta^2} = 0,$$

it follows that $J_\delta$ can be made to be arbitrarily close to $\mu_1/4$.

Remark 3.1. Clearly for $\alpha > \alpha_c$ the infimum is not attained, as was conjectured in \cite{EHL}.

A simple consequence of Theorem 2 is a lower bound for the principal eigenvalue of the Schrödinger operator on the circle.

**Corollary 3.** Consider the operator $H = -\frac{d^2}{dx^2} + V(x)$ defined on $(0, L)$ with periodic boundary conditions, and define $V_m = \inf_x V(x)$ and $I = \frac{1}{L} \int_0^L [V(x) - V_m]^{1/2} \, dx$.

Then

$$\lambda_0 \geq \begin{cases} V_m + I^2, & \text{if } I \leq \frac{\pi}{L} \\ V_m + \frac{\pi^2}{L}, & \text{if } I > \frac{\pi}{L} \end{cases}$$

with equality for $I < \pi/L$ if and only if $V$ is constant.

Proof. The first inequality follows directly by writing the eigenvalue problem as $-u'' + (V - V_m)u = (\lambda - V_m)u$ and applying the previous corollary with $\kappa = (V - V_m)/\alpha$. The second part is a consequence of the fact that for $\alpha$ larger than $\alpha_c$ the principal eigenvalue must be larger than $\alpha_c \kappa_0^2$.

Remark 3.2. It follows from Theorem 3 that the given inequalities are sharp in both cases.
4. Higher dimensions

In [EHL], the proof of the fact that for $\alpha$ smaller than $\alpha^*$ the constant potential is the unique global minimizer of $\Lambda_0$ relied on a result that is not available in higher dimensions. Namely, while in one dimension we have that

$$\int_{S^1} [(u - u_m)^2] ds \geq \frac{\mu_1}{4} \int_{S^1} (u - u_m)^2 ds,$$

where $u_m$ is the minimum of $u$ in $S^1$, from the results in [CF] it is known that there is no similar result in higher dimensions. More precisely, if we impose that a function $f$ be zero at a finite number of points of a compact manifold with dimension greater than or equal to two, then there is no relation of the form above with a positive constant on the right–hand side. This suggests that an argument similar to that used in the proof of Theorem 2 can now be used for all positive values of $\alpha$, and not just for $\alpha$ larger than $\alpha^*$. This is indeed the case, and we have the following

**Theorem 4.** Assume that $F(0)$ is a global minimum of $F$. Then, for $n$ greater than one, $\Lambda_j(\alpha) \equiv \mu_j - F(0)$ for all positive $\alpha$ and $j = 0, 1, \ldots$.

**Proof.** Fix a point $x_0$ in $M$ and denote by $B_\delta$ the geodesic ball centred at $x_0$ with radius $\delta$. Let now $\Omega_\delta = M \setminus B_\delta$ and define the potential

$$\kappa_\delta(x) = \begin{cases} \frac{\kappa_0}{|B_\delta|}, & x \in B_\delta(x_0), \\ 0, & x \in \Omega_\delta(x_0) \end{cases}$$

(As before, this is discontinuous but can be approximated by continuous functions without changing the results.) By subtracting $F(0)$ on both sides of the equation for the eigenvalues, we can, without loss of generality, take $F(0)$ to be zero. We are thus lead to the functional

$$J_\delta(u) = \int_M |\nabla u|^2 dv_g + \alpha F\left(\frac{\kappa_0}{|B_\delta|}\right) \int_{B_\delta} u^2 dv_g.$$

Consider now the auxiliary eigenvalue problem defined by

$$\begin{cases} -\Delta w = \mu w, & x \in \Omega_\delta \\ w = 0, & x \in \partial \Omega_\delta, \end{cases}$$

and denote its eigenvalues by $0 < \mu_0(\delta) < \mu_1(\delta) \leq \ldots$, with corresponding normalized eigenfunctions $v_{j\delta}$. From the results in [CF] we have that

$$\lim_{\delta \to 0^+} \mu_j(\delta) = \mu_j, \ j = 0, \ldots.$$

We now build test functions $u_{j\delta}$, $j = 0, \ldots$ defined by

$$u_{j\delta}(x) = \begin{cases} v_{j\delta}(x), & x \in \Omega_\delta \\ 0, & x \in B_\delta \end{cases},$$

for which

$$J_\delta(u_{j\delta}) = \int_{\Omega_\delta} |\nabla v_{j\delta}|^2 dv_g,$$
and, by the result from [CF] mentioned above, this converges to $\mu_j$, $j = 0, \ldots$, as $\delta$ goes to zero. Finally, note that for each $\delta$ the set $\{u_{j}\}_{j=0}^{\infty}$ satisfies the necessary orthogonality conditions, since this is the case for $\{v_{j}\}_{j=0}^{\infty}$.

A similar result will also hold in other cases, such as manifolds with boundary with Dirichlet or Neumann boundary conditions, for instance.

5. Concluding remarks

As was pointed out in [EHL] for the one-dimensional case, it is not difficult to see that for negative $\alpha$ the constant potential still maximizes the principal eigenvalue. It is also possible to show that in this case there is no lower bound on this eigenvalue, in the sense that there exist potentials $\kappa$ with fixed average $\kappa_0$ for which this eigenvalue can be made as large (in absolute value) as desired. It is not completely clear what happens to the supremum of the first eigenvalue for positive values of $\alpha$.

Regarding higher dimensions, it was shown that integral restrictions of this and similar type actually impose no restrictions at all as far as minimization is concerned, in the sense that it is possible to approximate the eigenvalues of the Laplacian as much as desired by potentials satisfying the given restrictions. Although we have seen that in this case the constant potential is never a global minimizer for positive $\alpha$, the results in Section 2 raise the question of whether or not it is a local minimizer for $\alpha < \alpha^*$.

We end by remarking that similar results to those in Sections 2 and 3 also hold in the case of manifolds with boundary and Neumann boundary conditions.

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