GENERALIZED HERMITE POLYNOMIALS

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The new method for obtaining a variety of extensions of Hermite polynomials is given. As a first example a family of orthogonal polynomial systems which includes the generalized Hermite polynomials is considered. Apparently, either these polynomials satisfy the differential equation of the second order obtained in this work or there is no differential equation of a finite order for these polynomials.

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1. Introduction

In our former work ([1]) we constructed an appropriate oscillator algebra \( A_\mu \) corresponding to the system of polynomials which are orthonormal with respect to a measure \( \mu \) in the space \( \mathbb{H}_x = L^2(R^1; \mu(dx)) \). By a standard manner the energy operator (hamiltonian) \( H_\mu = X_\mu^2 + P_\mu^2 \) was defined. The position operator \( X_\mu \) was introduced by the recurrent relations of the given polynomials system; the momentum operator \( P_\mu \) was determined as an unitary equivalent to the position operator \( Y_\mu \) in the dual space \( \mathbb{H}_y = L^2(R^1; \nu(dy)) \).

In ([1]) it was proved that the usual differential equations for the classical polynomials are equivalent to the equations of the form \( H_\mu \psi_n = \lambda_n \psi_n \), where the eigenvalues of the corresponding hamiltonian \( H_\mu \) denote by \( \lambda_n \). The central problem with a derivation of the differential equations was finding a representation of the annihilation operator \( a_\mu^- \) (or another "reducing" operator) of the algebra \( A_\mu \) by a differential operator in the space \( \mathbb{H}_x \). Unfortunately, these formulas ([2]) are rather complicated in the general case. Therefore our interest is in describing such orthogonal polynomials systems for which appropriate representations are simple. On this basis one can obtain some differential equations of a finite order (it is desirable that we have to deal only with differential equations of the second order).

The results of this work may be thought of as first step forward in this direction. From our point of view we consider a family of orthogonal polynomial systems which includes the generalized Hermite polynomials ([3]). These polynomials have been studied extensively in the monograph ([4]). Therefore the polynomials of the considered family is called Hermite-Chihara polynomials.

The paper is organized as follows. A generalized derivation operator is introduced in Sec.2. By these operators a family of the Hermite-Chihara polynomials is determined in Sec.3. More exactly the annihilation operator \( a_\mu^- \) of the algebra \( A_\mu \) corresponding to the system of the polynomials may be represented by a generalized derivation operator \( D_\vec{v} \). This operator is defined by a positive sequence \( \vec{v} \). In what follows we shall call this sequence \( \vec{v} \) a "governing sequence". In Sec.4 we construct the generators \( X_\mu, P_\mu \) and \( H_\mu \) of the algebra \( A_\mu \) corresponding to a system of the Hermite-Chihara polynomials. As an example of such polynomials we consider the "classical" Hermite-Chihara polynomials ([4]) in Sec.5. Moreover, in this section a new derivation of the well-known ([2],[4])

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differential equation for these polynomials is presented. Further, in Sec.6 we introduce a special family of orthogonal systems of Hermite-Chihara polynomials which includes the classical Hermite-Chihara polynomials. Furthermore, in this section we construct a "governing sequence" $\vec{v}$ of an appropriate generalized derivation operator $D_{\vec{v}}$. Then in Sec.7 we obtain a differential equation of the second order for above-mentioned polynomials by analogy with the derivation of the differential equation given in Sec.5. Finally, in the conclusion we consider the following conjecture. If the polynomials of a system of orthogonal Hermite-Chihara polynomials satisfy a differential equation of the second order, then these polynomials belong to the special family of orthogonal systems of Hermite-Chihara polynomials introduced in Sec.6. Moreover, the other Hermite-Chihara polynomials do not satisfy any differential equation of a finite order.

2. GENERALIZED DERIVATION OPERATOR

In this section we introduce a new class of differential operators (they are the infinite order in general case) which play a large role in the construction of the Hermite-Chihara polynomials. Let $\vec{v} = \{v_n\}_{n=0}^{\infty}$ be a monotone nondecreasing sequence:

$$1 = v_0 \leq v_1 \leq v_2 \leq \cdots \leq v_n \leq \ldots . \quad (2.1)$$

This sequence $\vec{v}$ define a linear operator $D_{\vec{v}}$ by the relations:

$$D_{\vec{v}}x^0 = 0, \quad D_{\vec{v}}x^n = v_{n-1}x^{n-1}, \quad n = 1, 2, \ldots , \quad (2.2)$$

on the set of the formal power series of real argument $x$.

We will seek for the operator $D_{\vec{v}}$ of the type

$$D_{\vec{v}} = \sum_{n,m=0}^{\infty} a_{nm}x^n \frac{d^m}{dx^m} . \quad (2.3)$$

Substituting (2.3) in (2.2), we get the following formula:

$$D_{\vec{v}} = \sum_{k=0}^{\infty} \varepsilon_k x^{k-1} \frac{d^k}{dx^k} , \quad (2.4)$$

The coefficients $\{\varepsilon_k\}_{k=0}^{\infty}$ are defined by the recurrent relations:

$$\varepsilon_1 = v_0 = 1, \quad \varepsilon_k = \frac{v_{k-1}}{k!} - \varepsilon_{k-1} - \frac{\varepsilon_{k-2}}{2!} - \cdots - \frac{\varepsilon_1}{(k-1)!}, \quad k = 1, 2, \ldots . \quad (2.5)$$

**Definition 2.1.** A differential operator $D_{\vec{v}}$ determined by formulas (2.4),(2.5) is called a generalized derivation operator induced of the sequence $\vec{v}$.

**Lemma 2.2.** For the order of a generalized derivation operator $D_{\vec{v}}$ defined by formulas (2.4),(2.3) to be finite it is necessary and sufficient that the following equalities:

$$\varepsilon_{k+1} = \varepsilon_{k+2} = \cdots = 0. \quad (2.6)$$

was valid.

To take three examples of generalized derivation operators of a finite order.

1. Let $k = 1$. There exist a unique solution of the system (2.6):

$$\vec{v} = \{n + 1\}_{n=0}^{\infty} . \quad (2.7)$$

The generalized derivation operator $D_{\vec{v}}$ corresponding to $\vec{v}$ take the following form:

$$D_{\vec{v}} = \frac{d}{dx} . \quad (2.8)$$
2. Let \( k = 2 \) and let \( v_1 \) to be a number such that \( v_1 \geq 1 \). There exist a one-parameter family of the solution \( \vec{v} = \{v_n\}_{n=0}^{\infty} \) of the system (2.6):

\[
v_0 = 1, \quad v_n = C_n^{n+1} v_1 - n^2 + 1, \quad n = 1, 2, \ldots.
\] (2.9)

The generalized derivation operator \( D_{\vec{v}} \) corresponding to \( \vec{v} \) take the following form:

\[
D_{\vec{v}} = \frac{d}{dx} + x\left(\frac{v_1}{2} - 1\right)\frac{d^2}{dx^2}.
\] (2.10)

If \( v_1 = 4 \), then from (2.9), (2.10) we have

\[
\vec{v} = \{(n + 1)^2\}_{n=0}^{\infty}, \quad D_{\vec{v}} = \frac{d}{dx} + x\frac{d^2}{dx^2}.
\] (2.11)

3. Let \( k = 3 \) and let \( v_1, v_2 \) to be some number such that \( 1 \leq v_1 \leq v_2 \). There exist a two-parameter family of the solution \( \vec{v} = \{v_n\}_{n=0}^{\infty} \) of the system (2.6):

\[
v_0 = 1 \leq v_1, \quad v_n = C_n^{n+1} v_2 - \frac{(n+1)n(n-2)}{2} v_1 + \frac{(n+1)(n-1)(n-2)}{2}, \quad n = 1, 2, \ldots.
\] (2.12)

The generalized derivation operator \( D_{\vec{v}} \) corresponding to \( \vec{v} \) take the following form:

\[
D_{\vec{v}} = \frac{d}{dx} + x\left(\frac{v_1}{2} - 1\right)\frac{d^2}{dx^2} + x^2\frac{v_2}{2} - 3v_1 + \frac{3}{3!} \frac{d^3}{dx^3},
\] (2.13)

If \( v_1 = 8, \quad v_2 = 27 \), then from (2.12) and (2.13) we have

\[
\vec{v} = \{(n + 1)^3\}_{n=0}^{\infty}, \quad D_{\vec{v}} = \frac{d}{dx} + x\frac{d^2}{dx^2} + x^2\frac{d^3}{dx^3}.
\] (2.14)

3. Hermit-Chihara Polynomials

Let \( \mu \) be a symmetric probability measure, i.e. the all odd moments of the measure \( \mu \) are vanish and \( \int_{-\infty}^{\infty} \mu(dx) = 1 \). In this section we consider a system of polynomials which are orthonormal with respect to the measure \( \mu \), such that there is a representation of the annihilation operator of the oscillator algebra \( A_\mu \) corresponding to this system by a generalized derivation operator.

Recall (\([4]\)) that the recurrent relations of a canonical orthonormal polynomials system \( \{\psi_n(x)\}_{n=0}^{\infty} \) take the following form:

\[
x \psi_n(x) = b_n \psi_{n+1}(x) + b_{n-1} \psi_{n-1}(x), \quad n \geq 1,
\] (3.1)

\[
\psi_0(x) = 1, \quad \psi_1(x) = \frac{x}{b_0}.
\] (3.2)

In (\([4]\)) it was described how to get the positive sequence \( \{b_n\}_{n=0}^{\infty} \) from the given sequence \( \{\mu_{2n}\}_{n=0}^{\infty} \) of even moments of a symmetric positive measure \( \mu \).

The question we are interested now is when for a canonical orthonormal polynomials system \( \{\psi_n(x)\}_{n=0}^{\infty} \) there are two sequences such that:

1. a positive sequence \( \vec{v} = \{v_n\}_{n=0}^{\infty} \) which satisfies (2.1);
2. a real sequence \( \vec{v} = \{v_n\}_{n=0}^{\infty} \) for which are hold the following relations:

\[
D_{\vec{v}} \psi_0 = 0, \quad D_{\vec{v}} \psi_n = v_n \psi_{n-1}, \quad n = 1, 2, \ldots,
\] (3.3)

where the generalized derivation operator \( D_{\vec{v}} \) is determined by formulas (2.4), (2.5).

We denote by \([n]\) the following symbol:

\[
[0] = 0, \quad [n] = \frac{b_{n-1}}{b_0}, \quad n = 1, 2, \ldots.
\] (3.4)
Let $J$ be a symmetric Jacobi matrix

$$J = \{b_{ij}\}_{i,j=0}^\infty$$

which has the positive elements $b_{i,i+1} = b_{i+1,i}, \ i = 0, 1, \ldots$ only distinct from zero. Then the polynomials of the first kind can be represented in the form (3.1):

$$\psi_n(x) = \frac{\epsilon(\frac{x}{2})}{\sqrt{[n]!}} b_0^{2m-n} \alpha_{2m-1,n-1} x^{n-2m}, \tag{3.5}$$

where the greatest integer function is denoted by $\epsilon(\cdot)$. The coefficients $\alpha_{2m-1,n-1}$ for any $n \geq 1, \ \epsilon(\frac{x}{2}) \geq m \geq 1$ are defined by the following equalities:

$$\alpha_{-1,n-1} = 0, \ \ \alpha_{2m-1,n-1} = \sum_{k_1=2m-1}^{n-1} [k_1] \sum_{k_2=2m-3}^{k_1-2} [k_2] \cdots \sum_{k_m=1}^{k_{m-1}-2} [k_m]. \tag{3.6}$$

Substituting (3.6), (3.5) and (3.4) into (3.3), it is easy to prove the following theorem (2).

**Theorem 3.1.** Let the orthonormal polynomial system $\{\psi_n(x)\}_{n=0}^\infty$ is defined by (3.6), (3.5) and (3.4). For existence two sequences $\bar{v} = \{v_n\}_{n=0}^\infty$ and $\bar{\gamma} = \{\gamma_n\}_{n=0}^\infty$ such that the conditions (2.3) are hold it is necessary and sufficient that

1. the sequence $\bar{v} = \{v_n\}_{n=0}^\infty$ satisfies (2.4) and the following conditions:

$$v_{n-2}v_{p-1} + v_{2p-3}v_{n-2p} = v_n v_{2p-3} + v_{2p-1} v_{n-2p}, \tag{3.7}$$

for any $n \geq 2, \ 2p \leq n$;

2. the coefficients $\alpha_{2m-1,n-1}$ take the following form

$$\alpha_{2m-1,n-1} = \frac{[2m-1]!!(v_{n-1})!}{(v_{2m-1})!(v_{n-2m-1})!}, \quad (v_k)! = v_0 v_1 \cdots v_k, \tag{3.8}$$

as $n \geq 1, \ 2m \leq n$ and regarding $(v_{-1})! = (v_0)! = 1$.

In this case the sequence $\bar{\gamma} = \{\gamma_n\}_{n=0}^\infty$ is defined by the following formulas:

$$\gamma_n = \sqrt{\frac{v_1 v_{n-1}}{b_0^2 (v_n - v_{n-2})}}, \quad n \geq 1. \tag{3.9}$$

Here we will not given the proof of this theorem (see [2]) to save room. However we present some formulas arising from the proof. These expressions relate the sequence $\bar{\gamma} = \{\gamma_n\}_{n=0}^\infty$ and the coefficients $\alpha_{2m-1,n-1}:

$$\gamma_1 = \frac{1}{b_0}, \quad b_0 \sqrt{2} \gamma_2 = \varepsilon_1 + \varepsilon_2, \tag{3.10}$$

$$\sqrt{2p+1} \gamma_{2p+1} = \frac{\alpha_{2p-1,2p} [2p-1]!!}{\alpha_{2p-1,2p+1}} \gamma_1, \tag{3.11}$$

$$\sqrt{2p+2} \gamma_{2p+2} = \frac{\alpha_{2p-1,2p} [2p]!!}{\alpha_{2p-1,2p+1}} \sqrt{2} \gamma_2, \tag{3.12}$$

where $\varepsilon_1$ and $\varepsilon_2$ are defined by (2.5).

Now we shall give the following definition.

**Definition 3.2.** The orthonormal polynomials system $\{\psi_n(x)\}_{n=0}^\infty$ completed in $H_\infty = L_r^2(R; \mu(dx))$ is called a system of Hermite-Chihara polynomials if these polynomials are defined by (3.6), (3.7) and (3.4).
Remark 3.3. It is clear that the Hermite polynomials fall in this class. Here
\[ \mathcal{v} = \{n + 1\}_{n=0}^{\infty}, \quad D\mathcal{v} = \frac{d}{dx}, \quad b_{n-1}^2 = \frac{n}{2}, \quad [n] = n, \quad n \geq 1. \]
According to (3.8), we have
\[ \alpha_{2m-1,n-1} = \frac{n!}{2^m m! (n-2m)!}. \tag{3.13} \]
Substituting (3.13) into (3.5), we obtain the usual form of the Hermite polynomials (see [2, 3]). In addition, \( \gamma_n = \sqrt{2n}, \quad n \geq 1 \) and then (3.3) is reduced to the usual rule of derivation for the Hermite polynomials:
\[ \frac{d}{dx} H_n(x) = 2n H_{n-1}(x). \tag{3.14} \]

Remark 3.4. According to the theorem 3.1, by any sequence \( \mathcal{v} = \{v_n\}_{n=0}^{\infty} \) complying with (2.1) and (3.7) we can write the the coefficients \( \alpha_{2m-1,n-1} \) which take the following form:
\[ [1] = 1, \quad [n] = \frac{v_{n-1}(v_n - v_{n-2})}{v_1}, \quad n \geq 2, \tag{3.15} \]
\[ b_{n-1}^2 = b_0^2 \frac{v_{n-1}(v_n - v_{n-2})}{v_1}, \quad n \geq 2. \tag{3.16} \]
The polynomials \( \psi_n(x) \) satisfy the recurrent relations (3.1) and (3.2). By solving the Hamburger moment problem of the Jacobi matrix \( J \), we obtain the symmetric probability measure \( \mu \) such that the polynomials of the system \( \{\psi_n(x)\}_{n=0}^{\infty} \) are orthonormal with respect to \( \mu \). If the moment problem for the Jacobi matrix \( J \) is a determined one, then the measure \( \mu \) is defined uniquely. Otherwise (when the moment problem for the Jacobi matrix \( J \) is a undetermined one) there is a infinite family of such measures (see [3]).

4. Oscillator algebra for the Hermite-Chihara polynomials

In this section we construct the generalized Heisenberg algebra \( A_{\mu} \) corresponding to the system of the Hermite-Chihara polynomials (see [1]).

Let \( \mathcal{v} = \{v_n\}_{n=0}^{\infty} \) be the positive sequence such that the conditions (2.1) and (3.7) are hold. Then the sequence \( \{b_n\}_{n=0}^{\infty} \) can be found by (3.16). Furthermore, we obtain the system of the Hermite-Chihara polynomials \( \{\psi_n(x)\}_{n=0}^{\infty} \) by the formulas (3.3) and (3.4). These polynomials satisfy the recurrent relations (3.1) and (3.2) with above-mentioned coefficients \( \{b_n\}_{n=0}^{\infty} \). Under the condition
\[ \sum_{n=0}^{\infty} b_n^{-1} = \frac{v_1}{b_0} \sum_{n=1}^{\infty} \frac{1}{\sqrt{v_n(v_{n+1} - v_{n-1})}} = \infty. \tag{4.1} \]
the moment problem for the corresponding Jacobi matrix is a determined one (see [3]). There is the only symmetric probability measure \( \mu \) such that the polynomials \( \{\psi_n(x)\}_{n=0}^{\infty} \) are orthonormal in the space \( H_x = L^2(R; \mu(dx)) \). In addition, the even moments \( \mu_{2n} \) of the measure \( \mu \) can be found from the following algebraic equations system (\( b_{-1} = 0, \quad n \geq 0 \))
\[ \sum_{m=0}^{\infty} \sum_{m=0}^{\infty} \frac{(-1)^{m+s}}{(b_{n-1})!} \alpha_{2m-1,n-1} \alpha_{2s-1,n-1} \mu_2 + 2n - 2m - 2s = b_{n-1}^2 + b_n^2. \tag{4.2} \]

It is easy to check that the condition (4.1) is correct for the "classical" Hermite-Chihara polynomials to be considered in the next section.
From theorem 3.1 it follows that there are the generalized derivation operator \( D_\vec{v} \) determined for given sequence \( \vec{v} \) by formulas (2.4), (2.5) and the sequence \( \vec{\gamma} = \{\gamma_n\}_{n=0}^\infty \):

\[
\gamma_n = \frac{v_{n-1}}{b_{n-1}}, \quad n \geq 1,
\]

such that the relations (3.3) are valid.

Using the methods of ([1]), we construct the canonical orthonormal polynomials system \( \{\psi_n(x)\}_{n=0}^\infty \) in the space \( H \), completed in the space \( \mathbb{C}^N \).

The operator-function \( f(N) \) acts on basis vectors \( \{\psi_n(x)\}_{n=0}^\infty \) by formulas:

\[
f(N)\psi_0 = 0, \quad f(N)\psi_1 = \sqrt{2b_0^2}\psi_1, \quad f(N)\psi_n = \sqrt{2b_0^2}\frac{v_n - v_{n-2}}{v_1}\psi_n,
\]

where \( n \geq 2 \). The position operator \( X_\mu \) is defined by the recurrence relations (3.1) and (3.2). Using \( X_\mu \) and \( a_\mu^- \), we determine by the well-known formulas (see [1]) the operators \( a_\mu^+, P_\mu \) (the momentum operator) and \( H_\mu \) (hamiltonian):

\[
P_\mu = i(\sqrt{2}a_\mu^- - X_\mu), \quad a_\mu^+ = \sqrt{2}X_\mu - a_\mu^-,
\]

\[
H_\mu = X_\mu^2 + P_\mu^2 = \sqrt{2}(a_\mu^-X_\mu + X_\mu a_\mu^+) = a_\mu^-a_\mu^+ + a_\mu^+a_\mu^-.
\]

We have the following commutation relation:

\[
[a_\mu^-, a_\mu^+] = 2(B(N + I) - B(N)).
\]

The operator-function \( f(N) \) acts on basis vectors by formulas:

\[
B(N)\psi_0 = 2b_0^2, \quad B(N)\psi_n = b_{n-1}^2\psi_n = b_0^2\frac{v_{n-1}(v_n - v_{n-2})}{v_1}\psi_n, \quad n \geq 1.
\]

Moreover, the ”energy levels” are

\[
\lambda_0 = 2b_0^2, \quad \lambda_n = 2(b_{n-1}^2 + b_n^2) = \frac{2b_0^2}{v_1}(v_nv_{n+1} - v_{n-1}v_{n-2}),
\]

where \( n \geq 1 \).

In what follows our prime interest is with the following question. Is it possible to get a differential equation of the second order for Hermite-Chihara polynomials from the equation \( H_\mu \psi_n = \lambda_n \psi_n \).

5. Classical Hermite-Chihara Polynomials

Now we consider a particular case of the Hermite-Chihara polynomials, namely, the well-known (see [3]) generalized Hermite polynomials which have been studied extensively in ([1]) (see also ([3])).

We denote by \( \mathbb{H}_\gamma \), the Hilbert space

\[
\mathbb{H}_\gamma = L^2(R; |x|^{\gamma}(\Gamma(\frac{1}{2}(\gamma + 1)))^{-1}\exp(-x^2)dx), \quad \gamma \geq -1.
\]

Using methods of ([1]), we construct the canonical orthonormal polynomials system \( \{\psi_n(x)\}_{n=0}^\infty \) completed in the space \( \mathbb{H}_\gamma \). The polynomials \( \psi_n(x) \) satisfy the recurrent relations (3.4)
and (3.2). The coefficients \( \{b_n\}_{n=0}^{\infty} \) are defined by formulas (3.16), where \( b_0 = \sqrt{\frac{\gamma+1}{2}} \) and the sequence \( \bar{v} = \{v_n\}_{n=0}^{\infty} \) is given by the following equalities:

\[
v_n = \begin{cases} \gamma+n+1 & n = 2m, \\ \frac{n+1}{\sqrt{n+1}} & n = 2m+1. \end{cases}
\] (5.2)

It is clear that \( v_0 = 1 \), \( v_1 = 2 \gamma+1 \). The coefficients \( \{b_n\}_{n=0}^{\infty} \) are defined by the formulas:

\[
b_{n-1} = \frac{1}{2} \left\{ \begin{array}{ll} \sqrt{n} & n = 2m, \\ \sqrt{n+\gamma} & n = 2m+1. \end{array} \right. \] (5.3)

The formulas (3.8), (3.5), (3.4) give a explicit form of the polynomials \( \psi_n(x) \). Recall that the polynomials

\[
K_n(x) = s_n \psi_n(x), \quad n \geq 0,
\]
as \( s_0 = 1 \) and \( s_n = (b_{n-1})! \), are named the generalized Hermite polynomials in ([4]) (see also ([8])). In what follows we shall call the polynomials \( \psi_n(x) \) as the "classical Hermite-Chihara polynomials". It is easy to prove that the family of these polynomials is a particular case of the more general class of Hermite-Chihara polynomials and that the generalized derivation operator \( D_{\bar{v}} \) corresponding to the given sequence \( \bar{v} \) is determined by formulas (2.4) and (2.5), where

\[
\varepsilon_1 = 1, \quad \varepsilon_m = \frac{(-2)^{m-1}}{m!} \frac{\gamma}{\gamma+1}, \quad m \geq 2.
\] (5.4)

In addition, the sequence \( \bar{\gamma} = \{\gamma_n\}_{n=0}^{\infty} \) appearing in (3.3) is defined by equalities:

\[
\gamma_n = \frac{\sqrt{2}}{\gamma+1} \left\{ \begin{array}{ll} \sqrt{n} & n = 2m, \\ \sqrt{n+\gamma} & n = 2m+1. \end{array} \right. \] (5.5)

Comparing (4.4), (4.5) and (3.3), we obtain

\[
a^{-\mu} = \frac{\gamma+1}{\sqrt{2}} D_{\bar{v}}. \] (5.6)

The following formulas are known ([4]) (see, also([8])):

\[
\frac{d}{dx} \psi_0 = 0, \quad \frac{d}{dx} \psi_n = \frac{n}{b_{n-1}} \psi_{n-1} + \frac{(n-1)\theta_n}{2b_{n-1}b_{n-2}} X^{-1} \psi_{n-2} = 2b_{n-2} \psi_{n-1} - \frac{\theta_n}{x} \psi_n, \quad n \geq 1,
\] (5.7)

where

\[
\theta_n = \theta_n(\gamma) = \gamma \frac{1 - (-1)^n}{2}. \] (5.8)

Taking into account how the annihilation operator \( a^{-\mu} \) and the number operator \( N \) act on the basis vectors \( \{\psi_n(x)\}_{n=0}^{\infty} \), it is easy to get from the relations (5.7)-(5.9) the following formula:

\[
X_{\mu} \frac{d}{dx} - N = (a^{-\mu})^2. \] (5.10)

Note also that the action of the position operator \( X_{\mu} \) on the basis vectors \( \{\psi_n(x)\}_{n=0}^{\infty} \) is defined by (3.1) and (5.3). Now we consider the operator

\[
\Theta_N = 2B(N) - N,
\] (5.11)
where the operator-function $B(N)$ is defined by (4.10). Using (5.9) and (5.11), we see that
\[ \Theta_N \psi_n = \theta_n \psi_n, \quad n \geq 1. \] (5.12)
Taking into account the equation $H_\mu \psi_n = \lambda_n \psi_n$, where a hamiltonian $H_\mu$ defined by (4.7), and the equality $a_\mu^+ a^-_\mu = 2B(N)$, we have the following relation:
\[ a^-_\mu a^+_\mu = 2B(N + I). \] (5.13)
Moreover, from (5.8) it follows that
\[ a^-_\mu = \frac{1}{\sqrt{2}} \frac{d}{dx} + \frac{1}{\sqrt{2}} X^{-1}_\mu \Theta_N. \] (5.14)
Now from (5.13), (5.14) and (4.7) we have
\[ (\frac{d}{dx} + X^{-1}_\mu \Theta_N)(X_\mu - \frac{1}{2} \frac{d}{dx} \Theta_N^{-1} X_\mu) = 2B(N + I). \] (5.15)
Multiplying both sides of (5.15) by $-2X_\mu$ from the left, we obtain:
\[ -2X_\mu(I + X_\mu \frac{d}{dx}) + X_\mu \frac{d^2}{dx^2} + X_\mu(-X^{-2}_\mu \Theta_N + X^{-1}_\mu \Theta_N) \]
\[ -2\Theta_N X_\mu + \Theta_N \frac{d}{dx} + \Theta_N X^{-1}_\mu \Theta_N = -2X_\mu 2B(N + I). \] (5.17)
It is not hard to prove that :
\[ \Theta_N X^{-1}_\mu \Theta_N \psi_n = 0, \quad \Theta_N X_\mu = X_\mu \Theta_{N+1}, \] (5.18)
\[ 2X_\mu(2B(N + I) - \Theta_{N+1}) - 2X_\mu = 2X_\mu N, \] (5.19)
\[ (\Theta_N \frac{d}{dx} + \frac{d}{dx} \Theta_N) \psi_n = \gamma \psi'_n. \] (5.20)
Applying both sides of (5.17) to $\psi_n$ and using (5.15)-(5.20), we get the following differential equation
\[ x\psi''_n + (\gamma - 2x^2)\psi'_n + (2nx - \frac{\theta_n}{x})\psi_n = 0, \quad n \geq 0, \] (5.21)
which is coincident with the well-known differential equation for the classical Hermite-Chihara polynomials [4] (see also [3]).

Remark 5.1. The generators $a^+_\mu, a^-_\mu$ of the generalized Heisenberg algebra $A_\mu$ corresponding to the classical Hermite-Chihara polynomials system subject to the following commutative relation (see [3]):
\[ [a^-_\mu, a^+_-\mu] = (\gamma + 1)I - 2\Theta_N. \] (5.22)
The "energy levels" of the associated oscillator are equal to:
\[ \lambda_0 = \gamma + 1, \quad \lambda_n = 2n + \gamma + 1, \quad n \geq 1. \] (5.23)
Finally, it follows from (4.7) and (5.14) that the momentum operator take the following form:
\[ P_\mu = i(\frac{d}{dx} + X^{-1}_\mu \Theta_N - X_\mu). \] (5.24)
6. Construction of the "governing sequence" $\vec{v}$ of a generalized derivation operator $D_{\vec{v}}$ for the classical Hermite-Chihara polynomials

Let $\{\psi_n(x)\}_{n=0}^{\infty}$ be a orthonormal Hermite-Chihara polynomials system. According to theorem 3.1, there is a sequence $\vec{v}$ such that (2.1) and (3.7) are hold. The formula (3.9) allows us to define the sequence $\vec{\gamma}$ by $\vec{v}$. Furthermore, the generalized derivation operator $D_{\vec{v}}$ corresponding to $\vec{v}$ is a reducing operator for the system $\{\psi_n(x)\}_{n=0}^{\infty}$, i.e. the equalities (3.3) are valid. In this section we obtain the exact condition on $\vec{v}$ which select some family of Hermite-Chihara polynomials. This family is a natural extension of the set of classical Hermite-Chihara polynomials. The associated set of $\vec{v}$ is a three-parameter family depending on the parameters $b_0, v_1$ and $v_2$. But it turn out that the parameter $v_1$ is unessential, so that the above-mentioned family is really a two-parameter one. According to (2.4) and (2.5), the generalized derivation operator $D_{\vec{v}}$ corresponding to a sequence $\vec{v}$ complying with (2.1) and (3.7) take the following form:

$$D_{\vec{v}} = X^{-1}(B_1 + \overline{B_1}) = \frac{d}{dx} + X^{-1}B_1, \quad B_1 = X \frac{d}{dx}.$$  

(6.1)

where

$$\overline{B_1} = \sum_{k=2}^{\infty} \varepsilon_k x^k \frac{d^k}{dx^k}. \quad (6.2)$$

The coefficients $\varepsilon_k$ in (6.2) are defined from given sequence $\vec{v}$ by the recurrent relations (2.3).

Remark 6.1. For the classical Hermite-Chihara polynomials it follows from (6.2) and (5.4) that

$$X \frac{d}{dx} + B_1^d = \delta(N),$$

(6.3)

where $\delta(N)$ is the projection on the subspace of the polynomials of odd degree:

$$\delta(N)x^n = \theta_n(1)x^n, \quad n \geq 0,$$

(see (5.9)) and hence

$$\delta(N)\psi_n^d = \theta_n(1)\psi_n^d.$$

We denote by $\{\psi_n(x)\}_{n=0}^{\infty}$ the orthonormal Hermite-Chihara polynomials system which is constructed according to remark 3.4 for given sequence $\vec{v}$. Obviously,

$$\overline{B_1} \psi_0 = \overline{B_1} \psi_1 = 0.$$

Now we shall restrict our consideration to a particular class of the Hermite-Chihara polynomials for which there are two real sequence $\overline{\vec{v}} = \{\overline{v}_n\}_{n=2}^{\infty}$ and $\overline{\vec{\beta}} = \{\overline{\beta}_n\}_{n=0}^{\infty}$ such that:

$$\overline{B_1} \psi_2 = \overline{\delta}_2 X \psi_1, \quad \overline{B_1} \psi_n = \overline{\delta}_n X \psi_{n-1} + \overline{\beta}_n \psi_{n-2}, \quad n \geq 3. \quad (6.4)$$

Replacing $\overline{\delta}_n$ by $\delta_n$ and $\overline{\beta}_n$ by $\beta_n$, we see from (5.1) and (3.3) that the condition (6.4) is valid for $B_1$ too. Then it is clear that the assumption (6.4) takes the place of the rule of derivation (5.8). Substituting (6.2) into (6.4), and taking into account (3.5), we obtain
the following relation:

\[
\sum_{k=2}^{n} \frac{\epsilon(n-k)}{k!} \frac{(-1)^{m}}{[n]!} b_{2m-n} \alpha_{2m-1,n-1} x^{n-2m} \frac{(n-m)!}{(n-2m-k)!} = \\
= \frac{\epsilon(n-k)}{k!} \frac{(-1)^{m}}{[n-2]!} b_{2m-n+2} \alpha_{2m-1,n-3} x^{n-2m-2} + \\
+ \alpha_{n} \sum_{m=0}^{n} \frac{(-1)^{m}}{[n-1]!} b_{2m-n+1} \alpha_{2m-1,n-2} x^{n-2m}, \quad n \geq 2.
\] (6.5)

Equating the coefficients at \(x^n\) in the both sides of (6.5), we get

\[
\alpha_{n} b_{n-1} = A_{n}(2), \quad n \geq 2,
\] (6.6)

where we used the notation

\[
A_{s}(m) = s! \sum_{k=m}^{s} \frac{\epsilon_k}{(s-k)!}, \quad s \geq m,
\] (6.7)

and the coefficients \(\epsilon_k\) are defined by formulas (2.5). For the classical Hermite-Chihara polynomials from (5.4), (5.5) and binomial formula it follows that (as \(n \geq 2\))

\[
\alpha_{n} = \frac{\sqrt{2\gamma}}{\gamma + 1} \left\{ \begin{array}{ll}
\sqrt{n} & n = 2m, \\
\sqrt{n+\gamma} & n = 2m + 1.
\end{array} \right.
\] (6.8)

In order to find the quantities \(\beta_n\) in (6.4) we equate the coefficients at \(x^t\) in the both sides of (6.5) (as \(0 \leq t < n\)). Obviously, a coefficients at \(x^t\) only distinct from zero when it is valid the following condition:

\[
n - t = 2p, \quad 1 \leq p \leq \epsilon(n/2).
\] (6.9)

We consider separately three cases \(t = 0, 1, 2\).

1. Let \(t = 0, \quad n = 2p\). We have \(\beta_{2p} = 0, \quad p \geq 1\).

2. Let \(t = 1, \quad n = 2p + 1\). We have

\[
\beta_{2p+1} = \frac{\sqrt{2p} \alpha_{2p-3,2p-2} \beta_{2p+1}}{b_{0} \alpha_{2p-1,2p-1}}, \quad p \geq 1.
\] (6.10)

3. Let \(t = 0, \quad n = 2p\). Regarding \(p \geq 1\), we have

\[
\alpha_{2p-1,2p+1} A_{2}(2) = -\beta_{2p+2} \sqrt{\frac{[2p+2][2p+1]}{\alpha_{2p-3,2p-1}} + \alpha_{2p-1,2p} A_{2p+2}(2)}.
\] (6.11)

Taking into account that \(\beta_{2p} = 0\), we have from here

\[
\alpha_{2p-1,2p+1} A_{2p+2}(2) - \alpha_{2p-1,2p+1} A_{2}(2) = 0, \quad p \geq 1.
\] (6.12)

Using (3.8), we simplify this relation:

\[
v_{1} A_{2p+2}(2) - v_{2p+1} A_{2}(2) = 0, \quad p \geq 1.
\] (6.13)

From the equalities (2.5) and the designation (6.7) it follows that:

\[
A_{k}(2) = v_{k-1} - k, \quad k \geq 2.
\] (6.14)

Substituting (6.14) into (6.13), we get

\[
v_{2p+1} = (p+1)v_{1}, \quad p \geq 1.
\] (6.15)
Now we consider the general case \( t \geq 3, \ n \geq 5 \) and \( 1 \leq p \leq \epsilon \left( \frac{n-3}{2} \right) \). From (6.13)-(6.17) we have:

\[
\alpha_{2p-1,n-1}A_{n-2p}(2) = -\beta_n \sqrt{n(n-1)}\alpha_{2p-3,n-2} + \alpha_{2p-1,n-2} A_n(2).
\] (6.16)

As \( n \geq 2 \) and \( 2p \leq n \), from the condition (3.5) of the paper (5) it follows that:

\[
\alpha_{2p-1,n-1} = [n-1]\alpha_{2p-3,n-2} + \alpha_{2p-1,n-2}.
\] (6.17)

Substituting (6.17) into (6.16), we get the following formula:

\[
\alpha_{2p-3,n-3}(n-1)A_{n-2p}(2) + \beta_n \sqrt{n(n-1)} = \alpha_{2p-1,n-2}(A_n(2) - A_{n-2p}(2)).
\] (6.18)

Combining (3.13), (3.14) and (3.16), we get (as \( p \geq 1 \) and \( n \geq 2p+1 \))

\[
\frac{\alpha_{2p-1,n-2}}{\alpha_{2p-3,n-3}} = \frac{v_{2p-1} - v_{2p-3}}{v_1 v_{2p-1}} v_{n-2} v_{n-2p-1},
\] (6.19)

as well as

\[
[n-1] = \frac{v_{n-2}(v_{n-1} - v_{n-3})}{v_1}.
\] (6.20)

Substituting (6.19) into (6.18), we get (as \( n \geq 5 \) and \( 1 \leq p \leq \epsilon \left( \frac{n-3}{2} \right) \)) the following formula:

\[
-\beta_n \sqrt{n(n-1)} v_{n-2} = \frac{v_{n-1} - v_{n-3} A_{n-2p}(2) - v_{n-2}(v_{2p-1} - v_{2p-3})}{v_1 v_{2p-1}} (A_n(2) - A_{n-2p}(2)).
\] (6.21)

Taking into account (3.4), the coefficient at \( A_{n-2p}(2) \) in the right side of (6.21) is equal to

\[
\frac{(v_{n-1} - v_{n-3}) v_{2p-1} + v_{n-2p-1}(v_{2p-1} - v_{2p-3})}{v_1 v_{2p-1}} = \frac{(v_{2p-1} - v_{2p-3}) v_{n-1}}{v_1 v_{2p-1}}.
\] (6.22)

As \( n \geq 5 \) and \( 1 \leq p \leq \epsilon \left( \frac{n-3}{2} \right) \), from (6.21) and (6.22) we get

\[
-\beta_n \sqrt{n(n-1)} v_{n-2} = \frac{2p-1}{v_1 v_{2p-1}} (v_{n-1} A_{n-2p}(2) - A_n(2) v_{n-2p-1}).
\] (6.23)

As \( n \geq 5 \) and \( 1 \leq p \leq \epsilon \left( \frac{n-3}{2} \right) \), taking into account (6.14) and (6.15), we can rewrite the right hand side of (6.23) in the following form:

\[
-\beta_n \sqrt{n(n-1)} v_{n-2} = \frac{n v_{n-2p-1} - (n-2p) v_{n-1}}{p v_1}.
\] (6.24)

Evidently, the condition (6.24) is valid as \( n = 2p \), if it is remembered that \( \beta_{2p} = 0 \) as well as (3.15). It remains to consider the case \( n = 2m + 1 \ (m \geq 2 \) and \( p \leq m - 1 \):)

\[
-\beta_{2m+1} \sqrt{[2m+1][2m]} v_{2m-1} = \frac{(2m+1) v_{2m-2p} - (2m-2p+1) v_{2m}}{p v_1}.
\] (6.25)

From (6.13), (6.14) and (6.19), as \( n = 2m + 1 \ , m \geq 2 \) and \( p \leq m - 1 \), we have

\[
\sqrt{[2m+1] = \frac{A_{2m+1}}{\sqrt{2m+1}} = \frac{v_{2m} - (2m + 1)}{\sqrt{2m+1}}.}
\] (6.26)
Substituting \((6.26)\) into \((6.25)\) and using \((6.15)\), we obtain the following relation:

\[-(v_{2m} - (2m + 1)) = \frac{m}{p}((2m + 1)v_2(m - p) - (2m - 2p + 1)v_{2m}),\]  

i.e. for any \(m \geq 2\) and \(p \leq m - 1\) it should be true the following equality:

\[(m - p)v_{2m} + p = mv_2(m - p).\] \((6.28)\)

It is readily seen that \((6.28)\) is valid if and only if:

\[v_{2m} = mv_2 - (m - 1), \quad m \geq 1.\] \((6.29)\)

So it is proved the following theorem.

**Theorem 6.2.** For the orthonormal Hermite-Chihara polynomials system constructed by the sequence \(\vec{v}\), submitting to \((2.1)\) and \((3.7)\), be satisfied \((6.4)\) it is necessary and sufficient that this system is obeying also the relations \((6.15)\) and \((6.29)\) at some \(v_1\) and \(v_2\) such that \(1 \leq v_1 \leq v_2\).

**Remark 6.3.** Thus we constructed the three-parameter Hermite-Chihara polynomials family (these parameters are \(b_0, v_1\) and \(v_2\)) for which hold the condition \((6.4)\). For the classical Hermite-Chihara polynomials valid the following relations:

\[v_1 = b_0^{-2} = \frac{2}{\gamma + 1}, \quad v_2 = 1 + v_1.\] \((6.30)\)

### 7. Deduction of the differential equation for the family of Hermite-Chihara polynomials

In this section we prove that any polynomial belonging to the three-parameter family considered above satisfies a the second order differential equation.

From \((3.3)\), \((6.1)\) and \((6.4)\) it follows that for any polynomials system \(\{\psi_n(x)\}_{n=0}^{\infty}\) belonging to the considered family the differentiation operator \(\frac{d}{dx}\) acts on this basis by the following formulas:

\[\frac{d}{dx}\psi_0 = 0, \quad \frac{d}{dx}\psi_1 = \frac{1}{b_0}, \quad \frac{d}{dx}\psi_n = (\gamma_n - \delta_n)\psi_{n-1} - \beta_n X^{-1}\mu X^{-2}, \quad n \geq 2.\] \((7.1)\)

Multiplying \((7.1)\) from the left by \(X_\mu\) and using \((3.1)\), we get

\[X_\mu \frac{d}{dx} - b_{n-1}(\gamma_n - \delta_n)\psi_n = (b_{n-2}(\gamma_n - \delta_n) - \beta_n)\psi_{n-2}, \quad n \geq 2.\] \((7.2)\)

Note that from \((3.9)\) and \((3.16)\) it follows the relation:

\[v_{n-1} = b_{n-1}\gamma_n, \quad n \geq 1.\] \((7.3)\)

Then from \((7.3)\), \((6.6)\) and \((6.14)\) we have as \(n \geq 2\)

\[b_{n-1}(\gamma_n - \delta_n) = v_{n-1} - A_n(2) = n,\] \((7.4)\)

and hence

\[b_{n-2}(\gamma_n - \delta_n) = n \frac{b_{n-2}}{b_{n-1}}, \quad n \geq 2.\] \((7.5)\)
From (7.1), (7.2), (7.4) and (7.5) we have

\[
\frac{d}{dx} \psi_0 = 0, \quad \frac{d}{dx} \psi_1 = \frac{1}{b_0},
\]

\[
(X_{\mu} \frac{d}{dx} - N)\psi_n = (-\beta n + n\frac{b_{n-2}}{b_{n-1}})\psi_{n-2}, \quad n \geq 2.
\]

Let us remark that from (6.6) and (6.10)

\[
\beta_{2p+1} = \frac{b_0 A_{2p+1}(2)\alpha_{2p-1,2p-1}}{b_{2p} \sqrt{2p}|\alpha_{2p-3,2p-2}|},
\]

By virtue of (6.19) as \( n = 2p + 1 \) and taking into account (6.15), we have

\[
\alpha_{2p-1,2p-1} - v_{2p-1} = v_{2p} - 3v_{2p-1} = 1.
\]

Further, from (7.7), (7.8), (6.14), (6.29) and (3.4) it follows that

\[
\beta_{2p+1} = b_0^2 \frac{v_{2p} - (2p+1)}{b_{2p} b_{2p-1}} = b_0^2 \frac{(v_2 - 3)p}{b_{2p} b_{2p-1}}.
\]

Note also that from (3.16), (6.29) and (6.15) it follows that

\[
b_{2p-1} = b_0^2 \frac{v_{2p-1} - v_{2p-2}}{v_1} = b_0^2 (v_2 - 1)p,
\]

\[
b_{2p} = b_0^2 \frac{v_{2p+1} - v_{2p}}{v_1} = b_0^2 v_{2p} = b_0^2 (pv_2 - (p - 1)).
\]

Taking into account that \( \beta_{2p} = 0 \), the right side of (7.6) takes the form:

\[
-\beta n + n\frac{b_{n-2}}{b_{n-1}} = \begin{cases} \frac{n\frac{b_{n-2}}{b_{n-1}}}{(n-1)\frac{b_{n-2}}{b_{n-1}}} & n = 2m, \\ \frac{n\frac{b_{n-2}}{b_{n-1}}}{(n-1)\frac{b_{n-2}}{b_{n-1}}} & n = 2m + 1. \end{cases}
\]

Then as \( n = 2p \)

\[
\frac{2pb_{2p-2}}{b_{2p-1}} = \frac{2pb_{2p-2}b_{2p-1}}{b_{2p-1}^2} = \frac{2b_{n-1}b_{n-2}}{b_0^2(v_2 - 1)},
\]

and as \( n = 2p + 1 \)

\[
\frac{2pb_{2p}}{b_{2p-1}} = \frac{2pb_{2p-2}b_{2p-1}}{b_{2p-1}^2} = \frac{2b_{n-1}b_{n-2}}{b_0^2(v_2 - 1)},
\]

so from (7.12), (7.13) and (7.6) it follows that

\[
X_\mu \frac{d}{dx} - N = c_1^{-1}(a_\mu^-)^2,
\]

where

\[
c_1 = b_0^2(v_2 - 1).
\]

We stress that, according to (3.31), the formula (7.14) is a extension of the one (5.10).

Next we will arguing by analogy with the derivation of the differential equation (5.21).

According to (4.7), (5.13) and using (7.14) as well as the following notation:

\[
\Delta_N = 2c_1^{-1}B(N) - N,
\]

we get:

\[
a_\mu^- a_\mu^+ = c_1 \left( \frac{d}{dx} + X_\mu^{-1}\Delta_N \right) (X_\mu - \frac{c_1}{2} \frac{d}{dx} + X_\mu^{-1}\Delta_N) = 2B(N + I).
\]
Taking into account the notation (7.15), we rewrite (7.10):

\[ 2b_n - 1^2 = \begin{cases} \ n c_1, & n = 2m, \\ \ n c_1 - c_1 + 2b_0^2, & n = 2m + 1. \end{cases} \quad (7.18) \]

From (7.16), (4.9), (7.4) and (7.18) we have:

\[ \Delta c_1^N = \alpha_n \psi_n, \quad (7.19) \]

where

\[ \alpha_n = \frac{3 - v_2}{v_2 - 1} \frac{1 - (-1)^n}{2}. \quad (7.20) \]

Then one rewrite the equation (7.17) in the form:

\[ \left( \frac{d}{dx} + \frac{X^{-1} \Delta c_1^N}{\mu} \right) \left( X^\mu - 2 \frac{X}{c_1} \frac{d}{dx} + X^{-1} \Delta c_1^N \right) = 2c_1^{-1} B(N + I). \quad (7.21) \]

Multiplying (7.21) by \(-\frac{2}{c_1} X\mu\) from the left, we have:

\[ -\frac{2}{c_1} X\mu(I + X\mu \frac{d}{dx}) + X\mu(-X^{-2} \Delta c_1^N + X^{-1} \frac{d}{dx} \Delta c_1^N) - \]
\[ -\frac{2}{c_1} \Delta c_1^N X\mu + \Delta c_1^N \frac{d}{dx} + \Delta c_1^N X^{-1} \Delta c_1^N = -\frac{2}{c_1} X\mu \frac{2}{c_1} B(N + I). \quad (7.22) \]

Applying (7.22) to \(\psi_n\) and using the following equalities:

\[ \Delta c_1^N X\mu^{-1} \Delta c_1^N = 0, \quad (7.23) \]
\[ \Delta c_1^N X\mu = X\mu \Delta c_1^N_{N+1}, \quad (7.24) \]
\[ \frac{2}{c_1} X\mu \left( \frac{2}{c_1} B(N + I) - \Delta c_1^N_{N+1} \right) = \frac{2}{c_1} X\mu = \frac{2}{c_1} X\mu N, \quad (7.25) \]
\[ (\Delta c_1^N \frac{d}{dx} + \Delta c_1^N \frac{d}{dx}) \psi_n = \frac{3 - v_2}{v_2 - 1} \psi_n', \quad (7.26) \]

we get the following differential equation (as all \(n \geq 0\)):

\[ x\psi_n'' + \frac{3 - v_2}{v_2 - 1} \frac{2}{b_0^2(v_2 - 1)} x^2 \psi_n' + \left( \frac{2}{b_0^2(v_2 - 1)} nx - \frac{\alpha_n}{x} \right) \psi_n = 0. \quad (7.27) \]

Because \(\alpha_n = \theta_n, \ c_1^d = 1\) it is clear that the equation (7.22) is a extension of the differential equation (5.21). Also, we obtain the following two-parameter differential equation (as any \(n \geq 0\)):

\[ x\psi_n'' + (\gamma - 2\alpha x^2) \psi_n' + (2\alpha n x - \frac{\theta_n(\gamma)}{x}) \psi_n = 0. \quad (7.28) \]

where

\[ \gamma = \frac{3 - v_2}{v_2 - 1}, \quad \alpha = \frac{1}{b_0^2(v_2 - 1)}, \quad \gamma > -1, \quad \alpha > 0, \quad (7.29) \]

for the Hermite-Chihara polynomials \(\{\psi_n(x)\}_{n=0}^{\infty}\) which are orthonormal with respect to the measure

\[ d\mu(x) = C|x|^\gamma \exp(-\alpha x^2) dx, \]
\[ C = \alpha^{-\gamma+\gamma/2+1/2} \frac{1}{\Gamma(\gamma/2+1)}, \quad (7.30) \]
The polynomials \( \{ \psi_n(x) \}_{n=0}^{\infty} \) satisfy the recurrence relations (3.1) with the coefficients:

\[
2b_{n-1}^2 = \begin{cases} 
\frac{n}{2^n} \alpha, & n = 2m, \\
\frac{n+1}{2^{n+1}} \alpha, & n = 2m + 1.
\end{cases}
\] (7.31)

Notice that the classical Hermite-Chihara polynomials correspond to the case \( \alpha = 1 \).

8. Conclusion

It is easily shown that the condition (6.4) is equivalent to the following conditions:

\[
\begin{align*}
 B_1 \psi_n &= a_n \psi_n + c_n \psi_{n-2}, \\
 B_1 \psi_n &= \overline{a_n} \psi_n + \overline{c_n} \psi_{n-2}, & n \geq 2,
\end{align*}
\] (8.1)

where from (6.1), (3.3) and (3.1) it follows that:

\[
\overline{a_2} = 0, \quad \overline{a_n} + a_n = \gamma_n b_{n-1}, \quad \overline{c_n} + c_n = \gamma_n b_{n-2}, & n \geq 2.
\] (8.2)

As the operator \( B_1 \) and hence the operator \( B_1 \) is a reduced operator (the lowering on the basis vectors shall be no more than two steps), so the differential equation (7.28) is a differential equation of the second order. Assume that (8.1) does not hold, i.e., we consider the general case:

\[
\begin{align*}
 B_1 \psi_n &= a_n \psi_n + c_n \psi_{n-2} + d_{n,2} \psi_{n-4} + \cdots + d_{n,\epsilon(n/2)} \psi_{n-2\epsilon(n/2)}, \\
 B_1 \psi_n &= \overline{a_n} \psi_n + \overline{c_n} \psi_{n-2} + d_{n,2} \psi_{n-4} + \cdots + d_{n,\epsilon(n/2)} \psi_{n-2\epsilon(n/2)},
\end{align*}
\] (8.3)

where (8.2) still stand as before. Besides, we have

\[
\overline{d_{n,s}} + d_{n,s} = 0, \quad s = 2, 3, \ldots, \epsilon(n/2),
\] (8.4)

with the coefficients \( \overline{d_{n,s}} \) (correspondingly \( d_{n,s} \)) do not all equal to zero. One can prove that if one of the coefficients vanish as fixed \( \overline{n} \), i.e.,

\[
d_{n,s} = 0, \quad s \geq 2, n \geq 2,
\] (8.5)

so it follows that for all \( n \geq 2 \) the same is true:

\[
d_{n,s} = 0.
\] (8.6)

In addition, as \( n = 2k \) from

\[
d_{n,2} = d_{n,3} = \cdots = d_{n,s} = 0
\] (8.7)

it follows that (6.15) is true for all \( 1 \leq p \leq s \). In the case \( n = 2m + 1 \) from (8.7) it follows that (6.29) is valid for all \( 1 \leq m \leq s \).

Furthermore, it seems that the following assumption hold. The Hermite-Chihara polynomials either satisfy the differential equation (7.28) of the second order (if the condition (8.1) is hold) or they do not satisfy any differential equation of a finite order (if the general condition (8.3) is hold). The proof of the promoted conjecture invites further investigation and this will be the object of another paper.

In conclusion we note that the description of the generalized Hahn-Hermite polynomials (see (7)) in the framework of the our scheme as thought is related either with a relaxation of the condition (5.3) or (what is more radical) with a relaxation of the condition (2.2). This will be discussed elsewhere.
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