ABSTRACT

Using the fully nonlinear and exact perturbation formulation with magnetohydrodynamics (MHD) in Minkowski background we derive first-order post-Newtonian (1PN) equations without imposing the slicing (temporal gauge) condition. The 1PN MHD formulation is complementary to our recently presented fully relativistic MHD combined with 0PN gravity available only in the maximal slicing. We present the 1PN MHD equations in two gauge conditions previously used in the literature and provide gauge transformation relations between different gauges. We derive the PN effects on MHD waves in a static homogeneous medium.

1. INTRODUCTION

Post-Newtonian (PN) approximation is a way of managing the relativistic effects of matter and gravity in the situation where the relativistic effects are weak, thus weak gravity and slow motion. In the PN approximation the relativistic effects are systematically treated as corrections to the well-known non-relativistic (Newtonian) limit. The PN approximation is based on the expansion in dimensionless parameters like $v^2/c^2$, $GM/(tc^2)$ and $\Phi/c^2$ with $v$, $M$, $\ell$ and $\Phi$ the characteristic velocity, mass, length scale, and the gravitational potential, respectively, involved in the system. The expansion on these parameters involving $c^{-2n}$-order is regarded as the $n$th-order PN ($n$PN) expansion (Poisson & Will 2014).

PN formulation of hydrodynamics was studied by Chandrasekhar and collaborators in a series of papers reaching up to 2.5PN order (Chandrasekhar 1965; Chandrasekhar & Nutku 1969; Chandrasekhar & Esposito 1970). 1PN formulation of magnetohydrodynamics (MHD) was presented by Greenberg (1971) and only recently another paper on the subject was presented by Nazari & Roshan (2018).

The fully nonlinear and exact perturbation (FNLE) formulation of Einstein’s gravity with MHD in the Minkowski background is presented recently in Noh, Hwang & Bucher (2019, NHB hereafter). The formulation is designed to produce nonlinear (higher order) perturbation equations in any temporal gauge (slicing, hypersurface) condition with ease. As the formulation is valid to fully nonlinear order and exact, it may have diverse applications. In NHB we showed that the fully relativistic (special relativistic) MHD can be combined with weak gravity consistently in a certain gauge condition.

In this work we will derive the 1PN approximation of MHD as a complementary formulation. While the fully relativistic MHD combined with weak gravity is possible only in a certain temporal gauge condition (the maximal slicing), our 1PN equations will be presented without fixing the slicing condition. In Hwang, Noh & Puetzfeld (2008, HNP hereafter) we presented 1PN hydrodynamics in cosmological context without fixing the temporal gauge. Here, we are extending the formulation to include the MHD, but in the Minkowski background.

Section 2 briefly introduces notations used in the FNLE formulation with MHD, and using the formulation a complete set of equations valid to 1PN order is derived in the Appendix. Section 3 presents the 1PN order MHD equations without fixing the temporal gauge condition; in a conventional notation, see Section 4. Section 5 provides the 1PN equations in two gauge conditions previously used in the literature, and shows the gauge transformation properties between different gauge conditions. Section 6 presents the PN corrections to the MHD waves in a static homogeneous medium without gravity. Section 7 is a discussion.

We adopt the cgs unit.

2. FULLY NONLINEAR AND EXACT PERTURBATIONS WITH MHD

Here we introduce our notations. Details on the FNLE formulation with MHD can be found in NHB. Our metric convention is

$$\tilde{g}_{00} = -(1 + 2\alpha), \quad \tilde{g}_{0i} = -\chi_i, \quad \tilde{g}_{ij} = (1 + 2\varphi)\delta_{ij},$$

(1)

where $\alpha$, $\varphi$ and $\chi_i$ are functions of spacetime with arbitrary amplitudes; the tildes indicate the covariant quantities. The spatial index of $\chi_i$ is raised and lowered using $\delta_{ij}$ as the metric. In this metric convention the spatial part of metric looks simple because we have ignored the transverse-tracefree (gravitational waves) part of the spatial metric (which is a serious assumption on two physical degrees of freedom), and (without losing generality) imposed a spatial gauge condition (fixing three coordinate degrees of freedom) which removes the spatial gauge mode completely to all perturbation orders; this statement is true as long as we simultaneously choose a temporal gauge condition which removes the temporal gauge mode completely; under these spatial and temporal gauge conditions all remaining variables
can be equivalently regarded as (spatially and temporally) gauge invariant ones to all perturbation orders (Bardeen 1988; Section VI of Noh & Hwang 2004; Hwang & Noh 2013).

The energy momentum tensor is introduced as
\[ \bar{T}_{ab} = \bar{\mu} \bar{u}_a \bar{u}_b + \bar{p} (\bar{\gamma}_{ab} + \bar{u}_a \bar{u}_b) + \bar{\pi}_{ab}, \]
where \(\bar{\mu}, \bar{p}\) and \(\bar{\pi}_{ab}\) are the energy density, pressure and the anisotropic stress (\(\bar{\pi}_{ab} \bar{u}^b = 0 \equiv \bar{\pi}^\alpha_{\alpha}\), \(\bar{\pi}_{ab} = \bar{\pi}_{ba}\)), respectively, based on the normalized time-like (\(\bar{u}^\alpha \bar{u}_\alpha \equiv -1\)) four-vector \(\bar{u}_a\) in the energy frame [setting the flux term to vanish as \(\bar{\varphi}_a \equiv \bar{T}_{cd} \bar{u}^c (\bar{\delta}_d + \bar{u}^d \bar{u}_d) \equiv 0\)]. We introduce the fluid velocity \(v_i\) as
\[ \bar{u}_i \equiv \frac{\gamma v_i}{c}, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{1 + 2\varphi}{c^2} v^2}} \quad \text{and} \quad v^2 \equiv v^i v_i. \]  \(\text{(3)}\)

The spatial index of \(v_i\) is raised and lowered using \(\delta_{ij}\) as the metric. We set
\[ \bar{\mu} \equiv \mu \equiv \rho c^2, \quad \bar{\rho} \equiv \bar{\varphi} \left( 1 + \frac{1}{c^2} \Psi \right), \quad \bar{\varphi} \equiv \bar{p}, \quad \bar{\pi}_{ij} \equiv \Pi_{ij}, \]  \(\text{(4)}\)
where \(\rho, \bar{\varphi}, \bar{\Psi}\) are the density, mass density and the internal energy density, respectively. The fluid quantities are functions of spacetime with arbitrary amplitudes; spatial indices of \(\Pi_{ij}\) are raised and lowered using \(\delta_{ij}\) as the metric.

The complete set of FNLE equations is presented in the Appendix of Hwang & Noh (2016) for a hydrodynamic fluid.

In the presence of electromagnetism, the energy-momentum tensor of electromagnetic field is
\[ \bar{T}_{ab}^{EM} = \frac{1}{4\pi} \left( \bar{F}_{ac} \bar{F}_{bc} - \frac{1}{4} \bar{g}_{cd} \bar{F}_{ac} \bar{F}_{bd} \right). \]  \(\text{(5)}\)
The electromagnetic tensor can be decomposed as
\[ \bar{F}_{ab} \equiv \bar{U}_a \bar{E}_b - \bar{U}_b \bar{E}_a - \bar{\eta}_{abcd} \bar{U}^c \bar{B}^d, \]
with \(\bar{E}_a \bar{U}^a \equiv 0 \equiv \bar{B}_a \bar{U}^a\); \(\bar{U}_a\) is a generic normalized time-like four-vector with \(\bar{U}^c \bar{U}_c \equiv -1\); it can be the fluid four-vector \(\bar{u}_a\) (comoving frame) or the normal four-vector \(\bar{n}_a\) (laboratory frame); for the normal four-vector, we have \(\bar{n}_i \equiv 0\). For fields in the laboratory frame we introduce
\[ \bar{E}_i^{(n)} \equiv E_i \equiv \mathbf{E}, \quad \bar{B}_i^{(n)} \equiv B_i \equiv \mathbf{B}, \]  \(\text{(7)}\)
where indices of \(E_i\) and \(B_i\) are raised and lowered using \(\delta_{ij}\) as the metric.

The Ohm’s law is expressed in the comoving frame as
\[ \bar{j}_a^{(u)} = \sigma \bar{E}_a^{(u)}, \]  \(\text{(8)}\)
with \(\sigma\) being the electric conductivity. Ideal MHD takes a perfectly conducting limit, \(\sigma \rightarrow \infty\) with \(\bar{E}_a^{(u)} = 0\), and \(\bar{j}_a^{(u)}\) is non-vanishing. From \(\bar{E}_a^{(u)} = 0\) we have the ideal MHD condition [see Equation (57) in NHB]
\[ \mathbf{E} = -\frac{1}{\sqrt{1 + 2\varphi}} \frac{1}{c} \mathbf{v} \times \mathbf{B}. \]  \(\text{(9)}\)

The complete set of FNLE equations with MHD is presented in NHB.

3. POST-NEWTONIAN APPROXIMATION WITH MHD

To 1PN order we set
\[ \alpha \equiv \frac{\Phi}{c^2}, \quad \varphi \equiv -\frac{\Psi}{c^2}, \quad \chi_i \equiv \frac{P_i}{c^3}. \]  \(\text{(10)}\)

Compared with notations used in Chandrasekhar (1965) and Chandrasekhar & Nutku (1969), we have
\[ \alpha \equiv \frac{\Phi}{c^2} = -\frac{1}{c^2} \left[ U + \frac{1}{c^2} (2\Psi - U^2) \right], \quad \varphi \equiv -\frac{\Psi}{c^2} = \frac{1}{c^2} V, \quad v_i = \varphi_i + \frac{1}{c^2} [(U + 2V) \varphi_i - P_i]. \]  \(\text{(11)}\)
Using FNLE notation we have
\[ \frac{\varphi_i}{c} \equiv \frac{dx^i}{dt} = \frac{d\varphi_i}{dx^i}, \]  \(\text{(12)}\)
where the index of \(\varphi_i\) is raised and lowered using \(\delta_{ij}\) as the metric. Thus, we have [see the Appendix D in Hwang & Noh (2013)]
\[ \bar{u}^i = \frac{\gamma \varphi_i}{N} \equiv \frac{v_i}{N}, \quad v^0 = \frac{\gamma}{N}, \quad v_i = \frac{1 + 2\varphi}{N} \varphi_i - \frac{c}{N} \chi_i, \quad \gamma \equiv \frac{1}{\sqrt{1 - \frac{1 + 2\varphi}{c^2} \frac{\Delta}{c^2}}}, \quad N = \sqrt{1 + 2\alpha + \frac{\chi}{1 + 2\varphi}}. \]  \(\text{(13)}\)
where $\gamma$ is the Lorentz factor and $N$ is the lapse function.

The complete set of Einstein equations, conservation equations and Maxwell equations are derived in the Appendix by reducing the FNLE equations presented in NHB to 1PN order. Equation (A5) gives

$$\Psi = \Phi,$$

(14)
to the 0PN order, thus $V = U$. Equation (A1) gives

$$\kappa = \frac{1}{c^2} \left( 3\dot{\Phi} - P_{,k}^k \right).$$

(15)

Using these, Einstein equations in (A3) and (A1) give

$$\Delta P_i - P_{,ki}^i + 4\Phi_{,i} = -16\pi G\overline{\rho} v_i,$$

(16)

$$\Delta \Phi = 4\pi G\overline{\rho} + \frac{1}{c^2} \left[ 4\pi G \left( \overline{\rho} \Pi + 2\overline{\rho} v^2 + 3p \right) - 3\dot{\Phi} + 2\Phi_{,k}^k + \dot{P}_{,k}^k + GB^2 \right].$$

(17)

The energy, momentum and mass conservation equations in (A9), (A7) and (A8), respectively, give

$$\frac{\partial}{\partial t} \left\{ \overline{\rho} \left( \Pi + v^2 - 3\Phi \right) + \frac{1}{8\pi} B^2 \right\}$$

$$+ \nabla^i \left\{ \overline{\rho} v_i + \frac{1}{c^2} \left[ \overline{\rho} \left( \Pi + v^2 \right) v_i + p v_i + \Pi_{ij} v^j + \frac{\overline{\rho} P_i}{4\pi} \left( \mathbf{v} \times \mathbf{B} \right) \times \mathbf{B}_i \right] \right\} = - \frac{1}{c^2} \overline{\rho} \nabla \cdot \Phi,$$

(18)

$$\frac{\partial}{\partial t} \left\{ \overline{\rho} v_i + \frac{1}{c^2} \left[ \overline{\rho} \left( \Pi + v^2 - 3\Phi \right) v_i + p v_i + \Pi_{ij} v^j - \frac{1}{4\pi} \left( \mathbf{v} \times \mathbf{B} \right)_i \left( \mathbf{v} \times \mathbf{B} \right)_j \right] \right\}$$

$$+ \nabla^j \left\{ \overline{\rho} v_i v_j + p \delta_{ij} + \Pi_{ij} + \frac{1}{c^2} \left[ \overline{\rho} \left( \Pi + v^2 \right) v_i v_j + p v_i v_j - 2 p \delta_{ij} \Phi + \overline{\rho} v_i P_j \right] \right\}$$

$$- \frac{1}{4\pi} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) - \frac{1}{4\pi c^2} \left( \mathbf{v} \times \mathbf{B} \right)_i \left( \mathbf{v} \times \mathbf{B} \right)_j - \frac{1}{2} \delta_{ij} |\mathbf{v} \times \mathbf{B}|^2 \right\}$$

$$= - \overline{\rho} \Phi_{,i} - \frac{1}{c^2} \left\{ \Phi_{,i} \left[ \overline{\rho} \left( \Pi + 2 v^2 - 3\Phi \right) + 3 p + \frac{B^2}{4\pi} \right] + \overline{\rho} v_i P_{,j}^j \right\},$$

(19)

$$\frac{\partial}{\partial t} \left( 1 + \frac{v^2}{2 c^2} - \frac{3\Phi}{c^2} \right) + \nabla^i \left( \overline{\rho} v_i \left( 1 + \frac{v^2}{2 c^2} \right) + \frac{\overline{\rho} P_i}{c^2} \right) = 0,$$

(20)

and Maxwell equations in (A9)-(A12) give

$$\nabla \cdot \left( 1 - \frac{\Phi}{c^2} \right) \mathbf{B} = 0,$$

(21)

$$\frac{\partial}{\partial t} \left( 1 - \frac{\Phi}{c^2} \right) \mathbf{B} = \nabla \times \left\{ \left[ \left( 1 + \frac{2\Phi}{c^2} \right) \mathbf{v} + \frac{1}{c^2} \mathbf{P} \right] \times \mathbf{B} \right\},$$

(22)

$$\nabla \cdot \left( 1 - \frac{\Phi}{c^2} \right) \mathbf{E} = - \frac{1}{c} \nabla \cdot \left( \mathbf{v} \times \mathbf{B} \right) - 4\pi \overline{\rho} \epsilon_{\text{em}} \left( 1 - \frac{3\Phi}{c^2} \right),$$

(23)

$$\frac{\partial}{\partial t} \left( 1 - \frac{\Phi}{c^2} \right) \mathbf{E} = - \frac{1}{c} \frac{\partial}{\partial t} \left( \mathbf{v} \times \mathbf{B} \right) = c \nabla \times \left[ \left( 1 + \frac{\Phi}{c^2} \right) \mathbf{B} \right] - 4\pi \left( \mathbf{j} + \frac{1}{c^2} \overline{\rho} \epsilon_{\text{em}} \mathbf{P} \right).$$

(24)

To 1PN order Equation (19) gives

$$\mathbf{E} = - \left( 1 + \frac{\Phi}{c^2} \right) \frac{1}{c} \mathbf{v} \times \mathbf{B}.$$ 

(25)

Using Maxwell’s equations the MHD contributions in Equations (18) and (19) can be written on the right-hand-sides, respectively, as

$$- \frac{1}{4\pi c^2} \left( \mathbf{v} \times \mathbf{B} \right) \cdot \left( \nabla \times \mathbf{B} \right),$$

$$\frac{1}{4\pi} \left\{ \left( \nabla \times \mathbf{B}_i \right) + \frac{1}{4\pi c^2} \left\{ \left( \mathbf{v} \times \mathbf{B} \right) \times \mathbf{B} \right\}_i + \frac{1}{4\pi} \left( \mathbf{v} \times \mathbf{B} \right) \nabla \cdot \left( \mathbf{v} \times \mathbf{B} \right) + \left( \mathbf{B} \times \left( \mathbf{B} \times \nabla \Phi \right) \right)_i \right\}.$$ 

(26)

To 0PN order, the conservation equations in (A8)-(A10) give

$$\overline{\rho} + \nabla \cdot (\overline{\rho} \mathbf{v}) = 0,$$

(27)

$$\overline{\rho} v_i + \nabla^j \left( \overline{\rho} v_i v_j + p \delta_{ij} + \Pi_{ij} \right) + \overline{\rho} \nabla_i \Phi = \frac{1}{4\pi} \left( \nabla \times \mathbf{B} \right)_i.$$ 

(28)
thus
\[ \dot{v}_i + v \cdot \nabla v_i + \frac{1}{\eta} \nabla^j (\delta_{ij} + \Pi_{ij}) + \nabla_i \Phi = \frac{1}{4\pi \eta} [(\nabla \times B) \times B]_i. \] (29)

Now, to 1PN order, from Equations (18) and (20) we have
\[ \dot{\varphi} + v \cdot \nabla \varphi + \Pi_{ij} v_{i,j} = 0. \] (30)

Thus, the gravity and MHD do not appear in the internal energy density conservation equation to 1PN order. From Equations (18) and (19) we can derive
\[ \left \{ \dot{\varphi} + \frac{1}{c^2} \left [ \varphi (\Pi + v^2 - 3\Phi) + p \right ] \right \} (\dot{v}_i + v \cdot \nabla v_i) + \left \{ \varphi + \frac{1}{c^2} \left [ \varphi (\Pi + 2v^2 - 4\Phi) + p \right ] \right \} \Phi_{,i} + \frac{1}{c^2} \varphi v \cdot (3\Phi \nabla v_i - v_i \nabla \Phi) + \frac{1}{c^2} \left [ v_j p^j_{,i,i} + P^j_{i,i,j} \right ] + p_{,i} + \Pi_{ij} + \frac{1}{c^2} \left [ p v_{i,j} - 2p_{,i,j} \Phi + (\Pi_{ij} v^j) \cdot v_i - v_i \left ( \Pi^j_{k} v^k \right )_{,j} \right ] = \frac{1}{4\pi} [ (\nabla \times B) \times B]_i. \] (31)

The above 1PN equations are presented without imposing the temporal gauge condition. The gauge transformation properties are studied in Section 6 of HNP. The general gauge conditions can be written as [Equation (210) in HNP]
\[ P^i_{,i} - n \dot{\Phi} = 0, \] (32)

with the real numbers \( n \) covering

Chandrasekhar (Standard PN) gauge, Maximal slicing : \( n = 3 \),
Harmonic gauge : \( n = 4 \),
Transverse–shear gauge : \( n = 0 \). (33)

The standard PN gauge was used by Chandrasekhar (1965) and is the same as the maximal slicing setting the trace of extrinsic curvature equal to zero (\( K_i^i = \kappa = 0 \)). The harmonic gauge condition is used in Section 9 of Weinberg (1972). In the transverse-shear gauge we have \( P^i_{,i} \equiv 0 \) where \( P_i \) is related to the shear of the normal frame, see Equation (42) in HNP. For various gauge conditions used in the literature in the hydrodynamic PN situations, see Blanchet, Damour & Schäfer (1990), Shibata and Asada (1995), Asada, Shibata and Futamase (1996), Asada and Futamase (1997), Racine and Flanagan (2005), and Poisson and Will (2014).

4. IN CONVENTIONAL PN NOTATION

In PN literature \( \nabla \) and \( U \) are used often; to 1PN order from Equation (11) we have
\[ v_i = u_i + \frac{1}{c^2} (3U u_i - P_i), \quad \Phi = -U - \frac{1}{c^2} (2\Upsilon - U^2), \] (34)

and the general PN gauge conditions in Equation (32) becomes
\[ P^i_{,i} + n \dot{U} = 0. \] (35)

Einstein’s equations in (16) and (17) give
\[ \Delta P_i - \left ( P^k_{,k} + 4\dot{U} \right )_{,i} = -16\pi G \varphi v_i, \] (36)
\[ \Delta U + 4\pi G \varphi = -\frac{1}{c^2} \left [ 2\Delta \Upsilon + 4\pi G \varphi \left ( \Pi + 2v^2 + 3\frac{p}{\varphi} + 2U + \frac{B^2}{4\pi \varphi} \right ) + 3\dot{U} + \dot{P}^k_{,k} \right ], \] (37)
and the conservation equations in (18) and (20) become

\[
\frac{\partial}{\partial t} \left\{ \bar{\mathbf{v}} + \frac{1}{c^2} \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 3U \right) + \frac{1}{8\pi} B^2 \right] \right\} \\
+ \nabla \cdot \left\{ \bar{\mathbf{v}} \mathbf{v} + \frac{1}{c^2} \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 3U \right) v_i + p v_i + \Pi_{ij} v_j - \frac{1}{4\pi} \left( (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \right)_i \right] \right\} = \frac{1}{c^2} \bar{\mathbf{v}} \cdot \nabla U, \tag{38}
\]

\[
\frac{\partial}{\partial t} \left\{ \bar{\mathbf{v}} + \frac{1}{c^2} \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 6U \right) v_i + p v_i + \Pi_{ij} v_j - \frac{1}{4\pi} \left( (\mathbf{v} \times \mathbf{B}) \times \mathbf{B} \right)_i \right] \right\} \\
+ \nabla \cdot \left\{ \bar{\mathbf{v}} \mathbf{v} + p \delta_{ij} + \Pi_{ij} + \frac{1}{c^2} \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 6U \right) v_i v_j + p v_i v_j + 2p \delta_{ij} U - \bar{\mathbf{v}} v_j P_i \right] \\
- \frac{1}{4\pi} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \right\} + \frac{1}{4\pi c^2} \left[ (\mathbf{v} \times \mathbf{B})_i (\mathbf{v} \times \mathbf{B})_j - \frac{1}{2} \delta_{ij} |\mathbf{v} \times \mathbf{B}|^2 \right] \right\} \\
= \bar{\mathbf{v}} U, + \frac{1}{c^2} \left[ 2 \bar{\mathbf{v}} \mathbf{T}_i + U_i \right] \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 2U \right) + 3p + \frac{B^2}{4\pi} \right] - \bar{\mathbf{v}} v_j P_{ij}, \tag{39}
\]

\[
\frac{\partial}{\partial t} \left[ \bar{\mathbf{v}} \left( 1 + \frac{v^2}{c^2} \right) \right] = \nabla \cdot \left[ \bar{\mathbf{v}} \right] \left( 1 + \frac{v^2}{c^2} \right) \mathbf{B}, \tag{40}
\]

The Maxwell’s equations in (21) and (22) give

\[
\nabla \cdot \left[ \left( 1 + \frac{U}{c^2} \right) \mathbf{B} \right] = 0, \tag{41}
\]

\[
\frac{\partial}{\partial t} \left[ \left( 1 + \frac{U}{c^2} \right) \mathbf{B} \right] = \nabla \times \left[ \left( 1 + \frac{U}{c^2} \right) \nabla \times \mathbf{B} \right], \tag{42}
\]

and Equation (31) gives

\[
\left\{ \bar{\mathbf{v}} + \frac{1}{c^2} \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 6U \right) + p \right] \right\} \left( \mathbf{v} + \mathbf{v} \cdot \nabla \mathbf{v} \right) + p, + \Pi_{ij} - \left\{ \bar{\mathbf{v}} + \frac{1}{c^2} \left[ \bar{\mathbf{v}} \left( \Pi + v^2 + 2U \right) + p \right] \right\} U, - \frac{1}{c^2} 2 \bar{\mathbf{v}} \mathbf{T}_i, \\
+ \frac{1}{c^2} \bar{\mathbf{v}} \left[ \left( 3 \bar{U} + 4 \mathbf{v} \cdot \nabla \mathbf{u} \right) \mathbf{v}_i - \bar{P}_i - v^2 \left( P_{ij} - P_{ji} \right) \right] + \frac{1}{c^2} \left[ \bar{P} v_i + 2p_i U + \left( \Pi_{ij} v^2 \right) - v_i \left( \Pi_{ij} v^2 \right) \right] \right\} \right\} \\
= \frac{1}{4\pi} \left[ \left( \nabla \times \mathbf{B} \right) \times \mathbf{B} \right], \tag{43}
\]

The gauge conditions directly affect only Einstein equations in (36) and (37).

By defining

\[
\varrho^* \equiv \bar{\mathbf{v}} \left[ 1 + \frac{1}{c^2} \left( \frac{1}{2} v^2 + 3U \right) \right], \tag{44}
\]

Equation (40) gives

\[
\frac{\partial}{\partial t} \varrho^* + \nabla \cdot \left( \varrho^* \nabla \right) = 0. \tag{45}
\]

Thus, according to Chandrasekhar (1965) “in the PN approximation, the mass defined in terms of the density \( \varrho^* \) is conserved.” In FNLE formulation, from the continuity equation, \( (\bar{\rho} \bar{u})_x = 0 \), using Equation (34) we have

\[
\varrho^* \equiv \sqrt{-\varrho} \sqrt{\varrho} = \sqrt{h}, \tag{46}
\]

where we used \( \sqrt{-g} = N \sqrt{h} \); \( \varrho \) is the determinant of \( g_{ab} \) and \( h \) is the determinant of the ADM intrinsic metric tensor, \( h_{ij} \equiv \varrho_{ij} \).

Chandrasekhar has similarly proved the PN order conservations of the total linear momentum, the total angular momentum and the total energy of the system and has introduced corresponding momentum and energy in the absence of MHD, see Equations (126), (141) and (165), and Equations (128) and (166) in Chandrasekhar (1965), and Chandrasekhar (1969). Corresponding conservation laws in the presence of MHD were studied in Section VII of Greenberg (1971). Although the mass conservation property shown above is independent of the temporal gauge condition (and independent of the presence of MHD), the momentum, angular momentum and energy conservation properties studied in Chandrasekhar (1965) and Greenberg (1971) are presented in the Chandrasekhar gauge.
5. COMPARISON WITH OTHER STUDIES

5.1. Chandrasekhar gauge (Standard PN gauge, Maximal slicing)

In the Chandrasekhar gauge we have $P^i_{\;i} = -3\dot{U}$. Equations \(46\) and \(47\) give

$$\Delta P^i = -16\pi G\theta v_i + \dot{U}_i,$$

(47)

$$\Delta U + 4\pi G\theta = -\frac{1}{c^2} \left[ 2\Delta \chi + 4\pi G\theta \left( \Pi + 2v^2 + \frac{3p}{\varrho} + 2U + \frac{B^2}{4\pi \varrho} \right) \right].$$

(48)

In the notation of Chandrasekhar (1965), we have

$$\chi = \Phi,$$

(49)

thus, metric becomes

$$\tilde{g}_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}(2\Phi - U^2), \quad \tilde{g}_{0i} = -\frac{1}{c^2} \left( 4U_i - \frac{1}{2}\chi, \chi \right), \quad \tilde{g}_{ij} = \left( 1 + \frac{2}{c^2}U \right)\delta_{ij}.$$  

(50)

In order to distinguish from our notation we put overlines in Chandrasekhar’s $\Phi$ and $\chi$. Chandrasekhar has defined $\tilde{\chi}$ so that we have \[Equation (44) in Chandrasekhar (1965)\]

$$\Delta \tilde{\chi} \equiv -2\dot{U}.$$  

(51)

Thus, the Chandrasekhar gauge condition gives $\nabla \cdot U = -\dot{U}$. Using Equation \(51\) and the gauge condition, Equation \(47\) gives \[Equation (45) in Chandrasekhar (1965)\]

$$\Delta U = -4\pi G\theta \nabla.$$  

(52)

To each PN order, Equation \(48\) gives \[Equations (3) and (41) in Chandrasekhar (1965)\]

$$\Delta U = -4\pi G\theta,$$

(53)

$$\Delta \Phi = -4\pi G\theta \left( \frac{1}{2} \Pi + v^2 + \frac{3p}{2\varrho} + U + \frac{B^2}{8\pi \varrho} \right) \equiv -4\pi G\theta \phi.$$  

(54)

The variable $U$ is the Newtonian potential and $\chi$, $U_i$ and $\Phi$ are post-Newtonian potentials introduced in Chandrasekhar (1965), now extended to include the MHD effect (Greenberg 1971). The potentials can be expressed in terms of integrals as \[Equations (69) and (82) in Chandrasekhar (1965)\]

$$U(t,x) = G \int \frac{\theta(t,x')v(t,x')}{|x-x'|}d^3x', \quad U(t,x) = G \int \frac{\theta(t,x')\phi(t,x')}{|x-x'|}d^3x', \quad \Phi(t,x) = G \int \frac{\theta(t,x')|x-x'|}{|x-x'|}d^3x', \quad \tilde{\chi}(t,x) = -G \int \frac{\theta(t,x')|x-x'|}{|x-x'|}d^3x'.$$

(55)

The gauge condition does not directly affect the conservation equations and Maxwell’s equations in \(48\)-\(49\).

5.2. Harmonic gauge

In the harmonic gauge we have $P^i_{\;i} = -4\dot{U}$. Equations \(46\) and \(47\) give

$$\Delta P^i = -16\pi G\theta v_i,$$

(56)

$$\Box U + 4\pi G\theta = -\frac{1}{c^2} \left[ 2\Delta \chi + 4\pi G\theta \left( \Pi + 2v^2 + \frac{3p}{\varrho} + 2U + \frac{B^2}{4\pi \varrho} \right) \right].$$

(57)

In this gauge the propagation speed of the potential $U$ is the speed of light, whereas in the Chandrasekhar gauge all potentials satisfy Poisson-like equations as in \(51\)-\(54\) with action-at-a-distance nature. Using $\tilde{\chi}$ defined in Equation \(51\), and by introducing

$$\chi = \Phi - \frac{1}{4}\tilde{\chi}, \quad P = 4U,$$

(58)

Equations \(51\) and \(54\) give exactly the same equations in \(52\)-\(53\), thus solutions in Equation \(53\) remain valid; we have $\nabla \cdot U = -\dot{U}$ in the harmonic gauge as well. The metric in the harmonic gauge becomes

$$\tilde{g}_{00} = -1 + \frac{2}{c^2}U + \frac{2}{c^4}(2\Phi - U^2), \quad \tilde{g}_{0i} = -\frac{4}{c^2}U_i, \quad \tilde{g}_{ij} = \left( 1 + \frac{2}{c^2}U \right)\delta_{ij}.$$  

(59)

The PN MHD formulation in the harmonic gauge was studied in Nazari & Roshan (2018).
5.3. Gauge transformation properties

The gauge transformation properties to 1PN order were studied in Section 6 of HNP. Here we summarize the gauge transformation properties in HNP and expand the case to include MHD. We consider a gauge transformation \( \tilde{x}^a = x^a + \xi^a(x^c) \) with \( x^0 = ct \) and
\[
\tilde{\xi}^0 = \frac{1}{c} \xi^{(2)0} + \frac{1}{c^3} \xi^{(4)0} + \ldots, \quad \tilde{\xi}^i = \frac{1}{c} \xi^{(2)i} + \ldots, \tag{60}
\]
where index of \( \xi^{(2)i} \) is raised and lowered by \( \delta_{ij} \). The spatial gauge condition taken in Equation (14) to simplify the space-space part of the metric leads to \( \xi^{(2)0} = 0 = \xi^{(2)i} \), see Equations (171) and (173) in HNP. To 1PN order we have
\[
\tilde{\Upsilon} = \Upsilon + \frac{1}{2} \xi^{(4)0}, \quad \tilde{P}_i = P_i - \xi^{(4)i}, \tag{61}
\]
and the other PN variables are gauge invariant, see Equation (180) in HNP.

The electromagnetic part is a new degree of freedom in addition to hydrodynamic case considered in HNP. To 1PN order, from Equations (6) and (7), we have
\[
\tilde{F}_{ij} = \left(1 - \frac{\Phi}{c^2}\right) \eta_{ij} B^k, \quad \tilde{F}_{0i} = - \left(1 + \frac{\Phi}{c^2}\right) E_i + \frac{1}{c^2} (P \times B)_i, \tag{62}
\]
with \( \Phi = -U \) here. Using the tensorial nature of \( \tilde{F}_{a0} \) and the gauge transformation property of the second-rank tensor presented in Equation (157) of HNP, we can show \( \tilde{F}_{ij} = \tilde{F}_{ij} \) and \( \tilde{F}_{0i} = \tilde{F}_{0i} \) to 1PN order. Thus we have
\[
\tilde{\mathbf{B}} = \mathbf{B}, \quad \tilde{\mathbf{E}} = \mathbf{E} + \frac{1}{c^3} \mathbf{B} \times \nabla \xi^{(4)0}. \tag{63}
\]

5.4. Gauge transformation between Chandrasekhar gauge and harmonic gauge

Using the gauge transformation properties, we can relate 1PN variables between the Chandrasekhar gauge (CG) and the harmonic gauge (HG). In \( \tilde{\mathbf{F}}_{CG} = \tilde{\mathbf{F}}_{HG} + \frac{1}{c^4} \xi^{(4)0}_{HG \rightarrow CG} \), we consider the hat coordinate to be the Chandrasekhar gauge and the non-hat coordinate to be the harmonic gauge. From
\[
\tilde{P}_i |_{CG} = 4 U_i - \frac{1}{2} \tilde{\chi}_{,ij}, \quad P_i |_{HG} = 4 U_i, \quad \tilde{\Upsilon} |_{CG} = \tilde{\Upsilon}, \quad \Upsilon |_{HG} = \tilde{\Upsilon} - \frac{1}{4} \tilde{\chi}, \tag{64}
\]
we have \( \xi^{(4)0}_{HG \rightarrow CG} = \frac{1}{2} \tilde{\chi} \). Thus we have
\[
\tilde{P}_i |_{CG} = P_i |_{HG} - \frac{1}{2} \tilde{\chi}_{,ij}, \quad \tilde{\Upsilon} |_{CG} = \Upsilon |_{HG} + \frac{1}{4} \tilde{\chi}, \quad \tilde{\mathbf{E}} |_{CG} = \mathbf{E} |_{HG} + \frac{1}{2 c^3} \mathbf{B} \times \nabla \tilde{\chi}, \tag{65}
\]
and other 1PN variables are gauge invariant.

6. PN MHD WAVES

Here we present 1PN corrections to the MHD waves. We follow Section 22 of Shu (1992) which present the case without PN correction.

We ignore the internal energy, stress and the gravity (thus \( \mathbf{v} = \mathbf{v} \)), and consider a static and homogeneous background medium with
\[
\tilde{\mathbf{v}}_0 = \text{constant}, \quad p_0 = \text{constant}, \quad \mathbf{B}_0 = \mathbf{B}_0 \tilde{n} = \text{constant}, \quad \mathbf{v}_0 = 0, \tag{66}
\]
where \( \tilde{n} \) is a constant unit vector. Introducing perturbations as
\[
\tilde{\mathbf{v}} = \mathbf{v}_0 + \delta \tilde{\mathbf{v}} = \mathbf{v}_0 (1 + \delta), \quad p = p_0 + \delta p, \quad \mathbf{B} = \mathbf{B}_0 + \delta \mathbf{B}, \tag{67}
\]
to the linear order perturbation, Equations (41)–(43) give
\[
\delta \ddot{\mathbf{v}} + \nabla \cdot \mathbf{v} = 0, \tag{68}
\]
\[
\left( \frac{\delta p_0}{c^2} + \frac{p_0}{c^2} \right) \dot{\mathbf{v}} + \nabla \delta p = \frac{1}{4 \pi} \left( \nabla \times \delta \mathbf{B} \right) \times \mathbf{B}_0 + \frac{1}{c^2} \left( \mathbf{v} \times \mathbf{B}_0 \right) \times \mathbf{B}_0, \tag{69}
\]
\[
\delta \mathbf{B} = \nabla \times (\mathbf{v} \times \mathbf{B}_0), \tag{70}
\]
\[
\nabla \cdot \delta \mathbf{B} = 0. \tag{71}
\]

The PN corrections appear only in the momentum conservation equation in (69); to 1PN order this can be written as
\[
\left( \frac{\delta p_0}{c^2} + \frac{p_0}{c^2} + \frac{B_0^2}{4 \pi c^2} \right) \dot{\mathbf{v}} = - \nabla \delta p + \frac{B_0}{4 \pi} \left( \nabla \times \delta \mathbf{B} \right) \times \tilde{n} - \frac{B_0^2}{4 \pi \delta_0 c^2} \tilde{n} \tilde{n} \cdot \nabla \delta p. \tag{72}
\]
We consider perturbation variables depending on Fourier expansion $e^{i(k \cdot x - \omega t)}$, thus

$$i \omega \delta = i k \cdot v,$$

$$i \omega \left(1 + \frac{p_0}{\rho_0 c^2} + \frac{c_A^2}{c^2}\right) v = i k c_s^2 \omega - i \frac{B_0}{4 \pi \rho_0} (k \times \delta B) \times \hat{n} + i \frac{c_A^2 c_s^2}{c^2} \hat{n} \cdot \kappa \delta,$$  

$$i \omega \delta B = -i B_0 k \times (v \times \hat{n}),$$

$$i k \cdot \delta B = 0,$$

where we introduced the adiabatic sound velocity $c_s$ and the Alfven velocity $c_A$ as

$$c_s^2 = \frac{\delta p}{\delta \rho}, c_A^2 = \frac{B_0^2}{4 \pi \rho_0}.$$  

Equation (75) implies Equation (76). Combining Equations (73)-(75), we have

$$\left[\omega^2 \left(1 + \frac{p_0}{\rho_0 c^2} + \frac{c_A^2}{c^2}\right) - c_A^2 (k \cdot \hat{n})^2\right] v = k \left[\left(c_s^2 + c_A^2\right) k \cdot v - c_A^2 \hat{n} \cdot \hat{n} \cdot \hat{n}\right] - \hat{n} c_A^2 k \cdot \hat{n} \cdot v \left(1 - \frac{c_s^2}{c^2}\right).$$

Following Shu (1992) we set the coordinate as

$$k \equiv k \hat{x}, \quad \hat{n} \equiv \cos \psi \hat{x} + \sin \psi \hat{y}, \quad v \equiv v_x \hat{x} + v_y \hat{y} + v_z \hat{z}.$$  

Thus, we have

$$\left[\omega^2 \left(1 + \frac{p_0}{\rho_0 c^2} + \frac{c_A^2}{c^2}\right) - k^2 c_A^2 \cos^2 \psi\right] v_z = 0,$$  

$$\left[\omega^2 \left(1 + \frac{p_0}{\rho_0 c^2} + \frac{c_A^2}{c^2}\right) - k^2 c_A^2 \sin^2 \psi + \frac{c_A^2 c_s^2}{c^2} \cos^2 \psi\right] v_x = \left[\omega^2 \left(1 + \frac{p_0}{\rho_0 c^2} + \frac{c_A^2}{c^2}\right) - k^2 c_A^2 \sin^2 \psi \cos \psi \right] v_y = 0.$$  

For $v_z \neq 0$ ($v$ perpendicular to $B_0 \cdot k$ plane), from Equation (81) we have the Alfven waves with the velocity

$$\frac{\omega^2}{k^2} = c_A^2 \cos^2 \psi \left(1 - \frac{p_0}{\rho_0 c^2} - \frac{c_s^2}{c^2}\right).$$  

Notice the PN corrections reduce the velocity.

For non-vanishing $v_x$ and $v_y$ ($v$ in $B_0 \cdot k$ plane), from Equation (81) we have

$$\left[\omega^2 \left(1 + \frac{p_0}{\rho_0 c^2} + \frac{c_A^2}{c^2}\right) \right] v_x = \frac{1}{2} \left[\left(c_s^2 + c_A^2 + \frac{c_A^2 c_s^2}{c^2} \cos^2 \psi\right) - 4 c_s^2 c_A^2 \cos^2 \psi \left(1 + \frac{c_s^2}{c^2}\right) \right].$$

The plus and minus signs in $\pm$ correspond to the fast and slow waves, respectively. For $k \parallel B_0 (\psi = 0^\circ)$, we have

$$\frac{\omega^2}{k^2} = c_s^2 \left(1 - \frac{p_0}{\rho_0 c^2}\right), \quad c_A^2 \left(1 - \frac{p_0}{\rho_0 c^2} - \frac{c_s^2}{c^2}\right),$$

with the faster (slower) mode the fast (slow) MHD waves (Shu 1992). For $k \perp B_0 (\psi = 90^\circ)$, we have

$$\frac{\omega^2}{k^2} = (c_s^2 + c_A^2) \left(1 - \frac{p_0}{\rho_0 c^2} - \frac{c_s^2}{c^2}\right),$$

with the fast mode the magnetosonic wave (Shu 1992) and the slow mode vanishing. Equations (82), (85) and (86) show that the PN effects of the pressure and the magnetic pressure, $p_M \equiv B^2/(8\pi)$, of the background tend to slowdown all the wave propagation velocities. The above analysis is gauge invariant.

7. DISCUSSION

We presented general relativistic MHD equations valid to 1PN order, (Sections 3-5). Derivation is presented in the Appendix using the FNLE formulation with MHD shown in NHB. Our 1PN-MHD formulation is complementary to the special relativistic (SR) MHD combined with the weak gravity presented also in NHB. Our 1PN-MHD considers 1PN order expansion for matter, field and gravity consistently. Whereas, the SR-MHD with weak gravity considers
fully relativistic (thus \(\gg PN\)) order in matter and field matched with non-relativistic (thus 0PN) order in gravity. It is not \textit{a priori} obvious that such an asymmetric combination is possible. In NHB we have shown that all equations in Einstein’s gravity are consistently valid with such a combination in the maximal slicing, see Hwang & Noh (2016) in the hydrodynamic situation. Our 1PN-MHD formulation is presented without imposing the temporal gauge condition; for general gauge conditions see Equation (22). 1PN approximation including the ideal MHD in Minkowski background is studied by Greenberg (1971) in the Chandrasekhar gauge and by Nazari & Roshan (2018) in the harmonic gauge. Comparisons are made in Section 5. The PN corrections to the well-known MHD waves in a static homogeneous medium without gravity are presented in Section 6; to 1PN order the gas pressure as well as the magnetic pressure tend to slow down the wave speeds.

Considering the fully nonlinear and exact nature of the original formulation in NHB, it is a trivial procedure to derive higher order PN expansion. The formulation presented in NHB took a special but unique spatial gauge condition without losing any generality or advantage, but \textit{ignored} the transverse-tracefree (TT) perturbation in the spatial metric; ignoring the TT mode is a serious physical restriction excluding the gravitational waves. But these two assumptions were completely relaxed in Gong et al (2017) in the cosmological context; by setting the scale factor to be unity and ignoring the cosmological constant, we recover the formulation in Minkowski background. Thus, our FNLE formulation may provide easier route to derive higher order PN expansion as well as higher order perturbation equations.

The geodesic equations for dust particles (time-like) and photons (null-like) are presented in Section 5 of HNP in the context of cosmology. The presence of MHD does not affect the geodesic equations.

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APPENDIX

FULLY NONLINEAR AND EXACT EQUATIONS TO 1PN ORDER

Using the 1PN notation in Equation (10), and the ideal MHD condition in Equation (9), to 1PN order Einstein’s and conservation equations in (79)-(86) of NHB give

\[
\kappa = \frac{1}{c^2} \left(3 \Psi - P_k, k\right), \tag{A1}
\]

\[-4\pi G\sqrt{\gamma} + \Delta \Psi = O(c^{-2}), \tag{A2}\]

\[
\kappa,i + \frac{3}{2c^2} \left(\Delta P_i + \frac{1}{3} \pi^k, k + 16\pi G\sqrt{\gamma}v_i\right) = 0, \tag{A3}\]

\[-4\pi G\sqrt{\gamma} + \Delta \Phi = -\kappa + \frac{1}{c^2} \left[4\pi G \left(\pi \Pi + 2\pi \gamma^2 + 3p\right) + 2(\Phi - \Psi) \Delta \Phi + \Phi \delta^k (\Phi + \Psi), k + GB^2\right], \tag{A4}\]

\[
\left(\nabla^i \nabla_j - \frac{1}{3} \delta^i_j \Delta\right) (\Phi - \Psi) = O(c^{-2}), \tag{A5}\]

\[
\frac{\partial}{\partial t} \left\{ \frac{\sqrt{\gamma}}{c^2} \left[\sqrt{\gamma} (\Pi + v^2 - 3\Psi) + \frac{1}{8\pi} B^2 \right] \right\}
+ \nabla^j \left\{ \sqrt{\gamma} v_i + \frac{1}{c^2} \left[\sqrt{\gamma} (\Pi + v^2 + \Phi - \Psi) v_i + pv_i + \Pi_{ij} v^j + \sqrt{\gamma} P_i - \frac{1}{4\pi} \left[(v \times B) \times B\right]_i\right]\right\} = -\frac{1}{c^2} \sqrt{\gamma} \cdot \nabla \Phi, \tag{A6}\]

\[
\frac{\partial}{\partial t} \left\{ \sqrt{\gamma} v_i + \frac{1}{c^2} \left[\sqrt{\gamma} (\Pi + v^2 - 3\Psi) v_i + pv_i + \Pi_{ij} v^j - \frac{1}{4\pi} [(v \times B) \times B]_i\right]\right\}
+ \nabla^j \left\{ \sqrt{\gamma} v_i v_j + p \delta_{ij} + \Pi_{ij} + \frac{1}{c^2} \left[\sqrt{\gamma} (\Pi + v^2 + \Phi - \Psi) v_i v_j + pv_i v_j + p \delta_{ij} (\Phi - 3\Psi) + \Pi_{ij} (\Phi - \Psi) + \sqrt{\gamma} P_{ij}\right]\right\}
- \frac{1}{4\pi} \left( B_i B_j - \frac{1}{2} \delta_{ij} B^2 \right) \left(1 + \frac{\Phi}{c^2} - \frac{\Psi}{c^2}\right) - \frac{1}{4\pi c^2} \left[(v \times B)_i (v \times B)_j - \frac{1}{2} \delta_{ij} |v \times B|^2 \right]\right\} = -\sqrt{\gamma} v_i, \tag{A7}\]

\[
\frac{\partial}{\partial t} \left[ \frac{\sqrt{\gamma}}{c^2} \left(1 + \frac{v^2}{2c^2} - 3\frac{\Psi}{c^2}\right) \right] + \nabla_i \left[ \sqrt{\gamma} v^i \left(1 + \frac{v^2}{2c^2} + \frac{\Phi}{c^2} - \frac{\Psi}{c^2}\right) + \sqrt{\gamma} P^i + \frac{B^2}{8\pi} (\Phi + \Psi) \right] = 0. \tag{A8}\]

We used Equation (A1) in deriving Equation (A8). We note that terms on the right-hand-sides of Equations (A2) and (A3) are 2PN order, thus can be ignored to 1PN order; the FNLE equations presented in NHB, being fully nonlinear,
can be expanded to any/all PN orders (except that we ignored the transverse-tracefree perturbation), and the correct 1PN orders in Equations (A2) and (A5) are limited by the spatial curvature terms $R_{ij}^{(h)}$, see Equations (54), (87), (89), (90), (94) and (97) in HNP.

The Maxwell’s equations in Equations (87)-(90) of NHB give

\[
\nabla \cdot \left[ \left( 1 - \frac{\Psi}{c^2} \right) B \right] = 0, \quad (A9)
\]

\[
\frac{\partial}{\partial t} \left[ \left( 1 - \frac{\Psi}{c^2} \right) B \right] = \nabla \times \left\{ \left( 1 + \frac{\Phi}{c^2} + \frac{\Psi}{c^2} \right) \nabla \frac{1}{c^2} \right\} \times B, \quad (A10)
\]

\[
\nabla \cdot \left[ \left( 1 - \frac{\Psi}{c^2} \right) E \right] = - \frac{1}{c} \nabla \cdot (v \times B) = 4\pi \rho_{em} \left( 1 - 3\frac{\Psi}{c^2} \right), \quad (A11)
\]

\[
\frac{\partial}{\partial t} \left[ \left( 1 - \frac{\Psi}{c^2} \right) E \right] = - \frac{1}{c} \frac{\partial}{\partial t} (v \times B) = c^2 \nabla \times \left[ \left( 1 + \frac{\Phi}{c^2} \right) B \right] - 4\pi \left[ j \left( 1 + \frac{\Phi}{c^2} - \frac{\Psi}{c^2} \right) + \frac{1}{c^2} \rho_{em} P \right]. \quad (A12)
\]

Using Maxwell’s equations, the MHD contributions in Equations (A6) and (A7) can be collected on the right-hand-sides, respectively, as

\[
- \frac{1}{4\pi c^2} (v \times B) : (\nabla \times B),
\]

\[
+ \frac{1}{4\pi} \left[ (\nabla \times B) \times B \right]_i \left( 1 + \frac{\Phi}{c^2} - \frac{\Psi}{c^2} \right) + \frac{1}{4\pi c^2} \left\{ \left[ (v \times B)^i \times B \right]_i + (v \times B)_i \nabla \cdot (v \times B) + [B \times (B \times \nabla \Phi)]_i \right\}. \quad (A13)
\]

In the absence of the MHD, using the notations in Equation (11) our equations above reproduce 1PN equations in HNP: using Equation (A1), Equations (A3), (A4), (A6), (A7) and (A8), respectively, give Equations (79), (78), (57), (58) and (62) in HNP; by setting the scale factor $a \equiv 1$ and setting the cosmological constant $\Lambda \equiv 0$ in HNP we have the 1PN formulation in Minkowski background.

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