Convergence Properties of the Heterogeneous Deffuant-Weisbuch Model *

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Abstract

The Deffuant-Weisbuch (DW) model is a bounded-confidence opinion dynamics model that has attracted much recent interest. Despite its simplicity and appeal, the DW model has proved technically hard to analyze and its most basic convergence properties, easy to observe numerically, are only conjectures.

This paper solves the convergence problem for the heterogeneous DW model. We establish that, for any positive confidence bounds and initial values, the opinion of each agent will converge to a limit value almost surely. Additionally, we show that the limiting opinions of any two agents either are the same or have a distance larger than the confidence bounds of the two agents. Moreover, we provide some sufficient conditions for the heterogeneous DW model to reach consensus. Finally, we show the mean-square convergence rate of the heterogeneous DW model is exponential.

Key words: Opinion dynamics, consensus, Deffuant model, gossip model, bounded confidence model

1 Introduction

The field of opinion dynamics studies the dynamical processes regarding the formation, diffusion, and evolution of public opinion about certain events and object of interest in social systems. The study of opinion dynamics can be traced back to the two-step communication flow model in (Katz and Lazarsfeld, 1955) and the social power and averaging model in (French Jr., 1956). The model by French Jr. (1956) was then elaborated by Harary (1959) and rediscovered by DeGroot (1974). Other notable developments include the model by Friedkin and Johnsen (1990) with attachment to initial opinions, a general influence network theory (Friedkin, 1998), social impact theory (Latané, 1981), and dynamic social impact theory (Latané, 1996). A comprehensive review of opinion dynamics models is given in the two tutorials Proskurnikov and Tempo (2017, 2018) and the textbook Bullo (2018).

In recent years, significant attention has focused on so-called bounded confidence (BC) models of opinion dynamics. In these models one individual is willing to accord influence to another only if their pair-wise opinion difference is below a threshold (i.e., the confidence bound). (Deffuant et al., 2000) propose their now well-known BC model called the Deffuant-Weisbuch (DW) model or Deffuant model. In this model a pair of individuals is selected randomly at each discrete time step and each individual updates its opinion if the other individual’s opinion lies within its confidence bound. A second well-known BC model is the Hegselmann-Krause (HK)
model (Hegselmann and Krause, 2002), where all individuals update their opinions synchronously by averaging the opinions of individuals within their confidence bounds.

As reported in (Lorenz, 2007, 2010), simulation results for the DW model have revealed numerous interesting phenomena such as consensus, polarization and fragmentation. However, the DW model is in general hard to analyze due to the nonlinear state-dependent inter-agent topology. Current analysis results focus on the homogeneous case in which all the agents have the same confidence bound. The convergence of the homogeneous DW model has been proved in (Lorenz, 2005) and its convergence rate is established in (Zhang and Chen, 2015). Some research has considered also modified DW models. For example, (Como and Fagnani, 2011) consider a generalized DW model with an interaction kernel and investigate its scaling limits when the number of agents grows to infinity; (Zhang and Hong, 2013) generalize the DW model by assuming that each agent can choose multiple neighbors to exchange opinion at each time step. Despite all this progress, the analysis of the heterogeneous DW model by assuming that each agent can choose multiple neighbors to exchange opinion at each time step. Despite all this progress, the analysis of the heterogeneous DW model is still incomplete in that its convergence properties are yet to be established.

This paper establishes the convergence properties of the heterogeneous DW model. We show that, for any positive confidence bounds and initial opinions, the opinion of each agent converges almost surely to a limiting value. Additionally, we prove that the limiting values of any two agents’ opinions are either identical or have a distance larger than the confidence bounds of the two agents. Moreover, we show that a sufficient, and in some cases also necessary, condition for almost sure consensus; the intuitive condition is expressed as a function of the largest confidence bound in the group. Finally, we show the exponential convergence of the mean square error.

The paper is organized as follows. Section 2 introduces the heterogeneous DW model and our main convergence results. Section 3 contains the proofs of our convergence results. Section 4 contains the analysis of the convergence rate and, finally, Section 5 concludes the paper.

2 The heterogeneous DW model and our main convergence results

Following (Lorenz, 2007), this paper considers the following basic DW model. In a group of $n \geq 3$ agents, we assume each agent $i \in \{1, \ldots, n\}$ has a real-valued opinion $x_i(t) \in \mathbb{R}$ at each discrete time $t \in \mathbb{Z}_{\geq 0}$. We let $x(t) := (x_1(t), \ldots, x_n(t))^T$ assume, without loss of generality, that $x(0) \in [0,1]^n$. We let $r_i > 0$ denote the confidence bound of the agent $i$ and we assume, without loss of generality,

$$r_1 \geq r_2 \geq \cdots \geq r_n > 0.$$

We let $\mathbb{1}_{\{\cdot\}}$ denote the indicator function, i.e., we let $\mathbb{1}_{\{\omega\}} = 1$ if the property $\omega$ holds true and $\mathbb{1}_{\{\omega\}} = 0$ otherwise. At each time $t \in \mathbb{Z}_{\geq 0}$, a pair $(i_t, j_t)$ is independently and uniformly selected from the set of all pairs $\mathcal{N} = \{(i, j) \mid i, j \in \{1, \ldots, n\}, i < j\}$. Subsequently, the opinions of the agents $i_t$ and $j_t$ are updated according to

$$x_{i_t}(t+1) = x_{i_t}(t) + \frac{1}{2} \mathbb{1}_{\{|x_{i_t}(t) - x_{j_t}(t)| \leq r_{i_t}\}} (x_{j_t}(t) - x_{i_t}(t)),$$

$$x_{j_t}(t+1) = x_{j_t}(t) + \frac{1}{2} \mathbb{1}_{\{|x_{i_t}(t) - x_{j_t}(t)| \leq r_{j_t}\}} (x_{i_t}(t) - x_{j_t}(t)),$$

whereas the other agents’ opinions remain unchanged:

$$x_k(t+1) = x_k(t), \text{ for } k \in \{1, \ldots, n\} \setminus \{i_t, j_t\}. \quad (2)$$

If $r_1 = \cdots = r_n$, the DW model is called homogeneous, otherwise heterogeneous.

Previous works (Lorenz, 2005) show that the homogeneous DW model (1)-(2) always converges to a limit opinion profile. Simulations reported in (Lorenz, 2007) show that this property holds also for the heterogeneous case; but a proof for this statement is lacking. We note that the original DW model (Defuian et al., 2000) contains a weighting factor $\mu \in (0,1)$ instead of $1/2$ factor in our protocol (1)-(2). Simulations in (Defuian et al., 2000; Weisbuch et al., 2002) show that the parameter $\mu$ affects only the convergence time and so previous works (Lorenz, 2007, 2010) simplified the model by setting $\mu := 1/2$. Following these previous works, also this paper adopts the $\mu := 1/2$ simplification.

Before stating our convergence results, we need to define the probability space of the DW model. If the initial state $x(0)$ is a deterministic vector, we let $\Omega = \mathcal{N}^\infty$ be the sample space, $\mathcal{F}$ be the Borel $\sigma$-algebra of $\Omega$, and $\mathbb{P}$ be the probability measure on $\mathcal{F}$. Then the probability space of the DW model is written as $(\Omega, \mathcal{F}, \mathbb{P})$. If the initial state is a random vector, we let $\Omega = [0,1]^n \times \mathcal{N}^\infty$ be the sample space and, similarly to the case of deterministic initial state, the probability space is defined by $(\Omega, \mathcal{F}, \mathbb{P})$.
Theorem 1 (Almost sure convergence of heterogeneous DW model) Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. For any initial state $x(0) \in [0, 1]^n$, there exists a random vector $x^* \in [0, 1]^n$ satisfying

(i) $x_i^* = x_j^*$ or $|x_i^* - x_j^*| > \max\{r_i, r_j\}$ for all $i \neq j$, and

(ii) $x(t)$ converges to $x^*$ almost surely (a.s.) as $t \to \infty$, that is,

$$\mathbb{P}\left(\omega \in \Omega : \lim_{t \to \infty} x(t)(\omega) = x^*(\omega)\right) = 1.$$ 

The proof of Theorem 1 is postponed to Section 3. Fig. 1 displays the simulation results for a heterogeneous DW model (1)-(2) with 8 agents and with confidence bounds $r_1, r_2, \ldots, r_8$ equal to .5, .41, .35, .24, .175, .165, .12, .047 respectively. Consistently with Theorem 1, Fig. 1 shows that the individual opinions converge to two distinct limit values and that the distance between the two values is larger than $r_1 = 0.5$.

Theorem 1 leads to two corollaries on convergence to consensus. By consensus we mean that all agents’ opinions converge to the same value.

Corollary 2 (Almost sure consensus for large confidence bound) Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. If the largest confidence bound satisfies $r_1 \geq 1$, then for any initial state $x(0) \in [0, 1]^n$ the system reaches consensus a.s.

Corollary 3 (Almost sure consensus if and only if large confidence bound) Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. Assume that the initial state $x(0)$ is randomly distributed in $[0, 1]^n$ and that its joint probability density has a lower bound $\rho_{\min} > 0$, that is, for any real numbers $a_i, b_i$,

$$i \in \{1, \ldots, n\}, \text{ with } 0 \leq a_i < b_i \leq 1,$$

$$\mathbb{P}\left(\bigcap_{i=1}^{n} \{x_i(0) \in [a_i, b_i]\}\right) \geq \rho_{\min} \prod_{i=1}^{n} (b_i - a_i). \quad (3)$$

Then the heterogeneous DW model reaches consensus almost surely if and only if $r_1 \geq 1$.

Corollary 3 provides a sufficient and necessary condition for almost sure consensus when the initial opinions are randomly distributed. However, for settings when almost sure consensus is not guaranteed, the probability of achieving consensus is unknown. In the remainder of this section, we provide simulation results for the consensus probability of the heterogeneous DW model. Let $n = 10$ and suppose that agent 1 has a maximal confidence bound $r_{\max}$ whose value is chosen over the set $\{\frac{i}{20} : i = 1, 2, \ldots, 20\}$. We approximate the consensus probability via the Monte Carlo method. We run 1000 samples for each value of $r_{\max}$. In each sample, we assume the initial opinions are independently and uniformly distributed on $[0, 1]$, while the confidence bounds of agents 2, 3, $\ldots$, 10 are independently and uniformly distributed on $[0, r_{\max}]$. Fig. 2 shows the estimated consensus probability of the heterogeneous DW model (1)-(2) as a function of the maximal confidence bound $r_{\max}$.

3 Proof of convergence results

The proof of Theorem 1 requires multiple steps. We adopt the method of “transforming the analysis of a stochastic system into the design of control algorithms” first proposed by (Chen, 2017). This method requires the construction of a new system called as DW-control system to help with the analysis of the DW model.
3.1 DW-control system and connection to DW model

Consider the DW protocol (1)-(2) where, at each time $t$, the pair $(i_t, j_t)$ is not selected randomly but instead treated as a control input. In other words, assume that $(i_t, j_t)$ is chosen from the set $\mathcal{N}$ arbitrarily as a control signal. We call such a control system the DW-control system.

Given $S \subseteq \mathbb{R}^n$, we say $S$ is reached at time $t$ if $x(t) \in S$ and is reached in the time interval $[t_1, t_2]$ if there exists $t \in [t_1, t_2]$ such that $x(t) \in S$.

**Definition 4** Let $S_1, S_2 \subseteq [0, 1]^n$ be two state sets. Under the DW-control system, $S_1$ is said to be (uniformly) finite-time reachable from $S_2$ if there exists a duration $t^* > 0$ such that for any $x(0) \in S_2$, we can find a sequence of pairs $(i_0^*, j_0^*), (i_1^*, j_1^*), \ldots, (i_{t^*}^*, j_{t^*}^*)$ for opinion update which guarantees $S_1$ is reached in the time interval $[0, t^*]$.

Based on these definitions we can get the following result.

**Lemma 5 (Connection between DW model and DW-control system)** Let $S \subseteq [0, 1]^n$ be a set of states. Assume $S$ is finite-time reachable from $[0, 1]^n$ under the DW-control system. Then, under the DW protocol, for any initial state $x(0) \in [0, 1]^n$, there exist constants $T > 0$ and $a \in (0, 1)$ such that

$$
P(\tau \geq t) \leq a^{t/T}, \quad \forall t \geq 1,
$$

where $\tau := \min \{ t' : x(t') \in S \}$ is the time when $S$ is firstly reached.

**PROOF.** First according to the rule of the DW protocol (1)-(2) we get $x(t) \in [0, 1]^n$ for all $t \geq 0$. Also, since $S$ is reached in finite time from $[0, 1]^n$ under the DW-control system, by Definition 4 there exist an integer $t^*$ such that for any $x(0) \in [0, 1]^n$, we can find a sequence of pairs $(i_0^*, j_0^*), (i_1^*, j_1^*), \ldots, (i_{t^*}^*, j_{t^*}^*)$ which guarantees $S$ is reached in $[0, t^*]$. From this and the definition of the DW-control system, for any $t \geq 0$ and $x(t) \in [0, 1]^n$, there exists a sequence of pairs $(i_t, j_t), (i_{t+1}, j_{t+1}), \ldots, (i_{t+t^*-1}, j_{t+t^*-1})$ such that $S$ is reached in $[t, t + t^*]$. Thus, under the DW protocol, for any $t \geq 0$ and $x(t) \in [0, 1]^n$ we have

$$
P\left(\{S \text{ is reached in } [t, t + t^*]\} | x(t)\right) \geq \mathbb{P}\left(\bigcap_{s=t}^{t+t^*-1} \{(i_s, j_s) = (i'_s, j'_s)\} | x(t)\right)
$$

$$
= \mathbb{P}\left((i_t, j_t) = (i'_t, j'_t)|x(t)\right)
$$

$$
\cdot \mathbb{P}\left(\bigcap_{s=t}^{t+t^*-1} \{(i_s, j_s) = (i'_s, j'_s)\} | x(t), (i_t, j_t) = (i'_t, j'_t)\right)
$$

$$
\cdots = \mathbb{P}\left((i_t, j_t) = (i'_t, j'_t)|x(t)\right)
$$

$$
\cdot \mathbb{P}\left(\{(i_{t+1}, j_{t+1}) = (i'_{t+1}, j'_{t+1})\} | x(t), (i_t, j_t) = (i'_t, j'_t)\right)
$$

$$
\cdots \mathbb{P}\left(\{(i_{t+t^*-1}, j_{t+t^*-1}) = (i'_{t+t^*-1}, j'_{t+t^*-1})\} | x(t), (i_t, j_t) = (i'_t, j'_t), s \in [t, t + t^* - 2]\right),
$$

where all the equations use Bayes’ Theorem. Because $(i_t, j_t)$ is uniformly and independently selected from the set $\mathcal{N}$, for any $x(s) \in [0, 1]^n$ we get

$$
P\left(\{(i_s, j_s) = (i'_s, j'_s)|x(s)\} = \frac{1}{|\mathcal{N}|} = \frac{2}{n(n - 1)^{t^*}}.
$$

where $|\mathcal{N}|$ denotes the cardinality of the set $\mathcal{N}$. Substituting (5) into (4) yields

$$
P\left(\{S \text{ is reached in } [t, t + t^*]\} | x(t)\right) \geq \frac{2^{t^*}}{n^{t^*}(n - 1)^{t^*}}.
$$

Set $E_t$ to be the event that $S$ is reached in $[t, t + t^*]$, and let $E^c_t$ be the complement set of $E_t$. For any integer $M > 0$ and $x(0) \in [0, 1]^n$, Bayes’ Theorem again and equation (6) imply

$$
P\left(\{S \text{ is not reached in } [0, (t^* + 1)M - 1]\} | x(0)\right)
$$

$$
= \mathbb{P}\left(\bigcap_{m=0}^{M-1} E^c_{m(t^*+1)} | x(0)\right)
$$

$$
= \mathbb{P}\left(E^c_0 | x(0)\right) \prod_{m=1}^{M-1} \mathbb{P}\left(E^c_{m(t^*+1)} | x(0), \bigcap_{0 \leq m' < m} E^c_{m'(t^*+1)}\right)
$$

$$
\leq \left(1 - \frac{2^{t^*}}{n^{t^*}(n - 1)^{t^*}}\right)^M.
$$

Let $a := 1 - \frac{2^{t^*}}{n^{t^*}(n - 1)^{t^*}}$. For any integer $M > 0$ and $x(0) \in [0, 1]^n$, by (7) we have

$$
P(\tau \geq (t^* + 1)M | x(0))
$$

$$
= \mathbb{P}\left(\{S \text{ is not reached in } [0, (t^* + 1)M - 1]\} | x(0)\right)
$$

$$
\leq a^M.
$$
Let $T := t^* + 1$. From (8) we can get
\[ P(\tau \geq t|x(0)) \leq P\left( \tau \geq \frac{t}{T} \right) |x(0)) \leq a^{t/T}. \]

According to Lemma 5, to prove the convergence of the DW model, we only need to design control algorithms for DW-control system such that a convergence set is reached. Before the design of such control algorithms we introduce some useful notions.

### 3.2 Maximal-confidence clusters and properties

Recall that we assume $r_1 \geq r_2 \geq \cdots \geq r_n > 0$. For any opinion state $x = (x_1, \ldots, x_n) \in [0,1]^n$, let $C_1(x) \subseteq \{1, \ldots, n\}$ be the set of the agents that can connect to agent 1 directly or indirectly with the confidence bound $r_1$, i.e., $i \in C_1(x)$ if and only if $|x_i - x_1| \leq r_1$ or there exists some agents $1', 2', \ldots, k' \in \{1, \ldots, n\}$ such that $|x_i - x_{1'}| \leq r_1, |x_j - x_{2'}| \leq r_1, \ldots, |x_{k'} - x_1| \leq r_1$. From this definition we have $1 \in C_1(x)$.

Set $\widetilde{C}_1(x) := \{1, \ldots, n\} \setminus C_1(x)$. If $\widetilde{C}_1(x)$ is not empty, we let $i_2 := \min_{i \in \widetilde{C}_1(x)} i$ and define $C_2(x) \subseteq \widetilde{C}_1(x)$ to be the set of the agents that can connect to agent $i_2$ directly or indirectly with the confidence bound $r_{i_2}$. Set $\widetilde{C}_2(x) := \{1, \ldots, n\} \setminus (C_1(x) \cup C_2(x))$. If $\widetilde{C}_2(x)$ is not empty, we let $i_3 := \min_{i \in \widetilde{C}_2(x)} i$ and define $C_3(x) \subseteq \widetilde{C}_2(x)$ to be the set of the agents that can connect to agent $i_3$ directly or indirectly with the confidence bound $r_{i_3}$. Repeat this process until there exists an integer $K$ such that $\widetilde{C}_K(x) = \emptyset$. We call the sets $C_1(x), C_2(x), \ldots, C_K(x)$ maximal-confidence (MC) clusters. Note that MC clusters are quite different from connected components in graph theory.

To illustrate the definition of MC clusters we give an example, visualized Fig. 3: Assume that $n = 7$ and that the agents are labeled by 1, 2, 3, 4, 5, 6, 7. We suppose $r_1 \geq r_2 \geq \cdots \geq r_7$. With the confidence bound $r_1$ the agent 1 can connect to agents 5 and 7, and the agent 7 can connect to agent 3; however agent 3 cannot connect to agent 2. Thus, the first MC cluster $C_1(x)$ is $\{1, 3, 5, 7\}$. The remaining agents are 2, 4, and 6. With the confidence $r_2$ the agent 2 can connect to agent 4, and the agent 4 can connect to agent 6, so the second MC cluster $C_2(x)$ is $\{2, 4, 6\}$.

The following lemma describes the distance between MC clusters.

**Lemma 6 (Distance between maximal-confidence clusters)** For any opinion state $x \in [0,1]^n$ and two different MC clusters $C_1(x)$ and $C_2(x)$, let $r_{ij}^{\text{max}} := \max_{k \in C_1(x) \cup C_2(x)} r_k$ be the maximal confidence bound of all agents in $C_1(x)$ and $C_2(x)$. Then, the opinion values of agents in $C_1(x)$ are all $r_{ij}^{\text{max}}$ bigger or smaller than those in $C_2(x)$, i.e.,

- $x_k - x_l > r_{ij}^{\text{max}} \quad \forall k \in C_1(x), l \in C_2(x)$,
- or $x_l - x_k > r_{ij}^{\text{max}} \quad \forall k \in C_1(x), l \in C_2(x)$.

**PROOF.** Without loss of generality we assume that $\max_{k \in C_1(x)} r_k = r_{ij}^{\text{max}}$. Let $x_{i_k}^{\text{min}} := \min_{k \in C_1(x)} x_k$ and $x_{i_k}^{\text{max}} := \max_{k \in C_1(x)} x_k$ denote the minimal and maximal opinion values of all agents in $C_1(x)$ respectively. For any $l \in C_1(x)$, if $x_l \in [x_{i_k}^{\text{min}} - r_{ij}^{\text{max}} - r_{ij}^{\text{max}} + r_{ij}^{\text{max}}]$, by the definition of the MC cluster we have $l \in C_1(x)$, which is contradictory with $l \in C_2(x)$. Thus, for any $l \in C_2(x)$, we get

\[ x_k - x_l > r_{ij}^{\text{max}} \quad \forall k \in C_1(x), l \in C_2(x) \quad (9) \]

or

\[ x_l - x_k > r_{ij}^{\text{max}} \quad \forall k \in C_1(x), l \in C_2(x) \quad (10) \]

Since $C_2(x)$ is also a MC cluster, there is no agent in $C_1(x)$ whose opinion value is located in the interval $[x_{i_k}^{\text{min}}, x_{i_k}^{\text{max}}]$. Thus, either (9) or (10) holds for all $l \in C_2(x)$. \(\Box\)

Under the DW protocol (1)-(2), the MC clusters have the convex property as follows.

**Lemma 7 (Convexity of maximal-confidence clusters)** Consider the DW protocol (1)-(2) with arbitrary initial state and update pairs $\{(i_t, j_t)\}_{t \geq 0}$. For any $t \geq 0$ and any MC cluster $C_i(x(t))$, the opinion values of all agents in $C_i(x(t))$ will always stay in the interval $[x_{i_k}^{\text{min}}(t), x_{i_k}^{\text{max}}(t)]$ at the time $s \geq t$, i.e.,

\[ x_{i_k}^{\text{min}}(t) \leq x_j(s) \leq x_{i_k}^{\text{max}}(t), \quad \forall j \in C_i(x(t)), s \geq t, \]

where $x_{i_k}^{\text{min}}(t) := \min_{k \in C_i(x(t))} x_k(t)$ and $x_{i_k}^{\text{max}}(t) := \max_{k \in C_i(x(t))} x_k(t)$ denote the minimal and maximal opinion values of all agents in $C_i(x(t))$ respectively.
PROOF. Assume that at time $t$ all MC clusters are $C_1 = C_1(x(t)), C_2 = C_2(x(t)), \ldots, C_K = C_K(x(t))$. By Lemma 6 we can order these clusters as

$$C_{j_1} < C_{j_2} < \cdots < C_{j_K},$$

and get, for $1 \leq k \leq K - 1$,

$$\min_{i \in C_{j_k+1}} x_i(t) - \max_{i \in C_{j_k}} x_i(t) > r^{k,k+1}, \quad (11)$$

where $C_i < C_j$ means that at time $t$ the opinion values of the agents in $C_i$ are all less than those in $C_j$, and $r^{k,k+1} := \max_{i \in C_{j_k} \cup C_{j_{k+1}}} r_i$.

By the DW protocol (1)-(2), if the update pair $(i_t, j_t)$ belongs to different MC clusters then from (11) we have $x_{i_t}(t + 1) = x_{i_t}(t)$ and $x_{j_t}(t + 1) = x_{j_t}(t)$; if $(i_t, j_t)$ belongs to a same MC cluster $C_{j_k}$ then $x_{i_t}(t + 1)$ and $x_{j_t}(t + 1)$ will stay in the interval $[x_{j_k}^{m}(t), x_{j_k}^{r}(t)]$. Thus, for $1 \leq k \leq K - 1$

$$\min_{i \in C_{j_k+1}} x_i(t + 1) - \max_{i \in C_{j_k}} x_i(t + 1) > r^{k,k+1}.$$ 

Repeating this process yields our result. \(\square\)

With the definition and properties of MC clusters we can design control algorithms and complete final proof of our results in the following subsection.

3.3 Design of control algorithms and final proofs

For any opinion state $x \in [0,1]^n$ and any MC cluster $C_i(x)$, we say that $C_i(x)$ is a complete cluster if any agent in $C_i(x)$ can interact with others with the minimal confidence bound of $C_i(x)$, i.e.,

$$\max_{j,k \in C_i(x)} |x_j - x_k| \leq \min_{j \in C_i(x)} r_j.$$

**Lemma 8** Let $t \geq 0$ and $x(t) \in [0,1]^n$ be arbitrarily given. Let $C_i(x(t))$ be an arbitrary MC cluster, in which the agents' maximal and minimal confidence bounds are $r^{i}_{\text{max}}$ and $r^{i}_{\text{min}}$ respectively. Assume

$$\max_{M,m \in C_i(x(t))} [x_M(t) - x_m(t)] > r^{i}_{\text{min}}. \quad (12)$$

Then, under the DW-control system, there is a sequence of agent pairs $(i'_1, j'_1), (i'_2, j'_2), \ldots, (i'_{t^*}, j'_{t^*})$ with

$$t^* \leq (|C_i(x(t))| - 1)^2 (1 + \lceil \log_2 [r^{i}_{\text{max}} / r^{i}_{\text{min}}] \rceil)$$

for opinion update, such that one of the following two results holds:

(i) the agents in $C_i(x(t))$ split into different MC clusters at time $t + t^*$; and

(ii) we have

$$\max_{M,m \in C_i(x(t))} [x_M(t + t^*) - x_m(t + t^*)] \leq \min_{M,m \in C_i(x(t))} [x_M(t) - x_m(t)] - r^{i}_{\text{min}} / 4.$$ 

The proof of Lemma 8 is quite complicated. We put it in Appendix A.

**Lemma 9** Let $t \geq 0$ and $x(t) \in [0,1]^n$ be arbitrarily given. Let $C_i(x(t))$ be an arbitrary MC cluster. Assume

$$\max_{M,m \in C_i(x(t))} [x_M(t) - x_m(t)] \leq \min_{m \in C_i(x(t))} r_m. \quad (13)$$

Then, under the DW-control system, there is a sequence of agent pairs $(i'_1, j'_1), (i'_2, j'_2), \ldots, (i'_{t^*}, j'_{t^*})$ with $t^* \leq |C_i(x(t))/2$ for opinion update, such that

$$\max_{M,m \in C_i(x(t))} [x_M(t + t^*) - x_m(t + t^*)] \leq \frac{2}{3} \max_{M,m \in C_i(x(t))} [x_M(t) - x_m(t)].$$

**PROOF.** The proof of this lemma is similar for the cases $t = 0, 1, 2, \ldots$. To simplify the exposition we consider only the case when $t = 0$. We set

$$x^{i}_{\text{min}}(0) = \min_{m \in C_i(x(0))} x_m(0),$$

$$x^{i}_{\text{max}}(0) = \max_{m \in C_i(x(0))} x_m(0).$$

Let $d(0) := x^{i}_{\text{max}}(0) - x^{i}_{\text{min}}(0)$.

Set

$$B(s) := \{m \in C_i(x(0)) : x_m(s) < x^{i}_{\text{min}}(0) + d(0)/3\},$$

and

$$\overline{B}(s) := \{m \in C_i(x(0)) : x_m(s) > x^{i}_{\text{max}}(0) - d(0)/3\}.$$ 

Take $I = |B(0)|$ and $J = |\overline{B}(0)|$. Without loss of generality we assume $I \leq J$.

Label the elements in $B(0)$ as $i_0, i_1, \ldots, i_{I-1}$, and the elements in $\overline{B}(0)$ as $j_0, j_1, \ldots, j_{J-1}$. For $0 \leq k \leq I - 1$, we choose $(i_k, j_k)$ as the agent pair for opinion update.
at time \( k \), then by the protocol (1)-(2) and (13) we get
\[
x_{i_k}(I) = x_{j_k}(I) = x_{i_k}(k+1) = \frac{x_{i_k}(k) + x_{j_k}(k)}{2} = \frac{x_{i_k}(0) + x_{j_k}(0)}{2}
\]
\[
\in \left( \frac{x^i_{\min}(0) + x^i_{\max}(0) - \frac{d(0)}{3}}{2}, \frac{x^i_{\min}(0) + \frac{d(0)}{3} + x^i_{\max}(0)}{2} \right),
\]
which proves \( B(I) = \emptyset \). Combining this equality with Lemma 7 we obtain
\[
\max_{M,m \in C_i(x(t))} [x_M(I) - x_m(I)] 
\leq \max_{1 \leq i \leq K, j \in C_i(x)} |x_j - x_k|.
\]
Finally, because \( I \leq |C_i(x(t))|/2 \), our result follows. □

For any opinion state \( x = (x_1, \ldots, x_n) \in [0,1]^n \), let \( D(x) \) denote the maximal diameter among all the MC clusters \( C_1(x), C_2(x), \ldots, C_K(x) \), i.e.,
\[
D(x) = \max_{1 \leq i \leq K} \max_{j \in C_i(x)} |x_j - x_k|.
\]

**Lemma 10** Consider the DW-control system. Let \( r_{\min} \) and \( r_{\max} \) be the maximal and minimal confidence bounds of all agents. Then for any initial state and constant \( \varepsilon > 0 \), there exists a sequence of agent pairs \((i_0' , j_0'), (i_1', j_1'), \ldots, (i_{t^*}, j_{t^*})\) with
\[
t^* \leq (n-1)^2 \left( 1 + \left[ \log_2 \left( \frac{r_{\max}}{r_{\min}} \right) \right] \right) \left( \frac{4(1 - r_{\min})}{r_{\min}} \right)
\]
\[
+ \frac{n}{2} \left[ -\log_2 \left( \frac{\varepsilon}{\log(3/2)} \right) \right].
\]
for opinion update such that \( D(x(t^*)) \leq \varepsilon \).

**Proof.** Assume there are \( K_t \) MC clusters \( C_1(x(t)), C_2(x(t)), \ldots, C_{K_t}(x(t)) \) at time \( t \). Using Lemma 8 repeatedly there exists a sequence of agent pairs \((i_0' , j_0'), (i_1', j_1'), \ldots, (i_{t_{T_t}}', j_{t_{T_t}}')\) for opinion update such that
\[
\max_{M,m \in C_i(x(T_{i(t))))} [x_M(T_{i(t)}) - x_m(T_{i(t)})] \leq \min_{m \in C_i(x(T_{i(t)}))} r_m
\]
for all \( 1 \leq i \leq K_{T_t} \). Since \((c_1-1)^2 + \cdots + (c_m-1)^2 \leq (c_1+\cdots+c_m-1)^2\) for any positive integers \( m, c_1, \ldots, c_m \), by Lemma 8 it can be computed that
\[
T_1 \leq (n-1)^2 \left( 1 + \left[ \log_2 \left( \frac{r_{\max}}{r_{\min}} \right) \right] \right) \left( \frac{4(1 - r_{\min})}{r_{\min}} \right).
\]
Further, using Lemma 9 repeatedly there exists a sequence of agent pairs \((i'_{T_1}, j_{T_1}), (i'_{T_1+1}, j_{T_1+1}), \ldots, (i'_{T_1+T_2-1}, j_{T_1+T_2-1})\) for opinion update such that
\[
D(x(T_1 + T_2)) \leq \varepsilon.
\]
Lemma 9 now implies \( T_2 \leq \frac{n}{2} \left[ -\log_2 \left( \frac{\varepsilon}{\log(3/2)} \right) \right] \). □

**Proof of Theorem 1** For any constant \( \varepsilon > 0 \), let \( S_\varepsilon \) be the state set defined by
\[
S_\varepsilon := \{ x \in [0,1]^n : D(x) \leq \varepsilon \},
\]
where \( D(x) \) is the maximal diameter of all MC clusters defined by (14). By Lemma 10, \( S_\varepsilon \) is finite-time reachable from \([0,1]^n\) under the DW-control system. Let \( \tau_\varepsilon \) be the time when \( S_\varepsilon \) is firstly reached under the DW protocol (1)-(2). By Lemma 5, \( \mathbb{P}(\tau_\varepsilon < \infty) = 1 \) for any \( \varepsilon > 0 \). By the convexity of MC clusters (Lemma 7) we have
\[
D(x(t)) \leq \varepsilon \text{ for all } t \geq \tau_\varepsilon.
\]
Let \( \varepsilon \to 0^+ \) we can get
\[
\mathbb{P} \left( \omega \in \Omega : \lim_{t \to +\infty} D(x(t)) (\omega) = 0 \right) = 1.
\]
From this and Lemma 7 we have that \( x(t) \) a.s. converges to a random vector \( x^* \). By Lemma 6 we obtain \( x^*_i = x^*_j \) or \( |x^*_i - x^*_j| \geq \max\{r_i, r_j\} \) for any \( i \neq j \). □

**Proof of Corollary 2** By Theorem 1 we have \( x(t) \) a.s. converges to a limit point \( x^* \in [0,1]^n \) which satisfies either \( |x^*_i - x^*_j| = 0 \) or \( |x^*_i - x^*_j| > r_1 \) for all \( 2 \leq i \leq n \). Because \( r_1 \geq 1 \), we have \( |x^*_i - x^*_j| = 0 \) for all \( 2 \leq i \leq n \), which indicates \( x^* \) is a consensus state. □

**Proof of Corollary 3** If \( r_1 \geq 1 \), then Corollary 2 implies that the system reaches consensus a.s.

If \( r_1 < 1 \), then equation (3) implies
\[
\mathbb{P} \left( x(0) \in \left[ 0, \frac{1-r_1}{3} \right], \bigcap_{i=2}^n \left\{ x_i(0) \in \left[ \frac{2+r_1}{3}, 1 \right] \right\} \right) \geq \rho_{\min} \left( \frac{1-r_1}{3} \right)^n.
\]
Also, if \( x_1(0) \in [0, \frac{1-c_1}{4}] \) and the event \( \bigcap_{i=2}^{n} \{ x_i(0) \in [\frac{2+i-1}{4}, 1] \} \) takes place, then \( |x_1(0) - x_i(0)| = \frac{1+2c_1}{4} > r_1 \) for \( 2 \leq i \leq n \). In turn, this implies that the system cannot reach consensus because the agent 1 can never interact with the agents \( 2, \ldots, n \).

\[ \square \]

### 4 Convergence rate of the heterogeneous DW model

The convergence rate has been studied for a homogeneous Deffuant-Weisbuch model in which every agent randomly chooses an objective for opinion update at each time step (Zhang and Chen, 2015). However, the system in (Zhang and Chen, 2015) is quite different from our protocol (1)-(2), so we need develop a different method to analyze the convergence rate.

Let \( m \) be a positive integer, and \( \{ W(k) \in [0,1]^{m \times m} \}_{k \geq 0} \) be a sequence of doubly stochastic matrices with independent and identical distribution. Suppose there exist constants \( \lambda, \delta, \varepsilon \in (0, 1) \) and positive integer \( L \) such that

\[
W_{ij}(k) \geq \lambda \quad \text{a.s. for all } k \geq 0, 1 \leq i \leq m,
\]

and

\[
\mathbb{P}\left( \min_{0 \leq s \leq k \leq L-1} \sum_{i=0}^{L-1} W_s(k) \geq \delta \right) \geq \varepsilon,
\]

where \( W_s(k) := \sum_{i,j=1}^{m} [W_{ij}(k) + W_{ji}(k)] \). With the above conditions on \( \{ W(k) \} \) we have the following lemma.

**Lemma 11** Let \( z(0) \in \mathbb{R}^m \) be a column vector and consider the dynamics \( z(k+1) = W(k)z(k) \) for \( k \geq 0 \). Then for all \( k \geq 0 \), we have

\[
\mathbb{E}[V(z(k+1))] \leq \mathbb{E}[V(z(k))]
\]

and

\[
\mathbb{E}[V(z(kL))] \leq V(z(0)) \left( 1 - \frac{\varepsilon \delta (1-\delta)^2}{m (m-1)^2} \right)^k,
\]

where \( V(z) := \frac{1}{m} \sum_{i=1}^{m} (z_i - \frac{z_1 + \cdots + z_m}{m})^2 \).

**Proof.** The inequality (15) follows immediately from Theorem 5 in (Touri and Nedić, 2011), while (16) follows immediately from the proof of Theorem 6 in (Touri and Nedić, 2011).

Given two sequences of positive numbers \( \{g_1(s)\}_{s=0}^{\infty} \) and \( \{g_2(s)\}_{s=0}^{\infty} \), we say \( g_1(s) = O(g_2(s)) \) if there exist a constants \( c > 0 \) such that \( g_1(s) \leq cg_2(s) \) for all \( s \geq 0 \). We show that the mean-square convergence rate of the heterogeneous DW model is exponential.

**Theorem 12 (Exponential convergence rate of heterogeneous DW model)** Consider the heterogeneous DW model (1)-(2) with positive confidence bounds. For any initial state \( x(0) \in [0,1]^n \), there exists a constant \( c > 0 \) such that

\[
\mathbb{E} \sum_{i=1}^{n} (x_i(t) - x_i^*)^2 = O \left( e^{-ct} \right),
\]

where \( x^* \) is the a.s. convergent limit of \( x(s) \) given by Theorem 1.

**Proof.** Similar to the proof of Theorem 1 we define \( S_{rn} := \{ x \in [0,1]^n : D(x) \leq r_n \} \), and \( \tau_{rn} \) to be the time when \( S_{rn} \) is firstly reached under the DW protocol (1)-(2). Here we recall that \( r_n \) is the smallest confidence bound among all agents. By Lemma 10, \( S_{rn} \) is finite-time reachable from \([0,1]^n \) under the DW-control system. So, under the DW protocol (1)-(2), by Lemma 5 there exist constants \( T > 0 \) and \( a \in (0, 1) \) such that

\[
\mathbb{P}(\tau_{rn} \geq t) \leq a^{\lfloor t/T \rfloor}, \quad \forall t \geq 1.
\]

Label the MC clusters as \( C_1, \ldots, C_K \) at time \( \tau_{rn} \). By Lemmas 6 and 7, for \( t \geq \tau_{rn} \) we have \( D(x(t)) \leq r_n \) and the MC cluster \( C_1, \ldots, C_K \) remains unchanged, i.e., if node \( i \) belongs to cluster \( C_j \) at time \( \tau_{rn} \) then it will always belong to \( C_j \) for \( t > \tau_{rn} \).

Next we consider the case when \( t \geq \tau_{rn} \). Let \( P(t) \in [0,1]^{n \times n} \) be a matrix defined by

\[
(P_{ij}(t), P_{jj}(t), P_{ij}(t), P_{ji}(t)) := \begin{cases} 
(1, 1, 1, 1), & \text{if } (i,j) \text{ is the opinion update pair at time } t \text{ and belongs a same MC cluster}, \\
(1, 1, 0, 0), & \text{otherwise}.
\end{cases}
\]

for all \( i < j \). By the protocol (1)-(2), and the facts that \( D(x(t)) \leq r_n \) and two agents in different MC clusters have no interaction, we can get \( P(t) \) is a doubly stochastic matrix and \( x(t+1) = P(t)x(t) \). Also, because \( C_1, \ldots, C_K \) remain unchanged, there exists a permutation matrix \( Q \in \{0,1\}^{n \times n} \) such that

\[
Q^T P(t) Q = \text{diag}(W^1(t), W^2(t), \ldots, W^K(t)) := W(t),
\]

where
where $W^k(t)$ is a $|C_k| \times |C_k|$ matrix corresponding to the MC cluster $C_k$. Let $z(t) := Q^T x(t)$ and $z^* := Q^T x^*$, we have

$$z(t + 1) = Q^T x(t + 1) = Q^T P(t)x(t) = Q^T P(t)Qz(t) = W(t)z(t) = W(t)\cdots W(\tau_{r_n})z(\tau_{r_n}) = \text{diag}(W(1)\cdots W(1)(\tau_{r_n}), \ldots, W^K(t)\cdots W^K(\tau_{r_n}))z(\tau_{r_n}).$$  

(20)

Set $I^0 := 0$ and $I^k := |C_1| + |C_2| + \cdots + |C_k|$ for $1 \leq k \leq K$. Let

$$z^k(t) := (z_{I^k-1+1}(t), z_{I^k-1+2}(t), \ldots, z_{I^k}(t))^T.$$  

(21)

By (20), for $1 \leq k \leq K$ we have

$$z^k(t + 1) = W^k(t) \cdots W^k(\tau_{r_n})z^k(\tau_{r_n}).$$  

(22)

We now check the properties of the sequence $\{W^k(t)\}_{t=\tau_{r_n}}^{r_n}$. First, by (18) and (19) we know $W^k(t)$ is a doubly stochastic matrix whose diagonal entries are not less than $1/2$. Also, because the opinion update agents are selected uniformly and independently at each time, we have $\{W^k(t)\}_{t=\tau_{r_n}}^{r_n}$ is an i.i.d. sequence. Now, equation (18) implies

$$\mathbb{P}\left(W^k_{ij}(t) = W^k_{ji}(t) = \frac{1}{2}\right) = \frac{1}{|N|} = \frac{2}{n(n-1)},$$

for all $1 \leq i < j \leq |C_k|$. Thus, there exists a positive constant $\varepsilon_n$ depending only on $n$, such that

$$\mathbb{P}\left(\min_{\emptyset \cup S \subseteq \{1, \ldots, |C_k|\}} \sum_{t=\tau_{r_n}}^{\tau_{r_n}+n-1} W^k_S(t) \geq \frac{1}{3} \right) \geq \varepsilon_n, \text{ for all } 1 \leq k \leq K.$$

By Lemma 11, for any $1 \leq k \leq K$, we obtain

$$\mathbb{E}\left[V(z^k(t))| t \geq 2\tau_{r_n}, C_1, \ldots, C_K\right] \leq \mathbb{E}\left[V\left(z^k(\tau_{r_n} + \left\lfloor \frac{t - \tau_{r_n}}{n} \right\rfloor n)\right)| t \geq 2\tau_{r_n}, C_1, \ldots, C_K\right] \leq \mathbb{E}\left[V\left(z^k(\tau_{r_n})\right)\left(1 - \frac{2\varepsilon_n}{27|C_k||C_k| - 1}\right)^{\left\lfloor \frac{t - \tau_{r_n}}{n}\right\rfloor}\right) | t \geq 2\tau_{r_n}, C_1, \ldots, C_K\right] \leq \left(1 - \frac{2\varepsilon_n}{27n(n-1)^2}\right)^{\left\lfloor \frac{t}{n}\right\rfloor}.$$  

(23)

By the convexity of MC clusters (Lemma 7), for any $I^{k-1} < j \leq I^k$ and $t \geq \tau_{r_n}$, the convergent limit $z_j^*$ satisfies

$$\min_{I^{k-1} < j \leq I^k} z_j(t) \leq z_j^* \leq \max_{I^{k-1} < j \leq I^k} z_j(t) \text{ a.s.}$$  

(24)

Let $\bar{z}^k(t) := \left(\frac{1}{|C_k|}, \ldots, \frac{1}{|C_k|}\right)z^k(t)$ be the average value of $z^k(t)$. Because

$$(z_i(t) - z_i^*)^2 = (z_i(t) - \bar{z}^k(t) + \bar{z}^k(t) - z_i^*)^2 \leq 2(z_i(t) - \bar{z}^k(t))^2 + 2(\bar{z}^k(t) - z_i^*)^2,$$

and by (24) we can get a.s.

$$\sum_{i=I^{k-1}+1}^{I^k} (\bar{z}^k(t) - z_i^*)^2 \leq \sum_{i=I^{k-1}+1}^{I^k} \max_{I^{k-1} < j \leq I^k} ((\bar{z}^k(t) - z_j(t))^2 \leq |C_k| \max_{I^{k-1} < j \leq I^k} ((\bar{z}^k(t) - z_j(t))^2 \leq |C_k| \sum_{j=I^{k-1}+1}^{I^k} (\bar{z}^k(t) - z_j(t))^2,$$

then, by (23) we have

$$\mathbb{E}\left[\sum_{i=I^{k-1}+1}^{I^k} (z_i(t) - z_i^*)^2| t \geq 2\tau_{r_n}, C_1, \ldots, C_K\right] \leq (2 + 2|C_k|)\mathbb{E}\left[\sum_{i=I^{k-1}+1}^{I^k} (z_i(t) - \bar{z}^k(t))^2| t \geq 2\tau_{r_n}, C_1, \ldots, C_K\right] = |C_k|(2 + 2|C_k|)\mathbb{E}[V(\bar{z}^k(t))| t \geq 2\tau_{r_n}, C_1, \ldots, C_K] \leq |C_k|(2 + 2|C_k|)\left(1 - \frac{2\varepsilon_n}{27n(n-1)^2}\right)^{\left\lfloor \frac{t}{n}\right\rfloor}.$$  

(25)

In turn, inequality (25) yields

$$\mathbb{E}\left[\sum_{i=1}^{n} (z_i(t) - z_i^*)^2| t \geq 2\tau_{r_n}, C_1, C_2, \ldots, C_K\right] \leq \mathbb{E}\left[\sum_{k=1}^{K} \sum_{i=I^{k-1}+1}^{I^k} (z_i(t) - z_i^*)^2| t \geq 2\tau_{r_n}, C_1, \ldots, C_K\right] \leq \sum_{k=1}^{K} |C_k|(2 + 2|C_k|)\left(1 - \frac{2\varepsilon_n}{27n(n-1)^2}\right)^{\left\lfloor \frac{t}{n}\right\rfloor} \leq 2n(n+1)\left(1 - \frac{2\varepsilon_n}{27n(n-1)^2}\right)^{\left\lfloor \frac{t}{n}\right\rfloor}.$$  

(26)
Also, because $Q^Ty$ is a permutation of $y$, we can get
\[
\sum_{1 \leq i \leq n} (z_i(t) - z_i^\ast)^2 = \sum_{1 \leq i \leq n} (x_i(t) - x_i^\ast)^2. \tag{27}
\]

By (27), (26) and the total probability formula we have
\[
E\left[\sum_{i=1}^{n} (x_i(t) - x_i^\ast)^2 \right] \mathbb{I}(t \geq 2\tau_r) = E\left[\sum_{i=1}^{n} (z_i(t) - z_i^\ast)^2 \right] \mathbb{I}(t \geq 2\tau_r)
\leq 2n(n+1) \left( 1 - \frac{2\tau}{27n(n-1)^2} \right)^{\frac{1}{2}}. \tag{28}
\]

Finally, using (17) and (28) and the total probability formula again, we have
\[
E\left[\sum_{i=1}^{n} (x_i(t) - x_i^\ast)^2 \right] = P\left(\tau_r > \frac{t}{2}\right) E\left[\sum_{i=1}^{n} (x_i(t) - x_i^\ast)^2 | \tau_r > \frac{t}{2}\right]
+ P\left(\tau_r \leq \frac{t}{2}\right) E\left[\sum_{i=1}^{n} (x_i(t) - x_i^\ast)^2 | \tau_r \leq \frac{t}{2}\right]
\leq a^{\lceil t/2T \rceil} n + 2n(n+1) \left( 1 - \frac{2\tau}{27n(n-1)^2} \right)^{\frac{1}{2}},
\]
which implies our result. \qed

5 Conclusions

Bounded confidence (BC) models of opinion dynamics adopt a mechanism whereby individuals are not willing to accept other opinions if these other opinions are beyond a certain confidence bound. These models have attracted significant mathematical and sociological attention in recent years. One well-known BC model is the Deffuant-Weisbuch (DW) model, in which a pair of agents is selected randomly at each time step, and each agent in the pair updates its opinion if the other agent’s opinion in the pair is within its confidence bound. Because the inter-agent topology of the DW model is coupled with the agents’ states, the heterogeneous DW model is hard to analyze. This paper proves the convergence of a heterogeneous DW model and shows the mean-square error is bounded by a negative exponential function of time.

As directions for future research, it remains to prove the convergence of the original heterogeneous DW model with an arbitrary weighting factor $\mu \in (0, 1) \setminus \{1/2\}$. For this original heterogeneous DW model, a more ingenious control design is required to establish that the DW-control system converges to a set with invariant topology in finite time.

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A. The proof of Lemma 8

The proof of this lemma is identical for all cases $t = 0, 1, 2, \ldots$. To simplify the exposition we consider only the case when $t = 0$.

Assume the agents $j$ and $k$ have the minimal and maximal opinions among $C_i(x(0))$ at time 0 respectively, i.e.,

\[
x_j(0) = \min_{m \in C_i(x(0))} x_m(0), \quad x_k(0) = \max_{m \in C_i(x(0))} x_m(0).
\]

Also, assume that the agent $l$ has the maximal confidence bound $r_{\text{max}}^l$ in $C_i(x(0))$.

We first consider the case when $x_l(0) \geq \frac{x_j(0) + x_k(0)}{2}$. From (12) we have

\[
x_l(0) \geq x_j(0) + r_{\text{min}}^l/2. \quad (A.1)
\]

Let

\[
\mathcal{A}(s) := \{m \in C_i(x(0)) : x_m(s) < x_j(0) + r_{\text{min}}^l/4\}.
\]

We aim to control the agent pairs for opinion update such that $\mathcal{A}(s)$ becomes empty in finite time. The control strategy can be divided into the following steps:

Step 1: Control the agent pairs for opinion update until one of the following two events happens:

(E1) The agents in $C_i(x(0))$ split into different MC clusters;

(E2) $|\mathcal{A}(s)| = |\mathcal{A}(0)| - 1$, where $| \cdot |$ denote the cardinality of a set.

Let $i_0'$ be the agent in $C_i(x(0))$ which has the smallest opinion within the confidence bound of agent $l$ at time 0, i.e.,

\[
i_0' = \arg \min_{m \in C_i(x(0))} \{x_m(0) : |x_l(0) - x_m(0)| \leq r_l\}.
\]

We continue our discussion by considering the following two cases:

Case I: $i_0' \in \mathcal{A}(0)$. Choose $(i_0', l)$ as the agent pair for opinion update at times $0, 1, \ldots, \lfloor \log_2 \left\lfloor \frac{x_l(0) - x_{i_0'}(0)}{r_{i_0'}^l} \right\rfloor \rfloor := T$. We can get $T \leq \left\lfloor \log_2 \left\lfloor nr_l/r_{i_0'}^l \right\rfloor \right\rfloor$ is uniformly bounded, and

\[
T - 1 < \log_2 \left[ \frac{x_l(0) - x_{i_0'}(0)}{r_{i_0'}^l} \right] \leq T
\]

\[
\iff 2^{T-1} < \left[ \frac{x_l(0) - x_{i_0'}(0)}{r_{i_0'}^l} \right] \leq 2^T
\]

\[
\iff 2^{T-1} r_{i_0'}^l < x_l(0) - x_{i_0'}(0) \leq 2^T r_{i_0'}^l. \quad (A.2)
\]

If $T = 0$, by (A.2) we have $x_l(0) - x_{i_0'}(0) \leq r_{i_0'}^l$, then by the protocol (1)-(2) and (A.1) we get

\[
x_{i_0'}(1) = x_l(1) = \frac{x_l(0) + x_{i_0'}(0)}{2} \geq \frac{x_l(0) + x_j(0)}{2} \geq x_j(0) + r_{\text{min}}^l/4,
\]

which implies $|\mathcal{A}(1)| = |\mathcal{A}(0)| - 1$. 

\[
\]
If $T \geq 1$, by (A.2) and using the protocol (1)-(2) repeatedly we can computer that

\[
\begin{cases}
x_{i_0}(s) = x_{i_0}(0) \\
x_i(s) = x_{i_0}(0) + \frac{1}{2} (x_l(0) - x_{i_0}(0))
\end{cases}, \quad s = 1, \ldots, T,
\]
and

\[
x_{i_0}(T + 1) = x_l(T + 1) = x_{i_0}(0) + \frac{1}{2} (x_l(0) - x_{i_0}(0)) > x_{i_0}(0) + r_{i_0}/4 \geq x_l(0) + r_{i_0}/4 \geq x_{i_0}(0) + r_{i_0}/4,
\]

which implies $|A(T + 1)| = |A(0)| - 1$.

**Case II**: $i_0 \not\in A(0)$. Choose $(i_0', l)$ as the agent pair for opinion update at times 0, 1, ..., $|\log_2 \left[ \frac{x_l(0) - x_{i_0}(0)}{r_{i_0}} \right]| := T_1$. Similar to (A.3) we get

\[
x_l(T_1 + 1) = x_{i_0}(T_1 + 1) = x_{i_0}(0) + \frac{1}{2T_1+1} (x_l(0) - x_{i_0}(0)) \geq x_{i_0}(0).
\]

Let $L_i(s)$ denote the set of the agents in $C_i(x(0))$ whose opinions at time $s$ are less than $x_l(s)$, i.e.,

\[
L_i(s) := \{ m \in C_i(x(0)) : x_m(s) < x_l(s) \}.
\]

By (A.4) and with the fact that all agents except $l$ and $i_0'$ keep their opinions invariant during the time $[0, T_1 + 1]$, we have

\[
|L_i(T_1 + 1)| \leq |L_i(0)| - 1. \tag{A.5}
\]

Let $i_1'$ be the agent in $C_i(x(0))$ which has the smallest opinion within the confidence bound of agent $l$ at time $T_1 + 1$, i.e.,

\[
i_1' = \arg \min_{m \in C_i(x(0))} \{ x_m(T_1 + 1) : x_l(T_1 + 1) - x_m(T_1 + 1) \leq r_l \}.
\]

If $x_{i_1'}(T_1 + 1) = x_l(T_1 + 1)$, the agents in $C_i(x(0))$ split into different MC clusters; otherwise, choose $(i_1', l)$ as the agent pair for opinion update at times $T_1 + 1, T_1 + 2, \ldots, T_1 + 1 + |\log_2 \left[ \frac{x_l(T_1 + 1) - x_{i_1'}(T_1 + 1)}{r_{i_1'}} \right]| := T_2$.

If $i_1' \not\in A(0)$, similar to case I we get $|A(T_2 + 1)| = |A(0)| - 1$.

If $i_1' \not\in A(0)$, similar to (A.5) we have

\[
|L_i(T_2 + 1)| \leq |L_i(T_1 + 1)| - 1. \tag{A.6}
\]

Repeat the above process until the agents in $C_i(x(0))$ split into different MC clusters, or $|A(T_p + 1)| = |A(0)| - 1$ for some positive integer $p$. By (A.5)-(A.6) we get that

\[
p \leq |L_i(0)| - |A(0)| + 1 \leq |C_i(x(0))| - |A(0)|.
\]

From this inequality and the definition of $T_1, T_2, \ldots$ we have

\[
T_p + 1 \leq (|C_i(x(0))| - |A(0)|) (1 + |\log_2 [r_{\max}^i / r_{\min}^i]|).
\]

Let $t_1$ be the minimal time such that E1 or E2 happens. By the discussion in Cases I and II we have

\[
t_1 \leq (|C_i(x(0))| - |A(0)|) (1 + |\log_2 [r_{\max}^i / r_{\min}^i]|). \tag{A.7}
\]

If E1 happens at time $t_1$, our result i) holds; otherwise, we need to carry out next step.

**Step 2:** For $s \geq t_1$ we control the agent $l$ moves toward the right until E1 or one of the following two events happens:

(E3) $x_l(s) \geq x_j(s) + r_{\min}^j/2$;

(E4) $\max_{m \in C_i(x(0))} x_m(s) \leq x_k(s) - r_{\min}^j/4$;

For $s \geq t_1$, let $i_s'$ be the agent in $C_i(x(0))$ which has the biggest opinion within the confidence bound of agent $l$ at time $s$, i.e.,

\[
i_s' = \arg \max_{m \in C_i(x(0))} \{ x_m(s) : |x_l(s) - x_m(s)| \leq r_l \}.
\]

Choose $(i_s', l)$ as the agent pair for opinion update, until at least one of the events E1, E3, and E4 happens. Let $t_2$ be the minimal time that E1, E3, or E4 happens. For $s \in [t_1, t_2)$, since E1 and E4 do not happen at time $s$,

\[
x_l(s + 1) = \frac{x_l(s) + x_{i_s'}(s)}{2} > x_l(s).
\]

By the similar method as Step 1, each agent in $C_i(x(0)) \setminus \{A(t_1) \cup \{l\} \}$ can be chosen at most $1 + |\log_2 [r_{\max}^i / r_{\min}^i]|$ times for opinion update during $[t_1, t_2)$. Then,

\[
t_2 - t_1 \leq (|C_i(x(0))| - |A(t_1)| - 1) (1 + |\log_2 [r_{\max}^i / r_{\min}^i]|) \leq (|C_i(x(0))| - |A(0)|) (1 + |\log_2 [r_{\max}^i / r_{\min}^i]|). \tag{A.8}
\]
If E4 happens, Lemma 7 implies

\[ \max_{M, m \in C_i(x(0))} [x_M(t_2) - x_m(t_2)] \]
\[ \leq \max_{M \in C_i(x(0))} x_M(t_2) - x_j(0) \leq x_k(0) - x_j(0) - r^{i}_{\min}/4, \]

which indicates our result ii) holds; if E1 happens, our result i) holds at time \( t_2 \); otherwise, we need to carry out next Step.

... ...

Step 2m + 1: For \( s \geq t_{2m} \), we use the similar control method as Step 1. Let \( t_{2m+1} \) be the minimal time such that E1 happens or \( |A(t_{2m+1})| = |A(t_{2m-1})| - 1 \). Similar to (A.7) we have

\[ t_{2m+1} - t_{2m} \]
\[ \leq (|C_i(x(0))| - |A(t_{2m-1})|) (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]) \]
\[ = (|C_i(x(0))| - |A(0)| + m) (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]). \]

Step 2m + 2: For \( s \geq t_{2m+1} \), we use the similar control method as Step 2. Let \( t_{2m+2} \) be the minimal time such that E1, E3, or E4 happens. Similar to (A.8) we have

\[ t_{2m+2} - t_{2m+1} \]
\[ \leq (|C_i(x(0))| - |A(t_{2m+1})| - 1) (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]) \]
\[ = (|C_i(x(0))| - |A(0)| + m) (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]). \]

The above process will end at Step 2|A(0)| − 1 because \( \overline{A}(t_{2|A(0)|-1}) = \emptyset \). By Lemma 7 and the definition of \( \overline{A}(s) \) we have

\[ \max_{M, m \in C_i(x(0))} [x_M(t_{2|A(0)|-1}) - x_m(t_{2|A(0)|-1})] \]
\[ \leq x_k(0) - \min_{m \in C_i(x(0))} x_m(t_{2|A(0)|-1}) \]
\[ \leq x_k(0) - x_j(0) - r^{i}_{\min}/4, \]

which indicates our result ii) holds when \( t^* = t_{2|A(0)|-1} \). Set \( t_0 := 0 \). By (A.10) and (A.11) we have

\[ t_{2|A(0)|-1} \]
\[ = \sum_{m=0}^{\overline{A}(0)-2} (t_{2m+2} - t_{2m}) + t_{2|A(0)|-1} - t_{2|A(0)|-2} \]
\[ \leq \left( \sum_{m=0}^{\overline{A}(0)-2} 2 (|C_i(x(0))| - |A(0)| + m) \right) \]
\[ + |C_i(x(0))| - 1 \right) (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]) \]
\[ = \left( (2|A(0)| - 1)|C_i(x(0))| + (-|A(0)| - 1)|A(0)| + 1 \right) \cdot (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]) \]
\[ \leq (|C_i(x(0))| - 1)^2 (1 + \log_2 [r^{i}_{\max}/r^{i}_{\min}]), \]

where the last inequality uses the fact that \(|A(0)| \leq |C_i(x(0))| - 1 |.

For the case when \( x_i(0) < \frac{x_i(0) + x_i(0)}{2} \), we can set

\[ \overline{A}(s) := \{ m \in C_i(x(0)) : x_m(s) > x_k(0) - r^{i}_{\min}/4 \} \]

and use the similar method as the case \( x_i(0) \geq \frac{x_i(0) + x_i(0)}{2} \) to control \( \overline{A}(s) \) becomes empty. \( \square \)