Deep Teams: Decentralized Decision Making with Finite and Infinite Number of Agents

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Abstract—Inspired by the concepts of deep learning in artificial intelligence and fairness in behavioural economics, we introduce deep teams in this paper. In such systems, agents are partitioned into a few sub-populations so that the dynamics and cost of agents in each sub-population is invariant to the indexing of agents. The goal of agents is to minimize a common cost function in such a manner that the agents in each sub-population are not discriminated or privileged by the way they are indexed. Two non-classical information structures are studied. In the first one, each agent observes its local state as well as the empirical distribution of the states of agents in each sub-population, called deep state, whereas in the second one, the deep states of a subset (possibly all) of sub-populations are not observed. Novel dynamic programs are developed to identify globally optimal and sub-optimal solutions for the first and second information structures, respectively. The computational complexity of finding the optimal solution in both space and time is polynomial (rather than exponential) with respect to the number of agents in each sub-population and is linear (rather than exponential) with respect to the control horizon. This complexity is further reduced in time by introducing a forward equation, that we call deep Chapman-Kolmogorov equation, described by multiple convolutional layers of Binomial probability distributions. Two different prices are defined for computation and communication, and it is shown that under mild assumptions they converge to zero as the quantization level and the number of agents tend to infinity. In addition, the main results are extended to the infinite-horizon discounted model and arbitrarily asymmetric cost function. Finally, a service-management example with 200 users is presented.

Index Terms—Team theory, deep structure, controlled Markov chains, large-scale systems, non-classical information.

I. INTRODUCTION

Team theory studies cooperative decision making and arises in numerous applications such as smart grids, swarm robotics, transportation networks, social networks, and emergent behaviours; to name only a few. Such applications often consist of a group of interconnected decision makers, modelled as Markov decision processes, that wish to accomplish a common task in the presence of limited computation and communication resources. Historically, team theory can be traced back to the seminal work of Radner [1] and Marschack and Radner [2] on static teams as well as Witsenhausen [3] and Ho [4] on dynamic teams. For a comprehensive literature overview, the interested reader is referred to [5] and references therein.

When centralized information structure is feasible (i.e., joint state is known to all decision makers), optimal solution is given by the celebrated dynamic programming principle [6]. In this case, the number of computational resources (in space and time) to identify the optimal solution increases exponentially with the number of decision makers, in general; a phenomenon known as the “curse of dimensionality”. For example, a centralized system consisting of 100 decision makers with binary states requires the computational resources of order $2^{100} \approx 10^{30}$, which is intractable. On the other hand, centralized information structure may not even be feasible due to limited communication resources, specially when the number of decision makers is large. In such cases, it is desired to have some form of decentralized information structure. However, a lower communication requirement comes at the cost of a harder optimization problem to solve because the decision makers in a decentralized structure may have different perception about the system [7]. Due to the high complexity of the decentralized control problems, an explicit optimal solution may be intractable for systems with more than two or three decision makers [8], [9]. As a result, after nearly 60 years of research, there is still a big gap between theory and practice.

In this paper, inspired by the concepts of deep learning in artificial intelligence [10] and fairness in behavioural economics [11], we introduce deep structured team (deep team for short) as an attempt to establish a bridge between team theory and its application. It is to be noted that deep learning and deep team share some resemblances; for example, they both involve multi-stage stochastic optimization problems over a number of i.i.d. random variables and more importantly, the optimal solution of a deep team resembles a deep neural network. However, deep team is conceptually different from deep learning because it is not a data-driven approach and its depth refers to the number of decision makers. In simple words, deep team (control) is an endeavour to provide a systematic framework to make the classical single-agent control algorithms deep with respect to the number of decision makers, to be applicable for large-scale control systems, by exploiting the notion of invariance principle. To this end, we borrow the notion of partially exchangeable systems, analogous to the notion of invariance of coordinates in physics. A multi-agent system is said to be partially exchangeable if the population of

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1In this paper, the computational resources in space and time respectively refer to the size of memory and the number of iterations that an algorithm needs to perform a task. It is to be noted that the computational complexity in time is different from the computational complexity in control horizon.
agents can be partitioned into a few sub-populations in such a way that the manner in which the agents are indexed in each sub-population does not matter. It is well known that Markov decision processes with partially exchangeable agents are equivalent to Markov decision processes coupled through the empirical distribution of states and actions of agents in each sub-population. Subsequently, without loss of generality, we restrict attention to the latter formulation in this paper. We consider two non-classical information structures: deep-state sharing and partial deep-state sharing, where deep state refers to the empirical distribution of the states of agents in each sub-population. Under deep-state sharing structure, each decision maker has access to its local state as well as the deep states of all sub-populations whereas under partial deep-state sharing, it has access to its local state and the deep states of only a subset (which can be empty) of sub-populations. Under these decentralized information structures, we study microscopic and macroscopic behaviors of the decision makers and identify globally optimal and sub-optimal fair strategies. This manuscript is a complete version of authors’ recent work in networked control systems [12]–[14] and is the generalization of the concept of mean-field teams introduced in [15].

In the context of game theory, mean-field games study the non-cooperative behavior of a large number of exchangeable players [16]–[18]. The solution concept is Nash equilibrium and the proof revolves around the fact that the effect of a single player on others is negligible when the number of players is sufficiently large such that it can be considered as infinite. This reduces the infinite-population game to a two-player game between a generic player and an infinite population, and since the deep state of the infinite population (i.e., mean-field) has deterministic dynamics, a Nash solution may be obtained in terms of the solution of two coupled forward-backward nonlinear partial differential equations (i.e., Fokker-Planck-Kolmogorov and Hamilton-Jacobi-Bellman equations). The existence of such a solution is established by imposing various Lipschitz-type fixed-point conditions (that generally hold for small time horizons) or monotonicity-type assumptions (that are often difficult to verify). For the special case of common cost function, mean-field games may be used to identify an approximate person-by-person (Nash-bargaining) solution. It is to be noted that person-by-person optimality is weaker than global optimality, in general. For example, consider a simple static two-agent control problem with the cost function \( \max(x_1, x_2) \), \( x_1, x_2 \in \mathbb{R}_{\geq 0} \). This problem has uncountably many person-by-person optimal solutions (e.g. \( x_1 = x_2 = x, \forall x \in \mathbb{R}_{\geq 0} \)) but admits a unique globally optimal solution (i.e. \( x_1 = x_2 = 0 \)).

Another relevant field of research that is closely related to mean-field games is mean-field-type control problems [19]–[21]. When the cost functions and control laws of players in this type of problem are identical, the infinite-population social optimization problem reduces to a single-agent stochastic optimal control problem, whose dynamics and cost depend on the distribution of the state (also known as McKean-Vlasov type). A solution to this problem is not known in general, as it is time-inconsistent [20], [22]. However, it is possible to find a person-by-person time-consistent solution by formulating the mean-field type control in terms of two coupled forward-backward equations, similar to those in mean-field games, with the distinction that the mean-field is replaced by the probability density function of the state of the generic player [22], [23].

Despite the differences between cooperative mean-field game and mean-field-type control, they both use the simplification afforded by the infinite-population model to propose strategies that are sub-optimal, fair, person-by-person, closed-loop in the local state and open-loop in the population state. To ensure that the infinite-population solution is a reasonable approximate solution for the finite population, the standard approach is to assume that the solution is a continuous function. Finding a numerical solution for the general case of nonlinear state dynamics, to the best of our knowledge, is still an open problem (specially when neither fixed-point nor monotonicity-type conditions are imposed on the model).

We take a different route in this paper and consider deep team problems that have finite state and action spaces with non-convex cost functions and possibly multiple person-by-person solutions. More precisely, we study an arbitrary number of agents (not necessarily large) wherein the effect of each agent on other agents is negligible. Note that this is a more challenging problem compared to the typical mean-field problems where the effect of a single agent on others is normally neglected. We propose a team-theoretic approach to identify the exact globally optimal fair strategy under deep-state sharing information structure (that is closed-loop in the population state) and a globally sub-optimal fair strategy under partial deep-state sharing (that includes open-loop strategies). The proposed dynamic programs are time-consistent for any number of agents and their minimization is carried out over the space of local control laws. In addition, we show that the performance gap between the optimal and sub-optimal solutions converges to zero as the number of agents goes to infinity, without imposing any continuity assumption on the solution. Furthermore, we propose a quantization technique for both finite- and infinite-population models to numerically compute the corresponding sub-optimal solutions without restricting to any fixed-point or monotonicity condition. It is to be noted that since deep team approach is neither based on the negligible effect of individual agents nor dependent on the future trajectory of the population state, its extension to major-minor and common-noise problems [24], [25] introduces no additional complication.

The rest of the paper is organized as follows. In Section II the problem is formulated and the main contributions of this paper are outlined. To identify an optimal solution under deep-state sharing and a sub-optimal one under partial deep-state sharing, two novel dynamic programs are proposed in Sections III and IV respectively. To alleviate the computational complexity of these dynamic programs, their quantized counterparts are presented in Section V. In Sections VI and VII the main results are analogously extended to the infinite-horizon discounted and arbitrarily asymmetric cost functions. The special case of major-minor deep teams is presented in Section VIII followed by a numerical example in Section IX. Finally, the paper is concluded in Section X.
II. Problem Formulation

Throughout the paper, \( \mathbb{N} \), \( \mathbb{R}_{\geq 0} \), and \( \mathbb{R}_{> 0} \) denote the set of natural numbers, non-negative real numbers, and positive real numbers, respectively. \( \mathbb{N}_k \) denotes the finite set of integers \( \{1, \ldots, k\} \). In addition, \( \mathbb{P}(\cdot) \) is the probability of a random variable; \( E[\cdot] \) is the expectation of an event; \( 1(\cdot) \) is the indicator function of a set; \( \|\cdot\| \) denotes the infinity norm of a vector; \( \xi(\cdot) \) is the empirical distribution of a set; \( \cdot \mid \cdot \) denotes the absolute value of a real number or the cardinality of a set, and \( \delta \) denotes a Dirac measure with a unit mass concentrated at one point. The short-hand notation \( x_{1:t} \) is used to denote vector \( (x_1, \ldots, x_t) \). For any pair of integers \( i, j \leq n \in \mathbb{N} \), \( \sigma_{i,j}(x) \) denotes the permuted version of vector \( x = (x_1, \ldots, x_n) \) such that the \( i \)-th element of \( \sigma_{i,j}(x) \) is \( x_j \). For any \( n \in \mathbb{N} \), \( \text{binompdf}(\cdot, n, p) \) denotes the binomial probability distribution of \( n \) trials with success probability \( p \in [0, 1] \). Furthermore, for any \( n, r \in \mathbb{N} \) and any finite set \( \mathcal{X} \), different spaces are defined as described in Table I. Let \( x_{1:n} \) denote a vector of \( n \in \mathbb{N} \) samples from set \( \mathcal{X} := \{a_1, \ldots, a_{|\mathcal{X}|}\} \), where \( x_t \in \mathcal{X}, i \in \mathbb{N}_n \). The empirical distribution function \( \xi : \prod_{t=1}^T \mathcal{X} \to \mathcal{E}_n(\mathcal{X}) \) is defined as a real-valued vector of size \( |\mathcal{X}| \) such that \( \xi(x_{1:n})(a_j) = \frac{1}{n} \sum_{i=1}^n 1(x_i = a_j), j \in \mathbb{N}_{|\mathcal{X}|} \).

**A. Model**

In practice, there are various applications in which the population of agents can be partitioned into a few sub-populations in such a way that the order of indexing of agents in each sub-population is not important. For example, in a smart grid, demands in a region may be classified as residential and commercial, and the numbering of the demands in each class does not affect the aggregate consumed energy. Similarly, in swarm robotics, robots may be categorized into a few groups with similar characteristics such as leaders and followers where neither the dynamics of motion nor the status of the swarm depends on the way the robots are indexed in each group. Such applications may be modelled as follows.

Consider a discrete-time control system consisting of a finite population of agents, where agents are partitioned into \( K \in \mathbb{N} \) disjoint sub-populations. Denote by \( \mathcal{K} \) the set of sub-populations, by \( \mathcal{X}^k \) the agents of sub-population \( k \in \mathcal{K} \), and by \( \mathcal{N} \) the entire population of agents; note that \( \mathcal{N} = \bigcup_{k \in \mathcal{K}} \mathcal{X}^k \).

Given the control horizon \( T \in \mathbb{N} \), let the state, action, and the noise of agent \( i \in \mathcal{N}^k \) of sub-population \( k \in \mathcal{K} \) at time \( t \in \mathbb{N}_T \) be denoted by \( x_i^t \in \mathcal{X}^k, u_i^t \in \mathcal{U}^k \), and \( w_i^t \in \mathcal{W} \), respectively. The spaces \( \mathcal{X}^k, \mathcal{U}^k \) and \( \mathcal{W} \) of every sub-population \( k \in \mathcal{K} \) are finite and do not depend on the size of sub-population \( k \in \mathcal{K} \), i.e., \( |\mathcal{X}^k| \). For the entire population, the joint state, joint action, and joint noise are analogously denoted by \( x_t = (x_i^t)_{i \in \mathcal{N}} \in \mathcal{X}, u_t = (u_i^t)_{i \in \mathcal{N}} \in \mathcal{U} \), and \( w_t = (w_i^t)_{i \in \mathcal{N}} \in \mathcal{W} \) at time \( t \in \mathbb{N}_T \).

Let \( \mathcal{D}_t \) denote the empirical distribution of the states and actions of sub-population \( k \in \mathcal{K} \) at time \( t \in \mathbb{N}_T \), i.e.,

\[
\mathcal{D}_t = \xi \left( (x_i^t, u_i^t)_{i \in \mathcal{N}^k} \right) \in \mathcal{E}_{|\mathcal{N}^k|}(\mathcal{X}^k \times \mathcal{U}^k).
\]

Similarly, let \( d_t^k \) denote the empirical distribution of the states of sub-population \( k \in \mathcal{K} \) at time \( t \in \mathbb{N}_T \), i.e.,

\[
d_t^k = \xi \left( (x_i^t)_{i \in \mathcal{N}^k} \right) \in \mathcal{E}_{|\mathcal{N}^k|}(\mathcal{X}^k).
\]

Define \( \mathcal{D}_t := (\mathcal{D}_t^1, \ldots, \mathcal{D}_t^K) \) and \( d_t := (d_t^1, \ldots, d_t^K) \). For ease of reference, the empirical distribution of states is called deep state in the sequel. Denote by \( \mathcal{E} \) the space of realizations \( \xi(\cdot) \), i.e., \( \xi(\cdot) \in \mathcal{E} := \prod_{k \in \mathcal{K}} \mathcal{E}_{|\mathcal{N}^k|}(\mathcal{X}^k), t \in \mathbb{N}_T \).

For any \( k \in \mathcal{K} \), the initial state of agent \( i \) of sub-population \( k \) is denoted by \( x_i^1 \in \mathcal{X}^k \), and at time \( t \in \mathbb{N}_T \) its state evolves as follows:

\[
x_{i+1}^t = f_t^k(x_i^t, u_i^t, \mathcal{D}_t^k, w_i^t),
\]

where the primitive random variables \( \{x_1, w_1, \ldots, w_T\} \) are defined on a common probability space and are mutually independent. The dynamics (3) can be equivalently represented in terms of the transition probability matrix such that:

\[
\mathbb{P}(x_{i+1}^t \mid x_i^t, u_i^t, \mathcal{D}_t^k) := \\
\sum_{w_i^t \in \mathcal{W}} 1(x_i^t = f_t^k(x_i^t, u_i^t, \mathcal{D}_t^k, w_i^t)) \mathbb{P}(w_i^t = w_i^t).
\]

In the sequel, we occasionally interchange the two equivalent representations (3) and (4), for ease of display. Let the per-step cost function be denoted by \( c_t(\mathcal{D}_t) \in \mathbb{R}_{\geq 0} \) at time \( t \). Note that the social cost function is a special case of the cost function considered above. To see this, let \( c_t^k(x_i^t, u_i^t, \mathcal{D}_t) \) denote the per-step cost of agent \( i \in \mathcal{N}^k \) of sub-population \( k \in \mathcal{K} \). Then, \( c_t(\mathcal{D}_t) := \sum_{k \in \mathcal{K}} \frac{1}{|\mathcal{N}_k|} \sum_{i \in \mathcal{N}_k} c_t^k(x_i^t, u_i^t, \mathcal{D}_t) \).

In this paper, we make the following three assumptions on the primitive random variables.

**Assumption 1.** For any sub-population \( k \in \mathcal{K} \), the primitive random variables \( (w_i^{t})_{i \in \mathcal{N}^k}, t \in \mathbb{N}_T \), are exchangeable.

**Assumption 2.** The primitive random variables \( (w_i^{t})_{i \in \mathcal{N}} \), \( t \in \mathbb{N}_T \), are mutually independent across agents. In addition, for each sub-population \( k \in \mathcal{K} \), random variables \( (w_i^{t})_{i \in \mathcal{N}^k} \) are identically distributed with probability mass functions \( \mathbb{P}_{|\mathcal{W}^k}. \)

**Remark 1.** Note that Assumptions 1 and 2 do not impose any restriction on the probability distribution of initial states.

**Assumption 3.** The primitive random variables \( (x_i^{t})_{i \in \mathcal{N}} \) are mutually independent across agents, and for each sub-population \( k \in \mathcal{K} \), random variables \( (x_i^{t})_{i \in \mathcal{N}^k} \) are identically distributed with probability mass functions \( \mathbb{P}_{X_i^k}. \)
Denote by $I_i^t \subseteq \{x_{1:t}, u_{1:t-1}\}$ the information set of agent $i \in \mathcal{N}$ at time $t \in \mathbb{N}_T$, i.e.,
\[ u_i^t = g_i^t(I_i^t), \quad (5) \]
where function $g_i^t$ is called the control law of agent $i$ at time $t \in \mathbb{N}_T$. The set of control laws $g := \{g_i^t\}_{i \in \mathcal{N}}^T$ is defined as the strategy of the system.

**Definition 1 (Partially exchangeable (fair) strategies).** A strategy $g$ is said to be partially exchangeable if for any pair of agents $(i, j) \in \mathcal{N}^2$ of any sub-population $k \in \mathcal{K}$ at any time $t \in \mathbb{N}_T$, $\sigma_{i,j}(g_i^t(I_i^t))_{s \in \mathcal{N}} = (g_{i,j}(\sigma_{i,j}I_i^t))_{s \in \mathcal{N}}$, i.e., exchanging agents $i$ and $j$ has no effect on the strategy.

**B. Admissible strategy**

The set of admissible strategies is defined as the set of partially exchangeable (fair) strategies, where no agent is privileged or discriminated by the way it is indexed in a sub-population. The restriction to fair strategies may be viewed from two angles: a constraint that must be satisfied or an assumption that is limiting. In this paper, we focus on the former case and present applications in which such a restriction is desirable and practical. It is substantiated by numerous experimental data in behavioural economics that humans make their decisions based not only on rationality but also on whether or not the decisions are fair [11], [25]. For example, an unfair resource allocation in a grid can lead to protest and anarchy among users, due to the discrimination against some users, even if such a strategy yields the lowest possible social cost function. This means that a Pareto-optimal unfair strategy is not necessarily a sustainable equilibrium. To learn more about the importance of fairness, the interested reader is referred to a pivotal counterexample in behavioural economics called ultimatum game [27] for a seemingly irrational behaviour. In addition, fair strategies are important in control theory because they provide robustness, where for instance it is often desirable to distribute the total load of a network in a fairly manner among servers in order to increase the life-time of the servers as well as the robustness of the network (in case a server fails). Furthermore, since finding a Pareto-optimal solution is computationally expensive, as described in the introduction section, optimal fair strategies are of particular interest in practice, as a simpler (yet more tractable) alternative.

Two decentralized information structures are investigated in this paper. The first one is referred to as the deep-state sharing (DSS), where for any $i \in \mathcal{N}$, agent $i$ at time $t \in \mathbb{N}_T$ observes its local state $x_i^t$ and the history of the deep states of a subset of sub-populations $\mathcal{S} \subseteq \mathcal{K}$, i.e.,
\[ I_i^t = \{x_i^t, d_{1:t}\}. \quad (DSS) \]
where $d_{1:t}$ is the control law of agent $i$ at time $t \in \mathbb{N}_T$.

In practice, there are different ways to share the deep state among agents. For example, in cellular communications, the deep state may be collected and transmitted to all agents by the base station, while in swarm robotics the deep state may be computed in a distributed manner using consensus-based algorithms. The second information structure is more general than DSS, and is called the partial deep-state sharing (PDSS), where for any $i \in \mathcal{N}$, agent $i$ at time $t \in \mathbb{N}_T$ observes its local state $x_i^t$ and the history of the deep states of a subset of sub-populations $\mathcal{S} \subseteq \mathcal{K}$, i.e.,
\[ J_N(g) = \mathbb{E}[\sum_{t=1}^T c_t(D_t)], \quad (6) \]
where the subscript $N$ denotes the dependence of the cost function to the number of agents, and the expectation is taken with respect to the probability measures induced by $g$.

**Problem 1.** For deep-state sharing information structure, find the optimal fair strategy $g^\ast$ such that for every fair strategy $g$, $J_N(g^\ast) \leq J_N(g)$.

Let $n$ be the size of the smallest sub-population whose deep state is not observed, i.e., $n := \min_{k \in \mathcal{S}_c} |\mathcal{N}^k|$.

**Problem 2.** For partial deep-state sharing information structure, find an $\varepsilon(n)$-optimal fair strategy $g^\ast$ such that $J_N(g^\ast) \leq J_N(g^\ast) + \varepsilon(n)$, where $\varepsilon(n) \in \mathbb{R}_{>0}$ and $\lim_{n \to \infty} \varepsilon(n) = 0$.

**Remark 2.** For an exchangeable system, Pareto-optimal and optimal fair strategies are not necessarily the same; however, for a linear quadratic model, they are identical under DSS [13].

Since both DSS and PDSS are non-classical information structures, the computational complexity of finding a solution to Problems 1 and 2 is NEXP, in general [7]. The main contributions of the present paper are spelled out below.

1) We develop a dynamic programming decomposition that identifies an optimal solution for Problem 1 (Theorem 2) and show that the computational complexity of finding the solution in both space and time is polynomial (rather than exponential) with respect to the number of agents in each sub-population and is linear (rather than exponential) with respect to the control horizon (Corollary 4).

2) Although polynomial complexity is much more efficient than the exponential one, it could still be high when the size of population is medium or large. For medium populations, we propose Theorems 3 and 4 to alleviate the computational complexity (in time and space). In particular, we discover the structure of the transition probability matrix of the deep state in Theorem 3 that we refer to as the Deep Chapman-Kolmogorov (DCK) equation, which has a structure analogous to that in convolutional neural networks (Subsection III-A).

3) For a large population, we consider Problem 2 because when some sub-populations are large, it may not be feasible to collect and share their deep states among agents. We show that such information sharing has a negligible effect on the optimal performance of the system. In particular, we develop a dynamic programming decomposition for Problem 2 that provides an $\varepsilon(n)$-optimal strategy, where $\varepsilon(n)$ converges to zero at the
rate $1/\sqrt{n}$ (Theorem 4). This dynamic program does not depend on the size of sub-populations $S^k$.  

4) For the numerical computation of the dynamic program of Theorem 4 it is required, in general, to solve a non-smooth non-convex optimization problem. We propose a quantized solution and prove that the quantization error converges to zero at a rate inversely proportional to the quantization level (Theorem 9). An immediate consequence of this result is that if the quantization level is greater than $\sqrt{n}$, then the quantized solution will converge to the optimal solution at the same rate that the unquantized solution does (Corollary 12).  

5) We extend our main results to infinite-horizon discounted cost (Theorems 7 and 8). It is shown that DSS strategy is stationary with respect to the observed deep states whereas PDSS strategy is not (Remark 12).  

We define a handful of short-hand notations to ease the exposition of the results and proofs in the sequel. Let $I := \prod_{k \in K} I_k(X^k)$, and given any subset $\mathcal{R} \subseteq \mathcal{K}$ and any scalar $r \in \mathbb{N}$, define the following spaces:

$$
\begin{align*}
\mathcal{P}_{\mathcal{R}} := \prod_{k \in \mathcal{R}} \mathcal{E}_{X^k}(X^k) \times \prod_{k \notin \mathcal{R}} \mathcal{P}(X^k), \\
\mathcal{Q}_{\mathcal{R}} := \prod_{k \in \mathcal{R}} \mathcal{E}_{X^k}(X^k) \times \prod_{k \notin \mathcal{R}} \mathcal{E}_{X^k}(X^k), \\
\mathcal{I}_{\mathcal{R}} := \prod_{k \in \mathcal{R}} I_k(X^k) \times \prod_{k \notin \mathcal{R}} \mathcal{E}_{X^k}(X^k),
\end{align*}
$$

where $\mathcal{E} \subset \mathcal{P} \subset \mathcal{I}$ and $\mathcal{Q}_{\mathcal{R}} \subset \mathcal{I}_{\mathcal{R}} \subset \mathcal{I}$. Denote by $Q : \mathcal{I}_{\mathcal{R}} \to \mathcal{Q}_{\mathcal{R}}$ the quantizer function that maps every point $z \in \mathcal{I}_{\mathcal{R}}$ to its nearest point $q \in \mathcal{Q}_{\mathcal{R}}$, i.e., $Q(z) = \arg \min_{q \in \mathcal{Q}_{\mathcal{R}}} \|z - q\|_2$, which implies that $\|z - Q(z)\|_2 \leq \frac{1}{2}, \forall z \in \mathcal{I}_{\mathcal{R}}$.  

Let $w^k : \mathcal{X}^k \to \mathcal{U}_k$ be the mapping from the local state space to the local action space of sub-population $k \in \mathcal{K}$, and $\gamma := \{\gamma_1, \ldots, \gamma_k\} \in \mathcal{G}$, where $\mathcal{G}$ denotes the space of all mappings $\gamma$. Let $w^k_0$ and $w_k$ denote the empirical distribution of the local noises of sub-population $k \in \mathcal{K}$ and the entire system, respectively, at time $t \in \mathbb{N}_T$, i.e.,

$$
\begin{align*}
w^k_0 := \xi((w^k_i)_{i \in \mathbb{N}^k}) \in \mathcal{E}_{X^k}(W^k), \\
w_k := (w^k_1, \ldots, w^k_k) \in \mathcal{W} := \prod_{k \in \mathcal{K}} \mathcal{E}_{X^k}(W^k).
\end{align*}
$$

Then, define the following functions at every time $t \in \mathbb{N}_T$ and for any $z = (z^1, \ldots, z^k) \in I$, $\gamma \in \mathcal{G}$, $w_0, w_k \in \mathcal{W}$, and $k \in \mathcal{K}$:  

1) For any $x \in X^k$ and $u \in \mathcal{U}_k$, $\phi^k(z, \gamma)(x, u) \in [0, 1]$ is defined as $\phi^k(x) := \xi(u \in \gamma_k(x))$, where its augmented form is:

$$
\phi(z, \gamma) := (\phi^1(z, \gamma), \ldots, \phi^k(z, \gamma)).
$$

2) For any $y \in X^k$, $\bar{f}^k_t(z, \gamma, w^k_k)(y)$ is defined as:

$$
\sum_{w_k \in \mathcal{W}_k} z^k(x) \xi(\bar{f}^k_t(x, \gamma_k(x), \phi(z, \gamma), w) = y) w^k_k(w),
$$

where its augmented form is:

$$
\bar{f}_t(z, \gamma, w_k) := (\bar{f}^1_t(z, \gamma, w^1_k), \ldots, \bar{f}^k_t(z, \gamma, w^k_k)).
$$

3) For any $y \in \mathcal{X}^k$, $\hat{f}^k_t(z, \gamma)(y)$ is defined as:

$$
\sum_{x \in \mathcal{X}^k} z^k(x) \pi^k(y | x, \gamma_k(x), \phi(z, \gamma)).
$$

4) For any per-step cost $c_t$, $c_t(z, \gamma) \in \mathbb{R}_{\geq 0}$ is defined as:

$$
c_t(\phi(z, \gamma)).
$$

III. MAIN RESULTS FOR PROBLEM 1

To find a solution to Problem 1, Witsenhausen’s standard form could be used to develop a dynamic programming decomposition [28]. However, the resultant dynamic program would be intractable, and since the size of its information state increases with time, it could not be extended to infinite horizon. Alternatively, one can use common information approach [29] to construct a dynamic program in terms of the conditional probability of the joint state, given common information, i.e., $P(x_t | d_{1:t})$. In such a case, it is shown in [30] by forward induction that if the initial states as well as noise processes are exchangeable, $P(x_t | d_{1:t})$ is also exchangeable under fair strategies, and hence can be represented by $d_t$. For the general case of non-exchangeable initial states, however, the result of [30] does not hold as $P(x_t | d_{1:t})$ is not necessarily exchangeable. Thus, we present a direct approach to obtain a dynamic programming decomposition in terms of $d_t$, regardless of the probability distribution of initial states. A salient feature of this method is to identify the structure of the transition probability matrix of deep state $d_t$, which proves to be useful not only for the numerical computations but also for the convergence analysis of Problem 2.

Lemma 1. When the attention is focused on fair strategies, the control laws of the DSS and PDSS strategies are identical in each sub-population, i.e., $g^k_t = g^k_t =: g^k_t, i, j \in N^k, k \in \mathcal{K}$.  

Proof. The proof follows directly from equation (5), the definitions of DSS and PDSS, and Definition 1.  

According to equation (5) and Lemma 1 for any $k \in \mathcal{K}$ and $i \in N^k$, the control law of agent $i$ of sub-population $k$ at time $t \in \mathbb{N}_T$ under DSS is $g^k_t : \mathcal{X}_t \times \mathcal{X}^k \to \mathcal{U}_k$, i.e.,

$$
g^k_t(x_t, x^k_t). (14)
$$

Using the information decomposition proposed in [29], we split $g^k_t$ into two parts for any $k \in \mathcal{K}$ and $t \in \mathbb{N}_T$. More precisely, define function $\psi^k_t : \mathcal{E}^k \to \mathcal{G}$ as follows:

$$
\psi^k_t(d_{1:t}) := g^k_t(\cdot, d_{1:t}). (15)
$$

Then, from (14) and (15), one has:

$$
u^k_i = \gamma^k_i(x^k_t). (16)
$$

where $\gamma^k_i : \mathcal{X}_t \to \mathcal{U}_k$ is defined by $\psi^k_t$, i.e.,

$$
\gamma^k_i := \psi^k_t(d_{1:t}). (17)
$$

According to (16) and (17), the action of agent $i$ of sub-population $k$ at time $t$ is determined by two functions $\psi^k_t$ and $\gamma^k_i$. In the sequel, we refer to the functions $\gamma_i := \{\gamma^k_i\}_{k \in \mathcal{K}}$ as the local laws and to $\psi_i := \{\psi^k_i\}_{k \in \mathcal{K}}$ as the global laws. From (1), (2) and (16), it follows that for any $k \in \mathcal{K}, t \in \mathbb{N}_T, x \in \mathcal{X}^k$ and $u \in \mathcal{U}_k$:

$$
\mathcal{D}^k_t(x, u) = \frac{1}{\mathcal{N}^k} \sum_{i \in \mathcal{N}^k} \mathbb{1}(x^k_i = x) \mathbb{1}(\gamma^k_i(x^k_i) = u). (18)
$$

Subsequently, it results from (6) and (18) that:

$$
\mathcal{D}_t = \phi(d_t, \gamma_t). (19)
$$
We show that the deep state evolves in a Markovian manner with respect to the local laws, which means that the history of the deep state except the most recent one can be ignored.

**Theorem 1.** Let Assumption 7 hold. For any \( k \in \mathcal{K} \) and \( t \in \{1,2\} \), the dynamics of the deep state of sub-population \( k \) at time \( t \) can be expressed by:

\[
d_{t+1}^{k} = \mathcal{F}_{t}^{k}(d_{t}, \gamma_{t}, w_{t}^{k}),
\]

where \( w_{t}^{k} \) and \( \mathcal{F}_{t}^{k} \) are defined by (8) and (19), respectively. In addition, the deep state of the entire population evolves as:

\[
d_{t+1} = \mathcal{F}_{t}(d_{t}, \gamma_{t}, w_{t}),
\]

where \( w_{t} \) and \( \mathcal{F}_{t} \) are given by (8) and (11), respectively.

**Proof.** According to (2), (3), (16) and (19), for any \( \gamma \in \mathcal{X}^{k} \), one has:

\[
d_{t+1}^{k}(y) = \frac{1}{|N|^{k}} \sum_{i \in N^{k}} \mathbb{I}(f_{t}^{k}(x_{i}^{k}, \gamma_{t}^{k}(x_{i}^{k}), d_{t}, w_{t}^{k}) = y).\]

(22)

For any \( j \in N_{\gamma}^{k} \), let \( \sigma_{j}(i) := i + j \) if \( i + j \leq |N^{k}| \); otherwise, \( \sigma_{j}(i) := i + j - |N^{k}| \). From Assumption 1 it follows that:

\[
d_{t+1}^{k}(y)^{\sigma_{j}(i)} = \frac{1}{|N|^{k}} \sum_{i \in N^{k}} \mathbb{I}(f_{t}^{k}(x_{i}^{k}, \gamma_{t}^{k}(x_{i}^{k}), d_{t}, w_{t}^{k}) = y).
\]

(23)

It is now possible to compute \(|N^{k}| d_{t+1}^{k}(y)\) by summing up (23) over all \( j \in N_{\gamma}^{k} \), which is almost surely equal to:

\[
\frac{1}{|N|^{k}} \sum_{j \in N_{\gamma}^{k}} \sum_{i \in N^{k}} \mathbb{I}(f_{t}^{k}(x_{i}^{k}, \gamma_{t}^{k}(x_{i}^{k}), d_{t}, w_{t}^{k}) = y) = \frac{1}{|N|^{k}} \sum_{w \in W^{k}} \sum_{x \in \mathcal{X}^{k}} \sum_{i \in N^{k}} \sum_{j \in N^{k}} \mathbb{I}(x_{i}^{k} = x) \mathbb{I}(w_{t}^{k} = w) \times \mathbb{I}(f_{t}^{k}(x, \gamma_{t}^{k}(x), d_{t}, w) = y) = \sum_{w \in W^{k}} \sum_{x \in \mathcal{X}^{k}} \sum_{i \in N^{k}} \mathbb{I}(x_{i}^{k} = x) \mathbb{I}(f_{t}^{k}(x, \gamma_{t}^{k}(x), d_{t}, w) = y) \times \frac{1}{|N|^{k}} \sum_{j \in N_{\gamma}^{k}} \sum_{i \in N^{k}} \mathbb{I}(w_{t}^{k} = w) = \frac{1}{|N|^{k}} \sum_{j \in N_{\gamma}^{k}} (a) \sum_{w \in W^{k}} \sum_{x \in \mathcal{X}^{k}} \sum_{i \in N^{k}} \mathbb{I}(x_{i}^{k} = x) \mathbb{I}(f_{t}^{k}(x, \gamma_{t}^{k}(x), d_{t}, w) = y) w_{t}^{k}(w).
\]

(b) \(|N^{k}| \sum_{w \in W^{k}} \sum_{x \in \mathcal{X}^{k}} d_{t}^{k}(x) \mathbb{I}(f_{t}^{k}(x, \gamma_{t}^{k}(x), d_{t}, w) = y) w_{t}^{k}(w),\]

where (a) and (b) follow from (8) and (2), respectively. Equation (20) follows now from equations (10) and (18). In addition, equation (21) follows from (8) and (11).

**Remark 3.** It is to be noted that the result of Theorem 1 holds irrespective of the global laws \( \psi_{1:t} \).

From Theorem 1 the transition probability matrix of the deep state of entire population can be presented as follows:

\[
P(d_{t+1} | d_{t}, \gamma_{t}) = \sum_{w \in W} \mathbb{I}(d_{t+1} = \mathcal{F}(d_{t}, \gamma_{t}, w)) \mathbb{P}(w_{t} = w).\]

(24)

A preliminary version of the next theorem was presented in [60].

**Theorem 2.** Let Assumption 7 hold. Then, strategy 26 is optimal for Problem 7.

**Proof.** The proof follows the fact that the deep state \( d_{t} \) is an information state for Problem 7. In particular, according to Theorem 1 \( d_{1:T} \) is a controlled Markov process with control actions \( \gamma_{1:T-1} \). Furthermore, according to equations (13) and (18), the per-step cost is a function of \( d_{t} \) and \( \gamma_{t} \). Thus, the dynamic programming decomposition follows from standard results in Markov decision theory [6].

The cardinality of \( \mathcal{E}_{|N^{k}|^{k}}(\mathcal{X}^{k}) \) is bounded polynomially in the number of the agents of sub-population \( k \in \mathcal{K} \),

\[
|\mathcal{E}_{|N^{k}|^{k}}(\mathcal{X}^{k})| \leq (|N^{k}|^{k} + 1)^{|X^{k}|^{k}}.
\]

Consequently, the space of dynamic program (25) increases at most polynomially with respect to the number of agents in each sub-population \( k \in \mathcal{K} \), and is independent of time.

**Remark 5.** The decomposition proposed in equation (25) is a non-standard dynamic program because the minimization is over local law \( \gamma_{t} \in \mathcal{G} \) rather than joint control action \( u_{t} \in \mathcal{U} \).

The optimal strategy of Theorem 2 can be implemented in a distributed manner since every agent can independently compute the dynamic program (25) and observe the deep state. According to (25), for any \( k \in \mathcal{K}, \gamma_{t} \in \mathcal{N}^{k} \) and \( t \in \{1,2\} \), the action (role) of agent \( i \) of sub-population \( k \) at time \( t \) is determined by three factors: (a) global law \( \psi_{t} \) that depends on the agents’ dynamics, per-step cost, and underlying probability distributions; (b) deep state \( d_{t} \) that provides the statistical
information on the states of agents, and (c) local state $x^i_t$ that is private information for agent $i$ and unknown to others.

In general, randomization can improve the performance of a fair strategy \[31\]. On the other hand, any parametrized randomized strategy can be formulated as a deterministic one wherein the randomization is embedded into the transition probability and cost function, and the action space is replaced by the parameter space. Therefore, finite parametrization does not lead to an enhancement of the formulation.

**Corollary 1.** The computational complexity of solving the dynamic program \[25\] in both space and time is polynomial with respect to the number of agents in each sub-population $|X^k|$, $k \in K$, and is linear with respect to the control horizon $T \in \mathbb{N}$.

**Proof.** The proof is presented in Appendix A. \[\square\]

**Remark 6.** It is to be noted that the fairness of admissible strategies is essential for establishing Corollary 1. In addition, although the computational complexity of dynamic program \[25\] is polynomial with respect to the number of agents, it is exponential with respect to the cardinality of the local state space, which means that the proposed strategy is rather tractable for small state spaces. When the model has a special structure, however, it is possible to relax the restriction to fair strategies as well as to small state spaces. Such special structures include, for example, linear quadratic model wherein the state space is an infinite set and agents are not necessarily exchangeable, i.e., optimal strategy is not necessarily fair \[32\].

For any realization $d_t \in E$, local law $\gamma_t \in G$, sub-population $k \in K$, and pair of states $x, y \in X^k$ at time $t \in \mathbb{Z}_T$, define $B^k_t(y, x, d_t, \gamma_t) \in \mathcal{P}(\{0, 1, \ldots, |X^k|d^k_t(x)\})$ as:

$$B^k_t(y, x, d_t, \gamma_t) := \text{binopdf}(d^k_t(x) = 0)\delta_0 + \text{binopdf}(d^k_t(x) > 0),$$

(28)

where $P^k$ and $\phi$ are given by \[4\] and \[18\], respectively. Furthermore, define $\hat{B}^k_t(y, d_t, \gamma_t) \in \mathcal{P}(\{0, 1, \ldots, |X^k|\})$ as the convolution of the vector-valued functions $B^k_t(y, x, d_t, \gamma_t)$ over all states $x \in X^k = \{x_1, \ldots, x_{|X^k|}\}$, i.e.,

$$\hat{B}^k_t(y, d_t, \gamma_t) := B^k_t(y, x_1, d_t, \gamma_t) \ast \ldots \ast B^k_t(y, x_{|X^k|}, d_t, \gamma_t).$$

(29)

**Theorem 3.** Let Assumption 2 hold. The transition probability matrix of the deep state can be computed efficiently in time as follows: for any $d_t \in E$, $\gamma_t \in G$, $k \in K$, $y \in X^k$ and $j \in \mathbb{N}_{|X^k|+1}$ at time $t \in \mathbb{N}_T$:

$$\mathbb{P}(d^k_{t+1}(y) = j - 1 \mid X^k) \mid d_t, \gamma_t) = \hat{B}^k_t(y, d_t, \gamma_t)(j).$$

(30)

**Proof.** From equations \[18\] and \[22\], it results that for any $d_t \in E$, $\gamma_t \in G$, $k \in K$ and $y \in X^k$ at time $t \in \mathbb{N}_T$,

$$|X^k|d^k_{t+1}(y) = \sum_{x \in X^k} \sum_{i \in X^k} \mathbb{1}(x^i_t = x) \times \mathbb{1}(f^k_t(x, \gamma_t^k(x), \phi(d_t, \gamma_t), w^i_t) = y).$$

(31)

For any sub-population $k \in K$ and any pair of states $x, y \in X^k$ at time $t \in \mathbb{N}_T$, define $\hat{B}^k_t(y, x, d_t, \gamma_t) \in \mathcal{P}(\{0, 1\})$ as the probability distribution function of the binary random variable $F^k_t(x, \gamma_t^k(x), \phi(d_t, \gamma_t), w^i_t) = y$. From \[3\], \[4\], and \[15\],

$$P(\mathbb{1}(F^k_t(x, \gamma_t^k(x), \phi(d_t, \gamma_t), w^i_t) = y) = 1 \mid d_t, \gamma_t) = \hat{B}^k_t(y, x, \gamma_t^k(x), \phi(d_t, \gamma_t)).$$

(32)

Thus, it follows from \[32\] that:

$$\hat{B}^k_t(y, x, d_t, \gamma_t) = (1 - \hat{B}^k_t(y, x, d_t, \gamma_t))\delta_0 + \hat{B}^k(y, x, \gamma_t(x), \phi(d_t, \gamma_t)).$$

(33)

Let $B^k_t(y, x, d_t, \gamma_t) \in \mathcal{P}(\{0, 1, \ldots, |X^k|d^k_t(x)\})$ denote the probability distribution function of the sum of $|X^k|d^k_t(x)$ binary random variables associated with state $x \in \mathcal{N}$, i.e.,

$$B^k_t(y, x, d_t, \gamma_t) = \mathbb{P}(\sum_{i \in X^k} \mathbb{1}(f^k_t(x, \gamma_t^k(x), \phi(d_t, \gamma_t), w_i^t) = y)).$$

Therefore, if $d^k_{t+1}(y) = 0$, it means $B^k_t(y, x, d_t, \gamma_t) = \delta_0$; otherwise, from equation \[33\], Assumption 2 and the fact that the probability distribution function of the sum of independent random variables is equal to the convolution of their probability distribution functions, we have

$$B^k_t(y, x, d_t, \gamma_t) = \hat{B}^k_t(y, x, d_t, \gamma_t) \ast \ldots \ast \hat{B}^k_t(y, x, d_t, \gamma_t).$$

A. Deep Chapman-Kolmogorov (DCK) equation

According to Theorem 3 the structure of the transition probability matrix of the deep state involves multiple convolution functions in \[29\] and several Binomial probability distributions in \[28\], triggered by some activation functions. This structure resembles a deep neural network with convolutional layers and Gaussian filters, which obeys a similar invariance feature called spatial invariance. Inspired by this resemblance, we refer to \[30\] as Deep Chapman-Kolmogorov (DCK) equation, which can be traced back to the seminal work of Russian mathematicians Bogolyubov and Krylov in 30s-40s; see \[33\].

In what follows, we present some of the special cases of DCK equation. Consider a homogeneous population with no control action, where $K$ is a singleton set. The following holds:

- For the single-agent case, the local state and deep state have equivalent information, meaning that the single-agent DCK can be represented equivalently in the form of the (classical) Chapman-Kolmogorov equation.
- For the infinite-population case with coupled dynamics, deep state $d_t$ appears in the transition probability matrix as a diffusion term; hence, the macroscopic DCK equation may be viewed as the discrete-time discrete-space...
counterpart of the Fokker-Planck-Kolmogorov and master equations in the continuous-time models. For the special case of decoupled dynamics, the macroscopic DCK equation simplifies to the classical Chapman-Kolmogorov equation, i.e., 
\[ d_{t+1}(y) = \sum_{x \in X} d_t(x) P(y \mid x) = \sum_{x \in X} P(y \mid x) P(x \mid z), \]
where \( d_{t+1} = \delta_z, y, z \in X \).
Consequently, the limit of deep state \( d_{t+1}(\cdot) \), as the population grows, which is in fact, mean-field, can be interpreted as the infinite-sample distribution of a generic random variable \( x_{t+1} \) with the transition probability 
\[ P(x_{t+1} = y \mid x_{t-1} = z). \]
For more details on the connection with deep neural networks, the interested reader is referred to [32], [34].

### IV. MAIN RESULTS FOR PROBLEM 2

So far, we have assumed that the deep states of all sub-populations are shared among agents. However, for large sub-populations it might be difficult to collect and share their deep states among agents. In such cases, we are interested in Problem 2 where the deep states of large sub-populations are not observed. We propose a sub-optimal strategy, where the optimality gap converges to zero as the size of sub-populations grows to infinity. We impose Assumptions 2 and 3 on the primitive random variables and the following two assumptions on the model.

**Assumption 4.** The deep states of sub-populations \( S^c \) do not affect the dynamics of the agents of sub-populations \( S \).

**Remark 7.** Notice that the deep states of sub-populations \( S \) can affect the dynamics of the agents of sub-populations \( S^c \). In addition, Assumption 4 automatically holds for \( S = \emptyset \) (i.e., when the information structure is completely decentralized).

An immediate implication of Assumption 4 is that the following relationship holds for \( f^k_t \) in (10) for any \( k \in S \), \( t \in \mathbb{N}_T \), \( z \in I, \gamma \in \mathcal{G} \) and \( \mathbf{w}_t \in \mathcal{W}^k \):
\[
\tilde{f}^k_t((x^k)_{k \in S}, \gamma, \mathbf{w}_t) = \tilde{f}^k_t(z, \gamma, \mathbf{w}_t^k).
\]

**Assumption 5.** For every \( x, y \in X^k \), \( u \in U^k \), \( k \in K \) and \( \mathbf{3}_1, \mathbf{3}_2 \in P(X^k \times U^k) \), there exist positive real constants \( H_{t,1}^k, H_{t,2}^k \) (independent of \( |X^k|, k \in K \)), such that
\[
|\tilde{S}_t(x, y, \mathbf{3}_1) - \tilde{S}_t(y, x, \mathbf{3}_2)| \leq H_{t,1}^k|\mathbf{3}_1 - \mathbf{3}_2|,
\]
\[
|\tilde{S}_t(x, y, \mathbf{3}_1) - \tilde{S}_t(x, y, \mathbf{3}_2)| \leq H_{t,2}^k|\mathbf{3}_1 - \mathbf{3}_2|.
\]

**Remark 8.** Any polynomial function of \( \mathbf{3} \) is a Lipschitz function in \( \mathbf{3} \) because \( \mathbf{3} \) is confined to a bounded interval. It is worth highlighting that any continuous function can be approximated as closely as desirable by polynomial functions according to the Weierstrass Approximation Theorem.

To distinguish from the optimal solution under DSS, we use superscript \( p \) to denote the parameters associated with the sub-optimal solution under PDSS. Denote by \( g^p \) the sub-optimal strategy and by \( \tilde{d}^p \in \mathcal{E} \) the deep state of the system under strategy \( g^p \) at \( t \in \mathbb{N}_T \). Similar to Section III, we split the strategy \( g^p \) into two parts as follows. For every sub-population \( k \in K \) and any \((d_{t+1}^p)_{k \in S}, t \in \mathbb{N}_T \), define:
\[
\begin{align*}
\psi^p_{t+1}((d_{t+1}^p)_{k \in S}, (d_{t+1}^p)_{k \in S}) := g^p_t(\cdot, (d_{t+1}^p)_{k \in S}),
\end{align*}
\]
where \( g^p_t(x_i) \). Therefore, the control action of agent \( i \in N^k \) of sub-population \( k \in K \) under PDSS is given by:
\[
u_t^i = g_t^p(x_i, (d_{t+1}^p)_{k \in S}) = \psi^p_t((d_{t+1}^p)_{k \in S})(x_i) = \gamma^i_t^p(x_i).
\]
Notice that the local laws in (36) are different from those in (16) because the information structures are different. However, the finite space \( \mathcal{G} \) is identical for both DSS and PDSS, i.e., \( \gamma^p_t := \{\gamma^p_1, \ldots, \gamma^p_K\} \in \mathcal{G} \). Since Theorem 1 holds regardless of the global laws \( \psi_{t+1} \) (Remark 3), it also holds for \( \psi_{t+1}^1, \ldots, \psi_{t+1}^K \) as a restriction function of \( \psi_{t+1} \). Hence, the dynamics of the deep state of sub-population \( k \in K \) at time \( t \) in \( \mathbb{N}_T \) can be written as follows, according to Theorem 1:
\[
\tilde{d}^p_{t+1} \overset{a.s.}{=} f_t^p(d_t^p, \gamma^p_t, \mathbf{w}_t^p).
\]
In the augmented form, we have
\[
\tilde{d}^p_{t+1} \overset{a.s.}{=} f_t^p(d_t^p, \gamma^p_t, \mathbf{w}_t^p).
\]

The remainder of this section is organized as follows. We first introduce a stochastic process that is controlled by local laws \( \gamma^p_t \) and is implementable under PDSS. Then, we propose a dynamic program based on this process and show that its solution is sub-optimal, by comparing its performance with the optimal solution of Theorem 2.

#### A. A mixed state

Let \( \tilde{m}_t^k \in \mathcal{P}(X^k) \) denote the mean-field of sub-population \( k \in S^c \) at time \( t \in \mathbb{N}_T \), where \( |X^k| \) is set to infinity. Define a mixed state \( \mathbf{p}_t \in \mathcal{P}_a \) consisting of the deep states of sub-populations \( S \) and the mean-fields of sub-populations \( S^c \), i.e.,
\[
p^k_t := \tilde{d}^p_{t+1} = d^k_t, \quad k \in S,
\]
\[
p^k_t := \tilde{m}^k_t = \mathbb{P}(X^k_t), \quad k \in S^c,
\]
and for \( t > 1 \), \( \mathbf{p}_t \) evolves under \( \gamma^p_t \in \mathcal{G} \) as follows:
\[
p^k_{t+1} := \begin{cases} f_t^p(\mathbf{p}_t, \gamma^p_t, \mathbf{w}_t^p), & k \in S, \\ f_t^k(\mathbf{p}_t, \gamma^p_t), & k \in S^c, \end{cases}
\]
where \( f_t^p \) and \( f_t^k \) are given by (11) and (12), respectively. Define by \( \tilde{f}_t^S, t \in \mathbb{N}_T \), the augmented form of (40) as follows:
\[
\tilde{p}_{t+1} = \tilde{f}_t^S(\mathbf{p}_t, \gamma^p_t, \mathbf{w}_t).
\]

**Proposition 1.** Under Assumptions 2 and 3 the stochastic process \( \mathbf{p}_{t+1} \) is adapted to the filtration \((\tilde{d}^p_{t+1})_{k \in S} \in \mathcal{E} \). In particular, \( \mathbf{p}_1 \) is specified by (38), and for any \( t \in \mathbb{N}_T \):
\[
p^k_{t+1} = d^p_{t+1} = d^k_t, \quad k \in S \quad \text{and} \quad p^k_{t+1} = f_t^k(\mathbf{p}_t, \gamma^p_t), \quad k \in S^c.
\]

**Proof.** The proof is presented in Appendix B.
B. Proposed dynamic program for Problem 2

We now propose a dynamic program for Problem 2. Define real-valued functions \( \{V^p_x, V^p_{x+1}\} \) backward in time such that \( V^p_{x+1}(p) = 0 \) for any \( p \in P_s \). Also, for any \( t \in N_T \) and \( p \in P_s \), define
\[
V^p_t(p) = \min_{\gamma \in \mathcal{G}} \left( \ell_t(p, \gamma) + \mathbb{E}[V^p_{t+1}(f^S_t(p, \gamma, w_t))] \right). \tag{42}
\]
The right-hand side of (42) admits at least one minimizer because \( P_s \in P_s \) can be expressed by (\( H_i^{p,k} \)) and the feasible set is finite. As a result, the minimization in (42) is a search over finite alphabets, meaning that it always has a minimizer (solution). With a slight abuse of notation, let \( \psi^p_t(p) := \{\psi_i^{p,k}(p), \ldots, \psi_K^{p,k}(p)\} \) be one of the minimizers of the right-hand side of (42). The proposed control law for agent \( i \in N^k \) of sub-population \( k \in K \) at time \( t \in N_T \) is given by:
\[
u^p_t = g^p_t(k_t, p_t) := \psi^p_t(k_t(p_t)^{x_t}), \quad x^p_t \in X^p, p_t \in P_s. \tag{43}
\]

Remark 9. An important feature of the dynamic program (42) is that it is independent of the size of sub-populations \( S^c \), and so is the strategy \( g^p \). This is due to the fact that \( P_s \) and \( \ell_t \) and \( \bar{f}^S_t \) given respectively by (7), (15), and (39) are all independent of \( |N^k|, k \in S^c \).

Compared to the DSS value function (25) and strategy (26), the source of error in the proposed PDSS value function (42) and strategy (43) is the substitution of the model with dynamics \( f^S_t \) and deep state process \( d^S_t \) by a model with dynamics \( \bar{f}^S_t \) and mixed-state process \( \bar{p}^S_t \), respectively. In what follows, we show that the propagation of such error in the dynamics and value functions are bounded by Lipschitz gains related to those introduced in Assumption 5.

C. Upper bounds on the dynamics and value functions

Lemma 2. Let Assumptions 2 hold. There exist positive constants \( \delta^S_i \), \( k \in K, t \in N_T \) (independent of \( |N^k| \)), such that for any \( z_1, z_2 \in \mathbb{I} \) and \( \gamma \in \mathcal{G} \),
\[
\|\bar{f}^S_t(z_1, \gamma) - \bar{f}^S_t(z_2, \gamma)\| \leq \delta^S_i \|z_1 - z_2\|,
\]
\[
\|\ell_t(z_1, \gamma) - \ell_t(z_2, \gamma)\| \leq \delta^S_i \|z_1 - z_2\|.
\]

Proof. The proof is presented in Appendix B.

Lemma 3. \( \text{[22 Lemma 2]} \) Consider a random vector \( w_{1:n} \) consisting of \( n \) i.i.d. random variables with probability function \( P_w \). Then, \( \mathbb{E}[\|w_{1:n} - P_w\|] \leq \frac{1}{\sqrt{n}} \).

Lemma 4. Let Assumption 5 hold. For any \( z_1, z_2 \in \mathbb{I} \), \( \gamma \in \mathcal{G} \) and \( k \in K \) at time \( t \in N_T \),
\[
\mathbb{E}[\|f^S_t(z_1, \gamma, w^k_t) - f^S_t(z_2, \gamma)\|] \leq H^S_i \|z_1 - z_2\| + O\left(\frac{1}{\sqrt{|N^k|}}\right),
\]

Proof. The proof is presented in Appendix C.

Lemma 5. Let Assumption 5 hold. For any \( z_1, z_2 \in \mathbb{I} \), \( \gamma \in \mathcal{G} \) and \( k \in K \) at time \( t \in N_T \),
\[
\mathbb{E}[\|f^S_t(z_1, \gamma, w^k_t) - f^S_t(z_2, \gamma, w^k_t)\|] \leq H^S_i \|z_1 - z_2\|.
\]

Proof. The proof is presented in Appendix D.

Lemma 6. Let Assumption 5 hold. Given any function \( f^S_t : \mathbb{I} \times \mathcal{G} \times \mathcal{W}_t \to \mathbb{I}, j \in \{1, 2\}, t \in N_T \), define real-valued functions \( \{f^S_t, \ldots, f^S_t\} \) such that for any \( z \in \mathbb{I}, V^S_{t+1}(z) = 0 \), and for any \( t \in N_T \) and \( z^i \in \mathbb{I} \):
\[
V^S_t(z^i) = \min_{\gamma \in \mathcal{G}} \left( \ell_i(z^i, \gamma) + \mathbb{E}[V^S_{t+1}(f^S_t(z^i, \gamma, w^k_t))] \right). \tag{44}
\]

Suppose there exist scalars \( H^S_i \in \mathbb{R}_{>0} \) and \( \Delta \in \mathbb{R}_{\geq 0} \) such that for every \( z^i, z^j \in \mathbb{I} \) and \( \gamma \in \mathcal{G} \),
\[
\mathbb{E}[\|f^S_t(z^i, \gamma, w^k_t) - f^S_t(z^j, \gamma, w^k_t)\|] \leq H^S_i \|z^i - z^j\| + \Delta. \tag{45}
\]

Then, at every time \( t \in N_T \),
\[
|V^S_t(z^i) - V^S_t(z^j)| \leq H^S_i \|z^i - z^j\|^2 + H^S_i \Delta, \tag{46}
\]
where \( H^S_{t+1} := H^S_t + 0 \), and \( H^S_t := H^S_{t+1} + H^S_{t+1} \). \( \tag{47} \)

Proof. The proof is presented in Appendix E.

Lemma 7. Consider Lemma 6 without \( \min \) operator in equation (44). Then, inequality (46) holds for any \( \gamma = \gamma^1 = \gamma^2 \).

Proof. The proof follows along the same lines of the proof of Lemma 6 with no minimization operator.

Lemma 8. \( \text{Under Assumption 2 for any} \ d \in \mathcal{E} \text{ and} \ p \in P_s \text{ at time} \ t \in N_T \),
\[
\mathbb{E}[\|f^S_t(d, \gamma, w^k_t) - f^S_t(p, \gamma, w^k_t)\|] \leq H^S_i \|d - p\| + O\left(\frac{1}{\sqrt{n}}\right).
\]

Proof. The proof follows from equations (11) and (46), and Lemmas 4 and 5 where \( n = \min_{k \in S^c} |N^k| \) in Problem 2.

D. Convergence result for Problem 2

Let \( g^* = \{(g^*_t)^k_{t \in T}\} \) and \( g^p = \{(g^p_t)^k_{t \in T}\} \) denote the strategies proposed in \( \text{(26)} \) and \( \text{(43)} \), respectively. Inspired by the notion of the price of information in \( \text{(35)} \), we define the price of information in a slightly different manner as follows.

Definition 2 (Price of information). The price of information (PoI) is defined as the loss of performance due to using the PDSS strategy \( g^p \) rather than the DSS strategy \( g^* \), i.e.,
\[
\text{PoI} := |J_N(g^p) - J_N(g^*)|. \tag{48}
\]

Theorem 4. Let Assumptions 2, 4 and 5 hold. The price of information (48) converges to zero at the rate \( 1/\sqrt{T} \), i.e.,
\[
\text{PoI} \leq (H^i + H^S_i) O\left(\frac{1}{\sqrt{T}}\right),
\]
where \( O\left(\frac{1}{\sqrt{T}}\right) \) is independent of the control horizon \( T \), and the positive constants \( H^i \) and \( H^S_i \) are computed recursively for \( t \in N_T \) from the Lipschitz constants of Lemma 2 as follows:
\[
H^i := H^i_{t+1} + H^S_i \quad \text{and} \quad H^S_i := H^i_{t+1} + H^S_{t+1}, \quad \text{where} \quad H^i_{t+1} := 0, \quad \text{Hence, strategy (43) is an} \epsilon(n)\text{-optimal strategy for Problem 2.}
\]

Proof. From the triangle inequality, \( |J_N(g^p) - J_N(g^*)| \) is upper-bounded by:
\[
|J_N(g^*) - \mathbb{E}[V^p_t(p_1)]| + |J_N(g^p) - \mathbb{E}[V^p_t(p_1)]|. \tag{49}
\]
The first term in (49) is associated with the difference between the optimal cost-to-go function under DSS, given by Theorem 2 and an approximate cost-to-go function proposed in (42). From the upper bounds developed in Subsection IV-C, it follows that:

\[ |J_N(g^*) - E[V^p_1(p_1)]| \leq \sum_{t=1}^{T} |\ell_t(p_t, \gamma_t^*)| + O(\frac{1}{\sqrt{n}}), \]

where (a) follows from Theorem 2, i.e., \( J_N(g^*) = E[V^p_1(d_1)] \); (b) follows from the fact that for every random variable \( Y \in \mathbb{R} \), \( E[|y|] \leq E[|y|] \); (c) follows from equations (25), (42), and Lemma 8 where \( H_2^3 := H_3^3 \) and \( \Delta = O(\frac{1}{\sqrt{n}}) \); (d) follows from Assumption 3, equation (39), and Lemma 3.

The second term in (49) corresponds to the difference between the performance of the system under the proposed strategy (43) and the approximate cost-to-go function defined in (42). Let \( \gamma_t^* = \gamma_t^*(p_t) \) be a minimizer of equation (42); then, equation (42) simplifies to:

\[ V^p_1(p_1) = E[\sum_{t=1}^{T} \ell_t(p_t, \gamma_t^*)] \quad (51) \]

According to equations (6), (13), (19) and (38), the performance of the proposed strategy (43) is given by:

\[ J(g^p) = E[\sum_{t=1}^{T} \ell_t(d^p_t, \gamma_t^p)] \quad (52) \]

From equations (38), (41), and Lemma 8 the approximation error between processes \( d^p_{t:T} \) and \( p_{1:T} \) is bounded as follows: for any \( d^p_t \in \mathcal{D} \) and \( p_t \in \mathcal{P} \),

\[ E[|d^p_{t+1} - p_{t+1}|] \leq H_3^2 |d^p_t - p_t| + O(\frac{1}{\sqrt{n}}), \quad t \in \mathbb{N}_T. \quad (53) \]

Therefore, it results from (51) and (52) that:

\[ |J_N(g^p) - E[V^p_1(p_1)]| \leq \sum_{t=1}^{T} |\ell_t(d^p_t, \gamma_t^p)| - \sum_{t=1}^{T} |\ell_t(p_t, \gamma_t^p)| \]

\[ \leq H_3^2 E[|d^p_t - p_t|] + H_2^6 O(\frac{1}{\sqrt{n}}) \leq (H_2^3 + H_3^3) O(\frac{1}{\sqrt{n}}), \quad (54) \]

where (a) follows from equation (53), Lemma 7 (where \( H_2^3 := H_3^3 + \Delta = O(\frac{1}{\sqrt{n}}) \)), and the fact that for every random variable \( Y \in \mathbb{R} \), \( E[|y|] \leq E[|y|] \); and (f) follows from Assumption 3, equation (39), and Lemma 3. The proof of Theorem 4 results from (49), (50), and (54).

V. QUANTIZATION RESULTS

In this section, we present quantized solutions for Problems 1 and 2 to reduce the complexity in space for dynamic programs (25) and (42) in Theorems 2 and 4, respectively.

Given a subset \( \mathcal{R} \subseteq \mathcal{K} \) of sub-populations and a quantization level \( r \in \mathbb{N} \), we are interested to find an \( \epsilon(r) \)-optimal solution for Problem 1 such that the computational complexity of finding such a solution in space is independent of the size of sub-populations \( \mathcal{R} \). To distinguish this solution from the optimal one under DSS, we use superscript \( q \) for the parameters associated with the quantized solution. Denote by \( g^q \) the strategy of the quantized solution and by \( d^q_t \in \mathcal{E} \) the deep state of the system under strategy \( g^q \) at time \( t \in \mathbb{N}_T \), where \( d^q_1 = d_1 \). We impose an assumption similar to Assumption 3 as follows.

**Assumption 6.** Assumption 2 holds for sub-populations \( \mathcal{R} \), where \( \mathcal{S}^c \) is substituted by \( \mathcal{R} \).

Define real-valued functions \( \{V^q_1, \ldots, V^q_T, V^q_{T+1}\} \) backward in time such that \( V^q_{T+1}(q) = 0 \) for any \( q \in \mathcal{Q}_c \). Also, for any \( t \in \mathbb{N}_T \) and \( q \in \mathcal{Q}_c \), define

\[ V^q_t(q) = \min_{\gamma \in \mathcal{G}} \{ \ell_t(q, \gamma) + E[V^q_{t+1}(Q(f_t(q, \gamma, w_t))] \}. \]

(55)

Denote by \( \psi^q_t(q) = \{\psi^q_{t,1}(q), \ldots, \psi^q_{t,K}(q)\} \) an argmin of the right-hand side of (55). Let agent \( i \in \mathcal{N}_k \) of sub-population \( k \in \mathcal{K} \) at time \( t \in \mathbb{N}_T \) take the following action:

\[ u^i_t = g^q_{t,k}(x^i_t, Q(d^q_t)) = \psi^q_{t,k}(Q(d^q_t))(x^i_t), \quad x^i_t \in \mathcal{X}_k, d^q_t \in \mathcal{E}. \]

(56)

Let \( g^q = \{g^q_{t,k} \in \mathcal{K}\}_{t=1}^{T} \) denote the strategy proposed in (56). We define the price of computation as follows.

**Definition 3 (Price of computation).** The price of computation (PoC) is defined as the loss of performance due to using the strategy \( g^q \) rather than the optimal strategy \( g^* \), i.e.,

\[ PoC := J_N(g^q) - J_N(g^*). \]

(57)

**Theorem 5.** Let Assumptions 2, 3, and 6 hold. The price of computation (57) converges to zero at the rate \( 1/r \), i.e.,

\[ PoC \leq (H_2^3 + H_3^3) \frac{1}{r}, \]

where \( H_2^3 \) and \( H_3^3 \) are given in Theorem 4. This implies that strategy (56) is an \( \epsilon(r) \)-optimal strategy for Problem 7.

**Proof.** The proof follows along the same steps of Theorem 4 with the distinction that the source of error is the quantization error, that is, bounded by \( \frac{\epsilon}{r^2} \) in the definition of the quantizer function \( Q \). Using the same upper bounds developed in Subsection IV-C, it can be shown that the propagation of the quantization error over the dynamics and value functions are bounded by the same Lipschitz gains introduced in Theorem 4.

In general, finding an exact solution to the dynamic program (42) is challenging because set \( \mathcal{P}_S \) is uncountable for \( S \neq K \). To address this challenge, we propose a quantized dynamic program similar to that of Theorem 3. Let \( \mathcal{R} \) in Theorem 5 be equal to \( S^c \) in Theorem 4. We are interested in finding an \( \epsilon(n, r) \)-optimal solution for Problem 2 such that not only the deep states of sub-populations \( S^c \) are not communicated among agents, but also the computational complexity of finding such a solution is independent of the size of sub-populations \( S^c \). We use superscript \( pq \) for the parameters associated to this solution.

Define real-valued functions \( \{V^{pq}_1, \ldots, V^{pq}_T, V^{pq}_{T+1}\} \) backward in time such that \( V^{pq}_{T+1}(q) = 0 \) for any \( q \in \mathcal{Q}_{S^c} \). Also, for any \( t \in \mathbb{N}_T \) and \( q \in \mathcal{Q}_{S^c} \), define

\[ V^{pq}_t(q) = \min_{\gamma \in \mathcal{G}} \{ \ell_t(q, \gamma) + E[V^{pq}_{t+1}(Q(f_t(q, \gamma, w_t))] \}. \]

(58)
Denote by $\psi^{pq,q}(q) = \{\psi^{pq,1}(q), \ldots, \psi^{pq,K}(q)\}$ an argmin of the right-hand side of (58). Let agent $i \in N^k$ of sub-population $k \in K$ at time $t \in \mathbb{N}_T$ take the following action:

$$g^i_{t}(x^i_t, Q(p)_t) := \psi^i_{t,q,k}(Q(p)_t)(x^i_t), \quad x^i_t \in X^k.$$  

**Theorem 6.** Let Assumptions 2, 3, 4, and 5 hold. Then, $g^{pq} := \{(g^{pq,k})_{k \in S} \}_{t=1}^{\infty}$ is an $\varepsilon(n, r)$-optimal solution for Problem 2.

$$|J(g^{pq}) - J(g^*)| \leq \varepsilon(n, r) = (H^5_{t} + H^6_{t})(\mathcal{O}(\frac{1}{\sqrt{n}}) + \frac{1}{r}),$$

where $H^5_{t}, H^6_{t},$ and $\mathcal{O}(\frac{1}{\sqrt{n}})$ are given in Theorem 4.

**Proof:** The proof follows from Theorems 3 and 5. To avoid repetition, the detailed proof is omitted.

**Corollary 2.** If $r \geq \sqrt{n}$, then the rate of convergence of the solution of Theorem 6 is the same as that of Theorem 4.

**VI. INFINITE HORIZON DISCOUNTED COST**

In this section, we generalize our main results to the infinite-horizon discounted cost problem. It is assumed that the dynamics and per-step cost in Problems 1 and 2 are time-homogeneous, and the information structures are the same as those in Subsection II-B. Given a discount factor $\beta \in (0, 1)$, the performance of any strategy $g$ is described by:

$$J^\beta_N(g) = \mathbb{E}_\tau \sum_{t=1}^{\infty} \beta^{t-1} c(D_t).$$

**Theorem 7.** Let Assumption 7 hold. The optimal solution of Problem 7 with the infinite-horizon discounted cost is obtained from the following Bellman equation, i.e., for any $d \in E$.

$$V^d_{\tau}(d) = \min_{\gamma \in \mathcal{G}} (\ell(d, \gamma) + \beta \mathbb{E}[V^d_{\tau+1}(\bar{f}(d, \gamma, w)])],$$  

(59)

where the above expectation is taken with respect to $w \in \mathcal{W}$. Let $\psi^d(d) = \{\psi^{d,1}(d), \ldots, \psi^{d,K}(d)\}$ be an argmin of the right-hand side of (59). Then, the optimal control law of agent $i \in N^k$ of sub-population $k \in K$ at time $t \in \mathbb{N}$ is given by:

$$g^{i,k}(x^i_t, d_t) := \psi^{i,k}(d_t)(x^i_t), \quad x^i_t \in X^k, d_t \in E.$$  

**Proof:** Consider the dynamic program of Theorem 2 for any finite horizon $T$. From (58), make a change of variable such that for any $d \in E$ and $t \in \mathbb{N}_T$,

$$W^d_{T}(d) := \beta^{T-t+1} V^d_{T-t+2}(d),$$  

(60)

where $W^d_{T}(d) := \beta^{T-T+1} V^d_{T+1}(d) = 0$. By simple algebraic manipulations and setting $t = 1$, we arrive at: $W^d_{1}(d) = \min_{\gamma \in \mathcal{G}} (\ell(d, \gamma) + \beta \mathbb{E}[W^d_{1+1}(\bar{f}(d, \gamma, w)])].$ Since the above Bellman operator is contractive [6], we have

$$\lim_{T \to \infty} W^d_{T} = W^d_{\infty} =: V^d.$$  

(61)

**Remark 10.** Although the computational complexity of finding the solution of Bellman equation (59) is polynomial with respect to the number of agents in each sub-population, it is exponential in time, in general; however, one can find an $\varepsilon$-optimal solution in polynomial time by using value iteration method, where $\varepsilon$ converges to zero exponentially [6].

For Problem 2 with the infinite-horizon discounted cost, we impose the following assumption on the model.

**Assumption 7.** It is assumed that $\beta H^3 < 1$, where $H^3$ is the Lipschitz constant in Lemma 2.

When the dynamics of agents are decoupled, Assumption 7 holds because $H^3 = 1$ satisfies Lemma 2. This is an immediate consequence of equation (13) and the fact that the probability of any event is upper-bounded by 1.

**Theorem 8.** Let Assumptions 2, 3, 4, 5, and 7 hold. Then, an $\varepsilon(n)$-optimal solution for Problem 2 with the infinite-horizon discounted cost function is identified by the following Bellman equation, i.e., for any $p \in \mathcal{P}_s$.

$$V^p(p) = \min_{\gamma \in \mathcal{G}} (\ell(p, \gamma) + \beta \mathbb{E}[V^p_{\tau+1}(\bar{f}^p(p, \gamma, w)])],$$  

(62)

where the above expectation is taken with respect to $w \in \mathcal{W}$. Let $\psi^p(p) = \{\psi^{p,1}(p), \ldots, \psi^{p,K}(p)\}$ be an argmin of the right-hand side of (62). The control law of agent $i \in N^k$ of sub-population $k \in K$ at time $t \in \mathbb{N}_T$ is given by:

$$u^i_t = g^{p,k}(x^i_t, p_t) = \psi^{p,k}(p_t)(x^i_t), \quad x^i_t \in X^k, p_t \in \mathcal{P}_s,$$

such that the optimality gap is bounded as follows:

$$|J^\beta_N(g^p) - J^\beta_N(g^*)| \leq \varepsilon(n) = \frac{H^4}{1 - \beta(1 - \beta H^3)^2} \mathcal{O}(\frac{1}{\sqrt{n}}),$$

where $H^3$ and $H^4$ are the Lipschitz constants in Lemma 2.

**Proof:** Consider the dynamic program of Theorem 2 for any finite horizon $T$. Make a change of variables such that for any $p \in \mathcal{P}_s$ and $t \in \mathbb{N}_T$,

$$W^p_T(p) := \beta^{T-t+1}V^p_{T-t+2}(p),$$

$$H^5_t := \beta^{T-t+1}H^5_{T-t+2}, H^6_t := \beta^{T-t+1}H^6_{T-t+2},$$

(63)

where $W^p_T(p) := \beta^{T-T+1}V^p_{T+1}(p) = 0, H^5_{T+1} := \beta^{T-H^5_{T+1}} = 0$ and $H^6_{T+1} := \beta^{T-H^6_{T+1}} = 0$. From Theorem 4, one may arrive at the following relations by simple algebraic manipulations and setting $t = 1$:

$$W^p_{T+1}(p) = \min_{\gamma \in \mathcal{G}} (\ell(p, \gamma) + \beta \mathbb{E}[W^p_{T+1}(\bar{f}^p(p, \gamma, w)])],$$

$$H^5_{T+1} = H^4 + \beta H^5_{T} H^3 = H^4 \sum_{t=1}^{T}(\beta H^3)^{t-1},$$

$$H^6_{T+1} = \beta(H^5_{T} + H^6_{T}) \leq H^5_{T+1} \beta^{T-t+1}.$$  

(64)

Since the above Bellman operator is contractive [6], we have

$$\lim_{T \to \infty} W^p_T = W^p_{\infty} =: V^p.$$  

(65)

In addition, from Lemmas 6 and 8 as well as equations (60) and (63), for any $d \in E$ and $p \in \mathcal{P}_s$,

$$\|W^p_{T+1}(d) - W^p_{T+1}(p)\| \leq \|V^d(d) - V^p(p)\| \leq H^5_{1}d - p + H^5_{1}O\left(\frac{1}{\sqrt{n}}\right) = H^5_{T+1}d - p + H^5_{T+1}O\left(\frac{1}{\sqrt{n}}\right).$$

(66)

Therefore, from equations (61), (65) and (66),

$$\|W^d_{\infty}(d) - W^p_{\infty}(p)\| \leq \|V^d(d) - V^p(p)\| \leq H^5_{T+1}d - p + H^5_{T+1}O\left(\frac{1}{\sqrt{n}}\right).$$  

(67)
where from Assumption 7 and equation (64): \( \hat{H}^0_\infty = H^4_\infty \) and \( \hat{H}^5_\infty \leq \frac{\beta H^4_\infty}{(1-\beta)(1-\beta H^4_\infty)} \). The rest of the proof follows along the same lines of the proof of Theorem 3 starting from the triangle inequality, where inequality (c) in (50) and inequality (e) in (64) are replaced by inequality (67).

\[ \Box \]

**Remark 11.** Since \( \mathcal{P}_s \) is a polish space, there always exists a minimizer \( \psi^p(p) \) for (62). Under Assumption 7 the dynamic programs in Theorems 5 and 6 can be extended to the infinite-horizon case by replacing \( (H^5_\infty + H^4_\infty) \) with \( (1-\beta)(1-\beta H^4_\infty) \).

**Remark 12.** Although strategy \( g^p \) in Theorem 8 is stationary with respect to \((x^i_t, p_i)\), it is not stationary with respect to \(x^i_t\), \( t \in \mathbb{N}_T \). This implies that the assumption of stationary strategy in (36) is rather restrictive.

**VII. ARBITRARILY ASYMMETRIC COST FUNCTION**

Let \( c_i(x^i_t, u^i_t) : X \times U \to \mathbb{R}_{\geq 0} \) be any arbitrarily-coupled (asymmetric) per-step cost function at time \( t \in \mathbb{N}_T \).

**Assumption 8.** For any sub-population \( k \in \mathcal{K} \), initial states \((x^i_0)_{i \in \mathcal{X}^k}_t\) are exchangeable.

**Proposition 2.** Let Assumptions 7 and 8 hold. When attention is restricted to fair strategies, there exists a function \( \bar{\eta}_t : \mathcal{E} \times \mathcal{G} \to \mathbb{R} \), independent of the global laws \( \psi^{1:t} \), such that: \( \bar{\eta}_t(d^i_t, \gamma_t) := c_i(x^i_t, u^i_t) | d^i_{1:t}, \gamma_{1:t} = \sum_{x \in X} \sum_{u \in U} \prod_{k \in \mathcal{K}} \prod_{i \in \mathcal{X}^k} \mathbb{I}(u^k = \gamma^k_t(x^i_t)) c_i(x, u) \mathbb{P}(x_t = x | d^i_{1:t}, \gamma_{1:t}) \).

**Proof:** The proof follows from a forward induction proposed in (30) Lemma 2 which shows that \( \bar{\mathbb{P}}(x_t = x | d^i_{1:t}, \gamma_{1:t}) \) is partially exchangeable, i.e., it is representable by \( d^i_t \).

According to Proposition 2 dynamic programs proposed in Sections 3, 4 extend naturally to any arbitrarily asymmetric cost function. Note that the complexity of computing \( \bar{\eta}_t(d^i_t, \gamma_t) \) in time is exponential with respect to the number of agents. However, this computation can be carried out off-line by machine learning methods or circumvented by reinforcement learning techniques (34), (37). In general, the exploration space of an arbitrarily asymmetric cost function grows exponentially with the number of agents while that of its deep state projection grows polynomially, according to Proposition 2 which is a considerable reduction in complexity.

**VIII. MAJOR-MINOR SETUP: A SPECIAL CASE**

Consider a special case where a sub-population has only one agent. In mean-field game theory, this case is known as major-minor mean-field game which was first introduced in (38). The sub-population with one agent is referred to as the major player because it can directly influence other players, called minor players, through its local state whereas other players can only influence the major player through their collective behaviour (mean-field). In this type of scenario, the classical mean-field game approach is not directly applicable as the mean-field of minor players is no longer deterministic (hence, unpredictable) due to the randomness of the major player’s state (24). A similar setup may be considered in deep teams with the distinction that no additional complication is introduced as the deep team solution is not in the form of coupled forward-backward equations, i.e., it is independent of the future trajectory of the deep state. To connect our results to major-minor setup, consider a sub-population \( k \in \mathcal{K} \) consisting of one agent, i.e., \( |\mathcal{X}^k| = 1 \). Since the local state and the deep state of this agent are identical, the split of strategy \( g^p \) in (35) can be done slightly different as follows: \( u^i_t = \psi^p_t((d^i_{1:t})_{k \in \mathcal{S}}) = g_t^k((d^i_{1:t})_{k \in \mathcal{S}}) \). In this case, the local law of major agent \( \gamma^k_t : \mathcal{X}^k \to U^k \) (that takes \( |U^k| \) values) simplifies to its local control action \( u^i_t \in U^k, i \in \mathcal{X}^k \), (that takes \( |U^k| \) values).

**IX. NUMERICAL EXAMPLE**

**Example 1.** Consider a company that provides a particular service (e.g., internet, electricity or cellular phone) for \( n \in \mathbb{N} \) users. Assume that each user \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N} \) makes an independent request with probability \( \mu \in (0, 1) \) to receive service. For simplicity, it is also assumed that each user is not allowed to make a new request until its current request is served. Therefore, the state of user \( i \) at time \( t \) is binary, i.e., \( x^i_t \in \mathbb{X} := \{0, 1\} \), where \( x^i_t = 1 \) means that user \( i \) has a request at time \( t \), and \( x^i_t = 0 \) means that it does not have a request at that time. The initial states of users are distributed identically and independently with respect to probability mass function \( \mathbb{P}_X \). Let \( d^i_t \in \mathcal{E} := \{0, \frac{1}{n}, \frac{2}{n}, \ldots, 1\} \) denote the empirical distribution of the requests at time \( t \), i.e., \( d^i_t = \frac{1}{n} \sum_{k=1}^{n} \mathbb{I}(x^k_t = 1) \). To serve its users, the company has \( h \in \mathbb{N} \) options which depend on different contracts and resources. Define the following for every \( u \in \mathbb{U} = \{1, \ldots, h\} \):

- **Participation rate:** This is the probability according to which a user is incentivized to delay its request. The company may use different contracts and price profiles to incentivize users to postpone their requests. For example in smart grids, the independent service operator may offer discounts to motivate users to delay their demands during peak-load time. Denote this probability by \( \alpha(u) \in [0, 1] \) and assume it is independent of the request probability; hence, the probability of receiving a request from a participant user is \( (1 - \alpha(u)) \mu \leq \mu \).
- **Service rate:** This is the probability according to which a request is served at each time instant. The company may use various suppliers (resources) to provide service; hence, the service rate may not be the same for different suppliers. Denote this probability by \( q(u) \in (0, 1) \).
- **Base price:** This is the price of serving a user when it has no request (i.e., \( x = 0 \)). This price is used for maintaining resources such as data storage and communications. Denote this price by \( c_{\text{base}}(u, 1 - d^i_t) \in \mathbb{R}_{\geq 0} \).
- **Service price:** This is the price of serving a user when it has a request (i.e., \( x = 1 \)). This price includes the price of service as well as the price of maintenance (but may not be a simple sum of the two). Denote this price by \( c_{\text{service}}(u, d^i_t) \in \mathbb{R}_{\geq 0} \).

Let \( u^i_t \in \mathbb{U} \) denote the option assigned to user \( i \in \mathbb{N}_n \) at time \( t \). Then, the transition probability matrix of user \( i \) under option \( u^i_t \) at time \( t \) can be written as follows:

\[
\mathbb{P}(x^i_{t+1} | x^i_t, u^i_t) = \begin{bmatrix}
1 - (1 - \alpha(u^i_t)) \mu & q(u^i_t) \\
(1 - \alpha(u^i_t)) \mu & 1 - q(u^i_t)
\end{bmatrix}.
\]
Alternatively, one can express the above dynamics in the form of (3), i.e.,
\[ x_{i+1} = (1-x_i)w_i(u_i) + x_i\tilde{w}_i(u_i), \]
where \( w_i(u_i), \tilde{w}_i(u_i) \in \{0, 1\} \) are the underlying Bernoulli random variables associated with the rates of request, participation and service such that:
\[ \mathbb{P}(w_i(u_i) = 1) = 1 - \alpha(u_i) \mu \]
and
\[ \mathbb{P}(\tilde{w}_i(u_i) = 1) = 1 - q(u_i). \]
In addition, the cost of serving user \( i \) at time \( t \) is given by:
\[ c(x_i^t, u_i^t, d_i) = (1-x_i^t)c_B(u_i^t, 1-d_i) + x_i^t cs(u_i^t, d_i). \]
It is desirable for the company to adapt its nominal capacity with the empirical distribution of requests.

Let \( x_i^0 \in \mathcal{X}^0 \) denote the nominal capacity of a server located in the company at time \( t \in \mathbb{N} \), where \( \mathcal{X}^0 \) is a finite set with values in \([0, 1]\). At each time instant, the server must decide the next nominal capacity. Let \( u_i^t \in \mathcal{U}^0 =: \mathcal{X}^0 \) denote the action of the server at time \( t \in \mathbb{N} \). Subsequently, the dynamics of the state of server evolves as follows:
\[ x_{i+1}^t = x_i^t w_i^t + u_i^t (1-w_i^t), \]
where \( \{w_i^t \in \{0, 1\}, t \in \mathbb{N}\} \) is a Bernoulli process that captures the probability of failure, i.e., \( w_i^0 = 1 \) implies that there is a fault at time \( t \), and \( w_i^0 = 0 \) implies that there is no fault. Denote by \( p_{w_i^0} \in [0, 1] \) the probability according to which a fault may occur. When a fault happens, no capacity is patched or dispatched. Define \( \ell : \mathcal{X}^0 \times \mathcal{U}^0 \to \mathbb{R}_{\geq 0} \) as the cost function of the server such that:
\[ \ell(x_i^t, u_i^t) := \ell_C(x_i^t) + \ell_P(u_i^0 - x_i^t), \]
where \( \ell_C(x_i^t) \in \mathbb{R}_{\geq 0} \) is the price of capacity \( x_i^t \in \mathcal{X}^0 \) and \( \ell_P(u_i^0 - x_i^t), \ell_P \in \mathbb{R}_{\geq 0} \), is the cost of patching and dispatching capacities \( x_i^t \) and \( u_i^0 \).

The objective of the company is to manage the users (consumption side) as well as the server (generation side) in such a way that the empirical distribution of requests is close to the nominal capacity while incurring the lowest possible price. More precisely, given \( \beta \in (0, 1) \) and \( \lambda \in \mathbb{R}_{\geq 0} \), it is desired to minimize the following cost
\[ \mathbb{E}
\[ \sum_{t=1}^{\infty} \beta^{-1} \left( \frac{1}{n} \sum_{i=1}^{\infty} c(x_i^t, u_i^t, d_i) + \ell(x_i^t, u_i^t) + \lambda(d_i - x_i^t)^2 \right) \], \]
where the first term is the average price of users, the second term is the cost of the server, and the third term is the penalty for the empirical distribution of requests deviating from nominal capacity. Since state space \( \mathcal{X} \) is binary, the empirical distribution of requests \( d_i \) is a sufficient statistic to identify the deep state, i.e., \((1-d_i, d_i), t \in \mathbb{N} \). Hence, to ease the presentation, \( d_i \) is referred to as the deep state hereafter.

To illustrate the results, suppose the company has 3 options (\( h = 3 \)). The first option, denoted by \( u = 1 \), is to provide service at a flat rate. The second option, denoted by \( u = 2 \), is to give 20\% discount to users in order to incentivize them to cooperate with the company. In such a case, users are motivated to delay their requests (with a probability directly dependent on the participation rate) and let the company serve their existing requests at a lower rate. The third option, denoted by \( u = 3 \), is that the company gets the desired service from another service company at a variable rate. In addition, the following parameters are chosen for the simulations:
\[ n = 200, \quad \beta = 0.8, \quad \mu = 0.8, \quad q = [0.1, 0.05, 0.2], \]
\[ \alpha = [0, 0.85, 0], \quad c_B(1, \cdot) = 0.59, \quad c_S(1, \cdot) = 0.65, \]
\[ c_B(2, \cdot) = (1+0.2)c_B(1, \cdot), \quad c_S(2, \cdot) = (1+0.2)c_S(1, \cdot), \]
\[ c_B(3,1-d) = 0.3(1+1-d), \quad c_S(3, d) = 0.5(1+d), \]
\[ \lambda = 15, \quad \ell_P = 0.5, \quad \mathcal{X}^0 = \mathcal{U}^0 = \{0, 1\}, \quad \ell_C = [0.02, 0.04, 0.07, 0.12, 0.15, 0.2], \quad p_{w, 0} = 0.05. \]

When there is no request at user \( i \in \mathbb{N}_n \) at time \( t \in \mathbb{N} \), the cheapest option for the user is chosen from Figure 1a.
at that time, and when there is a request from user $i$, the best affordable option for the user is selected from Figure 1b. In addition, the nominal capacity of the server is determined by Figure 1c, based on the current capacity and the deep state. Figure 2 demonstrates a trajectory of the deep state and nominal capacity. In Figure 3, the optimality gap between DSS and PDSS solutions are depicted with respect to the number of users, where the initial state of the server is 0.3 and the initial states of users are identically and independently distributed according to the probability distribution $[0.2, 0.8]$.

X. Conclusions

In this paper, we introduced deep teams and proposed novel dynamic programs to find optimal and sub-optimal solutions under two non-classical information structures, namely, deep-state sharing and partial deep-state sharing. In addition, we developed several results to alleviate the computational complexity of the proposed solutions when the population is medium or large. We then extended our results to the infinite-horizon discounted cost as well as arbitrarily coupled cost, and demonstrated their effectiveness by a numerical example in service management.

In practice, agents often have limited computation and communication resources. These practical limitations are the root causes of many challenges in team theory. We showed that in deep teams, the number of computational resources increases polynomially (rather than exponentially) with the number of agents in each sub-population. In addition, the size (length) of the shared information (i.e., deep state) among agents, on the other hand, is independent of the number of agents in each sub-population. Moreover, agents may decide not to communicate their states at all, in which case the error of such compromise was shown to be bounded by $O(1/\sqrt{n})$ and hence goes to zero as the number of agents $n$ increases. Furthermore, agents may use a quantized solution whose computational complexity in space is independent of the number of agents. We showed that the error of such compromise is bounded by $O(1/r)$, where the error tends to zero as the quantization level $r$ increases.

The main results of this paper can naturally be generalized to randomized strategies and partially observable deep states by simply replacing the action space and deep state with the space of probability measures and belief deep state, respectively. To further enhance the computational complexity of a deep team, one can use various approximation methods such as deep reinforcement learning to numerically solve the associated optimization problem. In such a case, the resultant design may be called doubly deep, which is not only deep in the number of decision makers but also deep in the number of hidden layers of the learned strategy. The authors have an upcoming paper on this subject.

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adapted to from identical initial values and evolve identically in time. Consequently, processes \(|G|\) occupies a certain amount of space and each iteration takes a certain amount of time. In addition, to compute the minimization in (35), \(|E|\) iterations are needed for searching over all possible cases. On the other hand, from (27), \(|E|\) and \(|W|\) are bounded by \(\prod_{k \in K} (|N^k| + 1)^{|X^k|}\) and \(\prod_{k \in K} (|N^k| + 1)^{|W^k|}\), respectively. Furthermore, \(|G| = \prod_{k \in K} |U^k| |X^k|\) is finite and independent of the number of agents in each sub-population as well as control horizon \(T\), which leads to the result.

APPENDIX B
PROOF OF COROLLARY [1]

The proof proceeds inductively. According to equation (39), Proposition [1] holds at initial time \(t = 1\). Suppose \(p_{1:t}\) is adapted to the filtration \((d_{1:k}^{W_k})_{k \in S}\) at time \(t\). From Assumption [4] and equation (33), the following equality holds for any \(d \in E, p \in P_s\) and \(\gamma \in G\) at any time \(t \in \mathbb{N}_T\),

\[
\tilde{f}_t^k ((d_{1:k}^{W_k})_{k \in S}, \gamma, w^k) = \tilde{f}_t^k (d, \gamma, w^k), \quad k \in S, \\
\tilde{f}_t^k ((p_{1:k}^{W_k})_{k \in S}, \gamma, w^k) = \tilde{f}_t^k (p, \gamma, w^k), \quad k \in S. \tag{68}
\]

Consequently, processes \(d_{1:k}^{W_k} + 1\) and \(p_{1:k}^{W_k} + 1\), \(k \in S\), start from identical initial values and evolve identically in time almost surely under identical local laws and noise processes, according to equations (40), (37), (39), (40), (41), (40), and Assumption [4]. Thus, \(f_{t+1}^{W_k} \overset{a.s.}{=} f_{t+1}^{p_k}\). For any \(k \in S^c, p_{t+1}^k\) evolves deterministically according to (40), where \(p_t\) and \(\gamma_t^k\) are adapted to \((d_{1:k}^{W_k})_{k \in S}\) from the induction assumption and (35), respectively. Therefore, \(p_{1:t+1}\) is adapted to \((d_{1:t+1}^{W_k})_{k \in S}\).

APPENDIX C
PROOF OF LEMMA [3]

It is well known that any linear combination and product of Lipschitz functions is also a Lipschitz function. According to (9), for any \(\gamma \in G\), function \(\phi(z, \gamma)\) is Lipschitz in \(z\). Consequently, for any \(y, x \in \mathcal{X}^k\) and any \(\gamma \in G\), the transition probability \(P^k_{(y|x, \gamma, \gamma)}(\Phi(z, \gamma))\) is Lipschitz in \(z\) due to Assumption [3]. Hence, function \(f_t^k(z, \gamma)\) given by (12), is Lipschitz in \(z\) because it is a linear combination of Lipschitz functions (and their products). Let \(H_3^{1:k}\) denote the corresponding Lipschitz constant, and define \(H_1^k\) as the maximum Lipschitz constant, i.e., \(H_1^k := \max_{k \in K} H_3^{1:k}\). Similarly, function \(\ell_t(z, \gamma)\), given by (13), is Lipschitz in \(z\) due to Assumption [5] on noting that the function \(\phi\) is Lipschitz.

APPENDIX D
PROOF OF LEMMA [4]

For every \(y \in \mathcal{X}^k, k \in K\), it follows that

\[
E[|f_t^k(z_{1}, \gamma, d_i(y)) - f_t^k(z_2, \gamma, \gamma)|] \\
= E[|f_t^k(z_{1}, \gamma, d_i(y)) + f_t^k(z_{1}, \gamma, \gamma) - f_t^k(z_2, \gamma, \gamma)|] \\
\leq E[|f_t^k(z_{1}, \gamma, d_i(y)) - f_t^k(z_1, \gamma, \gamma)|] \\
+ E[|f_t^k(z_1, \gamma, \gamma) - f_t^k(z_2, \gamma, \gamma)|] \\
\leq E[|\sum_{w \in \mathcal{W}_k} \sum_{x \in \mathcal{X}_k} z_k^t(1)f_t^k(x, \gamma, x, \gamma, \gamma, w) = y|] \\
\times (w_k^t(w) - P_{W_k}^t(w)) + H_1^k|z_1 - z_2| \\
\leq \sum_{w \in \mathcal{W}_k} \sum_{x \in \mathcal{X}_k} z_k^t(1)|E[w_k^t(w) - P_{W_k}^t(w)|] + H_1^k|z_1 - z_2| \\
\leq O(\frac{1}{\sqrt{|X^k|}}) + H_1^k|z_1 - z_2|,
\]

where (a) follows from the triangle inequality and the monotonicity of the expectation operator; (b) follows from equations (4), (10), (12) and Lemma [3] (c) follows from \(1(f_t^k(x, \gamma, x, \gamma, x, \gamma, w) = y) \leq 1\), the triangle inequality and the monotonicity of the expectation operator, and (d) from Lemma [3] \(z_k^t(1) \leq 1, \forall x \in \mathcal{X}^k\), and the fact that the cardinality of spaces \(\mathcal{X}^k\) and \(\mathcal{W}^k\) does not depend on \(|X^k|\).

APPENDIX E
PROOF OF LEMMA [5]

We first show that for any \(z \in \mathcal{Z}, \gamma \in \mathcal{G}, z_k \in \mathcal{M}_{\mathcal{N}^k}(\mathcal{W}^k)\) and \(y \in \mathcal{X}^k\) at time \(t \in \mathbb{N}_T\), \(f_t^k(z, \gamma, w^k)(y)\) is equal to:

\[
\sum_{w \in \mathcal{W}_k} \sum_{x \in \mathcal{X}_k} z_k^t(1)(f_t^k(x, \gamma, x, \gamma, \gamma, \gamma, w) = y) w_k^t(w) \\
\leq \max_{w \in \mathcal{W}_k} w_k^t(w) \sum_{x \in \mathcal{X}_k} \sum_{w \in \mathcal{W}_k} z_k^t(x) \\
x \times 1(f_t^k(x, \gamma, x, \gamma, \gamma, \gamma, \gamma, w) = y) w_k^t(w) \\
\leq \max_{w \in \mathcal{W}_k} w_k^t(w) \sum_{x \in \mathcal{X}_k} \sum_{w \in \mathcal{W}_k} z_k^t(x) \\
\times 1(f_t^k(x, \gamma, x, \gamma, \gamma, \gamma, \gamma, w) = y) w_k^t(w),
\]

where in (a) set \(\hat{\mathcal{W}}^k := \mathcal{W}^k \setminus \{w \in \mathcal{W}_k | P_{W_k}^t(w) = 0\}\) contains all realizations that have non-zero probability measures.
this equality holds a.s.), and (b) follows from \([4, 20, 21, 19]\). By similar argument for any \(y \in X^k\), we have
\[
f^k_t(z, \gamma, w^k_t)(y) \geq \min_{w \in \mathcal{W}^k} \left( \mathbb{E}_{W^k_t}(w) f^k_t(z, \gamma)(y) \right).
\] (70)

From (69) and (70), it results that for any \(z_1, z_2 \in \mathcal{I}\) and \(\gamma \in \mathcal{G}\) at time \(t \in \mathbb{N}_T\),
\[
\mathbb{E}\|f^k_t(z_1, \gamma, w^k_t) - f^k_t(z_2, \gamma, w^k_t)\| \\
\leq \mathbb{E}\max_{w \in \mathcal{W}^k} \left( \mathbb{E}_{W^k_t}(w) f^k_t(z_1, \gamma) - \min_{w \in \mathcal{W}^k} \mathbb{E}_{W^k_t}(w) f^k_t(z_2, \gamma) \right) \\
+ \mathbb{E}\left( \mathbb{E}\max_{w \in \mathcal{W}^k} \left( \mathbb{E}_{W^k_t}(w) f^k_t(z_1, \gamma) - \min_{w \in \mathcal{W}^k} \mathbb{E}_{W^k_t}(w) f^k_t(z_2, \gamma) \right) \right) \\
= \mathbb{E}\max_{w \in \mathcal{W}^k} \left( \mathbb{E}_{W^k_t}(w) f^k_t(z_1, \gamma) - \min_{w \in \mathcal{W}^k} \mathbb{E}_{W^k_t}(w) f^k_t(z_2, \gamma) \right) \\
+ \mathbb{E}\max_{w \in \mathcal{W}^k} \left( \mathbb{E}_{W^k_t}(w) f^k_t(z_2, \gamma) - \min_{w \in \mathcal{W}^k} \mathbb{E}_{W^k_t}(w) f^k_t(z_1, \gamma) \right) \\
\leq \|f^k_t(z_1, \gamma) - f^k_t(z_2, \gamma)\| H^k_t \|z_1 - z_2\|,
\]
where (c) follows from the triangle inequality and the monotonicity of the expectation operator, and (d) follows for every \(w \in \tilde{\mathcal{W}}^k\),
\[
\mathbb{E}\left[ \frac{w^k_t(w)}{P_{W^k_t}(w)} \right] = \mathbb{E}\left[ \sum_{w \in \mathcal{W}^k} \frac{w^k_t(w)}{P_{W^k_t}(w)} \right] = \frac{\mathcal{N}^k_t}{|P_{W^k_t}(w)|} = \frac{\mathcal{N}^k_t}{|W^k_t|} = 1,
\]
and (e) follows from Lemma 2.

APPENDIX F
PROOF OF LEMMA 6

The proof proceeds by backward induction. At \(t = T\),
\[
V^\gamma_T(z^1) = \min_{\gamma' \in \mathcal{G}} \mathcal{E}_T(z^1, \gamma') \\
\leq \min_{\gamma' \in \mathcal{G}} \left( \mathcal{E}_T(z^1, \gamma') - \mathcal{E}_T(z^2, \gamma') \right) + \mathcal{E}_T(z^2, \gamma') \\
\leq \min_{\gamma' \in \mathcal{G}} \left( H^\gamma_T \|z^1 - z^2\| + \mathcal{E}_T(z^2, \gamma') \right) \\
\leq H^\gamma_T \|z^1 - z^2\| + \min_{\gamma' \in \mathcal{G}} \mathcal{E}_T(z^2, \gamma') \\
= H^\gamma_T \|z^1 - z^2\| + V^\gamma_T(z^2),
\]
where (a) follows from the triangle inequality, the fact that \(\mathcal{E}_T(\cdot) \in \mathbb{R}_+\), and the monotonicity of the minimum operator; (b) follows from Lemma 2 and the monotonicity of the minimum operator; (c) follows from the fact that the space of minimization \(\mathcal{G}\) is the same for both \(\gamma\) and \(\gamma'\), and (d) follows from (44). Therefore, (46) holds at \(t = T\), where \(H^\gamma_T = H^\gamma_T = 0\). Now, assume that (46) holds at time \(t + 1\), i.e.,
\[
|V^\gamma_{t+1}(z^1) - V^\gamma_{t+1}(z^2)| \leq H^\gamma_{t+1} \|z^1 - z^2\| + H^\delta_{t+1} \Delta.
\]