Hyperbolic orbifolds of small volume

Mikhail Belolipetsky
IMPA

Friday, August 15. 16:00–16:45
2014 Seoul ICM
http://icm2014.org/
(IL5.2)
Volume in hyperbolic geometry

\( \mathcal{H}^n \) – the \textit{hyperbolic n-space}
(e.g. the upper half space with the hyperbolic metric \( ds^2 = \frac{dw^2}{y^2} \)).

Isom(\( \mathcal{H}^n \)) – the \textit{group of isometries} of \( \mathcal{H}^n \).
Volume in hyperbolic geometry

$\mathbb{H}^n$ – the *hyperbolic n-space*  
(e.g. the upper half space with the hyperbolic metric $ds^2 = \frac{dw^2}{y^2}$).

$\text{Isom}(\mathbb{H}^n)$ – the *group of isometries* of $\mathbb{H}^n$.

$\Gamma < \text{Isom}(\mathbb{H}^n)$, a discrete subgroup $\implies \mathcal{M} = \mathbb{H}^n / \Gamma$ is a  

*hyperbolic n-orbifold.*

$\mathcal{M}$ is a manifold $\iff \Gamma$ is torsion free.
Volume in hyperbolic geometry

$\mathcal{H}^n$ – the *hyperbolic* $n$-space
(e.g. the upper half space with the hyperbolic metric $ds^2 = \frac{dw^2}{y^2}$).

Isom($\mathcal{H}^n$) – the *group of isometries* of $\mathcal{H}^n$.

$\Gamma < $ Isom($\mathcal{H}^n$), a discrete subgroup $\implies \mathcal{M} = \mathcal{H}^n/\Gamma$ is a

*hyperbolic* $n$-*orbifold*.

$\mathcal{M}$ is a manifold $\iff$ $\Gamma$ is torsion free.

We will discuss *finite volume* hyperbolic $n$-manifolds and orbifolds.
Volume in hyperbolic geometry

For $n$ even:

\[ \text{Vol}(\mathcal{M}) = \frac{\text{Vol}(S^n)}{2} \cdot (-1)^{n/2} \chi(\mathcal{M}) \]  
(Chern–Gauss–Bonnet Theorem)
Volume in hyperbolic geometry

For $n$ even:

$$\text{Vol}(\mathcal{M}) = \frac{\text{Vol}(S^n)}{2} \cdot (-1)^{n/2} \chi(\mathcal{M}) \quad \text{(Chern–Gauss–Bonnet Theorem)}$$

For $n \geq 3$ finite volume hyperbolic $n$-orbifolds are rigid
(Mostow–Prasad rigidity) $\implies$ *volume is a topological invariant.*
Volume in hyperbolic geometry

For $n$ even:

$$\text{Vol}(\mathcal{M}) = \frac{\text{Vol}(S^n)}{2} \cdot (-1)^{n/2} \chi(\mathcal{M}) \quad (\text{Chern–Gauss–Bonnet Theorem})$$

For $n \geq 3$ finite volume hyperbolic $n$-orbifolds are rigid
(Mostow–Prasad rigidity) $\implies$ volume is a topological invariant.

If $\mathcal{M}$ is an oriented connected hyperbolic $n$-manifold,

$$\text{Vol}(\mathcal{M}) = v_n \|\mathcal{M}\| \quad (\text{Gromov–Thurston})$$

$\implies$ volume is a measure of complexity.
Volume in hyperbolic geometry

(Callahan–Dean–Weeks’ 1999)
Problem 23. (Thurston, Bull. AMS, 1982) Show that volumes of hyperbolic 3-manifolds are not all rationally related.
Problem 23. (Thurston, Bull. AMS, 1982) Show that volumes of hyperbolic 3-manifolds are not all rationally related.

For even $n$ the volumes are rationally related by the Gauss–Bonnet theorem.
Volume in hyperbolic geometry

**Problem 23.** (Thurston, Bull. AMS, 1982) Show that volumes of hyperbolic 3-manifolds are not all rationally related.

For even \( n \) the volumes are *rationally related* by the Gauss–Bonnet theorem.

The problem (restricted to arithmetic manifolds) is connected with difficult open problems in number theory about rational independence of certain Dedekind \( \zeta \)-values.
Volume in hyperbolic geometry

**Minimal Volume Problem.** Show that the volume of a hyperbolic \( n \)-orbifold is bounded below and find the minimal volume \( n \)-orbifolds and manifolds.

(2039) (Siegel, 1945) Raised the problem and solved it for \( n = 2 \).

(2039) (Kazhdan–Margulis, 1968) Proved the existence of the lower bound in general.

(2039) (B., B.–Emery) Minimal volume arithmetic hyperbolic \( n \)-orbifolds for \( n \geq 4 \).
Volume in hyperbolic geometry

**Minimal Volume Problem.** Show that the volume of a hyperbolic $n$-orbifold is bounded below and find the minimal volume $n$-orbifolds and manifolds.

- **(Siegel, 1945)** Raised the problem and solved it for $n = 2$. 
Volume in hyperbolic geometry

**Minimal Volume Problem.** Show that the volume of a hyperbolic $n$-orbifold is bounded below and find the minimal volume $n$-orbifolds and manifolds.

- *(Siegel, 1945)* Raised the problem and solved it for $n = 2$.

- *(Kazhdan–Margulis, 1968)* Proved the existence of the lower bound in general.
Volume in hyperbolic geometry

**Minimal Volume Problem.** Show that the volume of a hyperbolic $n$-orbifold is bounded below and find the minimal volume $n$-orbifolds and manifolds.

- **(Siegel, 1945)** Raised the problem and solved it for $n = 2$.

- **(Kazhdan–Margulis, 1968)** Proved the existence of the lower bound in general.

- **(B., B.–Emery)** Minimal volume *arithmetic* hyperbolic $n$-orbifolds for $n \geq 4$. 
Arithmeticity and volume: Example

$\mathcal{H}^2$ – the hyperbolic plane with the Poincaré metric.

$\text{Isom}^+(\mathcal{H}^2) = \text{PSL}(2, \mathbb{R})$.

$\Gamma = \text{PSL}(2, \mathbb{Z}) < \text{PSL}(2, \mathbb{R})$, a discrete subgroup.

$\Gamma$ acts on hyperbolic plane with $\mathcal{O} = \mathcal{H}^2 / \Gamma$. 

$\text{Vol}(\mathcal{O}) = \int\int F \, dx \, dy = -2\pi \chi(\mathcal{O}) = \frac{4\pi}{\pi^3} |\zeta(-1)| = \frac{\pi^3}{3}$. 

Arithmeticity and volume: Example

$\mathcal{H}^2$ – the hyperbolic plane with the Poincaré metric.

$\text{Isom}^+(\mathcal{H}^2) = \text{PSL}(2, \mathbb{R})$.

$\Gamma = \text{PSL}(2, \mathbb{Z}) < \text{PSL}(2, \mathbb{R})$, a discrete subgroup.

$\Gamma$ acts on hyperbolic plane with $\mathcal{O} = \mathcal{H}^2 / \Gamma$. 

\[ \text{Vol}(\mathcal{O}) = \int\int_{\mathcal{F}} dxdy = -2\pi \chi(\mathcal{O}) = \frac{1}{\pi} \prod_{\text{primes}} p^{3} \# \text{PSL}(2, \mathbb{F}_p) = 4\pi |\zeta(-1)| = \pi^3. \]
Arithmeticity and volume: Example

$\mathcal{H}^2$ – the hyperbolic plane with the Poincaré metric.

$\text{Isom}^+(\mathcal{H}^2) = \text{PSL}(2, \mathbb{R})$.

$\Gamma = \text{PSL}(2, \mathbb{Z}) < \text{PSL}(2, \mathbb{R})$, a discrete subgroup.

$\Gamma$ acts on hyperbolic plane with $\mathcal{O} = \mathcal{H}^2 / \Gamma$.

$\text{Vol}(\mathcal{O}) = \int\int_{\mathcal{F}} \frac{dx\,dy}{y^2} = -2\pi \chi(\mathcal{O})$

$$= \frac{1}{\pi} \prod_{\text{primes}} \frac{p^3}{\#\text{PSL}_2(\mathbb{F}_p)} = 4\pi |\zeta(-1)| = \frac{\pi}{3}. $$
Arithmeticity and volume: Definitions

Let $G$ be an algebraic group defined over a number field $k$.

Let $P = (P_v)_{v \in V_f}$ a collection of parahoric subgroups $P_v \subset G(k_v)$, where $v$ runs through all finite places of $k$ and $k_v$ denotes the non-archimedean completion of the field. The family $P$ is called **coherent** if $\prod_{v \in V_f} P_v$ is an open subgroup of the finite adele group $G(\mathbb{A}_f(k))$. The group

$$\Lambda = G(k) \cap \prod_{v \in V_f} P_v$$

is called the **principal arithmetic subgroup** of $G(k)$ associated to $P$. 

Example. $\text{SL}_n(\mathbb{Z}) = \text{SL}_n(\mathbb{Q}) \cap \prod_{p \text{ prime}} \text{SL}_n(\mathbb{Z}_p)$. Every maximal arithmetic subgroup is a normalizer of a principal arithmetic subgroup.
Arithmeticity and volume: Definitions

Let $G$ be an algebraic group defined over a number field $k$.

Let $P = (P_v)_{v \in V_f}$ a collection of parahoric subgroups $P_v \subset G(k_v)$, where $v$ runs through all finite places of $k$ and $k_v$ denotes the non-archimedean completion of the field. The family $P$ is called coherent if $\prod_{v \in V_f} P_v$ is an open subgroup of the finite adèle group $G(\mathbb{A}_f(k))$. The group

$$\Lambda = G(k) \cap \prod_{v \in V_f} P_v$$

is called the principal arithmetic subgroup of $G(k)$ associated to $P$.

Example. $\text{SL}_n(\mathbb{Z}) = \text{SL}_n(\mathbb{Q}) \cap \prod_{p \text{ prime}} \text{SL}_n(\mathbb{Z}_p)$,
**Arithmeticity and volume: Definitions**

Let $G$ be an algebraic group defined over a number field $k$.

Let $P = (P_v)_{v \in V_f}$ a collection of parahoric subgroups $P_v \subset G(k_v)$, where $v$ runs through all finite places of $k$ and $k_v$ denotes the non-archimedean completion of the field. The family $P$ is called **coherent** if $\prod_{v \in V_f} P_v$ is an open subgroup of the finite adèle group $G(\mathbb{A}_f(k))$. The group

$$\Lambda = G(k) \cap \prod_{v \in V_f} P_v$$

is called the **principal arithmetic subgroup** of $G(k)$ associated to $P$.

**Example.** $SL_n(\mathbb{Z}) = SL_n(\mathbb{Q}) \cap \prod_{p \text{ prime}} SL_n(\mathbb{Z}_p)$.

Every **maximal** arithmetic subgroup is a normalizer of a principal arithmetic subgroup.
If $\Gamma_1 < \Gamma_0$, then

$$\mathcal{O}_1 = \mathcal{H}^n / \Gamma_1$$

$$\mathcal{O}_0 = \mathcal{H}^n / \Gamma_0$$

Corollary. Minimal volume orbifolds correspond to maximal discrete subgroups.
Groups versus covers

If $\Gamma_1 < \Gamma_0$, then

\[ O_1 = \mathbb{H}^n / \Gamma_1 \]

\[ \downarrow \text{cover} \]

\[ O_0 = \mathbb{H}^n / \Gamma_0 \]

**Corollary.** Minimal volume orbifolds correspond to maximal discrete subgroups.
Arithmeticity and volume

Borel–Harish-Chandra Theorem. Arithmetic subgroups are discrete and have finite covolume.
Arithmeticity and volume

Borel–Harish-Chandra Theorem. Arithmetic subgroups are discrete and have finite covolume.

The volume of $G/\Gamma$ can be computed using volume formulas:
Arithmeticity and volume

**Borel–Harish-Chandra Theorem.** Arithmetic subgroups are discrete and have finite covolume.

The volume of $G/\Gamma$ can be computed using *volume formulas*:

- G. Harder, *A Gauss–Bonnet formula for discrete arithmetically defined groups* (Ann. Sci. École Norm. Sup., 1971)
- A. Borel, *Commensurability classes and volumes of hyperbolic 3-manifolds* (Ann. Scuola Norm. Sup. Pisa Cl. Sci., 1981)
- *G. Prasad, Volumes of S-arithmetic quotients of semi-simple groups* (Inst. Hautes Études Sci. Publ. Math., 1989)
- B. Gross, *On the motive of a reductive group* (Invent. Math., 1997)
Arithmeticity and volume

Borel–Harish-Chandra Theorem. Arithmetic subgroups are discrete and have finite covolume.

The volume of $G/\Gamma$ can be computed using volume formulas:

$$\mathcal{O} = \mathcal{H}^2 / \text{PSL}(2, \mathbb{Z})$$

$$\text{Vol}(\mathcal{O}) = \frac{1}{\pi} \prod_{\text{primes}} \frac{p^3}{\#\text{PSL}_2(\mathbb{F}_p)} = 4\pi |\zeta(-1)|$$
Results about minimal volume

$H = \text{PO}(n, 1)^\circ = \text{Isom}^+(\mathcal{H}^n)$
Results about minimal volume

\[ H = \text{PO}(n, 1)^\circ = \text{Isom}^+(\mathcal{H}^n) \]

**Theorem 1.** *(B.’2004, B.–Emery’2012)* For every dimension \( n \geq 4 \) there exists a **unique** cocompact arithmetic subgroup \( \Gamma_0^n < H \) of the smallest covolume. It is defined over \( k_0 = \mathbb{Q}[\sqrt{5}] \) and has

\[ \text{Vol}(\mathcal{H}^n/\Gamma_0^n) = \omega_c(n). \]
Results about minimal volume

\[ H = \text{PO}(n,1)^\circ = \text{Isom}^+(\mathbb{H}^n) \]

**Theorem 1. (B.’2004, B.–Emery’2012)** For every dimension \( n \geq 4 \) there exists a unique cocompact arithmetic subgroup \( \Gamma^n_0 < H \) of the smallest covolume. It is defined over \( k_0 = \mathbb{Q}[\sqrt{5}] \) and has

\[ \text{Vol}(\mathbb{H}^n/\Gamma^n_0) = \omega_c(n). \]

**Theorem 2. (B.’2004, B.–Emery’2012)** For every dimension \( n \geq 4 \) there exists a unique non-cocompact arithmetic subgroup \( \Gamma^n_1 < H \) of the smallest covolume. It is defined over \( k_1 = \mathbb{Q} \) and has

\[ \text{Vol}(\mathbb{H}^n/\Gamma^n_1) = \omega_{nc}(n). \]
\[ n = 2r, \text{ } r \text{ even:} \]
\[
\omega_c(n) = \frac{4 \cdot 5^{r^2 + r/2} \cdot (2\pi)^r}{(2r - 1)!!} \prod_{i=1}^{r} \frac{(2i - 1)!^2}{(2\pi)^{4i}} \zeta_k(2i); 
\]
\[ n = 2r, \text{ } r \text{ odd:} \]
\[
\omega_c(n) = \frac{2 \cdot 5^{r^2 + r/2} \cdot (2\pi)^r \cdot (4r - 1)}{(2r - 1)!!} \prod_{i=1}^{r} \frac{(2i - 1)!^2}{(2\pi)^{4i}} \zeta_k(2i); 
\]
\[ (B.'2004) \]
\[ n = 2r - 1: \]
\[
\omega_c(n) = \frac{5^{r^2 - r/2} \cdot 11^{r-1/2} \cdot (r - 1)!}{2^{2r-1} \pi^r} L_{\ell_0 | k_0}(r) \prod_{i=1}^{r-1} \frac{(2i - 1)!^2}{(2\pi)^{4i}} \zeta_k(2i), 
\]
\[ \text{where } k_0 = \mathbb{Q}[\sqrt{5}] \text{ and } l_0 \text{ is the quartic field with a defining polynomial } x^4 - x^3 + 2x - 1. \]
\[ (B.-Emery'2012) \]
\[ n = 2r, \text{ } r \text{ even}: \]
\[ \omega_c(n) = \frac{4 \cdot 5^{r^2+r/2} \cdot (2\pi)^r}{(2r - 1)!!} \prod_{i=1}^{r} \frac{(2i - 1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i); \]

\[ n = 2r, \text{ } r \text{ odd}: \]
\[ \omega_c(n) = \frac{2 \cdot 5^{r^2+r/2} \cdot (2\pi)^r \cdot (4r - 1)}{(2r - 1)!!} \prod_{i=1}^{r} \frac{(2i - 1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i); \]

\[ (B.'2004) \]

\[ n = 2r - 1: \]
\[ \omega_c(n) = \frac{5^{r^2-r/2} \cdot 11^{r-1/2} \cdot (r - 1)!}{2^{2r-1} \pi r} L_{l_0|k_0}(r) \prod_{i=1}^{r-1} \frac{(2i - 1)!^2}{(2\pi)^{4i}} \zeta_{k_0}(2i), \]

where \( k_0 = \mathbb{Q}[\sqrt{5}] \) and \( l_0 \) is the quartic field with a defining polynomial \( x^4 - x^3 + 2x - 1. \)

\[ (B.–Emery’2012) \]
\[ n = 2r, \ n \equiv 0, \ 1 \pmod{4}: \]
\[ \omega_{nc}(n) = \frac{4 \cdot (2\pi)^r}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \]

\[ n = 2r, \ n \equiv 2, \ 3 \pmod{4}: \]
\[ \omega_{nc}(n) = \frac{2 \cdot (2^r - 1) \cdot (2\pi)^r}{(2r-1)!!} \prod_{i=1}^{r} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \quad \text{(B.)} \]

\[ n = 2r - 1, \ r \text{ even}: \]
\[ \omega_{nc}(n) = \frac{3^{r-1/2}}{2^{r-1}} L_{\ell_1|\mathbb{Q}}(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i), \text{ where } \ell_1 = \mathbb{Q}[\sqrt{-3}]; \]

\[ n = 2r - 1, \ n \equiv 1 \pmod{4}: \]
\[ \omega_{nc}(n) = \frac{1}{2^{r-2}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \]

\[ n = 2r - 1, \ n \equiv 3 \pmod{4}: \]
\[ \omega_{nc}(n) = \frac{(2^r - 1)(2^{r-1} - 1)}{3 \cdot 2^{r-1}} \zeta(r) \prod_{i=1}^{r-1} \frac{(2i-1)!}{(2\pi)^{2i}} \zeta(2i); \quad \text{(B.–Emery)} \]
Proofs use

- Prasad’s volume formula
- Galois cohomology of algebraic groups
- Bruhat–Tits theory
- Bounds for discriminants and class numbers (Odlyzko bounds, Brauer–Siegel theorem, Zimmert’s bound for regulator)
Growth of minimal volume
The minimal volume compact/non-compact arithmetic hyperbolic n-orbifold in any dimension n is unique.

For $n \geq 2$ the compact arithmetic manifolds have $|\chi| > 2$ (in fact, this is true for all even $n \geq 6$—Emery'2014).

For $n \geq 5$ we have $\omega_c(n) > \omega(nc)(n)$ ("compact > open").
Corollaries

- The minimal volume compact/non-compact arithmetic hyperbolic $n$-orbifold in any dimension $n$ is unique.

- The values $\omega_c(n)$, $\omega_{nc}(n)$, and $\omega_c(n)/\omega_{nc}(n)$ grow super-exponentially.
Corollaries

- The minimal volume compact/non-compact arithmetic hyperbolic $n$-orbifold in any dimension $n$ is unique.

- The values $\omega_c(n)$, $\omega_{nc}(n)$, and $\omega_c(n)/\omega_{nc}(n)$ grow super-exponentially.

- For $n = 2r \geq 12$ the compact arithmetic manifolds have $|\chi| > 2$

  (in fact, this is true for all even $n \geq 6$ — Emery’2014).
Corollaries

- The minimal volume compact/non-compact arithmetic hyperbolic $n$-orbifold in any dimension $n$ is **unique**.

- The values $\omega_c(n)$, $\omega_{nc}(n)$, and $\omega_c(n)/\omega_{nc}(n)$ grow super-exponentially.

- For $n = 2r \geq 12$ the compact arithmetic manifolds have

  $$ |\chi| > 2 $$

  (in fact, this is true for all even $n \geq 6$ — Emery’2014).

- For $n \geq 5$ we have $\omega_c(n) > \omega_{nc}(n)$ ("compact > open").
**Conjecture.** (B.–Emery) Let $\mathcal{M}$ be a compact hyperbolic manifold of dimension $n \neq 3$. Then there exists a noncompact hyperbolic $n$-manifold $\mathcal{N}$ whose volume is smaller than the volume of $\mathcal{M}$. 

The conjecture is true for $n = 2$ – easy $n = 4$ – follows from Ratcliffe–Tschantz'2000 $n = 6$ – follows from Everitt–Ratcliffe-Tschantz'2012 arithmetic manifolds of dimension $n \geq 30$ (B.–Emery'2013)
**Conjecture.** (B.–Emery) Let $\mathcal{M}$ be a compact hyperbolic manifold of dimension $n \neq 3$. Then there exists a noncompact hyperbolic $n$-manifold $\mathcal{N}$ whose volume is smaller than the volume of $\mathcal{M}$.

The conjecture is *true* for

- $n = 2$ – easy
- $n = 4$ – follows from Ratcliffe–Tschantz’2000
- $n = 6$ – follows from Everitt–Ratcliffe-Tschantz’2012
- arithmetic manifolds of dimension $n \geq 30$ (B.–Emery’2013)
Lemma. (Margulis) For every dimension \( n \) there is a constant \( \mu = \mu_n > 0 \) such that for every discrete group \( \Gamma < \text{Isom}(\mathcal{H}^n) \) and every \( x \in \mathcal{H}^n \), the group

\[
\Gamma_\mu(x) = \langle \gamma \in \Gamma \mid \text{dist}(x, \gamma(x)) \leq \mu \rangle
\]

has an abelian subgroup of finite index.
Minimal volume without arithmeticity

**Lemma.** *(Margulis)* For every dimension $n$ there is a constant $\mu = \mu_n > 0$ such that for every discrete group $\Gamma < \text{Isom}(\mathcal{H}^n)$ and every $x \in \mathcal{H}^n$, the group

$$\Gamma_\mu(x) = \langle \gamma \in \Gamma \mid \text{dist}(x, \gamma(x)) \leq \mu \rangle$$

has an abelian subgroup of finite index.

**Theorem.** *(Gelander)* Given a hyperbolic $n$-orbifold $\mathcal{O}^n$, we have

$$\text{Vol}(\mathcal{O}^n) \geq \frac{2v(0.25\varepsilon)^2}{v(1.25\varepsilon)}, \quad \varepsilon = \min\{\frac{\mu_n}{10}, 1\}.$$
Lemma. (Margulis) For every dimension $n$ there is a constant $\mu = \mu_n > 0$ such that for every discrete group $\Gamma < \text{Isom}(\mathcal{H}^n)$ and every $x \in \mathcal{H}^n$, the group

$$\Gamma_\mu(x) = \langle \gamma \in \Gamma \mid \text{dist}(x, \gamma(x)) \leq \mu \rangle$$

has an abelian subgroup of finite index.

Theorem. (Gelander) Given a hyperbolic $n$-orbifold $\mathcal{O}^n$, we have

$$\text{Vol}(\mathcal{O}^n) \geq \frac{2v(0.25\varepsilon)^2}{v(1.25\varepsilon)}, \quad \varepsilon = \min\{\frac{\mu_n}{10}, 1\}.$$ 

Proposition. There exists a constant $C > 0$ such that $\mu_n \leq \frac{C}{\sqrt{n}}$. 
Corollary. The lower bound for the volume decreases super-exponentially with $n$. 
**Corollary.** The lower bound for the volume decreases super-exponentially with $n$.

**Remark.** The same is true for the bound of Adeboye–Wei obtained by quantifying the proof of the Kazhdan–Margulis Theorem.
Corollary. The lower bound for the volume decreases super-exponentially with $n$.

Remark. The same is true for the bound of Adeboye–Wei obtained by quantifying the proof of the Kazhdan–Margulis Theorem.

Conjecture. The minimal volume hyperbolic $n$-orbifold (manifold) is arithmetic.
**Corollary.** The lower bound for the volume decreases super-exponentially with $n$.

**Remark.** The same is true for the bound of Adeboye–Wei obtained by quantifying the proof of the Kazhdan–Margulis Theorem.

**Conjecture.** The minimal volume hyperbolic $n$-orbifold (manifold) is arithmetic.

It follows from the conjecture that we expect the minimal volume to grow super-exponentially but so far we can prove only super-exponentially decreasing bounds!
