Quantum teleportation of qudits by means of generalised quasi-Bell states of light

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Abstract

Quantum superpositions of coherent states are produced both in microwave and optical domains, and are considered realizations of the famous “Schrödinger cat” state. The recent progress shows an increase in the number of components and the number of modes involved. Our work creates a theoretical framework for treatment of multicomponent two-mode Schrödinger cat states. We consider a class of single-mode states, which are superpositions of \( N \) coherent states lying on a circle in the phase space. In this class we consider an orthonormal basis created by rotationally-invariant circular states (RICS). A two-mode extension of this basis is created by splitting a single-mode RICS on a balanced beam-splitter. We show that these states are generalizations of Bell states of two qubits to the case of \( N \)-level systems encoded into superpositions of coherent states on the circle, and we propose for them the name of generalized quasi-Bell states. We show that an exact probabilistic teleportation of an arbitrary superposition of coherent states on the circle (a qudit) is possible with such a state used as a shared resource. We calculate the probability of success for this type of teleportation and show that it approaches unity for the average number of photons in one component above \( N^2 \).

Keywords: Bell states, Entanglement, Qudits, Schrödinger cat, Teleportation

1. Introduction

Superpositions of two coherent states of an optical mode for several decades attract attention of the scientific community as optical realizations of the famous “Schrödinger cat” state of a quantum system \( \{ 1, 2, 3, 4, 5, 6, 7, 8, 9 \} \). There are two classes of these states with distinct properties and generation methods: (i) “even” and “odd” coherent states \( \{ 1, 2, 3, 4, 5, 6 \} \), and (ii) the Yurke-Stoler coherent state \( \{ 7, 8, 9 \} \). The states of the first class contain either even or odd number of photons and are generated for high-Q microwave cavity field by interaction with non-resonant Rydberg atoms \([3]\) or for traveling-wave optical field by photon subtraction from a squeezed state \([4, 5, 6]\). The states of the second class have a Poissonian distribution of photons and have been recently generated for microwave cavity field coupled to a superconducting qubit by nonlinear Kerr effect \([8, 9]\). The states of both classes have been extensively studied as models of decoherence \([2, 3, 10]\), sources of quantum instabilities \([11, 12]\) and resources for quantum computation \([13, 14, 15, 16]\).

Splitting these superpositions in two modes, one obtains entangled coherent states \( \{ 17, 18, 19, 20, 21, 22, 23 \} \). The states of the first class create in this way quasi-Bell states \([18]\), having the same structure as usual Bell states of two qubits, but with two non-orthogonal coherent states as basis for each mode. These states can be applied to quantum metrology \([21]\), quantum tomography \([22]\) and probabilistic quantum teleportation \([15, 20]\), which is a key element of coherent state quantum computation. While traveling-wave superpositions can be rather easily split by a beam-splitter, cavity fields require a more sophisticated technique, realized only recently for two coupled microwave cavities \([23]\).

Increasing the number \( N \) of coherent components in a single-mode superposition, one obtains “multiple component Schrödinger cats” \([24]\). The most attention has been attracted to coherent states placed equidistantly on a circle \( \{ 24, 25, 26 \} \), though other geometries are also possible \([27]\). Multicomponent extensions of the second class \([28]\) are known as “Kerr states” and can be generated by the same Kerr effect as their two-component variants, as has been demonstrated recently in the microwave domain \([8, 9]\). Production of multicomponent states of the first class has been proposed for cavity field \([26]\), but not yet reported. These states are highly important for studying decoherence \([29]\) and for application in quantum computation with qudits, quantum systems with the number of levels higher than two \([30, 31]\).

Splitting multiple component Schrödinger cat states in two modes, one naturally arrives at entangled coherent states of high dimension \([32, 33]\). The states of the first class create in this way the states which are generalizations of quasi-Bell states, and their structure has been recently analyzed in detail \([34]\). These states have been proposed for enhancing sensitivity in quantum metrology \([35]\). The states of the second class, entangled Kerr states, can be used for quantum teleportation of high dimensional systems \([32]\).

In this work we further develop the approach of Ref. \([34]\) and introduce explicitly entangled states of two optical modes, which we call “generalized quasi-Bell states” (GQBS). They are “quasi-Bell” because their basis vectors are not exactly orthogonal, the term having been coined in Ref. \([18]\) for the \( \mathbb{N} = 2 \)
case. At higher $N$ they generalise the quasi-Bell states to higher dimensions like generalised Bell states \cite{16,17} do for the Bell states. GQBS create a natural basis for quantum information processing with coherent states of light. In particular, we show that the protocol of probabilistic quantum teleportation works for these states much better than for the entangled Kerr states, for which it was originally suggested \cite{32}.

2. Generalized quasi-Bell states

We consider one mode of the electromagnetic field, for which we fix a set of $N$ coherent states $|\alpha_m\rangle$, $m = 0, 1, \ldots, N-1$, placed equidistantly on the circle of radius $|\alpha_0|$, i.e., $\alpha_m = \alpha_0 e^{-2\pi i m/N}$, where $\alpha_0$ is an arbitrary non-zero complex number (see Fig.1).

![Diagram](image)

Figure 1: Coherent states on the circle of radius $|\alpha_0|$. Each coherent state is represented by a circle of radius $\frac{1}{2}$, corresponding to the $\sigma$-area of its Wigner function.

We are interested in coherent superpositions of these states having equal weights and linear periodic relative phase:

$$|c_q\rangle = \frac{1}{\sqrt{g(q)}} \sum_{m=0}^{N-1} e^{i 2\pi m q/N} |\alpha_0 e^{-2\pi i m/N}\rangle,$$  

(1)

where $g(q)$ is the discrete Fourier transform,

$$\hat{g}(k) = \frac{1}{N} \sum_{m=0}^{N-1} g(m) e^{-2\pi i k m/N} = e^{-i |\alpha_0|^2} \sum_{k=0}^{\infty} \left| \frac{\alpha_0}{k + \frac{1}{N}} \right|^2,$$  

(2)

of the first column of the Gram matrix $G_{nm} = \langle \alpha_m | \alpha_n \rangle = g(m-n)$, defined as

$$g(m) = \exp \left[ |\alpha_0|^2 \left( e^{2\pi i m/N} - 1 \right) \right].$$  

(3)

Each state of the set $|\alpha_m\rangle$, $m = 0, 1, \ldots, N-1$ is produced from the previous one in this set by a rotation in phase space $|\alpha_m\rangle = U_N |\alpha_{m-1}\rangle$, described by the unitary operator

$$U_N = e^{-i 2\pi a \alpha_0 N},$$  

(4)

different eigenvalues, for which we let $0 \leq q \leq N - 1$. The corresponding eigenstates are given by Eq. (1):

$$U_N |c_q\rangle = e^{-i 2\pi q/N} |c_q\rangle.$$  

(5)

The last property allows us to call them rotationally-invariant circular states (RICS). They satisfy the orthonormality condition $\langle c_q | c_r \rangle = \delta_{q,r}$. Using the decomposition of a coherent state in the Fock basis, we rewrite the RICS, Eq. (1), in the form \cite{25}

$$|c_q\rangle = \frac{e^{-i \alpha_0 q/2}}{\sqrt{g(q)}} \sum_{l=0}^{\infty} \alpha_0^{q+lN} \sqrt{q + lN}! |q + lN\rangle,$$  

(6)

where $|q + lN\rangle$ is a Fock state with $q + lN$ photons. Eq. (6) shows that a RICS is a sum of Fock states with the number $m$ of photons such, that $(m \mod N) = q$. In the case of $N = 2$ this property is reduced to a fixed parity of the photon number, peculiar to “even” and “odd” coherent states. In general, RICS are highly nonclassical, their distance from the set of classical states \cite{38} is lower-bounded by $\sqrt{1 + 2(m-1)} - 1$, where $\langle m \rangle = \langle c_q | a^\dagger a | c_q \rangle$ is the mean photon number. Thus, with growing number of photons these states become harder to create and control. However, as we show, even for low values of the mean photon number they can be used for some protocols of quantum information processing.

The states, Eq. (6) have been recently considered in the case where all coherent states are almost orthogonal \cite{30}, which implies sufficiently high $|\alpha_0|$ and sufficiently low $N$. In this case they are referred to as “pseudo-number” states, while the coherent states $|\alpha_m\rangle$ as the corresponding “pseudo-phase” states. They create two complementary bases for an $N$-level system (qudit), encoded into a subspace of optical mode’s Hilbert space. Note, that the second basis is non-orthogonal for low values of $\alpha_0$.

Now we consider the RICS $|c_q\rangle$ with parameters $[N, \sqrt{g(q)}]$ at one input of a 50:50 beam-splitter with the vacuum at the other one. The state of the two output modes $A$ and $B$ of the beam-splitter is

$$|\Phi_{d0}\rangle_{AB} = \frac{1}{N} \sqrt{g(q)} \sum_{m=0}^{N-1} e^{i 2\pi m q/N} |\alpha_m\rangle_A |\alpha_m\rangle_B$$  

(7)

where $g(q) = \sum_{k=0}^{N-1} \sqrt{\hat{g}(k) \hat{g}(q-k)}$, where the second part represents a Schmidt decomposition with the Schmidt coefficients \cite{34}

$$\lambda_k(q) = \frac{\hat{g}(k) \hat{g}(q-k)}{\hat{g}(q)},$$  

(8)

and

$$\hat{g}(q) = \sum_{k=0}^{N-1} \hat{g}(k) \hat{g}(q-k) = \frac{1}{N} \sum_{m=0}^{N-1} g^2(m) e^{-2\pi i m q/N},$$  

(9)

from where the summation of the Schmidt coefficients to unity follows. Here and below all indexes and integer arguments are taken modulo $N$. 

\[2\]
This protocol was initially devised for an entangled Kerr state being an optical qudit, by the protocol proposed by van Enk [32].

Entangled two-mode states, GQBS, can be used for teleporting a superposition of states $|\alpha_m\rangle$ with arbitrary coefficients, known as “circular state” [34] and representing an optical qudit, by the protocol proposed by van Enk [32]. This protocol was initially devised for an entangled Kerr state $\rho_0$ with

$$|\Phi_{qp}\rangle_{AB} = \frac{1}{N} \sum_{m=0}^{N-1} \sum_{p=0}^{N-1} \alpha_{mp} |\alpha_m\rangle_A |\alpha_{m+p}\rangle_B$$

which is, on the one hand, an extension of the quasi-Bell states [18] to higher dimensions, and on the other hand, an extension of generalized Bell state [36,37] to non-orthogonal basis, and which we call GQBS. Two forms of this state, presented in Eq. (10), correspond to writing it in the non-orthogonal coherent basis with equal weights, or in the orthogonal RICS basis with non-equal weights.

Averaging out one mode one obtains the reduced density operator of the other one:

$$\rho_A(q) = \sum_{k=0}^{N-1} \lambda_k(q) |c_k\rangle_A \langle c_k|,$$

which tends to the completely undetermined qudit state $I/N$ in the limit of high $\alpha_0$. In this limit GQBS is maximally entangled, and its entanglement, defined as $E = -\text{Tr}[\rho_A \log_2 \rho_A]$, is $E = \log_2 N$. For lower values of $\alpha_0$ the reduced state, Eq. (11), is not the completely undetermined one and the GQBS is non-maximally entangled. Entanglement of the state $|\Phi_{qp}\rangle_{AB}$ is given in this case by the Shannon entropy of the Schmidt coefficients $\lambda_k(q)$

$$E(q) = -\sum_{k=0}^{N-1} \lambda_k(q) \log_2 \lambda_k(q),$$

and is a function of $q$ (but not $p$). Schmidt coefficients $\lambda_k(q)$ are shown in Fig. 2 for the case where each coherent state has on average just 1 photon.

The spread of coefficients in Fig. 2 increases with $q$, reaching its maximum at $q = N - 1$. Since the entanglement is given by the Shannon entropy of the Schmidt coefficients, it is minimal at $q = 0$ and maximal at $q = N - 1$. The same coefficients are shown in Fig. 3 for the case where each coherent state has on average 4 photons. The dependence of entanglement on $q$ is more complicated in this case. The maximum is reached for $q = 3$ and the corresponding state $|\Phi_{q3}\rangle$ for any $p$ is very close to a maximally entangled state, which is reflected in its entanglement 2.97, very close to $\log_2 N = 3$.

### 3. Teleportation of circular states of light

Quantum teleportation of optical states is a key quantum technique using an entangled state of two optical field as a resource [39,40]. Entangled two-mode states, GQBS, can be used for teleporting a superposition of states $|\alpha_m\rangle$ with arbitrary coefficients, known as “circular state” [34] and representing an optical qudit, by the protocol proposed by van Enk [32].
as a resource, but, as we will see, it works even better for a GQBS. Since a GQBS is in general non-maximally-entangled, the teleportation has a non-unit probability of success. Below we reproduce the description of the teleportation protocol from Ref. \[32\] with a replacement of the nonlocal state.

We suppose two parties, Alice and Bob, share a GQBS \( |\Phi_{00}\rangle_{AB} \) given by Eq. (17), with even number of components \( N = 2L \). Alice possesses in mode \( C \) a superposition of coherent states \( |\alpha_i\rangle \) with arbitrary coefficients \( Q_i \):

\[
|\psi\rangle_C = \sum_{i=0}^{N-1} Q_i |\alpha_i\rangle_C,
\]

that she wishes to teleport to Bob. Alice first uses beam splitters to make \( L = N/2 \) “dilated” copies of both the state to be teleported (ending up in modes \( C_k \) for \( k = 0, ..., L - 1 \)) and of her half of the entangled state (ending up in modes \( A_k \) for \( k = 0, ..., L - 1 \)) by the process

\[
|\alpha_m\rangle_{0L} \rightarrow |\alpha_m\rangle/\sqrt{L} |\alpha_m\rangle^\dagger_{L}.
\]

Then she applies the phase shift operator \( U^k_k \) to the mode \( A_k \) and, in order to perform her Bell measurement, subsequently combines the modes \( C_k \) and \( A_k \) on L 50:50 beam splitters. If we call the output modes \( G_k \) and \( H_k \) for \( k = 0, ..., L - 1 \), the resulting state is

\[
\frac{1}{\sqrt{\Omega(q)}} \sum_{m=0}^{N-1} e^{2\pi qm/N} \sum_{k=0}^{N-1} \frac{1}{\sqrt{L}} |\alpha_m + \alpha_{m+k}\rangle/\sqrt{L} H_k.
\]

Alice now performs photon-number measurements on all \( 2L = N \) output modes. She cannot find a nonzero number in every mode. But suppose she finds nonzero numbers of photons in all but one mode, say, mode \( H_k \). Then the only terms that survive the sums over \( m \) and \( l \) in Eq. (17) are those for which \( \alpha_i + \alpha_{m+k} = 0 \), that is \( e^{-i2\pi m/N} = e^{-i2\pi (m+k)/N} \) or \( m + k - l = \text{mod} N \). The state at Bob’s side reduces to

\[
|\psi\rangle_B = \sum_{k=0}^{N-1} e^{2\pi k(N-l)/N} Q_l |\alpha_l\rangle_{Ll-k+p}^\dagger_B.
\]

Alice communicates to Bob which mode contained no photons, and Bob then applies the appropriate unitary transformation. Here, with \( H_k \) being the empty mode, he applies \( U_N^{k-p-l} \) to his state to obtain

\[
|\psi\rangle_B = e^{-2\pi q(N-l)/N} \frac{1}{\sqrt{N}} \sum_{l=0}^{N-1} Q_l |\alpha_l\rangle_B.
\]

Since the states \( |\alpha_i\rangle \) are non-orthogonal, the phase factor under the sum cannot in the general case be removed by a unitary transformation. It means that for exact teleportation we need to choose \( q = 0 \) for the entangled state used as resource, which makes Bob’s state identical to Alice’s, \( |\psi\rangle_B = |\psi\rangle \). We see also, that the value of \( p \) does not affect the process of teleportation and is compensated at the final stage by the rotation operator, so that we can put \( p = 0 \) from the beginning, and consider the state \( |\Phi_{00}\rangle_{AB} \) as the optimal resource for quantum teleportation.

The analysis of the previous section (see also Ref. [34]) shows that the value of \( q = 0 \) does not always correspond to maximal entanglement for given \( N \) and \( \alpha_0 \), but employment of other GQBS for quantum teleportation is not possible if one requires an exact reproduction of the initial state. Occurrence of zero-photon measurement outcome in more than one mode leads to protocol failure, thus the described protocol is probabilistic.

The possibility of realizing an exact, although probabilistic, teleportation of an arbitrary circular state on the basis of GQBS is a serious advantage compared to the original teleportation scheme [32], where the final state contains phase factors quadratic in \( l \), which cannot be corrected by any unitary transformation.

4. Probability of success

Teleportation described in the previous section is probabilistic, it succeeds if a zero number of photons is found in one of the measured modes only. If zero photons is found in two or more modes, the protocol fails. The probability of success in general depends on the input state. Let us find this probability for the case where the state to be teleported is one of the states \( |\alpha_0\rangle \). Thanks to the rotational symmetry, we can choose this state as \( |\psi\rangle_C = |\alpha_0\rangle_C \) without loss of generality. The shared entangled state which we consider is \( |\Phi_{00}\rangle_{AB} \), the one providing an exact teleportation. For such a choice the multistate mode, Eq. (15), takes the form

\[
|\Psi\rangle = \frac{1}{\sqrt{\Omega(0)}} \sum_{m=0}^{N-1} |\alpha_m\rangle B \bigotimes_{l=0}^{L-1} \frac{|\alpha_l - \alpha_{m+k}\rangle}{\sqrt{N}} G_l \left| \alpha_l + \alpha_{m+k}\rangle H_k \right\rangle.
\]

The probability of obtaining zero photons in the mode \( H_k \) and non-zero number of photons in the other \( N - 1 \) measured modes is given by the average \( \langle |\Psi\rangle |\Gamma_{H_k}\rangle |\Psi_m\rangle \) of the projector

\[
\Gamma_{H_k} = \prod_{l=0}^{L-1} \Gamma_{H_k} \prod_{k=0}^{N-1} \Gamma_{H_k - l},
\]

where \( |\Pi\rangle = |0\rangle |0\rangle \) is the projector on the vacuum, and \( \Pi = I - |0\rangle |0\rangle \) is the projector on the non-vacuum state of the mode, whose label is indicated by the lower index. If we denote the summand in the right hand side of Eq. (18) by \( |\Psi_m\rangle \), then it is easy to see that only the state \( |\Psi_{L-k}\rangle \) makes a non-zero input to \( \langle |\Psi|_{H_k}\rangle |\Psi_m\rangle \). Indeed, each other state has vacuum in a mode different from \( H_k \) and gives zero when averaged with the projector \( \Pi \) of this mode. Thanks to the symmetry of all \( N \) modes the total success probability is \( N \) times the probability of the successful outcome in one mode. Finally we find

\[
P_{\text{success}} = \frac{N \langle |\Psi_{L-k}\rangle |\Gamma_{H_k}| |\Psi_{L-k}\rangle \rangle}{N \langle \Omega(0) \rangle} = \frac{1}{N \langle \Omega(0) \rangle} \prod_{l=1}^{N-1} (1 - e^{-\hbar_{00}q_N^l} \langle \Omega(0) \rangle).
\]

This probability is shown in Fig. [4] for different values of \( N \). We see that a practical value 0.2 of probability can be reached
for the case of 4 components for $|α_0| \approx 1.15$, available in the optical domain [3, 4], and for the case of 8 components for $|α_0| \approx 3.1$, available in the microwave domain [5].

\begin{align*}
|ϕ⟩ &= \sqrt{N} \sum_{q=0}^{N-1} e^{-i2πq|ϕ⟩} |ϕ⟩ \\
&= \sum_{m=0}^{N-1} \delta_{mk} |ϕ⟩ e^{-i2πm|ϕ⟩/N},
\end{align*}

(21)

where

\begin{align*}
\delta_{mk} &= \frac{1}{N^{3/2}} \sum_{q=0}^{N-1} \frac{1}{\sqrt{g(q)}} e^{i2π(m-k)q/N},
\end{align*}

(22)

is an almost diagonal matrix, close to the Kronecker delta $δ_{mk}$, with which it coincides at high $|α_0|$, where $g(q) \to 1/N$. The states $|ϕ⟩$ are mutually orthogonal $⟨ϕ|ϕ⟩ = δ_{ii}$ and create a mutually unbiased basis with RICS: $⟨ϕ_j|ϕ⟩ = 1/N$. Using these states as basis, we can construct the true generalised Bell states [6, 7] for two optical modes

\begin{align*}
|ϕ⟩_{AB} &= \frac{1}{\sqrt{N}} \sum_{m=0}^{N-1} e^{i2πq(ϕ−m)} |ϕ⟩_A |ϕ⟩_B \\
&= \frac{1}{\sqrt{N}} \sum_{k=0}^{N-1} e^{i2π(q−k)/N} |ϕ⟩_A |ϕ⟩_B.
\end{align*}

(23)

which can be used for standard deterministic teleportation of qudits. However, the way of generation of the state Eq. (23) is not clear, while a GQBS can be produced by beam-splitting a multicomponent cat state, as shown by Eq. (7).

\section{5. Pseudo-phase state}

In the regime where all coherent states in the superposition can be considered mutually orthogonal the set of these states is almost orthogonal and the teleportation becomes almost deterministic. However, with the growing number of components $N$, the coherent amplitude where the probability approaches 1 grows approximately as $|α_0| \approx N$.

In a similar way we can find the success probability of the original protocol of Ref. [32]. Expression for this probability coincides with Eq. (20) if we omit the normalisation factor before the product. However, since at $|α_0| > 1$ we have $N_{B1}(0) \approx 1$, for all practical values of the amplitude this probability is very close to that of the protocol based on GQBS.

\section{6. Conclusion}

We have introduced a class of entangled states of two optical modes, GQBS, which are direct generalizations of Bell states of two qubits to the case of $N$-level systems encoded into superpositions of coherent states on the circle. We have shown that these states can be written in orthogonal bases where they have non-uniform coefficients. We have shown that an exact probabilistic teleportation of circular states of optical mode is possible on the basis of these states and the GQBS $|ϕ⟩_{AB}$ is the optimal resource for this purpose. The probability of teleportation success is reasonably high for currently available sizes of cat states in the optical and microwave domains, which allows us to hope for a possible experimental realisation of the proposed teleportation protocol. We have also discussed the true generalised Bell states of two optical modes and shown that they are related to nonclassical pseudo-phase states. The obtained results can be useful for various schemes of quantum metrology and quantum information processing on the basis of information encoding in superpositions of coherent states of light.

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\section{References}

[1] V. V. Dodonov, I. A. Malkin, M. V. I., Even and odd coherent states and excitations of a singular oscillator, Physica (Utrecht) 72 (1992) 597–618.

[2] V. Bužek, A. Vidiella-Barranco, P. L. Knight, Superpositions of coherent states: Squeezing and dissipation, Phys. Rev. A 45 (1992) 6570–6585. doi:10.1103/PhysRevA.45.6570

[3] M. Brune, E. Hagley, J. Dreyer, X. Maitre, A. Maali, C. Wunderlich, J. M. Raimond, S. Haroche, Observing the progressive decoherence of the “meter” in a quantum measurement, Phys. Rev. Lett. 77 (1996) 4887–4890. doi:10.1103/PhysRevLett.77.4887

[4] A. Ourjoumtsev, R. Tualle-Brouri, J. Laurat, P. Grangier, Generating optical Schrödinger kittens for quantum information processing, Science 312 (2006) 83–86. doi:10.1126/science.1122858
