Disorder chaos in spin glasses

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Abstract. – We investigate numerically disorder chaos in spin glasses, i.e. the sensitivity of the ground state to small changes of the random couplings. Our study focuses on the Edwards-Anderson model in $d = 1, 2, 3$ and in mean-field. We find that in all cases, simple scaling laws, involving the size of the system and the strength of the perturbation, are obeyed. We characterize in detail the distribution of overlap between ground states and the geometrical properties of flipped spin clusters in both the weak and strong chaos regime. The possible relevance of these results to temperature chaos is discussed.

One of the most spectacular theoretical prediction about the glassy phase of disordered systems is its generic fragility to perturbations, in particular to temperature changes. ‘Temperature Chaos’ (TC), i.e, the chaotic change of the thermodynamically dominant configurations when temperature is slightly modified, has been first proposed in the context of the scaling theory of spin-glasses [1–4], and later extended to other disordered systems, such as pinned elastic objects [5]. Although the theoretical situation is well established in the latter, simpler case [6–8], the very existence of TC in 3d spin glasses is still a subject of controversy [9–12], in particular because it has never been directly observed in (static or dynamic) simulations. It is nevertheless an extremely acute issue, since TC could be relevant to interpret the spectacular rejuvenation and memory effects experimentally observed in spin-glasses [13–15] and in a host of other materials as well. Arguments for and against the relevance of TC scenario have been put forward in [16–21]. One reason for the long standing debate is that if TC occurs at all in spin-glasses, it does so on very large length scales, much larger than those accessible to present computers [10, 11]. However, it has been argued that TC can manifest itself even on small length scales, but only for rare regions of space [6]. This \textit{weak chaos regime} has in fact been argued to account in a quantitative way for experimental results [20]. It is therefore particularly important to clarify the chaotic behavior of spin glasses in that regime. Our strategy is to study a similar, but much stronger effect [22]: the chaotic dependence of the spin-glass ground state when the \textit{couplings} between the spins are slightly modified. This is referred to as disorder chaos (DC). Renormalization group arguments suggest that the two effects are
in fact deeply related and characterized by the same universal scaling function \([3, 5–7]\); only prefactors differ, to make the chaos length scale much larger in the TC case. In this letter, we follow early studies in \(2d\) \([23, 24]\), and study numerically DC (at zero temperature) in finite dimensional and mean field spin-glasses. A precise characterization of both weak and strong chaos regime is obtained, even using moderately small system sizes. We discuss in detail the geometrical aspects of DC, and show that the predictions of the scaling theory are in agreement with our numerical results, although some of the assumptions of the droplet theory must probably be revised, a conclusion already advocated in previous papers \([25,26]\). If TC in spin-glasses is indeed in the same universality class as DC, our results could shed light on the physics at play when the temperature is slightly changed in experiments. They also provide a useful guide to interpret numerical simulations on temperature chaos, which appear to be confined to the weak chaos regime.

\textit{Models and observables. –} We work with the Edwards-Anderson (EA) Hamiltonian

\[
H = \sum_{i,j} J_{ij} S_i S_j
\]

in finite dimension \(d = 2\) and \(d = 3\) with \(N = L^d\) Ising spins, and use periodic boundary conditions. We also study Ising spin glasses on a random graph (or Bethe Lattice) of fixed connectivity \(z = 3\). This system is known to behave like mean field models \([27]\) and in particular to display Replica Symmetry Breaking \([28]\). In order to study disorder chaos, we consider two copies of the system \(\{S^1\}, \{S^2\}\) and we modify the couplings \(J_{ij}\) in copy 2 as:

\[
J_{ij} \to J'_{ij} = \frac{J_{ij} + x_{ij} \Delta J}{\sqrt{1 + \Delta J^2}},
\]

where \(x_{ij}\) is a Gaussian random variable with zero mean and unit variance. With this definition, the original and the modified couplings share the same distribution. We compute the ground states of both systems using an optimization method called the Genetic Renormalization Algorithm \([29]\). We were able to compute 2000 instances for different \(\Delta J\)s up to \(L = 60\) in \(2d\), \(L = 10\) in \(3d\) and \(N = 448\) on random graphs. We will concentrate our analysis on the following observables:

\[
q = \left| \sum_{i=0}^{N} S_i^1 S_i^2 \right| \quad \text{and} \quad C(r, L) = \sum_{i=0}^{N} S_i^1 S_{i+r}^1 S_i^2 S_{i+r}^2,
\]

where \(q\) is the absolute value of the spin overlap between replicas and \(C\) is the four-point, two-replicas correlation function. The correlation function \(C\) therefore measures the similarity between the relative orientation of spins a distance \(r\) apart in the original and perturbed system. We will consider quantities averaged over disorder: \(\langle C(r, L) \rangle\) and \(\langle q(L) \rangle\), as well as the distribution of overlaps \(P_{L,\Delta J}(q)\).

\textit{Scaling predictions. –} Following the early arguments of Bray and Moore \([3]\), one can derive several scaling predictions. Consider the original system with unperturbed bonds; its ground state configuration(s) are \(C_1\) (and \(-C_1\)). According to the droplet theory, the low lying excitations are obtained by flipping connected, compact clusters of spins (droplets). A droplet of size \(\ell\) has a fractal surface of dimension \(d_s < d\), and its excitation energy \(E > 0\) is distributed as \(P(E, \ell) = \ell^{-\theta} \rho(E \ell^{-\theta})\), where \(\rho(x)\) is a scaling function assumed to be non zero at \(x = 0\) and which decays to zero for large \(x\). The energy exponent \(\theta\) is argued on general grounds to be such that \(0 < \theta \leq d_s/2\). Such a scaling law means that typical droplets of
size \( \xi \) have an energy that grows as \( \ell^\theta \), but that it is still possible to find rare excitations of size \( \xi \) with energy \( O(1) \). The probability to find such excitations however decays with \( \xi \) as \( 1/\ell^\theta \). Consider now a droplet of size \( \xi \) in this system, corresponding to a configuration \( C_2 \). The bonds are changed the excess energy of the droplet also changes. Obviously, this energy comes only from the bonds that differ in \( C_1 \) and \( C_2 \), and therefore only from the bonds which are on the surface of the droplet, the number of which is \( \propto \ell^d \). The contribution to the energy of a random perturbation of \( O(\Delta J) \) is thus the sum of \( \propto \ell^d \) independent random variables with random signs. i.e. a term of order \( \pm \Delta J \ell^d/\ell^{d/2} \). If the energy gained by \( C_2 \) is larger than its original excitation energy \( \ell^\theta \), then \( C_2 \) becomes the new ground state of the system. If \( \theta < \ell_s/2 \), this surely happens for large enough droplets, beyond the overlap length defined as:

\[
\ell_c(\Delta J) = \frac{1}{\Delta J^{1/\xi}} \quad \text{with} \quad \xi = \frac{d_s}{2} - \theta.
\]

When considering a system-size droplet (i.e. \( \ell = L \)), this argument suggests that the overlap in Eq.\( \text{(6)} \) obeys the following scaling law:

\[
\langle q(\Delta J, L) \rangle = F(L/\ell_c) = F(\Delta J^{1/\xi} L),
\]

where \( F(x) \) is a certain scaling function that we now discuss in both the strong and weak chaos limit. For small sizes or for small \( \Delta J \)'s, such that \( L \ll \ell_c \), the perturbation induced by the change in couplings is quite small so that typical droplets do not flip. However, there is still a non zero probability to find a large excitation with exceptionally low excess energy \([6,14]\). The probability to find a rare droplet of size \( L \), with an energy less than \( \Delta J L^{d_s/2} \) is \( p(\Delta J, L) \propto L^{-\theta} \int_0^{\Delta J L^{d_s/2}} \rho(EL^{-\theta})dE \sim \rho(0)\Delta J L^\xi \) (if \( \rho(0) \) is finite). If the ground state of the perturbed sample has \( \propto L^d \) spins flipped with respect to \( C_1 \) with probability \( p \), then \( 1 - \langle q \rangle \propto \rho(0)\Delta J L^\xi \) in the weak chaos limit. In the opposite regime (large size or large \( \Delta J \)), where \( L \gg \ell_c \), \( C_1 \) and \( C_2 \) should become almost independent at scales larger than \( \ell_c \), and the residual overlap should be of order \( (\ell_c/L)^{d/2} \). Therefore, one expects: \( F(x) \approx 1 - ax^\xi \) for \( x \ll 1 \) and \( F(x) \approx b/x^{d/2} \) for \( x \gg 1 \), where \( a \) and \( b \) are constants of order unity. This argument, as we show below, can in fact be generalized to estimate the full distribution of overlap \( P_bL,\Delta J(q) \), and not only its mean value.

**Numerical data.** Before turning to a numerical test of these predictions, we first review existing results on DC. The 1d chain can be exactly solved \([3]\), and the 2d model was extensively studied by numerical simulations \([23,24]\). In both cases, Eq.\( \text{(6)} \) is obeyed. There also exist very convincing Monte Carlo studies in 4d for both \( T = T_c \) and \( T < T_c \) \([22]\); these again show that DC is well described by Eq.\( \text{(6)} \). There are however no test in 3d nor in mean field (MF), nor systematic investigation of the shape of the scaling function \( F(x) \). Furthermore, nothing is known about the shape of \( P(q) \). We show our results for \( \langle q \rangle \) in Fig.\( \text{4} \) for 2d, 3d and MF. The scaling relations \( \text{(6)} \) works perfectly in all three cases, including MF. The values found for \( \xi \) are also in good agreement with known results for \( \theta \) and \( \ell_s \) in the three models. In particular: 1) It is widely accepted that the 2d model is described by the droplet theory with \( \theta \approx -0.29 \) and \( \ell_s \approx 1.3 \) \([30]\), which agree very well with the value \( \xi = 1 \) that gives the best collapse of our data in Fig.\( \text{4} \). 2) Mean field systems can also, in some sense, be described by a droplet theory with \( d = d_s \) and \( \theta = 0 \) \([31]\), so that, using \( N = L^d \), the scaling variable becomes \( \Delta J \sqrt{N} \). This is also working very well, demonstrating that, although the structure of mean field models is very complex \([28]\), simple scaling arguments are sometimes enough to understand its behavior \([10,31]\) (note that our definition \( \xi \) slightly differs in mean field and finite dimensional systems). 3) In 3d, the best collapse is obtained for \( \xi \approx 1.3 \). The standard
We find, as expected, $\langle q \rangle \propto \sqrt{\ell^d c/N}$ in the strong chaos regime and $1 - \langle q \rangle \propto \Delta J L^{1/\xi}$ in the weak chaos regime (see insets). Bottom right: Restricted overlap (Eq. (6)) for all models.

Droplet theory however gives $\theta \approx 0.2$ and $d_s \approx 2.6$, which would lead to $\xi_{dr} \approx 1.1$. Mean-field theory, on the other hand, predicts $\theta = 0$ and $d_s = d = 3$ so that $\xi_{mf} = 1.5$. Both theories fail to reproduce the value of $\xi$ favored by our data (at least for the small sizes we are able to deal with). However, for these small lattice sizes, the 3d EA model is best described by an intermediate, so-called TNT scenario [25], where $\theta \approx 0$ [25, 26] and $d_s \approx 2.6$ [25]. This is perfectly consistent with the value $\xi \approx 1.3$ found in Fig. 1. Fig. 1 also shows, in various insets, that the asymptotic behavior of $F(x)$ surmised above is correct.

Overlap distribution, weak and strong chaos regime. – We would like to go beyond the analysis of the average overlap to get an understanding of the distribution of events and their geometry. It is particularly important to study directly the weak chaos limit since two scenarios are in principle possible: either a small perturbation typically causes a small rearrangement, i.e., with probability of order unity, $O(\Delta J L^{1/\xi})$ spins flip; or a small perturbation typically leads to no rearrangement at all, except in rare cases, where if flips $O(L^d)$ spins with probability $O(\Delta J L^{1/\xi})$. A useful quantity to consider in numerical work is the following restricted average overlap, where we exclude the samples where almost no change is induced:

$$\langle q \rangle_{q<0.9} = \frac{\int_{0.9}^{1} qP(q) dq}{\int_{0}^{0.9} P(q) dq}.$$  

(6)
If $P(q)$ is peaked around its average value, $\langle q \rangle_{q<0.9}$ should be close to 0.9 for small $\Delta J L$, and progressively decrease away from that value. If on the other hand, rare events are dominant, $\langle q \rangle_{q<0.9}$ should be significantly smaller than 0.9 and constant as long as $\Delta J L$ is less than unity. Furthermore, if rare events induce a complete reshuffling of the spin configuration, as was conjectured in [14], then $\langle q \rangle_{q<0.9} \approx 0$. Fig. 1 (bottom right) unambiguously shows that the second scenario holds: in the weak chaos limit, $\langle q \rangle_{q<0.9}$ is found to be constant for $\Delta J L$ small, and close to 0.57, very different both from 0.9 and from 0. The full distribution $P(q)$ is plotted in Fig. 2 for different values of $\Delta J$. It is clear that, although for infinitesimal chaos almost all samples lead to $q = 1$ and for large chaos to $q = 0$, the distribution in the weak chaos regime is not simply a two-peak function with weight around $q = 0$ and $q = 1$. This means that in the rare cases where the perturbation is relevant, the new ground state typically retains a significant “backbone” of the previous ground state. This is illustrated by the 1d chain case. In the weak chaos limit, only the weakest bond is broken by the perturbation, so the overlap $q$ is uniformly distributed in $[0,1]$ (in which case $\langle q \rangle_{q<0.9} = 0.45$, see Fig. 1). For $d > 1$, a droplet like argument predicts $P(q)$ as follows: assume that droplets of size $\ell$ and $\ell/b$ can be considered as independent for some $b > 1$, and denote $r = 1 - q$ the fraction of flipped spins. Defining $\tilde{P}(r,L)$ as $P(r,L)$ without the $\delta(r)$ part, one can establish a recursion equation of the form: $\tilde{P}(r,L) = p[f(r) + b^{d-\xi}\tilde{P}(b^d r,L/b)]$, where $f$ is a regular function. From this, one deduces that $P(r,L)$ must behave as $r^{-\mu}$, with $\mu = 1 - \xi/d$. Using the values of $\xi$ reported above, we find $\mu = 0$ in $d = 1$ (which is the exact result), and $\mu \approx 0.5$ for $d = 2,3$ and MF. This prediction is however not in agreement with our numerical data (see Fig. 2 insets), which rather suggests $\mu \approx 1$, i.e. a stronger divergence of $P(r,L)$ at small $r$. Our data therefore suggests an excess number of small droplets compared to large ones, possibly related to the findings of [26], where fractal low energy clusters (corresponding to small $r$’s) were identified. Along these lines, it is interesting to compare the system size clusters generated in the weak chaos regime to those obtained using totally different and more specific methods [25]. Studying the geometrical properties of the weak chaos droplets, we computed directly the interface fractal dimension $d_s$ and found, as expected, that $d_s(2d) \approx 1.3$ and $d_s(3d) \approx 2.6$. We also studied the topological properties of excitations in 3d and found that most of them are spongy, winding around the lattice. Therefore, the clusters that are flipped in the weak chaos regime are similar to the large-scale low energy excitations obtained in [25].
Correlation functions. – The numerical study of the behavior of spatial correlation function between the two ground states is made difficult because there are now three relevant lengths in the problem: \( L, r \) and \( \ell_c \) so that \( C(r, \Delta J, L) \) may have non trivial finite size effects (see for instance [23] for 2d data). However, we will show that the most important features of the correlation function can still be fairly well understood. Scaling arguments suggest that the following form should hold:

\[
C(r, \Delta J, L) = \tilde{C} \left( \frac{r}{L \ell_c} \right). \tag{7}
\]

Our data in Fig. 3 is perfectly compatible with this scaling. Given this success, one would like to go beyond Eq. (7) and propose a specific form for \( \tilde{C} \). A factorized form does not seem to hold; rather, we found that the following form leads to a satisfactory collapse of our data both in the weak and strong chaos regimes (see Fig. 3):

\[
\tilde{C}(u, v) \approx \exp \left[ -Av^\xi \left( 1 - e^{-Bu^\xi} \right) \right], \tag{8}
\]

where \( A \) and \( B \) are fitting constant. In the limit \( L \to \infty, u \to 0 \) and \( \tilde{C} \) becomes a function of \( w = r/\ell_c \) only, of the form \( \exp -ABw^\xi \), which fits well the data, in particular in the weak chaos regime where it predicts that \( 1 - C(r) \propto r^\xi \Delta J \) as expected from the above arguments applied to droplets of size \( r \). In the long distance regime, this form of \( \tilde{C} \) suggests a super-exponential decay of the correlation, in agreement with the results on Migdal-Kadanoff lattices [32].

Conclusion. – We have studied numerically disorder chaos in different Ising spin glass models finding that, although the 1d and 2d models have a spin glass phase only at \( T = 0 \), the 3d model has a finite temperature transition and the mean field model is described by a complex hierarchy of states, DC could be understood in term of simple droplet-like scaling laws. In particular, we find that the weak chaos regime is dominated by rare events where large droplets are overturned, as conjectured in [6,14,20], with however an anomalous proliferation of small droplets. We also find once again strong indications that the droplets are not compact objects. Our study provides precise predictions that can be tested even on small spin-glass samples, to check whether temperature chaos is or not in the same universality class as DC.
It would be particularly interesting to reanalyze in this spirit the existing data of [9] for 3d and mean field models. If the same scaling functions are obtained, this would constitute compelling evidence for the presence of a chaotic temperature dependence in spin glasses. It is however possible that more complicated scenarios hold, for example temperature chaos but with continuously varying exponents. As far as experiments are concerned, our finding of a proliferation of small overturned droplets could play an important role in the quantitative interpretation of rejuvenation and memory effects in spin-glasses.

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