In this paper the focus is set on a modified Chua’s circuit model equation with saw-tooth function in place of piecewise linear function of Chua’s circuit displaying multiscroll chaotic attractors. We study the characteristic properties of first passage times ($t_{FPT}$) to $n$th scroll chaotic attractor, residence times ($t_{RT}$) on a scroll attractor and returned times ($t_{RET}$) to the middle-scroll attractor. $t_{FPT}$ exhibit a series of Gaussian-like distribution followed by a long tail continuous distribution. $t_{RT}$ and $t_{RET}$ show completely discrete distribution. Power-law variation of mean values of $t_{FPT}$, $t_{RT}$ and $t_{RET}$ with a control parameter is found. On the other hand, mean values of $t_{FPT}$ and $t_{RET}$ have linear dependence with the number of scroll attractors for fixed values of the control parameter. For the system with infinite scroll chaotic attractors normal diffusive motion occurs. In the normal diffusion process the mean square displacement grows linearly with time.

Keywords: Modified Chua’s circuit equation; multiscroll attractors; first passage times; residence times; returned times; diffusion.
A great deal of interest has been focused on the circuit design, identification of different forms of the piecewise linear function $f(x)$ and shapes of various types of scroll attractors that are generated. It is also important to analyze statistical dynamics of multiscroll chaotic attractors and also use nonlinear circuit models to explore statistical properties associated with the dynamics exhibited by them. For example, in the multiscroll chaotic systems, because a trajectory escapes from one scroll attractor to nearby scroll attractors, the first passage time (FPT), residence time (RT) and returned time (RET) of scroll attractors are important. The focus of the present work is to study the features of these quantities and diffusion dynamics in a modified Chua’s circuit model equation with multiscroll chaotic attractors. The FPT ($t_{\text{FPT}}$) is the time when a stochastic process $x(t)$ started at $t = 0$ from a given initial value within a domain $\Delta$ of its state space crosses it for the first time. For systems exhibiting multiscroll chaotic attractors or escape dynamics from one part of an attractor to another or from one potential well to another, FPT and mean FPT are important and their study has practical applications in many problems [Redner, 2001]. RT ($t_{\text{RT}}$) of a scroll attractor is defined as the time duration spent by a trajectory on it before passing to another scroll attractor. RET ($t_{\text{RET}}$) of a scroll attractor is the time taken by the system’s trajectory to re-enter it.

We consider the modified Chua’s circuit system [Yu et al., 2007a, 2007b]

$$\begin{align*}
\dot{x} &= \alpha [y - f(x)], \\
\dot{y} &= x - y + z, \\
\dot{z} &= -\beta y,
\end{align*}$$

where

$$f(x) = \xi x - \Delta A \sum_{j=0}^{N-1} \left( \text{sgn}[x + (2j + 1)A] + \text{sgn}[x - (2j + 1)A] \right)$$

with $\alpha, \beta, \xi, A > 0$, $N \geq 1$ and

$$\text{sgn}[x] = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x = 0 \\
-1, & \text{if } x < 0,
\end{cases}$$

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with $\alpha, \beta, \xi, A > 0$, $N \geq 1$ and

$$\text{sgn}[x] = \begin{cases} 
1, & \text{if } x > 0 \\
0, & \text{if } x = 0 \\
-1, & \text{if } x < 0,
\end{cases}$$
capable of generating \((2N+1)\)-scroll attractors for a range of fixed values of the parameters \(\alpha\) and \(\beta\). In the present work, for system (1), we consider the distributions and mean values of FPTs (on the breakpoints \(x^*\)) and variations of parameters in Eq. (1) as \(N = 2, \xi = 0.25\) and \(A = 0.5\). The breakpoints of \(f(x)\) are indicated by painted circles.

The equilibrium points of the system (1) are given by
\[
(x_c, y_c, z_c) = (0, 0, 0), \quad (\pm m, 0, \mp m),
\]
where \(m = 1, 2, \ldots, N\). (3)

The stability determining eigenvalues of the equilibrium points are the roots of the cubic equation
\[
\lambda^3 + (1 + \alpha f')\lambda^2 + (\alpha f' + \beta - \alpha)\lambda + \alpha \beta f' = 0,
\]
where \(f' = \frac{\partial f}{\partial x}\) and \(\alpha < \alpha_c = 7.74518\) all the equilibrium points are stable focus with one eigenvalue being real negative and the other two being complex conjugate with negative real part. As \(\alpha\) increases from a small value, the magnitude of the negative eigenvalue increases while the magnitude of the real part of the complex conjugate eigenvalues becomes positive. For \(\alpha > \alpha_c\), the two complex conjugate eigenvalues become pure imaginary. For \(\alpha > \alpha_c\), the real part of these two complex conjugate eigenvalues becomes positive. For \(\alpha > \alpha_c\) the equilibrium points are unstable. In the numerical simulation, we found sudden occurrence of chaotic motion (crisis) at \(\alpha = \alpha_c\). Figure 2 shows the phase portrait of the scroll chaotic attractors in \(x - y\) plane for \(\alpha = 10\) and for three values of \(N\).

Now, we present the features of FPT, mean FPT, RT, mean RT, RET and mean RET. First, we consider the case \(N = 1\). We calculate the FPT on the barrier (breakpoint) \(x^*\) for a set of random initial conditions chosen around the origin for \(\alpha > \alpha_c\). Figure 3(a) shows the numerically computed FPT on \(x^*\) for 500 random initial conditions with \(\alpha = 10\). FPT depends on the initial condition. FPTs are distributed over a range. Figure 3(b) shows the magnification of a part of Fig. 3(a). The probability distribution \(P(t_{\text{FPT}})\) is calculated using \(10^3\) FPTs and is depicted in Fig. 4. \(P\) is not a monotonically increasing or decreasing function of FPT. It has multiple peaks and a very long tail. The height of the peaks first increases with increase in the value of FPT, reaches a maximum value and then slowly decays for larger values of FPT. \(P\) is continuous for larger values of FPT. For smaller values of FPT, the distribution \(P\) shows a finite number of Gaussian-like profile. This is clearly seen in Figs. 3(b) and 4(b). We note that the peaks of the distribution occur at regular interval of \(t_{\text{FPT}}\). Between the first few hands there exists a range of time interval within which almost no \(t_{\text{FPT}}\) occurs.

2. Characteristics of FPT, RT and RET

For our numerical study we fix the values of the parameters in Eq. (1) as \(\xi = 0.25, A = 0.5, \beta = 16\) and vary the parameter \(\alpha\). The \(f(x)\) curve has discontinuities or breakpoints at
\[
x^* = \pm \left(m - \frac{1}{2}\right), \quad m = 1, 2, \ldots, N.
\]

The graph of \(f(x)\) given by Eq. (1d) with \(N = 2, \xi = 0.25\) and \(A = 0.5\). The breakpoints of \(f(x)\) are indicated by painted circles.

\[f(x) = \begin{cases} 0.1 - x^2, & -2 \leq x \leq 0, \\ 0.1 + x^2, & 0 < x \leq 2. \end{cases}\]
Fig. 2. Phase portrait of (a) three-scroll, (b) five-scroll and (c) eleven-scroll chaotic attractors in the $x-y$ plane. The values of the parameters in Eq. (1) are $\alpha = 1.0$, $\beta = 1.65$, $\xi = 0.25$ and $A = 0.5$.

The above character of $t_{FPT}$ is found for the barrier $x^* = -0.5$ and also for other values of $\alpha$.

In Fig. 2, we notice that the trajectories leave a scroll attractor through specific confined exit regions. Consider the middle-scroll confined between the breakpoints $x^* = -0.5$ and 0.5. Through one exit region trajectories enter into left-scroll while through the other they enter into right-scroll. A trajectory starting from the vicinity of an exit takes a minimum time to arrive at the neighborhood of the next exit. This minimum

Fig. 3. (a) Numerically computed FPTs on the barrier $x^* = 0.5$ of system (1) for $N = 1$ and $\alpha = 10$. (b) Magnification of part of (a) showing the finer details.

Fig. 4. (a) Plot of probability distribution of $t_{FPT}$ on the barrier $x^* = 0.5$ with $N = 1$ and $\alpha = 10$. (b) Magnification of part of $P$ in (a) corresponding to Fig. 3(b).
time is found to be nearly unity. Figure 5 shows two trajectories starting near the origin and leaving the middle-scroll after some time. In this figure, for each trajectory there are two vertical lines: the right-line denotes the time at which the trajectory leaves the middle-scroll and the left-line marks the latest time at which \( x(t) \) came near an exit region before leaving the scroll. The time difference between the two vertical lines is roughly unity.

It is interesting to point out the effect of adding an external periodic force on FPT. Suppose the system (1) is driven by the periodic force \( g \cos t \). We add this external force to Eq. (1a). Figure 6 shows \( t_{\text{FPT}} \) and \( P(t_{\text{FPT}}) \) for two values of \( g \). The continuous and non-Gaussian distribution observed in Fig. 4(a) for \( t_{\text{FPT}} \) > 12 disappear and a series of Gaussian-like distribution occurs even for small values of \( g \). This is evident in Fig. 6(b) where \( g = 0.1 \).

The number of peaks decreases with an increase in the value of \( g \) and \( P \) becomes more and more narrow and the peaks occur at regular intervals of time. The time intervals between successive peaks of \( P \) are almost the same and \( \approx 2\pi \), the period of the external force \( g \cos t \). The periodic nature of \( P(t_{\text{FPT}}) \) can be easily understood. The external force periodically oscillates at the barrier heights of the function \( f(x) \). The chance for crossing the critical value \( x^* \) is large when the barrier height at \( x^* \) becomes a minimum which happens once for each period of the external periodic force. This is the reason for the periodic occurrence of peaks. If the periodic force is replaced by a Gaussian white noise then the FPTs become random, the band-like structure would disappear and \( P \) becomes continuous. In a purely noise induced process the FPT distribution goes through a single maximum and then decays following either exponential or power-law relation. One can expect multipeaks in the presence of both periodic force and weak noise. Suppression of chaotic motion by the addition of weak periodic force and delay feedback is realized in several chaotic systems [Pyragas, 1992; Braiman & Goldhirsch, 1991]. In system (1) the effect of added periodic force is numerically studied for \( \alpha \in [0, 10] \) and \( g \in [0, 0.5] \). For \( \alpha > \alpha_c \), the suppression of chaos is not observed. For \( \alpha < \alpha_c \), the suppression of chaos is observed.

For each fixed value of \( \alpha < \alpha_c \), for \( g \) value less than a critical value there are three coexisting stable periodic orbits. They occur about the equilibrium points of the system. As \( g \) increases from a small value, the size of the orbit increases and above a certain critical value the system has a single periodic-\( T \) orbit enclosing all the three equilibrium points. Any route to chaotic motion is not found for the parametric choices considered here.

The mean FPT \( \langle t_{\text{MFPT}} \rangle \), the mean time the trajectory initially in the neighborhood of the origin takes to cross the target location \( x^* \) for the first time, is calculated for a range of values of \( \alpha > \alpha_c \). It is obtained by averaging over \( 10^5 \) FPTs. Numerically computed \( t_{\text{MFPT}} \) versus \( g \) is plotted in Fig. 7(a) with \( x^* = 0.5 \). It decreases rapidly with \( \alpha \) following the power-law relation: \( t_{\text{MFPT}} \propto 45.688(\alpha - \alpha_c)^{-1.003} \), i.e. \( t_{\text{MFPT}} \propto 1/(\alpha - \alpha_c) \).
In Fig. 2, we can clearly see that the trajectory moving on a scroll attractor leaves it when entering into a certain interval of \( y \) called turnstile. We call the length of this interval of \( y \) as exit length and denote it as \( y_{el} \). This quantity for the middle-scroll attractor is numerically calculated for a range of values of \( \alpha \) and the result is shown in Fig. 7(b). \( y_{el} \) is found to increase with \( \alpha - \alpha_c \) obeying the power-law relation

\[
y_{el} = C(\alpha - \alpha_c)^{\beta}.
\]

The power-law decay of MFPT with \( N \) in Fig. 8 is observed for each fixed value of \( \alpha \). We obtained

\[
t_{MFPT}(x^* = 0.5) = \begin{cases} 14.955N + 26.46, & \alpha = 8.5 \\ 12.167N + 15.753, & \alpha = 9 \\ 5.996N + 14.459, & \alpha = 10 \end{cases}
\]

In Fig. 8 we notice that for a fixed value of \( N \), \( t_{MFPT} \) decreases with increase in \( \alpha \). The variation of \( t_{MFPT} \) with \( \alpha \) follows the power-law relation while with \( N \) it follows the linear relation. The rate of variation of MFPT with \( N \) decreases with increase in \( \alpha \).

Next, we wish to present the results on residence time statistics. We denote \( t_{RT} \) as the time the trajectory resides on a scroll attractor before making a transition to any one of the adjacent attractors. We fix \( N = 1 \) and \( \alpha = 7.52 \) for which three-scroll chaotic attractor occurs. The residence times on the left-, middle- and right-scroll attractors are calculated. Unlike \( t_{FPT} \), the \( t_{RT} \)s on all the three scroll attractors exhibit a band-like structure over the entire possible range of values of \( t_{RT} \).

The middle-scroll attractor has more bands than the other two scroll attractors. This is because a trajectory can enter into the middle-scroll attractor either through the bottom of the attractor (when
it enters the attractor from the right-scroll attractor or through the top of the attractor (when it enters the attractor from the left-scroll attractor). But a trajectory can enter into the left (right)-scroll attractor from the middle-scroll attractor only through the bottom (top) side of the attractor. The probability distribution curves shown in Fig. 9 indicates that the bands of RT are very thin.

When the value of $\alpha$ increases, the band-like pattern of RT persists while its maximum value decreases. Figure 10 shows mean RT ($t_{MRT}$), $t_{MRT}$, versus $\alpha$. Though the number of bands of $t_{RT}$ on the middle-scroll attractor is larger than the other scroll attractors the MRTs on these three attractors are almost the same. For example, for $\alpha = 7.47$, $t_{MRT}$ for the middle and the other two attractors are 393.73 and 389.71, respectively. For $\alpha = 7.52$ the values of $t_{MRT}$ for these attractors are 110.54 and 109.75, respectively. The difference is less than 2%. Power-law dependence of MRT on $\alpha$ with the exponent value $\approx 1$ is observed: $t_{MRT} = 8.413(\alpha - \alpha_c)^{-0.962}$. RTs and MRT on a scroll attractor depend on $\alpha$ but are independent of $N$. Thus the MRT on a scroll attractor can be monitored by the control parameter $\alpha$.

We believe that the study of RT and MRT on multiscroll attractors may find practical applications. It is noteworthy to mention that RT based detection strategies for nonlinear sensors have been proposed [Gammaitoni & Bulsara, 2002; Bulsara et al., 2003; Dari et al., 2010]. This we have explored in system (1). Often a spectral technique is used to detect the presence of weak dc or low-frequency signal. An alternate approach in the case of low-frequency signal is the use of tuning of an internal or external noise to induce stochastic resonance. At an optimal noise intensity in a bistable system the signal-to-noise ratio measured at the low-frequency of the signal becomes maximum. To detect a weak dc signal one can use residence time asymmetry in noisy bistable devices. The effect of an additional dc signal is to skew the potential. In the absence of dc signal the RT distributions and MRTs in the two wells of the bistable double-well systems are identical. They will be different in the presence of additional dc signal and the difference in the MRT is shown to be proportional to the weak dc signal [Gammaitoni & Bulsara, 2002]. This characteristic feature can be used to identify the presence of a
weak dc signal. In the multiscroll circuit with $f(x)$ given by Eqs. (1d) and (1e) the MRTs of a trajectory in all the scrolls are almost identical. This property can also be explored for weak dc signal detection as shown below.

We introduce asymmetry in $f(x)$ given by Eqs. (1d) and (1e) by redefining it as

$$f(x) = \xi x - \xi A \sum_{j=0}^{N-1} \{ \text{sgn} [x + (2j+1)A] + \text{sgn} [x - (2j+1)A] + g \}, \quad (6a)$$

where

$$g = \begin{cases} d, & \text{if } x + (2j+1)A < 0 \\ 0, & \text{otherwise} \end{cases} \quad (6b)$$

Figure 11(a) shows the plot of $f(x)$ versus $x$ for $N = 1$ with $d = 0$ and 1. The effect of $d$ can be clearly seen. Such forms of $f(x)$ are studied both theoretically and experimentally [Aziz-Alaoui, 1999; Lu & Chen, 2006; Suykens & Vandewalle, 1993]. For $d = 0$ the heights of the barriers at the breakpoints $x^* = 0.5$ and $-0.5$ are the same. For $d = 1$ the barrier height at $x^* = 0.5$ remains the same while at $x^* = -0.5$, it is decreased. This asymmetry in the barrier heights can be used to detect the weak signal by measuring the difference in the MRTs on the left- and middle-scrolls with and without the dc signal. The MRT of a trajectory in the right-scroll remains the same for $d = 0$ and $d \neq 0$. We define $\Delta \text{MRT}_{\text{L}}(d = 0) - \Delta \text{MRT}(d)$. $\Delta \text{MRT}_L$ on left- and middle-scrolls that are calculated numerically for $\alpha = 7.46$ and 7.47 for a range of values of $d$. Figure 11(b) shows the variation of $\Delta \text{MRT}$ with $d$. Interestingly, it is found to show linear variation with $d$. From the predetermined relation between $\Delta \text{MRT}$ and $d$ we can make an estimate of the value of $d$ by calculating $\Delta \text{MRT}$.

We now turn to the discussion on returned time to the middle-scroll attractor ($t_{\text{RET}}$), the time taken by a trajectory to re-enter into the middle attractor after leaving it. For a trajectory starting with an initial condition near the origin $10^{12}$ RETs and the mean value of RETs denoted as $\Delta \text{MRT}$ are numerically calculated for various values of $\alpha$ and $N$. For $N = 1$ three discrete bands of RETs occur. The number of bands increases with increase in the value of $N$. $P(t_{\text{RET}})$ obtained with $10^{12}$ RETs for $N = 1$, 2 and 3 are reported in Fig. 12. Distribution of RETs is different from those of FPT and RT. The width of all the bands of RETs is very small. In Figs. 12(b) and 12(c) we can clearly see a series of Gaussian-like profile of $P$. The first profile is predominant over the others. The height of the profiles decays rapidly with $t_{\text{RET}}$. The above characteristics of RETs are found for higher values of $N$ also.

Next, we observe different kinds of dependence of $t_{\text{RET}}$ on $N$ and $\alpha$. In Fig. 13(a) we plotted $t_{\text{RET}}$ versus $N$ for various fixed values of $\alpha$. For each fixed value of $\alpha$ the $t_{\text{RET}}$ increases linearly with $N$. This is because the number of scroll attractors increases linearly with $N$. We obtained the
following linear fits:

\[ t_{\text{MRET}} = \begin{cases} 4.920N - 28.602 & \alpha = 7.8 \\ 30.243N - 18.995 & \alpha = 8.0 \\ 18.557N - 9.381 & \alpha = 8.5 \\ 12.639N - 7.926 & \alpha = 9.0 \\ 7.329N - 1.668 & \alpha = 10.0 \end{cases} \]

The rate of divergence of \( t_{\text{MRET}} \) with \( N \) decreases with increase in \( \alpha \). Regarding the dependence of \( t_{\text{MRET}} \) on \( \alpha \), we notice power-law variation in Fig. 13(b). The following result is obtained:

\[ N = 1 : t_{\text{MRET}} = 9.499(\alpha - \alpha_c)^{-0.923} \]
\[ N = 2 : t_{\text{MRET}} = 27.055(\alpha - \alpha_c)^{-0.943} \]

3. Normal Diffusion

In this section, we report the occurrence of normal diffusion in Eq. (1). When \( N = \infty \) the function \( f(x) \) has infinite number of sawtooth segments and for \( \alpha > \alpha_c \) the motion is not bounded to a finite range of \( x \) in the limit \( t \to \infty \). Figure 14 shows \( x \) versus \( t \) for \( \alpha = 8 \) and \( 9 \). For \( \alpha = 8 \) the divergence of \( x \) is very slow. We notice a relatively higher divergence
Fig. 14. \( x(t) \) versus \( t \) for two values of \( \alpha \) of system (1) when \( N = \infty \). The subplots (b) and (d) are magnification of a detailed evolution of \( x(t) \) of a part of the solution shown in (a) and (c) respectively.

In order to capture the type of diffusion, we calculate the mean square displacement over a set of \( M \) initial conditions given by

\[
\langle x^2(t) \rangle = \lim_{M \to \infty} \frac{1}{M} \sum_{i=1}^{M} [x^{(i)}(t) - \langle x^{(i)}(t) \rangle]^2, \tag{7}
\]

where \( x^{(i)} \) is the \( i \)th trajectory and \( \langle x^{(i)}(t) \rangle \) is the mean value of \( x^{(i)}(t) \). In our numerical simulation, \( M = 10^4 \) and the initial conditions are chosen around the origin. The variation of \( \langle x^2(t) \rangle \) against \( t \) is shown in Fig. 15 for three values of \( \alpha \). On the \( \log_{10} - \log_{10} \) scale the slopes of the best straight-line fits for \( \alpha = 8, 9 \) and 10 are \( \approx 1 \). In other words, \( \langle x^2(t) \rangle \sim t^{\mu} \) as \( t \to \infty \) with \( \mu = 1 \) for the system (1). A linear time dependence of \( \langle x^2(t) \rangle \) is the hallmark of normal diffusion, and such a diffusion process can be described by a classical random walk. The value of the exponent \( \mu \) is calculated for a range of values of \( \alpha \) in the interval \([\alpha_c, 11] \). In all the cases, \( \mu \) is found to be 1 and anomalous diffusion (\( \mu \neq 1 \)) is not found. We note that anomalous diffusion is found in several conservative systems when the phase space has accelerating modes or stochastic layers. In these systems when a trajectory comes closer to such an accelerating mode or a stochastic layer the system is accelerated rapidly and \( \langle x^2(t) \rangle \) diverges nonlinearly with time during the trapped times and moves chaotically otherwise. The combination of these two types of motion leads to anomalous diffusion. In certain dissipative chaotic diffusive systems, divergence of \( x \) takes place either during laminar intervals or chaotic evolution of a state variable. In the system (1) there is no laminar phase. The growth of the state variable \( x \) is due to the passage of the trajectory to the adjacent scroll.
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attractors along one direction and also because of the presence of infinite number of scroll attractors. This is clearly evident in Fig. 14(d). In this way diffusion in system (1) is different from the diffusion in systems like damped and forced pendulum.

The coefficient $D_\alpha$ in $\langle x^2(t) \rangle = D_\alpha t$ increases with increase in the value of $\alpha$. The values of $D_\alpha$ for $\alpha = 8, 9$ and $10$ are $0.018, 0.146$ and $0.346$, respectively. For a one-dimensional piecewise linear periodic map Schuster and Just [2005] showed that the presence of diffusion implies chaotic motion, and the scaling law associated with the diffusion coefficient $D_\alpha$ was obtained. For the conservative standard map, an analytical expression for momentum distribution at time $n$ for momentum scales larger than the control parameter $K$ is obtained [Ott, 1993]. Damped oscillatory variation of $D_\alpha$ with the control parameter $K$ about the quasilinear value of $D_\alpha$ has been reported [Rechester & White, 1980].

Normal diffusion can be characterized by kurtosis which is defined as

$$K = \frac{\langle (x - \langle x \rangle)^4 \rangle}{\langle (x - \langle x \rangle)^2 \rangle^2}$$

(8)

For normal diffusion $K = 3$ in the limit $t \to \infty$. In Fig. 16, the variation of $K$ is plotted for three values of $\alpha$. For large $t$ the value of $K$ is $\approx 3$ confirming normal diffusion.

We studied the occurrence of diffusion in two other multiscroll chaotic systems, one with $f(x) = \sum_{j=-\infty}^{\infty} (-1)^{j-1} \tanh[k(x-2j)]$ [Ozoguz et al., 2002; Salama et al., 2003], hyperbolic tangent function, realizable using a multicycle transconductors composed of alternating D-P cells and another one proposed by Tang and his co-workers [Tang et al., 2001] in which $f(x)$ is a sine function. Normal diffusion is found in both the systems.

4. Conclusion

In the present work we have considered the modified Chua’s circuit model equation [Eq. (1)] with sawtooth function for $f(x)$ proposed to generate multiscroll chaotic attractors. Attention has been paid to the study of characteristics of FPTs, RTs and RTs. We have presented the distribution of these quantities and explored the dependence of their mean values on the control parameter $\alpha$ and the number of scroll attractors. From our study, we can obtain some general results for infinitely large number of scroll attractors. Because the forms of function $f(x)$ associated with all scroll attractors are identical, the resident time characteristics of all the scroll attractors are almost the same within the numerical accuracy. Further, the returned time statistics of all the scroll attractors are also same. In the system with the infinite number of scroll attractors the motion is shown to be normal diffusion type. We wish to point out that the multiscroll systems and the features of multiscroll attractors can be used to emulate various logic gates and the ability to switch easily between different operational roles. In a recent work, Murali et al. [2009] have shown that the response of a bistable system with a two-square wave and appropriate noise intensity produces logic outputs (NOR/OR) and (NAND/AND). We believe that similar results can be realized if one replaces the bistable system by the multiscroll systems. Further, the different scrolls can be used to represent different logic states in which case MRT on a scroll attractor is important. In system (1) MRT on a scroll can be varied over a wide range of the control parameter $\alpha$ as shown in Fig. 10.

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