BIRTH-DEATH CHAINS ON A SPIDER: SPECTRAL ANALYSIS
AND REFLECTING-ABSORBING FACTORIZATION

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ABSTRACT. We consider discrete-time birth-death chains on a spider, i.e. a graph consisting of $N$ discrete half lines on the plane that are joined at the origin. This process can be identified with a discrete-time quasi-birth-death process on the state space $\mathbb{N}_0 \times \{1, 2, \ldots, N\}$, represented by a block tridiagonal transition probability matrix. We prove that we can analyze this process by using spectral methods and obtain the $n$-step transition probabilities in terms of a weight matrix and the corresponding matrix-valued orthogonal polynomials (the so-called Karlin-McGregor formula). We also study under what conditions we can get a reflecting-absorbing factorization of the birth-death chain on a spider which can be seen as a stochastic UL block factorization of the transition probability matrix of the quasi-birth-death process. With this factorization we can perform a discrete Darboux transformation and get new families of “almost” birth-death chains on a spider. The spectral matrix associated with the Darboux transformation will be a Geronimus transformation of the original spectral matrix. Finally, we apply our results to the random walk on a spider, i.e. with constant transition probabilities.

1. INTRODUCTION

In [19] (Section 4.2, Problem 1), Itô and McKean proposed an elementary but interesting diffusion process which they called skew Brownian motion. After that, in 1978, Walsh [29] characterized this process as a Brownian motion with excursions around zero in random directions on the plane which takes values in $[0, 2\pi)$ and call it a diffusion with a discontinuous local time. This diffusion is now called Walsh’s Brownian motion. Later, in 1989, Barlow, Pitman and Yor [1] considered this process as a motion which lives on $N$ half lines on the plane, called legs from now on, that are joined at the origin, called the body of the spider. More recently, in 2003, Evans and Sowers [8] considered the same construction but with different methods and used the name Walsh’s spider for this process. In a few words, the process behaves like a regular Brownian motion on each one of the legs and once it reaches the body it continues on any of the $N$ legs with a given probability. For some other results related with this process the reader can consult [15, 23, 26, 28].

If we replace the Brownian motions with simple symmetric random walks on the legs we get a discrete version of Walsh’s spider, also called a random walk on a spider. Hajri [14] studied this discrete version as an approximation of the Walsh’s spider. Some other results related with stochastic differential equations, limit theorems or weak convergence distributions can be found in [2, 4, 5, 22, 27]. In Example 3.5 of [6], Dette et al treated the discrete version of Walsh’s spider as a quasi-birth-death process with state space $\mathbb{N}_0 \times \{1, 2, \ldots, N\}$, which is a generalization of the birth-death process but allowing transitions between all the states of the second component (or phases). They studied the case of the symmetric random walk on a spider (which they call a tree) and gave an explicit expression of the corresponding spectral matrix, which was improved later by Grünbaum in [11] (see also [10]). For more information about quasi-birth-death processes the reader can consult [21, 25].

arXiv:2111.10450v1 [math.PR] 19 Nov 2021

Date: November 23, 2021.

2010 Mathematics Subject Classification. 60J10, 33C45, 42C05.

Key words and phrases. Birth-death chains. Matrix factorizations. Darboux transformations. Orthogonal polynomials. Geronimus and Christoffel transformations.

This work was partially supported by PAPIIT-DGAPA-UNAM grant IN104219 (México) and CONACYT grant A1-S-16202 (México).
In this paper we consider general discrete-time birth-death chains on a spider, allowing transition probabilities to depend on the state and on the leg in which the particle is living. This process behaves like a regular birth-death chain in each of the legs, but once it reaches the body of the spider it continues towards any of the \( N \) legs with a given probability (including remaining at the body). In Section 2 we make a precise definition of this process. In order to perform a spectral analysis it will be more convenient to treat this process as a quasi-birth-death process with state space \( \mathbb{N}_0 \times \{1, 2, \ldots, N\} \), where \( N \) is the number of legs. The transition probability matrix will be represented by a block tridiagonal matrix \( P \) (see (2.1) below). Using the results in [6] we prove in Proposition 2.1 that there always exists a weight matrix (or spectral matrix) \( W \) associated with \( P \) and therefore we can obtain an integral representation of the \( n \)-step transition probability matrix \( P^n \) in terms of this spectral matrix and the associated matrix-valued orthogonal polynomials, also known as the Karlin-McGregor formula (see [20, 6, 9] or more recently the monograph [7]). Using the \( 2 \times 2 \) block structure of the spectral matrix \( W \) we are able to get in Proposition 2.4 an explicit expression of the Stieltjes transform of \( W \) in terms of \( N \) scalar spectral measures (one for each leg), which will be very useful for the study of the example in Section 4.

Following our previous works in [17, 18], we consider in Section 3 a stochastic factorization of the birth-death chain on a spider into a reflecting and an absorbing birth-death chain to the state 0. That will mean that we will be looking for possible stochastic UL factorizations of the transition probability matrix \( P \) of the form \( P = PRPA \). The difference with [18] is that now we will have \( N \) free parameters, one for each leg, and we will show that each of these parameters must be bounded from below by certain continued fraction built from the transition probabilities of each leg (see Theorem 3.1 below). This new method of considering stochastic factorizations was introduced for the first time in [12] and later exploited in [13, 17, 18]. The motivation for these stochastic factorizations is to divide the probabilistic model of the original process into two different and simpler experiments. Once we have the conditions under we can perform a reflecting-absorbing factorization, we consider a discrete Darboux transformation, consisting in reversing the order of multiplication of the factors. Therefore we will get a family of Markov chains on a spider depending on \( N \) free parameters with transition probability matrix given by \( \tilde{P} = P\tilde{A}P\tilde{R} \). This new stochastic matrix \( \tilde{P} \) describes an “almost” birth-death chain on a spider since now there will be extra transition probabilities between the first states of each leg. However, the matrix \( \tilde{P} \) preserves the block tridiagonal structure and it will possible to derive the corresponding spectral matrix \( \tilde{W} \) in terms of a Geronimous transformation of \( W \) (see (3.15) below).

In Section 4 we apply our results to the random walk on a spider, i.e. with constant transition probabilities (including self-transitions on each state). We derive an explicit expression of the spectral matrix \( W \) computing first its Stieltjes transform from Proposition 2.4 and then using the Perron-Stieltjes inversion formula. This representation (as a \( 2 \times 2 \) block matrix) is different and simpler from the ones given in [6, 11]. We also apply Theorem 3.1 to study conditions under we get a stochastic reflecting-absorbing factorization and perform the discrete Darboux transformation. Finally, we get an explicit expression of the spectral matrix associated with the Darboux transformation \( \tilde{W} \).

2. Spectral analysis of birth-death chains on a spider

For \( N \in \mathbb{N} \) consider the spider graph given by

\[
\mathbb{S}_N := \{v_N(k, m), \ k \in \mathbb{N}_0, \ m = 1, \ldots, N\},
\]

where

\[
v_N(k, m) = k \exp \left( \frac{2\pi i(m-1)}{N} \right), \quad i = \sqrt{-1}.
\]

The number \( N \) is the number of legs of the spider \( \mathbb{S}_N \). If \( N = 1 \) then we go back to regular birth-death chains on \( \mathbb{N}_0 \), while if \( N = 2 \) we have a birth-death chain on \( \mathbb{Z} \) (see [20, 17, 18]). The point \( v_N(0) := \)
$v_N(0, m), m = 1, \ldots, N$, will be called the body of the spider. Consider a homogeneous discrete-time birth-death chain $\{S_n, n = 0, 1, \ldots\}$ on a spider $S_N$. The transition probabilities are given by

$$P \left[ S_{n+1} = v_N(0) \mid S_n = v_N(0) \right] = \alpha_0, \quad P \left[ S_{n+1} = v_N(1, m) \mid S_n = v_N(0) \right] = \alpha_m, \quad m = 1, \ldots, N,$$

where $\sum_{m=0}^{N} \alpha_m = 1$, and

$$P \left[ S_{n+1} = v_N(k+1, m) \mid S_n = v_N(k, m) \right] = a_{k,m},$$
$$P \left[ S_{n+1} = v_N(k, m) \mid S_n = v_N(k, m) \right] = b_{k,m},$$
$$P \left[ S_{n+1} = v_N(k-1, m) \mid S_n = v_N(k, m) \right] = c_{k,m},$$

where $a_{k,m} + b_{k,m} + c_{k,m} = 1$ for all $k \geq 1$ and $m = 1, \ldots, N$. A diagram of the probability transitions between the states of this process is given in Figure 1. As it was done in [6], the birth-death chain $\{S_n, n = 0, 1, \ldots\}$ on a spider $S_N$ can be seen as a quasi-birth-death process on the state space $\mathbb{N}_0 \times \{1, 2, \ldots, N\}$ (see [21, 25] for more information about quasi-birth-death processes). The labeling follows putting $v_N(0)$ as the origin 0. Then the first $N$ nodes on the first circle as $1, \ldots, N$, in a counter-clock wise fashion. The second circle with $N+1, \ldots, 2N$, and so on (see Figure 1). The transition probability matrix of the birth-death chain $\{S_n, n = 0, 1, \ldots\}$, seen as a quasi-birth-death process, is

$$P = \begin{pmatrix} B_0 & A_0 \\ C_1 & B_1 & A_1 \\ & C_2 & B_2 & A_2 \\ & & & & \ddots & \ddots & \ddots \end{pmatrix},$$

(2.1)
where

\[
B_0 = \begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\
c_{1,1} & b_{1,1} & 0 & \cdots & 0 \\
c_{1,2} & 0 & b_{1,2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c_{1,N-1} & 0 & 0 & \cdots & b_{1,N-1}
\end{pmatrix}, \quad A_0 = \text{diag}(\alpha_N, a_{1,1}, \ldots, a_{1,N-1}),
\]

(2.2)

and \(A_n, B_n, C_n\) are the diagonal matrices

\[
A_n = \text{diag}(a_{n,N}, a_{n+1,1}, \ldots, a_{n+1,N-1}), \quad B_n = \text{diag}(b_{n,N}, b_{n+1,1}, \ldots, b_{n+1,N-1}), \quad n \geq 1,
\]

(2.3)

\[
C_n = \text{diag}(c_{n,N}, c_{n+1,1}, \ldots, c_{n+1,N-1}), \quad n \geq 1.
\]

(2.4)

In the same fashion as in [6, 9] we consider the matrix-valued polynomials \((Q_n)_{n \geq 0}\) generated by the three-term recurrence relation

\[
xQ_n(x) = A_nQ_{n+1}(x) + B_nQ_n(x) + C_nQ_{n-1}(x), \quad n \geq 0,
\]

\[
Q_0(x) = I_N, \quad Q_{-1}(x) = 0,
\]

where \(I_N\) and \(\mathbf{0}\) denote the identity and the null matrix of dimension \(N \times N\), respectively (from now on whenever we write \(\mathbf{0}\) we will mean the null vector or matrix which dimension will be determined by the context). These matrix-valued polynomials can be written as

\[
Q_n(x) = \begin{pmatrix}
Q_{n,N}(x) & \alpha_1 Q_{n,N}^{(0)}(x) & \alpha_2 Q_{n,N}^{(0)}(x) & \cdots & \alpha_{N-1} Q_{n,N}^{(0)}(x) \\
Q_{n,1}^{(0)}(x) & Q_{n,1}(x) & 0 & \cdots & 0 \\
Q_{n,2}^{(0)}(x) & 0 & Q_{n,2}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
Q_{n,N-1}^{(0)}(x) & 0 & 0 & \cdots & Q_{n,N-1}(x)
\end{pmatrix}, \quad n \geq 0,
\]

(2.5)

where \(Q_{n,N}(x)\) satisfies the scalar-valued three-term recurrence relation (here \(a_{0,N} = \alpha_N\) and \(b_{0,N} = \alpha_0\))

\[
xQ_{n,N}(x) = a_{n,N}Q_{n+1,N}(x) + b_{n,N}Q_{n,N}(x) + c_{n,N}Q_{n-1,N}(x), \quad n \geq 0,
\]

\[
Q_{0,N}(x) = 1, \quad Q_{-1,N}(x) = 0,
\]

(2.6)

and \(Q_{n,N}^{(0)}\) will denote the corresponding associated polynomials (or 0-th associated polynomials). These are polynomials satisfying the same three-term recurrence relation [2.6] but with initial conditions \(Q_{0,N}^{(0)} = 0, Q_{1,N}^{(0)} = -1/\alpha_N\). Also \(Q_{n,k}(x), k = 1, \ldots, N - 1,\) satisfy the scalar-valued three-term recurrence relations

\[
xQ_{n,k}(x) = a_{n+1,k}Q_{n+1,k}(x) + b_{n+1,k}Q_{n,k}(x) + c_{n+1,k}Q_{n-1,k}(x), \quad n \geq 0,
\]

\[
Q_{0,k}(x) = 1, \quad Q_{-1,k}(x) = 0,
\]

(2.7)

and \(Q_{n,k}^{(0)}\), \(k = 1, \ldots, N - 1,\) will denote the corresponding associated polynomials with initial conditions \(Q_{0,k}^{(0)} = 0, Q_{1,k}^{(0)} = -c_{1,k}/a_{1,k}, k = 1, \ldots, N - 1.\) Observe that the associated polynomials have degree \(n - 1\). Therefore the matrix-valued polynomials \((Q_n)_{n \geq 0}\) satisfy \(\text{deg}(Q_n) = n\) and have nonsingular leading coefficient.

Let us now see that the discrete-time birth-death chain \(\{S_n, n = 0, 1, \ldots\}\) defined on the spider \(S_N\) and with transition probability matrix \(P\) [2.1] can be identified with a \(N \times N\) weight matrix \(W\) supported on the real line, i.e. a matrix of measures which is symmetric and nonnegative definite for any Borel set \(A\), i.e. \(W(A) \geq 0,\)
and with finite moments. The matrix-valued polynomials \((Q_n)_{n \geq 0}\) defined by (2.5) will be orthogonal with respect to \(W\) in the following sense

\[
\int_{-1}^{1} Q_n(x) dW(x) Q_m^T(x) = \|Q_n\|_W^2 \delta_{nm},
\]
as we will see in the following result.

**Proposition 2.1.** Let \(\{S_n, n = 0, 1, \ldots\}\) be a discrete-time birth-death chain on a spider \(S_N\) with transition probability matrix \(P\) (2.1). Then there exists a weight matrix \(W\) supported on the interval \([-1, 1]\) such that the polynomials \((Q_n)_{n \geq 0}\) defined by (2.5) are orthogonal with respect to \(W\).

**Proof.** For the existence and orthogonality we apply Theorem 2.1 of [6]. We need to define a sequence of nonsingular matrices \((T_n)_{n \geq 0}\) such that

\[
T_n T_n^T B_n = B_n^T T_n T_n, \quad n \geq 0,
\]

\[
T_n T_n^T A_n = C_{n+1}^T T_{n+1}^T, \quad n \geq 0,
\]

where the coefficients \((A_n)_{n \geq 0}\), \((B_n)_{n \geq 0}\) and \((C_n)_{n \geq 1}\) are defined by (2.2), (2.3) and (2.4), respectively. Let us define the following sequences

\[
\pi_{0,N} = 1, \quad \pi_{n,N} = \alpha_n \frac{a_{1,n} \cdots a_{n-1,N}}{c_{1,n} \cdots c_{n,N}}, \quad n \geq 1,
\]

\[
\pi_{1,k} = \frac{\alpha_k}{c_{1,k}}, \quad \pi_{n+1,k} = \alpha_k \frac{a_{1,k} \cdots a_{n,k}}{c_{1,k} \cdots c_{n+1,k}}, \quad n \geq 1, \quad k = 1, \ldots, N - 1.
\]

Then a straightforward computation shows that the diagonal matrices

\[
T_n = \text{diag} \left( \sqrt{\pi_{n,N}}, \sqrt{\pi_{n+1,1}}, \ldots, \sqrt{\pi_{n+1,N-1}} \right), \quad n \geq 0,
\]

satisfy the conditions (2.8). Finally the weight matrix \(W\) is supported on the interval \([-1, 1]\) as a consequence of Theorem 2.5 of [6]. \(\square\)

**Remark 2.2.** From (2.9) we can define the sequence of matrix-valued potential coefficients for the birth-death chain \(\{S_n, n = 0, 1, \ldots\}\) on a spider \(S_N\) as

\[
\Pi_n = T_n T_n^T = \text{diag} \left( \pi_{n,N}, \pi_{n+1,1}, \ldots, \pi_{n+1,N-1} \right), \quad n \geq 0.
\]

From formula (2.6) of [16] we have that this sequence can be identified with the inverse of the norms of the matrix-valued orthogonal polynomials \((Q_n)_{n \geq 0}\) defined by (2.5), i.e.

\[
\Pi_n = \left( \|Q_n\|^2_W \right)^{-1} = \left( \int_{-1}^{1} Q_n(x) dW(x) Q_n^T(x) \right)^{-1}, \quad n \geq 0.
\]

**Remark 2.3.** The existence of a weight matrix \(W\) for the birth-death chain \(\{S_n, n = 0, 1, \ldots\}\) on a spider \(S_N\) gives one way of computing the \((i,j)\)-block of the \(n\)-step transition probability matrix \(P^n\) by the so-called Karlin-McGregor formula. Indeed, using Theorem 2.3 of [6] we have that

\[
P_{ij}^n = \left( \int_{-1}^{1} x^n Q_i(x) dW(x) Q_j^T(x) \right) \Pi_j, \quad i, j \in \mathbb{N}_0,
\]

where \((Q_n)_{n \geq 0}\) and \((\Pi_n)_{n \geq 0}\) are defined by (2.5) and (2.10), respectively. Also, in [6], one can find some probabilistic results concerning recurrence or canonical moments of quasi-birth-death processes in terms of the spectral matrix \(W\). In particular, from Corollary 4.1 of [6], we have that the birth-death chain on a spider is recurrent if and only if

\[
e_j^T \left( \int_{-1}^{1} \frac{dW(x)}{1-x} \right) T_0^{-1} e_j = \infty,
\]
eliminating the first block row and column of $P$. And from Corollary 4.2 of [6] we have that the birth-death chain on a spider is positive recurrent if and only if one of the measures $e_j^T dW(x) T_0^{-1} e_j, j = 1, \ldots, N$, has a jump at the point 1.

The scalar-valued polynomials $(Q_{n,k})_{n \geq 0}, k = 1, \ldots, N$, and the corresponding associated polynomials $(Q_{n,k}^{(0)})_{n \geq 0}, k = 1, \ldots, N$, are defined in terms of regular three-term recurrence relations (see (2.6) and (2.7)). This means, using Favard’s theorem or the spectral theorem for orthogonal polynomials, that there exist spectral measures supported on the interval $[-1,1]$ (the corresponding Jacobi matrices are stochastic) such that these polynomials are orthogonal (see [20]). For $k = 1, \ldots, N$, let us denote $\omega_k$ and $\omega_k^{(0)}$ the spectral probability measures associated with the polynomials $(Q_{n,k})_{n \geq 0}$ and $(Q_{n,k}^{(0)})_{n \geq 0}$, respectively. For any measure $\omega$ supported on the real line let us define the Stieltjes transform of $\omega$ by

$$
B(z; \omega) = \int_\mathbb{R} \frac{d\omega(x)}{x-z}, \quad z \in \mathbb{C} \setminus \mathbb{R}.
$$

(2.12)

There is a very well-known connection between the Stieltjes transforms of $\omega_k$ and $\omega_k^{(0)}, k = 1, \ldots, N$, which can be found for instance in formula (6) of [20]. These formulas are given by

$$
B(z; \omega_N) = -\frac{1}{z - \alpha_0 + \alpha_1 c_1 N B(z; \omega_N^{(0)}),}
$$

$$
B(z; \omega_k) = -\frac{1}{z - b_1, k + a_1, k c_2, k B(z; \omega_k^{(0)}),} \quad k = 1, \ldots, N - 1.
$$

(2.13)

From Proposition 2.1 we know that any birth-death chain $\{S_n, n = 0,1, \ldots\}$ on a spider $S_N$ can be identified with some weight matrix $W$. Let us give one criterium to compute the Stieltjes transform of $W$ (entry by entry) in terms of the Stieltjes transforms of the measures $\omega_k, k = 1, \ldots, N$, associated with the polynomials $(Q_{n,k})_{n \geq 0}, k = 1, \ldots, N$. For that we need the following notation

$$
\tilde{\alpha} = (\alpha_1, \ldots, \alpha_N-1)^T, \quad \alpha_D = \text{diag}(\alpha_1, \alpha_2, \ldots, \alpha_N-1),
$$

$$
\tilde{c} = (c_1, \ldots, c_1, N-1)^T, \quad c_D = \text{diag}(c_1, c_1, 2, \ldots, c_1, N-1),
$$

$$
\tilde{\omega}(x) = (\omega_1(x), \ldots, \omega_N-1(x))^T, \quad \omega_D(x) = \text{diag}(\omega_1(x), \omega_2(x), \ldots, \omega_N-1(x)).
$$

(2.14)

(2.15)

Whenever we write $B(z; \tilde{\omega})$ or $B(z; \omega_D)$ we mean that we are taking the Stieltjes transform on each component/entry.

**Proposition 2.4.** Let $\{S_n, n = 0,1, \ldots\}$ be a birth-death chain on a spider $S_N$ with transition probability matrix $P$. The Stieltjes transform of the weight matrix $W$ obtained in Proposition 2.1 can be written as

$$
B(z; W) = \begin{pmatrix} 0 & 0 \\ 0 & -B(z; \omega_D) c_D \alpha_D^{-1} \end{pmatrix} + b(z) \begin{pmatrix} 1 & -\tilde{c}^T B(z; \omega_D) \\ -B(z; \omega_D) \tilde{c} & B(z; \omega_D) \tilde{c} \tilde{c}^T B(z; \omega_D) \end{pmatrix},
$$

(2.16)

where

$$
b(z) = \frac{1}{B(z; \omega_N) - \tilde{\alpha}^T B(z; \omega_D) \tilde{c}}.
$$

(2.17)

**Proof.** We will use Theorem 2.1 of [3], which gives a relation between the Stieltjes transform of $W$ and the Stieltjes transform of the spectral matrix $W^{(0)}$ of the 0-th associated process (built from the original one by eliminating the first block row and column of $P$). This relation is given by

$$
B(z; W) \Pi_0 = -\left[ z I_N - B_0 + A_0 B(z; W^{(0)}) \Pi_0^{(0)} C_1 \right]^{-1},
$$

where
where $\Pi_0^{(0)} = I_N$ and (see (2.10))

$$
\Pi_0 = \begin{pmatrix}
1 & \\
\alpha_D c_D^{-1}
\end{pmatrix}.
$$

(2.18)

Since $A_n, B_n, C_{n+1}, n \geq 1$, are diagonal matrices we have that $B(z; W^{(0)})$ is a diagonal matrix given by

$$
B(z; W^{(0)}) = \text{diag}(B(z; \omega_N^{(0)}), B(z; \omega_1^{(0)}), \ldots, B(z; \omega_{N-1}^{(0)})).
$$

Using the definition of $B_0, A_0$ and $C_1$ in (2.2) and (2.4) we can write the Stieltjes transform of $W$ in a $2 \times 2$ block matrix expression

$$
B(z; W) = -\begin{pmatrix}
M_{11} & M_{12} \\
M_{21} & M_{22}
\end{pmatrix}^{-1} \begin{pmatrix}
1 & \alpha_D c_D^{-1} \\
0 & -1
\end{pmatrix},
$$

where (see (2.13))

$$
M_{11} = z - \alpha_0 + \alpha_N c_{1,N} B(z; \omega_N^{(0)}) = -\frac{1}{B(z; \omega_N)}, \quad M_{12} = -\vec{\alpha}^T, \quad M_{21} = -\vec{c},
$$

$$
M_{22} = \text{diag}\left(z - b_{1,1} + a_{1,1} c_{2,1} B(z; \omega_1^{(0)}), \ldots, z - b_{1,N-1} + a_{1,N-1} c_{2,N-1} B(z; \omega_{N-1}^{(0)})\right) = -B(z; \omega_D)^{-1}.
$$

Using the well-known formula for the inverse of a $2 \times 2$ block matrix

$$
\begin{pmatrix}
A & B \\
C & D
\end{pmatrix}^{-1} = \begin{pmatrix}
(A - BD^{-1}C)^{-1} & -(A - BD^{-1}C)^{-1}BD^{-1} \\
-D^{-1}C(A - BD^{-1}C)^{-1} & D^{-1} + D^{-1}C(A - BD^{-1}C)^{-1}BD^{-1}
\end{pmatrix},
$$

in (2.19) and the fact that $\vec{\alpha}^T c_D \alpha_D^{-1} = \vec{c}^T$, we get (2.16) after some straightforward computations.

3. Reflecting-absorbing factorization for birth-death chains on a spider

In this section we decompose the discrete-time birth-death chain $\{S_n, n = 0, 1, \ldots\}$ on a spider $S_N$ described by $P$ (2.1) into two independent processes: the first one is a reflecting process from state 0 and the second one is an absorbing process to the state 0. Therefore we are looking for a factorization of the form $P = P_R P_A$, where

$$
P_R = \begin{pmatrix}
Y_0 & X_0 \\
Y_1 & X_1 \\
\vdots & \vdots
\end{pmatrix}, \quad P_A = \begin{pmatrix}
S_0 \\
R_1 \\
\vdots
\end{pmatrix},
$$

(3.1)

with blocks given by

$$
Y_0 = \begin{pmatrix}
\beta_0 & \beta_1 & \beta_2 & \cdots & \beta_{N-1} \\
0 & y_{1,1} & 0 & \cdots & 0 \\
0 & 0 & y_{1,2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & y_{1,N-1}
\end{pmatrix}, \quad S_0 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
0 & r_{1,1} & s_{1,1} & \cdots & 0 \\
0 & r_{1,2} & s_{1,2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
0 & 0 & 0 & \cdots & s_{1,N-1}
\end{pmatrix},
$$

(3.2)

$$
Y_n = \text{diag}(y_{n,N}, y_{n+1,1}, \ldots, y_{n+1,N-1}), \quad S_n = \text{diag}(s_{n,N}, s_{n+1,1}, \ldots, s_{n+1,N-1}), \quad n \geq 1,
$$

(3.3)
\[
X_0 = \text{diag}(\beta_N, x_{1,1}, \ldots, x_{1,N-1}), \quad X_n = \text{diag}(x_{n,N}, x_{n+1,1}, \ldots, x_{n+1,N-1}), \quad n \geq 1,
\]
\[
R_n = \text{diag}(r_{n,N}, r_{n+1,1}, \ldots, r_{n+1,N-1}), \quad n \geq 1,
\]
with the conditions that all these matrices are stochastic, i.e. \(\sum_{k=0}^{N} \beta_k = 1\) and
\[
x_{m,n} + y_{n,m} = 1, \quad n \geq 1, \quad m = 1, 2, \ldots, N, \quad (3.5)
\]
\[
r_{m,n} + s_{n,m} = 1, \quad n \geq 1, \quad m = 1, 2, \ldots, N. \quad (3.6)
\]
Diagrams of the possible transitions between the states of both birth-death chains are given in Figure 3. Observe that the reflecting-absorbing factorization \(P = P_R P_A\) is just a particular case of a block UL factorization (but not the only one). A simple computation from \(P = P_R P_A\) gives the following relations
\[
A_n = X_n S_{n+1}, \quad n \geq 0,
\]
\[
B_n = X_n R_{n+1} + Y_n S_n, \quad n \geq 0,
\]
\[
C_n = Y_n R_n, \quad n \geq 1. \quad (3.7)
\]
If we look entry by entry these relations can also be written as
\[
\alpha_0 = \beta_0 + \sum_{k=1}^{N} \beta_k r_{1,k},
\]
\[
\alpha_m = \beta_m s_{1,m}, \quad m = 1, 2, \ldots, N, \quad (3.8)
\]
\[
a_{n,m} = x_{n,m} s_{n+1,m}, \quad n \geq 1, \quad m = 1, 2, \ldots, N, \quad (3.9)
\]
\[
b_{n,m} = y_{n,m} s_{n,m} + x_{n,m} r_{n+1,m}, \quad n \geq 1, \quad m = 1, 2, \ldots, N,
\]
\[
c_{n,m} = y_{n,m} r_{n,m}, \quad n \geq 1, \quad m = 1, 2, \ldots, N. \quad (3.10)
\]
We can compute all the coefficients \(x_{m,n}, y_{m,n}, r_{m,n}, s_{m,n}\), in terms of \(N\) free parameters \(\beta_1, \ldots, \beta_N\), one for each leg. Indeed, if we fix \(\beta_m\) for \(m = 1, 2, \ldots, N\), we get \(s_{1,m}\), for \(m = 1, 2, \ldots, N\), from equation (3.8) and \(r_{1,m}\), for \(m = 1, 2, \ldots, N\), from equation (3.6). After this we get \(y_{1,m}\), for \(m = 1, 2, \ldots, N\), from equation (3.10).
and $x_{1,m}$ for $m = 1, 2, \ldots, N$, from equation (3.5). Then we get $s_{2,m}$ for $m = 1, 2, \ldots, N$ from equation (3.9) and so on using the same equations.

In the same fashion as in [12, 17, 18], let us now see under what conditions we can guarantee a stochastic reflecting-absorbing factorization. Let

$$H_m = \begin{pmatrix} \frac{\alpha_m}{1} & -\frac{c_{1,m}}{1} & -\frac{a_{1,m}}{1} & -\frac{c_{2,m}}{1} & \cdots \end{pmatrix}, \quad m = 1, 2, \ldots, N,$$

(3.11)

be the continued fraction with sequence of convergents given by

$$h_{n,m} = \frac{A_{n,m}}{B_{n,m}}, \quad n \geq 0, \quad m = 1, \ldots, N.$$  

(3.12)

The sequences $(A_{n,m})_{n \geq 0}$ and $(B_{n,m})_{n \geq 0}$ for every $m = 1, \ldots, N$, can be recursively obtained using the formulas

$$A_{2n,m} = A_{2n-1,m} - c_{n,m}A_{2n-2,m}, \quad n \geq 1, \quad A_{2n+1,m} = A_{2n,m} - a_{n,m}A_{2n-1,m}, \quad n \geq 0,$$

$$A_{-1,m} = -1, \quad A_{0,m} = 0,$$

$$B_{2n,m} = B_{2n-1,m} - c_{n,m}B_{2n-2,m}, \quad n \geq 1, \quad B_{2n+1,m} = B_{2n,m} - a_{n,m}B_{2n-1,m}, \quad n \geq 0,$$

$$B_{-1,m} = 0, \quad B_{0,m} = 1,$$

where here we are calling $a_{0,m} = \alpha_m$.

**Theorem 3.1.** Let $H_m, \ m = 1, 2, \ldots, N$, be the continued fractions defined by (3.11) with their corresponding sequences of convergents (3.12). Assume that

$$0 < A_{n,m} < B_{n,m}, \quad n \geq 0, \quad m = 1, \ldots, N.$$  

Then the continued fractions $H_m, \ m = 1, 2, \ldots, N$, are all convergent. Moreover, let $P = P_RP_A$ and assume that $\sum_{m=1}^{N} H_m < 1$. Then $P_R$ and $P_A$ are stochastic matrices if and only if

$$\beta_m \geq H_m, \quad m = 1, 2, \ldots, N.$$  

(3.13)

**Proof.** For $m = 1, 2, \ldots, N$, it is not hard to proof that

$$A_{2n,m}B_{2n+1,m} - B_{2n,m}A_{2n+1,m} = -\alpha_m c_{1,m}a_{1,m} \cdots a_{n,m}, \quad n \geq 0,$$

$$A_{2n+1,m}B_{2n+2,m} - B_{2n+1,m}A_{2n+2,m} = -\alpha_m c_{1,m}a_{1,m} \cdots c_{n+1,m}, \quad n \geq 0,$$

and consequently

$$h_{2n,m} - h_{2n+1,m} = \frac{A_{2n,m}}{B_{2n,m}} - \frac{A_{2n+1,m}}{B_{2n+1,m}} = -\frac{\alpha_m c_{1,m}a_{1,m} \cdots a_{n,m}}{B_{2n,m}B_{2n+1,m}} < 0, \quad n \geq 0,$$

$$h_{2n+1,m} - h_{2n+2,m} = \frac{A_{2n+1,m}}{B_{2n+1,m}} - \frac{A_{2n+2,m}}{B_{2n+2,m}} = -\frac{\alpha_m c_{1,m}a_{1,m} \cdots c_{n+1,m}}{B_{2n+2,m}B_{2n+1,m}} < 0, \quad n \geq 0.$$  

Therefore we conclude that

$$0 = h_{0,m} < \cdots < h_{2n,m} < h_{2n+1,m} < h_{2n+2,m} < \cdots < 1,$$

and then the sequences $(h_{n,m})_{n \geq 0}$ are all bounded and strictly increasing, so they converge to $H_m$ for every $m = 1, \ldots, N$. Now assume that $\sum_{m=1}^{N} H_m < 1$ and $P_R$ and $P_A$ are stochastic matrices. Then it is clear that

$$\beta_m > 0 = h_{0,m},$$

and using equation (3.8) we have

$$s_{1,m} = \frac{\alpha_m}{\beta_m} < 1 \iff \beta_m > \alpha_m = h_{1,m}.$$
Using now equations (3.10), (3.6) and (3.8) we have
\[ y_{1,m} = \frac{c_{1,m}}{r_{1,m}} \iff 1 - s_{1,m} > c_{1,m} \iff \beta_m > \frac{\alpha_m}{1 - c_{1,m}} = h_{2,m}, \]
and in general it can be shown that
\[ \beta_m > h_{n,m}. \]
Therefore \( 0 = h_{0,m} < h_{n,m} < H_m = \beta_m \). On the contrary, if (3.13) holds, in particular we have that \( \beta_m > h_{n,m} \) for every \( n \geq 0, m = 1, \ldots, N \). Following the same steps as before, using an argument of strong induction, will lead us to the fact that both \( P_R \) and \( P_L \) are stochastic matrices (see the proof of Proposition 2.1 of [18] for more details).

**Remark 3.2.** The reflecting-absorbing factorization \( P = P_R P_A \) is just one type of a stochastic block UL factorization of \( P \), but there can be more possibilities. Also we could have considered a stochastic block LU factorization of \( P \). As it was pointed out in [13], the different stochastic block factorizations of \( P \) may come with many degrees of freedom, and the analysis is more complicated than the case of classical birth-death chains.

### 3.1. Stochastic Darboux transformation and the associated spectral matrix

Once we have the conditions under we can perform a stochastic reflecting-absorbing factorization, it is possible to compute what is called a discrete Darboux transformation, consisting on inverting the order of the factors. The Darboux transformation has a long history but probably the first reference of a discrete Darboux transformation like we study here appeared in [24] in connection with the Toda lattice.

If \( P = P_R P_A \) as in (3.1), then, by inverting the order of the factors, we obtain another stochastic matrix of the form \( \tilde{P} = P_A P_R \), since the multiplication of two stochastic matrices is again a stochastic matrix. This new matrix preserves the block tridiagonal structure. Also \( \tilde{P} \) will be a family (depending on \( N \) free parameters \( \beta_1, \ldots, \beta_N \)) of Markov chains \( \{ \tilde{S}_n, n = 0, 1, \ldots \} \) on a spider \( S_N \) which is “almost” a family of birth-death chains. The only difference is that we will have extra transitions between the first states of each leg or, in other words, between the states \( 1, 2, \ldots, N \). If we call \( \tilde{B}_n, \tilde{A}_n, \tilde{C}_n+1, n \geq 0 \), the new coefficients of the block tridiagonal matrix \( \tilde{P} \), a direct computations shows
\[
\begin{align*}
\tilde{A}_n &= S_n X_n, & n \geq 0, \\
\tilde{B}_0 &= S_0 Y_0, & \tilde{B}_n &= R_n X_{n-1} + S_n Y_n, & n \geq 1, \\
\tilde{C}_n &= R_n Y_{n-1}, & n \geq 1. 
\end{align*}
\]
(3.14)

If we look entry by entry these relations can also be written as
\[
\begin{align*}
\tilde{\alpha}_m &= \beta_m, & m = 0, 1, \ldots, N, \\
\tilde{\alpha}_{n,m} &= s_{n,m} x_{n,m}, & n \geq 1, & m = 1, \ldots, N, \\
\tilde{b}_{1,m} &= r_{1,m} \beta_m + s_{1,m} y_{1,m}, & m = 1, \ldots, N, \\
\tilde{b}_{n,m} &= r_{n,m} x_{n-1,m} + s_{n,m} y_{n,m}, & n \geq 1, & m = 1, \ldots, N, \\
\tilde{c}_{1,m} &= r_{1,m} \beta_m, & m = 0, 1, \ldots, N, \\
\tilde{c}_{n,m} &= r_{n,m} y_{n-1,m}, & n \geq 1, & m = 1, \ldots, N.
\end{align*}
\]

We also have extra transition probabilities between the first states of each leg, given by
\[
\tilde{d}_{i,j} = \beta_j r_{i,j}, i,j = 1, \ldots, N, \quad i \neq j.
\]
A diagram of this process is similar to the one for the process described by \( P \) in Figure 1 but now we have to add probabilities between the first states of each leg. For instance, for \( N = 2 \) we get the diagram in page 10 of [18]. For \( N = 3 \) and \( N = 4 \) we have the diagrams in Figure 3. In general we have to add \( N(N - 1) \) extra transition probabilities between the first states of each leg.
In Proposition 2.1 we proved that all birth-death chains \( \{S_n, n = 0, 1, \ldots \} \) on a spider \( S_N \) have an associated weight matrix \( W \) supported on the interval \([-1, 1]\). Now we wonder if we can find the spectral matrix \( \tilde{W} \) associated with the Darboux transformation \( \tilde{P} \). This is possible using Remark 2.4 of [18]. The spectral matrix is then given by

\[
\tilde{W}(x) = S_0 \left( \frac{W(x)}{x} + \left( (\Pi_0 Y_0 S_0)^{-1} - M_{-1} \right) \delta_0(x) \right) S_0^T, \tag{3.15}
\]

where \( S_0 \) and \( Y_0 \) are given by (3.2), \( \Pi_0 \) is given by (2.10) (see also (2.18)), \( M_{-1} = \int_{-1}^{1} x^{-1} W(x) dx \) and \( \delta_0(x) \) is the Dirac delta at 0. A direct computation shows that

\[
\begin{pmatrix}
\alpha_0 + \alpha_N - \beta_N & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\
\alpha_1 & \frac{\alpha_1^2}{\beta_1 - \alpha_1} & 0 & \cdots & 0 \\
\alpha_2 & 0 & \frac{\alpha_2^2}{\beta_2 - \alpha_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{N-1} & 0 & 0 & \cdots & \frac{\alpha_{N-1}^2}{\beta_{N-1} - \alpha_{N-1}}
\end{pmatrix}
\]

\[
\Pi_0 Y_0 S_0 = \begin{pmatrix}
\alpha_0 + \alpha_N - \beta_N & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\
\alpha_1 & \frac{\alpha_1^2}{\beta_1 - \alpha_1} & 0 & \cdots & 0 \\
\alpha_2 & 0 & \frac{\alpha_2^2}{\beta_2 - \alpha_2} & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
\alpha_{N-1} & 0 & 0 & \cdots & \frac{\alpha_{N-1}^2}{\beta_{N-1} - \alpha_{N-1}}
\end{pmatrix}.
\]
Observe that this matrix is always symmetric. If we call \( X = ( \Pi_0 Y_0 S_0 )^{-1} \), it can be shown that the entries of \( X = (X_{ij}) \) for \( i \leq j \) (for \( i \geq j \) we have to change \( i \) by \( j \) since \( X \) is symmetric) are given by

\[
X_{ij} = \frac{1}{\beta_0} \begin{cases} 
1, & \text{if } i = j = 1, \\
1 - \frac{\alpha_{j-1}}{\alpha_{j-1}}, & \text{if } i = 1, j > 1, \\
\left(1 - \frac{\beta_{j-1}}{\alpha_{j-1}}\right) \left(1 - \frac{\beta_{j-1}}{\alpha_{j-1}}\right), & \text{if } i > 1, j > i, \\
\left(1 - \frac{\beta_{j-1}}{\alpha_{j-1}}\right) \left(1 - \frac{\beta_{j-1}}{\alpha_{j-1}} - \frac{\beta_{j-1}}{\alpha_{j-1}}\right), & \text{if } i > 1, i = j.
\end{cases}
\] (3.16)

However, it may be hard to compute the moment \( M_{-1} \). But observe that \( M_{-1} = B(0; W) \), where \( B(z; W) \) is the Stieltjes transform of \( W \) defined by (2.12). Using Proposition 2.4 we may have a way to compute explicitly \( B(z; W) \). In some cases, as we will see in the example of the next section, if will be possible to compute explicitly \( B(z; W) \) and therefore the moment \( M_{-1} \).

Finally, we can also compute the matrix-valued orthogonal polynomials \( \tilde{Q}_n \) associated with \( \tilde{W} \) using Theorem 2.3 of [1]. Consider the matrix-valued polynomials

\[
U_0(x) = S_0 Q_0(x) = S_0, \quad U_n(x) = R_n Q_{n-1}(x) + S_n Q_n(x), \quad n \geq 1,
\]

where \( (S_n)_{n \geq 0} \) and \( (R_n)_{n \geq 1} \) are defined by (3.2), (3.3) and (3.4), respectively. If we denote by \( Q = (Q_0^T, Q_1^T, \ldots)^T \) and \( U = (U_0^T, U_1^T, \ldots)^T \), then we have that \( U = P_A Q \), where \( P_A \) is given by (3.1). The matrix-valued orthogonal polynomials \( \tilde{Q}_n \) are then defined by

\[
\tilde{Q}_n(x) = U_n(x) S_0^{-1}, \quad n \geq 0.
\]

Since we have an explicit expression of the polynomials \( Q_n \) in (2.5) and

\[
S_0^{-1} = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
-\frac{r_{1,1}}{s_{1,1}} & 1 & 0 & \cdots & 0 \\
-\frac{r_{1,2}}{s_{1,2}} & 0 & \frac{1}{s_{1,2}} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
-\frac{r_{1,N-1}}{s_{1,N-1}} & 0 & \cdots & \frac{1}{s_{1,N-1}}
\end{pmatrix},
\] (3.17)

then we have that

\[
\tilde{Q}_n(x) = \begin{pmatrix}
R_{n,N}(x) & \frac{\alpha_1}{s_{1,1}} R_{n,N}^{(0)}(x) & \frac{\alpha_2}{s_{1,2}} R_{n,N}^{(0)}(x) & \cdots & \frac{\alpha_{N-1}}{s_{1,N-1}} R_{n,N}^{(0)}(x) \\
R_{n,1}(x) & R_{n,1}(x) & 0 & \cdots & 0 \\
R_{n,2}(x) & 0 & R_{n,2}(x) & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
R_{n,N}(x) & 0 & 0 & \cdots & R_{n,N}(x)
\end{pmatrix}, \quad n \geq 0,
\] (3.18)
Therefore, using (3.2), (3.17), (3.8) and (3.10), we obtain
\[\tilde{\Pi}_0 = \text{diag} \left( \beta_0, \frac{\beta_1}{r_{1,1}}, \ldots, \frac{\beta_{N-1}}{r_{1,N-1}} \right) = \text{diag} \left( \beta_0, \frac{\beta_1}{\beta_1 - \alpha_1}, \ldots, \frac{\beta_{N-1}}{\beta_{N-1} - \alpha_{N-1}} \right).\]

Therefore \(\tilde{\Pi}_n\) are always diagonal matrices. As a consequence we obtain the (diagonal) norms of the matrix-valued orthogonal polynomials \((\tilde{Q}_n)_{n \geq 0}\):
\[\tilde{\Pi}_n = \left( \|\tilde{Q}_n\|_W^2 \right)^{-1} = \left( \int_{-1}^{1} \tilde{Q}_n(x) d\tilde{W}(x) \tilde{Q}_n^T(x) \right)^{-1},\]
and the orthogonality relations
\[\int_{-1}^{1} \tilde{Q}_n(x) d\tilde{W}(x) \tilde{Q}_m^T(x) = \tilde{\Pi}_n^{-1} \delta_{n,m}.\]

4. An example: random walk on a spider

Consider the block tridiagonal transition probability matrix \(B_0\) with constant transition probability coefficients, i.e.
\[
B_0 = \begin{pmatrix}
\alpha_0 & \alpha_1 & \alpha_2 & \cdots & \alpha_{N-1} \\
c & b & 0 & \cdots & 0 \\
c & 0 & b & \cdots & 0 \\
\vdots & \vdots & \vdots & \ddots & \vdots \\
c & 0 & 0 & \cdots & b
\end{pmatrix}, \quad A_0 = \text{diag} (\alpha_N, a, \ldots, a), \quad A_n = aI_N, \quad B_n = bI_N, \quad C_n = cI_N, \quad n \geq 1,
\]
where \(\sum_{k=0}^{N} \alpha_k = 1\), and \(a + b + c = 1\). Observe that in this case the vector \(\hat{c}\) defined by (2.14) is given by \(\hat{c} = c \hat{e}_{N-1}\), where \(\hat{e}_k = (1, 1, \ldots, 1)^T\) is the vector of dimension \(k\) with all components equal to 1. Also the matrix \(c_D\) defined by (2.14) is given by \(c_D = c I_{N-1}\). The Stieltjes transforms of \(\omega_k^{(0)}\), \(k = 1, \ldots, N\), do not depend on \(k\) and are given by
\[
B(z; \omega_k^{(0)}) = \frac{b - z + \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2ac}, \quad \sigma_{\pm} = 1 - (\sqrt{a} \mp \sqrt{c})^2. \quad (4.1)
\]
Therefore, using the second formula in (2.13) and rationalizing, we obtain
\[ B(z; \omega_k) = -\frac{2}{z - b + \sqrt{(z - \sigma_+)(z - \sigma_-)}} = \frac{b - z + \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2ac}, \quad k = 1, \ldots, N - 1. \]

As a consequence
\[ B(z; \omega_D) = \frac{b - z + \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2ab} I_{N-1}. \]

On the other hand, using the first formula in (2.13) and (4.1), we obtain
\[ \frac{1}{B(z; \omega_N)} = -\left[ z - \alpha_0 + \alpha_N c B(z; \omega_N^0) \right] = -\left[ z - \alpha_0 + \frac{\alpha_N}{2a} \left( b - z + \sqrt{(z - \sigma_+)(z - \sigma_-)} \right) \right]. \]

Using the previous two formulas in (2.17) and the fact that \( \alpha_1 + \cdots + \alpha_{N-1} = 1 - \alpha_0 - \alpha_N \), we obtain an expression for \( b(z) \) in Proposition 2.4, given by
\[ b(z) = \frac{1}{\alpha_0 - \frac{b(1 - \alpha_0)}{2a} - \left( 1 - \frac{1 - \alpha_0}{2a} \right) z - \frac{1 - \alpha_0}{2a} \sqrt{(z - \sigma_+)(z - \sigma_-)}}. \]

After rationalizing we get
\[ b(z) = \frac{(1 - 2a - \alpha_0)z - b + \alpha_0(1 + a - c) + (1 - \alpha_0)\sqrt{(z - \sigma_+)(z - \sigma_-)}}{2(1 - z) \left[ (1 - \alpha_0)z + c - \alpha_0(1 - a + c - \alpha_0) \right]}. \]

Therefore we have all the functions necessary to compute the Stieltjes transform of \( W \), given by (2.16) of Proposition 2.4. After some computations we can write \( B(z; W) \) as
\[ B(z; W) = \begin{pmatrix} B_{11}(z; W) & B_{12}(z; W) e_N^T \alpha_D^{-1} \\ B_{21}(z; W) e_{N-1}^T & B_{22}(z; W) \end{pmatrix}, \tag{4.2} \]
where
\[ B_{11}(z; W) = b(z), \]
\[ B_{12}(z; W) = \frac{2c - \alpha_0(1 - a + c) + (b + \alpha_0)z - z^2 + (z - \alpha_0)\sqrt{(z - \sigma_+)(z - \sigma_-)}}{2(1 - z) \left[ (1 - \alpha_0)z + c - \alpha_0(1 - a + c - \alpha_0) \right]}, \]
\[ B_{22}(z; W) = \frac{b - z + \sqrt{(z - \sigma_+)(z - \sigma_-)}}{2a}, \]
\[ e_{N-1} \alpha_D^{-1} = \frac{p(z) + r(z)\sqrt{(z - \sigma_+)(z - \sigma_-)}}{2a(1 - z) \left[ (1 - \alpha_0)z + c - \alpha_0(1 - a + c - \alpha_0) \right]} e_{N-1}^T, \]
and
\[ p(z) = -z^3 + (\alpha_0 + 2b)z^2 - (\alpha_0(2 - 2a - c) + b^2 - 2ac - c)z - bc + \alpha_0(b - a(1 - a + c)), \]
\[ r(z) = z^2 - (\alpha_0 + b)z - c + \alpha_0(1 - a). \]

The weight matrix \( W \) will consist in the addition of an absolutely continuous part \( W_c \) and a discrete part \( W_d \), i.e. \( W = W_c + W_d \). On one hand, using the Perron-Stieltjes inversion formula, we get that \( W_c \) is given by
\[ W_c(x) = \begin{pmatrix} W_{11}(x) & W_{12}(x) e_N^T \alpha_D^{-1} \\ W_{21}(x) e_{N-1}^T & W_{22}(x) \end{pmatrix}, \quad x \in [\sigma_-, \sigma_+], \tag{4.3} \]
where

\[
W_{11}(x) = \frac{(1 - \alpha_0) \sqrt{(\sigma_+ - x)(x - \sigma_-)}}{2\pi(1 - x) [(1 - a - \alpha_0)x + c - \alpha_0(1 - a + c - \alpha_0)],}
\]

\[
W_{12}(x) = \frac{(x - \alpha_0) \sqrt{(\sigma_+ - x)(x - \sigma_-)}}{2\pi(1 - x) [(1 - a - \alpha_0)x + c - \alpha_0(1 - a + c - \alpha_0)],}
\]

\[
W_{22}(x) = \frac{\sqrt{(\sigma_+ - x)(x - \sigma_-)}}{2\pi a} \alpha_D^{-1} + \frac{r(x) \sqrt{(\sigma_+ - x)(x - \sigma_-)}}{2\pi a(1 - x) [(1 - a - \alpha_0)x + c - \alpha_0(1 - a + c - \alpha_0)]} e_{N-1} e_{N-1}^T.
\]

On the other hand, the discrete part \( W_d \) will be given by the behavior of the Stieltjes transform \( B(z; W) \) at its poles, given in this case by

\[ z_1 = 1, \quad z_2 = \frac{\alpha_0(1 - a + c - \alpha_0) - c}{1 - a - \alpha_0}. \]

After some computations we get

\[
W_d(x) = \frac{c - a}{c - a + 1 - \alpha_0} \delta_1(x) \bar{e}_N \bar{e}_N^T \chi_{(c > a)} + \frac{(1 - \alpha_0 - a)^2 - ac}{(1 - \alpha_0)(1 - a - \alpha_0 + c)} \delta_2(x) \bar{u}_N \bar{u}_N^T \chi_{(1 - \alpha_0 - a)^2 > ac},
\]

where \( \chi_A \) is the indicator function and

\[
\bar{u}_N = \left(1, \frac{c}{1 - \alpha_0 - a}, \ldots, \frac{c}{1 - \alpha_0 - a}\right)^T.
\]

From Remark 2.3 we can study recurrence for the random walk on a spider. There will be three cases:

- If \( a > c \), then we have that \([\sigma_-, \sigma_+] \subseteq [-1, 1]\). Therefore all integrals in (2.11) are bounded and the random walk on a spider is transient.
- If \( a = c \), then we have \([\sigma_-, \sigma_+] = [1 - 4a, 1]\). Therefore all integrals in (2.11) are divergent and the random walk on a spider is null recurrent (since there is no jump at the point 1).
- If \( a < c \), there will always be a jump at the point 1 (see (4.4)). Therefore the random walk on a spider is positive recurrent.

Finally, it is possible to see, by looking at the three-term recurrence relations (2.6) and (2.7), that the entries of the matrix-valued polynomials \( Q_n(x) \) in (2.5) are given by

\[
Q_{n,N}(x) = \frac{1}{\alpha_N} \left(\frac{c}{a}\right)^{n/2} \left[2(\alpha_N - a)T_n \left(\frac{x - b}{2\sqrt{ac}}\right) + (2\alpha_N - \alpha N)U_n \left(\frac{x - b}{2\sqrt{ac}}\right) + \alpha c(b - \alpha_0)U_{n-1} \left(\frac{x - b}{2\sqrt{ac}}\right)\right],
\]

\[
Q_{n,N}^{(0)}(x) = -\frac{1}{\alpha_N} \left(\frac{c}{a}\right)^{(n-1)/2} U_{n-1} \left(\frac{x - b}{2\sqrt{ac}}\right),
\]

\[
Q_{n,k}(x) = \left(\frac{c}{a}\right)^{n/2} U_n \left(\frac{x - b}{2\sqrt{ac}}\right), \quad Q_{n,k}^{(0)}(x) = -\left(\frac{c}{a}\right)^{(n+1)/2} U_{n-1} \left(\frac{x - b}{2\sqrt{ac}}\right), \quad k = 1, \ldots, N - 1,
\]

where \( (T_n) \) and \( (U_n) \) are the Chebychev polynomials of the first and second kind, respectively.

Let us now apply Theorem 3.1 and see under what conditions we can perform a reflecting-absorbing factorization of the form \( P = P_R P_A \) for this example as the one we saw in Section 3. The continued fractions in (3.11) can be written as \( H_m = \alpha_m/H \), where

\[
H = 1 - \frac{c}{1} - \frac{a}{1} - \frac{c}{1} - \frac{a}{1} - \ldots.
\]
$H$ is a continued fraction of period 2, and the explicit value is given by

$$H = \frac{1}{2} \left( 1 + a - c + \sqrt{(1 + c - a)^2 - 4c} \right),$$

as long as $a \leq (1 - \sqrt{7})^2$. Therefore, after rationalizing, we get

$$H_m = \frac{\alpha_m}{2a} \left( 1 + a - c - \sqrt{(1 + c - a)^2 - 4c} \right), \quad m = 1, \ldots, N.$$  

According to Theorem 3.1, the stochastic reflecting-absorbing factorization will be possible if and only if

$$\beta_m \geq \frac{\alpha_m}{2a} \left( 1 + a - c - \sqrt{(1 + c - a)^2 - 4c} \right), \quad m = 1, \ldots, N,$$  

and from $\sum_{m=1}^{N} H_m < 1$ we need to have

$$\alpha_0 > \frac{1}{2} \left( 1 + a - c - \sqrt{(1 + c - a)^2 - 4c} \right).$$

For instance, for $N = 3$, the set of values

$$a = \frac{1}{5}, \quad b = \frac{11}{20}, \quad c = \frac{1}{4}, \quad \alpha_0 = \frac{1}{2}, \quad \alpha_1 = \frac{1}{8}, \quad \alpha_2 = \frac{1}{6}, \quad \alpha_3 = \frac{5}{24},$$

gives a stochastic reflecting-absorbing factorization if and only if

$$\beta_1 \geq \frac{19 - \sqrt{11}}{64} \sim 0.196826 \ldots, \quad \beta_2 \geq \frac{19 - \sqrt{11}}{48} \sim 0.262434 \ldots, \quad \beta_3 \geq \frac{95 - 5\sqrt{11}}{192} \sim 0.328043 \ldots.$$

Under conditions (4.6) and (4.7) we can perform a stochastic discrete Darboux transformation given by $\tilde{P} = PA_P R$. This new block tridiagonal matrix $\tilde{P}$ gives rise to a family (depending on $N$ free parameters $\beta_1, \ldots, \beta_N$) of Markov chains $\{\tilde{S}_n, n = 0, 1, \ldots\}$ on a spider $S_N$ with coefficients given by (3.14). As we mentioned in Section 3.1, $\tilde{P}$ is “almost” a birth-death chain on a spider, since we have to add extra probability transitions between the first states of each leg (see Figure 3). A direct computation shows that the new coefficients $\tilde{A}_0$ and $\tilde{B}_0$ of $\tilde{P}$ (see (3.14)), which give the extra transitions between the first states of each leg, are given by

$$\tilde{B}_0 = \begin{pmatrix}
\beta_0 & \beta_1 & & & \\
\beta_0 (1 - \frac{\alpha_1}{\beta_1}) & \beta_1 - \alpha_1 + \frac{\alpha_0}{\beta_1 - \alpha_1} & \beta_2 (1 - \frac{\alpha_1}{\beta_1}) & & \\
\beta_0 (1 - \frac{\alpha_2}{\beta_2}) & \beta_1 (1 - \frac{\alpha_2}{\beta_2}) & \beta_2 - \alpha_2 + \frac{\alpha_0}{\beta_2 - \alpha_2} & \beta_3 (1 - \frac{\alpha_2}{\beta_2}) & \\
\vdots & \vdots & \vdots & \ddots & \ddots & \ddots \\
\beta_0 (1 - \frac{\alpha_{N-1}}{\beta_{N-1}}) & \beta_1 (1 - \frac{\alpha_{N-1}}{\beta_{N-1}}) & \cdots & \beta_{N-1} - \alpha_{N-1} + \frac{\alpha_0}{\beta_{N-1} - \alpha_{N-1}} & \\
\end{pmatrix},$$

$$\tilde{A}_0 = \begin{pmatrix}
\beta_N & 0 & & & \\
\beta_N (1 - \frac{\alpha_1}{\beta_1}) & \frac{\alpha_1}{\beta_1} - \frac{\alpha_0}{\beta_1 - \alpha_1} & 0 & & \\
\beta_N (1 - \frac{\alpha_2}{\beta_2}) & 0 & \frac{\alpha_2}{\beta_2} - \frac{\alpha_0}{\beta_2 - \alpha_2} & 0 & \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
\beta_N (1 - \frac{\alpha_{N-1}}{\beta_{N-1}}) & 0 & \cdots & \frac{\alpha_{N-1}}{\beta_{N-1} - \alpha_{N-1}} & 0 &
\end{pmatrix}.\]
Finally, the weight matrix $\tilde{W}$ associated with $\tilde{P}$ is given by \((3.15)\), i.e.

$$
\tilde{W}(x) = S_0 \left( \frac{W(x)}{x} + [X - M_{-1}] \delta_0(x) \right) S_0^T,
$$

where

$$
S_0 = \begin{pmatrix}
1 & 0 & 0 & \cdots & 0 \\
1 - \frac{\alpha_1}{\beta_1} & \frac{\alpha_1}{\beta_1} & 0 & \cdots & 0 \\
1 - \frac{\alpha_2}{\beta_2} & 0 & \frac{\alpha_2}{\beta_2} & \cdots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
1 - \frac{\alpha_{N-1}}{\beta_{N-1}} & 0 & \cdots & \frac{\alpha_{N-1}}{\beta_{N-1}} & \beta_{N-1} \\
\end{pmatrix},
$$

the weight matrix $W = W_c + W_d$ is given by \((4.3)\) and \((4.4)\), $X$ is the symmetric matrix given by \((3.16)\) and from \((4.2)\) we have

$$
M_{-1} = \begin{pmatrix}
\mu_{11} & \mu_{12} \tilde{e}_{N-1}^T \\
\mu_{12} \tilde{e}_{N-1} & \mu_{22} \\
\end{pmatrix},
$$

where

$$
\mu_{11} = \frac{\alpha_0(1 + a - c) - b - (1 - \alpha_0)\sqrt{\sigma_+ \sigma_-}}{2c[1 - \alpha_0(1 - a + c)]}, \quad \mu_{12} = \frac{2c - \alpha_0(1 - a + c) + \alpha_0\sqrt{\sigma_+ \sigma_-}}{2[c - \alpha_0(1 - a + c)]},
$$

$$
\mu_{22} = \frac{b - \sqrt{\sigma_+ \sigma_-} \alpha_D^{-1} + \alpha_0(b - a(1 - a + c)) - bc - (\alpha_0(1 - a) - c)\sqrt{\sigma_+ \sigma_-} \tilde{e}_{N-1}^T \tilde{e}_{N-1}}{2a[c - \alpha_0(1 - a + c)]}.
$$

The matrix-valued polynomials $(\tilde{Q}_n)_{n \geq 0}$ orthogonal with respect to $\tilde{W}$ can be computed from \((3.18)\) and can also be written as combinations of Chebyshev polynomials of the first and second kind (see \((4.5)\)).

**References**

[1] Barlow, M.T., Pitman, J.W. and Yor, M., *On Walsh’s Brownian motion*, in Azéma J., Yor M., Meyer P.A. (eds) Séminaire de Probabilités XXIII. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 1972 (1989), 275–293.

[2] Cherny, A., Shiryaev, A. and Yor, M., *Limit behavior of the “horizontal-vertical” random walk and some extension of the Donsker-Prokhorov invariance principle*, Teor. Veroyatnost. i Primenen., 47 (2002), 498–517; Theory Probab. Appl., 47 (2003), 377–394.

[3] Clayton, A., *Quasi-birth-death processes and matrix-valued orthogonal polynomials*, SIAM J. Matrix Anal. Appl. 31 (2010), 2239–2260.

[4] Csáki, E., Csörgő, M., Földes, A. and Révész, P., *Some limit theorems for heights of random walks on a spider*, J. Theor. Probab. 29 (2016), 1685–1709.

[5] Csáki, E., Csörgő, M., Földes, A. and Révész, P., *Limit theorems for local and occupation times of random walks and Brownian motion on a spider*, J. Theor. Probab. 32 (2019), 330–352.

[6] Dette, H., Reuther, B., Studden, W. and Zygmunt, M., *Matrix measures and random walks with a block tridiagonal transition matrix*, SIAM J. Matrix Anal. Appl. 29 (2006), 117–142.

[7] Domínguez de la Iglesia, M., *Orthogonal polynomials in the spectral analysis of Markov processes. Birth-death models and diffusion*, Encyclopedia of Mathematics and its Applications 181, Cambridge University Press, 2021.

[8] Evans, S.N. and Sowers, R.B., *Pinching and twisting Markov Processes*, Ann. Probab. 31 (2003), 486–527.

[9] Grünbaum, F.A., *Random walks and orthogonal polynomials: some challenges*, Probability, Geometry and Integrable Systems, MSRI Publication, volume 55, 2007.

[10] Grünbaum, F.A., *The Karlin-McGregor formula for a variant of a discrete version of Walsh’s spider*, J. Phys. A 42 (2009), no. 45, 454010, 10 pp.

[11] Grünbaum, F.A., *A spectral weight matrix for a discrete version of Walsh’s spider*, Operator theory: Advances and Applications, 202 (2010), 253–264.
[12] Grünbaum, F.A. and de la Iglesia, M.D., Stochastic LU factorizations, Darboux transformations and urn models, J. Appl. Prob. 55 (2018), 862–886.
[13] Grünbaum, F.A. and de la Iglesia, M.D., Stochastic Darboux transformations for quasi-birth-death processes and urn models, J. Math. Anal. Appl. 478 (2019), 634–654.
[14] Hajri, H., Discrete approximations to solution flows of Tanaka’s SDE related to Walsh Brownian motion, in Donati-Martin C., Lejay A., Rouault A. (eds) Séminaire de Probabilités XLIV. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg, 2046 (2012), 167–190.
[15] Harrison, J. and Shepp, L., On skew Brownian motion, Ann. Probab. 9 (1981), 309–313.
[16] de la Iglesia, M.D., A note on the invariant distribution of a quasi-birth-death process, J. Phys. A: Math. Theor. 44 (2011) 135201 (9pp).
[17] de la Iglesia, M.D. and Juarez, C., The spectral matrices associated with the stochastic Darboux transformations of random walks on the integers, J. Approx. Theory 258 (2020), 105458, 32 pp.
[18] de la Iglesia, M.D. and Juarez, C., Absorbing-reflecting factorizations for birth-death chains on the integers and their Darboux transformations, J. Approx. Theory 266 (2021), 105583, 27 pp.
[19] Itô, K. and McKean, H.P. Jr., Diffusion and their sample paths, Second printing, corrected. Die Grundlehren der mathematischen Wissenschaften, Band 125. Springer-Verlag, Berlin-New York, 1974.
[20] Karlin, S. and McGregor, J., Random walks, Illinois J. Math., 3 (1959), 66–81.
[21] Latouche, G. and Ramaswami, V., Introduction to Matrix Analytic Methods in Stochastic Modeling, ASA-SIAM Series on Statistics and Applied Probability, 1999.
[22] Le Gall J.F., One-dimensional stochastic differential equations involving the local times of the unknown process, in Truman A., Williams D. (eds) Stochastic Analysis and Applications. Lecture Notes in Mathematics, Springer, Berlin, Heidelberg. 1095 (1984), 51–82.
[23] Lejay, A., On the constructions of the skew Brownian motion, Probab. Surv. 3 (2006), 413–466.
[24] Matveev, V.B. and Salle, M.A., Differential-difference evolution equations II: Darboux transformation for the Toda lattice, Lett. Math. Phys. 3 (1979), 425–429.
[25] Neuts, M.F., Structured Stochastic Matrices of M/G/1 Type and Their Applications, Marcel Dekker, New York, 1989.
[26] Papanicolaou, V.G., Papageorgiou, E.G. and Lepipas, D.C., Random motion on simple graph, Methodol. Comput. Appl. Probab. 14 (2012), 285–297.
[27] Seol, Y., On weak limiting distributions for random walks on a spider, Symmetry 2020, 12 (12), 2000.
[28] Vakeroudis, S. and Yor, M., A scaling proof for Walsh’s Brownian motion extended arc-sine law, Electron. Commun. Probab. 17 (2012), 1–9.
[29] Walsh, J.B., A diffusion with discontinuous local time, Astérisque 52-53 (1978), 37–45.

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