On the space of Laplace transformable distributions

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Abstract
We show that the space $S'/(\Gamma_1)$ of Laplace transformable distributions, where $\Gamma_1 \subseteq \mathbb{R}^d$ is a non-empty convex open set, is an ultrabornological (PLS)-space. Moreover, we determine an explicit topological predual of $S'/(\Gamma_1)$.

Keywords
Laplace transform · Distributions · Ultrabornological (PLS)-spaces · Short-time Fourier transform

Mathematics Subject Classification
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1 Introduction
Schwartz introduced the space $S'/(\Gamma)$ of Laplace transformable distributions as

$$S'/(\Gamma) = \{ f \in \mathcal{D}'(\mathbb{R}^d) \mid e^{-\xi \cdot x} f(x) \in S'(\mathbb{R}^d) \ \forall \xi \in \Gamma \},$$

where $\Gamma \subseteq \mathbb{R}^d$ is a non-empty convex set [1, p. 303]. This space is endowed with the projective limit topology with respect to the mappings $S'/(\Gamma) \to S'(\mathbb{R}^d)$, $f \mapsto e^{-\xi \cdot x} f(x)$ for $\xi \in \Gamma$. The second author together with Kunzinger and Ortner [2] recently presented two new proofs of Schwartz’s exchange theorem for the Laplace transform of vector-valued distributions [3, Prop. 4.3, p. 186]. Their methods required them to show that $S'/(\Gamma)$ is complete, nuclear and dual-nuclear [2, Lemma 5]. Following a suggestion of Ortner, in this article, we further study the locally convex structure of the space $S'/(\Gamma)$.

In order to be able to apply functional analytic tools such as De Wilde’s open mapping and closed graph theorems [4, Theorem 24.30 and Theorem 24.31] or the theory of the derived projective limit functor [5], it is important to determine when a space is ultrabornological. This is usually straightforward if the space is given by a suitable inductive limit; in fact,
ultrabornological spaces are exactly the inductive limits of Banach spaces [4, Proposition 24.14]. The situation for projective limits, however, is more complicated. Particularly, this applies to the class of (PLS)-spaces (i.e., countable projective limits of (DFS)-spaces). The problem of ultrabornologicity has been extensively studied in this class, both from an abstract point of view as for concrete function and distribution spaces; see the survey article [6] of Domanski and the references therein.

In the last part of his doctoral thesis [7, Chap. II, Thm. 16, p. 131], Grothendieck showed that the convolutor space $O_C'$ is ultrabornological. He proved that $O_C'$ is isomorphic to a complemented subspace of the sequence space $s \hat{\otimes} s'$ and verified directly that the latter space is ultrabornological. Much later, a different proof was given by Larcher and Wengenroth using homological methods [8]. The first author and Vindas [9] extended this result to a considerably wider setting by studying the locally convex structure of a general class of weighted convolutor spaces. More precisely, they characterized when such spaces are ultrabornological and determined explicit topological preduals for them. One of their main tools is a topological description of these convolutor spaces in terms of the short-time Fourier transform (STFT).

In this work, we will identify $S'/(\Gamma)$ with a particular instance of the convolutor spaces considered in [9]. To this end, we make a detailed study of the mapping properties of the STFT on $S'(\Gamma)$. Once this identification has been established, we use Theorem 1.1 from [9] (see also Theorem 4.2 below) to show that $S'(\Gamma)$ is an ultrabornological (PLS)-space and that it admits a weighted (LF)-space of smooth functions on $\mathbb{R}^d$ as a topological predual.

2 Weighted spaces of continuous functions

For formulating the mapping properties of the STFT we recall the following notions from [9,10].

Each non-negative function $v$ on $\mathbb{R}^d$ defines a weighted seminorm on $C(\mathbb{R}^d)$ by

$$\|f\|_v := \sup_{x \in \mathbb{R}^d} |f(x)| v(x).$$

We endow the space

$$C_v(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) \mid \|f\|_v < \infty \}$$

with this seminorm; it is a Banach space if $v$ is positive and continuous. A pointwise decreasing sequence $V = (v_N)_{N \in \mathbb{N}}$ of positive continuous functions on $\mathbb{R}^d$ is called a decreasing weight system. With this, we define the $(LB)$-space

$$\mathcal{V}C(\mathbb{R}^d) := \lim_{N \to \infty} C_N(\mathbb{R}^d).$$

We consider the following condition on a decreasing weight system $\mathcal{V}$, see [10, p. 114]:

$$\forall N \in \mathbb{N} \exists M > N : \lim_{|x| \to \infty} \frac{v_M(x)}{v_N(x)} = 0.$$  (V)

The maximal Nachbin family associated with $\mathcal{V}$ is defined to be the family $\overline{V} = \overline{V(\mathcal{V})}$ consisting of all non-negative upper semicontinuous functions $v$ on $\mathbb{R}^d$ such that

$$\forall N \in \mathbb{N} : \sup_{x \in \mathbb{R}^d} \frac{v(x)}{v_N(x)} < \infty.$$
The projective hull of \( VC(\mathbb{R}^d) \) is defined as
\[
C \overline{V}(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^d) \mid \| f \|_v < \infty \ \forall v \in \overline{V} \}.
\]
and endowed with the locally convex topology generated by the system of seminorms \( \{ \| \cdot \|_v \mid v \in \overline{V} \} \). The spaces \( VC(\mathbb{R}^d) \) and \( C \overline{V}(\mathbb{R}^d) \) always coincide as sets and, if \( V \) satisfies condition (V), also as locally convex spaces [10, Thm. 1.3 (d), p. 118].

A pointwise increasing sequence \( \mathcal{W} = (w_N)_{N \in \mathbb{N}} \) of positive continuous functions on \( \mathbb{R}^d \) is called an increasing weight system. Given such a system, we define the Fréchet space
\[
WC(\mathbb{R}^d) := \lim_{N \to \infty} Cw_N(\mathbb{R}^d).
\]

We consider the following conditions on an increasing weight system \( \mathcal{W} \):
\[
\begin{align*}
\forall N \in \mathbb{N} \exists M > N & : \lim_{|x| \to \infty} \frac{w_N(x)}{w_M(x)} = 0, \quad (2.1) \\
\forall N \in \mathbb{N} \exists M > N & : \frac{w_N}{w_M} \in L^1(\mathbb{R}^d), \quad (2.2) \\
\forall N \in \mathbb{N} \exists M, M_2 \geq N \exists C > 0 \forall x, y \in \mathbb{R}^d : w_N(x + y) \leq C w_M(x) w_{M_2}(y). \quad (2.3)
\end{align*}
\]

In the next lemma, we obtain a concrete representation of the \( \varepsilon \)-tensor product of weighted spaces of continuous functions.

**Lemma 2.1** Let \( \mathcal{W} = (w_N)_{N \in \mathbb{N}} \) be an increasing weight system and \( \mathcal{V} = (v_n)_{n \in \mathbb{N}} \) a decreasing weight system satisfying (V). Then, we have the identification
\[
WC(\mathbb{R}^d)^{\otimes_{\varepsilon}} VC(\mathbb{R}^d) := \{ f \in C(\mathbb{R}^{2d}) \mid \forall N \in \mathbb{N} \exists n \in \mathbb{N} : \| f \|_{w_N \otimes v_n} < \infty \},
\]
where we set \( \| f \|_{w \otimes v} := \sup_{(x, \xi) \in \mathbb{R}^{2d}} | f(x, \xi) | w(x) v(\xi) \) for non-negative functions \( w, v \) on \( \mathbb{R}^d \). Moreover, \( f \in C(\mathbb{R}^{2d}) \) belongs to \( WC(\mathbb{R}^d)^{\otimes_{\varepsilon}} VC(\mathbb{R}^d) \) if and only if \( \| f \|_{w_N \otimes v} < \infty \) for all \( N \in \mathbb{N} \) and \( v \in \overline{V} \). Consequently, the topology of \( WC(\mathbb{R}^d)^{\otimes_{\varepsilon}} VC(\mathbb{R}^d) \) is generated by the system of seminorms \( \{ \| \cdot \|_{w_N \otimes v} \mid N \in \mathbb{N}, v \in \overline{V} \} \).

**Proof** This follows from the fact that the \( \varepsilon \)-tensor product commutes with projective limits and [10, Thm. 3.1 (c), p. 137].

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### 3 The short-time Fourier transform on \( \mathcal{D}'(\mathbb{R}^d) \)

The translation and modulation operators are denoted by \( T_x f(t) = f(t - x) \) and \( M_{\xi} f(t) = e^{2\pi i \xi \cdot t} f(t) \) for \( x, \xi \in \mathbb{R}^d \). The short-time Fourier transform (STFT) of a function \( f \in L^2(\mathbb{R}^d) \) with respect to a window function \( \psi \in L^2(\mathbb{R}^d) \) is defined as
\[
V_\psi f(x, \xi) := (f, M_{\xi} T_x \psi)_{L^2(d)} = \int_{\mathbb{R}^d} f(t) \overline{\psi(t - x)} e^{-2\pi i \xi \cdot t} \, dt, \quad (x, \xi) \in \mathbb{R}^{2d},
\]
where \( (\cdot, \cdot)_{L^2(d)} \) denotes the inner product on \( L^2(\mathbb{R}^d) \). We have that \( \| V_\psi f \|_{L^2(\mathbb{R}^{2d})} = \| \psi \|_{L^2} \| f \|_{L^2} \). In particular, the mapping \( V_\psi : L^2(\mathbb{R}^d) \to L^2(\mathbb{R}^{2d}) \) is continuous. The adjoint of \( V_\psi \) is given by the weak integral
\[
V_\psi^* F = \int_{\mathbb{R}^{2d}} \int_{\mathbb{R}^d} F(x, \xi) M_{\xi} T_x \psi \, dx \, d\xi, \quad F \in L^2(\mathbb{R}^{2d}).
\]
If \( \psi \neq 0 \) and \( \gamma \in L^2(\mathbb{R}^d) \) is a synthesis window for \( \psi \), that is, \( (\gamma, \psi)_{L^2} \neq 0 \), then
\[
\frac{1}{(\gamma, \psi)_{L^2}} V^*_\gamma \circ V_\psi = \text{id}_{L^2(\mathbb{R}^d)}.
\]

We refer to [11] for further properties of the STFT.

Next, we explain how the STFT can be extended to the space of distributions; see [9, Sect. 2] for details and proofs. We set \( \mathcal{V}_{\text{pol}} = ((1 + |\cdot|)^{-N})_{N \in \mathbb{N}} \). Fix a window function \( \psi \in \mathcal{D}(\mathbb{R}^d) \). For \( f \in \mathcal{D}'(\mathbb{R}^d) \) we define
\[
V_\psi f(x, \xi) := \langle f, \tilde{M}_\xi \hat{T}_x \psi \rangle, \quad (x, \xi) \in \mathbb{R}^{2d}.
\]

Clearly, \( V_\psi f \) is a continuous function on \( \mathbb{R}^{2d} \). In fact,
\[
V_\psi : \mathcal{D}'(\mathbb{R}^d) \to C(\mathbb{R}^d) \hat{\otimes}_\xi \mathcal{V}_{\text{pol}} C(\mathbb{R}^d)
\]
is a well-defined continuous mapping [9, Lemma 2.2]. We define the adjoint STFT of an element \( F \in C(\mathbb{R}^d) \hat{\otimes}_\xi \mathcal{V}_{\text{pol}} C(\mathbb{R}^d) \) as the distribution
\[
\langle V^*_\psi F, \varphi \rangle := \int \int_{\mathbb{R}^{2d}} F(x, \xi) V^*_\psi \varphi(x, -\xi) \, dx \, d\xi, \quad \varphi \in \mathcal{D}(\mathbb{R}^d).
\]

Then,
\[
V^*_\psi : C(\mathbb{R}^d) \hat{\otimes}_\xi \mathcal{V}_{\text{pol}} C(\mathbb{R}^d) \to \mathcal{D}'(\mathbb{R}^d)
\]
is a well-defined continuous mapping by [9, Prop. 2.2]. Finally, if \( \psi \neq 0 \) and \( \gamma \in \mathcal{D}(\mathbb{R}^d) \) is a synthesis window for \( \psi \), then the following reconstruction formula holds [9, Prop. 2.4]:
\[
\frac{1}{(\gamma, \psi)_{L^2}} V^*_\gamma \circ V_\psi = \text{id}_{\mathcal{D}'(\mathbb{R}^d)}.
\]

### 4 Duals of inductive limits of weighted spaces of smooth functions

Let \( v \) be a non-negative function on \( \mathbb{R}^d \) and \( n \in \mathbb{N} \). We define \( B^n_v(\mathbb{R}^d) \) as the seminormed space consisting of all \( \varphi \in C^v(\mathbb{R}^d) \) such that
\[
\| \varphi \|_{v, n} := \max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |\partial^\alpha \varphi(x)| \, v(x) < \infty.
\]

As before, \( B^n_v(\mathbb{R}^d) \) is a Banach space if \( v \) is positive and continuous. Let \( \mathcal{W} = (w_N)_{N \in \mathbb{N}} \) be an increasing weight system. We define the \( (\text{LF}) \)-space
\[
B_{\mathcal{W}^\circ}(\mathbb{R}^d) := \lim_{\rightarrow} \cup_{n \in \mathbb{N}} B^n_{w_N}(\mathbb{R}^d).
\]

We endow the dual space \( B'_{\mathcal{W}^\circ}(\mathbb{R}^d) := (B_{\mathcal{W}^\circ}(\mathbb{R}^d))^\prime \) with the strong topology. If \( \mathcal{W} \) satisfies (2.1), then \( \mathcal{D}(\mathbb{R}) \) is densely and continuously included in \( B_{\mathcal{W}^\circ}(\mathbb{R}^d) \) and therefore \( B'_{\mathcal{W}^\circ}(\mathbb{R}^d) \) is a vector subspace of \( \mathcal{D}'(\mathbb{R}^d) \).

On the other hand, we define the convolutor space
\[
O'_{\mathcal{C}, \mathcal{W}}(\mathbb{R}^d) := \{ f \in \mathcal{D}'(\mathbb{R}^d) \mid f \ast \varphi \in WC(\mathbb{R}^d) \forall \varphi \in \mathcal{D}(\mathbb{R}^d) \}.
\]

For \( f \in O'_{\mathcal{C}, \mathcal{W}}(\mathbb{R}^d) \) fixed, the mapping
\[
\mathcal{D}(\mathbb{R}^d) \to WC(\mathbb{R}^d), \quad \varphi \mapsto f \ast \varphi
\]
is continuous, as follows from the closed graph theorem. We endow \( \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) with the topology induced via the embedding

\[
\mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \to L_\beta(\mathcal{D}(\mathbb{R}^d), \mathcal{W}\mathcal{C}(\mathbb{R}^d)), \quad f \mapsto [\varphi \mapsto f * \varphi],
\]

where \( \beta \) denotes the topology of uniform convergence on bounded sets.

In [9] the structural and topological properties of the spaces \( \mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) \) and \( \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) are discussed. We now present the main results of this paper and refer to [9] for more details and proofs.

**Proposition 4.1** [9, Prop. 4.2] Let \( \mathcal{W} \) be an increasing weight system satisfying (2.1), (2.2) and (2.3) and let \( \psi \in \mathcal{D}(\mathbb{R}^d) \). Then, the mappings

\[
V_\psi : \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \to \mathcal{W}\mathcal{C}(\mathbb{R}^d) \hat{\otimes}_\varepsilon \mathcal{V}_{pol}(\mathbb{R}^d)
\]

and

\[
V_\psi^* : \mathcal{W}\mathcal{C}(\mathbb{R}^d) \hat{\otimes}_\varepsilon \mathcal{V}_{pol}(\mathbb{R}^d) \to \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d)
\]

are well-defined and continuous.

**Theorem 4.2** [9, Thm. 3.4, Thm. 4.6 and Thm. 4.15] Let \( \mathcal{W} = (w_N)_{N \in \mathbb{N}} \) be an increasing weight system satisfying (2.1), (2.2) and (2.3). Then, \( \mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) = \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) as sets and the inclusion mapping \( \mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) \to \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) is continuous. Moreover, the following statements are equivalent:

(i) \( \mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) = \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) as locally convex spaces.

(ii) \( \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) is an ultrabornological (PLS)-space.

(iii) The (LF)-space \( \mathcal{B}_{\mathcal{W}^o}(\mathbb{R}^d) \) is complete.

(iv) \( \mathcal{W} \) satisfies

\[
\forall N \in \mathbb{N} \exists M \geq N \forall P \geq M \exists \theta \in (0, 1) \exists C > 0 \forall x \in \mathbb{R}^d : w_N(x)^{1-\theta} w_P(x)^{\theta} \leq C w_M(x). \tag{4.1}
\]

**Remark 4.3** Condition (4.1) is closely connected with D. Vogt’s condition (\( \Omega \)) that plays an essential role in the structure and splitting theory for Fréchet spaces.

## 5 The space \( S'(\Gamma) \)

Our next goal is to characterize \( S'(\Gamma) \) in terms of the STFT.

Let \( \emptyset \neq \Gamma \subseteq \mathbb{R}^d \) be open and convex. We denote by \( \text{CCS}(\Gamma) \) the family of all non-empty compact convex subsets of \( \Gamma \) and by \( \mathcal{B}(S(\mathbb{R}^d)) \) the family of all bounded subsets of \( S(\mathbb{R}^d) \). The topology of \( S'(\Gamma) \) can easily be described by a system of concrete seminorms which essentially is due to Schwartz [1, p. 301]; for this, note that the system of convex hulls of finite sets is cofinal in \( \text{CCS}(\Gamma) \):

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1. To be precise, the spaces considered in [9], denoted there by \( (\mathcal{B}_{\mathcal{W}^o}(\mathbb{R}^d))^\prime \) and \( \mathcal{O}'_{C}(\mathcal{D}, L^1_{\mathcal{W}}) \), differ from \( \mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) \) and \( \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \) defined above. However, if \( \mathcal{W} \) satisfies (2.1), (2.2) and (2.3), then \( \mathcal{B}'_{\mathcal{W}}(\mathbb{R}^d) = (\mathcal{B}_{\mathcal{W}^o}(\mathbb{R}^d))^\prime \) and \( \mathcal{O}'_{C}(\mathcal{D}, L^1_{\mathcal{W}}) = \mathcal{O}'_{C, \mathcal{W}}(\mathbb{R}^d) \); the first equality is clear, while the second one follows from [9, Prop. 6.2]. Moreover, under these conditions, all statements and proofs from [9] remain valid if one replaces \( L^1_{\mathcal{W}}(\mathbb{R}^d) \) by \( \mathcal{W}\mathcal{C}(\mathbb{R}^d) \).
Lemma 5.1 [1, p. 301] Let \( \emptyset \neq \Gamma \subseteq \mathbb{R}^d \) be open and convex. For all \( K \in \text{CCS}(\Gamma) \) and \( B \in \mathcal{B}(S(\mathbb{R}^d)) \) we have that
\[
p_{K,B}(f) := \sup_{\eta \in K} \sup_{\phi \in B} \| e^{-\eta \cdot x} f(x) \phi(x) \| < \infty, \quad f \in S'(\Gamma).
\]
Moreover, the topology of \( S'(\Gamma) \) is generated by the system of seminorms \( \{ p_{K,B} \mid K \in \text{CCS}(\Gamma), B \in \mathcal{B}(S(\mathbb{R}^d)) \} \).

We need to introduce some additional terminology. Given a non-empty compact convex subset \( K \) of \( \mathbb{R}^d \), we define its supporting function as
\[
h_K(x) = \max_{\eta \in K} \eta \cdot x, \quad x \in \mathbb{R}^d.
\]
It is clear from the definition that \( h_K \) is subadditive and positive homogeneous of degree one. In particular, \( h_K \) is convex. Supporting functions have the following elementary properties.

Lemma 5.2 [12, Cor. 1.8.2 and Prop. 1.8.3] Let \( K_1 \) and \( K_2 \) be non-empty compact convex subsets of \( \mathbb{R}^d \).

(a) \( K_1 \subseteq K_2 \) if and only if \( h_{K_1}(x) \leq h_{K_2}(x) \) for all \( x \in \mathbb{R}^d \).

(b) \( h_{K_1+K_2}(x) = h_{K_1}(x) + h_{K_2}(x) \) for all \( x \in \mathbb{R}^d \).

Example 5.3 For \( r > 0 \) we have \( h_B(0,r)(x) = r |x| \) for all \( x \in \mathbb{R}^d \), where \( B(0,r) \) denotes the closed ball in \( \mathbb{R}^d \) centered at the origin with radius \( r \). Next, let \( K \) be a non-empty compact convex subset of \( \mathbb{R}^d \) and \( \varepsilon > 0 \). We set \( K_\varepsilon = K + B(0,\varepsilon) \). Lemma 5.2 and the above yield that \( h_{K_\varepsilon}(x) = h_K(x) + \varepsilon |x| \) for all \( x \in \mathbb{R}^d \).

Let \( \emptyset \neq \Gamma \subseteq \mathbb{R}^d \) be open and convex and let \( (K_N)_{N \in \mathbb{N}} \subset \text{CCS}(\Gamma) \) be such that \( K_N \subseteq K_{N+1} \) for all \( N \in \mathbb{N} \) and \( \Gamma = \bigcup_N K_N \). Lemma 5.2 yields that \( \mathcal{W} = (e^{h_{K_N}})_{N \in \mathbb{N}} \) is an increasing weight system. We set \( C_\Gamma(\mathbb{R}^d) := \mathcal{W}C(\mathbb{R}^d) \). Clearly, the definition of \( C_\Gamma(\mathbb{R}^d) \) is independent of the chosen sequence \( (K_N)_{N \in \mathbb{N}} \). The next result is the key observation of this article.

Proposition 5.4 Let \( \emptyset \neq \Gamma \subseteq \mathbb{R}^d \) be open and convex and let \( \psi \in \mathcal{D}(\mathbb{R}^d) \). Then, the mappings
\[
V_{\psi} : S'(\Gamma) \to C_\Gamma(\mathbb{R}^d) \hat{\otimes}_\mathfrak{c} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d)
\]
and
\[
V_{\psi}^* : C_\Gamma(\mathbb{R}^d) \hat{\otimes}_\mathfrak{c} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d) \to S'(\Gamma)
\]
are well-defined and continuous.

We need some preparation for the proof of Proposition 5.4. Firstly, Lemma 2.1 implies that the topology of \( C_\Gamma(\mathbb{R}^d) \hat{\otimes}_\mathfrak{c} \mathcal{V}_{\text{pol}} C(\mathbb{R}^d) \) is generated by the system of seminorms
\[
\| f \|_{K,v} := \sup_{(x,\xi) \in \mathbb{R}^{2d}} |f(x,\xi)| e^{h_K(x)\psi(\xi)} < \infty, \quad K \in \text{CCS}(\Gamma), v \in \overline{\mathcal{V}}(\mathcal{V}_{\text{pol}}).
\]
For \( k, n \in \mathbb{N} \) we write
\[
\| \phi \|_{S_k^n} := \max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} |\partial^\alpha \phi(x)| (1 + |x|)^k, \quad \phi \in \mathcal{S}(\mathbb{R}^d).
\]
The topology of \( \mathcal{S}(\mathbb{R}^d) \) is generated by the system of seminorms \( \{ \| \cdot \|_{S_k^n} \mid k, n \in \mathbb{N} \} \). We now give two technical lemmas.
Lemma 5.5 Let \( \psi \in \mathcal{D}(\mathbb{R}^d) \), \( K \subset \mathbb{R}^d \) be compact, \( v \in \overline{V}(\mathcal{V}_{pol}) \) and \( \varepsilon > 0 \). Then,
\[
\{ e^{\eta(t-x)}M_\varepsilon T_x \psi(t) e^{-\varepsilon|x|} v(\xi) \mid (x, \xi) \in \mathbb{R}^{2d}, \eta \in K \} \subset \mathcal{B}(S(\mathbb{R}_r^d)).
\]

Proof Choose \( r > 0 \) such that \( \text{supp } \psi \subset B(0, r) \) and \( R \geq 1 \) such that \( K \subset B(0, R) \). For all \( k, n \in \mathbb{N} \) we have that
\[
\sup_{(x, \xi)} \sup_{\eta \in K} e^{-\varepsilon|x|} v(\xi) \| e^{\eta(t-x)}M_\varepsilon T_x \psi(t) \| \leq \sup_{(x, \xi)} \sup_{\eta \in K} e^{-\varepsilon|x|} v(\xi).
\]
\[
\max_{|\alpha| \leq n} \sup_{x \in \mathbb{R}^d} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right) |\eta|^{\alpha - |\beta|} e^{\eta(t-x)} (2\pi |\xi|)^{|\gamma|} |\partial^{\beta - \gamma} \psi(t - x)| (1 + |t|)^k
\]
\[
\leq e^{R \varepsilon R} (8\pi R)^n \max_{|\alpha| \leq n} \| \partial^{\alpha} \psi \|_{L^\infty} (1 + r)^k \sup_{x \in \mathbb{R}^d} e^{-\varepsilon|x|} (1 + |x|)^k \sup_{\xi \in \mathbb{R}^d} v(\xi) (1 + |\xi|)^n
\]
\[
< \infty.
\]

Lemma 5.6 Let \( \psi \in \mathcal{D}(\mathbb{R}^d) \) and \( \eta \in \mathbb{R}^d \). Then, for all \( k, n \in \mathbb{N} \) and \( \varphi \in S(\mathbb{R}^d) \),
\[
\left| V_{\psi, t}(e^{-\eta \cdot t} \varphi(t))(x, -\xi) \right| \leq \frac{C_{\eta, k, n, \psi} e^{-\varepsilon x} \| \varphi \|_{S_{\delta}^n}}{(1 + |x|)^k (1 + |\xi|)^n}, \quad (x, \xi) \in \mathbb{R}^{2d},
\]
where
\[
C_{\eta, k, n, \psi} = 4^n (1 + \sqrt{d})^n \max_{|\alpha| \leq n} \| \partial^{\alpha} \psi \|_{L^\infty} \int_{\text{supp } \psi} e^{-\varepsilon t} (1 + |t|)^k \, dt.
\]
In particular, \( \sup_{\eta \in \mathbb{R}^d} C_{\eta, k, n, \psi} < \infty \) for all \( K \subset \mathbb{R}^d \) compact.

Proof We have that
\[
\left| V_{\psi, t}(e^{-\eta \cdot t} \varphi(t))(x, -\xi) \right| (1 + |x|)^k (1 + |\xi|)^n
\]
\[
\leq (1 + \sqrt{d})^n \max_{|\alpha| \leq n} \left| \xi^\alpha V_{\psi, t}(e^{-\eta \cdot t} \varphi(t))(x, -\xi) \right| (1 + |x|)^k
\]
\[
\leq (1 + \sqrt{d})^n (1 + |x|)^k \max_{|\alpha| \leq n} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right) |\eta|^{\beta - \gamma} |\partial^{\beta - \gamma} \psi(t - x)| \, dt
\]
\[
\leq (1 + \sqrt{d})^n (1 + |x|)^k \max_{|\alpha| \leq n} \sum_{\beta \leq \alpha} \sum_{\gamma \leq \beta} \left( \begin{array}{c} \alpha \\ \beta \\ \gamma \end{array} \right) |\eta|^{\beta - \gamma} |\partial^{\beta - \gamma} \psi(t + x)| \, dt
\]
\[
\leq C_{\eta, k, n, \psi} e^{-\varepsilon x} \| \varphi \|_{S_{\delta}^n}.
\]

Proof of Proposition 5.4 (i) \( V : S'(\Gamma) \rightarrow C^\Gamma(\mathbb{R}_r^d) \otimes \mathcal{V}_{pol} C(\mathbb{R}_r^d) \) is well-defined and continuous: Let \( K \in \text{CCS}(\Gamma) \) and \( v \in \overline{V}(\mathcal{V}_{pol}) \) be arbitrary. Choose \( \varepsilon > 0 \) so small that \( K_\varepsilon \subset \text{CCS}(\Gamma) \) and pick, for \( x \in \mathbb{R}^d \) fixed, \( \eta_x \in K \) such that \( h_{-K}(x) \leq (-\eta_x \cdot x) + 1 \). Example 5.3 implies that, for all \( f \in S'(\Gamma) \) and \( (x, \xi) \in \mathbb{R}^{2d} \),
Choose locally convex spaces and $\eta$ where $\eta$ is a well-defined continuous linear functional on $F \in C_\Gamma(\mathbb{R}_x^d) \otimes \mathcal{V}_\text{pol} C(\mathbb{R}_x^d)$, Lemma 5.6 implies that, for all $\eta \in \Gamma$, 

$$
\langle f_\eta, \varphi \rangle = \int \int_{\mathbb{R}_x^d} F(x, \xi) V_{\psi, t}(e^{-\eta \cdot \varphi}(t))(x, -\xi) \, dx \, d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}_x^d),
$$

is a well-defined continuous linear functional on $S(\mathbb{R}_x^d)$. Since $e^{-\eta \cdot t} V_{\psi} F(t) = f_\eta(t)|_{D(\mathbb{R}_x^d)}$, we obtain that $e^{-\eta \cdot t} V_{\psi} F(t) \in S'(\mathbb{R}_x^d)$ and that 

$$
\langle e^{-\eta \cdot t} V_{\psi} F(t), \varphi(t) \rangle = \int \int_{\mathbb{R}_x^d} F(x, \xi) V_{\psi, t}(e^{-\eta \cdot \varphi}(t))(x, -\xi) \, dx \, d\xi, \quad \varphi \in \mathcal{S}(\mathbb{R}_x^d).
$$

Next, we show that $V_{\psi}$ is continuous. Let $K \in \mathcal{C}_\Gamma(\mathbb{R}_x^d)$ and $B \in \mathcal{B}(\mathcal{S}(\mathbb{R}_x^d))$ be arbitrary. Choose $\varepsilon > 0$ so small that $K_\varepsilon \in \mathcal{C}_\Gamma(\mathbb{R}_x^d)$. Lemma 5.6 implies that there is $v \in \overline{\mathcal{V}(\mathcal{V}_\text{pol})}$ such that 

$$
|V_{\psi}(e^{-\eta \cdot \varphi}(t))(x, -\xi)| \leq e^{h-K(x)} v(\xi), \quad (x, \xi) \in \mathbb{R}_x^d,
$$

for all $\eta \in K$ and $\varphi \in B$. Set $w(\xi) = v(\xi)(1 + |\xi|)^{d+1} \in \overline{\mathcal{V}(\mathcal{V}_\text{pol})}$. Example 5.3 implies that, for all $F \in C_\Gamma(\mathbb{R}_x^d) \otimes \mathcal{V}_\text{pol} C(\mathbb{R}_x^d)$, 

$$
p_{K, B}(V_{\psi} F) \leq \sup_{\eta \in K} \sup_{\varphi \in B} \int \int_{\mathbb{R}_x^d} |F(x, \xi)| \left| V_{\psi, t}(e^{-\eta \cdot \varphi}(t))(x, -\xi) \right| \, dx \, d\xi \leq C \| F \|_{K_\varepsilon, w},
$$

where 

$$
C = \int_{\mathbb{R}_x^d} e^{-\varepsilon |x|} \, dx \int_{\mathbb{R}_x^d} \frac{1}{(1 + |\xi|)^{d+1}} \, d\xi.
$$

We now combine Theorem 4.1 with the results from Sect. 4 to study the space $S'(\Gamma)$. Let $\emptyset \neq \Gamma \subseteq \mathbb{R}_x^d$ be open and convex and let $(K_N)_{N \in \mathbb{N}} \subseteq \mathcal{C}_\Gamma(\mathbb{R}_x^d)$ be such that $K_N \subseteq K_{N+1}$ for all $N \in \mathbb{N}$ and $\Gamma = \bigcup_N K_N$. For $\mathcal{W} = (e^{h-K_N})_{N \in \mathbb{N}}$ we set $B'_{\mathcal{W}}(\mathbb{R}_x^d) := B'_{\mathcal{W}}(\mathbb{R}_x^d)$ and $O'_{C, \Gamma}(\mathbb{R}_x^d) = O'_{C, \mathcal{W}}(\mathbb{R}_x^d)$. Clearly, these definitions are independent of the chosen sequence $(K_N)_{N \in \mathbb{N}}$. We are ready to state and prove our main theorem.

**Theorem 5.7** Let $\emptyset \neq \Gamma \subseteq \mathbb{R}_x^d$ be open and convex. Then, $S'(\Gamma) = B'_{\mathcal{W}}(\mathbb{R}_x^d) = O'_{C, \Gamma}(\mathbb{R}_x^d)$ as locally convex spaces and $S'(\Gamma)$ is an ultrabornological (PLS)-space.

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Proof \( \{(K_N)_{N \in \mathbb{N}} \subset CCS(\Gamma) \) be such that \( K_N \subseteq K_{N+1} \) for all \( N \in \mathbb{N} \) and \( \Gamma = \bigcup_N K_N \). Set \( W = (e^{N-X_N})_{N \in \mathbb{N}} \). Lemma 5.2 and Example 5.3 imply that \( W \) satisfies (2.1), (2.2) and (2.3). Hence, in view of the reconstruction formula (3.1), the topological identity \( S'(\Gamma) = O'_{C,\Gamma}(\mathbb{R}^d) \) follows from Proposition 4.1 and Proposition 5.4. Since \( W \) also satisfies (4.1) (again by Lemma 5.2 and Example 5.3), the other statements are a direct consequence of Theorem 4.2.

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