Some properties of Fibonacci primes
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ABSTRACT
In this article we characterize the primes Fibonacci numbers of the form $x^2 + ry^2$, where $r = 1$, $r$ is a prime positive integer number or $r$ is a power of a prime positive integer, using techniques of combinatorics and numbers theory. We also evaluate some distances related to the Fibonacci numbers and function.

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KEYWORDS: Fibonacci numbers; quadratic fields.

1 Introduction
The primes natural numbers that can be written as $x^2 + ny^2$, for $n \in \mathbb{N}^*$, have been studied in [4]. Necessary and sufficient conditions for a prime $p$ to be written as $p = x^2 + ny^2$, with $n \in \mathbb{N}^*$, have been determined. One of the results of Cox’s book is:

Proposition 1.1 [4] Let $n$ be a square free positive integer which is not congruent to 3 modulo 4. Then there exists a monic irreducible polynomial $f \in \mathbb{Z}[X]$ of degree $h(\Delta)$, such that, if $p$ is an odd prime that doesn’t divide $n$ or the discriminant of $f$ and $E = HCF(K)$ is the Hilbert class field of $K = \mathbb{Q}(\sqrt{-n})$, the following statements are equivalent:

(i) $p = x^2 + ny^2$, for some $x, y \in \mathbb{N}$.
(ii) $p$ completely splits in $E$.
(iii) $\left(\frac{-n}{p}\right) = 1$ and the congruence $f(x) \equiv 0 \pmod{p}$ has solutions in $\mathbb{Z}$.
Moreover, $f$ is the minimal polynomial of a real algebraic integer $\alpha$ such that $E = K(\alpha)$.

In [6], [7] is given a characterization of some such primes $p$, when $n \equiv 3$ modulo 4 and the class number of the quadratic field $\mathbb{Q} \left(\sqrt{-n}\right)$ is 1, namely $n \in \{11, 19, 43, 67, 163\}$: $p$ is represented by $x^2 + ny^2$ if and only if the corresponding cubic field equation splits completely modulo $p$ if and only if the roots of the resolvent quadratic equation are cubic residues of $p$. The field equations and the corresponding root $\alpha_n$ can be taken as:
Concretely, the result from [7] is:

**Theorem 1.1** [7] For \(q \in \{11, 19, 43, 67, 163\}\) and for \(\alpha_q\) defined above, a prime positive integer number \(p \equiv 1\pmod{12}\) such that the Legendre symbol \(\left(\frac{p}{q}\right) = 1\) is represented by \(p = x^2 + qy^2\), if and only if the cubic character
\[\left(\frac{\alpha_q}{p}\right)_3 = 1.\]

In this paper we try to determine the primes Fibonacci numbers that can be written in the form \(x^2 + ry^2\), where \(r = 1\), \(r\) is a prime natural number or \(r\) is a power of a prime positive integer.

First, we recall some properties of quadratic fields which are necessary in our proofs.

**Proposition 1.2** [1] Let \(p, q\) be two distinct prime numbers, \(p \equiv q \equiv 1\pmod{4}\) and \(h\) the class number of the biquadratic field \(K = \mathbb{Q}(\sqrt{p}, \sqrt{q})\). If \(\left(\frac{p}{q}\right) = 1\), then \(h\) is odd if and only if \(\left(\frac{p}{q}\right)_4 \neq \left(\frac{q}{p}\right)_4\) (here \(\left(\frac{}{}\right)_4\) is the quartic character).

**Proposition 1.3** [4] Let \(K\) be an algebraic number field and \(P \in \text{Spec}(O_K)\). Then \(P\) completely splits in the ring of integers of the Hilbert class field of \(K\) if and only if \(P\) is a principal ideal in the ring \(O_K\).

**Proposition 1.4** Let \(p\) be a prime positive integer. Then:

(i) There exist integers \(x, y\) such that \(p = x^2 + y^2\) if and only if \(p = 2\) or the Legendre symbol \(\left(\frac{-1}{p}\right) = 1\);

(ii) There exist integers \(x, y\) such that \(p = x^2 + 2y^2\) if and only if the Legendre symbol \(\left(\frac{2}{p}\right) = 1\).

The following properties of Fibonacci numbers we will use in the following.
Proposition 1.5 [12]  

i) The cycle of the Fibonacci numbers mod 4 is

\[0, 1, 1, 2, 3, 1, (0, 1, ...),\]

so the cycle-length of the Fibonacci numbers mod 4 is 6.

ii) The cycle of the Fibonacci numbers mod 5 is

\[0, 1, 1, 2, 3, 0, 3, 1, 4, 0, 4, 4, 3, 2, 0, 2, 2, 4, 1, (0, 1, ...),\]

so the cycle-length of the Fibonacci numbers mod 5 is 20.

iii) The cycle of the Fibonacci numbers mod 20 is

\[0, 1, 1, 2, 3, 5, 8, 13, 1, 14, 15, 9, 4, 13, 17, 10, 7, 17, 4, 1, 5, 6, 11, 17, 8, 5, 13, 18, 11, 9, 0, 9, 9,\]

\[18, 7, 5, 12, 17, 9, 6, 15, 1, 16, 17, 13, 10, 3, 13, 16, 9, 5, 14, 19, 13, 12, 5, 17, 2, 19, 1, (0, 1, ...),\]

so the cycle-length of the Fibonacci numbers mod 20 is 60.

iv) The cycle of the Fibonacci numbers mod 8 is

\[0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, (0, 1, 1, 2, 3, 5, 0, 5, 5, 2, 7, 1, ...),\]

so the cycle-length of the Fibonacci numbers mod 8 is 12.

**Proposition 1.6 [12]** Let \((F_n)_{n \geq 0}\) be the Fibonacci sequence. If \(F_n\) is a prime number, then \(n\) is a prime number.

**Proposition 1.7 [12]** A Fibonacci number \(F_n\) is even if and only if \(n \equiv 0 \pmod{3}\).

**Theorem 1.2 (Legendre, Lagrange)** If \(p\) is a prime odd positive integer, then

\[F_p \equiv \left(\frac{p}{5}\right) \pmod{p}.\]

**Theorem 1.3 (Legendre, Lagrange)** Let \(p\) be a prime odd positive integer. Then

\[F_{p-1} \equiv \frac{1 - \left(\frac{p}{5}\right)}{2} \pmod{p}\]

and

\[F_{p+1} \equiv \frac{1 + \left(\frac{p}{5}\right)}{2} \pmod{p}.\]

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Theorem 1.4 (Z.H.Sun, Z.W.Sun) [10] Let \( p \not\in \{2, 5\} \) be a prime positive integer. Then

\[
F_{\frac{p-(\frac{p+3}{10})}{2}} \equiv \begin{cases} 
0 \pmod{p}, & \text{if } p \equiv 1 \pmod{4}, \\
2 (\frac{p+3}{10}) \cdot (\frac{p}{5}) \cdot 5^{\frac{p-3}{4}} \pmod{p}, & \text{if } p \equiv 3 \pmod{4} 
\end{cases}
\]

and

\[
\begin{cases} 
(\frac{-1}{p}) \cdot 5^{\frac{p-1}{4}} \pmod{p}, & \text{if } p \equiv 1 \pmod{4}, \\
(\frac{-1}{p}) \cdot 5^{\frac{p+3}{4}} \pmod{p}, & \text{if } p \equiv 3 \pmod{4}.
\end{cases}
\]

where \([x]\) is the integer part of \( x \).

2 Fibonacci primes of the form \( x^2 + r y^2 \)

Our first remark is:

Remark 2.1 Let \((F_n)_{n\geq 0}\) be the Fibonacci sequence.

i) If \( p \) is a prime Fibonacci number, \( p \neq 3 \), then there exist the integers \( x, y \) such that \( p = x^2 + y^2 \).

ii) If \( p \) is a prime Fibonacci number, \( p \equiv 1 \pmod{8} \), then there exist the integers \( x, y \) such that \( p = x^2 + 2y^2 \).

Proof. Case 1. \( p = 2 = F_3 \). We obtain \( 2 = 1^2 + 1^2 \).

Case 2. \( p = F_n \) is an odd prime number.

First, we show that there exist no prime Fibonacci number congruent with 3 \( \pmod{4} \). To prove it, we suppose, by reduction ad absurdum, that there exists a prime \( p = F_n \equiv 3 \pmod{4} \). Applying Proposition 1.5 (i) we obtain that \( n \in \{4, 10, 16, 22, \ldots\} \) and using Proposition 1.6 it results that \( F_n \) is not a prime number. So, \( p \equiv 1 \pmod{4} \). Applying Proposition 1.4 (i) it results that there exist integers \( x, y \) so that \( p = x^2 + y^2 \).

Another argument that can be used to prove this assertion is based on the relations: \( F_{2n+1} = F_n^2 + F_{n+1}^2 \), \( g.c.d(F_n, F_{n+1}) = 1 \). (see [3]).

Thus, all the odd prime Fibonacci numbers \( p = F_n \) are congruent with 1 \( \pmod{4} \) and are sums of two perfect squares.

ii) If \( p \equiv 1 \pmod{8} \), using the properties of Legendre symbol we have \( (\frac{-1}{p}) = 1 \), \( (\frac{2}{p}) = 1 \), so \( (\frac{-2}{p}) = 1 \). Applying Proposition 1.4 (ii) we obtain that there exist two integers \( x, y \) such that \( p = x^2 + 2y^2 \).
A natural idea is to ask ourselves if there exist Fibonacci numbers $F_p$ of the form $F_p = x^2 + p^2y^2$, where $p$ is a prime positive integer. The following result has been obtained:

**Proposition 2.1** (i) For each $p$, a prime odd positive integer, $p \equiv 1 \pmod{4}$, there exist integer numbers $x, y$ so that, the Fibonacci number $F_p$ can be written as $F_p = x^2 + p^2y^2$.

(ii) For each $p$, a prime odd positive integer $p \equiv 1 \pmod{4}$, with the property that the Fibonacci number $F_p$ is a prime number, there exist unique positive integer numbers $x, y$ so that the Fibonacci number $F_p$ can be written as: $F_p = x^2 + p^2y^2$.

**Proof. Case 1.** It is known that $F_{2n+1} = F_n^2 + F_{n+1}^2$, so, if $p$ is a prime odd positive integer, then $F_p = F_{\frac{p+1}{2}}^2 + F_{\frac{p-1}{2}}^2$. Since the Legendre symbol $(\frac{p}{5})$ is 1 when $p \equiv 1, 4 \pmod{5}$ and it is $-1$ when $p \equiv 2, 3 \pmod{5}$, we divide the proof in two subcases: 1: $p \equiv 1, 4 \pmod{5}$; 2. $p \equiv 2, 3 \pmod{5}$.

Subcase 1: $p \equiv 1, 4 \pmod{5}$.

Using the fact that $p \equiv 1 \pmod{4}$ and Chinese Theorem it results that $p \equiv 1, 9 \pmod{20}$. Applying Theorem 1.4, it results that $\frac{F_{p-1}}{2} \equiv 0 \pmod{p}$. Therefore, there exist integer numbers $x, y$, $x = \pm F_{\frac{p-1}{2}}$ and $y = \pm \frac{F_{p-1}}{p}$ such that $F_p = x^2 + p^2y^2$.

Subcase 2: $p \equiv 2, 3 \pmod{5}$.

Form $p \equiv 1 \pmod{4}$ and Chinese Theorem it results that $p \equiv 13, 17 \pmod{20}$. Applying Theorem 1.4, it results that $\frac{F_{p+1}}{2} \equiv 0 \pmod{p}$. We obtain that there exist integer numbers $x, y$, $x = \pm F_{\frac{p+1}{2}}$ and $y = \pm \frac{F_{p+1}}{p}$ such that $F_p = x^2 + p^2y^2$.

(ii) If moreover, the Fibonacci number $F_p$ is a prime number, applying (i) and the properties that $\mathbb{Z}[i]$ is a factorial ring and its group of units is $U(\mathbb{Z}[i]) = \{1, -1, i, -i\}$ and $F_p \equiv 1 \pmod{4}$ completely splits in the ring $\mathbb{Z}[i]$, it results that there exist unique positive integer numbers $x, y$, $x = F_{\frac{p+1}{2}}$ and $y = \frac{F_{p+1}}{p}$, when $p \equiv 1, 9 \pmod{20}$, respectively $x = F_{\frac{p-1}{2}}$ and $y = \frac{F_{p-1}}{p}$ when $p \equiv 13, 17 \pmod{20}$, such that $F_p = x^2 + p^2y^2$.

In the following we study the Fibonacci primes $F_p$ of the form $x^2 + py^2$.

A prime example of such a prime is $5 = F_5 = 0^2 + 5 \cdot (\pm 1)^2$.

We want determine the primes $F_p$ of the form $x^2 + py^2$, with $x, y$ positive integers.

With a simple computation in MAGMA sofware, we obtain:

```magma
R<x> := PolynomialRing(Integers());
f := x^2 + 29;
```
T:=Thue(f);
T;
Solutions(T,514229);
Submit
Thue object with form: $X^2 + 29Y^2$
$[-552, 85],$
$[552, -85],$
$[552, 85],$
$[-552, -85].$
So,

$$514229 = F_{29} = (\pm 552)^2 + 29 \cdot (\pm 85)^2.$$  

Analogous:

$$233 = F_{13} = (\pm 5)^2 + 13 \cdot (\pm 4)^2, \quad 1597 = F_{17} = (\pm 38)^2 + 17 \cdot (\pm 3)^2.$$  

We remark that in all these examples $p \equiv 1 \pmod{4}$. Therefore, the question that arises is: what happens when $p \equiv 3 \pmod{4}$? First, we tried to apply Theorem 1.1 for $p \in \{11, 19, 43, 67, 163\}$, but this was not possible because, using [12] we have: $F_{11}$ is not congruent with 1 (mod 12), $F_{19}$ is not a prime number, $F_{43}$ is not congruent with 1 (mod 12), $F_{67}$ and $F_{163}$ are not prime numbers.
The next result has been obtained:

**Proposition 2.2** If $p$ is a prime positive integer, $p \equiv 3$ or 7 (mod 20) then there exists no Fibonacci prime $F_p$ of the form $x^2 + py^2$.

**Proof.** Let $p$ be a prime positive integer, $p \equiv 3$ or 7 (mod 20). We suppose by reductio ad absurdum that there exists a prime positive Fibonacci number, $F_p$, such
that \( F_p = x^2 + py^2 \). Therefore, the Legendre symbol \( \left( \frac{F_p}{p} \right) \) = 1. But, applying Theorem 1.2 and the properties of Legendre’s symbol, we have:

\[
\left( \frac{F_p}{p} \right) = \left( \frac{\sqrt{p}}{p} \right) = \begin{cases} 
1, & \text{if } p \equiv 1 \text{ or } 4 \pmod{5}, \\
(-1)^{\frac{p-1}{2}}, & \text{if } p \equiv 2 \text{ or } 3 \pmod{5}.
\end{cases}
\]

Since \( p \equiv 3 \text{ or } 7 \pmod{20} \) it results \( p \equiv 2 \text{ or } 3 \pmod{5} \). So,

\[
\left( \frac{F_p}{p} \right) = \left( \frac{-1}{p} \right) = (-1)^{\frac{p-1}{2}} = -1.
\]

We obtain a contradiction with the fact that \( \left( \frac{F_p}{p} \right) = 1 \).

It is hard to know what happens when \( p \equiv 11 \text{ or } 19 \pmod{20} \). With a simple computation in MAGMA software, we obtain:

\[
F := \text{Fibonacci}(131);
\]

\[
R \langle x \rangle := \text{PolynomialRing}(\text{Integers}());
\]

\[
f := x^2 + 131; T := \text{Thue}(f);
\]

\[
T;
\]

\[
\text{Solutions}(T,F);
\]

\[
\text{Submit}
\]

\[
1066340417491710595814572169
\]

\[
\text{Thue object with form: } X^2 + 131Y^2
\]

\[
\]

So, \( F_{131} \) can not be written in the form \( x^2 + 131y^2 \).

Analogous: \( F_{359} \) can not be written in the form \( x^2 + 359y^2 \), but \( F_{2971} \) can be written in the form \( x^2 + 2971y^2 \).

Now, we study the prime Fibonacci numbers \( F_p \), with \( p \equiv 1 \pmod{4} \).

First, we give the following remark:

**Remark 2.2** Let \( K \) be the biquadratic field \( K = \mathbb{Q}(\sqrt{p}, \sqrt{F_p}) \) and let \( O_K \) be the ring of integers of this field. Then, the ideal \( (\sqrt{F_p} + y\sqrt{p})O_K \) is the square of an ideal of \( O_K \).
Proof. It is known that \( O_K \) is a Dedekind ring for every value of \( p \) and \( F_p \). Consider the Diophantine equation \( F_p = x^2 + py^2 \). Passing to ideals, this Diophantine equation becomes

\[
\left( \sqrt{F_p} - y\sqrt{p} \right) O_K \cdot \left( \sqrt{F_p} + y\sqrt{p} \right) O_K = x^2 O_K.
\]

It is easy to show that the ideals \( \left( \sqrt{F_p} - y\sqrt{p} \right) O_K \) and \( \left( \sqrt{F_p} + y\sqrt{p} \right) O_K \) are co-primes. Looking to the last form of our equation and applying a property of Dedekind rings, it results that there exist an ideal \( J \) in the ring \( O_K \) such that

\[
\left( \sqrt{F_p} + y\sqrt{p} \right) O_K = J^2 \tag{2.1}
\]

Next, we make a few observations about the last result obtained.

Since \( p \equiv F_p \equiv 1 \pmod{4} \) and \( \left( \frac{p}{F_p} \right) = 1 \), it is known \([8]\) that \( \left( \frac{p}{F_p} \right)_4 = \left( \frac{F_p}{p} \right)_4 \cdot \left( \epsilon_p \right)_4 \), where \( \epsilon_p \) is a fundamental unity of the field \( \mathbb{Q}(\sqrt{-p}) \).

If \( \left( \frac{p}{F_p} \right) = -1 \), then \( \left( \frac{p}{F_p} \right)_4 \neq \left( \frac{F_p}{p} \right)_4 \), and applying Proposition 1.2, it results that \( h_K \) is odd. Using a property of Dedekind ring we obtain that \( J \) is a principal ideal in the ring \( O_K \).

From the relation (2.1) it results that

\[
\sqrt{F_p} + y\sqrt{p} = u \left( a + b\sqrt{F_p} + c\sqrt{p} + d\sqrt{pF_p} \right)^2, \tag{2.2}
\]

where \( u \) is a unity in the ring \( O_K \).

But the fundamental system of unities of the ring \( O_K \) is \( \{ \epsilon_p, \epsilon_{F_p}, \sqrt{\epsilon_p F_p} \} \), when \( N (\epsilon_{pF_p}) = -1 \), respectively \( \{ \epsilon_p, \epsilon_{F_p}, \sqrt{\epsilon_p F_p} \} \), when \( N (\epsilon_{pF_p}) = 1 \).

Concluding, a characterization of primes Fibonacci numbers of the form \( F_p = x^2 + py^2 \), with \( p \equiv 1 \pmod{4} \) has been obtained. But, even if \( F_p \) is a prime integer number with special properties, it is hard to determine the solutions of (2.2). If \( h_K \) is an even number it is harder to determine the solutions of equation (2.2).

In the following we try to give another characterization of prime Fibonacci numbers of the form \( F_p = x^2 + py^2 \), when \( p \equiv 1 \pmod{4} \), so \( p \equiv 1 \) or \( 5 \pmod{12} \) using techniques of computational numbers theory.

A natural question is: How many primes Fibonacci numbers of the form \( p = x^2 + py^2 \) exist? From Proposition 1.1 it results that, when \( p \) is not congruent with 3 \( \pmod{4} \), a prim Fibonacci number has the form \( F_p = x^2 + py^2 \) if and only if \( F_p \) is completely splits in the ring of integer of the Hilbert class field for the quadratic field \( \mathbb{Q}(\sqrt{-p}) \).
If we denote by $L = \mathbb{Q}(\sqrt{-p})$, $P_L$ - the set of all finite primes of $L$, $HCF(L)$ - the Hilbert class field of $L$, $\delta$ - the Cebotarev density, $S$ - the set of prime from $\mathbb{N}$ which are completely split in $HCF(L)$, and $T$ - the set of prime Fibonacci numbers $F_p$ of the form $F_p = x^2 + py^2$, applying a result of [4], the theorem of transitivity of finite extensions and Proposition 1.1, we obtain that: $T \subset S$ and

$$\delta(S) = \frac{1}{[L : \mathbb{Q}] \cdot [HCF(L) : L]} = \frac{1}{2 \cdot [HCF(L) : L]} = \frac{1}{2 \cdot h_L},$$

where $h_L$ is the order of ideal class group of the ring of integers of $L$.

So $\delta(T) \leq \delta(S)$.

With a simple computation with MAGMA we obtain:

```plaintext
Q := Rationals();
Z := RingOfIntegers(Q);
Z;
L := QuadraticField(-50833);
L;
O_L := RingOfIntegers(L);
O_L;
ClassNumber(O_L);
```

Evaluate

Integer Ring
Quadratic Field with defining polynomial $x^2 + 50833$ over the Rational Field
Maximal Equation Order of $L$
128.
So, for $L = \mathbb{Q}(\sqrt{-50833})$, $h_L = 128$.

If we consider all prime Fibonacci numbers $F_p$, with $p \equiv 1$ or $5 \pmod{12}$ known up to now (see [12]) and we calculate the class number for the field $L = \mathbb{Q}(\sqrt{-p})$, using MAGMA, we obtain:

$$h_{\mathbb{Q}(\sqrt{-13})} = 2, h_{\mathbb{Q}(\sqrt{-17})} = 4, h_{\mathbb{Q}(\sqrt{-29})} = 6, h_{\mathbb{Q}(\sqrt{-41})} = 8, h_{\mathbb{Q}(\sqrt{-109})} = 20, h_{\mathbb{Q}(\sqrt{-5059})} = 30$$

$$h_{\mathbb{Q}(\sqrt{-50833})} = 128.$$  

We remark that, when a prime $p$, $p \equiv 1$ or $5 \pmod{12}$ increases, then $\delta(S)$ decreases.

Using the procedure (in MAGMA) described on the page 7 or the procedure described below, and applying Propositions 1.3 and 1.1, it results that the only prime Fibonacci numbers $F_p$ of the form $F_p = x^2 + py^2$, with $p < 10^4$ are $F_{13}, F_{17}, F_{29}, F_{2971}, F_{9311}, F_{9677}$.  

$\mathbb{Q} := \text{Rationals}();$
\[ \mathbb{Z} := \text{RingOfIntegers}(\mathbb{Q}); \]
\[ \mathbb{Q}[t] := \text{PolynomialRing}(\mathbb{Q}); \]
\[ f := t^2 + 17; \]
\[ K < a > := \text{NumberField}(f); \]
\[ a^2; \]
\[ O_L := \text{RingOfIntegers}(L); \]
\[ O_L; \]
\[ P := \text{ideal} < \mathbb{Z}|1597>; \]
\[ P; \]
\[ \text{IsPrime}(P); \]
\[ \text{Decomposition}(O_L, 1597); \]
\[ P_1 := \text{ideal} < O_L|1597, a + 545>; \]
\[ \text{IsPrime}(P_1); \]
\[ \text{IsPrincipal}(P_1); \]

**Evaluate**

\[ -17 \]
Maximal Equation Order with defining polynomial \( x^2 + 17 \) over \( \mathbb{Z} \)
Ideal of Integer Ring generated by 1597
true
\[ < \text{Prime Ideal of } O_L \]
Two element generators:
\[ [1597, 0] \]
\[ [545, 1], 1 > \]
\[ < \text{Prime Ideal of } O_L \]
Two element generators:
\[ [1597, 0] \]
\[ [1052, 1], 1 > \]
Ideal of \( O_L \)
Two element generators:
\[ [1597, 0] \]
\[ [113, -1] \]
true
true

Using Proposition 1.1 we obtain:

**Corollary 2.1** All Fibonacci primes \( F_p \), with \( p < 10^4 \), \( p \) is not congruent with 3 (mod 4) which splits completely in the ring of integers of the Hilbert class field for the quadratic field \( L = \mathbb{Q}(\sqrt{-p}) \) are \( F_{13}, F_{17}, F_{29}, F_{9677} \).
3 Geometric considerations concerning some prime Fibonacci numbers and Fibonacci functions

Knowing that the decomposition:

\[ p = x^2 + y^2, \quad x, y \in \mathbb{N}, \quad x \geq y \geq 0, \]  

(3.1)
is unique for \( p \), a prime Fibonacci number \( p \neq 3 \), we shall discuss the maximum distance between the prime Fibonacci numbers in the cycles given in Proposition 1.5. To do it, let us associate to each Fibonacci prime \( p > 0, \quad p \neq 3 \), the pair \((x, y)\), as in (3.1). If the pair \((x, y)\), corresponding to \( p \) is represented in plane, in a Cartesian system, with the origin \((0,0)\), and one denote by \( M(x, y) \) the point whose coordinates are \((x, y)\), then the euclidean norm of the vector \( \overrightarrow{OM} \) is exactly \( \sqrt{p} \).

If \( p_1 = x_1^2 + y_1^2, \quad x_1 \geq y_1 \geq 0 \) and \( p_2 = x_2^2 + y_2^2, \quad x_2 \geq y_2 \geq 0 \) are prime Fibonacci numbers, written as in (3.1), then, the function defined by:

\[ D(p_1, p_2) = \sqrt{(x_1 - x_2)^2 + (y_1 - y_2)^2} \]
is a metric that quantifies the "distance" between the prime Fibonacci numbers \( p_1 \) and \( p_2 \). \( D(p_1, p_2) \) is exactly the Euclidean distance between the points \( M_1(x_1, y_1) \) and \( M_2(x_2, y_2) \) associated to \( p_1 \) and \( p_2 \).

Let us consider the primes Fibonacci numbers \( p_1, \ p_2, \ldots, \ p_n \). We define the length of the path between \( p_1 \) and \( p_n \) passing through \( p_2, \ldots, p_{n-1} \) to be:

\[ L(p_1, ..., p_n) = D(p_1, p_2) + ... + D(p_{n-1}, p_n). \]

Using these definition we remark:

1. The length of the maximum path between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 4 is \( L_{max} = D(0, 1) + D(1, 2) = 2 \), because: \( 0 = 0^2 + 0^2 \) corresponds to \((0,0), 1 = 1^2 + 0^2 \) corresponds to \((1,0)\) and \( 2 = 1^2 + 1^2 \) corresponds to \((1,1)\).

The maximum distance between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 4 is \( \sqrt{2} \) (Fig.1).

2. The length of the maximum path between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 5 is 2: \( L_{max} = D(0, 1) + D(1, 2) = 2 \).

The maximum distance between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 4 is \( \sqrt{2} \) (Fig.1).

3. The length of the maximum path between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 20 is \( L_{max} = 3 + \sqrt{2} \) (Fig.2).
4. The length of the maximum path between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 8 is \(L_{\text{max}} = 3\) (Fig.3)

The maximum distance between two prime Fibonacci numbers \((\neq 3)\) in the cycle of the Fibonacci numbers mod 8 is \(\sqrt{5}\).

We remember that the Fibonacci numbers can be given by Binet’s Fibonacci number formula:

\[
F_n = \frac{(1 + \sqrt{5})^n - (1 - \sqrt{5})^n}{2^n\sqrt{5}} \iff F_n = \left[ \frac{1}{\sqrt{5}} \left( \frac{1 + \sqrt{5}}{2} \right)^n \right], \tag{3.2}
\]

where \([x]\) is the nearest integer function [7].

The Fibonacci numbers can be extended to negative integer numbers, according to:

\[
F_n = (-1)^{n+1}F_n, n \in \mathbb{N}
\]

and to a real number \(x\), by

\[
F_n = \frac{1}{\sqrt{5}} \left\{ \left( \frac{1 + \sqrt{5}}{2} \right)^x - \left( \frac{2}{1 + \sqrt{5}} \right)^x \cos(\pi x) \right\} \tag{3.3}
\]

It is well known that the Fibonacci sequence have fractal properties. In the following we determine the box - dimension and the information dimensions of the Fibonacci function, defined on a domain \([-a, a]\), where \(a > 0\). The software used was Benoit 1.3.1 (Copyright @ TruSoft Int’l Inc. 1997-1999).

The definitions and the procedures presented were used in the dimension calculation, by Benoit 1.3.1.

The box dimension is defined as the exponent \(D_b\) in the relationship: \(N(d) \approx d^{-D_b}\), where \(N(d)\) is the number of boxes of linear size \(d\) necessary to cover a data set of points distributed in a two-dimensional plane.

In practice, to measure \(D_b\) one counts the number of boxes of linear size \(d\) necessary to cover the set for a range of values of \(d\) and plot the logarithm of \(N(d)\) on the vertical axis versus the logarithm of \(d\) on the horizontal axis. If the set is indeed fractal, this plot will follow a straight line with a negative slope that equals \(-D_b\).

A choice to be made in this procedure is the range of values of \(d\). A conservative choice may be to use as the smallest \(d\) ten times the smallest distance between points in the set, and as the largest \(d\) the maximum distance between points in the set divided by ten.

In theory, for each box size, the grid should be overlaid in such a way that the minimum number of boxes is occupied. This is accomplished in Benoit by rotating the
grid for each box size through 90 degrees and plotting the minimum value of $N(d)$. Benoit permits the user to select the angular increments of rotation.

For a set of points composing a smooth curve, the information dimension $D_i$ is defined as in $I(d) \approx D_i \log(d)$, where $d$ is the linear size of the box and $I(d)$ the information entropy.

If the set is fractal, a plot of $I(d)$ versus the logarithm of $d$ will follow a straight line with a negative slope equal to $-D_i$ (Benoit 1.3.1).

For our purposes the Fibonacci function defined on symmetrical intervals with respect to the origin [-5, 5], [-10, 10],... [-30, 30] (Fig. 4) has been drawn and the two types of dimensions have been calculated.

The output (Figs. 5, 6) is displayed for the function defined on [-30, 30], and the following choices of the parameters:
- the side-length of the largest box: 343 pixels,
- the coefficient of box-decrease, i.e the factor by which the box sizes will be divided during the progression from the largest to the smallest box side-lengths: 1.1,
- the increment of grid rotation: $15^\circ$.

The number of box-sizes was automatically calculated as the ratio between the log(side-length of the largest box) and the log(coefficient of box-size decrease).

To obtain the maximum number of box sizes, i.e the best dimension estimation, the smallest coefficient of box decrease and the maximum side-length of largest box must be chosen.

Analyzing the two charts, we remark that the plots of $N(d)$, respectively $I(d)$ versus the logarithm of $d$ are distributed along a straight line, proving the fractal nature of the chart.

The values of box-dimension was about 1.86, with a standard deviation less than 0.27 and the information dimension was about 1.89, with a standard deviation less than 0.169.

The calculation has been done for different interval of definition of the Fibonacci function and the results obtained were very similar.

In a next article we shall present the results of the calculation of Hausdorff $h$-dimension of Fibonacci function, with respect to different choices of the measure function $h$.

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