A note on the universal separable Banach space with an unconditional Schauder basis with constant $K$

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Abstract

Using the technique of Fraïssé theory we construct a universal object in the class of separable Banach spaces with an unconditional Schauder basis with constant $K$.

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1 Introduction

A Banach space $X$ is complementably universal for a given class of Banach spaces if $X$ belongs to this class and every space from the class is isomorphic to a complemented subspace of $X$.

In 1969 Pełczyński [11] constructed a complementably universal Banach space for the class of Banach spaces with a Schauder basis. In 1971 Kadec [6] constructed a complementably universal Banach space for the class of spaces with the bounded approximation property (BAP). In the same year Pełczyński [9] showed that every Banach space with BAP is complemented in a space with a basis. Pełczyński & Wojtaszczyk [12] constructed in 1971 a universal Banach space for the class of spaces with a finite-dimensional decomposition. Applying Pełczyński’s decomposition argument [10], one immediately concludes that all three universal spaces are isomorphic. It is worth mentioning a negative result of Johnson & Szankowski [5] saying that no separable Banach space can be complementably universal for the class of all separable Banach spaces. The author in [4] presented a natural extension property that describes an isometric version

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of the Kadec-Pelczyński-Wojtaszczyk space. The constructed space is unique, up to isometry, for the class of Banach spaces with finite-dimensional decomposition and is isomorphic to the Kadec-Pelczyński-Wojtaszczyk space. In [1] we construct a Banach space with a normalized monotone unordered Schauder basis containing an \( \varepsilon \)-isometric copy of any Banach space with a normalized monotone unordered Schauder basis.

This note is generalization of the results from [1] to class of Banach space with unconditional normalized Schauder basis with constant \( K \).

2 Preliminaries

All Banach spaces considered in this paper are separable. Consequently, all Schauder bases in Banach spaces are countable.

2.1 Definitions

Let \( x_b : B \to X \) be a function from some set \( B \) to a Banach space \( X \). We say that a series \( \sum_{b \in B} x_b \) converges to an element \( x \in X \) if for any \( \varepsilon > 0 \) there exists a finite set \( F \subset B \) such that for any finite set \( E \subset B \) with \( F \subset E \) we get \( \| x - \sum_{b \in E} x_b \| < \varepsilon \). In this case we say that \( x \) is the sum of the series \( \sum_{b \in B} x_b \). A series \( \sum_{b \in B} x_b \) in a Banach space is convergent if and only if it is Cauchy in the sense that for every \( \varepsilon > 0 \) there exists a finite set \( F \subset B \) such that for any subsets \( C, D \subset B \) containing \( F \) it is easy to see that for any convergent series \( \sum_{b \in B} x_b \) and any \( C \subset B \) the subseries \( \sum_{b \in C} x_b \) is Cauchy and hence is convergent in the Banach space \( X \).

By an unconditional base in a Banach space \( X \) we understand any subset \( B \subset X \) such that for each element \( x \in X \) there exists a unique function \( x_b : B \to \mathbb{R} \) such that the series \( \sum_{b \in B} x_b \cdot b \) converges to \( x \). Equivalently, base in a Banach space is an unconditional if there exist a constant \( K \) such that \( \| \sum_{b \in C} x_b \| \leq K \| \sum_{b \in B} x_b \| \) for any convergent series \( \sum_{b \in B} x_b \) and its subseries with \( C \subset B \). In this case the smallest constant \( K \) is called basis constant. Unconditional basis with basis constant \( K = 1 \) are well-known as monotonic Schauder basis.

Let \( K \geq 1 \) be a fixed real number. By a \( K \)-based Banach space we understand a pair \((X, B_X)\) consisting of a Banach space \( X \) and a normalized unconditional Schauder basis \( B_X \) for \( X \) with basic constant \( \leq K \). A \( K \)-based Banach space \((X, B_X)\) is a subspace of a \( K \)-based Banach space \((Y, B_Y)\) if \( X \subseteq Y \) and \( B_X = X \cap B_Y \).

A finite dimensional \( K \)-based Banach \((X, B_X)\) is rational if its unit ball is a convex polyhedron spanned by finitely many vector whose coordinates in the basis \( B_X \) are rational.
2.2 Categories

Let $\mathcal{K}$ be a category. For two objects $A, B$ of the category $\mathcal{K}$, by $\mathcal{K}(A, B)$ we will denote the set of all $\mathcal{K}$-morphisms from $A$ to $B$. A subcategory of $\mathcal{K}$ is a category $\mathcal{L}$ such that each object of $\mathcal{L}$ is an object of $\mathcal{K}$ and each arrow of $\mathcal{L}$ is an arrow of $\mathcal{K}$.

A category $\mathcal{L}$ is cofinal in $\mathcal{K}$ if for every object $A$ of $\mathcal{K}$ there exists an object $B$ of $\mathcal{L}$ such that the set $\mathcal{K}(A, B)$ is nonempty. A category $\mathcal{K}$ has the amalgamation property if for every objects $A, B, C$ of $\mathcal{K}$ and for every morphisms $f \in \mathcal{K}(A, B)$, $g \in \mathcal{K}(A, C)$ we can find an object $D$ of $\mathcal{K}$ and morphisms $f' \in \mathcal{K}(B, D)$, $g' \in \mathcal{K}(C, D)$ such that $f' \circ f = g' \circ g$.

In this paper we shall work in the category $\mathcal{K}$, whose objects are $K$-based Banach spaces. For two $K$-based Banach spaces $(X, B_X)$, $(Y, B_Y)$, a morphism of category $\mathcal{K}$ is a linear continuous operator $T : X \to Y$ such that $T(B_X) \subseteq B_Y$. A morphism $T : X \to Y$ of the category $\mathcal{K}$ is called an isometry if $\|T(x)\|_Y = \|x\|_X$ for any $x \in X$.

2.3 Amalgamation

In this section we prove that the category $\mathcal{K}$ has the amalgamation property.

**Lemma 1. (Amalgamation Lemma)** Let $X, Y, Z$ be $K$-based Banach spaces and $j : Z \to X$, $i : Z \to Y$ be isometries. Then there exist a $K$-based Banach space $W$ and isometries $j' : Y \to W$ and $i' : X \to W$ such that $i' \circ j = j' \circ i$. Moreover, if the $K$-based Banach spaces $X, Y, Z$ are finite-dimensional (rational), then so is the $K$-based Banach space $W$.

**Proof.** Let $(X, B_X)$, $(Y, B_Y)$, $(Z, B_Z)$ be $K$-based Banach spaces. Without loss of generality we may assume that $Z = X \cap Y$, $B_Z = B_X \cap B_Y$ and the isometries $i, j$ are identity inclusions.

Consider the direct sum $X \oplus Y$ of the Banach space $X, Y$ endowed with the norm $\|(x, y)\| = \|x\|_X + \|y\|_Y$. Let $W = (X \oplus Y)/\Delta$ be the quotient by the subspace $\Delta = \{(z, -z) : z \in Z\}$.

We define linear operators $i' : X \to W$ and $j' : Y \to W$ by $i'(x) = (x, 0) + \Delta$ and $j'(y) = (0, y) + \Delta$.

Repeating the argument from Lemma 1 in [1] we can show that $i'$ and $j'$ are isometries.

We prove that the basis of $W$ is unconditional with basis constant $K$. Without loss of generality we may assume that $B_Z = B_X \cap B_Y$ and the isometries $i, j$ are identity inclusions.

We shall identify $X$ and $Y$ with their images $i'(X)$ and $j'(Y)$ in $W$. In this case the union $B_W = B_X \cup B_Y$ is a normalized Schauder basis for $W$.

Given any $w$ and a finite subset $D$ of $B_W$ we should prove the upper bound

$$\frac{1}{K} \left\| \sum_{b \in D} w_b \cdot b \right\| \leq \left\| \sum_{b \in B_W} w_b \cdot b \right\|,$$
where \((w_b)_{b \in B_W}\) are the coordinates of \(w = \sum_{b \in B_W} w_b \cdot b\) in the basis \(B_W\).

Write \(D = D_Z \cup D_X \cup D_Y\), where \(D_Z = D \cap B_Z = D \cap B_X \cap B_Y\), \(D_X = D \setminus B_Y\) and \(D_Y = D \setminus B_X\).

Taking into account that the norms of the \(K\)-based Banach spaces \(X, Y\) have basis constant \(K\), we obtain:

\[
\| \sum_{b \in B_W} w_b b \|_W = \inf \left\{ \| \sum_{b \in B_X \setminus B_Y} w_b b + \sum_{b \in B_X \cap B_Y} w' b \|_X + \| \sum_{b \in B_X \cap B_Y} w'' b + \sum_{b \in B_Y \setminus B_X} w_b b \|_Y : \right. 
\]

\[
\sum_{b \in B_X \cap B_Y} (w_b'' + w_b''')b = \sum_{b \in B_X \cap B_Y} w_b b \}
\geq \frac{1}{K} \inf \left\{ \| \sum_{b \in D_X} w_b b + \sum_{b \in D_Z} w' b \|_X + \| \sum_{b \in D_Z} w'' b + \sum_{b \in D_Y} w_b b \|_Y : \right. 
\]

\[
\sum_{b \in D_Z} (w_b' + w_b''')b = \sum_{b \in D_Z} w_b b \}
\geq \frac{1}{K} \inf \left\{ \| \sum_{b \in D_X} w_b b + \sum_{b \in B_X \cap B_Y} w_b b \|_X + \| \sum_{b \in B_X \cap B_Y} w'' b + \sum_{b \in B_Y \cap B_Y} w_b b \|_Y : \right. 
\]

\[
\sum_{b \in B_X \cap B_Y} (w_b' + w_b''')b = \sum_{b \in D_Z} w_b b \}
= \frac{1}{K} \| \sum_{b \in D} w_b b \|_W
\]

This completes the proof. \(\square\)

### 3 Rational universality

The inverse of a bijective \(\varepsilon\)-isometry is again an \(\varepsilon\)-isometry.

**Definition 1.** A \(K\)-based Banach space \(X\) is called *rationally universal* if each finite dimensional \(K\)-based subspace of \(X\) is rational and for any finite-dimensional rational \(K\)-based Banach space \(A\) and subspace \(A' \subset A\), any isometry \(f' : A' \to X\) can be extended to an isometry \(f : A \to X\).

Denote by \(\mathcal{F}\) the subcategory of \(\mathcal{R}\) whose objects are rational finite-dimensional \(K\)-based Banach spaces and morphisms are linear isometries of such spaces. Obviously, up to isomorphism the category \(\mathcal{F}\) contains countably many objects. By Lemma 1, the category \(\mathcal{F}\) has the amalgamation property. We now use the concepts from [7] for
constructing a “generic” sequence in \( F \). First of all, a sequence in a fixed category \( \mathcal{C} \) is formally a covariant functor from the set of natural numbers \( \omega \) into \( \mathcal{C} \). A sequence \((X_n)_{n \in \omega}\) of objects of the category \( F \) is called a chain if each space \( X_n \) is a subspace of the \( K \)-based Banach space \( X_{n+1} \). It is easy to see that each sequence in \( K \) is isomorphic to a chain \((X_n)_{n \in \omega}\) of finite-dimensional rational \( K \)-based Banach spaces.

**Definition 2.** A chain of \((U_n)_{n \in \omega}\) of objects of the category \( F \) is Fraïssé if for any \( n \in \omega \), and any morphism \( f : U_n \to Y \) of \( F \), there exist \( m > n \) and a morphism \( g : Y \to U_m \) of the category \( F \) such that \( g \circ f : U_n \to U_m \) is the identity inclusion of \( U_n \) to \( U_m \).

The name “Fraïssé sequence”, as in [7], is motivated by the model-theoretic theory of Fraïssé limits developed by Roland Fraïssé [3]. One of the results in [7] is that every countably cofinal category with amalgamation has a Fraïssé sequence. Applying this general result to our category \( F \) we get:

**Theorem 1 ([7]).** The category \( F \) has a Fraïssé sequence.

From now on, we fix a Fraïssé sequence \((U_n)_{n \in \omega}\) in \( F \), which can be assumed to be a chain of finite-dimensional rational \( K \)-based Banach spaces. Let \( \mathbb{U} \) be the completion of the union \( \bigcup_{n \in \omega} U_n \) and \( B_{\mathbb{U}} = \bigcup_{n \in \omega} B_{U_n} \).

**Lemma 2.** \((\mathbb{U}, B_{\mathbb{U}})\) is a \( K \)-based Banach space.

**Proof.** We have to prove that \( B_{\mathbb{U}} = \bigcup_{n \in \omega} B_{U_n} \) is a normalized unconditional Schauder basis with basis constant \( K \) for \( \mathbb{U} \). For each \( n \) the spaces \( U_n \) are \( K \)-based Banach spaces, so \( \|b\| = 1 \) for every \( b \in U_n \). This shows that \( B_{\mathbb{U}} \) is normalized. The fact that \( B_{\mathbb{U}} \) is an unconditional Schauder basis with basis constant \( K \) follows from Lemma 6.2 and Fact 6.3 in [2].

Definition 1 and the construction of the \( K \)-based Banach space \( \mathbb{U} \) implies the following theorem.

**Theorem 2.** The \( K \)-based Banach space \( \mathbb{U} \) is rationally universal.

The following theorem can be proved by a standard back-and-forth method, see Theorem 3 in [1].

**Theorem 3.** Any rationally universal \( K \)-based Banach spaces \( X, Y \) are isometric.

### 4 Almost universality

A linear operator \( f \) between Banach spaces \( X \) and \( Y \) is called an \( \varepsilon \)-isometry for a positive real number \( \varepsilon \), if

\[
(1 + \varepsilon)^{-1} \cdot \|x\|_X < \|f(x)\|_Y < (1 + \varepsilon) \cdot \|x\|_X
\]

for every \( x \in X \setminus \{0\} \).
Definition 3. A $K$-based Banach space $X$ called \textit{almost-universal} if for any $\varepsilon > 0$ and finite dimensional $K$-based Banach space $A$, any $\varepsilon$-isometry $f' : A' \to X$ defined on a subspace $A' \subset A$ can be extended to a $\varepsilon$-isometry $f : A \to X$.

The following three theorems can be proved by analogy with Theorems 4, 5, 6 in [1].

\textbf{Theorem 4.} Any rational universal $K$-based Banach space $X$ is almost-universal.

\textbf{Theorem 5.} Let $U$ and $V$ be almost-universal $K$-based Banach spaces and $\varepsilon > 0$. Each $\varepsilon$-isometry $f : X \to V$ defined on a finite-dimensional $K$-based subspace $X$ of the $K$-based Banach space $U$ can be extended to a bijective $\varepsilon$-isometry $\bar{f} : U \to V$.

\textbf{Theorem 6.} For any $\varepsilon > 0$, every $K$-based Banach space $X$ can be $\varepsilon$-isometrically embedded into the almost-universal $K$-based Banach space $U$.

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