The Dirac equation is one of the fundamental equations in theoretical physics that accounts fully for special relativity in the context of quantum mechanics for elementary spin-1/2 particles. The Dirac equation plays a key role to many exotic physical phenomena such as graphene, topological insulators and superconductors. These systems proved to be ideal testing grounds for theories of the coexistence of quantum and relativistic effects in condensed matter physics. More recently, with the advances of experimental and material science techniques, a collection of effects in different fields have been simulated using different physical platforms such as optical structures, metamaterials and ion traps.

The purpose of this letter is to demonstrate that optics can provide a fertile ground where physical phenomena described by the Dirac equation can be explored. In particular, we demonstrate that the TE polarized electromagnetic waves in one dimensional inhomogeneous media can be mapped into the Dirac equation in one dimension with a Lorentz scalar potential. By tailoring the refractive index we propose a optical structure that simulates a historically important relativistic model known as the Jackiw-Rebbi model. The model describes a one dimensional Dirac field coupled to a static background soliton field and is known as one of the earliest theoretical description of a topological insulator where the zero energy mode can be understood as the edge state. The Jackiw-Rebbi model can be equivalently thought of as the model describing a massless Dirac particle under a Lorentz scalar potential. In particular, the Jackiw-Rebbi model has been studied by Su, Shrieffer and Heeger in the continuum limit of polyacetylene.

To explore the connection between the Dirac equation and optical wave propagation in one dimension with an arbitrary refractive index distribution \( n(x) \) we consider TE waves propagating in the \( xz \) plane. Field modes propagating in this system are described by the following Helmholtz equation

\[
(\partial_{xx} + \partial_{zz} + k_0^2 n(x)) E_y(x, z) = 0,
\]

where \( k_0 \) is the vacuum wavenumber. TE modes governed by eq. (1) have the form \( E_y(x, z) = \psi_1(x) e^{i k_0 z} \), where \( \beta = k_0 n_0 \sin \theta \) is the propagation constant, \( n_0 \) is the constant background value of the refractive index at \( x \to \pm \infty \), \( \theta \) is the angle of incidence, and \( \psi_1 \) satisfy the following Schrodinger like equation

\[
\left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U_1(x) \right) \psi_1(x) = 0
\]  

where \( \hat{k} = i \partial_x, \, U_1(x) = -k_0 (n^2(x) - n_0^2 \sin^2 \theta) \). Let us now make the following transformation

\[
U_1(x) = \frac{1}{k_0} \left[ \left( \frac{k_0}{2} + S(x) \right)^2 - E^2 - \frac{dS}{dx} \right]
\]

where \( S(x) \) is a Lorentz scalar function and \( E \) is an auxiliary constant. Substituting eq. (3) into eq. (2) we have

\[
\psi_1'' - (k_0 S + S^2 - S') \psi_1 + \left[ E^2 - \left( \frac{k_0}{2} \right)^2 \right] \psi_1 = 0
\]

Adding and subtracting the term \( (k_0/2 + S) \psi_1' \) to the left hand side of eq. (4) we have

\[
[\psi_1' + (k_0/2 + S) \psi_1] - (k_0/2 + S) \psi_1' + \left[ E^2 - (k_0/2 + S)^2 \right] \psi_1 = 0
\]  

If we make the following substitution \( \psi_1' + (k_0/2 + S) \psi_1 = E \psi_2 \) into eq. (5) we end up with the following equation \( -\psi_2' + (k_0/2 + S) \psi_2 = E \psi_1 \). These two coupled differential equations can be written in the same mathematical form as the Dirac equation with \( c = \hbar = 1 \), i.e.

\[
\hat{H}_D \Psi = \left[ \sigma_y \hat{p} + \sigma_x \left( \frac{k_0}{2} + S(x) \right) \right] \Psi = E \Psi
\]

where

\[
\sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \quad \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \quad \text{and} \quad \Psi = \begin{pmatrix} \psi_1 \\ \psi_2 \end{pmatrix}.
\]

Equations (6) can be reduced to two uncoupled Schrödinger equations \( \hat{H}_i \psi_i = 0 \), for \( i = 1, 2 \), given by

\[
\hat{H}_i \psi_i = \left( \frac{-\hbar^2}{2m} \frac{\partial^2}{\partial x^2} + U_i(x) \right) \psi_i(x) = 0
\]
that there exists a zero mode with the following solutions which lacks a counterpart in From eq.(6) it follows that the Hamiltonian $H$ perpotential” $x$ is possible however for an eigenstate to be its own partner topology of the scalar field, whose existence is guaranteed by $E$ for $\pm$ $\sigma$, if this is the case then the state is topologically $m/\lambda$ zero energy mode and neither $\psi_1$ nor $\psi_2$ is normalizable.

We need to set $C_+ = 0$ in order to make the two-component spinor normalizable. Therefore, the wave function for the zero mode is given by

$$\Psi(x) = C_- \left( \frac{\cosh(\lambda x)}{\sqrt{m/\lambda}} \right)^m.$$ (14)

Substituting the “superpotential” into eq.(3) and using the fact that $U_1(x) = -k_0(n^2(x) - n_0^2 \sin^2 \theta)$ we can get the expression for the refractive index, i.e.

$$n^2(x) = \left( \frac{m}{k_0^2} + \frac{m^2}{k_0^2} \right) \frac{1}{\sec^2 \theta} - \frac{m^2}{k_0^2} \frac{1}{\sec^2 \theta}.$$ (15)

In Fig. (2) we show the real and imaginary parts of the refrac-

FIG. 1: Schematic diagram of the possible allignment of energy spectra of the Hamiltonians $H_D$, $H_1$ and $H_2$ are shown when (a) there exist a zero energy mode and $\psi_1$ is normalizable, (b) there exist a zero energy mode and $\psi_2$ is normalizable and (c) when there is no zero energy mode and neither $\psi_1$ nor $\psi_2$ is normalizable.

FIG. 2: Real and Imaginary parts of the refractive index profile obtained from the Jackiw-Rebbi model as a function of $k_0 x$ and the incidence angle $\theta$. Note how the real and imaginary part are both even functions with respect to the $x$ coordinate. We have used $\lambda = 2m$ and $m = k_0$.

| $H_2$ | $H_1$ | $H_2$ | $H_1$ | $H_2$ | $H_1$ | $H_2$ | $H_1$ |
|-------|-------|-------|-------|-------|-------|-------|-------|
| (a)   |       |       |       |       |       |       |       |
| (b)   |       |       |       |       |       |       |       |
| (c)   |       |       |       |       |       |       |       |

We can easily construct the zero energy mode by setting $E = 0$ in eq.(6) and solving for the uncoupled first order differential equations for $\psi_{1,2}$, i.e.

$$\psi_i = C_{\mp} \exp \left[ \mp \int \left( \frac{k_0}{2} + S(x) \right) dx \right]$$ (10)

where $C_{\mp}$ is a normalization constant and the double sign in eq.(10) is $-(+)$ for $i = 1(2)$. Note that $\psi_{1,2}$ cannot be both normalized. In the case when neither $\psi_1$ nor $\psi_2$ is normalizable there are no zero modes allowed, and $H_1, H_2$ share the same energy spectrum. In the case when $\psi_1$ is normalizable and $\psi_2$ is not, then there is a zero mode allowed and $H_1, H_2$ share the same energy spectrum except for the ground state of $H_1$. In Fig.(1) we show the energy spectra when there is or there is not a zero energy state for the Dirac Hamiltonian.

We are interested in the zero energy mode of the Jackiw-Rebbi model which is described by the following Dirac equation

$$\left( \partial_x + \phi(x) \begin{array}{c} 0 \\ -\partial_x + \phi(x) \end{array} \right) \begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} = 0$$ (12)

where $\phi(x) = m \tanh(\lambda x)$ corresponds to the soliton localised at $x = 0$, with $m > 0$ and $\lambda > 0$. If we take the “superpotential” $k_0/2 + S(x) = \phi(x)$ then we see from eq.(11) that there exists a zero mode with the following solutions

$$\psi_1(x) = C_- \left( \frac{\cosh(\lambda x)}{\sqrt{m/\lambda}} \right)^m \psi_2(x) = C_+ \left( \frac{\cosh(\lambda x)}{\sqrt{m/\lambda}} \right)^m.$$ (13)

where

$$U_2(x) = \frac{1}{k_0} \left[ \left( \frac{k_0}{2} + S(x) \right)^2 - E^2 + \frac{dS}{dx} \right].$$ (9)

Clearly, $\hat{H}_{1,2}$ are supersymmetric partner Hamiltonians which can be factorized as $\hat{H}_1 = \hat{A} \hat{A} - (E^2/k_0)$ and $\hat{H}_2 = \hat{A} \hat{A} - (E^2/k_0)$ where $\hat{A} = (\partial_x + k_0/2 + S(x))/\sqrt{k_0}$ and $\hat{A}1 = ( -\partial_x + k_0/2 + S(x))/\sqrt{k_0}$. Thus, if $E_1(2)$ is an eigenvalue of $H_1(H_2)$ with eigenfunction $\psi_1(\psi_2)$, the same eigenvalue is given for $H_2(H_1)$ with corresponding eigenfunction $\hat{A} \psi_1(\hat{A}^\dagger \psi_2)$. The only exception is the ground state of $\hat{H}_1$ which lacks a counterpart in $\hat{H}_2$.

From eq.(6) it follows that the Hamiltonian $H_D$ possesses a chiral symmetry defined by the operator $\sigma_z$, which anticommutes with the Hamiltonian, i.e. $\{ \hat{H}_D, \sigma_z \} = 0$. The chiral symmetry implies that eigenstates come in pairs with $\pm E$. It is possible however for an eigenstate to be its own partner for $E = 0$, if this is the case then the state is topologically protected. The resulting zero energy state is protected by the topology of the scalar field, whose existence is guaranteed by the index theorem, which is localised around the soliton.$^3$

We can easily construct the zero energy mode by setting $E = 0$ in eq.(6) and solving for the uncoupled first order differential equations for $\psi_{1,2}$, i.e.

$$\psi_i = C_{\mp} \exp \left[ \mp \frac{1}{2} \left( \frac{k_0}{2} + S(x) \right) dx \right]$$ (10)

where $C_{\mp}$ is a normalization constant and the double sign in eq.(10) is $-(+)$ for $i = 1(2)$. Note that $\psi_{1,2}$ cannot be both normalized. In the case when neither $\psi_1$ nor $\psi_2$ is normalizable there are no zero modes allowed, and $H_1, H_2$ share the same energy spectrum. In the case when $\psi_1$ is normalizable and $\psi_2$ is not, then there is a zero mode allowed and $H_1, H_2$ share the same energy spectrum except for the ground state of $H_1$. In Fig.(1) we show the energy spectra when there is or there is not a zero energy state for the Dirac Hamiltonian.

The existence of a zero mode then depends on the asymptotic behavior of $(k_0/2 + S(x))$, in general we have that$^{12}$

$$\frac{k_0/2 + S(\infty)}{k_0/2 + S(-\infty)} = \begin{array}{c} +1, \text{there is no zero mode,} \\
-1, \text{there exists a zero mode.} \end{array}$$ (11)

We are interested in the zero energy mode of the Jackiw-Rebbi model which is described by the following Dirac equation$^3$

$$\left( \partial_x + \phi(x) \begin{array}{c} 0 \\ -\partial_x + \phi(x) \end{array} \right) \begin{array}{c} \psi_1(x) \\ \psi_2(x) \end{array} = 0$$ (12)

where $\phi(x) = m \tanh(\lambda x)$ corresponds to the soliton localised at $x = 0$, with $m > 0$ and $\lambda > 0$. If we take the “superpotential” $k_0/2 + S(x) = \phi(x)$ then we see from eq.(11) that there exists a zero mode with the following solutions

$$\psi_1(x) = C_- \left( \frac{\cosh(\lambda x)}{\sqrt{m/\lambda}} \right)^m \psi_2(x) = C_+ \left( \frac{\cosh(\lambda x)}{\sqrt{m/\lambda}} \right)^m.$$ (13)
we see that \( n_0 = im \sec \theta / k_0 \), which means that the TE mode propagating in a optical structure with a refractive index given by eq. (15) which mimics the zero-mode state of the Jackiw-Rebbi model is an evanescent wave of the form 

\[ E_y(x, z) = \left[ \cosh(\lambda x) \right]^{-m/\lambda} e^{-m \tan \theta z} \]

(See Fig. 3). Note that if we have \( m < 0 \) the results remain exactly the same except that we must set \( C_- = 0 \) in order to make the two-component spinor normalizable.

It is well known that the refractive index is in general a complex valued function. Then, we can have two different “optical” potentials given by \[ \psi(x) = C_+ \sqrt{\text{sech}(2mx)} e^{i\varphi(x)} \]

(19)

where \( \varphi(x) = 2 \tan^{-1}(\text{tanh}(mx)) \) is a phase factor and \( C_+ \) is a normalization constant. Using the fact that \( U_2(x) = -k_0 n(x) \) we can get the expression for the complex refractive index, i.e.

\[ n^2(x) = \frac{3m^2}{k_0^2} \text{sech}^2(2mx) - \frac{m^2}{k_0^2} \sec^2 \theta + \frac{i}{2} \frac{2m^2}{k_0^2} \text{sech}(2mx) \text{tanh}(2mx). \]

(20)

In Fig. (4) we show the real and imaginary parts of the refractive index profile obtained from the complex Jackiw-Rebbi model as a function of \( k_0 x \) and the incidence angle \( \theta \). Note how the real part is even and the imaginary part is odd with respect to the \( x \) coordinate, which exhibits the \( \mathcal{PT} \) symmetric invariance of the refractive index.

Note that we have chosen \( a(x) = b'/2b \) in order for \( U_1(x) \) to be a real function and \( U_2(x) \) to be \( \mathcal{PT} \)-symmetric. Using eq. (10) it is clear that the normalizable zero energy state which corresponds to \( U_2(x) \) is given by

\[ \psi_2(x) = C_+ \sqrt{\text{sech}(2mx)} e^{i\varphi(x)} \]

(19)

FIG. 3: Intensity evolution of the TE polarized wave inside a waveguide with a refractive index which mimics the Jackiw-Rebbi model. We have set \( \lambda = 2m, m = k_0 \) and \( \theta = \pi/4 \).

FIG. 4: Real and Imaginary parts of the refractive index profile obtained from the complex Jackiw-Rebbi model as a function of \( k_0 x \) and the incidence angle \( \theta \). Note how the real part is even and the imaginary part is odd with respect to the \( x \) coordinate, which exhibits the \( \mathcal{PT} \) symmetric invariance of the refractive index.

In conclusion we have shown that TE polarized waves in one dimensional inhomogeneous settings can be used to simulate the dynamics of the Dirac equation in one space dimension with a Lorentz scalar potential. In particular, we demonstrate how the zero energy state of the Jackiw-Rebbi model can be
implemented in a designed optical set up with a specific refractive index. We have also shown that the zero energy state of the Jackiw-Rebbi model can be reproduced with a complex effective mass. Based on these findings, we have introduced an optical platform for engineering topological states in the optical domain by controlling the refractive index landscape, in particular we propose a way for directly realizing the Jackiw-Rebbi model which allows one to probe the topologically protected zero energy mode.

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1 Dirac, P.A.M., “The quantum theory of the electron”, Proc. R. Soc. A 117 610-624 (1928)
2 Novoselov, K.S. et al., “Two dimensional gas of massless Dirac fermions in graphene”, Nature 438, 197-200 (2005)
3 Hasan, M.Z. and Kane, C.L., “Topological insulators”, Rev. Mod. Phys. 82, 3045-3067 (2010)
4 Qi, X.L. and Zhang, S.C., “Topological insulators and superconductors”, Rev. Mod. Phys. 83, 1057-1110 (2011)
5 Mohammad-Ali Miri, Mathias Heinrich, Ramy El-Ganainy and Demetrios N. Christodoulides, “Supersymmetric Optical Structures”, Phys. Rev. Lett. 110, 233902 (2013)
6 Wei Tan, Yong Sun, Hong Chen and Shun-Qing Shen, “Photonic simulation of topological excitations in metamaterials”, Sci. Rep. 4, 3842 (2014)
7 Lamata, L., León, J., Schatz, T. and Solano, E., “Dirac equation and quantum relativistic effects in single trapped ion”, Phys. Rev. Lett. 98, 253005 (2007)
8 Jackiw, R. and Rebbi, C., “Solitons with fermion number”, Phys. Rev. D 13, 3398 (1976)
9 Su, W.P., Shrieffer, J.R. and Heeger, A.J., “Soliton excitations in polyacetylene”, Phys. Rev. B 22, 2099 (1980)
10 Yeh, P., Yariv A. and Hong C.S., “Electromagnetic propagation in periodic stratified media. I. General theory”, J. Opt. Soc. Am. 67, 423-438 (1977)
11 Mohammad-Ali Miri, Mathias Heinrich, and Demetrios N. Christodoulides, “SUSY-inspired one-dimensional transformation optics”, Optica 1, (2) 89 (2014)
12 Nogami, Y. and Toyama, F.M. “Supersymmetry aspects of the Dirac equation in one dimension with a Lorentz scalar potential”, Phys. Rev. A 47, (3) 1708 (1993)
13 Bagchi, B. and Roychoudhury, R., “A new PT-symmetric complex Hamiltonian with a real spectrum”, J. Phys. A: Math. Gen. 33, L1-L3 (2000)
14 Bender, C.M. and Boettcher, S., “Real spectra in Non-Hermitian Hamiltonians having PT symmetry”, Phys. Rev. Lett. 80, 5243 (1998)
15 Makris, K.G., El-Ganainy, R. and Christodoulides D.N., “Beam Dynamics in PT Symmetric Optical Lattices”, Phys. Rev. Lett. 100, 103904 (2008)
16 Makris, K.G., El-Ganainy, R., Christodoulides, D.N. and Musslimani, Z.H., “PT symmetric optical lattices”, J. Phys. A 81 063807 (2010)