On the Local Equilibrium Condition

Hermann Heßling
FHTW Berlin
University of Applied Sciences
Treskowallee 8
D–10318 Berlin
Germany

Abstract

A physical system should be in a local equilibrium if it cannot be distinguished from a global equilibrium by “infinitesimally localized measurements”. This seems to be a natural characterization of local equilibrium, however the problem is to give a precise meaning to the qualitative phrase “infinitesimally localized measurements”.

A solution is suggested in form of a Local Equilibrium Condition (LEC) which can be applied to non-interacting scalar quanta.

The Unruh temperature of massless quanta is derived by applying LEC to an arbitrary point inside the Rindler Wedge.

Massless quanta outside a hot sphere are analyzed. A stationary spherically symmetric local equilibrium does only exist according to LEC if the temperature is globally constant.

Using LEC a non-trivial stationary local equilibrium is found for rotating massless quanta between two concentric cylinders of different temperatures. This shows that quanta may behave like a fluid with a Bénard instability.
1 Introduction

An understanding of nature seems to be easier at very small length scales than at larger length scales. The better the resolution power of an observable, the less can be resolved: whatever the state of a physical system is, it cannot be distinguished from the vacuum state if the localization regions of the observables are shrunk to a point. This is the content of the Principle of Local Stability [16], [17]. What can be said about the state of a physical system if the localization region of a measurement is not completely shrunk to a point, but is “infinitesimally localized”? From general relativity we know that because of the Equivalence Principle, gravitation is locally constant. In [21] a formulation of a Quantum Equivalence Principle (QEP) was suggested. According to QEP the states a physical system are locally constant.

QEP was investigated in the Rindler space-time \(^1\) and it was shown that the Hawking–Bisognano–Wichman–Unruh temperature [19], [4], [5], [31] is a consequence of QEP [21].

There was a discussion in the literature on the value of the Hawking temperature of an extremal charged black hole. In [20] Hawking and collaborators investigated the action (on the tree level) and its topological behavior and claimed that an extremal charged black hole does not have a definite Hawking temperature, but equilibrium states of all temperatures may exist outside the horizon. On the other hand, Anderson and collaborators [2] calculated the thermal expectation value of the energy–momentum tensor on the horizon and found that the energy–momentum tensor is finite only for a vanishing Hawking temperature. The same result was derived by Moretti who showed that only a vanishing Hawking temperature is compatible with QEP [25].

In this article we continue the study of the short-distance behavior of states in relativistic quantum field theories. In section 2 it is investigated whether the Principle of Maximum Entropy can be used to characterize local equilibrium.\(^2\) The radiation of a hot sphere is treated within Relativistic Hydrodynamics in section 3. After having collected some facts about global equilibrium in section 4, we formulate a Local Equilibrium Condition (LEC) in section 5 and apply it to the massless Klein–Gordon field.

2 The Principle of Maximum Entropy

Global equilibrium states in non–relativistic quantum systems can be characterized by the extremalization of a certain functional: the entropy. The entropy of a state \(\langle \cdot \rangle = \text{Tr}\hat{\rho}(\cdot)\) is given by

\[
S = -\text{Tr}\hat{\rho} \ln \hat{\rho}
\]

\(^1\)The Rindler space-time is a wedge in the Minkowski space-time (\(|t| < x^{(1)}\)). It is a simple model of a black hole.

\(^2\) For different approaches to non–equilibrium see, e.g., [8], [30] and the literature cited therein.
where \( \hat{\rho} = \rho / \text{Tr} \rho \) is the normalized density matrix. Consider a system whose Hamilton operator \( H \) is the only constant of motion and whose total energy

\[
E = \langle H \rangle
\]

is fixed. It is a general experience that, whatever the state of the system is at some initial time, after a long time the system will become more and more stationary and finally will go over into a global equilibrium state. According to the Principle of Maximum Entropy the system will evolve most probably into a state of maximum entropy. To determine the most probable state one has to solve the following variational equation for the normalized density matrix:

\[
\delta \left( S + c(1 - \langle 1 \rangle) + \beta(E - \langle H \rangle) \right) = 0,
\]

where \( c \) and \( \beta \) are Lagrange multipliers for the normalization condition \( \text{Tr} \hat{\rho} = 1 \) and the constraint (2) respectively. One finds the Boltzmann equilibrium distribution

\[
\rho = e^{-\beta H}.
\]

It turns out that the Lagrange multiplier \( \beta \) is just the inverse temperature of the system.

To determine the state of a system with a non–uniform temperature it was suggested \cite{26}, \cite{32}, \cite{33} to apply the Principle of Maximum Entropy to a local form of the constraint (2):

\[
\epsilon(\vec{x}) = \langle H(\vec{x}) \rangle
\]

where \( H(\vec{x}) \) is the Hamilton density at time zero. This leads to a continuum of Lagrange multipliers \( \beta(\vec{x}) \) and the density matrix

\[
\rho_\beta(\vec{x}) = e^{-\int d^3x \beta(\vec{x}) H(\vec{x})}.
\]

\( 1/\beta(\vec{x}) \) is interpreted as the \textit{local temperature} of the system. For an introduction to this approach see, e.g., \cite{13}, \cite{34}, \cite{3} and the references therein.

Entropy is a probability measure for states, i.e. it assigns a positive number to each possible state of the system. The larger this number, the more likely is a state realized. From a formal point of view it is possible to apply the Principle of Maximum Entropy also to a system with a local constraint (3). But, from a physical point of view, the step from the global constraint (2) to the local constraint (3) is non–trivial and it is not clear whether also for local constraints the entropy (1) is the right probability measure to characterize the states which are realized physically. A further problem is to give a physical interpretation to the Lagrange multipliers \( \beta(\vec{x}) \).

Consider the simple but important example, where the inverse local temperature \( \beta(\vec{x}) \) is linear in one coordinate

\[
\beta(\vec{x}) = \beta_0 x^{(1)}. \tag{5}
\]
The Hamiltonian

\[ H = \int d^3x \ x^{(1)} \mathcal{H}(\vec{x}) \]  

is the generator of a boost transformation in \( x^{(1)} \)-direction, which transforms observables along orbits of constant acceleration in the Rindler spacetime. In [16], [21] the short distance behavior of the state given by Eq. (4) and Eq. (5) was analyzed for a Klein–Gordon field and it was found that the only temperature which is allowed physically, is \( \beta_0 = 2\pi \).

This result shows that the Principle of Maximum Entropy is not strong enough to exclude the unphysical temperatures \( 1/\beta_0 \neq 1/2\pi \). The root of the problem is that the state (3) is only the solution to a formal problem, namely to find a state which extremizes the entropy functional (1) in the case of a local constraint (3).

In summary, the Principle of Maximum Entropy despite its success in characterizing global equilibrium seems, in general not to be a good starting point to characterize local equilibrium.

3 Relativistic Hydrodynamics

Elements of relativistic hydrodynamics are briefly reviewed, see e. g. [27], and an ultrarelativistic perfect fluid outside a hot sphere is analyzed.

3.1 Thermodynamical Laws

The stress–energy tensor of a perfect fluid reads

\[ T^{\mu\nu} = (\rho + P)u^\mu u^\nu - P\eta^{\mu\nu} \]

where \( u \) is the normalized velocity 4–vector of the fluid. In the case of a radiation fluid the equation of state \( P = P(\rho) \) between its pressure \( P \) and its density \( \rho \) is given by

\[ P = \rho/3. \]

The stress–energy tensor is used to formulate the thermodynamical laws.

First Law of Thermodynamics: The stress–energy tensor is conserved

\[ \partial_\nu T^{\mu\nu}(x) = 0. \]

Second Law of Thermodynamics: Let \( S^\mu(x) \equiv T^{\mu\nu}(x)u_\nu(x)/T(x) \) be the entropy current four-vector with respect to the temperature \( T(x) \). The entropy production

\[ \sigma(x) \equiv \partial_\mu S^\mu(x) \geq 0 \]

cannot be negative.
The conservation equations of the stress–energy tensor are also known as the (relativistic) Navier–Stokes equations.

The minimum requirement for a fluid to be in a local equilibrium is a vanishing entropy production.

For a radiation fluid in a local equilibrium we obtain by taking the Navier-Stokes equations into account

\[ \partial_\mu \ln \left( \frac{P}{T^4} \right) u^\mu = 0. \tag{7} \]

### 3.2 Hot Sphere

Consider a hot sphere of radius \( r_0 \) with the surface temperature \( T_0 \) which is immersed in a radiation fluid of temperature \( T_\infty \). Can this system be in a stationary local equilibrium?

Before we consider this question within Relativistic Hydrodynamics, let us determine the temperature on a planet induced by the radiation of the sun.\(^3\)

The sun can be considered as a black sphere which radiates at a temperature \( T_0 \approx 5800 \text{K} \). The energy emitted per time through its surface reads according to the Stefan-Boltzmann law

\[ \left( \frac{E}{t} \right)_{\text{sun, emitted}} = \sigma_0 T_0^4 \left( 4\pi r_0^2 \right) \]

where \( \sigma_0 \) is the Stefan-Boltzmann constant. A planet with radius \( r_p \) at the distance \( r \) absorbs the following fraction of the radiation on the sunlit side

\[ \left( \frac{E}{t} \right)_{\text{planet, absorbed}} = \frac{\pi r_p^2}{4\pi r^2} \left( \frac{E}{t} \right)_{\text{sun, emitted}} \]

and reradiates to the space at the same rate

\[ \left( \frac{E}{t} \right)_{\text{planet, emitted}} = \sigma_0 T(r)^4 \left( 4\pi r_p^2 \right). \]

The equilibrium condition \( (E/t)_{\text{planet, absorbed}} = (E/t)_{\text{planet, emitted}} \) implies

\[ T(r) = T_0 \sqrt{\frac{r_0}{2r}}. \tag{8} \]

For the earth this gives \( T_{\text{earth}} \approx 280 \text{K} \), where the following astronomical data were used: radius of the sun \( \approx 7/3 \) lightseconds, distance between the earth and the sun \( \approx 500 \) lightseconds. This simple calculation leads to a surprisingly good result, although it neglects the important greenhouse effect caused by the atmosphere.

Let us introduce spherical coordinates

\[ ds^2 = dt^2 - dr^2 - r^2 d\theta^2 - r^2 \sin^2 \theta d\phi^2 \]

\(^3\) The following derivation is well-known and included for the convenience of the reader.
and make a rotationally symmetric ansatz for the equilibrium vector

\[ \beta^\mu \equiv v^\mu / T = (\beta^t, \beta^r, \beta^\vartheta, \beta^\varphi) = (A(r), B(r), 0, 0). \]

From Eq. (7) we obtain the following connection between the pressure and the temperature

\[ P \sim T^4 \]

i.e. the stress–energy tensor is of the form

\[ T^{\mu\nu}(x) \sim T^4(x) (4u^\mu(x)u^\nu(x) - \eta^{\mu\nu}). \]

Contracting the Navier-Stokes equations with \( \beta^\mu \) yields

\[ 3(T^6\beta^\nu)_{,\nu} = T^6_{,\nu}\beta^\nu. \quad (9) \]

With this result the Navier-Stokes equations read

\[ \frac{4}{3}T^6\beta^\nu \beta^\mu + 4T^6\beta^\nu \beta^\mu_{,\nu} - \eta^{\mu\nu}T^4_{,\nu} = 0. \quad (10) \]

Note that Eq. (9) can be reformulated as a conservation equation for the entropy current 4-vector

\[ (T^4\beta^\nu)_{,\nu} = 0 \quad (11) \]

showing that the entropy production of a radiation fluid is zero. The four Navier–Stokes equations (10) are equivalent to the equation

\[ 0 = (T^4B^2 + T^2)_{,r} \quad (12) \]

i.e.

\[ B^2 = \frac{C_1}{T^4} - \frac{1}{T^2}. \quad (13) \]

Together with the solution of Eq. (11)

\[ B = \frac{C_2}{r^2T^4}. \]

we obtain for the temperature outside the sphere

\[ \frac{C_1}{T^4} - \frac{1}{T^2} = \frac{C_2^2}{r^4T^8} \quad (14) \]

The integration constant \( C_1 \) turns out to be

\[ C_1 = T_\infty^2 \]
If $C_1$ is inserted into Eq. (13), it follows from the positivity of $B^2$

$$T_\infty \geq T(r)$$

i.e. in a local equilibrium the temperature of the quanta at infinity cannot be smaller than the temperature of the sphere, $T_\infty \geq T_0$.

This bound is not changed significantly if gravity is taken into account by using the Schwarzschild metric in the Navier-Stokes equations, since the right hand side of Eq. (13) is simply multiplied by the factor $[1 - r_{SS}/r]^{1/2}$ where $r_{SS} \approx 3$ km is the Schwarzschild radius of the sun.

To summarize, the radiation emitted by the sun cannot be understood as a stationary rotationally symmetric local equilibrium process.

However, in the case of a black hole, $r_0 = r_{SS}$, a local equilibrium may exist with a temperature which is zero everywhere outside the black hole. This follows from the fact that Eq. (14) generalizes to

$$\frac{C_1}{T^4} - \frac{r - r_{SS}}{r} \frac{1}{T^2} = \frac{C_2^2}{r^4T^{8}}.$$ 

(16)

Since a cold black hole is in contradiction to the Hawking effect [19], we conclude that the theory of Relativistic Hydrodynamics seems, in general also not to be a good starting point to characterize local equilibrium.

## 4 Global Equilibrium

### 4.1 Modular Evolution

In a quantum mechanical system of finite degrees of freedom the expectation value $\langle A \rangle$ of any observable $A$ in a state $\langle \cdot \rangle$ can be characterized by a density matrix $\rho$

$$\langle A \rangle = \frac{\text{Tr} \rho A}{\text{Tr} \rho}.$$ 

(17)

If one introduces the modular Hamiltonian $\tilde{H}$ via

$$e^{-\tilde{H}/T} = \rho$$

(18)

where the parameter $T$ is extracted for later convenience, the modular evolution

$$\alpha_\tau(A) = e^{i\tilde{H}\tau} A e^{-i\tilde{H}\tau}$$

(19)

can be defined.

Cyclicity of the trace gives the KMS–condition [23], [24]

$$\langle \alpha_\tau(A)B \rangle = \langle BA_{\tau+i/T}(A) \rangle.$$ 

(20)
In quantum field theory the right hand side of Eq. (17) does not exist, but the KMS–condition (20) can be used directly to characterize the state [15], [17].

By using the Fourier transformation of the KMS condition (20), it is possible to represent a state in terms of a commutator

\[
\langle B_{\alpha} \epsilon (A) \rangle = -\frac{1}{2\pi} \int d\omega \int d\tau \langle [B, \alpha_{\tau}(A)] \rangle e^{-i\omega(\tau - i\epsilon)} e^{-\omega/T} - 1
\]

\[= \frac{T}{2i} \int d\tau \langle [B, \alpha_{\tau}(A)] \rangle \coth(\pi T(\tau - i\epsilon)).
\]

(21)

Here, it is assumed that \(\langle B_{\alpha} \epsilon (A) \rangle\) goes to zero in the limit \(|\tau| \to \infty\). \(\epsilon\) is a tiny positive number which is necessary to give Eq. (21) a well–defined meaning in the sense of distributions and which finally goes to zero. The essential point is that the right hand side of Eq. (21) can be used to calculate states in linear field theories for a given modular Hamiltonian, since the commutator of bosonic fields and the anti–commutator of fermionic fields are multiples of the unit operator and thus independent of the state.

Global equilibrium states \(\langle \cdot \rangle_{eq}\) are characterizable by the modular Hamiltonian

\[\tilde{H} = u_{\mu} P^{\mu}\]

where \(P^{\mu}\) are the time and spatial translation generators and \(u\) is the equilibrium velocity

\[u^{\mu}u_{\mu} = 1.\]

The parameter \(T\) introduced in Eq. (18), can be identified with the temperature of the equilibrium state. It is convenient to introduce the equilibrium vector

\[\beta^{\mu} = u^{\mu}/T.\]

Often a co–moving coordinate system is chosen where the equilibrium velocity is at rest

\[u^{\mu} = (1, 0, 0, 0)\]

called rest system of the heat bath. With respect to this system the “fluid of quanta” appears to be stationary and spatially isotropic and the modular Hamiltonian reduces to the Hamilton operator.

---

4 Strictly speaking, the modular Hamiltonian \(\tilde{H}\) cannot be defined as the logarithm of the density matrix, as in Eq. (18), but the necessary technical modifications to make \(\tilde{H}\) well–defined (see e. g. [17]) are not important for our purpurses.

5 Otherwise, in Eq. (21), one can replace the left hand side by \(\langle B_{\alpha} \epsilon (A) \rangle - \langle B \rangle \langle A \rangle\); see e. g. [14], [17].
4.2 Massless Klein–Gordon Field

The massless Klein-Gordon field $\phi(x)$ in Minkowski spacetime is a solution of the wave equation

$$\eta^{\mu\nu} \partial_\mu \partial_\nu \phi(x) = \phi_{,tt} - \phi_{,x(1)x(1)} - \phi_{,x(2)x(2)} - \phi_{,x(3)x(3)} = 0.$$

The commutation relation reads

$$[\phi(x), \phi(x')] = -i D_u(x - x').$$

(23)

$D_u(x)$ is the massless Pauli–Jordan commutator function

$$D_u(x - x') = \frac{1}{4\pi} \sign(ux - ux') \delta(\sigma(x, x'))$$

where $u$ is a timelike vector and

$$2\sigma(x, x') = (t - t')^2 - (\vec{x} - \vec{x}')^2$$

is the square of the geodesic distance between the points $x$ and $x'$.

4.2.1 Thermal 2–Point Function

In this subsection we use Eq. (21) to calculate the 2–point function of a massless Klein–Gordon field $\phi(x)$ in a global equilibrium state.

The modular Hamiltonian (22) generates a timelike evolution in the direction of the equilibrium velocity $u$

$$\alpha_r \phi(x) \equiv \phi_r(x) = \phi(x + \tau u).$$

(24)

To calculate the 2–point function $\langle \phi(x') \phi(x) \rangle_{eq}$ of a global equilibrium state $\langle \cdot \rangle_{eq}$, we make use of Eq. (21) and the commutation relation (23) and obtain

$$\langle \phi(x') \phi_{i\epsilon}(x) \rangle_{eq} = \frac{T}{8\pi} \left( \frac{1}{\sigma_+} \coth(\pi T (\tau_+ - i\epsilon)) - \frac{1}{\sigma_-} \coth(\pi T (\tau_- - i\epsilon)) \right).$$

(25)

The times $\tau_\pm$ are implicitly given as the solutions of the equation

$$2\sigma_\tau(x, x') \equiv (\tau u - \Delta x)^2 = 0$$

(26)

where

$$\Delta x = x' - x.$$

$\tau_\pm$ mark the two times at which the orbit of the one–parametric diffeomorphism

$$x \rightarrow x + \tau u$$
becomes lightlike to the point $x'$. $\dot{\sigma}_\pm$ are the absolute values of the time derivative of Eq. (26) at the times $\tau_\pm$:

$$\dot{\sigma}_\pm = \left| \frac{d}{d\tau} \sigma(x, x') \right|_{\tau = \tau_\pm}.$$  \hspace{1cm} (27)

We find

$$\tau_\pm = u \cdot \Delta x \pm \dot{\sigma}$$  \hspace{1cm} (28)

and

$$\dot{\sigma}_+ = \dot{\sigma}_- = \sqrt{(u \cdot \Delta x)^2 - (\Delta x)^2} \equiv \dot{\sigma}.$$  \hspace{1cm} (29)

$\dot{\sigma}$ is real, because the equilibrium vector is timelike.

In summary, the thermal 2–point function of a massless Klein-Gordon field in a global equilibrium state is characterized by Eqs. (25), (28), (29).

### 4.2.2 Energy–Momentum Tensor

The canonical energy–momentum tensor of a massless Klein–Gordon field in the Minkowski spacetime is formally given by

$$T_{\mu\nu}(x) = \partial_\mu \phi(x) \partial_\nu \phi(x) - \frac{1}{2} \eta_{\mu\nu} \partial^\rho \phi(x) \partial_\rho \phi(x).$$  \hspace{1cm} (30)

It can be rewritten as

$$T_{\mu\nu}(x) = -\phi(x) \partial_\mu \partial_\nu \phi(x) + \left( \frac{1}{2} \partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \partial^\rho \partial_\rho \right) \phi^2(x).$$  \hspace{1cm} (31)

Because of the distributional character of quantum fields, the product of two fields at the same spacetime point has to be defined by a regularization prescription. According to the Principle of Local Definiteness physical states have the same “singularity structure” at short distances (see [17] for details) and, thus, different physical regularization prescriptions differ only in their finite parts.

We define the energy–momentum tensor of a state $\langle \cdot \rangle$ relative to the vacuum state $\langle 0 | \cdot | 0 \rangle$

$$T_{\mu\nu}(x) = D_{\mu\nu} \left( \langle \phi(x') \phi(x) \rangle - \langle 0 | \phi(x') \phi(x) | 0 \rangle \right)$$  \hspace{1cm} (32)

where we introduced the differential operator

$$D_{\mu\nu} = - \lim_{x' \to x} \partial_\mu \partial_\nu + \left( \frac{1}{2} \partial_\mu \partial_\nu - \frac{1}{4} \eta_{\mu\nu} \partial^\rho \partial_\rho \right) \lim_{x' \to x}.$$  \hspace{1cm} (33)

The differential operators $\partial_\mu \partial_\nu$ in $D_{\mu\nu}$ act on the point $x$, i. e. on only one point of the 2–point function, contrary to the standard point–splitting method which is
based on non–local differential operators \[11\], \[9\]. The details of the regularization (32), in particular the interplay between limiting processes und derivatives, originate in the second line of Eq. (31). Note that the energy-momentum tensor is symmetric.

The 2–point function of the vacuum state is roughly inverse proportional to the square of the geodesic distance, more precisely it reads

\[
\langle 0 | \phi(x') \phi_\epsilon(x) | 0 \rangle = \frac{1}{2\pi^2 \sigma_\epsilon(x', x)} - \frac{1}{2} \sigma_\epsilon(x', x) \quad (34)
\]

where

\[
2\sigma_\epsilon(x', x) = (x - x' + i\epsilon u)^2. \quad (35)
\]

is the square of the geodesic distance between the points \(x\) and \(x'\) modified by an \(i\epsilon\)–prescription, where \(u\) is a timelike vector which we identify with the equilibrium velocity.

To calculate the energy–momentum tensor, we expand the 2–point function of the equilibrium state (25) around \(x' \approx x\), i.e. \(\Delta x \approx 0\) or, equivalently, \(\tau_\pm \approx 0\). By using

\[
\coth \epsilon = \frac{1}{\epsilon} + \frac{\epsilon^3}{3} + \frac{\epsilon^5}{45} + O(\epsilon^5)
\]

we find

\[
\langle \phi(x') \phi_\epsilon(x) \rangle_{eq} = \frac{1}{4\pi^2 (\Delta x - i\epsilon u)^2} + \frac{1}{12\beta^2}
\]

\[
- \frac{\pi^2}{180} \left(T^6 4\beta_\mu \beta_\nu - \eta_{\mu\nu}T^4\right)(\Delta x)^\mu(\Delta x)^\nu + O(\Delta x)^3 \quad (36)
\]

where \(\beta_\mu = u^\mu / T\) is the equilibrium vector. Inserting in Eq. (32) the Eqs. (36) and (34) we obtain

\[
T_{\mu\nu}^{(eq)} = \frac{\pi^2}{90} \left(T^6 4\beta_\mu \beta_\nu - \eta_{\mu\nu}T^4\right) \quad (37)
\]

The energy momentum tensor of a global equilibrium is conserved

\[
\partial_\nu T_{\mu\nu}^{(eq)} (x) = 0 \quad (38)
\]

simply because the components of \(T_{\mu\nu}^{(eq)}\) are constant, and traceless

\[
T_{\mu\nu}^{(eq)} \mu \cdot (x) = 0. \quad (39)
\]

In the rest system of the heat bath

\[
\beta^\mu = \frac{1}{T}(1, 0, 0, 0)
\]
the energy–momentum tensor simplifies to
\[ T^{(eq)}_{\tilde{\mu} \tilde{\nu}} = \frac{\pi^2}{90} T^4 \text{diag}(3, 1, 1, 1) \]
i. e. the energy density is proportional to the fourth power of the temperature
\[ T^{(eq)}_{00} = \frac{\pi^2}{30} T^4 \]
which is the Stefan–Boltzmann law. In other words, Eq. (37) is a “heat bath frame–independent” formulation of the Stefan–Boltzmann law.

5 Local Equilibrium

Global equilibrium is well understood. Descriptions exist at any level of rigor. Local equilibrium, on the other hand, is much less studied despite the fact that it is much more often realized in nature than global equilibrium.

In this section we present a formulation of local equilibrium as a state which cannot be distinguished from a global equilibrium state by “infinitesimally localized measurements”.

5.1 Short Distance Behavior of States

If the localization region of an observable \( A \) is made smaller and smaller, the expectation value \( \langle A \rangle \) of an observable \( A \) in a local equilibrium state \( \langle \cdot \rangle \) should become more and more identical to the expectation value \( \langle A \rangle_{eq} \) of \( A \) in a global equilibrium state \( \langle \cdot \rangle_{eq} \). We assume that the global equilibrium is characterizable by the temperature \( T \) and the equilibrium velocity \( u \).

The shrinking of the localization region of an observable can be described by a one–parametric scaling procedure \( \delta_\lambda \) \cite{[14], [17]}. Let \( \phi(x) \) be a Klein-Gordon field. The one–parametric map \( \delta_\lambda \) scales the amplitude of the field \( \phi(x) \) by a factor \( N_\phi(\lambda) \) and shifts its localization point along a path \( \eta_\lambda x \)
\[ \delta_\lambda \phi(x) = N_\phi(\lambda) \phi(\eta_\lambda x). \]
The path \( \eta_\lambda x \) is defined as the 1–parametric diffeomorphism
\[ (\eta_\lambda x)^\mu = x^\mu + \lambda(x^\mu - x_0^\mu) \] (40)
which has the properties \( \eta_1 x = x \) and \( \eta_0 x = x_0 \). In the limit \( \lambda \to 0 \) the localization point \( x \) is scaled into the point \( x_0 \) along a straight line.

The scaling function \( N_\phi(\lambda) \) is determined relative to a state \( \langle \cdot \rangle \). It is positive and monotone and has to be adjusted in such a way that the scaling limit
\[ \lim_{\lambda \to 0} \langle \delta_\lambda (\psi_1(x_1) \ldots \psi_n(x_n)) \rangle \]
exists for all $n$–point functions and is non–vanishing for some. The linear scaling function $N(\lambda) = \lambda$ is a suitable scaling function for the Klein–Gordon field $\phi(x)$ in a thermal state, as can directly be seen by studying the short distance behavior of Eqs. (26). Each derivative increases the scaling function by one power in the scaling parameter, e. g. $N_{\partial_\mu \phi}(\lambda) = \lambda^2$.

In non–linear quantum field theories the scaling function can be determined by renormalization group techniques, e. g. [21].

An important statement about the scaling limit is made by the Principle of Local Stability (PLS) [16], [17]. The fact, that the scaling function of a thermal Klein–Gordon field does not depend on the temperature, is not an accident but a general property of physical states.

**PLS:** The scaling limit of any given physical state $\langle \cdot \rangle$ agrees with the scaling limit of the vacuum state $\langle 0 | \cdot | 0 \rangle$ in the Minkowski spacetime

$$\lim_{\lambda \to 0} \langle \delta_\lambda A \rangle = \lim_{\lambda \to 0} \langle 0 | \delta_\lambda A | 0 \rangle$$

for all observables $A$.

According to the Quantum Equivalence Principle (QEP) [21] the scaling limit does not change locally.

**QEP:** For physical states $\langle \cdot \rangle$ the expectation value of any scaled observable $\langle \delta_\lambda A \rangle$ is locally constant around the scaling point $x^*$. For linear theories and with respect to an inertial coordinate system this condition can be written as the extremum condition

$$\lim_{\lambda \to 0} \frac{d}{d\lambda} \langle \delta_\lambda A \rangle = 0.$$  \hspace{1cm} (41)

The scaling limit, $\lim_{\lambda \to 0} \langle \delta_\lambda A \rangle$, is a continuous function in the scaling point $x^*$.

In other words, the first non–trivial information about the state of a linear quantum field is beyond the first order in the scaling parameter $\lambda$. In the next subsection we investigate whether properties of local equilibrium become visible at the second order in the scaling parameter.

### 5.2 Local Equilibrium Condition

In studies of the local properties of a quantum statistical system people often consider the limit of “small, but not too small volumes”. Intuitively it seems to be clear that, if a volume is sufficiently small, the physical process of interest appears to be homogenous. On the other hand, if there are not enough particles included in a

---

6 If non–inertial coordinates are used around the scaling point, the derivative $d/d\lambda$ in Eq. (41) has to be replaced by a covariant derivative, see [21].
volume the “statistical average” is not well-defined. The existence of such volumes is expected phenomenologically. Analogously, in the coarse graining method the degrees of freedom of a system are integrated out up to a “characteristic length scale”; the remaining degrees of freedom are treated as phenomenological parameters.

Although the phrases “small, but not too small” or “characteristic length scale” are quite intuitive, a general strategy to characterize them conceptually well founded seems not to be known.

It is our point of view to look as closely as possible to the details of a system. Therefore, we propose to go over to the limit of infinitesimally small volumes or characteristic length scales, respectively. In the Local Equilibrium Condition (LEC) this idea is formulated quantitatively.

**LEC (Part 1):** A state $\langle \cdot \rangle$ is in a local equilibrium in a given point $x_\ast$ of a physical system, if it can be approximated by a global equilibrium state $\langle \cdot \rangle_{eq}$ of a certain temperature $T_\ast = T(x_\ast)$ and a certain equilibrium velocity $u_\ast = u(x_\ast)$ up to the second order in the scaling parameter $\lambda$

$$
\lim_{\lambda \to 0} \frac{d^2}{d\lambda^2} \left( \langle \delta_\lambda A \rangle - \langle \delta_\lambda A \rangle_{eq} \right) = 0.
$$

**LEC (Part 2):** The energy-momentum tensor of the state $\langle \cdot \rangle$ defined in Eq. (32), is identical to the energy-momentum tensor of the global equilibrium state

$$
T_{\mu\nu}(x_\ast) = T^{(eq)}_{\mu\nu}(x_\ast)
$$

in such a way that it fulfills the Navier-Stokes equations

$$
\partial^\nu T_{\mu\nu}(x_\ast) = 0
$$

and the entropy production is vanishing

$$
\sigma(x_\ast) \equiv \partial_\mu S^\mu(x_\ast) = 0
$$

where $S_\mu = T_{\mu\nu}\beta^\nu$ is the entropy current 4-vector.

The parameters of the local equilibrium, i.e. the local temperature $T_\ast$ and the local equilibrium velocity $u_\ast$, are functions of the scaling point $x_\ast$. These functions are also constrained by the field equations, as can be seen below.

To test whether LEC is sensible to characterize local equilibrium, we apply it to an important class of states, the Hadamard states.

### 5.2.1 Hadamard States of Massless Scalar Quanta

Hadamard states of a linear scalar field are quasifree states\footnote{A state is called quasifree, if its truncated $n$-point functions vanish for $n \neq 2$.} with a specific singularity structure: the 2-point function is identical with Hadamard’s fundamental
solution of the wave equation \[11\]

\[
\langle \phi(x') \phi_i(x) \rangle = \frac{-1}{8\pi^2} \left( \frac{U}{\sigma_{i\epsilon}} + V \ln \sigma_{i\epsilon} \right) + W
\]

where \(2\sigma_{i\epsilon}\) is the square of the geodesic distance between the points \(x'\) and \(x\), as defined in Eq. (33). \(U, V, W\) are regular functions in \(x\) and \(x'\). The information about the state is contained in \(W\); \(U\) and \(V\) are state–independent and are uniquely fixed by the geometry of the space-time. In a flat spacetime they read \[11\]

\[
U = 1, \quad V = -\frac{1}{2} m^2 + \frac{1}{8} m^2 \sigma + O(\sigma^{3/2})
\]

where \(m\) is the mass of the Klein–Gordon field which is assumed to be zero.

An application of the first part of LEC to Hadamard states yields

\[
W(x_*, x_*) = \frac{1}{12} T_2^2.
\]

To see this, make use of Eq. (36) and take into account that the state–independent singular parts, \(1/\sigma\) and \(\ln \sigma\), cancel because of the difference in \[42\], and that the scaled state–dependent part \(W(\eta_\lambda x', \eta_\lambda x)\) is regular in the limit \(\lambda \to 0\). This means that the derivative condition of LEC does not depend on the scaling function \(\eta_\lambda\) in the sense that Eq. (10) can be replaced by any one–parametric scaling diffeomorphism \(\eta_\lambda\) which has \(x_*\) as a fix–point. In other words, LEC is independent on the choice of the coordinate system around the the scaling point \(x_*\).

Hadamard states with a state–dependent function which is positive on the diagonal \(x = x' = x_*\)

\[
W(x_*, x_*) \geq 0
\]

have the local temperature

\[
T_* = \sqrt{12 W(x_*, x_*)}
\]

in the scaling point \(x\).

To study the implications of the field equations, we make the following ansatz for the 2–point function of a local equilibrium state

\[
\langle \phi(x') \phi(x) \rangle_{\text{leq}} = \frac{-1}{8\pi^2} \frac{1}{\sigma_{i\epsilon}} + \frac{1}{12} T^2(x)
\]

\[
+ W_\mu(x) \Delta x^\mu + \frac{1}{2} W_{\mu\nu}(x) \Delta x^\mu \Delta x^\nu + O(\Delta x^3)
\]

where we have already taken into account the result (47). The equation of motion for the field \(\phi(x')\) yields

\[
\partial_\mu \partial^\mu \langle \phi(x') \phi(x) \rangle = 0
\]
and it follows
\[ W^\mu_\mu(x) = 0 \] (49)
i.e. the coefficients \( W^\mu_\nu \) are traceless. From the field equation for \( \phi(x) \) we obtain an inhomogenous wave equation
\[ \partial_\mu \partial^\mu T^2(x) = 24W^\mu_\mu \] (50)
for the square of the local equilibrium temperature \( T(x) \).

The energy-momentum tensor takes the form
\[ T^\mu_\nu(x) = \frac{\pi^2}{90} \left( T^6(x)4\beta^\mu(x)\beta_\nu(x) - \eta^\mu_\nu T^4(x) \right) \] (51)
if the local equilibrium exists not only in a point but in a region of the spacetime. This follows from the second part of LEC and energy-momentum tensor Eq. (37) of the global equilibrium state. On the other hand we obtain
\[ T^\mu_\nu(x) = -\frac{1}{12} W^\mu_\nu(x) + W^{\mu,\nu}(x) + W^{\nu,\mu}(x) - \frac{1}{12} T^2(x)^,\mu^\nu - \frac{1}{48} \eta^\mu_\nu T^2(x)^,\rho \] by combining Eq. (52) and Eq. (48). The trace
\[ T^\mu_\mu(x) = -\frac{1}{12} T^2(x)^,\rho \]
has to vanish since the right hand side of Eq. (51) has this property, i.e. we arrive at a homogenous wave equation
\[ \partial^\mu \partial_\mu T^2(x) = 0 \] (52)
for the square of the local equilibrium temperature. Consequently the coefficients \( W^\mu_\nu(x) \) fulfill
\[ \partial_\mu W^\mu_\nu(x) = 0. \] (53)

Note that LEC does not lead to further constraints if it is applied to derivatives of fields. To see this, note that the 2-point function \( \langle \phi(x')\partial_\mu \phi(x) \rangle \) has the scaling function \( \lambda^3 \), i.e. the derivative condition in first part of LEC is not sensitive to it.

Now, we are in a position to clarify the meaning of an “infinitesimally localized measurement”: it is the measurement of any scaled observable in the vicinity of

---

8 This remark is of importance if LEC is applied to free photons since a basic photon observable, the field-stress tensor \( F^\mu_\nu \), is built from derivatives of the unphysical vector potential \( A_\mu \) whose thermal 2-point functions have the form of Hadamard states. However, it is possible to construct a photon observable in such a way that LEC is sensitive to it. LEC in free gauge theories will be considered in a separate publication.
a given spacetime point up to the second order in the scaling parameter \( \lambda \) and a
determination of the energy-momentum tensor in that point.

In summary, LEC leads to a set of constraints for the short distance behavior of
states. The local equilibrium vector \( \beta^\mu(x) \) has to fulfil the Navier-Stokes equations
and the square of the local temperature \( T^2(x) = 1/\beta^\mu(x)\beta_\mu(x) \) has to fulfil a wave
equation, i.e. we obtain an overdetermined system of five partial differential equations
for the four components of the local equilibrium vector. The wave equation (52) describes quantum effects not visible within Relativistic Hydrodynamics.

In [21] it was shown that the derivative condition (41) has to be modified if one
wants to formulate a Quantum Equivalence Principle for asymptotically free quantum
field theories, since in QCD the running coupling constant does not smoothly
go to zero in the short distance limit \( \lambda \to 0 \), but \( \sim 1/\ln \lambda \). Because of the same
reason the derivative condition in the first part of LEC, (12) needs a modification
if one wants to characterize local equilibrium states for self–interacting quantum
fields. As already mentioned in [21], the necessary modifications might be found by
calculating the short distance expansions of \( n \)-point functions via renormalization
group techniques. In this context the “wave front sets” [28] could be of importance
(see also [29], [3], [12] and the literature cited therein).

5.2.2 Killing Fields and the Unruh–Effect

The Navier-Stokes equations are solved if the local equilibrium vector \( \beta^\mu(x) \) is a
Killing field, i.e. if it is a solution of the equation

\[
\beta_{\mu,\nu} + \beta_{\nu,\mu} = 0.
\]

This can most easily be seen if Eq. (10) is rewritten as

\[
\begin{align*}
(\beta_{\mu,\nu} + \beta_{\nu,\mu})\beta^\nu + \beta_\mu \beta^\nu,\nu & = 3\beta_\mu (\beta_{\rho,\nu} + \beta_{\nu,\rho})\beta^\nu/\beta^2.
\end{align*}
\]

The simplest timelike Killing vector is

\[
\beta^\mu = \frac{1}{T_0}(1,0,0,0)
\]

where \( T_0 \) is a constant. It characterizes a global equilibrium with the temperature
\( T_0 \) in the Minkowski spacetime.

Not all Killing vectors are compatible to LEC and, thus, do correspond to a local
equilibrium. The non-stationary Killing vector

\[
\beta^\mu = \frac{1}{T}(x^{(1)}, t, 0, 0)
\]
is timelike in the Rindler wedge ($|t| < x^{(1)}$). It fulfills the wave equation condition (52) only if $T = 0$. The temperature $T = 0$ is not only the temperature of the Minkowskian vacuum state, but of special importance. It is exactly the Bisognano–Wichmann–Unruh temperature, see e. g. [17], although not written in its standard form but expressed with respect to an inertial time coordinate. We derived it by applying LEC to an arbitrary point inside the Rindler wedge. Thus, LEC is sensitive to physical structures which cannot be resolved by PLS or QEP.

In phenomenological fluid mechanics, a system with a vanishing entropy production is considered to be in an “(incomplete) equilibrium”, see e. g. [27]. According to this point of view, each local equilibrium vector $\beta^{\mu}(x)$ which is a Killing vector, would correspond to an “(incomplete) equilibrium”. Our derivation of the Bisognano–Wichmann–Unruh temperature shows that this point of view is not justified in general.

5.2.3 Hot Sphere

The radiation from a hot sphere was already considered in Sec. 3 within Relativistic Hydrodynamics.

We now show that, as a consequence of LEC, massless scalar quanta outside a hot sphere can only be in a stationary radially symmetric local equilibrium, if they are in a global equilibrium.

Because of the rotational symmetry we introduce a spherical coordinate system and assume that the temperature $T = T(r)$ is a function of only the radial coordinate. The statement follows from the fact that the wave equation (52) gives for the square of the local temperature

$$T^2(r) = \frac{a_0}{r} + a_1$$

where $a_0$ and $a_1$ are integration constants, which contradicts the prediction (14) of the Navier-Stokes equations, in general.

5.2.4 Hot Cylinder

Consider massless scalar quanta between two concentric cylinder with different surface temperatures.

It is convenient to introduce axially symmetric coordinates

$$ds^2 = dt^2 - dr^2 - dz^2 - r^2 d\phi^2.$$ 

For the equilibrium vector we make the ansatz

$$\beta^{\mu} \equiv v^{\mu}/T = (\beta^t, \beta^r, \beta^z, \beta^\phi) = (A(r), 0, 0, B(r)).$$

It is amazing to note that in the Eqs. (52) and (8) the local temperature shows the same dependence on the distance, $T(r) \sim r^{-1/2}$. 

18
The Navier-Stokes equations reduce to one equation
\[ B^2 = \frac{T_x^2}{2rT^4}. \] (56)

The wave equation (52) is solved by
\[ T^2(r) = T_0^2 - (T_0^2 - T_1^2) \frac{\ln r/r_0}{\ln r_1/r_0} \]
where we assumed that the cylinder of radius \( r_0 \) has the temperature \( T_0 \) and the cylinder of radius \( r_1 > r_0 \) the temperature \( T_1 \). If this result is inserted into (56) it follows from the condition that the components of the local equilibrium vector have to be real,
\[ T_0 \leq T_1 \]
i. e. the temperature of the outer cylinder cannot be smaller than the temperature of the inner cylinder in the case of a local equilibrium.

In summary, we have found a non-trivial stationary local equilibrium vector which characterizes rotating quanta around the symmetry axis since \( B = \beta \phi \) is non-vanishing.

With respect to this “structure generating behavior” the gas of scalar quanta seems to be comparable to a horizontal layer of fluid which is heated from the bottom. If the temperature of the fluid on the top surface is fixed and if the temperature gradient is stronger than a critical value, a Bénard instability is observed: the fluid starts to rotate in macroscopically regular patterns and, for example, horizontal roller can be observed [18].

This work is an extended version of [22] where it was suggested to characterize local equilibrium as a state which cannot be distinguished from a global equilibrium state within a sufficiently small spacetime region.

After we had derived the material presented here, Buchholz, Ojima and Roos published a general scheme to characterize local equilibrium [7]. Although based on a similar idea, they developed a somewhat complementary framework. Roughly speaking, while their theory covers the long distance aspects of local equilibrium, we concentrate on the short distance properties.

Acknowledgments

It is a pleasure to thank K.–H. Rehren for bringing the artikel [7] to our attention, D. Bucholz for a discussion on this artikel and L. Stuller for pointing out the Bénard effect to us.
References

[1] Adler, S. L., Lieberman, J. and Ng, Y. J. *Regularization of the Stress–Energy Tensor for Vector and Scalar Particles Propagating in a General Background Metric*. Ann. Phys. 106, 279 (1977).

[2] Anderson, P. R., Hiscock, W. A. and Loranz, D. J. *Semiclassical Stability of the Extreme Reisner–Nordström Black Hole*. Phys. Rev. Lett. 74, 4365 (1995).

[3] Bibilashvili, T. M. *Path Integral Determination of the Generating Functional for the Local Equilibrium Quantum Field Theory*. Phys. Lett. B 313, 119 (1993).

[4] Bisognano, J. J. and Wichmann, E. H. *On the Duality Condition for a Hermitean Scalar Field*. J. Math. Phys. 16, 985 (1975).

[5] Bisognano, J. J. and Wichmann, E. H. *On the Duality Condition for Quantum Fields*. J. Math. Phys. 17, 303 (1976).

[6] Brunetti, R. and Fredenhagen, K. *Microlocal Analysis and Interacting Quantum Field Theories: Renormalization on Physical Backgrounds*. Commun. Math. Phys. 208, 623 (2000).

[7] Buchholz, D., Ojima, I. and Roos, H. *Thermodynamic Properties of Nonequilibrium States in Quantum Field Theory*. hep-ph/0105051 (2001).

[8] Calzetta, E. and Hu, B. L. *Nonequilibrium Quantum Fields: Closed Time Path Effective Action, Wigner Function and Boltzmann Equation*. Phys. Rev. D 37, 2878 (1988).

[9] Christensen, S. M. *Vacuum Expectation Value of the Stress Tensor in an Arbitrary Curved Background; the Covariant Point–Separation Method*. Phys. Rev. D 14, 2490 (1976).

[10] Dütsch, M., Fredenhagen, K. *A Local (Perturbative) Construction of Observables in Gauge Theories: the Example of QED*. Comm. Math. Phys. 203, 71 (1999).

[11] DeWitt, B. S. and Brehme, R. W. *Radiation Damping in a Gravitational Field*. Ann. Phys. 9, 220 (1961).

[12] Duetsch, M. and Fredenhagen, K. *Algebraic Quantum Field Theory, Perturbation Theory, and the Loop Expansion*. Preprint, DESY-00-013, hep-ph/0004145 (2000).

[13] Elmfors, P. and Skagerstam B. S. *Quantum Fields at Local Equilibrium and Distortions of the Cosmic Microwave Background*. Ann. Phys. 217, 304 (1992).
[14] Fredenhagen, K. and Haag, R. Generally Covariant Quantum Field Theory and Scaling Limits. Commun. Math. Phys. 108, 91 (1987).

[15] Haag, R., Hugenholtz, N. M. and Winnink, M. On the Equilibrium States in Quantum Statistical Mechanics. Commun. Math. Phys. 5, 215 (1967).

[16] Haag, R., Narnhofer, H. and Stein, U. On Quantum Field Theories in Gravitational Background. Commun. Math. Phys. 94, 219 (1984).

[17] Haag, R. Local Quantum Physics. Berlin: Springer (1992).

[18] Haken, H. Synergetics. An Introduction. Third ed. Berlin: Springer (1983).

[19] Hawking, S. W. Particle Creation by Black Holes. Commun. Math. Phys. 43, 199 (1975).

[20] Hawking, S. W., Horowitz G. T., Ross, S. F. Entropy, Area, and Black Holes. Phys. Rev. D 51, 4302 (1995).

[21] Heßling, H. On the Quantum Equivalence Principle. Nucl. Phys. B415, 243 (1994).

[22] Heßling, H. On the Local Equilibrium Condition. Preprint, DESY-94-208 (1994).

[23] Kubo, R. Statistical Mechanical Theory of Irreversible Processes I. J. Math. Soc. Japan 12, 570 (1957).

[24] Martin, P. C. and Schwinger, J. Theory of Many Particle Systems: I. Phys. Rev. 115, 1342 (1959).

[25] Moretti, V. Wightman Functions’ Behavior on the Event Horizon of an Extremal Reissner–Nordstrom Black Hole. Class. Quant. Grav. 13, 985 (1996).

[26] Mori, H. Statistical–Mechanical Theory of Transport in Fluids. Phys. Rev. 112, 1829 (1958).

[27] Neugebauer, G. Relativistische Thermodynamik. Vieweg: Braunschweig (1981).

[28] Radzikowski, M. J. The Hadamard Condition and Kay’s conjecture in axiomatic quantum field theory on curved spacetime. PhD Thesis, Princeton Univ. (1992).

[29] Radzikowski, M. J. Micro–Local Approach to the Hadamard Condition in Quantum Field Theory on Curved Spacetime. Commun. Math. Phys. 179, 529 (1996).

[30] Umezawa H. and Yamanaka, Y. Thermal Degree of Freedom in Thermo Field Dynamics. Phys. Lett. A 155, 75 (1991).

[31] Unruh, W. G. Notes on Black Hole Evaporation. Phys. Rev. D 14, 870 (1976).
[32] Zubarev, D. N. Dokl. Akad. Nauk. USSR 140, 92 (1961).

[33] Zubarev, D. N. Nonequilibrium Statistical Thermodynamics. New York: Consultants Bureau (1974).

[34] Zubarev, D. N. and Tokarchuk, M. V. Nonequilibrium Thermofield Dynamics and the Nonequilibrium Statistical Operator Method. I. Basic Relations. Theor. Math. Phys. 88, 876 (1992).