LIE THEORY FOR NILPOTENT $L_\infty$-ALGEBRAS

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Dedicated to Ross Street on his sixtieth birthday

1. Introduction

Let $A$ be a differential graded (dg) commutative algebra over a field $K$ of characteristic 0. Let $\Omega_\bullet$ be the simplicial dg commutative algebra over $K$ whose $n$-simplices are the algebraic differential forms on the $n$-simplex $\Delta^n$. In [23], §8, Sullivan introduced a functor $A \mapsto \text{Spec}_\bullet(A) = \text{dAlg}(A, \Omega_\bullet)$ from dg commutative algebras to simplicial sets; here, $\text{dAlg}(A, B)$ is the set of morphisms of dg algebras from $A$ to $B$. (Sullivan use the notation $\langle A \rangle$ for this functor.) This functor generalizes the spectrum, in the sense that if $A$ is a commutative algebra, $\text{Spec}_\bullet(A)$ is the discrete simplicial set $\text{Spec}(A) = \text{Alg}(A, K)$, where $\text{Alg}(A, B)$ is the set of morphisms of algebras from $A$ to $B$.

If $E$ is a flat vector bundle on a manifold $M$, the complex of differential forms $(\Omega^\ast(M, E), d)$ is a dg module for the dg Lie algebra $\Omega^\ast(M, \text{End}(E))$; denote the action by $\rho$. To a one-form $\alpha \in \Omega^1(M, \text{End}(E))$ is associated a covariant derivative

$$\nabla = d + \rho(\alpha) : \Omega^\ast(M, E) \to \Omega^{\ast+1}(M, E).$$

The equation

$$\nabla^2 = \rho(d\alpha + \frac{1}{2}[\alpha, \alpha])$$

shows that $\nabla$ is a differential if and only if $\alpha$ satisfies the Maurer-Cartan equation

$$d\alpha + \frac{1}{2}[\alpha, \alpha] = 0.$$

This example, and others such as the deformation theory of complex manifolds of Kodaira and Spencer, motivates the introduction of the Maurer-Cartan set of a dg Lie algebra $g$ (Nijenhuis and Richardson [21]):

$$\text{MC}(g) = \{ \alpha \in g^1 \mid \delta\alpha + \frac{1}{2}[\alpha, \alpha] = 0 \}.$$

There is a close relationship between the Maurer-Cartan set and Sullivan’s functor $\text{Spec}_\bullet(A)$, which we now explain. The complex of Chevalley-Eilenberg cochains $C^\ast(g)$ of a dg Lie algebra $g$ is a dg commutative algebra whose underlying graded commutative algebra is the graded symmetric algebra $S(g[1]^\vee)$; here, $g[1]$ is the shifted cochain complex $(g[1])^i = g^{i+1}$, and $g[1]^\vee$ is its dual.

If $g$ is a dg Lie algebra and $\Omega$ is a dg commutative algebra, the tensor product complex $g \otimes \Omega$ carries a natural structure of a dg Lie algebra, with bracket

$$[x \otimes a, y \otimes b] = (-1)^{|a||b|}[x, y] ab.$$

**Proposition 1.1.** Let $g$ be a dg Lie algebra whose underlying cochain complex is bounded below and finite-dimensional in each degree. There is a natural identification between the $n$-simplices of $\text{Spec}_\bullet(C^\ast(g))$ and the Maurer-Cartan elements of $g \otimes \Omega_n$. 

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Proof. Under the stated hypotheses on $\mathfrak{g}$, there is a natural identification
\[ \mathcal{MC} (\mathfrak{g} \otimes \Omega) \cong d\text{Alg}(C^*(\mathfrak{g}), \Omega) \]
for any dg commutative algebra $\Omega$. Indeed, there is an inclusion
\[ d\text{Alg}(C^*(\mathfrak{g}), \Omega) \subset \text{Alg}(C^*(\mathfrak{g}), \Omega) = \text{Alg}(S(\mathfrak{g}[1]^\vee), \Omega) \cong (\mathfrak{g} \otimes \Omega)^1. \]
It is easily seen that a morphism in $\text{Alg}(C^*(\mathfrak{g}), \Omega)$ is compatible with the differentials on $C^*(\mathfrak{g})$ and $\Omega$ if and only if the corresponding element of $(\mathfrak{g} \otimes \Omega)^1$ satisfies the Maurer-Cartan equation. \qed

Motivated by this proposition, we introduce for any dg Lie algebra the simplicial set
\[ \mathcal{MC}_\bullet(g) = \mathcal{MC}(g \otimes \Omega_\bullet). \]
According to rational homotopy theory, the functor $g \mapsto \mathcal{MC}_\bullet(g)$ induces a correspondence between the homotopy theories of nilpotent dg Lie algebras over $\mathbb{Q}$ concentrated in degrees $(-\infty, 0]$ and nilpotent rational topological spaces. The simplicial set $\mathcal{MC}_\bullet(g)$ has been studied in great detail by Hinich [16]; he calls it the nerve of $\mathfrak{g}$ and denotes it by $\Sigma(\mathfrak{g})$.

However, the simplicial set $\mathcal{MC}_\bullet(g)$ is not the subject of this paper. Suppose that $\mathfrak{g}$ is a nilpotent Lie algebra and let $G$ be the simply-connected Lie group associated to $\mathfrak{g}$. The nerve $N_\bullet G$ of $G$ is substantially smaller than $\mathcal{MC}_\bullet(g)$, but they are homotopy equivalent. In this paper, we construct a natural homotopy equivalence
\[ (1.1) \quad N_\bullet G \hookrightarrow \mathcal{MC}_\bullet(g), \]
as a special case of a construction applicable to any nilpotent dg Lie algebra.

To motivate the construction of the embedding $(1.1)$, we may start by comparing the sets of 1-simplices of $N_\bullet G$ and of $\mathcal{MC}_\bullet(g)$. The Maurer-Cartan equation on $\mathfrak{g} \otimes \Omega_1$ is tautologically satisfied, since $\mathfrak{g} \otimes \Omega_1$ vanishes in degree 2; thus,
\[ \mathcal{MC}_1(g) \cong \mathfrak{g}[t] dt. \]
Let $\alpha \in \Omega^1(G, \mathfrak{g})$ be the unique left-invariant one-form whose value $\alpha(e) : T_e G \to \mathfrak{g}$ at the identity element $e \in G$ is the natural identification between the tangent space $T_e G$ of $G$ at $e$ and its Lie algebra $\mathfrak{g}$. Consider the path space
\[ P_\bullet G = \{ \tau \in \text{Mor}(\mathbb{A}^1, G) \mid \tau(0) = e \} \]
of algebraic morphisms from the affine line $\mathbb{A}^1$ to $G$. There is an isomorphism between $P_\bullet G$ and the set $\mathcal{MC}_1(g)$, induced by associating to a path $\tau : \mathbb{A}^1 \to G$ the one-form $\tau^* \alpha$.

There is a foliation of $P_\bullet G$, whose leaves are the fibres of the evaluation map $\tau \mapsto \tau(1)$, and whose leaf-space is $G$. Under the isomorphism between $P_\bullet G$ and $\mathcal{MC}_1(g)$, this foliation is simple to characterize: the tangent space to the leaf containing $\alpha \in \mathcal{MC}_1(g)$ is the image under the covariant derivative
\[ \nabla : \mathfrak{g} \otimes \Omega^0_1 \to \mathfrak{g} \otimes \Omega^1_1 \cong T_\alpha \mathcal{MC}_1(g) \]
of the subspace
\[ \{ x \in \mathfrak{g} \otimes \Omega^0_1 \mid x(0) = x(1) = 0 \}. \]
The exponential map
\[ \exp : \mathfrak{g} \to G \]
is a bijection for nilpotent Lie algebras; equivalently, each leaf of this foliation of $\mathcal{MC}_1(g)$ contains a unique constant one-form. The embedding $N_1 G \hookrightarrow \mathcal{MC}_1(g)$ is the inclusion of the constant one-forms into $\mathcal{MC}_1(g)$.

What is a correct analogue in higher dimensions for the condition that a one-form on $\Delta^1$ is constant? Dupont’s explicit proof of the de Rham theorem [7], [8], relies on a chain homotopy
is isomorphic to the nerve of the Deligne groupoid $C$. This homotopy induces maps $s_n : g \otimes \Omega^1 \rightarrow g \otimes \Omega_n$, and we impose the gauge condition $s_n \alpha = 0$, which when $n = 1$ is the condition that $\alpha$ is constant. The main theorem of this paper shows that the simplicial set

(1.2) $\gamma_\bullet (g) = \{ \alpha \in MC_\bullet (g) \mid s_\bullet \alpha = 0 \}$

is isomorphic to the nerve $N_s G$.

The key to the proof of this isomorphism is the verification that $\gamma_\bullet (g)$ is a Kan complex, that is, that it satisfies the extension condition in all dimensions. In fact, we give explicit formulas for the required extensions, which yield in particular a new approach to the Campbell-Hausdorff formula.

The definition of $\gamma_\bullet (g)$ works mutatis mutandi for nilpotent dg Lie algebras; we argue that $\gamma_\bullet (g)$ is a good generalization to the differential graded setting of the Lie group associated to a nilpotent Lie algebra. For example, when $g$ is a nilpotent dg Lie algebra concentrated in degrees $[0, \infty)$, $\gamma_\bullet (g)$ is isomorphic to the nerve of the Deligne groupoid $C(g)$.

Recall the definition of this groupoid (cf. Goldman and Millson [15]). Let $G$ be the nilpotent Lie group associated to the nilpotent Lie algebra $g^0 \subset g$. This Lie group acts on $MC(g)$ by the formula

(1.3) $e^X \cdot \alpha = \alpha - \sum_{n=0}^\infty \frac{\text{ad}(X)^n (\delta_\alpha X)}{(n+1)!}$.

The Deligne groupoid $C(g)$ of $g$ is the groupoid associated to this group action. There is a natural identification between $\pi_0 (MC_\bullet (g))$ and $\pi_0 (C(g)) = MC(g)/G$. Following Kodaira and Spencer, we see that this groupoid may be used to study the formal deformation theory of such geometric structures as complex structures on a manifold, holomorphic structures on a complex vector bundle over a complex manifold, and flat connections on a real vector bundle.

In all of these cases, the dg Lie algebra $g$ controlling the deformation theory is concentrated in degrees $[0, \infty)$, and the associated formal moduli space is $\pi_0 (MC_\bullet (g))$. On the other hand, in the deformation theory of Poisson structures on a manifold, the associated dg Lie algebra, known as the Schouten Lie algebra, is concentrated in degrees $[-1, \infty)$. Thus, the theory of the Deligne groupoid does not apply, and in fact the formal deformation theory is modelled by a 2-groupoid. (This 2-groupoid was constructed by Deligne [5], and, independently, in Section 2 of Getzler [13].)

The functor $\gamma_\bullet (g)$ allows the construction of a candidate Deligne $\ell$-groupoid, if the nilpotent dg Lie algebra $g$ is concentrated in degrees $(-\ell, \infty)$. We present the theory of $\ell$-groupoids in Section 2, following Duskin [9], [10] closely.

It seemed most natural in writing this paper to work from the outset with a generalization of dg Lie algebras called $L_\infty$-algebras. We recall the definition of $L_\infty$-algebras in Section 4; these are similar to dg Lie algebras, except that they have a graded antisymmetric bracket $[x_1, \ldots, x_k]$, of degree $2 - k$, for each $k$. In the setting of $L_\infty$-algebras, the definition of a Maurer-Cartan element becomes

$$\delta \alpha + \sum_{k=2}^\infty \frac{1}{k!} [\alpha, \ldots, \alpha] = 0.$$ 

Given a nilpotent $L_\infty$-algebra $g$, we define a simplicial set $\gamma_\bullet (g)$, whose $n$-simplices are Maurer-Cartan elements $\alpha \in g \otimes \Omega_n$ such that $s_n \alpha = 0$. We prove that $\gamma_\bullet (g)$ is a Kan complex, and that the inclusion $\gamma_\bullet (g) \hookrightarrow MC_\bullet (g)$ is a homotopy equivalence.

The Dold-Kan functor $K_\bullet (V)$ (Dold [4], Kan [17]) is a functor from positively graded chain complexes (or equivalently, negatively graded cochain complexes) to simplicial abelian groups. The set of $n$-simplices of $K_n (V)$ is the abelian group

(1.4) $K_n (V) = \text{Chain}(C_\ast (\Delta^n), V)$.
of morphisms of chain complexes from the complex $C_\bullet(\Delta^n)$ of normalized simplicial chains on the simplicial set $\Delta^n$ to $V$. Eilenberg-MacLane spaces are obtained when the chain complex is concentrated in a single degree (Eilenberg-MacLane [11]).

The functor $\gamma_\bullet(\mathfrak{g})$ is a nonabelian analogue of the Dold-Kan functor $K_\bullet(\mathfrak{V})$: if $\mathfrak{g}$ is an abelian dg Lie algebra and concentrated in degrees $(-\infty, 1]$, there is a natural isomorphism between $\gamma_\bullet(\mathfrak{g})$ and $K_\bullet(\mathfrak{g}[1])$, since (1.4) has the equivalent form

$$K_n(\mathfrak{V}) = Z^n(C^*(\Delta^n) \otimes V, d + \delta),$$

where $C^*(\Delta^n)$ is the complex of normalized simplicial cochains on the simplicial set $\Delta^n$.

The functor $\gamma_\bullet$ has many good features: it carries surjective morphisms of nilpotent $L_\infty$-algebras to fibrations of simplicial sets, and carries a large class of weak equivalences of $L_\infty$-algebras to homotopy equivalences. And of course, it yields generalizations of the Deligne groupoid, and of the Deligne 2-groupoid, for $L_\infty$-algebras. It shares with $MC_\bullet$ an additional property: there is an action of the symmetric group $S_{n+1}$ on the set of $n$-simplices $\gamma_n(\mathfrak{g})$ making $\gamma_\bullet$ into a functor from $L_\infty$-algebras to symmetric sets, in the sense of Fiedorowicz and Loday [12]. In order to simplify the discussion, we have not emphasized this point, but this perhaps indicates that the correct setting for $\ell$-groupoids is the category of symmetric sets.

2. Kan complexes and $\ell$-groupoids

Kan complexes are a natural non-abelian analogue of chain complexes: just as the homology groups of chain complexes are defined by imposing an equivalence relation on a subset of the chains, the homotopy groups of Kan complexes are defined by imposing an equivalence relation on a subset of the simplices.

Recall the definition of the category of simplicial sets. Let $\Delta$ be the category of finite non-empty totally ordered sets. This category $\Delta$ has a skeleton whose objects are the ordinals $[n] = (0 < 1 < \cdots < n)$; this skeleton is generated by the face maps $d_k : [n-1] \to [n], 0 \leq k \leq n$, which are the injective maps

$$d_k(i) = \begin{cases} i, & i < k, \\ i + 1, & i \geq k, \end{cases}$$

and the degeneracy maps $s_k : [n] \to [n-1], 0 \leq k \leq n - 1$, which are the surjective maps

$$s_k(i) = \begin{cases} i, & i \leq k, \\ i - 1, & i > k. \end{cases}$$

A simplicial set $X_\bullet$ is a contravariant functor from $\Delta$ to the category of sets. This amounts to a sequence of sets $X_n = X([n])$ indexed by the natural numbers $n \in \{0, 1, 2, \ldots\}$, and maps

$$\delta_k = X(d_k) : X_n \to X_{n-1}, \quad 0 \leq k \leq n,$$

$$\sigma_k = X(s_k) : X_{n-1} \to X_n, \quad 0 \leq k \leq n,$$

satisfying certain relations. (See May [21] for more details.) A degenerate simplex is one of the form $\sigma_i x$; a nondegenerate simplex is one which is not degenerate. Simplicial sets form a category; we denote by $sSet(X_\bullet, Y_\bullet)$ the set of morphisms between two simplicial sets $X_\bullet$ and $Y_\bullet$.

The geometric $n$-simplex $\Delta^n$ is the convex hull of the unit vectors $e_k$ in $\mathbb{R}^{n+1}$:

$$\Delta^n = \{(t_0, \ldots, t_n) \in [0, 1]^{n+1} \mid t_0 + \cdots + t_n = 1\}.$$  

Its $\binom{n+1}{k+1}$ faces of dimension $k$ are the convex hulls of the nonempty subsets of $\{e_0, \ldots, e_n\}$ of cardinality $k + 1$.  

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The $n$-simplex $\Delta^n$ is the representable simplicial set $\Delta^n = \Delta(-, [n])$. Thus, the nondegenerate simplices of $\Delta^n$ correspond to the faces of the geometric simplex $\Delta^n$. By the Yoneda lemma, $\operatorname{sSet}(\Delta^n, X_\bullet)$ is naturally isomorphic to $X_n$.

Let $\Delta[k]$ denote the full subcategory of $\Delta$ whose objects are the simplices $\{[i] | i \leq k\}$, and let $\sk_k$ be the restriction of a simplicial set from $\Delta^{op}$ to $\Delta[k]^{op}$. The functor $\sk_k$ has a right adjoint $\cosk_k$, called the $k$-coskeleton, and we have

$$\cosk_k(\sk_k(X))_n = \operatorname{sSet}(\sk_k(\Delta^n), X_\bullet).$$

For $0 \leq i \leq n$, let $\Lambda_i^n \subset \Delta^n$ be the union of the faces $d_k[\Delta^{n-1}] \subset \Delta^n$, $k \neq i$. An $n$-horn in $X_\bullet$ is a simplicial map from $\Lambda_i^n$ to $X_\bullet$, or equivalently, a sequence of elements

$$(x_0, \ldots, x_{i-1}, \ldots, x_{i+1}, \ldots, x_n) \in (X_{n-1})^n$$

such that $\partial_j x_k = \partial_{k-1} x_j$ for $0 \leq j < k \leq n$.

**Definition 2.1.** A map $f : X_\bullet \rightarrow Y_\bullet$ of simplicial sets is called a fibration if the maps

$$\xi^n_i : X_n \rightarrow \operatorname{sSet}(\Lambda_i^n, X_\bullet) \times_{\operatorname{sSet}(\Lambda_i^n, Y_\bullet)} Y_n$$

defined by

$$\xi^n_i (x) = (\partial_0 x, \ldots, \partial_{i-1} x, - \partial_{i+1} x, \ldots, \partial_n x) \times f(x)$$

are surjective for $n > 0$. A simplicial set $X_\bullet$ is a Kan complex if the map from $X_\bullet$ to the terminal object $\Delta^0$ is a fibration.

A groupoid is a small category with invertible morphisms. Denote the sets of objects and morphisms of a groupoid $G$ by $G_0$ and $G_1$, the source and target maps by $s : G_1 \rightarrow G_0$ and $t : G_0 \rightarrow G_1$, and the identity map by $e : G_0 \rightarrow G_1$. The nerve $N_\bullet G$ of a groupoid $G$ is the simplicial set whose 0-simplices are the objects $G_0$ of $G$, and whose $n$-simplices for $n > 0$ are the composable chains of $n$ morphisms in $G$:

$$N_n G = \{[g_1, \ldots, g_n] \in (G_1)^n \mid sg_i = tg_{i+1}\}.$$

The face and degeneracy maps are defined using the product and the identity of the groupoid:

$$\partial_k [g_1, \ldots, g_n] = \begin{cases} [g_2, \ldots, g_n], & k = 0, \\
[g_1, \ldots, g_k g_{k+1}, \ldots, g_n], & 0 < k < n, \\
[g_1, \ldots, g_{n-1}], & k = n, \end{cases}$$

$$\sigma_k [g_1, \ldots, g_{n-1}] = \begin{cases} [e t g_1, g_1, \ldots, g_{n-1}], & k = 0, \\
[g_1, \ldots, g_{k-1}, e t g_k, g_{k+1}, \ldots, g_{n-1}], & 0 < k < n, \\
[g_1, \ldots, g_n, e s g_{n-1}], & k = n. \end{cases}$$

The following characterization of the nerves of groupoids was discovered by Grothendieck; we sketch the proof.

**Proposition 2.2.** A simplicial set $X_\bullet$ is the nerve of a groupoid if and only if the maps

$$\xi^n_i : X_n \rightarrow \operatorname{sSet}(\Lambda^n_i, X_\bullet)$$

are bijective for all $n > 1$.

**Proof.** The nerve of a groupoid is a Kan complex; in fact, it is a very special kind of Kan complex, for which the maps $\xi^2_i$ are not just surjective, but bijective. The unique filler of the horn $(-, g, h)$ is the 2-simplex $[h, h^{-1}g]$, the unique filler of the horn $(g, - , h)$ is the 2-simplex $[h, g]$, and the unique filler of the horn $(g, h, -)$ is the 2-simplex $[h g^{-1}, g]$. Thus, the uniqueness of fillers in dimension 2 exactly captures the associativity of the groupoid and the existence of inverses.
The nerve of a groupoid is determined by its 2-skeleton, in the sense that
\[(2.5) \quad N_\bullet G \cong \cosk_2(N_\bullet G).\]
It follows from (2.5), and the bijectivity of the maps $\xi_i^n$, that the maps $\xi_i^n$ are bijective for all $n > 1$.

Conversely, given a Kan complex $X_\bullet$ such that $\xi_i^n$ is bijective for $n > 1$, we construct a groupoid $G$ such that $X_\bullet \cong N_\bullet G$: $G_i = X_i$ for $i = 0, 1$, $s = \partial_1 : G_1 \to G_0$, $t = \partial_0 : G_1 \to G_0$ and $e = \sigma_0 : G_0 \to G_1$.

Denote by $(x_0, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n) \in \text{sSet}(\Lambda_i^n, X)$.

Given a pair of morphisms $g_1, g_2 \in G_1$ such that $sg_1 = tg_2$, define their composition by the formula
$$g_1g_2 = \partial_1 \langle g_2, -, g_1 \rangle.$$ 

Given three morphisms $g_1, g_2, g_3 \in G_1$ such that $sg_1 = tg_2$ and $sg_2 = tg_3$, the 3-simplex $x = \langle g_1, g_2, g_3 \rangle \in X_3$ satisfies
$$g_1(g_2g_3) = \partial_1 \partial_2 x = \partial_1 \partial_1 x = (g_1g_2)g_3,$$ 

hence composition in $G_1$ is associative. For a picture of the 3-simplex $x$, see Figure 1.

**Figure 1.** The 3-simplex $[g_1, g_2, g_3]$

The inverse of a morphism $g \in G_1$ is defined by the formulas
$$g^{-1} = \partial_0 \langle -, etg, g \rangle = \partial_2 \langle g, e sg, - \rangle.$$ 

To see that these two expressions are equal, call them respectively $g^{-\ell}$ and $g^{-\rho}$, and use associativity:
$$g^{-\ell} = g^{-\ell} (gg^{-\rho}) = (g^{-\ell} g)g^{-\rho} = g^{-\rho}.$$ 

It follows easily that $(g^{-1})^{-1} = g$, that $g^{-1} g = e sg$ and $gg^{-1} = etg$, and that $sg^{-1} = tg$ and $tg^{-1} = sg$.

It is clear that
$$sh = \partial_1 \partial_2 \langle g, -, h \rangle = \partial_1 \partial_1 \langle g, -, h \rangle = s(gh)$$
and that
$$tg = \partial_0 \partial_0 \langle g, -, h \rangle = \partial_0 \partial_1 \langle g, -, h \rangle = t(gh).$$

We also see that
$$g = \partial_1 \sigma_1 [g] = \partial_1 [g, e sg] = g(e sg) = \partial_1 \sigma_0 [g] = \partial_1 [etg, g] = (etg)g.$$
Thus, \( G \) is a groupoid. Since \( \text{sk}_2(X_\bullet) \cong \text{sk}_2(N_\bullet G) \), we conclude by \(^{25}\) that \( X_\bullet \cong N_\bullet G \). \( \square \)

Duskin has defined a sequence of functors \( \Pi_\ell \) from the category of Kan complexes to itself, which give a functorial realization of the Postnikov tower. (See Duskin \(^{9}\) and Glenn \(^{14}\), and for a more extended discussion, Beke \(^{2}\).) Let \( \sim \) be the equivalence relation of homotopy relative to the boundary on the set \( X_\ell \) of \( \ell \)-simplices. Then \( \text{sk}_\ell(X_\bullet)/\sim_\ell \) is a well-defined \( \ell \)-truncated simplicial set, and there is a map of truncated simplicial sets

\[
\text{sk}_\ell(X_\bullet) \to \text{sk}_\ell(X_\bullet)/\sim_\ell,
\]

and by adjunction, a map of simplicial sets

\[
X_\bullet \to \text{cosk}_\ell(\text{sk}_\ell(X_\bullet)/\sim_\ell).
\]

Define \( \Pi_\ell(X_\bullet) \) to be the image of this map. Then the functor \( \Pi_\ell \) is an idempotent monad on the category of Kan complexes. If \( x_0 \in X_0 \), we have

\[
\pi_i(X_\bullet, x_0) = \begin{cases} 
\pi_i(\Pi_\ell(X_\bullet), x_0), & i \leq \ell, \\
0, & i > \ell.
\end{cases}
\]

Thus \( \Pi_\ell(X_\bullet) \) is a realization of the Postnikov \( \ell \)-section of the simplicial set \( X_\bullet \). For example, \( \Pi_0(X_\bullet) \) is the discrete simplicial set \( \pi_0(X_\bullet) \) and \( \Pi_1(X_\bullet) \) is the nerve of the fundamental groupoid of \( X_\bullet \). It is interesting to compare \( \Pi_\ell(X_\bullet) \) to other realizations of the Postnikov tower, such as \( \text{cosk}_{\ell+1}(\text{sk}_{\ell+1}(X_\bullet)) \): it is a more economic realization of this homotopy type, and has a more geometric character.

We now recall Duskin’s notion of higher groupoid: he calls these \( \ell \)-dimensional hypergroupoids, but we call them simply weak \( \ell \)-groupoids.

**Definition 2.3.** A Kan complex \( X_\bullet \) is a weak \( \ell \)-groupoid if \( \Pi_\ell(X_\bullet) = X_\bullet \), or equivalently, if the maps \( \xi_\ell^n \) are bijective for \( n > \ell \); it is a weak \( \ell \)-group if in addition it is reduced (has a single 0-simplex).

The 0-simplices of an \( \ell \)-groupoid are interpreted as its objects and the 1-simplices as its morphisms. The composition \( gh \) of a pair of 1-morphisms with \( \partial_1 g = \partial_0 h \) equals \( \partial_1 z \), where \( z \in X_2 \) is a filler of the horn

\[(g, -, h) \in \text{sSet}(\Delta_1^2, X_\bullet).
\]

If \( \ell > 1 \), this composition is not canonical — it depends on the choice of the filler \( z \in X_2 \) — but it is associative up to a homotopy, by the existence of fillers in dimension 3.

A weak 0-groupoid is a discrete set, while a weak 1-groupoid is the nerve of a groupoid, by Proposition \(^{22}\). In \(^{10}\), Duskin identifies weak 2-groupoids with the nerves of bigroupoids. A bigroupoid \( G \) is a bicategory whose 2-morphisms are invertible, and whose 1-morphisms are equivalences; the nerve \( N_\bullet G \) of \( G \) is a simplicial set whose 0-simplices are the objects of \( G \), whose 1-simplices are the morphism of \( G \), and whose 2-simplices \( x \) are the 2-morphisms with source \( \partial_2 x \circ \partial_0 x \) and target \( \partial_1 x \).

The singular complex of a topological space is the simplicial set

\[S_n(X) = \text{Map}(\Delta^n, X)\]

To see that this is a Kan complex, we observe that there is a continuous retraction from \( \Delta^n = |\Delta^n| \) to \( |\Lambda^n| \). The fundamental \( \ell \)-groupoid of a topological space \( X \) is the weak \( \ell \)-groupoid \( \Pi_\ell(S_\bullet(X)) \). For \( \ell = 0 \), this equals \( \pi_0(X) \), while for \( \ell = 1 \), it is the nerve of the fundamental groupoid of \( X \).

Often, weak \( \ell \)-groupoids come with explicit choices for fillers of horns: tentatively, we refer to such weak \( \ell \)-groupoids as \( \ell \)-groupoids. (Often, this term is used for what we call strict \( \ell \)-groupoids, but the latter are of little interest for \( \ell > 2 \).) We may axiomatize \( \ell \)-groupoids by a weakened form of the axioms for simplicial \( T \)-complexes, studied by Dakin \(^{4}\) and Ashley \(^{1}\).
Definition 2.4. An \(\ell\)-groupoid is a simplicial set \(X_\bullet\) together with a set of thin elements \(T_n \subseteq X_n\) for each \(n > 0\), satisfying the following conditions:

i) every degenerate simplex is thin;
ii) every horn has a unique thin filler;
iii) every \(n\)-simplex is thin if \(n > \ell\).

If \(g\) is an \(\ell\)-groupoid and \(n > \ell\), we denote by \(\langle x_0, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n \rangle\) the unique thin filler of the horn
\[(x_0, \ldots, x_{i-1}, -, x_{i+1}, \ldots, x_n) \in \text{sSet}(\Lambda^i_n, X_\bullet).\]

Definition 2.5. An \(\infty\)-groupoid is a simplicial set \(X_\bullet\) together with a set of thin elements \(T_n \subseteq X_n\) for each \(n > 0\), satisfying the following conditions:

i) every degenerate simplex is thin;
ii) every horn has a unique thin filler.

It is clear that every \(\ell\)-groupoid is a weak \(\ell\)-groupoid, and every \(\infty\)-groupoid is a Kan complex.

3. The simplicial de Rham theorem

Let \(\Omega_n\) be the free graded commutative algebra over \(K\) with generators \(t_i\) of degree 0 and \(dt_i\) of degree 1, and relations \(T_n = 0\) and \(dT_n = 0\), where \(T_n = t_0 + \cdots + t_n - 1\):
\[\Omega_n = K[t_0, \ldots, t_n, dt_0, \ldots, dt_n]/(T_n, dT_n).\]

There is a unique differential on \(\Omega_n\) such that \(d(t_i) = dt_i\), and \(d(dt_i) = 0\).

The dg commutative algebras \(\Omega_n\) are the components of a simplicial dg commutative algebra \(\Omega_\bullet\): the simplicial map \(f : [k] \to [n]\) acts by the formula
\[f^*t_i = \sum_{j(i)=i} t_j, \quad 0 \leq i \leq n.\]

Using the simplicial dg commutative algebra \(\Omega_\bullet\), we can define the dg commutative algebra of piecewise polynomial differential forms \(\Omega(X_\bullet)\) on a simplicial set \(X_\bullet\). (See Sullivan \[23\], Bousfield and Guggenheim \[3\], and Dupont \[7\], \[8\].)

Definition 3.1. The complex of differential forms \(\Omega(X_\bullet)\) on a simplicial set \(X_\bullet\) is the space
\[\Omega(X_\bullet) = \text{sSet}(X_\bullet, \Omega_\bullet)\]
of simplicial maps from \(X_\bullet\) to \(\Omega_\bullet\).

When \(K = \mathbb{R}\) is the field of real numbers, \(\Omega(X_\bullet)\) may be identified with the complex of differential forms on the realization \(|X_\bullet|\) which are polynomial on each geometric simplex of \(|X_\bullet|\).

The following lemma may be found in Bousfield and Guggenheim \[3\]: we learned this short proof from a referee.

Lemma 3.2. For each \(k \geq 0\), the simplicial abelian group \(\Omega^k_\bullet\) is contractible.

Proof. The homotopy groups of the simplicial set \(\Omega^k_\bullet\) equal the homology groups of the complex \(C_\bullet = \Omega^k_\bullet\) with differential
\[\partial = \sum_{i=0}^n (-1)^i \partial_i : C_n \to C_{n-1}.\]

Thus, to prove the lemma, it suffices to construct a contracting chain homotopy for the complex \(C_\bullet\).

For \(0 \leq i \leq n\), let \(\pi_i : \Delta^{n+1} \to \Delta^n\) be the affine map
\[\pi_i(t_0, \ldots, t_{n+1}) = (t_0, \ldots, t_{i-1}, t_i + t_{n+1}, t_{i+1}, \ldots, t_n).\]
Define a chain homotopy \( \eta : C_n \to C_{n+1} \) by
\[
\eta \omega = (-1)^{n+1} \sum_{i=0}^{n} t_i \pi_i^* \omega.
\]
For \( \omega \in \Omega^k_n \), we see that
\[
\partial_i \eta \omega =\begin{cases} 
-\eta \partial_i \omega, & 0 \leq i \leq n, \\
(-1)^{n+1} \omega, & i = n + 1.
\end{cases}
\]
It follows that \((\partial \eta + \eta \partial) \omega = \omega\). \(\square\)

Given a sequence \((i_0, \ldots, i_k)\) of elements of the set \(\{0, \ldots, n\}\), let
\[
I_{i_0 \ldots i_k} : \Omega_n \to K
\]
be the integral over the \(k\)-chain on the \(n\)-simplex spanned by the sequence of vertices \((e_{i_0}, \ldots, e_{i_k})\); this is defined by the explicit formula
\[
I_{i_0 \ldots i_k}(\omega_{i_0 \ldots i_k}) = \frac{a_1! \cdots a_k!}{(a_1 + \cdots + a_k + k)!}.
\]
Specializing \(K\) to the field of real numbers, this becomes the usual Riemann integral.

The space \(C_n\) of elementary forms is spanned by the differential forms
\[
\omega_{i_0 \ldots i_k} = k! \sum_{j=0}^{k} (-1)^j t_{i_j} dt_{i_0} \cdots \hat{dt}_{i_j} \cdots dt_{i_k}.
\]
(The coefficient \(k!\) normalizes the form in such a way that \(I_{i_0 \ldots i_k}(\omega_{i_0 \ldots i_k}) = 1\).) The spaces \(C_n\) are closed under the action of the exterior differential,
\[
d \omega_{i_0 \ldots i_k} = \sum_{i=0}^{n} \omega_{i_{i_0} \ldots i_{i_k}},
\]
and assemble to a simplicial subcomplex of \(\Omega_\ast\). The complex \(C_n\) is isomorphic to the complex of simplicial chains on \(\Delta^n\), and this isomorphism is compatible with the simplicial structure. In [24], Whitney constructs an explicit projection \(P_n\) from \(\Omega_n\) to \(C_n\):
\[
P_n \omega = \sum_{k=0}^{n} \sum_{i_0 < \cdots < i_k} \omega_{i_0 \ldots i_k} I_{i_0 \ldots i_k} (\omega).
\]
The projections \(P_n\) assemble to form a morphism of simplicial cochain complexes \(P_\ast : \Omega_\ast \to C_\ast\). If \(X_\ast\) is a simplicial set, the complex of elementary forms
\[
C(X_\ast) = \text{sSet}(X_\ast, C_\ast) \subset \Omega(X_\ast)
\]
on \(X_\ast\) is naturally isomorphic to the complex of normalized simplicial cochains.

**Definition 3.3.** A contraction is a simplicial endomorphism \(s_\ast : \Omega_\ast \to \Omega_{\ast-1}\) such that
\[
(3.7) \quad \text{id} - P_\ast = ds_\ast + s_\ast d.
\]
If \(X_\ast\) is a simplicial complex, a contraction \(s_\ast \) induces a chain homotopy \(s : \Omega_\ast(X_\ast) \to \Omega_{\ast-1}(X_\ast)\) between the complex of differential forms on \(X_\ast\) and the complex \(C(X_\ast)\) of simplicial cochains. In other words, a contraction is an explicit form of the de Rham theorem.

Next, we derive some simple properties of a contraction which we will need later. If \(a\) and \(b\) are operators on a chain complex homogeneous of degree \(k\) and \(\ell\) respectively, we denote by \([a, b]\) the graded commutator
\[
[a, b] = ab - (-1)^{k\ell} ba.
\]
In particular, if \( a \) is homogeneous of odd degree, then \( \frac{1}{2}[a,a] = a^2 \).

**Lemma 3.4.** Let \( s_\bullet \) be a contraction. Then

\begin{itemize}
  \item[i)] \( P_\bullet s_\bullet = 0 \);
  \item[ii)] \( s_\bullet P_\bullet = [d, (s_\bullet)^2] \).
\end{itemize}

**Proof.** To show that \( P_\bullet s_\bullet = 0 \), we must check that \( I_{i_0...i_k} \circ s_n = 0 \) for each sequence \((i_0 ... i_k)\). By the compatibility of \( s_\bullet \) with simplicial maps, this follows from the formula

\[ I_{i_0...i_k} \circ s_n = 0, \]

which is clear, since \( s_n \omega \) is a differential form on \( \Delta^k \) of degree less than \( k \).

The second part of the lemma is a simple calculation. \( \square \)

Dupont [7, 8], found an explicit contraction: we now recall his formula. Given \( 0 \leq i \leq n \), define the dilation map

\[ \varphi_i : [0,1] \times \Delta^n \to \Delta^n \]

by the formula

\[ \varphi_i(u,t) = ut + (1-u)e_i. \]

Let \( \pi_* : \Omega^*([0,1] \times \Delta^n) \to \Omega^{*-1}(\Delta^n) \) be integration along the fibers of the projection \( \pi : [0,1] \times \Delta^n \to \Delta^n \). Define the operator

\[ h^i_n : \Omega^*_n \to \Omega^{*-1}_n \]

by the formula

\begin{equation}
(3.8)
\end{equation}

\[ h^i_n \omega = \pi_* \varphi^*_i \omega, \]

Let \( \varepsilon^i_n : \Omega_n \to K \) be evaluation at the vertex \( e_i \). Stokes’s theorem implies the Poincaré lemma, that \( h^i_n \) is a chain homotopy between the identity and \( \varepsilon^i_n \):

\begin{equation}
(3.9)
\end{equation}

\[ dh^i_n + h^i_n d = \text{id}_n - \varepsilon^i_n. \]

The flow \( \varphi_i(u) \) is generated by the vector field

\[ E_i = \sum_{j=0}^n (t_j - \delta_{ij}) \partial_j. \]

Let \( \iota_\bullet \) be the contraction \( \iota(E_i) \); we have

\begin{equation}
(3.10)
\end{equation}

\[ \iota_j \varphi_i(u) = \varphi_i(u)(u t_j + (1-u) \iota_i), \]

and also

\begin{equation}
(3.11)
\end{equation}

\[ \iota_i \omega_{i_0...i_k} = \sum_{p=0}^k (-1)^{p-1} \delta_{i_p} \omega_{i_0...\hat{i}_p...i_k}. \]

The formula \( (3.8) \) for \( h^i_n \) may be written more explicitly as

\[ h^i_n = \int_0^1 u^{-1} \varphi_i(u) \iota_i du. \]

**Lemma 3.5.** \( h^i h^j + h^j h^i = 0 \)

**Proof.** Let \( \varphi_{ij} : [0,1] \times [0,1] \times \Delta^n \to \Delta^n \) be the map

\[ \varphi_{ij}(u,v,t) = uvt_k + (1-u)e_i + u(1-v)e_j. \]

Then we have

\[ h^i h^j \omega = \pi_* \varphi^*_{ij} \omega. \]
We have

\[ \varphi_{ji}(u, v) = \varphi_{ij}(\tilde{v}, \tilde{u}), \]

where \( \tilde{u} \) and \( \tilde{v} \) are determined implicitly by the equations

\[ (1 - u)v = 1 - \tilde{u}, \quad 1 - v = (1 - \tilde{v})\tilde{u}. \]

Since this change of variables is a diffeomorphism of the interior of the square \([0, 1] \times [0, 1]\), the lemma follows.

**Lemma 3.6.** \( I_{i_0\ldots i_k}(\omega) = (-1)^k \varepsilon_n^{i_k} h_n^{-1} \ldots h_n^{i_0} \omega \)

**Proof.** For \( k = 0 \), this holds by definition. We argue by induction on \( k \). We may assume that \( \omega \) has positive degree, and hence that \( \omega = d\nu \) is exact. By Stokes’s theorem,

\[ I_{i_0\ldots i_k}(d\nu) = \sum_{j=0}^k (-1)^{j-1} I_{i_0\ldots \hat{i}_j\ldots i_k}(\nu). \]

On the other hand, by (3.9), we have

\[
\varepsilon_n^{i_k} h_n^{i_{k-1}} \ldots h_n^{i_0} d\nu = \sum_{j=0}^{k-1} (-1)^j \varepsilon_n^{i_k} h_n^{i_{k-1}} \ldots [d, h_n^{i_j}] \ldots h_n^{i_0} \nu
\]

\[ = \sum_{j=0}^{k-1} (-1)^j \varepsilon_n^{i_k} h_n^{i_{k-1}} \ldots \varepsilon_n^{i_j} h_n^{i_0} \nu + (-1)^k \varepsilon_n^{i_k} \varepsilon_n^{i_{k-1}} h_n^{i_{k-2}} \ldots h_n^{i_0} \nu. \]

But \( \varepsilon_n^{i_k} \varepsilon_n^{i_{k-1}} = \varepsilon_n^{i_{k-1}} \).

The following theorem is due to Dupont.

**Theorem 3.7.** The operators

\[ s_n = \sum_{k=0}^{n-1} \sum_{i_0 < \ldots < i_k} \omega_{i_0\ldots i_k} h_n^{i_k} \ldots h_n^{i_0}, \quad n \geq 0, \]

form a contraction.

**Proof.** It is straightforward to check that \( s_n \) is simplicial. In the proof of (3.7), we abbreviate \( h_n^i \) to \( h^i \). In the definition of \( s_n \), we may take the upper limit of the sum over \( k \) to be \( n \). We now have

\[ [d, s_n] = \sum_{k=0}^{n-1} \sum_{i_0 < \ldots < i_k} \sum_{i \notin \{i_0, \ldots, i_k\}} \omega_{i_0\ldots i_k} h_n^{i_k} \ldots h_n^{i_0} \]

\[ + \sum_{k=0}^{n-1} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \ldots < i_k} \omega_{i_0\ldots i_k} h_n^{i_k} \ldots [d, h_n^{i_j}] \ldots h_n^{i_0}. \]

By (3.9), we have

\[ \sum_{k=0}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \ldots < i_k} \omega_{i_0\ldots i_k} h_n^{i_k} \ldots [d, h_n^{i_j}] \ldots h_n^{i_0} = \text{id} + \sum_{k=1}^{n} \sum_{j=0}^{k} (-1)^j \sum_{i_0 < \ldots < i_k} \omega_{i_0\ldots i_k} h_n^{i_k} \ldots \hat{h}_n^{i_j} \ldots h_n^{i_0} \]

\[ - \sum_{k=0}^{n} (-1)^k \sum_{i_0 < \ldots < i_k} \omega_{i_0\ldots i_k} \varepsilon_n^{i_k} h_n^{i_{k-1}} \ldots h_n^{i_0}. \]

The first term on the right-hand side equals the identity operator, the second cancels the first sum of (3.13), while by Lemma 3.6 the third sum equals \( P_n \).
We will need special class of contractions, which we call gauges.

**Definition 3.8.** A **gauge** is a contraction such that \((s_\bullet)^2 = 0\).

In fact, Dupont’s operator \(s_\bullet\) is a gauge. But by a trick of Lambe and Stasheff \[19\], any contraction gives rise to a gauge.

**Proposition 3.9.** If \(s_\bullet\) is a contraction, then the operator

\[ \tilde{s}_\bullet = s_\bullet d s_\bullet (\text{id} - P_\bullet) \]

is a gauge. If \(s_\bullet\) is a gauge, then \(\tilde{s}_\bullet = s_\bullet\).

**Proof.** Let \(\tilde{s}_\bullet\) be the contraction

\[ \tilde{s}_\bullet = s_\bullet (\text{id} - P_\bullet). \]

By construction, \(\tilde{s}_\bullet P_\bullet = 0\), hence by Lemma 3.4, \([d, (\tilde{s}_\bullet)^2] = 0\). Then \(\tilde{s}_\bullet = \tilde{s}_\bullet d \tilde{s}_\bullet\) is a contraction:

\[
\begin{align*}
[d, \tilde{s}_\bullet] &= [d, \tilde{s}_\bullet d \tilde{s}_\bullet] \\
&= [d, \tilde{s}_\bullet] d \tilde{s}_\bullet + \tilde{s}_\bullet d [d, \tilde{s}_\bullet] \\
&= (\text{id} - P_\bullet) d \tilde{s}_\bullet + \tilde{s}_\bullet d (\text{id} - P_\bullet) \\
&= d (\text{id} - P_\bullet) \tilde{s}_\bullet + \tilde{s}_\bullet (\text{id} - P_\bullet) d \\
&= [d, \tilde{s}_\bullet] = \text{id} - P_\bullet.
\end{align*}
\]

Since \(d(\tilde{s}_\bullet)^2 d = (\tilde{s}_\bullet)^2 d^2 = 0\), \(\tilde{s}_\bullet\) is a gauge:

\[
(\tilde{s}_\bullet)^2 = (\tilde{s}_\bullet d \tilde{s}_\bullet) (\tilde{s}_\bullet d \tilde{s}_\bullet) = \tilde{s}_\bullet d (\tilde{s}_\bullet)^2 d \tilde{s}_\bullet = 0.
\]

If \(s_\bullet\) happens to be a gauge, then by Lemma 3.4, \(s_\bullet P_\bullet = 0\). It follows that

\[
\begin{align*}
\tilde{s}_\bullet - s_\bullet &= s_\bullet (d s_\bullet (\text{id} - P_\bullet) - \text{id}) \\
&= s_\bullet (d s_\bullet - \text{id}) \\
&= -s_\bullet (s_\bullet d + P_\bullet) = -(s_\bullet)^2 d + s_\bullet P_\bullet = 0,
\end{align*}
\]

showing that \(\tilde{s}_\bullet = s_\bullet\). \(\square\)

We now turn to the proof that Dupont’s operator \(s_\bullet\) is a gauge. Denote by \(\varepsilon(\alpha)\) the operation of multiplication by a differential form \(\alpha\) on \(\Omega_n\).

**Lemma 3.10.** If \(i \notin \{i_0, \ldots, i_k\}\), then

\[
\varepsilon(\omega_{i_0 \ldots i_k}) h^i = (-1)^k h^i \left( \varepsilon(\omega_{i_0 \ldots i_k}) + \varepsilon(\omega_{i_0 \ldots i_k}) h^i \right).
\]

**Proof.** We have

\[
(-1)^k h^i \varepsilon(\omega_{i_0 \ldots i_k}) = (-1)^k \int_0^1 w^{-1} \varphi_i(w) t_i \varepsilon(\omega_{i_0 \ldots i_k}) dw
\]

\[
= \varepsilon(\omega_{i_0 \ldots i_k}) \int_0^1 w^k \varphi_i(w) t_i dw.
\]

On the other hand, by (3.11),

\[
(-1)^k h^i \varepsilon(\omega_{i_0 \ldots i_k}) h^i = (-1)^k \int_0^1 \int_0^1 (uv)^{-1} \varphi_i(u) t_i \varepsilon(\omega_{i_0 \ldots i_k}) \varphi_i(v) t_i dv du
\]

\[
= (k + 1) \int_0^1 \int_0^1 (uv)^{-1} \varphi_i(u) \varepsilon(\omega_{i_0 \ldots i_k}) \varphi_i(v) t_i dv du
\]

\[
= (k + 1) \varepsilon(\omega_{i_0 \ldots i_k}) \int_0^1 \int_0^1 u^k v^{-1} \varphi_i(uv) t_i dv du.
\]
Changing variables from \( u \) to \( w = uv \), we see that
\[
\int_0^1 \int_0^1 u^k v^{-1} \varphi_i(uv) \, dv \, du = \int_0^1 \left( \int_1^w v^{-k-2} \, dv \right) w^k \varphi_i(w) \, dw = (k + 1)^{-1} \int_0^1 (w^{-1} - w^k) \varphi_i(w) \, dw,
\]
establishing the lemma. \(\square\)

**Theorem 3.11.** The operator \( s_\bullet \) is a gauge.

**Proof.** By induction on \( k \), the above lemma shows that
\[
h^{i_k} \cdots h^{i_0} s = \sum_{\ell=0}^{n-1} (-1)^{k\ell+\ell} \sum_{\{j_0, \ldots, j_\ell\} = \emptyset, i_0 < \cdots < i_\ell, j_0 < \cdots < j_\ell} \omega_{j_0 \cdots j_\ell} h^{i_k} \cdots h^{i_0} h^{j_\ell} \cdots h^{j_0}.
\]
It follows that \( s^2 \) is given by the formula
\[
(3.14) \quad s^2 = \sum_{k, \ell=0}^{\infty} (-1)^{k\ell+\ell} \sum_{\{i_0, \ldots, i_k, j_0, \ldots, j_\ell\} = \emptyset, i_0 < \cdots < i_k, j_0 < \cdots < j_\ell} \omega_{i_0 \cdots i_k} \omega_{j_0 \cdots j_\ell} h^{i_k} \cdots h^{i_0} h^{j_\ell} \cdots h^{j_0}.
\]
We have
\[
\omega_{i_0 \cdots i_k} \omega_{j_0 \cdots j_\ell} h^{i_k} \cdots h^{i_0} h^{j_\ell} \cdots h^{j_0} = (-1)^{k\ell+(k+1)(\ell+1)} \omega_{j_0 \cdots j_\ell} \omega_{i_0 \cdots i_k} h^{j_\ell} \cdots h^{j_0} h^{i_k} \cdots h^{i_0}.
\]
The expression (3.14) changes sign on exchange of \((i_0, \ldots, i_k)\) and \((j_0, \ldots, j_\ell)\), and thus vanishes. \(\square\)

### 4. The Maurer-Cartan Set of an \( L_\infty \)-algebra

\( L_\infty \)-algebras are a generalization of dg Lie algebras in which the Jacobi rule is only satisfied up to a hierarchy of higher homotopies. In this section, we start by recalling the definition of \( L_\infty \)-algebras. Following Sullivan [23] and Hinich [16], we represent the homotopy type of an \( L_\infty \)-algebra \( g \) by the simplicial set \( \text{MC}_\bullet(g) = MC(g \otimes \Omega_\bullet) \). We prove that this is a Kan complex, and that under certain additional hypotheses, it is a homotopy invariant of the \( L_\infty \)-algebra \( g \).

An operation \([x_1, \ldots, x_k]\) on a graded vector space \( g \) is called graded antisymmetric if
\[
[x_1, \ldots, x_i, x_{i+1}, \ldots, x_k] + (-1)^{|x_i||x_{i+1}|} [x_1, \ldots, x_{i+1}, x_i, \ldots, x_k] = 0
\]
for all \( 1 \leq i \leq k - 1 \). Equivalently, \([x_1, \ldots, x_k]\) is a linear map from \( \Lambda^k g \) to \( g \), where \( \Lambda^k g \) is the \( k \)th exterior power of the graded vector space \( g \), that is, the \( k \)th symmetric power of \( s^{-1}g \).

**Definition 4.1.** An \( L_\infty \)-algebra is a graded vector space \( g \) with a sequence \([x_1, \ldots, x_k]\), \( k > 0 \) of graded antisymmetric operations of degree \( 2 - k \), or equivalently, homogeneous linear maps \( \Lambda^k g \to g \) of degree 2, such that for each \( n > 0 \), the \( n \)-Jacobi rule holds:
\[
\sum_{k=1}^{n} (-1)^k \sum_{\{i_1 < \cdots < i_k, j_1 < \cdots < j_{n-k}\}, \{i_1, \ldots, i_k\} \cup \{j_1, \ldots, j_{n-k}\} = \{1, \ldots, n\}} (-1)^\varepsilon \left[[x_{i_1}, \ldots, x_{i_k}], x_{j_1}, \ldots, x_{j_{n-k}}\right] = 0.
\]
Here, the sign \( (-1)^\varepsilon \) equals the product of the sign \( (-1)^\pi \) associated to the permutation
\[
\pi = \left(1 \atop i_1 \ldots i_k \atop j_1 \ldots j_{n-k}\right) k + 1 \ldots n
\]
with the sign associated by the Koszul sign convention to the action of \( \pi \) on the elements \([x_1, \ldots, x_n]\) of \( g \).
In terms of the graded symmetric operations
\[ \ell_k(y_1, \ldots, y_k) = (-1)^{\sum_{i=1}^k (k-i+1) |y_i|} s^{-1}[sy_1, \ldots, sy_k] \]
of degree 1 on the graded vector space \( s^{-1}g \), the Jacobi rule simplifies to become
\[ \sum_{k=1}^n \sum_{i_1 < \cdots < i_k < j_1 < \cdots < j_{n-k}} (-1)^{\xi} \{ \{ y_{i_1}, \ldots, y_{i_k}, y_{j_1}, \ldots, y_{j_{n-k}} \} \} = 0, \]
where \((-1)^{\xi}\) is the sign associated by the Koszul sign convention to the action of \( \pi \) on the elements \( (y_1, \ldots, y_n) \) of \( s^{-1}g \). This is a small modification of the conventions of Lada and Markl [18]; their operations \( l_k \) are related to ours by a sign
\[ l_k(x_1, \ldots, x_k) = (-1)^{(k+1)/2} [x_1, \ldots, x_k]. \]

The operation \( x \mapsto [x] \) makes the graded vector space \( g \) into a cochain complex, by the 1-Jacobi rule \( [[x]] = 0 \). Because of the special role played by the operation \([x]\), we denote it by \( \delta \). An \( L_\infty \)-algebra with \([x_1, \ldots, x_k] = 0\) for \( k > 2 \) is the same thing as a dg Lie algebra. A quasi-isomorphism of \( L_\infty \)-algebras is a quasi-isomorphism of the underlying cochain complexes.

The lower central filtration on an \( L_\infty \)-algebra \( g \) is the canonical decreasing filtration defined inductively by \( F_1^1g = g \) and, for \( i > 1 \),
\[ F^i g = \sum_{i_1 + \cdots + i_k = i} [F^{i_1} g, \ldots, F^{i_k} g]. \]

**Definition 4.2.** An \( L_\infty \)-algebra \( g \) is nilpotent if the lower central series terminates, that is, if \( F^i g = 0 \) for \( i \gg 0 \).

If \( g \) is a nilpotent \( L_\infty \)-algebra, the curvature
\[ \mathcal{F}(\alpha) = \delta \alpha + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} [\alpha^{\wedge \ell}] \in g^2 \]
is defined, and polynomial in \( \alpha \). If \( g \) is a dg Lie algebra, the curvature equals
\[ \mathcal{F}(\alpha) = \delta \alpha + \frac{1}{2} [\alpha, \alpha]; \]
this expression is familiar from the theory of connections on principal bundles.

**Definition 4.3.** The Maurer-Cartan set \( MC(g) \) of a nilpotent \( L_\infty \)-algebra \( g \) is the set of those \( \alpha \in g^1 \) satisfying the Maurer-Cartan equation
\[ \mathcal{F}(\alpha) = 0. \]

An \( L_\infty \)-algebra is abelian if the bracket \([x_1, \ldots, x_k]\) vanishes for \( k > 1 \). In this case, the Maurer-Cartan set is the set of 1-cocycles \( Z^1(g) \) of \( g \).

Let \( g \) be a nilpotent \( L_\infty \)-algebra. For any element \( \alpha \in g^1 \), the formula
\[ [x_1, \ldots, x_k]_\alpha = \sum_{\ell=0}^{\infty} \frac{1}{\ell!} [\alpha^{\wedge \ell}, x_1, \ldots, x_k] \]
defines a new sequence of brackets on \( g \), where \([\alpha^{\wedge \ell}, x_1, \ldots, x_k]\) is an abbreviation for
\[ \underbrace{[\alpha, \ldots, \alpha, x_1, \ldots, x_k]}_{\ell \text{ times}}. \]

**Proposition 4.4.** If \( \alpha \in MC(g) \), then the brackets \([x_1, \ldots, x_k]_\alpha\) make \( g \) into an \( L_\infty \)-algebra.
Proof. Applying the \((m + n)\)-Jacobi relation to the sequence \((\alpha^{\wedge m}, x_1, \ldots, x_n)\) and summing over \(m\), we obtain the \(n\)-Jacobi relation for the brackets \([x_1, \ldots, x_k]_\alpha\). □

Lemma 4.5. The curvature satisfies the Bianchi identity

\[
\delta F(\alpha) + \sum_{\ell=1}^{\infty} \frac{1}{\ell!} [\alpha^{\wedge \ell}, F(\alpha)] = 0.
\]

Proof. The \(n\)-Jacobi relation for \((\alpha^{\wedge n})\) shows that

\[
\sum_{\ell=0}^{n} \frac{1}{\ell!(n-\ell)!} [\alpha^{\wedge \ell}, [\alpha^{\wedge (n-\ell)}]] = 0.
\]

Summing over \(n > 0\), we obtain the lemma. □

If \(g\) is an \(L_\infty\)-algebra and \(\Omega\) is a dg commutative algebra, then the tensor product \(g \otimes \Omega\) is an \(L_\infty\)-algebra, with brackets

\[
\begin{align*}
[x \otimes a] &= [x] \otimes a + (-1)^{|x|} x \otimes da, \\
[x_1 \otimes a_1, \ldots, x_k \otimes a_k] &= (-1)^{\sum_{i<j} |x_i| |a_j|} [x_1, \ldots, x_k] \otimes a_1 \ldots a_k, \quad k \neq 1.
\end{align*}
\]

The functor \(MC(g)\) extends to a covariant functor \(MC(g, \Omega) = MC(g \otimes \Omega)\) from dg commutative algebras to sets, that is, a presheaf on the category of dg affine schemes over \(K\). If \(X_\bullet\) is a simplicial set, we have

\[
MC(g, \Omega(X_\bullet)) \cong sSet(X_\bullet, MC(g)).
\]

If \(g\) is a nilpotent \(L_\infty\)-algebra, let \(MC_\bullet(g)\) be the simplicial set

\[
MC_\bullet(g) = MC(g, \Omega_\bullet).
\]

In other words, the \(n\)-simplices of \(MC_\bullet(g)\) are differential forms \(\alpha\) on the \(n\)-simplex \(\Delta^n\), of the form

\[
\alpha = \sum_{i=0}^{n} \alpha_i
\]

where \(\alpha_i \in g^{1-i} \otimes \Omega^i(\Delta^n)\), such that

\[
(d + \delta)\alpha + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} [\alpha^{\wedge \ell}] = 0.
\]

Before developing the properties of this functor, we recall how it emerges naturally from Sullivan’s approach [23] to rational homotopy theory.

If \(g\) is an \(L_\infty\)-algebra which is finite-dimensional in each degree and bounded below, we may associate to it the dg commutative algebra \(C^*(g)\) of cochains. The underlying graded commutative algebra of \(C^*(g)\) is \(A g^\wedge = S(g[1]^\wedge)\), the free graded commutative algebra on the graded vector space \(g[1]^\wedge\), which equals \((g^{1-i})^\wedge\) in degree \(i\). The differential \(\delta\) of \(C^*(g)\) is determined by its restriction to the space of generators \(g[1]^\wedge \subset C^*(g)\), on which it equals the sum over \(k\) of the adjoints of the operations \(\ell_k\). The resulting graded derivation satisfies the equation \(\delta^2 = 0\) if and only if \(g\) is an \(L_\infty\)-algebra.

As explained in the introduction, the simplicial set

\[
Spec_\bullet(A) = dAlg(A, \Omega_\bullet).
\]

may be viewed as an analogue in homotopical algebra of the spectrum of a commutative algebra. Applied to \(C^*(g)\), we obtain a simplicial set \(Spec_\bullet(C^*(g))\) which has a natural identification with the simplicial set \(MC_\bullet(g)\).
The homotopy groups of a nilpotent $L_\infty$-algebra $\mathfrak{g}$ are defined as

$$\pi_i(\mathfrak{g}) = \pi_i(\text{MC}_* (\mathfrak{g})).$$

In particular, the set of components $\pi_0(\mathfrak{g})$ of $\mathfrak{g}$ is the quotient of $\text{MC}(\mathfrak{g})$ by the nilpotent group associated to the nilpotent Lie algebra $\mathfrak{g}^0$. This plays a prominent role in deformation theory: it is the moduli set of deformations of $\mathfrak{g}$.

In order to establish that $\text{MC}_*(\mathfrak{g})$ is a Kan complex, we use the Poincaré lemma. Let $0 \leq i \leq n$. By (3.9), we see that

$$\text{id}_n = \varepsilon^i_n + (d + \delta) h^i_n + h^i_n (d + \delta).$$

If $\alpha \in \text{MC}_n(\mathfrak{g})$, we see that

$$\alpha = \varepsilon^i_n \alpha + (d + \delta) h^i_n \alpha + h^i_n (d + \delta) \alpha$$

$$= \varepsilon^i_n \alpha + R^i_n \alpha - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} h^i_n [\alpha^\wedge \ell],$$

where $R^i_n = (d + \delta) h^i_n$. Introduce the space

$$\text{mc}_n(\mathfrak{g}) = \{(d + \delta) \alpha \mid \alpha \in (\mathfrak{g} \otimes \Omega)^0\}.$$

**Lemma 4.6.** Let $\mathfrak{g}$ be a nilpotent $L_\infty$-algebra. The map $\alpha \mapsto (\varepsilon^i_n \alpha, R^i_n \alpha)$ induces an isomorphism between $\text{MC}_n(\mathfrak{g})$ and $\text{MC}(\mathfrak{g}) \times \text{mc}_n(\mathfrak{g})$.

**Proof.** Given $\mu \in \text{MC}(\mathfrak{g})$ and $\nu \in \text{mc}_n(\mathfrak{g})$, let $\alpha_0 = \mu + \nu$ and define differential forms $(\alpha_k)_{k>0}$ inductively by the formula

$$(4.18) \quad \alpha_{k+1} = \alpha_0 - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} h^i_n [\alpha^\wedge \ell].$$

Then for all $k$, we have $\varepsilon^i_n \alpha_k = \mu$ and $R^i_n \alpha_k = \nu$. The sequence is eventually constant, since by induction, we see that

$$\alpha_{k+1} - \alpha_k = \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \sum_{j=1}^{\ell} h^i_n [\alpha^\wedge j-1, \alpha_{k-1} - \alpha_k, \alpha^\wedge \ell-j]$$

$$\in F^{k+1} \mathfrak{g} \otimes \Omega_n.$$
and hence that

\[ F(\alpha) = \delta \mu + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} [\alpha \wedge \ell] - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} (d + \delta) h^i_n[\alpha \wedge \ell] \]

\[ = F(\mu) + \sum_{\ell=2}^{\infty} \frac{1}{\ell!} h^i_n(d + \delta)[\alpha \wedge \ell] \]

\[ = F(\mu) + h^i_n(d + \delta)F(\alpha). \]

The Bianchi identity (4.16) implies that

\[ F(\alpha) = F(\mu) - \sum_{\ell=1}^{\infty} \frac{1}{\ell!} h^i_n[\alpha \wedge \ell, F(\alpha)] \]

\[ = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} h^i_n[\alpha \wedge \ell, F(\alpha)]. \]

The nilpotence of \( g \) implies that \( F(\alpha) = 0; \) it follows that \( \alpha \) is an element of \( \text{MC}_n(g) \) with \( \varepsilon^i_n\alpha = \mu \) and \( R^i_n\alpha = \nu. \)

If \( \alpha \) and \( \beta \) are a pair of elements of \( \text{MC}_n(g) \) such that \( \varepsilon^i_n\alpha = \varepsilon^i_n\beta \) and \( R^i_n\alpha = R^i_n\beta, \) then

\[ \alpha - \beta = -\sum_{\ell=2}^{\infty} \frac{1}{\ell!} \sum_{j=1}^{\ell} h^i_n[\alpha \wedge j-1, \alpha - \beta, \beta \wedge \ell-j]. \]

This shows, by induction, that \( \alpha - \beta \in F^i g \) for all \( i > 0, \) and hence, by the nilpotence of \( g, \) that \( \alpha = \beta. \)

The following result is due to Hinich [16] when \( g \) is a dg Lie algebra.

**Proposition 4.7.** If \( f: g \to h \) is a surjective morphism of nilpotent \( L_\infty \)-algebras, the induced morphism

\[ \text{MC}_*(f): \text{MC}_*(g) \to \text{MC}_*(h) \]

is a fibration of simplicial sets.

**Proof.** Let \( 0 \leq i \leq n. \) Given a horn

\[ \beta \in \text{sSet}(\Lambda^i_n, \text{MC}_n(g)) \]

and an \( n \)-simplex \( \gamma \in \text{MC}_n(h) \) such that \( \partial_j\gamma = f(\partial_j\beta) \) for \( j \neq i, \) we wish to construct an element \( \alpha \in f^{-1}(\gamma) \subset \text{MC}_n(g) \) such that

\[ \partial_j\alpha = \partial_j\beta \]

for \( j \neq i. \)

Since \( f: g \otimes \Omega \to h \otimes \Omega \) is a Kan fibration, there exists an extension \( \rho \in g \otimes \Omega_n \) of \( \beta \) of total degree \( 1 \) such that \( f(\rho) = \alpha. \) Let \( \alpha \) be the unique element of \( \text{MC}_n(g) \) such that \( \varepsilon^i_n\alpha = \varepsilon^i_n\rho \) and \( R^i_n\alpha = R^i_n\rho. \)

If \( j \neq i, \) we have \( \varepsilon^i_n\partial_j\alpha = \varepsilon^i_n\partial_j\beta \) and \( R^i_n\partial_j\alpha = R^i_n\partial_j\beta \) and hence, by Lemma 4.6, \( \partial_j\alpha = \partial_j\beta; \) thus, \( \alpha \) fills the horn \( \beta. \) We also have \( f(\varepsilon^i_n\alpha) = f(\varepsilon^i_n\rho) = \varepsilon^i_n\gamma \) and \( f(R^i_n\alpha) = f(R^i_n\rho) = R^i_n\gamma, \) hence \( f(\alpha) = \gamma. \)

The category of nilpotent \( L_\infty \)-algebras concentrated in degrees \((-\infty, 0]\) is a variant of Quillen’s model for rational homotopy of nilpotent spaces [22]. By the following theorem, the functor \( \text{MC}_*(g) \) carries quasi-isomorphisms of such \( L_\infty \)-algebras to homotopy equivalences of simplicial sets.
Theorem 4.8. If \( g \) and \( h \) are \( L_\infty \)-algebras concentrated in degrees \((-\infty, 0]\), and \( f : g \to h \) is a quasi-isomorphism, then

\[
MC_* (f) : MC_* (g) \to MC_* (h)
\]
is a homotopy equivalence.

Proof. Filter \( g \) by \( L_\infty \)-algebras \( F^j g \), where

\[
(F^{2j} g)^i = \begin{cases} 0, & \text{if } i + j > 0, \\ Z^{-j} (g), & \text{if } i + j = 0, \\ g^i, & \text{if } i + j < 0,
\end{cases}
\]
and similarly for \( h \). If \( j > k \), there is a morphism of fibrations of simplicial sets

\[
MC_*(F^j g) \to MC_*(F^k g) \to MC_*(F^k g/F^j g)
\]
\[
\downarrow \quad \downarrow \quad \downarrow
\]
\[
MC_*(F^j h) \to MC_*(F^k h) \to MC_*(F^k h/F^j h)
\]
We have

\[
MC_*(F^{2j} g/F^{2j+1} g) \cong MC_*(H^{-j} (g)) \cong MC_*(H^{-j} (h)) \cong MC_*(F^{2j} h/F^{2j+1} h).
\]
The simplicial sets

\[
MC_*(F^{2j+1} g/F^{2j+2} g) \cong B^{-j} (g) \otimes \Omega^{j+1}_*,
\]
and

\[
MC_*(F^{2j+1} h/F^{2j+2} h) \cong B^{-j} (h) \otimes \Omega^{j+1}_*
\]
are contractible by Lemma 3.2. The proposition follows. \qed

Let \( m \) be a nilpotent commutative ring; that is, \( m^{\ell+1} = 0 \) for some \( \ell \). If \( g \) is an \( L_\infty \)-algebra, then \( g \otimes m \) is nilpotent; this is the setting of formal deformation theory. In this context too, the functor \( MC_* (g, m) = MC_* (g \otimes m) \) takes quasi-isomorphisms of \( L_\infty \)-algebras to homotopy equivalences of simplicial sets.

Proposition 4.9. If \( f : g \to h \) is a quasi-isomorphism of \( L_\infty \)-algebras and \( m \) is a nilpotent commutative ring, then

\[
MC_* (f, m) : MC_* (g, m) \to MC_* (h, m)
\]
is a homotopy equivalence.

Proof. We argue by induction on the nilpotence length \( \ell \) of \( m \). There is a morphism of fibrations of simplicial sets

\[
MC_* (g, m^2) \to MC_* (g, m) \to MC_* (g \otimes m/m^2)
\]
\[
\downarrow \quad \downarrow \quad \downarrow
\]
\[
MC_* (h, m^2) \to MC_* (h, m) \to MC_* (h \otimes m/m^2)
\]
The abelian \( L_\infty \)-algebras \( g \otimes m/m^2 \) and \( h \otimes m/m^2 \) are quasi-isomorphic, hence the morphism

\[
MC_* (g \otimes m/m^2) \to MC_* (h \otimes m/m^2)
\]
is a homotopy equivalence. The result follows by induction on \( \ell \). \qed
5. The functor $\gamma_\bullet(g)$

In this section, we study the functor $\gamma_\bullet(g)$; we prove that it is homotopy equivalent to $\text{MC}_\bullet(g)$, and show that it specializes to the Deligne groupoid when $g$ is concentrated in degrees $[0, \infty]$. Fix a gauge $s_\bullet$, for example Dupont’s operator \( (3.12) \).

The simplicial set $\gamma_\bullet(g)$ associated to a nilpotent $L_\infty$-algebra is the simplicial subset of $\text{MC}_\bullet(g)$ consisting of those Maurer-Cartan forms annihilated by $s_\bullet$:

\[
\gamma_\bullet(g) = \{ \alpha \in \text{MC}_\bullet(g) \mid s_\bullet \alpha = 0 \}.
\]  

(5.19)

For any simplicial set $X_\bullet$, the set of simplicial maps $s\text{Set}(X_\bullet, \gamma_\bullet(g))$ equals the set of Maurer-Cartan elements $\alpha \in \text{MC}(g, X_\bullet)$ such that $s_\bullet \alpha = 0$. This is reminiscent of gauge conditions, such as the Coulomb gauge, in gauge theory.

**Proposition 5.1.** For abelian $g$, there is a natural isomorphism $\gamma_\bullet(g) \cong K_\bullet(g[1])$.

*Proof.* If $\alpha \in \gamma_n(g)$, then $(d + \delta)\alpha = s_n \alpha = 0$, hence by \( (3.7) \),

\[
\alpha = P_n \alpha + s_n (d + \delta) \alpha + (d + \delta) s_n \alpha = P_n \alpha.
\]

Thus $\gamma_n(g) \subset K_n(g[1])$. Conversely, if $\alpha \in K_n(g[1])$, then $P_n \alpha = \alpha$, hence $s_n \alpha = 0$. Thus $K_n(g[1]) \subset \gamma_n(g)$.

We show that $\gamma_\bullet(g)$ is an $\infty$-groupoid, and in particular, a Kan complex: the heart of the proof is an iteration, similar to the iteration \( (4.13) \), which solves the Maurer-Cartan equation on the $n$-simplex $\Delta^n$ in the gauge $s_n \alpha = 0$.

**Definition 5.2.** An $n$-simplex $\alpha \in \gamma_n(g)$ is thin if $I_{0 \ldots n}(\alpha) = 0$.

**Lemma 5.3.** Let $g$ be a nilpotent $L_\infty$-algebra. The map $\alpha \mapsto (\varepsilon^i_n \alpha, P_n R_n^i \alpha)$ induces an isomorphism between $\gamma_n(g)$ and $\text{MC}(g) \times P_n[\text{mc}_n(g)]$.

*Proof.* Let $0 \leq i \leq n$. By \( (3.7) \), we see that

\[
id_n = P_n + (d + \delta) s_n + s_n (d + \delta)
\]

\[= \varepsilon^i_n + (d + \delta) (P_n h^i_n + s_n) + (P_n h^i_n + s_n) (d + \delta).
\]

It follows that if $\alpha \in \gamma_n(g)$,

\[
\alpha = \varepsilon^i_n \alpha + P_n R_n^i \alpha - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} (P_n h_n^i + s_n) [\alpha^{\wedge \ell}].
\]  

(5.20)

Given $\mu \in \text{MC}(g)$ and $\nu \in P_n[\text{mc}_n(g)]$, let $\alpha_0 = \mu + \nu$ and define differential forms $(\alpha_k)_{k>0}$ inductively by the formula

\[
\alpha_k = \alpha_0 - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} (P_n h_n^i + s_n) [\alpha_k^{\wedge \ell}].
\]

Then for all $k$, we have $s_n \alpha_k = 0$, $\varepsilon^i_n \alpha_k = \mu$ and $P_n R_n^i \alpha_k = \nu$. The sequence $(\alpha_k)$ is eventually constant, since by induction, we see that

\[
\alpha_k - \alpha_{k-1} = \sum_{\ell=2}^{\infty} \frac{1}{\ell!} \sum_{j=1}^{\ell} (P_n h_n^i + s_n) [\alpha_{k-2}^{\wedge j-1}, \alpha_{k-2} - \alpha_{k-1}, \alpha_{k-1}^{\wedge j}] 
\]

\[\in F^k g \otimes \Omega_n.
\]

The limit

\[
\alpha = \lim_{k \to \infty} \alpha_k
\]
satisfies
\[ \alpha = \alpha_0 - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} (P h_n^\ell + s_n) [\alpha^{\wedge \ell}] . \]

By the same argument as in the proof of Lemma 4.6 it follows that
\[ \mathcal{F}(\alpha) = \mathcal{F}(\mu) - \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (P h_n^\ell + s_n) [\alpha^{\wedge \ell}, \mathcal{F}(\alpha)] \]
\[ = \sum_{\ell=1}^{\infty} \frac{1}{\ell!} (P h_n^\ell + s_n) [\alpha^{\wedge \ell}, \mathcal{F}(\alpha)] . \]

The nilpotence of \( g \) implies that \( \mathcal{F}(\alpha) = 0 \); it follows that \( \alpha \) is an element of \( \gamma_n(g) \) with \( \varepsilon_n^i \alpha = \mu \) and \( PR_n^i \alpha = \nu \).

If \( \alpha \) and \( \beta \) are a pair of elements of \( \gamma_n(g) \) such that \( \varepsilon_n^i \alpha = \varepsilon_n^i \beta \) and \( P_n R_n^i \alpha = P_n R_n^i \beta \), then
\[ \alpha - \beta = -\sum_{\ell=2}^{\infty} \frac{1}{\ell!} \sum_{j=1}^{\ell} (P_n h_n^i + s_n) [\alpha^{\wedge \ell-1}, \alpha - \beta, \beta^{\wedge \ell-j}] . \]

This shows, by induction, that \( \alpha - \beta \in F^i g \) for all \( i > 0 \), and hence, by the nilpotence of \( g \), that \( \alpha = \beta \).

**Theorem 5.4.** If \( g \) is a nilpotent \( L_\infty \)-algebra, \( \gamma_\bullet(g) \) is an \( \infty \)-groupoid. If \( g \) is concentrated in degrees \( (-\ell, \infty) \), respectively \( (-\ell, 0) \), then \( \gamma_\bullet(g) \) is an \( \ell \)-groupoid, resp. an \( \ell \)-group.

**Proof.** Let \( \beta \in sSet(\Lambda^n_\bullet, \gamma_\bullet(g)) \)

be a horn in \( \gamma_\bullet(g) \). The differential form
\[ \alpha_0 = \varepsilon_n^i \beta + (d + \delta) \sum_{k=1}^{n-1} \sum_{i_1 < \cdots < i_k \in i \not\in \{i_1, \ldots, i_k\}} \omega_{i_1 \cdots i_k} \otimes I_{i_1 \cdots i_k} (\beta) \in MC(g) \times P_n[\mathcal{MC}_n(g)] \]

satisfies \( I_{0 \ldots n}(\alpha_0) = 0 \). The solution \( \alpha \) of \( \gamma_n(g) \) of the equation
\[ \alpha = \alpha_0 - \sum_{\ell=2}^{\infty} \frac{1}{\ell!} (P_n h_n^\ell + s_n) [\alpha^{\wedge \ell}] \]

constructed in Lemma 5.3 is thin and \( \xi_n^i(\alpha) = \beta \). Thus \( \gamma_\bullet(g) \) is an \( \infty \)-groupoid.

If \( g^{1-n} = 0 \), it is clear that every \( n \)-simplex \( \alpha \in \gamma_n(g) \) is thin, while if \( g^1 = 0 \), then \( \gamma_\bullet(g) \) is reduced.

**Definition 5.5.** The \( n \)th generalized Campbell-Hausdorff series associated to the gauge \( s_\bullet \) is the function of \( \mu \in MC(g) \) and \( x_{i_1 \cdots i_k} \in g^{1-k} \), \( 1 \leq i_1 < \cdots < i_k \leq n \), let \( \alpha_n^\mu(x_{i_1 \cdots i_k}) \) be the solution of (5.20) with \( \varepsilon_n^i \alpha_n^\mu(x_{i_1 \cdots i_k}) = \mu \) and
\[ R_n^0 \alpha_n^\mu(x_{i_1 \cdots i_k}) = \sum_{k=1}^{n} \sum_{1 \leq i_1 < \cdots < i_k \leq n} \omega_{i_1 \cdots i_k} \otimes x_{i_1 \cdots i_k} . \]

**Definition 5.5.** The \( n \)th generalized Campbell-Hausdorff series associated to the gauge \( s_\bullet \) is the function of \( \mu \in MC(g) \) and \( x_{i_1 \cdots i_k} \in g^{1-k} \), \( 1 \leq i_1 < \cdots < i_k \leq n \), given by the formula
\[ \rho_n^\mu(x_{i_1 \cdots i_k}) = I_{1 \ldots n}(\alpha_n^\mu(x_{i_1 \cdots i_k})) \in g^{2-n} . \]
If \( \mathfrak{g} \) is concentrated in degrees \((-\infty, 0]\), then the Maurer-Cartan element \( \mu \) equals 0, and is omitted from the notation for \( \alpha_n(x_{i_1 \ldots i_k}) \) and \( \rho_n(x_{i_1 \ldots i_k}) \).

Since \( \alpha_2^\mu(x_1, x_2, x_{12}) \) is a flat connection 1-form on the 2-simplex, its monodromy around the boundary must be trivial. (The 2-simplex is simply connected.) In terms of the generalized Campbell-Hausdorff series \( \rho_2^\mu(x_1, x_2, x_{12}) \), this gives the equation

\[
e^{x_1} = e^{\rho_2^\mu(x_1, x_2, x_{12})} e^{x_2}
\]

in the Lie group associated to the nilpotent Lie algebra \( \mathfrak{g}^0 \). Thus, the simplicial set \( \gamma_\bullet(\mathfrak{g}) \) (indeed, its 2-skeleton) determines \( \rho_2^\mu(x_1, x_2, x_{12}) \) as a function of \( x_1, x_2 \) and \( x_{12} \). In the Dupont gauge, modulo terms involving more than two brackets, it equals

\[
\rho_2^\mu(x_1, x_2, x_{12}) = x_1 - x_2 + \frac{1}{2} [x_1, x_2]_\mu + \frac{1}{2} [x_1 + x_2, [x_1 + x_2]_\mu] + \frac{1}{6} ([x_1 + x_2, [x_1 + x_2]_\mu], x_1)_{\mu} + \frac{1}{12} ([x_1 + x_2, [x_1 + x_2]_\mu], x_2)_{\mu} + \cdots.
\]

If \( \mathfrak{g} \) is a dg Lie algebra, the thin 2-simplices define a composition on the 1-simplices of \( \gamma_\bullet(\mathfrak{g}) \) which is strictly associative.

**Proposition 5.6.** If \( \mathfrak{g} \) is a dg Lie algebra, the composition

\[
\rho_2^\mu(x_1, x_2) : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}
\]

is associative.

**Proof.** It suffices to show that \( \rho_2^\mu(x_1, x_2, x_3, x_i = 0) = 0 \), in other words, if three faces of a thin 3-simplex are thin, then the fourth is. The iteration leading to the solution \( \alpha \) of (5.20) with initial conditions

\[
\alpha_0 = \mu + (d + \delta)(t_1 x_1 + t_2 x_2 + t_3 x_3)
\]

lies in the space \( \mathfrak{g}^0 \otimes \Omega_3^1 \oplus \Omega_1 \otimes \Omega_3^0 \), hence \( I_{123}(\alpha) = 0 \). \( \square \)

In particular, if \( \mathfrak{g} \) is a dg Lie algebra concentrated in degrees \((-2, \infty) \), \( \gamma_\bullet(\mathfrak{g}) \) is the nerve of a strict 2-groupoid, that is, a groupoid enriched in groupoids; in this way, we see that \( \gamma_\bullet(\mathfrak{g}) \) generalizes the Deligne 2-groupoid (Deligne [5], Getzler [13]).

Although it is not hard to derive explicit formulas for the generalized Campbell-Hausdorff series up to any order, we do not know any closed formulas for them except when \( n = 1 \), in which case it is independent of the gauge. We now derive a closed formula for \( \rho_1^\mu(x) \), which resembles Cayley’s famous formula for the series solution of the ordinary differential equation \( x'(t) = f(x(t)) \). To each rooted tree, associate the word obtained by associating to a vertex with \( i \) branches the operation \( [x, a_1, \ldots, a_i]_\mu \). Multiply the resulting word by the number of total orders on the vertices of the tree such that each vertex precedes its parent. Let \( e^k_\mu(x) \) be the sum of these terms over all rooted trees with \( k \) vertices. For example, \( e^1_\mu(x) = [x]_\mu \), \( e^3_\mu(x) = [x, [x]_\mu]_\mu + [x, [x]_\mu, [x]_\mu]_\mu \). The coefficient of a tree \( T \) in \( e^k_\mu(x) \) equals the number of monotone orderings of its vertices, that is, total orderings such that each vertex is greater than its parent. See Figure 2 for the trees contributing to \( e^k_\mu(X) \) for small values of \( k \).

**Proposition 5.7.** The 1-simplex \( \alpha_1^\mu(x) \in \gamma_1(\mathfrak{g}) \) determined by \( \alpha \in MC(\mathfrak{g}) \) and \( x \in \mathfrak{g}^0 \) is given by the formula

\[
\alpha_1^\mu(x) = \alpha - \sum_{k=1}^{\infty} \frac{t_k}{k!} e^k_\mu(x) + x dt_0.
\]
**Proof.** To show that $\alpha_1^\mu(x) \in \gamma_1(\mathfrak{g})$, we must show that it satisfies the Maurer-Cartan equation. Let

$$\alpha(t) = \alpha - \sum_{k=1}^{\infty} \frac{t^k}{k!} \mathbf{e}^k_\mu(x).$$

It must be shown that

$$\alpha'(t) + \sum_{n=0}^{\infty} \frac{1}{n!} [\alpha(t)^n, x] = 0,$$

in other words, that

$$\mathbf{e}^{k+1}_\mu(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{n!} \sum_{k_1 + \ldots + k_n = k} \frac{k!}{k_1! \ldots k_n!} [\mathbf{e}^{k_1}_\mu(x), \ldots, \mathbf{e}^{k_n}_\mu(x), x]_\alpha$$

$$= \sum_{n=0}^{\infty} \sum_{k_1 + \ldots + k_n = k} \frac{1}{n!} \frac{k!}{k_1! \ldots k_n!} [x, \mathbf{e}^{k_1}_\mu(x), \ldots, \mathbf{e}^{k_n}_\mu(x)]_\alpha.$$

This is easily proved by induction on $k$. □

Proposition 5.7 implies the following formula for the generalized Campbell-Hausdorff series $\rho_1^\mu(x)$:

$$\rho_1^\mu(x) = \mu - \sum_{k=1}^{\infty} \frac{1}{k!} \mathbf{e}^k_\mu(x).$$
If \( g \) is a dg Lie algebra, only trees with vertices of valence 0 or 1 contribute to \( e^h_n(x) \), and we recover the formula \( \mathcal{L} \) figuring in the definition of the Deligne groupoid for dg Lie algebras.

There is also a relative version of Theorem 5.4, analogous to Theorem 4.7.

**Theorem 5.8.** If \( f : g \to h \) is a surjective morphism of nilpotent \( L_\infty \)-algebras, the induced morphism

\[
\gamma(f) : \gamma(g) \to \gamma(h)
\]

is a fibration of simplicial sets.

**Proof.** Let \( 0 \leq i \leq n \). Given a horn

\[
\beta \in \text{sSet}(\Lambda^i_n, \gamma(g))
\]

and an \( n \)-simplex \( \gamma \in \gamma_n(h) \) such that

\[
f(\partial_j \beta) = \partial_j \gamma
\]

for \( j \neq i \), our task is to construct an element \( \alpha \in f^{-1}(\gamma) \subset \gamma_n(g) \) such that

\[
\partial_j \alpha = \partial_j \beta
\]

if \( j \neq i \).

Choose a solution \( x \in g^{1-n} \) of the equation \( f(x) = I_{0\ldots n}(\gamma) \in h^{1-n} \). Let \( \alpha \) be the unique element of \( \gamma_n(g) \) such that \( \varepsilon^i_n \alpha = \varepsilon^i_n \beta \) and

\[
P_n R_n^i \alpha = (d + \delta) \left( \sum_{k=1}^{n-1} \sum_{i_1 < \ldots < i_k} \omega_{i_1 \ldots i_k} \otimes I_{i_1 \ldots i_k}(\beta) + (-1)^i \omega_{0\ldots \hat{i} \ldots n} \otimes x \right).
\]

If \( j \neq i \), we have \( \varepsilon^i_n \partial_j \alpha = \varepsilon^i_n \partial_j \beta \) and \( P_n R_n^i \partial_j \alpha = P_n R_n^i \partial_j \beta \) and hence, by Lemma 5.3 \( \partial_j \alpha = \partial_j \beta \); thus, \( \alpha \) fills the horn \( \beta \). We also have \( f(\varepsilon^i_n \alpha) = f(\varepsilon^i_n \beta) = \varepsilon^i_n \gamma \) and \( f(P_n R_n^i \alpha) = P_n R_n^i \gamma \), hence \( f(\alpha) = \gamma \). \( \square \)

**Corollary 5.9.** If \( g \) is a nilpotent \( L_\infty \)-algebra, the inclusion of simplicial sets

\[
\gamma_\bullet(g) \hookrightarrow \text{MC}_\bullet(g)
\]

is a homotopy equivalence; in other words, \( \pi_0(\gamma_\bullet(g)) \cong \pi_0(g) \), and for all 0-simplices \( \alpha_0 \in \text{MC}_0(g) = \text{MC}(g) \),

\[
\pi_i(\gamma_\bullet(g), \alpha_0) \cong \pi_i(g, \alpha_0), \quad i > 0.
\]

**Proof.** This is proved by induction on the nilpotence length \( \ell \) of \( g \). When \( g \) is abelian, \( \text{MC}_\bullet(g) \) and \( \gamma_\bullet(g) \) are simplicial abelian groups, and their quotient is the simplicial abelian group

\[
\text{MC}_n(g)/\gamma_n(g) \cong (d + \delta)s_n(g \otimes \Omega_n)^1.
\]

This simplicial abelian group is a retract of the contractible simplicial abelian group \( g \otimes \Omega_\bullet \), hence is itself contractible.

Let \( F^i g \) be the lower central series of \( g \). Given \( i > 0 \), we have a morphism of principal fibrations of simplicial sets

\[
\begin{array}{ccc}
\gamma_\bullet(F^{i+1}g) & \longrightarrow & \gamma_\bullet(F^i g) \\
\downarrow & & \downarrow \\
\text{MC}_\bullet(F^{i+1}g) & \longrightarrow & \text{MC}_\bullet(F^i g)
\end{array}
\]

Since \( F^i g/F^{i+1}g \) is abelian, we see that \( \gamma_\bullet(F^i g/F^{i+1}g) \cong \text{MC}_\bullet(F^i g/F^{i+1}g) \). The result follows by induction on \( \ell \). \( \square \)
When \( g \) is a nilpotent Lie algebra, the isomorphism

\[
\pi_0(\gamma \cdot (g)) \cong \pi_0(MC \cdot (g))
\]

is equivalent to the surjectivity of the exponential map. The above corollary may be viewed as a generalization of this fact.

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