What’s wrong with the Platonic ideal of space and time?

Lorenzo Sadun*

October 17, 2016

To our senses, space is smooth, 3-dimensional, and flat. We move in a continuum where all points are equal (space is “homogeneous”) and all directions are equal (“isotropic”, or “round”). If we head off in any direction, we keep on going, with no curving back on ourselves (“flat”). In short, we seem to live in a universe governed by Euclidean geometry.

Actually, almost everything I wrote in the previous paragraph was a lie. Our senses continually detect the difference between different directions. Things fall down, not up. The sun rises in the East, not in the West. We know that not all points are equal, and that Hawaii is a lot more pleasant than Antarctica. Curvature is all around us, from the hill my house sits on to the fact that we can fly around the world.

All the same, most of us still believe in 3-dimensional space, plus an added dimension (time) that describes how things change. We rely on a mathematical model of reality in which space is a Platonic ideal: smooth, homogeneous, isotropic, and flat. Everything that breaks that underlying symmetry is attributed to objects: a planet whose gravity causes things to fall in a preferred direction, and whose rotation makes another object appear to move through the sky, the hills and valleys of my home town, and lovely tropical islands. We think that space is simple, that objects are complicated,

*Department of Mathematics, University of Texas, Austin, TX 78712; sadun@math.utexas.edu
and that the job of scientists is to understand the messy behavior of objects against the perfect backdrop of space.

This idealization of perfect space and imperfect contents makes for a lovely theory, but is it correct? It was accepted almost without question for over 20 centuries, from the ancient Greeks through the Middle Ages and the Enlightenment. In the late 19th century, however, it started to break down. While the theory works very well to describe physics on many length scales, it gives results that are nonsensical, or at least that contradict experiment, when dealing with very big things, very small things, and very fast things.

In this essay, I'll touch briefly on the new theories that were developed to deal with these discrepancies — special relativity, general relativity, quantum mechanics and string theory. I'll then turn to the suggestion, popularized by Stephen Wolfram [11], that space and time aren’t smooth at all, but come in essentially discrete chunks. Using recent results from the theory of aperiodic tilings, I’ll argue that this last suggestion is not realistic, and will defend the conventional wisdom that Euclidean space and time, as modified slightly by 20th century physics, is still the best way to describe reality.

1 Relativity and the fall of Euclidean space

If space and time are absolutes, then how fast are we moving? After all, the earth rotates on its axis and revolves around the sun, the sun revolves around the center of the galaxy, and the galaxy tumbles through the universe. We must be moving, but in what direction, and how fast? In 1887, Michelson and Morley [7] tried to find out. They figured that light moving in the same direction as us would appear to be moving slower (since it has to catch up with us), that light moving in the opposite direction would appear to be moving faster, and that light moving perpendicular to our motion would have an intermediate speed. With a clever interferometry experiment, they measured these differences and got exactly zero, suggesting that we were not moving at all! How could that be?

Several complicated mechanisms were proposed for why the speed of light appeared to be the same in all directions. It took almost 20 years, until Einstein’s 1905 Special Theory of Relativity[4], for mankind to realize that
the laws of mechanics and electromagnetism, and hence the speed of light, really were the same relative to the earth, to the sun, and to the distant galaxies. Different observers moving relative to one another have different notions of space and different notions of time, but the same laws of physics. Space and time are not absolutes, but are only defined relative to an observer. Moreover, they are closely linked, and switching from one reference frame to another is like a rotation in a 4-dimensional space-time. As such, it is impossible to speak of the nature of space without also considering the nature of time, and vice-versa. To know one is to know both.

Einstein took things a step further in his 1915 General Theory of Relativity (GR) [5]. He proposed that space-time is not flat. Rather, the presence of mass, momentum and energy causes space to bend, and we perceive this bending as gravity. We can no longer place perfect space and imperfect matter in separate categories. Matter bends space and the geometry of space affects matter. If the distribution of matter isn’t uniform and isotropic, then neither is the geometry of space-time. Matter is lumpy, so space-time is bumpy.

Since that time, GR has been tested in numerous experiments, and has performed extremely well, most recently in the 2016 observation of gravitational radiation. GR may not be the ultimate theory and may require tweaking in the future (In particular, Andrei Sakharov [10] has argued that it is just the first term in an infinite series of corrections to Newton’s Laws), but it is hard to avoid the conclusion that, on extremely large length scales, the Platonic ideal of space-time just doesn’t work.

2 Quantum mechanics and the very small

A different challenge to classical physics came when studying very small distances. According to classical physics, a glowing hot object should emit a certain amount of long-wavelength infrared light. It should emit more

\footnote{When you rotate in the x-y plane, the new value of x depends on both the old values of x and the old value of y. Likewise, when you do a “Lorentz transformation” from one reference frame to another, the new position depends on both the old position and the old time, as does the new time.}
shorter-wavelength visible light, still more ultraviolet light, yet more x-rays, and so on. Not only is the bulk of the radiation supposed to be of such high frequency that a coal from your backyard grill would kill you, but the total amount of energy emitted per unit time is supposed to be infinite.

To explain why glowing coals aren’t lethal, Max Planck [8] proposed that light energy can only be emitted or absorbed in discrete chunks, called quanta. This theory of light, called quantum mechanics, was soon extended to all forms of matter and energy and then generalized to fields that describe the creation and annihilation of particles. This body of work took care of the “ultraviolet catastrophe” that puzzled Planck, but created other mysteries.

For one, Werner Heisenberg [6] observed that quantities that were once thought to be precise, like the position and momentum of a particle, are actually a bit fuzzy. There is uncertainty to position, there is uncertainty to momentum, and the product of the two uncertainties is at least Planck’s constant divided by $4\pi$. Likewise, there is uncertainty in energy and uncertainty in the time when things happen. By general relativity, the curvature of space is a function of mass and energy and momentum, but these quantities can’t be nailed down. So not only are particles fuzzy, but space-time itself is fuzzy.

Worse still, the infinities that appeared in the ultraviolet catastrophe aren’t completely tamed. By the uncertainty principle in energy, particles can blink in and out of existence. In many problems in quantum field theory, the effect of all these “virtual particles” could be infinite, which doesn’t make sense. To avoid these infinities, the laws of physics, and of geometry, have to become qualitatively different at the scale of the so-called “Planck length”. This is an incredibly small length of around $10^{-35}$ meters, or about a septillionth the radius of an atomic nucleus. (You could fit more Planck-length sized particles into a single proton than you could fit protons within a million-mile diameter ball.)

According to string theory, space-time isn’t 4 dimensional. It’s actually 10 dimensional (or 11 dimensional in some versions), with all but 4 of the dimensions wrapped up in a higher dimensional analogue of a surface, of size comparable to the Planck length. Just as we can treat a thin 3-dimensional filament, such as a human hair, as being effectively 1-dimensional, our thin 10 or 11 dimensional universe is effectively 4 dimensional.
A very different solution has been advocated by Stephen Wolfram [11]. He suggests that at very small length scales the universe is really 0-dimensional! His theory is that space and time are actually discrete, with the possible points ordered in a neat array. At each new time step, what is happening at each point in space depends only on what was happening at that point, and at all adjacent points, an instant earlier. That is, the universe is like a gigantic array of computers, each one updating based on what its neighbors are doing.

3 Life on the grid?

Such an array is called a “cellular automaton”. The past decades have seen an explosion of work on cellular automatons, including notable advances by Wolfram himself. The most famous example of a cellular automaton is John Conway’s Game of Life [1]. This game operates on a 2-dimensional grid of square “cells”, and time advances in discrete steps called “ticks”. At any given time, each cell is either alive or dead. At each tick of the clock, each live cell either survives or dies, depending on the number of live cells in the 8 positions around it, and each dead cell either stays dead or comes to life (“birth”) by a slightly different rule. This simple game exhibits amazingly complicated behavior, with intricate patterns propagating across the screen.

Could a 3-dimensional version of this sort of game be a model for the complex behavior of the real world? Wolfram says yes, but I say no. In the Game of Life, signals propagate at a maximum speed, just like the speed of light, but this speed depends on direction. Whether a cell at (0,0) is alive or dead at time 0 affects all the neighboring cells at time 1, all the cells around those at time 2, and so on. After $n$ time steps, the cells that are potentially influenced by the initial situation form a square with vertices at $(n,n)$, $(-n,n)$, $(-n,-n)$ and $(n,-n)$. Signals propagate fastest in the diagonal directions and slowest sideways or up-and-down. This contradicts the experimental fact that the speed of light is the same in all directions.

You might argue that this contradiction resulted from the details of the Game of Life, and that different rules might give a different speed of light. It’s true that more complicated rules can make things a bit more isotropic,
but they can’t make things completely round. As long as each cell has a finite number of neighbors, there will always be a finite number of directions in which information runs fastest, and intermediate directions in which information runs slower. Put another way, if the underlying geometry of space-time is a grid, then there will always be physical phenomena that reveal the underlying axes of the grid, in the same way that the facets of a crystal reveal the underlying arrangement of the atoms inside.

(An important caveat: Computers use grids to model continuous and isotropic systems all the time. However, these numerical models only work well when looking at patterns that move much slower than the maximum transmission speed of information, a.k.a. the speed of light. Cellular automata can accurately model a world governed by Newton’s laws, and can be very useful in understanding a cold weather front that is moving at 15 miles per hour, but they can’t handle extreme relativistic motion.)

4 Life in a raindrop?

The universe isn’t a grid, but can it still be discrete? Just because a crystal can’t be round doesn’t mean that we can’t make something round (or at least round to the naked eye) out of atoms. The raindrops falling outside my window say that you can! If you take a bunch of building blocks and assemble them randomly, as with the grains of sand in a sand pile or the water molecules in a raindrop, the resulting structure is unlikely to have any preferred directions.

However, random structures have their own problems. Imagine a small explosion in the middle of a sand pile. The sound from that explosion wouldn’t go straight to our ears, but instead would ricochet off of the various grains of sound in random directions. The sound would go in all directions at essentially the same speed, but different paths would take different amounts of time to reach us. What started out as a sharp BANG! would be heard as a not-so-sharp roar. The wave properties of sound (constructive and destructive interference) could reduce this effect but cannot eliminate it. Waves of different frequencies work their way through the maze at slightly different rates; in raindrops, this distortion causes rainbows. Waves propagating
through random media *always* get distorted and smeared.

However, signals from distant galaxies do *not* get blurred as they travel to us through empty space. The neutrino bursts from a supernova, or the gravitational waves from the merging of two black holes, travel for billions of years across the universe and then hit us in an instant. We don’t see any of the fuzziness that would be expected from random space-time.

In addition to the experimental evidence against random discrete space-time, such a model would raise as many additional metaphysical questions as it would answer. What determines the random structure at each point in space-time? The random arrangements of sand in a sand pile reflect the details of how the grains of sand were dropped and mixed, but there is no *process* by which space and time are created. Space and time just *are*. Einstein famously objected to the role of pure chance in quantum mechanics; this would be far worse.

## 5 Life in an aperiodic tiling?

Finally, we consider a possibility intermediate between random space-time and a regular periodic grid. It is possible to have order without periodicity. For instance, imagine a sequence of + signs and − signs. We start with a + sign and follow it with its opposite to get +−. We then follow this with the opposite of the pair, namely −+, to get +−++. We then follow this with the opposite of +−++, namely −−+−, to get +−−++−−++. Continuing the process forever, we get an infinite sequence, called the Thue-Morse sequence, with the magical property that no pattern within it (e.g., +−−++) ever repeats itself 3 times in a row. The Thue-Morse sequence is an example of *aperiodic order*, in which the arrangements follow precise rules but are not just the same pattern repeated over and over and over.

An interesting 2-dimensional aperiodic tiling is the *pinwheel tiling* invented by John Conway and Charles Radin. The basic tiles are right triangles with sides of length 1 and 2 and hypotenuse of length \( \sqrt{5} \). You can arrange five such tiles to make a bigger triangle of the same shape, which we call a *supertile of level 1*. We can then arrange five supertiles of level 1 to make a supertile of level 2, 5 of those to make a supertile of level 3, and so on.
This design is featured architecturally in Federation Square in Melbourne, Australia, and in the author’s home.

![Image of the author's bathroom floor](image)

Figure 1: The author’s bathroom floor. Note the light level-1 supertile sitting in the center of a mostly dark level-2 supertile, which is itself in the center of a level-3 supertile that extends beyond the frame.

The center tile of a supertile of level 1 is the same shape as the supertile, but is rotated by the angle \( \theta_0 = \tan^{-1}(1/2) \). If the supertile of level 1 is in the center of a supertile of level 2, then the center tile is rotated by \( 2\theta_0 \) relative to the level 2 supertile. Continuing the process, we get rotations by arbitrary multiples of \( \theta_0 \).

However, \( \theta_0 \) is an irrational number of degrees, so no multiple of \( \theta_0 \) will ever take you back exactly to the direction you started in. The pinwheel tiling has tiles pointing in infinitely many different directions, and all directions are equally likely. (In technical language, the distribution of directions is uniform on the circle.) While the tiles themselves are pointy triangles, the statistical properties of the pinwheel tiling are rotationally invariant, with no directions preferred over any others. Could the pinwheel tiling, or something like it, be
a discrete model for a seemingly isotropic universe?

The problem is that the rotational invariance only manifests itself in the limit of infinite size, and develops *incredibly* slowly. An $n$-th level supertile has $5^n$ tiles that appear in only $8n$ different directions, with a still smaller number of directions accounting for the vast majority of the tiles. If the tiles were the size of the Planck length, then a Milky Way Galaxy-sized supertile might have $10^{110} \sim 5^{160}$ tiles in it, but the bulk of those tiles would only be pointing in about 100 different directions. Even at astronomical length scales, space would not look isotropic.

Things are qualitatively the same for *all* 2-dimensional hierarchical tilings, and only slightly better in 3 dimensions. To get around the 2-dimensional limitations, Conway and Radin devised a 3-dimensional generalization of the pinwheel tiling, called the *quaquaversal* tiling (Latin for “every which way”) [2]. The number of relevant directions does grow faster than for the pinwheel, but a galaxy-sized supertile would still only feature a few thousand relevant directions [3].

6 Conclusions

The Platonic ideal of perfectly uniform and symmetric 3-dimensional space, coupled with perfectly uniform 1-dimensional time, did not stand up to 20th century physics. Special relativity shows that we can’t study space and time separately, but must instead think about 4-dimensional space-time. General relativity shows that space is not flat, but bends and curves in response to the matter that is in it. Quantum mechanics says that this matter is fundamentally uncertain, making the structure of space-time uncertain. Furthermore, something fundamentally different has to happen at the ultra-microscopic Planck length.

String theory says that, at the Planck scale, space-time is actually 10 or 11 dimensional, with all but 4 dimensions curled up into a tight ball. Many of us are very skeptical of string theory and open to alternatives, since there is absolutely no experimental evidence in string theory’s favor. (To be fair, there is almost no experimental evidence against it, either. We simply don’t know how to probe things that small.) However, the suggestion that the
universe is a gigantic automaton, with space-time being essentially discrete, doesn’t hold up, either.

If the universe were built on a lattice, then the directions of that lattice could be detected from physical phenomena occurring near the speed of light, and in particular by the propagation of light itself. If the universe were built with random local geometry, then light would not have a precise speed, and different parts of a signal would travel at slightly different speeds and directions, much as a prism splits light into differently colored beams. If the universe were modeled on an aperiodic tiling with rotational symmetry in a statistical sense, there would still be preferred directions at the scale of actual experiments.

All of the simple explanations have failed us. Platonic space-time works very well for day-to-day life, but the details of the actual universe are more complicated and mysterious that our human intelligences can currently fathom. Not because humans are stupid, but because we have the privilege of living in a universe of awe-inspiring subtlety and splendor.

Enjoy the ride.

References

[1] For a simple description of Conway’s Game of Life, see https://en.wikipedia.org/wiki/Game_of_Life.

[2] J. Conway and C. Radin, Quaquaversal Tilings and Rotations, Inventiones Mathematicae 132 (1998) 179–188.

[3] B. Draco, L. Sadun and D. Van Wieren, Growth Rates in the Quaquaversal Tiling, Discrete and Computational Geometry 23 (2000) 419–435. The first author is a purple stuffed dragon, but the other two authors are human.

[4] A. Einstein, Zur Elektrodynamik bewegter Körper, Annalen Den Physik, Bern (2005). An English translation (On the Electrodynamics of Moving Bodies) is found in Einstein’s 1923 book “The Principle of Relativity”.

10
[5] A. Einstein, *Die Feldgleichungen der Gravitation*, Sitzungsberichte der Preussischen Akademie der Wissenschaften zu Berlin: 844–847 (1915)

[6] W. Heisenberg, *Über den anschaulichen Inhalt der quantentheoretischen Kinematik und Mechanik* (About the intuitive content of quantum mechanics and kinematics) Zeitschrift für Physik 43 (1927) 172–198.

[7] A. Michelson and E. Morley, *On the Relative Motion of the Earth and the Luminiferous Ether*, American Journal of Science 34 (1887) 333–345.

[8] M. Planck, *Zur Theorie des Gesetzes der Energieverteilung im Normalpektrum* (On the Theory of the Energy Distribution Law of the Normal Spectrum), Verhandlungen der Deutschen Physikalischen Gesselschaft 2 (1900) 237.

[9] C. Radin, *The Pinwheel Tilings of the Plane*, Annals of Mathematics 139 (1994) 661–702.

[10] A.D. Sakharov, *Vacuum Quantum Fluctuations In Curved Space And The Theory Of Gravitation*, Sov. Phys. Dokl. 12 (1968) 1040 [Dokl. Akad. Nauk Ser. Fiz. 177 (1968) 70]. Reprinted in Gen. Rel. Grav. 32 (2000) 365-367.

[11] S. Wolfram, “A New Kind of Science”, ISBN-13: 978-1579550080 (2002)