Static force potential of non-abelian gauge theory at a finite box in Coulomb gauge

Tomohiro Furukawa\textsuperscript{b,*}, Keiichi Ishibashi\textsuperscript{b†}, H. Itoyama\textsuperscript{a,b,c‡} and Satoshi Kambayashi\textsuperscript{b§}

\textsuperscript{a}Nambu Yoichiro Institute of Theoretical and Experimental Physics (NITEP), Osaka City University
\textsuperscript{b}Department of Mathematics and Physics, Graduate School of Science, Osaka City University
\textsuperscript{c}Osaka City University Advanced Mathematical Institute (OCAMI)
3-3-138, Sugimoto, Sumiyoshi-ku, Osaka, 558-8585, Japan

Abstract

Force potential exerting between two classical static sources of pure non-abelian gauge theory in the Coulomb gauge is reconsidered at a periodic/twisted box of size $L^3$. Its perturbative behavior is examined by the short-distance expansion as well as by the derivative expansion. The latter expansion to one-loop order confirms the well-known change in the effective coupling constant at the Coulomb part as well as the Uehling potential while the former is given by the convolution of two Coulomb Green functions being non-singular at $x = y$. The effect of the twist comes in through its Green function of the sector.

\textsuperscript{*}e-mail: furukawa@sci.osaka-cu.ac.jp
\textsuperscript{†}e-mail: ishibashi.mathphys@gmail.com
\textsuperscript{‡}e-mail: itoyama@sci.osaka-cu.ac.jp
\textsuperscript{§}e-mail: kambayashi@zy.osaka-cu.ac.jp
1 Introduction

The force potential between two static classical sources is a classic object in quantum field theory since Yukawa. In theory where the gauge principle is operating, the computation of this quantity at the Coulomb gauge is a most straightforward one as the Coulomb potential is present in the interaction Hamiltonian as its instantaneous part\(^1\). In the covariant gauge, the Coulomb part and the longitudinal part come together in computation and one often derives the potential by comparing it with the nonrelativistic potential in quantum mechanics at the level of amplitudes.

Non-abelian gauge theory formulated in a finite box has been exploited in several directions both for the periodic boundary condition (see, for example, [12]) and for the twisted boundary conditions [13, 14, 15, 16, 17, 18, 19], combining them with several approximations\(^2\).

The goal of this paper is rather modest: we will reexamine the force potential of the non-abelian gauge theory in the Coulomb gauge at a finite periodic as well as twisted box of size \(L^3\) and determine its form both in the derivative expansion and in the short-distance expansion to one-loop order in old-fashioned perturbation theory. In the Coulomb gauge, the Hamiltonian acting on the reduced Hilbert space consists only of the physical degrees of freedom, all of the gauge degrees of freedom being eliminated. The momentum cutoff \(\Lambda\) can be introduced consistently with Ward-Slavnov-Taylor identity [6, 7] and this allows us to proceed to the straightforward short-distance expansion.

In the next section, we give several preliminaries to the subsequent sections. In particular, we present position space expression of the Coulomb Green function (the inverse of the Laplacian) for the periodic sector and that for the twisted sectors. In section three, we consider the case of (periodic) QED for comparison with the pure non-abelian case and illustrate the derivative and the short-distance expansions. Section four contains main results of our paper. We deal with the non-abelian case to confirm the asymptotic freedom from the effective coupling constant and to obtain the Uehling potential (see, for example, [23]) at the derivative expansion to one-loop order. The one-loop part of the short-distance expansion

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\(^1\)There is a vast amount of literature dealing with Coulomb gauge non-abelian gauge theory. We give here some of the references [1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 11]

\(^2\)For a review, see, for example, [20]. Also, for Witten index in supersymmetric gauge theories and its computation at finite volume, see [21, 22].
begins with $\Lambda^2/p^4$, which translates into $\Lambda^2 \int d^3z G(x - z)G(z - y)$ in position space, being non-singular at $x = y$. We determine the coefficient to one-loop order. The effect of the twist is seen through the phase factor of the Green function in the twisted sector by the Poisson resummation formula.

2 Some preliminaries

2.1 twisted boundary condition

While it is not a main scope of this paper, pure non-abelian gauge theory permits twisted as well as periodic boundary condition due to the presence of the center of $SU(N)$ group. In this subsection, we will briefly recall this well-known fact and treat the cases of periodic boundary condition and the twisted boundary conditions collectively.

Let $A_i(x, y, z) = \sum_a T^a A_i^a(x, y, z)$ be these spacial components of an $SU(N)$ gauge field, which is Lie algebra valued. As we work on Hamiltonian formalism, we will suppress time $t$ unless necessary. We adopt the twisted boundary condition of the following form:

$$A_i(x, y, z) = PA_i(x + L, y, z)P^{-1} = QA_i(x, y + L, z)Q^{-1} = A_i(x, y, z + L),$$  

(2.1)

where $P$ and $Q$ are the constant matrices which satisfy for $SU(N)$,

$$PQ = QPe^{2\pi i/N}. \tag{2.2}$$

An explicit representation for $P$ and $Q$ is

$$P = \alpha \begin{pmatrix} 0 & 1 & 0 \\ 0 & 1 & 0 \\ \vdots & \vdots & \ddots \\ 1 & 0 & 0 \end{pmatrix}, \quad Q = \beta \begin{pmatrix} 1 \\ e^{2\pi i/N} \\ e^{4\pi i/N} \\ \vdots \\ e^{2\pi (N-1)/N} \end{pmatrix}, \tag{2.3}$$

with $\det P = \det Q = 1$ [21]. In the next subsection and the subsequent ones, we will work on an explicit solution to this boundary condition in the case of $SU(2)$ only.

The extension to the explicit solution to the $SU(N)$ case ($N \geq 3$) is a straightforward eigenvalue problem in the linear algebra and will not be attempted here. In ’t Hooft terminology, the twisted boundary condition (2.1) describes one of the three twisted sectors with
a unit magnetic flux, the remaining two obtained by the cubic symmetry of the box. There are another three sectors (2.1) having the magnetic fluxes in two different directions and one sector with the magnetic fluxes in all three directions.

2.2 mode expansion and bracket notation

In order to avoid using plane wave expressions in most places, we will adopt the bracket notation. Let \( f(x) \) obey the twisted boundary condition labelled by \( \lambda \) and be expandable as Fourier series. Preparing the ket \(| f \rangle\) and the bra vector \( \langle x | \) in the coordinate representation such that \( \langle x | x' \rangle = \delta^{(3)}(x - x') \) and therefore \( \int d^3x' | x' \rangle \langle x' | = 1 \), we write

\[
f(x) = \langle x | f \rangle = \int d^3x' \langle x | x' \rangle f(x') = \langle x | \int d^3x' \langle x' | x' \rangle f(x'),
\]

while

\[
f(x) = \sum_{w \in \mathbb{Z}^3+\lambda} C_w^{(\lambda)} e^{\frac{2\pi i}{L} \cdot w \cdot x} \equiv \sum_{w \in \mathbb{Z}^3+\lambda} C_w^{(\lambda)} \langle x | w \rangle_{\lambda}.
\]

Here we have introduced the ket vector \(| w \rangle_{\lambda} \) in the momentum representation in the \( \lambda \) twisted sector. In the twisted sector,

\[
\frac{1}{L^3} \int d^3x e^{\frac{2\pi i}{L} \cdot (w - w') \cdot x} = \frac{1}{L^3} \int d^3x \delta^{(3)}(w' - x) \langle x | w \rangle_{\lambda} = \delta_{w,w'},
\]

still holds, so that

\[
C_w^{(\lambda)} = \frac{1}{L^3} \int d^3x' \langle w' | x' \rangle \langle x' | f \rangle.
\]

Plugging this into (2.5) and comparing with (2.4), we obtain

\[
\frac{1}{L^3} \sum_{w \in \mathbb{Z}^3+\lambda} | w \rangle_{\lambda} \langle w | = 1_{\lambda}.
\]

Here, we have denote by \( 1_{\lambda} \) the unit operator in the \( \lambda \) twisted sector.

Following the relativistic normalization seen in the standard textbook, we expand the gauge field \( A_t^a(x)_{\lambda(a)} \) belonging to the \( \lambda(a) \) twisted sector and its canonical conjugate \( \Pi_t^a(x)_{\lambda(a)} \) at \( t = 0 \) as

\[
A_t^a(x)_{\lambda(a)} = \sum_{w \in \mathbb{Z}^3+\lambda(a)} \frac{1}{\sqrt{2\omega(w)L^3}} \left( \alpha_t^a(w) \langle x | w \rangle_{\lambda(a)} + \text{h.c.} \right),
\]

\[
\Pi_t^a(x)_{\lambda(a)} = \sum_{w \in \mathbb{Z}^3+\lambda(a)} \sqrt{\frac{\omega(w)}{2L^3}} \left( -i \alpha_t^a(w) \langle x | w \rangle_{\lambda(a)} + \text{h.c.} \right),
\]

where \( \omega(w) = (2\pi)^3 L^3 \delta^{(3)}(w) \).
where $\omega(w) = \frac{2\pi}{L}|w|$. The solution to the twisted boundary condition (2.1) is

$$\lambda^{(a=1)} = \begin{pmatrix} 0 \\ 1/2 \\ 0 \end{pmatrix}, \quad \lambda^{(a=2)} = \begin{pmatrix} 1/2 \\ 1/2 \\ 0 \end{pmatrix}, \quad \lambda^{(a=3)} = \begin{pmatrix} 1/2 \\ 0 \\ 0 \end{pmatrix}. \quad (2.11)$$

Here, the column vectors refer to the $x, y, z$ components. Quantization in the Coulomb gauge contains only the transverse part of the gauge fields: the physically relevant part of the oscillators is

$$\alpha^{(tr)a}(w) = P(w)_{ij} \alpha_j^a(w), \quad P(w)_{ij} = \delta_{ij} - \frac{w_i w_j'}{w \cdot w}. \quad (2.12)$$

The canonical commutation relations are

$$\left[ \alpha^{(tr)a}(w), \alpha^{(tr)b\dagger}(w') \right] = \delta^{ab} \delta_{w,w'} P(w)_{ij}, \quad \left[ \alpha^{(tr)a}(w), \alpha^{(tr)b}(w') \right] = 0. \quad (2.13)$$

### 2.3 Green function

We will deal with the loop-corrected Coulomb force potential in the subsequent sections. We list here the Green function of the Laplacian in the $\lambda$ twisted sector:

$$G^\lambda(x|x') = (\partial_i \partial_i)^{-1} \langle x | x' \rangle = \langle x | \hat{\Delta}^{-1} | x' \rangle = \left( \frac{L}{2\pi} \right)^2 \left( \frac{1}{L^3} \sum_{w \in \mathbb{Z}^3, w \neq 0} (-1)^w \langle w | \lambda \langle w \rangle \rangle \right) \langle x' \rangle$$

$$= -\frac{1}{4\pi} \sum_{\ell \in \mathbb{Z}^3} e^{2\pi i \lambda \cdot \ell} \frac{1}{|x - x' + L\ell|}. \quad (2.14)$$

The last expansion is obtained from the Poisson resummation formula. In the case of the periodic sector $\lambda = 0$, we obtain

$$G^{\lambda=0}(x|x') = \left( \frac{L}{2\pi} \right)^2 \left( \frac{1}{L^3} \sum_{w \in \mathbb{Z}^3, w \neq 0} (-1)^w \langle w | \langle w \rangle \rangle \right) \langle x' \rangle = -\frac{1}{4\pi} \sum_{\ell \in \mathbb{Z}^3} \frac{1}{|x - x' + L\ell|}. \quad (2.15)$$

This agrees with the Green function in the periodic box$^3$.

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$^3$The charge neutrality condition for the total source is required in the periodic box by the Gauss’ law. This removes the zero-mode from our consideration.
The Green function $G^\lambda(x|x')$ in the limit $L \to \infty$ does not depend on the twist $\lambda$ and is simply
\[ G(x|x') = \frac{-1}{4\pi |x-x'|}. \] (2.16)

### 2.4 Coulomb gauge Hamiltonian

Here we just list the Hamiltonian
\[
H = \frac{1}{2} \sum_a \int d^3x \left( \mathcal{J}^{-1} \Pi_{i}^{(tr)a} \mathcal{J} \Pi_{i}^{(tr)a} + B_{i}^{(tr)a} B_{i}^{(tr)a} \right) + H_{\text{Coul}},
\] (2.17)

\[
H_{\text{Coul}} = \frac{g_{0}^{2}}{2} \sum_{a,b} \int d^3x d^3x' \mathcal{J}^{-1} \rho^a(x) \langle x|((\partial_{i} D_{i})^{-1}(-\partial^2)(\partial_{j} D_{j})^{-1})^{ab}|x'\rangle \mathcal{J} \rho^b(x'),
\] (2.18)

where $\Pi_{i}^{(tr)a}$, $B_{i}^{(tr)a}$, $D_{i}$, $\rho^a$ and $\mathcal{J}$ are respectively the conjugate momentum of the transverse gauge field $A_{i}^{(tr)a}$, the transverse magnetic field, covariant derivative, the current density and the Faddeev-Popov determinant $\mathcal{J} = \det(\partial_{i} D_{i})$. As we are in the Coulomb gauge, only the transverse parts of the gauge field $A_{i}$ (and its canonical conjugate $\Pi_{i}$) contribute to the Hamiltonian. For simplicity, we will omit the symbol (tr) in the following discussion. We expand the operator $((\partial_{i} D_{i})^{-1}(-\partial^2)(\partial_{j} D_{j})^{-1})^{ab}$ in the Coulomb potential part $H_{\text{Coul}}$ of the Hamiltonian by the coupling constant $g_{0}$ as follows:
\[
\langle x|((\partial_{i} D_{i})^{-1}(-\partial^2)(\partial_{j} D_{j})^{-1})^{ab}|x'\rangle = \langle x|(-\hat{\Delta}^{-1})^{ab} + 2g_{0}(\hat{\Delta}^{-1} \hat{\Omega} \hat{\Delta}^{-1})^{ab} - 3g_{0}^{2}(\hat{\Delta}^{-1} \hat{\Omega} \hat{\Delta}^{-1})^{ab} + O(g_{0}^{3})|x'\rangle,
\] (2.19)

where $(-\hat{\Delta}^{-1})^{ab}$, $\hat{\Omega}^{ab}$ are the operators respectively represented as
\[
\langle x|(-\hat{\Delta}^{-1})^{ab}|x'\rangle = \delta^{ab} G^\lambda(x|x'),
\] (2.20)

\[
\langle x|\hat{\Omega}^{ac}|x'\rangle = \delta^{(3)}(x-x') \epsilon^{abc} A_{i}^{b}(x') \frac{\partial}{\partial x'^{i}}.
\] (2.21)

In the Hamiltonian, we have included two of the classical external source terms in the current density $\rho^a(x)$
\[
\rho^a(x) = g_{0} \epsilon^{abc} A_{i}^{b}(x) \Pi_{i}^{c}(x) + \rho_{1,\text{ex}}^a(x) + \rho_{2,\text{ex}}^a(x),
\]
\[
\rho_{1,\text{ex}}^a(x) = \sum_{w \in \mathbb{Z}^{3} + \lambda(a)} \langle x|w\rangle \lambda(a) \rho_{1,\text{ex}}^a(w).
\] (2.22)
The two delta function sources localized at $x = x_1, x_2$ are respectively represented as
\[ \tilde{\rho}^a_{1,2,\text{ext}}(w) = q^a_{1,2} \frac{1}{L^3} \lambda^{(a)} \langle w | x_{1,2} \rangle, \] (2.23)
so
\[ \rho^a_{1,2,\text{ext}}(x) = q^a_{1,2} \frac{1}{L^3} \sum_{w \in \mathbb{Z}^3 + \lambda^{(a)}} \langle x | w \rangle \lambda^{(a)} \langle w | x_{1,2} \rangle = q^a_{1,2} \delta^{(3)}(x - x_{1,2}), \] (2.24)
in the $\lambda$ twisted sector.

We will be interested in the part of the vacuum energy which depends linearly upon both $\rho^a_{1,\text{ex}}(x)$ and $\rho^a_{2,\text{ex}}(x)$. Clearly, at the lowest classical level,
\[ E_{\text{tw}} = -g_0^2 \sum_a \int d^3 x d^3 x' \rho^a_{1,\text{ex}}(x) G^{\lambda^{(a)}}(x|x') \delta^{ab} \rho^b_{2,\text{ex}}(x') \]
\[ = -g_0^2 \sum_a q^a_1 q^a_2 G^{\lambda^{(a)}}(x_1|x_2). \] (2.25)

3 Case of QED

In this section, we obtain, to one-loop order, the interaction energy between the two external static sources with charges $q_1$ and $q_2$ for QED in the periodic box of size $L^3$ by old-fashioned perturbation theory well-known in quantum mechanics. We will confirm the UV divergence and the renormalization of the coupling constant and the Uehling potential in QED for the massless fermions at finite volume.

Let us first denote the free part and the interaction part of the Coulomb gauge Hamiltonian $H$ by $H^{(0)}$ and $H_{\text{int}}$ respectively:
\[ H_{\text{int}} \equiv H - H^{(0)}. \] (3.1)
The massless fermions are expanded as
\[ \psi(x) = \sum_{w \in \mathbb{Z}^3} \frac{1}{\sqrt{2 \omega(w) L^3}} \left( d^s(w) u^s(w) e^{i \frac{2\pi}{L} w \cdot x} + b^s(w) v^s(w) e^{-i \frac{2\pi}{L} w \cdot x} \right). \] (3.2)
\[ \bar{\psi}(x) = \sum_{w \in \mathbb{Z}^3} \frac{1}{\sqrt{2 \omega(w) L^3}} \left( b^s(w) \bar{u}^s(w) e^{i \frac{2\pi}{L} w \cdot x} + d^s(w) \bar{v}^s(w) e^{-i \frac{2\pi}{L} w \cdot x} \right). \] (3.3)
The Hamiltonian in this section includes the kinetic term for the fermions and their gauge interactions. The current density is
\[ \rho(x) = \rho_{1,\text{ex}}(x) + \rho_{2,\text{ex}}(x) + \bar{\psi} \gamma^0 \psi. \] (3.4)
The eigenstates and the eigenvalues for the Hamiltonian of $H$ and the free part of the Hamiltonian $H^{(0)}$ are respectively denoted by

$$H|N_\alpha; N_d, N_b\rangle = E_{N_\alpha; N_d, N_b}|N_\alpha; N_d, N_b\rangle,$$

(3.5)

$$H^{(0)}|N_\alpha; N_d, N_b\rangle = E^{(0)}_{N_\alpha; N_d, N_b}|N_\alpha; N_d, N_b\rangle,$$

(3.6)

$$H^{(0)}|0; 0\rangle = E^{(0)}_{0; 0}|0; 0\rangle = 0,$$

(3.7)

where $N_\alpha$ is the number of bosons and $N_d$ and $N_b$ are respectively the number of fermions and that of antifermions. As we are interested in the interaction energy between the two external sources, we ignore the zero point oscillation and set $E^{(0)}_{0; 0} = 0$.

As in quantum mechanics, the perturbative expansion of $E(r_{12})$ goes as

$$E(r_{12}) = E^{(i)}(r_{12}) + E^{(ii)}(r_{12}) + \cdots,$$

(3.8)

$$E^{(i)}(r_{12}) = \langle 0 | H_{\text{int}}(r_{12}) | 0 \rangle,$$

(3.9)

$$E^{(ii)}(r_{12}) = \sum_{N_\alpha; N_d, N_b \neq (0;0,0)} |\langle 0 | H_{\text{int}}(r_{12}) | N_\alpha; N_d, N_b \rangle|^2 - E^{(0)}_{N_\alpha; N_d, N_b},$$

(3.10)

After some calculation which we omit presenting here (it is a routine), we obtain the leading order $E^{(i)}(r_{12})$ and the second order $E^{(ii)}(r_{12})$ corrections respectively given by

$$E^{(i)}(r_{12}) = \frac{g_0^2 q_1 q_2}{4\pi^2 L} \sum_{\substack{n \in \mathbb{Z}^3 \backslash n \neq 0}} e^{i\frac{2\pi}{L} n \cdot (x_2 - x_1)} \frac{1}{n \cdot n} = \frac{1}{4\pi} \sum_{\ell \in \mathbb{Z}^3} \frac{g_0^2 q_1 q_2}{|x_1 - x_2 + \ell L|},$$

(3.11)

and

$$E^{(ii)}(r_{12}) = \frac{g_0^2 q_1 q_2}{4\pi^2 L} \sum_{\substack{n \in \mathbb{Z}^3 \backslash n \neq 0}} e^{i\frac{2\pi}{L} n \cdot (x_2 - x_1)} \left[g_0^2 \delta(n)\right],$$

(3.12)

$$\delta(n) = -\frac{1}{4\pi^3} \sum_{\substack{m \in \mathbb{Z}^3 \backslash m \neq 0}} \frac{1}{|m + n + m|} \left(1 - \frac{m \cdot (n + m)}{|m||n + m|}\right).$$

### 3.1 derivative expansion

Let us first consider the derivative expansion, which will be valid at the distance comparable to the size of the box, to evaluate the first quantum correction (3.12) to Coulomb potential.

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\(^4\)In this paper, we do not estimate the contributions coming from the zero momentum modes and possible infrared divergences associated with them.
in perturbation theory. The derivative expansion corresponds to the triple Taylor expansion with respect to \( n^i \) \((i = 1, 2, 3)\). We obtain

\[
\delta(n) = -\frac{1}{4\pi^3} \frac{1}{n \cdot n} f(n) = -\frac{1}{4\pi^3} \frac{1}{n \cdot n} \left( f(0) + \frac{1}{2!} \frac{\partial^2 f}{\partial n^i \partial n^j}(0)n^i n^j + \frac{1}{4!} \frac{\partial^4 f}{\partial n^i \partial n^j \partial n^k \partial n^\ell}(0)n^i n^j n^k n^\ell + \cdots \right) \tag{3.13}
\]

where we have omitted the terms odd in \( n^i \) as they cancel upon taking the summation over \( n^i \). Using the symmetry of cubic lattice, the above expansion (3.13) is written as

\[
\delta(n) = -\frac{1}{4\pi^3} \sum_{m \in \mathbb{Z}^3} \left[ \frac{1}{6|m|^3} + \frac{1}{120 |m|^5} + \mathcal{O}(|n|^4) \right]. \tag{3.14}
\]

Coming back to (3.12) and using the Poisson resummation formula, we obtain

\[
E^{(ii)}(r_{12}) = -\frac{1}{4\pi} \sum_{n \in \mathbb{Z}^3} \frac{g_0^2 q_1 q_2}{|x_1 - x_2 + nL|} \left( -\frac{g_0^2}{6\pi^2} \right) \frac{1}{4\pi} \sum_{m \in \mathbb{Z}^3, m \neq 0} \frac{1}{|m|^3}

- g_0^4 q_1 q_2 \frac{1}{(4\pi)^2} \frac{1}{15 (2\pi)^3} \left( L^2 \sum_{n \in \mathbb{Z}^3} \delta^{(3)}(x_1 - x_2 + nL) - \frac{1}{L} \right) \sum_{m \in \mathbb{Z}^3, m \neq 0} \frac{1}{|m|^5}

+ (\text{higher orders in the derivative expansion}). \tag{3.15}
\]

The higher orders will give (gaussian) width to the delta function potential. This expression is still at finite volume \( L^3 \).

Taking the large volume, up to the second order in perturbation theory, we obtain the derivative expansion of the loop-corrected Coulomb potential as the interaction energy between the two external charges \( q_1 \) and \( q_2 \):

\[
E(r_{12}) = \frac{1}{4\pi} \frac{q_1 q_2}{r_{12}} g_0^2 \left( 1 - \frac{g_0^2}{6\pi^2} \ln(\Lambda L) \right) - g_0^4 q_1 q_2 \frac{1}{(4\pi)^2} \frac{1}{15} L^2 \left( 1 - \frac{1}{(\Lambda L)^2} \right) \delta^{(3)}(x_1 - x_2), \tag{3.16}
\]

where \( \Lambda \) is the UV cutoff. The first term derives the positive \( \beta \)-function of QED at one-loop while the second term is the Uehling potential. Here, when we define the renormalized coupling constant \( g_L \) in the box \( L^3 \):

\[
g_L^2 = g_0^2 \left( 1 - \frac{g_0^2}{6\pi^2} \ln(\Lambda L) \right), \tag{3.17}
\]

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$g_L^2$ is written in $g_0^2$:

$$g_0^2 = g_L^2 \left( 1 + \frac{g_L^2}{6\pi^2} \ln(\Lambda L) \right) + \mathcal{O}(g_0^6).$$

By substituting this into (3.16), we can also write $E(r_{12})$ as

$$E(r_{12}) \simeq \frac{1}{4\pi} g_1 q_2 g_L^2 - \frac{g_1^2 q_1 q_2}{(4\pi)^2} \frac{1}{15} L^2 \left( 1 - \frac{1}{(\Lambda L)^2} \right) \delta^{(3)}(x_1 - x_2), \quad (3.18)$$

at the order $g_L^4$.

### 3.2 expansion at short-distance

Let us now probe the opposite limit to the last subsection. We will evaluate the interaction energy by the short-distance $r_{12} \ll L$ expansion. In this expansion, $E(r_{12})$ is expanded in $\Lambda/p$. Here, we take the limit $L \to \infty$ from the beginning.

We obtain

$$E(r_{12}) = g_0^2 q_1 q_2 \frac{1}{(2\pi)^3} \int d^3p \frac{1}{p \cdot p} e^{ip \cdot (x_1 - x_2)} \left[ 1 + g_0^2 \delta(p) \right], \quad (3.19)$$

$$\delta(p) = -\frac{1}{4\pi^3} \frac{1}{p \cdot p} \int d^3k \frac{1}{|k| + |p + k|} \left( 1 - \frac{k \cdot (p + k)}{|k||p + k|} \right). \quad (3.20)$$

Using the polar coordinates, $\delta(p)$ is further converted as

$$\delta(p) = -\frac{1}{2\pi^2} \frac{1}{p \cdot p} \int_0^\Lambda k^2 dk \int_0^\pi d\theta \frac{\sin \theta}{k + \sqrt{k^2 + p^2 + 2kp\cos \theta}} \left( 1 - \frac{k + p\cos \theta}{\sqrt{k^2 + p^2 + 2kp\cos \theta}} \right) \quad (3.21)$$

$$I(\Lambda, p, \theta) = \int_0^\Lambda k^2 dk \frac{1}{k + \sqrt{k^2 + p^2 + 2kp\cos \theta}} \left( 1 - \frac{k + p\cos \theta}{\sqrt{k^2 + p^2 + 2kp\cos \theta}} \right). \quad (3.22)$$

Expanding $I(\Lambda, p, \theta) / \Lambda^2$ in $\Lambda/p$, we obtain

$$I(\Lambda, p, \theta) = \frac{1 - \cos \theta}{3} + \frac{1 - \cos^2 \theta}{2} \Lambda^4 + \mathcal{O}\left( \frac{\Lambda^5}{p^5} \right). \quad (3.23)$$

(3.21) becomes

$$\delta(p) = -\frac{1}{2\pi^2} \int_0^\pi d\theta \sin \theta \left( \frac{1 - \cos \theta}{3} \left( \frac{\Lambda}{p} \right)^3 + \frac{1 + \cos^2 \theta}{2} \left( \frac{\Lambda}{p} \right)^4 + \mathcal{O}\left( \frac{\Lambda^5}{p^5} \right) \right) \quad (3.24)$$

$$= -\frac{1}{3\pi^2} \left( \frac{(\Lambda)^3}{p} - \left( \frac{\Lambda}{p} \right)^4 + \mathcal{O}\left( \frac{\Lambda^5}{p^5} \right) \right).$$
The loop-corrected potential as the interaction energy $E(r_{12})$ at short-distance expansion is thus given by

$$E(r_{12}) = \frac{g_0^2 q_1 q_2}{(2\pi)^3} \int d^3 p e^{i p (x_2 - x_2)} \left[ \frac{1}{|p|^2} - \frac{g_0^2}{3\pi^2} \left( \frac{\Lambda^3}{|p|^3} - \frac{\Lambda^4}{|p|^4} + O \left( \frac{\Lambda^5}{|p|^5} \right) \right) \right]. \quad (3.25)$$

4 static force potential in pure non-abelian gauge theory at a periodic and twisted box

Let us now turn to the case of pure non-abelian gauge theory at a twisted or periodic finite box of size $L^3$. Unlike the periodic one, the zero-mode is not present in the twisted sector. We manage to treat both cases at once in the notation in what follows.

Following the method in the abelian case, we will compute the interaction energy $E(r_{12})$ between the two external static sources of charge $q_1^a$ and $q_2^a$ in old-fashioned perturbation theory. Up to the second order, it reads $E(r_{12}) = E^{(i)}(r_{12}) + E^{(ii)}(r_{12})$:

$$E^{(i)}(r_{12}) = \sum_a \frac{g_0^2 q_1^a q_2^a}{4\pi^2 L} \sum_{n \in \mathbb{Z}^3 + \lambda(a)} \langle x_1 | n \rangle_{\lambda(a)} \frac{1}{n \cdot n} \left( g_0^2 + g_0^4 \delta'_{\lambda(a)}(n) \right) \langle n | x_2 \rangle, \quad (4.1)$$

$$E^{(ii)}(r_{12}) = \sum_a \frac{g_0^2 q_1^a q_2^a}{4\pi^2 L} \sum_{n \in \mathbb{Z}^3 + \lambda(a)} \langle x_1 | n \rangle_{\lambda(a)} \frac{1}{n \cdot n} \left( g_0^2 \delta''_{\lambda(a)}(n) \right) \langle n | x_2 \rangle, \quad (4.2)$$

where

$$\delta'_{\lambda(a)}(n) = \frac{3}{16\pi^3} \frac{1}{n \cdot n} \sum_{c \neq a} \sum_{m \in \mathbb{Z}^3 + \lambda(c)} \frac{(n + m)_i P(m)_{ij} n_j}{|n + m|^2 |m|}, \quad (4.3)$$

$$\delta''_{\lambda(a)}(n) = -\frac{1}{32\pi^3} \frac{1}{n \cdot n} \sum_{c \neq a} \sum_{m \in \mathbb{Z}^3 + \lambda(c)} \frac{(|m| - |n - m|)^2 P(m)_{ij} P(n - m)_{ij}}{|m| + |n - m| |m| |n - m|}. \quad (4.4)$$

Here we have used

$$\sum_{b,c} \epsilon^{bac} \epsilon^{bde} f^{(c)} = \sum_c (\delta^{ad} \delta^{cc} - \delta^{ac} \delta^{dc}) f^{(c)} = \delta^{ad} \sum_c (\delta^{cc} - \delta^{ac}) f^{(c)} = \delta^{ad} \sum_{c \neq a} f^{(c)}. \quad (4.5)$$

In the case of $SU(N)$, we need only to replace $\epsilon^{bac}$ by the structure constant. (The above derivation, of course, does not hold.) As for $E^{(i)}(r_{12})$, recall that $H_{\text{Coul}}$ in the non-abelian case in (2.18) takes a non-local expression in the position space and generates infinite series

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of gauge fields in coupling constant. It starts contributing already at the first order in perturbation theory. We have a result in $E^{(ii)}(r_{12})$ similar to that in the abelian case, replacing the fermionic intermediate states by the bosonic ones. Again, we have omitted presenting our calculation.

The interaction energy in the twisted box $L^3$ up to $g_0^4$ reads

$$E(r_{12}) = \sum_a \frac{g_0^2 g_1^a g_2^a}{4\pi^2 L} \sum_{\substack{n \in \mathbb{Z}^3 + \lambda^{(a)} \neq 0}} \langle x_1 | n \rangle_{\lambda^{(a)}} \frac{1}{n \cdot n} \left[ 1 + g_0^2 \left( \delta_{\lambda^{(a)}}(n) + \delta_{\lambda^{(a)}}''(n) \right) \right] \lambda^{(a)} \langle n | x_2 \rangle.$$  

(4.6)

The case of the periodic sector can be read off from this expression by setting $\lambda^{(a)} = 0$. The presence of $\delta_{\lambda}''(n)$ from the contribution $E^{(i)}(r_{12})$ is a unique feature of non-abelian gauge theory responsible for the asymptotic freedom.

4.1 derivative expansion

To evaluate the above result, we expand the quantum corrections $\delta_{\lambda^{(a)}}'(n) + \delta_{\lambda^{(a)}}''(n)$ in $n^i$ as in the abelian case:

$$\delta_{\lambda^{(a)}}'(n) + \delta_{\lambda^{(a)}}''(n) = \frac{1}{32\pi^3} \sum_{c \neq a} \sum_{\substack{m \in \mathbb{Z}^3 + \lambda^{(c)} \neq 0 \ n \neq 0}} \left[ \frac{1}{2!} \left( \frac{12}{|m|^3} - \frac{14(m \cdot n)^2}{|m|^5 |n|^2} \right) + \frac{1}{3!} \left( -\frac{66(m \cdot n)}{|m|^5} + \frac{57(m \cdot n)^3}{|m|^7 |n|^2} \right) \\
+ \frac{1}{4!} \left( -\frac{150|n|^4}{|m|^5} + \frac{822(m \cdot n)^2}{|m|^7} - \frac{714(m \cdot n)^4}{|m|^9 |n|^2} \right) + (\text{higher orders of } n^i) \right].$$  

(4.7)

In the twisted sectors, namely $\lambda \neq 0$, we cannot use the cubic symmetry to evaluate the parts involving $m \cdot n$ in the box $L^3$. Here we consider the case of the periodic box, namely the $\lambda = 0$ case only. Due to the cubic symmetry, the quantum corrections reduce to

$$\delta_{\lambda=0}'(n) + \delta_{\lambda=0}''(n) = \frac{1}{16\pi^3} \sum_{\substack{m \in \mathbb{Z}^3 \ m \neq 0 \ n \neq 0}} \left[ \frac{11}{3} \frac{1}{|m|^3} - \frac{47|n|^2}{60|m|^5} + O(|n|^4) \right].$$  

(4.8)
We obtain the interaction energy at the periodic box:

\[
E(r_{12}) = \frac{1}{4\pi} \sum_a \sum_{n \in \mathbb{Z}^3} \frac{g_0^2 q_1^a q_2^a}{x_1 - x_2 + nL} \left[ 1 + g_0^2 \frac{11}{12\pi^2} \frac{1}{4\pi} \sum_{m \in \mathbb{Z}^3, m \neq 0} \frac{6}{|m|^3} \right]
\]

\[
- \sum_a \frac{g_0^4 q_1^a q_2^a}{(4\pi)^2} \frac{47}{30} \frac{1}{(2\pi)^3} \left( L^2 \sum_{n \in \mathbb{Z}^3} \delta^{(3)}(x_1 - x_2 + nL) - \frac{1}{L^3} \right) \sum_{m \in \mathbb{Z}^3, m \neq 0} \frac{1}{|m|^5}
\]

\[+ \text{ (higher orders in the derivative expansion).} \quad (4.9)\]

For large \( L \), the static force potential is evaluated as follows up to the second order in perturbation theory,

\[
E(r_{12}) = \sum_a \frac{g_1^a q_2^a}{4\pi r_{12}} g_0^2 \left( 1 + g_0^2 \frac{11}{12\pi^2} \ln(\Lambda L) \right) - \sum_a \frac{g_0^4 q_1^a q_2^a}{(4\pi)^2} \frac{47}{30} L^2 \left( 1 - \frac{1}{(\Lambda L)^2} \right) \delta^{(3)}(x_1 - x_2), \quad (4.10)
\]

where \( \Lambda \) is the UV cutoff. The first term of (4.10) derives the negative \( \beta \)-function with the correct numerical coefficient\(^5\) and the second term is nothing but the Uehling potential.

Also, by defining the renormalized coupling constant \( g_L \) as

\[
g_L^2 = g_0^2 \left( 1 + g_0^2 \frac{11}{12\pi^2} \ln(\Lambda L) \right),
\]

we can also write \( E(r_{12}) \) as

\[
E(r_{12}) = \sum_a \frac{g_1^a q_2^a}{4\pi r_{12}} g_L^2 - \sum_a \frac{g_0^4 q_1^a q_2^a}{(4\pi)^2} \frac{47}{30} L^2 \left( 1 - \frac{1}{(\Lambda L)^2} \right) \delta^{(3)}(x_1 - x_2), \quad (4.11)
\]

at the order \( g_L^4 \).

\[4.2 \text{ expansion at short-distance}\]

Let us carry out the short-distance expansion as in the abelian case. Taking \( L \to \infty \) limit, we obtain

\[
E(r_{12}) = \sum_a \frac{g_0^2 g_1^a q_2^a}{(2\pi)^3} \int d^3p \frac{1}{p \cdot p} e^{ip \cdot (x_1 - x_2)} \left[ 1 + g_0^2 \left( \delta'(p) + \delta''(p) \right) \right], \quad (4.12)
\]

\[5\text{We regard this as support for the validity of the use of momentum cutoff to one-loop order. See [24] for further discussion on renormalization and renormalization group in this treatment.}\]
where

\[
\delta'(\mathbf{p}) = \frac{3}{8\pi^2} \frac{1}{\mathbf{p} \cdot \mathbf{p}} \int d^3 k \frac{(p + k) \cdot P(k)_{ij} p_j}{|p + k|^2 |\mathbf{k}|},
\]

(4.13)

\[
\delta''(\mathbf{p}) = -\frac{1}{16\pi^3} \frac{1}{\mathbf{p} \cdot \mathbf{p}} \int d^3 k \frac{[|k| - |p - k|]^2 P(k)_{ij} P(p - k)_{ij}}{|k| + |p - k| |\mathbf{k}| |p - \mathbf{k}|}.
\]

(4.14)

Using the polar coordinates, we obtain

\[
\begin{align*}
\delta'(\mathbf{p}) &= \frac{3}{4\pi^2} \frac{1}{\mathbf{p} \cdot \mathbf{p}} \int^\Lambda dk \int^\pi_0 d\theta \sin \theta \frac{kp^2(1 - \cos^2 \theta)}{p^2 + k^2 + 2pk \cos \theta} \\
&= \frac{3}{4\pi^2} \frac{1}{\mathbf{p} \cdot \mathbf{p}} \int^\pi_0 d\theta \sin \theta I'(\Lambda, p, \theta),
\end{align*}
\]

(4.15)

\[
\begin{align*}
\delta''(\mathbf{p}) &= -\frac{1}{8\pi^2} \frac{1}{\mathbf{p} \cdot \mathbf{p}} \int^\Lambda dk \int^\pi_0 d\theta \sin \theta \frac{k}{k + \sqrt{p^2 + k^2 - 2pk \cos \theta}} \\
&\quad \cdot \frac{(k - \sqrt{p^2 + k^2 - 2pk \cos \theta})^2}{\sqrt{p^2 + k^2 - 2pk \cos \theta}} \left(1 + \frac{(k - p \cos \theta)^2}{p^2 + k^2 - 2pk \cos \theta}\right) \\
&= -\frac{1}{8\pi^2} \frac{1}{\mathbf{p} \cdot \mathbf{p}} \int^\pi_0 d\theta \sin \theta I''(\Lambda, p, \theta).
\end{align*}
\]

(4.16)

Expanding in \(I'(\Lambda, p, \theta), I''(\Lambda, p, \theta)\) in \(\Lambda\):

\[
I'(\Lambda, p, \theta) = \frac{1 - \cos^2 \theta}{2} \Lambda^2 + \cdots,
\]

(4.17)

\[
I''(\Lambda, p, \theta) = \frac{1 + \cos^2 \theta}{2} \Lambda^2 + \cdots.
\]

(4.18)

We obtain

\[
E(r_{12}) = \sum_a \frac{g_0^2 q_1^a q_2^a}{(2\pi)^3} \int d^3 p e^{ip(x_1 - x_2)} \left[ \frac{1}{p^2} + g_0^2 \left(1 + \frac{\Lambda^2}{3\pi^2 p^4} + \mathcal{O}\left(\frac{\Lambda^3}{p^4}\right)\right) \right],
\]

(4.19)

We note that the \(1/p^4\)-term is written as

\[
\int d^3 z G(x_1|z) G(z|x_2) = \int \frac{d^3 p}{(2\pi)^3} \frac{1}{p^4} e^{ip(x_1 - x_2)}
\]

(4.20)

which is non-singular at \(x_1 = x_2\):

\[
\int d^3 z G(x|z) G(z|x') = \left(\frac{-1}{4\pi}\right)^2 \int d^3 z \frac{1}{|x - z|} \frac{1}{|z - x'|}.
\]

(4.21)

In terms of the Green functions, the interaction energy \(E(r_{12})\) is written as

\[
E(r_{12}) = -g_0^2 \sum_a q_1^a q_2^a \left[ G(x_1|x_2) - \frac{1}{3\pi^2 g_0^2 \Lambda^2} \int d^3 z G(x_1|z) G(z|x_2) + \mathcal{O}(\Lambda^3) \right].
\]

(4.22)
As is stated in the introduction, the expansion begins with the convolution of the two Coulomb Green functions and is non-singular at the short distance limit of the two external sources $x_1 = x_2$. Up to the same order in the expansion, this term does not appear in QED.

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**A Poisson resummation formula**

In this appendix, we give the proof for the Poisson resummation formula in three dimensions:

$$\sum_{n \in \mathbb{Z}^3} f(n) = \sum_{m \in \mathbb{Z}^3} \tilde{f}(2\pi m), \quad \text{(A.1)}$$

where $\tilde{f}$ is the Fourier transform of function $f$:

$$\tilde{f}(k) = \int d^3r f(r)e^{-ikr}.$$

The proof exploits the formula

$$\sum_{n \in \mathbb{Z}^3} e^{2\pi i k \cdot n} = \sum_{m \in \mathbb{Z}^3} \delta^{(3)}(k - m), \quad \text{(A.2)}$$

and goes as

$$\sum_{n \in \mathbb{Z}^3} f(n) = \sum_{n \in \mathbb{Z}^3} \int \frac{d^3k}{(2\pi)^3} \tilde{f}(k)e^{ik \cdot n} = \int \frac{d^3k}{(2\pi)^3} \tilde{f}(k) \sum_{n \in \mathbb{Z}^3} e^{ik \cdot n}$$

$$= \int \frac{d^3k}{(2\pi)^3} \tilde{f}(k) \sum_{m \in \mathbb{Z}^3} \delta^{(3)}(k - m) = \sum_{m \in \mathbb{Z}^3} \tilde{f}(2\pi m).$$

We can generalize (A.1) by considering $n \in \mathbb{Z}^3 + \lambda$ ($\lambda \in \mathbb{R}^3$) in the sum in the left hand side:

$$\sum_{n \in \mathbb{Z}^3 + \lambda} e^{2\pi i k \cdot n} = \sum_{m \in \mathbb{Z}^3} \delta^{(3)}(k - m)e^{2\pi i \lambda \cdot m}. \quad \text{(A.3)}$$
This leads us to
\[
\sum_{n \in \mathbb{Z}^3 + \lambda} f(n) = \sum_{m \in \mathbb{Z}^3} \tilde{f}(2\pi m) e^{2\pi i \lambda \cdot m},
\] (A.4)
which is exploited in the text.

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