On the Optimality of Opportunistic Interference Alignment in 3-Transmitter MIMO Interference Channels

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Abstract

In this paper, we propose opportunistic interference alignment (OIA) for three-transmitter multiple-input multiple-output (MIMO) interference channels (ICs). In the proposed OIA scheme, each transmitter has its own user group that consists of $K$ users, and each transmitter opportunistically selects the user whose received interference signals are most aligned. Thus, three-transmitter MIMO IC is opportunistically constructed by three transmitters and their selected users. Contrary to conventional IA, perfect channel information for all of the interference links is not required at the transmitter. Each user just needs to feed back one scalar value in the proposed OIA scheme. When the number of receive antennas is $N_R$ (with $N_R = 2M$) and the number of transmit antennas is $N_T$ (with $N_T \geq M$), we prove that each transmitter can achieve $M$ degrees of freedom (DoF) as the number of users in each group goes to infinity ($K \to \infty$), in comparison with the $2M^3$ DoF known to be achievable in the three-transmitter $M \times 2M$ MIMO IC. We also show that the user scaling $K \propto P^{\alpha M^2}$ enables the network to achieve $M$ DoF per transmitter where $P$ is transmit power, and $\alpha$ is the relative path loss of the interfering channel in decibels. The computational complexity of the proposed OIA scheme is analyzed and compared with conventional user selection schemes.

Index Terms

MIMO interference channel, interference alignment, opportunistic interference alignment, postprocessing
I. INTRODUCTION

Interference alignment (IA) has been touted as a key technology for handling interference in future wireless communications. Contrary to the conventional wisdom that orthogonalization is the best way to handle interference, [1] showed that IA can achieve a total of $\frac{N^2}{2}$ degrees of freedom (DoF) in an $N$-transmitter single-input single-output (SISO) interference channel (IC). Following this work, there have been many intensive studies on IA [1]–[7]. It has been recently shown that the achievable DoF at each transmitter in a $N$-transmitter $N_T \times N_R$ MIMO IC becomes

$$\frac{R \min(N_T, N_R)}{R + 1} \leq \frac{\text{DoF}_{\text{MIMO IC}}}{R + 1} \leq \frac{\max(N_T, N_R)}{R + 1}, \quad N > R$$

(1)

where $N_T$ and $N_R$ are the numbers of transmit and receive antennas, respectively, and $R = \left\lfloor \frac{\max(N_T, N_R)}{\min(N_T, N_R)} \right\rfloor$. In the $N > R$ case, symbol extensions are needed to achieve the DoF in time varying or frequency selective fading channels. Despite the promising aspects of IA, its implementation has many challenges. IA requires perfect global channel knowledge (including whole channel links) at the transmitter, which involves excessive signal overhead. Imperfect channel state information significantly degrades the gain of IA. The large computation complexity necessitated is also regarded as a big challenge for practical implementation. The sub-optimality of IA in the practical operating SNR region is another problem.

Recently, IA techniques to ameliorate these problems have been investigated. Iterative IA algorithms were proposed to optimize precoding matrix [5] and to reduce the global channel knowledge burden based on channel reciprocity [6]. To reduce computational complexity, Suh and Tse [7] proposed a subspace interference alignment technique for an uplink cellular network system. In [8], IA was opportunistically performed in MIMO cognitive radio networks, where secondary transmitters transmit their signals on only spatial dimensions not used by primary transmitters. IA with imperfect channel state information (CSI) was shown to achieve the same DoF as IA with perfect CSI if the feedback size per user is properly scaled [9], [10]. In [9], a DoF of $\frac{N^2}{2}$ was shown to be achieved with limited feedback in an $N$-transmitter frequency-selective SISO IC if the feedback size per user is maintained as $N(L - 1) \log P$ bits, where $L$ is the number of taps and $P$ is the total available transmit power. [10] showed that the maximum number of DoF is achievable with limited feedback in an $N$-user MIMO IC if the feedback size scales with the SNR. Also, full DoF is known to be achievable with analog feedback if the analog feedback power grows with the transmit power [11].
A. Opportunistic Interference Alignment (OIA)

Although there have been significant efforts to overcome the practical challenges of IA, the inherent shortcomings of IA highly motivate the development of more practical IA techniques that work well with limited feedback and reduce computational complexity while maintaining the promised gain. In this paper, we propose an opportunistic interference alignment (OIA) scheme in a three-transmitter MIMO IC with a generalized antenna configuration. In our system model, there are three transmitters and three user groups corresponding to the transmitters. Only a single user in each user group is served by each transmitter, so a three-transmitter IC is opportunistically constructed. In our OIA scheme, the concept of opportunistic beamforming is incorporated in IA. Each transmitter transmits with random beams and each user feeds back the distance between the two received interfering channels. The feedback information corresponds to a measure of interference alignment and is relatively small. At each transmitter, the user whose interfering channels are most aligned is selected and served. The proposed OIA scheme in this paper generalizes our preliminary study on OIA [12] by adopting multiple streams at each transmitter. Compared to this paper, [12] just considered two antennas at each transmitter and receiver, respectively, and assumed only a single stream from each transmitter.

Contrary to opportunistic beamforming in a MIMO broadcast channel [13], [14], random beams generated at a transmitter is not used for user selection at the transmitter, but helps other transmitters select their users. Once the users whose interfering channels are most aligned have been opportunistically selected, the selected users perform postprocessing with the equipped receive antennas. Because the postprocessing is performed after interference aligned user selection, only the selected user (as opposed to all users) needs to compute the postprocessing matrix. Correspondingly, the computational complexity is far reduced compared to conventional opportunistic user selection schemes where every user feeds back the signal-to-noise ratio (SNR), interference-to-noise ratio (INR), and the signal-to-interference-plus-noise ratio (SINR).

B. Contributions

We investigate the optimality of the OIA scheme in a three-transmitter $N_T \times N_R$ MIMO IC where $K$ users are associated with each transmitter and the selected users together with their transmitters construct a three-transmitter IC.

- If each receiver has $N_R = 2M$ antennas and the transmitter has more than $M$ antennas such that $N_T \geq M$, we prove that each transmitter achieves $\frac{N_T}{2} = M$ DoF without symbol extension by the OIA scheme as the
number of users in each user group, \( K \), goes to infinity. For a three-transmitter \( M \times 2M \) MIMO IC, \( \frac{2M}{3} \) DoF per user is known to be achievable (with symbol extension) according to the result given in (1). This result seems to be contradictory at first glance, but the required spatial dimensions for \( M \) data streams are secured through the user dimensions provided by the \( K \) users. This means that multiuser DoF are translated into IA spatial dimensions.

- We show that the number of users associated with each transmitter, \( K \), should scale as \( K \propto P^{\alpha M^2} \) to achieve \( M = \frac{N_R}{2} \) DoF per transmitter where \( \alpha \) is the relative path loss of the interfering channel in decibels.

- When \( K \) is finite, the achievable DoF of the OIA scheme is proved to be \( (1 - \alpha)M \). To achieve DoF \( M' \) such that \( (1 - \alpha)M \leq M' \leq M \), we show that the required number of users \( K \) should increase on the order of \( K \propto P^{(\alpha - \frac{M - M'}{M})M^2} \).

- Finally, we look into the practical advantages of the proposed OIA scheme; it is shown that the proposed OIA scheme significantly reduces not only the amount of feedback information but also computational complexity while achieving a notable rate improvement compared to conventional opportunistic user selection schemes.

C. Organization

The rest of this paper is organized as follows. In Section II, our system model is described. Preliminaries about the angles between two subspaces are provided in Section III. The proposed OIA scheme is described in Section IV, and the achievable rate and DoF are analyzed in Section V. Several conventional opportunistic user selection schemes are summarized and compared with OIA scheme in Section VI. The conclusions and comments on areas of future interest are given in Section VII.

D. Notations

Throughout the paper, \( A^\dagger \), \( \lambda_n(A) \), \( \mathbf{v}_n(A) \), \( tr(A) \) and \( \|A\|_F \) denote the conjugate transpose, \( n \)th largest eigenvalue, eigenvector corresponding to \( \lambda_n(A) \), trace, and Frobenius norm of matrix \( A \), respectively. Also, \( I_n \), \( \text{diag}(\cdot) \), \( \mathbb{C}^n \) and \( \mathbb{C}^{m \times n} \) indicate the \( n \times n \) identity matrix, a diagonal matrix whose diagonal elements are \( (\cdot) \), the \( n \)-dimensional complex space, and the set of \( m \times n \) complex matrices, respectively.

II. System Model

As depicted in Fig. [1], there are three transmitters having \( N_T \) antennas, and each transmitter has its own user group consisting of \( K \) users with \( N_R \) antennas each. In our system model, only a single user in each group is selected
and served by each transmitter. The transmitters and their selected users construct a three-transmitter MIMO IC. We assume that each transmitter sends $M (\leq N_T)$ streams and each user has $N_R (= 2M)$ antennas. The transmitters are assumed to have only partial CSI fed back from each user, and no information is shared among the transmitters.

In our system, user selection and transmission take the following four steps:

- **Step 1:** Each transmitter transmits with random beams.
- **Step 2:** Each user feeds back one analog value.
- **Step 3:** Each transmitter selects one user from its user group.
- **Step 4:** Each transmitter serves the selected user.

In the first step, the $i$th transmitter transmits its signal using a random beamforming matrix $W_i \in \mathbb{C}^{N_T \times M}$ such that $W_i^\dagger W_i = I_M$. The received signal at the $k$th user in the $i$th user group is modeled as

$$y_{i,k} = H_{i,k}^{(i)} W_i x_i + \sqrt{\gamma} \sum_{j=1,j\neq i}^{3} H_{i,k}^{(j)} W_j x_j + n_{i,k}$$

where $H_{i,k}^{(j)} \in \mathbb{C}^{N_R \times N_T}$ denotes the channel matrix from the $j$th transmitter to the $k$th user in the $i$th user group whose element is an independent and identically distributed (i.i.d.) and circularly symmetric complex Gaussian random variable with zero mean and unit variance. The transmit signal from the $i$th transmitter is denoted by $x_i \in \mathbb{C}^{M \times 1}$. Because a transmitter does not have CSI for power allocation, the same power is allocated for each stream and the transmitter power constraint is given by $E\{x_i x_i^\dagger\} = \frac{P}{M} I_M$. $n_{i,k} \in \mathbb{C}^{N_R \times 1}$ denotes Gaussian noise following a complex Gaussian distribution with zero mean and an identity covariance matrix $n_{i,k} \sim \mathcal{CN}(0, I_{N_R})$. $\gamma$ is a relative path loss from the interfering transmitter to each user and is assumed to be the same for all transmitters and users. We denote the received average interference power from each interfering transmitter as $P_I$ and define a constant $\alpha$ which is the relative propagation path loss of the interfering channel in decibels. Then, the received interfering power $P_I$ becomes $P_I = \gamma P = P^\alpha$, and it is satisfied that $\gamma = P^{\alpha - 1}$. We also assume the average interfering power is less than the desired signal power such that $P_I \leq P$ (equivalently, $0 \leq \alpha, \gamma \leq 1$). The effective channel from the $j$th transmitter to the $k$th user in the $i$th user group becomes $H_{i,k}^{(j)} \in \mathbb{C}^{N_R \times N_T}$. Because a random beamforming matrix $W_i$ is unitary and independent of $H_{i,k}^{(j)}$, each element of $H_{i,k}^{(j)}$ becomes an independent and identically distributed (i.i.d.) complex Gaussian random variable with zero mean and unit variance. At each user, the received signal can be equalized using the postprocessing matrix. We assume the postprocessing matrix at the $k$th user in the $i$th user group, denoted as $F_{i,k} \in \mathbb{C}^{M \times N_R}$, satisfies $F_{i,k} F_{i,k}^\dagger = I_M$. Then, the received signal after
postprocessing at the user becomes

\[ F_{i,k} y_{i,k} = F_{i,k} H_{i,k}^{(i)} x_i + \sqrt{\gamma} \sum_{j=1, j \neq i}^{3} F_{i,k} H_{i,k}^{(j)} x_j + F_{i,k} n_{i,k}, \]

and the achievable rate of the user is given by

\[ C_{i,k} = \log_2 \left| I_M + \frac{P}{M} F_{i,k} H_{i,k}^{(i)\dagger} F_{i,k}^\dagger \left( I_M + \frac{P^\alpha_M}{M} \sum_{j \neq i}^{M} F_{i,k} H_{i,k}^{(j)\dagger} H_{i,k}^{(j)} F_{i,k}^\dagger \right)^{-1} \right| \]

\[ = \log_2 \left| I_M + F_{i,k} \left( \frac{P}{M} F_{i,k} H_{i,k}^{(i)\dagger} H_{i,k}^{(i)} + \sum_{j \neq i}^{M} \frac{P^\alpha_M}{M} H_{i,k}^{(j)\dagger} H_{i,k}^{(j)} \right) F_{i,k}^\dagger \frac{I_M + \frac{P^\alpha_M}{M} \sum_{j \neq i}^{M} F_{i,k} H_{i,k}^{(j)\dagger} H_{i,k}^{(j)} F_{i,k}^\dagger}{I_M + \frac{P^\alpha_M}{M} \sum_{j \neq i}^{M} F_{i,k} H_{i,k}^{(j)\dagger} H_{i,k}^{(j)} F_{i,k}^\dagger} \right| \]  \hspace{1cm} (2)

In Step 2, each user feeds one analog value back to its own transmitter. The type of information needed to be fed back for user selection varies with the type of postprocessing and user selection scheme. In the OIA scheme, the chordal distance between the two subspaces spanned by the two interfering channels is used as feedback information. The details of feedback information construction will be given in Section IV. In Step 3, each transmitter selects a single user among the users in its user group. After user selection at each transmitter, the selected user is served by the random beams. If the \( k_i \)th user is selected in the \( i \)th transmitter, the average achievable sum rate of the system is obtained by

\[ E \sum_{i=1}^{3} C_{i,k_i}. \]

III. PRELIMINARIES – ANGLES BETWEEN TWO SUBSPACES SPANNED BY TWO INTERFERING CHANNELS

The Stiefel manifold and Grassmann manifold are widely used geometric concepts in communication systems [15]–[19]. The Stiefel manifold \( S_{N_R,M}(\mathbb{C}) \) (for \( N_R \geq M \)) is the set of all unitary matrices in \( \mathbb{C}^{N_R \times M} \), i.e.,

\[ S_{N_R,M}(\mathbb{C}) = \{ S \in \mathbb{C}^{N_R \times M} : S^\dagger S = I_M \}. \]

If the columns of two matrices \( S_1, S_2 \in S_{N_R,M}(\mathbb{C}) \) span the same subspace, \( S_1 \) and \( S_2 \) are defined to be equivalent [18]. As the quotient space of \( S_{N_R,M}(\mathbb{C}) \), the Grassmann manifold \( G_{N_R,M}(\mathbb{C}) \) is the set of all \( M \)-dimensional subspaces in an \( N_R \)-dimensional space \( \mathbb{C}^{N_R} \).

Consider two \( M \)-dimensional subspaces \( A, B \) in \( \mathbb{C}^{N_R} \), i.e., \( A, B \in G_{N_R,M} \). To define the angles between the subspaces, we use the principal angles, also called the canonical angles. The principal angles \( \theta_1, \ldots, \theta_M \in [0, \pi/2] \) between \( A \) and \( B \) are recursively defined by searching for \( N_R \)-dimensional vectors \( \{a_i\} \) and \( \{b_i\} \) such that [20] Chap. 12

\[ \cos \theta_i = \max_{a \in A} \max_{b \in B} |a^\dagger b| = |a_i^\dagger b_i| \]
subject to \(\|a\| = 1, \|b\| = 1, a^\dagger a_j = 0, b^\dagger b_j = 0 \ (1 \leq j \leq i - 1)\). The vectors \(\{a_i\}\) and \(\{b_i\}\) corresponding to the principal angles are called principal vectors of \(A\) and \(B\), respectively.

The distance between subspaces can be defined in many ways and chordal distance is arguably the most widely used one among them. The squared chordal distance between \(A\) and \(B\) (denoted by \(d^2_c(A, B)\)) is represented with the principal angles by

\[
d^2_c(A, B) \triangleq \sum_{i=1}^{M} \sin^2 \theta_i.
\]

Alternatively, the squared chordal distance can be represented with generator matrices; \(A \in \mathbb{C}^{N_R \times M}\) and \(B \in \mathbb{C}^{N_R \times M}\) are called generator matrices if their columns are orthonormal, respectively, i.e. \(A^\dagger A = B^\dagger B = I_M\), and span the subspaces \(A\) and \(B\), respectively. Thus, the generator matrices belong to the Stiefel manifold, i.e., \(A, B \in \mathcal{S}_{N_R, M}(\mathbb{C})\). Although generator matrices \(A\) and \(B\) are not unique, the generator matrices for \(A\) and \(B\) are equivalent in the Stiefel manifold, respectively. Also, the squared chordal distance between \(A\) and \(B\) is uniquely defined with arbitrary generator matrices \(A\) and \(B\) such that

\[
d^2_c(A, B) = \frac{1}{2} \|AA^\dagger - BB^\dagger\|^2_F = M - tr(A^\dagger BB^\dagger A).
\]

The principal angles and principal vectors can also be obtained from the generator matrices. Let the singular value decomposition (SVD) of \(A^\dagger B\) be denoted by \([20\text{ Chap. 12}]\)

\[
A^\dagger B = YDZ^\dagger
\]

where \(Y, Z \in \mathbb{C}^{M \times M}\) are unitary matrices and \(D = \text{diag}(\mu_1, \mu_2, \ldots, \mu_M)\) such that \(\mu_1 \geq \mu_2 \geq \ldots \geq \mu_M \geq 0\). Then, the \(i\)th singular value, \(\mu_i\), is given by

\[
\mu_i = \cos \theta_i
\]

where \(\theta_i\) is the \(i\)th principal angle and corresponding principal vectors \(a_i\) and \(b_i\) become, respectively,

\[
a_i = Ay_i, \quad b_i = Bz_i
\]

where \(y_i\) and \(z_i\) are the \(i\)th column vectors of \(Y\) and \(Z\), respectively.

In our system model, each user has two interfering channels, and each interfering channel spans \(M\)-dimensional subspace in \(\mathbb{C}^{N_R}\) with probability 1 because the elements of each \(N_R\)-dimensional interfering channel vector are
i.i.d. and circularly symmetric complex Gaussian random variables. Thus, the union of two subspaces formed by two interfering channels is a $2M (= N_R)$-dimensional space with probability 1 for the same reason.

**Lemma 1:** If two subspaces $A$ and $B$ are subspaces formed by interfering channels, the eigenvalues of $AA^\dagger + BB^\dagger$ can be represented with the principal angles such as

\[ 1 + \mu_1^2, \ldots, 1 + \mu_M^2, 1 - \mu_M^2, \ldots, 1 - \mu_1^2 \]  

where $A$ and $B$ are arbitrary generator matrices of $A$ and $B$, respectively.

**Proof:** See Appendix A.

### IV. Opportunistic Interference Alignment

This section describes the proposed OIA scheme. Without loss of generality, we consider the first transmitter and the users in the first user group. From (2), the achievable rate at the $k$th user in the first user group is given by

\[
C_{1,k} = \log_2 \frac{\left| \mathbf{I}_M + \mathbf{F}_{1,k} \left( \frac{\rho M}{\|\mathbf{H}_{1,k}^{(1)}\|} \mathbf{H}_{1,k}^{(1)\dagger} + \frac{\rho}{\|\mathbf{H}_{1,k}\|} \sum_{i=2}^{3} \mathbf{H}_{1,k}^{(i)} \mathbf{H}_{1,k}^{(i)\dagger} \right) \mathbf{F}_{1,k}^\dagger \right|}{\left| \mathbf{I}_M + \frac{\rho}{\|\mathbf{H}_{1,k}\|} \sum_{i=2}^{3} \mathbf{F}_{1,k} \mathbf{H}_{1,k}^{(i)} \mathbf{H}_{1,k}^{(i)\dagger} \mathbf{F}_{1,k}^\dagger \right|}. \tag{7}
\]

We decompose (7) into the rate gain term and the rate loss term denoted as $C_{1,k}^{\text{gain}}$ and $C_{1,k}^{\text{loss}}$, respectively, given as

\[
C_{1,k}^{\text{gain}} = \log_2 \left| \mathbf{I}_M + \mathbf{F}_{1,k} \left( \frac{\rho}{\|\mathbf{H}_{1,k}\|} \sum_{i=2}^{3} \mathbf{H}_{1,k}^{(i)} \mathbf{H}_{1,k}^{(i)\dagger} \right) \mathbf{F}_{1,k}^\dagger \right|, \tag{8}
\]

\[
C_{1,k}^{\text{loss}} = \log_2 \left| \mathbf{I}_M + \frac{\rho}{\|\mathbf{H}_{1,k}\|} \sum_{i=2}^{3} \mathbf{F}_{1,k} \mathbf{H}_{1,k}^{(i)} \mathbf{H}_{1,k}^{(i)\dagger} \mathbf{F}_{1,k}^\dagger \right|, \tag{9}
\]

and it is satisfied that $C_{1,k} = C_{1,k}^{\text{gain}} - C_{1,k}^{\text{loss}}$.

#### A. Feedback Information from Each User

If the interfering signals are approximately aligned in the $M$ dimensional space, the residual interference in the desired signal space will be negligible. The level of interference alignment can be effectively measured by the rate loss term after postprocessing when the user with the minimum rate loss is selected in the OIA scheme. If we assume that each user finds the postprocessing matrix to minimize the rate loss term, the minimum rate loss term at the $k$th user in the first user group becomes

\[
\min_{\mathbf{F}} C_{1,k}^{\text{loss}} = \min_{\mathbf{F}} \log_2 \left| \mathbf{I}_M + \frac{\rho}{\|\mathbf{H}_{1,k}\|} \sum_{i=2}^{3} \mathbf{F} \mathbf{H}_{1,k}^{(i)} \mathbf{H}_{1,k}^{(i)\dagger} \mathbf{F}^\dagger \right|,
\]

and the minimum occurs when

\[
\mathbf{F}_{1,k} = [v_{M+1}(\mathbf{B}_{1,k}), \ldots, v_{2M}(\mathbf{B}_{1,k})]^\dagger \tag{10}
\]
where $B_{1,k} = \sum_{i=2}^{3} H_{1,k}^{(i)} H_{1,k}^{(i)\dagger}$. If the user who has the minimum rate loss term is selected at the transmitter, the rate loss term at the selected user becomes

$$
\min_k \min_{F'} C_{1,k}^{\text{loss}} = \min_k \min_{F'} \log_2 \left| I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} F' H_{1,k}^{(i)} H_{1,k}^{(i)\dagger} F' \right|.
$$

In average sense, this rate loss term at the selected user is upper bounded as

$$
\mathbb{E}_{H_{1,k}^{(2)}, H_{1,k}^{(3)}} \left[ \min_k \min_{F'} C_{1,k}^{\text{loss}} \right] \leq \mathbb{E}_{H_{1,k}^{(2)}, H_{1,k}^{(3)}} \left[ \min_k \min_{F'} \mathbb{E}_{A_{1,k}, A_{1,k}} \left[ C_{1,k}^{\text{loss}} \right] \right] \leq \mathbb{E}_{H_{1,k}^{(2)}, H_{1,k}^{(3)}} \left[ \min_k M \log_2 \left( 1 + \frac{P^\alpha}{M} \sum_{m=M+1}^{2M} \lambda_m \left( C_{1,k} \right) \right) \right]
$$

where $A_{1,k}^{(i)}$ is a diagonal matrix comprising unordered eigenvalues of $H_{1,k}^{(i)} H_{1,k}^{(i)\dagger}$, $\tilde{H}_{1,k}^{(i)}$ is a unitary matrix constructed with corresponding eigenvectors, i.e. $H_{1,k}^{(i)} H_{1,k}^{(i)\dagger} = \tilde{H}_{1,k}^{(i)} A_{1,k}^{(i)} \tilde{H}_{1,k}^{(i)\dagger}$, and $C_{1,k} = \sum_{i=2}^{3} \tilde{H}_{1,k}^{(i)} \tilde{H}_{1,k}^{(i)\dagger}$. In the above equations, inequality $(a)$ holds because $\tilde{H}_{1,k}^{(i)}$ and $A_{1,k}^{(i)}$ are independent [19], and the average of the minimum values is smaller than the minimum of average values. Also, inequality $(b)$ is satisfied because

$$
\min_{F'} \mathbb{E}_{A_{1,k}, A_{1,k}} \left[ C_{1,k}^{\text{loss}} \right] = \min_{F'} \mathbb{E}_{A_{1,k}, A_{1,k}} \left[ \log_2 \left| I_M + \frac{P^\alpha}{M} F' \left( \sum_{i=2}^{3} \tilde{H}_{1,k}^{(i)} A_{1,k}^{(i)} \tilde{H}_{1,k}^{(i)\dagger} \right) F' \right| \right]
$$

$$
\leq \min_{F'} \log_2 \left| I_M + \frac{P^\alpha}{M} F' \left( \sum_{i=2}^{3} \tilde{H}_{1,k}^{(i)} \mathbb{E}[A_{1,k}^{(i)}] \tilde{H}_{1,k}^{(i)\dagger} \right) F' \right| \leq \log_2 \prod_{m=M+1}^{2M} \left( 1 + P^\alpha \lambda_m \left( C_{1,k} \right) \right)
$$

$$
\leq M \log_2 \left( 1 + \frac{P^\alpha}{M} \sum_{m=M+1}^{2M} \lambda_m \left( C_{1,k} \right) \right)
$$

where $\mathbb{E}[A_{1,k}^{(i)}] = M I_M$ [19]. In above equations, inequality $(c)$ is from Jensen’s inequality, and equality $(d)$ is when $F_{1,k} = [v_{M+1}(C_{1,k}), \ldots, v_{2M}(C_{1,k})]$. Also, inequality $(e)$ is from the concavity of logarithm function using Jensen’s inequality.

In order to avoid the large burden required by having all users compute their postprocessing matrices given by (10) for feedback and user selection, each user feeds back only the upper bound on its minimum rate loss given in (13) instead of the minimum rate loss itself. Correspondingly, each transmitter selects the user with the smallest rate loss bound. Note that the rate loss bound given in (13) does not depend on postprocessing allowing the computational complexity can be significantly reduced at each user. Consequently, the required feedback information from user $k$ associated with the first transmitter is given by

$$
\sum_{m=M+1}^{2M} \lambda_m \left( \tilde{H}_{1,k}^{(2)} \tilde{H}_{1,k}^{(2)\dagger} + \tilde{H}_{1,k}^{(3)} \tilde{H}_{1,k}^{(3)\dagger} \right).
$$
and the following theorem shows that the required feedback information is equivalent to the chordal distance between the two interfering channels.

**Theorem 1:** The required feedback information for the OIA scheme given in (14) is equivalent to the chordal distance between the two subspaces spanned by the two interfering channels such that

\[
\sum_{m=M+1}^{2M} \lambda_m \left( \tilde{H}_{1,k}^{(2)} \tilde{H}_{1,k}^{(2)\dagger} + H_{1,k}^{(3)} \tilde{H}_{1,k}^{(3)\dagger} \right) = d_c^2 \left( H_{1,k}^{(2)}, \tilde{H}_{1,k}^{(3)} \right).
\]

**Proof:** From Lemma 1, the eigenvalues of \( \tilde{H}_{1,k}^{(2)} \tilde{H}_{1,k}^{(2)\dagger} + H_{1,k}^{(3)} \tilde{H}_{1,k}^{(3)\dagger} \) in descending order become

\[
\frac{1 + \eta_1^2}{M}, \ldots, \frac{1 + \eta_M^2}{M}, \frac{1 - \eta_M^2}{M}, \ldots, \frac{1 - \eta_1^2}{M},
\]

where \( \eta_1 \geq \ldots \geq \eta_M \) correspond to the cosine of principal angles \( \phi_1, \ldots, \phi_M \) between the two subspaces generated by the columns of \( \tilde{H}_{1,k}^{(2)} \) and \( \tilde{H}_{1,k}^{(3)} \) (i.e., \( \eta_i = \cos \phi_i \)). Thus, the feedback information given in (14) becomes

\[
\sum_{i=1}^{M} (1 - \eta_i^2) = \sum_{i=1}^{M} \sin^2 \phi_i = d_c^2 \left( H_{1,k}^{(2)}, \tilde{H}_{1,k}^{(3)} \right)
\]

from the definition of chordal distance given in (3).

Thus, from the definition of chordal distance given in (4), the required feedback information from the \( k \)th user in the first user group becomes

\[
d_c^2 \left( H_{1,k}^{(2)}, \tilde{H}_{1,k}^{(3)} \right) = M - tr \left( \tilde{H}_{1,k}^{(2)\dagger} \tilde{H}_{1,k}^{(3)\dagger} \tilde{H}_{1,k}^{(2)} \tilde{H}_{1,k}^{(3)} \right)
\]

where \( \tilde{H}_{1,k}^{(2)\dagger} \) and \( \tilde{H}_{1,k}^{(3)\dagger} \) are the generator matrices of \( H_{1,k}^{(2)} \) and \( H_{1,k}^{(3)} \), respectively, and can be found via SVD or QR decomposition.

**B. User Selection at the Transmitter**

The first transmitter selects the user \( k_{1}^{\text{OIA}} \) who has the smallest feedback value such that

\[
k_{1}^{\text{OIA}} = \arg \min_{1 \leq k' \leq K} d_c^2 \left( H_{1,k'}^{(2)}, \tilde{H}_{1,k'}^{(3)} \right).
\]

Define a random variable \( D \) as the feedback value of the selected user which is equivalent to the chordal distance between the interfering channels of the selected user, i.e.,

\[
D = d_c^2 \left( H_{1,k_{1}^{\text{OIA}}}^{(2)}, \tilde{H}_{1,k_{1}^{\text{OIA}}}^{(3)} \right).
\]

Then, the average value of \( D \) is given in the following lemma.
Lemma 2: The average feedback value of the selected user, $E[D]$, is equivalent to the quantization error to quantize $\mathbf{H} \in \mathbb{C}^{N_R \times M}$ averaged over all possible random codebooks of size $K$ such that

$$E[D] = E\left[ \min_{\mathbf{W} \in \mathcal{C}_{\text{rand}}} d^2_2(\mathbf{H}, \mathbf{W}) \right]$$

(17)

where $\mathcal{C}_{\text{rand}} \subset \mathcal{G}_{N_R, M}(\mathbb{C})$ is a random codebook of size $K$ whose codewords are independently generated from the isotropic distribution on $\mathcal{G}_{N_R, M}(\mathbb{C})$.

Proof: See Appendix B. \hfill \blacksquare

It has previously been shown that the average quantization error to quantize a randomly distributed source on the Grassmann manifold $\mathcal{G}_{N_R, M}(\mathbb{C})$ with the random codebook $\mathcal{C}_{\text{rand}} \subset \mathcal{G}_{N_R, M}(\mathbb{C})$ of size $K$ is upper bounded by $\bar{D}$ such that [17]

$$E\left[ \min_{\mathbf{W} \in \mathcal{C}_{\text{rand}}} d^2_2(\mathbf{H}, \mathbf{W}) \right] \leq \bar{D},$$

where $\bar{D}$ is given by

$$\bar{D} = \frac{\Gamma(\frac{M}{2})}{M^2} (\eta K)^{-\frac{M}{2}} + M \exp \left[ - (\eta K)^{1-a} \right]$$

(18)

with $\eta = \frac{1}{\Gamma(M+1)} \prod_{i=1}^{M} \Gamma(2M-i+1) \Gamma(M-i+1)$, and $a \in (0, 1)$ is a real number chosen to satisfy $\eta K = 1$. Thus, we can conclude that the average feedback value from the selected user is upper bounded as

$$E[D] \leq \bar{D}.$$  

(19)

Note that the second term in (18) can be negligible for large $K$ [17]. For most engineering cases, the main order term of (18) is sufficiently accurate [17]–[19].

C. Postprocessing at the Selected User

As noted in Theorem [11] each user computes only the chordal distance between two interfering channels and feeds it back to its transmitter. Only the selected user, $k_{OIA}^1$, applies a postprocessing matrix to minimize the rate loss term in (10). The postprocessing matrix at the selected user becomes

$$F_{1,k_{OIA}^1} = \left[ v_{M+1}(B_{k_{OIA}^1}), \cdots, v_{2M}(B_{k_{OIA}^1}) \right] \dagger.$$  

(20)

where $B_{k_{OIA}^1} = H_{1,k_{OIA}^1} H_{1,k_{OIA}^1}^\dagger + H_{1,k_{OIA}^1} H_{1,k_{OIA}^1}^\dagger$.

It should be noted again that the complexity of the proposed OIA scheme can be significantly reduced because only the selected user applies a postprocessing matrix. The complexity of the OIA scheme is analyzed in the latter part.
V. Achievable Rate and Degrees of Freedom (DoF)

This section analyzes the achievable rate of the proposed OIA scheme and its DoF. Without loss of generality, the average achievable rate and a DoF of the first transmitter are derived. To simplify notations, we omit the user indices in the variables such as

\[ y \triangleq y_{1,k_1^{\text{OIA}}}, \quad H^{(i)} \triangleq H^{(i)}_{1,k_1^{\text{OIA}}}, \quad F \triangleq F_{1,k_1^{\text{OIA}}}, \quad n \triangleq n_{1,k_1^{\text{OIA}}}. \]

where \( F_{1,k_1} \) is the postprocessing matrix of the selected user given in (20). Using the simplified notations, the received signal after postprocessing at the selected user at the first transmitter is rewritten by

\[ Fy = FH^{(1)}x_1 + \sqrt{\gamma} \sum_{i=2}^{3} FH^{(i)}x_i + Fn. \]

When each group has \( K \) users, we denote the average achievable rate of the selected user using OIA scheme as \( R_{[K]}^{\text{OIA}} \), and the average rate gain term and the average rate loss term at this time as \( R_{[K]}^{\text{gain}} \) and \( R_{[K]}^{\text{loss}} \), respectively, i.e., \( R_{[K]}^{\text{OIA}} = EC_{1,k_1^{\text{OIA}}}, R_{[K]}^{\text{gain}} = EC_{1,k_1^{\text{OIA}}}^{\text{gain}}, \) and \( R_{[K]}^{\text{loss}} = EC_{1,k_1^{\text{OIA}}}^{\text{loss}}. \) Using the simplified notations, \( R_{[K]}^{\text{OIA}}, R_{[K]}^{\text{gain}}, \) and \( R_{[K]}^{\text{loss}} \) are represented as

\[
R_{[K]}^{\text{OIA}} = \mathbb{E} \log_2 \frac{I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} FH^{(i)}H^{(i)\dagger}F\dagger}{I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} FH^{(i)}H^{(i)\dagger}F\dagger},
\]

\[
R_{[K]}^{\text{gain}} = \mathbb{E} \log_2 \frac{I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} FH^{(i)}H^{(i)\dagger}F\dagger}{I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} FH^{(i)}H^{(i)\dagger}F\dagger},
\]

\[
R_{[K]}^{\text{loss}} = \mathbb{E} \log_2 \frac{I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} FH^{(i)}H^{(i)\dagger}F\dagger}{I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} FH^{(i)}H^{(i)\dagger}F\dagger},
\]

so it is satisfied that \( R_{[K]}^{\text{OIA}} = R_{[K]}^{\text{gain}} - R_{[K]}^{\text{loss}} \).

Theorem 2: When the number of users, \( K \), is finite, the achievable DoF of the proposed OIA scheme becomes \( (1 - \alpha)M \) such that

\[
\lim_{P \to \infty} \frac{R_{[K]}^{\text{OIA}}}{\log_2 P} = (1 - \alpha)M. \tag{23}
\]

Proof: We can directly derive the achievable DoF from (21) and (22). When \( K \) is finite, it is satisfied that \( \lim_{P \to \infty} \frac{R_{[K]}^{\text{gain}}}{\log_2 P} = M \) because the matrix scaled by \( P \) is dominant compared to the matrix scaled by \( P^\alpha \) (\( 0 \leq \alpha \leq 1 \)). Also, it is satisfied that \( \lim_{P \to \infty} \frac{R_{[K]}^{\text{loss}}}{\log_2 P} = \alpha M \) because \( H^{(2)}H^{(2)\dagger} + H^{(3)}H^{(3)\dagger} \) has non-zero \( M \) eigenvalues under the finite number of users with probability one. Thus, (23) holds from \( R_{[K]}^{\text{OIA}} = R_{[K]}^{\text{gain}} - R_{[K]}^{\text{loss}}. \)

Fig. 2 shows the average achievable rate of each user with the proposed OIA scheme according to \( \alpha \) when \( K = 10 \). As stated in Theorem 2, the achievable DoF of each user becomes \( (1 - \alpha)M \) when the number of users...
$K$ is finite. When $\alpha = 0$, the achievable rate linearly increases with $M$ so that full DoF $M$ is achieved only with finite $K$. When $\alpha = 1$, the achievable rate is saturated in the high SNR region and hence the achievable DoF becomes zero.

**Lemma 3:** When the number of users in the each group is $K$, the average rate loss term of the selected user, $R_{[K]}^{\text{loss}}$, is bounded as

$$R_{[K]}^{\text{loss}} \leq M \log_2 \left(1 + \frac{P^\alpha}{M \bar{D}}\right),$$

where $\bar{D}$ is given in (18).

**Proof:** See Appendix C.

**Theorem 3:** When the number of users per group goes to infinity ($K \to \infty$), the achievable rate of the selected user, $R_{[\infty]}^{\text{OIA}}$, behaves like

$$R_{[\infty]}^{\text{OIA}} = \frac{M}{N_R} \cdot C_{N_R \times \frac{N_T}{M} \times M}^{BD}(N_R \cdot P/M)$$

$$= \frac{1}{2} \cdot C_{2M \times 2 \times M}^{BD}(2P)$$

where $C_{N_R \times \frac{N_T}{M} \times M}^{BD}(N_R \cdot P/M)$ is the achievable rate of the block diagonalization system where a transmitter with $N_R$ antennas serves $\frac{N_T}{M}$ users with $M$ receive antennas each, and $M$ independent streams are transmitted for each user with power $\frac{P}{M}$ per stream under perfect CSIT.

**Proof:** See Appendix D.

**Corollary 1:** The achievable DoF of the selected user becomes $M$ as the number of users goes to infinity such that

$$\lim_{P \to \infty} \lim_{K \to \infty} \frac{R_{[K]}^{\text{OIA}}}{\log_2 P} = M.$$

**Proof:** Since the achievable DoF of a BD system is given by

$$\lim_{P \to \infty} \frac{C_{2M \times 2 \times M}^{BD}(2P)}{\log P} = 2M,$$

the equation (24) is easily obtained from Theorem 3.

Because the achievable DoF of each user in conventional IA is known to be at most $\frac{2M}{3}$ when $N_R = 2M$ and $N_T = M$, Theorem 2 and Corollary 1 appear contradictory at first glance. In case of conventional IA, only the spatial dimensions of the transmitter and the receiver are used for interference alignment. On the other hand, in the
TABLE I

REQUIRED NUMBER OF USERS TO MAINTAIN DOF $M = N_R/2$ WITH $R_{[^K]} \leq M$

| SNR  | $N_R = 2$          | $N_R = 4$          | $N_R = 6$          |
|------|-------------------|-------------------|-------------------|
| 5dB  | $(3.16)^\alpha$   | $(8.44)^\alpha$   | $(41.3)^\alpha$   |
| 10dB | $(10)^\alpha$     | $(8.44 \times 10^2)^\alpha$ | $(1.31 \times 10^6)^\alpha$ |
| 15dB | $(31.6)^\alpha$   | $(8.44 \times 10^3)^\alpha$ | $(4.13 \times 10^{10})^\alpha$ |
| 20dB | $(100)^\alpha$    | $(8.44 \times 10^6)^\alpha$ | $(1.31 \times 10^{15})^\alpha$ |
| 25dB | $(316)^\alpha$    | $(8.44 \times 10^8)^\alpha$ | $(4.13 \times 10^{19})^\alpha$ |

proposed OIA, additional user dimensions provided by $K$ users are also used to align interference. Therefore, the whole spatial dimensions $M$ can be secured and translated into achievable DoF.

**Lemma 4:** If we define $\Delta R(P)$ as the difference between the achievable rates for an infinite number of users and $K$ users, $\Delta R(P)$ is upper bounded on the average rate loss term (22) as

$$\Delta R(P) = R_{[\infty]}^{\text{OIA}} - R_{[K]}^{\text{OIA}} \leq R_{[K]}^{\text{loss}}.$$

**Proof:** It is satisfied that

$$\Delta R(P) = R_{[\infty]}^{\text{OIA}} - R_{[K]}^{\text{OIA}}$$

$$= \mathbb{E} \log_2 \left| I_M + \frac{P}{M} \mathbf{FH}^{(1)} \mathbf{H}^{(1)\dagger} \mathbf{F}^\dagger \right| - \mathbb{E} \log_2 \left| I_M + \frac{P}{M} \mathbf{FH}^{(1)} \mathbf{H}^{(1)\dagger} \mathbf{F}^\dagger + \frac{P^\alpha}{M} \sum_{i=2}^3 \mathbf{FH}^{(i)} \mathbf{H}^{(i)\dagger} \mathbf{P}^\dagger \right|$$

$$+ \mathbb{E} \log_2 \left| I_M + \frac{P^\alpha}{M} \sum_{i=2}^3 \mathbf{FH}^{(i)} \mathbf{H}^{(i)\dagger} \mathbf{P}^\dagger \right|$$

$$\leq \mathbb{E} \log_2 \left| I_M + \frac{P^\alpha}{M} \sum_{i=2}^3 \mathbf{FH}^{(i)} \mathbf{H}^{(i)\dagger} \mathbf{P}^\dagger \right|.$$  

This is because $\sum_{i=2}^3 \mathbf{H}^{(i)} \mathbf{H}^{(i)\dagger}$ is positive semi-definite so that

$$\lambda_m \left( \frac{P}{M} \mathbf{H}^{(1)} \mathbf{H}^{(1)\dagger} \right) \leq \lambda_m \left( \frac{P}{M} \mathbf{H}^{(1)} \mathbf{H}^{(1)\dagger} + \frac{P^\alpha}{M} \sum_{i=2}^3 \mathbf{H}^{(i)} \mathbf{H}^{(i)\dagger} \right).$$

Therefore, $\Delta R(P)$ is upper bounded by the average rate loss term. \hfill \blacksquare

**Theorem 4:** To achieve DoF $M$ per user, the number of users per each group, $K$, should increase as

$$K \gtrsim \frac{1}{\eta} \left[ \frac{\Gamma \left( \frac{1}{M^2} \right)}{M^3 - 2^{5/M - 1}} P^\alpha \right] M^2.$$  

(25)
That is, $K$ should scale like the interference power, $P^\alpha$, such that

$$K \propto P^{\alpha M^2}.$$  

Proof: See Appendix E.

In Table I the number of users to achieve DoF $M (= \frac{N_R}{2})$ per user is summarized when the average rate loss term is set to be less than $M$ (i.e., $R^\text{loss}_{[K]} = M$) which means a 3dB SNR gap between $R^\text{OIA}_{[\infty]}$ and $R^\text{OIA}_{[K]}$ in the high SNR region. When $N_R = 2$ and $\alpha = 0.8$, for example, 40 and 101 users are required at 20 and 25dB, respectively, to achieve DoF 1 ($= N_R/2$) per user. For the case that $N_R = 4$ and $\alpha = 0.5$, we need 85, 844 and 8438 users for 15dB, 20dB and 25dB, respectively, to achieve DoF of 2 ($= N_R/2$).

In Fig. 3 the achievable rate per transmitter with OIA schemes is plotted when the number of users is scaled. The antenna configuration and the relative path loss are given by $(N_T, M, N_R) = (1, 1, 2)$ and $\alpha = 1$, respectively. With the scaled number of users such as $K \propto P^{\alpha M^2}$, the achievable DoF is maintained as $M$. This is because the scaled number of users achieves a lower rate than an infinite number of users within a constant gap. Now the following theorem quantifies how the required number of users can be reduced as the target DoF per user decreases.

**Theorem 5:** To achieve DoF $M'$ per user such that

$$M - \alpha M \leq M' \leq M,$$

the number of users per each group, $K$, should be increased as

$$K \propto P^{(\alpha - \frac{M - M'}{M}) M^2}.$$  

Proof: If DoF $M'$ is achieved using OIA scheme with $K$ users, $M'$ can be represented as

$$\lim_{P \to \infty} \frac{R^\text{OIA}_{[K]}}{\log_2 P} = \lim_{P \to \infty} \frac{R^\text{OIA}_{[\infty]} - \Delta R(P)}{\log_2 P} = M - \lim_{P \to \infty} \frac{\Delta R(P)}{\log_2 P} = M'.$$

Thus, we need to find the number of users $K$ satisfying

$$\lim_{P \to \infty} \frac{\Delta R(P)}{\log_2 P} = M - M'.$$

$^1$The rate loss is set such that $\Delta R(P) = R^\text{OIA}_{[\infty]} - R^\text{OIA}_{[K]} \leq M \log_2(SNR) - M \log(SNR/2) = R^\text{loss}_{[K]}$.  



Since $\Delta R(P) \leq M \log_2 \left(1 + \frac{P^*}{M} \bar{D}\right)$,

$$\lim_{{P \to \infty}} \frac{M \log_2 \left(1 + \frac{P^*}{M} \bar{D}\right)}{\log_2 P} = M - M',$$

and substituting $\bar{D}$ in above equation yields

$$K \propto P^{(a - \frac{M - M'}{M}) M^2}.$$

VI. COMPARISON WITH CONVENTIONAL OPPORTUNISTIC USER SELECTION

OIA aligns interference by selecting the user whose interfering channels are most aligned. This section compares OIA scheme with conventional opportunistic user selection schemes, and the computational complexity of each scheme is analyzed.

A. Maximum SNR User Selection (MAX-SNR)

In maximum SNR user selection, each user maximizes the achievable rate ignoring the interfering channels. At the $k$th user in the first user group only considers its own channel and finds the postprocessing matrix such that

$$F_{1,k} = \arg \min_{F'} \log_2 \left| I_M + \frac{P}{M} F' H_{1,k}^{(1)} H_{1,k}^{(1)\dagger} F'\right|,$$

which becomes $F_{1,k} = [v_1(A_{1,k}), \ldots, v_M(A_{1,k})]^\dagger$ where $A_{1,k} = H_{1,k}^{(1)} H_{1,k}^{(1)\dagger}$. Using this simplification, the corresponding rate for the $k$th user is $\log_2 \prod_{m=1}^{M} \left(1 + \frac{P}{M} \lambda_m(A_{1,k})\right)$. Thus, the feedback information at the $k$th user becomes $\prod_{m=1}^{M} \left(1 + \frac{P}{M} \lambda_m(A_{1,k})\right)$, and the index of the selected user, $k_1$, becomes

$$k_1 = \arg \max_{1 \leq k' \leq K} \prod_{m=1}^{M} \left(1 + \frac{P}{M} \lambda_m(A_{1,k})\right).$$

B. Minimum INR User Selection (MIN-INR)

In minimum INR user selection, each user employs a postprocessing matrix that minimizes the rate loss term, $C_{1,k}^{loss}$, in (9). A postprocessing matrix at the $k$th user in the first user group is given by

$$F_{1,k}^{\text{INR}} = [v_{M+1}(B_{1,k}), \ldots, v_{2M}(B_{1,k})]^\dagger$$

where $B_{1,k} = H_{1,k}^{(2)} H_{1,k}^{(2)\dagger} + H_{1,k}^{(3)} H_{1,k}^{(3)\dagger}$. Then, corresponding $C_{1,k}^{loss}$ becomes

$$C_{1,k}^{loss} = \log_2 \prod_{m=M+1}^{2M} \left(1 + \frac{P}{M} \lambda_m(B_{1,k})\right).$$
Thus, the feedback information at user $k$ is $$\prod_{m=M+1}^{2M} \left( 1 + \frac{P\alpha}{M} \lambda_m(B_{1,k}) \right),$$ and the index of user selected at the first transmitter becomes

$$k_1 = \arg \min_{1 \leq k' \leq K} \prod_{m=M+1}^{2M} \left( 1 + \frac{P\alpha}{M} \lambda_m(B_{1,k}) \right).$$

C. Maximum SINR User Selection (MAX-SINR)

In maximum SINR user selection, the user offering the maximum rate is selected. Each user adopts the post-processing matrix to maximize the achievable rate, so the $k$th user at the first transmitter finds the postprocessing matrix to maximize $C_{1,k}$ given in (7).

When $M = 1$, the postprocessing vector $F_{1,k} \in \mathbb{C}^{N_R \times 1}$ at the $k$th user maximizing SINR such that

$$F_{1,k} = \arg \max_{F'} \frac{\left| F'(P_M H_{1,k} H_{1,k}^\dagger + \frac{P\alpha}{M} \sum_{i=2}^{3} H_{1,k}^{(i)} H_{1,k}^{(i)}\dagger) F'\right|}{1 + \frac{P\alpha}{M} \sum_{i=2}^{3} \left| F'H_{1,k}^{(i)} H_{1,k}^{(i)}\dagger F'\right|}$$

is obtained by solving a generalized eigenvalue problem and becomes

$$F_{1,k} = L_{1,k}^{-1} \cdot v_1 \left( I_{N_R} + \frac{P\alpha}{M} A_{1,k} L_{1,k}^{-1} \right)$$

where $L_{1,k}$ is Cholesky decomposition of positive semi-definite matrix $I_{N_R} + \frac{P\alpha}{M} B_{1,k}$ such that $L_{1,k}^\dagger L_{1,k} = I_{N_R} + \frac{P\alpha}{M} B_{1,k}$. At this time, corresponding $C_{1,k}$ becomes $\log_2 \left( 1 + \lambda_1 \left( (L_{1,k}^{-1})^\dagger \frac{P}{M} A_{1,k} L_{1,k}^{-1} \right) \right)$.

When $M \geq 2$, the postprocessing matrix $F_{1,k}$ at the $k$th user to maximize $C_{1,k}$ such that

$$F_{1,k} = \arg \max_{F'} \log_2 \frac{\left| I_M + F' \left( P_M H_{1,k} H_{1,k}^\dagger + \frac{P\alpha}{M} \sum_{i=2}^{3} H_{1,k}^{(i)} H_{1,k}^{(i)}\dagger \right) F'\right|}{\left| I_M + \frac{P\alpha}{M} \sum_{i=2}^{3} F'H_{1,k}^{(i)} H_{1,k}^{(i)}\dagger F'\right|}$$

is hard to find and generally unknown.

D. Time Division Multiplexing (TDM)

In this scheme, only one of the three transmitters serves its selected user at any point of time. Therefore, the selected user does not receive any interference from other transmitters. Each user finds the postprocessing matrix to maximize the achievable rate, so the postprocessing matrix at the transmitter is the same with the postprocessing matrix in MAX-SNR scheme given in (26). Also, the feedback information at the user and the user selection criterion are exactly the same as those of MAX-SNR scheme. Because only one selected user is exclusively served by the TDM approach, the achievable DoF per transmitter becomes $\frac{M}{3}$. 
TABLE II

The complexity of various operations for $G \in \mathbb{C}^{m \times n}$

| Operation                          | Complexity (flops) |
|------------------------------------|--------------------|
| $\alpha G, G + G$                  | $2mn$              |
| $\|G\|_F$                          | $4mn$              |
| $G \otimes G = GG^\dagger$        | $8mn^2 - 2mn$      |
| Gram-Schmidt Ortho.               | $8mn^2 - 2mn$      |
| Singular Value Decomp.            | $24m^2n + 48mn^2 + 54n^3$ |

TABLE III

The complexity of various schemes

| Scheme   | Complexity                                                                 | Ratio $(K, N_R \to \infty)$ |
|----------|----------------------------------------------------------------------------|-----------------------------|
| MAX-SNR  | $K \times (128N_R^3 - N_R^2 + \frac{3}{2}N_R)$                            | 98.4%                       |
| MIN-INR  | $K \times (130N_R^3 + 3N_R^2 + \frac{3}{2}N_R)$                           | 100%                        |
| OIA      | $K \times (8N_R^3 + 2N_R^2) + (130N_R^3 + 3N_R^2)$                       | 6.15%                       |

E. Complexity Analysis

The computational complexity of each scheme is represented by the number of floating point operations (flops) [20, Chap. 1]. An addition, multiplication, or division of real numbers is counted as one flop, so a complex addition and multiplication are counted as two flops and six flops, respectively. For an $m \times n$ complex matrix $G \in \mathbb{C}^{m \times n}$ ($m \geq n$), the flops required for several matrix operations are summarized in Table II where the operation $\otimes$ is defined as $G \otimes G = GG^\dagger$.

In MAX-SNR user selection, each user requires one $\otimes$ operation, a single SVD, $2M$ real additions and $M$ real multiplications to find feedback information. Correspondingly, the total computational complexity becomes $K \times (8N_RM^2 - 2N_RM) + K \times (24N_R^3 + 48N_R^3 + 54N_R^3) + K \times 3M = K \times (128N_R^3 - N_R^2 + \frac{3}{2}N_R)$ flops. In MIN-INR user selection, two $\otimes$ operations, two matrix scaling, a single matrix addition, a single SVD, $2M$ real additions, and $M$ real multiplications are required at each user to find the feedback information, so the total computational complexity becomes $K \times 2(8N_RM^2 - 2N_RM) + K \times 2N_R^2 + K \times 2N_R^2 + K \times (24N_R^3 + 48N_R^3 + 54N_R^3) + K \times 3M = K \times (130N_R^3 + 3N_R^2 + \frac{3}{2}N_R)$ flops. Note that the postprocessing matrix should be calculated to find feedback information both in MAX-SNR and MIN-INR schemes. On the other hand, the proposed OIA scheme requires two Gram-Schmidt orthogonalization, two $\otimes$ operations, one matrix addition, and a single $\| \cdot \|_F$ operation to construct feedback information. The selected user needs $130N_R^3 + 3N_R^2$ additional complexity to find the postprocessing.
matrix, $F^{\text{INR}}_k$. Therefore, the total complexity of OIA scheme becomes $K \times 4(8N_RM^2 - 2N_RM) + K \times 2N_R^2 + K \times 4N_R^2 + (130N_R^3 + 3N_R^2) = K \times (8N_R^3 + 2N_R^2) + (130N_R^3 + 3N_R^2)$. 

The computational complexity of various schemes is summarized in Table III and plotted according to the number of users, $K$, in Fig. 4 when $N_R = 4$. The complexity of the proposed OIA scheme is just about 6.15% of MIN-INR user selection scheme’s complexity when the number of receive antennas, $N_R$, and the number of users, $K$, are sufficiently large.

F. Performance Comparison

In Fig. 5, the proposed OIA scheme is compared with other user selection schemes in terms of achievable rate per transmitter when $(N_T, M, N_R) = (1, 1, 2)$. The relative interfering channel power $\alpha$, and the number of users, $K$, are set to be 1 and 50, respectively. It is shown that the proposed OIA scheme achieves a similar rate to the conventional MIN-INR user selection scheme while it requires much less computational complexity than MIN-INR user selection. It is also shown that the proposed OIA scheme significantly outperforms the conventional MAX-SNR user selection in high SNR region. For a finite number of users, the achievable rates using MAX-SINR, MIN-INR, and OIA schemes are saturated in high SNR region. On the other hand, the TDM scheme achieves a DoF of $\frac{1}{3}$ and outperforms OIA above 50dB SNR.

Fig. 6 shows sum rates of the proposed OIA scheme and the conventional user selection schemes for the antenna configuration $(N_T, M, N_R) = (2, 2, 4)$. The number of users, $K$, and $\alpha$ are set to be 50 and 0.7, respectively. Similar trends can be observed to Fig. 5 although the sum rates are greater than those in Fig. 5 due to the increased spatial dimensions. As predicted in Theorem 2, the achievable DoF becomes $0.3 (= 1 - \alpha)$ in high SNR region.

VII. CONCLUSION

We have proposed a novel opportunistic interference alignment (OIA) and analyzed its optimality in terms of the achievable DoF in the three-transmitter MIMO IC channels. When $N_R = 2M$ and $N_T \geq M$, the proposed OIA scheme has been shown to achieve $M$ DoF per user by opportunistically selecting the user whose received interference signals are most aligned with each other. Contrary to conventional IA which is known to achieve $\frac{2M}{3}$ DoF per user in a three-transmitter $M \times 2M$ MIMO IC, the proposed OIA scheme does not sacrifice the spatial dimensions in aligning interference signals and secures the full spatial DoF by exploiting the user dimensionality for interference alignment. Furthermore, the proposed OIA scheme does not require global channel knowledge at the transmitters but needs only scalar value feedback from each user for user selection. We have also proved that the
full DoF can be achieved even with a finite number of users if the number of users grows with an appropriate scale. Finally, the proposed OIA scheme has been shown to have advantages over conventional user selection schemes for interference mitigation in terms of both computational complexity and achievable rate.

APPENDIX A

PROOF OF LEMMA [1]

Using the unitary matrices \( Y \) and \( Z \) in [5], \( AA^\dagger + BB^\dagger \) can be rewritten as

\[
AA^\dagger + BB^\dagger = AY (AY)^\dagger + BZ (BZ)^\dagger
\]

\[
= \sum_{i=1}^{M} \left( a_i a_i^\dagger + b_i b_i^\dagger \right). \tag{A.1}
\]

Also, we decompose \( b_i \) as

\[
b_i = \cos \theta_i a_i + \sin \theta_i e_i, \tag{A.2}
\]

where \( \theta_i \) is the \( i \)th principal angle and \( e_i \) is a unit vector (\( \|e_i\| = 1 \)) orthogonal with \( a_i \) such that \( e_i \perp a_i \).

From the property of principal vectors, \( a_i \perp a_j \) and \( b_i \perp b_j \) for \( i \neq j \). Also, from the relationship between the principal angle and the principal vector given in (5), it is satisfied that

\[
(AY)^\dagger BZ = [a_1, \ldots, a_M]^\dagger [b_1, \ldots, b_M] = D
\]

where \( D = \text{diag}(\mu_1, \mu_2, \ldots, \mu_M) \) is diagonal matrix, so \( a_i \perp b_j \) for \( i \neq j \). Thus, it is satisfies that \( \text{span}(a_i, b_i) \perp \text{span}(a_j, b_j) \), for \( i \neq j \) and equivalently, \( \text{span}(a_i, e_i) \perp \text{span}(a_j, e_j) \) for \( i \neq j \). Using this and from the fact that \( a_i \perp e_i \), we can conclude that \( \{a_1, \ldots, a_M, e_1, \ldots, e_M\} \) becomes \( 2M \) orthonormal bases of \( \mathbb{C}^{2M} \).

Using (A.2), \( b_i b_i^\dagger \) becomes

\[
b_i b_i^\dagger = (\cos \theta_i a_i + \sin \theta_i e_i)(\cos \theta_i a_i + \sin \theta_i e_i)^\dagger
\]

\[
= \cos^2 \theta_i \cdot a_i a_i^\dagger + \sin^2 \theta_i \cdot e_i e_i^\dagger
\]

\[
= \mu_i^2 a_i a_i^\dagger + (1 - \mu_i^2) e_i e_i^\dagger,
\]

and (A.1) can be rewritten as

\[
AA^\dagger + BB^\dagger = \sum_{i=1}^{M} \left( a_i a_i^\dagger + b_i b_i^\dagger \right)
\]

\[
= \sum_{i=1}^{M} \left[ (1 + \mu_i^2) a_i a_i^\dagger + (1 - \mu_i^2) e_i e_i^\dagger \right].
\]

Thus, \( AA^\dagger + BB^\dagger \) has the eigenvectors \( \{a_i\} \) and \( \{e_i\} \), and has ordered eigenvalues \( \{\mu_i^2\} \).
Consider the unitary matrix $U_k = \tilde{H}_{1,k}^{(2)} H_{1,k}^{(2)\dagger}$ satisfying $U_k^\dagger U_k = U_k^\dagger U_k = I_{N_r}$. Because the chordal distance between two subspace is invariant with a rotation, the chordal distance between two subspaces spanned by two interfering channels at the $k$th user in the first user group satisfies

$$d_c^2(\tilde{H}_{1,k}^{(2)}, \tilde{H}_{1,k}^{(3)}) = d_c^2(U_k \tilde{H}_{1,k}^{(2)}, U_k \tilde{H}_{1,k}^{(3)})$$

$$= d_c^2(\tilde{H}_{1,k}^{(2)}, U_k \tilde{H}_{1,k}^{(3)}).$$

At the selected user $k_1^{\text{OIA}}$, the chordal distance between two interfering channels can be represented as

$$d_c^2(\tilde{H}_{1,k_1^{\text{OIA}}}^{(2)}, \tilde{H}_{1,k_1^{\text{OIA}}}^{(3)}) = \min_{k'} d_c^2(\tilde{H}_{1,k_1}^{(2)}, \tilde{H}_{1,k_1}^{(3)})$$

$$= \min_{k'} d_c^2(U_{k'} \tilde{H}_{1,k_1}^{(2)}, U_{k'} \tilde{H}_{1,k_1}^{(3)})$$

$$= \min_{k'} d_c^2(\tilde{H}_{1,k_1^{\text{OIA}}}^{(2)}, U_{k'} \tilde{H}_{1,k_1}^{(3)})$$

$$= \min_{\mathbf{w} \in C_{\text{rand}}} d_c^2(\tilde{H}_{1,k_1^{\text{OIA}}}^{(2)}, \mathbf{w}).$$ (B.1)

where $C_{\text{rand}} = \{ U_1 \tilde{H}_{1,1}^{(3)}, \ldots, U_K \tilde{H}_{1,K}^{(3)} \}$ is a random codebook of size $K$ whose elements are i.i.d. and isotropic in $G_{N_r,M}(\mathbb{C})$. After simple manipulations and taking average on both side of (B.1), we obtain (17).

APPENDIX C

PROOF OF LEMMA 3

Because $\mathbf{F}$ given in (20) minimizes the rate loss term, the average rate loss term $R_{[K]}^{\text{loss}}$ can be rewritten and upper bounded on

$$R_{[K]}^{\text{loss}} = \mathbb{E}_{\mathbf{H}(i)} \min_{\mathbf{F}} \log_2 |I_M + \frac{P_\alpha}{M} \sum_{i=2}^3 \mathbf{F}' \tilde{\mathbf{H}}^{(i)} \Lambda^{(i)} \tilde{\mathbf{H}}^{(i)\dagger} \mathbf{F}'\dagger|$$

$$\leq \mathbb{E}_{\mathbf{H}(i)} \min_{\mathbf{F}} \log_2 |I_M + \frac{P_\alpha}{M} \sum_{i=2}^3 \mathbf{F}' \tilde{\mathbf{H}}^{(i)} \mathbb{E}_{\Lambda^{(i)}} \{ \Lambda^{(i)} \tilde{\mathbf{H}}^{(i)\dagger} \mathbf{F}'\dagger \}$$

$$\leq \mathbb{E}_{\mathbf{H}(i)} \ M \log_2 \left( 1 + \frac{P_\alpha}{M} \sum_{m=M+1} \lambda_m (\mathbf{C}) \right)$$

$$\leq M \log_2 \left( 1 + \frac{P_\alpha}{M} \bar{D} \right)$$

where $\mathbf{C} = \sum_{i=2}^3 \tilde{\mathbf{H}}^{(i)} \tilde{\mathbf{H}}^{(i)\dagger}$. In above equations, inequality (a) is because of the independency between $\tilde{\mathbf{H}}^{(i)}$ and $\Lambda^{(i)}$ and Jensen’s inequality. In the OIA scheme, both the feedback information at each user (Theorem 1) and
the user selection \( \{15\} \) are independent with \( \Lambda^{(i)} \), so it is still satisfied that \( \mathbb{E}[\Lambda^{(i)}] = M I_M \). Thus, inequality (b) holds by the same reasons with \( \{11\} \rightarrow \{13\} \). The inequality (c) can be easily shown with Theorem \( \{14\} \rightarrow \{16\} \), Jensen’s inequality, and \( \{19\} \).

**APPENDIX D**

**PROOF OF THEOREM 3**

The term \( R_{[K]}^{\text{gain}} \) in \( \{8\} \) has lower and upper bounds such that

\[
R_{[K]}^{\text{gain}} \geq \mathbb{E} \log_2 \left| I_M + \frac{P}{M} F H(1)^\dagger H(1)^\dagger F \right| \tag{D.1}
\]

and

\[
R_{[K]}^{\text{gain}} \leq \mathbb{E} \log_2 \left| \left( I_M + \frac{P}{M} F H(1)^\dagger H(1)^\dagger F \right) \left( I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \right) \right| = \mathbb{E} \log_2 \left| I_M + \frac{P}{M} F H(1)^\dagger H(1)^\dagger F \right| + R_{[K]}^{\text{loss}}. \tag{D.2}
\]

Because both \( F H(1)^\dagger H(1)^\dagger F \) and \( \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \) are positive semi-definite matrices, the lower bound (D.1) is obtained by neglecting the interference term \( \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \). The upper bound given in (D.2) is obtained because

\[
\left| I_M + \frac{P}{M} F H(1)^\dagger H(1)^\dagger F \right| + \frac{P^\alpha}{M} \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \leq \left| I_M + H(1)^\dagger F \right| \left| I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \right|.
\]

Note that the inequality (a) holds since

\[
H(1)^\dagger F H(1) - H(1)^\dagger F \left( I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \right)^{-1} H(1)
\]

is positive semi-definite so that \( \lambda_m (H(1)^\dagger F H(1)) \) is bigger than \( \lambda_m \left( H(1)^\dagger F \left[ I_M + \frac{P^\alpha}{M} \sum_{i=2}^{3} F H(i)^\dagger H(i)^\dagger F \right]^{-1} H(1) \right) \) for all \( m \) \((1 \leq m \leq M)\).

In Lemma 3 the rate loss term, \( R_{[K]}^{\text{loss}} \), becomes zero as \( K \rightarrow \infty \) because \( \lim_{K \rightarrow \infty} \tilde{D} = 0 \). Applying the squeeze theorem for the two bounds, we can easily show that

\[
\lim_{K \rightarrow \infty} R_{[K]}^{\text{OIA}} = \mathbb{E} \log_2 \left| I_M + \frac{P}{M} F H(1)^\dagger H(1)^\dagger F \right|
\]

where \( F \) is independent of \( H(1) \) and uniformly distributed in \( \mathbb{C}^{M \times N_R} \) so that the distribution of \( F \) is the same with the distribution of the postprocessing matrix of a user in the \( N_R \times \frac{N_R}{M} \times M \) BD system.
APPENDIX E

PROOF OF THEOREM 4

As noted in Corollary 4, the DoF $M$ is achievable as $K \to \infty$. Thus, if we maintain $\Delta R(P)$ within a constant, we can achieve DoF $M$ even with a finite $K$. Using Lemma 3 and Lemma 4 we can make $\Delta R(P)$ less than $\delta$ such that

$$\Delta R(P) = R^{OIA}_{[\infty]} - R^{OIA}_{[K]} \leq R^{loss}_{[K]} \leq M \log_2 \left(1 + \frac{P^\alpha}{M \bar{D}}\right) \leq \delta.$$ 

Thus, we can achieve DoF $M$, if the upper bound of rate loss term is maintained within a constant $\delta$ such that

$$M \log_2 \left(1 + \frac{P^\alpha}{M \bar{D}}\right) \leq \delta.$$ 

Substitute the main order term of $\bar{D}$ in (18) in above equation, we obtain $K$ to maintain the DoF as in (25).

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Fig. 1. The system model considered in this paper is shown above. Each multiple antenna transmitter selects one user from its group.

Fig. 2. The achievable rate per user of OIA scheme varying $\alpha$ when $(N_T, M, N_R) = (2, 2, 4)$.
Fig. 3. The achievable rate per user of OIA scheme with scaling $K \propto P$ when $(N_T, M, N_R) = (1, 1, 2)$ and $\alpha = 1$.

Fig. 4. Complexity of various user selection scheme according to the number of users $K$ when $N_R = 4$. 
Fig. 5. The achievable rate per user of various user selection schemes according to the SNR when \((N_T, M, N_R) = (1, 1, 2), \alpha = 1\) and \(K = 50\).

Fig. 6. The achievable rate per user of various user selection schemes according to the SNR when \((N_T, M, N_R) = (2, 2, 4), \alpha = 0.7\) and \(K = 50\).