LETTER TO THE EDITOR

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1. Dedication. This Letter is dedicated to the 50th anniversary of unexpected death of Samuel Stanley Wilks. To exact distribution of His, Wilks's, statistics first author devoted his "Lambert W research" in 2000-2003.

2. Introduction. In 1938, Samuel Stanley Wilks proved the $\chi^2$-asymptotics of $-2\ln\Lambda$, where $\Lambda$ is the likelihood ratio statistics in regular exponential family (see Wilks S.S. (1938) [27]). But how does the exact CDF of $-2\ln\Lambda$ look like? Stehlík M. (2003) [20], derived the exact cumulative distribution function of $-2\ln\Lambda$ and decomposition of Kullback-Leibler-divergence ($\text{I}$-divergence) in the sense of Pázman A. (1993) [18], by substantial usage of Lambert W function, firstly introduced by Johann Heinrich Lambert in 1758 (see [17]), a contemporary of Euler. The paper by Goerg G. M. (2011) [11], "Lambert W random variables-a new family of generalized skewed distributions with applications to risk estimation", introduced a class of so called Lambert W×F random variables,

\[ Y_\gamma := X \exp(\gamma X), \quad (2.1) \]

where $\gamma \in R$ is skewness parameter and $X$ is continuous random variable. Stehlík M. (2003) [20] derived the exact distribution of Wilks statistics $-2\ln\Lambda$ to test for the scale hypothesis $H_0 : \theta = \theta_0$ versus $H_1 : \theta \neq \theta_0$ in the regular Gamma family and proven that Wilks statistics $-2\ln\Lambda$ is a function of a random variable

\[ G_u(X) = X - u \ln(X), \quad (2.2) \]

where $X$ is random variable from exponential family. Here, notice that

\[ -u \ln Y_{-\frac{1}{u}} = G_u(X), \text{ for } X > 0 \quad (2.3) \]

where $Y_\gamma$ of Goerg G. M. (2011) [11] is defined by (2.1) and $\gamma = -\frac{1}{u}$. The statistical application of the class (2.1) and "Lambert W function" is intrinsically related to $\text{I}$-divergence decompositions and the importance they play
in statistical inference. Stehlík M. (2003) [20] derived that Kullback-Leibler divergence in the sense of Pázman (1993) [18] has the form

$$I_N(y, \theta) = \sum_{i=1}^{N} \{G_u(\theta y_i) - G_u(u)\},$$

(2.4)

\(y = (y_1, ..., y_N)\). Notice, that \(I_1(X, 1) = G_u(X) - G_u(u)\), is the “basic” information of LR test, based on just a single random variable \(X\), directly relating nonlinearly transformed \(Y_\gamma\) of Goerg G. M. (2011) [11] to \(G_u(X)\) of Stehlík M. (2003) [20] (see (2.3)). In Stehlík M. (2006) [21] and Stehlík M. (2008) [22] extension of results to Weibull and generalized Gamma distributions (Ggds) was made. Considered Ggd covers for various choices of parameters of one-sided normal, \(\chi_2^n\), Weibull and in the limit a log-normal distribution. The LW function approach based on \(G_u(X)\) transformation was used for exact inference for Pareto heavy tailed distribution in Stehlík M. et al. (2010) [23]. The LW function approach and \(G_u(X)\) transformation was used fundamentally in Balakrishnan and Stehlík (2008) [1] for extension of results also to cases of Type I and Type II censored samples and missing data.

In this letter we discuss several important methodological and practical aspects of Lambert W variable. According Goerg G. M. (2011) [11] the Lambert W framework is a new generalized way to analyze skewed, heavy-tailed data. In the next two sections we discuss both, heavy-tails and skewness perspectives of this Lambert W framework.

In the next section “Heavy Tails: On three regimes of IGMM-algorithm”, based on heavy-tailedness we define three Regimes of Goerg G. M. (2011)’s Algorithm 3, and its implementation IGMM in R-package LambertW. However, current implementation of algorithm 3 cannot work in all three Regimes. In Regime III, where no moments of financial data exist, we show that IGMM is not working. Based on simple graphical method we give a practical guidelines how to discriminate between regimes. Also we provide tool based on robust tests for normality against heavy tails for a better linking of a given data to Regimes. The introduced methodology is illustrated on LATAM data, used by Goerg G. M. (2011). Also suggestion for correction of Algorithm 3 in Regime III is provided.

In the section ”Skewness: On asset Returns and t-distribution” we discussed difficulties with symmetrization of data, based on transformation introduced by Goerg G. M. (2011) [11].

3. Heavy Tails: On three regimes of IGMM-algorithm. In this section we describe three regimes of iterative method of moments introduced
by Goerg G. M. (2011) \cite{11} (IGMM-Method). The description is based on approximations by random walk, respectively to heavy-tailedness of input variable $U$. Such a description is important, in particular for applicability of Algorithm 3 to any financial data, e.g. LATAM data. The three regimes are defined as follows:

1. **Regime I:** distributions $U$ with finite mean and finite variance (here belongs e.g. student-$t_\nu$-distribution with $\nu > 2$)
2. **Regime II:** distributions $U$ with finite mean but infinite variance (here belongs e.g. student-$t_\nu$-distribution with $1 < \nu \leq 2$)
3. **Regime III:** distributions $U$ with infinite mean and infinite variance (here belongs e.g. student-$t_\nu$-distribution with $0 < \nu \leq 1$).

We are showing that algorithm which works in Regime I (because of Strong-Law of Large Numbers) cannot work well in Regime III, since statistical learning in Regime I is related to arithmetic mean, whereas in Regime III to harmonic mean (see Beran, Schell, and Stehlík (2014) \cite{2}). Before any further methodological discussion we provide illustration of computation with IGMM-method for the three regimes. Since in subexponential family Pareto tail is well fitting to student-$t_\nu$ (used also in Goerg G. M. (2011) \cite{11}), we simulate student-$t_\nu$-distribution with $\nu = 1, 1.5, 5$ degrees of freedom, to represent all three regimes. In these regimes we study sensitivity of parameter estimation of $\mu$, $\sigma$ and $\gamma$ of implemented function IGMM. The procedure for this sensitivity check is conducted as follows:

1. Simulating a sample $U$ from student or Pareto distribution for all three regimes
2. Transformation of $Y = U \cdot \exp(\gamma \cdot U)\sigma + \mu$ for all samples $U$
3. Estimation of parameters ($\mu, \sigma, \gamma$) for transformed samples by usage of IGMM.
4. Repeat steps 2.-3. for a different values of $\gamma$.

The calculated differences between the true values and their estimators are shown in Table 1. Higher degrees of freedom lead to better approximations of the parameters. Deviations are higher for increasing $\gamma$ and lower degrees of freedom. Due to increasing deviations for smaller $\nu$ it can be assumed that IGMM-method works acceptably for student-$t_\nu$-distribution of Regime I, deviations are larger for Regime II and astronomical deviations are received in Regime III. The similar results are obtained for Pareto distribution. Astronomical deviations of estimation with heavy-tailed distributions $\alpha = 1, 1.5$ are similar to those of student distributions of Regime II and III.

The astronomical discrepancies in Regime III (the case where no finite expectation exists), i.e. $0 < \nu \leq 1$ are theoretically explained by law of large
Estimation of parameters with IGMM() for $U$ having student-$t_\nu$ or Pareto-$\alpha$ distribution

| Student $t_\nu$-distribution | Pareto-$\alpha$ distribution |
|-----------------------------|----------------------------|
| $\nu$ | $\mu - \hat{\mu}$ | $\gamma - \hat{\gamma}$ | $\frac{s}{\hat{s}}$ | $\alpha$ | $\mu - \hat{\mu}$ | $\gamma - \hat{\gamma}$ | $\frac{s}{\hat{s}}$ |
| $\mu = 0.2$ | $\gamma = 0.1$ | $\sigma = 1.5$ | $\mu = 0.2$ | $\gamma = 0.1$ | $\sigma = 1.5$ |
| 5 | 0.0201 | 0.0182 | 1.2535 | 5 | 1.9479 | 0.3433 | 0.2331 |
| 1.5 | 0.6061 | 0.3993 | 5.5353 | 1.5 | -5.25-10^6 | 11.53 | 4.29-10^8 |
| 1 | -1.51-10^10 | 11.6649 | 0.0000 | 1 | 5.24-10^24 | 0.1504 | 1.12-10^26 |
| $\mu = 0.2$ | $\gamma = 0.3$ | $\sigma = 1.5$ | $\mu = 0.2$ | $\gamma = 0.2$ | $\sigma = 1.5$ |
| 5 | 0.0449 | 0.0531 | 1.2054 | 5 | 2.1923 | -0.2561 | 2.9713 |
| 1.5 | 1.58-10^12 | -0.0494 | 3.36-10^13 | 1.5 | 4.49-10^11 | 0.0501 | 9.54-10^12 |
| 1 | 2.84-10^34 | -0.0496 | 6.04-10^35 | 1 | 7.74-10^52 | 0.0504 | 1.64-10^54 |
| $\mu = 0.2$ | $\gamma = 0.5$ | $\sigma = 1.5$ | $\mu = 0.2$ | $\gamma = 0.25$ | $\sigma = 1.5$ |
| 5 | 0.1151 | 0.0445 | 1.2027 | 5 | 2.3260 | 0.2121 | 0.3343 |
| 1.5 | 1.29-10^23 | -0.2494 | 2.74-10^24 | 1.5 | 3.71-10^15 | 0.0004 | 7.89-10^16 |
| 1 | 8.99-10^59 | -0.2497 | 1.91-10^61 | 1 | 9.38-10^59 | 0.0005 | 1.99-10^68 |

numbers. Goerg G. M. (2011) [11] has used in his algorithm IGMM intuitively scaled score function, i.e. $\sigma S_\mu(X)$, of the normal distribution

$$S_\mu(X) := \frac{(X - \mu)}{\sigma^2},$$

where mean $\mu$ is taken as a parameter of interest and $\sigma$ is nuisance. Such an algorithm is working when both mean and variance are finite, i.e. for $\nu > 2$. However, when only mean is finite ($1 < \nu \leq 2$), the effect of nuisance $\sigma$ is well visible (see Table 1). In the case of heavy tailed student ($0 < \nu \leq 1$), where both mean and variance are infinite, the error converges in probability to infinity. This can be obtained by usage of e.g. Kolmogorov’s Strong Law of Large Numbers (LLN) (see e.g. [26]). If we have a sample from distribution with infinite mean (e.g. $t_1$), i.e. Lebesgue integral $\int_R |x|dF(x) = \infty$, then $\frac{1}{n}\sum_{i=1}^{n}X_i$ will have a finite limit for $n \to \infty$ with probability zero. Such random walk is introduced in step 8 (among others) of Algorithm 3 of Goerg G. M. (2011) [11], where sample mean and sample deviation updates scale and location parameters. Therefore we shall expect astronomical numbers in both differences of location parameters $\mu - \hat{\mu}$ and ratios of scales $\sigma/\hat{\sigma}$ with probability 1 (see e.g. rows $\nu = 1$ for Student $t_\nu$ and rows $\alpha = 1$ for Pareto($\alpha$) in Table 1). Such a divergence is not avoided by step 4, namely $||\tau(k) - \tau(k-1)|| > tol.$, in Algorithm 3 of Goerg G. M. (2011) [11].

Random walk of normal scores (3.1) is the reason for this behavior and it explains the astronomical errors of magnitude $10^{59}$ for $t_1$ distribution. Indeed, especially in financial returns (like Asset returns, discussed in section 7.2 of Goerg G. M. (2011) [11]) we shall expect heavy tailed data. Naturally following questions arise: What should be done in such cases? Can we define
some procedure how to check when we can apply IGMM? The answer to this questions is given in the next Sections 3.1 and 3.2.

3.1. Robust testing for normality against Pareto tail. First, we shall testify for the range of Pareto tail parameter $\alpha$ against light tailed normal distribution: for this purpose we need to apply a test for normality against Pareto tails. A consistent and robust test developed as a robust version of Jarque-Bera (JB) test based on the location functional is given by Stehlík et al (2012) [24]. This procedure recognizes in which regime we have our data. The developed test also works for arbitrary sample size, which is very practical for financial applications. For a specific alternatives, also robustified directed Lin-Mudholkar tests (see Stehlík, Thulin and Střelec (2014) [25]) have a good trade-off between power and robustness. Before using such a test one shall check for homogeneity in Pareto tail within our financial time series (this is a practical problem, because tail parameters typically varies during the series). Such testing procedure is developed, jointly with likelihood ratio test for simple hypothesis of the Pareto tail $\alpha = \alpha_0$ in Stehlík et al (2010) [23] by a substantial usage of Lambert W-random variable. We applied robustified JB test of Stehlík et al (2012) [24] for simulated data from Regime II and III, and received p-values 0. Thus it is not recommended to apply IGMM to these two regimes.

3.2. A graphical screening between regimes of IGMM. In the following, we show how the three Regimes of IGMM can be recognized based on t-Hill plots. t-Hill estimator is a robust but consistent Pareto tail estimator introduced in Fabián and Stehlík (2009) [6] and its consistency for iid sample was proven in Stehlík et al (2012) [25], whereas for dependent data in Jordanova, Dušek and Stehlík (2013) [16]. We base our regimes discrimination on robust t-Hill, so that regime boundaries are not influenced by possible outliers. However, to decrease variability (and increase efficiency) of specification of type of regime for a given data, we use flexible Harmonic mean estimator introduced in Beran, Schell and Stehlík (2014) [2].

To recall the Harmonic mean estimator, the next definition follows.

**Definition 1.** We suppose that $X_1, X_2, \ldots, X_n$ are possibly dependent copies of $X$ with d.f. $F$, upper order statistics

$$X_{(1,n)} \leq X_{(2,n)} \leq \ldots \leq X_{(n,n)}.$$ 

Let us denote $RV_\alpha$ the class of regularly varying functions at infinity, with an index of regular variation equal to $\alpha \in \mathbb{R}$, i.e. positive measurable functions
such that for all $x > 0$, $g(tx)/g(t) \to x^\alpha$, as $t \to \infty$.

(3.2) $1 - F \in RV_{-\alpha}, \quad \alpha > 0$.

Harmonic Moment tail Index Estimator has the form

$$H^*_k,n(\beta) = \frac{1}{\hat{\alpha}_{k,n}(\beta)} = \frac{1}{\beta - 1} \left\{ \frac{1}{k} \sum_{j=1}^{k} \left( \frac{X_{n-k,n}}{X_{n-j+1,n}} \right)^{\beta-1} - 1 \right\},$$

where $\beta > 0$ is tuning parameter.

For $\beta = 2$ we obtain t-Hill, for $\beta = 1$ we have Hill estimator (see Hill (1975) [14]). The tuning parameter $\beta$ is regulating the trade-off between efficiency and robustness. For $\beta > 1$ the effect of large contaminations is bounded, since the Harmonic Moment Tail Index Estimator benefits from the properties of the harmonic mean. However, a larger value of $\beta$ also implies an increased variance. For $\beta < 1$ the Harmonic Moment Tail Index Estimator also has a higher variance than Hill’s estimator.

**Remark 1.** *Remark on VAR for LATAM returns*

As the second example, Goerg G. M. (2011) [11] reexamines the LATAM returns. He assures that "a comparison of risk estimators (Value at Risk, VAR) demonstrates the suitability of the Lambert $W \times F$ distributions to model financial data." From the perspective of minimal mean square error, a Mean-of-order-$p$ (MOP) class of VAR estimators can have a mean square error smaller than that of classical extreme value index (EVI) estimators, not only around optimal levels, but for other levels too (see Gomes, Brilhante and Pestana (2014) [12]). MOP EVI-estimator $H_{k,n}^{(p)}$ was introduced in Brilhante, Gomes and Pestana (2012) [7]. Note that if we consider a generalization (motivated by robustness) to $p < 0$ of the MOP functionals $H_{k,n}^{(p)}$, we get the t-Hill estimator $H^*_k,n(2) = H_{k,n}^{(-1)}$. This is a VAR-justification of why to use t-Hill estimator for specification of boundaries of the Regimes. Such setup is also of interest for BASEL II (and higher) initiative in banking and audit.

Let $n$ be fixed as sample size. Analogously to the Hill plot we consider the set of points with coordinates

$$\left( k, \frac{1}{\hat{\alpha}_{k,n}(\beta)} \right), \quad k \in \{1, 2, ..., n\}.$$

Further on we call this plot "modified Hill plot”. Our graphical procedure is illustrated on discrimination between $t_1,t_{1.5}$ and $t_5$ in Figure 1. The three
colored areas representing Regimes are displayed in Figure 2. Therein also $H^*_k,n(1.001)$ (almost Hill-estimator) for the LATAM data is plotted. It is well visible, that LATAM data tail is substantially overlapping with Regime III, thus it is not recommended to process these data with IGMM.

(a) Comparison of three t-Hill plots lines  
(b) Convergence region of t-Hill plot: best estimation and distinguishing of 3 regimes

3.3. On Regime III of IGMM. As mentioned above, Normal score is working in Regime I, but not in Regime III. The classical score function as an indicator of the sensitivity of likelihood $L$, $S_\mu(X) = \frac{\partial}{\partial \theta} \log L(\theta; X)$, has been built for distributions with support on real line, having all moments (see Fisher (1925) [9]). In case of Regime 3 (no finite moments), we shall not only transform a random variable, but also appropriately transform its inference function. For classical transformed t-score results see Fabián (2001) [5] and Stehlík et al (2010) [23]. In this letter we consider only a semi-parametric setup. For a nonparametric analogy see Dobrovidov, Koshkin and Vasiliev (2012) [3] where scores $S_n = \frac{B^2}{A} \frac{\partial}{\partial x_n} \log f(x_n|x_{n-1}) + \frac{x_n}{A}$ are defined for a conditionally exponential family in the linear model $X_n = AS_n + B\eta_n$, where $A, B$ are known constants, $\eta_n$ is Gaussian noise, $(X_n, S_n), n > 1$ is a two-component Markov process, $(X_n)$ is an observable process and $(S_n)$ is an unobservable useful process.

In our setup, let $\mathcal{X}$ be the support of the distribution $F$ with density $f$, continuously differentiable according to $x \in \mathcal{X}$ and let $\eta : \mathcal{X} \rightarrow \mathbb{R}$ be given by Johnson (1949) [15] $\eta(x) = x$, if $\mathcal{X} = \mathbb{R}$, $\eta(x) = \log(x - a)$, if $\mathcal{X} = (a, \infty)$ and $\eta(x) = \log \frac{x}{1-x}$, if $\mathcal{X} = (0, 1)$. Then the transformation-based score or
shortly the \( t \)-score (see Fabián (2001) [5]) is defined by

\[
T(x) = -\frac{1}{f(x)} \frac{d}{dx} \left( \frac{1}{\eta'(x)} f(x) \right),
\]

which expresses a relative change of a "basic component of the density", i.e., density divided by the Jacobian of mapping \( \eta \).

It is clear that for Normal distribution, which is an archetypical distribution, we have \( \eta(x) = x, S(x, \theta) = \frac{d}{d\theta} \log f(x, \theta) \) and \( \hat{\theta} = \text{MLE} \), with MLE standing for maximum likelihood estimator, which is the solution of \( \sum_{i=1}^{n} S(X_i, \theta) = 0 \).

However, for the Pareto distribution we can consider two recently implemented approaches, namely:

- MLE, which is related to the “standard score” estimation with \( \eta(x) = x \) and \( S_F(X, \alpha) = \frac{1}{\alpha} - \log x \) and
- \( t \)-score estimation with \( \eta(x) = \log(x - 1) \) (see Stehlík M. et al. (2010) [23]). Notice that the MLE is not robust wrt right outliers, i.e. if \( X_i \rightarrow \infty \), then \( \hat{\alpha} \downarrow 0 \). For \( t \)-estimation we have

\[
S_F(x) = \alpha \left( 1 - \frac{\alpha + 1}{\alpha x} \right).
\]

Thus standard estimation \( \sum S_F(g)(X_i) = 0 \) gives us \( \hat{\alpha} = \frac{1}{x-1} \) (where
\[ x = \frac{\sum x_i}{\sum \frac{1}{x_i}} \] is harmonic mean) which is an estimator apparently robust against right-outliers.

Thus transformation of the data (e.g. by machinery of Lambert W variable), accompanied with a construction of proper score function transformation is the reasonable further research direction to regularize Algorithm 3 in Regime III.

4. **Skewness: On asset Returns and t-distribution.** Skewness and symmetry are fundamental objects of statistics and it is interesting to study their transformations. Symmetry itself is related to the nature of the problem and its permutation invariance, and cannot be obtained just by a simple transformation. Thus symmetry is one of the fundamental notions of non-parametric statistics and is fundament for typical value of Hartigan (1969) [13], studied in perspective of reflection groups in Francis, Stehlík and Wynn (2014) [10].

Goerg G. M. (2011) [11] defines a transformation \( Y_\gamma = U \cdot \exp(\gamma \cdot U)\sigma + \mu \) where \( Y_\gamma \) is skewed output and \( U \) is symmetrical input. It is true, that having a symmetric zero-mean \( U \), \( \gamma \neq 0 \) regulates the skewness. However, the inverse problem is much more delicate, as is demonstrated by the following simulations. In Section 7.2, ”Asset returns”, Goerg G. M. (2011) [11] used Kolmogorov-Smirnov (KS) test, and stated ”As a KS test cannot reject a student t-distribution..”. KS test implementation in R [19], (as function \( \text{ks.test()} \)) was also used in the function \( \text{ks.test.t()} \) which was introduced in Goerg G. M. (2011) [11] and in his package LambertW. However, parameters \( \hat{\tau}_{MLE} \) are estimated and thus, classical KS test cannot be used. There exist some more refined distribution theory for the KS test with estimated parameters (see Durbin (1973) [4]), but this is not implemented in \( \text{ks.test()} \), used in the function \( \text{ks.test.t()} \). The undesirable parameter dependence of such implementation can influence one of the goals of the paper, having a symmetric t-distribution input \( U \) and \( Y = (U \exp(\gamma U))\sigma_x + \mu_x \) being a skewed output.

The following example shows, that estimation of parameters affects this aim in an undesirable way. First, we simulated input variable \( U \) as a skewed t-distribution (see Fernandez and Steel (1998) [8]) with skew parameter \( \gamma^* \). Data was simulated with function \( \text{rskt(n, df, gamma^*)} \) of package \text{skewt}. The values for parameter \( \gamma^* \) and resulting skewness with four degrees of freedom can be found in left part of Table 2. Then we transformed data to \( Y = (U \exp(-bU))c + a \), where \( a, b \) and \( c \) have been chosen from grids \( a = \text{seq}(0,1,by = 0.01) \); \( b = \text{seq}(0,1,by = 0.01) \); \( c = \text{seq}(0.1,1.5,by = 0.01) \). Finally we estimated \( U \) and parameters by \text{IGMM} and conducted \text{ks.test.t()}.
from this package LambertW. This shows the effect of usage of \texttt{ks.test.t()} jointly with parameter estimation, which led to acceptance of skewed distributions as symmetric student distribution.

For $\gamma^*$ equal to 0.9 or 0.75 we received p-values of 0.502 and 0.269, and thus skewed distribution (skewness = -0.93 and -1.37) is accepted as symmetric student. First line of Table 2 presents simulation of t-distribution (skewness = -0.3415) and resulting p-value is correctly higher than 0.05. The same comparison was done for skewed normal distribution, which was simulated with \texttt{rsn(n, xi , omega, alpha)} from package \texttt{sn}. $\alpha$ is in this setting skewing parameter and its values can be seen in the first column of the right side of Table 2. Location (xi) and scaling parameters (omega), which are equivalent to mean and standard deviation, were chosen to be $\mu = 4$ and $\sigma = 2$. Skewness was compared for different $\alpha$ and p-values resulting from \texttt{ks.test.t()} are presented as before. For all listed cases we received p-values $p > 0.05$ and therefore skewed normal distributions were falsely assumed as symmetric t-distributions. Obviously p-values are decreasing for higher $\alpha$, but for all $\alpha \leq 8$ symmetric t-distribution cannot be rejected for simulated skewed normal distribution.

| Skewed t- and t-distribution | Skewed normal and normal distribution |
|------------------------------|---------------------------------------|
| skewness | p-value | $\gamma^*$ | skewness | p-value | $\alpha$ |
| -0.3415 | 0.2872 | 0.20 | 0.0130 | 0.1731 |
| -2.8894 | 0.0001 | 0.40 | -0.0892 | 0.0569 | 0.10 |
| -1.6850 | 0.0000 | 0.75 | 0.0277 | 0.5323 | 0.50 |
| -1.3785 | 0.2693 | 1.00 | 0.0810 | 0.6801 | 2.50 |
| -0.9304 | 0.5017 | | 0.8108 | 0.5035 | 5.00 |
| 0.8054 | 0.0924 | | 0.8054 | 0.0924 | 8.00 |
| 0.9391 | 0.0527 | | 0.9391 | 0.0527 | |

In order to check graphically for impact of $\gamma^*$ and $\alpha$ on skewed distributions, kernel density estimations were plotted in R [19]. This density estimation comparison in Figure 3(a) shows stronger skewed distributions for decreasing values of $\gamma$. Distributions were simulated with negative skewness in this example. Black density corresponds to student distribution and the others are computed with previously defined $\gamma^*$ values and show skewed t-distributions. A graphical comparison between skewed normal distributions and normal distribution is done in Figure 3(b). Increasing skewing parameter $\alpha$ leads to stronger skewness of the data and a shift to the right. In
contrast to the previous examples skewness is except for $\alpha = 0.1$ positive and increasing with $\alpha$.

(a) Skewed t- and t-distribution, four degrees of freedom
(b) Skewed normal and normal distribution, $\mu = 4$ and $\sigma = 2$

Fig 3. Comparison of skewed and unskewed t(df = 4)- and normal($\mu = 4, \sigma = 2$)-distribution

4.0.1. Auto-Correlation Rising from IGMM and LATAM data. We also checked auto-correlations resulting from estimation by Algorithm 3 for different distributions in a following simulation setup. We simulated standard Normal distribution, Weibull, Exponential and student-t distributions. In the next step IGMM was used to estimate parameters and as a consequence back-transformation with get.input was applied with estimated $\hat{\mu}$, $\hat{\sigma}$ and $\hat{\gamma}$. We observed significant auto-correlation for all 4 distributions. Also auto-correlation function of back-transformed series of LATAM has been observed to be significant (e.g. at lags 2, 7, 8, 13 and 30).

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ACF of back-transformed series of LATAM
ACF plot for N(0,1)

ACF plot for Exponential, rate = 3

ACF plot for Weibull, shape = 1, scale = 1.5

ACF plot for Student, nu = 5

ACF plot for Student, nu = 1.5

ACF plot for Student, nu = 5
Skewed t–distribution compared with t–distribution for df = 4
Comparison of skewed and normal distribution; mean = 4; sd = 2
Skewed $t$-distribution compared with $t$-distribution for df = 4
Comparison of skewed and normal distribution; mean = 4; sd = 2
3 Regimes of Algorithm by IGMM()
ACF of back-transformed series of LATAM
t-variant Hill-estimator compared for different df

- nu = 1
- nu = 1.5
- nu = 5
Convergence region for t–Hill estimator for different df
Comparison for \( \nu \)