Non-Geometric T-Duality as Higher Groupoid Bundles with Connections

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Abstract

We describe T-duality between general geometric and non-geometric backgrounds as higher groupoid bundles with connections. Our description extends the previous observation by Nikolaus and Waldorf that the topological aspects of geometric and half-geometric T-dualities can be described in terms of higher geometry. We extend their construction in two ways. First, we endow the higher geometries with adjusted connections, which allow us to discuss explicit formulas for the metric and the Kalb–Ramond field of a T-background. Second, we extend the principal 2-bundles to augmented 2-groupoid bundles, which accommodate the scalar fields arising in T-duality along several directions as well as $Q$- and $R$-fluxes. Our description is manifestly covariant under the full T-duality group $GO(n, n; \mathbb{Z})$, and it has interesting physical and mathematical implications.
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1. Introduction and results

T-duality is a crucial feature of string theory which sets it apart from field theories of point particles. In its simplest form, T-duality relates two string theories whose target spaces are of the form $X \times S^1$ for some Lorentzian manifold $X$, interchanging the momentum modes and the winding modes along the circle $S^1$. More complicated is the case in which a T-duality involves a non-trivial circle bundle that carries in addition a Kalb–Ramond two-form field $B$ describing a topologically non-trivial gerbe. Here, a T-duality can link target spaces with very different topologies. Even more generally, we can consider T-dualities along a torus fibration and introduce additional background $p$-form potentials with non-vanishing curvatures corresponding to (higher) gerbes. Such T-dualities can link fully geometric target spaces to non-geometric target spaces. A class of mildly non-geometric target spaces is known as T-folds. These are still locally geometric, but their local data are glued together by an element in the T-duality group $O(n,n;\mathbb{Z})$. T-dualities can, however, also produce $R$-spaces, for which there is not even a local geometric description. It is clear that a complete understanding of string theory requires an understanding of these non-geometric backgrounds. Given that T-duality is a duality on topologically non-trivial target spaces, it is particularly important not to work merely locally, as done in much of the literature. One of the aims of this paper is to provide a clean mathematical description of such non-geometric backgrounds arising in the context of non-trivial topologies.

By now, T-duality has attracted considerable mathematical interest due to its relation to a number of important mathematical constructions such as mirror symmetry and the Fourier–Mukai transform. The observation that T-duality can change the topology of the target space was linked in a formalism dubbed topological T-duality to the existence of a Gysin sequence [1, 2]. The latter provides an explicit relation between different topological classes, e.g. between the first Chern class of a torus fibration and the Dixmier–Douady class of a gerbe on its total space. Subsequent works have found interpretations of non-geometric backgrounds in terms of non-commutative [3] and non-associative geometries [4].

A useful geometric description of T-duality called T-correspondences was given in [5, 6]. Here, a T-background is defined as a torus bundle $P$ over a base manifold $X$ together with the Dixmier–Douady class $H$ of an abelian gerbe over $P$. A T-duality between two such pairs $(\tilde{P}, \tilde{H})$ and $(\hat{P}, \hat{H})$ is then formulated as a relation between the pullbacks of $\tilde{H}$ and $\hat{H}$ to the correspondence space $\tilde{P} \times_X \hat{P}$; the data $(\tilde{P}, \tilde{H})$, $(\hat{P}, \hat{H})$ and the relation are collectively called a T-duality triple. Quite recently, it was observed that T-backgrounds and indeed full T-duality triples can be represented by 2-stacks [7]. That is, a topological, geometric T-background can be equivalently seen as a principal 2-bundle or gerbe with a particular structure 2-group $\text{TB}_{n}^{F_{2}}$ that encodes both the torus directions as well as the gerbe part. The same holds for a T-duality triple between geometric T-backgrounds, where the structure group is denoted by $\text{TD}_{n}$. This structure group comes with two natural projections to $\text{TB}_{n}^{F_{2}}$, and the induced map on principal 2-bundles yields the data of a T-duality triple. Interestingly, even half-geometric T-dualities, which link geometric backgrounds with T-folds, can be captured in terms of principal 2-bundles. This opens up the exciting possibility that non-commutative and perhaps even non-associative geometries

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can be resolved into ordinary but higher geometries, which would clearly constitute a simplification: higher geometry enters the description of any geometric background anyway in the form of gerbes, and higher geometric objects are more readily derived than their non-commutative and, in particular, non-associative counterparts.

The appealing picture obtained in [7] poses three evident questions: First, can we extend the topological constructions to a more complete picture by adding a differential refinement in the form of connections? Second, can we extend the half-geometric correspondences further to the most general case of $R$-spaces? And third, is there a fully $O(n, n; \mathbb{Z})$-covariant formulation that manifests the action of the T-duality group on the components of this description? In this paper, we answer all of these questions affirmatively.

A differential refinement of the description of geometric T-duality triples in terms of principal 2-bundles should be straightforward by abstract nonsense. Unfortunately, this is complicated by the fact that the connections on principal 2-bundles as conventionally defined in the literature (see e.g. [8] and [9]) are not sufficiently general as they imply a particular flatness condition on the curvature. Instead, one has to work with adjusted connections as developed in [10, 11, 12, 13]. In particular, the finite form of differentially refined, adjusted cocycles was only identified very recently [13]. Using this technology, it is not hard to construct the relevant adjustment and to endow the principal 2-bundles describing geometric T-duality correspondences with adjusted connections.

A differential refinement in the half-geometric case, however, requires some more work. Recall that T-duality can be interpreted as a Kaluza–Klein reduction of the correspondence space, cf. [14, 15, 16, 17]. A Kalb–Ramond $B$-field on the correspondence space will thus give rise to scalar fields on the base space $X$ after T-duality, or dimensional reduction, along two directions. These scalar fields then take values in the Narain moduli space [18]

$$M_n = O(n, n; \mathbb{Z}) \setminus O(n, n; \mathbb{R}) / (O(n; \mathbb{R}) \times O(n; \mathbb{R})) =: O(n, n; \mathbb{Z}) \setminus \mathcal{Q}_n. \quad (1.1)$$

This makes it obvious that the principal 2-bundles used in [7] need to be extended to principal 2-groupoid bundles with structure 2-groupoid given by the 2-group $TD_n$ fibered over the Narain moduli space $M_n$.

As is often the case, it turns out to be convenient to replace the Narain moduli space by the action groupoid for the action of the T-duality group $O(n, n; \mathbb{Z})$,

$$O(n, n; \mathbb{Z}) \ltimes \mathcal{Q}_n \Rightarrow \mathcal{Q}_n. \quad (1.2)$$

The T-duality group $O(n, n; \mathbb{Z})$, however, should also induce some transformation on $TD_n$, and the construction of this transformation is (interestingly) non-trivial. In fact, $O(n, n; \mathbb{Z})$ by itself does not act on $TD_n$, and one has to introduce a larger 2-group\(^1\) $\mathcal{G}_0(n, n; \mathbb{Z})$. Once the action is derived, the relevant structure 2-groupoid $\mathcal{G}_n$ is evident, and the explicit cocycle description of principal $TD_n$-bundles can be given.

There is an evident continuation of our picture to $R$-spaces: the requirement for 0-form curvatures corresponding to (non-existent) $(-1)$-forms suggests augmenting the 2-groupoid $\mathcal{G}_n$ in the simplicial sense to an augmented 2-quasi-groupoid $\mathcal{G}_n^{\text{aug}}$. The

\(^1\)It has been recently shown that a 2-group of the form $\mathcal{G}_0(n, n; \mathbb{Z})$ is equivalent to the automorphism 2-group of $TD_n$ [19].
relevant space of \((-1)\)-simplices is then identified from the observation that R-fluxes are related to particular embedding tensors. Again, the relevant cocycles can be written down, and we obtain a description of T-duality correspondences between general R-spaces.\(^2\)

Throughout this paper, we work with affine torus bundles as defined in [21], extending the discussion of [19] for principal torus bundles even in the topological case. We argue that this forces us to work with the larger T-duality group\(^3\) \(\text{GO}(n, n; \mathbb{Z}) := O(n, n; \mathbb{Z}) \ltimes \mathbb{Z}_2\).
This group has been identified as relevant in the context of T-duality [22, 23], and it is a natural part of the automorphism 2-group of \(\text{TD}_n\) [7, 19].

We begin with an explicit construction of a 2-group of automorphisms \(\mathcal{H}(n, n; \mathbb{Z})\) of \(\text{TD}_n\) in section 3 together with the corresponding semidirect product 2-group \(\mathcal{H}(n, n; \mathbb{Z}) \ltimes \text{TD}_n\). The 2-group \(\mathcal{H}(n, n; \mathbb{Z})\) is equivalent as a 2-group to the full automorphism 2-group of \(\text{TD}_n\) constructed in [19].

We then give the explicit description of geometric T-dualities in terms of principal 2-bundles in section 4, extending the topological picture of [7] to general torus bundle and providing a differential refinement. We explicitly show how to treat the well-known case of the three-dimensional nilmanifold and how to recover the individual T-dual geometries from the principal 2-bundle data.

This picture is extended in section 5 to the case of T-folds. We review the arguments that T-duality is closely related to Kaluza–Klein theory and show how the group \(\text{TD}_n\) arises naturally from this perspective. We then construct the appropriate Lie 2-groupoid \(\mathcal{D}_n\) that governs T-dualities between T-folds. An explicit description of the cocycles of principal \(\mathcal{D}_n\)-bundles is given, and we discuss an explicit example of a T-fold in this context. We also show how the half-geometric T-dualities of [7] are subsumed in our construction.

The final extension to general T-dualities involving R-spaces is then made in section 6. To complete the picture, we recall that non-geometric fluxes are related to the embedding tensor in supergravity. That allows us to identify the correct representation of the R-fluxes, which we then adjoin as \((-1)\)-simplices to the simplicial form of the Lie 2-groupoid \(\mathcal{D}_n\).

The result is the augmented Lie 2-groupoid \(\mathcal{D}_n\text{aug}\), and it is not hard to write down the explicit cocycles for the principal \(\mathcal{D}_n\text{aug}\)-bundles describing general T-dualities. We also comment on explicit examples of R-spaces from this perspective.

We note that principal \(\mathcal{D}_n\text{aug}\)-bundles naturally truncate to principal \(\mathcal{D}_n\)-bundles describing T-dualities with T-folds and principal \(\text{TD}_n\)-bundles describing geometric T-dualities. Moreover, all constructions are manifestly \(\text{GO}(n, n; \mathbb{Z})\)-covariant: the action of the T-duality group is always explicit. In this sense, our approach is similar in spirit to double field theory.

Our construction only captures fields that have trivial dependence on the T-duality

\(^2\)The slides [20] mention a possible extension of the framework of [19] to R-spaces by using Lie 2-groupoids, but they do not give details; in particular, augmentation, which we see as necessary for a proper description of R-fluxes, is not mentioned.

\(^3\)Here and throughout this paper, \(\mathbb{Z}_2\) refers to the additive group of integers modulo 2, not to the 2-adic integers.
directions. This is simply due to the fact that we interpret T-duality as a Kaluza–Klein reduction from the correspondence space and ignore any massive modes arising therefrom. The latter encode the part of the geometry and dynamics that is not invariant under translation along the fibers, and these fields do not introduce new gauge symmetries. Since the compactification is toroidal, the truncation to massless fields is consistent.

There are a few open questions arising from our constructions. First of all, we observe that the T-duality group \( \text{GO}(n,n;\mathbb{Z}) \) does not act on the 2-group \( \text{TD}_n \), while the extension \( \mathcal{G}O(n,n;\mathbb{Z}) \) does. This leads to additional moduli in our description, which are then canceled by condition (5.28a) arising from demanding the existence of adjusted curvatures. Similarly, our cocycles for principal \( \mathcal{TD}^\text{aug}_n \)-bundles impose topological restrictions on the set of \( Q \)- and \( R \)-fluxes. It would be useful to understand both from a physical perspective. Second, it would be important to link our description of T-duality for non-geometric spaces to the descriptions available in the literature, in particular to [3, 4]. Third, it would be very interesting to relate our constructions much more closely to double field theory, in particular to the global constructions of [24] based on the formalism of [25]. Fourth, it may be possible to use our framework to make progress with the definition and the understanding of non-abelian T-duality. The issue here is that with the inclusion of non-abelian gauge groups, the relevant 2-groups including the gauge potential become more and more complicated, cf. [13]. In a related vein, while in this paper we restrict to the case of ungauged (super)gravities, it may be feasible to generalize our results to the gauged case with more non-trivial tensor hierarchies. Finally, all our constructions lift, in principle, readily to U-duality, and this is currently the focus of our attention [26]; see also [17, 27] for related work.

2. Lightning review: T-duality

In the following, we collect some basic results about T-duality from the literature; helpful reviews for further reading include [28, 29].

2.1. Topological T-duality

We start with a brief review of topological T-duality [1, 2] with an emphasis on the T-correspondences of [5, 6].

T-backgrounds. The low-energy sector of a geometric string theory background, or an \( \mathcal{N} = 0 \) supergravity background, is given by a smooth Riemannian manifold \((M,g)\) that carries an abelian gerbe \( \mathcal{G} \), whose connective structure provides the Kalb–Ramond field \( B \) [30, 31].\(^4\) Recall that abelian gerbes can be described in a geometrically appealing fashion as bundle gerbes [32, 33] or as central groupoid extensions; here, however, we will

\(^4\)Technically, the \( \mathcal{N} = 0 \) supergravity background also includes the dilaton \( \phi \). Its T-duality transformation, however, is trivial: the rescaled combination \( \exp(-2\phi) \sqrt{|\det g|} \) remains invariant. We therefore neglect it in this work. It will, however, become important in the extension to U-duality [26].
be using the equivalent but simpler Hitchin–Chatterjee gerbes \[34, 35\]. For us, a topological abelian gerbe is thus simply a cocycle in Čech cohomology \(h_{\text{top}} \in H^3(M, \mathbb{Z})\). It becomes differentially refined, i.e. equipped with a connection, if this Čech cocycle is extended to a cocycle in Deligne cohomology \(h_D \in H^3_D(M, \mathbb{Z})\). The cohomology class of \(h_{\text{top}}\) is called the Dixmier–Douady class of the gerbe; if the gerbe carries a connection with 2-form potential \(B\), then the image of \([h]\) in de Rham cohomology is the cohomology class of the 3-form curvature \(H = dB \in \Omega^3(M, \mathbb{Z})\) of the gerbe.

Most commonly, T-duality is defined for string theory backgrounds with a circle or, more generally, a torus action\(^5\) that preserves the metric and the curvature 3-form of the gerbe. We therefore focus on backgrounds containing a number of 1-cycles that are fibered as a torus bundle \(M = P\) over a base manifold \(X\). Recall that principal torus bundles are always oriented; we want to explicitly permit unoriented affine torus bundles as considered in [21]. As an additional geometric datum, there is an abelian gerbe \(G\) on the total space of this bundle. We call the triple \((X, P, G)\) a topological, geometric (toric) T-background; if both \(P\) and \(G\) carry connections and \(X\) carries a Riemannian structure, we speak of a differentially refined, geometric (toric) T-background, cf.\([6, 7]\). Note that a T-background is not necessarily a consistent background of supergravity or string theory.

**Classification.** There is now a useful classification of toric T-backgrounds. The Serre spectral sequence associated to the fibration \(\pi: P \to X\) defines a filtration

\[
\pi^*H^k(X) =: F^k \subset F^{k-1} \subset \cdots \subset F^0 := H^k(P)
\]

relating the cohomologies of the base \(X\) and the total space \(P\), cf.\([6, 21]\). In particular, the Dixmier–Douady class \(h \in H^3(P, \mathbb{Z})\) of a gerbe lies within the filtration \(F^3 \subset F^2 \subset F^1 \subset F^0\). If the gerbe carries a connection with curvature \(H\), then \(G\) belongs to \(F^i\) if some contractions of \(H\) with \(3-i\) vector fields along the fiber directions are non-trivial. We say that a toric T-background is (of type) \(F^i\) or simply an \(F^i\)-background if its Čech cocycle lies in \(F^i\) but not in \(F^{i+1}\).

This classification now allows us to make clear statements about the image of a toric T-background \((X, \hat{P}, \hat{G})\) under T-duality along fiber directions.

- **\(F^3\):** The gerbe \(\hat{G}\) is the pullback of a gerbe on \(X\) along \(\hat{\pi}: \hat{P} \to X\), and T-duality maps the toric T-background \((X, \hat{P}, \hat{G})\) to itself.

- **\(F^2\):** As shown in [6], this is the minimum requirement for having a geometric T-dual. A geometric T-duality relates a geometric toric T-background to another geometric toric T-background, \((X, \hat{P}, \hat{G}) \mapsto (X, \bar{P}, \bar{G})\), preserving the total dimension of \(\hat{P}\) but generically not the topologies of \(\hat{P}\) or \(\hat{G}\), cf. e.g. [1]. In particular, if \(\hat{P}\) is a principal \(T^n \cong U(1)^n\)-bundle, then so is \(\bar{P}\).

- **\(F^1\):** T-duality along the fibers maps such a toric T-background to a “non-geometric” background [22, 23, 36]. Locally, an \(F^1\)-background is always \(F^2\), and it has local

\(^5\)or, even more generally, a \(\text{GL}(n; \mathbb{Z}) \ltimes \mathbb{T}^n\)-action on a certain \(\text{GL}(n; \mathbb{Z})\)-fold cover of a possibly unoriented \(n\)-torus bundle; see section 4
T-duals, which can then be glued together into a T-fold. This was in particular the perspective adopted in [7]. Another possible interpretation is to regard certain T-folds as bundles of non-commutative tori, see e.g. [3].

$F^0$: The T-dual of an $F^0$-background is not even locally geometric, and the image is sometimes called an $R$-space and interpreted in terms of non-associative geometry, see [4, 37, 38] as well as [39, 29] for helpful reviews.

**Example: trivial fibration.** Let us briefly consider the simple case of a trivial torus bundle $P = X \times \mathbb{T}^n$ together with a gerbe $\mathcal{G}$ with curvature 3-form $H$. In this case, the Dixmier–Douady 3-class $h$ and, correspondingly, the curvature $H$ can be decomposed into four parts:

$$H \in H^3(X \times \mathbb{T}^n) \cong H^3(X) \oplus (H^2(X))^\oplus n \oplus (H^1(X))^\oplus n \oplus (H^0(X))^\oplus 3,$$

(2.1)

$$H = H^{(3)} + \sum_{i=1}^{n} H^{(2)}_i + \sum_{i,j=1}^{n} H^{(1)}_{ij} + \sum_{i,j,k=1}^{n} H^{(0)}_{ijk}.$$

In this decomposition, the 2-classes $H^{(2)}_i, \ldots, H^{(2)}_n \in H^2(X)$ dualize, under T-duality, to the Chern classes of a non-trivial torus bundle on $X$, thus to a geometric background. The 1-classes $H^{(1)}_{ij}$ correspond under T-duality to Q-fluxes. That is, the T-dual is formed by starting with a geometric universal cover and taking a possibly non-geometric quotient given by $\mathcal{O}(n, n; \mathbb{Z})$-transformations encoded by $H^{(1)}_{ij}$ to form T-folds. Finally, the 0-classes $H^{(0)}_{ijk}$ correspond to $R$-fluxes, which encode the degree to which even local geometry fails to exist.

**Topological geometric T-duality.** Let us consider the case of geometric T-duality in more detail and focus on the purely topological aspect. Topological T-duality [1, 2] is based on the existence of the Gysin sequence [40], see also [41, Prop. 14.33]. Given a principal torus bundle $\tilde{P} \to X$ with first Chern class $\tilde{F} \in H^2(X, \mathbb{Z})$, the following sequence is exact:

$$\ldots \to H^k(X, \mathbb{Z}) \xrightarrow{\tilde{\pi}^*} H^k(\tilde{P}, \mathbb{Z}) \xrightarrow{\tilde{\pi}^*} H^{k-1}(X, \mathbb{Z}) \xrightarrow{\tilde{F}^*} H^{k+1}(X, \mathbb{Z}) \to \ldots$$

(2.2)

For topological T-duality, we are interested in this segment for $k = 3$. Any 3-form $\tilde{H} \in H^3(P, \mathbb{Z})$ comes with an associated element $\tilde{F} := \tilde{\pi}_*\tilde{H} \in H^2(X, \mathbb{Z})$ with $\tilde{F} \smile \tilde{F} = 0$ in $H^4(X, \mathbb{Z})$. We can now consider a second torus bundle $\tilde{\pi}: \hat{P} \to X$ with first Chern class $\hat{F}$. Because $\hat{F} \smile \hat{F} = \hat{F} \smile \hat{F} = 0$, exactness of the Gysin sequence with hatted maps now shows that there is an $\tilde{H}$ such that $\tilde{\pi}_*\tilde{H} = \hat{F}$. Topological geometric T-duality is then the transition from $(\tilde{P}, \tilde{H})$ to $(\hat{P}, \hat{H})$. As shown in [1], this construction matches the various expectations from string theory considerations. It also extends to the case of affine torus bundles, and there is a corresponding Gysin sequence [21].
**T-duality correspondence.** We can arrive at a more geometric picture if we include the correspondence space \( \hat{\mathcal{P}} \times_X \hat{\mathcal{P}} \) and regard \( \hat{\mathcal{H}} \) and \( \hat{\mathcal{H}} \) as the Dixmier–Douady classes of some bundle gerbes \( \hat{\mathcal{G}} \) and \( \hat{\mathcal{G}} \), respectively. This then leads to the commutative diagram

\[
\begin{align*}
\mathcal{G}_C &= \hat{p}^* \hat{\mathcal{G}} \otimes \hat{p}^* \hat{\mathcal{G}}^{-1} \\
\hat{\mathcal{G}} &\xrightarrow{\hat{p}} \hat{\mathcal{P}} \xrightarrow{\hat{p}} \hat{\mathcal{P}} \\
X &\xrightarrow{\#} \hat{\mathcal{P}} \xleftarrow{\#} \hat{\mathcal{G}}
\end{align*}
\]

which is crucial in the definition of topological T-duality in terms of T-duality triples [6]. Such a T-duality triple is given by the data \((\hat{P}, \hat{H}), (\check{P}, \check{H}), u)\), where \(u\) is a trivialization of the gerbe \(\mathcal{G}_C\), relating it to the Poincaré bundle over the correspondence space, cf. also [42, Rem. 6.3] for a string theoretic interpretation.

**2.2. Differential refinement of topological T-duality**

In order to describe a geometric T-background \((X, P, \mathcal{G})\) completely, we need to provide a Riemannian metric on \(X\) and connections on \(P\) and \(\mathcal{G}\).

**Principal \(G\)-connection and Kaluza–Klein metric.** We can describe the connection on the principal \(U(1)^n\)-bundle \(P\) as a principal \(G\)-connection \(\theta\). Recall that such a connection is a \(t^n := \text{Lie}(U(1)^n) \cong \mathbb{R}^n\)-valued 1-form \(\theta \in \Omega^1(P, t^n)\) such that, for any fundamental vector field \(X_\xi \in \Gamma(TX)\) of \(\xi \in t^n\), the 1-form \(\theta\) is equivariant, \(\mathcal{L}_{X_\xi} \theta = 0\), and reproduces \(\xi\) in the sense that \(\iota_{X_\xi} \theta = \xi\).

Together with a Riemannian metric \(g\) on \(X\), the connection \(\theta\) induces the Kaluza–Klein metric \(\tilde{g}\) on \(P\) defined by

\[
\tilde{g} := \pi^* g + \theta^i \otimes \theta^i.
\]

In the case of an affine torus bundle, we use the connection on the corresponding principal \((\text{GL}(n; \mathbb{Z}) \ltimes U(1)^n)\)-bundle, which corresponds to locally defined \(u(1)^n\)-valued vector fields defined up to invertible integer linear transformations. This ambiguity, however, drops out of (2.4).

**The group \(O(n, n; \mathbb{Z})\).** T-duality is often presented as an involution given by a \(\mathbb{Z}_2\)-action. On a string background, this action maps the radius of the involved circle direction \(R\) to the inverse radius \(\frac{1}{R}\) and interchanges the momentum and the winding modes of the string. There is an additional freedom of reversing the sign in the latter interchange so

\[\text{We put } \alpha' = 1.\]
that the full T-duality group for T-duality along a circle direction should be identified with $\mathbb{Z}_2 \times \mathbb{Z}_2 \cong O(1, 1; \mathbb{Z})$.

For an $n$-dimensional torus $T^n$, this group is enlarged to the group $O(n, n; \mathbb{Z})$, see [43, 44]. Elements $g$ of $O(n, n; \mathbb{Z})$ are $2n \times 2n$ integer matrices that leave the form

$$\eta := \begin{pmatrix} 0 & 1_n \\ 1_n & 0 \end{pmatrix}$$

(2.5)

invariant in the sense that $g^T \eta g = \eta$, which in components becomes

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}, \quad A, B, C, D \in \text{Mat}(n; \mathbb{Z}),$$

(2.6)

$$A^T C + C^T A = B^T D + D^T B = 0, \quad A^T D + C^T B = 1_n.$$

The group $O(n, n; \mathbb{Z})$ is a subgroup of the larger group $GO(n, n; \mathbb{Z}) := O(n, n; \mathbb{Z}) \times \mathbb{Z}_2$ originally defined in [22, 23], which becomes relevant for T-duality with general torus bundles. For $n > 0$, this group can be identified with the $2n \times 2n$ integer matrices that leave $\eta$ invariant up to sign in the sense that $g^T \eta g = \eta$, which in components becomes

$$A^T C + C^T A = B^T D + D^T B = 0, \quad A^T D + C^T B = \pm 1_n.$$

(2.7)

For $n = 0$, we have $GO(0, 0; \mathbb{Z}) \cong \mathbb{Z}_2$. For future convenience, we introduce the indicator function $|−|: GO(n, n; \mathbb{Z}) \to \{0, 1\}$, which is simply the projection onto the $\mathbb{Z}_2$ component. In particular, $(-1)^{|g|} = +1$ for all $g \in O(n, n; \mathbb{Z})$.

Subgroups of $GO(n, n; \mathbb{Z})$. It is useful to introduce the following subgroups of the T-duality group $GO(n, n; \mathbb{Z})$, which together generate the entirety of $GO(n, n; \mathbb{Z})$, cf. [28]:

A) The subgroup $GL(n; \mathbb{Z}) \subset O(n, n; \mathbb{Z})$ of $A$-transformations consists of group elements

$$g_A = \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \quad \text{with} \quad A \in GL(n; \mathbb{Z}).$$

(2.8)

These transformations are simply the automorphism $\text{Aut}(T^n) \cong GL(n; \mathbb{Z})$ of the $n$-dimensional torus $T^n$ forming the fibers of the torus bundle, and it is therefore also sometimes called the geometric (sub)group.

B) The abelian torsion-free subgroup $o(n; \mathbb{Z}) \subset O(n, n; \mathbb{Z})$ of $B$-transformations consists of group elements

$$g_B = \begin{pmatrix} 1_n & B \\ 0 & 1_n \end{pmatrix} \quad \text{with} \quad B \in \{A \in \text{Mat}(n; \mathbb{Z}) \mid A^T = -A\}.$$

(2.9)

We note that if we tensor this subgroup with functions along the $n$-torus, then certain $B$-transformations are naturally identified with the 2-form $d\Lambda$ for a 1-form $\Lambda$ along the torus direction. The corresponding $B$-transformations then describe gauge transformations, as familiar from the Courant algebroid description.
\( \beta \) The abelian torsion-free subgroup \( \sigma(n; \mathbb{Z}) \subset O(n, n; \mathbb{Z}) \) of \( \beta \)-transformations consists of group elements

\[
g_B = \begin{pmatrix} \mathbb{1}_n & 0 \\ \beta & \mathbb{1}_n \end{pmatrix} \quad \text{with} \quad \beta \in \{ A \in \text{Mat}(n; \mathbb{Z}) \mid A^T = -A \} . \tag{2.10} \]

\( T_k \) The abelian torsion subgroup \( O(1, 1, \mathbb{Z})^n \cong (\mathbb{Z}_2)^{2n} \subset O(n, n; \mathbb{Z}) \) of the factorized dualities is generated by group elements

\[
g^\pm_{Tk} = \begin{pmatrix} \mathbb{1}_n - 1_k & \pm 1_k \\ \pm 1_k & \mathbb{1}_n - 1_k \end{pmatrix} \quad \text{with} \quad 1_k = \text{diag}(0, \ldots, 0, 1, 0, \ldots, 0) \, \text{for} \, k = 1, \ldots, n-1 . \tag{2.11} \]

These transformations can be identified with the involutions that are T-dualities along the \( k \)-th circle direction. Clearly, \((g^\pm_{Tk})^2 = \mathbb{1}_{2n}\).

\( G \) The abelian torsion subgroup \( \mathbb{Z}_2 \times \mathbb{Z}_2 \subset GO(n, n; \mathbb{Z}) \) consists of group elements

\[
g^s_1, s_2 \quad \text{with} \quad s_1, s_2 \in \{ \pm 1 \} . \tag{2.12} \]

Note that \((-1)^{|g^s_1, s_2|} = s_1 s_2 \). In other words, when \( s_1 s_2 = -1 \), then \( g^s_1, s_2 \not\in O(n, n; \mathbb{Z}) \).

**GO(n, n; \mathbb{Z}) versus O(n, n; \mathbb{Z}).** Throughout this paper, we will work with the larger T-duality group \( GO(n, n; \mathbb{Z}) \) instead of \( O(n, n; \mathbb{Z}) \). The difference between the two groups is that the former contains the additional generator \( g^+_{G} = \text{diag}(-1_n, 1_n) \). We will see in our later discussion that this group element flips the sign of the Kalb–Ramond field along the fiber directions of \( \tilde{P} \times X \). Identifying gerbes of opposite orientation becomes a necessity because working with general torus bundles implies that we also identify principal bundles with opposite orientation. As an example, consider a principal \( U(1) \)-bundle \( P \) regarded as a general circle bundle, i.e. a principal \( O(2) \)-bundle. A constant coboundary equal to the additional \( \mathbb{Z}_2 \)-factor in \( O(2) \) over \( U(1) \cong SO(2) \) now flips the orientation of \( P \), rendering \( P \) and its dual isomorphic. It is well-known that T-duality can interchange the topological invariant of the torus bundle with the topological invariant of the gerbe. Thus, working with general torus bundles implies that we have to enlarge the T-duality group from \( O(n, n; \mathbb{Z}) \) to \( GO(n, n; \mathbb{Z}) \). For further discussion, see also [22, 23].

**Non-geometric backgrounds.** If local descriptions of a T-background are glued together with elements of the geometric subgroup of the T-duality group, then we have a geometric T-background. T-folds\( ^7 \) [46, 47, 48] are T-backgrounds, most importantly torus fibrations with \( B \)-field, that are locally geometric, but whose local descriptions are glued
together by general elements of the T-duality group, i.e. $GO(n, n; \mathbb{Z})$ for torus fibrations. Therefore, T-folds always have a global double geometry [47]. Because some T-folds arise as T-duals of geometric T-backgrounds, it is clear that string theory is well-defined on these and, in particular, that they have to be included in the space of possible string backgrounds. There is, in fact, a constrained sigma model description of certain T-fold backgrounds [46]. We also note that a higher geometric local model of T-folds was given in [42].

A further generalization are the so-called $R$-spaces [49, 50] which do not even locally admit a geometric description.

**Vanishing Dixmier–Douady class.** In the case in which the Dixmier–Douady class of the gerbe vanishes, and, as a consequence, the Kalb–Ramond $B$-field is globally defined, we can combine it with the metric $g$ into the tensor $\mathcal{E} := g + B$; under an element $g \in GO(n, n; \mathbb{Z})$ of the form (2.6), $\mathcal{E}$ transforms in the Möbius-like, non-linear fashion

$$\tilde{\mathcal{E}} = g \triangleright \mathcal{E} := \frac{A \mathcal{E} + B}{C \mathcal{E} + D}.$$  

(2.13)

For the factorized dualities $T_k$, this transformation reproduces the Buscher rules [51, 52] for the transformations of the metric and $B$-field along a circle direction.

In order to render the above transformation linear, we can switch to the generalized metric [44, 43, 53, 54]

$$\mathcal{H} := \begin{pmatrix} g -Bg^{-1}B & Bg^{-1} \\ -g^{-1}B & g^{-1} \end{pmatrix},$$

(2.14)

satisfying $\mathcal{H}^{-1} = \eta \mathcal{H} \eta$. We then have a simple adjoint action of $O(n, n; \mathbb{Z})$ on $\mathcal{H}$,

$$\tilde{\mathcal{H}} = g \triangleright \mathcal{H} := g \mathcal{H} g^T.$$

(2.15)

In particular, the generalized metric is obtained as follows from $B$-transformations:

$$\mathcal{H} = \begin{pmatrix} \mathbb{1}_n & B \\ 0 & \mathbb{1}_n \end{pmatrix} \begin{pmatrix} g & 0 \\ 0 & g^{-1} \end{pmatrix} \begin{pmatrix} \mathbb{1}_n & B \\ 0 & \mathbb{1}_n \end{pmatrix}^T.$$

(2.16)

**Non-trivial Dixmier–Douady class.** In the case in which the $B$-field is not globally defined, we still have the group $GO(n, n; \mathbb{Z})$ as the relevant group of T-dualities. Its action on the data making up the differential refinement, however, is more complicated. In particular, the combination $\mathcal{E} := g + B$ can exist only locally (for T-folds) or not exist at all (for $R$-spaces). For further details, see e.g. [36, 48].

3. Higher geometric groups for T-duality

3.1. The Lie 2-group $TD_n$

**Generalities.** Recall that T-duality correspondences exist for toric T-backgrounds of type $F^2$. As shown in [7], there is a strict Lie 2-group $TB_{F^2}$ that represents these. In
other words, we can regard a toric T-background as a principal 2-bundle, i.e. a higher or categorified principal bundle with $TB_n^F$ as its structure 2-group. The interesting observation of [7] is then that not only the T-backgrounds but also the correspondence space and the gerbe $\mathcal{G}_C$ over it can be replaced by a principal 2-bundle. A T-duality correspondence then amounts to a double fibration or span of principal 2-bundles which is induced by an underlying span of Lie 2-groups. It is clear that the principal 2-bundle taking over the role of the correspondence space should describe the bundle $\tilde{P} \times_X \hat{P} \to X$, so the structure group should contain the abelian group $U(1)^{2n}$. As observed in [7], this group needs to be extended to a categorical torus, see [55], denoted by $TD_n$.

We will mostly use crossed modules of Lie groups in order to describe Lie 2-groups. Some background material and further pointers to the literature on higher groups and bundles are found in appendix B and appendix D.

The 2-group $TD_n$. We regard the abelian group $U(1)^{2n}$ as the quotient $\mathbb{R}^{2n}/\mathbb{Z}^{2n}$ and extend the corresponding action groupoid by a factor of $U(1)$, leading to the Lie groupoid

\[
\begin{array}{ccc}
\mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times U(1) & \longrightarrow & \mathbb{R}^{2n} \\
\xi & \mapsto & \xi - m_1 \\
(\xi, m_1, \phi_1) & \mapsto & (\xi - m_1, m_2, \phi_2) \\
(\xi, m_1 + m_2, \phi_1 + \phi_2) & \mapsto & (\xi, m_1, \phi_1)
\end{array}
\]

which becomes the (strict) Lie 2-group $TD_n$ together with the monoidal structure and inverse functor defined by

\[
(\xi_1, m_1, \phi_1) \otimes (\xi_2, m_2, \phi_2) := (\xi_1 + \xi_2, m_1 + m_2, \phi_1 + \phi_2 - \langle \xi_1, m_2 \rangle),
\]

\[
\mathrm{id}_\xi := (\xi, 0, 0), \quad (\xi, m, \phi)^{-1} := (\xi - m, -m, -\phi),
\]

(3.1a)

for $\xi, \xi_1, \xi_2 \in \mathbb{R}^{2n}, m, m_1, m_2 \in \mathbb{Z}^{2n}$ and $\phi, \phi_1, \phi_2 \in U(1)$. We will always use additive notation for elements in $\mathbb{R}/\mathbb{Z} \cong U(1)$. The binary bracket $\langle -, - \rangle$ is defined as

\[
\langle \xi_1, \xi_2 \rangle = \xi_1^T \begin{pmatrix} 0 & 0 \\ 1_n & 0 \end{pmatrix} \begin{pmatrix} \xi_2 \\ 0 \end{pmatrix} \quad \text{or} \quad \begin{pmatrix} \hat{\xi}_1 \\ \hat{\xi}_2 \end{pmatrix} = \xi_1 \xi_2
\]

(3.2)

for all $\xi_1, \xi_2 \in \mathbb{R}^{2n}$. This Lie 2-group corresponds to a crossed module of Lie groups,

\[
TD_n := (\mathbb{Z}^{2n} \times U(1) \xrightarrow{t} \mathbb{R}^{2n}),
\]

\[
t(m, \phi) := m,
\]

\[
\xi \triangleright (m, \phi) := (m, \phi - \langle \xi, m \rangle)
\]

(3.3)

with the group products in $\mathbb{Z}^{2n} \times U(1)$ and $\mathbb{R}^{2n}$ abelian and evident.

\[\text{---}\]

See appendix B for a review of higher groups and a summary of our conventions.
Lie 2-algebra of $TD_n$. We will also need the infinitesimal description of $TD_n$ in terms of the Lie 2-algebra $\mathfrak{t}_n$. The latter is given as the crossed module of Lie algebras

$$\mathfrak{t}_n = (\mathbb{R} \xrightarrow{t} \mathbb{R}^{2n}) ,$$

$$t(y) = 0 , \quad \xi \triangleright y = 0$$

for $y \in \mathbb{R}$ and $\xi \in \mathbb{R}^{2n}$. Weak morphisms of Lie 2-groups describing automorphisms of $TD_n$ then translate to invertible Lie 2-algebra morphisms $\phi : \mathfrak{t}_n \to \mathfrak{t}_n$ as defined in appendix C.

3.2. Automorphisms of $TD_n$

In [7], the authors announced the result that the group of isomorphism classes of objects $\pi_0(\text{Aut}(TD_n))$ in the 2-group of automorphisms given by crossed intertwiners $\text{Aut}(TD_n)$ is isomorphic to the group $GO(n,n;\mathbb{Z})$ defined in (2.7); the proof was given recently in [19]. In the following, we identify a subset of these automorphisms that will be suitable for our purposes.

Automorphism 2-functors. Weak morphisms $\Phi : TD_n \to TD_n$ can be equivalently regarded as weak 2-functors between the corresponding one-object 2-groupoids $\Phi : BTD_n \to BTD_n$ as defined in appendix A. Such a 2-functor consists of a functor $\Phi_1 : TD_n \to TD_n$ and a natural transformation given by a map $\Phi_2 : \mathbb{R}^{2n} \times \mathbb{R}^{2n} \to \mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times U(1)$ that satisfies the naturality and coherence conditions listed in (B.5). We start from the most general ansatz

$$\Phi_1(\xi,m,\phi) = (\Phi^0(\xi,m,\phi),\Phi^1(\xi,m,\phi),\Phi^2(\xi,m,\phi)) ,$$

$$\Phi_2(\xi_1,\xi_2) = (\Phi^0_2(\xi_1,\xi_2),\Phi^1_2(\xi_1,\xi_2),\Phi^2_2(\xi_1,\xi_2)) ,$$

(3.5)

where all components are evident smooth maps. The naturality condition reads as

$$\Phi_2(\xi_1,\xi_2) \circ (\Phi_1(\xi_1,m_1,\phi_1) \otimes \Phi_1(\xi_2,m_2,\phi_2))$$

$$= \Phi_1(\xi_1 + \xi_2,m_1 + m_2,\phi_1 + \phi_2 - \langle\xi_1,m_2\rangle) \circ \Phi_2(\xi_1 - m_1,\xi_2 - m_2) ,$$

(3.6)

and the coherence condition is

$$\Phi_2(\xi_1 + \xi_2,\xi_3) \tilde{\circ} (\Phi^0_2(\xi_1,\xi_2) + \Phi^1_2(\xi_1,\xi_2),\Phi^2_2(\xi_1,\xi_2))$$

$$= \Phi_2(\xi_1,\xi_2 + \xi_3) \tilde{\circ} (\Phi^0_2(\xi_1,\xi_1 + \xi_3),\Phi^1_2(\xi_1,\xi_2 + \xi_3),\Phi^2_2(\xi_1,\xi_2 + \xi_3) - \langle\Phi^0_1(\xi_1),\Phi^1_2(\xi_1,\xi_2 + \xi_3)\rangle) .$$

(3.7)

Because $\Phi_1$ is a functor, we have

$$\Phi^0_1(\xi,m,\phi) = \Phi^0(\xi) ,$$

$$\Phi^1_1(\xi,m,\phi) = \Phi^1(\xi,m,\phi) - \Phi^0(\xi - m) ,$$

$$\Phi^2_1(\xi,0,0) = 0 ,$$

(3.8)

$$\Phi^0_2(\xi_1,m_1 + m_2,\phi_1 + \phi_2) = \Phi^0_1(\xi_1,m_1,\phi_1) + \Phi^0_1(\xi - m_1,m_2,\phi_2) .$$

Applying the target map to both sides of (3.6) implies that $\Phi^0_2(\xi_1,\xi_2) = \Phi^0_1(\xi_1 + \xi_2)$. The composition on the left-hand side of (3.6) implies that $\Phi^1_2$ measures the failure of $\Phi^1_1$ to be additive:

$$\Phi^1_2(\xi_1,\xi_2) = \Phi^1_1(\xi_1 + \xi_2) - \Phi^0_1(\xi_1) - \Phi^0_1(\xi_2) .$$

(3.9)
The other composition is automatically satisfied, and the sources of both sides of (3.6) match.

We now restrict ourselves to a particular class of morphisms in which $\Phi_0^1$ is a group isomorphism on objects. This amounts to

$$
\Phi_0^1(\xi) = g\xi, \quad \Phi_1^1(\xi, m, \phi) = gm, \quad \Phi_2^1 = 0
$$

for $g \in \text{GL}(2n; \mathbb{Z})$, and we directly restrict $g$ further to be an element in $\text{GO}(n, n; \mathbb{Z})$. We note that the restriction to crossed intertwiners in [7] certainly implies this restriction. The coherence condition then reduces to

$$
\Phi_2^2(\xi_1, \xi_2) + \Phi_2^2(\xi_1 + \xi_2, \xi_3) = \Phi_2^2(\xi_1, \xi_2 + \xi_3) + \Phi_2^2(\xi_2, \xi_3),
$$

which implies

$$
\Phi_2^2(\xi_1, 0) = \Phi_2^2(0, \xi_2) = \Phi_2^2(0, 0).
$$

Also, the naturality condition (3.6) for $\xi_1 = m_1 = \phi_2 = 0$ reduces to

$$
\Phi_1^2(\xi_2, m_2, \phi_1) = \Phi_1^2(\xi_2, m_2, 0) + \Phi_1^2(0, 0, \phi_1),
$$

allowing us to split $\Phi_1^2$ into two components,

$$
\Phi_1^2(\xi_2, m_2, \phi_1) = \Phi_1^{21}(\xi_2, m_2) + \Phi_1^{22}(\phi_1)
$$

with $\Phi_1^{21}(0, 0) = 0$. Naturality for $\xi_1 = m_1 = 0$ then implies linearity of $\Phi_1^{22}$.

Further following [7], we restrict to morphisms with $\Phi_1^{21}(\xi, m) = \Phi_2^2(m, \xi)$ and assume $\Phi_2^2$ to be bilinear. This now completely solves the coherence relation (3.7). We further set $\Phi_1^2(\phi) = (-1)^{|\phi|} \phi$, reducing the naturality condition (3.6) to

$$
\Phi_2^2(\xi_1, m_2) - \Phi_2^2(m_2, \xi_1) = \langle g\xi_1, gm_2 \rangle - (-1)^{|\phi|} \langle \xi_1, m_2 \rangle.
$$

For an element $g$ parameterized as in (2.6), we have

$$
\Phi_2^2(\xi, m) - \Phi_2^2(m, \xi) = \xi^T \begin{pmatrix} C^T A & C^T B \\ D^T A - (-1)^{|g|} \mathbb{1}_n & D^T B \end{pmatrix} m
$$

$$
= \xi^T \begin{pmatrix} C^T A & C^T B \\ (-1)^{|g|} \mathbb{1}_n - (-1)^{|\phi|} \mathbb{1}_n - B^T C & D^T B \end{pmatrix} m.
$$

For convenience, we define the antisymmetric matrix

$$
\rho(g) := g^T \begin{pmatrix} 0_n & 0_n \\ 0_n & 0_n \end{pmatrix} g - (-1)^{|g|} \begin{pmatrix} 0_n & 0_n \\ 0_n & 0_n \end{pmatrix}
$$

$$
= \begin{pmatrix} C^T A & C^T B \\ D^T A - (-1)^{|g|} \mathbb{1}_n & D^T B \end{pmatrix} = \begin{pmatrix} C^T A & C^T B \\ -B^T C & D^T B \end{pmatrix}
$$

$$
= -\rho^T(g).
$$
We further decompose this matrix into its lower triangular part \( \rho_L(g) \) and its transpose,

\[
\rho(g) = \rho_L(g) - \rho_L(g)^T .
\]  

We then have

\[
\Phi_2^{\eta, \zeta}(\xi_1, \xi_2) := \xi_1^T (\rho_L(g) + \zeta) \xi_2
\]  

for \( \zeta \) an element of \( \text{Sym}(2n; \mathbb{Z}) \), the additive group of \( 2n \times 2n \)-dimensional symmetric matrices. Altogether, we have identified a subset of automorphisms of \( \mathbb{T}_D \), which is parameterized by \( \text{GO}(n, n; \mathbb{Z}) \times \text{Sym}(2n; \mathbb{Z}) \) according to

\[
\Phi_1^{\eta, \zeta}(\xi, m, \phi) = (g \xi, g m, (-1)^{|g|} \phi + m^T (\rho_L(g) + \zeta) \xi) ,
\]

\[
\Phi_2^{\eta, \zeta}(\xi_1, \xi_2) = (g (\xi_1 + \xi_2), 0, \xi_1^T (\rho_L(g) + \zeta) \xi_2) .
\]

**The group \( \text{GO}(n, n; \mathbb{Z}) \times \text{Sym}(2n; \mathbb{Z}) \).** We note that

\[
g_2^T \rho(g_1) g_2 + (-1)^{|g_1|} \rho(g_2) = \rho(g_1 g_2) ,
\]

and we measure the failure of the same relation to hold for the lower triangular part by the symmetric integer-valued matrix

\[
\sigma_L(g_1, g_2) := g_2^T \rho_L(g_1) g_2 + (-1)^{|g_1|} \rho_L(g_2) - \rho_L(g_1 g_2) \in \text{Sym}(2n; \mathbb{Z}) .
\]

This matrix-valued function satisfies

\[
- g_3^T \sigma_L(g_1, g_2) g_3 + \sigma_L(g_1, g_2 g_3) + (-1)^{|g_1|} \sigma_L(g_2, g_3) - \sigma_L(g_1 g_2, g_3) = 0 .
\]

Composition of the automorphisms (3.20) now induces a group structure on the space \( \text{GO}(n, n; \mathbb{Z}) \times \text{Sym}(2n; \mathbb{Z}) \) with the (associative) product given by

\[
(g_1, \zeta_1) \times (g_2, \zeta_2) = (g_1 g_2, g_2^T \zeta_1 g_2 + (-1)^{|g_1|} \zeta_2 + \sigma_L(g_1, g_2)) .
\]

One can lift this group to a 2-group by adding natural 2-transformations to these weak 2-endofunctors. For the purposes of this paper, however, we can work with a smaller 2-group acting on \( \mathbb{T}_D \).

### 3.3. 2-group action on \( \mathbb{T}_D \)

**The 2-group \( \mathcal{G}O(n, n; \mathbb{Z}) \).** The action of a 2-group on a 2-group is readily defined, see appendix B. An action of \( \text{GO}(n, n; \mathbb{Z}) \), trivially regarded as a (strict) 2-group, on \( \mathbb{T}_D \) that is an extension of the weak 2-functor (3.20) with \( \zeta = 0 \) does not exist directly, but the calculations involved in showing this make it evident that the slightly enlarged 2-group \( \mathcal{G}O(n, n; \mathbb{Z}) \) does allow for a 2-group action. This 2-group has underlying Lie groupoid

\[
\xymatrix{\text{GO}(n, n; \mathbb{Z}) \times \mathbb{Z}^{2n} \ar[r] & \text{GO}(n, n; \mathbb{Z}) \ar@/^/[r]^{(g, z_1)} \ar@/_/[r]_{(g, z_2)} & \text{GO}(n, n; \mathbb{Z}) \ar@{.>}[rr] \ar_\text{id}_g^{(g, 0)} \ar_{(g, z)^{-1}}^{(g, z_1 + z_2)} & g ,}
\]

\[
(3.25a)
\]

\[15\]
and the monoidal product and corresponding inverse are given by

\[(g_1, z_1) \otimes (g_2, z_2) = (g_1/g_2, z_1 + g_1 z_2) \quad \text{and} \quad \text{inv}(g, z) = (g^{-1}, -g^{-1} z) \quad (3.25b)\]

for all \(g, g_1, 2 \in \text{GO}(n, n; \mathbb{Z})\) and \(z, z_1, 2 \in \mathbb{Z}^{2n}\). The associator allowing for the existence of the action reads as

\[
a(g_1, g_2, g_3) = \left( g_1 g_2 g_3, \frac{(-1)^{|g_1 g_2 g_3|}}{2} g_1 g_2 g_3 \eta \times \left( g_3^T \text{diag}(\sigma_L(g_1, g_2)) \right.ight.
\]
\[\left. + \text{diag}(\sigma_L(g_1 g_2, g_3) - \sigma_L(g_1, g_2 g_3) - (-1)^{|g_1|} \sigma_L(g_2, g_3)) \right) \right)
\]
\[= \left( g_1 g_2 g_3, \frac{(-1)^{|g_1 g_2 g_3|}}{2} g_1 g_2 g_3 \eta \left( g_3^T \text{diag}(\sigma_L(g_1, g_2)) - \text{diag}(g_3^T \sigma_L(g_1, g_2)) \right) \right), \quad (3.25c)\]

where we have used (3.23) in the computations. Here, \(\text{diag} : \text{Sym}(2n; \mathbb{Z}) \rightarrow \mathbb{Z}^{2n}\) extracts the diagonal vector of a square matrix, and \(\eta\) is the usual \(O(n, n; \mathbb{Z})\)-invariant metric defined in (2.5). We note that indeed\(^9\)

\[
\frac{1}{2} (g_3 \text{diag}(\zeta) - \text{diag}(g_3^T \zeta g_3)) \in \mathbb{Z}^{2n} \quad (3.26)
\]

for any symmetric matrix \(\zeta \in \text{Sym}(2n; \mathbb{Z})\), and the associator is well-defined. The associator is fully encoded in a normalized cocycle \(\text{GO}(n, n; \mathbb{Z})^3 \rightarrow \mathbb{Z}^{2n}\): in particular, \(a(1, g_2, g_3), a(g_1, 1, g_3), \text{and} a(g_1, g_2, 1) \) are trivial, and

\[
\text{id}_{g_1} \otimes a(g_2, g_3, g_4) + a(g_1, g_2 g_3, g_4) + a(g_1, g_2, g_3) = a(g_1, g_2, g_3 g_4) + a(g_1 g_2, g_3, g_4). \quad (3.27)
\]

The 2-group \(\mathscr{H}(n, n; \mathbb{Z})\) is thus a special Lie 2-group in the sense of [56]. In particular, it is skeletal, i.e. isomorphic objects are equal. Moreover, it was shown in [19] that the automorphism 2-group of \(\text{TD}_n\) based on crossed intertwiners is equivalent to a 2-group with the same underlying groupoid as \(\mathscr{H}(n, n; \mathbb{Z})\): this 2-group is strictly equivalent to \(\mathscr{H}(n, n; \mathbb{Z})\).

**Action** \(\mathscr{H}(n, n; \mathbb{Z}) \curvearrowright \text{TD}_n\). The action \(\mathscr{H}(n, n; \mathbb{Z}) \curvearrowright \text{TD}_n\) is now given by the following data: the unital bifunctor

\[\triangleright : \mathscr{H}(n, n; \mathbb{Z}) \times \text{TD}_n \rightarrow \text{TD}_n, \quad (g, z) \times (\xi, m, \phi) \mapsto (g \xi, gm, (-1)^{|g|} \phi + m^T \rho_L(g) \xi + z^T \eta g \xi), \quad (3.28a)\]

the natural transformation

\[
\Upsilon_{\mathscr{H}(n, n; \mathbb{Z})} : (g_1 g_2) \triangleright \xi \xrightarrow{\cong} g_1 \triangleright (g_2 \triangleright \xi), \quad (3.28b)
\]

and the natural transformation

\[
\Upsilon_{\text{TD}_n} : (g \triangleright (\xi_1 + \xi_2)) \xrightarrow{\cong} (g \triangleright \xi_1) + (g \triangleright \xi_2), \quad (3.28c)
\]

\[\Upsilon_{\text{TD}_n}(g, \xi_1, \xi_2) := (g(\xi_1 + \xi_2), 0, -\xi_1^T \rho_L(g) \xi_2). \]

\(^9\)This follows from all terms proportional to off-diagonal elements of \(\zeta\) appearing twice, and all terms proportional to diagonal elements appearing with the even factor of the form \((g_i)_{ii}((g_3)_{ii} - 1)\).
One can check that these data satisfy all the relations required for a 2-group action. In particular, the functors $\Upsilon_{GO(n,n;\mathbb{Z})}$ and $\Upsilon_{TD_n}$ satisfy indeed the required coherence relations found in [57, Prop. 3.2]. In the underlying computations, we have to use the fact that

$$\frac{1}{2}m^T\zeta m + \frac{1}{2}\text{diag}(\zeta)^T m$$  \hspace{1cm} (3.29)

is an integer\(^{10}\) for all $\zeta \in \text{Sym}(2n;\mathbb{Z})$ and $m \in \mathbb{Z}^{2n}$.

**Action of subgroups.** Let us briefly go through the various subgroups of $GO(n,n;\mathbb{Z})$ we introduced in section 2.2 and list their action on $TD_n$.

A) $A$-transformations are parameterized by elements $A \in \text{GL}(n;\mathbb{Z}) \subset \text{O}(n,n;\mathbb{Z})$, and the corresponding automorphisms are strict:

$$\Phi^A_1(\xi, m, \phi) := \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} \xi , \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} m , \phi ,$$

$$\Phi^A_2(\xi_1, \xi_2) := \begin{pmatrix} A & 0 \\ 0 & (A^T)^{-1} \end{pmatrix} (\xi_1 + \xi_2) , 0 , 0 .$$ \hspace{1cm} (3.30a)

B) $B$-transformations are parameterized by an antisymmetric, integer-valued matrix $B$. The corresponding automorphisms read as

$$\Phi^B_1(\xi, m, \phi) := \begin{pmatrix} \mathbb{1}_n & -B \\ 0 & \mathbb{1}_n \end{pmatrix} \xi , \begin{pmatrix} \mathbb{1}_n & -B \\ 0 & \mathbb{1}_n \end{pmatrix} m , \phi - m^T B_L \hat{\xi} ,$$

$$\Phi^B_2(\xi_1, \xi_2) := \begin{pmatrix} \mathbb{1}_n & -B \\ 0 & \mathbb{1}_n \end{pmatrix} (\xi_1 + \xi_2) , 0 , -\xi_1^T B_L \hat{\xi_2} .$$ \hspace{1cm} (3.30b)

where $B = B_L - B_L^T$. This action was also defined, with minor differences in $\Phi_2$, in [7, Sect. 4.1].

$\beta$) $\beta$-transformations are parameterized by an antisymmetric, integer-valued matrix $\beta$. The corresponding automorphisms read as

$$\Phi^\beta_1(\xi, m, \phi) := \begin{pmatrix} \mathbb{1}_n & 0 \\ -\beta & \mathbb{1}_n \end{pmatrix} \xi , \begin{pmatrix} \mathbb{1}_n & 0 \\ -\beta & \mathbb{1}_n \end{pmatrix} m , \phi + m^T \beta_L \hat{\xi} ,$$

$$\Phi^\beta_2(\xi_1, \xi_2) := \begin{pmatrix} \mathbb{1}_n & 0 \\ -\beta & \mathbb{1}_n \end{pmatrix} (\xi_1 + \xi_2) , 0 , \xi_1^T \beta_L \hat{\xi_2} ,$$ \hspace{1cm} (3.30c)

where $\beta = \beta_L - \beta_L^T$.

\(^{10}\)This is clear from the fact that in the first term each summand involving off-diagonal components of $\zeta$ appears with a factor of 2 due to the symmetry of $\zeta$, and the second term then corrects the diagonal sum to an integer.
The generators of factorized dualities are parameterized by \( k \in \{1, \ldots, n\} \) and a sign. The corresponding automorphisms read as

\[
\Phi^{T_k}_{1}(\xi, m, \phi) := \begin{pmatrix} 1_n - 1_k & \pm 1_k \\ \pm 1_k & 1_n - 1_k \end{pmatrix} \xi, \begin{pmatrix} 1_n - 1_k & \pm 1_k \\ \pm 1_k & 1_n - 1_k \end{pmatrix} m, \phi - \langle m, 1_k \xi \rangle,
\]

\[
\Phi^{T_k}_{2}(\xi_1, \xi_2) := \begin{pmatrix} 1_n - 1_k & \pm 1_k \\ \pm 1_k & 1_n - 1_k \end{pmatrix} (\xi_1 + \xi_2), 0, -\langle \xi_1, 1_k \xi_2 \rangle.
\]

G) The \( G \)-transformations are parameterized by \( s_{1,2} \in \{\pm 1\} \), and the corresponding automorphisms are strict:

\[
\Phi^G_{1}(\xi, m, \phi) := \begin{pmatrix} s_1 1_n & 0 \\ 0 & s_2 1_n \end{pmatrix} \xi, \begin{pmatrix} s_1 1_n & 0 \\ 0 & s_2 1_n \end{pmatrix} m, s_1 s_2 \phi,
\]

\[
\Phi^G_{2}(\xi_1, \xi_2) := \begin{pmatrix} s_1 1_n & 0 \\ 0 & s_2 1_n \end{pmatrix} (\xi_1 + \xi_2), 0, 0.
\]

**The Lie 2-group \( \text{TD}^\infty_n \).** In order to capture T-duality involving general affine torus bundles, we have to extend the Lie 2-group \( \text{TD}_n \) by the action of \( \text{GL}(n; \mathbb{Z}) \subset \text{GO}(n, n; \mathbb{Z}) \) to the semidirect product

\[
\text{TD}^\infty_n := \text{GL}(n; \mathbb{Z}) \ltimes \text{TD}_n.
\]

Here, the action of \( \text{GL}(n; \mathbb{Z}) \) is that of the geometric subgroup on \( \text{GO}(n, n; \mathbb{Z}) \). We note that T-duality indeed just allows for an extension by \( \text{GL}(n; \mathbb{Z}) \subset \text{GO}(n, n; \mathbb{Z}) \) and not the perhaps expected group \( \text{GL}(2n; \mathbb{Z}) \). This amounts to a link between the orientations of the torus bundles \( \tilde{P} \) and \( P \) in (2.3).

Note that the group \( \text{GL}(n; \mathbb{Z}) \) indeed acts on \( \text{TD}_n \) without any need for a further extension, which is due to \( \rho_L(g) = 0 \) for \( g \in \text{GL}(n; \mathbb{Z}) \subset \text{GO}(n, n; \mathbb{Z}) \). The 2-group \( \text{TD}^\infty_n \) thus has underlying Lie groupoid

\[
\text{GL}(n; \mathbb{Z}) \times \mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times U(1) \xrightarrow{\text{id}} \text{GL}(n; \mathbb{Z}) \times \mathbb{R}^{2n},
\]

\[
(g, \xi) \xrightarrow{(g, \xi - m_1, \phi)} (g, \xi - m_1, \phi), \quad (g, \xi - m_1, \phi) \xrightarrow{(g, \xi, m, \phi)} (g, \xi, m, \phi), \quad \xi \xrightarrow{(g, \xi, m, \phi)} (g, \xi - m, \phi).
\]

and monoidal structure and inverse functor

\[
(g_1, \xi_1, m_1, \phi_1) \otimes (g_2, \xi_2, m_2, \phi_2) := (g_1 g_2, \xi_1 + g_1 \xi_2, m_1 + g_1 m_2, \phi_1 + \phi_2 - \langle \xi_1, g_1 m_2 \rangle),
\]

\[
\text{inv}(\xi, m, \phi) := (g^{-1}, -g^{-1} \xi, -g^{-1} m, -\phi). \]

(3.32b)
for \( g \in \text{GL}(n; \mathbb{Z}) \subset O(n, n; \mathbb{Z}) \), \( \xi, \xi_{1,2} \in \mathbb{R}^{2n} \), \( m, m_{1,2} \in \mathbb{Z}^{2n} \), \( \phi, \phi_{1,2} \in \text{U}(1) \). Like \( \text{T}_n \), \( \text{T}_n^\kappa \) is a strict Lie 2-group, and the corresponding crossed module of Lie groups is

\[
\text{T}_n^\kappa := (\mathbb{Z}^{2n} \times \text{U}(1) \to \text{GL}(n; \mathbb{Z}) \times \mathbb{R}^{2n}) \, ,
\]

(3.33)

The associated crossed module of Lie algebras is the same as \( \Omega_n \).

4. Geometric T-duality with principal 2-bundles

4.1. Topological T-duality correspondences as principal 2-bundles

In the following, we give the description of geometric T-duality correspondences in terms of principal 2-bundles, extending the description found in [7] to T-dualities involving affine torus bundles.

Span of principal 2-bundles. As mentioned above, it has been shown in [7] that T-duality correspondences can be formulated as spans of principal 2-bundles \( \mathcal{P}, \mathcal{P}^\prime, \) and \( \mathcal{P}_C \) over \( X \),

\[
\begin{array}{c}
\mathcal{P}_C \\
\mathcal{P} \\
\mathcal{P}^\prime
\end{array}
\]

(4.1)

which are induced by correspondences of Lie 2-groups. In the following, we review the cocycle description of the above principal 2-bundles as well as the projections \( \mathcal{P} \) and \( \mathcal{P}^\prime \) between them. We will always consider principal 2-bundles subordinate to a surjective submersion \( Y \to X \).

The principal 2-bundle \( \mathcal{P}_C \). The structure Lie 2-group of \( \mathcal{P}_C \) is the crossed module of Lie groups \( \text{T}_n^\kappa \) defined in (3.33). Correspondingly, the general cocycle relations for principal 2-bundle (D.2) specialize as follows:

\[
\begin{align*}
\mathcal{H} & = (m_{ijk}, \phi_{ijk}) \in C^\infty(Y^{[3]}, \mathbb{Z}^{2n} \times \mathbb{R}/\mathbb{Z}) \, , \\
\mathcal{G} & = (g_{ij}, \xi_{ij}) \in C^\infty(Y^{[2]}, \text{GL}(n; \mathbb{Z}) \times \mathbb{R}^{2n})
\end{align*}
\]

(4.2a)

with \( (ij) \in Y^{[2]} \) and \( (ijk) \in Y^{[3]} \), which satisfy

\[
\begin{align*}
\phi_{kli} + \phi_{ij} + \phi_{jkl} & = (\xi_{ij}, g_{ij}m_{jkl}) \\
m_{ikl} + m_{ijk} & = m_{ijl} + g_{ij}m_{jkl} \\
g_{ik} & = g_{ij}g_{jk} \\
\xi_{ik} & = m_{ijk} + \xi_{ij} + g_{ij}\xi_{jk}
\end{align*}
\]

(4.2b)

on \( Y^{[4]} \) and \( Y^{[3]} \), respectively.
Two such cocycles \((g, h)\) and \((\tilde{g}, \tilde{h})\) are considered equivalent if they are related by a coboundary consisting of maps
\[
b = (m_{ij}, \phi_{ij}) \in C^\infty(Y^{[2]}, \mathbb{R}/\mathbb{Z} \times \mathbb{Z}^{2n}) , \quad a = (g_i, \xi_i) \in C^\infty(Y, GL(n; \mathbb{Z}) \times \mathbb{R}^{2n}) ,
\]
that link the cocycles by the relations
\[
\phi_{ik} + \phi_{ijk} = \tilde{\phi}_{ijk} - \langle \xi_i, g_i m_{ijk} \rangle , \\
m_{ik} + m_{ijk} = g_i m_{ijk} + m_{ij} + g_{ij} m_{jk} , \\
g_{ij}^{-1} g_{ij} g_j , \\
\xi_i + g_i \tilde{\xi}_{ij} = m_{ij} + \xi_{ij} + g_{ij} \xi_j ,
\]
over \(Y^{[3]}\) and \(Y^{[2]}\), cf. the general formulas (D.3).

The flip morphism. The flip morphism [7] is simply the action of the concatenation of all factorized dualities \(T_i^+ \circ \cdots \circ T_n^+\), i.e. the \(GO(n, n; \mathbb{Z})\)-transformation \(g = \eta\). The corresponding automorphism reads as
\[
\Phi_1(g, \xi, m, \phi) := (\eta g \eta , \eta \xi , \eta m , \phi - \langle \xi, m \rangle) , \\
\Phi_2(g_1, \xi_1; g_2, \xi_2) := (\eta (\xi_1 + \xi_2) , 0 , \langle \xi_2, \xi_1 \rangle)) ,
\]
for all \(g \in GL(n; \mathbb{Z})\), \(\xi \in \mathbb{R}^n\), \(m \in \mathbb{Z}^n\), and \(f \in C^\infty(T^n, S^1)\), where \(s_\xi\) denotes the function \(s_\xi t \mapsto t - \xi\). The crossed module \(TB_n^{F2}\) defined in [7] is obtained by restricting to \(g = 1\).
The higher bundles $\hat{P}$ and $\hat{P}$ are correspondingly described by cocycles consisting of maps

$$h = (m_{ijk}, f_{ijk}) \in C^\infty(Y^{[3]}, \mathbb{Z} \times C^\infty(T^n, S^1)), \quad (4.7a)$$
$$g = (g_{ij}, \xi_{ij}) \in C^\infty(Y^{[2]}, GL(n; \mathbb{Z}) \times \mathbb{R}^n),$$

which satisfy

$$f_{ikl} + f_{ijk} = f_{ijl} + f_{jkl} \circ s_{g_{ij}} \xi_{ij},$$
$$m_{ikl} + m_{ijm} = g_{ij} m_{ijkl} + g_{ij} m_{ijkl} ,$$
$$g_{ij} = g_{ij} g_{jk},$$
$$\xi_{ik} = m_{ijk} + \xi_{ij} g_{ij} + \xi_{jk}.$$

Two such cocycles $(g, h)$ and $(\tilde{g}, \tilde{h})$ are considered equivalent if they are related by a coboundary consisting of maps

$$b = (m_{ij}, f_{ij}) \in C^\infty(Y^{[2]}, \mathbb{Z} \times C^\infty(T^n, S^1)), \quad (4.8a)$$
$$a = (g_{ij}, \xi_{ij}) \in C^\infty(Y, GL(n; \mathbb{Z}) \times \mathbb{R}^n),$$

that link the cocycles by the relations

$$f_{ik} + f_{ijk} = f_{ijk} \circ s_{g_{ij}} \xi_{jk} + f_{ij} + f_{jk} \circ s_{g_{ij}} \xi_{ij},$$
$$m_{ik} + m_{ijm} = g_{ij} m_{ijk} + m_{ij} + g_{ij} m_{jk} ,$$
$$\tilde{g}_{ij} = g_{ij}^{-1} g_{ij} g_{ij},$$
$$\xi_{ij} + g_{ij} \tilde{\xi}_{ij} = m_{ij} + \xi_{ij} + g_{ij} \xi_{ij}.$$

**The two projections.** It remains to specify the projections $\hat{\rho}: \mathcal{C} \to \hat{P}$ and $\hat{\phi}: \mathcal{C} \to \hat{P}$ in (4.1) which establish the span relating geometrically T-dual T-backgrounds to each other. The projection $\hat{\rho}$ is called the right-leg projection in [7]\(^{11}\), and it is given by a morphism of Lie 2-groups that induces the bundle map

$$\hat{\rho}: (g_{ij}, \xi_{ij}, m_{ijk}, \phi_{ijk}) = \left( \begin{pmatrix} \hat{g}_{ij} \\ \hat{\xi}_{ij} \end{pmatrix}, \begin{pmatrix} \hat{\xi}_{ij} \\ \hat{\xi}_{ij} \end{pmatrix}, \begin{pmatrix} \hat{m}_{ijk} \\ \hat{m}_{ijk} \end{pmatrix}, \hat{\phi} \right)$$
$$\mapsto (\hat{g}_{ij}, \hat{\xi}_{ij}, \hat{m}_{ijk}, c \mapsto \phi_{ijk} + \hat{m}_{ijk} T \hat{g}_{ij} c).$$

The projection $\hat{\phi}$ is obtained by concatenating the flip morphism from above with the right-leg projection. Explicitly, this amounts to the map

$$\hat{\phi}: (g_{ij}, \xi_{ij}, m_{ijk}, \phi_{ijk}) = \left( \begin{pmatrix} \hat{g}_{ij} \\ \hat{\xi}_{ij} \end{pmatrix}, \begin{pmatrix} \hat{\xi}_{ij} \\ \hat{\xi}_{ij} \end{pmatrix}, \begin{pmatrix} \hat{m}_{ijk} \\ \hat{m}_{ijk} \end{pmatrix}, \phi_{ijk} \right)$$
$$\mapsto \left( \hat{g}_{ij}, \hat{\xi}_{ij}, \hat{\xi}_{ij}, c \mapsto \phi_{ijk} + \hat{m}_{ijk} T \hat{g}_{ij} c + \hat{\xi}_{ij} \hat{\xi}_{ij} \right).$$

This completes the formulation of topological geometric T-correspondences in terms of spans of higher principal bundles: two principal $\mathbf{TB}^\infty_n$-bundles $\mathcal{P}$ and $\hat{P}$ form T-dual pairs if there is a double fibration (4.1) with projections (4.9) and (4.10).

\(^{11}\)Note that we interchanged right and left as compared to [7], so that the right-leg projection is the left projection in diagram (4.1).
Recovering T-backgrounds. Consider the image of the right leg projection $\hat{p}$ as given in (4.9), defining the principal $\mathcal{TB}_n^X$-bundle $\tilde{\mathcal{P}}$ subordinate to a surjective submersion $\sigma: Y \to X$. The triple $(\hat{g}_{ij}, \hat{\xi}_{ij}, \hat{m}_{ij})$ clearly defines an affine torus bundle, which we regard as a principal fiber bundle with fibers $\sigma_{TB}$ in (4.9), defining the principal cocycles describing the pullback 2-bundle $\tilde{\mathcal{P}}_{TB}$, where two points $(y, s, t), (y', s', t') \in V$ are equivalent if and only if
\[
\sigma(y) = \sigma(y'), \quad s = \hat{g}(y, y')s', \quad \text{and} \quad t - t' = \hat{g}(y, y')\hat{\xi}(y, y').
\] (4.12)
Correspondingly, we may cover $\tilde{\mathcal{P}}$ by the implied surjective submersion $V \to \tilde{\mathcal{P}}$. Consider now cocycles describing the pullback 2-bundle $\hat{\pi}^* \mathcal{P}$ subordinate to $V \to P$, recalling that any bundle naturally trivializes when pulled back over itself. We note that
\[
\tilde{\mathcal{P}} := (V/Z^n)/\sim \quad \text{with} \quad V := Y \times \text{GL}(n; \mathbb{Z}) \times \mathbb{R}^n,
\] (4.11)
where two points $(y, s, t), (y', s', t') \in V$ are equivalent if and only if
\[
\sigma(y) = \sigma(y'), \quad s = \hat{g}(y, y')s', \quad \text{and} \quad t - t' = \hat{g}(y, y')\hat{\xi}(y, y').
\] (4.12)
Correspondingly, we can define a coboundary $(g_i, \xi_i, m_{ij}, f_{ij})$ as in (4.8) by
\[
g_i := s_i, \quad \xi_i := t_i, \quad m_{ij} := t_i - t_j - \hat{g}_{ij}\hat{\xi}_{ij}, \quad f_{ij} := 0.
\] (4.14)
This coboundary induces a 2-bundle isomorphism which trivializes the $\tilde{\mathcal{P}}$ part in the cocycle:
\[
(\hat{g}_{ij}, \hat{\xi}_{ij}, \hat{m}_{ijk}, \hat{f}_{ijk}) \xrightarrow{\pi} (1, 0, 0, c \mapsto f_{ijk} + \hat{m}_{ijk}^Tg_i(c - t_i)).
\] (4.15)
We note that the part linear in $t_i$ satisfies
\[
-m_{ijk}^Tg_{it_i} + m_{ijkl}^Tg_{it_i} - m_{ijkl}^Tg_{it_j} + m_{ijkl}^Tg_{it_j} = m_{ijkl}^T(g_{it_i} - g_{it_j}) \in \mathbb{Z},
\] (4.16)
and thus it defines an abelian gerbe subordinate to the cover $V \to \tilde{\mathcal{P}}$. This is the abelian gerbe $\mathcal{G}$ that, together with $\tilde{\mathcal{P}}$, forms the toric T-background captured by $\mathcal{P}$. 

Equivalence of principal 2-bundles. We note that [7, Thm. 3.4.5] shows that the left leg projection yields a bijection between isomorphism classes of principal $\mathcal{TD}_n^X$-bundles and principal $\mathcal{TB}_n^{P2}$-bundles. In this sense, it is clear that no information is gained or lost by choosing to work with either $\mathcal{P}$ or $\mathcal{P}_C$, at least for principal torus bundles. This point will be important below as a differential refinement only exists on $\mathcal{P}_C$.

4.2. Differential refinement of $\mathcal{P}_C$

Adjustment data for $\mathcal{TD}_n^X$. In order to define a non-flat connection on a principal 2-bundle, one generically needs to lift the conventional definition in the literature to that of an adjusted one, see appendix D. As noted there, an adjustment for a crossed module
of Lie groups \( G = (H \to G, \triangleright) \) is really an algebraic datum, given by a map \( \kappa: G \times g \to h \), where \( g \) and \( h \) are the Lie algebras of \( G \) and \( H \), respectively. To qualify as an adjustment, the map \( \kappa \) has to satisfy the condition

\[
(g_2^{-1} g_1^{-1} \triangleright (h^{-1} (X \triangleright h))) + g_2^{-1} \triangleright \kappa(g_1, X) + \kappa(g_2, g_1^{-1} X g_1 - t(\kappa(g_1, X))) - \kappa(t(h)g_1 g_2, X) = 0
\]

for all \( g_1, g_2 \in G \), \( h \in H \), and \( X \in g \).

For the crossed module of Lie groups \( TD_n^\kappa \), a valid choice is

\[
\kappa: (g, \xi; X) \mapsto (0, -\langle X, \xi \rangle)
\]

for \( g \in GL(n; \mathbb{Z}) \) and \( \xi, X \in \mathbb{R}^{2n} \), as one can verify by straightforward computation.

No adjustment for \( TB_n^{F_2} \) or \( TB_n^\kappa \). For the general theory of principal 2-bundles with connection, it is interesting to note that there is, in fact, no adjustment for the crossed module of Lie groups \( TB_n^{F_2} \). The \( C^\infty(T^n, U(1)) \) part of condition (4.17) simplifies to

\[
f(c - X + \xi_1 + \xi_2) - f(c + \xi_1 + \xi_2) + \kappa_1(\xi_1, X)(c + \xi_2) + \kappa_1(\xi_2, X - \kappa_0(\xi_1, X))(c) - \kappa_1(m + \xi_1 + \xi_2, X)(c) = 0,
\]

where we have changed notation to \( g = \xi \) and split \( h = (m, f) \) and \( \kappa = (\kappa_0, \kappa_1) \). Clearly, there is no fixed \( \kappa_1 \) that can satisfy this equation for arbitrary \( f \). This problem persists for the Lie 2-group \( TB_n^\kappa \).

One can now speculate that the absence of an adjustment is due to the disconnected nature of the components of the 2-group. Regarding the situation at the level of Lie 2-algebras as done in [10] and in particular in [12], we would still conjecture that any crossed module of connected Lie groups admits an adjustment.

Differential refinement of \( \mathcal{P}_C \). We begin with the differential refinement of the principal \( TD_n^\kappa \)-bundle \( \mathcal{P}_C \) in the geometric T-duality span (4.1), making the adjusted cocycle formulas (D.2) explicit. Beyond the topological cocycle data \((g_{ij}, \xi_{ij}, m_{ijk}, \phi_{ijk})\), cf. (4.2), we have the 1- and 2-forms

\[
\Lambda \in \Omega^1(Y^{[2]}), \quad A \in \Omega^1(Y, \mathbb{R}^{2n}), \quad B \in \Omega^2(Y)
\]

satisfying the gluing relations

\[
\Lambda_{ik} = \Lambda_{jk} + \Lambda_{ij} + d\phi_{ijk} - (A_i, m_{ijk}),
\]

\[
A_j = g^{-1}_{ij} A_i + g^{-1}_{ij} d\xi_{ij},
\]

\[
B_j = B_i + d\Lambda_{ij} + d(A_i, \xi_{ij}).
\]

The adjusted curvature of this connection on \( \mathcal{P}_C \) is given by locally defined 2- and 3-forms

\[
F = dA \in \Omega^2(Y, \mathbb{R}^{2n}) \quad \text{and} \quad H = dB + (dA, A) \in \Omega^3(Y).
\]
Two differentially refined cocycles \((g, h, A, B)\) and \((\tilde{g}, \tilde{h}, \tilde{A}, \tilde{B})\) are equivalent if there is a differentially refined coboundary between them. Such a coboundary is given by the data \((\xi, m, \phi)\) of a topological coboundary, cf. (4.3), together with a 1-form
\[
\lambda \in \Omega^1(Y)
\]
such that
\[
\begin{align*}
\tilde{A}_{ij} &= \Lambda_{ij} + \lambda_j - \lambda_i - d\phi_{ij} - \langle A_i, m_{ij} \rangle, \\
\tilde{A}_i &= g_i^{-1}A_i + g_i^{-1}d\xi_i, \\
\tilde{B}_i &= B_i + d\lambda_i + \langle dA_i, \xi_i \rangle.
\end{align*}
\]

T-duality. Note that, when applying the flip morphism to a differentially refined cocycle describing a principal \(TD_n^*\)-bundle with connection, we can simply use (D.4) to identify the images of the cocycles describing the principal 2-bundle \(P_C\) under flips. Explicitly, we have the following maps:
\[
\begin{align*}
\xi_{ij} \mapsto \tilde{\xi}_{ij} &= \left(\hat{\xi}_{ij}, \tilde{\xi}_{ij}\right), & h_{ijk} \mapsto \tilde{h}_{ijk} &= h_{ijk} - \xi_{ij} \cdot \hat{\xi}_{ij}, \\
g_{ij} \mapsto \tilde{g}_{ij} &= \left(\hat{g}_{ij}, \tilde{g}_{ij}\right), & A_i \mapsto \tilde{A}_i &= \left(\hat{A}_i, \tilde{A}_i\right), \\
B_i \mapsto \tilde{B}_i &= B_i + \hat{A}_i \wedge \tilde{A}_i.
\end{align*}
\]
As a consequence, the curvatures are mapped to
\[
\begin{align*}
F_i \mapsto \tilde{F}_i &= \left(\hat{F}_i, \tilde{F}_i\right) = \left(\frac{d\hat{A}_i}{dA_i}\right) \quad \text{and} \quad H_i \mapsto \tilde{H}_i &= d\tilde{B}_i + (dA_i) \wedge \hat{A}_i,
\end{align*}
\]
and we note that the 3-form part of the curvature remains invariant: \(H_i = \tilde{H}_i\).

Due to the absence of an adjustment for \(TB_n^*\), however, we do not have analogues of the left- or right-leg projections that map to the principal 2-bundles \(\mathcal{P}\) and \(\hat{\mathcal{P}}\). Instead, we have to work over the correspondence space, but these cocycles contain all the required information, as we will show next.

Recovering the differentially refined T-backgrounds. Because we do not have the bundles \(\mathcal{P}\) or \(\hat{\mathcal{P}}\) at our disposal, we work with the bundle \(\mathcal{P}_C\). The cocycle data \((\hat{g}_{ij}, \xi_{ij}, m^\tau_{ijk}, A_i)\) and \((\tilde{g}_{ij}, \hat{\xi}_{ij}, \tilde{m}^\tau_{ijk}, \hat{A}_i)\) contained in a cocycle describing \(\mathcal{P}_C\) describe two affine torus bundles \(\hat{P}\) and \(\tilde{P}\) over \(X\) equipped with connections. It remains to recover the two gerbes \(\hat{\mathcal{G}}\) and \(\tilde{\mathcal{G}}\).

To this end, we pull back \(\mathcal{P}_C\) along \(\Pi = \hat{\pi} \circ \hat{\rho} = \tilde{\pi} \circ \tilde{\rho}\), so that the part corresponding to the affine torus bundles trivializes. Let \(\sigma: Y \to X\) be a surjective submersion. Then
the correspondence space forms an affine torus bundle, which we regard as a fiber bundle with fibers $GL(n; \mathbb{Z}) \times \mathbb{R}^{2n}$. This bundle can be identified with

$$
P \times X \cong \left( V/\mathbb{Z}^{2n} \right) / \sim \quad \text{with} \quad V := Y \times GL(n; \mathbb{Z}) \times \mathbb{R}^{2n},$$

(4.26)

where two points $(y, s, t), (y', s', t') \in V$ are equivalent if and only if

$$
\sigma(y) = \sigma(y'), \quad s = g(y, y')s', \quad \text{and} \quad t - t' = g(y, y')\xi(y, y').
$$

(4.27)

The fibered product over the correspondence space is then given by

$$
V^2 = \{(y_i, s_i, t_i, y_j, s_j, t_j) \in Y^2 \times GL(n; \mathbb{Z})^2 \times \mathbb{R}^{4n} | \sigma(y_i) = \sigma(y_j), \quad s_i = g_{ij}s_j, \quad t_i - t_j = g_{ij}\xi_{ij} \in \mathbb{Z}^n\},
$$

(4.28)

and we introduce the coboundary $(g_{ij}, \xi_{ij}, m_{ijk}, \phi_{ijk})$ with

$$
g_i := s_i, \quad \xi_i := t_i, \quad m_{ijk} := t_i - t_j - g_{ij}\xi_{ij}, \quad \phi_{ijk} := 0.
$$

(4.29)

This coboundary mostly trivializes the cocycle describing $\mathcal{P}_C$:

$$
(\xi_{ij}, m_{ijk}, \phi_{ijk}) \xrightarrow{\sim} (1, 0, 0, \phi_{ijk} + \langle \xi_{ij}, t_j - t_k - g_{ij}\xi_{jk} \rangle).
$$

(4.30)

Because the latter expression is a cocycle with $m_{ijk}$ and $\xi_{ij}$ trivial, it is the cocycle of an abelian gerbe. Furthermore, we note that this expression does not depend on $\hat{t}$; therefore, it is the pullback of a gerbe $\hat{\mathcal{G}}$ on $\hat{\mathcal{P}}$ along the map $\hat{\pi}$ in (2.3).

Let us now consider the differential refinement of the Čech cocycle $(1, 0, 0, \phi_{ijk} + \langle \xi_{ij}, t_j - t_k - g_{ij}\xi_{jk} \rangle)$. The data $(\Lambda, A, B) \in \Omega^1(V^2) \oplus \Omega^1(V, \mathbb{R}^{2n}) \oplus \Omega^2(V)$ satisfy the gluing relations

$$
\Lambda_{ik} = \Lambda_{jk} + \Lambda_{ij} + d\phi_{ijk}, \quad A_j = A_i, \quad B_j = B_i + d\Lambda_{ij}, \quad (4.31)
$$

and we note that $(\hat{\Lambda}, \hat{B})$ form the differential refinement of the gerbe $\hat{\pi}^*\hat{\mathcal{G}}$. In order to recover the gerbe $\hat{\mathcal{G}}$, we apply first the flip morphism to the cocycle data and then go through the same procedure.

### 4.3. Example: Geometric T-duality with nilmanifolds

**Topological T-duality for nilmanifolds.** An instructive example of geometric T-duality is that of geometric T-duality between three-dimensional nilmanifolds with fluxes, i.e. abelian gerbes with 3-form curvature $H$. Recall that a three-dimensional nilmanifold $N_k$ is a principal circle bundle over $T^2$ characterized by its first Chern number $k \in H^2(T^2, \mathbb{Z}) \cong \mathbb{Z}$. We can describe it as a quotient of $\mathbb{R}^3$ with coordinates $(x, y, z)$ by the relations

$$
(x, y, z) \sim (x, y + 1, z) \sim (x, y, z + 1) \sim (x + 1, y, z - ky),
$$

(4.32)
where $x$ and $y$ are coordinates on the base and $z$ is the fiber coordinate. Subordinate to the surjective submersion $\mathbb{R}^2 \to \mathbb{T}^2$, we can describe the principal circle bundle $N \to \mathbb{T}^2$ by a Čech cocycle given by the map $g: (\mathbb{R}^2)^{[2]} \to \mathbb{R}$ with

$$g(x, y; x', y') := k(x' - x)y . \tag{4.33}$$

A gerbe $\mathcal{G}_\ell$ on the nilmanifold $N$ is characterized by its Dixmier–Douady class $\ell \in H^3(N, \mathbb{Z}) \cong \mathbb{Z}$; subordinate to the surjective submersion $\mathbb{R}^3 \to N$, we can describe it by a Čech cocycle $h: (\mathbb{R}^2)^{[3]} \to \mathbb{R}$ with

$$h(x, y, z; x', y', z'; x'', y'', z'') = \ell x(y - y')(z' - z'') . \tag{4.34}$$

It is well-known that the T-background $(\mathbb{T}^2, N_k, \mathcal{G}_\ell)$ is an $F^2$-background, and (topological) geometric T-duality corresponds to the duality

$$(\mathbb{T}^2, N_k, \mathcal{G}_\ell) \leftrightarrow (\mathbb{T}^2, N_\ell, \mathcal{G}_k) . \tag{4.35}$$

**Differential refinement.** The connection on the principal circle bundle $N_k \to \mathbb{T}^2$ is given by 1-forms

$$A(x, y) = kx \, dy \in \Omega^1(\mathbb{R}^2) \tag{4.36a}$$

which are local pullbacks of the global 1-form $dz + kx \, dy$ on $N_k$. This leads to the Kaluza–Klein metric

$$g(x, y, z) = dx^2 + dy^2 + (dz + kx \, dy)^2 \tag{4.36b}$$
on the total space of $N_k$. Moreover, the gerbe $\mathcal{G}_\ell$ over $N_k$ is endowed with a connective structure given by the 2-form connection and the 1-form

$$B(x, y, z) = \ell x \, dy \wedge dz \in \Omega^2(\mathbb{R}^3) , \quad A(x, y, z; x', y', z') = \ell (x - x')ydz \in \Omega^2(\mathbb{R}^3) , \tag{4.36c}$$

and these data satisfy the cocycle conditions (D.2) for the 2-group $BU(1) = (U(1) \to *)$ in additive notation. The curvature of the gerbe $\mathcal{G}_\ell$ is the image of its Dixmier–Douady class in de Rham cohomology,

$$H = \ell dx \wedge dy \wedge dz . \tag{4.36d}$$

**Higher bundle description.** The topological T-duality (4.35) can be described by a principal $\text{TD}_1$-bundle $\mathcal{P}_C$ over $\mathbb{T}^2$ subordinate to the submersion $\mathbb{R}^2 \to \mathbb{T}^2$, and the cocycles (4.2) specialize as follows:

$$g = \begin{pmatrix} \hat{g}, \hat{\xi} \\ \hat{g}, \xi \end{pmatrix} , \quad \hat{g}(x, y; x', y') = \mathbbm{1} , \quad \hat{\xi}(x, y; x', y') = \ell(x' - x)y ,$$

$$m = \begin{pmatrix} \hat{m} \\ \hat{m} \end{pmatrix} , \quad \hat{m}(x, y; x', y'; x'', y'') = -\ell(x'' - x')(y' - y) ,$$

$$\phi(x, y; x', y'; x'', y'') = \frac{1}{2} k\ell \left( y' (xx'' - xx' - xx'') - (x'' - x')(y^2 - y'^2) x \right) . \tag{4.37a}$$

---

12We note that $(\mathbb{R}^2)^{[2]} = \mathbb{R}^2 \times_\mathbb{Z} \mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{Z}^2$ and $(\mathbb{R}^3)^{[3]} = \mathbb{R}^2 \times_\mathbb{Z} \mathbb{R}^2 \times_\mathbb{Z} \mathbb{R}^2 \cong \mathbb{R}^2 \times \mathbb{Z}^2 \times \mathbb{Z}^2$.

13We note that the structure group can be reduced to $\text{TD}_1$. 

26
The differential refinement (4.20) is given by

\[ A = \left( \frac{\hat{A}}{\check{A}} \right) = \left( \frac{kn \, dy}{\ell \, dx} \right), \]

\[ B(x, y) = 0, \]

\[ \Lambda(x, y; x', y') = \frac{1}{2} k \ell (xx' \, dy + (xy + x'y' + y^2(x' - x)) \, dx). \]

We note that under the flip homomorphism, the roles of \( k \) and \( \ell \) are interchanged, as expected.

**Span of principal 2-bundles.** Focusing on the topological part, the image of the projection \( \check{\pi} : \mathcal{P}_C \rightarrow \check{\mathcal{P}} \) in (4.1) is obtained from formula (4.9). It is given by the \( T^2 \)-bundle \( \check{\mathcal{P}} \) over \( T^2 \) which, subordinate again to the surjective submersion \( \mathbb{R}^2 \rightarrow T^2 \), is described by the cocycle

\[ f(x, y; x', y'; x'', y'') = \left( c \mapsto \frac{1}{2} k \ell \left( y'(xx'' - xx' - x'''x'') - (x'' - x')(y^2 - y^2)x \right) \right. \]

\[ m(x, y; x', y'; x'', y'') = -k(x'' - x')(y' - y), \]

\[ \xi(x, y; x', y') = k(x' - x)y. \]

On the other hand, we have the projection \( \hat{\pi} : \mathcal{P}_C \rightarrow \hat{\mathcal{P}} \) in (4.1), whose image is

\[ f(x, y; x', y'; x'', y'') = \left( c \mapsto \frac{1}{2} k \ell \left( y'(xx'' - xx' - x'''x'') - (x'' - x')(y^2 - y^2)x \right) \right. \]

\[ m(x, y; x', y'; x'', y'') = -\ell(x'' - x')(y' - y), \]

\[ \xi(x, y; x', y') = \ell(x' - x)y. \]

**Recovering the full T-backgrounds.** From the cocycle (4.37), we readily extract the cocycle data for the two circle bundles \( N_k \) and \( N_\ell \) described by \((\hat{\xi}, \hat{m}, \hat{A})\) and \((\check{\xi}, \check{m}, \check{A})\), respectively.

The gerbe cocycles are also extracted from (4.37) using the formulas (4.30) and (4.31). In the case of the gerbe \( \mathcal{G}_k \) over \( N_k \), we obtain the 2-form

\[ B = \langle dA, a \rangle = kdx \wedge dy \check{t}. \]

Identifying \( \check{t} \) with \( z \), this potential 2-form has curvature

\[ H = kdx \wedge dy \wedge dz, \]

which is indeed the curvature of the gerbe \( \mathcal{G}_k \) on \( \hat{P} = N_\ell \).
5. T-duality with T-folds and principal 2-groupoid bundles

The first step in generalizing geometric T-backgrounds is to consider T-folds, i.e. T-backgrounds which are locally geometric but globally glued together by general elements of the T-duality group. As before, we will restrict ourselves to affine torus bundles so that the T-duality group is $\text{GO}(n, n; \mathbb{Z})$. In order to discuss these fully, we first consider the mathematical description of Kaluza–Klein reduction that corresponds to double dimensional reduction.

5.1. Kaluza–Klein reductions yield principal 2-groupoid bundles

From our above discussion, it is clear that T-duality is intimately related to Kaluza–Klein reduction: the definition of topological T-duality via the Gysin sequence relies on fiber integration, and the metric on the total space of the principal torus bundle is defined as the Kaluza–Klein metric. For a more detailed discussion of this point with regards to double field theory, see also [14]. In order to push our analysis to the non-geometric situation, let us first develop some more mathematical tools for dealing with Kaluza–Klein reductions. This discussion will also explain the origin of the 2-group $\text{TD}_n$.

For a related but slightly different perspective that interprets double field theory as a Kaluza–Klein theory, see also [15, 16, 17].

**Kaluza–Klein reduction.** By a Kaluza–Klein reduction we mean the reduction of geometric structures on a principal torus bundle $P$ over a manifold $X$ to geometric structures on the manifold $X$. Mathematically, most geometric structures we want to reduce (as e.g. Riemannian metrics, gerbes, and principal bundles with connections) are given by functors that are represented by a classifying space $C$. An example would be principal $G$-bundles for $G$ some topological group, which are maps from $P$ to $C = BG$. In the following, we will focus on topological aspects, which suffices for the present section.

If $P = X \times \mathbb{T}^n$ is topologically trivial, then we have the usual currying relation

$$C^0(X \times \mathbb{T}^n, C) = C^0(X, C^0(\mathbb{T}^n, C)) ,$$

(5.1)

where $C^0(A, B)$ denotes the space of continuous maps from $A$ to $B$.\(^{14}\) This is due to the functors $\mathbb{T}^n \times -$ and $C^0(\mathbb{T}^n, -)$ forming an adjunction in a Cartesian-closed category. Taking homotopy classes,

$$[X \times \mathbb{T}^n, C] = [X, C^0(\mathbb{T}^n, C)] ,$$

(5.2)

we see that $C^0(\mathbb{T}^n, C)$ classifies $C$-objects on a trivial $n$-torus bundle. Note that, for $n = 1$, we obtain the maps from $X$ into the free loop space $LC := C^0(S^1, C)$ of $C$.\(^{14}\)

\(^{14}\)For this to hold, one must work in a Cartesian-closed category of topological spaces. The category of all topological spaces and continuous maps fails to be Cartesian-closed, but there are well-known fixes for this, e.g. working with compactly generated weakly Hausdorff spaces. For physical purposes, one might want to work with smooth maps rather than continuous ones; in that case one can use e.g. diffeological spaces, cf. [58] for a detailed discussion. Here, we neglect such technical details.
If $P$ is non-trivially fibered over $X$, then the above discussion holds only locally. In particular, the fibers can only be identified with $\mathbb{T}^n$ up to an action of $U(1)^n$, and we replace the mapping space $[\mathbb{T}^n, C]$ with the homotopy quotient space\(^{15}\)

$$C^0(\mathbb{T}^n, C) \sslash U(1)^n. \quad (5.3)$$

In the case $n = 1$, this quotient is also called the cyclic loop space, and the mapping

$$[P, C] \to [X, C^0(\mathbb{T}^n, C) \sslash U(1)^n] \quad (5.4)$$

is also called double dimensional reduction; see [59, 42] and also the corresponding nLab page\(^{16}\) for further details. We note that there is again an adjunction between the reduction functor (5.4) and the corresponding oxidation functor.

**The group $\text{TD}_1$.** In our double dimensional reduction, we will have to restrict ourselves to the zero modes along the fibers. This restriction, however, allows for much computational simplification, as we will see in the following.

For one-dimensional T-duality, we are interested in the case $n = 1$ and $C = \text{BBU}(1)$, the classifying space for abelian bundle gerbes\(^{17}\). Recall that for any higher group $G$, there is a homotopy equivalence between $LBG$ and the homotopy quotient $BG \sslash G$, which can be modeled by the corresponding action groupoid. In the case of $\text{BBU}(1)$, we thus identify

$$L\text{BBU}(1) \cong \text{BBU}(1) \times \text{BU}(1) \cong (U(1) \times U(1) \Rightarrow U(1) \Rightarrow *) . \quad (5.5)$$

The cyclic loop space $L\text{BBU}(1) \sslash U(1)$ is again a homotopy quotient, and we arrive at

$$(L\text{BBU}(1) \sslash U(1)) \cong \text{BU}(1) \times L\text{BBU}(1) \cong (U(1) \times U(1) \times U(1) \Rightarrow U(1) \times U(1) \Rightarrow *). \quad (5.6)$$

We note that the latter space is the classifying space of a smooth Lie 2-group $\mathcal{G}$,

$$L\text{BBU}(1) \sslash U(1) \cong B\mathcal{G}, \quad (5.7)$$

where

$$\mathcal{G} = \text{BU}(1) \times U(1) \times U(1). \quad (5.8)$$

Replacing the groups $U(1)$ with 2-groups $\mathbb{R} \times \mathbb{Z} \Rightarrow \mathbb{R}$ and taking the resulting crossed module of Lie groups, we arrive at the complex

$$\text{TD}_1 = (U(1) \times \mathbb{Z}^2 \xrightarrow{\theta} \mathbb{R}^2, \triangleright), \quad (5.9)$$

which underlies the 2-group $\text{TD}_1$. We note that the action $\triangleright$ has to be inferred to be that for $\text{TD}_1$.

---

\(^{15}\)This is not an ordinary quotient since the action of $U(1)^n$ has fixed points (on constant maps). Technically, such homotopy quotients can be realized in the category of topological spaces by using (topologically realized) classifying spaces to remove such fixed points. These details do not concern us, however: for our purposes, it is much more natural to model them in terms of higher group(oid)s as we explain.

\(^{16}\)https://ncatlab.org/nlab/show/geometry+of+physics---+fundamental+super+p-branes

\(^{17}\)This is simply the strict 2-category with a single object, a single 1-cell and $U(1)$ as its 2-cells.
Recovering the group \( \text{TD}_n \). We now readily iterate the above procedure. Because \( \mathcal{G} \) is a Cartesian product of groups, we can do this for each factor separately. Above we saw that, in each step,

\[
\mathcal{B} \mathcal{U}(1) \to \mathcal{B}(\mathcal{B} \mathcal{U}(1) \times \mathcal{U}(1) \times \mathcal{U}(1)).
\]

(5.10)

Similarly, it is easy to see that

\[
\mathcal{B} \mathcal{U}(1) \to \mathcal{B} \mathcal{U}(1) \times \mathcal{U}(1) \times \mathcal{B} \mathcal{U}(1),
\]

(5.11)

where the last factor, coming from the \( \mathcal{U}(1) \)-action of the cyclification, acts on the second, producing an image in the first. This is no longer the classifying space of a Lie group but a Lie groupoid whose set of objects is \( \mathcal{U}(1) \). We can consistently truncate to the first factor \( \mathcal{B} \mathcal{U}(1) \) in order to retain a group. Iterating this procedure \( n \) times and replacing \( \mathcal{U}(1) \)-factors with \( \mathbb{Z} \to \mathbb{R} \), we arrive at

\[
\text{TD}_n = \left( \mathcal{U}(1) \times \mathbb{Z}^{2n} \to \mathbb{R}^{2n}, \triangleright \right).
\]

(5.12)

Again, deriving the correct action \( \triangleright \) requires substantially more work.

**Lie Groupoids.** In the iteration procedure above, we truncated the part obtained from \( \mathcal{B} \mathcal{U}(1) \)-factors to preserve the 2-group structure. After two dimensional reductions, however, we ought to keep these groupoid parts. This is intuitively clear as a 2-form \( B \)-field, dimensionally reduced twice, will give rise to scalar fields, which should take values in the space of objects of this groupoid. We will develop this point in the following section. A further dimensional reduction can then be captured by an augmented groupoid, and we will discuss this later in section 6.

### 5.2. Lie 2-groupoid for T-duality with T-folds

**Narain moduli space.** The Lie 2-groupoids arising in the dimensional reduction come with a manifold of objects, which will be the target space of additional scalar fields arising after dimensional reduction. These have a 1-form field strength and correspond to 0-branes from a string theory perspective.

As is well-known, the moduli of the Riemannian metric and the Kalb–Ramond \( B \)-field on \( \mathbb{T}^n \) are given by the Narain moduli space [18]

\[
M_n = \mathcal{O}(n, n; \mathbb{Z}) \setminus \mathcal{O}(n, n; \mathbb{R}) / (\mathcal{O}(n; \mathbb{R}) \times \mathcal{O}(n; \mathbb{R})) =: \mathcal{O}(n, n; \mathbb{Z}) \setminus Q_n,
\]

(5.13)

which has dimension \( n^2 \). As argued above, we have to replace the T-duality group \( \mathcal{O}(n, n; \mathbb{Z}) \) by \( \mathcal{G} \mathcal{O}(n, n; \mathbb{Z}) \) to allow for general torus bundles. Correspondingly, we will work with the scalar manifold

\[
G M_n = \mathcal{G} \mathcal{O}(n, n; \mathbb{Z}) \setminus \mathcal{O}(n, n; \mathbb{R}) / (\mathcal{O}(n; \mathbb{R}) \times \mathcal{O}(n; \mathbb{R})) =: \mathcal{G} \mathcal{O}(n, n; \mathbb{Z}) \setminus Q_n.
\]

(5.14)
Lie 2-groupoid $\mathcal{T}_n$. We can replace the quotient $GM_n$ again by its action groupoid,
\[ \text{GO}(n, n; \mathbb{Z}) \times Q_n \rightrightarrows Q_n. \] (5.15)

The advantage of this replacement is that $GM_n$ generically has non-trivial 1-cycles, while $Q_n$ is contractible.\(^{18}\) We can then combine this Lie groupoid with the Lie 2-group $TD_n$, and it is evident how to do this: we need to extend $\text{GO}(n, n; \mathbb{Z})$ in the action groupoid by $\text{GO}(n, n; \mathbb{Z})$ and have it act diagonally on both $Q_n$ and $TD_n$.

This results in the Lie 2-groupoid $\mathcal{T}_n$ with the following 2-, 1-, and 0-cells:\(^{19}\)
\begin{align*}
(\mathcal{T}_n)_2 & = \text{GO}(n, n; \mathbb{Z}) \times \mathbb{Z}^{2n} \times \mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times U(1) \times Q_n, \\
(\mathcal{T}_n)_1 & = \text{GO}(n, n; \mathbb{Z}) \times \mathbb{R}^{2n} \times Q_n, \\
(\mathcal{T}_n)_0 & = Q_n. \\
\end{align*} (5.16)

The 2- and 1-morphisms read as
\[ (g, \xi, q) \leftrightarrow (g, \xi - m, q), \] (5.17)
and they compose vertically and horizontally according to
\[ (g, z_1, \xi, m_1, \phi_1, q) \circ (g, z_2, \xi - m_1, m_2, \phi_2, q) := (g, z_1 + z_2, \xi, m_1 + m_2, \phi_1 + \phi_2, q) \] (5.18)
and
\[ (g_1, z_1, \xi_1, m_1, \phi_1, q) \otimes (g_2, z_2, \xi_2, m_2, \phi_2, g_1^{-1}q) := \left( g_1g_2, z_1 + g_1z_2, \xi_1 + g_1\xi_2, m_1 + g_1m_2, \phi_1 + (-1)^{|g_1|}\phi_2 - \langle \xi_1, g_1m_2 \rangle + m_2^T \rho_L(g_1)\xi_2 + z_1^T \eta g_1\xi_2, q \right). \] (5.19)

Due to the expression in the last component, horizontal composition is no longer associative,

---

\(^{18}\)This is due to $O(n; \mathbb{R}) \times O(n; \mathbb{R})$ being a maximal compact subgroup of $O(n, n; \mathbb{R})$; the topology of a Lie group is essentially that of its maximal compact subgroup.

\(^{19}\)Note that $\mathcal{T}_n$ can be thought of as (i.e. it is equivalent to) a bundle of 2-groups, with fiber $TD_n$, on the orbifold $Q_n/\text{GO}(n, n; \mathbb{Z})$, up to the additional copy of $\mathbb{Z}^{2n}$ in $(\mathcal{T}_n)_2$. Around a non-contractible cycle labeled by $g \in \text{GO}(n, n; \mathbb{Z})$, the fiber $TD_n$ undergoes a monodromy given by $g$. This construction is the direct categorified analogue of a bundle of groups on an orbifold $\Sigma/\Gamma$, which arises e.g. in Yang–Mills-matter theories with gauge group $G$ with a scalar field taking values in a manifold $\Sigma$, for which a discrete subgroup $\Gamma \subset \text{Aut}(G)$ that acts on $\Sigma$ has been gauged.
and we have the associator

\[
\begin{align*}
\alpha(g_1, \xi_1, q_1; g_2, \xi_2, q_2; g_3, \xi_3; g_3) \\
= \left( \alpha(g_1, g_2, g_3), (\id_{\xi_1} \otimes \Upsilon_{T^{D_1}}(g_1, \xi_2, 2\xi_3)) \circ (\id_{\xi_1} \otimes \id_{g_1, \xi_2} \otimes \Upsilon_{\mathcal{O}(n, n; \mathbb{Z})}(g_1, g_2, \xi_3)) \right) \\
= \left( \alpha(g_1, g_2, g_3), (\xi_1 + g_1(\xi_2 + g_2\xi_3), 0, \xi_2^T \rho_L(g_1)g_2\xi_3) \\
\circ (\xi_1 + g_1\xi_2 + g_1g_2\xi_3, 0, \frac{1}{2}\xi_3^T \sigma_L(g_1, g_2)\xi_3 + \frac{1}{2}\text{diag}(\sigma_L(g_1, g_2))^T\xi_3), q_1 \right) \\
= \left( \alpha(g_1, g_2, g_3), \xi_1 + g_1(\xi_2 + g_2\xi_3), 0, \\
\xi_2^T \rho_L(g_1)g_2\xi_3 + \frac{1}{2}\xi_3^T \sigma_L(g_1, g_2)\xi_3 + \frac{1}{2}\text{diag}(\sigma_L(g_1, g_2))^T\xi_3), q_1 \right),
\end{align*}
\]

where \(\alpha(g_1, g_2, g_3)\) is the associator in \(\mathcal{O}(n, n; \mathbb{Z})\) defined in (3.25c), and \(q_{i+1} = g_i^{-1}q_i\), cf. (B.8c). Note that horizontal composition is still unital.

**T-duality as a gauge symmetry.** In our Lie 2-groupoid \(\mathcal{O}_n\), the T-duality group \(\mathcal{O}(n, n; \mathbb{Z})\) appears explicitly on par with the gauge group \(\mathbb{R}^{2n}/\mathbb{Z}^{2n}\). The T-duality group therefore is to be regarded as a gauge group. Because the group is discrete, there are no associated gauge potentials, but there are associated 2-groupoid bundle isomorphisms, effectively quotienting the space of inequivalent principal 2-groupoid bundles, while at the same time introducing new, topologically non-trivial bundles.

### 5.3. T-duality correspondences involving T-folds

A T-duality correspondence between T-folds is now a principal \(\mathcal{O}_n\)-bundle, and we develop the cocycle description of such a bundle in the following.

**Generalities on higher groupoid bundles.** Just as in the case of principal 2-bundles, higher groupoid bundles are most conveniently described in terms of cocycles, i.e. functors from the Čech groupoid of a surjective submersion to the higher groupoid itself.

Let \(M\) be a manifold and \(\sigma: Y \to M\) a surjective submersion, and let \(\check{\mathcal{C}}(Y \to M)\) be the corresponding Čech groupoid, cf. (D.1a), trivially regarded as a (strict) higher Lie groupoid. Let \(\mathcal{F}\) be a (higher) Lie groupoid, which we call the structure groupoid. A (higher) groupoid bundle over \(M\) subordinate to the surjective submersion \(\sigma\) with structure groupoid \(\mathcal{F}\) is then an (appropriately defined) higher functor

\[
\Phi: \check{\mathcal{C}}(Y \to M) \to \mathcal{F}.
\]

Groupoid bundle isomorphisms are given by (higher) natural transformations between two such functors, and the higher isomorphisms are then identified with modifications and higher transfors. Note that a groupoid bundle whose structure groupoid is the delooping \(BG\) of a Lie group \(G\) is just a principal \(G\)-bundle.

The definition of higher functors for Lie \(n\)-groupoids for \(n > 2\) is technically very involved, and it is a good idea to switch to the perspective of quasi-groupoids defined in
terms of Kan simplicial manifolds, cf. appendix E and e.g. [60]. The same holds for the definition of a differential refinement.

We remark that from a physical perspective, (1-)groupoid bundles are the geometric structures underlying gauged sigma models.

**Cocycle description of $\mathcal{T}D_n$-bundles.** We now specialize the above abstract discussion to the case of the structure groupoid $\mathcal{T}D_n$. This leads to the groupoid extension of the discussion in [61]. A $\mathcal{T}D_n$-bundle over a manifold $M$ subordinate to the surjective submersion $\sigma: Y \to M$ is then a weak 2-functor $\Phi: \mathcal{E}(Y \to M) \to \mathcal{T}D_n$. Such a functor is encoded in the data

\[
(g, z, \xi, m, \phi, q) \in C^\infty(Y^{[3]}, \text{GO}(n, n; \mathbb{Z}) \times \mathbb{Z}^{2n} \times \mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times U(1) \times Q_n),
\]

\[
(g, \xi, q) \in C^\infty(Y^{[2]}, \text{GO}(n, n; \mathbb{Z}) \times \mathbb{R}^{2n} \times Q_n),
\]

\[
q \in C^\infty(Y, Q_n),
\]

which define the natural isomorphism $\Phi_2$, the functor $\Phi_1$, and the function $\Phi_0$, respectively. On $Y^{[2]}$, we then have

\[
g_j = g_j^{-1} q_t
\]

with $(ij) \in Y^{[2]}$, while on $Y^{[3]}$, we deduce that

\[
e_{ijk} \circ (\text{id}_{d_{ij}} \otimes \text{id}_{d_{jk}}) = \text{id}_{d_{ik}} \circ e_{ijk}
\]

for $d_{ij} = (g_{ij}, \xi_{ij}, q_t)$, $e_{ijk} = (g_{ik}, z_{ijk}, \xi_{ik}, m_{ijk}, \phi_{ijk}, q_t)$, and $(ijk) \in Y^{[3]}$ or, equivalently,

\[
g_{ik} = g_{ij} g_{jk},
\]

\[
\xi_{ik} = m_{ijk} + \xi_{ij} + g_{ij} \xi_{jk}.
\]

Finally, on $Y^{[4]}$, we have

\[
e_{ikl} \circ (e_{ijk} \otimes \text{id}_{d_{kl}}) = e_{ijl} \circ (\text{id}_{d_{ij}} \otimes e_{jkl}) \circ a(d_{ij}, d_{jk}, d_{kl})
\]

or, equivalently,

\[
z_{ijk} + z_{ikl} = z_{ijl} + g_{ij} z_{jk} + g_{ik} z_{jl} + \frac{(-1)^{|g_{ij}|}}{2} g_{ik} \eta \text{ diag}(\sigma_L(g_{ij}, g_{jk})) + \frac{(-1)^{|g_{ij}|}}{2} g_{ij} \eta \text{ diag}(\sigma_L(g_{ik}, g_{kl}))
\]

\[
- \frac{(-1)^{|g_{ij}|}}{2} g_{il} \eta \text{ diag}(\sigma_L(g_{ij}, g_{jl})) - \frac{(-1)^{|g_{ij}|}}{2} g_{il} \eta \text{ diag}(\sigma_L(g_{jk}, g_{kl}))
\]

\[
m_{ijk} + m_{ikl} = m_{ijl} + g_{ij} m_{jkl},
\]

\[
\phi_{ijk} + \phi_{ikl} = \phi_{ijl} + (-1)^{|g_{ij}|} \phi_{jkl} - \langle \xi_{ij}, g_{ij} m_{jkl} \rangle + m_{jkl}^T \rho_L(g_{ij}) \xi_{jl} + \xi_{jk}^T \rho_L(g_{ij}) g_{jk} \xi_{kl}
\]

\[
+ \frac{1}{2} \xi_{kl}^T \sigma_L(g_{ij}, g_{jk}) \xi_{kl} + \frac{1}{2} \text{ diag}(\sigma_L(g_{ij}, g_{jk}))^T \xi_{kl} - z_{ijk}^T g_{ik} \xi_{kl},
\]

[20]see appendix A for definitions
where the functions $\rho_L$ and $\sigma_L$ were defined in (3.18) and (3.22), respectively. Note that the second equation is automatically satisfied due to (5.25).

Let us now differentially refine this topological21 cocycle data. The adjusted cocycle data only seem to exist if

$$z_{ijk} = (-1)^{|g_{ik}|} \frac{1}{2} g_{ik} \eta \text{diag}(\sigma_L(g_{ij}, g_{jk})).$$

(5.28a)

In this case, we have 1- and 2-forms

$$\Lambda \in \Omega^1(Y^{[2]}), \quad A \in \Omega^1(Y, \mathbb{R}^{2n}), \quad B \in \Omega^2(Y),$$

(5.28b)

which satisfy the gluing relations

$$\Lambda_{ik} = (-1)^{|g_{ij}|} \Lambda_{jk} + \Lambda_{ij} + d\phi_{ijk} - \langle A_i, m_{ijk} \rangle + \frac{1}{2} d \xi_{jk} \rho_L(g_{ij}) \xi_{jk} - A^T_k \eta g_{kj} \eta \rho_L(g_{ij}) \xi_{jk} - \frac{1}{2} A^T_j \rho_L(g_{ij}) \xi_{jk},$$

$$A_j = g_{ij}^{-1} A_i + g_{ij}^{-1} d \xi_{ij},$$

$$(-1)^{|g_{ij}|} B_j = B_i + d \Lambda_{ij} + \langle d A_i, \xi_{ij} \rangle - \frac{1}{2} A^T_j \rho(g_{ij}) A_j.$$

(5.28c)

We note that, for $g_{ij} \in \text{GL}(n; \mathbb{Z}) \subset \text{O}(n, n; \mathbb{Z})$, the relation (5.28a) is automatically satisfied for $z_{ijk} = 0$, and we recover the cocycle relations for differently refined principal $\text{T}^n_D$-bundles.

5.4. Example of a T-fold

3-dimensional example. Let us consider again the popular example of the three-dimensional nilmanifold defined in (4.32) and (4.36) with $k = 0$ and T-dualize along the $y$- and $z$-directions. In the T-correspondence, our base manifold $X$ is then simply the circle parameterized by $x$, and we consider a principal $\mathcal{PD}_n$-bundle over $S^1$ subordinate to the cover $\mathbb{R} \rightarrow S^1$.

Due to dimensionality, there can be no non-trivial triple intersections, and we can thus set $z_{ijk} = m_{ijk} = \phi_{ijk} = 0$. Since principal torus bundles over a circle are topologically trivial, we can trivialize $\hat{g}$ and $\hat{g}$. This leaves us with the scalars $(q_i): Y \rightarrow Q_2$ and the transition functions $(g_{ij}): Y^{[2]} \rightarrow \text{GO}(n, n; \mathbb{Z})$ with the only non-trivial cocycle conditions being

$$g_{ij} \circ q_j = q_i \quad \text{and} \quad g_{ij} g_{jk} = g_{ik}$$

(5.29)

for all $(ij) \in Y^{[2]}$ and $(ijk) \in Y^{[3]}$, together with the evenness condition

$$g_{jk} \eta \text{diag}(\sigma_L(g_{ij}, g_{jk})) \in 2\mathbb{Z}^{2n}$$

(5.30)

\footnote{We note that the “scalar part” of a topological groupoid bundle can already be considered as a part of the differential refinement. This is certainly more sensible from a physical perspective, where scalar fields arise from dimensionally reducing gauge potentials.}
required for adjustment. Modulo this evenness condition, these data are the same as those defining a groupoid bundle over $S^1$ with structure groupoid the action groupoid corresponding to the action $GO(2, 2; \mathbb{Z}) \rtimes Q_2$. This, in turn, is the same as a map of orbifolds $q: S^1 \to GM_2$.

Suppose now that the map $q$ factors through a map

$$q': S^1 \to \mathfrak{o}(2; \mathbb{Z}) \setminus Q_2,$$

where $\mathfrak{o}(2; \mathbb{Z})$ denotes the (abelian) subgroup of $\beta$-transformations in $GO(2, 2; \mathbb{Z})$. In terms of cocycle data, the maps $q_i$ are then glued together by transformations $g_{ij} \in \mathfrak{o}(2, \mathbb{Z})$. In this case, (5.30) holds automatically.

We recall that $Q_2 := O(2, 2; \mathbb{R})/O(2; \mathbb{R})^2$ is contractible and, furthermore, diffeomorphic to $\mathbb{R}^4$. This manifold is identified with the four scalars arising from the dimensional reduction of the metric and the Kalb–Ramond field:

$$\phi_{g_{yy}} = 0, \quad \phi_{g_{xy}} = 0, \quad \phi_{g_{xz}} = 1, \quad \text{and} \quad \phi_B = \ell x. \quad (5.32)$$

The $\beta$-transformation then corresponds to the matrix

$$g_{x+1,x} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & \ell & 1 & 0 \\ -\ell & 0 & 0 & 1 \end{pmatrix}, \quad (5.33)$$

and it acts on $\phi_B$ according to $\phi_B \to \phi_B + \ell$, leaving the other scalars unmodified.

Performing a T-duality transformation both directions will turn the $\beta$-transformation into a $B$-transformation, and the structure of the bundle can be encoded in an ordinary abelian gerbe over the nilmanifold with the transition 1-forms $\Lambda$ encoding the gluing.

**Special $\mathcal{TD}_n$-bundles.** Consider a $\mathcal{TD}_n$-bundle that is isomorphic to a $\mathcal{TD}_n$-bundle described by a cocycle whose underlying cocycle is such that the $(g_{ij})$ are purely $\beta$-transformations. We will call such a bundle a special $\mathcal{TD}_n$-bundle. In this case, the cocycle relations simplify in that $\sigma_L(g_1, g_2)$ vanishes so that $z_{ijk}$ can be put to zero. Moreover, the cocycle relations for the components $\hat{\xi}_{ij}$ and $\hat{A}_i$ in the cocycle data are precisely the same as those in the case of trivial $g_{ij}$. We thus recover an ordinary principal $U(1)^n$-bundle with connection over the base manifold.

**General picture.** In general, a $\mathcal{TD}_n$-groupoid bundle $\mathcal{P}$ describes a T-duality correspondence between two T-folds. If the $\mathcal{TD}_n$-groupoid bundle is special, then one of the T-backgrounds in the correspondence will descend from a $\mathcal{TD}_n$-groupoid bundle in which all $g_{ij}$ are merely $B$-transformations and thus fully geometrical. Generically, however, the T-dual of a T-fold does not have to be geometric.

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22The same holds when the $(g_{ij})$ are purely $B$-transformations, purely $A$-transformations, purely factorized dualities, or purely $G$-transformations.
5.5. Half-geometric T-correspondences as principal $\text{TD}_n^{1/2\text{geo}}$-bundles

Let us briefly compare our construction with that of [7], where the topological part of T-correspondences involving geometric T-backgrounds and T-folds was described in terms of higher geometry.

**Half-geometric T-correspondences.** We follow the nomenclature of [7] and call a T-correspondence involving a geometric background of type $F^1$ and a T-fold background half-geometric. Topological T-backgrounds of type $F^1$ have been shown to correspond to functors represented by the 2-group $\text{TB}^p_{F^1}$ [7]. This 2-group is given by the semidirect product of the strict 2-group $\text{TB}^p_{F^2}$ with the abelian subgroup of $\beta$-transformations $\text{o}(n;\mathbb{Z}) \subset \text{GO}(n,n;\mathbb{Z})$, 

$$\text{TB}^p_{F^1} := \text{o}(n;\mathbb{Z}) \ltimes \text{TB}^p_{F^2}. \quad (5.34)$$

For the definition of the action and the resulting semidirect product, see [7, App. A.4]; the corresponding cocycles of (topological) principal $\text{TB}^p_{F^1}$-bundles were also given in [7].

A half-geometric T-correspondence can then be described by a principal $\text{TD}_n^{1/2\text{geo}}$-bundle, where the structure 2-group is defined as

$$\text{TD}_n^{1/2\text{geo}} := \text{o}(n;\mathbb{Z}) \ltimes \text{TD}_n. \quad (5.35)$$

The left-leg projection is equivariant in a particular sense and induces a map $\hat{p} : \text{TD}_n^{1/2\text{geo}} \to \text{TB}^p_{F^1}$. The main result of [7] is that this map is a bijection and that every $F^1$-background is the image of the left-leg projection of a principal $\text{TD}_n^{1/2\text{geo}}$-bundle.

**Comparison to $\mathcal{G}(n,n;\mathbb{Z}) \ltimes \text{TD}_n$.** In our approach, the group $\text{TD}_n^{1/2\text{geo}}$ should be compared to the semidirect product 2-group

$$\mathcal{G}(n,n;\mathbb{Z}) \ltimes \text{TD}_n, \quad (5.36)$$

where the relevant group action was defined in (3.28). This 2-group is the group of morphisms with source a specific object $q \in Q_n$ in the Lie 2-groupoid $\mathcal{T}D_n$. There is now an evident inclusion

$$\text{TD}_n^{1/2\text{geo}} := \text{o}(n;\mathbb{Z}) \ltimes \text{TD}_n \hookrightarrow \mathcal{G}(n,n;\mathbb{Z}) \ltimes \text{TD}_n. \quad (5.37)$$

We note that a differential refinement of an $F^1$-background automatically induces scalar fields; in the presence of these, the 2-group $\text{TD}_n^{1/2\text{geo}}$ is not large enough to accommodate all required transformations. Conversely, however, it is clear that any topological $\text{TD}_n^{1/2\text{geo}}$-bundle embeds into a topological $\mathcal{T}D_n$-bundle. By [7, Thm. 4.2.2.], our construction certainly provides freedom to capture the topological part of any $F^1$-T-background, and there is a left-leg projection from principal $\text{TD}_n^{1/2\text{geo}}$-bundles to principal $\text{TB}^p_{F^1}$-bundles, cf. again [7].
6. T-duality with non-geometric backgrounds

We can now complete the final step, which consists in generalizing our description of T-duality to non-geometric T-backgrounds, i.e. T-backgrounds that are not even locally geometric. It is well known that Kaluza–Klein reductions with duality twists or on doubled twisted tori gives rise to $R$-flux [62, 63]; in order to describe general non-geometric backgrounds, we will have to incorporate non-trivial $R$-fluxes into our picture.

6.1. From Kaluza–Klein reduction to the tensor hierarchy

Adjustments and tensor hierarchies. As mentioned in appendix D, connections on higher principal bundles require an adjustment, which consists of an additional datum on the gauge group. This additional datum is readily obtained in the case in which the higher gauge algebra is derived from the differential graded vector space underlying a tensor hierarchy, cf. [12]. Of interest to us is the fact that Kaluza–Klein reductions lead to gauged supergravities in which non-geometric fluxes essentially define (at least parts of) the embedding tensor [64, 65, 66]. A general tensor hierarchy for double field theory was then defined in [67]. We note that also [16] argues that the gauge potential forms arising in a Kaluza–Klein interpretation of double field theory should be arranged into a tensor hierarchy.

Algebraic structure underlying tensor hierarchies. Let us briefly summarize the algebraic structure underlying the tensor hierarchies of gauged supergravity [68, 69] as explained in [12, 10]. In its simplest form, we start from a global symmetry group $G$ with Lie algebra $V_0 = \mathfrak{g}$ together with a representation $V_{-1}$. The embedding tensor is a linear map

$$\Theta: V_{-1} \to V_0$$

satisfying

$$[\Theta(v_1), \Theta(v_2)] = \Theta(\Theta(v_1)v_2) ,$$

which identifies a Lie subalgebra $\mathfrak{h} = \text{im}(\Theta)$. We can package these structures into a differential graded Lie algebra

$$V_\Theta = (V_{-1} \xrightarrow{\Theta} V_0) .$$

By [12, Prop. 4.1], this differential graded Lie algebra can be promoted to a weak Lie 2-algebra, which allows for an adjustment by [12, Thm. 6.2]. We note that the representation $V_{-1}$ can be enlarged to a graded vector space, giving rise to more general tensor hierarchies with underlying adjusted weak Lie $n$-algebras.

Besides the quadratic closure constraint (6.2), there is also a representation constraint that one imposes on the embedding tensor. This additional constraint can be seen as a requirement for supersymmetry [68, 70], for the locality of the action [71], or for anomaly cancellation [72]. While supersymmetry or chiral fields that could cause anomalies do not directly appear in the present paper, it will turn out that the representation constraint is nevertheless necessary here to obtain the correct set of $R$-fluxes.
Tensor hierarchy for $\text{GO}(n, n; \mathbb{Z})$. In our case, we set $G = \text{GO}(n, n; \mathbb{Z})$ and $V_1 = \mathbb{R}^{2n}$, the space in which the gauge potential 1-forms take values. Correspondingly, the embedding tensor is a map $\Theta: \mathbb{R}^{2n} \rightarrow \mathfrak{o}(n, n; \mathbb{R})$, which exponentiates to a map

$$\tilde{\Theta}: \mathbb{R}^{2n} \rightarrow \text{GO}(n, n; \mathbb{R}).$$

(6.4)

The map $\Theta$ needs to satisfy the closure constraint (6.2) as well as the representation constraint. We note that a generic map $\Theta$ is an element in the tensor product of the fundamental and the adjoint representation of $\mathfrak{o}(n, n; \mathbb{Z})$, which decomposes as follows:

$$\otimes = \oplus \oplus \oplus.$$  

(6.5)

Usually, the representation constraint is selected by requiring supersymmetry. Here, we have good heuristic reasons to impose the condition

$$\Theta \in \square$$

(6.6)

as we will argue now. Recall that $R$-flux corresponds to the 3-form flux $H$ wrapped around three of the $2n$ directions in the $2n$ torus fiber directions, subject to additional constraints reflecting the fact that the $2n$ coordinates cannot be regarded as geometric simultaneously. Under dimensional reduction, we expect the $R$-form fluxes for $(n + 1)$-dimensional torus fibers to originate from the $R$- and $Q$-fluxes of $n$-dimensional torus fibers, and this branching rule essentially fixes (6.6). We then have the following association of fluxes to $\text{GO}(n, n; \mathbb{Z})$-representations:

$$f\text{-flux } \leftrightarrow \square, \quad Q\text{-flux } \leftrightarrow \square, \quad R\text{-flux } \leftrightarrow \square.$$  

(6.7)

Here, “$f$-flux” refers to the curvatures of $\hat{A}$ and $\tilde{A}$, taking values in $\mathbb{Z}^{2n}$. It corresponds to $H$ wrapped around one of the $2n$ directions in the $2n$-torus fibers and forms the fundamental representation, while the $Q$-flux corresponds to $H$ wrapped around two of the $2n$ directions in the $2n$-torus fibers and, hence, to the adjoint representation, namely the linearization of the non-linear adjoint representation of $\text{GO}(n, n; \mathbb{Z})$ on itself.

A further check of our choice (6.6) comes from considering the embedding of T-duality into U-duality, for which supersymmetric arguments fully determine the representations. For the embedding $E_{7(7)} \supset \text{SO}(6, 6) \times \text{SL}(2)$, the representation $912$ of the embedding tensor for $E_{7(7)}$ decomposes as

$$912 \rightarrow (12, 2) \oplus (220, 2) \oplus \cdots,$$  

(6.8)

and we only find the representations

$$\square = 220 \quad \text{and} \quad \square = 12$$

(6.9)

of $\text{SO}(6, 6)$, but not

$$\square = 560.$$  

(6.10)
Note that we also have a second representation space $V_{-2} = U(1) \times \mathbb{Z}^{2n}$, and compatibility with this representation requires that the images of integer vectors should be a symmetry of the gauge 2-groupoid $TD_{2n}$. In particular, we need to impose that

$$\bar{\Theta} : \mathbb{Z}^{2n} \to GO(n, n; \mathbb{Z}) . \quad (6.11)$$

We will denote the set of embedding tensors satisfying the quadratic and linear representation constraints by $\tilde{R}_n$ and the subset that further satisfies this integrality constraint by $R_n \subset \tilde{R}_n$.

Exponentiation of the map $\bar{\Theta}$ implies that the image of $\bar{\Theta}$ lies in $SO^+(n, n; \mathbb{Z})$, the connected component of $GO(n, n; \mathbb{Z})$ containing the identity.

Finally, we will restrict ourselves in this paper to the case of ungauged (super)gravity theories, i.e. T-background configurations in which the 1-form potentials are purely abelian. This implies that the image of the embedding tensor $\Theta$ is an abelian Lie algebra. Hence, in the following, $\Theta$ and $\bar{\Theta}$ will always be Lie algebra and Lie group homomorphisms, respectively, with domains the abelian Lie algebra $\mathbb{R}^{2n}$ and Lie group $\mathbb{R}^{2n}$, respectively.

**Relation to R-fluxes.** We can now identify the discrete moduli in $\bar{\Theta}$ and relate them to R-fluxes. For $n = 0$ and $n = 1$, there are unique group homomorphisms $\bar{\Theta} : \mathbb{Z}^0 \to SO^+(0, 0; \mathbb{Z}) \cong 1$ and $\bar{\Theta} : \mathbb{Z} \to SO^+(1, 1; \mathbb{Z}) \cong 1$. Hence, there are no R-fluxes in either case.

For $n = 2$, there exist group homomorphisms $\bar{\Theta} : \mathbb{Z}^4 \to SO^+(2, 2; \mathbb{Z})$, for which we must check the representation constraint. Infinitesimally, $\Theta : \mathbb{R}^4 \to o(2, 2; \mathbb{R})$ forms a Lie algebra homomorphism. The image of this linearization is an abelian Lie subalgebra of $o(2, 2; \mathbb{R})$, whose dimension can be at most 2. If the dimension of the image is 0, this corresponds to trivial R-charge. If the dimension is 1, it is straightforward to check that the representation constraint fails. If the dimension is 2, the image corresponds to a pair of mutually commuting translations in $\mathbb{R}^{2,2}$; the requirement that the exponentiated rotations be integral implies that one of them can be taken to be along a spacelike 2-plane and the other along a timelike 2-plane. Thus, the putative R-flux can be put in a standard form, and the representation constraint can then be easily checked to fail. Hence, for $n = 2$ there are no non-trivial R-fluxes either.

In dimensions $n \geq 3$, however, non-trivial R-fluxes exist. One sufficient ansatz is to consider group homomorphisms

$$\mathbb{Z}^n \to o(n; \mathbb{Z}) , \quad (6.12)$$

where $o(n; \mathbb{Z})$ is the abelian group of $n \times n$ antisymmetric integer matrices, that are given by pairing with an $n$-dimensional totally antisymmetric integer 3-tensor, i.e. an element of

$$\{ f : \{1, \ldots, n\}^3 \to \mathbb{Z} \mid f_{ijk} = -f_{jik} = -f_{ikj} \} \cong \mathbb{Z}^3 \quad (6.13)$$

in which each of the $n$ generators map to linearly independent elements of $o(n; \mathbb{Z})$; this then defines a group homomorphism

$$\mathbb{Z}^{2n} \to o(n; \mathbb{Z}) \subset GO(n, n; \mathbb{Z}) \quad (6.14)$$

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in which elements of the other $\mathbb{Z}^n$ simply map to zero. This manifestly satisfies the representation constraint. Thus, in $n$ dimensions the set of $R$-fluxes contains at least $\mathbb{Z}^{(3)}$, which corresponds to the geometric cohomology 3-classes. Of course, there are many possible choices of the embedding $\mathfrak{o}(n; \mathbb{Z}) \hookrightarrow \text{GO}(n, n; \mathbb{Z})$ by conjugations; these correspond to the T-duality orbit of the geometric cohomology 3-classes.

6.2. The complete groupoid of T-duality

Motivation: augmented higher groupoids. For $n \leq 2$, the dimensional reduction of the Kalb–Ramond $B$-field and its field strength $H$ creates 1-forms and scalars with corresponding curvature 2- and 1-forms. These are accounted for in the gauge Lie 2-groupoid $\mathcal{T}D_n$. As evident from (2.1), dimensional reduction of $H$ with $n \geq 3$ will produce 0-forms, which, however, clearly cannot be seen as curvatures of non-existing $(-1)$-forms.

The groupoid picture, however, suggests a resolution. We note that 2-forms and their 3-form curvatures are essentially encoded by the 2-cells of $\mathcal{T}D_n$, while the 1-forms and their 2-form curvatures correspond to the 1-cells. The scalars are then encoded by the 0-cells.

We can regard the Lie 2-groupoid $\mathcal{T}D_n$ as a Lie 2-quasi-groupoid, i.e. a simplicial manifold satisfying the relevant Kan condition, cf. appendix E. In this context, there is the notion of augmented groupoid, which allows us to include $(-1)$-cells as the image of a single face map. Such an augmentation is quite natural: consider for example the Čech groupoid of a surjective submersion $\sigma: Y \to M$, cf. appendix D. We can augment the nerve of the Čech groupoid by $M$ and obtain the augmented quasi-groupoid

$$\mathcal{C}_{\text{aug}}(Y \to M) := \left( \ldots \xymatrix{ \cdots & Y^3 \ar[r] & Y^2 \ar[r] & Y \ar[r] & M } \right). \quad (6.15)$$

In order to capture all aspects of non-geometric T-duality, we evidently have to augment the Lie 2-groupoid $\mathcal{T}D_n$ by $R$, resulting in the augmented Lie 2-quasi-groupoid $\mathcal{T}D_n^{\text{aug}}$.

We note that the sequence of reductions indeed terminates here: 0-form curvatures do not reduce any further. Thus, the picture of augmented Lie groupoids is indeed sufficient for arbitrary $n$.

Augmented Lie 2-quasi-groupoid $\mathcal{T}D_n^{\text{aug}}$. In a first step, we enlarge the space of 0-cells in the Lie 2-groupoid $\mathcal{T}D_n$ from $Q_n$ to $Q_n \times R$ to incorporate the 0-form field strengths.\(^{23}\) The only additional datum we need is an action of the semidirect product group $\text{GO}(n, n; \mathbb{Z}) \ltimes \mathbb{R}^{2n}$ on the now enlarged space of scalars $Q_n \times R$, and we define\(^{24}\)

\[ \triangleright: (\text{GO}(n, n; \mathbb{Z}) \ltimes \mathbb{R}^{2n}) \times (Q_n \times R_n) \mapsto (Q_n \times R_n), \]

\[ (g, \xi; q, r) \mapsto ((g \triangleright r)(\xi)qq, g \triangleright r) = (gr(g^{-1}\xi)q, g \triangleright r), \quad (6.16) \]

\(^{23}\)Similar to $\mathcal{T}D_n$, this 2-groupoid can be thought of as a bundle of 2-groups, with fiber $TD_n$, on the disconnected orbifold $(Q_n \times R)/\text{GO}(n, n; \mathbb{Z})$.

\(^{24}\)The semidirect product $\text{GO}(n, n; \mathbb{Z}) \ltimes \mathbb{R}^{2n}$ is the usual one, i.e. $(g_1, \xi_1)(g_2, \xi_2) = (g_1g_2, \xi_1 + g_1\xi_2)$ for $g_{1,2} \in \text{GO}(n, n; \mathbb{Z})$ and $\xi_{1,2} \in \mathbb{R}^{2n}$.
where the action $g \triangleright r$ is defined as

$$ (g \triangleright r)(\zeta) := gr(g^{-1}\zeta)g^{-1} \quad (6.17) $$

for $\zeta \in \mathbb{Z}^{2n}$. This is indeed a group action as one readily verifies by direct computation:

$$ (g_1, \xi_1) \triangleright ((g_2, \xi_2) \triangleright (q, r)) = (g_1, \xi_1) \triangleright (g_2r(g_2^{-1}\xi_2)q, g_2 \triangleright r) $$
$$ = (g_1g_2 \triangleright r)(g_1^{-1}\xi_1)g_2r(g_2^{-1}\xi_2)q, g_1g_2 \triangleright r $$
$$ = (g_1g_2r((g_1g_2)^{-1}(\xi_1 + g_1\xi_2))q, g_1g_2 \triangleright r) $$
$$ = ((g_1, \xi_1)(g_2, \xi_2)) \triangleright (q, r) \ , \quad (6.18) $$

where we used that $r$ is a group homomorphism.

To incorporate “$(-1)$-form potentials,” we then turn it into a Lie 2-quasi-groupoid by constructing its Duskin nerve [73]. The full augmented Lie 2-quasi-groupoid $\mathcal{T}_n^{\text{aug}}$ is then obtained by augmenting it with the space $\bar{R}_n$ representing the “$(-1)$-form potentials.” The underlying simplicial manifold is given by

$$ \mathcal{T}_n^{\text{aug}} := \left( \ldots \rightarrow (\mathcal{T}_n^{\text{aug}})_2 \rightarrow (\mathcal{T}_n^{\text{aug}})_1 \rightarrow (\mathcal{T}_n^{\text{aug}})_0 \rightarrow (\mathcal{T}_n^{\text{aug}})_{-1} \right) \quad (6.19) $$

with

$$ (\mathcal{T}_n^{\text{aug}})_0 := Q_n \times R_n \quad \text{and} \quad (\mathcal{T}_n^{\text{aug}})_{-1} := \bar{R}_n \ . \quad (6.20) $$

The remaining sets $(\mathcal{T}_n^{\text{aug}})_i$ with $i \geq 1$ are those given by the Duskin nerve construction. This construction becomes rather technical; fortunately this intuitive picture of $\mathcal{T}_n^{\text{aug}}$ is sufficient for all our purposes.

### 6.3. $\mathcal{T}_n^{\text{aug}}$-bundles

**Topological cocycles.** Because of the simplicity of the augmentation, we can directly extend the topological cocycles of $\mathcal{T}_n$-bundles to cocycles of $\mathcal{T}_n^{\text{aug}}$-bundles. These are encoded in augmented simplicial maps from the augmented Čech 2-quasi-groupoid to the augmented Lie 2-quasi-groupoid $\mathcal{T}_n^{\text{aug}}$. Explicitly, we have the data

$$ (g, z, \xi, m, \phi, q, r) \in \mathcal{C}^{\infty}(Y[3], \text{GO}(n, n; \mathbb{Z}) \times \mathbb{Z}^{2n} \times \mathbb{R}^{2n} \times \mathbb{Z}^{2n} \times \mathbb{U}(1) \times Q_n \times R_n) \ , $$
$$ (g, \xi, q, r) \in \mathcal{C}^{\infty}(Y[2], \text{GO}(n, n; \mathbb{Z}) \times \mathbb{R}^{2n} \times Q_n \times R_n) \ , $$
$$ (q, r) \in \mathcal{C}^{\infty}(Y, Q_n \times R_n) \ , $$
$$ r \in \mathcal{C}^{\infty}(M, \bar{R}_n) \ , \quad (6.21) $$

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which satisfy the relations
\[
  r_i = r, \\
  (q_j, r_j) = (g_{ij}^{-1}, g_{ij}^{-1} \xi_{ij}) \cdot (q_i, r_i), \\
  (g_{ijk}, r_{ijk}) = (g_{i}, r_{i}), \\
  g_{ik} = g_{ij}g_{jk}, \\
  g_{ijk} = g_{ij}, \\
  \xi_{ij} = \xi_{ij} + g_{ij}\xi_{jk} + m_{ij}k, \\
  \xi_{ijk} = \xi_{ik} + m_{ijk}, \\
  m_{ij}k + m_{ikl} = m_{ijl} + g_{ij}m_{ijk}, \\
  \phi_{ijkl} + \phi_{ikl} = \phi_{ijl} + \langle g_{ij}, g_{ij}m_{ijkl} \rangle + m_{jkl}^T \rho_L(g_{ij})\xi_{jl} + \xi_{jk}^T \rho_L(g_{ij})g_{jk}\xi_{kl} + \frac{1}{2} \xi_{kl} \phi_L(g_{ij}, g_{jk})\xi_{kl},
\]
as well as the following condition required for adjustment:
\[
  z_{ijk} = \frac{(-1)^{|g_{ik}|}}{2} g_{ik} \eta \text{diag}(\sigma_L(g_{ij}, g_{jk})).
\]

**Differential refinement.** As we have only added discrete structures to our gauge Lie 2-groupoid that do not affect the continuous cocycles, the differential refinement (excluding the scalar fields) is the same as that of \(\mathcal{T}\mathcal{D}_n\). That is, we have 1- and 2-forms
\[
  \Lambda \in \Omega^1(\mathcal{Y}^2), \quad A \in \Omega^1(\mathcal{Y}, \mathbb{R}^{2n}), \quad B \in \Omega^2(\mathcal{Y}),
\]
satisfying the relations (5.28).

**Compatibility of Q- and R-fluxes.** Notice that, even though the \((-1)\)-form potentials \(r\) are a priori valued in the smooth space \(\mathcal{R}_n\) (similar to all other potentials), they are constrained to be quantized as elements of \(\mathcal{R}_n\) by the cocycle condition \(r = r_i \in \mathcal{R}_n\). This accords with the fact that \((-1)\)-form potentials do not encode independent local degrees of freedom, unlike potentials of higher form degrees.

The condition (6.22) implies a compatibility condition between the Q-flux and the R-flux: a generic Q-flux cannot coexist with a generic R-flux. If the Q-flux is given by \(g_{ij}\) taking values in some subgroup \(\Gamma \subset \text{GO}(n, n; \mathbb{Z})\), then the R-fluxes \(r\) are constrained to take values in the stabilizer subgroup of \(\Gamma\) under the group action (6.17). In particular, a vanishing Q-flux is compatible with arbitrary R-flux in \(\mathcal{R}_n\), whereas a generic Q-flux is compatible only with the trivial R-flux that is the constant map in \(\mathcal{R}_n\).

This has the following physical interpretation. The R-flux \(r\), regarded as the embedding tensor \(r: \mathbb{R}^{2n} \to \text{GO}(n, n; \mathbb{R})\), identifies the abelian gauge group \(\mathbb{R}^{2n}\) of the 1-forms with a subgroup of \(\text{GO}(n, n; \mathbb{R})\) given by the image of \(r\). In the presence of non-trivial Q-flux, as specified by \(g_{ij} \in \text{GO}(n, n; \mathbb{Z})\), however, this identification holds only locally, since \(g_{ij}\) acts non-trivially on the \(r_i\). In order for this identification to be globally well-defined, one requires that \(r\) be equivariant under the action of \(g_{ij}\), which is implied by (6.22). This renders the R-flux globally well-defined for each connected component of space-time (in the complement of domain walls).

It is illustrative to examine the compatibility between Q-flux and R-flux in the purely geometric case, i.e. when all fluxes correspond to the Kalb–Ramond 3-form flux \(H\) wrapped
around the $n$ geometric directions in an $n$-torus. The $Q$-flux being geometric in this sense corresponds to the ansatz $g_{ij} \in \mathfrak{o}(n; \mathbb{Z}) \subset \text{GO}(n, n; \mathbb{Z})$, and similarly $r_i$ must be a map $\mathbb{Z}^{2n} \to \text{GO}(n, n; \mathbb{Z})$ whose image lies in $\mathfrak{o}(n; \mathbb{Z})$. In such a case, since the adjoint action of $\mathfrak{o}(n; \mathbb{Z})$ on itself is trivial, the condition that $g^{-1}_{ij} \triangleright r = r$ corresponds to the condition that $r$ be constant on the orbits of the $\mathfrak{o}(n; \mathbb{Z})$-action on the domain $\mathbb{Z}^{2n}$, i.e.

$$r(\hat{a}, \check{a}) = r(\hat{a}, \check{a} + g \hat{a})$$

(6.25)

for $(\hat{a}, \check{a}) \in \mathbb{Z}^{2n}$ and arbitrary $g \in \mathfrak{o}(n; \mathbb{Z})$. The classification of such orbits is non-trivial. However, it is sufficient that $r$ satisfy the stronger condition

$$r(\hat{a}, \check{a}) = \hat{r}(\hat{a})$$

(6.26)

for some function $\hat{r} : \mathbb{Z}^n \to \mathfrak{o}(n; \mathbb{Z})$. After additionally imposing the representation constraint, such $R$-fluxes correspond to elements of $\mathbb{Z}^{(3)}$. Hence, we see that the purely geometric case is consistent with the constraints imposed by (6.22).

**Reductions of $\mathcal{T}_D^{\text{aug}}$.** Let us stress the obvious point that the augmented Lie quasi-groupoid $\mathcal{T}_D^{\text{aug}}$ naturally restricts to the higher Lie groupoids and higher Lie groups relevant to T-duality correspondences without $R$-fluxes or without $Q$- and $R$-fluxes, as expected. In particular, if the scalar fields are fixed to be constant, then $\mathcal{T}_D^{\text{aug}}$ effectively reduces to the 2-group $\mathcal{T}_D^\times$. At the other extreme, one may set the 1-form and 2-form fields to be trivial. Then one obtains a sigma model on the Narain moduli space $\text{GO}(n, n; \mathbb{Z}) \setminus (Q_n \times R_n)$ with constraints on the superselection sectors coming from the augmentation as discussed above; if we further turn off $R$-fluxes completely, this reduces to a sigma model on $GM_n = \text{GO}(n, n; \mathbb{Z}) \setminus Q_n$ subject to the minor constraint (5.30) on discrete moduli.

**Classification of branes.** The augmented Lie 2-quasi-groupoid $\mathcal{T}_D^{\text{aug}}$ that we obtained also leads to a natural classification of branes that appear in toroidal compactifications of string theory.

In general, a codimension $k$ brane can be stable if it couples to a $(k - 2)$-form potential magnetically, so that branes can be classified by classifying the corresponding $(k - 2)$-form potentials; a codimension $k$ brane can also be stable if it carries a non-trivial topological charge, i.e. one in which the scalar field exhibits a non-trivial monodromy around the $(k - 1)$-sphere around the brane; such codimension $k$ branes are classified by the homotopy group $\pi_{k-1}(\Sigma)$ of the manifold (or orbifold) that the scalars take values in. A higher gauge groupoid $\mathcal{G}$, describing both $p$-forms for $p > 0$ as well as scalars, unifies both of these conditions, such that codimension $k$ branes are uniformly classified by $\pi_{k-1}(\mathcal{G})$: the fact that $\pi_{k-1}(-)$ is abelian for $k \geq 3$ corresponds to the fact that three codimensions suffice to exclude anyonic statistics and, hence, non-abelian charges.
In the case of $\mathcal{D}^\text{aug}_n$, we have the following result:

\[
\begin{align*}
\pi_0(\mathcal{D}^\text{aug}_n) &= R_n, \\
\pi_1(\mathcal{D}^\text{aug}_n) &= \text{GO}(n, n; \mathbb{Z}), \\
\pi_2(\mathcal{D}^\text{aug}_n) &= \mathbb{Z}^{2n} \times \mathbb{Z}^{2n}, \\
\pi_3(\mathcal{D}^\text{aug}_n) &= \mathbb{Z}.
\end{align*}
\]

These deviate from the expectations somewhat due to the complications of adjustment, and we comment on each case in the following.

Codimension 1 branes, or domain walls, are labeled by the change in $R$-flux across them, which is thus labeled by an element of $R_n$. When the $R$-flux belongs to $\sigma(n; \mathbb{Z}) \subset \text{GO}(n, n; \mathbb{Z})$, the domain wall corresponds to an NS-brane wrapped around $n - 3$ directions of the $n$-torus. Note that the presence of generic domain walls may be incompatible with the presence of generic defect branes as explained above.

Codimension 2 branes, or defect branes [74], are labeled by a non-trivial monodromy of the $Q_n/\text{GO}(n, n; \mathbb{Z})$-valued scalar field around it, hence by $\text{GO}(n, n; \mathbb{Z})$. Upon dimensional uplift, these correspond to NS-branes wrapped around $n - 2$ directions of the $n$-torus, or to KK-branes, or to bound states of both, depending on the element of $\text{GO}(n, n; \mathbb{Z})$. Somewhat unexpectedly, the condition for the existence of an adjustment implies that the monodromy $g$ must satisfy, for every pair of integers $m_1, m_2$, the condition that

\[
g^{m_1 + m_2} \eta \text{diag}(\sigma_L(g^{m_1}, g^{m_2})) \in 2\mathbb{Z}^{2n},
\]

which is the special case of (5.30) for a codimension 2 brane. This condition always holds for $n = 1$, but it fails for generic $g \in \text{GO}(n, n; \mathbb{Z})$ for $n \geq 2$. However, in case $g$ belongs to one of the special subgroups discussed in section 3.3 — namely, the $\text{GL}(n; \mathbb{Z})$ subgroup of $A$-transformations, the $\sigma(n; \mathbb{Z})$ subgroup of $B$-transformations, the $\sigma(n; \mathbb{Z})$ subgroup of $\beta$-transformations, the subgroup of factorized dualities, or the $\mathbb{Z}_2 \times \mathbb{Z}_2$ subgroup generated by $\text{diag}(s_1, \ldots, s_1, -s_2, \ldots, -s_2)$ for $s_1, s_2 = \pm 1$ — the condition always holds. Thus, such “ordinary” codimension 2 branes do exist. The fact that the permitted $Q$-fluxes do not form a subgroup of $\text{GO}(n, n; \mathbb{Z})$ means that such defect branes are mutually non-local: the presence of one defect brane may forbid the presence of another defect brane somewhere else.

One expects a $2n$-plet of codimension 3 branes, corresponding to a single $\text{GO}(n, n; \mathbb{Z})$ orbit consisting of NS-branes wrapped around $n - 1$ directions and KK-branes. Hence, the presence of the additional copy of $\mathbb{Z}^{2n}$, which ultimately comes from the non-trivial 2-group structure of $\mathcal{G}(n, n; \mathbb{Z})$ in (3.25), comes as a surprise. However, the adjustment condition (5.28a) requires that this spurious charge be fixed by the $Q$-fluxes, such that the actual possible set of codimension 2 brane charges is simply labeled by $\mathbb{Z}^{2n}$ as expected.

The unique codimension 4 brane corresponds to an NS-brane fully wrapped around the $n$-torus, coupling magnetically to the Kalb–Ramond field.
6.4. Example: $R$-space

**Generic $R$-space.** A generic $R$-space is described as (part of) a full $\mathcal{T}\mathcal{D}_n^{\text{aug}}$-bundle over $X$. To render this data manageable, we can restrict ourselves to an $R$-space in which all fields are set to zero except for the scalar fields $(q, r) \in C^\infty(Y, Q_n \times R_n)$ and the monodromies $(g, \xi) \in C^\infty(Y^{[2]}, \text{GO}(n, n; \mathbb{Z}) \times \mathbb{R}^{2n})$ and $z \in C^\infty(Y^{[3]}, \mathbb{Z}^{2n})$. By (5.28a), the $g_{ij}$ define a $\text{GO}(n, n; \mathbb{Z})$-principal bundle that is even in the sense of (5.30). By the same equation, the $g_{ij}$ also fix the higher monodromies $z_{ijk}$. Note that we can set the connection data $(\Lambda, A, B)$ consistently to zero. In this case, the data $(g_{ij}, q_i, r_i)$ are equivalent (modulo the condition (5.30)) to that defining a cocycle of the action groupoid of $\text{GO}(n, n; \mathbb{Z})$ on $Q_n \times R_n$.

Thus, each connected component of the base manifold $X$ is associated to an element of $R_n$ — the $R$-flux. Within each connected component, then, one has additional $Q$-flux around non-contractible cycles; if the $R$-flux vanishes for a connected component $X_i$, then $Q$-flux is simply valued in terms of a group homomorphism $\pi_1(X_i) \to \text{GO}(n, n; \mathbb{Z})$ satisfying (5.30).

**Nilmanifold example.** Let us again consider the example of the nilmanifold, this time T-dualized completely, so that the base space is merely a point. Here, we can set all higher fields to zero except for $r$ and $q$, and these specify elements $r \in R_n$ and $q \in Q_n$.

**Acknowledgments**

We would like to thank Leron Borsten for helpful discussion as well as Konrad Waldorf for pointing out the problem with adding connections in the constructions of [7], which eventually led to this paper. This work was supported by the Leverhulme Research Project Grant RPG–2018–329 “The Mathematics of M5-Branes.”

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**Appendix**

A. Lie 2-groupoid basics

We will follow the conventions of [61] regarding 2-categories and speak of weak and strict 2-categories. The former are also known as bicategories; see [75] for the original definitions as well as [76, Chap. 7] for a textbook account.
**Weak 2-categories.** A weak 2-category \( \mathcal{B} \) consists of a collection of objects or 0-cells \( \mathcal{B}_0 \) and a category of morphisms \( \mathcal{B}(a,b) \) for every pair of objects \( a, b \in \mathcal{B}_0 \). Objects and morphisms in these categories are known as 1- and 2-cells, respectively. For each object \( a \in \mathcal{B}_0 \), there is an identity 1-cell \( \text{id}_a \). Composition of 2-cells is denoted by \( \circ \) and called vertical composition. Horizontal composition, on the other hand, is a collection of bifunctors \( \mathcal{B}(a,b) \otimes \mathcal{B}(b,c) \to \mathcal{B}(a,c) \) for all \( a, b, c \in \mathcal{B}_0 \). Horizontal composition is not strict and comes with a set of natural isomorphisms known as left- and right unitors,

\[
l: x \otimes \text{id}_a(x) \xrightarrow{\cong} x \quad \text{and} \quad r: \text{id}_a(x) \otimes x \xrightarrow{\cong} x ,
\]

as well as an associator,

\[
a: (x \otimes y) \otimes z \xrightarrow{\cong} x \otimes (y \otimes z) ,
\]

for all 1-cells \( x, y, z \). These morphisms satisfy coherence conditions known as the pentagon and triangle identities, see [61]. We will exclusively work with “unital” weak 2-categories that come with unital horizontal composition, reducing the coherence condition to the pentagon identity for the associator:

\[
\begin{array}{c}
((x \otimes y) \otimes u) \otimes v \\
\downarrow \text{id} \otimes a \\
(x \otimes (y \otimes u) \otimes v) \\
\downarrow a \\
x \otimes (y \otimes (u \otimes v))
\end{array}
\]

\[
\begin{array}{c}
(x \otimes y) \otimes (u \otimes v) \\
\downarrow a \\
(x \otimes (y \otimes u) \otimes v) \\
\downarrow \text{id} \otimes a \\
x \otimes ((y \otimes u) \otimes v)
\end{array}
\]

2-functors. Given two weak 2-categories \( \mathcal{B} \) and \( \tilde{\mathcal{B}} \), a unital lax 2-functor \( \Phi: \mathcal{B} \to \tilde{\mathcal{B}} \) consists of a function

\[
\Phi_0: \mathcal{B}_0 \to \tilde{\mathcal{B}}_0 ,
\]

a collection of functors

\[
\Phi_1^{ab}: \mathcal{B}(a,b) \to \tilde{\mathcal{B}}(\Phi_0(a), \Phi_0(b)) ,
\]

and a collection of natural transformations

\[
\Phi_2^{abc}: \Phi_1^{ab}(-) \otimes \Phi_1^{bc}(-) \Rightarrow \Phi_1^{ac}(- \otimes -)
\]
for all $a, b, c \in B_0$. The latter satisfy a coherence condition amounting to the commutative diagram

\[
\begin{array}{c}
\Phi_{abc}(x \otimes y) \otimes \Phi_{cd}(z) \\
\Phi_{abcd}((x \otimes y) \otimes z)
\end{array}
\]

If the natural transformations are natural isomorphisms, we speak of a weak 2-functor.

We note that 2-functors $\Psi: B_1 \to B_2$ and $\Phi: B_2 \to B_3$ compose as

\[
\Xi = \Phi \circ \Psi,
\]

\[
\Xi_0 = \Phi_0 \circ \Psi_0, \quad \Xi_{ab} = \Phi_{ab} \circ \Psi_{ab},
\]

\[
\Xi_{abc}((x, y)) = \Phi_{abc}(\Psi_{abc}(x, y)) \circ \Phi_{abc}(\Psi_{abc}(y)),
\]

where $\tilde{a} = \Psi_0(a)$, etc.

**Lie 2-groupoids.** A (weak) 2-groupoid is a weak 2-category in which all morphisms are equivalences. That is, all 2-cells are strictly invertible, and all 1-cells are invertible up to isomorphisms. A Lie 2-groupoid is then a 2-groupoid internal to a suitable category of smooth manifolds.\(^{25}\)

**B. Higher groups**

**2-groups.** A 2-group is a categorified group. In its most general form, a (weak) 2-group is a weak monoidal small category in which all morphisms are invertible and all objects are weakly invertible, cf. e.g. [56]. Equivalently, we can regard it as a monoidal category of morphisms contained in a pointed\(^{26}\) Lie 2-groupoid with a single 0-cell.

If the associator in a 2-group is trivial, then we obtain a strict 2-group, which can be regarded as a group object internal to $\text{Cat}$, the category of small categories. As shown in [77], strict 2-groups are equivalent to crossed modules of groups.

\(^{25}\)Note that the naive choice of smooth manifolds and smooth maps between these does not contain all pullbacks, which leads to problems in the composition. This is a well-known technicality, which can be resolved by working with diffeological spaces and which we ignore here.

\(^{26}\)Although the pointing is unique, one should use pointed (2-)functors between pointed (2-)groupoids in order to get the correct automorphisms, cf. the nLab page https://ncatlab.org/nlab/show/looping
Crossed modules of groups. A crossed module of groups $\mathcal{G} = (H \xrightarrow{t} G, \triangleright)$ is a pair of groups $G, H$ together with a group homomorphism $t: H \to G$ and an action of $G$ on $H$ by automorphisms $\triangleright: G \times H \to H$ such that, for all $g \in G$ and for all $h_{1,2} \in H$, we have
\[
 t(g \triangleright h_1) = gt(h_1)g^{-1} \quad \text{and} \quad t(h_1) \triangleright h_2 = h_1 h_2 h_1^{-1}. \tag{B.1}
\]

This has an evident specialization to crossed modules of Lie groups, and applying the tangent functor yields a crossed module of Lie algebras, which is equivalently a strict Lie 2-algebra or a strict 2-term $L_\infty$-algebra.

Crossed modules are indeed examples of weak Lie 2-groups, which are monoidal categories in which every object has a weak inverse and every morphism has an inverse [56]. Here, we will use the conventions of [61], under which the monoidal category $\mathcal{G}$ corresponding to a crossed module $\mathcal{G} = (H \xrightarrow{t} G, \triangleright)$ is defined as follows:

\[
 \begin{align*}
 G \ltimes H & \longrightarrow G, \\
 (g_1, h_1) \circ (t(h_1^{-1})g_1, h_2) & := (g_1, h_1 h_2), \\
 (g_1, h_1) \otimes (g_2, h_2) & := (g_1 g_2, (g_1 \triangleright h_2) h_1), \\
 \text{inv}(g_1, h_1) & := (g_1^{-1}, g_1^{-1} \triangleright h_1^{-1}).
\end{align*} \tag{B.2}
\]

Morphisms. A strict morphism of crossed modules of groups $\Phi: \mathcal{G} \to \tilde{\mathcal{G}}$ is simply a map
\[
 \Phi : (H \xrightarrow{t} G, \triangleright) \longrightarrow (\tilde{H} \xrightarrow{\tilde{t}} \tilde{G}, \tilde{\triangleright}) \tag{B.3}
\]
consisting of a pair of group homomorphisms $\Phi_0: G \to \tilde{G}$ and $\Phi_1: H \to \tilde{H}$ that are compatible with $t$ and $\triangleright$ in evident ways.

A very weak morphism of crossed modules of groups $\Phi: \mathcal{G} \to \tilde{\mathcal{G}}$, also known as a butterfly, cf. [78], is a commutative diagram of groups
\[
 \begin{array}{ccc}
 H_1 & \xrightarrow{\lambda_1} & E \\
 \downarrow{t_1} & \swarrow{\gamma_1} & \searrow{\gamma_2} \\
 G_1 & \xleftarrow{\lambda_2} & H_2 \\
 \downarrow{t_2} & \swarrow{\lambda_2} & \\
 G_2 & & 
\end{array} \tag{B.4}
\]

where $E$ is a group, $\lambda_{1,2}$ and $\gamma_{1,2}$ are group homomorphisms, the NE–SW diagonal is a short exact sequence (i.e. a group extension), and the NW–SE diagonal is a complex.

Between these two notions lies the notion of weak morphisms induced by a lax 2-functor of the corresponding two one-object Lie 2-groupoids whose categories of morphisms are $\mathcal{G}$ and $\tilde{\mathcal{G}}$, respectively, cf. [61], as defined in appendix A.\textsuperscript{27} Such a morphism $\Phi$ is thus encoded

\[\text{In this paper, we refrain from using the notion of crossed intertwiners developed in [7] for practical reasons.}\]
in a functor and a natural transformation,
\[
\Phi_1: \mathcal{G} \rightarrow \tilde{\mathcal{G}} \quad \text{and} \quad \Phi_2: \Phi_1(-) \tilde{\otimes} \Phi_1(-) \Rightarrow \Phi_1(- \otimes -). \quad (B.5a)
\]

Besides the naturality condition
\[
\Phi_2(g_1, g_2) \tilde{\circ}(\Phi_1(g_1, h_1) \tilde{\otimes} \Phi_1(g_2, h_2)) = \Phi_1((g_1, h_1) \otimes (g_2, h_2)) \tilde{\circ} \Phi_2(t(h_1^{-1})g_1, t(h_2^{-1})g_2) \quad (B.5b)
\]
for all \(g_{1,2} \in G\) and \(h_{1,2} \in H\), we have the coherence condition (A.2d) with trivial associators, resulting in
\[
\Phi_2(g_1 \otimes g_2, g_3) \tilde{\circ}(\Phi_2(g_1, g_2) \otimes \text{id}_{\Phi_1(g_3)}) = \Phi_2(g_1, g_2 \otimes g_3) \tilde{\circ} (\text{id}_{\Phi_1(g_1)} \otimes \Phi_2(g_2, g_3)) . \quad (B.5c)
\]

Strict morphisms are evidently included here as weak morphisms with \(\Phi_2\) trivial. Two weak morphisms of Lie 2-groups \(\Psi: \mathcal{G}_1 \rightarrow \mathcal{G}_2\) and \(\Phi: \mathcal{G}_2 \rightarrow \mathcal{G}_3\) compose into a morphism \(\Xi = \Phi \circ \Psi\) with
\[
\Xi_1(g, h) = \Phi_1(\Psi_1(g, h)), \\
\Xi_2(g_1, g_2) = \Phi_1(\Psi_2(g_1, g_2)) \circ \Phi_2(\Psi_1(g_1), \Psi_1(g_2)) \quad (B.6)
\]
for all \((g, h)\) in the morphisms of \(\mathcal{G}_1\), cf. (A.3).

The notion of weak morphisms is particularly useful for our discussion as it can be readily postcomposed with the lax 2-functors defining principal 2-bundles, cf. appendix D.

**2-group actions.** Any 2-group \(\mathcal{H}\) comes with a 2-group of automorphisms (or equivalences) \(\text{Aut}(\mathcal{H})\), having 2-group endomorphisms that are equivalences of categories as its objects and natural 2-transformations between these as its morphisms. An action of a (weak) 2-group \(\mathcal{G}\) on another 2-group \(\mathcal{H}\) is then readily defined as a homomorphism of 2-groups \(\Phi: \mathcal{G} \rightarrow \text{Aut}(\mathcal{H}) [79, 80]\).

Here, we will use the reformulation of [57, Prop. 3.2] for unital such actions. That is, a unital action of a 2-group \(\mathcal{G}\) on a 2-group \(\mathcal{H}\) is given by a bifunctor
\[
\triangleright: \mathcal{G} \times \mathcal{H} \rightarrow \mathcal{H} \quad (B.7a)
\]
and natural isomorphisms
\[
\Upsilon_{\mathcal{G}}: (g_1 \otimes g_2) \triangleright h \xrightarrow{\cong} g_1 \triangleright (g_2 \triangleright h) , \\
\Upsilon_{\mathcal{H}}: g \triangleright (h_1 \otimes h_2) \xrightarrow{\cong} (g \triangleright h_1) \otimes (g \triangleright h_2) \quad (B.7b)
\]
for all objects \(g, g_{1,2} \in \mathcal{G}\) and \(h, h_{1,2} \in \mathcal{H}\). These natural isomorphisms have to satisfy the coherence conditions listed in [57, Prop. 3.2]. We write \(\mathcal{G} \rhd \mathcal{H}\) for such an action. We also note that the proof of this proposition gives a helpful definition of the bifunctor \(\triangleright\) in terms of the homomorphism \(\mathcal{G} \rightarrow \text{Aut}(\mathcal{H})\).
Semidirect products. We further take [57, Def. 3.4] as our definition of a semidirect product of 2-groups. Given two weak 2-groups $G$ and $H$ with a unital action $G \acts H$, we define the semidirect product $G \ltimes H$ as the 2-group with underlying Lie groupoid $G \times H$ and monoidal product

$$ (G_1, H_1) \otimes (G_2, H_2) := (G_1 \otimes G_2, H_1 \otimes (G_1 \triangleright H_2)) $$

for all morphisms $G_{1,2}$ in $G$ and $H_{1,2}$ in $H$. The unit object is

$$ 1_{G \ltimes H} = (1_G, 1_H) $$

and the associator is given by

$$ a(g_1, h_1; g_2, h_2; g_3, h_3) := (a(g_1, g_2, g_3), (id_{h_1} \otimes \Upsilon^{-1}_H(g_1, h_2, g_2 \triangleright h_3)) \circ (id_{g_1} \otimes (id_{g_1 \triangleright h_2} \otimes \Upsilon(g_1, g_2, h_3))) \circ a(h_1, g_1 \triangleright h_2, (g_1 \otimes g_2) \triangleright h_3) $$

for all objects $g_{1,2,3} \in G$ and $h_{1,2,3} \in H$, where the inverse of $\Upsilon_H$ is with respect to vertical composition. We note that the definitions of group action and semidirect product we use here subsume those used in [7].

C. Higher Lie algebras

$L_\infty$-algebras. An $L_\infty$-algebra $\mathfrak{L}$ is a $\mathbb{Z}$-graded vector space $\mathfrak{L} = \bigoplus_{i \in \mathbb{Z}} \mathfrak{L}_i$ together with totally antisymmetric multilinear maps $\mu_k : \mathfrak{L}^\wedge k \to \mathfrak{L}$ of degree $|\mu_k| = 2 - k$ satisfying the homotopy Jacobi identity

$$ \sum_{i+j=n} \sum_{\sigma \in S_{ij}} \chi(\sigma; \ell_1, \ldots, \ell_n)(-1)^j \mu_{j+1}(\mu_i(\ell_{\sigma(1)}, \ldots, \ell_{\sigma(i)}), \ell_{\sigma(i+1)}, \ldots, \ell_{\sigma(n)}) = 0, $$

where the sum runs over all $(i, j)$-unshuffles and $\chi$ denotes the (graded) Koszul sign of the permutation of the arguments. An $L_\infty$-algebra concentrated in (i.e. non-trivial exclusively in) non-positive degrees is a model for a semistrict higher Lie algebra, and we use these two terms interchangeably. If all maps $\mu_k$ with $k > 2$ are trivial, we call the $L_\infty$-algebra strict.

Semistrict Lie 2-algebras. We will be particularly interested in the case of $L_\infty$-algebras concentrated in degrees $-1$ and $0$ that form models for semistrict Lie 2-algebras. They consist of two vector spaces $\mathfrak{L} = \mathfrak{L}_{-1} \oplus \mathfrak{L}_0$ together with maps

$$ \mu_1 : \mathfrak{L}_{-1} \to \mathfrak{L}_0, \quad \mu_2 : \mathfrak{L}_0 \wedge \mathfrak{L}_0 \to \mathfrak{L}_0, \quad \mu_2 : \mathfrak{L}_{-1} \wedge \mathfrak{L}_0 \to \mathfrak{L}_{-1}, \quad \mu_2 : \mathfrak{L}_0 \wedge \mathfrak{L}_{-1} \to \mathfrak{L}_0, \quad \mu_3 : \mathfrak{L}_0 \wedge \mathfrak{L}_0 \wedge \mathfrak{L}_0 \to \mathfrak{L}_{-1} $$

satisfying (C.1).
A morphism $\phi: \mathcal{L} \to \hat{\mathcal{L}}$ is given by linear maps
\[
\phi_0: \mathcal{L}_0 \to \mathcal{L}_0, \quad \phi_1: \mathcal{L}_{-1} \to \mathcal{L}_{-1}, \quad \phi_2: \mathcal{L}_0 \wedge \mathcal{L}_0 \to \mathcal{L}_{-1}
\]
such that
\[
0 = \phi_1(\mu_1(v_1)) - \tilde{\mu}_1(\phi_1(v_1)),
0 = \phi_1(\mu_2(w_1, w_2)) - \tilde{\mu}_1(\phi_2(w_1, w_2)) - \tilde{\mu}_2(\phi_1(w_1), \phi_1(w_2)),
0 = \phi_1(\mu_2(w_1, v_1)) + \phi_2(\mu_1(v_1), w_1) - \tilde{\mu}_2(\phi_1(w_1), \phi_1(v_1)),
0 = \phi_1(\mu_3(w_1, w_2, w_3)) - \phi_2(\mu_2(w_1, w_2), w_3) + \phi_2(\mu_2(w_1, w_3), w_2)
- \phi_2(\mu_2(w_2, w_3), w_1) - \tilde{\mu}_3(\phi_1(w_1), \phi_1(w_2), \phi_1(w_3))
+ \tilde{\mu}_2(\phi_1(v_1), \phi_2(w_2, w_3)) - \tilde{\mu}_2(\phi_1(w_2), \phi_2(w_1, w_3))
+ \tilde{\mu}_2(\phi_1(w_3), \phi_2(w_1, w_2))
\]
for all $v_1 \in \mathcal{L}_{-1}$ and $w_1, w_2, w_3 \in \mathcal{L}_0$.

**Crossed modules of Lie algebras.** Applying the tangent functor to a crossed module of Lie groups $\mathcal{G} = (H \xrightarrow{t} \mathcal{G}, \cdot)$, we obtain a crossed module of Lie algebras $\mathfrak{g} = (\mathfrak{h} \xrightarrow{t} \mathfrak{g}, \cdot)$, where $\mathfrak{g}$ and $\mathfrak{h}$ are the Lie algebras of $\mathcal{G}$ and $H$, respectively. Such a crossed module is equivalent to a strict 2-term $L_\infty$-algebra $\mathcal{L} = \mathcal{L}_{-1} \oplus \mathcal{L}_0$ under the relation
\[
\mathfrak{g} = \mathcal{L}_0, \quad \mathfrak{h} = \mathcal{L}_{-1},
[w_1, w_2] = \mu_2(w_1, w_2), \quad w \cdot v = \mu_2(w, v), \quad [v_1, v_2] = \mu_2(\mu_1(v_1), v_2),
\]
cf. also [81]. Applying the tangent functor to a morphism of Lie 2-groups yields a morphism of strict 2-term $L_\infty$-algebras where $\Phi_0$, $\Phi_1$, and $\Phi_2$ are the linearizations or differentials of $\Phi_0$, $\Phi_1$, and $\Phi_2$. The required properties of $\phi$ follow from those of $\Phi$.

**D. Principal 2-bundles with adjusted connection**

**Čech groupoid.** Consider a surjective submersion $\sigma: Y \to M$, which defines the Čech groupoid
\[
\check{\mathcal{C}}(Y \to M) := \left( Y^{[2]} \xrightarrow{\sim} Y \right), \quad y_1 \xleftarrow{(y_1, y_2)} y_2,
\]
where $Y^{[2]}$ is the fibered product
\[
Y^{[2]} = \{(y_1, y_2) \in Y \times Y \mid \sigma(y_1) = \sigma(y_2)\}.
\]
This groupoid trivially extends to higher groupoids with trivial $n$-morphisms for $n \geq 2$. For most purposes, one can restrict $Y$ to be an ordinary cover given in terms of open subsets of $\mathbb{R}^n$. In certain cases, it is more convenient to replace the Čech groupoid by a more general, higher groupoid giving rise to hypercovers, cf. e.g. [78].
Cocycle description. The cocycles of a higher principal bundle with higher structure group $G$ subordinate to the submersion $\sigma$ are then given by a higher functor from $\mathcal{C}(Y \to M)$ to the one-object groupoid $B\mathcal{G}$ whose (higher) category of morphisms is $\mathcal{G}$. These cocycles can be differentially refined to allow for a connection [61].

The introduction of a general connection\(^28\) requires a so-called adjustment, cf. [82, 10, 11, 13] for details, which leads to more general cocycles than those discussed in the original literature, e.g. [8, 9].

For $\mathcal{G}$ a 2-group given by a crossed module of Lie groups $\mathcal{G} := (H \to G, \triangleright)$ with corresponding Lie 2-algebra given by the crossed module of Lie algebras $\mathfrak{g} := (\mathfrak{h} \to \mathfrak{g}, \triangleright)$, this amounts to the following data [13]:

\begin{align}
  h &\in C^\infty(Y^{[3]}, H) , \\
  (g, \Lambda) &\in C^\infty(Y^{[2]}, G) \oplus \Omega^1(Y^{[2]}, \mathfrak{g}) , \\
  (A, B) &\in \Omega^1(Y, \mathfrak{g}) \oplus \Omega^2(Y, \mathfrak{h})
\end{align}

such that\(^29\)

\begin{align}
  h_{ikl}h_{ij} &= h_{ijl}(g_{ij} \triangleright h_{ikl}) , \\
  g_{ik} &= t(h_{ikl})g_{ij}g_{jk} , \\
  \Lambda_{ik} &= \Lambda_{jk} + g_{jk}^{-1} \triangleright \Lambda_{ij} - g_{ik}^{-1} \triangleright (h_{ijl} \nabla h_{ikl}) , \\
  A_j &= g_{ij}^{-1}A_i g_{ij} + g_{ij}^{-1}d g_{ij} - t(\Lambda_{ij}) , \\
  B_j &= g_{ij}^{-1} \triangleright B_i + d \Lambda_{ij} + A_j \triangleright \Lambda_{ij} + \frac{1}{2} [\Lambda_{ij}, \Lambda_{ij}] - \kappa(g_{ij}, F_i)
\end{align}

for all appropriate $(i, j, \ldots) \in Y^{[n]}$, where $\kappa$ is the contribution of the adjustment. The curvature of this principal 2-bundle is the sum of a 2-form $F$ and a 3-form $H$ and given by

\begin{align}
  F &:= dA + \frac{1}{2}[A, A] + t(B) \in \Omega^2(Y, G) , \\
  H &:= dB + A \triangleright B - \kappa(A, F) \in \Omega^3(Y, H) .
\end{align}

The adjustment function $\kappa$ is induced by a map

\begin{align}
  \kappa: G \times \mathfrak{g} &\to \mathfrak{h}
\end{align}

such that generically the equation

\begin{align}
  (g_2^{-1}g_1^{-1}) \triangleright (h^{-1}(X \triangleright h)) + g_2^{-1} \triangleright \kappa(g_1, X) \\
  + \kappa(g_2, g_1^{-1}X g_1 - t(\kappa(g_1, X))) - \kappa(t(h)g_1g_2, X) = 0
\end{align}

holds for all $g_{1,2} \in G$, $h \in H$, and $X \in \mathfrak{g}$. This condition implies directly that gauge transformations of the $B$-field glue together consistently.

The choice $\kappa = 0$ leads to the usual non-abelian gerbes with connection defined in [8, 9]. When gluing together gauge transformations, $X$ in (D.2c) is replaced by $F$; therefore, the choice $\kappa = 0$ generically requires the fake curvature condition $F = 0$.

Examples of adjusted connections are found in [13] and (in infinitesimal form) in [10].

\(^28\)We simply use the term connection to refer to what is sometimes in the abelian gerbe literature called a connective structure and a curving.

\(^29\)We note that our conventions are related to those in [83] by the map $B \mapsto -B$ and $H \mapsto -H$. 

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Bundle isomorphisms. Coboundaries are encoded in natural isomorphisms between two functors given in terms of the above cocycle data. They are encoded in maps

\[ b \in C^\infty(Y[2], H), \]
\[ (a, \lambda) \in C^\infty(Y, G) \oplus \Omega^1(Y, h), \]
\[(D.3a)\]

and two cocycles \((h, g, \Lambda, A, B)\) and \((\tilde{h}, \tilde{g}, \tilde{\Lambda}, \tilde{A}, \tilde{B})\) are equivalent if

\[
\tilde{h}_{ijk} = a_i^{-1} \triangleright (b_{ik} h_{ijk} (g_{ij} \triangleright b_{jk}^{-1}) b_{ij}^{-1}) ,
\]
\[\tilde{g}_{ij} = a_i^{-1} t(b_{ij}) g_{ij} a_j ,
\]
\[\tilde{\Lambda}_{ij} = a_j^{-1} \triangleright \Lambda_{ij} + \lambda_j - \tilde{g}_{ij}^{-1} \triangleright \lambda_i + (a_j^{-1} g_{ij}^{-1}) \triangleright (b_{ij}^{-1} \nabla_i b_{ij}) ,
\]
\[\tilde{A}_i = a_i^{-1} A_i a_i + a_i^{-1} d a_i - t(\lambda_i) ,
\]
\[\tilde{B}_i = a_i^{-1} \triangleright B_i + d \lambda_i + \tilde{A}_i \triangleright \lambda_i + \frac{1}{2} [\lambda_i, \lambda_i] - \kappa(a_i, F_i) .
\]
\[(D.3b)\]

Postcomposition with 2-group morphisms. Consider two crossed modules \(G = (H \xrightarrow{i} G, \triangleright)\) and \(\tilde{G} = (\tilde{H} \xrightarrow{\tilde{i}} \tilde{G}, \triangleright)\) as well as a principal \(G\)-bundle \(P\) with cocycles \((g, h)\). Then a 2-group morphism \(\Phi: \underline{G} \to \underline{\tilde{G}}\) yields a principal \(\tilde{G}\)-bundle with cocycles given by

\[
\tilde{g}_{ij} = \Phi_t(g_{ij}) \quad \text{and} \quad \tilde{h}_{ijk} = \Phi_{H}(h_{ijk}, m_{ij}, m_{jk}) \Phi_{H}^t(g_{ij}, g_{jk}) ,
\]
\[(D.4a)\]

where \(\Phi_{H}^t(h_{ijk})\) is the component of \(\Phi_t\) in \(\tilde{H}\). This follows abstractly from the interpretation of \(P\) and \(\Phi\) as weak 2-functors and the fact that 2-functors compose nicely, cf. (A.3).

For the connection part, we can generalize the discussion in [13] to apply the \(L_\infty\)-algebra morphism induced by the morphism \(\Phi\) to the local connection forms:

\[
A_i \mapsto \tilde{A}_i = \phi_0(A) , \quad B_i \mapsto \tilde{B}_i = \phi_1(B) + \frac{1}{2} \phi_2(A, A) .
\]
\[(D.4b)\]

This construction is familiar from homotopy Maurer–Cartan theory, cf. e.g. [84] for more details.

Interestingly, as observed in [13], this fully defines \(t(\Lambda_{ij})\) via the cocycle condition (D.2) relating \(A_i\) to \(A_j\). It remains to lift this to a full map \(\Lambda_{ij}: \Omega^1(Y[2], H)\), which is best done on a case-by-case basis.

Note that, in the case of morphisms \(\Phi: G \to \tilde{G}\) that are given by butterflies, the induced morphism of principal 2-bundles is more complicated and may require a refinement of the cover, cf. [85, § 4.2]. One such example is discussed in detail in [13].

E. Quasi-groupoids and augmentation

In the following, we briefly summarize some basic material on quasi-groupoids; for a detailed review in our conventions, see [60] and references therein.
Simplicial manifolds. Recall that a simplicial object in a category $C$ is a $C$-valued presheaf $X: \Delta^{\text{op}} \to C$ on the simplex category $\Delta$, which is (the skeleton of) the 1-category of finite non-empty well-ordered sets and order-preserving functions. Every finite well-ordered set is isomorphic to the ordinal $n = \{0, 1, \ldots, n-1\}$, and the image of $n$ is the set of $(n-1)$-simplices: $X_n := X(n+1)$. The images under $X$ of injective order-preserving maps $n \to n+1$ give the face maps $f^0_n, \ldots, f^{n-1}_n: X_n \to X_{n-1}$, and the images under $X$ of surjective order-preserving maps $n+2 \to n+1$ give the degeneracy maps $d^0_n, \ldots, d^{n-1}_n: X_n \to X_{n+1}$.

A simplicial set is then simply a simplicial object in $\text{Set}$, and a simplicial manifold is a simplicial object in a suitable category of smooth manifolds. Notice that every simplicial set can be trivially regarded as a discrete simplicial manifold. The standard simplicial $n$-simplex $\Delta^n$ is the simplicial set $\Delta^{\text{op}} \to \text{Set}$ represented by $n+1$.

Lie quasi-groupoids. An $(n,i)$-horn in $\Delta^n$ is the union of all faces of $\Delta^n$ except for the $i$th one. The $(n,i)$-horns of a simplicial manifold $M$ are the images of the $(n,i)$-horns of $\Delta^n$ in $M$. If every horn in $M$ can be filled to a simplex, and the face maps from the simplices in $M$ to the horns in $M$ are surjective submersions, we say that $M$ is a Kan simplicial manifold or a Lie quasi-groupoid. If all the fillers for $(m,i)$-horns are unique for every $m > n$ and $0 < i < m$, then we say that $M$ is a Lie $n$-quasi-groupoid.

From (higher) groupoids to quasi-groupoids. The nerve of any groupoid $\mathcal{G}$, i.e. the simplicial manifold whose 0-simplices are the objects of $\mathcal{G}$, whose 1-simplices are the morphisms of $\mathcal{G}$, and whose 2-simplices are the pairs of composable morphisms of $\mathcal{G}$, etc., is a 1-quasi-groupoid. In the case of 2-groupoids, the definition of a nerve is slightly more involved but similarly feasible. A common choice is the Duskin nerve [73], which also forms a 2-quasi-groupoid. A problem in this treatment of 2-groupoids is that the explicit expression of horizontal composition is replaced by the existence of a set of horn fillers. This, together with a vast redundancy of information in quasi-groupoids, is the reason for us not working with quasi-groupoids from the outset.

Augmentation. In the definition of simplicial objects, we restricted ourselves to non-empty well-ordered sets, but we can naturally extend this to all finite well-ordered sets, including $-1 = \emptyset$; we denote the resulting category by $\Delta_+$. An augmented simplicial object in $\mathcal{C}$ is then a $\mathcal{C}$-valued presheaf on $\Delta_+$. Concretely, an augmented simplicial object $X^+_\bullet$ is nothing but a triple $(X_\bullet, X_{-1}, f_0^{-1})$ where $X_\bullet$ is an (unaugmented) simplicial object, $X_{-1}$ is an object in $\mathcal{C}$, and $f_0^1: X_0 \to X_{-1}$ is a morphism in $\mathcal{C}$ such that

$$f_0^0 \circ f_1^1 = f_0^1 \circ f_1^1.$$  \hfill (E.1)

We define an augmented quasi-groupoid in a corresponding way. A particularly useful example is that of the augmented Čech groupoid (6.15).
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