Vacuum energy on orbifold factors of spheres

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March 28, 2022

Abstract

The vacuum energy is calculated for a free, conformally-coupled scalar field on the orbifold space-time $\mathbb{R} \times S^2/\Gamma$ where $\Gamma$ is a finite subgroup of $O(3)$ acting with fixed points. The energy vanishes when $\Gamma$ is composed of pure rotations but not otherwise. It is shown on general grounds that the same conclusion holds for all even-dimensional factored spheres and the vacuum energies are given as generalised Bernoulli functions. The relevant $\zeta$-functions are analysed in some detail and several identities are incidentally derived. The general discussion is presented in terms of finite reflection groups.
1 Introduction

Roughly speaking an orbifold, or a V-manifold \([1]\), is a quotient of a manifold by a group of symmetries with fixed points \([2]\). The orbifold is singular at the images of the fixed points where, for example, the Riemannian curvature would diverge. Orbifolds can provide useful approximations to more complicated manifolds, such as Calabi-Yau spaces, occurring in string compactifications \([3, 4, 5]\). They have also appeared in membrane compactification \([6]\). Some work has also been done on geometric quantisation on orbifolds \([7]\).

It is interesting to extend the usual theory of quantum fields on curved spaces to orbifolds. Some cases have already been considered. The ideal cosmic string is an example. We have earlier discussed some effects on the tetrahedron \([8]\).

In the present paper we consider a scalar field in the static space-time \(\mathbb{R} \times S^2/\Gamma\), where \(\Gamma\) is one of the finite subgroups of \(O(3)\) associated with the regular solids. The fundamental domain of \(\Gamma\) is a curvilinear triangle or quadrilateral on \(S^2\) depending on whether \(\Gamma\) does or does not include reflections. The discussion is then extended to higher spheres.

As usual, we consider free fields and, in the present work, as an example of a field theoretic calculation, we look at ways of evaluating the conformal scalar contribution to the total Casimir energy on the divided two-sphere. This will be sufficient to exhibit the various techniques and relations.

Recently, there have been calculations of the vacuum energies on the hyperbolic space-times, \(\mathbb{R} \times H^2/\Gamma\), Zerbini et al \([9]\).

We have already shown that the total vacuum energy vanishes on the full sphere \(\mathbb{R} \times S^2\) (and in fact on all even-dimensional spheres for conformal coupling). It will turn out that dividing by a purely rotational \(\Gamma\) does not alter this conclusion. Including reflections does produce a nonzero Casimir energy.

The analysis is presented in some calculational detail because we think the discussion of the relevant heat-kernels and \(\zeta\)-functions has some general methodological value. The calculation can be thought of as a rather explicit example of an equivariant situation, the general theory of which has been understood for a long time.

2 Vacuum energy

We consider an ultrastatic spacetime, \(\mathbb{R} \times M\), where the spatial section, \(M\), is \(d\)-dimensional. A scalar field satisfies the traditional equation of motion

\[
(\Box + \xi R + m^2) \varphi(x) = 0
\]  

with the D’Alembertian \(\Box = \partial_t^2 - \nabla^2\). \(\nabla^2\) is the usual Laplace-Beltrami operator on \(M\).
The general formula for the total vacuum, or Casimir, energy, \( E \), for scalar fields in a static space-time is given in the form we need in [10]. Thus

\[
E = \frac{1}{2} \lim_{s \to 1} L^{-2(s-1)} \zeta_d(s - 1/2) = \frac{1}{2} \zeta_d(-\frac{1}{2}),
\]

(2)

when this does not diverge, which it does not here. This will be discussed later. \( L \) is the scaling length.

The \( \zeta \)-function, \( \zeta_d \), is a Dirichlet series for the operator \((-\nabla^2 + \xi R + m^2)\) on \( M \); \( \zeta_d(s) = \sum \lambda^{-s} \). For constant scalar curvature, we can write

\[
\zeta_d(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau e^{-((m^2+\xi R)\tau)} \tau^{s-1} K_d(\tau)
\]

(3)

where \( K_d \) is the integrated coincidence limit of the Laplacian heat-kernel on \( M \).

Incidentally, in the present case, the vacuum energy is also the negative of the effective (one-loop) Lagrangian, \( L^{(1)} \), which is related to the effective action, \( W^{(1)} \), by \( W^{(1)} = \int dt L^{(1)} \).

3 The Selberg trace formula on \( S^2/\Gamma \)

On the unit two-sphere, \( \nabla^2 = -L^2 \) where \( L \) is the usual angular momentum operator. Then the defining equation for the heat kernel on \( S^2 \) is

\[
(\partial_\tau + L^2) K_{S^2}(r, r'; \tau) = \delta(\tau) \delta(r, r')
\]

(4)

which has the solution

\[
K_{S^2}(r, r'; \tau) = \sum_{l,m} e^{-l(l+1)\tau} Y_l^m(r) Y_l^m(r')
\]

\[
= \frac{1}{4\pi} \sum_{l=0}^\infty (2l + 1) e^{-l(l+1)\tau} P_l(\cos \Phi(r, r'))
\]

where \( \cos \Phi(r, r') = r \cdot r' \), \( r \) and \( r' \) being unit three-vectors.

Next, the Mehler-Dirichlet integral representation for the Legendre polynomial is used to get

\[
K_{S^2}(r, r'; \tau) = \frac{1}{2\sqrt{2\pi}} \sum_{l=0}^\infty (2l + 1) e^{-l(l+1)\tau} \int_0^\pi d\alpha \frac{\sin(l + \frac{1}{2})\alpha}{\sqrt{\cos \Phi - \cos \alpha}},
\]

(5)

which is quite standard e.g. [11].

We now restate our problem. The spatial manifold we want to examine initially is the two-sphere with a reduced symmetry, specifically \( M \cong S^2/\Gamma \) where \( \Gamma \) is a discrete subgroup of \( O(3) \). \( \Gamma \) is either a cyclic \((Cq)\), dihedral \((D_{2q})\), tetrahedral
(T), octahedral (O) or icosahedral group (Y). If one uses an extended group, Γ′ (i.e. one that includes a reflection), then \(S^2/\Gamma′\) is a fundamental spherical triangle in which the field satisfies Dirichlet or Neumann boundary conditions. For simplicity, we restrict the analysis initially to pure rotational \(\Gamma \subset \text{SO}(3)\).

The method of images is used, with an argument similar to that found in McKean and Lax & Phillips \[12, 13\], to find the heat kernel on the divided sphere. The idea is to write the pre-image sum as a sum over conjugacy classes. It is easy to show that

\[
K_{S^2/\Gamma}(\tau) = \int_{S^2/\Gamma} \sum_{\gamma \in \Gamma} K_{S^2}(r, \gamma r; \tau) \, dr
\]

where \(\gamma\) is a typical element, one of \(|\{\gamma\}|\), in the conjugacy class \(\{\gamma\}\). The elements of a class correspond to rotations through one fixed angle about a set of conjugate axes. For a given set of such axes, one corresponding class can be considered to be the primitive class, all other classes associated with these axes then being generated by this one. Thus the sum over all classes can be rewritten as a sum over primitive classes and the powers of these. Let \(q\) be the generic order of the rotation associated with the generic primitive class \(\{\hat{\gamma}\}\) so that \(\hat{\gamma}^q = \text{id}\). Then \(|\{\hat{\gamma}\}|\) is just the number, \(n_q\), of conjugate \(q\)-fold axes and we can write

\[
K_{S^2/\Gamma}(\tau) = \int_{S^2/\Gamma} K_{S^2}(r, r; \tau) \, dr + \frac{1}{|\Gamma|} \sum_{\gamma \in \Gamma} n_q \sum_{p=1}^{q-1} \int_{S^2} K_{S^2}(r, \hat{\gamma}^p r; \tau) \, dr,
\]

where \(\hat{\gamma}\) is the generic order of the rotation associated with the generic primitive class \(\{\hat{\gamma}\}\) so that \(\hat{\gamma}^q = \text{id}\). Then \(|\{\hat{\gamma}\}|\) is just the number, \(n_q\), of conjugate \(q\)-fold axes and we can write

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\]

We can think of this rather obvious equation as a simple application of Selberg’s trace formula to a compact, here spherical, domain.

4 Evaluation of the trace formula

We will now evaluate each term in the above trace formula, (6). The term corresponding to the identity element of the subgroup \(\Gamma\), from (3), is

\[
\frac{1}{2\pi^2} \sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\tau} \int_{S^2/\Gamma} \int_0^{\frac{\pi}{2}} d\alpha \chi_l(2\alpha),
\]

where \(\chi_l(2\theta) = \sin(2l+1)\theta / \sin \theta\) is the character of the \(l\)-representation of SO(3). Both integrals can be done to give

\[
\frac{1}{|\Gamma|} \sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\tau}.
\]
Up to scalings, this last quantity is nothing other than the extensively discussed rotational partition function of a gas of diatomic molecules \([14, 11, 15]\).

However we do not want this form for the first trace term (8), instead we shall leave it as follows

\[
\frac{2}{\pi |\Gamma|} \sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\tau} \int_0^{\pi} \sin(2\alpha) \chi_l(2\alpha).
\]  

Next we examine the second term in (7). A little geometry shows that, for \(\gamma = n(\omega) \in \text{SO}(3), \)

\[
\cos \Phi(r, \gamma r) = \cos \omega + (1 - \cos \omega)(r \cdot n)^2.
\]

The summand of the double sum is then

\[
\frac{1}{2\sqrt{2\pi}} \sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\tau} \times \int_{S^2} \int_0^{\pi} \sin(\alpha) \sin(l + \frac{1}{2} \alpha) \Re \frac{1}{\sqrt{c + (1 - c)(r \cdot n)^2 - \cos \alpha}},
\]

where \(c = \cos p\omega.\) We may choose \(n\) as the polar axis.

Pulling the space integral through, to change the order of integration, and setting \(\mu = \cos \theta,\) (10) becomes

\[
\frac{1}{2\sqrt{2\pi}} \sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\tau} \int_0^{\pi} \sin(\alpha) \sin(l + \frac{1}{2} \alpha) \Re \frac{2}{\sqrt{c - \cos \alpha + (1 - c)\mu^2}}.
\]

Since the radicand has to be positive, the two cases of \(c - \cos \alpha\) being either positive or negative must be considered. Define the quantity \(p\omega = p\omega\) if \(p\omega < \pi\) and \(p\omega = 2\pi - p\omega\) if \(p\omega \geq \pi.\) Hence

\[
\frac{1}{\pi} \frac{1}{\sqrt{1 - c}} \sum_{l=0}^{\infty} (2l + 1) e^{-l(l+1)\tau} \times \left\{ \int_0^{\infty} \sin(\alpha) \sin(l + \frac{1}{2} \alpha) \sqrt{Q(\alpha)} \Re \frac{1}{\sqrt{\mu^2 - Q(\alpha)}} \right\},
\]

where the terms in the braces can be evaluated to give

\[
\left\{ \int_0^{\infty} \sin(\alpha) \sin(l + \frac{1}{2} \alpha) \cosh \Phi - 1 \left(Q^{-\frac{1}{2}}(\alpha)\right) + \int_0^{\infty} \sin(\alpha) \sin(l + \frac{1}{2} \alpha) \sinh \Phi - 1 \left(Q^{-\frac{1}{2}}(\alpha)\right) \right\}.
\]
The functions $Q(\alpha)$ and $\bar{Q}(\alpha)$ are defined as $Q(\alpha) = (\cos \alpha - c)/(1 - c)$ and $\bar{Q}(\alpha) = (c - \cos \alpha)/(1 - c)$.

Integrating by parts leaves us with the desired, intermediate result,

$$
\frac{4}{\pi} \sum_{l=0}^{\infty} e^{-l(l+1)\tau} \int_{0}^{\pi/2} \frac{\cos(2l+1)\alpha \cos \alpha}{\cos 2\alpha - \cos p \omega} \, d\alpha.
$$

The principal part, denoted by the $P$, is necessary in order to remove the singularity at $\alpha = p\omega/2$ introduced by the partial integrations.

Finally, summing over the cyclic group, $\omega = 2\pi/q$, (excluding the identity, $p = q$), we get

$$
2\pi \sum_{l=0}^{\infty} e^{-l(l+1)\tau} \int_{0}^{\pi/2} \frac{\cos(2l+1)\alpha \cos \alpha}{\sin \alpha} \left[ \cot \alpha - q \cot q\alpha \right],
$$

using a standard identity for the trigonometric sum.

Noting the Jacobi theta-function relation,

$$
e^{-\tau/4} \sum_{l=0}^{\infty} e^{-l(l+1)\tau} \cos(2l+1)\alpha = \sqrt{\frac{\pi}{4\tau}} \sum_{n=-\infty}^{\infty} (-1)^n e^{-(\alpha+\pi n)^2/\tau},$$

and its derivative with respect to $\alpha$, we find the heat kernel, from (7), (9) and (12), to be

$$K_{S^2/T}(\tau \Gamma) = \frac{2\pi}{|\Gamma| (\pi \tau)^{3/2}} \sum_{l=0}^{\infty} (-1)^n \left\{ \int_{0}^{\pi/2} d\alpha \left[ \frac{\cos(2l+1)\alpha}{\sin \alpha} \right] e^{-(\alpha+\pi n)^2/\tau} \right\}.
$$

It is possible to subsume the outer sum into the integration range, producing the compact expression for the classical paths form of the trace formula,

$$K_{S^2/T}(\tau \Gamma) = \frac{2\pi}{|\Gamma| (\pi \tau)^{3/2}} \int_{\alpha}^{\infty} d\alpha \frac{e^{-\alpha^2/\tau}}{\sin \alpha} \left\{ \alpha + \frac{\tau}{2} \sum_{q} n_q \left[ \cot \alpha - q \cot q\alpha \right] \right\}.
$$

It should noted that this expression is akin to that obtained in the hyperbolic case by Donnelly [16] and later by Balazs & Voros [17].

It is better to avoid the principal part by introducing a contour, $C_\alpha$, located just above, or just below, the real $\alpha$ axis and writing for (13)

$$K_{S^2/T}(\tau \Gamma) = \frac{\pi}{|\Gamma| (\pi \tau)^{3/2}} \int_{C_\alpha} d\alpha \frac{e^{-\alpha^2/\tau}}{\sin \alpha} \left\{ \alpha + \frac{\tau}{2} \sum_{q} n_q \left[ \cot \alpha - q \cot q\alpha \right] \right\}.
$$

5
By expanding the sines as infinite sums of fractions and running $C_\alpha$ strictly along the real axis we can confirm that this integral is equal to the principal part form appearing in (13), (18).

In the form (14) it is possible to effect a partial integration to get

$$K_{S/\Gamma}(\tau) = \frac{\pi}{|\Gamma|} \frac{e^{-\tau^2/4}}{(\pi\tau)^{3/2}} \int_{C_\alpha} \frac{e^{-\alpha^2/\tau}}{\sin \alpha} \left\{ \alpha (1 - \sum \hat{\gamma}_n) - \frac{\tau}{2} \sum \hat{\gamma}_n q \cot q\alpha \right\} . \quad (15)$$

In the cyclic case, the term proportional to $\alpha$ goes away, leaving

$$K_q(\tau) = -\frac{e^{-\tau^2/4}}{2(\pi\tau)^{1/2}} \int_{C_\alpha} \frac{e^{-\alpha^2/\tau}}{\sin \alpha} \cot q\alpha , \quad (16)$$

and (13) can be written

$$K_{S/\Gamma}(\tau) = \frac{1}{|\Gamma|} \left[ \sum \hat{\gamma} q_n K_q(\tau) - (\sum \hat{\gamma}_n - 1) K_1(\tau) \right] \quad (17)$$

where $K_1(\tau) = K_{S^2}(\tau)$.

There are several possible uses of these formulae, for example we could make an asymptotic expansion to give a power series in the limit $\tau \to 0$,

$$K_{S/\Gamma}(\tau) \approx \frac{1}{\tau} \sum_{k=0,1, \ldots} C_k \tau^k . \quad (18)$$

This will be taken up later. The coefficients $C_k$ are related to values of the $\zeta$-function given in the next section.

## 5 The $\zeta$-function

Important to the present work is a generalization of Riemann’s zeta function. The incomplete Riemann $\zeta$-function (also known as Hurwitz’s $\zeta$-function) is defined by

$$\zeta_R(z, a) = \sum_{n=0}^{\infty} \frac{1}{(n + a)z} , \quad \text{for } \Re z > 1 .$$

The analytic continuation of this, for all $z$, is

$$\zeta_R(z, a) = -\frac{\Gamma(1-z)}{2\pi i} \int_{C_0} \frac{(-x)^{z-1}e^{-ax}}{1 - e^{-x}} dx ,$$

where $C_0$ is the standard Riemann-Hankel contour around the positive real axis [19]. In particular we will have cause to use the incomplete $\zeta$-function for the case $a = \frac{1}{2}$; so we give the contour representation

$$\zeta_R(z, \frac{1}{2}) = -\frac{2^{z-1}\Gamma(1-z)}{2\pi i} \int_{C_0} \frac{(-x)^{z-1}}{\sinh x} dx . \quad (19)$$
Turning now to the field equation (1), the case of conformal coupling \((m = 0, \xi = 1/8)\) is particularly simple. Since \(R = 2\) for our scale, conformal coupling is equivalent to using the operator \((L^2 + \frac{1}{4})\) when finding the heat kernel and amounts to dropping the \(\exp(\tau/4)\) factor in (15), which is the simplifying fact. Then the \(\zeta\)-function (3) is the Mellin transform of the integrated heat kernel (15), up to this factor. We have

\[
\zeta_{S^2/\Gamma}(s) = \frac{1}{|\Gamma| \sqrt{\pi \Gamma(s)}} \int_0^\infty dt \, t^{\frac{3}{2} - s} \int_{C_\alpha} \frac{e^{-\alpha^2 t}}{\sin \alpha} \alpha (1 - \sum \hat{\gamma} n_q) - \frac{\tau}{2} \sum \frac{q \alpha}{\gamma} \cot \frac{q \alpha}{\gamma} \right) .
\]

(20)

Note that we have set \(t = 1/\tau\). We will denote \(\zeta_{S^2/\Gamma}\) by \(\zeta_\Gamma\).

The immediate objective is to invert the order of integration in (20) so that the \(t\)-integral can be done. Some manoeuvring is necessary before this can be achieved because the \(t\)-integral will diverge when \(\alpha\) is inside the ‘light-cone’ i.e. when \(x^2 \leq y^2\), where \(\alpha = x + iy\). We always require \(\text{Re}(-\alpha^2 t) < 0\) for convergence.

In terms of the real and imaginary parts, \(\alpha = x + iy\) and \(t = u + iv\),

\[-\alpha^2 t = -(x^2 - y^2)u + 2xyv - i[(x^2 - y^2)v + 2xyu].\]

It is actually convenient to take the average of two contour integrals, one, \(C_\alpha\), above the real \(\alpha\)-axis and the other, \(C'_\alpha\), below. The \(q = 1\) integral (corresponding to the full sphere) will then be

\[
\zeta_1(s) = \frac{1}{2 \sqrt{\pi \Gamma(s)}} \int_0^\infty dt \, t^{\frac{3}{2} - s} \int_{C_\alpha + C'_\alpha} \frac{e^{-\alpha^2 t}}{\sin \alpha} \alpha .
\]

(21)

We study the behaviour as the \(\alpha\)- and \(t\)-contours are rotated.

Consider \(C_\alpha\) rotated anti-clockwise onto the line \(x = y\), then the \(t\)-integral would be along the negative imaginary axis of the complex \(t\)-plane. Moving \(C_\alpha\) further round to the vertical, \(x = 0\), would require the \(t\)-contour, \(C_t\), to run from the origin to \(-\infty\) below the negative real axis, which is a branch cut. Similarly, rotating \(C'_\alpha\) clockwise to the negative imaginary axis of \(\alpha\) implies that the corresponding contour \(C'_t\) runs above the branch cut from \(t = 0\) to \(t = -\infty\). As the \(\alpha\)-contours sweep passed the negative real axis, contributions are picked up from the poles but these cancel and so can be neglected. The next step is to combine the two \(t\)-contours, \(C_t\) and \(-C'_t\), to form one continuous path, \(C\), that loops clockwise round the negative \(t\)-axis, passing through the origin. This is possible by reversing the rotated \(C'_\alpha\) contour, which runs parallel, but opposite, to the rotated \(C_\alpha\) one.

These two rotated \(\alpha\)-contours are next both moved onto the \(y\)-axis. Redefining \(\alpha \rightarrow i \alpha\) and using the symmetry of the integrand in \(\alpha\) we find,
\[
\zeta_1(s) = \frac{1}{i\sqrt{\pi} \Gamma(s)} \int_C dt \; t^{s} \int_0^\infty d\alpha \; \frac{\alpha e^{\alpha^2 t}}{\sinh \alpha}. \tag{22}
\]

The inversion of the order of the \(\alpha\)- and \(t\)-integrations in (22) is effected by deforming the \(t\)-contour, \(C\), to run around the negative \(t\)-axis, avoiding the origin. Then we get

\[
\zeta_1(s) = \frac{1}{i\sqrt{\pi} \Gamma(s)} \int_0^\infty d\alpha \; \frac{\alpha}{\sinh \alpha} \int_C dt \; t^{s} e^{\alpha^2 t}.
\]

This should be compared with (21).

Integrating over \(t\) yields

\[
\zeta_1(s) = -\frac{2\Gamma\left(\frac{3}{2} - s\right)}{\sqrt{\pi} \Gamma(s)} \cos \pi s \int_0^\infty d\alpha \; \frac{\alpha^{2s-2}}{\sinh \alpha},
\]

where we must assume that \(\text{Re} \; s > 3/2\) in order that the \(\alpha\)-integration converge at the origin.

Then, using (13), we finish with

\[
\zeta_1(s) = -\frac{\Gamma\left(\frac{3}{2} - s\right) \Gamma(2s - 1) \cos \pi s}{\Gamma(s) 2^{2s-3} \sqrt{\pi}} \zeta_R(2s - 1, \frac{1}{2}) = 2 \zeta_R(2s - 1, \frac{1}{2}).
\]

This is the conformal \(\zeta\)-function on the undivided sphere and is well known, following immediately from the eigenvalues, \(l(l+1)+1/4 = (l+1/2)^2\) and degeneracies \(2(l+1/2)\),

\[
\zeta_{S^2}(s) = 2\zeta_R(2s - 1, \frac{1}{2}). \tag{23}
\]

(See Dowker [“Vacuum energies on spheres and in cubes”, unpublished 1983] and Camporesi [20].)

For the other terms in (20), it can similarly be shown that

\[
\frac{1}{\sqrt{\pi} \Gamma(s)} \int_0^\infty dt \; t^{-s-\frac{1}{2}} \int_0^\infty d\alpha \; e^{-\alpha^2 t} \cot q\alpha = \frac{2^{2s-1}}{\Gamma(2s)} \int_0^\infty d\alpha \; \frac{\alpha^{2s-1}}{\sinh \alpha} \coth q\alpha
\]

for \(\text{Re} \; s > 3/2\). Then the entire \(\zeta\)-function, (20), can be written,

\[
\zeta_{S^2,\Gamma}(s) = \frac{1}{|\Gamma| \Gamma(2s)} 2^{2s-1} \int_0^\infty d\alpha \; \frac{\alpha^{2s-2}}{\sinh \alpha} \left\{ (2s-1)(1 - \sum_n n_q) + \sum \gamma q_n \coth q\alpha \right\},
\]

valid for \(\text{Re} \; s > 3/2\). The same expression can be reached using the principal part form (13).

In the case of the cyclic group, \(\Gamma = \mathbb{Z}_q\); the first term in the braces in the integrand of (24) vanishes and the result simplifies to

\[
\zeta_{S^2,\mathbb{Z}_q}(s) = \frac{2^{2s-1}}{\Gamma(2s)} \int_0^\infty d\alpha \; \frac{\alpha^{2s-1}}{\sinh \alpha} \coth q\alpha. \tag{25}
\]
A simpler, eigenvalue derivation of this formula is given in the next section.

Calling the $\zeta$-function of (25) $\zeta_q(s)$ for short, the general expression (24) reads

$$\zeta_\Gamma(s) = \frac{1}{|\Gamma|} \left[ \sum_\gamma qn_\gamma \zeta_q(s) - (\sum_\gamma n_\gamma - 1)\zeta_1(s) \right]$$

(26)

where $\zeta_1(s) = \zeta_{S_2}(s)$ is given by (23).

For all $\Gamma$ except the cyclic groups, $\sum_\gamma n_\gamma - 1 = |\Gamma|/2$ and so we get

$$\zeta_\Gamma(s) = \frac{1}{2} \zeta_{S_2}(s) + \frac{1}{|\Gamma|} \sum_\gamma qn_\gamma \zeta_q(s)$$

for $\Gamma \neq \mathbb{Z}_q$.

We now look for an analytic continuation of (24). Expanding the hyperbolic cotangent in (25) in terms of exponential functions and integrating, yields [21, p361 (3.552.1)]

$$\zeta_q(s) = 2 \sum_{p=0}^{q-1} \sum_{m=0}^{\infty} \frac{m}{(qm + p + \frac{1}{2})^{2s}} + \zeta_R(2s, \frac{1}{2}).$$

Rearranging the sums,

$$\zeta_q(s) = \zeta_R(2s, \frac{1}{2}) + \frac{2}{q} \zeta_R(2s - 1, \frac{1}{2}) - \frac{1}{q^{2s+1}} \sum_{p=0}^{q-1} (2p + 1) \zeta_R(2s, \frac{2p+1}{2q}),$$

(27)

which would serve as an analytic continuation using the Hurwitz $\zeta$-function relation [19] [21, p1073 (9.521.2)].

To produce a functional relation directly for the continuation of $\zeta_q(s)$, we consider the integral

$$-\frac{2^{2s-1}\Gamma(1 - 2s)}{2\pi i} \int_{C_0} d\alpha \frac{(-\alpha)^{2s-1}}{\sin \alpha \coth \alpha}.$$

For Re $s > 3/2$, this equals the integral in (25) and so is an appropriate continuation.

For Re $s < 0$, a large loop can be added without penalty to the contour $C_0$ so forming a continuous path that ultimately surrounds all the poles on the imaginary axis. Then we get, from residues, and after rearranging the summations a little,
Therefore, comparing with (27), the functional relation that must exist is

\[
2^{1-2s} \Gamma(2s) \left\{ \zeta_R(2s, 1/2) - \frac{1}{q^{2s+1}} \sum_{p=0}^{q-1} (2p + 1) \zeta_R(2s, \frac{2p+1}{2}) \right\} = \frac{\pi^{2s}}{q \sin \pi s} \left\{ \sum_{p=1}^{q-1} \frac{1}{\sin \frac{\pi p}{q}} \left[ \zeta_R(1 - 2s, \frac{p}{q}) - 2^{2s} \zeta_R(1 - 2s, \frac{p+q}{2q}) \right] \right\}, \tag{28}
\]

which can be proved directly using the rational Hurwitz \( \zeta \)-functional relation and a couple of trigonometric sums found in Bromwich [24, p272 Ex18].

From (24), (27) and (28), there results the explicit forms

\[
\zeta_{\Gamma}(s) = 2 \frac{\zeta_R(2s - 1, 1/2) + 1}{\Gamma} \sum_{\gamma} q n_{\gamma} \left[ \zeta_R(2s, 1/2) - \frac{1}{q^{2s+1}} \sum_{p=0}^{q-1} (2p + 1) \zeta_R(2s, \frac{2p+1}{2q}) \right]
\]

or, equivalently,

\[
\zeta_{\Gamma}(s) = 2 \frac{\zeta_R(2s - 1, 1/2)}{\Gamma} + 2^{2s} \Gamma(1 - 2s) \cos \pi s \pi s \sum_{\gamma} q n_{\gamma} \sum_{p=1}^{q-1} \frac{1}{\sin \frac{\pi p}{q}} \left[ \zeta_R(1 - 2s, \frac{p}{q}) - 2^{2s} \zeta_R(1 - 2s, \frac{p+q}{2q}) \right]. \tag{29}
\]

Equations (29) and (30) also exhibit the analytic properties of the \( \zeta \)-function, \( \zeta_{\Gamma}(s) \), which has only the one pole at \( s = 1 \) of residue \( 1/|\Gamma| \) in accordance with general theory, which gives a residue of \( |S^2/\Gamma|/4\pi \). Further, the coefficients in the asymptotic expansion [18] are given by the values of the \( \zeta \)-function at negative integers. The formula is

\[
\zeta_{\Gamma}(-k) = (-1)^k k! C_{k+1} \tag{31}
\]

It will be enough to give these values in the cyclic case,

\[
\zeta_q(-k) = - \frac{1}{q(k+1)B_{2k+2}(1/2)} + \frac{q^{2k-1}}{2k+1} \sum_{p=0}^{q-1} (2p + 1) B_{2k+1}(2p + 1/2q) \tag{32}
\]

in terms of Bernoulli polynomials.

The important value \( \zeta_{\Gamma}(0) \) is given in general by

\[
\zeta_{\Gamma}(0) = C_{d/2} - n_0 \tag{33}
\]

where \( n_0 \) is the number of zero modes. For conformal coupling \( n_0 \) vanishes. From (29), or (27), we find \( (d = 2) \)

\[
\zeta_q(0) = \frac{2q^2 - 1}{12q}. \tag{33}
\]
We also note, most easily from (30), that $\zeta_\Gamma(s)$ has zeros at $s = -(2k + 1)/2$, with $k = 0, 1, \ldots$, in particular at $s = -1/2$. Thus, from (2), the vacuum energy vanishes, as mentioned earlier. It also vanishes on space-times with the structure $\mathbb{R} \times \mathbb{R}^{2k} \times S^2/\Gamma$. Of course, it must not be forgotten that this conclusion depends upon the assumption of conformal coupling.

Expressions for the coefficients $C_k$ can be found by expansion of the trigonometric functions in (13) or, more easily, in (16) in powers of $\alpha$. The result appears as a polynomial in $q$ whose coefficients are products of Bernoulli numbers. Equivalence with (32) devolves upon an interesting identity and is discussed in Appendix A.

6 A direct eigenvalue method

In this section we describe the more direct eigenvalue method of deriving the $\zeta$-function and will later enlarge the discussion to include the extended groups. These are easily allowed for by reinterpreting $\Gamma$ in (3) to be an extended group and inserting phase (sign) factors in the sum such that the direct elements, i.e. those with an even number of reflections, have a factor of +1 while those elements with an odd number of reflections have a factor of +1 or −1 depending, respectively, on whether we require Neumann or Dirichlet boundary conditions at the edges of the fundamental domain.

On $S^2/\Gamma$, the eigenvalues of $(L^2 + 1/4)$ are $n^2/4$ for $n = 2l + 1$ and $l = 0, 1, 2, \ldots$. If $d(l), \equiv d(n)$, is the degeneracy of each eigenvalue labelled by $l$ then, by definition, the conformal $\zeta$-function on $S^2/\Gamma$ is

$$\zeta_\Gamma(s) = 2^{2s} \sum_{n=1,3,\ldots} \frac{d(n)}{n^{2s}}. \quad (34)$$

From first principles $d(l)$ is the number of times the trivial representation of $\Gamma$ occurs in the $l$-representation of $SO(3)$, (cf [23])

$$d(l) = \frac{1}{|\Gamma|} \sum_\gamma \chi_l(\gamma) = \frac{2l + 1}{|\Gamma|} + \frac{1}{|\Gamma|} \sum_\hat{\gamma} n_q \sum_{p=1}^{q-1} \chi_l(\hat{\gamma}^p) \quad (35)$$

where the cyclic character is

$$\chi_l(\hat{\gamma}^p) = \frac{\sin((2l + 1)\pi p/q)}{\sin(\pi p/q)}. \quad (36)$$

Thus for $\Gamma = \mathbb{Z}_q$ (which is sufficient)

$$d_q(l) = \frac{1}{q} \sum_{p=0}^{q-1} \chi_l(\hat{\gamma}^p), \quad (37)$$
which reduces to $2[l/q] + 1$, where $[l/q]$ is the integer part of $l/q$.

Evaluation of the $\zeta$-function from (34) proceeds as follows.

\[
\zeta_q(s) = 2^{2s} \sum_{p=0}^{q-1} \frac{1}{\sin(\pi p/q)} \sum_{n=1,3,...} \frac{\sin(n\pi p/q)}{n^{2s}}
\]

\[
= 2^{2s} \sum_{p=0}^{q-1} \frac{1}{\sin(\pi p/q)} \sum_{n=1,2,...} \left( \frac{\sin(n\pi p/q)}{n^{2s}} - \frac{\sin(2n\pi p/q)}{(2n)^{2s}} \right). \tag{38}
\]

Separating off the $p = 0$ term and then setting $p \to q - p$ we find

\[
\zeta_q(s) = 2\frac{q}{q^{2s-1}} \zeta_R(2s-1, 1/2) + 2^{2s} \sum_{p=0}^{q-1} \frac{1}{\sin(\pi p/q)} \sum_{n=1,2,...} \frac{\sin(n\pi p/q)}{n^{2s}}. \tag{39}
\]

Next we recall Hurwitz's formula

\[
\sum_{n=1,2,...} \frac{\sin(2n\pi a)}{n^{2s}} = \frac{(2\pi)^{2s}}{4\Gamma(2s)\sin \pi s} (\zeta_R(1-2s,a) - \zeta_R(1-2s,1-a)) \tag{40}
\]

to rewrite (39) in terms of $\zeta_R$-functions,

\[
\zeta_q(s) = \frac{2}{q} \zeta_R(2s-1, 1/2)
\]

\[
+ \frac{(2\pi)^{2s}}{4\Gamma(2s)\sin \pi s} \sum_{p=1}^{q-1} \frac{2^{2s}}{\sin(\pi p/q)} \zeta_R(1-2s, \frac{p}{2q}) - \zeta_R(1-2s, \frac{p+q}{2q}). \tag{41}
\]

We could stop here but, if agreement with our previous result, (33), is required, the relation,

\[
\zeta_R(z, \frac{p}{2q}) = 2^z \zeta_R(z, \frac{p}{q}) - \zeta_R(z, \frac{p+q}{2q}),
\]

can be used.

It is easy to introduce a $U(1)$ twisting into the character formula, (37), by writing

\[
d_q(l, t) = 2 \sum_{p=0}^{q-1} \frac{\sin((2l+1)\pi p/q)}{\sin(\pi p/q)} \cos(2\pi pt/q) = d_q(l+t) + d_q(l-t), \tag{42}
\]

where $t$ is an integer between 0 and $q$. One can think of this as the effect of an Aharonov-Bohm flux along the rotation axis. The modification will clearly carry through unchanged into the final formula, like (11), for the $\zeta$-function which as a consequence will have the same zeros as before. The same conclusion holds for the twisting,

\[
d(l) = \frac{1}{|\Gamma|} \sum_{\gamma} \chi_l(\gamma) \chi_D^*(\gamma) = \frac{(2l+1)\dim D}{|\Gamma|} + \frac{1}{|\Gamma|} \sum_{\gamma} n_q \sum_{p=1}^{q-1} \chi_l(\gamma^p) \chi_D^*(\gamma^p), \tag{43}
\]
that results from taking the field to belong to any representation, $D$, of $\Gamma$.

It must be said that there is no substantial difference between the method of this section and that given in section 5. Not surprisingly, the integral (11) can be done to yield the expression (37) for $d_q(l)$. We have taken the route we have in order to exhibit various aspects of the analysis and to provide checks of the formulae.

It is interesting therefore to make contact with the work of Polya and Meyer [24] who gave Molien generating functions for the degeneracies of each harmonic basis of degree $l$ invariant under a point group of three-dimensional Euclidean space. (See also Laporte [25].)

The generating function is defined to be

$$h(\sigma) = \sum_{l=0}^{\infty} d(l)\sigma^l.$$  \hfill (44)

From (37), a geometric summation produces the cyclic form

$$h_q(\sigma) = \frac{(1 + \sigma)^{q-1}}{q} \sum_{p=0}^{q-1} \frac{1}{1 - 2\sigma \cos(2\pi p/q) + \sigma^2},$$

showing the connection with the Molien determinant expression [24]. A further summation yields, cf [24] Table 1,

$$h_q(\sigma) = \frac{1}{1 - \sigma} \frac{1 + \sigma^q}{1 - \sigma^q}.$$  \hfill (45)

Expanding gives again $d_q(l) = 2[l/q] + 1$.

It is amusing to note that the generating function is, to a factor, the integrated heat kernel of the (pseudo-) operator $(L^2 + 1/4)^{1/2}$,

$$K^{1/2}(\tau) \equiv \text{Tr} \left( \exp \left( -\tau \sqrt{L^2 + 1/4} \right) \right).$$

This can be seen by setting $\sigma = e^{-\tau}$ so that, making redefinitions where appropriate,

$$h(\tau) = \sum_l d(l)e^{-\tau l} = e^{\tau/2} \sum_{n=1,3,\ldots} d(n)e^{-\tau n/2}$$

$$= e^{\tau/2} \text{Tr} \left( \exp(-\tau \sqrt{L^2 + 1/4}) \right)$$

where the trace is over all eigenstates on $S^2/\Gamma$.

The $\zeta$-functions are related by

$$\zeta(s) \equiv \zeta^A(s) \equiv \text{Tr} A^{-s} = \text{Tr} (A^{1/2})^{2s} = \zeta^{A^{1/2}}(2s) \equiv \zeta^{1/2}(2s),$$  \hfill (46)
with
\[ \zeta^{1/2}(s) = \frac{1}{\Gamma(s)} \int_0^\infty d\tau \tau^{s-1} K^{1/2}(\tau) \]
where, by the above,
\[ K^{1/2}(\tau) = e^{-\tau/2} h(\tau). \]

For the cyclic case using (45)
\[ K_{q/2}^1(\tau) = \coth q\tau/2 \frac{2}{2\sinh\tau/2}. \]

Therefore, for \( \Re s > 3 \), setting \( \tau = 2\alpha \),
\[ \zeta_{q/2}^{1/2}(s) = \frac{2^{s-1}}{\Gamma(s)} \int_0^\infty d\alpha \frac{\alpha^{s-1}}{\sinh\alpha} \coth q\alpha, \]
and, using (46), we regain the answer given above, (25), as we must.

The general result (24) or (26) also follows using (35) and also is in accordance
with the equation on p.139 of [24]. The \( \zeta \)-functions for the finite subgroups
of \( O(3) \) thus take the compact integral form, written using the Hankel contour
so as to hold for all \( s \),
\[ \zeta_{r'}(s) = \frac{i\Gamma(1-2s)}{2\pi} \int_{C_0} d\tau (-\tau)^{2s-1} K_{r'}^{1/2}(\tau), \]
where the \( K_{r'}^{1/2}(\tau) \) are given by
\[ K_{r'}^{1/2}(\tau) = \frac{\cosh(d_0\tau/2)}{2\sinh(d_1\tau/2)\sinh(d_2\tau/2)}, \]
with the constants, \((d_0, d_1, d_2)\) equal to \((q, q, 1)\) for \( \mathbb{Z}_q \), \((2q+1, 2q, 2)\) for \( D_{2q} \),
\((6, 4, 3)\) for \( T \), \((9, 6, 4)\) for \( O \) and \((15, 10, 6)\) for \( Y \).

We have earlier shown that the \( \zeta_r(s) \) vanish at \( s = -(2k+1)/2, k = 0, 1, \ldots \).
This can also be seen very simply as follows.

At \( s = -(2k+1)/2 \), the integrand of (49) is single-valued and the contour can
be shrunk down to a small loop around the origin reducing the computation to
one of residues. The evenness of the integrand, (50), means that the residue at
the origin is zero without the need for any particular representation.

In contrast, the \( \zeta \)-functions for the extended groups do not vanish at \( s = -(2k+1)/2 \). In the case \( \mathbb{R} \times S^2/\Gamma' \) we have explicitly
\[ K_{r'}^{1/2}(\tau) = \frac{\exp(d_0\tau/2)}{4\sinh(d_1\tau/2)\sinh(d_2\tau/2)}, \]
with the corresponding constants, \((d_1, d_2, d_0)\), and find for the vacuum energy,
\[ E_{r'} = \pm \frac{d_0}{96d_1d_2}(d_0^2 - d_1^2 - d_2^2). \]
The sign depends on whether one chooses Dirichlet or Neumann conditions, which itself is determined by setting the twisting, $\chi(A)$, equal either to $\det A$ or to 1, as described earlier.

The specific numerical values are $\pm 11/192$ for $T'$, $\pm 29/256$ for $O'$ and $\pm 89/384$ for $Y'$. (See Table 1.)

7 Higher spheres and reflection groups

We enlarge the analysis to space-times of the form $\mathbb{R} \times \mathbb{R}^{2k} \times S^d/\Gamma$ and also give a more systematic account of the extended groups.

In place of (2) there is now the general equation

$$E = \frac{(-1)^k}{2(2\pi)^k(2k+1)!!} \zeta_d(-k - 1/2),$$

(53)

when finite, for the vacuum energy per unit volume in the $\mathbb{R}^{2k}$ space.

We assume that, for all $k$, the scalar field is conformally coupled in the static space-time, $\mathbb{R} \times S^d/\Gamma$, so that the relevant (conformal) eigenvalues are still $(n + d - 2)/2$, $n = 1, 3, \ldots$.

In the present case $\zeta_d$ is $\zeta_\Gamma$ given by (19) where the heat kernel is related to the generating function by

$$K^{1/2}(\tau) = e^{-(d-1)\tau/2} h(\tau).$$

(54)

From (19) it follows that

$$\zeta_\Gamma(-k - 1/2) = - \left( \frac{\partial}{\partial \tau} \right)^{2k+1} e^{-(d-1)\tau/2} h(\tau) \bigg|_{\tau=0}.$$

Hence the behaviour of the combination $e^{-(d-1)\tau/2} h(\tau) = \sigma^{(d-1)/2} h(\sigma)$ under $\tau \to -\tau$, i.e. under $\sigma \to 1/\sigma$, is relevant; if even, the residue is zero.

The structure of the (twisted) harmonic generating function, $h_\chi(\sigma)$, on factored spheres, $S^d/\Gamma$, is given by, [24, 26, 27],

$$h_\chi(\sigma) = \frac{1 - \sigma^2}{|\Gamma|} \sum_A \frac{\chi^*(A)}{\det (1 - \sigma A)},$$

(55)

which incorporates Molien’s theorem. The situation now is that $\Gamma$ is a finite subgroup (extended or not, as the case may be) of $O(d+1)$ and the $A$ are its $(d+1) \times (d+1)$ matrix representatives. A twisting, $\chi(A)$, has been included. $\chi(A)$ is shorthand for $\chi(\gamma)$ where $A$ represents $\gamma$. Further, since $A$ is real orthogonal, $\chi(A)$ could be replaced by its real part.
\[ h_\chi(\sigma) \] is a rational function but, at the moment, we do not need to be more precise than (55). Then it is easily shown that
\[
\sigma^{(d-1)/2}h_\chi(\sigma) - \sigma^{(1-d)/2}h_\chi(\sigma^{-1}) = \frac{\sigma^{(d-1)/2}(1-\sigma^2)}{|\Gamma|} \sum_A \chi^*(A)(1 - (-1)^d \det A) \frac{1}{\det (1 - \sigma A)}. \tag{56}
\]

For even-dimensional spheres, the sum in (56) reduces to one over those elements of \( \Gamma \) that contain an odd number of reflections so immediately we see that \( \sigma^{(d-1)/2}h_\chi(\sigma) \) is an even function when \( \Gamma \) contains only rotations. This shows quite generally that the \( \zeta \)-functions on such factored, even-dimensional spheres vanish at negative odd integers, (for conformal coupling of course). For odd spheres only the pure rotations contribute to (56).

The generating function (44) is an example of a Poincaré (or Hilbert) series and its derivation is an episode in Invariant Theory [28]. For a finite reflection group, which is another name for an extended group, the structure of the usual harmonic generating function is just (55) with \( \chi(A) \) equal to unity,
\[
h_N(\sigma) = \frac{(1 - \sigma^2)}{|\Gamma|} \sum_A \frac{1}{\det (1 - \sigma A)} = (1 - \sigma^2) \prod_{i=1}^{d+1} \frac{1}{(1 - \sigma^{d_i})}, \tag{57}
\]
where the \( d_i, i = 1, 2, \ldots, d + 1 \), are the degrees of the algebraically independent generating members of the invariant polynomial ring associated with the action of the group, \( \Gamma \), on the \( d + 1 \) dimensional vector space. Without loss of generality, \( \Gamma \) may be taken to be a subgroup of O\((d + 1)\).

Since there is always the basic invariant \( r^2 = x^2 + y^2 + z^2 + \cdots \), of degree 2, = \( d_{d+1} \), we see that the corresponding term, \( i = d + 1 \), in the denominator of (57) is cancelled by the ‘harmonic factor’, \( (1 - \sigma^2) \), leaving just the degrees \( (d_1, d_2, \ldots, d_d) \equiv d \).

The properties of reflection groups were worked out by Coxeter [29], Chevalley [30], Shephard and Todd [31], Steinberg [32] and Solomon [33]. Flatto gives a readable review in [34]. An introductory account can be found in Grove and Benson [35]. For a more advanced expository treatment see Humphreys [36]. Physicists might care to consult [37]. Carter, [38], contains useful algebraic information. Some standard, relevant facts follow.

For small \( d \), the degrees \( d_i \) can be determined by using an identity relating them to the geometrical properties of the group action. For example, it can be shown that \( \prod_i d_i \) is the order of the group and that \( \sum_i (d_i - 1) \) equals the number of reflections. In the two-sphere case these facts rapidly lead to the numbers cited earlier and we see that \( d_0 = d_1 + d_2 - 1 \) is the number of reflections.

Rewriting (57) in terms of the time \( \tau \) we find
\[
h_N(\tau) = e^{(d-1)\tau/2}e^{d_0\tau/2} \prod_{i=1}^{d} \frac{1}{2 \sinh(d_i \tau/2)}. \tag{58}
\]
Note that the effect of the conformal coupling term, \(e^{(d-1)\tau/2}\), is here associated with the number of reflection hypersurfaces, the number of ‘positive roots’. Using (54) we can check that (58) reproduces the heat kernel (51) on \(S^2/\Gamma'\).

On restricting the reflection group to its rotational subgroup, another generating invariant polynomial comes into play. This is the Jacobian, of degree \(d_0 = \sum_{i=1}^{d+1} (d_i - 1)\), of the original generating set. Under the action of \(\Gamma\), \(J\) changes to \((\det A)^{d_0} J\). It is an odd form and, as such, is the product of \(d_0\) linear factors, the vanishing of each of which is the equation of a reflecting hyperplane.

Multiplying invariant polynomials by \(J\) produces skew-invariant polynomials and conversely all such polynomials have this product form. Thus their generating function is

\[
\prod_{i=1}^{d+1} \frac{\sigma_{d_i-1}}{1 - \sigma_{d_i}} = \sigma_{d_0} \prod_{i=1}^{d+1} \frac{1}{1 - \sigma_{d_i}} = \frac{1}{|\Gamma|} \sum_A \det A \overline{\det (1 - \sigma A)}.
\] (59)

This can be referred to as the Dirichlet generating function since the odd elements of \(\Gamma\) enter with a minus sign, cf [27]. The harmonic generating function is

\[
h_D(\tau) = e^{(d-1)\tau/2} e^{-d_0\tau/2} \prod_{i=1}^{d} \frac{1}{2 \sinh(d_i \tau/2)}.
\] (60)

The corresponding Neumann function has already been given in (58).

Equation (49) with (51) is a particular case of Barnes’ \(\zeta\)-function [40] encountered in our earlier work [41]. We can write in the \(S^2\) case

\[
\zeta(\tau) = \zeta_2(2s, (d_1 + d_2 - d_0)/2 | d_1, d_2) = \zeta_2(2s, 1/2 | d_1, d_2).
\] (63)

The general definition of \(\zeta_d(s, * | *)\) is

\[
\zeta_d(s, a|d) = \frac{i\Gamma(1-s)}{2\pi} \int_L d\tau \frac{\exp(-a\tau)(-\tau)^{s-1}}{\prod_{i=1}^{d} (1 - \exp(-d_i\tau))} = \sum_{m=0}^{\infty} \frac{1}{(a + m.d)^s}, \quad \text{Re } s > d
\] (64)
where \( \mathbf{d} = (d_1, d_2, \ldots, d_d) \) and \( \mathbf{m} = (m_1, m_2, \ldots, m_d) \) with the \( m_i \) non-negative integers. The location of the infinite contour \( L \) depends on the relative positions of the numbers \( a \) and \( \mathbf{d} \). In the present case it can be taken to be the Hankel contour, \( C_0 \).

Barnes gives the values of \( \zeta_d(s, \ast | \ast) \) at all integral values of \( s \). In particular the values at the negative integers are the generalised Bernoulli functions which result from a residue evaluation via an expansion of the integrand. In particular

\[
\zeta_d(-q, a | \mathbf{d}) = \frac{(-1)^d q!}{(d + q)!} \prod d_i B_{d+q}(a | \mathbf{d}) \tag{65}
\]

in the notation of \([43]\). Here, and in the following, the sums and products run from \( i = 1 \) to \( i = d \).

The Bernoulli polynomials are developed most thoroughly by Nörlund \([44]\). As noted by Hirzebruch \([45]\) they are essentially the same as the Todd polynomials.

In the general case, constructing the heat kernel one gets

\[
K_{1/2}^{(N)} = e^{\pm d_0 \tau/2} \prod \frac{1}{2 \sinh(d_i \tau/2)}, \tag{66}
\]

According to \([54]\) with \([57]\), we find that the Neumann and Dirichlet \( \zeta \)-functionson \( S^d/\Gamma \) are given by \([64]\). Specifically,

\[
\zeta_N(s) = \zeta_d(2s, (d-1)/2 | \mathbf{d}), \tag{67}
\]

\[
\zeta_D(s) = \zeta_d(2s, \sum d_i - (d-1)/2 | \mathbf{d}), \tag{68}
\]

generalising the previous expression \([53]\).

We already know that the Neumann and Dirichlet vacuum energies on \( \mathbb{R} \times \mathbb{R} \times S^d/\Gamma \) are opposite in sign for even spheres and of the same sign for odd ones. Equivalently, we could use

\[
B_n^{(m)}(\sum d_i - x | \mathbf{d}) = (-1)^n B_n^{(m)}(x | \mathbf{d}) \tag{69}
\]

and the answer is

\[
E_{\Gamma} = -(\mp)^{d+1} \frac{(-1)^k k!}{|\Gamma| \pi^k (d + 2k + 1)!} B_{d+2k+1}^{(d)} ((d - 1)/2 | \mathbf{d}), \tag{70}
\]

where we have used \( 2 \prod d_i = |\Gamma| \). Here, and later, the upper sign is for Neumann and the lower for Dirichlet conditions and we recall that the field is conformally coupled in only \( (1 + d) \)-dimensions whatever the value of \( k \).

The vacuum energy on \( \mathbb{R} \times \mathbb{R} \times S^2/\Gamma \) is

\[
E_{s^2/\Gamma} = \pm \frac{(-1)^k k!}{\pi^k (2k + 3)!} B_{2k+3}^{(2)} (1/2 | d_1, d_2). \]
For \( k = 0 \), using

\[ B_3^{(2)}(\gamma | \alpha, \beta) = \frac{1}{4} (2\gamma - \alpha - \beta) (2\gamma(\gamma - \alpha - \beta) + \alpha\beta), \]

(52) can be checked.

The rather trivial case of the one-sphere, i.e. the space-time \( \mathbb{R} \times S^1/[q] \), gives the vacuum energy \( E = -q/24 \) for both Dirichlet and Neumann conditions. For the circle \( (q = 1) \), adding the two values produces the standard periodic result, \( E = -1/12 \).

For the regular tessellations of the three-sphere, we find the values in Table 1.

| Group \( \Gamma \) | Degrees \( d \) | Casimir energy \( E \) |
|-----------------|----------------|------------------|
| \([q]\)         | \((q)\)        | \(-q/24\)        |
| \([3, 3]\)      | \((3, 4)\)     | \(\mp 11/192\)  |
| \([3, 4]\)      | \((4, 6)\)     | \(\mp 29/256\)  |
| \([3, 5]\)      | \((6, 10)\)    | \(\mp 89/384\)  |
| \([3, 3, 3]\)   | \((3, 4, 5)\)  | \(-601/28800\)  |
| \([3, 3, 4]\)   | \((4, 6, 8)\)  | \(-3557/46080\) |
| \([3, 4, 3]\)   | \((6, 8, 12)\) | \(-69391/414720\)|
| \([3, 3, 5]\)   | \((12, 20, 30)\) | \(-3178447/5184000\) |

\( ^a \)Where there are two signs, the upper is for Neumann and the lower for Dirichlet boundary conditions.

Table 1: Vacuum energies for the reflection groups on \( S^1 \), \( S^2 \) and \( S^3 \).

8 The heat kernel expansion

In place of (18) we have

\[ K_{S^d/\Gamma}(\tau) \approx \frac{1}{\tau^{d/2}} \sum_{k=0,1/2,...}^{\infty} C_k \tau^k, \]

(71)

where any half odd-integral powers are due to the boundary conditions.
Equation (31) becomes

\[ \zeta_r(-k) = (-1)^k k! C_{k+d/2} \]  

(72)

and enables us to find \( C_{k+d/2} \), \( k \) an integer. A further result we need is the following.

On a \( d \)-dimensional manifold with boundary, \( \zeta_d(s) \) has poles at \( s = (d - m)/2 \), for \( m = 0, 1, \ldots, d - 1 \) and \( m = d + 1, d + 3, \ldots \), with residues

\[ \frac{1}{\Gamma((d - m)/2)} C_{m/2}. \]  

(73)

The possibly nonzero heat kernel coefficients are thus \( C_0, C_{1/2}, C_1, \ldots, C_{d/2} \) and \( C_{d/2+1}, C_{d/2+2}, \ldots \).

The importance of the coefficients, so far as field theory goes, is that in a space-time of the form \( \mathbb{R} \times \mathbb{R}^{2k} \times M_d, C_{(1+2k+d)/2} \) is proportional to the coefficient of the pole term in the effective Lagrangian and a nonzero value indicates an ultraviolet divergence. Our interest here lies in even \( 2k \) (most typically zero), since, even if no divergences appear, we do not wish to undertake the technical evaluation of the derivative of the \( \zeta \)-function that necessarily appears in the vacuum energy. (The result can be expressed as a multiple \( \Gamma \)-function.)

The issue of ultraviolet divergences thus amounts to the existence of a pole at \( s = -k - 1/2 \), as intimated in section 2. The Barnes \( \zeta \)-function \( \zeta_d(s, a \mid d) \), has poles only at positive \( s = 1, 2, \ldots, d - 1 \), so there is no ultraviolet divergence and no conformal anomaly. Expressions (2) and (53) are thus sensibly finite, although, de facto, evaluation is really enough, as we have seen.

We calculate the coefficients \( C_{d/2+k}, k = 1, 2, \ldots \), using (72) and (65). From (49), the residue this time involves the even part of \( K_{1/2}(\tau) \) and we need

\[ \sigma^{(d-1)/2} h_\lambda(\sigma) + \sigma^{(1-d)/2} h_\lambda(\sigma^{-1}) = \frac{\sigma^{(d-1)/2}}{|\Gamma|} \sum_A \chi^*(A) (1 + (-1)^d \det A) \det (1 - \sigma A). \]  

(74)

If \( d \) is even, only pure rotations elements survive while, for odd \( d \), only odd parity ones remain; in particular the identity does not contribute for odd \( d \).

From (57) and (58) expressions for the coefficients in terms of generalised Bernoulli numbers can be obtained

\[ C_{d/2+k} = (\mp)^d \frac{(-1)^k 2(2k)!}{|\Gamma|(d + 2k)! k!} B_{d+2k}^{(d)}((d - 1)/2 \mid d), \quad (k = 1, 2, \ldots). \]  

(75)

The coefficient \( C_{d/2} \) is given by (53)

\[ C_{d/2} = (\mp)^d \frac{2}{|\Gamma|d!} B_d^{(d)}((d - 1)/2 \mid d). \]  

(76)
The remaining coefficients can be found from the residues of the Barnes ζ-function

\[ C_{(d-k)/2} = (\mp)^{(d-k)} \frac{2\Gamma(k/2)}{|\Gamma|(k-1)!(d-k)!} B_{d-k}^{(d)}((d-1)/2 | d), \quad (k = 1, 2, \ldots, d). \]  

(77)

For odd-dimensional spheres, and purely rotational \( \Gamma \), we thus generalise the result that the heat kernel asymptotic expansion on the full sphere terminates, in this case with the \( C_{d/2-1/2} \) term.

A particularly interesting example is the three-sphere since we can make contact with the results of Ray [39] on the relevant generating functions.

The asymptotic expansion of the heat kernel on orbifolds has been discussed by Donnelly [16, 42], and Brüning & Heintze [26] amongst others. The particular case of factored spheres is considered in some detail in [26], the structure of the Poincaré series being presented more abstractly than in [24].

The fact that the eigenvalues are proportional to squares of integers means that (47) can be used immediately to give a series for the free energy, \( F \), of the finite temperature field theory. Thus, in general,

\[ F(\beta) = E + \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} K^{1/2}(m\beta) = E + \frac{1}{\beta} \sum_{m=1}^{\infty} \frac{1}{m} e^{\pm d_0 m\beta/2} \prod \frac{1}{2 \sinh(d_0 m\beta/2)}, \]  

(78)

where \( \beta = 1/kT \).

In the cyclic case, on \( S^2 \),

\[ F_q(\beta) = \frac{1}{2\beta} \sum_{m=1}^{\infty} \frac{\coth(qm\beta/2)}{m \sinh(m\beta/2)}. \]

9 Other couplings

It is also possible to perform the calculation for massive or for minimally coupled fields but then Bessel functions would arise and the analysis would be more complicated (cf. [4, 27]).

The minimally coupled case is important as it corresponds to conformal coupling on \( S^2/\Gamma \) regarded as a Euclideanised space-time. Interest would then centre on the evaluation of \( \zeta'(0) \) and of \( \zeta'(0) \), which is proportional to the conformal anomaly. In fact it is easy to evaluate this last quantity using the results of the present paper since there is no extra difficulty in finding the heat kernel expansion coefficients. The relation between the conformal heat kernel and that, \( K_m \), for the general equation of motion [11] is

\[ K_m(\tau) = \exp \tau (m^2 + \frac{d-1}{4}((4\xi - 1)d + 1)) K(\tau) \]  

(79)
which, upon expansion, allows the new coefficients to be determined in terms of the old in a well known way.

In the case of two dimensions, and minimal coupling, we find the conformal anomaly

\[ \zeta_{\nu}(0) = (C_1 - \frac{1}{4}C_0) = \frac{1}{360|\Gamma|}(12(d_1 + d_2)^2 - 4d_1d_2 + 60(d_1 + d_2) - 45). \quad (80) \]

The space-time region can be thought of as a curvilinear triangle, and the conformal anomaly can also be found by conformal transformation methods.

10 Conclusion

We have found that, if \( \Gamma \) contains only rotations, the total vacuum energy, for conformal coupling on the divided even-dimensional sphere, \( S^{2d}/\Gamma \), is the same as that on whole sphere, namely zero. One would expect, however, that the vacuum average of the local energy density, \( \langle T_{00}\rangle \), would be nonzero and would diverge as the fixed points are approached, just as for a cosmic string.

If \( \Gamma \) contains reflections, i.e. is an extended group, then the vacuum energy is nonzero. In this case, because of the Dirichlet or Neumann boundary conditions, we would expect the local density to diverge at the sides of the fundamental triangle.

The vacuum energy on an odd-dimensional sphere is always nonzero, since, this time, it is the pure rotation elements of \( \Gamma \) that contribute, and these must be present. The curiosity here is that Neumann and Dirichlet boundary conditions give the same result, just half that of the strictly rotational \( \Gamma \subset SO(d + 1) \) case, the half factor being simply a volume effect.

A number of extensions suggest themselves. It would be straightforward to evaluate the finite temperature corrections and also to include the effect of a monopole magnetic field. In the latter case, by adding, in an ad hoc fashion, an extra term proportional to the square of the magnetic field to the Hamiltonian, it is possible to make the eigenvalues perfect squares so that the generating function can again be used. The only difference is that the angular momentum quantum number starts at the monopole number.

A further extension that might be considered is that to the Dirac equation. However there appears to be an obstruction to setting up spin-half theory on \( S^2/\Gamma \), at least using images.
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A  A Bernoulli identity

In the limit as $\tau \to 0$, the heat kernel can be expanded as a power series:

$$K_{S^2/T}(\tau) \approx \frac{1}{\tau} \sum_{k=0}^{\infty} C_k \tau^k.$$  

We will just concentrate on the cyclic case, $\Gamma \cong \mathbb{Z}_q$.

As mentioned in section 5, the trigonometric functions in the integral representation of the heat kernel, (13) or (16) are written as power series in $\alpha$ involving Bernoulli numbers, and the integration then carried out. One finds, cf [17],

$$K_q(\tau) = \frac{1}{q\tau} \left\{ \sum_n (-1)^{n+1} (1 - 2^{1-2n}) \frac{B_{2n}}{n!} \tau^n + 2 \sum_{n,m} (-1)^{n+m+1} (1 - 2^{1-2n}) \frac{B_{2n} B_{2m}}{(2n)! (2m)! (n + m + 1)!} \right\}.$$  

Manipulation of the double sum yields the heat kernel coefficients, for $k \geq 0$,

$$C_k = \frac{(-1)^{k+1}}{q k!} \left\{ -B_{2k}(\frac{1}{2}) + \frac{1}{2k - 1} \sum_{m=0}^{k} \left( \frac{2k}{2m} \right) (q^{2m} - 1) B_{2k-2m}(\frac{1}{2}) B_{2m} \right\}.$$  

These coefficients give the values of the zeta function at negative integers, via (31).

The $C_k$ can also be found directly using the values of the Hurwitz zeta function, $\zeta_R(-k, b) = -B_{k+1}(b)/(k + 1)$ for $k \geq 0$, and the result is given in (32).

The relation needed to reconcile these two expressions is the Bernoulli sum identity,

$$\sum_{m=0}^{k} \left( \frac{2k}{2m} \right) (q^{2m} - 1) B_{2k-2m}(\frac{1}{2}) B_{2m} = k q^{2k-2} \sum_{p=0}^{q-1} (2p + 1) B_{2k-2}(\frac{2p+1}{2q}).$$  

It is simple to verify this relation for $k = 0, 1$ while for $k > 1$ we have recourse to Lemma 1 of Apostol, [16], which says that for $r > 1$, there is the identity,

$$2r \sum_{t=0}^{\nu-1} \frac{\nu}{t} B_{2r-t}(\frac{\nu}{r}) = \sum_{t=0}^{2r} \binom{2r}{t} B_t B_{2r-t} \nu^{1-t} + (2r - 1) \nu^{1-2r} B_{2r}.$$  

This can be used to get

$$\sum_{p=0}^{q-1} (2p + 1) B_m(\frac{2p+1}{2q}) = \frac{q}{2k} \sum_{m=0}^{k} \left( \frac{2k}{2m} \right) B_{2m} B_{2k-2m} q^{1-2m}(2^{2m} - 2)$$

$$+ \frac{2k - 1}{2k} q^{2-2k}(2^{2k} - 2) B_{2k}.$$  

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Thus
\[ k q^{2k-2} \sum_{p=0}^{q-1} (2p+1) B_{2k-1} \left( \frac{2p+1}{2q} \right) = (2k-1) B_{2k} \left( \frac{1}{2} \right) + \sum_{m=0}^{k} \binom{2k}{2m} q^{2k-2m} B_{2k-2m} B_{2m} \left( \frac{1}{2} \right), \]
and we are left with having to show that
\[ (2k - 1) B_{2k} \left( \frac{1}{2} \right) = - \sum_{m=0}^{k} \binom{2k}{2m} B_{2k-2m} \left( \frac{1}{2} \right) B_{2m} \equiv \sum_{m=0}^{2k} \binom{2k}{m} \left( 1 - 2^{1-m} \right) B_{2k-m} B_m. \]

Again the identity (82) from Apostol, \[46\], gives
\[ (2k B_{2k-1} - (2k - 1) B_{2k}) - 2 \left( \frac{2k}{4} B_{2k-1} \left( \frac{1}{2} \right) - \frac{2k - 1}{2^{2k}} B_{2k} \right) \]
for the right hand side of (83). This simplifies to
\[ (2k - 1)(2^{1-2k} - 1) B_{2k} \]
and so we have proved the Bernoulli sum, (81).