The Theory of Invariants and Interaction
Symmetries

J.S.Dowker

Department of Theoretical Physics,
The University of Manchester, Manchester, UK.

The background. Added 2001.

To judge by the recent appearance of a pedagogical textbook (Classical Invariant Theory by P.J.Olver, LMS, 1999) the constructional aspect of classical invariant theory has made a strong comeback, due, in no short measure, to the existence of powerful computer algebra packages and to applications in image recognition and computer aided design.

The present author was always particularly struck by the Gordan-Hilbert finite algebraic basis theorem which was the culminating swan-song (as it turned out) of 60 years of endeavour and, one might almost say, of hard labour. This eminent position has not been reflected by extensive applications in physics, where linear independence seems to rule the everyday roost.

In 1965 I attempted to apply the theorem to an approach to particle symmetries which has fallen by the wayside. Nevertheless, despite the naive setup (by today’s standards) and in view of the mentioned increased activity, I have decided to reissue the report produced at that time as I think it still contains something of value. No changes have been made.
The Theory of Invariants and Interaction Symmetries

J.S. Dowker

Department of Theoretical Physics,
The University, Manchester.

Recently a number of attempts have been made to derive elementary particle interaction symmetries, including Lorentz invariance, by essentially algebraic means. There seem to be various, more or less equivalent, ways of deriving the basic equations. In references [1-3] bootstraps are referred to, in references [4,5] appeal is made to the Shmushkevich principle while in references [6,7] the basic equations are derived from the compositeness conditions, $Z_3 = 0$, $Z_1 = 0$. However, in all derivations some approximation is needed, whether it be a perturbation expansion or a Born approximation. Now it turns out that only the lower perturbation approximations are needed. In other words we appear to have more information than is necessary and so we are led to the idea that perhaps the lower order equations are somehow exact.

In this report we should like to introduce some ideas which suggest a non-perturbative method for deriving coupling constant structure equations and also, less ambitiously, indicate why only lower order perturbation equations need to be considered. We should state at the beginning that no definite results have yet been achieved but we feel the technique is sufficiently of interest to merit some attention.

Since we are only interested in deriving the general structure of coupling constant equations and not in the derivation of symmetries from these equations we shall restrict ourselves to a system described by the Lagrangian

$$\mathcal{L} = \frac{1}{2} \partial^\mu \phi^i \partial_\mu \phi^i - \frac{1}{2} m^2 \phi^i \phi^i + \mathcal{L}'$$

(1)
where the interaction $\mathcal{L}'$ is given by

$$
\mathcal{L}' = \sum_{ijk} g_{ijk} \phi^i \phi^j \phi^k
$$

and $\phi$ is an $n$-component field and the $g_{ijk}$ are, of course, completely symmetric.

Let us write down the ‘$g$–structure’ equations which we are trying to derive,

$$
\sum_{jk} g_{ijk} g_{i'jk} = N \delta_{ii'}
$$

$$
\sum_{i'j'k'} g_{i'j'k} g_{ij'k'} g_{i'j'k'} = \lambda g_{ijk}
$$

We now observe, with Cutkosky [2], that (3) and (4) are covariant under orthogonal transformations $O$ $i.e.$ under the transformations

$$
g_{ijk} \rightarrow \tilde{g}_{ijk} = O_{ii'}O_{jj'}O_{kk'} g_{i'j'k'}
$$

where

$$
\tilde{O}O = O\tilde{O} = 1
$$

Let us now symbolically represent $g_{ijk}$ by a product of three ‘vector’ factors:

$$
g_{ijk} = \alpha_i \alpha_j \alpha_k
$$

so that the interaction becomes, symbolically, a perfect cube

$$
\mathcal{L}' = (\alpha_i \phi^i)^3 = (\tilde{\alpha}\phi)^3 \equiv [\alpha\phi]^3
$$

Equations (3) and (4) now read on multiplying through by as many $\phi$’s as are needed to contract all the spare indices

$$
[\alpha\phi][\beta\phi][\alpha\beta]^2 = N [\phi\phi]
$$

and

$$
[\alpha\phi][\beta\phi][\gamma\phi][\alpha\beta][\beta\gamma][\gamma\alpha] = \lambda [\alpha\phi]^3
$$

In accordance with the usual rules [9] no more than three similar vector symbols should occur in the same factor and this has been adhered to in (8) and (9). These equations are orthogonally covariant under the transformation

$$
\alpha \rightarrow \tilde{\alpha} = O\alpha
$$

$$
\phi \rightarrow \tilde{\phi} = O\phi
$$
There exists a considerable mathematical literature [9] concerned with invariants and invariant (covariant) equations. On scanning through the main results of the theory one comes across the fundamental Gordan-Hilbert expansion theorem, which says essentially that given any fundamental (invariant) ground form, e.g. $L'$, then any concomitant of this ground form can be expanded as a generalised polynomial in a finite set of so-called irreducible concomitants. A concomitant is any combination of coefficients, e.g. $g$’s, and variables, e.g. $\phi$’s, which, under the transformation in question, retains its form but is multiplied by some power of the determinant of the transformation. In particular an invariant is multiplied by the zeroth power, so that for orthogonal transformations all concomitants are, in effect, invariants.

Our method can now be expressed in the above language as follows. We take as our fundamental ground form the interaction $L'$ and assume it to be an invariant under the orthogonal group. The free Lagrangian, $L - L'$ is invariant under this group, and so therefore will be the $S$–matrix. If we expand the $S$–matrix in normal products of $\phi_i^0$, the interaction picture fields, thus:

\[ S_{\text{connected}} = 1 + \Phi_i : \phi_i^0 : + \Sigma_{ij} : \phi_i^0 \phi_j^0 : + \Gamma_{ijk} : \phi_i^0 \phi_j^0 \phi_k^0 : + \ldots \]

(suppressing space-time integrations) we can say that each term will be a concomitant of different order (i.e. power of $\phi_0$). Thus in particular we can say that $\Sigma_{ij} \phi_i^0 \phi_j^0$ will be expressible as a generalised polynomial of irreducible concomitants of total order two. If we were to write this out fully we could equate powers of $\phi_0$ and end up with an expression for $\Sigma_{ij}$ in terms of combinations of the $g_{ijk}$. The self consistency equations result now by putting

\[ \Sigma_{ij} \propto \delta_{ij} \]

and similarly

\[ \Gamma_{ijk} \propto g_{ijk} \]

The problems remaining are to determine the irreducible set of concomitants and to write down the appropriate generalised polynomial. The first problem is a purely mathematical one of considerable complexity, [9], which becomes virtually impenetrable for cubics (like $L'$) of order greater than three (ternary cubics). Littlewood [9] has derived, more or less, the complete set of irreducible concomitants for the orthogonal ternary cubic using non-symbolic methods. The case of the binary cubic (i.e. $\phi$–two component) is somewhat peculiar since equations (3) and (4) yield
\[ g_{ijk} = 0 \text{ (unless } \lambda = 0) \text{(see remarks later). It must be remarked that Littlewood’s analysis applied to the case when the ternary cubic is simple under the orthogonal group. This means that the } g \text{’s are traceless i.e. } [\alpha\alpha] = 0, \text{ (This follows from equations (3) and (4) which we are trying to derive). Presumably this restriction can be lifted using the Gordan–Capelli expansion and the Clebsch–Young theorem [9].}

However, even granted that we have a complete set at our disposal we still have to form the generalised polynomial of order two. This appears to be rather complicated due to the large number of irreducible concomitants (26 for the orthogonal ternary cubic). It is true that we can throw away some combinations if they contain a single \( \epsilon_{ijk} \) factor as such a term cannot arise in \( \Sigma_{ij} \) but the remainder still seems too complex to handle easily. However, it does seem that the Gordan-Hilbert theorem provides us in principle with a method of exhibiting the \( g \)-structure of \( \Sigma_{ij} \) and \( \Gamma_{ijk} \) in terms of expressions with an upper limit to their connected complexity.

If this is true, then it is reasonable to suppose that in some way the \( g \)-structure expressions occurring in the perturbation expansion are reducible above a certain stage. This follows from the so-called fundamental theorem [9], \textit{viz} Every form which can be derived from \( g_{ijk}, \epsilon_{ijk} \ldots \) and \( \phi^i \) by the usual tensor rules of contraction is a concomitant of the ground form \( \mathcal{L}' \). The terms in the perturbation expression fall into this class and since there are a finite number of irreducible concomitants, [11], clearly most of the perturbation terms will be reducible. This can be checked for the binary cubic by using the determinant identity

\[
2(ab)(bc)(cd)(da) = (ab)^2(cd)^2 + (bc)^2(da)^2 - (ac)^2(bd)^2
\]

(and extensions) where

\[
(ab) = a_1b_2 - a_2b_1
\]

and

\[
(ab)(cd) = [ac][bd] - [ad][bc]
\]

In this case the triangle vertex graph is reducible as can be seen by writing out equation (4) in full. (This is not true for the ternary case. The triangle expression occurs in the list of irreducible concomitants given by Littlewood, [9]).

The corresponding results for the ternary case have not been derived. The difficulty seems to be that classical invariant theory deals with quantities like \( (ab), (abc), \ldots \) \textit{i.e.} \( \epsilon^{ij}a_ib_j, \epsilon^{ijk}a_ibjc_k, \ldots \), (We see here a link between binary quantities and two-spinors, [10]), and all the methods of reduction are appropriately expressed.
However the perturbation series generates concomitants in terms of scalar products $[ab]$, i.e. $a_i b_i$. A translation of the reduction process into tensor or group theory language would seem to be necessary, [12] (cf Littlewood [9]).
(1) Chan, H-M, P.C. De Celles and J.E. Paton, *Phys. Rev. Lett.* **11** (1963) 521
(2) R.E. Cutkosky, *Phys. Rev.* **131** (1963) 1886; *Ann. Phys.* **23** (1963) 415
(3) E.C.G. Sudarshan, *Phys. Lett.* **9** (1964) 286
(4) E.C.G. Sudarshan, L.S. O’Raifeartaigh and T.S. Santhanam, *Phys. Rev.* **136** (1964) B1092
(5) E.C.G. Sudarshan, *Symmetry in Particle Physics*, APS, Chicago (1964)
(6) J.S. Dowker, Nuovo Cimento **34** (1964) 773
(7) J.S. Dowker and P.A. Cook, Nuovo Cimento, submitted 1965.
(8) J.S. Dowker and J.E. Paton, Nuovo Cimento **30** (1963) 540
(9) *e.g.* J.H. Grace and A. Young, *Algebra of Invariants*, Cambridge (1903)
    H.W. Turnbull, *Theory of Determinants*, Dover (New York 1960)
    E.B. Elliot, *The Algebra of Quantics*, Oxford (1908)
    R. Weitzenböck, *Invariantentheorie*, Groningen (1923)
    D.E. Littlewood, Phil. Trans. Roy. Soc. (A) **239** (1944) 305
    H. Weyl, *Classical Groups*, Princeton (1939)
    O.E. Glenn, *Theory of Invariants*, Boston (1915)
    A.I. Mal’cev, *Linear Algebra*, Freeman, San Francisco (1963)
(10) H. Weyl, *Theory of Groups and Quantum Mechanics*, Methuen, London.
(11) The irreducible set of concomitants is *algebraically* over-complete due to the existence of syzygies, *i.e.* identities between the concomitants, which allow one to eliminate, in a *non-rational* way, some of the concomitants. The consequence of syzygies for the perturbation series is unclear.
(12) Compare the Clebsch-Gordan series.