TOTALLY GEODESIC ORBITS OF ISOMETRIES

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Abstract. We study cohomogeneity one Riemannian manifolds and we establish some simple criteria to test when a singular orbit is totally geodesic. As an application, we classify compact, positively curved Riemannian manifolds which are acted on isometrically by a non semisimple Lie group with an hypersurface orbit.

1. Introduction.

We deal with complete Riemannian manifolds \((M, g)\) which are acted on isometrically by a Lie group \(G\), which is closed in the full isometry group of \((M, g)\) and which has a hypersurface orbit; we will say that the action of \(G\) is of cohomogeneity one. Such \(G\)-manifolds have been investigated by several authors (see e.g.\([AA]\),\([AA1]\) and \([AP]\) for a large bibliography); their large degree of symmetry allows to construct several interesting examples of geometric structures which are rare in the homogeneous case, like Kaehler-Einstein metrics, Riemannian metrics with exceptional holonomy.

For general results about cohomogeneity one Riemannian manifolds, we refer the reader to the basic papers \([AA]\) and \([AA1]\). We summarize here some basic facts.

Given a \(G\)-manifold \((M, g)\) of cohomogeneity one, then the orbit space \(M/G\) is a one dimensional manifold, possibly with boundary; the boundary points correspond to singular orbits, which can be one or two. In case \(M\) is compact, either all orbits are regular or we have exactly two singular orbits \(B_1, B_2\); if we fix a normal geodesic \(\gamma\), i.e. a geodesic \(\gamma : \mathbb{R} \rightarrow M\) which intersects every orbit orthogonally, we may denote by \(K\) the common stabilizer of all regular points of \(\gamma\) and by \(H, H'\) the two singular stabilizer, so that \(K \subset H \cap H'\) and \(B_1 = G/H, B_2 = G/H'\). The slice theorem ([Br]) asserts in this case that \(H/K\) and \(H'/K\) are diffeomorphic to spheres. We will call \(\theta\) the triple of subgroups \((H, K, H')\) which we have constructed using \(\gamma\); it is easy to see that, if we change the choice of the geodesic, the triple changes under conjugation by some element of \(G\). A triple \(\theta = (H, K, H')\) of subgroups of \(G\) will be called admissible if \(K \subset H \cap H'\) and \(H/K, H'/K\) are diffeomorphic to spheres. One of the main results in \([AA]\) states that there is a one to one correspondence between \(G\)-manifolds of cohomogeneity one with two singular orbits (up to \(G\)-diffeomorphism) and admissible triples (up to conjugation).

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Our first aim is to give some simple criterium to test when a singular orbit is totally geodesic; this is obtained in §2, where in Theorem 2.1 we find that a bound on the dimension of the singular orbit forces it to be totally geodesic. Another useful criterium is given essentially in Lemma 2.3, where we state that the singular orbit turns out to be totally geodesic as soon as the $G$-action on it has some non discrete kernel. As a first application of these results, we prove in Theorem 2.5 that a singular orbit which is, as a homogeneous space, a symmetric space of rank one is automatically totally geodesic, with the only exceptions of the two-dimensional sphere and the projective planes over $C$, $H$, $Ca$.

In §3, we apply the basic idea stated in Lemma 2.3 to the case of a compact positively curved Riemannian manifold which is of cohomogeneity one with respect to a compact, non semisimple Lie group of isometries.

Compact Riemannian manifolds of positive curvature with a non discrete isometry group have been investigated by Hsiang and Kleiner ([HK]) in the four-dimensional case; in [GS], Grove and Searle investigated the case of the action of some torus on positively curved manifolds and they established a classification theorem, up to diffeomorphism, when there exists a $T^1$-action or an $SU(2)$-action with fixed point set of codimension 2 or not less than 4 respectively. The problem of classifying compact, positively curved Riemannian manifolds has been solved in dimension 5 and 6 by Searle ([S]), giving a complete classification up to diffeomorphism.

Our main result is stated in Theorem 3.1, where we classify, for $G$ compact non semisimple, all possible $G$-manifolds of cohomogeneity one which can carry an invariant Riemannian metric with positive curvature. We find that $M$ must be diffeomorphic to a compact rank one symmetric space, with the exception of the Cayley projective plane.

2. Totally geodesic orbits.

Throughout the following, $(M, g)$ will be a (complete) Riemannian manifold which is acted on isometrically by a connected Lie group $G$; the group $G$ will be always supposed to be closed in the full group of isometries of $(M, g)$. We will denote by $M_{\text{reg}}$ the open dense subset of $M$ given by regular points for the $G$-action. Our first result on the geometry of singular orbits is as follows.

**Theorem 2.1.** Let $(M, g)$ be a Riemannian $G$-manifold, where $G$ is a connected Lie group acting (almost) effectively and isometrically on $(M, g)$ with cohomogeneity 1. Let $G/H$ be an isolated singular orbit and denote by $k$ the cohomogeneity of the $H$-action on $G/H$. If

$$\dim G/H < \frac{1}{2}\left[\dim M + k - 1\right],$$

then $G/H$ is totally geodesic.

**Proof.** We denote by $K$ a regular isotropy subgroup and we fix a singular point $q \in M$ such that the isotropy $G_q = H$ contains $K$. We now suppose that $G/H$ is not totally geodesic, so that there exists a tangent vector $v \in T_q(G/H)$ such that
the geodesic $\gamma : [0, \epsilon) \to M$ (for some $\epsilon \in \mathbb{R}^+$) starting from $q$ with initial vector $v$ is not entirely contained in $G/H$. Since $G/H$ is supposed to be an isolated singular orbit, it follows that $\gamma((0, \epsilon))$ intersects non trivially $M_{\text{reg}}$. We now denote by $i$ the isotropy representation of $H$ and consider the stabilizer $H_v = \{ h \in H; i(h)v = v \}$. It is clear that every element of $H_v$ fixes $\gamma$ pointwise, hence $H_v$ is contained in some conjugate of $K$. Hence $\dim H_v \leq \dim K$. We therefore get

$$\dim G/H - \dim H + \dim H_v \leq \dim G/H - \dim H + \dim K.$$ 

If we denote by $k$ the cohomogeneity of the $H$-action on $G/H$, this is equal to the cohomogeneity of the $H$-action on the tangent space $T_q(G/H)$ via isotropy representation and therefore

$$k \leq \dim G/H - \dim H/H_v \leq \dim G/H - \dim H/K.$$ 

We now recall that the cohomogeneity of the $G$-action on $M$ is equal to the cohomogeneity of the slice representation of $H$; since $H/K$ is a regular orbit for the slice representation of $H$, we have that

$$\dim H/K = \dim M - \dim G/H - p,$$

hence

$$k \leq 2 \dim G/H - \dim M + p$$

which contradicts our hypothesis. \(\square\)

When the action of the group $G$ has cohomogeneity one, the existence of totally geodesic orbit can be deduced by purely algebraic and topological assumptions, as the following proposition shows.

**Proposition 2.2.** Let $M$ be a compact manifold of positive Euler characteristic. If a compact, non semisimple Lie group $G$ acts almost effectively on $M$ with cohomogeneity one, then there exists at least one singular orbit which is totally geodesic with respect to any $G$-invariant Riemannian metric on $M$.

**Proof.** If the Euler characteristic $\chi(M)$ is positive, then the orbit space $M/G$ is homeomorphic to a closed interval $[0, 1]$ and there are exactly two singular orbits $B_1, B_2$; moreover we have that

$$\chi(M) = \chi(B_1) + \chi(B_2),$$

since a regular orbit is odd-dimensional (see e.g. [AP]). It then follows that at least one singular orbit, say $B_1 = G/H$, has positive characteristic; hence the stability subgroup $H$ is of maximal rank in $G$. If $G$ is not semisimple, it has some connected center $T$ of positive dimension, which is contained in $H$; therefore, the action of $G$ on $G/H$ has some non trivial kernel. The following useful Lemma will conclude the proof.
Lemma 2.3. If $G/H$ is not totally geodesic, then the action of $G$ on $G/H$ is almost effective (i.e. any ideal of $G$ which is contained in $H$ is finite)

Proof. We suppose that the representation $\iota$ of $H$ has some kernel $N$. If $G/H$ is not totally geodesic, then there exists a tangent vector $v \in T_p(G/H)$ such that the subgroup $H_v = \{ h \in H; \iota(h)v = v \}$ is contained in some conjugate $gKg^{-1}$ for some $g \in G$. But $N \subset H_v$, hence $N = g^{-1}Ng \subset K$. If we denote by $\nu$ the slice representation of $H$, we have that $\nu(N) \subset \nu(K)$ is a normal subgroup of $\nu(H)$; since $\nu(H)$ acts effectively on the unit sphere of the normal space $T_p(G/H)^\perp$, it follows that $N \subset \ker \nu$. But the action of $G$ on $M$ was supposed to be almost effective, so $N = N \cap \ker \nu$ is a finite subgroup of $G$. $\square$

Again, when the cohomogeneity of the $G$-action is 1 with one singular orbit $G/H$ and the cohomogeneity $k$ of the action of $H$ on $G/H$ is 1, that is when the singular orbit is a two point-homogeneous space, the estimate given in Corollary 2.2 can be refined. Actually, we may prove the following

Theorem 2.4. Let $(M, g)$ be a complete Riemannian manifold which is acted on almost effectively and isometrically by a connected Lie group $G$ with cohomogeneity one. If a singular orbit $G/H$ is a two-point-homogeneous space, then $G/H$ is totally geodesic in $M$ unless $G/H$ is the two dimensional sphere or a projective space of dimension 2 over $\mathbb{R}, \mathbb{C}, \mathbb{H}, \mathbb{Ca}$.

In Table 1 we indicate the exceptional homogeneous space $G/H$ for compact $G$, a compact Riemannian $G$-manifold $M$ of cohomogeneity one having $G/H$ as non totally geodesic singular orbit and the slice representation $\nu$ of $H$.

Remark. The manifold $M$ is not necessarily unique; but if $(N, h)$ is any Riemannian manifold which is acted on by $G$ with non totally geodesic singular orbit $G/H$, then a $G$-invariant tubular neighborhood of $G/H$ in $N$ is $G$-diffeomorphic to a $G$-invariant tubular neighborhood of $G/H$ in $M$. The embeddings of the projective spaces $G/H$ into $M$ are the Veronese embeddings

| $G/H$ | $G$ | $H$ | $M$ | $\nu$ |
|-------|-----|-----|-----|------|
| $S^2 \cong \mathbb{C}P^1$ | $SO(3), SU(2)$ | $SO(2), U(1)$ | $\mathbb{C}P^2$ | $\nu(e^{i\theta}) = e^{2i\theta}$ |
| $\mathbb{R}P^2$ | $SO(3)$ | $SO(1) \times O(2))$ | $S^4$ | $\nu : O(2) \to O(2)/\mathbb{Z}_2$ |
| $\mathbb{C}P^2$ | $SU(3)$ | $U(2)$ | $S^7$ | $\nu : U(2) \to SU(2)/\mathbb{Z}_2$ |
| $\mathbb{H}P^2$ | $Sp(3)$ | $Sp(1) \times Sp(2)$ | $S^{13}$ | $\nu : Sp(1) \cdot Sp(2) \to SO(5)$ |
| $\mathbb{C}aP^2$ | $F_4$ | $Spin(9)$ | $S^{25}$ | $\nu : Spin(9) \to SO(9)$ |

Table 1

Proof. We start recalling that if $p$ is a singular point with singular isotropy subgroup $H$, then the isotropy representation of $H$ at $p$ splits into the sum of two
representations $\nu$ and $\tau$, where $\nu$ is the slice representation and $\tau$ is the isotropy representation of $H$ on the homogeneous space $G/H$. The singular orbit $G/H$ is supposed to be a two-point-homogeneous space, on which $G/H$ group and $G/H$ act almost effectively by Lemma 2.4, if $G/H$ is not totally geodesic.

So, by the classification of two-point-homogeneous spaces, $G$ is a simple Lie group and $G/H$ is a symmetric space of rank one. We will analyze the case when $G$ is compact, while the not compact case follows by duality. Now, for each compact symmetric space of rank one (CROSS) $G/H$, we look at all possible representations $\nu : H \to O(V)$ such that $\nu(H)$ acts transitively and effectively on the unit sphere of $V$. Of course, we always have the choice $\nu = \tau$.

**Lemma 2.6.** If $\nu = \tau$, then $G/H$ is totally geodesic.

**Proof.** Indeed, we denote by $h$ the second fundamental form of $G/H$ at $p \in M$; we may consider $h$ as an element of $S^2(V^*) \otimes V$, which is acted on by $H$ in a natural way by $S^2\tau^* \otimes \tau$. Now, it easy to see that, for all $H$ appearing as isotropy subgroups of any CROSS, the space $S^2(V^*) \otimes V$ has no non trivial $H$-fixed vectors, so that $h = 0$. $\square$

We may now consider each CROSS separately; for each of them, we look for representations $\nu : H \to O(V)$, which are transitive on the unit sphere of $V$ and for which the $G$-manifold $G \times_{(H,\nu)} V$ admits a Riemannian metric such that $G/H$ is not totally geodesic. Throughout the following, we will keep the same notations, meaning $K$ for a regular isotropy subgroup and $H_v$ for a regular isotropy subgroup of $H$ for the action of $H$ on $G/H$; this means that, by the proof of the main theorem, $H_v$ must be contained in some conjugate of $K$. Moreover, we recall that we allow the action of $G$ on $G \times_{(H,\nu)} V$ to be almost effective.

1. $G/H = SO(n+1)/SO(n) = S^n$. In this case, we may suppose that $G = Spin(n+1)$ and $H = Spin(n)$ for $n \geq 3$ and $H = SO(2)$ for $n = 2$.

Then, any representation $\nu : H \to O(V)$ which is transitive on the unit sphere of $V$, factors through $\pi : Spin(n) \to SO(n)$ giving rise to the standard representation $\nu = \tau$ of $SO(n)$, unless $n = 2, 4, 5, 6, 7, 9$. We deal with each case separately.

(a) In case $n = 2$, we have the covering homomorphisms $\nu_k : SO(2) \to SO(2)$ given by the $k$-power, that is $\nu_k(e^{i\theta}) = e^{ik\theta}$. Again, if we denote by $h$ the fundamental form at $p$ and if we denote by $\sigma$ the element of $H = SO(2)$ such that $\nu_k(\sigma) = -Id$, then, for all tangent vectors $X, Y \in T_p(G/H)$, we have that $h(X, Y) = -h(\tau(\sigma)X, \tau(\sigma)Y)$. Now, it is elementary to see that if $k \neq 2$, then $h = 0$. We now construct an example of a $G = SO(3)$-action on some Riemannian manifold $M$ such that one singular orbit $G/H$ is $S^2$ and is not totally geodesic. In order to do this, we consider the action of $SO(3)$ on $\mathbb{C}P^2$ given by the standard inclusion $SO(3) \subset U(2)$; then the singular orbit $G/H$ is the orbit of $G$ through the point $[1, 0, 0]$, which is easily seen to be not totally geodesic.

(b) In case $n = 4$, we have that $Spin(4) \cong Sp(1) \times Sp(1)$ and we may take $\nu : Spin(4) \to Sp(1)$, with $V = \mathbb{H}$. In this case, $K$ can be taken to be one normal factor $Sp(1)$ of $Spin(4)$, while $H_v$ can be taken to be a copy of $Sp(1)$ embedded diagonally into $Spin(4)$. Now, $H_v$ is not contained in any conjugate $gKg^{-1}$ for
g ∈ Spin(5), as one can easily see by considering the projected subgroups π(K) and π(H_v) of SO(5), where π : Spin(5) → SO(5) is the standard covering map:

(c) When n = 5, 6, we have special isomorphisms, namely Spin(5) ≃ Sp(2) and Spin(6) ≃ SU(4), so that we can choose V = ℍ^2 and V = ℂ^4 respectively. Here Theorem 2.1 applies and the singular orbit is totally geodesic. When n = 7, 9, we have special representations of Spin(7) and Spin(9) which are transitive on the unit sphere of ℝ^8 and ℝ^16 resp. Again Theorem 2.1 applies.

(2) G/H = SU(n+1)/SU(1) × U(n)) = ℂP^n. Here we have G = SU(n+1) and H = U(n) and we have several possibilities for the representation ν.

(a) ν : U(n) → T^1/Z_n ≃ T^1. In this case K = ker ν = SU(n). If the orbit G/H should not be totally geodesic, then a regular isotropy subgroup H_v for the action of H on G/H should be contained in some conjugate of K; but this is easily seen to be impossible.

(b) We have ν_k : U(n) → U(n) given by ν_k(A) = (det A)^kA for k ∈ ℤ; they represent, up to equivalence, all possible coverings of U(n). But again an easy computation shows that the subgroup H_v is contained in some conjugate of K if and only if k = 1; so by Lemma 2.4, the orbit G/H is totally geodesic.

(c) When n = 2, we have ν : U(2) → SU(2)/Z_2 ≃ SO(3). In this case, in order to construct the manifold M, it is enough to consider the sphere S^7 ⊂ su(3), which is acted on by SU(3) by adjoint representation; this action has cohomogeneity one and one singular orbit is CP^2, which cannot be totally geodesic in S^7.

(3) G/H = Sp(n+1)/Sp(1) × Sp(n) ≃ ℍP^n. Here we may take the simply connected group G = Sp(n+1) and H = Sp(1) × Sp(n). We have to consider the following possibilities:

(a) ν : Sp(1) × Sp(n) → Sp(1)/Z_2 = SO(3). Here K = T^1 × Sp(n) and H_v = Sp(1) × Sp(n − 1), which is not contained in any conjugate of K. In this case the orbit G/H is totally geodesic.

(b) ν : Sp(1) × Sp(n) → Sp(n); in this case H_v = Sp(1) × Sp(n − 1) and K = Sp(1) × Sp(n − 1), where Sp(1)Λ is a copy of Sp(1) embedded diagonally into Sp(1) × Sp(1) ⊂ Sp(1) × Sp(n). Here again H_v is not contained in any conjugate of K, since, as subgroups of Sp(n + 1), we have that Fix(ℍ^n+1, H_v) is a quaternionic line, while Fix(ℍ^n+1, K) is reduced to {0}.

(c) ν : Sp(1) × Sp(2) → Sp(2)/Z_2 ≃ SO(5). In this case the manifold M can be chosen to be the sphere S^13; indeed the group Sp(3) admits a cohomogeneity two linear action given by λ^2ν − 1, with representation space ℝ^{14} (see [HL]); one singular orbit for the action on the unit sphere is a copy of the quaternionic projective plane ℍP^2 (Veronese embedding).

(4) G/H = F_4/Spin(9) = CaP^2. In this case the only representation ν different from τ is given by ν : Spin(9) → SO(9). If we now consider the irreducible linear representation φ of F_4 of dimension 26, it has cohomogeneity two (see [HL]) and one singular orbit for the F_4-action on S^25 is a copy of a Cayley projective plane (again Veronese embedding).

□

3. Manifolds of positive curvature.

In this section, we will use some ideas which were developed in the previous section and we will consider even dimensional, compact Riemannian manifolds (M, g)
of positive curvature, which are acted isometrically and (almost) effectively on by a compact, non semisimple Lie group $G$ of isometries with one hypersurface orbit.

Our aim is to prove the following

**Theorem 3.1.** Let $M^{2n}$ be an even dimensional, compact Riemannian manifold of positive sectional curvature. If a compact, non semisimple Lie group $G$ acts isometrically on $M^{2n}$ by cohomogeneity one, then

1. the dimension of the center of $G$ is one and the semisimple part of $G$ acts by cohomogeneity one;
2. the manifold $M$ is diffeomorphic to a rank one symmetric space with the exception of the Cayley projective plane.

We start proving some basic lemmata.

**Lemma 3.2.** Let $M^{2n}$ be an even dimensional, compact Riemannian manifold of positive sectional curvature. If a compact Lie group $G$ acts isometrically on $M^{2n}$ by cohomogeneity one, then $M^{2n}$ has positive Euler characteristic and there are exactly two singular orbits, one of which has positive Euler characteristic.

**Proof.** We note that $M^{2n}$ has finite fundamental group, so that there is no fibration $M^{2n} \to S^1$. It then follows that there are exactly two singular orbits, $B_1, B_2$. We now claim that at least one of these has positive Euler characteristic, which will conclude the proof. In fact, let $T$ be a maximal torus of $G$ and let $X$ be a Killing vector field on $M^{2n}$ so that $\{\exp(tX)\} = T$. By Berger’s Theorem (see e.g.[Ko]), the field $X$ has at least one zero $p \in M^{2n}$. We claim that we can suppose that $p$ is a singular point. Indeed, if $p$ is regular, then $X$ vanishes along the whole normal geodesic issuing from $p$, hence it vanishes at some singular point too. Therefore, we have that $T \subset G_p$ and $G_p$ has maximal rank in $G$. This implies that $\chi(G/G_p) > 0$. □

**Lemma 3.3.** Let $M^{2n}$ be an even dimensional, compact Riemannian manifold of positive sectional curvature. If a compact, non semisimple Lie group $G$ acts isometrically and almost effectively on $M^{2n}$ by cohomogeneity one, then

1. the dimension of the center $Z(G)$ is one;
2. there is a singular orbit which is totally geodesic and of positive Euler characteristic.

**Proof.** By Lemma 3.2, we know that there exists a singular orbit $B$ which is of positive Euler characteristic; hence, if we represent $B = G/H$, then $Z(G) \subset H$ and $B$ is totally geodesic by Lemma 2.3. Now $Z(G)$ acts trivially on $B$ and, if $\nu$ denotes the slice representation of $H$, then $\nu(Z(G))$ is at most one-dimensional; then almost effectiveness of the $G$-action implies that $\dim Z(G) = 1$. □

Throughout the following, we will always denote by $B$ the totally geodesic singular orbit with $\chi(B) > 0$. Moreover we will use the following results by Grove-Searle ([GS])
Theorem [GS]. Let $M^n$ be a simply connected, compact positively curved manifold.

1. If $T^1$ acts isometrically and effectively on $M^n$ with a fixed point set of codimension 2, then $M^n$ is diffeomorphic to $S^n$ or to $\mathbb{C}P^{n/2}$;
2. If $SU(2)$ acts isometrically and almost effectively on $M^n$ with fixed point set of codimension less or equal to 4, then $M^n$ is diffeomorphic to $S^n, \mathbb{C}P^{n/2}$ or $\mathbb{H}P^{n/4}$.

Since the manifold $(M, g)$ has finite fundamental group, we will always assume that $M$ is simply connected. Moreover, when the singular orbit $B$ is reduced to a point, i.e. when $G$ has a fixed point, we know from [AP], that the manifold is diffeomorphic to a CROSS (compact rank one symmetric space), while if the codimension of $B$ is 2, we may apply Theorem [GS], to get that $M$ is diffeomorphic to $S^n$ or to $\mathbb{C}P^{n/2}$. So, in the proof of Theorem, we will always assume that $\text{codim} B > 2$.

We may always decompose the group $G$ as $G = T^1 \cdot G_o \cdot G_1$, where $T^1 \cdot G_o$ is the kernel of the $G$-action on $B$; accordingly, we have that $H = T^1 \cdot G_o \cdot H_1$, where $G_1$ acts (almost) effectively on $B$ with $B = G_1/H_1$. We denote by $\nu$ the slice representation of $H$ and by gothic letters the Lie algebras.

Case $g_0 \neq 0$.

In this case we claim that there is an $SU(2)$-action whose fixed point set has codimension 4 in $M$, so that we can apply Theorem [GS].

Since the $G$-action on $M$ is almost effective, we have that $\nu|_{g_o + \mathbb{R}}$ is an isomorphism and since $\nu(g_o)$ has a non trivial centralizer, we have $g_0 \cong su(m)$ or $sp(m)$. We fix invariant complements $p$ and $m$ so that

$$g = h + m, \ g_1 = h_1 + m, \ h = \mathfrak{t} + p.$$ 

At a regular point $y \in M$, we have the identification (as $\mathfrak{t}$-modules)

$$T_yM = \mathbb{R} + p + m.$$ 

Moreover, $\nu(g_o)$ still acts transitively on the unit normal sphere, so we can consider the stabilizer $\mathfrak{t}_o \subset g_o$, with $\mathfrak{t}_o \cong su(m - 1), sp(m - 1)$. Now, if $\mathfrak{t}_o = \{0\}$, then $g_o \cong su(2)$ and $G_o$, which is locally isomorphic to $SU(2)$, fixes $B$ which has codimension 4.

If $\mathfrak{t}_o \neq \{0\}$, then $\mathfrak{t}_o$ acts trivially on $m$ and $p$ splits as $p = p_o + p_1$, where $p_o$ is a one- or three-dimensional trivial summand and $p_1$ is $\mathfrak{t}_o$-irreducible. In any case, there is a subalgebra $su(2) \subset \mathfrak{t}_o$, whose fixed point set in $p$ has codimension 4 in $p$ and we are done.

Case $g_o = \{0\}$.

In this case $G = T^1 \cdot G_1$ and $H = T^1 \cdot H_1$, where $G_1/H_1$ is even dimensional and carries a positively curved invariant metric. Moreover, $B$ is the fixed point set of $Z(G)$, hence it is simply connected by Synge Theorem ([Ko]). We give here the list
of all such pairs \((G_1, H_1)\) together with the dimension of the corresponding space \(G_1/H_1\); we call this the Wallach’s list ([Wal]):

| \(n\) | \(G_1\) | \(H_1\) | \(\text{dim}\) |
|-------|--------|--------|--------|
| 1     | \(SU(n+1)\) | \(U(n)\) | \(2n\) |
| 2     | \(SU(3)\) | \(T^2\) | \(6\) |
| 3     | \(Spin(2n+1)\) | \(Spin(2n)\) | \(2n\) |
| 4     | \(Sp(n)\) | \(Sp(1) \times Sp(n-1)\) | \(4(n-1)\) |
| 5     | \(Sp(n)\) | \(T^n \times Sp(n-1)\) | \(2(2n-1)\) |
| 6     | \(Sp(3)\) | \(Sp(1)^n\) | \(12\) |
| 7     | \(F_4\) | \(Spin(9)\) | \(16\) |
| 8     | \(F_4\) | \(Spin(8)\) | \(24\) |
| 9     | \(G_2\) | \(SU(3)\) | \(6\) |

**Wallach’s list**

Since we are supposing that \(\text{codim } B > 2\), we have that \(\nu|_{H_1}\) is not trivial, so \(h_1\) must contain some ideal isomorphic to \(\mathfrak{su}(m)\) or to \(\mathfrak{sp}(m)\). It then follows that we may exclude cases (2), (7), (8) and (3) (for \(n \neq 2, 3\)) from Wallach’s list.

Our next arguments will rely on the study of the second singular orbit \(B' = G/H'\), where \(K \subset H \cap H'\); we will denote by \(\nu' : H' \to O(V')\) the slice representation of \(H'\) on the normal space \(V'\). The following two observations will be useful.

**Fact 1.** The Lie algebra \(h'\) cannot be semisimple. Otherwise it would be contained in \(g_1\), while \(\mathfrak{k}\) always has a non trivial projection on \(j(g)\).

**Fact 2.** If \(h'\) has maximal rank, then \(\dim V' > \dim B\). Indeed, if \(h'\) has maximal rank, then \(B'\) is totally geodesic and, by Frankel Theorem ([Ko]), \(\dim B + \dim B' < \dim M\).

Now, for each case in Wallach’s list, we compute \(\mathfrak{k}\) and for each ideal \(n\) of \(\mathfrak{k}\), we look for subalgebras \(h'\) in \(g\) such that \(h'\) contains \(\mathfrak{k}\) and has \(n\) as an ideal with \((h'/n, \mathfrak{t}/n)\) belonging to Borel’s list.

Using Fact 1 and 2, it is not difficult to exclude case (9) and (3) for \(n = 2\).

We are left with the cases (1), (4), (5), (6). Case (1) is handled in the next

**Lemma 3.4.** If \(G_1 = SU(n)\), then the manifold \(M\) is diffeomorphic to a sphere, a complex or quaternionic projective space.

**Proof.** We have that \(G_1 = SU(m)\) and \(H_1 = U(m-1)\) for some \(m\). Since we are supposing that the codimension of \(B\) is bigger than 2, we have that \(\nu(H) = U(m-1)\) and \(\nu(H_1)\) still acts transitively on unit normal sphere. Note that \(\dim M = 4(m-1)\), which can be assumed to be bigger than 4, by the results of Hsiang and Kleiner ([HK]). So, the \(G_1\)-action on \(M\) is of cohomogeneity one. The slice representation \(\nu|_H\) can be assumed to be of the form

\[
\nu((e^{i\theta}, A)) = (\det A)^j A, \quad (e^{i\theta}, A) \in S(U(1) \times U(q)),
\]
where \( j \in \mathbb{Z} \). It then follows that we can fix \( K \) to be

\[
K = \left\{ \begin{pmatrix} e^{i\theta} & e^{i\phi} \\ 0 & A \end{pmatrix} \in SU(q+1); e^{i(j+1)\phi}(det A)^j = 1 \right\},
\]

so that \( \mathfrak{k} \cong \mathbb{R} + \mathfrak{su}(m-2) \). Now, for any ideal \( n' \subset \mathfrak{k} \), we determine all subalgebras \( \mathfrak{h}' \) of \( \mathfrak{g}_1 \) such that \( \mathfrak{k} \subset \mathfrak{h}' \) and \( (\mathfrak{h}'/n', \mathfrak{k}/n') \) belongs to Borel's list.

We have the following possibilities (here \( n' \) and \( \mathfrak{h}' \) are given up to isomorphism):

| \( n \) | \( n' \) | \( \mathfrak{h}' \) | \( \dim V' \) |
|---|---|---|---|
| 1 | \{0\} | \mathbb{R} + \mathfrak{su}(m-1) | 2q |
| 2 | \( \mathfrak{k} \) | \mathbb{R}^2 + \mathfrak{su}(m-2) | 2 |
| 3 | \( \mathfrak{k} \) | \( \mathbb{R} + \mathfrak{su}(2) + \mathfrak{su}(m-2) \) | 4 |
| 4 | \( \mathbb{R} \) | \( \mathbb{R} + \mathfrak{su}(m-1) \) | \( 2(m-1) \) |
| 5 | \( \mathbb{R}, m = 4 \) | \( \mathbb{R} + \mathfrak{so}(4) \) | 4 |
| 6 | \( \mathfrak{su}(m-2) \) | \( \mathfrak{u}(2) + \mathfrak{su}(m-2) \) | 4 |
| 7 | \( \mathfrak{su}(m-2) \) | \( \mathfrak{u}(2) + \mathfrak{su}(m-2) \) | 3 |

First of all we must exclude the case (1)-(5) of the previous table. In each case (1)-(5), the subalgebra \( \mathfrak{h}' \) is of maximal rank, hence \( \chi(B') > 0 \). On the other hand, we have that \( G = T^1 \cdot G_1 \) and, if \( \exp(tX) \in T^1 \), then \( \exp(tX) \) maps \( B' \) onto a \( G_1 \)-orbit; since \( B' \) is singular, we get that \( T^1 \) maps \( B' \) onto itself, hence \( G(B') = B' \). Since \( \chi(B') > 0 \), we may represent \( B' \) as \( B' = G/H' \) with \( H' \) of maximal rank in \( G \) so that \( B' \) is totally geodesic. But, it can be checked that \( \dim G/H + \dim G/H' \geq 4(m-1) = \dim M \) and by Frankel's theorem (see [Fr]) the two singular orbits should intersect, which is not the case.

So only case (5) is admissible. Moreover, since \( H \) is connected and \( H/K \cong S^{2m-3} \), with \( 2m - 3 > 1 \), we have that \( K \) is connected; since now \( H'/K \cong S^2 \), we get that \( H' \) is connected.

In order to avoid confusion, we put \( \mathfrak{h}' = \mathfrak{a} + \mathfrak{b} \), where \( \mathfrak{a}, \mathfrak{b} \) are ideals of \( \mathfrak{h}' \) which are isomorphic to \( \mathfrak{su}(2) \) and \( \mathfrak{su}(m-2) \) respectively.

We will now divide the discussion according to \( m \geq 4 \) or \( m = 3 \).

**Case** \( m \geq 4 \).

In this case \( \mathfrak{b} \neq \{0\} \) and \( \mathfrak{a} \) centralizes \( \mathfrak{su}(q-1) \) in \( \mathfrak{g}_1 \); so \( \mathfrak{a} \) is contained in the semisimple part of the centralizer \( C_{\mathfrak{a}}(\mathfrak{su}(m-2)) \), which is isomorphic to \( \mathfrak{su}(2) \). It then follows that \( \mathfrak{a} \) is embedded into \( \mathfrak{g}_1 \) as the set of all matrices

\[
\mathfrak{a} = \left\{ \begin{pmatrix} \mathfrak{su}(2) & 0 \\ 0 & 0 \end{pmatrix} \in \mathfrak{su}(m) \right\}
\]

and therefore

\[
\mathfrak{h}' = \left\{ \begin{pmatrix} \mathfrak{su}(2) & 0 \\ 0 & \mathfrak{su}(m-2) \end{pmatrix} \in \mathfrak{su}(m) \right\}.
\]
Since $\bar{K} \subset \bar{H} \cap \bar{H}^\prime$, we get that

$$\bar{K} = \{ \begin{pmatrix} e^{i\theta} & 0 \\ 0 & e^{-i\theta} \end{pmatrix} \} \in SU(m); \det A = 1,$$

which forces $j = -1$.

Therefore we have proved that the $G_1$-action on $M$ is associated to the triple $\theta = (H, K, H^\prime)$, which is uniquely given by

$$\theta = (S(U(1) \times U(m - 1)), S(U(1) \times U(1)) \times SU(m - 2), SU(2) \times SU(m - 2)).$$

On the other hand, if we consider the standard embedding $SU(m) \subset U(m) \subset Sp(m)$, then the group $SU(m)$ acts on the space $Sp(m)/Sp(1) \cdot Sp(m - 1) \cong \mathbb{H}^{m-1}$ by cohomogeneity one and associated triple of subgroups $\theta$. According to the main results in [AA], we conclude that $M$ is diffeomorphic to $\mathbb{H}^q$.

**Case** $m = 3$.

In this case $h^\prime \cong su(2)$ and

$$\mathfrak{k} = \{ \text{diag}(i\theta, ij\theta, -i(j + 1)\theta) \in su(3); \theta \in \mathbb{R} \}.$$

Up to conjugation there are exactly two imbeddings of $su(2)$ into $su(3)$, corresponding to a reducible of irreducible representation $\pi$ of $su(2)$ on $\mathbb{C}^3$; but in any case $\pi$ has 0 as a weight, so that we must have $j = 0$ or $j = -1$. Since the two subalgebras $\mathfrak{k}$ corresponding to $j = 0$ and $j = -1$ are conjugated by some element which normalizes $h$, we may suppose that $j = -1$. In this case, there is only one subalgebra $h^\prime$ isomorphic to $su(2)$ and containing $\mathfrak{k}$, so that, by the same argument as in the previous case, the manifold is diffeomorphic to $\mathbb{H}^2$. □

We now proceed excluding the remaining cases.

**Lemma 3.5.** Case (6) cannot occur.

**Proof.** In this case $G = T^1 \cdot Sp(3), H = T^1 \cdot Sp(1)^3$ with $\nu(H) = U(2)$. Using Fact 1 and 2, it is not difficult to see that the only possibility for $h^\prime$ is $h^\prime \cong \mathbb{R} \oplus sp(2)$.

We will now consider the action of the semisimple part of $G$, which still acts by cohomogeneity one with associated triple $\theta = (H, K, H^\prime) = (Sp(1)^3, Sp(1)^2, Sp(2))$.

We may write

$$sp(3) = \mathfrak{t} + m_o + m_1 + m_2 + m_3, \quad (3.1)$$

where $m_o$ is a trivial $ad(\mathfrak{t})$-module and $m_1$ are irreducible, mutually inequivalent $ad(\mathfrak{t})$-modules. We now fix a non zero vector $v \in m_1$ and consider a normal geodesic $\gamma : \mathbb{R} \to M$ w.r.t. a positively curved $G$-invariant metric $g$ on $M$; we choose $\gamma$ so that it induces the triple $\theta$.

First of all, we claim that the Killing vector field $\mathfrak{X}$ induced by $v$ on $M$ never vanishes along $\gamma$. In order to prove this, it is enough to check that $v$ does not belong.
to $Ad(W)\frako{h} \cap Ad(W)\frako{h'}$, where $W$ is the generalized Weyl group generated by the two geodesic symmetries $\sigma, \sigma'$. But we have that

$$Ad(\sigma)m_i = m_i, \ i = 1, 2 \text{ and } Ad(\sigma')m_1 = m_2, Ad(\sigma')m_2 = m_1.$$  

We now consider the smooth function $f(t) = \|X\|_{\gamma(t)}$ for $t \in \mathbb{R}$ and we claim that $f$ is a concave positive function, which is not possible. This will conclude our proof.

It will be enough to check that $f''(t) < 0$ for all $t$ such that $\gamma(t)$ is a regular point. First, we observe that (3.1) gives a decomposition of the tangent space to a regular orbit into $K$-irreducible, inequivalent submodules, so that the shape operator of the regular orbit hypersurface will preserve each submodule and will be a multiple of the identity operator on $m_1$. Therefore, if we denote by $D$ the Levi-Civita connection of $g$, we have that $D_{\gamma(t)}X$ is a multiple of $X_{\gamma(t)}$; we then have

$$2R_{X'\gamma'X\gamma} = 2\|D_{\gamma'}X\|^2 - \frac{d^2}{dt^2}f^2 = 2\frac{g(D_{\gamma'}X, X)^2}{f^2} - 2(f')^2 - 2ff'' = -2ff'' > 0,$$

since $g(D_{\gamma'}X, X) = f'$. □

**Lemma 3.6.** Case (3) for $n = 3$ cannot occur.

**Proof.** In this case we have $G = T^1 \times Spin(7)$ and $H = T^1 \times Spin(6)$, with $\nu(H) = U(4)$. Using Fact 1 and 2, it is easy to see that the only possibility for $\frako{h'}$ is $\frako{h'} \cong \mathbb{R} + su(4)$; moreover there is only one subalgebra of $su(7)$ which is isomorphic to $su(4)$ and which contains $\frako{f} \cong su(3)$. So, we have two totally geodesic singular orbits $B, B' \cong S^6$ in codimension 8 with $H = H'$. We now consider the normal space $V$ to the singular orbit $B$ at a point $p \in B$ and we claim that there exists an element $h \in H$ such that $\nu(h)$ is the identity, while $\tau(h) = Id$, where $\tau$ denotes the tangent isotropy representation.

Indeed, $B$ is the symmetric space $Spin(7)/Spin(6)$ and the symmetry $\sigma$ belongs to the center of $Spin(6)$, hence $\nu(\sigma)$ acts a scalar multiple of the identity on $V$; then there is an element $t \in T^1$ such that $\nu(t, \sigma) = Id$ and $\tau(t, \sigma) = -Id$ and we are done. This means that there exists a totally geodesic submanifold $F$ having $V$ as tangent space. The submanifold $F$ is of cohomogeneity one under the action of $H$ with two fixed points $p, p'$ as singular orbits. If now $\gamma$ denotes a normal geodesic issuing from $p$, then the parallel transport along $\gamma$ of the tangent space $T_pF = V$ will be tangent to $F$; since $T_{p'}F = V'$, where $V'$ is the normal space to $B'$ at $p'$, the parallel transport along $\gamma$ will send the tangent space $T_pB$ onto the tangent space $T_{p'}B'$. This is enough to apply Frankel’s argument ([Fr]) and get a contradiction. □

The last case is considered in the next

**Lemma 3.7.** Case (5) cannot occur.

**Proof.** Here we have $G = T^1 \times Sp(n)$ and $H = T^2 \times Sp(n - 1)$. We will indicate by $B$ the singular orbit $G/H$ and $B'$ the second singular orbit; moreover we will
indicate by $L$ the distance between the two singular orbits, so that the normal geodesic $\gamma(t)$ will intersect $B$ for $t \in 2\mathbb{Z}L$. Using Fact 1 and 2, it is easy to see that the only possibility for $h'$ is $h' \cong \mathbb{R} + sp(n - 2) + su(2)$, where the normal space $V'$ to the second singular orbit is of dimension 3. We decompose the Lie algebra $g$ as sum of $\mathfrak{k}$-modules, we have exactly four irreducible $\mathfrak{k}$-modules $\mathfrak{n}_i$, $i = 1, \ldots, 4$ of real dimension 2 and pairwise not equivalent. If we choose $X_i \in \mathfrak{n}_i$ for $i = 1, \ldots, 4$, then the functions $f_i(t) = ||X_i||_{\gamma(t)}$ must be smooth concave where they do not vanish, according to the same argument as in Lemma 3.5. Moreover three of them do not vanish for $t \in 2\mathbb{Z}L$. Since the geodesic symmetry $\sigma$ at $B$ preserves all these modules, the functions $f_i(t)$ which do not vanish for $t = 2\mathbb{Z}L$, must be even at such values of $t$. Using the fact that $\dim V' = 3$, it is now easy to see that at least one $f_i$ never vanishes, giving a contradiction. □

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