Investigation of Quantum Chaos in the Parametric Dependent System of Interacting oscillators

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Abstract

Formation of chaos in the parametric dependent system of interacting oscillators for the both classical and quantum cases has been investigated. Domain in which classical motion is chaotic is defined. It has been shown that for certain values of the parameters from this domain, form of the classical power spectrum is in a good agreement with the quantum band profile. Local density of states is calculated. The range in which application of perturbation theory is correct has been defined.

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1 Introduction. Problem statement.

At last time the investigation of parametric dependent Hamiltonians like $H(Q,P,\lambda)$ is a subject of great interest. Here $(Q,P)$ is a set of canonical coordinates and impulses, and $\lambda$ is a parameter describing connection of the system with the external field. Interest to such systems is motivated by mesoscopic physics [1]. Most of the works devoted to the parametric dependent systems are dealing with the investigation of following situation: For $\lambda = 0$ Hamiltonian is exactly integrable. For $\lambda > 0$ Hamiltonian $H(Q,P,\lambda)$ becomes non-integrable and for some defined values of $\lambda = \lambda_0$, solutions of classical equations generated by the Hamiltonian $H(Q,P,\lambda_0)$ display chaos. After this, the little variation of the parameter $\delta \lambda = \lambda - \lambda_0$ is of interest. Using the basis $|\Phi_n\rangle$ in which $\hat{H}(Q,P,\lambda_0)$ is diagonal, for small values of $\delta \lambda$, Hamiltonian $\hat{H}(Q,P,\lambda)$ can be written in the form of sum of two matrix

$$\hat{H} = E_0 + \delta \lambda B.$$ 

Here $E_0$ is the diagonal matrix, and $B$ is the banded matrix, off-diagonal elements of which are random numbers [2]. By way of numerical diagonalization it is possible to define eigenvectors $|\Phi_n(\lambda_0 + \delta \lambda)\rangle$, $|\Phi_n(\lambda_0)\rangle$ and eigenvalues $E_n(\lambda_0 + \delta \lambda)$, $E(\lambda_0)$ corresponding to the Hamiltonians $\hat{H}(Q,P,\lambda)$, $\hat{H}_0(Q,P,\lambda_0)$ appropriately. The main interest at this is connected with the dependence of parametric kernel

$$P(n|m) = |\langle \Phi_n(\lambda_0 + \delta \lambda)|\Phi_m(\lambda_0) \rangle|^2$$

from the parameter $\delta \lambda$. The value of $P(r) = \overline{P(n|n+r)}$ averaged over the $n$ can be considered as the local density of states. Descriptions of foregoing methods of solving of problems of quantum chaos are given in [3-5]. Goal of this paper is investigation of chaos in the system of interacting oscillators when both of the Hamiltonians $\hat{H}_0(Q,P,\lambda_0)$, $\hat{H}(Q,P,\lambda)$ display chaos. Situation when the Hamiltonian $\hat{H}_0$ is integrable has been studied by us earlier (see papers [6,7]).
2 Classical consideration. Model Hamiltonian.

Investigation of the following Hamiltonian is of interest for us

\[ H = H_0 + \lambda V, \]

\[ H_0 = \frac{p_1^2}{2} + \frac{p_2^2}{2} + \frac{p_3^2}{2} + \frac{1}{2}(q_1^2 + q_2^2 + q_3^2), \]

(1)

\[ V = q_1 q_2 q_3. \]

Here and below we work with the dimensionless quantities. Canonical equations for the Hamiltonian (1) are of form

\[ \dot{q}_1 = p_1, \]

\[ \dot{p}_1 = -q_1 - 2\lambda q_1 q_2 q_3, \]

\[ \dot{q}_2 = p_2, \]

\[ \dot{p}_2 = -q_2 - 2\lambda q_2 q_1 q_3, \]

\[ \dot{q}_3 = p_3, \]

\[ \dot{p}_3 = -q_3 - 2\lambda q_3 q_1 q_2. \]

Solutions for the set of equations in case of \( \lambda \neq 0 \) can be obtained only numerically. As is shown by numerical analyze, solutions of the set of equations (2) for the values of the parameters \( \lambda = 3.0, H = 4.2 \) are essentially chaotic (see Fig.1). For further calculations we have to obtain classical power spectrum with the purpose of its further comparison with the band profile. Let us construct correlation function for \( V(q_1 q_2 q_3) = -\frac{\partial H}{\partial \lambda} \)

\[ G(\tau) = \langle (V(t + \tau) - \bar{V}) (V(t) - \bar{V}) \rangle \]

(3)

where \( \langle \ldots \rangle \) means averaging over the time, \( \bar{V} \) is the averaged values of \( V \). We have to make Fourier transformation of the expression (3) to get classical power spectrum

\[ C(\omega) = \int e^{-i\omega t} G(\tau) d\tau. \]

(4)

In the practical calculation of (4) we use the method equivalent of (4)

\[ C(f) = \frac{1}{T} \left| \int_{-T/2}^{T/2} e^{-i\omega t} V(t) dt \right|, \]

(5)

here \( x \in [0, T] \) is time interval in which solution of the set of equations (2) is obtained, \( f = \frac{\omega}{2\pi} \). To get the power spectrum one needs to use discrete form of expression (5)

\[ C(k) = \frac{T\Delta}{2l + 1} \sum_{n=-l}^{l} |X(k + n)|^2, \]

(6)

here \( X(k) = \sum_{j=0}^{N-1} V(j) e^{i\frac{2\pi jk}{N}} \), \( k = fN\Delta, N \) is a number of dots dividing the interval \( T, \Delta = \frac{T}{N}, l \) is the averaging parameter [8,9]. Form of the classical power spectrum for the different values of the parameter \( l \) is given on the Figures Fig. 2, Fig.3. These data are confirming the fact that solutions of (2) are chaotic.
3 Quantum mechanical consideration

Our subsequent goal is the comparison of the band profile \( |B_{nm}|^2 \) considered as the function of \( \omega = \frac{E_n - E_m}{\hbar} \) with the classical band profile. Let us set about to the quantum-mechanical analysis of the Hamiltonian (1). Eigenvectors \( |\Psi_n\rangle \) of the basis, in which Hamiltonian \( H(P,Q,0) \) is diagonal can be found easily and are of the form [10]

\[
|\Psi_{n_1n_2n_3}(q_1q_2q_3)\rangle = \frac{H_{n_1}(q_1)H_{n_2}(q_2)H_{n_3}(q_3)}{\sqrt{2^{n_1+n_2+n_3}n_1!n_2!n_3!\pi^3}} e^{-\frac{1}{2}(q_1^2+q_2^2+q_3^2)},
\]

where \( H_n(q) \) denotes the Hermite polynomials [11]. Eigenvalue energy is given by the formula

\[
E_n = (n + \frac{3}{2}), \tag{8}
\]

where

\[
n = n_1 + n_2 + n_3. \tag{9}
\]

Condition (9) implies existence of the \((n+1)(n+2)\) fold degeneration for the given \( n \).

Diagonalization of the Hamiltonian (1) can be done only numerically. On the Figures 4, and 5 dependence of the energy eigenvalues \( E_n(\lambda) \) from the parameter \( \lambda \) are plotted. According to Fig.4 for the \( \lambda \neq 0 \) removal of degeneration takes place. In case of further increasing of the parameter \( \lambda \) up to the values characterizing the domain of chaotic motion \( \lambda = 3.0 \), as easy to see from Fig.5, repulsion of the energy terms happens. This fact is of great importance since the repulsion of the energy terms is a sine of emerging of quantum chaos [12]. For the defining of the band profile at first we need to find eigenvectors of the Hamiltonian \( H(P,Q,\lambda_0) \). This problem is possible to resolve by way of numerical diagonalization of the Hamiltonian \( H(P,Q,\lambda_0) \), at the values of \( \lambda_0 = 3.0 \) in the basis of functions (7).

\[
\langle \Psi_{n_1n_2n_3} | (H_0 + \lambda_0 V) | \Psi_{m_1m_2m_3}\rangle \tag{10}
\]

Since the matrix elements are satisfying the condition

\[
\langle n | V(q_1q_2q_3) | m \rangle = \langle n_1n_2n_3 | V(q_1q_2q_3) | m_1m_2m_3 \rangle = \langle n_1 | q_1^2 | m_1 \rangle \langle n_2 | q_2^2 | m_2 \rangle \langle n_3 | q_3^2 | m_3 \rangle,
\]

taking into account that

\[
q_{nm} = \frac{1}{\sqrt{2}} (\sqrt{n} \delta_{n-1,m} + \sqrt{n+1} \delta_{n+1,m}), \tag{11}
\]

and using the condition \((q^2)_{nm} = \sum_k q_{nk}q_{km}\) we get

\[
(q^2)_{nm} = \frac{1}{2} (\sqrt{n} \sqrt{n+1} \delta_{n-2,m} + (2n+1) \delta_{n,m} + \sqrt{n+1} \sqrt{n+2} \delta_{n+2,m}). \tag{12}
\]

In (11) and (12) \( \delta_{n,m} \) stands for Kronecker delta function. Substituting (11) and (12) into (10) after straightforward calculations one can obtain

\[
\langle n_1n_2n_3 | V | m_1m_2m_3 \rangle = \frac{1}{8} (\sqrt{n_1} \sqrt{n_1+1} \delta_{n_1-2,m_1} + (2n_1+1) \delta_{n_1,m_1} + \sqrt{n_1+1} \sqrt{n_1+2} \delta_{n_1+2,m_1}) \times \times (\sqrt{n_2} \sqrt{n_2+1} \delta_{n_2-2,m_2} + (2n_2+1) \delta_{n_2,m_2} + \sqrt{n_2+1} \sqrt{n_2+2} \delta_{n_2+2,m_2}) \times \times (\sqrt{n_3} \sqrt{n_3+1} \delta_{n_3-2,m_3} + (2n_3+1) \delta_{n_3,m_3} + \sqrt{n_3+1} \sqrt{n_3+2} \delta_{n_3+2,m_3}),
\]

\[
\langle n_1n_2n_3 | H_0 | m_1m_2m_3 \rangle = (n_1 + n_2 + n_3 + \frac{3}{2}) \delta_{n_1,m_1} \delta_{n_2,m_2} \delta_{n_3,m_3}. \tag{13}
\]

After forming from the matrix elements (13) matrix and making diagonalization one can obtain eigenvalues spectrum \( E_n(\lambda_0) \) and basis of eigenvectors \( \Phi_n(\lambda_0) \). Calculating matrix elements of \( V(q_1,q_2,q_3) \) in the basis \( \Phi_n(\lambda_0) \) we can get the band profile

\[
B_{nm}^2 = \frac{4\pi^2\hbar^2}{\lambda_0} |\langle \Phi_n(\lambda_0) | V(q_1,q_2,q_3) | \Phi_m(\lambda_0)\rangle|^2 \tag{14}
\]
Here \( \Delta_0 = a h^d \) is the mean level spacing, and \( d \) is the system’s dimension. Coefficient \( a \) is determined from the condition \( a = \frac{1}{D(E)} \), where \( D(E) = \frac{\delta N}{\delta E} \) is the density of states, and \( \delta N \) is the number of states in the energy interval \( \delta E = E(P, Q, \lambda_0 + \delta \lambda) - E(P, Q, \lambda_0) \). Density of states in it’s turn is possible to determine from the numerical diagonalization of the Hamiltonian \( H(P, Q, X_0) \), or from the integral

\[
D(E) = \frac{1}{(2\pi \hbar)^a} \frac{\partial}{\partial E} \int_{H(P, Q, \lambda) = E} dPdQ.
\]

Here the integral must be taken over the phase space enclosed by energy surface (see Fig.6) In our calculations we are interested in the domain in which classical motion is essentially chaotic: \( E(P, Q, \lambda_0) = 4.2; \delta \lambda = 0.1; \delta E = 0.1; \hbar = 0.1; a \approx 7.2 \). Taking smaller values of \( \hbar \) leads to the necessity of diagonalization of matrix larger then our. In our case \( \hbar = 0.1 \) matrix dimension is \( 455 \times 455 \). Band profile \( B_{nm}^2 \) as a function of \( \omega = \frac{E_n - E_m}{\hbar} \) is plotted on the Fig.7. According to Fig.7 band profile is in a good qualitative agreement with the classical power spectrum. Better quantitative agreement is possible to achieve for smaller values of \( \hbar \) \[1\] (since the quasi-classical approximation implies \( \hbar \to 0 \)). Another quantity we are interested in is the parametric kernel

\[
P(r) = \langle |\Phi_{n+r}(\lambda_0 + \delta \lambda)|^{2} \Phi_{n}(\lambda_0) \rangle^2,
\]  

(16)

where the averaging is done over the several \( n \) states. Evidently \[16\] determines correlation between the wave functions. The parametric kernel \[16\] frequently is named as the Local Density of States (LDS) \[3\]. We are interested in \( P(r) \), since with its help, is possible to determine the boundaries in which perturbation theory is valid. For that one has to compare parametric kernel \[16\] with the results for LDS obtained by means of perturbation theory \[5\]

\[
P_{prt}(r) = \frac{\delta x^2 |B_{nm}|^2}{\Gamma^2(\delta x) + (E_n - E_m)^2},
\]

(17)

where the parameter \( \Gamma(\delta x) \) for given \( \delta x \) must be defined from the normalization condition

\[
\sum_r P_{prt}(r) = 1.
\]

(18)

Results of numerical calculations for the values \( \delta x = 0.1 \) and \( \delta x = 0.2 \) are presented on the Figures Fig.8 and Fig.9. On basis of obtained results we conclude that boundary in which perturbation theory for our system is correct is limited by the condition \( \delta x \leq 1 \).
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Fig. 1 Phase trajectory projection on the plane \((p_1, q_1)\) obtained for the values of the parameters: 
\[ E = 4.2, \lambda = 3.0 \]. It is easy to see that trajectory is chaotic.

Fig. 2 Classical Power Spectrum plotted for the values of the parameters: 
\[ E = 4.2, \lambda = 3.0, l = 10 \]

Fig. 3 Classical Power Spectrum plotted for the values of the parameters: 
\[ E = 4.2, \lambda = 3.0, l = 3 \]
Fig. 4 Energy terms as a function of the parameter $\lambda$.

Fig. 5 Energy terms as a function of the parameter $\lambda$. For the values from the chaotic domain $\lambda \geq 3.0$, repulsion of the energy levels happens.
Fig. 6 Equipotential surface constraint by the condition \( H(P, Q, \lambda) = E = 4, 2 \)

Fig. 7 Quantum band profile \( B_{nm}^2 \) as a function of \( \omega = \frac{E_n - E_m}{\hbar} \) for the values of parameters \( E = 4.2, \lambda = 3.0, \delta \lambda = 0, 1 \). Evidently \( B_{nm}^2 \) is in a good qualitative agreement with the Classical Power Spectrum (see Fig. 2)
Fig. 8 Quantum Profile $P(r)$ (Local Density of States) and First-Order Perturbative Profile $P_{\text{prt}}(r)$ for the values of the parameter $\lambda = 0, 1$.

Fig. 9 Quantum Profile $P(r)$ (Local Density of States) and First-Order Perturbative Profile $P_{\text{prt}}(r)$ for the values of $\lambda = 0, 2$. Comparing Fig. 8 with Fig. 9 one could say that agreement between $P(r)$ and is better if $\lambda < 0, 1$. 