Analysis and Design of Reinforced Concrete Thin Cylindrical Shell

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Abstract. The need to come up with economical and efficient structural design led the engineers and researchers to focus more on shell structures. It is more durable, economical as it requires a minimum amount of material provides larger interior space and is aesthetic. A shell dominantly behaves as a membrane, though, at the edges, bending stresses get accumulated. Albeit several theories have been put forward, Schorer’s theory is eminent in the analysis of the long span thin cylindrical shells. This study is focused on the analysis of the stresses by utilizing the Fourier series and Schorer’s theory. Further, the shell is designed for the steel reinforcement as per the Concrete Reinforcing Steel Institute (CRSI) Design Handbook after calculating the final stress resultants and the detailing is also depicted.

Keywords: Long Cylindrical Shell; Membrane theory; Bending theory; Schorer’s theory in Matrix Notation, Fourier Series; Shell Reinforcement.

1. INTRODUCTION

Concrete Shells have been in use since ages due to their extraordinary characteristics such as lightweight, load carrying capacity, membrane action, and aesthetics [1]. A few of the common examples are roof-tops, domes, water tanks, cylindrical walls, etc. [2]. Further, thin cylindrical shells are of optimum use due to their low thickness to radius ratio, which efficaciously reduces the weight of the structure. Due to its geometry, thin cylindrical shells require very less material which reduces the amount of dead load. Long thin cylindrical Shells are preferred as they can span over large areas with supports only at the edges, thus making it aesthetically attractive [3].

The analysis of the stresses developed in a shell structure can be constructively inferred by broadly two methods: Analytical solution or Finite Element Analysis (FEA) of the shell structure. Though FEA is an advance and quicker method to analyze the shell structure, it is indispensable to comprehend the fundamental theories of the analysis of Shell structures [4]. The scope of the study is not limited to the analytical method only; instead, the results can later be juxtaposed with the FEA along with experimental analysis. However, this paper deals explicitly with the fundamental theories and analytical solution of the stresses developed in a Shell structure and then design as per stresses calculated.
Advantages of thin-shell structure: Shell structures have the edge over other structures because of the following points: Firstly, even if a shell structure fails under extreme conditions; the damage would be less detrimental compared to the failure of the framed structure owing to the lightweight nature of the shell structure. Secondly, As Shell structures are light in weight and can cover long spans without the aid of columns, it is considerably economical when compared to the conventional portal frames which require installation of heavy foundation thus increasing the cost. Thirdly, the curved geometry of the shell enables it to take up more load and can be spanned over a large length without the support of the interior columns leading to a large interior space. Last but not the least, as concrete can be cast or shaped in any geometry, thus RCC shell structure has a wide range of applications [5-7].

MEMBRANE THEORY: It is based on the exclusion of the bending stresses, thus making the structure statically determinate. Here, membrane force along transverse direction \( N_\varphi \)is transferred to the edge beam, and the bending moment along the same direction \( M_\varphi \)is transferred to the end frame. As bending effects usually get damped quickly away from the edge, a large portion of the shell behaves as a membrane. This is not always possible due to unfavorable loading and boundary conditions. Membrane theory can be applied to satisfying the following requirements: a) Uniform distribution of the loads over the shell’s surface. b) Bending stresses should not be developed and c) Uniform dissemination of the stresses throughout the thickness.

FOURIER SERIES: As solving the equations of plates and shells would be arduous, loading can be expressed in a series of equivalent sinusoidal form. This form of loading is known as the Fourier series loading. Fourier series has been used in this study for the calculations of membrane stresses which are given by equation (1), (2) and (3):

\[
N_\varphi = -\frac{4}{\pi} g a \cos(\varphi_c - \varphi) \cos \frac{\pi x}{l} \quad \text{(1)}
\]

\[
N_{x\varphi} = \frac{8}{\pi^2} g l \sin(\varphi_c - \varphi) \sin \frac{\pi x}{l} \quad \text{(2)}
\]

\[
N_x = -\frac{8}{\pi^3} g l^2 a \cos(\varphi_c - \varphi) \cos \frac{\pi x}{l} \quad \text{(3)}
\]

Where, \( N_{x\varphi} \) is the shear force, \( N_\varphi \) is the membrane force along the transverse direction, and \( N_x \) is the membrane force along the longitudinal direction.

BENDING THEORY: The bending behavior occurs only at the locations where membrane stresses are not able to withstand the applied loads. Thus, bending stresses are confined to a small part of the shell, leaving the rest of the shell free from bending stresses. This is a distinctive nature of Shell structures. In the worst-case scenario, if a shell undergoes inextensible deformation or buckling, bending stiffness becomes paramount to ensure the safety of the shell structure. In the practical world, no shell behaves completely as a membrane and hence the analysis has to be performed for both bending and membrane cases. This study is based on the Analysis of the stresses according to the Schorer’s theory.

SCHORER’S THEORY: Schorer’s theory was published in 1936, and it is quite simpler compared to the previous theories. The characteristic equation has roots that are not dependent on the shell dimensions. This makes it less strenuous. Schorer’s theory applies only to the long shells \( (l/a \geq \pi) \). Schorer assumed that the bending moment along longitudinal direction \( M_x \), radial shear \( Q_x \), and the twisting moment \( M_{x\varphi} \) are insignificant in the analysis. Another assumption implicit in his theory is that the tangential strain \( (c_\varphi) \) and the shear strain \( (\gamma_{xy}) \) are both small in comparison with the longitudinal strain \( (c_x = \frac{du}{dx}) \).

2. LITERATURE REVIEW

Love first developed the theory of thin shells in 1888. His theory was based on the Kirchhoff hypothesis for thin plate structures propounded in the mid-1800s [8]. Later, Koiter (1959) verified
Love’s assumptions and derived an applicable shell theory. Ever-since, several theories were posited, and most of them are based on the idea that the Membrane and Bending effects resist the deformations of shells caused by the loading. Amongst all the bending theories, Finsterwalder was the first to propound a theory that untangled the analysis by neglecting $M_x, Q_x,$ and $M_{x\varphi}$. It is only applicable to long shells. Further, Franz Dischinger (1935), Aas-Jakobsen (1937) and Wilhelm Flugge (1960) proposed similar theories and arrived at analogous yet non-identical characteristic equations. Theories proposed by scientists such as H. Lundgren (1960) and Ivar Holand (1957) are discrete to a certain degree. Donnell-Karman-Jenkins theory is considered to apply to the short shells ($l/a \leq 1.60$) and Schorer’s theory is only applicable to long shells.

However, since the late 1990s, the focus of the research has inclined more towards the Finite Element Analysis. Molyneaux and Long-Yuan Li (1996) analyzed a shell structure using nonlinear FE numerical methods. They found that the oval shape of the shell is the leading factor of its stability. Skallerud et al. (2001) focused more on the nonlinear response of shells under large rotations. Besides, Chang et al. (2008) have done a similar type of analysis but varied the material which opened up yet another aspect of the design. Later, Subramani and Sugathan (2012) have worked on the Software ANSYS to carry out the FEA of thin-walled shells. Myriads of research have been going on in this field, including the FEA performed by Harish et al. (2015) on doubly curved thin shells. The focus of this research is to analyze and design a shell structure using the analytical method to strengthen the basic concepts which can be compared with the FEA and experimental results of the same shell. It is pertinent to say that this paper is limited to the analytical method only and then design accordingly as per the stress resultants.

3. METHODOLOGY

Schorer’s differential equation for the long cylindrical shells is
\[ \omega'' + \frac{3}{k} \omega = 0 \quad -(4) \]
where \( k = \frac{d^2}{-12a^2} \)
\( d = \) thickness
\( a = \) radius

The characteristic equation then reduces to:
\[ m^8 + \frac{3}{k} m = 0 \quad -(5) \]
where \( \lambda_n = \frac{n\pi a}{l} \)

Eight roots of the equation (i) are:
\[ m_1 = \alpha_1 + i\beta_1, \quad m_2 = \alpha_1 - i\beta_1, \quad m_3 = -m_1 \]
\[ m_4 = \alpha_2 + i\beta_2, \quad m_5 = -m_2 \]
\[ m_6 = \alpha_2 - i\beta_2 \]

Where \( \alpha_1, \alpha_2, \beta_1 \) and \( \beta_2 \) are the coefficients for roots calculation.

To organize the lengthy calculations involved in the design of a cylindrical shell systematically, the MATRIX approach is best suited. Let “H” represent any shell action, be it stress resultant or displacement, as stated:
\[ H = M[e^{-a\varphi}\{B_2(A_2\cos\beta_2\varphi + B_3\sin\beta_2\varphi) + B_4(B_5\cos\beta_3\varphi - A_3\sin\beta_3\varphi)\} + e^{-a\varphi}\{B_6(C_2\cos\beta_2\varphi + D_2\sin\beta_2\varphi) + B_7(D_3\cos\beta_3\varphi - C_3\sin\beta_3\varphi)\}] \quad -(6) \]
Where M is a multiplier (Refer Table 1).
B₁, B₂, B₃, and B₄ are the numerical coefficients (Refer Table 2).
Aₙ, Bₙ, Cₙ, and Dₙ are constants, obtained after matrix operation.

Table 1. Multipliers ‘M’ as per the Schorer’s theory

| Quantity (Forces) | Multiplier (M) |
|-------------------|----------------|
| Nₓ                 | \(- \frac{D p^2}{\alpha^2} \left( \frac{\rho^2}{\lambda_n \alpha^2} \right) \cos \frac{\lambda_n \alpha}{\alpha} \) |
| Nₓα                | \(- \frac{D p^2}{\alpha^2} \left( \frac{\rho^2}{\lambda_n \alpha} \right) \sin \frac{\lambda_n \alpha}{\alpha} \) |
| Qₓ                  | \(- \frac{D p^2}{\alpha^2} \left( \frac{1}{\alpha \sqrt{2}} \right) \cos \frac{\lambda_n \alpha}{\alpha} \) |
| Nᵧ                 | \(- \frac{D p^2}{\alpha^2} \left( \frac{\rho^2}{\alpha^2} \right) \cos \frac{\lambda_n \alpha}{\alpha} \) |
| Mᵧ                 | \(- \frac{D p^2}{\alpha^2} \left( \frac{1}{\sqrt{2}} \right) \cos \frac{\lambda_n \alpha}{\alpha} \) |

Table 2: Coefficients as per the Schorer’s theory

| Quantity (Forces) | B₁ | B₂ | B₃ | B₄ |
|-------------------|----|----|----|----|
| Nₓ                | -1 | -1 | +1 | -1 |
| Nₓα               | +β₁| +α₁| -α₁| -β₁|
| Qₓ                | \(-α₁ + β₁\) | \(α₁ + β₁\) | \(α₁ + β₁\) | \(β₁ - α₁\) |
| [Nᵧ]              | 0  | +1 | 0  | -1 |
| Mᵧ                | +1 | -1 | -1 | +1 |

EXAMPLE FORMULATION USED FOR ANALYSIS AND DESIGN OF A SINGLE LONG CYLINDER SHELL WITH EDGE BEAMS

Sectional properties (Refer to Figure 1):
Span length (l) = 25 m,
Radius (a) = 7.5 m,
Thickness (d) = 0.075 m,
Semi-central angle (ф₁) = 40°,
Edge beam depth (2α₁) = 1.5 m,
Edge beam width (2b₁) = 0.225 m.

Loads:
Dead load = 1.8 kN/m² of shell surface,
Live load = 0.6kN/m² of shell surface,
For simplicity, total load, g = 2400 kN/m² of shell surface.

Parameters:
As per Aas and Jacobson,
\[ \rho = \frac{12a^4a^6}{14d^2} = 4.1872 \]

\[ k = \frac{4a^2}{14\rho^2} = 0.0506 \]

Here, \(4 < \rho < 7\),
\(0.03 < k < 0.12\)
So, the shell is long.
Hence, Schorer’s method can be used.

**Figure 1.** Shell with edge beam.

Membrane forces (as per Fourier series) are shown in Table 3. Bending forces (as per Schorer’s theory) (Refer Table 4). The multipliers and the corresponding coefficients as per the Schorer’s theory along with the constants \(A_n\), \(B_n\), \(C_n\), and \(D_n\) are solved through MATLAB by using the boundary conditions and matrix approach of representation of Schorer’s theory and substituted in equation (i) to get the stress resultant. While calculating the bending forces, the effect of both the edges is to be considered with the appropriate sign. The coefficients of \(A_n\), \(B_n\), \(C_n\), and \(D_n\) are assigned in a matrix “A” and the constants of the equation in a matrix “B”. The variables are calculated by inputting the formula \(X = A^{-1} * B\) in the MATLAB as:

\[ f \gg X = \text{inv}(A) * B \quad ---(7) \]

Final values of the stress resultants are shown in Table 5:

**Table 3. Membrane forces (N_\theta, N_x, \theta, and N_x)**

| \(\theta\) | \(N_\theta\) (N/m) at mid-span | \(N_{x\theta}\) (N/m) at the transverse | \(N_x\) (N/m) at midspan |
|---|---|---|---|
| 0° | -17565 | 31293 | -39590 |
| 5° | -18783 | 27924 | -42335 |
| 10° | -19858 | 24342 | -44757 |
| 15° | -20782 | 20544 | -46839 |
| 20° | -21547 | 16651 | -48564 |
| 25° | -22149 | 12600 | -49920 |
| 30° | -22582 | 8454 | -50896 |
| 35° | -22843 | 4243 | -51484 |
| 40° | -22930 | 0 | -51681 |
Table 4: Bending forces (\(N_\varnothing\), \(N_{\varnothing\varnothing}\), \(N_X\), and \(M_\varnothing\))

| \(\varnothing\) | \(N_\varnothing\) (N/m) at mid-span | \(N_{\varnothing\varnothing}\) (N/m) at transverse | \(N_X\) (N/m) at mid-span | \(M_\varnothing\) (Nm/m) at mid-span |
|-------------|-----------------|-----------------|-----------------|-----------------|
| 0°          | 55608           | 2039284         | -14205973       | 0               |
| 5°          | -67632          | 1004882         | -10960489       | -18067          |
| 10°         | -116850         | 238996          | -7702262        | -32515          |
| 15°         | -113947         | -265378         | -4623258        | -40548          |
| 20°         | -79629          | -530005         | -1889581        | -42230          |
| 25°         | -32287          | -588981         | 365201          | -39429          |
| 30°         | 12902           | -485953         | 2041750         | -34773          |
| 35°         | 44651           | -271370         | 3073044         | -30785          |
| 40°         | 56015           | 0               | 3420918         | -29243          |

Table 5: Final forces (\(N_\varnothing\), \(N_{\varnothing\varnothing}\), \(N_X\), and \(M_\varnothing\))

| \(\varnothing\) | \(N_\varnothing\) (N/m) at mid-span | \(N_{\varnothing\varnothing}\) (N/m) at transverse | \(N_X\) (N/m) at mid-span | \(M_\varnothing\) (Nm/m) at mid-span |
|-------------|-----------------|-----------------|-----------------|-----------------|
| 0°          | 38043           | 2070577         | -14245563       | 0               |
| 5°          | -86415          | 1032806         | -11002824       | -18067          |
| 10°         | -136708         | 263338          | -7747019        | -32515          |
| 15°         | -134729         | -244804         | -4670097        | -40548          |
| 20°         | -101176         | -513354         | -1938145        | -42230          |
| 25°         | -55436          | -576381         | 315281          | -39429          |
| 30°         | -9680           | -477499         | 1990854         | -34773          |
| 35°         | 21808           | -267127         | 3021560         | -30785          |
| 40°         | 33085           | 0               | 3369237         | -29243          |

DESIGN OF SHELL REINFORCEMENT

Since the resultant longitudinal force on the shell is zero, so, nominal longitudinal reinforcement of 8 Ø @ 250 mm c/c may be provided.

The maximum positive bending moment is 42230 Nm/m.

So, 42230 = 0.87 × 500 × \(A_{st}\)

Thus, \(A_{st} = 100\) mm²/m length.

So, 8 Ø @ 250 mm c/c may be provided as transverse reinforcement.

Maximum \(N_{\varnothing\varnothing}\) is 2070577 N/m.

So, 2070577 = 0.87 × 500 × \(A_{st}\)

Thus, \(A_{st} = 4760\) mm²/m length.

So, 25 Ø @ 100 mm c/c may be provided as transverse reinforcement and 16 Ø @ 200 mm c/c at the mid-span as diagonal reinforcement.

The edge beam shall be designed as a deep tensile beam.
4. Results and Discussions

![Graph a](image1)

![Graph b](image2)

![Graph c](image3)
The various values of $N_\theta$, $N_{x\theta}$, $N_x$, and $M_\theta$ have been plotted against the corresponding values of $\theta$ as shown in Figure 2. $N_\theta$ has the maximum value at $10^\circ$ central angle. As anticipated, $N_{x\theta}$ and $N_x$ is maximum when the central angle is $0^\circ$ while $M_\theta$ is maximum at $20^\circ$. Note that $M_\theta$ vanishes at $0^\circ$ central angle. $N_{x\theta}$ tends to zero near the edge of a shell.

In accord with the calculated stresses, the reinforcement detailing is shown in the following diagrams (Figure 3).

5. Conclusion

The study proves that a shell structure exhibits both membrane and bending behaviour. The analysis of the stresses has been performed by utilizing Fourier series and Schorer's theory. Further, the variation of the stresses at different central angles has been established. Schorer's theory is proven to be applicable in the analysis of long thin cylindrical shells. This study proves the necessity to be thorough with the analytical method of analysis to efficaciously perform the Finite Element Analysis. The
vision of the study is not restricted to the analytical solution only. A similar analysis can be performed, and the graphs can be plotted by carrying out the FEA of the shell structure and compared effectively.

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