Regularity theory for tangent-point energies:
The non-degenerate sub-critical case

Simon Blatt∗ Philipp Reiter∗∗

May 2, 2014

Abstract

In this article we introduce and investigate a new two-parameter family of knot energies $T^{p,q}$ that contains the tangent-point energies. These energies are obtained by decoupling the exponents in the numerator and denominator of the integrand in the original definition of the tangent-point energies.

We will first characterize the curves of finite energy $T^{p,q}$ in the sub-critical range $p \in (q + 2, 2q + 1)$ and see that those are all injective and regular curves in the Sobolev-Slobodeckii space $W^{p-1/q,q}(R/Z, R^n)$. We derive a formula for the first variation that turns out to be a non-degenerate elliptic operator for the special case $q = 2$ — a fact that seems not to be the case for the original tangent-point energies. This observation allows us to prove that stationary points of $T^{p,2} + \lambda$ length, $p \in (4, 5), \lambda > 0$, are smooth — so especially all local minimizers are smooth.

Contents

1 Introduction 2
2 Energy space 8
3 First variation 17
4 Bootstrapping 20
A Product and chain rule 25
B Finite-energy paths are embedded 26

∗Mathematics Institute, Zeeman Building, University of Warwick, Coventry CV4 7AL, United Kingdom, S.Blatt@warwick.ac.uk
∗∗Fakultät für Mathematik der Universität Duisburg-Essen, Forsthausweg 2, 47057 Duisburg, Germany, philipp.reiter@uni-due.de
1 Introduction

Strzelecki and von der Mosel [55] introduced us to the crew of a space shuttle traveling with constant speed through the universe on an unknown closed loop $\Gamma$ of length $L$. With the aid of their instruments they are able to measure at time $t$ the ratio of the squared distances $|\Gamma(s) - \Gamma(t)|^2$ from any previous position $\Gamma(s)$, $s \in [0, t]$, to the distance of the current tangent line $\ell(t) = \Gamma(t) + \varepsilon \Gamma'(t)$ from that previous position $\Gamma(s)$, i.e.

$$2r_\Gamma(t, s) := \frac{|\Gamma(s) - \Gamma(t)|^2}{\text{dist}(\ell(t), \Gamma(s))}.$$ 

Interestingly, the astronauts can gain essential topological information and regularity properties from the integral mean of a suitable inverse power of all these data, more precisely from

$$E_q(\Gamma) := \int_{[0, L]} \frac{ds}{r_\Gamma(t, s)} d^q, \quad q \geq 2. \tag{1.1}$$

During a hazardous maneuver in the southern Andromeda Galaxy the space craft unfortunately crashed, so the astronauts have to purchase a new one. The manufacturer meanwhile changed the model which now measures the ratio

$$r_\Gamma^{(p,q)}(t, s) := \frac{|\Gamma(s) - \Gamma(t)|^p}{\text{dist}(\ell(t), \Gamma(s))^q} \quad \text{for predefinable variables } p, q \geq 1 \tag{1.2}$$

and praises his innovation for giving more flexibility by choosing the “power parameters” $p$ and $q$. He promises that the integral

$$\text{TP}^{(p,q)}(\Gamma) := \int_{[0, L]} \frac{ds}{r_\Gamma^{(p,q)}(t, s)} \tag{1.3}$$

yields far more information on the topology and regularity of the loop $\Gamma$ and claims to have obtained particularly good results for $q = 2$ and $p$ somewhere between 4 and 5. Is he right?

We will see that for certain parameters the energy $\text{TP}^{(p,q)}$ is a knot energy. The notion of knot energies goes back to Fukuhara [21] and O’Hara [36]. The general idea is to search for a “nicely shaped” representative in a given knot class having strands being widely apart and being preferably smooth. More precisely, a knot energy is a functional that is (i) bounded below and (ii) self-repulsive (or, synonymously, self-avoiding), i.e. it blows up on embedded curves converging to a curve with a self-intersection (with respect to a suitable topology) [39, Def. 1.1].

Knot energies are the central object of the so-called geometric knot theory which aims at investigating geometric properties of a given knotted curve in order to gain information on its knot type. They also form a subfield of geometric curvature energies which include geometric integrals measuring smoothness and bending for objects that a priori do not have to be smooth.

Knot energies can help to model repulsive forces of fibres. The original Gedankenexperiment by Fukuhara [21] was the deformation of a thin fibre charged with electrons lying in a viscous liquid. There is indication for DNA molecules seeking to attain a minimum state of a suitable energy [35]. Attraction phenomena may also be modeled by a corresponding positive gradient flow [2].
The first knot energy on smooth curves goes back to O’Hara [36] who in 1991 defined the functional that was called Möbius energy later on by Freedman, He, and Wang [20]. It corresponds to the element $E^{α,1}$ of the two-parameter family of functionals

$$E^{α,β}(γ) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{1}{|γ(u + w) - γ(u)|^α - \frac{1}{D_γ(u + w, u)^3}} \left| γ'(u + w) \right| \left| γ'(u) \right| \, dw \, du$$

(1.4)

which O’Hara [37, 38] introduced shortly after. Here $α, β > 0$, and $γ \in C^{α,β}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. The quantity $D_γ(u + w, u)$ measures the intrinsic distance between $γ(u + w)$ and $γ(u)$ on the curve $γ$. Of particular interest is the subfamily

$$E^{α}(γ) := E^{α,1}(γ)$$

(1.5)

There are numerous contributions concerning topology [37, 38, 20], regularity [1, 36, 20, 29, 10, 44, 43], and the corresponding gradient flow [29, 9, 5]. Numerical experiments have been carried out in [34], error estimates have been obtained in [40, 41].

Another famous example of a knot energy is the reciprocal of thickness which can be characterized by means of the global radius of curvature $\varrho[γ]$ defined by Gonzalez and Maddocks [26]. This leads to the concept of ideal knots, minimizers of the ropelength (the quotient of length and thickness) within a prescribed isotopy class. Existence is discussed in [27, 16, 25] while the question of regularity turns out to be rather involved [46, 47, 15]. In fact, an explicit analytical characterization of the shape of a (non-trivial) ideal knot has not been found yet, so the state of the art is discretization and numerical visualization, cf. [3, 17, 18, 19, 22, 28, 48]. Maximizing length for prescribed thickness on the two-dimensional sphere $S^2$ leads to an interesting packing problem, see Gerlach and von der Mosel [23, 24].

Substituting some of the minimizations in the definition of thickness as proposed in [26, Sect. 6], one derives three families of integral-based energies, namely

$$\mathcal{U}_p(γ) := \left( \int_{\mathbb{R}/\mathbb{Z}} \inf_{\mathbb{R}/\mathbb{Z}} \varrho[γ](s, t)^p \right)^{1/p},$$

$$\mathcal{I}_p(γ) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{ds \, dt}{\varrho[γ](s, t)^p},$$

$$\mathcal{A}_p(γ) := \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \int_{\mathbb{R}/\mathbb{Z}} \frac{ds \, dt \, d\varrho}{R(s, t, \varrho)^p},$$

where $R(s, t, \varrho)$ denotes the radius of the circle passing through the three points $γ(s)$, $γ(t)$, $γ(\varrho)$. These functionals have been thoroughly investigated by Strzelecki and von der Mosel [54], Strzelecki, Szumalański and von der Mosel [49, 50], and Hermes [30].

The energy spaces are discussed in [7]. Energies for higher-dimensional objects are considered in Strzelecki and von der Mosel [52, 53, 56], Kolasiński [31, 32], and Kolasiński, Strzelecki, and von der Mosel [33].

The tangent-point energies (1.1) are a variant of these “three-point circle” based functionals. One just uses the radius of the smallest circle tangent to one point and going through another point on the curve instead of the radius of the smallest circle going through three points on the curve. The resulting energies already appeared as $U_{α, γ}^C[C]$ in the article by Gonzalez and Maddocks [26, Sect. 6]. Sullivan [57] used these functionals to approach ropelength. In contrast to these classical energies, the integrand of the generalized energies introduced in this article (1.3) bare such an appealing geometric interpretation. But we will see that they have nicer analytic properties, basically due to the fact that their first variation leads to non-degenerate elliptic operator.
Before presenting the results of this article, let us briefly review the main known results on the tangent-point energies defined in (1.1). The most striking observation Strzelecki and von der Mosel made in their seminal paper [55], is that if $E_q(\Gamma)$, $q \geq 2$, is finite then the image of $\Gamma$ is a one-dimensional topological manifold [55, Thms. 1.1 and 1.4] of class $C^{1,1-2/q}$ if $q > 2$ [55, Thm. 1.3]. The proof of this and all the other main results in the paper is based on exploiting a decay estimate of Jones’ beta numbers. Still for $q > 2$, they gave an explicit upper bound on the Hausdorff distance of two given curve in terms of their tangent-point energies implying ambient isotopy [55, Thm. 1.2]. Moreover, they could prove that in this case $E_q$ is a knot energy [55, Prop. 5.1], which improves an earlier result by Sullivan [57, Prop. 2.2] that requires higher regularity. The energy even is a strong knot energy, i.e. for given bounds on energy and length there are only finitely many knot types having a representative that satisfies these bounds.

In fact one can strengthen the above-mentioned result of Strzelecki and von der Mosel and show that $E_q(\Gamma)$ is finite if and only if the image of $\Gamma$ is an embedded manifold of class $W^{2-1/q,q} \subset C^{1,1-2/q}$, see [6, Cor. 1.2]. Results for higher-dimensional analogs to $E_q$ can be found in [51, 6, 33].

As $\TP^{(p,q)}$ is (increasing in $p$ and) decreasing in $q$, the results by Strzelecki and von der Mosel [55] immediately carry over to the $\TP^{(p,q)}$-functionals (1.3) with $p \geq 2q$, $p \geq 4$ via
\[
E_{p/2} = 2^{p/2} \TP^{(p,p/2)}, \quad p \geq 4.
\]

In fact we will show in Appendix B, that even for the full sub-critical range
\[
p \in (q + 2, 2q + 1), \quad q > 1
\]
the arguments in [55] can easily be adapted leading to self-repulsiveness of the energies and Hölder regularity of the first derivative. As in [55] this can be used to show for example that these energies are strong and that minimizers exist in every knot class.

Since the arguments in [55] are quite involved and technical, we will present a completely independent and fast approach to these type of questions for curves that are a priori injective, continuously differentiable and parametrized by arc-length. This approach is based on techniques developed in [6].

The first result we get for the subcritical range of parameters (1.7) is the following characterization of curves of finite energy among all injective $C^1$-curves parametrized by arc-length.

**Theorem 1.1 (Energy spaces).** Assume (1.7) and let $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ be an injective curve parametrized by arc-length. Then $\TP^{(p,q)}(\gamma) < \infty$ if and only if $\gamma \in W^{(p-1)/q,q}$. Moreover, one then has, for constants $C, \beta > 0$ depending on $p, q$ only,
\[
\|\gamma\|_{W^{(p-1)/q,q}} \leq C \left( \TP^{(p,q)}(\gamma) + \TP^{(p,q)}(\gamma^\beta) \right).
\]

**Remark 1.2 (Initial regularity).** Note that our method of proof works entirely without using the techniques by von der Mosel and Strzelecki [55] — if one always assumes curves to be continuously differentiable as stated in the preceding Theorem 1.1.

However, the requirement of initial $C^1$-regularity can be omitted as the image of finite-energy curves is an embedded $C^{1,\alpha}$-manifold by Theorem B.1, which is easily derived from [55]. This shows that the energy of an arbitrary absolutely continuous curve is finite if and only if its image is an embedded manifold of class $W^{(p-1)/q,q}$.
We will then show how to combine Theorem 1.1 with a bi-Lipschitz estimate to obtain the existence of minimizers in every knot class:

**Theorem 1.3 (Existence of minimizers within knot classes).** Assume (1.7). Then, in any knot class there is a minimizer of $\text{TP}^{p,q}$ for (1.7) among all injective, regular curves $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$.

In order to study stationary points of the energy, we derive a formula for the first variation on the space of injective and regular curves of finite energy. To shorten notation we abbreviate

$$\Delta \bullet := \bullet(u + w) - \bullet(u).$$

(1.9)

Let

$$P_{\gamma'(u) \alpha} := \left[ a \cdot \frac{\gamma'(u)}{|\gamma'(u)|} \right] \frac{\gamma'(u)}{|\gamma'(u)|}, \quad P_{\gamma'(u) \perp \alpha} := a - P_{\gamma'(u) \alpha}$$

for $a \in \mathbb{R}^n$ (1.10)

be the projection onto the tangential and normal part along $\gamma$ respectively.
Theorem 1.4 (First variation). For \( p, q \) satisfying (1.7) let \( \gamma \in W^{p-1/4,4}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be injective and parametrized by arc-length. Then, for any \( h \in W^{p-1/4,4}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \), the first variation of \( TP^{p,q} \) at \( \gamma \) in direction \( h \) exists and amounts to

\[
\delta TP^{p,q}(\gamma, h) = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left\{ q \left( \frac{\langle \triangle \gamma - wy'(u), \triangle h - \langle \triangle \gamma, y'(u) \rangle h'(u) \rangle}{|\triangle \gamma|^p} \right) |\triangle \gamma|^q \right\} \, du
\]

Be aware that, in contrast to O’Hara’s knot energies, we do not need a principal value to express the first variation here. The same situation applies to the integral Menger curvature functionals, see Hermes [30].

In this article, we only calculate the first variation at arc-length parametrized curves in order to make the proof as simple as possible. However, adapting the techniques from [11], one can even derive continuous differentiability of \( TP^{p,q} \) on the set of all injective regular curves in \( W^{p-1/4,4}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \).

For the non-degenerate case \( q = 2 \) we then finally study the regularity of stationary points of finite energy, i.e. curves \( \gamma \in W^{p-1/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) where

\[
p \in (4,5), \quad q = 2.
\]

We will see that those are smooth — which in a sense is a justification for inventing these new knot energies in the first place.

Theorem 1.5 (Stationary points of \( TP^{p,2p} \) are smooth).

For \( p \in (4,5) \), let \( \gamma \in W^{p-1/2,2}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be a stationary point of \( TP^{p,2p} \) with respect to fixed length, injective and parametrized by arc-length. Then \( \gamma \in C^\infty \).

Surprisingly, the proof of this theorem up to some new technical difficulties roughly follows the lines of the proof of the analogous result for O’Hara’s energies \( E^{p,1} \) for \( \alpha \in (2,3) \) — yet another indication that the two families of energies \( TP^{p,q} \) and \( E^{p,p} \) are not too different from the perspective of an analyst.

In Proposition 4.1 we will see that, for \( q = 2 \), the highest term in the Euler-Lagrange equation is an elliptic operator of order \( p - 1 \). We will show that the remainder consist of terms having a common form (Lemma 4.2) and is of lower order. This allows is to apply a bootstrapping argument to show that critical points are smooth and thus prove Theorem 1.5.

Let us stress once more that we do not expect the latter result to carry over to other parameters in (1.7). This is due to the fact that the first variation should then be a degenerate elliptic operator.

Remark 1.6 (The critical case \( p = q + 2 \)). Although we generally restrict to (1.7), our results partially also apply to the critical case \( p = q + 2 \).
This holds true for the characterization of energy spaces in Theorem 1.1 except for Estimate (1.8) and the derivation of the first variation in Theorem 1.4 where we additionally have to claim \( \gamma, h \in C^1 \).

However, the proofs of both Theorem 1.3 and Theorem 1.5 fundamentally rely on \( p > q + 2 \). In the light of corresponding results for the Möbius energy \( E^{2,1} \) [20, 12] we expect these situation to be much more involved.

To make the article as accessible as possible, we present the two main tools used in the bootstrapping argument, namely chain and product rules for fractional Sobolev spaces, in Appendix A. Furthermore, a sketch on how to prove that finiteness of the energy implies embeddedness (Theorem B.1) can be found in Appendix B.

Let us bring the energies into the form we will work with from now on. Observing that

\[
\text{dist}(f(u), \gamma(u + w)) = \left| P_{\gamma(u)}^\perp (\gamma(u + w) - \gamma(u)) \right| = \sqrt{ |(\gamma(u + w) - \gamma(u))|^2 - |(\gamma(u + w) - \gamma(u), \gamma'(u))|^2 }
\]

and taking into account absolutely continuous curves (of arbitrary regular parametrization), the functional (1.3) may be rewritten as

\[
TP^{(p,q)}(\gamma) = \int_{\mathbb{R}^n/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{P_{\gamma(u)}^\perp |(\gamma(u + w) - \gamma(u))|}{|\gamma(u + w) - \gamma(u)|^p} |\gamma'(u)| \, dw \, du. \tag{1.13}
\]

It will be crucial for the estimates later on, that

\[
P_{\gamma(u)}^\perp (\gamma(u + w) - \gamma(u)) = P_{\gamma(u)}^\perp (\gamma(u + w) - \gamma(u) - w \gamma'(u)), \tag{1.14}
\]

so, for \( \gamma \in C^{1,1} \), the integrand in (1.13) behaves like \( O \left( |w|^{2q-p} \right) \) as \( w \to 0 \).

We will use Sobolev-Slobodeckii spaces in the form they already appeared in [8]. For the readers’ convenience we briefly recall their definition and some basic properties. Let \( f \in L^2(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \). For \( s \in (0,1) \) and \( q \in [1, \infty) \) we define the seminorm

\[
[f]_{W^{s,q}} := \left( \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|f(u + w) - f(u)|^q}{|w|^{s+q}} \, dw \, da \right)^{1/q}. \tag{1.15}
\]

Now let \( W^{k,\ell}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), k \in \mathbb{N} \cup \{0\} \), denote the usual Sobolev space (recall \( W^{0,q} := L^q \)) and

\[
W^{k+s,\ell}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) := \left\{ f \in W^{k,\ell}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \mid \|f\|_{W^{k+s,\ell}} < \infty \right\}
\]

be equipped with the norm

\[
\|f\|_{W^{k+s,\ell}} := \|f\|_{W^{k,\ell}} + \left[ f^{(2)} \right]_{W^{s,\ell}}.
\]

Without further notice we will frequently use the embedding

\[
W^{k+s,\ell}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \hookrightarrow C^{k-s-1/\ell}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \quad s \in (q^{-1}, 1). \tag{1.16}
\]

We will denote by \( C_\gamma \) resp. \( W_\gamma \) injective (embedded) curves parametrized by arc-length and by \( W_\gamma \) regular curves. As usual, a curve is said to be regular if there is some \( c > 0 \) such that \( |\gamma'| \geq c \) a.e. Constants may change from line to line.
Acknowledgements. The first author was supported by Swiss National Science Foundation Grant Nr. 200020_125127 and the Leverhulm trust. The second author was supported by DFG Transregional Collaborative Research Centre SFB TR 71. This project was initiated during the ESF Research Conference ‘Knots and Links: From Form to Function’, 2 – 8 July 2011, at the Mathematical Research Center ‘Ennio De Giorgi’, Pisa, Italy.

2 Energy space

The main aim of this section is to characterize in some sense the domain of the energies $\text{TP}^{(p,q)}$ in the range

$$p \in (q + 2q + 1), \quad q > 1$$

and prove the existence of minimizers using this result.

We will see that these are the only parameters for which the energies are both self-repulsive and well-defined in the sense that there exist closed curves of finite energy, but not scaling invariant.

Remark 2.1 (Not a knot energy if $p < q + 2$). Let us give an example that shows that we do not get a bi-Lipschitz estimate for injective curves if $p < q + 2$. Consider the curves $u \mapsto (u, 0, 0)$ and $u \mapsto (0, u, \delta)$ for $u \in [-1, 1]$, $\delta > 0$. The interaction of these strands leads to the $\text{TP}^{(p,q)}$-value

$$2 \int_{[-1,1]^2} \frac{(u^2 + \delta^2)^{q/2}}{(u^2 + v^2 + \delta^2)^{p/2}} \, du \, dv \leq 2 \int_0^{\sqrt{2}} \int_0^{2\pi} \frac{(r^2 \sin^2 \varphi + \delta^2)^{q/2}}{(r^2 + \delta^2)^{p/2}} \, r \, d\varphi \, dr$$

$$\leq 4\pi \int_0^{\sqrt{2}} \frac{(r^2 + \delta^2)^{q/p - 1}}{r^2} \, r \, dr.$$  (2.1)

The integral on the right-hand side is bounded for $\delta \searrow 0$ if $p < q + 2$. Using Proposition 2.4 below and the monotonicity of $\text{TP}^{(p,q)}$ for fixed $q$, it is easy to join the endpoints of the two strands via suitable arcs producing a family of ‘figure eight’-like embedded smooth curves that does not lead to an energy blow-up as $\delta \searrow 0$. Clearly this does not meet the requirements for a knot energy as mentioned in the introduction.

The biggest difference here to the approach taken in [55] is that we will only look at curves parametrized by arc-length which are a priori $C^1$ and injective. It is surprising, that we will still be able to prove by rather simple means that the subset of these curves of bounded length and energy is compact in $C^1$ up to translations. This will follow from our classification of curves of finite energy and a bi-Lipschitz estimate which we will again prove using this classification.

We will use this together with the lower semi-continuity of the energies $\text{TP}^{(p,q)}$ with respect to convergence in $C^1$, to show that these are strong knot energies that can be minimized within each knot class — without using one of the basic tools in [55], the decay of Jones’ beta numbers. But as in [55] scaling is what makes our arguments work. More precisely: that things get punished more by the energy, if they happen on a small scale.

Remark 2.2 (Problem with non-injective curves). To see that considering just injective curves might be an idea, let us repeat an observation that was already made in [55]
for the classical tangent point energies. As the value $TP^{p,q}(\gamma)$ only depends on the image of $\gamma$ and multiplicities, it is easy to construct a non-injective curve parametrized by arc-length of finite energy that is moreover not $C^1$: Take e. g. an open $C^2$-curve defined on $[0, \frac{1}{2}]$. By Proposition 2.4 it has finite energy. Traversing it once, changing the direction at the end-point, and then traversing it in the opposite direction, produces a non-injective continuous parametrization on $\mathbb{R}/\mathbb{Z}$ of a one-dimensional manifold with boundary whose energy amounts to four times the original energy. By the same reasoning, passing a curve $k$-times results in an energy increase by the factor $k^2$.

To give a sufficient condition for an injective curve in $C^1$ parametrized by arc length — which will also turn out to be necessary — we will use the following easy result from [8]:

**Proposition 2.3 (Bi-Lipschitz continuity [8, Lem. 2.1]).**

Let $\gamma \in W^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$, $p \geq q + 2$. Then there is a constant $C_\gamma$ such that

$$\frac{|u|}{C_\gamma} \leq |\gamma(u + w) - \gamma(u)| \leq |w| \text{ for all } u \in \mathbb{R}/\mathbb{Z}, w \in [-\frac{1}{2}, \frac{1}{2}]. \tag{2.2}$$

In Proposition 2.7 below we will provide a uniform bi-Lipschitz estimate for curves of bounded $TP^{p,q}$-energy.

Now we are in the position to prove that curves in $W^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ have finite energy:

**Proposition 2.4 (Sufficient regularity condition for $p \in [q + 2, 2q + 1]$).**

If $\gamma \in W^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ is parametrized by arc-length, $q \geq 1$ and $p \in [q + 2, 2q + 1]$ then $TP^{p,q}(\gamma) < \infty$.

**Proof.** By (2.2) we derive as in [6]

$$TP^{p,q}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|P_{\gamma}(a) (\gamma(u + w) - \gamma(u))\|^q}{\|w\|^{p-q}} \, dw \, du \leq C_\gamma \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|P_{\gamma}(a) (\gamma'(u + w) - \gamma'(u))\|^q}{\|w\|^{p-q}} \, dw \, du \leq C_\gamma \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|\gamma'(u + w) - \gamma'(u)\|^q}{\|w\|^{p-q}} \, dw \, du \, d\theta \leq C_\gamma \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|\gamma'(u + w) - \gamma'(u)\|^q}{\|w\|^{p-q}} \, dw \, du \, d\theta \leq C_\gamma \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|\gamma'(u + w) - \gamma'(u)\|^q}{\|w\|^{p-q}} \, dw \, du \leq C_\gamma \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|\gamma'(u + w) - \gamma'(u)\|^q}{\|w\|^{p-q}} \, dw \, du \leq C_\gamma \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|\gamma'(u + w) - \gamma'(u)\|^q}{\|w\|^{p-q}} \, dw \, du \leq C_\gamma \int_{0}^{1} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\|\gamma'(u + w) - \gamma'(u)\|^q}{\|w\|^{p-q}} \, dw \, du = C_\gamma \|\gamma\|_{W^{(p-1)/q-1,q}}^q . \quad \Box$$

9
To get a classification of all finite-energy curves in $C^1_{in}$ we need to show that the inverse implication is true as well:

**Proposition 2.5 (Necessary regularity for finite energy).** Let $\gamma \in C^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^d)$ be injective and parametrized by arc-length with $TP^{p,q}(\gamma) < \infty$ for (1.7). Then $\gamma \in W^{p-1,q-1,d}$ and

$$[\gamma']^{q}_{w(p-1)/q-1,d} \leq C \left( TP^{p,q}(\gamma) + TP^{p,q}(\gamma)^{\beta} \right)$$

where $C$ and $\beta > 0$ depend only on $p, q$. Moreover,

$$\|\gamma'\|_{C^{p-2,q-1}} \leq C \left( TP^{p,q}(\gamma) + TP^{p,q}(\gamma)^{\beta} \right).$$

**Proof.** The estimate (2.4) immediately follows from (2.3) and Morrey’s embedding theorem for fractional Sobolev spaces.

The proof of (2.3) uses the techniques from [6]. By continuity we may choose some $\delta > 0$ such that

$$|\gamma'(u + w) - \gamma'(u)| \leq \frac{1}{2} \sqrt{2} \text{ for all } u \in \mathbb{R}/\mathbb{Z}, w \in [-\delta, \delta].$$

In fact we choose the biggest such constant, i.e., we assume that there are $u_0 \in \mathbb{R}/\mathbb{Z}, u_0 \in [-\delta, \delta]$ such that

$$|\gamma'(u_0 + w) - \gamma'(u_0)| = \frac{1}{2} \sqrt{2}. \quad (2.6)$$

This leads to

$$\left| P^\perp_{\gamma(u+w)} (\gamma(u + w) - \gamma(u)) - P^\perp_{\gamma(u)} (\gamma(u + w) - \gamma(u)) \right|^2$$

$$= \left| (\gamma(u + w) - \gamma(u), \gamma'(u + w)) \gamma'(u + w) - (\gamma(u), \gamma'(u)) \gamma'(u) \right|^2$$

$$= \left| (\gamma(u + w) - \gamma(u), \gamma'(u + w)) \gamma'(u + w) - (\gamma(u), \gamma'(u)) \gamma'(u) \right|^2$$

$$- 2 (\gamma(u + w) - \gamma(u), \gamma'(u + w)) (\gamma(u + w) - \gamma(u), \gamma'(u + w)) (\gamma'(u), \gamma'(u + w))$$

$$= \left| (\gamma(u) - \gamma(u), \gamma'(u)) \gamma'(u) - (\gamma(u + w) - \gamma(u), \gamma'(u + w)) \gamma'(u) \right|^2$$

$$+ (\gamma(u + w) - \gamma(u), \gamma'(u)) \gamma'(u) \gamma'(u + w) \right| \gamma'(u) - \gamma'(u + w) \right|^2$$

$$\geq \left| \gamma'(u) - \gamma'(u + w) \right|^2 \right|^2 a^2 \int_0^{1} \left| \gamma'(u + \theta_1 w), \gamma'(u) \right|^2 d\theta_1 \int_0^{1} \left| \gamma'(u + \theta_2 w), \gamma'(u) \right|^2 d\theta_2$$

$$\geq \frac{\sqrt{2}}{10} \left| \gamma'(u + w) - \gamma'(u) \right|^2.$$

This allows to estimate using $|\gamma(u + w) - \gamma(u)| \leq |w|$

$$TP^{p,q}(\gamma) = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left| P^\perp_{\gamma(u+w)} (\gamma(u + w) - \gamma(u)) \right|^q \text{d}w \text{d}u$$

$$= \frac{1}{2} \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left| P^\perp_{\gamma(u+w)} (\gamma(u + w) - \gamma(u)) \right|^q \text{d}w \text{d}u$$

$$\geq c_{p,q} \int_{\mathbb{R}/\mathbb{Z}} \int_{-\delta}^{\delta} \left| w^q \left| \gamma'(u + w) - \gamma'(u) \right|^q \text{d}w \text{d}u$$

$$/\sqrt{p} \sqrt{q} \sqrt{w^q}.$$
where

\[ \frac{1}{2r} \int_{B_r(x)} |y'(v) - y'_{B_r(x)}| \, dv \leq \frac{1}{4r^2} \int_{B_r(x)} \int_{B_r(x)} |y'(v) - y'(u)| \, du \, dv \]

which gives

\[ |y'(u) - y'(v)| \leq C \left( \text{TP}^{p,q}(y) + \left\| y' \right\|_{L^p} \int_0^{1/2} \frac{du}{u^{p-q}} \right) \]

Unfortunately, this last estimate gets worse as \( \delta \) gets small. We will derive a Morrey estimate for fractional Sobolev space to estimate \( \delta \) from below. More precisely, we will show

\[ \left\| y'(\cdot + w) - y'(\cdot) \right\|_{L^p} \leq C \text{TP}^{p,q}(y)^{1/q} |w|^\alpha \quad \text{for all } w \in [-\frac{1}{2}, \frac{1}{2}] \]

where \( \alpha = (p - 2)/q - 1 > 0 \). From (2.7) we infer

\[ \frac{1}{2} \sqrt[2]{\frac{1}{2}} \leq C \text{TP}^{p,q}(y)^{1/q} \delta^\alpha \]

which concludes the proof.

To complete the argument, we sketch the proof of the Morrey estimate stated above.

Let \( y'_{B_r(x)} \) denote the integral mean of \( y' \) over \( B_r(x) \). We calculate for \( x \in \mathbb{R}/\mathbb{Z} \) and \( r \in (0, \delta) \)

\[ \frac{1}{2r} \int_{B_r(x)} |y'(v) - y'_{B_r(x)}| \, dv \leq \frac{1}{4r^2} \int_{B_r(x)} \int_{B_r(x)} |y'(v) - y'(u)| \, du \, dv \]

which gives

\[ |y'(u) - y'(v)| \leq C \left( \text{TP}^{p,q}(y) + \left\| y' \right\|_{L^p} \int_0^{1/2} \frac{du}{u^{p-q}} \right) \]

The estimate (2.7) now follows from this by standard arguments due to Campanato [14].

We choose two Lebesgue points \( u, v \in \mathbb{R}/\mathbb{Z} \) of \( y' \) with \( r := |u - v| \in (0, \frac{1}{2}) \). Then

\[ |y'(u) - y'(v)| \leq \sum_{k=0}^{\infty} |y'_{B_2^{2^k}(u)} - y'_{B_2^{2^k}(v)}| + |y'_{B_2^{2^k}(u)} - y'_{B_2^{2^k}(v)}| + \sum_{k=0}^{\infty} |y'_{B_2^{2^k+1}(u)} - y'_{B_2^{2^k+1}(v)}| \]

Since

\[ |y'_{B_2^{2^k}(u)} - y'_{B_2^{2^k}(v)}| \leq \frac{\int_{B_2^{2^k}(y') - y'_{B_2^{2^k}(v)}} \int_{B_2^{2^k}(y') - y'_{B_2^{2^k}(v)}} \, dx}{|B_2^{2^k}(u) \cap B_2^{2^k}(v)|} \leq C |u - v|^{p-1} \text{TP}^{p,q}(y)^{1/q} \]

as \( r = |u - v| \) and for all \( y \in \mathbb{R}/\mathbb{Z}, R \in (0, \frac{1}{2}) \)

\[ |y'_{B_2^{2^k}(u)} - y'_{B_2^{2^k}(v)}| \leq \frac{\int_{B_2^{2^k}(y') - y'_{B_2^{2^k}(v)}} \int_{B_2^{2^k}(y') - y'_{B_2^{2^k}(v)}} \, dx}{R} \]

11
\[
\leq C R^\alpha TP^{(p,q)}(\gamma)^{1/q},
\]
we deduce that
\[
|\dot{\gamma}'(u) - \dot{\gamma}'(v)| \leq C \left( \sum_{k=0}^{\infty} 2^{-ka} + \sum_{k=0}^{\infty} 2^{-ka} \right) |u - v| TP^{(p,q)}(\gamma)^{1/q}.
\]
Thus
\[
|\dot{\gamma}'(u) - \dot{\gamma}'(v)| \leq C |u - v| TP^{(p,q)}(\gamma)^{1/q}
\]
for all Lebesgue points of \(\dot{\gamma}'\) with \(|u - v| \leq \frac{\delta}{2}\).

Since Lebesgue points are dense and using the triangle inequality this proves (2.7). \(\square\)

We assume \(p < 2q + 1\) in the last proposition mainly because (1.15) is not defined for \(s \geq 1\). For general \(p \geq 2q + 1\) we nevertheless still have
\[
\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left| \frac{\dot{\gamma}'(u + w) - \dot{\gamma}'(u)}{|u|^{p-q}} \right|^q \, dw \, du \leq C \left( TP^{(p,q)}(\dot{\gamma}) + TP^{(p,q)}(\dot{\gamma})^\beta \right).
\]
This enables us to derive the following result on what we want to call the singular range: For these parameters the integrand is so singular if it does not vanish completely, that the integral is either equal to 0 or infinite:

**Proposition 2.6 (Singular range \(p \geq 2q + 1\)).** For \(p \geq 2q + 1\), \(q > 1\), and an absolutely continuous \(\gamma : \mathbb{R}/\mathbb{Z} \to \mathbb{R}\) we have \(TP^{(p,q)}(\dot{\gamma}) < \infty\) if and only if the image of \(\dot{\gamma}\) lies on a straight line.

**Proof.** Applying (2.3*) to Brezis [13, Prop. 2] reveals that \(\dot{\gamma}'\) is constant. Hence, \(\gamma\) lies on a straight line. \(\square\)

Proposition 2.5 and Proposition 2.4 prove Theorem 1.1.

Using Proposition 2.5 together with the Arzelà-Ascoli theorem, we see that sets of curves in \(C_1^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)\) with a uniform bound on the energy are sequentially compact in \(C^1\). The next proposition will help us to show that the limits we get are injective curves:

**Proposition 2.7 (Uniform bi-Lipschitz estimate).** For every \(M < \infty\) and (1.7) there is a constant \(C(M, p, q) > 0\) such that the following is true: Every curve \(\gamma \in C_1^1(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)\) parametrized by arc-length with
\[
TP^{(p,q)}(\dot{\gamma}) \leq M
\]
satisfies the bi-Lipschitz estimate
\[
|u - v| \leq C(M, p, q) |\gamma(u) - \gamma(v)| \quad \text{for all } u, v \in \mathbb{R}/\mathbb{Z}.
\]

We will give an easy proof that essential boils down to combining the regularity we get form Proposition 2.5 with a subtle scaling argument. The following lemma will be
one of the essential parts in the proof. To be able to state it, we set for two arc-length parametrized curves $\gamma_i : I_i \to \mathbb{R}$, $i = 1, 2$, $I_1, I_2$ open intervals,

$$TP^{(p,q)}(\gamma_1, \gamma_2) := TP^{(p,q)}(\gamma_1) + TP^{(p,q)}(\gamma_2) + \int_{I_1} \int_{I_2} \left( \frac{\text{dist}(\ell(t), \gamma_2(s))^{q}}{|\gamma_1(s) - \gamma_2(t)|^p} + \frac{\text{dist}(\ell(t), \gamma_1(s))^{q}}{|\gamma_2(s) - \gamma_1(t)|^p} \right) \, ds \, dt,$$

where $\ell(t) = \gamma(t) + \mathbb{R}\gamma'(t)$ denotes line tangential to $\gamma_i$ at $\gamma_i(t)$, $i = 1, 2$.

We then have

**Lemma 2.8.** Let $\alpha \in (0, 1)$. For $\mu > 0$ we let $M_\mu$ denote the set of all pairs $(\gamma_1, \gamma_2)$ of curves $\gamma_i \in C^{1, \alpha}_b([-1, 1], \mathbb{R}^n)$ satisfying

(i) $|\gamma_1(0) - \gamma_2(0)| = 1$,

(ii) $\gamma_1'(0) \perp (\gamma_1(0) - \gamma_2(0)) \perp \gamma_2'(0)$,

(iii) $\|\gamma_i\|_{C^{1, \alpha}} \leq \mu$, $i = 1, 2$.

Then there is a $c = c(\alpha, \mu) > 0$ such that

$$TP^{(p,q)}(\gamma_1, \gamma_2) \geq c \quad \text{for all } (\gamma_1, \gamma_2) \in M_\mu.$$

**Proof.** It is easy to see that $TP^{(p,q)}(\gamma_1, \gamma_2)$ is zero if and only if both $\gamma_1$ and $\gamma_2$ are part of one single straight line. We will show that $TP^{(p,q)}(\cdot, \cdot)$ attains its minimum on $M_\mu$. As $M_\mu$ does not contain straight lines by (i), (ii), this minimum is strictly positive which thus proves the lemma.

Let $(\gamma_1^{(n)}, \gamma_2^{(n)})$ be a minimizing sequence in $M_\mu$, i.e., we have

$$\lim_{n \to \infty} TP^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) = \inf_{M_\mu} TP^{(p,q)}(\cdot, \cdot).$$

Subtracting $\gamma_1(0)$ from both curves, i.e., setting

$$\tilde{\gamma}_i^{(n)}(\tau) := \gamma_i^{(n)}(\tau) - \gamma_1(0), \quad i = 1, 2,$$

and using Arzelà-Ascoli we can pass to a subsequence such that

$$\tilde{\gamma}_i^{(n)} \to \tilde{\gamma}_i \quad \text{in } C^1.$$

Furthermore, $(\tilde{\gamma}_1, \tilde{\gamma}_2) \in M_\mu$ since $M_\mu$ is closed under convergence in $C^1$. Since, by Fatou's lemma, the functional $TP^{(p,q)}$ is lower semi-continuous with respect to $C^1$ convergence, we obtain

$$TP^{(p,q)}(\tilde{\gamma}_1, \tilde{\gamma}_2) \leq \liminf_{n \to \infty} TP^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) = \liminf_{n \to \infty} TP^{(p,q)}(\gamma_1^{(n)}, \gamma_2^{(n)}) = \inf_{M_\mu} TP^{(p,q)}(\cdot, \cdot).$$

Let us use this lemma to give the

**Proof of Proposition 2.7.** Applying Proposition 2.5 to (2.8) we obtain

$$\|\gamma_i'\|_{C^0} \leq C(M)$$
for $\alpha = \frac{\nu - 2}{q} - 1 > 0$. As an immediate consequence there is a $\delta = \delta(\alpha, M) > 0$ such that

$$|u - v| \leq 2|\gamma(u) - \gamma(v)|$$

for all $u, v \in \mathbb{R}/\mathbb{Z}$ with $|u - v| \leq \delta$. Let now

$$S := \inf \left\{ \left| \gamma(u) - \gamma(v) \right| \left| u, v \in \mathbb{R}/\mathbb{Z}, |u - v| \geq \delta \right\} \leq \frac{1}{2}.$$  

We will complete the proof by estimating $S$ from below. Using the compactness of $\{u, v \in \mathbb{R}/\mathbb{Z}, |u - v| \geq \delta\}$, there are $s, t \in \mathbb{R}/\mathbb{Z}$ with $|s - t| \geq \delta$ and

$$|\gamma(s) - \gamma(t)| = S.$$  

If now $|s - t| = \delta$ we get

$$2S = 2|\gamma(s) - \gamma(t)| \geq \delta$$

and hence

$$|u - v| \leq \frac{1}{2} \leq \frac{S}{\delta} \leq \frac{|\gamma(u) - \gamma(v)|}{\delta(\alpha, M)}$$

for all $u, v \in \mathbb{R}/\mathbb{Z}$ with $|u - v| \geq \delta$. This proves the proposition in this case. If on the other hand $|s - t| > \delta$ then we get using the minimality of $|\gamma(s) - \gamma(t)|$

$$\gamma'(s) \perp (\gamma(s) - \gamma(t)) \perp \gamma'(t).$$

We now set for $\tau \in [-1, 1]$

$$\gamma_1(\tau) := \frac{1}{S} \gamma(s + S \tau) \quad \text{and} \quad \gamma_2(\tau) := \frac{1}{S} \gamma(t + S \tau).$$

Since

$$\|\gamma'_1\|_{C^{0,\alpha}} \leq \|\gamma'\|_{C^{0,\alpha}} \quad (\leq C(M))$$

we can apply the Lemma 2.8 to get

$$TP^{(p, q)}(\gamma_1, \gamma_2) \geq c(\alpha, M) > 0.$$  

Together with

$$TP^{(p, q)}(\gamma_1, \gamma_2) \leq S^{p-q-2}TP^{(p, q)}(\gamma)$$

this leads to

$$S \geq \left( \frac{c(\alpha, M)}{TP^{(p, q)}(\gamma)} \right)^{\frac{1}{p-q-2}} \geq \left( \frac{c(\alpha, M)}{M} \right)^{\frac{1}{p-q-2}}.$$  

Hence,

$$|u - v| \leq \frac{1}{2} \leq \frac{|\gamma(u) - \gamma(v)|}{2S} \leq C(M, p, q) |\gamma(u) - \gamma(v)|$$

for all $u, v \in \mathbb{R}/\mathbb{Z}$ with $|u - v| \geq \delta$.  

We are now in the position to prove the following mighty
Theorem 2.9 (Compactness). For each $M < \infty$ the set
\[ A_M := \{ \gamma \in C^1_{\text{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \mid TP^{p,q}(\gamma) \leq M \} \]
is sequentially compact in $C^1$ up to translations.

Proof. By Proposition 2.5 there are $C(M) < \infty$ and $\alpha = \alpha(p, q) > 0$ such that
\[ \|\gamma\|_{C^\alpha} \leq C(M) \]
for all $\gamma \in A_M$ and hence
\[ \|\tilde{\gamma}\|_{C^\alpha} \leq C(M) + 1 \]
where $\tilde{\gamma}(u) := \gamma(u) - \gamma(0)$. Furthermore, from Proposition 2.7 we get the bi-Lipschitz estimate
\[ |u - v| \leq C(M, p, q)|\gamma(u) - \gamma(v)| \]
for all $u, v \in \mathbb{R}/\mathbb{Z}$. Let now $\gamma_n \in A_M$. Then
\[ \|\tilde{\gamma}_n\|_{C^\alpha} \leq C(M) + 1 \]
and hence after passing to suitable subsequence we have
\[ \tilde{\gamma}_n \to \gamma_0 \]
in $C^1$. Since $\gamma_n$ was parametrized by arc-length, $\gamma_0$ is still parametrized by arc-length and still
\[ |u - v| \leq C(M, p, q)|\gamma_0(u) - \gamma_0(v)| \]
for all $u, v \in \mathbb{R}/\mathbb{Z}$. So, especially, $\gamma_0 \in C^1_{\text{ia}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. From lower semi-continuity with respect to $C^1$ convergence we infer
\[ TP^{p,q}(\gamma_0) \leq \liminf_{n \to \infty} TP^{p,q}(\gamma_n) \leq M. \]
So $\gamma_0 \in A_M$. □

Let us conclude this section by deriving two simple corollaries of this sequential compactness and the lower semi-continuity of the energies with respect to $C^1$-convergence. The first one states that the tangent-point energies $TP^{p,q}$ are in fact knot energies as defined in the introduction. The second one, already stated in the introduction, ensures that there exist minimizers of the energies within every knot class — which are then smooth by Theorem 1.5.

Proposition 2.10 ($TP^{p,q}$ is a strong knot energy [55, Prop. 5.1, 5.2]).
Let (1.7) hold.
(i) If $(\gamma_k)_{k \geq 1} \subset W^{1,p-1/q,q}$ is a sequence uniformly converging to a non-injective curve $\gamma_0 \in C^{0,1}$ parametrized by arc-length then $TP^{p,q}(\gamma_k) \to \infty$.
(ii) For given $E, L > 0$ there are only finitely many knot types having a representative with $TP^{p,q} < E$ and length $= L$. 

15
Proof. The first statement immediately follows from the bi-Lipschitz estimate in Proposition 2.7, as a sequence with bounded energy would be sequentially compact in $C^1_{ia}$ and thus cannot uniformly converge to a non-injective curve.

To show the second statement, let us assume that it was wrong, i.e. that there are curves $(\gamma_n)_{n \in \mathbb{N}}$ of length $L$, all belonging to different knot classes, with energy less than $E$. Of course we can assume that $L = 1$. Theorem 2.9 tells us, that after suitable translations and going to a subsequence we can assume that there is a $\gamma_0 \in A_M$ such that $\gamma_n \to \gamma_0$ in $C^1$. As the intersection of every knot class with $C^1$ is an open set in $C^1$ [4, Cor. 1.5] (see [42] for an explicit construction), this implies that almost all $\gamma_n$ belong to the same knot class as $\gamma_0$, which is a contradiction.

Proof of Theorem 1.3. Let $(\gamma_k)_{k \in \mathbb{N}} \in C^1_{ia}$ be a minimal sequence for $\text{TP}^{(p,q)}$ in a given knot class $K$, i.e. let

$$\lim_{k \to \infty} \text{TP}^{(p,q)}(\gamma_k) = \inf_{C^1_{ia} \cap K} \text{TP}^{(p,q)}.$$

After passing to a subsequence and suitable translations, we hence get by Theorem 2.9 a $\gamma_0 \in C^1_{ia}$ with $\gamma_k \to \gamma_0$ in $C^1$. Again by [4, 42] the curve $\gamma_0$ belongs to the same knot class as the elements of the minimal sequence $(\gamma_k)_{k \in \mathbb{N}}$. The lower semi-continuity of $\text{TP}^{(p,q)}$ furthermore implies that

$$\inf_{C^1_{ia} \cap K} \text{TP}^{(p,q)} \leq \text{TP}^{(p,q)}(\gamma_0) \leq \lim_{k \to \infty} \text{TP}^{(p,q)}(\gamma_k) = \inf_{C^1_{ia} \cap K} \text{TP}^{(p,q)}.$$

Hence, $\gamma_0$ is the minimizer we have been searching for.

By the same reasoning one derives the existence of a global minimizer of $\text{TP}^{(p,q)}$.

Remark 2.11 (Strange range). On $p \in [2q+1, q+2)$, $p, q > 0$, see the hatched area in Figure 1, we find the strange behavior that there are no closed finite-energy $C^1$-curves while self-intersections, and in particular corners, are not penalized. So piecewise linear curves (polygons) have finite energy.

The latter can be seen by adapting the calculation (2.1). For the former we recall that a closed arc-length parametrized $C^2$-curve must have positive curvature $|\gamma''|$ at some point $u_0$ and by continuity there are $c, \delta > 0$ with $|\gamma''(u_0 + w)| \geq c > 0$ for all $w \in [-2\delta, 2\delta]$. As $\gamma'' \perp \gamma'$ we may lessen $\delta$ (if necessary) to obtain $|\langle \gamma''(u + w), \gamma'(u) \rangle| \leq \frac{1}{2} |\gamma''(u + w)|$ for all $u \in [u_0 - \delta, u_0 + \delta], w \in [-\delta, \delta]$. So $\text{TP}^{(p,q)}(\gamma)$ is bounded below by

$$\begin{align*}
\int_{u_0 - \delta}^{u_0 + \delta} \int_{w - \delta}^{w + \delta} u^{2q-1} & \left( \int_0^1 (1 - \theta)\gamma''(u + \theta w) d\theta \right)^2 \, dw \, du \\
&= \int_{u_0 - \delta}^{u_0 + \delta} \int_{w - \delta}^{w + \delta} u^{2q-1} \left( \frac{1}{2} \int_{[0,1]^2} (1 - \theta_1)(1 - \theta_2) \left| \gamma''(u + \theta_1 w) \right|^2 + \left| \gamma''(u + \theta_2 w) \right|^2 - \\
&\quad - \left| \gamma''(u + \theta_1 w) - \gamma''(u + \theta_2 w) \right|^2 - \left| \langle \gamma''(u + \theta_1 w), \gamma'(u) \rangle \right|^2 - \\
&\quad - \left| \langle \gamma''(u + \theta_2 w), \gamma'(u) \rangle \right|^2 + \left| \gamma''(u + \theta_1 w) - \gamma''(u + \theta_2 w), \gamma'(u) \rangle \right|^2 \right) \, d\theta_1 \, d\theta_2 \right)^{q/2} \, dw \, du.
\end{align*}$$
Lessening $\delta > 0$ once more, the term $|y''(u + \theta_1 w) - y''(u + \theta_2 w)|^2 \leq \omega^2 ||y''||^2_{L^2}$ can be made so small that the square bracket is $\geq \tilde{c} > 0$. This gives $\text{TP}^{[p,q]}(\gamma) = \infty$. 

3 First variation

Let turn to the proof of Theorem 1.4. In contrast to the investigation of O’Hara’s energies [11], we do not need to cut off the singular part in the energies. Instead, a straightforward calculation of the first variation using Lebesgue’s theorem of dominated convergence will prove that

$$\delta \text{TP}^{[p,q]}(\gamma, h) := \lim_{\tau \to 0} \frac{\text{TP}^{[p,q]}(\gamma + \tau h) - \text{TP}^{[p,q]}(\gamma)}{\tau}$$

exists.

For $\gamma \in W^{(p-1)/q,q}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ and $h \in W^{(p-1)/q,q}$ let

$$\gamma_\tau := \gamma + \tau h \quad \text{for any } \tau \in [-\tau_0, \tau_0]$$

where $\tau_0 \in (0, 1)$ is so small that

$$|\gamma'_\tau| \geq \frac{1}{2} \quad \text{on } \mathbb{R}/\mathbb{Z} \quad (3.1)$$

and each curve $\gamma + \tau h$, $\tau \in [-\tau_0, \tau_0]$, is still injective. Then, recalling (1.9),

$$\frac{\text{TP}^{[p,q]}(\gamma + \tau h) - \text{TP}^{[p,q]}(\gamma)}{\tau} = \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} I_\tau(u, w) \ dw \ du$$

where

$$I_\tau(u, w) := \frac{1}{\tau} \left( \frac{p_{\gamma_\tau(u)}(\Delta \gamma_\tau)}{\Delta \gamma_\tau^p} |\gamma'_\tau(u + w)| |\gamma'_\tau(u)| - \frac{p_{\gamma_\tau(u)}(\Delta \gamma)}{\Delta \gamma^p} |\gamma'(u + w)| |\gamma'(u)| \right).$$

To calculate the pointwise limit of $I_\tau(u, w)$ as $\tau \to 0$, we observe using $|y'| \equiv 1$ that

$$\left. \frac{d}{d\tau} \right|_{\tau=0} |\gamma'_\tau(u)| = \langle \gamma'(u), h'(u) \rangle,$$

$$\left. \frac{d}{d\tau} \right|_{\tau=0} |\gamma'_\tau(u)| = p_{\gamma'(u)}^\perp h'(u),$$

and

$$\left. \frac{d}{d\tau} \right|_{\tau=0} p_{\gamma'(u)}^\perp h'(u) = - \langle v, \gamma'_\tau(u) \rangle p_{\gamma'(u)}^\perp h'(u),$$

which gives

$$\langle p_{\gamma'(u)}^\perp \Delta \gamma_\tau - \langle \Delta \gamma, \gamma'(u) \rangle p_{\gamma'(u)}^\perp h'(u) \rangle$$

$$= \langle p_{\gamma'(u)}^\perp (\Delta \gamma - w \gamma'(u)) , \Delta h - \langle \Delta \gamma, \gamma'(u) \rangle h'(u) \rangle.$$

Hence,

$$\lim_{\tau \to 0} I_\tau(u, w)$$
We will give uniform majorants for these three terms. In order to treat the first term we first consider

\[
\left\langle P_{γ'(u)}^+ (Δγ - wγ'(u)) , Δh - \langle Δγ, γ'(u) \rangle h'(u) \right\rangle
\]

\[
- p \left\langle P_{γ'(u)}^+ (Δγ) \right\rangle \langle Δγ, Δh \rangle
\]

\[
+ \left| P_{γ'(u)}^+ (Δγ) \right|^q \left| Δγ \right|^p \cdot (\langle γ'(u), h'(u) \rangle + \langle γ'(u + w), h'(u + w) \rangle).
\]

We decompose

\[
I_1(u, w)
\]

\[
= \frac{1}{τ} \left( \frac{\left| P_{γ'(u)}^- (Δγ) \right|^q}{\left| Δγ \right|^p} \right) \left| \gamma'_τ(u + w) \right| \left| γ'_τ(u) \right| \left( \left\langle P_{γ'(u)}^+ (Δγ), γ'_τ(u) \right\rangle \right)
\]

\[
+ \left| P_{γ'(u)}^+ (Δγ) \right|^q \left( \frac{1}{\left| Δγ \right|^p} - \frac{1}{\left| Δγ \right|^p} \right) \left| γ'_τ(u + w) \right| \left| γ'_τ(u) \right|
\]

\[
+ \left| P_{γ'(u)}^+ (Δγ) \right|^q \left( \left| γ'_τ(u + w) \right| \left| γ'_τ(u) \right| - \left| γ'(u) \right| \left| γ'(u) \right| \right)
\]

\[
=: F_1 + F_2 + F_3.
\]

We will give uniform majorants for these three terms. In order to treat the first term we first consider

\[
P_{γ'(u)} a - P_{γ'(u)} a = \left\langle a, γ'_τ \right\rangle \gamma'_τ \left( \frac{1}{|γ'_τ|^2} - 1 \right) + \left\langle a, γ'_τ \right\rangle γ'_τ - \left\langle a, γ' \right\rangle γ'
\]

\[
= τ \left( - \left\langle a, γ'_τ \right\rangle γ'_τ \frac{2 \langle γ', h' \rangle + τ |h'|^2}{|γ'_τ|^2} + \left\langle a, h' \right\rangle γ'_τ + \left\langle a, γ' \right\rangle h' \right)
\]

which for \( a := Δγ - wγ'(u) \) gives

\[
P_{γ'(u)}^+ (Δγ) - P_{γ'(u)}^- (Δγ)
\]

\[
= P_{γ'(u)}^+ (Δγ - wγ'_τ(u)) - P_{γ'(u)}^+ (Δγ - wγ'(u))
\]

\[
= P_{γ'(u)}^+ (Δγ - wγ'(u)) + τ P_{γ'(u)}^+ (Δh - wh'(u)) - P_{γ'(u)}^+ (Δγ - wγ'(u))
\]

\[
= P_{γ'(u)}^+ (Δγ - wγ'(u)) - P_{γ'(u)}^+ (Δγ - wγ'(u)) + τ P_{γ'(u)}^+ (Δh - wh'(u))
\]

\[
= τ \left( \left\langle a, γ'_τ \right\rangle γ'_τ \frac{2 \langle γ', h' \rangle + τ |h'|^2}{|γ'_τ|^2} - \left\langle a, h' \right\rangle γ'_τ - \left\langle a, γ' \right\rangle h' + P_{γ'(u)}^+ (Δh - wh'(u)) \right).
\]

Recalling (3.1) and \( |γ'| = 1 \), we hence get a constant \( C \) depending on \( ||h'||_{L^∞} \) and \( τ_0 \) such that

\[
\left| P_{γ'(u)}^+ (Δγ) - P_{γ'(u)}^- (Δγ) \right| \leq C |τ| (|Δγ - wγ'(u)| + |Δh - wh'(u)|).
\]

By \( |a^b - b^a| \leq q |a - b| \max \left( a^{q-1}, b^{q-1} \right) \) for \( a, b \geq 0, q > 1 \) we deduce for \( C = C (||h'||_{L^∞}, q, τ_0) > 0 \)

\[
\left| P_{γ'(u)}^+ (Δγ) \right|^q - \left| P_{γ'(u)}^- (Δγ) \right|^q
\]
\[ \leq C |\tau| \left( \left| \Delta \gamma - w \gamma'(u) \right| + \left| \Delta h - w h'(u) \right| \right) \left( \left| \Delta \gamma_r - w \gamma'_r(u) \right|^q + \left| \Delta \gamma - w \gamma'(u) \right|^{q-1} \right) \]

\[ \leq C |\tau| \left( \left| \Delta \gamma - w \gamma'(u) \right| + \left| \Delta h - w h'(u) \right| \right) \left( \left| \Delta \gamma - w \gamma'(u) \right|^{q-1} + \left| \Delta h - w h'(u) \right|^{q} \right) \]

\[ \leq C |\tau| \left( \left| \Delta \gamma - w \gamma'(u) \right| + \left| \Delta h - w h'(u) \right| \right) \left( \left| \Delta \gamma - w \gamma'(u) \right|^{q} \right) \]

\[ \leq C |\tau| \frac{\int_0^1 (\gamma'(u + \theta w) - \gamma'(u)) \, d\theta}{|w|^{p-q}} + \int_0^1 (h'(u + \theta w) - h'(u)) \, d\theta \]

and hence, by Equations (2.2), (3.1),

\[ |F_1| \leq C \gamma \frac{\int_0^1 (\gamma'(u + \theta w) - \gamma'(u)) \, d\theta}{|w|^{p-q}} + \int_0^1 (h'(u + \theta w) - h'(u)) \, d\theta \].

Applying Jensen’s inequality, one sees

\[ \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} |F_1| \, dw \, du \]

\[ \leq C \gamma \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \int_0^1 \frac{|\gamma'(u + \theta w) - \gamma'(u)|^q + |h'(u + \theta w) - h'(u)|^q}{|w|^{p-q}} \, d\theta \, dw \, du \]

\[ \leq C \gamma \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(u + w) - \gamma'(u)|^q + |h'(u + w) - h'(u)|^q}{|w|^{p-q}} \, dw \, du \]

so we have found an \( L^1 \)-majorant for \( F_1 \).

The same conclusions lead to a majorant for the remaining terms to which we pass now. Using arc-length parametrization and (3.1) together with \(|a^{-p} - b^{-p}| \leq C_{a,b} |a - b| \) for \( a, b \geq \mu > 0 \), we compute

\[ |F_2| \leq C \frac{\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} |F_2| \, dw \, du}{|w|^{p-q}} \]

\[ \leq C \frac{\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left( \left| \Delta \gamma - w \gamma'(u) \right|^q \right) \frac{1}{|\Delta \gamma_r|^p} \, d\theta \, dw \, du}{|w|^{p-q}} \]

\[ \leq C \frac{\int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \left( \left| \gamma'(u + \theta w) - \gamma'(u) \right|^q \right) \frac{1}{|\Delta \gamma_r|^p} \, d\theta \, dw \, du}{|w|^{p-q}} \]

\[ \leq C \left\| h' \right\|_{L^1} \int_0^1 \left| \gamma'(u + \theta w) - \gamma'(u) \right|^q \, d\theta \frac{1}{|w|^{p-q}} \]

and get using a simple substitution

\[ \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} |F_2| \, dw \, du \leq C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(u + w) - \gamma'(u)|^q}{|w|^{p-q}} \, dw \, du \]

Finally

\[ |\gamma'_r - \gamma'_r| = |\gamma'_r| - 1 = \frac{|\gamma'_r|^2}{|\gamma'_r| + 1} - \frac{1}{|\gamma'_r| + 1} = \frac{2 (\gamma'_r h' + \tau |h'|^2)}{|\gamma'_r| + 1} \]

permits to proceed as in the proof of Proposition 2.4. We arrive at

\[ \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} |F_3| \, dw \, du \leq C \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{|\gamma'(u + w) - \gamma'(u)|^q}{|w|^{p-q}} \, dw \, du \].
Hence, the functions $I_f$ have a uniform $L^1$ majorant. Thus Lebesgue’s theorem of dominant convergence implies

$$\delta TP^{(p,q)}(\gamma, h)$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} q \left\{ \frac{P_{\gamma(w)}(\triangle \gamma - w \gamma'(u), \triangle h - \langle \triangle \gamma, \gamma'(u) \rangle h'(u))}{|\triangle \gamma|^p} \right\} |\gamma(r)| + \left\{ \frac{P_{\gamma(w)}(\langle \triangle \gamma, \triangle h \rangle)}{|\triangle \gamma|^q} \right\} |\gamma(r)| + \left\{ \frac{P_{\gamma(w)}(\langle \gamma'(u), h'(u) \rangle)}{|\gamma'(u)|^p \cdot |h'(u)|^q} \right\}.$$ 

**Remark 3.1 (Continuous differentiability).** Estimating more carefully, one can even show that $\delta TP^{(p,q)}(\cdot, \cdot)$ is continuous on $W^{p-1/q,q}_{\text{u}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ for $p \in (q+2, 2q+1)$. (3.2)

In contrast to O’Hara’s energies, even an explicit formula of the first variation can be given at arbitrary $\gamma \in W^{p-1/q,q}_{\text{u}}$ in the direction $h \in W^{p-1/q,q}$. In fact, we have

$$\delta TP^{(p,q)}(\gamma, h)$$

$$= \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} q \left\{ \frac{P_{\gamma(w)}(\triangle \gamma, \triangle h)}{|\triangle \gamma|^p} \right\} |\gamma(r)| + \left\{ \frac{P_{\gamma(w)}(\langle \gamma'(u), h'(u) \rangle)}{|\gamma'(u)|^p \cdot |h'(u)|^q} \right\}.$$ 

\[ \diamond \]

## 4 Bootstrapping

In this section we consider the non-degenerate sub-critical case

$$p \in (4, 5), \quad q = 2$$

(1.12)

which corresponds to the yellow line in Figure 1.

In order to start a bootstrapping process, we have to rearrange the Euler-Lagrange Equation for $TP^{(p,2)}$, exhibiting a gap of regularity between suitable terms. To this end, we decompose $\delta TP^{(p,2)}$ into the sum of a bilinear elliptic term $Q^{(p)}$ and a remainder term $R^{(p)}$ of lower order. The former is defined via

$$Q^{(p)}(f, g) := \int_{\mathbb{R}/\mathbb{Z}} \int_{-1/2}^{1/2} \frac{\langle \triangle f - uf'(u), \triangle g - wg'(u) \rangle}{|u|^p} \, dw \, du$$

for $f, g \in W^{p-1/2,2}_{\text{u}}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$. The operator $Q^{(p)}$ is characterized by the following
Using (1.11), we now consider the remainder term is finite by 
\( \mathcal{F} \) for 
\[ f \]
Proof. Since any \( L^2 \) function is uniquely determined by its Fourier series, we obtain for \( f, g \in W^{p-1/2,2} \)
\[ Q^{(p)}(f, g) = 2 \int_0^{1/2} \sum_{k \in \mathbb{Z}} \langle \hat{f}_k, \hat{g}_k \rangle_{C^d} \frac{F(2\pi k u)}{w^p} \, dw, \tag{4.1} \]
where \( F(x) := 2 - 2 \cos x - 2 \sin x + x^2 \), which turns out to be even and monotone increasing on \( \{ x \geq 0 \} \), for \( F(x) = 2x(1-\cos x) \geq 0 \). Now \( F(0) = 0 \) implies that \( F \) is non-negative on \( \mathbb{R} \), so \( Q^{(p)}(f, g) = \sum_{k \in \mathbb{Z}} \langle \hat{f}_k, \hat{g}_k \rangle_{C^d} \) where
\[ \hat{q}_k := 2 \int_0^{1/2} \frac{F(2\pi k u)}{w^p} \, dw = 2 |2\pi k|^{p-1} \int_0^{1/2} \frac{\hat{F}(x)}{x^p} \, dx \tag{4.2} \]
is finite by \( F(x) = O(x^4) \) as \( x \to 0 \). Obviously, \( \hat{q}_k \) is monotone increasing in \( |k| \), so \( \lim_{|k| \to \infty} \hat{q}_k k^{-p+1} \in (0, \infty) \) by \( F(x) = O(x^2) \) as \( |x| \to \infty \). \( \square \)

Using (1.11), we now consider the remainder term
\[ R^{(p)}(y, h) := \delta TP^{(p,2)}(y, h) - 2Q^{(p)}(y, h) \tag{4.3} \]
\[ = \int_{\mathbb{R}^2} \int_0^{-1/2} \left\{ 2 \left( P^{(p)}_{y,u}(\Delta \gamma - w \gamma') , \Delta h - (\Delta \gamma, \gamma') h' \right) \frac{1}{|\Delta \gamma|^p} - \frac{1}{w^p} \right\} \right. \
\[ + 2 \left( P^{(p)}_{y,u}(\Delta \gamma - w \gamma') , \Delta h - (\Delta \gamma, \gamma') h' \right) \right) = \left( \Delta \gamma - w \gamma', \Delta h - w h' \right) \right) \right) \frac{1}{|w|^p} \right\} \right. \
\[ - \left| P^{(p)}_{y,u}(\Delta \gamma) \right| (\Delta \gamma, \Delta h) \right) \right) \left\{ \left| \Delta \gamma \right|^{p+2} \right\} \right. \right) \left\{ \left( \gamma'(u), h'(u) \right) + (\gamma'(u+w), h'(u+w)) \right) \right\} \, dw \, da \right. \
= : R^{(p)}_1 + R^{(p)}_2 + R^{(p)}_3 + R^{(p)}_4. \]

Interestingly, all these terms have the same structure so we can treat them simultaneously. As in analysis the exact form of a multilinear mapping \( (\mathbb{R}^n)^N \to \mathbb{R} \) does not matter at all, let us introduce the \( \oplus \) notation which represents any sort of these operators, e. g., \( \langle (a \oplus b) c, d \rangle = a \oplus b \oplus c \oplus d \) for \( a, b, c, d \in \mathbb{R}^n \).
Lemma 4.2 (Structure of the remainder term). The term $R^{(p)}(\gamma, h)$ is a (finite) sum of expressions of type

$$\int_{\mathbb{R}^2} \int_{-1/2}^{1/2} \int_{[0,1]^3} g^{(p)}(u, w) \otimes h'(u + s_k w) \, d\theta_1 \cdots d\theta_K \, dw \, du$$

where

$$g^{(p)}(u, w) := G^{(p)} \left( \frac{\Delta \gamma}{w} \right) \frac{|\gamma'(u + s_1 w) - \gamma'(u + s_2 w)|^2}{|w|^{p-2}} \left( \sum_{i=3}^{k-1} \gamma'(u + s_i w) \right),$$

$G^{(p)}$ is some analytic function defined on $[c, \infty)$, and $s_i \in \{0, \theta_i\}$ for all $i = 1, \ldots, K$.

Proof. We begin with $R^{(p)}$. Using $P^\perp_{\gamma(w)} = 1 - \gamma \otimes \gamma$ where $1$ denotes the unit matrix, we obtain

$$w^{-2} \left( P^\perp_{\gamma(w)}(\Delta \gamma - w \gamma') \cdot \Delta h - \langle \Delta \gamma, \gamma' \rangle h' \right)$$

$$= \int_{[0,1]^3} (1 - \gamma \otimes \gamma) (\gamma'(u + \theta_1 w) - \gamma') \cdot h'(u + \theta_3 w) - \langle \gamma'(u + \theta_2 w), \gamma' \rangle h' \, d\theta_1 \, d\theta_2 \, d\theta_3.$$

As $|\gamma'| = 1$, we may add $\langle \gamma', \gamma' \rangle$ before $1$ and $h'(u + \theta_3 w)$. Expanding the three differences, we obtain eight terms all of which have the form

$$\int_{[0,1]^3} \gamma' \otimes \gamma' \otimes \gamma'(u + s_1 w) \otimes \gamma'(u + s_2 w) \otimes \gamma' \otimes h'(u + s_3 w) \, d\theta_1 \, d\theta_2 \, d\theta_3$$

where $s_i \in \{0, \theta_i\}$, $i = 1, 2, 3$. For the remaining factor we set $G^{(p)}(z) := \frac{z^{p-1}}{z-1}$ and compute, using $|\gamma'| = 1$ and $\langle a, b \rangle - 1 = -\frac{1}{2} |a - b|^2$ for $|a| = |b| = 1$,

$$w^{-2} \left( \frac{1}{|\Delta \gamma|^p} - \frac{1}{|w|^p} \right)$$

$$= |w|^{-2-p} \left( \frac{|\Delta \gamma|^2}{w} - 1 \right)$$

$$= G^{(p)} \left( \frac{\Delta \gamma}{w} \right) \frac{|\Delta \gamma|^2}{|w|^{p-2}} - 1$$

$$= G^{(p)} \left( \frac{\Delta \gamma}{w} \right) \frac{\int_{[0,1]^3} |\gamma'(u + \theta w)| \, d\theta_1 \, d\theta_2 \, d\theta_3}{|w|^{p-2}}$$

$$- \frac{1}{2} G^{(p)} \left( \frac{\Delta \gamma}{w} \right) \frac{\int_{[0,1]^3} |\gamma'(u + \theta_1 w) - \gamma'(u + \theta_2 w)|^2 \, d\theta_1 \, d\theta_2}{|w|^{p-2}}.$$

So $R^{(p)}$ has the desired form. The nominator of $R^{(p)}_2$ reads

$$\langle P^\perp_{\gamma(w)}(\Delta \gamma - w \gamma'), \Delta h - \langle \Delta \gamma, \gamma' \rangle h' \rangle - \langle \Delta \gamma - w \gamma', \Delta h - w h' \rangle$$

$$= \langle \Delta \gamma - w \gamma', w h' - \langle \Delta \gamma, \gamma' \rangle h' \rangle - \langle \Delta \gamma - w \gamma', \gamma' \rangle \langle \Delta h - \langle \Delta \gamma, \gamma' \rangle h', \gamma' \rangle$$

$$= \langle \Delta \gamma - w \gamma', (w \gamma' - \Delta \gamma, \gamma') h' \rangle - \langle \Delta \gamma - w \gamma', \gamma' \rangle \langle \Delta h - \langle \Delta \gamma, \gamma' \rangle h', \gamma' \rangle$$

$$= -\langle \Delta \gamma - w \gamma', \gamma' \rangle (\langle \Delta \gamma - w \gamma', h' \rangle + \langle \Delta h - \langle \Delta \gamma, \gamma' \rangle h', \gamma' \rangle)$$

...
To prove Proposition 4.3, we first note that, by partial integration, the terms of the regularity theorem which is deferred to the end of this section. This statement together with Proposition 4.1 immediately leads to the proof of the following auxiliary result.

\[ \int_0^1 \left( \gamma'(u + \theta_t w), \gamma'(u) \right) w \ dt_1. \]

where (\cdots) is a sum of terms of type \( \gamma'(u + s_2 w) \otimes \gamma' \otimes h'(u + s_3 w) \) with \( s_i \in \{0, \theta_t\}, i = 2, 3 \). For \( R'_4^{(p)} \) we set \( G^{(p)}(z) := |z|^{p-2} \) and obtain

\[
\frac{\left| P_{\gamma(u)}^{(1)} (\Delta \gamma) \right|^2}{|\Delta \gamma|^{p+2}} = \frac{\left| P_{\gamma(u)}^{(1)} (\Delta \gamma, \Delta h) \right|}{|\Delta \gamma|^{p+2}} \cdot \frac{\langle \gamma'(u + \theta_t w) - \gamma'(u + \theta_t w), \gamma'(u + \theta_t w) \rangle}{|w|^{p-2}}.
\]

The nominator reads

\[
\langle \gamma'(u + \theta_t w) - \gamma'(u + \theta_t w), \gamma', \gamma'(u + \theta_t w) \rangle = \langle \gamma'(u + \theta_t w), \gamma'(u + \theta_t w) \rangle - 1 + 1 - \langle \gamma'(u + \theta_t w), \gamma' \rangle + \langle \gamma'(u + \theta_t w), \gamma' \rangle (1 - \langle \gamma'(u + \theta_t w), \gamma' \rangle)
\]

\[
= -\frac{1}{2} \left| \gamma'(u + \theta_t w) - \gamma'(u + \theta_t w) \right|^2 + \frac{1}{2} \left| \gamma'(u + \theta_t w) - \gamma' \right|^2 + \frac{1}{2} \left| \gamma'(u + \theta_t w), \gamma \right| \left| \gamma'(u + \theta_t w) - \gamma' \right|^2.
\]

Finally, \( R'_4^{(p)} \) is treated similarly to \( R'_3^{(p)} \).

Our next task is to show that \( R'_4^{(p)} \) is in fact a lower-order term. More precisely, we have

**Proposition 4.3 (Regularity of the remainder term).** If \( \gamma \in W^\infty_{2,\sigma}(\mathbb{R}^p) \) for some \( \sigma \geq 0 \) then \( R'_4^{(p)}(\gamma, \cdot) \in \left(W^{3,2-\sigma+2\varepsilon,2}\right)^p \) for any \( \varepsilon > 0 \).

This statement together with Proposition 4.1 immediately leads to the proof of the regularity theorem which is deferred to the end of this section.

To prove Proposition 4.3, we first note that, by partial integration, the terms of \( R'_4^{(p)}(\gamma, h) \) may be transformed into

\[
\int_0^1 \int_{|\theta|\leq 1/2} \int_{|\theta|\leq 1/2} \left( (-\Delta)^{\sigma/2} g^{(p)}(\cdot, u) \right) (u) \otimes \left( (-\Delta)^{\sigma/2} h' \right) (u + s_k w) \ dw \ dt_1 \cdots \ dt_K
\]

\[
\leq \int_0^1 \int_{|\theta|\leq 1/2} \int_{|\theta|\leq 1/2} \left| g^{(p)}(\cdot, u) \right|_{W^{\sigma,1}(\mathbb{R}^p; \mathbb{R}^n)} \ dw \ dt_1 \cdots \ dt_K \left| (-\Delta)^{\sigma/2} h' \right|_{L^\infty(\mathbb{R}^p)}
\]

\[
\leq \int_0^1 \int_{|\theta|\leq 1/2} \int_{|\theta|\leq 1/2} \left| g^{(p)}(\cdot, u) \right|_{W^{\sigma,1}(\mathbb{R}^p; \mathbb{R}^n)} \ dw \ dt_1 \cdots \ dt_K \left| h \right|_{W^{3,2+\varepsilon,2}}
\]

where \( \sigma \in \mathbb{R}, \varepsilon > 0 \) can be chosen arbitrarily, and \( (-\Delta)^{\sigma/2} \) denotes the fractional Laplacian. We let \( \sigma := 0 \) if \( \sigma = 0 \) and \( \sigma := \sigma - \varepsilon/2 \) otherwise. Now the claim directly follows from the succeeding auxiliary result.
Lemma 4.4 (Regularity of the remainder integrand). Let $\gamma \in W^{(p-1)/2+\sigma,2}_{in}$.

- If $\sigma = 0$ then $g^{(p)} \in L^1(\mathbb{R}/\mathbb{Z} \times (-\frac{1}{2}, \frac{1}{2}), \mathbb{R}^n)$ and
- if $\sigma > 0$ then $(w \mapsto g^{(p)}(\cdot, w)) \in L^1((-\frac{1}{2}, \frac{1}{2}), W^{\beta,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n))$ for any $\beta < \sigma$.

The respective norms are bounded independently of $s_1, \ldots, s_K$.

Proof. Note that, by (2.2), the argument of $G^{(p)}$ is compact and bounded away from zero. Using arc-length parametrization, we immediately obtain

$$|g^{(p)}(u, w)| \leq C |\gamma'(u + s_1 w) - \gamma'(u + s_2 w)| \frac{1}{|u|^p}$$

which gives $\int_{\mathbb{R}/\mathbb{Z}} |g^{(p)}(u, w)| \, dw \, du \leq C \|\gamma\|_{W^{(p-3)/2,2}}$. To prove the second claim, we will derive a suitable bound on $\|g^{(p)}(\cdot, w)\|_{W^{\beta,2}}$ for some $r > 1$. We choose $q_1, \ldots, q_{K-1}$, which will be determined more precisely later on, such that

$$\sum_{i=1}^{K-1} \frac{1}{q_i} = \frac{1}{r}.$$

Lemma A.1 then leads to

$$\|g^{(p)}(\cdot, w)\|_{W^{\beta,2}} \leq C \left\| G^{(p)} \left( \frac{\Delta \gamma}{w} \right) \right\|_{W^{q_1,2}} \|\gamma'(\cdot + s_1 w) - \gamma'(\cdot + s_2 w)\|_{W^{q_2,2}} \prod_{j=3}^{K-1} \|\gamma\|_{W^{q_j,2}}.$$

For the second factor, we now choose $q_2 > r$ so small that $W^{q_2,2}$ embeds into $W^{\beta,2}$.

To this end, we set $\frac{1}{r} := 1 - (\sigma - \bar{\sigma})$ and $\frac{1}{q_1} := 1 - 2(\sigma - \bar{\sigma})$, and $q_i := \frac{K-1}{\sigma - \bar{\sigma}}$ for $i = 1, 3, 4, \ldots, K - 1$.

Then for the first factor we apply Lemma A.2. Recall that $G^{(p)}$ is analytic and its argument is bounded below away from zero and above by 1. We infer

$$\left\| G^{(p)} \left( \frac{\gamma'(\cdot + s_1 w) - \gamma'(\cdot)}{w} \right) \right\|_{W^{q_1,2}} \leq C \|\gamma\|_{W^{q_1,2}}.$$

The Sobolev embedding gives

$$\|\gamma\|_{W^{q_1,2}} \leq C \|\gamma\|_{W^{(p-1)/2+\sigma,2}} \leq C \quad \text{for } i = 1, 3, 4, \ldots, K - 1.$$

Together this leads to

$$\|g^{(p)}(\cdot, w)\|_{W^{\beta,2}} \leq C \left\| \frac{\gamma'(\cdot + s_1 w) - \gamma'(\cdot + s_2 w)}{w} \right\|_{W^{q_2,2}}$$

and finally

$$\int_{\mathbb{R}/\mathbb{Z}} |g^{(p)}(\cdot, w)| \, dw \leq C \|\gamma\|^2_{W^{(p-1)/2+\sigma,2}}. \quad \square$$
Proof of Theorem 1.5. We arrive at the Euler-Lagrange Equation
\[
\delta T^{p}(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} = 0
\] (4.4)
for any \( h \in C^\infty(\mathbb{R}/\mathbb{Z}) \) where \( \lambda \in \mathbb{R} \) is a Lagrange parameter stemming from the side condition (fixed length). Using (4.3) this reads
\[
2Q^{(p)}(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} + R^{(p)}(\gamma, h) = 0. \tag{4.5}
\]
Since first variation of the length functional satisfies
\[
\langle \gamma', h' \rangle_{L^2} = \sum_{k \in \mathbb{Z}} |2\pi k|^2 \langle \hat{\gamma}_k, \hat{h}_k \rangle_{L^2},
\]
we get using Proposition 4.1 that there is a \( c > 0 \) such that
\[
2Q^{(p)}(\gamma, h) + \lambda \langle \gamma', h' \rangle_{L^2} = \sum_{k \in \mathbb{Z}} \varrho_k \langle \hat{\gamma}_k, \hat{h}_k \rangle_{L^2} \tag{4.6}
\]
where
\[
\varrho_k = c |k|^{p-1} + o \left( |k|^{p-1} \right) \quad \text{as } |k| \to \infty.
\]
Assuming that \( \gamma \in W^{p-1/2+\sigma,2} \) for some \( \sigma \geq 0 \), we infer
\[
2Q^{(p)}(\gamma, \cdot) + \lambda \langle \gamma', \cdot \rangle_{L^2} \in \left( W^{3/2-\sigma+2,2} \right)^*.
\]
from applying Proposition 4.3 to (4.5). Equation (4.6) implies
\[
\varrho_k |k|^{-p+1} \text{ converges to a positive constant as } |k| \to \infty,
\]
we are led to
\[
\gamma \in W^{p-1 + \sigma + \frac{p+4}{2} - \varepsilon}.
\]
Choosing \( \varepsilon := \frac{p+4}{2} > 0 \), we gain a positive amount of regularity that does not depend on \( \sigma \). So by a simple iteration we get \( \gamma \in W^{s,2} \) for all \( s \geq 0 \).

\[\square\]

A Product and chain rule

As in [11], we make use of the following results which we briefly state for the readers’ convenience.

**Lemma A.1 (Product rule).** Let \( q_1, \ldots, q_k \in (1, \infty) \) with \( \sum_{i=1}^{k} \frac{1}{q_i} = \frac{1}{r} \in (1, \infty) \) and \( s > 0 \). Then, for \( f_i \in W^{s,q_i}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), i = 1, \ldots, k, \)
\[
\left\| \prod_{i=1}^{k} f_i \right\|_{W^{s,r}} \leq C_{k,s} \prod_{i=1}^{k} \left\| f_i \right\|_{W^{s,q_i}}.
\]

We also refer to Runst and Sickel [45, Lem. 5.3.7/1 (i)]. — For the following statement, one mainly has to treat \( \left\| (D^k \psi) \ast f \right\|_{W^{s,p}} \) for \( k \in \mathbb{N} \cup \{0\} \) and \( \sigma \in (0, 1) \) which is e. g. covered by [45, Thm. 5.3.6/1 (ii)].
Lemma A.2 (Chain rule). Let \( f \in W^{s,p}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), s > 0, p \in (1, \infty) \). If \( \psi \in C^\infty(\mathbb{R}) \) is globally Lipschitz continuous and \( \psi \) and all its derivatives vanish at 0 then \( \psi \circ f \in W^{s,p} \) and
\[
\|\psi \circ f\|_{W^{s,p}} \leq C \|\psi\|_{C^k} \|f\|_{W^{s,p}}
\]
where \( k \) is the smallest integer greater than or equal to \( s \).

B Finite-energy paths are embedded

Let us indicate how to adapt the respective arguments presented in [55, Sect. 2] to get

Theorem B.1 (Embeddedness for \( p \geq q + 2 \) [55, Thm. 1.1, Prop. 4.1]).
Let \( \gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) be parametrized by arc-length with \( TP^{p,q}(\gamma) < \infty \) for \( p \geq q + 2 \). Then the image of \( \gamma \) is a one-dimensional topological manifold, possibly with boundary, embedded in \( \mathbb{R}^n \). In the case that \( p > q + 2 \) this manifold is even of class \( C^{1+\kappa} \) for \( \kappa = \frac{p-q-2}{q+4} \).

To this end it is sufficient to change just a few lines in the proof of [55, Lem. 2.1]. However, we add some more details for the readers’ convenience.

We briefly introduce some notation that will be used in the statements below and refer to [55] for further details. The beta numbers introduced by Jones are defined via
\[
\beta_\gamma(x, r) := \inf \left\{ \sup_{y \in \text{image } \gamma \cap B_r(x)} \frac{\text{dist}(y, G)}{r} \mid G \text{ is a straight line through } x \right\}.
\]
For \( \gamma(s) \neq \gamma(t) \) we denote the straight line through \( \gamma(s) \) and \( \gamma(t) \) by
\[
G(s, t) := \gamma(t) + \mathbb{R} (\gamma(s) - \gamma(t)).
\]
The \( \delta \)-neighborhood of some closed set \( F \) is denoted by
\[
U_\delta(F) := \{ x \in \mathbb{R}^n \mid \text{dist}(x, F) < \delta \}.
\]

Lemma B.2 ([55, Lem. 2.1]). For \( p \geq q + 2 \) there is some \( c_{p,q} > 0 \) such that if \( \gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n), \epsilon \in (0, \frac{1}{10}), \) and \( d \in (0, \text{diam image } \gamma) \) satisfy
\[
e^{q+4} \delta^{2+q-p} \geq c_{p,q} TP^{p,q}(\gamma), \tag{B.1}
\]
then
\[
\text{image } \gamma \cap B_{2d}(\gamma(s)) \subset U_{20d}(G(s, t))
\]
holds for any two points \( \gamma(s), \gamma(t) \) with \( |\gamma(s) - \gamma(t)| = d \). In particular,
\[
\sup_{x \in \text{image } \gamma} \beta_\gamma(x, 2d) \leq 10\epsilon.
\]

Having this lemma, we follow exactly the line of argument in [55]. An immediate consequence of Lemma B.2 is then following corollary, which guarantees a certain decay of Jones’ beta numbers if \( p > q + 2 \):
Corollary B.3 ([55, Cor. 2.2]). For \( p \geq q + 2 \) there are \( \tilde{c}_{p,q}, \delta_{p,q} > 0 \) such that if

\[
\text{TP}^{(p,q)}(\gamma) \frac{1}{\sqrt[4]{\kappa^2}} d^p < \delta_{p,q} \quad \text{where} \quad \kappa = \frac{p-q-2}{q+4}
\]

for \( \gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n) \) and \( 0 < d \ll 1 \), then

\[
\sup_{\text{reim} \gamma} \beta_\gamma(x, 2d) \leq \tilde{c}_{p,q} \text{TP}^{(p,q)}(\gamma) \frac{1}{\sqrt[4]{\kappa^2}} d^p.
\]

Proof of Lemma B.2. As the quantities in the claim do not depend on the actual parametrization of \( \gamma \), we may assume that \( \gamma \) is parametrized by arc-length and set

\[
A_d(s, \varepsilon) := \{ \sigma \in \mathbb{R}/\mathbb{Z} \mid \gamma(\sigma) \in B_{\varepsilon d}(\gamma(s)) \},
\]

\[
X_d(s, t, \varepsilon) := \{ \sigma \in A_d(s, \varepsilon) \mid \exists \gamma'(\sigma) : (\gamma'(\sigma), \gamma(t) - \gamma(s)) \in \left[ \frac{\varepsilon}{10}, \pi - \frac{\varepsilon}{10} \right] \},
\]

\[
N_d(s, t, \varepsilon) := A_d(s, \varepsilon) \setminus X_d(s, t, \varepsilon).
\]

From [55, Eqn. (2.10), (2.11)] we infer for \( \sigma \in X_d(s, t, \varepsilon) \) and \( \tau \in A_d(t, \varepsilon) \)

\[
|\gamma(\sigma) - \gamma(\tau)| \in \left( d(1 - 2\varepsilon^2), d(1 + 2\varepsilon^2) \right), \quad \text{(B.2a)}
\]

\[
\text{dist}((\gamma(\tau), \ell(\sigma))) \geq \frac{ed}{25}. \quad \text{(B.2b)}
\]

From (1.2) we deduce

\[
\frac{1}{\ell_{\gamma}^{(p,q)}(\sigma, \tau)} \geq c(p, q)\varepsilon^d d^{p-2}.
\]

By \( |A_d(s, \varepsilon)| \geq 2\varepsilon^2 d \) we arrive at

\[
\text{TP}^{(p,q)}(\gamma) \geq \int_{X_d(s, t, \varepsilon) \setminus A_d(s, \varepsilon)} \frac{d\sigma d\tau}{\ell_{\gamma}^{(p,q)}(\sigma, \tau)} \geq c(p, q) \left| X_d(s, t, \varepsilon) \right| \varepsilon^{p+2} d^{4q-p}.
\]

As the assumption \( |X_d(s, t, \varepsilon)| \geq \frac{1}{4} \varepsilon^2 d \) is not consistent with (B.1) for a sufficiently large choice of \( c_{p,q} > 0 \), we obtain

\[
|N_d(s, t, \varepsilon)| \geq \frac{3}{2} \varepsilon^2 d. \quad \text{(B.3)}
\]

Supposing \( \gamma(\tau) \in B_{2d}(\gamma(s)) \setminus U_{20d}(G(s, t)) \), \( \sigma \in N_d(s, t, \varepsilon) \), and \( \tau_1 \in A_d(t, \varepsilon) \) yields by [55, Proof of Lemma 2.1, Step 2]

\[
\text{dist}(\gamma(\tau_1), \ell(\sigma)) \geq 18ed
\]

which again gives

\[
\frac{1}{\ell_{\gamma}^{(p,q)}(\sigma, \tau_1)} \geq c(p, q)\varepsilon^d d^{p-2}
\]

and

\[
\text{TP}^{(p,q)}(\gamma) \geq \int_{N_d(s, t, \varepsilon) \setminus A_d(s, \varepsilon)} \frac{d\sigma d\tau_1}{\ell_{\gamma}^{(p,q)}(\sigma, \tau_1)} \geq c(p, q) \left| N_d(s, t, \varepsilon) \right| \varepsilon^{p+2} d^{4q-p}.
\]

Applying (B.3) and increasing \( c_{p,q} \) if necessary, this contradicts (B.1). \( \Box \)
Revisiting the proof of Lemma B.2 we infer as in [55] the following result for the critical case $p = q + 2$:

**Lemma B.4 ([55, Lem. 2.3]).** There is some $c_q > 0$ with

$$
\sup_{x \in \text{image } \gamma} \beta_\gamma(x, 2d) \leq c_q \omega_q(d)
$$

for $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ parametrized by arc-length, $\text{TP}^{q+2,q}(\gamma) < \infty$, and $0 < d \ll 1$, where $\omega_q(d)$ denotes the supremum of

$$
\left( \int_{A \times B} \frac{ds dt}{\text{TP}^{q+2,q}(\gamma)(s, t)} \right)^{1/(q+4)}
$$

taken over all pairs of subsets $A, B \subset \mathbb{R}/\mathbb{Z}$ with $|A|, |B| \leq \frac{d}{100}$.

**Sketch of the proof of Theorem B.1.** According to [55, Theorem 1.4], the image of any arc-length parametrized curve $\gamma \in C^{0,1}(\mathbb{R}/\mathbb{Z}, \mathbb{R}^n)$ with

$$
\sup_{x \in \text{image } \gamma} \beta_\gamma(x, d) \leq \omega(d),
$$

where $\omega : [0, 1] \to [0, \infty)$ is some continuous non-decreasing function with $\omega(0) = 0$, is a one-dimensional submanifold of $\mathbb{R}^n$, possibly with boundary.

If now $p > q + 2$ we get from Lemma B.2 that

$$
\beta_\gamma(x, 2d) \leq Cd^n
$$

from which we deduce following exactly the arguments from [55, Section 4], that the image of $\gamma$ is a submanifold of class $C^\kappa$. □

**References**

[1] A. Abrams, J. Cantarella, J. H. G. Fu, M. Ghomi, and R. Howard. Circles minimize most knot energies. *Topology*, 42(2):381–394, 2003.

[2] W. Alt, D. Felix, Ph. Reiter, and H. von der Mosel. Energetics and dynamics of global integrals modeling interaction between stiff filaments. *J. Math. Biol.*, 59(3):377–414, 2009.

[3] T. Ashton, J. Cantarella, M. Piatek, and E. Rawdon. Self-contact sets for 50 tightly knotted and linked tubes. *Preprint*, 2005.

[4] S. Blatt. Note on continuously differentiable isotopies. *Report* 34, Institute for Mathematics, RWTH Aachen, August 2009.

[5] S. Blatt. The gradient flow of O’Hara’s knot energies. *Preprint*, 2010.

[6] S. Blatt. The energy spaces of the tangent point energies. *Preprint*, 2011.

[7] S. Blatt. A note on integral Menger curvature for curves. *Preprint*, 2011.
[8] S. Blatt. **Boundedness and regularizing effects of O’Hara’s knot energies.** *J. Knot Theory Ramifications*, 21(1):1250010, 9, 2012.

[9] S. Blatt. **The gradient flow of the Möbius energy near local minimizers.** *Calc. Var. Partial Differential Equations*, 43(3-4):403–439, 2012.

[10] S. Blatt and Ph. Reiter. **Does finite knot energy lead to differentiability?** *J. Knot Theory Ramifications*, 17(10):1281 – 1310, 2008.

[11] S. Blatt and Ph. Reiter. **Stationary points of O’Hara’s knot energies.** *manuscripta mathematica*, 2012. DOI: 10.1007/s00229-011-0528-8.

[12] S. Blatt, Ph. Reiter, and A. Schikorra. **Hard analysis meets critical knots (Stationary points of the Möbius energy are smooth).** *ArXiv e-prints*, Feb. 2012.

[13] H. Brezis. **How to recognize constant functions. A connection with Sobolev spaces.** *Uspekhi Mat. Nauk*, 57(4(346)):59–74, 2002.

[14] S. Campanato. Proprietà di hölderianità di alcune classi di funzioni. *Ann. Scuola Norm. Sup. Pisa (3)*, 17:175–188, 1963.

[15] J. Cantarella, J. H. G. Fu, R. Kusner, and J. M. Sullivan. **Ropelength Criticality.** *ArXiv e-prints*, Feb. 2011.

[16] J. Cantarella, R. B. Kusner, and J. M. Sullivan. On the minimum ropelength of knots and links. *Invent. Math.*, 150(2):257–286, 2002.

[17] J. Cantarella, M. Piatek, and E. Rawdon. Visualizing the tightening of knots. In *VIS ’05: Proceedings of the conference on Visualization ’05*, pages 575–582, Washington, DC, USA, 2005. IEEE Computer Society.

[18] M. Carlen and H. Gerlach. **Fourier approximation of symmetric ideal knots.** *Journal of Knot Theory and Its Ramifications*, 21(05):1250057, 2012.

[19] M. Carlen, B. Laurie, J. H. Maddocks, and J. Smutny. Biarcs, global radius of curvature, and the computation of ideal knot shapes. In *Physical and numerical models in knot theory*, volume 36 of *Ser. Knots Everything*, pages 75–108. World Sci. Publ., Singapore, 2005.

[20] M. H. Freedman, Z.-X. He, and Z. Wang. Möbius energy of knots and unknots. *Ann. of Math. (2)*, 139(1):1–50, 1994.

[21] S. Fukuhara. Energy of a knot. In *A fête of topology*, pages 443–451. Academic Press, Boston, MA, 1988.

[22] H. Gerlach. **Ideal Knots and Other Packing Problems of Tubes.** PhD thesis, EPF Lausanne, 2010.

[23] H. Gerlach and H. von der Mosel. **On sphere-filling ropes.** *Amer. Math. Monthly*, 118(10):863–876, 2011.

[24] H. Gerlach and H. von der Mosel. **What are the longest ropes on the unit sphere?** *Archive for Rational Mechanics and Analysis*, 201:303–342, 2011.

[25] O. Gonzalez and R. de la Llave. Existence of ideal knots. *J. Knot Theory Ramifications*, 12(1):123–133, 2003.

29
[26] O. Gonzalez and J. H. Maddocks. Global curvature, thickness, and the ideal shapes of knots. Proc. Natl. Acad. Sci. USA, 96(9):4769–4773 (electronic), 1999.

[27] O. Gonzalez, J. H. Maddocks, F. Schuricht, and H. von der Mosel. Global curvature and self-contact of nonlinearly elastic curves and rods. Calc. Var. Partial Differential Equations, 14(1):29–68, 2002.

[28] O. Gonzalez, J. H. Maddocks, and J. Smutny. Curves, circles, and spheres. In Physical knots: knotting, linking, and folding geometric objects in $\mathbb{R}^3$ (Las Vegas, NV, 2001), volume 304 of Contemp. Math., pages 195–215. Amer. Math. Soc., Providence, RI, 2002.

[29] Z.-X. He. The Euler-Lagrange equation and heat flow for the Möbius energy. Comm. Pure Appl. Math., 53(4):399–431, 2000.

[30] T. Hermes. Analysis of the first variation and a numerical gradient flow for integral Menger curvature. PhD thesis, RWTH Aachen, 2012. http://darwin.bth.rwth-aachen.de/opus3/volltexte/2012/4186.

[31] S. Kolański. Integral Menger curvature for sets of arbitrary dimension and codimension. ArXiv e-prints, 2010 – 2012.

[32] S. Kolański. Geometric Sobolev-like embedding using high-dimensional Menger-like curvature. ArXiv e-prints, May 2012.

[33] S. Kolański, P. Strzelecki, and H. von der Mosel. Characterizing $W^{2,p}$ submanifolds by $p$-integrability of global curvatures. ArXiv e-prints, Mar. 2012.

[34] R. B. Kusner and J. M. Sullivan. Möbius-invariant knot energies. In Ideal knots, volume 19 of Ser. Knots Everything, pages 315–352. World Sci. Publishing, River Edge, NJ, 1998.

[35] H. K. Moffatt. Pulling the knot tight. Nature, 384:114, 1996.

[36] J. O’Hara. Energy of a knot. Topology, 30(2):241–247, 1991.

[37] J. O’Hara. Family of energy functionals of knots. Topology Appl., 48(2):147–161, 1992.

[38] J. O’Hara. Energy functionals of knots. II. Topology Appl., 56(1):45–61, 1994.

[39] J. O’Hara. Energy of knots and conformal geometry, volume 33 of Series on Knots and Everything. World Scientific Publishing Co. Inc., River Edge, NJ, 2003.

[40] E. J. Rawdon and J. K. Simon. Polygonal approximation and energy of smooth knots. J. Knot Theory Ramifications, 15(4):429–451, 2006.

[41] E. J. Rawdon and J. Worthington. Error analysis of the minimum distance energy of a polygonal knot and the Möbius energy of an approximating curve. J. Knot Theory Ramifications, 19(8):975–1000, 2010.

[42] Ph. Reiter. All curves in a $C^1$-neighbourhood of a given embedded curve are isotopic. Report 4, Institute for Mathematics, RWTH Aachen, October 2005.
[43] Ph. Reiter. Regularity theory for the Möbius energy. *Commun. Pure Appl. Anal.*, 9(5):1463–1471, 2010.

[44] Ph. Reiter. Repulsive knot energies and pseudodifferential calculus for O’Hara’s knot energy family $E^{(\alpha)}$, $\alpha \in [2, 3)$. *Math. Nachr.*, 285(7):889–913, 2012.

[45] T. Runst and W. Sickel. *Sobolev spaces of fractional order, Nemytskij operators, and nonlinear partial differential equations*, volume 3 of de Gruyter Series in Nonlinear Analysis and Applications. Walter de Gruyter & Co., Berlin, 1996.

[46] F. Schuricht and H. von der Mosel. Euler-Lagrange equations for nonlinearly elastic rods with self-contact. *Arch. Ration. Mech. Anal.*, 168(1):35–82, 2003.

[47] F. Schuricht and H. von der Mosel. Characterization of ideal knots. *Calc. Var. Partial Differential Equations*, 19(3):281–305, 2004.

[48] J. Smutny. *Global Radii of Curvature, and the Biarc Approximation of Space Curves: In Pursuit of Ideal Knot Shapes*. PhD thesis, EPF Lausanne, 2004.

[49] P. Strzelecki, M. Szumańska, and H. von der Mosel. A geometric curvature double integral of Menger type for space curves. *Ann. Acad. Sci. Fenn. Math.*, 34(1):195–214, 2009.

[50] P. Strzelecki, M. Szumańska, and H. von der Mosel. Regularizing and self-avoidance effects of integral Menger curvature. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, IX(1):145–187, 2010.

[51] P. Strzelecki and H. von der Mosel. Tangent-point repulsive potentials for a class of non-smooth $m$-dimensional sets in $\mathbb{R}^n$. Part I: Smoothing and self-avoidance effects. *Journal of Geometric Analysis*. Published online: 08 December 2011.

[52] P. Strzelecki and H. von der Mosel. On a mathematical model for thick surfaces. In *Physical and numerical models in knot theory*, volume 36 of *Ser. Knots Everything*, pages 547–564. World Sci. Publ., Singapore, 2005.

[53] P. Strzelecki and H. von der Mosel. Global curvature for surfaces and area minimization under a thickness constraint. *Calc. Var. Partial Differential Equations*, 25(4):431–467, 2006.

[54] P. Strzelecki and H. von der Mosel. On rectifiable curves with $L^p$-bounds on global curvature: self-avoidance, regularity, and minimizing knots. *Math. Z.*, 257(1):107–130, 2007.

[55] P. Strzelecki and H. von der Mosel. Tangent-point self-avoidance energies for curves. *ArXiv e-prints*, June 2010. Published in *Journal of Knot Theory and Its Ramifications* 21(05):1250044, 2012.

[56] P. Strzelecki and H. von der Mosel. Integral Menger curvature for surfaces. *Advances in Mathematics*, 226(3):2233–2304, 2011.

[57] J. M. Sullivan. Approximating rope length by energy functions. In *Physical knots: knotting, linking, and folding geometric objects in $\mathbb{R}^3$ (Las Vegas, NV, 2001)*, volume 304 of *Contemp. Math.*, pages 181–186. Amer. Math. Soc., Providence, RI, 2002.