SUBSTITUTIONS TO LEBESGUE TYPE SPACE-FILLING CURVES

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Abstract. Lebesgue curve is a space-filling curve that fills the unit square through linear interpolation. We propose an algorithm to generate a space-filling curve from any given substitution (of any dimension) satisfying a mild condition. The constructed curves are defined by a linear interpolation method, and typically have geometric structures akin to the Lebesgue curve.

1. Introduction

A space-filling curve of the plane is a continuous mapping defined from the unit interval to a subset of the plane that has a positive area. One of the most common examples of a space-filling curve is given by Lebesgue in [2]. Lebesgue’s space-filling curve is formed by linear interpolation over the map \( \phi : \Gamma_c \mapsto [0, 1] \times [0, 1] \) given in (1.1), where \( \Gamma_c \) is the middle-third Cantor set. The map \( \phi \) is defined by the ternary representations of the points in the Cantor set, and is a continuous surjection onto the unit square.

\[(1.1) \quad \phi (0.3 (2 \cdot x_1)(2 \cdot x_2)(2 \cdot x_3) \ldots) = \begin{bmatrix} 0.2 x_1x_3x_5 \ldots \cr 0.2 x_2x_4x_6 \ldots \end{bmatrix}.
\]

The geometric interpretation of the Lebesgue’s curve is clarified by Sagan [3, 4], by means of approximating polygons; a notion defined by Wunderlich [6]. These approximating polygons are analogous to the iteration steps of the Morton order, first three of which are depicted in Figure 1. Using the approximating polygons, Sagan provided another proof in his book [5], indicating that the Lebesgue curve is a continuous surjection onto the unit square. This proof can be thought as a geometric construction of the Lebesgue curve.

**Figure 1.** First tree iterations of the Morton order
A tile $t$ consists of a subset of $\mathbb{R}^n$ and an assigned colour label. We denote the associated subset by $\text{supp} \ t$ and the label by $l(t)$. We assume that for every tile $t$, $\text{supp} \ t$ is homeomorphic to the closed unit disc. A substitution is a map defined over a collection of tiles such that it expands every tile by a fixed factor greater than 1 and divides each expanded tile into pieces, each of which is a translation of a tile. In this study we introduce an algorithm to produce space-filling curves from substitutions. In particular, for any given substitution satisfying a mild condition, we form a space-filling curve by mimicking the geometric construction of the Lebesgue’s curve. The main result of this study is the following theorem. The space-filling curves generated through this theorem are demonstrated at the end of the paper.

**Theorem 1.1.** Let $\mathcal{P}$ be a given finite collection of tiles in $\mathbb{R}^n$ for $n > 2$. Suppose $\omega$ is a substitution defined over $\mathcal{P}$ such that $\max \{ \text{diam}(t) : t \in \omega^n(p), \ p \in \mathcal{P} \} \to 0$ as $n \to \infty$, where $\text{diam}(t)$ denotes the diameter of $\text{supp} \ t$. Then for each $p \in \mathcal{P}$ there exists a Cantor set $\Gamma_{\omega,p} \subseteq [0,1]$ and a continuous surjection $f_{\omega,p} : \Gamma_{\omega,p} \to \text{supp} \ p$ such that $f_{\omega,p}$ extends to a space-filling curve $F_{\omega,p} : [0,1] \to \text{supp} \ p$ by linear interpolation.

The organisation of this paper is as follows. In Section 2 we provide the relevant preliminary definitions, and an example of a space-filling curve constructed through a substitution. In Section 3 we prove the main result, Theorem 1.1, and explain how the theorem can be regarded as an algorithm. Lastly, in Section 4 we present examples of space-filling curves which are formed by some of the known substitutions (or their variations).

2. **Methodology**

In this section we explain the main result of the paper with an example in detail.

2.1. **Substitutions.** Let $\mathbb{R}^n$ denote the $n$-dimensional Euclidean space for some $n \in \mathbb{Z}^+$. We define the following:

1. A tile $t$ consists of a subset of $\mathbb{R}^n$ that is homeomorphic to the closed unit disc, and a colour label $l(t)$ that distinguishes $t$ from any other identical sets. The associated subset of $t$ is denoted by $\text{supp} \ t$. We call $\text{supp} \ t$ the support of $t$.

2. For a given tile $t$ and a vector $x \in \mathbb{R}^n$ we obtain a new tile $t+x$ by the relations $\text{supp} \ (t+x) = (\text{supp} \ t) + x$ and $l(t+x) = l(t)$. We say $t+x$ is a translation of $t$.

3. A patch $P$ is a finite collection of tiles so that
   (i) $\bigcup_{t \in P} \text{supp} \ t$ is homeomorphic to the closed unit disc,
   (ii) $\text{int} \ (\text{supp} \ t) \cap \text{int} \ (\text{supp} \ t') = \emptyset$ for each distinct $t, t' \in P$.
   The support of a patch $P$ is the union of supports of its tiles. It is denoted by $\text{supp} \ P$, i.e. $\text{supp} \ P = \bigcup_{t \in P} \text{supp} \ t$.

**Definition 2.1.** Suppose $\mathcal{P}$ is a given collection of tiles. Let $\mathcal{P}^*$ denote the set of all patches consisting of tiles that are translations of tiles in $\mathcal{P}$. A map $\omega : \mathcal{P} \to \mathcal{P}^*$ is called a substitution if there exists $\lambda > 1$ such that $\text{supp} \ \omega(p) = \lambda \cdot \text{supp} \ \omega(p)$ for all $p \in \mathcal{P}$.

We call $\lambda$ the expansion factor of $\omega$. We say that $\omega$ is a finite substitution if, in addition, $\mathcal{P}$ has a finite size.
**Definition 2.2.** For a given substitution $\omega : \mathcal{P} \mapsto \mathcal{P}^*$, the patch $\omega^n(p)$ for $p \in \mathcal{P}$ and $n \in \mathbb{Z}^+$ is called an $n$-supertile of $\omega$.

An example of a substitution is given in Figure 2. It is defined over the two intervals $[0, (1 + \sqrt{5})/2]$ and $[0, 1]$ with labels $a, b$, respectively. The interval $[0, (1 + \sqrt{5})/2]$ with label $a$ is substituted into a patch consisting of two intervals $[0, (1 + \sqrt{5})/2]$ and $[(1 + \sqrt{5})/2, (3 + \sqrt{5})/2]$ with labels $a, b$, respectively. The interval $[0, 1]$ with label $b$ is substituted into a patch consisting of a single interval $[0, (1 + \sqrt{5})/2]$ with label $a$. This substitution is called the **Fibonacci substitution**. The expansion factor for the Fibonacci substitution is the golden mean $(1 + \sqrt{5})/2$. By taking the Cartesian product of two Fibonacci substitutions, we get another substitution, which is illustrated in Figure 3. Throughout the document we denote the Fibonacci substitution by $\mu$, the Cartesian product of two Fibonacci substitutions by $\nu$ and their associated domains by $\mathcal{P}_\mu$, $\mathcal{P}_\nu$, respectively. The domain $\mathcal{P}_\nu$ consists of four tiles $p_a, p_b, p_c, p_d$ such that $p_i$ is the tile with $l(p_i) = i$ for $i \in \{a, b, c, d\}$.

**Remark 2.3** (Powers of substitutions). Let $\omega : \mathcal{P} \mapsto \mathcal{P}^*$ be a given substitution with an expansion factor $\lambda$. The range of a substitution $\omega$ does not necessarily contain its domain. That is, $\omega^2$ may be not well-defined. Therefore, we define an extension

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The tile $p_b$ and $p_c$ are rotations of one another.
\( \omega' : \mathcal{P} + \mathbb{R}^n \mapsto \mathcal{P}^* \) by \( \omega'(p + x) = \omega(p) + \lambda \cdot x \) for \( p \in \mathcal{P} \) and \( x \in \mathbb{R}^n \). The powers of \( \omega' \) are well-defined. We use the maps \( \omega \) and \( \omega' \) interchangeably, when there is no confusion of taking powers of the substitutions.

2.2. Total Order Structures by \( \nu \). A fundamental ingredient of Sagan’s geometric construction of the Lebesgue curve is the approximating polygons. For that we define total orders over the supertiles of \( \nu \). These total order structures are the key feature to form such approximating polygons.

Assume that \( \nu(p_a) = \{a_1, a_2, a_3, a_4\} \), \( \nu(p_b) = \{b_1, b_2\} \), \( \nu(p_c) = \{c_1, c_2\} \) and \( \nu(p_d) = \{d_1\} \), as shown in Figure 4. The relations given in (2.1) - (2.4) indicate a total order \( \preceq \) on the collection \( \nu(p_i) \) for each \( i \in \{a, b, c, d\} \), respectively.

\[
\begin{align*}
(2.1) & \quad a_j \preceq_{a,1} a_k \text{ if } j \leq k \text{ for each } j, k \in \{1, 2, 3, 4\}.
(2.2) & \quad b_j \preceq_{b,1} b_k \text{ if } j \leq k \text{ for each } j, k \in \{1, 2\}.
(2.3) & \quad c_j \preceq_{c,1} c_k \text{ if } j \leq k \text{ for each } j, k \in \{1, 2\}.
(2.4) & \quad d_j \preceq_{d,1} d_k \text{ if } j \leq k \text{ for each } j, k \in \{1\}.
\end{align*}
\]

\[\text{Figure 4. The patches } \nu(p_a), \nu(p_b), \nu(p_c) \text{ and } \nu(p_d), \text{ from left to right.}\]

Note that \( \nu^m(p) = \bigcup_{t \in \nu(p)} \omega^{m-1}(t) \) for each \( m > 1 \) and \( p \in \mathcal{P}_\nu \). Therefore, the relations in (2.1) - (2.4) induce total orders for supertiles of \( \nu \) inductively. More precisely, suppose \( i \in \{a, b, c, d\} \) and \( m > 1 \) are fixed. Suppose further \( \preceq \) is a total order on the collection \( \nu^{m-1}(p_i) \). Define \( \preceq \) over the patch \( \nu^m(p_i) \) such that:

(i) If \( x, y \in \nu^{m-1}(t) \) for some \( t \in \nu(p_i) \), then \( x \preceq_{i,m} y \) whenever \( x \preceq_{i,m-1} y \).

(ii) If \( x \in \nu^{m-1}(t_1) \) and \( y \in \nu^{m-1}(t_2) \) for some distinct \( t_1, t_2 \in \nu(p_i) \), then \( x \preceq_{i,m} y \) whenever \( t_1 \preceq_{i,1} t_2 \).

The total orders between the tiles of 1-supertiles and 2-supertiles of \( \nu \) are shown in Figure 5. The associated order structures are elucidated by the numbers attached to the tiles in the figure.
2.3. **Partitions of the Unit Square by** $\nu$. The substitution $\nu$ induces a sequence $\{\lambda^{-k-1} \cdot \nu^k(p_a) : k = 0, 1, 2, \ldots\}$ of partitions of the unit square. In particular, $\lambda^{-k-1} \cdot (\text{supp} \nu^k(p_a)) = [0, 1] \times [0, 1]$ for every $k \in \mathbb{N}$. Each partition is a scaled copy of a supertile of $\nu$. Thus, the total orders introduced in Section 2.2 can be transferred over the rectangles in the partitions. We use these total orders in order to label the rectangles appearing in the partitions.

The partitions $\lambda^{-1} \cdot p_a, \lambda^{-2} \cdot \nu(p_a)$ and $\lambda^{-3} \cdot \nu^2(p_a)$ are demonstrated in Figure 6. For every $k \in \mathbb{N}$, $\lambda^{-k-1} \cdot \nu^k(p_a)$, consists of $\mathcal{F}^{k+2}_{k+2}$ rectangles, where $\mathcal{F}^{k+2}_{k+2}$ is the $(k + 2)$-th Fibonacci number. Denote these rectangles by $\mathcal{J}_k^1, \ldots, \mathcal{J}_k^{\mathcal{F}^{k+2}_{k+2}}$ such that

(i) $\bigcup_{i=1}^{\mathcal{F}^{k+2}_{k+2}} \lambda^{k+1} \cdot \mathcal{J}_k^i = \nu^k(p_a),$

(ii) $\lambda^{k+1} \cdot \mathcal{J}_k^i \lesssim_a \lambda^{k+1} \cdot \mathcal{J}_k^j$ if and only if $i \leq j$, for every $i, j \in \{1, \ldots, \mathcal{F}^{k+2}_{k+2}\}$.

The rectangles $\mathcal{J}_1^1, \mathcal{J}_1^1, \ldots, \mathcal{J}_1^1$ and $\mathcal{J}_2^1, \ldots, \mathcal{J}_2^9$ are demonstrated in Figure 7.

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**Figure 5.** The total orders between the tiles of 1-supertiles and 2-supertiles of $\nu$.

**Figure 6.** $\lambda^{-1} \cdot p_a, \lambda^{-2} \cdot \nu(p_a), \lambda^{-3} \cdot \nu^2(p_a)$, from left to right.
2.4. A Construction of a Cantor Set by $\nu$. We define a Cantor set according to the iteration rules shown in Figure 8. The rules demonstrate how to subdivide each interval type. More precisely, let $Q = \{a, b, c, d\}$ be a given set of labels and let $S = \{[x, y] : x, y \in \mathbb{R} \text{ with } x < y\}$ denote the set of non-trivial closed intervals in $\mathbb{R}$. Define

$$S^* = \left\{ \bigcup_{j=1}^{n} [x_j, y_j] : n \in \mathbb{Z}^+ \text{ and } x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n \right\},$$

$$R = \left\{ \bigcup_{j=1}^{n} [(I_j, p_j)] : \bigcup_{j=1}^{n} I_j \in R \text{ and } p_j \in Q \text{ for all } j \in \{1, \ldots, n\} \right\}.$$
The rules in Figure 8 can be characterised as a map \( \Psi : S \times Q \mapsto S^* \) by:

\[
\Psi([x, y], a) = \{(x, x_1), ([x_2, x_3], b), ([x_4, x_5], c), ([x_6, y], d)\}
\]

where \( x_i = x + \frac{y - x}{7} \cdot i \) for each \( i = 1, \ldots, 6 \),

\[
\Psi([x, y], b) = \left\{ \left( \left[ x, x + \frac{y - x}{3} \right], a \right), \left( \left[ x + \frac{2(y - x)}{3}, y \right], c \right) \right\},
\]

\[
\Psi([x, y], c) = \left\{ \left( \left[ x, x + \frac{y - x}{3} \right], a \right), \left( \left[ x + \frac{2(y - x)}{3}, y \right], b \right) \right\},
\]

\[
\Psi([x, y], d) = \left\{ \left( \left[ x + \frac{y - x}{2}, y \right], a \right) \right\},
\]

where \( x, y \in \mathbb{R} \) with \( x < y \). Extend \( \Psi \) to \( R \) by \( \Psi \left( \bigcup_{j=1}^{n} \{(I_j, p_j)\} \right) := \bigcup_{j=1}^{n} \Psi(I_j, p_j) \).

Define also \( \Phi : R \mapsto S^* \) by \( \Phi \left( \bigcup_{j=1}^{n} \{(I_j, p_j)\} \right) = \bigcup_{j=1}^{n} I_j \) for \( \bigcup_{j=1}^{n} \{(I_j, p_j)\} \in R \). Observe that \( \Gamma = \lim_{n \to \infty} \Phi (\Psi^n ([0, 1], a)) \) is a Cantor set.

\[
\begin{array}{c}
| & & & & & & & & \text{0} & & \text{1} \\
0 & & & & & & & & \text{1} & & \text{0} \\
\hline
\text{0} & & & & & & & & \text{1} & & \text{0} \\
\hline
\text{0} & & & & & & & & \text{1} & & \text{0} \\
\hline
0 & & & & & & & & \text{1} & & \text{0} \\
\hline
\end{array}
\]

\text{Figure 9. Construction steps of } \Gamma

For each \( n \in \mathbb{N} \), there exist \( \mathcal{F}_{k+2} \) many disjoint intervals in the \( n \)-th step\(^2\) of the construction of \( \Gamma \). Denote these intervals by \( \mathcal{I}_k^1, \ldots, \mathcal{I}_k^{2^{k+2}} \), from left to right respectively. The intervals \( \mathcal{I}_0^1, \mathcal{I}_1^1, \ldots, \mathcal{I}_1^4 \) and \( \mathcal{I}_2^1, \ldots, \mathcal{I}_2^2 \) are demonstrated in Figure 10.

2.5. A Space-Filling Curve by \( \nu \). There exist bijective correspondences between the intervals appearing in the construction steps of \( \Gamma \) and the rectangles in the partitions of the unit square. More precisely, for each \( n \in \mathbb{N} \), define \( f_n : \{\mathcal{I}_k^1 : k = 1, \ldots, \mathcal{F}_{n+2}\} \mapsto \{\mathcal{J}_n^k : k = 1, \ldots, \mathcal{F}_{n+2}^2\} \) by \( f_n(\mathcal{I}_k^1) = \mathcal{J}_n^k \) for \( k = 1, \ldots, \mathcal{F}_{n+2}. \) For every \( x \in \Gamma \) and

\( ^2\)With the convention that \( [0,1] \) is the 0-th step.
$n \in \mathbb{N}$, there exists $k_n \in \{1, \ldots, F_{n+2}^2\}$ such that $x \in I_{n}^{k_n}$. In particular, $\{x\} = \bigcap_{n=1}^{\infty} I_{n}^{k_n}$ by Cantor’s intersection theorem. By the same token, there exists $y \in [0, 1] \times [0, 1]$ such that $\{y\} = \bigcap_{n=1}^{\infty} J_{n}^{k_n}$. This process induces a surjection $f : \Gamma \mapsto [0, 1] \times [0, 1]$ such that $f(x) = y$ where $x$ and $y$ are as defined above. Next we prove that $f$ is continuous.

For each $n \in \mathbb{Z}^+$, define the following:

$$g_n = \min_{j \in \{1, \ldots, F_{n+2}^2\}} \{l(I_{n}^{j}) : l(I_{n}^{j}) \text{ denotes the length of } I_{n}^{j}\},$$

$$h_n = \max_{t \in \omega^n(p)} \{\text{diam } t : \text{diam } t \text{ denotes the diameter of supp } t\}.$$

We have that $(g_n, h_n) \to (0, 0)$ as $n \to \infty$. Let $x \in \Gamma$ be fixed and let $\epsilon > 0$ be given. Choose $N \in \mathbb{N}$ sufficiently large so that $h_N < \epsilon$. Set $\delta = g_N/2$. If $y \in \Gamma$ with $|x - y| < \delta$ then $x, y \in I_N^{j_0}$ for some $j_0 \in \{1, \ldots, F_{n+2}^2\}$. That is, $f(x), f(y) \in J_N^{j_0}$ and $||f(x) - f(y)|| < \epsilon$. Thus, $f$ is continuous, and extends to a space-filling curve $F : [0, 1] \mapsto [0, 1] \times [0, 1]$ by linear interpolation.

A Geometrisation of $F$. For each $n \in \mathbb{Z}^+$, puncture the centre of mass in every rectangle in $\lambda^{-n-1} \cdot \nu^n(p_0)$. Denote these points by $x(J_n^j)$ for $j = 1, \ldots, F_{n+2}^2$. Join the punctures $x(J_n^1), x(J_n^2), \ldots, x(J_n^{F_{n+2}^2})$ with straight lines, respectively. The constructed directed curve is called the $n$-th approximant of $F$. First four approximants of $F$ are shown in Figure 11.
3. The Algorithm

Proof of Theorem 1.1 Let \( \mathcal{P} \) be a finite collection of tiles. Suppose \( \omega : \mathcal{P} \mapsto \mathcal{P}^* \) is a substitution with an expansion factor \( \lambda > 1 \) such that \( \max \{ \text{diam}(t) : t \in \omega^n(q), \ q \in \mathcal{P} \} \to 0 \) as \( n \to \infty \). Assume without loss of generality \( \omega \) is a substitution in the plane and \( |\omega(q)| > 1 \) for all \( q \in \mathcal{P} \). Let \( p \in \mathcal{P} \) be fixed.

Define a bijection \( g_q : \omega(q) \mapsto \{1, \ldots, |\omega(q)|\} \), for every \( q \in \mathcal{P} \). For \( q \in \mathcal{P} \), \( g_q \) defines a total order \( \preceq_q \) between the tiles of \( \omega(q) \) such that \( x \preceq_q y \) whenever \( g_q(x) \leq g_q(y) \).

Extend these total orders to the supertiles of \( \omega \) inductively such that

(i) If \( x, y \in \omega^{n-1}(t) \) for some \( t \in \omega(q) \), then \( x \preceq_{q,n} y \) whenever \( x \preceq_{q,n-1} y \).
(ii) If \( x \in \omega^n(t_1) \) and \( y \in \omega^n(t_2) \) for some distinct \( t_1, t_2 \in \omega(q) \), then \( x \preceq y \) whenever \( t_1 \preceq t_2 \).

The total order \( \preceq \), for \( q \in \mathcal{P} \) and \( n \in \mathbb{Z}^+ \), can be transferred over the scaled patch \( \lambda^{-n} \cdot \omega^n(q) \). For each \( q \in \mathcal{P} \) and \( n \in \mathbb{Z}^+ \), label the scaled tiles in \( \lambda^{-n} \cdot \omega^n(q) \) by \( \mathcal{J}_{q,n}^1, \mathcal{J}_{q,n}^2, \ldots, \mathcal{J}_{q,n}^{q_n} \) such that \( \mathcal{J}_{q,n}^i \preceq \mathcal{J}_{q,n}^j \) if and only if \( i \leq j \).

Next we construct a Cantor set \( \Gamma_{\omega,p} \). Define the following:
\[
S = \{ [x, y] : x, y \in \mathbb{R} \text{ with } x < y \},
\]
\[
S^* = \left\{ \bigcup_{j=1}^{n} [x_j, y_j] : n \in \mathbb{Z}^+ \text{ and } x_1 < y_1 < x_2 < y_2 < \cdots < x_n < y_n \right\},
\]
\[
R = \left\{ \bigcup_{j=1}^{n} (I_j, q_j) \right\} : \bigcup_{j=1}^{n} I_j \in R \text{ and } q_j \in \mathcal{P} \text{ for all } j \in \{1, \ldots, n\} \right\}.
\]

Consider the map \( \Psi : S \times \mathcal{P} \mapsto S^* \) defined by \( \Psi([x, y], q) = \bigcup_{j=0}^{\lfloor \omega(q) \rfloor - 1} [x_{2j}, x_{2j+1}] \) where \( x_k = x + \frac{y-x}{2^{|\omega(q)|}} \cdot k \) for \( k = 0, 1, \ldots, 2^{|\omega(q)|} - 1 \). Extend \( \Psi \) to \( R \) by \( \Psi \left( \bigcup_{j=1}^{n} (I_j, q_j) \right) := \bigcup_{j=1}^{n} \Psi(I_j, q_j) \).

Define also \( \Phi : R \mapsto S^* \) by \( \Phi \left( \bigcup_{j=1}^{n} (I_j, q_j) \right) := \bigcup_{j=1}^{n} I_j \). Observe that \( \Gamma_{\omega,p} = \lim_{n \to \infty} \Phi (\Psi^n ([0,1], p)) \) is a Cantor set.

Denote the intervals appearing in the n-th step of the construction of \( \Gamma_{\omega,p} \) by \( \mathcal{I}_{p,n}^1, \ldots, \mathcal{I}_{p,n}^{\omega^n(p)} \), from left to right respectively. For each \( n \in \mathbb{Z}^+ \), define \( f_{p,n} : \{ \mathcal{I}_{p,n}^k : k = 1, \ldots, \omega^n(p) \} \mapsto \{ \mathcal{J}_{p,n}^k : k = 1, \ldots, \omega^n(p) \} \) by \( f_{p,n}(\mathcal{I}_{p,n}^k) = \mathcal{J}_{p,n}^k \) for \( k = 1, \ldots, \omega^n(p) \). For each \( x \in \Gamma_{\omega,p} \) and \( n \in \mathbb{N} \), there exists \( k_{p,n} \in \{1, \ldots, \omega^n(p)\} \) such that \( x \in \mathcal{I}_{p,n}^{k_{p,n}} \). In particular, \( \{x\} = \bigcap_{n=1}^{\infty} \mathcal{I}_{p,n}^{k_{p,n}} \) by Cantor’s intersection theorem.

By the same token, there exists \( y \in \text{supp } p \) such that \( \{y\} = \bigcap_{n=1}^{\infty} \mathcal{J}_{p,n}^{k_{p,n}} \). This process induces a surjection \( f_{\omega,p} : \Gamma_{\omega,p} \mapsto \text{supp } p \) such that \( f_{\omega,p}(x) = y \) where \( x \) and \( y \) are as defined above. Next we prove that \( f_{\omega,p} \) is continuous.

For each \( n \in \mathbb{Z}^+ \), define the following:
\[
g_{p,n} = \min_{j \in \{1, \ldots, \omega^n(p)\}} \{ l(\mathcal{I}_{p,n}^j) : l(\mathcal{I}_{p,n}^j) \text{ denotes the length of } \mathcal{I}_{p,n}^j \},
\]
\[
h_{p,n} = \max_{t \in \text{supp } p} \{ \text{diam } t : \text{diam } t \text{ denotes the diameter of supp } t \}.
\]

We have that \( (g_{p,n}, h_{p,n}) \to (0,0) \) as \( n \to \infty \). Choose \( x \in \Gamma_{\omega,p} \) and \( \epsilon > 0 \). Pick \( N_p \in \mathbb{N} \) sufficiently large so that \( h_{p,N_p} < \epsilon \). Set \( \delta = \frac{g_{p,N_p}}{2} \). If \( y \in \Gamma_{\omega,p} \) with \( |x - y| < \delta \) then \( x, y \in \mathcal{I}_{p,N_p}^{j_0} \) for some \( j_0 \in \{1, \ldots, \omega^{N_p}(p)\} \). That is, \( f(x), f(y) \in \mathcal{J}_{p,N_p}^{j_0} \) and

\(^3\text{With the convention that } [0,1] \text{ is the 0-th step.}\)
defined over two unit squares with labels $A,B$. A substitution is called 2-dimensional Thue-Morse substitution (Lebesgue curve)

$n$-th approximant of $F$ that $f$ extends to a space-filling curve $F_{\omega,p} : [0,1] \mapsto \text{supp } p$ by linear interpolation. ☐

**Corollary 3.1.** Let $\mathcal{P}$ be a given finite collection of tiles in $\mathbb{R}$. Suppose further $\omega : \mathcal{P} \mapsto \mathcal{P}^*$ is a substitution such that $\lim_{n \to \infty} \max \{ l(t) : t \in \omega^n(p), \ p \in \mathcal{P} \} = 0$, where $l(t)$ is the length of the interval $\text{supp } t$. Then for each $p \in \mathcal{P}$ there exists a Cantor set $\Gamma_{\omega,p} \subseteq [0,1]$ and a continuous surjection $f_{\omega,p} : \Gamma_{\omega,p} \mapsto \text{supp } \omega(p) \times \text{supp } \omega(p)$ such that $f_{\omega,p}$ extends to a space-filling curve $F_{\omega,p} : [0,1] \mapsto \text{supp } \omega(p) \times \text{supp } \omega(p)$ by linear interpolation.

Steps of the Algorithm. For every finite substitution $\omega : \mathcal{P} \mapsto \mathcal{P}^*$ satisfying the condition in Theorem 1.1 and a tile $p \in \mathcal{P}$, the following steps form a space-filling curve.

**Step – 1 :** Choose $k \in \mathbb{Z}^+$ such that $|\omega^k(q)| > 1$ for every $q \in \mathcal{P}$.

**Remark 3.2.** Replace $\omega$ with $\omega^k$ for the following steps. We assume without loss of generality $k = 1$ for the following steps.

**Step – 2 :** Define a bijection $g_q : \omega(q) \mapsto \{1, \ldots, |\omega(q)|\}$, for all $q \in \mathcal{P}$.

**Step – 3 :** The maps $\{g_q : q \in \mathcal{P}\}$ indicate total orders over the supertiles of $\omega$. Label the scaled tiles in $\lambda^{-n} \cdot \omega^n(p)$ for each $p \in \mathcal{P}$ and $n \in \mathbb{Z}^+$, according to the generated total order structures.

**Step – 4 :** Construct a Cantor set $\Gamma_{\omega,p}$.

**Step – 5 :** Define a bijection between the intervals appearing in the $n$-th construction step of $\Gamma_{\omega,p}$ and the scaled tiles in the collection $\lambda^{-n} \cdot \omega^n(p)$, for each $n \in \mathbb{Z}^+$.

**Step – 6 :** Construct a continuous surjection $f_{\omega,p} : \Gamma_{\omega,p} \mapsto \text{supp } p$ using the bijective correspondences described in Step - 5.

**Step – 7 :** Construct a space-filling curve $F_{\omega,p} : [0,1] \mapsto \text{supp } p$ by linear interpolation over $f_{\omega,p}$.

4. **Examples**

In this section we apply the algorithm given in Section 3 to some of the known substitutions. The substitutions provided in this section, as well as a vast collection of other substitutions, can be found at [1]. The generated space-filling curves are elucidated by their associated approximants.

**Definition 4.1.** Let $F_{\omega,p}$ be a space filling curve constructed by the algorithm in Section 3. With the same notations in the proof of Theorem 1.1, for each $n \in \mathbb{Z}^+$ and $j \in \{1, \ldots, |\omega^n(p)|\}$, denote the centre of mass of $J_{p,n}^j$ by $x(J_{p,n}^j)$. The directed curve formed by joining the points $x(J_{p,n}^1), x(J_{p,n}^2), \ldots, x(J_{p,n}^{|\omega^n(p)|})$ successively is called the $n$-th approximant of $F_{\omega,p}$.

**Example 4.2** (Lebesgue curve). Consider the substitution given in Figure 13. The substitution is called 2-dimensional Thue-Morse substitution (2DTM in short). It is defined over two unit squares with labels $A,B$. The expansion factor for this substitution is 2. Choose any of the tiles given in Figure 13 to input in the algorithm. Define an order structure over the 1-supertiles of 2DTM through the curves depicted in Figure 14 according to which tile is visited first by the curves. The associated orders...
are described by the numbers attached to the tiles in Figure 15. Then the space-filling curve generated by the algorithm is nothing but the Lebesgue curve.

![Figure 13. 2-dimensional Thue-Morse substitution](image)

![Figure 14. Total orders defined through curves for the 1-supertiles of 2DTM](image)

![Figure 15. Total orders over 1-supertiles of 2DTM](image)

For the rest of the examples we explain total order structures through directed curves. The associated total orders are defined according to which tile is visited first.

**Example 4.3 (2DTM curve).** Define another order structure over the 1-supertiles of 2DTM by the curves shown in Figure 16. There are two space-filling curves that can be produced by the algorithm. Choose the tile with the label \(A\). Let \(F_{tm}^A\) denote the space-filling curve formed by the algorithm from this tile. First four approximants of \(F_{tm}^A\) are shown in Figure 17.

![Figure 16. Total orders over 1-supertiles of 2DTM](image)
Example 4.4 (Square-Chair curve). The substitution given in Figure 13 is called the Square-Chair substitution. It is defined over four unit square tiles with an expansion factor 2. The curves in Figure 20 indicate total orders over the 1-superilites of the Square-Chair substitution. Choose the tile with the label $A$. Let $F^A_{sc}$ denote the space-filling curve generated by the algorithm from this tile. First four approximants of $F^A_{sc}$ are shown in Figure 21.

Figure 17. First four approximants of $F^A_{tm}$

Figure 18. 4th approximants of $F^A_{tm}$ and the Lebesgue curve, respectively

Figure 19. The Square-Chair substitution
Example 4.5 (Equithirds-variant). Consider the substitution depicted in Figure 22. It is defined over four tiles and their rotations, which are constructed by two different shapes, an equilateral triangle with side length 1 and an isosceles triangle with side lengths $1, 1, \sqrt{3}$. Its expansion factor is $\sqrt{3}$. The curves shown in Figure 23 describe total orders over its 1-supertiles. Let $F_{eq}^{i}$ denote the space-filling curve produced by the algorithm from the tile with the label $i$ for $i \in \{A^+, A^-, B^+, B^-\}$. First four approximants of $F_{eq}^{A^+}$ are shown in Figure 24 and first four approximants of $F_{eq}^{B^+}$ are shown in Figure 25. Observe that $F_{eq}^{B^+}(0) = F_{eq}^{B^+}(1)$. So, for illustration purposes, we modify the approximants of $F_{eq}^{B^+}$ to be closed curves. We connect the end points of its approximant curves with a straight line and fill the associated closed regions as demonstrated in Figure 26 and Figure 27.

Figure 20. Total orders defined over the 1-supertiles of the Square-Chair substitution

Figure 21. First four approximants of $F_{eq}^{A^+}$

Figure 22. Equithirds-variant
Figure 23. Total orders over the 1-supertiles of Equithirds-variant

Figure 24. First four approximants of $F_{eq}^{A^+}$

Figure 25. First four approximants of $F_{eq}^{B^+}$

Figure 26. 1st, 3rd, 5th and 7th approximants of $F_{eq}^{B^+}$ where the end points of the approximants are joined with a line and the associated closed regions are filled.

Figure 27. 2nd, 4th, 6th and 8th approximants of $F_{eq}^{B^+}$ where the end points of the approximants are joined with a line and the associated closed regions are filled.
Next we describe the geometry of approximants of $F_{eq}^{A^+}$. Let $p_{A^+}$ denote the rotated version of $p_{A^+}$ by $\pi$ such that their longer edges merge. Also, denote the space filling curves generated by these two tiles by $F_{eq}^{A^+}, F_{eq}^{A^+_r}$. Since $F_{eq}^{A^+}(0) = F_{eq}^{A^+_r}(1)$ and $F_{eq}^{A^+}(1) = F_{eq}^{A^+_r}(0)$, we can concatenate $F_{eq}^{A^+}$ with $F_{eq}^{A^+_r}$ in order to define another space-filling curve $F_{eq}^A$ so that $F_{eq}^A(0) = F_{eq}^A(1)$. Once again, we modify the approximants of $F_{eq}^A$ to be closed curves as in Figure 28.

**Figure 28.** First four approximants of $F_{eq}^A$ where their end points are joined with a line.

**Figure 29.** 1st, 3rd, 5th and 7th approximants of $F_{eq}^A$ where the end points of the approximants are joined with a line and the associated closed regions are filled.

**Figure 30.** 2nd, 4th, 6th and 8th approximants of $F_{eq}^A$ where the end points of the approximants are joined with a line and the associated closed regions are filled.

**Example 4.6** (Pinwheel-variant). The substitution in Figure 31 is defined over four tiles and their rotations. Every tile in the substitution is a right triangle with side lengths 1, 2, $\sqrt{5}$. The expansion factor for this substitution is $\sqrt{5}$. Let $R_1, R_2$ denote the regions, rhombus and rectangle, shown in Figure 33. The algorithm produces two space-filling curves $F_{pin}^1, F_{pin}^2$ over $R_1, R_2$, respectively, from the total orders given in Figure 32. Figure 34 indicates the first two approximants of $F_{pin}^1$ and $F_{pin}^2$. Observe that $F_{pin}^i(0) = F_{pin}^i(1)$ for $i = 1, 2$. 
Figure 31. Pinwheel-variant substitution

Figure 32. Total orders over 1-supertiles of the Pinwheel-variant substitution

Figure 33. The regions $R_1$ and $R_2$, from left to right respectively

Figure 34. First two approximants of $F_{pin}^1$ and $F_{pin}^2$

Figure 35. 1st, 3rd and 5th approximants of $F_{pin}^1$ - filled version
Example 4.7 (Penrose-Robinson-variant). Start with the substitution given in Figure 39. Its domain consists of 12 tiles and their rotations, which are congruent copy of two different shapes, an isosceles triangle with side lengths 1, 1, \((1 + \sqrt{5})/2\) and an isosceles triangle with side lengths 1, 1, \((\sqrt{5} - 1)/2\). The expansion factor for this substitution is the golden mean \((1 + \sqrt{5})/2\). Then the total orders described in Figure 40 induce space-filling curves \(F_{\text{star}}, F_{\text{deca}}\) that fill the supports of the patches shown in Figure 41 respectively, such that \(F_{\text{star}}(0) = F_{\text{star}}(1)\) and \(F_{\text{deca}}(0) = F_{\text{deca}}(1)\).
Figure 40. Total orders over 1-supertiles of the Penrose-Robinson-variant substitution

Figure 41. A star and a decagon.

Figure 42. First four approximants of $F_{\text{deca}}$ - filled version

Figure 43. First four approximants of $F_{\text{star}}$ - filled version
Figure 44. 6th approximants of $F_{\text{star}}$ and $F_{\text{deca}}$, from left to right.

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References

[1] Frettlöh D., Harriss E., Gähler F.: Tilings encyclopedia, https://tilings.math.uni-bielefeld.de/
[2] Lebesgue, H.: Leçons sur l’Intégration et la Recherche des Fonctions Primitives, 44-45. Gauthier-Villars, Paris (1904)
[3] Sagan, H.: Approximating Polygons for Lebesgue’s and Schoenberg’s Space-Filling Curves, American Mathematical Monthly, 93(5), 361-368 (1986)
[4] Sagan, H.: A Geometrization of Lebesgue’s Space-filling Curve, Mathematical Intelligencer, 15(4), 37-43 (1993)
[5] Sagan, H., Space-Filling Curves, Springer-Verlag, Berlin Heidelberg New York, (1994)
[6] Wunderlich, W.: Über Peano-Kurven, Elemente der Mathematik, 28, 1-10 (1973)

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