A SCHRODINGER FORMULATION OF BIANCHI I SCALAR FIELD COSMOLOGY

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Abstract: We show that the Bianchi I Einstein field equations in a perfect fluid scalar field cosmology are equivalent to a linear Schrödinger equation. This is achieved through a special case of the recent FLRW Schrödinger-type formulation, and provides an alternate method of obtaining exact solutions of the Bianchi I equations.

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1. Introduction

Recently, a correspondence was established between solutions of Einstein’s field equations in a Friedmann-Lemaître-Robertson-Walker (FLRW) universe and solutions of a particular nonlinear Schrödinger-type differential equation. That is, given a solution of the latter, a solution of the former can be constructed via the prescription given in [3] (and vice versa). Further motivated by a connection between Bianchi I and FLRW cosmologies seen in a paper by James E. Lidsey [6], an analogous linear Schrödinger formulation is demonstrated here for the anisotropic Bianchi I universe. The author would like to extend many thanks to Floyd L. Williams for his valued advice on the results presented here.

2. Einstein Equations

Consider the Einstein field equations $T_{ij} = K^2 G_{ij}$ for a perfect fluid Bianchi I universe with scalar field $\phi$, potential $V$ and metric $ds^2 = -dt^2 + X(t)^2 dx^2 + Y(t)^2 dy^2 + Z(t)^2 dz^2$. We will consider the case where the energy density and pressure are given solely by a scalar field contribution, i.e. no matter contribution. That is, $\rho = \dot{\phi}^2/2 + V \circ \phi$ and $p = \dot{\phi}^2/2 - V \circ \phi$. For a vanishing cosmological constant the equations take the form

$$\frac{\dot{X}\dot{Y}}{XY} + \frac{\dot{X}\dot{Z}}{XZ} + \frac{\dot{Y}\dot{Z}}{YZ} = K^2 \rho$$
$$\frac{\ddot{X}}{X} + \frac{\ddot{Y}}{Y} + \frac{\dot{X}\dot{Y}}{XY} = -K^2 \rho$$
$$\frac{\ddot{X}}{X} + \frac{\ddot{Z}}{Z} + \frac{\dot{X}\dot{Z}}{XZ} = -K^2 p$$
$$\frac{\ddot{Y}}{Y} + \frac{\ddot{Z}}{Z} + \frac{\dot{Y}\dot{Z}}{YZ} = -K^2 p$$
where $K^2 = 8\pi G$ and $G$ is Newton’s constant.

The fluid conservation equation can be derived from these equations and is

$$\dot{\rho} + \theta(\rho + p) = 0$$

where

$$\theta \equiv \left(\frac{\dot{X}}{X} + \frac{\dot{Y}}{Y} + \frac{\dot{Z}}{Z}\right)$$

is the expansion/contraction of volume. By definitions of $\rho$ and $p$, this equation reduces to the Klein-Gordon equation of motion

$$\ddot{\phi} + \theta \dot{\phi} + V' \circ \phi = 0.$$  

Note that for $\gamma = 2\dot{\phi}^2/[\dot{\phi}^2 + 2(V \circ \phi)]$ one has the equation of state $p = (\gamma - 1)\rho$.

### 3. Description of The Correspondence $(X, Y, Z, \phi, V) \longleftrightarrow u$

Similar to the formulation in Lidsey [6], we first describe how a solution to the Bianchi I equations (i)-(iv) can be used to derive a solution to the FLRW equations. Since [3] provides the FLRW-Schrödinger connection, this will motivate the Schrödinger-Bianchi I correspondence.

We begin by defining the quantities

$$\eta_1 \equiv \frac{\dot{X}}{X} - \frac{\dot{Y}}{Y}, \quad \eta_2 \equiv \frac{\dot{X}}{X} - \frac{\dot{Z}}{Z}, \quad \eta_3 \equiv \frac{\dot{Y}}{Y} - \frac{\dot{Z}}{Z}.$$  

(3.1)

Computing $\frac{1}{2}(i) - (ii) - (iii) - (iv)$ and using the definitions above, one can verify Raychaudhuri’s equation

$$\dot{\theta} + 2\mu^2 - \frac{1}{9} \theta^2 + \frac{K^2}{2} (3p + \rho) = 0$$

(3.2)

where $\mu$ is the shear scalar given by $\mu^2 \equiv \frac{1}{6} (\eta_1^2 + \eta_2^2 + \eta_3^2) + \frac{2}{9} \theta^2$. Also, rewriting equation (i) using the above notation,

$$\frac{5}{9} \theta^2 - \mu^2 = K^2 \rho.$$  

(3.3)

Remarkably, equations (3.2) and (3.3) are a special case of the FLRW field equations (see [3]) with the following substitutions:

$$n = 6, \quad k = 0, \quad a(t) = (XYZ)^{1/3}, \quad D = \frac{X^2Y^2Z^2}{6K^2} \left(\eta_1^2 + \eta_2^2 + \eta_3^2\right).$$

(3.4)

Note that [3] requires $D$ to be constant. By (i)-(iv) in (2.1) and the following lemma, one can easily show that each of the products $XYZ\eta_i$ for $i = 1, 2, 3$ is a constant function of $t$.

**Lemma.** For arbitrary functions $X(t), Y(t), Z(t) > 0$ and $f(t)$,

$$\dot{f} + \theta f = 0 \iff fXYZ \text{ is a constant function}$$

for $\theta$ as in (2.2) in terms of $X, Y, Z$. 
Therefore, given a quintet \((X,Y,Z,\phi,V)\) solution to (i)-(iv) in (2.1), one can construct a solution to the FLRW field equations by (3.4). By the converse of Theorem 1 from [3], one can then construct a solution to the time-independent linear Schrödinger equation
\[
u''(x) + [E - P(x)]\nu(x) = 0. \tag{3.5}
\]
with constant energy \(E\) and potential \(P(x)\). We will see that the Schrödinger solutions derived in this way will always be such that \(E < 0\).

In this paper, we show the direct correspondence \((X,Y,Z,\phi,V) \leftrightarrow u\) between solutions \((X,Y,Z,\phi,V)\) of (i)-(iv) in (2.1) and solutions \(u\) of (3.5). This correspondence provides an alternate method of solving Bianchi I field equations.

With the above notation in place, we can now state the main theorem:

**Theorem.** Let \(u(x)\) be a solution of equation (3.5), given \(E < 0\) and \(P(x)\). Then a solution \((X,Y,Z,\phi,V)\) of the Einstein equations (i)-(iv) in (2.1) can be constructed as follows. First choose functions \(\sigma(t), \psi(x)\) such that
\[
\dot{\sigma}(t) = u(\sigma(t)), \quad \psi'(x)^2 = \frac{2}{3K^2}P(x) \tag{3.6}
\]
and also constants \(c_1, c_2\) such that
\[
c_1^2 + c_1 c_2 + c_2^2 = -\frac{4E}{3}. \tag{3.7}
\]
Next define the functions
\[
R(t) = u(\sigma(t))^{-1/3} \tag{3.8}
\]
and
\[
\alpha(t) = \frac{c_1}{2}\sigma(t), \quad \beta(t) = \frac{c_2}{2}\sigma(t), \quad \gamma(t) = -\alpha(t) - \beta(t). \tag{3.9}
\]
Then the following quintet solves Einstein’s field equations (i)-(iv):
\[
X(t) = R(t)e^{\alpha(t)}, \quad Y(t) = R(t)e^{\beta(t)}, \quad Z(t) = R(t)e^{\gamma(t)}, \tag{3.10}
\]
\[
\phi(t) = \psi(\sigma(t)), \quad V = \frac{1}{3K^2}
\left((u')^2 + u^2[E - P]\right) \circ \psi^{-1}. \tag{3.11}
\]
Here, in fact, \((X,Y,Z,\phi,V)\) will also satisfy the equations
\[
\dot{\phi}^2 = \frac{2}{3K^2} \left(-\dot{\theta} + \frac{E}{X^2Y^2Z^2}\right) \tag{3.12}
\]
\[
V(\phi(t)) = \frac{1}{3K^2} \left(\theta'^2 + \dot{\theta}\right). \tag{3.13}
\]

Conversely, let \((X,Y,Z,\phi,V)\) be a solution of equations (i)-(iv) in (2.1), with \(\rho\) and \(p\) as before. Similar to (3.6), choose some solution \(\sigma(t)\) of the equation
\[
\dot{\sigma}(t) = \frac{1}{XYZ} \tag{3.14}
\]
Then equation (3.5) is satisfied for
\[
E = \frac{1}{2} X^2 Y^2 Z^2 \left(\eta_1^2 + \eta_2^2 + \eta_3^2\right), \tag{3.15}
\]
\[
P(x) = \frac{3}{2}K^2 \left[\dot{\phi}^2 X^2 Y^2 Z^2\right] \circ \sigma^{-1}(x), \tag{3.16}
\]
\[
u(x) = \left[\frac{1}{XYZ}\right] \circ \sigma^{-1}(x). \tag{3.17}
\]
Note that in (3.15), $E < 0$ and is constant by the same argument stated for $D$ above the lemma. The theorem therefore provides a concrete correspondence $(X, Y, Z, \phi, V) \leftrightarrow u$ between solutions $(X, Y, Z, \phi, V)$ of the field equations (i)-(iv) and solutions $u$ of the linear Schrödinger equation (3.5).

Remarks.

1. The case $P(x) = 0$:

   In most examples $P(x)$ is nonzero. However, if $P(x) = 0$ then the theorem must be stated carefully, as $\psi(x)$ is a constant function by (3.6) and has no inverse. Therefore the expression for $V$ in (3.11) has no meaning. In this case, we will show that the right-hand side of (3.13) is a constant function that will serve as our new definition for $V$. By (3.8)-(3.10), $u \circ \sigma = 1/(XYZ)$. Differentiating and using (3.6), $(u' \circ \sigma)(u \circ \sigma) = -\theta/(XYZ)$. That is, $u' \circ \sigma = -\theta$. Differentiating again, $(u'' \circ \sigma)(u \circ \sigma) = -\dot{\theta}$. Using these in (3.5), composed with $\sigma$ and multiplied by $u \circ \sigma$,

   $$-\dot{\theta} + \frac{E}{X^2Y^2Z^2} = 0. \quad (3.16)$$

   Therefore (3.12) is still valid, of course, with the left side equal to zero since $\phi$ is constant in this case by (3.11). Differentiating (3.16),

   $$-\ddot{\theta} - \frac{2E\theta}{X^2Y^2Z^2} = 0. \quad (3.17)$$

   Now, to show that (3.13) is constant, differentiate its right side to get

   $$\frac{d}{dt}\left\{\theta^2 + \dot{\theta}\right\} = 2\theta\dot{\theta} + \ddot{\theta}$$

   $$= 2\theta \left(\dot{\theta} - \frac{E}{X^2Y^2Z^2}\right) \quad \text{by (3.17)}$$

   $$= 0 \quad \text{by (3.16)}.$$

   Therefore in the case $P(x) = 0$ when the definition of $V$ in (3.11) no longer has meaning, we define $V(x) = V_0 \equiv (\theta^2 + \theta)/(3K^2)$ and $\phi(t) = \text{any constant}$; and we note that equations (3.12) and (3.13) still hold in this special case.

2. Equations (3.12) and (3.13) imply (i)-(iv) in (2.1) under a condition:

   By the comment preceding the lemma above, any solution to the equations (i)-(iv) will have the property that $XYZ\eta_i$ is constant for $\eta_i$ as in (3.1) and $i = 1, 2, 3$. Suppose we are given a priori (positive) functions $X, Y, Z$ with this property. Differentiating each of $XYZ\eta_i$ and setting equal to zero shows exactly that the left sides of (ii)-(iv) are equal to each other. Next, any three positive functions can be reparametrized as in (3.10) for $R(t) = (XYZ)^{1/3}$, $\alpha = \frac{1}{4} \ln \left(\frac{Y^2}{XZ}\right)$, $\beta = \frac{1}{4} \ln \left(\frac{X^2}{YZ}\right)$ and $\gamma = \frac{1}{4} \ln \left(\frac{Z^2}{XY}\right)$. Using these formulas and the constant quantities $XYZ\eta_i$, one can easily compute that each of $\dot{\alpha}, \dot{\beta}, \dot{\gamma}$ are scalar multiples of $1/XYZ$, therefore establishing (3.9) for $\sigma$ as in (3.14). Finally, under the condition that $c_1, c_2$ satisfy (3.7) for the constant $E < 0$ given by $X, Y, Z$ as in (3.15), one can show that equations (3.12) and (3.13) indeed compute $\phi$ and $V$ solving (i)-(iv) in terms of the given $X, Y, Z$. 
4. Examples

As an illustration of the theorem, take the solution \( u(x) = Ae^{-\sqrt{-Ex}} - Be^{\sqrt{-Ex}} \) for \( A, B > 0 \) to the equation (3.5) with \( E < 0 \) and \( P(x) = 0 \). Solving the differential equation (3.6) for \( \sigma \) using Mathematica, we obtain

\[
\sigma(t) = \frac{1}{2\sqrt{-E}} \ln \left[ \frac{A}{B} \tanh^2[\sqrt{-ABE}(t - c_0)] \right]
\]

(4.18)

for integration constant \( c_0 \). Also by (3.6), \( \psi(x) = \psi_0 \equiv \) any constant. Then by (3.8)

\[
R(t) = u \circ \sigma^{-1/3} = \left( \frac{1}{2\sqrt{AB}} \sinh[2\sqrt{-ABE}(t - c_0)] \right)^{1/3}.
\]

(4.19)

Further let \( c_1, c_2 \) be any constants such that (3.7) holds given the constant choice \( E \). We form \( X, Y, Z, \phi \) according to (3.9)-(3.11) and obtain

\[
X = \left( \frac{1}{2\sqrt{AB}} \sinh[2\sqrt{-ABE}(t - c_0)] \right)^{1/3} \left( \sqrt{\frac{A}{B}} \tanh[\sqrt{-ABE}(t - c_0)] \right)^{c_1/(2\sqrt{-E})}
\]

(4.20)

\[
Y = \left( \frac{1}{2\sqrt{AB}} \sinh[2\sqrt{-ABE}(t - c_0)] \right)^{1/3} \left( \sqrt{\frac{A}{B}} \tanh[\sqrt{-ABE}(t - c_0)] \right)^{c_2/(2\sqrt{-E})}
\]

\[
Z = \left( \frac{1}{2\sqrt{AB}} \sinh[2\sqrt{-ABE}(t - c_0)] \right)^{1/3} \left( \sqrt{\frac{A}{B}} \tanh[\sqrt{-ABE}(t - c_0)] \right)^{-(c_1+c_2)/(2\sqrt{-E})}
\]

for \( t > c_0 \) and \( \phi = \psi_0 \). Since \( P = 0 \), \( \psi^{-1} \) does not exist (see Remark 1) and we use (3.13) for the definition of \( V \) and obtain the constant \( V = V_0 \equiv -4ABE/(3K^2) \). The reader may compare this solution with a similar one in [1].

As another example, we will begin with the same assumptions on \( E, P \) and will obtain quite a different Einstein solution. That is, again let \( E < 0 \) and \( P(x) = 0 \), but take solution \( u(x) = Ae^{-\sqrt{-Ex}} \) to (3.5) with \( A > 0 \). Solving the differential equations in (3.6), we obtain

\[
\sigma(t) = \ln \left[ A\sqrt{-E}(t - c_0) \right] / \sqrt{-E}
\]

for \( t > c_0 \) and \( \psi(x) = \psi_0 \equiv \) any constant (therefore we will also have \( \phi = \psi_0 \) by (3.11)). By (3.8), \( R(t) = (\sqrt{-E}(t - c_0))^{1/3} \). Letting \( c_1, c_2 \) be any solution to (3.7), finally we compute \( X, Y, Z \) to be

\[
X = A^{c_1/(2\sqrt{-E})} \left( \sqrt{-E}(t - c_0) \right)^{c_1/(2\sqrt{-E})+1/3}
\]

(4.21)

\[
Y = A^{c_2/(2\sqrt{-E})} \left( \sqrt{-E}(t - c_0) \right)^{c_2/(2\sqrt{-E})+1/3}
\]

\[
Z = A^{-(c_1+c_2)/(2\sqrt{-E})} \left( \sqrt{-E}(t - c_0) \right)^{-(c_1+c_2)/(2\sqrt{-E})+1/3}
\]

by (3.9)-(3.10) for \( t > c_0 \). Again, since \( P = 0 \), \( \psi^{-1} \) does not exist and we use (3.13) as the definition for \( V \) and obtain \( V = 0 \). That is, this solution is vacuum.

To further demonstrate the utility of the theorem, we will take a trivial non-physical solution of (i)-(iv), and first use the converse theorem to map to a solution of (3.5). We will then apply the theorem a second time and map back to a solution of (i)-(iv) and will have produced a physically acceptable example. Begin by considering the vacuum (\( \phi = V = 0 \)) solution

\[
X = R_0e^{\alpha_0}, \quad Y = R_0e^{\beta_0}, \quad Z = R_0e^{\gamma_0}
\]

(4.22)
for constants $R_0 > 0, \alpha_0, \beta_0, \gamma_0$ such that $\alpha_0 + \beta_0 + \gamma_0 = 0$. By (3.1) and (3.15), $E = P = 0$ and $u(x) = u_0 \equiv (1/R_0^3)$. Clearly this is a solution to the linear Schrödinger equation (3.5). Note that since $u$ is constant and $\phi$ is zero, we did not need to compute $\sigma$. Now to map back to a solution of (i)-(iv), we solve (3.6) and use (3.11) so that $\sigma(t) = (1/R_0^3)t$ and $\psi = \phi = \alpha = 0$, $\beta = 0$, $\gamma = 0$. Now by (3.8)-(3.10),

$$X = R_0 e^{c_1 t/(2R_0^3)}, \quad Y = R_0 e^{c_2 t/(2R_0^3)}, \quad Z = R_0 e^{-(c_1 + c_2)t/(2R_0^3)}.$$  \hspace{1cm} (4.23)

Again by (3.13), $V = 0$.

As a final example, we take $u(x) = (1/x)e^{Ex^2/2}$, $E < 0$ and $P(x) = (2/x^2) + E^2x^2$ for $x > 0$. This Schrödinger solution was found using the techniques in [5]. Solving (3.6), we obtain $\sigma(t) = \sqrt{-2/E} \ln[-E(t - c_0)]$ for integration constant $c_0$ and

$$\psi(x) = \frac{1}{\sqrt{6K}} \left( \sqrt{2 + E^2x^4} + \sqrt{2} \ln \left[ \frac{x^2}{2 + \sqrt{4 + 2E^2x^2}} \right] \right).$$  \hspace{1cm} (4.24)

Graphing this function for a few values of $E$ indicates that the inverse exists, and we will denote it by $\psi^{-1}$. Calculating $X, Y, Z, \phi, V$ according to the theorem,

$$X = \left( (t - c_0) \sqrt{-2E \ln[-E(t - c_0)]} \right)^{1/3} e^{c_1/\sqrt{\ln[-E(t - c_0)]}/(-2E)} \hspace{1cm} (4.25)$$

$$Y = \left( (t - c_0) \sqrt{-2E \ln[-E(t - c_0)]} \right)^{1/3} e^{c_2/\sqrt{\ln[-E(t - c_0)]}/(-2E)}$$

$$Z = \left( (t - c_0) \sqrt{-2E \ln[-E(t - c_0)]} \right)^{1/3} e^{-(c_1 + c_2)/\sqrt{\ln[-E(t - c_0)]}/(-2E)}$$

$$V = \frac{1}{3K^2} \left[ \frac{e^{Ex^2}}{x^4} (1 + Ex^2) \right] \cdot \psi^{-1}(t)$$

$$\phi = \frac{1}{\sqrt{3K}} \left( \sqrt{1 + 2 \ln^2[-E(t - c_0)]} + \ln \left( \frac{-\ln[-E(t - c_0)]}{E + E \sqrt{1 + 2 \ln^2[-E(t - c_0)]}} \right) \right)$$

for $c_1, c_2$ satisfying (3.7) and $t > c_0$.

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