Spherical Quantum Chromodynamics of Heavy Quark Systems

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Abstract

We propose a model for Quantum Chromodynamics, obtained by ignoring the angular dependence of the gluon fields, which could qualitatively describe systems containing one heavy quark. This leads to a two dimensional gauge theory which has chiral symmetry and heavy quark symmetry. We show that in a light cone formalism, the Hamiltonian of this spherical QCD can be expressed entirely in terms of color singlet variables. Furthermore, in the large $N_c$ limit, it tends to a classical hadron theory. We derive an integral equation for the masses and wavefunctions of a heavy meson. This can be interpreted as a relativistic potential model. The integral equation is scale invariant, but renormalization of the coupling constant generates a scale. We compute the approximate beta function of the coupling constant, which has an ultraviolet stable fixed point at the origin.
1. Introduction

Two dimensional Quantum Chromodynamics (QCD) is fairly well understood in the $\frac{1}{N_c}$ expansion, following the work of 't Hooft, Witten and others [1], [2], [3]. Furthermore, it can be solved by numerically diagonalizing the hamiltonian [4]. The two methods are in satisfactory agreement with each other. In this paper we will study an approximation to four dimensional QCD in which the gluon field is independent of two angular variables, so that two dimensional methods can be applied. Although this could be studied just as a toy model for QCD, it might also be a model for hadrons containing one heavy quark. These systems have attracted much attention recently with the discovery of a new heavy quark symmetry [5], [6], [7]. Such a hadron is similar to an atom, the heavy quark being like the nucleus with the light quarks orbiting like electrons around it. The main difference is that the light quarks have to be described relativistically. Still, we should be able to treat these systems in the spirit of the Hartree–Fock or Thomas–Fermi methods [8]. The gluon field produced by the heavy quark is spherically symmetric. That of the light quarks is not spherically symmetric because the light quark wavefunctions depend on angles. Still, we should be able to do a ‘spherical averaging’ as in Hartree–Fock theory and approximate the current density of the light quarks by their average over the angular variables. The average of the angular components of the current density will vanish. This spherical averaging is equivalent to putting $A_\theta = A_\phi = 0$ and assuming that the remaining components $A$ are independent of the angles. This will lead to an effectively two dimensional theory (although without translation invariance) in light cone coordinates. This ‘Spherical QCD’ respects heavy quark symmetry, chiral symmetry of light quarks and scale invariance. (The scale invariance should be broken by quantum effects.) It should be possible to solve this theory numerically, using methods similar to those used for two dimensional QCD [4]. This is vastly simpler than solving the full $3 + 1$ dimensional problem numerically. In this paper, we will study this spherical QCD analytically using the $\frac{1}{N_c}$ expansion.

In this paper, we will view Spherical QCD as a model to study theoretical aspects of QCD such as the $\frac{1}{N_c}$ expansion and asymptotic freedom. We have a model that is solvable in the large $N_c$ limit, in which the beta function can be calculated in a non–perturbative approximation. Whether our approximate theory can make reliable numerical predictions can only be seen from future work.

't Hooft’s original approach to the $\frac{1}{N_c}$ expansion involved summing planar Feynman diagrams. This method is difficult to generalize to our situation. More recently, another method has been used to construct a bilocal quantum field theory of hadrons equivalent to
2DQCD for all $N_c$ [9]. The parameter $\frac{1}{N_c}$ plays the role of $\hbar$ in this hadron theory ($N_c$ must be an integer for topological reasons) so that the large $N_c$ limit of QCD corresponds to a classical theory of hadrons. The field equations of this theory are certain non-linear integral equations: the large $N_c$ limit of QCD is not a free field theory. Semiclassical expansion around the classical solutions will describe the hadron masses and wavefunctions in the $\frac{1}{N_c}$ expansion. The most obvious static solution is the vacuum, and small fluctuations around it (mesons) are described by a linear integral equation, in agreement with 't Hooft. This point of view also allows for baryons, which are then static solutions that deviate by a large amount from the vacuum [9].

This approach to two dimensional QCD can be generalized to Spherical QCD. One of our results is a linear integral equation for the heavy–light meson masses and wave functions. We can understand our result (just like 't Hooft’s integral equation) as a relativistic potential model. Ordinarily, such a one particle point of view would not be allowed in this highly relativistic situation due to effects of virtual light $q\bar{q}$ pairs. In the meson picture, these are the effects of virtual light mesons. However, in the large $N_c$ limit such effects are suppressed, so that a one particle description is allowed. Thus we will get a light quark moving in the field of the heavy quark. The Dirac equation can be separated in light cone coordinates, reducing the problem to a linear integral equation. (That light cone coordinates are better suited to nonperturbative problems in QCD have also been suggested for other reasons [10], [11].) It should be possible to fit meson masses to a potential with a small number of parameters as a phenomenological test of our approach.

In the next section (Sec.2) we will derive the action of spherical QCD; it will be convenient to use a light cone coordinate system. In Sec. 3 we will show that this theory can be written entirely in terms of a set of color singlet fields representing hadrons. The commutation relations and equations of motion of these variables are derived. In Sec. 4 we show that in the large $N_c$ limit, this tends to a classical theory whose Poisson brackets, constraints and equations of motion are derived. By expanding around the vacuum solution we get a linear integral equation for the masses and wavefunction of heavy mesons. In Sec 5. we have analysed the ground state of the wave equation in a variational approach. A renormalization of the coupling constant is necessary to keep the ground state energy finite as the cut–off is removed. We compute the approximate beta function and show that it has an ultraviolet stable fixed point at the origin. Finally, in the appendix we describe how the constraints on the color singlet variables can be derived, in a simplified context.
2. QCD in light cone coordinates

The action of four dimensional QCD with $N_c$ colors, $n_f$ massless quarks and $N_f$ heavy quarks of mass $M$ is

$$S = -\frac{1}{2g^2} \int \text{tr} F_{\mu\nu} F^{\mu\nu} d^4x + \int \bar{\psi}^a [i\gamma \cdot \partial + \gamma \cdot A] \psi_a d^4x$$
$$+ \int \bar{\Psi}^{A} [i\gamma \cdot \partial + \gamma \cdot A + M] \Psi_A d^4x,$$

where $A_\mu$ and $F_{\mu\nu}$ are $N_c \times N_c$ hermitian matrices. Here, the color indices are suppressed and $a = 1, \cdots n_f$, $A = 1, \cdots N_f$. We are interested in systems that contain only one heavy quark, which can be assumed to be at rest at the origin. Conventional spherical polar coordinates $t, r, \theta, \phi$ are cumbersome in this context due to the negative energy states of the light fermions and the dependence of the Dirac sea on the gluon fields. We find it most convenient to introduce a sort of light cone coordinate system $u, r, \theta, \phi$ centered at the heavy quark:

$$u = t + r, \quad ds^2 = du(du - 2dr) - r^2(d\theta^2 + \sin^2 \theta d\phi^2).$$

Note that this is not an orthogonal coordinate system; yet the Dirac equation is separable in this system. The surface $u =$constant is a past light cone with apex at the origin. We will use a canonical formalism in which initial data is given on this null surface. The vector field $\frac{\partial}{\partial u}$ is in fact just the usual time translation, so that its conjugate variable is energy. But $\frac{\partial}{\partial r}$ for fixed $u$ is not the same as the radial vector in spherical polar coordinates. Instead, it is a null vector pointing outward along the surface $u =$constant:

$$\left( \frac{\partial}{\partial r} \right)_t = \left( \frac{\partial}{\partial r} \right)_u + \frac{\partial}{\partial u}.$$  

We will expand all the fields in appropriate spherical harmonics in the angular variables. The essential approximation we will make is to include only the s-wave states of the gluon fields. This amounts to replacing the current density of the light quarks by its average over the angular variables, so that the mean gluon field they produce is spherically symmetric. (This concept of a ‘spherical average’ was introduced into atomic physics by Hartree–Fock and is extensively used in calculations of atomic energy levels and wavefunctions [8].) We will be able to eliminate the gluon fields and half the degrees of freedom of
the light quarks, to get a theory of quarks interacting through a selfconsistent potential. The resulting nonlocal lagrangian will have chiral symmetry, heavy quark symmetry and scale invariance, and defines our ‘Spherical QCD’.

Let us first simplify the gluon Lagrangian. After spherical averaging, the angular components of a spin one field must vanish, and $A_u$ and $A_r$ would be independent of the angles. Furthermore we can choose the gauge condition

$$A_r = 0,$$  \hspace{1cm} (3)

leaving only the component $A_u = A$. In this case the Yang–Mills field strength is

$$F_{\mu\nu} dx^\mu \wedge dx^\nu = \partial_r A_u dr \wedge du$$  \hspace{1cm} (4)

and the Yang–Mills action is

$$S_{YM} = \frac{1}{\alpha} \int \text{tr}[\partial_r A]^2 r^2 dr du.$$  \hspace{1cm} (5)

(Here $\alpha = \frac{g^2}{4\pi}$.) In this gauge there are no propagating components for the Yang–Mills fields since there is no derivative of $A$ with respect to $u$ in the action.

Let us now turn to the light quark action. The free Dirac operator in the usual spherical polar coordinates is \[12\]

$$i \gamma \cdot \partial = i \gamma_0 \left( \frac{\partial}{\partial \ell} + \alpha_r \left( \frac{\partial}{\partial r} + \frac{1}{r} - \frac{K}{r} \right) \right)$$  \hspace{1cm} (6)

where

$$K = \Sigma \cdot L + 1, \quad \alpha_r^2 = 1, \quad [\alpha_r, K]_+ = 0.$$  \hspace{1cm} (7)

As usual $\Sigma$ are the Dirac spin matrices, $\alpha_r = \gamma_0 \gamma_r$, and $L$ is the orbital angular momentum operator. In our coordinates,

$$i \gamma \cdot \partial = i \gamma_0 [(1 + \alpha_r) \frac{\partial}{\partial \ell} + \alpha_r \left( \frac{\partial}{\partial r} + \frac{1}{r} - \frac{K}{r} \right)].$$  \hspace{1cm} (8)

It will be convenient to diagonalize $\alpha_r$ and $\gamma_5$ (we define $\gamma_5$ such that $\gamma_5^2 = 1$):

$$\gamma_5 v_{k,\epsilon,\mu} = \epsilon v_{k,\epsilon,\mu} \quad \alpha_r v_{k,\epsilon,\mu} = \mu v_{k,\epsilon,\mu} \quad K v_{k,\epsilon,\mu} = k v_{k,\epsilon,-\mu}.$$  \hspace{1cm} (9)

The eigenvalue $k$ of $K$ is a positive integer. The spinorial harmonics $v_{k,\epsilon,\mu}$ are orthogonal and are normalized such that

$$\int d\Omega |v_{k,\epsilon,\mu}|^2 = 4\pi.$$  \hspace{1cm} (10)
Then $\psi$ can be expanded in this basis (suppressing color and flavor indices on $\psi$ and $\chi$):

$$\psi = \sum_{\mu=\pm 1, k>0, \epsilon=\pm 1} \frac{1}{\sqrt{(8\pi)r}} \chi_{k,\epsilon,\mu}(r) v_{k,\epsilon,\mu}. \quad (11)$$

The factor $\frac{1}{\sqrt{(8\pi)r}}$ is chosen so as to simplify later expressions. In this basis, $\alpha_r = \sigma_3, K = k\sigma_1$ and $-i\alpha_r K = k\sigma_2$. The free Dirac action is then

$$\int \bar{\psi}[i\gamma \cdot \partial] \psi d^4x = \frac{1}{2} \sum_{k,\epsilon} \int dr du \chi_{k\epsilon}^\dagger [i(1 + \sigma_3) \frac{\partial}{\partial u} + i\sigma_3 \frac{\partial}{\partial r} + \frac{k}{r}\sigma_2] \chi_{k\epsilon}, \quad (12)$$

where $\chi_{k,\epsilon} = \begin{pmatrix} \chi_{k,\epsilon,1} \\ \chi_{k,\epsilon,-1} \end{pmatrix}$. Including the gauge field amounts to replacing $\partial_u$ by $\partial_u - iA$:

$$\int \bar{\psi}[i\gamma \cdot \partial + \gamma \cdot A] \psi d^4x = \frac{1}{2} \sum_{k,\epsilon} \int dr du \chi_{k\epsilon}^\dagger [i(1 + \sigma_3)[\frac{\partial}{\partial u} - iA] + i\sigma_3 \frac{\partial}{\partial r} + \frac{k}{r}\sigma_2] \chi_{k\epsilon}. \quad (13)$$

The heavy quark action can be simplified by the transformation [5], [6]

$$\Psi \rightarrow e^{iMv \cdot x}\Psi, \quad (14)$$

so that in the limit $M \rightarrow \infty$

$$\int \bar{\Psi}^A[i\gamma \cdot \partial + \gamma \cdot A + M]\Psi_A d^4x \rightarrow i \int \bar{\Psi}^A v \cdot [\partial - iA] \Psi_A d^4x. \quad (15)$$

Here $v$ is the heavy quark velocity. Now we put

$$\Psi_A = \frac{1}{\sqrt{(4\pi)r}} Q_A; \quad (16)$$

the spinor $Q$ satisfies $\gamma \cdot vQ = Q$ [5], [6] and depends only on $r$. In our coordinate system, $v = (1, 0, 0, 0)$ so that this becomes just

$$i \int Q^\dagger [\partial_u - iA]Q d\epsilon. \quad (17)$$

(We may suppress flavor indices as well when they are not essential).

Thus the complete action is

$$S = i \int Q^\dagger [\partial_u - iA]Q d\epsilon + \frac{1}{\alpha} \int \text{tr}[\partial_r A]^2 r^2 d\epsilon$$

$$+ \frac{1}{2} \sum_{k,\epsilon} \int dr du \chi_{k\epsilon}^\dagger [i(1 + \sigma_3)[\frac{\partial}{\partial u} - iA] + i\sigma_3 \frac{\partial}{\partial r} + \frac{k}{r}\sigma_2] \chi_{k\epsilon}. \quad (18)$$
In this action, $A$ and $\chi_{k,\epsilon,-1}$ do not have derivatives with respect to $u$. Hence they do not propagate and can be eliminated by solving their equations of motion. We have,

$$\partial_r [r^2 \partial_r A_\alpha] = \alpha \rho_\alpha$$  \hspace{1cm} (18)

where,

$$\rho_\alpha = \sum_{k,\epsilon} \chi_{k,\epsilon,1}^\dagger t_\alpha \chi_{k,\epsilon,1} + Q_\alpha^\dagger t_\alpha Q,$$  \hspace{1cm} (19)

and $t_\alpha$ are the color matrices. We normalize them such that

$$\text{tr} t_\alpha t_\beta = \frac{1}{2} \delta_{\alpha\beta}.$$  \hspace{1cm} (20)

Half the light quark degrees of freedom also do not propagate:

$$-i \frac{\partial}{\partial r} \chi_{k,\epsilon,-1} + ik \frac{r}{r'} \chi_{k,\epsilon,1} = 0.$$  \hspace{1cm} (21)

Then,

$$A(r) = \alpha \int_0^\infty G(r,r') \rho(r') dr'$$  \hspace{1cm} (22)

where

$$G(r,r') = -\min\left(\frac{1}{r}, \frac{1}{r'}\right)$$  \hspace{1cm} (23)

and

$$\chi_{k,\epsilon,-1} = k \partial^{-1} \chi_{k,\epsilon,1} \bigg|_r = \int_r^\infty k \frac{1}{r'} \chi_{k,\epsilon,1} (r') dr'.$$  \hspace{1cm} (24)

The boundary condition are that as $r \to \infty$ the fields vanish. That is, there is no incoming radiation from past null infinity.

We can now eliminate $A, \chi_{k,\epsilon,-1}$ to get an action depending only on $\chi_{k,1}$ and $Q$: 

$$S = i \int Q^\dagger \dot{Q} dr du + i \sum_{k,\epsilon} \int dr du \chi_{k,1}^\dagger \dot{\chi}_{k,1} + \frac{1}{2} \sum_{k,\epsilon} \int dr du \chi_{k,1}^\dagger [i \partial_r + k^2 r^{-1} (i \partial_r)^{-1} r^{-1}] \chi_{k,1} - \frac{1}{2} \alpha \int dr' dr du \rho_\alpha (r) \rho_\alpha (r') G(r,r')$$

$$+ \alpha \int du dr' G(r,r') Q^\dagger (r) t_\alpha \rho_\alpha (r') Q (r) + \alpha \int du dr' G(r,r') \chi_{k,1}^\dagger (r) t_\alpha \rho_\alpha (r') \chi_{k,1} (r)$$

This action defines Spherical QCD. It is manifestly invariant under $U(n_f)_L \times U(n_f)_R$ (chiral symmetry), $SU(2N - f)$ (spin and flavor symmetry of heavy quarks) and scale
transformations (the only coupling constant $g^2$ is dimensionless). The first two terms simply say that $Q, \chi_{k\epsilon 1}$ are canonically conjugate to $\bar{Q}, \chi_{k\epsilon 1}^\dagger$.

$$[Q^\dagger(r), Q(r')]_+ = \delta(r - r'), \quad [\chi_{k\epsilon 1}^\dagger(r), \chi_{k'\epsilon'1}^\dagger(r')]_+ = \delta_{kk'}\delta_{\epsilon\epsilon'}\delta(r - r').$$ (25)

The remaining terms determine the Hamiltonian:

$$H = \sum_{k\epsilon} \int dr \chi_{k\epsilon 1}^\dagger h \chi_{k\epsilon 1} + \frac{1}{2} \alpha \int dr dr' \rho_\alpha(r) \rho_\alpha(r') G(r, r')$$

$$- \alpha \int dr dr' G(r, r') Q^\dagger(r) t_\alpha \rho_\alpha(r') Q(r) - \alpha \int dr dr' G(r, r') \chi_{k\epsilon 1}^\dagger(r) t_\alpha \rho_\alpha(r') \chi_{k\epsilon 1}(r),$$

where

$$h = -\frac{1}{2}(i\partial_r + k^2 r^{-1}(i\partial_r)^{-1} r^{-1})$$

is the single particle Hamiltonian of the light quarks. Since the Hamiltonian $H$ is conjugate to $u$, it is just the energy. The above anticommutation relations along with the Hamiltonian define Spherical QCD.

Apart from the fact that $h$ and $G$ have different explicit forms (and the presence of the heavy quarks), this is exactly like the Hamiltonian of 2D QCD, after eliminating the gluons and half the quark fields. Thus from this point, we can follow the method described in reference [9] to turn this into a theory of bilocal color singlet (meson) fields.

3. Hadron Theory

Let us define the color singlet bilinears,

$$H^I_i(r', r) = \frac{1}{N_c} Q^\dagger I(r') \chi_i(r), \quad H^I_i(r', r) = \frac{1}{N_c} \chi_i^\dagger(r') Q_i(r),$$

$$P^I_j(r, r') = \frac{1}{N_c} Q^\dagger I(r) Q_j(r'), \quad M^I_j(r, r') = \frac{1}{N_c} : \chi^\dagger_i(r) \chi_j(r') :$$

where $i = (k, \epsilon, a)$ ranges over spin, chirality and flavor of the light quark while $I$ labels the spin and flavor of the heavy quark. $P$ is normal ordered with respect to the trivial vacuum; after the transformations we made the heavy quark has no negative energy states. The normal ordering of $M$ is defined with respect to the vacuum obtained by filling the negative energy states of the one-particle Hamiltonian $h = -\frac{1}{2} [i\partial_r + k^2 r^{-1}(i\partial_r)^{-1} r^{-1}]$. In fact

$$\frac{1}{N_c} : \chi^\dagger_i(r) \chi_j(r') : = \frac{1}{N_c} \chi^\dagger_i(r) \chi_j(r') + \frac{1}{2} \epsilon(r, r') \delta_i^j + \frac{1}{2} \epsilon(r, r') \delta_i^j.$$
Here \( \epsilon(r, r') \) is the operator that is +1 on the eigenstates of \( h \) of positive energy and -1 on those of negative energy:

\[
\epsilon(r, r') = \int_0^\infty d\lambda u_\lambda(r)u_\lambda^*(r') - \int_{-\infty}^0 d\lambda u_\lambda(r)u_\lambda^*(r').
\]  

(26)

The eigenstates \( u_\lambda \) of \( h \) are defined as below:

\[
-\frac{1}{2}[i\partial_r + k^2r^{-1}(i\partial_r)^{-1}r^{-1}]u_\lambda(r) = \lambda u_\lambda(r)
\]

\[
\int_{-\infty}^\infty u_\lambda(r)u_\lambda^*(r')d\lambda = \delta(r - r').
\]

Also, we will often use the projection operators to the positive and negative energy states

\[
\delta_+(r, r') = \frac{1}{2}[\delta(r - r') + \epsilon(r, r')] = \int_0^\infty u_\lambda(r)u_\lambda^*(r')d\lambda
\]

(27)

\[
\delta_-(r, r') = \frac{1}{2}[\delta(r - r') - \epsilon(r, r')] = \int_{-\infty}^0 u_\lambda(r)u_\lambda^*(r')d\lambda.
\]

(28)

We will rewrite the commutation relations and the hamiltonian entirely in terms of these color singlet variables. The commutation relations are

\[
[H^I_i(r, r'), H^{J\dagger}_j(s, s')] = \frac{1}{N_c}[P^I_j(r, s')\delta^I_j\delta(r' - s) - M^I_i(s, r')\delta^I_j\delta(s' - r)
\]

\[
-\delta_-(s, r')\delta^I_j\delta^I_j\delta(s' - r)],
\]

\[
H^I_i(r, r'), M^I_i(s, s') = \frac{1}{N_c}H^I_k(r, s')\delta^I_j\delta(r' - s),
\]

\[
[H^I_i(r, r'), P^J_K(s, s')] = -\frac{1}{N_c}H^J_i(r, s')\delta^J_K\delta(r - s'),
\]

\[
H^{J\dagger}_j(r, r'), P^J_K(s, s') = \frac{1}{N_c}H^{J\dagger}_K(r, s')\delta^J_j\delta(r' - s),
\]

\[
[H^{J\dagger}_j(r, r'), M^J_K(s, s')] = -\frac{1}{N_c}H^{J\dagger}_K(s, r')\delta^J_j\delta(r - s'),
\]

\[
[P^J_i(r, r'), P^{J\dagger}_K(s, s')] = \frac{1}{N_c}[P^{J\dagger}_L(r, s')\delta^J_K\delta(r' - s) - P^J_K(s, r')\delta^J_j\delta(s' - r)],
\]

\[
[M^J_i(r, r'), M^J_K(s, s')] = \frac{1}{N_c}[M^J_L(r, s')\delta^J_j\delta(s - r') - M^J_K(s, r')\delta^J_j\delta(r - s'),
\]

\[
+\frac{1}{2}(\epsilon(s, r')\delta(s - r') + \epsilon(s, r')\delta(r - s'))\delta^J_j\delta^J_j].
\]

The c-number terms proportional to \( \epsilon(r, r') \) and \( \delta_- \) arises because of the normal ordering of \( M \) and \( P \).
The hamiltonian can be expressed in terms of these bilinears after a Fierz reordering using the identity for the color matrices:

\[ t^\rho_{\alpha q} t^r_{\alpha s} = \frac{1}{2} \delta^\rho_q \delta_r^r. \]  

(We are using the gauge group \( U(N_c) \) rather than \( SU(N_c) \), as is usual in large \( N_c \) approaches [1].) We get, with \( \tilde{\alpha} = \alpha N_c \)

\[
\begin{aligned}
\frac{H}{N_c} &= \int h(r, r') M_i^I(r, r') drdr' + \frac{1}{4} \tilde{\alpha} \int G(r, r') [P_j^I(r, r') P_j^I(r', r) \\
&+ H_i^I(r, r') H_i^I(r', r) + H_i^I(r, r') H_i^I(r', r) + M_j^i(r, r') M_j^i(r', r)] drdr'.
\end{aligned}
\]

It should be possible to find the eigenvalues of this hamiltonian by numerically diagonalizing a finite dimensional approximation to it. This is the method used in Ref. [4] to solve two dimensional QCD.

Now we note that \( N_c \) appears only as an overall constant in the commutation relations and the hamiltonian. In particular, the equations of motion are independent of \( N_c \). This means that the limit \( N_c \to \infty \) is a classical limit in which the commutators get replaced by Poisson brackets.

The large \( N_c \) limit of spherical QCD is a classical hadron theory. The semiclassical expansion around a solution of this classical theory will be equivalent to the \( \frac{1}{N_c} \) expansion of spherical QCD.

4. Classical Hadron Theory

The Poisson brackets obtained by the correspondence principle

\[
[A, B] = \frac{i}{N_c} \{A, B\}
\]  

(30)
where $\frac{1}{N_c}$ plays the role of $\hbar$. Thus,

\[
i\{H_i^I(r, r'), H_j^J(s, s')\} = P_j^I(r, s')\delta^j_\delta(s' - s) - M_i^J(s, r')\delta^j_\delta(s' - r)
- \delta_-(s, r')\delta^j_\delta\delta(s' - r),
\]

\[
i\{H_i^I(r, r'), M_j^J(s, s')\} = H_k^I(r, s')\delta^j_\delta\delta(r' - s),
\]

\[
i\{H_i^I(r, r'), P_j^K(s, s')\} = -H_i^I(s, r')\delta^j_\delta\delta(r - s'),
\]

\[
i\{I_i^I(r, r'), P_j^K(s, s')\} = H_k^I(r, s')\delta^j_\delta\delta(r' - s),
\]

\[
i\{I_i^I(r, r'), M_j^K(s, s')\} = -H_k^I(s, r')\delta^j_\delta\delta(r - s'),
\]

\[
i\{P_j^I(r, r'), P_j^K(s, s')\} = P_j^K(r, s')\delta^j_\delta\delta(r' - s) - P_j^K(s, r')\delta^j_\delta\delta(s' - r),
\]

\[
i\{M_j^I(r, r'), M_j^K(s, s')\} = M_j^I(r, s')\delta^j_\delta\delta(s - r') - M_k^I(s, r')\delta^j_\delta\delta(s' - r)
+ \frac{1}{2}(\epsilon(r, s')\delta(s - r') + \epsilon(s, r')\delta(r - s'))\delta^j_\delta\delta^i_\delta,
\]

and the hamiltonian

\[
H = \int h(r, r')M_i^I(r, r')drdr' + \frac{1}{2}\tilde{\alpha}\int G(r, r')\{P_j^I(r, r')P_j^I(r', r) + H_i^I(r, r')H_i^I(r', r) + M_i^J(r, r')M_i^J(r', r)]drdr'
\]

define a classical theory. The equations of motion of this classical theory can be worked out by a straightforward (but tedious) calculation of the Poisson brackets of the observables with the hamiltonian. We get

\[
i\frac{\partial H_i^I(r, r')}{\partial u} = \frac{1}{2}\tilde{\alpha}\int ds'H_i^I(r, s')G(s', s)\delta_-(s', r') + \int ds'H_i^I(r, s')h(s', r')
- \frac{1}{2}\tilde{\alpha}\int ds'G(r', s')[M_i^J(s', r')H_j^I(r, s') + P_j^I(r, s')H_i^I(s', r')]
+ \frac{1}{2}\tilde{\alpha}\int ds'G(s', r)[P_j^I(r, s')H_i^I(s', r') + M_i^J(s', r')H_j^I(r, s')],
\]

\[
i\frac{\partial M_i^J(r, r')}{\partial u} = \frac{1}{4}\tilde{\alpha}\int ds'[G(s', r')M_i^J(s', r')\epsilon(r, s') - G(r, s')M_i^J(r, s')\epsilon(s', r')]
- \frac{1}{2}\tilde{\alpha}\int ds'G(s', r')[M_i^J(s', r')M_i^J(r, s') + H_j^I(s', r')H_i^I(r, s')]
+ \frac{1}{2}\tilde{\alpha}\int ds'G(s', r')[M_i^J(s', r')M_i^J(r, s') + H_j^I(s', r')H_i^I(r, s')]
+ \int ds[M_i^J(r, s)h(s, r') - h(r, s)M_i^J(s, r')],
\]

\[
i\frac{\partial P_j^I(r, r')}{\partial u} = \frac{1}{2}\tilde{\alpha}\int ds'G(r, s')[P_K^I(r, s')P_j^K(s', r') + H_i^I(r, s')H_j^I(s', r')]
- \frac{1}{2}\tilde{\alpha}\int ds'G(s', r')[P_K^I(r, s')P_j^K(s', r') + H_i^I(r, s')H_j^I(s', r')].
\]
These equations of motion however do not describe the classical hadron theory completely. The bilinears satisfy some constraints, whose origin is described in the appendix in a simplified fermion theory. The constraint for $M$ has been derived in the appendix. It becomes in our present notation,

$$\int M_j^i(r, r') M_k^j(r', r'') dr' - \frac{1}{2} \int [\epsilon(r, r') M_k^j(r', r'') + M_k^i(r, r') \epsilon(r', r'')] dr' + \int H_i^j(r, r') H_k^j(r', r'') dr' = 0.$$  

Similar arguments show that

$$\int H_j^i(r, r') M_j^i(r', r'') dr' - \int H_j^i(r, r') \delta_+(r', r'') dr$$

$$+ \int P_j^i(r, r') H_j^i(r', r'') dr' = 0$$

and

$$\int H_j^i(r, r') H_j^i(r', r'') dr' + \int P_j^K(r, r') P_j^K(r', r'') dr' - P_j^K(r, r'') = 0$$  

(31)

The equations of motion and the constraints above define the classical hadron theory equivalent to spherical QCD in the large $N_c$ limit.

Any static solution to these equations of motion and constraints can be used as the starting point for a semiclassical expansion of spherical QCD. The most obvious solution is the vacuum:

$$M_j^i(r, r') = 0 \quad H_j^i(r, r') = 0 \quad P_j^i(r, r') = 0.$$  

(32)

The small fluctuations around this vacuum will describe the mesons in our approximation. We are currently interested in the heavy–light mesons described by the field $H_j^i$. (It should also be possible to get static solutions, solitons, that deviate from the vacuum by a finite amount, describing for example baryons with one heavy quark).

Expanding around the vacuum, we find that the equation for $H$ decouples from the others to linear order:

$$i \frac{\partial H_j^i(r, r')}{\partial u} = \frac{1}{2} \tilde{\alpha} \int ds' H_j^i(r, r') G(s', r) \delta_-(s', r') + \int ds' H_j^i(r, s') h(s', r').$$  

(33)

The constraint on $H$ becomes to linear order,

$$\int H_j^i(r, r') \delta_+(r', r'') dr' = 0.$$  

(34)
Now, \( H(r', r) \) is the wavefunction of the heavy–light meson; we should expect this to be the product of a heavy quark wavefunction concentrated at the origin, and a light quark wave function. The ansatz \( H^I_i(r', r) = \delta(r')\psi(r)c^I_i \) is therefore reasonable. The constant \( c^I_i \) will determine the internal quantum numbers (spin, flavor) that label the degenerate levels of the meson. We get

\[
\frac{i}{2} \frac{\partial \psi(r')}{\partial r} = \frac{1}{2} \tilde{\alpha} \int ds' \psi(s')G(0, s')\delta_-(s', r') + \int ds' \psi(s')h(s', r') \quad (35)
\]

Now recall that \( h(s', r') = -h(r', s') \), \( \delta_-(s', r') = \delta_+(r', s') \) so that

\[
-\frac{i}{2} \frac{\partial \psi(r')}{\partial r} = -\frac{1}{2} \tilde{\alpha} \int ds' \delta_+(r', s')\psi(s')G(s', 0) + \int ds' h(r', s')\psi(s') \quad (36)
\]

Thus we arrive at the equation for stationary states,

\[
-\frac{1}{2} [i\partial_r + k^2 r^{-1}(i\partial_r)^{-1} r^{-1}]\psi(r) + \int \delta_+(r, s)V(s)\psi(s)ds = E\psi \quad (37)
\]

Here, \( V(r) = -\frac{\tilde{\alpha}}{2r} \) is the Coulomb potential and \( E \) the binding energy of the meson. This is to be supplemented by the constraint

\[
\int \delta_-(r, s)\psi(s)ds = 0 \quad (38)
\]

The above pair of integral equations will determine the meson wavefunction and masses in our model. They describe the propagation of a light quark in the spherically symmetric potential created by the heavy quark. Since there are no gluon self–interactions within our approximations, this potential has turned out to be the Coulomb potential. Therefore the eigenvalue problem we get is scale invariant.

In fact we should expect the scale invariance to be broken by quantum effects; no scale invariant equation can have a discrete spectrum of bound states. In the next section we will perform a renormalization of the coupling constant which will make sure that the ground state energy is finite.

If we expand

\[
\psi(r) = \int_0^\infty u_\lambda(r)\tilde{\psi}(\lambda)d\lambda \quad (39)
\]

the constraint is automatically satisfied. The remaining equation can also be simplified,

\[
\lambda\tilde{\psi}(\lambda) + \frac{1}{2} \tilde{\alpha} \int_0^\infty K(\lambda, \lambda')\tilde{\psi}(\lambda')d\lambda' = E\tilde{\psi}(\lambda) \quad \text{for } \lambda > 0. \quad (40)
\]
Here
\[ K(\lambda, \lambda') = \int dr V(r) u^*_\lambda(r) u_{\lambda'}(r) \] (41)
is the integral kernel of the potential in the basis diagonalizing \( h \). This is the analogue of 't Hooft’s integral equation. Using the explicit form of the functions \( u \) we can simplify these equations further. (For example if \( k = 1 \) they are spherical Bessel functions; for arbitrary \( k \), \( u(r) \) can be obtained as a power series.) Our result is then similar to a Wiener–Hopf integral equation. The two dimensional analogue of this equation is discussed in ref. [13].

5. Renormalization and Beta Function

The eigenvalue equation of the free hamiltonian
\[ -\frac{1}{2}(i\partial_r + k^2r^{-1}(i\partial_r)^{-1}r^{-1})u(\lambda r) = \frac{\lambda}{2} u(\lambda r) \] (42)
can also written as the differential equation
\[ [-\partial_r(r\partial_r u) + \frac{k^2}{r}u] = -i\lambda \partial_r(\lambda u). \] (43)
The solutions are of the form \( e^{i\lambda r} \) times a polynomial in \( \frac{1}{i\lambda r} \). For the special case \( k = 1 \) which we will now study in detail,
\[ u(\lambda r) = \frac{1}{\sqrt{\pi}}[1 - \frac{1}{i\lambda r}]e^{i\lambda r}. \] (44)
These functions are orthonormal in the sense that
\[ \int_0^\infty u^*(\lambda r)u(\lambda' r)dr = \delta(\lambda - \lambda'). \] (45)
The solutions to the constraints are wavefunctions of positive radial momentum:
\[ \psi(r) = \int_0^\infty \tilde{\psi}(\lambda) u(\lambda r)d\lambda \] (46)
The integral equation is equivalent to minimizing the energy,
\[ \mathcal{E}[^{\tilde{\psi}}] = \int_0^\infty \lambda|\tilde{\psi}(\lambda)|^2d\lambda + \int_0^\infty V(r)|\psi(r)|^2dr \] (47)
where \( ||\psi||^2 = \int_0^\infty |\tilde{\psi}(\lambda)|^2d\lambda \).

Since it is difficult to solve this problem exactly, we will find the ground state energy by a variational principle.
However there is a problem: the ground state energy is divergent. The potential and kinetic energies are both of dimension 1 in energy units, so that with $\tilde{\psi}_\mu(\lambda) = \tilde{\psi}(\mu \lambda)$

$$E(\tilde{\psi}_\mu) = \frac{1}{\mu} E(\tilde{\psi})$$

(48)

for any $\mu > 0$. If there is one state with $E < 0$ (an example of which is given below), there are states of arbitrarily negative energy. We must introduce a cutoff and make the bare coupling constant depend on it such that the ground state energy remains finite as the cutoff is removed. If $a > 0$ is such a short distance cutoff, the regularized energy will be of the form

$$E(a, \tilde{\alpha}_0(a)) = a^{-1} E_1(\tilde{\alpha}_0(a))$$

(49)

by ordinary dimensional analysis. Define the beta function by

$$\beta(\tilde{\alpha}_0(a)) = -a \frac{\partial \tilde{\alpha}_0(a)}{\partial a}.$$  

(50)

The condition that $E$ be independent of $a$ is then determined the beta function:

$$\beta(\tilde{\alpha}_0) = -\frac{E_1(\tilde{\alpha}_0)}{E'_1(\tilde{\alpha}_0)}.$$  

(51)

Such a nonperturbative renormalization scheme was introduced by Thorn in a nonrelativistic context [14].

As in field theory, a naive short distance cutoff is not convenient. It is better to use instead an analogue of dimensional (or analytic) regularization. Define the regularized potential to be

$$V_0(r) = -\frac{\tilde{\alpha}_0}{2} \frac{\mu_0}{(\mu_0 r)^{1-\epsilon}}.$$  

(52)

We could then compute the regularized energy,

$$E(\mu_0, \tilde{\alpha}_0, \epsilon) = \mu_0 E_1(\alpha, \epsilon)$$  

(53)

from which the beta function can be determined.

We are not able to calculate the regularized ground state energy exactly; instead, we will estimate it by a variational principle. The ansatz

$$\tilde{\psi}(\lambda) = \lambda e^{-b\lambda} \quad \text{for} \quad \lambda > 0$$  

(54)
gives

\[ E \leq \frac{3}{2b} - \frac{\tilde{\alpha}_0}{\pi b} \{ \frac{4}{\epsilon} + 4 \log \, \mu_0 b - 3 \} + \, O(\epsilon) \]  

(55)

Minimizing the r.h.s. in the variational parameter \( b \) gives

\[ \log \mu_0 b = [1 - \frac{1}{\epsilon} + \frac{3\pi}{8}] + \frac{3\pi}{8\tilde{\alpha}_0} \]  

(56)

so that

\[ E_1(\tilde{\alpha}_0) \leq -\frac{8\tilde{\alpha}_0}{\pi b (\tilde{\alpha}_0)}. \]  

(57)

This gives,

\[ \beta(\tilde{\alpha}) \approx -\frac{8\tilde{\alpha}^2}{8\tilde{\alpha} + 3\pi}. \]  

(58)

This beta function has only one zero at \( \alpha = 0 \), near which

\[ \frac{\beta(\tilde{\alpha})}{\tilde{\alpha}} = -\frac{16}{3} \frac{\tilde{\alpha}}{2\pi} + \cdots \]  

(59)

so that our renormalized theory is asymptotically free. This can be compared with the well known one-loop result of perturbation theory [15], [16], [17], [18]

\[ \frac{\beta(\alpha)}{\alpha} = -\frac{11 N^2 - 1}{3} \frac{\alpha}{2N \pi} + \cdots \]  

(60)

Recalling that \( \tilde{\alpha} = N\alpha \) we see that (for large \( N \)) we have a 16 where the one-loop result has a 11; i.e., agreement to about 30%.

We found that a more general ansatz such as

\[ \tilde{\psi}(\lambda) = \lambda(1 + c\lambda)e^{-b\lambda} \]  

(61)

has a minimum at \( c = 0 \); i.e., does not improve the energy. The ansatz

\[ \tilde{\psi}(\lambda) = \lambda^c e^{-b\lambda} \]  

(62)

has a minimum at \( c = 1.17 \) and lowers the energy by less than a percent. To get a substantially better estimate of the ground state energy (and hence the beta function) we must solve the problem numerically. It is also of much interest to study the excited states. These issues are currently under study and we hope to report on them soon.
In this appendix we will describe how the large $N_c$ limit of fermion bilinears leads to a classical theory whose phase space is the Grassmannian. One can think of this as a quantum mechanical analogue of ‘bosonization’.

Let us consider a set of operators satisfying Canonical anticommutation relations (CAR):
\[ [\chi^\dagger_{\alpha a}, \chi_{\beta b}]_+ = \delta^a_\beta \delta^a_b \]
all other pairs of anticommutators being zero. Here $\alpha, \beta = 1 \cdots N_c$ while $a, b$ are all others quantum numbers such as spin, flavor, momentum etc. It will not matter to us for now what exactly they represent. For simplicity we will assume that they have a finite range. A representation for these CAR can be constructed on the Fermionic Fock space $F$ as usual.

Now consider the subspace $F_0$ of ‘color singlet’ states:
\[ \frac{1}{2}[\chi^\dagger_{\alpha a}, \chi_{\beta a}] |s> = 0 \quad \text{for} \quad |s> \in F_0. \]
(64)
(The operators $\frac{1}{2}[\chi^\dagger_{\alpha a}, \chi_{\beta a}]$ generate the algebra $U(N_c)$ of ‘color’.) In particular this condition implies that exactly half the one-particle states are occupied.

The color singlet bilinears
\[ \Phi^a_b = \frac{1}{2N_c}[\chi^\dagger_{\alpha a}, \chi_{\alpha b}] \]
map $F_0$ to itself. In this subspace, they obey the linear constraint
\[ \Phi^a_a = 0. \]
(66)

Also, they satisfy the algebra
\[ [\Phi^a_b, \Phi^c_d] = \frac{1}{N_c}[\delta^c_d \Phi^a_b - \delta^a_d \Phi^c_b]. \]
(67)

The meaning of this is that $N_c \Phi^a_b$ form a representation of the Lie algebra of the Unitary group on $F_0$. This is an irreducible representation so that the only operators in $F_0$ that commute with all the $\Phi^a_b$ are multiples of the identity.

The $\Phi^a_b$ form a complete set of observables of the quantum system whose Hilbert space is $F_0$. Within the space of color singlet states, any observable can be written in terms of
the $\Phi^a_b$. However, they are not independent variables; they satisfy a quadratic constraint between states in $\mathcal{F}_0$:

$$
\Phi^a_c \Phi^c_b = \left( \frac{1}{4} + \frac{1}{4N_c} \delta^a_c \right) \delta^a_b + \frac{1}{2N_c} \delta^c_c \Phi^a_b. \tag{68}
$$

To derive these one just has to reorder the factors so that the color singlet condition on the states can be used.

In the large $N_c$ limit this becomes just

$$
\Phi^a_c \Phi^c_b = \frac{1}{4} \delta^a_b. \tag{69}
$$

Now, if we define the normal ordered bilinear

$$
M^a_b = \Phi^a_b + \frac{1}{2} \delta^a_b
$$

we will get in the large $N_c$ limit the constraint

$$
M^a_c M^c_b - \frac{1}{2} (M^a_c \epsilon^c_b + \epsilon^a_c M^c_b) = 0. \tag{70}
$$

To derive the constraints in the text we must replace $\chi$ by $\begin{pmatrix} \chi \\ Q \end{pmatrix}$ and the matrix $\Phi$ by $\begin{pmatrix} M - \frac{1}{2} \epsilon & H^\dagger \\ H & P - \frac{1}{2} \delta \end{pmatrix}$.

In the large $N_c$ limit we get as dynamical variable a hermitian traceless matrix whose square is a multiple of the identity. The phase space, which is the set of all such matrices, is the ‘Grassmannian’. The commutation relations now are replaced by Poisson brackets on the classical variables. We can recover the finite $N_c$ theory by ‘quantizing’ this classical theory whose phase space is the Grassmannian. The wavefunctions of the quantum theory can be chosen to be some sort of functions on the Grassmannian. The precise statement is that the wavefunctions are holomorphic sections of a line bundle on the Grassmannian. (The Grassmannian is a complex manifold, so the concept of holomorphicity makes sense.)

This description of the wavefunctions is analogous to the coherent state picture of the harmonic oscillator. Line bundles on the Grassmannian are labelled by an integer (Chern class) which can be identified with $N_c$. For each $N_c$ therefore we have one quantum theory; the Hilbert space of this quantum theory can be shown to be just $\mathcal{F}_0$. This way we can recover the finite $N_c$ theory as a quantization of the classical theory on the Grassmannian. In particular this construction shows that there are no other constraints on the $\Phi^a_b$ in the large $N_c$ limit.
We will encounter some divergences in extending this to the infinite dimensional case. However once we deal with normal ordered operators, the divergences can be handled consistently. A more formal approach is to define the infinite dimensional Grassmannian with certain convergence conditions and to construct the space $\mathcal{F}_0$ as a space of sections of line bundles on it. \cite{19} \cite{20}. We do not need this construction for the purposes of this paper.

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