We introduce the definition of Haantjes manifolds with symmetry and explain why these manifolds appear in the theory of integrable systems of hydrodynamic type and in topological field theories.

Keywords: integrable system, WDVV equation, flat Riemannian metric

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1. Introduction

In 1968, Yano and Ako [1] significantly extended the work of Schouten and Nijenhuis on differential concomitants. Several years earlier, in 1940 and 1951, Schouten and Nijenhuis had discovered two remarkable differential concomitants, now called the Schouten bracket and the Nijenhuis torsion, respectively associated with a skew-symmetric tensor field of type (2, 0) and a tensor field of type (1, 1). In [1], Yano and Ako found the analogues of these concomitants for a wide class of higher-order tensor fields. In particular, they noted that if $C_{jk}(x)$ are the components of a tensor field $C$ of type (1, 2) on a manifold $M$, then the functions

$$[C, C]^m_{jkir} := \sum_{s=1}^{n} \left( C_{sj} \frac{\partial C^m_{ir}}{\partial x_k} + C_{sk} \frac{\partial C^m_{ir}}{\partial x_j} - C_{sr} \frac{\partial C^m_{jk}}{\partial x_l} - C_{sl} \frac{\partial C^m_{jr}}{\partial x_l} + \frac{\partial C^m_{jk}}{\partial x_s} C_{ir} - \frac{\partial C^m_{jr}}{\partial x_s} C_{sk} \right)$$

are the components of a tensor field $[C, C]$ of type (1, 4) if the components of $C$ satisfy the symmetry conditions

$$C^l_{jk} = C^l_{kj}$$

and the associativity conditions

$$\sum_{l=1}^{n} C^l_{jk} C_{im} = \sum_{l=1}^{n} C^l_{mk} C_{ij}.$$
could be taken to the end if the tensor field $C$ satisfied the algebraic constraints written above. The new object lacked any geometric interpretation, and it was consequently difficult to foresee possible uses for it. The Yano–Ako bracket therefore did not attract much attention and was rapidly forgotten.

A surprising application of the above bracket was found many years later in the theory of integrable systems of hydrodynamic type. In 2007, a study of the universal Whitham equations introduced by Krichever [2] led Boris Konopelchenko and me to consider a special class of deformations of associative and commutative algebras called coisotropic deformations [3]. We found that these deformations were controlled by a remarkable set of differential equations, which we called the central system. We were then unaware of the work of Yano and Ako but quickly realized that the central system is just the vanishing of the Yano–Ako bracket. The effect attracted our attention to the work of these authors and convinced us of the importance of their bracket in the theory of integrable systems of hydrodynamic type. Accordingly, we began to seek a geometric interpretation of their bracket. The outcome of the ensuing work is the concept of a Haantjes manifold discussed here.

Haantjes manifolds are a tool for analyzing the foundations of the theory of integrable systems of hydrodynamic type from a geometric standpoint. The concept of a Haantjes manifold can help to understand what is the minimal system of assumptions to be set at the basis of the theory and what is the role of each separate assumption. It can be regarded as a ramification of the concept of a bi-Hamiltonian manifold. The main novelty is the construction leading to a square of exact 1-forms. This construction extends the recursive procedures of bi-Hamiltonian geometry. The square of 1-forms recovers many interesting integrability conditions that had previously appeared in different contexts. Among them are the Yano–Ako equations of the theory of deformations, the WDVV equations of topological field theories [4], [5], the integrability conditions for the multiplicative structure of a Frobenius manifold [6], [7], and the integrability conditions of the theory of semi-Hamiltonian systems of hydrodynamic type [8]. Haantjes manifolds endowed with symmetries also have an interesting link to Riemannian geometry.

This paper comprises three sections. In Sec. 2, we present the concept of a Haantjes manifold. In Sec. 3, we show the links of these manifolds to the Yano–Ako equations, the WDVV equations, and the theory of integrable systems of hydrodynamic type. In Sec. 4, we finally establish the link to Riemannian geometry.

2. Haantjes manifolds

We consider a manifold $M$ of dimension $n$ equipped with an exact 1-form $dA$ and a tensor field $K: TM \to TM$ of type $(1,1)$. It is convenient to regard $K$ as a vector-valued 1-form on $M$ and let $d_K$ denote the derivation on forms associated with $K$ according to the theory of derivations of Frölicher and Nijenhuis [9]. The tensor field $K$ naturally acts on the 1-form $dA$ mapping it to a new 1-form denoted by $KdA$. For a wide choice of $dA$ and $K$, it often happens that the new 1-form $KdA$ is still exact. The exactness condition is the weak cohomological condition

$$dd_K A = 0,$$

which takes the form of an Euler–Poisson–Darboux system of partial differential equations in a suitable system of coordinates. Repeating the process, we find that the 1-form $K^2dA$ is seldom exact. There is a new strong obstruction, represented by 2-form vanishing:

$$d_Kd_K A = 0.$$  \hspace{1cm} (4)

We study this obstruction more attentively.
Lemma 1. We assume that the first iterated 1-form \( KdA \) is exact. Then the second iterated 1-form \( K^2dA \) is exact if and only if \( dKdKA = 0 \) or, equivalently, if and only if the 1-form \( dA \) annihilates the Nijenhuis torsion of \( K \) viewed as a vector-valued 2-form on \( M \).

**Proof.** Let \( \alpha \) be any 1-form on \( M \) and \( \alpha' = K\alpha \). Then the \( d \) and \( dK \) differentials of these 1-forms are related by the identities

\[
\begin{align*}
    da' (\xi, \eta) &= d\alpha (K\xi, \eta) + d\alpha (\xi, K\eta) - dK \alpha (\xi, \eta), \\
    dK \alpha' (\xi, \eta) &= d\alpha (K\xi, K\eta) + T_K(\xi, K\eta)).
\end{align*}
\]

Here, the symbol \( T_K(\xi, \eta) \) denotes the Nijenhuis torsion of \( K \) viewed as a vector-valued 2-form evaluated on an arbitrary pair of vector fields \( \xi \) and \( \eta \). These identities can be used as follows. We take \( \alpha = KdA = dKA \) in the first identity. This immediately shows that \( \alpha' = K^2dA \) is exact if and only if \( dKdKA = 0 \). We take \( \alpha = dA \) in the second identity. This immediately shows that \( dKdKA = 0 \) if and only if \( dA \) annihilates the torsion of \( K \), as claimed.

This lemma explains why the Nijenhuis torsion of \( K \) plays a prominent role in the theory of recursion operators. It points out that there are only two possibilities. If the Nijenhuis torsion of \( K \) vanishes, then there are no other obstructions to the iteration process. All iterated 1-forms \( KdA, K^2dA, K^3dA \), etc., are exact. If the Nijenhuis torsion of \( K \) does not vanish, then the iteration process ends after two steps because the 1-form \( K^3dA \) cannot be exact. A way of circumventing this obstruction is to reject the idea of a single recursion operator and consider a more general scheme with several recursion operators acting simultaneously. They should not be powers of a single recursion operator. To see how to manage the new situation and find the right conditions, we again consider the case of a single recursion operator from a different standpoint. We let the first \( n \) powers of \( K \) be denoted by the symbols

\[
    K_1 = Id, \quad K_2 = K, \quad K_3 = K^2, \quad \ldots, \quad K_n = K^{n-1},
\]

to prepare the transition to the general case. It is clear from what has already been said that the doubly iterated 1-forms \( K_1K_2dA \) are exact because \( K_1K_2 = K^{j+l-2} \). Of course, the triply iterated 1-forms \( K_1K_2K_3dA \) are also exact, but it is wise to ignore this fact. Indeed, insisting on it would lead again into the old case. Hence, the right idea is to work with a family of \( n \) distinct recursion operators \( (K_1 = Id, K_2, K_3, \ldots, K_n) \) that behave like the first \( n \) powers of \( K \) up to the second iteration without being powers of \( K \). This idea is formalized in the following definition of a Haantjes manifold.

**Definition 1.** We consider a manifold \( M \) of dimension \( n \) equipped with an exact 1-form \( dA \). We assume that \( (K_1 = Id, K_2, K_3, \ldots, K_n) \) are \( n \) pairwise commuting tensor fields of type \( (1,1) \) on \( M \):

\[
    K_jK_l = K_lK_j.
\]

The manifold \( M \) is a **Haantjes manifold** if all the doubly iterated 1-forms \( K_1K_2dA \) are closed and therefore locally exact. These forms constitute the square of 1-forms of the Haantjes manifold. Because we limit ourselves to studying the local geometry of the manifold, we always take the 1-forms \( K_1K_2dA \) to be exact. We set

\[
    K_jK_l dA = dA_{jl}
\]

call the scalar functions \( A_{jl} \) the **potential functions of the Haantjes manifold**. They form a symmetric matrix \( H \) called the **matrix potential of the manifold**.
It is clear from the preceding lemma that the recursion operators $K_j$ of a Haantjes manifold cannot be chosen arbitrarily. They must satisfy suitable integrability conditions, weaker than the vanishing of the Nijenhuis torsion, stemming from the exactness condition for the 1-forms $dA_j$. The discussion of the full set of integrability conditions is a delicate problem, which goes beyond the scope of this paper, but one basic condition must be mentioned.

**Proposition 1.** We assume that at least one of the recursion operators $K_j$ has real and distinct eigenvalues. Then the Haantjes torsion of all the recursion operators $K_j$ vanishes.

We recall that the Haantjes torsion of a tensor field $K$ of type $(1,1)$ is a vector-valued 2-form related to the Nijenhuis torsion of $K$ according to

$$H_K(\xi, \eta) = T_K(K\xi, K\eta) - KT_K(K\xi, \eta) - KT_K(\xi, K\eta) + K^2T_K(\xi, \eta).$$

The vanishing of the Haantjes torsion is the necessary and sufficient condition for the integrability of the eigendistributions of the tensor field $K$, as Haantjes showed in 1955 [10] (under the semisimplicity assumption stated above). Therefore, if at least one of the recursion operators of a Haantjes manifold has real and distinct eigenvalues, then on the manifold, there exists a privileged system of coordinates in which all the recursion operators become diagonal. These coordinates are usually called canonical coordinates (or Riemann invariants) in the theory of integrable systems of hydrodynamic type.

We do not prove the above proposition here, because the proof is long and because the result is not used in what follows. Its only use is to justify the name Haantjes manifolds given the manifolds defined above and to motivate introducing the notion of a weak Haantjes manifold.

**Definition 2.** A weak Haantjes manifold is a manifold $M$ equipped with a single exact 1-form $dA$ and with a single tensor field $K$ of type $(1,1)$ satisfying the three conditions

$$\text{Haantjes}(K) = 0, \quad (8)$$
$$dd_KA = 0, \quad (9)$$
$$d_Kd_KA = 0. \quad (10)$$

The weak Haantjes manifold seems to be the minimal and hence basic geometric structure underlying the theory of recursion operators. In many examples, we find that we can extend a weak Haantjes manifold to a full Haantjes manifold by recovering the missing $n-2$ recursion operators directly from $K$. This happens, for instance, when the Nijenhuis torsion of $K$ has particularly nice forms. In these cases, we can construct the missing tensor fields $K_j$ as polynomial functions of $K$. We are thus almost back to the initial situation where the Nijenhuis torsion of $K$ should vanish. The main difference is that we have lost the rule of powers, for which suitable polynomials in $K$ are substituted, constructed case by case such that the nonvanishing of the torsion of $K$ is compensated. The problem of extending a weak Haantjes manifold to a full Haantjes manifold is a fascinating problem to study, and it leads to many interesting results. But it requires going deeper into the problem of classifying weak Haantjes manifolds. At the moment, there is only a reasoned collection of examples. We hesitated to define a weak Haantjes manifold, but we note that if the condition that the tensor field $K$ can be diagonalized is added, then it is easy to find that the conditions $dd_KA = 0$ and $d_Kd_KA = 0$ permit recovering the definition of semi-Hamiltonian systems given by Tsarev in canonical coordinates. Hence, the definition of a weak Haantjes manifold can be seen as an intrinsic formulation of Tsarev’s theory. We are convinced that the concept of a weak Haantjes manifold has a central position in the present theory.
3. Three properties of Haantjes manifolds

Our purpose in this section is to outline the links between Haantjes manifolds, the Yano–Ako differential concomitant, topological field theories, and integrable systems of hydrodynamic type.

The relation to the Yano–Ako differential concomitant is quite simple. It is based on the remark that the recursion operators $K_j$ of a Haantjes manifold form an associative commutative algebra with unity. To prove this, we expand the 1-form $K_j K_l dA$ in the basis of 1-forms $dA_m = K_m dA$. We call its components $C_{jl}^m(A_i)$:

$$K_j K_l dA = \sum C_{jl}^m dA_m.$$  \hspace{1cm} (11)

Acting with the recursion operator $K_n$ on both sides of this equation, we obtain the identity

$$K_j K_l dA_n = \sum C_{jl}^m K_m dA_n,$$

which allows concluding that

$$K_j K_l = \sum C_{jl}^m K_m$$  \hspace{1cm} (12)

because the 1-forms $dA_n$ form a basis. This relation proves that the recursion operators of a Haantjes manifold form an associative algebra. To proceed toward the Yano–Ako equations, we note that the structure constants $C_{jl}^m(A_i)$ of this algebra are partial derivatives of the potential functions $A_{jl}$ with respect to the coordinates $A_m$, as shown by their definition. We write the Yano–Ako equations in these coordinates and see that all the terms cancel in pairs because of the above property. Hence, the Yano–Ako equations hold in the coordinates $A_m$. Because they are tensorial, they hold in any coordinate system. We conclude that the structure constants of the algebra of recursion operators of a Haantjes manifold satisfy the Yano–Ako equations (or central system in the terminology of [3]). This result provides a class of solutions of the Yano–Ako equations having a geometric meaning, but it does not yet completely solve the problem stated in Sec. 1. It remains unclear how exhaustive this class of solutions might be.

The relation to topological field theories involves the potential functions $A_{jl}$. They are scalar functions on the manifold and can accordingly be written in any coordinate system. Nevertheless, the recursion operators select a class of special coordinates on the Haantjes manifold, which makes a rather special property of these functions manifest. To develop this property, we need the concept of a generator of a Lenard chain.

**Definition 3.** A generator of a Lenard chain on a Haantjes manifold is a vector field $\xi$ such that the iterated vector fields $\xi_j = K_j \xi$ are linearly independent and commute in pairs.

For the moment, we assume that such a generator exists, and we note that it provides a distinguished system of coordinates on the Haantjes manifold because the vector fields $\xi_j$ commute. We call $t_j$ the corresponding coordinates: $\xi_j = \partial/\partial t_j$. We write the potential functions in these coordinates. We then have the following remarkable property.

**Proposition 2.** In the coordinates defined by the generator of a Lenard chain, the matrix of the potential functions of a Haantjes manifold is the Hessian matrix of a function $F(t_1, t_2, \ldots, t_n)$. This function is a solution of the (generalized) WDVV equations of topological field theories. Any solution of the WDVV equations can be thus obtained.

This proposition was proved in [11]. It subordinates the existence of the function $F(t_1, t_2, \ldots, t_n)$ to the existence of a generator of a Lenard chain. This problem leads to the theory of integrable systems of hydrodynamic type.
It is well known that there is a one-to-one correspondence between systems of equations of hydrodynamic type and tensor fields of type \((1, 1)\), such as \(K\). To pass from the tensor field to the differential equations, it suffices to introduce any coordinate system \(u^j\) on the manifold and consider the corresponding components of the tensor field \(K\) defined by

\[
K du^j = \sum K^j_l(u) du^l. \tag{13}
\]

The equations of hydrodynamic type are then written in the form

\[
\frac{\partial u^j}{\partial t} = \sum K^j_l(u) \frac{\partial u^l}{\partial x}. \tag{14}
\]

Inverting the steps, we easily pass from the differential equations to the tensor field \(K\). The tensorial character of \(K\) is guaranteed by the transformation law of the system of differential equations under a change of the unknown functions. It is fair to say that the tensor field \(K\) gives an intrinsic description of the differential equations, which allows controlling the properties of the equations in any coordinate system.

On a Haantjes manifold, we have \(n\) tensor fields \(K_j\) and therefore \(n\) systems of differential equations of hydrodynamic type, each composed of \(n\) differential equations.

**Proposition 3.** The \(n\) systems of differential equations of hydrodynamic type associated with the tensor fields \(K_1 = Id, K_2, \ldots, K_n\) on a Haantjes manifold are mutually compatible, and there hence exists a solution \(u^j(t_1, t_2, \ldots, t_n)\) common to all of them. Moreover, in the system of coordinates \(A_m\), the differential equations take the form of conservation laws.

**Proof.** To prove the compatibility of the \(n\) systems of PDEs, it is necessary and sufficient to prove that the tensor fields \(K_j\) satisfy the identity

\[
[K_j \xi, K_i \xi] - K_j[\xi, K_i \xi] - K_i[K_j \xi, \xi] = 0 \tag{15}
\]

for any choice of the vector field \(\xi\). This identity assures the equality of mixed second-order derivatives of the field functions \(u^j\) with respect to the independent variables \(t^k\) because the tensor fields \(K_j\) commute. To prove the identity, it is useful to evaluate the above vector expression on the basis of the differentials \(dA_m\) in order to use the basic relation \(K_j dA_m = dA_{jm}\). As before, we let \(\xi_j\) denote the vector field \(K_j \xi\), knowing that these vector fields do not commute, because \(\xi\) is not assumed to be a generator of a Lenard chain. We recall that \(\xi_j(A_{jm}) - \xi_j(A_{tm}) = 0\) because the tensor fields \(K_j\) commute. We then prove the identity as follows:

\[
dA_m([K_j \xi, K_i \xi] - K_j[\xi, K_i \xi] - K_i[K_j \xi, \xi]) =
\frac{\partial}{\partial t} = \xi_j \xi_i(A_{jm}) - \xi_j \xi_i(A_{tm}) =\]

\[
\xi_j \xi_i(A_{jm}) - \xi_j \xi_i(A_{jm}) + \xi_j \xi(A_{jm}) - \xi_j \xi(A_{tm}) + \xi_j \xi(A_{tm}) =\]

\[
\xi_j \xi(A_{tm}) - \xi_j \xi(A_{jm}) = 0.
\]

The existence of a common solution is thus established. To see that the differential equations can be written as conservation laws, it suffices to write them explicitly in the \(A_m\) coordinates and again use the basic relation \(K_j dA_m = dA_{jm}\). ■
We can now discuss the problem of the existence of the generators of Lenard chains on a Haantjes manifold. The tool is the common solution $u^j(t_1,t_2,\ldots,t_n)$ of the differential equations, whose existence has just been established. We regard this solution as the definition of a change of coordinates on the manifold $M$, from the old coordinates $u^j$ to the new coordinates $t^k$. We let $\partial/\partial t^k$ denote the vector fields of the corresponding basis in $TM$. It is almost a tautology to see that these vector fields form a Lenard chain because of the form of the differential equations. We can therefore say that there is a one-to-one correspondence between the solutions of the systems of differential equations of hydrodynamic type associated with the tensor fields $K_j$ and the Lenard chains of vector fields on a Haantjes manifold. Combined with Proposition 2, this remark shows that a solution of the WDVV equations is associated with any solution of the system of hydrodynamic type (and vice versa). This is one of the possible ways to introduce the Hirota tau function in this framework.

4. Haantjes manifolds with symmetry

There is a second class of vector fields worthy of attention on a Haantjes manifold in addition to the generators of Lenard chains. They are the conformal symmetries of the manifold.

**Definition 4.** A vector field $\xi$ such that the Lie derivatives of the 1-form $dA$ and of the tensor fields $K_j$ along $\xi$ are respectively multiples of $dA$ and $K_j$,

$$\text{Lie}_\xi(dA) = \alpha \cdot dA,$$

$$\text{Lie}_\xi(K_j) = \gamma_j \cdot K_j,$$

is called a *conformal symmetry of the Haantjes manifold*. It is a symmetry if the functions $\alpha$ and $\gamma_j$ vanish.

As before, we let $\xi_j$ denote the vector fields $K_j \xi$. They form a basis in $TM$ without defining a system of coordinates on $M$ because they do not commute. We use this basis, the conformal symmetry, and the potential functions $A_{jl}$ to define a second-order symmetric tensor field on $M$ by setting

$$g(\xi_j,\xi_l) = \xi(A_{jl}).$$

Explicitly, this means that the components of the tensor field $g$ on the basis $\xi_j$ are the derivatives of the potential functions $A_{jl}$ along the conformal symmetry $\xi$. In this section, we prove that this tensor field has the following remarkable property.

**Proposition 4.** We assume that the matrix $\xi(A_{jl})$ is nonsingular and that the functions $\alpha$ and $\gamma_j$ are constant. Then $g$ is a flat semi-Riemannian metric on $M$.

The proof of this proposition is split into four lemmas. The statement of these lemmas is made easier by introducing the symbols $g_{jl}$ for the components $g(\xi_j,\xi_l)$ of the metric and the functions

$$c_{jlm} := \xi_j(A_{tm}) = \xi_l(A_{mj}) = \xi_m(A_{jl})$$

as shorthand for the derivatives of the potential functions $A_{jl}$ along the vector fields $\xi_m$ of the basis generated by the conformal symmetry.

The first lemma yields an expression for the commutators of the vectors $\xi_j$ of the basis.

**Lemma 2.** We have the equality

$$[\xi_j,\xi_l] = (\gamma_l - \gamma_j) \sum c_{jlm} \frac{\partial}{\partial A_m}.$$
Proof. Because $\xi$ is a conformal symmetry,

$$[\xi, \xi_j] = \text{Lie}_\xi(K_j \xi) = \gamma_j \xi_j.$$ 

Consequently,

$$dA_m([\xi_j, \xi_l]) = \xi_j \xi_l(A_m) - \xi_l \xi_j(A_m) = \xi_j \xi(A_{lm}) - \xi_l \xi(A_{jm}) = [\xi_j, \xi_l](A_{lm}) - [\xi_l, \xi_j](A_{jm}) = (\gamma_l - \gamma_j) c_{jlm}.$$ 

The lemma is proved. ■

The second lemma specifies the value of the derivatives of the components of the metric along the vectors fields $\xi_j$ and also the value of the metric on the commutators $[\xi_j, \xi_l]$.

Lemma 3. We have the equalities

$$g(\xi_m, [\xi_j, \xi_l]) = (\gamma_l - \gamma_j) c_{jlm}, \quad (21)$$

$$\xi_m(g_{jl}) = (\alpha + \gamma_j + \gamma_l) c_{jlm}. \quad (22)$$

Proof. Equation (21) is a simple consequence of Lemma 2 and the formula

$$\xi_j = \sum g_{jm} \frac{\partial}{\partial A_m}, \quad (23)$$

which gives the expansion of the vector fields $\xi_j$ in the basis associated with the coordinates $A_m$. This expansion follows immediately from the definitions of the vector fields $\xi_j$ and the metric $g$.

To prove Eq. (22), we note that

$$\text{Lie}_\xi dA_j = \text{Lie}_\xi(K_j K_l dA) = (\alpha + \gamma_j + \gamma_l) dA_{jl}.$$ 

Consequently,

$$dg_{jl} = d\xi(A_{jl}) = \text{Lie}_\xi dA_{jl} = (\alpha + \gamma_j + \gamma_l) dA_{jl}.$$ 

The last equation gives the statement because

$$\xi_m(g_{jl}) = (\alpha + \gamma_j + \gamma_l) \xi_m(A_{jl}).$$ 

The lemma is proved. ■

Lemmas 2 and 3 allow computing the coefficients of the Levi-Civita connection of $g$ evaluated on the basis $\xi_j$. We must use the Koszul formula [12]:

$$2g(\nabla_{\xi_j} \xi_l, \xi_m) = \xi_j g(\xi_l, \xi_m) + \xi_l g(\xi_j, \xi_m) - \xi_m g(\xi_j, \xi_l) -$$

$$- g(\xi_j, [\xi_l, \xi_m]) + g(\xi_l, [\xi_m, \xi_j]) + g(\xi_m, [\xi_j, \xi_l]).$$

Lemma 4. The coefficients of the Levi-Civita connection on the basis $\xi_j$ are given by the formula

$$\nabla_{\xi_j} \xi_l = \left(\frac{\alpha}{2} + \gamma_l\right) \sum c_{jlm} \frac{\partial}{\partial A_m} \quad (24)$$

or, equivalently, by the formula

$$g(\nabla_{\xi_j} \xi_l, \xi_m) = \left(\frac{\alpha}{2} + \gamma_l\right) c_{jlm}. \quad (25)$$
Proof. The proof is a simple application of Lemmas 2 and 3 and the Koszul formula.

We can now finally compute the Riemann tensor

\[ R_{\xi_j \xi_l}(\xi_m) = (\nabla_{\xi_j} \nabla_{\xi_l} - \nabla_{\xi_l} \nabla_{\xi_j} - \nabla_{[\xi_j, \xi_l]})(\xi_m). \]

Lemma 5. The covariant components of the Riemann tensor in the basis \( \xi_j \) are given by

\[ R_{\alpha \beta \gamma \delta} = \left( \frac{\alpha}{2} + \gamma_\mu \right) \left( \frac{\alpha}{2} + \gamma_\nu \right) \sum_{s,t} g^{st}(c_{jms}c_{lpt} - c_{jpt}c_{lms}). \]

Proof. We split the computation of the Riemann tensor into two parts. First, we consider the term \( g(\nabla_{[\xi_j, \xi_l]}\xi_m, \xi_p) \). We obtain

\[ g(\nabla_{[\xi_j, \xi_l]}\xi_m, \xi_p) = \sum_s g(\nabla(\gamma_\mu - \gamma_\nu)(c_{jls} + \frac{1}{n_s})\xi_m, \xi_p) = \sum_s (\gamma_\mu - \gamma_\nu) c_{jls} g(\nabla g^{st}\xi_t, \xi_m, \xi_p) = \sum_{s,t} (\gamma_\mu - \gamma_\nu) c_{jls} g^{st} g(\nabla \xi_t, \xi_m, \xi_p) = \left( \frac{\alpha}{2} + \gamma_\mu \right) \sum_{s,t} g^{st} c_{jls} c_{lmp} \]

according to Lemma 4. We then consider the two remaining terms. Again using the properties of the connection 1-form, formalized by the Koszul axioms [12], and for the first time using the assumption that the functions \( \alpha \) and \( \gamma_j \) are constant, we obtain

\[ g(\nabla_{\xi_j} \nabla_{\xi_l} - \nabla_{\xi_l} \nabla_{\xi_j})(\xi_m, \xi_p) = \left( \frac{\alpha}{2} + \gamma_\mu \right) \left( \frac{\alpha}{2} + \gamma_\nu \right) \sum_{s,t} g^{st}(c_{lms}c_{jpt} - c_{jms}c_{lpt}). \]

We can simplify this expression by noting that

\[ \xi_j(c_{lmp}) - \xi_l(c_{jmp}) = (\gamma_\mu - \gamma_\nu) \sum_{s,t} c_{jls} g^{st} c_{lmp}. \]

Indeed,

\[ \xi_j(c_{lmp}) - \xi_l(c_{jmp}) = \xi_j(A_{mp}) - \xi_l(A_{mp}) = [\xi_j, \xi_l](A_{mp}) = \]

\[ = (\gamma_\mu - \gamma_\nu) \sum_s c_{jls} \frac{\partial A_{mp}}{\partial s} = (\gamma_\mu - \gamma_\nu) \sum_{s,t} c_{jls} g^{st} \xi_t(A_{mp}) = \]

\[ = (\gamma_\nu - \gamma_\mu) \sum_{s,t} c_{jls} g^{st} c_{lmp}. \]

Adding the two terms of the Riemann tensor with the proper signs, we finally obtain the sought expression of its covariant components.

The vanishing of the Riemann tensor is now a consequence of the fact that the recursion operators form a commutative associative algebra.
Proof of Proposition 4. The Riemann tensor contains the expression $\sum_{s,t} g^{st} c_{jms} c_{lpt} - c_{jpt} c_{lms}$.

We note that

$$\sum_s g^{st} c_{jms} = \sum_s g^{ls} \xi_s (A_{jm}) = \frac{\partial A_{jm}}{\partial A_t} = C^t_{jm}.$$ 

Therefore,

$$\sum_{s,t} g^{st} c_{jms} c_{ltp} = \sum_t C^t_{jm} \xi_t (A_{lt}) = \sum_{t,q} g^{pq} \frac{\partial A_{lt}}{\partial A_q} = \sum_{t,q} g^{pq} C^q_{lt} C^t_{jm},$$

and consequently

$$R_{mpjl} = \left( \frac{\alpha}{2} + \gamma_m \right) \left( \frac{\alpha}{2} + \gamma_p \right) \sum_{q,t} g^{pq} \left( C^q_{lt} C^t_{jm} - C^q_{jt} C^t_{lm} \right).$$

This expression vanishes because of associativity condition (3) satisfied by the structure constants of the algebra of the recursion operators. 

Because it is well-known that flat Riemannian metrics define Poisson brackets for systems of differential equations of hydrodynamic type, this result leads back to the bi-Hamiltonian setting, which was our point of departure. In some sense, the circle has been closed.

5. Concluding remarks

This paper aimed at explaining the role of Haantjes manifolds in the theory of integrable systems of hydrodynamic type and related fields. The main novelty presented here is the square of 1-forms $dA_{jl}$. It is a simple but nontrivial extension of the concept of bi-Hamiltonian recurrence, which so far seems to have passed unnoticed. As shown here, the square of 1-forms recovers many interesting integrability conditions that had already appeared in different contexts. Among them are the Yano–Ako equations of the theory of deformations, the WDVV equations of topological field theories, and the integrability conditions of the theory of semi-Hamiltonian systems of Tsarev. All these integrability conditions had already been thoroughly studied in the past, in particular, by Boris Dubrovin in his theory of Frobenius manifolds. Repetitions are therefore unavoidable. Nevertheless, we hope that the geometric framework of Haantjes manifolds provides a new view of old things and that it allows better seeing what is the minimal system of assumptions to be set at the basis of the theory and what is the role of each separate assumption. For instance, it shows that the role of the metric is not so essential in understanding the WDVV equations. We will discuss the points of contact with and the differences from the previous theories elsewhere.

Appendix

In this appendix, we recall the definition of the operator $d_K$ and exhibit a few of its interesting properties, to make the paper reasonably self-contained. We also take the opportunity to point out a very fine characterization of semisimple recursion operators with a vanishing Haantjes torsion discovered by Nijenhuis in 1955.

1. Definition of $d_K$. According to the theory of Frölicher and Nijenhuis, the differential operator $d_K$ is the unique derivation of degree one on the algebra of differential forms that satisfies the four conditions

$$d_K A = K dA, \quad d_K (\alpha + \beta) = d_K \alpha + d_K \beta,$$

$$d_K (\alpha \wedge \beta) = d_K \alpha \wedge \beta + (-1)^w \alpha \wedge d_K \beta, \quad d_K d + dd_K = 0.$$ 

This definition is rather abstract but is easily converted into a powerful algorithm for computing the differential $d_K$ in any concrete situation. First, we start by writing the differential form $\alpha$ as a sum of
products of 1-forms. We then use the second and third conditions to lead \( dK \) to act on any single 1-form appearing in \( \alpha \). By linearity, the problem reduces to evaluating the differential of simple 1-forms of the type \( \alpha = BdA \), where \( A \) and \( B \) are scalar functions. This problem is solved by the first and last conditions. We thus always end up evaluating only the differential of scalar functions.

We follow this procedure to prove the important identity

\[
d^2_K A(\xi, \eta) = dA(T_K(\xi, \eta)),
\]

which holds for any scalar function \( A \). We first note that

\[
d^2_K A = dK(d_K A) = dK \sum_l \frac{\partial A}{\partial x^l} d_K x^l = \sum_l dK \left( \frac{\partial A}{\partial x^l} \right) \wedge d_K x^l + \sum_l \frac{\partial A}{\partial x^l} d^2_K x^l =
\]

\[
= \sum_{l,m} \frac{\partial^2 A}{\partial x^l \partial x^m} d_K x^m \wedge d_K x^l + \sum_l \frac{\partial A}{\partial x^l} d^2_K x^l = \sum_l \frac{\partial A}{\partial x^l} d^2_K x^l.
\]

We then evaluate the differentials \( d^2_K x^j \) of the coordinate functions according to the above procedure:

\[
d^2_K x^j = d_K(K dx^j) = \sum_p d_K(K^j_p) dx^p = \sum_p d_K(K^j_p) \wedge dx^p - K^j_p d d_K x^p =
\]

\[
= \sum_{l,m,p} \left( \frac{\partial K^j_p}{\partial x^m} K^l_m - K^j_l \frac{\partial K^l_p}{\partial x^m} \right) dx^m \wedge dx^p.
\]

We conclude that

\[
d^2_K x^j = \sum_{l<m} T^j_{lm} dx^l \wedge dx^m,
\]

where \( T^j_{lm} \) are the components of the torsion tensor \( T_K \) of \( K \). Inverting this formula, we can write the torsion tensor of \( K \) in the form

\[
T_K = \sum_j d^2_K x^j \otimes \frac{\partial}{\partial x^j}.
\]

This implies that

\[
dA(T_K(\xi, \eta)) = \sum_l \frac{\partial A}{\partial x^l} d^2_K x^l(\xi, \eta).
\]

The comparison with \( d^2_K A \) proves the identity mentioned above.

2. Identities. In the study of the recurrence of exact 1-forms pursued in Sec. 2, we used identities (5) relating the differentials \( d \) and \( d_K \) of any 1-form \( \alpha \) to the differentials of its iterated 1-form \( \alpha' = K\alpha \). We now prove these identities. By linearity, it suffices to consider the special pair of 1-forms \( \alpha = BdA \) and \( \alpha' = Bd_K A \), where \( A \) and \( B \) are arbitrary functions. For \( d_K \alpha \), we have

\[
d_K \alpha = d_K B \wedge dA + B d_K dA = d_K B \wedge dA - B dd_K A =
\]

\[
ed_K B \wedge dA + dB \wedge d_K A - d(Bd_K A) = d_K B \wedge dA + dB \wedge d_K A - da'.
\]

Evaluated on two arbitrary vector fields \( \xi \) and \( \eta \), this equation gives

\[
d_K \alpha(\xi, \eta) = d\alpha(K\xi, \eta) + d\alpha(\xi, \eta) - da'(\xi, \eta).
\]
This is already the first identity. To prove the second identity, we consider
\[ d_K \alpha' = d_K B \wedge d_K A + B d_K^2 A. \]
Evaluated on an arbitrary pair of vector fields \( \xi \) and \( \eta \), this equation gives
\[ d_K \alpha'(\xi, \eta) = (dB \wedge dA)(K \xi, K \eta) + B dA(T_K(\xi, \eta)). \]
Because \( dB \wedge dA = d\alpha \), this equation can also be written as
\[ d_K \alpha'(\xi, \eta) = d\alpha(K \xi, K \eta) + \alpha(T_K(\xi, \eta)), \]
which is the second identity.

3. Recursion operators with vanishing Haantjes torsion. To conclude this appendix, we use the above formalism to write a result of Nijenhuis, concerning the recursion operators with a vanishing Haantjes torsion, in a form that is particularly terse and useful. From the preceding discussion, it is clear that the vanishing of the Haantjes torsion is an algebraic constraint on the Nijenhuis torsion that must be mirrored by the differential 2-form \( d^2 K B \) of any function \( B \). We assume that the recursion operator \( K \) has real and distinct eigenvalues. According to Nijenhuis (cf. Eq.(3.10) in [13]), there then exist at most \( n-1 \) 1-forms \( \alpha_0, \alpha_1, \ldots, \alpha_{n-2} \) such that
\[ d^2 K B = \alpha_0 \wedge dB + \alpha_1 \wedge K dB + \cdots + \alpha_{n-2} \wedge K^{n-2} dB. \]
The 1-forms are independent of the function \( B \). They generate a differential ideal that according to the result of Nijenhuis contains the differential \( d^2 K B \) of any scalar function \( B \). This ideal is certainly an important element of the geometry of the recursion operator, and studying it should provide clues for classifying the recursion operators with a vanishing Haantjes torsion. In this appendix, we give an example of such an ideal.

One of the simplest possible classes of recursion operators with a vanishing Haantjes torsion is certainly the class of operators whose ideal is generated by a single exact 1-form \( \alpha_0 = dA \). This class is not void. For instance, the recursion operators associated with the Coxeter groups of type \( A_n \) have this property. In this class of examples,
\[ d^2 K B = dA \wedge dB \]
for any function \( B \). Therefore, for \( B = A \), we obtain \( d^2 K A = 0 \). Hence, the function \( A \) characterizing the torsion of \( K \) satisfies the strong cohomological condition \( d_K d_K A = 0 \). There are cases where the function \( A \) also satisfies the weak cohomological condition \( ddK A = 0 \). These cases are clearly particularly remarkable. Indeed, without any additional assumption on \( K \), we can implement a recursive procedure that allows generating a sequence of functions \( A_i \) satisfying the same cohomological conditions as \( A \). The recurrence relation is dictated by the constraint \( d^2 K B = dA \wedge dB \) on the torsion of \( K \). Each function \( A_i \) in turn defines a new tensor field \( K_i \). It is the unique tensor field commuting with \( K \) and mapping the 1-form \( dA \) to the 1-form \( dA_i \). By this process, the single operator \( K \) generates an infinite sequence of operators \( K_i \). It turns out that these tensor fields satisfy the conditions defining a Haantjes manifold. This is a concrete example of how a weak Haantjes manifold can be prolonged into a Haantjes manifold when the Nijenhuis torsion of \( K \) has a “nice form,” as claimed in Sec. 2. When we discussed this subject informally there, we had this class of examples in mind. We hope that these short remarks might help to clarify the sense of that informal discussion.
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