Solving a perturbed amplitude-based model for phase retrieval

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Abstract—In this paper, we propose a new algorithm for solving phase retrieval problem, i.e., the reconstruction of a signal \( x \in \mathbb{H}^n \) (\( \mathbb{H} = \mathbb{R} \) or \( \mathbb{C} \)) from phaseless samples \( b_j = |\langle a_j, x \rangle|, \ j = 1, \ldots, m \). The proposed algorithm solves a perturbed amplitude-based model for phase retrieval and is correspondingly named as Perturbed Amplitude Flow (PAF). We prove that PAF can recover \( x \) (|\( c | = 1 \)) under \( O(n) \) Gaussian random measurements (optimal order of measurements). Unlike several existing \( O(n) \) algorithms that can be theoretically proven to recover only real signals, our algorithm works for both real and complex signals. Starting with a designed initial point, our PAF algorithm iteratively converges to the true solution at a linear rate. Besides, PAF algorithm enjoys both low computational complexity and the simplicity for implementation. The effectiveness and benefit of the proposed method can be validated by both the simulation studies and the experiment of recovering natural image.

Index Terms—Phase retrieval, Perturbed amplitude flow, Linear convergence.

I. INTRODUCTION

A. Problem Setup and Related Work

In this paper, we consider the well-known phase retrieval problem, which aims to recover a signal \( x \in \mathbb{H}^n \), where \( \mathbb{H} = \mathbb{R} \) or \( \mathbb{C} \), from phaseless measurements

\[ b_j = |\langle a_j, x \rangle|, \ j = 1, \ldots, m. \]

Here \( x \in \mathbb{H}^d \) is called the target signal or target vector and the vectors \( a_j \in \mathbb{H}^n \) for all \( j \) are called the measurement vectors. Phase retrieval has many applications in both science and engineering, such as X-ray crystallography [1], astronomy [2], optics [3], [4] and microscopy [5].

Due to the removal of phase information in the measurements \( |\langle a_j, x \rangle| \), we can only recover \( x \) up to a unimodular constant. It is also known that \( O(n) \) measurements can recover a signal \( x \in \mathbb{H}^n \). Particularly, it was shown that \( m \geq 2n - 1 \) and \( m \geq 4n - 4 \) generic measurements \( \{a_j\}_{j=1}^m \subset \mathbb{H}^n \) are sufficient to recover any \( x \in \mathbb{H}^n \) up to a unimodular constant for \( \mathbb{H} = \mathbb{R} \) and \( \mathbb{H} = \mathbb{C} \), respectively [6]–[8].

Many algorithms have been developed for the phase retrieval problem. They can be roughly divided into two categories: the convex methods and the non-convex ones. For convex methods, the general strategy is to lift the phase retrieval problem into a problem of recovering a rank one matrix and apply the semi-definite programming to solve it. The first such method, called PhaseLift [9]–[11], can achieve the exact recovery using \( m = \mathcal{O}(n) \) independent Gaussian random measurements \( a_j, j = 1, \ldots, m \). However, such an approach is computationally inefficient for large dimensional problems since semi-definite programming for \( n \times n \) matrices is slow for large \( n \). An alternative method called PhaseMax [12]–[14] aims to recover the signal \( x \) by solving the model

\[
\max_{\tilde{z}} \Re(\langle \tilde{z}, \tilde{x} \rangle) \text{ subject to } |\langle a_j, \tilde{z} \rangle| \leq b_j, \tag{1}
\]

where \( \tilde{z} \) is an approximation to the true signal \( x \). It is proved that this method is able to recover \( x \) with high probability when \( m \geq 4n/\theta \) where \( \theta = 1 - \frac{2}{3} \text{angle}(\tilde{z}, x) \). However, numerical experiments have shown that larger oversampling ratios \( m/n \) are often required for exact recovery, especially compared to several non-convex algorithms.

In a different direction a series of non-convex approaches have been proposed and studied. Among such schemes, early studies are based on the alternating projection approach, including the works by Gerchberg and Saxton [15] and Fienup [16]. These methods often perform well numerically but they lack theoretical foundation. Motivated by the success of alternating minimization, Netrapalli et al [17] developed the AltMinPhase method that is shown to achieve linear convergence with \( \mathcal{O}(n \log^3 n) \) Gaussian random measurements and resampling. More recently another framework was proposed, in which one starts from a “good” initial guess and try to iteratively refine it by solving the following intensity-based model [18], [19]

\[
\min_z g(z) := \frac{1}{4m} \sum_{j=1}^m |\langle a_j^*, z \rangle|^2 - b_j^2, \tag{2}
\]

or the amplitude-based model [20]–[23]

\[
\min_z f(z) := \frac{1}{2m} \sum_{j=1}^m |\langle a_j^*, z \rangle - b_j |^2, \tag{3}
\]

or the Poisson data model [24] derived from the maximum likelihood estimation

\[
\min_z h(z) := -\sum_{j=1}^m (b_j^4 \log(|\langle a_j^*, z \rangle|^2) - |\langle a_j^*, z \rangle|^2). \tag{4}
\]

Many algorithms have been developed for such a framework that can effectively solve the phase retrieval problem when the initial guess is sufficiently close to the true solution. Among those, Candès et al have developed the Wirtinger Flow...
method (WF) \cite{18} to recover \(x\) by solving the model \(2\) via the gradient descent algorithm. It achieves provable linear convergence with step size \(\sim O(1/n)\) and \(m = O(n \log n)\) Gaussian random measurements. The result is refined in real field \(\mathbb{H} = \mathbb{R}\) to achieve a reduction of measurements \(m = O(n)\) by solving model \(3\) via gradient descent \cite{22} or via truncated gradient descent \cite{21}, or by solving model \(4\) via modified gradient descent \cite{24}. In detail, Zhang et al. \cite{25} have proposed Reshaped Wirtinger Flow, which named as Amplitude Flow (AF) in this paper to coincide with the name in \cite{21}, to solve the model \(3\) by gradient descent. Wang et al. \cite{21} have proposed Truncated Amplitude Flow (TAF) to solve the model \(3\) by truncated gradient descent. Chen and Candes \cite{24} have designed Truncated Wirtinger Flow (TWF), which solves the model \(3\) by modified gradient descent. Although AF, TAF and TWF can achieve linear convergence under the optimal order of measurements, they all heavily rely on the fact that sign\((a_j, z)\) \(\in \{1, -1\}\), which only can be satisfied in the real field. For phase retrieval of complex signals, the theoretical results of AF, TAF and TWF have not been proved so far. Besides, both TAF and TWF all need truncation of weak samples in the gradient step, which is computationally inefficient. In this paper, we introduce a new perturbed amplitude-based model to address these theoretical deficiencies and limitations.

### B. Our Contribution: The Perturbed Amplitude Flow (PAF)

We propose the Perturbed Amplitude Flow (PAF) algorithm in this paper via the following model:

\[
\min_z f_\epsilon(z) := \min_z \frac{1}{2m} \sum_{j=1}^{m} \left( \sqrt{|a_j|^2 + \epsilon_j^2} - \sqrt{b_j^2 + \epsilon_j^2} \right)^2, \tag{5}
\]

where \(\epsilon = [\epsilon_1, \ldots, \epsilon_m] \in \mathbb{R}^m\) have prescribed value, with the requirement that

\[
\epsilon_j \neq 0 \quad \text{for all} \quad b_j \neq 0. \tag{6}
\]

Note that if \(b_j = 0\), then \(\left( \sqrt{|a_j|^2 + \epsilon_j^2} - \sqrt{b_j^2 + \epsilon_j^2} \right)^2\) is smooth regardless of the value of \(\epsilon_j\), even when \(\epsilon_j = 0\). The loss function \(f_\epsilon\) is thus smooth. When all \(\epsilon_j = 0\), we have the classic amplitude-based model \(3\). So we shall name this model as the perturbed amplitude-based model.

In the perturbed amplitude-based model \(5\), \(\epsilon\) plays a role similar to truncation while keeping the loss function smooth. With suitable choice of \(\epsilon\), one can control the size of the gradient. This is essential for avoiding the extreme large gradient components. More precisely, note that the Wirtinger derivative of \(f_\epsilon\) is

\[
\nabla f_\epsilon(z) = \frac{1}{m} \sum_{j=1}^{m} \left( 1 - \frac{\sqrt{b_j^2 + \epsilon_j^2}}{\sqrt{|a_j|^2 + \epsilon_j^2}} \right) a_j a_j^* z.
\]

As a result, the magnitude of \(\nabla f_\epsilon(z)\) is under control even when \(|a_j|^2 = 0\). This fact guarantees that the gradient satisfies the curvature condition, which we shall introduce in Lemma \ref{lem:curvature}. With the new model, numerical tests show that our proposed algorithm outperforms AF \((\epsilon = 0)\) in terms of success rate for real signals, as shown in Figure \ref{fig:success-rate}. Under the perturbed amplitude-based model \(5\), we also prove that with the vanilla gradient descent algorithm we can achieve linear convergence with \(m = O(n)\) measurements for both real and complex signals (see Section II). The result improves upon the WF method, which uses \(m = O(n \log n)\) measurements, or the AF, TAF, TWF methods, which can be theoretically proved only for real signals phase retrieval.

In the context of phase retrieval, most gradient descent algorithms are designed with the model \(3\) in mind because of the smoothness of the loss function \(g(z)\) in \(2\). But in practice, numerical experiments show that the non-smooth model \(3\) performs better than the smooth model \(2\). Thus to solve the model \(3\), sub-gradient are often employed. This is the case for the TAF \cite{21} method, which also uses truncation to avoid spurious generalized gradient components (which typically have components with large magnitude). While it is provable that TAF can recover real signals with \(m = O(n)\) measurements, extending such a result to complex signals are hard since the analysis heavily relies on the fact that sign\((a_j, z)\) \(\in \{1, -1\}\).

We call our algorithm the Perturbed Amplitude Flow (PAF). Compared with the previous algorithms for solving the model \(3\) or \(4\), the convergence result of PAF holds for both real and complex signals. Numerical experiments show that the proposed PAF method is slightly more efficient although comparable computationally with TAF, and significantly more efficient than TWF (see Section III). We believe the reason is that truncated methods, such as TAF, TWF, incur additional computational cost on measuring the gradient components.

### C. Notations

Let \(x \in \mathbb{H}^n\) (\(\mathbb{H} = \mathbb{C}\) or \(\mathbb{H} = \mathbb{R}\)) be the target signal. Throughout this paper, we assume that \(a_j \in \mathbb{H}^n, j = 1, \ldots, m\) are \(m\) independent and identically distributed standard Gaussian random measurement vectors, i.e. \(a_j \sim \mathcal{N}(0, I)\) for \(\mathbb{H} = \mathbb{R}\) and \(a_j \sim \mathcal{N}(0, I/2) + i\mathcal{N}(0, I/2)\) for \(\mathbb{H} = \mathbb{C}\). For each measurement \(a_j\), we obtain \(b_j = |a_j^* x|\). We shall attempt to recover the original signal \(x\) from \(b_j, j = 1, \ldots, m\) by solving the perturbed amplitude-based model \(5\). In this paper, we use \(C, c\) or the subscript/superscript form of them to represent constants and their values vary according to the context. Since for phase retrieval the best we can do is to recover the target signal \(x\) up to a global phase/sign, we use the following definition for distance between two vectors \(x, z \in \mathbb{H}^n\):

\[
\text{dist}(z, x) := \min_{\phi \in [0, 2\pi]} \|z - e^{i\phi} x\| := \|z - e^{i\phi(x)} x\|, \tag{7}
\]

where

\[
\phi(x) := \arg\min_{\phi \in [0, 2\pi]} \|z - e^{i\phi} x\|. \tag{8}
\]

For any \(\rho \geq 0\), we define the \(\rho\)-neighborhood of \(x\) as

\[
S_\rho(x) := \left\{ z \in \mathbb{C}^d : \text{dist}(z, x) \leq \rho \|x\| \right\}. \tag{9}
\]
II. Perturbed Amplitude Flow Algorithm

A. Initialization

To avoid iterations getting trapped in undesirable stationary points, a proper initialization is essential to any nonconvex optimization problem. To achieve this goal for nonconvex models in phase retrieval, many initialization methods have been proposed, such as the spectral initialization method [18], a modified spectral initialization method [24] and the null initialization method [21]. These methods are all based on finding the eigenvector corresponding to the largest eigenvalue of a specially designed Hermitian matrix.

Here we adopt the initialization strategy given in [19], which is shown to provide a good initial guess under $O(n)$ measurements. With this strategy, the initial guess $z_0$ is obtained by calculating the eigenvector corresponding to the largest eigenvalue of the Hermitian matrix

$$ Y = \frac{1}{m} \sum_{j=1}^{m} (\gamma - \exp(-b_j^2/\lambda^2)) a_j a_j^* $$

with $\gamma = 1/2$ for $\mathbb{H} = \mathbb{C}$ or $\gamma = 1/\sqrt{3}$ for $\mathbb{H} = \mathbb{R}$, and normalized to $\|z_0\| = \lambda$, where $\lambda$ is defined by

$$ \lambda^2 = \frac{1}{m} \sum_{j=1}^{m} b_j^2. $$

Lemma II.1 ([19]). Let $z_0$ be the above initial guess. For any $\xi > 0$, there exists a $C_\xi > 0$ such that for $m \geq C_\xi n$, dist$(z_0, x) \leq \xi \|x\|$ holds with probability at least $1 - 4 \exp(-c_\xi n)$.

B. Gradient Descent Iteration

After initialization to obtain $z_0$, we use gradient descent on the loss function $f_\epsilon$ given in [5] by

$$ f_\epsilon(z) := \frac{1}{2m} \sum_{j=1}^{m} \left( |a_j^*z|^2 + \epsilon_j^2 - \sqrt{b_j^2 + \epsilon_j^2} \right)^2, $$

to iteratively refine the estimation:

$$ z_{k+1} = z_k - \mu \nabla f_\epsilon(z_k), \quad (10) $$

where $\mu$ is the step size and $\nabla f_\epsilon(z)$ is the Wirtinger derivative of $f_\epsilon(z)$ in complex variables $z, \bar{z}$ which is defined as

$$ \nabla f_\epsilon(z) := \left( \frac{\partial f_\epsilon(z, \bar{z})}{\partial z} \right) \mid_{z=\text{constant}} \ast \left( \frac{\partial f_\epsilon(z, \bar{z})}{\partial \bar{z}} \right) \mid_{z=\text{constant}} = \frac{1}{m} \sum_{j=1}^{m} \left( 1 - \frac{\sqrt{b_j^2 + \epsilon_j^2}}{|a_j^*z|^2 + \epsilon_j^2} \right) a_j a_j^* z. $$

As simple as the scheme (10) may look, our main result proves that it can achieve linear convergence under the optimal order of measurements $m = O(n)$ by choosing $\epsilon = \sqrt{\alpha b}$ for an appropriately chosen parameter $\alpha > 0$.

Motivated by the technique used in WF, the proof of our main result is mainly based on the following two key lemmas, whose proofs are given in Section IV.

Lemma II.2. Let $x$ be the target signal and assume that $\epsilon$ satisfies [6]. For any $\delta > 0$, there exist constants $C_\delta, c_\delta > 0$ such that for $m \geq C_\delta n$, we have

$$ \|\nabla f_\epsilon(z)\| \leq (1 + \delta) \cdot \text{dist}(z, x) $$

with probability at least $1 - \exp(-c_\delta n)$.

This lemma implies that the gradient of $f_\epsilon$ is well controlled in the neighborhood of the target signal $x$.

Lemma II.3. Let $x$ be the target signal and assume that $\epsilon = \sqrt{\alpha b}$ with $0.37 \leq \alpha \leq 29$. There exist positive constants $C, c, \beta_\alpha$ depending on $\alpha$ such that for any $m \geq C n$ and $z \in S_x(1/10)$, we have

$$ \text{Re}(\langle \nabla f_\epsilon(z), z - xe^{i\phi_\alpha(z)} \rangle) \geq \beta_\alpha \cdot \text{dist}^2(z, x) \quad (12) $$

with probability at least $1 - \exp(-cn)$.

The constants in the lemma can in theory be explicitly estimated, although the theoretical estimates are typically “overkills” for practical applications, just like in other existing schemes. Later in Remark IV.1 we show more explicitly the relation between $\beta_\alpha$ and $\alpha$. Particularly, by setting $\alpha = 0.826$, $\beta_\alpha = 64/5945$ roughly reaches its largest value. For $\epsilon = \sqrt{\alpha b}$ with $\alpha \in [0.37, 29]$, Lemma II.3 guarantees sufficient descent along the search direction.

Set $h := e^{-i\phi_\alpha(z)} - x$ with $\rho = |h|$. Then

$$ \text{Re}(\langle \nabla f_\epsilon(z), z - xe^{i\phi_\alpha(z)} \rangle) = \frac{1}{m} \sum_{j=1}^{m} \left( 1 - \frac{\sqrt{b_j^2 + \epsilon_j^2}}{|a_j^*x + h|^2 + \epsilon_j^2} \right) (|a_j^*h|^2 + \text{Re}(h^*a_j a_j^*x)). $$

The main technique for proving Lemma II.3 is that we first fix one $z \in \mathbb{C}^n$ and then provide estimations separately for cases $|a_j^*h| \geq \rho|a_j^*x|$ and $|a_j^*h| < \rho|a_j^*x|$. An $\eta$-net argument is then used to obtain uniform control over all $z \in S_x(\rho)$.

Building on these two lemmas, we can now state and prove our main theorem, which establishes linear convergence of the PAF algorithm (10).

Theorem II.1. Under the conditions of Lemma II.3 let $\{z_k\}$ be the iterations given by (10) with $\mu = \beta_\alpha/1.001^2$. Assume that $z_0 \in S_x(1/10)$. Then there exist positive constants $C, c$ such that for $m \geq Cn$, with probability at least $1 - \exp(-cn)$ we have

$$ \text{dist}^2(z_{k+1}, x) \leq (1 - \beta_\alpha^2/1.001^2) \cdot \text{dist}^2(z_k, x). $$

In particular by taking $\alpha = 0.826$, with probability at least $1 - \exp(-cn)$ we have

$$ \text{dist}(z_k, x) \leq \frac{1}{1000} \left( 1 - 0.0107^2 \right)^{k/2} \cdot |x|. $$
Proof: According to the update rule (10), Lemma II.2 and Lemma II.3 for $m \geq Cn$, with probability at least $1 - \exp(-cn)$ we have

$$\begin{align*}
\text{dist}^2(z_{k+1}, x) &\leq \|z_{k+1} - xe^{i\phi(x)}(z_k)\|^2 \\
&= \|z_k - xe^{i\phi(x)}(z_k) - \mu \nabla f_c(z_k)\|^2 \\
&= \|z_k - xe^{i\phi(x)}(z_k)\|^2 \\
&\quad - 2\mu \text{Re}(\langle \nabla f_c(z_k), z_k - xe^{i\phi(x)}(z_k) \rangle) + \mu^2 \|\nabla f_c(z_k)\|^2 \\
&\leq \|z_k - xe^{i\phi(x)}(z_k)\|^2 - 2\mu \cdot \beta_\alpha \|z_k - xe^{i\phi(x)}(z_k)\|^2 \\
&\quad + \mu^2 \cdot 1.001^2 \|z_k - xe^{i\phi(x)}(z_k)\|^2 \\
&= (1 - \mu \cdot (2\beta_\alpha - 1.001^2)) \|z_k - xe^{i\phi(x)}(z_k)\|^2 \\
&= (1 - \beta_\alpha^2/1.001^2) \cdot \text{dist}^2(z_k, x).
\end{align*}$$

This establishes the linear convergence part of the theorem.

For the second part, we set $\alpha = 0.826$. Later in Remark IV.1 we show that one may take $\beta_\alpha = 64/5945$ in $\mu = \beta_\alpha/1.001^2$. Substituting these values in we thus obtain

$$\begin{align*}
\text{dist}(z_k, x) &\leq (1 - \beta_\alpha^2/1.001^2)^{1/2} \cdot \text{dist}(z_{k-1}, x) \\
&< (1 - 0.0107^2/1.001^2)^{1/2} \cdot \text{dist}(z_{k-1}, x) \\
&\leq \frac{1}{10} \left(1 - \frac{0.0107^2}{1.001^2}\right)^{k/2} \cdot \|x\|.
\end{align*}$$

As mentioned earlier, we can achieve $z_0 \in S_x(1/10)$ through initialization given in Lemma II.1, by setting $\xi = 1/10$. This also requires $m = O(n)$ measurements. Thus the combination of Lemma II.1 and Theorem II.1 yield linear convergence of the PAF algorithm.

III. NUMERICAL EXPERIMENTS

A. Simulation Study

To evaluate the performance of our PAF algorithm, we present a series of simulated tests and compare them with WF, TWF, AF and TAF. We perform all the simulations under the same initialization procedure. All experiments are carried out on Matlab 2018b with a 3.7 GHz Intel Core i7-8700K and 64 GB memory.

First we plot the relative error for the recovery of a complex-valued signal, in logarithmic scale versus the iteration count for WF, TWF, AF, TAF and our PAF. We choose $n = 512$ with $m = 4.5n$ and i.i.d. Gaussian random measurements $a_1, a_2, \ldots, a_m \in \mathbb{C}^n$. For the initialization, we follow the method in [19] with 50 power iterations. For the PAF algorithm we set $\epsilon = b$ and fix the step size $\mu = 2.5$. Note that AF is equivalent to our PAF algorithm with $\epsilon = 0$. We also consider the case where the measurements are contaminated by noise, i.e. $b = |Ax| + \omega$ where we set the noise to have distribution $\omega \sim N(0, I/10)$.

The results are plotted in Figure 1. It shows that PAF, TWF, TAF and AF, all of which converge linearly in theory, have comparable convergence rate. PAF seems to have a slight advantage possibly due to its ability to handle larger step size.

Next, we compare the empirical success rate of PAF with that of WF, TWF, TAF and AF. Here we set the maximum number of gradient-type iterations to $T = 2500$ for each scheme. In PAF, we set $n = 512$, $\epsilon = b$ and fix the step size to $\mu = 1$. We let $m/n$ vary from 1 to 6. A test is successful if the relative error is within $10^{-5}$ after the maximum number of iterations. For the test we compute the success rate by performing 100 random trials for each $m/n$. The results are given in Figure 2. Of particular note is that in the real case, PAF, TWF and TAF all perform better than AF, indicating the effectiveness of controlling the size of the gradient in all gradient descent algorithms for avoiding spurious stationary points. WF seems to lag behind other algorithms, unsurprisingly, as it agrees with the theoretical analysis.

![Figure 1](image-url)  
Fig. 1: Convergence experiments: Plot of relative error (log(10)) vs number of iterations for PAF (our algorithm), WF, TWF, TAF and AF method. Take $n = 512$, $m = 4.5n$. The figure (a) (for the exact measurements) and figure (b) (for noisy measurements) both show that our PAF method provides better solution and also converges faster.
Fig. 2: Success rate experiments: Empirical probability of successful test based on 100 random trails for different \( \frac{m}{n} \). Take \( n = 512 \) and change \( \frac{m}{n} \) between 1 and 6. The figures demonstrate that PAF, TWF, TAF and AF are better than WF in terms of the success rate.

B. Recovery of Natural Image

To show the efficiency and scalability of our algorithm, we use PAF to recover the Milky Way Galaxy image\(^1\) which is the image used in [18], [27] with the coded diffraction measurements. We denote the image by \( X \in \mathbb{R}^{1080 \times 1920 \times 3} \). This is a color image so it has three channels. Thus we actually perform phase retrieval for each of the three channels separately. Let \( x \) denote any of the color channels of \( X \). We have measurements

\[
\mathbf{b}^{(i)} = |FD^{(i)}x|, \quad 1 \leq i \leq L,
\]

where \( F \) denotes the \( n \times n \) discrete Fourier transform matrix, and \( D^{(i)} \) is a diagonal matrix having i.i.d. entries sampled from a distribution \( g \). Here we take the octanary pattern that \( g = g_1g_2 \), where \( g_1 \) and \( g_2 \) are independent with distributions

\[
g_1 = \begin{cases} 
1 & \text{with prob. } \frac{1}{4} \\
-1 & \text{with prob. } \frac{1}{4} \\
-i & \text{with prob. } \frac{1}{4} \\
i & \text{with prob. } \frac{1}{4}
\end{cases}
\]

and

\[
g_2 = \begin{cases} 
\frac{\sqrt{2}}{2} & \text{with prob. } \frac{4}{5} \\
\frac{\sqrt{3}}{2} & \text{with prob. } \frac{1}{5}
\end{cases}
\]

We set \( L = 20 \) and adopt the same initialization method for all schemes in our comparison. For each model, we record the time elapsed and the iterations needed to achieve relative error at \( 10^{-5} \) and \( 10^{-10} \), respectively. The results are shown in Table I. It is shown that PAF achieves the same level of precision and is comparable in efficiency with AF and TAF. Besides, note that it took PAF, TAF and AF the same number of iterations to achieve fixed relative error. Moreover, it’s reasonable that our PAF is a little bit slower than AF (\( \epsilon = 0 \)) with additional nonzero item \( \epsilon \). These three methods are significantly more efficient than WF and TWF.

### Table I

| Algorithm | Relative error | Iter | Time(s) |
|-----------|----------------|------|---------|
| WF        | \( 1 \times 10^{-5} \) | 172  | 407.07  |
|           | \( 1 \times 10^{-10} \) | 302  | 618.47  |
| TWF       | \( 1 \times 10^{-5} \) | 51   | 498.34  |
|           | \( 1 \times 10^{-10} \) | 118  | 985.96  |
| TAF       | \( 1 \times 10^{-5} \) | 37   | 211.80  |
|           | \( 1 \times 10^{-10} \) | 84   | 319.48  |
| AF        | \( 1 \times 10^{-5} \) | 37   | 193.08  |
|           | \( 1 \times 10^{-10} \) | 84   | 277.09  |
| PAF       | \( 1 \times 10^{-5} \) | 37   | 197.44  |
|           | \( 1 \times 10^{-10} \) | 84   | 287.16  |

\(^1\)Download from http://pics-about-space.com/milky-way-galaxy

Interestingly if we take a much smaller \( L = 6 \), while WF does not recover the target image, our PAF method actually performs better than with \( L = 20 \). It takes 300 iterations and computation time 136 sec to achieve recovery with a relative...
error of $5.05 \times 10^{-15}$ in Figure 5. While more iterations are taken here, the computational time is actually less because $L = 6$ is significantly smaller than $L = 20$.

IV. PROOF OF MAIN LEMMAS IN SECTION II-B

A. Proof of Lemma II.2

Proof: For each $z \in \mathbb{C}^n$, set $h = e^{-i\phi(x)}z - x$, where we recall that $\phi(x)$ is given in (8). Then $||h|| = \text{dist}(z, x)$. Denote $A = [a_1, \ldots, a_m]^* \in \mathbb{C}^{m \times n}$, $v = [v_1, v_2, \ldots, v_m]^T$ with $v_j = \left(1 - \frac{\sqrt{m} + \beta_j}{\sqrt{m} + \beta_j}ight)(a_j^*z)$. Note that we set $v_j = 0$ if $b_j = \epsilon_j = a_j^*z = 0$. Then $\nabla f_\epsilon(z) = \frac{1}{m} A^*v$. For any $\epsilon_j > 0$, we have

$$|v_j|^2 = \left|1 - \frac{\sqrt{b_j^2 + c_j^2}}{\sqrt{a_j^2 + c_j^2}}\right|^2 |a_j^*z|^2$$

$$= \frac{\left(\sqrt{a_j^2 + \epsilon_j^2} - \sqrt{a_j^2 + c_j^2}\right)^2}{|a_j^*z|^2 + \epsilon_j^2} |a_j^*z|^2$$

$$\leq \frac{\left(\sqrt{a_j^2} - \sqrt{a_j^2 + c_j^2}\right)^2}{|a_j^*z|^2}$$

$$= \frac{\left(\sqrt{a_j^2(x + h)^2} - \sqrt{a_j^2}^2\right)^2}{|a_j^*z|^2 + \epsilon_j^2}$$

$$\leq |a_j^*h|^2,$$

where the last inequality follows from the inequality $|\sqrt{t^2 + c^2} - \sqrt{a^2 + c^2}| \leq |t - s|$ for any $t, s, c \in \mathbb{R}$. According to Lemma A.1 (see the Appendix), for any $\delta' > 0$ and $m \geq C_{\delta'} n$ with a sufficiently large constant $C_{\delta'}$, the inequality

$$|v|^2 = \sum_{j=1}^{m} |v_j|^2 \leq \sum_{j=1}^{m} |a_j^*h|^2 \leq (1 + \delta') m||h||^2$$

holds with probability at least $1 - e^{-c_{\epsilon''} m}$ for some $c_{\epsilon''} > 0$. Also for the Gaussian random matrix $A$ and any $\delta'' > 0$, for $m \geq C_{\delta''} n$ we have $\|A^*\| \leq (1 + \delta'') \sqrt{m}$ with probability at least $1 - e^{-c_{\epsilon''} m}$ (28, Remark 5.40). These results together imply that

$$\|\nabla f_\epsilon(z)\| = \frac{1}{m} \|A^*v\|$$

$$\leq \frac{1}{m} \|A^*\| ||v||$$

$$\leq \sqrt{(1 + \delta')(1 + \delta'')} ||h||$$

$$\leq (1 + \delta)||h||$$

holds with probability at least $1 - \exp(-c_{\epsilon''} m)$ whenever $m \geq C_{\delta'} n$ for some $C_{\delta'}, C_{\delta''} > 0$. Here we choose $1 + \delta \geq \sqrt{(1 + \delta')(1 + \delta'')}$ and $C_{\delta'} \geq \max\{C_{\delta''}, C_{\delta'''}\}$.

B. Proof of Lemma II.3

Proof: Without loss of generality, we shall assume that the target signal $x$ has $||x|| = 1$. Again for each $z \in \mathbb{C}^n$ we set $h = e^{-i\phi(x)}z - x$, and denote $\hat{h} = h/||h||$. Definition 7 implies that $\text{Im}(h^*x) = 0$. Since $z \in \mathcal{S}_\delta(1/10)$, we have $\rho := ||h|| \leq 1/10$. Therefore

$$\Re\left(\nabla f_\epsilon(z), z - xe^{i\phi(x)}\right)$$

$$= \Re\left(\nabla f_\epsilon(z), e^{i\phi(x)}h\right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \left(1 - \frac{\sqrt{\beta_j^2 + \epsilon_j^2}}{\sqrt{a_j^2 + \epsilon_j^2}}\right) \Re\left((a_j^*z)e^{-i\phi(x)}(h^*a_j)\right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \left(\sqrt{a_j^2 x^2} + \epsilon_j^2 - \sqrt{a_j^2 x^2 + \epsilon_j^2} - \sqrt{a_j^2 x^2 + \epsilon_j^2}\right) \Re\left((a_j^*z)(x + h)(h^*a_j)\right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} (\sqrt{a_j^2 x^2} + \epsilon_j^2) \Re\left((a_j^*z)(x + h)(h^*a_j)\right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} \frac{2\Re(h^*a_j a_j^*x) + 3\Re(h^*a_j a_j^*x) |a_j^*h|^2 + |a_j^*h|^4}{\sqrt{a_j^2 x^2 + \epsilon_j^2}}$$

$$= \frac{1}{m} \sum_{j=1}^{m} T_j,$$

with $T_j$ being the $j$-th item of the summation. To simplify the statement, we use $d_j$ to denote the denominator of $T_j$, i.e.,

$$d_j = \sqrt{a_j^2 x^2 + \epsilon_j^2} \left(\sqrt{a_j^2 x^2 + \epsilon_j^2} + \sqrt{a_j^2 x^2 + \epsilon_j^2}\right).$$

We first consider $\hat{h} \in \mathbb{C}^n$ to be fixed. We divide it into two cases. In the first case we assume $\hat{h} = cx$ with $|c| = 1$. Here we have $\text{Im}(h^*x) = 0$, which implies $\hat{h} = \pm x$. Hence

$$d_j = \sqrt{a_j^2 x^2 + \epsilon_j^2} \left(\sqrt{a_j^2 x^2 + \epsilon_j^2} + \sqrt{a_j^2 x^2 + \epsilon_j^2}\right)$$

$$\leq (3 + 2 a + 1/2) |a_j^*x|^2$$

$$\leq (353/100 + 2 a)|a_j^*x|^2,$$

due to the facts that

$$|a_j^*x|^2 = |a_j^* (x + h)|^2 \leq 2(|a_j^*x|^2 + ||h||^2 \cdot |a_j^*x|^2),$$

and $a = \sqrt{a b}$ with $a + \sqrt{a b} \leq \frac{3}{2} a + \frac{1}{2} b$. Thus under the condition of $||h|| \leq \frac{1}{10}$, we obtain

$$T_j = \frac{(2 \pm 3||h|| + ||h||^2)|a_j^*x|^4}{d_j}$$

$$\geq \frac{(2 \pm 3||h|| + ||h||^2)|a_j^*x|^4}{353/100 + 2 a||h||^2}$$

$$\geq \frac{171}{353 + 200 a}||a_j^*x||^2||h||^2.$$

By Lemma A.1 of the Appendix, for $m \geq C_{\delta'} n$, with probability greater than $1 - \exp(-c_{\epsilon''} m)$ we have

$$\Re\left(\nabla f_\epsilon(z), z - xe^{i\phi(x)}\right)$$

$$= \frac{1}{m} \sum_{j=1}^{m} T_j$$

$$\geq \frac{1}{m} \sum_{j=1}^{m} \frac{171}{353 + 200 a}||a_j^*x||^2||h||^2$$

$$\geq \frac{171}{353 + 200 a}||h||^2.$$
For the second case $\tilde{h} \neq \pm x$, given the assumption $\|x\| = 1$ and $\|h\| = \rho$ we claim that
\[
\mathbb{P}(\rho|a_j^*x| > |a_j^*h|) = \mathbb{P}(\rho|a_j^*x| \leq |a_j^*h|) = 1/2. \tag{14}
\]
Indeed, for each measurement $a_j$ we have
\[
\mathbb{P}(\rho|a_j^*x| = |a_j^*h|) = \mathbb{P}(|a_j^*x| = |a_j^*h|) = 0. \tag{15}
\]
Also note that a Gaussian random measurement $a$ is rotational invariant, i.e. for any unitary matrix $O$, $Oa$ is also a Gaussian random measurement. Thus for fixed $x$ and $h$, we may without loss of generality assume that $\tilde{h} = e_1$ and $x = \sigma e_1 + \sqrt{1 - \sigma^2} e_2$, with $\sigma = h^* x \in \mathbb{R}$. This is because otherwise we can always find a unitary matrix to map $h, x$ to these two vectors. Set
\[
O := \begin{pmatrix} O_1 & 0 \\ 0 & O_2 \end{pmatrix} \in \mathbb{C}^{n \times n}
\]
where $O_2 \in \mathbb{C}^{(n-2) \times (n-2)}$ is unitary and
\[
O_1 = \begin{pmatrix} \sigma & \sqrt{1 - \sigma^2} \\ \sqrt{1 - \sigma^2} & -\sigma \end{pmatrix}.
\]
Then we have $Ox = \tilde{h}$ and $Oh = x$. Set $g := Oa$ and $g$ is a Gaussian random measurement. Consequently we have
\[
\mathbb{P}(|a^*x| > |a^*h|) = \mathbb{P}(|g^*Ox| > |g^*Oh|) = \mathbb{P}(|g^*h| > |g^*x|),
\]
which implies
\[
\mathbb{P}(\rho|a_j^*x| > |a_j^*h|) = \mathbb{P}(\rho|a_j^*x| < |a_j^*h|). \tag{16}
\]
Combining (15) and (16) we now obtain (14).

For each index set $I \subseteq \{1, 2, \ldots, m\}$, define a corresponding event
\[
\mathbb{E}_I := \{ \rho|a_j^*x| > |a_j^*h|, \forall j \in I; \rho|a_k^*x| \leq |a_k^*h|, \forall k \in I^c \}.
\]
According to (14), we know that the event $\mathbb{E}_I$ occurs with probability $1/2^m$. We assume that $I_0$ is an index set which satisfies $|I_0| \leq |I_0^c| \leq \frac{3n}{4}$. Then on event $\mathbb{E}_{I_0}$, $\Re\left(\langle \nabla f_\epsilon(x), z-e^{i\Phi_\epsilon(x)} \rangle \right)$ can be divided into two groups:
\[
m\Re\left(\langle \nabla f_\epsilon(x), z-e^{i\Phi_\epsilon(x)} \rangle \right) = \sum_{j \in I_0} T_j + \sum_{k \in I_0^c} T_k.
\]
For each group, we next provide an upper bound and a lower bound for the denominators $d_j$, $j = 1, \ldots, m$. Recall that $\epsilon = \sqrt{3/\alpha}(\alpha > 0)$. When $j \in I_0 = \{j : \rho|a_j^*x| > |a_j^*h|\}$ we have
\[
d_j = \sqrt{|a_j^*z|^2 + \epsilon_j^2} \left(\sqrt{|a_j^*z|^2 + \epsilon_j^2} + \sqrt{|a_j^*x|^2 + \epsilon_j^2} \right) \tag{17}
\]
\[
\leq \frac{3}{2} |a_j^*z|^2 + 2\epsilon_j + \frac{1}{2}|a_j^*x|^2
\]
\[
< \frac{3}{2} (1 + \rho^2)|a_j^*x|^2 + 2\alpha|a_j^*x|^2 + \frac{1}{2}|a_j^*x|^2
\]
\[
= \left(2\alpha + 2 + 3\rho + \frac{3}{\rho^2}\right)|a_j^*x|^2
\]
\[
= U_1 |a_j^*x|^2
\]
where $U_1 := 2\alpha + 2 + 3\rho + \frac{3}{\rho^2}$. Here the second inequality follows from $\epsilon_j^2 = \alpha|a_j^*x|^2$ and
\[
|a_j^*z|^2 = |a_j^*(x+h)|^2
\]
\[
\leq (|a_j^*x| + |a_j^*h|)^2
\]
\[
< (1 + \rho)^2|a_j^*x|^2.
\]
On the other hand, since
\[
|a_j^*z|^2 = |a_j^*(x+h)|^2
\]
\[
\geq (|a_j^*x| - |a_j^*h|)^2
\]
\[
> (1/\rho - 1)^2|a_j^*h|^2
\]
and $\epsilon_j^2 = \alpha|a_j^*x|^2 > (\alpha/\rho^2)|a_j^*h|^2$, we have
\[
d_j = \sqrt{|a_j^*z|^2 + \epsilon_j^2} \left(\sqrt{|a_j^*z|^2 + \epsilon_j^2} + \sqrt{|a_j^*x|^2 + \epsilon_j^2} \right) \tag{18}
\]
\[
> \sqrt{(1/\rho - 1)^2 + (\alpha/\rho^2)}(\sqrt{(1/\rho - 1)^2 + (\alpha/\rho^2)}
\]
\[
> (1/\rho - 1)^2|a_j^*h|^2
\]
\[
> \frac{1}{\rho^2}\left(1 - \frac{3}{2} \frac{1}{\rho^2}\right)|a_j^*h|^2
\]
\[
= L_1|a_j^*h|^2
\]
where $L_1 := \sqrt{1 - \frac{3}{2} \frac{1}{\rho^2}} + \alpha \left(1 - \frac{3}{2} \frac{1}{\rho^2}\right)/\rho^2$. Similarly, for $k \in I_0^c = \{k : \rho|a_k^*x| \leq |a_k^*h|\}$, we have
\[
|a_k^*z|^2 \leq (|a_k^*x| + |a_k^*h|)^2 \leq (1 + 1/\rho)^2|a_k^*h|^2,
\]
and hence
\[
d_k = \sqrt{|a_k^*z|^2 + \epsilon_k^2} \left(\sqrt{|a_k^*z|^2 + \epsilon_k^2} + \sqrt{|a_k^*x|^2 + \epsilon_k^2} \right) \tag{19}
\]
\[
\leq \frac{3}{2} |a_k^*z|^2 + 2\epsilon_k + \frac{1}{2}|a_k^*x|^2
\]
\[
\leq \frac{3}{2} (1/\rho + 1)^2|a_k^*h|^2 + (2\alpha + 1/2)/\rho^2 \cdot |a_k^*h|^2
\]
\[
= \left(\frac{2\alpha + 2 + 3\rho + 3}{\rho^2}\right)|a_k^*h|^2
\]
\[
= U_2|a_k^*h|^2
\]
where $U_2 := 2\alpha + 2 + 3\rho$, and
\[
d_k = \sqrt{|a_k^*z|^2 + \epsilon_k^2} \left(\sqrt{|a_k^*z|^2 + \epsilon_k^2} + \sqrt{|a_k^*x|^2 + \epsilon_k^2} \right) \tag{20}
\]
\[
\geq \epsilon_k \left(\frac{\alpha + \sqrt{\alpha}(1 + \alpha)}{\alpha + \sqrt{\alpha}(1 + \alpha)}\right)|a_k^*x|^2
\]
\[
= L_2|a_k^*x|^2
\]
where $L_2 := \alpha + \sqrt{\alpha}(1 + \alpha)$. 

\[\]
Based on (17), (18) and Lemma A.3, given any \( \delta > 0 \), for \( |I_0| \geq C_2(\delta) n \) we have with probability at least \( 1 - \exp \left( -c_1(\delta) \cdot |I_0| \right) \) the inequality

\[
\begin{align*}
\sum_{j \in I_0} T_j & = \sum_{j \in I_0} \left( \frac{(\sqrt{2} \text{Re}(\mathbf{h}^* a_j a_j^* x) + \frac{\delta}{\sqrt{2}} |a_j^* h|^2)}{d_j} - |a_j^* h|^4 \right) \\
& \geq \sum_{j \in I_0} \left( \frac{(\sqrt{2} \text{Re}(\mathbf{h}^* a_j a_j^* x) + \frac{\delta}{\sqrt{2}} |a_j^* h|^2)}{U_1 |a_j|^2} - |a_j^* h|^4 \right) \\
& \geq \sum_{j \in I_0} \left( \frac{2(\text{Re}(\mathbf{h}^* a_j a_j^* x))^2 - 3|a_j^* h|^2}{U_1 |a_j|^2} - |a_j^* h|^4 \right) \\
& \geq \sum_{j \in I_0} \left( \frac{2(|a_j^* h|^2)^2 - 3|a_j^* h|^2}{U_1 |a_j|^2} - |a_j^* h|^4 \right) \\
& \geq \sum_{j \in I_0} \left( \frac{\left| \frac{2}{U_1} |a_j|^2 - 3|a_j^* h|^2 - |a_j^* h|^4 \right|}{8L_1} \right) \\
& \geq |I_0| \cdot |h|^2 \left( \frac{2}{U_1} + \frac{7}{2U_1} \text{Re}^2(\mathbf{h}^* x) - \frac{3}{2U_1} |h|^2 - \frac{1}{16L_1} - \frac{\delta}{4} \right) \\
& \geq |I_0| \cdot |h|^2 \left( \frac{1}{4U_1} - \frac{3\rho}{2U_1} - \frac{1}{16L_1} - \frac{\delta}{4} \right) \\
& = |I_0| \cdot |h|^2 \cdot \varphi_1,
\end{align*}
\]

where \( \varphi_1 := \frac{1-6\rho}{4U_1} - \frac{1}{16L_1} - \frac{\delta}{4}. \) Here the fourth inequality comes from Lemma A.3.

Similarly, according to (19), (20) and Lemma A.3, for \( |I_0^c| \geq C_2(\delta) n \) we have with probability at least \( 1 - \exp \left( -c_2(\delta) \cdot |I_0^c| \right) \), the inequality

\[
\begin{align*}
\sum_{k \in I_0^c} T_k & = \sum_{k \in I_0^c} \left( \frac{\frac{1}{2} |a_k^* h|^4}{d_k} - \frac{(\text{Re}(\mathbf{h}^* a_k a_k^* x))^2}{4d_k} \right) \\
& \geq \sum_{k \in I_0^c} \left( \frac{\frac{1}{2} |a_k^* h|^4}{U_1 |a_k|^2} - \frac{(\text{Re}(\mathbf{h}^* a_k a_k^* x))^2}{4U_1 |a_k|^2} \right) \\
& \geq |I_0^c| \cdot |h|^2 \left( \frac{1}{8U_1} + \frac{7}{32} \text{Re}^2(\mathbf{h}^* x) + \frac{3}{8} |h|^2 \right) \\
& \geq |I_0^c| \cdot |h|^2 \left( \frac{3}{32} |h|^2 + \frac{1}{2U_1} - \frac{3}{32L_2} \right) \\
& \geq \left( \frac{9}{128L_2} \right)^{m/4} \text{Re}(\mathbf{h}^* x) + \frac{3}{32} \text{Re}^2(\mathbf{h}^* x) - \frac{3}{2U_1} - \frac{3}{32L_2} \right) \\
& \geq |I_0^c| \cdot |h|^2 \left( \frac{9}{32U_1 |h|^2} + \frac{1}{2U_1} - \frac{3}{32L_2} - \frac{\phi}{4} - \frac{\delta}{4} \right) \\
& = |I_0^c| \cdot |h|^2 \cdot \varphi_2,
\end{align*}
\]

where \( \varphi_2 := \frac{9}{32U_1 |h|^2} + \frac{1}{2U_1} - \frac{3}{32L_2} - \frac{\phi}{4} - \frac{\delta}{4}. \) The second inequality follows from the concentration inequalities given in Lemma A.3. The fourth inequality derives from the facts that \( \frac{1}{128L_2} - \frac{1}{128L_2} > 0 \) for any \( 0.37 \leq \alpha \leq 197 \) and \( \rho \leq 1/10. \)

Set \( \delta := 0.001. \) For arbitrary fixed \( \alpha \in [0.37, 197], \) a simple observation is that \( \varphi_2 \) and \( \varphi_2 \) are decreasing functions of \( \rho. \) So we next only consider \( \rho = 1/10. \) When \( 0.37 \leq \alpha \leq 197, \) we have

\[
\varphi_1 = \frac{1}{10U_1} - \frac{1}{16L_1} - \frac{\delta}{4} > 0
\]

and

\[
\varphi_2 = \frac{229}{8U_2} - \frac{3}{32L_2} + \frac{225}{4U_2^2} - \frac{\delta}{4} > 0,
\]

with \( \phi = \frac{9}{128L_2} - \frac{1575}{32U_2^2} < 0. \)

For sufficiently large constant \( C \geq 4 \max \{ C_1(\delta), C_2(\delta) \}, \) as long as \( m \geq Cn, \) we have \( |I_0^c| \geq m/4 \geq C_2(\delta) n. \) Thus with probability at least \( 1 - \exp(-c_3 m(\delta)/2m) \), we have

\[
\text{Re}(\nabla f_\epsilon(z), z - xc^{i\phi_\epsilon(x)}) = \frac{1}{m} \left( \sum_{j \in I_0} T_j + \sum_{k \in I_0^c} T_k \right)
\]

\[
\geq \frac{1}{m} \left( |I_0||h|^2 \cdot \varphi_1 + |I_0^c||h|^2 \cdot \varphi_2 \right)
\]

\[
\geq \frac{1}{m} \left( |m/4||h|^2 \cdot \varphi_1 + \frac{m}{4} ||h|^2 \cdot \varphi_2 \right)
\]

\[
= \left( \varphi_1 + \varphi_2 \right) ||h||^2 / 4.
\]

The number of the index sets \( I \) satisfying \( m/4 \leq |I| \leq 3m/4 \) is \( \sum_{k=m/4+1}^{3m/4} \binom{m}{k} \). So for fixed \( \mathbf{h}, \) when \( \mathbf{h} \neq \pm x, \) the inequality (23) holds with probability greater than \( \sum_{k=m/4+1}^{3m/4} \binom{m}{k}(1 - \exp(-c_3 m))/2^m. \) Note that

\[
\begin{align*}
\sum_{k=m/4+1}^{3m/4} \binom{m}{k} & = \sum_{k=0}^{m-1} \binom{m}{k} = \sum_{k=0}^{m-1} \binom{m}{k} \\
& = \sum_{k=0}^{m-1} \binom{m-1}{k} (m-1-k) (m-1-k) \\
& = \sum_{k=0}^{m-1} \binom{m}{k} (m-1-k) \sum_{k=0}^{m-1} \binom{(m-1)-k}{(m-1)-k} (m-1) \cdots (m-k) \\
& < \frac{m}{m-1} \sum_{k=0}^{m-1} \binom{(m-1)-k}{(m-1)-k} (m-1) \cdots (m-k) \\
& < \left( \frac{4e}{m} \right)^{m/4} \cdot 2^{m/2},
\end{align*}
\]

and \( (4e)^{1/4} < 2. \) Hence \( \sum_{k=m/4+1}^{3m/4} \binom{m}{k}2^m < c_0 m \) for some \( c_0 \in (0, 1), \) which implies that \( \sum_{k=m/4+1}^{3m/4} \binom{m}{k}(1 - \exp(-c_3 m))/2^m \geq 1 - \exp(-c_3 m). \) Moreover, for \( \alpha \in [0.37, 197] \) we have

\[
\frac{\varphi_1 + \varphi_2}{4} < \frac{171}{535} \cdot (1 - 0.001).
\]

Combining (13), (23) and (24) we obtain

\[
\text{Re}(\nabla f_\epsilon(z), z - xc^{i\phi_\epsilon(x)}) \geq \frac{\varphi_1 + \varphi_2}{4} ||h||^2
\]

with probability at least \( 1 - \exp(-c_3 m). \) Particularly, when \( \alpha \in [0.37, 29] \) we have \( \frac{\varphi_1 + \varphi_2}{4} > 0.001. \)
To complete the proof we will need to establish uniform bound over all vectors, so we adopt an \( \eta \)-net argument. Observe that

\[
\text{Re}\left( \langle \nabla f_{e}(z), z - xe^{i\phi_{x}(x)} \rangle \right) = \text{Re}\left( \langle \nabla f_{e}(e^{-i\phi_{x}(x)}z), e^{-i\phi_{x}(x)}z - x \rangle \right) = \text{Re}\left( \langle \nabla f_{e}(x + \rho \tilde{h}), \rho \tilde{h} \rangle \right).
\]

For any \( z \in \mathbb{C}^{n} \), which means for any \( \tilde{h} \) with \( \|\tilde{h}\| = 1 \) and \( \text{Im}(\tilde{h}^{*}x) = 0 \), we consider the function \( \text{Re}\left( \langle \nabla f_{e}(x + \rho \tilde{h}), \rho \tilde{h} \rangle \right) \) with \( \rho \leq 1/10 \). Suppose that \( \tilde{h}_{1}, \tilde{h}_{2} \in \mathbb{C}^{n} \) satisfy \( \|\tilde{h}_{1} - \tilde{h}_{2}\| \leq \eta \). When \( 0.37 \leq \alpha \leq 29 \) we have

\[
\left| \text{Re}\left( \langle \nabla f_{e}(x + \rho \tilde{h}_{1}), \rho \tilde{h}_{1} \rangle \right) - \text{Re}\left( \langle \nabla f_{e}(x + \rho \tilde{h}_{2}), \rho \tilde{h}_{2} \rangle \right) \right|
\leq \left| \text{Re}\left( \langle \nabla f_{e}(x + \rho \tilde{h}_{1}), \rho \tilde{h}_{1} \rangle \right) \right| + \left| \text{Re}\left( \langle \nabla f_{e}(x + \rho \tilde{h}_{2}), \rho \tilde{h}_{2} \rangle \right) \right|
= \rho \left\| \nabla f_{e}(x + \rho \tilde{h}_{1}) \right\| \cdot \| \eta + \| \nabla^{2} f_{e}(\xi) \| \cdot \rho^{2} \left\| \tilde{h}_{2} \right\| \cdot \eta \\
\leq 2\rho^{2} \tilde{h}_{1} \cdot \eta + 2\sqrt{\frac{1 + \alpha}{\alpha} \cdot \rho^{2} \left\| \tilde{h}_{2} \right\| \cdot \eta \\
= 2 \left( 1 + \sqrt{\frac{1 + \alpha}{\alpha}} \right) \cdot \eta \cdot \rho^{2} \\
< 6\eta \cdot \rho^{2}.
\]

where \( \xi \in \mathbb{C}^{n} \). Here the third inequality follows from Lemma [II.2] and Lemma [A.4]. Therefore for any \( \tilde{h}_{1} \) and \( \tilde{h}_{2} \) satisfying \( \|\tilde{h}_{1} - \tilde{h}_{2}\| \leq \eta := \frac{\delta}{2} \) with \( \delta = 0.001 \), let \( \eta \)-net \( N_{N_{\eta}} \) be an \( \eta \)-net for the unit sphere of \( \mathbb{C}^{n} \) with cardinality \( |N_{\eta}| \leq (1 + 2/\eta)^{2n} \). Then for all \( z \), \( 0.37 \leq \alpha \leq 29 \) and \( m = (C_{2} \cdot \eta^{-2} \cdot \log \eta^{-1})n \), with probability at least \( 1 - \exp(-\alpha n) \) we have

\[
\text{Re}\left( \langle \nabla f_{e}(z), z - xe^{i\phi_{x}(x)} \rangle \right) \geq \frac{(\varphi_{1} + \varphi_{2})/4 - \delta}{\|N_{\eta}\|} \geq \beta_{\alpha} \|z\|^{2} = \beta_{\alpha} \|\tilde{h}\|^{2}
\]

with \( \beta_{\alpha} = (\varphi_{1} + \varphi_{2})/4 - \delta > 0 \). According to Remark IV.1 when \( \alpha = 0.826 \), \( \beta_{\alpha} = 64/5945 \) approximately reaches its largest value.

**Remark IV.1.** According to the proof of Lemma [II.3] by taking \( \rho = 1/10 \) and \( \delta = 0.001 \), we have \( U_{1} = 2\alpha + 463/200 = 200\alpha + 463/2 \), \( L_{1} = 100\alpha + 81 + 100\sqrt{(\alpha + 1)/(\alpha + 0.81)} \) and \( L_{2} = x + \sqrt{x(1 + x)} \). Recall that

\[
\beta_{\alpha} = (\varphi_{1} + \varphi_{2})/4 - \delta \\
= \frac{1}{4U_{1}} + \frac{229}{32U_{2}} - \frac{1}{64L_{1}} - \frac{3}{128L_{2}} + \frac{225}{16U_{2}^{2} \phi} - \frac{9\delta}{8},
\]

with \( \phi = \frac{9}{128L_{2}} - \frac{1575}{32L_{2}} \). Figure 4 here shows the relationship between \( \beta_{\alpha} \) and \( \alpha \).

**Fig. 4:** The relationship between \( \beta_{\alpha} \) and \( \alpha (\log 10) \).

Particularly, when \( \alpha = 0.826 \), \( \beta_{\alpha} = 64/5945 \) roughly reaches its maximum.

**APPENDIX**

**AUXILIARY LEMMAS**

In previous sections we have applied concentration inequalities several times. They have played a key role in the proof of our results. Here we present these concentration inequalities used for the proof of Lemma [II.2] and Lemma [II.3].

**Lemma A.1.** ([9] Lemma 3.1.) Let \( a_{1}, a_{2}, \ldots, a_{m} \in \mathbb{C}^{n} \) be i.i.d. Gaussian random measurements. Fix any \( \delta \) in \( (0, 1/2) \) and assume \( m \geq 20\delta^{-2}n \). Then for all unit vectors \( u \in \mathbb{C}^{n} \),

\[
1 - \delta \leq \frac{1}{m} \sum_{j=1}^{m} |a_{j}^{*}u|^{2} \leq 1 + \delta
\]

holds with probability at least \( 1 - \exp(-mt^{2}/2) \), where \( \delta/4 = t^{2} + t \).

**Lemma A.2.** Let \( a \in \mathbb{C}^{n} \) be a Gaussian random measurement. Let \( x \in \mathbb{C}^{n} \) and \( \tilde{h} \in \mathbb{C}^{n} \) be two fixed vectors with \( \|x\| = \|\tilde{h}\| = 1 \), \( \text{Im}(\tilde{h}^{*}x) = 0 \) and \( \tilde{h} \neq \pm x \). Then we have

\[
\mathbb{E}\left( \text{Re}(\tilde{h}^{*}aa^{*}x) \cdot I_{\{\|a^{*}x\| > \|a^{*}\tilde{h}\|\}} \right) = \mathbb{E}\left( \text{Re}(\tilde{h}^{*}aa^{*}x) \cdot I_{\{\|a^{*}x\| \leq \|a^{*}\tilde{h}\|\}} \right) = \text{Re}(\tilde{h}^{*}x),
\]

\[
\frac{1}{2} \leq \mathbb{E}\left( \|a^{*}x\|^{2} \cdot I_{\{\|a^{*}x\| > \|a^{*}\tilde{h}\|\}} \right) \leq \frac{3}{4},
\]

\[
\frac{1}{4} \leq \mathbb{E}\left( \|a^{*}x\|^{2} \cdot I_{\{\|a^{*}x\| \leq \|a^{*}\tilde{h}\|\}} \right) \leq \frac{1}{2}.
\]

\[
\frac{1}{8} + \frac{7}{32} \text{Re}^{2}(\tilde{h}^{*}x) \leq \mathbb{E}\left( \left( \frac{\text{Re}(\tilde{h}^{*}aa^{*}x)}{|a^{*}x|^{2}} \right) \cdot I_{\{\|a^{*}x\| > \|a^{*}\tilde{h}\|\}} \right) \leq \frac{1}{4} + \frac{1}{4} \text{Re}^{2}(\tilde{h}^{*}x)
\]
and
\[
\frac{1}{4} + \frac{1}{4} \operatorname{Re}^2(\tilde{h}^* x) \leq \mathbb{E}\left(\frac{\left(\operatorname{Re}(\tilde{h}^* a^* x)\right)^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| \leq |a^* h|\}} \\
\leq \frac{3}{8} + \frac{9}{32} \operatorname{Re}^2(\tilde{h}^* x).
\]
(30)

Proof: Since the distribution of \(a\) is invariant by unitary transformation, we can take \(x = e_1\) and \(\tilde{h} = e_1 + \sqrt{1 - \sigma^2} e_2\), where \(\sigma = |x^* \tilde{h}| = \operatorname{Re}(x^* \tilde{h}) \in \mathbb{R}\) and \(|\sigma| < 1\). We use \(\xi_1, \xi_2, \xi_3, \xi_4\) to represent the real and imaginary parts of \(a_1\) and \(a_2\) respectively, which implies that the variables \(\xi_1, \xi_2, \xi_3, \xi_4\) are independent and obey normal distribution \(\mathcal{N}(0, 1/2)\). Then it follows that
\[
\mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x)) = \mathbb{E}(\sigma (\xi_1^2 + \xi_2^2) + \sqrt{1 - \sigma^2} (\xi_1 \xi_3 + \xi_2 \xi_4)) = \sigma = \operatorname{Re}(\tilde{h}^* x),
\]
\[
\mathbb{E}(|a^* x|^2) = \mathbb{E}(\xi_1^2 + \xi_2^2) = 1
\]
and
\[
\mathbb{E}\left(\frac{\left(\operatorname{Re}(\tilde{h}^* a^* x)\right)^2}{|a^* x|^2}\right) = \mathbb{E}\left(\frac{\sigma^2 (\xi_1^2 + \xi_2^2) + \sqrt{1 - \sigma^2}^2 (\xi_1 \xi_3 + \xi_2 \xi_4)^2}{\xi_1^2 + \xi_2^2}\right) = \frac{1}{2} + \frac{1}{2} \sigma^2.
\]

Since \(a\) is invariant by unitary transformation and \(x, \tilde{h}\) are two fixed vectors satisfying \(\tilde{h} \neq \pm x\), so we have
\[
\mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x) \cdot I_{\{|a^* x| \leq |a^* h|\}}) \\
= \mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x) \cdot I_{\{|a^* x| < |a^* h|\}}) \\
= \mathbb{E}(\operatorname{Re}(x^* g g^* h) \cdot I_{\{|g^* x| > |g^* h|\}}) \\
= \mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x) \cdot I_{\{|a^* x| > |a^* h|\}}).
\]

Here \(g := O a\) is a Gaussian random measurement with unitray matrix \(O\) satisfying \(O x = h\) and \(O h = x\). Then we obtain
\[
\mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x)) = \mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x) \cdot I_{\{|a^* x| \leq |a^* h|\}}) \\
+ \mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x) \cdot I_{\{|a^* x| > |a^* h|\}}) \\
= 2 \cdot \mathbb{E}(\operatorname{Re}(\tilde{h}^* a^* x) \cdot I_{\{|a^* x| > |a^* h|\}}),
\]
which implies \([26]\).

Similarly, we have
\[
\mathbb{E}(|a^* x|^2) \\
= \mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| > |a^* h|\}}) + \mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| \leq |a^* h|\}}) \\
= \mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| > |a^* h|\}}) + \mathbb{E}(|a^* h|^2 \cdot I_{\{|a^* x| > |a^* h|\}}),
\]
which implies
\[
\mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| > |a^* h|\}}) \geq \frac{1}{2} \mathbb{E}(|a^* x|^2) = \frac{1}{2},
\]
\[
\mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| \leq |a^* h|\}}) \leq \frac{1}{2}.
\]

And
\[
2 \cdot \mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| \leq |a^* h|\}} \\
\geq \mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| > |a^* h|\}} \\
+ \mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| \leq |a^* h|\}} \\
\geq 2 \cdot \mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| > |a^* h|\}}
\]
implies
\[
\mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| > |a^* h|\}} \leq \frac{1}{4} + \frac{1}{4} \operatorname{Re}^2(\tilde{h}^* x)
\]
and
\[
\mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| \leq |a^* h|\}} \geq \frac{1}{4} + \frac{1}{4} \operatorname{Re}^2(\tilde{h}^* x).
\]

Then to prove the inequalities \([27, 28, 29]\) and \([30]\), it’s sufficient to prove
\[
\frac{1}{8} + \frac{7}{32} \operatorname{Re}^2(\tilde{h}^* x) \leq \mathbb{E}\left(\frac{(\operatorname{Re}(\tilde{h}^* a^* x))^2}{|a^* x|^2}\right) \cdot I_{\{|a^* x| > |a^* h|\}}.
\]

Next, we commit to prove \([31]\) and \([32]\). Firstly, we take polar coordinates transformation:
\[
\begin{aligned}
\xi_1 &= r_1 \cos \theta_1 \\
\xi_2 &= r_1 \sin \theta_1 \\
\xi_3 &= (r_2 \cos \theta_2 - \sigma r_1 \cos \theta_1) / \sqrt{1 - \sigma^2} \\
\xi_4 &= (r_2 \sin \theta_2 - \sigma r_1 \sin \theta_1) / \sqrt{1 - \sigma^2}
\end{aligned}
\]
with \(r_1, r_2 \in (0, \infty), \theta_1, \theta_2 \in [0, 2\pi)\). Then we can write the expectation as
\[
\mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| > |a^* h|\}}) \\
= \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^\infty r_1^3 r_2 \exp\left(-\frac{r_1^2 + r_2^2}{1 - \sigma^2}\right) dr_1 dr_2 d\theta_1 d\theta_2 \\
= \frac{4}{k!^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^\infty \frac{\sigma^{2k+1} r_1^{2k+3} r_2^{2k+1}}{1 - \sigma^2} dr_1 dr_2 \\
= \sum_{k=0}^{\infty} \sigma^{2k} (1 - \sigma^2)^2 (k+1) \left(\frac{(2k+1)!!}{2k+1} \cdot \frac{k+1}{2}\right).
\]

It is an even function about \(\sigma\) and when \(\sigma \in [0, 1)\) the derivative
\[
d\mathbb{E}(|a^* x|^2 \cdot I_{\{|a^* x| > |a^* h|\}}) / d\sigma \\
= -\sigma - \sum_{k=1}^{\infty} \frac{(2k-1)!!}{2k+1} \sigma^{2k+1} \leq 0.
\]
Hence the expectation obtains its maximum at \( \sigma = 0 \), i.e.,
\[
E(\|x\|^2 \cdot I_{\{\|x\| > \|h\|\}})
\leq \frac{1}{\pi^2} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^r r_1 r_2 \exp \left( -\frac{r_1^2 + r_2^2}{2} \right) dr_2 dr_1 d\theta_1 d\theta_2
= \frac{3}{4}.
\]

Thus we have the inequality (31).

Using the same polar coordinates transformation, we know
\[
E\left( \frac{(\Re(\hat{h}^* a a^* x))^2}{\|x\|^2} \cdot I_{\{\|x\| > \|h\|\}} \right)
= \frac{1}{\pi} \int_0^{2\pi} \int_0^{2\pi} \int_0^\infty \int_0^1 r_1 r_2 \exp \left( -\frac{r_1^2 + r_2^2}{2} \right) \cdot \cos^2(\theta_1 - \theta_2) dr_2 dr_1 d\theta_1 d\theta_2
= \frac{1}{2} \sum_{k=0}^\infty \frac{1}{(2k)!} \frac{2^{2k}}{2k+2} \left( \frac{1}{1-2\sigma^2} \right)^{2k+2} \cdot \exp \left( -\frac{r_1^2 + r_2^2}{2(1-\sigma^2)} \right) dr_2 dr_1
= \frac{1}{2} \sum_{k=0}^\infty \sigma^{2k} (1-\sigma^2) \cdot \frac{1}{(k)!} \frac{2^{2k}}{2k+2} \left( (k+1)! \cdot \frac{(2k+7)!}{2k+1} \right)
= \frac{1}{8} + \frac{7}{32} \sigma^2 + \sum_{k=1}^\infty \frac{(2k+7)!(2k+1)!}{2k+2}(2k+4)(k+2) \cdot \sigma^{2k+2}.
\]

Thus we obtain (32). This completes the proof.

**Lemma A.3.** Let \( a_1, a_2, \ldots, a_m \in \mathbb{C}^n \) be i.i.d. Gaussian random measurements. Let \( x \in \mathbb{C}^n \) and \( h \in \mathbb{C}^n \) be two fixed vectors with \( \|x\| = \|h\| = 1, \text{Im}(\hat{h}^*x) = 0 \) and \( h \neq \pm x \). For any \( \delta > 0 \), there exist positive constants \( C_3, c_3 > 0 \) such that for any \( m \geq C_3 n \) the inequalities
\[
\left| \frac{1}{m} \sum_{j=1}^m \Re(\hat{h}^* a_j a_j^* x) \cdot I_{\{\|x\| > \|h\|\}} - \frac{1}{2} \Re(\hat{h}^* x) \right| \leq \delta,
\]
hold with probability at least \( 1 - \exp(-c_3 m) \).

**Proof:** For fixed \( h \) and \( x \), the following sets are all independent sub-exponential random variables
\[
\{ \Re(\hat{h}^* a_j a_j^* x) \cdot I_{\{\|x\| > \|h\|\}} , j = 1, \ldots, m \},
\{ \|x\|^2 \cdot I_{\{\|x\| > \|h\|\}} , j = 1, \ldots, m \},
\{ \|a_j\|^2 \cdot I_{\{\|a_j\| \leq \|h\|\}} , j = 1, \ldots, m \},
\{ \Re(\hat{h}^* a_j a_j^* x)^2 \cdot I_{\{\|x\| > \|h\|\}} , j = 1, \ldots, m \},
\{ \Re(\hat{h}^* a_j a_j^* x)^2 \cdot I_{\{\|a_j\| \leq \|h\|\}} , j = 1, \ldots, m \}.
\]
Recall that \( a = (a_1, \ldots, a_n) \in \mathbb{C}^n \sim \mathcal{N}(0, I/2) + i\mathcal{N}(0, I/2) \) is a Gaussian random measurement. Then based on Bernstein-type inequality, for any \( \delta > 0 \), the inequalities
\[
\left| \frac{1}{m} \sum_{j=1}^m \Re(\hat{h}^* a_j a_j^* x) \cdot I_{\{\|x\| > \|h\|\}} - \Re(\hat{h}^* a a^* x) \cdot I_{\{\|x\| > \|h\|\}} \right| \leq \delta,
\]
\[
\left| \frac{1}{m} \sum_{j=1}^m \|x\|^2 \cdot I_{\{\|x\| > \|h\|\}} - E(\|x\|^2 \cdot I_{\{\|x\| > \|h\|\}}) \right| \leq \delta,
\]
\[
\left| \frac{1}{m} \sum_{j=1}^m \|a_j\|^2 \cdot I_{\{\|a_j\| \leq \|h\|\}} - E(\|a_j\|^2 \cdot I_{\{\|a_j\| \leq \|h\|\}}) \right| \leq \delta,
\]
\[
\left| \frac{1}{m} \sum_{j=1}^m \Re(\hat{h}^* a_j a_j^* x)^2 \cdot I_{\{\|x\| > \|h\|\}} - \Re(\hat{h}^* a a^* x)^2 \cdot I_{\{\|x\| > \|h\|\}} \right| \leq \delta,
\]
\[
\left| \frac{1}{m} \sum_{j=1}^m \Re(\hat{h}^* a_j a_j^* x)^2 \cdot I_{\{\|a_j\| \leq \|h\|\}} - \Re(\hat{h}^* a a^* x)^2 \cdot I_{\{\|a_j\| \leq \|h\|\}} \right| \leq \delta
\]
hold with probability at least \( 1 - \exp(-c_3 m) \) provided \( m \geq C_3 n \), where \( C_3, c_3 \) are positive constants depending on \( \delta \). Then the inequalities (33), (34), (35), (36), (37) can be derived directly from the expectation bounds given in Lemma A.2.

The following lemma provides an upper bound for the operator norm of \( \nabla^2 f_e(z) \).

**Lemma A.4.** Set \( \epsilon = \sqrt{\alpha} b \). Then there exist constants \( C', c' > 0 \) such that for \( m \geq C'n \), \( \|\nabla^2 f_e(z)\| \leq 2\sqrt{1 + \alpha} \) holds with probability at least \( 1 - \exp(-c'm) \).
Proof: Recall that
\[
\nabla f_e(z) := \left( \frac{\partial f_e(z, z)}{\partial z} \bigg|_{z=\text{constant}} \right)^* = \frac{1}{m} \sum_{j=1}^{m} \left( 1 - \frac{\sqrt{b_j^2 + c_j}}{\sqrt{a_j^2 z^2 + c_j^2}} \right) a_j^* a_j^* z.
\]

Similarly, we obtain
\[
\nabla^2 f_e(z) = \frac{1}{m} \sum_{j=1}^{m} \left( 1 - \frac{\sqrt{b_j^2 + c_j^2}}{\sqrt{a_j^2 z^2 + c_j^2}} \right) a_j^* a_j^* + \frac{1}{m} \sum_{j=1}^{m} \frac{\sqrt{b_j^2 + c_j^2} a_j^* z^2}{2(\sqrt{a_j^2 z^2 + c_j^2})^{3/2}} a_j^* a_j^*.
\]

For any \( z \in \mathbb{C}^n \), we have
\[
\| \nabla^2 f_e(z) \| \leq \max_{y \in S^{n-1}} \frac{1}{m} \sum_{j=1}^{m} \left( 1 + \frac{\sqrt{b_j^2 + c_j^2}}{\sqrt{a_j^2 z^2 + c_j^2}} \right) |a_j^* y|^2 \leq \frac{2}{\alpha} \left( 1 + \frac{\alpha}{\alpha} \right) |a_j^* y|^2
\]
with probability at least \( 1 - \exp(-c'm) \) provided \( m \geq C'n \).

Here the third inequality is obtained by Lemma A.1.

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