Hopf bifurcation of viscous shock waves in compressible gas- and magnetohydrodynamics

Benjamin Texier* and Kevin Zumbrun†

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Abstract
Extending our previous results for artificial viscosity systems, we show, under suitable spectral hypotheses, that shock wave solutions of compressible Navier–Stokes (cNS) and magnetohydrodynamics (MHD) equations undergo Hopf bifurcation to nearby time-periodic solutions. The main new difficulty associated with physical viscosity and the corresponding absence of parabolic smoothing is the need to show that the difference between nonlinear and linearized solution operators is quadratically small in $H^s$ for data in $H^s$. We accomplish this by a novel energy estimate carried out in Lagrangian coordinates; interestingly, this estimate is false in Eulerian coordinates. At the same time, we greatly sharpen and simplify the analysis of the previous work.

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*Université Paris 7, Denis Diderot Institut de Mathmatiques de Jussieu, UMR 7586
Case 7012 2, place Jussieu; texier@math.jussieu.fr: Research of B.T. was partially supported under NSF grant number DMS-0300487.

†Indiana University, Bloomington, IN 47405; kzumbrun@indiana.edu: Research of K.Z. was partially supported under NSF grant number DMS-0300487.
1 Introduction

A well-known phenomenon in combustion is the appearance of “galloping”, “spinning”, and “cellular” instabilities of traveling detonation fronts, apparently corresponding with Hopf bifurcation of the background solution; see, e.g., [KS, LyZ1, LyZ2, TZ1, TZ2] and references therein. Mathematically, there are two distinct aspects of this phenomenon. The first is to verify the spectral scenario associated with Hopf bifurcation, consisting of a conjugate pair $\lambda_{\pm}(\epsilon) = \gamma(\epsilon) + i\tau(\epsilon)$ of complex eigenvalues of the linearized operator about the wave crossing the imaginary axis from stable (negative real part) to unstable side as bifurcation parameter $\epsilon$ varies from negative to positive values. Here, $\epsilon$ measures variation in physical parameters such as heat release, rate of reaction, or strength of the detonation.

The second issue is to show that this spectral scenario indeed corresponds at the nonlinear level to Hopf bifurcation: i.e., apparition of nearby time-periodic solutions $u^a$ of approximate period $2\pi/\tau(\epsilon)$, $\epsilon = \epsilon(a)$, branching from the steady solution at $\epsilon(0) = 0$. What makes this nontrivial is the absence of a spectral gap between zero and the essential spectrum of the linearized operator about the wave, which circumstance prevents the application of standard bifurcation theorems for PDE, as in, e.g., [C, He, MM, VT].
The first issue has been studied for detonations by numerical [KS], formal asymptotic [Er, FW, BMR], and Evans function [LyZ1, LyZ2] techniques. The second issue has been studied in a much more general setting in [TZ1, TZ2, SS]. In particular, it was shown in [TZ2] for smooth shock solutions of conservation laws with artificial (strictly parabolic) viscosity that a spectral Hopf scenario implies actual nonlinear Hopf bifurcation. This result was recently sharpened by Sandstede and Scheel [SS] to include also exponential localization of the perturbed solution (the result of [TZ2] asserts only $(1 + |x|)^{-1}$ decay) and exchange of spectral stability. The results of [SS] were obtained by quite different “spatial dynamics” techniques, viewing the problem as a spatial elliptic boundary value problem on the space of time-periodic functions, whereas the original results of [TZ2] were obtained by “temporal dynamics” techniques centered around the temporal evolution operator.

Our purpose in the present paper is to extend the latter, nonlinear, results to the physical (nonstrictly parabolic) systems of compressible gas- and magnetohydrodynamics (MHD) with either ideal or “real”, van der Waals-type, equation of state by a modification of the original, temporal dynamics argument of [TZ2]. At the same time, we show that exponential localization may by an improvement of the central cancellation estimate be obtained by temporal as well as spatial techniques, recovering the sharpened estimates of [SS]; moreover, this same improvement greatly simplifies the entire analysis. We expect, but have not verified that exchange of stability may be obtained by our techniques as well.

In contrast to the detonation case, this investigation is not driven by known physical phenomena, but by the common mathematical structure with detonation. That is, in this case the mathematics suggests a possible physical phenomenon, and not the reverse. It would be very interesting to look numerically or experimentally for its appearance, particularly in the rich settings of MHD or phase-transitional gas dynamics with van der Waals-type equation of state, where instability, similarly as in the detonation case (but in contrast to gas dynamics), frequently occurs [T].

**Remark 1.1.** A third approach to bifurcation in the absence of a spectral gap, besides those described in [TZ2, SS], may be found in a recent work of Kunze and Schneider [KuS] in which they analyze pitchfork bifurcation in the absence of a spectral gap using weighted-norm methods like those used to study stability for dispersive systems [PW]. This approach has been used to treat stability of dispersive–diffusive scalar undercompressive shocks [Do];
however, it does not appear to generalize to shock waves in the system case.

1.1 Equations and assumptions

Consider a one-parameter family of standing viscous shock solutions

\[ U(x, t) = \bar{U}^\varepsilon(x), \quad \lim_{z \to \pm\infty} \bar{U}^\varepsilon(z) = U^\varepsilon_\pm \quad \text{(constant for fixed } \varepsilon) \]

of a smoothly-varying family of conservation laws

\[ U_t = \mathcal{F}(\varepsilon, U) := (B(\varepsilon, U)U_x)_x - F(\varepsilon, U)_x, \quad U \in \mathbb{R}^n, \]

with associated linearized operators

\[ L_\varepsilon := \partial \mathcal{F}/\partial U|_{U = \bar{U}^\varepsilon} = \partial_x B^\varepsilon(x)\partial_x - \partial_x A^\varepsilon(x) \]

\[ B^\varepsilon(x) = B(\bar{U}^\varepsilon(x)), \quad A^\varepsilon(x) := F_U(\bar{U}^\varepsilon(x), \varepsilon). \]

Equations (1.2) are typically shifts \( B(\varepsilon, U) = B(U), \ F(\varepsilon, U) := f(U) - s(\varepsilon)U \) of a single equation

\[ U_t = (B(U)U_x)_x - f(u)_x \]

written in coordinates \( \tilde{x} = x - s(\varepsilon)t \) moving with traveling-wave solutions \( U(x, t) = \bar{U}^\varepsilon(x - s(\varepsilon)t) \) of varying speeds \( s(\varepsilon) \). Profiles \( \bar{U}^\varepsilon \) satisfy the standing-wave ODE

\[ B(U)U' = F(\varepsilon, U) - F(\varepsilon, U^\varepsilon_\pm). \]

Denote

\[ B^\varepsilon_\pm := \lim_{z \to \pm\infty} B^\varepsilon(z) = F_U(U^\varepsilon_\pm, \varepsilon), \quad A^\varepsilon_\pm := \lim_{z \to \pm\infty} A^\varepsilon(z) = F_U(U^\varepsilon_\pm, \varepsilon), \]

and \( A(\varepsilon, U) = F_u(\varepsilon, U) \).

We make the following structural assumptions.

(A1) \( U = \begin{pmatrix} U_1 \\ U_2 \end{pmatrix}, \quad B = \begin{pmatrix} 0 & 0 \\ 0 & b \end{pmatrix}, \quad b \text{ nonsingular, where } U \in \mathbb{R}^n, \]

\( U_1 \in \mathbb{R}^{n-r}, U_2 \in \mathbb{R}^r, \) and \( b \in \mathbb{R}^{r \times r}; \) moreover, the \( u \)-coordinate \( F_1(\varepsilon, U) \) of \( F \) is linear in \( U \) (strong block structure).

(A2) There exists a smooth, block-diagonal, positive definite matrix \( A^0(\varepsilon, U) \) such that \( A^0_{11}A_{11} \) is symmetric and \( A^0_{22}b \) is positive definite but not necessarily symmetric (symmetric hyperbolic-parabolicity).

To (A1)–(A2), we add the following more detailed hypotheses. Here and elsewhere, \( \sigma(M) \) denotes the spectrum of a matrix or linear operator \( M \).
\( F, B \in C^k, \ k \geq 5. \)

\( \sigma(A_{11}^\varepsilon) \) real, semisimple, nonzero, and constant multiplicity.

\( \sigma(A_{\pm}^\varepsilon) \) real, simple, and nonzero.

\( \mathbb{R} \sigma (i \xi_j dF(U_\pm) - \xi^2 B(U_\pm)) \leq -\frac{\theta |\xi|^2}{1+|\xi|^2}, \ \theta > 0, \) for \( \xi \in \mathbb{R}. \)

(H4) Considered as connecting orbits of (1.4), \( \bar{U}^\varepsilon \) lie in an \( \ell \)-dimensional manifold, \( \ell \geq 1, \) of solutions (1.1), obtained as a transversal intersection of the unstable manifold at \( U^-_\varepsilon \) and the stable manifold at \( U^+_\varepsilon. \)

Remark 1.2. Conditions (H1)–(H2) imply that \( U^-_\varepsilon \) and \( U^+_\varepsilon \) are nonhyperbolic rest points of ODE (1.4) expressed in terms of the \( v \)-coordinate, whence, by standard ODE theory,

\[
|\partial_\xi^\ell (\bar{U}^\varepsilon - U^-_\varepsilon)(x)| \leq Ce^{-\eta|x|}, \quad 0 \leq \ell \leq k+1,
\]

for \( x \geq 0, \) some \( \eta, \ C > 0; \) in particular, \( |(\bar{U}^\varepsilon)'(x)| \leq Ce^{-\eta|x|}. \) Condition (H4) implies in part that \( \bar{U}^\varepsilon \) is either of standard Lax type, \( \ell = 1, \) or nonclassical overcompressive type, \( \ell > 1, \) i.e., the hyperbolic convection matrices \( A_{\pm}^\varepsilon := F_U(U_\pm, \varepsilon) \) at \( \pm \infty \) have, respectively, \( p - \ell \) negative and \( n - p \) positive eigenvalues for \( 1 \leq \ell \leq p \leq n. \) In the Lax case, the \( (\ell = 1) \)-dimensional manifold of solutions (1.1) consist of translates of \( \bar{U}^\varepsilon, \) and \( \ell \) is the characteristic family associated with the shock. For further discussion, see [ZH, MaZ1, Z2].

Remark 1.3. Similarly as in [Z2, Z3], it is readily checked that the conclusions of this paper hold also with (A1)–(A2) replaced by the weakened assumptions that there exist an invertible change of coordinates \( U \to (U_1, W_2) \) for which \( F_1 \) is a linear function of \( U_1, W_2 \) with \( \partial_{U_1} F_1 \) symmetrizable and

\[
B(U)\partial_x U = \begin{pmatrix} 0 \\ \beta(U)\partial_x W_2 \end{pmatrix} \text{ with } \Re \sigma ((\partial W_2/\partial U_1)\beta) \leq -\theta < 0 \text{ and } (H1)
\]

modified as in [MaZ1] to account for nonvanishing of \( B_{21}, \) replacing \( A_{11} \) by \( A_* := A_{11} - B_{21} B_{22}^{-1} A_{12}; \) see Remarks 3.5 and 5.1. These assumptions are used to obtain the \( H^s \) energy estimates of Section 3.

Conditions (A1)–(A2) (or the alternatives described in Remark 3.5) are a slightly strengthened version of the corresponding hypotheses of [MaZ3, Z2, Z3] for general systems with “real”, or partially parabolic viscosity, the difference lying in the strengthened block structure condition (A1): in particular, the assumed linearity of the \( U_1 \) equation. Conditions (H0)–(H4) are
the same as those in [MaZ3, Z2, Z3]. The class of equations satisfying our assumptions, though not complete, is sufficiently broad to include many models of physical interest, in particular compressible Navier–Stokes equations and the equations of compressible magnetohydrodynamics (MHD), expressed in Lagrangian coordinates, with either ideal or “real” van der Waals-type equation of state; see Section 5 for further discussion. The role of assumptions (A1)–(A2) in our analysis is discussed further in Appendix A.

A simple example is the equations of isentropic gas dynamics written in Lagrangian coordinates:

\[ \begin{align*}
    v_t - u_x &= 0, \\
    u_t + p_x &= \left( \frac{\nu}{v} \right) u_x 
\end{align*} \tag{1.7} \]

where \( v > 0 \) denotes specific volume, \( u \) fluid velocity, \( p = p(v) \) pressure, and \( \nu > 0 \) the coefficient of viscosity, with \( p_v < 0 \) corresponding to hyperbolicity of the associated first-order system. Denoting \( U_1 = v, U_2 = u \), and considering a family of traveling-wave solutions \( U(x, t) = \tilde{U}(x - s(\varepsilon)t) \) with \( p_v(v^+_\varepsilon) < 0 \), we obtain after the change of coordinates \( x \to x - s(\varepsilon)t \) a family of stationary solutions \( U(x, t) = \tilde{U}(x) \) and equations

\[ \begin{align*}
    v_t - s(\varepsilon)v_x - u_x &= 0, \\
    u_t - s(\varepsilon)u_x + p_x &= \left( \frac{\nu}{v} \right) u_x 
\end{align*} \tag{1.8} \]

satisfying (A1)–(A2), (H0)–(H4).

**Remark 1.4.** Equations (1.7) written in Eulerian coordinates, or

\[ \begin{align*}
    \rho_t + (\rho u)_x &= 0, \\
    (\rho u)_t + (\rho u^2 + p)_x &= \nu u_{xx}
\end{align*} \tag{1.9} \]

where \( \rho = 1/v \) denotes density does not satisfy (A1)–(A2), (H0)–(H4), violating the second part of the strong block structure assumption (A1). (Here, \( U_1 = \rho, U_2 = m = \rho u \) is momentum, so that the first equation is still linear; however, the parabolic term \( u_{xx} = (m/\rho)_{xx} \) is not in diagonal form.) Indeed, we make crucial use in the analysis of the Lagrangian formulation, as discussed in Appendix A.

### 1.2 Spectral criteria

As discussed in [ZH, MaZ1, Z1, Z2, Z3], the linearized operators \( L_\varepsilon \) have no spectral gap, since essential spectrum accumulates at \( \lambda = 0 \) on the imaginary
axis. In this situation, standard stability and bifurcation criteria based on isolated spectra of $L_\varepsilon$ are replaced by generalized versions expressed in terms of an associated Evans function. The Evans function $D_\varepsilon(\lambda)$, defined as a Wronskian of functions spanning the decaying manifolds of solutions of the eigenvalue equation

$$
(L_\varepsilon - \lambda)u = 0
$$

associated with $L_\varepsilon$ at $x \to +\infty$ and $x \to -\infty$ is an analytic function with domain containing $\{\Re \lambda \geq 0\}$, whose zeroes away from the essential spectrum correspond in location and multiplicity with eigenvalues of $L_\varepsilon$. Its behavior is also closely linked with that of the resolvent kernel of $L_\varepsilon$, i.e., the Laplace transform with respect to time of the Green function $G$; see, e.g., [AGJ, GZ, ZH, ZS, Z1, Z2] for history and further details.

In [MaZ1] there was established the following stability criterion.

**Proposition 1.5 ([MaZ1]).** Let $\bar{U}_\varepsilon$, (1.2) be a family of traveling-waves and systems satisfying assumptions (A1)–(A2), (H0)–(H4). Then, $\bar{U}_\varepsilon$ is linearly stable from $L^1 \to L^p$ if and only if there exist precisely $\ell$ zeroes ($\ell$ as in (H4)) of $D_\varepsilon(\cdot)$ in the nonstable half-plane $\Re \lambda \geq 0$, necessarily at the origin $\lambda = 0$.

We define an analogous Hopf bifurcation criterion as

$$(D_\varepsilon) \quad \text{On a neighborhood of } \{\Re \lambda \geq 0\} \setminus \{0\}, \text{ the only zeroes of } D \text{ are (i) a zero of multiplicity } \ell \text{ at } \lambda = 0, \text{ and (ii) a crossing conjugate pair of zeroes } \lambda_\pm(\varepsilon) = \gamma(\varepsilon) + i\tau(\varepsilon) \text{ with } \gamma(0) = 0, \partial_\varepsilon \gamma(0) > 0, \text{ and } \tau(0) \neq 0.$$

**Remarks 1.6.** 1. In the simplest, Lax case, under (A1)–(A2), (H0)–(H4), simplicity of the root $\lambda = 0$, condition $D_\varepsilon(i)$ is equivalent to $\langle \pi^\varepsilon, (\bar{u}^\varepsilon)' \rangle = \pi^\varepsilon \cdot (\bar{u}_+^\varepsilon - \bar{u}_-^\varepsilon) \neq 0$ for $\pi^\varepsilon$ (constant) orthogonal to $S(A_\varepsilon^-) \cup U(A_\varepsilon^+)$ [GZ, MaZ2, Z1, Z2]. Under the normalization $\langle \pi^\varepsilon, (\bar{u}^\varepsilon)' \rangle = 1$, operator $\Pi_0 f := (\bar{u}^\varepsilon)'(\pi^\varepsilon, f)$ plays the role of a “generalized spectral projection” onto

$$\text{Ker}L(\varepsilon) = \text{Span}\{ (\bar{u}^\varepsilon)' \},$$

and $\pi^\varepsilon$ the role of a generalized left eigenfunction [MaZ1, ZH]. Note that $\pi^\varepsilon$ lies outside the domain of $\Pi_0$, a consequence of the absence of a spectral gap. A similar, but more complicated condition holds in the overcompressive case [MaZ1, Z1, Z2].

2. By Proposition 1.5, condition $D_\varepsilon$ with (A1)–(A2), (H0)–(H4) implies that $\bar{U}_\varepsilon$ is linearly stable for $\varepsilon < 0$ and unstable for $\varepsilon > 0$; that is, there is a transition from stability to instability at $\varepsilon = 0$.
1.3 Results

We introduce the following notation, to be used throughout the paper.

**Definition 1.7.** Let $B_2 \subset B_1$ and $X_2 \subset X_1$ denote the Banach spaces determined by norms $\|U\|_{B_1} := \|U\|_{H^1}$, $\|\partial_x U\|_{B_2} := \|\partial_x U\|_{B_1} + \|U\|_{L^1}$ and

\[
\begin{align*}
\|U\|_{X_1} &:= \|e^{\eta \langle x \rangle} U\|_{H^2}, \\
\|\partial_x U\|_{X_2} &:= \|\partial_x U\|_{X_1} + \|e^{2\eta \langle x \rangle} U\|_{H^1},
\end{align*}
\]  

(1.11)

where $\eta > 0$ and $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Our main result is the following theorem establishing Hopf bifurcation from the steady solution $\bar{U}^\varepsilon$ at $\varepsilon = 0$ under bifurcation criterion $(D_\varepsilon)$.

**Theorem 1.8.** Let $\bar{U}^\varepsilon$, (1.2) be a family of traveling-waves and systems satisfying assumptions (A1)-(A2), (H0)-(H4), and $(D_\varepsilon)$. Then, for $r \geq 0$, $\eta > 0$ sufficiently small and $C > 0$ sufficiently large, there is a $C^1$ function $\varepsilon(r)$, $\varepsilon(0) = 0$, and a $C^1$ family of time-periodic solutions $U^r(x, t)$ of (1.2) with $\varepsilon = \varepsilon(r)$, of period $T(r)$, $T(\cdot) \in C^1$, $T(0) = 2\pi/\tau(0)$, with

\[
\begin{align*}
C^{-1}r &\leq \|U^r - \bar{U}^\varepsilon\|_{X_1} \leq Cr
\end{align*}
\]  

(1.12)

for all $t$. For Lax shocks, up to fixed translations in $x$, $t$, these are the only time-periodic solutions nearby in $X_1$ with period $T \in [T_0, T_1]$ for any fixed $0 < T_0 < T_1 < +\infty$; if $U^\varepsilon_+ \neq U^\varepsilon_-$, they are the only nearby solutions of the more general form $U^r(x - \sigma^t t, t)$ with $U^r(x, \cdot)$ periodic. For overcompressive shocks, they are part of a $C^1$ $(\ell - 1)$-parameter family of solutions that are likewise unique up to translation in $x$, $t$.

**Remark 1.9.** Bound (1.12), by Sobolev embedding, includes also the result of exponential localization, $|U^r - \bar{U}^\varepsilon| \leq C r e^{-\eta |x|}$.

1.4 Analysis

Theorem 1.8 is proved using the general bifurcation framework established in [TZ2] together with pointwise Green function bounds established in [MaZ1] for general hyperbolic–parabolic systems. This is for the most part straightforward, demonstrating the power and flexibility of the frameworks set up in [TZ2] and [MaZ1]. However, there is an interesting and apparently general
difficulty associated with the absence of parabolic smoothing, namely, the
need to show that nonlinear source terms, defined as the difference between
nonlinear solutions and solutions of the linearized equations, are quadra-
tically small in the $X_1$ norm relative to the $X_1$ norm of initial data, the key
point being to control the high-derivative norm $\| \cdot \|_{H^s}$ despite apparent loss
of derivatives. We discuss this issue in detail in Section 3.

An analogous issue arises in the nonlinear stability theory in going from
the strictly parabolic to the hyperbolic–parabolic case, with the difficulty
again to control an apparent loss of derivatives. This is resolved in [MaZ2,
MaZ3] with an auxiliary nonlinear energy estimate. Here, as there, we ob-
tain the needed derivative control by auxiliary energy estimates (Prop. 3.2).
However, these are of a rather different type, being local rather than global
in time, and measuring variational rather than time-asymptotic properties.
They are also somewhat more delicate, depending strongly on the structure of
the hyperbolic part of the equations. In particular, they hold in Lagrangian
but not Eulerian coordinates, indicating the importance of nonlinear trans-
port effects; see the discussion of Appendix A.

A second new feature in the analysis is the incorporation of exponentially
weighted norms, yielding (1.12). We accomplish this by an improved way
of accounting cancellation, which at the same time greatly simplifies the
analysis. The key estimate in [TZ2] corresponds heuristically to showing
that the kernel

$$
K_y(x, y) := \sum_{j=0}^{\infty} K_y(x, jT; y)
$$

(1.13)

of the $y$-derivative of formal inverse

$$
(\text{Id} - e^{LT})^{-1} = \sum_{j=0}^{\infty} e^{LjT},
$$

(1.14)

$L := \partial_x^2 - a\partial_x$, converges in $L^\infty(L^2(x); y)$ uniformly for $0 < C^{-1} \leq T \leq C$, where

$$
K(x, t; y) := c_0 t^{-1/2} e^{(x-y-at)^2/4t}
$$

is a convected heat kernel, with $a > 0$, with decay of $K_y(x, y) = K_y(x - y, 0)$
in $|x - y|$ determining the ultimate decay rate for $|U^r - \bar{U}^\varepsilon|$.

Since $\|K(x, jT; y)\|_{L^2(x)} \equiv Ct^{-1/4}$, convergence in (1.13) cannot be abso-
lute, but must involve cancellation. This was detected in [TZ2] by using the
defining equation $K_t + aK_x - K_{xx} = 0$ to write

\[ \sum_{j=1}^{\infty} K_y(x, jT; y) \sim T^{-1} \int_T^{+\infty} K_y(x, t; y) dt \]

\[ = -a^{-1}T^{-1} \int_T^{+\infty} K_t(x, t; y) dt + a^{-1}T^{-1} \int_T^{+\infty} K_{yy}(x, t; y) dt \]

\[ = a^{-1}T^{-1} K(x, T; y) + a^{-1}T^{-1} \int_T^{+\infty} K_{yy}(x, t; y) dt, \]

and observing that $\int_T^{+\infty} K_{yy}(x, t; y) dt$ is convergent and $\leq C(1 + |x - y|)^{-1}$.

On the other hand, we could just as well have repeated the process to estimate

\[ \int_T^{+\infty} K_{yy}(x, t; y) dt = a^{-1}T^{-1} K_y(x, T; y) + a^{-1}T^{-1} \int_T^{+\infty} K_{yyy}(x, t; y) dt, \]

with $|\int_T^{+\infty} K_{yyy}(x, t; y) dt| \leq C(1 + |x - y|)^{-3/2}$. Continuing the process, we find that $K_y$ decays to any polynomial order, with asymptotic expansion

\[ K_y(x, y) = T^{-1} \sum_{j=0}^{\infty} a^{-j} \partial_y^j K(x, T; y), \]

each term of which exhibits exponential decay, but for which we have shown only polynomial spatial bounds and not convergence to zero of the truncation error of successive finite sums. Moreover, there is still the issue of the “continuation error” introduced at the first step of (1.15).

In the present analysis, we sidestep these issues, effectively summing to all orders expansion (1.17), by working at the level of the inverse Laplace transform formula

\[ K(x, y; t) = \oint_{\Gamma} e^{\lambda t} K_\lambda(x, y) d\lambda \]

used to obtain pointwise bounds for the actual, variable-coefficient problem, where $K_\lambda(x, y)$ is the resolvent kernel, or Laplace transform of $K(x, t; y)$, and $\Gamma$ is an appropriate sectorial contour contained strictly in the resolvent set of $L := \partial_x^2 - a\partial_x$ and lying in the strictly negative half-plane $\Re \lambda \leq -\eta_0 < 0$ for $|\lambda| \geq r.$
Contributions from $\lambda \geq r > 0$ are negligible, and likewise from $x < y$ (recall, $a > 0$), and so we may focus on the low-frequency regime $|\lambda| \leq r$ and $x > y$, where $K_\lambda(x, y) = c(\lambda)e^{\mu(\lambda)(x-y)}$ with

\begin{equation}
\begin{aligned}
c(\lambda) &= (c_0 + \lambda c_1 + \cdots) \\
\mu(\lambda) &= \lambda \tilde{\mu}(\lambda), \\
\tilde{\mu} &= -a^{-1} + d_1 \lambda + \cdots,
\end{aligned}
\end{equation}

and thus $\partial_y K_\lambda = \lambda \tilde{\mu}(\lambda) K_\lambda$. The assumption that $\Gamma$ lie in the resolvent set implies further that $\Re \mu \leq -\eta_0 < 0$ for $\lambda \in \Gamma \cap \{|\lambda| \leq r\}$.

Carrying out the sum (1.13) exactly, therefore, within Laplace inversion formula (1.18), on low frequencies $|\lambda| \leq r$ and $x > y$, we obtain a contribution to $K_y(x, y)$ of

\begin{equation}
\oint_{\Gamma \cap \{|\lambda| \leq r\}} \sum_{j=0}^{J} e^{\lambda j T} \partial_y \tilde{\mu}(\lambda) K_\lambda(x, y) d\lambda
\end{equation}

for some $C, \eta > 0$, by (1.19) and $\Re \mu \leq -\eta_0$. Not only does this argument yield exponential spatial decay bounds, but it turns out to be much easier than the previous argument to generalize to the full, variable-coefficient problem, thus streamlining and sharpening the analysis at the same time.

**Remark 1.10.** Accounting cancellation in this way on the whole solution at once is much simpler than breaking the solution into individual components and estimating each separately as was done in [TZ2]. Notice further that we have essentially performed the inversion (1.14) directly via the spectral resolution formula

\begin{equation}
(\text{Id} - e^{LT})^{-1} \partial_x = \oint_{\Gamma} (1 - e^{\lambda T})^{-1} \partial_y G_\lambda d\lambda,
\end{equation}

for some $C, \eta > 0$. Not only does this argument yield exponential spatial decay bounds, but it turns out to be much easier than the previous argument to generalize to the full, variable-coefficient problem, thus streamlining and sharpening the analysis at the same time.
which has an “elliptic” flavor perhaps somewhat analogous to the spatial dynamics point of view. However, in the full problem, there is a boundary contribution coming from the $J \rightarrow \infty$ term in (1.20) for certain (stationary eigen-) modes, that is not reflected in formula (1.21); see Remark 2.8.

1.5 Discussion and open problems

In the strictly parabolic case [TZ2], the analysis of the one-dimensional case extended easily to yield a corresponding multidimensional result for shocks propagating along a cylinder of finite cross-section with artificial (Neumann or periodic) boundary conditions. In the hyperbolic–parabolic case, however, our reliance on Lagrangian coordinates in deriving $H^s$ energy estimates limits us for the moment to one dimension; see Remark A.1 and the discussion of Appendix A. It would be very interesting (both physically and mathematically) to remove this restriction, perhaps by the incorporation of nonautonomous effects in the linearized equations as in [D].

A further issue when and whether bifurcation condition $(\mathcal{D}_\varepsilon)$ is actually satisfied in physically interesting situations. It may well be that $(\mathcal{D}_\varepsilon)$ does not occur at all for ideal gas dynamics, nor any other kind of instability. However, it seems quite possible that $(\mathcal{D}_\varepsilon)$ can occur in the richer setting of MHD, where Lax-type shock waves are known to be sometimes unstable even for an ideal gas equation of state [T], or for phase-transitional gas dynamics with a van der Waals-type equation of state. Numerical investigation of $(\mathcal{D}_\varepsilon)$ across a range of shock and detonation waves is an important direction for future investigation. Numerical evaluation of the Evans function may carried out in a well-conditioned fashioned, as described, e.g., in [Br1, Br2, BrZ, BDG, HuZ, BHR].

The analysis of the present paper serves also as a stepping-stone to the closely related but more complicated detonation case. With suitable elaboration, the argument extends to that case [TZ3], yielding Hopf bifurcation, or “galloping instability” of viscous strong detonation waves in one dimension. As in the shock case, this result of course is subject to verification of the spectral bifurcation hypothesis $(\mathcal{D}_\varepsilon)$. For detonation solutions of reacting gas dynamics, galloping is expected to occur, and Evans function results [LyZ1, LyZ2] show that instability if it occurs must be of a (possibly degenerate) Hopf type spectral configuration.

Finally, we mention the problem of determining stability of the bifurcating time-periodic waves whose existence is established in Theorem 1.8. In the
absence of a spectral gap, our method of analysis does not directly yield sta-

bility as in the finite-dimensional ODE case, but at best partial information
on the location of point spectrum associated with oscillatory modes, with
stability presumably corresponding to the standard condition $d\varepsilon/dr > 0$; see
[SS] for a corresponding result in the strictly parabolic case. The hope is
that one could combine such information with an analysis like that carried
out for stationary waves in [ZH, MaZ1], adapted from the autonomous to the
time-periodic setting: that is, a generalized Floquet analysis in the PDE set-
ning and in the absence of a spectral gap. This seems a particularly exciting
direction for further development of the theory.

**Plan of the paper.** We begin in Section 2 by recalling the linearized
bounds of [MaZ1, Z2, TZ2], and establishing the new bound described in
(1.20). In Section 3, we carry out the nonlinear energy estimates controlling
higher derivatives in nonlinear source terms. In Section 4, we prove the main
theorem, carrying out the bifurcation analysis following the framework of
[TZ2]. Finally, in Section 5, we show that our results apply to gas- and
magnetohydrodynamics.

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analysis was carried out.

## 2 Linearized estimates

Assuming (A1)–(A2), (H0)–(H4) (alternatively, the weakened hypotheses of
Remark 3.5), let $L_\varepsilon$ as in the introduction be the linearized operators (1.3),
and $\lambda_{\pm}(\varepsilon)$ the crossing eigenvalues of $(D_\varepsilon)$.

**Lemma 2.1** (TZ2). Associated with eigenvalues $\lambda_{\pm}(\varepsilon)$ of $L_\varepsilon$ are right and
left eigenfunctions $\phi_{\pm}^\varepsilon$ and $\tilde{\phi}_{\pm}^\varepsilon \in C^k(x, \varepsilon)$, $k \geq 2$, exponentially decaying in
up to $q$ derivatives as $x \to \pm \infty$, and $L_\varepsilon$-invariant projection

$$
\Pi f := \sum_{j=\pm} \phi_j^\varepsilon(x) \langle \tilde{\phi}_j^\varepsilon, f \rangle
$$

onto the total (oscillatory) eigenspace $\Sigma^\varepsilon := \text{Span}\{\phi_{\pm}^\varepsilon\}$, bounded from $L^q$ or
$B_2$ to $W^{k,p} \cap X_2$ for any $1 \leq q, p \leq \infty$. Moreover,

\begin{equation}
\phi^\varepsilon_\pm = \partial_x \Phi^\varepsilon_\pm,
\end{equation}

with $\Phi^\varepsilon \in C^{k+1}$ exponentially decaying in up to $k+1$ derivatives as $x \to \pm \infty$.

**Proof.** From simplicity of $\lambda_\pm$, and the fact [MaZ1] that they are bounded away from the essential spectrum of $L_\varepsilon$, we obtain either by standard spectral perturbation theory [Kat] or by direct Evans-function calculations [GJ1, GJ2, MaZ1, Z2] that there exist $\lambda_\pm(\cdot), \phi^\varepsilon_\pm(\cdot) \in L^2$ with the same smoothness $C^k(\varepsilon)$, $k \geq 5$, assumed on $F$. The exponential decay properties in $x$ then follow by standard asymptotic ODE theory; see, e.g., [GZ, Z2, Z3]. Finally, recall the observation of [ZH] that, by divergence form of $L_\varepsilon$, we may integrate $L_\varepsilon \phi = \lambda \phi$ from $x = -\infty$ to $x = +\infty$ to obtain $\lambda \int_{-\infty}^{+\infty} \phi(x) dx = 0$, and thereby (since $\lambda_\pm \neq 0$ by assumption)

\begin{equation}
\int_{-\infty}^{+\infty} \phi_\pm(x) dx = 0,
\end{equation}

from which we obtain by integration (2.2) with the stated properties of $\Phi_\pm$. From (2.2) and representation (2.1), we obtain by Hölder’s inequality the stated bounds on projection $\Pi$.

Defining $\tilde{\Pi}_\varepsilon := \text{Id} - \Pi^\varepsilon, \tilde{\Sigma}^\varepsilon := \text{Range} \tilde{\Pi}^\varepsilon, \text{and } \tilde{L}_\varepsilon := L_\varepsilon \tilde{\Pi}^\varepsilon$, denote by

\begin{equation}
G(x, t; y) := e^{L_\varepsilon t} \delta(y)(x)
\end{equation}

the Green kernel associated with the linearized solution operator $e^{L_\varepsilon t}$ of the linearized evolution equations $u_t = L_\varepsilon u$, and

\begin{equation}
\tilde{G}(x, t; y) := e^{\tilde{L}_\varepsilon t} \tilde{\Pi} \delta(y)(x)
\end{equation}

the Green kernel associated with the transverse linearized solution operator $e^{\tilde{L}_\varepsilon t} \tilde{\Pi}$. By direct computation, $G = O + \tilde{G}$, where

\begin{equation}
O(x, t; y) := e^{(\gamma(\varepsilon)+i\tau(\varepsilon)) t} \phi_+(x) \tilde{\phi}_+(y) + e^{(\gamma(\varepsilon)-i\tau(\varepsilon)) t} \phi_-(x) \tilde{\phi}_-(y).
\end{equation}
2.1 Short time estimates

Lemma 2.2. For $0 \leq t \leq T$, $1 \leq p \leq \infty$, $\eta > 0$, and some $C = C(T)$,

\begin{equation}
\| e^{L_{\varepsilon} t} f \|_{L^p}, \| e^{\tilde{L}_{\varepsilon} t} \tilde{\Pi} f \|_{L^p} \leq C \| f \|_{L^p},
\end{equation}

(2.7)

\begin{equation}
\| e^{\eta(x)} \int e^{L_{\varepsilon} t} \partial_z x \|_{L^p}, \| e^{\eta(x)} \int e^{\tilde{L}_{\varepsilon} t} \tilde{\Pi} \partial_z x \|_{L^p} \leq C \| e^{\eta(x)} f \|_{L^p},
\end{equation}

(2.8)

\begin{equation}
\| e^{\eta(x)} \int \partial_{x,t} e^{L_{\varepsilon} t} \partial_z x \|_{L^p}, \| e^{\eta(x)} \int \partial_{x,t} e^{\tilde{L}_{\varepsilon} t} \tilde{\Pi} \partial_z x \|_{L^p} \leq C \| e^{\eta(x)} f \|_{W^{2p}},
\end{equation}

(2.9)

where $\langle x \rangle := (1 + |x|^2)^{1/2}$.

Proof. From standard $C^0$ semigroup bound $|e^{L_{\varepsilon} t}|_{L^p \to L^p} \leq C$ and properties $e^{L_{\varepsilon} t} = e^{L_{\varepsilon} t} \Pi + e^{\tilde{L}_{\varepsilon} t} \tilde{\Pi}$ and $\| \Pi f \|_{L^p} \leq |f|_{L^p}$. We may obtain integrated bounds (2.8) ($r = 0$) using the divergence form of $L_{\varepsilon}$, by integrating the linearized equations with respect to $x$ to obtain linearized equations $U_t = \mathcal{L}_\varepsilon U$ for integrated variable

$$U(x, t, \varepsilon) := \int_{-\infty}^{x} u(z, t, \varepsilon) dz, \quad u(\cdot, t) := e^{L_{\varepsilon} t} \partial_x f,$$

with linearized operator $\mathcal{L}_\varepsilon := -A^\varepsilon(x) \partial_x + \partial_x^2$ of the same parabolic form as $L_{\varepsilon}$, then applying standard $C^0$ semigroup estimates (alternatively, more detailed pointwise bounds as in [MaZ1]) to bound

$$\| e^{\eta(x)} U(\cdot, t, \varepsilon) \|_{L^p} = \| e^{\eta(x)} e^{L_{\varepsilon} t} f \|_{L^p} \leq C \| e^{\eta(x)} f \|_{L^p},$$

the $e^{\tilde{L}_{\varepsilon} t} \tilde{\Pi}$ bound then following by relation $e^{L_{\varepsilon} t} = e^{L_{\varepsilon} t} \Pi + e^{\tilde{L}_{\varepsilon} t} \tilde{\Pi}$ together with $\| e^{\eta(x)} \int \Pi \partial_x f \|_{L^p} \leq |f|_{L^p}$. We obtain (2.8) ($r > 0$) by the change of variables $V = U e^{\eta(x)}$, which, since $\alpha := e^{\eta(x)}$ satisfies $|(d/dx)^k \alpha| \leq C |\alpha|$ for $k \geq 0$, converts the linearized equations to an equation with the same principal part plus lower-order terms with bounded coefficients.

Finally, (2.9) follows from $\partial_t e^{L(\varepsilon) t} = e^{L(\varepsilon) t} L(\varepsilon)$ and the variational equation $(\partial_t - L) \partial x U = (\partial_x L) U$, $U(t) := e^{L(t) t} U_0$, together with $\| L(\varepsilon) U_0 \|_{L^p}$, $\| L(\varepsilon) U \|_{L^p} \leq C \| U_0 \|_{W^{2,p}}$. \qed
2.2 Pointwise Green function bounds

We now develop the key cancellation estimates analogous to (1.20) of the introduction, adapting the pointwise semigroup methods of [ZH, MaZ1, Z2] to the present case. Our starting point is the inverse Laplace transform representation

\[
G(x, t; y) = \frac{1}{2\pi i} \text{P.V.} \int_{\eta-i\infty}^{\eta+i\infty} e^{\lambda t} G_\lambda(x, y) d\lambda,
\]

\(\eta > 0\) sufficiently large, established in [MaZ1].

Deforming the contour using analyticity of \(G_\lambda\) [MaZ1] across oscillatory eigenvalues \(\lambda_{\pm}(\varepsilon)\) we obtain \(G = \tilde{G} + O\), where \(O\), defined in (2.6), is the sum of the residues of the integrand at \(\lambda_{\pm}\), and, for \(\nu, \tau > 0\) sufficiently small,

\[
\tilde{G}(x, t; y) = \frac{1}{2\pi i} \oint_{\gamma} e^{\lambda t} G_\lambda(x, y) d\lambda
\]

\[
+ \frac{1}{2\pi i} \text{P.V.} \left( \int_{-\nu-i\infty}^{-\nu+ri} + \int_{-\nu+ri}^{-\nu-i\infty} \right) e^{\lambda t} G_\lambda(x, y) d\lambda
\]

\(=: \tilde{G}^I + \tilde{G}^{II}\),

where \(\gamma\) is the counterclockwise arc of circle \(\partial B(0, r)\) connecting \(-\nu - ri\) and \(-\nu + ri\), and \(G^I\) as in [Z3] is the low-frequency and \(G^{II}\) the high-frequency component of \(\tilde{G}\). Define associated solution operators \(\tilde{S}^I(t, \varepsilon)\) and \(\tilde{S}^{II}(t, \varepsilon)\) by

\[
\tilde{S}^\beta(t) f(x) := \int_{-\infty}^{+\infty} \tilde{G}^\beta(x, t; y) f(y) dy
\]

and

\[
\tilde{S} := \tilde{S}^I + \tilde{S}^{II}.
\]

Supressing the parameter \(\varepsilon\), denote by \(a_j^\pm, r_j^\pm, t_j^\pm\) the eigenvalues and associated right and left eigenvectors of \(A_\varepsilon^x = F_u(u_{\pm}^\varepsilon, \varepsilon)\). Following [MaZ1], let \(a_j^x(x)\), \(j = 1, \ldots, J \leq (n - r)\) denote the eigenvalues of

\[
A_* := A_{11} - B_{21} B_{22}^{-1} A_{12} = A_{11}.
\]

Let \(L(X, Y)\) denote the space of bounded linear operators from Banach space \(X\) to \(Y\), equipped with the usual operator norm \(|\cdot|_{L(X,Y)}\).
Proposition 2.3. Under assumptions (A1)–(A2), (H0)–(H4), (Dε) (alternatively, those of Remark 3.5), for 0 ≤ q, r, s, 0 ≤ 2q + s ≤ 4, and ν > 0, (2.13)

\[ \partial_\varepsilon^p \partial_T^r \partial_x^s \tilde{G}^{II}(x, t; y) = \sum_{p \leq (2q+r)+s} \left( \sum_{j=1}^J O(e^{-\nu t})\delta_x^{-\alpha_j t}(-y) + O(e^{-\nu(|x-y|+t)}) \right) \partial_y^p. \]

Proof. The case (q = r = 0) is immediate from the bounds of [MaZ1]. Derivatives with respect to ε may be converted using the variational equation (2.14) \((\lambda - L)(\partial_\varepsilon G_\lambda) = (\partial_\varepsilon L)G_\lambda\) into two spatial derivatives, thus extending to case (r = 0). Likewise, time-derivatives, appearing as powers of λ at the level of the resolvent kernel, have the effect of single spatial derivatives on “hyperbolic” blocks, and double spatial derivatives on “parabolic” blocks, in the notation of [MaZ1], yielding by the same estimates as in case (q = r = 0) the asserted bounds. This completes the proof of the general case.

\[ \sum_{j=0}^\infty S_{II}(jT) \] converges uniformly and absolutely in operator norm \(| \cdot |_{L(X_1, X_1)}\), for ε sufficiently small and T in any compact set, for any 0 < η < ν, to a limit that is \(C^1\) in \((\varepsilon, T)\) with respect to the operator norm \(L(B_1, B_1)\).

Proof. Straightforward computation using (2.13).

Corollary 2.4. Under assumptions (A1)–(A2), (H0)–(H4), (Dε) (alternatively, the weakened assumptions of Remark 3.5), \(\sum_{j=0}^\infty S_{II}(jT)\) converges uniformly (but conditionally) in the operator norm \(| \cdot |_{L(X_2, B_1)}\), for ε sufficiently small and T in any compact set bounded away from the origin, with limiting kernel satisfying for 0 ≤ q, r, s, 0 ≤ 2q + s ≤ 4, ν > 0,

(2.15)

\[ \partial_\varepsilon^p \partial_T^r \partial_x^s \sum_{j=0}^\infty \tilde{G}^{II}_y(x, jT; y) = O(e^{-\nu|x-y|} + e^{-\nu|x|}). \]

Proof. (Convergence). Since (by Sobolev embedding) \(\partial_x e^{2\eta|x|} \cdot \|L_\infty \subset X_2\), it is sufficient to show convergence in \(L(\partial_x e^{2\eta|x|} \cdot \|L_\infty, L^2\) or, equivalently, convergence in \(L(e^{2\eta|x|} \cdot \|L_\infty, L^2\) of the operator \(\sum_{j=0}^\infty S_{II} \partial_x\) with kernel

\[ \sum_{j=0}^J \tilde{G}^{II}_y(x, jT; y), \]

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where, by (2.11),

\[
\tilde{G}_y^I(x, jT; y) = \frac{1}{2\pi i} \oint_\gamma e^{\lambda jT} \partial_y \tilde{G}_\lambda(x, y) d\lambda,
\]

hence (summing under the integral as described in the introduction)

\[
\sum_{j=0}^J \tilde{G}_y^I(x, jT; y) = \frac{1}{2\pi i} \oint_\gamma \left( \frac{1}{1 - e^{\lambda T}} \right) \partial_y G_\lambda(x, y) d\lambda
\]

\[
- \frac{1}{2\pi i} \oint_\gamma e^{(j+1)T\lambda} \left( \frac{1}{1 - e^{\lambda T}} \right) \partial_y G_\lambda(x, y) d\lambda
\]

= : I + II.

Taking \( y \leq 0 \) for definiteness, recall from [MaZ1, Z2] that, for \( \lambda \) sufficiently small, \( \tilde{G}_\lambda(x, y) \) can in the Lax or overcompressive case be expanded analytically at \( \lambda = 0 \) as a sum of “excited” terms

\[
\lambda^{-1} \phi_j(x) l_k^{-T} e^{\lambda \tilde{\mu}_j^- (\lambda)y}, \quad \tilde{\mu}_j^- = -(a_j^-)^{-1} + O(\lambda),
\]

where \( \phi_j(x) = O(e^{-\nu|x|}) \) are stationary modes of the linearized equations; “scattering terms”

\[
r_j^\pm l_k^{-T} e^{\lambda \tilde{\mu}_j^\pm (\lambda)y + \lambda \tilde{\mu}_j^\pm (\lambda)x}, \quad \tilde{\mu}_j^\pm = -(a_j^\pm)^{-1} + O(\lambda);
\]

and error terms

\[
O(e^{-\nu(|x|+|y|)}
\]

and

\[
O(\lambda e^{\lambda \tilde{\mu}_j^- (\lambda)y + \lambda \tilde{\mu}_j^+ (\lambda)x}),
\]

with symmetric expansions for \( y \geq 0 \). These bounds may be converted by the Riemann Saddlepoint (~Stationary Phase) estimates to the pointwise Green function bounds of [ZH, MaZ1, Z2].

Taking \( y \)-derivatives, we find that

\[
\left( \frac{1}{1 - e^{\lambda T}} \right) \partial_y \tilde{G}_\lambda = -T^{-1}(\lambda^{-1} + O(1)) \partial_y \tilde{G}_\lambda
\]
expands as \((T a_k)^{-1}\) times the same excited and scattering terms \((2.18)\) and \((2.19)\), plus error terms of the same order \((2.20), (2.21)\), plus new pole terms of order

\[
(2.22) \quad \lambda^{-1}O(e^{-\nu(|x|+|y|)}).
\]

Thus, by the same Riemann saddlepoint estimates used to bound \(\tilde{G}^I(x, t; y)\) in [ZH, MaZ1, Z2], we find that the contribution to term \(II\) of all except the new terms \((2.22)\) satisfies exactly the same bounds as \(\tilde{G}^I(x, (J + 1)T; y)\), expanding for \(t = (J + 1)T\) as the sum of excited terms

\[
(2.23) \quad (T a_k)^{-1} \phi_j(x) l_k^{-t} \left( \text{erf} \left( \frac{y + a_k^+ t}{\sqrt{4\beta_k^- t}} \right) - \text{erf} \left( \frac{y - a_k^- t}{\sqrt{4\beta_k^- t}} \right) \right),
\]

\(\beta_k^+ > 0\), scattering terms

\[
\sum_{k=1}^{n} (1 + t)^{-1/2} O(e^{-(x-y-a_k^-)^2/Mt} e^{-\eta x^+})
+ \sum_{a_k^- > 0, a_j^- < 0} X_{\{|a_k^- t| \geq |y|\}} (1 + t)^{-1/2} O(e^{-(x-a_j^- (t-|y/a_k^-|))^2/Mt} e^{-\eta x^+}),
\]

\[
+ \sum_{a_k^- > 0, a_j^+ > 0} X_{\{|a_k^- t| \geq |y|\}} (1 + t)^{-1/2} O(e^{-(x-a_j^+ (t-|y/a_k^-|))^2/Mt} e^{-\eta x^-})
\]

bounded by convected heat kernels, where \(x^+\) (resp. \(x^-\)) denotes the positive (resp. negative) part of \(x\), and a negligible error

\[
(2.25) \quad O(e^{-\nu(|x-y|+t)}).
\]

Terms \((2.22)\) by the same argument give a time-independent contribution of

\[
(2.26) \quad \sum f_m(x) g_m(y) = O(e^{-\nu(|x|+|y|)})
\]

up to negligible error \((2.25)\).

Noting that all terms except \((2.23)\) and \((2.26)\) decay in \(L^2(x)\) at least at Gaussian rate \((1 + t)^{-1/4}\), independent of \(y\), while kernels \((2.23)\) integrated against an exponentially localized function converges exponentially to

\[
\phi_j(x) l_k^{-t} = O(e^{-\nu|x|}),
\]
we find that $II$ converges as $J \to \infty$ to a kernel with the claimed bound $O(e^{-\nu|x|} + e^{\nu|x-y|})$. Term $I$ (independent of $J$) by the elementary bound $|\partial_y G_{\lambda}(x,y)| \leq C e^{-\nu|x-y|}$ for $\lambda$ bounded away from essential spectrum of $L$, and the fact that $\gamma$ is bounded away from both $\lambda = 0$ and the essential spectrum of $L$, satisfies trivially the bound $I = O(e^{-\eta|x-y|})$, completing the proof of convergence while at the same time establishing (2.15) for $q, r, s = 0$.

(Derivative bounds) The remaining bounds (2.15)$(q, r, s \neq 0)$ follow easily, by the observation that the limiting summands in term $II$ are independent of $T$ and satisfy the same bounds after $x$- or $\varepsilon$- differentiation as before, while $I$, by (either using the variational equations (2.14) to convert $\varepsilon$-derivatives to two spatial derivatives as above, or estimating directly as in [TZ2])

$$\partial_x \partial_y G_{\lambda}(x,y) = O(e^{-\nu|x-y|})$$

and

$$|\partial_T \left( \frac{1}{1 - e^{\lambda T}} \right)| \leq C$$

for $\lambda \in \gamma$ likewise satisfies the same bounds after as before.

Remark 2.6. Note that, except for the excited term (2.23), the estimates on $II$ are rather crude compared to the detail in [MaZ1, Z2], while the estimate for $I$ is trivial. This represents a considerable simplification from [TZ2].

Remark 2.7. In the undercompressive case, there appear in the expansion of $\tilde{G}_{\lambda}$ additional terms of order $\lambda^{-1} e^{-\nu(|x|+|y|)}$ and $e^{\lambda \tilde{\mu}_j^{\pm}(\lambda)x} e^{-\nu|y|}$ that do not fit in our analysis. As discussed in [ZH, Z2], this case is essentially different.

Remark 2.8. Note that there is indeed a contribution from the limiting term $II$ at infinity, as mentioned in Remark 1.10.

Corollary 2.9. Under assumptions (A1)–(A2), (H0)–(H4), (D$_\varepsilon$) (alternatively, the weakened assumptions of Remark 3.5), $\sum_{j=0}^\infty S(jT)$ converges uniformly in operator norm $|\cdot|_{L(X_2,B_1)}$, for $\varepsilon$ sufficiently small and $T$ in any compact set bounded away from the origin, to a limit that is bounded in $|\cdot|_{L(X_2,X_1)\cap L(B_2,B_1)}$ and $C^1(\varepsilon, T)$ in $|\cdot|_{L(B_2,B_1)}$ on the subspace $X_2$.

Proof. By Corollary 2.4 and Proposition 2.5, it is sufficient to establish the second statement for $S^j$ using bounds (2.15). Further, since $\partial_x L^1 \subset B_2$ and
(by Sobolev embedding) \( \partial_x \| e^{2\eta(x)} \|_{L^\infty} \subset X_2 \), it is sufficient to show boundedness in \( L(\partial_x L^1, L^2) \) and \( L(\partial_x \| e^{2\eta(x)} \|_{L^\infty}, X_1) \), or, equivalently, boundedness in \( L(L^1, L^2) \) of the operator \( \partial(\varepsilon, T)^q \sum_{j=0}^\infty S_j \partial_x \), with kernel

\[
\sum_{j=0}^\infty \tilde{\partial}_{\varepsilon, T} G^I_\mu(x, jT; y),
\]

\( q = 0, 1 \), and boundedness in \( L(\| e^{2\eta(x)} \cdot \|_{L^\infty}, \| e^{\eta(x)} \cdot \|_{L^\infty}) \) of the operators \( \partial^r \sum_{j=0}^\infty S_j \partial_x \), \( 0 \leq r \leq 3 \) with kernels

\[
\partial^r \sum_{j=0}^\infty \tilde{G}^I_\mu(x, jT; y),
\]

both routine consequences of bound (2.15).

From Corollary 2.9 we obtain the following important conclusion.

**Proposition 2.10.** Under assumptions (A1)–(A2), (H0)–(H4), \((\mathcal{D}_\varepsilon)\) (alternatively, the weakened assumptions of Remark 3.5), for \( \varepsilon \) sufficiently small, and \( 0 < T_0 \leq T \leq T_1 \), \((\text{Id} - S(\varepsilon, T))\) has a right inverse

\[
(\text{Id} - S(\varepsilon, T))^{-1} \in L(X_2, X_1)
\]

that, restricted to its domain \( X_2 \), is \( C^1 \) in \( \varepsilon, T \) in the \( L(B_2, B_1) \) norm.

**Proof.** The first assertion follows by a standard telescoping sum argument, setting \((\text{Id} - S(\varepsilon, T))^{-1} := \sum_{j=0}^\infty S(\varepsilon, T)^j\), the second by the final assertion of Corollary 2.9. See [TZ2], Section 2, for further details. \( \square \)

### 3 Nonlinear energy estimates

We next carry out \( H^s \)-energy estimates on the perturbation equations

\[
U_t - L_\varepsilon U = Q_\varepsilon(U, U_x)_x
\]

of (1.2) about \( \bar{U}_\varepsilon \), to be used to control higher derivatives in the fixed point iteration used to carry out our Lyapunov–Schmidt reduction and bifurcation analyses.
By standard energy estimates, we have $H^s$ well-posedness,
\[
\|U(\cdot, T)\|_{H^s} \leq C\|U_0\|_{H^s}
\]
for $T$ uniformly bounded. Likewise, we have a formally quadratic linearized truncation error $|Q| = O(|U|\|U\|_1)\|U\|_1)$ for $|U| \leq C$. Our goal in this section, and what is far from evident in the absence of parabolic smoothing, is to establish a quadratic bound on the linearization error:
\[
(3.2) \quad \|U(\cdot, T) - e^{L_\epsilon T}U_0\|_{H^s} \leq C\|U_0\|_{H^s}^2.
\]
The corresponding bound does not hold for quasilinear hyperbolic equations, nor as discussed in Appendix A for systems of general hyperbolic-parabolic type, due to loss of derivatives. However, it follows easily for systems satisfying assumptions (A1)–(A2).

### 3.1 $H^s$ linearization error

By the strong block structure assumption (A1), we may write (3.1) more precisely as
\[
(3.3) \quad U_t + (A^\epsilon U)_x - (B^\epsilon U_x)_x = \partial_x \begin{pmatrix} 0 \\ q_1^\epsilon(U)(U, U) + \tilde{q}_2^\epsilon(U)(U, \partial_x U_2) \end{pmatrix},
\]
where $B^\epsilon = \begin{pmatrix} 0 & 0 \\ 0 & b^\epsilon \end{pmatrix}$ and $q_j(\cdot)$ are $C^{k-2}$ bilinear forms. Appealing to (A2), we may differentiate $\ell$ times and multiply by
\[
A^{0,\epsilon} := A^0(\bar{U}^\epsilon) = \begin{pmatrix} \bar{A}_{11}^{0,\epsilon} & 0 \\ 0 & \bar{A}_{22}^{0,\epsilon} \end{pmatrix}
\]
to obtain
\[
(3.4) \quad A^{0,\epsilon} \partial_\ell^x U_t + \tilde{A}^\epsilon \partial_\ell^x U_x + -(\bar{B}^\epsilon \partial_\ell^x U_{x+1})_x = \Theta_\ell + A^{0,\epsilon} \partial_\ell^x \begin{pmatrix} 0 \\ \tilde{q}_\ell^\epsilon \end{pmatrix},
\]
where $A^{0,\epsilon}$ is symmetric positive definite, $\tilde{A}_{11}$ is symmetric, $\bar{B}^\epsilon = \begin{pmatrix} 0 & 0 \\ 0 & \bar{b}^\epsilon \end{pmatrix}$ with $\bar{b}^\epsilon$ symmetric positive definite, and
\[
\|\Theta_\ell\|_{L^2} = O(\|U\|_{H^1}),
\]
\[
\tilde{q}_\ell = \partial_\ell^x \begin{pmatrix} q_1^\epsilon(U)(U, U) + \tilde{q}_2^\epsilon(U)(U, \partial_x U_2) \end{pmatrix}.
\]
Lemma 3.1. For $\ell \geq 1$, $\|\tilde{q}_\ell\|_{H^\ell} \leq C(\|U\|_{H^\ell} + \|U_2\|_{H^{\ell+1}})(\|U\|_{H^\ell} + \|U\|_{H^\ell})$.

Proof. Standard application of Moser’s inequality; see, e.g., [Ta, Z3]. □

Proposition 3.2. Assuming (A1), (A2), (H0), for $1 \leq s \leq k - 1$, $0 \leq T \leq T_0$ uniformly bounded, some $C = C(T_0) > 0$, and $U$ satisfying (3.1) with initial data $U(\cdot, 0) = U_0$ sufficiently small in $H^s$,

\begin{equation}
\|U(\cdot, T)\|_{H^s}^2 + \int_0^T \|U_2(\cdot, t)\|_{H^{s+1}}^2 dt \leq C\|U_0\|_{H^s}^2,
\end{equation}

(3.6)

\begin{equation}
\|U(\cdot, T) - e^{\mathcal{L}T}U_0\|_{H^s} \leq C\|U_0\|_{H^s}^2.
\end{equation}

(3.7)

Proof. By symmetric positive definiteness of $A^{0,\xi}$,

$$
\mathcal{E}(U) := (1/2) \sum_{\ell=1}^s (\partial_x^s U, A^{0,\xi} \partial_x^s U)
$$

defines a norm equivalent to $\| \cdot \|_{H^s}$, i.e., $\mathcal{E}(\cdot)^{1/2} \sim \| \cdot \|_{H^s}$. Applying (3.4), we find that

\begin{equation}
\partial_t \mathcal{E}(U) = -\langle \partial_x^s U, \tilde{A}^\xi \partial_x^{s+1} U \rangle + \langle \partial_x^s U, \partial_x (\tilde{B}^\xi \partial_x^s U) \rangle \\
+ \langle \partial_x^s U_2, A^{0,\xi}_{22} \partial_x \tilde{q}_s \rangle + O(\|U\|_{H^s}^2)
\end{equation}

(3.8)

\begin{align*}
= \langle \partial_x^s U, (1/2) \tilde{A}^\xi \partial_x^s U \rangle + \langle \partial_x^{s+1} U_2, \tilde{B}^\xi \partial_x^{s+1} U_2 \rangle \\
- \langle \partial_x^{s+1} U_2, A^{0,\xi}_{22} \tilde{q}_s \rangle - \langle \partial_x^s U_2, \partial_x A^{0,\xi}_{22} \tilde{q}_s \rangle + O(\|U\|_{H^s})
\leq -\theta \|U_2\|_{H^{s+1}}^2 + O(\|U\|_{H^s}^2)
\end{align*}

for some $\theta > 0$. So long as $\|U\|_{H^s}$ remains sufficiently small, this gives

\begin{align}
\partial_t \mathcal{E}(U) &\leq -\theta/2 \|U_2\|_{H^{s+1}}^2 + O(\|U\|_{H^s}^2) \\
&\leq -\theta/2 \|U_2\|_{H^{s+1}}^2 + C \mathcal{E},
\end{align}

(3.9)

from which (3.6) follows by Gronwall’s inequality, in the form

$$
\mathcal{E}(U(T)) + (\theta/2) \int_0^T \|U_2\|_{H^{s+1}}^2 dt \leq C_2 \mathcal{E}(U_0).
$$

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To obtain (3.7), observe that $V(\cdot, t) := U(\cdot, t) - e^{L_x t} U_0$ satisfies

$$
E_t + (A^\epsilon E)_x - (B^\epsilon E_x)_x = \partial_x \left( q^1_1(U)(U, U) + q^2_2(U)(U, \partial_x U_2) \right),
$$

with $E(\cdot, 0) = 0$, and thus

$$
A^{0,\epsilon} \partial_x^2 E_t + \tilde{A}^\epsilon \partial_x^2 E_x + - (\tilde{B}^\epsilon \partial_x^{\epsilon+1} E)_x = \Theta^E_t + A^{0,\epsilon} \partial_x \left( 0 \right),
$$

similarly as in (3.4), with $\|\Theta^E_t\|_{L^2} = O(\|E\|_{H^s})$. Thus, calculating as in (3.8), we obtain

$$
\partial_t \mathcal{E}(E) = - \langle \partial_x^2 E, \tilde{A}^\epsilon \partial_x^{\epsilon+1} E \rangle + \langle \partial_x^2 E, \partial_x (\tilde{B}^\epsilon \partial_x^\epsilon E) \rangle
+ \langle \partial_x^2 E_2, A^{0,\epsilon}_{22} \partial_x \tilde{q}_s \rangle + O(\|E\|_{H^s} \|E\|_{H^s} + \|U\|_{H^s}^2)
= \langle \partial_x^2 E, (1/2) \tilde{A}^\epsilon \partial_x^\epsilon E \rangle + \langle \partial_x^{\epsilon+1} E_2, \tilde{B}^\epsilon \partial_x^{\epsilon+1} E_2 \rangle
- \langle \partial_x^{\epsilon+1} E_2, A^{0,\epsilon}_{22} \tilde{q}_s \rangle - \langle \partial_x^2 U_2, \partial_x A^{0,\epsilon}_{22} \tilde{q}_s \rangle + O(\|E\|_{H^s} \|E\|_{H^s} + \|U\|_{H^s}^2)
\leq -\theta \|E_2\|_{H^{s+1}}^2 + O(\|E\|_{H^s} \|E\|_{H^s} + \|U\|_{H^s}^2)
+ O(\|E_2\|_{H^{s+1}} \|E_2\|_{H^{s+1}} \|U_2\|_{H^{s+1}} (\|U\|_{H^s} + \|U\|_{H^s}^2),

which, so long as $\|U\|_{H^s}$ remains sufficiently small, gives

$$
\partial_t \mathcal{E}(E) \leq -\theta \|E_2\|_{H^{s+1}}^2 + C \|E\|_{H^s}^2 + C \|U\|_{H^s}^2 \left( \|U\|_{H^s}^2 + \|U_2\|_{H^{s+1}}^2 \right).
$$

and thus, using $\|U\|_{H^s}(t) \leq C \|U_0\|_{H^s}$ by (3.6),

$$
\partial_t \mathcal{E}(E) \leq +CE(E) + C_2 \|U_0\|_{H^s}^2 \left( \|U\|_{H^s}^2 + \|U_2\|_{H^{s+1}}^2 \right).
$$

Applying Gronwall’s inequality, and using $E(0) = 0$, we thus obtain

$$
\mathcal{E}(E(T)) \leq C_3 \|U_0\|_{H^s}^2 \int_0^T \left( \|U\|_{H^s}^2 + \|U_2\|_{H^{s+1}}^2 \right) dt,
$$

yielding the result by (3.6) and $\mathcal{E}(E) \sim \|E\|_{H^s}^2$. 

\begin{remark}
Note that the above, finite-time estimate is considerably simpler than the “Kawashima-type” global-in-time estimates used in the nonlinear stability analysis [MaZ3, Z2, Z3], which require also additional assumptions of, among other things, symmetrizability and “genuine noncoupling” of $A^\epsilon \pm$ and $B^\epsilon \pm$.
\end{remark}
3.2 Weighted variational bounds

Define now
\[
N(U_0, \varepsilon, T) := U(\cdot, T) - e^{L(\varepsilon)T}U_0.
\]
where $U$ satisfies (3.1) with initial data $U_0$.

**Proposition 3.4.** Assuming (A1), (A2), (H0), for $1 \leq s \leq k - 1$, $0 \leq T \leq T_0$ uniformly bounded, some $C = C(T_0) > 0$, $N$ is uniformly bounded in $X_2$ and $C^1$ in $B_2$ with respect to $U_0$, $T$, $\varepsilon$ for $\|U_0\|_{X_1}$, $|\varepsilon|$ sufficiently small, with
\[
\|N(U_0, \varepsilon, T)\|_{X_2} \leq C\|U_0\|_{X_1}^2,
\]
(3.14)
\[
\|\partial_{u_0}N(U_0, \varepsilon, T)\|_{X_2} \leq C\|U_0\|_{X_1},
\]
\[
\|\partial_{T, \varepsilon}N(U_0, \varepsilon, T)\|_{B_2} \leq C\|U_0\|_{X_1}^2.
\]

**Proof.** Similarly as in the proof of (2.8)($r > 0$), we may make the change of variables $V = U\alpha(x)$, $\alpha(x) := e^{\eta(x)}$, $\langle x \rangle := (1 + |x|^2)^{1/2}$, to convert both (3.1) and the corresponding linearized equations to equations for which the same principal part plus lower-order terms with bounded coefficients and the nonlinear part is $\alpha^{-1}$ times the same principal part plus bounded factors times lower-order terms, thus recovering $\|U(\cdot, T) - e^{LT}U_0\|_{X_1} \leq C\|U_0\|_{X_1}^2$, or
\[
\|N(U_0, \varepsilon, T)\|_{X_1} \leq C\|U_0\|_{X_1}^2,
\]
(3.15)
by the same argument used to obtain (3.6). Here, we are using the fact that $\alpha$ satisfies both $|\alpha^{-1} \leq C|$ and $(|d/dx|^k \alpha) \leq C|\alpha|$ for $k \geq 0$. Bound (3.14)(i) then follows by (3.15) and the Duhamel formulation
\[
\int N = \int_0^T \left( \int e^{L(\varepsilon)(T-t)}\partial_x Q(U, U_x)(t) \right) dt,
\]
bounds (2.8) with $p = \infty$ and $r = 2$, and Sobolev embedding
\[
\|e^{\eta(x)}U(\cdot, t)\|_{L^\infty}, \|e^{\eta(x)}\partial_x U(\cdot, t)\|_{L^\infty} \leq \|e^{\eta(x)}U(\cdot, t)\|_{W^{1,\infty}}
\]
\[
\leq C\|e^{\eta(x)}U(\cdot, t)\|_{H^3}.
\]
(3.16)

Bound (3.14)(ii) follows, similarly, by repeating in weighted norm $X_1$ the arguments for (3.6) and (3.7) with $U$ replaced by $W := U^1 - U^2$, where $U^j$ are solutions of (3.1) with different initial data $U^1_0$ and $U^2_0$. Bound (3.14)(iii) follows by (3.14)(i) together with the equation for $F := W - e^{L(\varepsilon)T}W_0$. \qed
Remark 3.5. It is readily checked that bounds (3.6), (3.7), (3.14) hold also under the weakened assumptions described in Remark 1.3, in the favorable variables \((U_1, W_2)\), by rearranging (1.2) as

\[
\begin{pmatrix}
  U_1 \\
  W_2
\end{pmatrix}_t + \begin{pmatrix} A_{11}(U) & 0 \\
  0 & 0
\end{pmatrix} \begin{pmatrix}
  U_1 \\
  W_2
\end{pmatrix}_x = \partial_x \begin{pmatrix} 0 & 0 \\
  0 & (\partial W_2/\partial U_2) \beta(U)
\end{pmatrix} \begin{pmatrix}
  U_1 \\
  W_2
\end{pmatrix}_x + \begin{pmatrix} 0 \\
  O(|U_x|^2)
\end{pmatrix}
\]

similarly as in [Z2, Z3, GMWZ] and carrying out energy estimates as before. These then imply corresponding estimates in the original variables \((U_1, U_2)\), by the observation that, for \(\tilde{U}\) satisfying the linearized equations in \((U_1, U_2)\) coordinates, \((\tilde{U}_1, (\partial W_2/\partial U)(\tilde{U})\tilde{U})\) satisfies the linearized equations in \((U_1, W_2)\) coordinates, and, for \(U = \tilde{U} - \bar{U}\) satisfying the nonlinear perturbation equation in \((U_1, U_2)\) coordinates, \((\tilde{U}_1, W_2(\tilde{U})) - (\bar{U}_1, W_2(\bar{U})) = (U_1, (\partial W_2/\partial U)(\bar{U})\bar{U}) + O(|U|^2)\) satisfies the nonlinear perturbation equation in \((U_1, W_2)\) coordinates.

4 Bifurcation analysis

We now carry out the bifurcation analysis following the framework of [TZ2], with a few slight modifications to simplify the analysis.

4.1 Construction of the period map

Given a solution \(\tilde{U}\) of (1.2), define the perturbation variable

\[
U(x, t, \varepsilon) := \tilde{U}^\varepsilon(x, t) - \bar{U}^\varepsilon(x),
\]

satisfying nonlinear perturbation equations

\[
U_t - L_\varepsilon U = Q_\varepsilon(U, U_x)_x, \quad U(x, 0, \varepsilon) = U_0(x, \varepsilon).
\]

By (3.1), (3.3),

\[
Q_\varepsilon(U, \varepsilon) = O(|U|^2 + |U||U_x|)
\]

so long as \(U\) satisfies a uniform \(L^\infty\) bound: in particular, for \(\|U\|_{X_1} \leq C\).
Decomposing

\[ U = w_1 \phi_+ + w_2 \phi_- + v, \]

where \( w_1 \phi_+ + w_2 \phi_- \in \Sigma, v := \Pi U \in \tilde{\Sigma}, \) and coordinatizing as \((w, v)\), we obtain after a brief calculation

\[ \dot{w} = \left( \begin{array}{cc} \gamma(\varepsilon) & \tau(\varepsilon) \\ -\tau(\varepsilon) & \gamma(\varepsilon) \end{array} \right) w + N_w(w, v, \varepsilon), \]
\[ \dot{v} = \tilde{L}_\varepsilon v + N_v(w, v, \varepsilon), \]

where \( \tilde{L} = L\Pi \) and

\[ N_{w,1} \phi_+ + N_{w,2} \phi_- = \Pi \theta_\varepsilon(U, U_x), \]
\[ N_v = \Pi \theta_\varepsilon(U, U_x). \]

Lemma 4.1. \( N_w \) is \( C^k, k \geq 2, \) from \((w, v) \in \mathbb{R}^2 \times L^q \) to \( \mathbb{R}^2, \) for any \( 0 \leq q \leq \infty, \) with

\[ |N_v| \leq C(|w|^2 + \|v\|_{\mathbb{R}}^2). \]

\( N_v \) is \( C^k, k \geq 2, \) from \((w, v) \in \mathbb{R}^2 \times X_1 \) to \( X'_2 := \partial_x \{ f : \|e^{2\zeta(x)} f\|_{H^1} < +\infty \} \) and, for \( \|w\| + \|v\|_{X_1} \leq C, \) \( C^1 \) from \((w, v) \in \mathbb{R}^2 \times B_1 \) to \( \partial_x(L^1) \), with

\[ \|N_v\|_{X'_2} \leq C(|w|^2 + \|v\|_{X_1}^2) \quad \text{and} \quad \|DN_v\|_{\partial_x(L^1)} \leq C(|w| + \|v\|_{X_1}). \]

Proof. Direct calculation, using (4.3) and the II-bounds of Lemma 2.1. \( \Box \)

Proposition 4.2. For \( 0 \leq t \leq T, \) any fixed \( C, T > 0, \) some \( C > 0, \) and \( |a|, \|b\|_{X_1}, |\varepsilon| \) sufficiently small, system (4.5) with initial data \((w_0, v_0) = (a, b)\) sufficiently small in \( \mathbb{R}^2 \times X_1 \) possesses a solution

\[ (w, v)(a, b, \varepsilon, t) \in \mathbb{R}^2 \times X_1 \]

that is \( C^{k+1} \) in \( t \) and \( C^k \) in \((a, b, \varepsilon)\), \( k \geq 2, \) with respect to the weaker norm \( B_1, \) with

\[ C^{-1}|a| - C\|b\|_{X_1}^2 \leq |w(t)| \leq C(|a| + \|b\|_{X_1}^2), \]
\[ \|v(t)\|_{X_1} \leq C(\|b\|_{X_1} + |a|^2), \]

and

\[ |D(a, b)(w, v)(t)|_{\mathbb{R}^2 \times B_1 \to B_1} \leq C. \]
In particular, for \( \|b\|_{X_1} \leq C_1|a| \), all \( 0 \leq t \leq T \),
\[
(4.12) \quad \|v(t)\|_{X_1} \leq C|w(t)|.
\]
Likewise, for \( \|b\|_{X_1} \leq C_1|a|^2 \), all \( 0 \leq t \leq T \),
\[
(4.13) \quad \|v(t)\|_{X_1} \leq C|w(t)|^2.
\]

Proof. Existence and uniqueness follow by a standard Contraction–mapping argument, using a priori bounds (3.14), which also imply (4.10) (by decoupling of linear parts) and (4.11). Combining (4.10)(i)–(ii), we obtain evidently (4.12) and (4.13) for \(|a|\) sufficiently small.

Setting \( t = T \) in (4.9) and applying Duhamel’s principle/variation of constants, we may express the period map

\[
(4.14) \quad (a, b, \varepsilon) \to (\hat{a}, \hat{b}) := (w, v)(a, b, \varepsilon, T)
\]
as a discrete dynamical system

\[
(4.15) \quad \begin{align*}
\hat{a} &= R(\varepsilon, T)a + N_1(a, b, \varepsilon, T), \\
\hat{b} &= S(\varepsilon, T)b + N_2(a, b, \varepsilon, T)
\end{align*}
\]

with \( \varepsilon, T \in \mathbb{R}^1, a, N_1 \in \mathbb{R}^2 \) and \( b \in B_1, N_2 \in B_2 \), where

\[
(4.16) \quad \begin{align*}
R(\varepsilon, T) &:= e^{\gamma(\varepsilon)T(a,b,\varepsilon)}, \\
S(\varepsilon, T) &:= e^{\tilde{L}_\varepsilon T(a,b,\varepsilon)}
\end{align*}
\]
are the linearized solution operators in \( w, v \) and

\[
(4.17) \quad \begin{align*}
N_1(a, b, \varepsilon, T) &:= \int_0^T e^{\gamma(\varepsilon)(T-s)}N_w(wv, \varepsilon)(s)ds, \\
N_2(a, b, \varepsilon, T) &:= \int_0^T e^{\tilde{L}_\varepsilon(T-s)}N_v(w, v, \varepsilon)(s)ds
\end{align*}
\]
the differences between nonlinear and linear solution operators: equivalently,

\[
(4.18) \quad \begin{align*}
N_1(a, b, \varepsilon, T) &= \Pi N(a_1 \phi_+ + a_2 \phi_- + b, \varepsilon, T) \\
N_2(a, b, \varepsilon, T) &= (\text{Id} - \tilde{\Pi})N(a_1 \phi_+ + a_2 \phi_- + b, \varepsilon, T),
\end{align*}
\]
where $N$ is the linearization error $N$ defined in (3.13).

Evidently, periodic solutions of (4.5) with period $T$ correspond to fixed points of the period map (equilibria of (4.15)) or, equivalently, zeroes of the displacement map

\[
\Delta_1(a, b, \varepsilon, T) := (R(\varepsilon, T) - \text{Id})a + N_1(a, b, \varepsilon, T),
\]

\[
\Delta_2(a, b, \varepsilon, T) := (S(\varepsilon, T) - \text{Id})b + N_2(a, b, \varepsilon, T).
\]

### 4.2 Lyapunov–Schmidt reduction

We now carry out a nonstandard Lyapunov–Schmidt reduction following the “inverse temporal dynamics” framework of [TZ2], tailored for the situation that $(S(\varepsilon, T) - \text{Id})$ is not uniformly invertible, or, equivalently, $\sigma(\hat{L})$ is not bounded away from $\{j\pi/T\}, j \in \mathbb{Z}$. In the present situation, $\hat{L}$ has both an $\ell$-dimensional kernel (Lemma 4.5 below) and essential spectra accumulating at $\lambda = 0$, and no other purely imaginary spectra, so that $(S(\varepsilon, T) - \text{Id}) = (e^{LT}\hat{\Pi} - \text{Id})$ inherits the same properties; see [TZ2] for further discussion.

Our goal, and the central point of the analysis, is to solve $\Delta_2(a, b, \varepsilon, T) = 0$ for $b$ as a function of $(a, \varepsilon, T)$, eliminating the transverse variable and reducing to a standard planar bifurcation problem in the oscillatory variable $a$. A “forward” temporal dynamics technique would be to rewrite $\Delta_2 = 0$ as a fixed point map

\[
b = S(\varepsilon, T)b + N_2(a, b, \varepsilon, T),
\]

then to substitute for $T$ an arbitrarily large integer multiple $jT$. In the strictly stable case $\Re\sigma(\hat{L}) \leq -\eta < 0$, $|S(\varepsilon, jT)|_{L(X_1, X_1)} < 1/2$ for $j$ sufficiently large. Noting that $N_2$ is quadratic in its dependency, we would have therefore contractivity of (4.20) with respect to $b$, yielding the desired reduction. However, in the absence of a spectral gap between $\sigma(\hat{L})$ and the imaginary axis, $|S(\varepsilon, jT)|_{L(X_1, X_1)}$ does not decay, and may be always greater than unity; thus, this naive approach does not succeed.

The key idea in [TZ2] is to rewrite $\Delta_2 = 0$ instead in “backward” form

\[
b = (\text{Id} - S(\varepsilon, T))^{-1}N_2(a, b, \varepsilon, T),
\]

then show that $(\text{Id} - S)^{-1}$ is well-defined and bounded on $\text{Range}N_2$, thus obtaining contractivity by quadratic dependence of $N_2$. Since $(\text{Id} - S)^{-1}N_2$ is formally given by $\sum_{j=0}^{\infty}S^jN_2$ this amounts to establishing convergence: a
stability/cancellation estimate. Quite similar estimates appear in the non-linear stability theory, where the interaction of linearized evolution $S$ and nonlinear source $N_2$ are likewise crucial for decay. The formulation (4.21) can be viewed also as a “by-hand” version of the usual proof of the standard Implicit Function Theorem [TZ2].

**Lemma 4.3.** Assuming (A1), (A2), (H0)-(H4), $D_ε$, $N_1$ is quadratic order and $C^1$ from $\mathbb{R}^2 \times L^q \times \mathbb{R}^2 \to \mathbb{R}^1$ for any $1 \leq q \leq \infty$, and $N_2$ is quadratic order from $\mathbb{R}^2 \times X_1 \times \mathbb{R}^2 \to X_2$ and $C^1$ from $\mathbb{R}^2 \times B_1 \times \mathbb{R}^2 \to B_2$ for $\|b\|_{X_1}$ uniformly bounded, with

\begin{align}
|N_1(a,b,ε,T)|, \quad |\partial_{ε,T}N_1(a,b,ε,T)|_{L(\mathbb{R}^2,\mathbb{R}^2)} & \leq C(|a| + \|b\|_{B_1})^2, \\
|\partial_aN_1(a,b,ε,T)|_{L(\mathbb{R}^2,\mathbb{R}^2)} + |\partial_bN_1(a,b,ε,T)|_{L(\mathbb{B}_1,\mathbb{R}^2)} & \leq C(|a| + \|b\|_{B_1}),
\end{align}

(4.22)

\begin{align}
\|N_2(a,b,ε,T)\|_{X_2}, \quad |\partial_{ε,T}N_2(a,b,ε,T)|_{L(\mathbb{R}^2,\mathbb{B}_2)} & \leq C(|a| + \|b\|_{X_1})^2, \\
|\partial_aN_2(a,b,ε,T)|_{L(\mathbb{R}^2,\mathbb{B}_1)} + |\partial_bN_2(a,b,ε,T)|_{L(\mathbb{B}_1,\mathbb{R}^2)} & \leq C(|a| + \|b\|_{X_1}).
\end{align}

(4.23)

*Proof.* Bounds (4.22) follow from representation (4.18), variational bounds (3.14) of Proposition 3.4, and the Π-bounds of Lemma 2.1, from which we likewise obtain

\begin{align}
\|N_2(a,b,ε,T)\|_{X_1}, \quad |\partial_{ε,T}N_2(a,b,ε,T)|_{L(\mathbb{R}^2,\mathbb{B}_1)} & \leq C(|a| + \|b\|_{X_1})^2, \\
|\partial_aN_2(a,b,ε,T)|_{L(\mathbb{R}^2,\mathbb{B}_1)} + |\partial_bN_2(a,b,ε,T)|_{L(\mathbb{B}_1,\mathbb{R}^2)} & \leq C(|a| + \|b\|_{X_1}).
\end{align}

The remaining bounds

\begin{align}
\|N_2(a,b,ε,T)\|_{\partial_ε e^{-2ε}H^1}, \quad |\partial_{ε,T}N_2(a,b,ε,T)|_{L(\mathbb{R}^2,\partial_ε e^{-2ε}H^1)} & \leq C(|a| + \|b\|_{X_1})^2, \\
|\partial_aN_2(a,b,ε,T)|_{L(\mathbb{R}^2,\partial_a L^1)} + |\partial_bN_2(a,b,ε,T)|_{L(\mathbb{B}_1,\partial_a L^1)} & \leq C(|a| + \|b\|_{X_1}).
\end{align}

(4.24)

follow easily from Duhamel representation (4.17)(ii), bounds (4.10)(ii), (4.11), and (4.8), and bounds (2.8), (2.9) on the linearized solution operator $e^{Lt}Π$. □

**Corollary 4.4 ([TZ2]).** Assuming (A1), (A2), (H0)-(H4), $D_ε$, $\Delta_2(a,b,ε,T) = 0 \quad (a,b,ε,T) \in \mathbb{R}^2 \times X_1 \times \mathbb{R}^2$,

(4.24)

is equivalent to

\begin{align}
b = (\text{Id} - S(a,b,ε,T))^{-1}N_2(a,b,ε,T) + \omega
\end{align}

(4.25)

for

\begin{align}
\omega \in \text{Ker}(\text{Id} - S(a,b,ε,T)) \cap X_1.
\end{align}

(4.26)
Proof. Applying to the left of (4.24) the right inverse \((\text{Id} - S(\epsilon, a, b))^{-1}\) given by Proposition 2.10, we obtain by Lemma 4.3

\[
\tilde{b} := (\text{Id} - S(\epsilon, a, b))^{-1}(\text{Id} - S(\epsilon, a, b))b
= (\text{Id} - S(\epsilon, a, b))^{-1}N_2(\epsilon, a, b) \in X_1.
\]

Observing that \(\tilde{b} - b\) belongs to \(\text{Ker}(\text{Id} - S(\epsilon, a, b)) \cap X_1\) by the right inverse property, we obtain (4.25). Conversely, (4.25) implies (4.24) by application on the left of \((\text{Id} - S(a, b, \epsilon, T))\).

Lemma 4.5. The kernel of \((\text{Id} - S(\epsilon, T))\) is of fixed dimension \(\ell\) as in \((H4)\), is independent of \(T\), and has a smooth basis \(\omega = (\omega_1 \ldots \omega_{\ell})(\epsilon)\). In the Lax case, it is generated entirely by translation invariance.

Proof. This follows by the corresponding properties of \(\text{Ker} L\) assumed in \((H4)\).

Corollary 4.6. Assuming \((A1), (A2), (H0)-(H4), (D_\epsilon)\), the map

\[(4.27) \quad T(a, b, \epsilon, T, \alpha) := (\text{Id} - S(\epsilon, T))^{-1}N_2(a, b, \epsilon, T) + \omega(\epsilon)\alpha,\]

\((\text{Id} - S)^{-1} : X_1 \to X_2\) as defined in Proposition 2.10 is bounded from \(\mathbb{R}^2 \times X_1 \times \mathbb{R}^{2+\ell} \to X_1\) and \(C^1\) from \(\mathbb{R}^2 \times B_1 \times \mathbb{R}^{2+\ell} \to X_1\) for \(|\alpha|\) bounded and \(|a| + \|b\|_{X_1} + |(\epsilon, T)|\) sufficiently small, with

\[
\|T(a, b, \epsilon, T, \alpha)\|_{X_1} \leq C(|a| + \|b\|_{X_1}),
\]

\[
\|\partial_{a,b}T(a, b, \epsilon, T)\|_{L(B_1, B_1)} \leq C(|a| + \|b\|_{X_1}),
\]

\[
\|\partial_T T(a, b, \epsilon, T)\|_{L(B_1, B_1)} \leq C(|a|^2 + \|b\|_{X_1}^2),
\]

\[
\|\partial_{a}T(a, b, \epsilon, T)\|_{L(B_1, B_1)} \leq C(|a|^2 + \|b\|_{X_1}^2 + |\alpha|),
\]

\[
\|\partial_{\epsilon} T(a, b, \epsilon, T)\|_{L(B_1, B_1)} \leq C(|a|^2 + \|b\|_{X_1}^2 + 1).
\]

Proof. Immediate, from Proposition 2.10 and Lemmas 4.3 and 4.5.

Proposition 4.7 ([TZ2]). Under assumptions \((A1)-(A2), (H0)-(H4), (D_\epsilon)\) (alternatively, the weakened assumptions of Remark 3.5), there exists a function \(\beta(a, \epsilon, T, \alpha)\), bounded from \(\mathbb{R}^{4+\ell}\) to \(X_1\) and \(C^1\) from \(\mathbb{R}^{4+\ell}\) to \(B_1\), with

\[
\Delta_2(a, \beta(a, \epsilon, T, \alpha), \epsilon, T) \equiv 0,
\]

(4.29)
\[ \|\beta\|_{X_1}, \|\partial_{\epsilon,T}\beta\|_{L(R,B_1)} \leq C(|a|^2 + |\alpha|), \]
\[ \|\partial_a\beta\|_{L(R^2,B_1)} \leq C|a|, \]
\[ \|\partial_\alpha\beta\|_{L(R^2,B_1)} \leq C, \]

for \(|(a,\epsilon,T,\alpha)|\) sufficiently small. Moreover, for \(|(a,\epsilon,T)|, \|b\|_{X_1}\) sufficiently small, all solutions of (4.24) lie on the \(\ell\)-parameter manifold \(\{b = \beta(a,\epsilon,T,\alpha)\}\).

**Proof.** By Corollary 4.4, (4.24) is equivalent to the fixed-point problem
\[ b = \mathcal{T}(a,b,\epsilon,T,\alpha) \]
for some \(\alpha \in \mathbb{R}^\ell\). By (4.28)(i)–(ii), for \(|(a,\epsilon,T,\alpha)|\), sufficiently small, \(\mathcal{T}\) preserves a small ball in \(X_1\) on which it is contractive in \(b\) with respect to the weaker norm \(\| \cdot \|_{B_1}\). Observing that closed balls in \(X_1\) are closed also in \(B_1\), we may conclude by the contraction-mapping principle the existence of a unique solution \(\beta\), which, moreover, inherits the regularity of \(\mathcal{T}\) in its dependence on parameters \((a,\epsilon,T,\alpha)\). \(\square\)

**Remark 4.8.** At the expense of further bookkeeping, we may replace (1.11) by \(\|U\|_{X_1} := \|e^{\eta(x)}U\|_{H^4}, \|\partial_x U\|_{X_2} := \|\partial_x U\|_{X_1} + \|e^{2\eta(x)}U\|_{H^1}\) and carry one further derivative in \((a,\epsilon,T,\alpha)\) throughout the analysis, to obtain \(C^2\) dependence of \(\beta(a,\epsilon,T,\alpha)\). Indeed, strengthening (H0) to \(k = 2r + 1\) in (H0), we may obtain arbitrary smoothness \(C^r\) of reduction function (nullcline) \(\beta(\cdot)\).

### 4.3 Proof of the main theorem

The bifurcation analysis is straightforward now that we have reduced to a finite-dimensional problem, the only tricky point being to deal with the \(\ell\)-fold multiplicity of solutions (parametrized by \(\alpha\)). Define to this end
\[ \tilde{\beta}(a,\epsilon,T,\hat{\alpha}) := \beta(a,\epsilon,T,|a|\hat{\alpha}), \]
with \(\hat{\alpha}\) restricted to a ball in \(\mathbb{R}^\ell\), noting, by (4.30), that
\[ \|\tilde{\beta}\|_{X_1}, \|\partial_{\epsilon,T,\alpha}\tilde{\beta}\|_{L(R,B_1)} \leq C|a|, \]
with \(\tilde{\beta}\) Lipshitz in \((a,\epsilon,T,\hat{\alpha})\) and \(C^1\) away from \(a = 0\). Solutions \((w,v)\) of (4.5) originating at \((a,b) = (a,\tilde{\beta})\), by (4.12), remain for \(0 \leq t \leq T\) in a cone
\[ C := \{(w,v) : |v| \leq C_1|w|\}, \]
\(C_1 > 0\). Likewise, any periodic solution of (4.5) originating in \(C\), since it necessarily satisfies \(\Delta_2 = 0\), must originate from data \((a,b)\) of this form.
Proof of Theorem 1.8. Defining $b \equiv \tilde{\beta}(a, \varepsilon, T, \hat{\alpha})$, and recalling invariance of $\mathcal{C}$ under flow (4.5), we may view $v(t)$ as a multiple

$$v(t) = \gamma(a, \varepsilon, T, \hat{\alpha}, t)w(t)$$

of $w(t)$, where $\gamma$ is bounded, Lipshitz in all arguments, and $C^1$ away from $a = 0$. Substituting into (4.5)(i), we obtain a planar ODE

$$\dot{w} = \begin{pmatrix} \gamma(\varepsilon) & \tau(\varepsilon) \\ -\tau(\varepsilon) & \gamma(\varepsilon) \end{pmatrix} w + \tilde{N}(w, \varepsilon, T, t, \hat{\alpha}, a)$$

in approximate Hopf normal form, with nonlinearity $\tilde{N} := N_w(w, \varepsilon)$ now nonautonomous and depending on the additional parameters $(T, \hat{\alpha}, a)$, but still satisfying the key bounds

$$|\tilde{N}|, |\partial_{\varepsilon, T, \hat{\alpha}, a} \tilde{N}| \leq C|w|^2; \quad |\partial_w \tilde{N}| \leq C|w|$$

along with planar bifurcation criterion $(D_2)$(ii). From (4.36), we find that $\tilde{N}$ is $C^1$ in all arguments, also at $a = 0$. By standard arguments (see, e.g., [HK, TZ1]), we thus obtain a classical Hopf bifurcation in the variable $w$ with regularity $C^1$, yielding existence and uniqueness up to time-translates of an $\ell$-parameter family of solutions originating in $\mathcal{C}$, indexed by $r$ and $\delta$ with $r := a_1$ and (without loss of generality) $a_2 \equiv 0$. It remains only to establish uniqueness up to spatial translates.

In the Lax case, we observe, first, that, by dimensional considerations, the one-parameter family constructed must agree with the one-parameter family of spatial translates. Second, we argue as in [TZ2] that any periodic solution has a spatial translate originating in $\mathcal{C}$, yielding uniqueness up to translation among all solutions and not only those originating in $\mathcal{C}$; see Proposition 2.17 and Corollary 2.18 [TZ2] for further details.

In the overcompressive case, we observe, likewise, that the $\ell$-parameter family constructed consists of spatial translates of a smooth $(\ell - 1)$-parameter family of distinct orbits of traveling-wave ODE (1.4), then argue as in [TZ2] that any periodic solution has a spatial translate lying within a corresponding cone $\mathcal{C}'$ about some member of this $(\ell - 1)$-parameter family. Constructing solutions about each such member by the same technique, we thus obtain from their union an $\ell$-parameter family containing translates of all periodic solutions on an entire neighborhood of $\bar{U}$, and not only in cone $\mathcal{C}$. See Proposition 2.22 and Corollary 2.23 [TZ2] for further details. \qed
Remarks 4.9. The apparent restriction to $C^1$ regularity caused by factor $|a|$ in the definition of $\beta$ is illusory, since we restrict eventually to the ray $a_2 \equiv 0$ and $a_1 \geq 0$, on which $\tilde{\beta}$ is as smooth as $\beta$. By Remark 4.8, the latter may be made as smooth as desired by assuming sufficient regularity in (H0), and so we can carry out a bifurcation analysis to arbitrary desired regularity.

2. Restricting to the “central” solutions $\hat{\alpha} \equiv 0$ of the constructed cone of solutions yields $\tilde{\beta} = O(|a|^2)$, and thus the exact Hopf normal form

$$\dot{w} = \begin{pmatrix} \gamma(\varepsilon) & \tau(\varepsilon) \\ -\tau(\varepsilon) & \gamma(\varepsilon) \end{pmatrix} w + N_w(w,0,\varepsilon) + \tilde{M}(w,\varepsilon,T,t,\hat{\alpha},a),$$

where $\tilde{M} = O(|w|^3)$ and $\partial_{w,a} \tilde{M} = O(|w|^2)$. For these central solutions, we thus recover all of the standard description of Hopf bifurcation: in particular, that $\varepsilon(r)$ is a pitchfork bifurcation with $(d\varepsilon/dr)(0) = 0$ (hence $\text{sgn}(d^2\varepsilon/dr^2)(0)$ is expected to determine stability; see the final paragraph of Section 1.5).

5 Physical applications

We conclude by describing applications to various physical systems. To generate example systems and waves, it is easier to look for traveling-wave solutions $U(x,t) = \bar{U}(x-st)$ with possibly nonzero speed, to be transformed to steady solutions by the change of coordinates $x \rightarrow x-st$ as described in the introduction. Examples satisfying (A1)–(A2), (H0)–(H4) are:

1. The general Navier–Stokes equations of compressible gas dynamics, written in Lagrangian coordinates, appear as

$$\begin{cases}
v_t - u_x = 0, \\
u_t + p_x = (\nu/v) u_x, \\
(e + u^2/2)_t + (pu)_x = ((\kappa/v) T_x + (\mu/v) uu_x)_x,
\end{cases}$$

where $v > 0$ denotes specific volume, $u$ velocity, $e > 0$ internal energy, $T = T(v,e) > 0$ temperature, $p = p(v,e)$ pressure, and $\mu > 0$ and $\kappa > 0$ are coefficients of viscosity and heat conduction, respectively. For simplicity, assume an ideal temperature dependence

$$T = T(e),$$
independent of $v$. This assumption can be removed with further effort; see Remark 5.1. Defining $U_1 := (v), U_2 := (u, e + \frac{1}{2}v^2)$, we find that the single hyperbolic mode is $v$, and the associated equation is linear by inspection; likewise, the viscosity matrix $b = \begin{pmatrix} \nu/v & 0 \\ * & \kappa/\nu c \end{pmatrix}$ is lower triangular with positive real diagonal entries, hence its spectrum is positive real and (by Lyapunov’s Lemma) there exists $A^0_{22}$ symmetric positive definite such that $A^0_{22}b > 0$. Conditions (A1)–(A2) and (H1)–(H4) are thus satisfied under the mild assumptions of monotone temperature-dependence,

$$T_e > 0,$$

and thermodynamic stability of the endstates,

$$(p_v)_\pm < 0, (T_e)_\pm > 0;$$

see, e.g., [MaZ3, Z2, Z3] for further discussion. Notably, this allows the interesting case of a van der Waals-type equation of state, with some values of $v$

$$v \text{ along the connecting profile.}$$

2. Next, consider the equations of MHD:

$$\begin{cases}
  v_t - u_{tx} = 0, \\
  u_{1t} + (p + (1/2)\mu_0)(B_2^2 + B_3^2)_{2x} = ((\nu/v)u_{1x})_x, \\
  u_{2t} - ((1/\mu_0)B_1^*B_2)_x = ((\nu/v)u_{2x})_x, \\
  u_{3t} - ((1/\mu_0)B_1^*B_3)_x = ((\nu/v)u_{3x})_x, \\
  (vB_2)_t - (B_1^*u_2)_x = ((1/\sigma\mu_0)vB_2)_x, \\
  (vB_3)_t - (B_1^*u_3)_x = ((1/\sigma\mu_0)vB_3)_x, \\
  (e + (1/2)(u_1^2 + u_2^2 + u_3^2) + (1/2)\mu_0v(B_2^2 + B_3^2))_x \\
  + [(p + (1/2)\mu_0)(B_2^2 + B_3^2))u_1 - (1/\mu_0)B_1^*(B_2u_2 + B_3u_3)]_x \\
  = [(\nu/v)u_1u_{1x} + (\mu/v)(u_2u_{2x} + u_3u_{3x}) \\
  + (\kappa/v)T_x + (1/\sigma\mu_0^2v)(B_2B_{2x} + B_3B_{3x})]_x, \\
\end{cases}$$

where $v$ denotes specific volume, $u = (u_1, u_2, u_3)$ velocity, $p = P(v, e)$ pressure, $B = (B_1^*, B_2, B_3)$ magnetic induction, $B_1^*$ constant, $e$ internal energy, $T = T(v, e) > 0$ temperature, and $\mu > 0$ and $\nu > 0$ the two coefficients of viscosity, $\kappa > 0$ the coefficient of heat conduction, $\mu_0 > 0$ the magnetic permeability, and $\sigma > 0$ the electrical resistivity. Under assumptions (5.2), (5.3), and (5.4), conditions (A1)–(A2) are again satisfied, and conditions (H1)–(H4) are satisfied (see [MaZ3]) under the generically satisfied assumptions that the shock be of Lax or overcompressive type, the endstates $U^\pm_\pm$
be strictly hyperbolic, and the speed \(s\) be nonzero, i.e., the shock move with nonzero speed relative to the background fluid velocity, with \(U_1 := (v)\), \(U_2 := (u, B, e + |u|^2/2 + v|B|^2/2\mu_0)\). (For gas dynamics, only Lax-type shocks and nonzero speeds can occur, and all points \(U\) are strictly hyperbolic.)

3. (MHD with infinite resistivity/permeability) An interesting variation of (5.5) that is of interest in certain astrophysical parameter regimes is the limit in which either electrical resistivity \(\sigma\), magnetic permeability \(\mu_0\), or both, go to infinity, in which case the righthand sides of the fifth and sixth equations of (5.5) go to zero and there is a three-dimensional set of hyperbolic modes \((v, vB_2, vB_3)\) instead of the usual one. By inspection, the associated equations are still linear in the conservative variables. Likewise, (A1)–(A2), (H1)–(H4) hold under assumptions (5.2), (5.3), and (5.4) for nonzero speed Lax- or overcompressive-type shocks with strictly hyperbolic endstates.

4. (multi-species gas dynamics or MHD) Another simple example for which the hyperbolic modes are vectorial is the case of miscible, multi-species flow, neglecting species diffusion, in either gas- or magnetohydrodynamics. In this case, the hyperbolic modes consist of \(k\) copies of the hyperbolic modes for a single species, where \(k\) is the number of total species. Again, the associated equations are linear, and (A1)–(A2), (H1)–(H4) hold for nonzero speed Lax- or overcompressive-type shocks with strictly hyperbolic endstates, under assumptions (5.2), (5.3), and (5.4).

**Remark 5.1.** Assumption (5.2) may be removed in all cases by working in the more general framework described in Remarks 1.3 and 3.5, with variables \(U_1 = (v)\), \(W_2 = (u, T)\) (resp. \(W_2 = (u, B, T)\)).

### A  Lagrangian vs. Eulerian formulation

In this appendix, we discuss a bit further the role of structural assumptions (A1)–(A2) in the energy estimates of Section 3. Estimate (3.6) holds by a similar, quasilinear version of the argument of Proposition 3.2, for the general class of hyperbolic–parabolic systems considered in [Z3]. However, (3.7) is much more delicate, as are the related estimates (3.6), (3.7), and (3.14).

To see why, note that (3.7) is essentially a variational bound, measuring the difference between two solutions, and variational bounds typically cost an additional derivative. For example, in the quasilinear symmetric hyperbolic case, we have \(\|U(t)\|_{H^s} \leq C\|U_0\|_{H^s}\) by standard energy estimates, whereas
the same energy estimates applied to the difference between two solutions \(U^1\) and \(U^2\) yield, rather,

\[
\|U^1 - U^2\|_{H^s}(t) \leq C \|U^1_0 - U^2_0\|_{H^{s+1}}.
\]

In the linear case, \(\|U^1 - U^2\|_{H^s}(t) \leq C \|U^1_0 - U^2_0\|_{H^s}\) follows by superposition, and so this issue does not arise; this motivates the assumption of linearity of the hyperbolic part of (1.2) made in (A1).

Indeed, when (A1) fails, so, typically, does (3.7). For example, consider the Eulerian version (1.9) of the isentropic compressible Navier–Stokes equations, rewritten in quasilinear form

\[
\begin{aligned}
\rho_t + u\rho_x + \rho u_x &= 0, \\
u_t + uu_x + \rho^{-1}p'(\rho)\rho_x &= \nu\rho^{-1}u_{xx}.
\end{aligned}
\]

Perturbing about the constant solution \((\rho, u) \equiv (1, 0)\), and assuming without loss of generality that \(p'(1) = 1\), we obtain perturbation equations

\[
\begin{aligned}
\rho_t + u\rho_x + u_x &= -\rho u_x, \\
u_t + \rho_x - \nu u_{xx} &= O(|\rho|^2 + |u|^2 + |\rho||u_x|)_{x}
\end{aligned}
\]

so long as \(\|\rho\|_{L^\infty} \leq C\|\rho\|_{H^1}\) remains sufficiently small. Applying the same energy estimates as in the proof of Proposition 3.2, we easily obtain

\[
\|\rho - \rho_0\|_{H^s}(t) \leq C\|\rho_0\|_{H^s}.
\]

for \(s \geq 1\), similarly as in (3.6).

Now, consider the solution \(\hat{\rho}\) of

\[
\hat{\rho}_t + u\hat{\rho}_x + u_x = 0,
\]

with \(\rho, u\) as determined by (A.2). Defining \(\hat{E} := \hat{\rho} - \rho\), we have

\[
\hat{E}_t + u\hat{E}_x = \rho u_x.
\]

Applying the same energy estimates carried out on the variation \(E\) in the proof of Proposition 3.2, we thus obtain for \(s \geq 1\) that

\[
\|\hat{\rho} - \rho\|_{H^s}(t) \leq C\|(\rho_0, u_0)\|_{H^s}.
\]
On the other hand, defining \( \tilde{\rho} \) to be the solution of the linearized equation
\[
\tilde{\rho}_t + u_x = 0,
\]
we find
\[
(\text{A.5}) \quad \|\tilde{\rho}(t) - \rho_0\|_{H^s} \leq \int_0^t \|u\|_{H^{s+1}}(r) dr \leq t^{1/2} \left( \int_0^t \|u\|^2_{H^{s+1}}(r) dr \right)^{1/2} \leq C t^{1/2} \|(\rho_0, u_0)\|_{H^s}.
\]
Likewise, defining \( \bar{\rho} \) to be the solution of
\[
\bar{\rho}_t + u\bar{\rho}_x = 0,
\]
and \( \bar{e} := \bar{\rho} - \tilde{\rho} \) we find from similar energy estimates, together with
\[
\partial_t \|\partial_x^f \bar{e}\|_{L^2} = \partial_t \|\partial_x^f e\|_{L^2}^2 / 2 \|\partial_x^f \bar{e}\|_{L^2},
\]
that
\[
(\text{A.6}) \quad \|\bar{\rho} - \tilde{\rho}\|_{H^s}(t) \leq \int_0^t \|u\|_{H^{s+1}}(r) dr \leq C t^{1/2} \|(\rho_0, u_0)\|_{H^s}.
\]
Combining (A.3), (A.5), and (A.6), we obtain
\[
\|\rho - \tilde{\rho}\|_{H^s}(t) = \|\bar{\rho} - \rho_0\|_{H^s}(t) + O(\sqrt{t}) \|(\rho_0, u_0)\|_{H^s},
\]
which, for \( t \) sufficiently small, implies that \( \|\rho - \tilde{\rho}\|_{H^s}(t) \leq C \|(\rho_0, u_0)\|_{H^s}^2 \) only if \( \|\bar{\rho} - \rho_0\|_{H^s}(t) \leq C \|(\rho_0, u_0)\|_{H^s}^2 \). But, \( \tilde{\rho} \), as the solution to a simple transport equation, is exactly a displacement
\[
\tilde{\rho}(x, t) = \rho_0(X(x, t))
\]
of \( \rho_0 \), where \( X \) satisfies characteristic equation \( \partial_t X = u(X) \), and so in the absence of derivative bounds on \( \partial_x^f \rho_0 \), \( \|\tilde{\rho}(t) - \rho_0\|_{H^s} \) is in general no smaller than \( 2\|\rho_0\|_{H^s} \). Thus, \( \|\rho - \tilde{\rho}\|_{H^s}(t) \leq C \|(\rho_0, u_0)\|_{H^s}^2 \), in general does not hold in the Eulerian formulation.

The resolution of this apparent paradox is that the Lagrangian formulation incorporates nonlinear transport effects into the equations via a change of independent variables, so that \( \| \cdot \|_{H^s} \) compares different solutions along their respective particle-paths rather than at a fixed location, thus eliminating the principal contribution computed above.
Remark A.1. Some such superlinear bound as (3.2) appears necessary for bifurcation analysis based on temporal dynamics, at least as usually performed based on autonomous linearized equations. This is an obstruction to the application of invariant manifold/bifurcation techniques to hyperbolic systems such as the incompressible Euler equations [Li]. Likewise, for hyperbolic–parabolic systems, it limits us at present to the Lagrangian formulation and a single spatial dimension. For related discussion, see [Ho1, Ho2, HoZ, D].

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