Ensemble Copula Coupling as a Multivariate Discrete Copula Approach

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Abstract

In probability and statistics, copulas play important roles theoretically as well as to address a wide range of problems in various application areas. In this paper, we introduce the concept of multivariate discrete copulas, discuss their equivalence to stochastic arrays, and provide a multivariate discrete version of Sklar’s theorem. These results provide the theoretical frame for the ensemble copula coupling approach proposed by Schefzik et al. (2013) for the multivariate statistical postprocessing of weather forecasts made by ensemble systems.

Keywords and phrases: multivariate discrete copula, stochastic array, Sklar’s theorem, statistical ensemble postprocessing, ensemble copula coupling

1 Introduction

Originally introduced by Sklar (1959), copulas play an important role in probability and statistics whenever modeling of stochastic dependence is required. Roughly speaking, copulas are functions that link multivariate distribution functions to their univariate marginal distribution functions, as is manifested in the famous Sklar’s theorem (Sklar, 1959). The field of copulas has been developing rapidly over the last decades, and copulas have been applied to a wide range of problems in various areas such as climatology, meteorology and hydrology (Möller et al., 2013; Genest and Favre, 2007; Schölzel and Friederichs, 2008; Zhang et al., 2012) or econometrics, insurance and mathematical finance (Cherubini et al., 2004; Embrechts et al., 2003; Pflieger and Nešlehová, 2003; Genest et al., 2009). However, copulas are also of immense theoretical interest, due to their appealing mathematical properties. For a general overview of the mathematical theory of copulas, we refer to the textbooks by Joe (1997) and Nelsen (2006), as well as to the survey paper by Sempi (2011).

A special type of copulas are the so-called discrete copulas, whose properties have been studied by Kolesárová et al. (2006), Mayor et al. (2005), Mayor et al. (2007) and Mesiar (2005) in recent years. However, the discussion in the papers mentioned above focuses on the bivariate case, and it is natural to search for a treatment of the general multivariate situation. In what follows, we generalize both the notion of discrete copulas and the most important results in this context to the multivariate case, and show to what extent they build the theoretical frame of the ensemble copula coupling (ECC) approach recently proposed by Schefzik et al. (2013). ECC is a multivariate statistical postprocessing tool for ensemble weather forecasts in meteorology, which turns out to be based on the theoretical framework discussed here.

The remainder of this paper is organized as follows. In Section 2, we introduce the multivariate discrete
2 Multivariate discrete copulas

First, we transfer the notion of bivariate discrete copulas introduced by Kolesárová et al. (2006) to the general multivariate case. Although our new class of copulas turns out to be a special Fréchet class (Joe, 1997), we nevertheless give all relevant definitions in detail, as they provide the basic concepts required in the subsequent sections.

Let \( I_M := \{ 0, \frac{1}{M}, \frac{2}{M}, \ldots, \frac{M-1}{M}, 1 \} \), where \( M \in \mathbb{N} \), and \( \mathbb{R} = \mathbb{R} \cup \{-\infty, \infty\} \).

**Definition 2.1.** A function \( D : I_M^L \to [0, 1] \) is called a discrete copula on \( I_M^L := I_M \times \cdots \times I_M \) if it satisfies the following conditions:

- (D1) \( D \) is grounded in the sense that \( D\left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) = 0 \) if \( i_\ell = 0 \) for at least one \( \ell \in \{1, \ldots, L\} \).
- (D2) \( D(1, \ldots, 1, \frac{i}{M}, 1, \ldots, 1) = \frac{i}{M} \) for all \( \ell \in \{1, \ldots, L\} \).
- (D3) \( D \) is \( L \)-increasing in the sense that
  \[
  \Delta^{i_\ell}_{i_\ell-1} D\left( \frac{j_1}{M}, \ldots, \frac{j_L}{M} \right) := D\left( \frac{j_1}{M}, \ldots, \frac{j_{\ell-1}}{M}, \frac{i_\ell}{M}, \frac{j_{\ell+1}}{M}, \ldots, \frac{j_L}{M} \right) - D\left( \frac{j_1}{M}, \ldots, \frac{j_{\ell-1}}{M}, \frac{i_{\ell-1}}{M}, \frac{j_{\ell+1}}{M}, \ldots, \frac{j_L}{M} \right),
  \]

**Definition 2.2.** A discrete copula \( D : I_M^L \to [0, 1] \) is called irreducible if it has minimal range, that is, \( \text{Ran}(D) = I_M \).

**Definition 2.3.** A function \( D^* : J_M^{(1)} \times \cdots \times J_M^{(L)} \to [0, 1] \) with \( \{0, 1\} \subset J_M^{(1)}, \ldots, J_M^{(L)} \subset I_M \) is called a discrete subcopula if it satisfies the following conditions:

- (S1) \( D^*\left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) = 0 \) if \( i_\ell = 0 \) for at least one \( \ell \in \{1, \ldots, L\} \).
- (S2) \( D^*(1, \ldots, 1, \frac{i}{M}, 1, \ldots, 1) = \frac{i}{M} \) for all \( \frac{i}{M} \in J_M^{(\ell)} \).
- (S3) \( \Delta^{i_\ell}_{i_\ell-1} D^*\left( \frac{k_1}{M}, \ldots, \frac{k_L}{M} \right) := D^*\left( \frac{k_1}{M}, \ldots, \frac{k_{\ell-1}}{M}, \frac{i_\ell}{M}, \frac{k_{\ell+1}}{M}, \ldots, \frac{k_L}{M} \right) - D^*\left( \frac{k_1}{M}, \ldots, \frac{k_{\ell-1}}{M}, \frac{i_{\ell-1}}{M}, \frac{k_{\ell+1}}{M}, \ldots, \frac{k_L}{M} \right) \geq 0 \) for all \( \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right), \left( \frac{j_1}{M}, \ldots, \frac{j_L}{M} \right) \in J_M^{(1)} \times \cdots \times J_M^{(L)} \) such that \( i_\ell \leq j_\ell \) for all \( \ell \in \{1, \ldots, L\} \), where
  \[
  \Delta^{i_\ell}_{i_\ell-1} D^*\left( \frac{k_1}{M}, \ldots, \frac{k_L}{M} \right) := D^*\left( \frac{k_1}{M}, \ldots, \frac{k_{\ell-1}}{M}, \frac{i_\ell}{M}, \frac{k_{\ell+1}}{M}, \ldots, \frac{k_L}{M} \right) - D^*\left( \frac{k_1}{M}, \ldots, \frac{k_{\ell-1}}{M}, \frac{i_{\ell-1}}{M}, \frac{k_{\ell+1}}{M}, \ldots, \frac{k_L}{M} \right).
  \]

The definition of discrete (sub)copulas can be generalized in the following way: A discrete copula need not necessarily have domain \( I_M^L \), but can generally be defined on \( I_{M_1} \times \cdots \times I_{M_L} \), where \( M_1, \ldots, M_L \in \mathbb{N} \) might take distinct values. Then, the axioms (D1), (D2) and (D3) apply analogously to this case. Similarly, discrete subcopulas can generally be defined on \( J_{M_1}^{(1)} \times \cdots \times J_{M_L}^{(L)} \) for possibly distinct numbers.
Then, the empirical copula
\[ \Pi \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) := \prod_{\ell=1}^{L} \frac{i_\ell}{M} \]
is a discrete copula, the so-called product or independence copula.

(b) \[ \mathcal{M} \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) := \min \left\{ \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right\} \]
is an irreducible discrete copula.

Note that \( \Pi \) and \( \mathcal{M} \) are indeed multivariate discrete copulas because they represent the restrictions of two well-known standard copulas defined on \([0, 1]^L\) to the discrete set \( I_M^L \).

**Example 2.5.** Another example for an irreducible discrete copula is given by the so-called empirical copula, which will be very important with respect to the ECC approach, see Section 5. Let \( S = \{ (x^1_1, \ldots, x^L_1), \ldots, (x^1_M, \ldots, x^L_M) \} \), where \( x^\ell_m \in \mathbb{R} \) for all \( m \in \{1, \ldots, M\} \) and \( \ell \in \{1, \ldots, L\} \) with \( x^\ell_m \neq x^\ell_n \) for \( m \neq n \). That is, we assume for simplicity that there are no ties among the respective samples. Moreover, let \( x^\ell_{(1)} < \ldots < x^\ell_{(M)} \) be the (marginal) order statistics of the collections \( \{ x^1_1, \ldots, x^1_M \}, \ldots, \{ x^L_1, \ldots, x^L_M \} \), respectively.

Then, the empirical copula \( E_M : I_M^L \rightarrow I_M \) defined from \( S \) is given by

\[
E_M \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) := \begin{cases} 0 & \text{if } i_\ell = 0 \text{ for at least one } \ell \in \{1, \ldots, L\} \\ \frac{\# \{ (x^1_m, \ldots, x^L_m) \in S \mid x^\ell_m \leq x^\ell_{i_\ell}, \ldots, x^L_m \leq x^L_{i_L}\}}{M^L} & \text{if } i_\ell \in \{1, \ldots, M\} \text{ for all } \ell \in \{1, \ldots, L\} \end{cases}
\]

or, equivalently,

\[
E_M \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) := \frac{1}{M} \sum_{m=1}^{M} \mathbb{I} \{ \text{rk}(x^\ell_m) \leq i_1, \ldots, \text{rk}(x^L_m) \leq i_L \} = \frac{1}{M} \sum_{m=1}^{M} \prod_{\ell=1}^{L} \mathbb{I} \{ \text{rk}(x^\ell_m) \leq i_\ell \};
\]

where \( \text{rk}(x^\ell_m) \) denotes the rank of \( x^\ell_m \) in \( \{ x^\ell_1, \ldots, x^\ell_M \} \) for \( \ell \in \{1, \ldots, L\} \), and \( i_1, \ldots, i_L \in \{0, 1, \ldots, M\} \), compare Deheuvels (1979).

Obviously, the empirical copula is an irreducible discrete copula. Conversely, any irreducible discrete copula is the empirical copula of some set \( S \), as discussed in Example 3.4 (c) in Section 3.

### 3 A characterization of multivariate discrete copulas via stochastic arrays

According to Kolesárová et al. (2006) and Mayor et al. (2005), there is a one-to-one correspondence between discrete copulas and bistochastic matrices in the bivariate case. We now formulate a similar characterization for multivariate discrete copulas. To this end, the notion of stochastic arrays (Csima, 1970; Marchi and Tarazaga, 1979) turns out to be very useful.

**Definition 3.1.** An array \( A = (a_{i_1 \ldots i_L})_{i_1, \ldots, i_L=1}^{M} \) is called an \( L \)-dimensional stochastic array (or an \( L \)-stochastic matrix) of degree \( L - 1 \) if

(A1) \( a_{i_1 \ldots i_L} \geq 0 \) for all \( i_1, \ldots, i_L \in \{1, \ldots, M\} \)
In the situation of Corollary 3.3, we have

Example 3.4.

Essentially, Theorem 3.2 yields the following equivalences:

\[ M \\text{identified with} \quad \text{the probability mass function (pmf) (Xu, 1996)} \because \text{the stochastic array in Definition 3.1 can be stress that Theorem 3.2 can also be interpreted as a reformulation of the relation between the cdf} \]

As a special case, an \( L \)-dimensional stochastic array \( A \) is called an \( L \)-dimensional permutation array (or an \( L \)-permutation matrix) if the entries of \( A \) only take the values 0 and 1, that is, \( a_{i_1 \ldots i_L} \in \{0,1\} \) for all \( i_1, \ldots, i_L \in \{1, \ldots, M\} \).

**Theorem 3.2.** Let \( D : I^L_M \rightarrow [0, 1] \). Then, the following statements are equivalent:

1. \( D \) is a discrete copula.
2. There exists an \( L \)-dimensional stochastic array \( A = (a_{i_1 \ldots i_L})_{i_1 \ldots i_L=1}^M \) such that

\[
D\left(\frac{i_1}{M}, \ldots, \frac{i_L}{M}\right) = \frac{1}{M} \sum_{j_1=1}^{i_1} \cdots \sum_{j_L=1}^{i_L} a_{j_1 \ldots j_L} \tag{1}
\]

for \( i_1, \ldots, i_L \in \{0, 1, \ldots, M\} \).

**Corollary 3.3.** \( D \) is an irreducible discrete copula if and only if there is an \( L \)-dimensional permutation array \( A = (a_{i_1 \ldots i_L})_{i_1 \ldots i_L=1}^M \) such that (1) holds for \( i_1, \ldots, i_L \in \{0, 1, \ldots, M\} \).

The proof of Theorem 3.2 basically consists in showing the validity of the axioms (A1), (A2), (D1), (D2) and (D3) in Definitions 3.1 and 2.1, respectively. This is on the one hand straightforward, but on the other hand rather tedious, involving several calculations of multiple sums. We omit a detailed proof and stress that Theorem 3.2 can also be interpreted as a reformulation of the relation between the cdf and the probability mass function (pmf) (Xu, 1996) because the stochastic array in Definition 3.1 can be identified with \( M \) times the pmf.

Essentially, Theorem 3.2 yields the following equivalences:

\[
\text{discrete copula} \leftrightarrow \text{marginal distributions concentrated on} \quad \{\frac{1}{M}, \frac{2}{M}, \ldots, 1\} \leftrightarrow \text{probability masses on} \quad \{\frac{1}{M}, \frac{2}{M}, \ldots, 1\}^L \leftrightarrow \text{stochastic array}.
\]

In the situation of Corollary 3.3, we have

irreducible discrete copula \(\leftrightarrow\) empirical copula \(\leftrightarrow\) \(M\) point masses of \(\frac{1}{M}\) each \(\leftrightarrow\) permutation array \(\leftrightarrow\) Latin hypercube of order \(M\) in \(L\) dimensions (Gupta, 1974).

Illustrations of these equivalences are given in Section 5, where we discuss their relevance with respect to the ECC approach of Schefzik et al. (2013).

**Example 3.4.**

(a) The discrete product copula \( \Pi \left(\frac{i_1}{M}, \ldots, \frac{i_L}{M}\right) = {\prod_{\ell=1}^{L}} \frac{i_{\ell}}{M} \), where \( \left(\frac{i_1}{M}, \ldots, \frac{i_L}{M}\right) \in I^L_M \), in Example 2.4 (a) corresponds to the \( L \)-dimensional stochastic array \( A := (a_{i_1 \ldots i_L})_{i_1 \ldots i_L=1}^M \) whose entries are all equal to \(\frac{1}{M^L} \). Indeed,

\[
\frac{1}{M} \sum_{j_1=1}^{i_1} \cdots \sum_{j_L=1}^{i_L} \frac{1}{M^{L-1}} = \frac{1}{M} \sum_{j_1=1}^{i_1} \cdots \sum_{j_{L-1}=1}^{i_{L-1}} \frac{i_L}{M^{L-1}} = \frac{1}{M} \cdot \frac{i_1 \ldots i_{L-1} \cdot i_L}{M^{L-1}} = \frac{i_1 \ldots i_L}{M^L} = \prod_{\ell=1}^{L} \frac{i_{\ell}}{M} = \Pi \left(\frac{i_1}{M}, \ldots, \frac{i_L}{M}\right).
\]
(b) The irreducible discrete copula \( M \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) = \min \left\{ \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right\} \), where \( \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) \in I_M^L \), in Example 2.4 (b) corresponds to the \( L \)-dimensional identity stochastic array

\[
\mathbb{I} := (a_{i_1 \cdots i_L})_{i_1, \ldots, i_L = 1}^M, \quad \text{where} \quad a_{i_1 \cdots i_L} = \begin{cases} 
1 & \text{if } i_1 = \ldots = i_L \\
0 & \text{otherwise}.
\end{cases}
\]

Indeed, employing the definition and writing down the corresponding multiple sum explicitly yields

\[
\frac{1}{M} \sum_{j_1 = 1}^{i_1} \cdots \sum_{j_L = 1}^{i_L} a_{j_1 \cdots j_L} = \frac{1}{M} \cdot \min\{i_1, \ldots, i_L\} = \min \left\{ \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right\} = M \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right)
\]

(c) The empirical copula \( E_M \) in Example 2.5, which is an irreducible discrete copula, corresponds to the \( L \)-dimensional permutation array \( A = (a_{i_1 \cdots i_L})_{i_1, \ldots, i_L = 1}^M \) with

\[
a_{i_1 \cdots i_L} = \begin{cases} 
1 & \text{if } (x_{1(i_1)}, \ldots, x_{(i_L)}) \in S \\
0 & \text{if } (x_{1(i_1)}, \ldots, x_{(i_L)}) \notin S.
\end{cases}
\]

Conversely, for an irreducible discrete copula \( D \) with associated \( L \)-dimensional permutation array \( A = (a_{i_1 \cdots i_L})_{i_1, \ldots, i_L = 1}^M \), we consider the sets \( \mathcal{X}^1 = \{x_1^1 < \ldots < x_M^1\}, \ldots, \mathcal{X}^L = \{x_1^L < \ldots < x_M^L\} \). Then, \( D \) is the empirical copula of the set \( S = \{(x_{1(i_1)}, \ldots, x_{(i_L)})|a_{i_1 \cdots i_L} = 1\} \).

4 A multivariate discrete version of Sklar’s theorem

The most important result in the context of copulas is Sklar’s theorem, see Nelsen (2006) or Sklar (1959). Our goal is now to prove a multivariate discrete version of Sklar’s theorem, where the following extension lemma will play an essential role. A bivariate variant of this result has been shown by Mayor et al. (2007).

**Lemma 4.1.** (Extension lemma) For each irreducible discrete subcopula \( D^* : J_M^{(1)} \times \cdots \times J_M^{(L)} \rightarrow I_M \), there is an irreducible discrete copula \( D : I_M \times \cdots \times I_M \rightarrow I_M \) such that

\[
D|_{J_M^{(1)} \times \cdots \times J_M^{(L)}} = D^*,
\]

that is, the restriction of \( D \) to \( J_M^{(1)} \times \cdots \times J_M^{(L)} \) coincides with \( D^* \).

**Proof.** Let

\[
J_M^{(\ell)} := \left\{ \left. 0 = a_{0\ell}^{(\ell)} < a_{1\ell}^{(\ell)} < \ldots < a_{r_{\ell}^{(\ell)}}^{(\ell)} \frac{a_{r_{\ell}^{(\ell)} + 1}^{(\ell)}}{M} = 1 \right\} \quad \text{for all } \ell \in \{1, \ldots, L\},
\]

with the corresponding equivalent sets

\[
K_M^{(\ell)} := \{0 = a_{0\ell}^{(\ell)} < a_{1\ell}^{(\ell)} < \ldots < a_{r_{\ell}^{(\ell)}}^{(\ell)} \frac{a_{r_{\ell}^{(\ell)} + 1}^{(\ell)}}{M} = M\}.
\]

To get an irreducible discrete extension copula \( D \) of an irreducible discrete subcopula \( D^* \), according to Theorem 3.2, it suffices to construct an \( L \)-dimensional permutation array \( A \) such that each block specified by the points \( (a_{s_1}^{(1)}, a_{s_2}^{(2)}, \ldots, a_{s_L}^{(L)}) \) and \( (a_{s_1+1}^{(1)}, a_{s_2+1}^{(2)}, \ldots, a_{s_L+1}^{(L)}) \), which consists of the lines from \( a_{s_1}^{(1)} + 1 \) to \( a_{s_1+1}^{(1)} \), from \( a_{s_2}^{(2)} + 1 \) to \( a_{s_2+1}^{(2)} \), and so forth, up to the line from \( a_{s_L}^{(L)} + 1 \) to \( a_{s_L+1}^{(L)} \), contains a number of 1’s equal to the volume

\[
M \cdot \left( \Delta_{a_{s_L+1}^{(L)}}^{a_{s_L}^{(L)}} \frac{\Delta_{a_{s_1+1}^{(1)}}^{a_{s_1}^{(1)}}}{a_{s_1}^{(1)}} D^* \left( \frac{j_1}{M}, \ldots, \frac{j_L}{M} \right) \right).
\]
where \( s_\ell \in \{0, \ldots, r_\ell\} \) and \( \ell \in \{1, \ldots, L\} \).
To show the existence of such a permutation array, let \( k \in \{1, \ldots, L\} \) be fixed and consider the subarray specified by the lines \( a_k^{(k)} \) and \( a_{k+1}^{(k)} \) of the permutation array \( A \). This subarray contains all blocks determined by the points \( (a_1^{(L)}, \ldots, a_{s_1+1}^{(L)}) \) and \( (s_{s_1+1}^{(L)}, \ldots, a_{s_L+1}^{(L)}) \) for all \( s_\ell \in \{0, \ldots, r_\ell\} \), where \( \ell \in \{1, \ldots, L\} \setminus \{k\} \).

We need to show that the number \( a_{s_k+1}^{(k)} - a_s^{(k)} \) of lines in this subarray is equal to the number of 1’s corresponding to all those blocks. This indeed holds as

\[
\sum_{s_1=0}^{r_1} \cdots \sum_{s_{k-1}=0}^{r_{k-1}} \sum_{s_{k+1}=0}^{r_{k+1}} \cdots \sum_{s_L=0}^{r_L} M \cdot \left( \Delta_{a_k^{(L)}} a_k^{(1)} \Delta_{a_{k+1}^{(L)}} a_{k+1}^{(1)} D^* \left( \frac{j_1}{M}, \ldots, \frac{j_L}{M} \right) \right)
= M \cdot \sum_{s_1=0}^{r_1} \cdots \sum_{s_{k-1}=0}^{r_{k-1}} \sum_{s_{k+1}=0}^{r_{k+1}} \cdots \sum_{s_L=0}^{r_L} \left( \Delta_{a_k^{(L)}} a_k^{(1)} \Delta_{a_{k+1}^{(L)}} a_{k+1}^{(1)} D^* \left( \frac{j_1}{M}, \ldots, \frac{j_L}{M} \right) \right)
= \text{(S2)} M \cdot \left( a_{s_k+1}^{(k)} - a_k^{(k)} \right)
\]

To see equality \( (*) \) in this connection, we let \( \ell \in \{1, \ldots, L\} \setminus \{k\} \) be fixed and first consider the sum

\[
T := \sum_{s_\ell=0}^{r_\ell} \left( \Delta_{a_{s_\ell+1}^{(L)}} a_{s_\ell}^{(1)} \Delta_{a_{s_\ell+1}^{(L)}} a_{s_\ell+1}^{(1)} D^* \left( \frac{j_1}{M}, \ldots, \frac{j_L}{M} \right) \right).
\]

Writing down \( T \) explicitly yields that all of the \( (r_\ell + 1) \cdot 2^L \) terms \( D(\ldots, \cdot) \) of \( T \) cancel out except for those \( 2^L \) having a 0 or a 1 in the \( \ell \)-th component, which indeed occurs as \( a_0^{(\nu)} = 0 \) and \( a_{r_\nu+1}^{(\nu)} = M \) for \( \nu \in \{1, \ldots, L\} \). According to property (S1) in Definition 2.3, all the \( 2^L-1 \) terms having a 0 in the \( \ell \)-th component vanish, and it remains

\[
T = \Delta_{a_{s_{\ell}+1}^{(L)}} a_{s_{\ell}}^{(1)} \Delta_{a_{s_{\ell}+1}^{(L)}} a_{s_{\ell}+1}^{(1)} D^* \left( \frac{j_1}{M}, \ldots, \frac{j_{\ell-1}}{M}, 1, \frac{j_{\ell+1}}{M}, \ldots, \frac{j_L}{M} \right).
\]

By applying this iteratively and using again property (S1) in Definition 2.3, we finally see that all but two of the terms \( D(\ldots, \cdot) \) of \( S \) either vanish or cancel out, such that

\[
S = \Delta_{a_{s_k+1}^{(k)}} a_k^{(k)} D^* \left( 1, \ldots, 1, \frac{j_k}{M}, 1, \ldots, 1 \right),
\]

and \( (*) \) is thus shown.

Hence, we have proved that an irreducible discrete subcopula \( D^* \) can be extended to an irreducible discrete copula \( D \).

Note that the extension proposed in Lemma 4.1 is in general not uniquely determined. In the case of non-uniqueness, there are a largest and a smallest discrete extension copula \( D_{lar} \) and \( D_{sm} \), respectively, in the sense that

\[
D_{lar} \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) \geq D \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) \geq D_{sm} \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) \text{ for all } \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) \in I_M^L
\]
for any other discrete extension copula $D$ of $D^*$. 

With Lemma 4.1, we are now ready to state and prove a multivariate discrete version of Sklar’s theorem. For the bivariate case, such a result can be found in Mayor et al. (2007).

**Theorem 4.2.** (Sklar’s theorem in the multivariate discrete case)

1. Let $F_1, \ldots, F_L$ be distribution functions with $\text{Ran}(F_\ell) \subseteq I_M$ for all $\ell \in \{1, \ldots, L\}$. If $D$ is an irreducible discrete copula on $I_M$, then

   $$H(x_1, \ldots, x_L) = D(F_1(x_1), \ldots, F_L(x_L)) \quad \text{for} \quad (x_1, \ldots, x_L) \in \mathbb{R}_L$$

   (2)

   is a joint distribution function with $\text{Ran}(H) \subseteq I_M$, having $F_1, \ldots, F_L$ as marginal distribution functions.

2. Conversely, if $H$ is a joint distribution function with marginal distribution functions $F_1, \ldots, F_L$ and $\text{Ran}(H) \subseteq I_M$, there exists an irreducible discrete copula $D$ on $I_M$ such that

   $$H(x_1, \ldots, x_L) = D(F_1(x_1), \ldots, F_L(x_L)) \quad \text{for} \quad (x_1, \ldots, x_L) \in \mathbb{R}_L.$$ 

   Furthermore, $D$ is uniquely determined if and only if $\text{Ran}(F_\ell) = I_M$ for all $\ell \in \{1, \ldots, L\}$.

**Proof.**

1. This is just a special case of the common Sklar’s theorem.

2. Let $H$ be a finite $L$-dimensional joint distribution function with $\text{Ran}(H) \subseteq I_M$ having one-dimensional marginal distribution functions $F_1, \ldots, F_L$. Set

   $$J_M^{(\ell)} := \left\{ \frac{i_\ell}{M} \in I_M \left| \frac{i_\ell}{M} \in \text{Ran}(F_\ell) \right. \right\} \supseteq \{0, 1\}$$

   for all $\ell \in \{1, \ldots, L\}$ and define

   $$D^*: J_M^{(1)} \times \cdots \times J_M^{(L)} \rightarrow I_M, \quad D^* \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) := H(x_1, \ldots, x_L),$$

   where $x_\ell$ satisfies $F_\ell(x_\ell) = \frac{i_\ell}{M}$ for all $\ell \in \{1, \ldots, L\}$. We now show that $D^*$ is indeed an irreducible discrete subcopula. First, $\text{Ran}(H) \subseteq I_M$ by assumption, and $D^*$ is well-defined, due to the well-known fact that $H(x_1, \ldots, x_L) = H(y_1, \ldots, y_L)$ for points $(x_1, \ldots, x_L) \in \mathbb{R}_L$ and $(y_1, \ldots, y_L) \in \mathbb{R}_L$ such that $F(x_\ell) = F(y_\ell)$ for all $\ell \in \{1, \ldots, L\}$. Furthermore, the axioms (S1), (S2) and (S3) for discrete subcopulas in Definition 2.3 are fulfilled:

   **(S1)** Let $i_\ell = 0$ for an $\ell \in \{1, \ldots, L\}$. Then, $D^*\left( \frac{i_1}{M}, \ldots, \frac{i_{\ell-1}}{M}, 0, \frac{i_{\ell+1}}{M}, \ldots, \frac{i_L}{M} \right) = H(x_1, \ldots, x_L)$ with $F_\ell(x_\ell) = \frac{0}{M} = 0$ and $F_k(x_k) = \frac{i_k}{M}$ for all $k \in \{1, \ldots, L\} \setminus \{\ell\}$. However, $F_\ell(x_\ell) = H(\infty, \ldots, x_\ell, \infty, \ldots, \infty) = 0$, and since $H$ is non-decreasing in each argument, we have $H(x_1, \ldots, x_\ell, \ldots, x_L) = 0$, and hence $D^*\left( \frac{i_1}{M}, \ldots, \frac{i_{\ell-1}}{M}, 0, \frac{i_{\ell+1}}{M}, \ldots, \frac{i_L}{M} \right) = 0$ for all $\frac{i_\ell}{M} \in J_M^{(k)}$, $k \in \{1, \ldots, L\} \setminus \{\ell\}$. Clearly, this is also true if $i_\ell = 0$ for two or more $\ell \in \{1, \ldots, L\}$.

   **(S2)** $D^*(1, \ldots, 1, \frac{i_\ell}{M}, 1, \ldots, 1) = H(x_1, \ldots, x_L)$ with $F_\ell(x_\ell) = \frac{i_\ell}{M}$ and $F_k(x_k) = 1$ for all $k \in \{1, \ldots, L\} \setminus \{\ell\}$. Set $x_\ell := \infty$ for all $k \in \{1, \ldots, L\} \setminus \{\ell\}$. Then, $D^*\left( \frac{i_1}{M}, \ldots, 1, \frac{i_\ell}{M}, 1, \ldots, 1 \right) = H(\infty, \ldots, \infty, x_\ell, \infty, \ldots, \infty) = F_\ell(x_\ell) = \frac{i_\ell}{M}$ for all $\frac{i_\ell}{M} \in J_M^{(\ell)}$. 

7
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In this section, we relate our concepts to the ensemble copula coupling (ECC) approach of Schefzik et al. (2013), which is discussed in the following.
5 Ensemble copula coupling: An application of multivariate discrete cop-
ulas in meteorology
In this section, we relate our concepts to the ensemble copula coupling (ECC) approach of Schefzik et al. (2013), which is a multivariate statistical postprocessing technique for ensemble weather forecasts, and deepen the theoretical considerations in Section 4.2 in Schefzik et al. (2013).
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tions and/or in details of the parameterized numerical representation of the atmosphere (Gneiting and Raftery, 2005). However, ensemble forecasts often reveal biases and dispersion errors. It is thus common that they get statistically postprocessed in order to correct these shortcomings. Ensemble predictions and
their postprocessing lead to probabilistic forecasts in form of predictive probability distributions over future weather quantities, where forecast distributions of good quality are characterized by sharpness subject to calibration (Gneiting et al., 2007). Several ensemble postprocessing methods have been proposed, yet many of them, such as Bayesian model averaging (BMA: Raftery et al. (2005)) or ensemble model output statistics (EMOS; Gneiting et al. (2005)), only apply to a single weather quantity at a single location for a single prediction horizon. In many applications, however, it is crucial to account for spatial, temporal and inter-variable dependence structures, as in air traffic management or ship routeing, for instance.
To address this, ECC as introduced by Schefzik et al. (2013) offers a simple yet powerful tool, which in
a nutshell performs as follows: For each weather variable i ∈ {1, . . . , I}, location j ∈ {1, . . . , J} and
prediction horizon k ∈ {1, . . . , K} separately, we are given the M forecasts xℓ i , . . . , xℓ M of the original
unprocessed raw ensemble, where ℓ := (i, j, k), ℓ ∈ {1, . . . , L}, and L = I × J × K. For each fixed

(S3) To show that D∗ is L-increasing, we use the L-increasingness of H as a finite distribution function and obtain
\[ \Delta_{i_2}^L \cdots \Delta_{i_1}^L D^*(\frac{k_1}{M}, \ldots, \frac{k_L}{M}) = \Delta_{i_2}^L \cdots \Delta_{i_1}^L H(u_1, \ldots, u_L) \geq 0 \]
with \( \frac{k_i}{M} \geq \frac{i_i}{M} \) for all \( i_i, k_i \in J_M^{(\ell)} \) and \( \frac{k_i}{M} \in J_M^{(\ell)} \), where \( F_{\ell}(x_{\ell}) = \frac{i_{\ell}}{M} \) and \( F_{\ell}(y_{\ell}) = \frac{j_{\ell}}{M} \) for all \( x_{\ell} \in \mathbb{R}, y_{\ell} \in \mathbb{R} \) and \( \ell \in \{1, \ldots, L\} \). This means that D∗ is L-increasing.
Thus, D∗ is indeed a subcopula.
According to Lemma 4.1, D∗ can therefore be extended to a discrete copula D, which obviously satisfies
\[ D(F_1(x_1), \ldots, F_L(x_L)) = D\left(\frac{i_1}{M}, \ldots, \frac{i_L}{M}\right) = D^*\left(\frac{i_1}{M}, \ldots, \frac{i_L}{M}\right) = H(x_1, \ldots, x_L) \]
for \((x_1, \ldots, x_L) \in \mathbb{R}^L\). Hence, \( H(x_1, \ldots, x_L) = D(F_1(x_1), \ldots, F_L(x_L)) \).
The last issue left to prove is that D is uniquely determined if and only if Ran(Fℓ) = I_M for all \( \ell \in \{1, \ldots, L\} \). Assume that D∗ : J_M^{(1)} \times \cdots \times J_M^{(L)} \to I_M can be extended in only a single way to a discrete copula D. Then, due to the unique extension, we have \( D_{\text{lar}} = D = D_{\text{sm}}, \) where \( D_{\text{lar}} \) and \( D_{\text{sm}} \) denote the largest and the smallest discrete extension copulas, respectively. However, this only holds if \( J_M^{(1)} = \cdots = J_M^{(L)} \), that is, if Ran(Fℓ) = I_M for all \( \ell \in \{1, \ldots, L\} \). Conversely, if Ran(Fℓ) = I_M for all \( \ell \in \{1, \ldots, L\} \), then the discrete subcopula D∗ has domain \( I_M^L \), and thus we have \( D = D^* \).

Theorem 4.2 is especially tailored to and suitable for situations in which dealing with empirical copulas of data with no ties matters. This is for instance the case in the ECC approach proposed by Schefzik et al. (2013), which is discussed in the following.

5 Ensemble copula coupling: An application of multivariate discrete cop-
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In this section, we relate our concepts to the ensemble copula coupling (ECC) approach of Schefzik et al. (2013), which is a multivariate statistical postprocessing technique for ensemble weather forecasts, and deepen the theoretical considerations in Section 4.2 in Schefzik et al. (2013).
In state of the art meteorological practice, weather forecasts are derived from ensemble prediction sys-
tems, which comprise multiple runs of numerical weather prediction models differing in the initial condi-
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To address this, ECC as introduced by Schefzik et al. (2013) offers a simple yet powerful tool, which in
a nutshell performs as follows: For each weather variable i ∈ {1, . . . , I}, location j ∈ {1, . . . , J} and
prediction horizon k ∈ {1, . . . , K} separately, we are given the M forecasts xℓ i , . . . , xℓ M of the original
unprocessed raw ensemble, where ℓ := (i, j, k), ℓ ∈ {1, . . . , L}, and L = I × J × K. For each fixed
ℓ, let $\sigma_\ell(m) := \text{rk}(x_\ell^f)$ for $m \in \{1, \ldots, M\}$ be the permutation of $\{1, \ldots, M\}$ induced by the order statistics $x_{(1)}^\ell \leq \cdots \leq x_{(M)}^\ell$ of the raw ensemble, with any ties resolved at random. In a first step, we employ state-of-the-art univariate postprocessing methods such as BMA or EMOS to obtain calibrated and sharp predictive cdfs $F_{X_\ell}^\ell$, $\ell \in \{1, \ldots, L\}$, for each variable, location and look-ahead time individually. Then, we draw $M$ samples $\tilde{x}_1^\ell, \ldots, \tilde{x}_M^\ell$ from $F_{X_\ell}^\ell$ for each $\ell \in \{1, \ldots, L\}$. This can be done, for instance, by taking the equally spaced $\frac{m-1}{M}$-quantiles, where $m \in \{1, \ldots, M\}$, of each predictive cdf $F_{X_\ell}^\ell$, $\ell \in \{1, \ldots, L\}$. In the final ECC step, the $M$ samples (quantiles) are rearranged with respect to the ranks the ensemble members are assigned within the raw ensemble in order to retain the spatial, temporal and inter-variable rank dependence structure and to capture the flow dependence of the raw ensemble. That is, the ECC ensemble consists of $\hat{x}_1^\ell := \tilde{x}_{\sigma_\ell(1)}^\ell, \ldots, \hat{x}_M^\ell := \tilde{x}_{\sigma_\ell(M)}^\ell$ for each $\ell \in \{1, \ldots, L\}$.

Schefzik et al. (2013) and Schuhen et al. (2012) show in several case studies that ECC is well-performing in the sense that the ECC ensemble exhibits better stochastic characteristics than the unprocessed raw ensemble.

Although our concepts have been discussed for the general multivariate case, we now consider for illustrative purposes a bivariate ($L = 2$) example in the first row of Figure 1, with 24 hour ahead forecasts for temperature at Berlin and Hamburg, based on the $M = 50$-member European Centre for Medium-Range Weather Forecasts (ECMWF) ensemble (Molteni et al., 1996; Buizza, 2006) and valid on 27 June 2010, 2:00 am local time. Univariate postprocessing is performed by BMA. In the first panel of the first row, the unprocessed raw ensemble forecasts are shown, while the plot in the middle presents the independently site-by-site postprocessed ensemble, in which the bivariate rank order characteristics of the unprocessed forecasts from the left pattern are lost, even though biases and dispersion errors have been corrected. Finally, the postprocessed ECC ensemble in the right panel corrects biases and dispersion errors as the independently postprocessed ensemble does, but also takes account of the rank dependence structure given by the raw ensemble.

As indicated by its name, ECC has strong connections to copulas, particularly to the notions and results given by the raw ensemble.

To this end, we stick to the notation employed above and let $X_1, \ldots, X_L$ be discrete random variables that can take values in $\{x_1^1, \ldots, x_M^1\}, \ldots, \{x_1^L, \ldots, x_M^L\}$, respectively, where $x_1^1, \ldots, x_M^L$ are the $M$ raw ensemble forecasts for fixed $\ell \in \{1, \ldots, L\}$, that is, for fixed weather quantity, location and look-ahead time. For convenience, we assume that there are no ties among the corresponding values. Considering the multivariate random vector $X := (X_1, \ldots, X_L)$, the margins $X_1, \ldots, X_L$ are uniformly distributed with step $\frac{1}{M}$, and their corresponding univariate cdfs $F_{X_1}, \ldots, F_{X_L}$ hence take values in $I_M$, that is, $\text{Ran}(F_{X_1}) = \cdots = \text{Ran}(F_{X_L}) = I_M$. Moreover, we have $\text{Ran}(H) = I_M$ for the multivariate cdf $H$ of $X$.

According to the multivariate discrete version of Sklar’s theorem tailored to the ECC framework here, compare Theorem 4.2, there exists a uniquely determined irreducible discrete copula $D : I_M^L \rightarrow I_M$ such that

$$H(u_1, \ldots, u_L) = D(F_{X_1}(u_1), \ldots, F_{X_L}(u_L))$$

for $(u_1, \ldots, u_L) \in \mathbb{R}_+^L$.

That is, the multivariate distribution is connected to its univariate margins via $D$. Following and generalizing the statistical interpretation of discrete copulas for the bivariate case by Mesiar (2005),

$$D \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) = \mathbb{P}(H \in [-\infty, u_1] \times \cdots \times [-\infty, u_L]),$$

where $u_1, \ldots, u_L \in \mathbb{R}$ such that $F_{X_1}(u_1) = \mathbb{P}(X_1 \leq u_1) = \frac{i_1}{M}, \ldots, F_{X_L}(u_L) = \mathbb{P}(X_L \leq u_L) = \frac{i_L}{M}$, that is, $D \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right) = \mathbb{P}(X_1 \leq u_1, \ldots, X_L \leq u_L)$.

To describe the discrete probability distribution of the random vector $X$, we set $\alpha_{i_1, \ldots, i_L} := \text{P}(X_1 = x_{i_1}^{(1)}), \ldots, X_L = x_{i_L}^{(L)}), \ldots, x_{(i_L)}^\ell, \ell \in \{1, \ldots, L\}$, denote the corresponding order statistics from samples describing the values of $X_1, \ldots, X_L$. Obviously, $\alpha_{i_1, \ldots, i_L} \in \{0, \frac{1}{M}\}$ for all $i_1, \ldots, i_L \in \{1, \ldots, M\}$. Hence, $a_{i_1, \ldots, i_L} := M\alpha_{i_1, \ldots, i_L} \in \{0, 1\}$ for $i_1, \ldots, i_L \in \{1, \ldots, M\}$.
Figure 1: Different ensemble prediction approaches comprising the (a) raw, (b) independently postprocessed and (c) ECC ensemble. First row: Scatterplots with corresponding marginal histograms of 24 hour ahead temperature forecasts (in °C) at Berlin and Hamburg, valid on 27 June 2010, 2:00 am local time, based on the 50-member ECMWF ensemble. The red dots show the forecasts of the corresponding ensemble members, and the verifying observation is indicated by the blue cross. Second row: Perspective plots of the corresponding empirical copulas. Third row: Contour plots of the corresponding empirical copulas. Fourth row: Corresponding Latin squares.
\{1, \ldots, M\}, A := (a_{i_1 \ldots i_L})_{i_1, \ldots, i_L = 1}^M is a permutation array, and
\[
\frac{1}{M} \sum_{j_1=1}^{i_1} \cdots \sum_{j_L=1}^{i_L} a_{j_1 \ldots j_L} = D \left( \frac{i_1}{M}, \ldots, \frac{i_L}{M} \right),
\]
which is in accordance to Theorem 3.2. Analogously, the same considerations hold for both the independently postprocessed ensemble consisting of the \(M\) samples \(\tilde{x}_1^L, \ldots, \tilde{x}_M^L, \ell \in \{1, \ldots, L\}\), and the ECC ensemble \(\hat{x}_1^L, \ldots, \hat{x}_M^L, \ell \in \{1, \ldots, L\}\), that is,
\[
\hat{H}(u_1, \ldots, u_L) = \hat{D}(\hat{F}_{\tilde{X}_1}^L(u_1), \ldots, \hat{F}_{\tilde{X}_L}^L(u_L)) \quad \text{for} \quad (u_1, \ldots, u_L) \in \mathbb{R}^L
\]
and
\[
\hat{H}(u_1, \ldots, u_L) = \hat{D}(\hat{F}_{\tilde{X}_1}^L(u_1), \ldots, \hat{F}_{\tilde{X}_L}^L(u_L)) \quad \text{for} \quad (u_1, \ldots, u_L) \in \mathbb{R}^L,
\]
in obvious notation. Although both the independently postprocessed and the ECC ensemble have the same marginal distributions, that is, \(F_{\tilde{X}_1}^L = F_{\tilde{X}_1}^L, \ldots, F_{\tilde{X}_L}^L = F_{\tilde{X}_L}^L\), as is illustrated by the marginal histograms in the first row of our example in Figure 1, they differ drastically in their multivariate rank dependence structure. Since the ECC ensemble is designed in the manner that it inherits the rank dependence pattern from the raw ensemble, the considerations above yield that \(D = \hat{D}\). Thus, the raw and the ECC ensemble are associated with the same irreducible multivariate discrete copula modeling the dependence. This is visualized in the second and third row of Figure 1, where the perspective and contour plots, respectively, of the empirical copulas linked to the different ensembles in our illustrative example are shown, both suitably indicating rather high dependence. On the other hand, the perspective and contour plots of the empirical copula associated with the independently postprocessed ensemble in the mid-panel of Figure 1 are not far away from those of the independence copula \(I\) introduced in Section 2. According to the equivalences in Section 3, the raw and the ECC ensemble are also related to the same Latin square of order \(M = 50\), as can be seen in the fourth row in Figure 1.

Hence, ECC indeed can be considered as a copula approach, as it comes up with a postprocessed, discrete \(L\)-dimensional distribution, which is by Theorem 4.2 constructed from the \(L\) univariate predictive cdfs \(F_{\tilde{X}_1}^L, \ldots, F_{\tilde{X}_L}^L\) obtained by the postprocessing and the empirical copula \(D\) induced by the unprocessed raw ensemble. Conversely, each multivariate distribution with fixed univariate margins yields a uniquely determined empirical copula \(\hat{D}\), which defines the rank dependence structure in our setting.

Although several multivariate copula-based methods for discrete data have been proposed, for example recently by Panagiotelis et al. (2012) using vine and pair copulas, we feel that our discrete copula approach still provides an appropriate and useful alternative to these methods. ECC is especially valuable when being faced with extremely high-dimensional data, as is the case in weather forecasting, where one has to deal with several millions of variables. Since its crucial reordering step is computationally non-expensive, one of the major advantages of ECC is that it practically comes for free, once the univariate postprocessing is done. However, BMA and EMOS as univariate postprocessing methods are already implemented efficiently in the R packages \texttt{ensembleBMA} and \texttt{ensembleMOS}, respectively, which are freely available at \url{http://cran.r-project.org}. Hence, with the discrete copula-based non-parametric ECC approach, we can circumvent the problems that arise when using parametric methods, such as computational unfeasibility. In addition, ECC offers a simple and intuitive, yet powerful technique that goes without complex modeling or sophisticated parameter fitting in multivariate copula models, which work well in comparably low dimensional settings (Möller et al., 2013; Schözel and Friederichs, 2008), but tend to fail in very high dimensions. The notion of discrete copulas arises naturally in the context of the ECC approach. Furthermore, as documented in Section 4.4 in Schefzik et al. (2013), the discrete copula notion presented in the paper at hand can be interpreted as an overarching concept and theoretical frame not only for ECC, but also for other ensemble postprocessing methods that have recently appeared in the meteorological literature, and applies in other settings as well.
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