The characteristic cycle and the singular support of a constructible sheaf

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Abstract

We define the characteristic cycle of an étale sheaf as a cycle on the cotangent bundle of a smooth variety in positive characteristic using the singular support recently defined by Beilinson. We prove a formula à la Milnor for the total dimension of the space of vanishing cycles and an index formula computing the Euler-Poincaré characteristic, generalizing the Grothendieck-Ogg-Shafarevich formula to higher dimension.

An essential ingredient of the construction and the proof is a partial generalization to higher dimension of the semi-continuity of the Swan conductor due to Deligne-Laumon. In the proof of the index formula, we use a description of the characteristic cycle in terms of ramification theory, established earlier for surfaces.

As is observed by Deligne in [14], a strong analogy between the wild ramification of ℓ-adic sheaf in positive characteristic and the irregular singularity of partial differential equation on a complex manifold suggests to define the characteristic cycle of an $F_\ell$-sheaf as a cycle on the cotangent bundle of a smooth variety in positive characteristic $p \neq \ell$.

Recently, Beilinson [7] defined the singular support as a closed conical subset of the cotangent bundle that controls the local acyclicity of morphisms. We define the characteristic cycle as a $\mathbb{Z}_p[[1]]$-linear combination of its irreducible components, characterized by a Milnor formula

$$\dim \text{tot } \phi_u(j^*F, f) = (\text{Char } \mathcal{F}, df)_{T^*W,u}$$

for the total dimension of the space of vanishing cycles. The Milnor formula proved by Deligne [11] is the case where the sheaf is constant.

We also prove an index formula

$$\chi(X, \mathcal{F}) = (\text{Char } \mathcal{F}, T_X^*X)_{T^*X}.$$ computing the Euler-Poincaré characteristic. The Grothendieck-Ogg-Shafarevich formula [17] is the case where the variety is a curve. Roughly speaking, we carry out the program described in [14]. Some of the key arguments in a previous article [30] where we studied sheaves on surfaces are generalized to higher dimension.

To define the characteristic cycle, it suffices to determine the coefficient of each irreducible component of the singular support. We do this by imposing the Milnor formula (3.11) for morphisms defined by pencils by choosing an embedding to a projective space.
To prove that the coefficients are independent of the choice and that the characteristic cycle satisfies the Milnor formula (3.11) in general, we use the continuity Proposition 1.18 of the total dimension of the space of vanishing cycles. This is a partial generalization to higher dimension of the semi-continuity of Swan conductor [25].

The index formula Theorem 4.21 computing the Euler-Poincaré characteristic is deduced from the compatibility Theorem 4.4 [14, 2e Conjecture, p. 10] of the construction of characteristic cycles with the pull-back by properly $C$-transversal morphisms (Definition 4.1) for the singular support $C$. The compatibility Theorem 4.4 implies a description Theorem 4.6 of the characteristic cycle in terms of ramification theory [1], [29]. In the tamely ramified case, the description has been proved by a different method in [31]. The compatibility Theorem 4.4 is proved using the special case of Theorem 4.6 for surfaces proved earlier in [30, Proposition 3.20].

The results from Section 1 necessary for the definition of characteristic cycle Theorem 3.4 in Section 3 are the continuity Proposition 1.18 together with an example of flat function Lemma 1.3 and an elementary fact in Lemma 1.2.3 to deduce an equality of flat functions. In the proof of Theorem 4.4 and consequently of Theorem 4.21 in Section 4, the formalism of vanishing cycles with general base scheme [21], [27] studied in Section 1 plays an essential role. The contents of Section 5 where we establish a characterization Proposition 5.6 of the singular support in terms of the $\mathcal{F}$-transversality introduced in Definition 5.1 depends only on the first two subsections in Section 2 where we recall basic properties of the singular support from [7].

We describe briefly the content of each section. In Section 1.1, we introduce and study flat functions on a scheme quasi-finite over a base scheme, used to formulate the partial generalization of the semi-continuity of Swan conductor to higher dimension. After briefly recalling the generalization of the formalism of vanishing cycles with general base scheme and its relation with local acyclicity, we recall from [27] basic properties Proposition 1.8 in the case where the locus of non local acyclicity is quasi-finite, in Section 1.2. We recall and reformulate the semi-continuity of Swan conductor from [25] using the formalism of vanishing cycles with general base scheme and give a partial generalization to higher dimension in Section 1.3.

We briefly recall definitions and results from [7] in Section 2. We give a description Proposition 2.29 of the singular support in terms of ramification theory using a characterization Corollary 2.18 of the singular support given in Section 2.3.

We define the characteristic cycle as characterized by the Milnor formula (3.11) in Section 3.2. We fix some terminology and notation to formulate the Milnor formula in Section 3.1. We first define a candidate by imposing Milnor formulas for morphisms defined by pencils for a fixed embedding to a projective space. Then, we prove that the definition is independent of the choice of immersion to a projective space by establishing a stability of vanishing cycles using the continuity Proposition 1.18. We establish some elementary properties of characteristic cycles in Section 3.3.

We state the compatibility Theorem 4.4 of the construction of characteristic cycles with properly $C$-transversal morphisms for the singular support $C$ in Section 4.1. We also state a description of the characteristic cycle Theorem 4.6 in terms of ramification theory. In Section 4.5, we prove Theorem 4.4 using the constructions in Sections 4.3 and 4.4 and we deduce Theorem 4.6 from Theorem 4.4. In Section 4.2, we study the dimension of the fiber product appearing in the definition Definition 4.1 of proper $C$-transversality. Finally in Section 4.6, we deduce an index formula Theorem 4.21 computing the Euler number.
from Theorem 4.4.

We introduce the notion of $F$-transversality in Definition 5.1 using a canonical morphism (5.2) and establish a characterization Proposition 5.6 of the singular support in terms of the $F$-transversality in Section 5.

The author thanks Pierre Deligne for sending him an unpublished notes [14]. This article is the result of an attempt to understand its contents. The author thanks Luc Illusie for sending him an unpublished preprint [22] and for the introduction to the formalism of generalized vanishing cycles. The author thanks Alexander Beilinson for the preprint [7], an earlier suggestion of use of Radon transform [8] and for discussions. The author thanks Ofer Gabber for suggesting an improvement on the statement of Proposition 5.6.

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1 Vanishing topos and the semi-continuity of the Swan conductor

1.1 Calculus on vanishing topos

Let \( f : X \to S \) be a morphism of schemes. For the definition of the vanishing topos \( X \times_S S \) and the morphisms

\[
\begin{array}{ccc}
X & \overset{\Psi_f}{\longrightarrow} & X \times_S S \\
\downarrow & & \downarrow p_2 \\
S & \overset{p_1}{\longrightarrow} & X
\end{array}
\]

of toposes, we refer to [21, 1.1, 4.1, 4.3] and [22, 1.1]. For a geometric point \( x \) of a scheme \( X \), we assume in this article that the residue field of \( x \) is a separable closure of the residue field at the image of \( x \) in \( X \), if we do not say otherwise explicitly.

For a point \( s \in S \), let \( S^h_s = \text{Spec} \mathcal{O}^h_{S,s} \) denote the henselization. Then, in

\[
s \times_S S \leftarrow s \times_S S^h_s \longrightarrow S^h_s
\]

the first canonical morphism is an isomorphism by [21, Proposition 1.11] and the second projection is also an isomorphism by the definition [21, 1.1] of the oriented product. Similarly and more generally, for an extension \( L \) of the residue field \( k(s) \) and \( t = \text{Spec} L \), the vanishing topos \( t \times_S S \) is canonically identified with the spectrum \( \text{Spec} A \) of the unramified extension \( A \) of \( \mathcal{O}^h_{S,s} \) with residue field the separable closure of \( k(s) \) in \( L \). In particular, for a geometric point \( x \) of \( X \), the fiber of \( p_1 : X \times_S S \to X \) at \( x \) is the vanishing topos \( x \times_S S \) and is canonically identified with the strict localization \( S_{(x)} \) at the geometric point \( s = f(x) \) of \( S \) defined by the image of \( x \) (cf. [22, (1.8.2)]).

A point on the topos \( X \times_S S \) is defined by a triple denoted \( x \leftarrow t \) consisting of a geometric point \( x \) of \( X \), a geometric point \( t \) of \( S \) and a specialization \( s = f(x) \leftarrow t \) namely a geometric point \( S_{(x)} \leftarrow t \) of the strict localization lifting \( S \leftarrow t \). The fiber of the canonical morphism \( X \times_S S \to S \times_S S \) at a point \( s \leftarrow t \) is canonically identified with the geometric fiber \( X_s \). The fiber products \( X_{(x)} \times_{S_{(x)}} S_{(t)} \) and \( X_{(x)} \times_{S_{(x)}} t \) are called the Milnor tube and the Milnor fiber respectively.

For a commutative diagram

\[
\begin{array}{ccc}
X & \overset{f}{\longrightarrow} & Y \\
p \downarrow & & \downarrow g \\
S & \overset{g}{\longrightarrow} & Y
\end{array}
\]

do morphisms of schemes, the morphism \( \overset{g}{\longrightarrow} : X \times_Y Y \to X \times_S S \) is defined by functoriality and we have a canonical isomorphism \( \Psi_p \to \overset{g}{\longrightarrow} \circ \Psi_f \). On the fibers of a geometric point \( x \) of \( X \), the morphism \( \overset{g}{\longrightarrow} \) induces a morphism

\[
g_{(x)} : Y_{(y)} = x \times_Y Y \to S_{(x)} = x \times_S S
\]

on the strict localizations at \( y = f(x) \) and \( s = p(x) \) [22, (1.7.3)]. In particular for \( Y = X \), we have a canonical isomorphism \( \overset{\Psi_p}{\longrightarrow} \to \overset{g}{\longrightarrow} \circ \Psi_{\text{ld}} \).
Let $\Lambda$ be a finite local ring with residue field of characteristic $\ell$ invertible on $S$. Let $D^+(-)$ denote the derived category of complexes of $\Lambda$-modules bounded below and let $D^b(-)$ denote the subcategory consisting of complexes with bounded cohomology. In the following, we assume that $S$ and $X$ are quasi-compact and quasi-separated. We say that an object of $D^b(X_{\neq S}\to S)$ is constructible if there exist locally finite partitions $X = \bigsqcup_{x} X_{x}$ and $S = \bigsqcup_{\beta} S_{\beta}$ by locally closed subschemes such that the restrictions to $X_{x} \times_{S} S_{\beta}$ of cohomology sheaves are locally constant and constructible [22, 1.3].

Let $D^b_c(-)$ denote the subcategory of $D^b(-)$ consisting of constructible objects and let $D_{ctf}(-) \subset D^b_c(-)$ denote the subcategory of $D^b(-)$ consisting of objects of finite tor-dimension. If $\Lambda$ is a field, we have $D_{ctf}(-) = D^b_c(-)$.

We canonically identify a function on the underlying set of a scheme $X$ with the function on the set of isomorphism classes of geometric points $x$ of $X$. Similarly, we call a function on the set of isomorphism classes of points $x \leftarrow t$ of $X \times_{S} S$ a function on $X \times_{S} S$. We say that a function on $X \times_{S} S$ is a constructible function if there exist locally finite partitions $X = \bigsqcup_{x} X_{x}$ and $S = \bigsqcup_{\beta} S_{\beta}$ as above such that the restrictions to $X_{x} \times_{S} S_{\beta}$ are locally constant.

For an object $K$ of $D_{ctf}(X_{\neq S}\to S)$, the rank function $\dim K_{x \leftarrow t}$ is defined as a constructible function on $X \times_{S} S$. If $\Lambda$ is a field, we have $\dim K_{x \leftarrow t} = \sum q (-1)^{q} \dim H^{q} K_{x \leftarrow t}$. In general, if $\Lambda_0$ denotes the residue field of $\Lambda$, we have $\dim_{\Lambda} K = \dim_{\Lambda_0} K \otimes_{\Lambda_0} \Lambda_0$.

**Definition 1.1.** Let $Z$ be a quasi-finite scheme of finite type over $S$ and let $\varphi: Z \to \mathbb{Q}$ be a function. We define the derivative $\delta(\varphi)$ of $\varphi$ as a function on $Z_{\neq S} \to S$ by

$$
\delta(\varphi)(x \leftarrow t) = \varphi(x) - \sum_{z \in Z(s)_{\neq S(s)} \leftarrow t} \varphi(z) \tag{1.3}
$$

where $s = f(x)$. If the derivative $\delta(\varphi)$ is 0 (resp. $\delta(\varphi) \geq 0$), we say that the function $\varphi$ is flat (resp. increasing) over $S$. If the morphism $f: Z \to S$ is finite, we define a function $f_{*}\varphi$ on $S$ by

$$
f_{*}\varphi(s) = \sum_{x \in Z_{s}} \varphi(x) \tag{1.4}
$$

If $\varphi$ is constructible, the function $f_{*}\varphi$ is also constructible.

**Lemma 1.2.** Let $S$ be a noetherian scheme, $Z$ a quasi-finite scheme of finite type over $S$ and $\varphi: Z \to \mathbb{Q}$ be a function.

1. Assume that $Z$ is étale over $S$. Then $\varphi$ is locally constant (resp. upper semi-continuous) if and only if it is flat over $S$ (resp. constructible and increasing over $S$).

2. The function $\varphi$ is constructible if and only if its derivative $\delta(\varphi): Z_{\neq S} \to \mathbb{Q}$ defined in (1.3) is constructible. Consequently, if $\varphi$ is flat over $S$, then $\varphi$ is constructible.

3. Assume that $\varphi$ is flat over $S$. Then, $\varphi = 0$ if and only if $\varphi(x) = 0$ for the generic point $x$ of every irreducible component of $Z$.

4. Assume that the morphism $f: Z \to S$ is finite. If $\varphi$ is constructible and is increasing over $S$, then the function $f_{*}\varphi$ on $S$ is upper semi-continuous. The function $\varphi$ is flat over $S$ if and only if $f_{*}\varphi$ is locally constant.
\textbf{Proof.} 1. Since the question is étale local on \( Z \), we may assume that \( Z \to S \) is an isomorphism. Then the assertion is clear.

2. Assume \( \delta(\varphi) \) is constructible. By noetherian induction, it suffices to show the following: For every geometric point \( t \) of \( S \) and the closure \( T \subset S \) of its image, there exists a dense open subset \( V \subset T \) such that \( \varphi \) is locally constant on \( Z \times_S V \). Replacing \( S \) by \( T \), it suffices to consider the case where \( t \) dominates the generic point of an irreducible scheme \( S \). For a geometric point \( x \) of \( Z \) above \( t \), we have \( \delta(\varphi)(x \leftarrow t) = 0 \). By further replacing \( X \) by an étale neighborhood of \( x \), it suffices to consider the case where \( Z \) is étale over \( S \) and \( \delta(\varphi) = 0 \). Then, by 1., \( \varphi \) is locally constant and hence constructible.

Assume \( \varphi \) is constructible. For a closed subset \( T \) of \( S \), the subtopos \( (Z - Z \times_S T) \times_S T \) is empty. Hence, by noetherian induction, it suffices to show the following: For every geometric point \( t \) of \( S \) and the closure \( T \subset S \) of its image, there exists a dense open subset \( V \subset T \) such that \( \delta(\varphi) \) is locally constant on \( (Z \times_S V) \times_S V \). Similarly as above, it suffices to consider the case where \( Z \) is étale over \( S \) and \( \varphi \) is locally constant. Then, by 1., \( \delta(\varphi) = 0 \) and is constructible.

3. By noetherian induction, a function flat over \( S \) is uniquely determined at the values of the generic points of irreducible components.

4. If \( f: Z \to S \) is finite, for a specialization \( s \leftarrow t \), we have \( \sum_{x \in Z} \delta(\varphi)(x \leftarrow t) = f_*\varphi(s) - f_*\varphi(t) = \delta(f_*\varphi)(s \leftarrow t) \). Hence we may assume \( Z = S \) and then the assertion is clear. \hfill \Box

We give an example of flat function. Let \( S \) be a locally noetherian scheme, \( X \) be a scheme of finite type over \( S \) and \( Z \subset X \) be a closed subscheme quasi-finite over \( S \). Let \( A \) be a complex of \( \mathcal{O}_X \)-modules such that the cohomology sheave \( \mathcal{H}^q(A) \) are coherent \( \mathcal{O}_X \)-modules supported on \( Z \) for all \( q \) and that \( A \) is of finite tor-dimension as a complex of \( \mathcal{O}_S \)-modules. For a geometric point \( z \) of \( Z \) and its image \( s \) in \( S \), let \( \mathcal{O}_{X,z} \) and \( \mathcal{O}_{S,s} \) denote the strict localizations and \( k(s) \) the separably closed residue field of \( \mathcal{O}_{S,s} \). Then, the \( \mathcal{O}_{X,z} \)-modules \( \text{Tor}_q^{\mathcal{O}_{S,s}}(A_z, k(s)) \) are of finite length and are 0 except for finitely many \( q \). We define a function \( \varphi_A: Z \to \mathbb{Z} \) by

\[
\varphi_A(z) = \sum_q (-1)^q \dim_{k(s)} \text{Tor}_q^{\mathcal{O}_{S,s}}(A_z, k(s)).
\]

\textbf{Lemma 1.3.} Let schemes \( Z \subset X \to S \) and a complex \( A \) be as above.

1. The function \( \varphi_A: Z \to \mathbb{Z} \) defined by (1.5) is flat over \( S \) and constructible.

2. Suppose that \( S \) and \( Z \) are integral and that the image of the generic point \( \xi \) of \( Z \) is the generic point \( \eta \) of \( S \). If \( A = \mathcal{O}_Z \), the value of the function \( \varphi_A \) at a geometric point of \( Z \) above \( \xi \) is the inseparable degree \([k(\xi): k(\eta)]_{\text{insep}}\).

\textbf{Proof.} 1. Since the assertion is étale local on \( Z \), we may assume that \( Z \) is finite over \( S \), that \( X \) and \( S \) are affine and that \( z \) is the unique point in the geometric fiber \( Z \times_S \text{Spec} \ k(s) \). Then, the complex \( Rf_*A \) is a perfect complex of \( \mathcal{O}_S \)-modules and \( \varphi_A(z) \) equal the rank of \( Rf_*A \). Hence, the assertion follows.

2. By the same argument as in the proof of 1., it is reduced to the case where \( S = \text{Spec} \ k(\eta) \) and \( Z = \text{Spec} \ k(\xi) \) and the assertion follows. \hfill \Box

We generalize the definition of derivative to functions on vanishing topos.
Definition 1.4. Let

\[ Z \xrightarrow{f} Y \]
\[ p \downarrow \quad g \]
\[ \quad S \]

be a commutative diagram of morphisms of schemes such that \( Z \) is quasi-finite over \( S \). Let \( \psi: Z \times_Y Y \rightarrow Q \) be a function such that \( \psi(x \leftarrow w) = 0 \) unless \( w \) is not supported on the image of \( f(x): Z(x) \rightarrow Y(y) \) where \( y = f(x) \). We define the derivative \( \delta(\psi) \) as a function on \( Z \times_S S \rightarrow Z \) by

\[ \delta(\psi)(x \leftarrow t) = \psi(x \leftarrow y) - \sum_{w \in Y(y) \times_S S(t)} \psi(x \leftarrow w) \]

where \( s = p(x) \). The sum on the right hand side is a finite sum by the assumption that \( Z \) is quasi-finite over \( S \) and the assumption on the support of \( \psi \). We say that \( \psi \) is flat over \( S \) if \( \delta(\psi) = 0 \).

If \( Z = Y \), we recover the definition (1.3) by applying (1.7) to the pull-back \( p_2^* \varphi: Z \times_Z Z \rightarrow Z \) by \( p_2: Z \times_Z Z \rightarrow Z \).

The following elementary Lemma will be used in the proof of a generalization of the continuity of the Swan conductor.

Lemma 1.5. Let the assumption on the diagram (1.6) be as in Definition 1.4 and let \( \varphi \) be a function on \( Z \). We define a function \( \psi \) on \( Z \times_Y Y \) by

\[ \psi(x \leftarrow w) = \sum_{z \in Z(x) \times Y(y) w} \varphi(z) \]

where \( y = f(x) \). Then the derivative \( \delta(\varphi) \) on \( Z \times_S S \) defined by (1.3) equals \( \delta(\psi) \) defined by (1.7).

Proof. It follows from \( \psi(x \leftarrow y) = \varphi(x) \) and \( Z(x) \times Y(y) t = \coprod_{w \in Y(y) \times_S S(t)} (Z(x) \times Y(y) w) \). Note that except for finitely many geometric points \( w \) of \( Y(y) \times_S S(t) \) those supported on the image of \( Z(x) \), the fiber \( Z(x) \times Y(y) w \) is empty.

1.2 Nearby cycles and the local acyclicity

For a morphism \( f: X \rightarrow S \), the morphism \( \Psi_f: X \rightarrow X \times_S S \) defines the nearby cycles functor \( R\Psi_f : D^+(X) \rightarrow D^+(X \times_S S) \). The canonical morphism \( p_1^* \rightarrow R\Psi_f \) of functors is defined by adjunction and by the isomorphism \( id \rightarrow p_1 \circ \Psi_f \). The cone of the morphism \( p_1^* \rightarrow R\Psi_f \) defines the vanishing cycles functor \( R\Phi_f : D^+(X) \rightarrow D^+(X \times_S S) \). If \( S \) is the spectrum of a henselian discrete valuation ring and if \( s, \eta \) denote its closed and generic points, we recover the classical construction of complexes \( \psi, \phi \) of nearby cycles and vanishing cycles as the restrictions to \( X_s \times_S \eta \) of \( R\Psi_f \) and \( R\Phi_f \) respectively.
Recall that \( f : X \to S \) is said to be locally acyclic relatively to \( \mathcal{K} \) \cite[Definition 2.12]{13} if the canonical morphism
\[
\mathcal{K}_x \longrightarrow R\Gamma(X(x) \times_{S(x)} t, \mathcal{K}|_{X(x) \times_{S(x)} t})
\]
is an isomorphism for every \( x \leftarrow t \). Recall that \( f : X \to S \) is said to be universally locally acyclic relatively to \( \mathcal{K} \), if for every morphism \( S' \to S \), its base change is locally acyclic relatively to the pull-back of \( \mathcal{K} \).

**Lemma 1.6** (cf. \cite[Proposition 7.6.2]{15}). Let \( f : X \to S \) be a morphism of finite type and let \( \mathcal{K} \in D^b(X) \) be a complex of finite tor-dimension.

1. Suppose that \( \mathcal{K} \) is of tor-amplitude \([a, b]\) and that \( f : X \to S \) is of relative dimension \( d \). Then, for points \( x \leftarrow t \) of \( X \times_S S \), the complex \( R\Gamma(X(x) \times_{S(x)} t, \mathcal{K}|_{X(x) \times_{S(x)} t}) \) of \( \Lambda \)-modules is of tor-amplitude \([a, b + d]\) and, for a \( \Lambda \)-module \( M \), the canonical morphism
\[
R\Gamma(X(x) \times_{S(x)} t, \mathcal{K}|_{X(x) \times_{S(x)} t}) \otimes^\Lambda_M \to R\Gamma(X(x) \times_{S(x)} t, \mathcal{K}|_{X(x) \times_{S(x)} t} \otimes^\Lambda_M M)
\]
is an isomorphism.

2. Let \( \Lambda_0 \) be the residue field of \( \Lambda \). Then, \( f : X \to S \) is locally acyclic (resp. universally locally acyclic) relatively to \( \mathcal{K} \) if and only if it is so relatively to \( \mathcal{K}_0 = \mathcal{K} \otimes^\Lambda \Lambda_0 \).

**Proof.** 1. By the assumption that \( f : X \to S \) is of finite type and of relative dimension \( d \), the functor \( R\Gamma(X(x) \times_{S(x)} t, \mathcal{K}) \) is of cohomological dimension \( \leq d \) by \cite[Colloraire 3.2]{3}. Hence, similarly as \cite[(4.9.1)]{12}, the canonical morphism (1.10) is an isomorphism. Thus, the complex \( R\Gamma(X(x) \times_{S(x)} t, \mathcal{K}|_{X(x) \times_{S(x)} t}) \) is of tor-amplitude \([a, b + d]\).

2. It suffices to show the assertion for local acyclicity. If the canonical morphism (1.9) is an isomorphism for \( x \leftarrow t \), then (1.9) for \( \mathcal{K}_0 \) is an isomorphism by the isomorphism (1.10) for \( M = \Lambda_0 \).

To show the converse, let \( I^\bullet \) be a filtration by ideals of \( \Lambda \) such that \( \text{Gr} \Lambda \) is a \( \Lambda_0 \)-vector space. Then, \( I^\bullet \) defines a filtration on \( \mathcal{K} \) and a canonical isomorphism \( \text{Gr} \mathcal{K} \to \mathcal{K}_0 \otimes_{\Lambda_0} \text{Gr} \Lambda \). Hence if (1.9) is an isomorphism for \( \mathcal{K}_0 \) then (1.9) for \( \mathcal{K} \) is an isomorphism. \( \square \)

We consider a commutative diagram
\[
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p & \downarrow{g} & \downarrow{\Psi_f} \\
S & & \Psi_p
\end{array}
\]
of schemes. The canonical isomorphism \( \Psi_p \circ \Psi_f \to \Psi_p \) induces an isomorphism of functors
\[
R\Psi_p \circ R\Psi_f \to R\Psi_p
\]

For an object \( \mathcal{K} \) of \( D^+(X \times_S Y) \) and a geometric point \( x \) of \( X \), the restriction of \( R\Psi_p \mathcal{K} \) on \( x \times_S S = S(x) \) for \( s = f(x) \) is canonically identified with \( R\Psi(x)_*(\mathcal{K}|_{Y(y)}) \) for \( y = f(x) \) in the notation of (1.2) by \cite[(1.9.2)]{22}. For the stalk at a point \( x \leftarrow t \) of \( X \times_S S \), this identification gives a canonical isomorphism
\[
R\Psi_p \mathcal{K}_{x \leftarrow t} \to R\Gamma(Y(y) \times_{S(y)} S(t), \mathcal{K}|_{Y(y) \times_{S(y)} S(t)}).
\]
For an object $K$ of $\mathcal{D}^+(X)$, (1.13) applied to $Y = X$ gives a canonical identification

\begin{equation}
R\Psi_f K_{x \times -} \to R\Gamma(X_\times \times_{S(t)} S(t), \mathcal{K}|_{X_\times \times_{S(t)} S(t)})
\end{equation}

with the cohomology of the Milnor tube [22, (1.1.15)].

A cartesian diagram

\[
\begin{array}{ccc}
X & \to & X_T \\
\downarrow f & & \downarrow f_T \\
S & \to & T
\end{array}
\]

of schemes defines a 2-commutative diagram

\[
\begin{array}{ccc}
X_T & \to & X_T \times_T T \\
\downarrow i & & \downarrow i \\
X & \to & X
\end{array}
\]

and the base change morphisms define a morphism

\[
\begin{array}{c}
\to i_p^* \\
\sim \downarrow \\
\to i (R\Psi_f) \\
\sim \downarrow \\
\to i (R\Phi_f)
\end{array}
\]

\begin{equation}
(1.15)
\end{equation}

of distinguished triangles of functors. For an object $K$ of $\mathcal{D}^+(X)$, we say that the formation of $R\Psi_f K$ commutes with the base change $T \to S$ if the middle vertical arrow defines an isomorphism $i (R\Psi_f)K \to (R\Psi_f) i^* K$.

For a point $x \leftarrow t$ of $X \times_S S$, if $T \subset S$ denotes the closure of the image of $t$ in $S$, the left square of (1.15) induces a commutative diagram

\begin{equation}
(1.16)
\end{equation}

\[
\begin{array}{ccc}
(p_1^* K)_{x \times -} = K_x & \to & R\Psi_f K_{x \times -} = R\Gamma(X_\times \times_{S(t)} S(t), \mathcal{K}|_{X_\times \times_{S(t)} S(t)}) \\
\downarrow & & \downarrow \\
R\Psi_f (\mathcal{K}|_{X_T})_{x \times -} = R\Gamma(\mathcal{K}|_{X_T} \times_{S(t)} t, \mathcal{K}|_{X_T \times_{S(t)} S(t)})
\end{array}
\]

The vertical arrow is the canonical morphism from the cohomology of the Milnor tube to that of the Milnor fiber and the slant arrow is the canonical morphism (1.9). Recall that we assume that the residue field of $t$ is a separable closure of the residue field at the image of $S_{(s)}$.

We interpret the local acyclicity in terms of vanishing topos.

**Lemma 1.7.** Let $f : X \to S$ be a morphism of schemes. Then, for an object $K$ of $\mathcal{D}^+(X)$, the conditions (1) and (2) in 1. and 2. below are equivalent to each other respectively.

1. (1) For every point $x \leftarrow t$ of $X \times_S S$, the vertical arrow in (1.16) is an isomorphism.
2. (2) The formation of $R\Psi_f K$ commutes with finite base change $T \to S$.

2. ([19, Corollaire 2.6]) (1) The morphism $f : X \to S$ is (resp. universally) locally acyclic relatively to $K$.

(2) The canonical morphism $p_1^* K \to R\Psi_f K$ is an isomorphism and the formation of $R\Psi_f K$ commutes with finite (resp. arbitrary) base change $T \to S$.
Structibility by \([27, 8.1, 10.5]\) as noted after \([22, \text{Theorem 1.3.1}]\).

By the strict localization \(S\) where the condition (1) implies the condition (2).

2. The condition (1) is equivalent to that the slant arrow in (1.16) is an isomorphism for every geometric point \(x \leftarrow t\) of \(X \times S\) (resp. after any base change \(T \rightarrow S\)). Hence the condition (2) implies the condition (1) by (2)\(\Rightarrow\)(1) in 1. and the commutativity of the diagram (1.16).

Conversely, by \([19, \text{Corollaire 2.6}]\), if the condition (1) is satisfied, the formation of \(Rf_{(x)*}(\mathcal{K}|_{X(x)})\) commutes with finite base change for every geometric point \(x\) of \(X\) where \(f_{(x)}: X(x) \rightarrow S(s)\) is the morphism on the strict localizations induced by \(f\). Since the vertical arrow in (1.16) is the stalk \(Rf_{(x)*}(\mathcal{K}|_{X(x)})_t \rightarrow R(f_T)_*(\mathcal{K}|_{(X_T)_T})_t\) of the base change morphism, the condition (1) implies the condition (2) by (1)\(\Rightarrow\)(2) in 1. and the commutativity of the diagram (1.16).

Proposition 1.8. Let \(f: X \rightarrow S\) be a morphism of finite type of schemes and let \(Z \subset X\) be a closed subscheme quasi-finite over \(S\). Let \(\mathcal{K}\) be an object of \(D^b_c(X)\) such that the restriction of \(f: X \rightarrow S\) to the complement \(X - Z\) is (resp. universally) locally acyclic relatively to the restriction of \(\mathcal{K}\).

1. (cf. \([27, \text{Proposition 6.1}]\)) \(R\Psi_f\mathcal{K}\) and \(R\Phi_f\mathcal{K}\) are constructible and their formations commute with finite (resp. arbitrary) base change. The constructible object \(R\Phi_f\mathcal{K}\) is supported on \(Z \times S\).

2. Let \(x\) be a geometric point of \(X\) and \(s = f(x)\) be the geometric point of \(S\) defined by the image of \(x\) by \(f\). Let \(t\) and \(u\) be geometric points of \(S(s)\) and \(t \leftarrow u\) be a specialization. Then, there exists a distinguished triangle

\[
(1.17) \quad R\Psi_f\mathcal{K}_{x \leftarrow t} \rightarrow R\Psi_f\mathcal{K}_{x \leftarrow u} \rightarrow \bigoplus_{z \in (Z \times X(x)) \times S(s), t} R\Phi_f\mathcal{K}_{z \leftarrow u}
\]

where \(R\Psi_f\mathcal{K}_{x \leftarrow t} \rightarrow R\Psi_f\mathcal{K}_{x \leftarrow u}\) is the cospecialization.

The commutativity of the formation of \(R\Psi_f\mathcal{K}\) with any base change implies its constructibility by \([27, 8.1, 10.5]\) as noted after \([22, \text{Theorem 1.3.1}]\).

Proof. 1. The constructibility is proved by taking a compactification in \([27, \text{Proposition 6.1}]\). The commutativity with base change is proved similarly by taking a compactification and applying the proper base change theorem.

The assertion on the support of \(R\Phi_f\mathcal{K}\) follows from Lemma 1.7.2 (1)\(\Rightarrow\)(2).

2. Let \(t\) and \(u\) be geometric points of \(S(s)\) and \(t \leftarrow u\) be a specialization. By replacing \(S\) by the strict localization \(S(s)\) and shrinking \(X\), we may assume that \(S = S(s)\), that \(X\) is affine and that \(Z = Z \times X(x)\) is finite over \(S\).

We consider the diagram

\[
\begin{array}{ccc}
S & \xrightarrow{j} & S(s) \\
\downarrow{i_s} & & \downarrow{k} \\
S & \xleftarrow{i_t} & u
\end{array}
\]

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and let the morphisms obtained by the base change \( X \to S \) denoted by the same letters, by abuse of notation. Similarly as the sliced vanishing cycles in the proof of [27, Proposition 6.1], we consider an object \( \Phi_{t-u} \mathcal{K} \) on \( X \times S \) fitting in the distinguished triangle \( j^* \mathcal{K} \to Rk_*(j \circ k)^* \mathcal{K} \to \Phi_{t-u} \mathcal{K} \to \). Since the formation of \( R\Phi_j \mathcal{K} \) commutes with finite base change by 1., we have a distinguished triangle (1.17) with the third term replaced by \( \Delta_x = (Rj_* \Phi_{t-u} \mathcal{K})_x \). Further, the third term itself is canonically isomorphic to the direct sum of \( \Delta_z = (\Phi_{t-u} \mathcal{K})_z \) for \( z \in Z_i \).

Since \( R\Phi_j \mathcal{K} \) is acyclic outside \( Z \times SS \) by 1., the canonical morphisms \( i^*_s j^* \mathcal{K} \to i^*_s Rj_* j^* \mathcal{K} \) and \( i^*_s \mathcal{K} \to i^*_s R(j \circ k)_s (j \circ k)^* \mathcal{K} \) are isomorphisms on \( X_s - Z_s \). Hence, the restriction \( i^*_s Rj_* \Phi_{t-u} \mathcal{K} \) is acyclic on \( X_s - Z_s \). Similarly, the restriction \( i^*_t \Phi_{t-u} \mathcal{K} \) is acyclic on \( X_t - Z_t \).

We take a compactification \( \bar{X} \) of \( X \) and an extension \( \bar{K} \) of \( K \) to \( \bar{X} \). Define \( \Phi_{t-u} \bar{K} \) on \( \bar{X} \times S \) similarly as \( \Phi_{t-u} \mathcal{K} \) and set \( Y = \bar{X} - X \). By the proper base change theorem, the canonical morphisms \( R\Gamma(\bar{X}_s, i^*_s Rj_* \Phi_{t-u} \bar{K}) \leftarrow R\Gamma(\bar{X}, Rj_* \Phi_{t-u} \bar{K}) \to R\Gamma(\bar{X} \times S S(d), \Phi_{t-u} \bar{K}) \to R\Gamma(\bar{X}_t, i^*_t \Phi_{t-u} \bar{K}) \) are isomorphisms and similarly for the restrictions to \( Y \).

Hence, we obtain a commutative diagram

\[
\begin{array}{ccc}
\Delta_x = & R\Gamma(Z_s, i^*_s Rj_* \Phi_{t-u} \mathcal{K}) & \longrightarrow R\Gamma_c(X_s, i^*_s Rj_* \Phi_{t-u} \mathcal{K}) \\
& \downarrow & \downarrow \\
\bigoplus_{z \in Z_i} \Delta_z = & R\Gamma(Z_t, i^*_t \Phi_{t-u} \mathcal{K}) & \longrightarrow R\Gamma_c(X_t, i^*_t \Phi_{t-u} \mathcal{K})
\end{array}
\]

of isomorphisms and the assertion follows.

\[ \square \]

**Corollary 1.9.** We keep the assumptions in Proposition 1.8 and let \( x \) be a geometric point of \( X \) and \( s = f(x) \) be the geometric point of \( S \) defined by the image of \( x \) by \( f \) as in Proposition 1.8.2. Then, the restriction of \( R\Psi_j \mathcal{K} \) on \( x \times S S = S(d) \) is locally constant and constructible outside the image of the finite scheme \( Z \times X X(x) \) for every \( q \).

**Proof.** Let \( t \) and \( u \) be geometric points of \( S(d) \) not in the image of \( Z \times X X(x) \) and \( t \leftarrow u \) be a specialization. Since \( R\Psi_j \mathcal{K} \) is constructible, it suffices to show that the cospecialization morphism \( R\Psi_j \mathcal{K}_{x=t} \to R\Psi_j \mathcal{K}_{x-u} \) is an isomorphism. Then by the assumption on the local acyclicity, the complex \( \Phi_{t-u} \mathcal{K} \) in the proof of Proposition 1.8.2 is acyclic. Hence the assertion follows from (1.17) and the isomorphism \( R\Phi_j \mathcal{K}_{x=t} \to (\Phi_{t-u} \mathcal{K})_x \).

\[ \square \]

**Corollary 1.10.** We keep the assumptions in Proposition 1.8. We further assume that \( \mathcal{K} \) is of finite tor-dimension and that \( S \) and \( X \) are quasi-compact and quasi-separated.

1. The complexes \( R\Psi_j \mathcal{K} \) and \( R\Phi_j \mathcal{K} \) are of finite tor-dimension. Consequently, the functions \( \dim R\Psi_j \mathcal{K} \) and \( \dim R\Phi_j \mathcal{K} \) are defined and constructible.

2. Define a constructible function \( \delta_{\mathcal{K}} \) on \( X \times S \) supported on \( Z \times S \) by \( \delta_{\mathcal{K}}(x \leftarrow t) = \dim R\Phi_j \mathcal{K}_{x=t} \). Assume that \( R\Phi_j \mathcal{K} \) is acyclic except at degree 0. Then, we have \( \delta_{\mathcal{K}} \geq 0 \) and the equality \( \delta_{\mathcal{K}} = 0 \) is equivalent to the condition that the morphism \( f \) is (resp. universally) locally acyclic relatively to \( \mathcal{K} \).

**Proof.** 1. By Proposition 1.8.1, Lemma 1.7.1 and Lemma 1.6.1, the complex \( R\Psi_j \mathcal{K} \) is of finite tor-dimension and hence \( R\Phi_j \mathcal{K} \) is also of finite tor-dimension. Since they are constructible by Proposition 1.8.1, the functions \( \dim R\Psi_j \mathcal{K} \) and \( \dim R\Phi_j \mathcal{K} \) are defined and constructible.

2. The positivity \( \delta_{\mathcal{K}} \geq 0 \) follows from the assumption that \( R\Phi_j \mathcal{K} \) is acyclic except at degree 0. Further the equality \( \delta_{\mathcal{K}} = 0 \) is equivalent to \( R\Phi_j \mathcal{K} = 0 \). Since the formation
of $R\Psi_fK$ commutes with finite (resp. arbitrary) base change by Proposition 1.8.1, it is further equivalent to the condition that the morphism $f$ is (resp. universally) locally acyclic relatively to $K$ by Lemma 1.7.2.

**Lemma 1.11.** The assumption that $R\Phi fK$ is acyclic except at degree 0 in Corollary 1.10.2 is satisfied if the following conditions are satisfied: The scheme $S$ is noetherian, the restriction of $f: X \to S$ to $X - Z$ is universally locally acyclic relatively to the restriction of $K$ and the following condition ($P$) is satisfied.

($P$) For every morphism $T \to S$ from the spectrum $T$ of a discrete valuation ring, the pull-back of $K[1]$ to $X_T$ is perverse.

**Proof.** Let $x \leftarrow t$ be a point of $X \times_S S$ and let $T \to S$ be a morphism from the spectrum $T$ of a discrete valuation ring such that the image of $T \to S$ is the same as that of $\{f(x), t\}$. Since the formation of $R\Phi fK$ commutes with arbitrary base change by Proposition 1.8.1, the base change morphism $R\Phi fK_{x \leftarrow t} \to R\Phi f_{p,T}(K|_{X_T})_{x \leftarrow t}$ is an isomorphism. The complex $R\Phi f_{p,T}(K|_{X_T})$ is a perverse sheaf by the assumption ($P$) and by the theorem of Gabber [20, Corollaire 4.6]. Since $R\Phi f_{p,T}(K|_{X_T})$ vanishes outside the closed fiber $Z_s$, this implies that the complex $R\Phi fK$ is acyclic except at degree 0.

The condition ($P$) is satisfied if $f: X \to S$ is smooth of relative dimension $d$ and $K = j!*F[d]$ for the open immersion $j: U \to X$ of the complement $U = X - D$ of a Cartier divisor $D$ and a locally constant sheaf $F$ on $U$.

We study the local acyclicity of a complex on the vanishing topos.

**Definition 1.12.** Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p \downarrow & & \downarrow g \\
S & \xleftarrow{p} & Y
\end{array}
$$

be a commutative diagram of schemes and $K$ be a constructible complex of $\Lambda$-modules on $X \times_Y Y$. We say that $g: Y \to S$ is locally acyclic relatively to $K$ if for every point $x \leftarrow t$ of $X \times_S S$ and for $y = f(x)$ and $s = p(x)$, the canonical morphism

$$K_{x \leftarrow y} = R\Gamma(Y(y), K) \to R\Gamma(Y(y) \times_S t, K)$$

is an isomorphism where the fiber $x \leftarrow y$ of $p_1: X \times_S S \to X$ is identified with $Y(y)$.

**Lemma 1.13.** Let

$$
\begin{array}{ccc}
X & \xrightarrow{f} & Y \\
p \downarrow & & \downarrow g \\
S & \xleftarrow{p} & Y
\end{array}
$$

be a commutative diagram of schemes and $K$ be a constructible complex of $\Lambda$-modules on $X$. Assume that the formation of $R\Psi fK$ commutes with finite base change. Then, the following conditions are equivalent:

1. $p: X \to S$ is locally acyclic relatively to $K$.
2. $g: Y \to S$ is locally acyclic relatively to $R\Psi fK$. 

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Proof. For a point \( x \leftarrow t \) of \( X \times_S S \), set \( y = f(x) \) and \( s = p(x) \) and consider the commutative diagram

\[
\begin{array}{c}
\mathcal{K}_x \\
\downarrow \\
R\Psi_f \mathcal{K}_x \leftarrow y \\
\end{array} \longrightarrow \begin{array}{c}
R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{K}) \\
\uparrow \\
R\Gamma(Y_{(y)} \times_{S_{(s)}} t, R\Psi_f \mathcal{K}).
\end{array}
\]

The left vertical arrow is an isomorphism since \( y = f(x) \). The condition (1) (resp. (2)) is equivalent to that the top (resp. bottom) horizontal arrow is an isomorphism for every \( x \leftarrow t \).

Let \( T \subset S \) denote the reduced closed subscheme whose underlying set is the closure of the image of \( t \) and let \( f_T : X_T \rightarrow Y_T \) denote the base change of \( f \) by the closed immersion \( T \rightarrow S \). Then the right vertical arrow is identified with the base change morphism \( R\Gamma(Y_{(y)} \times_{S_{(s)}} t, R\Psi_f \mathcal{K}) \rightarrow R\Gamma(Y_{T,(y)} \times_{T_{(s)}} t, R\Psi_f \mathcal{K}|_{X_T}) = R\Gamma(X_{(x)} \times_{S_{(s)}} t, \mathcal{K}) \). By the assumption that the formulation of \( R\Psi_s \mathcal{K} \) commutes with the finite base change, the right vertical arrow is an isomorphism for every \( x \leftarrow t \). Hence the assertion follows. \( \square \)

### 1.3 Semi-continuity of the Swan conductor

In this subsection, we assume that \( \Lambda \) is a field for simplicity. If \( \Lambda \) is not a field, the same results hold without modifications for constructible complexes of \( \Lambda \)-modules of finite tor-dimension, by considering the tensor product with the residue field.

We reformulate the main result of Deligne-Laumon in [25] in Proposition 1.14 below. Let \( f : X \rightarrow S \) be a flat morphism of relative dimension 1 and let \( Z \subset X \) be a closed subscheme. Assume that \( X - Z \) is smooth over \( S \) and that \( Z \) is quasi-finite over \( S \). Let \( \mathcal{K} \) be a constructible complex of \( \Lambda \)-modules on \( X \) such that the restrictions of the cohomology sheaves on \( X - Z \) are locally constant.

Let \( s \rightarrow S \) be a geometric point with algebraically closed residue field. For a geometric point \( x \) of \( Z \) above \( s \), the normalization of the strict localization \( X_{s,(x)} \) is the disjoint union \( \coprod_i X_i \) of finitely many spectra of strictly local discrete valuation rings with residue fields equal to that of \( s \). Let \( K_i \) denote the fraction field of \( X_i \) for each component \( i \) and let \( \bar{\eta}_i = \text{Spec } K_i \rightarrow X_i \) denote the geometric generic point defined by a separable closure. For a \( \Lambda \)-representation \( V \) of the absolute Galois group \( G_{K_i} = \text{Gal}(\bar{K}_i/K_i) \), the Swan conductor \( Sw_{K_i} V \in \mathbb{N} \) is defined [25] and the total dimension is defined as the sum \( \dim \text{tot}_{K_i} V = \dim V + Sw_{K_i} V \).

The stalk \( H^q(K)_{\bar{\eta}_i} \) for each integer \( q \) defines a \( \Lambda \)-representation of the absolute Galois group \( G_{K_i} \) and hence the total dimension \( \dim \text{tot}_{K_i} \mathcal{K}_{\bar{\eta}_i} \) is defined as the alternating sum \( \sum_q (-1)^q \dim \text{tot}_{K_i} H^q(K)_{\bar{\eta}_i} \). We define the Artin conductor by

\[
(1.18) \quad a_x(K|_{X_s}) = \sum_i \dim \text{tot}_{K_i} \mathcal{K}_{\bar{\eta}_i} - \dim \mathcal{K}_x.
\]

We define a function \( \varphi_\mathcal{K} \) on \( X \) supported on \( Z \) by

\[
(1.19) \quad \varphi_\mathcal{K}(x) = a_x(K|_{X_s})
\]

for \( s = f(x) \). The derivative \( \delta(\varphi_\mathcal{K}) \) on \( X \times_S S \) is defined by (1.3).
Proposition 1.14 ([25, Théorème 2.1.1]). Let $S$ be a noetherian scheme and $f : X \to S$ be a flat morphism of relative dimension 1. Let $Z \subset X$ be a closed subscheme quasi-finite over $S$ such that $U = X - Z$ is smooth over $S$. Let $\mathcal{K}$ be a constructible complex of $\Lambda$-modules on $X$ such that the restrictions of cohomology sheaves on $X - Z$ are locally constant.

1. The objects $R\Psi_f \mathcal{K}$ and $R\Phi_f \mathcal{K}$ are constructible and their formations commutes with any base change. The function $\varphi_{\mathcal{K}}$ (1.19) satisfies

$$\dim R\Phi_f \mathcal{K}_{x - t} = \delta(\varphi_{\mathcal{K}})(x \leftarrow t)$$

and is constructible.

2. Assume $\mathcal{K} = j_* \mathcal{F}[1]$ for the open immersion $j : U = X - Z \to X$ and a locally constant constructible sheaf $\mathcal{F}$ on $U$ and that $Z$ is flat over $S$. Then, we have $\delta(\varphi_{\mathcal{K}}) \geq 0$. The function $\varphi_{\mathcal{K}}$ is flat over $S$ if and only if $f : X \to S$ is universally locally acyclic relatively to $\mathcal{K} = j_* \mathcal{F}[1]$.

Proof. We sketch and/or recall an outline of proof with some simplifications.

1. The constructibility of $R\Psi_f \mathcal{K}$ and $R\Phi_f \mathcal{K}$ and the commutativity with base change follow from Proposition 1.8.1 and the local acyclicity of smooth morphism.

By devissage, it suffices to show (1.20) in the case where $\mathcal{K} = j_* \mathcal{F}[1]$ for the open immersion $j : U = X - Z \to X$ and a locally constant sheaf $\mathcal{F}$ on $U$. By the commutativity with base change, the equality (1.20) is reduced to the case where $S$ is the spectrum of a complete discrete valuation ring with algebraically closed residue field. Further by base change and the normalization, we may assume that $X$ is normal and that its generic fiber is smooth.

In this case, (1.20) was first proved in [25], under an extra assumption that $X$ is smooth, by constructing a good compactification using a deformation argument. Later it was reproved together with a generalization in [24, Remark (4.6)] using the semi-stable reduction theorem of curves without using the deformation argument.

Since $R\Phi_f \mathcal{K}$ is constructible, the equality (1.20) implies that the function $\delta(\varphi_{\mathcal{K}})$ on $Z \times_S S$ is constructible. Hence $\varphi_{\mathcal{K}}$ on $Z$ is constructible by Lemma 1.2.2.

2. The complex $R\Phi_f \mathcal{K}$ is acyclic except at degree 0 by Lemma 1.11. Hence the assertions follow from the equality (1.20) and Corollary 1.10.2.

Corollary 1.15. Assume further that $Z$ is finite and flat over $S$ and that $\mathcal{K} = j_* \mathcal{F}$ for a locally constant sheaf $\mathcal{F}$ on $U$. Then, the function $f_* \varphi_{\mathcal{K}}$ (1.4) on $S$ is lower semi-continuous. The function $f_* \varphi_{\mathcal{K}}$ is locally constant if and only if $f : X \to S$ is universally locally acyclic relatively to $\mathcal{K} = j_* \mathcal{F}$.

Proof. It follows from Proposition 1.14, Lemma 1.2.4 and Corollary 1.10.2. The lower semi-continuity replaces the upper semi-continuity because of the shift [1] in Proposition 1.14.2.

We give a slight generalization of Proposition 1.14. Let

$$Z \xrightarrow{\iota} X \xrightarrow{f} Y \xrightarrow{g} S$$

(1.21)
be a commutative diagram of morphisms of finite type of noetherian schemes such that $g: Y \to S$ is flat of relative dimension 1 and that $Z \subset X$ is a closed subscheme quasi-finite over $S$. For geometric point $x \to X$, we set $y = f(x)$ and $s = p(x)$ and define $T(x) \subset Y$ to be the image of the finite scheme $Z \times_X X_y$ over $S_y$ by $f_y: X_y \to Y_y$. Assume that, for every geometric point $x \to X$, the complement $Y_y - T(x)$ is essentially smooth over $S_y$.

Let $\mathcal{L}$ be an object of $D^b_c(X \times_Y Y)$ such that, for every geometric point $x \to X$, the restrictions of cohomology sheaves on $Y_y - T(x) \subset Y_y = X \times_Y Y$ are locally constant. Then, similarly as (1.19), we define a function $\psi_{\mathcal{L}}$ on $X \times_S Y$ by

$$\psi_{\mathcal{L}}(x \leftarrow w) = a_w(\mathcal{L}|_{Y_y \times S_y})$$

where $y = f(x), s = p(x)$ and $t = g(w)$. We also define a function $\delta(\psi_{\mathcal{L}})$ on $X \times_S S$ by (1.7).

**Proposition 1.16.** Let the notation be as above. Let $\mathcal{L}$ be an object of $D^b_c(X \times_Y Y)$ and $x \leftarrow t$ be a point of $X \times_S S$. Set $y = f(x)$ and $s = p(x)$ and assume that the restriction of cohomology sheaf $H^q\mathcal{L}$ on $Y_y - T(x)$ is locally constant for every $q$. Then, we have

$$\dim R^{q'} g_* \mathcal{L}_{x \leftarrow t} \to \dim R^{q'} g_* \mathcal{L}_{x \leftarrow s} = \delta(\psi_{\mathcal{L}})(x \leftarrow t).$$

**Proof.** By the canonical isomorphisms $R^{q'} g_* \mathcal{L}_{x \leftarrow t} \to R\Gamma(Y_y \times_S S(t), \mathcal{L}|_{Y_y \times S_S(t)})$ and $R^{q'} g_* \mathcal{L}_{x \leftarrow s} \to R\Gamma(Y_y, \mathcal{L}|_{Y_y}) = L_y$ (1.13), we obtain a distinguished triangle $R^{q'} g_* \mathcal{L}_{x \leftarrow t} \to R^{q'} g_* \mathcal{L}_{x \leftarrow s} \to R\Phi_{g/y}(\mathcal{L}|_{Y_y})_{y \leftarrow t} \to$. Hence it follows from Proposition 1.14.1. 

In fact, (1.20) is a special case of (1.24) below where $X = Y$.

**Corollary 1.17.** We keep the notation in Proposition 1.16. Let $\mathcal{K}$ be an object of $D^b_c(X)$ such that $\mathcal{L} = R\Psi f^* \mathcal{K}$ is an object of $D^b_c(X \times_Y Y)$. Assume that $\mathcal{L}$ and a point $x \leftarrow t$ of $X \times_S S$ satisfies the condition in Proposition 1.16. Then, we have

$$\dim R\Phi_{f/y} \mathcal{K}_{x \leftarrow t} = \delta(\psi_{\mathcal{L}})(x \leftarrow t).$$

**Proof.** By the isomorphisms $R\Psi_{f/y} \mathcal{K} \to R^{q'} g_* \mathcal{L}$ and $R^{q'} g_* \mathcal{L}_{x \leftarrow s} \to \mathcal{L}_y \to \mathcal{K}_x$, we obtain a distinguished triangle $R^{q'} g_* \mathcal{L}_{x \leftarrow s} \to R^{q'} g_* \mathcal{L}_{x \leftarrow t} \to R\Phi_{f/y} \mathcal{K}_{x \leftarrow t} \to$. Hence it follows from (1.23).

We consider the diagram (1.21) satisfying the condition there and assume further that $g: Y \to S$ is smooth. Let $\mathcal{K}$ be an object of $D^b_c(X)$ and assume that $p: X \to S$ is locally acyclic relatively to $\mathcal{K}$ and that the restriction of $f: X \to Y$ to the complement $X - Z$ is locally acyclic relatively to the restriction of $\mathcal{K}$.

We define a function $\varphi_{\mathcal{K}, f}$ on $Z$ as follows. For a geometric point $x$ of $Z$, set $y = f(x)$ and let $s = p(x) \to S$ be a geometric point with algebraically closed residue field. The base change $f_s: X_s \to Y_s$ of $f: X \to Y$ is a morphism to a smooth curve over the algebraically closed field $k(s)$. The strict localization $Y_{s(y)}$ is the spectrum of a strictly local discrete valuation ring with residue field $k(y) = k(s)$ since $Y_s$ is a smooth curve over $s$. Let $u$ be the geometric generic point of $Y_{s(y)}$. The cohomology of the stalk of the vanishing
cycles complex $\phi_z(K|_{X_z}, f_s)$ define $\Lambda$-representation of the absolute Galois group $G_{K_s}$ of the fraction field $K_u$ of $O_{Y,(y)}$ and hence the total dimension $\dim \text{tot}_y \phi_z(K|_{X_z}, f_s)$ is defined as the alternating sum. Similarly as (1.19), we define a function $\varphi_{K,f}$ on $Z$ by

\begin{equation}
(1.25) \quad \varphi_{K,f}(x) = \dim \text{tot}_y \phi_z(K|_{X_z}, f_s)
\end{equation}

**Proposition 1.18.** Let

\begin{equation}
(1.21) \quad Z \xrightarrow{c} X \xrightarrow{f} Y
\end{equation}

be a commutative diagram of morphisms of finite type of noetherian schemes such that $g: Y \to S$ is smooth of relative dimension 1 and that $Z \subset X$ is a closed subscheme quasi-finite over $S$.

Let $K$ be an object of $D^b(X)$ and assume that $p: X \to S$ is locally acyclic relatively to $K$ and that the restriction of $f: X \to Y$ to the complement $X - Z$ is locally acyclic relatively to the restriction of $K$. Then, the function $\varphi_{K,f}$ (1.25) on $Z$ is constructible and flat over $S$. If $Z$ is étale over $S$, it is locally constant.

**Proof.** The complex $R\Phi_fK$ is constructible and its construction commutes with base change by Proposition 1.8.1. Hence, we have a canonical isomorphism $\phi_z(K|_{X_z}, f_s) \to R\Phi_fK_{x \leftarrow u}$ and

$$\varphi_{K,f}(x) = \dim \text{tot}_y \phi_z(K|_{X_z}, f_s) = \dim \text{tot}_y R\Phi_fK_{x \leftarrow u}.$$  

We apply Proposition 1.16 to $L = R\Psi_fK$. The assumption in Proposition 1.16 that $\mathcal{H}^qL = R^q\Psi_fK$ on $Y_{(y)} - T_{(x)} \subset Y_{(y)} = x \times_Y Y$ is locally constant for every $q$ is satisfied for every geometric point $x$ of $X$ by Corollary 1.9. Hence the function $\psi_L$ (1.22) for $L = R\Psi_fK$ is defined as a function on $X \times_Y Y$.

In order to apply Lemma 1.5, we show

$$\psi_L(x \leftarrow w) = \sum_{z \in Z(x) \times_Y Y_{(y)} w} \varphi_{K,f}(z)$$

for a point $x \leftarrow w$ of $Z \times_Y Y$ such that $w$ is supported on the image $T_{(x)} \subset Y_{(y)}$ of $Z_{(x)}$. By the assumption that $Y \to S$ is smooth, the Milnor fiber $Y_{(w)} \times_S t$ is the spectrum of a discrete valuation ring. Let $u$ be its geometric point dominating the generic point regarded as a geometric point of $Y_{(w)}$.

By (1.22) and (1.18), we have

$$\psi_L(x \leftarrow w) = \dim \text{tot}_w(R\Psi_fK_{x \leftarrow u}) - \dim(R\Psi_fK_{x \leftarrow w}).$$

We apply Proposition 1.8.2 to $f: X \to Y$ and specializations $y \leftarrow w \leftarrow u$ to compute the right hand side. Then, the distinguished triangle (1.17) implies that the right hand side equals $\sum_{z \in Z(x) \times_Y Y_{(y)} w} \dim \text{tot}_w(R\Phi_fK_{z \leftarrow u}) = \sum_{z \in Z(x) \times_Y Y_{(y)} w} \varphi_{K,f}(z)$ as required.

Therefore, by applying Lemma 1.5, we obtain $\delta(\psi_L) = \delta(\varphi_{K,f})$ as functions on $Z \times_S S$. Since $R\Phi_pK = 0$, the function $\psi_L$ is flat over $S$ by (1.24). Hence, the function $\varphi_{K,f}$ is also flat over $S$. Since it is flat over $S$, the function $\varphi_{K,f}$ is constructible by Lemma 1.2.2.

If $Z$ is finite over $S$, the function $p_*\varphi_{K,f}$ is locally constant by Lemma 1.2.4. \qed
2 Singular support

We recall definitions and results from [7] in the first three subsections. We give a description of singular support of the 0-extension of a locally constant sheaf on the complement of a divisor with simple normal crossings under a certain assumption in Section 2.4.

2.1 \( C \)-transversality

We say that a closed subset \( C \subset E \) of a vector bundle \( E \) over \( X \) is conical if it is stable under the action of \( G_m \). Equivalently, it is defined by a graded ideal \( I \) of the graded algebra \( \mathcal{O}_X^\vee \), if the vector bundle \( E = \mathbf{V}(\mathcal{E}) \) is associated to a locally free \( \mathcal{O}_X \)-module \( \mathcal{E} \). For a closed conical subset \( C \subset E \), we call its intersection \( B \) with the 0-section regarded as a closed subset of \( X \) the base of \( C \).

Let \( \mathbf{P}(C) = \text{Proj}_X(S^*\mathcal{E}^\vee/I) \subset \mathbf{P}(E) = \text{Proj}_X S^*\mathcal{E}^\vee \) denote the projectivization. The projectivization \( \mathbf{P}(C) \) is empty if and only if \( C \) is a subset of the 0-section. The projectivization \( \mathbf{P}(C) \) itself does not determine \( C \) uniquely but the pair with the base \( B \) determine \( C \) uniquely.

We study the intersection of a closed conical subset with the inverse image of the 0-section by a morphism of vector bundles. For a morphism \( X \to Y \) of locally noetherian schemes and a closed subset \( Z \subset X \), we say that the restriction of \( X \to Y \) on \( Z \) is finite (resp. proper) if for every closed subscheme structure or equivalently for the reduced closed subscheme structure on \( Z \), the induced morphism \( Z \to Y \) is finite (resp. proper).

Lemma 2.1. Let \( E \to F \) be a morphism of vector bundles over a locally noetherian scheme \( X \). For a closed conical subset \( C \subset E \), the following conditions are equivalent.

1. The intersection of \( C \) with the inverse image of the 0-section of \( F \) by \( E \to F \) is a subset of the 0-section of \( E \).

2. The restriction of \( E \to F \) on \( C \) is finite.

Proof. (1)\( \Rightarrow \) (2): By replacing \( E, C \) and \( f : E \to F \) by \( E \oplus F \), the image \( g(C) \) of \( C \) by the graph \( g = (\text{id}, f) : E \to E \oplus F \) of \( f \) and \( \text{pr}_2 : E \oplus F \to F \), we may assume that \( E \to F \) is a surjection. We regard \( E \) as the complement \( \mathbf{P}(E \oplus \mathcal{A}^1) - \mathbf{P}(E) \) of the associated projective space bundle \( \mathbf{P}(E) \) in \( \mathbf{P}(E \oplus \mathcal{A}^1) \) and similarly for \( F \). The closure \( \overline{C} \subset \mathbf{P}(E \oplus \mathcal{A}^1) \) of \( C \subset E \) is the projectivization \( \mathbf{P}(C \oplus \mathcal{A}^1) \). Let \( \mathbf{P}(E \oplus \mathcal{A}^1)' \to \mathbf{P}(E \oplus \mathcal{A}^1) \) be the blow-up at the projectivization \( \mathbf{P}(K) \subset \mathbf{P}(E) \subset \mathbf{P}(E \oplus \mathcal{A}^1) \) of the kernel \( K \) of the surjection \( E \to F \). Then the morphism \( E \to F \) is extended to \( \mathbf{P}(E \oplus \mathcal{A}^1)' \to \mathbf{P}(E \oplus \mathcal{A}^1) \).

The condition (1) means that the closure \( \overline{C} \subset \mathbf{P}(E \oplus \mathcal{A}^1) \) of \( C \subset E \) does not meet \( \mathbf{P}(K) \). Hence \( \overline{C} \) defines a closed subset of \( \mathbf{P}(E \oplus \mathcal{A}^1)' \) and is proper over \( \mathbf{P}(F \oplus \mathcal{A}^1) \). Thus the intersection \( C = \overline{C} \cap E \) is proper and affine over \( F \) and is finite over \( F \).

(2)\( \Rightarrow \) (1): The intersection \( C \cap K \subset E \) is a closed conical subset finite over \( X \). Hence, it is a subset of the 0-section. \( \square \)

In the rest of this article, \( k \) denotes a field of characteristic \( p \geq 0 \) and \( X \) denotes a smooth scheme over \( k \), unless otherwise stated. The cotangent bundle \( T^*_X \) is the covariant vector bundle over \( X \) associated to the locally free \( \mathcal{O}_X \)-module \( \Omega^1_{X/k} \). The 0-section of \( T^*_X \) is identified with the conormal bundle \( T^*_X \) of \( X \subset X \).

For a closed conical subset \( C \subset T^*_X \), we define the condition for a morphism coming into \( X \) to be \( C \)-transversal.
Definition 2.2. Let $X$ be a smooth scheme over a field $k$ and let $C \subset T^*X$ be a closed conical subset of the cotangent bundle. Let $h: W \to X$ be a morphism of smooth schemes over $k$. Define

$$h^*C = W \times_X C \subset W \times_X T^*X$$

(2.1)

to be the pull-back of $C$ and let $K \subset W \times_X T^*X$ be the inverse image of the 0-section by the canonical morphism $dh: W \times_X T^*X \to T^*W$.

1. For a point $w \in W$, we say that $h: W \to X$ is $C$-transversal at $w$ if the fiber $(h^*C \cap K) \times_W w$ of the intersection is a subset of the 0-section $W \times_X T^*_X X \subset W \times_X T^*X$. We say that $h: W \to X$ is $C$-transversal if the intersection $h^*C \cap K$ is a subset of the 0-section $W \times_X T^*_X X \subset W \times_X T^*X$.

2. If $h: W \to X$ is $C$-transversal, we define a closed conical subset

$$h^oC \subset T^*W$$

(2.2)

to be the image of $h^*C$ by $W \times_X T^*X \to T^*W$.

By Lemma 2.1, if $h: W \to X$ is $C$-transversal, then $h^oC$ is a closed conical subset of $T^*W$.

Lemma 2.3. Let $X$ be a smooth scheme over a field $k$ and let $C \subset T^*X$ be a closed conical subset of the cotangent bundle. Let $h: W \to X$ be a morphism of smooth schemes over $k$.

1. If $h$ is smooth, then $h$ is $C$-transversal and the canonical morphism $h^*S \to h^oC$ is an isomorphism.

2. If $C \subset T^*_X X$ is a subset of the 0-section, then $h$ is $C$-transversal.

3. (cf. [7, Lemma 2.2 (i)]) Assume that $h$ is $C$-transversal. For a morphism $g: V \to W$ of smooth schemes over $k$, the following conditions are equivalent:

(1) $g: V \to W$ is $h^oC$-transversal.

(2) The composition $h \circ g: V \to X$ is $C$-transversal.

4. ([7, Lemma 1.1 (i)]) The subset of $W$ consisting of points $w \in W$ where $h: W \to X$ is $C$-transversal is an open subset of $W$.

5. Let $D = \bigcup_{i=1}^m D_i$ be a divisor with simple normal crossings of $X$ relatively to $X \to \text{Spec } k$ and let $C = \bigcup_{I \subset \{1, \ldots, m\}} T^*_D X$ be the union of the conormal bundles of the intersections $D_I = \bigcap_{i \in I} D_i$ of irreducible components for all subsets $I \subset \{1, \ldots, m\}$ of indices. Then, $h: W \to X$ is $C$-transversal if and only if $h^*D = D \times_X W$ is a divisor with simple normal crossings relatively to $W \to \text{Spec } k$ and $h^*D_i \subset W$ are smooth divisors for $i = 1, \ldots, m$.

The assertion 3 shows that $C$-transversal morphisms have similar properties as étale morphisms. For $\mathcal{F}$-transversal morphisms introduced in Definition 5.1, properties corresponding to 1-3 will be proved in Lemma 5.2.

Proof. 1. Since the canonical morphism $dh: W \times_X T^*X \to T^*W$ is a closed immersion, the intersection $h^*C \cap K$ is a subset of the 0-section $K \subset W \times_X T^*X$ and the morphism $h^*C \to h^oC$ is an isomorphism.

2. Since $h^*C \subset W \times_X T^*X$ is a subset of the 0-section, the assertion follows.

3. Since the surjection $h^*C \to h^oC$ is finite, the restriction of $V \times_X T^*X \to T^*V$ to $(hg)^*C = g^*(h^*C)$ is finite if and only if the restriction of $V \times_W T^*W \to T^*V$ to $g^*(h^oC)$ is finite. Hence the assertion follows from Lemma 2.1.
4. The complement of the subset is the image of the closed subset $\mathbb{P}(h^*C \cap K) \subset \mathbb{P}(W \times_X T^*X)$ of a projective space bundle over $W$.

5. The $C$-transversality is equivalent to the injectivity of $W \times_X T^*_D, X \to T^*W$ for all $I \subset \{1, \ldots, m\}$. Hence the assertion follows.

For a closed conical subset $C \subset T^*X$, we define the condition for a morphism going out of $X$ to be $C$-transversal.

**Definition 2.4.** Let $X$ be a smooth scheme over a field $k$ and let $C \subset T^*X$ be a closed conical subset of the cotangent bundle. Let $f : X \to Y$ be a morphism of smooth schemes over $k$.

1. For a point $x \in X$, we say that $f : X \to Y$ is $C$-transversal at $x$ if the fiber $df^{-1}(C) \times_X x$ of the inverse image of $C$ by the canonical morphism $df : X \times_Y T^*Y \to T^*X$ is a subset of the $0$-section $X \times_Y T^*_Y \subset X \times_Y T^*Y$.

We say that $f : X \to Y$ is $C$-transversal if the inverse image $df^{-1}(C)$ of $C$ by the canonical morphism $df : X \times_Y T^*Y \to T^*X$ is a subset of the $0$-section $X \times_Y T^*_Y \subset X \times_Y T^*Y$.

2. We say that the pair of morphisms $h : W \to X$ and $f : W \to Y$ of smooth schemes over $k$ is $C$-transversal if $h : W \to X$ is $C$-transversal and if $f : W \to Y$ is $h^*C$-transversal.

**Lemma 2.5.** Let $X$ be a smooth scheme over a field $k$ and let $C \subset T^*X$ be a closed conical subset of the cotangent bundle. Let $f : X \to Y$ be a morphism of smooth schemes over $k$.

1. For $Y = \text{Spec } k$, the canonical morphism $f : X \to \text{Spec } k$ is $C$-transversal.

2. Assume that $C$ is the $0$-section $T^*_X X \subset T^*X$. Then, $f$ is $C$-transversal if and only if $f$ is smooth.

3. Assume that $f$ is étale. Then, $f$ is $C$-transversal if and only if $C$ is a subset of the $0$-section $T^*_X X \subset T^*X$.

4. Assume that $f : X \to Y$ is $C$-transversal and let $g : Y \to Z$ be a smooth morphism. Then, the composition $g \circ f : X \to Z$ is $C$-transversal.

5. (Lemma 1.1 (i)) The subset of $X$ consisting of points $x \in X$ where $f : X \to Y$ is $C$-transversal is an open subset of $Y$.

6. Assume that $f : X \to Y$ is $C$-transversal. Then, the morphism $f : X \to Y$ is smooth on a neighborhood of the base $B$ of $C$.

7. Assume that $Y$ is a curve and let $x \in X$ be a point. Then, $f : X \to Y$ is not $C$-transversal at $x$ if and only if the image of the fiber $(X \times_Y T^*Y) \times_X x$ by the canonical morphism $df : X \times_Y T^*Y \to T^*X$ is a subset of the fiber $C \times_X x$.

8. Let $D = \bigcup_{i=1}^m D_i$ be a divisor with simple normal crossings of $X$ relatively to $X \to \text{Spec } k$ and let $C = \bigcup_{I \subset \{1, \ldots, m\}} T^*_D X$ be the union of the conormal bundles of the intersections $D_I = \bigcap_{i \in I} D_i$ of irreducible components for all subsets $I \subset \{1, \ldots, m\}$ of indices. Then, $f : X \to Y$ is $C$-transversal if and only if $D \subset X$ has simple normal crossings relatively to $f : X \to Y$.

In next subsection, we will see that the property 1 is related to the generic local acyclicity [13, Corollaire 2.16]. The property 2 is related to the local acyclicity of smooth morphism (see also Lemma 2.11.1). The property 3 is related to the characterization of locally constant sheaves [2, Proposition 2.11] (see also Lemma 2.11.3). The property 4 is related to [19, Corollaire 2.7].

**Proof.** 1. Since $T^*Y = 0$ for $Y = \text{Spec } k$, the assertion follows.
2. Assume that $C$ is the 0-section $T^*_XY \subset T^*X$. Then, $f: X \to Y$ is $C$-transversal if and only if the canonical morphism $X \times_Y T^*Y \to T^*X$ is an injection. Hence, this is equivalent to that $f$ is smooth.

3. Assume that $f$ is étale. Then, since $X \times_Y T^*Y \to T^*X$ is an isomorphism, the morphism $f$ is $C$-transversal if and only if $C$ is a subset of the 0-section $T^*_XY \subset T^*X$.

4. Assume that $g: Y \to Z$ is smooth. Then, since $X \times_Z T^*Z \to X \times_Y T^*Y$ is an injection, the $C$-transversality for $g$ implies that for $g \circ f$.

5. The complement of the subset is the image of the closed subset $\mathbf{P}(df^{-1}C) \subset \mathbf{P}(X \times_Y T^*Y)$ of a projective space bundle over $X$.

6. If $f: X \to Y$ is $C$-transversal, then $df: X \times_Y T^*Y \to T^*X$ is an injection on a neighborhood of $B$ and hence $f: X \to Y$ on a neighborhood of $B$.

7. Since the fiber $(X \times_Y T^*Y) \times_X x$ is a line and the inverse image of the fiber $C \times_X x$ is its conical subset, the inverse image of $C \times_X x$ is not the subset of the 0-section if and only if it is equal to the line $(X \times_Y T^*Y) \times_X x$ itself.

8. The $C$-transversality is equivalent to the injectivity of $D_I \times_Y T^*Y \to T^*D_I$ for all subsets $I \subset \{1, \ldots, m\}$. Hence, it is equivalent to the smoothness of $D_I \to Y$ for $I \subset \{1, \ldots, m\}$. Thus the assertion follows.

**Lemma 2.6.** Let $C \subset T^*X$ be a closed conical subset and let

$$
\begin{array}{ccc}
X & \leftarrow^h & W \\
\downarrow f & & \downarrow g \\
Y & \leftarrow & Z
\end{array}
$$

be a cartesian diagram of smooth schemes over $k$.

1. Assume that the horizontal arrows are regular immersions of the same codimension. Then, the following conditions are equivalent:

   (1) The morphism $f: X \to Y$ is $C$-transversal on a neighborhood of $W$.

   (2) The immersion $h: W \to X$ is $C$-transversal and the morphism $g: W \to Z$ is $h^*C$-transversal.

2. Assume that $f: X \to Y$ is smooth and $C$-transversal. Then, the morphism $h: W \to X$ is $C$-transversal and $g: W \to Z$ is $h^*C$-transversal.

**Proof.** 1. We have a commutative diagram

$$
\begin{array}{cccccc}
0 & \longrightarrow & T^*_W X & \longrightarrow & W \times_X T^*X & \longrightarrow & T^*W & \longrightarrow & 0 \\
\uparrow & & \uparrow & & \uparrow & & \uparrow & & \\
0 & \longrightarrow & W \times_Z T^*_Z Y & \longrightarrow & W \times_Y T^*Y & \longrightarrow & W \times_Z T^*Z & \longrightarrow & 0
\end{array}
$$

of exact sequences of vector bundles on $W$. The condition (1) is equivalent to that the inverse image in $W \times_Y T^*Y$ of $h^*C \subset W \times_X T^*X$ by the middle vertical arrow is a subset of the 0-section, by Lemma 2.3.4. The condition (2) is equivalent to that the inverse image in $T^*_W X$ of $h^*C \subset W \times_X T^*X$ by the upper left horizontal arrow and the inverse image in $W \times_Z T^*Z$ of $h^*C \subset T^*W$ by the right vertical arrow are subsets of the 0-sections. Since the left vertical arrow is an isomorphism, the conditions (1) and (2) are equivalent.
2. First, we assume $Z \rightarrow Y$ is smooth. Then $h$ is smooth and $C$-transversal. Further in the commutative diagram

$$
\begin{array}{ccc}
W \times_Y T^*Y & \longrightarrow & W \times_X T^*X \\
\downarrow & & \downarrow \\
W \times_Z T^*Z & \longrightarrow & T^*W
\end{array}
$$

of canonical injections, $W \times_Y T^*Y$ is identified with the intersection of $W \times_X T^*X$ and of $W \times_Z T^*Z$ in $T^*W$. Thus the inverse image of $h \circ C \subset W \times_X T^*X \subset T^*W$ in $W \times_Z T^*Z$ is the same as the inverse image in $W \times_Y T^*Y$ and hence $g$ is $h \circ C$-transversal.

If $Z \rightarrow Y$ is an immersion, the assertion follows from 1. In general, it suffices to decompose $Z \rightarrow Y$ as the composition $Z \rightarrow Z \times Y \rightarrow Y$ of the graph and the projection by Lemma 2.3.3 \hspace{1cm} \Box

For a closed conical subset $C \subset T^*X$ and a morphism $f : X \rightarrow Y$ satisfying a certain conditions of smooth schemes over $k$, we will define a closed conical subset $f_1C \subset T^*Y$.

**Definition 2.7.** Let $f : X \rightarrow Y$ be a morphism of smooth schemes over $k$ and $C \subset T^*X$ be a closed conical subset. We assume that the following condition is satisfied.

(Q) For each irreducible component $P$ of the inverse image $df^{-1}(C)$ of $C$ by the canonical morphism $df : X \times_Y T^*Y \rightarrow T^*X$, if $P$ is not a subset of the 0-section $X \times_Y T^*_Y Y$, the morphism $f : X \rightarrow Y$ is proper on the base $Q \subset X$ of $P$.

For each irreducible component $P$ of the inverse image $df^{-1}(C)$, we define a closed conical subset $f_1P \subset T^*Y$ to be the image of $P$ by the projection $X \times_Y T^*Y \rightarrow T^*Y$ if $f : X \rightarrow Y$ is proper on the base $Q$ of $P$ and to be the 0-section of $T^*Y$ supported on the closure $\overline{f(Q)}$ if otherwise. We define a closed conical subset

$$(2.3) \quad f_1C \subset T^*Y$$

to be the union of $f_1P \subset T^*Y$. If $f : X \rightarrow Y$ is proper on the base $C \cap T^*_X X \subset X$, we let

$$(2.4) \quad f_0C \subset T^*Y$$
denote $f_1C$.

If $f : X \rightarrow Y$ is a proper morphism of smooth schemes over $k$, then $f_0C \subset T^*Y$ is the image by the proper morphism $X \times_Y T^*Y \rightarrow T^*Y$ of the inverse image $df^{-1}(C) \subset X \times_Y T^*Y$ by $df : X \times_Y T^*Y \rightarrow T^*X$ of $C \subset T^*X$.

**Lemma 2.8.** Let $f : X \rightarrow Y$ be a morphism of smooth schemes over $k$. Let $C \subset T^*X$ be a closed conical subset satisfying the condition (Q) in Definition 2.7. Let $g : Y \rightarrow Z$ be a morphism of smooth schemes over $k$. If $g : Y \rightarrow Z$ is $f_1C$-transversal, then the composition $g \circ f : X \rightarrow Z$ is $C$-transversal.

**Proof.** For each irreducible component $P$ of the inverse image $df^{-1}(C) \subset X \times_Y T^*Y$, it suffices to show that the inverse image of $P$ by $X \times_Z T^*Z \rightarrow X \times_Y T^*Y$ is a subset of the 0-section assuming that $g : Y \rightarrow Z$ is $f_1P$-transversal.

If $f : X \rightarrow Y$ is proper on the base $Q$ of $P$, then the inverse image of $f_1P \subset T^*Y$ by $Y \times_Z T^*Z \rightarrow T^*Y$ is a subset of the 0-section and hence the inverse image of $P$ by $X \times_Z T^*Z \rightarrow X \times_Y T^*Y$ is a subset of the 0-section. If otherwise, then $g : Y \rightarrow Z$ is smooth on a neighborhood of $f(Q)$ by Lemma 2.3.6 and the inverse image of $P = Q$ by $X \times_Z T^*Z \rightarrow X \times_Y T^*Y$ is also a subset of the 0-section. \hspace{1cm} \Box
2.2 Singular support

We recall definitions and results from [7]. We assume that $X$ is a smooth scheme over a field $k$.

Let $\Lambda$ be a finite local ring such that the characteristic $\ell$ of the residue field is invertible in $k$ and let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on the étale site of $X$. Recall that a complex $\mathcal{F}$ of $\Lambda$-modules is constructible if every cohomology sheaf $\mathcal{H}^q\mathcal{F}$ is constructible and if $\mathcal{H}^q\mathcal{F} = 0$ except for finitely many $q$. We say that a constructible complex $\mathcal{F}$ of $\Lambda$-modules is locally constant if every cohomology sheaf is locally constant.

We say that a constructible complex $\mathcal{F}$ of $\Lambda$-modules is of finite tor-dimension if there exists an integer $a$ such that $\mathcal{H}^q(\mathcal{F} \otimes^\Lambda_k M) = 0$ for every $q < a$ and for every $\Lambda$-module $M$. The dualizing complex $\mathcal{K}_X$ defined as $Rf^!\Lambda$ for the canonical morphism $f: X \to \text{Spec } k$ is canonically isomorphic to $\Lambda(n)[2n]$ if every irreducible component of $X$ is of dimension $n$.

For a constructible complex $\mathcal{F}$ of finite tor-dimension, the dual $D_X\mathcal{F} = R\mathcal{H}\text{om}(\mathcal{F}, \mathcal{K}_X)$ is also constructible of finite tor-dimension.

**Definition 2.9.** Let $X$ be a smooth scheme over a field $k$ and let $C \subset T^*X$ be a closed conical subset. Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

1. [7, 1.2] We say that $\mathcal{F}$ is micro-supported on $C$ if for every $C$-transversal pair $(f, h)$ of morphisms $h: W \to X$ and $f: W \to Y$ of smooth schemes, the morphism $f: W \to Y$ is locally acyclic relatively to $h^*\mathcal{F}$.

2. [7, 1.4] We say that $\mathcal{F}$ is weakly micro-supported on $C$ if for every $C$-transversal pair $(f, j)$ of an étale morphism $j: W \to X$ satisfying the condition (W) below and a morphism $f: W \to Y = A^1_k$, the morphism $f: W \to Y$ is locally acyclic relatively to $h^*\mathcal{F}$:

(W) If $k$ is infinite, then we assume $j: W \to X$ to be an open immersion. If $k$ is finite, then we assume that $j: W \to X$ is the composition of an open immersion $V \to X$ with $W = V_k = V \times_k k' \to V$ for a finite extension $k'$ of $k$.

**Lemma 2.10.** Let $C$ and $C'$ be closed conical subsets of $T^*X$ and $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

1. Assume that $C \subset C'$. If $\mathcal{F}$ is micro-supported (resp. weakly micro-supported) on $C$, then $\mathcal{F}$ is micro-supported (resp. weakly micro-supported) on $C'$.

2. The complex $\mathcal{F}$ is weakly micro-supported on $C$ if and only if $C$ contains the image of the fibers $(U \times_Y T^*Y) \times_U x$ for all étale morphisms $j: U \to X$ and morphisms $f: U \to Y$ to smooth curves satisfying the condition (W) in Definition 2.9 and for closed points $x$ of $U$ where $f_*(j^*\mathcal{F}, f) \neq 0$.

3. If $\mathcal{F}$ is weakly micro-supported on $C$, then the base $B$ of $C$ contains the support of $\mathcal{F}$ as a subset.

4. We consider the following conditions for a morphism $h: W \to X$ of smooth schemes over $k$:

   (1) $\mathcal{F}$ is micro-supported on $C$.

   (2) $h^*\mathcal{F}$ is micro-supported on $h^*C$.

   If $h$ is $C$-transversal, we have (1)$\Rightarrow$(2). If $h$ is étale and surjective, conversely we have (2)$\Rightarrow$(1).

5. ([7, Lemma 2.1 (ii)]) Assume that $\mathcal{F}$ is micro-supported on $C$. Then, for $C$-transversal pair $(f, h)$ of morphisms $h: W \to X$ and $f: W \to Y$ of smooth schemes, the morphism $f: W \to Y$ is universally locally acyclic relatively to $h^*\mathcal{F}$.
6. ([7, Lemma 2.2 (ii)]) Assume \( \mathcal{F} \) is micro-supported (resp. weakly micro-supported) on \( C \) and let \( f : X \to Y \) be a proper morphism of smooth schemes over \( k \). Then \( Rf_*\mathcal{F} \) is micro-supported (resp. weakly micro-supported) on \( f_*C \).

7. Let \( \Lambda_0 \) be the residue field of \( \Lambda \). Then, \( \mathcal{F} \) is micro-supported on \( C \) if and only if \( \mathcal{F} \otimes^L_{A} \Lambda_0 \) is micro-supported on \( C \).

Proof. 1. If a pair \((f, h)\) of morphisms \( h : W \to X \) and \( f : X \to Y \) is \( C' \)-transversal, then \((f, h)\) is \( C \)-transversal. Hence the assertion follows.

2. Since the vanishing cycles complex \( \phi(j^*\mathcal{F}, f) \) is constructible, it follows from Lemma 2.5.7.

3. Define an open subset \( U = X - B \) to be the complement of the base \( B \) of \( C \). Let \( j : U \to X \) be the open immersion and \( f : U \to \mathbf{A}^1 \) be the morphism collapsing to 0. Then the pair \((f, j)\) is \( C \)-transversal. Hence the morphism \( f : U \to \mathbf{A}^1 \) is locally acyclic relatively to \( j^*\mathcal{F} \) and we have \( \mathcal{F}|_U = 0 \). Thus the base \( B \) contains \( \text{supp} \ \mathcal{F} \).

4. (1)\(\Rightarrow\)(2): Assume \( h : W \to X \) is \( C \)-transversal. If the pair \((f, g)\) of morphism \( g : V \to W \) and \( f : V \to Y \) of smooth schemes over \( k \) is \( h^0C \)-transversal, then the pair \((f, h \circ g)\) is \( C \)-transversal by Lemma 2.3.3. Hence the assertion follows.

(2)\(\Rightarrow\)(1): The local acyclicity is an \( \acute{e} \text{tale} \) local condition.

7. It follows from Lemma 1.6.2. \( \Box \)

By Lemma 2.10.2, there exists a smallest closed conical subset \( C \subset T^*X \) such that \( \mathcal{F} \) is weakly micro-supported on \( C \). This smallest \( C \) is called the weak singular support and is denoted by \( SS^w\mathcal{F} \). For a closed conical subset \( C \) of \( T^*X \), a constructible complex \( \mathcal{F} \) is weakly micro-supported on \( C \) if and only if \( SS^w\mathcal{F} \) is a subset of \( C \). By Lemma 2.10.2 and 3, the base of \( SS^w\mathcal{F} \) equals the support of \( \mathcal{F} \).

If there exists a smallest closed conical subset \( C \subset T^*X \) such that \( \mathcal{F} \) is micro-supported on \( C \), then we call such \( C \) the singular support of \( \mathcal{F} \) and let it denoted by \( SS\mathcal{F} \). If the singular support exists, we have \( SS^w\mathcal{F} \subset SS\mathcal{F} \) by definition. In fact, Theorem 2.12 below includes the equality \( SS^w\mathcal{F} = SS\mathcal{F} \).

**Lemma 2.11.** Let \( X \) be a smooth scheme over \( k \) and \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on \( X \).

1. ([7, Lemma 2.1 (iii)]) The following conditions are equivalent:
   1. \( \mathcal{F} \) is micro-supported on the 0-section \( T^*_X X \).
   2. \( \mathcal{F} \) is locally constant.

   If \( \mathcal{F} \) is locally constant and if the support of \( \mathcal{F} \) is \( X \), then the singular support \( SS\mathcal{F} \) exists and both \( SS\mathcal{F} \) and \( SS^w\mathcal{F} \) are equal to the 0-section \( T^*_X X \).

2. Assume \( \dim X = 1 \). Let \( U \subset X \) be the largest open subset where the restriction \( \mathcal{F}|_U \) is locally constant. Then, \( \mathcal{F} \) is micro-supported on the union

\[
T^*_X X \cup \bigcup_{x \in X - U} (T^*X \times_X x)
\]

of the 0-section and the fibers of the complement. Further if \( X = \text{supp} \ \mathcal{F} \), then the singular support \( SS\mathcal{F} \) and \( SS^w\mathcal{F} \) equal the closed conical subset (2.5).

3. Let \( D = \bigcup_{i=1}^m D_i \) be a divisor with simple normal crossings of \( X \) relatively to \( k \). Let \( \mathcal{G} \) be a locally constant constructible sheaf of \( \Lambda \)-modules on \( U = X - D \) and \( j : U \to X \) be the open immersion. Then \( \mathcal{F} = j_!\mathcal{G} \) is micro-supported on the union

\[
\bigcup_{I \subset \{1, \ldots, m\}} T^*_{D_I} X
\]
of the conormal bundles of the intersections $D_I = \bigcap_{i \in I} D_i$ of irreducible components for all subsets $I \subset \{1, \ldots, m\}$ of indices.

4. Assume that $\mathcal{F}$ is micro-supported on $C$ and let $f: X \to Y$ be a $C$-transversal proper morphism. Then $Rf_*\mathcal{F}$ is locally constant.

5. ([7, Lemma 2.5 (i)]) For a closed immersion $i: X \to Y$ of smooth schemes over $k$, we have $SS^w i_* \mathcal{F} = i_* SS^w \mathcal{F}$.

**Proof.** 1. (1)$\Rightarrow$(2) Since the identity $id_X: X \to X$ is $T^*_X X$-transversal by Lemma 2.5.3, the identity $id_X: X \to X$ is locally acyclic relatively to $\mathcal{F}$. This means that $\mathcal{F}$ is locally constant by [2, Proposition 2.11].

(2)$\Rightarrow$(1) Since a $T^*_X X$-transversal morphism $f: X \to Y$ of smooth schemes is smooth by Lemma 2.5.2, it follows from the local acyclicity of smooth morphism [4, Théorème 2.1].

If $\mathcal{F}$ is locally constant and if the support of $\mathcal{F}$ is $X$, Lemma 2.10.3 and (2)$\Rightarrow$(1) implies that the singular support $SS \mathcal{F}$ exists and that the equality $SS \mathcal{F} = SS^w \mathcal{F} = T^*_X X$ holds.

2. It follows from 1. that $\mathcal{F}$ is micro-supported on the union in (2.5).

Assume that $\mathcal{F}$ is weakly micro-supported on $C$. If $\mathcal{F}$ is not locally constant at $x$, then $\phi_x (\mathcal{F}, id_X) \neq 0$. Hence $C$ contains the fiber $x \times_X T^*_X X$ by Lemma 2.10.2. If supp $\mathcal{F} = X$, then $C$ contains the 0-section $T^*_X X$ by Lemma 2.10.3.

3. By Lemma 2.3.5, it suffices to show that any $C$-transversal morphism $f: X \to Y$ of smooth schemes is universally locally acyclic relatively to $\mathcal{F}$. By Lemma 2.5.8, the $C$-transversality of $f: X \to Y$ is equivalent to that the divisor $D$ has simple normal crossing relatively to $Y$. Hence, the assertion follows from [19, 1.3.3 (i)].

4. Since $f$ is proper, $Rf_* \mathcal{F}$ is micro-supported on $f_* C$ by Lemma 2.10.6. Since $f$ is $C$-transversal, $f_* C \subset T^* Y$ is a subset of the 0-section. Hence $Rf_* \mathcal{F}$ is locally constant by 1 and Lemma 2.10.1.  \hfill $\square$

**Theorem 2.12.** Let $X$ be a smooth scheme of finite type over a field $k$ and let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$.

1. ([7, Theorem 1.2 (i)]) The singular support $SS \mathcal{F}$ exists.

2. ([7, Theorem 1.2 (ii)]) If every irreducible component of $X$ is of dimension $n$, then every irreducible component of $SS \mathcal{F}$ is of dimension $n = \dim X$.

3. ([7, Theorem 1.4]) We have $SS \mathcal{F} = SS^w \mathcal{F}$.

4. ([7, Theorem 1.3 (ii)]) For an exact sequence $0 \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to 0$ of perverse sheaves, we have

\begin{equation}
SS \mathcal{F} = SS \mathcal{F}' \cup SS \mathcal{F}''.
\end{equation}

We have

\begin{equation}
SS \mathcal{F} = \bigcup_q SS^q H^q \mathcal{F}.
\end{equation}

**Corollary 2.13.** 1. For a closed conical subset $C$ of $T^* X$, the following conditions are equivalent:

(1) $\mathcal{F}$ is micro-supported on $C$.

(2) $\mathcal{F}$ is weakly micro-supported on $C$.

2. ([7, Theorem 1.3 (i)]) For a smooth morphism $h: W \to X$, we have $SS h^* \mathcal{F} = h^* SS \mathcal{F}$. 

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3. ([7, Lemma 2.1 (i)]) The base of \( SS\mathcal{F} = SS^w\mathcal{F} \) equals the support of \( \mathcal{F} \).

4. Let \( \Lambda_0 \) be the residue field of \( \Lambda \) and set \( \mathcal{F}_0 = \mathcal{F} \otimes_\Lambda \Lambda_0 \). Then, we have \( SS\mathcal{F} = SS\mathcal{F}_0 \).

**Proof.** 1. By Theorem 2.12.3, both conditions are equivalent to that \( SS\mathcal{F} \) is a subset of \( \mathcal{C} \).

By Lemma 2.10.4 (2) \( \Rightarrow \) (1) and by 1, we have \( SSh^*\mathcal{F} = h^*SS\mathcal{F} \) for an étale morphism \( h: W \to X \). Hence we may assume \( W = A^n \times X \). We have \( SSh^*\mathcal{F} \subset h^*SS\mathcal{F} \) by Lemma 2.10.4 (1) \( \Rightarrow \) (2). Hence the 0-section \( i: X \to W = A^n \) is \( SSh^*\mathcal{F} \)-transversal and further we have \( SS\mathcal{F} \subset i^*SSh^*\mathcal{F} \) by Lemma 2.10.4 (1) \( \Rightarrow \) (2). Thus we have \( SSh^*\mathcal{F} = h^*SS\mathcal{F} \).

4. It follows from Lemma 1.6.2. \( \square \)

We make an analogous definition to Definition 2.9 for complex on vanishing topos.

**Definition 2.14.** Let \( f: X \to Y \) be a morphism of smooth schemes over \( k \) and \( \mathcal{G} \) be a constructible complex of \( \Lambda \)-modules on \( X \times_Y Y \). Let \( C \subset T^*Y \) be a closed conical subset. We say that \( \mathcal{G} \) is weakly micro-supported on \( C \) if for every \( C \)-transversal pair of étale morphism \( h: V \to Y \) and morphism \( g: V \to Z \) to a smooth curve \( Z \), the morphism \( g \) is locally acyclic relatively to the restriction of \( \mathcal{G} \) to \( X_V \times_Y V \) where \( X_V = X \times_Y V \), in the sense of Definition 1.12.

**Lemma 2.15.** Let \( f: X \to Y \) be a morphism of smooth schemes over \( k \). Let \( C \) be a closed conical subset of \( T^*X \) satisfying the condition (Q) in Definition 2.7. Assume that the formation of \( R\Psi_f\mathcal{F} \) commutes with finite base change. If \( \mathcal{F} \) is weakly micro-supported on \( C \) (in the sense of Definition 2.9) for \( \mathcal{F} \), then \( R\Psi_f\mathcal{F} \) is weakly micro-supported on \( f_!C \) (in the sense of Definition 2.14).

**Proof.** Let \( (g, h) \) be a \( f_!C \)-transversal pair of étale morphism \( h: V \to Y \) and morphism \( g: V \to Z \) to a smooth curve \( Z \). We show that the morphism \( \overset{\sim}{g}: X_V \times_Y V \to X_V \times_Z Z \) is locally acyclic relatively to the restriction of \( \mathcal{G} = R\Psi_f\mathcal{F} \). By replacing \( X \) and \( Y \) by \( X_V \) and \( V \), we may assume \( V = Y \).

By Lemma 2.8.1, the composition \( g \circ f: X \to Z \) is \( C \)-transversal. Hence \( g \circ f: X \to Z \) is locally acyclic relatively to \( \mathcal{F} \). Thus, by the assumption that the formation of \( R\Psi_f\mathcal{F} \) commutes with finite base change and by Lemma 1.13, the morphism \( g: Y \to Z \) is locally acyclic relatively to \( R\Psi_f\mathcal{F} \). \( \square \)

### 2.3 The universal family of hyperplane sections

Assume that \( X \) smooth over \( k \) is quasi-projective and let \( \mathcal{L} \) be an ample invertible \( \mathcal{O}_X \)-module. Let \( E \subset \Gamma(X, \mathcal{L}) \) be a sub \( k \)-vector space of finite dimension such that the canonical morphism \( E \otimes_k \mathcal{O}_X \to \mathcal{L} \) is a surjection and induces an immersion

\[
(2.9) \quad i: X \to P = P(E^\vee) = \text{Proj} S^*E.
\]

We use a contra-Grothendieck notation for a projective space \( P(E)(k) = (E - \{0\})/k^\times \).

The dual \( P^\vee = P(E) \) of \( P \) parametrizes hyperplanes \( H \subset P \). The universal hyperplane \( Q = \{(x, H) \mid x \in H\} \subset P \times P^\vee \) is defined by the identity \( id \in \text{End}(E) \) regarded as a section \( F \in \Gamma(P \times P^\vee, \mathcal{O}(1,1)) = E \otimes E^\vee \). By the canonical injection \( \Omega^1_{P/k}(1) \to E \otimes \mathcal{O}_P \), the universal hyperplane \( Q \) is identified with the covariant projective space bundle \( P(T^*P) \).
associated to the cotangent bundle \( T^*\mathbb{P} \). The image of the conormal bundle \( T_Q^*(\mathbb{P} \times \mathbb{P}^\vee) \rightarrow Q \times_{\mathbb{P} \times \mathbb{P}^\vee} T^*(\mathbb{P} \times \mathbb{P}^\vee) \rightarrow Q \times_{\mathbb{P}^\vee} T^*\mathbb{P} \) by the projection is identified with the universal sub line bundle of the pull-back \( Q \times_{\mathbb{P}^\vee} T^*\mathbb{P} \) on \( Q = \mathbb{P}(T^*\mathbb{P}) \).

The fibered product \( X \times_{\mathbb{P}^\vee} Q = \mathbb{P}(X \times_{\mathbb{P}^\vee} T^*\mathbb{P}) \) is the intersection of \( X \times \mathbb{P}^\vee \) with \( Q \) in \( \mathbb{P} \times \mathbb{P}^\vee \) and is the universal family of hyperplane sections. We consider the universal family of hyperplane sections

\[
(2.10) \quad X \leftarrow^p X \times_{\mathbb{P}^\vee} Q \rightarrow^p \mathbb{P}^\vee = \mathbb{P}(E).
\]

Let \( C \subset T^*X \) be a closed conical subset. Define a closed conical subset \( \widetilde{C} \subset X \times_{\mathbb{P}^\vee} T^*\mathbb{P} \) to be the pull-back of \( C \) by the surjection \( X \times_{\mathbb{P}^\vee} T^*\mathbb{P} \rightarrow T^*X \) and let

\[
(2.11) \quad \mathbb{P}(\widetilde{C}) \subset \mathbb{P}(X \times_{\mathbb{P}^\vee} T^*\mathbb{P}) = X \times_{\mathbb{P}^\vee} Q
\]

be the projectivization. The subset \( \mathbb{P}(\widetilde{C}) \subset X \times_{\mathbb{P}^\vee} Q \subset X \times \mathbb{P}^\vee \) consists of the points \((x, H)\) such that the fiber \( T_{X \times_{\mathbb{P}^\vee} Q}(X \times \mathbb{P}^\vee) \times_{X \times_{\mathbb{P}^\vee} Q} (x, H) \subset (X \times_{\mathbb{P}^\vee} T^*\mathbb{P}) \times_{X \times_{\mathbb{P}^\vee} Q} x \) is a subset of \( \widetilde{C} \) since the conormal bundle \( T_{X \times_{\mathbb{P}^\vee} Q}(X \times \mathbb{P}^\vee) \subset X \times_{\mathbb{P}^\vee} T^*\mathbb{P} \) is the universal sub line bundle on the projective bundle

\[
X \times_{\mathbb{P}^\vee} Q = \mathbb{P}(X \times_{\mathbb{P}^\vee} T^*\mathbb{P}).
\]

If \( V \subset \mathbb{P} \) is an open subscheme such that \( i: X \rightarrow \mathbb{P} \) induces a closed immersion \( i^\circ: X \rightarrow V \), then \( C \subset X \times_{\mathbb{P}^\vee} T^*\mathbb{P} = X \times_{\mathbb{V}} T^*V \subset T^*V \) is identified with \( i^\circ C \) \((2.4)\).

**Lemma 2.16.** Let \( C \subset T^*X \) be a closed conical subset.

1. The complement \( X \times_{\mathbb{P}^\vee} Q - \mathbb{P}(\widetilde{C}) \) is the largest open subset where the pair \( X \leftarrow X \times_{\mathbb{P}^\vee} Q \rightarrow \mathbb{P}^\vee \) is \( C \)-transversal.

2. Assume \( \mathcal{F} \) is micro-supported on \( C \). Then, \( p^\vee: X \times_{\mathbb{P}^\vee} Q = \mathbb{P}(X \times_{\mathbb{P}^\vee} T^*\mathbb{P}) \rightarrow \mathbb{P}^\vee \) is universally locally acyclic relatively to \( p^*\mathcal{F} \) on the complement of the projectivization \( \mathbb{P}(\widetilde{C}) \subset \mathbb{P}(X \times_{\mathbb{P}^\vee} T^*\mathbb{P}) \).

**Proof.** 1. Since \( p: X \times_{\mathbb{P}^\vee} Q \rightarrow X \) is smooth, the pair \( X \leftarrow X \times_{\mathbb{P}^\vee} Q \rightarrow \mathbb{P}^\vee \) is \( C \)-transversal at \((x, H)\) \(\in X \times_{\mathbb{P}^\vee} Q \) if and only if \( X \times_{\mathbb{P}^\vee} Q \rightarrow \mathbb{P}^\vee \) is \( p^\circ C \)-transversal at \((x, H)\). The latter condition is equivalent to that the fiber at \((x, H)\) of the inverse image of the pull-back \( p^\circ C \subset T^*(X \times_{\mathbb{P}^\vee} Q) \) by \( (X \times_{\mathbb{P}^\vee} Q) \times_{X \times_{\mathbb{P}^\vee} Q} T^*X \) is a subset of the 0-section.

Note that since the subscheme \( X \times_{\mathbb{P}^\vee} Q \subset X \times \mathbb{P}^\vee \) is a divisor smooth over \( X \), the canonical morphism \( T_{X \times_{\mathbb{P}^\vee} Q}(X \times \mathbb{P}^\vee) \rightarrow (X \times_{\mathbb{P}^\vee} Q) \times_{X \times \mathbb{P}^\vee} T^*\mathbb{P}^\vee \) is an injection. Hence, by the isomorphism

\[
\operatorname{Coker}(T_{X \times_{\mathbb{P}^\vee} Q}(X \times \mathbb{P}^\vee) \rightarrow ((X \times_{\mathbb{P}^\vee} Q) \times_{X \times_{\mathbb{P}^\vee} Q} T^*(X \times_{\mathbb{P}^\vee} Q))) \rightarrow T^*(X \times_{\mathbb{P}^\vee} Q),
\]

the condition above is further equivalent to that the fiber at \((x, H)\) of the conormal line bundle \( T_{X \times_{\mathbb{P}^\vee} Q}(X \times \mathbb{P}^\vee) \) is not sent into the pull-back \( p^\circ C \subset (X \times_{\mathbb{P}^\vee} Q) \times_{X \times_{\mathbb{P}^\vee} Q} T^*X \). Or further equivalently, the fiber at \((x, H)\) of the conormal bundle \( T_Q^*(\mathbb{P} \times \mathbb{P}^\vee) \) is not sent into the pull-back of \( \widetilde{C} \subset T^*\mathbb{P} \). Since the conormal bundle \( T_Q^*(\mathbb{P} \times \mathbb{P}^\vee) \subset Q \times_{\mathbb{P}^\vee} T^*\mathbb{P} \) is the universal sub line bundle on the projective bundle \( Q = \mathbb{P}(T^*\mathbb{P}) \), the condition is equivalent to \((x, H) \notin \mathbb{P}(\widetilde{C})\). □

2. Clear from 1 and Lemma 2.10.5.

More precisely, the following holds.
Theorem 2.17 (cf. [7, Theorem 3.2, Lemma 3.3]). Let \( i: X \to P \) be an immersion and let \( \tilde{C} \subset X \times_P T^*P \) be the inverse image of the singular support \( C = SSF \subset T^*X \) by the surjection \( X \times_P T^*P \to T^*X \). Then, the complement \( U = X \times_P P - P(\tilde{C}) \) of the projectivization \( P(\tilde{C}) \subset P(X \times_P T^*P) = X \times_P Q \) is the largest open subset where \( p^\vee: X \times_P Q = P(X \times_P T^*P) \to P^\vee \) is universally locally acyclic relatively to \( p^*F \).

Proof. First, we consider the case where \( X = P \) is a projective space. Applying [7, Theorem 3.3] to the Radon transform \( RF \) [9], we obtain \( E_{p^\vee}(p^*R^iR(F)) = P(SSR(F)) \) in the notation loc. cit. Since, \( R^iR(F) \) is isomorphic to \( F \) except locally constant sheaf [9], we have \( E_{p^\vee}(p^*R^iR(F)) = E_{p^\vee}(p^*F) \). By [7, Lemma 3.3], we have \( P(SSF) = P(SSR(F)) \subset Q \). Hence the assertion follows for \( X = P \).

Let \( V \subset P \) be an open subset including \( X \) as a closed subset and let \( i^\#: X \to V \) be the closed immersion. Since the assertion is proved for \( P \), it holds also for an open subscheme \( V \). Since \( SS(i^\#:F) = P(\tilde{C}) \), the complement \( V \times_P Q - P(\tilde{C}) \) is the largest open subset of \( V \times_P Q \) where \( p^\vee: V \times_P Q \to P^\vee \) is universally locally acyclic relatively to \( p^*i^\#F \). Since \( p^*i^\#F = 0 \) outside \( X \times_P Q \), the assertion follows.

Corollary 2.18. Let \( i: X \to P \) be an immersion and \( C \subset T^*X \) be a closed conical subset such that the base \( B \subset X \) contains the support of \( F \). Assume that there exists a closed subset \( Z \subset X \times_P Q = P(X \times_P T^*P) \) of codimension \( > \dim X \) such that \( p^\vee: X \times_P Q = P(X \times_P T^*P) \to P^\vee \) is locally acyclic on the complement of \( P(\tilde{C}) \cup Z \subset P(X \times_P T^*P) \). Then \( F \) is micro-supported on \( C \).

Proof. Let \( C_0 = SSF \) denote the singular support. By Corollary 2.13.3, the base \( B_0 \) of \( C_0 \) equals the support of \( F \). By Theorem 2.17, we have \( P(\tilde{C}) \cup Z \subset P(\tilde{C}_0) \). Since \( Z \subset X \times_P Q \) is of codimension \( > \dim X \) by the assumption and every irreducible component of \( P(\tilde{C}_0) \subset X \times_P Q \) is of codimension \( \dim X \) by Theorem 2.12.2, we have \( P(\tilde{C}) \subset P(\tilde{C}_0) \). Thus, we have \( C - B \subset C_0 - B_0 \). Since \( B \supset B_0 \), we have \( C \supset C_0 = SSF \) and \( F \) is micro-supported on \( C \) by Lemma 2.10.1.

We recall from [7] a definition and properties.

Definition 2.19 (cf. [7, 4.3]). Let \( f: X \to S \) be a morphism of separated schemes of finite type over a field \( k \) and \( Y, Z \subset X \) be closed subsets. We say that \( Y \) and \( Z \) well intersect with respect to \( f \) if we have

\[
\dim(Y \times_S Z - Y \times_X Z) \leq \dim Y + \dim Z - \dim S.
\]

We slightly modified the original definition by replacing the equality by an inequality.

Lemma 2.20 (cf. [7, Lemma 4.3]). Let \( Y, Z \subset X \) be irreducible closed subsets. Assume that \( Y \) and \( Z \) well intersect with respect to \( f \) and we have \( \max(\dim Y, \dim Z) < \dim S \).

1. If \( Y = Z \), then \( Y \to \overline{f(Y)} \) is generically radical.
2. If \( Y \not\subset Z \), then we have \( f(Y) \not\subset \overline{f(Y)} \cap f(Z) \) and \( f(Y) \not\subset f(Z) \).

Proof. 1. We have \( \dim(Y \times_S Y - Y) \leq 2 \dim Y - \dim S < \dim Y \).

2. By 1, we have \( \dim Y = \dim \overline{f(Y)} \). We have \( \dim \overline{f(Y)} \cap f(Z) \leq \dim Y \times_S Z \leq \max(\dim(Y \times_S Z - Y \times_X Z), \dim Y \times_X Z) \). Since \( Y \) and \( Z \) assumed to well intersect, we have \( \dim(Y \times_S Z - Y \times_X Z) \leq \dim Y + \dim Z - \dim S < \dim Y \). If \( Y \not\subset Z \), we have \( \dim Y \times_X Z < \dim Y \). Thus, we have \( \dim \overline{f(Y)} \cap f(Z) < \dim Y = \dim \overline{f(Y)} \).

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If we had $f(Y) \subset f(Z)$, we would have $f(Y) \subset f(Y) \cap f(Z)$ and a contradiction. □

For a subspace $E \subset \Gamma(X, \mathcal{L})$ of finite dimension defining an immersion $X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee)$, we consider the following condition:

(E) For every pair of distinct closed points $u \neq v$ of the base change $X_\bar{k}$ to an algebraic closure $\bar{k}$ of $k$, the composition

$$E \subset \Gamma(X, \mathcal{L}) \otimes_k \bar{k} \rightarrow \mathcal{L}_u/m_u^2\mathcal{L}_u \oplus \mathcal{L}_v/m_v^2\mathcal{L}_v$$

is a surjection.

For an integer $d \geq 1$, let $E^{(d)} \subset \Gamma(X, \mathcal{L}^{\otimes d})$ denote the image of $S^d E \rightarrow \Gamma(X, \mathcal{L}^{\otimes d})$ induced by the inclusion $E \rightarrow \Gamma(X, \mathcal{L})$.

**Lemma 2.21.** Let $X$ be a quasi-projective scheme over a field $k$ and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Assume that $E \subset \Gamma(X, \mathcal{L})$ defines an immersion $X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee)$. For $d \geq 3$, the subspace $E^{(d)} \subset \Gamma(X, \mathcal{L}^{\otimes d})$ satisfies the condition (E) above.

**Proof.** We may assume $k = \bar{k}$, $X = \mathbb{P}^n$, $\mathcal{L} = \mathcal{O}(1)$, $E = \Gamma(X, \mathcal{L})$ and $u = (0, \cdots, 0, 1), v = (1, 0, \cdots, 0)$. Then, the assertion is clear. □

For an immersion $i : X \rightarrow \mathbf{P}$ to a projective space and a closed conical subset $C \subset T^*X$, we consider the following condition:

(C) For every irreducible component $C_a \subset T^*X$ of $C = \bigcup_a C_a$, the inverse image $\tilde{C}_a \subset X \times_p T^*\mathbf{P}$ is not a subset of the 0-section.

If $X$ is connected and if $C = SSF$, then the condition (C) means that the support of $i_*\mathcal{F}$ is not equal to $\mathbf{P}$ by Corollary 2.13.3. If the condition (C) is satisfied, there is a one-to-one correspondence between the irreducible components of $C$ and those of $\mathbf{P}(\tilde{C})$ sending $C_a$ to $\mathbf{P}(\tilde{C}_a)$.

**Proposition 2.22 ([7, Proposition 4.4]).** Let $X$ be a quasi-projective smooth scheme of dimension $n$ over a field $k$ and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee)$ and satisfying the condition (E) before Lemma 2.21. Let $C \subset T^*X$ be a closed conical subset satisfying the condition (C) above. Then, for irreducible components $C_a, C_b$ of $C$, the projectivizations $\mathbf{P}(\tilde{C}_a), \mathbf{P}(\tilde{C}_b) \subset \mathbf{P}(X \times_p T^*\mathbf{P}) = X \times_p Q$ well intersects with itself with respect to $p^\vee : X \times_p Q \rightarrow \mathbf{P}^\vee$.

The proof is the same as that of [7, Proposition 4.4].

**Corollary 2.23.** Let $X$ be a quasi-projective smooth scheme of dimension $n$ over a field $k$ and $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \rightarrow \mathbf{P} = \mathbf{P}(E^\vee)$ and satisfying the condition (E) before Lemma 2.21. Let $C \subset T^*X$ be a closed conical subset satisfying the condition (C) before Proposition 2.22. Assume that every irreducible component $C_a$ of $C = \bigcup_a C_a$ is of dimension $n$.

1. For every irreducible component $C_a$ of $C$, the restriction $\mathbf{P}(\tilde{C}_a) \rightarrow \mathbf{P}^\vee$ of $p^\vee : X \times_p Q \rightarrow \mathbf{P}^\vee$ is generically radicial and the closure $D_a = p^\vee(\mathbf{P}(\tilde{C}_a)) \subset \mathbf{P}^\vee$ is a divisor.
2. For distinct irreducible components $C_a \neq C_b$ of $C$, we have $D_a \neq D_b$. 

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Proposition 2.22 and Lemma 2.20.1. (E)

there exist dense open subsets
For an irreducible component
the singular support (\[7, 4.7 (iii), (iv)\])

\[ x \]

and for every line
D
X
\[ \text{is of relative dimension } n \]

2. Similarly as the proof of 1, the assertion follows from Proposition 2.22 and Lemma 2.20.2.

\[ \square \]

\[ \text{Theorem 2.24 ([7, Theorem 1.6])}. \text{ Let } X \text{ be a projective smooth scheme of dimension } n \text{ over a field } k \text{ and } L \text{ be an ample invertible } \mathcal{O}_X \text{-module. Let } E \subset \Gamma(X, L) \text{ be a subspace of finite dimension defining a closed immersion } X \to P = P(E^*) \text{ and satisfying the condition (E) before Lemma 2.21. Then, } D = p^*(P(SS\mathcal{F})) \subset P^\vee \text{ is a divisor and the complement } P^\vee - D \text{ is the largest open subset where } Rp^*_p^*\mathcal{F} \text{ is locally constant.} \]

Proof. It suffices to apply [7, Theorem 1.6] to \( i_*\mathcal{F} \).

\[ \square \]

Corollary 2.25. Let \( \mathcal{F} \) be a constructible complex of finite tor-dimension. Then, we have

\[ \text{(2.13)} \]

\[ SS\mathcal{F} = SSD_X\mathcal{F}. \]

Proof. Since the assertion is local on \( X \), we may take a closed immersion \( i: X \to U \) and an open immersion \( j: U \to P \) to a projective space. By Lemma 2.11.5 and Corollary 2.13.2, we may assume \( X \) is projective.

We take a projective embedding \( i: X \to P \) defined by \( E \) satisfying the conditions (E) before Lemma 2.21 and (C) before Proposition 2.22. Let \( C = SS\mathcal{F} \) and \( C' = SSD_X\mathcal{F} \) be the singular supports and let \( D = p^*(P(\widetilde{C})), D' = p^*(P(\widetilde{C}')) \subset P^\vee \) be the images. Then, by Theorem 2.24, the complement \( P^\vee - D \) is the largest open subset where the (shifted and twisted) Radon transform \( Rp^*_p^*\mathcal{F} \) is locally constant. Similarly, the complement \( P^\vee - D' \) is the largest open subset where \( Rp^*_p^*D_X\mathcal{F} \) is locally constant. Since the Radon transform commutes with duality up to shift and twist, we have \( D = D' \). Hence we have \( C = C' \) by Corollary 2.23.2.

\[ \square \]

In the proof of Proposition 3.17, we will use the following fact proved in the course of the proof of [7, Theorem 1.4]. For a line \( L \subset P^\vee \), the axis \( A_L \subset P \) is a linear subspace of codimension 2 defined as the intersections of hyperplanes parametrized by \( L \). On the complement \( X_L^\circ = X - (X \cap A_L) \), a canonical morphism

\[ \text{(2.14)} \]

\[ p^\circ_L: X_L^\circ \longrightarrow L \]

is defined by sending a point \( x \in X_L^\circ \) to the unique hyperplane \( H \in L \) containing \( x \).

Lemma 2.26 ([7, 4.7 (iii), (iv)]). Let \( i: X \to P \) be an immersion satisfying the condition (E) before Lemma 2.21. Let \( \mathcal{F} \) be a perverse sheaf of \( \Lambda \)-modules on \( X \) and assume that the singular support \( C = SS\mathcal{F} \subset T^*X \) satisfies the condition (C) before Proposition 2.22. For an irreducible component \( C_a \) of \( C \), let \( D_a = p^*(P(\widetilde{C_a})) \subset P^\vee \) denote the image. Then there exist dense open subsets \( D_a^\circ \subset D_a \) satisfying the following conditions:

For \( C_a \neq C_b \), we have \( D_a^\circ \cap D_b = \emptyset \). The inverse image \( P(\widetilde{C_a})^\circ = P(\widetilde{C_a}) \times_{D_a} D_a^\circ \) is radicial over \( D_a^\circ \). For every \( (x, H) \in P(\widetilde{C_a})^\circ \) and for every line \( L \subset P^\vee \) such that \( x \in X_L^\circ \), that \( L \) meets \( D_a^\circ \) properly at \( H = p_a^\circ(x) \) and that the tangent line \( TL \times_L \{H\} \) of \( L \) at \( H = p_a^\circ(x) \in P^\vee \) is not perpendicular to the fiber \( T^*_Q(P \times P^\vee) \times_Q (x, H) \subset T^*P^\vee \times_{P^\vee} \{H\} \), we have \( \varphi^{-1}_x(\mathcal{F}, p_a^\circ) \neq 0 \) and \( \varphi^q_x(\mathcal{F}, p_a^\circ) = 0 \) for \( q \neq -1 \).

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2.4 Singular support and ramification

We assume $k$ is perfect. In this section, we describe the singular support $SS\mathcal{F}$ of $\mathcal{F} = j^!\mathcal{G}$ for a locally constant sheaf $\mathcal{G}$ on the complement $U = X - D$ of a divisor $D$ with simple normal crossings of a smooth scheme $X$ over a perfect field $k$ and the open immersion $j: U \to X$. First, we study the tamely ramified case.

Proposition 2.27. Let $\mathcal{G} \neq 0$ be a locally constant constructible sheaf of $\Lambda$-modules on the complement $U = X - D$ of a divisor $D = \bigcup_{i=1}^{m} D_i$ with simple normal crossings of a smooth scheme $X$ over a perfect field $k$. Assume that $\mathcal{G}$ is tamely ramified along $D$. For the open immersion $j: U \to X$ and $\mathcal{F} = j^!\mathcal{G}$, we have

$$SS\mathcal{F} = \bigcup_{I \subset \{1, \ldots, m\}} T_{D_i}^*X$$

Proof. By Lemma 2.11.3, we have $SS\mathcal{F} \subset \bigcup_{I \subset \{1, \ldots, m\}} T_{D_i}^*X$. By Corollary 2.13.2, it is reduced to the case where $X = \mathbb{A}^n$ and $U = \mathcal{G}_{\mathbb{A}^n}^m$. By the induction on $n = \dim X$ and further by the compatibility with smooth pull-back, it suffices to show that the fiber $T_0X$ at the origin $0 \in X = \mathbb{A}^n = \text{Spec } k[T_1, \ldots, T_n]$ is a subset of $SS^nX$. If $n = 0$, since $\mathcal{G} \neq 0$, we have $SS\mathcal{F} = T^*X$ by Lemma 2.10.3.

Assume $n > 0$. Let $D_i = (T_i = 0) \subset X$ and $C = \bigcup_{I \subset \{1, \ldots, n\}} T_{D_i}^*X$ be the union except the fiber $T_0X$. Since the morphism $f: X \to \text{Spec } k[T]$ defined by $T \mapsto T_1 + \cdots + T_n$ is $C$-transversal, it suffices to show the following. \hfill $\square$

Lemma 2.28. Let $S = \text{Spec } \mathcal{O}_K$ be the spectrum of a henselian discrete valuation ring with algebraically residue field $k$, $X$ be a smooth scheme of finite type of relative dimension $n - 1$ over $S$ and $D$ be a divisor of $X$ with simple normal crossings. Let $x$ be a closed point of the closed fiber of $X$ contained in $D$. Let $t_1, \ldots, t_n \in \mathfrak{m}_x$ be elements of the maximal ideal such that $\mathfrak{m}_1, \ldots, \mathfrak{m}_n \in \mathfrak{m}_x/\mathfrak{m}_x^2$ is a basis. Assume that $D$ is defined by $t_1 \cdots t_d$ and that the class of a uniformizer $\pi$ of $S$ in $\mathfrak{m}_x/\mathfrak{m}_x^2$ is not contained in any subspaces generated by $n - 1$ elements of the basis $\mathfrak{m}_1, \ldots, \mathfrak{m}_n$.

Let $\Lambda$ be a finite local ring with residue characteristic $\ell$ invertible on $S$ and let $\mathcal{G}$ be a locally constant constructible sheaf of $\Lambda$-modules on the complement $U = X - D$ tamely ramified along $D$. Let $j: U \to X$ denote the open immersion.

1. On a neighborhood of $x$, the complex $\phi(j^!\mathcal{G})$ is acyclic except at $x$ and at degree $n - 1$ and the action of the inertia group $I_K = \text{Gal}(\bar{K}/K)$ on $\phi_x^{n-1}(j^!\mathcal{G})$ is tamely ramified.

2. If $\mathcal{G}$ is a sheaf of free $\Lambda$-modules, then the $\Lambda$-module $\phi_x^{n-1}(j^!\mathcal{G})$ is free of rank rank $\mathcal{G}$.

Proof. 1. We may write $\pi = \sum_{i=1}^{n} u_i t_i$ in $\mathfrak{m}_x$ and $u_i$ are invertible. By replacing $t_i$ by $u_i t_i$, we may assume that $X$ is étale over $\text{Spec } \mathcal{O}_K[t_1, \ldots, t_n]/(\pi - (t_1 + \cdots + t_n))$. By Abhyankar’s lemma, we may assume that $\mathcal{G}$ is trivialized by the abelian covering $s_i^n = t_i$ for an integer $m$ invertible on $S$. Since the assertion is étale local on $X$, we may assume $X = \text{Spec } \mathcal{O}_K[t_1, \ldots, t_n]/(\pi - (t_1 + \cdots + t_n))$. Hence by [19, 1.3.3 (i)], the complex $\phi(j^!\mathcal{G})$ is acyclic outside $x$. Since the complex $\phi(j^!\mathcal{G})[n - 1]$ is a perverse sheaf by [20, Corollaire 4.6], the complex $\phi(j^!\mathcal{G})$ is acyclic except at degree $n - 1$.

Let $p: X' \to X$ be the blow-up at $x$ and $j': U \to X'$ be the open immersion. Let $D'$ be the proper transform of $D$ and $E$ be the exceptional divisor. Then, the union of $D'$ with the closed fiber $X'_e$ has simple normal crossings. Hence, the action of the inertia group $I_K$ on $\phi(j'_!\mathcal{G})$ is tamely ramified by [28, Proposition 6] and $\phi_x(j'_!\mathcal{G}) = R\Gamma(E, \phi(j'_!\mathcal{G}))$ is also tamely ramified.
2. By 1, we may assume $\Lambda$ is a field. We consider the stratification of $E$ defined by the intersections with the intersections of irreducible components of $D'$. Then, on each stratum, the restriction of the cohomology sheaves $\phi^g(j_!G)$ are locally constant and is tamely ramified along the boundary by [28, Proposition 6]. Further, the alternating sum of the rank is 0 except for $E^o = E - (E \cap D')$ and equals rank $G$ on $E^o$. Hence, we have

$$\dim \phi_x(j_!G) = \chi(E, \phi(j_!G)) = \chi_c(E^o, \phi(j_!G)) = \text{rank } G \cdot \chi_c(E^o).$$

Since $E^o = E_{n-1}^o \subset G_{m-1}^o = \text{Spec}[t_1^\pm 1, \ldots, t_{n-1}^\pm 1]$ is the complement of the intersection with the hyperplane $\sum t_i + 1 = 0$ and the intersection is isomorphic to $E_{n-2}^o$, we have $\chi_c(E_{n-1}^o) = \chi_c(G_{m-1}^o) - \chi_c(E_{n-2}^o) = (-1)^{n-1}$ by induction on $n$. Thus the assertion follows.

To give a description of the singular support in some wildly ramified case using ramification theory, we briefly recall ramification theory [1], [29]. Let $K$ be a henselian discrete valuation field with residue field of characteristic $p > 0$ and $G_K = \text{Gal}(K_{\text{sep}}/K)$ be the absolute Galois group. Then, the (non-logarithmic) filtration $(G_K^n)_{n \geq 1}$ by ramification groups is defined in [1, Definition 3.4]. It is a decreasing filtration by closed normal subgroups indexed by rational numbers $\geq 1$.

For a real number $r \geq 1$, we define subgroups $G_K^r+ \subset G_K^r$ by $G_K^r+ = \bigcup_{s < r} G_K^s$ and $G_K^r = \bigcap_{s > r} G_K^s$. It is proved in [1, Proposition 3.7 (1)] that $G_1^r$ is the inertia group $I = \text{Ker}(G_K \to G_F)$ where $G_F$ denotes the absolute Galois group of the residue field $F$ and $G_K^{1+}$ is the wild inertia group $P$ that is the pro-$p$ Sylow subgroup of $I$. It is also proved in [1, Theorem 3.8] that $G_K^r = G_K^1$ for rational numbers $r > 1$ and $G_K^r = G_K^{r+}$ for irrational numbers $r > 1$.

Let $\Lambda$ be a finite local ring with residue characteristic $\neq p$ and let $V$ be a continuous representation of $G_K$ on a $\Lambda$-vector space of finite dimension. Then, since $P = G_K^{1+}$ is a pro-$p$ group and since $G_K^r = G_K^1$ for rational $r$ and $G_K^r = G_K^{r+}$ for irrational $r$, there exists a unique decomposition $V = \bigoplus_{r \geq 1} V(r)$ called the slope decomposition characterized by the condition that the $G_K^r$-fixed part $V(r)$ of $V$ is equal to the sum $\bigoplus_{s \leq r} V(s)$.

We study a geometric case where $X$ is a smooth scheme over a perfect field $k$ of characteristic $p > 0$. Let $D$ be a reduced and irreducible divisor and $U = X - D$ be the complement. Let $G$ be a locally constant constructible sheaf of $\Lambda$-vector spaces on $U$. Let $\xi$ be the generic point of an irreducible component of $D$. Then, the local ring $\mathcal{O}_{X, \xi}$ is a discrete valuation ring and the fraction field $K$ of its henselization is called the local field at $\xi$. The stalk of $\mathcal{H}^dG$ at the geometric point of $U$ defined by a separable closure $K_{\text{sep}}$ defines a $\Lambda$-vector space $V^q$ with an action of the absolute Galois group $G_K$.

For a rational number $r > 1$, the graded quotient $\text{Gr}^r G_K = G_K^r/G_K^{r+}$ is a profinite abelian group annihilated by $p$ [29, Corollary 2.28] and its dual group is related to differential forms as follows. We define ideals $m_{K_{\text{sep}}}^{(r)}$ and $m_{K_{\text{sep}}}^{(r+)}$ of the valuation ring $\mathcal{O}_{K_{\text{sep}}}$ by $m_{K_{\text{sep}}}^{(r)} = \{x \in K_{\text{sep}} \mid \text{ord}_K x \geq r\}$ and $m_{K_{\text{sep}}}^{(r+)} = \{x \in K_{\text{sep}} \mid \text{ord}_K x > r\}$ where $\text{ord}_K$ denotes the valuation normalized by $\text{ord}_K(\pi) = 1$ for a uniformizer $\pi$ of $K$. The residue field $\bar{F}$ of $\mathcal{O}_{K_{\text{sep}}}$ is an algebraic closure of $F$ and the quotient $m_{K_{\text{sep}}}^{(r)}/m_{K_{\text{sep}}}^{(r+)}$ is an $\bar{F}$-vector space of dimension 1. A canonical injection

$$(2.16) \quad \text{ch}: \text{Hom}_{F_p}(\text{Gr}^r G_K, F_p) \to \text{Hom}_{\bar{F}}(m_{K_{\text{sep}}}^{(r)}/m_{K_{\text{sep}}}^{(r+)}, \Omega^1_{X/k, \xi} \otimes \bar{F})$$

is also defined [29, Corollary 2.28].
We say that the ramification of $\mathcal{G}$ along $D$ is isoclinic of slope $r \geq 1$ if $V = V^{(r)}$ in the slope decomposition. The ramification of $\mathcal{G}$ along $D$ is isoclinic of slope 1 if and only if the corresponding Galois representation $V$ is tamely ramified. Assume that the ramification of $\mathcal{G}$ along $D$ is isoclinic of slope $r > 1$. Assume also that $\Lambda$ contains a primitive $p$-th root of 1 and identify $\mathbf{F}_p$ with a subgroup of $\lambda^*$. Then, $V = V^{(r)}$ is further decomposed by characters $V = \bigoplus_{\chi : \text{Gr}^r G_K \to \mathbf{F}_p} \chi^{\oplus m(\chi)}$. For a character $\chi$ appearing in the decomposition, the twisted differential form $ch(\chi)$ defined on a finite covering of a dense open scheme of $D$ is called a characteristic form of $\mathcal{G}$.

Assume that $U = X - D$ is the complement of a divisor with simple normal crossings $D$ and let $D_1, \ldots, D_m$ be the irreducible components of $D$. We say the ramification of $\mathcal{G}$ along $D$ is isoclinic of slope $R = \sum_i r_i D_i$ if the ramification of $\mathcal{G}$ along $D_i$ is isoclinic of slope $r_i$ for every irreducible component $D_i$ of $D$.

In [29, Definition 3.1], we define the condition for ramification of $\mathcal{G}$ along $D$ to be non-degenerate. The condition implies that the characteristic forms are extended to differential forms on the boundary without zero. We say that the ramification of $\mathcal{G}$ is non-degenerate along $D$ if it admits étale locally a direct sum decomposition $\mathcal{G} = \bigoplus_j \mathcal{G}_j$ such that each $\mathcal{G}_j$ is isoclinic of slope $R_j \geq D$ for a $\mathbf{Q}$-linear combination $R_j$ of irreducible components of $D$ and that the ramification of $\mathcal{G}_j$ is non-degenerate along $D$ at multiplicity $R_j$. Note that there exists a closed subset of codimension at least 2 such that on its complement, the ramification of $\mathcal{G}$ along $D$ is non-degenerate.

We introduce a slightly stronger condition that implies local acyclicity. We say that the ramification of $\mathcal{G}$ is strongly non-degenerate along $D$ if it satisfies the condition above with $R_j \geq D$ replaced by $R_j = D$ or $R_j > D$. The inequality $R_j > D$ means that the coefficient in $R_j$ of every irreducible component of $D$ is $> 1$. Note that if the ramification of $\mathcal{G}$ along $D$ is non-degenerate, on the complement of the singular locus of $D$, the ramification of $\mathcal{G}$ along $D$ is strongly non-degenerate.

Let $j : U = X - D \to X$ denote the open immersion. We define a closed conical subset $S(j_! \mathcal{G}) \subset T^*X$ following the definition of the characteristic cycle given [29, Definition 3.5] in the strongly non-degenerate case. We will show that $S(j_! \mathcal{G})$ is in fact equal to the singular support in Proposition 2.29 below.

Assume first that the ramification of $\mathcal{G}$ along $D$ is isoclinic of slope $R = D$. Then, the locally constant sheaf $\mathcal{G}$ on $U$ is tamely ramified along $D$. In this case, let the singular support $SS_j \mathcal{G}$ (2.15) denoted by

$$S(j_! \mathcal{G}) = \bigcup_i T^*_{D_i} X$$

(2.17)

where $T^*_{D_i} X$ denotes the conormal bundle of the intersection $D_i = \bigcap_I D_i$ for a set of indices $I \subset \{1, \ldots, m\}$.

Assume the ramification of $\mathcal{G}$ along $D$ is isoclinic of slope $R = \sum_i r_i D_i > D = \sum_i D_i$. For each irreducible component, we have a decomposition by characters $V = \bigoplus_{\chi : \text{Gr}^r G_K \to \mathbf{F}_p} \chi^{\oplus m(\chi)}$. Further, the characteristic form of each character $\chi$ appearing in the decomposition defines a sub line bundle $L_\chi$ of the pull-back $D_\chi \times_X T^*X$ of the cotangent bundle to a finite covering $\pi_\chi : D_\chi \to D_i$ by the non-degenerate assumption. Then, define a closed conical subset $C = S(j_! \mathcal{G}) \subset T^*X$ in the case $R > D$ by

$$S(j_! \mathcal{G}) = T^*_X \cup \bigcup_{\chi} \pi_\chi(L_\chi)$$

(2.18)
In the general strongly non-degenerate case, we define a closed conical subset \( C = S(j_! G) \subset T^* X \) by the additivity and étale descent.

**Proposition 2.29** ([29, Proposition 3.15]). Assume that the ramification of a locally constant constructible sheaf \( G \) of \( \Lambda \)-modules on the complement \( U = X - D \) of a divisor with simple normal crossings is strongly non-degenerate along \( D \). Then, \( j_! G \) is micro-supported on \( S(j_! G) \) defined by (2.17), (2.18), the additivity and by étale descent. In other words, we have an inclusion \( S S j_! G \subset S(j_! G) \).

We will show the equality \( S S j_! G = S(j_! G) \) as a consequence of the equality \( \text{Char} j_! G = C(j_! G) \) proved in Theorem 4.6.

**Proof.** Since the assertion is étale local, we may assume that \( G \) is isoclinic of slope \( R = D \) or \( R > D \) and that the ramification of \( G \) is non-degenerate along \( D \) at multiplicity \( R \). If \( R = D \), it is proved in Proposition 2.27. To prove the case \( R > D \), we use the following Lemma.

**Lemma 2.30.** Let \( E \subset \Gamma(X, \mathcal{L}) \) be a subspace of finite dimension defining an immersion \( X \to \mathbf{P} = \mathbf{P}(E^\vee) \). Let \( T \subset X \) be an integral closed subscheme and define a subspace \( E' = \text{Ker}(E \to \Gamma(T, \mathcal{L} \otimes_{\mathcal{O}_X} \mathcal{O}_T)) \) and \( \mathbf{P}^{\vee} = \mathbf{P}(E') \subset \mathbf{P}^\vee = \mathbf{P}(E) \).

1. \( T \times \mathbf{P}^{\vee} \subset X \times \mathbf{P}^\vee \) is contained in \( T \times \mathbf{P} Q \subset X \times \mathbf{P} Q \) and the complement \( (T \times \mathbf{P} Q) - (T \times \mathbf{P}^\vee) \) is the largest open subscheme where \( T \times \mathbf{P} Q \to \mathbf{P}^\vee \) is flat.

2. The codimension of \( E' \subset E \) is strictly larger than \( \dim T \).

**Proof.** 1. For a hyperplane \( H \subset \mathbf{P} \), we have \( T \subset H \) if \( H \in \mathbf{P}^\vee \) and \( T \cap H \) is a divisor of \( T \) if otherwise. Hence we have \( T \times \mathbf{P}^\vee \subset T \times \mathbf{P} Q \) and on the complement of \( T \times \mathbf{P}^\vee \), the divisor \( T \times \mathbf{P} Q \) of \( T \times \mathbf{P}^\vee \) is flat over \( \mathbf{P}^\vee \).

2. The immersion \( X \to \mathbf{P} \) induces an immersion \( T \to \mathbf{P}(E/E') \). Hence we have \( \dim T < \dim E/E' \). \( \square \)

We assume \( R > D \). Since the assertion is local on \( X \), we may assume that \( X \) is quasi-projective. Let \( i: X \to \mathbf{P} \) be an immersion defined satisfying the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22. For irreducible component \( D_i \) of \( D \), let \( \mathbf{P}_i^\vee = \mathbf{P}(E_i) \subset \mathbf{P}^\vee = \mathbf{P}(E) \) be the subspace defined by \( E_i = \ker(E \to \Gamma(D_i, \mathcal{L})) \).

Let \( Z \subset X \times \mathbf{P} Q \) be the union \( Z \subset \bigcup_{i=1, \ldots, m} D_i \times \mathbf{P}_i^\vee \). Then, by Lemma 2.30.1, the divisor \( D \times \mathbf{P} Q \subset X \times \mathbf{P} Q \) with simple normal crossings is flat over \( \mathbf{P}^\vee \) outside \( Z \). Hence, by [29, Proposition 3.15], \( p^\vee: X \times \mathbf{P} Q \to \mathbf{P}^\vee \) is universally locally acyclic relatively to the pull-back of \( j_! G \) outside the union \( Z \cup \mathbf{P}(\mathcal{C}) \) for \( C = S(j_! F) \). Further by Lemma 2.30.2, the closed subset \( Z \subset X \times \mathbf{P} Q \) is of codimension \( \geq \dim X \). Hence the assertion follows by Corollary 2.18. \( \square \)

### 3 Characteristic cycle and the Milnor formula

In this section, \( X \) denotes a smooth scheme over a perfect field \( k \). Assume that every irreducible component of \( X \) is of dimension \( n \), unless otherwise stated.
3.1 Isolated characteristic point and the morphism defined by a pencil

The characteristic cycle will be defined as a cycle characterized by the Milnor formula at isolated characteristic points. We will introduce the notion of isolated characteristic point and study the intersection number appearing in the Milnor formula.

**Definition 3.1.** Let $X$ be a smooth scheme over a field $k$. Let $C \subset T^*X$ be a closed conical subset of the cotangent bundle $T^*X$. Let $h : W \to X$ be an étale morphism and $f : W \to Y$ be a morphism over $k$ to a smooth curve over $k$.

1. We say that a closed point $u$ of $W$ is an isolated characteristic point of $f : W \to Y$ with respect to $C$ if the pair $X \leftarrow W - \{u\} \to Y$ is $C$-transversal.

2. Assume that every irreducible component of $X$ and every irreducible component $C_a$ of $C$ are of dimension $n$. Let $u$ be an isolated characteristic point of $f : W \to Y$ with respect to $C$ and $A = \sum a_m [C_a]$ be a linear combination of irreducible components of $C$. Then, we define the intersection number

$$\text{(3.1)}$$

$$(A, df)_{T^*W, u}$$

as the intersection number $\sum a_m (j^*C_a, f^*\omega)_{T^*W, u}$ supported on the fiber of $u$ for the section $f^*\omega$ of $T^*W$ defined by the pull-back of a basis $\omega$ of $T^*Y$ on a neighborhood of $f(u) \in Y$.

The cotangent bundle $T^*W$ is of dimension $2n$ and its closed subsets $j^*C_a, f^*\omega$ are of dimension $n$. Their intersections $j^*C_a \cap f^*\omega$ consist of at most a unique isolated point $f^*\omega(u) \in T^*_u W$ on the fiber of $u$ and the intersection numbers $(j^*C_a, f^*\omega)_{T^*W, u}$ are defined. Further, since $C$ is conical, the intersection numbers $(j^*C_a, f^*\omega)_{T^*W, u}$ are independent of the choice of basis $\omega$ and the intersection number $(A, df)_{T^*W, u}$ is well-defined.

We compute the intersection number (3.1) using the universal family of morphisms defined by pencils. Let $E$ be a $k$-vector space of finite dimension and $P = P(E^\vee)$ and its dual $P^\vee = P(E)$ be as in Section 2.3. Let $G = \text{Gr}(1, P^\vee)$ be the Grassmannian variety parametrizing lines in $P^\vee$. The universal line $D \subset P^\vee \times G$ is canonically identified with the flag variety parametrizing pairs $(H, L)$ of points $H$ of $P^\vee$ and lines $L$ passing through $H$. It is the same as the flag variety $\text{Fl}(1, 2, E)$ parametrizing pairs of a line and a plane including the line in $E$.

The projective space $P^\vee$ and the Grassmannian variety $G$ are also equal to the Grassmannian varieties $\text{Gr}(1, E)$ and $\text{Gr}(2, E)$ parametrizing lines and planes in $E$ respectively. The projections $P^\vee \leftarrow D \to G$ sending a pair $(H, L)$ to the line $L$ and to the hyperplane $H$ are the canonical morphisms $\text{Gr}(1, E) \leftarrow \text{Fl}(1, 2, E) \to \text{Gr}(2, E)$. By the projection $D \to P^\vee$, it is also canonically identified with the projective space bundle associated to the tangent bundle $D = P(TP^\vee)$ by sending a line passing through a point to the tangent line of the line at the point. Let $A \subset P \times G$ be the universal family of the intersections of hyperplanes parametrized by lines. The scheme $A$ is also canonically identified with the Grassmannian bundle $\text{Gr}(2, T^*P)$ over $P$.

Let $X$ be a smooth scheme over $k$ and let $i : X \to P$ be an immersion. We construct a commutative diagram

$$\begin{array}{ccc}
X \times_P Q & \xleftarrow{p^\vee} & (X \times G)' \\
\downarrow & & \downarrow \\
P^\vee & \xleftarrow{D} & G.
\end{array}$$

(3.2)
Lemma 2.3.3. Hence, for the intersection numbers.

\((X \times G)^\circ = (X \times G) - (X \times_p A)\).

For a line \(L \subseteq P^\vee\), the morphism \(p_L : X_L \to L\) defined by pencil is defined by the cartesian diagram

\[
\begin{array}{ccc}
X_L & \longrightarrow & X \times_p Q \\
\downarrow p_L & & \downarrow p^\vee \\
L & \longrightarrow & P^\vee.
\end{array}
\]

It is also the fiber of the middle vertical arrow \((X \times G)^\prime \to D\) of (3.2) at the point of \(G\) corresponding to \(L\). For a line \(L\), the axis \(A_L \subseteq P\) is the intersection of hyperplanes parametrized by \(L\). If the axis \(A_L\) meets \(X\) properly, the scheme \(X_L\) is the blow-up of \(X\) at the intersection \(X \cap A_L\). The morphism \(p_L^\prime : X_L^\prime \to L\) (2.14) is the restriction of \(p_L : X_L \to L\) to the complement \(X_L^\circ = X - (X \cap A_L)\). The complement \(X_L^\circ\) is identified with \(X_L \cap (X \times G)^\circ\).

**Proposition 3.2.** Let \(C \subseteq T^*X\) be a closed conical subset and \(P(\widetilde{C}) \subseteq X \times_p Q = P(X \times_p T^*P)\) be the projectivization. Assume that every irreducible component of \(X\) and every irreducible component \(C_a\) of \(C\) are of dimension \(n\). Let \(L \subseteq P^\vee\) be a line and \(p_L^\circ : X_L^\circ \to L\) be the morphism defined by the pencil.

1. The complement \(X_L^\circ = (X_L^\circ \cap P(\widetilde{C})) \subseteq X_L^\circ\) is the largest open subset where \(p_L^\circ : X_L^\circ \to L\) is \(C\)-transversal.

2. Let \(u \in X_L^\circ\) be a closed point such that \(u\) is an isolated characteristic point of \(p_L^\circ : X_L^\circ \to L\). Then \(u\) is an isolated point of the intersection \(X_L^\circ \cap P(\widetilde{C}) \subseteq X \times_p Q\) and we have an equality

\[
(C, d_{p_L^\circ})_{T^*X,u} = (P(\widetilde{C}), X_L^\circ)_{X \times_p Q,u}
\]

of the intersection numbers.

3. Assume that \(k\) is algebraically closed and that \(C\) is irreducible. Suppose that \(p^\vee : X \times_p Q \to P^\vee\) is generically finite on \(P(\widetilde{C})\) and let \(\xi \in P(\widetilde{C})\) and \(\eta \in \Delta = p^\vee(P(\widetilde{C})) \subseteq P^\vee\) denote the generic point. Then, there exists a smooth dense open subscheme \(\Delta^\circ \subseteq \Delta\) satisfying the following condition:

For a line \(L \subseteq P^\vee\) meeting \(\Delta\) transversally at \(H \in \Delta^\circ\) and for an isolated characteristic point \(u\) of \(p_L^\circ : X_L^\circ \to L\) such that \((u, H) \in P(\widetilde{C})^\circ = P(\widetilde{C}) \times \Delta \Delta^\circ\), the both sides of (3.3) are equal to the inseparable degree \([\xi : \eta]_{\text{insep}}\).

Since \(P(\widetilde{C}) \subseteq X \times_p Q\) is of codimension \(n\), the intersection product in the right hand side of (3.3) is defined.

**Proof.** 1. Since \(X \times_p Q \to X\) is smooth, the immersion \(X_L^\circ \to X \times_p Q\) is \(C\)-transversal by Lemma 2.3.3. Hence, for \(x \in X_L^\circ\), the morphism \(p_L^\circ : X_L^\circ \to L\) is \(C\)-transversal at \(x\) and
only if \((x, p_L(x)) \in X \times P Q\) is not contained in \(P(\tilde{C})\) by Lemma 2.16.1 and by Lemma 2.6.1 applied to the cartesian diagram (3.4). Hence the assertion follows.

2. The point \(u\) is an isolated point of the intersection \(X_L^o \cap P(\tilde{C}) \subset X \times P Q\) by 1. Let \(\tilde{p}_L^o: P_L^o = P - A_L \to L\) denote the morphism \(p_L^o\) defined with \(X\) replaced by \(P\). Let \(\omega\) be a basis of \(T^*L\) on the image \(p_L(u)\) and let \(\tilde{p}_L^o\) denote the section \(X_L^o \to X_L^o \times P T^*P\) defined on a neighborhood of \(u\) by the pull-back of \(\omega\) by \(\tilde{p}_L^o\). Then, we have

\[
(C, dp_L^o)_{T^*X,u} = (C, \tilde{p}_L^o(\omega))_{T^*X,u} = (\tilde{C}, \tilde{p}_L^o(\omega)_{X \times P T^*P,x} = (P(\tilde{C}), \tilde{p}_L^o(\omega)_{X \times P Q,x}).
\]

The graph \(P_L^o \to P \times L\) of \(\tilde{p}_L^o\) is a regular immersion of codimension 1 and defines an exact sequence

\[
0 \to P_L^o \times_Q T^*_Q(P \times P^\vee) \to (P_L^o \times_P T^*P) \times_{P_L^o} (P_L^o \times_L T^*L) \to T^*P_L^o \to 0
\]

of vector bundles on \(P_L^o\). Since \(T^*_Q(P \times P^\vee) \to Q \times_P T^*P\) is the universal sub line bundle on \(Q = P(T^*P)\), the exact sequence (3.7) implies that \(P_L^o \times_L T^*L \to T^*P_L^o\) defines the restriction of the universal sub line bundle on \(P_L^o \subset Q\). Hence the image of the section \(\tilde{p}_L^o: X_L^o \to X \times_P Q = P(X \times_P T^*P)\) equals \(X_L^o \subset X \times_P Q\) and (3.6) implies (3.5).

3. Let \(\Delta^o \subset \Delta\) be a smooth dense open subscheme such that the base change \(P(\tilde{C})^o = P(\tilde{C}) \times_\Delta \Delta^o \to \Delta^o\) is the composition of a finite flat radical morphism of degree \([\xi : \eta]_{\text{insep}}\) and a finite étale morphism. Then, for a line \(L\) and \((u, H) \in P(\tilde{C})^o\) as in the assumption, the intersection number \((P(\tilde{C}), X_L^o)_{X \times_P Q,u}\) equals the degree of the localization at \(u\) of the fiber of the finite flat morphism \(P(\tilde{C})^o \to \Delta^o\) at \(H\) and is equal to \([\xi, \eta]_{\text{insep}}\).

We show that the intersection number \((C, p_L^o)_{T^*X,u}\) defines a flat function on the universal family. Define an open subset

\[
(X \times G)^\vee \subset (X \times G)^o
\]

of \((X \times G)^o \subset (X \times G)^\vee\) (3.3) to be the largest open subset such that the inverse image

\[
Z(\tilde{C}) = P(\tilde{C}) \times_{X \times_P Q} (X \times G)^\vee
\]

is quasi-finite over \(G\).

**Lemma 3.3.** 1. For the pair \((u, L) \in (X \times G)^o\) of a line \(L \subset P^\vee\) and \(u \in X_L^o\), the following conditions are equivalent:

1. The pair \((u, L) \in (X \times G)^o\) is a point of \(Z(\tilde{C})\).
2. \(u \in X_L^o\) is an isolated characteristic point of \(p_L^o: X_L^o \to L\) with respect to \(C\).

2. Let \(C \subset T^*X\) be a closed conical subset. Assume that every irreducible component of \(X\) and every irreducible component \(C_a\) of \(C = \bigcup_a C_a\) are of dimension \(n\). Then, for a linear combination \(A = \sum a m_a[C_a]\), the intersection number

\[
(A, dp_L^o)_{T^*X,u}
\]

regarded as a function on \((u, L) \in Z(\tilde{C})\) is constructible and flat (Definition 1.1) over \(G\).

**Proof.** 1. Clear from Proposition 3.2.1.

2. Define a cycle \(P(\tilde{A})\) of \(P(X \times_P T^*P) = X \times_P Q\) supported on \(P(\tilde{C})\) by \(P(\tilde{A}) = \sum a m_a[P(\tilde{C}_a)]\). By Proposition 3.2.2, it suffices to show that \((P(\tilde{A}), X_L^o)_{X \times_P Q,u}\) regarded as a function \(\varphi_A\) on \(Z(\tilde{C})\) is constructible and flat over \(G\).
It suffices to show the case where $A = C_a$. Since $G$ is regular, the complex of $\mathcal{O}_{(X \times G)^\circ}$-modules $\mathcal{O}_{P(C_a)} \otimes_{\mathcal{O}_{(X \times G)^\circ}} \mathcal{O}_{(X \times G)^\circ}$ is of finite tor-dimension as a complex of $\mathcal{O}_G$-modules. Hence the function $\varphi_A$ on $Z(\widetilde{C})$ is constructible and flat over $G$ by Lemma 1.3.1.

3.2 Definition of characteristic cycle and the Milnor formula

We state and prove the existence of characteristic cycle satisfying the Milnor formula.

**Theorem 3.4** (cf. [14, Principe p. 7]). Let $X$ be a smooth scheme over a perfect field $k$ of characteristic $p > 0$ (resp. $p = 0$) and $\mathcal{F}$ be a constructible complex of $A$-modules of finite tor-dimension on $X$. Let $C = \bigcup_a C_a$ be a closed conical subset of the cotangent bundle $T^*X$ such that $\mathcal{F}$ is micro-supported on $C$. Assume that every irreducible component of $X$ and every irreducible component $C_a$ of $C$ is of dimension $n$. Then, there exists a unique $\mathbb{Z}[1/p]$-linear (resp. $\mathbb{Z}$-linear) combination $\text{Char}_C \mathcal{F} = \sum_a m_a[C_a]$ satisfying the following condition:

$$\dim \text{tot } \phi_u(j^* \mathcal{F}, f) = (\text{Char}_C \mathcal{F}, df)_{T^*W;u}.$$ (3.11)

The proof of Theorem 3.4 will occupy the rest of this subsection. Let $\Lambda_0$ denote the residue field of the finite local ring $\Lambda$ and set $\mathcal{F}_0 = \mathcal{F} \otimes_{\Lambda} \Lambda_0$. Then, we have $\dim \text{tot } \phi_u(j^* \mathcal{F}, f) = \dim \text{tot } \phi_u(j^* \mathcal{F}_0, f)$ and $\mathcal{F}_0$ is micro-supported on $C$ if and only if $\mathcal{F}$ is micro-supported on $C$ by Lemma 2.10.7. Thus, to show Theorem 3.4, it suffices to consider the case where $\Lambda$ is a field.

**Corollary 3.5.** The linear combination $\text{Char}_C \mathcal{F}$ is independent of $C$ on which $\mathcal{F}$ is micro-supported.

**Proof.** It suffices to show $\text{Char}_C \mathcal{F} = \text{Char}_{C'} \mathcal{F}$ assuming $C \subset C'$. A pair $X \leftarrow W \rightarrow Y$ of étale morphism $W \rightarrow X$ and a morphism to a curve $Y$ with an isolated characteristic point $u$ with respect $C'$ has isolated characteristic point $u$ with respect $C \subset C'$. Hence by the uniqueness, we have $\text{Char}_C \mathcal{F} = \text{Char}_{C'} \mathcal{F}$. 

**Definition 3.6.** We define the characteristic cycle $\text{Char} \mathcal{F}$ to be $\text{Char}_C \mathcal{F}$ independent of $C$ on which $\mathcal{F}$ is micro-supported.

**Conjecture 3.7.** The characteristic cycle $\text{Char} \mathcal{F}$ has integral coefficients.

The Milnor formula [11] and (3.11) imply that for the constant sheaf $\Lambda$, we have

$$\text{Char } \Lambda = (-1)^n \cdot [T^*_X X].$$

We will give more examples and properties in Sections 3.3 and 4.1.

First, we define a linear combination $\text{Char}_p \mathcal{F} = \sum_a m_a[C_a]$ using an embedding $X \rightarrow \mathbb{P} = \mathbb{P}(E)$ as in Section 3.1 satisfying the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22, that may a priori depend on the choice. Then we will prove that it satisfies the Milnor formula for morphisms defined by pencils.

Assume that $X$ is quasi-projective over $k$ and let $\mathcal{L}$ be an ample invertible $\mathcal{O}_X$-module. Let $E \subset \Gamma(X, \mathcal{L})$ be a subspace of finite dimension defining an immersion $X \rightarrow \mathbb{P} =
$\mathbf{P}(E^\vee)$ and satisfying the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22. Recall that $Z(\tilde{C}) \subset (X \times G)^\vee$ defined in (3.9) and (3.8) fits in a commutative diagram

$$
\begin{array}{ccc}
Z(\tilde{C}) & \longrightarrow & (X \times G)^\vee \\
\downarrow & & \downarrow f \\
\mathbf{P}(\tilde{C}) & \longrightarrow & X \times_P Q \\
\downarrow & & \downarrow \\
& & G \\
\end{array}
$$

(3.12)

where the left square is cartesian.

In order to define a candidate $\text{Char}_E^F$ for $\text{Char}_C^F$, we introduce a function $\varphi_F$ on $Z(\tilde{C}) \subset (X \times G)^\vee$ using the diagram

$$
\begin{array}{ccc}
Z(\tilde{C}) & \longrightarrow & (X \times G)^\vee \\
\downarrow p^\vee & & \downarrow f \\
& & D \\
\downarrow g & & \\
& & G \\
\end{array}
$$

as in (1.21). For a point $L$ of $G$, the fiber of $f: (X \times G)^\vee \to D$ is a restriction of $p^\vee_L: X^\circ \to L$. We define $\varphi_F$ by

$$
\varphi_F(z) = \dim \text{tot}_u \phi_u(F, p^\circ_L)
$$

(3.14)

as (1.25) for a point $z \in Z(\tilde{C})$ corresponding to the pair $(u, L)$ of an isolated characteristic point $u$ of the morphism $p^\circ_L: X^\circ \to L$ defined by $L \subset \mathbf{P}^\vee$.

**Lemma 3.8.** The function $\varphi_F$ is constructible and flat over $G$.

*Proof.* We apply Proposition 1.18 to the diagram (3.13). The morphism $f: (X \times G)^\vee \to D$ is locally acyclic relatively to the pull-back of $\mathcal{F}$ on the complement of $Z(\tilde{C})$ by Lemma 2.16.2 and the left cartesian square in (3.12). The morphism $p^\vee: (X \times G)^\vee \to D$ is locally acyclic relatively to the pull-back of $\mathcal{F}$ by the generic universal local acyclicity [13, Théorème 2.13]. Since $Z(\tilde{C})$ is quasi-finite over $G$, the function $\varphi_F$ is constructible and flat over $G$ by Proposition 1.18.

There exists a dense open subscheme $Z(\tilde{C})^\circ \subset Z(\tilde{C})$ where the function $\varphi_F$ is locally constant. For each irreducible component $C_a$ of $C = \bigcup_a C_a$, the restriction $\mathbf{P}(\tilde{C}_a) \to D_a = p^\vee(\mathbf{P}(\tilde{C}_a))$ of $p^\vee: X \times_P Q \to \mathbf{P}^\vee$ is generically finite by Corollary 2.23.1 since $E$ is assumed to satisfy the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22. Thus the function $\varphi_F$ is constant on a dense open subscheme $Z(\tilde{C}_a)^\circ = Z(\tilde{C}_a) \cap Z(\tilde{C})^\circ \subset Z(\tilde{C}_a)$. Define an integer $\varphi_a$ to be the value of $\varphi_F$ on $Z(\tilde{C}_a)^\circ$ and let $\xi_a \in \mathbf{P}(\tilde{C}_a)$ and $\eta_a \in D_a$ be the generic points. We define

$$
\text{Char}_E^F = - \sum_a \frac{\varphi_a [\xi_a : \eta_a]_{\text{insep}}}{[C_a]}
$$

(3.15)

Since the inseparable degree $[\xi_a : \eta_a]_{\text{insep}}$ is a power of $p$ if $p > 0$ (resp. is 1 if $p = 0$), the coefficients in $\text{Char}_E^F$ are in $\mathbf{Z}_{\frac{1}{p}}$ (resp. in $\mathbf{Z}$).

We show that $\text{Char}_E^F$ satisfies the Milnor formula (3.11) for the morphism defined by pencil.
Proposition 3.9. Assume that \( \mathcal{F} \) is micro-supported on \( C = \bigcup_a C_a \subseteq T^*X \). Let \( \mathcal{L} \) be an ample invertible \( \mathcal{O}_X \)-module and \( E \subseteq \Gamma(X, \mathcal{L}) \) be a \( k \)-vector space of finite dimension defining an immersion \( X \to \mathbb{P} = \mathbb{P}(E^\vee) \). Assume that \( C \) satisfies the condition \( \text{C} \) before Proposition 2.22.

Then, \( A = \text{Char}^E_C \mathcal{F} \) (3.15) is a unique linear combination of irreducible components \( C_a \) satisfying the following condition: For every line \( L \subseteq \mathbb{P}^\vee \) and every isolated characteristic point \( u \in p_1^\times \colon X^\circ \subseteq L \to L \) with respect to \( C \), we have a Minnig formula

\[
(3.16) \quad -\dim \text{tot} \phi_a(j^* \mathcal{F}, p_1^\circ) = (A, dp_1^\circ)_{T^*X, u}.
\]

Proof. Let \( L \subseteq \mathbb{P}^\vee \) be a line and \( u \in X^\circ \subseteq L \) be an isolated characteristic point of \( p_1^\times : X^\circ \subseteq L \). Then we have \((u, L) \in Z(\widehat{C})\) by Lemma 3.3.1.

Shrinking \( Z(\widehat{C}) \) if necessary, we may assume that \( Z(\widehat{C})^0 = \bigsqcup_a Z(\widehat{C}_a)^0 \). If \((u, L) \in Z(\widehat{C}_a)^0 \), the left hand side of (3.16) is \(-\varphi_a \) by the definition of \( \varphi_a \). By Proposition 3.2.3, the left hand side of (3.16) is \(- \frac{\varphi_a}{\text{indep} \cdot [\xi_a : \eta_a]} \). Hence the equality (3.16) holds on the dense open subset \( Z(\widehat{C})^0 \subseteq Z(\widehat{C}) \). This also proves the uniqueness of \( A \).

The left hand side of (3.16) is the function \( \varphi_\mathcal{F} \) on \( Z(\widehat{C}) \) and is constructible and flat over \( G \) by Lemma 3.8. The right hand side of (3.16) is also a function on \( Z(\widehat{C}) \) flat over \( G \) by Lemma 3.3.2. Hence the equality (3.16) holds for every \((u, L) \in Z(\widehat{C})\) by Lemma 1.2.3. \( \square \)

The following interpretation in terms of the ramification theory is not used in the sequel. The local ring of \( \mathbb{P}^\vee \) at the generic point \( \eta_a \) of \( \Delta_a \) is a discrete valuation ring \( \mathcal{O}_{K_a} \). Hence the total dimension of the Galois representation of \( \phi_{\xi_a}(p^* \mathcal{F}, p^\vee) \) is defined.

Lemma 3.10.

\[
\varphi_a = -\dim \text{tot} \eta_a \phi_{\xi_a}(p^* \mathcal{F}, p^\vee)
\]

Proof. It follows from [29, Corollary 3.9.2] and \( \mathbf{D} = \mathbb{P}(TP^\vee) \). \( \square \)

To deduce Theorem 3.4 from Proposition 3.9, we prove the following consequence of Theorem 3.4 using the flatness of the Swan conductor Proposition 1.18. For morphisms \( f, g : X \to Y \) of schemes and a closed subscheme \( Z \subseteq X \) defined by the ideal sheaf \( \mathcal{I}_Z \subseteq \mathcal{O}_X \), we say that \( g \equiv f \) mod \( \mathcal{I}_Z \) if their restrictions to \( Z \) are the same.

Proposition 3.11. Let \( X \) be a smooth scheme over a perfect field \( k \) and let \( C \subseteq T^*X \) be a closed conical subset. Let \( f : X \to Y \) be a morphism to a smooth curve \( Y \) such that \( u \in X \) is an isolated characteristic point of \( f \) with respect to \( C \).

Regard \( C \subseteq T^*X \) as a reduced closed subscheme and regard \( X \) as a closed subscheme of \( T^*X \) by the section \( df : X \to T^*X \) defined by the pull-back of a basis of \( T^*Y \) at the image \( f(u) \). Let \( N \geq 2 \) be an integer such that \( m_a^{N-2} \subseteq \mathcal{O}_{X, u} \) annihilates the local ring at \( u \) of the fiber product \( X \times_{T^*X} C \). Let \( V \to X \) be an étale neighborhood of \( u \) and \( g : V \to Y \) be a morphism satisfying

\[
g \equiv f \mod m_a^N.
\]

1. The closed point \( u \in V \) is an isolated characteristic point of \( g \) with respect to \( C \).

2. Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules of finite tor-dimension on \( X \). Assume that \( \mathcal{F} \) is micro-supported on \( C \). Then, we have

\[
(3.17) \quad \dim \text{tot} \phi_a(\mathcal{F}, f) = \dim \text{tot} \phi_a(\mathcal{F}, g).
\]
3. Assume that every irreducible component of $X$ and every irreducible component $C_a$ of $C = \bigcup_a C_a$ is of dimension $n$. Then, we have

$$(C_a, df)_{T^*X,u} = (C_a, dg)_{T^*V,u}$$

for every irreducible component $C_a$ of $C$.

**Proof.** By taking an étale morphism to $\mathbb{A}^1_k$ on a neighborhood of $f(u) \in Y$ and by replacing $f$ by the composition, we may assume $Y = \mathbb{A}^1_k = \text{Spec } k[t]$ and $f(u) = 0$.

1. Since $dg \equiv df \mod m_u^{N+1}$, the ideal $m_u^{N+2}$ also annihilates the local ring at $u$ of the fiber product $V \times_{T^*X} C$ with respect to the section $dg: V \to T^*V$ by Nakayama’s lemma. Hence $u$ is also an isolated characteristic point of $g: V \to Y$.

2. By replacing $X$ by $V$ and $k$ by an algebraic closure, we may assume $X = V$ and $k$ is algebraically closed. Define a commutative diagram

$$
\begin{align*}
\begin{array}{ccc}
X \times \mathbb{A}^1 & \xrightarrow{h} & Y \times \mathbb{A}^1 \\
\downarrow \text{pr}_2 & & \downarrow \text{pr}_2 \\
\mathbb{A}^1 = \text{Spec } k[s] & & 
\end{array}
\end{align*}
$$

by $h = (1-s)f + sg$. It is a homotopy connecting $f$ to $g$. By the assumption on $g$, we have $h \equiv f \mod m_u^N$. Hence there exists a neighborhood $W \subset X \times \mathbb{A}^1$ of $\{u\} \times \mathbb{A}^1$ such that $h: X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$ over $\mathbb{A}^1$ is $C$-transversal on $W - (\{u\} \times \mathbb{A}^1)$. Thus, the morphism $h: X \times \mathbb{A}^1 \to Y \times \mathbb{A}^1$ is locally acyclic on $W - (\{u\} \times \mathbb{A}^1)$ relatively to the pull-back $\text{pr}_1^*F$.

We apply Proposition 1.18 to the restriction of the diagram (3.18) and the restriction of $\text{pr}_1^*F$ on $W \subset X \times \mathbb{A}^1$. The projection $\text{pr}_2: X \times \mathbb{A}^1 \to \mathbb{A}^1$ is locally acyclic relatively to the pull-back of $F$ by the generic universal local acyclicity [13, Théorème 2.13]. Hence by Proposition 1.18, the function $\varphi_{\text{pr}_1^*F,h}$ on $Z$ is locally constant. Since $Z = \{u\} \times \mathbb{A}^1$ is connected, it is a constant function and we obtain $\varphi_{\text{pr}_1^*F,h}(0) = \varphi_{\text{pr}_1^*F,h}(1)$ namely the equality (3.17).

2. The localizations of $X \times_{T^*X} C$ at $u$ defined by $df$ and by $dg$ are isomorphic to each other. Hence, we have an equality $(C_a, df)_{T^*X,u} = (C_a, dg)_{T^*V,u}$ for every irreducible component $C_a$ of $C = \bigcup_a C_a$. \hfill $\Box$

Combining Proposition 3.9 with Proposition 3.11, we prove (3.11) with $E$ replaced by $E^{(n)}$ for sufficiently large $n$.

**Corollary 3.12.** Assume that $F$ is micro-supported on a closed conical subset $C \subset T^*X$. Assume that every irreducible component of $X$ and every irreducible component of $C$ is of dimension $n$. Let $L$ be an ample invertible $O_X$-module and $E \subset \Gamma(X, L)$ be a $k$-vector space of finite dimension defining an immersion $X \to \mathbb{P} = \mathbb{P}(E^*)$.

Let $f: V \to Y$ be a morphism to a smooth curve over $k$ defined on an étale neighborhood $V$ of a closed point $u$. Assume that $u$ is an isolated characteristic point of $f$ with respect to $C$. Then, there exists an integer $M \geq 1$ such that for every $n \geq M$, we have

$$
- \dim \text{tot}\phi_u(F, f) = (\text{Char}_C^{E^{(n)}} F, df)_{T^*V,u}.
$$

To prove Corollary 3.12, we show the existence of a pencil $L$ such that the morphism $p_L^*: X_L \to L$ approximates $f$ sufficiently.
Lemma 3.13. Let \( u \) be a closed point of \( X, N \geq 1 \) be an integer and \( f \in \mathcal{O}_{X,u}/m_u^N \). Then, for \( n \geq N \), there exist \( l_0, l_\infty \in E^{(n)} = \Gamma(X, \mathcal{L}^{\otimes n}) \) such that \( l_\infty(u) \neq 0 \) and

\[
(3.20) \quad l_0/l_\infty \equiv f \mod m_u^N.
\]

Proof. Since \( E \) defines an immersion \( X \to \mathbf{P} \), the composition \( E \subset \Gamma(X, \mathcal{L}) \to \mathcal{L}_u/m_u^2 \mathcal{L}_u \) is a surjection. Hence for an integer \( n \geq 1 \), the composition \( E^{(n)} \subset \Gamma(X, \mathcal{L}^{\otimes n}) \to \mathcal{L}_u^{\otimes n}/m_u^{n+1} \mathcal{L}_u^{\otimes n} \) is also a surjection. Take a section \( l \in E \) such that \( l(u) \neq 0 \). Then, it suffices to identify \( \mathcal{O}_{X,u}/m_u^n \) with \( \mathcal{L}^{\otimes n}/m_u^n \mathcal{L}^{\otimes n} \) by the local basis \( l_\infty = l^{\otimes n} \in E^{(n)} \) at \( u \). \( \square \)

Proof of Corollary 3.12. Let \( N \geq 2 \) be an integer as in Proposition 3.11 for \( f \) and \( u \) and set \( M = \max(N, 3) \). By Lemma 2.21, for \( n \geq M \geq 3 \), the subspace \( E^{(n)} \subset \Gamma(X, \mathcal{L}^{\otimes n}) \) satisfies the condition (E) before Lemma 2.21. The condition (C) before Proposition 2.22 is also satisfied. By Lemma 3.13, for \( n \geq M \geq N \), there exists \( l_0, l_\infty \in E^{(n)} \) such that the morphism \( p_L : X^L \to L \) for the line \( L \) spanned by \( l_0, l_\infty \) satisfies \( p_L^\flat = f \equiv f \) mod \( m_u^N \). Then, we have (3.16) with \( A \) replaced by \( \text{Char}_C^{E^{(n)}} \mathcal{F} \) by Proposition 3.9. Further, the both sides of (3.16) are equal to the corresponding sides of (3.19) by Proposition 3.11. Thus the assertion follows. \( \square \)

Proof of Theorem 3.4. Since the assertion is local on \( X \), we may assume that \( X \) is quasi-projective. Further, we may assume \( k \) is algebraically closed. By Lemma 2.21, Proposition 3.9 and Corollary 3.12, it suffices to show that the characteristic cycle \( \text{Char}_C^{E^{(n)}} \mathcal{F} \) is independent of the choice of \( E \subset \Gamma(X, \mathcal{L}) \) defining an immersion \( X \to \mathbf{P}(E') \) and satisfying the condition (E) before Lemma 2.21.

Let \( E \subset \Gamma(X, \mathcal{L}) \) be a subspace satisfying the condition (E). Let \( E' \subset \Gamma(X, \mathcal{L}') \) be another subspace defining an immersion \( X \to \mathbf{P}(E') \) and \( C_a \) be an irreducible component of \( C = \bigcup_a C_a \). It suffices to show that the coefficient \( m_a \) of \( C_a \) in \( \text{Char}_C^{E^{(n)}} \mathcal{F} \) equals the coefficient \( m_a^{(n)} \) of \( C_a \) in \( \text{Char}_C^{E^{(n)}} \mathcal{F} \) for sufficiently large \( n \geq 1 \).

Let \( (u, L) \in \mathbf{Z}(\mathbf{C}_a)^2 \) be a point as in the proof of Proposition 3.9 and let \( N \geq 2 \) be an integer as in Proposition 3.11 for \( p_L^\flat \) and \( u \). Then, as in the proof of Corollary 3.12, there exists an integer \( M \geq 1 \) such that for every \( n \geq M \), there exists a line \( L' \subset \mathbf{P}(E'^{(n)}) \) such that \( p_{L'}^\flat = p_L^\flat \) mod \( m_u^N \). We have \( \dim \text{tot}_u \phi(F, p_{L'}) = \dim \text{tot}_u \phi(F, p_L^\flat) \). Therefore, we have \( C_a, dp_L)_{T^*X^L_u} = (C_b, dp_L')_{T^*X^L_{u'}} \neq 0 \). Hence, we have \( m_a = m_a^{(n)} \).

3.3 Elementary properties of characteristic cycles

We keep assuming that \( k \) is perfect and \( X \) is smooth of dimension \( n \) over \( k \). Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules of finite tor-dimension on \( X \).

**Lemma 3.14.** 1. If \( \mathcal{F} \) is locally constant, we have

\[
(3.21) \quad \text{Char} \mathcal{F} = (-1)^n \text{rank} \mathcal{F} \cdot [T^*_X X].
\]

2. For an étale morphism \( j : U \to X \), we have

\[
(3.22) \quad \text{Char} j^* \mathcal{F} = j^* \text{Char} \mathcal{F}.
\]
3. Assume that \( \dim X = 1 \) and let \( U \subset X \) be a dense open subscheme where \( \mathcal{F} \) is locally constant. For \( x \in X - U \), let \( \bar{\eta}_x \) denote a geometric generic point of the strict localization at a geometric point \( \bar{x} \) above \( x \) and let

\[
a_x(\mathcal{F}) = \text{rank} \mathcal{F}_{\bar{\eta}_x} - \text{rank} \mathcal{F}_x + \text{Sw}_x \mathcal{F}_{\bar{\eta}_x}
\]

be the Artin conductor. Then, we have

\[
\text{Char} \mathcal{F} = -\left( \text{rank} \mathcal{F} \cdot [T_x^*X] + \sum_{x \in X - U} a_x(\mathcal{F}) \cdot [T_x^*X] \right)
\]

Proof. 1. It follows from the Milnor formula [11]. We will give another proof in the proof of Theorem 4.6.

2. Since the characterization (3.11) is an étale local condition, the assertion follows.

3. By Lemma 2.11.2, it suffices to determine the coefficients. For the 0-section \( T^*_X \), it follows from 1 and 2. For the fibers, since \( \dim \text{tot}_x \varphi_x(\mathcal{F}, \text{id}) = a_x(\mathcal{F}) \), it follows from the Milnor formula (3.11) for the identity \( X \to X \).

For surfaces, the characteristic cycle is studied in [30].

**Definition 3.15.** Let \( i: X \to Y \) be a closed immersion of smooth schemes over \( k \) and let

\[
T^*X \leftarrow X \times_Y T^*Y \to T^*Y
\]

be the canonical morphisms. Let \( C \subset T^*X \) be a closed conical subset. Assume that every irreducible component of \( X \) and every irreducible component \( C_a \) of \( C = \bigcup_a C_a \) is of dimension \( n \) and that every irreducible component of \( Y \) is of dimension \( m \). Then, for a linear combination \( A = \sum a_m C_a \), we define \( i_* A \) to be \((-1)^{n-m}\) times the push-forward by the second arrow \( X \times_Y T^*Y \to T^*Y \) in (3.25) of the pull-back of \( A \) by the first arrow \( X \times_Y T^*Y \to T^*X \) in the sense of intersection theory.

**Lemma 3.16.** Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules of finite tor-dimension on \( X \).

1. For a distinguished triangle \( \to \mathcal{F}' \to \mathcal{F} \to \mathcal{F}'' \to \) in \( D_{ctf}(X, \Lambda) \), we have

\[
\text{Char} \mathcal{F} = \text{Char} \mathcal{F}' + \text{Char} \mathcal{F}''.
\]

2. For a closed immersion \( i: X \to Y \) of smooth schemes over \( k \), we have

\[
\text{Char} i_* \mathcal{F} = i_* \text{Char} \mathcal{F}.
\]

3. For a morphism \( f: X \to Y \) of smooth schemes over \( k \), we have

\[
\text{Char} Rf_* \mathcal{F} = \text{Char} Rf_* \mathcal{F}.
\]

4. We have

\[
\text{Char} D_X \mathcal{F} = \text{Char} \mathcal{F}.
\]
**Proof.** 1. By the characterization of characteristic cycle by the Milnor formula (3.11), it follows from the additivity of the total dimension.

2. Let $C \subset T^*X$ be the singular support of $\mathcal{F}$. Then, $i_*\mathcal{F}$ is micro-supported on $i_*C \subset T^*Y$ by Lemma 2.10.6. Let $Y \to P$ be an immersion satisfying the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22 for $i_*C$. Then, by the description of the characteristic cycle $\text{Char} \mathcal{F} = \text{Char}^{E}_{\mathcal{C}} \mathcal{F}$ in (3.15), it follows from the canonical isomorphism $\phi(\mathcal{F}, \rho^*_L \circ i) \to \phi(i_*\mathcal{F}, \rho^*_L)$ for the morphism $\rho^*_L : Y^0 \to L$ defined by a pencil $L$.

3. By 1, it follows from [26].

4. We have $SS\mathcal{F} = SS\mathcal{D}_X \mathcal{F}$ by Corollary 2.25. By 2 and Lemma 3.14.2, we may assume $X$ is projective as in the proof of Corollary 2.25. Let $C = SS\mathcal{F} = SS\mathcal{D}_X \mathcal{F}$ be the singular support. Let $X \to P$ be a closed immersion satisfying the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22. Then, for a point $(u, L)$ in the dense open subset $\mathcal{P}(\mathcal{C})^0 \subset \mathcal{P}(\mathcal{C}) \subset X \times_P Q$ and $v = p_L(u) \in L$, we have $\dim \text{tot} \phi_u(\mathcal{F}, \rho^*_{L}) = a_v(Rp^*_L \mathcal{F})|_L$ and similarly for $D_X \mathcal{F}$. Since $a_v(Rp^*_L \mathcal{F})|_L = a_v(D_L(Rp^*_L \mathcal{F}))|_L = a_v(Rp^*_L \mathcal{D}_X \mathcal{F})|_L$, it follows from the description of the characteristic cycle $\text{Char} \mathcal{F} = \text{Char}^{E}_{\mathcal{C}} \mathcal{F}$ in (3.15). □

For the residue field $\Lambda_0$ of $\Lambda$, a constructible complex $\mathcal{F}$ of $\Lambda$-modules of finite tor-dimension on $X$ is a perverse sheaf if and only if $\mathcal{F} \otimes^L_{\Lambda} \Lambda_0$ is a perverse sheaf.

**Proposition 3.17.** Assume $\mathcal{F}$ is a perverse sheaf on $X$.

1. ([14, Question p. 7]) We have

\begin{equation}
\text{Char} \mathcal{F} \geq 0
\end{equation}

2. The support of $\text{Char} \mathcal{F}$ equals $SS\mathcal{F}$.

**Proof.** By the description of the characteristic cycle $\text{Char} \mathcal{F} = \text{Char}^{E}_{\mathcal{C}} \mathcal{F}$ in (3.15), it follows from Lemma 2.26. □

**Corollary 3.18.** Let $j : U = X - D \to X$ be the open immersion of the complement of a Cartier divisor. Then, for a perverse sheaf $\mathcal{F}$ of $\Lambda$-modules on $U$, we have

\begin{equation}
SSR_j \mathcal{F} = SS_j \mathcal{F}.
\end{equation}

**Proof.** Since the open immersion $j : U = X - D \to X$ is an affine morphism, $Rj_* \mathcal{F}$ and $j_* \mathcal{F}$ are perverse sheaves on $X$ by [6, Corollaire 4.1.10]. Hence, it follows from Lemma 3.16.3 and Proposition 3.17.2. □

We show a generalization of the Milnor formula for nearby cycles over general base scheme.

**Definition 3.19.** Let $f : X \to Y$ be a morphism of smooth schemes over $k$ and $C \subset T^*X$ be a closed conical subset. Assume that every irreducible component of $X$ and every irreducible component $C_a$ of $C = \bigcup_a C_a$ are of dimension $n$ and that every irreducible component of $Y$ is of dimension $m$. We assume that $C$ satisfies the following condition (P) below stronger than (Q) in Definition 2.7:

(P) For every irreducible component $P$ of the inverse image of $df^{-1}(C)$ by the canonical morphism $df : X \times_Y T^*Y \to T^*X$, if $P$ is not a subset of the 0-section $X \times_Y T^*_Y$, then $f : X \to Y$ is finite on the base $Q \subset X$ of $P$ and $\dim P = \dim Y = m$.

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For a linear combination \( A = \sum a[m_a[C_a]] \), we define a linear combination

\[
(3.32) \quad f! A = \sum_a \sum_{P \in df^{-1}(C_a)} m_a \cdot f! P
\]

of irreducible closed conical subsets of \( T^*Y \) of dimension \( m \) as follows. If an irreducible component \( P \subset df^{-1}(C_a) \) is not contained in the 0-section, we define the coefficient of \( P \) as the pull-back of \( C_a \) by the first arrow \( X \times_Y T^*Y \to T^*X \) in (3.25) in the sense of intersection theory and the coefficient of \( f! P \) as the push-forward by the second arrow \( X \times_Y T^*Y \to T^*Y \). If otherwise, we define the coefficient of \( f! P \) to be 0.

For irreducible component \( P \subset df^{-1}(C_a) \), we have \( \dim P \geq \dim Y \) (cf. Proposition 4.9.2). If \( Y \) is a curve, the condition (P) is equivalent to that \( f: X \to Y \) has at most isolated characteristic points with respect to \( C \).

**Lemma 3.20.** Let \( f: X \to Y \) be a morphism of smooth schemes over \( k \). Let \( C \subset T^*X \) be a closed conical subset satisfying the condition (P) in Definition 3.19 and let \( f_! C \subset T^*Y \) be as defined in Definition 2.7. Define a closed subset \( B \subset X \) finite over \( Y \) to be the union of bases \( Q \) for \( P \subset df^{-1}(C_a) \) not contained in the 0-section.

Let \( g: Y \to Z \) be a smooth morphism to a smooth curve over \( k \). Let \( y \) be an isolated characteristic point of \( g: Y \to Z \) with respect to \( f_! C \) and let \( x \in B \cap f^{-1}(y) \) be a point in the fiber.

1. \( x \) is an isolated characteristic point of the composition \( g \circ f: X \to Z \) with respect to \( C \).

2. Let \( F \) be a constructible complex of \( \Lambda \)-modules of finite tor-dimension on \( X \) micro-supported on \( C \subset T^*X \). Then, \( R\Psi_f F \) is constructible and is of finite tor-dimension.

Assume that \( x \) is a unique point in the fiber \( B_y \) and let \( z = g(y) \) be the image. Then, the cycles \( f_! \text{Char} \ F \) satisfies the Milnor formula,

\[
(3.33) \quad -\dim \text{tot}\, \phi_y(R\Psi_f F \mid_{x \times_Y Y}, g) = (f_! \text{Char} \ F, dg)_{T^*Y, Y}
\]

**Proof.** 1. By Lemma 2.8, the composition \( g \circ f: X \to Z \) is \( C \)-transversal on the complement of \( f^{-1}(y) \). Since \( f: X \to Y \) is \( C \)-transversal on the complement of \( B \) and \( g: Y \to Z \) is smooth, the composition \( g \circ f: X \to Z \) is also \( C \)-transversal on the complement of \( B \) by Lemma 2.5.4. Since \( B \) is finite over \( Y \), the intersection \( B \cap f^{-1}(y) \) is finite and the assertion follows.

2. Since \( f: X \to Y \) is \( C \)-transversal on the complement of \( B \) finite over \( Y \), the complex \( R\Psi_f F \) is constructible and is of finite tor-dimension by Proposition 1.8.1 and Corollary 1.10.1.

Let \( p: X \to Z \) denote the composition \( g \circ f: X \to Z \). Then, by 1 and by the Milnor formula (3.11), we have

\[
(3.34) \quad -\dim \text{tot}\, \phi_x(F, p) = (\text{Char} \ F, dp)_{T^*X, X}
\]

For the left hand side of (3.33), we have a canonical isomorphism \( R\bar{g}_{(y)*}(R\Psi_f F \mid_{x \times_Y Y}) \to R\Psi_p F \mid_{x \times_C C} \) by (1.12). This implies the equality

\[
(3.35) \quad \dim \text{tot}\, \phi_y(R\Psi_f F \mid_{x \times_Y Y}, g) = a_z(R\bar{g}_{(y)*}(R\Psi_f F \mid_{x \times_Y Y})) = a_z(R\Psi_p F \mid_{x \times_C C}) = \dim \text{tot}\, \phi_x(F, p)
\]
where $\alpha$ denotes the Artin conductor (3.23).

We have an equality $(f_! \text{Char} \mathcal{F}, dg)_{T^*Y, y} = (\text{Char} \mathcal{F}, dp)_{T^*X, x}$ on the right hand side of (3.33), by the assumption that $x$ is the unique point in the fiber $B_y$. Hence the equality (3.34) is equivalent to the Milnor formula (3.33).

Lemma 3.20 can be reformulated by introducing the characteristic cycle $\text{Char} R\Psi f \mathcal{F}$ as follows. Since $R\Psi f \mathcal{F}|_{x \times Y}$ on $Y_{(y)}$ is constructible of finite tor-dimension, we may regard it as the pull-back of a constructible sheaf on an étale neighborhood of $y = f(x)$ and define the characteristic cycle

$$\text{Char} R\Psi f \mathcal{F}|_{x \times Y}$$

as the pull-back. Since the construction of the characteristic cycle is compatible with the pull-back by étale morphism, it is well-defined. For example, if $Y$ is a curve, we have

$$\text{Char} R\Psi f \mathcal{F}|_{x \times Y} = -(\text{rank } \psi_u(f, f)[T^*_Y Y_{(y)}] + \dim \text{tot}_y \phi_u(f, f)[T^*_y Y_{(y)}]).$$

**Proposition 3.21.** Let the notation and the assumptions be as in Lemma 3.20. Then, we have

$$\text{Char} R\Psi f \mathcal{F}|_{x \times Y} = (f_! \text{Char} \mathcal{F})|_{x \times Y} \mod \langle T^*_Y Y_{(y)} \rangle$$

where the congruence means that the both cycles have the same coefficients except that of the 0-section.

By (3.37), the equality (3.38) for a curve $Y$ is equivalent to the Milnor formula (3.11).

**Proof.** Let $C = SS \mathcal{F}$ be the singular support. Then $R\Psi f \mathcal{F}$ is weakly micro-supported on $f_!C$ by Lemma 2.15. Hence it suffices to compare the coefficient of each irreducible component $C_a$ of $f_!C$ except that for the 0-section.

As in the proof of Lemma 2.26, locally on $Y$, there exists a smooth morphism $Y \to Z$ to a smooth curve with isolated characteristic point with respect to $f_!C$ that is $C_b$-transversal for every irreducible component $C_b$ of $f_!C$ different from $C_a$ but is not $C_a$-transversal. Thus, it follows from Lemma 3.20.

4 Pull-back of characteristic cycle and the index formula

We prove that the construction of the characteristic cycles is compatible with the pull-back by properly transversal morphisms. We will derive from this in Section 4.6 an index formula for the Euler number.

In this section, we assume that irreducible components of a smooth scheme over $k$ have the same dimension unless otherwise stated. We also assume that irreducible components of a closed conical subset of the cotangent of a smooth scheme over $k$ have the same dimension as the base scheme.
4.1 Pull-back of characteristic cycle

In this subsection, we assume that $X$ and $W$ are smooth schemes over a perfect field $k$. We assume that every irreducible component of $X$ (resp. of $W$) is of dimension $n$ (resp. $m$) and that every irreducible component of a closed conical subset $C \subset T^*X$ is of dimension $n$. We assume that a constructible complex $F$ of $\Lambda$-modules on $X$ is of finite tor-dimension.

**Definition 4.1.** Let $X$ and $W$ be smooth schemes over a perfect field $k$ and let $C \subset T^*X$ be a closed conical subset. Assume that every irreducible component of $X$ and every irreducible component of $C$ are of dimension $n$ and that every irreducible component of $W$ is of dimension $m$.

1. We say that a $C$-transversal morphism $h : W \to X$ over $k$ is properly $C$-transversal if every irreducible component of $h^*C = W \times_X C$ is of dimension $m$.

2. Let $h : W \to X$ be a properly $C$-transversal morphism and let

\begin{equation}
(4.1) \quad T^*W \longleftarrow W \times_X T^*X \longrightarrow T^*X
\end{equation}

be the canonical morphisms. Then, for a linear combination $A = \sum a[C_a]$ of irreducible components of $C = \bigcup C_a$, we define $h^!A$ to be $(-1)^{n-m}$-times the push-forward by the first arrow $W \times_X T^*X \to T^*W$ in (4.1) of the pull-back of $A$ by the second arrow $W \times_X T^*X \to T^*X$ in the sense of intersection theory.

**Lemma 4.2.** Let $h : W \to X$ be a morphism of smooth scheme over $k$ and $C \subset T^*X$ be a closed conical subset. Assume that every irreducible component of $X$ is of dimension $n$ and that every irreducible component of $W$ is of dimension $m = n - c$. Let $\dim_{\text{h}(W)} C$ denote the minimum of $\dim C \cap U$ where $U$ runs through open neighborhood of the image $h(W)$.

1. For $h^*C = W \times_X C$, we have $\dim h^*C \geq \dim \text{h}(W) - c$.

2. Let $g : V \to W$ be a morphism of smooth scheme over $k$. Assume that every irreducible component of $C$ is of dimension $n$ and that every irreducible component of $V$ is of dimension $l = m - c'$. Assume that $h : W \to X$ is $C$-transversal and that $h \circ g : V \to X$ is properly $C$-transversal. Then, $h : W \to X$ is properly $C$-transversal on a neighborhood of the image $g(V)$.

**Proof.** 1. If $h$ is smooth, we have $\dim h^*C = \dim_{\text{h}(W)} C - c$. Hence, it suffices to consider the case where $h$ is a regular immersion of codimension $c$. Then it follows from [16, Chap. 0 Proposition (16.3.1)] (cf. Proposition 4.9.2).

2. By 1, we have $\dim g^*(h^*C) \geq \dim g(V) h^*C - c' \geq n - c - c'$. By the assumption that $h \circ g : V \to X$ is properly $C$-transversal, we have an equality $\dim g^*(h^*C) = n - c - c'$. Hence, we have $\dim g(V) h^*C - c' = n - c$ and $h : W \to X$ is properly $C$-transversal on a neighborhood of $g(V)$. \hfill $\square$

If $h : W \to X$ is properly $C$-transversal, then every irreducible component of $h^*C$ is of dimension $\dim W$ by Lemma 2.1. A smooth morphism $h : W \to X$ is properly $C$-transversal by Lemma 2.3.1.

**Example 4.3.** ([29, Example 2.18]) Let $p > 2$ and $X = \mathbb{A}^2 = \Spec k[x, y] \supset U = \mathbb{G}_m \times \mathbb{A}^1 = \Spec k[x^{\pm 1}, y]$. Let $\mathcal{G}$ be a locally free sheaf of rank $1$ on $U$ defined by the Artin-Schreier equation $t^p - t = y/x^p$. Then, the singular support $C = SS_j \mathcal{G}$ for the open immersion $j : U \to X$ equals the union of the $0$-section $T^*_X X$ with the line bundle $\langle dy \rangle_D$.
on the y-axis $D = X - U$ spanned by the section $dy$. Hence, the immersion $D \to X$ is $C$-transversal but is not properly $C$-transversal.

**Theorem 4.4.** Let $X$ and $W$ be smooth schemes over a perfect field $k$ and let $C \subset T^*X$ be a closed conical subset. Assume that every irreducible component of $X$ and every irreducible component of $C$ is of dimension $n$ and that every irreducible component of $W$ is of dimension $m$.

Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$ of finite tor-dimension micro-supported on $C \subset T^*X$ and let $h: W \to X$ be a properly $C$-transversal morphism. Then, we have

\begin{equation}
\text{Char } h^*\mathcal{F} = h^!\text{Char } \mathcal{F}.
\end{equation}

We first prove Theorem 4.4 for a properly $C$-transversal immersion $i: W \to X$ of a smooth hyperplane section $W = X \cap H$ assuming $X \subset \mathbb{P}^n$ is embedded in a projective space. The first and the crucial step of the proof is the following Milnor formula for an isolated characteristic point of a morphism defined by a pencil.

**Proposition 4.5.** Let $L$ be an ample invertible $\mathcal{O}_X$-module, a subspace $E \subset \Gamma(X, L)$ of finite dimension defining an immersion $X \to \mathbb{P} = \mathbb{P}(E^\vee)$ satisfying the condition (E) before Lemma 2.21 and $H \subset \mathbb{P}$ be a hyperplane. Assume that $W = X \cap H$ is smooth, that the immersion $i: W \to X$ is properly $C$-transversal and that $C = \text{SS} \mathcal{F}$ satisfies the condition (C) before Proposition 2.22.

Let $u \in W$ be a closed point and $p^0_L: W^0_L \to L$ be a morphism defined by a pencil $L \subset H^\vee$ such that $u$ is an isolated characteristic point of $p^0_L: W^0_L \to L$. Then, we have

\begin{equation}
- \dim \text{tot } \phi_u(i^*\mathcal{F}, p^0_L) = (i^!\text{Char } \mathcal{F}, dp^0_L)_T^*W, u.
\end{equation}

We sketch an outline of the proof of Theorem 4.4 given in Section 4.5. We will state a description of characteristic cycle in terms of ramification theory as Theorem 4.6 below. At the end of this subsection, we deduce a special case of Theorem 4.4 stated as Proposition 4.7 below from the dimension 2 case of Theorem 4.6 proved in [30, Proposition 3.20]. In Section 4.5, we first prove Proposition 4.5 above using Proposition 4.7.

The proof of Proposition 4.5 is similar to that of Theorem 3.4 with the universal family of morphisms defined by pencils replaced by the universal family of morphisms defined by pencils on hyperplane sections, studied in Section 4.4. As preliminaries of the proof, we study the locus where an immersion is not properly $C$-transversal and the locus where the universal family is not $C$-transversal in Sections 4.2 and 4.3 respectively.

In Section 4.5, we deduce Theorem 4.4 from Proposition 4.5 and then deduce Theorem 4.6 from Theorem 4.4. In summary, the proof follows the logical diagram [30, Proposition 3.20] (= dimension 2 case of Theorem 4.6) $\Rightarrow$ Proposition 4.7 $\Rightarrow$ Proposition 4.5 $\Rightarrow$ Theorem 4.4 $\Rightarrow$ Theorem 4.6.

To formulate Theorem 4.6, we briefly recall the definition of the characteristic cycle [29, Definition 3.5] in the strongly non-degenerate case. We use the notation in Section 2.4.

Let $X$ be a smooth scheme of dimension $n$ over a perfect field $k$ and $U = X - D$ be the complement of a divisor $D$ with normal crossings. Let $j: U \to X$ be the open immersion and $\mathcal{G}$ be a locally constant constructible sheaf of free $\Lambda$-modules on $U$. Assume that the ramification of $\mathcal{G}$ along $D$ is strongly non-degenerate.
Assume first that $R = D$. Then, the locally constant sheaf $G$ on $U$ is tamely ramified along $D$ and we define a linear combination by

\[(4.4) \quad C(j_!G) = (-1)^n \text{rank } G \cdot \sum_I [T_{D_I}X]\]

where $T_{D_I}X$ denotes the conormal bundle of the intersection $D_I = \bigcap_i D_i$ for sets of indices $I \subset \{1, \ldots, m\}$.

Assume $R = \sum_i r_i D_i > D = \sum_i D_i$. For each irreducible component, we have a decomposition by characters $V = \bigoplus_{\chi} G^* \rightarrow F_x \chi^{m(\chi)}$. Further, the characteristic form of each character $\chi$ appearing in the decomposition defines a sub line bundle $L_\chi$ of the pull-back $D_\chi \times_X T^*_X$ of the cotangent bundle to a finite covering $\pi_\chi: D_\chi \rightarrow D_i$ by the non-degenerate assumption. Then we define

\[(4.5) \quad C(j_!G) = (-1)^n \left(\text{rank } G \cdot [T^*_X X] + \sum_i \sum_\chi r_i \cdot m(\chi) [D_\chi : D_i] \pi_\chi[L_\chi] \right).\]

In the general strongly non-degenerate case, we define $C(j_!G)$ by the additivity and étale descent.

**Theorem 4.6.** Let $X$ be a smooth scheme of dimension $n$ over a perfect field $k$ and let $j: U \rightarrow X$ be the open immersion of the complement $U = X - D$ of a divisor $D$ with simple normal crossings. Let $G$ be a locally constant constructible sheaf of free $\Lambda$-modules on $U$ such that the ramification along $D$ is strongly non-degenerate. Then we have

\[(4.6) \quad \text{Char } j_!G = C(j_!G),\]

\[(4.7) \quad \text{SS } j_!G = S(j_!G).\]

In other words, $C(j_!G)$ defined by (4.4), (4.5), the additivity and by étale descent using ramification theory satisfies the Milnor formula (3.11).

Theorem 4.6 is proved for dimension $\leq 1$ in Lemma 3.14.1 and 2. Recall that Theorem 4.6 is proved in dimension 2 in [30, Proposition 3.20] using a global argument, as in [14]. We will prove the general case in Section 4.5 as a consequence of Theorem 4.4. The tamely ramified case of Theorem 4.6 has been proved in [31] by a different method. Theorem 4.6 gives an affirmative answer to [29, Conjecture 3.16].

The dimension 2 case of Theorem 4.6 implies the following special case of Theorem 4.4, that will be used in the proof of Proposition 4.5.

**Proposition 4.7.** Let $X$ be a smooth scheme over a perfect field $k$ and $D \subset X$ be a smooth divisor. Let $F$ be a constructible sheaf of $\Lambda$-modules of finite tor-dimension on $X$. Assume that the restrictions $F|_D$ and $F|_U$ on $U = X - D$ are locally constant and that the ramification of every cohomology sheaf $H^q F|_U$ along $D$ is strongly non-degenerate. Define $C = T^*_D U \cap S(j_!F)$ to be the union with the conormal bundle where $j: U \rightarrow X$ denotes the open immersion.

Let

\[\begin{array}{ccc}
V & \xrightarrow{i} & X \\
\downarrow \sigma & & \downarrow \pi \\
W & \xrightarrow{h} & 
\end{array}\]
be immersions of a smooth curve $V$ and a smooth surface $W$. Assume that the immersion $i: V \to X$ is $C$-transversal.

1. The immersion $i: V \to X$ is properly $C$-transversal and $h: W \to X$ is properly $C$-transversal on a neighborhood of $V$.

2. We have

\begin{equation}
\text{Char } i^* \mathcal{F} = g^! \text{Char } h^* \mathcal{F}.
\end{equation}

**Proof.** 1. Since $i: V \to X$ is $T^*_D X$-transversal, the curve $V$ intersects the divisor $D$ transversely by Lemma 2.3.5. Hence the $C$-transversality of $i: V \to X$ implies the proper $C$-transversality. Since $i: V \to X$ is properly $C$-transversal, $g: W \to X$ is also properly $C$-transversal on a neighborhood $V$ by Lemma 4.2.2.

2. After replacing $W$ by a neighborhood $V$, we may assume that $g: W \to X$ is properly $C$-transversal. Hence the surface $W$ also intersects the divisor $D$ transversely by Lemma 2.3.5. We may assume $\Lambda$ is a field.

We show the equality (4.9) first assuming $\mathcal{F} = j_! \mathcal{G}$ for a locally constant constructible sheaf $\mathcal{G}$ on $U$ such that its ramification along $D$ is non-degenerate. Since $i: V \to X$ and $g: W \to X$ are properly $C$-transversal, the ramification of the restrictions $\mathcal{G}_V = \mathcal{G}|_{V \cap U}$ and $\mathcal{G}_W = \mathcal{G}|_{W \cap U}$ along $D \cap V$ and $D \cap W$ respectively are non-degenerate and we have $C(j_V \mathcal{G}_V) = i^! C(j_! \mathcal{G})$ and $C(j_W \mathcal{G}_W) = g^! C(j_! \mathcal{G})$ by [29, Proposition 3.8] where $j_V: V \cap U \to V$ and $j_W: W \cap U \to W$ denote the open immersions. Since Theorem 4.6 is proved in dimension $\leq 2$ in [30, Proposition 3.20], we have $\text{Char } i^* \mathcal{F} = C(j_V \mathcal{G}_V)$ and $\text{Char } h^* \mathcal{F} = C(j_W \mathcal{G}_W)$. Thus, the equality (4.9) in this case follows from $i^! C(j_! \mathcal{G}) = g^! h^! C(j_! \mathcal{G})$.

Next, we show the case where $\mathcal{F} = i_{D*} \mathcal{G}$ for the immersion $i_D: D \to X$ and a locally constant constructible sheaf $\mathcal{G}$ of $\Lambda$-modules on $D$. In this case, by Lemmas 3.14.1 and 3.16.2, we have $\text{Char } i^* \mathcal{F} = -\text{rank } \mathcal{G} \cdot [T^*_D V]$ and $\text{Char } h^* \mathcal{F} = \text{rank } \mathcal{G} \cdot [T^*_D W]$ and the assertion follows. In general, it follows by additivity.

\hfill \Box

### 4.2 Proper intersection

In this section, from Definition 4.8 to Corollary 4.10, we do not assume that the field $k$ is perfect or the scheme $X$ is smooth over $k$.

**Definition 4.8.** Let $X$ be a scheme of finite type over a field $k$.

1. Let $C \to X$ be a morphism of schemes of finite type. We say that a regular immersion $W \to X$ of codimension $c$ meets $C$ properly if for every irreducible component $P$ of $C$ and for every irreducible component $Q$ of $P \times_X W$, we have $\dim Q = \dim P - c$.

2. Let $(B_j)_{j \geq 0}$ be a decreasing sequence of closed subsets of $X$ indexed by integers $i \geq 0$ and let $n \geq 0$ be an integer. We say that $(B_j)_{j \geq 0}$ is of dimension $\leq n - \bullet$ if $\dim B_j \leq n - j$ for every $i \geq 0$.

Assume that $(B_j)_{j \geq 0}$ is of dimension $\leq n - \bullet$. We say that a regular immersion $W \to X$ of codimension $c$ meets $(B_j)_{j \geq 0}$ $n$-semi properly if $(B_j \times_X W)_{i \geq 0}$ is of dimension $\leq n - c - \bullet$.

Recall that a scheme is said to be equidimensional of dimension $n$ if its every irreducible component is of dimension $n$. The inequality $\dim C \leq n$ is equivalent to that every irreducible component of $C$ is of dimension $\leq n$. The inequality $\dim C \geq n$ is equivalent to the existence of irreducible component of $C$ of dimension $\geq n$. 

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Proposition 4.9. Let $C \to X$ be a morphism of schemes of finite type over a field $k$ and define subsets $B_j \subset X$ by $B_j = \{ x \in X \mid \dim C \times_X x \geq j \}$.

1. Assume that the subsets $B_j \subset X$ are closed for all $j \geq 0$. Then for an integer $n \geq 0$, the following conditions are equivalent:
   
   (1) $\dim C \leq n$.
   
   (2) The sequence $(B_j)_{j \geq 0}$ is of dimension $\leq n - \bullet$.

2. Assume that $C$ is equidimensional of dimension $n$. Then for a regular immersion $W \to X$ of codimension $c$, the following conditions are equivalent:

   (1) $W \to X$ meets $C$ properly.
   
   (2) $\dim C \times_X W \leq n - c$.

   If we further assume that $B_j \subset X$ are closed for all $j \geq 0$, then $(B_j)_{j \geq 0}$ is of dimension $\leq n - \bullet$ and the conditions (1) and (2) are equivalent to the following condition:

   (3) $W \to X$ meets $(B_j)_{j \geq 0}$ $n$-semi properly.

If $C$ is proper over $X$, the assumption in 1 that $B_j \subset X$ are closed is satisfied by [16, Corollaire (13.1.6)].

Proof. 1. $(2) \Rightarrow (1)$: Let $P$ be an irreducible component of $C$ and $Q \subset X$ be the closure of the image. Set $j = \dim P - \dim Q$. Then, we have $Q \subset B_j$ and (2) implies $\dim P = \dim Q + j \leq n - j + j = n$.

$(1) \Rightarrow (2)$: Let $Q$ be an irreducible component of $B_j$ and $P$ be an irreducible component of $C \times_X Q$ dominating $Q$. Then (1) implies $n \geq \dim P \geq \dim Q + j$.

2. $(1) \Rightarrow (2)$: By the assumption, (1) implies that for every irreducible component $P$ of $C$ and every irreducible component $Q$ of $P \times_X W$, we have $\dim Q = n - c$. Since $C \times_X W$ is the union of $P \times_X W$, the assertion follows.

$(2) \Rightarrow (1)$: By the assumption that every irreducible component $P$ of $C$ is of dimension $n$ and by [16, Chap. 0 Proposition (16.3.1)], for every irreducible component $Q$ of $P \times_X W$, we have $\dim Q \geq \dim P - c = n - c$. Hence (2) implies the equality.

If $B_j \subset X$ are closed for all $j \geq 0$, then $(B_j)_{j \geq 0}$ is of dimension $\leq n - \bullet$ by 1. The condition (2) is equivalent to (3) by 1 applied to $C \times_X W$. 

Let $E$ be a vector bundle on a scheme $X$. Recall that a closed subset $C$ of $E$ is said to be conical if it is stable under multiplication. The assumption of Proposition 4.9.1 is satisfied if $C$ is a closed conical subset of a vector bundle.

Corollary 4.10. Let $X$ be a scheme of finite type over $k$ and let $C$ be a closed conical subset of a vector bundle $E$ over $k$. Assume that every irreducible component of $X$ and every irreducible component of $C$ are of dimension $n$. For $j \geq 0$, define subsets $B_j$ of the base $B = C \cap X$ of $C$ by $B_j = \{ x \in X \mid \dim C \times_X x \geq j \}$ as in Proposition 4.9.

1. The subsets $B_j$ are closed subsets of $X$ and $(B_j)_{j \geq 0}$ is of dimension $n - \bullet$.

2. Let $W \to X$ be a regular immersion of codimension $c$. Then, the following conditions are equivalent:

   (1) $W \to X$ meets $C$ properly.
   
   (2) $W \to X$ meets $(B_j)_{j \geq 0}$ $n$-semi properly.

Proof. 1. Let $\pi : C \to X$ denote the projection. Then, the subsets $C_j = \{ s \in C \mid \dim_x \pi^{-1}(\pi(s)) \geq j \}$ are closed by [16, Théorème (13.1.3)]. Since $C$ is assumed to be a closed conical subset, the subsets $C_j$ are also closed conical subsets of $E$. Hence $B_j$ are their bases and are closed.
The sequence \((B_j)_{j \geq 0}\) is of dimension \(n - \bullet\) by Proposition 4.9.1.

2. It follows from 1 and Proposition 4.9.2. \(\square\)

Let \(X\) be a smooth scheme over a field \(k\) and let \(C \subset T^*X\) be a closed conical subset. We assume that every irreducible component of \(X\) and every irreducible component of \(C\) are of dimension \(n\). Let \(i\): \(X \to \mathbb{P} = \mathbb{P}(E^\vee)\) be an immersion. For \(j \geq 0\), define closed subsets \(B_j\) of the base \(B = C \cap T^*_X X \subset X\) by

\[
B_j = \{x \in X \mid \dim C \times_X x \geq j\}
\]

as in Proposition 4.9.

**Proposition 4.11.** Let \(H \subset \mathbb{P}\) be a hyperplane and regard the hyperplane section \(W = X \cap H\) as the fiber of \(p^\vee\): \(X \times_{\mathbb{P}} Q \to \mathbb{P}\) at the point \(H \in \mathbb{P}\). Define open subsets \(W_2 \subset W_1 \subset W\) by

\[
W_1 = W - W \cap \mathbb{P}(T^*_X \mathbb{P} \cup \widetilde{C}),
\]

\[
W_2 = W_1 - W_1 \cap \bigcup_{j \geq 0} \bigcup_{T \subset W, T \subset B_j, \dim T = n - j} T
\]

where \(T\) runs through irreducible components of \(B_j\) satisfying \(T \subset W\) and \(\dim T = n - j\).

1. The open subset \(W_1 \subset W\) is the largest smooth open subset of \(W\) where the immersion \(i\): \(W \to X\) is \(C\)-transversal.

2. The open subset \(W_2 \subset W\) is the largest smooth open subset of \(W\) where the immersion \(i\): \(W \to X\) is properly \(C\)-transversal.

**Proof.** 1. The complement \(W_0 = W - W \cap \mathbb{P}(T^*_X \mathbb{P})\) is the largest smooth open subset of \(W\). Since \(p\): \(X \times_{\mathbb{P}} Q \to X\) is smooth, the largest open subset of \(W_0\) where the immersion \(i\): \(W \to X\) is \(C\)-transversal is the the largest one where the immersion \(W_0 \to X \times_{\mathbb{P}} Q\) is \(p^C\)-transversal by Lemma 2.3.3. Since \(W_0 \to \{H\}\) is smooth, it further equals the intersection with the largest open subset of \(X \times_{\mathbb{P}} Q\) where \(p^\vee\): \(X \times_{\mathbb{P}} Q \to \mathbb{P}\) is \(p\)-\(C\)-transversal by Lemma 2.5.1 and Lemma 2.6.1 applied to the cartesian diagram

\[
\begin{array}{ccc}
X \times_{\mathbb{P}} Q & \leftarrow & W \\
\downarrow & & \downarrow \\
\mathbb{P} & \leftarrow & \{H\}
\end{array}
\]

Hence the assertion follows from Lemma 2.16.1.

2. For \(j \geq 0\), the largest open subset of \(W\) where the immersion \(i\): \(W \to X\) of codimension 1 meets \((B_j)_{j \geq 0}\) \(n\)-properly is the complement in \(W\) of the union of irreducible components \(T\) of dimension \(n - j\) of \(B_j\) contained in \(W\). Thus, it follows from 1 and Corollary 4.10. \(\square\)

For a closed subset \(T \subset X\), we regard it as a reduced subscheme of \(X\) and define a subspace \(E_T = \text{Ker}(E \to \Gamma(T, \mathcal{L} \otimes \mathcal{O}_T)) \subset E\) and

\[
P_T^\vee = \mathbb{P}(E_T) \subset \mathbb{P} = \mathbb{P}(E).
\]

Since \(H \in \mathbb{P}_T^\vee\) implies \(T \subset X \cap H\), we have \(T \times \mathbb{P}_T^\vee \subset T \times \mathbb{P} \subset X \times \mathbb{P}^\vee\). Define a closed subset \(\mathbb{P}(\bar{C})^+ \subset X \times_{\mathbb{P}} Q = \mathbb{P}(X \times_{\mathbb{P}} T^*\mathbb{P}) \subset X \times \mathbb{P}^\vee\) as the union

\[
\mathbb{P}(\bar{C})^+ = \mathbb{P}(T_X^* \mathbb{P} \cup \bar{C}) \cup \bigcup_{j \geq 0} \bigcup_{\dim T = n - j} (T \times \mathbb{P}_T^\vee)
\]
where $T$ runs through irreducible components of $B_j$ such that $\dim T = n - j$.

**Corollary 4.12.** Let $H \subset P$ be a hyperplane and regard the hyperplane section $W = X \cap H$ as the fiber of $X \times_P Q \to P^\vee$ at the point $H \in P^\vee$. Then, the complement $W - W \cap P(\overline{C})^+$ is the largest smooth open subset of $W$ where the immersion $h: W \to X$ is properly $C$-transversal.

**Proof.** For $T$ reduced, the condition $T \subset W$ is equivalent to $H \overset{\sim}{\to} T$. Hence $W_2$ in Proposition 4.11.2 equals the complement $W \setminus W \cap P(\overline{C})^+$. \hfill $\square$

### 4.3 The universal family of linear sections of codimension 2

Assume that $X$ is quasi-projective and let $E \subset \Gamma(X, L)$ be a subspace of finite dimension defining a closed immersion $X \to P = P(E^\vee)$. We identify the Grassmannian variety $G = \text{Gr}(2, E)$ parametrizing subspaces of dimension 2 of $E$ with the Grassmannian variety $G = \text{Gr}(1, P^\vee)$ parametrizing lines in $P^\vee$. The universal family $A \subset P \times G$ of linear subspace of codimension 2 of $P = P(E^\vee)$ consists of pairs $(x, L)$ of a point $x$ of the axis $A_L \subset P$ of a line $L \subset P^\vee$. The intersection $X \times_P A = (X \times G) \cap A$ is canonically identified with the bundle $\text{Gr}(2, X \times_P T^*P)$ of Grassmannian varieties parametrizing rank 2 subbundles.

We canonically identify the fiber product $X \times_P A \times G D$ with the flag bundle $\text{Fl}(1, 2, X \times_P T^*P)$ parametrizing pairs of sub line bundles and rank 2 subbundles of $X \times_P T^*P$ with inclusions. We consider the commutative diagram

$$
\begin{array}{ccc}
X \times_P Q & \leftarrow & X \times_P A \times_G D \\
\downarrow & & \downarrow \\
P^\vee & \leftarrow & D \\
\downarrow & & \downarrow \\
\text{Gr}(1, E) & \leftarrow & \text{Fl}(1, 2, E) \\
\downarrow & & \downarrow \\
\text{Gr}(2, E). & & \\
\end{array}
$$

(4.15)

defined as

$$
\begin{array}{ccc}
\text{Gr}(1, X \times_P T^*P) & \leftarrow & \text{Fl}(1, 2, X \times_P T^*P) \\
\downarrow & & \downarrow \\
\text{Gr}(1, E) & \leftarrow & \text{Fl}(1, 2, E) \\
\downarrow & & \downarrow \\
\text{Gr}(2, E). & & \\
\end{array}
$$

(4.16)

The horizontal arrows are forgetful morphisms and the vertical arrows are induced by the canonical injection $\Omega^1_P \to E \otimes \mathcal{O}_P$. The right square is cartesian.

Let $C \subset T^*X$ be a closed conical subset. Define a closed subset

$$
R(\tilde{C}) \subset X \times_P A = \text{Gr}(2, X \times_P T^*P) \subset X \times G
$$

(4.17)

to be the subset consisting of $(x, V)$ such that the intersection $V \cap (x \times X \tilde{C}) \subset x \times X T^*X$ is not a subset of 0. We also define a closed subset

$$
Q(\tilde{C}) \subset X \times_P A \times_G D
$$

(4.18)

to be the inverse image of $P(\tilde{C}) \subset X \times_P Q$ by the upper left horizontal arrow in (4.15). The subset $R(\tilde{C}) \subset X \times_P A$ is the image of $Q(\tilde{C})$ by the upper right arrow $X \times_P A \times_G D \to X \times_P A$ of (4.15).

Similarly as Lemma 2.16, we have the following.
Lemma 4.13. Let $C \subset T^*X$ be a conical closed subset. The complement $X \times_p A - R(\tilde{C})$ is the largest open subset where the pair $X \leftarrow X \times_p A \rightarrow G$ is $C$-transversal.

Proof. The proof is similar to that of Lemma 2.16. Since the projection $p: X \times_p A \rightarrow X$ is smooth, the pair $X \leftarrow X \times_p A \rightarrow G$ is $C$-transversal at $(x, V) \in X \times_p A$ if and only if $X \times_p A \rightarrow G$ is $p^*C$-transversal at $(x, V)$. The latter condition is equivalent to that the fiber at $(x, V)$ of the inverse image of the pull-back $p^*C \subset T^*(X \times_p A)$ by $(X \times_p A) \times_G T^*G \rightarrow T^*(X \times_p A)$ is a subset of the 0-section.

Note that since the subscheme $X \times_p A \subset X \times G$ is smooth over $X$, the canonical morphism $T^*_{X \times_p A}(X \times G) \rightarrow (X \times_p A) \times_G T^*G$ is an injection. Hence, by the isomorphism

$$C \text{oker}(T^*_{X \times_p A}(X \times G) \rightarrow ((X \times_p A) \times_X T^*X) \times_{X \times_p A}((X \times_p A) \times_G T^*G)) \rightarrow T^*(X \times_p A),$$

the condition above is further equivalent to that the fiber at $(x, V)$ of the inverse image to the conormal bundle $T^*_{X \times_p A}(X \times G)$ of the pull-back $p^*C \subset (X \times_p A) \times_X T^*X$ is not a subset of the 0-section. Since the conormal bundle $T^*_{X \times_p A}(X \times G) \subset X \times_p T^*P$ is the universal sub vector bundle of rank 2 on the Grassmanian bundle $X \times_p A = \text{Gr}(2, X \times_p T^*P)$, the condition is equivalent to $(x, V) \notin R(\tilde{C})$.

We relate the construction of $R(\tilde{C})$ with that of $i^!C$. Let $H \subset P$ be a hyperplane and assume that $W = X \cap H$ is smooth. Then, the surjection $W \times_p T^*P \rightarrow W \times_H T^*H$ induces a closed immersion

$$(4.19) \quad i^!: P(W \times_H T^*H) \longrightarrow X \times_p A = \text{Gr}(2, X \times_p T^*P).$$

Proposition 4.14. Let $C = \bigcup_a C_a \subset T^*X$ be a closed conical subset and let $H \subset P$ be a hyperplane. Assume that every irreducible component of $X$ and every irreducible component $C_a$ of $C$ are of dimension $n$ and that the immersion $i: W = X \cap H \rightarrow X$ is properly $C$-transversal.

Let $A = \sum_a m_a[C_a]$ be a linear combination of irreducible components of $C$. We define $i^!A$ as in Definition 4.1.2 and a cycle $P(i^!A)$ on $P(W \times_H T^*H)$ as in the proof of Lemma 3.3.2. We also define a cycle $R(A)$ on $X \times_p A$ to be the image of $P(\tilde{A}) = \sum_a m_a[P(\tilde{C}_a)]$ by the upper line

$$(4.20) \quad P(X \times_p T^*P) = X \times_p Q \leftarrow X \times_p A \times_G D \longrightarrow X \times_p A$$

of (4.15) regarded as an algebraic correspondence in the sense of intersection theory. Then, for the pull-back $i^*R(\tilde{A})$ in the sense of intersection theory by the immersion $i$ (4.19), we have

$$(4.21) \quad P(i^!A) = -i^*R(\tilde{A}).$$

Proof. We construct a commutative diagram

$$P(W \times_p T^*P) \leftarrow P(W \times_p T^*P)' \longrightarrow P(W \times_H T^*H)$$

$$(4.22) \quad X \times_p Q \leftarrow X \times_p A \times_G D \longrightarrow X \times_p A$$

where the lower line is the same as (4.20) and the right column is (4.19). The left column is the canonical inclusion $P(W \times_p T^*P) \rightarrow P(X \times_p T^*P) = X \times_p Q$. The upper left
Proof. The blow-up at the section \( W = P(W \times_H T_H^* P) \to P(W \times_P T^* P) \) and the upper right arrow is induced by the surjection \( W \times_H T^* H \to W \times_H T^* H \). Since the blow-up \( P(W \times_P T^* P) \) parametrizes flags \( L \subset V \) of sub line bundles \( L \) and sub vector bundles \( V \) of \( W \times_P T^* P \) such that \( W \times_H T_H^* P \subset V \), the right square is cartesian.

By the assumption that \( i: W \to X \) is \( C \)-transversal, the projectivization \( P(\widetilde{\mathcal{O}_C}) \subset P(W \times_P T^* P) \) does not meet the section \( W = P(W \times_H T_H^* P) \to P(W \times_P T^* P) \) and is canonically lifted to the blow-up \( P(W \times_P T^* P)' \). Hence the cycle \( P(\widetilde{\mathcal{O}}) \) on \( P(W \times_H T^* H) \) equals the image of the cycle \( -P(\widetilde{\mathcal{O}}) \) on \( X \times_P Q \) by (4.22) via upper left regarded as an algebraic correspondence. The sign comes from \((-1)^{n-m}\) in Definition 4.1 since the regular immersion \( i: W \to X \) is of codimension 1. Since the right square is cartesian, it is the same as the image via lower right and the assertion follows. \( \Box \)

Similarly as Corollary 2.23, the following holds.

Lemma 4.15. Assume that \( X \) is quasi-projective over \( k \) and let \( \mathcal{L} \) be an ample invertible \( \mathcal{O}_X \)-module. Let \( E \subset \Gamma(X, \mathcal{L}) \) be a subspace of finite dimension defining an immersion \( X \to P = P(E) \) and satisfying the condition (E) before Lemma 2.21. Let \( C \subset T^* X \) be a closed conical subset satisfying the condition (C) before Proposition 2.22.

1. For irreducible components \( C_a, C_b \) of \( C \), the closed subsets \( Q(\widetilde{C}_a), Q(\widetilde{C}_b) \subset X \times_P A \times_G D \) well intersects with respect to \( X \times_P A \times_G D \to G \).

2. Assume that every irreducible component of \( X \) and every irreducible component of \( C \) are of dimension \( n \). Then, \( R(\widetilde{C}) \subset X \times_P A \) is of codimension \( n-1 \) and the restriction of \( X \times_P A \to G \) to \( R(\widetilde{C}) \) is generically radicial.

Proof. 1. Since \( D \) is a projective space bundle over \( P^v \) of relative dimension \( \dim P - 1 \) and is a \( P^1 \)-bundle over \( G \), we have \( \dim D = \dim G = 2 \dim P^v - \dim D = 1 \) and

\[
\dim Q(\widetilde{C}_a) + \dim Q(\widetilde{C}_b) - \dim G = \dim P(\widetilde{C}_a) + \dim P(\widetilde{C}_b) + 2(\dim D - \dim P^v) = \dim G = \dim P(\widetilde{C}_a) + \dim P(\widetilde{C}_b).
\]

We regard \( Q(\widetilde{C}_a), Q(\widetilde{C}_b) \) as closed subsets of \( X \times D \) by the closed immersion \( X \times_P A \times_G D \subset X \times D \) and \( Q(\widetilde{C}_a) \times_G Q(\widetilde{C}_b) \) as a closed subset of \( X \times X \times D \times_G D \). Since the intersection of \( Q(\widetilde{C}_a) \times_G Q(\widetilde{C}_b) \) with the diagonals \( X \times D \subset X \times X \times D \subset X \times X \times D \times_G D \) are \( Q(\widetilde{C}_a) \cap Q(\widetilde{C}_b) \subset Q(\widetilde{C}_a) \times_D Q(\widetilde{C}_b) \subset Q(\widetilde{C}_a) \times_G Q(\widetilde{C}_b) \), we have

\[
(4.23) \quad \dim(Q(\widetilde{C}_a) \times_G Q(\widetilde{C}_b) - Q(\widetilde{C}_a) \cap Q(\widetilde{C}_b)) = \max(\dim(Q(\widetilde{C}_a) \times_G Q(\widetilde{C}_b) - Q(\widetilde{C}_a) \times_D Q(\widetilde{C}_b)),
\]

\[
\dim(Q(\widetilde{C}_a) \times_D Q(\widetilde{C}_b) - Q(\widetilde{C}_a) \cap Q(\widetilde{C}_b)) \).
\]

Thus, it suffices to show that the right hand side of (4.23) is \( \leq \dim P(\widetilde{C}_a) + \dim P(\widetilde{C}_b) \).

We consider the inverse images of \( P(\widetilde{C}_a), P(\widetilde{C}_b) \subset X \times X \times P^v \times P^v \) by the arrows in the cartesian diagram

\[
\begin{array}{ccc}
X \times D & \overset{c}{\longrightarrow} & X \times X \times D \\
\downarrow & & \downarrow \\
X \times P^v & \overset{c}{\longrightarrow} & X \times X \times P^v \\
\end{array}
\]

\[
\begin{array}{ccc}
X \times P^v & \overset{c}{\longrightarrow} & X \times X \times P^v \\
\downarrow & & \downarrow \\
X \times D & \overset{c}{\longrightarrow} & X \times X \times D
\end{array}
\]

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Since the forgetful morphism $D \times_G D \to P^v \times P^v$ induces an isomorphism $D \times_G D - D \to P^v \times P^v - P^v$, we have an isomorphism $Q(\tilde{C}_a) \times_G Q(\tilde{C}_b) - Q(\tilde{C}_a) \times_D Q(\tilde{C}_b) \to P(\tilde{C}_a) \times P(\tilde{C}_b) - P(\tilde{C}_a) \times_P^v P(\tilde{C}_b)$ and

$$\dim(Q(\tilde{C}_a) \times_G Q(\tilde{C}_b) - Q(\tilde{C}_a) \times_D Q(\tilde{C}_b)) \leq \dim P(\tilde{C}_a) + \dim P(\tilde{C}_b).$$

Since $D \to P^v$ is smooth and $P(\tilde{C}_a)$ and $P(\tilde{C}_b)$ well intersects with respect to $X \times_P Q \to P^v$ by Proposition 2.22, we have

$$\dim(Q(\tilde{C}_a) \times_D Q(\tilde{C}_b) - Q(\tilde{C}_a) \cap Q(\tilde{C}_b)) = \dim(P(\tilde{C}_a) \times_P^v P(\tilde{C}_b) - P(\tilde{C}_a) \cap P(\tilde{C}_b)) + \dim D - \dim P^v$$

$$\leq \dim P(\tilde{C}_a) + \dim P(\tilde{C}_b) + \dim D - 2 \dim P^v = \dim P(\tilde{C}_a) + \dim P(\tilde{C}_b) - 1.$$

Thus we have the required inequality.

2. By 1, $Q(\tilde{C})$ is generically radicial over $G$. Hence its image $R(\tilde{C})$ is also generically radicial over $G$.

Since $P(\tilde{C}) \subset X \times_P Q$ is of codimension $n$, its inverse image $Q(\tilde{C}) \subset X \times_P A \times_G D$ is of also codimension $n$. Since $D \to G$ is a $P^1$-bundle and $Q(\tilde{C}) \to R(\tilde{C})$ is generically finite, the image $R(\tilde{C}) \subset X \times_P A$ is of codimension $n - 1$. \hfill $\square$

### 4.4 Morphism defined by a pencil on a hyperplane section

Let $H \subset P = P(E^\vee)$ be a hyperplane. It corresponds to a line $E_H \subset E$ and the dual projective space $H^\vee$ is the projective space $P(E')$ of the quotient space $E' = E/E_H$. The surjection $E \to E'$ induces a closed immersion

$$(4.24) \quad H^\vee = P(E') \longrightarrow G = \text{Gr}(2, E).$$

Let $L \subset H^\vee = P(E')$ be a line. Let $W = X \cap H$ be the hyperplane section and $p_L : W_L \to L$ be the morphism defined by the pencil $L$. The canonical morphism $W_L \to W$ is an isomorphism on the complement $W_L^\circ = W - W \cap A_L$ of the intersection with the axis of the pencil. The restriction of $p_L$ is denoted by $p_L^\circ : W_L^\circ \to L$. The composition of the immersions $L \to H^\vee \to G$ defines a cartesian diagram

$$\begin{array}{ccc}
W_L & \longrightarrow & X \times_P A \\
p_L \downarrow & & \downarrow \\
L & \longrightarrow & G.
\end{array}$$

(4.25)

Similarly as Proposition 3.2, the following holds.

**Lemma 4.16.** Let $H \subset P = P(E^\vee)$ be a hyperplane and let $L \subset H^\vee$ be a line. Assume that $W = X \cap H$ is smooth and let $i : W \to X$ be the closed immersion. We regard $W$ as the fiber of $p^\vee : X \times_P Q \to P^v$ at $H$ regarded as a point of $P^v$ and $W_L$ as the inverse image by $X \times_P A \to G$ of $L \subset G$. Let $W_L^\circ - W_L^\circ \cap P(\tilde{C})^+ \subset W_L^\circ$ be the largest open subset where the restriction of the immersion $i : W \to X$ is properly $C$-transversal (Proposition 4.11).

Then, the complement of the inverse image of $R(\tilde{C}) \subset X \times_P A$ is the largest open subset of $W_L^\circ - W_L^\circ \cap P(\tilde{C})^+$ where the morphism $p_L^\circ : W_L^\circ - W_L^\circ \cap P(\tilde{C})^+ \to L$ is $i^* C$-transversal.
Proof. Since the projection \( p: X \times_p A \to X \) is smooth, the immersion \( W^\circ_L \to X \times_p A \) is properly \( p''C \)-transversal on the complement \( W^\circ_L - W^\circ_L \cap P(\tilde{C})^+ \) by Lemma 2.3.3. By applying Lemma 2.6.1 to the cartesian diagram (4.25), the assertion follows from Lemma 4.13.

We construct the universal family of the diagram (4.25). Let \( B = \text{Fl}(1, 3, E) \) and \( C = \text{Fl}(1, 2, 3, E) \) denote the flag varieties parametrizing pairs of a line and a subspace of dimension 3 with inclusion and triples of a line, a plane and a subspace of dimension 3 with inclusions respectively. The scheme \( C \) is a \( \mathbb{P}^1 \)-bundle over \( B \).

For a pair \( (H, L) \) of a hyperplane \( H \subset P \) and a line \( L \subset H^\vee \) in the dual projective space, let \( E_H \subset E \) be the line corresponding to \( H \) and \( E_L \subset E \) be the subspace of dimension 3 corresponding to the subspace \( E' = E/E_H \) of dimension 2 corresponding to \( L \). Thus such a pair \( (H, L) \) corresponds to a point of \( B \) defined by the flag \( E_H \subset E_L \subset E \). The fiber of \( C \) at the point of \( B \) corresponding to \( (H, L) \) is the line \( L \).

Similarly as the diagram (3.2), we consider a commutative diagram

\[
\begin{array}{cccc}
P(\tilde{C})^+ \subset X \times_p Q & \xleftarrow{\quad} & W_B & \xleftarrow{\quad} & W'_B & \longrightarrow & X \times_p A \supset R(\tilde{C}) \\
\downarrow & & \downarrow & & \downarrow & & \downarrow \\
P^\vee & \xleftarrow{\quad} & B & \xleftarrow{\quad} & C & \longrightarrow & G
\end{array}
\]

(4.26)

where the left and the right squares are cartesian. The bottom horizontal arrows

\[
\begin{array}{ccc}
P(E) & \xleftarrow{\quad} & \text{Fl}(1, 3, E) \\
\downarrow & & \downarrow \\
\text{Fl}(1, 3, E) & \xleftarrow{\quad} & \text{Fl}(1, 2, 3, E) & \longrightarrow & \text{Gr}(2, E)
\end{array}
\]

(4.27)

and the top middle arrow

\[
\begin{array}{ccc}
P(X \times_p T^*P) \times_{P(E)} \text{Fl}(1, 3, E) & \xleftarrow{\quad} & \text{Gr}(2, X \times_p T^*P) \times_{\text{Gr}(2, E)} \text{Fl}(1, 2, 3, E)
\end{array}
\]

are the forgetful morphisms.

Define a morphism \( \text{Fl}(1, 3, X \times_p T^*P) \to W_B \) to be the canonical morphism \( \text{Fl}(1, 3, X \times_p T^*P) \to P(X \times_p T^*P) \times_{P(E)} \text{Fl}(1, 3, E) \). Since \( X \times_p T^*P \) is a twist of a sub vector bundle of codimension 1 of \( E \times X \), the morphism \( \text{Fl}(1, 3, X \times_p T^*P) \to W_B \) is a regular immersion of codimension 2 = 3 − 1. Similarly the morphism \( \text{Fl}(1, 2, 3, X \times_p T^*P) \to W'_B \) defined as the canonical morphism \( \text{Fl}(1, 2, 3, X \times_p T^*P) \to \text{Gr}(2, X \times_p T^*P) \times_{\text{Gr}(2, E)} \text{Fl}(1, 2, 3, E) \) is a regular immersion of codimension 1. The top middle arrow is the blow-up of \( W_B \) at \( \text{Fl}(1, 3, X \times_p T^*P) \) and induces an isomorphism

\[
(4.28) \quad W^\circ_B = W_B - \text{Fl}(1, 3, X \times_p T^*P) \xleftarrow{\quad} W'_B - \text{Fl}(1, 2, 3, X \times_p T^*P).
\]

At the point of \( B \) corresponding to \( (H, L) \), the fiber at \( (H, L) \in B \) of the middle square of (4.26) together with the inclusion \( W^\circ_B \subset W_B \) is

\[
W^\circ_L \subset W \xleftarrow{\quad} W_L \\
\downarrow \quad \downarrow \\
\{(H, L)\} \xleftarrow{\quad} L.
\]

Define an open subscheme

\[
(4.29) \quad W^\circ_B \subset W^\circ_B
\]
to be the largest one satisfying the following conditions: Its image in \( X \times_p Q \) does not intersect \( \mathbf{P}(\widetilde{C})^+ \) defined in (4.14) and the inverse image

\[(4.30) \quad \mathbf{Y}(\widetilde{C}) \subset W_B^2\]

of \( \mathbf{R}(\widetilde{C}) \subset X \times_p A \) (4.17) is quasi-finite over \( G \).

Similarly as Lemma 3.3, we have the following.

**Lemma 4.17.** Let \( C \subset T^*X \) be a closed conical subset and assume that every irreducible component \( C_a \) of \( C = \bigcup_a C_a \) is of dimension \( n \).

1. For the triple \((u, H, L)\) of a hyperplane \( H \subset P \), a line \( L \subset H \) and \( u \in W_L^0 \), the following conditions are equivalent:
   
   (1) The pair \((u, H, L)\) regarded as a point of \( W_B^0 \) is a point of \( \mathbf{Y}(\widetilde{C}) \).
   
   (2) The hyperplane section \( W = X \cap H \) is smooth at \( u \), the immersion \( W \to X \) is properly \( C \)-transversal at \( u \) and \( u \in W_L^0 \) is an isolated characteristic point of \( p_u^0 : W_L^0 \to L \) with respect to \( i^*C \).

2. Let \( W = X \cap H \) be the hyperplane section and \( p_u^0 : W_L^0 \to L \) be the morphism defined by pencil for \((H, L) \in B \). Then, for a linear combination \( A = \sum_a m_a[C_a] \), the intersection number

\[(4.31) \quad (i^!A, dp_L^o)_{T^*W,u}\]

regarded as a function on \((u, H, L) \in \mathbf{Y}(\widetilde{C})\) is constructible and flat over \( B \).

**Proof.** 1. It follows from Lemma 4.16.

2. The surjections \( E \to E' \) and \( W \times_p T^*P \to W \times_H T^*H \) define a cartesian diagram

\[(4.32) \quad \begin{array}{ccc}
X \times_p A & \xleftarrow{i^!} & P(W \times_H T^*H) & \xleftarrow{\ast} & W_L \\
\downarrow & & \downarrow & & \downarrow \\
G & \leftarrow & H^v & \leftarrow & L
\end{array}\]

dividing (4.25). Let \( i^0 : W_L^0 \to X \) be the immersion and define the cycle \( P(i^{-1}_!A) \) on \( P(W_L^0 \times_H T^*H) \) as in Proposition 4.14. Applying Proposition 3.2 to the right square in (4.32), we obtain

\[(i^!A, dp_L^o)_{T^*W,u} = (P(i^{-1}_!A), W_L^0)_{P(W \times_H T^*H),u}.\]

By Proposition 4.14, we have

\[(P(i^!A), W_L^0)_{P(W \times_H T^*H),u} = -(R(\tilde{A}), W_L)_{X \times_p A, u}\]

where the cycle \( R(\tilde{A}) \) on \( X \times_p A \) supported on \( R(\widetilde{C}) \) is defined as in Proposition 4.14. Thus, it suffices to show that \((R(\tilde{A}), W_L)_{X \times_p A, u}\) regarded as a function \( \varphi_A \) on \( \mathbf{Y}(\widetilde{C}) \) is constructible and flat over \( B \).

It suffices to show the case where \( A = C_a \). Since \( B \) is regular, the complex \( \mathcal{O}_{\mathbf{R}(C_a)} \otimes^L_{X \times_p A} \mathcal{O}_{W_B^0} \) of \( \mathcal{O}_{W_B^0} \)-modules is of finite tor-dimension as a complex of \( \mathcal{O}_B \)-module. Hence the function \( \varphi_A \) on \( \mathbf{Y}(\widetilde{C}) \) is constructible and flat over \( B \) by Lemma 1.3.1.

The following elementary lemma will be used in the proof of Theorem 4.4 to verify that the assumption in Proposition 4.7 is satisfied on a dense open subset of \( \mathbf{Y}(\widetilde{C}) \) for the universal family.
Lemma 4.18. Let $T^*(C/B)$ denote the relative cotangent bundle.

1. The morphism $C \times_G T^*G \to T^*(C/B)$ defined by the lower line $B \leftarrow C \to G$ of (4.26) is a surjection and defines a morphism

$$C = P(T(C/B)) \longrightarrow P(TG)$$

over $G$ of the projectivized (relative) tangent bundles.

2. For every point of $G$, the fiber of the image of (4.33) is not contained in any finitely many union of hyperplanes.

The morphism (4.33) maps a point of a fiber $L$ of the $P^1$-bundle $C$ over $B$ to the tangent line of the line $L \subset G$ at the point.

Proof. 1. Let $L \subset W \subset E \times B$ denote the universal sub line bundle and the universal sub vector bundle of rank 3 on $B = \text{Fl}(1,3,E)$ and let $V \subset E \times G$ denote the universal sub bundle of rank 2 on $G = \text{Gr}(2,E)$. Let $L_C \subset V_C \subset W_C \subset E \times C$ denote their pull-backs on $C = \text{Fl}(1,2,3,E)$. Then, the relative cotangent bundle $T^*(C/B)$ is canonically identified with the Hom-bundle $\text{Hom}(W_C/V_C, V_C/L_C)$ and the cotangent bundle $T^*G$ is canonically identified with $\text{Hom}((E \times G)/V, V)$ respectively.

Under these identifications, the canonical morphism $C \times_G T^*G \to T^*(C/B)$ is the morphism $\text{Hom}((E \times C)/V_C, V_C) \to \text{Hom}(W_C/V_C, V_C/L_C)$ induced by the injection $W_C/V_C \to (E \times C)/V_C$ and the surjection $V_C \to V_C/L_C$. Hence $C \times_G T^*G \to T^*(C/B)$ is a surjection and induces (4.33).

2. At the point of $G$ corresponding to a subspace $V \subset E$ of dimension 2, the fiber of (4.33) is identified with the Segre embedding $P(V) \times P(E/V) \to P(\text{Hom}(V,E/V))$ sending a pair of lines $L \subset V,W/V \subset E/V$ to the line $\text{Hom}(V/L,W/V) \subset \text{Hom}(V,E/V)$. Hence the assertion follows.

\[ \square \]

4.5 Proof of Theorem 4.4

To deduce Proposition 4.5 from Proposition 4.7, we construct the diagram (4.34) below. Let $(H,L)$ be the pair of a hyperplane $H \subset P = P(E^\vee)$ and a line $L \subset H^\vee$. Let $E_H \subset E$ be the line corresponding to $H$ and let $E_L$ be the subspace of $E$ of dimension 3 containing $E_H$ corresponding to the line $L \subset H^\vee = P(E/E_H)$. Let $P = P(E_L^\vee) = \text{Gr}(2,E_L) \subset G = \text{Gr}(2,E)$ denote the projective plane parametrizing planes in $E_L \subset E$. The line $L$ is canonically identified with the line in $P$ consisting of planes in $E_L$ containing $E_H$.

We define $f: X_P \to P$ by the cartesian diagram

$$
\begin{array}{ccc}
W_L & \xrightarrow{i} & X_P \\
\downarrow g=pr_L & \quad & \downarrow j \\
L & \xrightarrow{h} & P \\
\end{array}
\xrightarrow{p} G
$$

dividing (4.25). The composition $X_P \to X \times_P A \to X$ with the projection is an isomorphism on the complement $X_P^0 = X \setminus (X \cap A_P)$ of the intersection with the linear subspace $A_P \subset P$ of codimension 3 orthogonal to $E_L \subset E$. The restriction of $f$ to $X_P^0$ maps a point $x$ to the unique linear subspaces of codimension 2 parametrized by $P$ and containing $x$. We have $W_L^0 = W_L \cap X_P^0$. The composition of the lower line is the restriction to $L \subset C$.
of the canonical morphism $C \to G$ where $L$ is identified with the fiber of the morphism $C \to B$ at $(H,L)$.

Let $u \in W^o_L$ be an isolated characteristic point of $p^o_L : W^o_L \to L$ with respect to $i^oC$. Let $v = g(u) \in L$ be the image of $u$ and $t$ be the local coordinate of $L$ at $v$. By Lemma 4.16, $u$ is an isolated point of the intersection $W_L \cap R(\tilde{C})$. Hence, on a neighborhood of $u$, the intersection $X_P \cap R(\tilde{C})$ is quasi-finite over $P$ and is of dimension $\leq 1$ by [16, Chap. 0 Proposition (16.3.1)]. Thus by Lemma 4.13, the complex $R\Psi_f\mathcal{F}$ is constructible on a neighborhood of $u \leftarrow v$ in $X_P \times_P P$ and the characteristic cycle $\text{Char} \ R\Psi_f\mathcal{F}|_{u \times_P P}$ is defined as in (3.36).

**Lemma 4.19.** We have

\begin{equation}
(4.35) \quad (h^i \text{Char} \ R\Psi_f\mathcal{F}|_{u \times_P P}, dt)_{T^*L, u} = (i^i \text{Char} \mathcal{F}, dg)_{T^*W, u}.
\end{equation}

**Proof.** We show the equality (4.35) by applying Proposition 3.21 to the morphism $f' : X' \to P'$ in the diagram (4.36) below. Since the intersection $B = X_P \cap R(\tilde{C})$ is quasi-finite over $P$ and is of dimension $\leq 1$ on a neighborhood of $u$, there exists an étale neighborhood $P' \to P$ of $v$ and an open neighborhood $X' \subset X_P \times_P P'$ of $u$, such that the inverse image $B' = B \times_{X_P} X'$ is finite over $P'$ and $\dim B' \leq 1$ and that $u$ is the unique point of the fiber $B'_v$. Define

\begin{equation}
(4.36) \quad \begin{array}{cc}
W' & \xrightarrow{v'} X' \\
g' \downarrow & \downarrow f' \\
L' & \xrightarrow{k'} P'
\end{array}
\end{equation}

from the left square of (4.34) by taking the base changes of the lines.

For the right hand side of (4.35), we have $(i^i \text{Char} \mathcal{F}, dg)_{T^*W, u} = (g'^i \text{Char} \mathcal{F}, dt)_{T^*L', v}$ since $u$ is the unique point of the inverse image. By the cartesian diagram (4.36), we have

\begin{equation}
(4.37) \quad g'^i \text{Char} \mathcal{F} = h^i f^! \text{Char} \mathcal{F}.
\end{equation}

We show that the condition (P) in Definition 3.19 is satisfied. Since the immersion $W \to X$ is $C$-transversal, the fiber at $u$ of the inverse image $df^{-1}(C)$ of $C$ by $X_P \times_P T^* P \to T^* X_P$ does not contain the fiber of $X_P \times_P T^*_LP$ and is of dimension at most 1. Since $\dim B' \leq 1$, an irreducible component of $df^{-1}(C)$ not contained in the 0-section is of dimension at most $2 = \dim P$ and the condition (P) is satisfied.

Thus applying Proposition 3.21, we obtain

\begin{equation}
(4.38) \quad (f^! \text{Char} \mathcal{F})|_{u \times_P P'} = \text{Char} \ R\Psi_f\mathcal{F}|_{u \times_P P'} \mod \langle T^*_P(v), P(v) \rangle.
\end{equation}

Since $h^i[T^*_P(v), P(v)] = -T^*_P(v), L(v)$ and the section $dt$ does not meet the 0-section of $T^* L$, the equality (4.37) and the congruence (4.38) imply the equality (4.35). \hfill \Box

An isolated characteristic point $u$ of $p^o_L : W^o_L \to L$ corresponds to a point $(u,H,L)$ of $\mathcal{Y}(\tilde{C}) \subset W^o_B$ such that $W = X \cap H$, by Lemma 4.17.1. We prove (4.3) first for $(u,H,L)$ on a dense open subset of $\mathcal{Y}(\tilde{C})$ by applying Proposition 4.7. We will then deduce the general case by using Lemma 1.2.3, similarly as in the proof of Proposition 3.9.
Lemma 4.20. There exists a dense open subset $U \subset Y(\tilde{C})$ such that for $(u, H, L) \in U$, we have

\begin{equation}
\text{Char}(R\Psi g^*F)|_{u \times L} = h^i \text{Char } R\Psi f^*F|_{u \times P}. 
\end{equation}

Proof. We will prove this by applying Proposition 4.7 to the lower line $L \to P \to G$ of (4.34). Define an open subset $(X \times \mathbb{P} \mathbb{A})^\square \subset X \times \mathbb{P} \mathbb{A}$ to be the largest one such that the intersection $R(\tilde{C})^\square = R(\tilde{C}) \cap (X \times \mathbb{P} \mathbb{A})^\square$ is quasi-finite over $G$. Let $\tilde{f}: (X \times \mathbb{P} \mathbb{A})^\square \to G$ denote the canonical morphism. The complex $R\Psi f^*F$ on $(X \times \mathbb{P} \mathbb{A})^\square \times_G G$ is constructible and its formation commutes with base change by Proposition 1.8.1.

It suffices to show that (4.39) holds on a dense open subset of $Y(\tilde{C}_a)$ for each irreducible component $C_a$ of $C$. Let $t$ be the generic point of the irreducible component $R(\tilde{C}_a)$ of $R(\tilde{C})$ and let $s \in G$ be the image. The local ring $\mathcal{O}_{G, s}$ is a discrete valuation ring and the residue field $k(t)$ is a purely inseparable extension of $k(s)$ by Lemma 4.15.2.

As is remarked at the beginning of Section 1.1, the vanishing topos $t \times_G G \subset (X \times \mathbb{P} \mathbb{A})^\square \times_G G$ is canonically identified with the henselization $\text{Spec } \mathcal{O}_{G, s}^h$. Hence, there exist an étale neighborhood $G_a^\circ \to G$ of $t$ such that the closure $D_a$ of $t$ is a smooth divisor, a constructible complex $K$ on $G_a^\circ$ such that the restrictions on $D_a$ and on $G_a^\circ - D_a$ are locally constant and an isomorphism on $((X \times \mathbb{P} \mathbb{A})^\square \times_G G) \times G G_a^\circ$ of the pull-backs of $R\Psi f^*F$ and $K_a$.

Further replacing $G_a^\circ$ by a neighborhood of $t$ if necessary, we may assume that the ramification of each cohomology sheaf of the restriction of $K_a$ on $G_a^\circ - D_a$ is non-degenerate along $D_a$. Let $j_a^*: G_a^\circ - D_a \to G_a^\circ$ be the open immersion and define a closed conical subset $C_a' = T_{D_a} G_a^\circ \cup S(j_a^*j_a^* K_a) \subset T^* G_a^\circ$ as in Proposition 4.7. Then, for a point $(u, H, L)$ of $Y(\tilde{C}_a)$ such that the immersion $L \times G G_a^\circ \subset G_a^\circ$ is $C_a'$-transversal at $p_L^*(u)$, we have

\[
\text{Char } ((l \circ h)^* R\Psi f^*F)|_{u \times L} = h^i \text{Char } (l^* R\Psi f^*F)|_{u \times P}
\]

where $l: P \to G$ denotes the lower right arrow in (4.34), by Proposition 4.7 applied to $L \subset P \subset G$. Since the formation of $R\Psi f^*F$ commutes with base change on $(X \times \mathbb{P} \mathbb{A})^\square \times_G G$, the equality (4.39) holds.

Since $C_a' \subset T^* G_a^\circ$ is the union of the 0-section and the images of line bundles on $D_a$, there exists a dense open subset $U_a$ of $Y(\tilde{C}_a)$ such that for $(u, H, L) \in U_a$, the tangent line of $L$ at $p_L^*(u) \in L \subset H' \subset G$ is not annihilated by any non-zero differential form in the fiber $C \times_G u \subset T^* G \times_G u$ by Lemma 4.18. Hence for $(u, H, L) \in U_a$, the immersion $L \times G G_a^\circ \subset G_a^\circ$ is $C_a'$-transversal at $p_L^*(u)$ and the equality (4.39) holds.

Proof of Proposition 4.5. We regard the both sides of (4.3) as functions on $Y(\tilde{C})$. By (3.37), by taking the coefficients of the fiber $T^*_a L$ we have

\[- \dim \text{tot}_u \phi(i^* F, p_L) = (\text{Char}(R\Psi g^* F)|_{u \times L}, dt)_{T^* L, v}. \]

Hence by Lemma 4.20, the equality

\begin{equation}
- \dim \text{tot}_u \phi(i^* F, p_L) = (h^i \text{Char } R\Psi f^* F|_{u \times P}, dt)_{T^* L, v}
\end{equation}

holds on a dense open subset of $Y(\tilde{C})$. Thus, by Lemma 4.19, the equality (4.3) holds on the same dense open subset of $Y(\tilde{C})$. 

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We show that the equality (4.3) holds on the whole $Y(\tilde{C})$ by using the universal family

$$
Y(\tilde{C}) \xleftarrow{\subset} W_B^\circ \xrightarrow{\tilde{g}} C
$$

of $p_L: W_L^\circ \rightarrow L$. We regard the left hand side of (4.3) as a function $\varphi_{i,\mathcal{F}}$ on $Y(\tilde{C}) \subset W_B^\circ$ defined by

$$
(4.41) \quad \varphi_{i,\mathcal{F}}(z) = \dim \text{tot}_u \phi_u(i^*\mathcal{F}, p_L^\circ)
$$

for $z \in Y(\tilde{C})$ corresponding to $(u, H, L)$ where $u$ is an isolated characteristic point of $p_L^\circ: W_L^\circ \rightarrow L$ and $W = X \cap H$.

We show that $\varphi_{i,\mathcal{F}}$ is constructible and flat over $B$. We apply Proposition 1.18 to the commutative diagram (4.41) as in the proof Lemma 3.8. By Lemma 4.16.1. and 2., the morphism $p^\circ: W_B^\circ \rightarrow B$ is locally acyclic relatively to the pull-back of $\mathcal{F}$ and $\tilde{g}: W_B^\circ \rightarrow C$ is locally acyclic relatively to the pull-back of $\mathcal{F}$ on the complement of $Y(\tilde{C})$ respectively. Since $Y(\tilde{C})$ is quasi-finite over $B$, the function $\varphi_{i,\mathcal{F}}$ is constructible and flat over $B$ by Proposition 1.18.

By Lemma 4.17.2, the right hand side of (4.3) is also a constructible function on $Y(\tilde{C})$ flat over $B$. Since the both sides of (4.3) are constructible functions on $Y(\tilde{C})$ flat over $B$ and the equality has been proved to hold on a dense open subset of $Y(\tilde{C})$, we have a equality of the functions on $Y(\tilde{C})$ by Lemma 1.2.3. \hfill \Box

**Proof of Theorem 4.4.** First, we prove the case where $h = i: W \rightarrow X$ is a closed immersion of smooth divisor. If $\dim X = 1$, then $\mathcal{F}$ is locally constant on a neighborhood of $i(W) \subset X$ and the assertion follows. Hence, we may assume $\dim X \geq 2$.

Assume that $\mathcal{F}$ is micro-supported on $C \subset T^*X$. Then by Lemma 2.10.4, the pull-back $i^*\mathcal{F}$ is micro-supported on $i^*C \subset T^*W$. Since the characteristic cycle is characterized by the Milnor formula, it suffices to show that $i^!\text{Char } \mathcal{F}$ satisfies the Milnor formula. Hence the question is local and we may assume that $X$ is affine.

Further, we may assume that $W = X \cap H$ is a hyperplane section for an embedding $X \rightarrow P$ satisfying the condition (E) before Lemma 2.21 and the condition (C) before Proposition 2.22. The induced immersion $W \rightarrow H \subset P$ is defined by $\text{Im}(E \rightarrow \Gamma(X, \mathcal{L}) \rightarrow \Gamma(W, \mathcal{L} \otimes \mathcal{O}_W))$ also satisfies the conditions (E) and (C) loc. cit. By the uniqueness in Proposition 3.9, it suffices to show that $A = i^!\text{Char } \mathcal{F}$ satisfies the Milnor formula (3.16) for morphisms $p^\circ_L: W^\circ_L \rightarrow L$ defined by pencils. Thus, the case where $h = i: W \rightarrow X$ is the immersion of a smooth divisor follows from Proposition 4.5.

We prove Theorem 4.4 for a properly $C$-transversal regular immersion $i = h: W \rightarrow X$. We prove the assertion on the codimension $c$ of the immersion. Since the case $c = 1$ is proved, we may assume $c \geq 2$. Since the assertion is local on $W$, we may assume that there exists a smooth divisor $V \subset X$ containing $W$. Replacing $X$ by a neighborhood of $V$ if necessary, we may assume that the immersion $j: V \rightarrow X$ is properly $C$-transversal by Lemma 4.2.2. Then, the immersion $i': W \rightarrow V$ is properly $j^\circ C$-transversal by Lemma 2.3.3 and by the induction hypothesis, we have $\text{Char } i^*\mathcal{F} = i'^!\text{Char } j^*\mathcal{F} = i'^!j^!\text{Char } \mathcal{F} = i^!\text{Char } \mathcal{F}$.

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We prove Theorem 4.4 for a smooth morphism \( h: W \to X \). Since \( SSh^*F = h^*SSF \) by Corollary 2.13.2, it suffices to show the equalities of the coefficients for each irreducible component. Since \( \text{Char} h^*F = h^!\text{Char} F \) for étale morphism \( h: W \to X \), we may assume \( W = \mathbb{A}^n \times X \). Since the 0-section \( i: X \to W = \mathbb{A}^n \times X \) is properly \( SSh^*F = h^!SSF \)-transversal, we have \( \text{Char} F = i^!\text{Char} h^*F \). Hence we have \( \text{Char} h^*F = h^!\text{Char} F \).

We show Theorem 4.4 for a general properly \( C \)-transversal morphism \( h: W \to X \). The morphism \( h \) is the composition of the graph \( g: W \to W \times X \) and the projection \( p: W \times X \to X \). Since the projection \( W \times X \to X \) is smooth and the graph \( g: W \to W \times X \) is a properly \( p^oC \)-transversal immersion, we have \( \text{Char} h^*F = g^!\text{Char} p^*F = g^!p^!\text{Char} F = h^!\text{Char} F \).

**Proof of Theorem 4.6.** It suffices to show the equality of the coefficients for each irreducible component of \( S = SSF \). By the additivity of characteristic cycles and the compatibility with étale pull-back, it suffices to show the tamely ramified case and the totally wildly ramified case separately.

First, we prove the tamely ramified case. It suffices to determine the coefficients of \( [T_{D_1}X] \) by induction on the number \( d \) of elements of \( I \). By Theorem 4.4, we may assume \( X = \mathbb{A}^n \) and \( D \) is the complement of \( G^m_\circ \). If \( n = 0 \) and \( X \) consists of a single point, we have \( \text{Char} F = \text{rank} F \cdot [T_{D_1}X] \). If \( n \geq 1 \), it follows from Lemma 2.28.

We prove the totally wildly ramified case. If \( \dim X = 0 \), it is proved above. It follows from (3.24) if \( \dim X \leq 1 \). It suffices to compare the coefficients in \( \text{Char} j_!G \) and \( C(j_!G) \) assuming \( \dim X \geq 2 \). Since the assertion is étale local, we may assume that \( C(j_!G) \) has a unique irreducible component different from the 0-section. By Theorem 4.4 and Proposition [29, Proposition 3.8], for every properly \( C \)-transversal immersion \( i: W \to X \) of a smooth curve, we have \( i^!\text{Char} j_!G = \text{Char} i^*j_!G = C(i^*j_!G) = i^!C(j_!G) \). Hence it is reduced to the case \( \dim X = 1 \).

Since \( j_!G[n] \) is a perverse sheaf, the equality (4.7) follows from (4.6) and Proposition 3.17.2.

**4.6 The index formula**

We state and prove the index formula for the Euler-Poincaré characteristic.

**Theorem 4.21.** Let \( X \) be a projective smooth variety over an algebraically closed field. Then, we have

\[
\chi(X, F) = (\text{Char} F, T^*_X X)_{T^*X}.
\]

As a preparation of the proof, first we prove some formulas for intersection number.

**Lemma 4.22.** Assume \( X \) is proper and smooth over \( k \). Let \( C = \bigcup_a C_a \subset T^*X \) be a closed conical subset. Assume that every irreducible component of \( X \) and of \( C \) are of dimension \( n \). Let \( A = \sum m_a C_a \) be a linear combination.

1. Let \( f: X \to Y \) be a proper flat morphism to a proper smooth and geometrically connected curve of genus \( g \). Assume that \( f \) has at most isolated characteristic point with respect to \( C \) and that there exists a dense open subset \( V \subset Y \) such that \( f_V: X \times_Y V \to V \) is smooth and for closed points \( y \in V \) the immersion \( i: W = X_y \to X \) of the fiber is properly \( C \)-transversal. Then, we have

\[
(A, T^*_X X)_{T^*X} = (2 - 2g)(i^!A, T^*_W W)_{T^*W} + \sum_x (A, df)_{T^*X,x}.
\]
where \( x \) runs through isolated characteristic points.

2. Let \( V \subset X \) be a smooth closed subscheme of codimension 2 such that the immersion \( i: V \to X \) is properly \( C \)-transversal. Then the blow-up \( \pi: W \to X \) is properly \( C \)-transversal and we have

\[
(4.44) \quad (\pi^! A, T^*_W W)_{T^*W} = (A, T^*_X X)_{T^*X} + (i^! A, T^*_V V)_{T^*V}.
\]

Proof. 1. First, we show the case where \( f: X \to Y \) is the identity of \( Y \). It suffices to show the cases where \( A = T^*_y Y \) is the fiber of a closed point \( y \) and \( A = T^*_y Y \) is the 0-section respectively. In the first case, the both sides of (4.43) equal 1. In the second case, the both sides equal \( \deg \Omega^1_{Y/k} = 2g - 2 \). Note that we have a \(-\) sign in the definition of \( i^! A \) in Definition 4.1 since the regular immersion \( i \) is of codimension 1.

We show the general case. By the assumption that \( i: W = X_y \to X \) is \( C \)-transversal for \( y \in V \), the closed conical subset \( f_\circ C \subset T^*Y \) (Definition 2.7) is of dimension 1. We define a cycle \( f_\circ A \) supported on \( f_\circ C \) by the morphisms

\[
T^*Y \xleftarrow{f \times \text{id}} X \times_Y T^*Y \xrightarrow{df} T^*X
\]

regarded as an algebraic correspondence. Namely, first we take the pull-back \( df^! A \) by the right arrow in the sense of intersection theory and then take the push-forward \( f_\circ A = (f \times \text{id})_*(df^! A) \) by the left arrow in the sense of intersection theory. Then, we have

\[
(A, T^*_X X)_{T^*X} = (df^! A, X \times_Y T^*_Y Y)_{X \times_Y T^*Y} = (f_\circ A, T^*_Y Y)_{T^*Y}
\]

and \( \sum_{x \in f^{-1}(y)} (A, df)_{T^*X,x} = (f_\circ A, d\text{id}_y)_{T^*Y,y} \) for closed point \( y \in Y \) by the projection formula. By the assumption that \( i: W = X_y \to X \) is properly \( C \)-transversal for \( y \in V \), we have \( (i^! A, T^*_W W)_{T^*W} = i^!_y (f_\circ A) \) for the regular immersion \( i_y: Y \to Y \) of codimension 1. Hence it suffices to apply the formula (4.43) for \( Y \) and \( f_\circ A \).

2. We consider the commutative diagram

\[
\begin{array}{ccc}
E \times_X T^*X & \longrightarrow & E \times_V T^*V \\
\downarrow & & \downarrow \\
E \times_W T^*W & \longrightarrow & T^*E
\end{array}
\]

of morphisms of vector bundles. Since the exceptional divisor \( E = V \times_X W \) is smooth over \( V \), the right vertical arrow is injective. Hence, the kernel of the left vertical arrow is a subset of the kernel of the upper horizontal arrow. Thus, if the immersion \( V \to X \) is \( C \)-transversal then \( \pi: W \to X \) is \( C \)-transversal.

Assume that the immersion \( V \to X \) is properly \( C \)-transversal. For \( j \geq 0 \), let \( B_j \subset C \cap T^*_X X \subset X \) be the closed subset defined in (4.10). Then, the regular immersion \( V \to X \) of codimension 2 meets \( (B_j)_{j \geq 0} \) n-semi properly. Since \( E \to V \) is smooth of relative dimension 1, we have \( \dim B_j \times_X E \leq n - j - 1 \). Hence, we have \( \dim B_j \times_X W \leq n - j - 1 \) and \( \pi: W \to X \) is properly \( C \)-transversal by Proposition 4.9.1.

We show (4.44). Let \( K \subset E \times_X T^*X \subset W \times_X T^*X \) denote the kernel of the canonical morphism \( E \times_X T^*X \to E \times_W T^*W \) of vector bundles. We regard the diagram

\[
T^*X \xleftarrow{\pi \times \text{id}} W \times_X T^*X \xrightarrow{d\pi} T^*W
\]
as an algebraic correspondence. Then, since \(dr^!(T^\ast_W W) = W \times X T^\ast_X X + K\), we have

\[
(\pi^! A, T^\ast_W W)_{T \cdot W} = (\pi^! A, W \times X T^\ast_X X)_{W \times X T \cdot X} + (\pi^! A, K)_{W \times X T \cdot X}
\]

and the first term on the right hand side equals \((A, T^\ast_X X)_{T \cdot X}\) by the projection formula. By the exact sequence \(0 \rightarrow K \rightarrow E \times_V T^\ast_V X \rightarrow T^\ast_E W \rightarrow 0\) and the excess intersection formula, the second term equals \((i^! A, T^\ast_V V)_{T \cdot V} \cdot \deg(T^\ast_E W)\) where \(\deg(T^\ast_E W)\) denotes the degree of the line bundle of a \(P^1\)-bundle \(E\) over \(V\). Since \(\deg(T^\ast_E W) = 1\) and the codimension 2 of the immersion \(V \rightarrow X\) is even, the equality (4.44) follows. \(\square\)

As a second preparation for the proof of Theorem 4.21, we show the existence of a good pencil. Assume that every irreducible component of a smooth scheme \(X\) over \(k\) and that every irreducible component of a closed conical subset \(C \subset T^\ast X\) are of dimension \(n\). For each \(j \geq 0\), let \(B_j\) be the closed subset of the base \(B = C \cap T^\ast_X X \subset X\) defined in (4.10) and define a closed subset

\[
(4.45) \quad (B_j \times_P A)^2 \subset B_j \times_P A
\]

by \((B_j \times_P A)^2 = \{(x, L) \in B_j \times_P A \subset B_j \times G | \dim_x B_j \cap A_L > n - j - 2\}\).

**Proposition 4.23.** Let \(X \rightarrow P\) be an immersion and \(C \subset T^\ast X\) be a conical closed subset. Assume that every irreducible component of \(X\) and of \(C\) are of dimension \(n\). Let \(L \subset P\) be a line such that the axis \(A_L \subset P\) of \(L\) meets \(X\) transversely.

1. The immersion \(V = X \cap A_L \rightarrow X\) is \(C\)-transversal if and only if the point of \(G\) corresponding to \(L\) is not contained in the image of \(R(C) \subset X \times_P A\) by \(X \times_P A \rightarrow G\).

2. The immersion \(V \rightarrow X\) is properly \(C\)-transversal if and only if the point of \(G\) corresponding to \(L\) is not contained in the image by \(X \times_P A \rightarrow G\) of the union \(R(C) \cup \bigcup_{j \geq 0} (B_j \times_P A)^2 \subset X \times_P A\).

**Proof.** The proof is similar to that of Proposition 4.11.

1. Since \(p : X \times_P A \rightarrow X\) is smooth, the immersion \(V = X \cap A_L \rightarrow X\) is \(C\)-transversal if and only if the immersion \(V \rightarrow X \times_P A\) is \(p^! C\)-transversal by Lemma 2.3.3. We consider the cartesian diagram

\[
\begin{array}{ccc}
X \times_P A & \leftarrow & V \\
\downarrow & & \downarrow \\
G & \leftarrow & \{L\}
\end{array}
\]

Since the right vertical arrow \(V \rightarrow \{L\}\) in (4.46) is smooth, this condition is equivalent to that the left vertical arrow \(X \times_P A \rightarrow G\) is \(p^! C\)-transversal on a neighborhood of \(V\) by Lemma 2.5.1 and Lemma 2.6.1. The condition is further equivalent to that the point of \(G\) corresponding to \(L\) is not contained in the image of \(R(C)\) by Lemma 4.13.

2. Suppose that the immersion \(V \rightarrow X\) is \(C\)-transversal. Then it is properly \(C\)-transversal if and only if \(V\) regarded as the fiber of \(X \times_P A \rightarrow G\) at \(L\) does not meet the union of \((B_j \times_P A)^2\) by Corollary 4.10. \(\square\)

**Corollary 4.24.** Assume further that \(X\) is projective.

1. There exists a dense open subset \(U \subset P^\vee\) consisting of hyperplanes \(H \subset P\) satisfying the following condition: The hyperplane \(H \subset P\) meets \(X\) transversely and the immersion \(i : W = X \cap H \rightarrow X\) is properly \(C\)-transversal.
2. There exists a dense open subset $U' \subset G$ consisting of lines $L \subset P^V$ satisfying the following conditions: The axis $A_L \subset P$ of $L$ meets $X$ transversely, the immersion $i': V = X \cap A_L \to X$ is properly $C$-transversal and the morphism $p_L: X_L \to L$ has at most isolated characteristic points with respect to $\pi^oC$ where $\pi: X_L \to X$ denotes the blow-up at $V = X \cap A_L \subset X$.

Proof. 1. By Bertini, for hyperplanes $H \subset P$ corresponding to points in the complement of the image of $P(T_X^pP)$ by $p^v: X \times P Q \to P^V$, the intersection $W = X \cap H$ is smooth. Further, by Proposition 4.11, the hyperplanes $H \subset P$ such that the immersion $i: W \to X$ is properly $C$-transversal form a dense open subset $U \subset P^V$.

2. By Bertini, the lines $L \subset P^V$ such that the axis $A_L \subset P$ intersects $X$ transversely form a dense open subset $U' \subset G$. We show that the image of $R(\tilde C)$ by $X \times P A \to G$ is not dense. Since $P(\tilde C) \subset X \times P Q$ is of codimension $n$, its inverse image $Q(\tilde C) \subset X \times P Q \times D$ is also of codimension $n$. Since $D$ is a $P^1$-bundle over $G$ the subset $R(\tilde C) \subset X \times_P A$ is of codimension $\geq n - 1$. Since $\dim X \times_P A = \dim G + n - 2$, the image of $R(\tilde C)$ by $X \times_P A \to G$ is not dense, as claimed.

Since $\dim B_j \leq n - j$ for $j \geq 0$ by Corollary 4.10.1, the image of $(B_j \times P A)^\sharp \subset X \times_P A$ by $X \times_P A \to G$ are not dense further by Bertini. Hence the lines $L \subset P^V$ such that the immersion $V = X \cap A_L \to X$ is properly $C$-transversal form a dense open subset $U'_1 \subset U'_2 \subset G$ by Proposition 4.23.2.

Let $L$ be a line corresponding to a point of $U'_1 \subset G$. Then, the blow-up $\pi: X_L \to X$ is properly $C$-transversal by Lemma 4.22.2. Since $p: X \times_P Q \to X$ is smooth, the immersion $X_L \to X \times_P Q$ is $p^oC$-transversal. Hence, by Lemma 2.6.1 applied to the cartesian diagram

$$X \times_P Q \leftarrow X_L \quad \begin{array}{c}\downarrow p^v \quad \downarrow p_L \\ P^V \to L \end{array}$$

and by Lemma 2.16.1, the complement $X_L = X_L \cap P(\tilde C)$ is the largest open subset where $p_L: X_L \to L$ is $\pi^oC$-transversal. Thus, $p_L: X_L \to L$ has at most isolated characteristic points if and only if the intersection $X_L \cap P(\tilde C)$ is finite.

There exists a closed subset $\Delta' \subset \Delta$ of the image $\Delta \subset P^V$ of $P(\tilde C) \subset X \times_P Q$ by $X \times_P T^*P \to P^V$ such that $P(\tilde C) \to \Delta$ is finite outside $\Delta' \subset \Delta$ and $\Delta' \subset P^V$ is of codimension $\geq 2$ since $P(\tilde C) \subset X \times_P Q$ is of codimension $n$ and $X \times_P T^*P \to P^V$ is of relative dimension $n - 1$. Hence, the lines $L \subset P^V$ such that $P(\tilde C) \times_P L = X_L \cap P(\tilde C)$ is finite form a dense open subset $U'_2 \subset U'_2 \subset G$. It suffices to set $U' = U'_1 \cap U'_2$. □

Proof of Theorem 4.21. We prove the assertion by induction on $\dim X$. If $\dim X = 0$, both sides are rank $F$.

Assume $\dim X \geq 1$ and let $C = SSF$ denote the singular support. Take a closed immersion $X \to P$ and let $U \subset P^V$ and $U'' \subset G$ be the dense open subsets in Proposition 4.23. Let $U'' \subset G$ be the image of the inverse image by $P \leftarrow D \to G$ of $U \subset P$ and take a line $L \subset P^V$ corresponding to a point $L \in G$ in the intersection $U' \cap U'' \subset G$.

For a point of the line $L \subset P^V$ contained in the intersection $L \cap U \neq \emptyset$, the corresponding hyperplane $H \subset P$ meets $X$ transversely and the immersion $i: W = X \cap H \to X$ is properly $C$-transversal. The axis $A_L \subset P$ of $L$ also meets $X$ transversely and the immersion $i': V = X \cap A_L \to X$ is properly $C$-transversal. Hence the blow-up $\pi: X_L \to X$ at
$X \cap A_L \subset X$ is properly $C$-transversal by Lemma 4.22.2. The morphism $p_L : X_L \to L$ has at most isolated characteristic points with respect to $\pi^C$.

Then, since the immersions $i : W = X \cap H \to X$ and $i' : V = X \cap A_L \to X$ are properly $C$-transversal, we have $i^!\Char F = \Char i^*F$ and $i'^!\Char F = \Char i'^*F$ by Theorem 4.4. By the induction hypothesis, we have

\begin{align}
(4.47) \quad \chi(W, F) &= (i^!\Char F, T^*_W W)_{T^* W}, \quad \chi(V, F) = (i'^!\Char F, T^*_V V)_{T^* V}.
\end{align}

By the projection formula, we have

\begin{align}
(4.48) \quad \chi(X, F) &= \chi(X_L, F) - \chi(V, F)
\end{align}

since $A_L$ meets $X$ transversely. By applying the Grothendieck-Ogg-Shafarevich formula [17] to $Rp_L_* F$, we have

\begin{align}
(4.49) \quad \chi(X_L, F) &= \chi(L, Rp_L_* F) = 2\chi(W, F) - \sum_u \dim \text{tot}_u \phi_u(F, p_L)
\end{align}

where $u$ runs through isolated characteristic points of $p_L : X_L \to L$ since $p_L : X_L \to L$ assumed to have at most isolated characteristic points. By the Milnor formula (3.11), we have

\begin{align}
(4.50) \quad -\dim \text{tot}_u \phi_u(F, p_L) &= (\Char F, dp_L)_{T^* X, u}.
\end{align}

Substituting the first equality of (4.47) and (4.50) to (4.49), we obtain the first equality in

\begin{align*}
\chi(X_L, F) &= 2(i^!\Char F, T^*_W W)_{T^* W} + \sum_u (\Char F, dp_L)_{T^* X, u} \\
&= (\pi^!\Char F, T^*_X X_L)_{T^* X_L}.
\end{align*}

The second equality follows from Lemma 4.22.1. Substituting this and the second equality of (4.47) to (4.48), we obtain the first equality in

\begin{align*}
\chi(X, F) &= (\pi^!\Char F, T^*_X X_L)_{T^* X_L} - (i'^!\Char F, T^*_V V)_{T^* V} \\
&= (\Char F, T^*_X X)_{T^* X}.
\end{align*}

The second equality follows from Lemma 4.22.2 and the assertion follows.

\hfill $\Box$

5 \quad $F$-transversality and singular support

5.1 \quad $F$-transversality

Let $k$ be a field and $\Lambda$ be a finite local ring such that the residue characteristic $\ell$ is invertible in $k$. Let $F$ and $G$ be constructible complexes of $\Lambda$-modules on $X$ and assume that $G$ is of finite tor-dimension. For a morphism $h : W \to X$ of separated schemes of finite type over $k$ the canonical morphism

\begin{align}
(5.1) \quad h^* F \otimes^\Lambda R^! h^* G &\to R^! h^*(F \otimes^\Lambda G)
\end{align}
is defined as the adjoint of the composition
\[
Rh(h^* \mathcal{F} \otimes_{\Lambda}^L Rh^! \mathcal{G}) \to \mathcal{F} \otimes_{\Lambda}^L Rh^! \mathcal{G} \to \mathcal{F} \otimes_{\Lambda}^L \mathcal{G}
\]
of the inverse of the isomorphism of projection formula [12, (4.9.1)] and the morphism induced by the adjunction \( Rh \mathcal{G} \to \mathcal{G} \).

**Definition 5.1.** Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on a separated scheme \( X \) of finite type over \( k \). We say that a morphism \( h: W \to X \) of separated schemes of finite type over \( k \) is \( \mathcal{F} \)-transversal if the canonical morphism
\[
(5.2) \quad h^* \mathcal{F} \otimes_{\Lambda}^L Rh^! \Lambda \to Rh^! \mathcal{F}
\]
defined as (5.1) for \( \mathcal{G} = \Lambda \) is an isomorphism.

**Lemma 5.2.** Let \( \mathcal{F} \) be a constructible complex of \( \Lambda \)-modules on a separated scheme \( X \) of finite type over \( k \) and let \( h: W \to X \) be a morphism of separated schemes of finite type over \( k \).

1. If \( h \) is smooth, then \( h: W \to X \) is \( \mathcal{F} \)-transversal.
2. If \( \mathcal{F} \) is locally constant, then \( h: W \to X \) is \( \mathcal{F} \)-transversal.
3. Assume that \( h: W \to X \) is \( \mathcal{F} \)-transversal and that \( Rh^! \Lambda \) is isomorphic to \( \Lambda(c)[2c] \) for a locally constant function \( c \) on \( W \). Then for a morphism \( g: V \to W \) of separated schemes of finite type over \( k \), the following conditions are equivalent:
   1. \( g: V \to W \) is \( h^* \mathcal{F} \)-transversal.
   2. The composition \( h \circ g: V \to W \) is \( \mathcal{F} \)-transversal.
4. Let \( \Lambda_0 \) be the residue field of \( \Lambda \). Assume that \( \mathcal{F} \) is of finite tor-dimension and set \( \mathcal{F}_0 = \mathcal{F} \otimes_{\Lambda}^L \Lambda_0 \). Then, \( h: W \to X \) is \( \mathcal{F} \)-transversal if and only if it is \( \mathcal{F}_0 \)-transversal.

**Proof.** 1. Poincaré duality [10, Théorème 3.2.5].

2. Since the assertion is étale local on \( X \), it is reduced to the case where \( \mathcal{F} = \Lambda \) by devissage.

3. We have a commutative diagram
\[
\begin{array}{ccc}
(h \circ g)^* \mathcal{F} \otimes^L R(h \circ g)^! \Lambda & \longrightarrow & R(h \circ g)^! \mathcal{F} = Rg^! Rh^! \mathcal{F} \\
\uparrow & & \uparrow \\
g^* h^* \mathcal{F} \otimes^L Rg^! Rh^! \Lambda & & Rg^!(h^* \mathcal{F} \otimes^L Rh^! \Lambda) \\
\uparrow & & \uparrow \\
g^* h^* \mathcal{F} \otimes^L Rg^! \Lambda \otimes^L g^* Rh^! \Lambda & \longrightarrow & Rg^! h^* \mathcal{F} \otimes^L g^* Rh^! \Lambda
\end{array}
\]
where the arrows are defined by (5.1). Since \( Rh^! \Lambda \) is assumed to be isomorphic to \( \Lambda(c)[2c] \) for a locally constant function \( c \) on \( W \), the lower vertical arrows are isomorphisms. By the assumption that \( h: W \to X \) is \( \mathcal{F} \)-transversal, the upper right vertical arrow is an isomorphism. The condition (1) means that the top horizontal arrow is an isomorphism and the condition (3) is equivalent to that the bottom horizontal arrow is an isomorphism by the assumption that \( Rh^! \Lambda \) is isomorphic to \( \Lambda(c)[2c] \). Hence the equivalence follows.

4. Similarly as Lemma 1.6.1, the canonical morphism \( Rf^! \mathcal{F} \otimes_{\Lambda}^L \Lambda_0 \to Rf^! \mathcal{F}_0 \) is an isomorphism. Hence, similarly as the proof of Lemma 1.6.2, the morphism (5.2) for \( \mathcal{F} \) is an isomorphism if and only if (5.2) for \( \mathcal{F}_0 \) is an isomorphism. \( \square \)

Recall that the dual \( D_X \mathcal{F} \) is defined as \( RHom_{\Lambda}(\mathcal{F}, K_X) \) where \( K_X = Ra^! \Lambda \) for the structure morphism \( a: X \to \text{Spec } k \).
**Proposition 5.3.** Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules of finite tor-dimension on a separated scheme $X$ of finite type over $k$.

1. Let $h: W \to X$ be a morphism of separated schemes of finite type over $k$ and assume that $Rh^1\Lambda$ is isomorphic to $\Lambda(c)[2c]$ for a locally constant function $c$ on $W$. Then, the following conditions are equivalent:
   
   (1) The morphism $h: W \to X$ is $\mathcal{F}$-transversal.
   
   (2) The morphism $h: W \to X$ is $D_X\mathcal{F}$-transversal.

Further if $h: W \to X$ is $K_X$-transversal, they are equivalent to the following condition:

(3) The canonical morphism $h^* R\text{Hom}_X(\mathcal{F}, K_X) \to R\text{Hom}_W(h^* \mathcal{F}, h^* K_X)$ is an isomorphism.

2. Let $\mathcal{G}$ be a constructible complex of $\Lambda$-modules on a scheme $X$ of finite type over $k$. Then, the following conditions are equivalent:

   (1) The diagonal morphism $\delta: X \to X \times X$ is $R\text{Hom}_{X \times X}(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{G})$-transversal.

   (2) The canonical morphism $\mathcal{G} \otimes^L R\text{Hom}_X(\mathcal{F}, \Lambda) \to R\text{Hom}_X(\mathcal{F}, \mathcal{G})$ is an isomorphism.

The assumptions in Proposition 5.3.1 are satisfied if $X$ and $W$ are smooth over $k$. The canonical morphism in Proposition 5.3.1(3) is an analogue of $h^* \text{Sol} \mathcal{M} \to \text{Sol} h^* \mathcal{M}$ for a $\mathcal{D}$-module $\mathcal{M}$.

**Proof.** 1. We consider the commutative diagram

$$
\begin{array}{c}
D_W Rh^1 \mathcal{F} \\
\uparrow \\
D_W(h^* \mathcal{F} \otimes^L Rh^1 \Lambda) \\
\downarrow \\
h^* D_X \mathcal{F} \\
\rightarrow R\text{Hom}_W(Rh^1 \Lambda, Rh^1 D_X \mathcal{F})
\end{array}
$$

defined as follows. The top horizontal arrow is the dual of $h^* \mathcal{F} \otimes^L Rh^1 \Lambda \to Rh^1 \mathcal{F}$ (5.2) and the bottom horizontal arrow is the adjoint of $h^* D_X \mathcal{F} \otimes^L Rh^1 \Lambda \to Rh^1 D_X \mathcal{F}$ (5.2) for $D_X \mathcal{F}$. The left vertical arrow is a canonical isomorphism and the right vertical arrow is

$$
D_W(h^* \mathcal{F} \otimes^L Rh^1 \Lambda) = R\text{Hom}_W(h^* \mathcal{F} \otimes^L Rh^1 \Lambda, K_W)
$$

(5.3)

$$
\rightarrow R\text{Hom}_W(Rh^1 \Lambda, R\text{Hom}_W(h^* \mathcal{F}, \Lambda))
$$

$$
= R\text{Hom}_W(Rh^1 \Lambda, D_W h^* \mathcal{F}) = R\text{Hom}_W(Rh^1 \Lambda, Rh^1 D_X \mathcal{F}).
$$

The condition (1) is equivalent to that the top horizontal arrow is an isomorphism. The condition (2) is equivalent to that the bottom horizontal arrow is an isomorphism by the assumption that $Rh^1 \Lambda$ is isomorphic to $\Lambda(c)[2c]$ for an integer $c$. Hence the conditions (1) and (2) are equivalent.

Further assume that $h: W \to X$ is $K_X$-transversal. Then the isomorphism $h^* K_X \otimes^L Rh^1 \Lambda \to Rh^1 K_X = K_W$ (5.2) for $K_X$ induces an isomorphism

$$
R\text{Hom}_W(Rh^1 \Lambda, R\text{Hom}_W(h^* \mathcal{F}, \Lambda))
$$

$$
\leftrightarrow R\text{Hom}_W(Rh^1 \Lambda, R\text{Hom}_W(h^* \mathcal{F}, h^* K_X \otimes^L Rh^1 \Lambda)) = R\text{Hom}_W(h^* \mathcal{F}, h^* K_X)
$$

for (5.3). Since $h^* D_X \mathcal{F} = h^* R\text{Hom}_X(\mathcal{F}, K_X)$, the assertion follows.

2. We consider the commutative diagram

$$
\begin{array}{c}
\delta^* R\text{Hom}_X(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{G}) \otimes^L R\delta^1 \Lambda \\
\uparrow \\
R\delta^1 R\text{Hom}_X(\text{pr}_2^* \mathcal{F}, \text{pr}_1^* \mathcal{G}) \\
\rightarrow R\text{Hom}_X(\mathcal{G} \otimes^L \mathcal{F}, \mathcal{G})
\end{array}
$$

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defined as follows. The top horizontal arrow is the morphism (5.2) for $\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_X(\text{pr}_2^*F, \text{pr}_1^*G)$ and the bottom horizontal arrow is the canonical morphism in the condition (2). The canonical isomorphism $\mathcal{G} \otimes^L D_X F \to \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{X \times X}(\text{pr}_2^*F, \text{pr}_1^*G)$ induces an isomorphism $\mathcal{G} \otimes^L \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_X(F, K_X) = \delta^*(\mathcal{G} \otimes^L D_X F) \to \mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}_{X \times X}(\text{pr}_2^*F, \text{pr}_1^*G)$. This together with the canonical isomorphism $\mathcal{R}\delta^! \otimes K_X \to \Lambda$ defines the left vertical arrow. The right vertical arrow is defined by [10, 3.1.12.2]. Since the condition (1) is equivalent to that the top horizontal arrow is an isomorphism, the assertion follows.

**Proposition 5.4.** Let

$$
\begin{array}{ccc}
W & \xrightarrow{h} & X \\
j' \uparrow & & \uparrow j \\
V & \xrightarrow{h'} & U
\end{array}
$$

be a cartesian diagram of separated schemes of finite type over $k$ such that the vertical arrows are open immersions. Let $\mathcal{G}$ be a constructible complex of $\Lambda$-modules on $U$. We consider the conditions:

1. The morphism $h: W \to X$ is $j_!G$-transversal.
2. The morphism $h': V \to U$ is $G$-transversal.

1. The condition (1) implies (2). Conversely, if $Rh^1\Lambda$ is isomorphic to $\Lambda(c)[2c]$ for an integer $c$ and if the canonical morphisms

$$
(5.4) \\
j_!G \to Rj_!G, \quad j_!^*h^*\mathcal{G} \to Rj'_!h'^*\mathcal{G}
$$

are isomorphism, the condition (2) implies (1).

2. Assume that $\mathcal{G}$ is of finite tor-dimension on $U$. Then, the condition (1) is equivalent to the combination of (2) and the following condition:

3. The base change morphism

$$
(5.5) \\
h^*Rj_*\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(\mathcal{G}, K_U) \to Rj'_!h^*\mathcal{R}\mathcal{H}\mathcal{o}\mathcal{m}(\mathcal{G}, K_U)
$$

is an isomorphism.

**Proof.** 1. The implication (1)$\Rightarrow$(2) is clear.

We consider the commutative diagram

$$
\begin{array}{ccc}
j_!^*h^*\mathcal{G} \otimes^L Rh^1\Lambda & \xrightarrow{h^* j_!\mathcal{G} \otimes^L Rh^1\Lambda} & Rh^1j_!\mathcal{G} \\
\downarrow & & \downarrow \\
Rj'_!h^*\mathcal{G} \otimes^L Rh^1\Lambda & \xrightarrow{Rh^1j_*\mathcal{G}} & Rj'_!Rh^1\mathcal{G}
\end{array}
$$

defined as follows. The top right horizontal arrow and the bottom horizontal arrow are defined by (5.2). The upper vertical arrows are induced by the canonical morphisms (5.4). The top left horizontal arrow is induced by the isomorphism $j_!^*h^* \to h^* j_!$ and is an isomorphism. The slant arrow is defined as the adjoint of the isomorphism $j_!^*(Rj'_!h^*\mathcal{G} \otimes^L Rh^1\Lambda) \to Rj'_!(h^*\mathcal{G} \otimes^L Rh^1\Lambda)$ and is an isomorphism if the assumption on $Rh^1\Lambda$ is satisfied.
The lower right vertical arrow is the adjoint of the isomorphism $j^*Rh \to Rh'j'^*$ and is an isomorphism. Thus under the assumptions, the implication $(2) \Rightarrow (1)$ holds.

2. Since the condition (1) implies (2), it suffices to show that (5.5) is an isomorphism if and only if the condition (3) for $\mathcal{F} = j_!\mathcal{G}$ in Proposition 5.3.1 is satisfied, assuming (2). We consider the commutative diagram

$$
\begin{array}{ccc}
\mathcal{H}om_X(j_!\mathcal{G}, \mathcal{K}_X) & \xrightarrow{(3)} & \mathcal{H}om_W(h^*j_!\mathcal{G}, h^*\mathcal{K}_X) \\
\downarrow & & \downarrow \\
\mathcal{H}om_U(j'_!h^*\mathcal{G}, h^*\mathcal{K}_U) & \xrightarrow{(5.5)} & \mathcal{H}om_V(h^*\mathcal{G}, h^*\mathcal{K}_U)
\end{array}
$$

defined as follows. The upper left and the lower right vertical arrows are the adjunction morphisms and are isomorphisms. The upper right vertical arrow is induced by the isomorphism $h^*j_!\mathcal{G} \to j'_!h^*\mathcal{G}$ and is an isomorphism.

The top horizontal arrow (3) is the canonical morphism in the condition (3) in Proposition 5.3.1 for $\mathcal{F} = j_!\mathcal{G}$. The lower one is induced by that for $\mathcal{G}$ and is an isomorphism if (2) is satisfied, by Proposition 5.3.1 (1)$\Rightarrow$(3). Thus the assertion follows. □

5.2 Singular support and $\mathcal{F}$-transversality

First, we study the relation between local acyclicity and $\mathcal{F}$-transversality.

Lemma 5.5. Let $f: X \to Y$ be a smooth morphism of separated schemes of finite type over a field $k$ and $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. We consider the following conditions:

1. The morphism $f: X \to Y$ is locally acyclic relatively to $\mathcal{F}$.
2. For every quasi-finite morphism $g: Y' \to Y$ of separated schemes of finite type over $k$ and for every closed immersion $i: Z \to Y'$ and the cartesian diagram

$$
\begin{array}{ccc}
X & \xleftarrow{g'} & X' \\
\downarrow & & \downarrow \\
Y & \xleftarrow{g} & Z
\end{array}
$$

the immersion $i': X'_Z \to X'$ is $g'^*\mathcal{F}$-transversal.

1. (cf. [19, Proposition 2.10]) If $\mathcal{F}$ is of finite tor-dimension, then the condition (1) implies (2).
2. Assume that $Y$ is a normal curve. Then, the condition (2) restricted to quasi-finite flat and generically étale morphisms $g: Y' \to Y$ of normal curves and closed immersions $i: y' \to Y'$ of closed points implies (1).

Proof. 1. Since the local acyclicity is preserved by quasi-finite base change, we may assume $Y'' = Y$. The morphism $f: X \to Y$ is strongly locally acyclic relatively to $\mathcal{F}$ by [15, 70]...
Then, by [19, Proposition 2.10], the canonical morphism $\mathcal{F} \otimes^L f^*Rj_Y^*\Lambda \rightarrow Rj_*j^*\mathcal{F}$ is an isomorphism.

Since $f: X \rightarrow Y$ is assumed to be smooth, the base change morphism $f^*Rj_Y^*\Lambda \rightarrow Rj_*\Lambda$ is an isomorphism. Hence the canonical morphism $\mathcal{F} \otimes^L Rj_*\Lambda \rightarrow Rj_*j^*\mathcal{F}$ is an isomorphism by the smooth base change theorem [5, Corollaire 1.2]. By the distinguished triangle $\triangleright i_i Rj^! \rightarrow \text{id} \rightarrow Rj_*j^* \rightarrow \triangleright$, the canonical morphism $i_i^*\mathcal{F} \otimes^L Rj_*\Lambda \rightarrow Rj^! \mathcal{F}$ is an isomorphism.

2. We may assume $k$ is separably closed. For a closed point $y \in Y$, let $i_i: X \times_Y y \rightarrow X$ be the closed immersion, let $j_j: X \times_Y (Y - y) \rightarrow X$ be the open immersion of the complement and let $\psi_y$ denotes the classical nearby cycles functor for the base change of $X \rightarrow Y$ by the strict localization $Y(y) \rightarrow Y$. It suffices to show that the canonical morphism $i_i^*\mathcal{F} \rightarrow \psi_y \mathcal{F}$ is an isomorphism.

Let $(g: (Y', y') \rightarrow (Y, y))$ be a system of quasi-finite flat and generically étale morphisms of pointed normal curves such that the fraction field of $\lim \mathcal{O}_{Y', y'}$ is a separable closure of the function field of $Y$. Let $g': X' = X \times_Y Y' \rightarrow X$ be the base change of $g$, let $i_i': X'_y = X \times_Y y' \rightarrow X'$ be the closed immersion and let $j_j': X' - X'_y \rightarrow X'$ be the open immersion of the complement. Then, we have an isomorphism $\lim i_i^*g'^*j_{j'}^*g'^* \rightarrow \psi_y$ of functors.

Since $i_i^*g'^*j_{j'}^*g'^* \rightarrow X'$ is assumed to be $g'^*\mathcal{F}$-transversal, the canonical morphism $i_i^*g'^*j_{j'}^*g'^* \otimes^L Rj_{j'}^* \Lambda \rightarrow Rj_{j'}^* \psi_y$ is an isomorphism. By the distinguished triangle $\triangleright i_i^*g'^*j_{j'}^*g'^* \rightarrow \text{id} \rightarrow Rj_{j'}^*j_{j'}^* \rightarrow \triangleright$, the canonical morphism $i_i^*g'^* \otimes^L i_i^*Rj_{j'}^*\Lambda \rightarrow i_i^*Rj_{j'}^*j_{j'}^*g'^* \mathcal{F}$ is an isomorphism. Thus in the commutative diagram

$$
\begin{array}{ccc}
 i_i^*\mathcal{F} \otimes^L \lim i_i^*Rj_{j'}^*\Lambda & \longrightarrow & \lim i_i^*Rj_{j'}^*j_{j'}^*g'^* \mathcal{F} \\
 \downarrow & & \downarrow \\
 i_i^*\mathcal{F} \otimes^L \psi_y \Lambda & \longrightarrow & \psi_y \mathcal{F}, \\
\end{array}
$$

every arrow is an isomorphism. Since $f: X \rightarrow Y$ is smooth, the morphism $f: X \rightarrow Y$ is locally acyclic relatively to $\Lambda$ and the canonical morphism $\Lambda \rightarrow \psi_y \Lambda$ is an isomorphism by the local acyclicity of smooth morphism [4, Théorème 2.1]. Thus, the canonical morphism $i_i^*\mathcal{F} \rightarrow \psi_y \mathcal{F}$ is an isomorphism as required.

\begin{proposition}
Let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules on $X$. For a conical closed subset $C \subset T^*X$, we consider the following conditions.

1. $\mathcal{F}$ is micro-supported on $C$.

2. The support of $\mathcal{F}$ is a subset of the base $B = C \cap T^*_X X \subset X$ of $C$ and every $C$-transversal morphism $h: W \rightarrow X$ of smooth separated schemes of finite type is $\mathcal{F}$-transversal.

1. If $\mathcal{F}$ is of finite tor-dimension, then the condition (1) implies (2).

2. If $k$ is perfect, the condition (2) implies (1).
\end{proposition}

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Proof. 1. By Lemma 2.11.3, the condition that $\mathcal{F}$ is micro-supported on $C$ implies that the support of $\mathcal{F}$ is a subset of the base $B$ of $C$.

We show that a $C$-transversal morphism $h: W \to X$ of smooth separated schemes of finite type is $\mathcal{F}$-transversal assuming that $\mathcal{F}$ is micro-supported on $C$. By applying Lemmas 2.3.3 and 5.2.3 to the graph $W \to W \times X \to X$ of $h$ and by replacing $h: W \to X, C$ and $\mathcal{F}$ by $W \to W \times X, \text{pr}_2^* C$ and $\text{pr}_2^* \mathcal{F}$, we may assume that $h$ is an immersion. Since the assertion is local on $W$, we may assume that there exists a cartesian diagram

\[
\begin{array}{ccc}
W & \xrightarrow{h} & X \\
\downarrow & & \downarrow f \\
0 = y & \longrightarrow & Y = \mathbb{A}^d
\end{array}
\]

where $f: X \to Y$ is smooth. Since $h: W \to X$ is $C$-transversal, by Lemma 2.5.1, we may assume that $f: X \to Y$ is $C$-transversal.

Since $\mathcal{F}$ is assumed to be micro-supported on $C$, the morphism $f: X \to Y$ is locally acyclic relatively to $\mathcal{F}$. Hence the assertion follows from Lemma 5.5.1.

2. By Corollary 2.13 (2)⇒(1), it suffices to show that $\mathcal{F}$ is weakly micro-supported on $C$ assuming the condition (2) is satisfied. In other words, it suffices to show that, after replacing $X$ by an étale scheme over $X$, a $C$-transversal morphism $f: X \to Y$ to a smooth curve $Y$ is locally acyclic relatively to $\mathcal{F}$.

Since the base $B$ of $C$ contains the support of $\mathcal{F}$, after replacing $X$ by a neighborhood of the support of $\mathcal{F}$, we may assume that $f: X \to Y$ is smooth by Lemma 2.5.6. We show that the condition (2) in Lemma 5.5 is satisfied.

Let $g: Y' \to Y$ be a morphism of smooth curves and $y'$ be a closed point of $Y'$. Then, since $f: X \to Y$ is smooth and $C$-transversal, the base change $g': X' \to X$ and its composition $i: X'_{y'} \to X$ with the immersion $i': X'_{y'} \to X'$ of the fiber are $C$-transversal by Lemma 2.6.2. Thus, the condition (2) implies that $g': X' \to X$ and $i: X'_{y'} \to X$ are $\mathcal{F}$-transversal. Hence by Lemma 5.2.3, the immersion $i': X'_{y'} \to X'$ is $g'^* \mathcal{F}$-transversal. Thus the condition (2) in Lemma 5.5 is satisfied.

By Lemma 5.5.2, the morphism $f: X \to Y$ is locally acyclic relatively to $\mathcal{F}$. \hfill $\square$

Corollary 5.7. Assume that $k$ is perfect and let $\mathcal{F}$ be a constructible complex of $\Lambda$-modules of finite tor-dimension on $X$. Then, the following conditions are equivalent:

1. $\mathcal{F}$ is locally constant.
2. Every morphism $f: W \to X$ of smooth schemes is $\mathcal{F}$-transversal.

Proof. By Lemma 2.11, the condition (1) is equivalent to that $\mathcal{F}$ is micro-supported on the 0-section $T^*_X X$. Hence, it follows from Proposition 5.6. \hfill $\square$

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