LARGE N, SUPERSYMMETRY … AND QCD

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This paper consists of two (still only vaguely) related parts: in the first, we briefly review work done in the past three years on the “planar equivalence” between a class of non-supersymmetric theories (including limiting cases of QCD) and their corresponding supersymmetric “parents”; in the second, we present details of a new formulation of planar quantum mechanics and illustrate its effectiveness in an intriguing supersymmetric example.

1 Introduction

Lattice Gauge Theory (LGT) – an approach where Adriano has distinguished himself with many important achievements – is the main tool available at present for extracting quantitative results from QCD. This approach has established unambiguously some key non-perturbative features of QCD, such as confinement, chiral symmetry breaking, the absence of a $U(1)$-Nambu–Goldstone (NG) boson, and more. In spite of these uncontested successes, LGT remains limited, in its predictive power, by its range of applicability (in terms of number and kind of external particles and of the accessible kinematical regions) and by the fact that most of its results, being mere numerical computer outputs, often offer limited theoretical insight.

Few techniques are available today for the analytic study of non-perturbative QCD. Among these, the method of effective chiral lagrangians, based on the general concepts of spontaneous symmetry breaking (SSB) and (pseudo) Nambu–Goldstone bosons is one of the most widely used. Even though successful, it is also limited in scope to the spectrum and interactions of light NG bosons. At the other extreme, heavy quark effective theories have been of great value for describing the behaviour of hadrons carrying charm and beauty.

A very different kind of analytic tools is given, finally, by large-$N$ expansions of various kinds, an approach that is a priori of much wider applicability. Formulated more than three decades ago,[1,2] large-$N$ limits still remain elusive while attracting the attention of many theorists.[3] The relation between the large-$N$ classification of diagrams and loop expansions in string theory only enhances the interest in this approach. Unfortunately, with the exception of some quantitative results for the $\eta'$ mass and interactions,[4] large-$N$ predictions have been mostly qualitative in nature.

The situation concerning quantitative analytic results in strongly coupled gauge theories improves considerably if one is willing to consider supersymmetric variants – or
extensions– of QCD. Predictivity is increased dramatically with the number $N$ of supersymmetries: while for $N = 1$ this is basically limited to holomorphic quantities, such as the superpotential or gauge kinetic terms, for $N = 2$ it extends much further as in the famous Seiberg-Witten solution. Finally, for $N = 4$, we have the much celebrated ADS/CFT correspondence between a gauge theory on the boundary and a gravity theory in the bulk of anti de-Sitter space.

In view of the above discussion, one may argue that combining large-$N$ ideas and supersymmetry should provide as much an analytic predictive power as one may hope for. However, both large-$N$ and supersymmetry appear to take us further and further away from our initial goal: a non-perturbative understanding of ($N = 3$, non-supersymmetric) QCD.

In the first part of this contribution we will summarize work done during the past three years on a new large-$N$ expansion that has the virtue of connecting (a sector of) ordinary QCD to a supersymmetric “cousin”, and thus of allowing us to derive both qualitative and quantitative properties of the former. In the second part we apply this new large-$N$ methodology to quantum mechanical gauge systems similar to those obtained by dimensional reduction of fully space-extended field theories. In particular, we define and solve (numerically as well as analytically) a seemingly simple supersymmetric system, which exhibits several intriguing features such as a phase transition in the ’t Hooft coupling and a new form of strong–weak duality. This should provide, eventually, a good and simple laboratory for testing the idea of planar equivalence. It should also allow an extension of recent studies of Supersymmetric Quantum Mechanics to the $N \to \infty$ limit, and to make a possible contact with M-theory.

In a sense, therefore, the two parts of this contribution to Adriano’s fest try to fit with the two keywords in its title: Physics and Beauty. Achieving their synthesis represents, needless to say, our (yet unattained) final goal.

2 Large-$N$ expansions: Who needs another one?

As we mentioned in the introduction, very few techniques are currently available for analytic studies of (non-supersymmetric) gauge theories such as QCD at a non-perturbative level. Among the most promising ones, large-$N$ expansions play a special role, particularly because of their conjectured connection to string theories (see e.g. [10]).

The simplest and oldest large-$N$ expansion in QCD is the one suggested in 1974 by ’t Hooft. It considers the limit $N_c \to \infty$ while keeping the ’t Hooft coupling $\lambda \equiv g^2 N_c$, as well as the number of quark flavours $N_f$, fixed. Only quenched planar diagrams survive to leading order. Quark loops are suppressed by a power of $(N_f/N_c)$ per loop, while non-planar diagrams are suppressed by a factor of $1/N_c^2$ per handle. This ’t Hooft expansion led to a number of notable successes in issues such as justifying the validity of the OZI rule and in the $\eta'$ mass formula. Unfortunately, nobody succeeded in fully solving QCD even to leading order in this expansion.

For questions where quark loops are important, e.g. in processes with a large number of produced hadrons, a better approximation to full QCD is provided by the topological expansion (TE), where $N_f/N_c$, rather than $N_f$ itself, is kept fixed in the large-$N_c$ limit. Thus, at leading order, the TE keeps all planar diagrams, including quark loops. This is
easily seen by slightly modifying \(^2\)‘t Hooft’s double-line notation by adding a flavour line to the single colour line for quarks. In the leading (planar) diagrams the quark loops are “empty” inside, since gluons do not couple to flavour. Needless to say, obtaining analytic results in the TE is even harder than in the case of the original ‘t Hooft expansion.

In Ref.\(^7\) a new large-\(N_c\) expansion, which shares some advantages with the TE while retaining a significant predictive power, was proposed (for a review, see\(^11\)). Its basic idea is actually quite simple. Let us start from ordinary (\(N_c = 3\)) QCD with \(N_f\) quark flavours. Quarks can be described by a Dirac field transforming in the fundamental representation of SU(3)\(^\text{colour}\), or, equivalently, in the two-index antisymmetric representation (plus their complex conjugates). In extrapolating from \(N_c = 3\) to arbitrary \(N_c\), the former alternative leads to the ‘t Hooft limit. The new expansion explores instead the latter alternative, by representing the quark of a given flavour by a Dirac field in the two-index antisymmetric representation. In that case, taking the \(N_c \to \infty\) limit at \(g^2 N_c\) and \(N_f\) fixed does not decouple the quark loops since, for large \(N_c\), the number of degrees of freedom in the antisymmetric field scales like the one in the adjoint, i.e. as \(N_c^2\). For reasons explained in\(^7\), it has been referred to as the orientifold large-\(N_c\) limit. The leading order of this new expansion corresponds to the sum of all planar diagrams, in the same way as in TE, but with the crucial difference that quark loops are now “filled”, because the second line in the fermion propagator is also, now, a colour line.

The orientifold large-\(N_c\) limit is, therefore, unquenched. Its ‘t Hooft-notation diagrams look, modulo reversal of some arrows in the fermion loops, precisely as those of an SU(\(N_c\)) gauge theory with \(N_f\) Majorana fields in the adjoint representation, a theory that we may call \textit{adjoint QCD}. Adjoint QCD can be seen, in turn, as a softly broken version of supersymmetric Yang–Mills theory (SYM) with \(N_f - 1\) additional adjoint chiral superfields. The soft-breaking terms are just large mass terms for the scalar fields that make them decouple.

2.1 Proofs of planar equivalence

The planar equivalence between \(\mathcal{N} = 1\) SYM theory and orientifold field theories amounts to the following statement: an SU(\(N_c\)) gauge theory with two Weyl fermions in the two-index antisymmetric representation (i.e. one Dirac antisymmetric fermion) is equivalent, in a bosonic subsector and as \(N_c \to \infty\), to \(\mathcal{N} = 1\) gluodynamics.

In Ref.\(^7\), a perturbative proof of the planar equivalence was provided and a non-perturbative extension was outlined. In a subsequent paper\(^12\) a refined non-perturbative proof of planar equivalence (extended to \(N_f > 1\)) was given. Its basic idea is the comparison of generating functionals of appropriate gauge-invariant correlators in the parent and daughter theories. This is done by, first integrating out their respective fermions in a fixed gauge background and, then, averaging over the gauge field itself. The first step produces a fermionic determinant, which, in turn, can be expanded as a sum of Wilson loops computed for different representations. Using some group-theory identities as well as certain factorization properties of Wilson loops at large \(N\), it was possible to relate the generating functionals of the parent and daughter theories for a subset of suitably identified sources. Recently, a lattice version of the non-perturbative proof of\(^12\) has been given\(^13\), together with suggestions on how to check the equivalence by realistic lattice simulations.

It is worth noting that the two theories are not fully identical. In particular the colour-
singlet spectrum of the orientifold theory consists only of bosons and does not include composite fermions at \( N_c \to \infty \).

2.2 SUSY relics in QCD

We will now use planar equivalence to make predictions\[^{14}\] for one-flavour QCD, keeping in mind that they are expected to be valid up to corrections of the order of \( 1/N_c = 1/3 \) (barring large numerical coefficients):

(i) Confinement with a mass gap. Here we assume that large-\( N_c \) \( \mathcal{N} = 1 \) gluodynamics is a confining theory with a mass gap. Alternatively, if we start from the statement that one-flavour QCD confines, we arrive at the statement that \( \mathcal{N} = 1 \) SYM theory shares that property, while the mass gaps are dynamically generated in both theories.

(ii) Degeneracy in the colour-singlet bosonic spectrum. Even/odd parity mesons (typically mixtures of fermionic and gluonic colour-singlet states) are expected to be degenerate. In particular,

\[
\frac{m_{\eta'}^2}{m_\sigma} = 1 + O(1/N_c), \quad \text{one-flavour QCD,}
\]

where \( \eta' \) and \( \sigma \) stand for \( 0^- \) and \( 0^+ \) mesons, respectively. This follows from the exact degeneracy in \( \mathcal{N} = 1 \) SYM theory. Note that the \( \sigma \) meson is stable in this theory, as there are no pions. This prediction should be taken with care (i.e. a rather large numerical coefficient in front of \( 1/N_c \) may occur), since the \( \eta' \) mass is given by the anomaly (the WV formula\[^{4}\]), whereas the \( \sigma \) mass is more “dynamical.” The degeneracy between even and odd-parity mesons should improve at higher levels on the expected Regge trajectory.

In order to check this relation in real QCD, let us assume that the \( \sigma \) mass is not very sensitive to the number of flavours. On the other hand, according to the WV formula\[^{4}\] and neglecting quark masses, the \( \eta' \) mass scales like \( \sqrt{N_f} \); we can therefore extrapolate relation \((1)\) to obtain a prediction for real QCD

\[
m_{\eta'} \sim \sqrt{3}m_\sigma. \tag{2}
\]

Although the \( \sigma \) bump is very broad, it is encouraging that the above relation is indeed in qualitative agreement with the position of the enhancement in the appropriate \( \pi \pi \) channel.

(iii) Bifermion condensate. \( \mathcal{N} = 1 \) \( SU(N_c) \) gluodynamics has a bifermion condensate\[^{15}\] that can take \( N_c \) distinct values:

\[
\langle \lambda \lambda \rangle_k \sim M_{uv}^3 e^{-\tau/N_c} e^{2i\pi k/N_c} = c\Lambda^3 e^{i(\theta+2\pi k)/N_c}, \quad k = 0, 1, \ldots, N_c - 1, \tag{3}
\]

with

\[
\tau = \frac{8\pi^2}{g^2} - i\theta
\]

and \( c \) a calculable numerical coefficient. The finite-\( N_c \) orientifold field theory is non-supersymmetric, and here we expect (taking account of pre-asymptotic \( 1/N_c \) corrections) \( N_c - 2 \) degenerate vacua with

\[
\langle \bar{\Psi}_L \Psi_R \rangle_{k'} \sim M_{uv}^3 \exp \left\{ -\frac{8\pi^2}{g^2(N_c + 4/9)} + \frac{\theta + 2\pi k'}{N_c - 2} \right\}
\]
\[ \sim c' \Lambda^3 \exp \left\{ i \frac{\theta + 2\pi k'}{N_c - 2} \right\}, \quad k' = 0, 1, \ldots, N_c - 3. \] (4)

The term \(4/9\) in (4) is due to the one-loop \(\beta\) function of the orientifold field theory, \(b = 3N_c + \frac{4}{3}\), while \(N_c - 2\) in (4) is twice the dual Coxeter number of the antisymmetric representation (fixing the coefficient of the axial anomaly). Finally, \(c'\) is a normalization factor that we will discuss below.

### 2.3 Analytic estimate of the quark condensate in QCD

The relation between the quark condensate in one-flavour QCD and the gluino condensate in SYM theory can be pushed further\(^{16}\) by appealing to the fact that the quark condensate of QCD\(_{OR}\) must agree with the gluino condensate at \(N = \infty\) and must vanish at \(N = 2\). Thus, its value at \(N = 3\) (which is nothing but the quark condensate of one-flavour QCD) can be obtained by interpolating between these two values. The final outcome of such an analysis\(^{16}\) is the following analytic formula:

\[
\langle \bar{\Psi} \Psi \rangle_{\mu} = -\frac{3}{2\pi^2} \mu^3 (\lambda(\mu)) \left( \frac{\lambda_0}{\beta_0} \right) \frac{9}{\beta_1 (1/3)} \exp \left( -\frac{3}{\beta_0 \lambda(\mu)} \right) k(1/3), \quad (5)
\]

where \(\mu\) is the renormalization scale for the condensate, \(\beta_0\) and \(\beta_1\) are the one- and two-loop \(\beta\)-function coefficients, \(\gamma\) is the one-loop anomalous dimension of the condensate, and \(k(1/N)\) stems for further, unidentified \(1/N\) corrections.

In a recent paper\(^{17}\) an attempt was made to extend these considerations to an arbitrary number of quark flavours. The idea (actually first proposed for different reasons in\(^{18}\)) is to add to QCD\(_{OR}\) \(n_f\) quarks in the (traditional) fundamental representation of SU\((N)\), so that they become irrelevant in the large-\(N\) limit. This new class of theories has been dubbed QCD\(_{OR}'\). At \(N = 3\), such a theory is nothing but QCD with \(N_f\) flavours where \(N_f = n_f + 1\). One has to determine which sectors of QCD\(_{OR}'\) can be mapped into corresponding quantities of SYM theory, in particular since the former theory has a NG sector which is absent in SYM. When this is done\(^{17}\) one obtains an interesting generalization of (5) to \(N_f > 1\) and can even argue that the connections should be particularly accurate for the “realistic” case of \(N_f = 3\). For three-flavour QCD the relevant values are \(\beta_0 = 9\), \(\beta_1 = 32\), \(\gamma = 4\).

In order to calculate the condensate (5) we need to know the value of the ’t Hooft coupling \(\lambda\) at a scale \(\mu\) that we choose to be \(\mu = 2\text{ GeV}\). The Particle Data Group\(^{19}\) quotes \(\alpha_s(2\text{ GeV}) = 0.31 \pm 0.01\), which corresponds to \(\lambda(2\text{ GeV}) = 0.148 \pm 0.010\). We therefore choose to plot the function

\[
\langle \bar{\Psi} \Psi \rangle_{2\text{ GeV}} = -\frac{3}{2\pi^2} 2^3 \lambda \frac{34}{27} \exp \left( -\frac{1}{\lambda} \right),
\]

in a range of \(\lambda\). Comparison between the above theoretical prediction and present determinations of the condensate is shown in Fig. 1. Clearly the result supports the validity of planar equivalence within the expected precision at \(N = 3\).
Figure 1: The quark condensate expressed as $-\langle y \text{ MeV}\rangle^3$ as a function of the ’t Hooft coupling $\lambda$. The $\pm 1\sigma$ range of the coupling, $0.138 < \lambda < 0.158$, and the lattice estimate $-\langle 259 \pm 27 \text{ MeV}\rangle^3$ define the shaded region.

3 Planar Quantum Mechanics

In this second part of the paper, we turn from QFT to the simpler case of Quantum Mechanics and try to define a “Planar Quantum Mechanics” that hopefully mimicks the large-$N$ limit of a system whose degrees of freedom are $N \times N$ (bosonic or fermionic) matrices. The hope is that, eventually, this technique can be extended to QFT and provide a new handle on solving interesting gauge theories –such as YM, SYM and QCD– in the large-$N$ limit. It may also provide useful information on whether (and how) “planar equivalence” may or may not work.

3.1 Fock states, and matrix elements at large $N$

Our quantum mechanical system is defined by a Hamiltonian $H$, which is polynomial in $N \times N$-matrix annihilation and in creation operators $a_{ij}, a^\dagger_{ij}$. Symmetry under the $U(N)$ transformations is ensured by taking $H$ to be the trace of such a polynomial. Creation and annihilation operators satisfy standard commutation relations

$$[a_{ij}, a^\dagger_{kl}] = \delta_{il}\delta_{jk}.$$  \hspace{1cm} (7)
We shall work in the eigenbasis of the $U(N)$-invariant number operator $B = \text{Tr}(a^\dagger a)$. Therefore basis states are constructed by acting on the Fock vacuum with the invariant building blocks, or “bricks”, $\text{Tr}[(a^\dagger)^n]$, $n = 1, 2, \ldots, N$. Following the Cayley–Hamilton theorem there are $N$ independent bricks for the $U(N)$ group. A general state is a product of all possible bricks and their powers. It will prove useful to define a cut Hilbert space $H_B$, i.e. a space restricted to states with no more than $B$ bosonic quanta. Such a space is spanned by all polynomials of bricks with maximum $B$-th order. Obviously there are many states with a fixed number of bosonic quanta $n$.

In the large-$N$ limit, things simplify considerably: for given $n$, there is only one relevant state. It is created by the single-trace operator:

$$|n\rangle = \frac{1}{\mathcal{N}_n} \text{Tr}[(a^\dagger)^n]|0\rangle,$$

(8)

where $\mathcal{N}_n$ is a suitable normalization factor. All states that are created by products of traces are non-leading in the sense that they give rise to non-leading matrix elements. A second simplification is that leading operators also have a single trace form. These two basic rules of the Planar Calculus are best illustrated by the explicit examples below.

Normalization of planar states

The normalization factor $\mathcal{N}_n$ in (8) reads

$$\mathcal{N}_n^2 = \langle 0|\text{Tr}[a^n]\text{Tr}[(a^\dagger)^n]|0\rangle = \langle 0|(12)(23)...(n1)[1'2'][2'3']...[n'1']|0\rangle,$$

(9)

where we have simplified the notation, e.g. $a_{i_1,i_2} \rightarrow (12)$ and $a_{i_7,i_8} \rightarrow [7'8']$ and all indices are summed. Using (7) it is easy to see that the maximal power of $N$ is achieved by contracting indices between () and [] starting from the middle and working outwards. This produces the famous planar diagrams. One way to picture it is to represent ()()...() as a circle with $n$ pairs of short lines, representing indices, pointing inside and []][[ as a smaller circle with lines pointing outside. Leading contributions result when opposite pairs of lines match: this gives $n$ loops, i.e. a factor $N^n$. Because of the cyclic properties of a trace in (9), one gets $n$ such contributions. Moreover, according to (7), each contraction gives a factor 1. The final result reads

$$\mathcal{N}_n = \begin{cases} \sqrt{n}N^{(n/2)}, & n > 0; \\ 1, & n = 0. \end{cases}$$

(10)

Example of matrix elements

As another example, we calculate the matrix element of a typical term in a generic Hamiltonian:

$$H_{n+2,n} = g^2 \langle n + 2|\text{Tr}[a^\dagger a^\dagger a a]|n\rangle.$$

(11)

Consider a state

$$\text{Tr}[a^\dagger a^\dagger a^\dagger a]|(a^\dagger)^n\rangle = [12][23][34](41)[1'2'][2'3'][3'4']...[n'1']|0\rangle = n[1'2][23][3'4']...[n'1']|0\rangle,$$

(12)

$n$ equal terms result from commuting one annihilator, (41), all the way until it hits the vacuum state. Each contraction (7) gives just a factor 1. Pictorially, one can attach a small circle with 3 creators and one annihilator to a big circle with $n$ pairs of indices
in $n$ equivalent ways. The resulting state is proportional to $|n + 2\rangle$. Collecting the normalization factors gives

$$H_{n+2,n} = g^2 n \frac{1}{N_n} \langle n + 2 | Tr[(a^\dagger)^{(n+2)}]|0\rangle = g^2 n \frac{\sqrt{(n+2)/nN}}{N_{n+2}} \times$$

$$\langle n + 2 | Tr[(a^\dagger)^{(n+2)}]|0\rangle = g^2 N \sqrt{n(n+2)}, \quad n > 0. \quad (13)$$

Indeed, such an interaction term depends only on $\lambda = g^2 N$, showing the relevance of the t’Hooft coupling. To complete this example, we also calculate a matrix element of the conjugate operator

$$H_{n-2,n} = g^2 \langle n - 2 | Tr[a^\dagger a a a]|n\rangle = \frac{1}{N_n} \langle n - 2 | Tr[a^\dagger a a a]|(a^\dagger)^n]|0\rangle. \quad (14)$$

The leading contribution comes from the three subsequent contractions of adjacent $a$’s and $a^\dagger$’s, which give a factor $N^2$. Again this can be done in $n$ ways. Taking into account the renormalization $N_n \to N_{(n-2)}$ gives finally

$$H_{n-2,n} = g^2 N \sqrt{n(n-2)}, \quad n > 2, \quad (15)$$

which again shows the t’Hooft scaling and, upon the replacement $n \to n + 2$, confirms the hermiticity of the planar hamiltonian.

### 3.2 An anharmonic oscillator

Let us now apply planar rules to find the spectrum of the $U(\infty)$-invariant anharmonic oscillator. The Hamiltonian

$$H = Tr[p^2] + g^2 Tr[x^4] \equiv T + V \quad (16)$$

is bounded from below and represents a perfectly well defined system per se. On the other hand, there exists a whole family of similar Hamiltonians (supersymmetric or not), which have been obtained from gauge-invariant field theories by reducing the system to a single point in space. For example, reduced two-dimensional Yang–Mills gluodynamics is described solely by the kinetic term of Eq. (16) with a global $U(N)$ invariance as the reminder of the local gauge symmetry of the original, space-extended theory.

As a first step, introduce matrix creation and annihilation operators

$$x = \frac{1}{\sqrt{2}} (a + a^\dagger), \quad p = \frac{1}{\sqrt{2i}} (a - a^\dagger), \quad (17)$$

and rewrite each term of $H$ in the normal-ordered form:

$$H = \frac{1}{2} N^2 + \frac{1}{2} N^3 g^2$$

$$+ \left( -\frac{1}{2} + N g^2 \right) (a^\dagger^2 + a^2) + (1 + 2g^2 N) a^\dagger a$$

$$+ \frac{g^2}{4} (a^\dagger^4 + a^4) + g^2 (a^\dagger^2 a^2 + g^2 (a^\dagger^3 a + a^\dagger a^3) \quad (18)$$
A note on the normal ordering may be useful here. For higher powers of $a$'s and $a^\dagger$'s the normal ordering may spoil the group structure. For example

$$ a_{ik}a_{kj}a_{jl}a_{li} := a^\dagger_{kj}a^\dagger_{li}a_{ik}a_{jl}, \quad (19) $$

is not a trace, hence it does not contribute to the planar limit. On the other hand

$$ a^\dagger_{ik}a_{kj}a_{jl}a_{li} := a^\dagger_{ik}a^\dagger_{li}a_{kj}a_{jl}, \quad (20) $$

can be brought to the trace form - it preserves the group structure - and consequently contributes to the leading behavior. As an example consider one of the quartic terms in the expansion of the potential in Eq. (16).

$$ (2, 2) = Tr[a^2a^\dagger^2 + aa^\dagger aa^\dagger + a^\dagger a + a^\dagger a]^2] $$

Out of the six quartic terms only four preserve the trace structure after bringing them to the normal ordered form. The remaining two do not, hence they are non-leading.

The calculation of all matrix elements of (18) closely follows the earlier examples. After some algebra one obtains

$$<n|H|n > = \frac{N^2}{2}(1 + \lambda) + \left\{ \begin{array}{ll} (1 + 3\lambda)n, & n \geq 2, \\ (1 + 2\lambda)n, & n < 2, \end{array} \right. $$

$$<n + 2|H|n > =<n|H|n + 2 > = \left\{ \begin{array}{ll} \left(-\frac{1}{2} + 2\lambda\right) \sqrt{n(n + 2)}, & n > 0, \\ \left(-\frac{1}{2} + \lambda\right) \sqrt{2N}, & n = 0, \end{array} \right. $$

$$<n + 4|H|n > =<n|H|n + 4 > = \left\{ \begin{array}{ll} \frac{1}{4}\lambda\sqrt{n(n + 4)}, & n > 0, \\ \frac{1}{2}\lambda N, & n = 0. \end{array} \right. $$

This result illustrates an important feature of all large-$N$ calculations (also valid in QFT). Namely, not all matrix elements scale with $\lambda$. There exists a class of "superleading" contributions which are divergent in the ’t Hooft limit. They result from vacuum diagrams and should be treated separately. At the moment, we just neglect them, i.e. we use in the following the subtracted Hamiltonian $H - \frac{N^2(1 + \lambda)}{2}$ and ignore the first row and column where the non-scaling vacuum diagrams contribute only.

We may now go ahead and diagonalize the so modified Hamiltonian $\tilde{H}$. To this end we introduce a cut-off $n \leq B$, find numerically the spectrum of $\tilde{H}$, and increase $B$ until the convergence is reached.

Our results are shown in Fig. 2 and can be summarized as follows.

- Satisfactory convergence of lower eigenenergies, with increasing cut-off, is achieved for $\lambda < \sim 4$. In this region the system resembles an effective harmonic oscillator with almost (but not exactly) equidistant "infinite volume" levels. This seems to be a general feature of the large-$N$ dynamics.

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*We are grateful to E. Onofri for useful discussions on this issue.
Figure 2: Cut-off dependence of the spectrum of the anharmonic oscillator in the even-parity sector.
With $\lambda$ approaching $\lambda_c \sim 4$ the mass gap vanishes and all levels fall towards zero with increasing cut-off. Such a behaviour has been studied before and is the characteristic sign of the continuous spectrum in the infinite-volume limit. For $\lambda > \lambda_c$ all levels fall with the cut-off towards $-\infty$; $\bar{H}$ appears to be unbounded from below.

The last point may seem unnatural, signalling some trouble for the planar approach. In fact it provides one more argument that a quantum mechanics turns into a field theory in the large-$N$ limit. Namely, bound states with negative energies are of course quite common in both systems. However only a field theory, with its multiparticle sectors, can account for an infinite series of negative states.

Nevertheless this emergence of the negative eigenenergies provides an important lesson (and a warning) about possible intricacies of the planar limit. The original Hamiltonian (16) is perfectly well defined and positive for any finite $N$. This raises the question of the relation between the "true" large-$N$ limit, defined as solving non-perturbatively a system at finite $N$ and then taking $N$ large, and the planar limit discussed here. There may be many reasons for the non-commutativity of the two, the most obvious one being the ad hoc subtraction of superleading contributions. The model discussed hereafter does not have this deficiency.

3.3 A very symmetric supersymmetric system

We now add fermionic degrees of freedom described by matrix creation and annihilation operators $f^\dagger$, $f$. $\{f_{ij}, f_{kl}\} = \delta_{ij}\delta_{jk}$. To avoid vacuum diagrams, we require that supersymmetry charges (hence also a Hamiltonian) annihilate the empty Fock state. A simple choice is

$$Q = \text{Tr}[f a^\dagger(1 + ga)] \quad Q^\dagger = \text{Tr}[f^\dagger(1 + ga)a] \quad H = \{Q, Q^\dagger\} = H_B + H_F,$$

or explicitly (a trace is always implied)

$$H_B = a^\dagger a + g(a^\dagger a + a^\dagger a^2) + g^2a^\dagger a^2,$$

$$H_F = f^\dagger f + g(f^\dagger f(a^\dagger + a) + f^\dagger(a^\dagger + a)f) + g^2(f^\dagger a f a^\dagger + f^\dagger a^\dagger f + f^\dagger f a^\dagger a + f^\dagger a f a).$$

This Hamiltonian conserves the fermion number $F = f^\dagger f$. In each fermionic sector the planar basis $\{|F,n\rangle\}$ is now created by the single trace with $F$ fermionic and $n$ bosonic creation operators. It is now a simple, but somewhat tedious, exercise to calculate the matrix elements of $H$ in this representation. We obtain for the first two fermionic sectors:

$$\langle 0, n | H | 0, n \rangle = (1 + \lambda)n - \lambda n_1,$$

$$\langle 0, n + 1 | H | 0, n \rangle = \langle 0, n | H | 0, n + 1 \rangle = \sqrt{\lambda} \sqrt{n(n + 1)},$$

$$\langle 1, n | H | 1, n \rangle = (1 + \lambda)(n + 1) + \lambda,$$

$$\langle 1, n + 1 | H | 1, n \rangle = \langle 1, n | H | 1, n + 1 \rangle = \sqrt{\lambda}(2 + n).$$

The numerical calculation of the spectrum proceeds in the same way as for the anharmonic oscillator (see Fig. 3).

JW thanks P. Menotti for a very instructive discussion on this subject.
Figure 3: Cut-off dependence of the spectrum of $H$, in the $F = 0$ sector, in a range of values for $\lambda$. 

\[ E \] vs. \[ B \] for different values of $\lambda$: 
- $\lambda = 0.2$ 
- $\lambda = 0.4$ 
- $\lambda = 0.7$ 
- $\lambda = 1.0$ 
- $\lambda = 2.0$ 
- $\lambda = 7.0$
Figure 4: First 10 energy levels of $H$, in $F = 0$ and $F = 1$ sectors, at $\lambda = 0.5$.

Now, however, the salient features of this system are:

- The spectrum is non-negative for all values of the ’t Hooft coupling. This confirms the suspicion of vacuum diagrams being responsible for the earlier problems.

- For $\lambda$ away from 1 the eigenenergies (at infinite cut-off) are again almost equally separated with a common mass gap depending on $\lambda$.

- There is a phase transition at $\lambda_c = 1$. At this point, and only there, the spectrum loses its mass gap and becomes continuous.

- In the vicinity of $\lambda_c$ we observe a critical slowing down: higher cut-offs are required to achieve satisfactory convergence.

- There is an excellent boson–fermion degeneracy, see Fig. 4. Supersymmetry is unbroken at infinite cut-off but is quite badly broken at finite cut-off near the critical point $\lambda_c$.

- In both phases there is an unpaired bosonic SUSY vacuum with zero energy. This is the empty Fock state, which, by construction, is annihilated by our Hamiltonian.

- While moving across the transition point, at finite cut-off, members of supermultiplets rearrange; see Fig. 5. In particular a second ground state with zero energy appears in the strong coupling phase.

- Some symmetry between the strong and weak coupling regions emerges.

It turns out that the problem can be solved analytically, offering better understanding of these features. We shall now discuss each one of them in turn.
Supersymmetry

With the rules of Sect.2 the matrix representation of supersymmetry charges is easily obtained in the planar limit:

\[ \langle 1, n - 1 | Q^\dagger | 0, n \rangle = \sqrt{n}, \quad \langle 1, n - 2 | Q^\dagger | 0, n \rangle = b \sqrt{n}. \]  

(27)

Obviously the \( Q^\dagger \) generator has matrix elements only between \( F = 0 \) and \( F = 1 \) sectors. One can now readily check the sum rule

\[ \sum_m \langle 0, n' | Q | 1, m \rangle \langle 1, m | Q^\dagger | 0, n \rangle = \langle 0, n' | H | 0, n \rangle, \]  

(28)

which follows from \( H = \{ Q, Q^\dagger \} \). In our representation its test becomes a simple exercise in matrix multiplication. Notice that even the exceptional diagonal contribution at \( n = 1 \) is correctly reproduced by the anticommutator of SUSY charges.

A little more involved test employs the identity

\[ \langle 0, n' | H^2 | 0, n \rangle = \langle 0, n' | QHQ^\dagger | 0, n \rangle, \]  

(29)

which again turns into straightforward matrix algebra. Notice that the sum rule resulting from (29) relates matrix elements of \( H \) in different fermionic sectors.

We conclude that the planar approximation preserves supersymmetry. On the other hand the cut-off in terms of the number of bosonic quanta explicitly breaks SUSY. Consequently the fermion–boson degeneracy is observed numerically only at sufficiently large cut-offs. It is conceivable that SUSY invariant cut-offs can be found25.

The second ground state

To understand the existence of the second massless state, let us introduce the composite creation and annihilation operators \( a_n^\dagger \) and \( a_n \), which create the planar basis in the new Hilbert space:

\[ |0, n\rangle = a_n^\dagger |0\rangle. \]  

(30)

In terms of these operators the \( F = 0 \) Hamiltonian reads

\[ H^{(F=0)} = a_1^\dagger a_1 + \sum_{n=1}^{\infty} n(1 + b^2)a_n^\dagger a_n + \left( \sum_{n=1}^{\infty} b \sqrt{n(n+1)} a_n a_{n+1} + \text{h.c.} \right). \]  

(31)

Then one can formally construct the following state

\[ |0\rangle_2 = \sum_{n=1}^{\infty} \left( \frac{-1}{b} \right)^n a_n^\dagger \sqrt{n} |0\rangle_1, \]  

(32)

which is annihilated by \( Q^\dagger \), Eq. (27), hence also by the Hamiltonian (25). It is normalizable only for \( \lambda > 1 \), thereby explaining the puzzle found numerically. The emergence of this second ground state in the strong coupling phase causes the Witten index26 to jump by one unit (in the sectors with \( F = 0, 1 \)) across the phase transition. According to the Feynman–Kac relation, the thermal partition function also reveals such a discontinuity at zero temperature. However, since higher levels collapse to zero at \( \lambda_c \), there is also a \( \delta \)-like contribution at this point: c.f. Fig. 6.
Figure 5: Rearrangement of the $F = 0$ and $F = 1$ spectra, around $\lambda = 1$, for increasing cut-offs.
Figure 6: $\lambda$ dependence of the Witten index and the partition function, at $\beta = 6$, for different cut-offs.

**Strong–weak duality**

The qualitative symmetry of the energy levels between the strong and weak coupling phases, seen in Fig. 5, finds its explanation in a rather intriguing exact duality between the two regimes. Consider the Hamiltonian (26) in the $F = 1$ sector. It is evident that the “reduced” Hamiltonian

$$\bar{H} = \frac{1}{b}(H - b^2)$$

is symmetric under the replacement $b \to 1/b$. Therefore the eigenenergies of $H$ satisfy

$$\frac{1}{b} \left( E_{n}^{(F=1)}(b) - b^2 \right) = b \left( E_{n}^{(F=1)}(1/b) - \frac{1}{b^2} \right).$$

The same reciprocity relation in the $F = 0$ sector does not hold at the operator level, c.f. (25). On the other hand, because of the supersymmetry, the duality works there as well. However, since there exists a second bosonic vacuum, in this sector duality relates eigenenergies with different indices:

$$\frac{1}{b} \left( E_{n}^{(F=0)}(b) - b^2 \right) = b \left( E_{n+1}^{(F=0)}(1/b) - \frac{1}{b^2} \right), \quad b < 1. \tag{34}$$

**The analytic solution**

Finally, and rather surprisingly, it turns out that the Hamiltonian (25) can be analytically diagonalized. More details can be found in 20; here we outline only the main steps of the derivation.

1. The $F = 0$ Hamiltonian is strikingly simple when written in terms of the linear combinations of $a_n$:

$$H = \sum_{n=1}^{\infty} B_n^\dagger B_n, \quad B_n = \sqrt{n}a_n + b\sqrt{n+1}a_{n+1},$$
2. The states created by the $B_n^\dagger$ operators

$$\left| B_n \right\rangle \equiv B_n^\dagger \left| 0 \right\rangle = \sqrt{n}\left| n \right\rangle + b\sqrt{n + 1}\left| n + 1 \right\rangle. \quad (35)$$

form a non-orthonormal basis. We shall find the eigenstates $\left| \psi \right\rangle$ of the reduced Hamiltonian by expanding them into the $\left| B_n \right\rangle$ basis and constructing the generating function $f(x)$ for the expansion coefficients:

$$\left| \psi \right\rangle = \sum_{n=0}^{\infty} c_n \left| B_n \right\rangle \leftrightarrow f(x) = \sum_{n=0}^{\infty} c_n x^n. \quad (36)$$

3. The simple way in which $\bar{H}$ acts on the $\left| B_n \right\rangle$ basis implies the following first-order differential equation for $f(x)$

$$\bar{H} \left| \psi \right\rangle = \epsilon \left| \psi \right\rangle \leftrightarrow w(x)f'(x) + xf(x) - bf(0) - f'(0) = \epsilon f(x), \quad (37)$$

where $w(x) = (x + b)(x + 1/b)$, and the eigenvalues of $H$ are simply related to those of $\bar{H}$: $E = b(\epsilon + b)$.

4. A solution of (37) is straightforward (see again 20 for details). The final expression for the generating function reads

$$f(x) = \begin{cases} \frac{1}{\alpha} \frac{1}{x + 1/b} F(1, \alpha; 1 + \alpha; \frac{x + 1/b}{x + 1/b}), & b < 1, \\ \frac{1}{1 - \alpha} \frac{1}{x + b} F(1, 1 - \alpha; 2 - \alpha; \frac{x + 1/b}{x + b}), & b > 1, \end{cases} \quad \alpha = \frac{\epsilon + b}{b - 1/b}, \quad (38)$$

where $F(a, b, c; x)$ is the standard geometric function, and the quantization condition,

$$(\epsilon + b)f(0) = 0, \quad (39)$$
determines the discrete series of eigenenergies in the $F = 0$ sector.

This analytic solution nicely explains all results discussed above. In particular:

- The absolute value of $f(0)$ reveals a series of zeros in $\alpha$, c.f. Fig. 7 which agree with the large cut-off limit of the numerical eigenvalues of (25).

- The duality among massive eigenvalues follows immediately from (39) and the symmetry of (38) under the substitution $b \rightarrow 1/b$ and $\alpha \rightarrow 1 - \alpha$.

- The collapse of the eigenenergies to zero at the critical point is evident from $E = \alpha(b^2 - 1)$, and so is the fact that the roots $\alpha_n$ are bounded by nearby poles of the $\beta$ function.

- The generating function of the second ground state can be easily obtained by setting $\alpha = 0$ in Eq. (38) for $b > 1$. This gives

$$f_0(x) = \frac{1}{1 + bx} \log \frac{b + x}{b - 1/b}, \quad b > 1, \quad (40)$$

which indeed corresponds to the expansion (32). On the other hand, one cannot set $\alpha = 0$ in the $b < 1$ solution – the second vacuum does not exist for $b < 1$. 


Figure 7: The quantization condition as a function of $\alpha$. The first four zeros are clearly visible. To see higher zeros the $\alpha$ resolution of the plot must be increased.

Given (38) and (39) one can study the flow of the eigenvalues in both phases. Perhaps the most interesting is the behaviour in the vicinity of the critical point. From the known asymptotics of the hypergeometric functions around $\lambda = 1$ we conclude that the first eigenvalue tends to zero as $-(1 - \lambda)/\log(1 - \lambda)$ when $\lambda \to 1^-$, which results in a vanishing first – and infinite second – derivative.

One can also quantitatively study the free energy near the phase transition, which appears to be stronger than in the Gross–Witten model\textsuperscript{27}.

4 Higher-$F$ sectors

The structure of higher-$F$ sectors, rather than being a simple repetition of the $F = 0, 1$ sectors, appears to exhibit novel interesting features. This already follows quite directly from looking at the free theory ($\lambda = 0$). Even in that case the structure of the multiplets, and the way they should rearrange into supermultiplets, is highly non-trivial. This is best illustrated in a plot that resembles the old Chew–Frautschi plot (CFP), in which angular momentum is plotted against squared mass\textsuperscript{[4]}.

[4] Linearity of the (Regge) trajectories in this plot was perhaps the first evidence for a string-like structure of hadrons, with the inverse of the Regge slope reinterpreted as the string tension.
achieved non-trivially by having some of the $F = 3$ states pair with those with $F = 4$. Once more, this leaves some of the $F = 4$ states unpaired but, continuing the exercise upwards in the CFP, shows that, eventually, every state finds its supersymmetric partner.

An amusing mathematical digression can be made at this point: our states are, in many ways, what mathematicians would call “binary necklaces” because they have a cyclic structure and are made out of just two kinds of “beads”, a bosonic and a fermionic one. One immediately finds on the simplest examples that, in general, necklaces of a given (even) length do not split in an equal number of necklaces with even or odd fermions (there are always more bosonic than fermionic necklaces). This would immediately violate SUSY, except that Pauli’s principle comes to the rescue by killing a subset of bosonic necklaces. Such a result, which can obviously be formulated as a purely mathematical combinatorial problem, appears to be new.

Coming back to our CFP, we may discuss further which linear combinations of the degenerate states of a given $F$ and $E$ form SUSY doublets. The task is facilitated by introducing another conserved quantity:

$$C \equiv [Q^\dagger, Q], \quad [C, H] = 0 \quad (41)$$

and by noting that $C^2 = H^2$. Clearly the eigenstates can be classified according to their “$C$-parity” i.e. according to whether $C = \pm E$. We may even consider the combination of $C$ and fermion number $C_F = (-1)^F C$. States with $C = +1 (-1)$ are clearly those annihilated by $Q^\dagger (Q)$. Most of the states turn out to have $C_F = -1$ (all of them in the $F = 0, 1$ sectors), but there also are some “unnatural” ones with $C_F = +1$, which are very important for the full matching of bosons and fermions discussed above. While the analysis of generic values of $F$ needs a more systematic –and probably computer-based– approach, we have been able to deal by brute force with the $F = 2, 3$ sectors confirming all the results discussed above. There is, however, another surprise: As

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*GV wishes to thank Professor J.C. Yoccoz for this private information.*
\( \lambda \to 1 \), all states again go to \( E = 0 \); at \( \lambda > 1 \), two states with \( F = 2 \) stay at \( E = 0 \), while all other states move up to positive \( E \). The question of whether new \( E = 0 \) states keep popping up at large \( \lambda \) in even higher-\( F \) sectors is currently being explored.

5 Outlook

We would like to conclude this paper by outlining a possible programme for connecting its two parts in a reasonably near future.

A first step would consist in generalizing our SQM model to the case in which bosons and fermions belong to different representations of the symmetry group \( U(N) \). In the case of bosons in the adjoint representation and fermions in the two-index (anti)symmetric representation (plus its complex conjugate), it would be interesting to check whether (at least parts of) the bosonic spectra of the latter theories coincide with those of the supersymmetric case in the planar limit.

Next, we would like to extend our planar calculus to systems with rotational symmetry, such as those obtained by dimensional reduction of (supersymmetric) Yang–Mills theories in \( D = 3 + 1 \) and \( D = 9 + 1 \) dimensions. This basically requires generalization of the planar rules for more than one species of bosons and fermions. A good step in this direction is being made by studying our supersymmetric quantum mechanical model with an arbitrary number of fermions.\(^{29}\)

Should these first two steps succeed, the next one would be to extend the whole approach to the planar Hamiltonians of full fledged quantum field theories, with or without supersymmetry, and in particular to the orientifold large-\( N \) limit of QCD described in the first part of this paper.

Finally, comparing the \( N = \infty \) results with those that can be obtained at \( N = 3 \) by the method of Ref.\(^{21}\) would give an estimate of how good the orientifold \( 1/N \) expansion is at \( N = 3 \) and of the extent to which supersymmetric gauge theories can make accurate predictions for actual QCD.

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