RAMANUJAN CONGRUENCES FOR SIEGEL MODULAR FORMS

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Abstract. We determine conditions for the existence and non-existence of Ramanujan-type congruences for Jacobi forms. We extend these results to Siegel modular forms of degree 2 and as an application, we establish Ramanujan-type congruences for explicit examples of Siegel modular forms.

1. Introduction and statement of results

Congruences in the coefficients of automorphic forms have been the subject of much study. A famous early example involves the partition function \( p(n) \) which counts the number of ways of writing \( n \) as a sum of non-increasing positive integers. Ramanujan established

\[
\begin{align*}
p(5n+4) & \equiv 0 \pmod{5} \\
p(7n+5) & \equiv 0 \pmod{7} \\
p(11n+6) & \equiv 0 \pmod{11},
\end{align*}
\]

which are now simply called Ramanujan congruences. More generally, an elliptic modular form with Fourier coefficients \( a(n) \) is said to have a Ramanujan-type congruence at \( b \pmod{p} \) if \( a(pn+b) \equiv 0 \pmod{p} \), where \( p \) is a prime. Ahlgren and Boylan \cite{ahlgren} build on work by Kiming and Olsson \cite{kiming} to prove that (1.1) are the only such congruences for the partition function. Nevertheless, congruences of non-Ramanujan-type also exist, as Ono \cite{ono} demonstrates. (See also Chapter 5 of Ono \cite{ono} for an account of congruences for the partition function.) The existence and non-existence of Ramanujan-type congruences for elliptic modular forms have recently been studied by Cooper, Wage, and Wang \cite{cooper} and Sinick \cite{sinick}. See also \cite{cooper2}, which generalizes \cite{ahlgren} to provide a method to find all Ramanujan-type congruences in certain weakly holomorphic modular forms.

In this paper, we investigate Ramanujan-type congruences for Siegel modular forms of degree 2. Throughout, \( Z := (\tfrac{z}{\tau}, \tfrac{z'}{\tau'}) \) is a variable in the Siegel upper half space of degree 2, \( q := e^{2\pi i \tau} \), \( \zeta := e^{2\pi i z} \), \( q' := e^{2\pi i z'} \), and \( \mathbb{D} := (2\pi i)^{-2} \left( 4 \frac{\partial}{\partial \tau} \frac{\partial}{\partial \tau'} - \frac{\partial^2}{\partial z \partial z'} \right) \) is the generalized theta operator, which acts on Fourier expansions of Siegel modular
forms as follows:

$$\mathbb{D} \left( \sum_{\substack{T \geq 0 \ T\text{ even}}} a(T)e^{\pi i \text{tr}(TZ)} \right) = \sum_{\substack{T \geq 0 \ T\text{ even}}} \det(T)a(T)e^{\pi i \text{tr}(TZ)},$$

where \( \text{tr} \) denotes the trace, and where the sum is over all symmetric, semi-positive definite, integral, and even \( 2 \times 2 \) matrices. Additionally, we always let \( p \geq 5 \) be a prime and (for simplicity) we always assume that the weight \( k \) is an even integer.

**Definition 1.1.** A Siegel modular form \( F = \sum a(T)e^{\pi i \text{tr}(TZ)} \) with \( p \)-integral rational coefficients has a Ramanujan-type congruence at \( b \) \((mod \ p)\) if \( a(T) \equiv 0 \ (mod \ p) \) for all \( T \) with \( \det T \equiv b \ (mod \ p) \).

Note that such congruences at \( 0 \) \((mod \ p)\) have already been studied in [3] and our main result in this paper complements [3] by giving the case \( b \neq 0 \) \((mod \ p)\).

**Theorem 1.2.** Let \( F(Z) = \sum_{n,r,m \in \mathbb{Z}} A(n,r,m)q^n q^r q^m \) be a Siegel modular form of degree 2 and even weight \( k \) with \( p \)-integral rational coefficients and let \( b \neq 0 \) \((mod \ p)\). Then \( F \) has a Ramanujan-type congruence at \( b \) \((mod \ p)\) if and only if

$$D^{p+1 \over 2}(F) \equiv -\left( {b \over p} \right) D(F) \ (mod \ p),$$

where \( \left( {b \over p} \right) \) is the Legendre symbol. Moreover, if \( p > k \), \( p \neq 2k - 1 \), and there exists an \( A(n,r,m) \) with \( p \nmid \gcd(n,m) \) such that \( A(n,r,m) \neq 0 \) \((mod \ p)\), then \( F \) does not have a Ramanujan-type congruence at \( b \) \((mod \ p)\).

**Remarks:**

1. If \( F \) in Theorem 1.2 has a Ramanujan-type congruence at \( b \neq 0 \) \((mod \ p)\), then it also has such congruences at \( b' \) \((mod \ p)\) whenever \( \left( {b \over p} \right) = \left( {b' \over p} \right) \), i.e., there are \( \frac{p+1}{2} \) or \( p-1 \) such congruences.
2. The condition \( p \neq 2k - 1 \) in the second part of Theorem 1.2 is necessary since there are Siegel modular forms \( F \) of weight \( \frac{p+1}{2} \) such that \( F \neq 0 \) \((mod \ p)\) and \( D(F) \equiv 0 \) \((mod \ p)\). For example, let \( F \) be the Siegel Eisenstein series of weight 4 normalized by \( a((0 \ 0 \ 0 \ 0)) = 1 \) and take \( p = 7 \). Such Siegel modular forms satisfy (1.2) for any \( b \) and hence have Ramanujan-type congruences at all \( b \neq 0 \) \((mod \ p)\). The condition that there exists an \( A(n,r,m) \neq 0 \) \((mod \ p)\) where \( p \nmid \gcd(n,m) \) is also necessary since there exist Siegel modular forms \( F \) of weight \( p-1 \) such that \( F \equiv 1 \) \((mod \ p)\) (see Theorem 4.5 of [12]). Such forms have Ramanujan-type congruences at all \( b \neq 0 \) \((mod \ p)\).

In Section 2, we investigate congruences of Jacobi forms and, in particular, we establish criteria for the existence and non-existence of Ramanujan-type congruences for Jacobi forms. In Section 3, we use such congruences for Jacobi forms to prove
Theorem 1.2. Using our results, it is now a finite computation to find Ramanujan-type congruences at all $b \not\equiv 0 \pmod{p}$ for any Siegel modular form. We give several explicit examples. Finally, we present a construction of Siegel modular forms that have Ramanujan-type congruences at $b \pmod{p}$ for arbitrary primes $p \geq 5$.

2. Congruences and filtrations of Jacobi forms

Let $J_{k,m}$ be the vector space of Jacobi forms of even weight $k$ and index $m$ (for details on Jacobi forms, see Eichler and Zagier [6]). The heat operator $L_m := (2\pi i)^{-2} \left( 8\pi im \frac{\partial}{\partial \tau} - \frac{\partial^2}{\partial z^2} \right)$ is a natural tool in the theory of Jacobi forms and plays an important role in this Section. In particular, if $\phi = \sum c(n, r) q^n \zeta^r$, then

\begin{equation}
L_m \phi := L_m(\phi) = \sum (4nm - r^2)c(n, r) q^n \zeta^r.
\end{equation}

Set

$\tilde{J}_{k,m} := \{ \phi \pmod{p} : \phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}_p[[q, \zeta]] \}$,

where $\mathbb{Z}_p := \mathbb{Z}_p \cap \mathbb{Q}$ denotes the local ring of $p$-integral rational numbers. If $\phi \in \tilde{J}_{k,m}$, then we denote its filtration modulo $p$ by

$\Omega(\phi) := \inf \left\{ k : \phi \pmod{p} \in \tilde{J}_{k,m} \right\}$.

Recall the following facts on Jacobi forms modulo $p$:

**Proposition 2.1** (Sofer [21]). Let $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]]$ and $\psi(\tau, z) \in J_{k',m'} \cap \mathbb{Z}[[q, \zeta]]$ such that $0 \not\equiv \phi \equiv \psi \pmod{p}$. Then $k \equiv k' \pmod{p - 1}$ and $m = m'$.

**Proposition 2.2** ([18]). If $\phi(\tau, z) \in J_{k,m} \cap \mathbb{Z}[[q, \zeta]]$, then $L_m \phi \pmod{p} \in \tilde{J}_{k+p+1,m}$.

Moreover, we have

$\Omega(L_m \phi) \leq \Omega(\phi) + p + 1$

with equality if and only if $p \nmid (2\Omega(\phi) - 1)m$.

We will now explore Ramanujan-type congruences for Jacobi forms.

**Definition 2.3.** For $\phi(\tau, z) = \sum c(n, r) q^n \zeta^r \in \tilde{J}_{k,m}$, we say that $\phi$ has a Ramanujan-type congruence at $b \pmod{p}$ if $c(n, r) \equiv 0 \pmod{p}$ whenever $4nm - r^2 \equiv b \pmod{p}$.

Equation (2.1) implies that a Jacobi form $\phi$ has a Ramanujan-type congruence at $0 \pmod{p}$ if and only if $L_m^{p-1} \phi \equiv \phi \pmod{p}$. More generally, $\phi$ has a Ramanujan-type congruence at $b \pmod{p}$ if and only if

$L_m^{p-1} \left( q^{-\frac{b}{4m}} \phi \right) \equiv q^{-\frac{b}{4m}} \phi \pmod{p}$.

Ramanujan-type congruences at $0 \pmod{p}$ for Jacobi forms have been considered in [17] [18]. The following proposition determines when Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ for Jacobi forms exist.
Let \( \phi \in J_{k,m} \) and \( b \not\equiv 0 \pmod{p} \). Then \( \phi \) has a Ramanujan-type congruence at \( b \pmod{p} \) if and only if \( L_m^{p+1} \cdot \phi \equiv -\left(\frac{b}{p}\right) L_m \phi \pmod{p} \).

**Proof:** If \( \phi \in \mathbb{Z}[[q, \zeta]] \) and \( f \in \mathbb{Z}[[q]] \), then \( L_m(f \phi) = L_m(f) \phi + f L_m(\phi) \). This implies

\[
L_m^{p-1} \left( q^{-\frac{b}{4m}} \phi \right) = \sum_{i=0}^{p-1} \left( \frac{p-1}{i} \right) L_m^{p-1-i} \left( q^{-\frac{b}{4m}} \right) L_m^i \phi \\
= \sum_{i=0}^{p-1} \left( \frac{p-1}{i} \right) (-b)^{p-1-i} q^{-\frac{b}{4m}} L_m^i \phi \\
\equiv q^{-\frac{b}{4m}} \sum_{i=0}^{p-1} b^{p-1-i} L_m^i \phi \pmod{p}.
\]

In particular, \( \phi \) has a Ramanujan-type congruence at \( b \not\equiv 0 \pmod{p} \) if and only if

\[
(2.2) \quad 0 \equiv \sum_{i=1}^{p-1} b^{p-1-i} L_m^i \phi \pmod{p}.
\]

Let \( M_k^{(1)} \) denote the space of elliptic modular forms of weight \( k \). Recall that every even weight \( \phi \in J_{k,m} \) with \( p \)-integral coefficients can be written as

\[
\phi = \sum_{j=0}^{m} f_j(\phi_{-2,1})^j(\phi_{0,1})^{m-j},
\]

where \( \phi_{-2,1}(\tau, z) \in \mathbb{Z}[[q, \zeta]] \) and \( \phi_{0,1}(\tau, z) \in \mathbb{Z}[[q, \zeta]] \) are weak Jacobi forms of index 1 and weights \(-2\) and 0, respectively, and where each \( f_j \in M_k^{(1)} \) has \( p \)-integral rational coefficients and is uniquely determined (see §8 and §9 of [6] for details and also for the corresponding result for Jacobi forms of odd weight). Furthermore, by Proposition 2.2 for every \( i \) there exists \( \psi_i \in J_{k+i(p+1),m} \) such that \( L_m^i \phi \equiv \psi_i \pmod{p} \). Hence there exist \( F_{i,j} \in M_{k+i(p+1)+2j}^{(1)} \) with \( p \)-integral rational coefficients such that

\[
L_m^i \phi \equiv \psi_i \equiv \sum_{j=0}^{m} F_{i,j}(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \pmod{p}
\]

and hence (2.2) is equivalent to

\[
0 \equiv \sum_{j=0}^{m} \left( \sum_{i=1}^{p-1} b^{p-1-i} F_{i,j} \right) (\phi_{-2,1})^j(\phi_{0,1})^{m-j} \pmod{p}.
\]

Since \( (\phi_{-2,1})^j(\phi_{0,1})^{m-j} \) are linearly independent over \( M_k^{(1)} \), we deduce that (2.2) is equivalent to \( \sum_{i=1}^{p-1} b^{p-1-i} F_{i,j} \equiv 0 \pmod{p} \) for every \( j \). Elliptic modular forms
modulo $p$ have a natural direct sum decomposition (see Section 3 of [22] or Theorem 2 of [19]) graded by their weights modulo $p - 1$. Thus (2.2) is equivalent to

$$0 \equiv b^{p-1-i}F_{i,j} + b^{(p-1)/2-i}F_{i+(p-1)/2,j} \pmod{p}$$

and hence also

$$F_{i+(p-1)/2,j} \equiv -\left(\frac{b}{p}\right)F_{i,j} \pmod{p}$$

for all $0 \leq j \leq m$ and $1 \leq i \leq \frac{p-1}{2}$. This implies, for all $1 \leq i \leq \frac{p-1}{2},$

$$L_m^{i+\frac{p-1}{2}} \phi \equiv \sum_{j=0}^{m} F_{i+\frac{p-1}{2},j}(\phi_{-2,1})^j(\phi_{0,1})^{m-j} \pmod{p}$$

We conclude that

$$L_m^{i+\frac{p-1}{2}} \phi \equiv -\left(\frac{b}{p}\right)L_m \phi \pmod{p},$$

which completes the proof. $\square$

By (2.1), $L_m^0 \phi \equiv L_m \phi \pmod{p}$. We call $L_m \phi, L_m^2 \phi, \ldots, L_m^{p-1} \phi$ the heat cycle of $\phi$ and we say that $\phi$ is in its own heat cycle whenever $L_m^{p-1} \phi \equiv \phi \pmod{p}$. Assume $L_m \phi \not\equiv 0 \pmod{p}$ and $p \nmid m$. By Proposition 2.2 applying $L_m$ to $\phi$ increases the filtration of $\phi$ by $p + 1$ except when $\Omega(\phi) \equiv \frac{p+1}{2} \pmod{p}$. If $\Omega(L_m^i \phi) \equiv \frac{p+1}{2} \pmod{p}$, then call $L_m^i \phi$ a high point and $L_m^{i+1} \phi$ a low point of the heat cycle. By Propositions 2.1 and 2.2

$$\Omega(L_m^{i+1} \phi) = \Omega(L_m^i \phi) + p + 1 - s(p - 1) \tag{2.3}$$

where $s \geq 1$ if and only if $L_m^i \phi$ is a high point and $s = 0$ otherwise. The structure of the heat cycle of a Jacobi form is similar to the structure of the theta cycle of a modular form (see §7 of [8]). We will now prove a few basic properties:

**Lemma 2.5.** Let $\phi \in \tilde{J}_{k,m}$ with $p \nmid m$ a prime such that $L_m \phi \not\equiv 0 \pmod{p}$.

1. If $j \geq 1$, then $\Omega(L_m^i \phi) \not\equiv \frac{p+3}{2} \pmod{p}$.
2. The heat cycle of $\phi$ has a single low point if and only if there is some $j \geq 1$ with $\Omega(L_m^i \phi) \equiv \frac{p+5}{2} \pmod{p}$. Furthermore, $L_m^i \phi$ is the low point.
3. If $j \geq 1$, then $\Omega(L_m^{i+1} \phi) \neq \Omega(L_m^i \phi) + 2$.
4. The heat cycle of $\phi$ either has one or two high points.
Proof: (1) If \( \Omega\left( L_i^m \phi \right) \equiv \frac{p+3}{2} \pmod{p} \), then by (2.3) for \( 1 \leq n \leq p-1 \) we have
\[
\Omega\left( L_i^m + n \phi \right) = \Omega\left( L_i^m \phi \right) + n(p+1).
\]
In particular, \( L_i^{p-1} \phi \not\equiv L_i^m \phi \pmod{p} \), which is impossible.

(2) If \( \Omega\left( L_i^m \phi \right) \equiv \frac{p+5}{2} \pmod{p} \), then by (2.3), for \( 1 \leq n \leq p-2 \) we have
\[
\Omega\left( L_i^m + n \phi \right) = \Omega\left( L_i^m \phi \right) + n(p+1)
\] and
\[
\Omega\left( L_i^m \phi \right) = \Omega\left( L_i^{p-1} \phi \right) = \Omega\left( L_i^m \phi \right) + (p-1)(p+1) - s(p-1)
\]
where \( s \) must be \( p+1 \) and there can be no other low point. On the other hand, if there is a single low point, then the filtration must increase \( p-2 \) consecutive times. The only way this is possible is if the low point has filtration \( \frac{p+5}{2} \pmod{p} \).

(3) By Proposition 2.2, \( \Omega\left( L_i^{m+1} \phi \right) = \Omega\left( L_i^m \phi \right) + 2 \) can only happen when \( \Omega\left( L_i^m \phi \right) \equiv \frac{p+1}{2} \pmod{p} \). Suppose \( \Omega\left( L_i^{m+1} \phi \right) = \Omega\left( L_i^m \phi \right) + 2 = \frac{p+5}{2} \pmod{p} \). By part (2), this implies that the filtration increases \( p-2 \) more times before falling. Hence \( L_i^{p-1} \phi \not\equiv L_i^m \phi \pmod{p} \), which is impossible.

(4) Suppose there are \( t \geq 2 \) high points \( L_i^j \phi \) where \( 1 \leq i_1 < \cdots < i_t \leq p-1 \).

By (2.3) and part (3) above, there are \( s_j \geq 2 \) such that
\[
\Omega\left( L_i^{m+1} \phi \right) = \Omega\left( L_i^m \phi \right) + p+1 - s_j(p-1).
\]
Hence
\[
\Omega\left( L_i^m \phi \right) = \Omega\left( L_i^p \phi \right) = \Omega\left( L_i^m \phi \right) + (p-1)(p+1) - \sum_{j=1}^{t} s_j(p-1),
\]
and so \( \sum s_j = p+1 \). By (2.4), \( \Omega\left( L_i^{m+1} \phi \right) \equiv \frac{p+1}{2} + 1 + s_j \pmod{p} \) and so there will be \( p-1 - s_j \) increases before the next fall. That is, for \( 1 \leq j \leq t \), \( i_{j+1} - i_j = p - s_j \) where we take \( i_{t+1} = i_1 + p - 1 \) for convenience. Thus
\[
p-1 = i_{t+1} - i_1 = \sum_{j=1}^{t} (i_{j+1} - i_j) = \sum_{j=1}^{t} (p - s_j) = tp - (p + 1),
\]
i.e., \( t = 2 \). We conclude that the heat cycle of \( \phi \) has at most two (i.e., one or two) high points.

\[ \square \]

The following Corollary of Proposition (2.4) is a key ingredient in the proof of Proposition (2.7) below.
Corollary 2.6. If \( \phi \in \tilde{J}_{k,m} \) has a Ramanujan-type congruence at \( b \not\equiv 0 \pmod{p} \) and \( L_m \phi \not\equiv 0 \pmod{p} \), then the heat cycle of \( \phi \) has two low points which both have filtration congruent to 2 (mod \( p \)).

Proof: Since \( L_m^{p+1} \phi \equiv -(\frac{b}{p}) L_m \phi \pmod{p} \), we have \( \Omega \left( L_m^{p+1} \phi \right) = \Omega (L_m \phi) = \Omega (L_m \phi) \). Hence there is a fall in the first half of the heat cycle and in the second half of the heat cycle. Furthermore, after a low point, the filtration increases \( \frac{p-3}{2} \) times and then falls once. Thus, the filtration of the low points is 2 (mod \( p \)).

Our final result in this section gives the non-existence of Ramanujan-type congruences of Jacobi forms.

Proposition 2.7. Let \( \phi \in \tilde{J}_{k,m} \) where \( k \geq 4 \), \( L_m \phi \not\equiv 0 \pmod{p} \) and let \( b \not\equiv 0 \pmod{p} \). If \( p > k \) and \( p \nmid m \), then \( \phi \) does not have a Ramanujan-type congruence at \( b \pmod{p} \).

Proof: Assume that \( \phi \) has a Ramanujan-type congruence at \( b \pmod{p} \). First suppose \( k = \frac{p+1}{2} \). Then \( \Omega (\phi) = \frac{p+1}{2} \) and so we must have \( s \geq 1 \) in (2.3). Since we need \( \Omega (L_m \phi) \geq 0 \), we must have \( s = 1 \) and hence \( \Omega (L_m \phi) = \frac{p+5}{2} \). But by Lemma 2.5, this implies there is only one low point, contrary to Corollary 2.6.

Now suppose \( k \neq \frac{p+1}{2} \). Then \( \Omega (L_m \phi) = k + p + 1 \). There must be a low point of the heat cycle with filtration either \( k + p + 1 \) or \( k \). By Corollary 2.6, either \( k + 1 \equiv 2 \pmod{p} \) or \( k \equiv 2 \pmod{p} \). Both of these alternatives are impossible since \( p > k \geq 4 \).

3. Proof of Theorem 1.2 and examples

We employ the Fourier-Jacobi expansion of a Siegel modular form (as in [3]) to prove Theorem 1.2. Let \( M_k^{(2)} \) denote the vector space of Siegel modular forms of degree 2 and even weight \( k \) (for details on Siegel modular forms, see for example Freitag [7] or Klingen [10]).

Proof of Theorem 1.2. Let \( F \in M_k^{(2)} \) be as in Theorem 1.2 with Fourier-Jacobi expansion \( F(\tau, z, \tau') = \sum_{m=0}^{\infty} \phi_m(\tau, z) e^{2\pi im\tau'} \), i.e., \( \phi_m \in J_{k,m} \). Let \( b \not\equiv 0 \pmod{p} \).

Then \( F \) has a Ramanujan-type congruence at \( b \pmod{p} \) if and only if \( \phi_m \) has a Ramanujan-type congruence at \( b \). By Proposition 2.4, it is equivalent that for all \( m \)

\[
L_m^{p+1} \phi_m \equiv -(\frac{b}{p}) L_m \phi_m \pmod{p},
\]

which is equivalent to (1.2), since

\[
D(F) = \sum_{m=0}^{\infty} L_m (\phi_m(\tau, z)) e^{2\pi im\tau'}.
\]
Now we turn to the second part of Theorem 1.2. Here we assume that \( p > k, p \neq 2k-1 \), and that there exists an \( A(n,r,m) \) with \( p \nmid \gcd(n,m) \) such that \( A(n,r,m) \equiv 0 \) (mod \( p \)). Suppose that \( F \) has a Ramanujan-type congruence at \( b \) (mod \( p \)). Then all Fourier-Jacobi coefficients \( \phi_m \) have such a congruence at \( b \). We would like to apply Proposition 2.7. First, \( k \geq 4 \), since \( F \) is non-constant and \( M_k^{(2)} \subset \mathbb{C} \) if \( k < 4 \). Moreover, if \( \phi_m \equiv 0 \) (mod \( p \)) with \( p \nmid m \), then \( \Omega(\phi_m) = k \) by Proposition 2.2. In particular, \( L_m \phi_m \equiv 0 \) (mod \( p \)) and \( \Omega(L_m \phi_m) = k + p + 1 \) by Proposition 2.2. We conclude that \( F \) does not have a Ramanujan-type congruence at \( b \) (mod \( p \)).

We will use Theorem 1.2 to discuss Ramanujan-type congruences for explicit examples of Siegel modular forms after reviewing a few facts on Siegel modular forms modulo \( p \).

\[
\tilde{M}_k^{(2)} := \left\{ F \pmod{p} : F(Z) = \sum a(T) e^{\pi i \text{tr}(TZ)} \in M_k^{(2)} \text{ where } a(T) \in \mathbb{Z}(p) \right\}.
\]

Recall the following two theorems on Siegel modular forms modulo \( p \):

**Theorem 3.1** (Nagaoka [12]). There exists an \( E \in M_{p-1}^{(2)} \) with \( p \)-integral rational coefficients such that \( E \equiv 1 \) (mod \( p \)). Furthermore, if \( F_1 \in M_{k_1}^{(2)} \) and \( F_2 \in M_{k_2}^{(2)} \) have \( p \)-integral rational coefficients where \( 0 \neq F_1 \equiv F_2 \) (mod \( p \)), then \( k_1 \equiv k_2 \) (mod \( p - 1 \)).

**Theorem 3.2** (Böcherer and Nagaoka [2]). If \( F \in \tilde{M}_k^{(2)} \), then \( \mathbb{D}(F) \in \tilde{M}_{k+p+1}^{(2)} \).

Theorems 3.1 and 3.2 imply that the \( G := \mathbb{D}_{\tilde{M}_k^{(2)}}(F) + \left( \frac{b}{p} \right) \mathbb{D}(F) \in \tilde{M}_{k+(p+1)^2}^{(2)} \).

**Theorem 1.2** states that \( F \in \tilde{M}_k^{(2)} \) has a Ramanujan-type congruence at \( b \neq 0 \) (mod \( p \)) if and only if \( G \equiv 0 \) (mod \( p \)) in (3.1). One can apply the following analog of Sturm’s theorem for Siegel modular forms of degree 2 to verify that \( G \equiv 0 \) (mod \( p \)) in (3.1) for concrete examples of Siegel modular forms.

**Theorem 3.3** (Poor and Yuen [15]). Let \( F = \sum a(T) e^{\pi i \text{tr}(TZ)} \in M_k^{(2)} \) be such that for all \( T \) with dyadic trace \( w(T) \leq \frac{k}{2} \) one has that \( a(T) \in \mathbb{Z}(p) \) and \( a(T) \equiv 0 \) (mod \( p \)). Then \( F \equiv 0 \) (mod \( p \)).
Remark: If $T = \left( \begin{smallmatrix} a & b \\ c & d \end{smallmatrix} \right) > 0$ is Minkowski reduced (i.e., $2|b| \leq a \leq c$), then $w(T) = a + c - |b|$. For more details on the dyadic trace $w(T)$, see Poor and Yuen [16].

The following table gives all Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ for Siegel cusp forms of weight 20 or less when $p \geq 5$. Let $E_4, E_6, \chi_{10}$, and $\chi_{12}$ denote the usual generators of $M_k(2)$ of weights 4, 6, 10, and 12, respectively, where the Eisenstein series $E_4$ and $E_6$ are normalized by $a(\left( \begin{smallmatrix} 0 & 0 \\ 0 & 0 \end{smallmatrix} \right)) = 1$ and where the cusp forms $\chi_{10}$ and $\chi_{12}$ are normalized by $a(\left( \begin{smallmatrix} 2 & 1 \\ 1 & 2 \end{smallmatrix} \right)) = 1$. Cris Poor and David Yuen kindly provided Fourier coefficients up to dyadic trace $w(T) = 74$ of the basis vectors for $M_k^2(2)$ with $k \leq 20$. We used Magma to check that $G \equiv 0 \pmod{p}$ in (3.1) for each of the forms in (3.2) below. It is not difficult to verify that (up to scalar multiplication) no further Ramanujan-type congruences at $b \not\equiv 0 \pmod{p}$ exist for Siegel cusp forms of weights 20 or less.

(3.2)

| Expression                        | $b \not\equiv 0 \pmod{p}$ |
|-----------------------------------|-----------------------------|
| $\chi_{12}$                       | $b \equiv 1, 4 \pmod{5}$ and $b \equiv 2, 6, 7, 8, 10 \pmod{11}$ |
| $E_4\chi_{12}$                    | $b \equiv 1, 4 \pmod{5}$    |
| $E_4\chi_{12} - E_6\chi_{10}$     | $b \equiv 3, 5, 6 \pmod{7}$ |
| $E_6\chi_{12}$                    | $b \equiv 1, 4 \pmod{5}$    |
| $E_4^2\chi_{10} + 7E_6\chi_{12}$ | $b \equiv 1, 2, 4, 8, 9, 13, 15, 16 \pmod{17}$ |
| $E_4^2\chi_{12}$                  | $b \equiv 1, 4 \pmod{5}$    |
| $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ | $b \equiv 2, 3, 8, 10, 12, 13, 14, 15, 18 \pmod{19}$ |

Remarks:

1. For $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ modulo 19 we have $G \in \tilde{M}_{220}^2(\mathbb{Z})$ in (3.1) and we really do need Fourier coefficients up to dyadic trace $w(T) = \frac{290}{3}$, i.e., up to 74 in Theorem 3.3 to prove that $G \equiv 0 \pmod{19}$.

2. For Siegel modular forms in the Maass Spezialschar one could decide the existence and non-existence of their Ramanujan-type congruences also using Propositions 2.4 and 2.7 in combination with Maass’ lift [11] (see also §6 of [6]). However, Theorem 1.2 is an essential tool in establishing such results for Siegel modular forms that are not in the Maass Spezialschar, such as $E_4^2\chi_{12}$ and $\chi_{10}^2 + 2E_4^2\chi_{12} - 2E_4E_6\chi_{10}$ for example.

The following construction generates infinitely many Siegel modular forms with Ramanujan-type congruences. Note that this construction also works for elliptic
modular forms and for Jacobi forms by replacing $\mathbb{D}$ with $\Theta := \frac{1}{2\pi i} \frac{d}{dz}$ and $L_m$, respectively. For any $F \in \tilde{M}_k^{(2)}$ and any prime $p \geq 5$, set

$$
F_0 := F - p^{p-1} F \in \tilde{M}_{k+p^2}^{(2)}
$$

$$
F_{+1} := \frac{1}{2} \left( p^{p-1} F + \mathbb{D}^{p-2} F \right) \in \tilde{M}_{k+p^2}^{(2)}
$$

$$
F_{-1} := \frac{1}{2} \left( p^{p-1} F - \mathbb{D}^{p-2} F \right) \in \tilde{M}_{k+p^2}^{(2)}.
$$

Clearly $F = F_0 + F_{+1} + F_{-1}$ and if $F = \sum a(T)e^{\pi i tr(TZ)}$, then for $s = 0, \pm 1$, one finds that

$$
F_s = \sum_{(\det(TZ)/p)=s} a(T)e^{\pi i tr(TZ)}.
$$

Hence $F_s$ has Ramanujan-type congruences at all $b$ with $\left( \frac{b}{p} \right) \neq s$. For example, if $F := \chi_{10}^2$, then a computation (in combination with Theorem 3.3) reveals that

$$
F_0 \equiv 3E_4 \chi_{10}^2 + 2E_4^2 E_6 \chi_{10} \chi_{12} \pmod{5}
$$

$$
F_{+1} \equiv E_4^2 \chi_{10}^2 + 4E_4 \chi_{10} \chi_{12} + 4E_4^2 \chi_{12}^2 + 2E_4 E_6 \chi_{10} \chi_{12} + 3E_4^3 E_6 \chi_{10} \chi_{12} \pmod{5}
$$

$$
F_{-1} \equiv 3E_4 \chi_{10}^2 \chi_{12} + 3E_4^2 \chi_{12}^2 + E_4^2 E_6 \chi_{10} \chi_{12} + 2E_4^3 E_6^2 \chi_{10} \chi_{12} \pmod{5}.
$$

Since $E_4 \equiv 1 \pmod{5}$, we actually have $F_0 \in \tilde{M}_{28}^{(2)}$ and $F_{\pm 1} \in \tilde{M}_{32}^{(2)}$.

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References

[1] Ahlgren, S., and Boylan, M. Arithmetic properties of the partition function. Invent. Math. 153, no. 3 (2003), 487–502.

[2] Böcherer, S., and Nagaoka, S. On mod $p$ properties of Siegel modular forms. Math. Ann. 338, 2 (2007), 421–433.

[3] Choi, D., Choie, Y., and Richter, O. Congruences for Siegel modular forms. Preprint.

[4] Cooper, Y., Wage, N., and Wang, I. Congruences for modular forms of non-positive weight. Int. J. Number Theory 4, no. 1 (2008), 1–13.

[5] Dewar, M. Non-existence of Ramanujan congruences in modular forms of level four. Preprint.

[6] Eichler, M., and Zagier, D. The theory of Jacobi forms. Birkhäuser, Boston, 1985.

[7] Freitag, E. Siegelsche Modulfunktionen. Springer, Berlin, Heidelberg, New York, 1983.

[8] Jochnowitz, N. A study of the local components of the Hecke algebra mod $l$. Trans. Amer. Math. Soc. 270, 1 (1982), 253–267.

[9] Kiming, I., and Olsson, J. Congruences like Ramanujan’s for powers of the partition function. Archiv Math. 59, 4 (1992), 348–360.

[10] Klingen, H. Introductory lectures on Siegel modular forms, vol. 20 of Cambridge Studies in Advanced Mathematics. Cambridge University Press, 1990.

[11] Maass, H. Über eine Spezialschar von Modulformen zweiten Grades. Invent. Math. 52, no. 1 (1979), 95–104.

[12] Nagaoka, S. Note on mod $p$ Siegel modular forms. Math. Z. 235, 2 (2000), 405–420.
[13] Ono, K. Distribution of the partition function modulo m. *Ann. of Math. (2)* **151**, 1 (2000), 293–307.

[14] Ono, K. *The web of modularity: Arithmetic of the coefficients of modular forms and q-series*, vol. 102 of *CBMS Regional Conference Series in Mathematics*. Published for the Conference Board of the Mathematical Sciences, Washington, DC, 2004.

[15] Poor, C., and Yuen, D. Paramodular cusp forms. Preprint.

[16] Poor, C., and Yuen, D. Linear dependence among Siegel modular forms. *Math. Ann.* **318**, no. 2 (2000), 205–234.

[17] Richter, O. On congruences of Jacobi forms. *Proc. Amer. Math. Soc.* **136**, no. 8 (2008), 2729–2734.

[18] Richter, O. The action of the heat operator on Jacobi forms. *Proc. Amer. Math. Soc.* **137**, no. 3 (2009), 869–875.

[19] Serre, J.-P. *Congruences et formes modulaires*, in: Séminaire Bourbaki, 24ème année (1971/1972). Lecture Notes in Math. **317**. Springer, 1973, pp. 319–338.

[20] Sinick, J. Ramanujan congruences for a class of eta quotients. To appear in International Journal of Number Theory.

[21] Sofer, A. p-adic aspects of Jacobi forms. *J. Number Theory* **63**, no. 2 (1997), 191–202.

[22] Swinnerton-Dyer, H. P. F. *On l-adic representations and congruences for coefficients of modular forms*, in: Modular functions of one variable III. Lecture Notes in Math. **350**. Springer, 1973, pp. 1–55.

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