SASAKI-EINSTEIN METRICS AND K-STABILITY

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Abstract. We show that a polarized affine variety admits a Ricci flat Kähler cone metric if and only if it is K-stable. This generalizes Chen-Donaldson-Sun’s solution of the Yau-Tian-Donaldson conjecture to Kähler cones, or equivalently, Sasakian manifolds. As an application we show that the five-sphere admits infinitely many families of Sasaki-Einstein metrics.

1. Introduction

The existence of Kähler-Einstein metrics is a fundamental problem in Kähler geometry. If $M$ is a compact complex manifold with $c_1(M) = 0$ or $c_1(M) < 0$, then the work of Yau [79] shows that $M$ admits Kähler-Einstein metrics with zero or negative Ricci curvature. The case when $c_1(M) > 0$ is more subtle, and the Yau-Tian-Donaldson conjecture [81, 77, 37], proved by Chen-Donaldson-Sun [22, 23, 24], relates the existence of a Kähler-Einstein metric on $M$ to the K-stability of $M$, which is a certain algebro-geometric condition. Our goal in the present paper is to generalize this result to the setting of Kähler cones, giving a criterion for the existence of a Ricci flat Kähler cone metric, or equivalently, a Sasaki-Einstein metric on the link. The question of existence of such metrics has received increasing attention in the physics community through their connection to the AdS/CFT correspondence (see e.g. [63, 55]), and we anticipate further developments along these lines (see e.g. [28]).

Quite generally, if $X$ is an affine variety with an isolated singular point, one can ask whether $X$ admits a Ricci flat Kähler cone metric. We will address this question under the extra assumption that we fix the vector field on $X$ that gives the homothetic scaling on the cone. More precisely, suppose that $X \subset \mathbb{C}^N$ is an affine variety, with an isolated singular point at the origin, invariant under the action of a torus $T \subset U(N)$, which for simplicity we assume to be diagonal. We call $\xi \in \mathfrak{t}$ a polarization of $X$ if it acts with positive weights on the coordinate functions, i.e. the corresponding holomorphic vector field $\xi$ satisfies $L_\xi(z_i) = i a_i z_i$ with $a_i > 0$. We then seek a Kähler Ricci flat metric $\omega$ on $X$ such that $L_{-i\xi}\omega = 2\omega$. We say that such a metric $\omega$ is a Ricci flat Kähler cone metric on the pair $(X, \xi)$. Such a Ricci
flat Kähler cone metric can only exist on $X$ if the pair $(X, \xi)$ is a normalized Fano cone singularity, in the terminology of Definition 2.1 below.

Our main result is the following.

**Theorem 1.1.** Let $(X, \xi)$ be a normalized Fano cone singularity. Then $(X, \xi)$ admits a Ricci flat Kähler cone metric if and only if it is K-stable.

We will give a precise definition of K-stability in this setting below. For now let us say that if $(X, \xi)$ does not admit a Ricci flat Kähler cone metric, then there exists an embedding $X \hookrightarrow \mathbb{C}^N'$, a corresponding embedding of the torus $T \subset U(N')$, and a one-parameter subgroup $\lambda : \mathbb{C}^* \to GL(N')^T$ generated by a vector field $w$ with the following properties:

1. The limit $Y = \lim_{t \to 0} \lambda(t) \cdot X$ is normal.
2. The Futaki invariant $\text{Fut}(Y, \xi, w) \leq 0$, and $Y \not\sim X$ if equality holds.

While in principle $T$ just needs to be a torus of automorphisms of $X$ for which $\xi \in \mathfrak{t}$, in practice it is useful to choose a maximal such torus. In fact as in [30] we can also obtain an equivariant version of the theorem for the action of any compact group on $X$, but to simplify the exposition we will mostly focus on the case of a torus.

Usually there are infinitely many such degenerations that one needs to check in order to determine whether a pair $(X, \xi)$ is K-stable, and so there does not seem to be an effective way to test K-stability. This become possible, however, in certain situations with large symmetry group, where there are only a finite number of possible normal limits $Y$ under equivariant degenerations of $X$. Just as in [30], a simple example is when $X$ is toric, in which case if we work equivariantly with respect to the maximal torus $T$, then we necessarily have $Y \cong X$, and we only need to test the Futaki invariants $\text{Fut}(X, \xi, \eta)$ for $\eta \in \mathfrak{t}$. If this vanishes for all $\eta \in \mathfrak{t}$, then $(X, \xi)$ admits a Ricci flat Kahler cone metric, recovering the result of Futaki-Ono-Wang [44].

A more general situation is when $X$ has a complexity-one action of a torus $T$, i.e. $\dim T = \dim X - 1$. In this case, using the methods in Ilten-Süß [52] we can still effectively test K-stability, by checking a finite number of degenerations. In Section 8 we will apply these techniques to several explicit families of hypersurface singularities. For example, we study

$$Z_{BP}(p, q) = \{x^2 + y^2 + z^p + w^q = 0\} \subset \mathbb{C}^4,$$

where $p, q > 1$. We show the following.

**Theorem 1.2.** For a suitable choice of $\xi$ the pair $(Z_{BP}(p, q), \xi)$ admits a Ricci flat Kähler cone metric if and only if $2p > q$ and $2q > p$. As a consequence $S^5$ admits infinitely many families of Sasaki-Einstein metrics.

The necessary conditions $2p > q$ and $2q > p$ follow from the Lichnerowicz obstruction of Gauntlett-Martelli-Sparks-Yau [46], while the existence result was only known previously for $(p, q) = (2, 2)$ and $(2, 3)$, where the latter was shown by Li-Sun [61].
To date, many Sasaki-Einstein manifolds have been found by employing estimates for the \(\alpha\)-invariant \([76, 34]\). For example, the affine varieties \(Z_{BP}(p, q)\) are a special case of the Brieskorn-Pham singularities, which have been thoroughly studied in the literature. Boyer-Galicki-Kollár \([15]\) used estimates for the \(\alpha\)-invariant of Brieskorn-Pham singularities to produce 68 distinct Sasaki-Einstein metrics on \(S^5\), as well as SE metrics on all 28 oriented diffeomorphism types of \(S^7\), and the the standard and Kervaire spheres \(S^{4m+1}\). Note that previously infinitely many Einstein (not Sasakian) metrics on spheres in dimensions 5 to 9 were constructed by Böhm \([11]\).

Estimates for the \(\alpha\)-invariant were also used by Boyer-Galicki \([12, 13]\), Boyer-Nakamaye \([17]\), Kollár-Johnson \([54]\), Ghigi-Kollár \([47]\), Kollár \([58, 56]\) and others to produce many infinite families of Sasaki-Einstein metrics in dimensions 5 and 7, and higher. For example, \(#k(S^2 \times S^3)\) is known to admit infinite families of Sasaki-Einstein metrics for any \(k \geq 1\). We refer the reader to \([14]\) for a thorough discussion of these results. We note that Kollár has classified the possible topologies of Sasaki-Einstein manifolds \([50, 57, 59]\). For example it is known that for affine varieties of complex dimension 3 with a 2-torus action, the only possible topologies of the links are \(S^5\) and \(k\#(S^2 \times S^3)\) for any \(k \geq 1\) (see \([14\), Proposition 10.2.27]).

Our techniques also produce new infinite families of distinct Sasaki-Einstein metrics on \(k\#(S^2 \times S^3)\) for all \(k \geq 1\), and hence cover all possible topologies that can occur with a 2-torus action.

We expect that many more examples can be found along the same lines. A particularly interesting problem is to find Sasaki-Einstein metrics with irregular Reeb vector fields. Remarkably, the first examples of irregular Sasaki-Einstein metrics were discovered by Gauntlett-Martelli-Spark-Waldram \([45]\) by explicitly writing down the metric in coordinates. We expect K-stability to be particularly useful for finding irregular Sasaki-Einstein manifolds in real dimension 5, since if the cone \(X\) has \(\dim_C X = 3\), and \(\xi\) is an irregular Reeb field, then \(X\) admits a complexity-one action of a 2-torus. In particular, using the methods of Ilten-Süß \([52]\) we can effectively test whether \((X, \xi)\) admits a Ricci flat Kähler cone metric.

The overall strategy of our proof is the same as that of Chen-Donaldson-Sun \([22, 23, 24]\), as adapted in \([75, 80]\) to the smooth continuity method. We will set up this continuity method in Section 2 where we also give the precise definition of K-stability based on our previous work \([27]\), extending the definition of Ross-Thomas \([67]\) from the quasi-regular case. The main technical results are contained in Sections 3 and 4. In Section 3 we discuss weak solutions of the equations along the continuity method, which is analogous to the theory of singular Kähler-Einstein metrics, as was studied by Eyssidieux-Guedj-Zeriahi \([43]\). Much of this discussion, such as the convexity of the Ding functional due to Berndtsson \([9]\), extends to the case of cones without substantial difficulties. In Section 4 we generalize the partial \(C^0\)-estimate along the smooth continuity method from \([75]\) to the setting of cones. The main new technical difficulty is that in the method of \([75]\)
the strict positivity of the Ricci curvature was a crucial ingredient, while in our setting the Ricci curvature on a cone is never strictly positive. Instead we need to exploit the transverse Kähler structure, which does have strictly positive Ricci curvature. The proof of Theorem 1.1 is given in Section 5, primarily along the lines of the argument in [30]. In Section 6 we collect some more algebraic results, with the goal of establishing the equality between the differential geometric and the algebraic definitions of the Futaki invariant. In Section 7 we prove the other implication in Theorem 1.1 along the lines of the work of Berman [6]. In Section 8 we give some example calculations of K-stability, including the proof of Theorem 1.2 and we finish with some further discussion and questions in Section 9.

2. Basic definitions

In this section we fix some basic definitions, and set up the continuity method that we would like to use to find Ricci flat Kähler cone metrics. The continuity method is equivalent to the usual continuity path for finding Kähler-Einstein metrics, but it involves a scaling to ensure that we have metrics of non-negative Ricci curvature on our cones.

Definition 2.1. A polarized affine variety of dimension $n$ is a triple $(X, T, \xi)$, where $X$ is a normal affine variety, $\dim \mathbb{C} X = n$, $T$ is a torus of automorphisms of $X$, and $\xi \in \mathfrak{t}$ acts on the ring of functions of $X$ with positive weights in the following sense. We have a decomposition $R(X) = \bigoplus_{\chi \in \mathfrak{t}^*} R_\chi(X)$ under the torus action into weight spaces, and we require that $\chi(\xi) > 0$ for all non-zero $\chi$, for which $R_\chi$ is non-trivial. Often we simply speak of a pair $(X, \xi)$, where $\xi$ is a vector field on $X$ generating a compact torus of automorphisms, and then $T$ is understood to be this torus. We call $\xi$ a Reeb field or polarization on $X$. We denote by $\mathcal{C}_R \subset \mathfrak{t}$ the cone of Reeb fields.

We say that the pair $(X, \xi)$ is a Fano cone singularity, if $X$ is $\mathbb{Q}$-Gorenstein, together with a trivializing section $\Omega$ of $mK_X$ for some $m > 0$, such that $L_\xi \Omega = i \lambda \Omega$ for some $\lambda > 0$. The last condition is equivalent to $X$ having log-terminal singularities (see Section 6). The Fano cone singularity $(X, \xi)$ is normalized if $\lambda = nm$.

The basic example is obtained by taking a Fano manifold $M$, and letting $X$ be the total space of $mK_M$, with the zero section blown down, for some $m$ such that $-mK_M$ is very ample. In other words $X$ is the cone over $M$ under a projective embedding by $-mK_M$.

In [27] we defined a notion of K-semistability for a pair $(X, \xi)$, in terms of test-configurations for $X$ that commute with a torus $T$ whose Lie algebra contains $\xi$. Here we give a very similar definition, which is adapted to our work here, but is closer in spirit to the definition of K-stability by Tian [77], which only allows test-configurations with normal central fibers. In addition,
in view of possible future applications we work equivariantly for a compact group acting on $X$ in analogy with [30].

Suppose that $(X, \xi)$ is a normalized Fano cone singularity of dimension $n$, with $X$ only having an isolated singularity, and $G$ is a compact group of automorphisms of $X$, such that $\xi$ is in the center of its Lie algebra. In applications we will take $G$ to be a maximal torus of automorphisms, containing the torus generated by $\xi$.

A $G$-equivariant special degeneration (or test-configuration) of $X$ consists of an embedding $X \to \mathbb{C}^N$ such that $G$ acts linearly through an embedding $G \subset U(N)$, together with a one-parameter subgroup $\lambda : \mathbb{C}^* \to GL(N)^G$ commuting with $G$, such that $\lambda(S^1) \subset U(N)$ and $Y = \lim_{t \to 0} \lambda(t) \cdot X$ is normal. In this case $(Y, \xi)$ is also a normalized Fano cone singularity, together with a $\mathbb{C}^*$-action given by $\lambda$ commuting with $\xi$. Let us write $T$ for the torus generated by $\lambda$ and $\xi$. By a slight abuse of notation we will denote by $\lambda \in \mathfrak{t}$ the generator of the corresponding $S^1$-action, and note that for small $s \in \mathbb{R}$ the pairs $(Y, \xi + s\lambda)$ are also Fano cone singularities (which may not be normalized). We showed in [27] that the index character

$$F(\xi, t) = \sum_{\lambda \in \mathfrak{t}^*} e^{-t\lambda(\xi)} \dim R_{\lambda}(Y)$$

can be extended meromorphically to a neighborhood of the origin, and we can define functions $a_i(\xi)$ by

$$F(\xi, t) = \frac{a_0(\xi)(n-1)!}{t^n} + \frac{a_1(\xi)(n-2)!}{t^{n-1}} + O(t^{2-n}).$$

As a matter of notation we will write

$$D_\lambda a_i(\xi) = \left. \frac{d}{ds} \right|_{s=0} a_i(\xi + s\lambda).$$

**Definition 2.2.** The Futaki invariant of a special degeneration as above is defined to be

$$\text{Fut}(X, \xi, \lambda) = \frac{a_0(\xi)}{n-1} D_\lambda \left( \frac{a_1}{a_0} \right)(\xi) + \frac{a_1(\xi)D_\lambda a_0(\xi)}{n(n-1)a_0(\xi)},$$

where each $a_i$ is computed on the variety $Y$.

A normalized Fano cone singularity $(X, \xi)$ is called $G$-equivariantly K-stable, if for all special degenerations as above, we have $\text{Fut}(X, \xi, \lambda) \geq 0$, and equality holds only if $(Y, \xi)$ is isomorphic to $(X, \xi)$.

Our main result, Theorem 1.1 then says that $X$ admits a Ricci flat Kähler cone metric with homotheties given by $-J\xi$, if and only if $(X, \xi)$ is equivariantly K-stable.

Let us digress briefly on the Futaki invariant in Definition 2.2. As remarked above, given a special test configuration generated by a $\mathbb{C}^*$ action $\lambda$, the central fiber $Y$ is again a $\mathbb{Q}$-Gorenstein variety with log terminal singularities, and hence $(Y, \xi + s\lambda)$ is a Fano cone singularity, which is not necessarily normalized. As we will see in Proposition 5.4 the normalized
Reeb vector fields form a linear subspace $\Sigma_Y$ of the Reeb cone of $Y$ defined by the linear equation

$$\frac{a_1(w)}{a_0(w)} = \frac{n(n-1)}{2}.$$ 

Let $N := \nabla(a_1/a_0)$ denote the normal vector to the normalized hyperplane $\Sigma_Y$.

First consider the case when $Y \cong X$, and that $\lambda$ is generated by $w \in \text{Lie}(T)$. Assume that $w$ is tangent to $\Sigma_X$, then the Futaki invariant is just $(1/2)D_w a_0(\xi)$. Since we can also consider $-w$, this implies that if $(X, \xi)$ is $K$-stable, then $\xi$ must be an extremal value of $a_0$, on $\Sigma_X$. By [65], $a_0$ is a convex function on $\Sigma_X$ which has a unique minimum. Since $a_0$ can be interpreted as the volume of the link, this is called “volume minimization”, and was discovered in fundamental work of Martelli-Sparks-Yau [65].

Now suppose we have a nontrivial test configuration, so that $Y \not\cong X$, and $\lambda$ is generated by $w$ and suppose that $\xi$ is the Reeb vector field minimizing the volume. We can compute the Futaki invariant by the formula

$$\frac{1}{2} D_{w'} a_0(\xi) = \text{Fut}(X, \xi, \lambda), \quad \text{if} \quad w' = w - 2N \cdot \frac{N \cdot w}{n(n-1)}\xi,$$

where now $w'$ is normalized, but it may not generate a test configuration if $\xi$ is irrational. This observation extends the interpretation of stability as volume minimization [65, 64, 27] from trivial test configurations to all test configurations, and will be useful in Section 6. Note that when $Y \not\cong X$ we cannot replace $w$ with $-w$, since this will change the central fiber of the test configuration. These observations have applications in conformal field theory, where the AdS/CFT correspondence provides an interpretation of $K$-stability as a maximization problem for the central charge of the dual conformal field theory [28].

We next set up the relevant continuity method for polarized affine manifolds. Suppose that $(X, \xi)$ is normalized Fano. Fixing any metric $\alpha$ on $(X, \xi)$, our continuity method is to find metrics $\omega_t$ on $(X, \xi)$ satisfying

$$(2.2) \quad \text{Ric}(\omega_t) = 2n[t\omega_t^\tau + (1-t)\alpha^\tau] - 2n\omega_t^\tau,$$

where $\omega_t^\tau, \alpha^\tau$ denote the transverse metrics induced by $\omega_t, \alpha$. In terms of the transverse metrics induced on the Reeb foliation of the link $L = \{r_t = 1\}$ the method of continuity is

$$\text{Ric}^\tau(\omega_t^\tau) = 2n[t\omega_t^\tau + (1-t)\alpha^\tau].$$

In particular, $(2.2)$ is the natural lift to the cone of the continuity method for Kähler-Einstein metrics (see, for example, [4]). We will call Equation $(2.2)$ the twisted equation, with twisting form $\alpha^\tau$.

**Proposition 2.3.** Let $I = \{t \in [0,1] : (2.2) \text{ has a solution } \}$. Then $I$ is non-empty, and open.
The non-emptiness follows from the transverse version of Yau’s theorem \cite{Yau}, due to El-Kacimi Alaoui \cite{El-Kacimi}, while the openness is also analogous to the Kähler case as in Aubin \cite{Aubin}.

As in the compact Kähler case, we must study the Gromov-Hausdorff limit of a sequence \((X, \omega_t)\) as \(t \to T\). For this it is convenient to do a scaling of the Reeb fields to ensure that we have metrics with Ricci curvature bounded below. Let us denote the radial function of \(\omega_t\) by \(r_t\), and define \(\tilde{r}_t = r_t^1\); in the Sasakian literature, this is often referred to as a \(D\)-homothetic transformation. It is straightforward to verify that \(\tilde{\omega}_t = \sqrt{-1} \partial \bar{\partial} \tilde{r}_t\) satisfies 

\[
\text{Ric}(\tilde{\omega}_t) = 2n \frac{1 - t}{t} \alpha^r,
\]

i.e. the Ricci curvature is non-negative.

3. Weak solutions, twisted Futaki invariants and the Ding functional

The key result that we will ultimately need is that in the context of the continuity method defined in the previous section, as \(t \to \sup I\), we can extract a limit that is a normalized Fano cone singularity \((Y, \xi)\) together with a transverse positive current \(\beta^T\), and a weak solution \(\omega_T\) on \((Y, \xi)\) of the equation

\[
\text{Ric}(\omega_T) = 2n(1 - T)[\beta^T - \omega_T^T].
\]

In this section we will give a precise definition of such weak solutions, and describe how in analogy with the compact Kähler case, the existence of such a metric implies the reductivity of a certain automorphism group, and the vanishing of a twisted Futaki invariant.

We first define the weak solutions of the twisted equation. We assume that \((Y, \xi)\) is a normalized Fano cone singularity, so we have a non-vanishing global holomorphic section \(\Omega\) of \(mK_Y\) for some \(m \geq 1\), with \(L_\xi(\Omega) = \text{imm}\Omega\).

This gives rise to the volume form

\[
dV = \text{i}^{n^2}(\Omega \wedge \Omega)^{1/m},
\]

which satisfies \(L_\xi dV = 2ndV\). This volume form is uniquely defined up to a constant multiple, and we will call it the canonical volume form on a normalized Fano cone singularity.

Suppose that we have an embedding \(Y \to \mathbb{C}^N\), such that the Reeb field (or rather the torus it generates) acts diagonally, and \(Y\) is not contained in a linear subspace. Then \(\xi\) defines a Reeb field on \(\mathbb{C}^N\) and so we can then fix a smooth reference radial function \(\hat{r}\) on \(\mathbb{C}^N\) which is compatible with this Reeb field \cite{Reeb}. In the presence of the action of a torus \(T\), we can take our embedding to be \(T\)-equivariant as well, where \(T\) acts diagonally.

The space of transverse psh potentials is the space of basic functions \(\varphi\) (i.e. \(L_\xi \varphi = L_\xi \varphi = 0\)) such that \(r_\varphi = \hat{r} e^{\hat{r}}\) is psh. Recall that a psh function on a normal variety can always be viewed as the restriction of a psh function from an ambient space, after embedding \cite{Demailly} Theorem 1.10. In particular,
$r_\varphi$ is always the restriction of a psh function defined in a neighborhood of the origin. For smooth such $\varphi$ we write

$$\omega_\varphi = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} r_\varphi^2,$$

and we suppose that we have a twisting form $\beta^\tau$ given as

$$\beta^\tau = \sqrt{-1} \partial \bar{\partial} \log r_\psi,$$

where $\psi$ is also a transverse psh potential. If $Y, \beta^\tau$ were smooth, then the twisted equation

$$(3.1) \quad \text{Ric}(\omega_\varphi) = 2n(1-t)[\beta^\tau - \omega^\tau]$$

could be written on the level of volume forms as

$$(3.2) \quad \omega^n_\varphi = Ce^{2n(1-t)[\varphi - \psi]} dV,$$

and we can use this latter formulation to define solutions of the twisted equation in a weak sense (see [43] for the analogous definition of weak Kähler-Einstein metrics). More precisely, given $\psi, t$, a weak solution of (3.1) is a continuous transverse psh potential $\varphi$ such that (3.2) holds as an equality of measures. In particular this implies that $e^{-2n(1-t)\psi} dV$ must be integrable in a neighborhood of the cone vertex.

**Definition 3.1.** Let $(Y, \xi)$ be a normalized Fano cone singularity together with a reference radial function $\hat{r}$ as above. Suppose that $\psi$ is a transverse psh potential such that $e^{-2n(1-t)\psi} dV$ is integrable in a neighborhood of the vertex for some $t \in [0, 1]$. We say that $(Y, \xi, (1-t)\beta^\tau)$ has a continuous transverse psh potential $\varphi$ satisfying

$$\omega^n_\varphi = Ce^{2n(1-t)[\varphi - \psi]} dV$$

in the sense of measures. Often we will write $(Y, \xi, (1-t)\beta^\tau)$ when the twisting form $\beta^\tau$ is more natural than its potential $\psi$.

As in [30], Remark 4 (see also [7, Proposition 3.8]), it is enough to check that outside of a closed set $\Sigma$ with vanishing $(2n - 2)$-dimensional Hausdorff measure, $\omega^n_\varphi$ defines a singular metric $e^{-\tilde{f}}$ on $K_Y$ with $f \in L^1_{\text{loc}}$, and in addition, on $Y \setminus \Sigma$

$$\sqrt{-1} \partial \bar{\partial} f = 2n(1-t)[\beta^\tau - \omega^\tau_\varphi].$$

The main properties of weak solutions of the twisted equation are the reductivity of the automorphism group, and the vanishing of the twisted Futaki invariant, analogous to Propositions 7, 8 in [30]. Let us first define the relevant automorphism group, or rather its Lie algebra.

**Definition 3.2.** Suppose we have a triple $(Y, \xi, \beta^\tau)$ as above. We define $\mathfrak{g}_{Y, \xi}$ to be the space of holomorphic vector fields on $Y$ (more precisely on the regular part), commuting with $\xi$. We then define

$$\mathfrak{g}_{Y, \xi, \beta^\tau} = \{ w \in \mathfrak{g}_{Y, \xi} : \iota_w \beta^\tau = 0 \}.$$
As in [30], note that $g_{Y,\xi,\tau}^{\beta}$ is in general smaller than the space of holomorphic vector fields $w$ on $Y$ commuting with $\xi$, and preserving $\beta^{\tau}$ in the sense that $L_{w}\beta^{\tau} = 0$. For instance if $\beta^{\tau}$ is a transverse Kähler form, then $g_{Y,\xi,\tau}^{\beta}$ is spanned by $\Xi$, where $\text{Im} \Xi = \xi$.

The following is analogous to [30, Proposition 7], the special case of which, without the twisting form, has been shown in Donaldson-Sun [41]. The proof is based on a uniqueness theorem due to Berndtsson [10], generalizing the classical Bando-Mabuchi theorem [5], further extended by Boucksom-Eyssidieux-Guedj-Zeriahi [7], Berman-Witt-Nyström [8], Chen-Donaldson-Sun [24], Yi [83].

**Proposition 3.3.** Suppose that $(Y,\xi, (1-t)\beta^{\tau})$ admits a solution $\omega_{t}$ of the twisted equation. Then $g_{Y,\xi,\tau}^{\beta}$ is reductive. Moreover if $G$ is a compact group of biholomorphisms of $Y$, commuting with $\xi$ and fixing $\omega_{t}$, then the centralizer $(g_{Y,\xi,\tau}^{\beta})^{G}$ is also reductive.

Next we define the twisted Futaki invariant. Suppose that $w$ is a vector field commuting with $\xi$, that preserves the radial function $\hat{r}$ above. In addition suppose that $w$ is the real part of a holomorphic vector field in $g_{Y,\xi,\tau}^{\beta}$. The transverse Hamiltonian $\theta_{w}$ is defined by letting $\theta_{w}\hat{r}^{2}$ be a Hamiltonian for $w$, i.e. satisfying the equation

$$
(3.3) \quad \theta_{w}\hat{r}^{2} = \iota_{\theta_{w}} d\left(\frac{1}{2} \hat{r}^{2}\right).
$$

Note that with this convention the transverse Hamiltonian of the Reeb field $\xi$ is $\theta_{\xi} = -1$. We then define the twisted Futaki invariant to be

$$
\text{Fut}_{Y,\xi,\beta^{\tau}}(w) = \frac{t}{\nu} \int_{Y} \theta_{w} e^{-\frac{1}{2} \hat{r}^{2}} \omega^{n} - t \frac{\int_{Y} \theta_{w} e^{-2n(1-t)\psi} e^{-\frac{1}{2} \hat{r}^{2}} dV}{\int_{Y} e^{-2n(1-t)\psi} e^{-\frac{1}{2} \hat{r}^{2}} dV}.
$$

Proposition 6.8 will show that for $t = 1$, this definition agrees with the algebraic definition given in Definition 2.2. Below we will also give a different formula for the twisted Futaki invariant when $t \neq 1$.

We have the following, analogous to [30, Proposition 8].

**Proposition 3.4.** If $(Y,\xi)$ admits a weak solution of the twisted equation above, then $\text{Fut}_{Y,\xi,\beta^{\tau}}(w) = 0$ for all vector fields $w$ as above.

The proofs of Propositions 3.3, 3.4 both follow from convexity properties of the twisted Ding functional along weak geodesic segments, based essentially on work of Berndtsson [10]. The arguments follow those in [10] (see also [30, Section 6]) closely together with the discussion in Donaldson-Sun [41] on extending these to the setting of cones. See also Guan-Zhang [49] for geodesics of Sasakian metrics.

**Definition 3.5.** Suppose that $(Y,\xi)$ is a normalized Fano cone singularity, and $\psi$ is a transverse psh potential as above (relative to a reference radial
function), such that $e^{-2n(1-t)\psi} dV$ is locally integrable. The twisted Ding functional $D_{(1-t)\psi}$ is defined for continuous transverse psh potentials $\varphi$ by

$$D_{(1-t)\psi}(\varphi) = -tE(\varphi) - \frac{1}{2n} \log \int_Y e^{2n(1-t)(\varphi-\psi)} e^{-\frac{1}{2}r^2} dV.$$ 

Here $E$ is defined by its variation:

$$\delta E(\varphi) = \frac{1}{V(\xi)} \int_Y \dot{\varphi} e^{-\frac{1}{2}r^2} \omega^n.$$

The properties of the twisted Ding functional in this setting follow calculations analogous to those in the compact Kähler case that was studied in [30] (see Ding-Tian [36], Berndtsson [10], Chen-Donaldson-Sun [24] for earlier work), using some additional identities that are valid on cones. An alternative approach is to work with pluripotential techniques on the link, as developed recently by van Coevering [78], but we prefer to work directly on the cone. As a sample of the calculations involved we have the following simple result.

**Lemma 3.6.** $E(\varphi)$ is well defined.

**Proof.** Let us consider first the restriction of $E$ to smooth transverse psh potentials $\varphi$. We show that the 1-form defined by $\delta E$ is closed. Note that the variation of $\frac{1}{2} r^2$ is $\delta \left( \frac{1}{2} r^2 \right) = (\delta \varphi) r^2$. The differential of $\delta E$ is the map

$$(\psi_1, \psi_2) \mapsto -\int_Y \psi_1 \psi_2 r^2 e^{-\frac{1}{2}r^2} \omega^n + n \int_Y \psi_1 e^{-\frac{1}{2}r^2} \sqrt{-1} \partial \bar{\partial} (\psi_2 r^2) \wedge \omega^{n-1},$$

and we need to show that this is symmetric in $\psi_1, \psi_2$. We have

$$\int_Y \psi_1 \Delta (\psi_2 r^2) e^{-\frac{1}{2}r^2} \omega^n = \int_Y \left[ \psi_1 \Delta \psi_2 r^2 + \psi_1 \nabla \psi_2 \cdot \nabla r^2 + \psi_1 \nabla r^2 \Delta \psi_2 \right] e^{-\frac{1}{2}r^2} \omega^n = \int_Y \left[ -\frac{1}{2} \nabla \psi_1 \cdot \nabla \psi_2 r^2 + 2 n \psi_1 \psi_2 \right] e^{-\frac{1}{2}r^2} \omega^n,$$

where the integration by parts is justified since $r^2 = O(\hat{r}^2)$ and $\nabla r^2 = O(\hat{r})$, and we used

$$\nabla \psi_2 \cdot \nabla r^2 = 0,$$

$$\Delta \frac{1}{2} r^2 = n.$$

We obtain that the differential of $\delta E$ is

$$(\psi_1, \psi_2) \mapsto n \int_Y \nabla \psi_1 \cdot \nabla \psi_2 e^{-\frac{1}{2}r^2} \omega^n,$$

where we used the formula

$$\int_Y f r^2 e^{-\frac{1}{2}r^2} dV = 2n \int_Y f e^{-\frac{1}{2}r^2} dV,$$

for any basic function $f$.

This shows that $E$ is well defined on the space of smooth transverse psh potentials. We can then extend $E$ to the space of continuous transverse psh potentials by continuity, since the formula for the variation of $E$ implies that $E$ is uniformly continuous for the $L^\infty$ norm on potentials. \qed
The critical points of the twisted Ding functional are given by solutions of the twisted equation. To see this, note that the variation of $D(1-t)\psi(\varphi)$ is given by

$$
\delta D(1-t)\psi(\varphi) = -\frac{t}{V} \int_Y \phi e^{-\frac{1}{2}r^2_\varphi} \omega^n_\varphi
- \frac{1}{2n} \int_Y 2n(1-t)\dot{\varphi} - \dot{\varphi}r^2_\varphi e^{2n(1-t)[\varphi - \psi]} e^{-\frac{1}{2}r^2_\varphi} dV
$$

$$
= -\frac{t}{V} \int_Y \phi e^{-\frac{1}{2}r^2_\varphi} \omega^n_\varphi + \frac{t}{2} \int_Y e^{2n(1-t)[\varphi - \psi]} e^{-\frac{1}{2}r^2_\varphi} dV,
$$

where we used (3.4) again. It follows that critical points satisfy

$$
\omega^n_\varphi = Ce^{2n(1-t)[\varphi - \psi]} dV,
$$

which is what we wanted.

In addition from this calculation of the variation we see that the variation of $D(1-t)\psi$ along a suitable 1-parameter family of biholomorphisms recovers the twisted Futaki invariant. Suppose that $w$ is a vector field as above, and let $f_s : Y \to Y$ denote the 1-parameter group of biholomorphisms generated by $-Jw$. We claim that the twisted Futaki invariant is given by the variation of the twisted Ding functional along $f_s$. Writing $\varphi_s$ for the induced family of potentials, we have

$$
\frac{1}{2} r^2_\varphi e^{2\varphi_s} = f^*_s(\frac{1}{2} r^2 e^{2\varphi}),
$$

and so

$$
\dot{\varphi}_s r^2_\varphi = -\iota_J w d(\frac{1}{2} r^2).\n$$

We obtain that $\dot{\varphi} = -\theta_w$ in terms of the transverse Hamiltonian of $w$ as in (3.3). It follows that

$$
\left. \frac{d}{ds} \right|_{s=0} D(1-t)\psi(\varphi_s) = \frac{t}{V} \int_Y \theta_w e^{-\frac{1}{2}r^2_\varphi} \omega^n_\varphi - \frac{t}{2} \int_Y e^{2n(1-t)[\varphi - \psi]} e^{-\frac{1}{2}r^2_\varphi} dV
$$

$$
= \frac{t}{V} \int_Y \theta_w e^{-\frac{1}{2}r^2_\varphi} \omega^n_\varphi + \frac{t}{2} \int_Y e^{2n(1-t)[\varphi - \psi]} e^{-\frac{1}{2}r^2_\varphi} dV,
$$

which when $\varphi = 0$ is just the twisted Futaki invariant as we have defined it above. We can rewrite this in a different form, as in the proof of Proposition 8 in [30].

**Proposition 3.7.** The twisted Futaki invariant is given by

$$
\text{Fut}_{Y,\xi}(1-t)\psi(w) = \text{Fut}_{Y,\xi}(w) - n(n-1) \frac{1-t}{V} \int_Y \theta_w e^{-\frac{1}{2}r^2} \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-1},
$$

where

$$
\text{Fut}_{Y,\xi}(w) = \frac{1}{V} \int_Y \theta_w e^{-\frac{1}{2}r^2} \omega^n - \frac{\int_Y \theta_w e^{-\frac{1}{2}r^2} dV}{\int Y e^{-\frac{1}{2}r^2} dV},
$$

(3.5)
is the “untwisted” Futaki invariant. Note that here, as before, we are assuming that $\iota_{Jw}\sqrt{-1}\partial\overline{\partial}\psi = 0$, since $w$ is the real part of a holomorphic vector field in $\mathfrak{g}_{Y,\xi,\beta^r}$.

Note that in addition when $w$ is normalized, i.e. $L_w\Omega = 0$, then we have

$$\text{Fut}_{Y,\xi}(w) = \frac{1}{V} \int_Y \theta_w e^{-\frac{1}{2}r^2} \omega^n,$$

since in this case we have

$$\int_Y \theta_w e^{-\frac{1}{2}r^2} dV = 0,$$

as can be seen by considering the variation of the integral $\int_Y e^{-\frac{1}{2}r^2} dV$ along the flow generated by $Jw$.

**Proof of Proposition 3.7** We define

$$I(\varphi) = \frac{1}{V} \int_Y \log \frac{\left(\int_Y e^{-\frac{1}{2}r^2} dV\right)^{-1} e^{-\frac{1}{2}r^2} dV}{\left(\int_Y e^{2n(1-t)\varphi - \frac{1}{2}r^2} dV\right)^{-1} e^{2n(1-t)\varphi - \frac{1}{2}r^2} dV} e^{-\frac{1}{2}r^2} \omega^n_{\varphi}.$$

Differentiating along the one-parameter group generated by $Jw$ we must get zero. To see this, we use that all the terms in the integral are invariant under biholomorphisms up to constant factors, and these constants cancel. For instance that fact that $\iota_{Jw}\beta^r = 0$ implies that $L_{Jw}\psi$ is a constant. The result of the differentiation is the following.

$$2n \int_Y \phi e^{-\frac{1}{2}r^2} dV \int_Y e^{-\frac{1}{2}r^2} dV - 2nt \int_Y \phi e^{2n(1-t)\varphi - \frac{1}{2}r^2} dV \int_Y e^{2n(1-t)\varphi - \frac{1}{2}r^2} dV - 2n \frac{1-t}{V} \int_Y \phi e^{-\frac{1}{2}r^2} \omega^n_{\varphi} - 2n \frac{1}{V} \int_Y (\varphi - \psi) \left[ -\dot{\phi} r^2_{\varphi} + \Delta(\dot{\phi}^2_{\varphi}) \right] e^{-\frac{1}{2}r^2} \omega^n_{\varphi} = 0.$$

Similar calculations to before, using also that $r^2_{\varphi}\Delta(\varphi - \psi)$ is a basic function, show that

$$\int_Y (\varphi - \psi) \left[ -\dot{\phi} r^2_{\varphi} + \Delta(\dot{\phi}^2_{\varphi}) \right] e^{-\frac{1}{2}r^2} \omega^n_{\varphi} = 2n(n-1) \int_Y e^{-\frac{1}{2}r^2} \varphi \sqrt{-1}\partial\overline{\partial}(\varphi - \psi) \wedge \omega^{n-1}_{\varphi}.$$

We obtain

$$\int_Y \phi e^{-\frac{1}{2}r^2} dV \int_Y e^{-\frac{1}{2}r^2} dV - t \int_Y \phi e^{2n(1-t)\varphi - \frac{1}{2}r^2} dV \int_Y e^{2n(1-t)\varphi - \frac{1}{2}r^2} dV = \frac{1-t}{V} \int_Y \phi e^{-\frac{1}{2}r^2} \omega^n_{\varphi} + n(n-1) \frac{1-t}{V} \int_Y e^{-\frac{1}{2}r^2} \varphi \sqrt{-1}\partial\overline{\partial}(\varphi - \psi) \wedge \omega^{n-1}_{\varphi}.$$

From this, using that at the reference metric $\varphi = 0$ we have $\dot{\varphi} = \theta_w$, we obtain the required formula.
We can write the twisting term above in a more intrinsic way as follows:

\[- \int_Y \theta w e^{-\frac{1}{2} \hat{r}^2} \sqrt{-1} \partial \bar{\partial} \psi \wedge \omega^{n-1} = - \int_Y \theta w e^{-\frac{1}{2} \hat{r}^2} (\beta^\tau - \omega^\tau) \wedge \omega^{n-1},\]

recalling that \(\beta^\tau = \sqrt{-1} \partial \bar{\partial} \log r\psi\) and \(\omega^\tau = \sqrt{-1} \partial \bar{\partial} \log \hat{r}\).

As in [30] we use the twisted Futaki invariant to define a notion of (equivariant) twisted stability following also Dervan [35]. We will also choose a twisting form of a special form for which we will calculate an alternative formula for the twisted Futaki invariant.

Suppose that \((X, \xi)\) is a normalized Fano cone singularity, smooth away from the vertex, and write \(R(X)\) for the coordinate ring as before. We also denote by \(R_{<D}(X)\) the direct sum of weight spaces for the action of \(T\) with weights \(\chi \neq 0\) such that \(\chi(\xi) < D\). Suppose that \(D\) is large enough so that the functions in \(R_{<D}(X)\) give an embedding \(X \hookrightarrow \mathbb{C}^N\). It will be convenient to separate the spaces corresponding to different characters as

\[\mathbb{C}^N = \mathbb{C}^{N_1} \times \ldots \times \mathbb{C}^{N_m},\]

and so \(\xi\) acts diagonally on the coordinate functions with weights \(a_1, \ldots, a_m > 0\).

A test-configuration for \(X\) commuting with \(T\) is given by a one-parameter subgroup \(\lambda : \mathbb{C}^* \to GL(N)^T\), and we assume that \(\lambda(S^1) \subset U(N)^T\), generated by a vector field \(w\). Note that \(GL(N)^T\) is simply the product of \(GL(N_i)\) for \(i = 1, \ldots, m\). By a further unitary change of basis we will assume that \(\lambda\) is diagonal. We define

\[Y = \lim_{t \to 0} \lambda(t) \cdot X,\]

and we suppose that \(Y\) is normal, and \(\mathbb{Q}\)-Gorenstein. We define the limiting current

\[\beta^\tau = \lim_{t \to 0} \lambda(t) \cdot \alpha^\tau,\]

whose existence follows from the proof of Proposition 3.8 below, see [319]. The twisted Futaki invariant of the corresponding test-configuration is then defined by

\[\Fut_{X, \xi, (1-t)\alpha^\tau}(w) = \Fut_{Y, \xi, (1-t)\beta^\tau}(w).\]

Note here that \(w\) is not tangent to \(X\), but it is tangent to \(Y\), and it is the real part of a holomorphic vector field in \(g_{Y, \xi, \beta^\tau}\).

As in [30], a crucial role is played by an alternative formula for this twisted Futaki invariant. We assume that \(\xi\) is quasi-regular. There is then a constant \(M\) such that \(\frac{M}{a_i} \in \mathbb{Z}\). We define a reference radial function analogous to the Fubini-Study metric for projective varieties, with radial function \(\hat{r}\) given by

\[(3.7) \quad \hat{r}^2 = \left[ \sum_{i=1}^m \left( \sum_{j=1}^{N_i} |z_j^{(i)}|^2 \right)^{M/a_i} \right]^{1/M},\]
where the \( \{ z_j^{(i)} \} \) form an orthonormal basis for \( \mathbb{C}^N \). We take the background metric to be \( \omega = \frac{1}{2} \sqrt{-1 \partial \overline{\partial} r^2} \), and the transverse form \( \alpha^\tau = \sqrt{-1 \partial \overline{\partial} \log \hat{r}} \), i.e. \( \psi = 0 \) relative to this radial function. To see that \( \omega \) is indeed a metric, note first that for any cone metric \( \sqrt{-1 \partial \overline{\partial} r^2} \) and \( \gamma > 0 \), the forms \( \sqrt{-1 \partial \overline{\partial} r^2} \gamma \) also define cone metrics, with Reeb fields obtained by scaling. In this way, 
\[
\sum_{i=1}^{m} \left( \sum_{j=1}^{N_i} |z_j^{(i)}|^2 \right)^{M/a_i} \]
defines a product metric on \( \mathbb{C}^N \), and it follows that \( \omega \) is also a cone metric.

In addition \( \hat{r} \) is preserved by the action of \( U(N)^T \). With this setup we have the following.

**Proposition 3.8.**

\[
\text{Fut}_{X, \xi, (1-t)\alpha^\tau}(w) = \text{Fut}_{X, \xi}(w) + c(n) \frac{1-t}{V} \int_Y (\max Y \theta_w - \theta_w)e^{-\frac{1}{2} \hat{r}^2} \omega^n, \]
where \( \omega = \sqrt{-1 \partial \overline{\partial} \frac{1}{2} \hat{r}^2} \), and \( c(n) \) is a dimensional constant.

**Proof.** The proof follows a similar argument to that in [30, Proposition 11], expressing \( \alpha^\tau \) as an average of currents of integration along hypersurfaces in \( X \). One new difficulty is that the limit \( Y \) may be contained in a coordinate hyperplane.

We will use hypersurfaces defined by functions of the form
\[
f_\mu = \sum_{j=1}^{B} \mu_j u_j^{M/b_j},
\]
where the \( u_j \) are monomials in the \( z_i \) (including each \( z_i \) as well), the \( b_j \) are corresponding weights, and we think of \( \mu \in \mathbb{P} = \mathbb{P}^{B-1} \). It may happen that \( X \) is contained in some of these hypersurfaces, but there is a linear subspace \( E \subset \mathbb{P} \) such that for \( \mu \in \mathbb{P} \setminus E \), the function \( f_\mu \) does not vanish on \( X \). Let us write \( V_\mu = X \cap f_\mu^{-1}(0) \), which may have multiplicity. By Shiffman-Zelditch [70, Lemma 3.1], whose proof is entirely local, we have that on \( \mathbb{C}^N \),
\[
2\pi \int_{\mathbb{P}} [f_\mu^{-1}(0)] \, d\mu = \sqrt{-1 \partial \overline{\partial}} \log \left( \sum_{j=1}^{B} |u_j|^{2M/b_j} \right),
\]
in the sense of distributions, for the standard probability measure \( d\mu \) on the projective space \( \mathbb{P} \). Multiplying out the \( M/a_i \) power in (3.7) we see that for suitable choices of the \( u_j \) restricted to \( X \) we will have
\[
\alpha^\tau = \frac{\pi}{M} \int_{\mathbb{P} \setminus E} [V_\mu] \, d\mu = \frac{1}{2M} \sqrt{-1 \partial \overline{\partial}} \log \left( \sum_{j=1}^{B} |u_j|^{2M/b_j} \right).
\]
We can then compute $\beta^\tau$ on $Y$ by taking the limits of the currents $[V_\mu]$ under the $\mathbb{C}^*$-action $\lambda$. We will do this by computing the limit of the underlying schemes. Suppose that $\lambda(t)$ acts on the $u_j$ diagonally with entries $t^{w_j}$. Then we have

$$\lambda(t) \cdot f_\mu = \sum_{j=1}^B \frac{w_j t^{w_j}}{\mu_j} u_j^{M/b_j}.$$ 

If $I$ denotes the homogeneous ideal defining $X$, then $V_\mu$ is defined by the ideal $I + (f_\mu)$. The weights of the $\mathbb{C}^*$-action $\lambda$ define a partial order on the monomials, and the limit $\lambda(t) \cdot V_\mu$ has ideal defined by the lowest weight parts of elements of $I + (f_\mu)$, i.e. the initial ideal $\text{in}_\lambda(I + (f_\mu))$. In the same way the limiting variety $Y$ is defined by the ideal $\text{in}_\lambda(I)$.

Suppose that we find a function $g_\mu \in \text{in}_\lambda(I + (f_\mu))$ such that $g_\mu \notin \text{in}_\lambda(I)$, and in terms of the grading of the polynomial ring given by $\xi$ the function $g_\mu$ has the same degree as the $f_\mu$ (i.e. degree $M$). In this case, since the coordinate ring of $Y$ is an integral domain (we have assumed that $Y$ is reduced and irreducible), we will see that the Hilbert functions of the ideals $I + (f_\mu)$ and $\text{in}_\lambda(I) + (g_\mu)$ coincide, and so $\lim_{t \to 0} V_\mu$ is the hypersurface in $Y$ cut out by $g_\mu$. The key information that we need is the weight $\Lambda$ of the $\lambda$-action on $g_\mu$. We will determine this for generic $\mu$ and in fact we claim that generically $\Lambda = -M \max_Y \theta_w$. 

For simplicity let us assume that the (relative) weights $w_i/b_i$ are ordered so that $w_1/b_1 \leq w_2/b_2 \leq \ldots \leq w_B/b_B$. Let $c$ be the smallest index so that $u_c$ does not vanish on $Y$. Suppose that $\mu$ is chosen to be in general position in the sense that we cannot write $f_\mu = h + f'_\mu$, where $h \in I$, and $f'_\mu$ has strictly larger weights than $u_c$. If we write $u_{c'}, u_{c'+1}, \ldots, u_B$ for the monomials with strictly larger weight (we have $c' \geq c+1$, but this inequality may be strict), then the condition is that $f_\mu \notin I + (u_{c'}, \ldots, u_B)$, or in other words that $f_\mu$ does not vanish on the intersection of $X$ with the subspace $H = \{u_{c'}, \ldots, u_B = 0\}$. If this intersection were just the origin, then we would have $u_c \in I + (u_{c'}, \ldots, u_B)$, so that $u_c \in \text{in}_\lambda(I)$. But then $u_c$ vanishes on $Y$ contrary to our assumption. This means that the intersection $X \cap H$ is non-trivial, and so the space of $\mu$ for which $f_\mu$ vanishes on $X \cap H$ is contained in a hyperplane in $\mathbb{P}$. We can then enlarge the subspace $E$ above by this hyperplane, and focus on $\mu \in \mathbb{P} \setminus E$.

By assumption $u_1, \ldots, u_{c-1} \in \text{in}_\lambda(I)$, and so $I$ must contain $u_i^{M/b_i}$ modulo higher weight terms, for $i = 1, \ldots, c-1$. It follows that $I$ contains

$$\sum_{i=1}^{c-1} \mu_i u_i^{M/b_i}$$

modulo higher weight terms, but it cannot contain

$$\sum_{i=1}^{c'-1} \mu_i u_i^{M/b_i},$$
modulo higher weight terms, by our genericity assumption. It follows that
\[ \text{in}_\lambda(I + (f_\mu)) = \text{in}_\lambda(I + (f'_\mu)), \]
where
\[ f'_\mu = \sum_{i=0}^{B} \mu'_i u_i^{M/b_i}, \]
for some new coefficients \( \mu'_i \) with at least one of \( \mu'_c, \mu'_{c+1}, \ldots, \mu'_{c-1} \neq 0 \). We can then let \( g_\mu \) be the part of \( f'_\mu \) which has the same weight as \( u_c^{M/b_c} \) under \( \lambda \), i.e. \( \Lambda = Mw_c/b_c \).

Since the transverse Hamiltonian is
\[ \theta_w = \frac{\sum_{i=1}^{B} \mu'_i |u_i|^{2M/b_i}}{\sum_{i=1}^{B} |u_i|^{2M/b_i}}, \]
and \( u_1, \ldots, u_{c-1} \) vanish on \( Y \), but \( u_c \) does not, we have \( \frac{Mw_c}{b_c} = -M \max_y \theta_w \), and so the weight of \( g_\mu \) is \( \Lambda = -M \max_y \theta_w \) as we claimed.

Let us write \( Y_\mu = Y \cap g_\mu^{-1}(0) \). By the discussion above we then have
\[ \beta^\tau = \frac{\pi}{M} \int_{E \setminus Y_\mu} \mu d\mu, \]
where recall that now \( E \) is a suitable union of two hyperplanes. It follows that
\[ \int_Y \theta_w e^{-\frac{1}{4}\hat{r}^2} \beta^\tau \wedge \omega^{n-1} = \frac{\pi}{M} \int_{Y_\mu} \int_Y \theta_w e^{-\frac{1}{4}\hat{r}^2} \omega^{n-1}. \]

From Propositions 6.6 and 6.7 we have
\[ \int_{Y_\mu} \theta_w e^{-\frac{1}{4}\hat{r}^2} \omega^{n-1} = (2\pi)^{n-1} \left( \begin{array}{c} n-2 \end{array} \right)^2 D_w a_0(Y_\mu, \xi), \]
where as in (2.1), the notation \( D_w \) refers to varying the Reeb field \( \xi \) in the direction of \( w \). We have that \( \xi \) acts on \( g_\mu \) with weight \( M \), while \( w \) acts with weight \( -M \max_y \theta_w \). The index characters of \( Y \) and \( Y_\mu \) are related by
\[ F_{Y_\mu}(t, \xi + sw) = F_Y(t, \xi + sw) \left[ 1 - e^{-tM(1-s \max_y \theta_w)} \right], \]
and so, differentiating with respect to \( s \), at \( s = 0 \), we obtain
\[ D_w F_{Y_\mu}(t, \xi) = D_w F_Y(t, \xi) \left[ 1 - e^{-tM} \right] - F_Y(t, \xi) t M \max_y \theta_w. \]

Comparing the leading terms in the Laurent expansion, we have
\[ (n-2)! D_w a_0(Y_\mu, \xi) = M(n-1)! D_w a_0(Y, \xi) - M \max_y \theta_w (n-1)! a_0(Y, \xi). \]

It then follows from (3.10), (3.11), and Proposition 6.7 that
\[ \int_Y \theta_w e^{-\frac{1}{4}\hat{r}^2} \beta^\tau \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2(n-1)} \int_Y \left[ n \theta_w - \max_y \theta_w \right] \frac{\omega^n}{n!}. \]

At the same time we have
\[ \omega^\tau \wedge \omega^{n-1} = \frac{1}{r^2} \frac{n-1}{n} \omega^n, \]
and so
\[ \int_Y \theta_w e^{-\frac{t}{2} \omega^\tau} \wedge \frac{\omega^{n-1}}{(n-1)!} = \frac{1}{2} \int_Y \theta_w e^{-\frac{t}{2} \omega^n} \frac{\omega^n}{n!}. \]
Combining these formulas we obtain the required result. \qed

**Remark 3.9.** We remark that \( \max_Y \theta_w \) and the integral of \( \theta_w \) on \( Y \) depend only on the induced action on \( Y \) and can be computed from the weights of the action. Indeed, in the argument above we have seen that \( -\max_Y \theta_w \) is the minimal relative weight \( w_j/b_j \) of a monomial \( u_j \) that does not vanish on \( Y \), under the action \( \lambda \) relative to the action of the Reeb field. This is the same as the minimal relative weight \( w_i/a_i \) of a coordinate function \( z_i \) which does not vanish on \( Y \), and since the \( z_i \) generate the ring of algebraic functions of \( Y \) which vanish at the vertex, this is simply the minimum relative weight of any such function on \( Y \). More precisely, if \( f \) is any algebraic function on \( Y \) vanishing at the vertex which is in a weight space of the torus spanned by \( \lambda \) and \( \xi \), then the relative weight is the quotient of the weights of these two \( C^* \)-actions, and \( -\max_Y \theta_w \) is the minimum of this quotient over all such \( f \). At the same time the integral of \( \theta_w \) on \( Y \) can be interpreted as the variation of the \( a_0 \) coefficient in the Hilbert series of \( Y \), by Propositions 6.6 and 6.7.

From the above proof we also see the following.

**Proposition 3.10.** In the above setup, given \( X \) and \( \lambda \), there is a union of 2 hyperplanes \( E \subset \mathbb{P} \), such that if \( \mu \in \mathbb{P} \setminus E \) and \( V_\mu = X \cap f_\mu^{-1}(0) \), then
\[ \text{Fut}_{X,\xi,(1-t)\alpha^\tau}(w) = \text{Fut}_{X,\xi,\frac{t}{\pi} f_\mu}(w). \]

In other words, when we want to compute the twisted Futaki invariant, we can replace \( \alpha^\tau \) by a current of integration along a suitable hypersurface on \( X \), as long as this hypersurface, as a point in a projective space \( \mathbb{P} \), is not contained in a certain union of 2 hyperplanes. It is important to emphasize that these 2 hyperplanes can depend on the choice of \( X \) and \( \lambda \).

### 4. The partial \( C^0 \) estimate

Our goal in this section is to prove the partial \( C^0 \)-estimate, Theorem 4.7 below, for cone metrics satisfying a Ricci curvature equation. A special case of this will be the partial \( C^0 \) estimate along the continuity method. The partial \( C^0 \)-estimate was introduced by Tian [76] in his study of Kähler-Einstein metrics on complex surfaces, and he conjectured a general version for compact Kähler manifolds with a positive lower bound on the Ricci curvature. For Kähler-Einstein metrics in arbitrary dimension this estimate was obtained by Donaldson-Sun [40], using the Cheeger-Colding convergence theory [20] under Ricci curvature bounds, together with the Hörmander technique [51] for constructing holomorphic functions. Many more general results followed this development (see [23, 24, 66, 75, 21, 53]).

Our method will be fairly close to that in [75] for the smooth continuity method. The main difficulty is that along our continuity method the Ricci
curvature of the cone metrics is only non-negative, while in the approach of [75] it is important to treat the Ricci form as a metric. On the other hand it is not clear how to extend the Cheeger-Colding theory to the transverse Kähler structure, which does have strictly positive Ricci curvature. We therefore use the convergence theory on the level of the cones, but at certain crucial steps we invoke the positivity of the Ricci curvature of the transverse metric.

We suppose that \((X,\xi)\) is a normalized Fano cone singularity, and \(\alpha\) is a smooth Kähler cone metric on \((X,\xi)\). From the discussion in Section 2 we know that we have a family of metrics \(\omega_t\) on \((X,t^{-1}\xi)\), satisfying
\[
\text{Ric}(\omega_t) = 2n \frac{1-t}{t} \alpha_t^\tau,
\]
for \(t \in [t_0, T]\), with \(T \leq 1\) and \(t_0 > 0\).

Since we will also have to obtain uniform estimates while varying the Reeb field, we suppose more generally that we have a family of Reeb fields \(\xi_t\), and metrics \(\alpha_t, \omega_t\) on \((X,\xi_t)\) satisfying
\[
(4.1) \quad \text{Ric}(\omega_t) = 2nc_t \alpha_t^\tau
\]
where \(0 \leq c_t \leq c_0 < \infty\), and the pairs \((\xi_t, \alpha_t)\) move in a bounded family in the following sense.

**Definition 4.1.** We say that data \((\xi_t, \alpha_t)\) consisting of Reeb vector fields and a compatible cone metrics on \(X\) are in a \(C^2\) bounded family if the \(\xi_t\) are in a compact subset of \(C^\infty_R\), and the metrics \(\alpha_t\) are locally uniformly equivalent to a fixed reference cone metric, and locally bounded in \(C^2\) when measured with respect to this reference metric.

Along our continuity method we will have uniform constants \(\kappa, C_L\) so that
- \((X, \omega_t)\) are uniformly non-collapsed. That is, \(\text{Vol}(B_1(0,\omega_t)) > \kappa > 0\), where \(0 \in X\) denotes the cone point.
- \(\text{Ric}(\omega_t) \geq 0\) on \(X\), and the corresponding Sasakian metric \(g_t\) on the link \(L\) satisfies \(\text{Ric}(g_t) = (2n - 2)g_t + (2n - 2)c_t \alpha_t^\tau\),
- \(\text{diam}(L, g_t) < C_L\) for some controlled constant \(C_L\).

The first point follows from the lower bound for \(t_0\). The second point is the formula relating the Ricci curvature on the cone to the Ricci curvature on \(L\), while the last point follows from Myers’ Theorem. Let \(r_t\) denote the radial function of \(\omega_t\), then the Bishop-Gromov comparison theorem implies the metrics \(\omega_t\) are uniformly non-collapsed on the annuli \(\{1/2 < r_t < 2\}\) and so results of Croke [29] and Yau (see, e.g. [80, page 9]) imply

**Lemma 4.2.** For metrics \(\omega_t\) satisfying (4.1) there is a uniform Sobolev inequality on the set \(\text{Ann} := \{1/2 < r_t < 2\}\). That is, there exists a constant \(C(\kappa, C_L)\) independent of \(t\) so that for any \(W^{1,2}\) function \(f\) on \(\text{Ann}\) we have
\[
C^{-1} \left( \int_{\text{Ann}} |f|^2 \omega_t^n \right)^{n-1} \leq \left( \int_{\text{Ann}} |\nabla f|^2 \omega_t^n \right)^{1/2} + \left( \int_{\text{Ann}} |f|^2 \omega_t^n \right)^{1/2}
\]
Our eventual estimates will depend only on the dimension, the non-collapsing condition, and a bound on the geometry of $\alpha$, which roughly speaking says that we have good control of the transverse metric $\alpha^\tau$ on sufficiently small balls (see Definition 4.5 for a precise statement). As motivation consider the following analogous property of a compact Kähler manifold, which is easily proven by covering the manifold with sufficiently small coordinate balls.

**Lemma 4.3.** Let $(M, \omega)$ be a compact Kähler manifold, and let $g$ be the associated Kähler metric. Then for any constant $K > 0$ sufficiently large the following holds: if $B \subset M$ is any $g$-ball of radius smaller than $K^{-1}$, then there exist holomorphic coordinates $\{z_1, \ldots, z_n\}$ defined on $B$ such that

$$\frac{1}{2} \delta_{ij} < g_{ij} < 2 \delta_{ij}, \quad \|g_{ij}\|_{C^2(g_{Euc})} < K.$$ 

Furthermore, $K$ can be chosen to be uniform over $C^2$ bounded families of metrics.

We need a generalization of this to the transverse Kähler structure defined by $\alpha^\tau$. Let us denote by $Q = TX/C\xi$, where by $C\xi \subset TX$ we denote the complex subbundle spanned by $\xi$. Note that $Q$ has a natural integrable complex structure, and $\alpha^\tau$ defines a Kähler form on it (see Boyer-Galicki [14]).

**Definition 4.4.** Let $B \subset C^{n-1}$ be a ball. We say that an immersion $F : B \rightarrow X$ is a $\xi$-transverse immersion if for any point $p \in B$, the image $dF_p(T_pB)$ is transverse to $C\xi$. In particular, we get an induced vector space isomorphism

$$dF_p : T_pB \rightarrow Q_{F(p)}.$$ 

We say that $F$ is a transverse holomorphic immersion if $dF_p$ is complex linear with respect to the standard complex structure on $B$, and the transverse complex structure on $Q$.

For any transverse holomorphic immersion $F$, we obtain a Kähler metric $F^*\alpha^\tau$ on $B$, compatible with the standard structure on $B$. Note that we have a $S_\xi$-action on $X$ induced by $\xi$, and for any smooth $h : B \rightarrow C$, the maps $F$ and $h \cdot F$ induce the same Kähler structure on $B$. In particular we can assume that $F : B \rightarrow L$ maps into the unit link $L \subset X$.

**Definition 4.5.** We say that $\alpha$ (or equivalently $\alpha^\tau$) has geometry bounded by $K$ if the following holds. Let $F : B \rightarrow L$ be a transversal holomorphic immersion as above, where $B \subset C^{n-1}$ denotes a ball. We write $g = F^*\alpha^\tau$ for the induced Kähler metric on $B$. If $(B, g)$ has diameter at most $K^{-1}$, then there are holomorphic functions $z_1, \ldots, z_{n-1}$ on $B$ (not necessarily giving an embedding of $B$ into $C^{n-1}$, but an immersion), such that

$$\frac{1}{2} \delta_{ij} < g_{ij} < 2 \delta_{ij}, \quad \|g_{ij}\|_{C^2(B, g_{Euc})} < K.$$
Here \( g_{ij} = g(\partial/\partial z^i, \partial/\partial \bar{z}^j) \), and \( g_{\text{Euc}} = \delta_{ij} \) is the pull-back of the Euclidean metric by the map \((z_1, \ldots, z_{n-1}) : B \to \mathbb{C}^{n-1}\).

We now prove the analog of Lemma 4.3.

**Proposition 4.6.** For \( K > 0 \) sufficiently large, depending on \( \alpha \), the geometry of \( \alpha \) is bounded by \( K \). Moreover, \( K \) can be chosen uniformly over bounded families.

**Proof.** We consider the case of a fixed metric \( \alpha \). We first cover \( L \) by a finite number of adapted charts \( V_i \). This means that on such a \( V \) we have coordinates

\[(x, z_1, \ldots, z_{n-1}) : V \to \mathbb{R} \times \mathbb{C}^{n-1},\]

in which the Reeb field \( \xi \) is given by \( \partial/\partial x \) and the Sasakian metric \( \alpha \) agrees with the Euclidean metric at the origin. If we denote by \( \alpha_0 \) the metric on the slice \( \{x = 0\} \), then \( \alpha_0 = \alpha^\tau \) is a Kähler metric which in the coordinates \( z_i \) agrees with the Euclidean metric at the origin. By using a cover by smaller charts if necessary, we can assume that in these coordinates \( \alpha_0 \) satisfies

\[
\frac{1}{2} \delta_{ij} < \alpha_0 < 2 \delta_{ij} \quad \text{and} \quad \|\alpha_0\|_{C^2} < K, \quad \text{for some} \ K \ (\text{independent of the chart}).
\]

Increasing \( K \) if necessary, we can ensure that every \( \alpha \)-ball of radius \( K^{-1} \) in \( L \) is contained in one of our adapted charts \( V_i \).

Fixing again one of our charts \( V \), suppose that we have a transverse holomorphic immersion \( f : B \to V \), transverse to \( \partial/\partial x \). Then the induced Kähler structure on \( B \) is simply the pullback \((\pi \circ f)^* \alpha_0\), where \( \pi \) is the projection in \( V \) onto the \( \{x = 0\} \) slice. The holomorphic functions \( z_i \circ \pi \circ f \) then satisfy our requirements.

To prove the proposition it would suffice to show that if \( F : B \to L \) is any transversal holomorphic immersion such that \( (B, F^* \alpha^\tau) \) has diameter smaller than \( K^{-1} \), then it is contained in one of the \( V_i \). This is clearly impossible, since given such an immersion one could easily stretch the immersion by the Reeb flow to obtain a new immersion which is not contained in a ball of radius \( K^{-1} \). Instead, given such an immersion \( F \), we will construct a new, “equivalent” immersion \( f : B \to L \) whose image lies in one of our adapted charts \( V \). Since the Reeb vector field is real holomorphic, we can flow our transverse holomorphic immersion \( F \) to a new transverse holomorphic immersion along the Reeb field. Writing \( \Phi : L \times \mathbb{R} \to L \) for the Reeb action, we are looking for a smooth function \( a : B \to \mathbb{R} \) such that the image of

\[f : B \to L\]

\[p \mapsto \Phi(F(p), a(p))\]

lies in one of our adapted charts. We can choose the function \( a \), so that \( f \) maps radial rays \( \gamma \) from the origin in \( B \) to curves \( f(\gamma) \) in \( L \) that are orthogonal to \( \xi \). The length of \( f(\gamma) \) with respect to \( \alpha \) is then equal to its transversal length—that is, its length in \((B, g)\). By assumption the diameter of \((B, g)\) is at most \( K^{-1} \), and so the image \( f(B) \) must be contained in an \( \alpha \)-ball of radius \( K^{-1} \), and so it is contained in one of our adapted charts.
This completes the proof of bounded geometry of a fixed metric. Moreover it is clear from the above argument that we can choose a uniform \( K \) for metrics in a bounded family. \( \square \)

We now state the main result that we will prove in this section. Recall that we write \( R_\chi(X) \) for the part of the ring of functions of \( X \) on which the torus \( T \) acts by the character \( \chi \). For any \( D > 0 \) let us write

\[
R_{\leq D}(X) = \bigoplus_{0 < \chi(\xi) < D} R_\chi(X).
\]

Suppose that we have a sequence of solutions \( \omega_k \) on \((X,\xi_k)\) of \( \text{Ric}(\omega_k) = 2n c_k \alpha_\tau k \) for \( c_k \in [0,c_0] \) for some \( c_0 > 0 \). Choosing \( L^2 \) orthonormal bases of \( R_{\leq D} \) with respect to \( \omega_k \) we obtain a sequence of maps \( F_k : X \to \mathbb{C}^N \).

**Theorem 4.7.** There exists a constant \( D \), depending on the dimension, the non-collapsing constant and the bound \( K \) on the geometry of \( \alpha \), such that each \( F_k \) is an embedding, and up to choosing a subsequence we have \( F_k(X) \to Y \) in the sense of currents, where \( Y \) is a normal, \( Q \)-Gorenstein variety with a Reeb field \( \xi = \lim \xi_k \). In addition \((F_k)_*(\alpha_k^\tau) \to \beta^\tau\) for a positive transverse current on \( Y \), and \((Y,T^{-1}\xi,(1-T)\beta^\tau)\) admits a weak solution of the twisted equation, where \( t_k \to T \).

We will spend the rest of this section proving this result, based on work of Donaldson-Sun \[40\], Chen-Donaldson-Sun \[23, 24\] as well as the second author \[75\]. A key ingredient in the work of Chen-Donaldson-Sun is to make use of the Hörmander technique for producing holomorphic sections of positive line bundles. In our setting we will use the Hörmander technique to produce holomorphic functions on our affine varieties. The following estimate holds on non-compact manifolds (see Demailly \[31, \text{Theorem 4.1}\] or \[9, \text{Theorem 6.2}\]).

**Theorem 4.8.** Let \( L \) be a holomorphic line bundle endowed with a metric \( e^{-\varphi} \) over a complex manifold \( X \) which has some complete Kähler metric. Assume the metric \( e^{-\varphi} \) has strictly positive curvature, and that

\[
\sqrt{-1} \partial \bar{\partial} \varphi \geq c\omega
\]

where \( \omega \) is some Kähler form on \( X \) (not necessarily complete) and \( c > 0 \).

Let \( f \) be a \( \bar{\partial} \)-closed \((n,q)\) form \((q > 0)\) with values in \( L \). Then there is an \((n,q-1)\) form \( u \) with values in \( L \) such that \( \bar{\partial} u = f \), and

\[
\|u\|^2_{L^2(X,e^{-\varphi},\omega)} \leq \frac{1}{cq} \|f\|^2_{L^2(X,e^{-\varphi},\omega)},
\]

provided the right hand side is finite.

Using a resolution of singularities (see Saper \[68, \text{Example 9.4}\]) we know that \( X \setminus \{0\} \) admits a complete Kähler metric since \( 0 \in X \) is an isolated singular point. Hence Theorem 4.8 applies in our setting.
We are going to apply the Hörmander theorem to \( L = \mathcal{O}_X \otimes K_X^{-1} \simeq K_X^{-1} \), where we recall that \( \mathcal{O}_X \) is endowed with the metric \( e^{-\frac{1}{2}r^2} \) and we use the corresponding volume form \( \omega^n \) as a metric on \( K_X^{-1} \). The reason we make this choice for \( L \) is that we have isomorphisms

\[
\Lambda^{0,1}(K_X^{-1}) \cong \Lambda^{0,1} \otimes K_X \otimes K_X^{-1} \cong \Lambda^{0,1} K_X^{-1} \otimes K_X \cong \mathcal{O}_X.
\]

In particular, Theorem 4.8 implies that we can solve the \( \overline{\partial} \) equation on \( \mathcal{O}_X \).

Following the ideas of Donaldson-Sun [40], we can then use the Hörmander technique to transplant holomorphic functions from tangent cones of Gromov-Hausdorff limits to our non-compact cone manifold \((X, \omega)\). In order to do this, we need to ensure that there exist iterated tangent cones which are “good”.

**Definition 4.9.** Suppose that \((Z, d_Z)\) is a Gromov-Hausdorff limit of \((X, \omega_t)\) as \( t \to T \leq 1 \), and suppose that \( C(Y) \) is an (iterated) tangent cone at \( p \in Z \). We say that the tangent cone is good if:

1. The regular set \( Y_{\text{reg}} \subset Y \) is open in \( Y \) and smooth.
2. The distance function on \( C(Y_{\text{reg}}) \) is induced by a Ricci flat cone metric, and on \( C(Y_{\text{reg}}) \) the scaled up metrics along our sequence converge in \( L^p_{\text{loc}} \) for all \( p \) to this Ricci flat metric.
3. For all \( \delta > 0 \) there is a Lipschitz function \( g \) defined on \( Y \) which is identically 1 on a neighborhood of \( Y_{\text{sing}} = Y \setminus Y_{\text{reg}} \), with support contained in the \( \delta \) neighborhood of \( Y_{\text{sing}} \) and with \( \| \nabla g \|_{L^2} \leq \delta \), where the \( L^2 \) norm is with respect to the Sasaki-Einstein metric on \( Y_{\text{reg}} \).

Suppose that \( \omega_t \) are Kähler cone metrics on \( X \), solving (4.1) where \((\xi_t, \alpha_t)\) move in a bounded family. Suppose that, along a subsequence \((X, \omega_{t_i})\) converge in the Gromov-Hausdorff sense to \((Z, d_Z)\). If we can show that each tangent cone of \((Z, d_Z)\) is good, then the techniques of [40], together with the above remarks, will imply that there is a number \( \varepsilon_0 \) depending only on \( \alpha_t \) with the following effect: Let \( r_t \) be the radial function for \( \omega_t \). For any point \( p \in L = \{ r_t = 1 \} \) there is a holomorphic function \( f \in \mathcal{O}_X \) with \( \| f \|_{L^2(e^{-\frac{1}{2}r^2})} = 1 \), and \( | f(p) | > \varepsilon_0 \).

At this point we will need to pass from arbitrary holomorphic functions to those with polynomial growth. This is done in section 4.4 essentially by truncating the Taylor series of \( f \) at a sufficiently high (but controlled) order. Putting all of these results together with techniques from Donaldson-Sun [41] will imply Theorem 4.7. With this discussion, we state our first goal:

**Proposition 4.10.** Suppose \( \omega_t \) are solutions of (4.1) with data \((\alpha_t, \xi_t)\) which moves in a bounded family. If \((Z, d_Z)\) is any Gromov-Hausdorff limit of a sequence \((X, \omega_{t_i})\), then \( Z \) has good tangent cones.
4.1. Gromov-Hausdorff Convergence. We will now specialize to sequences $(X, \omega_i, \alpha_i, \xi_i)$, where
\[(\xi_i, \alpha_i) \longrightarrow (\xi, \alpha)\]
in the $C^2$ topology for a fixed background metric, and
\[(4.2) \quad \text{Ric}(\omega_i) = 2nc_i\alpha_i^\tau\]
with $c_i \longrightarrow c$.

Suppose we have a sequence of metrics solving (4.2), with $c_i \rightarrow c$. Since the links $(L, g_i)$ have bounded diameter, positive Ricci curvature, and are uniformly non-collapsed, we can take a Gromov-Hausdorff limit
\[ (L, g_i) \overset{d_{GH}}{\longrightarrow} (Z, d). \]
At the same time we will have convergence of the cones
\[ (X, \omega_i) \overset{d_{GH}}{\longrightarrow} (C(Z), \hat{d}), \]
in the pointed Gromov-Hausdorff topology, where we can identify $Z$ with the unit link in $C(Z)$.

To understand iterated tangent cones in the space $C(Z)$, for any $p \in Z$ we need to study very small balls centered around $p \in C(Z)$, scaled to unit size. This in turn means that we need to study small balls centered at points on the unit link in $(C(L), \omega_i)$, scaled to unit size. Such a ball $B$ has the following structure: $B$ is the unit ball with respect to a Kähler metric $\omega$, satisfying the equation
\[(4.3) \quad \text{Ric}(\omega) = c\alpha^\tau, \]
and a uniform non-collapsing condition $\text{Vol}(B, \omega) > K^{-1} > 0$. There is a holomorphic vector field $\Xi$ on $B$, whose imaginary part is the Reeb field $\xi$ scaled down, satisfying
\[ 1 - \delta < |\Xi|_\omega < 1 + \delta, \quad L_\Xi \omega = \lambda \omega \]
for some $\lambda \leq \delta \leq 1/2$. If the ball that we scaled up is sufficiently small, then $\delta$ can be taken to be arbitrarily small. Finally $\alpha^\tau$ is a closed, non-negative $(1, 1)$-form, vanishing along $\Xi$, and defining a transverse Kähler metric with bounded geometry on $TB/C\xi$ in the sense of Definition 4.5.

There are two different cases to study, depending on whether $c_i$ are bounded away from zero.

4.2. The case $c_i$ are bounded away from zero. Fix $i$, and suppress the index. Scaling $\alpha^\tau$ by a bounded factor we can rewrite Equation (4.3) as
\[ \text{Ric}(\omega) = \alpha^\tau. \]

The next proposition, which is based on [75, Proposition 8] shows that when $(B, \omega)$ is close to the Euclidean ball in the Gromov-Hausdorff sense, then on a smaller ball the Ricci curvature is bounded. The quantity $I(B)$, as defined in [28, 75] is
\[ I(B) = \inf_{B(x, r) \subset B} VR(x, r), \]
where $VR(x, r)$ is the ratio of the volumes of the ball $B(x, r)$ and the Euclidean ball $rB^{2n}$.

**Proposition 4.11.** There is a $\delta = \delta(K) > 0$, depending on the bound $K$ for the geometry of $\alpha$, such that if $1 - I(B) < \delta$, then $\text{Ric}(\omega) < 5\omega$ on $\frac{1}{2}B$.

**Proof.** The difference with [75, Proposition 8] is that $\alpha^\tau$ is not strictly positive, however it is strictly positive on slices transverse to $\Xi$, and it is invariant under the flow of $\Xi$, since this flow simply scales $\omega$.

As in [75, Proposition 8], if $\text{Ric}(\omega)$ is not bounded by $5\omega$ on $0.5B$, then we can find a small ball inside $B$, which when scaled to unit size ($\tilde{B}$, $\tilde{\omega}$) satisfies $\alpha^\tau \leq \tilde{\omega}$, and in addition there is a unit vector $v$ at the origin (with respect to $\tilde{\omega}$) such that

$$\alpha^\tau(v, \bar{v}) \geq \frac{1}{2}.$$

The equation for $\omega$ implies that on $\tilde{B}$ the metric $\tilde{\omega}$ has bounded Ricci curvature, and so if $\delta$ is sufficiently small, then Anderson’s result [3] implies that we have holomorphic coordinates $z^1, \ldots, z^n$ on the ball $\theta\tilde{B}$, with respect to which $\tilde{\omega}$ is close to the Euclidean metric in $C^{1,\alpha}$. In these coordinates the holomorphic vector field $\Xi$ will satisfy $1/4 < |\Xi|_{\text{Euc}} < 4$, and so by rotating the coordinates and shrinking $\theta$ we can assume that on $\theta\tilde{B}$ the vector field $\Xi$ is very close to $\partial z^1$. In particular $\alpha(\partial z^1, \tilde{w})$ is very small for any unit vector $w$. It follows that the slice $U = \{z^1 = 0\} \cap \theta\tilde{B}$ is transverse to $\Xi$, and so $(U, \alpha^\tau)$ is a Kähler manifold with bounded geometry. The inequality $\alpha^\tau \leq \tilde{\omega}$ implies that the diameter of $(U, \alpha^\tau)$ is at most $\theta$, so shrinking $\theta$ further if necessary, we have holomorphic functions $w^2, \ldots, w^n$ on $U$, defining local coordinates near each point, in which the components of $\alpha^\tau$ are controlled in $C^2$.

The vector $v$ may not be tangent to the slice $U$, but we may simply discard its $\partial z^1$-component, while still having $\alpha^\tau(v, \bar{v}) > \frac{1}{2}$. Rotating the $z^2, \ldots, z^n$ coordinates, we can then assume that

$$\alpha^\tau(\partial z^2, \partial z^2) > \frac{1}{4}.$$

We now have that the components of $\alpha^\tau$ in the $z^i$ coordinates have bounded derivatives along the slice $U$, but in addition $\alpha^\tau$ is also constant along the flow of $\Xi$, which is very close to $\partial_{z^0}$. It follows that just as in [76] Proposition 8 we can obtain a spherical sector in which the Ricci curvature of $\tilde{\omega}$ is strictly positive. Applying the Bishop-Gromov volume comparison we get a contradiction to $1 - I(B) < \delta$ if $\delta$ is sufficiently small. □

**Corollary 4.12.** If we have solutions of $(X, g_i)$ of (4.2) with $c_i \geq c > 0$, and if $(B(p_i, 1), g_i) \subset X$ converge in the Gromov-Hausdorff sense to the Euclidean ball, then the convergence is $C^{1,\alpha}$ on compact sets. In particular, if $(B(p_i, 1), g_i) \rightarrow Z$, then the regular set in $Z$ is open, and then convergence on the regular set is locally $C^{1,\alpha}$. 
Lemma 4.13. In the above setting, for every \( \varepsilon \) to a holomorphic vector field on \( 1 \) \( C \) to get a uniform Ricci bound, and then apply Anderson’s result in \([3]\) to get \( C^{1,\alpha} \) convergence to a Euclidean ball.

Proof. We combine Colding’s volume convergence \([26]\) with Proposition \([4.11]\) to get a uniform Ricci bound, and then apply Anderson’s result in \([3]\) to get \( C^{1,\alpha} \) convergence to a Euclidean ball.

Now assume that we have a sequence of balls as above such that \( (B(p_i, 1), g_i) \to Z \), with \( p_i \to p \), and a tangent cone at \( p \in Z \) is of the form \( C_{\gamma} \times C^{n-1} \). As in \([75, \text{Proposition 11}]\) we have \( \gamma \in (\gamma_1, \gamma_2) \) for some \( 0 < \gamma_1 < \gamma_2 < 1 \). The results of \([23]\) apply, and in particular, arguing as in \([23, \text{Section 2.5}]\), after scaling up the \( \omega_i \):

\[
\tilde{\omega}_i = k \omega_i, \quad \tilde{\xi}_i = \frac{1}{\sqrt{k}} \xi_i
\]

we can view \( \tilde{\omega}_i \) as a metric on the unit Euclidean ball \( B^{2n} \) with coordinates \( (u, v_1, \ldots, v_{n-1}) \), in which \( \tilde{\omega}_i \) is close to the model cone metric

\[
\eta_\gamma = \sqrt{-1} \frac{du \wedge dv}{|u|^{2-2\gamma}} + \sum_{i=1}^{n-1} dv_i \wedge dv_i.
\]

More precisely, if we scale by a large integer \( k \), and take \( i \) large depending on \( k \), we have, for some fixed constant \( C \):

- \( \tilde{\omega}_i = \sqrt{-1} \partial \bar{\partial} \varphi_i \) with \( 0 \leq \varphi_i \leq C \)
- \( \omega_{Euc} < C \tilde{\omega}_i \)
- Given \( \delta > 0 \) and a compact set \( K \subset B^{2n} \setminus \{ u = 0 \} \) we can suppose (by taking \( i \) large once \( k \) is taken sufficiently large) that \( |\tilde{\omega}_i - \eta_\gamma|_{C^{1,\alpha}(K, g_{Euc})} < \delta \).

Lemma 4.13. In the above setting, for every \( \varepsilon \in (0, 1/2) \) we have \( |\tilde{\xi}_i|_{g_{Euc}} > \frac{1}{2} \) on the set \( \{ |u| = \varepsilon \} \) provided \( i \), and the scaling factor \( k \) are sufficiently large.

Proof. We fix \( \varepsilon > 0 \), and suppose that the conclusion is false. Then we have a sequence of metrics \( \tilde{\omega}_i \) and holomorphic vector fields \( \tilde{\xi}_i \) on \( B^{2n} \) (with scaling factors \( k \to \infty \)) converging in \( C^{1,\alpha} \) to the standard cone metric \( \eta_\gamma \) locally away from \( \{ u = 0 \} \), and \( \tilde{\xi}_i \) satisfy

\[
|\tilde{\xi}_i|_{g_{Euc}} \leq C \left( 1 - \frac{1}{k} \right) \frac{1}{\sqrt{k}} \tilde{\omega}_i < 1 + \frac{1}{k}, \quad L_{\tilde{\xi}_i} \tilde{\omega}_i = \lambda \tilde{\omega}_i,
\]

where \( \lambda < 1/k \). Since the \( \tilde{\xi}_i \) are holomorphic and bounded we obtain uniform \( C^{3,\alpha} \) estimates on \( \frac{1}{2} B^{2n} \). We can choose a subsequence so that \( \tilde{\xi}_i \) converges to a holomorphic vector field \( \tilde{\xi} \) in \( C^{3}(g_{Euc}) \) on \( \frac{1}{2} B \), and \( \tilde{\omega}_i \to \eta_\gamma \) on \( \{ |u| \geq \varepsilon/2 \} \cap \frac{1}{2} B \). Furthermore, we have

\[
L_{\tilde{\xi}} \eta_\gamma = 0, \quad |\tilde{\xi}|_{\eta_\gamma} = 1.
\]

By direct computation one verifies that the only holomorphic vector fields on \( \frac{1}{2} B \) which are Killing for \( \eta_\gamma \) on \( \{ |u| \geq \varepsilon/2 \} \) \( \cap \frac{1}{2} B \) with unit length are the translations in the \( v_i \) directions. In particular \( |\tilde{\xi}|_{Euc} = |\tilde{\xi}|_{\eta_\gamma} \). The result then follows from the convergence \( \tilde{\omega}_i \to \eta_\gamma \).
Proposition 4.14. There is a constant $c_0 > 0$, such that if $(B(p,1),\omega)$ is sufficiently close to the unit ball in the cone $\mathbb{C}_\gamma \times \mathbb{C}^{n-1}$ with $\gamma \in (\gamma_1, \gamma_2)$, then
\[ \int_{B(p,1)} \alpha^r \wedge \omega^{n-1} > c_0. \]

Proof. We argue by contradiction and assume there is no such $c_0$. Then we will have a sequence $B(p_i,1) \to B(0,1)$, where 0 is the vertex in the cone $\mathbb{C}_\gamma \times \mathbb{C}^{n-1}$, and such that
\[ \lim_{i \to \infty} \int_{B(p_i,1)} \alpha^r_i \wedge \omega_i^{n-1} = 0. \]

As discussed above, we can then find a small $r_0 > 0$ such that for sufficiently large $i$, the scaled up metric $r_0^{-2}\omega_i$ can be thought of as a metric on a set containing the Euclidean unit ball $B^{2n}$, such that $\omega_{Euc} < C r_0^{-2}\omega_i$ on $B^{2n}$.

By the previous lemma we can assume (choosing $r_0$ smaller if necessary), that the rescaled Reeb vector field $\Xi_i$ is a perturbation of the vector field $\partial/\partial z^0$ for one of the holomorphic coordinates $z_0$ on $B^{2n}$, and so the sets $U_c = \{ z_0 = c \}$ are transverse to $\Xi_i$ for $|c| < 1/2$, say.

Suppose that
\[ \int_{B^{2n}} \alpha_i^r \wedge \omega_{Euc}^{n-1} < \varepsilon_1, \]
for some $\varepsilon_1 > 0$. Then we must have
\[ \int_{U_c} \alpha_i^r \wedge \omega_{Euc}^{n-2} < C_1 \varepsilon_1 \]
for at least one $c$ with $|c| < 1/2$, and a uniform constant $C_1$. The form $\alpha_i^r$ defines a Kähler metric on $U_c$ with bounded curvature, so the $\varepsilon$-regularity theorem of Schoen-Uhlenbeck [69] (see also [75, Proposition 7]) implies that if $C_1 \varepsilon_1 < \varepsilon_0$, then we must have $\alpha_i^r < C' \omega_{Euc} < C' C r_0^{-2}\omega_i$ on $U_c$ (evaluated on vectors tangent to this slice). Since $\Xi_i$ is a small perturbation of $\partial/\partial z^0$, and $\alpha_i^r$ vanishes along $\Xi_i$ we obtain a bound for $\alpha^r$ on $\frac{1}{2} B^{2n}$. This implies a uniform bound for $\text{Ric}(\omega_i)$ on this ball, independent of $i$. The result of Cheeger-Colding-Tian [19] implies that no conical singularity can then form, which is a contradiction.

As a result, we must have
\[ \int_{B^{2n}} \alpha_i^r \wedge \omega_{Euc}^{n-1} \geq \varepsilon_1, \]
for sufficiently large $i$, where $\varepsilon_1 = C_1^{-1} \varepsilon_0$. This in turn implies
\[ \int_{B^{2n}} \alpha_i^r \wedge (r_0^{-2}\omega_i)^{n-1} \geq C^{-n-1} \varepsilon_1, \]
contradicting (4.4). $\square$

We also have the following, whose proof is the same as that of [75, Proposition 13].
Proposition 4.15. There is a constant $A > 0$, such that if $(B(p,1), \omega)$ is sufficiently close to either the Euclidean unit ball or the unit ball in the cone $C_\gamma \times \mathbb{C}^{n-1}$ with $\gamma \in (\gamma_1, \gamma_2)$, then
\[
\int_{B(p, \frac{1}{2})} \alpha^r \wedge \omega^{n-1} < A.
\]

We can now show that the iterated tangent cones are good, similarly to Chen-Donaldson-Sun [23], or [75]. The argument in the published version of [75] was incomplete, but it is corrected in the latest version on the arXiv, and that argument can be used verbatim in our setting, using the estimates on the “densities” given by Propositions 4.14 and 4.15.

4.3. The case $c_i \to 0$. In this case we study non-collapsed balls $B = B(p,1)$ with metrics $\omega$ satisfying
\[
\text{Ric}(\omega) = c_i \alpha^r,
\]
where $c_i \to 0$. As above, the additional structure is the (rescaled) Reeb field $\Xi$, which is a holomorphic vector field satisfying $\frac{1}{2} < |\Xi|_\omega < 2$ and $L_\Xi \omega = \lambda \omega$, for some $\lambda \leq 2$. Note that as we scale up the metric, we must scale down the Reeb field, and so scale down $\lambda$. The form $\alpha^r$ defines a transverse Kähler metric on $TB/C\xi$, with bounded geometry as before. Once again, the difficulty when compared to [75] is that $\alpha^r$ is not strictly positive, and so the $\varepsilon$-regularity theorem for harmonic maps cannot be applied. Our strategy, as above, is to find transverse slices.

We first need a slight refinement of [24, Proposition 1], which will allow us to control the Reeb field $\Xi$. First we recall some definitions. For a subset $A$ in a $2n$-dimensional length space $P$, and for $\eta < 1$, let $m(\eta, A)$ be the infimum of those $M$ for which $A$ can be covered by $Mr^{2-2n}$ balls of radius $r$ for all $\eta \leq r < 1$.

For $x \in B$ and $r, \delta > 0$ a holomorphic map $\Gamma : B(x, r) \to \mathbb{C}^n$ is called an $(r, \delta)$-chart centered at $x$ if
\begin{itemize}
  \item $\Gamma(x) = 0$,
  \item $\Gamma$ is a homeomorphism onto its image,
  \item For all $x', x'' \in B(x, r)$ we have $|d(x', x'') - d(\Gamma(x'), \Gamma(x''))| \leq \delta$,
  \item For some fixed $p > 2n$, we have $\|\Gamma_*(\omega) - \omega_{Euc}\|_{L^p} \leq \delta$.
\end{itemize}

With these definitions, we need the following slight modification of [24, Proposition 1].

Proposition 4.16. Given $M, c$ there are $\rho(M), \eta(M, c), \delta(M, c) > 0$ with the following effect. Suppose that $1 - I(B) < \delta$ and $W \subset B$ is a subset with $m(\eta, W) < M$, such that for any $x \in B \setminus W$ there is a $(c\eta, \delta)$-chart centered at $x$. There is a constant $C$ depending only on the dimension, such that:

1. There is a holomorphic map $F : B(p, \rho) \to \mathbb{C}^n$ which is a homeomorphism to its image, $|\nabla F| < C$, and its image lies between $0.9\rho B^{2n}$ and $1.1\rho B^{2n}$.

2. There is a local Kähler potential $\varphi$ for $\omega$ on $B(p, \rho)$ with $|\varphi| \rho^{-2} < C$. 

The slices \( \{ z_n = l \} \cap C^{-1} \rho B^{2n} \) are transverse to the Reeb field for \( |l| < C^{-1} \rho \).

The way this proposition is used is that under the assumption s we can use \( F \) to think of \( \omega \) as a metric on the Euclidean ball \( 0.9 \rho B^{2n} \), and because of the gradient bound for \( F \) we have \( \omega_{Euc} < C_1 \omega \). The new statement is (3), which will essentially follow if we can show that \( |\Xi|_{\omega_{Euc}} \) is not too small near the origin. In fact we will show that given any \( \kappa > 0 \), the \( \eta, \delta \) in the proposition can be chosen so that there exists a point \( q \in \kappa \rho B^{2n} \) at which \( F_*(\omega) < 2 \omega_{Euc} \).

In order to prove this claim, we recall briefly the proof of [24, Proposition 1]. For this, let \( \Omega(W, s) \subset B \) denote the set of points in \( B \) at a distance greater than \( s \) from \( W \) and from the boundary of \( B \). One part of the proof of the proposition is to produce an embedding \( \varphi : \Omega(W, r) \to B^{2n} \), for which \( ||\varphi_*(\omega) - \omega_{Euc}||_{L^p} \leq \tilde{\theta}, |\varphi_*(J) - J_{Euc}| \leq \tilde{\theta} \) and \( d(x, \varphi(x)) \leq \tilde{\theta} \), where \( r, \tilde{\theta} > 0 \) can be chosen a priori, and \( d \) denotes the distance function realizing the Gromov-Hausdorff distance from \( B \) to \( B^{2n} \). In addition \( \omega \) is the curvature of a metric on the trivial bundle on \( B \), and \( \varphi \) can be lifted to a bundle map from the trivial bundle on \( B \) to the trivial bundle on \( B^{2n} \), which almost identifies the corresponding connections (c.f. [24, Proposition 4]).

The holomorphic function \( F \) is now obtained by taking the holomorphic functions \( 1, z_1, \ldots, z_n \) on \( B^{2n} \), and applying suitable cutoff functions to obtain approximately holomorphic functions (sections of the trivial bundle) \( \sigma, \sigma_1, \ldots, \sigma_n \) over \( B \), vanishing on \( \Omega(W, r) \), and near \( \partial B \). Using the Hörmander \( L^2 \)-estimate these can be projected to holomorphic sections \( s, s_1, \ldots, s_n \) (globally on our cone \( X \)), and \( s_i/s \) give the components of \( F \). The result of Proposition 1 is then obtained by choosing the parameters in the cutoff functions in a suitable way.

In order to obtain (3), we just note that given \( \kappa > 0 \) we simply need to choose \( r \) much smaller than \( \kappa \rho \), so that we can find some \( q \in \kappa \rho B \), which is contained in a definite ball \( B_q \) disjoint from \( \Omega(W, 2r) \), say. If \( \tilde{\theta} \) above is sufficiently small, then on this ball \( B_q \) the geometry of \( \omega \) will be almost identical in an \( L^p \)-sense to the Euclidean geometry. In particular the holomorphic function \( F \) will be very close to the identity map on \( \frac{1}{2} B_q \). There will then exist a point \( q' \in \frac{1}{2} B_q \) at which \( F_*(\omega) < 2 \omega_{Euc} \). Note that \( \Xi \) gives a holomorphic vector field on \( 0.9 \rho B^{2n} \), and it has bounded length with respect to the Euclidean metric. In particular on \( 0.8 \rho B^{2n} \) the derivatives of the components of \( \Xi \) are bounded. On the other hand we know that we can choose a point \( q' \) very close to the origin, where \( |\Xi|_{\omega_{Euc}} > 1/4 \). Rotating coordinates, we can assume that the \( \partial/\partial z_n \) component of \( \Xi \) is non-zero inside \( C_2^{-1} \rho B^{2n} \). This implies our claim (3).

Given this result, the rest of the argument is quite similar to that in [75]. We give the required modifications of the proofs. The following is analogous to [75, Proposition 16].
Proposition 4.17. Given $M$, suppose that the ball $B$ satisfies the hypotheses of Proposition 4.16 for some $c > 0$. There are $A, \kappa > 0$ depending on $M$, such that if 
\[
\int_B \alpha^\tau \wedge \omega^{n-1} < \kappa,
\]
then $\alpha^\tau < A\kappa$ on $\frac{1}{3}C_2(K)^{-1}\rho B$, with $\rho = \rho(M), C_2(K)$ from Proposition 4.16.

Proof. The assumption implies that 
\[
\int_{0.9\rho B^{2n}} \alpha^\tau \wedge \omega^{n-1} < C_1\kappa,
\]
for some $C_1$. We argue similarly to the proof of Proposition 4.14. There is a slice $U_l = \{z_n = l\} \cap C_2(K)^{-1}\rho B^{2n}$, with $|l| < \frac{1}{2}K^{-1}\rho B^{2n}$ on which we have 
\[
\int_{U_l} \alpha^\tau \wedge \omega^{n-1} < C_3\kappa.
\]
The $\varepsilon$-regularity implies that if $\kappa$ is sufficiently small, then on a slightly smaller set we have $\alpha^\tau < C_4\omega_{Euc} < C'_4\omega$. The flow of the Reeb field preserves $\alpha^\tau$, and acts on $\omega$ by a controlled scaling factor, so we obtain the required estimate on a ball of a definite size, $\frac{1}{3}C_2(K)^{-1}\rho B^{2n}$. \hfill \Box

For any ball $B(q, r) \subset B$ we define 
\[
V(q, r) = r^{2-2n} \int_{B(q, r)} \alpha^\tau \wedge \omega^{n-1}.
\]
We have the following.

Proposition 4.18. There are $\delta, \varepsilon > 0$ depending on $K$ satisfying the following: if $1 - I(B) < \delta$ and 
\[
\sup_{B(q, r) \subset B} V(q, r) < \varepsilon,
\]
then $\alpha^\tau \leq 4\omega$ on $\frac{1}{2}B$.

Proof. The proof follows the argument of [75, Proposition 17], together with the idea we used in the proof of 4.14 to make use of the vector field $\Xi$. \hfill \Box

There is one other time when the strict positivity of $\alpha$ is used in [75], namely in the proof of [75, Proposition 19] where the $\varepsilon$-regularity is used again. In the present setting the same argument can be used, just like in the proof of Proposition 4.17 above. The remainder of the proof of Proposition 4.10 is identical to the argument in [75].
4.4. **Polynomial growth holomorphic functions.** In this section we assume that we have a Kähler cone \((C(L), \omega)\), which has non-negative Ricci curvature, and is non-collapsed (i.e. we have a lower bound on the volume of \(L\)). In addition we assume that there is a constant \(\varepsilon_0 > 0\) with the following property: for every \(x \in L\) there is a holomorphic function \(f\) on \(C(L)\) such that \(f(0) = 0\), \(\|f\|_{L^2} = 1\), and \(|f(x)|^2 > \varepsilon_0\). Here the \(L^2\)-norm is with respect to the weight \(e^{-\frac{1}{2}r^2}\) as before. This extra property holds for any family of solutions of (4.1) with data moving in a bounded family by using the results of the previous section on having good tangent cones. Our goal is to show that up to replacing \(\varepsilon_0\) by a smaller constant, we can take \(f\) to have polynomial growth with controlled degree. More precisely we have the following.

**Proposition 4.19.** There are constants \(D, \varepsilon_1 > 0\), depending on \(\varepsilon_0\) and the lower bound on the volume of \(L\), such that the following holds: if \(x \in L\), then there is a holomorphic function \(f\) on \(C(Y)\) with \(f(0) = 0\), \(\|f\|_{L^2} = 1\), \(|f(x)|^2 > \varepsilon_1\), and in addition \(|f| = O(r^D)\).

**Proof.** First let us write \(H\) for the space of \(L^2\) holomorphic functions on \(C(L)\), which we can decompose into weight spaces under the torus action

\[
H = \bigoplus_{\chi \in t^*} H_{\chi},
\]

where infinite convergent sums are allowed. In addition we have the Reeb field \(\xi \in t\).

As we mentioned above, we already have an \(f\) with the required properties, except for the growth condition. To restate our goal, we are trying to construct an \(f'\) that also satisfies the growth condition, which is equivalent to

\[f \in \bigoplus_{\chi \neq 0 \atop \langle \chi, \xi \rangle < D} H_{\chi}.
\]

Lemma 4.21 below shows that we have a constant \(C\) such that

\[\#(C \cap \{\chi \in t^* : \langle \chi, \xi \rangle \in (w - 1, w]\}) < C5^w
\]

for all \(w \geq 1\). Note that this estimate is far from optimal, but it is enough for our purposes here.

Suppose that we have \(f\) such that \(f(0) = 0\), \(|f(x)|^2 > \varepsilon_0\) and \(\|f\|_{L^2} = 1\). We write

\[f = \sum_{w=1}^{\infty} \sum_{i=1}^{N_w} f_{w,i},\]

where each \(f_{w,i}\) is in a weight space \(H_{\chi}\) with \(\langle \chi, \xi \rangle \in (w - 1, w]\). We have \(N_w < C5^w\).

From Lemma 4.20 below we have that on \(L\)

\[|f_{w,i}|^2 \frac{w!}{2^w} < C\|f_{w,i}\|_{L^2}^2,
\]

\[\sum_{w=1}^{\infty} \sum_{i=1}^{N_w} \left|f_{w,i}\right|^2 \frac{w!}{2^w} < C\sum_{w=1}^{\infty} \sum_{i=1}^{N_w} \|f_{w,i}\|_{L^2}^2 \leq C\|f\|_{L^2}^2 = 1.
\]
and so for any $D > 0$ we have

$$\left| \sum_{w=D}^{\infty} \sum_{i=1}^{N_w} f_{w,i} \right|^2 \leq \left( \sum_{w=D}^{\infty} \sum_{i=1}^{N_w} \frac{2^w}{w!} \right) \left( \sum_{w=D}^{\infty} \sum_{i=1}^{N_w} |f_{w,i}|^2 \frac{w!}{2^w} \right)$$

$$\leq C \|f\|_{L^2}^2 \sum_{w=D}^{\infty} \frac{10^w}{w!}.$$ 

Since the series on the right converges, and $|f(x)|^2 > \varepsilon_0$, we can choose $D$ sufficiently large so that at $x$ we have

$$\left| \sum_{w=1}^{D} \sum_{i=1}^{N_w} f_{w,i} \right|^2 > \frac{\varepsilon_0}{2}.$$ 

We can then let

$$f' = \sum_{w=1}^{D} \sum_{i=1}^{N_w} f_{w,i},$$

and $f'\|f'\|^{-1}_{L^2}$ will satisfy the required properties. □

**Lemma 4.20.** Suppose that $f \in \mathcal{H}_\chi$, and let $w = \lceil \langle \chi, \xi \rangle \rceil$. Then we have

$$|f|^2 < \frac{2^w C \|f\|_{L^2}^2}{w!} \text{ on } L.$$ 

**Proof.** Assume $\|f\|_{L^2} = 1$. We have (with dimensional factors $c_n$)

$$1 = \|f\|_{L^2}^2 \geq c_n \|f\|_{L^2(L)}^2 \int_1^{\infty} r^{2w+2n-3} e^{-\frac{1}{2} r^2} dr$$

$$\geq c'_n 2^w (w + n - 2)! \|f\|_{L^2(L)}^2.$$ 

The $L^2$ norm on the link gives a $C^0$-estimate on the half ball around the cone vertex, and then using the assumed growth rate on $f$ we obtain a $C^0$-estimate on the link $L$:

$$\sup_L |f|^2 < \frac{2^w C}{(w + n - 2)!}.$$ 

□

We have also used the following simple estimate on the dimension of the space of holomorphic functions of polynomial growth.

**Lemma 4.21.** For an integer $w \geq 1$, let us write $\mathcal{H}_w$ for the space of holomorphic functions satisfying the growth condition $|f| = O(r^w)$ as $r \to \infty$. We then have

$$\dim \mathcal{H}_w < C 5^w.$$
Proof. This follows a standard argument using the previous lemma. Let \( \{f_1, \ldots, f_N\} \) be an \( L^2 \)-orthonormal basis for \( \mathcal{H}_w \), and define the function \( B \) by

\[
B(x) = \sum_{i=1}^{N} |f_i(x)|^2 e^{-\frac{1}{2}r^2}.
\]

The previous lemma implies that

\[
B \leq \frac{2^w C}{w!} \max\{1, r^{2w}\},
\]

and so

\[
\int_{C(L)} B \omega^n \leq C 5^w.
\]

On the other hand, by definition the integral of \( B \) is the dimension of \( \mathcal{H}_w \).

\[\square\]

4.5. Synthesis. Let us recall where we stand. Suppose we have a sequence of cone Kähler metrics \( \omega_t \) on \( (X, t^{-1}_k \xi_k) \), with radial functions \( r_k \), satisfying the equations

\[
\text{Ric}(\omega_t) = 1 - t_k \alpha_k,
\]

and \( t_k \to T \), \( \xi_k \to \xi \) and \( \alpha_k \to \alpha \). In addition the sequence \( (X, \omega_t) \) converges in the Gromov-Hausdorff sense to \( (Z, d_Z) \). Our work so far leads to the following "partial \( C^0 \)-estimate".

**Proposition 4.22.** There exists a constant \( D > 0 \) depending only on the non-collapsing constant, the dimension and a bound for the geometry of \( (\xi_k, \alpha_k) \) such that if \( \{f_1^{(k)}, \ldots, f_N^{(k)}\} \) denotes an \( L^2 \)-orthonormal basis of \( \mathcal{R}_{<D}(X) \), then the map \( F_k : X \to \mathbb{C}^N \) whose components are given by the \( f_i^{(k)} \) gives an embedding of \( X \). Furthermore, there is a uniform constant \( C \) so that

\[
C^{-1} < |F_k| < C, \quad C^{-1} F_k^* \omega_{Euc} < \omega_t,
\]

on the set \( \{1/2 < r_k < 2\} \).

Proof. In Sections 4.2 and 4.3 we have shown that the iterated tangent cones in suitable Gromov-Hausdorff limits are good, and so the arguments of Donaldson-Sun \[40\] apply: for each \( x \in L \), we can find a holomorphic function \( f \) on \( X \) with polynomial growth of bounded degree, unit \( L^2 \)-norm on \( L \), and \( |f(x)|^2 > \varepsilon_1 \) for a fixed number \( \varepsilon_1 \) (independent of \( k \)). Increasing \( D \) if necessary, we can assume that \( F_k \) gives an embedding (note that \( X \) is fixed, only the metric is changing). It then follows directly that \( C^{-1} < |F_k| \) on \( L \), while the bound on the growth rate implies that the same estimate (with different \( C \)) also holds on the annulus \( 1/2 < r < 2 \). A uniform bound \( |F_k| < C \) and derivative bound \( |\nabla F_k|_{\omega_{\tau_k}} < C \) on the annulus follows by using Moser iteration, and this implies the estimate \( C^{-1} F_k^* \omega_{Euc} < \omega_t \). \[\square\]
Arguing as in Donaldson-Sun [11, Section 2] we can deduce the convergence of the affine varieties

$$F_k(X) \longrightarrow Y$$

where $Y$ is a normal, $\mathbb{Q}$-Gorenstein affine variety with Reeb vector field $\xi$, and furthermore $Y$ is homeomorphic to $Z$. In order to finish the proof of Theorem 4.7 it suffices to prove the last statement regarding convergence to a weak solution of the twisted equation, after possibly passing to a further subsequence.

Consider the pushed forward forms $(F_k)_* \alpha^\tau$. We would like to take a weak limit of these forms. As a first step we prove the volume is bounded below. Suppose that $\alpha = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \log r$ is a cone metric, with transverse form $\alpha^\tau = \sqrt{-1} \partial \bar{\partial} \log r$. Recall [71] we have the form $\eta := \sqrt{-1}(\partial - \partial) \log r$. In terms of $\eta$ we have $\alpha^\tau = \frac{1}{2} d\eta$, and $\alpha = \frac{1}{2} d(r^2 \eta)$. Then we compute

$$\alpha^\tau \wedge \alpha^{n-1} = 2^{1-n}(n-1)r^{2n-3} dr \wedge \eta \wedge (d\eta)^{n-1}.$$

Writing everything on the cone we get

$$\int_{\{r \leq 1\}} \alpha^\tau \wedge \alpha^{n-1} = 2^{1-n}(n-1) \int_0^1 r^{2n-3} dr \int_{r=1} \eta \wedge (d\eta)^{n-1} = c_n Vol(L, \xi) > 0$$

where $Vol(L, \xi)$ is the volume of the link, which is a topological invariant depending only on the Reeb field. Furthermore, if $\omega$ is another metric compatible with $\xi$, with radial function $\tilde{r}$ then we can write $r = e^\psi \tilde{r}$ for a function $\psi$ which is independent of $r$, and $L_\xi \psi = 0$. Then it is easy to check that

$$\int_{\{\tilde{r} \leq 1\}} \alpha^\tau \wedge \omega^{n-1} = c_n Vol(L, \xi).$$

Now suppose we have sequence of maps $F_k : X \rightarrow \mathbb{C}^N$ as above and consider the closed positive currents

$$(F_k)_* \alpha^\tau \wedge [F_k(X)].$$

Let $A := \{ p \in \mathbb{C}^N : M^{-1} < |p| < M \}$ for some constant $M > 0$. Let $A' := F_k^{-1}(A \cap F_k(X))$. Suppose that $\nu$ is a smooth positive $(n-1,n-1)$ form with compact support in $A$. Then we have

$$\int_{\mathbb{C}^N} (F_k)_* \alpha^\tau \wedge [F_k(X)] \wedge \nu = \int_{A'} \alpha^\tau \wedge F_k^* \nu$$

We can find a constant $C_1$ such that on $A$ we have $\nu \leq C_1 \omega_{Euc}^{n-1}$ as $(n-1,n-1)$ forms. Furthermore, by the properties of $F_k$ we have

$$F_k^* \omega_{Euc} < C \omega_k$$

on $A'$, where $\omega_k$ is our metric, and the partial $C^0$ estimate implies that

$$\{ 1/2 < r_{t_k} < 2 \} \subset A' \subset \{ r < C' \}$$
for some constant $C'$, provided $M$ is sufficiently large. The above discussion implies a uniform upper bound

$$\int_{A'} \alpha^r \wedge F_k^* \nu < C,$$

with the constant $C$ depending only on the form $\nu$. It follows that $(F_k)_* \alpha^r \wedge [F_k(X)]$ converges weakly to a closed positive current $\beta^r$ on $Y = \lim_k F_k(X)$.

By arguing as in [30] we can show that $Y$ admits a weak solution $\omega_T$ of the equation

$$\text{Ric} (\omega_T) = \frac{1 - T}{T} \beta^r.$$

Note that the proof of this result in [30] is essentially local, working in neighborhoods of points $p$ in the limit $Y$ where the complex structure of $Y$ is smooth, and where in terms of the metric structure we have a tangent cone of the form $\mathbb{C}^n$ or $\mathbb{C}^{n-1} \times \mathbb{C}_\gamma$. On this set we have good local coordinates, and so we can study the limiting equation, while at the same time the complement of this set has Hausdorff codimension greater than two, and therefore it can be ignored in terms of writing down weak solutions of the twisted equation, as in [30, Remark 4]. This completes the proof of Theorem 4.7.

5. Proof of the main result

In this section we prove on direction of Theorem 1.1. Recall that we have a normalized Fano cone singularity $(X, \xi)$ together with the action of a torus $T$, whose Lie algebra contains $\xi$. We assume that $(X, \xi)$ is $T$-equivariantly K-stable, and our goal is to show that $(X, \xi)$ admits a Ricci flat Kähler cone metric. The proof naturally splits into two cases depending on whether $(X, \xi)$ is quasi-regular, or irregular. We will first focus on the former, and then we will deal with irregular $\xi$ by approximating it with a sequence of quasi-regular Reeb fields.

5.1. The quasi-regular case. Suppose that $(X, \xi)$ is quasi-regular. We first fix a $T$-invariant transverse Kähler metric $\alpha^r$, using an embedding $X \to \mathbb{C}^N$ by a collection of (non-constant) holomorphic functions just as in Equation (3.7). We can then solve the continuity method (2.2) up to some time $T \leq 1$.

According to Theorem 4.7 there is a number $D$, depending on the bound on the geometry of the twisting form $\alpha^r$ and on the pair $(X, \xi)$ through the non-collapsing condition, so that using orthonormal bases of holomorphic functions with growth rates less than $D$ we obtain embeddings $F_k : X \to \mathbb{C}^N$, such that $F_k(X) \to Y$ with a normal limit space $Y$. In addition we have convergence of twisting forms $(F_k)_* \alpha^r \to \beta^r$ and the limit $Y$ admits a weak solution of the twisted equation

$$\text{Ric} (\omega_T) = \frac{1 - T}{T} \beta^r.$$
In order to apply the results from Section 3 we need to make sure that $\alpha^\tau$ can be written as an integral over currents of integration (with respect to a positive measure) over suitable hypersurfaces. The problem with $\alpha^\tau$ defined as in (3.7) is that it does not use all the functions in $R_{<D}(X)$ and so in the integral expression (3.8) we are not using a positive measure on $P$. To fix this, we will modify $\alpha^\tau$ slightly by adding small terms corresponding to the remaining functions in $R_{<D}(X)$, and so that the new transverse metric still has the same bound on its geometry. In particular the above discussion still holds with the same constant $D$. Let us write $\alpha^\tau = \sqrt{-1} \partial \bar{\partial} \log \hat{R}$, and we will call the perturbed radial function $\hat{r}$.

Let us decompose $R_{<D}(X) = R_1 \oplus \ldots \oplus R_m \oplus R_{m+1} \oplus \ldots \oplus R_k$ into weight spaces of $T$, where $\alpha^\tau$ is defined as in (3.7) using bases of $R_i$ for $i \leq m$. Suppose that $\xi$ acts on the functions in $R_i$ with weight $a_i$ as before, and recall that we chose $M$ so that $M/a_i \in \mathbb{Z}$ for $i \leq m$. We also choose $K$ so that $K/a_i \in \mathbb{Z}$ for all $i$, and then for $\delta > 0$ we define a new radial function $\hat{r}$ by

$$\hat{r}^{2K} = \left[ \sum_{i=1}^{m} \left( \sum_{j=1}^{N_i} |z_j^{(i)}|^2 \right)^{M/a_i} \right]^{K/M} + \delta \sum_{i=m+1}^{k} \left( \sum_{j=1}^{N_i} |z_j^{(i)}|^2 \right)^{K/a_i}.$$

Here the $z_j^{(i)}$ form a basis for $R_i$. As $\delta \to 0$, we recover the original radial function $\hat{R}$, and so for sufficiently small $\delta$ the transverse metric $\sqrt{-1} \partial \bar{\partial} \log \hat{r}$ has the same bounded geometry as $\sqrt{-1} \partial \bar{\partial} \log \hat{R}$, but at the same time the methods of Section 3.1 in particular Proposition 3.8 can be applied.

At this point we are in essentially the same setup as in [30], Section 3.1], and can follow the argument there closely. We have a sequence $\rho_k \in GL(N)^T$, such that $F_k = \rho_k \circ F_1$. For simplicity of notation let us write $F_1(X) = X$ and $(F_1)_*(\alpha^\tau) = \alpha^\tau$. Then in the notation of Section 3 on $X$ we have

$$(5.1) \quad \alpha^\tau = \frac{\pi}{M} \int_{P \setminus F} [V_\mu] d\mu,$$

where $F$ is a hyperplane in the projective space $P$ and each $V_\mu$ is a hypersurface in $X$. Similarly to [30] Lemma 14) we can choose a subsequence of the $\rho_k$, such that $\rho_k(V_\mu)$ converges for all $\mu$. Let us define

$$\rho_\infty(V_\mu) = \lim_{k \to \infty} \rho_k(V_\mu).$$
Generalizing Definition 3.2, for any (1, 1)-current on $Y$ we let $\mathfrak{g}_{Y,\xi,\lambda}$ denote the holomorphic vector fields on $Y$, commuting with $\xi$ such that $\iota_\xi \lambda = 0$. Then as in [30, Lemma 15] we have

**Lemma 5.1.** We can find $\mu_1, \ldots, \mu_d$ for some $d$, such that  

$$\mathfrak{g}_{Y,\xi,\beta^\tau} = \bigcap_{i=1}^d \mathfrak{g}_{Y,\xi,[(\rho_\infty(V_{\mu_i}))]}.$$

The proof is the same as in [30], except instead of the Fubini-Study volume form we use $e^{-\frac{1}{2}\beta^2} \omega^n$, where $\omega = \frac{1}{2} i \partial \bar{\partial} r^2$ as usual.

By Lemma 5.1 and Proposition 3.3 we can choose hypersurfaces $V'_1, \ldots, V'_d$ in $\mathbb{P} \setminus F$ such that the automorphism group of the $(d+1)$-tuple $(Y, \rho_\infty(V'_1), \ldots, \rho_\infty(V'_d))$ is reductive, by Proposition 3.3. Since $Y$ may be contained in a hyperplane, the stabilizer of this $(d+1)$-tuple in the multigraded Hilbert scheme, under the action of $GL(N)^T$ may contain extra factors of $GL(k_i)$ for suitable $k_i$, but this product is still reductive. At this point, we can use the Luna slice theorem (see [62] and also [39, 24]) to find a $\mathbb{C}^*$-subgroup $\lambda(t) \subset GL(N)^T$ with $\lambda(S^1) \subset U(N)^T$, and an element $g \in GL(N)^T$ such that

$$(Y, \rho_\infty(V'_1), \ldots, \rho_\infty(V'_d)) = \lim_{t \to 0} \lambda(t)g \cdot \left( X, V'_1, \ldots, V'_d \right).$$

More generally, enlarging $F$ by a set of measure zero, if $V_1, \ldots, V_K$ are hypersurfaces in $\mathbb{P} \setminus F$, then the stabilizer of the $(1 + d + K)$-tuple

$$(Y, \rho_\infty(V'_1), \ldots, \rho_\infty(V'_d))_{i=1, \ldots, d, j=1, \ldots, K}$$

is unchanged, and so the Luna slice theorem provides a corresponding $\mathbb{C}^*$-subgroup $\lambda(t)$ and element $g$. These will satisfy

$$Y = \lim_{t \to 0} \lambda(t)g \cdot X,$$

$$\rho_\infty(V_i) = \lim_{t \to 0} \lambda(t)g \cdot V_i.$$

Note that we do not necessarily have $\beta^\tau = \lim_{t \to 0} \lambda(t)g \cdot \alpha^\tau$, and that the one-parameter subgroup $\lambda(t)$ and $g$ may depend on the choice of $V_1, \ldots, V_K$. Note also that the vector field $w$ on $Y$ induced by the $\mathbb{C}^*$ subgroup $\lambda$ will stabilize $\rho_\infty(V'_i)$ for $1 \leq i \leq d$, and hence by Lemma 5.1 we have $w \in \mathfrak{g}_{Y,\xi,\beta^\tau}$.

It is important to choose $V_1, \ldots, V_K$ above correctly, and we will discuss how this is done shortly, but for the time being, let us assume we have a $\mathbb{C}^*$ subgroup $\lambda$ generated by a vector field $w$ commuting with $\xi$. Let $\theta_w$ denote the transverse Hamiltonian with respect to the radial function $r$. We will assume that $\theta_w$ is normalized so that

$$\int_Y \theta_we^{-\frac{1}{2}\beta^2} \omega^n = 0,$$

and write $\|w\| = \sup_Y |\theta_w|$. Note that any two choices of norm are equivalent on the finite dimensional space of holomorphic vector fields on $Y$ commuting with $\xi$. In addition we cannot have $\theta_w = 0$ on $Y$, unless $\lambda$ already acts trivially on $X$. To see this note that $\lambda$ induces a filtration on the coordinate
ring of $X$, and the coordinate ring of $Y$ is the associated graded ring, with
the action $\lambda$ on $Y$ being induced by the corresponding grading. If this
grading is trivial, then the original filtration must have been trivial.

As in [30], the idea is to use Proposition 3.10 to estimate the twisted
Futaki invariant of the test-configuration $\lambda(t)$ for $g \cdot X$. In order to apply
Proposition 3.10 we must ensure that our hypersurfaces avoid the set $E$,
which is a union two hyperplanes in $P$. On other other hand, if no $N + 1$
of our hypersurfaces $V_1, \ldots, V_K$ are on a hyperplane in $P$, then at most $2N$
of them can be contained in $E$. So once we choose $K$ very large compared
to $N$, then we get a good approximation to the twisted Futaki invariant by
replacing $g \cdot \alpha^\tau$ by the average of $g \cdot [V_i]$.

We now describe how to choose the $V_1, \ldots, V_K$. For simplicity of notation
we will suppose that $g$ is the identity. From (5.1) we have that on $Y$
\[
\beta^\tau = \frac{\pi}{M} \int_{P \setminus F} [\rho_{\infty}(V_\mu)] d\mu.
\]
In addition since $(Y, \xi, (1 - T)^{\beta^\tau})$ admits a weak solution of the twisted
equation, from Proposition 3.4 we have
\[
\text{Fut}_{Y, \xi, (1 - T)^{\beta^\tau}}(w) = 0
\]
for all $w \in g_{Y, \xi, \beta^\tau}$. By the formula for the twisted Futaki invariant in
Proposition 3.7 this means that
\[
\text{Fut}_{Y, \xi}(w) - c_n \frac{1 - T}{V} \int_Y \theta_\omega e^{-\frac{1}{2} r^2} \beta^\tau \wedge \omega^{n-1} = 0,
\]
and so if we define the function $h : P \setminus F \to \mathbb{R}$ to be
\[
h_w(\mu) = \frac{\pi}{M} \int_{\rho_{\infty}(V_\mu)} \theta_\omega e^{-\frac{1}{2} r^2} \omega^{n-1},
\]
then we have
\[
\text{Fut}_{Y, \xi}(w) = c_n \frac{1 - T}{V} \int_{P \setminus F} h_w(\mu) d\mu.
\]
Since the possible $w$ form a finite dimensional space, for any $\varepsilon > 0$ we can
choose a large $K_0$ with the following effect: for any $K \geq K_0$ we can find
hypersurfaces $V_1, \ldots, V_K$ with no $N + 1$ on a hyperplane in $P$, such that
\[
(5.2) \quad \text{Fut}_{Y, \xi}(w) \leq \varepsilon \|w\| + c_n \frac{1 - T}{V} \frac{1}{K} \sum_{i=1}^{K} \pi \frac{\pi}{M} \int_{\rho_{\infty}(V_i)} \theta_\omega e^{-\frac{1}{2} r^2} \omega^{n-1}.
\]
To see this more precisely, we choose a basis $w^1, \ldots, w^K$ for $g_{Y, \xi, \beta^\tau}$, and then
using Lusin’s theorem applied to each $h_{w^i}$ we can find a compact set $B \subset P \setminus F$ with arbitrarily small complemen, on which each $h_{w^i}$ is continuous.
We can then approximate the integrals on $B$ using Riemann sums over
discrete finite sets.
Assume $T < 1$. There is a constant $\delta > 0$ so that

$$\int_Y (\max_Y \theta_w - \theta_w) e^{-\frac{1}{2} \hat{r}^2} \omega^n > \delta \|w\|$$

since the left hand side is a norm on the finite dimensional vector space $g_{Y,\xi,\beta,\tau}$. Choose $0 < \varepsilon < 4^{-1} c(n) (1 - T) \delta$ where $c(n)$ is the constant from Proposition 3.10. Take $K$ sufficiently large as above, and so that $K > 2N\varepsilon^{-1}$. Using the hypersurfaces $V_1, \ldots, V_K$ we now find $\lambda$ as discussed before (and for simplicity assume $g$ is the identity). It follows that $\rho_\infty(V_i) = \lambda(t) \cdot V_i$.

We now use Proposition 3.10, which implies that in the sum in (5.2), for all but $2N$ of the integrals we have

$$\pi \int_{\rho_\infty(V_i)} \theta_w e^{-\frac{1}{2} \hat{r}^2} \omega^{n-1} = -c(n) \int_Y \max_Y \theta_w e^{-\frac{1}{2} \hat{r}^2} \omega^n.$$

From our choice of $K$ we have

$$\text{Fut}_{Y,\xi}(w) \leq 2\varepsilon \|w\| - c(n) \frac{1 - T}{V} \int_Y \max_Y \theta_w e^{-\frac{1}{2} \hat{r}^2} \omega^n$$

(5.3)

$$\leq -c(n) \frac{1 - T}{2} \max_Y \theta_w.$$

This is a contradiction if $(X, \xi)$ is $K$-stable.

If $T = 1$, then $(Y, \xi)$ admits a weak Ricci flat metric, and then as in [30] or Donaldson-Sun [41, Section 3.3], we obtain a test-configuration for $(X, \xi)$ with vanishing Futaki invariant.

5.2. The irregular case. Suppose now that $(X, \xi)$ is irregular, and let $\xi_k \to \xi$ be a sequence of normalized quasi-regular Reeb fields approximating $\xi$. Let $\alpha^\tau_k \to \alpha^\tau$ be a sequence of compatible transverse Kähler metrics, obtained by restricting suitable reference forms under an embedding $X \to \mathbf{C}^M$. Note that we may not be able to choose the $\alpha^\tau_k$ to be exactly of the form considered in the previous section. On the other hand it is easy to show in an identical way to the argument in [74] that if $(X, \xi, (1 - t)\alpha^\tau)$ admits a solution of the twisted equation, then so does $(X, \xi, (1 - t)\tilde{\alpha}^\tau)$ for any $\tilde{\alpha}^\tau$ in the same transverse cohomology class as $\alpha^\tau$.

For each $k$ we obtain a $T_k < 1$, such that we can solve

$$\text{Ric}(\omega^{(k)}_t) = 2n(1 - t)[\alpha^\tau_k - \omega^{(k),\tau}_t]$$

on $(X, \xi_k)$ for $t \in [0, T_k]$. Note that if we choose $\xi$ to be the Reeb field with minimal volume, then necessarily $T_k < 1$, since each $(X, \xi_k)$ will be strictly unstable if $\xi_k \neq \xi$. Indeed by [65] the Reeb field with minimal volume is unique and a Sasaki-Einstein metric can exist only for that Reeb field.

**Proposition 5.2.** If $(X, \xi)$ is $K$-stable, then $T_k \to 1$ as $k \to \infty$. 
Proof. Let us suppose that $\limsup T_k < 1$. From [5.3], for each $k$ we obtain a vector field $w_k$, inducing a test-configuration for $X$ with central fiber $Y_k$, such that
\begin{equation}
\text{Fut}_{Y_k, \xi_k}(w_k) \leq -\delta \max_{Y_k} \theta_{w_k},
\end{equation}
for some $\delta > 0$. Applying Theorem 4.7 to a diagonal sequence, we can assume that $Y_k \to Y$, and moreover that $Y$ is normal.

Our goal is to show that for sufficiently large $k$ we have
\[ \text{Fut}_{Y_k, \xi}(w_k) < 0, \]
which will contradict the K-stability of $(X, \xi)$. Recall that we have normalized each $\theta_{w_k}$ to have zero integral on $Y_k$. It is worth pointing out that the background metrics on $\mathbb{C}^N$ are also varying with $k$, and so the transverse Hamiltonians are all computed with different metrics, but these background metrics also converge as $k \to \infty$ to a metric for the limiting Reeb field $\xi$. In addition as discussed in Remark 3.9, the normalization of $w_k$ and $\max_{Y_k} \theta_{w_k}$ only depends on the induced action on $Y_k$, and not for instance on the background metric used to compute the Hamiltonians.

By (3.5) the Futaki invariant is
\[ \text{Fut}_{Y_k, \xi}(w_k) = \frac{\int_{Y_k} \theta_{w_k} e^{-\frac{1}{2} f_k^2} dV_k}{\int_{Y_k} e^{-\frac{1}{2} f_k^2} dV_k}, \]
where $dV_k$ is the canonical volume form on $Y_k$. Note that these can be scaled so that they converge to the canonical volume form on $Y$. In addition if we scale each $w_k$ so that $\text{osc}_{Y_k} \theta_{w_k} = 1$, then we can extract a limit $\theta_w$ on $Y$, corresponding to a vector field $w$. Note that $w$ may not be defined on all of $\mathbb{C}^N$ if $Y$ is contained in a hyperplane, since in this case our scaling might make some of the weights of $w_k$ become unbounded. However we must still have $\text{osc}_Y \theta_w = 1$.

From our normalization this implies a positive lower bound for $\max_Y \theta_w$ and hence a uniform positive lower bound for $\max_{Y_k} \theta_{w_k}$. From (5.4) we obtain
\[ \text{Fut}_{Y_k, \xi_k}(w_k) \leq -\delta'. \]
Now the required result follows, since both $\text{Fut}_{Y_k, \xi_k}(w_k)$ and $\text{Fut}_{Y, \xi}(w)$ converge to $\text{Fut}_{Y, \xi}(w)$, which we have just seen must be negative. \[ \square \]

We can now choose metrics $\omega_{\tau_k}^{(k)}$ on $(X, \xi_k)$, satisfying
\[ \text{Ric}(\omega_{\tau_k}^{(k)}) = 2n(1 - t_k)\alpha_{\tau_k} - \omega_{\tau_k}^{(k)} \tau, \]
with $t_k = T_k - k^{-1}$. We can apply the same arguments as above to the sequence $(X, \omega_{\tau_k}^{(k)})$, to obtain a normal limit space $Y$, which admits a weak Ricci flat metric, since $t_k \to 1$. As before, we can then realize $Y$ as the central fiber of a test-configuration for $X$, contradicting the assumption that $(X, \xi)$ is K-stable, unless $Y \cong X$. But if $Y \cong X$, then we have obtained the desired Ricci flat Kähler cone metric on $X$. 


6. The algebraic Futaki invariant

In this section we collect some results of a more algebraic nature. First we will consider the normalization (or gauge fixing) condition for a Fano cone singularity \((X, T, \xi)\), and show that normalized Reeb fields \(\xi\) form a linear subspace of the Lie algebra of \(T\). We then discuss the relation between the algebro-geometric definition of the Futaki invariant in Definition 2.2 and the differential geometric definition in Equation 3.5. The analogous result in the smooth projective case was shown by Donaldson [38, Proposition 2.2.2], using the equivariant Riemann-Roch formula. Here our approach is more algebraic in order to avoid having to resolve possible singularities.

6.1. The Gauge Fixing Condition. We begin with a lemma alluded to in Section 2.

Lemma 6.1. Suppose \((X, T, \xi)\) is \(\mathbb{Q}\)-Gorenstein with an isolated singularity at the origin. Suppose \(\Omega \in \Gamma(X, mK_X)\) is a non-vanishing section with \(L_\xi \Omega = i\lambda \Omega\) for some \(\lambda \in \mathbb{R}\). Then \(X\) has log-terminal singularities at 0 if and only if \(\lambda > 0\).

Proof. Define a volume form \(dV = i^n a (\Omega \wedge \overline{\Omega})^{1/m}\). By [43, Lemma 6.4] it suffices to determine conditions for \(dV\) to have finite volume in a neighborhood of 0. Let \(r\) be a radial function for \(\xi\). Using the flow by \(-J_\xi\), we can write

\[
\int_{\{2^{-k} \leq r < 2^{-(k-1)}\}} dV = e^{-\lambda k \log 2} \int_{\{\frac{1}{2} \leq r < 1\}} dV
\]

and so we get

\[
\int_{\{0 < r < 1\}} dV = \sum_{k=0}^{\infty} e^{-\lambda k \log 2} \int_{\{\frac{1}{2} \leq r < 1\}} dV
\]

which proves the lemma. \(\square\)

Note that the proof implies slightly more. Namely, that if \((X, T, \xi)\) is a normal, \(\mathbb{Q}\)-Gorenstein, with \(\Omega\) as above, then \(X\) is log terminal if and only if \(\lambda > 0\), and \(X\) is log-terminal at all points away from the origin.

Suppose we have an \(n\)-dimensional polarized affine variety \((X, T, \xi)\) which is a Fano cone singularity, so \(X\) is normal \(\mathbb{Q}\)-Gorenstein with log-terminal singularities. In addition suppose that \(m \in \mathbb{N}\) is minimal such that \(mK_X\) is trivial. Let \(R\) be the coordinate ring of \(X\), which is an integral domain since \(X\) is a variety. Furthermore, since \(X\) has log-terminal singularities, it is known that \(R\) is Cohen-Macaulay. When \(X\) is Gorenstein with an isolated singularity at the cone point, Martelli-Sarkis-Yau [65] showed the existence of a \(T\)-equivariant trivialization of \(K_X\). Their argument applies verbatim to the singular, \(\mathbb{Q}\)-Gorenstein case. We include the short proof for the reader’s convenience.
**Lemma 6.2.** There exists a unique, up to scale, non-vanishing section $\Omega \in \Gamma(X, mK_X)$, and a unique linear function $\ell : \mathcal{C}_R \to \mathbb{R}_{>0}$ with the property that, for any Reeb field $\xi \in \mathcal{C}_R$ we have

$$L_{\xi} \Omega = \ell(\xi) \Omega,$$

In particular, for any $c > 0$ the set $\{ \xi \in \mathcal{C}_R : \ell(\xi) = c \}$ defines an affine hyperplane in $t$ intersecting $\mathcal{C}_R$ in a set of codimension 1.

**Proof.** Fix a non-vanishing section $\Omega \in \Gamma(X, mK_X)$, and $\xi \in \mathcal{C}_R$. Since $mK_X$ is trivial we have

$$L_{\xi} \Omega = k(z) \Omega$$

for some function $k(z) \in \mathbb{R}$, the coordinate ring of $X$. We decompose $k(z)$ into its graded pieces according to the grading defined by $\xi$:

$$k(z) = \sum_{\alpha \in t^*} k_\alpha(z).$$

This decomposition converges in $L^2_{loc}$ and therefore also locally uniformly. Indeed, we can restrict $k$ to the link of $X$, and consider its weight decomposition in $L^2$ for the torus action.

Since $X$ has log-terminal singularities, we can assume that $k_0 \in \mathbb{R}_{>0}$. We project to the degree zero part to conclude. Explicitly, define

$$f(z) = \sum_{\alpha \in t^* - \{0\}} \frac{1}{\alpha(\xi)} k_\alpha(z)$$

then $\tilde{\Omega} := e^{-f(z)} \Omega$ satisfies our requirements. The last two claims are clear from the linearity of the projection and the positivity of $k_0$. The uniqueness follows from the fact that $\xi$ induces a positive grading, and hence the only homogeneous, holomorphic, non-vanishing holomorphic functions on $X$ are constant. $\square$

The above discussion makes it possible to introduce the gauge fixing condition, as in Martelli-Sparks-Yau [65].

**Definition 6.3.** We say that $\xi \in \mathcal{C}_R$ satisfies the gauge fixing condition, (or is normalized) if

$$L_{\xi} \Omega = i m \Omega$$

where $\Omega$ is as in Lemma 6.2.

An important point for us is that the gauge fixing condition can in fact be read off from the index character. This observation is implicit in the work of Martelli-Sparks-Yau [65] when $X$ is Gorenstein with an isolated log-terminal singularity. We now extend this to the case of general $\mathbb{Q}$-Gorenstein affine varieties with log-terminal singularities. To this end, fix a trivializing section $\Omega \in \Gamma(X, mK_X)$ and suppose that $L_{\xi} \Omega = i \lambda \Omega$ for some $\lambda > 0$. Then we have an isomorphism of graded $R$-modules

$$R(-\lambda) \simeq \Gamma(X, mK_X)$$
by the map \( f \mapsto f\Omega \). The index character of the ring \( R \), denoted \( F_R(\xi, t) \), expands as a Laurent series

\[
F_R(\xi, t) = \frac{a_0(\xi)(n-1)!}{t^n} + \frac{a_1(\xi)(n-2)!}{t^{n-1}} + O(t^{2-n}).
\]

We have the following

**Proposition 6.4.** In the above setting, the coefficients \( a_0(\xi), a_1(\xi) \) satisfy

\[
a_1(\xi) = \frac{\lambda(n-1)}{2m}.
\]

In particular, the set \( \{ \xi \in C_R : 2a_1(\xi) = n(n-1)a_0(\xi) \} \) is an affine subset of \( C_R \) with codimension 1, which agrees with the normalized Reeb fields.

**Proof.** The proof is essentially a computation in commutative algebra. Recall that a \( \mathbb{Q} \)-Gorenstein ring \( R \) with log-terminal singularities is Cohen-Macaulay. The main tool in the proof is a duality equation due to Stanley [72], [18, Corollary 4.4.6], which says that if \( R \) is a Cohen-Macaulay, positively graded \( \mathbb{C} \)-algebra of dimension \( n \) with canonical module \( \Omega_R \), then the Hilbert series satisfies

\[
H_{\Omega_R}(s) = (-1)^n H_R(s^{-1})
\]
as rational functions, or in terms of the index character we have

\[
F_{\Omega_R}(t) = (-1)^n F_R(-t).
\]

(6.2)

Roughly speaking this is a form of Serre duality. Let us first explain the proof in the easier case that \( R \) is Gorenstein, so that \( m = 1 \). Then the isomorphism in (6.1) becomes

\[
R(-\lambda) \cong \Omega_R,
\]

and so, in particular \( F_{\Omega_R}(t) = e^{-\lambda t} F_R(t) \). Combining this with (6.2) gives

\[
F_R(t) = (-1)^n e^{\lambda t} F_R(-t).
\]

In terms of the index character this implies

\[
\frac{a_0(n-1)!}{t^n} + \frac{a_1(n-2)!}{t^{n-1}} = (1 + \lambda t) \left( \frac{a_0(n-1)!}{t^n} - \frac{a_1(n-2)!}{t^{n-1}} \right) + O(t^{2-n}).
\]

Comparing coefficients we get that

\[
2a_1(n-2)! = \lambda a_0(n-1)!
\]

which proves the proposition in the Gorenstein case.

We now consider the case when \( X \) is a \( \mathbb{Q} \)-Gorenstein. Since the coordinate ring \( R \) is a Cohen-Macaulay integral domain, [18, Proposition 3.3.18] says that there is a homogeneous ideal \( I \subset R \) such that \( \Omega_R \cong I \) as \( R \)-modules. In the current case \( X \) is normal and affine, so the canonical sheaf is given by \( K_X = i_* K_U \), where \( i : U = X_{reg} \hookrightarrow X \) and \( K_U \) is the canonical sheaf of \( U \). Then

\[
\Omega_R = \Gamma(X, K_X)
\]
as $R$-modules. Thanks to the fact that $X$ is $\mathbb{Q}$-Gorenstein we have

$$I^{(m)} \simeq R \cdot \sigma \simeq R(−\lambda)$$

where $\sigma$ is the trivializing section of $\Gamma(X, mK_X)$, and $I^{(m)}$ denotes the $m$–th symbolic power of $I$. Since $I$ is a reflexive $R$-module of rank one, the symbolic power may be defined as

$$I^{(m)} = \left(I \otimes_R I \cdots \otimes_R I\right)^{**} = (I^{\otimes m})^{**},$$

where if $M$ is an $R$-module, then $M^* = \text{Hom}_R(M, R)$ denotes the dual.

Geometrically, we have an exact sequence of sheaves on $X$

$$0 \to K \to K \otimes m_X \to (K \otimes m_X)^{**} \to Q \to 0.$$

Since $K_X$ is reflexive and locally free on $X_{reg}$, the sheaves $K, Q$ are supported on a subvariety of codimension at least 2. By Serre’s criterion for affineness, we have $H^1(X, F) = 0$ for any coherent sheaf $F$. In particular, by taking global sections, using [51, Proposition 5.2] we get an exact sequence of graded $R$-modules

$$0 \to K := \Gamma(X, K) \to I^{\otimes m} \to I^{(m)} \to Q := \Gamma(X, Q) \to 0.$$ 

On the level of Hilbert series this implies that

$$H_{I^{(m)}}(s) = H_{I^{\otimes m}}(s) + H_{Q}(s) - H_{K}(s).$$

Since $Q, K$ are supported in codimension 2, their associated index characters satisfy

$$F_Q(t) = O(t^{2-n}), \quad F_K(t) = O(t^{2-n}).$$

We now consider the index character (or Hilbert series) of $I^{\otimes m}$. We need the following lemma.

**Lemma 6.5.** Suppose that $M, N$ are graded $R$-modules, with $M$ free in codimension one. Then

$$F_{M \otimes_R N}(t) = \frac{F_M(t)F_N(t)}{F_R(t)} + O(t^{2-n}).$$

**Proof.** Let us take a free resolution of $N$, i.e. a complex

$$0 \to E_k \to E_{k-1} \to \ldots \to E_0 \to 0,$$

whose only cohomology is $H^0 = N$. Tensoring with $M$ we obtain a complex

$$0 \to M \otimes_R E_k \to \ldots \to M \otimes_R E_0 \to 0,$$

whose cohomology is $H^i = \text{Tor}^R_i(M, N)$. It is easy to check that the alternating sum of index characters of a complex is the same as the alternating sum of the index characters of its cohomology, i.e. we have

$$\sum_{i=0}^k (-1)^i F_{M \otimes_R E_i}(t) = \sum_{i=0}^k (-1)^i F_{\text{Tor}^R_i(M, N)}(t).$$
Since $M$ is free in codimension 1, we have that $\text{Tor}_R^i(M, N)$ is supported in codimension 2 for $i > 0$, i.e. its index character is of order $t^{2-n}$. It follows that
\[
F_{\otimes R N}(t) = \sum_{i=0}^{k} (-1)^i \frac{F_M(t)F_{E_i}(t)}{F_R(t)} + O(t^{2-n})
\]
\[
= \frac{F_M(t)}{F_R(t)} \sum_{i=0}^{k} (-1)^i F_{E_i}(t) + O(t^{2-n})
\]
\[
= \frac{F_M(t)F_N(t)}{F_R(t)} + O(t^{2-n}),
\]
which is what we wanted to prove. \(\square\)

From this lemma a simple induction gives
\[
F_I \otimes m(t) = F_{\Omega_R}(t)^m m F_R(t)^{m-1} + O(t^{2-n}).
\]
As remarked earlier, the $\mathbb{Q}$-Gorenstein assumption implies that $I^{(m)} \simeq R(-\lambda)$, and so
\[
e^{-t\lambda} F_R(t) = \frac{F_{\Omega_R}(t)^m}{F_R(t)^{m-1}} + O(t^{2-n}).
\]
Expanding this equation to order $t^{1-n}$ we obtain
\[
\frac{a_1}{a_0} = \frac{\lambda(n-1)}{2m},
\]
as required. \(\square\)

6.2. **The Futaki invariant.** Suppose that $(X, \xi)$ is a normalized Fano cone singularity and we have a test-configuration $\lambda$ for $X$ with central fiber $Y$. Recall that $\lambda$ gives a $\mathbb{C}^*$-action on $Y$ generated by a vector field $w$. We have given two versions of the Futaki invariant of this test-configuration. One was purely algebraic in terms of the weights of the action $\lambda$ on the coordinate ring of $Y$, in Definition 2.2. As discussed below that definition, for small $s$ we can consider the Reeb field $\xi + sw$ on $Y$, and we can normalize the test-configuration (i.e. modify the vector field $w$ by adding a multiple of $\xi$) in such a way that
\[
\frac{a_1(Y, \xi + sw)}{a_0(Y, \xi + sw)} = \frac{n(n-1)}{2}.
\]
Under this normalization the Futaki invariant is given by
\[
\text{Fut}(Y, \xi, w) = \frac{1}{2} D_w a_0(Y, \xi).
\]
At the same time, from Proposition 6.4 we see that this normalization is equivalent to requiring $L_w dV = 0$, where $dV$ is the canonical volume form.
on $Y$. The differential geometric definition of the Futaki invariant in that case is given in Equation 3.6 by

\[(6.5) \quad \text{Fut}_Y(\xi)(w) = \frac{1}{V} \int_Y \theta_w e^{-\frac{1}{2} r^2} \omega^n,\]

where $\omega = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} r^2$ is a suitable reference metric with Reeb field $\xi$, the function $\theta_w$ is the transverse Hamiltonian of $w$ and $V$ is the volume of $(Y, \xi)$. To relate this to the algebraic definition we have the following two results.

**Proposition 6.6.** Suppose $X \subset \mathbb{C}^N$ is a polarized affine variety with Reeb field $\xi$, which has weights $w_i$ on the coordinates $z_i$, i.e. $\xi$ is the imaginary part of $\sum_{i=1}^{\infty} w_i z_i \partial / \partial z_i$.

Let

\[\hat{r}^2 = \sum_{i=1}^{N} |z_i|^{2/w_i},\]

and $\omega = \frac{1}{2} \sqrt{-1} \partial \bar{\partial} \hat{r}^2$. We have

\[(6.6) \quad a_0(n-1)! = \frac{1}{(2\pi)^n} \int_X e^{-\frac{1}{2} \hat{r}^2} \omega^n / n!,\]

where $a_0$ is defined by the index character

\[F(\xi, t) = \frac{a_0(n-1)!}{t^n} + O(t^{-n+1}).\]

**Proof.** We can choose a generic $\mathbb{C}^*$-action $\lambda(t)$ commuting with $\xi$, and degenerate $X$ to $Y = \lim_{t \to 0} \lambda(t) \cdot X$. This will not affect the integral in (6.6), since the integral over $\lambda(t) \cdot X$ is the same as the integral over $X$ using a different metric. The volume, however, is a function of just the Reeb field. At the same time the index character is also unchanged in passing to the limit since $Y$ is a flat limit.

For a generic $\mathbb{C}^*$-action the top dimensional part of $Y$ is a union of $n$-dimensional coordinate subspaces with multiplicity. The leading term in the index character will be the sum of the corresponding terms for these subspaces, with multiplicity, and the limiting integral on $Y$ is also given by a corresponding sum. As such, we only need to check the formula on $\mathbb{C}^n$ for a given Reeb field. But both the index character and the integral is multiplicative when taking products of varieties, so it is enough to do the calculation for $\mathbb{C}$, with a Reeb field

\[\xi = \text{Im} \left( wz \partial / \partial z \right),\]
with corresponding radial function \( r^2 = |z|^{2/w} \). The metric is then
\[
\omega = \frac{i}{2} \cdot \frac{1}{w^2} |z|^{2/w - 2} dz \wedge d\bar{z},
\]
so a calculation gives
\[
\int_C e^{-\frac{i}{2} r^2} \omega = \frac{2\pi}{w}.
\]
At the same time the index character is
\[
\sum_{k=0}^{\infty} e^{-tkw} = \frac{1}{1 - e^{-tw}} = \frac{1}{tw} + O(1),
\]
from which the result follows.

We also have the following formula for the variation of the volume as we vary the Reeb field.

**Proposition 6.7.** Suppose that we consider a variation \( \delta \xi = w \) of the Reeb field. The corresponding variation in the volume
\[
V(\xi) = \int_X e^{-\frac{i}{2} r^2} \omega^n/n!
\]
is given by
\[
\delta V(\xi) = n \int_X \theta w e^{-\frac{i}{2} r^2} \omega^n/n!.
\]

**Proof.** This result was shown by Martelli-Sparks-Yau [65], see also Donaldson-Sun [41]. In comparing these formulas recall that by our convention \( \theta \xi = -1 \). The result and its proof are valid even if \( X \) is not normal, interpreting the integral as just a sum of integrals on the \( n \)-dimensional components of \( X \), with multiplicity.

Using these results we can now compare the definitions (6.4) and (6.5) to see that the two Futaki invariants agree up to a dimensional constant, obtaining the following.

**Proposition 6.8.** We have \( \text{Fut}_{X,\xi}(w) = c(n)\text{Fut}(X,\xi,\lambda) \) in terms of Definition 2.2, where \( \lambda \) is the \( \mathbb{C}^* \)-action generated by \( w \), and \( c(n) > 0 \) is a dimensional constant.

7. K-stability of affine varieties with Ricci-flat Kähler cone metrics

The main theorem of this section is

**Theorem 7.1.** Suppose that \( (X,\xi) \) admits a Ricci-flat Kähler cone metric. Then \( (X,\xi) \) is K-stable.
The idea of the proof follows work of Berman [6], and goes as follows. First, we will show that the Ding functional is convex along (sub)-geodesics. Geodesics in the space of Sasakian metrics have been studied by Guan-Zhang [49], but we will need a slightly different formulation than the one given there. Since the Ricci-flat Kähler cone metric is a critical point of the Ding functional, the strict convexity along geodesics means that, along a (sub)-geodesic \( \varphi_s \) emanating from a Sasaki-Einstein potential \( \varphi_{SE} \) we have

\[
\frac{d}{ds} D(\varphi_s) \geq 0
\]

and the limit slope \( \lim_{s \to \infty} \frac{d}{ds} D(\varphi_s) \) exists in \([0, \infty]\). Furthermore, a result of Berndtsson [10] says that the limit must be strictly positive unless the geodesic was generated by a real holomorphic vector field (see [41] for the generalization of Berndtsson’s result to our setting). Next, we show that any special degeneration gives rise to a (sub)-geodesic, and we show that the limit slope is precisely the Futaki invariant.

Let \((X, \xi)\) be a polarized cone, which we assume is \(Q\)-Gorenstein and log-terminal. Recall that we have defined the Ding functional

\[
D(\varphi) = -E(\varphi) - \frac{1}{2n} \log \int_X e^{-\frac{1}{2} r^2} \omega^n
\]

where \(dV = (\Omega \wedge \Omega')^{1/m} \) and \(\Omega\) is a \(T\)-equivariant trivialization of \(mK_X\), \(r : X \to \mathbb{R}_+\) is a radial function compatible with \(\xi\), and \(r\varphi = e^{\varphi} r\) where \(\varphi\) is basic, and independent of \(r\). As before, the function \(E\) is defined by its variation

\[
\delta E(\varphi) = \frac{1}{V(\xi)} \int_X \varphi e^{-\frac{1}{2} r^2} \omega^n
\]

Our goal is to compute the second variation of \(E(\varphi_s)\). For our computation, we will assume that \(\varphi_s\) is a smooth variation. Dropping the \(V(\xi)\) term for convenience an easy computation shows that

\[
-\frac{d}{ds} E(\varphi) = \frac{1}{2n} \int_X \varphi r^2 e^{-\frac{1}{2} r^2} \omega^n = \frac{1}{2n} \int_X \left( \frac{r^2}{2} \right) e^{-\frac{1}{2} r^2} \omega^n
\]

Let us suppress the dependence on \(\varphi\) to ease notation. Then we have

\[
-2^n + n \frac{d^2}{ds^2} E = \int_X \left[ \frac{\dot{r}^2}{2} - \frac{(\ddot{r}^2)^2}{4} \right] e^{-\frac{1}{2} r^2} (\sqrt{-1 \partial \bar{\partial} r^2})^n + n \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1 \partial \bar{\partial} r^2} \wedge (\sqrt{-1 \partial \bar{\partial} r^2})^{n-1}.
\]
Let us manipulate the last term. Integrating by parts gives
\[
\begin{align*}
\int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
= -n \int_X e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
+ \frac{n}{2} \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1}
\end{align*}
\]
We focus now on the second term of this expression. Integration by parts gives
\[
\begin{align*}
\frac{n}{2} \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
= \frac{n}{2} \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
- \frac{n}{4} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
+ \frac{n}{2} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
= -n \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
+ \frac{n}{4} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
- \frac{n}{2} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} (\sqrt{-1} \partial \partial r^2)^{n}.
\end{align*}
\]
Thus, we get
\[
\frac{n}{2} \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} \\
= \frac{n}{8} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} - \frac{n}{4} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} (\sqrt{-1} \partial \partial r^2)^n
\]
Now, since \(\omega^r = \sqrt{-1} \partial \partial \log r^2\) satisfies \((\omega^r)^n = 0\), a direct computation shows that
\[
\partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} = \frac{r^2}{n} (\sqrt{-1} \partial \partial r^2)^{n}.
\]
From this observation, an easy computation shows that
\[
\frac{n}{8} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} = \frac{1}{8} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} r^2 (\sqrt{-1} \partial \partial r^2)^n \\
= \frac{n+2}{4} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} (\sqrt{-1} \partial \partial r^2)^n,
\]
and hence
\[
\frac{n}{2} \int_X r^2 e^{-\frac{1}{2} r^2} \sqrt{-1} \partial r^2 \wedge (\sqrt{-1} \partial \partial r^2)^{n-1} = \frac{1}{2} \int_X (r^2)^2 e^{-\frac{1}{2} r^2} (\sqrt{-1} \partial \partial r^2)^n.
\]
Plugging this back into the equation (7.1) we get
\[ -2^{n+2} n \frac{d^2}{ds^2} E = \int_X e^{-\frac{1}{2} r^2} [\sqrt{-1} \partial \bar{\partial} r^2 - n \sqrt{-1} \partial r^2 \wedge \bar{\partial} r^2] \wedge (\sqrt{-1} \partial \bar{\partial} r^2)^{n-1}. \]

A geodesic is a path \( \{ \varphi_s \} \) along which the function \( E \) is affine. In particular, we can write the geodesic equation as
\[ [\sqrt{-1} \partial \bar{\partial} r^2 - n \sqrt{-1} \partial r^2 \wedge \bar{\partial} r^2] \wedge (\sqrt{-1} \partial \bar{\partial} r^2)^{n-1} = 0. \]

We will also make use of subgeodesics, along which the functional \( E \) is concave; that is, paths \( \varphi_s \) satisfying
\[ [\sqrt{-1} \partial \bar{\partial} r^2 - n \sqrt{-1} \partial r^2 \wedge \bar{\partial} r^2] \wedge (\sqrt{-1} \partial \bar{\partial} r^2)^{n-1} \geq 0. \]

Another standard computation shows that if we introduce a holomorphic coordinate \( \tau \) with \( |\tau| = e^{-s} \), then the (sub)-geodesic equation can be written as
\[ (\sqrt{-1} \partial \bar{\partial} r^2)^{n+1} = 0 \]
where now \( D, \bar{D} \) denote the \( \partial, \bar{\partial} \) operators in the variables \( \tau, z \) jointly. In particular, a subgeodesic is nothing but a family of radial functions \( r(z, \tau) : X \to \mathbb{R}_+ \), all compatible with \( \xi \), and such that \( \sqrt{-1} \partial \bar{\partial} r^2 \geq 0 \).

Furthermore, the above description of the geodesic equation makes it clear that we can produce (weak) geodesics using the standard techniques of envelopes and subsolutions, but we will not need this here.

In order to obtain the convexity of the Ding functional, we also need the convexity of the second term in its definition. Berndtsson’s theorem gives this and even more; we refer the reader to [1] for an extension of Berndtsson’s theorem to our setting.

**Proposition 7.2.** Let \( r(x, \tau) : X \times \Delta^* \to \mathbb{R}_{>0} \) be a path of radial functions compatible with \( \xi \), and \( S^1 \) invariant. Suppose that \( \sqrt{-1} \partial \bar{\partial} r \geq 0 \), and that \( D(r_s) \) is affine \( (s = -\log |\tau|) \). Then there exists a holomorphic vector field \( \xi \) on \( X \), commuting with \( \xi, r \partial_r \) so that \( r_s = F_s^* r(x, 0) \) where
\[ F_s = \exp(s \text{Re}(\xi)). \]

We now explain how a special degeneration gives rise to a subgeodesic. Let \( T \subset \text{Aut}(X) \) be a torus containing \( \xi \). Recall that a special degeneration consists of an embedding \( X \to \mathbb{C}^N \), which we may assume is not contained in a linear subspace, and such that \( T \subset \text{Aut}(X) \) acts linearly and diagonally through and embedding \( T \subset U(N) \), together with a one-parameter subgroup \( \lambda : \mathbb{C}^* \to GL(N)^T \) commuting with \( T \), and such that \( \lambda(S^1) \subset U(N) \), and \( Y = \lim_{t \to 0} \lambda(t) \cdot X \) is normal. We may package this as an affine scheme \( \mathcal{Y} \subset \mathbb{C}^N \times \mathbb{C} \), together with a \( \mathbb{C}^* \) equivariant projection
\[ \pi : \mathcal{Y} \to \mathbb{C} \]
where the \( \mathbb{C}^* \) action is by \( \lambda \); we will usually restrict our attention to \( \pi^{-1}(\Delta) \) where \( \Delta \) is the closed unit disk. Abusing notation, we will also denote this by \( \mathcal{Y} \). By definition \( \xi \in u(N) \) induces a Reeb vector field on \( \mathbb{C}^N \), and hence
we may find \( r_0 : \mathbb{C}^N \to \mathbb{R}_+ \), a \( U(N) \) invariant radial function compatible with \( \xi \). Let \( p_1 : \mathbb{C}^N \times \mathbb{C} \to \mathbb{C}^N \), and consider \( p_1^* r_0 : \mathbb{C}^N \times \mathbb{C} \to \mathbb{R}_+ \). The \( \mathbb{C}^* \) action allows us to identify \( \mathcal{Y}^* := \pi^{-1}(\Delta^*) \) with \( X \times \Delta^* \), and hence \( p_1^* r_0 \) induces a radial function \( r(\tau) \) compatible with \( \xi \) on each fiber of \( X \times \Delta^* \).

By the \( U(N) \) invariance of \( r_0 \), the function \( r(\tau) \) is \( S^1 \)-invariant, and since the map \( X \times \Delta^* \to \mathcal{Y}^* \) is holomorphic we have

\[
\sqrt{-1} D \bar{D} r(\tau) \geq 0.
\]

We can therefore write

\[
r(\tau) = r_0 e^{\psi(|\tau|)}.
\]

Let \( \varphi \) be any potential of a radial function on \( X \). By taking \( A, C \) large and \( \varepsilon \) small we define

\[
\Phi(z, \tau) = \Phi(z, |\tau|) := \begin{cases} \max \{ \varphi(z) + A \log |\tau|, \psi(z, |\tau|) - C \} & |\tau| \geq \frac{1}{2} \\
\psi(z, |\tau|) - C & |\tau| \leq \frac{1}{2} \end{cases}
\]

where \( \max \) is the regularized maximum \( [33, \text{Section 5.E.}] \). A few words are in order about how to choose \( A, C \). First, we choose \( C \gg 1 \) large so that \( \psi(z, \tau) - C < \varphi(z) - 100 \) on \( X \times \{1\} \). Next we choose \( A \) large so that

\[
\varphi(z) + A \log |\tau| \leq \psi(z, |\tau|) - C - 100
\]

for all \( \frac{1}{2} < |\tau| < \frac{3}{4} \), and finally choose \( 0 < \varepsilon \ll 1 \). Clearly \( A, C \) exist since \( \varphi, \psi(\tau) \) are smooth on \( X \), and uniformly bounded for \( \tau \) in any compact subset of \( \Delta^* \). By our choices and the properties of the regularized maximum we obtain that for every \( \tau \in \Delta^* \), \( r_{\Phi} := r e^{\Phi(\tau)} \) defines a smooth radial function on \( X \) compatible with \( \xi \), and furthermore, \( r e^{\Phi(\tau)} \) is plurisubharmonic on \( X \times \Delta^* \). In particular, \( \Phi(\tau) \) defines a subgeodesic emanating from \( \varphi \).

We now compute the limit slope of the Ding functional along \( \Phi(\tau) \). Note that \( \Phi(\tau) \) for \( |\tau| < \frac{1}{4} \) depends only on \( r_0 \) and the test configuration, and not on the initial data. Since we are computing the limit

\[
\lim_{s \to \infty} \frac{d}{ds} D(\Phi(e^{-s}))
\]

we can assume that in fact \( \Phi(\tau) = \psi(\tau) \). Before proceeding, we need a preliminary lemma.

**Lemma 7.3.** The total space of the special degeneration \( \mathcal{Y} \) is a polarized cone. In particular, it is \( \mathbb{Q} \)-Gorenstein, with log-terminal singularities and admits a Reeb vector field.

**Proof.** That \( \mathcal{Y} \) is \( \mathbb{Q} \)-Gorenstein and log-terminal follow from the fact that \( \mathcal{Y} \) is flat over \( \mathbb{C} \) and every fiber is normal, \( \mathbb{Q} \)-Gorenstein with log-terminal singularities. We only need to show that \( \mathcal{Y} \) has a Reeb vector field. Let \( \eta \) be the generator of the \( \mathbb{C}^* \) action defining the test configuration. By definition \( \eta \) acts on the coordinate \( t \) on \( \mathbb{C} \) with weight one, and commutes with \( \xi \). Hence for \( s \) sufficiently small, \( \xi + s \eta \) is a Reeb vector field for \( \mathcal{Y} \). \( \square \)
Let \( T' \) be the torus in \( \text{Aut}(\mathcal{Y}) \) containing \( T \), and \( \eta \), the generator of the \( \mathbb{C}^* \) action defining the special degeneration. It follows from Lemma 6.2 that we can choose \( \hat{\Omega} \) a \( T' \)-equivariant trivializing section of \( mK_\mathcal{Y} \) so that 
\[
\tau \in \mathbb{C} 
\]
where \( X_t = \pi^{-1}(t) \). By the uniqueness part of Lemma 6.2 we must have
\[
(\lambda^{-1}(\tau))^* \Omega = c(\tau) t \frac{d}{d\tau} \hat{\Omega}
\]
where \( c(\tau) \) is non-vanishing holomorphic function, constant on the fibers. In particular, on the level of volume forms we have
\[
(\lambda^{-1}(\tau))^* dV = |c(\tau)|^{2/m} dV_{X_t} := |c(\tau)|^{2/m} \left( t \frac{d}{d\tau} \hat{\Omega} \wedge t \frac{d}{d\tau} \hat{\Omega} \right).
\]
We now compute the limit slope of the Ding functional. As explained above, it suffices to compute the limit slope of \( D(r_s) \), where \( s = -\log |\tau| \), and \( r_s = \lambda(e^{-s})^* p_1 r_0 \). Recall that
\[
D(r_s) = -E(r_s) - \frac{1}{2n} \log \int_X e^{-\frac{1}{2} r_s^2} dV.
\]
Let us first focus on the \( E \) term. By definition we have
\[
-\frac{d}{ds} E(r_s) = -\frac{1}{V(\xi)} \int_X \frac{d}{ds} \log \left( \frac{r_s}{r_0} \right) e^{-\frac{1}{2} r_s^2} \omega^n
\]
We now use the biholomorphism \( \lambda(e^{-s}) \) to push the integral forward to \( X_{e^{-s}} \). Note that
\[
\lambda(e^{-s}) \frac{d}{ds} \log \frac{\lambda(e^{-s})^* r_0}{r_0} = -\theta_\lambda
\]
where \( \theta_\lambda \) is the Hamiltonian function, with respect to \( r_0 \), of the real part of the holomorphic vector field generating the action of \( \lambda \) on \( \mathbb{C}^N \). Thus we have
\[
-\frac{d}{ds} E(r_s) = \frac{1}{V(\xi)} \int_{X_{e^{-s}}} \theta_\lambda e^{-\frac{1}{2} r_s^2} \omega^n.
\]
Since \( \mathcal{Y} \) is flat over \( \mathbb{C} \), the current of integration \([X_{e^{-s}}]\) converges to \([X_0]\) weakly and we obtain
\[
\lim_{s \to \infty} -\frac{d}{ds} E(r_s) = \frac{1}{V(\xi)} \int_{X_0} \theta_\lambda e^{-\frac{1}{2} r_s^2} \omega^n,
\]
which is justified by the weak convergence since \( \theta_\lambda e^{-\frac{1}{2} r_s^2} \omega^n \) is a smooth \((n,n)\) form defined on the ambient space \( \mathbb{C}^N \).

We now compute the contribution of the second term. Specifically, we are computing
\[
-\frac{d}{ds} \frac{1}{2n} \log \int_X e^{-\frac{1}{2} r_s^2} dV
\]
\[
= \frac{1}{2n} \frac{\int_X r_s^2 \frac{d}{ds} \log (r_s) e^{-\frac{1}{2} r_s^2} dV}{\int_X e^{-\frac{1}{2} r_s^2} dV}
\]
Pulling this back to $X_{\tau}$ and using that $(\lambda^{-1}(\tau))^*dV = |c(\tau)|^{2/m}dV_{X_{\tau}}$, we get

$$-\frac{d}{ds} \frac{1}{2n} \log \int_X e^{-\frac{1}{2}r^2} dV = \frac{1}{2n} \int_{X_{\tau}} r^2 \theta_{\lambda} e^{-\frac{1}{2}r^2} dV_{\tau}.$$ 

Since that $L_{\xi}dV_{\tau} = \sqrt{-1}2ndV_{\tau}$ and $\theta_{\lambda}$ is basic we obtain

$$\int_{X_{\tau}} r^2 \theta_{\lambda} e^{-\frac{1}{2}r^2} dV_{\tau} = 2n \int_{X_{\tau}} \theta_{\lambda} e^{-\frac{1}{2}r^2} dV_{\tau}.$$ 

Putting everything together we have

$$-\frac{d}{ds} \frac{1}{2n} \log \int_X e^{-\frac{1}{2}r^2} dV = \frac{1}{2n} \int_{X_{\tau}} \theta_{\lambda} e^{-\frac{1}{2}r^2} dV_{\tau}.$$ 

Now, by definition $\Omega$ is a non-vanishing, holomorphic section of $mK_{Y}$, which is in particular smooth on $Y_{reg}$. Thanks to the fact that $X_0 = \pi^{-1}(0)$ is reduced and normal, Hartog’s theorem implies that

$$\lim_{\tau \to 0} \int_{X_{\tau}} \theta_{\lambda} e^{-\frac{1}{2}r^2} w = \frac{1}{2n} \int_{X_0} \theta_{\lambda} e^{-\frac{1}{2}r^2} dV_{0}.$$ 

Putting everything together we get

$$\lim_{s \to \infty} \frac{d}{ds} D(r_s) = \int_{X_0} \theta_{\lambda} e^{-\frac{1}{2}r^2} \omega^n = \frac{1}{2n} \int_{X_0} \theta_{\lambda} e^{-\frac{1}{2}r^2} dV.$$ 

By Proposition 6.8 this is (up to a positive constant $c(n)$), the algebraic Futaki invariant of the test configuration $(Y, \lambda)$. From the convexity of the Ding functional we conclude

$$Fut(Y, \lambda, \xi) \geq 0.$$ 

If $Fut(Y, \lambda, \xi) = 0$, then we must have that $\frac{d}{ds} D(r_s) = 0$ identically, and then Berndtsson’s theorem implies that $r_s = F_s^* r_0$ on $X$, where

$$F_s = \exp(sV)$$ 

and $V$ is the real part holomorphic vector field on $X$ commuting with $\xi$. Consider the map

$$\rho := \lambda(\tau) \circ F_{-1}^{-1} : X \times \Delta^* \to Y^*,$$
and \( \rho_\tau = \rho(\cdot, \tau) \) for \( \tau \in \Delta^* \). By definition we have \( \rho^*_\tau r_0 = r_0 \), and \( \rho_\tau \star \xi = \xi \).

Since \( r_0 \) is the potential for a Kähler cone metric on \( \mathbb{C}^N \) compatible with \( \xi \), this implies that for any compact set \( K \subset X \), the image \( \rho(K \times (\frac{1}{2} \Delta \setminus \{0\})) \) is compact in \( \mathcal{Y} \). By Riemann's extension theorem \( \rho \) extends to a map

\[
\rho : X \times \Delta \rightarrow \mathcal{Y}
\]

which is an isomorphism away from \( \tau = 0 \). The same argument applied to \( \rho^{-1} \) shows that \( \rho : X \times \Delta \rightarrow \mathcal{Y} \) is an isomorphism, and so \( \mathcal{Y} \) is a trivial test-configuration. This completes the proof of Theorem 7.1.

8. Examples and Applications

In this section we check \( K \)-stability for a family of hypersurfaces of dimension 3 admitting a 2 torus action. Rational hypersurface singularities in \( \mathbb{C}^4 \) admitting a \( \mathbb{C}^* \) action were classified by Yau-Yu [82]. In the terminology of the Yau-Yu classification, will study the links of type \( I-I\text{II} \) admitting a \( T^2 \) action. For this section we will describe a holomorphic vector field in terms of its \( S^1 \) action by specifying the weights; in particular, a vector \((a_1, a_2, a_3, a_4)\) should be understood to act on \( \mathbb{C}^2(z_1, z_2, z_3, z_4) \) by acting on \( z_i \) with weight \( a_i \). With this notation, the main theorem of this section is

**Theorem 8.1.** The following affine affine varieties admit conical Ricci flat Kähler metrics with respect to the given Reeb field:

(I) The Brieskorn-Pham singularity

\[
Z_{BP}(p,q) := \{uv + z^p + w^q = 0\} \subset \mathbb{C}^4_{(u,v,z,w)}
\]

\[
\xi = \frac{3}{2(p+q)}(pq, pq, 2q, 2p)
\]

if \( 2p > q \) and \( 2q > p \). Topologically, the link is \#\( m(S^2 \times S^3) \), where \( m = \gcd(p,q) - 1 \) and \( \#0(S^2 \times S^3) = S^5 \).

(II) The Yau-Yu singularities of type II

\[
Z_{II}(p,q) := \{uv + z^p + zw^q = 0\} \subset \mathbb{C}^4_{(u,v,z,w)}
\]

\[
\xi = \frac{3}{2(q+p-1)}(qp, qp, 2q, 2(p-1))
\]

provided \( 3(p-1) > (q + p - 1) \) and \( 2qp + 1 > p^2 + q \). Topologically, the link is \#\( m(S^2 \times S^3) \), where \( m = \gcd(p-1,q) \).

(III) The Yau-Yu singularities of type III

\[
Z_{III}(p,q) := \{uv + z^p w + zw^q = 0\} \subset \mathbb{C}^4_{(u,v,z,w)}
\]

\[
\xi = \frac{3}{2(p+q-2)}(pq - 1, pq - 1, 2(q - 1), 2(p - 1))
\]

provided \( 3(p-1)^2(q-1) > (p+q-2)(pq-2p+1) \), and \( 3(q-1)^2(p-1) > (p+q-2)(pq-2q+1) \). Topologically, the link is \#\( m(S^2 \times S^3) \) where \( m = \gcd(p-1,q-1) + 1 \).
In particular, there exists:

- One infinite family of inequivalent Sasaki-Einstein metrics on $S^5$,
- Two distinct infinite families of inequivalent Sasaki-Einstein metrics on $S^2 \times S^3$,
- Three distinct families of inequivalent Sasaki-Einstein metrics on $#^m(S^2 \times S^3)$ for any $m \geq 2$.

The proof of this theorem will occupy the remainder of this section. We use the theory of polyhedral divisors. The original paper in this area is the work of Altmann-Hausen \[1\], and there is a nice survey by Altmann-Ilt-Petersen-Süß \[2\]. For applications of this theory to K-stability of Fano manifolds, see Ilten-Süß \[52\].

8.1. The Brieskorn-Pham singularities. We begin with the Brieskorn-Pham (BP) singularities, denoted $Z_{BP}(p,q)$ above, which appear as type I singularities in the Yau-Yu classification. In order to avoid the trivial cases, we will assume that $\max\{p,q\} > 2$. Let $\gcd(p,q) = m$, and choose relatively prime integers $a, b \in \mathbb{Z}$ such that

$$\frac{aq}{m} - \frac{bp}{m} = 1.$$ 

The affine variety admits 2 torus action, which is generated by $\mathbb{C}^*$ actions with weights $(1, -1, 0, 0)$, and $(0, \frac{pq}{m}, \frac{q}{m}, \frac{p}{m})$ on $(u, v, z, w)$ respectively. Let $t$ denote the Lie algebra of the compact torus, equipped with a basis $\{e_1, e_2\}$ corresponding to the above actions. The Reeb cone inside of $t$ is given by

$$C_R = \{ (x, y) := xe_1 + ye_2 \mid x > 0, \quad xpq - ym > 0 \}.$$ 

A simple symmetry argument shows that the Reeb field minimizing the volume of the link is a multiple of $(2, pq)$, which corresponds to the $\mathbb{C}^*$ action $(pq, p, 2q, 2p)$. Let $F : \mathbb{Z}^2 \to \mathbb{Z}^4$ be the inclusion of the algebra $t$ into the lie algebra of the diagonal torus acting on $\mathbb{C}^4$, equipped with the standard basis. With these choices, $F$ is represented by the matrix

$$F = \begin{bmatrix} 0 & 1 \\ \frac{pq}{m} & -1 \\ \frac{q}{m} & 0 \\ \frac{p}{m} & 0 \end{bmatrix}.$$

Let $P : \mathbb{Z}^4 \to \mathbb{Z}^2$ be the orthogonal projection to the cokernel of $F$. We have

$$P = \begin{bmatrix} 0 & 0 & \frac{-p}{m} & \frac{q}{m} \\ 1 & 1 & -p & 0 \end{bmatrix}.$$ 

So we have an exact sequence

$$0 \to \mathbb{Z}^2 \xrightarrow{F} \mathbb{Z}^4 \xrightarrow{P} \mathbb{Z}^2 \to 0.$$ 

we choose a splitting $s : \mathbb{Z}^4 \to \mathbb{Z}^2$ such that $s \circ F = 1$,

$$s = \begin{bmatrix} 0 & 0 & a & -b \\ 1 & 0 & 0 & 0 \end{bmatrix}.$$
In the language of [1], the tail cone is given by
\[ \sigma := s(F(Q^2) \cap Q_{\geq 0}^4) = \text{cone} \left[ \begin{pmatrix} 1 \\ 0 \\ \frac{pq}{m} \end{pmatrix} \right]. \]

It is straightforward to check that \( \sigma = \overline{\text{Reeb}} \) is the closure of the Reeb cone. The 2 torus action on \( \mathbb{C}^4 \) induces a fibration of \( \mathbb{C}^4 \) over a surface whose fibers are non-compact toric surfaces. Using the techniques of [1] we can describe this fibration in terms of combinatorial data called a p-divisor. We first compute the base of the fibration, which is given by the fan \( \Sigma \) with maximal cones
\[ \text{cone} \left[ \begin{pmatrix} 0 \\ 1 \\ -m \end{pmatrix} \right], \quad \text{cone} \left[ \begin{pmatrix} 1 \\ 0 \\ -1 \end{pmatrix} \right], \quad \text{cone} \left[ \begin{pmatrix} 0 \\ 1 \end{pmatrix} \right], \]
which is the fan corresponding to the projective space \( \mathbb{P}^2_{[1:1:m]} \). The rays of this fan are the columns of \( P \). The p-divisor is then a formal finite sum
\[ \sum \Delta_\rho \otimes D_\rho \]
where \( D_\rho \) is a divisor on \( \mathbb{P}^2_{[1:1:m]} \) and \( \Delta_\rho \) is a convex polytope with tail cone \( \sigma \). For the case at hand we have
\[ \Delta_{(1,0)} = (-bm/q, 0) + \sigma \]
\[ \Delta_{(-1,-m)} = (am/p, 0) + \sigma \]
\[ \Delta_{(0,1)} = \{0\} \times [0,1] + \sigma, \]
where \( \Delta_\rho = s(P^{-1}(\rho) \cap Q_{\geq 0}^4) \). This restricts to define a p-divisor on the curve \( C := \{X^m + Y^m + Z = 0\} \subset \mathbb{P}^2_{[1:1:m]} \), which is precisely the base of the induced fibration of \( Z_{BP} \). This curve intersects the divisor \( Z = 0 \) at \( m \) points, so the polytope \( \Delta_{(0,1)} \) will appear \( m \) times. Let \( \sigma^\vee \) denote the dual of the tail cone as a subset of \( t^\vee \) the dual of the lie algebra. In this case \( \sigma^\vee \) is described explicitly by
\[ \sigma^\vee = \text{cone} \left[ \begin{pmatrix} 0 \\ 1 \\ \frac{pq}{m} \end{pmatrix} \right]. \]

For each p-divisor \( \Delta_\rho \) we get a function \( \Psi : \sigma^\vee \to \mathbb{R} \) defined by \( \Psi_\rho(w) = \min_{u \in \Delta_\rho} \langle w, u \rangle \), where \( \langle \cdot, \cdot \rangle \) denotes the natural pairing between \( t \) and \( t^\vee \). In our case we get
\[ \Psi_{(1,0)}(s,t) = -\frac{bm}{q} s \]
\[ \Psi_{(-1,-m)}(s,t) = \frac{am}{p} s \]
\[ \Psi_{(0,1)}(s,t) = \min\{t,0\}, \]
where again \( \Psi_{(0,1)} \) is repeated \( m \) times. If \( \frac{am}{p} \) is an integer then we obtain an equivalent p-divisor by replacing \( \Psi_{(-1,-m)} \) by zero, and replacing \( \Psi_{(1,0)} \) by \( (\frac{am}{p} - \frac{bm}{q}) s \). This new p-divisor only has two distinct polytopes, and so from the description of \( T \)-equivariant test configurations with normal
central fiber due to Ilten-Suss [52] we see that there are at most two non-trivial test configurations. The same applies when $\frac{bn}{q}$ is an integer. When neither is an integer, then both $\Psi_{(1,0)}$ and $\Psi_{(0,1)}$ have non-integral slope, and so again from [52, Proposition 4.2] we obtain at most two non-trivial test configurations. These can be obtained from the methods of [52], but here we can simply guess the test configurations. They are

$$X_1 = \{(uv + (t \cdot z)p + w^q = 0)\},$$

$$X_2 = \{(uv + z^p + (t \cdot w)^q = 0)\}.$$

We will compute the Futaki invariant for $X_1$, the other case being identical.

The special fiber of $X_1$ is $Z_0 := \text{Spec} \mathbb{C}[u,v,z,w]$ polarized by the Reeb vector field $((pq,pq,2q,2p))$. Since $Z_{BP}(p,q)$ is a hypersurface, it is straightforward to check (see, e.g. [65]) that $(Z_{BP}(p,q), \xi)$ is normalized Fano if

$$\xi = \frac{3}{2(p + q)}(pq,pq,2q,2p).$$

On the central fiber there is a new $\mathbb{C}^*$ action corresponding to the one parameter subgroup induced by $\eta = (0,0,1,0)$. The index character for the Reeb field $\xi + s\eta$ can be computed directly. Let $\lambda = 3/2(p + q)$, then

$$F(\xi + s\eta, t) = \frac{1 - e^{-2pq\lambda t}}{(1 - e^{-pq\lambda t})^2(1 - e^{-(2q\lambda - s)t})(1 - e^{-2p\lambda t})} = \frac{1}{\lambda^2 p^2 q(2q\lambda - s)t^2} + \frac{2p\lambda + 2q\lambda - s}{2\lambda^2 p^2 q(2q\lambda - s)t^2} + O(t^{-1})$$

From this we read off

$$a_0(\xi - s\eta) = \frac{1}{2\lambda^2 p^2 q(2q\lambda - s)}; \quad \left(\frac{a_1}{a_0}\right)(\xi - s\eta) = \lambda(2p + 2q) - s$$

and so

$$D_\eta a_0(\xi) = -\frac{1}{2\lambda^2 p^2 q(2q\lambda)^2}, \quad \frac{1}{2}D_\eta a_0(\xi) = 1.$$

By definition, the Futaki invariant is

$$Fut(X_1, \xi) = \frac{a_0(\xi)}{2}D_\eta \left(\frac{a_1}{a_0}\right)(\xi) + \frac{1}{2}D_\eta a_0(\xi)$$

$$= \frac{a_0(\xi)}{2} \left(1 - \frac{1}{2q\lambda}\right)$$

$$= \frac{a_0(\xi)}{2} \left(\frac{2q - p}{3q}\right),$$

which is positive if and only if $2q > p$. Similarly for the other test-configuration the condition is $2p > q$, and so we obtain that $(Z_{BP}(p,q), \xi)$ is K-stable, and
hence admits a conical Ricci-flat Kähler metric if and only if
\[ 2p > q, \quad 2q > p, \]
which is precisely the Lichnerowicz obstruction discovered by Gauntlett-Martelli-Sparks-Yau \[46\]. In dimension 5 there is standard machinery for computing the topology of links of isolated hypersurface singularities, see \[14\] for a complete description. In particular, it is straightforward to compute that the link of \( Z(p,q) \) is topologically \( \#(\gcd(p,q)-1)S^2 \times S^3 \). In particular, whenever \( \gcd(p,q) = 1, \) and \( 2p > q, 2q > p \) we obtain a Sasaki-Einstein metric on \( S^5 \). Furthermore, as a function of \( p,q \), the (unnormalized) volume of \( (Z_{BP}(p,q), \xi) \) is given by
\[ a_0(\xi) = \frac{2(p+q)^3}{27p^2q^2}, \]
and hence infinitely many of these metrics are inequivalent. For example, fix a positive integer \( m \) and let \( p > 2 \). Then the affine varieties \( Z_{BP}(pm, (p-1)m) \) are K-stable, and the link is topologically \( \#m(S^2 \times S^3) \). Furthermore, the volume is given by
\[ \text{Vol}(Z_{BP}(pm, (p-1)m), \xi) = \frac{2(2p-1)^3}{27mp^2(p-1)^2} \]
which is a strictly decreasing function as \( p \to \infty \). By taking a sequence of primes going to infinity we obtain the existence of infinite families of inequivalent, non-toric Sasaki-Einstein metrics on \( \#m(S^2 \times S^3) \) for any \( m \geq 0 \) (where \( m = 0 \) means \( S^5 \)). Furthermore, we note that \( Z_{BP}(2,3) \) is also K-stable, confirming the result of Li-Sun \[61\] that the \( A_2 \) singularity admits a Ricci-flat cone metric.

8.2. The Yau-Yu singularities of type II. One can apply similar techniques to treat the Yau-Yu links of type II and III. We mention these applications briefly. Consider the family of hypersurface singularities described by \( Z_{II}(p,q) \) which admits a two torus generated by the \( \mathbb{C}^* \) actions with weights \( (0, pq, q, p-1) \) and \( (1, -1, 0, 0) \). An easy symmetry argument shows that the normalized Reeb field minimizing the volume is given by
\[ \xi = \frac{3}{2(q+p-1)}(qp, qp, 2q, 2(p-1)), \quad \text{Vol}(Z_{II}(p,q), \xi) = \frac{2(p+q-1)^3}{27pq^2(p-1)} \]
By similar techniques used for the Brieskorn-Pham links one can show that there are only two \( T \)-equivariant test configurations. The first of these test configurations is
\[ \mathcal{X}_1 := \{ uv + z^p + z(t \cdot w)^q = 0 \} \]
which is induced by the \( \mathbb{C}^* \) action with weights \( (0, 0, 1) \). A straightforward computation using the index character yields
\[ \text{Fut}(\mathcal{X}_1, \xi) > 0 \iff \frac{3(p-1)}{(q+p-1)} > 1. \]
The second test configuration is more interesting, given by
\[ X_2 := \{ uv + t^{pq}w^p + wz^q = 0 \} \]
which is induced by the \( C^* \) action with weights \((0, 0, q, −1)\). Computing the Futaki invariant yields
\[ \text{Fut}(X_2, \xi) > 0 \iff 2qp + 1 > p^2 + q \]
Note, in particular, that this obstruction is strictly stronger than the Lichnerowicz. For example, the affine variety \( Z_{II}(6, 3) \) is not obstructed by the Lichnerowicz bound, but is destabilized by the test configuration \( X_2 \). Topologically, the link of \( Z_{II}(p, q) \) is \# \( \gcd(p − 1, q)(S^2 \times S^3) \). If \( p \geq 2 \) is a prime number then one can easily check that
\[ Z_{II}(m(p − 1) + 1, mp) \]
is \( K \)-stable, and hence generates a Sasaki-Einstein metric on \# \( m(S^2 \times S^3) \) with volume
\[ \text{Vol}(Z_{II}(m(p − 1) + 1, mp), \xi) = \frac{2(2p - 1)^3}{27p^2(p - 1)(m(p - 1) + 1)}. \]
Taking a sequence of primes going to \( \infty \) yields a second infinite sequence of distinct Sasaki-Einstein metrics on \# \( m(S^2 \times S^3) \) for any \( m \geq 1 \).

8.3. The Yau-Yu links of type III. Finally, a similar analysis works for the Yau-Yu links of type III, given by
\[ Z_{III}(p, q) = \{ uv + z^{pq}w + zw^{pq} = 0 \} \subset \mathbb{C}^4_{(u,v,z,w)} \]
which have a 2 torus action generated by the \( C^* \) actions with weights \((0, (pq − 1), (q − 1), (p − 1))\) and \((1, −1, 0, 0)\). The critical Reeb field is then
\[ \xi = \frac{3}{2(p + q − 2)}(pq - 1, pq - 1, 2(q - 1), 2(p - 1)). \]
There are two non-trivial \( T \)-equivariant test configurations generated by the \( C^* \) actions with weights \((0, 0, −1, p)\) and \((0, 0, q, −1)\). Computing the Futaki invariants as above we find that the link of \( Z_{III}(p, q) \) admits a Sasaki-Einstein metric if and only if
\[ 3(p − 1)^2(q − 1) > (p + q − 2)(pq − 2p + 1) \]
\[ 3(q − 1)^2(p − 1) > (p + q − 2)(pq − 2q + 1). \]
If we let \( m = \gcd(p − 1, q − 1) + 1 \), then using [14] Chapter 9 one can check that the link of \( Z_{III}(p, q) \) is topologically \# \( m(S^2 \times S^3) \). As before, we obtain a third infinite family of distinct Sasaki-Einstein metrics on \# \( m(S^2 \times S^3) \) for any \( m \in \mathbb{N} \) with \( m \geq 2 \).
9. Further Discussion

The results contained in this paper motivate the following picture, which is the Sasakian analog of the general picture described in \[73\]. Fix a polarized affine variety \((X, T, \xi)\) of dimension \(n\), where \(\xi \in t\) is normalized, and has minimal volume. We try to find a Ricci flat Kähler cone metric compatible with \(\xi\) by deforming along the method of continuity. If \((X, \xi)\) is \(K\)-stable, then we succeed. If not, then the method of continuity breaks at some time \(T_1 \leq 1\), and we get a test configuration with central fiber a normal, polarized affine variety \((Y_1, T_1, \xi)\), where \(T_1\) is the torus generated by \(T\) and the vector field \(w_1\) giving the test-configuration. In particular \(\dim T_1 = \dim T + 1\), but it is possible that \(Y_1 \cong X\) if the torus \(T\) that we started with was not maximal.

The test configuration is destabilizing, and so by the discussion in Section 2 if \(w_1\) is normalized, then we have

\[D_w a_0(Y_1, \xi) \leq 0,\]

with strict inequality if \(Y_1 \cong X\). We can now repeat the volume minimization for \((Y_1, T_1, \xi)\) to obtain a new Reeb field \(\xi_1\). We expect that it will be possible to restart the method of continuity with the data \((Y_1, T_1, \xi_1)\). Assuming that the results here carry over to the case of non-isolated singularities, we can repeat the above process to get

\[X \to Y_1 \to \cdots \to Y_k := Y,\]

where the final \((Y, \xi_k)\) is \(K\)-stable, since after finitely many steps we must reach a toric variety. Once the variety is toric, then it is automatically \(K\)-stable, after volume minimization, since there are no non-trivial toric test-configurations with normal central fiber. Note that it was previously shown by Futaki-Ono-Wang \[44\] that toric Fano cone singularities with an isolated singularity admit Ricci flat cone metrics, and we expect the same to hold when there are singularities away from the cone point. It then follows that given any \((X, T, \xi)\), it should be possible to deform \(X\) to a \(K\)-stable affine variety \((Y, T', \xi')\) by at most \(n - 1\) test-configurations.

It is natural to wonder whether this process can be made canonical and it seems reasonable to expect that the \(K\)-stable variety \((Y, \xi_k)\) is canonically associated to \((X, \xi)\). In view of the discussion in Donaldson-Sun \[41\] Section 3.3 and the example of Hein-Naber mentioned there, we expect that the Ricci flat cone metric on \((Y, \xi_k)\) is the metric tangent cone at the vertex of any Ricci flat Kähler metric on a neighborhood of the vertex on \(X\).

One can also ask for a more algebraic description of each \(Y_i\) in the sequence above, and for this at each step it would be necessary to distinguish one particular destabilizing test-configuration. Motivated by conformal field theory (see \[28\]) the natural way to choose between any two destabilizing test-configurations with central fibers \(Y_1, Y_2\) is to compare their volumes, after volume minimization. That is, we repeat the volume minimization on \(Y_i\) and get new polarized affine varieties \((Y_1, \xi_1)\) and \((Y_2, \xi_2)\). We choose \(Y_1\)
over $Y_2$ if
\[ \text{Vol}(Y_1, \xi_1) > \text{Vol}(Y_2, \xi_2), \]
and vice versa, where the volume can be computed algebraically from the index character. When equality occurs, it may be that either $Y_1 \cong Y_2$ or there is a test configuration taking $(Y_1, \xi_1)$ to $(Y_2, \xi_2)$ or vice versa. These statements are confirmed to some extent by example calculations, but so far there is still little evidence for them.

In a less speculative vein there are many interesting questions regarding the existence of Sasaki-Einstein metrics on various manifolds. From Cho-Futaki-Ono [25] we know that $m\#(S^2 \times S^3)$ admits infinitely many irregular Sasaki-Einstein metrics for $m \geq 1$, and we have shown that $S^5$ admits infinitely many quasi-regular Sasaki-Einstein metrics. It is natural to ask therefore:

**Question 9.1.** Does there exist an irregular Sasaki-Einstein metric on $S^5$? More generally can we classify all Sasaki-Einstein metrics on $S^5$ with an isometric 2-torus action?

The combinatorial description of T-varieties should help with this classification, as long as one develops a method for reading off the topology of the link from the p-divisor (see the work [60] in this direction). A more thorough study should also lead to higher dimensional existence results, in particular on odd dimensional spheres. We expect the following.

**Conjecture 9.2.** There are infinitely many families of Sasaki-Einstein metrics on $S^{2n+1}$ for all $n$.

The same question can be asked for exotic spheres which bound parallelizable manifolds, and the existence of Sasaki-Einstein metrics on these was conjectured by Boyer-Galicki-Kollár [15]. This conjecture was verified up to dimension 15 by Boyer-Galicki-Kollár-Thomas [16].

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