The Geometry of $p$-Adic Fractal Strings: A Comparative Survey

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Abstract. We give a brief overview of the theory of complex dimensions of real (archimedean) fractal strings via an illustrative example, the ordinary Cantor string, and a detailed survey of the theory of $p$-adic (nonarchimedean) fractal strings and their complex dimensions. Moreover, we present an explicit volume formula for the tubular neighborhood of a $p$-adic fractal string $L_p$, expressed in terms of the underlying complex dimensions. Special attention will be focused on $p$-adic self-similar strings, in which the nonarchimedean theory takes a more natural form than its archimedean counterpart. In contrast with the archimedean setting, all $p$-adic self-similar strings are lattice and hence, their complex dimensions (as well as their zeros) are periodically distributed along finitely many vertical lines. The general theory is illustrated by some simple examples, the nonarchimedean Cantor, Euler, and Fibonacci strings. Throughout this comparative survey of the archimedean and nonarchimedean theories of fractal (and possibly, self-similar) strings, we discuss analogies and differences between the real and $p$-adic situations. We close this paper by proposing several directions for future research, including seemingly new and challenging problems in $p$-adic (or rather, nonarchimedean) harmonic and functional analysis, as well as spectral theory.

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References

Nature is an infinite sphere of which the center is everywhere and the circumference nowhere.

Blaise Pascal

1. Introduction

In this survey, we present aspects of a geometric theory of $p$-adic (or nonarchimedean) fractal strings, that is, bounded open subsets of the $p$-adic line $Q_p$ having a fractal subset of $Q_p$ for “boundary”. This theory, developed by Michel Lapidus and L’Hûng in [27, 28], as well as by those same authors and Machiel van Frankenhuysen in [29], extends in a natural way the theory of real (or archimedean) fractal strings and their complex dimensions developed in [38, 39], and building on [24, 30, 35], for example. Following [27, 28, 29], we introduce suitable geometric zeta functions, the poles of which play the role for $p$-adic fractal strings of the complex dimensions for the standard real fractal strings. Furthermore, we discuss the analogies and the differences between the real and $p$-adic fractal strings.

More specifically, we recall the definition of $p$-adic self-similar strings introduced in [28]; furthermore, we show (as in [28]) that all $p$-adic self-similar strings are lattice (in a strong sense) and deduce from this fact the simple periodic structure of their complex dimensions. We also discuss the explicit fractal tube formulas obtained in [29], both in the general case of (languid) $p$-adic fractal strings and that of $p$-adic self-similar strings. Throughout this paper, these various results are illustrated in the case of suitable nonarchimedean analogs of the Cantor and the Fibonacci strings (which are both self-similar), as well as in the case of a new (and non self-similar) $p$-adic fractal string, namely, the $p$-adic Euler string introduced in [29]. Some particular attention is devoted to the nonarchimedean (or 3-adic) Cantor string (introduced and studied in [27]), an appropriate counterpart of the archimedean Cantor string, whose ‘metric’ boundary is the nonarchimedean (or 3-adic) Cantor set ([27]), a suitable $p$-adic analog of the classic ternary Cantor set.

We note that $p$-adic (or nonarchimedean) analysis has been used in various areas of mathematics (such as representation theory, number theory and arithmetic geometry), as well as (more speculatively) of mathematical and theoretical physics (such as quantum mechanics, relativity theory, quantum field theory, statistical and condensed matter physics, string theory and cosmology); see, e.g., [5, 6, 9, 46, 51] and the relevant references therein. In particular, it is believed by some authors that $p$-adic numbers (or, more generally, nonarchimedean fields) can be used to describe the geometry of spacetime at very high energies and hence, very small scales (i.e., below the Planck or the string scale); see, e.g., [52]. Furthermore, several physicists and mathematical physicists have suggested that the small scale structure of spacetime may be fractal; see, e.g., [13, 17, 25, 42, 53].

On the other hand, in the recent book [24], it has been suggested that fractal strings and their quantization, fractal membranes, may be related to aspects of string theory and that $p$-adic (and possibly, adèlic) analogs of these notions would be useful in this context in order to better understand the underlying (noncommutative) spacetimes and their moduli spaces ([25, 31]). The theory of $p$-adic fractal strings, once suitably ‘quantized’, may be helpful in further developing some of
these ideas and eventually providing a framework for unifying the real and $p$-adic fractal strings and membranes.

The rest of this paper is organized as follows:

In §2 we give a brief survey of the main pertinent properties of the Cantor string and set, both in the real (or archimedean) and the $3$-adic (or nonarchimedean) situations. This will serve, in particular, as a pedagogical introduction to the general theory of real and $p$-adic fractal (and possibly self-similar) strings.

In §3 we recall the definition of an arbitrary $p$-adic fractal string, along with some of the key pertaining notions (geometric zeta function and complex dimensions, as well as Minkowski dimension and content). We also discuss the more technical question of how to suitably define and calculate the volume (i.e., $p$-adic Haar measure) of the ‘inner’ $\varepsilon$-neighborhood (or inner tube) of a $p$-adic fractal string (§3.2). In §3.3, we then use these results to express this volume as an infinite sum over the underlying complex dimensions, thereby obtaining a nonarchimedean analog of the ‘fractal tube formula’ of [38, 39]. We illustrate this formula in §3.4 by providing (as well as deriving via a direct computation) the fractal tube formula for the $p$-adic Euler string, the definition of which is given in §3.1.

In §4 we focus on the important special class of $p$-adic self-similar strings, of which the 3-adic Cantor string and the 2-adic Fibonacci string are among the simplest examples. After having explained their construction in §4.1 via an iterated function (or self-similar) system, we provide (in §4.2–4.4) a detailed study of their geometric zeta functions and complex dimensions. It turns out that due to the discreteness of the valuation group of $\mathbb{Q}_p$, all $p$-adic self-similar strings are ‘strongly lattice’ (§4.3), from which it follows that their complex dimensions (along with the zeros of their geometric zeta functions) are periodically distributed along finitely many vertical lines (§4.4). It follows that (under mild assumptions) the fractal tube formula of a $p$-adic self-similar string involves finitely many (multiplicative) periodic functions, one for each ‘line’ of complex dimensions. We describe such explicit tube formulas in some detail in §4.5 and also obtain in §4.6 an explicit expression for the average Minkowski content of a $p$-adic self-similar string (and the associated nonarchimedean self-similar set).

Throughout this expository paper and comparative survey, we illustrate some of the main results by discussing the examples of the nonarchimedean Cantor, Euler and Fibonacci strings. Moreover, we point out the main analogies and differences between the archimedean and nonarchimedean theories of fractal strings.

Finally, in §5 we conclude this paper by proposing several possible research directions for future work in this new field. This includes, in particular, a possible extension of the nonarchimedean theory of fractal strings and their tube formulas to Berkovich spaces, along with seemingly new and quite challenging problems in nonarchimedean spectral, harmonic and functional analysis.

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1 Most of the proofs given in this paper will be concentrated in §3.2 because they truly depend on the nonarchimedean (specifically, $p$-adic) nature of the underlying geometry.

2 This is not necessarily the case for a general ‘lattice’ archimedean self-similar string. Moreover, a generic archimedean self-similar string is ‘nonlattice’. It follows that the theory of $p$-adic self-similar fractal strings is more natural as well as simpler than its archimedean counterpart.
2. Archimedean vs. Nonarchimedean Cantor Set and String

In this section, we briefly recall, for a simple but important example, some of the main notions pertaining to the theory of real and $p$-adic fractal strings. Namely, the geometric zeta function, the complex dimensions, and the ‘fractal tube formula’ that expresses the volume of the inner $\varepsilon$-neighborhoods of a suitable boundary of the string as a ‘fractal power series’ with exponents involving the underlying complex dimensions.

In the archimedean case, the classic example is the real Cantor string (§2.2), whose boundary (or associated self-similar set) is the ternary Cantor set (§2.1); cf. [39, Ch. 1]. The nonarchimedean counterpart of the real Cantor set or string is the 3-adic Cantor set or string, discussed in §2.3 or §2.4 respectively (and both introduced in [27]). In particular, the 3-adic Cantor set is the nonarchimedean self-similar set naturally associated with the 3-adic Cantor string (which is a special case of $p$-adic self-similar string, in the sense of [28] and §4 below).

Finally, a comparative study of the archimedean and nonarchimedean Cantor strings and their respective fractal tube formulas is provided in §2.5. It will help us preview some of the main analogies and differences between the real and $p$-adic theory of fractal (and possibly, self-similar) strings, as further discussed and developed in §3 and §4.

2.1. Archimedean (or Ternary) Cantor Set. The classical archimedean (or ternary) Cantor set, denoted by $C$, is the set that remains after iteratively removing the open middle third subinterval(s) from the closed unit interval $C_0 = [0, 1]$. The construction is illustrated in Figure 1. There, for each $n \geq 0$, $C_n$ is the compact set defined as the union of $2^n$ compact intervals of length $3^{-n}$ and endpoints the ternary points of ‘scale’ $n$ (i.e., of the form $\frac{3k + j}{3^n}$, with $k \in \mathbb{N}$ and $j = 1, 2$). Hence, the archimedean Cantor set $C$ is equal to $\bigcap_{n=0}^{\infty} C_n$.

\[
\begin{array}{cccc}
C_0 & & & \\
C_1 & & & \\
C_2 & & & \\
C_n & & & \\
\end{array}
\]

\text{Figure 1. Construction of the archimedean Cantor set } C = \bigcap_{n=0}^{\infty} C_n.

For comparison with our results in the nonarchimedean case, we state the following well-known results (see, e.g., [12, Ch. 9] and [15, p. 50]):

\textbf{Theorem 2.1.} The archimedean Cantor set $C$ is self-similar. More specifically, it{ is} the unique nonempty, compact invariant set in $[0, 1] \subset \mathbb{R}$ generated by the iterated function system (IFS) $\Phi = \{\Phi_1, \Phi_2\}$ of affine similarity contraction mappings of $[0, 1]$ into itself, where

\[\Phi_1(x) = \frac{x}{3} \text{ and } \Phi_2(x) = \frac{x}{3} + \frac{2}{3} \text{.}\]
The archimedean Cantor set is characterized by the ternary expansion of its elements as
\[ C = \{ \tau \in [0,1] : \tau = a_0 + a_13^{-1} + a_23^{-2} + \cdots, a_j \in \{0,2\}, \forall j \geq 0 \}. \]

We note that, as usual, we choose the nonrepeating ternary expansion here (so that none of the coefficient \( a_j \) is equal to 1, and hence, the sequence of digits does not end with \( \bar{1} \), where the overbar indicates that 1 is repeated ad infinitum). Such a precaution will not be needed in §2.3 for the elements of \( \mathbb{Q}_3 \) because the 3-adic expansion is unique.

2.2. Archimedean (or Real) Cantor String. The ordinary archimedean (or real) Cantor string \( \mathcal{CS} \) is defined as the complement of the ternary Cantor set in the closed unit interval \([0,1]\). By construction, the topological boundary of \( \mathcal{CS} \) is the ternary Cantor set \( C \). The Cantor string is one of the simplest and most important examples in the research monographs [38, 39] by Lapidus and van Frankenhuijsen. Indeed, it is used throughout those books to illustrate and motivate the general theory; see also, e.g., [23] and [35]. From the point of view of the theory of fractal strings and their complex dimensions [38, 39], it suffices to consider the sequence \( \{l_n\}_{n \in \mathbb{N}} \) of lengths associated to \( \mathcal{CS} \). More specifically, these are the distinct lengths of the intervals of which the bounded open set \( \mathcal{CS} \subset \mathbb{R} \) is composed, counted according to their multiplicities. Accordingly, the archimedean Cantor string consists of \( m_1 = 1 \) interval of length \( l_1 = 1/3 \), \( m_2 = 2 \) intervals of length \( l_2 = 1/9 \), \( m_3 = 4 \) intervals of length \( l_3 = 1/27 \), and so on; see Figure 2.

\[ l_1 \ l_2 \ l_2 \ l_3 \ l_3 \ l_3 \ l_n \]

\[ \cdots \]

Figure 2. The archimedean Cantor string \( \mathcal{CS} \) (above); the Cantor string viewed as a fractal harp (below).

That is,
\[ \mathcal{C} = \Phi_1(C) \cup \Phi_2(C). \]
Important information about the geometry of CS, e.g., the Minkowski dimension and the Minkowski measurability, is contained in its geometric zeta function

\begin{equation}
\zeta_{CS}(s) := \sum_{n=1}^{\infty} m_n \cdot l_n^s = \sum_{n=1}^{\infty} \frac{2^{n-1}}{3^n} \cdot l_n^s = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \quad \text{for } \Re(s) > D,
\end{equation}

where \(D = \log 2 / \log 3 = \log 2 \) is the Minkowski dimension of the ternary Cantor set.\(^7\) In addition, \(\zeta_{CS}\) can be extended to a meromorphic function on the entire complex plane \(\mathbb{C}\), as given by the last expression in (2.1). The corresponding set of poles of \(\zeta_{CS}\) is then given by

\begin{equation}
D_{CS} = \{D + iv \mid v \in \mathbb{Z}\},
\end{equation}

where \(i := \sqrt{-1}\) and \(p := 2\pi / \log 3\) is the oscillatory period of \(CS\). The set \(D_{CS}\) is called the set of complex dimensions of the real Cantor string; see Figure 5 in §2.4.

For \(\varepsilon > 0\), let \(V_{CS}(\varepsilon)\) be the volume of the inner tubular neighborhood of the boundary of the real Cantor string, i.e., \(\partial(CS) = C\), with radius \(\varepsilon\):

\begin{equation}
V_{CS}(\varepsilon) = \mu_L(\{x \in CS \mid d(x, C) < \varepsilon\}),
\end{equation}

where \(\mu_L\) is the one-dimensional Lebesgue measure on \(\mathbb{R}\). Then it can be computed directly (as in [39, §1.12]) to depend only on the lengths of \(CS\) and to be given by

\begin{equation}
V_{CS}(\varepsilon) = \frac{1}{2\log 3} \sum_{\omega \in D_{CS}} \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} - 2\varepsilon,
\end{equation}

where \(D_{CS}\) is as in (2.2).

The general theme of the monographs [38, 39] is that the complex dimensions describe oscillations in the geometry and the spectrum of a fractal string. In particular, due to the presence of nonreal complex dimensions on the vertical line \(\Re(s) = D\), there are oscillations of order \(D\) in the geometry of \(CS\) and therefore its boundary, the ternary Cantor set, is not Minkowski measurable; see [34, 39] §1.1.2).

2.3. Nonarchimedean (or 3-Adic) Cantor Set. Our goal in this section is to provide a natural nonarchimedean (or \(p\)-adic) analog of the classic ternary Cantor set \(C\) and to show that it satisfies a counterpart of many of the key properties of \(C\) in this nonarchimedean context. Furthermore, we will show in [24] that the corresponding \(p\)-adic fractal string, called the nonarchimedean (or 3-adic) Cantor string and denoted by \(CS_3\), is an exact analog of the ordinary archimedean Cantor string \(CS\), a central example in the theory of real fractal strings and their complex dimensions [38, 39]. The nonarchimedean Cantor set and string were both introduced and studied in detail in [27].

We begin by recalling a few simple facts concerning the field of \(p\)-adic numbers \(\mathbb{Q}_p\), equipped with the standard \(p\)-adic absolute value \(|\cdot|_p\) and associated topology; see, e.g., [24, 38, 39].\(^8\) As is well known, every \(z \in \mathbb{Q}_p\) has a unique representation as a convergent infinite series in \((\mathbb{Q}_p, |\cdot|_p)\):

\[z = a_v p^v + \cdots + a_0 + a_1 p + a_2 p^2 + \cdots,\]

\(^7\)Throughout this paper, \(\log t\) denotes the natural logarithm of \(t > 0\).

\(^8\)Here and thereafter, \(|\cdot|_p\) is normalized in the usual way; namely, \(|p^k|_p = p^{-k}\), for any \(k \in \mathbb{Z}\).
for some \( v \in \mathbb{Z} \) and \( a_j \in \{0, 1, \ldots, p-1\} \) for all \( j \geq v \). An important subset of \( \mathbb{Q}_p \) is the unit ball, \( \mathbb{Z}_p = \{ x \in \mathbb{Q}_p : |x|_p \leq 1 \} \), which can also be represented as follows:

\[
\mathbb{Z}_p = \{ a_0 + a_1 p + a_2 p^2 + \cdots : a_j \in \{0, 1, \ldots, p-1\}, \forall j \geq 0 \}.
\]

Using this \( p \)-adic expansion, we can easily see that

\[
\mathbb{Z}_p = \bigcup_{c=0}^{p-1} (c + p\mathbb{Z}_p),
\]

where \( c + p\mathbb{Z}_p = \{ y \in \mathbb{Q}_p : |y - c|_p \leq p^{-1} \} \) is the \( p \)-adic ball (or interval) of center \( c \) and radius \( p^{-1} \).

A remarkable property of the nonarchimedean ‘unit interval’ \( \mathbb{Z}_p \) of \( \mathbb{Q}_p \), which does not have any analog for the archimedean unit interval \([0, 1]\) of \( \mathbb{R} \), is that \( \mathbb{Z}_p \) is a ring, and in particular, is stable under addition; see, e.g., [16] for a thorough discussion of this point. Indeed, \( |a+b|_p \leq \max\{|a|_p, |b|_p\} \leq 1 \), if \( a, b \in \mathbb{Z}_p \). Moreover, \( \mathbb{Z}_p \) is a compact group and as such, admits a unique translation invariant measure, to be also denoted by \( \mu_H \), which is the restriction to \( \mathbb{Z}_p \) of Haar measure \( \mu_H \) on \( \mathbb{Q}_p \).

Finally, note that unlike its real counterpart \([0, 1]\) (or \([-1, 1]\)), \( \mathbb{Z}_p \) is not connected; actually, it is totally disconnected. This well known fact, combined with the ‘self-duplicating property’ (2.5), will naturally lead us to suitably modify many of the definitions and results of the standard theory of ordinary real fractal strings.

Consider the ring of 3-adic integers \( \mathbb{Z}_3 \). In a procedure reminiscent of the construction of the classic ternary Cantor set (see [2, 1]), we construct the nonarchimedean (or 3-adic) Cantor set as follows. First, we subdivide \( T_0 = \mathbb{Z}_3 \) into 3 equally long subintervals. We then remove the “middle third” subinterval \( 1 + 3\mathbb{Z}_3 \) and call \( T_1 \) the remaining set: \( T_1 = 0 + 3\mathbb{Z}_3 \cup 2 + 3\mathbb{Z}_3 \). We then repeat this process with each of the remaining subintervals, i.e., with \( 0 + 3\mathbb{Z}_3 \) and \( 2 + 3\mathbb{Z}_3 \). Finally, we define the nonarchimedean Cantor set \( C_3 \) to be \( \bigcap_{n=0}^{\infty} T_n \); see Figure 3. Here, for each \( n \geq 0 \), the compact set \( T_n \) is the union of \( 2^n \) 3-adic intervals of scale \( n \) (i.e., of radius or diameter \( 3^{-n} \)). Note that \( C_3 \) is compact, as the intersection of a decreasing sequence of compact subsets of \( \mathbb{Z}_3 \).

The nonarchimedean analog of Theorem 2.1 is then given by Theorem 2.3.
Theorem 2.3. The nonarchimedean Cantor set $C_3$ is self-similar. More specifically, it is the unique nonempty, compact invariant set in $\mathbb{Z}_3 \subset \mathbb{Q}_3$ generated by the IFS $\Phi = \{ \Phi_1, \Phi_2 \}$ of affine similarity contraction mappings of $\mathbb{Z}_3$ into itself, where
\[
\Phi_1(x) = 3x \quad \text{and} \quad \Phi_2(x) = 3x + 2.
\]
That is,
\[
C_3 = \Phi_1(C_3) \cup \Phi_2(C_3).
\]

The next result is also a counterpart of a well known property of the ternary Cantor set (which we have omitted to recall in \S 2.1, by necessity of concision). It is a simple consequence of the self-similarity of the nonarchimedean Cantor set $C_3$, as expressed by Theorem 2.3, and provides a useful introduction to the more general notion of $p$-adic self-similar string to be discussed in \S 4.

Theorem 2.4. Let $W_\alpha = \{1, 2\}^\alpha$ be the set of all finite words, on two symbols, of a given length $\alpha \geq 0$. Then
\[
C_3 = \bigcap_{\alpha=0}^{\infty} \bigcup_{w \in W_\alpha} \Phi_w(\mathbb{Z}_3),
\]
where $\Phi_w := \Phi_{w_\alpha} \circ \cdots \circ \Phi_{w_1}$ for $w = (w_1, \ldots, w_\alpha) \in W_\alpha$ and the maps $\Phi_{w_j}$ are as in Equation (2.6).

The following result is the nonarchimedean analog of Theorem 2.2.

Theorem 2.5. The nonarchimedean Cantor set is characterized by the 3-adic expansion of its elements. That is,
\[
C_3 = \{ \tau \in \mathbb{Z}_3 \mid \tau = a_0 + a_1 3 + a_2 3^2 + \cdots, a_j \in \{0, 2\}, \forall j \geq 0 \}.
\]

Theorem 2.6. The ternary Cantor set $C$ and the nonarchimedean Cantor set $C_3$ are homeomorphic.

**Proof.** Let $\gamma : C \to C_3$ be the map sending
\[
\sum_{j=0}^{\infty} a_j 3^{-j} \mapsto \sum_{j=0}^{\infty} a_j 3^j,
\]
where $a_j \in \{0, 2\}, \forall j \geq 0$. The map $\gamma$ is clearly a bijection and continuous. Furthermore, it respects the self-similarity of $C$ and $C_3$, and hence is a homeomorphism.
where $a_j \in \{0, 2\}$ for all $j \geq 0$. We note that on the left-hand side of (2.7), we use the ternary expansion in $\mathbb{R}$, whereas on the right-hand side we use the 3-adic expansion in $\mathbb{Q}_3$. Then, in light of Theorems 2.2 and 2.5, $\gamma$ is a continuous bijective map from $\mathcal{C}$ onto $\mathcal{C}_3$. Since both $\mathcal{C}$ and $\mathcal{C}_3$ are compact spaces in their respective natural metric topologies, $\gamma$ is a homeomorphism. □

Remark 2.7. In view of Theorem 2.6, like its archimedean counterpart, the nonarchimedean Cantor set $\mathcal{C}_3$ is compact, totally disconnected, uncountably infinite and has no isolated points. In particular, it is a perfect and complete metric space; furthermore, its topological dimension is 0.

2.4. Nonarchimedean (or 3-Adic) Cantor String. The nonarchimedean (or 3-adic) Cantor string $\mathcal{CS}_3$ is defined to be

$$\mathcal{CS}_3 := (1 + 3\mathbb{Z}_3) \cup (3 + 9\mathbb{Z}_3) \cup (5 + 9\mathbb{Z}_3) \cup \cdots = \mathbb{Z}_3 \setminus \mathcal{C}_3,$$

the complement of $\mathcal{C}_3$ in $\mathbb{Z}_3$; see the “middle” parts of Figure 4. Therefore, by analogy with the relationship between the archimedean Cantor set and Cantor string, the nonarchimedean Cantor set $\mathcal{C}_3$ can be thought of as a kind of “boundary” of the nonarchimedean Cantor string. Certainly, $\mathcal{C}_3$ is not the topological boundary of $\mathcal{CS}_3$ because the latter boundary is empty.

As was alluded to earlier, since $\mathbb{Q}_p$ is a locally compact group, there is a unique translation invariant positive measure on $\mathbb{Q}_p$, called Haar measure and denoted by $\mu_H$, normalized so that $\mu_H(\mathbb{Z}_p) = 1$ and hence $\mu_H(a + p^k\mathbb{Z}_p) = p^{-k}$, for any $a \in \mathbb{Q}_p$ and $k \in \mathbb{Z}$; see [41, 42, 43]. As in the real case in §2.2, we may identify $\mathcal{CS}_3$ with the sequence of lengths $l_n = 3^{-n}$, counted with multiplicities $m_n = 2^{n-1}$, for all $n \geq 1$.

\begin{center}
\begin{tabular}{c|c|c}
 & 0 & 1 \\
\hline
$\mathcal{D}_{\mathcal{C}}$ & $\mathcal{D}_{\mathcal{CS}_3}$ & $\mathcal{D}$ \\
\hline
0 & $\circ$ & $\circ$ \\
10 & $\circ$ & $\circ$ \\
$\mathcal{P}$ & $\circ$ & $\circ$ \\
\end{tabular}
\end{center}

Figure 5. The set of complex dimensions, $\mathcal{D}_{\mathcal{CS}} = \mathcal{D}_{\mathcal{CS}_3}$, of the archimedean and nonarchimedean Cantor strings, $\mathcal{C}$ and $\mathcal{CS}_3$. 
Clearly, these lengths are given by the Haar measure of the $2^{n-1}$ 3-adic intervals \( \{ I_{n,q} \}_{q=1}^{2^{n-1}} \) of scale \( n \) (and hence, of length \( 3^{-n} \)) composing the level \( n \) approximation to \( \mathcal{C}S_3 \). In the sequel, in agreement with the general definition of the geometric zeta function of a \( p \)-adic fractal string to be given in \( \S \, 3 \), \( \zeta_{\mathcal{C}S_3}(s) \) is initially defined by the following convergent Dirichlet series:

\[
\zeta_{\mathcal{C}S_3}(s) := \sum_{n=1}^{\infty} \frac{2^{n-1} \mu_H(I_{n,q})}{3^n},
\]

for \( \Re(s) > \log_3 2 \).

The following theorem provides the exact analog of Equations (2.1) and (2.2):

**Theorem 2.8.** The geometric zeta function of the nonarchimedean Cantor string is meromorphic in all of \( \mathbb{C} \) and is given by

\[
\zeta_{\mathcal{C}S_3}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}}, \quad \text{for } s \in \mathbb{C}.
\]

Hence, the set of complex dimensions of \( \mathcal{C}S_3 \) is given by

\[
\mathcal{D}_{\mathcal{C}S_3} = \{ D + ivp \mid v \in \mathbb{Z} \},
\]

where \( D = \log_3 2 \) is the dimension of \( \mathcal{C}S_3 \) and \( p = 2\pi / \log 3 \) is its oscillatory period; see Figure 5.

**Remark 2.9.** It is proved in [29] that \( D \) is the Minkowski dimension of \( \mathcal{C}S_3 \subset \mathbb{Z}_3 \); see Theorem 3.21 below. Clearly, it follows from the above computation that \( D \) is also the abscissa of convergence of the Dirichlet series initially defining \( \zeta_{\mathcal{C}S_3} \).

**Remark 2.10.** We will see in Example 4.5 that \( \mathcal{C}S_3 \) is a 3-adic self-similar string (in the sense of [28] and [41], with associated nonarchimedean self-similar set \( \mathcal{C} \) and such that \( \mathcal{C}S_3 = \mathbb{Z}_3 \setminus \mathcal{C}_3 \), the complement of the 3-adic Cantor set in \( \mathbb{Z}_3 \).

The following result is the analog (for the nonarchimedean Cantor string \( \mathcal{C}S_3 \)) of Theorem 2.4. It provides a precise description of \( \mathcal{C}S_3 \) as a countable disjoint union of 3-adic intervals. It also admits an archimedean counterpart, for the ternary Cantor string \( \mathcal{C} \) (which we have omitted to state in \( \S \, 2.2 \)). As we shall see in the more general context of \( \S \, 4 \), the property described in that theorem follows from the self-similarity of the nonarchimedean Cantor string \( \mathcal{C}S_3 \); see Figure 6.

**Theorem 2.11.** With the same notation as in Theorem 2.4, we have that

\[
\mathcal{C}S_3 = \bigcup_{\alpha=0}^{\infty} \bigcup_{w \in W_\alpha} \Phi_w(1 + 3\mathbb{Z}_3).
\]

The counterpart for the nonarchimedean Cantor string \( \mathcal{C}S_3 \) of the explicit tube formula (2.4) for the archimedean Cantor string \( \mathcal{C} \) is given by

\[
V_{\mathcal{C}S_3}(\varepsilon) = \frac{1}{6 \log 3} \sum_{\omega \in \mathcal{D}_{\mathcal{C}S_3}} \frac{\varepsilon^{1-\omega}}{1 - \omega},
\]

where \( \mathcal{D}_{\mathcal{C}S_3} \) is given by (2.10).

**Remark 2.12.** With the exception of the fractal tube formula (2.11) for \( \mathcal{C}S_3 \), which is derived in [29], all of the results stated in [2] are obtained in [27], where the interested reader can find their detailed proofs.
Remark 2.13. The precise definition of the volume of the tubular neighborhood of a \( p \)-adic fractal string (and in particular, of the 3-adic Cantor string) will be given in Definition 3.15 (which makes use of Definition 3.11) of §3.2.

2.5. A Comparative Study of the Real and 3-Adic Cantor Strings. A glance at Equations (2.2) and (2.10) shows that the archimedean and nonarchimedean Cantor strings \( CS \) and \( CS_3 \) have the exact same set of complex dimensions:

\[
D_{CS} = D_{CS_3} = \{ D + i\nu \mid \nu \in \mathbb{Z} \};
\]

in particular, they have the same Minkowski dimension \( D = \log_3 2 \) and the same oscillatory period \( p = 2\pi / \log 3 \). In the present case, the complex dimension are in arithmetic progression (with period \( p \)) along a single vertical line, \( \Re(s) = D \), and they are simple (i.e., they are simple poles of the geometric zeta function).

We will see in §4.3 and §4.4 that \( p \)-adic self-similar strings (of which \( CS_3 \) is the simplest, nontrivial example) are always lattice, in a strong sense, which implies that their complex dimensions are periodically distributed along finitely many vertical lines, beginning with the rightmost line \( \Re(s) = D \), where \( D \) is both the abscissa of convergence and the Minkowski dimension of the string.

We next focus our attention on the fractal tube formulas for \( CS \) and \( CS_3 \), as given by (2.4) and (2.11), respectively. In each case, one sums over the complex dimensions \( \omega \) in \( D_{CS} = D_{CS_3} \) a certain expression of \( \varepsilon \) and \( \omega \); namely,

\[
(2\varepsilon)^{1-\omega} \quad \text{or} \quad \varepsilon^{1-\omega},
\]

which can be interpreted as the residue at \( s = \omega \) of the so-called ‘tubular zeta function’ of \( CS \) or \( CS_3 \), respectively (cf. [32–34] and Remark 3.29 in §3.5).

Alternatively, since \( \zeta_{CS}(s) = \zeta_{CS_3}(s) \), in light of (2.1) and (2.9),

\[
\text{res}(\zeta_{CS}; \omega) = \text{res}(\zeta_{CS_3}; \omega) = \frac{1}{2\log 3},
\]

\footnote{11}{For now, we neglect the lower order term \( 2\varepsilon \) in (2.3), which does not have a counterpart in (2.10).}

\footnote{12}{Here and thereafter, we denote by \( \text{res}(f(s); \omega) \) (or \( \text{res}(f; \omega) \), when no ambiguity may arise) the residue of a meromorphic function \( f \) at the pole \( s = \omega \).}
for every $\omega \in D_{\mathcal{CS}} = D_{\mathcal{CS}_3}$, we can rewrite (2.4) and (2.11) as follows:

(2.14) \[ V_{\mathcal{CS}}(\varepsilon) + 2\varepsilon = \sum_{\omega \in D_{\mathcal{CS}}} res(\zeta_{\mathcal{CS}}; \omega) \frac{(2\varepsilon)^{1-\omega}}{\omega(1-\omega)} \]

and

(2.15) \[ V_{\mathcal{CS}_3}(\varepsilon) = 3^{-1} \sum_{\omega \in D_{\mathcal{CS}_3}} res(\zeta_{\mathcal{CS}_3}; \omega) \frac{\varepsilon^{1-\omega}}{1-\omega} \]

(recall that for $\mathcal{CS}_3$, $p = 3$ is the underlying prime).

Finally, we provide an additional reformulation of the tube formulas (2.4) and (2.11). This reformulation will make apparent the role played by the complex dimensions. Namely, the real parts of the complex dimensions govern the amplitudes of the underlying oscillations, while their imaginary parts are directly linked with the frequencies of these oscillations. More specifically, (2.4) or (2.14) implies that

(2.16) \[ (2\varepsilon)^{-1-D}(V_{\mathcal{CS}}(\varepsilon) + o(1)) = G_{\mathcal{CS}}(\log 3 \varepsilon^{-1}), \]

where $o(1) \to 0$ as $\varepsilon \to 0^+$ and

(2.17) \[ G_{\mathcal{CS}}(x) := \frac{1}{2\log 3} \sum_{n \in \mathbb{Z}} e^{2\pi inx} \frac{e^{2\pi inp}}{(D + inp)(1 - D - inp)} \]

is a bounded, nonconstant periodic function of period 1 on $\mathbb{R}$.\footnote{Actually, $G_{\mathcal{CS}}$ is bounded away from zero and from infinity; see [35] and [39] Fig. 2.6 and §2.3.1.}

Similarly, (2.11) or (2.15) becomes

(2.18) \[ \varepsilon^{-(1-D)}V_{\mathcal{CS}_3}(\varepsilon) = G_{\mathcal{CS}_3}(\log 3 \varepsilon^{-1}), \]

where

(2.19) \[ G_{\mathcal{CS}_3}(x) := \frac{3^{-1}}{2\log 3} \sum_{n \in \mathbb{Z}} e^{2\pi inx} \frac{1}{1 - D - inp} \]

is a nonconstant periodic function of period 1 on $\mathbb{R}$.

In light of (2.16)–(2.17) and (2.18)–(2.19), it is clear that neither the limit (as $\varepsilon \to 0^+$) of $\varepsilon^{-(1-D)}V_{\mathcal{CS}}(\varepsilon)$ nor the limit of $\varepsilon^{-(1-D)}V_{\mathcal{CS}_3}(\varepsilon)$ exists. Hence, neither the archimedean Cantor string $\mathcal{CS}$ nor the nonarchimedean Cantor string $\mathcal{CS}_3$ is Minkowski measurable (see [23, 29, 35, 39] and §3.3). However, by suitably averaging the left-hand side of (2.16) over a large number of periods of $G_{\mathcal{CS}}$, one can show that the appropriately defined average Minkowski content of $\mathcal{CS}$ exists and is given by

\[ M_{av}(\mathcal{CS}) = \frac{2^{-D}}{(1 - D) \log 2}; \]

cf. [39] Rem. 8.35.\footnote{The minor discrepancy between the value of $M_{av}(\mathcal{CS})$ given here and that of [39] Rem. 8.35 is due to the fact that $\mathcal{CS}$ is defined as in [39] §1.1.2 and not as in [39] §2.3.1; in particular, it has total length 1 rather than 3.} Similarly, by suitably averaging the left-hand side of (2.18) over infinitely many periods of $G_{\mathcal{CS}_3}$, one shows that the average Minkowski content of $\mathcal{CS}_3$ exists and is given by

\[ M_{av}(\mathcal{CS}_3) = \frac{1}{6(\log 3 - \log 2)}. \]
3. $p$-Adic Fractal Strings

Let $\Omega$ be a bounded open subset of $\mathbb{Q}_p$. Then it can be decomposed into a countable union of disjoint open balls with radius $p^{-n_j}$ centered at $a_j \in \mathbb{Q}_p$,

$$a_j + p^{n_j} \mathbb{Z}_p = B(a_j, p^{-n_j}) = \{ x \in \mathbb{Q}_p \mid |x - a_j|_p \leq p^{-n_j} \},$$

where $n_j \in \mathbb{Z}$ and $j \in \mathbb{N}^*$. There may be many different such decompositions since each ball can always be decomposed into smaller disjoint balls; see Equation (2.5). However, there is a canonical decomposition of $\Omega$ into disjoint balls with respect to a suitable equivalence relation, as we now explain.

**Definition 3.1.** Let $U$ be an open subset of $\mathbb{Q}_p$. Given $x, y \in U$, we write that $x \sim y$ if and only if there is a ball $B \subseteq U$ such that $x, y \in B$.

It is clear from the definition that the relation $\sim$ is reflexive and symmetric. To prove the transitivity, let $x \sim y$ and $y \sim z$. Then there are balls $B_1$ containing $x, y$ and $B_2$ containing $y, z$. Thus $y \in B_1 \cap B_2$; so it follows from the ultrametricity of $\mathbb{Q}_p$ that either $B_1 \subseteq B_2$ or $B_2 \subseteq B_1$. In any case, $x$ and $z$ are contained in the same ball; so $x \sim z$. Hence, the above relation $\sim$ is indeed an equivalence relation on the open set $U$. By a standard argument (and since $\mathbb{Q}$ is dense in $\mathbb{Q}_p$), one shows that there are at most countably many equivalence classes.

**Remark 3.2.** (Convex components) The equivalence classes of $\sim$ can be thought of as the ‘convex components’ of $U$. They are an appropriate substitute in the present nonarchimedean context for the notion of connected components, which is not useful in $\mathbb{Q}_p$ since $\mathbb{Z}_p$ (and hence, every interval) is totally disconnected. Note that given any $x \in U$, the equivalence class (i.e., the convex component) of $x$ is the largest ball containing $x$ (or equivalently, centered at $x$) and contained in $U$.

**Definition 3.3.** A $p$-adic (or nonarchimedean) fractal string $L_p$ is a bounded open subset $\Omega$ of $\mathbb{Q}_p$.

Thus it can be written, relative to the above equivalence relation, canonically as a disjoint union of intervals or balls:

$$L_p = \bigcup_{j=1}^{\infty} (a_j + p^{n_j} \mathbb{Z}_p) = \bigcup_{j=1}^{\infty} B(a_j, p^{-n_j}).$$

Here, $B(a_j, p^{-n_j})$ is the largest ball centered at $a_j$ and contained in $\Omega$. We may assume that the lengths (i.e., Haar measure) of the intervals $a_j + p^{n_j} \mathbb{Z}_p$ are nonincreasing, by reindexing if necessary. That is,

$$p^{-n_1} \geq p^{-n_2} \geq p^{-n_3} \geq \cdots > 0.$$ 

**Definition 3.4.** The geometric zeta function of a $p$-adic fractal string $L_p$ is defined as

$$\zeta_{L_p}(s) = \sum_{j=1}^{\infty} \left( \mu_H(a_j + p^{n_j} \mathbb{Z}_p) \right)^s = \sum_{j=1}^{\infty} p^{-n_j s}$$

for $\Re(s)$ sufficiently large.

---

16We shall often call a $p$-adic ball an *interval*. By ‘ball’ here, we mean a metrically closed and hence, topologically open (and closed) ball.
**Remark 3.5.** The geometric zeta function $\zeta_{\mathcal{L}_p}$ is well defined since the decomposition of $\mathcal{L}_p$ into the disjoint intervals $a_j + p^n, \mathbb{Z}_p$ is unique. Indeed, these intervals are the equivalence classes of which the open set $\Omega$ (defining $\mathcal{L}_p$) is composed. In other words, they are the $p$-adic “convex components” (rather than the connected components) of $\Omega$. Note that in the real (or archimedean) case, there is no difference between the convex or connected components of $\Omega$, and hence the above construction would lead to the same sequence of lengths as in [39, §1.2].

![Figure 7. The screen S and the window W.](image)

The **screen** $S$ is the graph of a real-valued, bounded and Lipschitz continuous function $S(t)$:

$$S = \{S(t) + it \mid t \in \mathbb{R}\}.$$  

The **window** $W$ is the part of the complex plane to the right of the screen $S$: (see Figure 7):

$$W = \{s \in \mathbb{C} \mid \Re(s) \geq S(\Im(s))\}.$$  

Let

$$\inf S = \inf_{t \in \mathbb{R}} S(t) \quad \text{and} \quad \sup S = \sup_{t \in \mathbb{R}} S(t),$$

and assume that $\sup S \leq \sigma$, where $\sigma = \sigma_{\mathcal{L}_p}$ is the abscissa of convergence of $\mathcal{L}_p$ (to be precisely defined in (3.4) below).

**Definition 3.6.** If $\zeta_{\mathcal{L}_p}$ has a meromorphic continuation to an open connected neighborhood of $W \subseteq \mathbb{C}$, then

$$D_{\mathcal{L}_p}(W) = \{\omega \in W \mid \omega \text{ is a pole of } \zeta_{\mathcal{L}_p}\}$$

is called the set of **visible complex dimensions** of $\mathcal{L}_p$. If no ambiguity may arise or if $W = \mathbb{C}$, we simply write $D_{\mathcal{L}_p} = D_{\mathcal{L}_p}(W)$ and call it the set of **complex dimensions** of $\mathcal{L}_p$.

---

\[18\] With the vertical and horizontal axes interchanged.
Moreover, the abscissa of convergence of \( L_p \) (or rather, of the Dirichlet series initially defining \( \zeta_{L_p} \) in Equation (3.2)) is denoted by \( \sigma_{L_p} \). Recall that it is defined by\(^{19}\)

\[
\sigma_{L_p} = \inf \left\{ \alpha \in \mathbb{R} \mid \sum_{j=1}^{\infty} p^{-nj \alpha} < \infty \right\}.
\]

**Remark 3.7.** In particular, if \( \zeta_{L_p} \) is entire (which occurs only in the trivial case when \( L_p \) is given by a finite union of intervals), then \( \sigma_{L_p} = -\infty \). Otherwise, \( \sigma_{L_p} \geq 0 \) (since \( L_p \) is composed of infinitely many intervals) and we will see in Theorem 3.21 that \( \sigma_{L_p} < \infty \) since \( \sigma_{L_p} \leq D_M \leq 1 \), where \( D_M = D_M_{L_p} \) is the Minkowski dimension of \( L_p \). Furthermore, it will follow from Theorem 3.21 that for a nontrivial \( p \)-adic fractal string, \( \sigma_{L_p} = D_M \). This is the case, for example, for the 3-adic Cantor string introduced in (2.4) for which \( \sigma = D_M = \log_3 2 \).

Observe that since \( D_{L_p}(W) \) is defined as a subset of the poles of a meromorphic function, it is at most countable.

Finally, we note that it is well known that \( \zeta_{L_p} \) is holomorphic for \( \Re(s) > \sigma_{L_p} \); see, e.g., \(^{50}\). Hence,

\[
D_{L_p} \subset \{ s \in \mathbb{C} \mid \Re(s) \leq \sigma_{L_p} \}.
\]

**Remark 3.8 (Archimedean fractal strings).** Archimedean or real fractal strings are defined as bounded open subsets of the real line \( \mathbb{R} = \mathbb{Q}_\infty \). They were initially defined in \(^{35}\), following an early example in \(^{22}\), and have been used extensively in a variety of settings; see, e.g., \(^{11, 18–19, 22–24, 26, 30–37, 45}\) and the books \(^{38, 39, 26}\). Since an open set \( \Omega \subset \mathbb{R} \) is canonically equal to the disjoint union of finitely or countably many open and bounded intervals (namely, its connected components), say \( \Omega = \bigcup_{j=1}^{\infty} I_j \), we may also describe a real fractal string by a sequence of lengths \( L = \{l_j\}_{j=1}^{\infty} \), where \( l_j = \mu_L(I_j) \) is the length or 1-dimensional Lebesgue measure of the interval \( I_j \), written in nonincreasing order:\(^{20}\)

\[
l_1 \geq l_2 \geq l_3 \geq \ldots.
\]

Note that since \( \mu_L(\Omega) < \infty, l_j \to 0 \) as \( j \to \infty \) (except in the trivial case when \( \Omega \) consists of finitely many intervals)\(^{21}\).

All the definitions given above for \( p \)-adic fractal strings have a natural counterpart for real fractal strings. For instance, the geometric zeta function of \( L \) is initially defined by

\[
\zeta_L(s) = \sum_{j=1}^{\infty} (\mu_L(I_j))^s = \sum_{j=1}^{\infty} l_j^s,
\]

for \( \Re(s) > \sigma_L \), the abscissa of convergence of \( L \), and for a given screen \( S \) and associated window \( W \), the set \( D_L = D_L(W) \) of visible complex dimensions of \( L \) is given exactly as in \(^{46, 48}\) of Definition 3.6 except with \( L_p \) and \( \zeta_{L_p} \) replaced with \( L \) and \( \zeta_L \), respectively. Similarly, \( \sigma_L \), the abscissa of convergence of \( L \) is given as in \(^{44, 47}\), except with the lengths of \( L \) instead of those of \( L_p \). Moreover, it follows from

\(^{19}\)See, e.g., \(^{50}\).

\(^{20}\)A justification for this identification is provided by the formula for the volume \( V_L(\varepsilon) \) of \( \varepsilon \)-inner tubes of \( \Omega \), as given by Equation (3.2) below.

\(^{21}\)Also observe that the 1-dimensional Lebesgue measure \( \mu_L \) is nothing but the Haar measure on \( \mathbb{R} = \mathbb{Q}_\infty \), normalized so that \( \mu_L([0, 1]) = 1 \).
that for any nontrivial real fractal string \( L \), we have \( \sigma_L = D_M \), the Minkowski dimension of \( L \) (i.e., of its topological boundary \( \partial \Omega \)).

We refer the interested reader to the research monographs [38, 39] for a full development of the theory of real fractal strings and their complex dimensions.

3.1. \( p \)-Adic Euler String. The following \( p \)-adic Euler string is a new example of \( p \)-adic fractal string, which is not self-similar (in the sense of §4). It is a natural \( p \)-adic counterpart of the elementary prime string, which is the local constituent of the completed harmonic string; cf. [39, §4.2.1].

Let \( X = p^{-1} \mathbb{Z}_p \). Then, by the ‘self-duplication’ formula (2.5),

\[
X = \bigcup_{\xi=0}^{p-1} (\xi p^{-1} + \mathbb{Z}_p).
\]

We now keep the first subinterval \( \mathbb{Z}_p \), and then decompose the next subinterval further. That is, we write

\[
p^{-1} + \mathbb{Z}_p = \bigcup_{\xi=0}^{p-1} (p^{-1} + \xi + p\mathbb{Z}_p).
\]

Again, iterating this process, we keep the first subinterval \( p^{-1} + p\mathbb{Z}_p \) in the above decomposition and decompose the next subinterval, \( p^{-1} + 1 + p\mathbb{Z}_p \). Continuing in this fashion, we obtain an infinite sequence of disjoint subintervals \( \{a_n + p^n\mathbb{Z}_p\}_{n=0}^\infty \), where \( \{a_n\}_{n=0}^\infty \) satisfies the following initial condition and recurrence relation:

\[
a_0 = 0 \quad \text{and} \quad a_n = a_{n-1} + p^{n-2} \quad \text{for all } n \geq 1.
\]

We call the corresponding \( p \)-adic fractal string,

\[
\mathcal{E}_p = \bigcup_{n=0}^\infty (a_n + p^n\mathbb{Z}_p),
\]

the \( p \)-adic Euler string. (See Figure 8.)

\[\begin{align*}
p^{-1}\mathbb{Z}_p & \quad \mid \quad \mid \quad \mid \\
\mathbb{Z}_p & \quad p^{-1} + \mathbb{Z}_p & \cdots & \quad (p-1)p^{-1} + \mathbb{Z}_p \\
p^{-1} + p\mathbb{Z}_p & \quad p^{-1} + 1 + p\mathbb{Z}_p & \cdots & \quad p^{-1} + p - 1 + p\mathbb{Z}_p \\
\vdots & \quad \vdots & \cdots & \quad \vdots
\end{align*}\]

\textbf{Figure 8.} Construction of the \( p \)-adic Euler string \( \mathcal{E}_p \).
The geometric zeta function of the \( p \)-adic Euler string \( \mathcal{E}_p \) is

\[
\zeta_{\mathcal{E}_p}(s) = \sum_{n=0}^{\infty} (\mu_H(a_n + p^n \mathbb{Z}_p))^s = \sum_{n=0}^{\infty} p^{-ns} = \frac{1}{1 - p^{-s}}, \quad \text{for } \Re(s) > 0.
\]

Therefore, \( \zeta_{\mathcal{E}_p} \) has a meromorphic extension to all of \( \mathbb{C} \) given by the last expression, which is the classic \( p \)-th Euler factor:

\[
\zeta_{\mathcal{E}_p}(s) = \frac{1}{1 - p^{-s}}, \quad \text{for } s \in \mathbb{C}.
\]

Hence, the set of complex dimensions of \( \mathcal{E}_p \) is given by

\[
D_{\mathcal{E}_p} = \{ D + i\nu \mid \nu \in \mathbb{Z} \},
\]

where \( D = \sigma = 0 \) and \( p = 2\pi/\log p \).

**Remark 3.9** (Ad`elic Euler string). Note that \( \zeta_{\mathcal{E}_p} \) is the \( p \)-th Euler factor of the Riemann zeta function; i.e.,

\[
\prod_{p<\infty} \zeta_{\mathcal{E}_p}(s) = \prod_{p<\infty} \frac{1}{1 - p^{-s}} = \sum_{n=1}^{\infty} \frac{1}{n^s} = \zeta(s) \quad \text{for } \Re(s) > 1.
\]

Recall that the meromorphic continuation \( \xi \) of the Riemann zeta function \( \zeta \) has the same (critical) zeros as \( \zeta \) and satisfies the functional equation \( \xi(s) = \xi(1-s) \).

We hope to form a certain ‘ad`elic product’ over all \( p \)-adic Euler strings (including the prime at infinity) so that the geometric zeta function of the resulting ad`elic Euler string \( \mathcal{E} \) is the completed Riemann zeta function. Formally, the ad`elic Euler string may be written as

\[
\mathcal{E} = \bigotimes_{p\leq \infty} \mathcal{E}_p
\]

and its geometric zeta function \( \zeta_{\mathcal{E}}(s) \) would then coincide with the completed Riemann zeta function \( \xi \) (see [47] and, e.g., [10]):

\[
\zeta_{\mathcal{E}}(s) = \xi(s) := \pi^{-s/2} \Gamma(s/2) \prod_{p<\infty} \frac{1}{1 - p^{-s}}.
\]

**Remark 3.10** (Comparison with the archimedean theory). From the geometric point of view, the nonarchimedean Euler string \( \mathcal{E}_p \) is more natural than its archimedean counterpart, the \( p \)-th elementary prime string \( h_p \), described in [39 §4.2.1]. Indeed, as we have just seen, \( \mathcal{E}_p \) has a very simple geometric definition. Since, by construction, \( \mathcal{E}_p \) and \( h_p \) have the same sequence of lengths \( \{p^{-n}\}_{n=0}^{\infty} \), they have the same geometric zeta function, namely, the \( p \)-th Euler factor

\[
\zeta_p(s) := \frac{1}{1 - p^{-s}}
\]

of the Riemann zeta function \( \zeta(s) \), and hence, the same set of complex dimensions

\[
D_p = \left\{ iv \frac{2\pi}{\log p} \mid \nu \in \mathbb{Z} \right\}.
\]

An ‘ad`elic version’ of the ‘harmonic string’ \( h \), a generalized fractal string whose geometric zeta function is \( \zeta_h(s) = \zeta(s) \), or rather, of its completion \( \tilde{h} \) (so that \( \zeta_{\tilde{h}}(s) = \xi(s) \)), is provided in [39 §4.2.1]. In particular, with each term being
interpreted as a positive measure on \((0, \infty)\) and the symbol \(*\) denoting multiplicative convolution on \((0, \infty)\), we have that
\[
(3.10) \quad h = *_{p<\infty}h_p \quad \text{and} \quad \tilde{h} = *_{p\leq \infty}h_p.
\]

Furthermore, a noncommutative geometric version of this construction is provided in \cite{29} in terms of the ‘prime fractal membrane’; see especially, \cite{29}, Chaps. 3 and 4, along with \cite{31}. Heuristically, a ‘fractal membrane’ (as introduced in \cite{29}) is a kind of adelic, noncommutative torus of infinite genus. It can also be thought of as a ‘quantized fractal string’; see \cite{25} Chap. 3. It is rigorously constructed in \cite{31} using Dirac-type operators, Fock spaces, Toeplitz algebras, and associated spectral triples (in the sense of \cite{4}); see also \cite{29} §4.2. We hope in the future to obtain a suitable nonarchimedean version of that construction. It is possible that in the process, we will establish contact with the physically motivated work in \cite{5} involving \(p\)-adic quantum mechanics.

### 3.2. Volume of Thin Inner Tubes

In this section, based on a part of \cite{29}, we provide a suitable analog in the \(p\)-adic case of the ‘boundary’ of a fractal string and of the associated inner tubes (or “inner \(\varepsilon\)-neighborhoods”). Moreover, we give the \(p\)-adic counterpart of the expression that yields the volume of the inner tubes (see Theorem 3.16). This result serves as a starting point in \cite{29} for proving the corresponding explicit tube formula.

**Definition 3.11.** Given a point \(a \in \mathbb{Q}_p\) and a positive real number \(r > 0\), let \(B = B(a, r) = \{x \in \mathbb{Q}_p \mid |x - a|_p \leq r\}\) be a metrically closed ball in \(\mathbb{Q}_p\), as above.\(^{24}\)
We call \(S = S(a, r) = \{x \in \mathbb{Q}_p \mid |x - a|_p = r\}\) the sphere of \(B\).\(^{25}\)
Let \(\mathcal{L}_p = \bigcup_{j=1}^{\infty} B(a_j, r_j)\) be a \(p\)-adic fractal string. We then define the metric boundary \(\beta \mathcal{L}_p\) of \(\mathcal{L}_p\) to be the disjoint union of the corresponding spheres, i.e.,
\[
\beta \mathcal{L}_p = \bigcup_{j=1}^{\infty} S(a_j, r_j).
\]

Given a real number \(\varepsilon > 0\), define the thick \(p\)-adic ‘inner \(\varepsilon\)-neighborhood’ (or ‘inner tube’) of \(\mathcal{L}_p\) to be
\[
(3.11) \quad \mathcal{N}_\varepsilon = \mathcal{N}_\varepsilon(\mathcal{L}_p) := \{x \in \mathcal{L}_p \mid d_p(x, \beta \mathcal{L}_p) < \varepsilon\},
\]
where \(d_p(x, E) = \inf \{|x - y|_p \mid y \in E\}\) is the \(p\)-adic distance of \(x \in \mathbb{Q}_p\) to a subset \(E \subset \mathbb{Q}_p\). Then the volume \(\mathcal{V}_\mathcal{L}_p(\varepsilon)\) of the thick inner \(\varepsilon\)-neighborhood of \(\mathcal{L}_p\) is defined to be the Haar measure of \(\mathcal{N}_\varepsilon\), i.e., \(\mathcal{V}_\mathcal{L}_p(\varepsilon) = \mu_H(\mathcal{N}_\varepsilon)\).

**Lemma 3.12.** Let \(B = B(a, r)\) and \(S = S(a, r)\), as in Definition 3.11. Then, for any positive number \(\varepsilon < r\), we have
\[
(3.12) \quad \mathcal{N}_\varepsilon(B) := \{x \in B \mid d_p(x, S) < \varepsilon\} = S.
\]
Hence, if \(r = p^{-m}\) for some \(m \in \mathbb{Z}\), then for all \(\varepsilon < r\),
\[
(3.13) \quad \mu_H(\{x \in B \mid d_p(x, S) < \varepsilon\}) = \mu_H(S) = (1 - p^{-1})p^{-m}.
\]

\(^{24}\)Recall that it follows from the ultrametricity of \(|\cdot|_p\) that \(B\) is topologically both closed and open (i.e., clopen) in \(\mathbb{Q}_p\).

\(^{25}\)In our sense, \(S\) also coincides with the ‘metric boundary’ of \(B\), as given in this definition.
Proof. (i) Clearly \( S \subseteq \{ x \in B \mid d_p(x, S) < \varepsilon \} \) since for any \( x \in S \), \( d_p(x, S) = 0 \). Next, fix \( \varepsilon \) with \( 0 < \varepsilon < r \) and let \( x \in B \) be such that \( d_p(x, S) < \varepsilon \). Then there must exist \( y \in S \) such that \( |x - y|_p < \varepsilon \). But, since \( |y - a|_p = r \), we deduce from the fact that every “triangle” in \( \mathbb{Q}_p \) is isosceles \([21, \text{p. 6}]\) that \( |x - a|_p = |y - a|_p \) and thus \( x \in S \). This completes the proof of (3.14).

(ii) We next establish formula (3.13). In light of Equation (3.12), it suffices to show that

\[
\mu_H(S) = (1 - p^{-1})p^{-m}.
\]

Let \( S^1 = S(0,1) = \{ x \in \mathbb{Q}_p \mid |x|_p = 1 \} \) denote the unit sphere in \( \mathbb{Q}_p \). Since \( S = S(a, p^{-m}) = a + p^mS^1 \), we have that \( \mu_H(S) = \mu_H(S^1)p^{-m} \). Next we note that

\[
B(0,1) = \bigcup_{m \geq 0} S(0,p^{-m})
\]

is a disjoint union. Hence, by taking the Haar measure of \( B(0,1) \), we deduce that

\[
1 = \left( \sum_{m=0}^{\infty} p^{-m} \right) \mu_H(S^1) = \frac{1}{1 - p^{-1}}\mu_H(S^1),
\]

from which (3.14) and hence, in light of part (i), (3.13) follows.

Theorem 3.13 (Volume of thick inner tubes). Let \( \mathcal{L}_p = \bigcup_{j=1}^{\infty} B(a_j, p^{-n_j}) \) be a \( p \)-adic fractal string. Then, for any \( \varepsilon > 0 \), we have

\[
\mathcal{V}_{\mathcal{L}_p}(\varepsilon) = (1 - p^{-1}) \sum_{j=1}^{k} p^{-n_j} + \sum_{j > k} p^{-n_j}
\]

where \( k = k(\varepsilon) \) is the largest integer such that \( p^{-n_k} \geq \varepsilon \).

Sketch of the proof. In light of the definition of \( \mathcal{N}_\varepsilon = \mathcal{N}_\varepsilon(\mathcal{L}_p) \) given in Equation (3.11) and the definition of \( k \) given in the theorem, we have that

\[
\mathcal{N}_\varepsilon = \bigcup_{j=1}^{k} S_j \cup \bigcup_{j > k} B_j,
\]

where \( B_j := B(a_j, p^{-n_j}) \) and \( S_j := S(a_j, p^{-n_j}) \) for each \( j \geq 1 \).

We then apply Lemma 3.12 to deduce the expression of \( \mathcal{V}_{\mathcal{L}_p}(\varepsilon) = \mu_H(\mathcal{N}_\varepsilon) \) stated in Equations (3.16) and (3.17).

Note that \( \zeta_{\mathcal{L}_p}(1) = \sum_{j=1}^{\infty} p^{-n_j} \) is the volume of \( \mathcal{L}_p \) (or rather, of the bounded open subset \( \Omega \) of \( \mathbb{Q}_p \) representing \( \mathcal{L}_p \)):

\[
\zeta_{\mathcal{L}_p}(1) = \mu_H(\mathcal{L}_p) < \infty.
\]

It is clearly independent of the choice of \( \Omega \) representing \( \mathcal{L}_p \), and so is \( \mathcal{V}_{\mathcal{L}_p}(\varepsilon) \) in light of either (3.16) or (3.17).

Corollary 3.14. The following limit exists in \( (0, \infty) \):

\[
\lim_{\varepsilon \to 0^+} \mathcal{V}_{\mathcal{L}_p}(\varepsilon) = \mu_H(\beta \mathcal{L}_p) = (1 - p^{-1})\zeta_{\mathcal{L}_p}(1).
\]
This follows by letting $\varepsilon \to 0^+$ in either (3.16) and (3.17) and noting that
$k = k(\varepsilon) \to \infty$.

Corollary 3.14 combined with the fact that $\beta L_p \subset N_\varepsilon(L_p)$ for any $\varepsilon > 0$, naturally leads us to introduce the following definition.

**Definition 3.15.** Given $\varepsilon > 0$, the thin $p$-adic ‘inner $\varepsilon$-neighborhood’ (or ‘inner tube’) of $L_p$ is given by

$$(3.19) \quad N_\varepsilon = N_\varepsilon(L_p) := N_\varepsilon(L_p) \setminus \beta L_p.$$ 

Then, in light of Corollary 3.14 the volume $V_{L_p}(\varepsilon)$ of the thin inner $\varepsilon$-neighborhood of $L_p$ is defined to be the Haar measure of $N_\varepsilon$ and is given by

$$(3.20) \quad V_{L_p}(\varepsilon) := \mu_H(N_\varepsilon) = V_{L_p}(\varepsilon) - \mu_H(\beta L_p).$$

Note that, by construction, we now have $\lim_{\varepsilon \to 0^+} V_{L_p}(\varepsilon) = 0$.

We next state the counterpart (for thin inner tubes) of Theorem 3.13, which is the key result that will enable us to obtain an appropriate $p$-adic analog of the fractal tube formula as well as of the notion of Minkowski dimension and content (see (3.3)).

**Theorem 3.16 (Volume of thin inner tubes).** Let $L_p = \bigcup_{j=1}^{\infty} B(a_j, p^{-n_j})$ be a $p$-adic fractal string. Then, for any $\varepsilon > 0$, we have

$$(3.21) \quad V_{L_p}(\varepsilon) = p^{-1} \sum_{j > k} p^{-n_j} = p^{-1} \sum_{j : p^{-n_j} < \varepsilon} p^{-n_j}$$

$$(3.22) \quad = p^{-1} \left( \zeta_{L_p}(1) - \sum_{j=1}^{k} p^{-n_j} \right),$$

where $k = k(\varepsilon)$ is the largest integer such that $p^{-n_k} \geq \varepsilon$, as before.

**Remark 3.17.** Observe that because the center $a$ of a $p$-adic ball $B = B(a, p^{-n})$ can be chosen arbitrarily without changing its radius $p^{-n}$, the metric boundary of a ball, $\beta B = S = S(a, p^{-n})$, may depend on the choice of $a$. Note, however, that in view of Equation (3.13) in Lemma 3.12 its volume $\mu_H(S)$ depends only on the radius of $B$. Similarly, even though the decomposition of a $p$-adic fractal string $\Omega$ (i.e., $L_p$) into maximal balls $B_j = B_j(a_j, p^{-n_j})$ is canonical, ‘the’ metric boundary of $L_p$, $\beta L_p = \bigcup_{j=1}^{\infty} S(a_j, r_j)$, may in general depend on the choice of the centers $a_j$. However, according to Corollary 3.14 $\mu_H(\beta L_p)$ is independent of this choice and hence, neither $V_{L_p}(\varepsilon) = \mu_H(N_\varepsilon(L_p))$ nor $V_{L_p}(\varepsilon) = \mu_H(N_\varepsilon(L_p))$ depends on the choice of the centers. Indeed, in light of Theorem 3.13 and Theorem 3.16 $V_{L_p}(\varepsilon)$ and $V_{L_p}(\varepsilon)$ depend only on the choice of the $p$-adic lengths $p^{-n_j}$, and hence solely on the $p$-adic fractal string $L_p$, viewed as a nonincreasing sequence of positive numbers, and not on the representation $\Omega$, let alone on the choice of the centers of the balls of which $\Omega$ is composed. Although it is not entirely analogous to it, this situation is somewhat reminiscent of the fact that the volume $V_{\Omega}(\varepsilon)$ of the inner $\varepsilon$-neighborhoods of an archimedean fractal string depends only on its lengths $\{l_j\}_{j=1}^{\infty}$ and not on the representative $\Omega$ of $\mathcal{L}$ as a bounded open set; see Equation (3.23) and the discussion surrounding it in Remark 3.18.

**Remark 3.18 (Comparison between the archimedean and the nonarchimedean cases).** Recall that $V_{L_p}(\varepsilon)$ does not tend to zero as $\varepsilon \to 0^+$, but that instead it
tends to the positive number \((1 - p^{-1})\zeta_{C_p}(1)\), whereas \(V_{E_p}(\varepsilon)\) does tend to zero. This is the reason why the Minkowski dimension must be defined in terms of \(V_{E_p}(\varepsilon)\) (as will be done in \([39]\)) rather than in terms of \(V_{E_p}(\varepsilon)\). Indeed, if \(V_{E_p}(\varepsilon)\) were used instead, then every \(p\)-adic fractal string would have Minkowski dimension 1. This would be the case even for a trivial \(p\)-adic fractal string composed of a single interval, for example. This is also why, in the \(p\)-adic case, we will focus only on the tube formula for \(V_{E_p}(\varepsilon)\) rather than for \(V_{E_p}(\varepsilon)\), although the latter could be obtained by means of the same techniques.

Note the difference between the expressions for \(V_L(\varepsilon)\) in the case of an archimedean fractal string \(L\) and for its nonarchimedean thin (resp., thick) counterpart \(V_{E_p}(\varepsilon)\) (resp., \(V_{E_p}(\varepsilon)\)) in the case of a \(p\)-adic fractal string \(E_p\). Compare Equation (8.1) of \([39]\) (which was first obtained in \([35]\)),

\[
V_L(\varepsilon) = \sum_{j \mid l_j \geq 2\varepsilon} 2\varepsilon + \sum_{j \mid l_j < 2\varepsilon} l_j,
\]

with Equations (3.21)–(3.22) in Theorem 3.16. (Here, we are using the notation of Remark 3.8 to which the reader is referred to for a brief introduction to real fractal strings.) It follows, in particular, that \(V_L(\varepsilon)\) is a continuous function of \(\varepsilon\) on \((0, \infty)\), whereas \(V_{E_p}(\varepsilon)\) (and hence also \(V_{E_p}(\varepsilon)\)) is discontinuous (because it is a step function with jump discontinuities at each point \(p^{-n_j}\), for \(j = 1, 2, \ldots\)). The above discrepancies between the archimedean and the nonarchimedean cases help explain why the tube formula for real and \(p\)-adic fractal strings have a similar form, but with different expressions for the corresponding ‘tubular zeta function’ (in the sense of \([32\), \([33]\)). We note that a minor aspect of these discrepancies is that \(2\varepsilon\) is now replaced by \(\varepsilon\). Interestingly, this is due to the fact that the unit interval \([0, 1]\) has inradius 1/2 in \(\mathbb{R} = \mathbb{Q}_\infty\) whereas \(\mathbb{Z}_p\) has inradius 1 in \(\mathbb{Q}_p\).²²

Finally, we note that for an archimedean fractal string \(L\), there is no reason to distinguish between the ‘thin volume’ \(V_L\) and the ‘thick volume’ \(V_L\), as we now explain. Indeed, the archimedean analogue \(\beta L\) of the metric boundary is a countable set, and hence has measure zero, no matter which realization \(\Omega\) one chooses for \(L\). More specifically, in the notation of Remark 3.8, \(\beta L\) consists of all the endpoints of the open intervals \(I_j\) (the connected components of \(\Omega\), or equivalently, its convex components). Hence, \(\mu_L(\beta L) = 0\) and so

\[
V_L(\varepsilon) := V_L(\varepsilon) - \mu_L(\beta L) = V_L(\varepsilon),
\]

as claimed.

The reason that \(\beta L\) is the ternary Cantor string \(CS\), then \(\beta L\) is the countable set consisting of all the endpoints of the ‘deleted intervals’ in the construction of the real Cantor set \(C\) (see \([21\); in other words, \(\beta L\) is the set \(T\) of ternary points (which has measure zero because it is countable). Hence, the metric boundary \(\beta L\) of \(CS\) is dense in \(\partial L\), the topological boundary of \(CS\), and which in the present case, coincides with the ternary Cantor set \(C\). Also note that the fact that \(C = \partial L\) (and not \(T = \beta L\)) has measure zero is purely coincidental and completely irrelevant here. Indeed, the same type of argument would apply if \(L\) were any archimedean fractal string, even if \(\mu_L(\partial L) > 0\) as is the case for example, if \(\partial L\) is a ‘fat Cantor set’ (i.e., a Cantor set of positive measure) or, more generally, if \(\partial L\) is a ‘fat fractal’

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²²Recall that the inradius of a subset \(E\) of a metric space is the supremum of the radii of the balls entirely contained in \(E\).
The underlying reason is that in the archimedean case, the topological boundary $\partial \mathcal{L} = \partial \Omega$ is disjoint from $\Omega$ (since $\Omega$ is open), and hence, does not play any role in the computation of $V_\mathcal{L}(\varepsilon)$ or of $V_\mathcal{C}(\varepsilon)$. By contrast, it is not true that the metric boundary $\beta \mathcal{L}$ and $\Omega$ are disjoint (since, in fact, $\beta \mathcal{L} \subset \Omega$), but what is remarkable is that the Minkowski dimension of $\beta \mathcal{L}$ coincides with that of its closure, and hence (in most cases of interest), with $D_{\mathcal{M}, \mathcal{L}}$; see [22] and the relevant references therein.

As a first application of Theorem 3.16, we can obtain, via a direct computation, a tube formula for the $p$-adic Euler string $\mathcal{E}_p$; that is, an explicit formula for the volume of the thin inner $\varepsilon$-neighborhood, $V_{\mathcal{E}_p}(\varepsilon)$, as given in Definition 3.15. Later on, we will similarly obtain a tube formula for the nonarchimedean counterpart of the Cantor string, as discussed in §2.4. We will also show how to recover these results from the general theory developed in the next section.

Example 3.19. (Explicit and exact tube formula for the $p$-adic Euler string $\mathcal{E}_p$). Let $\mathcal{E}_p$ be the $p$-adic Euler string defined in §3.1. Given $\varepsilon > 0$, let $k$ be the largest integer such that $\mu_H(a_k + p^k \mathbb{Z}_p) = p^{-k} \geq \varepsilon$; then $k = \lfloor \log_p \varepsilon^{-1} \rfloor$. Thus, by Equation (3.21) of Theorem 3.16 we have successively:

$$V_{\mathcal{E}_p}(\varepsilon) = p^{-1} \sum_{n=k+1}^{\infty} p^{-n} = \frac{p^{-1}}{p-1} p^{-k} = \frac{p^{-1}}{p-1} p^{-\log_p \varepsilon^{-1}} \left( \frac{1}{p} \right)^{-\lfloor \log_p \varepsilon^{-1} \rfloor} = \frac{p^{-1}}{p-1} \log p \sum_{n \in \mathbb{Z}} \varepsilon^{1-np} \frac{1}{1-np} = \frac{1}{p \log p} \sum_{\omega \in \mathcal{D}_{\mathcal{E}_p}} \varepsilon^{1-\omega} \frac{1}{1-\omega}.$$  

(3.24)

We now explain some of the steps above. In the third equality, we have written that $k = \log_p \varepsilon^{-1} - \lfloor \log_p \varepsilon^{-1} \rfloor$. Furthermore, in the next to last equality, we have appealed to the Fourier series expansion for $b^{-\{x\}}$ given by

$$b^{-\{x\}}(x) = \frac{b-1}{b} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i n x}}{\log b + 2\pi i n},$$

(3.25)

for $b = p^{-1}$ and $x = \log_p \varepsilon^{-1}$. (See [39, Eq. (1.13)].) Finally, in the last equality, we have used Equation (3.7) for the set of complex dimensions $\mathcal{D}_{\mathcal{E}_p}$ of $\mathcal{E}_p$.

3.3. Minkowski Dimension. In the sequel, the (inner) Minkowski dimension and the (inner) Minkowski content of a $p$-adic fractal string $\mathcal{L}_p$ (or, equivalently, of its metric boundary $\beta \mathcal{L}_p$, see Definition 3.11) is defined exactly as the corresponding notion for a real fractal string (see [39, Defn. 1.2]), except for the fact that we use

---

29Here, for $x \in \mathbb{R}$, we write $x = [x] + \{x\}$, where $[x]$ is the integer part and $\{x\}$ is the fractional part of $x$; i.e., $x \in \mathbb{Z}$ and $0 \leq x < 1$. 

---
the definition of \( V(\varepsilon) = V_{\mathcal{L}_p}(\varepsilon) \) provided in Equation (3.20) of §3.2. (For reasons that will be clear to the reader later on in this section, we denote by \( D_M = D_{M,\mathcal{L}_p} \) instead of by \( D = D_{\mathcal{L}_p} \) the Minkowski dimension of \( \mathcal{L}_p \).) More specifically, the Minkowski dimension of \( \mathcal{L}_p \) is given by

\[
D_M = D_{M,\mathcal{L}_p} := \inf \left\{ \alpha \geq 0 \mid V_{\mathcal{L}_p}(\varepsilon) = O(\varepsilon^{1-\alpha}) \text{ as } \varepsilon \to 0^+ \right\}.
\]

Furthermore, \( \mathcal{L}_p \) is said to be Minkowski measurable, with Minkowski content \( \mathcal{M} \), if the limit

\[
\mathcal{M} = \lim_{\varepsilon \to 0^+} V_{\mathcal{L}_p}(\varepsilon) \varepsilon^{-(1-D_M)}
\]

exists in \((0, \infty)\).

**Remark 3.20.** Note that since \( V_{\mathcal{L}_p}(\varepsilon) = V_{\mathcal{L}_p}(\varepsilon) - \mu_H(\beta \mathcal{L}_p) \), the above definition of the Minkowski dimension is somewhat analogous to that of “exterior dimension”, which is sometimes used in the archimedean case to measure the roughness of a ‘fat fractal’ (i.e., a fractal with positive Lebesgue measure). The notion of exterior dimension has been useful in the study of aspects of chaotic nonlinear dynamics; see, e.g., [13] and the survey article [43].

The following theorem (from [29]) is the exact \( p \)-adic analog of [39, Thm. 1.10] (first observed in [23], using a result of Besicovitch and Taylor [2]).

**Theorem 3.21.** Let \( \mathcal{L}_p \) be a nontrivial \( p \)-adic fractal string. Then the abscissa of convergence \( \sigma_{\mathcal{L}_p} \) of the geometric zeta function \( \zeta_{\mathcal{L}_p} \) coincides with the Minkowski dimension \( D_M \). That is, \( \sigma_{\mathcal{L}_p} = D_M \).

**Remark 3.22.** For any \( p \)-adic fractal string, we have \( 0 \leq D_M \leq 1 \). Indeed, by definition, \( D_M \geq 0 \); furthermore, in light of (3.26), \( D_M \leq 1 \) since a \( p \)-adic fractal string is a bounded open set, and hence, has finite volume.

The next corollary follows by combining Theorem 3.21 and Remark 3.22.

**Corollary 3.23.** Let \( \mathcal{L}_p \) be a nontrivial \( p \)-adic fractal string. Then \( 0 \leq \sigma \leq 1 \).

**Remark 3.24.** The proof of \( \sigma_{\mathcal{L}_p} \leq D_M \) in Theorem 3.21 is obtained by adapting the first part of the proof of [39, Thm. 1.10]. In light of the new form of \( V_{\mathcal{L}_p}(\varepsilon) \) given by Theorem 3.16, however, it does not seem possible to prove the reverse inequality \( \sigma_{\mathcal{L}_p} \geq D_M \) by simply adapting the second part of the proof of [39, Thm. 1.10]. Nevertheless, this latter inequality follows from the following new integral representation of the geometric zeta function \( \zeta_{\mathcal{L}_p} \) in Lemma 3.25, a specification to \( p \)-adic fractal strings of a general lemma in [29].

**Lemma 3.25.** Let \( \mathcal{L}_p \) be a \( p \)-adic fractal string, then

\[
\zeta_{\mathcal{L}_p}(s) = \zeta_{\mathcal{L}_p}(1) l_1^{s-1} + p(1-s) \int_0^{l_1} V_{\mathcal{L}_p}(\varepsilon) \varepsilon^{s-2} d\varepsilon,
\]

where \( l_1 = p^{-n_1} \) as in (3.1). Furthermore, the integral converges exactly when \( \zeta_{\mathcal{L}_p}(s) = \sum_{j=1}^{\infty} p^{-n_j s} \) converges.

\[30\text{Recall that as was explained towards the end of Remark 3.16 in the archimedean case, } V(\varepsilon) = V_{\mathcal{L}_p}(\varepsilon) \text{ is the same, whether it is defined by the analog for } \mathcal{L} \text{ of (2.29) or by the counterpart of (7.20) in Definition 4.14.}
\]

\[31\text{Note that like in [39, Thm. 1.10], we need to assume that } \mathcal{L}_p \text{ has infinitely many lengths since in the latter case, we have } \sigma_{\mathcal{L}_p} = -\infty.
\]

\[32\text{i.e., } \mathcal{L}_p \text{ is not given by a finite union of intervals.}
\]
3.4. Languid and Strongly Languid p-Adic Fractal Strings. In [38, Def. 3.26] we will obtain explicit tube formulas for p-adic fractal strings, with and without error term. (See Theorem 3.28 and Corollary 3.30.) We will then apply the tube formula without error term (the strongly languid case of Theorem 3.28) to p-adic self-similar strings in §3.5 and will apply its corollary to the p-adic Euler string discussed in §3.1 and revisited in Example 3.32 (at the end of §3.5).

In order to state the explicit formulas with (or without) error term, we need to assume the following technical hypotheses (see [39, Defns. 5.2 and 5.3] and recall Definition 3.26. A p-adic fractal string \( L_p \) is said to be languid if its geometric zeta function \( \zeta_{L_p} \) satisfies the following growth conditions: There exist real constants \( \kappa \) and \( C > 0 \) and a two-sided sequence \( \{T_n\}_{n \in \mathbb{Z}} \) of real numbers such that \( T_{n-1} < 0 < T_n \) for \( n \geq 1 \), and

\[
\lim_{n \to \infty} T_n = \infty, \quad \lim_{n \to \infty} T_{-n} = -\infty, \quad \lim_{n \to \infty} \frac{T_n}{|T_{-n}|} = 1,
\]

such that

- **L1** For all \( n \in \mathbb{Z} \) and all \( u \geq S(T_n) \),
  \[
  |\zeta_{L_p}(u + iT_n)| \leq C(|T_n| + 1)^\kappa,
  \]

- **L2** For all \( t \in \mathbb{R} \), \( |t| \geq 1 \),
  \[
  |\zeta_{L_p}(S(t) + it)| \leq C|t|^{\kappa}.
  \]

We say that \( L_p \) is strongly languid if its geometric zeta function \( \zeta_{L_p} \) satisfies the following conditions, in addition to L1 with \( S(t) \equiv -\infty \): There exists a sequence of screens \( S_m : t \mapsto S_m(t) \) for \( m \geq 1 \), \( t \in \mathbb{R} \), with \( \sup S_m \to -\infty \) as \( m \to \infty \) and with a uniform Lipschitz bound \( \sup_{m \geq 1} ||S_m||_{Lip} < \infty \), such that

- **L2’** There exist constants \( A, C > 0 \) such that for all \( t \in \mathbb{R} \) and \( m \geq 1 \),
  \[
  |\zeta_{L_p}(S_m(t) + it)| \leq CA^{|S_m(t)|}(|t| + 1)^\kappa.
  \]

**Remark 3.27.** (a) Intuitively, hypothesis L1 is a polynomial growth condition along horizontal lines (necessarily avoiding the poles of \( \zeta_{L_p} \)), while hypothesis L2 is a polynomial growth condition along the vertical direction of the screen.

- (b) Clearly, condition L2’ is stronger than L2. Therefore, if \( L_p \) is strongly languid then it is also languid (for each screen \( S_m \) separately).

- (c) Moreover, if \( L_p \) is languid for some \( \kappa \), then it is also languid for every larger value of \( \kappa \). The same is also true for strongly languid strings.

- (d) Finally, hypotheses L1 and L2 require that \( \zeta_{L_p} \) has an analytic (i.e., meromorphic) continuation to an open, connected neighborhood of \( \Re(s) \geq \sigma_{L_p} \), while L2’ requires that \( \zeta_{L_p} \) has a meromorphic continuation to all of \( \mathbb{C} \).

3.5. Explicit Tube Formulas for p-Adic Fractal Strings. The following result is the counterpart in this context of Theorem 8.1 of [39], the distributional tube formula for real fractal strings. It is established in [29] by using, in particular, the extended distributional explicit formula of [39, Thms. 5.26 and 5.27], along with the expression for the volume of thin inner \( \varepsilon \)-tubes obtained in Theorem 3.10.

**Theorem 3.28** (p-Adic explicit tube formula). (i) Let \( L_p \) be a languid p-adic fractal string for some real exponent \( \kappa \) and a screen \( S \) that lies strictly to the left
of the vertical line $\Re(s) = 1$. Further assume that $\sigma_{L_p} < 1$ \footnote{Recall from Corollary \ref{cor:1} that we always have $\sigma_{L_p} \leq 1$. Moreover, we will see in Remark \ref{rem:1} II that if $L_p$ is self-similar, then $\sigma_{L_p} < 1$.}. Then the volume of the thin inner $\varepsilon$-neighborhood of $L_p$ is given by the following distributional explicit formula, on test functions in $D(0, \infty)$ \footnote{Here, $D(0, \infty)$ is the space of $C^\infty$ functions with compact support in $(0, \infty)$.}:

\begin{equation}
V_{L_p}(\varepsilon) = \sum_{\omega \in D_{L_p}(W)} \text{res}\left(\frac{p^{-1}\zeta_{L_p}(s)\varepsilon^{1-s}}{1-s}; \omega\right) + R_p(\varepsilon),
\end{equation}

where $D_{L_p}(W)$ is the set of visible complex dimensions of $L_p$ (as given in Definition \ref{def:2}). Here, the distributional error term is given by

\begin{equation}
R_p(\varepsilon) = \frac{1}{2\pi i} \int_S \frac{p^{-1}\zeta_{L_p}(s)\varepsilon^{1-s}}{1-s} ds
\end{equation}

and is estimated distributionally \footnote{As in \cite{39} Defn. 5.29.} by

\begin{equation}
R_p(\varepsilon) = O(\varepsilon^{1-\sup S}), \quad \text{as } \varepsilon \to 0^+.
\end{equation}

(ii) Moreover, if $L_p$ is strongly languid (as in the second part of Definition \ref{def:2}), then we can take $W = \mathbb{C}$ and $R_p(\varepsilon) \equiv 0$, provided we apply this formula to test functions supported on compact subsets of $[0, A]$. The resulting explicit formula without error term is often called an exact tube formula in this case.

Remark 3.29. We may rewrite the (typically infinite) sum in (3.29) as follows:

\begin{equation}
\sum_{\omega \in D_{L_p}(W)} \text{res}(\zeta_{L_p}(\varepsilon; s); s = \omega),
\end{equation}

where (by analogy with the definitions and results in \ref{32}, \ref{34}),

\begin{equation}
\zeta_{L_p}(\varepsilon; s) := \frac{p^{-1}\zeta_{L_p}(s)\varepsilon^{1-s}}{1-s}
\end{equation}

is called the nonarchimedean tubular zeta function of the $p$-adic fractal string $L_p$.

By contrast, the archimedean tubular zeta function (in the present one-dimensional situation) of a real fractal string $L$ is given by

\begin{equation}
\zeta_{L}(\varepsilon; s) := \frac{\zeta_L(s)(2\varepsilon)^{1-s}}{s(1-s)},
\end{equation}

and the analog of the above sum in the archimedean tube formula of \ref{33} (as rewritten in \ref{34}) is given as in \ref{35}, \ref{36}, except with $L_p$ replaced by $L$ and with $D_{L}(W) \cup \{0\}$ instead of $D_{L_p}(W)$. Note that $\zeta_L(\varepsilon; s)$ typically has a pole at $s = 0$, whereas $\zeta_{L_p}(\varepsilon; s)$ doesn’t.

Corollary 3.30 (p-Adic fractal tube formula). If, in addition to the hypotheses in Theorem 3.28, we assume that all the visible complex dimensions of $L_p$ are simple, then

\begin{equation}
V_{L_p}(\varepsilon) = \sum_{\omega \in D_{L_p}(W)} c_\omega \varepsilon^{1-\omega} + R_p(\varepsilon),
\end{equation}

\footnote{As in \cite{39} Defn. 5.29.}
where $c_\omega = p^{-1} \text{res} (\zeta_{L_p}; \omega)$. Here, the error term $\mathcal{R}_p$ is given by (3.30) and is estimated by (3.31) in the languid case. Furthermore, we have $\mathcal{R}_p(\varepsilon) \equiv 0$ in the strongly languid case provided we choose $W = \mathbb{C}$.

**Remark 3.31.** In [39, Ch. 8], under different sets of assumptions, both distributional and pointwise tube formulas are obtained for archimedean fractal strings (and also, for archimedean self-similar fractal strings). (See, in particular, Theorems 8.1 and 8.7, along with §8.4 in [39].) At least for now, in the nonarchimedean case, we limit ourselves to discussing distributional explicit tube formulas. We expect, however, that under appropriate hypotheses, one should be able to obtain a pointwise fractal tube formula for $p$-adic fractal strings and especially, for $p$-adic self-similar strings. In fact, for the simple examples of the nonarchimedean Cantor, Euler and Fibonacci strings, the direct derivation of the fractal tube formula (3.36) yields a formula that is valid pointwise and not just distributionally. (See, in particular, Examples 4.6, 4.19 and 4.26.) We leave the consideration of such possible extensions to a future work.

**Example 3.32** (Fractal tube formula for the $p$-adic Euler string). We now explain how to recover from Theorem 3.28 (or Corollary 3.30) the tube formula for the Euler string $E_p$ obtained via a direct computation in Example 3.19 of §3.2. Indeed, it follows from Corollary 3.30 (applied with $W = \mathbb{C}$) that

\[
V_{E_p}(\varepsilon) = \frac{1}{p} \sum_{\omega \in \mathcal{D}_{E_p}} \text{res}(\zeta_{E_p}; \omega) \frac{\varepsilon^{1-\omega}}{1-\omega},
\]

which is exactly the expression obtained for $V_{E_p}(\varepsilon)$ in formula (3.24) of Example 3.19 since

\[
\text{res}(\zeta_{E_p}; \omega) = \frac{1}{\log p}
\]

for all $\omega \in \mathcal{D}_{E_p}$. (This follows easily from the expression of $\zeta_{E_p}$ obtained in Equation (3.6).) Note that Corollary 3.30 can be applied here in the strongly languid case when $W = \mathbb{C}$ and $\mathcal{R}_p(\varepsilon) \equiv 0$ since, in light of the discussion in §3.1 all the complex dimensions of $E_p$ are simple and $\zeta_{E_p}$ is clearly strongly languid of order $\kappa := 0$ and with the constant $A := p^{-1}$. Furthermore, formula (3.36) can be rewritten in the following more concrete form:

\[
V_{E_p}(\varepsilon) = \frac{1}{p \log p} \sum_{n \in \mathbb{Z}} \varepsilon^{1-inp} \frac{1}{1-inp},
\]

since $\mathcal{D}_{E_p} = \{inp : n \in \mathbb{Z}\}$ and $p = 2\pi/\log p$ (as in Equation (3.7) of §3.1).

Finally, note that since the series

\[
\sum_{n \in \mathbb{Z}} \varepsilon^{1-inp} \frac{1}{1-inp}
\]

converges pointwise because the associated Fourier series $\sum_{n \in \mathbb{Z}} \frac{2\pi inx}{\varepsilon^{1-inp}}$ is pointwise convergent on $\mathbb{R}$, the $p$-adic fractal tube formulas (3.36)–(3.37) actually converge pointwise rather than just distributionally.
4. Nonarchimedean Self-similar Strings

Nonarchimedean (or $p$-adic) self-similar strings form an important class of $p$-adic fractal strings. In this section, we first recall the construction of these strings, as provided in [28]; see §3.1. Furthermore, we give an explicit expression for their geometric zeta functions and deduce from it the periodic structure of their poles (or complex dimensions) and zeros, as obtained in [28]; see §4.3. Moreover, in §4.4 we deduce from the results of §3.3 and §4.2 the special form of the fractal tube formula for $p$-adic self-similar strings, as obtained in [29]. Finally, in §4.6 we apply this latter result in order to calculate the average Minkowski content of such strings, as is also done in [29].

4.1. Geometric Construction. Before explaining how to construct arbitrary $p$-adic self-similar strings, we need to introduce a definition and a few facts pertaining to $p$-adic similarity transformations.

Definition 4.1. A map $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$ is called a similarity contraction mapping of $\mathbb{Z}_p$ if there is a real number $r \in (0, 1)$ such that $|\Phi(x) - \Phi(y)|_p = r \cdot |x - y|_p$, for all $x, y \in \mathbb{Z}_p$.

Unlike in Euclidean space (and in the real line $\mathbb{R}$, in particular), it is not true that every similarity transformation of $\mathbb{Q}_p$ (or of $\mathbb{Z}_p$) is necessarily affine. Actually, in the nonarchimedean world (for example, in $\mathbb{Q}_p$ with $d \geq 1$), and in the $p$-adic line $\mathbb{Q}_p$, in particular, there are a lot of similarities which are not affine. However, it is known (see, e.g., [49]) that every analytic similarity must be affine. Hence, from now, we are working with a similarity contraction mapping $\Phi : \mathbb{Z}_p \to \mathbb{Z}_p$ that is affine. Thus we assume that there exist constants $a, b \in \mathbb{Z}_p$ with $|a|_p < 1$ such that $\Phi(x) = ax + b$ for all $x \in \mathbb{Z}_p$. Regarding the scaling factor $a$ of the contraction, it is well known that it can be written as $a = u \cdot p^n$, for some unit $u \in \mathbb{Z}_p$ (i.e., $|u|_p = 1$) and $n \in \mathbb{N}^*$ (see [41]). Then $r = |a|_p = p^{-n}$. We summarize this fact in the following lemma:

Lemma 4.2. Let $\Phi(x) = ax + b$ be an affine similarity contraction mapping of $\mathbb{Z}_p$ with the scaling ratio $r$. Then $b \in \mathbb{Z}_p$ and $a \in p\mathbb{Z}_p$, and the scaling factor is $r = |a|_p = p^{-n}$ for some $n \in \mathbb{N}^*$.

For simplicity, let us take the unit interval (or ball) $\mathbb{Z}_p$ in $\mathbb{Q}_p$ and construct a $p$-adic (or nonarchimedean) self-similar string $\mathcal{L}_p$ as follows (see [28]).

Let $N \geq 2$ be an integer and $\Phi_1, \ldots, \Phi_N : \mathbb{Z}_p \to \mathbb{Z}_p$ be $N$ affine similarity contraction mappings with the respective scaling ratios $r_1, \ldots, r_N \in (0, 1)$ satisfying $1 > r_1 \geq r_2 \geq \cdots \geq r_N > 0$; see Figure 9. Assume that

\[
\sum_{j=1}^{N} r_j < 1, \tag{4.2}
\]

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38 This construction is the nonarchimedean analog of the geometric construction of real (or archimedean) self-similar strings carried out in [29] §2.1.

39 Here, a map $f : \mathbb{Q}_p \to \mathbb{Q}_p$ is said to be analytic if it admits a convergent power series expansion about 0, and with coefficients in $\mathbb{Q}_p$, that is convergent in all of $\mathbb{Q}_p$.

40 In the sequel, $\mathcal{L}_p$ is interchangeably called a $p$-adic or nonarchimedean self-similar string.
and the images $\Phi_j(Z_p)$ of $Z_p$ do not overlap, i.e., $\Phi_j(Z_p) \cap \Phi_l(Z_p) = \emptyset$ for all $j \neq l$.

Note that it follows from Equation (4.2) that $\bigcup_{j=1}^N \Phi_j(Z_p)$ is not all of $Z_p$. We therefore have the following (nontrivial) decomposition of $Z_p$ into disjoint $p$-adic intervals:

$$Z_p = \bigcup_{j=1}^N \Phi_j(Z_p) \cup \bigcup_{k=1}^K G_k,$$

where $G_k$ is defined below.

In a procedure reminiscent of the construction of the ternary Cantor set in §2.1 or of the 3-adic Cantor set in §2.3, we then subdivide the interval $Z_p$ by means of the subintervals $\Phi_j(Z_p)$. Then the convex components of $Z_p \setminus \bigcup_{j=1}^N \Phi_j(Z_p)$ are the first substrings of the $p$-adic self-similar string $L_p$, say $G_1, G_2, \ldots, G_K$, with $K \geq 1$. These intervals $G_k$ are called the generators, the deleted intervals in the first generation of the construction of $L_p$.43 The length of each $G_k$ is denoted by $g_k$; so that $g_k = \mu_H(G_k)^{44}$ Without loss of generality, we may assume that the lengths $g_1, g_2, \ldots, g_K$ of the first substrings (i.e., intervals) of $L_p$ satisfy

$$1 > g_1 \geq g_2 \geq \cdots \geq g_K > 0.$$

It follows from Equation (4.3) and the additivity of Haar measure $\mu_H$ that

$$\sum_{j=1}^N r_j + \sum_{k=1}^K g_k = 1.$$

We then repeat this process with each of the remaining subintervals $\Phi_j(Z_p)$ of $Z_p$, for $j = 1, 2, \ldots, N$. And so on, ad infinitum. As a result, we obtain a $p$-adic

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42 We choose the convex components instead of the connected components because $Z_p$ is totally disconnected. Naturally, no such distinction is necessary in the archimedean case; cf. [39, §2.1.1]. Here and elsewhere in this paper, a subset $E$ of $\mathbb{Q}_p$ is said to be ‘convex’ if for every $x, y \in E$, the $p$-adic segment $\{tx + (1-t)y : t \in Z_p\}$ lies entirely in $E$.

43 Their archimedean counterparts are called ‘gaps’ in [39, Ch. 2 and §8.4], where archimedean self-similar strings are introduced.

44 We note that the lengths $g_k$ ($k = 1, 2, \ldots, K$) will sometimes be called the (nonarchimedean) ‘gaps’ or ‘gap sizes’ in the sequel.
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Self-similar string $\mathcal{L}_p = l_1, l_2, l_3, \ldots$, consisting of intervals of length $l_n$ given by

$$r_{\nu_1} r_{\nu_2} \cdots r_{\nu_q} g_k,$$

for $k = 1, \ldots, K$ and all choices of $q \in \mathbb{N}$ and $\nu_1, \ldots, \nu_q \in \{1, \ldots, N\}$. Thus, the lengths are of the form $r_1^{e_1} \cdots r_N^{e_N} g_k$ with $e_1, \ldots, e_N \in \mathbb{N}$ (but not all zero).

In [28], the classic notion of self-similarity is extended to the nonarchimedean setting, much as in [20], where the underlying complete metric space is allowed to be arbitrary. We note that the next result follows by applying the classic Contraction Mapping Principle to the complete metric space of all nonempty compact subsets of $\mathbb{Z}_p$.

**Theorem 4.3** ($p$-Adic self-similar set). There is a unique nonempty compact subset $\mathcal{S}_p$ of $\mathbb{Z}_p$ such that

$$\mathcal{S}_p = \bigcup_{j=1}^N \Phi_j(\mathcal{S}_p).$$

The set $\mathcal{S}_p$ is called the $p$-adic self-similar set associated with the self-similar system $\Phi = \{\Phi_1, \ldots, \Phi_N\}$. (It is also called the $\Phi$-invariant set.)

The relationship between the $p$-adic self-similar string $\mathcal{L}_p$ and the above $p$-adic self-similar set $\mathcal{S}_p$ is given by the following theorem, also obtained in [28].

**Theorem 4.4.** (i) $\mathcal{L}_p = \mathbb{Z}_p \setminus \mathcal{S}_p$, the complement of $\mathcal{S}_p$ in $\mathbb{Z}_p$.

(ii) $\mathcal{L}_p = \bigcup_{n=0}^\infty \bigcup_{w \in W_\alpha} \bigcup_{k=1}^{K_w} \Phi_w(G_k)$, while $\mathcal{S}_p = \bigcap_{n=0}^\infty \bigcup_{w \in W_\alpha} \Phi_w(\mathbb{Z}_p)$, where $W_\alpha = \{1, 2, \ldots, N\}^\alpha$ denotes the set of all finite words on $N$ symbols, of length $\alpha$, and $\Phi_w := \Phi_{w_n} \circ \cdots \circ \Phi_{w_1}$ for $w = (w_1, \ldots, w_\alpha) \in W_\alpha$.

![Figure 10. Construction of the nonarchimedean Cantor string CS$_3$ via an IFS.](image)

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45Recall that $\mathbb{Z}_p$ is complete since it is a compact metric space (see [28]).

46In Theorem 4.3, $\mathcal{L}_p$ is not viewed as a sequence of lengths but is viewed instead as the open set which is canonically given by a disjoint union of intervals (its $p$-adic convex components), as described in the above construction of a $p$-adic self-similar string.

47By convention, $\Phi_w(G_k) = \emptyset$ if $w \in W_{\alpha-1}$.
Example 4.5 (Nonarchimedean Cantor string as a 3-adic self-similar string). In this example, we review the construction (given in [24]) of the nonarchimedean Cantor string $CS_3$, as introduced in [27] and revisited in [28]. Our main point here is to stress the fact that $CS_3$ is a special case of a $p$-adic self-similar string, as constructed just above, and to prepare the reader for more general results about nonarchimedean self-similar strings, as given in the rest of §4.

Let $\Phi_1, \Phi_2 : \mathbb{Z}_3 \rightarrow \mathbb{Z}_3$ be the two affine similarity contraction mappings of $\mathbb{Z}_3$ given by

$$\Phi_1(x) = 3x \quad \text{and} \quad \Phi_2(x) = 2 + 3x,$$

with the same scaling ratio $r = 1/3$ (i.e., $r_1 = r_2 = 1/3$). By analogy with the construction of the real Cantor string (see §2.2), subdivide the interval $\mathbb{Z}_3$ into subintervals $\Phi_1(\mathbb{Z}_3) = 0 + 3\mathbb{Z}_3$ and $\Phi_2(\mathbb{Z}_3) = 2 + 3\mathbb{Z}_3$.

The remaining (3-adic) convex component

$$\mathbb{Z}_3 \setminus \bigcup_{j=1}^{2} \Phi_j(\mathbb{Z}_3) = 1 + 3\mathbb{Z}_3 = G$$

is the first substring of a 3-adic self-similar string, called the nonarchimedean Cantor string and denoted by $CS_3$ [27]. (Of course, this is the same $p$-adic fractal string $CS_3$ as the one constructed in §2.4.)

By repeating this process with the remaining subintervals $\Phi_j(\mathbb{Z}_3)$, for $j = 1, 2$, and continuing on, ad infinitum, we eventually obtain a sequence $CS_3 = l_1, l_2, l_3, \ldots$, associated with the open set resulting from this construction and consisting of intervals of lengths $l_v = 3^{-v}$ with multiplicities $m_v = 2^{v-1}$, for $v \in \mathbb{N}^*$. As we have seen in §2.4, and as follows from this construction (see Figure 10 and Equation (4.7), along with part (ii) of Theorem 4.4), the nonarchimedean Cantor string $CS_3$ can also be written as

$$CS_3 = (1 + 3\mathbb{Z}_3) \cup (3 + 9\mathbb{Z}_3) \cup (5 + 9\mathbb{Z}_3) \cup \cdots.$$

By definition, the geometric zeta function of $CS_3$ is given by

$$\zeta_{CS_3}(s) = \frac{(\mu_H(1 + 3\mathbb{Z}_3))^s + (\mu_H(3 + 9\mathbb{Z}_3))^s + (\mu_H(5 + 9\mathbb{Z}_3))^s + \cdots}{2^{v-1} \cdot 3^{-s}} \quad \text{for} \quad \Re(s) > \log 3.$$

Hence, by analytic continuation, the meromorphic extension of $\zeta_{CS_3}$ to the entire complex plane $\mathbb{C}$ exists and is given by

$$\zeta_{CS_3}(s) = \frac{3^{-s}}{1 - 2 \cdot 3^{-s}} \quad \text{for} \quad s \in \mathbb{C},$$

with poles at

$$\omega = \frac{\log 2}{\log 3} + in \frac{2\pi}{\log 3}, \quad n \in \mathbb{Z}.$$

Therefore, we recover the fact that the set of complex dimensions of $CS_3$ is given by

$$D_{CS_3} = \{D + ip \mid n \in \mathbb{Z}\},$$

where $D = \log_3 2$ is the dimension of $CS_3$ and $p = 2\pi / \log 3$ is its oscillatory period. (This terminology will be explained in §4.3 and §4.4.) Naturally, Equations (4.9)
and (4.10) are in agreement with the statement of Theorem 2.8. Finally, note that 
\( \zeta_{CS_3}(s) = \frac{z}{1 - 2z} \).

We refer the interested reader to [27] as well as to §2.3–2.5 above for additional information concerning the nonarchimedean Cantor string \( CS_3 \) and the associated nonarchimedean Cantor set \( C_3 \). We just mention here that in light of part (i) of Theorem 4.4, we can recover the 3-adic Cantor set \( C_3 \) as the complement of the 3-adic Cantor string \( CS_3 \) in the unit interval (and vice-versa):

\[
CS_3 = \mathbb{Z}_3 \setminus C_3, \quad \text{and so} \quad C_3 = \mathbb{Z}_3 \setminus CS_3.
\]

Example 4.6 (The explicit tube formula for the nonarchimedean Cantor string). In this example, we explain how to derive the exact fractal tube formula for \( CS_3 \) as stated in Equation (2.11), in two different ways:

(i) First, via a direct computation (much as we derived the tube formula for the \( p \)-adic Euler string in Example 3.19).

(ii) Second, as a special case of the general \( p \)-adic tube formula obtained in §3.5.

Let \( \varepsilon > 0 \). Then, by Theorem 3.16, we have

\[
V_{CS_3}(\varepsilon) = \frac{1}{3} \sum_{n=k+1}^{\infty} \frac{2n-1}{3^n} = \frac{1}{3} \left( \frac{2}{3} \right)^k,
\]

where \( k := \lfloor \log_3 \varepsilon^{-1} \rfloor \). Let \( x := \log_3 \varepsilon^{-1} = k + \{x\} \), where \( \{x\} \) is the fractional part of \( x \). Then a simple computation shows that \( \left( \frac{2}{3} \right)^x = \varepsilon^{1-D} \) and \( \varepsilon^{2\pi i x} = \varepsilon^{-ip} \), with \( D = \log_3 2 \) and \( p = 2\pi / \log 3 \) as in Example 4.5. Using the Fourier expansion for \( b^{-x} \), as given by Equation (3.25), for \( b = 3^{-1} \) and the above value of \( x \), we obtain an expansion in terms of the complex dimensions \( \omega = D + ip \) of \( CS_3 \):

\[
V_{CS_3}(\varepsilon) = \frac{3^{-1}}{2 \log 3} \sum_{n \in \mathbb{Z}} \frac{\varepsilon^{1-D-ip}}{1 - D - ip} \xi^{-\omega} = \frac{3^{-1}}{2 \log 3} \sum_{\omega \in \mathcal{D}_{CS_3}} \frac{\varepsilon^{1-\omega}}{1 - \omega},
\]

since \( \mathcal{D}_{CS_3} \) is given by (4.10). Next, using Equation (4.9), we see that

\[
\text{res}(\zeta_{CS_3}; \omega) = \frac{1}{2 \log 3},
\]

independently of \( \omega \in \mathcal{D}_{CS_3} \), and so the exact fractal tube formula for the nonarchimedean Cantor string is found to be

\[
V_{CS_3}(\varepsilon) = \frac{1}{3} \sum_{\omega \in \mathcal{D}_{CS_3}} \text{res}(\zeta_{CS_3}; \omega) \varepsilon^{1-\omega} \frac{1-\omega}{1 - \omega}.
\]

Note that since \( CS_3 \) has simple complex dimensions, we may also apply Corollary 3.30 (in the strongly languid case when \( W = \mathbb{C} \)) in order to precisely recover Equation (4.14). (Alternatively, we could use Theorem 4.21 in §4.5 below.)
We may rewrite (4.13) or (4.14) in the following form (which agrees with the tube formula to be obtained in Theorem 4.21):

\[
V_{CS_3}(\varepsilon) = \varepsilon^{1-D} G_{CS_3}(\log_3 \varepsilon^{-1}),
\]

where (as in Equation (2.19) of §2.5) \(G_{CS_3}\) is the nonconstant periodic function (of period 1) on \(\mathbb{R}\) given by

\[
G_{CS_3}(x) := \frac{1}{6 \log 3} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi inx}}{1 - D - inp}.
\]

Finally, we note that since the Fourier series

\[
\sum_{n \in \mathbb{Z}} \frac{e^{2\pi inx}}{1 - D - inp}
\]

is pointwise convergent on \(\mathbb{R}\), the above direct computation of \(V_{CS_3}(\varepsilon)\) shows that (4.13) and (4.14) actually hold pointwise rather than distributionally.

### 4.2. Geometric Zeta Function of \(p\)-Adic Self-Similar Strings.

In this section, as well as in §4.3 and §4.4, we will survey results obtained in [28] about the geometric zeta functions and the complex dimensions of \(p\)-adic self-similar strings.

In the next theorem, we provide a first expression for the geometric zeta function of a nonarchimedean self-similar string. At first sight, this expression is almost identical to the one obtained in the archimedean case in [39, Thm. 2.4]. Later on, however, we will see that unlike in the archimedean case where the situation is considerably more subtle and complicated (cf. [39, Thms. 2.17 and 3.6]), this expression can be significantly simplified since the two potentially transcendental functions appearing in the denominator and numerator of Equation (4.15) below can always be made rational; see Theorem 4.14 in §4.3.

**Theorem 4.7.** Let \(L_p\) be a \(p\)-adic self-similar string with scaling ratios \(\{r_j\}_{j=1}^N\) and gaps \(\{g_k\}_{k=1}^K\), as in the above construction. Then the geometric zeta function of \(L_p\) has a meromorphic extension to the whole complex plane \(\mathbb{C}\) and is given by

\[
\zeta_{L_p}(s) = \frac{\sum_{k=1}^K g_k^s}{1 - \sum_{j=1}^N r_j^s}, \quad \text{for } s \in \mathbb{C}.
\]

**Corollary 4.8.** The set of complex dimensions of a \(p\)-adic self-similar string \(L_p\) is contained in the set of complex solutions \(\omega\) of the Moran equation \(\sum_{j=1}^N r_j^\omega = 1\). If the string has a single generator (i.e., if \(K = 1\)), then this inclusion is an equality. \(^{50}\)

**Definition 4.9.** A \(p\)-adic self-similar string \(L_p\) is said to be lattice (or nonlattice) if the multiplicative group generated by the scaling ratios \(r_1, r_2, \ldots, r_N\) is discrete (or dense) in \((0, \infty)\). \(^{51}\)

**Theorem 4.10.** Every \(p\)-adic self-similar string is lattice.

\(^{50}\) See Examples 4.5, 4.26 and Theorem 4.14.

\(^{51}\) More precisely, much as in the archimedean case in [39] Ch. 3, we say that \(L_p\) is lattice if the multiplicative group generated by the distinct scaling ratios of \(L_p\) is a subgroup of \((0, \infty)\) of rank one (i.e., if it is a free abelian group). As it turns out, in the present case of \(p\)-adic self-similar strings, it does not matter whether we use this slightly refined definition or the one given in Definition 4.9; see Remark 4.11.
Remark 4.11. Theorem 4.10 follows from the fact that all the scaling ratios $r_j$ must belong to the multiplicative group $p^\mathbb{Z}$. In fact, much more is true since the gaps $g_k$ must also belong to $p^\mathbb{Z}$, as will be discussed below in more detail in §4.3. It follows that $p$-adic self-similar strings are lattice strings in a very strong sense, namely, their geometric zeta functions are rational functions of a suitable variable $z$ (see Theorem 4.14 below).

Remark 4.12. Theorem 4.10 is in sharp contrast with the usual theory of real (or archimedean) self-similar strings developed in [39, Chs. 2 and 3]. Indeed, there are both lattice and nonlattice strings in the archimedean case. Furthermore, generically, archimedean self-similar strings are nonlattice. Moreover, it is shown in [39, Ch. 3] by using Diophantine approximation that every nonlattice string in $\mathbb{R} = \mathbb{Q}_\infty$ can be approximated by a sequence of lattice strings with oscillatory periods increasing to infinity. It follows that the complex dimensions of an archimedean nonlattice string are quasiperiodically distributed (in a very precise sense, that is explained in loc. cit.) because the complex dimensions of archimedean lattice strings are periodically distributed along finitely many vertical lines. Clearly, there is nothing of this kind in the nonarchimedean case since $p$-adic self-similar strings are necessarily lattice.

Remark 4.13. The $p$-adic Euler string $E_p$, discussed in §3.1, is not self-similar because $E_p$ has dimension $D = 0$, whereas the requirement that $N \geq 2$ in the definition of a $p$-adic self-similar string implies that $D > 0$ for any $p$-adic self-similar string.

4.3. $p$-Adic Self-Similar Strings Are Strongly Lattice. A small modification of the above argument enables us to show that every $p$-adic self-similar string is lattice in a much stronger sense, as we now explain. It will follow (see Theorem 4.16) that not only the poles (i.e., the complex dimensions of $L_p$) but also the zeros of $\zeta_{L_p}$ are periodically distributed. Accordingly, we will say that $p$-adic self-similar strings are strongly lattice.

We introduce some necessary notation. First, by Lemma 4.2 we write

$$r_j = p^{-n_j}, \quad \text{with } n_j \in \mathbb{N}^* \text{ for } j = 1, 2, \ldots, N.$$  

Second, we write

$$g_k = \sum_H(G_k) = p^{-m_k}, \quad \text{with } m_k \in \mathbb{N}^* \text{ for } k = 1, 2, \ldots, K.$$  

Third, let

$$d = \gcd\{n_1, \ldots, n_N, m_1, \ldots, m_K\}.$$  

Then there exist positive integers $n'_j$ and $m'_k$ such that

$$n_j = dn'_j \quad \text{and} \quad m_k = dm'_k \quad \text{for } j = 1, \ldots, N \quad \text{and} \quad k = 1, \ldots, K.$$  

Finally, we set

$$p^d = 1/r.$$  

Note that by construction, $r_j = r^{n'_j}$ and $g_k = r^{m'_k}$ for $j = 1, \ldots, N$ and $k = 1, \ldots, K$. Hence, $r$ is the multiplicative generator in $(0,1)$ of the rank one group generated by $\{r_1, \ldots, r_N, g_1, \ldots, g_K\}$ (or, equivalently, by either $\{r_1, \ldots, r_N\}$ or $\{g_1, \ldots, g_K\}$).
Without loss of generality, we may assume that the scaling ratios \( r_j \) and the gaps \( g_k \) are written in nonincreasing order as in Equations (4.1) and (4.4), respectively; so that
\[
0 < n'_1 \leq n'_2 \leq \cdots \leq n'_N \quad \text{and} \quad 0 < m'_1 \leq m'_2 \leq \cdots \leq m'_K.
\]

**Theorem 4.14.** Let \( L_p \) be a \( p \)-adic self-similar string and \( z = r^s \), with \( r = p^{-d} \) as in Equation (4.17). Then the geometric zeta function \( \zeta_{L_p} \) of \( L_p \) is a rational function in \( z \). Specifically,
\[
\zeta_{L_p}(s) = \frac{\sum_{k=1}^{K} z^{m'_k}}{1 - \sum_{j=1}^{N} z^{n'_j}},
\]
where \( m'_k, n'_j \in \mathbb{N}^* \) are given by Equation (4.16).

**Definition 4.15.** Let \( p = \frac{2\pi}{\sigma \log p} \). Then \( p \) is called the oscillatory period of \( L_p \).

### 4.4. Periodicity of the Poles and the Zeros of \( \zeta_{L_p} \)

The following result (also from [28]) is the nonarchimedean counterpart of [39, Thms. 2.17 and 3.6], which provide the rather subtle structure of the complex dimensions of archimedean self-similar strings. It is significantly simpler, however, due in part to the fact that nonlattice \( p \)-adic self-similar strings do not exist.

To avoid any confusion, we stress that in the statement of the next theorem, \( \zeta_{L_p} \) is viewed as a function of the original complex variable \( s \). Moreover, it follows from Theorem 3.21 in §3.3 above and from Theorem 4.16 below that \( D \), the dimension of \( L_p \), defined as the abscissa of convergence of the Dirichlet series originally defining \( \zeta_{L_p} \) (and sometimes also denoted by \( \sigma = \sigma_{L_p} \) here) coincides with \( \delta \) and the Minkowski dimension \( D_M = D_{M,L_p} \) of \( L_p \):
\[
D = D_M = \sigma = \delta,
\]
where \( \delta \) is the similarity dimension of \( L_p \), i.e., the unique real solution of the Moran equation \( \sum_{j=1}^{N} r_j = 1 \). Hence, in the present case of \( p \)-adic self-similar strings, there is no need to distinguish between these various notions of ‘fractal dimensions’. (See Remark 4.18 below for more information.)

**Theorem 4.16 (Structure of the complex dimensions).** Let \( L_p \) be a nontrivial \( p \)-adic self-similar string. Then

(i) The complex dimensions of \( L_p \) and the zeros of \( \zeta_{L_p} \) are periodically distributed along finitely many vertical lines, with period \( p \), the oscillatory period of \( L_p \) (as given in Definition 4.15).

(ii) Furthermore, along a given vertical line, each pole (respectively, each zero) of \( \zeta_{L_p} \) has the same multiplicity.

(iii) Finally, the dimension \( D \) of \( L_p \) is the only complex dimension that is located on the real axis \( \mathbb{R} \). Moreover, \( D \) is simple \( \mathbb{C} \) and is located on the right most vertical line. That is, \( D \) is equal to the maximum of the real parts of the complex dimensions.

**Remark 4.17.** As will be apparent to the expert reader, the situation described above—specifically, the rationality of the zeta function in the variable \( z = r^s \), with \( r = p^{-d} \), and the ensuing periodicity of the poles and the zeros—is analogous to the

\[\text{By contrast, it is immediate to check that there are no real zeros (still in the } s \text{ variable).} \]
\[\text{i.e., } D \text{ is a simple pole of } \zeta_{L_p}. \]
one encountered for a curve (or more generally, a variety) over a finite field \( \mathbb{F}_p \); see, e.g., Chapter 3 of [44]. In this analogy, the prime number \( p \) naturally corresponds to the characteristic of the finite field, and \( p^d = r^{-1} \) is the analog of the cardinality of the field.

**Remark 4.18.** By Theorem 3.21 \( D \) is also the inner Minkowski dimension \( D_M \) of the self-similar set associated with the present self-similar system \( \Phi \): \( D = D_M = \sigma \), the abscissa of convergence of the geometric zeta function. Moreover, in light of (4.15) and part (iii) of Theorem 4.16 above, it always coincides with the similarity dimension \( \delta \) of \( \Phi \). Namely, \( D \) is the unique real solution of the Moran equation

\[
\sum_{j=1}^{N} r_j^D = 1.
\]

As a result, \( D = \sigma = D_M = \delta < 1 \), since by assumption (see Equation (4.2) above), \( \sum_{j=1}^{N} r_j < 1 \). This last observation will enable us, in particular, to apply the fractal tube formula (Theorem 3.28 and Corollary 3.30) to any \( p \)-adic self-similar string.

We next supplement the above results by establishing a theorem obtained in [29] and which will be very useful to us in §4.5 in order to simplify the tube formula associated with a \( p \)-adic self-similar string.

According to part (i) of Theorem 4.16 there exist finitely many poles

\[ \omega_1, \ldots, \omega_q, \]

with \( \omega_1 = D \) and \( \Re(\omega_q) \leq \cdots \leq \Re(\omega_2) < D \), such that

\[ D_{L_p} = \{ \omega_u + \text{inp} \mid n \in \mathbb{Z}, \ u = 1, \ldots, q \}. \]

Furthermore, each complex dimension \( D + \text{inp} \) is simple (by parts (ii) and (iii) of Theorem 4.16) and the residue of \( \zeta_{L_p}(s) \) at \( s = D + \text{inp} \) is independent of \( n \in \mathbb{Z} \) and equal to

\[
\text{res}(\zeta_{L_p}; D + \text{inp}) = \frac{\sum_{k=1}^{K} r_{m_k}^{D} \omega_u}{\log r^{-1} \sum_{j=1}^{N} n_j^{r} r_{n_j}^{D} \omega_u}.
\]

The latter fact (concerning residues) is an immediate consequence of the following result from [29]:

**Theorem 4.19.** (i) For each \( v = 1, \ldots, q \), the principal part of the Laurent series of \( \zeta_{L_p}(s) \) at \( s = \omega_v + \text{inp} \) does not depend on \( n \in \mathbb{Z} \).

(ii) Moreover, let \( v \in \{1, \ldots, q\} \) be such that \( \omega_v \) (and hence also \( \omega_u + \text{inp}, \) for every \( n \in \mathbb{Z} \), by part (ii) of Theorem 4.17) is simple. Then the residue of \( \zeta_{L_p}(s) \) at \( s = \omega_v + \text{inp} \) is independent of \( n \in \mathbb{Z} \) and

\[
\text{res}(\zeta_{L_p}; \omega_v + \text{inp}) = \frac{\sum_{k=1}^{K} r_{m_k}^{D} \omega_v}{\log r^{-1} \sum_{j=1}^{N} n_j^{r} r_{n_j}^{D} \omega_v}.
\]

In particular, this is the case for \( \omega_1 = D \).

Note that by contrast, in the lattice case of the archimedean theory of self-similar strings developed in [39] Chs. 2 and 3, we had to assume that the gap sizes (and not just the scaling ratios) are integral powers of \( r \) in order to obtain the counterpart of Theorem 4.19.
Remark 4.20 (Comparison with the archimedean case). Part (i) of Theorem 4.16 along with Theorem [4.14] shows that in some very explicit sense, the theory of \( p \)-adic self-similar strings is both simpler and more natural than its archimedean counterpart. Indeed, not only is it the case that every \( p \)-adic self-similar string \( \mathcal{L}_p \) is lattice, but both the zeros and poles of \( \zeta_{\mathcal{L}_p}(s) \) are periodically distributed along vertical lines, with the same period. By contrast, even if an archimedean self-similar string \( \mathcal{L} \) is assumed to be ‘lattice’, then the zeros of \( \zeta_{\mathcal{L}}(s) \) are usually not periodically distributed because the multiplicative group generated by the distinct gap sizes need not be of rank one; see [39, Chs. 2 and 3].

4.5. Exact Tube Formulas for \( p \)-Adic Self-Similar Strings. In view of Equation (4.15), it follows from the argument given at the beginning of [39, §6.4] that every \( p \)-adic self-similar string \( \mathcal{L}_p \) is strongly languid, with \( \kappa = 0 \) and \( A = r_N g_K^{-1} \), in the notation of the latter part of Definition 4.20. Indeed, Equation (4.15) implies that \( |\zeta_{\mathcal{L}_p}(s)| \ll (r_N^{-1} g_K)^{-|\Re(s)|} \), as \( \Re(s) \to -\infty \). Hence, we can apply the distributional tube formula without error term (i.e., the last part of Theorem 3.28 and of Corollary 3.30) with \( W = \mathbb{C} \). Since by Theorem 4.10 \( \mathcal{L}_p \) is a lattice string, we obtain (in light of Theorems 4.14, 4.16, and 4.19) the following simpler analogue of Theorem 8.25 in [39], established in [29].

Theorem 4.21. Let \( \mathcal{L}_p \) be a \( p \)-adic self-similar string with multiplicative generator \( r \). Assume that all the complex dimensions of \( \mathcal{L}_p \) are simple. Then, for all \( \varepsilon \) with \( 0 < \varepsilon < g_K r_N^{-1} \), the volume \( V_{\mathcal{L}_p}(\varepsilon) \) is given by the following exact distributional tube formula:

\[
V_{\mathcal{L}_p}(\varepsilon) = \sum_{u=1}^{q} \varepsilon^{1-\omega_u} G_u(\log_{1/r} \varepsilon^{-1}),
\]

where \( 1/r = p^k \) (as in Equation (4.17)), and for each \( u = 1, \ldots, q \), \( G_u \) is a real-valued periodic function of period 1 on \( \mathbb{R} \) corresponding to the line of complex dimensions through \( \omega_u \) \( (\omega_1 = D > \Re(\omega_2) \geq \cdots \geq \Re(\omega_q)) \), and is given by the following (conditionally and also distributionally convergent) Fourier series:

\[
G_u(x) = \frac{\res(\zeta_{\mathcal{L}_p}; \omega_u)}{p} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi i nx}}{1 - \omega_u - inp},
\]

where (as in Equation (4.22) of Theorem 4.19),

\[
\res(\zeta_{\mathcal{L}_p}; \omega_u) = \frac{\sum_{k=1}^{K} p^{m_k} \omega_u}{\log r^{-1} \sum_{j=1}^{N} n_j^{p^m} \omega_u}.
\]

Moreover, \( G_u \) is nonconstant and bounded.

Remark 4.22. In comparing our results with the corresponding results in Chapter 2 and §8.4 of [39], obtained for real self-similar fractal strings, the reader should keep in mind the following two facts: (i) the simplification brought upon by the “strong lattice property” of \( p \)-adic self-similar strings; see Theorem 4.19 and Remark 4.20 above. (ii) By construction, any \( p \)-adic self-similar string \( \mathcal{L}_p \) (as defined in this section) has total length \( L \) equal to one: \( L = \mu_H(\mathcal{L}_p) = \zeta_{\mathcal{L}_p}(1) = \mu_H(\mathbb{Z}_p) = 1 \). Indeed,

\[57\]We note that instead, we could more generally apply parts (i) and (ii) of Theorem 4.10 in order to obtain a distributional tube formula with or without error term, valid without assuming that all of the complex dimensions of \( \mathcal{L}_p \) are simple. This observation is used in Remark 4.29.
for notational simplicity, we have assumed that the similarity transformations \( \Phi_j \) 
\((j = 1, \ldots, N)\) are self-maps of the ‘unit interval’ \( \mathbb{Z}_p \), rather than of an arbitrary ‘interval’ of length \( L \) in \( \mathbb{Q}_p \).

**Remark 4.23** (Truncated tube formula with error term). The analog of Corollary 8.27 in \[39\] holds in the present context, with \( 2\varepsilon \) replaced by \( \varepsilon \) and with \( L := 1 \); see the previous remark. In particular, in light of the method of proof of *loc. cit.*, we have the following ‘truncated tube formula’:

\[
V(\varepsilon) = \varepsilon^{1-D} G(\log_{1/\varepsilon} \varepsilon^{-1}) + E(\varepsilon),
\]

where \( G = G_1 \) is the nonconstant, bounded periodic function of period 1 given by Equation (4.24) of Theorem 4.21 (with \( u = 1 \) and \( \omega_1 = D \)). Here, \( E(\varepsilon) \) is an error term that can be estimated much as in *loc. cit.* In particular, \( E(\varepsilon) = o(1) \) and, moreover, there exists \( \delta > 0 \) such that \( \varepsilon^{-(1-D)} E(\varepsilon) = O(\varepsilon^\delta), \) as \( \varepsilon \to 0^+ \).

Furthermore, since we limit ourselves here to the first line of complex dimensions, and since those complex dimensions are always simple (by part (iii) of Theorem 4.16), we do not have to assume (as in Theorem 4.21) that all the complex dimensions of \( L_p \) are simple in order for Equation (4.25) and the corresponding error estimate for \( E(\varepsilon) \) to be valid. (This latter fact can be used to give a direct proof of the equality \( D = D_M = \sigma \) for any nontrivial \( p \)-adic self-similar string.)

More specifically, we note that Equation (4.25) and the corresponding error estimate for \( E(\varepsilon) \) follow from part (i) of Theorem 3.28 (the explicit tube formula with error term, applied to a suitable window), along with the fact that the complex dimensions on the rightmost vertical line \( \Re(s) = D \) are simple (according to parts (ii) and (iii) of Theorem 4.16, from \[28\]).

**Remark 4.24.** Note that in light of Remark 4.18, we have \( D = \sigma < 1 \) for any nontrivial \( p \)-adic self-similar string \( L_p \). Hence, we can also apply the distributional tube formula in the general case (when the complex dimensions of \( L_p \) are not necessarily simple) or, in the present special case of simple complex dimensions (Corollary 3.30), to obtain a distributional tube formula in this situation, as claimed in Theorem 4.21.

The next result follows immediately from the truncated tube formula provided in Remark 4.23 along with the corresponding error estimate.

**Theorem 4.25.** A \( p \)-adic self-similar string is never Minkowski measurable because it always has multiplicatively periodic oscillations of order \( D \) in its geometry.

**Example 4.26** (Nonarchimedean Fibonacci string and its fractal tube formula). Let \( \Phi_1 \) and \( \Phi_2 \) be the two affine similarity contraction mappings of \( \mathbb{Z}_2 \) given by

\[
\Phi_1(x) = 2x \quad \text{and} \quad \Phi_2(x) = 1 + 4x,
\]

with the respective scaling ratios \( r_1 = 1/2 \) and \( r_2 = 1/4 \). The associated 2-adic self-similar string (introduced in \[28\]) with generator \( G = 3 + 4\mathbb{Z}_2 \) is called the *nonarchimedean Fibonacci string* and denoted by \( \mathcal{FS}_2 \) (compare with the archimedean counterpart discussed in \[39\] §2.3.2). It is given by the sequence \( \mathcal{FS}_2 = \{l_1, l_2, l_3, \ldots \} \) and consists (for \( m = 1, 2, \ldots \)) of intervals of lengths \( l_m = 2^{-(m+1)} \) with multiplicities \( f_m \), the Fibonacci numbers. Alternatively, the nonarchimedean Fibonacci numbers are defined by the recursive formula: \( f_{m+1} = f_m + f_{m-1}, f_0 = 0 \) and \( f_1 = 1 \).

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58 These numbers are defined by the recursive formula: \( f_{m+1} = f_m + f_{m-1}, f_0 = 0 \) and \( f_1 = 1 \).
The bounded open subset of $\mathbb{Z}_2$ given by the following disjoint union of 2-adic intervals (necessarily its 2-adic convex components):

$$\mathcal{FS}_2 = (3 + 4\mathbb{Z}_2) \cup (6 + 8\mathbb{Z}_2) \cup (12 + 16\mathbb{Z}_2) \cup (13 + 16\mathbb{Z}_2) \cup \cdots.$$ 

By Theorem 4.7, the geometric zeta function of $\mathcal{FS}_2$ is given (almost exactly as for the archimedean Fibonacci string, cf. loc. cit.) by

$$\zeta_{\mathcal{FS}_2}(s) = \frac{4^{-s}}{1 - 2^{-s} - 4^{-s}}.$$ 

Hence, the set of complex dimensions of $\mathcal{FS}_2$ is given by

$$D_{\mathcal{FS}_2} = \{D + inp \mid n \in \mathbb{Z}\} \cup \{-D + i(n + 1/2)p \mid n \in \mathbb{Z}\}$$

with $D = \log_2 \phi$, where $\phi = (1 + \sqrt{5})/2$ is the golden ratio, and $p = 2\pi / \log 2$, the oscillatory period of $\mathcal{FS}_2$; see Figure 11. We refer the interested reader to [28] for additional information concerning the nonarchimedean Fibonacci string.

Note that $\zeta_{\mathcal{FS}_2}$ does not have any zero (in the variable $s$) since the equation $4^{-s} = 0$ does not have any solution. Moreover, in agreement with Theorem 4.14, $\zeta_{\mathcal{FS}_2}$ is a rational function of $z = 2^{-s}$, i.e.,

$$\zeta_{\mathcal{FS}_2}(s) = \frac{z^2}{1 - z - z^2}.$$ 

The minor difference between the two geometric zeta functions is due to the fact that the real Fibonacci string $\mathcal{FS}$ in [39] §2.3.2 and Exple. 8.32 has total length 4 whereas the present 2-adic Fibonacci string $\mathcal{FS}_2$ has total length 1.
Since, in light of (4.28), the complex dimensions of $F\mathcal{S}_2$ are simple, we may apply either Corollary 3.30 or Theorem 4.21 in order to obtain the following exact fractal tube formula for the nonarchimedean Fibonacci string:

$$V_{F\mathcal{S}_2}(\varepsilon) = \frac{1}{2} \sum_{\omega \in D_{F\mathcal{S}_2}} \text{res}(\zeta_{F\mathcal{S}_2}; \omega) \frac{\varepsilon^{1-\omega}}{1-\omega}$$

$$= \varepsilon^{1-D}G_1(\log_2 \varepsilon^{-1}) + \varepsilon^{1+D-i\pi/2}G_2(\log_2 \varepsilon^{-1}),$$

where $G_1$ and $G_2$ are bounded periodic functions of period 1 on $\mathbb{R}$ given by their respective (conditionally convergent) Fourier series:

(4.29) $$G_1(x) = \phi^{-2} \frac{\phi + 2}{10 \log 2} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi inx}}{1 - D - ip}$$

and

(4.30) $$G_2(x) = \phi^2 \frac{3 - \phi}{10 \log 2} \sum_{n \in \mathbb{Z}} \frac{e^{2\pi inx}}{1 + D - i(n + 1/2)p}.$$

Note that the above Fourier series for $G_1$ and $G_2$ are conditionally (and also distributionally) convergent, for all $x \in \mathbb{R}$. Furthermore, the explicit fractal tube formula $V_{F\mathcal{S}_2}(\varepsilon)$ for $F\mathcal{S}_2$ actually holds pointwise and not just distributionally, as the interested reader may verify via a direct computation (much as in Example 4.6 for the 3-adic Cantor string $C\mathcal{S}_3$).

### 4.6. The Average Minkowski Content

According to Theorem 4.25, a $p$-adic self-similar string does not have a well-defined Minkowski content, because it is not Minkowski measurable. Nevertheless, as we shall see in Theorem 4.28 below, it does have a suitable ‘average content’ $M_{\text{av}}$, in the following sense:

**Definition 4.27.** Let $L_p$ be a $p$-adic fractal string of dimension $D$. The average Minkowski content, $M_{\text{av}}$, is defined by the logarithmic Cesaro average

$$M_{\text{av}} = M_{\text{av}}(L_p) := \lim_{T \to \infty} \frac{1}{\log T} \int_{1/T}^1 \varepsilon^{-(1-D)}V_{L_p}(\varepsilon) \frac{d\varepsilon}{\varepsilon},$$

provided this limit exists and is a finite positive real number.

**Theorem 4.28.** Let $L_p$ be a $p$-adic self-similar string of dimension $D$. Then the average Minkowski content of $L_p$ exists and is given by the finite positive number

$$M_{\text{av}} = \frac{1}{p(1-D)} \text{res}(\zeta_{L_p}; D) = \frac{1}{p(1-D)} \sum_{k=1}^K \frac{r^{m_k}D}{\log r^{-1} \sum_{j=1}^N \nu_j r^{n_j}D}.$$

**Remark 4.29.** Definition 4.27 and Theorem 4.28 are the exact nonarchimedean counterpart of [39], Definition 8.29 and Theorem 8.30. Furthermore, Theorem 4.28 (which is obtained in [29]) follows from the ‘truncated explicit tube formula’ given by Equation (4.25) in Remark 4.23 along with the corresponding error estimate.

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61 We leave it as an exercise for the interested reader to verify the computations below or to obtain a direct derivation of $V_{F\mathcal{S}_2}(\varepsilon)$, much as we have done for $V_{E_p}(\varepsilon)$ and $V_{C\mathcal{S}_3}(\varepsilon)$ in Examples 3.19 and 4.6, respectively.
**Example 4.30** (Nonarchimedean Cantor string). The average Minkowski content of the nonarchimedean Cantor string $CS_3$ is given by

$$M_{av}(CS_3) = \frac{1}{6(\log 3 - \log 2)}.$$  

Indeed, we have seen in Examples 4.13 and 4.10 that $D = \log_3 2$, $\text{res}(\zeta_{CS_3}; D) = 1/2 \log 3$ and $p = 3$.

**Example 4.31** (Nonarchimedean Fibonacci string). The average Minkowski content of the nonarchimedean Fibonacci string $FS_2$ is given by

$$M_{av}(FS_2) = \frac{1}{2(\phi + 2)(\log 2 - \log \phi)},$$

where $\phi = \frac{1+\sqrt{5}}{2}$ is the golden ratio (and so $\phi^{-1} = \frac{\sqrt{5}-1}{2}$). Indeed, with $p = 2$ and $D = \log_2 \phi$, one can verify that

$$\text{res}(\zeta_{FS_2}; D) = \frac{1}{(\phi + 2) \log 2}.$$  

Hence, the above expression for $M_{av} = M_{av}(FS_2)$ follows from Theorem 4.28. Furthermore, note that $\log 2 - \log \phi = \log(\sqrt{5} - 1)$. Hence, $M_{av}$ can be rewritten as follows:

$$M_{av} = \frac{1}{(5 + \sqrt{5}) \log(\sqrt{5} - 1)}.$$

**Remark 4.32.** Even though the $p$-adic Euler string $E_p$ is not self-similar, we can still use the same methods as those used to prove Theorem 4.28, along with the fractal tube formula (3.24), in order to calculate its average Minkowski content; we leave the easy verification to the reader. Clearly, $E_p$ is not Minkowski measurable since, in light of Equation (3.7), it has nonreal complex dimension on the real line $\Re(s) = D = 0$. This is also apparent from the fractal tube formula (3.24).

## 5. Concluding Comments

We close this paper with some comments regarding several possible directions for future research in this area. We hope to address these issues in later work.

### 5.1. Adelic Fractal Strings and Their Spectra.

It would be interesting to unify the archimedean and nonarchimedean settings by appropriately defining adelic fractal strings, and then studying the associated spectral zeta functions (as is done for standard archimedean fractal strings in [22, 25] and [30, 35, 36, 38, 39]).

To this aim, the spectrum of these adelic fractal strings should be suitably defined and its study may benefit from Dragovich’s work [5] on adelic quantum harmonic oscillators. In the process of defining these adelic fractal strings, we expect to make contact with the notion of a fractal membrane (or “quantized fractal string”) introduced in [20, Ch. 3] and rigorously constructed in [91] as a Connes-type noncommutative geometric space; see also [25, §4.2]. The aforementioned spectral zeta function of an adelic fractal string would then be viewed as the (completed) spectral partition function of the associated fractal membrane, in the sense of [20]. (See also Remark [3.10] and [5.4].)
5.2. Nonarchimedean Fractal Strings in Berkovich Space. As we have seen in §3.2, there can only exist lattice $p$-adic self-similar strings, because of the discreteness of the valuation group of $\mathbb{Q}_p$. However, in the archimedean setting, there are both lattice and nonlattice self-similar strings. We expect that by suitably extending the notion of $p$-adic self-similar string to Berkovich’s $p$-adic analytic space $\mathbb{A}^1_\mathbb{Q}$, it can be shown that $p$-adic self-similar strings are generically nonlattice in this broader setting. Furthermore, we conjecture that every nonlattice string in the Berkovich projective line can be approximated by lattice strings with increasingly large oscillatory periods (much as occurs in the archimedean case [39, Ch. 3]). Finally, we expect that, by contrast with what happens for $p$-adic fractal strings, the volume $V_{L_p}(\varepsilon)$ will be a continuous function of $\varepsilon$ in this context. (Compare with Remark 3.18.)

5.3. Higher-Dimensional Fractal Tube Formula. We expect that the higher-dimensional tube formulas obtained by Lapidus and Pearse in [32, 33] (as well as, more generally, by those same authors and Winter in [31]) for archimedean self-similar systems and the associated tilings in $\mathbb{R}^d$ have a natural nonarchimedean counterpart in the $d$-dimensional $p$-adic space $\mathbb{Q}_p^d$, for any integer $d \geq 1$. In the latter $p$-adic case, the corresponding ‘tubular zeta function’ $\zeta_{T_p}(\varepsilon; s)$ (when $d = 1$, see Remark 3.29) should have a more complicated expression than in the one-dimensional situation, and should involve both the inner radii and the ‘curvature’ of the generators (see [32, 34] for the archimedean case.) of the tiling (or $p$-adic fractal spray) $T_p$. Moreover, by analogy with what is expected to happen in the Euclidean case, the coefficients of the resulting higher-dimensional tube formula should have an appropriate interpretation in terms of yet to be suitably ‘nonarchimedean fractal curvatures’ associated with each complex and integral dimension of $T_p$. Finally, by analogy with the archimedean case (for $d \geq 1$, see [32 and 34]), the $p$-adic higher-dimensional fractal tube formula should take the same form as in Equation (3.32), except with $\zeta_{L_p}(\varepsilon; s)$ given by a different expression from the one in (3.33) where $d = 1$, and with $D_{L_p}(W)$ replaced by $D_{L_p}(W) \cup \{0, 1, \ldots, d\}$, as well as (for nonarchimedean self-similar tilings) with $W = \mathbb{C}$ and $\mathcal{R}_p(\varepsilon) \equiv 0$ in the counterpart of Equation (3.33) or (3.34). In the future, we plan to investigate the above problems along with related question pertaining to fractal geometry and geometric measure theory in nonarchimedean spaces.

5.4. Towards Nonarchimedean Bergman Spaces and Toeplitz Algebras. We close this discussion by pointing out a long-term problem that involves challenging and seemingly wide open questions in nonarchimedean harmonic and functional analysis.

In [25], fractal membranes were introduced as suitable quantized analogues of fractal strings. They were viewed heuristically as infinite, adelic noncommutative tori but were also proposed to be properly defined as noncommutative spaces (in the sense of Connes, [4]).

A rigorous construction of archimedean fractal membranes is provided by Lapidus and Nest in [31]; see also [25 §4.2.1]. It involves, in particular, Toeplitz operators acting on Bergman spaces (in the standard setting of archimedean complex, harmonic and functional analysis). The resulting Toeplitz algebra is the $C^*$-algebra $\mathcal{T}$ which is represented on a suitable Hilbert space $\mathcal{H}$ (an infinite tensor product of
Bergman spaces, one space for each ‘circle’ in the underlying adèlic infinite dimensional torus representing the fractal membrane or, equivalently, one space for each interval of the original fractal string being ‘quantized’).

Then, the noncommutative space representing the given (archimedean) fractal membrane in a ‘spectral triple’

\[(5.1) \quad ST = \{A, \mathcal{H}, D\},\]

where $D$ is a suitable Dirac-type operator acting on $\mathcal{H}$. (Here, $D$ is a specific unbounded self-adjoint operator with compact resolvents and bounded commutators with the elements of a suitable dense subalgebra of $A$.) We refer to [25, §4.2.1] for an outline of the construction of $ST$ and to [31] for a precise description.

From our present perspective, the challenge alluded to at the beginning of this subsection consists in obtaining an appropriate nonarchimedean counterpart of the above construction, and thereby of the notion of a fractal membrane [64]. (Various aspects of this problem are closely connected with the problems discussed in §§5.1 and 5.2 above.)

At a more modest (but already quite nontrivial level), we must begin by obtaining appropriate nonarchimedean analogs of classical notions in archimedean harmonic and functional analysis, including especially Bergman spaces [8], as well as Toeplitz operators and the associated Toeplitz algebras [3]. As we suggested in §§2.2 for different, but related reasons, it would likely be helpful in this context to work with nonarchimedean Berkovich-type spaces [1, 7] rather than with the traditional $p$-adic spaces. It does not seem that much information is available on this subject in the literature on nonarchimedean functional analysis and operator theory, but it would be certainly be interesting to investigate aspects of this problem in the future. We invite the interested reader to do so as well.

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\begin{center}
\begin{tikzpicture}
\node (z3) at (0,0) {$Z_3$};
\node (z0) at (-1,-2) {$Z_0$};
\node (z1) at (0,-2) {$Z_1$};
\node (z2) at (1,-2) {$Z_2$};
\node (z012) at (-1,-4) {1 2 3};
\node (z123) at (0,-4) {4 5};
\node (z234) at (1,-4) {6};
\node (dots) at (-1,-3) {\text{dots}};
\node (dots2) at (0,-3) {\text{dots}};
\node (dots3) at (1,-3) {\text{dots}};
\end{tikzpicture}
\end{center}
