Spacelike thermal correlators are almost time-independent

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ABSTRACT: We show that, in relativistic field theories, the thermal correlation function of $N$ bosonic operators $\langle O_1(x_1,t_1)O_2(x_2,t_2)\ldots O_N(x_N,t_N)\rangle_T$ at sufficiently spacelike-separated points shows exponentially weak dependence on the time variables $t_1,\ldots,t_N$, when the space separations are held fixed. For classical thermal field theory, the time dependence vanishes when all points are spacelike separated.

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1 Introduction

In this brief note, we present a property of relativistic quantum field theories, in equilibrium at nonvanishing temperature $T$. Even though we are not aware of any new, physically relevant applications of this property, we find it sufficiently striking, and not well known nor appreciated by the community, that we thought it appropriate to point it out.

To motivate the discussion, consider the simplest relativistic field theory: free massless one-component scalar field theory. The ordering-averaged two-point correlation function of the scalar field, in vacuum and expressed in coordinate space, is

$$\frac{1}{2}\left\langle \phi(x,t)\phi(0,0) + \phi(0,0)\phi(x,t) \right\rangle_{T=0} = \frac{1}{4\pi^2(x^2 - t^2)}.$$  \hspace{1cm} (1.1)

As is well known, the correlation function only depends on spatial separation $x$ and time difference $t$ through the invariant length squared $x^2 - t^2$, so the correlation function varies as one changes $t$ at fixed $x$, but not as one moves along a hyperboloid.

The introduction of temperature selects a preferred axis, destroying this property. A straightforward calculation (see below) shows that the correlator becomes

$$\frac{1}{2}\left\langle \phi(x,t)\phi(0,0) + \phi(0,0)\phi(x,t) \right\rangle_{T\neq0} = \frac{T}{8\pi x} \left( \frac{1 + e^{-2\pi T(x-t)}}{1 - e^{-2\pi T(x-t)}} + \frac{1 + e^{-2\pi T(x+t)}}{1 - e^{-2\pi T(x+t)}} \right). \hspace{1cm} (1.2)$$

![Figure 1](image-url)

**Figure 1.** The free scalar theory correlation function, at fixed separation and as a function of time, in vacuum and at two temperatures. At the higher temperature the correlator rapidly approaches zero in the timelike region and becomes exponentially flat in the spacelike region.
This correlator is illustrated in Figure 1. Several features are clear. Compared to the vacuum, the thermal correlation function is larger in the spacelike region and smaller in the timelike region. It now decays exponentially at large times. This property is not generic; it depends on our choice to examine a field operator in a free and massless theory. But another feature turns out to be generic: the correlator becomes very flat near \( t = 0 \).

In fact, closer examination of Eq. (1.2) shows that the correlator is exponentially weakly \( t \) dependent when both \( x - t \) and \( x + t \) are large compared to \( 1/2\pi T \). The remainder of the paper proves that this property holds for arbitrary \( N \)-point functions in a general interacting theory, provided that points are sufficiently spacelike separated from each other.

2 Points on a hyperplane and perturbation theory

In Ref. [1] one of us considered the correlation function of \( N \) operators which all lie on a single spacelike hyperplane, \( t = vz \) (or \( x^0 = vx^3 \)) with \(-1 < v < 1\). Although that reference does not remark on it specifically, the analysis there shows that the time-dependence of such correlators is exponentially weak for sufficiently strong spacelike separation, to all orders in perturbation theory. Here we will reproduce this result, and the formalism of [1], in a slightly simpler way.

First consider the free scalar field correlator we already considered, at finite temperature and at equal time. It can be computed within the Matsubara formalism. We will assume that the reader has some familiarity with this approach; for a review, see [2–4]. The density matrix \( \rho = \frac{1}{Z} e^{-\beta H} \) is expressed as a path integral in terms of a fictitious Euclidean time \( \tau \) which is periodic with period \( \beta = 1/T \); \( Z = Tr e^{-\beta H} \) is the partition function, as usual. The equal-time correlation functions of quantum field theory coincide with the equal-\( \tau \) correlation functions in this Euclidean path integral. Furthermore, we can consider unequal \( \tau \) values if we want. Perturbatively this is done by Fourier transforming the \( \tau \) direction into a discrete frequency sum – the Matsubara sum – with discrete frequencies \( \omega_n = 2\pi n T \). These behave as heavy modes with exponentially decaying spatial correlations.

Let us illustrate this for the scalar two-point function we considered above, at unequal \( z \) and \( \tau \):

\[
\langle \phi(z, \tau)\phi(0, 0) \rangle = \frac{T}{4\pi z} + T \sum_{n\neq 0} \frac{1}{4\pi z} e^{-2\pi n|Tz|} \times e^{2\pi n\tau},
\]

where we have separately written the zero and nonzero frequency contributions, and \( 1/4\pi z \) and \( e^{-2\pi nTz}/4\pi z \) are the spatial correlation functions, in 3 dimensions, of a massless field and a field of mass \( m = 2\pi n T \), respectively. The unequal-time correlation function is just the continuation of the unequal-\( \tau \) correlation function from imaginary to real time, \( \tau \rightarrow it \):

\[
\langle \phi(z, t)\phi(0, 0) \rangle = \frac{T}{4\pi z} \left( 1 + \sum_{n>0} e^{-2\pi nTz} e^{-2\pi nTt} + \sum_{n>0} e^{-2\pi nTz} e^{2\pi nTt} \right).
\]

The second sum is the negative \( n \) values in Eq. (2.1), relabeled to make \( n \) positive. For \(|t| < z\) the sums converge, for larger \( t \) we must rely on analytical continuation. This leads directly to Eq. (1.2).
This generalizes to an interacting theory with nontrivial two-point function $G(z, \tau)$ as follows. Write the Euclidean frequency-momentum domain correlator as $G(p, \omega_n)$; then the unequal time correlation function is found by Fourier transforming to $(z, \tau)$ but replacing $\tau \rightarrow it$:

$$
\langle \phi(z, t)\phi(0, 0) \rangle = T \sum_n \int \frac{d^3p}{(2\pi)^3} e^{ip_zz} e^{-\omega nt} G(p, \omega_n) .
$$

The contribution from the $n$'th mode decays like $e^{-m_n|z|}$, where $m_n$, sometimes called the $n$'th screening mass, parametrizes the position of the first complex-$p$ singularity of $G(p, \omega_n)$. Significantly, $m_n \geq 2\pi|n/T$ for all $n$ in any relativistic theory.\footnote{The proof is a combination of two standard facts: the Euclidean correlator with positive $\eta$ coincides with the retarded correlator: $G(p, \omega_n) = G_R(p, i\omega_n)$ \cite{1}; by causality, the retarded correlator is analytic when $\text{Im}(p)\equiv v$ is future timelike \cite{5}. Thus $G(p, \omega_n)$ is analytic when $|\text{Im}(p)| < 2\pi|T|$ \cite{1}.}

Therefore, all time-dependent contributions decay at least as fast as $e^{-2\pi|T(z-|t|)|}$, and the time-independent term dominates away from the light cone (if $m_0 < m_n$ for all $n \neq 0$, which we generally expect). For fermions, only half-integer $n$ contributes so all modes are time-dependent.

Similarly, for an $N$-point function and its Fourier transform in terms of the first $N-1$ coordinates, we have

$$
\langle O_1(x_1, t_1)O_2(x_2, t_2)\ldots O_N(0, 0) \rangle = T^{N-1} \sum_{n_1, n_2, \ldots, n_{N-1}} \int \frac{d^3p_1 \ldots d^3p_{N-1}}{(2\pi)^{3(N-1)}} \times
$$

$$
G^{O_1O_2\ldots O_N}(p_1, \omega_{n_1}; p_2, \omega_{n_2}; \ldots)e^{i\sum_i \vec{p}_i \cdot \vec{x}_i - \sum_i \omega_{n_i} t_i} .
$$

Here $\omega_{n_i} = 2\pi n_i T$ is the Matsubara frequency associated with the $i$'th operator. Analyzing this requires more care since the analyticity properties of $G^{O_1O_2\ldots O_N}$ are less well understood, and the sum does not converge in all of Minkowski space (it certainly diverges whenever two points become timelike). Let us thus restrict to the case where all operators lie on a spacelike hypersurface with $t_i = vz_i$. We can rewrite this expression as

$$
e^{i\sum_i \vec{p}_i \cdot \vec{x}_i - \sum_i \omega_{n_i} t_i} \rightarrow e^{i\sum_i \vec{p}_i \cdot \vec{x}_i} , \quad \vec{p}_z \equiv p_z + iv\omega_n .
$$

This imaginary shift in $p_z$ can be removed by shifting the integration variable by $p_z \rightarrow p'_z \equiv p_z - iv\omega_n$. This shift never encounters singularities in $G^{O_1O_2\ldots O_N}$, provided $|v| < 1$, as discussed in \cite{1}. After this shift, all reference to time is hidden in this shift of the momentum integration variable, or equivalently, in a shift to the $p_z$ arguments of the momentum-space correlation function: $G^{O_1O_2\ldots O_N}(p'_1, \omega_{n_1}; p'_2, \omega_{n_2}; \ldots)$. That is, we can formulate the Matsubara formalism for the case of a spacelike hyperplane by replacing $p_z \rightarrow p'_z = p_z - iv\omega_n$ on all propagators (and, in a derivative-coupled theory, all momentum-dependent vertices). This is equivalent to the formulation of Ref. \cite{1}, which arrived at this result somewhat differently.

It is instructive to consider a generic diagram contributing to the correlation function in perturbation theory, as depicted in Figure 2. If all the external lines have zero Matsubara frequency (all $n_i = 0$ in Eq. (2.4)) then there is no shift in the $p_z$ variables. Such a shift can still occur in a loop momentum, but no singularities are encountered if we shift the
loop integration variable back to real $q_z$, restoring precisely the equal-time perturbative expansion. Therefore, the contribution from zero external Matsubara frequencies, for $|v| < 1$, is identical to that at $v = 0$, that is, equal time. Any time dependence must arise when one or more external Matsubara frequencies are nonzero.

But if an external Matsubara frequency is nonzero, then momentum conservation at all vertices requires that this frequency flows through the diagram to another external line (to another operator). There are a series of propagators carrying this nonzero frequency from the incoming to the outgoing operator, as indicated in Figure 2 by the series of red lines. Each nonzero-frequency propagator connects spacelike separated points, and is suppressed by $\exp(-2\pi n|n|T(|\Delta x| \mp v\Delta z))$. On a spacelike hyperplane, the sum of the spacelike lengths of a series of segments is at least as large as the spacelike length of the straight-line path; so this leads to a suppression of at least $\exp(-2\pi n|n|T(|x| \mp v|t|))$. Therefore the time dependence is exponentially suppressed provided that all pairs of external points have $|x| - |t| \gg 1/2\pi T$. In this argument it was essential that an Euclidean formalism applies when all operators lie on a spacelike hypersurface.

3 General case

So far we have shown that $N$-point functions of bosonic operators in relativistic, thermal quantum field theory have exponentially small time dependence to all orders in perturbation theory, provided that

- all coordinates are “strongly” spacelike separated, $|\Delta x| - |\Delta t| \gg 1/2\pi T$, and
- all coordinates lie on a spacelike hyperplane.

But does this hold fully nonperturbatively? And is there anything special about spacelike hyperplanes, or does this hold more generally, provided that some operator is spacelike separated from others?

To address this, consider $N$ operators $O_1(x_1, t_1)$ through $O_N(x_N, t_N)$. We will assume that $x_1$ is far enough from other points that there is some range of $t_1$ values where $(x_1, t_1)$
is spacelike separated from every other \((x_1, t_1)\). Now consider the dependence on \(t_1\) of the following two functions:

\[
G_S(t_1) \equiv Z^{-1/2} \text{Tr} e^{-\beta H} \left( \mathcal{O}_1(x_1, t_1) \mathcal{O}_{\text{others}} + \mathcal{O}_{\text{others}} \mathcal{O}_1(x_1, t_1) \right), \\
G_\sigma(t_1) \equiv -i Z^{-1} \text{Tr} e^{-\beta H} \left( \mathcal{O}_1(x_1, t_1) \mathcal{O}_{\text{others}} - \mathcal{O}_{\text{others}} \mathcal{O}_1(x_1, t_1) \right).
\]  

(3.1)

Here \(\mathcal{O}_{\text{others}}\) is the product of the other operators, taken in some arbitrary but fixed order. We recognize \(G_S(t_1)\) as the symmetrized function, and \(G_\sigma(t_1)\) as the commutator or spectral function. We can Fourier transform each function, and in Fourier space they obey a

\[
G^>(\omega) \equiv \int dt \ e^{i\omega t} \text{Tr} e^{-\beta H} \mathcal{O}_1(t) \mathcal{O}_{\text{others}} = e^{\beta\omega} \int dt \ e^{i\omega t} \text{Tr} e^{-\beta H} \mathcal{O}_{\text{others}} \mathcal{O}_1(t) \equiv e^{\beta\omega} G^<(\omega)
\]

(3.2)

and therefore

\[
G_S(\omega) = \frac{e^{\beta\omega} + 1}{2} G^<(\omega), \quad G_\sigma(\omega) = -i(e^{\beta\omega} - 1) G^<(\omega), \quad \text{and hence}
\]

\[
G_S(\omega) = -i \left( n_b(\omega) + \frac{1}{2} \right) G_\sigma(\omega),
\]

(3.3)

where \(n_b(\omega) = 1/(e^{\omega/T} - 1)\) is the Bose-Einstein distribution function. These statements are nonperturbative, relying only on energy conservation and the form of the thermal density matrix.

In the high temperature or low-frequency limit we can approximate this as

\[
G_S(\omega) \simeq -\frac{T}{\omega} G_\sigma(\omega) \quad \text{or, in the time domain,} \quad G_\sigma(t) = -\frac{1}{T} \frac{d}{dt} G_S(t).
\]

(3.4)

In this approximation, the spectral function is \((-1/T)\) times the time derivative of the symmetrized function. But causality ensures that the spectral function vanishes unless \((x_1, t_1)\) is null- or timelike- related to at least one of the other operators. Therefore, in the time range where \((x_1, t_1)\) is spacelike separated from all others, the correlator \(G_S(t)\) is independent of the time \(t_1\). This strict high-temperature limit is equivalent to the classical field approximation.

Relaxing the high-temperature approximation, we can Fourier transform Eq. (3.3) to find a kind of dispersive representation:

\[
G_S(t_1) = T \int dt_1' G_\sigma(t_1') \left( \frac{1 + e^{-2\pi T(t_1'-t_1)}}{1 - e^{-2\pi T(t_1'-t_1)}} \right).
\]

(3.5)

The quantity in parenthesis, which is the Fourier transform of \(-i \left( n_b(\omega) + \frac{1}{2} \right)\), is constant up to exponentially small corrections away from \((t_1' - t_1) = 0\). Therefore, if \(G_\sigma(t_1')\) vanishes

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2Here is a quick derivation of the KMS relation. Write \(\mathcal{O}_1(t) = e^{iH t} \mathcal{O}_1 e^{-iH \tau} \) and insert a complete set of (energy eigen)states \(\sum_m |m\rangle \langle m|\) to its left and \(\sum_n |n\rangle \langle n|\) to its right. Then the lefthand side of Eq. (3.2) becomes \(\sum_{mn} \int dt e^{i\omega t} e^{i(E_m - E_n)\tau} e^{-\beta E_m} \langle m| \mathcal{O}_1 |n\rangle \langle n| \mathcal{O}_{\text{others}} |m\rangle\). The right-hand side is the same except with \(e^{-\beta E_m} \rightarrow e^{-\beta E_n}\). Performing the time integral introduces \(\delta(\omega + E_m - E_n)\), allowing us to rewrite \(e^{-\beta E_m} = e^{\beta\omega} e^{-\beta E_n}\). The relation then follows.
in some window around \( t'_1 = t_1 \), the time dependence of \( G_S(t_1) \) is exponentially small. But \( G_S(t'_1) \) is nonvanishing only if \((x_1, t'_1)\) is null- or timelike-separated from at least one other operator. Therefore, when \((x_1, t_1)\) is strongly spacelike separated from all other points, the time dependence of \( G_S(t_1) \) is exponentially suppressed by \( 2\pi T \) times the distance to the closest lightcone. Note that a stronger suppression is possible, as illustrated above in the two-point case where the exponent features \( m_1 \geq 2\pi T \).

4 Discussion

We have shown that spacelike correlation functions in relativistic, thermal field theories have exponentially weak dependence on the relative times of the operators, suppressed by at least \( \exp(-2\pi T |x - t|) \), with \( |x - t| \) the time separation from the nearest light cone and \( T \) the temperature. This property does not depend on perturbation theory; indeed, it follows only from causality and the KMS relation.

In a strong coupling scenario, this result may not be too surprising. Then one expects static correlations to decay with a mass \( m_0 \sim 2\pi T \times O(1) \) [6], so they may already be small in regimes where they dominate over time-dependent correlations. Similarly, our result has no implications for equilibrium hydrodynamics (thermal hydrodynamical fluctuations); while hydrodynamical correlators are highly nontrivial in the timelike region, they generally vanish at spacelike separation. In theories near a critical point, there are strong long-range fluctuations, and in this case our result provides nontrivial information about the time dependence of correlation functions. Namely, at a separation \( x \lesssim \xi \) with \( \xi \) the correlation length, there is a time scale \( t \in [-x, x] \) over which correlators do not evolve. This is unexpected. We emphasize, however, that it does not contradict any standard results regarding dynamical critical fluctuations, which pertain to time scales of order \( t \sim x^z T^{z-1} \) with \( z > 1 \) the dynamical critical exponent [7] and \( 1/T \) an estimate for the UV scale below which criticality arguments become applicable. Therefore, the standard analysis of dynamical critical scaling involves time scales which are parametrically longer than the time scale where our result is valid.

We find the generality of our results surprising, and thought that they were worth presenting, even though we are not yet aware of applications, other than those to jet quenching at weak coupling which are explored in Ref. [1].

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