Maximal $L^1$-regularity for parabolic initial-boundary value problems with inhomogeneous data

TAKAYOSHI OGAWA AND SENJO SHIMIZU

Abstract. End-point maximal $L^1$-regularity for parabolic initial-boundary value problems is considered. For the inhomogeneous Dirichlet and Neumann data, maximal $L^1$-regularity for initial-boundary value problems is established in time end-point case upon the homogeneous Besov space $\dot{B}^s_{p,1}(\mathbb{R}^n_+)$ with $1 < p < \infty$ and $-1 + 1/p < s \leq 0$ as well as optimal trace estimates. The main estimates obtained here are sharp in the sense of trace estimates and it is not available by known theory on the class of UMD Banach spaces. We utilize a method of harmonic analysis, in particular, the almost orthogonal properties between the boundary potentials of the Dirichlet and the Neumann boundary data and the Littlewood-Paley dyadic decomposition of unity in the Besov and the Lizorkin-Triebel spaces.

1. Introduction and main results

In this paper, we are concerned with maximal $L^1$-regularity for initial-boundary value problems of parabolic equations in the half-space $\mathbb{R}^n_+$ with inhomogeneous boundary data.

Let $X$ be a Banach space and $A$ be a closed linear operator in $X$ with a densely defined domain $D(A)$. For an initial data $u_0 \in X$ and an external force $f \in L^\rho(0, T; X)$ ($1 \leq \rho \leq \infty$), let $u$ be a solution to the abstract Cauchy problem:

$$\begin{cases}
\frac{d}{dt} u + Au = f, \quad t > 0, \\
u(0) = u_0.
\end{cases} \quad (1.1)$$

Then, $A$ has maximal $L^\rho$-regularity if there exists a unique solution $u$ of (1.1) such that $\frac{d}{dt} u, Au \in L^\rho(0, T; X)$ satisfy the estimate:

$$\left\| \frac{d}{dt} u \right\|_{L^\rho(0, T; X)} + \left\| Au \right\|_{L^\rho(0, T; X)} \leq C \left( \left\| u_0 \right\|_{(X, D(A))_{1-1/\rho}^{1-1/\rho}} + \| f \|_{L^\rho(0, T; X)} \right),$$

under the restriction $u_0 \in (X, D(A))_{1-1/\rho}^{1-1/\rho}$, where $(X, D(A))_{1-1/\rho}^{1-1/\rho}$ denotes the real interpolation space between $X$ and $D(A)$, and $C$ is a positive constant independent

Mathematics Subject Classification: Primary 35K20; Secondary 42B25

Keywords: Parabolic equations with variable coefficients, Maximal $L^1$-regularity, End-point estimate, Initial-boundary value problems, The Dirichlet problem, The Neumann problem.
of \(u_0\) and \(f\). Maximal regularity for parabolic equations was first considered by Ladyzhenskaya–Solonnikov–Ural’tseva [29]. Then, the research of maximal regularity has developed immensely in these last few decades by many authors; [5,8,9,13,16–19,21,22,25,31,43,44]. In the general framework on Banach spaces \(X\) that satisfy the unconditional martingale differences (called as UMD), the general theory of maximal regularity was well established, especially by Amann [2,3], Denk–Hieber–Prüss [14,15], Weis [53] (see also [26,28,40]).

On the other hand, maximal regularity on non-UMD Banach spaces, for instance, non-reflexive Banach space such as \(L^1\) or \(L^\infty\) requires independent arguments. For example, one can observe some results for maximal \(L^1\)-regularity for the Cauchy problem on the homogenous Banach spaces in Chemin [7]. Danchin [10,11], Giga–Saal [20], Iwabuchi [23], Ogawa–Shimizu [33,34] in various non-UMD settings. In general, maximal time \(L^1\)-regularity fails over the Lebesgue spaces in spatial variables and we need to introduce restrictive function classes such as homogeneous and inhomogeneous Besov spaces for the spatial variables. Hence, maximal \(L^1\)-regularity for the initial boundary value problems is not well established in the general framework.

1.1. The Dirichlet boundary condition

We first recall non-endpoint maximal regularity for the initial boundary value problems to parabolic equations. Let \(I = (0, T)\) with \(0 < T \leq \infty\). Let \(u\) be a solution of the initial-boundary value problem of the second-order parabolic equation with variable coefficients and the inhomogeneous Dirichlet boundary condition in the half-space \(\mathbb{R}^n_+ = \{x = (x', x_n); x' \in \mathbb{R}^{n-1}, x_n > 0\}:

\[
\begin{cases}
\partial_t u - \sum_{1 \leq i,j \leq n} a_{ij}(t, x) \partial_i \partial_j u = f, & t \in I, \quad x \in \mathbb{R}^n_+ , \\
u(t, x', x_n)|_{x_n = 0} = g(t, x'), & t \in I, \quad x' \in \mathbb{R}^{n-1}, \\
u(t, x)|_{t = 0} = u_0(x), & x \in \mathbb{R}^n_+.
\end{cases}
\] (1.2)

where \(\partial_t\) and \(\partial_i \equiv \partial_{x_i}\) are partial derivatives with respect to \(t\) and \(x_i\), \(u = u(t, x)\) denotes the unknown function, \(u_0 = u_0(x)\), \(f = f(t, x)\) and \(g = g(t, x')\) are given initial, external force and boundary data, respectively. The coefficient matrix \(\{a_{ij}(t, x)\}_{1 \leq i,j \leq n}\) satisfies uniformly elliptic condition and have enough regularity. Namely \(\{a_{ij}\}\) is a real-valued symmetric matrix such that for some constant \(c > 0\),

\[
\sum_{1 \leq i,j \leq n} a_{ij}(t, x) \xi_i \xi_j \geq c|\xi|^2 \quad \text{for all } \xi \in \mathbb{R}^n \text{ with sufficient regularity in both } t \text{ and } x.
\]

For \(1 \leq \rho \leq \infty\) and a Banach space \(X\), we denote the Lebesgue-Bochner space \(L^\rho(I; X)\) and the inhomogeneous and homogeneous Sobolev-Bochner spaces as \(W^{1,\rho}(I; X)\), \(\dot{W}^{1,\rho}(I; X)\), respectively. We denote a set of all continuous bounded \(X\)-valued functions over an interval \(I\) by \(C_b(I; X)\) and on \(\Omega \subset \mathbb{R}^n\) by \(C_b(\Omega)\). We also denote a set of all \(X\)-valued functions which is bounded uniformly continuous over an interval \(I\) by \(BUC(I; X)\) and on \(\Omega \subset \mathbb{R}^n\) by \(BUC(\Omega)\).
In this context, the following results were obtained by Weidemaier [51] and Denk–Hieber–Prüss [15].

**Proposition 1.1** (The Dirichlet boundary condition [15,51]). Let $1 < \rho < \infty$ with $1 - 1/(2p) \neq 1/\rho$, $I = (0, T)$ with $T < \infty$. Then, the problem (1.2) admits a unique solution $u \in W^{1,\rho}(I; L^p(\mathbb{R}^n_+)) \cap L^\rho(I; W^{2,p}(\mathbb{R}^n_+))$ if and only if

\[
\begin{align*}
    f &\in L^\rho(I; L^p(\mathbb{R}^n_+)), \\
    u_0 &\in B^{2(1-1/\rho)}_{p,\rho}(\mathbb{R}^n_+), \\
    g &\in F^{1-1/2p}_{\rho,p}(I; L^\rho(\mathbb{R}^n-1)) \cap L^\rho(I; B^{2-1/p}_{p,p}(\mathbb{R}^{n-1})), \\
    \text{if } 1 - 1/(2p) > 1/\rho, \text{ then } u_0(x', x_n)|_{x_n=0} = g(t, x')|_{t=0}. 
\end{align*}
\]

Besides there exists a constant $C_T > 0$ depending on $n$, $p$, $\rho$, $\{a_{ij}\}$, $T$ such that the solution $u$ is subject to the inequality:

\[
\begin{align*}
    \|\partial_i u\|_{L^\rho(I; L^p(\mathbb{R}^n_+))} + \|\nabla^2 u\|_{L^\rho(I; L^p(\mathbb{R}^n_+))} \\
    \leq C_T \left(\|u_0\|_{B^{2(1-1/2p)}_{p,\rho}(\mathbb{R}^n_+)} + \|f\|_{L^\rho(I; L^p(\mathbb{R}^n_+))} + \|g\|_{F^{1-1/2p}_{\rho,p}(I; L^p(\mathbb{R}^{n-1}))} \right) \\
    + \|g\|_{L^p(I; B^{2-1/p}_{p,p}(\mathbb{R}^{n-1}))},
\end{align*}
\]

where $|\nabla^2 u| = (\sum_{1 \leq i, j \leq n} |\partial_i \partial_j u|^2)^{1/2}$, $L^\rho(I; X)$ denotes the $\rho$-th powered Lebesgue–Bochner space upon a Banach space $X$ and $B^{2-1/p}_{p,p}(\mathbb{R}^{n-1})$ and $F^{1-1/2p}_{\rho,p}(I; X)$ denote the interpolation spaces of the Besov and the Lizorkin–Triebel type, respectively.

Weidemaier [49] first obtained a trace theorem for functions in anisotropic Sobolev spaces. Then, he extended his result to a boundary trace of a solution of parabolic equations in the Bochner space and obtained the optimal trace estimates ([50–52]) with introducing the Lizorkin–Triebel space in the time variable. In the proof of the results, he employed a solution formula with respect to the time variable and the proof is involved the maximal function for a test function. The results in [48–52] are obtained under the restriction of exponents of the Bochner spaces to space $\rho$ and time $\rho$ variables as $\frac{3}{2} < p \leq \rho < \infty$ of for the Dirichlet problem and $3 < p \leq \rho < \infty$ for the Neumann problem, respectively. Denk–Hieber–Prüss [15] obtained the above necessary and sufficient condition of unique existence of solutions to initial-boundary value problems including higher order parabolic operators subject to general boundary conditions in a domain $\Omega$ in $\mathbb{R}^n$ with a compact boundary. Their ingenious idea is regarding the problem as an evolution equation with respect to the spatial variable $x_n$ and the boundary condition as an initial data. However, the proof in [15] is based on the vector valued version of Mikhlin’s Fourier multiplier theorem, and accordingly the result is restricted in the cases $1 < \rho < \infty$. Their result is essentially in a time local estimate because the elliptic operator considered there has strictly positive spectrum, and hence, the boundary conditions are limited in the inhomogeneous real interpolation spaces.

In this paper, we show *time global maximal $L^1$-regularity* for initial-boundary value problems of the evolution equation of parabolic type in both the inhomogeneous Dirichlet and the Neumann boundary conditions. For 0-Dirichlet boundary
data, Danchin–Mucha [12] obtained maximal $L^1$-regularity for the initial-boundary value problem (1.2) in the half-space with $a_{ij}(t, x) = \delta_{ij}$ and $g(t, x') = 0$. On the other hand, maximal $L^1$-regularity for the case of nonzero boundary condition requires very different treatment and it is far from obvious since the time exponent is the end-point and it is neither clear nor straightforward if a natural extension from the known result Proposition 1.1 holds in a natural exponent. Here, we consider inhomogeneous initial-boundary value problems in both the Drichlet and the Neumann boundary conditions and show that a natural extension from Proposition 1.1 does not hold in general (namely the conditions for the boundary data in (1.3) with $\rho = 1$) and we explicitly prove it by showing the end-point time exponent invites the end-point interpolation exponent similar to the case of the initial value problem.

Before stating our results, we define Besov spaces and Lizorkin–Triebel spaces in the half-space and the half-line. Since the global estimate requires the base space for spatial interpolation exponent similar to the case of the initial value problem.

Definition (The Besov and the Lizorkin–Triebel spaces). Let $s \in \mathbb{R}$, $1 \leq p, \sigma \leq \infty$. Let $\{\phi_j\}_{j \in \mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity for $x \in \mathbb{R}^n$, namely $\hat{\phi}$ is the Fourier transform of a smooth radial function $\phi \in \mathcal{F}_0$ with $\hat{\phi}(\xi) \geq 0$ and $\text{supp} \, \hat{\phi} \subset \{\xi \in \mathbb{R}^n \mid 2^{-1} < |\xi| < 2\}$, and

$$
\hat{\phi}_j(\xi) = \hat{\phi}(2^{-j} \xi), \quad \sum_{j \in \mathbb{Z}} \hat{\phi}_j(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n \setminus \{0\}, \quad j \in \mathbb{Z}
$$

and

$$
\hat{\phi}_0(\xi) + \sum_{j \geq 1} \hat{\phi}_j(\xi) = 1 \quad \text{for any } \xi \in \mathbb{R}^n,
$$

where $\hat{\phi}_0(\xi) \equiv \hat{\xi}(|\xi|)$ with a low frequency cutoff $\hat{\xi}(r) = 1$ for $0 \leq r < 1$ and $\hat{\xi}(r) = 0$ for $2 < r$. For $s \in \mathbb{R}$ and $1 \leq p, \sigma \leq \infty$, $\tilde{B}^s_{p, \sigma} (\mathbb{R}^n)$ be the homogeneous Besov space with norm

$$
\| \tilde{f} \|_{\tilde{B}^s_{p, \sigma}} \equiv \begin{cases} 
\left( \sum_{j \in \mathbb{Z}} 2^{sj} \| \phi_j * \tilde{f} \|_p^\sigma \right)^{1/\sigma}, & 1 \leq \sigma < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj} \| \phi_j * \tilde{f} \|_p^\sigma, & \sigma = \infty,
\end{cases}
$$

where $f * g$ denotes the convolution between Schwartz class functions $f$ and $g \in \mathcal{S}(\mathbb{R}^n)$ given by

$$
f * g(x) = c_n^{-1} \int_{\mathbb{R}^n} f(x - y)g(y)dy, \quad c_n = (2\pi)^{-n/2}
$$

and for $f, g \in \mathcal{S}'$ as the distribution sense, where $\mathcal{S}'$ is the tempered distributions. $B^s_{p, \sigma} (\mathbb{R}^n)$ be the inhomogeneous Besov space with norm

$$
\| \tilde{f} \|_{B^s_{p, \sigma}} \equiv \begin{cases} 
\left( \| \phi_0 * \tilde{f} \|_p + \sum_{j \in \mathbb{Z}} 2^{sj} \| \phi_j * \tilde{f} \|_p^\sigma \right)^{1/\sigma}, & 1 \leq \sigma < \infty, \\
\| \phi_0 * \tilde{f} \|_p + \sup_{j \in \mathbb{Z}} 2^{sj} \| \phi_j * \tilde{f} \|_p^\sigma, & \sigma = \infty.
\end{cases}
$$
For $s \in \mathbb{R}$, $1 \leq p < \infty$ and $1 \leq \sigma \leq \infty$, $\dot{F}^s_{p, \sigma}(\mathbb{R}^n)$ be the homogeneous Lizorkin–Triebel space with norm

\[
\| \tilde{f} \|_{\dot{F}^s_{p, \sigma}} \equiv \left\{ \begin{array}{ll}
\left\| \left( \sum_{j \in \mathbb{Z}} 2^{s\sigma j} |\phi_j \ast \tilde{f}(\cdot)|^\sigma \right)^{1/\sigma} \right\|_p, & 1 \leq \sigma < \infty, \\
\sup_{j \in \mathbb{Z}} 2^{sj}|\phi_j \ast \tilde{f}(\cdot)| \|_p, & \sigma = \infty.
\end{array} \right.
\]

We define the homogeneous Besov space $\dot{B}^s_{p, \sigma}(\mathbb{R}^n_+)$ as the set of all measurable functions $f$ in $\mathbb{R}^n_+$ satisfying

\[
\| f \|_{\dot{B}^s_{p, \sigma}(\mathbb{R}^n_+)} \equiv \inf \left\{ \| \tilde{f} \|_{\dot{B}^s_{p, \sigma}(\mathbb{R}^n)} < \infty; \ \tilde{f} = \left\{ \begin{array}{ll}
f(x', x_n) & (x_n > 0) \\
\text{any extension} & (x_n < 0)
\end{array} \right\} \right\},
\]

\[
\tilde{f} = \sum_{j \in \mathbb{Z}} \phi_j \ast \tilde{f} \text{ in } S'.
\]

(1.5)

Analogously, we define the inhomogeneous Besov space $B^s_{p, \sigma}(\mathbb{R}^n_+)$ as the set of all measurable functions $f$ in $\mathbb{R}^n_+$ satisfying

\[
\| f \|_{B^s_{p, \sigma}(\mathbb{R}^n_+)} \equiv \inf \left\{ \| \tilde{f} \|_{B^s_{p, \sigma}(\mathbb{R}^n)} < \infty; \ \tilde{f} = \left\{ \begin{array}{ll}
f(x', x_n) & (x_n > 0) \\
\text{any extension} & (x_n < 0)
\end{array} \right\} \right\}.
\]

Definition (The Bochner–Lizorkin–Triebel spaces). Let $s \in \mathbb{R}$, $1 \leq p, \sigma \leq \infty$ and $X(\mathbb{R}^n_+)$ be a Banach space on $\mathbb{R}^n_+$ with the norm $\| \cdot \|_X$. Let $\{\psi_k\}_{k \in \mathbb{Z}}$ be the Littlewood-Paley dyadic decomposition of unity for $t \in \mathbb{R}$. For $s \in \mathbb{R}$ and $1 \leq p < \infty$, $\dot{F}^s_{p, \sigma}(\mathbb{R}; X)$ be the Bochner–Lizorkin–Triebel space with norm

\[
\| \tilde{f} \|_{\dot{F}^s_{p, \sigma}(\mathbb{R}; X)} \equiv \left\{ \begin{array}{ll}
\left\| \left( \sum_{k \in \mathbb{Z}} 2^{s\sigma k} \| \psi_k \ast \tilde{f}(t, \cdot) \|_X^\sigma \right)^{1/\sigma} \right\|_{L^p(\mathbb{R})}, & 1 \leq \sigma < \infty, \\
\sup_{k \in \mathbb{Z}} 2^k \| \psi_k \ast \tilde{f}(t, \cdot) \|_X \|_{L^p(\mathbb{R})}, & \sigma = \infty.
\end{array} \right.
\]

Analogously above, we define the Bochner–Lizorkin–Triebel spaces $\dot{F}^s_{p, \sigma}(I; X)$ as the set of all measurable functions $f$ on $X$ satisfying

\[
\| f \|_{\dot{F}^s_{p, \sigma}(I; X)} \equiv \inf \left\{ \| \tilde{f} \|_{\dot{F}^s_{p, \sigma}(\mathbb{R}; X)} < \infty; \ \tilde{f} = \left\{ \begin{array}{ll}
f(t, x) & (t \in I) \\
\text{any extension} & (t \in \mathbb{R} \setminus I)
\end{array} \right\} \right\}.
\]

We note that all the spaces of homogeneous type are understood as the Banach spaces by introducing the quotient spaces identifying all polynomial differences.

We assume that the real-valued coefficients $\{a_{ij}(t, x)\}_{1 \leq i, j \leq n}$ satisfy the following conditions.
Assumption 1. For $1 \leq i, j \leq n$ and $(t, x) \in I \times \mathbb{R}^n_+$,

1. $a_{ij}(t, x) = \delta_{ij} + \tilde{a}_{ij} + b_{ij}(t, x)$, where $\delta_{ij}$ and $\tilde{a}_{ij}$ denote the Kronecker delta and components of a positive constant matrix, respectively,
2. $\tilde{a}_{ij} = \tilde{a}_{ji}$ and $b_{ij}(t, x) = b_{ji}(t, x)$ for all $t > 0$ and $x \in \mathbb{R}^n_+$,
3. there exists a constant $c > 0$ such that for any $\xi \in \mathbb{R}^n$,
   \[ \sum_{1 \leq i, j \leq n} a_{ij}(t, x)\xi_i \xi_j \geq c|\xi|^2, \quad (t, x) \in I \times \mathbb{R}^n_+, \]
4. $b_{ij} \in \text{BUC}(\mathbb{R}^n_+; \dot{B}^{\frac{n}{p}}_{q,1}(\mathbb{R}^n_+))$ for some $1 \leq q < \infty$, where $\text{BUC}(I; X)$ denotes a set of all bounded uniformly continuous functions on $I$.

It is known that for $1 \leq q < \infty$, $\dot{B}^{\frac{n}{p}}_{q,1}(\mathbb{R}^n_+)$ satisfies $S_0(\mathbb{R}^n) \hookrightarrow \dot{B}^{\frac{n}{p}}_{q,1}(\mathbb{R}^n_+) \hookrightarrow C_v(\mathbb{R}^n)$, where $S_0$ denotes the rapidly decreasing smooth functions with vanishing at the origin of its Fourier transform and $C_v$ denotes the set of all continuous functions with vanishing at infinity, respectively (cf. [34, Proposition 2.3]). It also satisfies the product inequality, namely the following inequality holds:

\[ \|b_{ij}h\|_{\dot{B}^{\frac{n}{p}}_{q,1}} \leq C\|b_{ij}\|_{\dot{B}^{\frac{n}{p}}_{q,1}}\|h\|_{\dot{B}^{\frac{n}{p}}_{q,1}}, \quad 1 \leq p \leq \infty, \quad -\frac{n}{q} < s \leq 0 \]

(cf. Adibi-Paicu [1, 4]). Hence to ensure the uniformly elliptic condition (3) in Assumption 1, we split the coefficients $a_{ij}(t, x)$ into the constant parts $\delta_{ij} + \tilde{a}_{ij}$ and the decreasing functions $b_{ij}(t, x)$ as $|x| \to \infty$.

Then our main result for the end-point case of maximal regularity to the problem (1.2) now read as the following:

**Theorem 1.2** (The Dirichlet boundary condition). Let $1 < p < \infty$, $-1 + 1/p < s \leq 0$ and assume that the coefficients $\{a_{ij}\}_{1 \leq i, j \leq n}$ satisfy Assumption 1.

1. Suppose that $b_{ij}(t, x) \equiv 0$ for all $1 \leq i, j \leq n$. Then, the problem (1.2) admits a unique solution

\[ u \in \dot{W}^{1,1}((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+))), \quad \Delta u \in L^1((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+))), \]

if and only if the external, initial and boundary data in (1.2) satisfy

\[ f \in L^1((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+))), \quad u_0 \in \dot{B}^s_p(\mathbb{R}^n_+), \quad g \in \dot{F}^{1-1/2p}_{1,1}((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+))) \cap L^1((\mathbb{R}^n_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^n_+))), \]

respectively. Besides, the solution $u$ satisfies the following estimate for some constant $C_M > 0$ depending only on $p$, $s$ and $n$

\[ \|\partial_t u\|_{L^1((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+)))} + \|\nabla^2 u\|_{L^1((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+)))} \leq C_M \left(\|u_0\|_{\dot{B}^s_p(\mathbb{R}^n_+)} + \|f\|_{L^1((\mathbb{R}^n_+; \dot{B}^s_p(\mathbb{R}^n_+)))} + \|g\|_{\dot{F}^{1-1/2p}_{1,1}((\mathbb{R}^n_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^n_+)))} \right). \]
(2) Let \( 1 \leq q < \infty, -n/q < s \leq 0 \) and let \( I = (0, T) \) for \( T < \infty \). Then, the problem (1.2) admits a unique solution

\[
\dot{u} \in \dot{W}^{1,1}(I; \dot{B}^{s}_{p,1}(\mathbb{R}^{n}_{+})), \quad \Delta u \in L^{1}(\mathbb{R}^{n}_{+}; \dot{B}^{s}_{p,1}(\mathbb{R}^{n}_{+})),
\]

if and only if the external force, the initial data and the boundary data satisfy (1.6) and (1.7) with replacing \( \mathbb{R}^{+} \) into \( I \). Besides, the solution \( u \) satisfies the following estimate for some constant \( C_{M} = C_{M}(n, p, q, \{ \tilde{a}_{ij} \}) > 0 \)

\[
\| \partial_{t} u \|_{L^{1}(I; \dot{B}^{s}_{p,1}(\mathbb{R}^{n}_{+}))} + \| \nabla^{2} u \|_{L^{1}(I; \dot{B}^{s}_{p,1}(\mathbb{R}^{n}_{+}))} \\
\leq C_{M} \int_{0}^{T} e^{\mu(T-s)} \| f(s) \|_{\dot{B}^{s}_{p,1}} ds \\
+ C_{M} \left\{ 1 + \max_{1 \leq i, j \leq n} \| b_{ij} \|_{L^{\infty}(I; \dot{B}^{s/q}_{p,1}(\mathbb{R}^{n}_{+})) (e^{\mu T} - 1)} \right\} \\
\times \left( \| u_{0} \|_{\dot{B}^{s}_{p,1}} + \| g \|_{L^{1}(I; \dot{B}^{s-1/p}_{p,1}(\mathbb{R}^{n}_{+}) + \| g \|_{L^{1}(I; \dot{B}^{s+1/p}_{p,1}(\mathbb{R}^{n}_{+}))} \right), \quad (1.8)
\]

where \( \mu = C_{M}^{2} \log(1 + C_{M}) \).

Remarks. (i) The solution in Theorem 1.2 satisfies

\[
u \in C_{b}\left([0, T); \dot{B}^{s}_{p,1}(\mathbb{R}^{n}_{+})\right)
\]

with \( T = \infty \) for the solution \( u \) in (1) and with \( T < \infty \) in (2). The linear evolution generated by the elliptic operator generates \( C_{0} \)-semigroup in \( \dot{B}^{s}_{p,1}(\mathbb{R}^{n}_{+}) \) and the estimate of maximal \( L^{1} \)-regularity ensures that the absolute continuity in \( t \) to the solution.

(ii) Since \( 1 - 1/(2p) < 1 \) for all \( 1 < p < \infty \), the compatibility condition

\[
u_{0}(x', x_{n})|_{x_{n}=0} = g(t, x')|_{t=0}
\]

is not necessarily required for (1.8).

(iii) If \( p = \infty \), the corresponding result holds for the homogeneous Besov space \( \dot{B}^{s}_{\infty,1}(\mathbb{R}^{n}) \equiv \overline{C_{0}^{\infty}(\mathbb{R}^{n})} \dot{B}^{s}_{\infty,1}(\mathbb{R}^{n}) \), where \( C_{0}^{\infty}(\mathbb{R}^{n}) \) denotes all compactly supported smooth functions with vanishing at the origin of its Fourier image and

\[
\| f \|_{\dot{B}^{s}_{\infty,1}(\mathbb{R}^{n})} \equiv \inf \left\{ \| \tilde{f} \|_{\dot{B}^{s}_{p,1}(\mathbb{R}^{n})} < \infty; \& \tilde{f} = \begin{cases} f(x', x_{n}) & (x_{n} > 0) \\ \text{any extension} & (x_{n} \leq 0) \end{cases} \right\},
\]

\[
\tilde{f} = \sum_{j \in \mathbb{Z}} \phi_{j} * \tilde{f} \text{ in } S',
\]

instead of the Besov space \( \dot{B}^{s}_{\infty,1}(\mathbb{R}^{n}_{+}) \) with imposing the compatibility condition (1.9) if \( (s, p) = (0, \infty) \). Note that \( \dot{B}^{0}_{\infty,1}(\mathbb{R}^{n}_{+}) \subset C_{b}(\mathbb{R}^{n}_{+}) \) for the endpoint case \( (s, p) = (0, \infty) \).
We only show the estimate for \( \mathbb{R}_+^n \) in time but a similar estimate for the finite time interval \( I = (0, T) \) with \( T < \infty \) is also available. In such a case, the restriction on the initial data \( u_0 \) can be relaxed into the class of inhomogeneous Besov spaces \( B^s_{p,1}(\mathbb{R}_+^n) \supset \dot{B}^s_{p,1}(\mathbb{R}_+^n) \) and the constant appeared in the estimate can be estimated as \( C_M \simeq \mathcal{O}(\log T) \) \( (T \to \infty) \) (see [34]).

The function class for the \( x \)-variable in Theorem 1.2 is restricted in \( \dot{B}^s_{p,1}(\mathbb{R}_+^n) \subseteq \dot{W}^{s,p}(\mathbb{R}_+^n) \) and this restriction is necessary for obtaining maximal \( L^1 \)-regularity (see [34]). Connecting this fact, the conditions on the coefficients \( a_{ij}(t, x) \) are so far best available. In general, the elliptic and parabolic type estimates in \( L^p \)-setting allow us to treat much general coefficients such as \( a_{ij}(t, x) \in VMO(I \times \mathbb{R}^n) \) for the whole space case, where \( VMO \) stands for the vanishing mean oscillation (see e.g. Krylov [27]). However, since maximal regularity in \( L^1 \) generally fails for the Lebesgue spaces \( L^p \), we need to restrict the spatial function class to the Besov space \( \dot{B}^0_{p,1} \subseteq L^p \) even for the whole space \( \mathbb{R}_+^n \) case. Such a restriction limits the condition on the coefficients as given in Assumption 1.

On the other hand, Theorem 1.2 does not cover the end-point spatial exponent \( p = 1 \) nor \( p = \infty \). Instead of the above result, we show a substituting estimate holds in \( L^p(\mathbb{R}_+^n) \).

**Theorem 1.3** (The Dirichlet boundary condition). Assume that the coefficients \( \{a_{ij}\}_{1 \leq i, j \leq n} \) be constants that satisfy Assumption 1, i.e., \( b_{ij}(t, x) \equiv 0 \).

1. Let \( 1 \leq p < \infty \). If the external force, the initial data and the boundary data satisfy

\[
\begin{align*}
  f &\in L^1(\mathbb{R}_+; \dot{B}^0_{p,1}(\mathbb{R}_+^n)), \quad u_0 \in \dot{B}^0_{p,1}(\mathbb{R}_+^n), \\
  g &\in \dot{F}^{1-1/p}_{1,1}(\mathbb{R}_+; L^p(\mathbb{R}^n-1)) \cap L^1(\mathbb{R}_+; B^{2-1/p}_{p,1}(\mathbb{R}^n-1)),
\end{align*}
\]

(1.10) then there exists a unique solution \( u \) to (1.2) in

\[
W^{1,1}(\mathbb{R}_+; L^p(\mathbb{R}_+^n)) \cap L^1(\mathbb{R}_+; \dot{W}^{2,p}(\mathbb{R}_+^n))
\]

and which satisfies the following estimate:

\[
\| \partial_t u \|_{L^1(\mathbb{R}_+; L^p(\mathbb{R}_+^n))} + \| \nabla^2 u \|_{L^1(\mathbb{R}_+; L^p(\mathbb{R}_+^n))} \leq C \left( \| u_0 \|_{\dot{B}^0_{p,1}(\mathbb{R}_+^n)} + \| f \|_{L^1(\mathbb{R}_+; \dot{B}^0_{p,1}(\mathbb{R}_+^n))} + \| g \|_{\dot{F}^{1-1/p}_{1,1}(\mathbb{R}_+; L^p(\mathbb{R}^n-1))} \right),
\]

(1.11)

where \( C \) is depending only on \( p \) and \( n \).

2. For \( p = \infty \), if the external force, the initial and the boundary data satisfy

\[
\begin{align*}
  f &\in L^1(\mathbb{R}_+; \dot{B}^0_{\infty,1}(\mathbb{R}_+^n)), \quad u_0 \in \dot{B}^0_{\infty,1}(\mathbb{R}_+^n), \\
  g &\in \dot{F}^{1,1}(\mathbb{R}_+; C^1(\mathbb{R}^n-1)) \cap L^1(\mathbb{R}_+; B^{2,1}_{\infty,1}(\mathbb{R}^n-1)),
\end{align*}
\]

with imposing the compatibility condition (1.9), then there exists a unique solution \( u \) to (1.2) in

\[
W^{1,1}(\mathbb{R}_+; L^\infty(\mathbb{R}_+^n)) \cap L^1(\mathbb{R}_+; \dot{W}^{2,\infty}(\mathbb{R}_+^n))
\]
and the corresponding estimate to (1.11) holds as
\[
\| \partial_t u \|_{L^1(\mathbb{R}^+; L^\infty(\mathbb{R}^n_+))} + \| \nabla^2 u \|_{L^1(\mathbb{R}^+; L^\infty(\mathbb{R}^n_+))} \\
\leq C \left( \| u_0 \|_{B^{0,1}_{\infty,1}(\mathbb{R}^n_+)} + \| f \|_{L^1(\mathbb{R}^+; B^{0,1}_{\infty,1}(\mathbb{R}^n_+))} + \| g \|_{F^{1,1}_{1,1}(\mathbb{R}^+; \mathcal{C}_v(\mathbb{R}^{n-1}))} \right) \\
+ \| g \|_{L^1(\mathbb{R}^+; \dot{B}^{2,1}_{\infty,1}(\mathbb{R}^{n-1}))}.
\]

The main difference between Theorem 1.2 and 1.3 is not only on the condition of the end-point exponents \( p = 1, \infty \) but also the difference of boundary regularity. The estimate in Theorem 1.3 only requires the Lebesgue and the inhomogeneous Besov regularity in the spatial direction and this is closer result to the known estimate in (1.3) due to Denk–Hieber–Prüss [15]. Note that the inhomogeneous Besov space \( B^{0,1}_{p,1}(\mathbb{R}^n_+) \) is slightly wider than the one for homogeneous space \( \dot{B}^{0,1}_{p,1}(\mathbb{R}^n_+) \). However, it does not stand for the strict sense of maximal regularity since the regularity for the initial and external force is more than the regularity for the solution itself by \( \dot{B}^{0,1}_{p,1}(\mathbb{R}^n_+) \subsetneq L^p(\mathbb{R}^n_+) \) for all \( 1 \leq p \leq \infty \).

Remarks. (i) In the second statement of Theorem 1.3, \( F^{1,1}_{1,1}(\mathbb{R}^+; \mathcal{C}_v(\mathbb{R}^{n-1})) \) is embedded into the continuous functions \( C_b([0, \infty); \mathcal{C}_v(\mathbb{R}^{n-1})) \) and the compatibility condition (1.9) is required.

(ii) Shibata–Shimizu [41,42] showed a boundary estimate by extending the boundary data \( g \) into \( \tilde{g} : \mathbb{R}^+ \times \mathbb{R}^n_+ \to \mathbb{R} \) and assume that the extended function satisfies
\[
\tilde{g} \in W^{1,\rho}(\mathbb{R}^+; L^p(\mathbb{R}^n_+)) \cap L^p(\mathbb{R}^+; W^{2,p}(\mathbb{R}^n_+))
\]
for \( 1 < \rho, p < \infty \). However, the estimate there does not include the endpoint exponent \( \rho = 1 \).

(iii) In the case \( 1 < \rho \leq \infty \), one can extend Proposition 1.1 into a time global version. Indeed, the case \( p = 1 \) for (1) in Theorem 1.3 is the corresponding estimate to Proposition 1.1.

1.2. The Neumann boundary condition

Similar to the initial-boundary value problem with the Dirichlet condition, we consider the initial-boundary value problem of the Neumann boundary condition. We assume Assumption 1 for the coefficients \( \{a_{ij}\}_{1 \leq i, j \leq n} \).

\[
\begin{aligned}
\partial_t u - \sum_{1 \leq i, j \leq n} a_{ij}(t, x) \partial_i \partial_j u &= f, \quad t \in I, \ x \in \mathbb{R}^n_+, \\
\partial_n u(t, x', x_n) \big|_{x_n=0} &= g(t, x'), \quad t \in I, \ x' \in \mathbb{R}^{n-1}, \\
u(t, x) \big|_{t=0} &= u_0(x), \quad x \in \mathbb{R}^n_+,
\end{aligned}
\]

(1.12)

where \( x = (x', x_n) \in \mathbb{R}^n_+ \) and \( \partial_n \) denotes the normal derivative \( \partial/\partial x_n \) at any boundary point of \( \mathbb{R}^n_+ \). Denk-Hieber-Prüss [15] showed the following:
Proposition 1.4 (The Neumann boundary condition \([15]\)). Let \(1 < \rho, p < \infty\) with \(1/2 - 1/(2p) \neq 1/\rho, T < \infty\) and set \(I = (0, T)\) for \(T < \infty\). The initial-boundary value problem \((1.12)\) has a unique solution \(u\) in \(W^{1,\rho}(I; L^p(\mathbb{R}^n_+)) \cap L^\rho(I; W^{2,p}(\mathbb{R}^n_+))\) if and only if
\[
\begin{align*}
    f & \in L^\rho(I; L^p(\mathbb{R}^n_+)), \quad u_0 \in \dot{B}^{2(1-1/\rho)}_{p,\rho}(\mathbb{R}^n_+), \\
    g & \in F_{1,p}(I; L^p(\mathbb{R}^{n-1}_+)) \cap L^\rho(I; \dot{B}^{1-1/p}_{p,p}(\mathbb{R}^{n-1})),
\end{align*}
\]
if \(1/2 - 1/(2p) > 1/\rho\), then assume further that the compatibility condition
\[
    (\partial_n u_0)(x', x_n)|_{x_n=0} = g(t, x')|_{t=0}.
\]

For the case of Neumann boundary problem \((1.12)\), we obtain end-point maximal \(L^1\)-maximal regularity as follows:

Theorem 1.5 (The Neumann boundary condition). Let \(1 < p < \infty, -1 + 1/p < s \leq 0\) and assume that the coefficients \(\{a_{ij}\}_{1 \leq i, j \leq n}\) satisfy Assumption 1.

(1) Suppose that \(b_{ij}(t, x) \equiv 0\) for all \(1 \leq i, j \leq n\). Then, the problem \((1.12)\) admits a unique solution
\[
    u \in \dot{W}^{1,1}(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \Delta u \in L^1(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)),
\]
if and only if the external, initial and boundary data in \((1.12)\) satisfy
\[
\begin{align*}
    f & \in L^1(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad u_0 \in \dot{B}^s_{p,1}(\mathbb{R}^n_+), \\
    g & \in F_{1,1}(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1}))) \cap L^1(\mathbb{R}^+_+; \dot{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1})))
\end{align*}
\]
respectively. Moreover end-point maximal \(L^1\)-regularity holds:
\[
\begin{align*}
    \|\partial_t u\|_{L^1(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|\nabla^2 u\|_{L^1(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} & \\
    \leq C(\|u_0\|_{\dot{B}^s_{p,1}(\mathbb{R}^n_+)} + \|f\|_{L^1(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \|g\|_{F_{1,1}(\mathbb{R}^+_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1})))} \\
    & \quad + \|g\|_{L^1(\mathbb{R}^+_+; \dot{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1})))},
\end{align*}
\]
where \(C\) is depending only on \(p, s\) and \(n\).

(2) Let \(1 \leq q < \infty\) and \(-n/q < s \leq 0\). For any \(T < \infty\), let \(I = (0, T)\). Then, there exists a unique solution \(u\) to \((1.12)\)
\[
    u \in \dot{W}^{1,1}(I; \dot{B}^s_{p,1}(\mathbb{R}^n_+)), \quad \Delta u \in L^1(I; \dot{B}^s_{p,1}(\mathbb{R}^n_+)),
\]
if and only if the external force, the initial data and the boundary data satisfy \((1.13)\) and \((1.14)\) with replacing \(\mathbb{R}^+_+\) into \(I\). Besides, the solution \(u\) satisfies the following estimate for some constant \(C_M = C_M(n, p, q, \{|\tilde{a}_{ij}\}|)>0\)
\[ \| \partial_t u \|_{L^1(I; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \| \nabla^2 u \|_{L^1(I; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} \leq C_M \int_0^T e^{\mu(T-s)} \| f(s) \|_{\dot{B}^s_{p,1}} \, ds \]
\[ + C_M \left\{ 1 + \max_{1 \leq i, j \leq n} \| b_{ij} \|_{L^\infty(I; \dot{B}^{s/n}_{q,1})(e^{\mu T} - 1)} \right\} \times (\| u_0 \|_{\dot{B}^s_{p,1}} + \| g \|_{\dot{F}^{1/2-1/2p}_{1,1}(I; \dot{B}^s_{p,1}(\mathbb{R}^n_+))} + \| g \|_{L^1(I; \dot{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}_+))}), \]  
(1.15)

where \( \mu = C_M^2 \log(1 + C_M) \).

For the case of the Neumann boundary condition, we have an analogous compensated end-point result as in Theorem 1.3 with \( L^p \) type boundary data.

**Theorem 1.6** (The Neumann boundary condition). Assume that the coefficients \( \{a_{ij}\}_{1 \leq i, j \leq n} \) satisfy Assumption 1 and be constants, i.e., \( b_{ij}(t, x) \equiv 0 \).

(1) Let \( 1 \leq p < \infty \). Assume that the external, initial and boundary data satisfy
\[ f \in L^1(\mathbb{R}^+_+; \dot{B}^0_{p,1}(\mathbb{R}^n_+)), \quad u_0 \in \dot{B}^0_{p,1}(\mathbb{R}^n_+), \]  
(1.16)
\[ g \in \dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}^+_+; L^p(\mathbb{R}^{n-1}_+)) \cap L^1(\mathbb{R}^+_+; B^{1-1/p}_{p,1}(\mathbb{R}^{n-1}_+)), \]  
(1.17)
then there exists a unique solution \( u \) to (1.12) in
\[ \dot{W}^{1,1}(\mathbb{R}^+_+; L^p(\mathbb{R}^n_+)) \cap L^1(\mathbb{R}^+_+; \dot{W}^{2,p}(\mathbb{R}^n_+)) \]
and which satisfies the following estimate:
\[ \| \partial_t u \|_{L^1(\mathbb{R}^+_+; L^p(\mathbb{R}^n_+))} + \| \nabla^2 u \|_{L^1(\mathbb{R}^+_+; L^p(\mathbb{R}^n_+))} \leq C (\| u_0 \|_{\dot{B}^0_{p,1}(\mathbb{R}^n_+)} + \| f \|_{L^1(\mathbb{R}^+_+; \dot{B}^0_{p,1}(\mathbb{R}^n_+))} + \| g \|_{\dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}^+_+; L^p(\mathbb{R}^{n-1}_+))}) \]
\[ + \| g \|_{L^1(\mathbb{R}^+_+; B^{1-1/p}_{p,1}(\mathbb{R}^{n-1}_+))}, \]
where \( C \) is depending only on \( p \) and \( n \).

(2) For \( p = \infty \), the corresponding result to (1) holds, i.e.,
\[ f \in L^1(\mathbb{R}^+_+; \dot{B}^0_{\infty,1}(\mathbb{R}^n_+)), \quad u_0 \in \dot{B}^0_{\infty,1}(\mathbb{R}^n_+), \]
\[ g \in \dot{F}^{1/2}_{1,1}(\mathbb{R}^+_+; C_v(\mathbb{R}^{n-1}_+)) \cap L^1(\mathbb{R}^+_+; \dot{B}^1_{\infty,1}(\mathbb{R}^{n-1}_+)), \]
then there exists a unique solution \( u \) to (1.12) in
\[ \dot{W}^{1,1}(\mathbb{R}^+_+; L^\infty(\mathbb{R}^n_+)) \cap L^1(\mathbb{R}^+_+; \dot{W}^{2,\infty}(\mathbb{R}^n_+)) \]
and which satisfies the following estimate:
\[ \| \partial_t u \|_{L^1(\mathbb{R}^+_+; L^\infty(\mathbb{R}^n_+))} + \| \nabla^2 u \|_{L^1(\mathbb{R}^+_+; L^\infty(\mathbb{R}^n_+))} \leq C (\| u_0 \|_{\dot{B}^0_{\infty,1}(\mathbb{R}^n_+)} + \| f \|_{L^1(\mathbb{R}^+_+; \dot{B}^0_{\infty,1}(\mathbb{R}^n_+))} + \| g \|_{\dot{F}^{1/2}_{1,1}(\mathbb{R}^+_+; C_v(\mathbb{R}^{n-1}_+))}) \]
\[ + \| g \|_{L^1(\mathbb{R}^+_+; \dot{B}^1_{\infty,1}(\mathbb{R}^{n-1}_+))}, \]
where \( C \) is depending only on \( n \).
Remark. In the case \( p = \infty \), the compatibility condition (1.9) appeared in Theorem 1.3 (2) is redundant in Theorem 1.6 (2) since the regularity for the boundary data (1.17) is weaker than the case in Theorem 1.3 and the boundary data is not the continuous function in \( t \)-variable.

The rest of this paper is organized as follows. We present the basic formulation for the proof in particular the reduction to the boundary value problems of the heat equations in the next section. We construct explicit solution formulas of the fundamental solutions in Sect. 3. Section 4 is devoted to prove for the Dirichlet condition case and Sect. 5 for the Neumann boundary condition case. In Sect. 6, we devote the proof of the key estimate almost orthogonality (2.12). Finally, we show the optimality of the main result by showing the sharp boundary trace estimate in Sect. 7.

Throughout this paper we use the following notations. Let \( \mathbb{R}_+ = (0, \infty) \) and \( \mathbb{R}^n_+ \) denote the \( n \)-dimensional Euclidean half-space; \( \{(x', x_n); x' \in \mathbb{R}^{n-1}, x_n \in \mathbb{R}_+\} \). For \( x \in \mathbb{R}^n \), \( \langle x \rangle \equiv (1 + |x|^2)^{1/2} \). The Fourier and the inverse Fourier transforms are defined for any rapidly decreasing function \( f \in \mathcal{S}(\mathbb{R}^n) \) with \( c_n = (2\pi)^{-n/2} \) by

\[
\hat{f}(\xi) = \mathcal{F}[f](\xi) \equiv c_n \int_{\mathbb{R}^n} e^{-ix\cdot\xi} f(x) dx, \quad \mathcal{F}^{-1}[f](x) \equiv c_n \int_{\mathbb{R}^n} e^{ix\cdot\xi} f(\xi) d\xi.
\]

For \( f \in \mathcal{S}'(\mathbb{R}^n) \), we also denote \( \hat{f}(\xi') = \mathcal{F}[f](\xi') \). For any functions \( f = f(t, x', x_n) \) and \( g = g(t, x', x_n) \), \( f \ast g \), \( f \ast g \) and \( f \ast g \) stand for the convolution between \( f \) and \( g \) with respect to the variable indicated under \( \ast \), respectively. In the summation \( \sum_{k \in \mathbb{Z}} \), the parameter \( k \) runs for all integers \( k \in \mathbb{Z} \) and for \( \sum_{k \leq j} \), \( k \) runs for all integers less than or equal to \( j \in \mathbb{Z} \). In the norm for the Bochner spaces on \( \mathcal{F}_{p, \rho}(I; X(\mathbb{R}^{n-1})) \), we use

\[
\|f\|_{\mathcal{F}_{p, \rho}(I; X)} = \|\hat{f}\|_{\mathcal{F}_{p, \rho}(I; X(\mathbb{R}^{n-1}))}
\]

unless it may cause any confusion. For \( a \in \mathbb{R}^n \), we denote \( B_R(a) \) as the open ball centered at \( a \) with its radius \( R > 0 \). We also denote the compliment of \( B_R(0) \) by \( B_R^c \). Various constants are simply denoted by \( C \) unless otherwise stated.

2. Reduction to the heat equation and outline of the proofs

The outline of the proof of Theorems 1.2 and 1.3 is summarized as follows: We decompose the initial-boundary value problem (1.2) into the following three problems and reduce the problem into the inhomogeneous boundary value problem with 0 initial and external force:

\[
\begin{aligned}
\partial_t u_1 - \Delta u_1 &= 0, \\
u_1(t, x) \big|_{t=0} &= \begin{cases} u_0(x', x_n), & x_n > 0, \\
-u_0(\xi'-x_n), & x_n \leq 0, \end{cases} & t \in I, & x \in \mathbb{R}^n, \\
\partial_t u_2 - \Delta u_2 &= 0, \\
u_2(t, x') \big|_{x_n=0} &= g(t, x') - u_1(t, x', x_n) \big|_{x_n=0}, & t \in I, & x' \in \mathbb{R}^{n-1}, \\
u_2(t, x) \big|_{t=0} &= 0, & x \in \mathbb{R}^n_+.
\end{aligned}
\]

(2.1)
\[
\begin{aligned}
\partial_t u_3 - \sum_{i,j} a_{ij}(t, x) \partial_i \partial_j u_3 &= f + \sum_{i,j} (\tilde{a}_{ij} + b_{ij}(t, x)) \partial_i \partial_j (u_1 + u_2) \equiv F, \quad t \in I, \ x \in \mathbb{R}_+^n,
\left. u_3(t, x) \right|_{x=0} = 0,
u_3(t, x)|_{t=0} = 0,
\end{aligned}
\]

where the coefficients \( \tilde{a}_{ij} \) and \( b_{ij}(t, x) \) are defined in Assumption 1 (1) and \( f = f(t, x) \), \( g = g(t, x') \), \( u_0(x) \) are given external force, the Dirichlet boundary data, and initial data, respectively.

Then
\[
u(t, x', x_n) = u_1(t, x', x_n)|_{x_n>0} + u_2(t, x', x_n) + u_3(t, x', x_n)
\]
is the solution of (1.2).

The external force \( F \) in (2.3) contains not only the given data \( f \) but the solutions \( u_1 \) and \( u_2 \) to (2.1) and (2.2) and we need to verify the regularity of \( u_1 \) and \( u_2 \) in order to solve (2.3). Indeed, if \( u_1 \) and \( u_2 \) have maximal \( L^1 \)-regularity: For \( 1 < p < \infty \) and 

\[
u_1, u_2 \in W^{1,1}(\mathbb{R}_+^n; \dot{B}^{s}_{p,1}(\mathbb{R}_+^n)), \quad \Delta u_1, \Delta u_2 \in L^1(\mathbb{R}_+^n; \dot{B}^{s}_{p,1}(\mathbb{R}_+^n)),
\]

then under the assumption \( b_{ij} \in BUC(\mathbb{R}_+^n; \dot{B}^{s}_{q,1}(\mathbb{R}_+^n)) \) (\( 1 \leq q < -\frac{n}{s} \)) it holds that

\[
\left\| \sum_{i,j} \left( \tilde{a}_{ij} + b_{ij}(t, x) \right) \partial_i \partial_j (u_1 + u_2) \right\|_{L^1(\mathbb{R}_+^n; \dot{B}^{s}_{p,1}(\mathbb{R}_+^n))} \leq C(1 + \sup \|b_{ij}(t, \cdot)\|_{\dot{B}^{s}_{q,1}(\mathbb{R}_+^n)}) \left( \|\Delta u_1\|_{L^1(\mathbb{R}_+^n; \dot{B}^{s}_{p,1})} + \|\Delta u_2\|_{L^1(\mathbb{R}_+^n; \dot{B}^{s}_{p,1})} \right).
\]

We set \( g - u_1 \) as \( h \) and assume that it is given. When \( (s, p) = (0, \infty) \), the compatibility condition \( h(t, x') = g(t, x') - u_1(t, x', 0) \) on \( t = 0 \) is required and it coincides with \( u_2(t, x)|_{t=0} = 0 \), namely

\[
g(t, x')|_{t=0} = u_0(x', x_n)|_{x_n=0}.
\]

The requirement for the compatibility condition is natural for maximal \( L^p \)-regularity in the cases \( 1 < p < \infty \). However, such a restriction is not adopted for maximal \( L^1 \)-regularity except the endpoint case \( (s, p) = (0, \infty) \).

For the problem (2.3), we first notice that assuming the regularity (2.4), the external force \( F \) is in \( L^1(I; \dot{B}^{s}_{p,1}) \) under the regularity assumption on \( b_{ij}(t, x) \) and \( \tilde{a}_{ij} \) being constant coefficients. Then, we extend the problem (2.3) into the whole space by an appropriate extension of data and coefficients and maximal regularity (1.8) follows from the estimate for the Cauchy problem in \( \mathbb{R}_+^n \) shown in [34]. Note that the result in [34] treats only the case \( \tilde{a}_{ij} = 0 \); however, the analogous result follows for the constant positive coefficient case.

Therefore, our main issue is to consider the initial boundary value problem (2.2).

\[
\begin{aligned}
\partial_t u - \Delta u &= 0, \quad t \in I, \ x \in \mathbb{R}_+^n,
u(t, x', x_n)|_{x_n=0} = h(t, x'), \quad t \in I, \ x' \in \mathbb{R}_+^{n-1},\quad u(t, x)|_{t=0} = 0,
\end{aligned}
\]
where the boundary function \( h(t, x') \) is given by the function after a proper linear transformed function of \( g(t, x') - u_1(t, x', x_n)|_{x_n = 0} \) in (2.2). Once we obtain maximal \( L^1 \)-regularity to (2.5) with the boundary trace, then the original problem can be reduced into the initial value problem, and it can be reduced into the Cauchy problem in the whole Euclidian space \( \mathbb{R}^n \). Note that the solution \( u_1(t, x) \) that has regularity (2.4) has the boundary trace estimate (see Theorem 7.1 in Section 7) and the condition on the boundary data \( h \) is the same as the condition on the original data \( g \). In what follows, we rewrite \( u_1 \) into \( u \) and consider the initial-boundary value problem (2.5).

If we obtain the following theorem, then the main result Theorem 1.2 also follows:

**Theorem 2.1** (Maximal \( L^1 \)-regularity by the Dirichlet boundary data). Let \( 1 < p < \infty \) and \( -1 + 1/p < s \leq 0 \). there exists a unique solution

\[
 u \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n)), \quad \Delta u \in L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n)),
\]

to (2.5) if and only if \( h \) satisfies

\[
 h \in \dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1})). \tag{2.6}
\]

Besides the solution \( u \) is subject to the estimate:

\[
 \| \partial_t u \|_{L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n))} + \| \nabla^2 u \|_{L^1(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^n))} \\
 \leq C \left( \| h \|_{\dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+; \dot{B}^s_{p,1}(\mathbb{R}^{n-1}))} + \| h \|_{L^1(\mathbb{R}_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^{n-1}))} \right),
\]

where \( C \) is depending only on \( p, s \) and \( n \).

When \( p = \infty \), the analogous result holds under arranging the function classes as in Theorem 1.3 with the compatibility condition

\[
 h(t, x')|_{t=0} = 0. \tag{2.7}
\]

Similarly, Theorem 1.3 can be reduced into the following:

**Theorem 2.2** (\( L^p \)-estimate by the Dirichlet boundary data). (1) Let \( 1 \leq p < \infty \). If \( h \) satisfies

\[
 h \in \dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; B^{2-1/p}_{p,1}(\mathbb{R}^{n-1})),
\]

then there exists a unique solution \( u \) to (2.5) which satisfies the following estimate:

\[
 \| \partial_t u \|_{L^1(\mathbb{R}_+; L^p(\mathbb{R}^n))} + \| \nabla^2 u \|_{L^1(\mathbb{R}_+; L^p(\mathbb{R}^n))} \\
 \leq C \left( \| h \|_{\dot{F}^{1-1/2p}_{1,1}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1}))} + \| h \|_{L^1(\mathbb{R}_+; B^{2-1/p}_{p,1}(\mathbb{R}^{n-1}))} \right),
\]

where \( C \) is depending only on \( p \) and \( n \).

(2) For \( p = \infty \), the corresponding result to (1) holds under the analogous arrangement for the function classes as in Theorem 1.3 with imposing the compatibility condition (2.7).
For the case of the Neumann boundary condition, we decompose the problem (1.12) into the following problems and reduce the boundary condition into the case of heat equation.

\[
\begin{aligned}
\partial_t u_1 - \Delta u_1 &= 0, \quad t \in I, \ x \in \mathbb{R}^n,
\end{aligned}
\]
\[
\begin{aligned}
0 &= \begin{cases} u_0(x', x_n), & x_n > 0, \ x \in \mathbb{R}^n, \\
u_0(x', -x_n), & x_n \leq 0, \ x \in \mathbb{R}^n, 
\end{cases}
\end{aligned}
\]
\[
\begin{aligned}
\partial_t u_2 - \Delta u_2 &= 0, \quad t \in I, \ x \in \mathbb{R}^n,
\end{aligned}
\]
\[
\begin{aligned}
\partial_n u_2(t, x') &\big|_{x_n=0} = g(t, x') - \partial_n u_1(t, x', x_n) \big|_{x_n=0}, \ t \in I, \ x' \in \mathbb{R}^{n-1}, \ x \in \mathbb{R}^n,
\end{aligned}
\]
\[
\begin{aligned}
u_2(t, x) \big|_{t=0} = 0,
\end{aligned}
\]
\[
\begin{aligned}
\partial_t u_3 - \sum_{1 \leq i, j \leq n} a_{ij} \partial_i \partial_j u_3 &\equiv F, \ t \in I, \ x \in \mathbb{R}^n,
\end{aligned}
\]
\[
\begin{aligned}
\partial_n u_3(t, x', x_n) \big|_{x_n=0} = 0, \ t \in I, \ x' \in \mathbb{R}^{n-1}, \ x \in \mathbb{R}^n,
\end{aligned}
\]
\[
\begin{aligned}
u_3(t, x) \big|_{t=0} = 0,
\end{aligned}
\]
where \( F \) is similarly defined as in (2.3) and \( f = f(t, x), \ g = g(t, x') \) are given external data and the Neumann boundary condition, respectively, and \( u_0(x) \) is the initial data. Therefore, the problem can be reduced by setting \( h(t, x') = g(t, x') - \partial_n u_1(t, x', x_n) \big|_{x_n=0} \) into the following problem:

\[
\begin{aligned}
\partial_t u - \Delta u &= 0, \quad t \in I, \ x \in \mathbb{R}^n,
\end{aligned}
\]
\[
\begin{aligned}
\partial_n u(t, x', x_n) \big|_{x_n=0} &= h(t, x'), \ t \in I, \ x' \in \mathbb{R}^{n-1},
\end{aligned}
\]
\[
\begin{aligned}
u(t, x) \big|_{t=0} = 0,
\end{aligned}
\]

Then, the following result yields our main result for the Neumann problem Theorem 1.5.

**Theorem 2.3** (Maximal \( L^1 \)-regularity by the Neumann boundary data). Let \( 1 < p < \infty \) and \( -1 + 1/p < s \leq 0 \). There exists a unique solution

\[
u \in \dot{W}^{1, 1}(\mathbb{R}_+; \dot{B}^s_{p, 1}(\mathbb{R}^n_+)), \quad \Delta u \in L^1(\mathbb{R}_+; \dot{B}^s_{p, 1}(\mathbb{R}^n_+)),
\]

to (2.8) if and only if

\[
h \in F^{1/2 - 1/p, 1/2 - 1/p}_1(\mathbb{R}_+; \dot{B}^s_{p, 1}(\mathbb{R}^n_+)) \cap L^1(\mathbb{R}_+; \dot{B}^{s+1 - 1/p}_{p, 1}(\mathbb{R}^n_+)).
\]

Besides, it holds the estimate:

\[
\begin{aligned}
\|\partial_t u\|_{L^1(\mathbb{R}_+; \dot{B}^s_{p, 1}(\mathbb{R}^n_+))} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+; \dot{B}^s_{p, 1}(\mathbb{R}^n_+))} \\
\leq C(\|h\|_{F^{1/2 - 1/p, 1/2 - 1/p}_1(\mathbb{R}_+; \dot{B}^s_{p, 1}(\mathbb{R}^n_+))} + \|h\|_{L^1(\mathbb{R}_+; \dot{B}^{s+1 - 1/p}_{p, 1}(\mathbb{R}^n_+))}),
\end{aligned}
\]

where \( C \) is depending only on \( p, s \) and \( n \).

When \( p = \infty \), the analogous result holds under arranging the function classes as in Theorem 1.6.
For the $L^p$-estimate in Theorem 1.6 is reduced into the following:

**Theorem 2.4** \((L^p\text{-estimate by the Neumann boundary data).} \) (1) Let \(1 \leq p < \infty\). If the boundary data \(h\) satisfies
\[
h \in \dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}_+; L^p(\mathbb{R}^{n-1})) \cap L^1(\mathbb{R}_+; B^1_{p,1}(\mathbb{R}^{n-1})),
\]
then there exists a unique solution \(u\) to (2.8) which fulfills the following estimate:
\[
\|\partial_t u\|_{L^1(\mathbb{R}_+; L^p(\mathbb{R}^n))} + \|\nabla^2 u\|_{L^1(\mathbb{R}_+; L^p(\mathbb{R}^n))} \\
\leq C\left(\|h\|_{\dot{F}^{1/2-1/2p}_{1,1}(\mathbb{R}_+; B^1_{p,1}(\mathbb{R}^{n-1}))} + \|h\|_{L^1(\mathbb{R}_+; B^1_{p,1}(\mathbb{R}^{n-1}))}\right),
\]
where \(C\) is depending only on \(p\) and \(n\).

(2) For \(p = \infty\) the corresponding result to (1) holds under the analogous arrangement for the function classes as in Theorem 1.6.

Therefore, in order to obtain main results Theorems 1.2–1.3 and Theorems 1.5–1.6, it is enough to show Theorems 2.1–2.2 and Theorems 2.3–2.4, respectively.

To show Theorem 2.1, we first apply the Laplace transform with respect to \(t\), the partial Fourier transform with respect to \(x'\) and we obtain the solution formula of (2.5) as
\[
\Delta u(t, x', x_n) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^n} \Psi_D(t - s, x' - y', \eta) h(s, y')dy'ds \bigg|_{\eta=x_n}
\]
by using the boundary potential term:
\[
\Psi_D(t, x', \eta) = \frac{c_{n-1}}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}^{n-1}} e^{\lambda t + ix' \cdot \xi'} \lambda e^{-\sqrt{\lambda + |\xi'|^2} \eta} d\xi' d\lambda, \tag{2.10}
\]
where \(c_{n-1} = (2\pi)^{-(n-1)/2}\) and \(\Gamma\) is a pass parallel to the imaginary axis. We then extend the boundary data \(h(t, x')\) into \(t < 0\) by the zero extension and it enable us to formulate the above formula by the Fourier transform. The key idea to derive the boundary estimate is applying an almost orthogonal estimate between the boundary potential \(\psi_D\) in (2.10) and the Littlewood-Paley dyadic decompositions of unity in both space \(x'\) and time \(t\) variables; \(\{\phi_j(x')\}_{j \in \mathbb{Z}}, \{\psi_k(t)\}_{k \in \mathbb{Z}}\). From (2.10), let \(\eta > 0\) as a spectral parameter (such as \(\eta \simeq 2^{-\ell}\) with \(\ell \in \mathbb{Z}\)) and consider
\[
\Psi_D(t, x', \eta)^2 = \frac{c_{n-1}}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}^{n-1}} e^{\lambda t + ix' \cdot \xi'} \eta^2 e^{-\sqrt{\lambda + |\xi'|^2} \eta} d\xi' d\lambda
\]
and the almost orthogonality between the boundary potential \(\Psi_D(t, x', \eta)^2\) and the Littlewood-Paley dyadic decompositions can be shown. Indeed, by setting
\[
\Psi_{D,k,j}(t, x', \eta) = \int_{\mathbb{R}_+} \int_{\mathbb{R}^{n-1}} \Psi_D(t - s, x' - y', \eta) \psi_k(s) \phi_j(y')dy'ds \tag{2.11}
\]
for \( \eta > 0 \), the almost orthogonal property is presented in two-way estimates separated by the time-like estimate and space-like regions. If \( \eta \in [2^{-\ell}, 2^{-\ell+1}) \) for \( \ell \in \mathbb{Z} \), then

\[
\| \Psi_{D, k, j}(t, \cdot, \eta) \|^2_{L^1(\mathbb{R}^{n-1})} \leq \begin{cases} 
C_n 2^{k-2\ell} (1 + 2^{(n+2)(k-2\ell)}) e^{-2^{1/2(k-2\ell)} 2^k (2^j t)^{-2}}, & k \geq 2j, \\
C_n 2^{j-2\ell} (1 + 2^{(n+2)(j-2\ell)}) e^{-2^{j-\ell} 2^{k} (2^j t)^{-2}}, & k < 2j,
\end{cases}
\tag{2.12}
\]

where \( \langle t \rangle = \left(1 + |t|^2\right)^{1/2} \) which are essentially shown in Lemma 6.1 in Section 6. Then, our main strategy to show maximal \( L^1 \)-regularity for the boundary term (2.5) relies on the above estimate (2.12) and the proof can be complete after exchanging the solution potential \( \Psi_1 \) into the Littlewood-Paley dyadic decompositions \( \phi_j \) and \( \psi_k \).

In order to complete such procedure, the above type estimate (2.12) plays a key role. The only difference between the case of Dirichlet boundary condition and the Neumann boundary condition is to the regularity of the boundary data. This is because the solution can be realized by the potential term

\[
\Psi_N(t, x', \eta) = -\frac{c_{n-1}}{2\pi i} \int \int \int e^{\lambda t + ix' \cdot \xi'} \frac{\lambda}{\sqrt{\lambda + |\xi'|^2}} e^{-\sqrt{\lambda + |\xi'|^2} \eta} d\xi' d\lambda,
\]

instead of (2.10).

3. Boundary potentials

3.1. The Dirichlet boundary potential and the compatibility condition

In this subsection, we derive the exact solution formula of (2.5). Let \( h = h(t, x') \) be the boundary data extended into \( t < 0 \) by the zero extension. We apply the Laplace transform \( \mathcal{L} \) in time and the Fourier transform in \( \mathbb{R}^{n-1} \)-dimensional spatial variables to the equation (2.5). Noting \( \hat{u}(0, \xi', x_n) = 0 \), we obtain that

\[
\begin{cases}
(\lambda + |\xi'|^2 - \partial^2_n) \widehat{u}(\lambda, \xi', x_n) = 0, \\
\widehat{u}(\lambda, \xi', 0) = \widehat{h}(\lambda, \xi').
\end{cases}
\]

Then, it follows that

\[
\widehat{u}(\lambda, \xi', x_n) = \widehat{h}(\lambda, \xi') e^{-\sqrt{\lambda + |\xi'|^2} x_n}.
\]

Hence, the solution is expressed by

\[
u(t, x) = \frac{c_{n-1}}{2\pi i} \int \int e^{\lambda t} \int \int e^{ix' \cdot \xi'} \widehat{h}(\lambda, \xi') e^{-\sqrt{\lambda + |\xi'|^2} x_n} d\xi' d\lambda,
\tag{3.1}
\]

where \( c_{n-1} = \frac{2\pi}{(2\pi)^{n-1}} \) and \( \Gamma \) denotes an integral path in holomorphic domain parallel to the imaginary axis \( \text{Re} \lambda > 0 \). The solution (3.1) satisfies the heat equation
and the boundary condition \( u(t, x', 0) = h(t, x') \). From (3.1), the Green function \( G_D(t, x) \) of the initial-boundary value problem (2.5) is identified as

\[
G_D(t, x', x_n) = \frac{c_{n-1}}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}^{n-1}} e^{\lambda t} e^{ix' \cdot \xi'} e^{-\sqrt{\lambda + |\xi'|^2} x_n} d\xi' d\lambda. \tag{3.2}
\]

Introducing the Dirichlet boundary potential by

\[
\Psi_D(t, x', \eta) = \frac{c_{n-1}}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}^{n-1}} e^{\lambda t} e^{i x' \cdot \xi'} \lambda e^{-\sqrt{\lambda + |\xi'|^2} \eta} d\xi' d\lambda, \tag{3.3}
\]

where we set \( \eta = x_n \), we decompose this boundary potential (3.3) by a combination of two families of the Littlewood-Paley dyadic decomposition of unity. If we put an extra-parameter \( \eta \simeq 2^{-\ell} \) to \( \Psi_D \) with \( \ell \in \mathbb{Z} \) which is a substitution of the boundary parameter \( x_n \), then we find that \( \Psi_D \eta^2 \) can be expressed by the time Littlewood-Paley decomposition \( \{\psi_k(t)\}_k \) and the space Littlewood-Paley decomposition \( \{\phi_j(x')\}_j \) and the relations between the parameters \( \ell \) and \( k, j \) are explicitly estimated. Such estimates stand for the almost orthogonality between \( \{\Psi_D(t, x', \eta)\eta^2|_{\eta = 2^{-\ell}}\}_{\ell \in \mathbb{Z}} \) and \( \{\psi_k(t), \phi_j(x')\}_{k, j \in \mathbb{Z}} \). Here, we notice that from (3.1)-(3.3), the potential \( \Psi_D \) represents the solution operated by the Laplace operator, namely

\[
\Psi_D(t, x', \eta) \equiv \Delta G_D(t, x', \eta). \tag{3.4}
\]

Then, it follows that the potential \( \Psi_D \) represents

\[
\Delta u(t, x', \eta) = \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} \Psi_D(t - s, x' - y', \eta) h(s, y') ds dy'.
\]

If we consider the case when \( 1 - \frac{1}{2p} < \frac{1}{\rho} \), then the class \( F^{1-\frac{1}{2p}}_{\rho, p} (\mathbb{R}_+; L^q (\mathbb{R}^{n-1})) \) of the boundary data with \( p > 1 \) is embedded into the class \( C_b(I; L^q (\mathbb{R}^{n-1})) \) and the data have to satisfy continuity at \( t = 0 \) as in Proposition 1.1. On the other hand, if we consider maximal \( L^1 \) regularity, the class \( F^{1-\frac{1}{2p}}_{1, 1} (\mathbb{R}_+; \check{B}^{s}_{p, 1}(\mathbb{R}^{n-1})) \) is not embedded into \( C_b(\mathbb{R}_+; \check{B}^{s}_{p, 1}(\mathbb{R}^{n-1})) \) because \( 1 - \frac{1}{2p} < 1 \) under \( p < \infty \), it does not necessarily require continuity at \( t = 0 \). This shows that it is not necessary to require the compatibility condition \( g(t, x')|_{t=0} = u_0(x', x_n)|_{x_n=0} \) point-wisely when \( p < \infty \). In order to obtain maximal \( L^p \) regularity for \( 1 < p < \infty \), the boundary data is extended to the zero extension for \( t < 0 \).

From (3.3), we change the integral path into \( \Gamma = \gamma + \Gamma_\varepsilon \rightarrow \Gamma_\varepsilon \) with

\[
\Gamma_\varepsilon = L_\varepsilon \cup C_\varepsilon,
\]

where

\[
L_\varepsilon = \{ \lambda = i \tau; \tau \in (-\infty, \varepsilon) \cup (\varepsilon, \infty) \}, \quad C_\varepsilon = \{ \lambda = \varepsilon e^{i \theta}; \varepsilon > 0, \theta : -\frac{\pi}{2} \rightarrow \frac{\pi}{2} \}. \tag{3.5}
\]
Then by setting $\lambda \in \gamma + \Gamma_\varepsilon$,

$$
\Psi_D(t, x', \eta) = \lim_{\gamma \to 0} c_n + i^{-1} \int_{\gamma + L_\varepsilon} e^{\lambda t + i x' \cdot \xi'} \lambda e^{-\sqrt{\lambda^2 + |\xi'|^2} \eta} d\xi' d\lambda
$$

Then by setting $\lambda \in \gamma + \Gamma_\varepsilon$, $\eta$.

We emphasize that for $-\frac{\pi}{4} < \frac{1}{2} \tan^{-1} \frac{\tau}{|\xi'|^2} < \frac{\pi}{4}$ the exponent appeared in (3.7) is negative definite unless $(\tau, \xi') = (0, 0)$ and hence $\lim_{\varepsilon \to 0} I_{L_\varepsilon}$ converges.
\[
\lim_{\varepsilon \to 0} I_{L_{\varepsilon}} = \lim_{\varepsilon \to 0} \lim_{\gamma \to 0} c_{n+1} \frac{1}{i} e^{\gamma t} \int_{L_{\varepsilon}} \int_{\mathbb{R}^{n-1}} e^{i\tau + i\xi \cdot \xi'} (\gamma + i\tau) e^{-\sqrt{\gamma^2 + |\xi'|^2} \eta} d\xi' i d\tau \\
= \lim_{\varepsilon \to 0} c_{n+1} \int_{L_{\varepsilon}} \int_{\mathbb{R}^{n-1}} e^{i\tau + i\xi \cdot \xi'} i \tau e^{-\sqrt{i\tau + |\xi'|^2} \eta} d\xi' d\tau \\
= c_{n+1} \int_{\mathbb{R} \setminus \{0\}} \int_{\mathbb{R}^{n-1}} e^{i\tau + i\xi \cdot \xi'} i \tau e^{-\sqrt{i\tau + |\xi'|^2} \eta} d\xi' d\tau. \tag{3.8}
\]

For the convergence of the second term \(I_{C_{\varepsilon}}\) of the right hand side of (3.6), we see by noting
\[
\left| \tan^{-1} \left( \frac{\varepsilon \sin \theta}{\varepsilon \cos \theta + |\xi'|^2} \right) \right| \leq \left| \tan^{-1} \left( \frac{\sin \theta}{\cos \theta} \right) \right| = |\theta|, \quad -\frac{\pi}{2} \leq \theta \leq \frac{\pi}{2}
\]
that for \(\eta > 0\)
\[
\lim_{\varepsilon \to 0} |I_{C_{\varepsilon}}| \leq \lim_{\varepsilon \to 0} c_{n+1} \frac{1}{i} \int_{C_{\varepsilon}} \int_{\mathbb{R}^{n-1}} e^{\lambda t + i\xi \cdot \xi'} \lambda e^{-\sqrt{\lambda^2 + |\xi'|^2} \eta} d\xi' d\lambda \\
\leq \lim_{\varepsilon \to 0} c_{n+1} \int_{\mathbb{R} \setminus \{0\}} \int_{\mathbb{R}^{n-1}} \frac{e^{\varepsilon t \cos \theta \xi' \cdot \xi'}}{\varepsilon^2 \eta^2} e^{-\sqrt{\varepsilon^2 \eta^2} \xi' \cdot \xi'} d\xi' \cdot \exp \left( -\eta \left( (\varepsilon \cos \theta + |\xi'|^2\right)^2 + \varepsilon^2 \sin^2 \theta \right)^{1/4} \\
\quad \times \cos \left( \frac{1}{2} \tan^{-1} \left( \frac{\varepsilon \sin \theta}{\varepsilon \cos \theta + |\xi'|^2} \right) \right) d\theta d\xi' \\
\leq \lim_{\varepsilon \to 0} c_{n+1} \int_{\mathbb{R} \setminus \{0\}} \int_{\mathbb{R}^{n-1}} \frac{\varepsilon^2 t \cos \theta \xi' \cdot \xi'}{\varepsilon^2 \eta^2} \exp \left( -\frac{\sqrt{\varepsilon^2 \eta \xi' \cdot \xi'}}{2} \right) d\xi' \int_{-\pi}^{\pi} e^{\varepsilon t \cos \theta} d\theta \\
\leq \lim_{\varepsilon \to 0} c_{n+1} \varepsilon^2 \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{\sqrt{\varepsilon^2 \eta \xi' \cdot \xi'}}{2} \right) d\xi' \int_{-\pi}^{\pi} e^{\varepsilon t \cos \theta} d\theta \\
\leq c_{n+1} \lim_{\varepsilon \to 0} \varepsilon^2 \left( \int_{\mathbb{R}^{n-1}} \exp \left( -\frac{\sqrt{\varepsilon^2 \eta \xi' \cdot \xi'}}{2} \right) d\xi' \right) \left( \int_{-1}^{1} \frac{e^{\varepsilon t |\xi'| \cos \theta}}{\sqrt{1 - \xi'^2 \cos^2 \theta}} d\xi' \right) = 0. \tag{3.10}
\]
Therefore, by passing \(\varepsilon \to 0\) in (3.6), we obtain the following formula from (3.8) and (3.10):
\[
\Psi_D(t, x', \eta) = c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i\tau + i\xi \cdot \xi'} i \tau e^{-\sqrt{i\tau + |\xi'|^2} \eta} d\xi' d\tau. \tag{3.11}
\]

**Remark.** We note that from (3.1) and similar way in (3.3)-(3.10), the Green’s function (3.2) is expressed by the Fourier inverse transform as
\[
G_D(t, x', \eta) = c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i\tau + i\xi \cdot \xi'} e^{-\sqrt{i\tau + |\xi'|^2} \eta} d\xi' d\tau.
\]
Therefore, the potential function \(\Psi_D\) given by (3.11) is understood as the Green’s function operated by the Laplacian;
\[
\Psi_D(t, x', \eta) = \Delta(x', \eta) G_D(t, x', \eta).
\]
Note that the 0-initial condition for $G_D(t, x', \eta)\big|_{t=0} = 0$ is fulfilled by the Cauchy integral theorem on the same complex path (3.5) avoiding the branch cut at the negative real-line and passing the limit $\gamma \to 0$ and $\epsilon \to 0$ along the analogous estimates (3.8) and (3.10).

3.2. The Neumann boundary potential and the compatibility condition

Following the method of the case of the Dirichlet boundary condition, we consider the initial-boundary value problem (2.8). Then, we deduce the boundary potential (the Green’s function) that yields the solution to (2.8). We apply the Laplace transform in time and the Fourier transform in $\mathbb{R}^{n-1}$-dimensional spatial variables and noting

$$\hat{\mathcal{L}}u(0, \xi', x_n) = 0$$

have

$$\left\{ \begin{array}{l}
(\lambda + |\xi'|^2 - \partial_n^2)\hat{\mathcal{L}}u = 0,
\partial_n\hat{\mathcal{L}}u(\lambda, \xi', 0) = \hat{\mathcal{L}}h(\lambda, \xi').
\end{array} \right.$$ (3.12)

Then, we write it explicitly,

$$u(t, x) = -\frac{c_{n-1}}{2\pi i} \int_{\Gamma} e^{\lambda t} \int_{\mathbb{R}^{n-1}} e^{ix' \cdot \xi'} \hat{\mathcal{L}}h(\lambda, \xi') \frac{1}{\sqrt{\lambda + |\xi'|^2}} e^{-\sqrt{\lambda + |\xi'|^2}x_n d\xi' d\lambda}, \quad (3.13)$$

where $c_{n-1} = (2\pi)^{-\frac{n-1}{2}}$ and $\Gamma$ is a proper path on the analytic region. From (3.12), the Green’s function $G_N(t, x)$ of the initial-boundary value problem (2.8) is given by

$$G_N(t, x', x_n) = -\frac{c_{n-1}}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}^{n-1}} e^{\lambda t} e^{ix' \cdot \xi'} \frac{1}{\sqrt{\lambda + |\xi'|^2}} e^{-\sqrt{\lambda + |\xi'|^2}x_n d\xi' d\lambda}. \quad (3.13)$$

One can choose as the parallel line to the imaginary axis in $\text{Re}\lambda > 0$. Let

$$\Psi_N(t, x', \eta) = -\frac{c_{n-1}}{2\pi i} \int_{\Gamma} \int_{\mathbb{R}^{n-1}} e^{\lambda t} e^{ix' \cdot \xi'} \frac{\lambda}{\sqrt{\lambda + |\xi'|^2}} e^{-\sqrt{\lambda + |\xi'|^2}x_n d\xi' d\lambda}, \quad (3.14)$$

where $\Gamma$ is a proper integral path basically parallel to the imaginary axis. For the case of maximal $L^1$-regularity, the boundary regularity is $F_{1, p}^{1- \frac{1}{p}} (I; L^p(\mathbb{R}^{n-1}))$ and the continuity in time direction does not hold; hence, the compatibility condition is redundant. Along the Dirichlet boundary case before, the boundary condition can be prolonged to $t < 0$ by zero extension and hence the solution is understood by zero extension. From (3.14), we pass the integral path (3.5) into $\Gamma = \gamma + \Gamma_\epsilon \to \Gamma_\epsilon$ with

$$\Gamma_\epsilon = L_\epsilon \cup C_\epsilon$$

by $\gamma \to 0$. Then,
\[ \Psi_N(t, x', \eta) = \lim_{\gamma \to 0} c_{n+1} i \int_{\gamma + L_x} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{\lambda}{\sqrt{\lambda + |\xi'|^2}} e^{-\sqrt{\lambda + |\xi'|^2} \eta} d\xi' d\lambda \]

\[ + \lim_{\gamma \to 0} c_{n+1} i \int_{\gamma + C_x} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{\lambda}{\sqrt{\lambda + |\xi'|^2}} e^{-\sqrt{\lambda + |\xi'|^2} \eta} d\xi' d\lambda \]

\[ = II_{L_x} + II_{C_x}. \quad (3.15) \]

The first term of the right hand side of (3.15) is converging even \( \gamma \to 0 \) if we observe the real part of the exponential integrant is for the case \( \gamma = 0 \) that

\[ \frac{i \tau}{\sqrt{i \tau + |\xi'|^2}} = \frac{i \tau e^{\frac{i}{2} \tan^{-1} \frac{\tau}{|\xi'|^2}}}{\left( \tau^2 + |\xi'|^4 \right)^{1/4}}, \quad (3.16) \]

and (3.3). Then, we obtain that

\[ \sqrt{i \tau + |\xi'|^2} e^{-\sqrt{i \tau + |\xi'|^2} \eta} = \frac{\tau}{(\tau^2 + |\xi'|^4)^{1/4}} \exp \left( - (\tau^2 + |\xi'|^4)^{1/2} \eta \cos \left( \frac{1}{2} \tan^{-1} \frac{\tau}{|\xi'|^2} \right) \right). \quad (3.17) \]

Under the condition on the argument \(-\frac{\pi}{4} < \frac{1}{2} \tan^{-1} \frac{\tau}{|\xi'|^2} < \frac{\pi}{4}\) the real part of the exponential function in (3.16) is integrable unless \((\tau, \xi) = (0, 0)\). We observe that for \(-\frac{\pi}{4} < \frac{1}{2} \tan^{-1} \frac{\tau}{|\xi'|^2} < \frac{\pi}{4}\) (3.17) is integrable unless \((\tau, \xi') = (0, 0)\) and hence \(II_{L_x}\) converges, i.e.,

\[ \lim_{\epsilon \to 0} II_{L_x} = -c_{n+1} \int_{[R \setminus [0]} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{i \tau}{\sqrt{i \tau + |\xi'|^2}} e^{-\sqrt{i \tau + |\xi'|^2} \eta} d\xi' d\tau. \quad (3.18) \]

For the convergence of the second term \(I_{C_x}\) of the right hand side of (3.15), we see by noting (3.9) and \(\eta > 0\) that

\[ \lim_{\epsilon \to 0} II_{C_x} \leq \lim_{\epsilon \to 0} c_{n+1} \int_{[R^{n-1} \setminus [0]} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{e^{i \theta}}{\sqrt{\epsilon e^{i \theta} + |\xi'|^2}} e^{-\sqrt{\epsilon e^{i \theta} + |\xi'|^2} \eta} i e^{i \theta} d\epsilon d\theta |d\xi'| \]

\[ \leq \lim_{\epsilon \to 0} c_{n+1} \int_{[R^{n-1} \setminus [0]} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{e^{i \theta}}{\epsilon \cos \theta + |\xi'|^2} \exp \left( - \eta \left( \epsilon \cos \theta + |\xi'|^2 \right)^{1/4} \cos \left( \frac{1}{2} \tan^{-1} \left( \frac{\epsilon \sin \theta}{\epsilon \cos \theta + |\xi'|^2} \right) \right) \right) \right) d\epsilon d\xi'. \]

\[ \leq \lim_{\epsilon \to 0} c_{n+1} \int_{[R^{n-1} \setminus [0]} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{e^{i \theta}}{\epsilon \cos \theta + |\xi'|^2} \exp \left( - \eta |\xi'| \cos \left( \frac{\theta}{2} \right) \right) \right) d\epsilon d\xi'. \]

\[ \leq c_{n+1} \lim_{\epsilon \to 0} \epsilon^{\frac{1}{2}} \left( \int_{[R^{n-1} \setminus [0]} \exp \left( - \frac{\sqrt{2}}{2} \eta |\xi'| \right) d\xi' \right) \left( \int_{-\frac{1}{2}}^{\frac{1}{2}} \frac{e^{i \xi}}{\sqrt{|\xi| \sqrt{1 - \xi^2}}} d\xi \right) = 0. \quad (3.19) \]

Then, it follows from (3.18) and (3.19) that

\[ \Psi_N(t, x', \eta) = -c_{n+1} \int_{[R} \int_{[R^{n-1} \setminus [0]} e^{i \tau \cdot x' - \xi'} \frac{i \tau}{\sqrt{i \tau + |\xi'|^2}} e^{-\sqrt{i \tau + |\xi'|^2} \eta} d\xi' d\tau. \quad (3.20) \]
We notice that the potential of the solution operated by the Laplace operator is given by

$$\Psi_N(t, x', \eta) \equiv \Delta G_N(t, x', \eta),$$  (3.21)

where

$$G_N(t, x', x_n) = -c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i\tau + ix' \cdot \xi'} \frac{1}{\sqrt{i \tau + |\xi'|^2}} e^{-\sqrt{2 \cdot |\eta|}} d\xi' d\tau. \quad (3.22)$$

We then regard this boundary potential as a role of Littlewood-Paley dyadic decomposition of unity and the main argument for the proof consists on exchanging the boundary potential into the standard Littlewood-Paley decomposition \{\psi_k\}_{k \in \mathbb{Z}} and \{\phi_j\}_{j \in \mathbb{Z}}.

4. The Dirichlet boundary condition

4.1. The Besov spaces on the half-spaces

First, we recall the summary for the Besov spaces over the half-space on the Euclidean space \( \mathbb{R}^n_+ \).

**Definition.** Let \( 1 \leq p < \infty \) and \( 1 \leq \sigma < \infty \) with \( s \geq 0 \). Let

$$\overset{\circ}{\dot{B}}^s_{p, \sigma} (\mathbb{R}^n_+) = C_0^\infty (\mathbb{R}^n_+) \dot{B}^s_{p, \sigma} (\mathbb{R}^n_+),$$

$$\dot{B}^s_{p, \sigma} (\mathbb{R}^n_+) = \{ f \in \dot{B}^s_{p, \sigma} (\mathbb{R}^n); \ \text{supp} f \subset \mathbb{R}^n_+ \} \dot{B}^s_{p, \sigma} (\mathbb{R}^n),$$

by the Besov norm \( \dot{B}^s_{p, \sigma} (\mathbb{R}^n_+) \) (see Triebel [46] Section 2.9.3). It is shown that the above-defined space coincides the space \( \dot{B}^s_{p, \sigma} (\mathbb{R}^n_+) \) defined by the restriction in (1.5). Namely, the following proposition is shown by Triebel [46] and Danchin–Mucha [12].

**Proposition 4.1** ([12,46]). Let \( 1 < p < \infty \).

(1) For \( 0 < s, 1 \leq \sigma < \infty \),

$$\dot{B}^{-s} \overset{\circ}{\dot{B}}^s_{p, \sigma} (\mathbb{R}^n_+) \simeq (\dot{B}^s_{p, \sigma} (\mathbb{R}^n_+))^\ast.$$

(2) For \( -\infty < s \leq \frac{1}{p} \) and for \( 1 < \sigma < \infty \),

$$\overset{\circ}{\dot{B}}^s_{p, \sigma} (\mathbb{R}^n_+) \simeq \dot{B}^s_{p, \sigma} (\mathbb{R}^n_+).$$

(3) For \( -\infty < s < \frac{1}{p} \) and \( \sigma = 1 \),

$$\overset{\circ}{\dot{B}}^s_{p, 1} (\mathbb{R}^n_+) \simeq \dot{B}^s_{p, 1} (\mathbb{R}^n_+).$$
(4) For $-1 + \frac{1}{p} < s < \frac{1}{p}$ and $1 \leq \sigma < \infty$,

$$\tilde{B}^s_{p,\sigma}(\mathbb{R}^n_+) \simeq \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+).$$

We consider the restriction operator $R_0$ by multiplying a cutoff function

$$\chi_{\mathbb{R}^n_+}(x) = \begin{cases} 1, & \text{in } \mathbb{R}^n_+, \\ 0, & \text{in } \mathbb{R}^n. \end{cases}$$

i.e., for $f \in \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$, set $R_0 f = \chi_{\mathbb{R}^n_+}(x) f(x)$ in $\dot{B}^s_{p,\sigma}(\mathbb{R}^n)$ if $s > 0$ and it is understood as a distribution sense. Let the extension operator $E_0$ from $\tilde{B}^s_{p,\sigma}(\mathbb{R}^n_+)$ given by the zero-extension, i.e., for any $f \in B^s_{p,\sigma}(\mathbb{R}^n_+)$, set

$$E_0 f = \begin{cases} f(x), & \text{in } \mathbb{R}^n_+, \\ 0, & \text{in } \mathbb{R}^n. \end{cases}$$

One can find that those operators are basic tools to recognize the homogeneous Besov spaces. Using Proposition 4.1, the following statement is a variant introduced by Triebel [46, p.228]

**Proposition 4.2.** Let $1 \leq p < \infty$, $1 \leq \sigma < \infty$ and $-1 + \frac{1}{p} < s < \frac{1}{p}$. It holds that

$$R_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),$$

$$E_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n),$$

are linear bounded operators. Besides, it holds that

$$R_0 E_0 = \text{Id} : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),$$

where $\text{Id}$ denotes the identity operator. Namely, $E_0$ and $R_0$ are a retraction and a co-retraction, respectively.

The proof of Proposition 4.2 is along the same line of the proof in [46]. Note that the spaces are homogeneous Besov spaces and then the arrangement appears in Proposition 3 in Danchin–Mucha [12] is required.

**Proof of Proposition 4.2.** It is clear that both

$$R_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+),$$

$$E_0 : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n)$$

are linear operators and

$$R_0 E_0 = \text{Id} : \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+) \to \dot{B}^s_{p,\sigma}(\mathbb{R}^n_+).$$
then it sufficient to show that both operators are bounded. Then, they are retraction and co-retraction, respectively.

To see the first operator \((4.1)\) is bounded, let \(f \in \dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)\) and we show that

\[
\|R_0 f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)} = \inf \|R_0 f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)} \leq \|\chi_{\mathbb{R}_+^n} f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)} \leq \|f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)}
\]

is valid under the restriction \(s > 0\). Let \(-1/p' < s < 0\) and \(f \in \dot{B}_{p,\sigma}^{s'}(\mathbb{R}^n)\). Suppose that \(\phi \in C_0^\infty(\mathbb{R}^n)\), Then

\[
|\langle \chi_{\mathbb{R}_+^n} f, \phi \rangle| = |\langle f, \chi_{\mathbb{R}_+^n} \phi \rangle| \leq \|f \|_{\dot{B}_{p,\sigma}^{s'}(\mathbb{R}^n)} \|\chi_{\mathbb{R}_+^n} \phi \|_{\dot{B}_{p',\sigma'}^{-s'}(\mathbb{R}^n)} \leq \|f \|_{\dot{B}_{p,\sigma}^{s'}(\mathbb{R}^n)},
\]

where \(0 < -s < 1/p'\) and the last inequality follow from the pointwise sense. Thus from the definition of the norm in \(\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)\), it holds similarly to the above that

\[
\|R_0 f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)} \leq \|\chi_{\mathbb{R}_+^n} f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)} = \sup_{\phi \in \dot{B}_{p',\sigma'}^{-s'}(\mathbb{R}^n) \setminus \{0\}} \|\phi \|_{\dot{B}_{p',\sigma'}^{-s'}(\mathbb{R}^n)} \leq \|f \|_{\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)}.
\]

To see the second bound \((4.2)\), we introduce a quotient space \(\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)/\sim\), where we identify all \(f \in \dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)\) that coincides on \(\mathbb{R}_+^n\). Then, the restriction operator \(R_0\) is one to one mapping from \(\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)/\sim\) onto \(\dot{B}_{p,\sigma}^{s}(\mathbb{R}_+^n)\), and then the extension operator \(E_0\) is an inverse operator of \(R_0\). Thus, the open mapping theorem implies the required boundedness directly.

In what follows, we restrict ourselves to the regularity range of the Besov spaces \(\dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)\) in \(-1 + 1/p < s < 1/p\) for \(1 < p < \infty\). Hence, we freely use the above-mentioned results. As a consequence, any component in \(\dot{B}_{p,\sigma}^{s}(\mathbb{R}_+^n)\) under such restriction on \(s\) and \(p\) can be extended into the one over whole space \(\mathbb{R}^n\) and conversely the component \(f \in \dot{B}_{p,\sigma}^{s}(\mathbb{R}^n)\) is restricted into the one over the half-space \(\mathbb{R}_+^n\). We frequently use those facts without noticing for every case below.

### 4.2. The L-P decomposition with separation of variables

In order to split the variables \(x' \in \mathbb{R}^{n-1}\) and \(x_n \in \mathbb{R}_+\), we introduce an \(x'\)-parallel decomposition and an \(x_n\)-parallel decomposition by Littlewood-Paley type. In what follows, \(\eta \in \mathbb{R}_+\) denotes a parameter for \(x_n\)-axis in \(\mathbb{R}_+^n\). We introduce \(\{\Phi_m\}_{m \in \mathbb{Z}}\) as a Littlewood-Paley dyadic frequency decomposition of unity in separated variables \((\xi', \xi_n) \in \mathbb{R}^{n-1} \times \mathbb{R}^\ast_+\).

**Definition** *(The Littlewood-Paley decomposition of separated variables).* For \(m \in \mathbb{Z}\), let

\[
\hat{\zeta}_m(\xi_n) = \begin{cases} 
1, & 0 \leq |\xi_n| < 2^m, \\
\text{smooth}, & 2^m \leq |\xi_n| < 2^{m+1}, \\
0, & 2^{m+1} \leq |\xi_n|. 
\end{cases}
\]

\[
\hat{\zeta}_m(\xi_n) = \hat{\zeta}_{m-1}(\xi_n) + \Phi_m(\xi_n)
\]
(one can choose \( \hat{\zeta}_m(r) = \sum_{\ell \leq m-1} \hat{\phi}_\ell(r) + \hat{\phi}_{-\infty}(r) \) with a correction distribution \( \hat{\phi}_{-\infty}(r) \) at \( r = 0 \) and set
\[
\hat{\Phi}_m(\xi) \equiv \hat{\phi}_m(|\xi'|) \otimes \hat{\zeta}_{m-1}(\xi_n) + \hat{\zeta}_m(|\xi'|) \otimes \hat{\phi}_m(\xi_n). \tag{4.7}
\]
Then, it is obvious from Fig. 2 (restricted on the upper half region in \( \mathbb{R}^n \)) that
\[
\sum_{m \in \mathbb{Z}} \hat{\Phi}_m(\xi) = 1, \quad \xi = (\xi', \xi_n) \in \mathbb{R}^n \setminus \{0\}. \tag{4.8}
\]
Indeed, from (4.6) and (4.7),
\[
\sum_{m \in \mathbb{Z}} \hat{\Phi}_m(\xi)
= \sum_{m \in \mathbb{Z}} \hat{\phi}_m(|\xi'|) \otimes \sum_{-\infty \leq \ell \leq m-1} \hat{\phi}_\ell(\xi_n) + \sum_{m \in \mathbb{Z}} \left( \sum_{-\infty \leq \ell \leq m} \hat{\phi}_\ell(|\xi'|) \right) \otimes \hat{\phi}_m(\xi_n)
= \sum_{m \in \mathbb{Z}} \hat{\phi}_m(|\xi'|) \otimes \sum_{-\infty \leq \ell \leq m-1} \hat{\phi}_\ell(\xi_n) + \sum_{m \in \mathbb{Z}} \sum_{\ell \leq m} \hat{\phi}_\ell(|\xi'|) \otimes \hat{\phi}_m(\xi_n)
+ \sum_{m \in \mathbb{Z}} \hat{\phi}_{-\infty}(|\xi'|) \otimes \hat{\phi}_m(\xi_n)
= \sum_{m \in \mathbb{Z}} \hat{\phi}_m(|\xi'|) \otimes \left( \sum_{-\infty \leq \ell \leq m-1} \hat{\phi}_\ell(\xi_n) + \sum_{\ell \geq m} \hat{\phi}_\ell(\xi_n) \right) + \hat{\phi}_{-\infty}(|\xi'|) \otimes \sum_{m \in \mathbb{Z}} \hat{\phi}_m(\xi_n)
= \sum_{m \in \mathbb{Z}} \hat{\phi}_m(|\xi'|) \otimes \left( \sum_{\ell \leq m} \hat{\phi}_\ell(\xi_n) + \hat{\phi}_{-\infty}(|\xi'|) \right) + \hat{\phi}_{-\infty}(|\xi'|) \otimes \sum_{m \in \mathbb{Z}} \hat{\phi}_m(\xi_n)
\otimes \sum_{\ell \in \mathbb{Z} \cup \{-\infty\}} \hat{\phi}_m(\xi_n) - \hat{\phi}_{-\infty}(|\xi'|) \otimes \hat{\phi}_{-\infty}(\xi_n)
\begin{align*}
&= \left( \sum_{m \in \mathbb{Z} \cup \{-\infty\}} \hat{\phi}_m (|\xi'|) \otimes 1 \right) - \hat{\phi}_{-\infty} (|\xi'|) \otimes \hat{\phi}_{-\infty} (\xi_n) \\
&= 1 - \hat{\phi}_{-\infty} (|\xi'|) \otimes \hat{\phi}_{-\infty} (\xi_n).
\end{align*}

**Definition** (Varieties of the Littlewood-Paley dyadic decompositions). Let \((\tau, \xi', \xi_n) \in \mathbb{R} \times \mathbb{R}^{n-1} \times \mathbb{R}\) be Fourier adjoint variables corresponding to \((t, x', \eta) \in \mathbb{R}_+ \times \mathbb{R}^{n-1} \times \mathbb{R}_+\).

- \(\{\Phi_m(x)\}_{m \in \mathbb{Z}}\): the standard (annulus type) Littlewood-Paley dyadic decomposition by \(x = (x', \eta) \in \mathbb{R}_+^n\).
- \(\{\hat{\Phi}_m(x)\}_{m \in \mathbb{Z}}\): the Littlewood-Paley dyadic decomposition over \(x = (x', \eta) \in \mathbb{R}_+^n\) given by (4.7).
- \(\{\psi_k(\tilde{\tau})\}_{k \in \mathbb{Z}}\): the Littlewood-Paley dyadic decompositions in \(\tilde{\tau} \in \mathbb{R}\).
- \(\{\phi_j(x')\}_{j \in \mathbb{Z}}\) and \(\{\phi_j(\tilde{\eta})\}_{j \in \mathbb{Z}}\): the standard (annulus type) Littlewood-Paley dyadic decompositions in \(x' \in \mathbb{R}^{n-1}\) and \(\tilde{\eta} \in \mathbb{R}\), respectively.
- \(\{\zeta_m(x')\}_{m \in \mathbb{Z}}\) and \(\{\zeta_m(\tilde{\eta})\}_{m \in \mathbb{Z}}\): the lower frequency smooth cutoff given by (4.6), respectively.
- Let \(\tilde{\phi}_j = \phi_{j-1} + \phi_j + \phi_{j+1}\) be the Littlewood-Paley dyadic decompositions with its \(j\)-neighborhood.
- Since all the above-defined decompositions are even functions, we identify \(\tilde{\tau} \in \mathbb{R}\) and \(\tilde{\eta} \in \mathbb{R}\) with \(|\tilde{\tau}| = t > 0\) and \(|\tilde{\eta}| = \eta > 0\), respectively.

Then, it is easy to see that the norm of the Besov spaces on \(\mathbb{R}_+^n\) defined by \(\{\Phi_m\}_m\) is equivalent to the one from the Littlewood-Paley decomposition of direct sum type, \(\{\Phi_m\}_m\):

\[
\|\Delta u(t)\|_{\dot{B}^s_{p,1}(\mathbb{R}_+^n)} = \sum_{m \in \mathbb{Z}} 2^{sm} \|\Phi_m * \Delta u(t)\|_{L^p(\mathbb{R}_+^n)} \\
\leq 3C \sum_{m \in \mathbb{Z}} 2^{sm} \|\bar{\Phi}_m * \Delta u(t)\|_{L^p(\mathbb{R}_+^n)}. \tag{4.9}
\]

4.3. Separation on the Dirichlet potential

In order to show the sufficiency part of Theorem 2.1, we first decompose the solution potential \(\Psi_D\) defined in (3.11). From the solution formula with the Green function (3.4) and (3.11), we see that

\[
\bar{\Phi}_m * (\Psi_D(t, x', \eta)) \\
= \left( \phi_m (|x'|) \otimes \zeta_{m-1}(\eta) \right) * \Psi_D(t, x', \eta) + \left( \zeta_m (|x'|) \otimes \phi_m (\eta) \right) * \Psi_D(t, x', \eta) \\
= \zeta_{m-1}(\eta) * \left( \phi_m (|x'|) * \Psi_D(t, x', \eta) \right) + \phi_m (\eta) * \left( \zeta_m (|x'|) * \Psi_D(t, x', \eta) \right)
\]
\[
\gamma \left( t, x', \eta \right) \equiv \left\| P \sum_{j=1}^{n} \left( \phi_j (x') \ast \Psi_D (t, x', \eta) \right) \right\|_{L^p (\mathbb{R}^n)}.
\]

Applying the Hausdorff–Young inequality to the first term of the right hand side of (4.10) and restrict the range of the summation of \( j \) by \( \zeta_m (x') \ast \), we have from (4.9) that

\[
\int_0^\infty \| \Delta u (t) \|_{\dot{B}^s_{p,1} (\mathbb{R}^n_+)} \, dt \\
\leq C \sum_{m \in \mathbb{Z}} \sum_{j \leq m+1} \left( \int_{\mathbb{R}^n_+} \left\| \psi_k (t) \ast \phi_j (y') \ast h (t, y') \right\|_{L^p (\mathbb{R}^n_+)} \, dt \right) \left\| \psi_k (t) \ast \phi_j (y') \ast h (t, y') \right\|_{L^p (\mathbb{R}^n_+)} \\
\leq C \sum_{m \in \mathbb{Z}} \sum_{j \leq m+1} \| \psi_k (t) \ast \phi_j (y') \ast h (t, y') \|_{L^p (\mathbb{R}^n_+)} \left\| \psi_k (t) \ast \phi_j (y') \ast h (t, y') \right\|_{L^p (\mathbb{R}^n_+)} \\
\leq \| P^D_1 \|_{L^1_1 (\mathbb{R}^+) + \| P^D_2 \|_{L^1_1 (\mathbb{R}^+)}},
\]

where the first term of the right hand side of (4.11) includes \( \phi_m (x') \), once the outer decomposition \( \sum_{m \in \mathbb{Z}} \) is fixed then the inner decomposition \{ \phi_j (x') \ast \} \( j \in \mathbb{Z} \) is restricted into only \( |j - m| \leq 1 \) and the summation for \( j \) disappears.

4.4. Time-space splitting argument

We separate the estimate of (4.11) into two regions; one is time-dominated area and the other is space-dominated area (Fig. 3).

- The relation between each variables:
  In order to prove Theorem 2.1, it is enough to prove Lemma 4.3.

**Lemma 4.3.** Let \( 1 < p \leq \infty \). The term \( P^D_1 \) defined in (4.11) is estimated as follows:

\[
\| P^D_1 \|_{L^1_1 (\mathbb{R}^+)} \leq C \left( \| \hat{h} \|_{F^L_{1,1} (\mathbb{R}^+; \dot{B}^s_{p,1} (\mathbb{R}^n_+))} + \| \hat{h} \|_{L^1 (\mathbb{R}^+; \dot{B}^{L+2/p - \delta - 1} (\mathbb{R}^n_+))} \right).
\]

(4.12)
Simultaneously the term $P^D_2$ defined in (4.11) is estimated as follows:

$$\|P^D_2\|_{L^1(\mathbb{R}^+)} \leq C \left( \|h\|_{L^{1+1/2p}(\mathbb{R}^+; \dot{B}_{p,1}^{s}(\mathbb{R}^n-1))} + \|h\|_{L^1(\mathbb{R}^+; \dot{B}_{p,1}^{s+2-1/p}(\mathbb{R}^n-1))} \right).$$  (4.13)

**Proof of Lemma 4.3.** We split the data $h$ into the time-dominated region and the space-dominated region.

$$h(t, x') = \sum_{k \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \psi_k(t) * \phi_j(x') * h(t, x')$$

$$= \sum_{k \in \mathbb{Z}} \sum_{j \leq k} \psi_k(t) * \phi_j(x') * h(t, x') + \sum_{k \in \mathbb{Z}} \sum_{j > k} \psi_k(t) * \phi_j(x') * h(t, x').$$  (4.14)

Letting $h(m, x') \equiv \tilde{\phi}_m * h(t, x') (m \in \mathbb{Z})$, we proceed

$$P^D_1 = C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \in \mathbb{Z}} \sum_{|j-m| \leq 1, 2j < k} \Psi_D(t, x', \eta) * \psi_k(t) * \phi_j(x') * h(t, x') \right\|^p_{L^p(\mathbb{R}^n-1)} \, d\eta \right)^{1/p}$$

$$+ C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \left\| \sum_{j \in \mathbb{Z}} \sum_{|j-m| \leq 1, k < 2j} \Psi_D(t, x', \eta) * \psi_k(t) * \phi_j(x') * h(t, x') \right\|^p_{L^p(\mathbb{R}^n-1)} \, d\eta \right)^{1/p}$$

$$\leq C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \geq 2m} \Psi_D(t, x', \eta) * \psi_k(t) * \phi_m(x') * h_m(t, x') \right\|^p_{L^p_x} \, d\eta \right)^{1/p}$$

$$+ C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \left\| \sum_{k < 2m} \Psi_D(t, x', \eta) * \psi_k(t) * \phi_m(x') * h_m(t, x') \right\|^p_{L^p_x} \, d\eta \right)^{1/p}$$

$$= L_1 + L_2.$$  (4.15)
Setting
\[ \Psi_{D,k,m}(t, x', \eta) \equiv \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \psi_{t-k} \phi_{m}(y') dy' ds, \quad (4.16) \]
we see that \( L_1 \) is the time-dominated region and applying the Minkowski and the Hausdorff–Young inequality with using (4.16), we have
\[ L_1 \leq \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \Psi_{D,k,m}(t, x', \eta) \right) \left( \sum_{k \geq 2m} \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \psi_{t-k} \phi_{m}(y') dy' ds \right)^{p} \right)^{1/p} \]
\[ \leq C \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \Psi_{D,k,m}(t, \eta) \right) \left( \int_{\mathbb{R}} \| \psi_{t-k} \phi_{m}(y') \|_{L_{p}^{p}} ds \right)^{p} \eta^{1/p} \]  

(4.17)

Then by the almost orthogonal estimate between the boundary potential \( \Psi_D \) and the Littlewood-Paley decomposition \( \psi_k \) in time, namely we invoke Lemma 6.1, for any \( t, \eta \in \mathbb{R}^+ \),
\[ \| \Psi_{D,k,m}(t, \cdot, \eta) \|_{L_{x'}^{p}} \leq \begin{cases} C 2^k \exp(-2^k \eta) \frac{2^k}{(2^k t)^{2}}, & k \geq 2m, \\ C 2^k \exp(-2^{m-1} \eta) \frac{2^k}{(2^k t)^{2}}, & k < 2m. \end{cases} \]  

(4.18)

Noting the restriction \( k \geq 2m \) on the time-dominated region, we apply (4.18) to (4.17) and obtain that
\[ \| L_1 \|_{L_{x'}^{1}(\mathbb{R}^+)} \leq C \sum_{m \in \mathbb{Z}} 2^{2m} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \psi_{t-k} \phi_{m}(y') \| \phi_{m}(y') \|_{L_{p}^{p}} ds \eta^{1/p} \]  

(4.19)

Meanwhile, the second term \( L_2 \) is the space-dominated region and letting \( h_m(t, x') \equiv \tilde{\phi}_m \ast h(t, x') \), we apply again the Minkowski inequality, the Hausdorff–Young inequality and the estimate (4.16),
\[
\|L_2\|_{L^1_t(\mathbb{R}^n_+)} \leq C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^n_+} \left\{ \sum_{k=2m}^{2m} \left(2^k e^{-2^{m-1} \eta} \right)^{\frac{2}{k}} \|h_m(\cdot, \cdot)\|_{L^p_{x'}} \right\}^p \, d\eta \right)^{1/p} \\
= C \sum_{m \in \mathbb{Z}} 2^{sm} \sum_{k < 2m} \left( \int_{\mathbb{R}^n_+} \left(2^k (t-s)\right)^{\frac{2}{k}} \|h_m(\cdot, \cdot)\|_{L^p_{x'}} \, ds \right)^{1/p} \\
\leq C \sum_{m \in \mathbb{Z}} 2^{sm + 2m^2 - \frac{m}{p}} \sum_{k < 2m} \left( \int_{\mathbb{R}^n_+} \left(2^k (t-s)\right)^{\frac{2}{k}} \|h_m(\cdot, \cdot)\|_{L^p_{x'}} \, ds \right)^{1/p} \\
\leq C \sum_{m \in \mathbb{Z}} 2^{(s+2-\frac{1}{p})m} \sum_{k < 2m} \left(2^k \|h_m(\cdot, \cdot)\|_{L^p_{x'}} \right)^{1/p} \\
\leq C \|h\|_{L^1_t(\mathbb{R}^n_+; B^{s+2-1/p}_{p,1}(\mathbb{R}^{n+1}_{x'})}, \\
\]  

From (4.15), (4.19) and (4.20), the estimate (4.12) is shown.\(^1\)

We then prove (4.13). By (4.14), we split \(P^D_2\) into the time-like region and the space-like region;

\[
P^D_2 = C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^n_+} \phi_m(\eta) \ast \Psi_D(t-s, x', \eta) \right) \ast_{(t,x')} \\
\times \left\{ \sum_{k \in \mathbb{Z}, j \leq m+1} \psi_k(t) \ast \phi_j(y') \ast h(t, y') \right\} \|
\times \left\{ \int_{\mathbb{R}^n_+} \exp(-p 2^{m-1} \eta) \, d\eta \right\}^{1/p} \\
\leq C \sum_{m \in \mathbb{Z}} 2^{sm} \left\{ \sum_{k \geq 2m, j \geq m+1} \phi_m(\eta) \ast \Psi_D(t, x', \eta) \ast_{(t,x')} \psi_k(t) \ast \phi_j(x') \ast h(t, x') \right\} \|
\times \left\{ \int_{\mathbb{R}^n_+} \exp(-p 2^{m-1} \eta) \, d\eta \right\}^{1/p} \\
\leq C \sum_{m \in \mathbb{Z}} 2^{sm} \left\{ \sum_{k \leq 2m, j \leq k} \phi_m(\eta) \ast \Psi_D(t, x', \eta) \ast_{(t,x')} \psi_k(t) \ast \phi_j(x') \ast h(t, x') \right\} \|
\times \left\{ \int_{\mathbb{R}^n_+} \exp(-p 2^{m-1} \eta) \, d\eta \right\}^{1/p} \\
\leq C \sum_{m \in \mathbb{Z}} 2^{sm} \left\{ \sum_{k \leq 2m, \min(k, 2m+2)} \phi_m(\eta) \ast \Psi_D(t, x', \eta) \ast_{(t,x')} \psi_k(t) \ast \phi_j(x') \ast h(t, x') \right\} \|
\times \left\{ \int_{\mathbb{R}^n_+} \exp(-p 2^{m-1} \eta) \, d\eta \right\}^{1/p} \\
\equiv M_1 + M_2. \\
\]  

The first term \(M_1\) of the right hand side is the time-dominated part, letting \(h_j \equiv \hat{\phi}_j \ast h\), we apply the Minkowski inequality and the Hausdorff–Young inequality with

\(^1\)Up to this level, there is no restriction on \(p\) nor \(s\).
(4.16) to see we use the almost orthogonal estimate (4.18) between \( \psi_m \) and \( \Psi_{D,k,m} \) (Lemma 6.2): For any \( N \in \mathbb{N} \),

\[
\| \phi_m(\eta) \ast \Psi_{D,k,j}(t, \cdot, \eta) \|_{L^1_{x,t} \left( \mathbb{R}^{n+1}_x \right)} \leq \begin{cases} 
CN 2^k 2^{-\frac{1}{2} - m} \frac{2^k}{(2 \min(\frac{1}{2}, m) \eta)^N (2k t)^2}, & k \geq 2j, \\
CN 2^k 2^{-|j - m|} \frac{2^k}{(2^j \eta)^N (2k t)^2}, & k < 2j 
\end{cases}
\]

for some \( CN > 0 \). Then setting \( 2m \eta = \tilde{\eta} \) and shifting \((m, k, j) \to (m', k, j)\) by \( m - \frac{k}{2} = m' \), the first term of the right hand side of (4.21) can be estimated as follows:

\[
\| M_1 \|_{L^1_t \left( \mathbb{R}_t \right)} \leq \| \sum_{m \in \mathbb{Z}} 2^m \left( \int_{\mathbb{R}_t} \left( \sum_{k \in \mathbb{Z}} 2^{\frac{1}{2} - m} \frac{2^k}{(2^k (t - s))^2} \sum_{2j \leq k} \left\| \psi_k \ast h_j(s, \cdot) \right\|_{L^p_{x,t}} ds \right) \frac{1}{(\langle \tilde{\eta} \rangle)^N} \right \|^p_{L^1_t \left( \mathbb{R}_t \right)}
\]

\[
\leq C \| \sum_{m \in \mathbb{Z}} 2^{m} \left( \int_{\mathbb{R}_t} \left( \sum_{k \in \mathbb{Z}} 2^{\frac{1}{2} - m} \frac{2^k}{(2^k (t - s))^2} \sum_{2j \leq k} \left\| \psi_k \ast h_j(s, \cdot) \right\|_{L^p_{x,t}} ds \right) \frac{1}{(\langle \tilde{\eta} \rangle)^N} \right \|^p_{L^1_t \left( \mathbb{R}_t \right)}
\]

\[
\leq C \| \sum_{m \in \mathbb{Z}} 2^{m} \left( \int_{\mathbb{R}_t} \left( \sum_{k \in \mathbb{Z}} 2^{\frac{1}{2} - m} \frac{2^k}{(2^k (t - s))^2} \sum_{2j \leq k} \left\| \psi_k \ast h_j(s, \cdot) \right\|_{L^p_{x,t}} ds \right) \frac{1}{(\langle \tilde{\eta} \rangle)^N} \right \|^p_{L^1_t \left( \mathbb{R}_t \right)}
\]

where we used the fact that \( |s - 1/p| < 1 \), i.e., \(-1 + 1/p < s < 1 + 1/p\) and \( s \leq 0 \) at the second line from the bottom.
On the other hand for the estimate $M_2$, we proceed a similar way to treat $M_1$. Exchanging the order of the summation of $j$ and $k$ and setting $h_j(t) \equiv \phi_j * h(t)$, it follows by changing $(m, k, j) \rightarrow (m, k, j)$ with $m - j = m'$ and (4.21) that

$$
\| M_2 \|_{L^1} \leq \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \left\{ \sum_{j \in \mathbb{Z}} \sum_{k < 2j} \int_{\mathbb{R}^+} \phi_m(\eta) * \Psi_{D,k,j}(t-s,x',\eta) \right\} \right)_{L^p_{x'}} \times \left\| h_j(s, \cdot) \right\|_{L^p_{x'}} d\eta \right)^{1/p} L^1_{\mathbb{R}^+}.
$$

(changing the variable $2^j \eta = \tilde{\eta}$)

$$
\leq C \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \sum_{j \in \mathbb{Z}} \sum_{k < 2j} 2^{k-2|j-m|} \int_{\mathbb{R}^+} \frac{2^k}{(2^k(t-s))^2} \left\| h_j(s, \cdot) \right\|_{L^p_{x'}} d\eta \left( \int_{\mathbb{R}^+} \frac{1}{(2^j|\eta|)^{pN}} d\eta \right) \right)^{1/p} L^1_{\mathbb{R}^+}.
$$

Here, we used the fact that $|s| < 1$ for convergence of the summation on $m'$ In the last estimate, we exchange the order of the integration in time and the summation of
ℓ′ and use the Hausdorff–Young inequality to remove the convolution with the time potential term and then recovers the time integration outside. From (4.21), (4.22) and (4.23), the estimate (4.13) is shown. This completes the proof of Lemma 4.3.

4.5. Proof of Theorem 2.2

For the proof of Theorem 2.2, it is simpler than the proof of Theorem 2.1 since it does not involve the Littlewood-Paley decomposition in \((x′, η)\)-variable. We recall 
\(I_{−ℓ} = [2^{−ℓ}, 2^{−ℓ+1})\) for all \(ℓ \in \mathbb{Z}\). We again split the case for the time-dominated part and space-dominated part and proceed the estimate:

\[
\int_0^\infty \| \Delta u(t) \|_{L^p(\mathbb{R}^n_+)} \, dt \\
= \| (\int_{\mathbb{R}^n_+} (\int_0^\infty |\Psi_D(t − s, x′ − y′, η)h(s, y')dsdy'|^p d\eta)^{1/p} \|_{L^1(\mathbb{R}^n_+)} \\
= \| \left( \sum_{\ell \in \mathbb{Z}} \int_{\mathbb{R}^n_+} (\int_0^\infty |\Psi_D(t − s, x′ − y′, η)h(s, y')dsdy'|^p d\eta)^{1/p} \right) \|_{L^1(\mathbb{R}^n_+)} \\
\leq C \left( \sum_{\ell \in \mathbb{Z}} 2^{−\ell} \| \Psi_D(t, x′, η) \|_{\eta \geq 2^{−\ell} \ast (t, x')} \|_{L^p(\mathbb{R}^n_+)} \right)^{1/p} \|_{L^1(\mathbb{R}^n_+)},
\]

(4.24)

where we split the last term of (4.24) into two parts:

\[
\left\| \Psi_D(t, x′, η) \right\|_{\eta \geq 2^{−\ell} \ast (t, x')} \|_{L^p(\mathbb{R}^n_+)} \\
\leq \left\| \Psi_D(t, x′, η) \right\|_{\eta \geq 2^{−\ell} \ast (t, x')} \sum_{k \in \mathbb{Z}} \sum_{0 \leq j \leq k/2} \| \psi_k(t) \ast \phi_j(x′) \ast h(t, x′) \|_{L^p_{x'}} \\\n+ \left\| \sum_{j \geq 0} \sum_{0 \leq k \leq j/2} \psi_k(t) \ast \phi_j(x′) \ast h(t, x′) \right\|_{L^p_{x'}} \\
\leq \sum_{k \in \mathbb{Z}} \left\| \int_{\mathbb{R}^n_+} \sum_{0 \leq j \leq k/2} \Psi_{D,k,j}(t − r, x′ − y′) \left[ \psi_k(t) \ast \phi_j(y′) \right] h(r, y') \, dr \, dy' \right\|_{L^p_{x'}} \\
+ \sum_{j \geq 0} \left\| \int_{\mathbb{R}^n_+} \sum_{k \leq j/2} \Psi_{D,k,j}(t − s, x′ − z′) \left[ \phi_j(x'') \ast \psi_k(s) \ast h(s, z') \right] \, ds \, dz' \right\|_{L^p_{x'}} \\
\leq \sum_{k \in \mathbb{Z}} \sup_{j \in \mathbb{Z}} \left\| \Psi_{D,k,j}(t − r, \cdot) \right\|_{L^1_{x'}} \left\| \sum_{0 \leq j \leq k/2} \phi_j(x′) \ast \left( \psi_k \ast h(r, \cdot) \right) \right\|_{L^p_{x'}} \, dr \\
+ \sum_{j \geq 0} \left\| \sum_{k \leq j/2} \Psi_{D,k,j}(t − s, \cdot) \right\|_{L^1_{x'}} \left\| \phi_j(x′) \ast h(s, \cdot) \right\|_{L^p_{x'}} \, ds \\
eq D_{\ell}^1 + D_{\ell}^2.
\]

(4.25)

Here, the notation \(j = \widehat{0}\) for the Littlewood-Paley decomposition is given in (1.4). Thus from (4.24), (4.25), we estimate the following two terms:
\[
\int_0^\infty \|\Delta u(t)\|_{L^p(\mathbb{R}^n)} \, dt
\]

\[
\leq C \left( \sum_{\ell \in \mathbb{Z}} 2^{-\ell} \|\Psi_D(t, x', \eta) \ast h(t, x')\|_{L_p^p(\mathbb{R}^{n-1})} \right)^{1/p} \leq L^1(\mathbb{R}^+).
\]

\[
\leq C \left\| \{D^1_{\ell}\}_{\ell \in \mathbb{Z}} \right\|_{L^p(\mathbb{R}^+)} + C \left\| \{D^2_{\ell}\}_{\ell \in \mathbb{Z}} \right\|_{L^p(\mathbb{R}^+)}.
\]

We now see that the following proposition is enough to ensure that Theorem 2.2 hold.

**Proposition 4.4.** Let \( 1 \leq p \leq \infty \). Let \( D^1_{\ell} \) and \( D^2_{\ell} \) be defined in (4.25). Then, there exist constants \( C > 0 \) such that the following estimate holds:

\[
\left\| \{D^1_{\ell}\}_{\ell \in \mathbb{Z}} \right\|_{L^p(\mathbb{R}^+)} \leq C \left\| h \right\|_{F_{1,1}^{1,1/2}(\mathbb{R}^+; L^p(\mathbb{R}^{n-1}))},
\]

\[
\left\| \{D^2_{\ell}\}_{\ell \in \mathbb{Z}} \right\|_{L^p(\mathbb{R}^+)} \leq C \left\| h \right\|_{L^1(\mathbb{R}^+; H^{-1,1/2}(\mathbb{R}^{n-1}))}.
\]

**Proof of Proposition 4.4.** The estimates (4.26) and (4.27) are proved in the similar manner as (4.12) and (4.13) in the case \( 1 \leq p < \infty \). Setting \( k - 2\ell = -2\ell' \) to change \((\ell, k, j) \to (\ell', k, j),\)

\[
\left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{-\ell} \left| D^1_{\ell} \right|^p \right)^{1/p} \right\|_{L^1(\mathbb{R}^+)} \]

\[
\leq C \left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{(1-\frac{1}{p})\ell} \left( \sum_{k \in \mathbb{Z}} \int_{\mathbb{R}^+} \chi_{k,\ell,j}(t-r, \cdot) \right) \right)^p \right\|_{L^1(\mathbb{R}^+)}
\]

\[
\leq \frac{2^k}{2^{(1-\frac{1}{p})\ell}} \left\| \left( \sum_{k \in \mathbb{Z}} \psi_k \ast h(\cdot) \right)^p \right\|_{L^p(\mathbb{R}^+)}
\]

\[
= C \left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{(1-\frac{1}{p})\ell} \phi_{\ell} \ast e^{-2^{-\ell}} \right)^p \right\|_{L^1(\mathbb{R}^+)}.
\]
\[
\begin{align*}
&\leq C \sum_{k \in \mathbb{Z}} 2^{l - \frac{1}{p} k} \left\| \int_{\mathbb{R}} \frac{2^k}{(2^k(t-r))^2} \psi_k * h(r, \cdot) \, dr \right\|_{L^p_t(L^1_r)} \\
&\leq C \sum_{k \in \mathbb{Z}} 2^{l - \frac{1}{p} k} \left\| \psi_k * h(t, \cdot) \right\|_{L^p_t(L^1_r)} \\
&\leq C \left\| \sum_{k \in \mathbb{Z}} 2^{(1 - \frac{1}{p}) k} \psi_k * h(t, \cdot) \right\|_{L^p_t(L^1_r)} \\
&\leq C \left\| h \right\|_{L^{\frac{1}{1+1/p}(\mathbb{R}_+; L^\infty_r)}},
\end{align*}
\]

(4.28)

For the case \( p = \infty \) in (4.26), setting \( k - 2\ell = -2\ell' \) and changing \( \ell \to \ell' \), we have

\[
\left\| \sup_{\ell \in \mathbb{Z}} |D^\ell_j| \right\|_{L^1_t(L^1_r)} \\
\leq C \left\| \sup_{\ell' \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{\ell' + 2\ell} (2^{k-2\ell} e^{-2\frac{1}{2}(k-2\ell)}) \\
\times \int_{\mathbb{R}} \frac{2^k}{(2^k(t-r))^2} \sum_{0 \leq j \leq k/2} \phi_j * h_k(r, \cdot) \, dr \right\|_{L^1_t(L^1_r)} \\
= C \left\| \sup_{\ell' \in \mathbb{Z}} \sum_{k \in \mathbb{Z}} 2^{\ell' + 2\ell} (2^{k-2\ell} e^{-2\frac{1}{2}(k-2\ell)}) \\
\times \int_{\mathbb{R}} \frac{2^k}{(2^k(t-r))^2} \sum_{0 \leq j \leq k/2} \phi_j * h_k(r, \cdot) \, dr \right\|_{L^1_t(L^1_r)} \\
= C \left\| \left\| e^{-\frac{1}{2} \ell'} \right\|_{L^\infty_t(L^1_r)} \sum_{k \in \mathbb{Z}} 2^k \int_{\mathbb{R}} \frac{2^k}{(2^k(t-r))^2} \left\| h_k(r, \cdot) \right\|_{L^\infty_r} \, dr \right\|_{L^1_t(L^1_r)} \\
\leq C \left\| \left\| e^{-\frac{1}{2} \ell'} \right\|_{L^\infty_t(L^1_r)} \sum_{k \in \mathbb{Z}} 2^{(1 - \frac{1}{p}) \ell'} \left( \sum_{j \geq 0} 2^{2(j-\ell) e^{-2j-\ell}} \right) \\
\times \int_{\mathbb{R}} \frac{2^k}{(2^k(t-r))^2} \left\| h_j(r, \cdot) \right\|_{L^p_r} \, dr \right\|_{L^1_t(L^1_r)} \leq C \left\| h \right\|_{L^{1/(1+1/p)}(\mathbb{R}_+; L^\infty_r)}.
\]

To show the estimate (4.27) for \( 1 \leq p < \infty \), we derive it by setting \( g_j(s, y') = \phi_j * g(s, y') \) and then \( j - \ell = -\ell' \) and \( (\ell, k, j) \to (\ell', k, j) \) that

\[
\left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{(-\frac{1}{p}) \ell |D^\ell_j|^p} \right)^{1/p} \right\|_{L^1_t(L^1_r)} \\
\leq C \left\| \sum_{\ell \in \mathbb{Z}} 2^{(-\frac{1}{p}) \ell} \left( \int_{\mathbb{R}} \sum_{j \geq 0} 2^{2(j-\ell) e^{-2j-\ell}} \right) \left\| D_{\mathbb{Z},k,j}(t-r, \cdot) \right\|_{L^p_t(L^1_r)} \left\| h_j(r, \cdot) \right\|_{L^p_r} \, dr \right\|_{L^1_t(L^1_r)} \\
\leq C \left\| \sum_{\ell \in \mathbb{Z}} 2^{(2 - \frac{1}{p}) \ell} \left( \sum_{j \geq 0} 2^{2(j-\ell) e^{-2j-\ell}} \right) \\
\times \int_{\mathbb{R}} \frac{2^k}{(2^k(t-r))^2} \left\| h_j(r, \cdot) \right\|_{L^p_r} \, dr \right\|_{L^1_t(L^1_r)} \\
= C \left\| \sum_{\ell' \in \mathbb{Z}} (2^{-\frac{1}{p} \ell'} e^{-2\ell'})^p \right\|_{L^{1/p}_t(L^1_r)}.
\]
Green’s function is slightly different from the one for the Dirichlet case. However as

For the Neumann boundary condition, the boundary potential associated to the

For the case $p = \infty$ in (4.27), by setting $j - \ell = -\ell'$ to change $(\ell, j) \to (\ell', j)$, we see that

This concludes the proof of Proposition 4.4. \hfill \square

5. The Neumann boundary condition

5.1. The Neumann boundary condition and the boundary trace

For the Neumann boundary condition, the boundary potential associated to the Green’s function is slightly different from the one for the Dirichlet case. However as
we see the explicit form of the potential function, it simply adjusts the order of the derivative and the estimate is very similar to the Dirichlet case. From (3.21), (3.20) and (4.9),

\[
\Phi_m^{*}_{(x',\eta)}(\eta^{-1}\Psi_N(t, x', \eta))
\]

\[
= -c_n^2 \zeta_m - (\eta) \int_{\mathbb{R}^{n-1}} \int_{\mathbb{R}} e^{i\tau + ix' \cdot \xi'} \phi_m(|\xi'|) \frac{\tau}{\sqrt{i \tau + |\xi'|^2}} e^{-\sqrt{i \tau + |\xi'|^2} \eta} d\xi' d\tau
\]

\[
- c_n^2 \phi_m(\eta) \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i\tau + ix' \cdot \xi'} \zeta_m(|\xi'|) \frac{\tau}{\sqrt{i \tau + |\xi'|^2}} e^{-\sqrt{i \tau + |\xi'|^2} \eta} d\xi' d\tau.
\]

(5.1)

To estimate the Besov-norm of the solution, we use the Littlewood-Paley decomposition for direct sum type (4.7). The estimates for the right hand side of (5.1) follow very similar manner to the case of the Dirichlet boundary condition.

\[
\int_0^{\infty} \| \Delta u(t) \|_{L^p(R^n)} dt
\]

\[
\leq C \left\| \sum_{m \in \mathbb{Z}} 2^m \left( \int_{\mathbb{R}^+} \| \Psi_N(t, x', \eta) \|_{L^p(\mathbb{R}^{n-1})} \right) \frac{p}{p-1} \right\|_{L_1^1(\mathbb{R}^+)}
\]

\[
+ C \left\| \sum_{m \in \mathbb{Z}} 2^m \left( \int_{\mathbb{R}^+} \| \phi_m(\eta) \|_{L^p(\mathbb{R}^{n-1})} \right) \frac{p}{p-1} \right\|_{L_1^1(\mathbb{R}^+)}
\]

\[
\times \sum_{k \in \mathbb{Z}} \sum_{j \leq m} \| \psi_k(t) \|_{L^p(\mathbb{R}^{n-1})} \frac{p}{p-1} \right\|_{L_1^1(\mathbb{R}^+)}
\]

\[
\equiv \| P_1^N \|_{L_1^1(\mathbb{R}^+)} + \| P_2^N \|_{L_1^1(\mathbb{R}^+)}. \quad (5.2)
\]

5.2. Proof of Theorem 2.3

We prove the sufficiency part of Theorem 2.3, which is reduced to prove Lemma 5.1. For the Neumann case, we use the time-space splitting argument same as the Dirichlet case. We split each of two terms in the right hand side of (5.2) into time-dominated area and space-dominated area.

**Lemma 5.1.** Let \( 1 < p < \infty \) and \(-1 + \frac{1}{p} < s \leq 0\). There exists a constant \( C > 0 \) such that \( P_1^N \) defined in (5.2) satisfies

\[
\| P_1^N \|_{L_1^1(\mathbb{R}^+)} \leq C \left( \| h \|_{\dot{F}_{1,1}^{1/2-1/2p}(\mathbb{R}^n; \dot{B}_{p,1}^{s}(\mathbb{R}^{n-1}))} + \| h \|_{L^1(\mathbb{R}^+; \dot{B}_{p,1}^{s+1-1/p}(\mathbb{R}^{n-1}))} \right). \quad (5.3)
\]

Similarly \( P_2^N \) defined in (5.2) satisfies the following estimate:

\[
\| P_2^N \|_{L_1^1(\mathbb{R}^+)} \leq C \left( \| h \|_{\dot{F}_{1,1}^{1/2-1/2p}(\mathbb{R}^n; \dot{B}_{p,1}^{s}(\mathbb{R}^{n-1}))} + \| h \|_{L^1(\mathbb{R}^+; \dot{B}_{p,1}^{s+1-1/p}(\mathbb{R}^{n-1}))} \right). \quad (5.4)
\]
Proof of Lemma 5.1. Introducing
\[ \Psi_{N,k,j}(t,x',\eta) \equiv \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \Psi_N(t-s,x'-y',\eta) \psi_k(s) \psi_j(y')dy'ds, \]
we apply (4.14) to \( P_1^N \) and set \( h_m \equiv \bar{\phi}_m \ast h \). Similar to the estimates (4.15) and (4.17) in the Dirichlet boundary case, we obtain that
\[ P_1^N \leq C \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \in \mathbb{Z}} \sum_{2j \leq k} \Psi_N(t,x',\eta) \ast \psi_k(t) \ast \phi_m(x') \ast h_j(t,x') \right\|_p^{1/p} \right)^p \]
\[ + C \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \in \mathbb{Z}} \sum_{2j > k} \Psi_N(t,x',\eta) \ast \psi_k(t) \ast \phi_m(x') \ast h_j(t,x') \right\|_p^{1/p} \right)^p \]
\[ \leq C \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \geq 2m} \Psi_N(t,x',\eta) \ast \psi_k(t) \ast \phi_m(x') \ast h_m(t,x') \right\|_p^{1/p} \right)^p \]
\[ + C \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \geq 2m} \left\| \Psi_{N,k,m}(t-s,x',\eta) \right\|_{L^p_1} \left\| \psi_k \ast h_m(s,\cdot) \right\|_{L^p_1} fds \right\|_p^{1/p} \right)^p \]
\[ = L_1 + L_2. \] (5.5)

Then similar to the Dirichlet boundary condition, we invoke Lemma 6.5 in Sect. 6 and it implies
\[ \left\| \Psi_{N,k,m}(t,\cdot) \right\|_{L^1_{x'}} \leq \begin{cases} C_n 2^k e^{-2^{\frac{k}{2}} \eta} \frac{2^k}{(2^k t)^2}, & k \geq 2m, \\ C_n 2^k e^{-2^{m-\frac{k}{2}} \eta} \frac{2^k}{(2^k t)^2}, & k < 2m. \end{cases} \] (5.6)

Applying (5.6) to (5.5), by the similar manner as the proof of (4.19) in Lemma 4.3, we have for the first term of the right hand side that
\[ \left\| L_1 \right\|_{L^1_1(\mathbb{R}^+)} \]
\[ \leq C \sum_{m \in \mathbb{Z}} 2^{2m} \left( \int_{\mathbb{R}^+} \left\| \sum_{k \geq 2m} \left\{ 2^k e^{-2^{\frac{k}{2}} \eta} \right\} \frac{2^k}{(2^k t-s)^2} \left\| \psi_k \ast h_m(s,\cdot) \right\|_{L^p_1} fds \right\|_p^{1/p} \right)^p \]
\[
\|L_2\|_{L^1_I(\mathbb{R}^+)} \leq C\left\| \sum_{m \in \mathbb{Z}} 2^{sm} \sum_{k < 2m} \left( 2 \frac{t}{2} e^{-2m\eta} \int_{\mathbb{R}^+} \frac{2^k h_m(s, x')}{\langle 2^k(t-s) \rangle^2} \|h_m(s, x')\|_{L^p_{x'}} \, ds \right) \right\|_{L^1_I(\mathbb{R}^+)}^{1/p} + C \left\| \sum_{m \in \mathbb{Z}} 2^{sm} \sum_{k < 2m} \left( 2 \frac{t}{2} e^{-2m\eta} \int_{\mathbb{R}^+} \frac{2^k h_m(s, x')}{\langle 2^k(t-s) \rangle^2} \|h_m(s, x')\|_{L^p_{x'}} \, ds \right) \right\|_{L^1_I(\mathbb{R}^+)}^{1/p} 
\]

(5.8)

To see (5.4), we split in a similar way to (4.21) again and obtain that

\[
P_2^N \leq C \left\| \sum_{m \in \mathbb{Z}} 2^{sm} \left\{ \sum_{k \in \mathbb{Z} \cap [2^j, 2^{j+1})} \left( \phi_m(\eta) \ast \psi_N(t, x', \eta) \right) \right\} \right\|_{L^p_{t,x'}}^{1/p} + C \left\| \sum_{m \in \mathbb{Z}} 2^{sm} \left\{ \sum_{k \in \mathbb{Z} \cap [2^j, 2^{j+1})} \left( \phi_m(\eta) \ast \psi_N(t, x', \eta) \right) \right\} \right\|_{L^p_{t,x'}}^{1/p} 
\]

(5.9)

We use the almost orthogonal estimate (5.6) between \(\psi_N\) and \(\psi_N, k, j\). From (5.9),

\[
\left\| M_1 + M_2 \right\|_{L^1_I(\mathbb{R}^+)} \leq C\left\| \sum_{m \in \mathbb{Z}} 2^{sm} \left( \int_{\mathbb{R}^+} \left\{ \sum_{k \in \mathbb{Z} \cap [2^j, 2^{j+1})} \left( \phi_m(\eta) \ast \psi_N(t, x', \eta) \right) \right\} \right\|_{L^1_I(\mathbb{R}^+)}^{1/p} \right\|_{L^1_I(\mathbb{R}^+)}^{1/p} 
\]

(5.10)
\[
\times 2^{-\frac{1}{2} \min(\frac{k}{2},m)} \left( \int_{\mathbb{R}_+} \frac{1}{|\eta|^p} d\eta \right)^{1/p} \left\| \mathcal{J}_1 \right\|_{L_1(\mathbb{R}_+)}.
\]

\[
\leq C \left\| \sum_{k \in \mathbb{Z}} 2^{\frac{k}{2}} \left( \sum_{m \in \mathbb{Z}} 2^{m} \sum_{j \leq k} 2^{-|m|/2} \right) \right\|_{L_1(\mathbb{R}_+)}
\]

\[
= C \left\| \psi_k \ast h_j(s,\cdot) \right\|_{L_p^p(\mathbb{R}_+)}
\]

(5.10)

under the condition \(-1 + 1/p < s \leq 0\). Setting \(h_j(t) \equiv \phi_j^{(x')} h(t)\) in (4.21) and changing the order of summation between \(j\) and \(k\), we have similar to the estimate for \(M_1\) by the Minkowski and the Hausdorff–Young inequalities with (4.16) that

\[
\left\| M_2 \right\|_{L_1(\mathbb{R}_+)}
\]

\[
\leq C \left( \int_{\mathbb{R}_+} \sum_{j \in \mathbb{Z}} \sum_{k \leq j} \frac{C_\phi \left( 2^{2(j-k-|m|)} - 2^k \right)}{(2|\eta|)^N} \right) \frac{2^k}{(2^k (t+s))^2} \left\| h_j(s,\cdot) \right\|_{L_p^p(\mathbb{R}_+)}
\]

\[
\times \left( \int_{\mathbb{R}_+} \frac{1}{|\eta|^p} d\eta \right)^{1/p} \left\| \mathcal{J}_1 \right\|_{L_1(\mathbb{R}_+)}
\]

\[
\leq C \left( \int_{\mathbb{R}_+} \sum_{j \in \mathbb{Z}} \sum_{k \leq j} \sum_{m \in \mathbb{Z}} 2^{-|m|/2} \right) \frac{2^k}{(2^k (t+s))^2} \left\| h_j(s,\cdot) \right\|_{L_p^p(\mathbb{R}_+)}
\]

\[
\times 2^{-\frac{1}{2} \min(\frac{k}{2},m)} \left( \int_{\mathbb{R}_+} \frac{1}{|\eta|^p} d\eta \right)^{1/p} \left\| \mathcal{J}_1 \right\|_{L_1(\mathbb{R}_+)}
\]

(5.11)

under the condition \(|s| < 1\). From (5.10) and (5.11), we conclude the proof of (5.4).

\[\square\]

5.3. Proof of Theorem 2.4

The proof of Theorem 2.4 can be done by a similar way to the case of the Dirichlet boundary condition. Indeed, we obtain a similar estimate to (5.3):

\[
\int_0^\infty \left\| \Delta u(t) \right\|_{L_p^p(\mathbb{R}_+)} dt
\]

\[
= \left\| \left( \int_{\mathbb{R}_+} \int_0^\infty \int_{\mathbb{R}^{d+1}} \Psi_N(t-s, x', \eta) h(s, y') ds dy' d\eta \right)^{1/p} \right\|_{L_p^p(\mathbb{R}_+)}
\]

\[
\times \left( \sum_{k \in \mathbb{Z}} 2^{-\frac{k}{2}} \left( \sum_{m \in \mathbb{Z}} 2^{-|m|/2} \right) \right) \left\| \mathcal{J}_1 \right\|_{L_1(\mathbb{R}_+)}
\]

(5.12)

then we split the last term into two terms:
From (5.12) and (5.13), we reduce the estimate for the following:

\[
\int_0^\infty \| \Delta u(t) \|_{L^p(\mathbb{R}^n_+)} \, dt
\]

\[
= \left\| \left( \sum_{\ell \in \mathbb{Z}} 2^{-\frac{\ell}{p}} \left\| \Psi_N(t, x', \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right) \cdot h(t, x') \right\|_{L^p(\mathbb{R}^n_+)}^{1/p} \left\| \psi_{\ell,j} \right\|_{L^p(\mathbb{R}^n_+)}^{1/p} \left\| h(s, \cdot) \right\|_{L^p(\mathbb{R}^n_+)} \, ds
\]

\[
= N_1^\ell + N_2^\ell.
\]  

(5.13)

The proof of Theorem 2.4 is now reduced to show the following proposition.

**Proposition 5.2.** Let \( 1 \leq p \leq \infty \). For \( N_1^\ell \) and \( N_2^\ell \) given by (5.13), the following estimates hold.

\[
\left\| \left\| \{N_1^\ell\} \| \ell^{-1/p} \right\|_{L^1(\mathbb{R}^n_+)} \leq C \left\| h \right\|_{L^{1/2,1/2}_1(\mathbb{R}_+;L^p(\mathbb{R}^{n-1}))},
\]

\[
\left\| \left\| \{N_2^\ell\} \| \ell^{-1/p} \right\|_{L^1(\mathbb{R}^n_+)} \leq C \left\| h \right\|_{L^1(\mathbb{R}_+;L^{1/2,1/2}_{p,1}(\mathbb{R}^{n-1}))}.
\]

The proof of Proposition 5.2 can be shown in a similar way to Proposition 4.4.

6. Almost orthogonality

In this section, we prove the almost orthogonality (2.12) (or (4.18) in Sect. 4) between the boundary potential \( \Psi_D \) for the Dirichlet boundary case and \( \Psi_N \) for the Neumann boundary case with the time and space Littlewood-Paley decomposition \( \{\psi_k\}_{k \in \mathbb{Z}} \) and \( \{\phi_j\}_{j \in \mathbb{Z}} \). The difficulty is that the fundamental solution \( \Psi_D(t, x', \eta) \) and \( \Psi_N(t, x', \eta) \) made by heat kernel is time and space convolution.
6.1. The Dirichlet potential case

**Lemma 6.1** (Crucial potential orthogonality). For $k, j, \ell \in \mathbb{Z}$ let $\{\psi_k(t)\}_{k \in \mathbb{Z}}$ and $\{\phi_j(x)\}_{j \in \mathbb{Z}}$ be the time and the space Littlewood-Paley dyadic decomposition and let $\Psi_D(t, x', \eta)$ be the boundary potential defined in (3.11). Set

$$
\Psi_{D,k,j}(t, x', \eta) \equiv \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \Psi_D(t - s, x' - y', \eta) \psi_k(s) \phi_j(y') dy' ds,
$$

(6.1)

for $\eta > 0$. Then, there exists a constant $C_n > 0$ depending only on the dimension $n$ satisfying

$$
\|\Psi_{D,k,j}(t, \cdot, \eta)\|_{L^1_{x'}} \leq \begin{cases} 
C_n 2^k (1 + (2^j \eta)^{n+2}) e^{-2^j \eta} \frac{2^k}{(2^k t)^2}, & k \geq 2j, \\
C_n 2^k (1 + (2^j \eta)^{n+2}) e^{-2^j \eta} \frac{2^k}{(2^k t)^2}, & k < 2j.
\end{cases}
$$

(6.2)

**Proof of Lemma 6.1.** We consider a time-like estimate $k > 2j$. Taking $\zeta'$-space cutoff, and using the change of variables $\tau = 2^k \sigma$, $\xi' = 2^j \zeta'$ we have

$$
\|\Psi_{D,k,j}(t, \cdot, \eta)\|_{L^1_{x'}} = \left\| c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{it\tau + ix' \cdot \xi'} \right\|_{L^1_{x'}}
$$

$$
\times \exp\left( -\sqrt{i} \tau + |\xi'|^2 \eta \right) \hat{\psi}(2^{-k} \tau) \hat{\phi}(2^{-j} \xi') d\xi' d\tau
$$

$$
= c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i2^{k} t \sigma + i2^j x' \cdot \zeta'} 2^{k} \sigma
$$

$$
\times \exp\left( -\sqrt{2^k i \sigma + 2^j |\zeta'|^2 \eta} \right) \hat{\psi}(\sigma) \hat{\phi}(\zeta') 2^{(n-1)j} d\zeta' \cdot 2^k d\sigma
$$

$$
= c_{n+1} 2^k \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i2^{k} t \sigma + i2^j x' \cdot \zeta'} \right\|_{L^1_{x'}}
$$

$$
\times \exp\left( -\frac{2^j \eta}{\sqrt{i \sigma + 2^j - k |\zeta'|^2}} \right) \hat{\psi}(\sigma) \hat{\phi}(\zeta') 2^{(n-1)j} d\zeta' \cdot 2^k d\sigma
$$

(6.3)

by setting $x' = 2^{-j} y'$. In the last equality. Using the identity

$$
e^{i(2^k t \sigma + y' \cdot \zeta')} = -\frac{1}{i 2^k t} \frac{1}{|y'|^2} \Delta_{\zeta'} \partial_\sigma e^{i(2^k t \sigma + y' \cdot \zeta')},
$$

(6.4)

and integrating by parts with respect to $\sigma$ and $\zeta'$, setting $p(\sigma, \zeta'; 2^j - k) \equiv \sqrt{i \sigma + 2^j - k |\zeta'|^2}$, we proceed...
\[ \| \Psi_{D,k,j}(t, \cdot) \|_{L^1_{\xi'}(B^c_{r_j})} \]
\[ = c_{n+1}2^{|2j-k|} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \frac{1}{i 2^j \xi} \left( 1 - \frac{1}{|y'|^2} \right) e^{i (2^j k \sigma + y' \cdot \xi')} \]
\[ \times \Delta_{\xi'} \frac{\partial}{\partial \sigma} \left[ \exp \left( -2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \right) \widehat{\psi}(\sigma) \hat{\phi}(\xi') \right] d\xi'd\sigma \|_{L^1_{\xi'}(B^c_{r_j})}. \]

Here,
\[ \frac{\partial}{\partial \sigma} \left[ \exp \left( -2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \right) \widehat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\xi') \right] \]
\[ = \exp \left( -2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \right) \times \left\{ \frac{-2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \widehat{\psi}(\sigma) 2^k \sigma + \widehat{\psi}'(\sigma) 2^k \sigma + \widehat{\psi}(\sigma) 2^k \right\} \hat{\phi}(\xi'). \]

In order to take the second derivative for space, setting \( r = |\xi'| \) and using the relation \( \Delta_{\xi'} = \partial_r^2 + \frac{n-2}{r} \partial_r \), we have by \( \hat{\phi}(\xi') = \hat{\phi}(|\xi'|) \) that
\[ \Delta_{\xi'} \frac{\partial}{\partial \sigma} \left[ \exp \left( -2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \right) \widehat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\xi') \right] \]
\[ = \Delta_{\xi'} \exp \left( -2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \right) \times \left\{ \frac{-2 \frac{k}{j} \eta \sigma(\sigma', \zeta'; 2^{2j-k}) \widehat{\psi}(\sigma) 2^k \sigma + \widehat{\psi}'(\sigma) 2^k \sigma + \widehat{\psi}(\sigma) 2^k \right\} \hat{\phi}(\xi'). \]
\[ + 2^{k \frac{1}{2}} n \left( -3(2^{2j-k} i) |\zeta'|^2 + 2^{2j-k} i p(\sigma, \zeta', 2^{2j-k})^2 \overline{\psi}(\sigma) 2^k \sigma \overline{\phi}(\zeta') \right) \]
\[ + 2^{k \frac{1}{2}} n \left( 2^{2j-k} i |\zeta'| \frac{2^k \psi(\sigma) 2^k \sigma \overline{\phi}(\zeta')}{|\zeta'|} \right) \]
\[ + (n - 2) \exp \left( - 2^{k \frac{1}{2}} n p(\sigma, \zeta'; 2^{2j-k}) \right) \]
\[ \times \left[ \left( -2^{k \frac{1}{2}} n \frac{2^{2j-k}}{p(\sigma, \zeta'; 2^{2j-k})} \overline{\phi}(\zeta') + \frac{\overline{\phi}'(\zeta')}{|\zeta'|} \right) \frac{-2^{k \frac{1}{2}} n i}{2p(\sigma, \zeta'; 2^{2j-k})} \psi(\sigma) 2^k \sigma \right] \]
\[ + \frac{2^{2j-k} i}{2p(\sigma, \zeta'; 2^{2j-k})^3} \psi(\sigma) 2^k \sigma \overline{\phi}(\zeta'). \]

Summarizing the results, all terms have
\[ \exp \left( -2^{k \frac{1}{2}} n i \sigma + 2^{2j-k} |\zeta'|^2 \right) = \exp \left( -2^{k \frac{1}{2}} n p(\sigma, \zeta'; 2^{2j-k}) \right) \]
amely \( \overline{\psi}(\sigma) \), \( \overline{\phi}(\zeta') \) or their derivatives and the order of derivatives are the order of partial derivatives with respect to \( \sigma \) and \( \zeta' \). And \( 2^{k \frac{1}{2}} n \) arises the same order of partial derivatives at most and arises one time at least. The function in denominator
\[ p(\sigma, \zeta'; 2^{2j-k}) = \sqrt{i \sigma + 2^{2j-k} |\zeta'|^2} \] is estimated as
\[ 2^{-1/2} \leq \left| p(\sigma, \zeta', 2^{2j-k}) \right| = (\sigma^2 + 2^{2(2j-k)} |\zeta'|^4)^{1/4} \leq 20^{1/4}, \]
thanks to the cutoff functions \( \overline{\psi}(\sigma) \) and \( \overline{\phi}(\zeta') \) or its derivative. Therefore, if we differentiate it \( n + 2 \) times, then the terms involving \( p(\sigma, \zeta'; 2^{2j-k}) \) are estimated from below by \( 2^{n+2} \). Moreover \( p(\sigma, \zeta'; 2^{2j-k}) \) which arises by each derivative contains the surplus scale parameter \( 2^{2j-k} \) and it is estimated from above by 1 because of the restriction \( k \geq 2j \). Thus for the functions
\[ k_{k,j}(\sigma, \zeta', \eta) \equiv \exp \left( -2^{k \frac{1}{2}} n p(\sigma, \zeta'; 2^{2j-k}) \right) \overline{\psi}(\sigma) 2^k \sigma \overline{\phi}(\zeta'), \]
\[ K_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^{2j-k}) \equiv \exp \left( 2^{k \frac{1}{2}} n p(\sigma, \zeta'; 2^{2j-k}) \right) D_{n,\sigma,\zeta'}^{n+2} k_{k,j}(\sigma, \zeta', \eta), \]
where we set
\[ D_{n,\zeta'}^{n+2} \equiv (1 - \Delta_{\zeta'} \frac{1}{2} \frac{\partial^2}{\partial \sigma^2}). \]
under the condition \( k \geq 2j \) namely \( 2^{2j-k} \leq 1 \), we obtain the following estimate: For \( \eta > 0 \)
\[ \| \Psi_{D,k}(\sigma, \zeta') \|_{\mathcal{L}_1^1} \]
\[ = c_{n+2}^{2n} \left( \frac{1}{(\eta')^2} \right)^{n+2} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{(2^{2j-k} \sigma + y' \zeta')} \exp \left( -2^{k \frac{1}{2}} n p(\sigma, \zeta'; 2^{2j-k}) \right) \]
\[ \times K_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^{j-k})d\zeta'd\sigma \bigg|_{L_{p'}}^1 \]

\[ = c_{n+1}^{k+1} \exp(-2^{k-1}) \eta \frac{2^k}{(2^k t)^2} \times \left| \left( \frac{1}{|Y'|^2} \right)^{\frac{k}{2}} \int_{R} \int_{R^{n-1}} e^{(i\xi i + y' \cdot \xi')} e^{-\frac{i}{\eta} \phi(\sigma, \zeta', 2^{j-k})} K_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^{j-k})d\zeta'd\sigma \right|_{L_{p'}}^1 \]

\[ \leq C2^k \exp(-2^{k-1}) \eta \frac{2^k}{(2^k t)^2} \times \int_{R} \int_{R^{n-1}} e^{-\frac{i}{\eta} \phi(\sigma, \zeta', 2^{j-k})} \left| K_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^{j-k}) \right| d\zeta'd\sigma \]

\[ \leq C2^k \exp(-2^{k-1}) \eta \left( 1 + C(2^k \eta)^{n+2} \right) \frac{2^k}{(2^k t)^2}, \]

arranging the suffixes we obtain (6.2).

Next we consider the case \( k \leq 2j \). Setting \( \eta = 2^{-\ell} \) and using the change of variables \( \tau = 2^k \sigma', \xi' = 2^j \xi' \)

\[ \| \Psi_{D,k,j}(t, \cdot) \|_{L_{p'}^1} \]

\[ = \left\| c_{n+1} \int_{R} \int_{R^{n-1}} e^{i(\tau + x \cdot \xi') \cdot \zeta'} \tau \exp \left( -\sqrt{i \tau + |\xi'|^2} \eta \right) \tilde{\psi}(2^-k \tau) \phi(2^{-j} \xi')d\zeta'd\tau \right\|_{L_{p'}^1} \]

\[ = \left\| c_{n+1} \int_{R} \int_{R^{n-1}} e^{i(\xi i + 2^j \sigma - 2^j \xi' \cdot \tau) \cdot \sigma} \right\|_{L_{p'}^1} \]

\[ \times \left\| \exp \left( -\sqrt{2^j i \sigma + 2^j |\xi'|^2} \eta \right) \tilde{\psi}(\sigma) \phi(2^j \xi') \right\|_{L_{p'}^1} \]

\[ = 2^k \left\| c_{n+1} \int_{R} \int_{R^{n-1}} e^{i(\xi i + 2^j \sigma - 2^j \xi' \cdot \tau) \cdot \sigma} \exp \left( -2^j \eta \sqrt{2^{k-2} i \sigma + |\xi'|^2} \right) \tilde{\psi}(\sigma) \phi(2^k \sigma) d\zeta'd\sigma \right\|_{L_{p'}^1, (B_{\xi'}^j)} \]

where we set \( \gamma' = 2^{-j} y' \) in the last equality. Noting the identity (6.4) and integrating by parts with respect to \( \sigma \) and \( \xi' \),

\[ \| \Psi_{D,k,j}(t, \cdot) \|_{L_{p'}^1, (B_{\xi'}^j)} \]

\[ = 2^k \left\| c_{n+1} \int_{R} \int_{R^{n-1}} \frac{1}{i \xi i \sigma} \exp \left( -2^j \eta \sqrt{2^{k-2} i \sigma + |\xi'|^2} \right) \tilde{\psi}(\sigma) \phi(2^k \sigma) d\zeta'd\sigma \right\|_{L_{p'}^1, (B_{\xi'}^j)} \]

Set \( q(\sigma, \xi'; 2^{k-2} j \sigma) = \sqrt{2^{k-2} j \sigma + |\xi'|^2} \). Then, we have

\[ \frac{\partial}{\partial \sigma} \left[ \exp \left( -2^j \eta q(\sigma, \xi'; 2^{k-2} j \sigma) \right) \tilde{\psi}(\sigma) 2^k \sigma \phi(\xi') \right] \]

\[ = \exp \left( -2^j \eta q(\sigma, \xi'; 2^{k-2} j \sigma) \right) \left\{ -2^j \eta \frac{2^{k-2} j \sigma}{2q(\sigma, \xi'; 2^{k-2} j \sigma)} \tilde{\psi}(\sigma) 2^k \sigma \right. \]

\[ + \tilde{\psi}(\sigma) 2^k \sigma + \hat{\psi}(\sigma) 2^k \right\} \phi(\xi'). \]
In order to take second derivative with respect to space, setting \( r = |\zeta'| \) and using the relation \( \Delta_{\zeta'} = \partial_r^2 + \frac{n-2}{r^2} \partial_r \), we have

\[
\Delta_{\zeta'} \frac{\partial}{\partial \sigma} \left[ \exp \left( -2j \eta \sqrt{2k^{-2}j i \sigma + |\zeta'|^2} \right) \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') \right] = \Delta_{\zeta'} \exp \left( -2j \eta q(\sigma, \zeta'; 2k^{-2}j) \right) \left\{ \frac{2k^{2}j i}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') + \hat{\psi}'(\sigma) 2k^{\frac{1}{2}} \hat{\phi}(\zeta') \right\} \\
= \left( \partial_r + \frac{n-2}{r} \right) \exp \left( -2j \eta q(\sigma, \zeta'; 2k^{-2}j) \right) \\
\times \left\{ \left( -\frac{2j \eta |\zeta'|}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}(\zeta') + \hat{\phi}'(\zeta') \right) \left( -\frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') + \hat{\psi}'(\sigma) 2k^{\frac{1}{2}} + \hat{\psi}(\sigma) 2k^{\frac{1}{2}} \right) \right\} \\
+ \frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') \\
+ \left( -\frac{2j \eta}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}(\zeta') + \frac{2j \eta |\zeta'|}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}'(\zeta') \right) \\
\times \left( -\frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}'(\zeta') - \frac{2j \eta |\zeta'|}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}'(\zeta') + \hat{\phi}''(\zeta') \right) \\
\times \left( -\frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}'(\zeta') + \hat{\psi}'(\sigma) 2k^{\frac{1}{2}} \hat{\phi}(\zeta') \right) \\
+ \left( -\frac{2j \eta}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}'(\zeta') \right) \left( \frac{2j \eta 2k^{-2}j |\zeta'|}{2q(\sigma, \zeta'; 2^{2j^{-2}j}k)^{3}} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \right) \\
+ \frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') \\
\times \left( -\frac{2j \eta}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}(\zeta') + \frac{2j \eta |\zeta'|}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}'(\zeta') \right) \\
\times \left( -\frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') + \hat{\psi}'(\sigma) 2k^{\frac{1}{2}} \hat{\phi}(\zeta') \right) \\
\times \left( -\frac{2j \eta}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}(\zeta') + \frac{2j \eta |\zeta'|}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}'(\zeta') \right) \\
\times \left( -\frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') + \hat{\psi}'(\sigma) 2k^{\frac{1}{2}} \hat{\phi}(\zeta') \right) \\
\times \left( -\frac{2j \eta}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}(\zeta') \right) \left( \frac{2j \eta 2k^{-2}j |\zeta'|}{2q(\sigma, \zeta'; 2^{2j^{-2}j}k)^{5}} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \hat{\phi}(\zeta') \right) \\
+ (n-2) \exp \left( -2j \eta q(\sigma, \zeta'; 2k^{-2}j) \right) \\
\times \left\{ \left( -\frac{2j \eta}{q(\sigma, \zeta'; 2k^{-2}j)} \hat{\phi}(\zeta') + \frac{\hat{\phi}'(\zeta')}{|\zeta'|} \right) \left( -\frac{2j \eta}{2q(\sigma, \zeta'; 2k^{-2}j)} \hat{\psi}(\sigma)^2k^{\frac{1}{2}} \right) \right\}
\]
which is estimated from above by 1 by $k$

As the case for $k > 2j$ before, we may summarize that all terms are including

$$
\exp \left( -2^j \eta \sqrt{2^{k-2}j i \sigma + |\zeta'|^2} \right) = \exp \left( -2^j \eta q(\sigma, \zeta'; 2^{k-2}j) \right)
$$

$\hat{\psi}(\sigma)$, $\hat{\phi}(\zeta')$ or their derivatives, and the order of derivatives is the same as the order of partial derivatives of $\sigma$ and $\zeta'$. And $2^j \eta$ is multiplied by the same order of the partial derivatives at most, and one time at least. In the denominator, the function $q(\sigma, \zeta'; 2^{k-2}j) = \sqrt{2^{k-2}j i \sigma + |\zeta'|^2}$ is estimated from below by $2^{-1}$ thanks to the cutoff function $\hat{\phi}(\zeta')$ or its derivative when $j \geq 1$. Thus, if we derive it $n + 2$ times, the terms involving $q(\sigma, \zeta'; 2^{k-2}j)$ are estimated from below by $2^{n+2}$. Moreover, $q(\sigma, \zeta'; 2^{k-2}j)$ which arises by derivative contains surplus scale parameter $2^{k-2}j$ which is estimated from above by 1 by $k < 2j$. Setting

$$
\frac{\hat{\psi}(\sigma) k}{2g(\sigma, \zeta'; 2^{k-2}j)} \frac{2^{k-2}j}{3}
$$

we obtain

$$
\left\| \Psi_{D,k,j}(t, \cdot) \right\|_{L^1_\eta} = c_{n+1} 2^k \left( \frac{1}{|\gamma'|^2} \right)^\frac{n}{2} \frac{2^n}{(2^k t)^\frac{n}{2}} \int _{\mathbb{R}^{n+1}} e^{i(2^k t \sigma + y' \cdot \zeta')}(1
$$

$$
- \Delta \zeta' \left( 1 - \frac{\partial^2}{\partial \sigma^2} \right) h_{k,j}(\sigma, \zeta', \eta) d \zeta' d \sigma \right\|_{L^1_\eta}
$$

$$
= c_{n+1} 2^k \left( \frac{1}{|\gamma'|^2} \right)^\frac{n}{2} \frac{2^n}{(2^k t)^\frac{n}{2}} \int _{\mathbb{R}^{n+1}} e^{i(2^k t \sigma + y' \cdot \zeta')}(1 - \Delta \zeta') \left( 1 - \frac{\partial^2}{\partial \sigma^2} \right) h_{k,j}(\sigma, \zeta', \eta) d \zeta' d \sigma \right\|_{L^1_\eta}
$$

$$
= c_{n+2} 2^n \eta^2 \exp \left( -2j - 1 \eta \right) \frac{2^n}{(2^k t)^\frac{n}{2}} \times \left\| \left( \frac{1}{|\gamma'|^2} \right)^\frac{n}{2} \int _{\mathbb{R}^{n+1}} e^{i(2^k t \sigma + y' \cdot \zeta')}(1 - \Delta \zeta') \left( 1 - \frac{\partial^2}{\partial \sigma^2} \right) h_{k,j}(\sigma, \zeta', \eta) d \zeta' d \sigma \right\|_{L^1_\eta}
$$

$$
= c_{n+1} 2^k \eta^2 \exp \left( -2j - 1 \eta \right) \frac{2^n}{(2^k t)^\frac{n}{2}} \times \left\| \left( \frac{1}{|\gamma'|^2} \right)^\frac{n}{2} \int _{\mathbb{R}^{n+1}} e^{i(2^k t \sigma + y' \cdot \zeta')}(1 - \Delta \zeta') \left( 1 - \frac{\partial^2}{\partial \sigma^2} \right) h_{k,j}(\sigma, \zeta', \eta) d \zeta' d \sigma \right\|_{L^1_\eta}
$$
\[
\leq C2^k \exp(-2^{-j-1} \eta) \frac{2^k}{(2^k t)^2} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{\left(-2^j \eta\left(\eta^2 + 2^{k-2j}\right)\right)} \left| H^{n+2}_{k,j}(\sigma, \zeta, \eta, 2^{k-2j})\right| d\zeta d\sigma
\]
\[
\leq C2^k \exp(-2^{-j-1} \eta) (1 + C(2^{j} \eta)^n+2) \frac{2^k}{(2^k t)^2}.
\]
which completes the proof of Lemma 6.1.

6.2. The second orthogonal estimate

For the estimating the term \( P_D^2 \) in (4.21), it involving an \( \eta \)-convolution between the potential \( \Psi_D \) and \( \phi_m(\eta) \). We show the following second orthogonal estimate:

**Lemma 6.2.** (Potential orthogonality 2) Let \( k, j, m \in \mathbb{Z} \) and assume \( j \leq m + 1 \). Let \( \Psi_D(t, x', \eta) \) be the potential of the solution for the Dirichlet data defined by (3.11) and let \{\( \psi_k(t) \)\}_{k \in \mathbb{Z}} \text{and} \{\phi_j(x)\}_{j \in \mathbb{Z}} \) be a spatial and time Littlewood-Paley decomposition. Let \( \Psi_{D,k,j}(t, x', \eta) \) be defined by (6.1). Then for any \( N \in \mathbb{N} \), there exists a constant \( C_N > 0 \) such that

\[
\| (\phi_m * \Psi_{D,k,j})(t, \cdot, \eta) \|_{L^{1}_{x'}} \leq \begin{cases} 
C N 2^k \frac{2^{-|\frac{j}{2}-m|}}{(2 \min(\frac{j}{2}, m) \eta)^N} \frac{2^k}{(2^k t)^2}, & k \geq 2j, \\
C N 2^k \frac{2^{-|j-m|}}{(2^j \eta)^N} \frac{2^k}{(2^k t)^2}, & k < 2j.
\end{cases}
\]

**Proof of Lemma 6.2.** Assuming \( 2j < k \), we show the \( t \)(time) dominated estimate. Changing \( \tau = 2^k \sigma, \xi' = 2^j \xi' \), as we have seen in (6.3), it also follows that

\[
\| (\phi_m * \Psi_{D,k,j})(t, \cdot, \eta) \|_{L^{1}_{x'}} = \left\| c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i(2^k t_1 + 2^j x' \cdot \xi') i 2^k} \left( \phi_m * \exp \left(-\sqrt{i \sigma + 2^j |\xi'|^2 \eta}\right)\right) \widehat{\psi}(2^{-k} \tau) \widehat{\phi}(2^{-j} \xi') d\xi' d\tau \right\|_{L^{1}_{x'}}
\]
\[
= c_{n+1} \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i(2^k t_1 + 2^j x' \cdot \xi') i 2^k \sigma} \times \left( \phi_m * \exp \left(-\sqrt{i \sigma + 2^j |\xi'|^2 \eta}\right)\right) \widehat{\psi}(\sigma) \widehat{\phi}(\xi') 2^{(n-1)j} d\xi' \cdot 2^k d\sigma \right\|_{L^{1}_{x'}}
\]
\[
= c_{n+1} \left\| \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i(2^k t_1 + 2^j x' \cdot \xi') i 2^k \sigma} \times \left( \phi_m * \exp \left(-\sqrt{i \sigma + 2^j |\xi'|^2 \eta}\right)\right) \sigma \widehat{\psi}(\sigma) \widehat{\phi}(\xi') d\xi' \cdot 2^k d\sigma \right\|_{L^{1}_{x'}}.
\]
Then using
\[ e^{i(2^t \sigma + y')\zeta'} = -\frac{1}{i2^t} \frac{1}{|y'|^2} \Delta_{\zeta'} \partial_\alpha e^{i(2^t \sigma + y')\zeta'} \]
we apply the integration by parts by \( \sigma, \zeta' \) as before. Then, the following lemma is a crucial step.

**Lemma 6.3.** Let \( k, j, m \in \mathbb{Z} \) and assume \( j < m + 1 \) and \( 2^{-1} \leq \sigma, |\zeta'| \leq 2 \). For \( \alpha = \alpha_1 + \alpha_2, 0 \leq \alpha_1 \leq 1, 0 \leq \alpha_2 \leq n \), the following estimates hold:

\[
\left| \phi_m \left( \eta \right) \left( \sum_{|\alpha| \leq n+2} C_{n,\alpha} \partial_\alpha^{\alpha_1} \partial_{\zeta'}^{\alpha_2} \exp \left( - 2^{\frac{k}{2}} \eta i \sigma + 2^j \zeta'^{2} \right) \right) \right| \leq \frac{C N^{2-|\frac{j}{2} - m|}}{(2^{\min\{\frac{k}{2}, m\}} \eta)^N}, \quad k \geq 2j,
\]

\[
\left| \phi_m \left( \eta \right) \left( \sum_{|\alpha| \leq n+2} C_{n,\alpha} \partial_\alpha^{\alpha_1} \partial_{\zeta'}^{\alpha_2} \exp \left( - 2^{\frac{j}{2}} \eta i 2^j \sigma + |\zeta'|^2 \right) \right) \right| \leq \frac{C N^{2-|j - m|}}{(2 \eta)^N}, \quad k < 2j.
\]

**Proof of Lemma 6.3.** We first show the lemma for the case of \( k > 2j \). For \( p(\sigma, \zeta', 2^{j-k}) = \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \), let

\[
P_\alpha(\sigma, \zeta', 2^{j/2} \eta) = \exp \left( 2^{\frac{k}{2}} \eta \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \right) \sum_{|\alpha| \leq n+2} C_{n,\alpha} \partial_\alpha^{\alpha_1} \partial_{\zeta'}^{\alpha_2} \exp \left( - 2^{\frac{\eta}{2}} \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \right).
\]

Then, \( P_\alpha \) is a polynomial of \( 2^{\frac{k}{2}} \eta \) and \( |p(\sigma, \zeta', 2^{j-k})|^{-1} = \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \) and by \((6.5)\), it holds that

\[
|P_\alpha(\sigma, \zeta', 2^{j/2} \eta)| \leq C \left( 1 + (2^{\frac{k}{2}} \eta)^{n+2} \right).
\]

We also set

\[
\begin{cases}
\tilde{\theta} = 2^{\frac{k}{2}} \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \theta = 2^{\frac{k}{2}} p \theta, \\
\tilde{\eta} = 2^{\frac{k}{2}} \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \eta = 2^{\frac{k}{2}} p \eta.
\end{cases}
\]

**Step 1:** The case \( k \geq 2m \geq 2j \). Using \((6.5)\),

\[
\left| \int_{\mathbb{R}^+} \phi_m(\eta - \theta)\left(P_\alpha(\sigma, \zeta', 2^{k/2} \eta) \exp \left( - 2^{\frac{k}{2}} \theta \sqrt{i \sigma + 2^{j-k} |\zeta'|^2} \right) \right) d\theta \right|
\]

\[
= \left| \int_{\mathbb{R}^+} 2^{2m} \phi(2m 2^{-\frac{k}{2}} \sqrt{i \sigma + 2^{j-k} |\zeta'|^2}^{-1} \tilde{\eta} - \tilde{\theta}) \tilde{P}_\alpha(\tilde{\theta}) \exp \left( - \tilde{\theta} 2^{-\frac{k}{2}} \sqrt{i \sigma + 2^{j-k} |\zeta'|^2}^{-1} \tilde{\theta} \right) d\tilde{\theta} \right|
\]

\[
\leq C 2^{\frac{2m-k}{2}} \int_{\mathbb{R}} \phi(2m 2^{-\frac{k}{2}} \sqrt{i \sigma + 2^{j-k} |\zeta'|^2}^{-1} \tilde{\eta} - \tilde{\theta}) \tilde{P}_\alpha(\tilde{\theta}) \exp \left( - |\tilde{\theta}| \right) d\tilde{\theta}
\]
Hence by (6.12), (6.14) and (6.15), we obtain (6.10); where we set \( \tilde{P}_\alpha(\tilde{\theta}) \) as a polynomial of \( 2^\frac{k}{2}\tilde{\theta} \) and has the following estimate form (6.10);
\[
|\tilde{P}_\alpha(\tilde{\theta})| \leq C_\alpha (1 + |2^\frac{k}{2}\tilde{\theta}|)^{n+2}
\] under the restrictions \( 2^{-1} \leq \sigma < 2, 2^{-1} < |\zeta'| \leq 2 \) and \( k \geq 2j \).

Then, the first term of the right hand side of (6.12) is estimated by using (6.13) as follows:
\[
I \leq C_2^{m-\frac{k}{2}} \exp \left( -\frac{1}{4}|\tilde{\eta}| \right) \int_{|\tilde{\theta}| > \frac{1}{2}|\tilde{\eta}|} \frac{C_N}{\left(2^\frac{k}{2}|i\sigma + 2^j\zeta'|^2|\eta| \right)^N} \tilde{P}_\alpha(\tilde{\theta}) \exp \left( -\frac{1}{2}|\tilde{\theta}| \right) d\tilde{\theta}
\]
\[
\leq C_2^{m-\frac{k}{2}} \frac{C_N}{\left(2^\frac{k}{2}|i\sigma + 2^j\zeta'|^2|\eta| \right)^N} \int_{|\tilde{\theta}| > \frac{1}{2}|\tilde{\eta}|} \tilde{P}_\alpha(\tilde{\theta}) \exp \left( -\frac{1}{2}|\tilde{\theta}| \right) d\tilde{\theta}
\]
\[
\leq C_2^{m-\frac{k}{2}} \frac{C_N2^{m-\frac{k}{2}}}{(2^\frac{k}{2}|\eta|)^N} \leq C_N2^{m-\frac{k}{2}} \frac{C_N2^{m-\frac{k}{2}}}{(2^m\eta)^N}.
\]

by \( k \geq 2m \). For the second term in (6.12), we note that \( |\tilde{\theta}| < \frac{1}{2} |\tilde{\eta}| \) implies \( |\tilde{\eta} - \tilde{\theta}| \geq |\tilde{\eta}| - |\tilde{\theta}| \geq \frac{1}{2} |\tilde{\eta}| = \frac{1}{2} |\tilde{\eta}| \), and it follows that
\[
II \leq C_2^{m-\frac{k}{2}} \int_{|\tilde{\theta}| \leq \frac{1}{2}|\tilde{\eta}|} \frac{C_N}{\left(2^{m-1}\frac{k}{2}|i\sigma + 2^{j-k}\zeta'|^2|\eta| \right)^N} \tilde{P}_\alpha(\tilde{\theta}) \exp \left( -|\tilde{\theta}| \right) d\tilde{\theta}
\]
\[
\leq C_2^{m-\frac{k}{2}} \frac{C_N2^{m-\frac{k}{2}}}{(2^{m-1}\frac{k}{2}|\eta|)^N} \int_{\mathbb{R}} \tilde{P}_\alpha(\tilde{\theta}) \exp \left( -|\tilde{\theta}| \right) d\tilde{\theta}
\]
\[
\leq C_2^{m-\frac{k}{2}} \frac{C_N2^{m-\frac{k}{2}}}{(2^{m-1}\frac{k}{2}|\eta|)^N}.
\]

Hence by (6.12), (6.14) and (6.15), we obtain
\[
\left| \int_{\mathbb{R}^+} \phi_m(\eta - \theta) \left( P_\alpha(\tau, \zeta, 2^{k/2}\theta) \exp \left( -2^\frac{k}{2}\theta \sqrt{i\sigma + 2^j\zeta'|^2} \right) \right) d\theta \right| \leq C_2^{m-\frac{k}{2}} \frac{C_N2^{m-\frac{k}{2}}}{(2^{m-1}\frac{k}{2}|\eta|)^N}.
\]
and

\[
\left| \int_{\mathbb{R}^+} \phi_m(\theta) \left( P_\alpha(\tau, \xi', 2^{k/2}(\eta - \theta)) \exp \left( -2^{k \frac{1}{2}} (\eta - \theta) \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right) \right) d\theta \right|
\]

\[
= \left| \int_{\mathbb{R}^+} \phi_m(\theta) \left( P_\alpha(\tau, \xi', 2^{k/2}(\eta - \theta)) \exp \left( -2^{k \frac{1}{2}} (\eta - \theta) \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right) \right.
\]

\[
- P_\alpha(\tau, \xi', 2^{k/2}\eta) \exp \left( -2^{k \frac{1}{2}} \eta \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right) \right) d\theta \right|
\]

\[
= \left| \int_{\mathbb{R}^+} \phi_m(\theta) \left( \int_0^1 \frac{d}{dv} P_\alpha(\tau, \xi', 2^{k/2}(\eta - v\theta)) \right.
\]

\[
\times \exp \left( -2^{k \frac{1}{2}} (\eta - v\theta) \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right) d\theta dv \right|
\]

\[
\leq \left| \int_0^1 \int_{|\bar{\theta}|} \left| \phi(2m \frac{\tau}{2^{k \frac{1}{2}}} \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right| ^{-1} \left| \left( \eta - v\bar{\theta} \right) \right| ^{2m \frac{\kappa}{2} \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right| ^{-1} \left| \bar{\theta} \right| \right.
\]

\[
\times \tilde{P}_\alpha(\tau, \xi', (\eta - v\bar{\theta})) \exp \left( - |\eta - v\bar{\theta}| \right) d\bar{\theta} dv \left. \right|
\]

\[
+ \int_0^1 \int_{|\bar{\theta}| \leq \frac{1}{2} |\bar{\theta}|} \left| \phi(2m \frac{\tau}{2^{k \frac{1}{2}}} \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right| ^{-1} \left| \left( \eta - v\bar{\theta} \right) \right| ^{2m \frac{\kappa}{2} \sqrt{i\sigma + 2^{j-\kappa} |\xi'|^2} \right| ^{-1} \left| \bar{\theta} \right| \right.
\]

\[
\times \tilde{P}_\alpha(\tau, \xi', (\eta - v\bar{\theta})) \exp \left( - |\eta - v\bar{\theta}| \right) d\bar{\theta} dv \left. \right|
\]

\[
\equiv III + IV.
\]

(6.17)

where we use the variables defined in (6.11) and set

\[
\tilde{P}_\alpha(\tau, \xi', (\eta - v\bar{\theta})) = P_\alpha(\tau, \xi', 2^{k/2}(\eta - \theta)) - \partial_\mu P_\alpha(\tau, \xi', \mu) \bigg|_{\mu = 2^{k \frac{1}{2}} (\eta - v\theta)}
\]

\[
= P_\alpha(\tau, \xi', |p|^{-1}(\eta - v\bar{\theta})) - \partial_\mu P_\alpha(\tau, \xi', \mu) \bigg|_{\mu = |p|^{-1}(\eta - v\bar{\theta})}.
\]

Since \( \phi \) is rapidly decreasing function, for any \( N \in \mathbb{N} \), there exists a constant \( C_N > 0 \) and the first term of the right hand side of (6.17) is
\( I \leq \int_0^1 \int \frac{C_N|2^m2^{\frac{-k}{2}}\sqrt{i\sigma + 2^j \cdot |\xi'|^2\cdot \bar{\theta}|}}{(2^m2^{\frac{-k}{2}}\sqrt{i\sigma + 2^j \cdot |\xi'|^2\cdot \bar{\theta})^N} \int_0^1 \int_{|\bar{\theta}| < \frac{1}{2}|\bar{\theta}|} \left| \bar{P}_\alpha(\tau, \xi', (\bar{\eta} - v\bar{\theta})) \right| d\bar{\theta} d\nu \)

\( \leq C_N 2^{-m+\frac{k}{2}} \int_0^1 \int \frac{2^m2^{\frac{-k}{2}}\sqrt{i\sigma + 2^j \cdot |\xi'|^2\cdot \bar{\theta}|}}{(2^m2^{\frac{-k}{2}}\sqrt{i\sigma + 2^j \cdot |\xi'|^2\cdot \bar{\theta})^N} \int_0^1 \int_{|\bar{\theta}| < \frac{1}{2}|\bar{\theta}|} \left| \bar{P}_\alpha(\tau, \xi', (\bar{\eta} - v\bar{\theta})) \right| d\bar{\theta} d\nu \)

\( \leq C_N 2^{-m+\frac{k}{2}} \int_0^1 \int \frac{1}{\bar{\theta}} \exp \left( -2^{-1}2^{-m+\frac{k}{2}}|p|\bar{\eta} - v\bar{\theta}) \right) d\bar{\theta} d\nu \)

\( \leq C_N 2^{-m+\frac{k}{2}} \int_0^1 \int \frac{1}{\bar{\theta}} \exp \left( -2^{-1}2^{-m+\frac{k}{2}}|p|\bar{\eta} - v\bar{\theta}) \right) d\bar{\theta} d\nu \)

For the second term in (6.17), we prepare the following simple estimate:

**Lemma 6.4.** For \( N = 2, 3, \ldots \) and \( a > 0 \),

\[
\int_{|x| \leq a} \frac{dx}{(1 + |x|^2)^N} \leq \frac{\sqrt{2\pi} a}{(1 + |a|^2)^{1/2}}.
\]

**Proof of Lemma 6.4.** Applying the integration by parts,

\[
\int_{|x| \leq a} \frac{dx}{(1 + |x|^2)^N} \leq \int_{|x| \leq a} \frac{dx}{1 + |x|^2} = \frac{2\tan^{-1} a}{(1 + |a|^2)^{1/2}}
\]

while

\[
\int_{|x| \leq a} \frac{dx}{(1 + |x|^2)^{N/2}} \leq \frac{dx}{1 + |x|^2} = \frac{2\tan^{-1} a}{(1 + |a|^2)^{1/2}}
\]

This shows the estimate. □

Using \( |\bar{\eta} - v\bar{\theta}| \geq |\bar{\eta}| - |\bar{\theta}| \geq \frac{1}{2}|\bar{\eta}| \) under the restriction \( |\bar{\theta}| \leq \frac{1}{2}|\bar{\eta}| \) and the estimates (6.5), (6.10) and (6.19),
\[ \leq C_N 2^{-m+\frac{k}{2}} \exp \left( -2^{-2} 2^{-m+\frac{k}{2}} |p| |\bar{\eta}| \right) \int_{|\bar{\theta}| < \frac{1}{2} |\bar{\eta}|} \frac{1}{\langle \bar{\eta} \rangle^N} d\bar{\theta} \]
\[ \leq C_N 2^{-m+\frac{k}{2}} \exp \left( -2^{-2} 2^{-m+\frac{k}{2}} |p| |\bar{\eta}| \right) \frac{|\bar{\eta}|}{(1 + |\bar{\eta}|^2)^{1/2}} \]
\[ \leq C_N 2^{-m+\frac{k}{2}} \exp \left( -2^{-2} |\bar{\eta}| \right) \]
\[ \leq \frac{C_N 2^{-m+\frac{k}{2}}}{(2^{\frac{k}{2}} \eta)^{N}} \quad \text{for } N \geq 2. \]  

(6.20)

Hence, we obtain from (6.18) and (6.20) that
\[ \left| \int_{\mathbb{R}^+} \phi_m(\eta - \theta) \left( P_\alpha(\tau, \xi', 2^{k/2} \theta) \exp \left( -2^k \theta \sqrt{i\sigma + 2^{2j-k} |\xi'|^2} \right) \right) d\theta \right| \leq \frac{C_N 2^{-m+\frac{k}{2}}}{(2^{\frac{k}{2}} \eta)^{N}}. \]

(6.21)

The estimates (6.16) and (6.21) yield (6.7).

\langle Step 3 \rangle  \quad \text{The case } k < 2j:

To show (6.8), we use \( q(\sigma, \zeta', 2^{2j-k}) \equiv \sqrt{i2^{k-2j} \sigma + |\zeta'|^2} \) instead of (6.9) and let
\[ Q_\alpha(\sigma, \zeta', \eta, 2^j) = \exp \left( 2^j \eta \sqrt{i2^{k-2j} \sigma + |\zeta'|^2} \right) \sum_{|\alpha| \leq n+2} C_{n, \alpha} \partial^{\alpha_1}_\sigma \theta^{\alpha_2}_{\zeta'} \exp \left( -2^j \eta \sqrt{i2^{k-2j} \sigma + |\zeta'|^2} \right). \]

Since \( |Q_\alpha| \) is a polynomial of \( \eta \) and \( |q(\sigma, \zeta', 2^{2j-k})|^{-1} \), it follows from the assumption that
\[ 2^{-1} \leq |q(\sigma, \zeta', 2^{k-2j})| = \left( 2^{2(k-2j)} \sigma^2 + |\zeta'|^4 \right)^{1/4} \leq 20^{1/4}. \]

Hence as in the previous step,
\[ |Q_\alpha(\tau, \xi', 2^j \eta)| \leq C \left( 1 + (2^j \eta)^{n+2} \right). \]

All the other estimate is very similar to the case of \( \langle Step 2 \rangle \) and all the terms involving \( 2^{\frac{k}{2}} \) are arranged into \( 2^j \). The estimate corresponding to the case \( \langle Step 1 \rangle \) is redundant by the assumption \( j < m \). \( \square \)

**Proof of Lemma 6.2, continued.** The proof of Lemma 6.2 goes in a similar way to the case of Lemma 6.1. After integrating by parts, the integrable factors \( \langle r \rangle^{-2} \) and \( \langle \gamma' \rangle^{-n} \) appear and then estimate the integrant to obtain the desired estimate in Lemma 6.3 for the case \( 2j < k \). The other case \( 2j \geq k \) is also obtained from (6.8) in Lemma 6.3. This complete the proof of Lemma 6.2. \( \square \)

6.3. The Neumann potential case

The almost orthogonal estimate for the Neumann boundary potential is very similar to the case of Dirichlet potential case except the order of the derivative. The following lemma shows the estimates (5.6) hold valid.
Lemma 6.5 (A crucial potential orthogonality). For \( k, j, \ell \in \mathbb{Z} \) let \( \{\psi_k(t)\}_{k \in \mathbb{Z}} \) and \( \{\phi_j(x)\}_{j \in \mathbb{Z}} \) be the time and the space Littlewood-Paley dyadic decomposition and let \( \Psi_N(t, x', \eta) \) be the boundary potential defined in (3.20). Set

\[
\Psi_{N,k,j}(t, x', \eta) \equiv \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} \Psi_N(t - s, x' - y', \eta) \psi_k(s) \phi_j(y') dy'ds
\]

for \( \eta = x_n \in I_\ell = [2^{-\ell}, 2^{-\ell+1}) \). Then, there exists a constant \( C_n > 0 \) depending only on the dimension \( n \) satisfying

\[
\|\Psi_{N,k,j}(t, \cdot, \eta)\|_{L^1_t} \leq \begin{cases} 
C_n 2^{2j} (1 + (2^j \eta)^{n+2}) \frac{2^k}{(2^k t)^2}, & k \geq 2j, \\
C_n 2^{2j} (1 + (2^j \eta)^{n+2}) \frac{2^k}{(2^k t)^2}, & k < 2j,
\end{cases}
\]

(6.22)

and

\[
\|\phi_m * \Psi_{N,k,j}(t, \cdot, \eta)\|_{L^1_t} \leq \begin{cases} 
C_n 2^{2j} \frac{2^{-|\frac{j}{2}-m|}}{\langle \min(\frac{j}{2}, m) \rangle^{N}} \frac{2^k}{(2^k t)^2}, & k \geq 2j, \\
C_n 2^{2j} \frac{2^{-|j-m|}}{\langle 2^j \eta \rangle^{N}} \frac{2^k}{(2^k t)^2}, & k < 2j.
\end{cases}
\]

(6.23)

The proof for Lemma 6.5 is shown in a parallel way to the proof of Lemma 6.1. The only difference stems from the difference of two potentials \( \Psi_D \) and \( \Psi_N \) of boundary data and the difference from (6.3) reflects the order of spatial derivatives appearing the Fourier image of the following expression:

\[
\Psi_{N,k,j}(t, x', \eta)
\]

\[
= -c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i\tau + ix' \cdot \xi'} \frac{i\tau}{\sqrt{i\tau + |\xi'|^2}} \exp \left( -\sqrt{i\tau + |\xi'|^2} \eta \right) \tilde{\psi}(2^{-k} \tau) \tilde{\phi}(2^{-j} \xi') d\xi' d\tau.
\]

Proof of Lemma 6.5. We only show the out-lined proof of (6.22). The other estimate (6.23) follows very much in a similar way to the proof of the Dirichlet case in Lemma 6.2.

We consider a time-like estimate \( k \geq 2j \) as is in the Dirichlet case. Taking \( \xi' \)-space cutoff, we have by using the change of variables \( \tau = 2^k \sigma, \xi' = 2^j \xi' \) that

\[
\|\Psi_{N,k,j}(t, \cdot, \eta)\|_{L^1_t} \leq \left| c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i\tau + ix' \cdot \xi'} \frac{i\tau}{\sqrt{i\tau + |\xi'|^2}} \exp \left( -\sqrt{i\tau + |\xi'|^2} \eta \right) \tilde{\psi}(2^{-k} \tau) \tilde{\phi}(2^{-j} \xi') d\xi' d\tau \right|_{L^1_t}
\]

\[
= 2^j \left| c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^{n-1}} e^{i2^k \sigma + iy' \cdot \xi'} \frac{\sigma}{\sqrt{i\sigma + 2^2 j^{-k} |\xi'|^2}} \exp \left( -2^j \eta \sqrt{i\sigma + 2^2 j^{-k} |\xi'|^2} \right) \tilde{\phi}(\sigma) \tilde{\phi}(\xi') d\xi' \cdot 2^k d\sigma \right|_{L^1_{\xi'}}
\]
where we set \( x' = 2^{-j} y' \) in the last line. Using (6.4) and integrating by parts, we see by setting \( p(\sigma, \zeta'; 2^{2j-k}) \equiv \sqrt{i\sigma + 2^{2j-k}|\zeta'|^2} \) that

\[
\|\Psi_{N,k,j}(t, \cdot; \eta)\|_{L^1_t(B^n_j)} = 2^{\frac{1}{2}} \left[ c_{n+1} \int_{\mathbb{R}} \int_{\mathbb{R}^n-1} \frac{1}{i2^k t} \left( -\frac{1}{|y'|^2} \right) e^{i(2^k t\sigma + y' \cdot \zeta')} \right] x \frac{\partial}{\partial \sigma} \left[ \frac{1}{p(\sigma, \zeta'; 2^{2j-k})} \exp \left( -2^{\frac{k}{2}} \eta p(\sigma, \zeta'; 2^{2j-k}) \right) \hat{\psi}(\sigma) \hat{\phi}(\zeta') 2^k \sigma \right] d\zeta' d\sigma \bigg|_{L^1_t(B^n_j)}.
\]

Since

\[
\frac{\partial}{\partial \sigma} \left[ \frac{1}{p(\sigma, \zeta'; 2^{2j-k})} \exp \left( -2^{\frac{k}{2}} \eta p(\sigma, \zeta'; 2^{2j-k}) \right) \hat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\zeta') \right] = \frac{1}{p(\sigma, \zeta'; 2^{2j-k})} \exp \left( -2^{\frac{k}{2}} \eta p(\sigma, \zeta'; 2^{2j-k}) \right) \frac{\partial}{\partial \sigma} \left[ \hat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\zeta') \right]
\]

\[
\times \left\{ -\frac{2^{\frac{k}{2}} \eta i}{2p(\sigma, \zeta'; 2^{2j-k})} \hat{\psi}(\sigma) 2^k \sigma - \frac{i}{2p(\sigma, \zeta'; 2^{2j-k})^2} \hat{\psi}(\sigma) 2^k \sigma \right. \\
+ \left. \hat{\psi}'(\sigma) 2^k \sigma + \hat{\psi}(\sigma) 2^k \right\} \hat{\phi}(\zeta').
\]

we take second derivative by setting \( r = |\zeta'| \) and \( \Delta \zeta' = \partial_r^2 + \frac{n-2}{r} \partial_r \) to have

\[
\Delta \zeta' \frac{\partial}{\partial \sigma} \left[ \frac{1}{p(\sigma, \zeta'; 2^{2j-k})} \exp \left( -2^{\frac{k}{2}} \eta p(\sigma, \zeta'; 2^{2j-k}) \right) \hat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\zeta') \right] = \left( \partial_r + \frac{n-2}{r} \right) \exp \left( -2^{\frac{k}{2}} \eta p(\sigma, \zeta'; 2^{2j-k}) \right) p(\sigma, \zeta'; 2^{2j-k})
\]

\[
\times \left[ \left( -\frac{2^{\frac{k}{2}} \eta 2^{2j-k} |\zeta'|}{p(\sigma, \zeta'; 2^{2j-k})} \hat{\phi}(\zeta') - \frac{2^{2j-k} |\zeta'|}{p(\sigma, \zeta'; 2^{2j-k})^2} \hat{\phi}(\zeta') + \hat{\phi}'(\zeta') \right) \right]
\]

\[
\times \left[ \left( -\frac{2^{\frac{k}{2}} \eta i}{2p(\sigma, \zeta'; 2^{2j-k})} \hat{\psi}(\sigma) 2^k \sigma - \frac{i}{2p(\sigma, \zeta'; 2^{2j-k})^2} \hat{\psi}(\sigma) 2^k \sigma \right. \\
+ \left. \hat{\psi}'(\sigma) 2^k \sigma + \hat{\psi}(\sigma) 2^k \right) \right.
\]

\[
+ \left. \left( \frac{2^{\frac{k}{2}} \eta 2^{2j-k} |\zeta'|}{2p(\sigma, \zeta'; 2^{2j-k})^3} + \frac{2^{2j-k} |\zeta'|}{p(\sigma, \zeta'; 2^{2j-k})^4} \right) \hat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\zeta') \right].
\]

As the Dirichlet boundary case, all terms again have \( \exp \left( -2^{\frac{k}{2}} \eta p(\sigma, \zeta'; 2^{2j-k}) \right) \), with \( \hat{\psi}(\sigma), \hat{\phi}(\zeta') \) or their derivatives. The function in denominator \( p(\sigma, \zeta'; 2^{2j-k}) = \sqrt{i\sigma + 2^{2j-k}|\zeta'|^2} \) is estimated from below by \( 2^{-1} \) thanks to the cutoff function \( \hat{\phi}(\zeta') \) or its derivative. Therefore, if we derivate \( n + 2 \) times, then it is estimated from below by \( 2^{n+2} \). Moreover, \( p(\sigma, \zeta'; 2^{2j-k}) \) which arises by derivative contains surplus scale parameter \( 2^{2j-k} \) and it is estimated from above by \( 1 \) by \( k \geq 2j \). Thus for the functions
Here, we set \( J. \text{ Evol. Equ. Maximal} \)
\( \tau = k \)
where we put following estimate:
\[
\| \Psi_{N,k,j}(t, \cdot, \eta) \|_{L_1^\prime}
= 2^{\frac{1}{2}} \left\| \left( \frac{1}{|y|^2} \right)^{\frac{1}{2}} \frac{2^k}{(2^k t)^2} c_{n+1} \int_{\mathbb{R}^{n-1}} e^{i(2^k \sigma + y' \cdot \zeta')} (1 - \Delta_{\zeta'}^x)^{\frac{1}{2}} \left( 1 - \frac{\partial^2}{\partial \sigma^2} \right) k_{k,j}(\sigma, \zeta', \eta) d\sigma \right\|_{L_1^\prime}
= C 2^{\frac{1}{2}} \exp(-2^{\frac{1}{2} - 1} \eta) \frac{2^k}{(2^k t)^2} \times \left\| \left( \frac{1}{|y|^2} \right)^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} e^{i(2^k \sigma + y' \cdot \zeta')} e^{-2^{\frac{1}{2} \eta} \left( p(\sigma, \zeta', 2^j - k) - \frac{1}{2} \right)} K_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^j - k) d\sigma \right\|_{L_1^\prime}
\leq C 2^{\frac{1}{2}} \exp(-2^{\frac{1}{2} - 1} \eta) \frac{2^k}{(2^k t)^2} \times \left\| \left( \frac{1}{|y|^2} \right)^{\frac{1}{2}} \int_{\mathbb{R}^{n-1}} e^{-2^{\frac{1}{2} \eta} \left( R e^{2(\sigma + 2^j - k) |\zeta'|^2} \right)} K_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^j - k) d\sigma \right\|_{L_1^\prime}
\leq C 2^{\frac{1}{2}} \exp(-2^{\frac{1}{2} - 1} \eta) (1 + 2^{\frac{1}{2} \eta} |\eta|^{n+2}) \frac{2^k}{(2^k t)^2}.
\]

For the case of the space-like region \( k < 2j \), we proceed similar way. By changing \( \tau = 2^k \sigma, \zeta' = 2^j \zeta' \),
\[
\| \Psi_{N,k,j}(t, \cdot, \eta) \|_{L_1^\prime}
= \left\| c_{n+1} \int_{\mathbb{R}^{n-1}} e^{i(\tau + y' \cdot \zeta')} \frac{\tau}{\sqrt{i \tau + |\xi|^2}} \exp \left( -\sqrt{i \tau + |\xi|^2} \eta \right) \hat{\psi}(2^k \tau) \hat{\phi}(2^{-j} \zeta') d\xi d\tau \right\|_{L_1^\prime}
= 2^{\frac{1}{2}} \left\| c_{n+1} \int_{\mathbb{R}^{n-1}} e^{i2^k \sigma + y' \cdot \zeta'} \frac{\sigma}{\sqrt{i \sigma + 2^j - k |\zeta'|^2}} \exp \left( -2^k \eta \sqrt{2^{k-j} |\sigma| + |\zeta'|^2} \right) \times \hat{\psi}(\sigma) \hat{\phi}(\zeta') d\zeta' \cdot 2^k d\sigma \right\|_{L_1^\prime}
\]

Here, we set \( x' = 2^{-j} y' \) in the last line. Using (6.4) and integrating by parts,
\[
\| \Psi_{N,k,j}(t, \cdot, \eta) \|_{L_1^\prime(B_{2^k}^c)}
= 2^{\frac{1}{2}} \left\| c_{n+1} \int_{\mathbb{R}^{n-1}} \frac{1}{2^k t} e^{i(2^k \sigma + y' \cdot \zeta')} \times \Delta_{\zeta'} \frac{\partial}{\partial \sigma} \left[ \frac{\sigma}{\sqrt{i \sigma + 2^j - k |\zeta'|^2}} \exp \left( -2^k \eta \sqrt{2^{k-j} |\sigma| + |\zeta'|^2} \right) \right] d\zeta' d\sigma \right\|_{L_1^\prime(B_{2^k}^c)}.
\]
Setting

\[
\begin{align*}
    p(\sigma, \zeta'; 2^{j-k}) &= \sqrt{i\sigma + 2^{j-k}|\zeta'|^2}, \\
    q(\sigma, \zeta'; 2^{k-2j}) &= \sqrt{2^{k-2j}i\sigma + |\zeta'|^2},
\end{align*}
\]

we see that

\[
\frac{\partial}{\partial \sigma} \left[ \exp\left(-2^j \eta q(\sigma, \zeta'; 2^{k-2j})\right) \frac{p(\sigma, \zeta'; 2^{2j-k})}{p(\sigma, \zeta'; 2^{2j-k})} \hat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\zeta') \right]
\]

\[
= \frac{\exp\left(-2^j \eta q(\sigma, \zeta'; 2^{k-2j})\right)}{p(\sigma, \zeta'; 2^{2j-k})} \times \left[ \left( -\frac{2^j \eta 2^{k-2j}i}{2q(\sigma, \zeta'; 2^{2j-k})} - \frac{i}{2p(\sigma, \zeta'; 2^{2j-k})^2} \right) \hat{\psi}(\sigma) 2^k \sigma \\
+ \hat{\psi}'(\sigma) 2^k \sigma + \hat{\psi}(\sigma) 2^k \hat{\phi}(\zeta') \right].
\]

Taking the second derivative in \( r = |\zeta'| \)

\[
\Delta_{\zeta'} \frac{\partial}{\partial \sigma} \left[ \exp\left(-2^j \eta \sqrt{2^{k-2j}i\sigma + |\zeta'|^2}\right) \frac{\sigma}{\sqrt{i\sigma + 2^{2j-k}|\zeta'|^2}} \hat{\psi}(\sigma) 2^k \sigma \hat{\phi}(\zeta') \right]
\]

\[
= \Delta_{\zeta'} \frac{\exp\left(-2^j \eta q(\sigma, \zeta'; 2^{k-2j})\right)}{p(\sigma, \zeta'; 2^{2j-k})} \times \left[ \left( -\frac{2^j \eta 2^{k-2j}i}{2q(\sigma, \zeta'; 2^{2j-k})} - \frac{i}{2p(\sigma, \zeta'; 2^{2j-k})^2} \right) \hat{\psi}(\sigma) 2^k \sigma \\
+ \hat{\psi}'(\sigma) 2^k \sigma + \hat{\psi}(\sigma) 2^k \hat{\phi}(\zeta') \right].
\]

Similarly before all terms again have \( \exp\left(-2^\frac{k}{2} \eta q(\sigma, \zeta'; 2^{k-2j})\right) \), \( \hat{\psi}(\sigma) \), \( \hat{\phi}(\zeta') \) or their derivatives. The denominators \( q(\sigma, \zeta'; 2^{k-2j}) \) and \( p(\sigma, \zeta', 2^{2j-k}) \) are estimated from below by \( 2^{-1} \) thanks to the cutoff functions \( \hat{\psi}(\sigma) \) and \( \hat{\phi}(\zeta') \) or its derivative. Moreover, \( p(\sigma, \zeta'; 2^{2j-k}) \) is estimated from above by \( 1 \) by \( k \leq 2j \). Thus introducing

\[
\hat{h}_{k,j}(\sigma, \zeta', \eta) \equiv \exp\left(-2^j \eta \sqrt{2^{k-2j}i\sigma + |\zeta'|^2}\right) \frac{2^k \sigma \hat{\psi}(\sigma) \hat{\phi}(\zeta')}{\sqrt{i\sigma + 2^{2j-k}|\zeta'|^2}}
\]

\[
\hat{H}_{k,j}^{n+2}(\sigma, \zeta', \eta, 2^{k-2j}) \equiv \exp\left(-2^j \eta \sqrt{2^{k-2j}i\sigma + |\zeta'|^2}\right) \frac{D_{\sigma, \zeta'}^{n+2} \hat{h}_{k,j}(\sigma, \zeta', \eta)}{\sqrt{i\sigma + 2^{2j-k}|\zeta'|^2}}
\]

we obtain
\[ \| \Psi_{N,k,j}(t, \cdot, \eta) \|_{L^1_j} = 2^k \left( \frac{1}{(y')^2} \right)^{\frac{3}{2}} \frac{2^k}{(2^{k+1}t)^2} e^{i(2^{k+1}t + y')^2} (1) \]

\[ -\Delta \zeta (1 - \frac{\partial^2}{\partial \sigma^2}) \left( \widehat{b_{L,k,j}}(\sigma, \zeta', \eta) d\sigma \right) \leq C \frac{2^k}{(2^{k+1}t)^2} e^{(2^{k+1}t - y')^2} \left( \widehat{b_{L,k,j}}(\sigma, \zeta', \eta, 2^{k+2}) \right) d\sigma \]

This completes the proof of (6.22) in Lemma 6.5.

7. Optimal boundary trace estimates

In this section, we show the optimality for the boundary trace estimate shown in Theorem 2.1 and 2.3. This shows that the condition on the boundary data in those theorems are not only the sufficient condition but also the necessary condition (see for more detailed estimates for the boundary trace [24,32])

**Theorem 7.1** (The Dirichlet boundary trace). Let \( 1 \leq p \leq \infty \) and \( s \in \mathbb{R} \). There exists a constant \( C > 0 \) such that for all function \( u = u(t, x', \eta) \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+)) \), \( \Delta u(t, x', \eta) \in L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+)) \) with satisfying \( u(0, x', \eta) = 0 \) almost everywhere in \( (x', \eta) \in \mathbb{R}^n_+ \), the following estimate holds:

\[
\sup_{\eta \in \mathbb{R}_+} \left( \| u(\cdot, \cdot, \eta) \|_{F^{1-1/2p}_{1,1}(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))} + \| u(\cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^n_+))} \right) 
\leq C \left( \| \partial_t u \|_{L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))} + \| \nabla^2 u \|_{L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))} \right). \tag{7.1} \]

From Theorem 7.1, the following corollary immediately follows:

**Corollary 7.2** (Sharp boundary trace for Dirichlet data). For \( 1 \leq p \leq \infty \), there exists a constant \( C > 0 \) such that the solution \( u \) to the initial-boundary value problem (2.5) with

\[
u \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+)), \quad \Delta u \in L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+)) \]

gives the following estimate on the boundary condition \( h \):

\[
\| h \|_{F^{1-1/2p}_{1,1}(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))} + \| h \|_{L^1(\mathbb{R}_+; \dot{B}^{s+2-1/p}_{p,1}(\mathbb{R}^n_+))} \leq \begin{cases} C \| \nabla^2 u \|_{L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))}, \\ C \| \partial_t u \|_{L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+))}. \end{cases} \]

**Proof of Theorem 7.1.** For \( 1 \leq p \leq \infty \), we assume that \( u \in \dot{W}^{1,1}(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+)) \), \( \Delta u \in L^1(\mathbb{R}_+; \dot{B}^{s}_{p,1}(\mathbb{R}^n_+)) \).
\[ \|u(\cdot, \cdot, \eta)\|_{\dot{H}_{1,1}^{1-1/2p}(\mathbb{R}_+; \dot{B}_{p,1}^{s}((0,1)^{-1}))} \leq \left\| \sum_{j \in \mathbb{Z}} \sum_{k \geq 2j} 2^{(1/2)p}k^{2} \psi_k \ast \phi_j \ast u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \|
abla u(t, \cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+)} \] 
\[ + \left\| \sum_{j \in \mathbb{Z}} \sum_{k \leq 2j} 2^{(1/2)p}k^{2} \psi_k \ast \phi_j \ast u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \|
abla u(t, \cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+)} \equiv I + II. \]

(7.2)

Using the assumption \(u(0, x', \eta) = 0\) almost everywhere,
\[
\psi_k(t) \ast u(t, x', \eta) = - \int_{\mathbb{R}_+} \partial_s \left( \int_s^\infty \psi_k(t - r)dr \right)u(s, x', \eta)ds
\]
\[= - \left[ \left( \int_s^\infty \psi_k(t - r)dr \right)u(s, x', \eta) \right]_{s=0}^\infty
\]
\[+ \int_{\mathbb{R}_+} \int_s^\infty \psi_k(t - r)dr \partial_s u(s, x', \eta)ds
\]
\[= \partial_t^{-1} \psi_k (t) \ast \partial_t u(t, x', \eta), \quad (7.3)
\]

where we set
\[\partial_t^{-1} \psi_k (t - s) \equiv \int_{-\infty}^{t-s} \psi_k(r)dr = \int_s^{\infty} \psi_k(t - r)dr. \quad (7.4)
\]

Here, we recall the Littlewood-Paley decomposition of direct sum type defined in (4.7). Since \(\partial_t^{-1} \psi_k = 2^{-k} (\partial_t^{-1} \psi)k\) is also rapidly decreasing smooth function, \(\xi_j(\eta) \ast \xi_{j-1}(\eta)\) by definition (4.6) and using (4.8), the Hausdorff–Young inequality with (7.3), there exists a constant \(C > 0\) such that
\[I = \left\| \sum_{j \in \mathbb{Z}} \sum_{k \geq 2j} 2^{(1/2)p}k^{2} \psi_k \ast \phi_j \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \|
abla u(t, \cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+)} \]
\[\leq C \left\| \sum_{j \in \mathbb{Z}} 2^{j} \sum_{k \geq 2j} 2^{(1/2)p}k^{2} \psi_k \ast \phi_j \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \|
abla u(t, \cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+)} \]
\[\leq C \left\| \sum_{j \in \mathbb{Z}} 2^{j} \sum_{k \geq 2j} 2^{-k} \left\| \int_{\mathbb{R}_+} \frac{2^k}{(2^k(t-s))^2} \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \left\| \sum_{j \in \mathbb{Z}} 2^{j} \sum_{k \geq 2j} 2^{-k} \left\| \int_{\mathbb{R}_+} \frac{2^k}{(2^k(t-s))^2} \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \left\| \sum_{j \in \mathbb{Z}} 2^{j} \sum_{k \geq 2j} 2^{-k} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \xi_j(\eta) \ast \xi_{j-1}(\eta) \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \left\| \sum_{j \in \mathbb{Z}} 2^{j} \sum_{k \geq 2j} 2^{-k} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \xi_j(\eta) \ast \xi_{j-1}(\eta) \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \sum_{j \in \mathbb{Z}} 2^{j} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \xi_j(\eta) \ast \xi_{j-1}(\eta) \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \sum_{j \in \mathbb{Z}} 2^{j} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \xi_j(\eta) \ast \xi_{j-1}(\eta) \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \sum_{j \in \mathbb{Z}} 2^{j} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \sum_{j \in \mathbb{Z}} 2^{j} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \sum_{j \in \mathbb{Z}} 2^{j} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \sum_{j \in \mathbb{Z}} 2^{j} \left\| \sum_{m \in \mathbb{Z}} \Phi_m \ast \phi_j \ast \partial_t u(t, \cdot, \cdot, \eta) \right\|_{L^p(\mathbb{R}_+^{n-1})} \right\|_{L^1(\mathbb{R}_+)} \]
\[\leq C \|
abla u(t, \cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+; \dot{B}_{1,1}^{s}((0,1)^{-1}))}. \quad (7.5)
\]
On the other hand, the second term of the right hand side of (7.2) can be treated by using the Minkowski inequality that

$$11 = \left\| \sum_{j \in \mathbb{Z}} \sum_{k \leq 2j} 2^{(1-1/2)p} k 2^j \left\| \psi_k * \phi_j * u(t, \cdot, \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right\|_{L_1^1(\mathbb{R}^+)}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^j \sum_{k \leq 2j} 2^{(1-1/2)p} k \left\| \psi_k * \phi_j * u(t, \cdot, \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right\|_{L_1^1(\mathbb{R}^+)}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^j \sum_{k \leq 2j} 2^{(1-1/2)p} k \left\| \sum_{m \in \mathbb{Z}} \Phi_m * \zeta_j(\eta) * \zeta_{j-1}(\eta) * \zeta_{j-1}(\eta) * \phi_j * u(t, \cdot, \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right\|_{L_1^1(\mathbb{R}^+)}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^{(1-1/2)p} j \left\| \zeta_j(\eta) * \phi_j * u(t, \cdot, \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right\|_{L_1^1(\mathbb{R}^+)}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^{(1-1/2)p} j \left\| (-\Delta)^{-1/2} \Phi_j(\cdot, \eta) * (t, \cdot) \right\|_{L^p(\mathbb{R}^+)} \right\|_{L_1^1(\mathbb{R}^+)}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^{(1-1/2)p} j \left\| \zeta_j(\eta) * \Phi_j(\cdot, \eta) \right\|_{L^p(\mathbb{R}^+)} \right\|_{L_1^1(\mathbb{R}^+)}$$

(7.6)

One can apply the similar treatment for the spatial direction, for $1 \leq p \leq \infty$. Let $\Delta u \in L^1(\mathbb{R}^+_+; \mathcal{B}_{p,1}^5(\mathbb{R}^n_+))$ and $\eta \in I_{-\ell} = \{2^{-\ell}, 2^{-\ell+1}\}$ with $\ell \in \mathbb{Z}$, we insert the unity $\sum_{m \in \mathbb{Z}} \Phi_m(\cdot, \eta)$ to the estimate to see

$$\left\| u(t, \cdot, \eta) \right\|_{L^1(\mathbb{R}^+_+; \mathcal{B}_{p,1}^{5+1/2 - 1/p}(\mathbb{R}^{n-1}_+))}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^j 2^{(1-1/2)p} j \left\| \Phi_j(\cdot, \eta) * \zeta_{j-1}(\eta) * \Phi_m(\cdot, \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right\|_{L_1^1(\mathbb{R}^+)}$$

$$\leq C \sum_{j \in \mathbb{Z}} 2^j \left\| \Phi_j(\cdot, \eta) * \Phi_m(\cdot, \eta) \right\|_{L^p(\mathbb{R}^{n-1})} \right\|_{L_1^1(\mathbb{R}^+)}$$

(7.7)

Combining the estimates (7.2), (7.5), (7.6) and (7.7), we conclude the estimate (7.1).

\[\square\]

**Theorem 7.3** (Sharp boundary derivative trace). For $1 \leq p \leq \infty$ and $s \in \mathbb{R}$, there exists a constant $C > 0$ such that for all function $u = u(t, x', \eta) \in \dot{W}^{1,1}(\mathbb{R}^+_+; \mathcal{B}_{p,1}^s(\mathbb{R}^n_+))$, we have
Δu ∈ L^1(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+)) with \partial_\eta u(0, x', \eta) = 0, it holds

\[
\sup_{\eta \in \mathbb{R}_+} \left( \| \partial_\eta u(\cdot, \cdot, \eta) \|_{F_{1,1}^{1/2-1/2p}(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^{n-1}_+))} + \| \partial_\eta u(\cdot, \cdot, \eta) \|_{L^1(\mathbb{R}_+; \hat{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}_+))} \right)
\leq C \left( \| \partial_\eta u \|_{L^1(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+))} + \| \nabla^2 u \|_{L^1(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+))} \right).
\]

(7.8)

Similar to the boundary trace estimate, Theorem 7.3 immediately implies the optimality of the boundary derivative trace estimate for the Neumann boundary condition.

**Corollary 7.4** (Sharp boundary trace for Neumann data). For \(1 \leq p \leq \infty\) and \(s \in \mathbb{R}\), let \(u\) be a solution to the Cauchy problem (2.8) with

\[
\tilde{W}^{1,1}(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+)) \cap L^1(\mathbb{R}_+; \hat{B}^{s+2}_{p,1}(\mathbb{R}^n_+)),
\]

then there exists a constant \(C > 0\) such that the boundary condition has to satisfy

\[
\| h \|_{F_{1,1}^{1/2-1/2p}(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^{n-1}_+))} + \| h \|_{L^1(\mathbb{R}_+; \hat{B}^{s+1-1/p}_{p,1}(\mathbb{R}^{n-1}_+))} \leq \begin{cases} 
C \| \nabla^2 u \|_{L^1(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+))}, \\ 
C \| \partial_\eta u \|_{L^1(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+))}.
\end{cases}
\]

(7.10)

**Proof of Theorem 7.3.** The proof is very similar to the proof for Theorem 7.1. For \(1 \leq p \leq \infty\) and \(s \in \mathbb{R}\), assume \(u \in \tilde{W}^{1,1}(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+)), \Delta u \in L^1(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+)).\) Then

\[
\| \partial_\eta u(\cdot, \cdot, \eta) \|_{F_{1,1}^{1/2-1/2p}(\mathbb{R}_+; \hat{B}^s_{p,1}(\mathbb{R}^n_+))}
\leq \| \sum_{j \in \mathbb{Z}} 2^{sj} \sum_{k \geq 2j} 2^{(1/2-1/2p)k} \| \psi_k \ast \phi_{j}^{(x')} \partial_\eta u(t, \cdot, \cdot, \eta) \|_{L^p(\mathbb{R}^{n-1}_+)} \|_{L^1(\mathbb{R}_+)}
+ \| \sum_{j \in \mathbb{Z}} 2^{sj} \sum_{k \geq 2j} 2^{(1/2-1/2p)k} \| \psi_k \ast \phi_{j}^{(x')} \partial_\eta u(t, \cdot, \cdot, \eta) \|_{L^p(\mathbb{R}^{n-1}_+)} \|_{L^1(\mathbb{R}_+)}
\equiv I + II.
\]

(7.9)

For all \(j \in \mathbb{Z},\)

\[
\partial_\eta u(0, x', \eta) = 0, \quad (x', \eta) \in \mathbb{R}^{n-1} \times \mathbb{R}_+,
\]

(7.10)

and it follows from (7.4) that

\[
\psi_k(t) \ast \partial_\eta u(t, x', \eta) = \partial_t^{-1} \psi_k(t) \ast \partial_\eta \partial_t u(t, x', \eta).
\]

(7.11)

Hence from (7.10) and (7.11) and using the Hausdorff–Young inequality

\[
I = \left\| \sum_{j \in \mathbb{Z}} 2^{sj} \sum_{k \geq 2j} 2^{(1/2-1/2p)k} \| \partial_t^{-1} \psi_k \ast \phi_{j}^{(x')} \partial_\eta \partial_t u(t, x, \eta) \|_{L^p(\mathbb{R}^{n-1}_+)} \|_{L^1(\mathbb{R}_+)}
\leq C \left\| \sum_{j \in \mathbb{Z}} 2^{sj} \sum_{k \geq 2j} 2^{-(\frac{1}{2} + \frac{1}{2p})k} \int_{\mathbb{R}_+} \| \phi_{j}^{(x')} \ast \partial_\eta \partial_t u(s, x, \eta) \|_{L^p(\mathbb{R}^{n-1}_+)} \|_{L^1(\mathbb{R}_+)} \right\|
\]

(7.12)
Combining the estimates (7.9), (7.12), (7.13) and (7.14), we obtain the result (7.8). This completes the proof.
8. Concluding remarks

Since we establish maximal $L^1$-regularity for the inhomogeneous boundary data of Dirichlet and Neumann boundary conditions, the estimate can be generalized into other boundary conditions. For instance, one can generalize to the case of the oblique boundary condition:

\[
\begin{cases}
\partial_t u - \Delta u = f(t, x), & t > 0, \quad x \in \mathbb{R}^n_+,
\quad b \cdot \nabla u |_{x_n=0} = h(t, x'), & t > 0, \quad x' \in \mathbb{R}^{n-1},
\quad u |_{t=0} = u_0, & x \in \mathbb{R}^n_+,
\end{cases}
\]  

(8.1)

where $b = (b', b_n)$ is a given constant vector in $\mathbb{R}^n$ with $b_n \neq 0$. We also obtain that the integral kernel to the Laplacian of the solution $\Psi_{ob}$ is given by

\[
\Psi_{ob}(t, x) = c_{n+1} \int_{\mathbb{R}^n_+} \int_{\mathbb{R}^{n-1}} e^{i\tau + i x' \cdot \xi'} \frac{i\tau}{ib' \cdot \xi' - b_n \sqrt{i\tau + |\xi'|^2} x_n} e^{-\sqrt{i\tau + |\xi'|^2} x_n} d\xi' d\tau.
\]  

(8.2)

Then, the analogous estimate to the boundary potential in Theorem 2.4 can be obtained once we establish the almost orthogonal estimate as in Lemma 6.5. The detailed estimate is shown in the forthcoming paper [37].

For a general domain $\Omega \subset \mathbb{R}^n$, for instance bounded domain with smooth boundary, we may generalize our results. In such a case, by standard decompositions of unity near the boundary $\partial \Omega$, it may be reduced into a problem in the half-space by a smooth diffeomorphism. Then if we establish maximal $L^1$-regularity for the parabolic initial-boundary value problems with lower order spatial derivative terms, we may extend the estimate for general domain cases. Further application to the nonlinear problem is also available. We discuss such an application to the fluid mechanics in a forthcoming paper [36] (cf. [35]).

Acknowledgements

The authors are grateful to the anonymous referee for valuable suggestions and comments that improve the presentation of this paper largely. The first author is partially supported by JSPS grant-in-aid for Scientific Research (S) #19H05597, Scientific Research (B) #18H01131 and Challenging Research (Pioneering) #20K20284. The second author is partially supported by JSPS grant-in-aid for Scientific Research (B) #16H03945 and (B) #21H00992 and Fostering Joint International Research (B) #18KK0072.

Declarations

Conflict of interest The authors declare that there is no conflict of interest.
Open Access. This article is licensed under a Creative Commons Attribution 4.0 International License, which permits use, sharing, adaptation, distribution and reproduction in any medium or format, as long as you give appropriate credit to the original author(s) and the source, provide a link to the Creative Commons licence, and indicate if changes were made. The images or other third party material in this article are included in the article’s Creative Commons license, unless indicated otherwise in a credit line to the material. If material is not included in the article’s Creative Commons license and your intended use is not permitted by statutory regulation or exceeds the permitted use, you will need to obtain permission directly from the copyright holder. To view a copy of this licence, visit http://creativecommons.org/licenses/by/4.0/.

Publisher’s Note Springer Nature remains neutral with regard to jurisdictional claims in published maps and institutional affiliations.

REFERENCES

[1] Abidi, H., Paicu, M. Existence globale pour un fluide inhomogène. Ann. Inst. Fourier (Grenoble) 57 (2007) 883–917.

[2] Amann, H., Linear and Quasilinear Parabolic Problems. Vol I Abstract Linear Theory, Monographs in Math. Vol 89. Birkhäuser Verlag, Basel-Boston-Berlin, 1995.

[3] Amann, H., Linear and Quasilinear Parabolic Problems. Vol II: Function Spaces, Monographs in Math. Vol 106, Birkhäuser Verlag, Basel-Boston-Berlin, 2019.

[4] Bahouri, H., Chemin, J-Y., Danchin, R., Fourier Analysis and Nonlinear Partial Differential Equations. Grundlehren der mathematische Wissenshaften 343, Springer-Verlag, Berlin-Heidelberg-Dordrecht-London-New York 2011.

[5] Benedek, A., Calderón, A.P., Panzone, R., Convolution operators on Banach space valued functions. Proc. Nat. Acad. Sci. USA 48 (1962) 356–365.

[6] Bergh, J., Löfström, J., Interpolation Spaces; an introduction, Springer-Verlag, Berlin, 1976.

[7] Chemin, J.-Y., Théorèmes d’unicité pour le système de Navier-Stokes tridimensionnel, J. Anal. Math., 77 (1999), 27–50.

[8] Clément, Ph., Prüss, J., Global existence for a semilinear parabolic Volterra equation, Math. Z., 209 (1992) 17–26.

[9] Coulhon, T., Lamberton, D., Régularité $L^p$ pour les équations dévolution. In: Séminaire d’analyse fonctionnelle 1984-85, Publications mathématiques de l’Université Paris VII, 26 (1987) 141–153.

[10] Danchin, R., Density-dependent incompressible viscous fluids in critical spaces, Proc. Roy Soc. Edinburgh 133A (2003), 1311–1334.

[11] Danchin, R., Well-posedness in critical spaces for barotropic viscous fluids with truly not constant density, Comm. Partial Differential Equations, 32 (2007) 1373–1397.

[12] Danchin, R., Mucha, P. B., A critical functional framework for the inhomogeneous Navier-Stokes equations in the half-space, J. Funct. Anal., 256 (2009) 881–927.

[13] Da Prato, G., Grisvard, P., Sommes d’opérateurs linéaires et équations différentielles opérationnelles, J. Math. Pure Appl. 54 (1975) 305–387.

[14] Denk, R., Hieber, M., Prüss, J., $R$-boundedness, Fourier multipliers and problems of elliptic and parabolic type, Memoirs of AMS, 166, No. 788 (2003).

[15] Denk, R., Hieber, M., Prüss, J., Optimal $L_p$-$L_q$-regularity for parabolic problems with inhomogeneous boundary data, Math. Z., 257 (2007) 193–224.

[16] De Simon, L., Un’applicazione della teoria degli integrali allo studio delle equazioni differenziali astratta del primo ordine, Rend. Sem. Mat. Univ. Padova, 34 (1964) 157–162.

[17] Dore, G., $L^p$ regularity for abstract differential equations. In: Functional Analysis and Related Topics, H.Komatsu (ed.), Lecture Notes in Math., 1540, Springer (1993).

[18] Dore, G., Venni, A., On the closedness of the sum of two closed operators, Math. Z., 196 (1987) 189–201.

[19] Duong, X.T., $H_\infty$ functional calculus of second order elliptic partial differential operators on $L^p$ spaces, In: Miniconference on Operators in Analysis, 1989. Proc. Centre Math. Anal. ANU, Canberra, 24 (1990) 91–102.
Giga, Y., Saal, J., $L^1$ maximal regularity for the Laplacian and applications, Discrete Cont. Dyn. Syst. 1 (2011) 495–504.

Giga, Y., Sohr, H, Abstract $L^p$ estimates for the Cauchy problem with applications to the Navier-Stokes equations in exterior domains, J. Funct. Anal., 102 (1991) 72–94.

Hieber, M., Prüss, J., Heat kernels and maximal $L^p$-$L^q$ estimates for parabolic evolution equations, Comm. P.D.E., 22 (1997), 1674-1669.

Iwabuchi, T., Global solutions for the critical Burgers equation in the Besov spaces and the large time behavior, Ann. I. H. Poincaré, (2015), 687–713.

Johnsen, J., Sickel, W., On the trace problem for Lizorkin-Triebel spaces with mixed norms, Math. Nachr. 281 (2008), 669–696.

Kalton, N., Lancien, G., A solution to the problem of the $L^p$-maximal regularity, Math. Z., 235 (2000) 559–568.

Kalton, N., Weis, L., The $H^\infty$-calculus and sums of closed operators, Math. Ann., 321 (2001) 319–345.

Krylov, N.V., Parabolic and elliptic equations with VMO coefficients, Comm. Partial Differential Equations, 32 (2007), 453–475.

Kunstmann, P.C., Weis, L., Maximal $L^p$-regularity for parabolic equations, Fourier multiplier theorems and $H^\infty$-functional calculus, M. Iannelli, R. Nagel and S. Piazzera (ed.) Functional Analytic Methods for Evolution Equations, Lecture Notes in Mathematics 1899, Springer-Verlag Berlin Heidelberg, 2004.

Ladyzhenskaya, O.A., Solonnikov, V.A., Ural’tseva, N.N., Linear and quasilinear equations of parabolic type, Amer. Math. Soc. Transl. Math. Monographs, Providence, R.I., 1968.

Lizorkin, P.I., Properties of functions of class $\Lambda^{p, q}_{\omega}$, Trudy Mat. Inst. Steklov, 131 (1974), 158–181.

McIntosh, A., Yagi, A., Operators of type $\omega$ without a bounded $H^\infty$-functional calculus, In: Mini-conference on Operators in Analysis, 1989. Proc. Centre Math. Anal. ANU, Canberra, 24 (1990) 159–172.

Meyries, M., Veraar, M. C., Traces and embeddings of anisotropic function spaces, Math. Ann. 360 (2014), 571–606.

Ogawa, T., Shimizu, S., End-point maximal regularity and its application to two-dimensional Keller-Segel system, Math. Z., 264 (2010) 601–628.

Ogawa, T., Shimizu, S., End-point maximal $L^1$-regularity for a Cauchy problem to parabolic equations with variable coefficient, Math. Ann., 365 (2016) 661–705.

Ogawa, T., Shimizu, S., Maximal $L^1$-regularity of the heat equation and application to a free boundary problem of the Navier–Stokes near the half-space, J. Elliptic Parabol. Equ., 7 (2021) 509–535.

Ogawa, T., Shimizu, S., Maximal $L^1$ regularity and free boundary value problems for the incompressible Navier-Stokes equations in critical spaces, preprint (2021).

Ogawa, T., Shimizu, S., End-point maximal $L^1$-regularity for an initial boundary value problem of the heat equations under the oblique boundary condition, in preparation.

Peetre, J., On spaces of Triebel-Lizorkin type, Ark. Mat. 13 (1975) 123–130.

Peetre, J., New thoughts on Besov spaces, Duke University Mathematics Series, No.1, Duke University, Durham, N., C., 50 1976.

Prüss, J., Simonett, G., Moving Interfaces and Quasi-linear Parabolic Differential Equations, Monographs in Math. 105, Birkhäuser, Basel (2016).

Shibata, Y., Shimizu, S., On the free boundary problem for the Navier–Stokes equations, Differential Integral Equations 20 no. 3 (2007), 241–276.

Shibata, Y., Shimizu, S., On the $L_p$-$L_q$ maximal regularity of the Neumann problem for the Stokes equations in a bounded domain, J. reine angew. Math. 615 (2008), 157–209.

Sobolevskii, P.E., Fractional powers of coercively positive sums of operators, Dokl. Akad Izv.11 (1977) 1323–1358.

Solonnikov, V. A., A priori estimates for a solutions of second-order equations of parabolic type, Trudy Mat. Inst. Stekelov., 70 (1964), 133-212, (English translation; Transl Amer. Math. Soc. 65 (1967), 51–137.)

Triebel, H., Spaces of distributions of Besov type in Euclidean n-space, Duality, interpolation, Ark. Mat. 11 (1973), 13–64.
[46] Triebel, H., *Interpolation Theory, Function spaces, Differential Operators*, North-Holland, Amsterdam - New York - Oxford, 1978.

[47] Triebel, H., *Theory of Function Spaces*, Birkhäuser, Basel, 1983.

[48] Weidemaier, P., *Refinement of an Lp-estimate of Solonnikov for a parabolic equation of the second order with conormal boundary condition*, Math. Z., 199 (1988) 589–604.

[49] Weidemaier, P., *On the trace theory for functions in Sobolev spaces with mixed Lp-norm*, Czech. Math. J. 44 (1994), 7–20.

[50] Weidemaier, P., *Maximal regularity for parabolic equations with inhomogeneous boundary conditions in Sobolev spaces with mixed Lp-norm*, Electron. Res. Announc. Amer. Math. Soc. 8 (2002) 47–51.

[51] Weidemaier, P., *Vector-valued Lizorkin-Triebel spaces and sharp trace theory for functions in Sobolev spaces with mixed Lp-norm for parabolic problem*, Sbornik: Math. 196 (2005), 777–790.

[52] Weidemaier, P., *On Lp-estimate of optimal type for the parabolic oblique derivative problem with VMO-coefficients– A refined version*, Progress in Nonlinear Differential Equations and Their Applications vol. 64, 529–536, Birkhäuser Verlag Basel, 2005.

[53] Weis, L., *Operator-valued Fourier multiplier theorems and maximal Lp-regularity*, Math. Ann., 319 (2001) 735–758.

Takayoshi Ogawa  
Mathematical Institute/Research Alliance Center of Mathematical Science  
Tohoku University  
Sendai 980-8578  
Japan  
E-mail: takayoshi.ogawa.c8@tohoku.ac.jp

Senjo Shimizu  
Graduate School of Human and Environmental Studies  
Kyoto University  
Kyoto 606-8501  
Japan  
E-mail: shimizu.senjo.5s@kyoto-u.ac.jp

Accepted: 19 October 2021