5-Dimensional Covariance and Generation of Solutions of Einstein Equations

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Abstract

A generation procedure, based on the 5-dimensional covariance of the Kaluza-Klein theory, is developed. The procedure allows one to obtain exact solutions of the 4-dimensional Einstein equations with electromagnetic and scalar fields from vacuum 5-dimensional solutions using special 5-dimensional coordinate transformations. Relations between the physical properties of the resulting solutions and invariant geometrical properties of the generating Killing vectors are found out.

1 Introduction

Searches for exact solutions of the Einstein equations are stimulated by the internal logic of General Relativity (GR). A special role in this problem is played by the so-called generation methods which give new solutions without solving the Einstein equations. As a rule, they are based on some original solution and special kinds of transformations of the metric, dynamical Lagrangian variables and coordinates [1].

In the present paper a generation method based on the classical version of the Kaluza-Klein theory (KKT) is worked out. Theories of the Kaluza-Klein type have recently gained a new physical content. It concerns both mathematical and physical aspects of the theory, which were out of consideration earlier [2, 3, 4]. The method to be proposed is related to the well-known property of the KKT, which takes place in any version of the theory due to its multidimensional covariance: the 4-dimensional interpretation of multidimensional geometrical objects depends on the choice of the multidimensional coordinate system [5]. Thus by general coordinate transformations one can generate new properties of 4-dimensional space-times, though the multidimensional world remains the same. This fact has been used by a number of authors. For instance, in [7] the electromagnetic field generation procedure in 4-D vacuum by a special 5-D coordinate transformation has been proposed. In [8] a linear transformation in the $(x^0,x^5)$-plane generating an exact solutions with electromagnetic and scalar fields from the Schwarzschild metric has been used. In [9] these transformations have been applied to Kramer’s metric. In [10] an explicit form of 5-D coordinate transformations which transform the flat 5-D metric into a cosmological metric of Friedmann type is given. The present approach includes the methods proposed in [7, 8, 10], and partially in [9], as special cases.

The idea of our method has a clear geometric meaning. We assume (which is usual in KKT) that the invisibility of the 5th dimension is due to an isometry of the 5-dimensional space-time.

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Then the set of integral curves of the Killing vector field can be associated with the coordinate lines of $x^5$. If 5-D space-time admits more than one Killing vector, the following uncertainty appears: which Killing vector should be connected with a 5-D coordinate system? If we first connect it with one Killing vector field, then, by a suitable coordinate transformation with another one we get a transition between different 4-D physical worlds.

Section 2 presents a sketch of the classical KKT.

In Section 3 the class of admissible 5-D coordinate transformations is generalized. In addition, we prove a number of propositions connecting the invariant properties of Killing vectors with the physical properties of the resulting 4-dimensional solutions.

In Section 4 the method is applied to a flat metric.

## 2 Kaluza-Klein theory and monad formalism

The Kaluza-Klein theory is a direct 5-dimensional generalization of GR. It was developed for geometrical unification of gravity and electromagnetism \[8, 9\]. A (1+4)-splitting procedure (monad formalism) worked out later gives an invariant description of the fifth dimension and allows one to interpret 5-dimensional objects in terms of 4-dimensional ones \[10\]. A modern presentation of the classical 5-dimensional KKT together with the (1+4)-splitting formalism can be found in \[5\]. Some necessary features of the theory are presented below.

The 5-dimensional metric $G$ defined on a Riemannian space $V_5$ is the basic geometric object in KKT. It can be taken in the form

$$G = -\lambda \otimes \lambda + \tilde{g}$$

(1)

where $\lambda = \lambda_A dx^A$ is the 1-form determining a (1+4)-splitting of the 5-dimensional space ($A = 0, 1, 2, 3, 5$), $\tilde{g}$ is the local metric of a 4-dimensional hypersurface orthogonal to $\lambda$. These objects satisfy the following conditions:

$$\lambda_A \lambda^A = -1; \quad \lambda^A \tilde{g}_{AB} = 0.$$  

The tilde over $g$ distinguishes the geometric 4-dimensional metric and the physical (observable) one (see below).

To shorten the presentation, we make some assumptions which simplify the mathematical formulae and “tune” the KKT to the problem of finding exact 4-dimensional solutions:\[1\]

1. The coordinate lines of $x^5$ can be associated with a vector field $\lambda$ which corresponds to its special gauge:

$$\lambda^A = \frac{G^A_5}{\sqrt{-G_{55}}} \rightarrow \lambda^A = \frac{G_{A5}}{\sqrt{-G_{55}}}.$$  

(2)

2. We accept the standard KKT argumentation: the unobservability of the 5th dimension can be explained by cylindricity of 5-dimensional world in the fifth coordinate. In the formalism to be presented this means that $\lambda$ is a Killing vector in $V_5$. In the gauge chosen the cylindricity condition takes the form

$$\partial_5 G_{AB} = 0.$$  

(3)

\[1\]A general approach is developed in \[5\].

\[2\]The compactification assumption can be accepted as well, but here it is not necessary.
3. The class of admissible coordinate transformations which do not violate (3) includes purely 4-dimensional general coordinate transformations and gauge transformations:

\[ x^5 = x^5 + f(x^0, x^1, x^2, x^3). \]

4. Only the 4-dimensional space-time section tensors projected onto \( \vec{\lambda} \) or orthogonal to \( \vec{\lambda} \) have a physical meaning.

5. The fifth direction should be spacelike to get a correct sign of the geometrized energy-momentum tensor of the electromagnetic field in the right-hand side of the projected 5-dimensional Einstein equations. The relation between the 4-vector potential \( A \) and the 4-dimensional part of \( \vec{\lambda} \) has the form

\[ \lambda_{\mu} = (2\sqrt{k/c^2})\varphi A_{\mu}, \]

where \( \varphi = \sqrt{-G_{55}} \) is the geometrized scalar field, \( k \) is the Newtonian gravitational constant, \( c \) is the velocity of light and \( \mu = 0, 1, 2, 3 \).

6. The 5-dimensional Einstein equations are vacuum:

\[ (^5)G_{AB} = 0, \]

where \((^5)G_{AB}\) is the 5-dimensional Einstein tensor. This assumption makes the theory more economic and close to its modern versions \([\text{2, 4]}\).

7. The observable 4-dimensional metric is not the projector \( \tilde{g} \) but the conformally transformed tensor \( g \), related to \( \tilde{g} \) by the expression

\[ \tilde{g} = \varphi^2 g. \]

Taking into account all these items, the projected equations (7) take the form

\[ (^4)G_{\mu\nu} = \alpha T_{\mu\nu}^{(\text{em})} + 3(\phi_{,\mu,\nu} - \phi_{,\nu,\mu} - g_{\mu\nu}(\nabla^2 \phi + (\nabla \phi)^2)); \]

\[ F_{\mu\nu} - 3\phi_{,\mu}F_{\alpha\mu} = 0; \]

\[ \nabla^2 \phi + (\nabla \phi)^2 - \frac{1}{6}^4R - \frac{3\alpha}{8\pi}F^2 = 0. \]

Here: \((^4)G_{\mu\nu}\) is the 4-dimensional Einstein tensor built up from the observable metric \( g \);

\( F_{\mu\nu} = A_{\nu,\mu} - A_{\mu,\nu} \) is the geometrized electromagnetic field tensor;

\( T_{\mu\nu}^{(\text{em})} = -(1/4\pi)(F_{\mu} \wedge F_{\nu} - (1/4)g_{\mu\nu}F^2) \)

is the canonical energy-momentum tensor of the electromagnetic field;

\( F^2 = F_{\alpha\beta}F^{\alpha\beta} \);

\( \phi = \ln \varphi = \ln \sqrt{-G_{55}} \) is a new notation for the geometric scalar field;

\( \nabla^2 = g^{\alpha\beta}\nabla_\alpha \nabla_\beta \);

\( \nabla_\alpha \) is the covariant derivative corresponding to \( g \).

The ten equations (5) are the \((\mu\nu)\)-projection of (3), the four equations (6) are the \((\mu5)\)-projection of (3) and eq. (9) is the \((55)\)-projection.

Thus in the present version of the KKT any vacuum solution of the 5-dimensional Einstein equations, which is cylindrical in the 5th coordinate, corresponds to the set (4)–(9) of the Einstein-Maxwell-scalar field equations in 4-dimensional GR.
3 5-dimensional transformations and generation theorems

Consider the general 5-dimensional coordinate transformations

\[ x'^A = x^A'(x^A) \]  

(10)

and choose from those which preserve the cylindricity condition. This means that the transformed metric \( G_{A'B'} \) satisfies the cylindricity condition in the new coordinates \( x'A' \): \( \partial_{B'} G_{A'B'} = 0 \).

To find out whether such transformations do exist, let us formulate a theorem. The following chain of equalities takes place:

\[ \alpha_A^A \equiv \partial x'^A / \partial x^A \]

— Jacobi matrix elements;

\[ \alpha'_A \equiv \partial x^A / \partial x'^A \]

— inverse Jacobi matrix elements;

\[ (\alpha'^A_A \alpha_A^B = \delta'_B^B); \quad s^A \equiv \alpha_A^A \]

— vector-column of the inverse Jacobi matrix. Hereafter it is assumed that a sufficiently smooth reversible coordinate transformation (10) is given.

**Theorem 1** The cylindricity condition (3) is satisfied in the coordinate system \( x^A' \) if and only if the vector-column \( s^A \) is a Killing vector of the original metric.

**Proof.** Let the transformed metric satisfy the cylindricity condition \( G_{A'B',5'} = 0 \). Then the following chain of equalities takes place:

\[
G_{A'B',5'} = (G_{AB} \alpha^A_{A'} \alpha^B_{B'}),_{5'} = G_{AB,5'} \alpha^A_{A'} \alpha^B_{B'} + G_{AB} \alpha^A_{A'} \alpha^B_{B'},_{5'} + G_{AB} \alpha^A_{A'} \alpha^B_{B'} = 0.
\]

Here we have used the symmetry properties of the Jacobi matrix derivatives, the definition of \( s^A \) and a substitution of variables. Due to the nondegeneracy of the transformation, the last equality gives the Killing equations for \( s^A \). The necessity is proved. If the 5-dimensional metric admits a Killing vector, then, identifying its components with \( \alpha^A_{5'} \) and repeating the above formulae backward, we get the cylindricity condition for the transformed metric.

Hereafter we will call the Killing vector that has appeared in the theorem a generating Killing vector. A 5-dimensional space-time can admit several generating Killing vectors. Let us note that the conditions of the theorem do not include the cylindricity of the original metric.

Using Theorem 1, let us prove the following useful propositions:

1. **In the new coordinate system the vector \( s \) has the components** \( s'^A = \alpha'^A_A s^A = \alpha'^A_A \alpha^A_{5'} = \delta^A_{5'} \). This circumstance makes clear the geometric meaning of the method discussed in the Introduction.

2. **The generating Killing covector in the new coordinate system becomes proportional to the vector potential of a new geometrical electromagnetic field.** This can be easily seen from the fact that \( s'^A = \delta^A_{5'} \). Lowering the index \( A' \) using the new metric \( G_{A'B'} \), we get \( s_{5'} = G_{5'B'} \sim A_{5'} \) (see (3) and (2)). This fact leads us to the hypothesis that the proposed method is related to the Mitskievich-Horský generation procedure, which is purely 4-dimensional.
3. A general form of admissible transformations is given by the expressions

\[ x^{\mu'} = F^{\mu'}(\varphi^\nu); \quad x^5' = F^5'(\varphi^\nu, \varphi^5) \]

where \{\varphi^A\} is a set of independent functions defining the set of integral curves of a generating Killing vector, \( F^A' \) is a set of arbitrary independent differentiable functions of its arguments. This proposition directly follows from the set of linear partial differential equations obtained from the equality:

\[ s^A' = \alpha_A^B s^B = \delta^A_5'. \]  \hspace{1cm} (11)

The functions \( \varphi^A \) are independent integrals of the corresponding set of characteristics equations.

4. The linearity of the Killing equations leads to the fact that any linear superposition of Killing vectors with constant coefficients can be considered as a generating Killing vector.

5. The new scalar field \( \varphi' \) is related to a generating Killing vector in the following way: \( G_{55'} = -\varphi'^2 = \hat{s} \cdot \hat{s}' \). In the new coordinate system \( \hat{s}' \cdot \hat{s}' = G_{AA'} \delta^A_5' \delta^B_5' = G_{55'} \). The proposition follows from the invariance of the norm.

This last proposition allows one to get rid of the scalar field in the transformed set of equations (7)-(9), which has been noted by Rosly. In \[7\] he has formulated and proved the following theorem:

**Theorem 2 (Rosly)** If a 5-dimensional metric admits a space-like Killing vector with a constant norm, then the admissible coordinate transformation generated by this vector gives a 4-dimensional vacuum solution of the Einstein-Maxwell equations. The electromagnetic field obtained in this way satisfies the condition \( F^2 = 0 \).

The theorem directly follows from proposition 5 and eqs. (7)-(9) with \( \phi = \text{const.} \)

4 Generation of exact solution with a scalar field

Let us demonstrate the present procedure taking the flat 5-dimensional metric

\[ ds^2_{(5)} = dt^2 - dx^2 - dy^2 - dz^2 - dv^2. \] \hspace{1cm} (12)

as the original one. It has a complete 15-parameter isometry group.

Let us take a generating vector in the following form: \( \hat{s} = x\partial_y - y\partial_x + \alpha^0\partial_t + \alpha^z\partial_z + \alpha^v\partial_v \), which is a linear superposition of the rotation generator in the \((xy)\) plane and translations along the \(t, z, v\) axes. Solution of eqs. (11) gives the following transformations (up to 4-dimensional and gauge transformations):

\[
\begin{align*}
t &= t' + av' \ , \\
x &= x' \sin v' \\
y &= y' \cos v' \\
z &= z' + bv' \\
v &= y'.
\end{align*}
\]

Here \(a\) and \(b\) are arbitrary constants. The metric (12) transformed according to (11) takes the form (we omit accents at the new coordinates):

\[ ds^2_{(5)} = dt^2 - dx^2 - dy^2 - dz^2 - (\Delta + x^2)dv^2 + 2dv(a\ dt - b\ dz) \]
where $\Delta = b^2 - a^2$. The 1-forms $\lambda$ and $A$ from (2) and (4) have the form

\[
\lambda = \frac{1}{\sqrt{\Delta + x^2}}(a \, dt - b \, dz - (\Delta + x^2) \, dv); \quad A = \frac{1}{2(\Delta + x^2)}(a \, dt - b \, dz).
\]

The physical metric can be found from (1) by the conformal transformation (6). It has the form

\[
\frac{1}{X^2}(4) \quad ds^2 = \frac{1}{X^2}[X_b \, dt^2 - X(dx^2 + dy^2) - X_a \, dz^2 - 2ab \, dt \, dz],
\]

where $X = \Delta + x^2$; $X_a = x^2 - a^2$; $X_b = x^2 + b^2$. Taking an external differential of the 1-form $A$, we get the electromagnetic field 2-form $F$:

\[
F = dA = x(\Delta + x^2)^2(adt \wedge dx + bdx \wedge dz).
\]

The electric current density 4-vector (8) is

\[
\vec{j} = -\frac{3}{4\pi}\phi_{\alpha}F^{\alpha\mu}\partial_{\mu} = -\frac{3}{4\pi}(a\partial_t + b\partial_z).
\]

Assuming that the electric charge is carried by the geometrized matter, i.e. $\vec{j} = \sigma \vec{u}_c$ where $\sigma$ is the electric charge density and $\vec{u}_c$ is the 4-velocity of charged matter, we get:

\[
\vec{j} \cdot \vec{j} = \sigma^2 = \frac{9\Delta}{16\pi^2X^2}; \quad \sigma = \frac{3}{4\pi X \sqrt{-\Delta}}; \quad \vec{u}_c = \vec{j}/\sigma = -\frac{X}{\sqrt{-\Delta}}(a\partial_t + b\partial_z).
\]

From the expression for $\sigma$ one can get the following restriction on the parameters $a$ and $b$: $\Delta < 0$, or $|a| > |b|$.

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