ON THE FIELDS OF DEFINITION OF HODGE LOCI

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Abstract. Given a polarizable variation of \( \mathbb{Z} \)-Hodge structure \( V \) over a smooth quasi-projective complex variety \( S \), Cattani, Deligne and Kaplan proved that the Hodge locus of closed points \( s \in S \) such that \( V_s \) admits exceptional Hodge tensors is a countable union of strict closed irreducible algebraic subvarieties of \( S \), called the special subvarieties of \( S \) for \( V \).

When \( V \) is moreover defined over a number field \( L \subset \mathbb{C} \) i.e. both \( S \) and the filtered algebraic module with integrable connection \( (V, F^\bullet, \nabla) \) associated with \( V \) are defined over \( L \), any special subvariety of \( S \) for \( V \) is conjectured to be defined over \( \mathbb{Q} \), and its Gal(\( \mathbb{Q} \)/\( L \))-conjugates to be again special subvarieties for \( V \). In the geometric case this follows from the conjecture that Hodge classes are absolute Hodge.

We prove that if \( S \) is defined over a number field \( L \) then any special subvariety of \( S \) for \( V \) which is weakly non-factor is defined over \( \mathbb{Q} \); and that its Gal(\( \mathbb{Q} \)/\( L \))-conjugates are special if moreover \( V \) is defined over \( L \). The non-factor condition roughly means that the special subvariety cannot be non-trivially Hodge-theoretically deformed inside a larger special subvariety.

Our result implies that if \( S \) is defined over a number field \( L \subset \mathbb{C} \) and if the adjoint group of the generic Mumford-Tate group of \( V \) is simple then any strict special subvariety of \( S \) for \( V \) with non-trivial algebraic monodromy and which is maximal for these properties is defined over \( \overline{\mathbb{Q}} \) and that its Gal(\( \overline{\mathbb{Q}} \)/\( L \))-conjugates are special if moreover \( V \) is defined over \( L \). It also implies that special subvarieties for \( Z \)-VHSs defined over a number field are defined over \( \mathbb{Q} \) if and only if it holds true for special points.

1. Introduction

1.1. Hodge loci. The main object of study in this article are Hodge loci. Let us start by recalling their definition in the geometric case, where their behaviour is predicted by the Hodge conjecture.

1.1.1. The geometric motivation. Let \( f : X \to S \) be a smooth projective morphism of smooth irreducible complex quasi-projective varieties and let \( k \) a positive integer. The Betti and De Rham incarnation of the \( 2k \)-th cohomology of the fibers of \( f \) give rise to a weight zero polarizable variation of Hodge structure \( (V := R^{2k}f_*\mathbb{Z}(k), V := R^{2k}f_*\Omega^\bullet_{X/S}, F^\bullet, \nabla) \) on \( S \). Here \( V \) is the local system on the complex manifold \( S^{an} \) associated to \( S \) parametrizing the \( 2k \)-th Betti cohomology of the fibers of \( f \); \( V \) is the corresponding algebraic vector bundle, endowed with its flat Gauss-Manin connection; and \( F^\bullet \) is the Hodge filtration on \( V \) induced by the stupid filtration on the algebraic De Rham complex \( \Omega^\bullet_{X/S} \). In this situation one defines the locus of exceptional Hodge classes \( \text{Hod}(V) \subset V^{an} \) as the set of Hodge classes \( \lambda \in F^0V^{an} \cap V_Q \) whose orbit under monodromy is infinite, and the Hodge locus \( \text{HL}(S,V) \) as its projection in \( S^{an} \). Thus
Theorem 1.1 says that \( \text{Hod}(\mathcal{V}) \) implies, in addition to \( \star \). In that case one easily checks, refining Weil’s argument, that the Hodge conjecture.

\[ \text{corresponding special subvarieties.} \]

1.2.1. Algebraicity of Hodge loci. More generally let \( (\mathcal{V}, \mathcal{V}^\bullet, \nabla) \) be any polarizable variation of \( \mathbb{Z} \)-Hodge structure (ZVHS) on a smooth complex irreducible algebraic variety \( S \). Thus \( \mathcal{V} \) is a finite rank \( \mathbb{Z}_{\text{an}} \)-local system on the complex manifold \( S_{\text{an}} \); and \( (\mathcal{V}, \mathcal{V}^\bullet, \nabla) \) is the unique regular algebraic module with integrable connection on \( S \) whose analytification is \( \mathcal{V} \otimes_{\mathbb{Z}_{\text{an}}} \mathcal{O}_{S_{\text{an}}} \) endowed with its Hodge filtration \( \mathcal{F}^\bullet \) and the holomorphic flat connection \( \nabla_{\text{an}} \) defined by \( \mathcal{V} \), see \([\text{Sc73}, (4.13)]\). We will abbreviate the ZVHS \( (\mathcal{V}, \mathcal{V}^\bullet, \nabla) \) simply by \( \mathcal{V} \). If we define the locus of exceptional Hodge classes \( \text{Hod}(\mathcal{V}) \subset \mathcal{V} \) and the Hodge locus \( \text{HL}(S, \mathcal{V}) \subset S \) as in the geometric case, Cattani, Deligne and Kaplan \([\text{CDK95}]\) proved a vast generalization of Weil’s expectation:

**Theorem 1.1.** (Cattani-Deligne-Kaplan) Let \( \mathcal{V} \) be a ZVHS on a smooth complex quasi-projective variety \( S \). Then \( \star \) holds true.

From now on we do not distinguish a complex algebraic variety \( X \) from its associated complex analytic space \( X_{\text{an}} \), the meaning being clear from the context. It will be convenient for us to work in the following more general tensorial setting. Let \( \mathcal{V}^\otimes \) be the infinite direct sum of ZVHS \( \bigoplus_{a,b \in \mathbb{N}} \mathcal{V}^{\otimes a} \otimes (\mathcal{V}^\vee)^{\otimes b} \), where \( \mathcal{V}^\vee \) denotes the ZVHS dual to \( \mathcal{V} \); and let \( (\mathcal{V}^\otimes, \mathcal{F}^\bullet) \) be the corresponding filtered algebraic vector bundle of infinite rank. We denote by \( \text{Hod}(\mathcal{V}^\otimes) \subset \mathcal{V}^\otimes \) and \( \text{HL}(S, \mathcal{V}^\otimes) \subset S \) the corresponding locus of Hodge tensors and the tensorial Hodge locus respectively. Thus \( \text{HL}(S, \mathcal{V}^\otimes) \) is the subset of points \( s \) in \( S_{\text{an}} \) for which the Hodge structure \( \mathcal{V}_s \) admits more Hodge tensors than the very general fiber \( \mathcal{V}_s \). Theorem 1.1 says that \( \text{Hod}(\mathcal{V}^\otimes) \) and \( \text{HL}(S, \mathcal{V}^\otimes) \) are countable unions of closed irreducible subvarieties of \( \mathcal{V}^\otimes \) and \( S \) respectively, called the special subvarieties of \( \mathcal{V}^\otimes \) and \( S \) for \( \mathcal{V} \). We refer to \([\text{BKT18}]\) for a simplified proof of the statement for \( \text{HL}(S, \mathcal{V}^\otimes) \) using o-minimal geometry.

1.2. Fields of definition of Hodge loci. The question we attack in this paper is the relation between the field of definition of the ZVHS \( \mathcal{V} \) and the fields of definition of the corresponding special subvarieties.

1.2.1. The geometric case. Once again the geometric case again provides us with a motivation and a heuristic. Suppose that \( f : X \to S \) is defined over a number field \( L \subset \mathbb{C} \). In that case one easily checks, refining Weil’s argument, that the Hodge conjecture implies, in addition to \( \star \):

| (*) | The locus of Hodge classes \( \text{Hod}(\mathcal{V}) \) is a countable union of closed irreducible algebraic subvarieties of \( \mathcal{V} \). The restriction of \( f \) to any such subvariety of \( \mathcal{V} \) is finite over its image. In particular the Hodge locus \( \text{HL}(S, \mathcal{V}) \) is a countable union of closed irreducible algebraic subvarieties of \( S \). |
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## Table

| \( \star \) | The locus of Hodge classes \( \text{Hod}(\mathcal{V}) \) is a countable union of closed irreducible algebraic subvarieties of \( \mathcal{V} \). The restriction of \( f \) to any such subvariety of \( \mathcal{V} \) is finite over its image. In particular the Hodge locus \( \text{HL}(S, \mathcal{V}) \) is a countable union of closed irreducible algebraic subvarieties of \( S \). |

### Notes

- **Theorem 1.1.** (Cattani-Deligne-Kaplan) Let \( \mathcal{V} \) be a ZVHS on a smooth complex quasi-projective variety \( S \). Then \( \star \) holds true.

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- **1.2.1. The geometric case.** Once again the geometric case again provides us with a motivation and a heuristic. Suppose that \( f : X \to S \) is defined over a number field \( L \subset \mathbb{C} \). In that case one easily checks, refining Weil’s argument, that the Hodge conjecture implies, in addition to \( \star \):

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### References

- [Weil79]
- [Sc73]
- [CDK95]
- [BKT18]
(⋆⋆) (a) each irreducible component of \(\text{Hod}(V)\), respectively \(\text{HL}(S,V)\), is defined over a finite extension of \(L\).
(b) each of the finitely many \(\text{Gal}(\overline{\mathbb{Q}}/L)\)-conjugates of such a component is again an irreducible component of \(\text{Hod}(V)\), respectively \(\text{HL}(S,V)\).

Remark 1.2. Of course (⋆⋆) for \(\text{Hod}(V)\) implies (⋆⋆) for \(\text{HL}(S,V)\), and is a priori strictly stronger.

Remark 1.3. The full Hodge conjecture is not needed to expect (⋆⋆) to hold. As proven by Voisin [Voi07, Lemma 1.4], the property (⋆⋆) for \(\text{Hod}(V)\) is equivalent to the conjecture that Hodge classes in the fibers of \(f\) are (de Rham) absolute Hodge classes. We won’t use the notion of absolute Hodge classes in this article and refer the interested reader to [ChSc14] for a survey.

1.2.2. Variations of Hodge structure defined over a number field. Let us now turn to general \(\mathbb{Z}\)VHS.

Definition 1.4. We say that a \(\mathbb{Z}\)VHS \(V\) is defined over a number field \(L \subset \mathbb{C}\) if \(S, V, F^\bullet\) and \(\nabla\) are defined over \(L\): \(S = S_K \otimes_K \mathbb{C}, V = V_K \otimes_K \mathbb{C}, F^\bullet V = (F^\bullet_K V_K) \otimes_K \mathbb{C}\) and \(\nabla = \nabla_K \otimes_K \mathbb{C}\) with the obvious compatibilities.

In the same way the property (⋆), which is implied by the Hodge conjecture in the geometric case, was proven to be true for a general \(\mathbb{Z}\)VHS, we expect the property (⋆⋆), which is implied by the Hodge conjecture in the geometric case, to hold true for any \(\mathbb{Z}\)VHS \(V\), namely:

Conjecture 1.5. Let \(V\) be a \(\mathbb{Z}\)VHS defined over a number field \(L \subset \mathbb{C}\). Then:
(a) any special subvariety of \(\mathcal{V}^\circ\), resp. \(S\), for \(V\) is defined over a finite extension of \(L\).
(b) any of the finitely many \(\text{Gal}(\overline{\mathbb{Q}}/L)\)-conjugates of a special subvariety of \(\mathcal{V}^\circ\), resp. \(S\), for \(V\) is a special subvariety of \(\mathcal{V}^\circ\), resp. \(S\), for \(V\).

Remark 1.6. Simpson conjectures that any \(\mathbb{Z}\)VHS defined over a number field \(L \subset \mathbb{C}\) ought to be motivic: there should exist a \(\overline{\mathbb{Q}}\)-Zariski-open subset \(U \subset S\) such that the restriction of \(V\) to \(U\) is a direct factor of a geometric \(\mathbb{Z}\)VHS on \(U\), see [Si90, “Standard conjecture” p.372]. Thus Conjecture 1.5 would follow from Simpson’s “standard conjecture” and (⋆⋆) in the geometric case. Of course Simpson’s standard conjecture seems unreachable with current techniques.

Let us mention the few results in the direction of Conjecture 1.5 we are aware of:

Suppose we are in the geometric situation of a morphism \(f : X \to S\) defined over \(\mathbb{Q}\). In [Voi07, Theor. 0.6] (see also [Voi13, Theor. 7.8]), Voisin proves the following:

(1) for \(\text{Hdg}(V)\): let \(Z \subset V\) is an irreducible component of \(\text{Hod}(V)\) through a Hodge class \(\alpha \in H^{2k}(X_0, Z(k))_{\text{prim}}\) such that the only constant sub-QVHS of the base change of \(V_\mathbb{Q}\) to \(Z\) is \(\mathbb{Q} \cdot \alpha\). Then \(Z\) is defined over \(\overline{\mathbb{Q}}\).

(2) for \(\text{HL}(S,V)\): under the weaker assumption that any constant sub-QVHS of the base change of \(V_\mathbb{Q}\) to \(Z\) is purely of type \((0,0)\), the irreducible component \(p(Z)\) of \(\text{HL}(S,V)\) is defined over \(\overline{\mathbb{Q}}\) and its \(\text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})\)-translates are still special subvarieties of \(S\) for \(V^\circ\).
In the case of a general ZVHS Saito and Schnell [SaSc16] prove:

(1) for Hod(V): if V is defined over a number field then a special subvariety of V for V is defined over \( \overline{\mathbb{Q}} \) it contains a \( \overline{\mathbb{Q}} \)-point of V.

(2) for HL(S, V): without assuming that V is defined over \( \overline{\mathbb{Q}} \) but only assuming that S is defined over a number field \( L \), then a special subvariety of S for V is defined over \( \overline{L} \) if and only if it contains a \( \overline{\mathbb{Q}} \)-point of S. This generalizes the well-known fact that the special subvarieties of Shimura varieties are defined over \( \mathbb{Q} \) (as any special subvariety of a Shimura variety contains a CM-point, and CM-points are defined over \( \mathbb{Q} \)).

**Remark 1.7.** These results seem to indicate a significant gap in difficulty between Conjecture 1.5 for Hod(V) and Conjecture 1.5 for HL(S, V). Saito and Schnell’s result (2), which only requires S to be defined over \( \overline{\mathbb{Q}} \), looks particularly surprising. They also seem to indicate that the statement (b) in Conjecture 1.5 goes deeper than (a). In particular Saito and Schnell’s result (2) says nothing about Galois conjugates.

**Remark 1.8.** Voisin’s and Saito-Schnell’s criteria look difficult to check in practice. Even in explicit examples one usually knows very little about the geometry of a special variety Y. In Voisin’s case one would need to control the Hodge structure on the cohomology of a smooth compactification of \( X \) base-changed to \( Z \). In Saito-Schnell’s case there is in general no natural source of \( \overline{\mathbb{Q}} \)-points (like the CM points in the Shimura case).

### 1.3. Main results.

All results in this paper concern Conjecture 1.5 for HL(S, V). We provide a simple geometric criterion for a special subvariety of S for V to be defined over \( \overline{\mathbb{Q}} \) and its Galois conjugates to be special.

Let us first recall the notion of algebraic monodromy group.

**Definition 1.9.** Let S be a smooth irreducible complex algebraic variety, let \( k \) be a field and \( V \) a \( k \)-local system (of finite rank) on \( S^{an} \) (in our case \( k \) will be \( \mathbb{Q} \) or \( \mathbb{C} \)). Given an irreducible closed subvariety \( Y \subset S \), the algebraic monodromy group \( H_Y \) of \( Y \) for \( V \) is the \( k \)-algebraic group connected component of the Tannaka group of the category \( \langle V|_{Y^{nor}} \rangle_{k_{loc}} \) of \( k \)-local systems on (the normalisation of) \( Y \) tensorially generated by the restriction of \( V \) and its dual.

Equivalently \( H_Y \) is the connected component of the Zariski-closure of the monodromy \( \rho : \pi_1(Y^{nor, an}) \to GL(V_k) \) of the local system \( V|_{Y^{nor}} \).

**Definition 1.10.** Let S be a smooth irreducible complex algebraic variety and \( V \) a \( k \)-local system on \( S^{an} \). Let \( Y \subset S \) be an irreducible closed subvariety. We say that \( Y \) is weakly non-factor for \( V \) if it is not contained in a closed irreducible \( Z \subset S \) such that the \( k \)-algebraic monodromy group \( H_Y \) is a strict normal subgroup of \( H_Z \). We say that \( Y \) is positive dimensional for \( V \) if \( H_Y \neq \{1\} \).

**Remark 1.11.** If \( V \) is a \( k \)-local system on S, \( Y \subset S \) is a closed irreducible subvariety, and \( k' \) is a field extension of \( k \), the \( k' \)-algebraic monodromy group \( H_Y(V \otimes_k k') \) is the base change \( H_Y(V) \otimes_k k' \). Thus being weakly non-factor for \( V \) and positive dimensional for \( V \) is equivalent to being weakly non-factor for \( V \otimes_k k' \) and positive dimensional for \( V \otimes_k k' \) respectively.

Our main result in this paper is the following:
Theorem 1.12. Let $\mathcal{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety $S$.

(a) if $S$ is defined over a number field $L$ then any special subvariety of $S$ for $\mathcal{V}$ which is weakly non-factor for $\mathcal{V}_\mathbb{Q}$ is defined over a finite extension of $L$;
(b) if moreover $\mathcal{V}$ is defined over $L$ then the finitely many $\text{Gal}(\overline{\mathbb{Q}}/L)$-translates of such a special subvariety are also special, weakly non-factor subvarieties of $S$ for $\mathcal{V}$.

As a first corollary we obtain Conjecture 1.5 for maximal strict special subvarieties of $S$ under a simplicity assumption on the generic Mumford-Tate group:

Corollary 1.13. Let $\mathcal{V}$ be a polarized variation of $\mathbb{Z}$-Hodge structure on a smooth quasi-projective variety $S$, whose adjoint generic Mumford-Tate group $G^\text{ad}_S$ is simple. Then:

(a) if $S$ is defined over a number field $L$ then any strict special subvariety $Y \subset S$ for $\mathcal{V}$, which is positive dimensional for $\mathcal{V}$ and maximal for these properties, is defined over $\mathbb{Q}$.
(b) if $\mathcal{V}$ is moreover defined over $L$ then the finitely many $\text{Gal}(\overline{\mathbb{Q}}/L)$-translates of such a special subvariety are special subvarieties of $S$ for $\mathcal{V}$.

Theorem 1.12 also enables to reduce the Conjecture 1.5(a) for $\text{HL}(S, \mathcal{V})$ to the case of points:

Corollary 1.14. Special subvarieties for $\mathbb{Z}$-VHSs defined over $\mathbb{Q}$ are defined over $\overline{\mathbb{Q}}$ if and only if it holds true for special points.

2. ZVHS versus local systems, Mumford-Tate group versus monodromy, special versus weakly special

In this section we recall the geometric background providing the intuition for Theorem 1.12, namely the geometry of special subvarieties and their generalization, the weakly special subvarieties. We refer to [K17] and [KO19] for details.

Let $\mathcal{V}_\mathbb{Q}$ be a $\mathbb{Q}$-local system on $S$ and $Y \subset S$ an irreducible closed subvariety. In Definition 1.9 we recalled the definition of the algebraic monodromy group $H_Y$ for $\mathcal{V}_\mathbb{Q}$. Suppose now that $\mathcal{V}_\mathbb{Q}$ underlies a $\mathbb{Z}$-VHS $\mathcal{V}$ over $S$. In addition to $H_Y$, which depends only on the underlying local system, one attaches a more subtle invariant to $Y$ and $\mathcal{V}$: the generic Mumford-Tate group $G_Y$ i.e. the Tannaka group of the category $\langle \mathcal{V}_{(\text{Y}\text{an})} \rangle_{\text{QVHS}}$ of $\mathbb{Q}$-VHS on the normalisation of $Y$ tensorially generated by the restriction of $\mathcal{V}$ and its dual. This group is usually much harder to compute than $H_Y$ as its definition is not purely geometric. The $\mathbb{Z}$-VHS $\mathcal{V}$ is completely described by its complex analytic period map $\Phi_S : S^{\text{an}} \to X_S := \Gamma \backslash D_S$. Here $D_S$ denotes the Mumford-Tate domain associated to the generic Mumford-Tate group $G_S$ of $(S, \mathcal{V})$. $\Gamma_S \subset G_S(\mathbb{Q})$ is an arithmetic lattice and the complex analytic quotient $X_S$ is called the Hodge variety associated to $\mathcal{V}$. The special subvarieties of the Hodge variety $X_S$ and their generalisation, the weakly special subvarieties of $X_S$ are defined purely in group-theoretic terms, see [KO19, Def. 3.1]. One proves that the special subvarieties of $S$ for $\mathcal{V}$ are precisely the irreducible components of the $\Phi_S$-preimage of the special subvarieties of $X_S$, thus obtaining the following characterization, see [KO19, Def. 1.2].
Proposition 2.1. Let $\mathcal{V}$ be a $\mathbb{Z}$-VHS on $S$. A special subvariety of $S$ for $\mathcal{V}$ is a closed irreducible algebraic subvariety $Y \subset S$ maximal among the closed irreducible algebraic subvarieties of $S$ with generic Mumford-Tate group $G_Y$.

Similarly, one defines a generalisation of the special subvarieties of $X_S$, the so-called weakly special subvarieties of $X_S$, purely in group-theoretic terms see [KO19, Def. 3.1]. The weakly special subvarieties of $S$ for $\mathcal{V}$, which generalize the special ones, are defined as the irreducible components of the $\Phi_S$-preimage of the weakly special subvarieties of $X_S$. Again one obtains the following characterization, see [KO19, Cor. 3.14]:

Proposition 2.2. Let $\mathcal{V}$ be a $\mathbb{Z}$-VHS on $S$. A weakly special subvariety $Y \subset S$ for $\mathcal{V}$ is a closed irreducible algebraic subvariety $Y$ of $S$ maximal among the closed irreducible algebraic subvarieties of $S$ with algebraic monodromy group $H_Y$.

A posteriori Proposition 2.2 offers an alternative definition of the weakly special subvarieties of $S$ for a $\mathbb{Z}$-VHS $\mathcal{V}$. It is important for us to notice that this alternative definition of the weakly special subvarieties of $S$ for $\mathcal{V}$ makes sense for $\mathcal{V}$ any $k$-local system on $S^{an}$, $k$ a field:

Definition 2.3. Let $k$ be a field and let $\mathcal{V}$ be a $k$-local system on $S$. We define a weakly special subvariety $Y \subset S$ for $\mathcal{V}$ to be a closed irreducible algebraic subvariety $Y$ of $S$ maximal among the closed irreducible algebraic subvarieties of $S$ with algebraic monodromy group $H_Y$.

Remark 2.4. Following Remark 1.11 $Y$ being weakly special for $\mathcal{V}$ is equivalent to $Y$ being weakly special for $\mathcal{V} \otimes_k k'$.

For $\mathcal{V}$ a ZVHS and $Y \subset S$ an irreducible closed subvariety there exists a unique weakly special subvariety $\langle Y \rangle_{ws}$ with algebraic monodromy group $H_Y$ and a unique special subvariety $\langle Y \rangle_s$ with generic Mumford-Tate group $G_Y$ containing $Y$, see [KO19, 2.1.4]:

$$Y \subset \langle Y \rangle_{ws} \subset \langle Y \rangle_s \subset S.$$ When $\mathcal{V}$ is a mere local system there exists by definition a weakly special subvariety with algebraic monodromy group $H_Y$ and containing $Y$ but its uniqueness is not clear to us.

Let us now recall that for $\mathcal{V}$ a ZVHS special subvarieties of $S$ for $\mathcal{V}$ can be thought of as families of weakly special subvarieties. Indeed let $Y \subset S$ be a weakly special subvariety. A fundamental result of Deligne-André [An92, Theor.1] states that the group $H_Y$ is normal in (the derived group of) $G_Y$. Following [KO19, Prop. 2.13], the decomposition $G_Y^{\text{ad}} = H_Y^{\text{ad}} \times G_Y^{\text{ad}}$ induces a product decomposition $X_Y = wX_Y \times X_Y'$, where $X_Y$ is the smallest special subvariety of $X_S$ containing $\Phi_S(Y)$ and $Y$ is (an irreducible component of) $\Phi_S^{-1}(wX_Y \times \{x'_0\})$ for a certain point $x'_0 \in X_Y'$, and a weakly special subvariety $wX_Y$ of $X_S$. All the (irreducible components of) the preimages $\Phi_S^{-1}(wX_Y \times \{x'\})$, $x' \in X_Y'$, are weakly special subvarieties of $S$ for $\mathcal{V}$ that can be thought as Hodge theoretic deformations of $Y$. In particular, there are only countably many special subvarieties of $S$ for $\mathcal{V}$, while there are uncountably many weakly special ones, organized in countably many “product families”.
We can now make a few remarks on the notion of weakly non-factor subvarieties defined in Definition 1.10:

1. For $V$ a local system a closed irreducible subvariety $Y \subset S$ is weakly non-factor if and only if any weakly special subvariety $Y \subset Z \subset S$ with $H^*_Z = H^*_Y$ is weakly non-factor. When $V$ is a ZVHS it amounts to saying that the weakly special closure $(Y)_{ws} \subset S$ is weakly non-factor.

2. Let $V$ be a ZVHS. Given a closed irreducible subvariety $Y \subset S$, let $wX_Y \subset X_S$ be the smallest weakly special subvariety containing $\Phi_S(Y)$. It follows from the above description of the weakly special subvarieties that $Y$ is weakly non-factor for $V$ if and only if there does not exist $Y \subset Z \subset S$, with $Z$ closed irreducible, such that $wX_Z = wX_Y \times wX' \subset X_S$ with $wX'$ a positive dimensional weakly special subvariety of $X_S$. The “weakly non-factor” condition is thus a Hodge theoretic rigidity of $Y$. In particular one obtains the following:

**Lemma 2.5.** Let $V$ be a ZVHS on $S$. Any weakly non-factor, weakly special subvariety of $S$ is special.

3. The terminology “weakly non-factor” generalizes the terminology “non-factor” introduced by Ullmo [Ull07] for special subvarieties of Shimura varieties.

4. For $V$ a non-isotrivial local system on $S$, it follows from the definition that for any weakly non-factor subvariety $Y \subset S$ the algebraic monodromy group $H^*_Y$ is non-trivial. When $V$ is moreover a ZVHS this last condition is equivalent to saying that $Y$ is positive dimensional for $V$ in the sense of [KO19]: its image $\Phi_S(Y)$ is not a point.

Given $S$ a smooth complex quasi-projective variety and $V$ a complex local system, we say that $V$ is defined over a number field $L \subset \mathbb{C}$ if both $S$ and the algebraic module with integrable connection $(V, \nabla)$ corresponding to $V$ under the Deligne-Riemann-Hilbert correspondence (see (3.1) below) are defined over $L$. Theorem 1.12 then follows immediately from Lemma 2.5 and the general result on local systems:

**Theorem 2.6.** Let $S$ be a smooth complex quasi-projective variety and $V$ a complex local system on $S^{an}$.

(a) Suppose that $S$ is defined over a number field $L$. Then any weakly special, weakly non-factor subvariety of $S$ for $V$ is defined over a finite extension of $L$;

(b) if moreover $V$ is defined over $L$, then any $\text{Gal}(\overline{\mathbb{Q}}/L)$-translates of a weakly special, resp. weakly non-factor, subvariety of $S$ for $V$ is a weakly special, resp. weakly non-factor, subvariety of $S$ for $V$.

3. Proof of the main results

3.1. **Proof of Theorem 2.6(b).**

Let $S$ be a smooth complex quasi-projective variety, $\text{Loc}_C(S^{an})$ the category of complex local systems of finite rank on $S^{an}$, $\text{MIC}(S^{an})$ the category of holomorphic modules with integrable connection on $S^{an}$ and $\text{MIC}_r(S)$ the category of algebraic modules with regular integrable connection on $S$. Following Deligne [De70, Theor.5.9], the analytification
functor $\text{MIC}_r(S) \to \text{MIC}(S^{\text{an}})$ is an equivalence of tensor categories. Composed with the Riemann-Hilbert correspondence this provide an equivalence of tensor categories

$$\text{(3.1)} \quad \text{MIC}_r(S) \xrightarrow{\sim} \text{Loc}_C(S^{\text{an}}).$$

Let $\mathcal{V} \in \text{Loc}_C(S^{\text{an}})$. Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism. Let $S^{\sigma} := S \times_{\mathbb{C}, \sigma} \mathbb{C}$ be the twist of $S$ under $\sigma$. We denote by $\mathcal{V}^{\sigma} \in \text{Loc}_C((S^{\sigma})^{\text{an}})$ the image of $\mathcal{V}$ under the composition of equivalence of (Tannakian) categories

$$\text{(3.2)} \quad \text{Loc}_C(S^{\text{an}}) \overset{\tau^{-1}}{\sim} \text{MIC}_r(S) \overset{\times_{\mathbb{C}, \sigma} \mathbb{C}}{\sim} \text{MIC}_r(S^{\sigma}) \xrightarrow{\sim} \text{Loc}_C((S^{\sigma})^{\text{an}}).$$

Theorem 2.6(b) then follows immediately from the following more general:

**Proposition 3.1.** Let $S$ be a smooth complex quasi-projective variety and $\mathcal{V} \in \text{Loc}_C(S^{\text{an}})$. Let $\sigma : \mathbb{C} \to \mathbb{C}$ be a field automorphism. Let $Y \subset S$ be a closed irreducible subvariety with Galois twist $Y^{\sigma} \subset S^{\sigma}$.

1. the complex algebraic monodromy group $H_Y$ of $Y$ with respect to $\mathcal{V}$ is canonically isomorphic to the complex algebraic monodromy group $H_{Y^{\sigma}}$ of $Y^{\sigma}$ with respect to $\mathcal{V}^{\sigma}$.
2. $Y$ is weakly special for $\mathcal{V}$ if and only if $Y^{\sigma}$ is weakly special for $\mathcal{V}^{\sigma}$.
3. $Y$ is weakly non-factor for $\mathcal{V}$ if and only if $Y^{\sigma}$ is weakly non-factor for $\mathcal{V}^{\sigma}$.

Proof. Let us first assume that $Y$ is smooth. In that case the equivalence of tensor categories (3.2) $\text{Loc}_C(Y^{\text{an}}) \overset{\tau}{\sim} \text{Loc}_C((Y^{\sigma})^{\text{an}})$ restricts to an equivalence of tensor categories

$$\langle \mathcal{V}_{|Y} \rangle^\circ \overset{\tau}{\sim} \langle \mathcal{V}_{|Y}^{\sigma} \rangle^\circ.$$

Taking (the connected component of the identity of) their Tannaka groups we obtain a canonical isomorphism

$$H_Y \simeq H_{Y^{\sigma}},$$

thus proving Proposition 3.1(1) in that case.

When $Y$ is not smooth, we consider a desingularisation $Y^s \xrightarrow{p} Y_{\text{nor}} \xrightarrow{\pi} Y$. Notice that $(Y^s)^{\sigma}$ is a desingularisation of $(Y^{\text{nor}})^{\sigma} = (Y^{\sigma})^{\text{nor}}$. Notice moreover that the algebraic monodromy groups of $(p \circ \pi)^*\mathcal{V}_{|Y}$ and $\pi^*\mathcal{V}_{Y}$ coincide, as $p_* : \pi_1(Y^s) \to \pi_1(Y^{\text{nor}})$ is surjective. Arguing as above for $Y^s$ and $(Y^s)^{\sigma}$ proves Proposition 3.1(1) in general.

Suppose now that $Y \subset S$ is a closed irreducible subvariety. If $Y^{\sigma}$ is not weakly special for $\mathcal{V}$ there exists $Z \supset Y^{\sigma}$ a closed irreducible subvariety of $S^{\sigma}$ containing $Y^{\sigma}$ strictly and such that $H_Z = H_{Y^{\sigma}}$. But then $Z^{\sigma^{-1}}$ is a closed irreducible subvariety of $S$ containing $Y$ strictly, and such that $H_{Z^{\sigma^{-1}}} = H_Y$ by Proposition 3.1(1). It follows that $Y$ is not weakly special. This proves Proposition 3.1(2).

The argument for Proposition 3.1(3) is similar. We are reduced to showing that for $S$ a smooth complex quasi-projective variety, $\mathcal{V} \in \text{Loc}_C(S^{\text{an}})$, $\sigma : \mathbb{C} \to \mathbb{C}$ a field automorphism and $Y \subset S$ a closed irreducible subvariety with Galois twist $Y^{\sigma} \subset S^{\sigma}$, then $H_Y$ is normal in $H_S$ if and only if $H_{Y^{\sigma}}$ is normal in $H_{S^{\sigma}}$. Consider the tannakian subcategory $\mathcal{T}$ of $\text{Loc}_C(S^{\text{an}})$ consisting of the local systems which are trivial in restriction to $Y^{\text{an}}$. Applying $\sigma$ we obtain that $\mathcal{T}^{\sigma}$ is the tannakian subcategory of $\text{Loc}_C((S^{\sigma})^{\text{an}})$ of local systems that are trivial on $(Y^{\sigma})^{\text{an}}$. But as a result of the tannakian formalism the
Tannaka group of \( T \), resp. \( T' \), are the normal closures of \( H_Y \) and \( H_{Y'} \) in \( H_S \) and \( H_{S'} \) respectively. Hence the result.

\[ \square \]

3.2. Proof of Theorem 1.12 when \( V \) is defined over a number field.

Although this is not logically necessary, let us notice that Theorem 1.12 in the case where \( V \) is defined over a number field \( L \) follows from Theorem 2.6(b). Indeed when \( V \) is a ZVHS, weakly special weakly non-factor subvarieties of \( S \) for \( V \) are special subvarieties of \( S \) for \( V \) by Lemma 2.5. Applying Theorem 2.6(b), it follows that the \( \text{Aut}(\mathbb{C}/L) \)-translates of any special, weakly non-factor, subvariety of \( S \) for \( V \) is special (and weakly non-factor). But special subvarieties of \( S \) for \( V \) form a countable set. It follows immediately that any special, weakly non-factor, subvariety of \( S \) for \( V \) is defined over \( \overline{\mathbb{Q}} \) (see for instance [Voi13, Claim p.25]).

\[ \square \]

3.3. Proof of Theorem 2.6(a).

Let us now prove Theorem 2.6(a), hence finish the proof of Theorem 1.12. Let \( S \) be a complex irreducible smooth quasi-projective variety and \( V \) a complex local system on \( S^{an} \). Suppose that \( S \) is defined over a number field \( L \subset \mathbb{C} \). Let \( Y \subset S \) be a weakly special subvariety of \( S \) for \( V \) which is weakly non-factor. Let us show that \( Y \) is defined over \( \overline{\mathbb{Q}} \).

Let \( Z \subset S \) be the \( \overline{\mathbb{Q}} \)-Zariski-closure of \( Y \), i.e. the smallest closed subvariety of \( S \) defined over \( \overline{\mathbb{Q}} \) and containing \( Y \). Thus \( Z \) is irreducible.

The subset \( Z^0 \subset Z \) of smooth points is \( \overline{\mathbb{Q}} \)-Zariski-open (meaning that \( Z - Z^0 \) is a closed subvariety of \( Z \) defined over \( \overline{\mathbb{Q}} \) and dense. Notice that \( Y \cap Z^0 \) is Zariski-open in \( Y \) (otherwise \( Y \) would be contained in the closed subvariety \( Z - Z^0 \) defined over \( \overline{\mathbb{Q}} \), in contradiction to the \( \overline{\mathbb{Q}} \)-Zariski-density of \( Y \) in \( Z \)); moreover the fact that \( Y \subset S \) is weakly special, resp. weakly non-factor for \( (S,V) \) implies that \( Y^0 := Y \cap Z^0 \) is weakly special, resp. weakly non-factor for \( (Z^0, V|_{Z^0}) \). Replacing \( Y \subset S \) by \( Y^0 \subset Z^0 \) if necessary, we can without loss of generality assume that \( Y \) is \( \overline{\mathbb{Q}} \)-Zariski-dense in \( S \). We are reduced to proving that \( Y = S \), or equivalently that \( H_Y = H_S \). This follows immediately from the Proposition 3.2 below, of independent interest.

\[ \square \]

**Proposition 3.2.** Let \( S \) be a smooth complex quasi-projective variety, \( V \) a complex local system on \( S^{an} \) and let \( Y \subset S \) be a closed irreducible weakly non-factor subvariety for \( V \). Suppose that \( S \) is defined over \( \overline{\mathbb{Q}} \) and that \( Y \) is \( \overline{\mathbb{Q}} \)-Zariski-dense in \( S \). Then \( H_Y = H_S \).

**Proof.** Let \( \mathcal{Y} \) be “the” spread of \( Y \) with respect to \( S \). Let us recall its definition. Let \( K \subset \mathbb{C} \) be the minimal field of definition of \( Y \), see [Gro65, Cor. 4.8.11]. This is the smallest subfield \( \overline{\mathbb{Q}} \subset K \subset \mathbb{C} \) such that \( Y \) is defined over \( K \); there exists a \( K \)-scheme of finite type \( Y_K \) such that \( Y = Y_K \otimes_K \mathbb{C} \). Let \( \mathcal{Y}_R \) be a \( R \)-model of \( Y_K = \mathcal{Y}_R \otimes_R K \). The morphism \( \mathcal{Y}_R \to \text{Spec } R \) induces a morphism of complex varieties \( \mathcal{Y} := \mathcal{Y}_R \otimes_{\overline{\mathbb{Q}}} \mathbb{C} \to T := \text{Spec } (R \otimes_{\overline{\mathbb{Q}}} \mathbb{C}) \), defined over \( \overline{\mathbb{Q}} \). Notice that the complex dimension of \( T \) is the
transcendence degree of $K$ over $\mathbb{Q}$. The natural closed immersion $Y_R \subset S \otimes_{\mathbb{Q}} R$ makes $\mathcal{Y}$ a closed irreducible variety

$$\mathcal{Y} \subset S \times \mathbb{C} T$$

defined over $\mathbb{Q}$, with induced projections $p : \mathcal{Y} \to S$ and $\pi : \mathcal{Y} \to T$, both defined over $\mathbb{Q}$, such that $\mathcal{Y}_{t_0} := \pi^{-1}(x_0) \simeq Y$ where $t_0 \in T(\mathbb{C})$ is the closed point given by $R \subset K \subset \mathbb{C}$. By construction the morphism $p$ is dominant. The variety $\mathcal{Y}$ is called “the” spread of $Y$. It depends on the choice of $R$ but different choices give rise to birational varieties $\mathcal{Y}$s. Shrinking Spec $R$ if necessary, we can assume without loss of generality that $T$ is smooth.

Let $\mathcal{Y}^0 \subset \mathcal{Y}$ be the $\mathbb{Q}$-Zariski-open dense subset of smooth points. As $p$ is dominant, the fact that $Y \subset S$ is weakly non-factor for $(S, \mathcal{V})$ implies that $\mathcal{Y}^0 := \mathcal{Y}^0 \cap Y \subset \mathcal{Y}^0$ is weakly non-factor for $(\mathcal{Y}^0, p^{-1}(\mathcal{V})|_{\mathcal{Y}^0})$. As $H_{\mathcal{Y}^0} = H_\mathcal{Y}$ and $H_{\mathcal{Y}^0} = H_S$, to show that $H_\mathcal{Y} = H_S$ we are reduced, replacing $S$ by $Y$ and $Y$ by $\mathcal{Y}^0 \cap Y$ if necessary, to the situation where there exists a morphism $\pi : S \to T$ defined over $\mathbb{Q}$ such that $Y = S_{t_0} \subset S$ and $Y$ is weakly non-factor for $(S, \mathcal{V})$.

It follows from [GM88, Theorem p.57] that there exist finite Whitney stratifications $(S_i)$ of $S$ and $(T_i)_{1 \leq d}$ of $T$ by locally closed algebraic subsets $T_i$ of dimension $l$ ($d = \dim T$) such that for each connected component $Z$ (stratum) of $T_i$, $\pi^{-1}(Z)^{an}$ is a topological fibre bundle over $Z^{an}$, and a union of connected components of strata of $(S_i^{an})$, each mapped submersively to $Z^{an}$ (moreover, for all $t \in Z^{an}$, there exists an open neighbourhood $U(t)$ in $Z^{an}$ and a stratum preserving homeomorphism $h : \pi^{-1}(U) \simeq \pi^{-1}(t) \times U$ such that $\pi_{|\pi^{-1}(U)} = p_U \circ h$, where $p_U$ denotes the projection to $U$). These Whitney stratifications can be chosen defined over $\mathbb{Q}$ (meaning that the closure of each stratum is defined over $\mathbb{Q}$): see [Tei82], [Ar13, 3.1.9].

It follows from the minimality of $K$ that $t_0$ belongs to the unique open stratum $T_d$, $d = \dim T$. Without loss of generality we can and will assume from now on that $T = T_d$. In particular $S^{an}$ is a topological fibre bundle over $T^{an}$.

If follows that the image of $\pi_1(Y^{an})$ in $\pi_1(S^{an})$ is a normal subgroup. Hence $H_\mathcal{Y}$ is a normal subgroup of $H_S$. As $Y \subset S$ is weakly non-factor it follows that $H_\mathcal{Y} = H_S$.

$\square$

3.4. Proof of Corollary 1.13.

Let $S$, $\mathcal{V}$ and $Y$ as in the statement of Corollary 1.13. Let us show that $Y$ is weakly non-factor. Let $Z \subset S$ be a closed irreducible subvariety of $S$ containing $Y$ strictly, and such that $H_\mathcal{Y}$ is is a strict normal subgroup of $H_Z$. As the special closure $\langle Z \rangle_s$ of $Z$ is a special subvariety of $(S, \mathcal{V})$ containing $Y$, it follows from the maximality of $Y$ that $\langle Z \rangle_s = S$. As $H_Z$ is normal (see [An92, Theor.1]) in the algebraic group $G^\text{der}_Z = G^\text{der}_S$ which is assumed to be simple, it follows that either $H_Z = \{1\}$ or $H_Z = H_S = G^\text{der}_S$. As $H_\mathcal{Y}$ is a strict normal subgroup of $H_Z$, necessarily $H_\mathcal{Y} = \{1\}$ (and $H_Z = H_S$). This is impossible as $Y$ is positive dimensional for $\mathcal{V}$. Hence such a $Z$ does not exist and $Y$ is weakly non-factor. The conclusion then follows from Theorem 1.12.

$\square$
3.5. Proof of Corollary 1.14.

Let us suppose that the special points for $\mathbb{Z}VHS$’s defined over $\overline{\mathbb{Q}}$ are defined over $\mathbb{Q}$. Let $\mathbb{V} \to S^{an}$ be a $\mathbb{Z}VHS$ defined over $\mathbb{Q}$ and let $Y$ be a special subvariety of $S$ for $\mathbb{V}$. Let us show that $Y$ is defined over $\overline{\mathbb{Q}}$.

Suppose for the sake of contradiction that $Y$ is not defined over $\overline{\mathbb{Q}}$. Let $Z \subset S$ be the $\overline{\mathbb{Q}}$-Zariski closure of $Y$ in $S$. Again, replacing $S$ by the $\overline{\mathbb{Q}}$-Zariski open subset of smooth points $Z^0$ of $Z$ and $Y$ by $Y^0 := Z^0 \cap Y$ we can without loss of generality assume that $Z = S$ is smooth. Arguing as in the proof of Theorem 2.6(a) we may assume that $H_Y$ is a strict normal subgroup of $H_S$, hence of $G_S$.

It follows that there exist a finite collection of natural integers $a_i, b_i, 1 \leq i \leq n$ such that the $\mathbb{Z}VHS$ $\mathbb{V}' := (\bigoplus_{1 \leq i \leq n} \mathbb{V}^{\otimes a_i} \otimes (\mathbb{V}')^{\otimes b_i})^{H_Y}$ consisting of the $H_Y$-invariant vectors in $\bigoplus_{1 \leq i \leq n} \mathbb{V}^{\otimes a_i} \otimes (\mathbb{V}')^{\otimes b_i}$ has generic Mumford-Tate group $G'_S = G_S/H_Y$ and algebraic monodromy group $H'_S := H_S/H_Y$. Writing $(G'_S = G_S/H_Y, D'_S := D_S/H_Y)$ for the quotient Hodge datum of $(G_S, D_S)$ by $H_Y$ and $\pi : X_S \to X'_S$ the induced projection of Hodge varieties, the period map for $\mathbb{V}'$ is $\Phi'_S := \pi \circ \Phi_S : S^{an} \to X'_S$. The special subvariety $Y$ of $S$ for $\mathbb{V}$ is still a special subvariety of $S$ for $\mathbb{V}'$ and its image $\Phi'_S(Y)$ is a point.

Following [BBT18, Theor.1.1] there exists a factorisation

$$\Phi'_S = \Psi \circ q,$$

where $q : S \to B$ is a proper morphism of quasi-projective varieties defined over $\overline{\mathbb{Q}}$ and $\Psi : B \to X'$ is a quasi-finite period map. This means that $\mathbb{V}' = q^*\mathbb{V}'_B$ for a $\mathbb{Z}VHS$ $\mathbb{V}'_B$, and that $b_0 := q(Y)$ is a special point of $B$ for $\mathbb{V}'_B$.

It follows from Lemma 3.3 below that the $\mathbb{Z}VHS$ $\mathbb{V}'$ can be defined over $\overline{\mathbb{Q}}$. It then follows from Lemma 3.4 below that $\mathbb{V}'_B$ is also defined over $\overline{\mathbb{Q}}$. Under our assumption that special points of $\mathbb{Z}VHS$ defined over $\overline{\mathbb{Q}}$ are defined over $\overline{\mathbb{Q}}$ one concludes that the special point $b_0$ of $B$ for $\mathbb{V}'_B$ is defined over $\overline{\mathbb{Q}}$. But then the irreducible component $Y$ of $q^{-1}(b_0)$ is also defined over $\overline{\mathbb{Q}}$, a contradiction.

This finishes the proof of Corollary 1.14. \hfill $\square$

Lemma 3.3. Let $\mathbb{V}$ be a $\mathbb{Z}VHS$ and $\mathbb{V}'$ a sub-$\mathbb{Z}VHS$. If $\mathbb{V}$ is definable over $\overline{\mathbb{Q}}$ then there exists a $\overline{\mathbb{Q}}$-structure on $\mathbb{V}$ and $\mathbb{V}'$ such that the projection $\mathbb{V} \to \mathbb{V}'$ is defined over $\overline{\mathbb{Q}}$.

Proof. Let $E$ be the finite dimensional $\overline{\mathbb{Q}}$-algebra of $\nabla$-flat $F^\bullet$-preserving algebraic sections over $S$ of $\mathbb{V}'_S \otimes \mathbb{V}'_S$. Each invertible element of $E_\mathbb{C} := E \otimes_{\overline{\mathbb{Q}}} \mathbb{C}$ defines a natural $\overline{\mathbb{Q}}$-structure on $\mathbb{V}_S$, $F^\bullet$ and $\nabla$, the original one $(\mathbb{V}_S, F^\bullet_S, \nabla_S)$ being preserved exactly by the invertible elements of $E$.

Let $J$ be the Jacobson radical of $E$. Let us choose $T \subset E$ a (semi-simple) splitting of the projection $E \to E/J$. As the category of polarizable $\mathbb{Q}VHS$ is abelian semi-simple the finite dimensional complex algebra $\text{Hom}_{\mathbb{Z}VHS}(\mathbb{V}, \mathbb{V}) \otimes \mathbb{C}$ is semi-simple. Under the Riemann-Hilbert correspondence it identifies with a semi-simple subalgebra $A \subset E_\mathbb{C}$.
Following a classical result of Wedderburn-Malcev there exists an element \( j \in J_C := J \otimes_{\mathbb{Q}} \mathbb{C} \) such that \((1 + j)A(1 + j)^{-1} \subset T_C\).

Let \( e_C \in A \) be the idempotent of \( S_C \) corresponding to the projection of \( \text{ZVHS} \pi : \mathbb{V} \to \mathbb{V}' \) under the Riemann-Hilbert correspondence. As \( T_C \) is semi-simple, hence a product of matrix algebras, any idempotent of \( T_C \) is conjugated to an idempotent in \( S \). Thus there exist an invertible element \( f \in T_C \) and \( e \in T \) such that \((1 + j)e_C(1 + j)^{-1} = f^{-1}ef\).

If we endow \( (\mathbb{V}, f^*, \nabla) \) with the \( \mathbb{Q} \)-structure defined by the element \( f(1 + j) \in E_C \) it follows that the image of \( \pi : \mathbb{V} \to \mathbb{V}' \) under the Riemann-Hilbert correspondence is defined over \( \mathbb{Q} \) for this new \( \mathbb{Q} \)-structure. Hence the result.

\[ \square \]

**Lemma 3.4.** Let \( f : S \to B \) be a proper morphism of \( \mathbb{Q} \)-varieties defined over \( \mathbb{Q} \), such that \( f_*O_S = O_B \). Let \( \mathbb{V}_B \) be a \( \text{ZVHS} \) on \( B \). If the \( \text{ZVHS} \mathbb{V}_S := f^*\mathbb{V}_B \) on \( S \) is definable over \( \mathbb{Q} \), then \( \mathbb{V}_B \) is also definable over \( \mathbb{Q} \).

**Proof.** Let \( (\mathbb{V}_S := f^*\mathbb{V}_B, F^*_S := f^*F^*_B, \nabla_S := f^*\nabla_B) \) be the De Rham incarnation of \( \mathbb{V}_S \). It follows from the projection formula and the assumption \( f_*O_S = O_B \) that

\[ f_*\mathbb{V}_S = f_*(f^*\mathbb{V}_B \otimes_{O_S} O_S) = \mathbb{V}_B \otimes_{O_B} f_*O_S = \mathbb{V}_B. \]

It follows easily that \( F^*_B = f_*F^*_S \) and \( \nabla_B = f_*\nabla_S \). As \( f, F^*_S \) and \( \nabla_S \) are defined over \( \mathbb{Q} \), it follows that \( F^*_B \) and \( \nabla_B \) are defined over \( \mathbb{Q} \).

\[ \square \]

**Remark 3.5.** The companion statement to Corollary 1.14 that conjugates of special varieties for \( \text{ZVHSs} \) defined over a number field are special if and only if it holds true for special points would follow from a version of Lemma 3.3 over a fixed number field \( L \) rather than over \( \mathbb{Q} \), but this last version is not clear to us.

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