The initial singularity in solutions of the Einstein-Vlasov system of Bianchi type I

Alan D. Rendall
Institut des Hautes Etudes Scientifiques
35 Route de Chartres
91440 Bures-sur-Yvette
France

Abstract
The dynamics of solutions of the Einstein-Vlasov system with Bianchi I symmetry is discussed in the case of massive or massless particles. It is shown that in the case of massive particles the solutions are asymptotic to isotropic dust solutions at late times. The initial singularity is more difficult to analyse. It is shown that the asymptotic behaviour there must be one of a small set of possibilities but it is not clear whether all of these possibilities are realized. One solution is exhibited in the case of massless particles which behaves quite differently near the singularity from any Bianchi I solution with perfect fluid as matter model. In particular the matter is not dynamically negligible near the singularity for this solution.

1. Introduction

The simplest of all anisotropic cosmological models are those of Bianchi type I. They are the spacetimes which admit a three-dimensional abelian symmetry group whose orbits are spacelike. (For general information on Bianchi models see [1].) Just how simple their dynamics is depends significantly on the nature of the matter content of the spacetime. For a perfect fluid with a linear equation of state it has been known for a long time how to analyse the dynamics [2,3]. For a non-interacting mixture of two fluids with linear equations of state the time evolution is also well understood and is asymptotic near the singularity and at large times to that of a single fluid [4]. The case of a fluid with nonlinear equation of state is discussed in an appendix to the present paper. The dynamics does not differ much from the picture in the linear case. When a magnetic field is added to the fluid things are already more complicated. In fact, as was shown by Collins [5], a Bianchi type I model with fluid and magnetic field resembles a model of the more complicated Bianchi type II with fluid alone. It is also interesting to note that models of type V\textsubscript{I} with perfect fluid and a magnetic field have a dynamical behaviour resembling the notoriously complicated ‘Mixmaster’ behaviour of Bianchi type IX models [6]. Thus changing the matter model can have effects on the complexity of the dynamics comparable with those
encountered when passing to more general symmetry types.

A matter model for which the details of the global dynamics of Bianchi type I space-times has not previously been studied mathematically is the collisionless gas, described by the Vlasov equation. The only general facts which are known are that, with an appropriate choice of time orientation,

(i) the spacetime is future geodesically complete (when maximally extended towards the future)
(ii) there is a crushing singularity in the past where, except in the vacuum case, the curvature invariant \( R_{\alpha\beta}R^{\alpha\beta} \) tends to infinity

These fundamental facts were proved in [7], where it was shown that they hold for any Bianchi type other than IX and for a general class of matter models. The aim of this paper is to refine (i) and (ii) in the case of Bianchi type I symmetry and matter described by the Vlasov equation so as to get more detailed information about the asymptotics of the expanding phase and the nature of the initial singularity. An aspect of the situation which makes this more difficult than in the case of many other matter models is that for general initial data it is not possible to derive an explicit closed system of ordinary differential equations which describes the dynamics. This is because certain integrals which occur can not be evaluated. In one special case, where massless particles are considered and the initial phase space density has the form of the characteristic function of a ball, these integrals have been computed by Lukash and Starobinski[8]. However, the explicit expressions they obtain are sufficiently complicated that they do not seem to make a rigorous analysis of the global dynamics any easier. On the other hand they would probably be useful for numerical calculations since they would allow costly numerical evaluation of integrals to be avoided.

The dynamics at late times of the models with massive particles can be described precisely. All solutions become isotropic and can be approximated by dust solutions in this limit (Theorem 5.4). On the other hand, the results of this paper do not give a complete picture of the dynamics near the initial singularity of the spacetimes being studied. They merely reduce the possible types of asymptotic behaviour to a small number of alternatives. Improving on this is likely to require new techniques. These results leave open the possibility that Bianchi I spacetimes with a matter content described by kinetic theory may show complicated oscillatory behaviour and thus be very different from those with other types of matter content studied up to now. The mechanism which allows for this complexity is simply the presence of anisotropy in the pressure which may respond to changes in the geometry. It may be that the only reason that the dynamics is so simple in the case of a perfect fluid is that this mechanism is excluded by a special symmetry assumption (the isotropy of the pressure). The one conclusion which emerges and which applies to all solutions considered here is that the ratio of the mean pressure to the energy density tends to one third as the singularity is approached. This means that in a certain weak sense the dynamics for particles of unit mass is approximated near the singularity by that for massless particles. For this reason both cases are often considered together in the following, although the main emphasis is on the case \( m = 1 \).

The results will now be summarized. There are broadly speaking two possible types of asymptotic behaviour of solutions of the Einstein-Vlasov system with Bianchi I symmetry
near the singularity. They will be referred to as convergent and oscillatory. Let $\lambda_i$ denote the eigenvalues of the second fundamental form of the homogeneous hypersurfaces. Then the mean curvature of the homogeneous hypersurfaces is given by $H = \lambda_1 + \lambda_2 + \lambda_3$. Let $p_i = \lambda_i/H$. In the convergent type the $p_i$ tend to limits as the singularity is approached. There are three different cases, depending on these limiting values. The first case is that where the limiting values are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$. The well known homogeneous and isotropic solutions of the Einstein-Vlasov system[9] are of this type. The second is that the limiting values are $(0, \frac{1}{2}, \frac{1}{2})$ or some permutation thereof. The existence of solutions of this kind in the case of massless particles is shown in Section 6. These limiting values of the Kasner exponents are not realized by any Bianchi type I spacetime when the matter model is a perfect fluid (see appendix). The third is that the limiting values satisfy the Kasner relation $p_1^2 + p_2^2 + p_3^2 = 1$. Any solution for which one of the eigenvalues becomes negative at some time has this asymptotic behaviour and so there are plenty of examples. This is proved in Theorem 5.1. Note that the special case of this result when two of the $p_i$ are equal is closely related to the homogeneous special case of a result of Rein[10] for plane symmetric spacetimes. In the oscillatory type the $p_i$ undergo infinitely many oscillations, in a sense which will now be specified. There are two cases to be considered, according to whether two of the eigenvalues are always equal or not. Consider first the case where two eigenvalues are equal and suppose without loss of generality that $\lambda_2 = \lambda_3$. Associate to any solution a string of symbols (which may be finite or infinite, depending on the solution) as follows. Moving backwards from some fixed time add an $x$ to the string each time that $\lambda_1 - \lambda_2$ changes from being $\leq 0$ to being $> 0$ and add a $y$ each time it changes from being $> 0$ to being $< 0$. That this makes sense follows from the fact, proved in Section 2, that the set of times where $\lambda_1 = \lambda_2$ can have no limit point unless this equality holds at all times. Thus a finite time interval can only contribute a finite number of symbols. The solution is said to undergo infinitely many oscillations if the resulting string of symbols is infinite. Similarly, if there is some time where all eigenvalues are distinct then a string of symbols is associated to the solution by adding $x$, $y$ or $z$ each time $\lambda_1$, $\lambda_2$ or $\lambda_3$ respectively becomes strictly larger than the other two eigenvalues.

Unfortunately it could not be shown whether any oscillatory solutions exist. If they did then the behaviour of Bianchi I models with kinetic theory as matter model would be much more complicated than in the case of a perfect fluid. If it could be shown that they existed the question would remain whether the sequences of symbols they produce have some regularity or whether they are chaotic.

To each type of solution discussed above corresponds a characteristic behaviour of the pressures. The solutions considered in the following all have diagonal energy-momentum tensors and so three pressures $P_i$ are defined by three diagonal components. The quantities $R_i = P_i/\rho$ must have a sum which converges to unity at the singularity. When the limiting values of the $p_i$ are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or $(0, \frac{1}{2}, \frac{1}{2})$ then the limiting values of the $R_i$ are $(\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ or $(0, \frac{1}{2}, \frac{1}{2})$ respectively. When the sum of the squares of the limiting values of the $p_i$ is equal to unity then the $R_i$ tend to $(0, \frac{1}{2}, \frac{1}{2})$ or a permutation thereof, unless one $p_i$ has limiting value zero. In the latter case the $R_i$ tend to $(1, 0, 0)$ or a permutation thereof.

The paper is organized as follows. In Section 2 some basic facts about the solutions are collected. Section 3 is concerned with a simplified system which in some cases models
the asymptotic behaviour of the solutions of the original system. Some estimates for the pressures are derived in Section 4. Section 5 contains the main results. Section 6 contains proofs of the existence or non-existence of solutions with certain kinds of asymptotic behaviour.

2. Basic facts

The Einstein-Vlasov system for particles of mass \( m \geq 0 \) can be written in the following form in the case of Bianchi type I symmetry:

\[
-k_{ij}k^{ij} + H^2 = 16\pi \rho \tag{2.1}
\]

\[
T_{0i} = 0 \tag{2.2}
\]

\[
\partial_t g_{ij} = -2k_{ij} \tag{2.3}
\]

\[
\partial_t k_{ij} = Hk_{ij} - 2k_{il}k^{lj} - 8\pi T_{ij} - 4\pi \rho g_{ij} + 4\pi \text{tr}Tg_{ij} \tag{2.4}
\]

\[
\partial f/\partial t + 2k_{ij} v^j \partial f/\partial v^i = 0 \tag{2.5}
\]

\[
\rho = \int f(t, v^k)(m^2 + g_{rs}v^rv^s)^{1/2}(\det g)^{1/2}dv^1dv^2dv^3 \tag{2.6}
\]

\[
T_{ij} = \int f(t, v^k)v_iv_j(m^2 + g_{rs}v^rv^s)^{-1/2}(\det g)^{1/2}dv^1dv^2dv^3 \tag{2.7}
\]

Here \( g_{ij} \) is the induced metric of the homogeneous hypersurfaces, \( k_{ij} \) is the second fundamental form, \( f \) is the phase space density of particles, \( \rho \) is the energy density, \( T_{0i} \) and \( T_{ij} \) are components of the energy-momentum tensor and \( H \) is the mean curvature \( g^{ij}k_{ij} \).

With the exception of \( f \) all these quantities depend only on the time coordinate \( t \). This time coordinate is Gaussian i.e. it is constant on each homogeneous hypersurface and defines a parametrization by proper time when restricted to any geodesic normal to these hypersurfaces. The spacetime metric is of the form

\[
ds^2 = -dt^2 + g_{ij}(t)dx^idx^j \tag{2.8}
\]

In the following it is always assumed, when talking about solutions of (2.1)-(2.7), that the function \( f(t, v) \) is non-negative and has compact support for each fixed \( t \). It is assumed that \( f \) is \( C^1 \) except in Section 3, where \( f \) may be a distribution. The case of primary interest here is the case \( m = 1 \). Since, however, solutions of (2.1)-(2.7) with \( m = 1 \) resemble solutions with \( m = 0 \) close to the singularity, it is useful to also allow the case \( m = 0 \) from the beginning.

For a given Bianchi I geometry the Vlasov equation can be solved explicitly. (This is not possible for the other Bianchi types. The reason is explained in [11].) The result is that if \( f \) is expressed in terms of the covariant components \( v_i \) then it is independent of time. This means that if \( t_0 \) is some fixed time and \( f_0(v_i) = f(t_0, v_i) \) then (2.6) can be rewritten as

\[
\rho = \int f_0(v_i)(m^2 + g^{rs}v_rv_s)^{1/2}(\det g)^{-1/2}dv_1dv_2dv_3 \tag{2.9}
\]

and (2.7) can be rewritten in a similar way. The explicit solution allows certain special subclasses of solutions of (2.1)-(2.7) to be identified. The first of these will be referred to
as reflection-symmetric and is defined by the conditions that

\[ f_0(v_1, v_2, v_3) = f_0(-v_1, -v_2, v_3) = f_0(v_1, -v_2, -v_3). \]  \hspace{1cm} (2.10)

and that the initial values of \( g_{ij} \) and \( k_{ij} \) are diagonal. Equation (2.10) implies that \( T_{ij} \) is diagonal. It then follows from (2.3) and (2.4) that \( g_{ij} \) and \( k_{ij} \) are always diagonal. The second case, which will be referred to as LRS (locally rotationally symmetric) is obtained by supplementing (2.10) by the conditions that

\[ f_0(v_1, v_2, v_3) = F((v_1)^2 + (v_2)^2, v_3) \]  \hspace{1cm} (2.11)

for some function \( F \) and that \( g_{11} = g_{22}, k_{11} = k_{22} \) initially. Equation (2.11) implies that \( T_{11} = T_{22} \) and it follows from (2.3) and (2.4) that \( g_{11} = g_{22} \) and \( k_{11} = k_{22} \) everywhere. A solution will also be called LRS if it satisfies the definition obtained from that just given by a permutation of the indices 1, 2, 3. A solution will be called isotropic if

\[ f_0(v_1, v_2, v_3) = F((v_1)^2 + (v_2)^2 + (v_3)^2) \]  \hspace{1cm} (2.12)

and if \( g_{ij} \) and \( k_{ij} \) are proportional to \( \delta_{ij} \) on the initial hypersurface and hence everywhere. If a solution satisfies the conditions on \( g_{ij} \) and \( k_{ij} \) in the definition of reflection-symmetric, LRS or isotropic for all \( t \) in some interval, but no assumption is made on \( f \) then the solution will be said to have reflection-symmetric, LRS or isotropic geometry respectively on that interval. It follows from (2.4) that \( T_{ij} \) has a corresponding symmetry property.

In this paper only reflection-symmetric solutions of (2.1)-(2.7) are considered. The alternative notation \( a^2 = g_{11}/g_{11}(t_0), b^2 = g_{22}/g_{22}(t_0), c^2 = g_{33}/g_{33}(t_0) \) is used when it is convenient. Yet another form of (2.6) can then be obtained by doing a change of variables in (2.9).

\[ \rho = \int f_0(aw_1, bw_2, cw_3)(m^2 + \delta^{rs}w_rw_s)^{1/2}dw_1dw_2dw_3 \]  \hspace{1cm} (2.13)

The geometric interpretation of the \( w_i \) is that they are the components of the momentum in an orthonormal frame. Similarly:

\[ T^i_i = \int f_0(aw_1, bw_2, cw_3)w_i^2(m^2 + \delta^{rs}w_rw_s)^{-1/2}dw_1dw_2dw_3 \]  \hspace{1cm} (2.14)

In order to study the dynamics of the solutions of the system (2.1)-(2.7) in detail it is useful to introduce certain dimensionless variables which remain finite at the singularity. It follows from (2.1) that \( H \) never vanishes except in the case of flat spacetime, which is excluded from consideration in the following. By replacing \( t \) by \(-t\) if necessary, it can be arranged that \( H < 0 \) everywhere. It will be assumed that this has been done. Define

\[ \hat{k}_{ij} = k_{ij}/H \] \hspace{1cm} (2.15)

\[ \hat{\rho} = \rho/H^2 \] \hspace{1cm} (2.16)

\[ \hat{T}_{ij} = T_{ij}/H^2 \] \hspace{1cm} (2.17)

\[ \tau(t) = -\int_{t_0}^{t} H(t')dt' \] \hspace{1cm} (2.18)
In terms of these variables equations (2.1) and (2.4) become:

\[- \hat{k}^i_j \hat{k}^j_i + 1 = 16\pi \hat{\rho} \quad (2.19)\]
\[\partial_\tau \hat{k}^i_j = -12\pi \hat{\rho} (\hat{k}^i_j - \frac{1}{3} \delta^i_j) + 8\pi \hat{T}^i_j + 4\pi (\hat{k}^i_j - \delta^i_j) \text{tr} \hat{T} \quad (2.20)\]

The following lemma provides some information about the range of \( \tau \).

**Lemma 2.1** Let a solution of (2.1)-(2.7) with \( m = 0 \) or \( m = 1 \) be given which is the maximal globally hyperbolic development of initial data on the hypersurface \( t = t_0 \). Then \( H(t) \) is a monotonically defined on an interval \((t_1, \infty)\). By translating \( t \) it can be assumed that \( t_1 = 0 \). Then \( \lim_{t \to 0} H = -\infty \) and \( \lim_{t \to \infty} H = 0 \). Moreover, \(-3/t \leq H(t) \leq -1/t\).

**Proof** That \( H(t) \) is monotonically defined on an interval of the form \((t_1, \infty)\) with \( \lim_{t \to t_1} H = -\infty \) was shown for the case \( m = 1 \) in [12]. Essentially the same argument applies for \( m = 0 \). In the latter case the coefficients of the characteristic system are only Lipschitz instead of \( C^1 \) but this causes no problems. Now it follows from (2.4) that \( \frac{1}{3} H^2 \leq \partial_\tau H \leq H^2 \). Comparing the solution with the ordinary differential equations corresponding to these inequalities then gives the desired estimates (cf. [7]).

This result implies that the integral defining \( \tau \) diverges as \( t \to 0 \) and as \( t \to \infty \). Hence the solution of (2.19)-(2.20) exists globally in \( \tau \).

The solution of (2.19)-(2.20) of course contains only a small part of the information of that contained in the solution of (2.1)-(2.7). The former is only a projection of the latter. Nevertheless it will be seen that a lot of information about the solution of the full equations can be obtained by studying this projection. Consider now the set \( K \) of triples of real numbers \( \hat{k}^1_i, \hat{k}^2_i, \hat{k}^3_i \) which satisfy \( \sum_i (\hat{k}^i_i)^2 \leq 1 \) and \( \sum_i \hat{k}^i_i = 1 \). This is a compact subset of \( \mathbb{R}^3 \). In fact it is a disc in a plane. A solution of (2.19)-(2.20) defines a point of \( K \) at each time \( \tau \). It is on the boundary of \( K \) in the plane if \( f \) is identically zero and in the interior otherwise. This point depending on \( \tau \), considered as a mapping from \( \mathbb{R} \) to \( K \), will be referred to as the projection of the given solution. The projection of a vacuum solution is constant. Let \( C \) denote the point \((\frac{1}{3}, \frac{1}{3}, \frac{1}{3})\). If a solution has isotropic geometry on a time interval then its projection lies at the point \( C \) during this time. Conversely, if its projection lies at \( C \) on a given time interval then it can be made to have isotropic geometry by a time-independent rescaling of the spatial coordinates. Let \( L_1, L_2 \) and \( L_3 \) be the subsets of \( K \) defined by \( \hat{k}_2^2 = \hat{k}_3^3 = \hat{k}_1^1 \) and \( \hat{k}_1^1 = \hat{k}_2^2 \) respectively. A solution has LRS geometry on a time interval (up to a constant rescaling of the spatial coordinates as above) if and only if its projection lies on one of the lines \( L_1, L_2 \) or \( L_3 \) during this time. Let \( L_i^+ \) denote the open half of \( L_i \) which ends at the point with coordinates \((-\frac{2}{3}, \frac{2}{3}, \frac{2}{3})\) or a permutation thereof and let \( L_i^- \) denote the opposite half-line, which ends at the point with coordinates \((1, 0, 0)\) or a permutation thereof. Let \( V_i^+ \) and \( V_i^- \) denote these endpoints. Let \( A_1 \) be the open region bounded by \( L_2^+, L_3^+ \) and the boundary of \( K \) and let \( A_2 \) and \( A_3 \) be defined by cyclically permuting \((1, 2, 3)\) in this definition. Let \( B_i \) be the subset of \( K \) where \( \hat{k}_i^i \leq 0 \).
The components of the metric satisfy the evolution equations
\[
\frac{dg_{ii}}{d\tau} = 2 \hat{k}_i^i g_{ii}, \quad \frac{d}{d\tau}(g_{ii}/g_{jj}) = 2(\hat{k}_i^i - \hat{k}_j^j)(g_{ii}/g_{jj})
\]
which imply that \(g_{ii}\) or their ratios increase or decrease exponentially if certain sign conditions are satisfied by the \(\hat{k}_i^i\). There are of course corresponding statements for the scale factors \(a, b\) and \(c\). Given an initial datum \(f_0\) for \(f\) the quantities \(\rho\) and \(T_{ii}\) are determined uniquely by the \(g_{ii}\) by means of equations (2.13) and (2.14). Thus (2.19), (2.20) and (2.21), together with the equation
\[
\partial_\tau H = -H(1 - 12\pi \hat{\rho} + 4\pi \text{tr}\hat{T})
\]
derived from (2.4) form a closed system of ordinary differential equations, which formally determine the quantities \(\hat{k}_i^i, g_{ii}\) and \(H\) as functions of \(\tau\) in terms of initial data. If the coefficients of this system were locally Lipschitz, it would follow from the standard uniqueness theorem for ordinary differential equations that they determine them uniquely. It will now be shown that in fact for \(m = 1\) this dependence is analytic. To do this it is convenient to use the expressions for \(\rho\) and \(T_{ii}\) of the type (2.9). Analyticity is a consequence of the following lemma:

**Lemma 2.2** Let \(W\) be a mapping of \(U \times \mathbb{R}^3\) to \(\mathbb{R}\), where \(U\) is an open subset of \(\mathbb{R}^3\). Suppose that \(W\) extends to a \(C^1\) mapping \(\tilde{W}\) of \(\tilde{U} \times \mathbb{R}^3\) to \(C\), where \(\tilde{U}\) is an open neighbourhood of \(U\) in \(\mathbb{C}^3\), and that \(\tilde{W}(\cdot, y)\) satisfies the Cauchy-Riemann equations for each fixed \(y \in \mathbb{R}^3\). Finally suppose that each \(z \in \tilde{W}\) has an open neighbourhood \(V\) such that the supports of the functions \(\tilde{W}(z, \cdot)\) are contained in a common compact subset \(K\) of \(\mathbb{R}^3\). Then the function \(F(x) = \int_{\mathbb{R}^3} W(x, y)dy\) is analytic.

**Proof** It suffices to show that the function \(\tilde{F}(z) = \int_{\mathbb{R}^3} \tilde{W}(z, y)dy\) is complex analytic and this is true if \(\tilde{F}\) is \(C^1\) and satisfies the Cauchy-Riemann equations [13]. The assumptions on the smoothness and support of \(\tilde{W}\) justify differentiation under the integral and so the Cauchy-Riemann equations for \(\tilde{F}\) follow from the Cauchy-Riemann equations for \(\tilde{W}\).

A consequence of the analyticity of the coefficients in the system of ordinary differential equations is that the solutions are analytic. It follows that if a solution with \(m = 1\) has LRS or isotropic geometry on some non-empty open time interval then it must have LRS or isotropic geometry respectively for all values of \(\tau\). The same conclusion holds if there is a sequence of times having a limit point where the geometry is LRS or isotropic, respectively.

### 3. The asymptotic system

In this section a certain system of ordinary differential equations is introduced and the qualitative behaviour of its solutions analysed. This system is used later to study the asymptotic behaviour of solutions of the system (2.1)-(2.7). This system can be obtained formally from (2.1)-(2.7) by replacing the \(C^1\) function \(f(t, v_1, v_2, v_3)\) by a measure of the form \(f(t, v_1)\delta(v_2 - \bar{v}_2(t))\delta(v_3 - \bar{v}_3(t))\) where \(\delta\) is a Dirac measure and taking \(m = 0\). Solutions of this system can be interpreted as certain distributional solutions of the Einstein-Vlasov system for massless particles. These are intermediate between smooth solutions and the even more singular solutions which are in one to one correspondence with dust solutions.
(For the correspondence between dust and distributional solutions of the Vlasov equation see [14].) The mathematical results which will now be derived are independent of this interpretation.

The system of ODE’s to be considered is the special case of the equations (2.19) and (2.20) obtained by setting \( \hat{T}_1 = \hat{T}_2 = 0 \) and \( \hat{T}_3 = \hat{\rho} \). Note that, in contrast to the general case of (2.19) and (2.20), these specialized equations suffice to determine all unknowns occurring in them from initial data. The explicit form of (2.20) in this case is:

\[
\begin{align*}
\partial_\tau \hat{k}_1^1 &= -8\pi \hat{\rho} \hat{k}_1^1 \\
\partial_\tau \hat{k}_2^2 &= -8\pi \hat{\rho} \hat{k}_2^2 \\
\partial_\tau \hat{k}_3^3 &= -8\pi \hat{\rho} (\hat{k}_3^3 - 1)
\end{align*}
\]

If initial data are chosen at some time which satisfy the condition \( \sum_i (\hat{k}_i^i) = 1 \) then the solution also satisfies it. Only solutions with this property are considered here. It follows from (3.1) and (3.2) that \( \partial_\tau (\hat{k}_2^2/\hat{k}_1^1) = 0 \) whenever \( \hat{k}_1^1 \neq 0 \). Let \( r \) be the constant value of \( \hat{k}_2^2/\hat{k}_1^1 \). Then

\[
\begin{align*}
\hat{k}_2^2 &= r \hat{k}_1^1 \\
\hat{k}_3^3 &= 1 - (1 + r) \hat{k}_1^1
\end{align*}
\]

Substituting (2.19), (3.4) and (3.5) into (3.1) gives:

\[
\partial_\tau \hat{k}_1^1 = (\hat{k}_1^1)^2 \left[ (1 + r + r^2) \hat{k}_1^1 - (1 + r) \right]
\]

**Proposition 3.1** Let \((\hat{k}_1^1, \hat{k}_2^2, \hat{k}_3^3)\) be a solution of (3.1)-(3.3) satisfying \( \sum_i (\hat{k}_i^i) = 1 \) and \( \sum_i (\hat{k}_i^i)^2 < 1 \). Define \( r = \hat{k}_2^2/\hat{k}_1^1 \) whenever \( \hat{k}_1^1 \neq 0 \). Then:

(i) if \( \hat{k}_1^1 \) is zero initially it is always zero

(ii) if \( \hat{k}_1^1 \) is initially (and hence always) non-zero then \( r \) is constant

(iii) when \( \hat{k}_1^1 \neq 0 \) it is a monotonic function with \( \lim_{\tau \to -\infty} \hat{k}_1^1 = (1 + r)/(1 + r + r^2) \) and \( \lim_{\tau \to \infty} \hat{k}_1^1 = 0 \)

(iv) in that case \( \lim_{\tau \to -\infty} (\hat{k}_3^3 - \hat{k}_2^2) = -r(r + 2)/(1 + r + r^2) \)

**Proof** Statement (i) follows from (3.1). Statement (ii) has been demonstrated above. Statement (iii) follows from (3.6). The last conclusion is then an immediate consequence of the definitions.

Of course analogous statements hold if \( \hat{k}_1^1 \) and \( \hat{k}_2^2 \) are interchanged since these two quantities occur symmetrically in the hypotheses.

4. Pressure estimates

The results of this section are all variations on the theme that if the spacetime is expanding in a certain direction then the pressure in that direction tends to decrease. It is assumed throughout that the geometry is reflection-symmetric. A solution of (2.1)-(2.7) with \( m = 0 \) satisfies \( \rho = \text{tr}T \). This condition never holds when \( m = 1 \) but the next lemma shows that it does hold asymptotically.
Lemma 4.1 Suppose that some solution of (2.1)-(2.7) with $m = 1$ is defined on an interval $(-\infty, \tau_1)$, with $f$ not identically zero. Then $\lim_{\tau \to -\infty} \frac{\text{tr} T}{\rho} = 1$.

**Proof** It was shown in [7] that $\lim_{\tau \to -\infty} \rho = \infty$. Thus the result will follow if it can be shown that $\rho \to \infty$ implies $\text{tr} T / \rho \to 1$. To do this choose some radius $L > 0$ and write the $\rho = \rho_1 + \rho_2$ and $\text{tr} T = (\text{tr} T)_1 + (\text{tr} T)_2$ where the first summand is the integral over the region $|w| < L$ of the integrand in (2.13) or (2.14) respectively and the second is the integral over the complementary region. Using the fact that $(1+x^2)^{1/2} - x^2/(1+x^2)^{1/2} = (1+x^2)^{-1/2}$ it can be seen that $\rho_2 - (\text{tr} T)_2 \leq (1 + L^2)^{-1}\rho_2$. Hence

$$\text{(tr} T)_2 \geq \frac{L^2}{1 + L^2} \rho_2 \quad (4.1)$$

and

$$\text{tr} T \geq \text{tr} T_2 \geq \frac{L^2}{1 + L^2} \left(\rho - (4\pi/3)L^3(1 + L^2)^{1/2}\|f_0\|_\infty\right) \quad (4.2)$$

By choosing $L$ sufficiently large the quantity $L^2/(1 + L^2)$ can be made as close to unity as desired. For fixed $L$ the quantity in brackets on the right hand side of (4.2) approaches $\rho$ as $\rho$ becomes large. This suffices to give the conclusion of the lemma.

For a given initial datum $f_0$ the equation (2.14) defines the pressures $T_i^j$ as functions of $a, b$ and $c$. The following results concern the qualitative behaviour of these functions.

**Lemma 4.2** If $f_0$ is not identically zero and $a \leq C' \min\{1, b, c\}$ for some constant $C' > 0$ there exists a constant $C > 0$ such that $T_1^1 \geq Ca^{-2}b^{-1}c^{-1}$ and $T_2^2 / T_1^1 \leq C(a/b)^{4/3}$. In the case $m = 0$ the conclusion holds under the weaker hypothesis that $a \leq C' \min\{b, c\}$

**Proof** Let $p$ be a point of $R^3$ where $f_0 \neq 0$ whose first coordinate $w_1$ is non-zero. Let $\delta$ be a positive number such that $f$ is bounded below by some positive constant $\eta$ on the closed cube $W$ of side $2\delta$ centred at $p$ and such that $w_1$ does not vanish anywhere on this cube. Consider now the image $W'$ of the cube $W$ under the mapping $(w_1, w_2, w_3) \mapsto (a^{-1}w_1, b^{-1}w_2, c^{-1}w_3)$. On $W$ the functions $w_2/w_1$ and $w_3/w_1$ are bounded. Under the assumptions of the lemma they are bounded by the same constant on $W'$. It follows that $|w_1|/(1 + |w|^2)^{1/2}$ is bounded below on any such cube by a positive constant which is independent of $a, b$ and $c$ which satisfy the hypotheses of the lemma. The integral defining $T_1^1$ can be bounded from below by the integral of the same quantity over $W$. It follows that

$$T_1^1 \geq C\eta a^{-2}b^{-1}c^{-1} \quad (4.3)$$

and this proves the first part of the lemma. To get a lower bound for $T_1^1 / T_2^2$ the domain of integration in the definition of these two quantities will be divided into the regions $|w_2| > R|w_1|$ and $|w_2| < R|w_1|$, where $R$ is a positive number which will be specified later. Corresponding to this decomposition of the domain of integration, there are decompositions $T_1^1 = T_{11} + T_{12}$ and $T_2^2 = T_{21} + T_{22}$. The volume of the region where $|w_2| > R|w_1|$ and $f(aw_1, bw_2, cw_3) \neq 0$ can be bounded by an expression of the form $CR^{-1}c^{-1}b^{-2}$ and so $T_{21} \leq CR^{-1}c^{-1}b^{-3}$. On the other hand, $T_{22} \leq R^2 T_{12}$. Thus

$$T_2^2 \leq CR^{-1}c^{-1}b^{-3} + R^2 T_{12} \leq (CR^{-1}(a/b)^2 + R^2) T_1^1 \quad (4.4)$$

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where in the last step (4.3) has been used. Choosing $R = (a/b)^{2/3}$ gives $T_2^3 \leq C(a/b)^{1/3}T_1^1$ and this proves the result for $T_2^2/T_1^1$.

**Lemma 4.3** Suppose that some solution of (2.1)-(2.7) is defined on the interval $(-\infty, \tau_1)$, with $f$ not identically zero. If $\hat{k}_1^1 - \hat{k}_2^2 \geq A$ and $\hat{k}_1^1 - \hat{k}_3^3 \geq A$ on this interval for some $A > 0$ then $T_i^1/T_1^1 \leq Ce^{A\tau/3}$ for $i = 2, 3$

**Proof** It suffices to note that under the assumptions of this lemma there will be a time interval $(-\infty, \tau_2)$ where the hypotheses of Lemma 4.2 hold, so that (4.4) can be applied.

The time derivatives of the quantities $\hat{T}_i^1$ cannot in general be expressed in terms of the dimensionless quantities (2.9)-(2.11) so as to get a closed system of ordinary differential equations. However they can be estimated in terms of these quantities. Note first that

$$d\hat{T}_i^1/d\tau = -H^{-3}dT_i^1/dt + 2\hat{T}_i^1[1 - 12\pi \hat{\rho} + 4\pi \text{tr}\hat{T}]$$

(4.5)

Next, a change of variables in (2.7) gives in the diagonal case:

$$T_i^1 = g^{ii}\int f_0(v_1, v_2, v_3)(v_i)^2(m^2 + g^{rs}s_r s_s)^{-1/2}(\det g)^{-1/2}dv_1dv_2dv_3$$

(4.6)

Hence

$$dT_1^1/dt = (3k_1^1 + k_2^2 + k_3^3)T_1^1$$

$$+ g^{11}\int f_0(v_1, v_2, v_3)(v_1)^2F(v_1, v_2, v_3)(m^2 + g^{rs}s_r s_s)^{-3/2}(\det g)^{-1/2}dv_1dv_2dv_3$$

(4.7)

where

$$F(v_1, v_2, v_3) = (-g^{11}k_1^1(v_1)^2 - g^{22}k_2^2(v_2)^2 - g^{33}k_3^3(v_3)^2)$$

(4.8)

Note now that

$$|F(v_1, v_2, v_3)| \leq (|k_1^1| + |k_2^2| + |k_3^3|)(m^2 + g^{rs}s_r s_s)$$

(4.9)

and so the integral in (4.7) can be bounded in modulus by $3HT_1^1$. Putting this information into (4.5) gives the desired bound.

**Lemma 4.4** Consider a maximally extended solution of (2.1)-(2.7) with $m = 1$ and $f$ not identically zero. If $g_{ii} \to \infty$ as $\tau \to \infty$ then $\lim_{\tau \to \infty} T_i^1/\rho = 0$. If on the other hand $g_{11}$ is bounded above and all $g_{ii}$ are bounded below by a positive constant on an interval of the form $[\tau_1, \infty)$ then $T_i^1/\rho$ is bounded below by a positive constant on that interval.

**Proof** It follows from (2.13) and (2.14) that $T_1^1 \leq Ca^{-2}\rho$ and this proves the first statement. To get the other conclusion, choose a cube $C_1$ as in the proof of Lemma 4.2. Then $T_1^1 \geq Ca^{-1}b^{-1}c^{-1}$ while $\rho \leq Ca^{-1}b^{-1}c^{-1}$. Hence $T_1^1/\rho \geq C > 0$.

5. The main results

**Lemma 5.1** (Compactness lemma) Let a sequence of reflection-symmetric global solutions of equations (2.1)-(2.7) be given. Then there exists a subsequence such that $\hat{k}_j^i$ and $\hat{\rho}$
converge uniformly on compact sets of $\mathbb{R}$. $\hat{T}_i^j$ also converges uniformly on compact subsets (after possibly passing to a subsequence once more) and the limiting quantities satisfy (2.19) and (2.20).

**Proof** The quantities $\hat{k}_j^i$ are contained in the compact set $K$ and so are, in particular, uniformly bounded. By (2.19) $\hat{\rho}$ is uniformly bounded. It follows that $\hat{T}_i^j$ is uniformly bounded. Equation (2.20) now shows that $\partial_{\tau}\hat{k}_j^i$ is uniformly bounded. By Ascoli’s theorem there exists a subsequence such that $\hat{k}_j^i$ converges uniformly on the interval $[-1, 1]$. Applying the theorem again shows that this subsequence has a subsequence such that $\hat{k}_j^i$ converges uniformly on $[-2, 2]$. Continuing in this way we obtain a collection of subsequences indexed by a positive integer $n$ with the properties that for the $n$th subsequence $\hat{k}_j^i$ converges uniformly on $[-n, n]$ and each sequence is a subsequence of the previous one. The diagonal sequence has the property that $\hat{k}_j^i$ converges on each compact subset of the real line. By the Hamiltonian constraint $\hat{\rho}$ also converges uniformly on compact sets along this subsequence. In the diagonal case the derivatives $\partial_{\tau}\hat{T}_i^j$ are bounded, as was shown in Section 4 and applying Ascoli’s theorem as before gives the remaining conclusions.

**Theorem 5.1** Let a global solution of equations (2.1)-(2.7) be given for which $f$ is not identically zero. If at some time $\tau_1$ the projection of the solution lies in the set $B_i$ for some $i$ then:

(i) the projection lies in $B_i$ for all $\tau \leq \tau_1$

(ii) if there exists some $\tau_2 > \tau_1$ such that the projection of the solution lies in the complement of $B_i$ then it lies in the complement of all $B_j$ for $\tau > \tau_2$.

(iii) as $\tau \to -\infty$ the projection converges to a point of the boundary which is not one of the points $V_i^-$.

**Proof** Suppose without loss of generality that $i = 1$. It follows from (2.20) that

$$\partial_{\tau}\hat{k}_1^1 = -4\pi(\hat{\rho} - \text{tr}\hat{T})(3\hat{k}_1^1 - 1) - 8\pi\text{tr}\hat{T}\hat{k}_1^1 + 8\pi\hat{T}_1^1$$

(5.1)

If at some time $\hat{k}_1^1 \leq 0$ then the first and second terms on the right hand side of (5.1) are non-negative while the third term is positive. Hence $\partial_{\tau}\hat{k}_1^1 > 0$. This implies the first conclusion of the theorem. Moreover it means that if the projection once leaves $B_1$ it can never reenter it. A similar statement of course applies to any other $B_j$ and this gives (ii).

To prove (iii) note first that $\partial_{\tau}\hat{k}_1^1$ is bounded below by a positive constant as long as $\hat{\rho}$ is. This shows that $\lim \inf_{\tau \to -\infty} \hat{\rho} = 0$. Equation (5.1) also implies that the integral of $\hat{\rho}$ on the interval $(-\infty, \tau_i]$ must be finite so that for each $i$ the integral of the right hand side of (2.20) is absolutely convergent. Hence each $\hat{k}_j^i$ tends to a limit as $\tau \to -\infty$. By what has already been said it can only be a point of the boundary of $K$. The monotonicity of $\hat{k}_1^1$ shows that this limit cannot be one of the points $V_i^-$. 

**Theorem 5.2** Let a global solution of equations (2.1)-(2.7) be given for which $f$ is not identically zero. If at some time $\tau_1$ the projection of the solution lies in the set $A_i$ for some $i$ then as $\tau$ decreases either:

(i) the projection converges to a point of the boundary of $K$ as $\tau \to -\infty$ or

(ii) it reaches $L_j^+$ for some $j$ or $C$ at a finite time before $\tau_1$ or

(iii) it stays in $A_i$ for all $\tau < \tau_1$ and it has a point of one of the lines $L_j^+$ or the point $C$ as an accumulation point.
Proof Suppose without loss of generality that \( i = 1 \). When the projection lies in \( A_1 \) the inequalities \( k_1^1 > k_2^1, k_1^1 > k_3^1 \) and \( \hat{k}_1 > 1/3 \) hold. Suppose that on the time interval \((-\infty, \tau_1)\) the inequalities \( \hat{k}_1^1 - \hat{k}_2^1 \geq A \) and \( \hat{k}_1^1 - \hat{k}_3^1 \geq A \) are satisfied for some \( A > 0 \). Then by Lemma 4.3 it follows that on this time interval \( T_1^1 / T_2^1 \) and \( T_1^3 / T_3^3 \) can be bounded below by a decreasing function which tends to \( \infty \) as \( \tau \to -\infty \). Moreover, by Lemma 4.1 \( trT/\rho \to 1 \) as \( \tau \to -\infty \). Now define a sequence of solutions of (2.1)-(2.7) by \( \hat{u}_n(\tau) = u(\tau - n), \tau \in (-\infty, \tau_1) \), where \( u \) denotes any of the functions which make up the solution and \( n \) is a positive integer. By Lemma 5.1 there exists a subsequence such that \( \hat{k}_j^1 \) and \( \hat{\rho} \) converge uniformly on compact subsets. By the statements made above \( trT \) must tend to the same limit as \( \hat{T} \) and \( \hat{T}_n \). The solution of the asymptotic system obtained inherits the properties that \( k_1^1 \geq k_2^1 \) and \( k_1^1 \geq k_3^1 \). The only solutions of the asymptotic system which satisfy these inequalities on an interval of the form \((-\infty, \tau_1)\) are the vacuum solutions. If for some choice of subsequence this vacuum solution is not that corresponding to the point \( V_1^- \) then the projection of the original solution must converge to that point, by Theorem 5.1. Otherwise every subsequence of the sequence of translated solutions has a subsequence which converges to the same solution of the asymptotic system. Hence the whole sequence converges to this solution and the projection of the original solution converges to \( V_1^- \). In both cases the solution of the original system converges a point of the boundary of \( K \).

It remains to consider the case where the above estimate is not satisfied for any \( A > 1 \). If the solution does not reach \( L_j^+ \) for some \( j \) or \( C \) in finite time then it stays in \( A \), for all \( \tau < \tau_1 \). Then it must have as an accumulation point either a point on \( L_j^+ \) for some \( j \), \( C \) or \( V_j^+ \) for some \( j \). In the first two cases this gives case (iii) of the conclusion of the theorem. In the third case the solution enters \( B_j \) and so by Theorem 5.1 case (i) of the conclusion holds.

In case (iii) of this theorem we can also consider a limit of translates of the solutions whose existence is guaranteed by Lemma 5.1. If the ratios \( a/b \) and \( a/c \) tended to zero as \( \tau \to -\infty \) for the original solution then by Lemma 4.2 the ratios \( T_1^1 / T_2^2 \) and \( T_1^3 / T_3^3 \) would tend to infinity and the solution would belong to case (i). Thus in case (iii) it can be assumed without loss of generality (after possibly interchanging the indices 2 and 3) that \( b/a \) is bounded as \( \tau \to -\infty \). Hence \( f_{-\infty}^{\tau_1}(\hat{k}_1^1 - \hat{k}_2^1) \) is finite. Since \( \partial_{\tau} (\hat{k}_1^1 - \hat{k}_2^1) \) is bounded it follows that \( \hat{k}_1^1 - \hat{k}_2^1 \to 0 \) as \( \tau \to -\infty \). Hence the solution obtained as a limit of translates has LRS geometry. It also satisfies \( \rho = trT \). Information about the asymptotics of LRS solutions can thus be used to obtain information about the asymptotics of the solutions which fit into case (iii) of Theorem 5.2 but do not fit into case (i).

**Theorem 5.3** Let a solution of equations (2.19)-(2.20) be given which satifies the LRS condition \( k_2^1 = k_3^1 \). If at some time \( \tau_1 \) the projection satisfies \( \hat{k}_1^1 < 1/3 \) then either:

(i) the projection of the solution tends to the point \((-\frac{1}{3}, \frac{2}{3}, \frac{2}{3})\) as \( \tau \to -\infty \),

(ii) it tends to the point \((0, \frac{1}{2}, \frac{1}{2})\) as \( \tau \to -\infty \),

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(iii) it reaches the point \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) at a finite time before \( \tau_1 \) or
(iv) it tends to the point \( (\frac{1}{3}, \frac{1}{3}, \frac{1}{3}) \) as \( \tau \to -\infty \).

**Proof** Suppose first that \( \hat{k}_1^i \leq 1/3 - A \) for some \( A > 0 \). Then by Lemma 4.2 the ratio of \( T_2^i = T_3^i \) to \( T_1^i \) increases without limit. Passing to a limit of translates in the familiar way gives a solution of the equation \( \partial_\tau \hat{k}_1^i = -8\pi \hat{k}_1^i \hat{\rho} \) for which \( \hat{k}_1^i \) satisfies the same inequality as before. There are only two such solutions, namely that for which \( \hat{\rho} = 0 \) and that for which \( \hat{k}_1^i = 0 \). In the first of these cases the original solution must enter the region \( B_1 \) and hence by Theorem 5.1 belong to case (i) of the conclusion of the present theorem. The only way of avoiding this is if, no matter which subsequence is chosen, the limiting value of \( \hat{k}_1^i \) is zero. Hence the projection of original solution must converge to the point \( (0, \frac{1}{3}, \frac{1}{3}) \).

Thus the solution belongs to case (ii) of the conclusion. Now consider the case where there is no \( A > 0 \) with the given property. If, despite this, the ratio of \( T_2^i \) to \( T_1^i \) tends to infinity we can argue as before to show that the solution belongs to case (i) or (ii). If, on the other hand this ratio remains bounded then then ratio \( a/b \) must remain bounded and hence if \( \hat{k}_1^i \) remains smaller than 1/3 for ever then \( \int_{-\infty}^{\tau_1} (1/3 - \hat{k}_1^i) \) is finite. It then follows as in the discussion following the proof of Theorem 5.2 that \( \hat{k}_1^i \to 1/3 \). Thus the solution either belongs to case (iii) or case (iv).

**Theorem 5.4** Let a solution of the equations (2.1)-(2.7) with \( m = 1 \) and \( f \) not identically zero be given. Then \( \hat{k}_1^i \to 1/3 \) and \( T_1^i/\rho \to 0 \) for each \( i \) as \( \tau \to \infty \).

**Proof** In [12] it was shown that the scale factors \( a, b \) and \( c \) are bounded below by a positive constant on any interval of the form \([\tau_1, \infty)\). Using (5.1) this statement can be strengthened. Suppose that \( \hat{k}_1^i \) were negative on an interval of the form \([\tau_1, \infty)\). Then it would follow, as in the proof of Theorem 5.1, that the integral of \( \hat{\rho} \) on this interval was finite. But \( \hat{\rho} \) is increasing on this interval, a contradiction. It follows that each \( \hat{k}_1^i \) must become zero after a finite time and once this happens it must immediately become positive and stay positive. Thus, for \( \tau_1 \) sufficiently large \( a, b \) and \( c \) are increasing. Consider now the behaviour of the quantity \( \min\{a, b, c\} \). Suppose first that it tends to infinity as \( \tau \to \infty \). Then by Lemma 4.4 the ratios \( T_i^i/\rho \) tend to zero as \( \tau \to \infty \). Construct a limit of translates as in the proof of Theorem 5.2 except that this time the translations should be done in the opposite direction. Then the limiting solution satisfies \( \partial_\tau \hat{k}_1^i = -12\pi \hat{\rho}(\hat{k}_1^i - \frac{1}{3}) \).

This is the equation which is satisfied by a Bianchi I solution of the Einstein equations coupled to dust. It is well known and also easy to see directly that in the case of dust each \( \hat{k}_1^i \) converges to 1/3 as \( \tau \to \infty \). Because a convergent subsequence can be extracted from any subsequence of the sequence of translates by integers it follows that \( \hat{k}_1^i \to 1/3 \) for the original solution of the Einstein-Vlasov system as well. Next consider the case where \( a \) is bounded on an interval of the form \([\tau_1, \infty)\) while \( \min\{b, c\} \to \infty \) as \( \tau \to \infty \). Then by Lemma 4.4 the ratios \( T_2^i/\rho \) and \( T_3^i/\rho \) converge to zero as \( \tau \to \infty \) while \( T_1^i/\rho \) remains bounded away from zero. Equation (5.1) implies that \( \partial_\tau \hat{k}_1^i \) is bounded below by a positive constant if \( \hat{k}_1^i < 1/3 \) and \( T_2^i/T_1^i \) and \( T_3^i/T_1^i \) are less than \( 1 - A \) for some constant \( A > 0 \). However this contradicts the boundedness of \( a \). Since the volume tends to infinity as \( \tau \to \infty \) at least one of \( a, b \) or \( c \) must tend to infinity. It follows that to complete the proof we may assume without loss of generality that \( a \) and \( b \) are bounded while \( c \) tends to

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infinity. By Lemma 4.4 $T_3^3/\rho \to 0$ while $T_1^1/\rho$ and $T_2^2/\rho$ are bounded below by a positive constant. Now the integral $\int_{\tau_1}^\infty \dot{k}_i^1(\tau)d\tau$ is finite for $i = 1,2$ and $\partial_\tau \dot{k}_i^1$ is bounded. Hence $k_1^1$ and $k_2^2$ tend to zero as $\tau \to \infty$ and $\dot{k}_3^3 \to 1$. But the given behaviour of the pressures shows that for $\dot{k}_3^3 \geq 1/3$ and sufficiently large times $\partial_\tau \dot{k}_3^3$ is negative, a contradiction. This completes the proof.

6. A compactification

This section is devoted to a finer examination of LRS solutions of the Einstein-Vlasov system with massless particles. For LRS solutions with $k_2^2 = k_3^3$ let $k = \dot{k}_1^1$, $q = b/a$, $Q = T_1^1/\rho$. Then $\dot{k}_3^3 = \frac{1}{2}(1-k)$ and $\dot{\rho} = (1/16\pi)(\frac{1}{2} + k - \frac{3}{2}k^2)$. Then the essential equations describing the dynamics are:

\[ \partial_\tau k = \frac{1}{4}(1+3k)(1-k)(Q-k) \tag{6.1} \]
\[ \partial_\tau q = \frac{1}{2}(1-3k)q \tag{6.2} \]

The quantity $Q$ can be expressed entirely in terms of $q$ and the initial data as follows:

\[ Q = q^2 \left[ \frac{\int f_0(v_1)v_1^2(q^2v_2^2 + v_2^2 + v_3^2)^{-1/2}dv_1dv_2dv_3}{\int f_0(v_1)(q^2v_1^2 + v_2^2 + v_3^2)^{1/2}dv_1dv_2dv_3} \right] \tag{6.3} \]

Substituting (6.3) into (6.1) makes the equations (6.1) and (6.2) into an autonomous system of ordinary differential equations for $k$ and $q$. Lemma 4.2 shows that $Q(q) = O(q^{4/3})$ as $q \to 0$. This means that the system (6.1)-(6.2) can be extended in a $C^1$ manner to the boundary $q = 0$. Moreover, $Q$ does not contribute to the linearization of the extended system at the critical point $q = 0$, $k = 0$. The eigenvectors of the linearization are directed along the $k$ and $q$ axes with eigenvalues $-1/4$ and $1/2$ respectively. It follows (see e.g. [15]) that the dynamical system has an unstable manifold which is a curve tangent to the $k$ axis at the point $(0,0)$. This shows that for any initial value $f_0$ it is possible to find LRS solutions of the Einstein-Vlasov system where the distribution function has the initial value $f_0$ and where the quantities $\dot{k}_i^1$ converge to $(0,\frac{1}{2},\frac{1}{2})$ as $\tau \to -\infty$. Note that the stable manifold is just the $k$ axis and so does not give rise to any smooth solutions of the Einstein-Vlasov system. The information about the linearization also determines the nature of the phase portrait near the singular point and shows that there are solutions for which $\dot{k}_1^1$ approaches zero but turns back before reaching it. A typical feature of Bianchi models is that the matter becomes dynamically negligible near the singularity. No attempt will be made here to make this notion precise but one aspect of it is that the projection of the solution should tend to a point of the boundary of $K$ as $\tau \to -\infty$. The solution whose existence has just been shown is an exception to the rule. For anisotropic Bianchi I models with a perfect fluid no exceptional solutions of this kind exist. However they do occur for other Bianchi types[16].

The critical points of the system (6.1)-(6.2) in the region where $0 < q$ and $-1/3 < k < 1$ are the points of the form $(1/3, q_0)$ where $q_0$ has the property that $Q(q_0) = 1/3$. Differentiating (6.3) and estimating the result in an elementary way leads to the inequality $qQ' \geq Q(1-Q)$. This shows that the function $Q$ is strictly increasing for $q > 0$. Taking
account of the limiting values of $Q$, it follows that there is precisely one value $q_0$ for which $Q(q_0) = 1/3$. Moreover, at this point $qQ' \geq 2/9$. The eigenvalues of the linearization at the corresponding critical point are $-\frac{1}{6} \pm \frac{1}{2} \sqrt{1 - 2q_0 Q'(q_0)}$. They both have negative real parts and the critical point is a sink. In particular no solution emerges from this critical point. Thus it is seen that if the projection of any LRS solution for massless particles approaches the point $C$ as $\tau \to -\infty$ then the projection must stay at $C$ for all time i.e. the solution must have isotropic geometry. This should be compared with the results of Newman[17] on isotropic singularities in solutions of the Einstein equations coupled to a radiation fluid.

**Appendix: Fluids with nonlinear equation of state**

Consider a perfect fluid with equation of state $p = f(\rho)$ which satisfies the following general assumptions:

(i) $f$ is a continuous function from $[0, \infty)$ to itself with $f(0) = 0$ which is $C^1$ for $\rho > 0$
(ii) $0 \leq f'(\rho) \leq 1$ for all $\rho > 0$
(iii) there exists a constant $C < 1$ such that $p \leq C\rho$ for $\rho < 1$

Assumptions (i) and (ii) are standard. The third assumption is, when (i) and (ii) are satisfied, equivalent to the assumption made in [7] that the solution is not asymptotically stiff at low densities. In the case of a linear equation of state $f(\rho) = k\rho$ the assumptions (i)-(iii) are satisfied if and only if $0 \leq k < 1$. In a Bianchi I spacetime it follows from the momentum constraint (2.2) that the four-velocity of the fluid is orthogonal to the hypersurfaces of homogeneity. Hence the energy density $\rho$ measured by an observer whose wordline is orthogonal to the hypersurfaces of homogeneity is the same as that measured by a comoving observer. Equations (2.19) and (2.20) are valid as in the case of the Einstein-Vlasov system. For a fluid it can be assumed without loss of generality that the solution is reflection-symmetric because given any initial data, it suffices to do a linear transformation of the coordinates which simultaneously diagonalizes the metric and second fundamental form in order to transform the given data to data for a reflection-symmetric spacetime.

In the case of a fluid $T^i_j = p\delta^i_j$ and hence $\dot{T}^i_j = \dot{p}\delta^i_j$, where $\dot{p} = p/H^2$. If the equation of state is linear then $\dot{p}$ can be expressed as a function of $\dot{\rho}$ alone and (2.19) and (2.20) then reduce to a system of ordinary differential equations which suffice to determine $\dot{k}^i_j$ from initial data. For a nonlinear equation of state this is no longer the case. The equations (2.19) and (2.20) no longer form a closed system and must instead be considered as the projection of a bigger system, as in the case of the Vlasov equation. This is one reason why the linear case has been studied preferentially in the literature. Nevertheless it turns out that the projection can be analysed very effectively in the general case.

The first question which needs to be addressed is that of global existence in $\tau$ i.e. the equivalent of Lemma 2.1 for a fluid. This follows from the results of [7]. The assumption (iii) has been used at this stage. A direct calculation shows that the quantity $(\dot{k}_1^1 - \dot{k}_3^3)/(\dot{k}_1^1 - \dot{k}_3^3)$ is independent of $\tau$ whenever $\dot{k}_1^1 - \dot{k}_3^3$ is non-zero. Moreover if $\dot{k}_1^1 - \dot{k}_3^3$ is zero at some time it remains zero. Hence the projection of each solution is constrained to move on a straight line in $K$ passing through the centre $C$. This is already a much stronger statement than could be proved in the case of the Vlasov equation. To find out how the
projection moves on this straight line, the time derivative of the dimensionless version of
the density will be calculated. For a fluid it is given by

$$\partial_\tau \hat{\rho} = (\hat{\rho} - \hat{p})(1 - 24\pi \hat{\rho})$$  \hspace{1cm} (A1)

Noting that the Hamiltonian constraint implies that $24\pi \rho \leq 1$ with equality only at the
point $C$, it can now be seen that the projection moves monotonically from the boundary of
$K$ at $\tau = -\infty$ to the centre $C$ of $K$ at $\tau = \infty$. This qualitative behaviour is independent of
the equation of state satisfying (i)-(iii). The only difference is in the speed with which the
projection moves along the radial line at different times. Equation (A1) also makes clear
that this picture changes completely if the equations of state considered here are replaced
by the limiting case of a stiff fluid, $p = \rho$.

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