q-VOLKENBORN INTEGRATION AND ITS APPLICATIONS

TAEKYUN KIM
Department of Mathematics Educations, Kongju
National University, Kongju 314-701, S. Korea
e-mail: tkim@kongju.ac.kr

Abstract. The main purpose of this paper is to present a systemic study of some families of multiple $q$-Euler numbers and polynomials. In particular, by using the $q$-Volkenborn integration on $\mathbb{Z}_p$, we construct $p$-adic $q$-Euler numbers and polynomials of higher order. We also define new generating functions of multiple $q$-Euler numbers and polynomials. Furthermore, we construct Euler $q$-Zeta function.

1. Introduction

For any complex number $z$, it is well known that the familiar Euler polynomials $E_n(z)$ are defined by means of the following generating function, cf.[3, 5, 6, 9, 13]:

$$F(z, t) = \frac{2}{e^t + 1} e^{zt} = \sum_{n=0}^{\infty} E_n(z) \frac{t^n}{n!}, \quad (|t| < \pi).$$ (1)

We note that, by substituting $z = 0$ into (1), $E_n(0) = E_n$ is the familiar $n$-th Euler number defined by

$$G(t) = F(0, t) = \frac{2}{e^t + 1} = \sum_{n=0}^{\infty} E_n \frac{t^n}{n!}, \quad (|t| < \pi), \text{ cf.[4, 5]}. $$

By the meaning of the generalization of $E_n$, Frobenius-Euler numbers and polynomials are also defined by

$$\frac{1-u}{e^t - u} = \sum_{n=0}^{\infty} H_n(u) \frac{t^n}{n!}, \text{ and } \frac{1-u}{e^{xt} - u} = \sum_{n=0}^{\infty} H_n(u, x) \frac{t^n}{n!} \quad (u \in \mathbb{C} \text{ with } |u| > 1), \text{ cf.[16]}. $$
Over five decades ago, Calitz [2, 3] defined $q$-extension of Frobenius-Euler numbers and polynomials and proved properties analogous to those satisfied $H_n(u)$ and $H_n(u, x)$. Recently, Satoh [14, 15] used these properties, especially the so-called distribution relation for the $q$-Frobenius-Euler polynomials, in order to construct the corresponding $q$-extension of the $p$-adic measure and to define a $q$-extension of $p$-adic $l$-function $l_{p,q}(s,u)$.

Let $p$ be a fixed odd prime in this paper. Throughout this paper, the symbols $\mathbb{Z}$, $\mathbb{Z}_p$, $\mathbb{Q}_p$, $\mathbb{C}$ and $\mathbb{C}_p$, denote the ring of rational integers, the ring of $p$-adic integers, the field of $p$-adic numbers, the complex number field, and the completion of the algebraic closure of $\mathbb{Q}_p$, respectively. Let $\nu_p(p)$ be the normalized exponential valuation of $\mathbb{C}_p$ with $|p|_p = p^{-\nu_p(p)} = p^{-1}$. When one speaks of $q$-extension, $q$ can be regarded as an indeterminate, a complex number $q \in \mathbb{C}$, or a $p$-adic number $q \in \mathbb{C}_p$; it is always clear from the context. If $q \in \mathbb{C}$, then one usually assumes that $|q| < 1$. If $q \in \mathbb{C}_p$, then one usually assumes that $|q - 1|_p < p^{-\frac{1}{2}}$, and hence $q^x = \exp(x \log q)$ for $x \in \mathbb{Z}_p$. In this paper, we use the below notation

$$[x]_q = \frac{1 - q^x}{1 - q}, \quad (a : q)_n = (1 - a)(1 - aq) \cdots (1 - aq^{n-1}), \text{ cf.}[6, 7, 8, 9, 10, 11, 14].$$

Note that $\lim_{q \to 1}[x]_q = x$ for any $x$ with $|x|_p \leq 1$ in the $p$-adic case. For a fixed positive integer $d$ with $(p,d) = 1$, set

$$X = X_d = \lim_{N \to \infty} \mathbb{Z}/dp^N,$$

$$X_1 = \mathbb{Z}_p, X^* = \bigcup_{0 < a < dp} a + dp\mathbb{Z}_p,$$

$$a + dp^N\mathbb{Z}_p = \{x \in X | x \equiv a \pmod{p^N}\},$$

where $a \in \mathbb{Z}$ satisfies the condition $0 \leq a < dp^N$, (cf.[10,11]). We say that $f$ is a uniformly differentiable function at a point $a \in \mathbb{Z}_p$, and write $f \in UD(\mathbb{Z}_p)$, if the difference quotients $F_f(x,y) = \frac{f(x)-f(y)}{x-y}$ have a limit $f'(a)$ as $(x,y) \to (a,a)$, cf.[11]. For $f \in UD(\mathbb{Z}_p)$, let us begin with the expression

$$\frac{1}{[p^N]_q} \sum_{0 \leq j < p^N} q^j f(j) = \sum_{0 \leq j < p^N} f(j) \mu_q(j + p^N \mathbb{Z}_p), \text{ cf.}[7, 8, 9, 11],$$

which represents a $q$-analogue of Riemann sums for $f$. The integral of $f$ on $\mathbb{Z}_p$ is defined as the limit of those sums (as $n \to \infty$) if this limit exists. The $q$-Volkenborn integral of a function $f \in UD(\mathbb{Z}_p)$ is defined by

$$I_q(f) = \int_X f(x)d\mu_q(x) = \int_{X_d} f(x)d\mu_q(x) = \lim_{N \to \infty} \frac{1}{[dp^N]_q} \sum_{0}^{dp^N-1} f(x)q^x.$$
Recently, we considered another construction of a $q$-Eulerian numbers, which are different than Carlitz’s $q$-Eulerian numbers as follows, cf.[6, 12, 13]:

$$F_q(x, t) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^n e^{(n+x)t} = \sum_{n=0}^{\infty} E_{n,q}(x) \frac{t^n}{n!}.$$  

Thus we have

$$E_{n,q}(x) = \frac{[2]_q}{(1-q)_n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}},$$

$$E_{n,q}(x) = \frac{[2]_q}{(1-q)_n} \sum_{l=0}^{n} \binom{n}{l} \frac{(-1)^l}{1+q^{l+1}} q^{lx},$$

where $\binom{n}{l}$ is a binomial coefficient, cf.[13].

Note that $\lim_{q \to 1} E_{n,q} = E_n$ and $\lim_{q \to 1} E_{n,q}(x) = E_n(x)$. In [12], we also proved that $q$-Eulerian polynomial $E_{n,q}(x)$ can be represented by $q$-Volkenborn integral as follows:

$$\int_{X_d} [x + x_1]^k d\mu_{-q}(x_1) = \int_{Z_p} [x + x_1]^k d\mu_{-q}(x) = E_{k,q}(x), \text{ for } k, d \in \mathbb{N},$$

where $\mu_{-q}(x + p^N \mathbb{Z}_p) = \frac{q^x [2]_q}{1+q^{p^N}} (-1)^x$.

The purpose of this paper is to present a systemic study of some families of multiple $q$-Euler numbers and polynomials. In particular, by using the $q$-Volkenborn integration on $\mathbb{Z}_p$, we construct $p$-adic $q$-Euler numbers and polynomials of higher order. We also define new generating function of these $q$-Euler numbers and polynomials of higher order. Furthermore, we construct Euler $q$-$\zeta$-function. From section 2 to section 5, we assume that $q \in \mathbb{C}_p$ with $|1-q|_p < p^{-\frac{1}{p-1}}$.

2. $q$-Euler numbers and polynomials associated with an invariant $p$-adic $q$-integrals on $\mathbb{Z}_p$

Let $h \in \mathbb{Z}$, $k \in \mathbb{N} = \{1, 2, 3, \ldots \}$, and let us consider the extended higher-order $q$-Euler numbers as follows:

$$E_{m,q}^{(h,k)} = \underbrace{\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p}}_{k \text{ times}} [x_1 + x_2 + \cdots + x_k]_q^m q^{x_1(h-1)+\cdots+x_k(h-k)} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).$$

Then we have

$$E_{m,q}^{(h,k)} = \frac{[2]_q^k}{(1-q)_m} \sum_{l=0}^{m} \binom{m}{l} \frac{(-1)^l}{(-q^{h+l} : q^{-1})_k}.$$
From the definition of $E_{m,q}^{(h,k)}$, we can easily derive the below:

$$E_{m,q}^{(h,k)} = E_{m,q}^{(h-1,k)} + (q-1)E_{m+1,q}^{(h-1,k)}, \quad (m \geq 0).$$

It is easy to show that

$$\int_{Z_p} \cdots \int_{Z_p} q^{k+1(m-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1})$$

$$= \sum_{j=1}^{m} \binom{m}{j} (q-1)^j \int_{Z_p} \cdots \int_{Z_p} [\sum_{l=1}^{k+1} x_l]_q^j q^{-\sum_{j=1}^k jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}), \quad (2)$$

and we also get

$$\int_{Z_p} \cdots \int_{Z_p} q^{k+1(m-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_{k+1}) = \frac{[2]_{q+1}}{(-q^m : q^{-1})_{k+1}}. \quad (3)$$

From (2) and (3), we can derive the below proposition.

**Proposition 1.** For $m, k \in \mathbb{N}$, we have

$$\sum_{j=0}^{m} \binom{m}{j} (q-1)^j E_{m-i+j,q}^{(0,k+1)} = \frac{[2]_{q+1}}{(-q^m : q^{-1})_{k+1}},$$

$$E_{m,q}^{(h,k)} = \frac{[2]_{q}}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} \frac{(-1)^l}{(-q^{h+l} : q^{-1})_{k}}.$$

**Remark.** Note that $E_{n,q}^{(1,1)} = E_{n,q}$, where $E_{n,q}$ are the $q$-Euler numbers (see [13]).

From the definition of $E_{m,q}^{(h,k)}$, we can derive

$$\sum_{j=0}^{i} \binom{i}{j} (q-1)^j E_{m-i+j,q}^{(h-1,k)} = \sum_{j=0}^{i-1} \binom{i-1}{j} (q-1)^j E_{m+j-i,q}^{(h,k)},$$

for $m \geq i$. By simple calculation, we easily see that

$$\sum_{j=0}^{m} \binom{m}{j} (q-1)^j E_{j,q}^{(h,1)} = \int_{Z_p} q^{m-h} x^{(h-1)x} d\mu_{-q}(x) = \frac{[2]_{q}}{[2]_{q^{m-h}}}. \quad (4)$$

Furthermore, we can give the following relation for the $q$-Euler numbers, $E_{m,q}^{(0,h)}$:

$$\sum_{j=0}^{m} \binom{m}{j} (q-1)^j E_{j,q}^{(0,k)} = \frac{[2]_{q}}{(-q^m : q^{-1})_{k}}. \quad (4)$$

4
3. Polynomials $E_{n, q}^{(0, k)}(x)$

We now define the polynomials $E_{n, q}^{(0, k)}(x)$ (in $q^x$) by

$$E_{n, q}^{(0, k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ x_1 + x_2 + \cdots + x_k \right]_{q^m} q^{\sum_{j=1}^{k} jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).$$

Thus, we have

$$(q - 1)^m E_{m, q}^{(0, k)}(x) = [2]_q^k \sum_{j=0}^{m} \binom{m}{j} q^{jx} (-1)^{m-j} \frac{1}{(-q^j : q^{-1})_k}. \quad (5)$$

It is not difficult to show that

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{m} (m-j)x_j + mx} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = q^{mx} [2]_q^k \frac{1}{(-q^m : q^{-1})_k},$$

and

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} q^{\sum_{j=1}^{m} (m-j)x_j + mx} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \sum_{j=0}^{m} \binom{m}{j} (q - 1)^j E_{j, q}^{(0, k)}(x).$$

Therefore we obtain the following.

**Lemma 2.** For $m, k \in \mathbb{N}$, we have

$$\sum_{j=0}^{m} \binom{m}{j} (q - 1)^j E_{j, q}^{(0, k)}(x) = \frac{q^{mx} [2]_q^k}{(-q^m : q^{-1})_k},$$

$$E_{m, q}^{(0, k)}(x) = \frac{[2]_q^k}{(1-q)^m} \sum_{j=0}^{m} \binom{m}{j} q^{jx} (-1)^{j} \frac{1}{(-q^j : q^{-1})_k}. \quad (6)$$

Let $l \in \mathbb{N}$ with $l \equiv 1 \pmod{2}$. Then we get easily

$$\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^{k} x_j \right]_{q^m} q^{-\sum_{j=1}^{k} jx_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)$$

$$= \frac{[l]_q^m}{[l]_{-q}^{i_1, \ldots, i_k = 0}} \sum_{i_{j=2}^{k} = (j-1)i_j} (-1)^{\sum_{j=1}^{k} i_j} \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ x + \sum_{j=1}^{k} i_j x_j \right]_{q^m} q^{-\sum_{j=1}^{k} jx_j} d\mu_{-q^i}(x_1) \cdots d\mu_{-q^i}(x_k).$$

From this, we can derive the following “multiplication formula”:
Theorem 3. Let \( l \) be an odd positive integer. Then

\[
E_{m,q}^{(0,k)}(x) = \frac{[l]^m}{[l]^q} \sum_{i_1, \ldots, i_k=0}^{l-1} q^{-\sum_{j=2}^k (j-1)i_j} (-1)^{\sum_{i=1}^k i_i} E_{m,q}^{(0,k)}(\frac{x + i_1 + \cdots + i_k}{l}).
\] (7)

Moreover,

\[
E_{m,q}^{(0,k)}(lx) = \frac{[l]^m}{[l]^q} \sum_{i_1, \ldots, i_k=0}^{l-1} q^{-\sum_{j=2}^k (j-1)i_j} (-1)^{\sum_{i=1}^k i_i} E_{m,q}^{(0,k)}(\frac{x + i_1 + \cdots + i_k}{l}).
\] (8)

From (4) and (5), we can also derive the below expression for \( E_{n,q}^{(0,k)}(x) \):

\[
E_{m,q}^{(0,k)}(x) = \sum_{i=0}^m \binom{m}{i} E_{i,q}^{(0,k)}(x) [x]_q^{m-i} q^{ix},
\]

whence also

\[
E_{m,q}^{(0,k)}(x + y) = \sum_{j=0}^m \binom{m}{j} [y]_q^{m-j} q^{jy} E_{j,q}^{(0,k)}(x).
\] (10)

4. Polynomials \( E_{m,q}^{(h,1)}(x) \)

Let us define

\[
E_{m,q}^{(h,1)}(x) = \int_{\mathbb{Z}_p} [x + x_1]^m q^{x_1(h-1)} d\mu_{-q}(x_1).
\] (11)

Then we have

\[
E_{m,q}^{(h,1)}(x) = \frac{[2]_q}{(1-q)^m} \sum_{l=0}^m \binom{m}{l} (-1)^l q^{lx} \frac{1}{(1 + q^{l+h})}.
\]

By simple calculation of \( q \)-Volkenvorn integral, we note that

\[
q^x \int_{\mathbb{Z}_p} [x + x_1]^m q^{x_1(h-1)} d\mu_{-q}(x_1) = (q - 1) \int_{\mathbb{Z}_p} [x + x_1]^m q^{x_1(h-2)} d\mu_{-q}(x_1)
\]

\[
+ \int_{\mathbb{Z}_p} [x + x_1]^m q^{x_1(h-2)} d\mu_{-q}(x_1).
\]

Thus, we have

\[
q^x E_{m,q}^{(h,1)}(x) = (q - 1) E_{m+1,q}^{(h-1,1)}(x) + E_{m,q}^{(h-1,1)}(x).
\] (12)
It is easy to show that
\[ \int_{\mathbb{Z}_q} [x + x_1]^m q^{(h-1)x_1} d\mu_{-q}(x_1) = \sum_{j=0}^{m-1} \binom{m}{j} q^{m-j} \int_{\mathbb{Z}_q} [x_1]^j q^{(h-1)x_1} d\mu_{-q}(x_1). \]
This is equivalent to
\[ E_{m,q}^{(h,1)}(x) = \sum_{j=0}^{m-1} \binom{m}{j} q^{m-j} \int_{\mathbb{Z}_q} [x_1]^j q^{x_1} d\mu_{-q}(x_1), \]
where we use the technique method notation by replacing \( (E_q^{(h,1)})^n \) by \( E_{n,q}^{(h,1)} \), symbolically.

From (11), we can derive
\[ q^h E_{m,q}^{(h,1)}(x + 1) + E_{m,q}^{(h,1)}(x) = [2]q[x]_q^m. \] (13)

For \( x = 0 \) in (13), this gives
\[ q^h (qE_{m,q}^{(h,1)} + 1) + E_{m,q}^{(h,1)} = \delta_{0,k}, \] (14)
where \( \delta_{0,k} \) is Kronecker symbol. By the simple calculation of \( q \)-Volkenborn integration, we easily see that
\[ \int_{\mathbb{Z}_q} q^{x_1(h-1)} d\mu_{-q}(x_1) = \frac{[2]_q}{[2]q^h}. \]
Thus, we have \( E_{0,q}^{(h,1)} = \frac{[2]_q}{[2]q^h} \). From the definition of \( q \)-Euler polynomials, we can derive
\[ \int_{\mathbb{Z}_q} [1 - x + x_1]^m q^{-x_1(h-1)} d\mu_{-q}(x_1) = q^{m+h-1} (-1)^m E_{m,q}^{(h,1)}(x). \]

Therefore we obtain the below “complementary formula”:

**Theorem 4.** For \( m \in \mathbb{N}, n \in \mathbb{Z} \), we have
\[ E_{m,q-1}^{(h,1)}(1 - x) = (-1)^m q^{m+h-1} E_{m,q}^{(h,1)}(x). \] (15)

In particular, for \( x = 1 \), we see that
\[ E_{m,q-1}^{(h,1)}(0) = (-1)^m q^{m+h-1} E_{m,q}^{(h,1)}(1) = (-1)^{m-1} q^{m-1} E_{m,q}^{(h,1)}, \] for \( m \geq 1 \). (16)

For \( l \in \mathbb{N} \) with \( l \equiv 1 \pmod{2} \), we have
\[ \int_{\mathbb{Z}_q} q^{(h-1)x_1} [x + x_1]^m q^{x_1(h-1)} d\mu_{-q}(x_1) = [l]_q \sum_{i=0}^{l-1} q^h (-1)^i \int_{\mathbb{Z}_q} \left[ \frac{x + i}{l} + x_1 \right]^m q^{x_1(h-1)} d\mu_{-q}(x_1). \]

Thus, we can also obtain the following:
Theorem 5. (Multiplication formula) For \( l \in \mathbb{N} \) with \( l \equiv 1 \pmod{2} \), we have
\[
\frac{[2]_q}{[l]_q^m} \sum_{i=0}^{l-1} q^{hi} (-1)^i E_{m,q}^{(h,1)} \left( \frac{x+i}{l} \right) = E_{m,q}^{(h,1)}(x).
\]
Furthermore,
\[
\frac{[2]_q}{[l]_q^m} \sum_{i=0}^{l-1} q^{hi} (-1)^i E_{m,q}^{(h,1)}(x + \frac{i}{l}) = E_{m,q}^{(h,1)}(lx).
\]

5. Polynomials \( E_{m,q}^{(h,k)}(x) \) and \( h = k \)

It is now easy to combine the above results and define the new polynomials as follows:

\[
E_{m,q}^{(h,k)}(x) = \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + x_1 + \cdots + x_k] q^{(h-1)x_1 + \cdots + (h-k)x_k} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
\]

Thus, we note that
\[
(q-1)^m E_{m,q}^{(h,k)}(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} q^{x_j} \left[ \frac{[2]_q^k}{(-q^{j+k} : q-1)^k} \right].
\]  

(17)

We may now mention the following formulas which are easy to prove.

\[
q^h E_{m,q}^{(h,k)}(x + 1) + E_{m,q}^{(h,k)}(x) = [2]_q E_{m,q}^{(h-1,k-1)}(x),
\]  

(18)

and

\[
q^x E_{m,q}^{(h+1,k)}(x) = (q-1) E_{m+1,q}^{(h,k)}(x) + E_{m,q}^{(h,k)}(x).
\]  

(19)

Let \( l \in \mathbb{N} \) with \( l \equiv 1 \pmod{2} \). Then we note that

\[
\int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + \sum_{j=1}^{k} x_j] q^{\sum_{j=1}^{k} (h-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
\]

\[
= \frac{[l]_q^m}{[l]_q^k} \sum_{i_1, \ldots, i_k=0}^{l-1} q^{\sum_{j=1}^{k} i_j - \sum_{j=2}^{k} (j-1)i_j} (-1)^{\sum_{j=1}^{k} i_j}
\]

\[
\cdot \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ \frac{x + \sum_{j=1}^{k} j}{l} + \sum_{j=1}^{k} x_j \right] q^{\sum_{j=1}^{k} (h-j)x_j} d\mu_{-q^i}(x_1) \cdots d\mu_{-q^i}(x_k).
\]

Therefore we obtain the following:
Theorem 6. (Distribution for q-Euler polynomials) For \( l \in \mathbb{N} \) with \( l \equiv 1 \pmod{2} \). Then we have

\[
E_{m,q}^{(h,k)}(lx) = \left\lfloor \frac{m}{l} \right\rfloor l^m \sum_{i_1, \ldots, i_k=0}^{l-1} q^{\sum_{j=1}^{k} i_j - \sum_{j=2}^{k} (j-1)i_j} (-1)^{\sum_{j=1}^{k} i_j} E_{m,q}^{(h,k)} \left( x + \frac{i_1 + \cdots + i_k}{l} \right).
\]

It is interesting to consider the case \( h = k \), which leads to the desired extension of the q-Euler numbers of higher order, cf.[1]. We shall denote the polynomials in this special case by \( E_{m,q}^{(k,k)}(x) := E_{m,q}^{(k)}(x) \). Then we have

\[
(q - 1)^m E_{m,q}^{(k)}(x) = \sum_{j=0}^{m} \binom{m}{j} (-1)^{m-j} q^j x \frac{[2]^k}{(-q^{j+k} : q^{-1})_k},
\]

and

\[
E_{m,q-1}^{(k)}(k - x) = (-1)^m q^{m+(k\underline{2})} E_{m,q}^{(k)}(x).
\]

For \( x = k \) in (22), we see that

\[
E_{m,q-1}^{(k)}(0) = (-1)^m q^{m+(k\underline{2})} E_{m,q}^{(k)}(k).
\]

From (18), we can derive the below formula:

\[
q^k E_{m,q}^{(k)}(x + 1) + E_{m,q}^{(k)}(x) = [2]_q E_{m,q}^{(k-1)}(x).
\]

Putting \( x = 0 \) in (17), we obtain

\[
(q - 1)^m E_{m,q}^{(k)} = \sum_{i=0}^{m} \binom{m}{i} (-1)^{m-i} \frac{[2]^k}{(-q^{i+k} : q^{-1})_k}.
\]

Note that

\[
\sum_{i=0}^{m} \binom{m}{i} (q - 1)^i \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} \left[ x_1 + \cdots + x_k \right]^i q^{\sum_{j=1}^{k-1} (k-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
\]

\[
= \frac{[2]^k}{(-q^{m+k} : q^{-1})_k}.
\]

From this, we can easily derive

\[
\sum_{i=0}^{m} \binom{m}{i} (q - 1)^i E_{i,q}^{(k)} = \frac{[2]^k}{(-q^{m+k} : q^{-1})_k}.
\]
and so it follows
\[ E_{m,q}^{(k)}(x) = (q^x E_q^{(k)} + [x]_q)_m, \quad m \geq 1, \quad (27) \]

where we use the technique method notation by replacing \((E_q^{(k)})^n\) by \(E_{n,q}^{(k)}\), symbolically. In particular, from (24), we have
\[ q^k(qE_q^{(k)} + 1)^m + E_{m,q}^{(k)} = [2]_q E_{m,q}^{(k-1)}. \quad (28) \]

It is easy to see that
\[
\int_{Z_p} \cdots \int_{Z_p} q^{(k-1)x_1 + \cdots + x_{k-1}} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) = \frac{[2]_q^k}{(-q^k : q^{-1})_k}.
\]

Thus, we note that \(E_{0,q}^{(k)} = \frac{[2]_q^k}{(-q^k : q^{-1})_k}\).

6. Generating function for \(q\)-Euler polynomials

An obvious generating function for \(q\)-Euler polynomials is obtained, from (17), by
\[
[2]_q^k e^{\sum_{j=0}^\infty \frac{(-1)^j}{(-q^j : q^{-1})_k} q^j x \left( \frac{1}{1 - q} \right)^j \frac{t^j}{j!}} = \sum_{n=0}^\infty E_{n,q}^{(h,k)} \frac{t^n}{n!}.
\]

From (17), we can also derive the below formula:
\[ q^{h-k} E_{m,q}^{(h,k+1)}(x + 1) = [2]_q E_{m,q}^{(h,k)}(x) - E_{m,q}^{(h,k+1)}(x). \quad (30) \]

Again from (21) and (25), we get easily
\[
\int_{Z_p} \cdots \int_{Z_p} [x + \sum_{j=1}^k x_j]_q^m q^{\sum_{j=1}^{k-1} (k-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k)
\]
\[
= \sum_{j=0}^m \binom{m}{j} q^{x_j} \int_{Z_p} \cdots \int_{Z_p} [x_k]_q^j [x + \sum_{j=1}^{k-1} x_j]_q^{n-j} q^{\sum_{l=1}^{k-1} (k+j-l)x_l} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k).
\]
Thus, we note that
\[
E^{(k)}_{m,q}(x) = \sum_{j=0}^{m} \binom{m}{j} q^{xj} E^{(1)}_{j,q} E^{(k+j,k-1)}_{m-j,q}(x).
\] (31)

Take \( x = 0 \) in (31), we have
\[
E^{(k)}_{m,q} = \sum_{i=0}^{m} \binom{m}{i} E^{(1)}_{j,q} E^{(k+j,k-1)}_{m-j,q}.
\] (32)

So, for \( k = 2 \),
\[
E^{(2)}_{m,q} = \sum_{i=0}^{m} \binom{m}{i} E^{(j+2,1)}_{j,q} E^{(j+1)}_{m-j,q}.
\]

It is not difficult to show that
\[
\int_{\mathbb{Z}} x^m q^{hx} d\mu_q(x) = \sum_{j=0}^{h} \binom{h}{j} (q-1)^j \int_{\mathbb{Z}} x^m q^{h+j} d\mu_q(x), \text{ for } h \in \mathbb{N}.
\]

From this, we can derive the below:
\[
E^{(h+1,1)}_{m,q} = \sum_{j=0}^{h} \binom{h}{j} (q-1)^j E^{(j+1)}_{m+j,q}, \text{ for } h \in \mathbb{N}.
\] (33)

By (32) and (33), we easily see that
\[
E^{(2)}_{m,q} = \sum_{j=0}^{m} \binom{m}{j} E^{(j+2,1)}_{j,q} E^{(j+1)}_{m-j,q}.
\] (34)

By (34), for \( q = 1 \), we note that
\[
E^{(2)}_{m} = \sum_{j=0}^{m} \binom{m}{j} E^{(j+1)}_{j} E^{(j+1)}_{m-j}, \text{ where } \left( \frac{2}{e^t + 1} \right)^k = \sum_{n=0}^{\infty} E^{(n)}_{n} \frac{t^n}{n!}.
\]

It is easy to show that
\[
[x + x_1 + \cdots + x_k]^m = \sum_{j=0}^{m} \binom{m}{j} [x_1 + x]^{m-j} q^j [x_2 + \cdots + x_k]^j.
\]
By using this, we get easily

\[ \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x + \sum_{j=1}^{k} x_j]_q^m q^{\sum_{j=1}^{k-1} (k-j)x_j} d\mu_{-q}(x_1) \cdots d\mu_{-q}(x_k) \]

\[ = \sum_{j=0}^{m} \binom{m}{j} q^{jx} \int_{\mathbb{Z}_p} [x + x_1]_q^{m-j} q^{(k+j-1)x_1} d\mu_{-q}(x_1) \]

\[ \cdot \int_{\mathbb{Z}_p} \cdots \int_{\mathbb{Z}_p} [x_2 + \cdots + x_k]_q^{k-1} (k-j)x_j d\mu_{-q}(x_2) \cdots d\mu_{-q}(x_k). \]

Therefore we obtain the following:

**Theorem 7.** For \( m, k \in \mathbb{N} \), we have

\[ E_{m,q}^{(k)}(x) = \sum_{j=0}^{m} \binom{m}{j} q^{jx} E_{m-j,q}^{(k+j,1)}(x) E_{j,q}^{(k-1)}. \] (35)

Indeed for \( x = 0 \),

\[ E_{m,q}^{(k)} = \sum_{j=0}^{m} \binom{m}{j} E_{m-j,q}^{(k+j,1)} E_{j,q}^{(k-1)} \]

\[ = \sum_{j=0}^{m} \binom{m}{j} E_{j,q}^{(k-1)} \sum_{j=0}^{k+j} (q - 1)^i \binom{k+j-1}{i} E_{m-j+i,q}^{(1)}. \] (36)

As for \( q = 1 \), we get the below formula

\[ E_{m}^{(k)} = \sum_{j=0}^{m} \binom{m}{j} E_{j}^{(k-1)} E_{m-j}^{(1)}. \]

7. q-Euler zeta function in \( \mathbb{C} \)

In this section, we assume that \( q \in \mathbb{C} \) with \( |q| < 1 \). From section 4, we note that

\[ E_{m,q}^{(h,1)}(x) = \frac{[2]_q}{(1-q)^m} \sum_{l=0}^{m} \binom{m}{l} q^{lx} (-1)^l \frac{1}{1 + q^{l+h}} \]

\[ = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{nh} [n + x]_q^n. \] (37)

Thus, we can define q-Euler zeta function:
Definition 8. For \( s, q \in \mathbb{C} \) with \( |q| < 1 \), define

\[
\zeta_{E,q}^h(s, x) = [2]_q \sum_{n=0}^{\infty} \frac{(-1)^n q^{nh}}{(n+x)_q^s},
\]

where \( x \in \mathbb{R} \) with \( 0 < x \leq 1 \).

Note that \( \zeta_{E,q}^h(-m, x) = E_{m,q}^{(h,1)}(x) \), for \( m \in \mathbb{N} \). Let

\[
F_q(t, x) = \sum_{n=0}^{\infty} E_{n,q}^{(h,1)}(x) \frac{t^n}{n!}.
\]

Then we have

\[
F_q(t, x) = [2]_q e^{\frac{t}{1-q}} \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{-\frac{q^{n+x}}{1-q}t}
\]

\[
= [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t}, \quad \text{for } h \in \mathbb{Z}.
\]

Therefore we obtain the following

**Lemma 9.** For \( h \in \mathbb{Z} \), we have

\[
F_q(t, x) = [2]_q \sum_{n=0}^{\infty} (-1)^n q^{hn} e^{[n+x]_q t}
\]

\[
= \sum_{n=0}^{\infty} E_{n,q}^{(h,1)}(x) \frac{t^n}{n!}.
\]

(38)

Let \( \Gamma(s) \) be the gamma function. Then we easily see that

\[
\frac{1}{\Gamma(s)} \int_0^{\infty} t^{s-1} F_q(-t, x) dt = \zeta_{E,q}^h(s, x), \quad \text{for } s \in \mathbb{C}.
\]

(39)

From (38) and (39), we can also derive the below Eq.(40):

(40)

\[
\zeta_{E,q}^h(-n, x) = E_{n,q}^{(h,1)}(x), \quad \text{for } n \in \mathbb{N}.
\]
References

[1] E. Boros, K. Elbassioni, V. Gurvich, L. Khachiyan, K. Makino, An intersection inequality for discrete distributions and related generation problems, Lecture Notes in Comput. Sci. 2719 (2003), 543-555.
[2] L. Carlitz, q-Bernoulli numbers and polynomials, Duke Math. J. 15 (1948), 987-1000.
[3] L. Carlitz, q-Bernoulli and Eulerian numbers, Trans. Amer. Math. Soc. 76 (1954), 332-350.
[4] E. Deeba and D. Rodriguez, Stirling’s series and Bernoulli numbers, Amer. Math. Monthly 98 (1991), 423-426.
[5] T. Howard, Applications of a recurrences formula for the Bernoulli numbers, J. Number Theory 52 (1995), 157-172.
[6] T. Kim, L. C. Jang, H. K. Park, A note on q-Euler and Genocchi numbers, Proc. Japan Academy 77 (2001), 139-141.
[7] T. Kim, Power series and asymptotic series associated with the q-analog of the two-variable p-adic L-function, Russ. J. Math. Phys. 12 (2005), 189-196.
[8] T. Kim, Non-Archimedean q-integrals associated with multiple Changhee q-Bernoulli polynomials, Russ. J. Math. Phys. 10 (2003), 91-98.
[9] T. Kim, An invariant p-adic integral associated with Daehee numbers, Integral Transforms and special functions 13 (2002), 65-69.
[10] T. Kim, On a q-analogue of the p-adic log gamma functions and related integrals, J. Number Theory 76 (1999), 320-329.
[11] T. Kim, q-Volkenborn integration, Russ. J. Math. Phys. 9 (2002), 288-299.
[12] T. Kim, A note on q-Volkenborn integration, Proc. Jangjeon Math. Soc. 8 (2005), 13-17.
[13] T. Kim, q-Euler and Genocchi numbers, arXiv: math. NT/0506278 vol 1 14 June (2005).
[14] J. Satoh, q-analogue of Riemann’s ζ-function and q-Euler numbers, J. Number Theory 31 (1989), 346-362.
[15] J. Satoh, Sums products of two q-Bernoulli numbers, J. Number Theory 74 (1999), 173-180.
[16] J. Shiratani and S. Yamamoto, On a p-adic interpolating function for the Euler numbers and its derivatives, Mem. Fac. Sci. Kyushu Univ. Math. 39 (1985), 113-125.