A Relation between $\theta_{13}$ and the Leptonic Dirac CP Phase in $SO(10)$ Lopsided Models

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It is shown that in $SO(10)$ models where the large solar and atmospheric neutrino angles come from the charged-lepton mass matrix being “lopsided”, there is a characteristic relation between the $13$ mixing angle of the neutrinos and the size of the Dirac CP-violating phase in the lepton sector. This is illustrated in a recently proposed realistic and predictive model.

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The symmetries of the Standard Model do not constrain the masses and mixings of the quarks and leptons, which are therefore free parameters of that model. The best hope for obtaining predictions (and testably precise postdictions) of these quantities seems to lie with the more powerful symmetries of grand unified theories. The greatest degree of predictivity comes from the unification group $SO(10)$, since it relates the mass matrices of the up-type quarks, down-type quarks, and charged leptons, and the Dirac mass matrix of the neutrinos. (These four mass matrices will be denoted here by $M_U$, $M_D$, $M_L$, and $M_N$, respectively.) An obstacle to making predictions, however, is the fact that in the usual “type I see-saw” set-up the mass matrix of the observed light neutrinos $M_\nu$ depends on the Majorana mass matrix of the right-handed neutrinos $M_R$ through the well-known see-saw formula $M_\nu = -M_N M_R^{-1} M_T$. And because $M_R$ tends in most models to be only loosely related by $SO(10)$ symmetry, if at all, to the other mass matrices, our experimental knowledge of the properties of the quarks and charged leptons gives no information about $M_\nu$. Since $M_R$ is unconstrained and is a symmetric three-by-three complex matrix, it introduces many free parameters into the calculation of the light neutrino masses and mixing angles.

In this letter, we discuss a simple class of $SO(10)$ models, well-motivated on other grounds, in which the number of free parameters coming from $M_R$ is much reduced and where it is therefore possible to get definite predictions for the neutrino mixing matrix $U_{MNS}$, including a testable relation between two Standard Model quantities that have not yet been measured, namely $\theta_{13}$ and $\delta_{\text{CP}}$. (We denote by $\delta_{\text{CP}}$ the “Dirac” CP-violating phase of the lepton sector. For a review of leptonic CP violation, see [1].)

The neutrino mixing matrix is given by the standard formula $U_{MNS} = U_L U_L^\dagger$, where $U_L$ is the unitary transformation of the left-handed charged leptons required to diagonalize the charged-lepton mass matrix $M_L$, and $U_\nu$ is the unitary matrix that diagonalizes the mass matrix of the observed light neutrinos $M_\nu$. Typically, the matrix $U_L$ is highly constrained or even known in $SO(10)$ models, because the unified symmetry relates $M_L$ to the quark mass matrices; but in most models $U_\nu$ is poorly constrained or unknown, because of its dependence on $M_R$. What is different about the models we are discussing in this paper is that the neutrino Dirac mass matrix $M_N$ is assumed to have negligibly small elements in its first row and column (in the “original basis” defined by the flavor symmetries of the model). This obviously implies that $M_\nu = -M_N M_R^{-1} M_T^T$ also has negligibly small first row and column, which means that $U_\nu$ is in effect a $U(2)$ rather than a $U(3)$ rotation. As such, it contains only one real rotation angle and three complex phases, of which two phases do not contribute to low-energy physics. In other words, in the kind of model we shall discuss, only two parameters that depend on $M_R$ actually come into the computation of the lepton mixing matrix $U_{MNS}$, namely one angle and one phase that we shall call $\theta_\nu$ and $\phi_\nu$.

The crucial assumption that the first row and column of $M_N$ are very small is motivated by two observations. First, in $SO(10)$ models there tends to be a close relationship between $M_N$ and the mass matrix of the up-type quarks $M_U$. Second, in many models $M_U$ has very small elements in its first row and column to account for the extreme smallness of its smallest eigenvalue compared to its largest: $m_u/m_t \sim 10^{-5}$, which is much less than the corresponding ratios for $M_D$ and $M_L$: $m_d/m_b \sim 10^{-3}$ and $m_e/m_\tau \sim 0.3 \times 10^{-3}$.

If, as we are assuming, $U_\nu$ is to a good approximation a $U(2)$ rotation of the second and third families, then the large solar neutrino angle, which involves the first family, must come from the $U_L$ rather than from $U_\nu$. That is, the solar neutrino angle must come from the diagonalization of $M_L$ rather than $M_\nu$. But this means that in the original basis $M_L$ has large off-diagonal elements, which is the distinguishing feature of so-called “lopsided models” [2, 3, 4, 5]. In particular, one is naturally led to models of the “doubly lopsided” form [6, 7, 8, 9, 10, 11].

The basic idea of “lopsided models” is that large neutrino mixing angles are caused by large asymmetrical off-diagonal elements in $M_L$. All lopsided models explain the large atmospheric angle by the $13$ element of $M_L$ being large, i.e. as large as the $33$ element. In doubly lopsided models the $13$ element of $M_L$ is also assumed to be large to explain the large solar angle. If these large elements arise in an $SU(5)$-
invariant way (i.e. from effective operators of the form $C_110, 5_1(5_H) + C_210, 3_3(5_H) + C_310, 5_9(5_H)$), then the matrices $M_L$ and $M_D$ have the form

$$M_L = \begin{pmatrix} -C_1 & -C_2 & -C_3 \\ -C_1 & -C_2 & -C_3 \\ -C_1 & -C_2 & -C_3 \end{pmatrix} v_d, \quad M_D = \begin{pmatrix} -C_1 & -C_2 & -C_3 \\ C_1 & C_2 & C_3 \\ -C_1 & -C_2 & -C_3 \end{pmatrix} v_d$$

where the dashes indicate elements much smaller than the $C_i$. (The convention we use is that the left-handed fermions multiply the mass matrix from the left, and the right-handed fermions multiply it from the right.) The forms in Eq. (1) reflect the well-known fact that $SU(5)$ relates $M_L$ to the transpose of $M_D$. The reason for this left-right transposition is that the 10’s of $SU(5)$ contain the left-handed down-type quarks and right-handed charged leptons, while the 5’s contain the right-handed down-type quarks and left-handed charged leptons. That is why the large lopsided mass-matrix elements $C_{1,2}$ produce large mixing of the left-handed leptons but of the right-handed quarks, which accounts for the fact that the MNS angles are big and the CKM angles are small.

The large elements of $M_L$ can be diagonalized by two successive rotations of the left-handed charged leptons:

$$\begin{pmatrix} \begin{pmatrix} -C_1 & -C_2 & -C_3 \\ -C_1 & -C_2 & -C_3 \\ -C_1 & -C_2 & -C_3 \end{pmatrix} U_{12} (\theta_s) & - \begin{pmatrix} -C_1 & -C_2 & -C_3 \\ -C_1 & -C_2 & -C_3 \\ -C_1 & -C_2 & -C_3 \end{pmatrix} U_{23} (\theta_a) \end{pmatrix} \rightarrow \begin{pmatrix} 0 \\ -\sqrt{C_1^2 + C_2^2} \\ -C_3 \end{pmatrix}$$

where $C \equiv \sqrt{C_1^2 + C_2^2 + C_3^2}$, $\tan \theta_s = C_1/C_2$ and $\tan \theta_a = \sqrt{C_1^2 + C_2^2}/C_3$. Another rotation of the left-handed charged leptons (call it $U_{13} (\eta)$) is required to eliminate the small 12 element that remains after the first two rotations. (The small elements that remain below the main diagonal are eliminated by rotations of the right-handed leptons.) Thus $U_L$ has the form $U_L = U_{13} (\eta) U_{12} (\theta_s) U_{23} (\theta_a)$.

The magnitude of the third rotation angle, $\eta$, depends on the relative magnitudes of the small elements of $M_L$ that are denoted by dashes in Eq. (1). If, as in the models we shall be considering, the 32 element of $M_L$ is much larger than the 22 and 12 elements, then the angle $\eta$ is small, and $U_L$ has the approximate form

$$U_L \cong \begin{pmatrix} 1 & 0 & 0 \\ 0 & c_s & s_s \\ 0 & -s_s & c_s \end{pmatrix} \begin{pmatrix} c_s & s_s & 0 \\ -s_s & c_s & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} c_s & s_s & 0 \\ -c_s s_s & c_a c_s & s_a \\ c_a s_s & -s_a c_s & c_a \end{pmatrix}$$

where $s_a \equiv \sin \theta_a$, $c_a \equiv \cos \theta_a$, $s_s \equiv \sin \theta_s$, and $c_s \equiv \cos \theta_s$. If the angles in $U_{13}$ are small (as they will be in the models we are considering, because of the hierarchical nature of $M_L$), then $U_{MNS} = U_{13} U_{13}^T$ will be approximately given by Eq. (3). One sees from this that the doubly lopsided structure accounts in a simple and natural way for the “bi-large” form of $U_{MNS}$, i.e. the form in which the solar and atmospheric angles are large, but the 13 mixing, $U_{e3}$, is small. Note, however, that the angles $\theta_s$ and $\theta_a$ in Eq. (3) are not exactly equal to the atmospheric and solar neutrino angles, which we will denote by $\theta_{atm}$ and $\theta_{sol}$, since the latter get small contributions from $\eta$ and from the angles in $U_\nu$.

In the models we are discussing, $U_L$ can be determined from the known quark masses, charged lepton masses and the CKM angles. That means that $U_{MNS}$ depends only on the two unknown parameters $\theta_\nu$ and $\phi_\nu$ coming from $U_\nu$. Since the solar neutrino angle $\theta_{sol}$ tends to be quite insensitive to these parameters, as will be seen, one has three observable quantities in $U_{MNS}$, $(\theta_{atm}, \theta_{13},$ and $\delta_{lep})$ being calculable in terms of just two free parameters, thus yielding one prediction, which can be expressed as a relation between $\theta_{13}$ and $\delta_{lep}$.

To see what kind of relation one expects, let us neglect $\eta$ and approximate $U_L$ by the simple form in Eq. (3). Then

$$U_{MNS} \cong \begin{pmatrix} c_s & s_s & 0 \\ -c_s s_s & c_a c_s & s_a \\ 0 & 0 & 1 \end{pmatrix}$$

$$\begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}$$

where $s_s \equiv \sin \theta_\nu$, and $c_s \equiv \cos \theta_\nu$. Multiplying this out, one obtains (for small $\theta_\nu$):

$$\sin \theta_{sol} \cong c_s s_s \cong \sin \theta_s,$$

$$\sin \theta_{atm} \cong |c_s s_s + s_a c_a e^{i \phi_\nu}| \cong \sin \theta_a + \cos \theta_a \cos \theta_{sol} \sin \theta_\nu \cos \phi_\nu,$$

$$\sin \delta_{lep} \cong \phi_\nu.$$
FIG. 1: The relation between sin δ_{lep} and sin θ_{13} given in Eq. 7 for Δ = 0.25.

the mass hierarchy among the families arises radiatively; i.e. the masses of the second and third families arise at tree level and the masses of the first family from loop diagrams. The details of this model are set forth in other papers [8, 11]; here, we merely summarize. The mass matrices in this model have the approximate form (we use slightly different notation than in [11]):

\[
\begin{align*}
\frac{M_U}{v_u} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & \frac{1}{\pi} \\ 0 & \frac{1}{\pi} & 1 \end{pmatrix}, \\
\frac{M_D}{v_d} &= \begin{pmatrix} 0 & 0 & \delta \\ 0 & \delta_H & \frac{1}{\pi} \delta' \\ fC_1 & fC_2 - \frac{1}{\pi} \delta' & 1 \end{pmatrix}, \\
\frac{M_N}{v_N} &= \begin{pmatrix} 0 & 0 & 0 \\ 0 & -\epsilon & 1 \\ 0 & \epsilon & 1 \end{pmatrix}, \\
\frac{M_{\nu}}{v_{\nu}} &= \begin{pmatrix} 0 & 0 & C_1 \\ 0 & f_H \delta_H & \epsilon - \epsilon' \delta \\ 3\delta & \epsilon + 3\delta' & 1 \end{pmatrix}
\end{align*}
\]

(8)

where \( \delta' \equiv (C_2/C_1)\delta \). The parameters denoted by \( \epsilon, \delta, \) and \( \delta_H \) are small, so that \( M_L \) and \( M_D \) have the forms given in Eq. (1) (with \( v_d \) and \( v_u \) scaled to make \( C_3 = 1 \)), and \( M_U \) and \( M_N \) indeed have approximately vanishing first row and column.

The elements in these matrices denoted by \( \epsilon, \delta, \) and \( C_i, i = 1, 2 \) arise at tree-level from three effective operators: \( O_1 = 16_316_310_H, O_2 = 16_316_310_H 45_H/M_2, \) and \( O_3 = c_i 16_316_316_H 16_H/M_3 \) (i=1,2), respectively. Some of the structure of these tree-level elements is easily understood group-theoretically. The vacuum expectation value (VEV) of the adjoint Higgs field \( \langle 45_H \rangle \) in \( O_2 \) is proportional to the \( SO(10) \) generator \( B - L \) and gives the factor of \( -\frac{1}{\epsilon} \) in the \( \epsilon \) terms of the quark matrices relative to the lepton matrices. That factor is responsible for the well-known Georgi-Jarlskog relation of quark to lepton masses [12]. In \( O_3 \), the fact that the Higgs fields are in spinors (16) of \( SO(10) \), which contain 3 but not 5 of \( SU(5) \), explains why this operator contributes the elements \( C_i \) only to \( M_D \) and \( M_L \), and not to \( M_U \) or \( M_N \).

The elements denoted by \( \delta, \delta', \) and \( \delta_H \) arise from one-loop diagrams. \( f \) and \( f_H \) are factors reflecting the breaking of \( SO(10) \) (or more precisely, of \( SU(4)_c \)). The parameters \( v_d \) and \( v_u \) set the overall scales of the mass matrices of the \( I_3 = -\frac{1}{2} \) and \( I_3 = +\frac{1}{2} \) fermions, and have a small ratio \( (v_d/v_u \approx 0.9 \times 10^{-2}) \) that is responsible for the small ratio of \( m_d \) to \( m_u \). \( (v_d \) and \( v_u \) come respectively from the VEVs of the \( SU(5) \) 3 and 5 in the \( SO(10) \) 10 of Higgs fields.) Aside from this ratio of VEVs, all dimensionless parameters of the model that come into the quark and lepton mass ratios and mixing angles are of order 1 if they arise at tree-level, and of order 1/16\( \pi^2 \) if they arise at one-loop level: a fit of the data [11] gives \( \epsilon \approx 0.189, C_1 \approx 1.03, C_2 \approx -1.51, f \approx 0.566, f_H \approx 0.208, \delta \approx 2.29(16\pi^2)^{-1}, \) and \( \delta_H \approx 2.66(16\pi^2)^{-1} \). Some of these parameters are complex. There are four physical phases, but of these only two have a significant effect on the fit, and these also are of order 1: \( \arg(\epsilon) \approx 1.52 \) rad and \( \arg(\delta_H) \approx 0.514 \) rad.

In spite of the fact that the dimensionless parameters of the model have “natural” values, there is a good fit with 11 parameters to 14 measured quantities that span a very wide range: namely the quark masses, charged lepton masses, CKM parameters, and the solar and atmospheric angles. Of course, from the four Dirac mass matrices in Eq. (8), it is not the angles in \( U_{\alpha\nu} \) that are predicted, but the angles in \( U_L \). From the fit to the data performed in [11], the best fit value of \( (U_{1L})_{23} \) comes out to be 0.891, whereas the experimental central value of \( (U_{M\nu N\alpha S})_{23} \equiv \sin \theta_{\alpha\nu \text{ atm}} \) is about 0.71.

Since the matrices \( M_U \) and \( M_N \) in Eq. (8) have vanishing first row and column, the mass of the up quark is zero at this level. The up quark mass can be fit by a 11 element of order \( 10^{-5} \). That is too small to be a one-loop effect, but it is the right magnitude to be a two-loop or three-loop effect. One expects from \( SO(10) \) symmetry that in \( M_N/v_u \) there would also be a 11 element of order \( 10^{-5} \). That should have negligible effect on the predictions for the neutrino mixing parameters that will be presented below.

To obtain predictions for the angles \( \theta_{13} \) and \( \delta_{\text{lep}} \), we fix the parameters appearing in Eq. (8) to the values that give the best \( \chi^2 \) fit to the following set of measured parameters: quark masses, CKM angles, CKM phase, charged lepton masses, and solar neutrino angle. This numerical fit was done in [11], and the details can be found in that paper. We then scan over the possible values of \( \theta_{\nu} \) and \( \phi_{\nu} \) for those that give a particular value of the atmospheric neutrino angle \( \theta_{\text{atm}} \) and plot the resulting points in the \( \theta_{13}-\delta_{\text{lep}} \) plane. This is shown in Fig. 2, where the best-fit points coalesce to form the dark curves in the center of the shaded bands. These curves, as expected, are similar to the one shown in Fig. 1. Each shaded band represents a different value of \( \sin^2 \theta_{\text{atm}} \). The central curve in each band corresponds to the parameters that give the best \( \chi^2 \) for the set of measured parameters, which is 4.5. The shaded region corresponds to fits for which \( \chi^2 \leq 6.5 \).
FIG. 2: The actual relation between \( \sin \delta_{lep} \) and \( \sin \theta_{13} \) for various values of \( \sin^2 \theta_{atom} \) (the numbers in the shaded band) in a realistic model.

The major source of the uncertainty shown by the shaded bands is the phase of the parameter \( f_H \) in Eq. (8). This phase has no direct effect on the quark masses and mixing angles and a relatively weak effect on most of the leptonic quantities due to the fact that \( f_H \) is such a small parameter. (Of course, \( \arg f_H \) does have an indirect effect on the quark masses and mixing angles, since the effects on the very precisely known charged lepton masses from varying \( \arg f_H \) have to be compensated in the \( \chi^2 \) fit by changes in the other parameters.) Since \( \arg f_H \) can vary considerably without harming the \( \chi^2 \) fit to the other quantities, it has a significant effect on \( \theta_{13} \) and \( \delta_{lep} \).

In conclusion, doubly lopsided models based on \( SO(10) \) in which the Dirac neutrino mass matrix has very small first row and column can give interesting and testable predictions for the two as-yet-unknown parameters of the Standard Model, \( \theta_{13} \) and \( \delta_{lep} \). In particular, there is a fairly precise relation between these two quantities, such that the smaller \( \theta_{13} \) is the smaller is \( \delta_{lep} \), with a lower limit on \( \theta_{13} \), as Figs. 1 and 2 show. These features have been illustrated in a particular realistic model, which is non-supersymmetric and has a radiative fermion mass hierarchy. However, qualitatively similar predictions should also be obtainable from doubly lopsided \( SO(10) \) models that are supersymmetric and that have tree-level hierarchies. The predicted relation between \( \theta_{13} \) and \( \delta_{lep} \) will become more precise as the quark masses, CKM angles, CKM phase, and solar and atmospheric neutrino angles are determined with more precision. If they were known perfectly, the predicted relation would be a single sharp curve like the one shown in Fig. 1. One sees, then, that the rigorous testing of models of quark and lepton masses will require progress along a broad front.

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