SMOOTH SQUAREFREE AND SQUARE-FULL INTEGERS IN ARITHMETIC PROGRESSIONS

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Abstract. We obtain new lower bounds on the number of smooth squarefree integers up to \(x\) in residue classes modulo a prime \(p\), relatively large compared to \(x\), which in some ranges of \(p\) and \(x\) improve that of A. Balog and C. Pomerance (1992). We also estimate the smallest squarefull number in almost all residue classes modulo a prime \(p\).

1. Introduction

1.1. Background and motivation. Let \(p\) be a prime. For any integer \(n \geq 2\) we denote \(P^+(n)\) to be the largest prime factor of \(n\). For any positive real number \(y\), we say that an integer is \(y\)-smooth if \(P^+(n) \leq y\).

Studying the distribution of \(y\)-smooth numbers \(n \leq x\) in progressions modulo an integer \(q \geq 2\) has always been a very active subject of research, see [3, 8, 11, 16, 18] and references therein. For instance, as pointed out in [18], a very good level of distribution would imply the truth of Vinogradov’s conjecture about the smallest quadratic non-residue.

As usual, we denote by \(\psi(x, y; p, a)\) the number of positive integers \(n \leq x\) which are \(y\)-smooth and satisfy \(n \equiv a \mod p\). Furthermore, we use \(\psi^\#(x, y; p, a)\) for the number of those integers which are also squarefree.

Due to its link with Euclidean prime generators, the positivity of \(\psi^\#(x, y; p, a)\) in the special case of \(y = p\) is of special interest, see [5]. Thus, following Booker and Pomerance [5], we use \(M(p)\) to denote the least \(x\) such that \(\psi^\#(x, p; p, a) > 0\) for every integer \(a\). The quantity \(M(p)\) has been considered in [16], where in particular the conjecture of Booker and Pomerance [5] that \(M(p) = p^{O(1)}\) is established in a
stronger form
\[ M(p) \leq p^{3/2 + o(1)}, \]
for all primes \( p \), and
\[ M(p) \leq p^{4/3 + o(1)}, \]
for all, but a set of primes \( p \) of relative zero density.

Here we use similar ideas to obtain lower bounds on \( \psi^\sharp(x, y; p, a) \) of essentially the right order of magnitude in a broader range of \( y \). These bounds, even without taking into account the squarefreeness condition, that is, using
\[ \psi(x, y; p, a) \geq \psi^\sharp(x, y; p, a), \]
improve the range in which the result of Balog and Pomerance [3] applies.

Subsequently, we also address a question about squarefull numbers in arithmetic progressions (that is numbers, which are divisible by squares of all its prime divisors). This question is significantly less studied, see however [7, 15, 17]. In particular, Chan [7] obtained an asymptotic formula for the number of squarefull numbers in an arithmetic progression, however, due to a rather complicated structure of the main term, it is not immediately clear when the main term starts to exceed the error term. Here we consider a Linnik-type version of this question. Namely, using very different arguments compared to the case of square-free numbers (and also to [7]), we investigate the quantity \( F(a, p) \) which is defined as the smallest positive squarefull number \( n \equiv a \mod p \).

1.2. Results for squarefree numbers. We start with a lower bound on \( \psi^\sharp(x, y; p, a) \) which holds for any prime \( p \).

**Theorem 1.1.** For any fixed real numbers \( \alpha \) and \( \beta \) with \( \beta \in (23/24, 1] \) and \( \alpha \in (9/2 - 3\beta, 3\beta] \), for \( x = p^{\alpha + o(1)} \) and \( y = p^{\beta + o(1)} \) we have
\[ \psi^\sharp(x, y; p, a) \geq x^{1+o(1)}/p \]
as \( p \to \infty \).

Taking \( y = p^\beta \) with \( 23/24 < \beta \leq 1 \) and \( q = p \) in the main result of Balog & Pomerance [3] gives the existence of a \( p^\beta \)-smooth integer (not necessary squarefree) \( n \leq p^{\max\{3\beta/2, 3/4 + \beta\} + o(1)} = p^{3/4 + \beta + o(1)} \) since \( \beta \leq 1 \). We notice that
\[ 9/2 - 3\beta < 3/4 + \beta, \]
under the condition \( 23/24 < \beta \). Therefore, Theorem 1.1 always improves on the bound given by the main result of Balog & Pomerance [3]. We remark that removing the squarefreeness condition does not help us to improve on Theorem 1.1 due to the method used.
We also obtain a result for almost all primes. Firstly, we define the interval
\[ \mathcal{I}(\beta) = (\alpha_0(\beta), \beta + 1) \]
where
\[
\alpha_0(\beta) = \begin{cases} 
5(2 - \beta)/3 & \text{if } \beta \in (7/8, 13/14], \\
12 - 11\beta & \text{if } \beta \in (13/14, 17/18], \\
(7 - 4\beta)/2 & \text{if } \beta \in (17/18, 25/26], \\
16 - 15\beta & \text{if } \beta \in (25/26, 31/32], \\
(18 - 11\beta)/5 & \text{if } \beta \in (31/32, 41/42], \\
20 - 19\beta & \text{if } \beta \in (41/42, 49/50], \\
(11 - 7\beta)/3 & \text{if } \beta \in (49/50, 61/62], \\
24 - 23\beta & \text{if } \beta \in (61/62, 68/69], \\
4/3 & \text{if } \beta \in (68/69, 1]. 
\end{cases}
\]

**Theorem 1.2.** Fix real numbers \( \alpha \) and \( \beta \) such that \( \beta \in (7/8, 1] \) and \( \alpha \in \mathcal{I}(\beta) \). Letting \( x = Q^{\alpha + o(1)} \) and \( y = Q^{\beta + o(1)} \), as \( Q \to \infty \), we have
\[ \psi^\flat(x, y; p, a) \geq x^{1+o(1)}/p \]
for all but \( o(Q/\log Q) \) primes \( p \in [Q, 2Q] \).

1.3. **Results for squarefull numbers.** First we observe that if \( a \) is a quadratic residue modulo \( p \) (or \( a = 0 \)), then \( a \equiv b^2 \mod p \) for some integer \( b \in [0, p - 1] \) and so we have trivially \( F(a, p) \leq (p - 1)^2 \) in this case.

To estimate \( F(a, p) \) for a quadratic non-residue \( a \) we denote
\[ \eta_0 = \frac{1}{4\sqrt{e}} \]
and recall that by the classical bound of Burgess [6] on the smallest quadratic non-residue \( n_p \) we have
\[ n_p \leq p^{\eta} \]
for any \( \eta > \eta_0 \) and a sufficiently large \( p \). Noticing that \( an_p^{-3} \) is a quadratic residue modulo \( p \), we now obtain \( F(a, p) \leq n_p^3(p - 1)^2 \).

Hence, we have the trivial bound \( F(a, p) \leq p^{2+3\eta+o(1)} \) for any \( a \), which we unfortunately do not know how to improve. However, we remark that assuming the Vinogradov’s conjecture that \( n_p \leq p^{o(1)} \) (which is implied by the Generalised Riemann Hypothesis in the stronger form \( n_p \ll \log^2 p \) proved by Ankeny [2], see also [12, Section 5.9] for a discussion), we have the bound \( F(a, p) \leq p^{2+o(1)} \). Even though we cannot reach such a bound, we obtain an unconditional better bound for almost all \( a \in \{0, \ldots, p - 1\} \).
We also note that from a result on counting squarefull integers \[19\], for any set \(A\) of \(A\) distinct residues modulo \(p\) we have
\[
\max_{a \in A} F(a, p) \gg A^2,
\]
where, as usual, we use \(A \ll B\) and \(B \gg A\) as an equivalent to the inequality \(|A| \leq cB\) with some constant \(c > 0\), which occasionally, where obvious, may depend on the real parameter \(\varepsilon > 0\). We slightly refine this result:

**Theorem 1.3.** For all but \(o(p)\) quadratic non-residues \(a \in [0, p - 1]\), we have
\[
p^{2n_p} f(p) \ll F(a, p) \leq p^{2 + \eta_0 + o(1)}
\]
for any function \(f(p)\) such that \(f(p) \to 0\) as \(p \to \infty\) and
\[
\max_{a \mod p} F(a, p) \gg p^{2n_p},
\]
where \(n_p\) denotes the least quadratic non-residue modulo \(p\).

Using the lower bound in Theorem 1.3, together with an unconditional result of Graham and Ringrose \[10\] on primes with large values of \(n_p\) and a conditional result on the Generalised Riemann Hypothesis (GRH) of Montgomery \[14\], we immediately derive

**Corollary 1.4.** For infinitely many primes \(p\) we have
\[
\max_{a \mod p} F(a, p) \gg \begin{cases} 
p^{2} (\log p)(\log \log \log p), & \text{unconditionally}, \\
p^{2} (\log p)(\log \log p), & \text{under the GRH}. \end{cases}
\]

In Section 2, we collect some results which will be used to prove the main results in Section 3.

## 2. Preparation lemmas

### 2.1. Exponential sums with reciprocals of primes

As usual, we define \(e_p(z) = \exp(2i\pi z/p)\). For an integer \(k\) with \(\gcd(k, p) = 1\) we use \(k^{-1}\) to denote the multiplicative inverse of \(k\) modulo \(p\), that is, the unique integer with
\[
k^{-1} \equiv 1 \mod p \quad \text{and} \quad 1 \leq k < p.
\]

It is convenient to introduce the quantity
\[
B(p, L) = \begin{cases} 
L^{3/2} p^{1/8}, & \text{if } L < p^{1/3}, \\
L^{15/8}, & \text{if } p^{1/3} \leq L < p.
\end{cases}
\]

The following bound of the double exponential sum over primes is a combination of \[16, \text{Lemma 3.5}\] (for \(L \leq p^{1/3}\)) and of \[9, \text{Lemma 2.4}\] (for \(p^{1/3} \leq L < p\)).
Lemma 2.1. For any real $L \leq p$, we have
\[
\max_{\gcd(a,p) = 1} \left| \sum_{\ell_1, \ell_2 \in \mathcal{L}} \varepsilon_p(a \ell_1 \ell_2) \right| \leq B(p, L)p^{o(1)},
\]
as $p \to \infty$, where $\mathcal{L}$ is the set of primes $\ell \in [L, 2L]$.

2.2. Some congruences with products of primes. We denote $N_{a,p}(L, h)$ to be the number of solutions to the congruence
\[
\ell_1 \ell_2 u \equiv a \mod p, \quad \ell_1, \ell_2 \in \mathcal{L}, \quad 1 \leq u \leq h,
\]
where $h$ and $L$ are two positive real numbers and $\mathcal{L}$ is the set of primes $\ell \in [L, 2L]$.

We now use Lemma 2.1 to derive an analogue of [16, Lemma 3.10] (which also applies to $L \geq p^{1/3}$).

Lemma 2.2. For any integer $a$ and prime $p$ with $\gcd(a, p) = 1$ and reals $1 \leq h, L < p$, we have
\[
N_{a,p}(L, h) = \frac{K^2 h}{p} + O \left( B(p, L)p^{o(1)} \right),
\]
where $K = \#\mathcal{L}$ is the cardinality of $\mathcal{L}$ and $B(p, L)$ is defined by (2.1).

We also recall that by [16, Lemma 3.12] we have:

Lemma 2.3. For any integer $a$ and prime $p$ with $\gcd(a, p) = 1$ and reals $1 \leq h, L < p$ we have
\[
N_{a,p}(L, h) \leq \left( \frac{L^2 h}{p} + 1 \right)p^{o(1)}.
\]

Furthermore, let $N_{a,p}^\sharp$ count the number of squarefree solutions to the congruence (2.2). Following the proof of [16, Theorem 1.4], but using a more general bound of Lemma 2.2 instead of [16, Lemma 3.10] as well as Lemma 2.3 (exactly as in [16]), we derive

Lemma 2.4. For any integer $a$ and prime $p$ with $\gcd(a, p) = 1$ and reals $h, D$ and $L$ with
\[
1 \leq L, h < p \quad \text{and} \quad 1 \leq D \leq h^{1/2},
\]
we have
\[
N_{a,p}^\sharp(L, h) = \frac{K^2 h}{\zeta(2)p} + O \left( \left( \frac{L^2 h}{DP} + DB(p, L) + h^{1/2} \right)p^{o(1)} \right),
\]
where $K = \#\mathcal{L}$ is the cardinality of $\mathcal{L}$ and $B(p, L)$ is defined by (2.1).
We also need the bound of [16, Lemma 3.14] on the number of solutions $Q_{a,p}(L, h)$ to the congruence

$$\ell_1 \ell_2^2 v \equiv a \mod p, \quad \ell_1, \ell_2 \in L, \ 1 \leq v \leq h.$$  

**Lemma 2.5.** For any integer $a$ and prime $p$ with $\gcd(a, p) = 1$ and reals $1 \leq L, h \leq p$ with $2Lh \leq p$ we have

$$Q_{a,p}(L, h) \leq \left(\frac{Lh}{p} + 1\right)Lp^{o(1)}.$$

It is shown in [16, Lemma 3.11], that for almost all primes $p$, the asymptotic formula of Lemma 2.2 can be improved as follows.

**Lemma 2.6.** As $Q \to \infty$ for any fixed integer $k \geq 1$, for $1 \leq L \leq Q$ for all but $o(Q/\log Q)$ primes $p \in [Q, 2Q]$, for any integer $a$ with $\gcd(a, p) = 1$ and real $h$ with $1 \leq h \leq p$, we have

$$N_{a,p}(L, h) = \frac{K^2h}{p} + O((L^{(3k-1)/2k}p^{1/2k} + L^{(4k-1)/(2k)})p^{o(1)}),$$

where $K = \#L$ is the cardinality of $L$.

Finally, we also recall that by [16, Lemma 3.13] we have:

**Lemma 2.7.** As $Q \to \infty$, for all but $o(Q/\log Q)$ primes $p \in [Q, 2Q]$, for any integer $a$, and reals $1 \leq F, L, h \leq p$ with $F, L^2h < p$, for the sum

$$R_{a,p}(F, L, h) = \sum_{F \leq d \leq 2F} N_{ad^{-2},p}(L, h)$$

we have

$$R_{a,p}(F, L, h) \leq \max\{F(L^2h)^{1/4}p^{-1/4}, F^{1/2}(L^2h)^{1/4}\}p^{o(1)}.$$  

2.3. **Moments of character sums.** Let $\Omega_p$ denote the set of all Dirichlet characters modulo $p$ and let $\Omega_p^* = \Omega_p \setminus \{\chi_0\}$ denote the set of all non-principal Dirichlet characters modulo $p$.

We need the following result of Ayyad, Cochrane and Zheng [1, Theorem 2], see also [13] for a slightly sharper bound (which however does not change our final result).

**Lemma 2.8.** For any integer $K \geq 1$, we have

$$\sum_{\chi \in \Omega_p^*} \left| \sum_{1 \leq n \leq K} \chi(n) \right|^4 \leq K^2p^{1+o(1)}.$$
2.4. Quadratic non-residues in short intervals. Let $T_p(K)$ denote the number of quadratic non-residues modulo $p$ in the interval $[1, K]$.

We need an extension of (1.2). The following bound is given in [4, Theorem 2.1].

**Lemma 2.9.** For any real $\eta > \eta_0$, where $\eta_0$ is given by (1.1), there is a constant $c > 0$, such that for a sufficiently large $p$ and $K \geq p^{\eta_0}$ we have

$$T_p(K) \geq cK.$$

3. Proofs of main results

3.1. **Proof of Theorem 1.1.** For a sufficiently small $\varepsilon > 0$, we set

$$L = p^{(\alpha-\beta)/2\varepsilon/2} \quad \text{and} \quad h = p^\beta.$$

Since $L \leq h \leq y$, $N_{a,p}^\sharp(L, h)$ counts a subset of $y$-smooth integers in an arithmetic progression. Noticing that $L^2h \leq p^{\alpha-\varepsilon} = x^{1-\varepsilon+o(1)}$, we see that for a sufficiently large $p$ we have

$$\psi^\sharp(x, y; p, a) \geq N_{a,p}^\sharp(L, h) + O \left( (h/p + 1) Lp^{o(1)} \right)$$

where we estimated the contribution coming from non-squarefree products $\ell_1\ell_2u$ (precisely products with $\ell_1 = \ell_2$ or with $\ell_1 \mid u$ or with $\ell_2 \mid u$) using Lemma 2.5 with $h/L$ replacing $h$ as in the end of the proof of [16, Theorem 1.4].

We use a crude estimate for the main term:

$$L^2h/p (\log L)^2 = p^{\alpha-1+o(1)}.$$

Choosing

$$D = p^{\varepsilon/2}$$

and using Lemma 2.4, we derive

$$\psi^\sharp(x, y; p, a) \gg \frac{K^2h}{p} + O \left( (p^{\varepsilon/2}B(p,L) + L + h^{1/2}) p^{o(1)} \right)$$

since $B(p,L)$ dominates $L$ and the main term (3.2) dominates the first error term $L^2h(Dp)^{-1}$ in Lemma 2.4.

To begin, we remark that the term $h^{1/2}$ in (3.3) is dominated by the main term due to the inequality $\alpha - 1 > 9/2 - 3\beta - 1 > \beta/2$ for $\beta \leq 1$.

We split the discussion on the contribution of $B(p,L)$ into two cases depending on $\alpha$. 
Firstly, suppose that \( \alpha \in (9/2 - 3\beta, 2/3 + \beta] \). Since \( \alpha \leq 2/3 + \beta \), this implies \( L < p^{1/3} \) and hence \( B(p, L) = L^{3/2}p^{1/8} \) by (2.1). Therefore, recalling (3.3) and (3.2), we obtain

\[
(3.4) \quad \psi^x(x, y; p, a) \gg p^{\alpha - 1 - \varepsilon + o(1)} + O(p^{3(\alpha - \beta)/4 - \varepsilon/4 + 1/8 + o(1)}).
\]

For \( \varepsilon \) sufficiently small, we have \( \alpha > 9/2 - 3\beta + 3\varepsilon \) which implies that the main term dominates trivially the remainder term in (3.4).

Secondly, assume that \( \alpha \in (2/3 + \beta, 3\beta] \). In particular, since \( \beta \leq 1 \) we have

\[
2/3 + \beta + \varepsilon \leq \alpha < 3\beta < 2 + \beta + \varepsilon,
\]

for \( \varepsilon > 0 \) chosen sufficiently small. Hence \( p^{1/3} \leq L < p \) and we have \( B(p, L) = L^{15/8} \) by (2.1). Therefore, recalling (3.3) and (3.2), we obtain

\[
(3.5) \quad \psi^x(x, y; p, a) \geq p^{\alpha - 1 - \varepsilon + o(1)} + O(p^{15(\alpha - \beta)/16 - 7\varepsilon/16 + o(1)}).
\]

Notice that we have

\[
\alpha > 2/3 + \beta \geq 16 - 15\beta + 9\varepsilon + o(1)
\]

when \( \beta \in (23/24, 1] \) and \( \varepsilon > 0 \) is sufficiently small. It follows that the main term dominates the remainder term in (3.5). Therefore, in all cases we conclude

\[
\psi^x(x, y; p, a) \geq p^{\alpha - \varepsilon - 1 + o(1)}.
\]

Since this is valid for all sufficiently small \( \varepsilon > 0 \), the result follows.

3.2. **Proof of Theorem 1.2.** We follow the proof of [16, Theorem 1.6]. For \( \varepsilon > 0 \), we set

\[
(3.6) \quad L = Q^{(\alpha - \beta)/2 - \varepsilon/2}, \quad h = Q^\beta, \quad D = Q^{\varepsilon/2}, \quad E = Q^{(\alpha - 1)/2}.
\]

We note that \((\alpha - 1)/2 > 0\) for \( \alpha \in I(\beta) \) and so \( D < E \) if \( \varepsilon > 0 \) is sufficiently small. We also have \( E < h^{1/2} \) since \( \alpha < \beta + 1 \).

Since \( \alpha < \beta + 1 \leq 3\beta \) in the range \( \beta \in (7/8, 1] \), we get \( L \leq h \). In particular, we have as before the inequality (3.1).

By inclusion and exclusion, we have

\[
(3.7) \quad N_{\alpha, p}^x(L, h) = \sum_{d \leq h^{1/2}} \mu(d)N_{\alpha - 2, p}(L, h/d^2) = \Sigma_1 + \Sigma_2 + \Sigma_3,
\]
\[ \Sigma_1 = \sum_{d \leq D} \mu(d) N_{ad^2, p}(L, h/d^2), \]
\[ \Sigma_2 = \sum_{D < d \leq E} \mu(d) N_{ad^2, p}(L, h/d^2), \]
\[ \Sigma_3 = \sum_{E < d \leq h^{1/2}} \mu(d) N_{ad^2, p}(L, h/d^2). \]

To abstain from clutter, all the bounds below are valid for all but \( o(Q/\log Q) \) primes \( p \in [Q, 2Q] \).

Since \( \alpha < \beta + 1 < 2 + \beta + \varepsilon + o(1) \) and \( \beta \leq 1 \), we obtain respectively \( L \leq Q \) and \( h \leq p \). By Lemma 2.6

\[ \Sigma_1 = \frac{K^2 h}{\zeta(2)p} \]
\[ + O \left( \frac{K^2 h}{Dp} + D \left( L^{(3k-1)/(2k)} p^{1/(2k)} + L^{(4k-1)/(2k)} \right) p^o(1) \right) \]

for any fixed positive integer \( k \).

By Lemma 2.3 with \( h/d^2 \) replacing \( h \) there, we have

\[ \Sigma_2 \leq \left( \frac{L^2 h}{Dp} + E \right) p^o(1). \]

We split \( \Sigma_3 \) into \( O(\log p) \) sums with intervals of the form \([F, 2F]\) where \( E \leq F \leq h^{1/2} \).

From the choice of \( E \) in (3.6) we see that

\[ L^2 h/d^2 \leq L^2 h/F^2 \leq L^2 h/E^2 < p, \]

hence by Lemma 2.7

\[ R_{a,p}(F, L, h/F^2) \leq \max \{ F(L^2 h/F^2)^{1/4} p^{-1/4}, F^{1/2}(L^2 h/F^2)^{1/4} \} p^o(1) \]
\[ = (L^2 h)^{1/4} p^o(1) \]

since \( F \leq h^{1/2} \leq p^{1/2} \) and so

\[ \Sigma_3 \leq (L^2 h)^{1/4} p^o(1). \]

Substituting (3.8), (3.9) and (3.10) in (3.7), we obtain

\[ N_{a,p}^2(L, h) = \frac{K^2 h}{\zeta(2)p} + O \left( R p^o(1) \right) \]

where we set

\[ R = D \left( L^{(3k-1)/(2k)} p^{1/(2k)} + L^{(4k-1)/(2k)} \right) + \frac{L^2 h}{Dp} + E + (L^2 h)^{1/4} \]
and the main term verifies an analogue of (3.2), precisely,

$$\frac{K^2 h}{p} \sim \frac{L^2 h}{p (\log L)^2} = Q^{\alpha - 1 - \varepsilon + o(1)}.$$  

Notice that the choice of $E$ in (3.6) implies that $E$ is smaller than the main term (3.11). We now see from (3.11) that if

$$\alpha - 1 > \max \left\{ \frac{3k - 1}{2k} \alpha - \beta, \frac{1}{2k}, \frac{4k - 1}{2k} \alpha - \beta, \frac{\alpha}{4}, \frac{\alpha - \beta}{2} \right\}$$

for some positive integer $k$, then for a sufficiently small $\varepsilon$ the main term dominates the remainder term in (3.1) and the result follows.

Rearranging (3.12) gives

$$\alpha > \max \left\{ \frac{(1 - 3k)\beta + 2 + 4k}{k + 1}, (1 - 4k)\beta + 4k, 4/3, 2 - \beta \right\}. $$

First, we remark that $2 - \beta \leq (1 - 4k)\beta + 4k$ since $\beta \leq 1$ and we can discard $2 - \beta$ from the maximum in (3.13).

Furthermore, for $k \leq 5$, we see that $4/3$ is dominated by the first term of the right hand side of (3.13). In this case, a quick computation shows that

$$\frac{(1 - 3k)\beta + 2 + 4k}{k + 1} \geq (1 - 4k)\beta + 4k$$

if and only if $\beta \geq 1 - 1/2k^2$. Thus, in the interval $[1 - 1/2k^2, 1 - 1/2(k + 1)^2]$, the maximum is given either by $(1 - 3k)\beta + 2 + 4k) / (k + 1)$

or by $(1 - 4m)\beta + 4m$ with $m \geq k + 1$. Since the function $f(z) = (1 - 4z)\beta + 4z$ is a monotonically increasing function of $z$, we check only the case $m = k + 1$ and verify that

$$\frac{(1 - 3k)\beta + 2 + 4k}{k + 1} \geq f(k + 1) = (1 - 4(k + 1))\beta + 4(k + 1)$$

if and only if

$$\beta \geq \beta_0(k)$$

where

$$\beta_0(k) = 1 - \frac{1}{2(k^2 + k + 1)}.$$  

Splitting the interval

$$\mathcal{I}_k = \left( 1 - \frac{1}{2k^2}, 1 - \frac{1}{2(k + 1)^2} \right]$$
into two intervals as follows

\[ I_k = \left(1 - \frac{1}{2k^2}, 1 - \frac{1}{2(k^2 + k + 1)}\right) \]

\[ \cup \left(1 - \frac{1}{2(k^2 + k + 1)}, 1 - \frac{1}{2(k + 1)^2}\right) \]

and recalling that \( k \leq 5 \), we deduce after short computations the result for \( \beta \leq \beta_0(5) = 61/62 \).

For \( k \geq 6 \), noticing that \((1 - 4\beta + 4k \geq 4/3 \text{ in the range})\)

\[ \beta \leq 1 - \frac{1}{3(4k - 1)}, \]

we also deduce the case \( \beta \in (61/62, 68/69] \).

For the remaining case \( \beta \in (68/69, 1] \) and \( k \geq 6 \), we see that

\[ \frac{(1 - 3k)\beta + 2 + 4k}{k + 1} \leq 4/3. \]

Based on the above argument, we now give explicit choices of \( k \) and corresponding intervals which optimise our bound.

- If \( \beta \in (7/8, 13/14] \), we take \( k = 2 \) and (3.13) simplifies to
  \[ \alpha > \max \{5(2 - \beta)/3, 8 - 7\beta, 4/3, 2 - \beta\} = 5(2 - \beta)/3. \]

- If \( \beta \in (13/14, 17/18] \), we take \( k = 3 \) and (3.13) simplifies to
  \[ \alpha > \max \{(7 - 4\beta)/2, 12 - 11\beta, 4/3, 2 - \beta\} = 12 - 11\beta. \]

- If \( \beta \in (17/18, 25/26] \), we take \( k = 3 \) and (3.13) simplifies to
  \[ \alpha > \max \{(7 - 4\beta)/2, 12 - 11\beta, 4/3, 2 - \beta\} = (7 - 4\beta)/2. \]

- If \( \beta \in (25/26, 31/32] \), we take \( k = 4 \) and (3.13) simplifies to
  \[ \alpha > \max \{(18 - 11\beta)/5, 16 - 15\beta, 4/3, 2 - \beta\} = 16 - 15\beta. \]

- If \( \beta \in (31/32, 41/42] \), we take \( k = 4 \) and (3.13) simplifies to
  \[ \alpha > \max \{(18 - 11\beta)/5, 16 - 15\beta, 4/3, 2 - \beta\} = (18 - 11\beta)/5. \]

- If \( \beta \in (41/42, 49/50] \), we take \( k = 5 \) and (3.13) simplifies to
  \[ \alpha > \max \{(11 - 7\beta)/3, 20 - 19\beta, 4/3, 2 - \beta\} = 20 - 19\beta. \]

- If \( \beta \in (49/50, 61/62] \), we take \( k = 5 \) and (3.13) simplifies to
  \[ \alpha > \max \{(11 - 7\beta)/3, 20 - 19\beta, 4/3, 2 - \beta\} = (11 - 7\beta)/3. \]

- If \( \beta \in (61/62, 68/69] \), we take \( k = 6 \) and (3.13) simplifies to
  \[ \alpha > \max \{(26 - 17\beta)/7, 24 - 23\beta, 4/3, 2 - \beta\} = 24 - 23\beta. \]
• If $\beta \in (68/69, 1]$, we take $k = 6$ and (3.13) simplifies to 
\[
\alpha > \max \{(26 - 17\beta)/7, 24 - 23\beta, 4/3, 2 - \beta\} = 4/3.
\]

Therefore in all cases, where we also recall the condition $\alpha < \beta + 1$, we have 
\[
\psi^\#(x, y; p, a) \geq p^{\alpha - 1 - \epsilon + o(1)}.
\]

Since this is true for all $\epsilon > 0$, the result follows immediately.

### 3.3. Proof of Theorem 1.3.

Let $M$ be a parameter which will be fixed later. We introduce the subset of residues modulo $p$
\[
S = \{a : a \text{ quadratic non-residue such that } F(a, p) \leq M\}.
\]

Firstly, we remark that every squarefull integer $n$ can be written as $n = r^2 s$ with $s \mid r$. Furthermore, if $a$ is a quadratic non-residue, we notice that $s$ has to be a quadratic non-residue in this representation; in particular $s \geq n_p$.

Let us count the number of products $r^2 s \leq M$ with $s \mid r$ and $s \geq n_p$ the smallest quadratic non-residue modulo $p$. Noticing that $r \leq (M/s)^{1/2}$, we have at most $M^{1/2}s^{-3/2}$ possible values of $r$. Thus the number of different products $r^2 s$ is bounded by
\[
\sum_{s \geq n_p} M^{1/2}s^{-3/2} \ll M^{1/2}n_p^{-1/2}.
\]

This implies
\[
\#S \ll M^{1/2}n_p^{-1/2}.
\]

Setting $M = p^2n_pf(p)$, we get $\#S = o(p)$ which concludes the proof. The assertion $\max_{a \mod p} F(a, p) \gg p^2n_p$ follows by the same argument by setting
\[
S = \{a : a \text{ quadratic non-residue}\} \quad \text{and} \quad M = \max_{a \in S} F(a, p).
\]

So we now turn our attention to the upper bound.

Clearly, if $a \equiv u^2 \mod p$, $0 \leq u < p$, is quadratic residue (or $a = 0$), then $F(a, p) \leq u^2 \leq p^2$.

We now fix some $\epsilon > 0$ and denote by $A$ the set of quadratic non-residues for which $F(a, p) \geq p^{2+\eta_0+\epsilon}$.

It is enough to show that the cardinality of $A$ satisfies
\[
\#A = o(p).
\]

We set
\[
K = \left[p^{\eta_0+\epsilon/2}\right] \quad \text{and} \quad U = \left[p^{1-\eta_0}\right]
\]
and let \( \mathcal{N} \) be the set of quadratic non-residues in the interval \([1, K]\). In particular

\[
\#\mathcal{N} = T(K).
\]

Clearly for \( a \in \mathcal{A} \) the congruence

\[
(3.15) \quad a \equiv n^3 u^2 \mod p, \ n \in \mathcal{N}, \ 1 \leq u \leq U,
\]

has no solution. Thus expressing the number of solutions to \((3.15)\) via characters we see that

\[
\sum_{n \in \mathcal{N}} \sum_{1 \leq u \leq U} \frac{1}{p-1} \sum_{\chi \in \Omega_p} \chi (a^{-1} n^3 u^2) = 0.
\]

Summing over all \( a \in \mathcal{A} \) and using the multiplicativity of characters, we arrive to

\[
(3.16) \quad \sum_{\chi \in \Omega_p} \sum_{a \in \mathcal{A}} \chi (a) \sum_{n \in \mathcal{N}} \chi (n)^3 \sum_{u=1}^{U} \chi (u)^2 = 0,
\]

where \( \overline{\chi} \) denotes the complex conjugate character of \( \chi \).

Now, the contribution to \((3.16)\) from the principal character is obviously \( \#\mathcal{A} T(K) U \).

Furthermore, since all elements of \( \mathcal{A} \) are quadratic non-residues, the contribution to \((3.16)\) from the quadratic character, that is, from the Legendre symbol is

\[
\sum_{a \in \mathcal{A}} (a \mod p) \sum_{n \in \mathcal{N}} \left( \frac{n}{p} \right)^3 \sum_{u=1}^{U} \left( \frac{u}{p} \right)^2 = \sum_{a \in \mathcal{A}} (-1) \sum_{n \in \mathcal{N}} (-1)^3 \sum_{u=1}^{U} \chi (u)^2 = \#\mathcal{A} T(K) U.
\]

This allows us to write \((3.16)\) as

\[
(3.17) \quad 2 \#\mathcal{A} T(K) U = - \sum_{\chi \in \Omega_p^*} \sum_{a \in \mathcal{A}} \overline{\chi}(a) \sum_{n \in \mathcal{N}} \chi (n)^3 \sum_{u=1}^{U} \chi (u)^2
\]

with \( \Omega_p^* \) being the subset of \( \Omega_p \) where we removed the quadratic character.

Now, for \( \chi \in \Omega_p^* \) we have by definition \( \chi^2 \neq \chi_0 \). Furthermore, each character from \( \Omega_p^* \) occurs at most twice as \( \chi^2 \) and each character from \( \Omega_p \) (including also \( \chi_0 \) in this case) occurs at most three times as \( \chi^3 \) for \( \chi \in \Omega_p^* \).

Using the Hölder inequality, we now derive from \((3.17)\) that

\[
(3.18) \quad 2 \#\mathcal{A} T(K) U \leq \Sigma_1^{1/2} \Sigma_2^{1/4} \Sigma_3^{1/4}
\]
where

\[ \Sigma_1 = \sum_{\chi \in \Omega_p^\prime} \left| \sum_{a \in A} \chi(a) \right|^2 \leq \sum_{\chi \in \Omega_p} \left| \sum_{a \in A} \chi(a) \right|^2, \]

\[ \Sigma_2 = \sum_{\chi \in \Omega_p^\prime} \left| \sum_{n \in \mathcal{N}} \chi(n) \right|^4 \leq 3 \sum_{\chi \in \Omega_p} \left| \sum_{n \in \mathcal{N}} \chi(n) \right|^4, \]

\[ \Sigma_3 = \sum_{\chi \in \Omega_p^\prime} \left| \sum_{u=1}^{U} \chi(u) \right|^4 \leq 2 \sum_{\chi \in \Omega_p^\prime} \left| \sum_{u=1}^{U} \chi(u) \right|^4, \]

and the upper bounds come from the discussion above. We now see by the orthogonality of characters that we have

\[ \Sigma_1 \leq (p - 1) \# \mathcal{A}. \]  

(3.19)

For \( \Sigma_2 \), using again the orthogonality of characters, we write

\[ \Sigma_2 = 3(p - 1) \# \left\{ (n_1, n_2, n_3, n_4) \in \mathcal{N} : n_1 n_2 \equiv n_3 n_4 \mod p \right\} \]

\[ \leq 3(p - 1) \# \left\{ (n_1, n_2, n_3, n_4) \in \lfloor 1, K \rfloor : n_1 n_2 \equiv n_3 n_4 \mod p \right\} \]

\[ = 3 \sum_{\chi \in \Omega_p^\prime} \left| \sum_{n=1}^{K} \chi(n) \right|^4 = 3K^4 + \sum_{\chi \in \Omega_p^\prime} \left| \sum_{n=1}^{K} \chi(n) \right|^4. \]

Applying Lemma 2.8 and using that \( K^2 \leq p \) provided that \( \varepsilon \) is small enough, we derive

\[ \Sigma_2 \leq K^2 p^{1+o(1)}. \]  

(3.20)

Finally, we also estimate \( \Sigma_3 \), directly by Lemma 2.8 getting

\[ \Sigma_3 \leq U^2 p^{1+o(1)}. \]  

(3.21)

Substituting (3.19), (3.20) and (3.21) in (3.18), we now derive

\[ \# \mathcal{A} T(K) U \leq (\# \mathcal{A})^{1/2} K^{1/2} U^{1/2} p^{1+o(1)} \]

which together with Lemma 2.9 yields

\[ \# \mathcal{A} \leq K^{-1} U^{-1} p^{2+o(1)} = p^{1-\varepsilon/2+o(1)}. \]

We now see that (3.14) holds which concludes the proof.

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