Simple geometric approximations for global atmospheres on moderately oblate planets

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Certain geometric approximations such as the widely used traditional shallow-atmosphere, spherical-geoid (TSA-SG) and the deep-atmosphere, spherical-geoid (DA-SG) approximations boil down to the specification of a spatial metric tensor. In order to eliminate the leading-order errors due to the SG and TSA approximations, a sequence of three metric geometric approximations of increasing accuracy at high altitudes is obtained.

Their metric tensors possess a simple, closed-form analytical expression. The approximations capture to leading order the oblateness of the planet, the widening of atmospheric columns with height, the horizontal and vertical variations of gravity and the non-traditional part of the Coriolis force. Furthermore, for the first two approximations, the horizontal metric is conformal (proportional) to the spherical metric, which simplifies analytical and numerical formulations of the equations of motion.

I. INTRODUCTION

Due to centrifugal effects, rotating planets are oblate rather than spherical. More generally, geoids (isosurfaces of geopotential \( \Phi \)) are quite accurately approximated as oblate ellipsoids. Departure from sphericity is measured by the flattening \( \varepsilon \) (also called ellipticity or oblateness) of a reference ellipsoid which approximates either a geoid close to the solid surface of a rocky planet or a constant pressure surface, typically at 1000 hPa. Flattening \( \varepsilon \), defined as the relative difference between the semi-major and semi-minor axes \( a \) and \( b = a(1 - \varepsilon) \) of the reference ellipsoid, reaches about 1/10 for giant planets of the Solar system [3], but is only about \( \varepsilon \approx 1/300 \ll 1 \) for Earth. In Cartesian coordinates \((x, y, z)\) centered on the planet’s center of mass, with the \( z \) axis aligned with the axis of rotation of the planet, an expression accurate to \( O(\varepsilon) \) of the geopotential is [12] :

\[
\Phi = \Phi_0 \frac{a}{r} \left[ 1 - \left( \varepsilon - \frac{m}{2} \right) \left( \frac{r}{a} \right)^2 \left( \sin^2 \chi - \frac{1}{3} \right) + \frac{m}{2} \left( \frac{r}{a} \right)^3 \cos^2 \chi \right]
\]

where \( \lambda, \chi, r \) are spherical coordinates such that \((x, y, z) = (r \cos \lambda \cos \chi, r \sin \lambda \cos \chi, r \sin \chi)\),

\[
\Phi_0 = \frac{\Gamma M_E}{a}
\]

with \( \Gamma \) Newton’s gravitational constant, \( M_E \) the mass of the Earth or planet and

\[
m = \frac{\Omega^2 a^2}{\Phi_0} = O(\varepsilon).
\]

The ratio \( m/\varepsilon \) is \( O(1) \) and its precise value depends on the repartition of mass inside the planet (for instance \( m/\varepsilon = 4/5 \) for a uniform-density planet).

Because \( \varepsilon \ll 1 \), it is quite accurate to neglect the non-sphericity of geoids when formulating the equations of atmospheric or oceanic motion for theoretical or numerical purposes. This omission of \( O(\varepsilon) \) terms in the equations of motion defines the spherical-geoid (SG) approximation. If no further approximations are made, the resulting equations are the so-called deep-atmosphere (DA) equations of motion [11]. Assuming spherical geoids simplifies the expression of metric coefficients involved in the equations of motion and suppresses the horizontal variations of the acceleration of gravity. Another parameter whose smallness is often used to further simplify equations of motion is the relative thickness of the atmosphere \( \tau = H/a = c^2/\Phi_0 \) where \( c \) is a typical speed of sound and \( H = c^2/(\Phi_0/a) \) is the height scale. On Earth \( \tau \approx 1/500 = O(\varepsilon) \). Loosely speaking, neglecting terms of order \( O(\tau) \) in the equations of motion defines the traditional shallow-atmosphere (TSA) approximation (see Tort and Dubos [3] for a more in-depth discussion).

Although the SG and TSA approximations are quite accurate, it may be desirable to avoid the errors they incur. Bénard [2] has evaluated global forecast errors due to the SG approximation using idealized shallow-water numerical experiments. An important conclusion is that these errors could be comparable in magnitude to those due to the shallow-atmosphere approximation. Indeed, TSA errors are formally \( O(\tau) \) while SG errors are \( O(\varepsilon) \), and \( \tau \sim \varepsilon \).
Therefore, solving DA-SG equations of motion may not be really more accurate that solving the TSA-SG equations until the SG approximation is also relaxed.

A handful of operational numerical models of the global atmosphere do relax the TSA approximation and solve the deep-atmosphere equations \[10, 14\]. But to the best of the author’s knowledge, a spherical geoid and horizontally-uniform gravity are assumed in all current operational atmospheric models. One reason is of course that SG errors are in principle small. In fact they might not be that small on some planets, especially gaseous giants. Conversely on Earth, where these errors are presumably very small indeed, the expected quality of numerical modelling is also very high. Another compelling reason to retain the SG approximation is that, to date, no system of three-dimensional equations of motion has been proposed that would include the effects of oblateness in a fully satisfactory way and would also be straightforward to implement in existing models.

Previous efforts to relax the SG approximation tackle two aspects of the problem. One aspect is the construction of orthogonal geopotential coordinates, that is of a mapping \((\lambda, \phi, \Phi) \mapsto r = (x, y, z)\) where \(\lambda\) is longitude and \(\phi\) is a latitudinal coordinate. This mapping should be orthogonal (or at least such that \(\partial r/\partial \Phi\) be orthogonal to \(\partial r/\partial \phi\) and \(\partial r/\partial \lambda)\) in order to avoid the pollution of the horizontal momentum budget by large gravitational terms that would be compensated by a similarly large pressure gradient, obscuring the interesting dynamics and potentially causing numerical inaccuracies. Even given the relatively simple, closed-form expression \([1]\), constructing orthogonal geopotential coordinates is a non-trivial problem. Previous attempts rely on geometrical constructions of families of ellipsoids \([1, 7]\) but the actual computation of the mapping involve expansions in series or some sort of iterative procedure. Recently, Tort and Dubos \([3]\) (hereafter TD14) devised a perturbative procedure to construct a mapping \(r(\lambda, \phi, \Phi)\) that is only quasi-orthogonal, i.e. \(\partial r/\partial \Phi \cdot \partial r/\partial \lambda = 0\) but \(\partial r/\partial \Phi \cdot \partial r/\partial \phi = O(\varepsilon^n)\) for \(n\) as large as desired. They obtain a simple, closed-form expression for \(n = 2\).

The second aspect of the problem is to formulate the equations of motion in coordinates \((\lambda, \phi, \Phi)\). Gates \([4]\) starts from the equations of motion in Cartesian coordinates. He then relates the orthonormal basis associated to coordinates \((x, y, z)\) to the local orthonormal basis associated to \((\lambda, \phi, \Phi)\), obtains formulae for the spatial derivatives of the latter, and finally substitutes into the equations of fluid motion in Cartesian coordinates. This procedure is not only very tedious, it also misses a fundamental point: ultimately, it is not so much the mapping \(r(\lambda, \phi, \Phi)\) that matters. In fact, as shown by White and Wood \([13]\) for orthogonal geopotential coordinates and by TD14 for general non-orthogonal coordinates, knowing only the metric tensor \(g_{ij} = \partial_i \partial_r \cdot \partial_j r\) (where \(i, j = \lambda, \phi, \Phi\)) and the covariant components of planetary velocity \(R_i = \Omega \times e_z \cdot \partial_i r\) (where \(i = \lambda, \phi, \Phi\)) is sufficient to formulate the equations of motion in coordinates \((\lambda, \phi, \Phi)\). All that needs to be done is to derive explicitly \(g_{i\lambda}\) and \(R_i\) and inject them into the TD14 generic form of the equations of motion.

Because the TD14 mapping is not strictly orthogonal, the resulting equations of motion would have \(O(\varepsilon^n)\) gravitational terms in the horizontal momentum budget. If one is willing to accept errors of order \(O(\varepsilon^n)\) (instead of the \(O(\varepsilon)\) errors involved in the SG approximation), these terms can be neglected. More precisely, following the procedure advocated by TD14 for the sake of dynamical consistency (in the sense defined by \([11]\)), \(g_{ij}\) and \(R_i\) can be approximated to accuracy \(O(\varepsilon^{n-1})\). By neglecting more generally all \(O(\varepsilon^n)\) terms in \(g_{ij}\) and \(R_i\), the expression of \(g_{ij}\) is further simplified.

The main purpose of this work is to provide a practical way to incorporate the effects of planetary oblateness into the equations of atmospheric and oceanic motion. To this end, the simplest possible expressions of \(g_{ij}\) and \(R_i\) that take into account planetary oblateness to leading order are sought. Following the procedure sketched above, expressions of \(g_{ij}\) accurate to \(O(\varepsilon)\) are obtained in Appendix C. In section 2, further simplifications to these expressions are devised. These simplified expressions are still accurate to \(O(\varepsilon)\) at altitudes \(O(\varepsilon a)\), where most of the atmospheric mass resides if \(\tau \leq O(\varepsilon)\), but less accurate at altitudes \(O(a)\). Section 3 discusses practical issues, including values of defining parameters for Earth. A brief section 4 concludes.

II. SIMPLE GEOMETRIC APPROXIMATIONS

A. Procedure

Notice that \([1]\) uses the same sign convention as White et al. \([12]\), while the opposite convention is most often used. It is convenient and more conventional to define a geopotential \(\xi\) that increases with height and vanishes on the reference ellipsoid:

\[
\xi = \Phi_a - \Phi
\]
where $\Phi_0$ is the value of $\Phi$ on the reference ellipsoid. For the sake of conciseness, units such that $a = 1$ and $\Phi_0 = 1$ are adopted in this section and in the appendices. Standard units are restored at the end of this section.

Tort and Dubos \cite{ref} define nearly-orthogonal coordinates where the third coordinate $\xi$ is, as above, a function of $\Phi$ only (see Appendix B). As done in appendix C, it is possible to use their derivation to obtain the corresponding metric tensor truncated to $O(\epsilon)$, which by design is orthogonal, i.e. the squared length $dl^2$ associated to variations of the coordinates $\lambda, \phi, \xi$ is of the form:

$$dl^2 = h_\lambda^2 d\lambda^2 + h_\phi^2 d\phi^2 + g^{-2} d\xi^2$$

(2)

where $g(\phi, \xi) = \|\nabla \Phi\| = \|\nabla \xi\|$ is the local value of gravity and $\phi$ is a coordinate akin to latitude, to be specified more precisely below. In order to obtain the simplest possible expressions of $h_\lambda$, $h_\phi$, $g$ that capture effects neglected in the TSA-SG approximation, we distinguish the low-altitude region $\xi = O(\epsilon)$ from the high-altitude region $\xi = O(1)$. Simpler expressions, accurate only for $\xi = O(\epsilon)$ are found by expanding $h_\lambda$, $h_\phi$, $g$ in powers of $\xi$ then truncating to first order in $\epsilon$. This expansion is done formally in Appendix D, but here we follow a shortcut: knowing that this approximate orthogonal metric exists, we construct it step-by-step by invoking physical arguments. While the resulting expressions are as simple as one can get to capture the effects of oblateness with $O(\epsilon)$ accuracy for $\xi = O(\epsilon)$, they are fully inaccurate at high altitudes $\xi = O(1)$. Noticing that the deep-atmosphere expressions for $h_\lambda$, $h_\phi$, $g$ are accurate to $O(1)$ at such altitudes, the previously obtained expressions are modified in order to restore $O(1)$ accuracy at $\xi = O(1)$ without loss of accuracy at $\xi = O(\epsilon)$.

**B. Horizontal metric on the reference ellipsoid**

The value $\Phi_0$ of $\Phi$ on the reference ellipsoid is obtained by letting $(\lambda, \phi, r) = (0, 0, 1)$ in:

$$\Phi = r^{-1} - (\epsilon - \frac{m}{2}) r^{-3} \left( \sin^2 \chi - \frac{1}{3} \right) + \frac{m}{2} r^2 \cos^2 \chi,$$

(3)

yielding

$$\Phi_0 = 1 + \frac{\epsilon + m}{3}.$$

Letting $\chi = \pi/2$ and $\Phi = \Phi_0$ in (3) yields $r = 1 - \epsilon + O(\epsilon^2)$, confirming that the semi-minor axis of the reference ellipsoid is indeed $1 - \epsilon$. On this ellipsoid, there exist many orthogonal coordinate systems $(\lambda, \phi)$ which are similar to spherical coordinates in that $\lambda$ is longitude ($\lambda = \text{cst}$ is a great circle) and $\phi$ is akin to latitude ($\phi = \text{cst}$ is a circle parallel to the Equator). Appendix A recalls the definition of several variants of latitude (reduced latitude, geodetic latitude and conformal latitude) which coincide on a perfect sphere but differ for finite flattening. Rather than the commonly used *geodetic latitude*, we choose to define $\phi$ as the *conformal latitude*. Using this latitudinal coordinate, the scale factors are, to $O(\epsilon)$ accuracy:

$$h_\phi = 1 - \epsilon \sin^2 \phi,$$

$$h_\lambda = h_\phi \cos \phi$$

(4)

With (4), $dl^2 = h_\lambda^2 d\lambda^2 + h_\phi^2 d\phi^2$ is said to be conformal with respect to the strictly spherical metric $d\phi^2 + \cos^2 \phi d\lambda^2$ because it differs from the latter only by the multiplicative factor $h_\phi^2$. This property simplifies numerical formulations of differential operators (see 3.3).

**C. Gravity on the reference ellipsoid**

We have decided to use the conformal latitude $\phi$ of the reference ellipsoid as latitudinal coordinate. There is no guarantee that the TD14 latitudinal coordinate coincides with $\phi$ and it is in fact demonstrated in Appendix C that they differ slightly. Even so, due to zonal symmetry, there exists a change of latitudinal coordinate that, on the reference ellipsoid, maps the TD14 latitude to conformal latitude. Such a change of coordinate preserves the orthogonal character of the metric, hence (2) is still valid.
Using (3) one obtains the values of $g$ at the Poles and Equator of the reference ellipsoid:

$$g_P = \frac{\partial \Phi}{\partial r} \bigg|_{\chi=\pi/2, r=1-\varepsilon} = 1 + m$$

$$g_E = \frac{\partial \Phi}{\partial r} \bigg|_{\chi=0, r=1} = 1 - \frac{3}{2}m + \varepsilon$$

In between, $g$ varies with latitude as (see (C9) in Appendix C):

$$g(\phi) = 1 + m - \left(\frac{5}{2}m - \varepsilon\right) \cos^2 \phi.$$  \hfill (5)

Notice especially that $g_P/g_E = 1 + \left(\frac{5}{2}\right)m - \varepsilon + O(\varepsilon^2)$ as expected \cite{12}.

D. Widening of atmospheric columns with height

Now consider a horizontal displacement along a geoid $\xi = \text{cst} \neq 0$. The squared displacement equals $dl^2 = h_\lambda^2 d\lambda^2 + h_\phi^2 d\phi^2$ where $h_\lambda, h_\phi$ are given at $\xi = 0$ by (4). For $\xi = O(\varepsilon)$,

$$h_\phi = 1 - \varepsilon \sin^2 \phi + \left. \frac{\partial h_\phi}{\partial \xi} \right|_{\xi=0} \xi + O(\varepsilon^2)$$

$$h_\lambda = (1 - \varepsilon \sin^2 \phi) \cos \lambda + \left. \frac{\partial h_\lambda}{\partial \xi} \right|_{\xi=0} \xi + O(\varepsilon^2).$$

The corrections $(\partial h_\phi/\partial \xi)\xi, (\partial h_\lambda/\partial \xi)\xi$ express the widening of atmospheric columns with height. Their dependence on $\varepsilon$ can, at this order of accuracy, be neglected. Hence $\partial h_\phi/\partial \xi, \partial h_\lambda/\partial \xi$ can be obtained under the assumption of spherical geoids. In that case the distance from the center of the planet is $\Phi^{-1} = (1 - \xi)^{-1}$, $h_\lambda = h_\phi \cos \phi$ and $h_\phi = (1 - \xi)^{-1} = 1 + \xi + O(\varepsilon^2)$. Adding the $O(1)$ term to (4) yields:

$$h_\phi = 1 + \xi - \varepsilon \sin^2 \phi,$$

$$h_\lambda = h_\phi \cos \phi.$$  \hfill (6)

Clearly, (6) is inaccurate for $\xi = O(1)$. Indeed in this region of high altitudes, $h_\phi$ should coincide at $O(1)$ with its deep-atmosphere expression $h_\phi = \Phi^{-1} = (1 - \xi)^{-1} + O(\varepsilon)$. Hence, the expressions:

$$h_\phi = (1 - \xi)^{-1}(1 - \varepsilon \sin^2 \phi),$$

$$h_\lambda = h_\phi \cos \phi.$$  \hfill (7)

restore $O(1)$ accuracy where $\xi = O(1)$ without loss of accuracy for $\xi = O(\varepsilon)$, since they coincide with (6) at $O(\varepsilon)$ for $\xi = O(\varepsilon)$.

E. Vertical variation of gravity

Similarly, variations of $g$ with height, which are a $O(\varepsilon)$ effect for $\xi = O(\varepsilon)$, can be obtained to $O(\varepsilon)$ accuracy in the purely spherical case. In that case $g = r^{-2} = \Phi^2 = (1 - \xi)^2 = 1 - 2\xi + O(\varepsilon^2)$. Adding the $O(\xi)$ term to (5) yields:

$$g = 1 - 2\xi + m - \left(\frac{5}{2}m - \varepsilon\right) \cos^2 \phi$$

and demanding $O(1)$ accuracy for $\xi = O(1)$ yields:

$$g = (1 - \xi)^2 \left(1 + m - \left(\frac{5}{2}m - \varepsilon\right) \cos^2 \phi\right)$$

(9)
F. Full expressions

In summary, (2,6,8) define a spatial metric that is accurate to $O(\varepsilon)$ only up to $\xi = O(\varepsilon)$. (2,7,9) define a metric that is accurate to $O(\varepsilon)$ for $\xi = O(\varepsilon)$ and accurate to $O(1)$ for $\xi = O(1)$. Both metrics are horizontally conformal to the spherical metric. For $O(\varepsilon)$ accuracy at $\xi = O(1)$, one should use (C1,C6,C2,C3) with definitions (B4,B5,B8,C7,C8) (Appendix C). This more accurate metric is not horizontally conformal to the spherical metric. Restoring usual units, the full expressions of $h_\lambda$, $h_\phi$ and $g$ are:

• approximation I (least accurate):

\[
\begin{align*}
  h_\lambda &= h_\phi \cos \lambda \\
  h_\phi &= a \left( 1 + \frac{\xi}{\Phi_0} - \varepsilon \sin^2 \phi \right) \\
  g &= \frac{\Phi_0}{a} \left[ 1 - \frac{2\xi}{\Phi_0} + m - \left( \frac{5}{2}m - \varepsilon \right) \cos^2 \phi \right]
\end{align*}
\]

• approximation II (intermediate)

\[
\begin{align*}
  h_\lambda &= h_\phi \cos \lambda \\
  h_\phi &= a \left( 1 - \frac{\xi}{\Phi_0} \right)^{-1} (1 - \varepsilon \sin^2 \phi) \\
  g &= \frac{\Phi_0}{a} \left( 1 - \frac{\xi}{\Phi_0} \right)^2 \left[ 1 + m - \left( \frac{5}{2}m - \varepsilon \right) \cos^2 \phi \right]
\end{align*}
\]

• approximation III (most accurate)

\[
\begin{align*}
  h_\lambda &= a \left( R_E(\Phi) - \Delta R(\Phi) \sin^2 \phi + \Delta \phi(\Phi) \sin^2 \phi \right) \cos \phi, \\
  h_\phi &= a \left( R_E(\Phi) - \Delta R(\Phi) \sin^2 \phi - \Delta \phi(\Phi) \cos 2\phi \right), \\
  g &= \frac{\Phi_0}{a} \left( g_E(\Phi) + \Delta g(\Phi) \sin^2 \phi \right),
\end{align*}
\]

where $\Phi = 1 + \frac{\varepsilon + m}{3} - \frac{\xi}{\Phi_0}$,
\[
\begin{align*}
  R_E(\Phi) &= \Phi^{-1} + \frac{1}{3} \left( \varepsilon - \frac{m}{2} \right) \Phi + \frac{m}{2} \Phi^{-4}, \\
  \Delta R(\Phi) &= \left( \varepsilon - \frac{m}{2} \right) \Phi + \frac{m}{2} \Phi^{-4}, \\
  \Delta \phi(\Phi) &= \frac{5m}{6} - \varepsilon + \left( \varepsilon - \frac{m}{2} \right) \Phi - \frac{m}{3} \Phi^{-4}, \\
  g_E(\Phi) &= \Phi^2 + \frac{1}{3} \left( \varepsilon - \frac{m}{2} \right) \Phi + 2m \Phi^{-1}, \\
  \Delta g(\Phi) &= -\left( \varepsilon - \frac{m}{2} \right) \Phi + 2m \Phi^{-1}.
\end{align*}
\]

III. PRACTICAL CONSIDERATIONS

A. Defining parameters and their values

The shape of the Earth and its gravitational field are known with a very high accuracy, and geodetic systems have been very precisely defined, among which the World Geodetic System 1984 (WGS84) is perhaps the most relevant at global scale [6]. For terrestrial applications, it is therefore desirable to specify values of the parameters $a, \Phi_0, \Omega, \ldots$ that match as closely as possible those of WGS84, some of which are listed in table I. A perfect match is not possible since WGS84 has better than $O(\varepsilon)$ accuracy.
\[
\begin{array}{|c|c|c|c|}
\hline
\text{Unit} & \text{Earth} & \text{Jupiter} & \text{Saturn} \\
\hline
a & 6378.137 & 71492 & 60268 \\
b & 6356.752 & 66854 & 54364 \\
\Gamma M_E & 3.986004418 \times 10^{14} & 12.6687 \times 10^{16} & 3.7931 \times 10^{16} \\
T & 23.93447 & 9.9250 & 10.656 \\
\hline
\end{array}
\]

Table I: Terrestrial value of important physical quantities according to the WGS84 geodetic system [6].

\[
\Omega = 2\pi/T = 7.292115 \times 10^{-5} \text{rad} \cdot s^{-1}
\]

Table II: Proposed parameters for Earth, Jupiter and Saturn. Values of \(a, b, \Gamma M_E\) and sidereal period \(T\) closely match those of the WGS84 system for Earth [6]. For Jupiter they are taken from [7] and for Saturn from [8] (slightly update values of \(a\) and \(b\) for Saturn are given in [9]). Other quantities \(\Omega, \varepsilon, m, g_F, g_E\) are deduced from \(a, b, \Gamma M_E, T\). While actual implementations should compute them to machine accuracy, rounded values are presented in the table.

We suggest to take as fundamental parameters the semi-major and semi-minor axes \(a, b\), the gravitational parameter \(\Gamma M_E\) and the sidereal period \(T\) closely match those of the WGS84 system for Earth [6]. Table [11] presents possible values for \(a, b, \Gamma M_E\) and \(T\) for the Earth, Saturn and Jupiter. The Earth values of \(a, b, \Gamma M_E\) and \(T\) have been rounded to 1 m accuracy. In addition to \(a, b, \Gamma M_E, T\), the derived quantities \(\Omega, \varepsilon, m, g_E\) and \(g_p\) are computed. It is seen that the terrestrial values of gravity at the Equator and Poles match very well, although not perfectly, those of WGS84. Furthermore large horizontal variations of gravity occur on Saturn and Jupiter.

Similarly, one has to define precisely the relation between \(\phi\) and geodetic latitude \(\phi_g\). Since geographical databases typically refer to \(\phi_g\), conversions between \(\phi_g\) and \(\phi\) are required when processing inputs and outputs of a numerical model. There exists a closed-form relationship between \(\phi_g\) and the conformal latitude, but it is rather involved and not easily invertible. The approximate relationship \(\phi = \phi_g - 2\varepsilon \cos \phi_g \sin \phi_g\) could be used as a definition of \(\phi\), which should probably be called pseudo-conformal latitude, since it differs slightly from the true conformal latitude. To obtain \(\phi_g\) from \(\phi\), one may iterate \(\phi_g \leftarrow \phi - 2\varepsilon \sin \phi_g \cos \phi_g\) from an initial value \(\phi_g \leftarrow \phi\), or use a few Newton iterations.

B. Equations of motion

\[\Omega, h_\lambda, h_\phi\text{ and } g\text{ being fully specified, White and Wood [11] provide equations (A.10-A.12) which prognose the “physical” velocity components } (u, v, w) = (h_\lambda u^\lambda, h_\phi u^\phi, g^{-1} \cdot u^s) \text{ where the contravariant velocity components } (u^\lambda, u^\phi, u^s) \text{ are defined as the Lagrangian derivatives of } (\lambda, \phi, \xi). \text{ As noted in TD14, these equations are a special case of those derived in TD14 in non-orthogonal curvilinear coordinates, provided the Jacobian } J \text{ (converting between density } \rho \text{ and pseudo-density } \mu = \rho J \text{ involved in the flux-form mass budget) is defined as}
\]

\[
J = h_\lambda h_\phi g^{-1}.
\]

(10)

and the covariant components of planetary velocity are defined as \((R^\lambda, R^\phi, R^s) = (\Omega, 0, 0)\) hence \((R_\ell) = (g_\ell R^\ell) = (\Omega h_\lambda^2, 0, 0)\):

\[
R^\lambda = \Omega h_\lambda^2.
\]

(11)
Importantly, definition (11) implies that $R_\lambda$ depends on the vertical coordinate $\xi$. This property restores the non-traditional part of the Coriolis force, which is neglected in the TSA-SG approximation, for which $R_\lambda = \Omega a^2 \cos^2 \phi$ does not depend on the vertical coordinate $\xi$.

For practical purposes, especially for numerical modeling, it may be preferable to use other forms than that given in White and Wood [13], forms that would prognose covariant or contravariant components, such as the flux form or curl form found in TD14, and/or use non-Eulerian coordinates (e.g. [3]).

### C. Benefit of horizontally conformal coordinates

While approximations I and II are less accurate than approximation III, their expression is simpler. Especially, $h_\lambda = h_\phi \cos \phi$. Practical benefits of this property are discussed here. Consider as an illustrative example, simpler than the full equations of fluid motion, the Poisson problem:

$$\Delta p = f$$

where

$$\Delta = \frac{1}{J} \left( \frac{\partial}{\partial \lambda} (J h_\lambda^{-2} \frac{\partial}{\partial \lambda} + \frac{\partial}{\partial \phi} (J h_\phi^{-2} \frac{\partial}{\partial \phi} + \frac{\partial}{\partial \xi} (J g^2 \frac{\partial}{\partial \xi}) \right)$$

The fields $p$ and $f$, rather than functions of $\lambda, \phi, \xi$, can be regarded as functions of $x$ and $\xi$, where $x$ is the position on the unit sphere corresponding to longitude $\lambda$ and latitude $\phi$. If $h_\lambda = h_\phi \cos \phi$, (13) can be rewritten as:

$$\Delta = \frac{1}{J} \left( \nabla \cdot (J h_\phi^{-2} \nabla) + \frac{\partial}{\partial \xi} \left( J g^2 \frac{\partial}{\partial \xi} \right) \right)$$

where $\tilde{J} = g^{-1} h_\phi^2$ and $\nabla \cdot$ and $\nabla$ are the divergence and gradient operators on the unit sphere. In TSA-SG geometry, $g, h_\phi = a$ and $\tilde{J} = g^{-1} a^2$ are constants while in the DA-SG geometry they depend on altitude but not on latitude. Hence the main change introduced by approximations I and II is that metric factors become latitude-dependent. No additional operator is introduced beyond $\nabla, \nabla \cdot, \nabla \times$ and the $\bot$ operator that rotates a vector tangent to the unit sphere by an angle of $\pi/2$. Upgrading a numerical solver for the TSA-SG or DA-SG equations to geometries I or II should be essentially a matter of replacing constant or height-dependent metric factors by similar factors that depend also on latitude.

### IV. CONCLUSION

A sequence of three increasingly accurate metric tensors have been obtained, that capture to leading order the oblateness of the planet, the widening of atmospheric columns with height, the horizontal and vertical variations of gravity and the non-traditional part of the Coriolis force. Given the presumably small magnitude of the errors due to the SG approximation, these geometric approximations have been developed with simplicity and ease of numerical implementation in mind. Hopefully this work will facilitate the development of even more accurate global atmospheric solvers.

As a final remark, notice that the geometric approximations I-III could be applied as well to ocean modeling. In this case, since $H/a \sim 10^{-4}$ is an order of magnitude smaller than $\varepsilon$, vertical variations of $h_\lambda, h_\phi$ and $g$ may be neglected without much loss of accuracy. Expressions for $h_\lambda, h_\phi, g$ would then reduce to (15). On the other hand, in order to retain the non-traditional component of the Coriolis force, one would expand expression (11) of $R_\lambda$ to first order in $\xi$ as in Tort and Dubos [3], yielding $R_\lambda = \Omega a^2 (1 + 2\xi/\Phi_0) \cos^2 \lambda$. Notice that this $R_\lambda \neq \Omega h_\lambda^2$, so that the equations of motion should be obtained from TD14 rather than White and Wood [13].

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[2] Bénard P. 2015. An assessment of global forecast errors due to the spherical geopotential approximation in the shallow-water case: Errors with spherical geopotential approximation. *Quat. J. Roy. Met. Soc.* **141**(686): 195–206.
[3] Dubos T, Tort M. 2014. Equations of atmospheric motion in Non-Eulerian vertical coordinates: Vector-Invariant form and Quasi-Hamiltonian formulation. *Mon. Wea. Rev.* **142**(10): 3860–3880.
Appendix A: Latitudes on an ellipsoid

A simple set of orthogonal longitude-latitude coordinates \((\lambda, \phi_p)\) on an ellipsoid of semi-major axis 1 and semi-minor axis \(1 - \varepsilon\) is defined by the mapping:

\[
(x, y, z) = (\cos \lambda \cos \phi_p, \sin \lambda \cos \phi_p, (1 - \varepsilon) \sin \phi_p).
\] (A1)

Coordinate \(\phi_p\) is then called the \textit{reduced or parametric latitude}. The metric in these coordinates is, assuming \(\varepsilon \ll 1\),

\[
dl^2 = \dl^2_{\text{parametric}} + O(\varepsilon^2)\,
\]

\[
\dl^2_{\text{parametric}} = \cos^2 \phi_p \dl^2 + (1 - 2\varepsilon \cos^2 \phi_p) \, d\phi_p^2.
\]

The \textit{geodetic latitude} \(\phi_g\) is defined as the angle between the symmetry axis \((0, 0, 1)\) and the poleward horizontal vector. Using (A1), one finds for \(\varepsilon \ll 1\):

\[
\phi_p = \phi_g - \varepsilon \cos \phi_g \sin \phi_g + O(\varepsilon^2)
\]

\[
\phi_g = \phi_p + \varepsilon \cos \phi_p \sin \phi_p + O(\varepsilon^2).
\] (A2)

It is seen that \(\phi_p\) and \(\phi_g\) differ by the small latitude-dependent angle \(\varepsilon \cos \phi_p \sin \phi_p\). Different flavors of latitude differ by a similar amount, with different constant prefactors. Consider especially the latitude \(\phi_c\) such that

\[
\phi_p = \phi_c + \varepsilon \cos \phi_c \sin \phi_c.
\] (A3)

Using \(d\phi_p = (1 + \varepsilon \cos 2\phi_c) \, d\phi_c\), \(\cos \phi_p = (1 - \varepsilon \sin^2 \phi_c) \cos \phi_c + O(\varepsilon^2)\) one finds \(d\ell^2 = d\ell^2_{\text{conf}} + O(\varepsilon^2)\) where:

\[
\dl^2_{\text{conf}} = (1 - \varepsilon \sin^2 \phi_c)^2 \left( \cos^2 \phi_c \dl^2 + d\phi_c^2 \right).
\]

The metric \(d\ell^2_{\text{conf}}\) is said to be conformal to the spherical metric \(\cos^2 \phi \dl^2 + d\phi^2\) because they are related by a scaling factor, here \((1 - \varepsilon \sin^2 \phi_c)^2\). Accordingly, \(\phi_c\) coincides, with accuracy \(O(\varepsilon)\), with the so-called \textit{conformal latitude}, and we shall conflate the two here. Comparing (A2) and (A3), one finds that conformal latitude \(\phi_c\) and geodetic latitude \(\phi_g\) are related by:

\[
\phi_c = \phi_g - 2\varepsilon \cos \phi_g \sin \phi_g + O(\varepsilon^2)
\]

\[
\phi_g = \phi_c + 2\varepsilon \cos \phi_c \sin \phi_c + O(\varepsilon^2).
\]
Finally, let us consider a general ellipsoid, of semi-major axis \( A \) and semi-minor axis \( A - \Delta A \) (with \( \varepsilon = \Delta A / A \ll 1 \)), and a latitudinal coordinate \( \phi_X \) on this ellipsoid. Scaling the above expressions by \( A \):

\[
dl^2 = h_X^2 d\phi_X^2 + h_\lambda^2 d\lambda^2 + O(\varepsilon^2)
\]

where \( h_X = A - \Delta A \sin^2 \phi_X - A \Delta \phi \cos \phi_X \) \( (A5) \)

\[
h_\lambda = (A - \Delta A \sin^2 \phi_X + A \Delta \phi \sin^2 \phi_X) \cos \phi_X \)
\]

\[
\phi_X = \phi_c + \Delta \phi \cos \phi_c \sin \phi_c \)
\]

where \( \Delta \phi = O(\varepsilon) \) is a constant that defines the relationship between \( \phi_X \) and \( \phi_c \). Conversely, if the metric coefficients \( h_X, h_\lambda \) can be written in the form \( (A4,A7) \), the parameters \( A, \Delta A, \Delta \phi \) are readily identified and coordinate \( \phi_X \) can be related to \( \phi_c \).

**Appendix B: TD14 mapping**

We reproduce here, with adapted notations, the TD14 construction of quasi-orthogonal coordinates \( (\lambda, \beta, \Phi) \) with \( \lambda \) longitude, \( \beta \) a latitude to defined precisely later, and \( \Phi \) geopotential. These coordinates map to the position in 3D Cartesian space:

\[
r = \left( \Phi^{-1} - \frac{\partial \psi}{\partial \Phi} \right) e_r - \varepsilon \Phi^{-1} \frac{\partial \psi}{\partial \beta} e_\beta
\]

where \( e_r = (\cos \lambda \cos \beta, \sin \lambda \cos \beta, \sin \beta) \),

\[
e_\beta = (-\cos \lambda \sin \beta, -\sin \lambda \sin \beta, \cos \beta)
\]

where again we assume \( a = 1 \) and \( \Phi_0 = 1 \), and \( \psi \) is to be determined below. The form adopted in \( (B1) \) ensures that \( (\partial r / \partial \beta) \cdot (\partial r / \partial \Phi) = O(\varepsilon^2) \). Now \( (B1) \) implies \( \chi - \beta = O(\varepsilon) \) and

\[
r = \Phi^{-1} - \varepsilon \frac{\partial \psi}{\partial \Phi} + O(\varepsilon^2).
\]

Using \( \chi - \beta = O(\varepsilon) \), \( (B3) \) yields:

\[
r = R(\Phi) + O(\varepsilon^2)
\]

where

\[
R(\Phi) = R_E(\Phi) - \sin^2 \beta \Delta R(\Phi),
\]

\[
R_E(\Phi) = \Phi^{-1} + \frac{1}{3} \left( \varepsilon - \frac{m}{2} \right) \Phi + \frac{m}{2} \Phi^{-4},
\]

\[
\Delta R(\Phi) = \left( \varepsilon - \frac{m}{2} \right) \Phi + \frac{m}{2} \Phi^{-4}.
\]

Substituting \( (B3) \) in \( (B2) \) then yields \( \psi \) hence \( \partial \psi / \partial \beta \):

\[
\varepsilon \frac{\partial \psi}{\partial \Phi} = \left( \varepsilon - \frac{m}{2} \right) \left( \sin^2 \beta - \frac{1}{3} \right) \Phi - \frac{m}{2} \Phi^{-4} \cos^2 \beta,
\]

\[
\varepsilon \psi = \frac{1}{2} \left( \varepsilon - \frac{m}{2} \right) \left( \sin^2 \beta - \frac{1}{3} \right) \Phi^2 + \frac{m}{6} \Phi^{-3} \cos^2 \beta,
\]

\[
\varepsilon \Phi^{-1} \frac{\partial \psi}{\partial \beta} = X(\Phi) \cos \beta \sin \beta,
\]

where \( X(\Phi) = \left( \varepsilon - \frac{m}{2} \right) \Phi - \frac{m}{3} \Phi^{-4} \).

\( (B6,B8) \) specify completely \( (B1) \).
Appendix C: TD14 metric

Rather than the mapping \((\lambda, \beta, \Phi) \mapsto \mathbf{r}\), the equations of motion involve the metric \(dl^2\). By design \(dl^2\) is, to \(O(\varepsilon)\) accuracy, orthogonal, i.e. \(dl^2 = dl^2_{TD} + O(\varepsilon^2)\) with:

\[
dl^2_{TD} = h_\lambda^2 d\lambda^2 + h_\beta^2 d\beta^2 + g^{-2} d\Phi^2
\]

(C1)

where the scale factors \(h_\lambda, h_\beta\) and \(h_\Phi = g^{-1}\) are obtained now. Using

\[
r = (R_E(\Phi) - \sin^2 \beta \Delta R(\Phi)) \mathbf{e}_r - X(\Phi) \cos \beta \sin \beta \mathbf{e}_\beta
\]

and

\[
\frac{\partial \mathbf{e}_r}{\partial \lambda} = \cos \beta \mathbf{e}_\lambda, \quad \frac{\partial \mathbf{e}_r}{\partial \beta} = \mathbf{e}_\beta, \quad \frac{\partial \mathbf{e}_r}{\partial \Phi} = -\sin \beta \mathbf{e}_\lambda
\]

one finds

\[
\frac{\partial r}{\partial \lambda} = (R_E(\Phi) - \Delta R(\Phi) \sin^2 \beta + X(\Phi) \sin^2 \beta) \cos \beta \mathbf{e}_\lambda
\]

\[
\frac{\partial \mathbf{r}}{\partial \beta} = -2 \Delta R(\Phi) \sin \beta \cos \beta \mathbf{e}_\mathbf{r}
\]

\[
\frac{\partial \mathbf{r}}{\partial \Phi} = -\left(1 + (\varepsilon - \frac{5m}{2}) \sin \phi \cos \phi + O(\varepsilon^2)\right) \mathbf{e}_\beta \\
+ \frac{dX}{d\Phi} \cos \beta \sin \beta \mathbf{e}_\mathbf{r}
\]

Focusing first on the horizontal metric :

\[
h_\beta = R_E(\Phi) - \Delta R(\Phi) \sin^2 \beta + X(\Phi) \cos 2\beta
\]

(C2)

\[
h_\lambda = (R_E(\Phi) - \Delta R(\Phi) \sin^2 \beta + X(\Phi) \sin^2 \beta) \cos \beta.
\]

(C3)

Comparing (C2,C3) to (A3,A7) and noting that \(R_E^{-1} X = \Phi X + O(\varepsilon)\), one concludes that the geoid \(\Phi = \text{cst}\) is an ellipsoid of semi-major (resp. semi-minor) axis \(R_E(\Phi)\) (resp. \(R_E(\Phi) - \Delta R(\Phi)\)) and that the coordinate \(\beta\) is related to the conformal latitude \(\phi^\circ\) on that ellipsoid by:

\[
\beta = \phi^\circ + \Delta \phi \sin \phi^\circ \cos \phi^\circ + O(\varepsilon^2), \quad \Delta \phi = \Phi X(\Phi)
\]

(C4)

to \(O(\varepsilon)\) accuracy. Especially on the reference ellipsoid \(\Phi = 1 + (\varepsilon + m)/3\), :

\[
\beta = \phi + \left(\varepsilon - \frac{5m}{6}\right) \sin \phi \cos \phi + O(\varepsilon^2).
\]

(C5)

where \(\phi\) is defined, as in the main text, as the conformal latitude on the reference ellipsoid. Focusing next on gravity :

\[
g^{-1} = -\frac{dR_E}{d\Phi} - \frac{d\Delta R}{d\Phi} \sin^2 \beta,
\]

\[
g = g_E(\Phi) + \Delta g(\Phi) \sin^2 \beta + O(\varepsilon^2),
\]

(C6)

where \(g_E(\Phi) = \Phi^2 + \frac{1}{3} \left(\varepsilon - \frac{m}{2}\right) \Phi^4 + 2m\Phi^{-1}\)

\[
\Delta g(\Phi) = -\left(\varepsilon - \frac{m}{2}\right) \Phi^4 + 2m\Phi^{-1}.
\]

(C7)

(C8)

Especially, on the reference ellipsoid :

\[
g(\Phi, \beta) = 1 + m - \left(\varepsilon - \frac{5}{2} m - \frac{m}{2}\right) \cos^2 \beta + O(\varepsilon^2).
\]

(C9)

To \(O(\varepsilon)\) accuracy, (CHC2|C3|C6) with definitions (B4|B5|B8|C7|C8) specify the metric associated to the TD14 mapping, while (C5) relates latitude \(\beta\) to the conformal latitude on the reference ellipsoid.
Appendix D: Near-surface metric

In order to simplify the metric near the reference ellipsoid, we can finally change coordinates from \((\lambda, \beta, \Phi)\) to \((\lambda, \phi, \Phi)\), i.e. use the conformal latitude on the reference ellipsoid as latitudinal coordinate. In these coordinates, with \(O(\varepsilon)\) accuracy:

\[
\begin{align*}
    h_\lambda &= (R_E(\Phi) - \Delta R(\Phi) \sin^2 \phi + (X(\Phi) - X_a) \sin^2 \phi) \cos \phi \\
    h_\phi &= R_E(\Phi) - \Delta R(\Phi) \sin^2 \phi - (X(\Phi) - X_a) \cos 2\phi \\
    g &= g_E(\Phi) - \Delta g(\Phi) \sin^2 \phi
\end{align*}
\]

where \(X_a = X(\Phi_a)\). Let us now consider near-surface region, where \(\xi = \Phi_a - \Phi = O(\varepsilon)\). Then

\[
\begin{align*}
    X(\Phi) - X_a &= -\xi \frac{dX}{d\Phi}(\Phi_a) + \cdots = O(\varepsilon^2) \\
    \Delta R(\Phi) - \Delta R(\Phi_a) &= -\xi \frac{d\Delta R}{d\Phi}(\Phi_a) + \cdots = O(\varepsilon^2) \\
    \Phi^4 \frac{d\Delta R}{d\Phi}(\Phi) - \Phi_a^4 \frac{d\Delta R}{d\Phi}(\Phi_a) &= O(\varepsilon^2)
\end{align*}
\]

(note that \(X, \Delta R = O(\varepsilon)\)) so that:

\[
\begin{align*}
    h_\lambda &= \left(1 - \xi \frac{dR_E}{d\Phi} - \varepsilon \sin^2 \phi \right) \cos \phi + O(\varepsilon^2) \\
    h_\phi &= 1 - \xi \frac{dR_E}{d\Phi} - \varepsilon \sin^2 \phi + O(\varepsilon^2) \\
    g &= g(\Phi_a, \phi) - \xi \frac{dg_E}{d\Phi} + O(\varepsilon^2)
\end{align*}
\]

Since \(\xi = O(\varepsilon)\), it is sufficient to evaluate \(\frac{dR_E}{d\Phi}\) and \(\frac{dg_E}{d\Phi}\) to \(O(1)\) accuracy, i.e. \(-\frac{dR_E}{d\Phi} = \Phi_a^{-2} = 1 + O(\varepsilon), \quad \frac{dg_E}{d\Phi} = 2\Phi_a = 2 + O(\varepsilon)\), yielding:

\[
\begin{align*}
    h_\lambda &= (1 + \xi - \varepsilon \sin^2 \phi) \cos \phi + O(\varepsilon^2) \\
    h_\phi &= 1 + \xi - \varepsilon \sin^2 \phi + O(\varepsilon^2) \\
    g &= 1 - 2\xi + m - \left(\frac{5}{2}m - \varepsilon\right) \cos^2 \phi + O(\varepsilon^2).
\end{align*}
\]