A STRUCTURE THEOREM OF DIRAC-HARMONIC MAPS
BETWEEN SPHERES

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ABSTRACT. For an arbitrary Dirac-harmonic map \((\phi, \psi)\) between compact oriented Riemannian surfaces, we shall study the zeros of \(|\psi|\). With the aid of Bochner-type formulas, we explore the relationship between the order of the zeros of \(|\psi|\) and the genus of \(M\) and \(N\). On the basis, we could clarify all of nontrivial Dirac-harmonic maps from \(S^2\) to \(S^2\).

1. Introduction

Let \((M, h)\) be an \(m\)-dimensional Riemannian spin manifold; \(\text{Spin} M\) denotes the Spin-bundle on \(M\), and \(\eta : \text{Spin} M \rightarrow \text{SOM}\) is the bundle map, where \(\text{SOM}\) denotes the tangent orthonormal frame bundle on \(M\). Denote by \(\Sigma M\) the spinor bundle associated to \(\text{Spin} M\), i.e. \(\Sigma M = \text{Spin} M \times \rho \Sigma_m\), where \(\rho : \text{Spin} m \rightarrow \Sigma_m\) is the standard representation. On \(\Sigma M\) we can choose an Hermitian product \(\langle , \rangle\), such that

\[
\langle X \cdot \psi, \xi \rangle = -\langle \psi, X \cdot \xi \rangle \quad X \in \Gamma(TM), \psi, \xi \in \Gamma(\Sigma M).
\]

Here

\[
(1.2) \quad m : X \otimes \psi \mapsto X \cdot \psi
\]

is the Clifford multiplication. There is a connection on \(\Sigma M\) induced by the Levi-Civita connection of \(\text{SOM}\); denote it by \(\nabla\); and it is well known that \(\nabla\) is compatible with \(\langle , \rangle\). Let \(\phi\) be a smooth map from \(M\) to another Riemannian manifold \((N, g)\) of dimension \(n \geq 2\). Denote by \(\phi^{-1}TN\) the pull-back bundle of \(TN\) and by \(\Sigma M \otimes \phi^{-1}TN\) the twisted bundle. On it there is a metric induced from those on \(\Sigma M\) and \(\phi^{-1}TN\). Similarly we have a natural connection \(\tilde{\nabla}\) on \(\Sigma M \otimes \phi^{-1}TN\) induced from those on \(\Sigma M\) and \(\phi^{-1}TN\). Based on it, we can define the Dirac operator along the map \(\phi\) by

\[
(1.3) \quad \slashed{D} \psi = m \circ \tilde{\nabla} \psi.
\]

Here \(\psi\) is a smooth section of \(\Sigma M \otimes \phi^{-1}TN\).

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In [1], Q. Chen, J. Jost, J. Li and G. Wang introduced a functional that couples the nonlinear sigma model with a spinor field:

\[ L(\phi, \psi) = \int_M \left[ |d\phi|^2 + \langle \psi, D\psi \rangle \right] * 1 \]

The critical points of the functional is called Dirac-harmonic maps. In the paper, some geometric and analytic aspects of such maps were studied, especially a removable singularity theorem was established. Later in [2], [3], [8] and [9], another structure theorem of Dirac-harmonic maps from \(S^2\) to \(M\) using the above denotation. In Section 3, we assume formulas of \(\psi\) and \(M\) is a Dirac-harmonic map from \(S\) of \(\log |\cdot|\) using the above denotation. In Section 3., we assume that \(\phi, \psi\) is a harmonic map. In [1], the authors constructed non-trivial Dirac-harmonic maps \((\phi, \psi)\) from \(S^2\) to \(S^2\), where \(\phi\) is a (possible branched) conformal map, \(\psi\) could be written in the form

\[ \psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha), \]

\(\{e_\alpha : \alpha = 1, 2\}\) is a local orthonormal frame field on \(S^2\), and \(\Psi\) is a twistor spinor. It is natural to ask whether there exists another form of Dirac-harmonic maps from \(S^2\) to \(S^2\). And furthermore, is there a Dirac-harmonic map \((\phi, \psi)\) such that \(\phi\) is not a harmonic map?

In the theory of harmonic maps between two compact Riemannian surfaces, Bochner formulas of \(\log |\partial u|\) and \(\log |\bar{\partial} u|\) play an important role \((u\) denotes a harmonic map). From it several interesting formulas easily follow, which tell us the relationship between the order of the zeros of \(\log |\partial u|\) and \(\log |\bar{\partial} u|\) and the genus of \(M\) and \(N\); and moreover we can obtain some uniqueness theorems and non-existence theorems (see [6] Chapter I). This phenomenon motives us to study the zeros of \(|\psi|\).

Now we give a brief outline of the paper. In Section 2, the subjects we study are general Dirac-harmonic maps. In the viewpoint that \(\Sigma M \otimes \phi^{-1}TN = \Sigma M \otimes (\phi^{-1}TN)^C\), \(\psi\) could be written as \(\psi = \psi^j \otimes W_j\), where \(\{W_1, \cdots, W_n\}\) is a local complex tangent frame field on \(N\); and we derive the Euler-Lagrange equation of \(L\) by using the above denotation. In Section 3, we assume \(M\) and \(N\) to be oriented Riemannian surfaces; the equations of harmonic spinor \(\psi\) along \(\phi\) in the local complex coordinates are derived, which imply that the zeros of \(|\pi^+_0(\psi)|\) are isolated, unless \(|\pi^+_1(\psi)|\) is identically zero, so are \(|\pi^+_0(\psi)|\), \(|\pi^-_1(\psi)|\) and \(|\pi^-_0(\psi)|\). (Here the definition of \(\pi^+_1, \pi^+_0, \pi^-_1, \pi^-_0\) is introduced in Section 3.) In Section 4-5, under the further assumption that \(M\) and \(N\) are both compact, we derive several Bochner formulas of \(\log |\pi^+_1(\psi)|, \log |\pi^+_0(\psi)|, \log |\pi^-_1(\psi)|, \log |\pi^-_0(\psi)|\) on the basis of Weitzenböck-type formulas of \(\psi\) and furthermore give the proof of main theorems as follows, including a structure theorem of Dirac-harmonic maps from \(S^2\) to \(S^2\). (In the process, it is necessary to use the results in Section 2-3.)

**Theorem 1.1.** \(M\) and \(N\) are both compact oriented Riemannian surfaces, and \((\phi, \psi)\) is a Dirac-harmonic map from \(M\) to \(N\). If \(g_M = 0\) or \(|g_M - 1| < |\deg(\phi)||2g_N - 2|\), then \(\phi\) has to be a harmonic map.
Theorem 1.2. If $M = N = S^2$ equipped with arbitrary metric, $(\phi, \psi)$ is a non-trivial Dirac-harmonic map from $M$ to $N$, then $\phi$ has to be holomorphic or anti-holomorphic, $\psi$ could be written in the form

\begin{equation}
\psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha),
\end{equation}

where $\Psi$ is a twistor spinor (possibly with isolated singularities).

Please note that here and in the sequel we use the summation convention and agree the range of indices:

\[ 1 \leq i, j, k \leq n; \quad 1 \leq \alpha, \beta \leq m. \]

We refer to [5] and [4] for more background material on spin structures and Dirac operators.

2. Euler-Lagrange equations of Dirac-harmonic maps

Denote the complexification of $\phi^{-1}TN$ by $(\phi^{-1}TN)^C$. Obviously $\Sigma M \otimes \phi^{-1}TN \subset \Sigma M \otimes (\phi^{-1}TN)^C$. On the other hand, for any $\psi \in \Sigma M$, $X + \sqrt{-1}Y \in (\phi^{-1}TN)^C$ (here $X, Y \in \phi^{-1}TN$),

\[ \psi \otimes (X + \sqrt{-1}Y) = \psi \otimes X + \sqrt{-1}\psi \otimes Y \in \Sigma M \otimes \phi^{-1}TN; \]

which implies $\Sigma M \otimes (\phi^{-1}TN)^C \subset \Sigma M \otimes \phi^{-1}TN$. Hence $\Sigma M \otimes (\phi^{-1}TN)^C = \Sigma M \otimes \phi^{-1}TN$. The pull-back metric $\phi^{-1}g$ on $\phi^{-1}TN$ could be naturally extended to a Hermitian product on $(\phi^{-1}TN)^C$; and there is a natural Hermitian product on $\Sigma M \otimes (\phi^{-1}TN)^C$ induced from those on $\Sigma M$ and $(\phi^{-1}TN)^C$, which is also denoted by $\langle \ , \ \rangle$.

For each point $x \in M$, we can choose $\{W_i \in \Gamma(TU) : 1 \leq i \leq n\}$, where $U$ is a neighborhood of $\phi(x)$, such that

\begin{equation}
(T_{\phi(y)}N)^C = \bigoplus_{i=1}^{n} \mathbb{C}W_i(\phi(y)) \quad y \in \phi^{-1}(U),
\end{equation}

then on $\phi^{-1}(U)$, $\psi$ could be expressed by

\begin{equation}
\psi(y) = \psi_j(y) \otimes W_i(\phi(y)),
\end{equation}

where $\psi_1, \ldots, \psi_n \in \Gamma(\Sigma(\phi^{-1}(U)))$. We shall derive the Euler-Lagrange equations for $L$ by using the above denotation.

Proposition 2.1. Let $\{e_\alpha : 1 \leq \alpha \leq m\}$ be a local tangent orthonormal frame field, then the Euler-Lagrange equations for $L$ are

\begin{align}
\mathcal{D} \psi &= 0 \\
\tau(\phi) &= -\langle \psi^j, e_\alpha \cdot \psi^k \rangle R_{W_j, W_k}^N \phi_*(e_\alpha).
\end{align}
Proof. At first, we consider a family of $\psi_t$ with $\frac{d\psi_t}{dt} = \eta$ at $t = 0$ and fix $\phi$. Since $\mathcal{D}$ is formally self-adjoint (see [1]), we have

$$\frac{dL}{dt}\bigg|_{t=0} = \int_M \langle \eta, \mathcal{D} \psi \rangle + \langle \psi, \mathcal{D} \eta \rangle = \int_M \langle \eta, \mathcal{D} \psi \rangle + \langle \mathcal{D} \psi, \eta \rangle = 2 \int_M \text{Re} \langle \eta, \mathcal{D} \psi \rangle.$$

(2.5)

Since $\eta$ could be chosen arbitrarily, (2.3) is easily followed.

Now we consider a variation $\phi_t$ ($t \in (-\varepsilon, \varepsilon)$) of $\phi$ such that $\phi_t = \phi$ outside a compact set $K \subset \partial^{-1}(U)$ and $\phi_t(K) \subset U$; denote $\psi(y) = \psi^j(y) \otimes W_j(\phi(y))$ for each $y \in U$, then we define $\psi_t(y) = \psi^j(y) \otimes W_j(\phi_t(y))$. Denote $\xi = \frac{d\phi_t}{dt}\big|_{t=0}$.

Obviously

$$\frac{dL}{dt}\bigg|_{t=0} = \int_M \frac{d}{dt}\bigg|_{t=0} |d\phi_t|^2 + \int_M \frac{d}{dt}\bigg|_{t=0} \langle \psi, \mathcal{D} \psi \rangle = I + II,$$

and

$$I = -2 \int_M \langle \xi, \tau(\phi) \rangle.$$

(2.6)

Here $\tau(\phi)$ denotes the tension field of $\phi$. Since $\psi = \psi^j \otimes W_j$, we have

$$\mathcal{D} \psi = \partial \psi^k \otimes W_k + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} W_k,$$

where $\partial$ denotes the usual Dirac operator. Then

$$\frac{d}{dt}\bigg|_{t=0} \mathcal{D} \psi = \partial \psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} \nabla_{e_\alpha} W_k$$

$$= e_\alpha \cdot \nabla_{e_\alpha} \psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} \nabla_{\frac{\partial}{\partial t}} W_k + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} W_k$$

$$= e_\alpha \cdot \nabla_{e_\alpha} (\psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k) + e_\alpha \cdot \psi^k \otimes \nabla_{e_\alpha} W_k$$

$$= \mathcal{D} (\psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k) + e_\alpha \cdot \psi^k \otimes R^N_{\phi_t(e_\alpha), \xi} W_k.$$

Please note that here and in the following text $R_{XY} = -[\nabla_X, \nabla_Y] + \nabla_{[X,Y]}$. In conjunction with (2.3), we have

$$II = \int_M \langle \psi, \frac{d}{dt}\bigg|_{t=0} \mathcal{D} \psi \rangle + \langle \mathcal{D} \psi, \frac{d}{dt}\bigg|_{t=0} \mathcal{D} \psi \rangle$$

$$= \int_M \langle \psi, \mathcal{D} (\psi^k \otimes \nabla_{\frac{\partial}{\partial t}} W_k) + e_\alpha \cdot \psi^k \otimes R^N_{\phi_t(e_\alpha), \xi} W_k \rangle$$

$$= \int_M \langle \psi^j, e_\alpha \cdot \psi^k \rangle \langle R^N_{\phi_t(e_\alpha), \xi} W_k, W_j \rangle + \int_M \langle \psi^j, e_\alpha \cdot \psi^k \rangle \langle W_j, R^N_{\phi_t(e_\alpha), \xi} W_k \rangle$$

$$= \int_M \langle \psi^j, e_\alpha \cdot \psi^k \rangle \langle R^N_{\phi_t(e_\alpha), \xi} W_k, W_j \rangle$$

$$= -\int_M \langle \psi^j, e_\alpha \cdot \psi^k \rangle \langle R^N_{W_j, W_k}, \phi_t(e_\alpha), \xi \rangle.$$

(2.9)
Substituting (2.7) and (2.9) into (2.6) yields
\begin{equation}
\frac{dL}{dt} = - \int_M \left(2\tau(\phi) + \langle \psi^j, e_\alpha \cdot \psi^k \rangle R_{\bar{w},w}^N \phi_*(e_\alpha), \xi \right).
\end{equation}

Thereby (2.4) follows.

\[\square\]

**Remark 2.1.** The Euler-Lagrange equations of \(L\) was firstly derived in [1]. But our denotation is different.

3. Zeros of harmonic spinor fields

In this section, \(M\) and \(N\) are both oriented Riemannian surfaces. Then \(\Sigma M = \Sigma^+ M \oplus \Sigma^- M\), where
\begin{equation}
\Sigma^\pm M = \{ \xi \in \Sigma M : \sqrt{-1} e_1 \cdot e_2 \cdot \xi = \pm 1 \}.
\end{equation}

(Here \(\{e_1, e_2\}\) is an orthonormal basis of \(T_\xi \Sigma M\), and \(\pi\) denotes the bundle projection of \(\Sigma M\) onto \(M\).) In conjunction with \((\phi^{-1}TN)^C = \phi^{-1}(T^{(1,0)}N) \oplus \phi^{-1}(T^{(0,1)}N)\), we have
\begin{equation}
\begin{split}
\Sigma M \otimes (\phi^{-1}TN)^C &= (\Sigma^+ M \otimes \phi^{-1}(T^{(1,0)}N)) \oplus (\Sigma^+ M \otimes \phi^{-1}(T^{(0,1)}N)) \\
&\quad \oplus (\Sigma^- M \otimes \phi^{-1}(T^{(1,0)}N)) \oplus (\Sigma^- M \otimes \phi^{-1}(T^{(0,1)}N)).
\end{split}
\end{equation}

Denote by \(\pi^+_1, \pi^+_0, \pi^-_1, \pi^-_0\) the projections of \(\Sigma M \otimes (\phi^{-1}TN)^C\) onto the subbundles, respectively. Let \(X\) be a tangent vector field on \(M\), then \(\nabla_X\) keeps \(\Gamma(\Sigma^\pm M),\nabla_X(\phi^{-1}(T^{(1,0)}N))\) and \(\Gamma(\phi^{-1}(T^{(0,1)}N))\) invariant, and \(X \cdot \Sigma^\pm M \subset \Sigma^\mp M\); therefore
\begin{equation}
\begin{split}
\mathcal{D} \left( \Gamma(\Sigma^+ M \otimes \phi^{-1}(T^{(1,0)}N)) \right) &\subset \Gamma(\Sigma^+ M \otimes \phi^{-1}(T^{(0,1)}N)), \\
\mathcal{D} \left( \Gamma(\Sigma^- M \otimes \phi^{-1}(T^{(0,1)}N)) \right) &\subset \Gamma(\Sigma^- M \otimes \phi^{-1}(T^{(0,1)}N)).
\end{split}
\end{equation}

Hence \(\mathcal{D} \psi = 0\) yields that \(\pi^+_1(\psi), \pi^+_0(\psi), \pi^-_1(\psi), \pi^-_0(\psi)\) are all harmonic spinor fields along \(\phi\).

Let \(\psi \in \Gamma(\Sigma^+ M \otimes \phi^{-1}(T^{(1,0)}N))\) be harmonic, we shall derive the equation of \(\psi\) in local complex coordinates.

Let \(z = x + \sqrt{-1} y, w = u + \sqrt{-1} v\) be complex coordinates of \(M, N\), respectively. Then the metric of \(M, N\) are of the forms \(\lambda(z)|dz|^2, \rho(w)|dw|^2\), respectively. Denote
\begin{equation}
s = \{e_1, e_2\}, \quad \text{where } e_1 = \lambda^{-\frac{1}{2}} \frac{\partial}{\partial x}, \quad e_2 = \lambda^{-\frac{1}{2}} \frac{\partial}{\partial y},
\end{equation}
then \(s\) is a local tangent orthonormal frame bundle; i.e. \(s\) is a smooth section of \(\text{SOU}, U \subset M\). Let \(\bar{s} \in \Gamma(\text{SpinU})\) be a lift of \(s\), i.e. \(\eta \circ \bar{s} = s\). Denote
\begin{equation}
\psi^+ = [\bar{s}, \sigma], \quad \psi^- = e_1 \cdot \psi^+,
\end{equation}
where $\sigma$ is a unit vector in $\Sigma^+_2$, then from (3.1),

\begin{align*}
e_1 \cdot \psi^+ &= \psi^-, & e_1 \cdot \psi^- &= -\psi^+, \\
e_2 \cdot \psi^+ &= \sqrt{-1} \psi^- , & e_2 \cdot \psi^- &= \sqrt{-1} \psi^+.
\end{align*}

And furthermore,

\begin{align*}
\frac{\partial}{\partial z} \cdot \psi^+ &= \lambda^\frac{1}{2} \psi^-, & \frac{\partial}{\partial z} \cdot \psi^- &= 0, \\
\frac{\partial}{\partial \bar{z}} \cdot \psi^+ &= 0, & \frac{\partial}{\partial \bar{z}} \cdot \psi^- &= -\lambda^\frac{1}{2} \psi^+.
\end{align*}

By the definition of the connection on $\Sigma M$, we have

\begin{align*}
\partial \psi^+ &= e_\alpha \cdot \nabla_{e_\alpha} \psi^+ = \frac{1}{2} e_\alpha \cdot (\nabla_{e_\alpha} e_1, e_2) e_1 \cdot e_2 \cdot \psi^+ \\
&= \frac{1}{2} (\nabla_{e_1} e_1, e_2) e_1 \cdot e_2 \cdot \psi^+ + \frac{1}{2} (\nabla_{e_2} e_1, e_2) e_2 \cdot e_1 \cdot e_2 \cdot \psi^+ \\
&= \frac{1}{2} \frac{\partial \lambda^\frac{1}{2}}{\partial y} (-\sqrt{-1}) \psi^- - \frac{1}{2} \frac{\partial \lambda^\frac{1}{2}}{\partial x} \psi^- \\
&= -\frac{\partial \lambda^\frac{1}{2}}{\partial z} \psi^-.
\end{align*}

Let $f$ be a smooth function on $U$, such that $\psi = f \psi^+ \otimes \frac{\partial}{\partial w}$, then

\begin{align*}
0 &= \partial \psi = \partial (f \psi^+ \otimes \frac{\partial}{\partial w}) \\
&= \partial \psi^+ \otimes f \frac{\partial}{\partial w} + \frac{2}{\lambda} \left( \frac{\partial}{\partial z} \cdot \psi^+ \otimes \nabla \frac{\partial}{\partial w} (f \frac{\partial}{\partial w}) + \frac{\partial}{\partial \bar{z}} \cdot \psi^+ \otimes \nabla \frac{\partial}{\partial w} (f \frac{\partial}{\partial w}) \right) \\
&= -\frac{\partial \lambda^\frac{1}{2}}{\partial z} \psi^- \otimes f \frac{\partial}{\partial w} + 2 \lambda^\frac{1}{2} \psi^- \otimes \left( \frac{\partial f}{\partial \bar{z}} \frac{\partial}{\partial w} + f \nabla^N \frac{\partial}{\partial w} + \frac{\partial}{\partial w} \frac{\partial}{\partial \bar{w}} \right) \\
&= 2 \lambda^\frac{1}{2} \left( \frac{1}{4} \frac{\partial \log \lambda}{\partial z} f + \frac{\partial \log \rho}{\partial w} \frac{\partial f}{\partial \bar{z}} \right) \psi^- \otimes \frac{\partial}{\partial w}.
\end{align*}

Thereby we get the equation of $f$ as follows:

\begin{align*}
0 &= \partial f = \partial \left( f \psi^+ \otimes \frac{\partial}{\partial w} \right) \\
&= \frac{\partial f}{\partial z} + \left( \frac{1}{4} \frac{\partial \log \lambda}{\partial z} + \frac{\partial \log \rho}{\partial w} \frac{\partial}{\partial \bar{z}} \right) f = 0.
\end{align*}

From it, we can prove the following proposition.

**Proposition 3.1.** If $\psi$ is a harmonic spinor field along $\phi$, which is a smooth map between two oriented Riemannian surfaces, then $|\pi^+_1(\psi)|$ is identically zero or it has isolated zeroes. So are $|\pi^-_1(\psi)|, |\pi^+_0(\psi)|$ and $|\pi^-_0(\psi)|$.

**Proof.** As we have seen, $\pi^+_1(\psi)$ is harmonic whenever $\psi$ is harmonic. Denote $\pi^+_1(\psi) = f \psi^+ \otimes \frac{\partial}{\partial w}$, then $f$ satisfies (3.10). Denote

$$h(z) = \frac{1}{4} \frac{\partial \log \lambda}{\partial z} + \frac{\partial \log \rho}{\partial w} \frac{\partial}{\partial \bar{z}}.$$
then \( \frac{\partial f}{\partial \bar{z}} + h f = 0 \). Let \( \zeta \) be a local solution of \( \frac{\partial h}{\partial \bar{z}} = h \), then

\[
\frac{\partial (f e^\zeta)}{\partial \bar{z}} = -h f e^\zeta + h f e^\zeta = 0.
\]

i.e. \( f e^\zeta \) is holomorphic. Hence the conclusion follows from the well-known fact that the zeros of a holomorphic function are isolated, unless it is identically zero. And the proof for \( |\pi^+_0 (\psi)|, |\pi^-_1 (\psi)| \) and \( |\pi^-_0 (\psi)| \) is similar.

□

4. Weitzenböck-type formulas and Bochner-type formulas

For a spinor field \( \psi \) along a map \( \phi : M \to N \), where \( (M^m, h) \) is a Riemannian spin manifold, we can proceed as [1] Proposition 3.4 to have the following Weitzenböck-type formula.

\[
\mathcal{D}^2 \psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{4} S \psi + \frac{1}{2} \sum_{\alpha \neq \beta} e_\beta \cdot e_\alpha \cdot \psi^j \otimes R^N_{\phi_* (e_\alpha), \phi_* (e_\beta)} W_j.
\]

Here \( \psi = \psi^j \otimes W_j \), \( \{ e_\alpha \} \) is a local tangent orthonormal frame field on \( M \) such that \( \nabla e_\alpha = 0 \) at the considered point, and \( S \) is the scalar curvature. When \( m = 2 \), \( S = 2K_M \), where \( K_m \) denote the Gauss curvature of \( M \), then

\[
\mathcal{D}^2 \psi = -\tilde{\nabla}_{e_\alpha} \tilde{\nabla}_{e_\alpha} \psi + \frac{1}{2} K_M \psi + e_2 \cdot e_1 \cdot \psi^j \otimes R^N_{\phi_* (e_1), \phi_* (e_2)} W_j.
\]

Now we assume \( M, N \) are both oriented Riemannian surfaces; \( \psi \in \Gamma (\Sigma^+ M \otimes \phi^{-1} (T^{(1,0)} N)) \) and \( \mathcal{D} \psi = 0 \). For arbitrary \( x \in M \), let \( r > 0 \) such that \( \exp_x : B(r) \to M \) is injective; denote \( U = \exp_x (B(r)) \), then we can define a local section of \( \Sigma^+ M \) (denoted by \( \psi^+ \)) and a local section of \( \phi^{-1} (T^{(1,0)} N) \) (denoted by \( W \)) on \( U \), such that for any geodesic \( \gamma \) starting from \( x \),

\[
\nabla_\gamma \psi^+ = \nabla_\gamma W = 0,
\]

and \( \langle \psi^+, \psi^+ \rangle = 1, \langle W, W \rangle = 1 \); hence at \( x \),

\[
\nabla_{e_\alpha} \psi^+ = \nabla_{e_\alpha} \nabla_{e_\alpha} \psi^+ = \nabla_{e_\alpha} W = \nabla_{e_\alpha} \nabla_{e_\alpha} W = 0.
\]

\( \psi \) could locally be expressed by \( \psi = f \psi^+ \otimes W \), then at \( x \),

\[
\tilde{\nabla}_{e_\alpha} \nabla_{e_\alpha} \psi = \Delta f \psi^+ \otimes W.
\]

Denote \( W = \frac{\sqrt{2}}{2} (V_1 - \sqrt{-1} V_2) \), then \( V_2 = J^N V_1 \), where \( J^N \) is the complex structure on \( N \), and \( g(V_i, V_j) = \delta_{ij} \). Denote

\[
\phi_* e_i = \phi_{ij} V_j,
\]
then
\[ R_{\phi^*, e_1, e_2}^N W = R_{\phi_{11} V_{11} + \phi_{12} V_{12}, \phi_{21} V_{21} + \phi_{22} V_{22}}^N \left( \frac{\sqrt{2}}{2} (V_1 - \sqrt{-1} V_2) \right) \]
(4.4)
\[ = \det(\phi_{ij}) R_{V_{11}, V_{12}}^N \left( \frac{\sqrt{2}}{2} (V_1 - \sqrt{-1} V_2) \right) \]
\[ = \sqrt{-1} J(\phi) K_N W. \]

Here \( J(\phi) \) denotes the Jacobian of \( \phi \) and \( K_N \) denotes the Gauss curvature of \( N \).

Substituting (4.3) and (4.4) into (4.2) yields
\[ 0 = \mathcal{D}^2 \psi = -\tilde{\nabla}_{e_a} \tilde{\nabla}_{e_b} \psi + \frac{1}{2} K_M \psi + e_2 \cdot e_1 \cdot \psi^+ \otimes R_{\phi^*, e_1, e_2}^N W \]
(4.5)
\[ = (-\Delta f + \frac{1}{2} K_M f - K_N J(\phi) f) \psi^+ \otimes W. \]
i.e.
\[ \Delta f = \frac{1}{2} K_M f - K_N J(\phi) f \quad \text{at } x. \]

Furthermore,
\[ \Delta |\psi|^2 = \Delta |f|^2 = \bar{f} \Delta f + f \Delta \bar{f} + 2|f|^2 \]
(4.6)
\[ = K_M |\psi|^2 - 2K_N J(\phi) |\psi|^2 + 2|\nabla \psi|^2. \]

From
\[ 0 = e_1 \cdot \mathcal{D} \psi = e_1 \cdot e_1 \cdot \nabla_{e_1} \psi + e_1 \cdot e_2 \cdot \nabla_{e_2} \psi \]
(4.7)
\[ = -(\nabla_{e_1} f) \psi^+ \otimes W + (\nabla_{e_2} f) e_1 \cdot e_2 \cdot \psi^+ \otimes W \]
\[ = -(\nabla_{e_1} f + \sqrt{-1} \nabla_{e_2} f) \psi^+ \otimes W \]
we have
\[ \nabla_Z f = 0. \]

Here \( Z = \frac{\sqrt{2}}{2} (e_1 - \sqrt{-1} e_2) \) and \( \bar{Z} = \frac{\sqrt{2}}{2} (e_1 + \sqrt{-1} e_2) \), which satisfy \( h(Z, \bar{Z}) = 1, h(Z, Z) = h(\bar{Z}, \bar{Z}) = 0 \). Then \( \nabla f = (\nabla_Z f) \bar{Z} \) and
\[ |\nabla \psi|^2 = |\nabla f|^2 = |\nabla_Z f|^2. \]

Furthermore, from
\[ \nabla |\psi|^2 = \nabla |f|^2 = f \nabla \bar{f} + \bar{f} \nabla f \]
(4.8)
\[ = f (\nabla_Z \bar{f}) Z + \bar{f} (\nabla_Z f) \bar{Z} \]
we arrive at
\[ |\nabla |\psi|^2|^2 = 2 |f|^2 |\nabla_Z f|^2 = 2 |\psi|^2 |\nabla \psi|^2. \]

Substituting (4.12) into (4.7) yields
\[ \Delta |\psi|^2 = K_M |\psi|^2 - 2K_N J(\phi) |\psi|^2 + \frac{|\nabla |\psi|^2|^2}{|\psi|^2}. \]
(4.9)
And at last we derive the following Bochner-type formula
\[(4.14) \Delta \log |\psi| = \frac{1}{2}K_M - K_N J(\phi).\]

Similarly, when \(\psi \in \Sigma^+ M \otimes \phi^{-1}(T^{(0,1)} N), \Sigma^- M \otimes \phi^{-1}(T^{(1,0)} N)\) or \(\Sigma^+ M \otimes \phi^{-1}(T^{(0,1)} N)\), the corresponding Bochner-type formulas could be derived. We write those results as the following theorem.

**Theorem 4.1.** Let \(M\) and \(N\) are both oriented Riemannian surfaces. If \(\psi\) is a harmonic spinor field along \(\phi : M \to N\), then \(\log |\pi_1^+(\psi)|\), \(\log |\pi_0^+(\psi)|\), \(\log |\pi_1^-(\psi)|\) and \(\log |\pi_0^- (\psi)|\) satisfy Bochner-type formulas as follows:

\[
(4.15) \quad \Delta \log |\pi_1^+(\psi)| = \frac{1}{2}K_M - K_N J(\phi),
\]

\[
(4.16) \quad \Delta \log |\pi_0^+(\psi)| = \frac{1}{2}K_M + K_N J(\phi),
\]

\[
(4.17) \quad \Delta \log |\pi_1^-(\psi)| = \frac{1}{2}K_M + K_N J(\phi),
\]

\[
(4.18) \quad \Delta \log |\pi_0^- (\psi)| = \frac{1}{2}K_M - K_N J(\phi).
\]

When \(M\) and \(N\) are both compact, since the zeros of \(|\pi_1^+(\psi)|\) are isolated, there exist a finite number of zeros \(p_1, \ldots, p_k\) in \(M\). And similarly the zeros of \(|\pi_0^+(\psi)|\), \(|\pi_1^- (\psi)|\) and \(|\pi_0^- (\psi)|\) are finite. Integrating both side of (4.15)-(4.18) on \(M\), in conjunction with divergence theorem and Gauss-Bonnet formula, we can proceed as [6]pp. 11-12 to get the proposition:

**Theorem 4.2.** Let \(M\) and \(N\) are both compact oriented Riemannian surfaces, \(\psi\) is a harmonic spinor field along \(\phi : M \to N\). If \(|\pi_1^+(\psi)|\) is not identically zero, then
\[
(4.19) \quad \sum_{p \in M, |\pi_1^+(\psi)| (p) = 0} n_p^+ = g_M - 1 - \deg(\phi)(2g_N - 2).
\]

If \(|\pi_0^+ (\psi)|\) is not identically zero, then
\[
(4.20) \quad \sum_{p \in M, |\pi_0^+(\psi)| (p) = 0} m_p^+ = g_M - 1 + \deg(\phi)(2g_N - 2).
\]

If \(|\pi_1^- (\psi)|\) is not identically zero, then
\[
(4.21) \quad \sum_{p \in M, |\pi_1^-(\psi)| (p) = 0} n_p^- = g_M - 1 + \deg(\phi)(2g_N - 2).
\]

If \(|\pi_0^- (\psi)|\) is not identically zero, then
\[
(4.22) \quad \sum_{p \in M, |\pi_0^-(\psi)| (p) = 0} m_p^- = g_M - 1 - \deg(\phi)(2g_N - 2).
\]

Here \(n_p^+, m_p^+, n_p^-, m_p^-\) are respectively the order of \(|\pi_1^+(\psi)|\), \(|\pi_0^+ (\psi)|\), \(|\pi_1^- (\psi)|\), \(|\pi_0^- (\psi)|\) at \(p\); \(\deg(\phi)\) denotes the degree of mapping; \(g_M\) and \(g_N\) are genus of \(M\) and \(N\), respectively.
5. Proof of Main Theorems

In conjunction with Proposition 2.1 and Theorem 4.2, it is not difficult to obtain:

**Theorem 5.1.** $M$ and $N$ are both compact oriented Riemannian surfaces, and $(\phi, \psi)$ is a Dirac-harmonic map from $M$ to $N$. If $g_M = 0$ or $|g_M - 1| < |\deg(\phi)||2g_N - 2|$, then $\phi$ has to be a harmonic map.

**Proof.** If $g_M = 0$ or $|g_M - 1| < |\deg(\phi)||2g_N - 2|$, then $g_M - 1 - \deg(\phi)(2g_N - 2) < 0$ or $g_M - 1 + \deg(\phi)(2g_N - 2) < 0$. Hence from Theorem 4.2, either $\pi_1^+(\psi) = \pi_0^-(\psi) = 0$ or $\pi_0^+(\psi) = \pi_1^-(\psi) = 0$ could be obtained. If $\pi_1^+(\psi) = \pi_0^-(\psi) = 0$, then there exist smooth functions $f$ and $g$, such that

$$\psi = f\psi^+ \otimes \frac{\partial}{\partial \bar{w}} + g\psi^- \otimes \frac{\partial}{\partial w},$$

where the definition of $\psi^+, \psi^-, \frac{\partial}{\partial \bar{w}}, \frac{\partial}{\partial w}$ is similar to Section 3. Hence by (2.4),

$$\tau(\phi) = -\frac{1}{2}(f\psi^+, e_\alpha \cdot f\psi^+) R^N_{\bar{w}w} \frac{\partial}{\partial \bar{w}} \phi_*(e_\alpha) - \frac{1}{2}(g\psi^-, e_\alpha \cdot g\psi^-) R^N_{\bar{w}w} \frac{\partial}{\partial \bar{w}} \phi_*(e_\alpha)$$

$$- \frac{1}{2}(f\psi^+, e_\alpha \cdot g\psi^-) R^N_{\bar{w}w} \frac{\partial}{\partial \bar{w}} \phi_*(e_\alpha) - \frac{1}{2}(g\psi^-, e_\alpha \cdot f\psi^+) R^N_{\bar{w}w} \frac{\partial}{\partial \bar{w}} \phi_*(e_\alpha) = 0.$$

So $\phi$ is a harmonic map. When $\pi_0^+(\psi) = \pi_1^-(\psi) = 0$, the proof is similar.

□

It is well known that $S^2$ is biholomorphically isomorphic to $\mathbb{C}^* = \mathbb{C} \cup \{\infty\}$; hence in the following text we identify $S^2$ and $\mathbb{C}^*$. Let $h = \lambda(z)|dz|^2$ be a metric on $\mathbb{C}^*$; denote $\tilde{z} = z^{-1}$, then

$$\lambda(z)|dz|^2 = \lambda(\tilde{z}(z))|\tilde{z}|^{-4}d\tilde{z}^2;$$

hence $\lambda(\tilde{z}(z))|\tilde{z}|^{-4}$ is regular at $\tilde{z} = 0$; then there exists a constant $c > 0$, such that

$$\lim_{z \to \infty} \lambda(z)|z|^4 = c.$$

And the definition of $e_1, e_2, \psi^+, \psi^-$ is similar to Section 3.

**Theorem 5.2.** If $M = S^2 = \mathbb{C}^*$ equipped with metric $h = \lambda(z)|dz|^2$ and $N = S^2$ equipped with arbitrary metric, $(\phi, \psi)$ is a nontrivial Dirac-harmonic map from $M$ to $N$, then $\phi$ has to be holomorphic or anti-holomorphic, $\psi$ could be written in the form

$$\psi = e_\alpha \cdot \Psi \otimes \phi_*(e_\alpha);$$

and there exist two meromorphic function $u_1, u_2$ on $\mathbb{C}^*$ such that

$$\Psi = \bar{u}_1 \lambda^\frac{1}{2} \psi^+ + u_2 \lambda^\frac{1}{2} \psi^-;$$

if $u_i (i = 1 \text{ or } 2)$ has a pole of order $k$ at $z_0 \in \mathbb{C}$, then $|d\phi|(z_0) = 0$ and the order of $|d\phi|$ at $z_0$ is no less than $k$; if $\infty$ is a pole of order $k \geq 2$, then $|d\phi|(\infty) = 0$ and the order of $|d\phi|$ at $\infty$ is no less than $k - 1$. And vice versa.
Proof. By Theorem 5.1, \( \phi \) has to be a harmonic map. It is well known that when \( \deg \phi = 0 \), \( \phi \) is a constant mapping; when \( \deg \phi \geq 1 \), \( \phi \) is holomorphic; and when \( \deg \phi \leq -1 \), \( \phi \) is anti-holomorphic (cf. [6] pp. 11-12). From Theorem 4.2, when \( \deg \phi = 0 \), \( |\pi^-_1(\psi)|, |\pi^+_0(\psi)|, |\pi^-_0(\psi)|, |\pi^+_1(\psi)| \) are all identically zero, hence \( \psi = 0 \); it is a trivial solution of (2.3)-(2.4). When \( \deg \phi \geq 1 \), we have \( |\pi^+_0(\psi)| = |\pi^-_1(\psi)| = 0 \); since \( \phi \) is holomorphic, \( \phi^* e_1 - \sqrt{-1} \phi^* e_2 \in T^{1,0} N, \phi^* e_1 + \sqrt{-1} \phi^* e_2 \in T^{0,1} N \), there exist two functions \( f, g \) (possibly with isolated singularities), such that

\[
\psi = f \psi^+ \otimes (\phi^*(e_1) - \sqrt{-1} \phi^*(e_2)) + g \psi^- \otimes (\phi^*(e_1) + \sqrt{-1} \phi^*(e_2)).
\]

From (3.6), it is easy to obtain

\[
\psi = e_{\alpha} \cdot \Psi \otimes \phi^*(e_{\alpha});
\]
where \( \Psi = g \psi^+ - f \psi^- \). When \( \deg \phi \leq -1 \), similarly we can construct a spinor \( \Psi \) (possibly with isolated singularities) satisfying (5.6).

By [1] Proposition 2.2, \( \Psi \) is a twistor spinor, i.e.

\[
\nabla_v \Psi + \frac{1}{2} v \cdot \bar{\theta} \Psi = 0
\]

for any \( v \in T_p S^2 \), where \( p \) is an arbitrary regular point of \( \Psi \). If \( \Psi \in \Gamma(\Sigma^+ M) \), then from (3.1), (5.7) is equivalent to

\[
\nabla_{\partial z} \Psi = 0.
\]

Denote

\[
\Psi_0^+ = \lambda^{\frac{1}{2}} \psi^+,
\]
then from

\[
\nabla_{\partial z} \psi^+ = \frac{1}{2} (\nabla_{\partial z} e_1, e_2) e_1 \cdot e_2 \cdot \psi^+
= -\frac{1}{4} \sqrt{-1} \lambda^{\frac{1}{2}} (\nabla e_{1-\sqrt{-1}} e_2) \psi^+
= -\frac{1}{4} \sqrt{-1} \lambda^{\frac{1}{2}} \left( \frac{\partial (\lambda^{-\frac{1}{2}})}{\partial y} + \sqrt{-1} \frac{\partial (\lambda^{-\frac{1}{2}})}{\partial x} \right) \psi^+
= -\frac{1}{4} \frac{\partial \log \lambda}{\partial z} \psi^+;
\]

we have

\[
\nabla_{\partial z} \Psi_0^+ = \frac{1}{4} \frac{\partial \log \lambda}{\partial z} \lambda^{\frac{1}{2}} \psi^+ - \frac{1}{4} \frac{\partial \log \lambda}{\partial z} \lambda^{\frac{1}{2}} \psi^+ = 0.
\]
Let \( u_1 \) be a function on \( C^* \) (possibly with isolated singularities) such that

\[
\Psi = \bar{u}_1 \Phi_0^+,
\]
then it is easy to obtain \( \frac{\partial u_1}{\partial z} = 0 \). Similarly, if \( \Psi \in \Gamma(\Sigma^- M) \) is a twistor spinor, then we could obtain

\[
\Psi = u_2 \Phi_0^-,
\]
where $u_2$ is a meromorphic function and $\Phi_0^- = \lambda^4 \psi^-$. Thereby (5.4) follows. The last statement is followed from $\lambda > 0$ on $\mathbb{C}$ and (5.2).

On the other hand, let $\psi = \epsilon_a \cdot \Psi \otimes \phi_\ast (e_a)$, where $\phi$ is holomorphic or antiholomorphic, and $\Psi$ is a twistor spinor (possibly with isolated singularities) satisfying (5.4). Denote by $z_1, \ldots, z_l \in \mathbb{C}^\ast$ the singularities of $\Psi$, then it is easily to check that $(\phi, \psi)$ satisfies (2.3) and (2.4) on $\mathbb{C}^\ast - \{ z_1, \ldots, z_l \}$, and by the assumption on the poles of $u_i$ and the zeros of $|d\phi|, |\psi|$ is bounded on $\mathbb{C}^\ast - \{ z_1, \ldots, z_l \}$. Hence the energy of $(\phi, \psi)$ on $\mathbb{C}^\ast$

$$E(\phi, \psi, \mathbb{C}^\ast) = \int_{\mathbb{C}^\ast} (|d\phi|^2 + |\psi|^4)$$

is finite. The removable singularity theorem (see [1]) yields that $(\phi, \psi)$ is a Dirac-harmonic map from $M$ to $N$.

\[ \square \]

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