Cold Fermi gases: a new perspective on spin-charge separation

Corinna Kollath\textsuperscript{1} and Ulrich Schollwöck\textsuperscript{2}

\textsuperscript{1} DPMC-MaNEP, University of Geneva, 24 Quai Ernest-Ansermet, CH-1211 Geneva, Switzerland
\textsuperscript{2} Institute for Theoretical Physics C, RWTH Aachen, D-52056 Aachen, Germany
E-mail: kollath@physics.unige.ch

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Abstract. In recent years, ultracold bosonic and fermionic atom gases in optical lattices have allowed us to realize experimentally phenomena originally predicted or observed in condensed matter physics with unprecedented control and purity. Among others, proposals have been made to demonstrate a key feature of one dimensional (1D) quantum systems, the so-called spin-charge separation, which has been quite elusive in condensed matter systems. However, as we want to discuss in this paper using the time-dependent density-matrix renormalization group (DMRG), beyond a mere demonstration of spin-charge separation cold atom gases will also shed light on previously inaccessible aspects of this phenomenon. Moreover, the previous proposals for experimental realizations will be revisited taking into account the modelling of an external magnetic field.
1. Introduction

Increasingly, the dimensionality of quantum systems has been the focus of theoretical and experimental research. Theoretically, the insight that the physics of low-dimensional quantum systems might be qualitatively different from that of three dimensional (3D) quantum systems has come from various directions. On the one hand, it has been known since the work of Hohenberg [1] and Mermin and Wagner [2] that superfluidity and ordered phases with a multicomponent order parameter are excluded in 1D and 2D quantum systems at any finite temperature. Thermal phase transitions are replaced by quantum phase transitions at zero temperature. This phenomenon is due to the very strong thermal and quantum fluctuations in low-dimensional quantum systems. In field theoretical language, this observation corresponds to the fact that field theories for these systems are at or below the lower critical dimension. Experimentally, the preparation of condensed matter systems of reduced dimensionality at very low temperature has always been plagued by the presence of parasitic interactions of higher dimensionality: 1D systems are typically obtained as almost non-interacting arrays of parallel 1D systems ('tubes') in a 3D bulk. The interactions between tubes is never completely absent and changes the effective dimensionality of the system as soon as temperature is sufficiently low for these interactions to survive thermal fluctuations. Moreover, in condensed matter systems the determination of the precise Hamiltonian is a very hard task such that the values of interactions are never exactly known. Control of the microscopic interactions is very limited. The recent progress in the area of ultracold atomic gases, in particular ultracold gases in optical lattices, has opened the possibility to study strongly correlated systems in a new context. In these systems, the microscopic parameters of the Hamiltonian can be calculated virtually exactly and can be controlled to a high precision: optical lattices allow us to strongly vary the kinetic energy of the gas, Feshbach resonances allow us to strongly vary its interaction energy. Either way, the ratio of the two energetic contributions can be varied over orders of magnitude. This has allowed us, in a by now classic experiment [3] on cold bosons, to freely tune between the two predicted quantum phases, a superfluid and a Mott insulating phase, which are separated by a key quantum phase transition, the so-called Mott transition. It is interesting to note that the peculiarities of the experimental setups, in particular the presence of a confining trap potential, do lead to the observation of interesting new aspects: while one does not observe a true phase transition in the statistical physics sense, new
patterns of coexistence emerge (e.g. the famous ‘wedding cake’ structure). Unlike in condensed matter physics, it is relatively straightforward to map the dynamics of the quantum system. Furthermore, the crossover from three, to quasi-2D and quasi-1D systems can also be investigated in a controlled fashion using anisotropic optical lattices and the behaviour in different dimensions can be compared directly [4, 5].

In the meantime, cold Fermi gases in optical lattices are also an object of intense study, where the focus has been both on $^40$K and $^6$Li gases [6, 7]. Here, a key problem is that it is much more difficult to approach truly low temperatures. In cold Fermi gases, an ‘atomic quantum wire’ configuration in an array of thousands of parallel atom waveguides was realized by the application of two strong perpendicular laser beams [8].

Beyond the pre-eminence of quantum phase transitions in low dimensions, a key feature of 1D quantum many-body systems is their universality. Conventionally, universality is about the observation that a wide range of physical systems shares the same critical exponents close to phase transitions provided they agree in the dimensionality of the system and the symmetry associated with the order parameter of the transition. In one dimension, universality means that independent of their bosonic or fermionic nature, the low-energy behaviour of such systems can be described by the Luttinger liquid theory [9]–[11], which predicts that there exist a few so-called Luttinger liquid parameters that lead to quantitatively identical low-energy behaviour in all physical quantities in different physical systems provided these few parameters take identical values. Perhaps the most fascinating universal phenomenon in 1D systems is that of spin-charge separation: in an electronic Luttinger liquid—upon the insertion of a single electron—the excitations of spin and charge completely decouple and propagate with different velocities. This is in striking contrast to Fermi liquids, where the elementary quasiparticle excitations carry both charge and spin, moving at the Fermi velocity. The quest for an experimental demonstration of this phenomenon has been an important topic of condensed matter physics for more than 20 years. Attempts have focused on photoemission from 1D metallic wires on surfaces (for a different view, see [12]), on 1D organic wires [13] and on carbon nanotubes [14]. Truly convincing evidence was obtained via tunnelling experiments in quantum wires in semiconductors, where the two branches of the spectral functions associated with spin-charge separation were observed [15]. However, there have been proposals to find this evidence also in experiments on cold Fermi gases [16]–[19]. These proposals have focused on a proof of principle. In this paper, we want to discuss beyond a mere demonstration what new aspects and perspectives of spin-charge separation so far unavailable in condensed matter systems can be observed in cold atoms.

The paper is organized as follows: first, we want to recall briefly the salient points of the theoretical description of spin-charge separation (section 2) and the experimental proposal how to demonstrate spin-charge separation in cold atom gases (section 3). We then move on to discuss the new aspects of spin-charge separation in this type of experiment (section 4).

### 2. Spin-charge separation: the theoretical basics

To get a first understanding of the phenomenon of spin-charge separation, let us consider the Hubbard model on a 1D lattice,

\[
\hat{H} = -J \sum_{i,\sigma} (c_{i,\sigma}^\dagger c_{i+1,\sigma} + \text{h.c.}) + U \sum_i n_{i,\uparrow} n_{i,\downarrow} + \epsilon \sum_{i,\sigma} n_{i,\sigma}.
\]
The next crucial step is to introduce (bosonic or fermionic) field operators $\psi^\dagger(x)$, $\psi(x)$ with $n(x) = \sum_{k=1}^{N} \delta(x - x_k)$ for particles at positions $x_k$ is then encoded by some monotonically increasing field $\Phi(x)$ which equals multiples of $2\pi$ at particle positions: $\Phi(x_k) = 2\pi k$. Using the identities $\delta(f(x)) = \sum_{x: \ker f} \delta(x - x_k)/|f'(x_k)|$ and $\sum_x e^{ipx} = 2\pi \sum_k \delta(x - 2\pi k)$, where $p$ and $k$ run over all integers, one finds $n(x) = (\nabla \Phi(x)/2\pi) \sum_p e^{ip\Phi(x)}$. It is now customary to express particle positions with respect to their ‘crystalline’ rest positions $x_k^0 = n_0 k$, where $n_0 = N/L$ is the mean density, by introducing a field $2\phi(x) = 2\pi n_0 x - \Phi(x)$, leading to the starting expression

$$n(x) = (n_0 - \nabla \phi(x)/\pi) \sum_p e^{2ip(n_0x - \phi(x))}. \quad (2)$$

The next crucial step is to introduce (bosonic or fermionic) field operators $\psi^\dagger(x)$, $\psi(x)$ with $n(x) = \psi^\dagger(x)\psi(x)$. One makes an amplitude-phase ansatz $\psi^\dagger(x) = \sqrt{n(x)} e^{-i\theta(x)}$, introducing a second field $\theta(x)$. The field operators of course have to obey the usual commutator rules, e.g. in the bosonic case $[\psi(x), \psi^\dagger(x')] = \delta(x - x')$. After some algebra, one finds that this implies a commutator rule $[\nabla \phi(x)/\pi, \theta(x')] = -i\delta(x - x')$, or upon partial integration, $[\phi(x), \nabla \theta(x')/\pi] = i\delta(x - x')$, which is just the commutator relation between a field and its canonical momentum. One can therefore identify the canonical momentum of the field $\phi(x)$ as $\Pi(x) = \nabla \theta(x)/\pi$. Then the bosonic and fermionic field creation operators read

$$\psi_B^\dagger(x) = \sqrt{n_0 - \nabla \phi(x)/\pi} \sum_p e^{ip(n_0x - \phi(x))} e^{-i\theta(x)}. \quad (3)$$

$$\psi_F^\dagger(x) = \sqrt{n_0 - \nabla \phi(x)/\pi} \sum_p e^{i(p+1)(n_0x - \phi(x))} e^{-i\theta(x)}. \quad (4)$$

The ‘+1’ in the fermionic expression ensures that the bosonic commutator relation is turned into the proper fermionic anticommutator relation. It is important to note that the small-$p$ terms are sensitive to slow variations in the fields, the large-$p$ terms to fast variations. If one is interested in the low-energy limit, corresponding to slowly varying fields, one neglects all but the smallest-$p$ contributions.

With equations (2)–(4), we have a translation dictionary of Hamiltonians expressed in field operators and densities into the new fields, which obey bosonic commutator relations independent
of the original statistics—hence the name bosonization. What really matters, however, is that Hamiltonians become generically quadratic in the new fields (and thus easy to solve), whereas interactions in the old field operators are typically quartic (which is hard to solve). For example, if one considers the low-energy limit by ignoring all but the \( p = 0 \) term, a contact density–density interaction translates as \( \int dx \, n(x)^2 \to \int (\nabla \phi(x))^2 \). A kinetic energy term would translate as \( \int dx \, \psi^\dagger \nabla \psi \to \int dx \, n_0 (\nabla \theta)^2 \), if we consider only the slowest varying terms, i.e. the low-energy limit. Collecting all terms and using the canonical momentum, the most general low-energy Hamiltonian reads

\[
H = \frac{v}{2\pi} \int dx \left[ K (\pi \Pi(x))^2 + \frac{1}{K} (\nabla \phi(x))^2 \right],
\]

where the two coefficients of the quadratic terms are rearranged as \( vK \) and \( v/K \), introducing the velocity \( v \) and the Luttinger liquid parameter \( K \), which completely characterize low-energy physics both for fermionic and bosonic 1D systems.

To encode fermionic spin, one carries out the above procedure for each spin species \( \uparrow, \downarrow \) separately, introducing bosonic fields \( \phi_{\uparrow, \downarrow}(x) \). After working out kinetic terms and interactions (now of the form \( \int dx \, n_\uparrow(x) n_\downarrow(x) \)), it is easy to see that upon introducing the combinations \( \phi_\rho = (\phi_\uparrow + \phi_\downarrow)/\sqrt{2} \) and \( \phi_\sigma = (\phi_\uparrow - \phi_\downarrow)/\sqrt{2} \), encoding the charge and spin density respectively, spin and charge degrees of freedom decouple. One obtains two free bosonic Hamiltonians,

\[
H_\nu = \frac{v_\nu}{2\pi} \int dx \left[ K_\nu (\pi \Pi(x))^2 + \frac{1}{K_\nu} (\nabla \phi_\nu(x))^2 \right],
\]

with \( \nu = \rho, \sigma \), velocities \( v_\nu \) and Luttinger liquid parameters \( K_\nu \). It is crucial to note that both spin and charge sectors of the Hamiltonian decouple completely, leading to an independent existence of spin and charge degrees of freedom. The excitations are called spinons and holons, respectively. In general, they will move at different velocities, leading to the separation of spin and charge degrees of freedom. This is the phenomenon referred to as spin-charge separation. Its special importance derives from the observation that it has no counterpart in higher dimensional quantum systems.

An important question we still have to address is whether all of low-energy physics is captured adequately by retaining the very lowest \( p \)-terms only for the case of interest to us here, which is fermions with spin 1/2. Consider again first spinless fermions. At density \( n_0 \), the Fermi momentum is \( k_F = n_0 \pi \). Low-energy excitations will have momenta around \( \pm k_F \), which one customarily refers to as left and right movers. Considering (4), the terms \( p = 0, -1 \) pick out just these movers, and one may introduce at this order left and right mover field operators and densities as \( \psi(x) = \psi_\uparrow(x) e^{-ik_F x} + \psi_\downarrow(x) e^{ik_F x} \) and \( n(x) = n_\uparrow(x) + n_\downarrow(x) \). The densities can be related back to our original fields as \( \nabla \phi(x) = -\pi [\rho_\uparrow(x) + \rho_\downarrow(x)] \) and \( \nabla \theta(x) = \pi [\rho_\uparrow(x) - \rho_\downarrow(x)] \). This discussion generalizes immediately to fermions with spin, i.e. we introduce left movers with spin-up, spin-down, etc.

Expressing the contact interaction between spin-up and spin-down fermions in terms of spinful movers, it turns out that a term appears which is not yet captured by our previous purely quadratic Hamiltonian. It is due to the physical process of a left and a right mover of different spin-state scattering off each other, while exchanging spin, such as \( \psi_\uparrow^\dagger \psi_\downarrow \psi_\uparrow^\dagger \psi_\downarrow \).
Expressing this type of scattering process in terms of the bosonic fields, one finds a contribution proportional to

$$\int dx \sum_{\sigma = \uparrow, \downarrow} e^{-2i\phi_{\sigma}(x)} e^{2i\phi_{\sigma}(x)} + \text{h.c.} \sim \int dx \cos \sqrt{8} \phi_{\sigma}(x). \quad (7)$$

Summarizing the bosonization procedure, the terms of the Hubbard Hamiltonian can be written as

$$H = H_\rho + H_\sigma + g \int dx \cos(\sqrt{8} \phi_{\sigma}(x)), \quad (8)$$

where $g \propto U$ (the exact expression will not matter in the following). At half-filling, there is an additional contribution to the charge sector due to Umklapp processes, because now the scattering of two left movers into two right movers is allowed, transferring momentum to the lattice. With a similar calculation as in the preceding paragraph, one finds

$$H \rightarrow H - g \int dx \cos(\sqrt{8} \phi_{\rho}(x)). \quad (9)$$

Let us now analyse this Hamiltonian. One may show that neglecting the cos-terms, the correlators of the two free Hamiltonians follow power laws; both the spin and charge sector are critical (gapless), with long-ranged fluctuations. This picture may be modified by the additional cosine terms, because their energy would be minimized by pinning the $\phi$-fields in a minimum of the cosine term. Upon a renormalization group analysis, it turns out that these cosine terms are relevant for $K_\nu < 1$, marginally irrelevant for $K_\nu = 1$, irrelevant for $K_\nu > 1$. When it is relevant (for $K_\nu < 1$), the field of the corresponding sector will be essentially pinned with only short-ranged fluctuations. It therefore develops an excitation gap and correlators will follow exponential laws. For $K_\nu = 1$ there will be logarithmic corrections to criticality, whereas for $K_\nu > 1$ on sufficiently long length scales (sufficiently low energies) these terms may be neglected completely.

In the isotropic ($SU(2)$ rotation-invariant) Hubbard model as discussed here, $K_\sigma = 1$. The cosine term in the spin sector will therefore be marginally irrelevant. The determination of the velocities and $K_\rho$ is more involved since due to the various approximations the values that would be read off directly from the bosonized Hamiltonian are highly approximative only. If we restrict our attention to the repulsive Hubbard model, the velocities and $K_\rho$ can be determined exactly from the Bethe ansatz [20]. $K_\rho$ takes values $1/2 \leq K_\rho \leq 1$. The value $K_\rho = 1$ is taken at arbitrary filling in the non-interacting case, while the value $K_\rho = 1/2$ is taken in the interacting case in the limits of $U \rightarrow \infty$, or $n = N/L = 1$, or $n \rightarrow 0$. This means that apart from the interaction free case, the cosine term in the charge sector will always be relevant if it appears, i.e. at half-filling.

In agreement with the exact determination of the ground state of the 1D Hubbard model by the Bethe ansatz [20], bosonization therefore shows that two different phases occur: a liquid phase and a Mott-insulating phase. The Mott-insulating phase exists at half-filling for arbitrary repulsion (i.e. any $K_\nu < 1$) between the fermions, otherwise it is in the liquid phase. Away from half-filling, the charge liquid phase occurs for arbitrary interactions. Therefore only the commensurate–incommensurate phase transition in the charge sector exists for repulsive interactions. The lowest excitations in the system are two collective modes: the spin and the charge mode. Both modes
are gapless in the liquid phase, whereas in the Mott-insulating phase the charge mode opens up a gap, while the spin mode remains gapless.

If we introduce the useful exponent
\[ \gamma_\nu = (K_\nu + K_\nu^{-1} - 2)/8, \tag{10} \]
we can note that \( \gamma_\sigma = 0 \). At the same time, \( 0 \leq \gamma_\rho \leq 1/16 \).

It is now possible to calculate Green’s functions and derived quantities for this type of Hamiltonian. Essentially, due to the nature of available precision measurement techniques (photoemission, tunnelling) in condensed matter setups, all successfully realized or proposed experiments probe the spectral density
\[ \rho(k, \omega) \sim (\omega - v_\sigma k)^{\gamma_\sigma - 1/2} |\omega - v_\rho k|^{\gamma_\rho/2 - 1/2} (\omega + v_\rho k)^{\gamma_\rho/2}. \tag{11} \]

A signature of the spin-charge separation can be seen in the single particle spectral function by the presence of two power-law singularities at \( \omega = v_\sigma k \) and \( \omega = v_\rho k \) for the spin and charge excitations respectively \([10, 21, 22]\). For \( v_\sigma \rightarrow v_\rho \), they coalesce but still stay power-law singularities. By comparison, in a Fermi liquid in higher dimensions, one would find in the low-energy limit a single peak corresponding to a quasiparticle of energy \( \omega \), momentum \( k \), and a finite lifetime \( \tau \) which would be of Lorentzian shape
\[ \rho(k, \omega) \sim \frac{\tau^{-1}}{(\omega - v_F k)^2 + \tau^{-2}}, \tag{12} \]
where \( v_F \) is the Fermi velocity and the lifetime \( \tau \) diverges for \( \omega \rightarrow 0 \). In the low-energy limit, the quasiparticle peak therefore turns into a \( \delta \)-peak. Note that in addition to the quasiparticle excitations collective spin and charge modes exist in higher dimensions.

The most successful condensed matter experiment on spin-charge separation to date \([15]\) studies the tunnelling between two quantum wires at a voltage bias exposed to a perpendicular magnetic field, while the voltage bias defines the energy \( \hbar \omega \) of excitations transferred from one wire to the other, the magnetic field defines a momentum \( \hbar k \). Measuring tunnelling rates as a function of \( k \) and \( \omega \) gives essentially access to \( \rho(k, \omega) \), allowing to plot the single-particle dispersion. Can we learn more from a cold atom setup? As all the above analytics has been in the context of the lowest-energy excitations, what will remain relevant in the cold atoms, where, as we will see, it is mandatory to go to higher energy excitations to get experimentally observable signals?

### 3. Spin-charge separation in cold Fermi gases

Despite the fact that ultracold atom gases are extremely well modelled by the Hubbard model, there are some further advantages, but also disadvantages to this realization as opposed to a condensed matter one. In a realization with cold gases, the spin degrees of freedom are represented by two different hyperfine levels of the fermionic atoms which are addressed individually by external laser fields. Similar to bosons in an optical lattice \([23]\), the ratio \( u = U/J \) between the on-site repulsion \( U \) and the hopping \( J \) can be changed experimentally by varying the depth \( V_0 \) of the optical lattice or by means of a Feshbach resonance. The chemical potential term
∑_i ϵ_i n_i, ϵ = −μ, is modified to be site-dependent, i.e. ∑_i ϵ_i n_i, in order to describe the harmonic confinement potential which we can model by ϵ_i = −μ + V_t(i − L/2 + 0.5)^2 E_r, where E_r is the recoil energy, and V_t adjusts the strength of the confining potential. To model the situation of a condensed matter system without external magnetic field, an equal number of spin-up and spin-down fermions N^↑ = N^↓ = N/2 is assumed. This ratio can be prepared in the experiments with a high accuracy by the application of a sequence of two radio-frequency pulses. (In [6] a mixture with 50 ± 4% of the F = 9/2 and m_F = −9/2 state and the F = 9/2 and m_F = −7/2 state of ^40K atoms has been prepared.) In such systems two different excitations can be prepared, single-particle excitations and collective excitations like density perturbations. Experimentally density perturbation can be generated for example by perturbing the system locally by time-dependent potentials, i.e. by switching on and off focused potentials coupling to the local density, generated by a blue- or red-detuned laser beam tightly focused perpendicular to an array of atomic wires, which generates locally repulsive or attractive potentials for the atoms in the wires. In principle, even though up to now difficult in the current realizations, it is possible to use hyperfine level-selective (or, ‘spin’ selective) potentials by choosing appropriate hyperfine states. We therefore generalize ϵ_i → ϵ_i^↑, ϵ_i^↓. A typical local laser-induced potential would be of Gaussian form

\[ f(j, t) = W_t \exp^{-\left(j-j_0\right)^2/(2\sigma^2)}, \] (13)

with a time-dependent strength W_t and a waist size of σ ≈ 2. To create a localized spin- and charge density perturbation the local potentials are switched on slowly, such that the system is in the ground state of the system with the additional potential at time t = 0. At t = 0 the localized potential is removed abruptly which generates the density perturbation. To generate a single particle excitation, which would correspond most closely to the condensed matter point of view, is a stronger challenge for experiments. This could be realized by transferring one particle of either of the two hyperfine states into a third non-interacting hyperfine state using a weak radiofrequency field.

The result of this local ‘pumping’ now has to be probed elsewhere in the system. Techniques for local ‘spin’-selective density probes of some finite resolution, e.g. ten lattice sites, are available, with the major limitation being the waist of the laser beam involved. For example, at a certain time an additional trap could be applied to freeze the number of particles in a certain region, while all other particles are released. Then the number of particles left in the region could be imaged spin-selectively using a magnetic field gradient. As opposed to the condensed matter setup, these measurements will not directly probe the spectral function, but the real time evolution of the density. Strictly speaking, real evidence for spin-charge separation can only be obtained using an excitation with single-particle character, since generally collective excitations of spin and charge move in any dimension with distinct velocities. Since experimentally the creation of a single particle excitation is involved in cold atoms, we propose to use the possibility to tune between one and higher dimension experimentally to check the single particle character of the excitation. If the proposed experiment is repeated in higher dimensions no separation of spin and charge should be seen. Instead the excitation should keep its quasi-particle character for a long time.

A key problem in the systems of cold atoms is given by the fact that one is observing on a number of 1D wires at the same time that may not have identical filling. As ν depend on density, this will lead to a smearing out of the experimental signal. It was therefore proposed [17] to exploit the coexistence of both a conducting and a Mott-insulating phase in a harmonic trap potential to create the perturbation relatively close to the Mott-insulating phase and to observe
the reflection of the charge perturbation by the Mott insulator and the transmission of the spin perturbation through it. This is sketched in figure 1. By this the charge density is the same in most of the wires and the problem of the presence of several wires is turned into an advantage, since the signal is enhanced.

In the first proposal to observe spin-charge separation in ultracold atoms Recati et al [16] applied a hydrodynamic description of the Luttinger liquid and used the local density approximation to account for the presence of a harmonic confining potential. Within this analysis, only relatively weak and broad perturbations can be treated. In practice, the perturbations due to an external laser field are quite strong, typically of the order of the recoil energy \( E_r \) and thus clearly require a non-perturbative treatment. In addition, the effect of boundaries, where the local density approximation breaks down, are of crucial importance, in the setup described above. Moreover, with typically less than 100 atoms per atomic wire [8], experimentally observable effects require to use stronger and more localized perturbations. For a quantitative description of spin-charge separation in 1D cold Fermi gases, it is thus necessary to use a microscopic description and properly treat the inhomogeneous case with realistic system sizes. One also has to wonder whether the low-energy long-wavelength predictions of bosonization remain meaningful at all. At the same time, the good tunability in time gives direct access to the real-time dynamics of strongly correlated systems, a subject hardly studied so far. The theoretical treatment of the real-time evolution of strongly correlated systems is very challenging; the most powerful method currently available is the adaptive time-dependent density-matrix renormalization group (adaptive t-DMRG) [24] which is an efficient implementation of Vidal’s time-evolving block-decimation (TEBD) algorithm [25] in the DMRG framework [26]. The adaptive t-DMRG has previously been applied to study density perturbations in bosonic 1D condensates over a large range of interaction strengths [27]. For detailed descriptions of this method, we refer to [24, 28, 29]; a detailed error analysis is shown in Gobert et al [30]. In our calculations here the length of the chains was chosen up to \( L = 128 \) sites, keeping of the order of several hundred
DMRG states. In the present calculation, where no exact solution is available for comparison, an error analysis has been carried out by varying the number of DMRG states as well as the size of the small finite-time steps used, as the method would be exact in the limit of infinitely many states and infinitesimally small time steps. It turns out that the maximal deviations between the charge and spin density results for the different number of DMRG states agreed up to $10^{-3}$.

4. New aspects of spin-charge separation

4.1. Separation in real versus momentum space

The power-law singularities of the single particle spectral density suggest that spin-charge separation is a phenomenon best visible in frequency and momentum space and might not have a very clear counterpart in time and real space. However, this would only be true if we had $\delta$-like singularities; as the spin and charge excitations are only partially localized in frequency and momentum space, they will also delocalize only partially in time and real space. Our starting perturbation—the single particle excitation—is localized in real space. Hence, delocalization will take the form of a spreading out of our excitation and a corresponding decay of its amplitude.

Analytically, we face—among other things—the difficulty of an adequate description of the wavefunction at $t = 0$. This problem is not present in our numerical analysis, where the time-evolution of a homogeneous system is determined in which one particle with spin-up is added at site $j$ at time $t = 0$ to the ground state of the Hubbard model as determined numerically by DMRG. In figure 2, snapshots of the real-time evolution of the density distributions under the Hubbard Hamiltonian (1) are shown. Obviously, spin-charge separation clearly is a phenomenon extremely well visible also in time and real space. For small times $t = 0.2$ in figure 2 the added particle results in a peak in the total density, the spin-up density, and in the spin density at site $j$. After some time each of the peaks in the spin and the charge density split up into two peaks which move into opposite direction whereas the peak in the spin-up density performs a complicated time-evolution induced by the interaction between the spin-up and spin-down fermions. At longer times ($t = 7.2$ in figure 2) a clear separation of the spin and the charge density excitations in real space can be seen. At the same time, the numerics indicates a stronger spread in the spin perturbation.
Figure 3. Decay of the amplitude of the charge density perturbation (shown as the maximum density of the perturbation) for different strength of the initial perturbation in a system of $L = 72$ sites with $N = 56$ particles, and an interaction strength of $u = 2$ in the presence of a spin perturbation (■) and without the presence of a spin perturbation (●). In the left figure, the initial perturbation increases density, in the right figure, it reduces density. The small oscillations result from the discreteness of the underlying lattice, since the maximum value of the density does not always fall on to a lattice site. In the presence of a spin perturbation the decay of the amplitude is slower. Time is measure in units of $\hbar/J$.

If we assume that in momentum space the perturbation corresponds to a relatively compact object, we may analyse group and phase velocities. As it turns out, the relationships $\omega = v_p k$ for low energies would imply that the propagation is free of dispersion. Spread must therefore come from (i) the power-laws in the singularities themselves which indicate a power-law decay of excitations in a Luttinger liquid or (ii) a strong effect of the local density on $v_p$, which we will show in the following. On the short time and length scales considered for the cold atom setups, we expect that (ii) is much more relevant, even if one may speculate that—as the holon exponent is more negative and the spectral function more singular—this implies that spread and amplitude decay are stronger for the spin part of the perturbation.

4.2. Charge and spin perturbations away from low energies

A big advantage of the cold atom setup is that it is possible to excite charge and spin degrees of freedom separately, e.g. by creating corresponding density perturbations. These perturbations have the additional convenience that the amplitude and the width of the perturbation, which are fixed in the case of a single particle excitation, can be controlled (and adapted to experimental needs). Thereby cold atoms open the possibility to study the properties of the characteristic excitations predicted for these systems in a well controllable fashion. The need for considering spin and charge perturbations separately becomes clear, if one compares the time-evolution for a system, where (i) both spin and charge perturbations are created at the same time and space and (ii) only a perturbation in charge is created. In figure 3, the decay of the charge amplitude is shown for the situation (i) and (ii). Compared to the decay of the pure charge perturbation the decay of the charge perturbation in the presence of a spin perturbation is slowed down. This
Figure 4. Decay of the amplitude of the charge density perturbation for different strength of the initial perturbation in a system of $L = 72$ sites with $N = 56$ particles, and an interaction strength of $u = 4$. Time is measured in units of $\hbar/J$. The small oscillations result from the discreteness of the underlying lattice, since the maximum value of the density does not always fall on to a lattice site. Taken from [18].

means that the charge and spin are interacting and not totally decoupled. We still find that for longer times the separation takes place as predicted by bosonization, but interactions have to be considered at small times.

4.2.1. Charge density perturbation. Our calculations show that the pure density perturbation is not a stable excitation of the system, but changes its form during the time-evolution. In particular, a decay of the amplitude of the perturbation is seen. The timescale of the decay depends on its initial amplitude and the interaction strength in the system (cf [27]). In figure 4(a), the decay of the amplitude is shown for two different initial values. The fast decrease for small times simply corresponds to the splitting of the density perturbations into two oppositely moving perturbations and has no deeper physical meaning. The decay of the already separated perturbations is then seen to be very slow for small initial amplitude and becomes faster for larger initial amplitudes.

In the limit of an infinitesimal perturbation much broader than the average interparticle spacing, both spin and charge velocities are known analytically from the Bethe ansatz [20]. The velocities depend on the charge density in the system and on the interaction strength. In figure 5, the dependence of the charge velocity on the background density is shown for two different values of the interaction strength. To compare our numerical findings to the exact charge velocity, we create pure charge density perturbations, by applying a potential of the form $\varepsilon_{j,\uparrow}(t) = \varepsilon_{j,\downarrow}(t) = 2\eta f(j, t)$. The parameter $\eta$ determines the strength of the potential and thereby the amplitude of the perturbation. The potential is assumed to have been switched on slowly enough for equilibration, and is then switched off suddenly at time $t = 0$.

The charge velocity is determined from the propagation of the maximum (minimum) of the charge density perturbation for bright (amplitude $\eta_c > 0$) and grey ($\eta_c < 0$) perturbations, respectively. $\eta_c$ is the difference between the maximum (minimum) $n_c$ of the charge density.
perturbation and the background density $n_0$, i.e. $\eta_c = n_c - n_0$. In figure 5, the charge velocities for various background densities $n_0$ and perturbation amplitudes $\eta_c$ are shown and compared to the results for infinitesimal perturbations found by the Bethe ansatz. We find good agreement, if we plot the charge velocity versus the charge density at the maximum (minimum), i.e. $n_c = n_0 + \eta_c$.

The velocity of the maximum (minimum) of the wavepacket is therefore mainly determined by the value of the charge density at the maximum (minimum), and is robust against separate changes of the background density $n_0$ and the height of the perturbation $\eta_c$. This behaviour was found even up to strong perturbations $\eta_c \approx \pm 0.1$ which corresponds to 20% of the charge density (figure 5).

4.2.2. Spin density perturbation. In contrast to the case of pure charge density perturbations the creation of pure spin density perturbations by the means of a potential which couples to the $\uparrow$ and $\downarrow$ densities is much more involved due to the presence of the interaction between both spin species. We applied a potential of the form $\epsilon_{j,\uparrow}(t) = -\epsilon_{j,\downarrow}(t) = 2\eta f(j, t)$ which couples directly to the spin density and only indirectly to the charge density. We calculated again the time-evolution for times after switching off the potential and determine the velocity of the spin perturbation from the propagation of the maximum (minimum) of the wavepacket. We found that the velocity of the spin perturbation varies strongly with its amplitude (figure 6), but only with its absolute value. For decreasing amplitude, its velocity approaches the value obtained by the Bethe ansatz for an infinitesimal spin perturbation. This different amplitude dependence of the spin velocity compared to the dependence of the charge velocity may have several origins. It might be due to the intrinsic difference between charge and spin degrees of freedom (cf next paragraph), a locally created imbalance between the spin up and spin down particles (cf subsection 4.3), or the interaction between the residual charge perturbation and the spin perturbation (seen above).
Figure 6. Spin velocity for different perturbation strength. Numerical results of the adaptive t-DMRG for finite perturbations are compared to the Bethe ansatz results for infinitesimal perturbations. Here, $\eta_s$ is the amplitude of the spin density wave and $n_0$ is the background charge density. The lower bound for the amplitude of the spin perturbation is given by the accuracy in the calculated densities. Taken from [18].

At first glance, the major difference between the spin and charge sectors of the bosonized Hamiltonian is the presence of the marginally irrelevant cosine operator in the spin sector whose influence might still be appreciable on the length scales under study. In order to check this assumption, we considered a spinless fermion model at half-filling with nearest-neighbour interaction which is equivalent to a spin-1/2 chain at magnetization 0. In terms of a spin-1/2 chain, our hopping $J = J_{xy}/2$ and $V = J_z$. This model can also be bosonized, yielding

$$H = \frac{v}{2\pi} \int \mathrm{d}x \left[ K (\pi \Pi(x))^2 + K^{-1} (\nabla \phi(x))^2 \right] - \tilde{g} \int \mathrm{d}x \cos 4\phi(x), \quad (14)$$

with $\tilde{g}$ proportional to $V$. The cosine term is relevant for $K > 1/2$, irrelevant for $K < 1/2$, and marginally irrelevant for $K = 1/2$, the difference to the Hubbard model being due to the prefactor in the cosine (4 versus $2\sqrt{2}$). For $J_{xy} = J_z = 1$ (or $V = 2J$) and magnetization zero, one obtains the marginally irrelevant case $K = 1/2$. The amplitude dependence of the velocity is shown in figure 7 for the marginally irrelevant case of $V = 2J$ and for $V = 1.5J$. In both cases, we find that the velocity depends mainly on the absolute value of the perturbation. This is analogous to the amplitude dependence of the spin velocity in the Hubbard model and contrarily to the amplitude dependence of the charge velocity for the Hubbard model which depends as well on the sign of the perturbation. In the case of $V = 2J$ in which the cosine term is marginally irrelevant the determination of the time-evolution of the perturbation is complicated by the presence of strong Friedel oscillations. However within the given limitations, we cannot observe a similar strong velocity dependence on the amplitude as for the spin velocity in the Hubbard model. Therefore we conclude that the observed behaviour for the spin velocity seems not to be associated with the presence of an irrelevant operator in the bosonization description which is the same for both models.
4.3. Unequal particle species: external fields

One difference to condensed matter systems is in cold gases that on the relevant timescales there is no flipping process between two hyperfine states present, i.e. the total number of $\uparrow$ and $\downarrow$ particles is conserved in the system. This enables the creation of systems, in which an imbalance between the number of $\uparrow$ and $\downarrow$ particles exists for example by pumping a part of the atoms from one hyperfine level to the other by RF radiation. Recently, experimental realization [31, 32] started intense investigations of the physics in these systems. Here, we will discuss the effects of the imbalance in the particle numbers, i.e. $N_\uparrow \neq N_\downarrow$ at constant $N$ (constant charge density) on spin-charge separation.

For strongly different populations there will be a notable effect in that the two populations will have strongly different Fermi momenta and velocities as sketched in figure 8. If we assume that a linearization can be performed around each of the Fermi surfaces for the two species, we...
Figure 9. Snapshots of the time-evolution of the spin and charge density distributions. The total number of particles is fixed to \( N = 28 \) and \( M \) denotes the difference \( N_\uparrow - N_\downarrow \), i.e. \( M = 0 \) means \( N_\uparrow = N_\downarrow = 14 \) and \( M = 12 \) means \( N_\uparrow = 20 \) and \( N_\downarrow = 8 \). The solid arrow marks the position of the charge wave for equal numbers of particles. The dashed arrow marks the coupling between the charge and the spin for \( M = 12 \).

Obtain additionally to the usual Hamiltonian a term of the form

\[
(v_{F,\uparrow} - v_{F,\downarrow})(\pi \Pi \Phi + \nabla \phi \nabla \phi)
\]

which couples the charge and spin modes, and no longer allows a total decoupling of these modes. Spin and charge are no longer good quantities to describe the system. Instead we may diagonalize the free bosonic Hamiltonians with independent modes \( \phi_1 = \alpha \phi_\uparrow + \beta \phi_\downarrow \), \( \phi_2 = \beta \phi_\uparrow - \alpha \phi_\downarrow \), but with \( \alpha, \beta \neq 1/\sqrt{2} \). Spin and charge degrees of freedom hence are coupled again.

We performed simulations with imbalances in the number of \( \uparrow \) and \( \downarrow \) particles to probe, if the description by spin and charge is still suitable. We applied a Gaussian potential, which couples directly to the charge and only indirectly to the spin. In the case of equal number of \( \uparrow \) and \( \downarrow \) particles by this only a charge perturbation is created, whereas for an imbalance of the number of \( \uparrow \) and \( \downarrow \) particles as well a perturbation in the spin density is induced. This can be seen in the first snapshot in figure 9, in which the density distribution is shown at a short time after the creation of the perturbation for an equal number of \( \uparrow \) and \( \downarrow \) particles and for \( N_\uparrow = 20 \) and \( N_\downarrow = 8 \). The solid arrow marks the grey density perturbation. The following snapshots in figure 9 present the time-evolution of the perturbations at later times. In the case of an equal number of \( \uparrow \) and \( \downarrow \) particles the charge perturbation splits up into two waves which move into opposite direction (marked by solid arrow). In contrast for an imbalance in \( \uparrow \) and \( \downarrow \) particles (\( M = 12 \)) no clear evolution of the charge and spin perturbations takes place. In the charge density two waves move outwards, but the density in the centre does not restore the average density value. We attribute this remaining charge perturbation (marked by dashed arrow) in the centre to the interaction between spin and charge. Tracking the time-evolution for different imbalances of the magnetization, we found that for small imbalances two separate waves in spin and charge can be seen (not shown here), whereas for large differences the evolution of spin and charge influence each other strongly, such that spin charge separation breaks down. Even though the number of particles of one species becomes relatively small for strong imbalance (\( N_\downarrow = 8 \)), we expect our
findings to hold as well for larger system sizes: for equal number of particles the phenomenon of spin-charge separation has been observed even for a smaller number of particles (previous studies based on exact diagonalization took three particles in a lattice with 16 sites in [33]), even though the exact velocities can deviate.

5. Conclusion

We have performed numerical simulations of the time-evolution of single particle and density excitations in the 1D Hubbard model to simulate possible experiments on spin-charge separation in cold fermions. Whereas typical condensed matter experiments focus on the spectral function in frequency and momentum space, here the focus is on real time and space. The separation of spin and charge can be clearly seen in the time and real space domain even far beyond the low-energy regime where it is predicted by the Luttinger liquid description. The possibility to address charge and spin individually in cold atoms reveals significant coupling between both degrees of freedom on short timescales, as was to be expected, but also a non-trivial difference between charge and spin in the dependence of the velocities on perturbation amplitudes. Ultracold atoms also allow us to observe the breakdown of spin-charge separation away from equal numbers for spin-up and spin-down particles which would correspond to an external field in condensed matter setups.

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