Unitarity Flow in 2+1 Dimensional Massive Gravity

Sinan Sevim and M. Salih Zoğ
Department of Physics, Istanbul Technical University,
Maslak 34469 Istanbul, Turkey

(Dated: January 3, 2020)

We analyse the most general case of 3rd order Chern-Simons-Like theories of massive 3D gravity. Results show the conditions for finding the unitary regions on the parameter space. There exists (n − 1)th order theories on the boundary of a unitary nth order model on parameter space under certain conditions, as the first example recently demonstrated from the bi-metric generalization of EMG. We investigated the mechanism that causes this type of transition for 3rd order models. Hamiltonian analysis of the theory also presents that ghost and no-ghost regions can be separated by Chern-Simons theories.

I. INTRODUCTION

General Relativity (GR) can be considered as a consistent theory of non-linearly interacting massless spin-2 particles, gravitons, on 4-dimensional space-time. Consideration of GR in 3-dimensions (3D) yields a trivial theory which contains no local degree of freedom. However, a degree of freedom is attainable by adding a gravitational Chern-Simons term to Einstein-Hilbert action in 3D which called Topological Massive Gravity (TMG). This modification generates a massive bulk mode in the linearized theory which emerges from the 3rd order derivative acting on the metric tensor. The new degree of freedom corresponds to helicity ±2 or -2 massive graviton, depends on the overall sign factor, with positive energy and obeys causality.

As seen in TMG, the number of degrees of freedom in the theory can be increased by adding higher-order derivative terms to Einstein-Hilbert action. An example of this is New Massive Gravity (NMG) which is a parity-preserving 4th order model and its linearization gives Fierz-Pauli theory which possesses 2 massive graviton modes with helicity ±2. If one abandons the parity-preserving condition on NMG the theory can be enlarged into General Massive Gravity (GMG) that has ±2 helicity states with different masses and gives both TMG and NMG as a limit. Other than NMG and GMG, there is another 4th order theory called, Exotic Massive Gravity (EMG) which is a parity odd theory with an intriguing feature called third-way consistency. Instead of adding higher-order derivative terms, a bi-metric theory can be formulated. In the 3D case, the first-order formulation of the bi-metric theory is known as Zwei-Dreibein Gravity (ZGD) that has a limiting case to NMG.

A motivating reason to study 3D gravity is its profound relation with 2D Conformal Field Theory (CFT). The relation has been shown that under certain conditions 3D gravity theory generates two copies of Virasoro algebra on the boundary which is dual to the bulk theory known as AdS/CFT correspondence. This connection permits us to work on the boundary CFT instead of bulk space, however, there exists a complication known as Bulk/Boundary clash. Even though TMG, NMG, and GMG are seen as unitary theories, their dual CFT’s are non-unitary due to the negative central charges. An attempt to have a unitary CFT from these theories ends up with a non-unitary Bulk theory which contains negative energy gravitons. The solution for this problem is proposed in a novel manner by adding a specific combination of higher-order curvature terms to TMG is called Minimal Massive Gravity (MMG).

Non-dynamical nature of GR in 3D permits it to be seen as topological theory and hence a Chern-Simons gauge theory. Chern-Simons (CS) formulation allows us to have a toy model for quantum gravity and gather valuable knowledge to get insight on 4D. CS formulation is also a convenient tool to form higher-order theories in first-order formalism. In this case, the new theory is called ‘CS-like’ and contains a degree of freedom instead of being a topological theory. In CS-like terminology order of theory is usually referred to as flavor and higher-order theories can be achieved by adding a new flavor to the theory. In TMG case the theory can be constructed by adding an extra flavor to Einstein-Cartan action, which only contains vielbein (e) and dual-spin connection (ω), to form a 3rd order or 3-flavor theory.

In this work, the most general 3-flavor case is analysed. By most general we mean, a model with a Lagrangian which includes all the possible terms with arbitrary coefficients, up to 3 derivatives. At this point, equations of motion are not “simply solvable” like the cases, one may need to use infinite serial expansions for several regions. However, generic cases have more opportunities to become a unitary model.

In our case, there are 10 linearly independent terms in Lagrangian with their arbitrary coefficients which can be seen as parameters. Since we only consider the models that contain AdS background solution, two of these coefficients become dependent via additional restrictions. Therefore; our parameter space is 8-dimensional and every theory represented here by a point. Between two points (or models) there exist infinitely many trajecto-
ries as usual, thus on these paths, we can observe a theory smoothly flows to the other one. During the flow, the model passes from different segments with various characteristics, including unitarity. Here, we conclude that unitary and non-unitary regions are separated by 7-dimensional hyper-surfaces which can intersect with a flow at a transition point. These hyper-surfaces contain lower-order theories under certain conditions as the first example recently demonstrated by [10]. Here, we will investigate the general properties of the $n \equiv 3$ model by means of bulk/boundary unitarity and local degrees of freedom. After the study of the theory, we are expecting to observe $3 \to 2$ transitions at critical surfaces.

II. CHERN-SIMONS-LIKE FORMALISM

3D gravity models can be expressed in terms of Lorentz vector valued one forms $a^r$ where in 3-flavor case $r = 1, 2, 3$ and we have $a^r = (e, f, \omega)$. Here $f$ is an auxiliary field and $e, \omega$ denotes the dreibein and dual spin connection, respectively. Defining the torsion and dual curvature 2-forms,

$$T(\omega) = D(\omega) e = de + \omega \times e, \quad R(\omega) = d\omega + \frac{1}{2} \omega \times \omega \quad (1)$$

by the exterior derivative. One can construct the Lagrangian 3-form with a flavor metric $g_{rs}$ and a coefficient tensor $f_{rst}$ as mentioned in [13].

$$L = \frac{1}{2} g_{rs} a^r \cdot da^s + \frac{1}{6} f_{rst} a^r \cdot (a^s \times a^t) \quad (2)$$

where dot and cross product notation is used for 3-dimensional Lorentz vectors.

$$(a^r \times a^s)^k = \epsilon^{ijk} a^r_i a^s_j \quad (3)$$

Following the procedure introduced in [6], most general model in 3-flavor theory have

$$L = \frac{a_1}{2} e \cdot T + a_2 e \cdot Df + a_3 e \cdot R + \frac{a_4}{2} f \cdot Df +$$

$$a_5 f \cdot R + \bar{a_6} L_{LCS} + \frac{a_7}{6} e \cdot e \times e + \frac{a_8}{2} e \cdot f \times f \quad (4)$$

where the Lorentz Chern-Simon 3-form defined as $L_{LCS} = \frac{1}{2}\omega \cdot (d\omega + \frac{1}{2} \omega \times \omega)$. One can see the components of coefficient tensor as, for instance; $f_{122} = a_8$. Therefore, all the calculations after this step will stand on these $a_\gamma$-values. Although this method seems natural, it causes the later expressions to be extremely long and unclear. Passing to a new set of coefficients can ease them all.

$$\delta e : a_1 T + a_2 Df + a_3 R + \frac{a_7}{2} e \times e + \frac{a_8}{2} f \times f + a_9 e \times f = 0$$

$$\delta \omega : a_3 T + a_5 Df + a_6 R + \frac{a_1}{2} e \times e + \frac{a_4}{2} f \times f + a_2 e \times f = 0$$

$$\delta f : a_2 T + a_4 Df + a_5 R + \frac{a_9}{2} e \times e + \frac{a_{10}}{2} f \times f + a_8 e \times f = 0 \quad (5)$$

Equations of motion for (4) can be written as expression (5) has a general form too, which comes from variation of $L$.

$$E_r = g_{rs} da^s + \frac{1}{2} f_{rst} a^r \times a^t \quad (6)$$

One can raise the indice of $E_r$ with our non-inevitable flavor metric,

$$E^p = g^{pr} E_r = da^p + \frac{1}{2} f^{pr} a^r \times a^t \quad (7)$$

Notice that, now, the derivatives are alone and coefficients are $f_{rst}$ instead of $f_{rst}$. We can define a new set of coefficients to write equation of motions by using (6)

$$T + \frac{1}{2} a_1 e \times e + a_2 e \times f + \frac{1}{2} a_3 f \times f = 0$$

$$Df + \frac{1}{2} a_4 e \times e + a_5 e \times f + \frac{1}{2} a_6 f \times f = 0 \quad (8)$$

$$R + \frac{1}{2} a_7 e \times e + a_8 e \times f + \frac{1}{2} a_9 f \times f = 0$$

Where $\alpha$’s are

$$f_{11}^1 = \alpha_1, \quad f_{12}^1 = \alpha_2, \quad f_{22}^1 = \alpha_3 \quad etc. \quad (9)$$

Thanks to these new $\alpha$ parameters our separated equations of motions can be written in an elegant fashion. One can check necessity of the $\alpha$’s by just looking at

$$\alpha_1 = \frac{a_1 a_2 a_5 - a_1 a_3 a_4 - a_2 a_6 a_9 + a_3 a_5 a_9 + a_4 a_6 a_7 - a_2^2 a_7}{det(g)} \quad (10)$$

where $det(g)$ is determinant of flavor metric.

A. Linearization about AdS

In AdS background, our auxiliary flavor $f_{\mu\nu}$ has to be proportional to the dreibein, therefore;

$$e = \bar{e}, \quad f = \bar{f} = c \bar{e}, \quad \omega = \bar{\omega} \quad (11)$$

where $c$ is the proportionality constant and determined by the”AdS conditions”which we will investigate in a general sense. According to above definitions, one can write the first order fluctuations as

$$e = \bar{e} + \kappa k, \quad f = c(\bar{e} + \kappa k) + \kappa p, \quad \omega = \bar{\omega} + \kappa v \quad (12)$$

With the help of $\alpha_i$ coefficients linearized equation of motions from (8) is derived.

$$\bar{D} k + \bar{e} \times v + (\alpha_3 c + \alpha_2) \bar{e} \times p = 0$$

$$\bar{D} p + M \bar{e} \times p = 0 \quad (13)$$

$$\bar{D} v + \frac{1}{L^2} \bar{e} \times k + (\alpha_9 c + \alpha_8) \bar{e} \times p = 0$$
where $M = \alpha_5 + (\alpha_6 - \alpha_2)c - \alpha_3 c^2$ is the mass parameter. This can be tested through some of the well-known 3D models, such as TMG [2] and MMG [12]. In TMG case, it can be directly seen that (13) gives the correct mass parameter. However, MMG needs to redefine the spin connection $\Omega = \omega + \alpha f$ as in [12].

One can notice that in terms of $\alpha_i$-coefficients, our mass parameter will be horrific. In addition to being simple, the main advantage of $\alpha_i$ formalism is its prediction of $M$’s denominator. Since all $\alpha_i$ have flavor metric determinant on the denominator, then $M$’s denominator must be proportional to $\text{det}(g)$. This feature is crucial for the analysis of unitarity as it will be shown later. By using the expressions of torsion and curvature 2-forms in the AdS background,

\[ \tilde{T} \equiv D(\bar{\omega}) \bar{e} = 0, \quad \tilde{R} \equiv R(\bar{\omega}) = \frac{1}{2} \Delta \bar{e} \times \bar{e} \]  

Following relations which we refer here as "AdS conditions" can be derived from equations of motion.

\[ \frac{1}{2} \alpha_1 + \alpha_2 c + \frac{1}{2} \alpha_3 c^2 = 0 \]
\[ \frac{1}{2} \alpha_4 + \alpha_5 c + \frac{1}{2} \alpha_6 c^2 = 0, \]
\[ \frac{1}{2} \alpha_7 + \alpha_8 c + \frac{1}{2} \alpha_9 c^2 = 1 - \frac{1}{2l^2} \]  

B. Unitarity Conditions

In this section we will examine the bulk and boundary unitarity of the most general 3-flavor model. Our conditions are the positivity of central charges (if one considers the Brown-Henneaux condition for AdS3), absence of tachyons and ghost degrees of freedom.

For boundary, we have to first obtain the central charges. By the method introduced in [17] one can find,

\[ C_\pm = \frac{3l^3}{2G} (-a_3 + a_5) \pm \frac{1}{l} a_6 \]  

Therefore, $C_\pm > 0$ is the sufficient condition for boundary unitarity.

For bulk, one has to find the Fierz-Pauli mass $M$ which is related to the mass parameter we found in [13] as $M^2 = M^2 + \Lambda$. After this observation, no-tachyon condition $M^2 > 0$ becomes $M^2 > 1$ which, now, can be connected to the $\alpha_i$ coefficients. Absence of ghost degrees of freedom needs a little more effort as it has to diagonalize the [13] and its Lagrangian. By the redefinitions,

\[ k = l(f_+ - f_-) + \frac{l^2(M\alpha_3c + M\alpha_2 + \alpha_6 c) }{l^2M^2 - 1} p \\
 v = f_+ + f_- + \frac{\alpha_9 c^2 M + \alpha_8 c^2 M + \alpha_3 c + \alpha_2 }{l^2M^2 - 1} p \]  

[13] can be written in a more simple form.

\[ \tilde{D}f_+ + \frac{1}{l^2} \bar{e} \times f_+ = 0 \]
\[ \tilde{D}f_- + \frac{1}{l^2} \bar{e} \times f_- = 0 \]
\[ \tilde{D}p + M\bar{e} \times p = 0 \]  

which also admits that the 2nd order perturbative Lagrangian becomes also diagonal.

\[ L^{(2)} = l a_+ f_+ \cdot (\tilde{D} f_+ + \frac{1}{l^2} \bar{e} \times f_+) - l a_- f_- \cdot (\tilde{D} f_- - \frac{1}{l^2} \bar{e} \times f_-) \]
\[ - \frac{k}{M^2} \bar{p} \cdot (\tilde{D} p + M\bar{e} \times p) \]  

where $a_\pm = -\frac{2G}{l} C_\pm$ and $K = \frac{M \text{det}(g)}{2 \alpha_6}$ are written in well-known identities.

C. Hamiltonian Analysis

3rd order theories can have either 0 or 1 degree of freedom (d.o.f.) which can be counted by analysing their primary and secondary constraints (In this section we will follow the method introduced in [13]). Firstly, we have to look at the Poisson brackets of primary constraints $\phi_r$ of the theory. An easy way to compute them is defining the Poisson bracket matrix which has the form,

\[ \mathcal{P}_{rs}^{ab} = f_{ij}^{b[rs]} \eta^{ij} \Delta^{pq} + 2 f_{ij}^{b[ij]} \eta^{ij} (V^{ab})^{pq} \]

where the objects $V^{pq}_{ab}$ and $\Delta^{pq}$ have the definitions,

\[ V^{pq}_{ab} = \varepsilon^{ij} a^p_{ia} a^q_{jb} \quad \Delta^{pq} = \varepsilon^{ij} a^p_i a^q_j \]  

Here, $\Delta^{ee}$ and $\Delta^{ff}$ is identically zero and the remaining $\Delta^{ef}$ can be investigated through the integrability conditions.

To drive the integrability conditions, we have to take the covariant exterior derivative of $\phi$ and insert the below identities.

\[ D(\omega)T(\omega) = R(\omega) \times e, \quad D(\omega)R(\omega) = 0 \]  

Then one can show that these conditions and the invertibility of dreibein impose $\Delta^{ef} = 0$ which turns out to be an additional or a secondary constraint.

\[ \psi = \Delta^{ef} = 0 \]

and has the Poisson brackets with primary constraints as

\[ \{ \phi[\xi], \psi \}_{\text{P.B.}} = P_\delta \epsilon^{i\delta} (\xi^f D_i e_j - \xi^e D_i f_j) \]
\[ + (\alpha_3 \xi^c + \alpha_6 \xi^e) e_i \times e_j + ((-\alpha_1 + \alpha_5) \xi^e + (-\alpha_2 + \alpha_6) \xi^f) e_i \times f_j \]  

\[ + (\alpha_2 + \alpha_6) \xi^e f_i \times f_j + (\alpha_6 \xi^c + \alpha_3 \xi^f) f_i \times f_j \]
where $P_0$ is an $a_i$ dependent coefficient (which is assumed nonzero). Calculation of (20), now follows as,

\[
(P_{ab})_{rs} = P_0 \begin{pmatrix}
(Q_{ab})_{rs} & 0 \\
0 & 0
\end{pmatrix} = P_0 \begin{pmatrix}
V_{ab}^f & V_{ab}^e \\
V_{ab}^e & V_{ab}^f
\end{pmatrix}
\]

(25)

The rank of matrix $P_{ab}$ (is also the same with the rank of $Q_{ab}$) is an important quantity to count degree of freedom. Notice that $Q_{ab}$ is $6 \times 6$ and has the rank of 2 by a Mathematica calculation, but it does not stand for all the analysis. To complete it, we have to also consider the Poisson brackets between primary and secondary constraints. Now we are extending our $Q$-matrix to $7 \times 7$ with these extra relation and looking for rank again. Taking 24 as a $6 \times 1$ vector and adding it and its trasposes results as

\[
Q = \begin{pmatrix}
Q \\
-v^T \\
v
\end{pmatrix}, \quad v = \begin{pmatrix}
\{\phi_e, \psi\}_P \\
\{\phi_f, \psi\}_P
\end{pmatrix}
\]

(26)

The rank of $Q$ is now 4 unless the rare possibility of satisfying simultaneously the conditions below,

\[
\alpha_4 = \alpha_3 = 0, \quad \alpha_2 = 2\alpha_5, \quad \alpha_6 = 2\alpha_2
\]

(27)

In this situation, we have 4 second class and $10 - 4 = 6$ first-class constraints. Dimension of physical phase space can be counted as $3 \times 6 - 6 \times 2 - 4 = 2$. Then we can conclude that our degree of freedom is one. On the other hand, if 27 satisfied, rank of $Q$ reduces to 2 and now there are 2 second class, $10 - 2 = 8$ first class constraints. Therefore; phase space dimension is $3 \times 6 - 8 \times 2 - 2 = 0$ and theory propagates no bulk degree of freedom.

Notice that together with the AdS 13 and rank-2 24 conditions imposes our mass parameter $M$ to be zero. Since $M$ and $K$ are proportional to each other, one can think that as mass changes sign, no-ghost condition $K > 0$ will break down. This can be concluded that Chern-Simon theories may exist on the boundaries which separate the ghost and no-ghost regions on the parameter space.

### III. UNITARITY FLOW

Our main interest in this paper is the flows between unitary and non-unitary regions on the parameter space. This space is 8-dimensional, because; there are 10 $a_i$-parameters and 3 AdS conditions restrict 3 of them but regains 1. Therefore; one can conclude that the hypersurface which separates the unitary and non-unitary regions are 7-dimensional. As first discovered in 16, this surface may contain an $(n - 1)$-flavor model. Here, we will investigate the generic fall of $3^{rd}$ order theories to $2^{nd}$ ones.

First off all, we need to define what reducing to a lower order is, in a mathematical way. For instance; consider the GMG Lagrangian $L_{GMG}$,

\[
L_{GMG} = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e + h \cdot D e
\]

(28)

Notice that as $m \to \infty$, the terms containing auxiliary field $f_{\mu \nu}$ decouples and theory falls down to 3-flavor. Vanishing the terms' coefficients like this one is a way but not the most general one. For example; one can provide a fall by adding terms:

\[
L'_{GMG} = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e + h \cdot D e
\]

(29)

This is a modification of GMG for no purpose. One can write the same Lagrangian with defining a new field $q_{\mu \nu} = h_{\mu \nu} + f_{\mu \nu}$:

\[
L'_{GMG} = -\sigma e \cdot R + \frac{\Lambda_0}{6} e \cdot e + q \cdot D e
\]

(30)

Therefore; 29 is not actually a $4^{th}$ order model despite the fact that it has 4 different fields.

During a flow, coefficients of terms of a Lagrangian smoothly change and at some critical surfaces, theories fall to lower-order ones and we get situations like 30. The main reason for this comes from the flavor metric which is first mentioned in 2. As the columns of this tensor become linearly dependent on each other, different fields can be expressed in terms of the others. We know determinant $det(g)$ goes to zero where metric degenerates, so it can be our condition for a fall:

\[
det(g) = 0
\]

(31)

At these critical surfaces where determinant vanishes, theory can degenerate into a $(n - 1)$-flavor model or a $(n - 2)$ one or totally collapse since there is no such $1^{st}$ order theory in 3D gravity.

Now we are expecting some of the unitarity conditions to break down while $det(g)$ goes to zero, so where the flow and hyper-surface intersect, we can observe the fall. Since the mass parameter has the determinant in its denominator, it changes sign while $det(g)$ passes through zero value but $M^2$ does not. Therefore the no-tachyon condition $M^{2r^2} > 1$ is not a good candidate. The positivity of central charges does not depend on $det(g)$ since they are not proportional. Nevertheless, one can maintain proportionality with imposing some conditions as

\[
a_5 = +l(a_4 c + a_2) \to det(g) \propto C_+
\]

\[
a_5 = -l(a_4 c + a_2) \to det(g) \propto C_-
\]

(32)
So where we lose the positivity of one of the central charges, determinant also vanishes. However, these types of degenerations completely collapse the theory which is not exactly what we want.

Now we expect that the break down of remaining no ghost condition \( K > 0 \) can provide a \( 3 \rightarrow 2 \) flavor fall at the transition point. Since \( K = \frac{M_{det(g)}}{2\pi a_2 a_3 \dots} \), and mass parameter has \( det(g) \) in its denominator, this condition also seems independent of metric determinant. But in practice, we observe some of the flows depend on \( det(g) \) thus there must be some regions where dependence regained. Actually, these are the ones where some factors of \( K \) are proportional to the determinant. Therefore; numerator or the denominator of \( K \) can have the same roots, thus the same critical points with \( det(g) \).

At this point let us discuss the model when metric degenerates, so rewrite \( \text{(33)} \) with the metric determinant. Since all \( \alpha_i \)-coefficients has \( det(g) \) in their denominator, after multiplication we will have

\[
det(g)T + \frac{1}{2} \text{Num}(\alpha_1)e \times e + \text{Num}(\alpha_2)e \times f + \frac{1}{2} \text{Num}(\alpha_3)f \times f = 0
\]

\[
det(g)Df + \frac{1}{2} \text{Num}(\alpha_4)e \times e + \text{Num}(\alpha_5)e \times f + \frac{1}{2} \text{Num}(\alpha_6)f \times f = 0
\]

\[
det(g)R + \frac{1}{2} \text{Num}(\alpha_7)e \times e + \text{Num}(\alpha_8)e \times f + \frac{1}{2} \text{Num}(\alpha_9)f \times f = 0
\]

where \( \text{Num}(\alpha_i) \) denotes the numerator of \( \alpha_i \)-coefficients. As \( det(g) \rightarrow 0 \) all of \( \text{(33)} \) takes the form:

\[
\frac{1}{2} \alpha e \times e + \beta e \times f + \frac{1}{2} \gamma f \times f = 0
\]

Now, we have to expand \( e, f \) to infinite series around the AdS background in order to find \( f_{\mu \nu} \), because; we have \( f \times f \) term in \( \text{(34)} \).

\[
e = \bar{e} + \kappa k + \frac{\kappa^2}{2!} k' + \frac{\kappa^3}{3!} k'' + \cdots = \bar{e} + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} k^{(n-1)}
\]

\[
f = c \left( \bar{e} + \kappa k + \frac{\kappa^2}{2!} k' + \frac{\kappa^3}{3!} k'' + \cdots \right) + \kappa P + \frac{\kappa^2}{2!} P' + \cdots
\]

\[
= c \left( \bar{e} + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} k^{(n-1)} \right) + \left( \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} P^{(n-1)} \right)
\]

where primes do not denote derivatives but higher order fluctuations. Substituting these series into \( \text{(34)} \) one can show that

\[
p_{\mu \nu} = p'_{\mu \nu} = p''_{\mu \nu} = \cdots = 0
\]

therefore; our auxiliary field becomes proportional to the dreibein; in other words, it just imposes a constraint \( \text{(34)} \).

\[
f = c \left( \bar{e} + \sum_{n=1}^{\infty} \frac{\kappa^n}{n!} k^{(n-1)} \right) = ce
\]

This result match with the fact that as \( det(g) \rightarrow 0 \), flavor number decreases. When metric degenerates, flavors become linearly dependent of each other.

Let us examine this result through an example. We will write two random sets of \( a_i \)-parameters as first set corresponds to a unitary model and the other one is non-unitary. Of course; these sets are two points at the 8-dimensional parameter space and can be connected via infinitely many trajectories. As coefficients flow thorough one of these paths, \( c \) also changes smoothly and its value obtained by the AdS conditions. These conditions can be expressed with \( \alpha_i \)-coefficients as

\[
a_1 + a_4 e^2 + 2 a_2 c = \frac{a_6}{\ell^2}
\]

\[
a_7 + a_8 e^2 + 2 a_9 c = \frac{a_3}{\ell^2}
\]

\[
a_9 + a_{10} e^2 + 2 a_{12} c = \frac{a_5}{\ell^2}
\]

Here we will solve \( c \) from the first equation of \( \text{(38)} \) by choice, and the remaining two will be used to find \( a_7, a_9 \). Since these two can be calculated completely by the others, parameter space dimension reduces to 8 from 10.

Notice that, \( \text{(38)} \) is a list of quadratic equations which make \( c \) complex at some regions. Seeing that some of the important quantities such as central charges and mass parameter depend on \( c \), for simplicity, we set \( a_4 = 0 \) through the flow in this example. Thus,

\[
c = \frac{a_6 - a_{12} l^2}{2 a_{12} l^2} \in \mathbb{R}
\]

Now, denominator of \( M \) becomes \( \alpha_3^2 \ det(g) \) and proportional to the flavor metric determinant as it claimed before. As a result, \( a_2 \) becomes a factor of \( K \) and if we yield \( a_1 a_5 = 0 \) (at transition point), then \( a_2 \) is also a factor of \( det(g) \).

Our random point in unitary region of parameter space has the coordinates;

\[
a_1 = 6, \ a_2 = 6, \ a_3 = -12, \ a_4 = 0, \ a_5 = 12, \ a_6 = -16, \ a_8 = -20, \ a_{10} = 13
\]

in AdS units (\( l = 1 \)). The 3D gravity model with the below coordinates has positive central charges and does not include tachyons but it has a ghost degree of freedom, thus it is non-unitary.

\[
a_1^{**} = 22, \ a_2^{**} = -20, \ a_3^{**} = 0, \ a_4^{**} = 0, \ a_5^{**} = -40, \ a_6^{**} = -23, \ a_8^{**} = 13, \ a_{10}^{**} = 11
\]

In our example, the trajectory that will connect these two random points will be parametrized by the flow:

\[
a_i(\lambda) = (1 - \lambda^2) a_i^* + \lambda a_i^{**}
\]
So as $\lambda$ changes 0 to 1 theory flows through the non-unitary point from the unitary one. Then we can observe the change of central charges, metric determinant and $K$ through the path.

\[ L = \frac{a_1}{2} e \cdot T + a_3 e \cdot R + a_6 L_{LCS} + \frac{a_7}{6} e \times e + \frac{a_8}{2} e \cdot f \times f + \frac{a_9}{2} e \times f + \frac{a_{10}}{6} f \cdot f \times f \]  

(43)

with equations of motion:

\[ \delta e : a_1 T + a_3 R + \frac{a_7}{2} e \times e + \frac{a_8}{2} f \times f + a_9 e \times f = 0 \]

\[ \delta \omega : a_3 T + a_6 R + \frac{a_1}{2} e \times e = 0 \]

\[ \delta f : \frac{a_9}{2} e \times e + \frac{a_{10}}{2} f \times f + a_8 e \times f = 0 \]  

(44)

Notice that, variation according to $f$ comes in the form of \( \delta e \), therefore; its solution gives $f = ce$ and it is a non-dynamical field. Now imposing this result into the (43) results in

\[ L = \frac{a_1}{2} e \cdot T + a_3 e \cdot R + a_6 L_{LCS} + \frac{1}{6} (a_7 + 3c^2 a_8 + 3ca_9 + c^3 a_{10}) e \cdot e \times e \]  

(45)

Remember that $a_7, a_9$ are not free parameters, they are expressed by \( a_8 \). Imposing back their definitions and $\Lambda = -\frac{1}{\ell}$ now gives;

\[ L = -a_6 \frac{\Lambda}{2} e \cdot T + a_3 e \cdot R + a_6 L_{LCS} - a_3 \frac{\Lambda}{6} e \cdot e \times e \]  

(46)

where $a_1 = \frac{有望}{有望}$ also by \(有望 \). Thus, equations of motion become

\[ \delta e : -a_6 \Lambda T + a_3 R + a_6 L_{LCS} - a_3 \frac{\Lambda}{2} e \times e = 0, \]

\[ \delta \omega : a_3 T + a_6 R - a_6 \frac{\Lambda}{2} e \times e = 0 \]  

(47)

Here, one can rearrange (47) with separating the curvatures. Finally we got

\[ a_3 \delta e + a_6 \Lambda \delta \omega : \left( a_3^2 + a_6^2 \Lambda \right) \left( R - \frac{\Lambda}{2} e \times e \right) = 0 \]

\[ a_3 \delta \omega - a_6 \delta e : \left( a_3^2 + a_6^2 \Lambda \right) T = 0 \]  

(48)

which is exactly the same equations as Einstein-Cartan theory in 3D has.

IV. DISCUSSION & CONCLUSION

In this work we presented the most general form of $n = 3$ Chern-Simons-like theory and derived the unitarity condition for both bulk and boundary. This classification shows us there is a hyper-surface in the parameter space which separates unitary and non-unitary regions and between them there exists a $n = 2$ model under certain
restrictions. One can write a flow between these regions and along this trajectory, at some transition point the theory degenerates into a lower flavor. Despite the fact that all study has made for 3rd order 3D massive gravity, the ideas of degeneration and transitions are quite general and can have applications for any theory which has a matrix of its kinetic terms.

The generic case also presents that the regions which contain ghost degrees of freedom and the regions that do not can be separated by Chern-Simons theories. These types of models have zero mass parameters and their constraints sufficiently construct a Lie algebra.

Acknowledgements: We thank Mehmet Ozkan for useful discussions. The work of M.S.Z. and S.S. is partially supported by TUBITAK grant 118F091.

[1] S. Deser, R. Jackiw, and G. ’t Hooft, Annals Phys. 152, 220 (1984).
[2] S. Deser, R. Jackiw, and S. Templeton, [Annals Phys. 140, 372 (1982) [Annals Phys.281,409(2000)].
[3] E. A. Bergshoeff, O. Hohm, and P. K. Townsend, Phys. Rev. Lett. 102, 201301 (2009) arXiv:0901.1766 [hep-th].
[4] M. Fierz and W. Pauli, Proc. Roy. Soc. Lond. A173, 211 (1939).
[5] W. Merbis, Chern-Simons-like Theories of Gravity, Ph.D. thesis, Groningen U. (2014), arXiv:1411.6888 [hep-th].
[6] M. Ozkan, Y. Pang, and P. K. Townsend, JHEP 08, 035 (2018) arXiv:1806.04179 [hep-th].
[7] E. Bergshoeff, W. Merbis, A. J. Routh, and P. K. Townsend, Int. J. Mod. Phys. D24, 1544015 (2015), arXiv:1506.05949 [gr-qc].
[8] E. A. Bergshoeff, S. de Haan, O. Hohm, W. Merbis, and P. K. Townsend, Phys. Rev. Lett. 111, 111102 (2013), [Erratum: Phys. Rev. Lett.111,no.25,259902(2013)] arXiv:1307.2774 [hep-th].
[9] J. D. Brown and M. Henneaux, Commun. Math. Phys. 104, 207 (1986).
[10] J. M. Maldacena, Int. J. Theor. Phys. 38, 1113 (1999). [Adv. Theor. Math. Phys.2,231(1998)], arXiv:hep-th/9711200 [hep-th].
[11] E. Witten, Adv. Theor. Math. Phys. 2, 253 (1998), arXiv:hep-th/9802150 [hep-th].
[12] E. Bergshoeff, O. Hohm, W. Merbis, A. J. Routh, and P. K. Townsend, Class. Quant. Grav. 31, 145008 (2014), arXiv:1404.2867 [hep-th].
[13] E. Witten, Nucl. Phys. B311, 46 (1988).
[14] A. Achucarro and P. K. Townsend, Phys. Lett. B180, 89 (1986), [732(1987)].
[15] E. A. Bergshoeff, O. Hohm, W. Merbis, A. J. Routh, and P. K. Townsend, Proceedings of the 7th Aegean Summer School : Beyond Einstein's theory of gravity. Modifications of Einstein’s Theory of Gravity at Large Distances. : Paros, Greece, September 23-28, 2013, Lect. Notes Phys. 892, 181 (2015), arXiv:1402.1688 [hep-th].
[16] M. Ozkan, Y. Pang, and U. Zorba, Phys. Rev. Lett. 123, 031303 (2019), arXiv:1905.00438 [hep-th].
[17] E. A. Bergshoeff, W. Merbis, and P. K. Townsend, Phys. Rev. D86, 084035 (2012), arXiv:1208.0038 [hep-th].