The Averaging Problem in Cosmology

A Thesis

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by

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Synopsis

Introduction

A central assumption in modern cosmology is that the universe on large scales is homogeneous and isotropic \[3\]. This assumption leads to tremendous simplifications in the application of general relativity (GR) to cosmology, since it reduces the ten independent components of the metric of spacetime \(g_{ab}(t, \vec{x})\) to essentially a single function of time \(a(t)\) known as the scale factor. In the early days of modern cosmology, beginning with stalwarts such as Einstein and deSitter, the assumption of homogeneity and isotropy was largely motivated on grounds of simplicity and aesthetic appeal. In recent times however, it has become possible to confront this assumption with observations, which remarkably appears to be justified to a large extent (based on observations of the cosmic microwave background (CMB) radiation \[5\], and on analyses of galaxy surveys \[6, 7\]). This indicates that a model based on essentially a single function of time might in fact go a long way in furthering our understanding of the behaviour of the universe.

Of course the real universe is not homogeneous; we see a rich variety of structure around us from stellar systems to galaxies to clusters of galaxies and even larger structures. The study of the large scale structure (LSS) in the universe has a long history going back several decades \[10\]. Perhaps one of the biggest successes of cosmological theory based on GR, has been the explanation of how statistical properties of the LSS arise \[11\]. The relevant calculations are largely based on linear perturbation theory, in which one describes inhomogeneities in the universe as perturbations around the smooth solution characterised by the scale factor \(a(t)\) and expands the Einstein equations as a series in these small perturbations. While such a treatment has met with great success in the description of the statistical properties of the tiny fluctuations (anisotropies) in the temperature of the CMB, there are two causes for concern.

The first is a purely theoretical issue, and is the basis of this work. The idea that the large scale universe is homogeneous and isotropic necessarily entails an implicit notion of averaging on these large scales. In other words, what one is really saying is, “When the spatially fluctuating parts of the solution of GR describing our universe are averaged out, what is left is the homogeneous and isotropic solution of Einstein’s equations\[1\]. The immediately obvious problem with this statement

\[1\]This solution is known as the Friedmann-Lemaître-Robertson-Walker (FLRW) solution after those who first studied it.
is that the details of the averaging operation are not at all clear, and indeed are usually never
specified. A bigger problem is one noted by Ellis [17], and can be stated in the following symbolic
way. If \( g \) denotes the metric, \( \Gamma \) the Christoffel connection and \( E[g] \) the Einstein tensor for the
metric \( g \), then we have the relations

\[
\Gamma \sim \partial g \ ; \ E[g] \sim \partial \Gamma + \Gamma^2,
\]

with \( \partial \) denoting spacetime derivatives. The Einstein equations are therefore

\[
E[g] = T,
\]

with \( T \) denoting the energy-momentum tensor of the matter components. Now, irrespective of any
details of the averaging operation, one notes that

\[
E[\langle g \rangle] - \langle E[g] \rangle \sim \langle \Gamma \rangle^2 - \langle \Gamma^2 \rangle \neq 0,
\]

with the angular brackets denoting the averaging. The FLRW solution would amount to solving
the equations \( E[\langle g \rangle] = \langle T \rangle \). In general therefore, it is not true that averaging out the fluctuating
inhomogeneities leaves behind the FLRW solution, since what we are actually left with is

\[
E[\langle g \rangle] = \langle T \rangle - C \ ; \ C \sim \langle \Gamma^2 \rangle - \langle \Gamma \rangle^2,
\]

and the homogeneous solution that we are looking for will depend on the details of the correction
terms \( C \).

The second cause for concern comes from observations. It has now been established beyond a
reasonable doubt, that the FLRW metric confronted with observations indicates an accelerating
scale factor [18]. Conventional sources of energy such as radiation and nonrelativistic matter cannot
explain the acceleration, and it is now common to attribute this effect to a hitherto unknown
“Dark Energy”, which in its simplest form is a cosmological constant. The true nature of this
additional component in the cosmological equations, is perhaps the most challenging puzzle facing
both theorists and observers today. A huge amount of research has gone into (a) explaining the value
that a cosmological constant term must take to explain data or (b) assuming a zero cosmological
constant, constructing models of a dynamical dark energy which explains the observed acceleration
[19]. It is fair to say however that there is no theoretical consensus on what the origin of Dark
Energy is.

Since we have seen above that the effects of averaging lead to some extra, as yet unknown terms
in the equations, it is natural to ask whether these two issues are connected. Could the acceleration
of the universe be explained by the effects of averaging inhomogeneities (“backreaction”) in the
universe? Regardless of the answer to this question, what is the nature and magnitude of this
backreaction? The purpose of this thesis is to answer these questions as rigorously as possible.
The conventional wisdom, and loopholes

One should note that the conventional wisdom on the issue of backreaction in the sense described in the previous section, is that this effect can never be significant. It is of importance therefore to understand this argument and its shortcomings. The argument goes as follows [27]. One starts by assuming that inhomogeneities in the universe can be described in the Newtonian approximation of GR, by the gravitational potential $\Phi(t, \vec{x})$ with $|\Phi| \ll 1$, which satisfies the Poisson equation $\nabla^2 \Phi = 4\pi G a^2 \delta \rho$ where $\delta \rho(t, \vec{x})$ is the fluctuation of matter density about the mean homogeneous value $\bar{\rho}(t)$, and can in general have a large value. (E.g., in clusters of galaxies one finds $\delta \approx \delta \rho / \bar{\rho} \sim 10^2$, and the ratio increases on smaller length scales). One then argues that the universe we observe does seem to be very well-described by the above model, and effects of averaging this model can only arise at second order in $\Phi$ and should hence be extremely small.

There is a loophole in this argument though. The catch is that the background expansion $a(t)$ is defined completely ignoring the backreaction, which is an integrated effect with contributions from a large range of length scales. This means that the following possibility cannot be a priori ruled out: Initial conditions are specified as a perturbation around a specific FLRW solution, but the integrated effect of the backreaction grows (with time) in such a manner as to effectively yield a late time solution which is a perturbation around a different FLRW model. Indeed, there are calculations in the literature that do indicate such a possibility being realised [28, 31, 34].

We therefore see the need to actually perform a rigorous calculation that will describe the time evolution of the backreaction, and thereby either confirm or overthrow the conventional wisdom.

Averaging schemes

A major hurdle in computing the effects of averaging has been the lack of reliable averaging procedures which can be used in GR, mainly because defining and physically interpreting averaging operations suitable for tensors, is a challenging prospect. A number of authors have attempted to solve this problem, both in the specific context of cosmology and also as a more general problem of the mathematics of GR (see, e.g. Refs. [22, 23, 24, 25, 26]). To date, the most promising averaging schemes have been the spatial averaging of scalars due to Buchert [40, 41], and the fully covariant tensor averaging due to Zalaletdinov [43].

Buchert’s scalar averaging deals with a chosen $3 + 1$ splitting of spacetime, and only averages two of the Einstein equations. This averaging scheme is simple to implement and intuitively easy to grasp, however it is ultimately difficult to interpret its physical significance. Zalaletdinov’s scheme on the other hand, is technically challenging to handle, since its averaging operation can deal with full-fledged tensors at the cost of introducing some new mathematical structures into the problem. The appeal of this scheme lies in the fact that ultimately one has in hand an object which can legitimately be called the “averaged metric” on an “averaged manifold”.

In this thesis we use both these schemes to address certain specific questions concerning the
backreaction problem. As we discuss below however, ultimately we rely upon Zalaletdinov’s scheme to make realistic statements regarding the nature of cosmological backreaction.

Can backreaction ever be large?  
A Paranjape and T P Singh, Class. Quant. Grav. 23, 6955 (2006).

One question to ask in the context of the conventional wisdom presented above, is whether it is technically possible within GR to have a situation in which the backreaction dominates the averaged expansion. We answer this in the affirmative by studying a toy model. We use the exact, spherically symmetric Lemaitre-Tolman-Bondi (LTB) solution of GR, to construct a parametrized toy model for a curvature dominated universe, i.e. – a spacetime in which the spatial curvature of the 3-dimensional slices dominates over the contribution of the (nonrelativistic) matter. Using Buchert’s averaging scheme we find that the effective scale factor obtained in this toy model, does in fact accelerate for a wide class of parameter values. Now one needs to ask whether this can happen in the real universe, for which we turn to Zalaletdinov’s approach.

Simplifying Zalaletdinov’s framework  
A Paranjape and T P Singh, Phys. Rev. D76, 044006 (2007).

As it stands, Zalaletdinov’s averaging framework deals with averaging an arbitrary spacetime, and due to its generality it is technically challenging and difficult to work with. By restricting its application to cosmology and requiring consistency with basic cosmological assumptions, we find that we can simplify this framework and bring it to a form which can be readily applied to perform calculations. Doing this also clarifies the nature of the backreaction as being a physically relevant quantity on the same footing as the scale factor of the FLRW spacetime. An important point is that for cosmology one must necessarily consider the spatial averaging limit of Zalaletdinov’s 4-dimensional spacetime averaging. In this limit we further highlight the similarities and differences between Zalaletdinov’s and Buchert’s approaches, and the fact that the structure of the correction terms in both approaches is very similar, being essentially the same as expected from the heuristic arguments of Ellis discussed earlier.

There is a significant difference between the original philosophy of the averaging formalism, common to both the Buchert and Zalaletdinov schemes, and the manner in which we employ Zalaletdinov’s averaging. The original idea as developed by these authors was to construct a framework which would independently describe a suitably defined averaged dynamics, with no reference to the inhomogeneous spacetime whose average leads to this dynamics. So, for example, Zalaletdinov formulates a new theory of gravity (named Macroscopic Gravity or MG) which attempts to describe the dynamics of an averaged manifold, with no recourse to the underlying manifold which is described by the usual Einstein equations. The backreaction in this approach is actually a new field in the problem which satisfies its own equations and whose dynamics must be solved for simulta-
neously with that of other fields such as the averaged metric and the averaged energy-momentum tensor for matter.

Our approach to the backreaction issue is different: We consider it central to be able to self-consistently describe both the inhomogeneous geometry as well as its averaged counterpart. We find this necessary since modern cosmology crucially relies on observations of inhomogeneities around us, and ignoring the evolution of inhomogeneities when solving for the averaged dynamics does not appear to be satisfactory. Put another way, when faced with a solution of the averaged dynamics, we find it essential to answer the question “which (if any) inhomogeneous solution could lead to this averaged homogeneous solution?” All our calculations therefore focus on solving for the averaged dynamics of specific inhomogeneities, which we attempt to keep as realistic as possible.

**Backreaction in (linear) perturbation theory**

**A Paranjape, Phys. Rev. D78, 063522 (2008).**

We apply the simplified version of Zalaletdinov’s scheme to the problem of calculating the backreaction in the perturbative framework and determining its evolution. We discuss the issue of a possible *gauge dependence* of the backreaction, which is essentially the problem that in the perturbative framework one must be careful to distinguish physical effects from artifacts of choosing a specific coordinate system. We show how the backreaction can be calculated in a gauge independent manner, although one is forced to make certain choices concerning the averaging operation itself, which are not fixed by Zalaletdinov’s formalism. Once the formalism is developed, we are left with expressions for the backreaction that are valid whenever the *metric* of spacetime can be described as a perturbation around its FLRW form, regardless of the magnitude of matter density fluctuations.

To deal with the issue of self-consistency, we propose an iterative procedure to compute the backreaction. Since order-of-magnitude estimates of the backreaction indicate that the effect is expected to be small in the early universe (around the epoch of last scattering say), we begin with a “zeroth iteration” in which the backreaction is assumed to vanish entirely. This of course is simply the setup for the standard treatment of cosmology, in which the evolution of inhomogeneities can be numerically evaluated. We do this and consequently obtain a first estimate for the backreaction using the formalism developed earlier. We find that the magnitude of the backreaction in this first estimate remains negligible ($\sim 10^{-4}$ at present epoch in dimensionless units) compared to the background contribution at all times, and its evolution indicates that continuing to further iterations would not lead to any instability; the final answer is expected to converge to a form very close to the original “zeroth order” choice for the background.
The nonlinear regime

The preceding arguments however are valid in the linear regime of perturbation theory where matter fluctuations are small. However, the structure of the integrands of the backreaction functions indicate that the contribution from length scales where matter fluctuations have become nonlinear, should in fact also remain negligible. Yet one would like to see this in an actual nonlinear calculation rather than relying on heuristic arguments. Specifically, in the nonlinear regime when matter fluctuations are large, one needs to address two issues: (a) Is a perturbative expansion in the metric still valid? (b) If so, then is the contribution to the backreaction from nonlinear scales in fact negligible?

A toy model for structure formation

A Paranjape and T P Singh, JCAP03(2008)023;

It has been claimed in the literature [37] that perturbation theory does not give correct insight into the problem of structure formation in the late universe, and that when one studies simple but nonperturbative examples of structure formation, the contribution of backreaction to the averaged dynamics is in fact large. These claims are clearly contradictory to the conventional wisdom and to the arguments presented earlier. We attempt to sort out this debate by studying a toy model of structure formation, using the spherically symmetric LTB solution.

The matter source in the LTB solution is pressureless “dust”, which is sufficient for our purposes since we wish to enquire whether a universe dominated by nonrelativistic matter can have a large backreaction component in realistic situations. We assume initial conditions to be a perturbation around an FLRW model without dark energy. Our model contains an inner spherical overdense ball surrounded by an underdense shell, outside which we take the matter density to be homogeneous. The overdense ball initially expands but eventually turns around and begins collapsing, mimicking for example the infalling region outside clusters of galaxies. This happens because this inner region satisfies equations which are identical to those of a “closed” FLRW model in which the universe eventually recollapses. Naively one would expect that the underdense region behaves like the “open” FLRW models which expand forever, however the situation is more involved. It turns out that imposing appropriate matching conditions at the boundary of the over- and underdense regions, implies that a section of the underdense region immediately surrounding the overdense ball, will in fact eventually collapse.

This result has interesting consequences. By simply ignoring this part of the underdense shell that eventually turns around, we can show using our model that arguments such as in Ref. [37] which ignore matching conditions across boundaries can lead to an accelerating effective scale factor. However, as soon as this region is correctly accounted for, the acceleration disappears.

Consistently with this result, we show that one can rewrite the nonperturbative LTB solution as a perturbation around the same FLRW model we started with, provided the peculiar velocity of
the dust remains nonrelativistic. This is exactly what one expects from standard textbook results concerning cosmology in the Newtonian limit of GR. The small parameter governing the linear perturbation theory valid in the early universe is the magnitude of the matter density contrast $\delta$, while the small parameter relevant for the late time Newtonian limit is the nonrelativistic peculiar velocity $v$. Our calculation is an explicit demonstration in a physically clear and simple setting, of how the perturbation theory in $\delta$ becomes a perturbation theory in $v$.

**Backreaction in the nonlinear regime**  
A Paranjape and T P Singh, Phys. Rev. Lett. 101, 181101 (2008).

As a final step, for completeness we compute the backreaction in our model of structure formation described above. The formalism described in the preceding sections can be applied to this model since we can bring the metric of this model to the perturbed FLRW form. There are some subtleties regarding the numerical calculations, since the coordinates that are natural to the model are not natural to the backreaction formalism, but these can be handled in a straightforward manner. As expected, we find that the backreaction for such a model of structure formation is in fact negligible. The significance of this calculation is that it is the first one in which the backreaction has been calculated as a physically meaningful quantity even in the late time nonlinear phase of the cosmological evolution.

**Conclusions and Outlook**

The question of whether backreaction from averaging of inhomogeneities can lead to significant effects, has generated a heated debate in the literature. There have been conflicting results on issues such as the stability of perturbation theory in the presence of these corrections. Our calculations using Zalaletdinov’s covariant averaging scheme applied to both linear perturbation theory and toy models of nonlinear structure formation, form the first systematic demonstration that perturbation theory is in fact stable against corrections due to backreaction, and that backreaction cannot explain the late time acceleration of the universe. In principle such calculations can be extended to numerical simulations of structure formation, however that is beyond the scope of this work.

Given that cosmological data is rapidly increasing in quantity and improving in quality, it will soon become possible to determine cosmological parameters with percent level accuracy, and even perform tests of fundamental assumptions such as the Copernican principle. In this context, it becomes interesting to ask whether the contribution of the backreaction, while very small compared to the background, could be tested or used to improve parameter estimation. This remains a subject for future work.

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2The peculiar velocity is defined as the difference between the physical velocity and the Hubble flow of the dust element.
List of publications

• Publications contributing to this thesis

I. “The Possibility of Cosmic Acceleration via Spatial Averaging in Lemaitre-Tolman-Bondi Models”, Aseem Paranjape and T. P. Singh, *Class. Quant. Grav.* **23**, 6955 (2006) [arXiv:astro-ph/0605195].

II. “The Spatial Averaging Limit of Covariant Macroscopic Gravity – Scalar Corrections to the Cosmological Equations”, Aseem Paranjape and T. P. Singh, *Phys. Rev.* **D76**, 044006 (2007) [arXiv:gr-qc/0703106].

III. “Backreaction of Cosmological Perturbations in Covariant Macroscopic Gravity”, Aseem Paranjape, *Phys. Rev.* **D78**, 063522 (2008) [arXiv:0806.2755].

IV. “Structure Formation, Backreaction and Weak Gravitational Fields”, Aseem Paranjape and T. P. Singh, *JCAP* **03**(2008)023 [arXiv:0801.1546].

V. “Cosmic Inhomogeneities and the Average Cosmological Dynamics”, Aseem Paranjape and T. P. Singh, *Phys. Rev. Lett.* **101**, 181101 (2008) [arXiv:0806.3497].

• Other publications

1. “Nonlinear Structure Formation, Backreaction and Weak Gravitational Fields”, Aseem Paranjape, arXiv:0811.2619 (2008), talk presented at CRAL-IPNL Conference on Dark Energy and Dark Matter, Observations, Experiments and Theories, July, 2008, Lyon Center for Astrophysics Research (CRAL), Lyon, France; to appear in the Conference Proceedings.

2. “A Covariant Road to Spatial Averaging in Cosmology – Scalar Corrections to the Cosmological Equations”, Aseem Paranjape, *Int. J. Mod. Phys.* **D17**, 597 (2008) [arXiv:0705.2380]. (This essay received an Honourable Mention in the Gravity Research Foundation’s Essay Competition, 2007.)

3. “Entropy of Null Surfaces and Dynamics of Spacetime”, T. Padmanabhan and Aseem Paranjape, *Phys. Rev.* **D75**, 064004 (2007) [arXiv:gr-qc/0701003].

4. “Explicit Cosmological Coarse Graining via Spatial Averaging”, Aseem Paranjape and T. P. Singh, *Gen. Rel. Grav.* **40**, 139 (2008) [arXiv:astro-ph/0609481].

5. “Thermodynamic route to field equations in Lanczos-Lovelock gravity”, Aseem Paranjape, Sudipta Sarkar and T. Padmanabhan, *Phys. Rev.* **D74**, 104015 (2006) [arXiv:hep-th/0607240].

6. “Embedding diagrams for the Reissner-Nordstrom space-time”, Aseem Paranjape and Naresh Dadhich, *Gen. Rel. Grav.* **36**, 1189 (2004) [arXiv:gr-qc/0307056]. (This paper was the result of part of the work done under the JNCASR Summer Research Fellowship, April – June 2003.)
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Chapter 1

Introduction

Our understanding of the universe has undergone dramatic changes in the last century. Edwin Hubble’s discovery in 1924 that stars known as Cepheid variables could be found in the Andromeda nebula and appeared fainter than Cepheids in the Milky Way, established that such nebulae were not part of the Milky Way but were in fact distant galaxies themselves. And his demonstration 5 years later that these galaxies appear to be receding from the Milky Way at speeds proportional to their distances, has found its way into popular consciousness as the maxim “the universe is expanding” [1]. On the theoretical front, Einstein’s general relativity had at this time quickly gained acceptance as a fundamental theory of gravity, and its application to cosmology was being studied by several workers including Einstein himself, deSitter, Lemaître, Friedmann among others. The simple models of a universe described by the homogeneous and isotropic geometries characterised by the Friedmann-Lemaître-Robertson-Walker (FLRW) metric, were very successful at describing the then limited amount of cosmological observations. The decades since have seen the emergence of the highly successful Big Bang model of cosmology, which posits that the universe went through a very hot dense phase at early times and cooled as it expanded, with tiny fluctuating inhomogeneities in the past that have grown to form structures such as galaxies today [2].

A central assumption in this (widely accepted) model of cosmology is that the universe on large scales is homogeneous and isotropic [3]. This assumption leads to tremendous simplifications in the application of general relativity (GR) to cosmology, since it reduces the ten independent components of the metric of spacetime $g_{ab}(t, \vec{x})$ to essentially a single function of time $a(t)$ known as the scale factor. In the early days of 20th century cosmology, the assumption of homogeneity and isotropy was largely motivated on grounds of simplicity and aesthetic appeal. In recent times however, it has become possible to confront this assumption with observations, which remarkably appears to be justified to a large extent (based on observations of the CMB radiation [4] [5], and on analyses of galaxy surveys [6] [7] [8], although see Ref. [9]). This indicates that a model based on essentially a single function of time might in fact go a long way in furthering our understanding of the behaviour of the universe.
Of course the real universe is not homogeneous; we see a rich variety of structure around us from stellar systems to galaxies to clusters of galaxies and even larger structures [10]. Perhaps one of the biggest successes of cosmological theory based on GR, has been the explanation of how statistical properties of the large scale structure arise [11]. The relevant calculations are largely based on linear perturbation theory (i.e. linearizing Einstein’s equations around the smooth FLRW solution) which is valid at all length scales of interest at early times and on large scales at late times [12, 13]. Dynamics on small scales at late times involves nonlinear theory, and is dealt with using approximation schemes such as the Press-Schechter formalism and its extensions [14], “Newtonian” nonlinear perturbation analyses [15] and numerical simulations [16]. While such treatments have met with great success in the description of the statistical properties of the anisotropies in the temperature of the CMB, as well as of the inhomogeneous distribution of galaxies, there are two causes for concern.

The first is a purely theoretical issue, and is the basis of this work. The idea that the large scale universe is homogeneous and isotropic necessarily entails an implicit notion of averaging on these large scales. In other words, what one is really saying is, “When the spatially fluctuating parts of the solution of GR describing our universe are averaged out, what is left is the homogeneous and isotropic FLRW solution of Einstein’s equations”. The immediately obvious problem with this statement is that the details of the averaging operation are not at all clear, and indeed are usually never specified. A bigger problem is one noted by Ellis [17], and can be stated in the following symbolic way. If $g$ denotes the metric, $\Gamma$ the Christoffel connection and $E[g]$ the Einstein tensor for the metric $g$, then we have the relations

$$\Gamma \sim \partial g \ ; \ E[g] \sim \partial \Gamma + \Gamma^2,$$

(1.1)

with $\partial$ denoting spacetime derivatives. The Einstein equations are therefore

$$E[g] = T,$$

(1.2)

with $T$ denoting the energy-momentum tensor of the matter components. Now, irrespective of any details of the averaging operation, one notes that

$$E[\langle g \rangle] - \langle E[g] \rangle \sim \langle \Gamma \rangle^2 - \langle \Gamma^2 \rangle \neq 0,$$

(1.3)

with the angular brackets denoting the averaging. The FLRW solution would amount to solving the equations $E[\langle g \rangle] = \langle T \rangle$. In general therefore, it is not true that averaging out the fluctuating inhomogeneities leaves behind the FLRW solution, since what we are actually left with is

$$E[\langle g \rangle] = \langle T \rangle - C \ ; \ C \sim \langle \Gamma^2 \rangle - \langle \Gamma \rangle^2,$$

(1.4)

and the homogeneous solution that we are looking for will depend on the details of the correction
terms $C$.

The second cause for concern comes from observations. It has now been established beyond a reasonable doubt, that the FLRW metric confronted with observations indicates an accelerating scale factor \[18\]. Conventional sources of energy such as radiation and nonrelativistic matter cannot explain the acceleration, and it is now common to attribute this effect to a hitherto unknown “dark energy”, which in its simplest form is a cosmological constant. The true nature of this additional component in the cosmological equations, is perhaps the most challenging puzzle facing both theorists and observers today. A huge amount of research has gone into (a) explaining the value that a cosmological constant term must take to explain data or (b) assuming a zero cosmological constant, constructing models of a dynamical dark energy which explains the observed acceleration \[19\]. It is fair to say however that there is no theoretical consensus on what the origin of dark energy is. Since we have seen above that the effects of averaging lead to some extra, as yet unknown terms in the equations, it is natural to ask whether these two issues are connected. Could the acceleration of the universe be explained by the effects of averaging inhomogeneities (“backreaction”) in the universe? Regardless of the answer to this question, what is the nature and magnitude of this backreaction? The purpose of this thesis is to answer these questions as rigorously as possible.

1.1 History of the averaging problem

The problem of averaging in general relativity has a history going back even further than Ellis’ work of 1984. In the context of gravitational radiation, the problem of second order effects of gravity waves on the large scale background metric of spacetime was studied by Isaacson in the 1960’s \[20\] in the “short-wavelength” approximation. Isaacson used an averaging operation which he called the “BH assumption” after Brill and Hartle \[21\], which was suited to studying the effects of perturbative gravity waves in a spacetime region encompassing many wavelengths. An attempt to generalize Isaacson’s results was made by Noonan \[22\], who introduced a different averaging procedure which was also constructed for situations where inhomogeneities were perturbative in nature. Interest in the cosmological consequences of such an averaging picked up only after Ellis very clearly laid down the problems and possibilities that open up when the idea of averaging in general relativity is taken seriously. An example is the work of Futamase \[23\], who introduced a spatial averaging procedure after performing a $3 + 1$ splitting of spacetime, and computed backreaction terms arising from averaging second order perturbations, finding them to be negligibly small (see also Ref. \[24\]). Another example is the work of Boersma \[25\], who attempted to construct a gauge-invariant (i.e. coordinate independent) averaging procedure in perturbation theory, and also estimated that backreaction effects remain negligibly small at the present epoch. (For other work on the averaging problem, see Ref. \[26\].)

It may seem intuitively obvious that perturbatively small inhomogeneities can only lead to negligibly small backreaction effects. Indeed, this has been the conventional wisdom on this sub-
ject, and has recently been spelt out by Ishibashi and Wald [27]. One starts by assuming that inhomogeneities in the universe can be described in the Newtonian approximation of GR, by the gravitational potential $\Phi(t, \vec{x})$ with $|\Phi| \ll 1$, which satisfies the Poisson equation $\nabla^2 \Phi = 4\pi G a^2 \delta \rho$ where $\delta \rho(t, \vec{x})$ is the fluctuation of matter density about the mean homogeneous value $\bar{\rho}(t)$, and can in general have a large value. (E.g., in clusters of galaxies one finds $\delta \equiv \delta \rho/\bar{\rho} \sim 10^2$, and the ratio increases on smaller length scales). One then argues that the universe we observe does seem to be very well-described by the above model, and effects of averaging this model can only arise at second order in $\Phi$ and should hence be extremely small.

There is a loophole in this argument though. The catch is that the background expansion $a(t)$ is defined completely ignoring the backreaction, which is an integrated effect with contributions from a large range of length scales. This means that the following possibility cannot be $a priori$ ruled out: Initial conditions are specified as a perturbation around a specific FLRW solution, but the integrated effect of the backreaction grows (with time) in such a manner as to effectively yield a late time solution which is a perturbation around a different FLRW model. Indeed, there are calculations in the literature that do indicate that this may happen. For example, Martineau and Brandenberger [28] showed in a toy model that long wavelength fluctuations can give rise to a backreaction contribution which has a late-time effective equation of state similar to a cosmological constant. Their calculations were based on the averaging procedure developed by Abramo et al. [29] in the context of backreaction in inflationary cosmology. Other claims to solving the dark energy problem using backreaction from long wavelength fluctuations were made by Barausse et al. [30] and Kolb et al. [31]. It is fair to say however, that such claims have been controversial. A number of authors have argued that when effects of long wavelength fluctuations are suitably “renormalized” and the background suitably redefined, the backreaction cannot lead to acceleration of the scale factor [32]. Nevertheless, what is definitely true is that the idea of backreaction of cosmological fluctuations has generated a lively debate in the community [27, 33, 34, 35].

In this thesis we will not deal with the effects of long wavelength fluctuations, although we will see that certain assumptions need to be made in order to define a self-consistent perturbation theory in the presence of an averaging operation. A separate and equally interesting question, which will be the main focus of this work, is whether cosmological perturbation theory is stable in the presence of the backreaction contribution. There are results in the literature which indicate that this might not be the case, and that the backreaction can grow with time in such a manner that at late times (when matter fluctuations have become nonlinear) perturbation theory in the metric also no longer holds [34] (see also Ref. [36]). In the same vein, there are arguments using nonperturbative toy models of gravitational collapse and nonlinear structure formation, which suggest that perturbation theory may not give correct insight into gravitational dynamics at late times in cosmology [37]. If these results are relevant for the real world, then it not only means that the conventional wisdom is badly failing, but in fact implies that all of late-time cosmology must be reworked from scratch (see, e.g. Ref. [38]; also see however Ref. [39]). On the other hand, if these results are for some
reason or other not realistic, then it is important ask what is wrong with such arguments, and further what the correct approach to the problem is.

Clearly, to make any headway in this problem, it is first essential to have a reliable averaging scheme at hand. Since the questions one is asking deal with the stability of cosmological perturbation theory, this averaging scheme needs to be inherently nonperturbative, i.e. the validity of perturbation theory should not be a prerequisite to defining the averaging prescription. This thesis will deal with two averaging schemes present in the literature: the spatial averaging of scalars defined by Buchert [40, 41, 42], and the fully covariant tensor averaging defined by Zalaletdinov [43, 44, 45]. Some very interesting early work on possible nonperturbative effects of averaging was by Buchert and Ehlers [46], followed up by Ref. [47], in the context of spatial averaging in Newtonian cosmology. Buchert’s averaging operation in general relativity has since been used by several authors to explore the effects of backreaction in various situations [38, 48, 49, 50, 51], including the perturbative contexts mentioned above [31, 36], and has also been compared against observations [52]. As we shall see later, this averaging scheme has an appealing simplicity of implementation, which could be a reason for the amount of attention it has received. In contrast, Zalaletdinov’s averaging scheme (which was developed earlier than Buchert’s work) is technically rather challenging to handle and involves a fair amount of complicated algebra. Its strength however lies in the fact that it is a fully covariant prescription which, at the end of the day, yields an object which can be legitimately called the “averaged metric” on an “averaged manifold”. This ultimately allows us to make physically clear statements regarding the backreaction, which is difficult to do in Buchert’s scheme as it stands. While Zalaletdinov’s scheme has not received the same amount of attention as Buchert’s, there has been a series of very interesting results derived in this framework by Zalaletdinov and coworkers [53].

1.1.1 The “Special Observer” assumption

The idea of using inhomogeneities to explain the dark energy problem has generated a flurry of research in the backreaction problem in recent years, as we saw above (see also Ref. [54]). It is important to also mention another approach which has gained popularity in this context, namely that of ascribing the dark energy phenomenon to light propagation effects in an inhomogeneous universe [55]. The central idea here is that light propagation through an inhomogeneous underdensity or “void” can be significantly different from that in a homogeneous space. In fact, it is possible to show that luminosity distance data from supernovae can always be fit by modelling ourselves as observers in a void with a suitable density profile. Typically however, the (usually spherical) voids invoked for this purpose are very large (in the range of $\sim 200h^{-1}\text{Mpc}$ to $\sim 1h^{-1}\text{Gpc}$ in diameter), and are difficult to reconcile with the typical sizes of voids seen in galaxy surveys, which are in the range of 30-50$h^{-1}\text{Mpc}$, with some “supervoids” reaching $\sim 100h^{-1}\text{Mpc}$ [56]. Nevertheless, this idea has been rather vigorously investigated in the last several years. Since this thesis will not directly deal with this approach to the dark energy problem, we will simply point the reader to
a list of references \cite{57} dealing with the study of light propagation in an inhomogeneous universe and of supernovae data and the CMB from the point of view of void-based observers. Unlike the backreaction issue which requires mainly theoretical work, a detailed description of the inhomogeneous universe belongs squarely in the regime of observational cosmology \cite{58}. Due to the obvious observational difficulties involved in such a program (for example due to the lack of homogeneous samples of galaxy data), this approach at present is largely restricted to being an exercise in building toy models of the local large scale structure \cite{57}. As a final comment on this topic, we note that this “non-Copernican” approach (even at the level of building toy models) is amenable to observational verification or disproof in the coming generation of surveys, as pointed out by Ref. \cite{60}.

With this introduction, our main results (the thesis of the thesis!) are:

- **Although technically possible, in the real world backreaction does not significantly affect the expansion history of the universe.**

- **Cosmological perturbation theory is stable against backreaction effects, well into the nonlinear regime.**

- **Dark energy cannot therefore be an effect of the backreaction of inhomogeneities.**

The outline of this thesis is as follows:

Chapters 2 and 3 will deal respectively with Buchert’s and Zalaletdinov’s averaging schemes. We will first describe Buchert’s scheme in chapter 2 and show using a toy model of a spherically symmetric inhomogeneous spacetime (the Lemaître-Tolman-Bondi or LTB solution \cite{61}), how an averaged effective description of the spacetime can have an accelerating scale factor even when the underlying exact solution has no exotic elements. This calculation will be based on Paranjape and Singh, CQG (2006) (henceforth Paper 1). We will then turn to Zalaletdinov’s formalism in chapter 3 and give a pedagogical introduction to his 4-dimensional covariant averaging scheme and the derivation of the equations in his effective theory of Macroscopic Gravity (MG). We will then specialize this formalism for use in cosmology and emphasize the need for a spatial averaging limit of this averaging. By explicitly writing out the backreaction terms in a 3 + 1-splitting of spacetime, we will also be in a position to give a detailed comparison between Zalaletdinov’s formalism and Buchert’s spatial averaging, and to demonstrate the physical relevance of the backreaction terms. This part will be based on Paranjape and Singh, PRD (2007) (henceforth Paper 2).

In chapter 4 we will use the spatial averaging limit of Zalaletdinov’s scheme in the setting of cosmological perturbation theory. We will show how the leading order contribution to the backreaction can be calculated in a gauge invariant manner, and derive expressions for the backreaction which are valid whenever the metric of spacetime can be written as a perturbation around the FLRW form. We will also discuss the issue of self-consistency of the backreaction calculation, and
propose an iterative scheme to calculate the backreaction. As concrete examples we will perform the first iteration in such a process, for some well-studied models of perturbative inhomogeneities in the linear regime. This set of calculations will be based on Paranjape, PRD (2008) (henceforth Paper 3).

Chapter 5 will deal with the regime when matter fluctuations have become large, so that linear perturbation theory no longer holds. We will use the LTB solution once more to describe a semi-realistic situation of spherical collapse, which we will follow well into the nonlinear regime. Initially working in the Buchert formalism, we will emphasize the importance of correctly accounting for boundary conditions when building models, by showing how spurious results can be obtained for the averaged expansion, by ignoring boundary conditions. We will also show how the late time nonperturbative behaviour in our LTB model can be recast in the perturbed FLRW form, by a straightforward coordinate transformation, provided matter velocities remain non-relativistic. This will demonstrate in a clear and unambiguous manner, how a perturbation theory in the density contrast \( \delta \) at early times becomes a perturbation theory in peculiar velocity \( v \) at late times. These results will be based on Paranjape and Singh, JCAP (2008) (henceforth Paper 4). Finally, to complete the picture, we will apply the formalism developed in chapter 3, to the toy model described above, and explicitly show that the backreaction in the nonlinear regime of structure formation does remain small in this model. This calculation will be based on Paranjape and Singh, PRL (2008) (henceforth Paper 5).

We will conclude in chapter 6 with a summary and a brief comparison with other work in the literature. Additionally, in the Appendices we have collected some results which will be used in the main text. Appendix A describes the homogeneous and isotropic FLRW cosmology and serves to fix notation. Appendix B describes the LTB metric which is a solution of the Einstein equations sourced by a spherically symmetric pressureless fluid. Appendix C contains proofs of some results quoted in chapter 2. Throughout this work we shall set the speed of light \( c \) to unity, and use the metric signature \((- + + +)\). Lowercase Latin indices \( a, b, \ldots i, j, \ldots \) will take values 0, 1, 2, 3, while uppercase indices \( A, B, \ldots I, J, \ldots \) will take spatial values 1, 2, 3. The Hubble constant \( H_0 \) is parametrized when necessary in usual astronomers’ units as \( H_0 = 100 h \text{km s}^{-1} \text{Mpc}^{-1} \).
Chapter 2

Averaging schemes : Buchert’s spatial averaging

This chapter and the next describe the averaging schemes we will use for the backreaction calculations. In this chapter we deal with Buchert’s spatial averaging scheme and show in a toy example how a large, acceleration-inducing backreaction can arise even in the absence of any exotic dark energy.

2.1 Buchert’s spatial averaging of scalars

The most straightforward and intuitively clear application of Buchert’s spatial averaging is in the case when the matter source is a pressureless “dust” with an energy-momentum tensor $T^{ab} = \rho u^a u^b$, with $u^a$ the dust 4-velocity which satisfies $u_a u^a = -1$. Assuming further that the dust is irrotational, the 4-velocity will be orthogonal to 3-dimensional spatial sections and the metric can be written in “synchronous and comoving” coordinates (in which $u^a = (1, \vec{0})$) \[62\] as,

$$ds^2 = -dt^2 + h_{AB}(t, \vec{x}) dx^A dx^B.$$ (2.1)

The expansion tensor $\Theta^A_B$ is given by $\Theta^A_B = (1/2) h^{AC} \dot{h}_{CB}$ where the dot refers to a derivative with respect to time $t$. The traceless symmetric shear tensor is defined as $\sigma^A_B = \Theta^A_B - (\Theta/3) \delta^A_B$ where $\Theta = \Theta_A^A$ is the expansion scalar. The Einstein equations can be split into a set of scalar equations and a set of vector and traceless tensor equations. The scalar equations are the Hamiltonian constraint (2.2a) and the evolution equation for $\Theta$ (2.2b),

$$R + \frac{2}{3} \Theta^2 - 2\sigma^2 = 16\pi G \rho ,$$ (2.2a)

$$R + \dot{\Theta} + \Theta^2 = 12\pi G \rho ,$$ (2.2b)

8
where \((3)\mathcal{R}\) is the Ricci scalar of the 3-dimensional hypersurface of constant \(t\) and \(\sigma^2\) is the rate of shear defined by \(\sigma^2 \equiv (1/2)\sigma_B^A\sigma^A_B\). Eqns. \((2.2a)\) and \((2.2b)\) can be combined to give Raychaudhuri’s equation

\[
\dot{\Theta} + \frac{1}{3} \Theta^2 + 2\sigma^2 + 4\pi G \rho = 0.
\] 

(2.3)

The continuity equation \(\dot{\rho} = -\Theta \rho\) which gives the evolution of \(\rho\), is consistent with Eqns. \((2.2a), (2.2b)\). We only consider the scalar equations, since the spatial average of a scalar quantity can be defined in a gauge covariant manner within a given foliation of spacetime. For the spacetime described by \((2.1)\), the spatial average of a scalar \(\Psi(t, \vec{x})\) over a comoving domain \(\mathcal{D}\) at time \(t\) is defined by

\[
\langle \Psi \rangle_{\mathcal{D}} = \frac{1}{V_{\mathcal{D}}} \int_{\mathcal{D}} d^3x \sqrt{h} \, \Psi,
\]

(2.4)

where \(h\) is the determinant of the 3-metric \(h_{AB}\) and \(V_{\mathcal{D}}\) is the volume of the comoving domain given by \(V_{\mathcal{D}} = \int_{\mathcal{D}} d^3x \sqrt{h}\). The following commutation relation then holds \([10]\)

\[
\langle \Psi \rangle_{\mathcal{D}} - \langle \dot{\Psi} \rangle_{\mathcal{D}} = \langle \Psi \Theta \rangle_{\mathcal{D}} - \langle \Psi \rangle_{\mathcal{D}} \langle \Theta \rangle_{\mathcal{D}},
\]

(2.5)

which yields for the expansion scalar \(\Theta\)

\[
\langle \Theta \rangle_{\mathcal{D}} - \langle \dot{\Theta} \rangle_{\mathcal{D}} = \langle \Theta^2 \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}}^2.
\]

(2.6)

Introducing the dimensionless scale factor \(a_{\mathcal{D}} \equiv (V_{\mathcal{D}}/V_{\mathcal{D}_0})^{1/3}\) normalized by the volume of the domain \(\mathcal{D}\) at some initial time \(t_{in}\), we can average the scalar Einstein equations \((2.2a), (2.2b)\) and the continuity equation to obtain \([10]\)

\[
\left( \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right)^2 = \frac{8\pi G}{3} \langle \rho \rangle_{\mathcal{D}} - \frac{1}{6} \left( Q_{\mathcal{D}} + \langle \mathcal{R} \rangle_{\mathcal{D}} \right),
\]

\[
\left( \frac{\ddot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \right) = -\frac{4\pi G}{3} \langle \rho \rangle_{\mathcal{D}} + \frac{1}{3} Q_{\mathcal{D}},
\]

\[
\langle \rho \rangle_{\mathcal{D}} = -\langle \Theta \rangle_{\mathcal{D}} \langle \rho \rangle_{\mathcal{D}} = -3 \frac{\dot{a}_{\mathcal{D}}}{a_{\mathcal{D}}} \langle \rho \rangle_{\mathcal{D}}.
\]

(2.7a, b, c)

Here \(\langle \mathcal{R} \rangle_{\mathcal{D}}\), the average of the spatial Ricci scalar \((3)\mathcal{R}\), is a domain dependent spatial constant. The ‘backreaction’ \(Q_{\mathcal{D}}\) is given by

\[
Q_{\mathcal{D}} \equiv \frac{2}{3} \left( \langle \Theta^2 \rangle_{\mathcal{D}} - \langle \Theta \rangle_{\mathcal{D}}^2 \right) - 2\langle \sigma^2 \rangle_{\mathcal{D}},
\]

(2.8)

and is also a spatial constant. The last equation \((2.7c)\) simply reflects the fact that the mass contained in a comoving domain is constant by construction : since \(\Theta = \partial_t \ln \sqrt{h}\), the local continuity equation \(\dot{\rho} = -\Theta \rho\) can be solved to give \(\rho \sqrt{h} = \rho_0 \sqrt{h_0}\) where the subscript 0 refers to some arbitrary reference time \(t_0\). The mass \(M_{\mathcal{D}}\) contained in a comoving domain \(\mathcal{D}\) is then
\[ M_D = \int_D \rho \sqrt{h} d^3x = \int_D \rho_0 \sqrt{h_0} d^3x = \text{constant}. \]

Hence

\[ \langle \rho \rangle_D = \frac{M_D}{V_{Din} a_D^3} \quad (2.9) \]

which is precisely what is implied by Eqn. (2.7c). Equations (2.7a), (2.7b) can be thought of as “modified Friedmann equations” (compare Eqns. (A.3)), with the modifications arising due to the presence of the backreaction \( Q_D \) and the fact that the averaged Ricci curvature in general need not evolve like \( \sim a_D^{-2} \) as in the FLRW case.

A necessary condition for (2.7b) to integrate to (2.7a) takes the form of the following differential equation involving \( Q_D \) and \( \langle R \rangle_D \)

\[ \dot{Q}_D + 6 \frac{\dot{a}_D}{a_D} Q_D + \langle R \rangle_D + 2 \frac{\dot{a}_D}{a_D} \langle R \rangle_D = 0, \quad (2.10) \]

which is a very interesting equation because it shows that the evolution of the backreaction is intimately tied to that of the average spatial curvature. Scaling solutions for this equation have been explored by Buchert, Larena and Alimi [50], a simple example being \( \langle R \rangle_D \propto a_D^{-2}, \ Q_D \propto a_D^{-6} \).

Clearly the FLRW solution with \( Q_D = 0 \) is a special case. In this thesis we will mainly be concerned with the behaviour of the backreaction arising from explicitly averaging an inhomogeneous spacetime, and will therefore not discuss these scaling solutions which make no reference to the underlying inhomogeneous geometry. We note that the criterion to be met in order for the effective scale factor \( a_D \) to accelerate, is

\[ Q_D > 4\pi G \langle \rho \rangle_D. \quad (2.11) \]

Equations (2.7) and (2.10) describe the essence of Buchert’s averaging formalism, for the simplest case of irrotational dust. We note that the remaining eight Einstein equations for the inhomogeneous geometry, which are not scalar equations, are not averaged. These are the five evolution equations for the trace-free part of the shear,

\[ \partial_t (\sigma^A_B) = -\Theta \sigma^A_B - R^A_B + \frac{2}{3} \delta^A_B \left( \sigma^2 - \frac{1}{3} \Theta^2 + 8\pi G \rho \right). \quad (2.12) \]

and the three equations relating the spatial variation of the shear and the expansion,

\[ \sigma^A_B \parallel_A = \frac{2}{3} \Theta \parallel_B. \quad (2.13) \]

Here, \( R^A_B \) is the spatial Ricci tensor and a \( \parallel \) denotes covariant derivative with respect to the 3-metric. Later, when we apply Zalaletdinov’s formalism to cosmology, we will see that accounting for the full set of Einstein’s equations when performing the averaging, leads to additional constraints on the inhomogeneous geometry, which are not present in Buchert’s formalism.

For completeness, and to enable a fuller comparison with Zalaletdinov’s formalism, we also
display the results of applying Buchert’s averaging to the case of a perfect fluid with energy-momentum tensor $T^{ab} = (\rho + p)u^au^b + pg^{ab}$ [11]. In this case comoving coordinates are not in general synchronous, and for an irrotational perfect fluid in comoving coordinates, the metric takes the form

$$ds^2 = -N^2dt^2 + h_{AB}dx^Adx^B.$$  \hspace{1cm} (2.14)

The averaged scalar Einstein equations for the scale factor $a_D$ are

$$3\frac{\partial^2 a_D}{a_D} + 4\pi G\langle N^2 (\rho + 3p) \rangle_D = \bar{Q}_D + \bar{P}_D,$$ \hspace{1cm} (2.15)

$$6H_D^2 - 16\pi G\langle N^2 \rho \rangle_D = -\bar{Q}_D - \langle N^2 R \rangle_D; \quad H_D = \frac{\partial a_D}{a_D},$$ \hspace{1cm} (2.16)

where the kinematical backreaction $\bar{Q}_D$ is given by

$$\bar{Q}_D = \frac{2}{3} \left( \langle (N\Theta)^2 \rangle_D - \langle N\Theta \rangle_D^2 \right) - 2\langle N^2 \sigma^2 \rangle_D,$$ \hspace{1cm} (2.17)

and the dynamical backreaction $\bar{P}_D$ is given by

$$\bar{P}_D = \langle N^2 A \rangle_D + \langle \Theta \partial_t N \rangle_D,$$ \hspace{1cm} (2.18)

where $A = \nabla_j (u^i \nabla_i u^j)$ is the 4-divergence of the 4-acceleration of the fluid. Eqn. (2.16) follows as an integral from Eqn. (2.15) if and only if the relation

$$\partial_t \bar{Q}_D + 6H_D \bar{Q}_D + \partial_t \langle N^2 R \rangle_D + 2H_D \langle N^2 R \rangle_D + 4H_D \bar{P}_D$$

$$- 16\pi G \left[ \partial_t \langle N^2 \rho \rangle_D + 3H_D \langle N^2 (\rho + p) \rangle_D \right] = 0,$$ \hspace{1cm} (2.19)

is satisfied. There are also the unaveraged equations (which we do not display here) for the shear, analogous to the shear equations (2.12) and (2.13) for dust.

### 2.2 Acceleration from averaging

In this section we use the spherically symmetric dust solution of Einstein’s equations (the Lemaître-Tolman-Bondi or LTB solution described in Appendix B) to construct an explicit example where Buchert’s effective scale factor accelerates even though the underlying model has no cosmological constant or dark energy. Our model contains a single underdense and “curvature-dominated” region (in a sense to be described below), whose evolution we study at late times.
2.2.1 Late time and curvature dominated unbound models

Consider the LTB metric (B.1)

$$ds^2 = -dt^2 + \frac{R^2(t,r)}{1 - k(r)r^2}dr^2 + R^2(t,r) \left(d\theta^2 + \sin^2 \theta d\phi^2 \right),$$

which satisfies

$$\dot{R}^2(t,r) = \frac{2GM(r)}{R(t,r)} - k(r)r^2,$$  \hspace{1cm} (2.21)

for the specific case $k(r) < 0$ in the region of interest, so that the solution is (B.4a)

$$R \approx r \sqrt{-k(r)} (t - t_0(r)),$$  \hspace{1cm} (2.22)

Although it is straightforward to numerically evaluate $R(t,r)$ for any given choice of the free functions, we would like to try and analytically simplify these expressions as far as possible. Since $R(t,r)$ in the unbound case is an increasing function of time for arbitrary $k(r) < 0$, Eqn. (2.21) shows that the late time solution (after neglecting the $1/R$ term in the equation) can be expressed as

$$R \simeq r \sqrt{-k(r)} (t - t_0(r)),$$  \hspace{1cm} (2.23)

[This leading order solution can also be derived from an asymptotic expansion of the solution for large $\eta$, see below.] The function $t_0(r)$ can be obtained using the closed form expression for $t(R)$ obtained by integrating Eqn. (2.22) \[65\] and the scaling $R_{in}(r) = r$, as

$$t_0(r) = t_{in} - r \left(\frac{r}{g} \right)^{1/2} F \left( \frac{r}{g} \right); \quad F(x) \equiv \frac{1}{x}(1 + x)^{1/2} - \frac{1}{x^{3/2}} \sinh^{-1} \left(x^{1/2}\right),$$  \hspace{1cm} (2.24)

where we have defined

$$g(r) \equiv \frac{2GM(r)}{-k(r)r^2}.$$  \hspace{1cm} (2.25)

If we further assume that the matter contribution encoded in $M(r)$ is negligible compared to that of the spatial curvature as encoded in $k(r)$, i.e. if $|g(r)/r| \ll 1$, then the expression for $t_0(r)$ also simplifies at the leading order to

$$t_0(r) \simeq t_{in} - \frac{1}{\sqrt{-k(r)}}.$$  \hspace{1cm} (2.26)

The leading order unbound LTB solution in this late time, negligible matter limit is then given by

$$R(t,r) = r \left[\lambda_r + \sqrt{-k(r)} (t - \lambda_t t_{in})\right],$$  \hspace{1cm} (2.27)

\[1\] We are assuming that $g(r)/r$ remains finite at all $r$. In particular as $r \rightarrow 0$ this implies $k(r \rightarrow 0) \sim r^\mu$; $\mu \leq 0$ (see Eqn. (B.6)).
where we have introduced two placeholders \( \lambda_r \) and \( \lambda_t \) (ultimately set to unity) which will remind us that we are working with a late time solution with large \( t \). Note that Eqn. (2.27) (with \( \lambda_r = \lambda_t = 1 \)), is the exact solution Eqn. (2.21) in the special case \( M(r) \to 0 \) for all \( r \). This actually corresponds to Minkowski spacetime, with the corresponding Riemann tensor being exactly zero. The constant time 3-spaces are hypersurfaces of negative curvature, with the 3-curvature being determined by the function \( k(r) \). The ‘FLRW’ limit of this solution is in fact the Milne universe; the solution (2.27) could hence be thought of as the ‘LTB’ type generalization of the Milne universe. Although we will use this form of the solution to draw conclusions regarding acceleration of Buchert’s \( a_D \), we will later argue that these conclusions are not altered by the presence of a nonzero but small amount of matter. Although it may appear at this stage that requiring \( M \to 0 \) renders the late time approximation redundant, we will see below that additionaly imposing the late time approximation allows us to write down a fairly straightforward sufficient condition for acceleration of \( a_D \), which would not be possible with only the \( M \to 0 \) condition.

For the metric (2.20), the volume of a spherical comoving domain of radius \( r_D \) is

\[
V_D = 4\pi \int_0^{r_D} \frac{R'R^2}{\sqrt{1-k(r)r^2}} \, dr .
\]

(2.28)

Substituting the solution (2.27) in this expression, we find

\[
V_D = (t - \lambda_t t_{in})^3 I_k + \lambda_r (t - \lambda_t t_{in})^2 I_{kr} + \lambda_t^2 (t - \lambda_t t_{in}) I_{kr^2} + \lambda_r^2 I_{r^2} ,
\]

(2.29)

where we have defined the domain dependent integrals

\[
I_k = 2\pi \int_0^{r_D} \frac{\sqrt{-kr^2}(-kr^2)'}{\sqrt{1-kr^2}} \, dr \quad ; \quad I_{kr} = 4\pi \int_0^{r_D} \frac{(-kr^3)'}{\sqrt{1-kr^2}} \, dr ,
\]

\[
I_{kr^2} = 4\pi \int_0^{r_D} \frac{(r^3 \cdot \sqrt{-kr^2})'}{\sqrt{1-kr^2}} \, dr \quad ; \quad I_{r^2} = 4\pi \int_0^{r_D} \frac{r^2}{\sqrt{1-kr^2}} \, dr .
\]

(2.30)

The sum of the exponents of \( \lambda_r \) and \( \lambda_t \) in each term in (2.29) indicates the relative order of that term with respect to the leading \( t^3 \) term. This approach of treating some terms as small compared to others is valid since the various integrals which multiply the powers of \( t \), are all finite and non-zero. Expanding \( V_D \) in powers of \( \lambda_t, \lambda_r \), we find for the effective scale factor,

\[
3 \frac{a_D}{\dot{a}_D} = \frac{\ddot{V}_D}{V_D} - \frac{2}{3} \left( \frac{\dot{V}_D}{V_D} \right)^2 = 2\lambda_t^2 \frac{I_{kr^2}}{I_k t^4} \left( I_{kr} - \frac{1}{3 I_k} (I_{kr})^2 \right) + \mathcal{O}(3) ,
\]

(2.31)

where \( \mathcal{O}(3) \) represents terms involving \( \lambda^m \lambda^n \) (i.e. containing \( 1/t^{m+n} \)) with \( m + n \geq 3 \).

We see that the generic late time (i.e. \( t \to \infty \)) behaviour of the unbound models under consideration is \( \ddot{a}_D \to 0 \), and that deviations from zero are small, being a second order effect. Whether the approach to \( \ddot{a}_D = 0 \) is via an accelerating or decelerating phase, depends upon the
Figure 2.1: The models described by $k(r) = -1/(1 + r^a)$. (a) The scaled function $P/I_k$. (b) $P/I_k$ plotted against $a$ for specific values of $r_D$.

relative magnitudes of the domain integrals involved. A sufficient condition for an unbound model with negligible matter to accelerate at late times, is

$$P \equiv I_{kr^2} - \frac{1}{3I_k} (I_{kr})^2 > 0.$$  \hfill (2.32)

As an explicit example, consider a model with $k(r)$ given by

$$k(r) = -\frac{1}{1 + r^a}; \quad 0 < a < 2,$$  \hfill (2.33)

in arbitrary units. The condition $0 < a < 2$ ensures that the regularity conditions of Appendix B are satisfied. The function $P/I_k$ for these models, which controls the magnitude of the late time acceleration (see Eqn. (2.31)) is shown in Fig. 2.1 against $r_D$ and $a$. For clarity, in the second panel we have shown $P/I_k$ against $a$ for specific values of $r_D$. We find that $P/I_k$ is positive everywhere in the region shown. To explicitly demonstrate acceleration, we plot the evolution of the dimensionless quantity $q_D$ defined by

$$q_D \equiv -\frac{\ddot{a}_D a_D}{\dot{a}_D^2} = 2 - 3\frac{\ddot{V}_D V_D}{(V_D)^2},$$  \hfill (2.34)

for various fixed values of $a$ and $r_D$, using the full expression for $V_D$ in (2.29). The results are shown in Fig. 2.2. We have used units in which $t_{in} = 1$, and have displayed the evolution for times $t > 100 t_{in}$.

Even though the acceleration condition (2.32) is strictly derived for the case $M(r) = 0$, one can easily see that it remains valid at late enough times even in the presence of a small amount of matter. Introducing another place holder $\epsilon$ to keep track of the smallness of the function $g(r)$ defined in
Eqn. (2.23), one sees that by waiting long enough, the solution for \( R(t,r) \) will be approximately given by Eqn. (2.23), with \( t_0(r) \) given by Eqn. (2.24). The expression for the volume \( V_D \) will have the same form as Eqn. (2.29), but the integrals involved will be different due to the presence of terms involving \( \epsilon \neq 0 \). A condition similar to (2.32), say \( P(\epsilon) > 0 \), will then be obtained for late time acceleration, with different integrals involved in the definition of the functional \( P(\epsilon) \). The point to note is that \( P(\epsilon) = P + \text{terms containing } \epsilon \) with \( P \) defined as in Eqn. (2.32), and for small enough \( \epsilon \), \( P(\epsilon) \) will be positive whenever \( P \) is positive\(^2\). Hence the acceleration condition is robust against adding a small but nonzero amount of matter.

**Chapter summary and discussion:**

This chapter dealt with details of Buchert’s scheme for spatially averaging scalar quantities, and the effective cosmological equations it leads to. Using exactly solvable toy models of inhomogeneities, which were explicitly averaged using this scheme, we saw how an accelerating effective scale factor can arise even in situations where there is no exotic matter component.

However, the fact that Buchert’s scheme deals with only two of the ten Einstein equations, makes it difficult to relate the effective scale factor \( a_D \) with observations. In particular, it is not clear whether or not \( a_D \) should replace the usual scale factor in the FLRW metric. We see this difficulty of interpretation as arising from the inherent non-covariant structure of Buchert’s averaging scheme. To get around this problem we will study a different averaging scheme in the next chapter, namely Zalaletdinov’s fully covariant Macroscopic Gravity. This scheme will allow us to deal with objects which are structurally similar to Buchert’s scale factor and backreaction, while being easier to interpret in a physically clear manner.

\(^2\)More precisely, we will have \( P(\epsilon) = P + O(\epsilon \ln \epsilon) \).
Chapter 3

Averaging schemes : Zalaletdinov’s covariant Macroscopic Gravity

In this chapter we turn to the averaging defined by Zalaletdinov [43], which is a 4-dimensional generally covariant procedure. This averaging is used on the Einstein equations and, together with some additional assumptions, leads to what Zalaletdinov has called Macroscopic Gravity (henceforth MG). After introducing MG, we will describe its spatial averaging limit as discussed in Paper 2.

3.1 A covariant averaging scheme

The starting point in any covariant averaging scheme has to be the question: “How does one average tensors while retaining their transformation properties under coordinate changes?” If the averaging operation is to involve an integral over a spacetime region, then clearly only scalar objects can be averaged, since they change only trivially under coordinate transformations. To define a scheme for general tensors then, it is essential to introduce some additional structure in the formalism. The most convenient option is to introduce a bivector $W^a_{\alpha}(x', x)$ which transforms as a vector at event $x'$ and as a co-vector at event $x$. In Zalaletdinov’s scheme one postulates the existence of such a bivector, requires it to have certain desirable properties, and then explicitly constructs an object which has all these properties. We will see how this is done in what follows. [Throughout this chapter primed indices (e.g. $P^a'$) will refer to “primed events” ($x'$) and unprimed indices to unprimed events.]

To begin with, we require that this bivector be idempotent (i.e. “square to itself”)

$$W^a_{\alpha}(x', x'')W^a_{\alpha}(x'', x) = W^a_{\alpha}(x', x),$$

and have the coincidence limit

$$\lim_{x' \to x} W^a_{\alpha}(x', x) = \delta^a_{\alpha},$$

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This ensures that }W^j_{\alpha}(x', x)\text{ has the inverse operator }W^j_{\alpha}(x, x')\text{ (which is easily seen by taking the }x \to x'\text{ limit in Eqn. (3.1) and using the condition (3.2)). The bivector is then used to define the "bilocal extension" of a general tensorial object (denoted by an overtilde) : for a vector }P^a(x)\text{ this takes the form}

\[ \tilde{P}^a(x', x) = W^a_{\alpha}(x, x')P^\alpha(x'), \quad (3.3) \]

with the obvious generalisation to higher rank objects. Notice that the bilocal extension as defined above transforms like the original tensor at the event }x\text{, but as a }scalar\text{ at the event }x'.\text{ This allows us to define the "average" of }P^a(x)\text{ over a 4-dimensional spacetime region }\Sigma\text{ with a supporting point }x\text{, as}

\[ \bar{P}^a(x) = \langle \tilde{P}^a \rangle_{ST} = \frac{1}{V_\Sigma} \int_{\Sigma} d^4x' \sqrt{-g} \tilde{P}^a(x', x) ; \quad V_\Sigma = \int_{\Sigma} d^4x' \sqrt{-g}, \quad (3.4) \]

the subscript }ST\text{ standing for 'spacetime'. While this averaged tensor has the correct transformation properties at the event }x\text{, in order to be a local }function\text{ of its argument, one needs to ensure that }\bar{P}^a(x)\text{ has appropriate differential properties. Since the }x\text{ dependence of this object arises not only from the explicit appearance of }W^j_{\alpha}(x', x)\text{ but also through the dependence of the domain }\Sigma\text{ on the support point, in order to correctly calculate the derivative of }\bar{P}^a(x)\text{ we need to specify how neighbouring domains are related to each other.}

This is done as follows : The same bivector }W^j_{\alpha}\text{ is used to specify a }Lie\ dragging\text{ of the averaging region }\Sigma\text{, ensuring that the volumes of the averaging regions constructed at nearby supporting points are coordinated in a well defined manner (which motivates the terminology "coordination bivector" for }W^j_{\alpha}\text{, which we will follow henceforth). Suppose }x^a\text{ and }x^a + \xi^a\Delta \lambda\text{ are the coordinates of two support points, where }\Delta \lambda\text{ is a small change in the parameter along the integral curve of a given vector field }\xi^a.\text{ Symbolically denote the two points as }x\text{ and }x + \xi\Delta \lambda.\text{ Then the averaging region at }x + \xi\Delta \lambda\text{ is defined in terms of the averaging region }\Sigma(x)\text{ at }x\text{, by transporting every point }x' \in \Sigma(x)\text{ around }x\text{ along the appropriate integral curve of a }new\ bilocal\ vector\ field }S^a_{x'}\text{ defined as }S^a_{x'}(x', x) = W^a_{\alpha}(x', x)\xi^\alpha(x),\text{ thereby constructing the averaging region }\Sigma(x, \Delta \lambda)\text{ with support point }x + \xi\Delta \lambda.\text{ }

We can now evaluate the Lie derivative of }\bar{P}^a(x)\text{ along the vector field }\xi^a\text{, by first noting that the Lie derivative of the volume }V_{\Sigma(x)}\text{ is}

\[ \frac{d}{d\lambda}V_{\Sigma(x)} = \xi^a(x)\langle W^a_{\alpha}\rangle_{ST} V_{\Sigma(x)}, \quad (3.5) \]

where the semicolon denotes a covariant derivative. An easy way to see this is to note that since

\[ \Sigma(x, \Delta \lambda) = \{ y^a' \mid y^a' = x^a' + S^a_{x'}(x', x)\Delta \lambda ; x' \in \Sigma(x) \}, \quad (3.6) \]
or symbolically \( y' = x' + S \Delta \lambda \), we will have
\[
\sqrt{-g(y')} = \sqrt{-g(x')} \left( 1 + \Delta \lambda S \partial_{a'} \ln \sqrt{-g(x')} \right) ; \quad d^4 y' = d^4 x' \left( 1 + \Delta \lambda \partial_a S \right),
\]
and the result follows from writing
\[
\frac{d}{d\lambda} V_{\Sigma(x)} = \lim_{\Delta \lambda \to 0} \frac{1}{\Delta \lambda} \left( V_{\Sigma(x, \Delta \lambda)} - V_{\Sigma(x)} \right),
\]
and using \( S_{a'} = W_{[a'} \xi_{b]} \). For the derivative of \( \bar{P}^a(x) \), we need \( \bar{P}^a(y = x + \xi \Delta \lambda) \). Explicitly we have
\[
\bar{P}^a(x + \xi \Delta \lambda) = \frac{1}{V_{\Sigma(x, \Delta \lambda)}} \int_{\Sigma(x, \Delta \lambda)} d^4 y' \sqrt{-g(y')} \bar{P}^a(y', x + \xi \Delta \lambda),
\]
which after writing \( y' = x' + S \Delta \lambda \) with \( x' \in \Sigma(x) \) and using (3.7), finally gives
\[
\frac{d}{d\lambda} \bar{P}^a = \lim_{\Delta \lambda \to 0} \frac{1}{\Delta \lambda} \left( \bar{P}^a(x + \xi \Delta \lambda) - \bar{P}^a(x) \right)
\]
\[
= \xi^b(x) \left[ (\partial_b \bar{P}^a)_{ST} + \langle W_{b[j'} \bar{P}^a \rangle_{ST} - \langle W_{b[j'}^j \rangle_{ST} \bar{P}^a \right],
\]
where we have defined the “bilocal partial derivative”
\[
\partial_b \equiv \partial_b + W_{b[j'} \partial_{j'}.
\]
Since the vector field \( \xi^a \) is arbitrary, Eqn. (3.10) gives us an expression for the partial derivative \( \partial_b \bar{P}^a \) (recalling that \( d \bar{P}^a / d\lambda = \xi^b \partial_b \bar{P}^a - \bar{P}^b \partial_b \xi^a \)). Requiring that partial derivatives of \( \bar{P}^a(x) \) commute, leads to the following condition
\[
\partial_b \partial_c \bar{P}^a = \langle \partial_b \partial_c \bar{P}^a \rangle_{ST} + \langle \bar{P}^a \partial_b W_{c[j']^k}^k \rangle_{ST} - \langle \partial_b W_{c[j']^k}^k, \rangle_{ST} \bar{P}^a = 0.
\]
Straightforward algebra shows that
\[
\partial_b \partial_c \bar{P}^a = (\partial_b W_{c[j']}^k) \partial_{k'} \bar{P}^a,
\]
\[
\partial_b W_{c[j']^k}^k = (\partial_b W_{c[j']}^k), k',
\]
and hence the necessary and sufficient condition for (3.12) to hold, is
\[
\partial_b W_{c[j']}^k = W_{[b, c]}^k + W_{[b, j']}^k W_{c[j']} = 0,
\]
where underlined indices are not antisymmetrized.

At this stage it is convenient to re-express results in the language of differential forms, since
it will make the algebra that follows concise and readable. The original papers by Zalaletdinov introduce a full-fledged bilocal exterior calculus, involving \((k, l')\)-forms which are \(k\)-forms at \(x\) and \(l\)-forms at \(x'\). In what follows however, we will almost exclusively only need to deal with \((k, 0')\)-forms that are differential forms at a single event \(x\), although they may be bilocal in their functional dependence on events \(x\) and \(x'\). This is because we will only deal with bilocal extensions of local \(k\)-forms, which are defined to be \((k, 0')\)-forms.

For example, the bilocal extension of a 2-form \(\alpha\) is defined as

\[
\tilde{\alpha}(x', x) = \frac{1}{2!} \alpha_{j'k'}(x') \mathcal{W}_j^k(x', x) \mathcal{W}_b^{k'}(x', x) dx^a \wedge dx^b, \tag{3.16}
\]

which is a \((2, 0')\)-form. We define the bilocal exterior derivative as

\[
d = d + d'_{\mathcal{W}}, \tag{3.17}
\]

where \(d\) is the usual exterior derivative which “differentiates and antisymmetrizes at \(x\)”, and the “shifted” exterior derivative \(d'_{\mathcal{W}}\) “differentiates at \(x'\) but antisymmetrizes at \(x\)” so that, say for the \((2, 0')\)-form \(\tilde{\alpha}\), we have

\[
d'_{\mathcal{W}} \tilde{\alpha} = \frac{1}{2!} \tilde{\alpha}_{ab,j'}(x', x) \mathcal{W}_j^c(x', x) dx^c \wedge dx^a \wedge dx^b. \tag{3.18}
\]

For a bilocal function \(f(x, x')\) we will have \(df = \partial_a f dx^a\). In this language, the preceding results for differential conditions on the coordination bivector can be generalised to arbitrary tensor-valued \(k\)-forms. In particular for a vector-valued \(k\)-form \(p^a(x)\) whose bilocal extension is the \((k, 0')\)-form \(\tilde{p}^a(x', x)\), we have

\[
d \tilde{p}^a = (d \tilde{p}^a)_{ST} + (\text{div}_e \mathcal{W} \wedge \tilde{p}^a)_{ST} - (\text{div}_e \mathcal{W})_{ST} \wedge \tilde{p}^a, \tag{3.19}
\]

where we have defined,

\[
\text{div}_e \mathcal{W} = \mathcal{W}^j_{a,j'} dx^a = \left( \mathcal{W}^j_{a,j'} + \mathcal{W}^j_{a,b} \partial_j \ln \sqrt{-g'} \right) dx^a. \tag{3.20}
\]

The condition \((3.15)\) for the averaged object to be a local function of its argument, reduces to

\[
d \mathcal{W}^{j'} = 0; \quad \mathcal{W}^{j'} = \mathcal{W}^{j'}_a dx^a, \tag{3.21}
\]

Equation \((3.15)\) shows that it is desirable to choose a coordination bivector which satisfies

\[
\text{div}_e \mathcal{W} = 0, \tag{3.22}
\]
since firstly, this allows us to commute the exterior derivative with the averaging according to

$$d\tilde{p}^\alpha = (d\tilde{p}^\alpha)_{ST}, \quad (3.23)$$

and secondly, it implies that the volume of the averaging region is held constant during the coordination (see Eqn. (3.5)), and is therefore a free parameter in the formalism.

Mars and Zalaletdinov [44] show in their Theorems 1, 3 and 4, that firstly, the general solution of Eqn. (3.21) for an idempotent coordination bivector is given by

$$\mathcal{W}^{a'}_j(x', x) = f^{a'}_m(x') f^{-1}_j(x), \quad (3.24)$$

where \(f^a_m(x)\partial_n = f_m\) is any vector basis satisfying the commutation relations

$$[f_i, f_j] = C^k_{ij} f_k; \quad C^k_{ij} = \text{constant}, \quad (3.25)$$

and secondly, that Eqn. (3.22) with the coordination bivector given by Eqn. (3.24) is always integrable on a differentiable manifold with a given volume \(n\)-form. The proofs given in Ref. [44] are very clear, although somewhat lengthy, and we will hence omit them here. Further, these authors also show that for the special class of bivectors for which \(C^k_{ij} = 0\), the vectors \(\{f_k\}\) form a coordinate basis, with ‘proper’ coordinate functions \(\phi^m(x)\) say, so that

$$f^a_m(x(\phi^a)) = \frac{\partial x^a}{\partial \phi^m}; \quad f^{-1}_j(x) = \frac{\partial \phi^m}{\partial x^j}, \quad (3.26)$$

and satisfying Eqn. (3.22) makes this proper coordinate system volume preserving, with \(g(\phi^m) = \text{constant}\). When expressed in terms of such a volume preserving coordinate (VPC) system, the coordination bivector takes its most simple form, namely

$$\mathcal{W}^{a'}_j(x', x) |_{\text{proper}} = \delta^{a'}_j. \quad (3.27)$$

Volume preserving coordinates in fact form a large class in themselves, generalizing the Cartesian coordinate system of Minkowski spacetime. For a discussion on the properties of VPCs and the associated bivectors \(\mathcal{W}^{a'}_j\), see Sec. 8 of Ref. [44]. The problem of defining an averaging operator has now been reduced to the far simpler problem of choosing a specific VPC system which then fixes the coordination bivector. We emphasize that the averaging is still fully covariant; choosing a coordination bivector is distinct from choosing a coordinate system to perform calculations in. This freedom in defining the coordination bivector leads to a lack of uniqueness of the average in the formalism as it stands. We will return to this issue when we study the backreaction arising from perturbative inhomogeneities in chapter 4.
3.2 The averaged manifold

With the averaging operation in place, we can turn to the description of an averaged geometry. Let us begin by recalling some standard results from differential geometry. Given a differentiable manifold \( M \) endowed with a metric \( g_{ab} \) of Lorentzian signature \((- + + +)\), the connection 1-forms \( \omega^a_b \) are defined by the action of the exterior derivative on the basis vectors \( e^a \):\

\[
de_a = e_a \omega^a_b = e_a (\Gamma^a_{bc}) \, dx^c \Rightarrow (dx^a, de_b) = \omega^a_b,
\]

(3.28)

where \( \Gamma^a_{bc} \) are the Christoffel symbols and the parentheses represent the inner product with \((dx^a, e^b) = \delta^a_b\). We define the “exterior covariant derivative” \( D_\omega \) associated with the connection \( \omega^a_b \), as follows: for a \( k \)-form \( p^a_b \),

\[
D_\omega p^a_b = dp^a_b - \omega^k_b \wedge p^a_k + \omega^a_k \wedge p^k_b,
\]

(3.29)

with an obvious generalisation to higher rank objects. The compatibility between the metric and the connection on \( M \) is expressed by the condition

\[
D_\omega g_{ab} = dg_{ab} - g_{ak} \omega^k_b - g_{bk} \omega^k_a = 0,
\]

(3.30)

with a similar condition for the inverse metric \( g^{ab} \). The Cartan structure equations are given by

\[
\omega^a_b \wedge dx^b = 0,
\]

(3.31a)

\[
d\omega^a_b + \omega^a_c \wedge \omega^c_b = r^a_b,
\]

(3.31b)

where Eqn. (3.31a) expresses the symmetry of the connection \( \omega^a_b \) and \( r^a_b \) is the curvature 2-form on \( M \) which defines the Riemann curvature tensor via \( r^a_b = (1/2!) g^{aci} \omega^i_c \wedge dx^d \). Finally, the structure equations (3.31) and the metric compatibility condition (3.30) are supplemented by their respective integrability conditions, given by equations (3.32a), (3.32b) and (3.32c)

\[
\omega^a_b \wedge dx^b = 0,
\]

(3.32a)

\[
dr^a_b - \omega^a_c \wedge r^c_b + \omega^a_c \wedge r^c_b = 0,
\]

(3.32b)

\[
g_{ak} r^k_b + g_{bk} r^k_a = 0.
\]

(3.32c)

Note that Eqn. (3.32a) corresponds to the cyclic identity \( r^a_{[bcd]} = 0 \) and Eqn. (3.32b) to the Bianchi identity \( r^a_{b[jk]} = 0 \).

Consider now the bilocal version of the basis vector \( e_a \), given by \( W_a = \omega^a_{b'} e_{b'} \), which is a scalar at \( x \) and a vector at \( x' \), and whose inverse is the \((0, 1')\)-form \( W^{-1b} = W^b_{a'} dx^{a'} \). The construction of an averaged manifold begins with the observation that the \((1, 0')\)-form defined by

\[
\Omega^a_b(x, x') \equiv (W^{-1a}, dW_b),
\]

(3.33)
transforms like a connection under coordinate transformations at \( x \) and, equally importantly, as a scalar at \( x' \), and further has the coincidence limit
\[
\lim_{x' \to x} \Omega^a_b(x, x') = \omega^a_b(x) .
\]
Explicitly, we have
\[
\Omega^a_b(x, x') = \left[ W^a_i(x, x') W^j_b(x', x) \Gamma^k_{ij} \Gamma^l_{kl}(x') + W^a_i(x, x') \partial_b W^j_i(x', x) \right] dx^c
\equiv \Gamma^a_{bc}(x, x') dx^c ,
\]
where \( \Gamma^a_{bc} \) is symmetric in \( (bc) \) due to the condition \((3.21)\) or \((3.15)\). This explicit form can be used to directly check the transformation properties at \( x \) and \( x' \), and \( \Omega^a_b(x, x') \) can therefore be considered as the bilocal extension of the connection \( \omega^a_b(x) \).

The transformation properties of \( \Omega^a_b \) imply that its average \( \bar{\Omega}^a_b \)
\[
\bar{\Omega}^a_b \equiv \langle \Omega^a_b \rangle ,
\]
has the transformation properties of a local connection, and the coincidence limit \((3.34)\) implies the limit \( \lim_{V\Sigma \to 0} \bar{\Omega}^a_b(x) = \omega^a_b(x) \). This leads to the key idea of MG, which is that \( \Omega^a_b \), is defined as the connection 1-form on a new, averaged manifold \( \tilde{\mathcal{M}} \). As a set, \( \tilde{\mathcal{M}} \) is identical to \( \mathcal{M} \); however for consistency in the definition of averaged quantities, the guiding principle one adopts is that each averaging domain \( \Sigma \) is effectively treated as a single point on \( \tilde{\mathcal{M}} \). This ensures, for example, that the averaging operation is idempotent \([44]\).

The goal now is to average out the bilocal extensions of the structure equations \((3.31)\) and the compatibility condition \((3.30)\) and their integrability conditions \((3.32)\), and to express them in terms of appropriate differential forms defined on \( \mathcal{M} \). The bilocal extensions of Eqns. \((3.31)\) and \((3.30)\), are respectively given by
\[
\Omega^a_b \wedge dx^b = 0 ,
\]
\[
d\Omega^a_b + \Omega^c_b \wedge \Omega^a_c = \Gamma^a_{bc} ,
\]
\[
\mathbf{D}_\Omega \tilde{g}_{ab} = d\tilde{g}_{ab} - \tilde{g}_{ak} \Omega^k_b - \tilde{g}_{bk} \Omega^k_a = 0 ,
\]
where, in the last equation, \( \mathbf{D}_\Omega \) is the bilocal covariant exterior derivative associated with the bilocal connection \( \Omega^a_b \). The integrability conditions of Eqns. \((3.37)\) are given by the bilocal
extensions of Eqns. (3.32),

\[
\tau^a_{\ b} \wedge dx^b = 0, \quad (3.38a)
\]

\[
D_\Omega \tau^a_{\ b} = d\tau^a_{\ b} - \Omega^c_{\ b} \wedge \tau^a_{\ c} + \tau^a_{\ c} \wedge \tau^c_{\ b} = 0, \quad (3.38b)
\]

\[
\bar{\tau}^k_{\ b} + \bar{\tau}^k_{\ a} = 0. \quad (3.38c)
\]

In the above equations, the \((2, 0')\)-form \(\tau^a_{\ b}\) is the bilocal extension of the curvature 2-form constructed according to the rules set out in Eqns. (3.16) and (3.3). Similarly the \((0, 0')\)-form \(\bar{g}_{ab}\) is the bilocal extension of the metric. To proceed with the averaging, a correlation 2-form is defined

\[
Z^{\ a \ i}_{\ b \ j} = \langle \Omega^a_{\ b} \wedge \Omega^i_{\ j} \rangle_{ST} - \bar{\Omega}^a_{\ b} \wedge \bar{\Omega}^i_{\ j}. \quad (3.39)
\]

The average of the curvature 2-form \(\tau^a_{\ b}\) on \(\mathcal{M}\) is denoted \(R^a_{\ b} \equiv \langle \tau^a_{\ b} \rangle_{ST}\), and the curvature 2-form on the averaged manifold \(\bar{\mathcal{M}}\) is denoted \(M^a_{\ b}\),

\[
M^a_{\ b} = d\Omega^a_{\ b} + \Omega^c_{\ k} \wedge \Omega^k_{\ b}. \quad (3.40)
\]

Equations (3.37a) and (3.37b) then average out to give

\[
\bar{\Omega}^a_{\ c} \wedge dx^c = 0, \quad (3.41a)
\]

\[
M^a_{\ b} = R^a_{\ b} - Z^{\ a \ c}_{\ b \ c}. \quad (3.41b)
\]

The averages of Eqn. (3.38a) and the identity \(\tau^a_{\ a} = 0\) give us

\[
R^a_{\ b} \wedge dx^b = 0 \ ; \quad R^a_{\ a} = 0, \quad (3.42)
\]

and the symmetry of \(\Omega^a_{\ b}\) and hence of \(\bar{\Omega}^a_{\ b}\) give us

\[
Z^{\ a \ c}_{\ b \ c} \wedge dx^b = 0 \ ; \quad Z^{\ a \ c}_{\ c \ a} = 0. \quad (3.43)
\]

This ensures that the curvature 2-form \(M^a_{\ b}\) satisfies the correct algebraic identities

\[
M^a_{\ b} \wedge dx^b = 0 \ ; \quad M^a_{\ a} = 0, \quad (3.44)
\]

The formalism at this stage becomes somewhat complicated. The reason is that there is no simple way of averaging out equations (3.37c), (3.38b) and (3.38c), which become

\[
d\bar{g}_{ab} - \langle g_{ak} \Omega^k_{\ b} \rangle_{ST} - \langle \bar{g}_{bk} \Omega^k_{\ a} \rangle_{ST} = 0, \quad (3.45a)
\]

\[
dR^a_{\ b} - \langle \Omega^c_{\ b} \wedge \bar{\tau}^a_{\ c} \rangle_{ST} + \langle \Omega^c_{\ c} \wedge \bar{\tau}^a_{\ b} \rangle_{ST} = 0, \quad (3.45b)
\]

\[
\langle \bar{g}_{ak} \bar{\tau}^k_{\ b} \rangle_{ST} + \langle \bar{g}_{bk} \bar{\tau}^k_{\ a} \rangle_{ST} = 0, \quad (3.45c)
\]
where \( \bar{g}_{ab} = \langle \bar{g}_{ab} \rangle_{ST} \). What we need are “splitting rules” for the various products appearing inside the averager in these equations. Of these, Eqn. (3.45b) can be split by noting that the exterior covariant derivative of the correlation 2-form becomes

\[
\bar{D}\bar{\Omega}Z^{a\ i}_{b\ j} = -2\mathbb{P}Y^{a\ m\ i}_{m\ b\ j} + 2\mathbb{P}\left(\langle \bar{r}^{a\ i}_{b\ j} \wedge \Omega^{i}_{j} \rangle_{ST} - R^{a\ i}_{b\ j} \wedge \bar{\Omega}^{j}_{i}\right),
\]

(3.46)

where the symbol \( \mathbb{P} \) permutes the free indices in, say \( K^{a\ i\ m}_{b\ j\ n} \) pairwise according to

\[
\mathbb{P}(K^{a\ i\ m}_{b\ j\ n}) = \frac{1}{3!}(K^{a\ i\ m}_{b\ j\ n} - K^{i\ a\ m}_{b\ j\ n} + K^{i\ m\ a}_{b\ j\ n}),
\]

and any summed indices are ignored, and the correlation 3-form is defined as

\[
Y^{a\ i\ m\ n}_{b\ j\ n} = \langle \Omega^{a}_{b} \wedge \Omega^{i\ m}_{j\ n} \rangle_{ST} - 3\mathbb{P}(Z^{a\ i\ m}_{b\ j\ n} - \Omega^{a}_{b} \wedge \bar{\Omega}^{i\ j}_{m\ n} - \bar{\Omega}^{m\ i}_{a\ j} \wedge \Omega^{a\ m}_{b\ n}).
\]

(3.47)

Tracing Eqn. (3.46) on the indices \( b \) and \( i \) kills the term involving the correlation 3-form due to the presence of the permutation symbol, leaving behind

\[
-\langle \bar{r}^{a\ i}_{i\ i} \wedge \Omega^{i}_{b\ b} \rangle_{ST} + \langle \bar{r}^{i\ i}_{b\ b} \wedge \Omega^{a\ i}_{i\ i} \rangle_{ST} = -R^{a\ i}_{i\ i} \wedge \bar{\Omega}^{i\ i}_{i\ i} + R^{i\ i}_{b\ b} \wedge \Omega^{a\ i}_{i\ i} - \bar{D}\bar{\Omega}Z^{a\ i}_{i\ i},
\]

(3.48)

which averages out Eqn. (3.45b) to give the Bianchi identity for the curvature 2-form \( M^{a\ b} \),

\[
\bar{D}\bar{\Omega}M^{a\ b}_{b\ j} = dM^{a\ b}_{b\ j} - \bar{\Omega}^{k\ j}_{b\ j} \wedge M^{a\ k}_{b\ k} + \Omega^{k\ j}_{b\ j} \wedge M^{a\ k}_{b\ k} = 0.
\]

(3.49)

This was achieved at the cost of introducing a new object, the correlation 3-form \( Y^{a\ i\ m\ n}_{b\ j\ n} \), which fixes the differential properties of the correlation 2-form \( Z^{a\ i}_{b\ j} \) and hence of the 2-form \( R^{a\ i}_{b\ j} \). The differential properties of this 3-form are in turn fixed by introducing a correlation 4-form in an analogous manner. Due to the 4-dimensionality of spacetime there are no higher correlation \( p \)-forms that need to be defined.

In practice, it is cumbersome to keep track of the correlation 3-form and 4-form, and furthermore it is only the correlation 2-form which will appear in the averaged Einstein equations. We will ultimately be interested in explicit calculations of the correlation objects in a perturbative setting at leading order, and will hence ignore the 3-form and 4-form. Remarkably, it is also possible to self-consistently ignore these forms in the nonperturbative setting [43] as follows: We set the 3-form and 4-form to zero and impose the conditions

\[
\bar{D}\bar{\Omega}Z^{a\ i}_{b\ j} = 0 = \bar{D}\bar{\Omega}R^{a\ i}_{b\ j},
\]

(3.50)

with the second equality required since Eqn. (3.49) holds. It can be shown [43] that requiring the 4-form to vanish also imposes the condition

\[
\mathbb{P}\left( Z^{a\ c\ i}_{b\ d\ j} \wedge Z^{j\ i\ k}_{k\ t\ l} \right) = 0.
\]

(3.51)
The integrability condition for $\mathbf{D}_\Omega \mathbf{Z}^a_{b \ j} = 0$ is

$$
P \left( \mathbf{R}^a_{\ c} \wedge \mathbf{Z}^c_{b \ j} - \mathbf{Z}^a_{b \ k} \wedge \mathbf{R}^k_{\ j} \right) = 0 ,$$

(3.52)

which also contains the integrability condition for $\mathbf{D}_\Omega \mathbf{R}^a_b = 0$.

So far we have only managed to average out Eqn. (3.45b). To make progress with Eqns. (3.45a) and (3.45c) we need additional assumptions. Zalaletdinov argues [43] that for a class of slowly varying tensor fields (tensor-valued $k$-forms) $\mathbf{c}^m_{\ n\ldots}$ on $\mathcal{M}$ such as the metric and other covariantly constant tensors, and Killing tensors, etc., the following assumptions may be reasonable

$$
\langle \Omega^a_{\ b} \wedge \mathbf{c}^m_{\ n\ldots} \rangle_{ST} = \bar{\Omega}^a_{\ b} \wedge \bar{\mathbf{c}}^m_{\ n\ldots} ,
$$

(3.53a)

$$
\langle \Omega^a_{\ b} \wedge \Omega^i_{\ j} \wedge \mathbf{c}^m_{\ n\ldots} \rangle_{ST} = \langle \Omega^a_{\ b} \wedge \Omega^i_{\ j} \rangle_{ST} \wedge \mathbf{c}^m_{\ n\ldots} .
$$

(3.53b)

Then Eqn. (3.45a) and its analogue for $g^{ab}$ average out to give

$$
\mathbf{D}_\Omega \bar{g}_{ab} = 0 ; \quad \mathbf{D}_\Omega \bar{g}^{ab} = 0 .
$$

(3.54)

Further, for a general slowly varying object $\mathbf{c}^m_{\ n\ldots}$, the following identity holds

$$
\langle \mathbf{R}^a_{\ b} \wedge \mathbf{c}^m_{\ n\ldots} \rangle_{ST} - \mathbf{R}^a_{\ b} \wedge \mathbf{c}^m_{\ n\ldots} = \langle \Omega^a_{\ b} \wedge \mathbf{D}_\Omega \mathbf{c}^m_{\ n\ldots} \rangle_{ST} + \bar{\Omega}^a_{\ b} \wedge \mathbf{D}_\Omega \mathbf{c}^m_{\ n\ldots}
\hspace{1cm} = -\mathbf{Z}^a_{\ b \ j} \wedge \mathbf{c}^m_{\ n\ldots} - \ldots + \mathbf{Z}^a_{\ b \ n} \wedge \mathbf{c}^m_{\ j\ldots} + \ldots ,
$$

(3.55)

which follows from differentiating Eqn. (3.53a), and which averages out Eqn. (3.45c) (and its analogue for $g^{ab}$) to give

$$
\bar{g}_{ab} \mathbf{M}^k_{\ b} + \bar{g}_{bb} \mathbf{M}^k_{\ a} = 0 ; \quad \mathbf{M}^a_{\ k} \bar{g}^{k\ b} + \mathbf{M}^a_{\ k} \bar{g}^{k\ b} = 0 .
$$

(3.56)

Eqn. (3.54) allows one to choose $G_{ab} = \bar{g}_{ab}$, where $G_{ab}$ is the metric on the averaged manifold $\bar{\mathcal{M}}$. In general however, we have $G^{ab} \neq \bar{g}^{ab}$, and one defines the tensor $U_{ab} \equiv \bar{g}^{ab} - \bar{g}^{ab}$ to keep track of this difference. However, we shall see later that when the averaged manifold is highly symmetric, as in the case of a manifold with homogeneous and isotropic spatial sections which we will consider, one finds that $U_{ab} = 0$.

### 3.3 Averaging Einstein’s equations

In the general case, it turns out that Eqn. (3.55) is all that is needed to average out the Einstein equations

$$
g^{a\ k} r_{k \ b} - \frac{1}{2} \bar{g}^{a\ b} g_{ij} r_{ij} = -\kappa t^{(mic)}_b ,
$$

(3.57)
where $\kappa = 8\pi G_N$, $t^a_{\text{mic}}$ is the microscopic energy momentum tensor of the matter distribution, and the Ricci tensor $r_{ab}$ on $\mathcal{M}$ is defined according to the sign convention $r_{ab} = r^j_{\ abj}$. The averaging leads to the equations

$$G^{ab}M_{\ b} - \frac{1}{2}\delta^a_b G^i_j M_{ij} = -\kappa \langle \tilde{t}^a_{\ (\text{mic})} \rangle_{ST} + \left( Z^a_{\ ijb} - \frac{1}{2}\delta^a_b Z^m_{\ ijm} \right) \bar{g}^{ij} - \left( U^{ak}M_{\ b} - \frac{1}{2}\delta^a_b U^{ij}M_{ij} \right),$$

(3.58)

where $M_{ab} = M^j_{\ abj}$ is the Ricci tensor on $\hat{\mathcal{M}}$ and we have defined

$$Z^a_{\ ijb} = 2Z^a_{\ ikj} \ ; \ Z^a_{\ b} = Z^a_{\ bnj} \ dx^m \wedge dx^n. \ (3.59)$$

The averaged equations (3.58) differ from the usual Einstein equations by the correlation tensor which we define as

$$C^a_{\ b} = \left( Z^a_{\ ijb} - \frac{1}{2}\delta^a_b Z^m_{\ ijm} \right) \bar{g}^{ij} - \left( U^{ak}M_{\ b} - \frac{1}{2}\delta^a_b U^{ij}M_{ij} \right). \ (3.60)$$

Hence, denoting the Einstein tensor on $\hat{\mathcal{M}}$ as $E^a_{\ b}$, and defining the tensor $T^a_{\ b}$ via

$$T^a_{\ b} = \langle \tilde{t}^a_{\ (\text{mic})} \rangle_{ST}, \quad (3.61)$$

the averaged Einstein equations read

$$E^a_{\ b} = -\kappa T^a_{\ b} + C^a_{\ b}. \quad (3.62)$$

Since the left hand side of Eqn. (3.62) is covariantly conserved by construction ($E^a_{\ b;} = 0$), where the semicolon denotes covariant differentiation with respect to the connection on $\mathcal{M}$, in general one has

$$(-\kappa T^a_{\ b} + C^a_{\ b})_{;a} = 0, \quad (3.63)$$

with no condition on $T^a_{\ b}$ and $C^a_{\ b}$ separately. If, however, we assume that $D_{\Omega}Z^a_{\ b\ j} = 0$, it follows that

$$C^a_{\ b;\ a} = 0, \quad (3.64)$$

which implies that the averaged energy-momentum tensor $T^a_{\ b}$ is also covariantly conserved. In our explicit calculations in the perturbative setting in chapter 4, we will see that this condition does not hold in general.

It can also be shown that in 4 dimensions, the 720 $a$ priori independent components of $Z^a_{\ bm\ jn}$ are subject to 680 constraints arising from Eqns. (3.52) and (3.51). This leaves 40 independent components which combine to give the 10 independent components of the correlation tensor $C^a_{\ b}$. The conditions in Eqns. (3.52) and (3.51) do not constrain the components of $C^a_{\ b}$, which follows from considering the structure of those equations.
3.4 A 3 + 1 spacetime splitting and the spatial averaging limit

We are now in a position to apply the MG formalism to the problem of cosmology. The main idea we wish to emphasize is that in the cosmological context, it is essential to consider a spatial averaging limit of the covariant averaging used in MG. The simplest way to see this is to note that the homogeneous and isotropic FLRW spacetime must be left invariant under the averaging operation, and this is only possible if the averaging is tuned to the uniquely defined spatial slices of constant curvature in the FLRW spacetime. We will elaborate on this below. Our main motivation here is to spell out all the assumptions usually made in the standard approach to cosmology, and ask what they imply in the context of the averaging paradigm. It has been suggested that the observationally relevant averaging must necessarily be performed on the light cone [67]. This would not preserve the symmetries of the FLRW spacetime. Our point of view is that one needs a theoretically self-consistent mathematical framework in which to study cosmological expansion, structure formation, etc. We will be conservative in our approach and only make the minimum assumptions necessary in order to make standard cosmology compatible with the averaging paradigm.

We start with the assumption that Einstein’s equations are to be imposed on length scales $L_{\text{inhom}}$ where stars are pointlike objects, and that there exists a length scale $L_{\text{FLR}}$ such that averaging on this length scale yields a geometry which has homogeneous and isotropic spatial sections. We expect $L_{\text{FLR}} \gtrsim 100 h^{-1} \text{Mpc}$ and we assume $L_{\text{inhom}} \ll L_{\text{FLR}} \ll L_{\text{Hubble}}$ where $L_{\text{Hubble}}$ is the length scale of the observable universe. In other words, we will assume that the averaged manifold $\bar{M}$ admits a preferred, hypersurface-orthogonal unit timelike vector field $\bar{v}^a$, which defines 3-dimensional spacelike hypersurfaces of constant curvature, and that $\bar{v}^a$ is tangent to the trajectories of observers who see an isotropic cosmic background radiation. For simplicity we will work with the special case where these spatial sections on $\bar{M}$ are flat. One can then choose coordinates $(t, x^A), A = 1, 2, 3,$ on $\bar{M}$ such that the spatial line element takes the form

$$\left(\bar{M}\right) ds_{\text{spatial}}^2 = a^2(t)\delta_{AB}dx^A dx^B, \quad (3.65)$$

where $\delta_{AB} = 1$ for $A = B,$ and 0 otherwise, and we have $\bar{v}^a = (\bar{v}^t, 0, 0, 0)$ so that the spatial coordinates are comoving with the preferred observers. The vector field $\bar{v}^a$ also defines a proper time (the cosmic time) $\tau$ such that $\partial_{\tau} = \bar{v}^a \partial_a = \bar{v}^t \partial_t$. We will further assume that the averaged energy-momentum tensor $T^a_b$ can be written in the form of a perfect fluid, as

$$T^a_b = \rho \bar{v}^a \bar{v}_b + p \pi^a_b, \quad (3.66)$$

where the projection operator $\pi^a_b$ is defined as

$$\pi^a_b = \delta^a_b + \bar{v}^a \bar{v}_b, \quad (3.67)$$

1 Appendix A describes the main features of the FLRW spacetime.
and satisfies $\pi_b^a \bar{v}^b = 0 = \pi_b^a \bar{v}_a$, $\pi_b^a \bar{v}_b = \pi_b^b$, and $\rho$ and $p$ are the homogeneous energy density and pressure respectively, as measured by observers moving on trajectories (in $\bar{M}$) with the tangent vector field $\bar{v}^a$, 

$$\rho \equiv T^a_b \bar{v}^b \bar{v}_a \quad ; \quad p \equiv \frac{1}{3} \pi^b_a T^a_b. \quad (3.68)$$

An important consequence of the above assumptions is that the correlation tensor $C^a_b$, when expressed in terms of the natural coordinates adapted to the spatial sections defined by the vector field $\bar{v}^a$, is homogeneous and depends only on the time coordinate. This is clear when the modified Einstein equations (3.62) are written in these natural coordinates.

Using the vector field $\bar{v}^a$, the (FLRW) Einstein tensor $E^a_b$ can be written as 

$$E^a_b = j_1(x) \bar{v}^a \bar{v}_b + j_2(x) \pi^a_b,$$

$$j_1(x) \equiv E^a_b \bar{v}^b \bar{v}_a \quad ; \quad j_2(x) \equiv \frac{1}{3} \left( \pi^b_a E^a_b \right), \quad (3.69)$$

where $j_1(x)$ and $j_2(x)$ are scalar functions whose form depends upon the coordinates used. The remaining components given by $\pi^b_a E^a_b \bar{v}_a$ and the traceless part of $\pi^a_i \pi^b_k E^a_b$, vanish identically. Since the energy-momentum tensor $T^a_b$ in Eqn. (3.66) also has an identical structure, this structure is therefore also imposed on the correlation tensor $C^a_b$. Namely, only the components $C^a_b \bar{v}^b \bar{v}_a$ and the trace $\pi^b_a C^a_b$ are relevant to the dynamics of the averaged metric. The remaining components, namely $\pi^b_k C^a_b \bar{v}_a$ and the traceless part of $\pi^a_i \pi^b_k C^a_b$, must vanish. This is a condition on the underlying inhomogeneous geometry, irrespective of the coordinates used to describe the geometry on either $\mathcal{M}$ or $\bar{M}$, and is clearly a consequence of demanding that the averaged geometry have the symmetries of the FLRW spacetime.

This leads us to the crucial question of the choice of gauge for the underlying geometry: namely, what choice of spatial sections for the inhomogeneous geometry, will lead to the spatial sections of the FLRW metric in the comoving coordinates defined in Eqn. (3.65)? Since the matter distribution at scale $L_{inhom}$ need not be pressure-free (or, indeed, even of the perfect fluid form), there is clearly no natural choice of gauge available, although locally, a synchronous reference can always be chosen. We note that there must be at least one choice of gauge in which the averaged metric has spatial sections in the form (3.65) – this is simply a refinement of the Cosmological Principle, and of the Weyl postulate, according to which the universe is homogeneous and isotropic on large scales, and individual galaxies are considered as the “observers” travelling on trajectories with tangent $\bar{v}^a$. In the averaging approach, it makes more sense to replace “individual galaxies” with the averaging domains considered as physically infinitesimal cells – the “points” of the averaged manifold $\bar{M}$. This is physically reasonable since we know after all, that individual galaxies exhibit peculiar motions, undergo mergers and so on. This idea is also more in keeping with the notion that the universe is homogeneous and isotropic only on the largest scales, which are much larger than the scale of individual galaxies.

Consider any $3 + 1$ spacetime splitting in the form of a lapse function $N(t, x^J)$, a shift vector
$N^A(t, x^I)$, and a metric for the 3-geometry $h_{AB}(t, x^I)$, so that the line element on $\mathcal{M}$ can be written as

$$
(M)ds^2 = -(N^2 - N_A N^A) dt^2 + 2N_B dx^B dt + h_{AB} dx^A dx^B,
$$

(3.70)

where $N_A = h_{AB} N^B$. At first sight, it might seem reasonable to leave the choice of gauge arbitrary. One could then formally consider a coordination bivector given by the Eqns. (3.24) and (3.26), with $x^i$ denoting the coordinates in the chosen gauge and $\phi^m$ the VPCs; and demand for example, that the metric (3.70) (with say $N^A = 0$) average out to the FLRW form (with a nonsynchronous time coordinate in general). This would imply

$$
G_{00} = \langle \bar{g}_{00} \rangle_{ST} = -f^2(t) ; \quad G_{0A} = \langle \bar{g}_{0A} \rangle_{ST} = 0 ; \quad G_{AB} = \langle \bar{g}_{AB} \rangle_{ST} = a^2(t) \delta_{AB}.
$$

(3.71)

Note that the condition on the bilocal extension $\bar{g}_{0A}(x', x)$ is in general nontrivial even when the components $g_{0A}(x)$ are chosen to be zero. In the Appendix (C.1) we show that with the above assumptions, for a general lapse function $N$, the conditions $D_{\bar{t}A} \bar{g}^{ab} = 0$ (Eqn. (3.54)) also allow us to choose

$$
U^{ij} \equiv \bar{g}^{ij} - G^{ij} = 0.
$$

(3.72)

However, it turns out that if we make the assumption that the spatial sections on $\mathcal{M}$ leading to the spatial metric (3.65) on $\bar{\mathcal{M}}$, are spatial sections in a volume preserving gauge, then the correlation terms simplify greatly. This is not surprising since the MG formalism is nicely adapted to the choice of volume preserving coordinates. Moreover, as we will see in chapter 4, at least in the perturbative context a modified version of this “VP gauge assumption” in fact becomes a necessity in order to consistently set up the formalism. We will therefore introduce spatial averaging in MG by making the VP gauge assumption, and will then calculate the correlation terms and display the modified equations resulting from this choice of gauge. Following that calculation we will also show how the correlation terms can be generalized to the case where the gauge in the inhomogeneous metric is (formally) left unspecified.

To begin our first calculation, we perform a coordinate transformation and shift to the gauge wherein the new lapse function $N$ is given by $N = 1/\sqrt{h}$ where $h$ is the determinant of the new 3-metric, denoted $h_{AB}$. In general, one will now be left with a non-zero shift vector $N^A$; however, the condition $N \sqrt{h} = 1$ ensures that the coordinates we are now using are volume preserving, since the metric determinant is given by $g = -N^2 h = -1 = \text{constant}$. We denote these volume preserving coordinates (VPCs) by $(\bar{t}, x) = (\bar{t}, x^A) = (\bar{t}, x, y, z)$, and will assume that the spatial coordinates are noncompact. For simplicity, we make the added assumption that $N^A = 0$ in the inhomogenous geometry so that $g_{\bar{t}\bar{t}} = -N^2 = -1/h$ and $g_{\bar{t}A} = 0$. The line element for the

\footnote{We are making the assumption $N^A = 0$ in the volume preserving gauge for algebraic convenience only – in our case this assumption cannot be justified by the absence of vorticity. A more detailed (and complicated) analysis should retain an arbitrary $N^A$ in the inhomogeneous geometry, and make assumptions about its average – such as $\langle N_A \rangle = 0$ for example.}
inhomogenous manifold $\mathcal{M}$ becomes

$$\langle \mathcal{M} \rangle ds^2 = -\frac{d\bar{t}^2}{h(\bar{t}, \mathbf{x})} + h_{AB}(\bar{t}, \mathbf{x})dx^Adx^B.$$  \hspace{1cm} (3.73)

Note that in this gauge, the average takes on a particularly simple form: for a tensor $p^i_j(x)$, with a spacetime averaging domain given by the “cuboid” $\Sigma$ defined by

$$\Sigma = \{(\bar{t}', x^A) \mid \bar{t} - L/2 < \bar{t}' < \bar{t} + T/2, x^A - L/2 < x^A < x^A + L/2; A = 1, 2, 3\},$$  \hspace{1cm} (3.74)

where $T$ and $L$ are averaging time and length scales respectively, the average is given by

$$\langle \tilde{p}^i_j \rangle_{ST}(\bar{t}, \mathbf{x}) = \langle p^i_j \rangle_{ST}(\bar{t}, \mathbf{x}) = \frac{1}{T^3 L^3} \int_{\bar{t} - T/2}^{\bar{t} + T/2} d\bar{t}' \int_{x - L/2}^{x + L/2} dx' \int_{y - L/2}^{y + L/2} dy' \int_{z - L/2}^{z + L/2} dz' \left[p^i_j(\bar{t}', x', y', z') \right].$$  \hspace{1cm} (3.75)

We define the “spatial averaging limit” as the limit $T \to 0$ (or $T \ll L_{\text{Hubble}}$) which is interpreted as providing a definition of the average on a spatial domain corresponding to a “thin” time slice, the averaging operation now being given by

$$\langle p^i_j \rangle(\bar{t}, \mathbf{x}) = \frac{1}{L^3} \int_{x - L/2}^{x + L/2} dx' \int_{y - L/2}^{y + L/2} dy' \int_{z - L/2}^{z + L/2} dz' \left[p^i_j(\bar{t}, \mathbf{x}') \right] + O(TL^{-1}_{\text{Hubble}}).$$  \hspace{1cm} (3.76)

(Note the time dependence of the integrand.) Henceforth, averaging will refer to spatial averaging, and will be denoted by $\langle \ldots \rangle$, in contrast to the spacetime averaging considered thus far (denoted by $\langle \ldots \rangle_{ST}$). The significance of introducing a spatial averaging in this manner is that the construction of spatial averaging is not isolated from spacetime averaging, but is a special limiting case of the latter and is, in fact, still a fully covariant operation.

For the volume preserving gauge, the averaging assumption (3.71) reduces to

$$G_{H} = \langle g_{\bar{t}\bar{t}} \rangle = \langle -\frac{1}{h} \rangle = -f^2(\bar{t}) \quad ; \quad G_{AB} = \langle h_{AB} \rangle = a^2(\bar{t})\delta_{AB},$$  \hspace{1cm} (3.77)

where $\bar{a}$ and $f$ are some functions of the time coordinate alone. A few remarks are in order on this particular choice of assumptions. Apart from the fact that the spacetime averaging operation takes on its simplest possible form (3.75) in this gauge and allows a transparent definition of the spatial averaging limit, it can also be shown that the assumptions in Eqn. (3.71) are sufficient to establish the following relations:

$$f^2(\bar{t}) = \langle \frac{1}{h} \rangle = \langle \frac{1}{\bar{a}} \rangle = \frac{1}{\bar{a}}.$$  \hspace{1cm} (3.78)

Here the second equality arises from the condition $\bar{g}^{ij} = C^{ij}$ which can be assumed whenever the

---

3The choice of a cube with sides of length $L$ as the spatial averaging domain was arbitrary, and is in fact not essential for any of the calculations to follow. In particular, all calculations can be performed with a spatial domain of arbitrary shape. We will only use the cube for definiteness and simplicity in displaying equations.
The averaged metric is of the FLR W form (see Appendix (C.1)). The last equality follows on considering the conditions $\langle \tilde{\Gamma}^a_{bc} \rangle = (\text{FLR W}) \Gamma^a_{bc}$ in obvious notation, (the basic assumption of the MG averaging scheme), details of which can be found in the Appendix (C.2). Eqn. (3.78) reduces the line element on $\bar{M}$ to the form

$$\langle \bar{\mathcal{M}} \rangle ds^2 = -\frac{dt^2}{\bar{a}^6(t)} + \bar{a}^2(\bar{t})\delta_{AB}dx^Adx^B.$$  

(3.79)

The line element in Eqn. (3.79) clearly corresponds to the FLR W metric in a volume preserving gauge which differs from the standard synchronous and comoving gauge, only by a redefinition of the time coordinate. The vector field $\bar{v}^a$ introduced at the beginning of this section and which defines the FLR W spatial sections, is now given by

$$\bar{v}^a = (\bar{a}^3, 0, 0, 0) ; \quad \bar{v}_a = G_{ab}\bar{v}^b = \left(-\frac{1}{\bar{a}^3}, 0, 0, 0, \right).$$  

(3.80)

Note that $\bar{v}^a$ is not in general the average of the vector field $u^a = (\sqrt{h}, 0, 0, 0)$ which defines the $3 + 1$ splitting on $\mathcal{M}$, but (at least in the volume preserving gauge) is related to it by

$$\bar{v}^a = \frac{\bar{a}^3}{\langle \sqrt{h} \rangle} \langle u^a \rangle.$$  

(3.81)

(A simple relation such as (3.81) cannot in general be written for an arbitrary gauge.)

As mentioned earlier, the spatial averaging limit of the covariant MG averaging is important because we want the homogeneous and isotropic FLRW geometry to average to itself. Since the FLRW geometry has a preferred set of spatial sections, one therefore needs to average over these sections. Further, since the FLRW metric adapted to its preferred spatial sections depends on the time coordinate, it is also essential that the spacetime average involve a time range that is short compared to the scale over which say the scale factor changes significantly. Clearly then, averaging the FLRW metric (denoted $^{(\text{FLRW})}g_{ab}$) given in Eqn. (3.79) will strictly yield the same metric only in the limit $T \to 0$. Namely, for the cuboid $\bar{\Sigma}$ defined in Eqn. (3.74)

$$\langle (^{(\text{FLRW})}g_{ab}) \rangle = \lim_{T \to 0} \frac{1}{T^3} \int_{\bar{\Sigma}} dt' d^3x' (^{(\text{FLRW})}g_{ab})(t', x') = (^{(\text{FLRW})}g_{ab}),$$  

(3.82)

which should be clear from the definition of the metric. The result $\langle (^{(\text{FLRW})}g_{ab}) \rangle = (^{(\text{FLRW})}g_{ab})$ in the spatial averaging limit can also be shown to hold for the FLRW metric in synchronous gauge, where the coordination bivector $\mathcal{W}^a_j$ can be easily computed using the transformation from the VPCs $(\bar{t}, x^A)$ to the synchronous coordinates $(\tau, y^A)$ given by

$$\tau = \int \frac{dt}{\bar{a}^3(t)} ; \quad y^A = x^A.$$  

(3.83)

The transformation (3.83) will also later allow us to write the averaged equations in the synchronous
gauge for the averaged geometry.

3.5 The correlation 2-form and the averaged field equations

3.5.1 Results for the Volume Preserving Gauge

In any gauge with $N^A = 0$, the expansion tensor $\Theta^A_B$ is given by

$$\Theta^A_B \equiv \frac{1}{2N} h^{AC} \partial_C h_{CB} .$$

(3.84)

The notation is the same as used in chapter 2, and the expansion scalar $\Theta$, shear tensor $\sigma^A_B$ and shear scalar $\sigma^2$ are defined as before. Note that

$$\sigma^2 = \frac{1}{2} \Theta^A_B \Theta^B_A - \frac{1}{6} \Theta^2 ; \quad \Theta^2 - \Theta^A_B \Theta^B_A = \frac{2}{3} \Theta^2 - 2\sigma^2 .$$

(3.85)

The connection 1-forms $\omega^i_j = \Gamma^i_{jk} dx^k$ in terms of the expansion tensor, are listed below for an arbitrary lapse function $N$ :

$$\omega^0_0 = \partial_t (\ln N) \bar{d}t + \partial_A (\ln N) dx^A ; \quad \omega^A_0 = N^2 h^{AC} \partial_C (\ln N) \bar{d}t + N \Theta^A_B dx^B ,$$

$$\omega^0_A = \partial_A (\ln N) \bar{d}t + \frac{1}{N} \Theta_{AB} dx^B ; \quad \omega^A_B = N \Theta^A_B dx^C ,$$

(3.86)

where $(3)\Gamma^A_{BC}$ is the Christoffel symbol built from the 3-metric $h_{AB}$ and its inverse. Specializing now to the volume preserving gauge ($N = h^{-1/2}$), the bilocal extensions $\Omega^i_j$ of the connection 1-forms are trivial and are simply given by

$$\Omega^i_j (x', x) = \Gamma^i_{jk} dx^k .$$

(3.87)

Since $G_{ab} = \bar{g}_{ab}$, the connection 1-forms $\bar{\Omega}^i_j$ for the averaged manifold $\bar{M}$ are constructed using the FLRW metric in volume preserving gauge given in Eqn. (3.79), and are given by

$$\bar{\Omega}^0_0 = -3H \bar{d}t ; \quad \bar{\Omega}^A_0 = H \delta^A_B dx^B ,$$

$$\bar{\Omega}^0_A = \bar{a} \delta^A_B dx^B ; \quad \bar{\Omega}^A_B = H \delta^A_B \bar{d}t ,$$

(3.88)

where we have defined

$$H \equiv \frac{1}{\bar{a}} \frac{d\bar{a}}{dt} .$$

(3.89)

Using Eqns. (3.80) with $N = h^{-1/2}$, Eqn. (3.87) and Eqns. (3.88), we can now easily construct the correlation 2-form $Z^i_{b,j}$ defined in Eqn. (3.39). For completeness, we will display all the nontrivial components $Z^i_{b,j}$, although not all of them will be relevant for the final equations. The condition $N = h^{-1/2}$ has the effect that several of the Christoffel symbols (which can be read off from

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Eqn. (3.86) become related to each other. For example, we have $\Gamma_{00}^0 = -\partial_0 (\ln \sqrt{h}) = -\Gamma_{0A}^A = -(1/\sqrt{h}) \Theta$, and so on. Denoting the spatial average of some quantity $p_j$ as simply $\langle p_j \rangle$, we have

$$Z_{0}^{0,A} = -Z_{0}^{0,0}$$

$$= \left[ (\Theta \Theta_{AJ}) + \left( \partial_A (\ln \sqrt{h}) \partial_J (\ln \sqrt{h}) \right) - 3\tilde{a}^8 H^2 \delta_{AJ} \right] dx^J \wedge d\tilde{t}$$

$$+ \left( \sqrt{h} \Theta_{AJ} \partial_K (\ln \sqrt{h}) \right) dx^J \wedge dx^K,$$

(3.90a)

$$Z_{0}^{0,A} = -Z_{0}^{0,0}$$

$$= \left[ \left( \frac{1}{h} \Theta \Theta_{J} \right) + \left( \frac{1}{h} \tilde{h}^{AB} \tilde{h} \partial_K (\ln \sqrt{h}) \right) \right] dx^J \wedge d\tilde{t}$$

$$\left( \frac{1}{h} \sqrt{h} \Theta_{J} \partial_K (\ln \sqrt{h}) \right) dx^J \wedge dx^K,$$

(3.90b)

$$Z_{0}^{A,B} = -Z_{0}^{0,0}$$

$$= \left[ \left( \frac{1}{\sqrt{h}} \Theta (3)_{B}^{A} \right) \right] dx^J \wedge d\tilde{t}$$

$$\left( \frac{1}{\sqrt{h}} \Theta (3)_{B}^{A} \right) dx^J \wedge dx^K,$$

(3.90c)

$$Z_{0}^{A,B} = \left[ \left( \frac{1}{\sqrt{h}} \Theta (3)_{B}^{A} \right) \right] dx^J \wedge d\tilde{t}$$

$$\left( \frac{1}{\sqrt{h}} \Theta (3)_{B}^{A} \right) dx^J \wedge dx^K,$$

(3.90d)

$$Z_{0}^{B,C} = -Z_{0}^{0,0}$$

$$= \left[ \left( \frac{1}{\sqrt{h}} \Theta (3)_{C}^{A} \right) \right] dx^J \wedge d\tilde{t}$$

$$\left( \frac{1}{\sqrt{h}} \Theta (3)_{C}^{A} \right) dx^J \wedge dx^K,$$

(3.90e)

where we have used the relation $(3)^{B}_{J} = \partial_B (\ln \sqrt{h})$.

It is now straightforward to use the relations in Eqn. (3.59) (note the unconventional normalization of the 2-form) to read off the components $Z_{bm}^{i,j}$ and hence perform the required summations to construct $Z_{i,j}^{a}$. This, together with the fact that $\tilde{g}_{ab} = G_{ab}$ (see Appendix (C.1)), allows us to
construct the correlation tensor \( C^a_b \) defined in Eqn. (3.60)

\[
C^a_b = \left( Z^a_{ijb} - \frac{1}{2} \delta^a_b Z^m_{ijm} \right) G^{ij}.
\] (3.91)

Now, the components of the Einstein tensor \( E^a_b \) for the averaged spacetime with metric (3.79) are given by

\[
E^t_t = 3\dot{a}^6 H^2; \quad E^t_A = 0 = E^B_t,
\]

\[
E^A_B = \dot{a}^6 \delta^A_B \left[ 2 \left( \frac{\ddot{a}}{a} + 3H^2 \right) + H^2 \right],
\] (3.92)

where the peculiar splitting of terms in the last equation is for later convenience. Recall that the overdot denotes a derivative with respect to the VPC time \( \bar{t} \), not synchronous time. In terms of the coordinate independent objects introduced in Eqn. (3.69), we have

\[
\dot{j}_1(x) = -3\dot{a} H^2; \quad \dot{j}_2(x) = \dot{a}^6 \left[ 2 \left( \frac{\ddot{a}}{a} + 3H^2 \right) + H^2 \right].
\] (3.93)

From the averaged Einstein equations in (3.62) we next construct the scalar equations which in the standard case would correspond to the Friedmann equation and the Raychaudhuri equation. These correspond to the Einstein tensor components,

\[
E^a_b \bar{v}^b \bar{v}_a = \dot{j}_1(x) \quad \pi^b_a E^a_b + E^a_b \bar{v}^b \bar{v}_a = 3\dot{j}_2(x) + \dot{j}_1(x),
\] (3.94)

and are given by

\[
3\dot{a}^6 H^2 = (\kappa T^a_b - C^a_b) \bar{v}_a \bar{v}^b = \kappa \bar{\rho} - \frac{1}{2} \left[ Q^{(1)} + S^{(1)} \right],
\] (3.95a)

\[
6\dot{a}^6 \left( \frac{\ddot{a}}{a} + 3H^2 \right) = (\kappa T^a_b + C^a_b) \left( \bar{v}_a \bar{v}^b + \pi^b_a \right) = -\kappa (\bar{\rho} + 3\bar{\rho}) + 2 \left[ Q^{(1)} + Q^{(2)} + S^{(2)} \right].
\] (3.95b)

Here Eqn. (3.95a) is the modified Friedmann equation and Eqn. (3.95b) the modified Raychaudhuri equation (in the volume preserving gauge on \( \mathcal{M} \)). We have used Eqn. (3.68), with the overbar on \( \rho \) and \( p \) reminding us that they are expressed in terms of the nonsynchronous time \( \bar{t} \), and we have
defined the correlation terms

\[ Q^{(1)} = \bar{a}^6 \left[ \frac{2}{3} \left( \frac{1}{h} \Theta^2 - \frac{1}{\bar{a}^6} (F \Theta^2) \right) - 2 \left( \frac{1}{h} \sigma^2 \right) \right] ; \quad \frac{1}{\bar{a}^6} (F \Theta^2) = (3H)^2 , \]  

(3.96a)

\[ S^{(1)} = \frac{1}{\bar{a}^2} \delta^{AB} \left[ \left( (3) \Gamma^J_{AC} (3) \Gamma^C_{BJ} \right) - \langle \partial_A (\ln \sqrt{h}) \partial_B (\ln \sqrt{h}) \rangle \right] , \]  

(3.96b)

\[ Q^{(2)} = \bar{a}^6 \left( \frac{1}{h} \Theta^A \Theta^B \right) - \frac{1}{\bar{a}^2} \delta^{AB} \left( \Theta_A \Theta^J_B \right) , \]  

(3.96c)

\[ S^{(2)} = \bar{a}^6 \left( \frac{1}{h} h^A_B \partial_A (\ln \sqrt{h}) \partial_B (\ln \sqrt{h}) \right) - \frac{1}{\bar{a}^2} \delta^{AB} \left( \partial_A (\ln \sqrt{h}) \partial_B (\ln \sqrt{h}) \right) . \]  

(3.96d)

We have used the second relation in Eqn. (3.85) in defining \( Q^{(1)} \). \( Q^{(1)} \) and \( Q^{(2)} \) are correlations of the extrinsic curvature, whereas \( S^{(1)} \) and \( S^{(2)} \) are correlations restricted to the intrinsic 3-geometry of the spatial slices of \( M \). Since the components of \( C^a_b \) are not explicitly constrained by Eqns. (3.51) and (3.52), we can treat the combinations \((1/2)(Q^{(1)} + S^{(1)}) = -C^0_0 \) and \( 2(Q^{(1)} + Q^{(2)} + S^{(2)}) = (C^A_A - C^0_0) \) as independent, subject only to the differential constraints (3.64) which follow if we assume \( D^i_0 Z^a_{b\ j} = 0 \). We will return to these below.

As discussed earlier, the remaining components of \( C^a_b \) must be set to zero, giving constraints on the underlying inhomogeneous geometry. In coordinate independent language, these constraints read

\[ \pi^b_k C^a_b v_a = 0 = \pi^k_a C^a_b v^b ; \quad \pi^i_a \pi^k_b C^a_b - \frac{1}{3} \pi^i_a \left( \pi^b_a C^a_b \right) = 0 . \]  

(3.97)

Eqns. (3.97) reduce to the following for our specific choice of volume preserving coordinates,

\[ C^0_A = 0 ; \quad C^A_0 = 0 ; \quad C^A_B - \frac{1}{3} \delta^A_B (C^J_J) = 0 , \]  

(3.98)
or, in full detail,

\[
\begin{align*}
\left[ a^6 \left( \frac{h^{JK}}{h} \sqrt{h} \Theta_{J\alpha\partial K} (\ln \sqrt{h}) \right) - \langle h^{JK} \rangle \langle \sqrt{h} \Theta_{J\alpha\partial K} (\ln \sqrt{h}) \rangle \right] \\
+ \left[ \langle h^{JK} \rangle \langle \sqrt{h} \Theta_{JB}^{\langle 3 \rangle} \Gamma_{\alpha K}^{B} \rangle - a^6 \left( \frac{h^{JK}}{h} \sqrt{h} \Theta_{J\alpha\partial K} (\ln \sqrt{h}) \right) \right] = 0 , \quad (3.99a)
\end{align*}
\]

\[
\begin{align*}
\left[ a^6 \left( \frac{h^{JK}}{h} \right) \frac{1}{\sqrt{h}} \Theta_{J\alpha\partial K} (\ln \sqrt{h}) - \langle h^{JK} \rangle \left( \frac{1}{\sqrt{h}} \Theta_{J\alpha\partial K} (\ln \sqrt{h}) \right) \right] \\
+ \left[ \langle h^{JK} \rangle \left( \frac{1}{\sqrt{h}} \Theta_{K}^{\langle 3 \rangle} \Gamma_{JB}^{B} \right) - a^6 \left( \frac{h^{JA}}{h} \sqrt{h} \Theta_{K}^{\langle 3 \rangle} \Gamma_{JB}^{B} \right) \right] = 0 , \quad (3.99b)
\end{align*}
\]

\[
\begin{align*}
\left[ a^6 \left( \frac{h^{JK}}{h} \Theta_{J\alpha\partial K} (\ln \sqrt{h}) \right) - \langle h^{JK} \rangle \langle \Theta_{\alpha\beta\partial K} (\ln \sqrt{h}) \rangle \right] \\
+ a^6 \left[ \left( \frac{1}{h} \right) h^{AC} \partial_{C} (\ln \sqrt{h}) \partial_{B} (\ln \sqrt{h}) \right] \\
- \langle h^{JK} \rangle \langle \langle 3 \rangle \Gamma_{JC}^{\langle 3 \rangle} \Gamma_{KB}^{C} \rangle = \frac{1}{3} \delta_{AB} \left( Q^{(2)} - S^{(1)} + S^{(2)} \right) , \quad (3.99c)
\end{align*}
\]

where \( \langle h^{JK} \rangle = G^{JK} = (1/a^2) \delta^{JK} \). Eqns. \( (3.86) \) with the choice \( g_{00} = -N^2 = -h^{-1} \) show that all the terms paired within square brackets in Eqns. \( (3.99) \) above, as also the correlations \( Q^{(2)} \) and \( S^{(2)} \) defined in Eqns. \( (3.96c) \) and \( (3.96d) \), are of the form

\[
\frac{1}{\langle g_{00} \rangle} \langle g_{00} g^{AB} \Gamma_{b_1 c_1 j_1 k_1}^{a_1 i_1} \rangle - \langle g^{AB} \rangle \langle \Gamma_{b_2 c_2 j_2 k_2}^{a_2 i_2} \rangle . \quad (3.100)
\]

The assumption in Eqn. \( (3.53b) \) shows that one can write

\[
\langle g_{00} g^{AB} \Gamma_{bc}^{a} \Gamma_{jk}^{i} \rangle = \langle g_{00} g^{AB} \rangle \langle \Gamma_{bc}^{a} \Gamma_{jk}^{i} \rangle = -\left( \frac{h^{AB}}{h} \right) \langle \Gamma_{bc}^{a} \Gamma_{jk}^{i} \rangle . \quad (3.101)
\]

An interesting point is that the VPC assumption \( N = h^{-1/2} \) also allows us to assume \( \langle h^{AB}/h \rangle = \langle h^{AB} \rangle (1/h) \) consistently with the formalism (details in Appendix \( (C.2) \)). Using Eqn. \( (3.78) \) this gives us

\[
\left( \frac{h^{AB}}{h} \right) = \frac{1}{a^6} \langle h^{AB} \rangle . \quad (3.102)
\]

This leads to some remarkable cancellations in Eqns. \( (3.99) \), and also shows that the correlation terms \( Q^{(2)} \) and \( S^{(2)} \) in fact vanish,

\[
Q^{(2)} = 0 = S^{(2)} . \quad (3.103)
\]
Eqns. (3.99) simplify to give
\[\delta^J K \left[ \langle \sqrt{\hbar} \Theta_{JK}^{(3)} \Gamma^B_{AK} \rangle - \langle \sqrt{\hbar} \Theta_{JK}^{(3)} \Gamma^B_{AB} \rangle \right] = 0, \quad (3.104a)\]
\[\delta^J K \left( \frac{1}{\sqrt{\hbar}} \Theta^B_K^{(3)} \Gamma^A_{JB} \right) - \delta^A J \left( \frac{1}{\sqrt{\hbar}} \Theta^K_J^{(3)} \Gamma^B_{JB} \right) = 0, \quad (3.104b)\]
\[\delta^J K \left( \Gamma^A_{JC}^{(3)} \Gamma^C_{KB} \right) - \delta^A J \left( \Gamma^C_{JC}^{(3)} \Gamma^K_{JB} \right) = \frac{1}{3} \delta_B^A \left( \bar{a}^2 S^{(1)} \right), \quad (3.104c)\]

These simplifications are solely a consequence of assuming that the inhomogeneous metric in the volume preserving gauge averages out to give the FLRW metric in standard form. In general, these simplifications will not occur when the standard FLRW metric arises from an arbitrary choice of gauge for the inhomogeneous metric.

In order to come as close as possible to the standard approach in cosmology, we will now rewrite the scalar equations (3.95) (which are the cosmologically relevant ones) after performing the transformation given in Eqn. (3.83) in order to get the FLRW metric to the form
\[\bar{M}^A \delta_{AB} dy^A dy^B; \quad a(\tau) = \bar{a}(\bar{t}(\tau)). \quad (3.105)\]

Since Eqns. (3.95) are scalar equations, this transformation only has the effect of re-expressing all the terms as functions of the synchronous time \(\tau\). Although the transformation will change the explicit form of the coordination bivector \(W_a'\), this change involves only the time coordinate, and in the spatial averaging limit there is no difference between averages computed in the VPCs and those computed after the time redefinition. This again emphasizes the importance of the spatial averaging limit of spacetime averaging, if we are to succeed operationally in explicitly displaying the correlations as corrections to the standard cosmological equations. The correlation terms in Eqns. (3.96) are therefore still interpreted with respect to the volume preserving gauge, but are treated as functions of \(\tau\). For the scale factor on the other hand, we have
\[\bar{a}^3 H = \frac{1}{a} \frac{da}{d\tau} \equiv H_{FLRW}; \quad \bar{a}^6 \left( \frac{\bar{a}}{a} + 3H^2 \right) = \frac{1}{a} \frac{d^2 a}{d\tau^2}. \quad (3.106)\]

Further writing
\[\rho(\tau) = \bar{\rho}(\bar{t}(\tau)); \quad p(\tau) = \bar{p}(\bar{t}(\tau)), \quad (3.107)\]
equations (3.95) become
\[H^2_{FLRW} = \frac{8\pi G N}{3} \rho - \frac{1}{6} \left[ Q^{(1)} + S^{(1)} \right], \quad (3.108a)\]
\[\frac{1}{a} \frac{d^2 a}{d\tau^2} = -\frac{4\pi G N}{3} (\rho + 3p) + \frac{1}{3} Q^{(1)}. \quad (3.108b)\]
We emphasize that the quantities \( Q^{(1)} \) and \( S^{(1)} \), defined in Eqns. (3.96a) and (3.96b) as correlations in the volume preserving gauge, are to be thought of as functions of the synchronous time \( \tau \), where the synchronous time coordinate itself was defined after the spatial averaging. Such an identification is justified since we are dealing with scalar combinations of these quantities. Note that \( Q^{(1)} \) and \( S^{(1)} \) can be treated independently, apart from the constraints imposed by Eqn. (3.64), which we turn to next. These conservation conditions can be decomposed into a scalar part and a 3-vector part, given respectively by

\[
\bar{v}^b C_{b,a} = 0 ; \quad \pi_k C_{b,a} = 0 .
\]

In the synchronous gauge (3.105) for the FLRW metric, the scalar equation reads

\[
\left( \partial_\tau Q^{(1)} + 6H_{\text{FLRW}} Q^{(1)} \right) + \left( \partial_\tau S^{(1)} + 2H_{\text{FLRW}} S^{(1)} \right) = 0 .
\]

We recall that this equation is a consequence of setting the correlation 3-form and the correlation 4-form to zero, and it relates the evolution of \( Q^{(1)} \) and \( S^{(1)} \). The 3-vector equation (on imposing the first set of conditions in Eqn. (3.97)) simply gives \( \partial_\tau C_{A}^\tau = 0 \), so that \( C_{A}^\tau = 0 = \text{constant} \), which also implies that \( C_{A}^0 = 0 = \text{constant} \) and hence this equation gives nothing new. (We have used the relations \( C_{0}^0 = C_{0}^\tau \), \( C_{A}^0 = \bar{a}^3 C_{A}^\tau \) and \( C_{A}^0 = (1/\bar{a}^3)C_{A}^\tau \) where 0 denotes the nonsynchronous time coordinate \( \bar{t} \).)

The cosmological equations (3.108), along with the constraint equations (3.104) and (3.110) are the key results of this section. Subject to the acceptance of the volume preserving gauge on the underlying manifold \( \mathcal{M} \) they can in principle be used to study the role of the correction terms resulting from spatial averaging.

### 3.5.2 Results for an arbitrary gauge choice

In this subsection, we will display the results obtained on assuming that the metric

\[
(\mathcal{M}) ds^2 = -N^2(t, \mathbf{x}) dt^2 + h_{AB}(t, \mathbf{x}) dx^A dx^B ,
\]

averages out to the FLRW metric in standard form with a nonsynchronous time coordinate \( t \) in general, to give

\[
(\mathcal{M}) ds^2 = -f^2(t) dt^2 + \bar{a}^2(t) \delta_{AB} dx^A dx^B .
\]

In other words, we are assuming that the relations in Eqn. (3.71) hold. Note that the averaging operation is no longer trivial, although we are still assuming an averaging on domains corresponding to “thin” time slices. We again split the averaged Einstein equations into scalar equations, and 3-vector and traceless 3-tensor equations. After transforming to the synchronous time coordinate \( \tau \), now defined by

\[
\tau = \int^t f(t') dt' ,
\]

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and again defining \( H \equiv (1/a)(da/dt) \) and \( H_{\text{FLRW}} \equiv (1/a)(da/d\tau) \) with \( a(\tau) = \bar{a}(t(\tau)) \), the modified Friedmann and Raychaudhuri equations read

\[
H_{\text{FLRW}}^2 = \frac{8\pi G_N}{3} \rho - \frac{1}{6} \left[ \hat{\rho}^{(1)} + \hat{S}^{(1)} \right], \tag{3.114a}
\]

\[
\frac{1}{f^2} \frac{d^2 a}{d\tau^2} = -\frac{4\pi G_N}{3} (\rho + 3p) + \frac{1}{3} \left[ \hat{\rho}^{(1)} + \hat{\rho}^{(2)} + \hat{S}^{(2)} \right], \tag{3.114b}
\]

where the correlation terms are now defined using the relations,

\[
\hat{\rho}^{(1)} = \frac{1}{f^2} \left[ \langle \bar{\Gamma}^A_{0A} \bar{\Gamma}^B_{0B} \rangle - \langle \bar{\Gamma}^A_{0B} \bar{\Gamma}^B_{0A} \rangle - 6H^2 \right], \tag{3.115a}
\]

\[
\hat{S}^{(1)} = \langle \bar{g}^{JK} \rangle \left[ \langle \bar{\Gamma}^A_{JBK} \bar{\Gamma}^B_{KA} \rangle - \langle \bar{\Gamma}^A_{JA} \bar{\Gamma}^B_{KB} \rangle \right], \tag{3.115b}
\]

\[
\hat{\rho}^{(2)} + \hat{\rho}^{(1)} = -\frac{1}{f^2} \langle \bar{\Gamma}^A_{0A} \bar{\Gamma}^0_{00} \rangle - \langle \bar{g}^{JK} \rangle \langle \bar{\Gamma}^0_{JA} \bar{\Gamma}^A_{0K} \rangle + \frac{3H}{f^2} (\partial_t (\ln f) + H), \tag{3.115c}
\]

\[
\hat{S}^{(2)} = \frac{1}{f^2} \langle \bar{\Gamma}^A_0 \bar{\Gamma}^0_{A0} \rangle + \langle \bar{g}^{JK} \rangle \langle \bar{\Gamma}^0_{JA} \bar{\Gamma}^A_{0K} \rangle. \tag{3.115d}
\]

We emphasize that averaging here refers to spatial averaging. Also \( \langle \bar{g}^{JK} \rangle = G^{JK} = (1/a^2)\delta^{JK} \), and the index 0 refers to the nonsynchronous time \( t \). It is easy to check using Eqn. (3.86), that \( \hat{\rho}^{(1)} \) and \( \hat{\rho}^{(1)} + \hat{\rho}^{(2)} \) correspond to correlations of (the bilocal extensions of) the extrinsic curvature with itself and with the time derivative of the lapse function. \( \hat{S}^{(1)} \) corresponds to correlations between the Christoffel symbols of the 3-geometry, and \( \hat{S}^{(2)} \) to correlations of the spatial derivative of the lapse function with itself and with the Christoffel symbols of the 3-geometry. Due to the way we have defined these correlations, one can also check that when the lapse function satisfies \( N\sqrt{h} = 1 \) (so that the averaging becomes trivial), we have \( \hat{\rho}^{(1)} = Q^{(1)}, \hat{S}^{(1)} = S^{(1)} \), and \( \hat{\rho}^{(2)} = 0 = \hat{S}^{(2)} \), where \( Q^{(1)} \) and \( S^{(1)} \) were defined in Eqns. (3.96). The 3-vector and traceless 3-tensor equations become

\[
\frac{1}{f^2} \left[ \langle \bar{\Gamma}^0_{0A} \bar{\Gamma}^B_{00} \rangle - \langle \bar{\Gamma}^0_{0B} \bar{\Gamma}^B_{0A} \rangle \right] + \langle \bar{g}^{JK} \rangle \left[ \langle \bar{\Gamma}^A_{JBK} \bar{\Gamma}^B_{KA} \rangle - \langle \bar{\Gamma}^A_{JA} \bar{\Gamma}^B_{KB} \rangle \right] = 0, \tag{3.116a}
\]

\[
\frac{1}{f^2} \left[ \langle \bar{\Gamma}^A_{00} \bar{\Gamma}^B_{00} \rangle - \langle \bar{\Gamma}^B_{00} \bar{\Gamma}^A_{00} \rangle \right] + \langle \bar{g}^{JK} \rangle \left[ \langle \bar{\Gamma}^A_{JB} \bar{\Gamma}^B_{0K} \rangle - \langle \bar{\Gamma}^A_{J0} \bar{\Gamma}^B_{KB} \rangle \right] = 0, \tag{3.116b}
\]

\[
\frac{1}{f^2} \left[ \langle \bar{\Gamma}^A_{B0} \bar{\Gamma}^m_{0m} \rangle - \langle \bar{\Gamma}^m_{B0} \bar{\Gamma}^A_{0m} \rangle \right] + \langle \bar{g}^{JK} \rangle \left[ \langle \bar{\Gamma}^A_{jm} \bar{\Gamma}^m_{KB} \rangle - \langle \bar{\Gamma}^A_{j0} \bar{\Gamma}^m_{KB} \rangle \right] = -\frac{1}{3} \delta^B_A \left[ \hat{\rho}^{(2)} - \hat{S}^{(2)} - \hat{S}^{(1)} - \frac{9H}{f^2} \left( H + \frac{1}{3} \partial_t (\ln f) \right) \right], \tag{3.116c}
\]
where the lower case index $m$ in the last equation runs over all spacetime indices $0, 1, 2, 3$, with the index 0 referring to the nonsynchronous time $t$. It is easy to check that Eqns. (3.104) reduce to Eqns. (3.111) with the choice $N = h^{-1/2}$. The condition $C_{a b}^a = 0$ has the scalar part,

$$\left( \partial_\tau \tilde{\mathcal{P}}^{(1)} + 6 H_{\text{FLRW}} \tilde{\mathcal{P}}^{(1)} \right) + \left( \partial_\tau \tilde{\mathcal{S}}^{(1)} + 2 H_{\text{FLRW}} \tilde{\mathcal{S}}^{(1)} \right) + 4 H_{\text{FLRW}} \left( \tilde{\mathcal{P}}^{(2)} + \tilde{\mathcal{S}}^{(2)} \right) = 0,$$

(3.117)

while the 3-vector part, as before, gives nothing new and simply states $\partial_\tau C_A^A = 0$.

We can now state the main result of this section as follows: Having assumed that the FLRW spatial sections arise as the average of some gauge choice with lapse function $N(t, \mathbf{x})$, spatial 3-metric $h_{AB}(t, \mathbf{x})$ and shift vector $N^A$ set to zero for convenience, we can construct the scalar quantities $C_{b}^a \bar{v}^b \bar{v}_a$ and $\pi_{a}^b C_{b}^a + C_{b}^a \bar{v}^b \bar{v}_a$ which, in coordinates natural to the FLRW metric take the form,

$$C_{b}^a \bar{v}^b \bar{v}_a = \frac{1}{2} \left[ \tilde{\mathcal{P}}^{(1)} + \tilde{\mathcal{S}}^{(1)} \right]; \quad \pi_{a}^b C_{b}^a + C_{b}^a \bar{v}^b \bar{v}_a = 2 \left[ \tilde{\mathcal{P}}^{(1)} + \tilde{\mathcal{P}}^{(2)} + \tilde{\mathcal{S}}^{(2)} \right],$$

(3.118)

with the various quantities being defined in Eqns. (3.115). These scalars modify the usual cosmological equations as shown in Eqns. (3.114), and are themselves subject to the differential conditions (3.117). In addition, for consistency of our assumptions with the formalism, the underlying inhomogeneous metric is also subject to the conditions (3.116).

The combinations on the right hand sides of the relations (3.118) can clearly be treated independently, apart from the conditions (3.117). Further, since the correlation 2-form has 40 independent components $Z_{b m j n}^a$ after imposing all algebraic constraints, and since none of the four quantities $\tilde{\mathcal{P}}^{(1)}, \tilde{\mathcal{P}}^{(2)}, \tilde{\mathcal{S}}^{(1)}$ and $\tilde{\mathcal{S}}^{(2)}$ are trivially related by these constraints, one can always treat these four functions independently of each other, subject only to the constraint in Eqn. (3.117). In general this constraint may also not be satisfied, and the correlation tensor may not be independently covariantly conserved. In chapter 4 we will see the explicit time dependence of these correlation objects in the perturbative setting in cosmology.

### 3.6 Comparing the approaches of Buchert and Zalaletdinov

An important motivation in studying the spatial averaging limit of MG was to be able to compare its results with those of Buchert’s spatial averaging. Buchert’s averaging is the only approach apart from Zalaletdinov’s MG, which is capable of treating inhomogeneities in a nonperturbative manner, although it is limited to using only scalar quantities within a chosen $3 + 1$ splitting of spacetime. Buchert takes the trace of the Einstein equations in the inhomogeneous geometry, and averages these inhomogeneous scalar equations. In the context of Zalaletdinov’s MG however, we have used the existence of the vector field $\bar{v}^a$ in the FLRW spacetime to construct scalar equations after averaging the full Einstein equations. As far as observations are concerned, it has been noted by Buchert and Carfora [42], that the spatially averaged matter density $\langle \rho \rangle_D$ defined by Buchert is not the appropriate observationally relevant quantity – the “observed” matter density (and pressure)
is actually defined in a homogeneous space. Since we have done precisely this in Eqn. (3.68), we are directly dealing with the appropriate observationally relevant quantity in the MG framework.

Another important difference between the two approaches is the averaging operation itself. Buchert’s spatial average, given by (2.4), is different from the averaging operation we have been using (given by Eqn. (3.76) using the volume preserving gauge), which is a limit of a spacetime averaging defined using the coordination bivector \( \mathcal{W}_j^i \). Further, since Buchert only averages two of the Einstein equations, a major difference between the two schemes is the presence of the constraints (3.104) on the underlying geometry. These are in general nontrivial and hence indicate that it is not sufficient to assume that the metric of the inhomogeneous manifold averages out to the FLRW form.

Most importantly though, Buchert’s averaging scheme by itself does not incorporate the concept of an averaged manifold \( \bar{\mathcal{M}} \) (although the work of Buchert and Carfora [42] does deal with 3-spaces of constant curvature). The question then arises as to how one should interpret Buchert’s \( a_D \). If one does not wish to identify \( a_D \) with the scale factor in FLRW cosmology, one is compelled to develop a whole new set of ideas in order to try and compare theory with observation. On the other hand, if one does (naively) identify \( a_D \) with the scale factor, comparison with standard cosmology becomes more convenient, but this introduces a possible inconsistency since we know from Zalaletdinov’s approach that additional constraints need to be satisfied. It is not clear how one should account for these constraints since the non-scalar Einstein equations are not averaged in Buchert’s approach.

It is our understanding that the MG approach is a complete and self-consistent scheme which allows us to meaningfully pose questions in the averaging paradigm, which are directly interpretable in terms of standard cosmological ideas. The Buchert approach on the other hand, is harder to interpret. In the MG approach, there are no unaveraged shear equations, because the trace of the Einstein equations has been taken after performing the averaging on the underlying geometry. Since the averaged geometry is FLRW, its shear is zero by definition. There is a natural metric on the averaged manifold by construction, the FLRW metric. The correlations satisfy additional constraints, given by Eqns. (3.104). Thus, once a gauge has been chosen and if one can overcome the computational complexity of the averaging operation, the cosmological equations derived by us in the MG approach are complete and ready for application, without any further caveats.

In spite of these differences, our equations (3.108) and (3.110) for the volume preserving gauge are strikingly similar to Buchert’s effective FLRW equations and their integrability condition in the dust case; and in the case of general \( N \), the role of Buchert’s dynamical backreaction \( \bar{P}_D \) in Eqns. (2.15) and (2.19) is identical to that of our combination of \( (\tilde{P}^{(2)} + \tilde{S}^{(2)}) \) in Eqns. (3.114b) and (3.117). Concentrating on the volume preserving case, the structure of the correlation \( Q^{(1)} \) is identical to Buchert’s kinematical backreaction \( \bar{Q}_D \) (or \( Q_D \) in the general case). The correlation \( S^{(1)} \) appears in place of the averaged 3-Ricci scalar \( \langle R \rangle_D \) in Buchert’s dust equations. This is not unreasonable since Buchert’s \( \langle R \rangle_D \) can be thought of as \( \langle R \rangle_D = 6k_D/a_D^2 + \) corrections, where \( 6k_D/a_D^2 \) represents the 3-Ricci scalar on the averaged manifold which in our case is zero, and
hence $\mathcal{S}^{(1)}$ represents the corrections due to averaging. Further, these similarities are in spite of the fact that our correlations were defined assuming that a *volume preserving* gauge averages out to the FLRW 3-metric in standard form, whereas Buchert’s averaging is most naturally adapted to beginning with a *synchronous* gauge. This remarkable feature, at least to our understanding, does not seem to have any deeper meaning – it simply seems to arise from the structure of the Einstein equations themselves, together with our assumption $D_\Omega Z^i_{b\ j} = 0$. In the absence of this latter condition, one would have to consider the correlation 3- and 4-forms mentioned earlier, and the structure of the correlation terms and their “conservation” equations would be far more complicated. In the rest of this thesis we will restrict our calculations to Zalaletdinov’s approach.

**Chapter summary and discussion:**

This chapter dealt with Zalaletdinov’s fully covariant framework for averaging Einstein’s equations, named Macroscopic Gravity (MG). Although the details of this approach are rather involved, its strength lies in the fact that one can speak in terms of a physically relevant “averaged metric” whose behaviour is governed by a set of modified Einstein equations. We argued that in the context of cosmology, the standard assumptions regarding the large scale homogeneity and isotropy of the universe, translate to the requirement of taking a *spatial averaging limit* of the four-dimensional averaging in MG. The resulting modified cosmological equations were then compared with the effective equations derived by Buchert, which we discussed in the previous chapter.

We wish to emphasize an issue which is of importance in understanding the approach we will take in subsequent chapters, which will deal with Zalaletdinov’s averaging. There is a significant difference between the original philosophy of the averaging formalism, common to both the Buchert and Zalaletdinov schemes, and the manner in which we employ Zalaletdinov’s averaging. The original idea as developed by these authors was to construct a framework which would independently describe a suitably defined averaged dynamics, with no reference to the inhomogeneous spacetime whose average leads to this dynamics. Zalaletdinov’s MG is therefore a new *theory* of gravity describing the dynamics of an averaged manifold, with no recourse to the underlying manifold which is described by the usual Einstein equations. The backreaction $C_\Omega^a$ in this approach is

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4 An entirely different outlook towards his approach has been emphasized to us by Buchert [68]. According to Buchert, the absence of an averaged manifold $\mathcal{M}$ is not to be thought of as a ‘caveat’, but as a feature deliberately retained ‘on purpose’. The actual inhomogeneous universe is regarded by Buchert as the only fundamental entity, and the introduction of an averaged universe is in fact regarded as an unphysical and unnecessary approximation. As we mentioned earlier, this is probably the most important difference between MG and Buchert’s approach. In the latter, contact with observations is to be made by constructing averaged quantities, such as the scalars defined earlier in this section, and by introducing the expansion factor $a_\Omega$. The assertion here is that the averaging of geometry, as discussed in MG or in the Renormalization Group approach of Buchert and Carfora [42], is not an indispensable step in comparing the inhomogeneous universe with actual observations. The need for averaging of geometry is to be physically separated from simply looking at effective properties (such as the constructed scalars) which can be defined for any inhomogeneous metric. Averaging of geometry becomes relevant if (i) an observer insists on interpreting the data in a FLRW template model, so that (s)he needs a mapping from the actual inhomogeneous slice and its average properties to the corresponding properties in this template, or (ii) one desires a mock metric, to sort of have a thermodynamic effective metric to approximate the real one.
actually a new field in the problem which satisfies its own equations and whose dynamics must be solved for simultaneously with that of other fields such as the averaged metric and the averaged energy-momentum tensor for matter.

Our approach to the backreaction issue is different: We consider it central to be able to self-consistently describe both the inhomogeneous geometry as well as its averaged counterpart. We find this necessary since modern cosmology crucially relies on observations of inhomogeneities around us, and ignoring the evolution of inhomogeneities when solving for the averaged dynamics does not appear to be satisfactory. Put another way, when faced with a solution of the averaged dynamics, we find it essential to answer the question “which (if any) inhomogeneous solution could lead to this averaged homogeneous solution?” All our subsequent calculations will therefore focus on solving for the averaged dynamics of specific inhomogeneities, which we attempt to keep as realistic as possible. We will start with an application of the spatial averaging limit of MG, to linear perturbation theory in cosmology, in the next chapter. The reader is referred to papers by Zalaletdinov and co-workers in Ref. [53], for applications of MG as a stand-alone theory for an averaged manifold.
Chapter 4

Backreaction in linear perturbation theory

In this chapter we will adapt the MG formalism and its spatial averaging limit to the specific case of linear cosmological perturbation theory (PT). The motivation behind this exercise is to verify the self-consistency of cosmological PT in the presence of a backreaction due to averaging. In other words, we wish to ask whether cosmological PT is stable against the inclusion of dynamical backreaction terms, or whether a runaway process can render the PT invalid.

This chapter is organised as follows: In Sec. 4.1 we collect some useful results from linear cosmological perturbation theory. Sec. 4.2 presents details of the MG averaging procedure adapted to cosmological PT, including general expressions for the leading order backreaction terms, with a discussion of gauge related issues and the definition of the averaging operator. The main results are in Sec. 4.3 where we derive final expressions for the backreaction, both in real space and Fourier space, which can be directly utilised in model calculations. These expressions use a few simplifying restrictions which can be lifted if necessary in a completely straightforward manner. Sec. 4.4 contains example calculations in first order PT, which show that the magnitude of the backreaction is, as expected, negligible compared to the homogeneous energy density of matter in the radiation dominated era and for a significant part of the matter dominated era. Throughout the chapter, a prime refers to a derivative with respect to conformal time unless stated otherwise, and we will assume that the metric of the universe is a perturbation around the FLRW metric given by

$$ds^2 = a^2(\eta) \left( -d\eta^2 + \gamma_{AB} dx^A dx^B \right).$$  \hspace{1cm} (4.1)

Here $a$ is the scale factor and $\eta$ is the conformal time coordinate related to cosmic time $\tau$ by the differential relation

$$d\tau = a(\eta)d\eta.$$ \hspace{1cm} (4.2)

In Eqn. (4.1) we have allowed the spatial metric to have the general form $a^2 \gamma_{AB}$ where $\gamma_{AB}$ is the
metric of a 3-space of constant curvature. For the calculations in this chapter we will assume a flat FLRW background in coordinates such that $\gamma_{AB} = \delta_{AB}$; however for future reference we shall present certain expressions in terms of the more general spatial metric. Defining

$$H = \frac{1}{a} \frac{da}{d\eta} \equiv \frac{a'}{a},$$

and setting $f = a$ in Eqs. (3.115), the correction terms in the modified Friedmann and Raychaudhuri equations (3.114) now read

$$p^{(1)} = \frac{1}{a^2} \left[ \langle \tilde{\Gamma}_0^A \tilde{\Gamma}_0^B \rangle - \langle \tilde{\Gamma}_0^A \tilde{\Gamma}_0^B \rangle - 6H^2 \right],$$

$$s^{(1)} = \langle \tilde{g}^{JK} \rangle \left[ \langle \tilde{\Gamma}_J^B \tilde{\Gamma}_K^A \rangle - \langle \tilde{\Gamma}_J^A \tilde{\Gamma}_K^B \rangle \right],$$

$$p^{(2)} + p^{(1)} = -\frac{1}{a^2} \langle \tilde{\Gamma}_0^A \tilde{\Gamma}_0^0 \rangle - \langle \tilde{g}^{JK} \rangle \langle \tilde{\Gamma}_0^0 \tilde{\Gamma}_J^A \rangle + \frac{6H^2}{a^2},$$

$$s^{(2)} = \frac{1}{a^2} \langle \tilde{\Gamma}_0^0 \tilde{\Gamma}_0^A \rangle + \langle \tilde{g}^{JK} \rangle \langle \tilde{\Gamma}_0^0 \tilde{\Gamma}_J^A \rangle,$$

where we have dropped the tildes for convenience. The averaging in Eqs. (4.4) is assumed to be a spatial averaging in an unspecified spatial slicing in the inhomogeneous manifold $M$; in Sec. 4.2 we will specify the averaging procedure more exactly. In addition, the “cross-correlation” constraints (3.116) with $f = a$, $t = \eta$ and $H = \mathcal{H}$ must also be satisfied by the inhomogeneities. Note that we are not imposing the conservation condition (3.64).

Before we move on to deriving formulae for the correlation terms (4.4) in terms of perturbation functions in the metric, there is one issue which merits discussion. The cosmological perturbation setting, together with the paradigm of averaging, presents us with a rather peculiar situation. On the one hand, the time evolution of the scale factor is needed in order to solve the equations satisfied by the perturbations. Indeed, the standard practice is to fix the time evolution of the background once and for all, and to use this in solving for the evolution of the perturbations. On the other hand, the evolution of the perturbations (i.e. – the inhomogeneities) is needed to compute the correlation terms appearing in Eqs. (3.114). Until these terms are known, the behaviour with time of the scale factor cannot be determined; and until we know the scale factor as a function of time, we cannot solve for the perturbations. Note that this is a generic feature independent of all details of the averaging procedure.

It would appear therefore, that we have reached an impasse. To clear this hurdle, one can try the following iterative approach: Symbolically denote the background as $a$, the inhomogeneities as $\varphi$, and the correlation objects as $C$. Note that $a, \varphi$ and $C$ all refer to functions of time. We start
with a chosen background, say a standard flat FLRW background with radiation, baryons and cold
dark matter (CDM), and solve for the perturbations in the usual way, *without* accounting for the
correlation terms $C$. In other words, for this “zeroth iteration”, we artificially set $C$ to zero and
obtain $a^{(0)}$ and $\varphi^{(0)}$ using the standard approach (see e.g. Ref. [12]). Clearly, since the “true”
background (say $a_*$) satisfies Eqns. (3.114) *with* a nonzero $C$, we have in general $a^{(0)} \neq a_*$. Now,
using the solution $\varphi^{(0)}$, we can calculate the zeroth iteration correlation objects $C^{(0)}$ by applying
the prescription to be developed later in this chapter. As a first correction to the solution $a^{(0)}$,
we now solve for a new background $a^{(1)}$, with the *known functions* $C^{(0)}$ acting as sources in Eqns.
(3.114). This first iteration will then yield a solution $\varphi^{(1)}$ for the inhomogeneities, and hence a new
set of correlation terms $C^{(1)}$, and this procedure can be repeatedly applied. Pictorially,
\[
a^{(0)} \rightarrow \varphi^{(0)} \rightarrow C^{(0)} \rightarrow a^{(1)} \rightarrow \varphi^{(1)} \rightarrow \ldots \tag{4.5}
\]
As for convergence, if perturbation theory *is* in fact a good approximation to the real universe,
then one can expect that the correlation terms will tend to be small compared to other background
objects, and will therefore not affect the background significantly at each iteration, leading to rapid
convergence. On the other hand, if the correlation terms are large, this procedure may not converge
and one might expect a breakdown of the perturbative picture itself. We will see that in the linear
regime of cosmological perturbation theory, the correlation terms do in fact remain negligibly small.

4.1 Metric perturbations in cosmology

For ready reference, in this subsection we present expressions for the metric, its inverse, and the
Christoffel connection in *first order* cosmological PT, in an arbitrary, unfixed gauge. The notation
we use is similar to that used in Ref. [63]. We will also give expressions for the first order gauge
transformations of the perturbation functions (see e.g. Ref. [64]). The first order perturbed FLRW
metric in an arbitrary gauge and in terms of conformal time $\eta$, can be written as
\[
ds^2 = a^2(\eta) \left[ -(1 + 2\varphi) d\eta^2 + 2\omega_A dx^A d\eta + ((1 - 2\psi) \gamma_{AB} + \chi_{AB}) dx^A dx^B \right] \tag{4.6}
\]
The functions $\varphi$ and $\psi$ are scalars under spatial coordinate transformations. The functions $\omega_A$ and
$\chi_{AB}$ can be decomposed as follows
\[
\omega_A = \partial_A \omega + \hat{\omega}_A ; \quad \chi_{AB} = D_{AB} \chi + 2\nabla(A \hat{\chi}_B) + \hat{\chi}_{AB}, \tag{4.7}
\]
where the parentheses indicate symmetrization; $D_{AB}$ is the tracefree second derivative defined by
\[
D_{AB} \equiv \nabla_A \nabla_B - (1/3) \gamma_{AB} \nabla^2 ; \quad \nabla^2 \equiv \gamma^{AB} \nabla_A \nabla_B, \tag{4.8}
\]
with $\nabla_A$ the covariant spatial derivative compatible with $\gamma_{AB}$; and $\hat{\omega}_A$, $\hat{\chi}_A$ and $\hat{\chi}_{AB}$ satisfy

$$\nabla_A \hat{\omega}^A = 0 = \nabla_A \hat{\chi}^A; \quad \nabla_A \hat{\chi}^A_B = 0 = \hat{\chi}^A_A,$$

(4.9)

where spatial indices are raised and lowered using $\gamma_{AB}$ and its inverse $\gamma^{AB}$. From their definitions it is clear that $\varphi$, $\psi$, $\omega$ and $\chi$ each correspond to one scalar degree of freedom, the transverse 3-vectors $\hat{\omega}_A$ and $\hat{\chi}_A$ each correspond to two functional degrees of freedom, and the transverse tracefree 3-tensor $\hat{\chi}_{AB}$ corresponds also to two functional degrees of freedom. This totals to 10 degrees of freedom, of which 4 are coordinate degrees of freedom which can be arbitrarily fixed, which is what one means by a gauge choice. For example, the conformal Newtonian or longitudinal or Poisson gauge \[13, 64\] is defined by the conditions

$$\omega = 0 = \chi; \quad \hat{\chi}^A = 0.$$

(4.10)

For the metric (4.6) we have at first order,

$$\sqrt{-\det g} = a(\eta)^4 (1 + \varphi - 3\psi).$$

(4.11)

The inverse of metric (4.6), correct to first order, has the components

$$g^{00} = -\frac{1}{a^2} (1 - 2\varphi); \quad g^{0A} = \frac{1}{a^2} \omega^A;$$

$$g^{AB} = \frac{1}{a^2} \left( (1 + 2\psi) \gamma^{AB} - \chi^{AB} \right).$$

(4.12)

With $\mathcal{H} = (a'/a)$, the first order accurate Christoffel symbols are

$$\Gamma^0_{00} = \mathcal{H} + \varphi'; \quad \Gamma^0_{0A} = \partial_A \varphi + \mathcal{H} \omega_A; \quad \Gamma^A_{00} = \partial^A \varphi + \omega^A + \mathcal{H} \omega^A,$$

$$\Gamma^0_{AB} = (\mathcal{H} - \psi') \gamma^B_A - \nabla (\omega^A B) + \frac{1}{2} \chi^A_B + \mathcal{H} \chi_{AB},$$

$$\Gamma^A_{0B} = (\mathcal{H} - \psi') \delta^A_B + \frac{1}{2} \left( \nabla_B \omega^A - \nabla^A \omega_B \right) + \frac{1}{2} \chi^A_B,$$

$$\Gamma^A_{BC} = (3)\Gamma^A_{BC} - (\delta^A_B \partial_C \psi + \delta^A_C \partial_B \psi - \gamma_{BC} \partial^A \psi) - \mathcal{H} \omega^A \gamma_{BC} + \frac{1}{2} \left( \nabla_C \chi^A_B + \nabla_B \chi^A_C - \nabla^A \chi_{BC} \right),$$

(4.13)

where \((3)\Gamma^A_{BC}\) denotes the Christoffel connection associated with the homogeneous 3-metric $\gamma_{AB}$.

### 4.1.1 Gauge transformations

While the concept of gauge transformations can be described in a rather sophisticated language using pullback operators between manifolds \[63\], for our purposes it suffices to implement a gauge transformation using the simpler notion of an infinitesimal coordinate transformation (also known
as the “passive” point of view). Hence, denoting the coordinates and perturbation functions in the new gauge with a tilde (i.e. \( \tilde{x}^a, \tilde{\varphi}, \tilde{\omega}_A, \) and so on), we have

\[
\tilde{x}^a = x^a + \xi^a(x) \quad ; \quad x^a = \tilde{x}^a - \xi^a,
\]

(4.14)

where the infinitesimal 4-vector \( \xi^a \) can be decomposed as

\[
\xi^a = (\xi^0, \xi^A) = (\alpha, \partial^A\beta + d^A) ,
\]

(4.15)

where \( \alpha \) and \( \beta \) are scalars and \( d^A \) is a transverse 3-vector satisfying \( \nabla_A d^A = 0 \).

It is then easy to show that if this transformation is assumed to change the metric (4.6) by changing only the perturbation functions but leaving the background intact (a so-called “steady” coordinate transformation), then the old perturbations and the new are related by [64]

\[
\varphi = \tilde{\varphi} + \alpha' + \mathcal{H}\alpha ,
\]

\[
\psi = \tilde{\psi} - \frac{1}{3} \nabla^2 \beta - \mathcal{H}\alpha,
\]

\[
\omega = \tilde{\omega} - \alpha + \beta',
\]

\[
\tilde{\omega}^A = \tilde{\omega}^A + d^A ,
\]

\[
\chi = \tilde{\chi} + 2\beta ,
\]

\[
\tilde{\chi}^A = \tilde{\chi}^A + d^A ,
\]

\[
\tilde{\chi}_{AB} = \tilde{\chi}_{AB} .
\]

(4.16)

The last equality shows that the transverse tracefree tensor perturbations are gauge invariant. They correspond to gravitational waves.

### 4.2 The Averaging Operation and Gauge Related Issues

In this section, we will describe the details of the MG (spatial) averaging procedure adapted to the setting of cosmological PT.

#### 4.2.1 Volume Preserving (VP) Gauges and the Correlation Scalars

It will greatly simplify the discussion if we start with symbolic calculations which allow us to see the broad structure of the objects we are after. Since the correlation objects in Eqns. (3.114) depend only on derivatives of the metric, we will primarily deal with metric fluctuations; matter perturbations will only come into play when solving for the actual dynamics of the system. Before dealing with the issue of which gauge to choose in order to set the condition (3.36) with the average connection taken to be the FLRW one, we will show that irrespective of this choice, the leading order contribution to the correlations requires knowledge of only first order perturbation functions.
We will use the following symbolic notation:

- Inhomogeneous connection: $\Gamma$
- FLRW connection: $\Gamma_F$
- Perturbation in the connection: $\delta \Gamma \equiv \Gamma - \Gamma_F = \delta \Gamma^{(1)} + \delta \Gamma^{(2)} + \ldots$
- Coordination bivector: $W \equiv 1 + \delta W = 1 + \delta W^{(1)} + \delta W^{(2)} + \ldots$
- Bilocal extension of the connection: $\tilde{\Gamma}$
- Inhomogeneous part of the bilocal extension of the connection: $\tilde{\delta} \Gamma \equiv \tilde{\Gamma} - \Gamma_F = \tilde{\delta} \Gamma^{(1)} + \tilde{\delta} \Gamma^{(2)} + \ldots$
- Correlation object: $C$

The integer superscripts denote the order of perturbation. The form of the coordination bivector arises from the fact that in perturbation theory, *in the spatial averaging limit*, a transformation from an arbitrary gauge to a VP one can be achieved by an infinitesimal coordinate transformation. By a VP gauge we mean a gauge in which the metric determinant is independent of the spatial coordinates to the relevant order in PT, but may be a function of time. It can be shown that such a function of time (which will typically be some power of the scale factor), is completely consistent with all definitions and requirements of MG in the spatial averaging limit. An easy way of seeing this is to note that in any averaged quantity, the metric determinant appears in two integrals, one in the numerator and the other in the denominator (which gives the normalising volume). In the “thin time slicing” approximation we are using to define the averaging, any overall time dependent factor in the metric determinant therefore cancels out. Also, a fully volume preserving coordinate system can clearly be obtained from any VP gauge as defined above, by a suitable rescaling of the time coordinate. It is not hard to show that in the thin time slicing approximation, this gives the same coordination bivector $W_{b}'(x',x)$ as the VP gauge definition above.

To see that first order perturbations are sufficient to calculate $C$ to leading order, we only have to note that the background connection $\Gamma_F$ satisfies

$$\langle \Gamma_F \rangle = \Gamma_F,$$ (4.17)

and that the structure of the correlation is $C = \langle \tilde{\Gamma}^2 \rangle - \langle \tilde{\Gamma} \rangle^2$, which then leads to

$$C = \langle \tilde{\delta} \Gamma^2 \rangle - \langle \tilde{\delta} \Gamma \rangle^2,$$ (4.18)

which is exact. Clearly, the correlation is quadratic in the perturbation as expected, and hence to leading order, $\tilde{\delta} \Gamma$ above can be replaced by $\tilde{\delta} \Gamma^{(1)}$. 
Eqns. (4.17) and (4.18) treat the averaging operation at a conceptual level only. To make progress however, we also need to prescribe how to practically impose the averaging assumption

$$\langle \tilde{\Gamma} \rangle = \Gamma_F \quad \text{i.e.} \quad \langle \tilde{\delta} \Gamma \rangle = 0,$$

(4.19)

in any given perturbative context. This requires some discussion since, for example, the bilocal extension of the connection $\tilde{\Gamma}$ has the structure

$$\tilde{\Gamma} = W^{-1} \Gamma W^2 + W^{-1} (\partial + W \partial') W,$$

(4.20)

where $\partial$ is a derivative at $x$ and $\partial'$ a derivative at $x'$ (see Eqn. (3.35)). The actual MG averaging operation in general is therefore a rather involved procedure. Additionally, it is also necessary to address certain gauge related issues.

To clarify the situation, let us start with a fictitious setting in which the geometry has exactly the flat FLRW form, with no physical perturbations. Clearly, if we work in the standard comoving coordinates in which the metric $\gamma_{AB}$ of Eqn. (4.1) is simply $\gamma_{AB} = \delta_{AB}$, then since these coordinates are volume preserving in the sense described above, the coordination bivector becomes trivial. The averaging involves a simple integration over 3-space, and we can easily see that Eqn. (4.17) is explicitly recovered.

Now suppose that we perform an infinitesimal coordinate transformation, after imposing Eqn. (4.17). Since the averaging operation is covariant, then from the point of view of a general coordinate transformation, both sides of Eqn. (4.17) will be affected in the same way. However, suppose that we had performed the transformation before imposing Eqn. (4.17). In the language of cosmological PT, we would then be dealing with some “pure gauge” perturbations around the fixed, spatially homogeneous background. If we did not know that these perturbations were pure gauge, we might naively construct the nontrivial coordination bivector for this metric, compute the bilocal extension of the connection according to Eqn. (4.20) and try to impose Eqn. (4.19). This would be incorrect since these perturbations were arbitrarily generated and need not average to zero (for example they could be positive definite functions). In order to maintain consistency, it is then necessary to ensure in practice that the averaging condition (4.19) is applied only to gauge invariant fluctuations (which is rather obvious in hindsight).

There is another problem associated with the structure of the coordination bivector, even when there are real, gauge invariant inhomogeneities present. Note from Eqns. (3.24) and (3.26) that the coordination bivector has the structure

$$W = \frac{\partial x}{\partial x_V} \bigg|_{x'} \frac{\partial x_V}{\partial x} \bigg|_x,$$

(4.21)

where $x$ denotes the coordinates we are working in and $x_V$ a set of VPCs. In perturbation theory
(in the spatial averaging limit) we will have, at leading order,

\[ x = x_V - \xi \quad ; \quad x_V = x + \xi, \]  

(4.22)

where \( \xi \) symbolically denotes an infinitesimal 4-vector defining the transformation, and hence

\[ (\partial x_V)/(\partial x) = 1 + \partial \xi, \]  

(4.23)

and so on, which gives us

\[ W = 1 - (\partial \xi)|_{x'} + (\partial \xi)|_x + \ldots = 1 + \delta W^{(1)} + \ldots \]  

(4.24)

Now when we compute a quantity such as \( \langle \Gamma_F \delta W^{(1)} \rangle \) which appears in the expression for \( \langle \tilde{\Gamma} \rangle \), we will be left with a fluctuating (\( \vec{x} \)-dependent) term of the form \( \Gamma_F(\langle \partial \xi \rangle - \partial \xi) \), where \( \vec{x} \) denotes the 3 spatial coordinates. Hence if we try to impose Eqn. (4.19) we will ultimately be left with equations of the type

\[ \langle f(\vec{x}) \rangle - f(\vec{x}) = 0, \]  

(4.25)

for some functions derived from the inhomogeneities which we have collectively denoted \( f \). In other words, consistency would seem to demand that the inhomogeneities vanish in this coordinate system, which is of course not desirable.

It therefore appears that we are forced to impose Eqn. (4.19) \textit{in a volume preserving gauge}, since by definition, only in such a gauge will we have \( W = 1 \) exactly. We emphasize that this is a purely practical aspect related to defining the averaging operation, and is completely decoupled from, e.g. the choice of gauge made when studying the time evolution of perturbations. We are in no way breaking the usual notion of gauge invariance by choosing an averaging operator. The conditions Eqn. (4.25) now reduce to the form

\[ \langle f_{VPC}(\vec{x}) \rangle = 0, \]  

(4.26)

which are far more natural than Eqn. (4.25). The averaging condition is now unambiguous, \textit{but depends on a choice of the VP gauge which defines the averaging operation}, an issue we shall discuss in the next subsection. For now, all we can assert is that this VP gauge must be such that in the \textit{absence} of gauge invariant fluctuations, it must reduce to the standard comoving (volume preserving) coordinates of the background geometry as in Eqn. (4.1). This of course is simply the statement that the VP gauge must be \textit{well defined} and must not contain any residual degrees of freedom.

The averaging operation now takes on an almost trivial form as we have seen in chapter 3 – to leading order, for any quantity \( f(\eta, \vec{x}) \) (with or without indices), the spatial average of \( f \) in a VP
gauge is given by
\[ \langle f \rangle(\eta, \vec{x}) = \frac{1}{V_L} \int_{V(\vec{x})} d^3 y f(\eta, y), \]
\[ (4.27) \]
where the integral is over a spatial domain \( V(\vec{x}) \) with a constant volume \( V_L \). The spatial coordinates are the comoving coordinates of the background metric, and at leading order the boundaries of \( V(\vec{x}) \) can be specified in a straightforward manner as, e.g.,
\[ V(\vec{x}) = \{ \vec{y} \mid x^A - L/2 < y^A < x^A + L/2, A = 1, 2, 3, \}, \]
\[ (4.28) \]
where \( L \) is a comoving scale over which the averaging is performed (in which case \( V_L = L^3 \)). The averaging definition can be written more compactly in terms of a window function \( W_L(\vec{x}, \vec{y}) \) as
\[ \langle f \rangle(\eta, \vec{x}) = \int d^3 y W_L(\vec{x}, \vec{y}) f(\eta, \vec{y}); \quad \int d^3 y W_L(\vec{x}, \vec{y}) = 1, \]
\[ (4.29) \]
where \( W_L(\vec{x}, \vec{y}) \) vanishes everywhere except in the region \( V(\vec{x}) \), with the integrals now being over all space. This expression will come in handy when working in Fourier space, as we shall do in later sections.

A couple of comments are in order at this stage. Firstly, we have not specified the magnitude of the averaging scale \( L \). The general philosophy is that this scale must be large enough that a single averaging domain encompasses several realisations of the random inhomogeneous fluctuations, and small enough that the observable universe contains a large number of averaging domains. However, as we will show later in Sec. 4.3 if one is ultimately interested in quantities which are formally averaged over an ensemble of realisations of the universe (as is usually done in interpreting observations), then the actual value of the averaging scale becomes irrelevant.

This brings us to the second issue. The above discussion is valid only in the situation where there are no fluctuations at arbitrarily large length scales, since in the presence of such fluctuations the averaging condition (4.19) loses meaning (in such a situation it would be impossible to isolate the background from the perturbation by an averaging operation on any finite length scale). Indeed, we shall see a manifestation of this restriction in Sec. 4.3 where the correlation scalars will be seen to diverge in the presence of a nonzero amplitude at arbitrarily large scales, of the power spectrum of metric fluctuations.

We will end this subsection by explicitly writing out the averaging condition in an “unfixed VP” gauge, to be defined below, and also writing the correlation terms appearing in Eqn. (3.114), in this gauge. As we can see from Eqn. (4.11), the basic condition to be satisfied by a VP gauge is
\[ \tilde{\varphi} = 3 \tilde{\psi}. \]
\[ (4.30) \]
Hereafter, all VP gauge quantities will be denoted using a tilde. This should not be confused with the similar notation that was used so far for the bilocal extension, which will not be needed in the
rest of the chapter. \( \tilde{\varphi} \) and \( \tilde{\psi} \) are the scalar potentials appearing in the perturbed FLRW metric \((4.6)\). The single condition \((4.30)\) leaves 3 degrees of freedom to be fixed, in order to completely specify the VP gauge one is working with. The MG formalism by itself does not prescribe a method to choose a particular VPC system; in fact this freedom of choice of VPCs is an inherent part of the formalism. We shall return to this issue in the next subsection. For now we define the “unfixed VP (uVP) gauge” by the single requirement \((4.30)\), with 3 unfixed degrees of freedom, and present the expressions for the averaging condition and the correlation scalars, with this choice.

It is straightforward to determine the consequences of requiring Eqn. \((3.36)\) to hold, with the right hand side corresponding to the FLRW connection in conformal coordinates, and remembering that the coordination bivector (in the spatial averaging limit) is now simply a Kronecker delta. Together with some additional reasonable requirements, namely

\[
\langle \nabla^2 s \rangle = 0 = \langle \nabla^2 \partial_A s \rangle,
\]

for any scalar \(s(\eta, \vec{x})\), the averaging condition in the uVP gauge reduces to

\[
\langle \tilde{\psi} \rangle = 0 ; \quad \langle \partial_A \tilde{\psi} \rangle = 0 = \langle \tilde{\psi}' \rangle, \\
\langle \tilde{\omega}_A \rangle = 0 = \langle \tilde{\omega}'_A \rangle ; \quad \langle \tilde{\chi}'_{AB} \rangle = 0, \\
\langle \nabla_C \tilde{\chi}^A_B \rangle + \langle \nabla_B \tilde{\chi}^A_C \rangle - \langle \nabla^A \tilde{\chi}_{BC} \rangle = 0, \\
\langle \nabla_A \tilde{\omega}_B \rangle = \langle \nabla_B \tilde{\omega}_A \rangle = \mathcal{H}(\tilde{\chi}_{AB}),
\]

where we have used the expressions in Eqn. \((4.13)\) with the uVP condition \((4.30)\). We will also make the additional reasonable requirement that

\[
\langle \tilde{\chi}_{AB} \rangle = 0,
\]

using which it is easy to see that the perturbed FLRW metric \((4.6)\) and its inverse \((4.12)\), in the uVP gauge, both on averaging reduce to their respective homogeneous counterparts, namely

\[
\langle g_{ab} \rangle = g^{(FLRW)}_{ab} ; \quad \langle g^{ab} \rangle = g^{ab}_{(FLRW)}.
\]
Using these results, the expressions \( (4.4) \) simplify to give, in the uVP gauge,

\[
P^{(1)}(1) = \frac{1}{a^2} \left[ 6\langle (\psi')^2 \rangle + \langle \nabla[A\tilde{\omega}_{B}]\nabla[A\tilde{\omega}_{B}] \rangle - \frac{1}{4} \langle \hat{\chi}'_{AB}\hat{\chi}^{ABr} \rangle \right],
\]

\[
S^{(1)} = \frac{1}{a^2} \left[ -10\langle \partial_{A}\tilde{\psi}\partial_{A}\tilde{\psi} \rangle - 2\langle \partial_{A}\tilde{\psi}\nabla_{B}\hat{\chi}^{AB} \rangle + \frac{1}{4} \langle \nabla_{B}\hat{\chi}^{AC} (2\nabla_{A}\hat{\chi}_{BC} - \nabla_{B}\hat{\chi}_{AC}) \rangle \right],
\]

\[
P^{(1)} + P^{(2)} = \frac{1}{a^2} \left[ 6\langle (\psi')^2 \rangle - 24\mathcal{H}\langle \tilde{\psi}'\tilde{\psi} \rangle - \langle \tilde{\psi}'\nabla^2\hat{\omega} \rangle + \frac{1}{2} \langle \hat{\chi}'_{AB}\nabla^A\hat{\omega}^B \rangle - \frac{1}{4} \langle \hat{\chi}'_{AB} (\hat{\chi}^{ABr} + 2\mathcal{H}\hat{\chi}^{AB}) \rangle \right],
\]

\[
S^{(2)} = \frac{1}{a^2} \left[ 3\langle \hat{\omega}^A\partial_A\tilde{\psi} \rangle + \mathcal{H}\langle \hat{\omega}^A\tilde{\omega}'_A \rangle \right],
\]

where square brackets denote antisymmetrization.

### 4.2.2 Choice of VP Gauge

In this subsection we will prescribe a choice for the VP gauge which defines the averaging operation. In general, the class of volume preserving coordinate systems for any spacetime, is very large (see Ref. [44] for a detailed characterisation). We have so far managed to pare it down by requiring that the VP gauge we choose should reduce to the standard FLRW coordinates in the absence of fluctuations. It turns out to be somewhat difficult to go beyond this step, since there does not appear to be any unambiguously clear guiding principle governing this choice. We will therefore motivate a choice for the VP gauge based on certain details of cosmological PT which one knows from the standard treatments of the subject.

In particular, we shall make use of certain nice properties of the conformal Newtonian or longitudinal or Poisson gauge, which is defined by the conditions \( (4.10) \) [64] (henceforth we shall refer to this gauge as the cN gauge for short). Since this gauge is well defined and has no residual degrees of freedom, all the nonzero perturbation functions in the cN gauge, namely \( \varphi, \psi, \hat{\omega}_A \) and \( \hat{\chi}_{AB} \) in the notation of Appendix C, are equal to gauge invariant objects. This is trivially true for \( \hat{\chi}_{AB} \), as seen in the last equation in \( (4.16) \). For the rest, note that in any arbitrary unfixed gauge, the following combinations are gauge invariant at first order

\[
\Phi_B = \varphi + \frac{1}{a}\partial_\eta \left[ a \left( \omega - \frac{1}{2}\chi \right) \right],
\]

\[
\Psi_B = \psi - \mathcal{H} \left( \omega - \frac{1}{2}\chi \right) + \frac{1}{6} \nabla^2\chi,
\]

\[
\hat{V}_A = \hat{\omega}_A - \hat{\chi}'_A,
\]

(4.36)
which can be easily checked using Eqns. (4.16), and in the cN gauge, \( \omega, \chi \) and \( \hat{\chi}_A \) all vanish. Here \( \Phi_B \) and \( \Psi_B \) are the Bardeen potentials \(^{[69]}\) (upto a sign), and \( \Psi_B \) in particular has the physical interpretation of giving the gauge invariant curvature perturbation, which is the quantity on which initial conditions are imposed post inflation \(^{[70]}\).

Additionally, it is also known that the cN gauge for the metric remains stable even during structure formation, when matter inhomogeneities have become completely nonlinear\(^{[71]}\). We believe that this is a strong argument in favour of using the cN gauge to define a VP gauge which will then define the averaging operation in the perturbative context. This will ensure that this “truncated” averaging operation, defined for first order PT, will remain valid at leading order even during the nonlinear epochs of structure formation.

To implement this in practice, consider a transformation from the cN gauge to the uVP gauge defined by Eqn. (4.30). The transformation equations (4.16) reduce to

\[
\begin{align*}
\alpha' + 4H\alpha + \nabla^2 \beta &= \varphi - 3\psi, \\
\tilde{\psi} &= \frac{1}{3} \varphi - \alpha' - H\alpha, \\
\tilde{\omega} &= \alpha - \beta', \\
\tilde{\omega}^A &= \hat{\omega}^A - d^A, \\
\tilde{\chi} &= -2\beta, \\
\tilde{\hat{\chi}}^A &= -d^A, \\
\tilde{\hat{\chi}}_{AB} &= \hat{\chi}_{AB}.
\end{align*}
\]

(4.37)

Recall that to completely specify a VP gauge, we need to fix 3 degrees of freedom in the uVP gauge. Our requirement regarding the “well defined”-ness of the VP gauge, forces us to set \( d^A = 0 \), and to choose \( \alpha \) and \( \beta \) such that they vanish in the case where \( \varphi = 0 = \psi \).

This has fixed 2 degrees of freedom, in addition to the condition (4.30) which is just the definition of the uVP gauge, and has hence not yielded a uniquely specified VP gauge. To do this, we shall make the following additional requirement. Since we are dealing with a spatial averaging, it seems reasonable to require that the VP gauge being used to define the averaging, should be “as close as possible” to the cN gauge in terms of time slicing, and for this reason we shall set the function \( \alpha \) to zero. To summarize, the VP gauge chosen is defined in terms of the gauge transformation functions \( \xi^a = (\alpha, \partial A^a + d^A) \) between the cN gauge and the VP gauge, by the following relations

\[
\alpha = 0 = d^A,
\]

(4.38)

\(^{[27]}\) See Ref. \(^{[27]}\) for an intuitive description of why this is so. We will also see an explicit demonstration in a toy model of structure formation in chapter 5.
where the function $\beta$ is restricted not to contain any nontrivial solution of the homogeneous (Laplace) equation $\nabla^2 \beta = 0$.

Having made this choice for the VP gauge, we are now assured that all averaged quantities which we compute are gauge invariant: our choice ensures that the averaging procedure does not introduce any pure gauge modes, and the philosophy of “steady” coordinate transformations ensures that all background objects are, by assumption, unaffected by gauge transformations. In particular, the correlation objects in Eqns. (4.4) are all gauge invariant. This is different from the gauge invariance conditions derived in Ref. [29], where the background was also taken to change under gauge transformations at second order in the perturbations. It is at present not clear how these results are related to ours.

### 4.3 The Correlation Scalars

With the VP gauge choice defined by Eqns. (4.39), it is straightforward to rewrite the correlation objects in Eqns. (4.35) (which are in the uVP gauge) in terms of the perturbation functions in the cN gauge. We will restrict the subsequent calculations in this chapter to the case where there are no transverse vector perturbations, i.e.,

$$\tilde{\omega}_A = 0,$$

(4.40)

in the cN gauge. This is a reasonable choice since such vector perturbations, even if they are excited in the initial conditions, decay rapidly and do not source the other perturbations at first order [12]. In addition, for simplicity we will choose to ignore the gauge invariant tensor perturbations as well,

$$\tilde{\chi}_{AB} = 0.$$

(4.41)
In terms of the scalar perturbations in the cN gauge, for a flat FLRW background, the correlation objects reduce to

\[ P^{(1)} = \frac{1}{a^2} \left[ 2\langle (\psi')^2 \rangle + \langle (\varphi' - \psi')^2 \rangle - \langle (\nabla_A \nabla_B \beta') (\nabla^A \nabla^B \beta') \rangle \right], \]  

\[ S^{(1)} = -\frac{1}{a^2} \left[ 6\langle \partial_A \psi \partial^A \psi \rangle + \langle \partial_A (\varphi - \psi) \partial^A (\varphi - \psi) \rangle - \langle (\nabla_A \nabla_B \nabla_C \beta)(\nabla^A \nabla^B \nabla^C \beta) \rangle \right], \]  

\[ P^{(1)} + P^{(2)} = \frac{1}{a^2} \left[ \langle \varphi' (\varphi' - \psi') \rangle - 6H \{ \langle \varphi' \varphi \rangle - \langle \psi' \psi \rangle + \langle \psi' (\varphi - \psi) \rangle \rangle \right. \]
\[ \left. + \langle \psi' (\varphi' - \psi') \rangle + \langle (\nabla_A \nabla_B \beta)(\nabla^A \nabla^B \beta') \rangle \right], \]  

\[ S^{(2)} = -\frac{1}{a^2} \left[ \langle \partial^A \beta'' (\partial_A \varphi - H \partial_A \beta') \rangle \right], \]

where \( \beta \) is defined in Eqn. (4.39b).

Since we are working with a flat FLRW background, it becomes convenient to transform our expressions in terms of Fourier space variables. This will also highlight the problem with large scale fluctuations which was mentioned in Sec. 4.2. We will use the following Fourier transform conventions: For any scalar function \( f(\eta, \vec{x}) \), its Fourier transform \( f_k(\eta) \) satisfies

\[ f(\eta, \vec{x}) = \int \frac{d^3k}{(2\pi)^3} e^{i\vec{k} \cdot \vec{x}} f_k(\eta), \]
\[ f_k(\eta) = \int d^3x e^{-i\vec{k} \cdot \vec{x}} f(\eta, \vec{x}). \]  

Consider an average of a generic quadratic product of two scalars \( f^{(1)}(\vec{x}) \) and \( f^{(2)}(\vec{x}) \) where we have suppressed the time dependence since it simply goes along for a ride. Using the definition, and keeping in mind that the scalars are real, it is easy to show that we have

\[ \langle f^{(1)}(\vec{x}) f^{(2)}(\vec{x}) \rangle = \int \frac{d^3k_1 d^3k_2}{(2\pi)^6} W_L(\vec{k}_1 - \vec{k}_2, \vec{x}) f_{k_1}^{(1)*} f_{k_2}^{(2)}, \]

where \( W_L(\vec{k}, \vec{x}) \) is the Fourier transform of the window function \( W_L(\vec{x}, \vec{y}) \) on the variable \( \vec{y} \), and the asterisk denotes a complex conjugate.

In the present context, the functions \( f^{(1)} \) and \( f^{(2)} \) will typically be derived in terms of the initial random fluctuations in the metric \( \varphi_{k_i} \) which are assumed to be drawn from a statistically homogeneous and isotropic Gaussian distribution with some given power spectrum. In order to ultimately make contact with observations, it seems necessary to perform a formal ensemble average.
over all possible realisations of this initial distribution of fluctuations. The statistical homogeneity and isotropy of the initial distribution implies that the functions \( f^{(1)} \) and \( f^{(2)} \) will satisfy a relation of the type

\[
[f^{(1)}_{\vec{k}_1} f^{(2)\ast}_{\vec{k}_2}]_{\text{ens}} = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) P_{f_1 f_2}(|\vec{k}_1|),
\]

for some function \( P_{f_1 f_2}(k, \eta) \) which is derivable in terms of the initial power spectrum of metric fluctuations, and where \([...]_{\text{ens}}\) denotes an ensemble average and \( \delta^{(3)}(\vec{k}) \) is the Dirac delta distribution.

Applying an ensemble average to Eqn. (4.44) introduces a Dirac delta which forces \( \vec{k}_1 = \vec{k}_2 \). Further, the normalisation condition on the window function in Eqn. (4.29) implies that we have

\[
W_L(\vec{k} = 0, \vec{x}) = 1,
\]

which means that all dependence on the averaging scale and domain drops out, and we are left with

\[
[\langle f^{(1)} f^{(2)} \rangle]_{\text{ens}} = \int \frac{d^3k}{(2\pi)^3} P_{f_1 f_2}(k).
\]

Note however, that the right hand side of Eqn. (4.47) is precisely what we would have obtained, had we treated the spatial average \( \langle ... \rangle \) to be the ensemble average \([...]_{\text{ens}}\) to begin with. Therefore for all practical purposes, we are justified in replacing all the spatial averages in the expressions for the correlation scalars (4.42), by ensemble averages.

It is convenient to define the transfer function \( \Phi_k(\eta) \) via the relation

\[
\varphi_{\vec{k}}(\eta) = \varphi_{\vec{k}i} \Phi_k(\eta).
\]

For the calculations in this chapter, we shall assume that the cN gauge scalars \( \varphi(\eta, \vec{x}) \) and \( \psi(\eta, \vec{x}) \) are equal

\[
\varphi(\eta, \vec{x}) = \psi(\eta, \vec{x}),
\]

a choice which is valid in first order PT when anisotropic stresses are negligible (see Ref. [12]). This simplifies many of the expressions we are dealing with. The Fourier transform of \( \beta \) can be written, using Eqns. (4.39b) and (4.49), as

\[
\beta_{\vec{k}}(\eta) = \frac{2}{k^2} \varphi_{\vec{k}}(\eta).
\]

Finally, in terms of the transfer function \( \Phi_k(\eta) \) and the initial power spectrum of metric fluctuations defined by

\[
[\varphi_{\vec{k}_1i} \varphi_{\vec{k}_2i}^\ast]_{\text{ens}} = (2\pi)^3 \delta^{(3)}(\vec{k}_1 - \vec{k}_2) P_{\varphi i}(k_1),
\]

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the correlation scalars (4.42) can be written as

\[ \mathcal{P}^{(1)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) \left( \Phi_k' \right)^2, \tag{4.52a} \]

\[ \mathcal{S}^{(1)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) \left( k^2 \Phi_k^2 \right), \tag{4.52b} \]

\[ \mathcal{P}^{(1)} + \mathcal{P}^{(2)} = -\frac{8\mathcal{H}}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) \left( \Phi_k \Phi_k' \right), \tag{4.52c} \]

\[ \mathcal{S}^{(2)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^2 P_{\varphi i}(k) \Phi_k'' \left( \Phi_k - \frac{2\mathcal{H}}{k^2} \Phi_k' \right). \tag{4.52d} \]

These expressions highlight the problem of having a finite amplitude for fluctuations at arbitrarily large length scales \((k \to 0)\), which was mentioned in Sec. 4.2. As a concrete example, consider the frequently discussed Harrison-Zel’dovich scale invariant spectrum \([71]\) which satisfies the condition

\[ k^3 P_{\varphi i}(k) = \text{constant}. \tag{4.53} \]

Eqns. (4.52) now show that if the transfer function \(\Phi_k(\eta)\) has a finite time derivative at large scales (as it does in the standard scenarios – see the next section), then the correlation objects \(\mathcal{P}^{(1)}, \mathcal{P}^{(2)}\) and \(\mathcal{S}^{(2)}\) all diverge due to contributions from the \(k \to 0\) regime. This demonstrates the importance of having an initial power spectrum in which the amplitude dies down sufficiently rapidly on large length scales (which is a known issue, see Ref. [70]). Keeping this in mind, we shall concentrate on initial power spectra which display a long wavelength cutoff. Models of inflation leading to such power spectra have been discussed in the literature [72], and more encouragingly, analyses of WMAP data seem to indicate that such a cutoff in the initial power spectrum is in fact realised in the universe [73].

A final comment before proceeding to explicit calculations: In addition to picking up nontrivial correlation corrections in the cosmological equations, the averaging formalism also requires that the “cross-correlation” constraints in Eqns. (3.116) be satisfied. It is straightforward to show that the statistical homogeneity and isotropy of the metric fluctuations implies that these constraints are identically satisfied, for all types of perturbations (scalar, vector and tensor). At the lowest order therefore, these constraints do not impose any additional conditions on the perturbation theory, which is reassuring.
4.4 Worked out examples

We will now turn to some explicit calculations of the backreaction, which will show that the magnitude of the effect remains negligibly small for most of the evolution duration in which linear PT is valid. At early times, linear PT is valid at practically all scales including the smallest scales at which we wish to apply general relativity. As matter fluctuations grow, the small length scales progressively approach nonlinearity, and linear PT breaks down at these scales. As we will see, however, by the time a particular length scale becomes nonlinear, its contribution to the amplitude of the metric fluctuations correspondingly becomes negligible. In practice therefore, one can extend the linear calculation well into the matter dominated era, with the expectation that the order of magnitude of the various integrals will not change significantly due to nonlinear effects.

The model we will use is the standard Cold Dark Matter (sCDM) model consisting of radiation and CDM [12]. We will neglect the contribution of baryons, and at the end we shall discuss the effects this may have on the final results. We shall also discuss, without explicit calculation, the effects which the introduction of a cosmological constant is likely to have. In the following, $\Omega_r$ and $\Omega_m$ denote the density parameters of radiation and CDM respectively at the present epoch $\tau_0$, with $\tau$ denoting cosmic time. $\Omega_r$ is assumed to contain contributions from photons and 3 species of massless, out-of-equilibrium neutrinos. At the “zeroth iteration” we have

$$\left(\frac{1}{a} \frac{da}{d\tau}\right)^2 = H^2(a) = H_0^2 \left[ \frac{\Omega_m}{a^3} + \frac{\Omega_r}{a^4} \right],$$

(4.54)

where $H_0$ is the standard Hubble constant, the scale factor is normalised so that $a(\tau_0) = 1$, and $\mathcal{H}$ and $H$ are related by

$$\mathcal{H}(a) = a H(a).$$

(4.55)

The comoving wavenumber corresponding to the scale which enters at the matter radiation equality epoch, is given by

$$k_{eq} = a_{eq} H(a_{eq}) = H_0 \left( \frac{2 \Omega_m^2}{\Omega_r} \right)^{1/2} \sim H_0 \cdot 10^{5/2},$$

(4.56)

where we have set (see Refs. [74] [12] for details)

$$\Omega_r = \Omega_{\text{photon}} + 3 \Omega_{\text{neutrino}}$$

$$= \Omega_{\text{photon}} \left( 1 + 3 \cdot \frac{7}{8} \left( \frac{4}{11} \right)^{4/3} \right)$$

$$= 4.15 \times 10^{-5} h^{-2},$$

(4.57)

where $h$ is the dimensionless Hubble parameter defined by $H_0 = 100h \text{ km/s/Mpc}$. For all calculations we shall set $h = 0.72$ [75].
4.4.1 EdS background and non-evolving potentials

Before dealing with the full model (which requires a numerical evolution) let us consider the simpler situation, described by an Einstein-deSitter (EdS) background, with negligible radiation and a nonevolving potential \( \varphi = \varphi(\vec{x}) \) (which is a consistent solution of the Einstein equations in the sCDM model at least at subhorizon scales at late times [12]). Although not fully accurate, this example requires some very simple integrals and will help to give us a feel for the structure and magnitude of the backreaction.

With a constant potential, the only correlation object which survives is \( S^{(1)} \), which evolves like \( \sim a^{-2} \), where the scale factor refers to the “zeroth iteration”. The constant of proportionality can be written in terms of the BBKS transfer function \( T_{BBKS}(k/k_{eq}) \) [76, 12], to give

\[
S^{(1)} = -\frac{2}{a^2} \int \frac{dk}{2\pi^2} k^4 P_{\varphi i}(k) T_{BBKS}^2(k/k_{eq}) , \tag{4.58}
\]

where we have [76]

\[
T_{BBKS}(x) = \frac{\ln [1 + 0.171x]}{(0.171x)} \left[ 1 + 0.284x + (1.18x)^2 + (0.399x)^3 + (0.490x)^4 \right]^{-0.25} , \tag{4.59}
\]

where \( x \equiv (k/k_{eq}) \).

The integral in Eqn. (4.58) is well-behaved even in the presence of power at arbitrarily large scales, for a (nearly) scale invariant spectrum. Since we are only looking for an estimate, we shall evaluate the integral in the absence of a large scale cutoff, and leave a more accurate calculation for the next subsection. For the initial spectrum given by

\[
\frac{k^3 P_{\varphi i}(k)}{2\pi^2} = A(k/H_0)^{n_s-1} , \tag{4.60}
\]

where the scalar spectral index \( n_s \) is close to unity, the integral in Eqn. (4.58) can be easily performed numerically and has the order of magnitude

\[
\int \frac{dk}{2\pi^2} k^4 P_{\varphi i}(k) T_{BBKS}^2(k/k_{eq}) \sim A(k_{eq})^2 \sim A H_0^2 \cdot 10^5 , \tag{4.61}
\]

up to a numerical prefactor of order 1. Since the amplitude of the power spectrum is \( A \sim 10^{-9} \) [77], the overall contribution of the backreaction is

\[
\frac{S^{(1)}}{H_0^2} \sim -\frac{1}{a^2} (10^{-4}) . \tag{4.62}
\]

Now, as long as the correlation objects give a negligible backreaction to the usual background quantities, when we proceed with the next iteration, the effect of the backreaction on the evolution of the perturbations will also remain negligible (at least at the leading order). Hence in practice
there will be essentially no difference between the zeroth iteration and first iteration perturbation functions. This amounts to saying that when the backreaction is negligible, convergence to the “true” solution for the scale factor at the leading order, is essentially achieved in a single calculation. We will discuss the issue of convergence in somewhat more detail at the end of the chapter.

4.4.2 Radiation and CDM without baryons

Let us now turn to the full sCDM model (without baryons). An analytical discussion of this model in various regions of \((k, \eta)\)-space, can be found e.g. in Ref. \([12]\). Since we are interested in integrals over \(k\) across a range of epochs \(\eta\), it is most convenient to solve this model numerically. It is further convenient to use \((\ln a)\) in place of \(\eta\), as the variable with which to advance the solution. Also, it is useful to introduce transfer functions like \(\Phi_k(\eta)\) for all the relevant perturbation functions in exactly the same manner (see Eqn. \((4.48)\)), namely by pulling out a factor of \(\varphi_{ki}\), since the initial conditions are completely specified by the initial metric perturbation. For a generic perturbation function \(s_k(\eta)\) (other than the metric fluctuation \(\varphi_k\)) the transfer function corresponding to \(s\) will be denoted by a caret, so that

\[
\hat{s}_k(\eta) = \varphi_{ki} \hat{s}_k(\eta). \tag{4.63}
\]

The relevant Einstein equations can be brought to the following closed set of first order ordinary differential equations (adapted from Eqns. \((7.11)-(7.15)\) of Ref. \([12]\),

\[
\frac{\partial \Phi_k}{\partial (\ln a)} = - \left[ \left( 1 + \frac{K^2}{3E^2} \right) \Phi_k + \frac{1}{2E^2a} \left( \Omega_m \hat{\delta}_k + \frac{4}{a} \Omega_r \hat{\Theta}_0k \right) \right], \tag{4.64a}
\]

\[
\frac{\partial \hat{\delta}_k}{\partial (\ln a)} = - \frac{K}{E} \hat{V}_k + 3 \frac{\partial \Phi_k}{\partial (\ln a)}, \tag{4.64b}
\]

\[
\frac{\partial \hat{\Theta}_{0k}}{\partial (\ln a)} = - \frac{K}{E} \hat{\Theta}_{1k} + \frac{\partial \Phi_k}{\partial (\ln a)}, \tag{4.64c}
\]

\[
\frac{\partial \hat{\Theta}_{1k}}{\partial (\ln a)} = \frac{K}{3E} \left( \hat{\Theta}_{0k} + \Phi_k \right), \tag{4.64d}
\]

\[
\frac{\partial \hat{V}_k}{\partial (\ln a)} = - \hat{V}_k + \frac{K}{E} \Phi_k. \tag{4.64e}
\]

Here we have introduced the dimensionless variables

\[
K \equiv \frac{k}{H_0} \; ; \; E(a) \equiv \frac{\mathcal{H}(a)}{H_0} = \frac{\mathcal{H}(a)}{H_0}, \tag{4.65}
\]

and the various perturbation functions are defined as follows : \(\delta_k\) is the \(k\)-space density contrast of CDM, \(\Theta_{0k}\) and \(\Theta_{1k}\) are the monopole and dipole moments respectively of the \(k\)-space temperature fluctuation of radiation, and \((-iV_k)\) is the \(k\)-space peculiar velocity scalar potential of CDM (i.e., the real space peculiar velocity is \(v_A = \partial_A v\) where \(v\) is the Fourier transform of \((-iV_k))\).

Assuming adiabatic perturbations, the initial conditions satisfied by the transfer functions at
\( a = a_i \) are (adapted from Ch. 6 of Ref. [12])

\[
\Phi_k(a_i) = 1 \quad ; \quad \dot{\delta}_k(a_i) = -\frac{3}{2} \quad ; \quad \dot{\Theta}_0k(a_i) = -\frac{1}{2} \quad ; \quad \dot{V}_k(a_i) = 3\dot{\Theta}_{1k}(a_i) = \frac{1}{2} \frac{K}{E(a_i)}.
\] (4.66)

We choose \( a_i = 10^{-16} \), which corresponds to an initial background radiation temperature of \( T \sim 10^3 \text{GeV} \). While this is not as far back in the past as the energy scale of inflation (which is closer to \( \sim 10^{15} \text{Gev} \)), it is on the edge of the energy scale where known physics begins [70]. This makes Eqn. (4.57) unrealistic since we have ignored all of Big Bang nucleosynthesis and also the fact that neutrinos were in equilibrium with other species at temperatures higher than about 1MeV. However the modifications due to these additional details are not expected to drastically change the final results, and these assumptions lead to some simplifications in the code used. The goal here is only to demonstrate an application of the formalism; more realistic calculations accounting for the effects of baryons can also be performed (see, e.g. Behrend et al. [49] who incorporate these effects for the post-recombination era, albeit in the Buchert formalism).

In order to partially account for the fact that inflationary initial conditions are actually set much earlier than \( a = 10^{-16} \), we impose an absolute small wavelength cutoff at the scale which enters the horizon at the initial epoch which we have chosen. In the above notation this corresponds to setting \( K_{max} = E(a_i) \sim 10^{13} \). This makes sense since scales satisfying \( K \gg K_{max} \) have already entered the horizon and decayed considerably by the epoch \( a = 10^{-16} \). There is a source of error due to ignoring scales \( K \gtrsim K_{max} \) which have not yet decayed significantly, but this error rapidly decreases with time as progressively larger length scales enter the horizon and decay. [In fact, in practice to compute the integrals at any given epoch \( a = a_* \), one only needs to have followed the evolution of modes with \( K < \sim 5000E(a_*) \): more on this in the next subsection.] More important is the cutoff at long wavelengths, which we set at \( K_{min} = 1 \) (corresponding to \( k_{min} = H_0 \)), which is firstly a natural choice given that \( H_0^{-1} \) is the only large scale in the system, and is secondly also guided by analyses of CMB data which have detected such a cutoff [73]. We will see that reducing \( K_{min} \) even by a few orders of magnitude, does not affect the final qualitative results significantly.

**Numerical Results**

Equations (4.64) with initial conditions (4.66) were solved with a standard 4th order Runge-Kutta integrator with adaptive stepsize control (based on the algorithm given in Ref. [78]). For the integrals in Eqns. (4.52), only the function \( \Phi_k(a) \) needs to be tracked accurately. Hence, although \( \dot{\Theta}_0k \) and \( \dot{\Theta}_{1k} \) are difficult to follow accurately beyond the matter radiation equality \( a_{eq} = (\Omega_r/\Omega_m) \simeq 8 \times 10^{-5} \) due to rapid oscillations, the integrals can still be reliably computed since \( \dot{\Theta}_0k \) and \( \dot{\Theta}_{1k} \) do not significantly affect the evolution of \( \Phi_k \) in the matter dominated era (as seen in Eqn. (4.64a)).

To see that known results are being reproduced by the code, consider Figs. 4.1(a) and 4.1(b) as examples. Fig. 4.1(a) shows the evolution of two scales corresponding to \( K = 1 \) (\( k = H_0 \text{Mpc}^{-1} \)) and \( K = 0.01 \). The first enters the horizon at the present epoch, while the second remains super-
(a) The evolution of two large scale modes. Also shown is the Kodama-Sasaki analytical solution in the large scale limit \( k \eta \ll 1 \), given by Eqn. (4.67).

\[ \Phi_k(y) = \frac{1}{10y^3} \left[ 16\sqrt{1+y + 9y^3 + 2y^2 - 8y - 16} \right], \]  

(4.67)

where \( y \equiv a/a_{eq} \), and this function is also shown. Clearly all the curves in Fig. 4.1(a) are practically identical. Fig. 4.1(b) shows the function \( \Phi_k \) normalised by its constant value at large scales, at the epoch \( a = 500a_{eq} \). The dotted line is the BBKS transfer function (4.59).

Figure 4.1: Numerical results for the transfer function.

Horizon for the entire evolution, satisfying \( k \eta \ll 1 \). In this limit an analytical solution exists in the sCDM model, due to Kodama and Sasaki \[79, 12\], given by

\[ \Phi_{\text{analytic}}(a) \]

The evolution of two large scale modes. Also shown is the Kodama-Sasaki analytical solution in the large scale limit \( k \eta \ll 1 \), given by Eqn. (4.67).

(b) The transfer function \( \Phi_k \) normalised by its constant value at large scales, at the epoch \( a = 500a_{eq} \). The dotted line is the BBKS transfer function (4.59).

To numerically estimate the integrals in Eqns. (4.52), the values of \( \Phi_k \) and its first and second derivatives with respect to (\( \ln a \)) are needed across a range of \( K \) values. For reference, note that the following relations hold for a generic function of time \( w(\eta) \),

\[ \frac{dw}{d\eta} = a \frac{dw}{da} = \mathcal{H} \frac{dw}{d(\ln a)}. \]  

(4.68)

Based on the earlier discussion, the initial power spectrum \( P_{\varphi i}(k) \) is taken to satisfy

\[ \frac{k^3 P_{\varphi i}(k)}{2\pi^2} = A, \quad \text{for} \quad H_0 < k < k_{\text{max}} = \mathcal{H}(a_i), \]  

(4.69)

and zero otherwise, and we set

\[ A = 1.0 \times 10^{-9}, \]  

(4.70)

which, for the sCDM model follows from the convention (see Eqn.(6.100) of Ref. [12]) \( A = (5\delta_H/3)^2 \) with \( \delta_H \approx 2 \times 10^{-5} \) (see, e.g. Ref. [77]).
(a) The dimensionless integrand of $S^{(1)}$, namely the function $(k/H_0)^2\Phi_k^2$, at three sample values of the scale factor. The function dies down rapidly for large $k$, with the value at some $k$ being progressively smaller with increasing scale factor. The declining behaviour of the curves for $a = a_{eq}$ and $a = 200a_{eq}$ extrapolates to large $k$.

(b) The dimensionless CDM density contrast. Together with Fig. 4.2(a) this shows that nonlinear scales do not impact the backreaction integrals significantly.

Figure 4.2: Backreaction and nonlinearity.

Consider Figs. 4.2(a) and 4.2(b) which highlight two issues discussed earlier. Fig. 4.2(a) shows the integrand of $S^{(1)}$ at three sample epochs, and we see that the integrand dies down rapidly at increasingly smaller $k$ values for progressively later epochs. (The other integrands, not displayed here, also show this rapid decline for large $k$.) [We have not shown the integrand at the later two epochs for all values of $k$ since this was computationally expensive, but the declining trend of the curves can be extrapolated to large $k$, which is well understood analytically [12].] This justifies the statement in the beginning of this section, that at any epoch $a_*$ it is sufficient to have followed the evolution of scales satisfying $K < 5000E(a_*)$ for computing the integrals. Secondly, Fig. 4.2(b) shows the behaviour of $k^{3/2}|\delta_k| = A^{1/2}|\hat{\delta}_k|$ at the same three epochs, and comparing with Fig. 4.2(a) we see that at any epoch, the region of $k$-space where linear PT has broken down, does not contribute significantly to the integrals. This is in line with the conjecture in Ref. [80] that the effects of the backreaction should remain small since the mass contained in the nonlinear scales is subdominant. We will return to this issue in chapter 5.

Due to the structure of the integrals and the chosen initial power spectrum, it is convenient to compute the integrands in Eqns. 1.52 equally spaced in $(\ln K)$, and then perform the integrals using the extended Simpson’s rule [78]. If $2^N + 1$ points are used to evaluate a given integral, resulting in a value $I_N$ say, then the error can be estimated by computing the integral with $2^{N-1} + 1$ points to get $I_{N-1}$, and estimating the relative error as $|I_{N-1}/I_N| - 1$. With $N = 10$, the estimated errors in all the integrals at all epochs were typically less than 0.1%. A bigger error is incurred in computing the integrand itself at any given epoch, leading to estimated errors of order $\sim 1\%$ in $S^{(1)}$, $P^{(1)}$ and $P^{(1)} + P^{(2)}$, with a larger error in $S^{(2)}$ as explained below.

65
The second derivative $\partial^2 \Phi_k / \partial (\ln a)^2$ proves to be difficult to track numerically. At early times, when most scales are superhorizon, the Kodama-Sasaki analytical solution (4.67) is a good approximation for most values of $k$. Using this one can see that at early times the value of the derivative is numerically very small, and is difficult to reliably estimate due to roundoff errors. For this reason the integral $S^{(2)}$ could not be accurately estimated at early times. However, the structure of the integrand of $S^{(2)}$ shows that the largest contribution comes from large (superhorizon) scales (the small scales being subdominant due to the presence of $\Phi_k$ and $1/k^2$). An analysis using the Kodama-Sasaki solution then shows in a fairly straightforward manner that the behaviour of the backreaction term is $|S^{(2)}/H^2| \sim 10^{-6} (a/a_{eq})(H_0/k_{min})^2$ for our choices of parameters, where $(a/a_{eq}) \ll 1$. At intermediate times around $a \sim a_{eq}$ and later, although it becomes computationally expensive to obtain convergent values for the second derivative at all relevant scales\textsuperscript{3}, moderately good accuracy (1-5%) can be achieved.

The results are shown in Fig. 4.3 in which the magnitudes of the correlation integrals of Eqn. (4.52), normalised by the Hubble parameter squared $H^2(a) = (\mathcal{H}/a)^2$ are plotted as a function of the scale factor in a log-log plot. The values for $S^{(2)}$ are shown only for epochs later than $a \simeq 0.01 a_{eq} \sim 10^{-6}$. We see that at all epochs, the correlation terms remain negligible compared to the chosen zeroth iteration background. Also, in the radiation dominated epoch all the correlation scalars (except $S^{(2)}$ whose evolution couldn’t be accurately obtained) track the $\sim a^{-4}$ behaviour of the background radiation density (see also Ref. [28]). The discussion above shows however that the

\textsuperscript{3}Convergence was tested by varying a global parameter which dynamically controls the stepsize during evolution (by stepsize doubling/halving, see Ref. [78]). The integrals other than $S^{(2)}$ show convergence at 3 or more significant digits for all epochs, whereas convergence can be obtained for $S^{(2)}$ only at epochs sufficiently close to matter radiation equality, and there only for 1-2 significant digits, by setting stringent conditions on stepsize doubling.
magnitude of $S^{(2)}$ is far smaller than the other backreaction functions at early times, for a cutoff at $k_{\text{min}} = H_0$. On the other hand, in the matter dominated epoch $S^{(1)}$ dominates the backreaction and settles into a curvature-like $\sim a^{-2}$ behaviour (note that in the matter dominated epoch we have $H^2 \sim a^{-3}$). As for the signs of the correlations, $S^{(1)}$, $S^{(2)}$, and $P^{(1)}$ are negative throughout the evolution while $P^{(1)} + P^{(2)}$ is positive throughout.

Finally, a few comments regarding the effects of ignoring baryons, nonlinear corrections, etc. Including baryons in the problem (with a background density parameter of $\Omega_b \simeq 0.05$) will lead to a significant suppression of small scale power (by introducing pressure terms which will tend to wipe out inhomogeneities) and also a small suppression of large scale power. This effect causes a (downward) change in the late time transfer function of roughly 15-20% \cite{12}, and therefore cannot increase the contribution of the backreaction. Quasi-linear corrections can lead to significant changes in the transfer function, but do not cause shifts by several orders of magnitude (see Ref. \cite{81} and references therein). Hence accounting for changes due to quasi-linear behaviour will also not increase the magnitude of the backreaction by a large amount. As for effects from fully nonlinear scales, we have seen that these can be expected to remain small (see also chapter 5).

Adding a cosmological constant (and retaining a flat background geometry) will change the qualitative features of the correlation functions by shifting the scale $k_{\text{eq}}$ (due to a reduced $\Omega_m$, which will also increase the power spectrum amplitude \cite{12}, but again not by orders of magnitude). Also, the late time behaviour of the correlation scalars will be affected since the potential $\Phi_k$ will decay at late times instead of remaining constant (see also below). Regardless, the backreaction is expected to remain small even in this case (which is also indicated by the calculations of Behrend et al. \cite{49} in the Buchert framework).

**Chapter summary and discussion:**

This chapter showed how the spatial averaging limit of MG can be adapted to the case of linear perturbation theory (PT) in cosmology, to calculate the backreaction in gauge invariant manner. In doing so we also saw the significance of volume preserving gauges in defining self-consistent averaging operators. The formalism leaves some freedom in the choice of the averaging operator (via a choice of a volume preserving gauge), and therefore we cannot claim that our results are unique down to all numerical factors. However, since the final explicit form of the late time backreaction in Fig. 4.3 matches closely with the (essentially order of magnitude) estimate of Eqn. (4.62) which is actually independent of any details of the averaging procedure, we do expect our results to be qualitatively robust.

Fig. 4.3 also shows that in the absence of a cosmological constant, the backreaction after a single iteration of the procedure outlined in section 4.1, tracks the radiation density in the radiation dominated era, and essentially behaves like a curvature term in the matter dominated era. Two issues arise from this behaviour. Denote the corrections to the Friedmann equation (A.3a) and the acceleration equation (A.3b) as $C_F$ and $C_{\text{acc}}$ respectively. Then firstly, we find that in the radiation
dominated era, although both $C_F$ and $C_{acc}$ behave like $\sim a^{-4}$, their numerical coefficients do not combine so as to preserve the conservation criterion $\text{(4.64)}$ separately. Since the backreaction must now necessarily couple to the background radiation density, this points to a very tiny gravitationally induced correction in the equation of state for radiation. This effect can be traced back essentially to the presence of a small but non-zero correlation 3-form and 4-form, arising from higher order perturbative effects. In the matter dominated era, at least in the “zeroth” iteration, this effect seems to be highly suppressed since we now have $C_F \sim a^{-2}$ and $|C_{acc}| \ll |C_F|$, which is approximately consistent with $\text{(3.64)}$.

The second issue concerns what happens at higher iterations, and is important from the point of view of obtaining a convergent answer for the backreaction. The basic cycle that one needs to keep in mind is that the backreaction affects $H^2$, which affects the equations for the density and metric perturbations, which in turn define the backreaction. Consider the situation in the matter dominated era, which is easier to handle since firstly only one term $S^{(1)}$ contributes to the backreaction and secondly the linear PT solution has a simple analytic form. The estimate in Eqn. $\text{(4.62)}$ shows that most of the contribution comes from (quasi)linear subhorizon scales $k \sim k_{eq}$ for which the Poisson equation holds, so that if the density contrast behaves like $\delta_k \sim D(a)$ then the metric transfer function behaves like $\Phi_k \sim D/a$. Standard linear PT $\text{[12]}$ tells us that $D(a)$ is the so-called growth function which can be written as $D \sim E \int da/(aE)^3$ up to some numerical coefficient, where $E \equiv H/H_0$. An analysis similar to the one leading to Eqn. $\text{(4.62)}$ then shows that we should expect $C_F \sim a^{-2}(D/a)^2$ at late times. For a flat universe without a cosmological constant, $D(a) = a$ and we recover the single iteration result that we have been discussing so far.

The crucial thing to note is that since the backreaction affects only the background equations and not the perturbation equations, $D(a)$ is completely determined by the Hubble parameter $H(a) = H_0E(a)$, so that at any iteration $i$ we will have

$$ (E^{(i+1)})^2 = \Omega_m^{(i+1)}a^{-3} + \epsilon_{\text{bkrxn}}^{(i)}a^{-2}(D^{(i)}/a)^2 ; \quad D^{(i)}(a) \sim E^{(i)} \int da/(aE^{(i)})^3, $$

where we expect $\epsilon_{\text{bkrxn}}^{(i)} \sim 10^{-4}$. This immediately suggests that the limit of this series is the solution of the integral equation

$$ E^2 = \Omega_m a^{-3} + \epsilon_{\text{bkrxn}} a^{-2}(D/a)^2 ; \quad D(a) \sim E \int da/(aE)^3. $$

This equation can in principle be solved perturbatively by exploiting the smallness of the parameter $\epsilon_{\text{bkrxn}}$, and we expect the solution to be close to the “zeroth” iteration answer $D \sim a$. To understand why, notice that at the zeroth iteration we found $S^{(1)}$ to be negative, so that the first iteration Hubble parameter $E^{(1)}(a)$ is effectively that of an open universe with a small negative curvature. Standard analysis shows that the growth factor in an open universe is suppressed compared to that in a flat matter dominated one, and hence the Hubble parameter at the second iteration $E^{(2)}(a)$ will
have a slightly smaller contribution from the backreaction than $E^{(1)}(a)$. This will correspondingly slightly enhance the contribution of the backreaction to $E^{(3)}(a)$ over the contribution to $E^{(2)}(a)$, and so on until the solution converges.

This convergent solution will, like the radiation dominated case, mildly violate the conservation criterion Eqn. (4.62). Further, the analysis above generalises to the case when the cosmological constant is nonzero. In this case the late time growth factor is suppressed compared to the EdS case even at the zeroth iteration [12], and the convergent solution will violate the conservation criterion by an amount comparable to the backreaction itself. What is important however is that in all cases, the backreaction as well as the violation of matter conservation remain negligibly small, approximately at the level of one part in $10^4$. This analysis ignored all contributions from scales which have become fully nonlinear in the matter density contrast at late times. The reasoning was that these scales are not expected to contribute significantly to the backreaction due to a suppression in the transfer function $\Phi_k$. In the next chapter, we will confirm this expectation in the context of a toy model of nonlinear structure formation.
Chapter 5

Nonlinear structure formation and backreaction

We have seen so far that applying the averaging framework during epochs when linear perturbation
theory (PT) is expected to be valid, leads to only negligible modifications in the standard cosmological equations. This is not unexpected, since the estimate in e.g. Eqn. (4.62) does not depend
crucially on the details of the averaging procedure, and is therefore robust. Most of the interest
in the backreaction issue however, has been from the point of view of late time cosmology when
matter fluctuations at least have become nonlinear on small scales. It has been claimed using some
simple models of structure formation [37] that using the perturbed FLRW ansatz for the metric
is no longer a good approximation in this regime, and that one should expect backreaction effects
to grow large at these times. We will investigate this issue in this chapter, in the framework of
an exact toy model of structure formation based on the LTB solutions of general relativity (see
Appendix B).

We have already seen in our linear PT calculation that if the metric at late times continues
to be of the perturbed FLRW form (4.6), then backreaction effects of the nonlinear scales are in
fact likely to be small, which follows from studying the structure of the integrands of the various
backreaction functionals (see the discussion of Figs. 4.2(a) and 4.2(b)). Order of magnitude argu-
ments such as those in Ref. [27] suggest that the perturbed FLRW metric does remain a good
approximation at late times. In this chapter we will see an explicit demonstration that a fully
relativistic, highly nonlinear collapsing system can be described by a perturbed FLRW metric,
provided peculiar velocities of the matter remain nonrelativistic. Further, in our model (which will
satisfy this condition) we can then use the formalism developed in chapter 4 to explicitly compute
the backreaction. We will see that the backreaction in this case does remain small.
5.1 Spherical Collapse: Setting up the model

Our model is based on the LTB solution described in Appendix B, and is completely determined once the initial conditions are specified.

5.1.1 Initial conditions

While choosing the initial density, velocity and coordinate scaling profiles, we make the important assumption that at initial time, a well-defined global background FLRW solution can be identified, with scale factor \( a(t) \), Hubble parameter \( H(t) \) and density \( \rho_b(t) \). This is reasonable since the CMB data (combined with the Copernican principle) assure us that inhomogeneities at the last scattering epoch were at the level of 10 parts per million. This assumption plays a crucial role in deciding which regions are overdense and will eventually collapse, and which regions will keep expanding.

- **Initial density profile** \( \rho(t_i, r) \):
  
  The initial density is chosen to be
  
  \[
  \rho(t_i, r) = \rho_{bi} \begin{cases} 
  (1 + \delta_s), & r < r_s \\
  (1 - \delta_v), & r_s < r < r_v \\
  1, & r > r_v, 
  \end{cases} \tag{5.1}
  \]

  where \( \rho_{bi} = \rho_b(t_i) \). Initially, the region \( r < r_s \) is assumed to contain a tiny overdensity and the region \( r_s < r < r_v \), an underdensity. In other words,

  \[
  0 < \delta_s, \delta_v \ll 1. \tag{5.2}
  \]

  The discontinuities in the initial density profile can be smoothed out by replacing the step functions appropriately. We will not do this here, since the step functions make calculations very simple. This is not expected to affect the qualitative features of our final results.

- **Initial conditions on scaling and velocities**:
  
  We match the initial velocity and coordinate scaling to the global background solution, by requiring

  \[
  R(t_i, r) = a_i r, \tag{5.3}
  \]

  \[
  \dot{R}(t_i, r) = a_i H_i r, \tag{5.4}
  \]

  with \( a_i \) and \( H_i \) denoting the initial values of the scale factor and Hubble parameter respectively of the global background. This amounts to setting the initial velocities to match the Hubble flow, ignoring initial peculiar velocities. This is only a convenient choice and the introduction of initial peculiar velocities is not expected to modify our final results qualitatively.
For the FLRW background we consider an Einstein-deSitter (EdS) solution with scale factor and Hubble parameter given by

\[ a(t) = \left( \frac{t}{t_0} \right)^{2/3} \; ; \; t_0 = 2/(3H_0), \quad (5.5) \]
\[ H(t) \equiv \dot{a}/a = 2/(3t), \quad (5.6) \]

with \( t_0 \) denoting the present epoch. \( a_i \) fixes the initial time as

\[ t_i = 2/(3H_0)a_i^{3/2}. \quad (5.7) \]

We will always use \( a_i = 10^{-3} \), so that the initial conditions are being set around the CMB last scattering epoch; in general \( a_i \) must be treated as one of the parameters in the problem. The initial EdS background density is given in terms of \( H_0 \) and \( a_i \) as

\[ \rho_{bi} = \frac{3}{8\pi G}H_0^2a_i^{-3}. \quad (5.8) \]

5.1.2 Mass function \( M(r) \) and curvature function \( k(r) \)

We now have enough information to fix \( M(r) \) and \( k(r) \). Using Eqn. (B.2b) at initial time together with the scaling in Eqn. (5.3) gives us

\[
GM(r) = \frac{1}{2}H_0^2r^3 \begin{cases} 
1 + \delta_s, & 0 < r < r_s \\
1 + \delta_v \left( (r_c/r)^3 - 1 \right), & r_s < r < r_v \\
1 + \left( \delta_v/r^3 \right) \left( r_c^3 - r_v^3 \right), & r > r_v,
\end{cases}
\]

where we have defined a “critical” radius \( r_c \) by the equation

\[ \left( \frac{r_c}{r_s} \right)^3 = 1 + \frac{\delta_s}{\delta_v}. \quad (5.10) \]

The significance of \( r_c \) will become apparent shortly. Using the initial conditions Eqns. (5.3) and (5.4) in the evolution equation (B.2a) at initial time, gives

\[ k(r)r^2 = \frac{2GM(r)}{a_i r} - a_i^2H_i^2r^2, \quad (5.11) \]

with \( H_i^2 = H_0^2a_i^{-3} \), and hence

\[
 k(r) = \frac{H_0^2}{a_i} \begin{cases} 
\delta_s, & r < r_s \\
\delta_v \left( (r_c/r)^3 - 1 \right), & r_s < r < r_v \\
\left( \delta_v/r^3 \right) \left( r_c^3 - r_v^3 \right), & r > r_v.
\end{cases}
\]

(5.12)
The significance of $r_c$ is now clarified. Since $\delta_s, \delta_v > 0$, we have $r_c > r_*$ by definition (Eqn. [5.10]). The following possibilities arise:

- If $r_c > r_v$, then $k(r) > 0$ for all $r$, and every shell will ultimately collapse, including the “void” region $r_* < r < r_v$.
- If $r_c < r_v$, then $k(r) > 0$ for $r < r_c$ and changes sign at $r = r_c$. Hence, the region $r_* < r < r_c$ will collapse even though it is underdense, while the region $r > r_c$ will expand forever.
- If $r_c = r_v$, then the “void” exactly compensates for the overdensity, and the universe is exactly EdS for $r > r_v$. $[GM(r) = (1/2)H_0^2 r^3$ and $k(r) = 0.]$ Also the “void” will eventually collapse.

Clearly the most interesting case for us is the one with $r_c < r_v$, and we will hence make this choice for our model. We realize that the model as it stands is not a very realistic depiction of the (nearly spherical) voids we see in our Universe [56], since these voids are seen to be surrounded by “walls” of matter. However, our goal is to describe two regions, one of which collapses while the other expands ever more rapidly, and our model is capable of doing so while retaining its fully relativistic character.

Although we have set up the model for all values of the radial coordinate $r$, hereon we will concentrate on the region $0 < r < r_v$. One reason is that most of the interesting dynamics takes place in this region. Another is that the region $r > r_v$ develops shell-crossing singularities due to the sharp rise in density across $r = r_v$. A more realistic model would be able to incorporate the pressures that are expected to build up when a shell-crossing occurs [82], but the LTB model is limited in this respect due to its pressureless character. We will therefore ignore the region $r > r_v$.

### 5.1.3 The solution in the region $0 < r < r_v$

The region of interest can be split into three parts: region 1 = \{0 < r < r_*\}, region 2 = \{r_* < r < r_c\} and region 3 = \{r_c < r < r_v\}. The solution in the three regions is as follows:

- **region 1** ($0 < r < r_*$):

  
  \[
  R = \frac{1}{2} \left( \frac{a_i}{\delta_*} \right) r (1 + \delta_*) (1 - \cos u),
  \]
  
  \[
  u - \sin u = \frac{2H_0}{1 + \delta_*} \left( \frac{\delta_*}{a_i} \right)^{3/2} (t - t_i) + (u_i - \sin u_i),
  \]

  \[
  1 - \cos u_i = \frac{2\delta_*}{1 + \delta_*},
  \]

  \[
  R^2 R' = \frac{R^3}{r}.
  \]
For regions 2 and 3, it is convenient to define a function $\epsilon(r)$ as

$$
\epsilon(r) \equiv \delta_v \left( \frac{r_v}{r} \right)^3 - 1 = \frac{a_i}{H_0 k(r)}, \quad r_s < r < r_v.
$$

(5.14)

- **region 2** ($r_s < r < r_c$): 

$$
R = \frac{1}{2} \left( \frac{a_i}{\epsilon} \right) r (1 + \epsilon)(1 - \cos \alpha),
$$

(5.15a)

$$
\alpha - \sin \alpha = \frac{2H_0}{1 + \epsilon} \left( \frac{\epsilon}{a_i} \right)^{3/2} (t - t_i) + (\alpha_i - \sin \alpha_i),
$$

(5.15b)

$$
1 - \cos \alpha(r) = \frac{2\epsilon}{1 + \epsilon},
$$

(5.15c)

$$
R^2 R' = \frac{R^3}{r} \left( 1 - \frac{r e'}{\epsilon (1 + \epsilon)} \right) \left\{ 1 - \frac{\epsilon^{3/2}}{(1 - \cos \alpha)^2} \left[ H_i(t - t_i) \sin \alpha \left( \frac{3 + \epsilon}{1 + \epsilon} \right) 
        + \frac{4\epsilon^{1/2}}{(1 + \epsilon)^2} \left( \frac{\sin \alpha}{\sin \alpha_i} \right) \right] \right\}.
$$

(5.15d)

- **region 3** ($r_c < r < r_v$): 

$$
R = \frac{1}{2} \left( \frac{a_i}{|\epsilon|} \right) r (1 + \epsilon)(\cosh \eta - 1),
$$

(5.16a)

$$
\sinh \eta - \eta = \frac{2H_0}{1 + \epsilon} \left( \frac{|\epsilon|}{a_i} \right)^{3/2} (t - t_i) + (\sinh \eta_i - \eta_i),
$$

(5.16b)

$$
cosh \eta_i(r) - 1 = \frac{2|\epsilon|}{1 + \epsilon},
$$

(5.16c)

$$
R^2 R' = \frac{R^3}{r} \left( 1 - \frac{r e'}{\epsilon (1 + \epsilon)} \right) \left\{ 1 - \frac{|\epsilon|^{3/2}}{(\cosh \eta - 1)^2} \left[ H_i(t - t_i) \sinh \eta \left( \frac{3 + \epsilon}{1 + \epsilon} \right) 
        + \frac{4|\epsilon|^{1/2}}{(1 + \epsilon)^2} \left( \frac{\sinh \eta}{\sinh \eta_i} \right) \right] \right\}.
$$

(5.16d)

The crossover from region 1 to region 2 is discontinuous in $R'$ (but not in $R$) due to our discontinuous choice of initial density. Smoothing out the density will also smooth out $R'$. The crossover from region 2 to region 3 can be shown to be smooth, by considering the limits $r \to r_c^-$ and $r \to r_c^+$ or equivalently $\epsilon \to 0^-$ and $\epsilon \to 0^+$. Note that the results in Eqns. (5.13), (5.15) and (5.16) are exact, and do not involve any perturbative expansions in $\delta_s$ or $\delta_v$, even though these parameters are small.
Table 5.1: Values of various parameters used in generating plots.

| Parameter name | Parameter value |
|----------------|-----------------|
| $a_i$          | 0.001           |
| $H_0$          | $1/13.59 \text{Gyr}^{-1}$ (= 72 km/s/Mpc) |
| $t_0$          | $2/(3H_0) = 9.06 \text{Gyr}$ |
| $c$            | $306.6 \text{MpcGyr}^{-1}$ |
| $\delta_s$     | $1.25a_i(3\pi/4)^{2/3} = 2.21 \times 10^{-3}$ |
| $\delta_v$     | 0.005           |
| $r_*$          | $0.004c/H_0 = 16.7 \text{Mpc}$ |
| $t_{\text{turn}}/t_0$ | 0.72 |
| $r_c$          | $r_*(1 + \delta_s/\delta_v)^{1/3} = 18.8 \text{Mpc}$ |
| $r_v$          | $1.25r_c = 23.5 \text{Mpc}$ |
| $R(t_0, r_*)$  | 6.8 Mpc        |
| $R(t_0, r_v)$  | 33.3 Mpc       |

5.1.4 Behaviour of the model

Each shell in the inner, homogeneous and overdense region 1 behaves as a closed FLRW universe, expanding out to a maximum radius $R_{\text{max}}(r)$ given by

$$R_{\text{max}}(r) = \frac{a_i}{\delta_s} r (1 + \delta_s).$$

(5.17)

All the inner shells reach their maximum radius and turn around at the same time $t_{\text{turn}}$ given by

$$t_{\text{turn}} = t_i + \frac{1 + \delta_s}{2H_0} \left( \frac{a_i}{\delta_s} \right)^{3/2} \left( \pi - (u_i - \sin u_i) \right) \approx t_0 \left( \frac{3\pi}{4} \right) \left( \frac{a_i}{\delta_s} \right)^{3/2},$$

(5.18)

where we have used the smallness of $a_i$ and $\delta_s$ to make the last approximation. By appropriately choosing a value of $\delta_s$, we can arrange for the turnaround of region 1 to occur either before or after the present epoch.

In Table 5.1 we have listed the parameter values which we will use frequently in displaying plots. Along with the parameter set $\{a_i, H_0, \delta_s, \delta_v, r_*, r_v\}$, we have also listed the values of the derived quantities $\{r_c, t_i, t_0, t_{\text{turn}}\}$ and speed of light $c$ in units of MpcGyr$^{-1}$. We have also shown the values of the present day physical area radius $R(t_0, r)$ at $r = r_*$ and $r = r_v$. The density contrasts are to be understood to reflect the inhomogeneities in the dark matter density close to last scattering, and not the inhomogeneities of the baryons which were much smaller [12]. In Fig. 5.1 we have shown the evolution of the density contrast $\delta(t, r)$ defined in the usual way by

$$1 + \delta(t, r) = \frac{\rho(t, r)}{\rho_0(t)},$$

(5.19)
for the parameter choices of Table 5.1 for which one has $t_{\text{turn}}/t_0 \simeq 0.72$, so that the collapse is well under way in region 1 at the present epoch. The two panels show the contrast for two representative values of $r$, one in region 1 and the other in region 3.

5.1.5 Aside : Acceleration from initial conditions

It is interesting to note that our model is capable of qualitatively reproducing results derived by earlier by Räsänen [37] in the context of a very simple model of structure formation. Räsänen’s model can be summarized as follows: one considers two disjoint regions, one overdense and the other completely empty, each evolving according to the FLRW evolution equations. (The embedding of these regions in an FLRW background, and the behaviour of the region between these two regions, is not considered.) The scale factor in the overdense region therefore behaves as $a_1 \propto (1 - \cos u)$ with $t \propto (u - \sin u)$, and the scale factor in the empty region behaves as $a_2 \propto t$. It is then straightforward to show that if one defines a volume averaged scale factor by $a^3 \equiv a^3_1 + a^3_2$, then the effective deceleration parameter given by $q \equiv -(\ddot{a}/\dot{a})^2$ becomes negative (indicating acceleration) around the time that the overdense region turns around and starts collapsing.

If we define the volume of each of our three comoving regions separately, as

$$V_1 \equiv 4\pi \int_0^{r_*} \frac{R^2 R'}{\sqrt{1-k(r)r^2}} dr ; \quad V_2 \equiv 4\pi \int_{r_*}^{r_c} \frac{R^2 R'}{\sqrt{1-k(r)r^2}} dr ; \quad V_3 \equiv 4\pi \int_{r_c}^{r_v} \frac{R^2 R'}{\sqrt{1-k(r)r^2}} dr ,$$

then the total volume of the region can be used to define a “Buchert-style” volume averaged scale factor as

$$a(t) \equiv \left( \frac{V(t)}{V(t_0)} \right)^{1/3} ; \quad V(t) \equiv V_1(t) + V_2(t) + V_3(t) ,$$
and hence an effective deceleration parameter \( q \) given by

\[
q \equiv -\frac{\ddot{a}}{a^2} = 2 - 3 \frac{\dot{V}V}{V^2},
\]

whereas Räsanen’s model can be mimicked more closely by ours, if we simply remove the region 2, by hand. By doing so we are left with two disjoint regions, each spherically symmetric, one of which is collapsing and the other expanding ever rapidly and becoming ever emptier. There is no physical reason to throw away region 2 in this manner, but for the sake of comparison we will define a “modified” scale factor \( a_{\text{mod}} \) and its corresponding deceleration parameter \( q_{\text{mod}} \) by

\[
a_{\text{mod}}(t) \equiv \left( \frac{V_1(t) + V_3(t)}{V_1(t_0) + V_3(t_0)} \right)^{1/3}; \quad q_{\text{mod}} \equiv -\frac{\ddot{a}_{\text{mod}} a_{\text{mod}}}{\dot{a}_{\text{mod}}^2}.
\]

In Fig. 5.2 we plot \( q(t) \) and \( q_{\text{mod}}(t) \), for several sets of initial conditions which are close to our “base set” listed in Table 5.1 (except for Fig. 5.2(d) which has a large value for \( \delta v \)). The various initial conditions correspond to turnaround times that are slightly greater than, or slightly less than, or significantly less than the present epoch. The results are therefore valid regardless of whether the collapse has just begun or is well under way at the present epoch. We see that while the modified scale factor does accelerate as in Räsanen’s model, the scale factor \( a(t) \) does not show this effect. The reason for this can be understood as follows. The region 2 is of a rather peculiar nature – it is underdense initially and becomes emptier with time, however its evolution is closely linked to that of the overdense region 1. Namely, the whole of region 2 (except its boundary at \( r = r_c \)), is dragged along with region 1 and eventually turns around, instead of expanding away to infinity like its counterpart region 3. Now, if one ignores region 2, then Räsanen’s arguments about the remaining two regions stand – one region is contracting and the other is expanding faster than the global mean, and this stand-off leads to an acceleration of the effective scale factor \( a_{\text{mod}} \), as we see in the plots of Fig. 5.2. But if we account for region 2 as well, then we bring in a counter-balancing influence of a large underdense volume which is expanding slower than average, and this reduces the accelerating influence of region 3 to the point of making the effect completely disappear. Note that at late times, the volume of region 1 contributes negligibly to the total volume, and the volumes of regions 2 and 3 are comparable.

We wish to highlight two points. First, it is very important to note the role played by the initial conditions in this entire exercise. The function \( k(r) \) is defined in a continuous fashion once the initial density, velocity and coordinate scaling are given, and \( k(r) \) then decides which shells will eventually collapse and which will not. The continuity of \( k(r) \) assures us that in models such as ours, with an overdensity surrounded by an underdensity, the underdense region will always contain a subregion in which \( k(r) > 0 \). We see therefore that the existence of region 2, is a generic feature

\[1\text{Both curves in Fig. 5.2(d) begin at } q \sim 0.5 \text{ at } t = t_i. \text{ To enhance the contrast between the curves, we have plotted them for times } t > 0.15\Delta t_{\text{turn}}. \text{ The remaining plots (Figs. 5.2(a)–5.2(c)) are plotted starting from } t = t_i.\]
not restricted to our specific choice of discontinuous initial density or vanishing initial peculiar velocities. Further, as we see in Fig. 5.2(d), it is possible to make $q$ deviate even more significantly from the EdS value than the $\sim 10\%$ effect of the first three figures, but this requires an unnaturally high value of $\delta_v \gtrsim 0.01$ (the figure has $\delta_v = 0.05$), which contradicts CMB data. Secondly, one may argue about the “naturalness” of choosing one set of regions over another set, in order to compute volumes. But this itself places the physicality of the acceleration effect into question – if one has to judiciously choose a specific set of averaging domains in order to obtain acceleration on average, then the effect would appear to be an artifact of this choice rather than something which observers would see.

5.2 Transforming to Perturbed FLRW form

We now turn to the main calculation of this section. We ask whether the LTB metric (B.1) for our model can be brought to the perturbed FLRW form with scalar perturbations, at any arbitrary stage of the collapse. Namely, we want a coordinate transformation $(t, r) \rightarrow (\tau, \tilde{r})$ such that the
metric in the new coordinates is

\[ ds^2 = -(1 + 2\varphi)d\tau^2 + a^2(\tau)(1 - 2\psi) \left( d\tilde{r}^2 + \tilde{r}^2d\Omega^2 \right), \tag{5.24} \]

with at least the conditions

\[ |\varphi| \ll 1; \quad |\psi| \ll 1, \tag{5.25} \]

being satisfied. We will ignore conditions on the derivatives of \( \psi \) and \( \varphi \) for now (see the end of Section 3.2). The scale factor is the EdS solution, with \( \tau \) as the argument. The coordinate \( \tilde{r} \) is comoving with the (fictitious) background Hubble flow, but not with the matter itself. On physical grounds we expect that this transformation should be possible as long as the gravitational field is weak and matter velocity is small. We will see below that this is exactly what happens. In the new coordinates, all matter shells labelled by \( \tilde{r} \) expand with the Hubble flow, with a superimposed peculiar velocity.

Since we want \( \tilde{r} \) to be comoving with the background, the natural choice for this coordinate would be \( \tilde{r} \sim R/a \), at least at early times. Also, we need to account for the local spatial curvature induced by the initial conditions. As an ansatz for the coordinate transformation therefore, we consider the equations

\[ \tilde{r} = \frac{R(t,r)}{a(t)} (1 + \xi(t,r)), \tag{5.26a} \]

\[ \tau = t + \xi^0(t,r), \tag{5.26b} \]

where \( \xi(t,r) \) and \( \xi^0(t,r) \) are expected to satisfy

\[ |\xi| \ll 1; \quad |\xi^0H| \ll 1. \tag{5.27} \]

This form of the transformation keeps us close to the standard gauge transformation of cosmological perturbation theory, while still accounting for the deviations in the evolution from the background FLRW, caused by structure formation. We will show that a self-consistent transformation exists, which preserves the conditions (5.25) and (5.27) for most of the evolution. We will use the metric transformation rule given by

\[ \tilde{g}_{ab}(\tilde{x}) \frac{\partial \tilde{x}^a}{\partial x^i} \frac{\partial \tilde{x}^b}{\partial x^j} = g_{ij}(x), \tag{5.28} \]

and expand to leading order in the small functions \( \xi \), \( \xi^0H \), \( \varphi \), \( \psi \) and also \( k(r)r^2 \) which, as we see from Eqn. (5.12), remains small in the entire region of interest. The relations in Eqn. (5.28) must be analysed for the cases \( (ij) = \{(tt), (tr), (rr), (\theta\theta)\} \), in each of the three regions. (The remaining cases can be shown to lead to trivial or non-independent relations.) The analysis is similar to the standard gauge transformation analysis in relativistic perturbation theory [12]. Since the calculations involved are straightforward but tedious, we will only present an outline of the calculation and highlight certain issues. At the end we will present equations for all three regions.
and numerically show that the transformation is well-behaved in the regime of interest.

The case \((ij) = (\theta\theta)\) is easily analysed and leads to

\[
\psi = \xi^0 H + \xi, \tag{5.29}
\]

The cases \((ij) = (tr)\) and \((tt)\) both require \(|\partial_t \tilde{r}| \ll 1\) for consistency (since the RHS of Eqn. (5.28) in these cases has no zero order term to balance a large \(\partial_t \tilde{r}\)). Note that since \(t\) is the proper time of each matter shell, the quantity \(\partial_t \tilde{r}\) is simply the velocity of matter in the \((\tau, \tilde{r})\) frame (which is comoving with the Hubble flow). In other words,

\[
\tilde{v} \equiv \frac{\partial \tilde{r}}{\partial t}, \tag{5.30}
\]

is the radial comoving peculiar velocity of the matter shells in the \((\tau, \tilde{r})\) frame. We will soon see that whereas the quantities \(\xi\) and \(\xi^0\) behave roughly as \(\sim (H_0 r)^2\), the peculiar velocity \(a \tilde{v}\) behaves roughly as \(\sim (H_0 r)\). We will therefore treat \((a \tilde{v})^2\) as a small quantity of the same order as \(\xi\), etc.

The case \((ij) = (tr)\) then leads to

\[
\xi^0 \tilde{v} = a \tilde{v} R', \tag{5.31}
\]

the case \((ij) = (tt)\) gives

\[
\varphi = -\dot{\xi}^0 + \frac{1}{2} (a \tilde{v})^2, \tag{5.32}
\]

and the case \((ij) = (rr)\) gives

\[
\xi' = \frac{1}{2} \left( k(r) r^2 + (a \tilde{v})^2 \right) \left( \frac{R'}{R} \right). \tag{5.33}
\]

The equations (5.29), (5.33), (5.31), and (5.32) are valid in the entire range \(0 < r < r_v\), provided the peculiar velocity remains small in magnitude. The comoving peculiar velocity is given by

\[
\tilde{v} = \partial_t \left( \frac{R}{a} \right), \tag{5.34}
\]

where we have assumed for consistency that \(|\partial_t (R/a)| \ll 1\) and have dropped the term \((R/a) \dot{\xi}\) since it is expected to be of higher order than \(\partial_t (R/a)\). (This can be seen from simple dimensional considerations – we have \(\partial_t (R/a) \sim HR/a\), and since, from Eqns. (5.33) and (5.12), \(\xi \sim (HR)^2\), we also have \((R/a) \dot{\xi} \sim (HR)^3/a\).) We will see that these conditions do indeed hold for most of the evolution, throughout the region of interest.

---

1This corrects an error in Eqn. 35 of Paper 4. I am grateful to Karel Van Acoleyen for pointing this out to me.
Since region 1 corresponds to a homogeneous solution, the integrals in Eqns. (5.31) and (5.33) can be analytically performed. Since $R$ has the structure $R = ry_1(u(t))$, we have $a\tilde{v} = r(\dot{y}_1 - y_1 H)$ and Eqn. (5.31) then leads to

$$\xi_0 = \frac{1}{2} \tilde{v} R,$$

after setting an arbitrary function of time to zero, while Eqn. (5.33) gives

$$\xi = \frac{r^2}{4} \left[ \frac{\delta_s}{a_i} H_0^2 + (\dot{y}_1 - y_1 H)^2 \right],$$

after setting another arbitrary function of time to zero\(^3\). The peculiar velocity can be explicitly calculated to be

$$a(t)\tilde{v}(t,r) = (H_0 r)^{1/2} \frac{\sin u}{1 - \cos u} - \frac{2}{3} \frac{1 - \cos u}{u - \sin u + B},$$

where the various functions are defined in Eqns. (5.13), and we have defined the constant $B$ by

$$B = \frac{2H_0 t_i}{1 + \delta_s} \left( \frac{\delta_s}{a_i} \right)^{3/2} - (u_i - \sin u_i).$$

In the rest of this section we will use the parameter values listed in Table 5.1. In Fig. 5.3 we have plotted $a\tilde{v}/(H_0 r)$ in region 1. We see that this dimensionless quantity remains of order $\sim 1$.

\(^3\)Note that it might be more meaningful to fix the two arbitrary functions of time $\xi(t,0)$ and $\xi^0(t,0)$, by requiring that $\xi(t,r_c)$ and $\xi^0(t,r_c)$ vanish. This would be in line with the shell $r = r_c$ expanding like the flat EdS background. However, this complicates some of the expressions we evaluate, and does not change the order of magnitude of any of the final results. Hence we will continue to assume that the transformation functions $\xi$ and $\xi^0$ vanish at $r = 0$ rather than at $r = r_c$. 

Figure 5.3: The quantity $a\tilde{v}/(H_0 r)$ in region 1, plotted using parameter values from Table 5.1. Since $(H_0 r_*/c) \sim 0.001$, the peculiar velocity $a\tilde{v}$ remains small.
throughout the evolution. For our choice of \((H_0 r_*/c) \sim 10^{-3}\), which corresponds to an overdensity spanning a few Mpc today, the peculiar velocity is of order \(\sim 10^{-3}\) in region 1.

Direct calculation shows that \(\varphi\) and \(\psi\) are equal and given by

\[
\varphi = \psi = \frac{1}{4} \left( \frac{\delta_s}{a_i} \right) (H_0 r)^2 \left[ 2 \left( \frac{1}{1 - \cos u} - \frac{4}{9} \frac{(1 - \cos u)^2}{(u - \sin u + B)^2} \right) \right],
\]

It is not hard to see that for our parameter choices, \(\varphi\) and \(\psi\) remain of order \(\sim (H_0 r)^2 \sim 10^{-6}\) for most of the evolution (the \(1/(1 - \cos u)\) factor will start becoming significant only at very late times which are larger than \(t_0\) for our parameter choices). The fact that \(\varphi = \psi\) is in fact a general result which follows from the absence of anisotropic stresses in the problem. It can be shown (see e.g. Refs. [35, 12]) that the difference between \(\varphi\) and \(\psi\) is governed by the stress-tensor component \(T_{\theta\theta}\) which vanishes for the spherically symmetric dust we are considering. This is fortunate, since the form of \(\xi\) in regions 2 and 3 is complicated, and is cumbersome to evaluate numerically. All we need however is \((a\tilde{v})\) which can be directly evaluated, and \(\xi^0\) which can be found after one integration in (5.31). These are sufficient to determine \(\varphi\) and the form for \(\psi\) immediately follows, assuming that \(\varphi\) and \(\psi\) vanish at the same radius \(r\) (which in our case is \(r = 0\)).

### 5.2.2 The transformation in regions 2 and 3

For the calculation in regions 2 and 3, the integrals involved cannot be computed analytically. We will therefore display the expressions we obtain for \(\tilde{v}\) and \(\xi^0\), and plot the results of numerically computing \(\varphi = \psi\) from these quantities.

- **region 2** \((r_*/r < r_c)\):

  In region 2 we have

  \[
  \tilde{v} = \frac{R}{a} [C(t, r) - H],
  \]

  \[
  \xi^0(t, r) = \xi^0(t, r_*) + a(t) \int_{r_*}^r \tilde{v}(t, \bar{r}) R'(t, \bar{r}) d\bar{r},
  \]

  where \(\xi^0(t, r_*)\) is computed from Eqn. (5.35) at \(r = r_*\), and we have defined

  \[
  C(t, r) \equiv \frac{H_i \sin \alpha}{(1 - \cos \alpha)^2} \frac{2\varepsilon^{3/2}}{1 + \varepsilon}.
  \]

  \(\varphi\) must now be computed using (5.32).

---

4This is independent of whether we use \((t, r)\) or \((\tau, \bar{r})\), since the angular coordinates are not affected by this transformation.
Figure 5.4: The peculiar velocity $a\tilde{v}/c$ in regions 2 and 3 using parameter values from Table 5.1.

- region 3 ($r_c < r < r_v$):

  The analysis is very similar to that in region 2. We find

  \[ \tilde{v} = \frac{R}{a} [D(t, r) - H], \]  
  \[ \xi^0(t, r) = \xi^0(t, r_c) + a(t) \int_{r_c}^{r} \tilde{v}(t, \bar{r}) R'(t, \bar{r}) d\bar{r}, \]  

  where $\xi^0(t, r_c)$ is obtained from (5.41), evaluated in the limit $r \to r_c^-$, and we have defined

  \[ D(t, r) \equiv \frac{H_i \sinh \eta}{(\cosh \eta - 1)^2} \frac{2|\varepsilon|^{3/2}}{1 + \varepsilon}, \]  

  In Fig. 5.4 we have plotted the velocity $a\tilde{v}/c$ in regions 2 and 3 for a range of time. It can be shown that at the order of approximation we are working at, $a\tilde{v}$ changes sign at $r = r_c^5$. In Fig. 5.5 we plot $\varphi$. We see that this function is well behaved and remains small for the entire region of interest (in space and time). Hence the perturbed FLRW picture is indeed valid for this system, even though each region by itself appears to be very different from FLRW in the synchronous coordinates comoving with the matter. Due to numerical difficulties close to the initial time $t = t_i$, we have plotted the time axis starting from $t = 50t_i$. Note that the magnitude of $\varphi$ is sensitive to the overall size of the region, determined by the value of $R(t, r_v)$. For our parameter choices given in Table 5.4, the size of the region at the present epoch is $\sim 33$Mpc, which is a typical size for observed voids. The dependence is roughly $(HR)^2$, and hence a void which is about 10 times larger in length scale than the above value, would have metric functions about 100 times larger.

5Recall $\varepsilon(r_c) = 0$ and hence this shell expands exactly like the EdS background. The metric in the $(\tau, \bar{r})$ coordinates will not be exactly EdS at $r = r_c$, due to our unusual choice of normalisation for $\xi$ and $\xi^0$ at $r = 0$. This does not pose any problem for our conclusions.
Figure 5.5: The metric function $\varphi(t, r)$ in regions 2 and 3 using parameter values from Table 5.1. The time axis begins at $t = 50t_i$.

We end this subsection by noting the following. It is known that simply having a metric of the form (5.24) with only the magnitude of the perturbations being small, is not enough to guarantee consistency with Einstein’s equations written as a perturbation series; additional constraints on the derivatives of these functions must be satisfied. These constraints, given in e.g. Ref. [27], take the form (for the metric (5.24) with $\tilde{\psi} = -\varphi$),

$$\left| \frac{\partial \varphi}{\partial t} \right|^2 \ll \frac{1}{a^2} \nabla^\alpha \varphi \nabla_\alpha \varphi, \quad (\nabla^\alpha \varphi \nabla_\alpha \varphi)^2 \ll \left( \nabla^\alpha \nabla^\beta \varphi \right) \nabla_\alpha \nabla_\beta \varphi,$$

(5.46)

where $\alpha, \beta = 1, 2, 3$, and $\nabla_\alpha$ is the spatial covariant derivative associated with the flat 3-space metric. On dimensional grounds, treating $\varphi \sim (HR)^2 \ll 1$, $\partial_t \sim H$ and $\nabla \sim aR^{-1}$, it is easy to see that these constraints will be satisfied by our solution. This should also be expected since we started from an exact solution of the Einstein equations and performed a self-consistent coordinate transformation.

5.2.3 The magnitude of the backreaction

One can now legitimately ask the question, “How large is the effect of the small metric inhomogeneities?” Naively, one would argue that small inhomogeneities must lead to small effects. Indeed, the question of the magnitude of the backreaction in the Newtonianly perturbed FLRW setting has been investigated by Behrend, et al. [49] in the linear and quasilinear regimes, and they find that corrections to the FLRW equations remain at the level of one part in $10^5$. However, what we are dealing with is a situation in which the matter perturbations are completely nonlinear, and it is not a priori clear that the same arguments would carry through. Indeed, we saw in section 2 that the deceleration parameter $q$ deviated from its EdS value by about $\sim 10\%$. Here we give an argument based on dimensional considerations supplemented with realistic numbers, which will show that this effect is scale dependent, and is not expected to be present if a sufficiently large averaging scale is chosen.
In the following we will work at the present epoch $t_0$. Consider a model situation similar to the one we have been considering so far, such that at present epoch the physical extent of the overdense region is $R_\ast$, and that of the underdense is $R_v$. For order of magnitude estimates, we assume that in the perturbed FLRW metric (5.24) (which is valid for this system provided $H_0 R_v \ll 1$), $\varphi \sim -\tilde{\psi}$. Also assume that the density contrast in the overdense region is $\delta_{\ast 0}$ and that in the underdense region is $\delta_{v 0}$, where we take $\delta_{\ast 0}$ and $\delta_{v 0}$ to be constant in space, which is fine for an order of magnitude estimate. The backreaction in the Buchert approach contains, among other terms, the spatial average of the quantity $\nabla^2 \varphi$ which appears in the spatial curvature \[40, 49\], where $\nabla^2$ is the Laplacian operator for the flat 3-space metric. The spatial curvature has the structure

$$R \sim \frac{1}{a^2} \left[ (\#_1) \nabla^2 \varphi + (\#_2) \varphi \nabla^2 \varphi + (\#_3) (\nabla \varphi)^2 \right],$$

(5.47)

where $\#_1, \#_2, \#_3$ are constants whose values are irrelevant for this order of magnitude argument. Due to the Einstein equations in the small scale Newtonian approximation, the leading order effect in the nonlinear regime, comes from $\nabla^2 \varphi$ which satisfies

$$\nabla^2 \varphi \sim \begin{cases} H_0^2 \delta_{\ast 0}, & \text{overdense region,} \\ H_0^2 \delta_{v 0}, & \text{underdense region.} \end{cases}$$

(5.48)

Consider the situation when, at present epoch, $R_\ast \sim 6\text{Mpc}$, $R_v \sim 30\text{Mpc}$, $\delta_{\ast 0} \sim 10^2$ and $\delta_{v 0} \sim -0.9$. These are typical numbers for clusters of galaxies and voids. It is straightforward to now show that the spatial average of $\nabla^2 \varphi$ over a domain comprising the overdense and underdense region, works out to be

$$\langle \nabla^2 \varphi \rangle \sim \frac{H_0^2}{R_\ast^2 + R_v^3} \left[ R_\ast^3 \delta_{\ast 0} + R_v^3 \delta_{v 0} \right],$$

$$\simeq -0.1 H_0^2.$$  

(5.49)

It would appear therefore, that this spatial average of $\nabla^2 \varphi$ (which is usually neglected) thus turns out to be a significant contributor to the backreaction. (In fact it is the most significant contributor, since the other terms are clearly of at least one higher order in the small quantity $(H_0 R_v)^2$, for such a model.)

As we now argue, however, the above effect can be deceptive, and is really scale dependent. Let the initial density contrasts in the overdense and underdense regions be $\delta_{\ast i}$ and $\delta_{v i}$ respectively, so that $\delta_{\ast i}, |\delta_{v i}| \ll 1$. If $M_{\ast i}$, $M_{v i}$, $M_\ast$ and $M_v$ are the masses at initial time and today, in the overdense and underdense region respectively, and $\rho_i$ and $\rho_0$ are the values of the background density at initial time and today, then at initial time

$$M_{\ast i} \approx \rho_i (a_i R_\ast)^3 = \rho_0 R_\ast^3, \quad M_{v i} \approx \rho_0 R_v^3,$$

(5.50)
and at present time,

\[ M_s = \rho_0(1 + \delta_s)R_s^3 > M_{si}, \quad M_v = \rho_0(1 + \delta_v)R_v^3 < M_{vi}. \]  \hspace{1cm} (5.51)

We now make the crucial observation that if the averaging scale is large enough, and we are counting several such “pairs” of overdense and underdense regions, then the mass ejected from the underdense region must have all gone into the overdense region. It is then easy to show, that

\[ \delta_s R_s^3 \approx -\delta_v R_v^3, \]  \hspace{1cm} (5.52)

which means that, just like in the linear theory, the average of \( \nabla^2 \phi \) is expected to be negligible on such a scale. In the real universe, we do expect that the averaging scale must be at least of the order of the homogeneity scale, and on such a scale we will be sampling several pairs of overdense and underdense regions. The only cumulative effects that may arise with such a choice of scale are from terms such as \( (\nabla \phi)^2 \), which as we mentioned earlier, are of one higher order in the perturbation and will give effects of the size \( \sim H_0^2(H_0 R_v)^2 \ll H_0^2 \). (For a demonstration of the scale dependence of the effect, see e.g. the work of Li and Schwarz, the first paper in Ref. [52].)

5.3 Backreaction during nonlinear growth of structure

We can do better than the estimates for the magnitude of the backreaction during late stages of structure formation. In chapter 4 we have already developed a formalism in place to calculate the backreaction whenever the metric has the perturbed FLRW form (irrespective of matter inhomogeneities). We can use this procedure on our LTB model in the \((\tau, \tilde{r})\) coordinates to explicitly evaluate the backreaction functions.

The expressions for the backreaction in Eqns. (4.42) were derived under the requirement that the averaging operation be free of gauge related ambiguities, in linear perturbation theory. However, the actual conditions used to derive Eqns. (4.42) only depended on the fact that one is working with leading order effects in the metric perturbations. In particular, a key step was the transformation between the metric in the conformal Newtonian gauge (in Cartesian spatial coordinates) and the corresponding volume preserving form. In the present context, the same transformation remains valid at the leading order, and hence the expressions (4.42) for the backreaction are physically relevant here as well. We emphasize that this truncated averaging operation remains valid even at late times since the weak field approximation for gravity works well during the nonlinear phase of structure formation.

Since our numerical results are in terms of the LTB variables \((t, r)\), where \(r\) “comoves” with the matter but not with the FLRW background, we need to reexpress the averaging operation (4.27) in terms of these variables. It is easy to show that, at the leading order, the average of a scalar
s(t,r) defined in Eqn. (4.27) can be written as

$$\langle s \rangle = \frac{3}{(a(t)L)^3} \int_0^{r_L(t)} s R^2 R' dr,$$

where the function $r_L(t)$ solves the equation

$$R(t,r_L(t)) = a(t)L.$$

Eqn. (5.53) gives the average of $s$ over a single domain centered at the origin, which is what we will restrict ourselves to in this section. There are two reasons behind this choice: firstly this is the most natural choice given the symmetry of the system, and secondly since our model is constructed as a “typical representative” of nonlinear inhomogeneities, it makes sense to use averages over the single central region as representative of more general averages. As discussed in section 5.1, we are constrained to consider values $r < r_v$, due to unphysical shell crossing singularities in the region beyond. For this reason the largest value of $L$ which we can choose is $L = r_v$, which then ensures $r_L(t) < r_v$ since $r_L(t)$ is a decreasing function for this choice. This gives us an averaging scale of $L = 23.5$ Mpc (comoving with the FLRW background), which is smaller than the more realistic expected value of $\sim 100h^{-1}$Mpc. One consequence is that our model does not strictly satisfy the condition that the potentials $\varphi$ and $\psi$ and their spatial and time derivatives should average to zero, as was assumed in chapter 4. One can check that the actual average values of the form $\langle (\partial_A \varphi)^2 \rangle$ are small ($\lesssim 10\%$) compared to terms like $\langle (\partial_A \varphi)^2 \rangle$ which are needed in the backreaction calculations, for all times, although it turns out that the time derivatives satisfy $\langle \dot{\varphi}^2 \rangle \sim \langle (\varphi)^2 \rangle$, throughout the evolution. However, since the averaging scale chosen here is large enough to encompass all the inhomogeneity of this system, we expect that our estimates for the backreaction functions in Eqns. (4.42) are fairly representative.

Consider now the function $\beta$. Since $\beta$ satisfies the Poisson equation $\nabla^2 \beta = \varphi - 3\psi$ on a flat 3-space background, with no nontrivial solutions of the corresponding homogeneous equation allowed, we can directly write the solution for $\beta$ in terms of the background radial coordinate $\tilde{r}$ as,

$$\beta(\tau, \tilde{r}) = -\frac{1}{4\pi} \int |\vec{q}(\tau, \vec{y})| d^3 y$$

$$\quad = -\frac{1}{\tilde{r}} \int_0^\tilde{r} q y^2 dy - \int_\tilde{r}^\infty q y dy,$$

where $q \equiv -2\varphi$ (we have set $\varphi = \psi$ at leading order) and the integration is over the spatial coordinates comoving with the background. The following relations turn out to be useful in the
calculations,
\[ \partial_t \beta = \frac{1}{\tilde{r}^2} \int_0^{\tilde{r}} q y^2 dy, \quad (5.56a) \]
\[ \partial_t^2 \beta = q - \frac{2}{\tilde{r}} \partial_t \beta, \quad (5.56b) \]
\[ \partial_t^3 \beta = \frac{6}{\tilde{r}^2} \partial_t \beta - \frac{2}{\tilde{r}} q + \partial_t q. \quad (5.56c) \]

We will need the quantities \( \partial_t \beta, \partial_t^2 \beta \) and \( \partial_t^3 \beta \) as functions of the LTB variables \((t, r)\), which can be done by replacing \( \tau \) and \( \tilde{r} \) at leading order by \( t \) and \( R(t, r)/a(t) \) respectively. This gives us (treating \( q \) as a function of \( t \) and \( r \)),
\[ (\partial_t \beta)(t, r) = \frac{1}{aR^2} \int_0^{R} q R^2 R' dr, \quad (5.57a) \]
\[ (\partial_t^2 \beta)(t, r) = q - \frac{2}{R^3} \int_0^{R} q R^2 R' dr, \quad (5.57b) \]
\[ (\partial_t^3 \beta)(t, r) = \frac{6a}{R^4} \int_0^{R} q R^2 R' dr - \frac{2a}{R} q + \frac{a}{R} q', \quad (5.57c) \]

where in the last equation we have used the fact that, at leading order,
\[ \partial_t q = \left( \frac{a}{R'} \right) q'. \]

Also, noting that the time derivatives in Eqns. (4.42) are taken keeping the coordinate \( \tilde{r} \) fixed, we have at the leading order, and in terms of \( t \approx \tau \)
\[ (\partial_t \dot{\beta})(t, r) = \frac{1}{aR^2} \int_0^{R} \dot{q} R^2 R' dr, \quad (5.58a) \]
\[ (\partial_t \ddot{\beta})(t, r) = \frac{1}{aR^2} \int_0^{R} \ddot{q} R^2 R' dr, \quad (5.58b) \]
\[ (\partial_t^2 \dot{\beta})(t, r) = \dot{q} - \frac{2}{R^3} \int_0^{R} \dot{q} R^2 R' dr, \quad (5.58c) \]

which follow from Eqns. (5.56). The expressions in Eqns. (4.42), rewritten in terms of the LTB proper time \( t \) and valid at leading order in the various small quantities, reduce to
\[ P^{(1)} = \left[ 2\langle (\dot{\varphi})^2 \rangle - \langle (\partial_t \dot{\beta})^2 \rangle - 2\langle (1/\tilde{r}^2)(\partial_t \dot{\beta}) \rangle \right], \]
\[ S^{(1)} = -\frac{1}{a^2} \left[ 6\langle (\partial_t \dot{\beta})^2 \rangle - \langle (\partial_t^2 \beta)^2 \rangle - 6\langle (\partial_t \beta - \tilde{r}\partial_t^2 \beta)^2 / \tilde{r}^4 \rangle \right], \]
\[ P^{(1)} + P^{(2)} = -2H \left[ \langle (\partial_t \dot{\beta})(\partial_t^2 \dot{\beta}) \rangle + 2\langle (1/\tilde{r}^2)(\partial_t \beta)(\partial_t \dot{\beta}) \rangle \right], \]
\[ S^{(2)} = \langle (\partial_t \ddot{\beta}) + H\partial_t \dot{\beta})(a^2 H\partial_t \dot{\beta} - \partial_t \varphi) \rangle, \quad (5.59) \]

where the angular brackets are now defined by Eqn. (5.53) and the various integrands can be read off using Eqns. (5.57), (5.58) and the results \( \partial_t \varphi \approx (a/R') \varphi' \) and \( \tilde{r} \approx (R/a) \), at leading order. Figs.
Figure 5.6: The evolution of $|S^{(1)}|$, normalised by $6H^2$. Also shown is a hypothetical curvature-like correction, evolving like $\sim a^{-2}$.

5.6 and 5.7 show results of numerical calculations performed with Mathematica. Fig. 5.6 shows the evolution of $-S^{(1)}/(6H^2)$, the dominant correction, as a function of the scale factor. The dotted line shows a hypothetical curvature-like correction. Clearly the evolution of the actual backreaction is more complicated, due to significant evolution of $\varphi$. Note that the largest value of $|S^{(1)}/H^2|$ computed here is $\sim 10^{-6}$, whereas estimates using linear theory in chapter 4 suggested that this value should be around $\sim 10^{-4}$. This discrepancy highlights an issue we noticed earlier in chapter 4, namely that nonlinear inhomogeneities on small scales do not contribute significantly to the backreaction. Our model has no large scale inhomogeneities and underestimates the backreaction. Reassuringly, accounting for the deficit only requires a calculation in linear theory, such as the one in chapter 4. Fig. 5.7 shows the evolution of the remaining integrals, also normalised by $6H^2$. An initial rapid decay of $P^{(1)}/H^2$ starting from values of $\sim 10^{-8}$ has not been shown, in order to enhance the contrast in the late time behaviour of the three functions. The other functions remain subdominant compared to $P^{(1)}$ at the early times not shown.

This completes the picture of the effects of backreaction in the cosmic expansion history. Our covariant and self-consistent calculation of the backreaction in this spherical collapse model establishes that inhomogeneities have an insignificant impact on the average cosmological dynamics. In particular, the observed cosmic acceleration cannot be explained by the averaging of inhomogeneities. Our nonlinear dust model can be regarded as representing a realistic situation, because it has an overdensity-void structure, and departure from sphericity, tidal interactions, and second order corrections are not expected to introduce any significant change in the results. What appears true in general is that as long as peculiar velocities remain small, as seems to be the case in the
real universe, a description as a perturbed FLRW model is valid, and this keeps the backreaction small.

Chapter summary and discussion:

In this chapter we addressed the question of whether or not the perturbed FLRW form for the metric remains a valid approximation at late times in the cosmological history. Heuristic arguments such as those presented by Ishibashi and Wald [27] indicate that this should in fact be the case. We have studied this issue in the context of an exactly solvable, fully relativistic toy model which allows us to track the behaviour of both the metric and matter perturbations unambiguously, well into the regime where the matter perturbations have become completely nonlinear.

In these late stages of structure formation, a perturbation theory in the density contrast $\delta$ is no longer valid. As expected from standard Newtonian analyses however [11], we found that a perturbative expansion in peculiar velocities remains valid until fairly late times. Our model parameters were chosen to reflect typical inhomogeneities on scales of tens of Mpc. While the perturbative description of our model eventually breaks down (see e.g. the $(1 - \cos u)^{-1}$ factor in the expression (5.39) for $\varphi$), the model itself is not expected to reflect realistic conditions at very late times. For example, real clusters of galaxies eventually achieve virial equilibrium due to random motions of the galaxies, rather than collapsing to a singularity. Our pressureless model cannot accommodate such a behaviour and must be abandoned beyond the point where virialisation is expected to occur. Heuristic reasoning further suggests that in the virial phase, the peculiar velocity with respect to the Hubble flow will essentially be $|a\ddot{v}| \sim HR_{\text{vir}}$ where $R_{\text{vir}}$ is the virial
radius, and the backreaction which is controlled by $(a\dot{v})^2$ should therefore decrease with time in this phase. Our model parameters were chosen so that the physical radius of the overdensity at $t = t_0$ is close to typical virial radii for corresponding realistic clusters, and it is therefore meaningful to stop the model evolution at $t = t_0$.

Additionally we also computed the backreaction in our model using the formalism developed in chapter 4. We emphasize that this formalism is fully covariant, and is guaranteed to yield results which are coordinate invariant at the leading order. The metric perturbation function $\varphi$ does not satisfy the equations of linear perturbation theory at late times, and neither is it expected to. This is reflected in the complicated behaviour of $S^{(1)}$ seen in Fig. However it is still true that the metric can be brought to the perturbed FLRW form, which is all that is required for a reliable calculation of the backreaction. And finally, since the nonlinear inhomogeneities do not contribute as much to the backreaction as those on scales $k \sim k_{eq}$, our entire discussion regarding convergence of the iterative calculation (see chapter 4 summary and discussion) is expected to remain approximately valid even after accounting for nonlinearity. This is important since it establishes that cosmological perturbation theory is stable against including the effects of the backreaction from averaging inhomogeneities.
Chapter 6

Conclusions

Backreaction as an explanation for the late time cosmic acceleration would have truly been the most conservative solution to the dark energy problem. Not only would it have resolved the discrepancy between observed data and what is generally considered to be “ordinary physics”, but more importantly it would have obviated the need for statements such as “We do not understand what 70% of the universe is made of”. Needless to say, this approach has captured the imagination of many cosmologists, and the (possibly incomplete) list of references cited in the Introduction is testimony to this fact. Due to the technically challenging nature of the problem however, it is very important to proceed systematically and rigorously while determining the size and nature of the effects of backreaction. This is especially true since order of magnitude estimates on the one hand indicate that the effect can never be large [27], while simple toy models indicate exactly the opposite [37]. It has been our goal in this thesis to provide a reliable and self-consistent calculation to estimate the nature of the backreaction.

We have used a fully covariant averaging formalism, adapted to the specific needs of cosmology, and developed it further to allow estimates of the effects of perturbative metric inhomogeneities in a gauge invariant manner. We have used this formalism in the linear regime of cosmological perturbations and have shown that linear PT is stable against the inclusion of backreaction effects. We have also demonstrated using simple but realistic toy models of nonlinear structure formation, that (a) a description of the universe in terms of the perturbed FLRW metric is indeed valid at late times as order of magnitude estimates suggest, and (b) the backreaction due to these perturbative metric inhomogeneities remains small, irrespective of whether the matter inhomogeneities are in the linear or nonlinear regime. One might argue that more precise nonlinear calculations, e.g. in numerical simulations, might enhance the effect. While this might be true, this enhancement is not expected to be by orders of magnitude, since the transfer functions routinely found in nonlinear calculations do not deviate drastically enough from the linear theory ones [15], to lead to an order

\footnote{Including dark matter would take this number up to approximately 95%. If we further take into account the fact that the only component which we directly measure with great precision is the CMB radiation, then one might say that we truly understand only $\sim 10^{-4}$ of the universe!}
unity backreaction. The cosmological expansion history therefore appears to be largely insensitive to the presence of the backreaction terms.

Of course, this means that backreaction from averaging cannot solve the dark energy problem. There have been claims in the literature which suggest otherwise; we will comment on a few of these in what follows. Kolb et al. [51] study a model with very large nonlinear inhomogeneities (a $\sim 1$ Gpc sized void) to claim that in such a case the perturbed FLRW form for the metric cannot be recovered at all times, and that backreaction effects can be large. Our results do not contradict this, since we have seen that the quantity $HR$ controls the late time perturbative expansion, and a $\sim 1$ Gpc size inhomogeneity with a density contrast of order unity today will imply a breakdown in the perturbation theory. What is important to bear in mind however, is whether such large inhomogeneities are generic. If the universe is dominated by Gpc sized voids then our analysis will indeed break down; but this does not appear to be the case observationally.

Wiltshire [38] models the scale dependence of inhomogeneities in a nonperturbative manner using Buchert’s formalism. The assumption is that the clocks of observers in voids run at very different rates from those of observers in the “walls” surrounding voids. The argument then is that since the universe today is dominated by voids (of sizes ranging from 30-50$h^{-1}$ Mpc [56]), “wall” observers such as ourselves are atypical. The difference in clock rates between average observers and wall-observers is then fitted to data and can account for several sets of observations. This was a rather simplistic picture of Wiltshire’s model, which is actually far more involved in its construction. However, Wiltshire’s basic final result, the clock rate difference mentioned above, at least superficially does not agree with our results, since in our toy model we explicitly see that clock rate differences for any pair of observers are governed by the metric potential $\varphi$ which remains perturbatively small. It is not clear where this discrepancy arises from, and this matter is further complicated by the fact that Wiltshire’s model does not have a concrete setup in which to follow the evolution of inhomogeneities.

To date, perhaps the most physically clear attempts to explain e.g. supernova data without invoking dark energy, have involved the “non-Copernican” models which place us at a special location in the universe [57]. At the risk of some repetition, we will briefly discuss a few issues that arise in this context. As we discussed in the Introduction, a lot of attention has been focused on studying light propagation in the inhomogeneous spacetime of a void modelled by the LTB metric. These calculations involve standard GR, without any averaging and the associated complications. As far as supernova data are concerned, the calculations involve light propagation in a known metric, and the two functional degrees of freedom in LTB models can be suitably employed to construct a void geometry (which will typically span several hundred $h^{-1}$ Mpc) which fits the supernova data without any dark energy/cosmological constant. This apparently straightforward result leads to a host of questions. Do such “super”-voids exist in the universe? Galaxy surveys at least do not show any evidence of this, although some analyses of CMB data, specifically using the integrated Sachs-Wolfe effect, have thrown up some interesting results [83]. Even if such voids
exist, is there any independent evidence that we reside close to the center of one? (Were we not close to the center, the CMB dipole would have been much higher.) Current data is not sufficient to answer this question, although future surveys are likely to make some headway in this issue [60]. On the theoretical front, how does one analyze the growth of structure in a geometry which contains a nonperturbative void? This question has begun to receive some attention only recently (see the papers by Zibin, Moss and Scott, and Clifton, Ferreira and Zuntz, in Ref. [57]). The jury is therefore still out on whether or not “dark energy” is simply a consequence of our being in a special location. The most encouraging aspect of the “void instead of dark energy” scenario is that it is amenable to observational testing, which is very likely to occur in the foreseeable future.

To conclude, backreaction from inhomogeneities cannot solve the dark energy problem. Void-like inhomogeneities, while having the potential to solve this problem, await further observational evidence. And for now, we still do not understand what (at least) 70% of the universe is made of. The future continues to hold significant challenges for cosmology and theoretical physics.
Appendix A

Basics of FLRW cosmology

The unique line element for a spacetime which admits homogeneous and isotropic spatial sections (i.e. 3-surfaces of constant curvature) is given by

\[ ds^2 = -d\tau^2 + a^2(\tau) \left( \frac{dr^2}{1 - kr^2} + r^2 d\theta^2 + r^2 \sin^2 \theta d\phi^2 \right). \]  

(A.1)

Here \( k \) is a parameter related to the spatial curvature of 3-space \((^3R = 6k/a^2)\) and \( a(\tau) \) is the single dynamical function - the scale factor - which describes the evolution of the universe. This thesis will mostly deal with the case \( k = 0 \) corresponding to spatially flat sections. The line element is written in coordinates which are comoving with those observers who see a homogeneous and isotropic 3-space. The coordinate \( \tau \) is called cosmic time. The subscript 0 will refer to the present epoch \( \tau_0 \) and the scale factor is always normalized so that \( a(\tau_0) \equiv a_0 = 1 \).

The energy-momentum tensor of matter is taken to describe a perfect fluid, which is homogeneous and isotropic in its rest frame (which therefore coincides with the comoving reference frame of Eqn. (A.1)). This has the form

\[ T^a_{\, b} \equiv \text{diag} (-\rho(\tau), p(\tau), p(\tau), p(\tau)), \]  

(A.2)

with \( \rho(\tau) \) the energy density and \( p(\tau) \) the pressure as measured in the fluid rest frame.

The Einstein equations \( E_{ab} = 8\pi GT_{ab} \), with \( E_{ab} \) the Einstein tensor constructed using \( (A.1) \) and \( T_{ab} \) as given above, reduce to

\[ \frac{\dot{a}}{a} \left( \frac{\dot{a}}{a} \right)^2 \frac{8\pi G}{3} \rho, \]  

\[ \frac{\ddot{a}}{a} = - \frac{4\pi G}{3} (\rho + 3p), \]  

(A.3a)

(A.3b)

with the overdot referring to a time derivative \( \dot{} \equiv \partial_\tau \). We will refer to Eqns. (A.3a) and (A.3b) as

\[^1\text{Eqn. (A.3b) is actually obtained after substituting (A.3a) in the Einstein equation for } \dot{a}.\]
the Friedmann equation and the acceleration equation respectively. Together with an equation of state $p = p(\rho)$, these equations can be solved to get an expression for $\rho(a)$ and $a(\tau)$. (One could also use the Friedmann equation (A.3a) in conjunction with the continuity equation $d(\rho a^3) + pd(a^3) = 0$ which follows from $T_{\mu\nu}^{\alpha} = 0$, to solve for $\rho(a)$ and $a(\tau)$.) For example, with the simple assumption $p = w\rho$ for constant $w$, it can be shown that $\rho \propto a^{-3(1+w)}$, and further if $k = 0$ then

$$a(\tau) \propto \tau^\frac{2}{3(1+w)}, \quad w \neq -1$$

$$\propto \exp(\mathcal{H} \tau), \quad w = -1,$$

where $H$ is constant. Some commonly occurring values of the equation of state parameter $w$ are $w = 0$ (pressureless matter or dust) and $w = 1/3$ (radiation). The case $w = -1$ corresponds to the cosmological constant $\Lambda$, which gives rise to a constant energy density $\rho_\Lambda = \Lambda/(8\pi G)$, and consequently a value of $H = \sqrt{\Lambda/3}$ in Eqn. (A.4) above.

In general we define $H \equiv (\dot{a}/a)$, called the Hubble parameter. Its value at the present epoch ($H_0$) is called the Hubble constant, and is usually parametrized as $H_0 = 100h$ km s$^{-1}$ Mpc$^{-1}$, with the current consensus on its numerical value being $0.5 \lesssim h \lesssim 0.8$. Define the critical density $\rho_c$ at the present epoch as $\rho_c \equiv (3H_0^2)/(8\pi G)$, and also the quantities $\Omega_i \equiv \rho_i/\rho_c$ where $\rho_i$ is the value of the density of the $i^{th}$ matter component, at the present epoch. Here $i$ labels the components radiation ($R$), baryons ($b$), dark matter ($DM$) and an additional “dark energy” represented by the cosmological constant ($\Lambda$). The total energy density is simply $\rho(a) = \sum_i \rho_i(a)$ and we can write $\Omega(a) \equiv \rho(a)/\rho_c$, with the value of $\Omega(a)$ at the present epoch being $\Omega \equiv \Omega(a = 1) = \sum_i \Omega_i$. The Friedmann equation (A.3a) can be written as

$$H^2(a) = H_0^2 \left[ (1 - \Omega) a^{-2} + \Omega_R a^{-4} + (\Omega_b + \Omega_{DM}) a^{-3} + \Omega_\Lambda \right],$$

where the cases $k = 0, k > 0, k < 0$ correspond respectively to $\Omega = 1, \Omega > 1, \Omega < 1$, and in general one would have $\Omega_\Lambda = \Omega_\Lambda(a)$. The cosmological redshift $z$ of a source which emits light of wavelength $\lambda_{em}$ at time $\tau_{em}$ and is observed “here-and-now” at time $\tau_0$ with wavelength $\lambda_0$, is

$$1 + z \equiv \frac{\lambda_0}{\lambda_{em}} = \frac{1}{a(\tau_{em})}.$$  \hspace{1cm} (A.6)

Using this in (A.5) gives an expression for $H(z)$. Often it is useful to work in terms of conformal time $\eta$ defined by

$$\eta = \int^\tau d\tilde{\tau} \frac{a'}{a(\tilde{\tau})},$$

with the corresponding Hubble parameter

$$\mathcal{H} \equiv \frac{a'}{a} \equiv \frac{1}{a} \frac{da}{d\eta} = aH(a).$$  \hspace{1cm} (A.8)
Appendix B

The Lemaître-Tolman-Bondi solution

In this appendix we describe the spherically symmetric Lemaître-Tolman-Bondi (LTB) solution of the Einstein equations for matter comprising a pressureless “dust” [61]. For an arbitrary dust configuration, the metric can always be expressed in coordinates which are “synchronous” (g_{00} = −1; g_{0A} = 0) and “comoving” (world lines of the fluid elements are orthogonal to 3-space) [62]. Specifically, the LTB metric is given in the synchronous and comoving gauge, by

$$ds^2 = -dt^2 + \frac{R'^2(t,r)}{1 - k(r)r^2}dr^2 + R^2(t,r)(d\theta^2 + \sin^2 \theta d\phi^2).$$  \hspace{1cm} \text{(B.1)}

Throughout this appendix, a prime and a dot will refer to partial derivatives with respect to \(r\) and \(t\) respectively. The Einstein equations simplify to

$$\dot{R}^2(t,r) = \frac{2GM(r)}{R(t,r)} - k(r)r^2, $$ \hspace{1cm} \text{(B.2a)}

$$4\pi\rho(t,r) = \frac{M'(r)}{R'(t,r)R^2(t,r)}.$$ \hspace{1cm} \text{(B.2b)}

Surfaces of constant \(r\) are 2−spheres having area \(4\pi R^2(t,r)\). \(\rho(t,r)\) is the energy density of dust, while \(k(r)\) and \(M(r)\) are arbitrary functions that arise on integrating the dynamical equations. Solutions can be found for three cases \(k(r) < 0\), \(k(r) = 0\) and \(k(r) > 0\). The solution for \(k(r) = 0\) (the marginally bound case) has the particularly simple form

$$R(t,r) = \left(\frac{9GM(r)}{2}\right)^{1/3} (t - t_0(r))^{2/3}, \text{ for } k(r) = 0.$$  \hspace{1cm} \text{(B.3)}

Here \(t_0(r)\) is another arbitrary function arising from integration. The solution describes an expanding region, with the initial time \(t_{in}\) chosen such that \(t > t_{in} \geq t_0(r)\) for all \(r\). For the other two
In the marginally bound case is interpreted as the mass contained in a comoving shell), and then decreases back to 0 in a finite time. Use this freedom to set \(t_0(r) = \frac{GM(r)}{(k(r)r^2)^{3/2}} (\sinh \eta - \eta)\), for \(k(r) < 0\). \(R = \frac{GM(r)}{k(r)r^2} (1 - \cos \eta)\); \(M(r)\) decreases back to 0 in a finite time. In the bound case \((k(r) > 0)\), \(R(t, r)\) increases to a maximum value \(R_{\text{max}}(r)\) for each shell \(r\) and then decreases back to 0 in a finite time.

In all cases, there are two physically different free functions, although three arbitrary functions \(k, M\) and \(t_0\) appear. One of the three represents the freedom to rescale the coordinate \(r\). We use this freedom to set \(R(t_{in}, r) = R_{in}(r) = r\). To completely specify the solution, we specify the initial density \(\rho_{in}(r)\) and the function \(k(r)\) (which can be related to the initial velocity profile \(\dot{R}_{in}(r)\) using Eqn. (B.2a) evaluated at initial time). This specifies \(M(r) = 4\pi \int_0^r \rho_{in}(\tilde{r})\tilde{r}^2d\tilde{r}\) (which in the marginally bound case is interpreted as the mass contained in a comoving shell), and \(t_0(r)\) can be solved for using Eqns. (B.3), (B.4a) or (B.4b) as the case may be, at time \(t = t_{in}\). The FLRW solution is a special case and is recovered by setting \(\rho_{in} = \text{constant}, k = \text{constant}\).

### B.1 Regularity conditions

It is useful to keep in mind certain regularity conditions when constructing LTB models. In any LTB model, the functions \(M(r)\) and \(k(r)\) are to be specified by initial conditions at \(t = t_{in}\), and the choice of scaling \(R(r, t_{in}) = r\) fixes \(t_0(r)\) as

\[
t_0(r) = t_{in} - \frac{GM(r)}{(k(r)r^2)^{3/2}} S_{in}(r)\quad;\quad C_{in}(r) = \frac{|k(r)r^3|}{GM(r)},
\]

where \(S_{in}(r) = (\sinh \eta_{in}(r) - \eta_{in}(r))\) and \(C_{in}(r) = \cosh \eta_{in}(r) - 1\) for \(k(r) < 0\); \(S_{in}(r) = (\eta_{in}(r) - \sin \eta_{in}(r))\) and \(C_{in}(r) = 1 - \cos \eta_{in}(r)\) for \(k(r) > 0\).

The regularity conditions imposed on this model, and their consequences, are as follows

- **No evolution at the symmetry centre:**
  This is required in order to maintain spherical symmetry about the same point at all times, and translates as \(\dot{R}(t, 0) = 0\) for all \(t\). The right hand side of Eqn. (B.2a) must therefore vanish in the limit \(r \to 0\). Since the functions involved are non-negative, we assume that we can write

\[
|k(r \to 0)| \sim r^\mu,\; \mu > -2;\quad M(r \to 0) \sim r^\alpha;\quad R(t, r \to 0) \sim r^\beta f(t),\; \alpha > \beta \geq 0.
\]

\(^1\)In the main text we also use a slightly modified rescaling when convenient.
Consistency requires $\beta$ to be constant, and our scaling choice further requires $\beta = 1$. We do not require the exponents $\mu$ and $\alpha$ to necessarily be integers.

- **No shell-crossing singularities:**
  Physically, we demand that an outer shell (labelled by a larger value of $r$) have a larger area radius $R$ than an inner shell, at any time $t$. Unphysical shell-crossing singularities arise when this condition is not met. Mathematically, this requires
  \[ R'(t, r) > 0 \quad \text{for all } r, \text{ for all } t. \] (B.7)

- **Regularity of energy density:**
  We demand that the energy density $\rho(t, r)$ remain finite and strictly positive for all values of $r$ and $t$. Combining this with Eqns. (B.2b) and (B.7) gives (assuming that $R'$ is finite for all $r$ and since $\beta = 1$)
  \[ \lim_{r \to 0} \rho(t, r) = \text{finite} \Rightarrow \alpha - 1 - 2\beta = 0 \Rightarrow \alpha = 3. \] (B.8)

- **No trapped shells:**
  In order for an expanding shell to not be trapped initially, it must satisfy the condition $r > 2GM(r)$. Near the regular center, this condition is automatically satisfied independent of the exact form of $M(r)$, since there $M \sim r^3$. 
Appendix C

Cosmology in MG

In this appendix we give proofs of several results that were used in chapter 2.

C.1 Analysis of $\nabla \bar{\Omega} \bar{g}^{ab} = 0$

We start with the metric

$$(M) ds^2 = g_{00}(t, x) dt^2 + g_{AB}(t, x) dx^A dx^B,$$  \hspace{1cm} (C.1)

on $M$ and assume that it averages out to the FLRW form (Eqn. (3.71)):

$$G_{00} = \langle \bar{g}_{00} \rangle = -f^2(t) ; \ G_{0A} = \langle \bar{g}_{0A} \rangle = 0 ; \ G_{AB} = \langle \bar{g}_{AB} \rangle = a^2(t) \delta_{AB}. \hspace{1cm} (C.2)$$

We will analyze the second relation of Eqn. (3.54) and show that it leads to the result

$$U^{ij} \equiv \bar{g}^{ij} - G^{ij} = 0,$$

where $\bar{\Omega}^a_b$ refers to the connection 1-forms associated with $G_{ij}$ given by

$$\bar{\Omega}^0_0 = \partial_t \ln f dt ; \ \bar{\Omega}^A_0 = H \delta^A_B dx^B,$$

$$\bar{\Omega}^0_A = \frac{a^2}{f^2} H \delta_{AB} dx^B ; \ \bar{\Omega}^A_B = H \delta^A_B dt,$$  \hspace{1cm} (C.3)

where $H = (1/a)(da/dt)$ for this section. We have

$$d \bar{g}^{ab} + \bar{\Omega}^a_j \bar{g}^{jb} + \bar{\Omega}^b_j \bar{g}^{aj} = 0. \hspace{1cm} (C.4)$$

Consider the three cases ($a = b = 0$), ($a = 0, b = B$) and ($a = A, b = B$) in turn. The first case gives

$$d \bar{g}^{00} + 2\bar{\Omega}^0_0 \bar{g}^{00} + 2\bar{\Omega}^0_A \bar{g}^{0A} = 0, \hspace{1cm} (C.5)$$
which reduces to
\[
[\partial_t \bar{g}^{00} + 2 \partial_t (\ln f) \bar{g}^{00}] \, dt + \left[ \partial_A \bar{g}^{00} + 2 \frac{a^2}{f^2} H \delta_{AB} \bar{g}^{0B} \right] \, dx^A = 0 . \quad \text{(C.6)}
\]

We can conclude that
\[
\bar{g}^{00}(t, x) = - \frac{k(x)}{f^2(t)} , \quad \text{(C.7a)}
\]
\[
\partial_A k(x) = 2 a^2 H \delta_{AB} \bar{g}^{0B} . \quad \text{(C.7b)}
\]

where \( k(x) \) is a positive definite function (so that the metric signature is preserved) which arises as an integration constant and is constrained by Eqn. \((C.7b)\). The second case \((a = 0, b = B)\) leads to
\[
[\partial_t \bar{g}^{0B} + \partial_t \ln(a^2) \bar{g}^{0B}] \, dt + \left[ \partial_J \bar{g}^{0B} + \frac{a^2}{f^2} H \delta_{AJ} \bar{g}^{AB} + \frac{k(x)}{f^2} H \delta^B_J \right] \, dx^J = 0 , \quad \text{(C.8)}
\]
which gives us
\[
\bar{g}^{0B} = \frac{m^B(x)}{a(t) f(t)} , \quad \text{(C.9a)}
\]
\[
\frac{1}{a f} \partial_J m^B(x) + \frac{a^2}{f^2} H \delta_{AJ} \bar{g}^{AB} - \frac{k(x)}{f^2} H \delta^B_J = 0 . \quad \text{(C.9b)}
\]

where \( m^B(x) \) is a 3-vector that arises as a constant of integration like \( k(x) \), and is constrained by Eqn. \((C.9b)\). Finally, the last case \((a = A, b = B)\) leads to
\[
[\partial_t \bar{g}^{AB} + 2 H \bar{g}^{AB}] \, dt + \left[ \partial_J \bar{g}^{AB} + \frac{1}{a f} H (\delta^A_J m^B(x) + \delta^B_J m^A(x)) \right] \, dx^J = 0 , \quad \text{(C.10)}
\]
which gives us
\[
\bar{g}^{AB} = \frac{1}{a^2(t)} s^{AB}(x) , \quad \text{(C.11a)}
\]
\[
\frac{1}{a^2} \partial_J s^{AB}(x) + \frac{1}{a f} H (\delta^A_J m^B(x) + \delta^B_J m^A(x)) = 0 . \quad \text{(C.11b)}
\]

Here \( s^{AB}(x) \) is another constant of integration, a symmetric 3-tensor. Now, since the left hand side of Eqn. \((C.7b)\) is independent of time, either the time dependence of the right hand side must cancel, or both sides must vanish. For the time dependence to cancel, we need \( f \propto (da/dt) \) which is not expected \( a \text{ priori} \). Therefore both sides of Eqn. \((C.7b)\) must vanish, which immediately tells us that the vector \( m^B(x) \) must vanish, and the function \( k(x) \) must be a constant,
\[
k(x) = k = \text{constant} ; \quad m^B(x) = 0 . \quad \text{(C.12)}
\]

Equations \((C.9b)\) and \((C.11b)\) then give us
\[
s^{AB}(x) = k \delta^{AB} , \quad \text{(C.13)}
\]
with the same constant $k$ as in Eqn. (C.12). Finally, putting everything together we find

$$\bar{g}^{00} = -\frac{k}{f^2}; \quad \bar{g}^{0A} = 0; \quad \bar{g}^{AB} = \frac{k}{a^2} \delta^{AB},$$

$$\Rightarrow \bar{g}^{ij} = kG^{ij}. \quad (C.14)$$

The constant $k$ is not constrained by any of the equations and appears to be a free parameter in the theory. The modified Einstein equations (3.58) show that $k$ can be absorbed into the averaged energy momentum tensor. We will for simplicity assume $k$ to be unity thereby obtaining, as required

$$U^{ij} \equiv \bar{g}^{ij} - G^{ij} = 0. \quad (C.15)$$

### C.2 Analysis of the condition $\langle \Gamma^a_{bc} \rangle = (\text{FLRW}) \Gamma^a_{bc}$

Here we will assume that the line element on $\mathcal{M}$ is in the volume preserving gauge

$$(\mathcal{M}) ds^2 = -\frac{dt^2}{h(t, \mathbf{x})} + h_{AB}(t, \mathbf{x}) dx^A dx^B,$$  \hspace{1cm} (C.16)

so that the averaging is trivial, and the metric and averages out to the FLRW line element on $\mathcal{M}$ given by

$$(\mathcal{M}) ds^2 = -\frac{dt^2}{\langle h(t) \rangle} + \bar{a}^2(t) \delta_{AB} dx^A dx^B,$$  \hspace{1cm} (C.17)

where we used the condition $\langle 1/h \rangle = 1/\langle h \rangle$ that follows from $\bar{g}^{00} = G^{00}$. The conditions $\langle \Gamma^a_{bc} \rangle = (\text{FLRW}) \Gamma^a_{bc}$ then result in the following:

$$\Gamma^0_{00} : \langle \partial_t (\ln \sqrt{h}) \rangle = \partial_t (\ln \sqrt{\langle h \rangle}), \quad (C.18a)$$
$$\Gamma^0_{0A} : \langle \partial_A (\ln \sqrt{h}) \rangle = 0, \quad (C.18b)$$
$$\Gamma^A_{00} : \langle \frac{h^{AB}}{h} \partial_B (\ln \sqrt{h}) \rangle = 0, \quad (C.18c)$$
$$\Gamma^A_{0B} : \langle \frac{1}{\sqrt{h}} \Theta^A_B \rangle = H \delta^A_B, \quad (C.18d)$$
$$\Gamma^A_{AB} : \langle \sqrt{h} \Theta_{AB} \rangle = \langle h \rangle \bar{a}^2 H \delta_{AB}, \quad (C.18e)$$
$$\Gamma^A_{BC} : \langle (3) \Gamma^A_{BC} \rangle = 0. \quad (C.18f)$$

Eqns. (C.18b) and (C.18f) are consistent with each other since $(3) \Gamma^A_{BA} = \partial_B (\ln \sqrt{h})$, and Eqn. (C.18c) is consistent with the assumption Eqn. (3.53a). The trace of Eqn. (C.18d) gives $\langle (1/\sqrt{h}) \Theta \rangle = 3H$. However we have $(1/\sqrt{h}) \Theta = N \Theta = \partial_t (\ln \sqrt{h})$, and combined with Eqn. (C.18a) this gives

$$\frac{1}{2} \partial_t (\ln \langle h \rangle) = 3 \partial_t (\ln \bar{a}) \Rightarrow \langle h \rangle = \bar{a}^6, \quad (C.19)$$

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where we have set an arbitrary proportionality constant (representing rescaling of the time coordinate by a constant) to unity. This establishes the last equality in Eqn. (3.78).

Finally, consider the trace \( \langle \frac{h^{AB}}{\langle h \rangle} \rangle \langle \sqrt{\hbar} \Theta_{AB} \rangle \) : using the condition \( \tilde{g}^{AB} = G^{AB} \), Eqn. (C.18a) and the trace of Eqn. (C.18d), this gives us

\[
\langle \frac{h^{AB}}{\langle h \rangle} \rangle \langle \sqrt{\hbar} \Theta_{AB} \rangle = \frac{1}{\langle h \rangle} \frac{\delta^{AB}}{a^2} \langle \sqrt{\hbar} \Theta_{AB} \rangle = 3H = \frac{1}{\sqrt{\hbar}} \Theta = \langle \frac{h^{AB}}{\hbar} \rangle \langle \sqrt{\hbar} \Theta_{AB} \rangle.
\]

(C.20)

On using the condition (3.53a) this leads to

\[
\left( \frac{\langle h^{AB} \rangle}{\langle h \rangle} - \frac{h^{AB}}{\langle h \rangle} \right) \langle \Gamma^0_{AB} \rangle = 0,
\]

(C.21)

which is consistent with the assumption

\[
\frac{\langle h^{AB} \rangle}{\langle h \rangle} = \langle \frac{h^{AB}}{\hbar} \rangle.
\]

(C.22)
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