Abstract: In these lectures I review some recent progresses in counting the number of microstates of AdS supersymmetric black holes in dimensions greater than four using holography. The counting is obtained by applying localization and matrix model techniques to the dual field theory. I cover in details the case of dyonic AdS$_4$ black holes, corresponding to a twisted compactification of the dual field theory, and I discuss the state of the art for rotating AdS$_5$ black holes.
1 Introduction

One of the great successes of string theory is the microscopic explanation of the entropy of certain asymptotically flat black holes. The first result was obtained in [1], more than twenty years ago, and has been followed by an immense literature, which would be too long to refer to. No similar results exist for asymptotically AdS black holes until very recently. Since holography suggests that the microstates of the black hole correspond to states in a dual conformal field theory, the AdS/CFT
correspondence is the natural setting where to explain the black hole entropy in terms of a microscopical theory. In the past, various attempts have been made to derive the entropy of a class of rotating black holes in AdS$_5$ in terms of states of the dual $\mathcal{N} = 4$ super-Yang-Mills (SYM) theory in the large $N$ limit, but none was completely successful. The more recent advent of localization techniques for supersymmetric quantum field theories, in the spirit of [2], opens a new perspective on this problem. In these lectures we discuss how to use localization to derive the entropy for a class of supersymmetric magnetically charged black holes in AdS$_4$ and discuss the current status for other black holes appearing in holography, including those in AdS$_5$. We will work in dimension greater than four. AdS$_3$ is somehow special, and well-studied in the literature, and it will not be discussed in these notes.

The first microscopic counting for AdS black holes in dimension greater than four was performed in [3], considering asymptotically AdS$_4$ static supersymmetric black holes. One of the main characteristics of this class of black holes is the presence of magnetic charges that correspond to a topological twist in the dual field theory. The black holes considered in [3] can be embedded in M theory and are asymptotic to AdS$_4 \times S^7$. They are dual to a topologically twisted compactification of the ABJM theory in three dimensions [4], and their entropy scales as $N^{3/2}$ at large $N$, as familiar from three-dimensional holography. In these lectures, we will focus on this example as a prototype for many similar computations. The entropy can be extracted from the (regularized) Witten index of the quantum mechanics obtained by compactifying ABJM on a Riemann surface $\Sigma_g$. Holographically, the quantum mechanics describes the physics of the near horizon geometry AdS$_2 \times \Sigma_g$ of the black holes. The index can be computed, using localization, as the three-dimensional supersymmetric partition function of ABJM on $\Sigma_g \times S^1$, topologically twisted along the Riemann surface $\Sigma_g$. From this perspective, this computation can be generalized to more general domain wall solutions interpolating between AdS$_{d+n}$ and AdS$_d \times M_n$, with a topological twist along the $n$-dimensional compact manifold $M_n$, thus providing general tests of holography.

In the last part of these lectures, we also discuss the analogous problem for supersymmetric rotating electrically charged black holes in AdS$_d$. The main difference with the previous case is the absence of a topological twist. The entropy for such black holes should be obtained by counting states with given electric charge and spin in the dual field theory and the natural observable to consider is the superconformal index, which receives contributions precisely from the BPS states of the theory. We will discuss recent progresses in this direction.

Most of the relevant field theory computations are performed using localization. This allows to reduce exact path integrals in quantum field theory to matrix models, which can be solved in the large $N$ limit combining standard and more recent techniques. One successful approach for the physics of black holes, that works both in four [3, 5] and five dimensions [6], involves writing the matrix model partition
function as a sum of Bethe vacua [7] of an auxiliary theory. Some technical aspects of this approach are discussed in section 3.

Let us stress that many derivations of the entropy for asymptotically flat black holes involve the use of the Cardy formula for the asymptotic growing of states of a two-dimensional CFT. In the localization approach for AdS black holes, we directly count the number of microstates using an index. We will briefly make contact with the Cardy approach in section 5 where we discuss black strings.

These lectures assume some familiarity with supersymmetry and the main examples of holographic dualities in various dimensions. We assume that the reader knows that $\mathcal{N} = 4$ SYM in four dimensions is dual to AdS$_5 \times S^5$, the ABJM theory to AdS$_4 \times S^7$ and the so-called (2, 0) theory in six dimensions to AdS$_7 \times S^4$. Some preliminary exposure to localization computation\(^1\) would be also useful although not necessary. We review instead in section 2 the elements of gauged supergravity that are needed for these lectures.

The lectures are organized as follows. In section 2 we give a general overview of the various classes of supersymmetric black holes that are relevant for holography, stressing that they fall into two main classes, distinguished by the presence or absence of magnetic charges (or more precisely of a twist). In the second part of section 2 we focus in more details on four dimensions and we review elements of gauged supergravity, in particular the attractor mechanism, that will be used in the following. In section 3 and 4 we discuss in details the main example, dyonic black holes asymptotic to AdS$_4 \times S^7$ and dual to a twisted compactification of ABJM. In section 3 we discuss the field theory aspects of the story, introducing the topologically twisted index and showing how to evaluate it using localization. In section 4 we perform the large $N$ limit of the resulting matrix model and we compare with gravity. In section 5 we discuss black string solutions interpolating between AdS$_5$ and AdS$_3 \times \Sigma_g$, as a prototype of more general domain walls interpolating between AdS spaces that can be studied with these techniques. Finally, in section 6 we discuss the state of the art for rotating electrically charged black holes in AdS$_d$.

2 AdS Black Holes in $d \geq 4$

In these lectures we are interested in supersymmetric black holes that can be embedded in string theory or M-theory and are asymptotic to AdS$_d$ vacua with a known field theory dual. There are many such black holes that can be embedded in maximally supersymmetric backgrounds. For example, we can find supersymmetric black holes in AdS$_5 \times S^5$, AdS$_4 \times S^7$ and AdS$_7 \times S^4$, whose dual field theories are well known. They are all characterized by a set of charges and angular momenta. Supersymmetric black holes have zero temperature. They also satisfy a BPS bound that

\(^1\)We refer to [8] for a nice introduction and [9] for a more comprehensive review.
relates their mass to the other conserved charges. In the limit where gravity is weakly
coupled, the entropy of a black hole can be computed with the Bekenstein-Hawking
formula
\[ S = \frac{A}{4G_N}, \tag{2.1} \]
where \( A \) is the area of the horizon and \( G_N \) is the Newton constant.

We first discuss some general features of these black holes and their holographic
interpretation, and then we focus on a special class of four-dimensional static black
holes that will play an important role in the rest of these lectures.

2.1 AdS Black Holes and holography

For the purposes of holography, we can divide the known supersymmetric black holes
in dimension \( d \geq 4 \) into two main classes, distinguished by the existence (or absence)
of magnetic charges and a topological twist.

2.1.1 Electrically charged rotating black holes

The first class of black holes consists of supersymmetric electrically charged rotating
black holes. The most famous examples are the type IIB supergravity black holes
asymptotic to \( \text{AdS}_5 \times S^5 \) found in [10–14]. They depend on two angular momenta
corresponding to two Cartan isometries of \( \text{AdS}_5 \)
\[ (j_1, j_2) \quad U(1)^2 \subset SO(4) \subset SO(2,4), \tag{2.2} \]
and three electric charges under the Cartan isometries of \( S^5 \)
\[ (q_1, q_2, q_3) \quad U(1)^3 \subset SO(6), \tag{2.3} \]
with a constraint.\(^2\) The electric charges parameterize rotations in the internal space
\( S^5 \). These black holes preserve two real supercharges out of the original thirty-two
of type IIB supergravity on \( \text{AdS}_5 \times S^5 \). The five-dimensional part of the metric is
asymptotic to \( \text{AdS}_5 \) with \( \mathbb{R} \times S^3 \) as conformal boundary. As well known, type IIB
string theory on \( \text{AdS}_5 \times S^5 \) is dual to \( \mathcal{N} = 4 \) SYM in four dimensions. It is then a
natural expectation that the black holes correspond holographically to an ensemble
of states of \( \mathcal{N} = 4 \) SYM on \( \mathbb{R} \times S^3 \) that preserve the same supersymmetries and have
the same electric charges and the same angular momenta. It is also natural to expect
that, by counting all the 1/16 BPS states of \( \mathcal{N} = 4 \) SYM on \( \mathbb{R} \times S^3 \) with electric
charges \( (q_1, q_2, q_3) \) and spin \( (j_1, j_2) \), we should be able to reproduce the entropy of
these black holes. We will work under these assumptions. We are interested in

\(^2\)Curiously, the most famous supersymmetric solutions depend only on four parameters, because
the BPS conditions impose a relation among the charges, \( f(j_1, j_2, q_1, q_2, q_3) = 0 \). More recently,
BPS hairy black holes with diverging tidal forces but depending on all the charges have been found
[15, 16].
macroscopic black holes whose entropy, when expressed in terms of field theory data, scales as $O(N^2)$, where $N$ is the number of colors of the dual field theory.

The situation is analogous in other dimensions [17–20]. Consider the maximally supersymmetric backgrounds $\text{AdS}_4 \times S^7$ and $\text{AdS}_7 \times S^4$ in M-theory. The isometry of $\text{AdS}_4 \times S^7$ is $SO(2,3) \times SO(8)$ and we can find electrically charged rotating black holes depending on one angular momentum $j$ and four electric charges $(q_1, q_2, q_3, q_4)$ with a constraint. They preserve two real supercharges. We expect to reproduce the entropy of such black holes by counting all $1/16$ BPS states of the dual field theory on $\mathbb{R} \times S^2$ with the same quantum numbers. As well-known, the dual of M-theory on $\text{AdS}_4 \times S^7$ is the three-dimensional ABJM theory at Chern-Simons level $k = 1$ [4]. The entropy of these black holes scales as $O(N^{3/2})$. Similarly, since the isometry of $\text{AdS}_7 \times S^4$ is $SO(2,6) \times SO(5)$, there are black holes depending on three angular momenta $(j_1, j_2, j_3)$ and two electric charges $(q_1, q_2)$ with a constraint, again preserving two real supercharges. In this case, the entropy, which scales as $O(N^3)$, should be reproduced by counting states in the $\mathcal{N} = (2,0)$ theory in six dimensions [21] on $\mathbb{R} \times S^5$.

Notice that all these supersymmetric black holes rotate. If we turn off the angular momenta $j_i$, we find singularities.

In principle, although there are not so many examples in the literature, we expect the existence of similar supersymmetric black holes in more general type II or M-theory backgrounds with an $\text{AdS}_d$ vacuum, rotating in $\text{AdS}_d$ and charged under the isometries of the compactification manifold. The holographic interpretation is similar. For example, for type IIB black holes in $\text{AdS}_5 \times SE_5$ [22], where $SE_5$ is a five-dimensional Sasaki-Einstein manifold, we should try to match the entropy by counting $1/4$ BPS states of the dual $\mathcal{N} = 1$ superconformal field theory on $\mathbb{R} \times S^3$.

2.1.2 Magnetically charged black holes

The second class of black holes contains the magnetically charged ones. We should more properly refer to such black holes as solutions where supersymmetry is realized with a topological twist, as we will see. Although there are examples in higher dimensions, we will focus on four dimensions, where these black holes arise naturally. Indeed we can have both magnetic and electric charges in $d = 4$ and it is natural to consider dyonic black holes. For this class of solutions, we can also turn off rotation and have static supersymmetric black holes. There exists an entire family of static BPS black holes in $\text{AdS}_4 \times S^7$ depending on electric as well as magnetic charges.

3Actually, only subsets of such family of black holes, where some of the charges are identified, have been studied in the literature. Similarly, only black holes with equal charges or equal momenta in $\text{AdS}_7$ have been studied. In both cases, we expect a family with at least four independent parameters to exist.

4There are exotic $d$-dimensional solutions with horizon $\text{AdS}_2 \times M_{d-2}$, where $M_{d-2}$ is a compact manifold, with non-zero fluxes of the gauge fields on $M_{d-2}$. 
under the abelian $U(1)^4 \subset SO(8)$ isometries of $S^7$. Supersymmetry requires a linear constraint among the magnetic charges $p_i$ and a non-linear one among the charges, $f(p_a, q_a) = 0$, so that we have a six-dimensional family of dyonic black holes.\footnote{These BPS black holes have been found in $\mathcal{N} = 2$ gauged supergravity in $d = 4$ with vector multiplets and later uplifted to M-theory. The first example, with a hyperbolic horizon, was found in \cite{28} in minimal gauged supergravity. The first spherically symmetric example was found in \cite{23}, further discussed in \cite{24, 25} and generalized to the dyonic case in \cite{26, 27}.} We expect to be able to turn on continuously the angular momentum and to find a seven-dimensional family of supersymmetric black holes depending on one angular momentum $j_1$ in $\text{AdS}_4$ and electric and magnetic charges, $(q_1, q_2, q_3, q_4)$ and $(p_1, p_2, p_3, p_4)$, subject to two constraints. These black holes have been recently found in \cite{29}.\footnote{For other examples, see also \cite{30}.}

In general, $\text{AdS}_4$ black holes with magnetic charges are qualitatively different from those with zero magnetic charge, as first noticed in \cite{31} and elaborated in \cite{32, 33}. The difference is well explained using holography. Consider the black holes as solutions of an effective four-dimensional theory with a $\text{AdS}_4$ vacuum dual to a boundary conformal field theory (CFT). For most of these lectures, the CFT will be ABJM, but the following arguments apply to black holes in general compactifications and more general CFTs with at least $\mathcal{N} = 2$ supersymmetry. For our purposes, we need the terms of the effective theory describing the dynamics of the metric and of the vector fields $A_{\mu}^a$ corresponding to the abelian isometries of the internal manifold. Their dynamics will be described by an Einstein-Maxwell theory

\[
\mathcal{L} = \sqrt{g} \left( R + g_{ab}(\phi_i)F^{\mu a}_{\mu}F^{\nu b}_{\nu} + \ldots \right)
\]

where, in general, the matrix of coupling constants depends on the scalar fields $\phi_i$ of the theory. According to the rules of holography, gauge fields in the bulk correspond to global symmetries in the boundary CFT. For example, in the case of $\text{AdS}_4 \times S^7$, we have four fields $A_{\mu}^a$ corresponding to the abelian isometries $U(1)^4 \subset SO(8)$ and they couple to the field theory conserved currents $J^{\mu a}$

\[
\int d^4 x A_{\mu}^a J^{\mu a}
\]

associated to the Cartan generators of the $SO(8)$ R-symmetry of ABJM at Chern-Simons level $k = 1$. Focusing, for simplicity, on the static case, one finds that, near the boundary, the black hole solutions behave as

\[
ds^2 = \frac{dr^2}{r^2} + r^2 ds^2_{\mathcal{M}_3} + \ldots,
\]

\[
A_{\mu}^a(x, r) = \hat{A}_{\mu}^a(x) + \ldots,
\]

\[
(\ref{2.4})
\]
where $r$ is some large radial coordinate, the ellipsis refers to terms suppressed by inverse powers of $r$, and $x$ are coordinates on the boundary manifold. For spherically symmetric black holes the boundary manifold is $\mathcal{M}_3 = \mathbb{R} \times S^2$. However we can have more exotic solutions with horizon $\text{AdS}_2 \times \Sigma_g$, where $\Sigma_g$ is a Riemann surface of genus $g$, and, in this case, $\mathcal{M}_3 = \mathbb{R} \times \Sigma_g$. Holography tells us that we should interpret (2.7) as the dual of our CFT defined on the curved manifold $\mathcal{M}_3$. What can we say about $A_\mu^a$? Recall again the basic rules of the AdS/CFT correspondence [34]. Any field $\phi(x,r)$ in AdS is associated with an operator $O(x)$ in the dual CFT. If we have an expansion

$$\phi(x, r) = r^{\alpha_1} \phi_0(x) + r^{\alpha_2} \phi_1(x),$$

(2.8)

of the solution of the second order equations of motion for $\phi$, with $\alpha_i$ related to the conformal dimension of $O$, we interpret the non-normalizable piece, $\phi_0(x)$, as a deformation of the original CFT with the corresponding operator $O$, $\mathcal{L}_{\text{CFT}}(x) \rightarrow \mathcal{L}_{\text{CFT}}(x) + \phi_0(x)O(x)$, (2.9)

while we interpret the normalizable one, $\phi_1(x)$, as a vacuum expectation value (vev) for $O$, $\langle O \rangle \neq 0$. More precisely, if $\phi_0(x) \neq 0$, we are deforming the CFT with $O$; if $\phi_0(x) = 0$ and $\phi_1(x) \neq 0$ we have a state of the CFT with non zero vev for $O$. There are situations where both modes $\phi_0(x)$ and $\phi_1(x)$ are normalizable (or better have finite energy). In this case there are different possible quantizations of the same theory and we have to choose who plays the role of $\phi_0$. Massless vector fields in AdS$_4$ allow for different types of quantizations, related to electric/magnetic duality in the bulk, and this leads to interesting applications, but this is not strictly the most important point. What is important is that, in the expansion (2.7) for $A_\mu^a$ both leading and sub-leading terms are turned on. The field $A_\mu^a$ has a leading contribution for $r \gg 1$ that approaches a constant value on the boundary $\mathcal{M}_3 = \mathbb{R} \times \Sigma_g$, corresponding to the magnetic charge of the black hole

$$\frac{1}{2\pi} \int_{\Sigma_g} F^a = p^a,$$

(2.10)

and sub-leading terms (the ellipsis in (2.7)) that encode information about the electric charges. This means that a dyonic black hole is holographically dual to a deformation of the dual CFT. In the natural quantization of the theory, the non-zero value of $A_\mu^a$ at the boundary corresponds to the deformation

$$\mathcal{L}_{\text{CFT}}(x) \rightarrow \mathcal{L}_{\text{CFT}}(x) + \hat{A}_\mu^a(x)J^{\mu a}(x).$$

(2.11)

This deformation in field theory is equivalent to turning on a background gauge field for a global symmetry. For example, on $S^2$ we would turn on a background which is
just the familiar Dirac monopole $\hat{A}_\mu^a = -\frac{1}{2} p^\rho \cos \theta d\phi$. On a torus $T^2$ we would just turn on a background constant magnetic field. Fields satisfying (2.10) can be also written explicitly for all $\Sigma_g$ but their expression is not particular illuminating. We will see in section 3 that the deformation (2.11) is compatible with supersymmetry.

To understand better what is going on, it is useful to have a look at how supersymmetry is preserved in the presence of magnetic charges. We will be very schematic here just to convey the general idea. Consider the case where our effective theory is some $\mathcal{N} = 2$ gauged supergravity in four dimensions. For the solution to be supersymmetric, all fermion variations in the black hole background must be zero. The gravitino variation in $\mathcal{N} = 2$ gauged supergravity is schematically given by

$$\delta \psi_\mu = \partial_\mu \epsilon + \frac{1}{4} \omega^{ab}_\mu \Gamma_{ab} \epsilon - i A_R^\mu \epsilon + \ldots .$$

(2.12)

Here $A_R^\mu$ is the graviphoton field, holographically dual to the $U(1)$ R-symmetry of the theory. In general, $A_R^\mu$ is a linear combination of the vector fields $A_a^\mu$ corresponding to the isometries of the internal manifold. The important point is that magnetically charged static black holes satisfy the BPS condition $\delta \psi_\mu = 0$ by cancelling the spin connection with a background field for the R-symmetry. More precisely, we can regard the spin connection $\omega_\mu$ along $\Sigma_g$ as a $U(1)$ gauge field. An explicit computation shows that $\omega_\mu$ is just a monopole of charge $2 - 2g$, as the familiar relation $\frac{1}{2\pi} \int_{\Sigma_g} R = 2 - 2g$, with $R = d\omega$, clearly shows. Since $A_R^\mu$ is a linear combination of the $A_a^\mu$, it is also a monopole, with charge given by a linear combination of the $p^a$. By appropriately choosing this linear combination and the spinor $\epsilon$, we can cancel the second and the third term on the right hand side in (2.12). We will come back to more precise expressions in section 3. For the dyonic static black holes, restricting the index $\mu$ to lie along $\Sigma_g$, one discovers that the ellipsis cancels independently and we are left with the equation

$$\delta \psi_\mu = \partial_\mu \epsilon = 0 .$$

(2.13)

This equation is solved by taking $\epsilon$ constant along $\Sigma_g$. One also finds that the other components of the supersymmetry variations imply that $\epsilon$ is time-independent but, in general, has a non-trivial profile in $r$.

This discussion can be also applied to the dual CFT. By restricting the variations to the boundary, we see that the field theory on $\mathbb{R} \times \Sigma_g$ is invariant under supersymmetry transformations with a constant spinor. Also on the boundary the same mechanism is at work: we are turning on a magnetic background for the R-symmetry that compensates the spin connection. In quantum field theory, this construction is well-known [35]. It is called topological twist, as we will discuss in details in section 3. The conclusion is that the dual CFT is deformed by the presence of magnetic fluxes for all the global and R-symmetries, and, in particular, it is topologically twisted by

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7This is the linear constraint on the magnetic charges of the black hole that we mentioned before.
the magnetic flux for the R-symmetry. Notice that our argument was based on $\mathcal{N} = 2$ supersymmetry with a $U(1)$ R-symmetry. Theories can have a larger R-symmetry group, like ABJM, or many flavor symmetries. In these cases, the choice of a $U(1)$ R-symmetry is not unique. Each choice corresponds to a different twist. We can indeed think of the magnetic charges $p^a$ as parameterizing a family of inequivalent twists. It is important to remember, however, that a linear combination of the $p^a$ is fixed by the condition that the background for the selected $U(1)$ R-symmetry cancels the spin connection. There are only $n_V - 1$ independent magnetic charges, where $n_V$ is the number of massless vectors. This number is $n_V = 4$ for ABJM.

The interpretation of the general rotating dyonic black hole is more complicated but similar in spirit. We can have rotation only in the spherically symmetric case, where $j$ is the spin along $S^2$. Rotations in the bulk correspond to turning on an Omega-background [36] in the boundary theory on $S^2$. The theory is still topologically twisted.

All this should be contrasted with the case of electrically charged rotating black holes, with no magnetic charge, where the Killing spinors $\epsilon(x)$ are non-trivial functions of all coordinates and there is no explicit cancellations between the spin connection and the R-symmetry. This is the reason for the differences among the two classes of black holes in AdS$_4$. It can be expressed more formally in a difference between the supersymmetry algebra, as discussed in [32, 33].

Now it is clear what we should do in order to compute the entropy of the magnetically charged black holes using field theory: enumerate all the states with electric charges $q_i$ and angular momentum $j$ and the right amount of supersymmetry in the twisted CFT on $\mathbb{R} \times \Sigma_g$. The theory is topologically twisted$^8$ by the magnetic background for a $U(1)$ R-symmetry and possibly deformed by magnetic fluxes for all other global symmetries.

### 2.1.3 Computing the entropy

It is reasonable to expect that we can recover the entropy of the two classes of AdS$_d$ black holes by enumerating BPS states in the dual field theory on $\mathbb{R} \times \mathcal{M}_{d-2}$. This is equivalent to knowing the grand canonical BPS partition function

$$Z(\Delta_a, \omega_i) = \text{Tr} \left|_{Q=0} \right. e^{i(\Delta_a R_a + \omega_i J_i)} = \sum_{q_a, j_i} c(q_a, j_i) e^{i(\Delta_a q_a + \omega_i j_i)},$$

(2.14)

where the trace is taken over the Hilbert space of states on $\mathcal{M}_{d-2}$ that preserves the same amount of supersymmetry of the black hole, $R_a$ and $J_i$ are field theory charge operators associated with the global symmetries and the angular momenta, respectively, and $\Delta_a$ and $\omega_i$ conjugated chemical potentials. In practical applications, $Z$ is a function of complex chemical potentials and converges in an appropriate

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$^8$And also Omega-deformed if there is rotation.
domain of the complex plane for the fugacities \( y_a = e^{i\bar{\Delta}_a} \), \( \zeta_i = e^{i\omega_i} \). In the previous formula, \( c(q_a, j_i) \) is the number of supersymmetric states of electric charge \( q_a \) and angular momentum \( j_i \). For magnetically charged black holes, the trace must be taken in the topologically twisted theory in the presence of magnetic fluxes for the global symmetries. Notice that electric and magnetic charges enter in an asymmetric way in this construction. The magnetic charges \( p_a \) enter explicitly as a set of couplings in the Lagrangian of the deformed CFT, while the electric charges \( q_a \) are introduced through chemical potentials.

The grand canonical partition function (2.14) also enumerates the BPS states in the dual gravity theory. Our working assumption is that, for large charges (scaling with appropriate powers of \( N \)) the supersymmetric density of states is dominated by the macroscopic black holes we discussed before. Under this assumption, by the very definition of entropy, the entropy of the black hole is given by

\[
S(q_a, j_i) = \log c(q_a, j_i),
\]

where the dependence on the magnetic charges \( p_a \), if present, is hidden in the form of the function \( c \). If \( \mathcal{Z}(\Delta_a, \omega_i) \) is known, the entropy can be extracted as a Fourier or Laplace coefficient

\[
e^{S(q_a, j_i)} = c(q_a, j_i) = \int \frac{d\Delta_a}{2\pi} \frac{d\omega_i}{2\pi} \mathcal{Z}(\Delta_a, \omega_i) e^{-(\Delta_a q_a + \omega_i j_i)},
\]

with an appropriate integration contour. In the limit of large charges, in many circumstances, this can be evaluated by a saddle point approximation

\[
S(q_a, j_i) = \log \mathcal{Z}(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i) \bigg|_{\bar{\Delta}_a, \bar{\omega}_i},
\]

where \( \bar{\Delta}_a \) and \( \bar{\omega}_i \) are obtained by extremizing the functional

\[
\mathcal{I}(\Delta_a, \omega_i) = \log \mathcal{Z}(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i),
\]

with respect to \( \Delta_a \) and \( \omega_i \),

\[
\partial_{\Delta_a} \mathcal{I}(\Delta_a, \omega_i) = \partial_{\omega_i} \mathcal{I}(\Delta_a, \omega_i) = 0 \bigg|_{\Delta_a, \omega_i}.
\]

The saddle point values of \( \bar{\Delta}_a \) and \( \bar{\omega}_i \) could be complex, but the final value for \( S(q_a, j_i) \) must be real. We see that the entropy is just the Legendre transform of the logarithm of the partition function.\(^9\) The entropy is also obtained via a Legendre transform in

\(^9\) Recall that we are interested in extremal supersymmetric black holes. In particular, they have zero temperature. The standard thermodynamics relation for the grand canonical partition function of a system with temperature \( T \), \( \log \mathcal{Z} = -(E - TS - i\bar{\Delta}_a q_a - i\bar{\omega}_i j_i)/T \), looks singular in the zero-temperature limit. However, we are dealing with BPS black holes, where all the states satisfy some BPS condition \( E = \mu_a q_a + \nu_i j_i \), with numbers \( \mu_a \) and \( \nu_i \) of order one. When we take the zero temperature limit, we need also to scale \( \bar{\Delta}_a(T) = -i\mu_a + \Delta_a T \) and \( \omega_i(T) = -i\nu_i + \omega_i T \).

In this way we obtain the Legendre transform \( S = \log \mathcal{Z} - i\bar{\Delta}_a q_a - i\bar{\omega}_i j_i \). For explicit examples of this zero-temperature limit from the gravitational point of view see [37], and, in particular, for AdS\(_5\) black holes see [38, 39].
many other approaches, as the OSV conjecture [40] and the Sen’s quantum entropy functional [41, 42] for asymptotically flat black holes.

So far everything was simple. The problem is that $Z(\Delta_a, \omega_i)$ is too hard to compute, in general. For electrically charged rotating black holes in $\text{AdS}_5 \times S^5$, computing $Z(\Delta_a, \omega_i)$ would correspond to enumerate all the $1/16$ BPS states of $\mathcal{N} = 4$ SYM. For comparison, in a four-dimensional theory with $\mathcal{N} = 1$ supersymmetry, this would correspond to count all the $1/4$ BPS states. While almost everything is known about the counting of $1/2$ BPS states in an $\mathcal{N} = 1$ theory [43, 44], the analogous problem for $1/4$ BPS states is still open.

What we can instead compute is a supersymmetric index

$$Z_{\text{index}}(\Delta_a, \omega_i) = \text{Tr} \left|_{\mathcal{Q} = 0} \right. (-1)^F e^{i(\Delta_a R_a + \omega_i J_i)} ,$$

(2.20)

with the insertion of the fermionic number $(-1)^F$. Choosing a supercharge $\mathcal{Q}$ and assuming, for simplicity, that the algebra of supersymmetry is $\{Q^\dagger, Q\} = H$, standard arguments tell us that we can also write

$$Z_{\text{index}}(\Delta_a, \omega_i) = \text{Tr} (-1)^F e^{-\beta H} e^{i(\Delta_a R_a + \omega_i J_i)} = Z_{\text{susy}}^{S^1 \times M_{d-1}}(\Delta_a, \omega_i) ,$$

(2.21)

if $R_a$ and $J_i$ commute with $\mathcal{Q}$. In the first step of the previous identification we used the fact that states with $\mathcal{Q} \neq 0$ do not contribute to the trace, since bosonic and fermionic states are paired and contribute with opposite sign.\footnote{This is the logic of the Witten index [45]. For every bosonic state $|\Omega\rangle$ with energy $E \neq 0$ there is a fermionic state $\mathcal{Q}|\Omega\rangle$ with the same energy. Since these states have opposite value of $(-1)^F$ but equal value of $H$ and all other conserved charges, their contribution cancels in the trace in (2.21). The argument does not apply to states with $E = 0$ since the supersymmetry algebra implies that $\mathcal{Q}|\Omega\rangle = 0$. Therefore, only the ground states, which have zero energy and are annihilated by $\mathcal{Q}$, contribute to the Witten index. In general, we will have a more complicated algebra $\{Q^\dagger, Q\} = H - \mu_a R_a - \nu_i J_i$. The same argument applies with $e^{-\beta H}$ in the trace replaced by $e^{-\beta (Q^\dagger, Q)}$. In this case only the BPS states $\mathcal{Q}|\Omega\rangle = 0$, satisfying $E = \mu_a q_a + \nu_i j_i$, contribute to the index. See also the discussion in section 3.3.1.}\footnote{Remember that the finite temperature partition function $\text{Tr} e^{-\beta H}$ can be expressed as an Euclidean partition function with time compactified on a circle of radius $\beta$ and periodic boundary conditions for bosons and anti-periodic boundary conditions for fermions. In a supersymmetric partition functions, bosons and fermions should have the same boundary conditions and this is enforced by the fermionic number $(-1)^F$.} More impor-

In general, $Z_{\text{index}}(\Delta_a, \omega_i) \neq Z(\Delta_a, \omega_i)$. First of all, the index can accommodate fugacities only for the conserved charges that commute with the selected $\mathcal{Q}$ and, in general, contains less parameters than the BPS partition function.\footnote{We will see that this is not a problem for the black holes considered in this paper that also have a constraint among charges.}
tantly, the entropy should count all the BPS ground states of the theory, while the index counts bosonic ground states with a plus and fermionic ground states with a minus. However, it may happen that, for particular theories, the majority of ground states are of definite statistic. In this case there is no cancellation between bosonic and fermionic ground states and the index correctly reproduces the entropy in a suitable limit. This typically happens for asymptotically flat black holes in the limit of large charges, and we may hope that the same is true for asymptotically AdS black holes in the large $N$ limit. In the case of certain asymptotically flat spherically symmetric black holes, there is an extra symmetry that implies $(-1)^F = 1$ on the relevant set of states [42] and one can prove that $Z_{\text{index}}(\Delta_a, \omega_i) = Z(\Delta_a, \omega_i)$. Unfortunately, no similar argument can be used for asymptotically AdS black holes with the same level of rigor and we need to rely on an explicit computation. Notice also that, for asymptotically flat black holes, the entropy and the index coincide at leading order in the charges but are in general different when corrections are included [46].

As we will see in the rest of these lectures

- For magnetically charged black holes in AdS$_4$, the dual field theory is topologically twisted. $Z_{\text{index}}(\Delta_a, \omega_i)$ is the so-called topologically twisted index that we define in section 3. In this case there is no cancellation between bosonic and fermionic ground states and the index correctly reproduces the entropy at large $N$, as we will see in section 4.

- For electrically charged rotating black holes, we are just counting states of the CFT on $\mathbb{R} \times S^{d-1}$. $Z_{\text{index}}(\Delta_a, \omega_i)$ is the superconformal index, which is known to have large cancellations between bosons and fermions for real values of the fugacities [43]. This would suggest that we really need to compute the original BPS partition function $Z(\Delta_a, \omega_i)$. However, some recent results suggest otherwise. As originally pointed out in [47, 48] the entropy for electrically charged rotating black holes in AdS$_5 \times S^5$ and AdS$_7 \times S^7$ can be written, in the limit of large $N$, as the Legendre transform of a very simple function related to the anomalies of the dual field theory. Various derivations of the entropy, confirming this picture, either studying the index for complex values of the chemical potentials or invoking a regularization ambiguity in the relation between $Z_{\text{index}}$ and $Z_{\text{susy}}$ in (2.21), have been proposed [6, 38, 39, 50]. We will discuss these results in section 6.

2.2 Entropy and the attractor mechanism

In the limit where gravity is weakly coupled, we can compute the entropy of a black hole from the area of the horizon using the Bekenstein-Hawking formula (2.1). The area can be extracted from the explicit solution of the relevant gauged supergravity,
which typically contains many scalars $X^I(r)$ varying with the radial distance. In principle, the area may depend on many parameters including asymptotic moduli. However, the microscopic entropy of our black holes is just a function of the conserved charges $q_a$ and $j_i$. This is explicitly realized through an \textit{attractor mechanism}: independently of the asymptotic moduli, the scalar fields approach a value at the horizon, $X^I_i(q_a, j_i)$, that is a function of the conserved charges only. Moreover, this mechanism often allows to express the area, and therefore the entropy, in terms of the values of the scalar fields at the horizon with simple algebraic equations. This is the case of the attractor mechanism for $\mathcal{N} = 2$ supergravity discovered in [51, 52]. It is also the idea behind Sen’s quantum entropy function [53] that allows to find the entropy of black holes with AdS$_2$ horizon including higher derivative corrections. In all these cases, one can define some sort of entropy functional $S(X^I, q_a, J_i)$, which is a function of the conserved charges and the \textit{horizon value} $X^I$ of the scalar fields, and whose extremization with respect to $X^I$ reproduces the entropy. This extremization is the gravity analog of (2.17). It will play an important role in the interpretation of the field theory result leading to the black hole entropy, and therefore we will review it in this section. Unfortunately, the attractor mechanism for supersymmetric AdS black holes is fully understood only for magnetically charged or dyonic black holes in four dimensions, and we will focus on this case. We will see in section 6 that one can write a proposal for entropy functionals also for rotating electrically charged black holes in various dimensions. This section serves also to introduce the very basic formalism of $\mathcal{N} = 2$ gauged supergravity and to give some more details on the dyonic black holes embeddable in AdS$_4 \times S^7$.

The attractor mechanism for AdS$_4$ static black holes in $\mathcal{N} = 2$ gauged supergravity was derived in [23, 24, 54].\textsuperscript{14} We start by briefly discussing the main features of $\mathcal{N} = 2$ supergravity in four dimensions. The $\mathcal{N} = 2$ supergravity multiplets are:

- the graviton multiplet, whose bosonic components are the metric $g_{\mu\nu}$ and a vector field $A^0_\mu$, called graviphoton;
- the vector multiplet, whose bosonic components are a vector $A^i_\mu$ and a complex scalar $z$;
- the hypermultiplet, whose bosonic components are four real scalars $q^a$.

For simplicity, we will restrict to $\mathcal{N} = 2$ gauged supergravities with $n_V$ vector multiplets and no hypermultiplets.\textsuperscript{15} This will be enough to describe the black holes dual to ABJM. The theory contains $n_V + 1$ vector multiplets $A^1_\mu$ and $n_V$ complex

\textsuperscript{14}For some recent progress for dyonic rotating black holes see [29].
\textsuperscript{15}Theory with hypermultiplets have been considered for matching the entropy of black holes in massive type IIA and other models [55–57].
scalar fields \( z_i \), where \( \Lambda = 0, 1, \ldots, n_V \) and \( i = 1, \ldots, n_V \). The Lagrangian is completely specified by supersymmetry in terms of a holomorphic prepotential \( F(\Lambda) \), which is a homogeneous function of degree two, and a vector of magnetic and electric Fayet-Iliopoulos (FI) parameters \((g^\Lambda, g_\Lambda)\). \( X^\Lambda(z_i) \) are a set of \( n_V + 1 \) homogeneous coordinates on the scalar manifold. The theory is invariant under rescaling of the \( X^\Lambda \) and one can identify the physical scalar fields with \( z_i = X^i/X^0 \), for example.\(^{16}\)

It is also convenient to define \( F_\Lambda \equiv \partial_\Lambda F \). The theory is fully covariant under a \( Sp(2n_V + 2) \) group of electric/magnetic dualities acting on \((X^\Lambda, F_\Lambda)\) and \((g^\Lambda, g_\Lambda)\) as symplectic vectors.

The action of the bosonic part of the theory reads \([58]\)

\[
S^{(4)} = \frac{1}{8\pi G_N^{(4)}} \int_{\mathbb{R}^3} \left[ \frac{1}{2} R^{(4)} + \frac{1}{2} \text{Im} \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma + \frac{1}{2} \text{Re} \mathcal{N}_{\Lambda\Sigma} F^\Lambda \wedge F^\Sigma + g_{ij} \partial_i \mathcal{K}(z, \bar{z}) \right].
\]

The metric on the scalar manifold is given by

\[
g_{ij} = \partial_i \partial_j \mathcal{K}(z, \bar{z}). \quad (2.22)
\]

Here, \( \mathcal{K}(z, \bar{z}) \) is the Kähler potential and it reads

\[
e^{-\mathcal{K}(z, \bar{z})} = i \left( \bar{X}^\Lambda \mathcal{F}_\Lambda - X^\Lambda \bar{\mathcal{F}}_\Lambda \right). \quad (2.23)
\]

The matrix \( \mathcal{N}_{\Lambda\Sigma} \) of the gauge kinetic term is a function of the vector multiplet scalars and is given by

\[
\mathcal{N}_{\Lambda\Sigma} = \bar{\mathcal{F}}_{\Lambda\Sigma} + 2 \text{Im} \mathcal{F}_{\Lambda\Delta} \text{Im} \mathcal{F}_{\Sigma\Theta} X^\Delta X^\Theta / \text{Im} \mathcal{F}_{\Delta\Theta} X^\Delta X^\Theta. \quad (2.24)
\]

Finally, the scalar potential reads

\[
V(z, \bar{z}) = g^{ij} D_i \mathcal{L} D_j \bar{\mathcal{L}} - 3 |\mathcal{L}|^2, \quad (2.25)
\]

where \( \mathcal{L} = e^{\mathcal{K}/2} \left( X^\Lambda g_\Lambda - \mathcal{F}_\Lambda g^\Lambda \right) \) and \( D_i \mathcal{L} = \partial_i \mathcal{L} + \partial_i \mathcal{K} \mathcal{L}/2 \).

The ansatz for a static dyonic black hole with horizon \( \Sigma_g \) is of the form\(^{17}\)

\[
ds^2 = -e^{2U(r)} dt^2 + e^{-2U(r)} \left( dr^2 + V(r)^2 ds^2_{\Sigma_g} \right)
\]

\[
A^\Lambda = a \left( r \right) dt + a_1 \left( r \right) A_{\Sigma_g},
\]

where \( A_{\Sigma_g} \) is the gauge potential for a magnetic flux on \( \Sigma_g \). For example, for \( \Sigma_g = S^2 \)

we can take \( A_{S^2} = - \cos \theta d\phi. \) We assume that the scalar fields \( z^i \) have only radial

\(^{16}\)Other choices of gauge fixing for the rescaling symmetry are possible, corresponding to field redefinitions.

\(^{17}\)We normalize the metric on \( \Sigma_g \) such that the scalar curvature is \( 2\kappa \), where \( \kappa = 1 \) for \( S^2 \), \( \kappa = 0 \) for \( T^2 \), and \( \kappa = -1 \) for \( \Sigma_g \) with \( g > 1 \). The volume is then \( \text{Vol}(\Sigma_g) = 2\pi \eta \) where \( \eta = 2|g - 1| \) for \( g \neq 1 \) and \( \eta = 1 \) for \( g = 1 \).
dependence. We are interested in solutions that are asymptotic to AdS
4
for large
values of the radial coordinate, and this requires
\[ e^U(r) \sim r, \quad V(r) \sim r^2, \quad r \gg 1, \] (2.26)
and approach a regular horizon AdS
2
× Σg at some fixed value \( r = r_0 \),
\[ e^U(r) \sim r - r_0, \quad V(r) \sim e^{U(r)}, \quad r \sim r_0. \] (2.27)
Notice that we can also interpret these black holes as domain walls interpolating
between AdS
4
and AdS
2
× Σg. We will come back to the field theory interpretation
of this observation in section 3.

There are two conserved quantities
\[ \int_{\Sigma_g} F^\Lambda = \text{Vol}(\Sigma_g) p^\Lambda, \quad \int_{\Sigma_g} G_\Lambda = \text{Vol}(\Sigma_g) q_\Lambda, \] (2.28)
where \( G_\Lambda = 8\pi G_N \delta(L d\text{vol})/\delta F^\Lambda \), corresponding to the magnetic and electric
charges of the black hole. Under \( Sp(2nV + 2) \) they transform as a symplectic vector
\((p^\Lambda, q_\Lambda)\). In a frame with purely electric gauging \( g_\Lambda \), the magnetic and electric charges
are quantized as follows
\[ \text{Vol}(\Sigma_g) p^\Lambda g_\Lambda \in 2\pi \mathbb{Z}, \quad \frac{\text{Vol}(\Sigma_g) q_\Lambda}{4G_N^{(4)} g_\Lambda} \in 2\pi \mathbb{Z}, \] (2.29)
not summed over \( \Lambda \).

As already mentioned, the Killing spinors \( \epsilon_A, A = 1, 2 \), for static black holes
only depend on the radial coordinate. The BPS equations give a set of ordinary
differential equations for the functions \( U, V, a_0, a_1, z_i, \epsilon_A \) that are explicitly given in
[23–27]. For our purposes, the only important point is that the gravitino variation
contains, among other pieces,
\[ \delta \psi_\mu A = \partial_\mu \epsilon_A + \frac{1}{4} \omega^a_{\mu b} \Gamma^a_{ab} \epsilon_A + g_\Lambda A^\Lambda_{\mu} \delta A^B \epsilon^B \epsilon_B + \ldots. \] (2.30)
The vanishing of this variation, when the index \( \mu \) is restricted to \( \Sigma_g \), requires that
\( A^\Lambda_\mu \) cancels the spin connection and one obtains
\[ \sum_\Lambda g_\Lambda p^\Lambda = -\kappa. \] (2.31)
We see that, as already discussed, a linear combination of the magnetic charges is
fixed by the twist. In a general theory with also magnetic FI the previous condition
is replaced by
\[ \sum_\Lambda (g_\Lambda p^\Lambda - g^A q_\Lambda) = -\kappa \] (2.32)
which is manifestly symplectic invariant.

It has been noticed in [24] that the BPS equations of gauged supergravity for the near-horizon geometry can be put in the form of attractor equations. The BPS equations are indeed completely equivalent to the extremization of the quantity

$$I_{\text{sugra}}(X^\Lambda) = -i \frac{\text{Vol}(\Sigma_g) q^\Lambda X^\Lambda - p^\Lambda F_{\Lambda}}{4G_N^{(4)}} g^\Lambda X^\Lambda - g^\Lambda F_{\Lambda},$$

with respect to the horizon-value of the symplectic sections $X^\Lambda$, combined with the requirement that the value of $I_{\text{sugra}}$ at the critical point $\bar{X}^\Lambda$ is real. In general, in gauged supergravity, $F(X^\Lambda)$ is a homogeneous function of degree two, so we can equivalently define $\hat{Y}^\Lambda = X^\Lambda / (g^{\Sigma} X^\Sigma - g^{\Sigma} F_{\Sigma})$ and extremize

$$I_{\text{sugra}}(\hat{Y}^\Lambda) = i \frac{\text{Vol}(\Sigma_g)}{4G_N^{(4)}} \left( p^\Lambda F_{\Lambda}(\hat{Y}) - q_{\Lambda} \hat{Y}^\Lambda \right).$$

The extremization of (2.33) gives a set of algebraic equations for the value of the physical scalars $z^i$ at the horizon, and the entropy of the black hole is given by evaluating the functional (2.33) at its extremum

$$S_{\text{BH}}(p^\Lambda, q_{\Lambda}) = I_{\text{sugra}}(\bar{X}^\Lambda).$$

Not for all choices of prepotential $F$ and charges $(p^\Lambda, q_{\Lambda})$ we can find a regular black hole in AdS$_4$. However, as already mentioned, for the gauged supergravity that arises as a consistent truncation of M-theory on AdS$_4 \times S^7$, there is a six-dimensional family of regular black holes [23–27]. The $\mathcal{N} = 2$ gauged supergravity contains four vectors parameterizing the Cartan of $SO(8)$. In the language of gauged supergravity, one is the graviphoton and the other three give rise to a model with three vector multiplets, $n_V = 3$. By explicitly reducing M-theory on AdS$_4 \times S^7$, one can determine the prepotential

$$F = -2i \sqrt{X^0 X^1 X^2 X^3},$$

and the FI, $g_{\Lambda} \equiv g$, $g^\Lambda = 0$, that are purely electric. With these notations the AdS$_4$ vacuum has radius $L^2 = 1/2g^2$. We can introduce four magnetic and electric charges $(p^\Lambda, q_{\Lambda})$. However, the twisting condition (2.31) constraints linearly one magnetic charge. For purely magnetic black holes [23], this is the only constraint and we find a three-dimensional family of solutions. They are particularly simple since all the scalars $z_i$ are real. We can write, for example, the solution for a black hole with $S^2$ horizon

$$ds^2 = -\frac{1}{2} e^{\kappa(X)} \left( r - \frac{c}{r^2} \right)^2 dt^2 + 2 e^{-\kappa(X)} \left( r - \frac{c}{r^2} \right) dr^2 + 2 e^{-\kappa(X)} r^2 (d\theta^2 + \sin^2 \theta d\phi^2),$$

$$F_{\theta\phi}^\Lambda = p^\Lambda \sin \theta$$

See for example [59].
where the real sections are given by $X^\Lambda = \frac{1}{4} - \frac{\beta^\Lambda}{\gamma}$ and the parameters $\beta^\Lambda$ and $c$ are determined in terms of the magnetic charges by

$$c = 4\left(\beta^0_0 + \beta^1_1 + \beta^2_2 + \beta^3_3\right) - \frac{1}{2}, \quad -\sqrt{2}p^\Lambda - \frac{1}{2} = 16\beta^2_\Lambda - 4\sum_\Sigma \beta^2_\Sigma, \quad \sum_\Lambda \beta_\Lambda = 0$$

where we also set $L = 1$ or, equivalently, $g = 1/\sqrt{2}$. The generic dyonic black holes found in [26, 27] are more complicated and we will not report here the form of the solution. For dyonic black holes there is an extra constraint on the charges following from the requirement that the entropy (2.35) is a real number. This constraint is highly non-linear in the charges and leaves a six-dimensional family of black holes.

With the gauged supergravity data, in principle, is very simple to compute the entropy of all these black holes, even without knowing the explicit form of the metric. (2.34) gives a set of algebraic equations for $X^\Lambda$ and (2.35) gives the entropy. Notice, however, that the final expression is quite complicated, especially if compared with simple forms of the entropy as a function of charges that one can find for some asymptotically flat black holes. In the simplest case of a purely magnetic black hole with $p^1 = p^2 = p^3 \equiv -p/(2g)$, $p^0 = (3p - 2)/(2g)$ and horizon $S^2$, the dependence of the entropy on the charges is of the form

$$S_{BH} \sim \sqrt{-1 + 6p - 6p^2 + \sqrt{(6p - 1)(-1 + 2p)^3}}. \quad (2.38)$$

### 3 The topologically twisted index

As we saw, magnetically charged black holes in AdS$_4$ are dual to topologically twisted CFT$_3$. In this section, we discuss the topologically twisted index in three dimensions, $Z_{\text{index}}(\Delta_a, \omega_i)$, defined as the supersymmetric partition function $Z_{\Sigma_g \times S^1}^{\text{susy}}(\Delta_a, \omega_i)$ with a topological A-twist along $\Sigma_g$. We will discuss the case of a generic $\mathcal{N} = 2$ Yang-Mills-Chern-Simons theory in three dimensions with an R-symmetry and we will specialize to the ABJM theory in section 4.

The index can be computed in many different ways. We will discuss the localization approach here, following [60, 61]. The index has been first derived by topological field theory arguments in various examples in [62, 63] and discussed in general in [64]. In this second approach, further discussed and generalized in [65–71], the index is written as a sum of contributions coming from the Bethe vacua, the critical points of the twisted superpotential of the two-dimensional theory obtained by compactifying on $S^1$ [7, 72]. We will discuss the connections between the two approaches in section 3.3.2.

#### 3.1 The topological twist

Consider an $\mathcal{N} = 2$ theory in three dimensions. The $\mathcal{N} = 2$ supersymmetry multiplets are:
• the vector multiplet, \((A_\mu, \lambda, \sigma, D)\), where \(\lambda\) is a Dirac spinor and \(\sigma\) and \(D\) are real scalars. \(D\) is an auxiliary field;

• the chiral multiplet, \((\phi, \psi, F)\), where \(\psi\) is a Dirac spinor and \(\phi\) and \(F\) are complex scalars. \(F\) is an auxiliary field.

These multiplets can be obtained by dimensional reduction from the corresponding \(\mathcal{N} = 1\) multiplets in four dimensions. We assume that the theory has an R-symmetry\(^{19}\)

\[
\lambda \rightarrow e^{-i\alpha} \lambda, \quad (\phi, \psi, F) \rightarrow (e^{i r_\phi \alpha} \phi, e^{i (r_\phi - 1) \alpha} \psi, e^{i (r_\phi - 2) \alpha} F),
\]

with charges \(r_\phi\) for the chiral fields. The charges should be integrally quantized, as we will discuss in the following. We want to define a supersymmetric theory on \(\Sigma \times S^1\) using a non-trivial background for the R-symmetry. Let us start, for simplicity, with the case of \(S^2 \times S^1\) with metric

\[
ds^2 = R^2 (d\theta^2 + \sin \theta^2 d\phi^2) + \beta^2 dt^2,
\]

and a non-trivial R-symmetry background field \(A^R_\mu\).

To see if supersymmetry is preserved we can use the approach of [73]. We can promote the metric \(g_{\mu\nu}\) and the R-symmetry background field \(A^R_\mu\) to dynamical fields by coupling the theory to supergravity, using an appropriate off-shell formulation. The theory coupled to gravity is invariant under supersymmetry transformations with a local spinorial parameter \(\epsilon(x)\). We can recover the rigid theory by freezing the supergravity multiplet to a background value. This can be done, for example, by sending the Planck mass of the supergravity theory to infinity. In this process we set all supergravity fermionic fields to zero, while keeping a non-trivial background for the metric and \(A^R_\mu\), and possibly some auxiliary fields. The rigid theory so obtained is no more invariant under local supersymmetries. However, it is still invariant under those transformations that preserve the background fields. Using an off-shell formulation, the variation of the bosonic supergravity fields is automatically zero since it is proportional to the supergravity fermions that vanish in the background. On the other hand, the vanishing of the fermionic variations gives a differential equation for \(\epsilon(x)\). The solutions to this equation determine the rigid supersymmetries that are preserved by the curved background.\(^{20}\)

In any supergravity with a R-symmetry gauge field, with all the other supergravity fields set to zero, the fermionic variations have the universal form

\[
\delta \psi_\mu = D_\mu \epsilon = \partial_\mu \epsilon + \frac{1}{4} \omega^a_{\mu} \gamma_a \epsilon + i A^R_\mu \epsilon = 0,
\]

\(^{19}\)The sign of the charges is somehow unconventional (for example, \(\lambda\) and \(\epsilon\) have charge \(-1\)) but we keep it for consistency with [60, 61].

\(^{20}\)These conditions impose constraints on the manifold and the choice of background fields. For examples related to our context see [73–77].
In three dimensions, we can choose $\gamma_a = \sigma_a$, where $\sigma_a$ are the Pauli matrices. The non-trivial components of the spin connection are easily computed\(^{21}\) to be $\omega_{12} = -\cos \theta d\phi$. If we take $\gamma_3 \epsilon = \epsilon$, so that $\gamma_{12} \epsilon = i \epsilon$, we see that the background field

$$A^R = \frac{1}{2} \cos \theta d\phi$$

(3.4)

precisely cancels the spin connection. The equation reduces to

$$\delta \psi_{\mu} = \partial_{\mu} \epsilon = 0,$$

(3.5)

which is solved by a constant spinor $\epsilon$. We thus see that the background (3.4) allows to define a supersymmetric theory on $S^2 \times S^1$. The generalization of the above discussion to $\Sigma_g \times S^1$ is straightforward: we just turn on a background for the $R$-symmetry with $A^R = -\omega/2$ and everything else works in the same way. Notice that the previous discussion closely parallels the supergravity analysis of the supersymmetries preserved by a magnetically charged black hole in section 2.1.2.\(^{22}\)

This way of preserving supersymmetry corresponds to a topological twist along $\Sigma_g$ in the sense of [35, 78]. To make contact with the language used in [35, 78], we can interpret the background field for $A^R$ as effectively changing the spin of the fields in the theory, thus transforming $\epsilon$ into a scalar.

A supersymmetric Lagrangian for a Yang-Mills-Chern-Simons theory with gauge group $G$ and chiral matter transforming in a representation $\mathcal{R}$ on $\Sigma_g \times S^1$ can be written as $\mathcal{L} = \mathcal{L}_{YM} + \mathcal{L}_{CS} + \mathcal{L}_{\text{mat}} + \mathcal{L}_{W}$ with\(^{23}\)

$$\mathcal{L}_{YM} = \text{Tr} \left[ \frac{1}{4} F_{\mu \nu} F^{\mu \nu} + \frac{1}{2} D_\mu \sigma D^\mu \sigma + \frac{1}{2} D^2 - i \frac{1}{2} \lambda^\dagger \gamma^\mu D_\mu \lambda - i \frac{1}{2} \lambda^\dagger [\sigma, \lambda] \right]$$

$$\mathcal{L}_{CS} = -\frac{ik}{4\pi} \text{Tr} \left[ \epsilon^{\mu \nu \rho} \left( A_\mu \partial_\nu A_\rho - \frac{2i}{3} A_\mu A_\nu A_\rho \right) + \lambda^\dagger \lambda + 2D \sigma \right]$$

$$\mathcal{L}_{\text{mat}} = D_\mu \phi_i^\dagger D^\mu \phi_i + \phi_i^\dagger (\sigma^2 + i D - r_{\phi} F_{12}^R) \phi_i + F_i^+ F_i$$

$$+ i \psi_i^\dagger (\gamma^\mu D_\mu - \sigma) \psi_i - i \psi_i^\dagger \lambda \phi_i + i \phi_i^\dagger \lambda^\dagger \psi_i$$

(3.6)

$$\mathcal{L}_W = i \left( \frac{\partial W}{\partial \Phi_i} F_i - \frac{1}{2} \partial^2 W \psi_j^\dagger \psi_i + \frac{\partial W}{\partial \Phi_i} F_i^\dagger - \frac{1}{2} \partial^2 W \psi_i^\dagger \psi_j \psi_j^\dagger \psi_i \right),$$

where the superpotential $W(\phi_i)$ is a holomorphic function of $R$-charge two and the fields $A_\mu, \sigma, D$ act on the matter fields in the appropriate representation. Here the derivative $D_\mu$ is covariantized with respect to the spin and gauge connection and

\(21\)We use the frame $e^1 = Rd\theta, e^2 = R \sin \theta d\phi$ and $e^3 = \beta dt$.

\(22\)This is not a coincidence [74]: when holography applies, solving the Killing spinor equations in bulk near the AdS boundary gives a set of constraints on the boundary theory that are equivalent to those obtained with the approach proposed in [73].

\(23\)This can be obtained for example by taking the rigid limit of supergravity in the background (3.2), as suggested in [73].

– 19 –
also to the R-symmetry background $A^R$. As usual in Euclidean signature, fields and
their conjugate, $\phi$ and $\phi^\dagger$ for example, should be considered as independent variables.
Notice that a vev for the scalar field $\sigma$ gives mass to the matter fields $\phi_i$ and $\psi_i$. This
kind of coupling is typical of three dimensions and called real mass to distinguish it from the mass terms that can be introduced through the superpotential $W$. One
can easily check that the Lagrangian is invariant under the following supersymmetry
transformations

$$
QA_\mu = \frac{i}{2} \lambda^\dagger \gamma_\mu \epsilon \\
\tilde{Q}A_\mu = \frac{i}{2} \tilde{\epsilon}^\dagger \gamma_\mu \lambda \\
QD = -\frac{i}{2} D_\mu \lambda^\dagger \gamma^\mu \epsilon + \frac{i}{2} [\lambda^\dagger \epsilon, \sigma] \\
\tilde{Q}D = \frac{i}{2} \tilde{\epsilon}^\dagger \gamma^\mu D_\mu \lambda + \frac{i}{2} [\sigma, \tilde{\epsilon}^\dagger \lambda]
$$

for the vector multiplet fields and

$$
Q\phi = 0 \\
\tilde{Q}\phi = -\tilde{\epsilon}^\dagger \psi \\
Q\phi^\dagger = \psi^\dagger \epsilon \\
\tilde{Q}\phi^\dagger = 0
$$

for the chiral multiplets. To future purposes, we also define $Q = Q + \tilde{Q}$.

Notice that the Lagrangian (3.6) and the transformations of supersymmetry are
almost identical to the flat space ones with the further covariantization with respect
to the metric and the background R-symmetry. This is not always the case in curved
space, where extra terms should be included to maintain supersymmetry. In general,
a Lagrangian is not invariant under flat space supersymmetry transformations when
defined on a curved space because covariant derivatives do not commute anymore.
With a topological twist, the spinor $\epsilon$ is covariantly constant and this problem is
milder.

### 3.2 The localization formula

The topologically twisted index is just the $\Sigma_g \times S^1$ path integral of the theory discussed
in the previous section. We can evaluate it using localization. The basic idea of
localization is simple. Let us review it briefly, referring to [8, 9] for more details.\textsuperscript{24}
In a theory with a fermionic symmetry squaring to zero (or to a bosonic symmetry
of the theory)\textsuperscript{25} we can deform the action with a $Q$-exact term, $QV$, where $V$ is a

\textsuperscript{24}These references cover the developments after [2]. The idea of localization in physics is much
older and it has been applied to many system since [79].

\textsuperscript{25}In our case $Q^2$ is a linear combination of a gauge transformation and a rotation along $S^1$. 
fermionic functional invariant under all the symmetries. The new action is still $\mathcal{Q}$ invariant and the path integral independent of the deformation

$$Z^{\text{susy}}(t) = \int e^{-S_{\text{YM}} + t\mathcal{Q}V}; \quad \frac{d}{dt} Z^{\text{susy}} = \int e^{-S_{\text{YM}} + t\mathcal{Q}V} \mathcal{Q}V = 0,$$

since $\mathcal{Q}$ acts as a total derivative. The path integral can be then computed for $t \to \infty$ and, if we choose $V$ cleverly and we are lucky, it reduces to a sum over saddle points of a classical contribution and a one-loop determinant,

$$Z^{\text{susy}}(t = \infty) = \sum_{\text{saddle points}} e^{-S_{\text{classical}}} \frac{\det \text{fermions}}{\det \text{bosons}}.$$  \hspace{1cm} (3.8)

In many supersymmetric gauge theories on Euclidean manifolds, this approach successfully reduces the path integral to the evaluation of a matrix model.

In our theory, $\mathcal{L}_{\text{YM}}, \mathcal{L}_{\text{mat}}$ and $\mathcal{L}_W$ are not only $\mathcal{Q}$-closed but also $\mathcal{Q}$-exact. For example, up to total derivatives,

$$\epsilon^\dagger \epsilon \mathcal{L}_{\text{YM}} = \mathcal{Q} \mathcal{Q} \text{Tr} \left( \frac{1}{2} \lambda^\dagger \lambda + 2D\sigma \right).$$  \hspace{1cm} (3.9)

Therefore we can put arbitrary coefficients in front of the various $\mathcal{Q}$-exact terms in the Lagrangian

$$\mathcal{L} = \frac{1}{g^2} \mathcal{L}_{\text{YM}} + \mathcal{L}_{\text{CS}} + \frac{1}{\lambda^2} \mathcal{L}_{\text{mat}} + \frac{1}{\eta^2} \mathcal{L}_W,$$

and the path integral is independent of $g, \lambda, \eta$. We can then take the limit $g, \lambda, \eta \to 0$ and evaluate the path integral in the saddle point approximation. As usual, in localization, the superpotential is $\mathcal{Q}$-exact. This means that the partition function is independent of the precise form of the interactions in the Lagrangian. The superpotential is important nevertheless for determining the global symmetries of the theory, which enter in the partition function through chemical potentials.

We now give the localization formula for the topologically twisted index. Since the computation is complicated and subtle, we just provide the final formula, referring to [8, 9] for a general introduction to localization and to [60, 61] for the details of this particular computation.

The path integral of the topologically twisted theory on $\Sigma_g \times S^1$ for a $\mathcal{N} = 2$ supersymmetric gauge theory with gauge group $G$ can be written as a contour integral

$$Z_{\Sigma_g \times S^1} = \frac{1}{|W|} \sum_{m \in \Gamma} \oint_{C} Z_{\text{int}}(u, m),$$  \hspace{1cm} (3.11)

of a meromorphic form of Cartan-valued variables $u$, summed over a lattice $\Gamma$ of magnetic fluxes. $W$ is just the order of the Weyl group of $G$. We will explain all the ingredients in the following, referring to [60, 61] for proofs. Notice that from now on we drop the superscript $\text{susy}$ from the partition functions.
3.2.1 The BPS locus

We have a family of saddle points labeled by the vev of the scalar field \( \sigma \), the value of the Wilson line \( A_t \) along \( S^1 \) and a quantized magnetic flux \( m \) along \( \Sigma_g \). As standard in localization computation, these saddle points can be found as the locus where the fermionic variations vanish. The gaugino BPS equations read

\[
Q\lambda = \left( \frac{1}{2} \gamma^{\mu\nu} F_{\mu\nu} - D \right) \epsilon + i \gamma^{\mu} \epsilon D_{\mu} \sigma = 0 ,
\]

and are solved by setting the two terms on the right-hand side to zero. The second term in (3.12), \( D_\mu \sigma \), vanishes for constant commuting adjoint fields \( \sigma \) and \( A_t \). With a gauge transformation, we can put them in the Cartan subalgebra. We can combine these fields in a complex Cartan-valued quantity

\[
u = A_t + i\beta \sigma .
\]

The Wilson line \( A_t \) is periodic, invariant under a shift of any element \( \chi \) of the co-root lattice \( \Gamma \), \( A_t \sim A_t + 2\pi \chi \).\(^{26}\) the physical object being the holonomy \( e^{iA_t} \). So \( u \) naturally lives on a cylinder and it is natural to define the quantity \( x = e^{iu} \). For a \( U(1) \) theory, \( u \) lives on the cylinder \( S^1 \times \mathbb{R} \) and \( x \) on the punctured plane \( \mathbb{C}^* \). The first term in (3.12), using \( \gamma_{12} \epsilon = i \epsilon \), implies that the auxiliary field \( D \) is proportional to the gauge field strength along \( \Sigma_g \), \( D = iF_{12} \), and both live in the Cartan subalgebra. The curvature along \( \Sigma_g \) is quantized

\[
\frac{1}{2\pi} \int_{\Sigma_g} F = m \in \Gamma
\]

where \( \Gamma \) is again the co-root lattice. For a \( U(1) \) theory \( m \) is just an integer.

The path integral involves a sum over saddle points and is therefore given as an integral over \( u \) and a sum over the magnetic fluxes \( m \). Both variables live in the Cartan subalgebra and are only defined up to an action of the Weyl group, the surviving gauge symmetry. This explains the factor \( 1/|W| \) in (3.11). For a \( U(N) \) theory the co-root lattice is just \( \Gamma = \mathbb{Z}^N \) and the Weyl group is the permutation group of \( N \) elements with \(|W| = N!\).

3.2.2 The integrand

The contribution of the saddle point to the classical action only comes from the Chern-Simons term\(^{27}\)

\[
Z_{\text{CS}}^{\text{class}}(u) = x^{km} = \prod_{i=1}^{r} x_i^{km},
\]

\(^{26}\)The co-root lattice is defined by the requirement that \( e^{2\pi i \chi} \) acts as the identity on any representation of the group \( G \), and defines the weight lattice of the Langland (or S-dual) group \( \hat{G} \).

\(^{27}\)This expression follows from a holomorphic recombination of the terms \( A_t \wedge F_{12} \) and \( \sigma D \) in the Chern-Simons action, using \( D = iF_{12} \) and \( u = A_t + i\beta \sigma \).
where \( x = e^{iu} \).

The one-loop determinant receives contributions from the vector multiplets and the chiral multiplets. The vector multiplet contribution is

\[
Z_{\text{gauge}}^{1\text{-loop}}(u) = \prod_{\alpha \in G} (1 - x^\alpha)^{1-g} (i \, du)^r
\]

where \( \alpha \) are the roots of \( G \) and, for convenience, we included the integration measure \((du)^r\) in this expression. The chiral multiplet contribution is

\[
Z_{\text{chiral}}^{1\text{-loop}}(u, m) = \prod_{\rho \in \mathcal{R}} \left( \frac{x^\rho/2}{1 - x^\rho} \right)^{(\rho(m) + (g - 1)(r_\phi - 1))}
\]

\( (3.16) \)

where \( \mathcal{R} \) is the representation under the gauge group \( G \), \( \rho \) are the corresponding weights and \( r_\phi \) is the R-charge of the field. These expressions arise by taking ratios of determinants for fermionic and bosonic modes, computed by expanding in modes on \( \Sigma_g \times S^1 \). Due to supersymmetry, most of the modes cancel between bosons and fermions and we are left with the contribution of a set of zero-modes (indeed a convenient way to perform this computation is via an index theorem). For a chiral multiplet these zero modes contribute

\[
\prod_{\rho \in \mathcal{R}} \prod_{n = -\infty}^{\infty} \left( \frac{2\pi i}{\beta} n + i \rho \left( \frac{A_t}{\beta} + i \sigma \right) \right)^{-((\rho(m) + (g - 1)(r_\phi - 1))} \cdot (3.18)
\]

The term in bracket represents the mass of a chiral multiplet mode due to the coupling to \( \sigma \), which acts as a real mass, to the Wilson line \( A_t \) and to the KK momentum \( n \) along the circle \( S^1 \). The exponent is the multiplicity of the zero-mode that can be easily obtained using the Riemann-Roch theorem. Notice that this multiplicity must be an integer and therefore the R-charges \( r_\phi \) must be quantized.\(^{28}\)

This infinite product needs to be regularized. In (3.17) we chose a parity invariant regularization. There are other possible ones.\(^{29}\)

The full integrand is

\[
Z_{\text{int}}(u, m) = Z_{\text{pert}}(u, m) \left( \frac{\det \partial^2 \log Z_{\text{pert}}(u, m)}{\partial u_a \partial m_b} \right)^g
\]

\( (3.19) \)

where

\[
Z_{\text{pert}}(u, m) = Z_{\text{class}}^{\text{CS}}(u, m) Z_{\text{gauge}}^{1\text{-loop}}(u) Z_{\text{chiral}}^{1\text{-loop}}(u, m).
\]

\( (3.20) \)

The determinant term exists only on a Riemann surface of genus \( g > 0 \) and arise from the integration of the extra \( g \) fermionic zero-modes existing on these surfaces.

\(^{28}\)On \( S^2 \times S^1 \) the R-charges \( r_\phi \) must be integer. In the case of a higher genus Riemann surface it is enough to require that the quantity \((1 - g)(r_\phi - 1)\) is an integer.

\(^{29}\)Parity acts as \( u \rightarrow -u \) and (3.17) is obviously invariant. A gauge invariant regularization breaking parity is used in [67, 69–71]. The latter has the advantage of clarifying subtle sign issues and simplifying the mapping of parameters between dual theories. However, even for theories with zero CS, in the gauge invariant regularization one has to introduce extra effective CS contact terms, which makes the physical interpretation less transparent.
3.2.3 The contour

The integrand (3.19) is a meromorphic form in the Cartan variables \( u \) with poles at \( x^p = 1 \), the points in the BPS locus where chiral multiplets become massless, and at the boundaries \( x = 0 \) and \( x = \infty \) of the moduli space. The partition function is obtained by using the residue theorem. Supersymmetry will choose the correct integration contour and tell us which poles to include. One might hope that we need to integrate over some simple contour, like the unit circle in the plane \( x \), but one actually discovers that the contour is highly non-trivial and depends on the charges of the matter fields. For example, for a \( U(1) \) theory with chiral fields of charge \( Q_i \) and Chern-Simons level \( k \), defining the effective Chern-Simons level \(^{30}\)

\[
k_{\text{eff}}(\sigma) = k + \frac{1}{2} \sum_i Q_i^2 \text{sign}(Q_i \sigma),
\]

(3.21)

the rule is to take the residues of the poles created by fields with positive charge \( Q_i > 0 \), the residue at the origin \( x = 0 \) if \( k_{\text{eff}}(\infty) < 0 \) and the residue at infinity \( x = \infty \) if \( k_{\text{eff}}(-\infty) > 0 \). The rule for a generic gauge group can be written in terms of the so-called Jeffrey-Kirwan (JK) residue [80], a prescription for dealing with poles arising from multiple intersecting hyperplane singularities. To explain it properly will lead us too far and we refer to [60, 61] for details. The JK residue also appears in localization computations for elliptic genera in two dimensions, quantum mechanics, and various other partition functions [81–83].

The reader may object that we are supposed to integrate over the BPS locus, which is the whole complex plane, and not to perform a contour integral in \( u \). Luckily again supersymmetry comes to a rescue. On \( \Sigma_g \times S^1 \) there are gaugino zero-modes that contribute an extra term in the integrand in addition to the one-loop determinant. It turns out that the full integrand is a total derivative in \( \bar{u} \) and we can reduce the integral over the \( u \) plane to a contour integral around the singularities.

3.2.4 Adding flavor fugacities

If the theory has a flavor symmetry group \( F \) acting on the chiral fields, we can introduce extra parameters in a supersymmetric way. We can just gauge the flavor symmetry and then freeze all the bosonic fields to background values that have vanishing fermionic variations. These background bosonic fields will correspond to supersymmetric couplings in the Lagrangian. The analysis of fermionic variations is identical to the one performed in section 3.2.1 for gauge symmetries. We need to solve (3.12) for a background multiplet, \((A^F_\mu, \lambda^F, \sigma^F, D^F)\). The result is that we can turn on in a supersymmetric way a constant value for \( \sigma^F \) and \( A^F_\mu \) which we combine into a complex quantity \( u^F = A^F_\mu + i\beta \sigma^F \), and a magnetic flux \( m^F \) with

\(^{30}\)This is actually the Chern-Simons level that one sees at one-loop after integrating out the matter fields (they have mass \( \sigma \) at a generic point of the BPS locus).
$D^F = i m_F$. $\sigma^F$ appears in the Lagrangian as a real mass for the chiral fields. In three dimensions, any gauge theory with a $U(1)$ factor with field strength $F$ has also a topological symmetry associated with the current $J = *F$, which is automatically conserved. We can similarly introduce parameters $u_T$ and $m_T$ for the topological symmetry.

The path integral is then a function of $x_F, x_T$ and $m_F, m_T$. In the localization formula we just need to replace the one-loop determinant of a chiral field with

$$Z_{\text{chiral}}^{1-\text{loop}}(u, m; u_F, m_F) = \prod_{\rho \in \mathbb{R}} \left( \frac{x^{\rho/2} F^\nu/2}{1 - x^\rho x_F^\nu} \right)^{\rho(m) + \nu(m^F) + (g-1)(r_\phi-1)}$$

where $x_F = e^{i u_F}$ and $\nu$ is the weight of the chiral field under the flavor symmetry $F$. There is no modification to the vector multiplet determinant. A $U(1)$ topological symmetry just contributes a classical term

$$x^{m_F} x_T^{m_T}$$

to the classical action.

### 3.2.5 The trace interpretation

As any path integral that involves an $S^1$ factor, the topologically twisted index can be written as a trace\(^{\text{31}}\)

$$Z_{\Sigma \times S^1}(x_G, m_G) = \text{Tr}(-1)^F e^{i A_G^i J^G} e^{-\beta H_g}$$

where $H_g$ is the Hamiltonian of the topologically twisted theory on $\Sigma_g$, in the presence of magnetic fluxes $m_G = (m_F, m_T)$ and a supersymmetric background $x_G = (x_F, x_T)$ for the global symmetries, whose conserved charges have been denoted as $J^G$. The Hamiltonian $H_g$ explicitly depends on the magnetic fluxes $m_G$ and the real masses $\sigma^G$. If sufficiently real masses are turned on, the spectrum of $H_g$ is discrete and the trace is well-defined.

### 3.2.6 An Example: SQED

To explain all the ingredients, we can give a simple example of the final formula, using supersymmetric QED. This is a $U(1)$ theory with two chiral multiplets $Q$ and $\tilde{Q}$ of charges $\pm 1$ (electron and positron), and no Chern-Simons couplings. Since there is no superpotential, we have many possible choices of integer R-charges for the fields. We choose to assign R-charge $+1$ to both $Q$ and $\tilde{Q}$. There is an axial flavor symmetry $U(1)_A$ acting on $Q$ and $\tilde{Q}$ with equal charges and a topological symmetry

\(^{\text{31}}\)See footnote 11. The factor $e^{i A_G^i J^G}$ represents the insertion of a Wilson line $A_G^i$.\[\]
The charges of the chiral fields are

\[
\begin{array}{cccc}
Q & U(1)_g & U(1)_T & U(1)_A & U(1)_R \\
\hat{Q} & 1 & 0 & 1 & 1 \\
\end{array}
\]  

(3.25)

The topological symmetry acts only on non-perturbative states constructed with monopole operators. We introduce a gauge variable \(x\), with associated magnetic flux \(m\), and flavor and topological variables \(y = x_F\) and \(\xi = x_T\), with associated background fluxes \(n = m_F\) and \(t = m_T\). According to our rules, the partition function on \(S^2 \times S^1\) is

\[
Z(y, \xi, n, t) = \sum_{m \in \mathbb{Z}} \int \frac{dx}{2\pi i x} x^i (-\xi)^m \left(\frac{x^1 y^1}{1 - xy}\right)^{m+n} \left(\frac{x^{-1} y^{-1}}{1 - x^{-1} y}\right)^{-m+n} 
\]  

(3.26)

where we included an extra \((-1)^m\), which can be reabsorbed in the definition of \(\xi\), for later convenience.\(^\text{32}\) Notice that gauge and flavor variables enter in a similar way in this formula. The main difference is that \(x\) and \(m\) are integrated and summed over, while \(y, \xi\) and \(n, t\) are background parameters.

Our prescription instructs us to take the residues from the field \(Q\) with positive gauge charge, whose pole is at \(x = \frac{1}{y}\). By computing residues and resumming the result, one finds

\[
Z(y, \xi, n, t) = \left(\frac{y}{1 - y^2}\right)^{2n-1} \left(\frac{\xi \frac{1}{2} y^{-\frac{1}{2}}}{1 - \xi y^{-1}}\right)^{i-n+1} \left(\frac{\xi^{-\frac{1}{2}} y^{-\frac{1}{2}}}{1 - \xi^{-1} y^{-1}}\right)^{-i-n+1} 
\]  

(3.27)

One recognizes here the product of three factors of the form (3.22) that we can associate with free chiral multiplets. Indeed it is well known that the mirror theory to SQED is a Wess-Zumino model with fields \(M, T, \tilde{T}\) and a cubic superpotential \(W = MT \tilde{T}\) [84].

### 3.3 Interpretation of the localization formula

We can give an interpretation of the localization formula for the theory on \(\Sigma_g \times S^1\) in two different ways that correspond to two different dimensional reductions of the three-dimensional theory. Compactification on \(\Sigma_g\) gives rise to a quantum mechanics and compactification on \(S^1\) to a two-dimensional \((2, 2)\) supersymmetric theory.

#### 3.3.1 Reduction to quantum mechanics

Compactifying on \(\Sigma_g\), we obtain a supersymmetric quantum mechanics describing an infinite number of KK modes on \(\Sigma_g\). These are particles living on the Riemann

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\(^{32}\)For a more careful discussion of sign ambiguities see [69]. They will not play any important role in these lectures.
surface in the presence of a magnetic field for the R-symmetry and magnetic fluxes \( m_G \) for the global symmetries. These magnetic fields create Landau levels. The trace (3.24) can be interpreted as the Witten index \([45]\) of this quantum mechanics. Let us understand this concept better.

The quantum mechanics in question has \( \mathcal{N} = 2 \) supersymmetry. With no background for the global symmetries, the algebra of supersymmetry is simply \( \{\bar{Q}, Q\} = H_g \), where \( Q \) is a complex supercharge. The index is just

\[
\text{Tr}(-1)^F e^{-\beta H_g},
\]

and, according to standard arguments, is independent of \( \beta \). Indeed, any state \( \psi \) with \( H_g \neq 0 \) has a non-zero partner \( Q\psi \) with the same energy and opposite statistic and their contributions cancel in the trace. Therefore the only contribution comes from ground states.\(^{33}\) The index is then clearly independent of \( \beta \) and it is an integer counting the number of ground states with signs (plus for bosonic ones, minus for fermionic ones). When we turn on backgrounds for the global symmetries, the supersymmetry algebra is modified to \( \{\bar{Q}, Q\} = H_g - \sigma^G J^G \), where \( J^G \) is the conserved charge associated to the global symmetry.\(^{34}\) Using (3.24), we can now write the index as follows

\[
\text{Tr}(-1)^F e^{iA_G^G J^G} e^{-\beta H_g} = \text{Tr}(-1)^F e^{i(A_G^G + i\beta \sigma^G) J^G} e^{-\beta \{\bar{Q}, Q\}} = \sum_n g(n) x_G^n,
\]

where \( g(n) \) is the number of supersymmetric states, \( Q\psi = 0 \), with charge \( n \) under the global symmetry and, as usual, \( x_G = e^{i(A_G^G + i\beta \sigma^G)} \). In deriving this expression, we used again the fact that states with \( Q\psi \neq 0 \) are paired by supersymmetry and have the same energy and charge. This time, the states that contribute to the trace are chiral states with \( H_g = \sigma^G J^G \). In this way, we have obtained an equivariant index, where the supersymmetric states are graded according to their charge by powers of the fugacity \( x_G \). Notice also that this argument shows that the topologically twisted index is an holomorphic function of the fugacities, as we already found using localization.

Let us also notice that the integrand of the localization formula has a simple Hamiltonian interpretation. There are two type of multiplets in the \( \mathcal{N} = 2 \) quantum mechanics we are discussing: the chiral multiplet containing a complex scalar \( \phi \) and a spinor as dynamical fields, and the Fermi multiplet containing only a spinor \([82]\). The Landau levels on \( \Sigma_g \) give rise to zero-modes with multiplicities dictated by the Riemann-Roch theorem. One can see that the zero-modes organize themselves into \( \rho(m) + (g - 1)(r_{\phi} - 1) \) chiral multiplets if \( \rho(m) + (g - 1)(r_{\phi} - 1) > 0 \), and

\(^{33}\)By the algebra of supersymmetry \( H_g \psi = 0 \) is equivalent to \( Q\psi = 0 \) and, therefore, ground states not necessarily have a partner.

\(^{34}\)See, for example, \([82]\) or appendix C of [3].
Fermi multiplets if $\rho(m) + (g - 1)(r_\phi - 1) < 0$, where for simplicity we set the flavor fugacities to zero. We can now compute the index for Fermi and chiral multiplets. For the Fermi multiplet the Hilbert space is a fermionic Fock space, and assigning charge $-\frac{\rho}{2}$ and fermion number 0 to the vacuum, the index is

$$\frac{1 - x^\rho}{x^{\frac{\rho}{2}}}.$$  \hspace{1cm} (3.30)

For the chiral multiplet the Hilbert space is the product of a bosonic Fock space generated by $\phi, \phi^\dagger$ and a fermionic Fock space; assigning fermion number 1 to the vacuum, the index is

$$(-x^{-\frac{\rho}{2}} + x^{\frac{\rho}{2}}) \sum_{n=0}^{\infty} x^{n\rho} \sum_{m=0}^{\infty} x^{-m\rho} = \frac{x^{\rho}}{1 - x^\rho}.$$  \hspace{1cm} (3.31)

Raising these quantities to a power corresponding to the multiplicity, and taking into account the different signs for the two types of multiplets, we recover exactly the contribution (3.17) of a three-dimensional chiral multiplet to the partition function.

In particular, the localization formula for the topologically twisted index is just the sum over many topological sectors labelled by $m$ of the localization formula for the Witten index of $\mathcal{N} = 2$ quantum mechanics found in [82].

### 3.3.2 Reduction to two dimensions

We can alternatively reduce our three-dimensional theory on $S^1$ and obtain a $(2,2)$ supersymmetric theory containing all the KK modes on $S^1$. At a generic point of the Coulomb branch where $\sigma \neq 0$, all the non-Cartan gauge bosons and the chiral multiplets are massive\(^{35}\) and we can integrate them out and write a Lagrangian for the Cartan modes of the vector multiplets. In two dimensions, a vector multiplet can be described using a twisted chiral multiplet $\Sigma$ and its interactions is described by a twisted superpotential $\int d\theta^+ d\bar{\theta}^- W$.\(^{36}\)

It is interesting to observe that such twisted superpotential $W$ enters explicitly in the integrand of the localization formula [67, 69]. Indeed, the dependence on the gauge flux $m$ can be explicitly written as

$$\sum_{m \in \Gamma} \int \frac{dx_i}{2\pi i x_i} Q(x) e^{i m_i \frac{\partial W}{\partial x_i}},$$  \hspace{1cm} (3.32)

where $Q(x)$ is a meromorphic function independent of $m$. The function $W$, up to an overall normalization and sign ambiguities that we fix for convenience, is given by\(^{37}\)

$$W(u) = \frac{k}{2} \sum_i u_i^2 + \sum_{n \in \mathbb{N}} \left( \frac{1}{2} g_2(\rho(u)) - \text{Li}_2(e^{i \rho(u)}) \right),$$  \hspace{1cm} (3.33)

\(^{35}\)Due to the KK mass or their coupling to $\sigma$.

\(^{36}\)Σ has a scalar as its lowest component and it satisfies $D_+ \Sigma = D_- \Sigma = 0$. See [85].

\(^{37}\)Polylogarithms $\text{Li}_s(z)$ are defined by $\text{Li}_s(z) = \sum_{n=1}^{\infty} \frac{z^n}{n^s}$ for $|z| < 1$ and by analytic continuation outside the disk. Notice, in particular, that $\text{Li}_1(z) = -\log(1-z)$. For $s \geq 1$, there exists a the branch
with \( g_2(u) = \frac{u^2}{2} - \pi u + \frac{\pi^2}{3} \). As argued in [7, 64], this can be interpreted as the effective twisted superpotential of the two-dimensional theory obtained by compactifying on \( S^1 \). The first term in (3.33) is the classical contribution coming from the CS term, and the second is the sum of all the perturbative contributions of massive fields, including the infinite tower of KK modes. Indeed, a one-loop diagram for a mode of mass \( m \) contributes a term proportional to \( i(\Sigma + m)(\log(\Sigma + m)/2\pi - 1) \) to \( \mathcal{W} \) and there are no higher order corrections [85]. The contribution of the KK modes of a chiral multiplet, whose mass depends on \( \sigma \), the Wilson line \( A_t \) and the KK momentum \( n \), can be resummed to

\[
i \sum_{n \in \mathbb{Z}} (u + 2\pi n) \left( \log \frac{u + 2\pi n}{2\pi} - 1 \right) = -\text{Li}_2(e^{i\rho(u)}). \tag{3.34}
\]

The other term in the round bracket in (3.33) is local and it is due to our choice of a parity invariant regularization.

Using the asymptotic expansion of the polylogarithms, we find that the content of the bracket in (3.33) behaves, for large \( \sigma \), as

\[
\rho(u)^2 \frac{\text{sign}(\rho(\sigma))}{4}. \tag{3.35}
\]

This can be interpreted as a one-loop effective Chern-Simons term obtained by integrating out a field of mass \( \rho(\sigma) \).[39] The Jeffrey-Kirwan prescription typically selects poles in the integrand of the localization formula that are contained in a half-lattice \( m_i > M \) (or \( m_i < M \)) for some cut-off \( M \). We can then use the geometric series to resum the integrand in (3.32)

\[
\int \frac{dx_i}{2\pi i x_i} \frac{Q(x)}{\prod_i \left(1 - e^{iQ(x)/2\pi} \right)}, \tag{3.36}
\]

and evaluate the index by taking the residues at the poles

\[
\exp \left( i \frac{\partial \mathcal{W}}{\partial u_i} \right) = 1. \tag{3.37}
\]

These are the so-called Bethe vacua of the two-dimensional theory. They play an important role in the Bethe/gauge correspondence [7, 72].

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[38] For a choice of a gauge invariant regularization and an extensive discussion of other issues related to definition of \( \mathcal{W} \) see [69].

[39] For a field of mass \( m \) and charge \( Q_i \) the one-loop effective Chern-Simons term is \( k_{\text{eff}} = k + \frac{1}{2}Q_i^2 \text{sign}(m) \).
We then find the following general characterization of the topologically twisted index as a sum over Bethe vacua

\[ Z_{\Sigma_g \times S^1} = \sum_{x^*} \frac{Q(x^*)}{\det_{ij}(-\partial^2_{u_iu_j}W(x^*))} , \]

where \( x^* \) are the solutions of (3.37). The expression for the topologically twisted index as a sum over Bethe vacua was first derived by topological field theory arguments in [62–64, 66]. In the context of localization, this expression for the index has been derived and generalized in [67, 69–71]. The expression in (3.38) can be also written as

\[ Z_{\Sigma_g \times S^1} = \sum_{x^*} \mathcal{H}(x^*)^{g-1} , \]

in terms of a handle-guing operator \( \mathcal{H}(x) = e^{\Omega(x)} \det_{ij} \partial_{u_iu_j} W(x) \) and an effective dilaton \( \Omega(x) \) whose complete characterization in terms of field theory data can be found in the above mentioned papers. Here we just notice that, for genus \( g > 0 \), the Hessian of \( W \) enters at the power \( g-1 \). Indeed, the determinant in (3.19) contributes \( g \) extra powers of the Hessian that combine with the contribution in (3.38).

A very interesting result of [69–71] is that there exists a generalization of formula (3.39) to three-dimensional manifolds that are not a direct product. For example, the supersymmetric partition function on a three-manifold \( M_3 \) which is an \( S^1 \) fibration of Chern class \( p \) over a Riemann surface \( \Sigma_g \) can be written as a sum over the very same set of Bethe vacua,

\[ Z_{M_3} = \sum_{x^*} \mathcal{F}(x^*)^p \mathcal{H}(x^*)^{g-1} , \]

for a fibering operator \( \mathcal{F}(x) \) that can be expressed in terms of field theory data. There exists a similar result for the partition function on more general three-dimensional manifolds and also for some selected four-dimensional ones [69–71]. The particular case of the formula for the four-dimensional superconformal index plays a role in the physics of AdS\(_5\) black holes [6, 86], as discussed in section 6.2.

Let us give a couple of examples of Bethe vacua. For a pure \( \mathcal{N} = 2 \) Chern-Simons theory with gauge group \( SU(2) \), the expression for the partition function (3.19) is

\[ Z = \frac{(-1)^{g-1}}{2} \sum_{m \in \mathbb{Z}} \int_{jk} \frac{dx}{2\pi i x} (2k)^{\theta} x^{2k} \left[ \left( 1 - x^2 \right)^2 \right]^{1-g} , \]

where we used \( x_i = (x, 1/x) \) and \( m_i = (m, -m) \). The twisted superpotential receives contribution only from the classical action: \( W = \sum_i k u_i^2 / 2 = k u^2 \). The Bethe vacua (3.37) are then \( x^{2k} = 1 \) with solutions the \( 2k \)-roots of unity. Formula (3.38) gives, up to an ambiguous sign,

\[ Z = \left( \frac{\bar{k} + 2}{2} \right)^{g-1} \sum_{j=1}^{k+1} \left( \sin \frac{\pi i}{k+2} j \right)^{2-2g} , \]
where $\bar{k} = k - 2$. This is the well-known Verlinde formula for the CS partition function on $\Sigma_g \times S^1$.\textsuperscript{40} Notice that the root $x = 1$ is not included in the sum: as a general rule, the Bethe vacua that are also zeros of the Vandermonde determinant are not physical.

In the presence of matter, the Bethe equations (3.37) are more complicated. In the SQED example discussed in (3.2.6), the Bethe equation is

$$\frac{\xi(y-x)}{1-xy} = 1,$$

(3.43)

with solution $x = (1 - \xi y)/(y - \xi)$. It is easy to see that (3.38) correctly reproduces (3.27). For a general theory with gauge group $G$, the Bethe equations (3.37) cannot be analytically solved.

4 The entropy of dyonic AdS$_4$ black holes

In this section we consider the large $N$ limit of the topologically twisted index for the ABJM theory in three dimensions and we will derive microscopically the entropy of a family of dyonic black holes that can be embedded in AdS$_4 \times S^7$ [3].

The ABJM theory describes the low-energy dynamics of $N$ M2-branes on $\mathbb{C}^4/\mathbb{Z}_k$ [4]. It is a three-dimensional supersymmetric Chern-Simons-matter theory with gauge group $U(N)_k \times U(N)_{-k}$ (the subscripts are the CS levels) and matter in bifundamental representation. The matter content, in $\mathcal{N} = 2$ notations, is described by the quiver diagram

\begin{center}
\begin{tikzpicture}
\node (Nk) at (0,0) {$N$};
\node (N-k) at (2,0) {$N$};
\node (Ai) at (1,1.5) {$A_i$};
\node (Bi) at (1,-1.5) {$B_j$};
\path[->] (Nk) edge[bend right] (Ai);
\path[->] (N-k) edge[bend right] (Bi);
\path[->] (Ai) edge[bend right] (Bi);
\end{tikzpicture}
\end{center}

where $i, j = 1, 2$ and nodes represent gauge groups and arrows represent bifundamental chiral multiplets. The theory has a quartic superpotential

$$W = \text{Tr} (A_1 B_1 A_2 B_2 - A_1 B_2 A_2 B_1).$$

(4.1)

The ABJM theory has a number of interesting properties:

- the theory has $\mathcal{N} = 6$ superconformal symmetry, non-perturbatively enhanced to $\mathcal{N} = 8$ for $k = 1, 2$;

\textsuperscript{40}Since $\sigma$ and $\lambda$ are massive and free, they can be integrated out leading to a shift in the CS coupling. An $\mathcal{N} = 2$ Chern-Simons theory is thus equivalent to a bosonic CS theory with level $k = k - 2$. 
• it has an SU(4) R-symmetry, enhanced to SO(8) for \( k = 1, 2 \);

• for \( N \gg k^5 \) the theory is well-described by a weakly coupled M-theory background, \( \text{AdS}_4 \times S^7/Z_k \);

• the free energy on \( S^3 \) can be computed using localization and scales as \( O(N^{3/2}) \) in the M-theory limit \[87\]

\[
F_{S^3} = \log Z_{S^3} = \frac{\pi \sqrt{2}}{3} \sqrt{k} N^{3/2}.
\]

(4.2)

For a review of these properties see \[8, 88\].

We will consider the case \( k = 1 \) where the theory has maximal supersymmetry, SO(8) R-symmetry and is dual to AdS\(_4 \times S^7\). The four abelian symmetries of the theory, \( U(1)^4 \subset SO(8) \) correspond, in \( \mathcal{N} = 2 \) notation, to an R-symmetry and three global symmetries. There are many choices of \( U(1) \) R-symmetry corresponding to different decompositions \( SO(8) \rightarrow U(1)_R \times U(1)_S \). In order to write the index we need to select one with integer charges. Introducing a natural basis of \( U(1) \) R-symmetries,

\[
\begin{array}{c|cccc}
R_1 & R_2 & R_3 & R_4 \\
\hline
A_1 & 2 & 0 & 0 & 0 \\
A_2 & 0 & 2 & 0 & 0 \\
B_1 & 0 & 0 & 2 & 0 \\
B_2 & 0 & 0 & 0 & 2 \\
\end{array}
\]

(4.3)

we can for example choose the R-symmetry \( \sum_a R_a/2 \) that has integer charges. The remaining three \( U(1) \)s combine to give three flavor symmetries, say \( (R_a - R_4)/2 \) for \( a = 1, 2, 3 \).

Our general rules for the index allow to introduce a number of independent fluxes and fugacities equal to the number of global symmetries. We then introduce three magnetic fluxes \( p \) and three fugacities \( y \) for the three flavor symmetries of ABJM. It will be convenient to choose a redundant but democratic parameterization of these quantities. We assign a flux and a fugacity, \( p_a \) and \( y_a \) with \( a = 1, 2, 3, 4 \), to each of

\[\text{We are cheating a little bit here. The ABJM theory has also two topological symmetries, } T_1 \text{ and } T_2, \text{ associated with the two } U(1) \text{ gauge groups. This apparently makes a total of five } U(1) \text{ global symmetries. However } T_1 + T_2 \text{ is decoupled, and the baryonic symmetry that rotates } A_i \text{ and } B_i \text{ with opposite charges is actually gauged. More precisely, due the CS term, a linear combination of } T_1 - T_2 \text{ and the baryonic symmetry differ by a gauge transformation and are therefore equivalent. In the index we could introduce extra fluxes and fugacities for } T_1 \text{ and } T_2 \text{ but these can be re-absorbed by a shift of the fluxes and a rescaling of the integration variables. See } [3] \text{ for more details.} \]
the fields $A_1, A_2, B_1, B_2$ in the order indicated. The index is given by

$$Z = \frac{1}{(N!)^2} \sum_{m, \tilde{m} \in \mathbb{Z}^N} \int \prod_{i=1}^N \frac{dx_i}{2\pi i x_i} \frac{d\tilde{x}_i}{2\pi i \tilde{x}_i} \prod_{i \neq j}^N \left(1 - \frac{x_i}{x_j}\right) \left(1 - \frac{\tilde{x}_i}{\tilde{x}_j}\right) \times \prod_{i,j}^N \prod_{a=1,2} \left(\frac{\sqrt{\frac{x_i}{\tilde{x}_j} y_a}}{1 - \frac{x_i}{\tilde{x}_j} y_a}\right)^{m_i - \tilde{m}_j - p_a + 1 - g} \prod_{a=3,4} \left(\frac{\sqrt{\frac{\tilde{x}_i}{x_j} y_a}}{1 - \frac{\tilde{x}_i}{x_j} y_a}\right)^{\tilde{m}_j - m_i - p_a + 1 - g} \left(\det_{AB} \frac{\partial^2 W}{\partial u_A \partial u_B}\right)^g ,$$

(4.4)

where we used the rules of section 3.2.2. Notice that the Hessian of $W$ should be computed using the $2N$ variables $u_A = (u_i, \tilde{u}_i)$. We have also included the CS term $k$ in order to make clear where the terms come from but soon we will set $k = 1$. As already said, the fugacities are not independent. Since the superpotential (4.1) must have charge zero under a global symmetry we must set

$$\prod_{a=1}^4 y_a = 1.$$  

(4.5)

This translates into a constraint for the corresponding (complexified) chemical potentials $\Delta_a$, $y_a = e^{i\Delta_a}$, $\sum_a \Delta_a \in 2\pi \mathbb{Z}$,  

(4.6)

since the $\Delta_a$ are only defined modulo $2\pi$.\footnote{In the notation of section 3.2.4, $\Delta_a = A_{fa}^\Lambda + i \beta \sigma_a^F$, where $A_{fa}^\Lambda$ and $\sigma_a^F$ are the backgrounds for the $a$-th symmetry. The periodicity of $\Delta_a$ is due to the periodicity of the Wilson line $A_{fa}^\Lambda$.} Similarly, the four fluxes $p_a$ are not independent. To understand our parameterization, let us compare the chiral fields contributions in (4.4) with (3.22). Identifying exponents we have

$$- p_a + 1 - g = m_a^F + (g - 1)(r_a - 1),$$

(4.7)

where $m_a^F$ is an assignment of background fluxes for the global symmetry and $r_a$ the R-charge of the $a$-th field. Since $W$ has charge zero under global symmetries and charge two under R-symmetries, we have $\sum_{a=1}^4 m_a^F = 0$ and $\sum_{a=1}^4 r_a = 2$, so that

$$\sum_{a=1}^4 p_a = 2(1 - g).$$

(4.8)

The dependence of our index on three magnetic fluxes and three fugacities fits well with the family of black holes discussed in section 2.2 that have three magnetic and three electric charges. (4.8) is clearly the analog of (2.31) and already suggests the following identification between parameters $p^A \rightarrow -\kappa p_a/(2g(1 - g))$.  


The index can be written as a sum over Bethe vacua (3.38). The twisted superpotential \((3.33)\) reads
\[
W = \sum_{i=1}^{N} \frac{k}{2} (\bar{u}_{i}^{2} - u_{i}^{2}) + \sum_{i,j=1}^{N} \left[ \sum_{a=3,4} \text{Li}_{2} \left( e^{i(\bar{u}_{j} - u_{i} + \Delta_{a})} \right) - \sum_{a=1,2} \text{Li}_{2} \left( e^{i(\bar{u}_{j} - u_{i} - \Delta_{a})} \right) \right] \tag{4.9}
\]
and the Bethe vacua equations are
\[
x_{i}^{k} \prod_{j=1}^{N} \frac{(1 - y_{3} \frac{x_{j}}{x_{i}})(1 - y_{4} \frac{x_{j}}{x_{i}})}{(1 - y_{1} \frac{x_{j}}{x_{i}})(1 - y_{2} \frac{x_{j}}{x_{i}})} = x_{j}^{k} \prod_{i=1}^{N} \frac{(1 - y_{3} \frac{x_{i}}{x_{j}})(1 - y_{4} \frac{x_{i}}{x_{j}})}{(1 - y_{1} \frac{x_{i}}{x_{j}})(1 - y_{2} \frac{x_{i}}{x_{j}})} = 1 . \tag{4.10}
\]
In the large \(N\) limit we expect that just one Bethe vacuum dominates the partition function.

### 4.1 The ABJM Bethe vacua in the large \(N\) limit

We want to study the solutions of (4.10) in the large \(N\) limit [3]. By running numerics, one discovers that the imaginary parts of the solutions \(u_{i}\) and \(\bar{u}_{i}\) grow with \(N\),
\[
u_{i} = i N^{\alpha} t_{i} + \nu_{i} \quad \bar{\nu}_{i} = i N^{\alpha} t_{i} + \bar{\nu}_{i} \tag{4.11}
\]
and are equal for the two sets, while the real parts remain bounded. As usual, in the large \(N\) limit, the distributions of \(u_{i}\) and \(\bar{u}_{i}\) become almost continuous and we introduce a parameter \(t(i/N) = t_{i}\), defined in an interval \([t_{-}, t_{+}]\). We also introduce two functions of \(t, v(t)\) and \(\bar{v}(t)\), defined implicitly by \(v(i/N) = \nu_{i}\), \(\bar{v}(i/N) = \bar{\nu}_{i}\), and a normalized density
\[
\rho(t) = \frac{1}{N} \frac{dt}{dt}, \quad \int_{t_{-}}^{t_{+}} \rho(t)dt = 1 . \tag{4.12}
\]

The interesting feature of this model is that \(W\) becomes a local functional,\(^{44}\)
\[
W[\rho(t), \delta v(t)] = i N^{1+\alpha} \int dt t \rho(t) \delta v(t) + i N^{2-\alpha} \int dt \rho(t)^{2} g_{3}(-\epsilon_{a} \delta v(t) + \Delta_{a}) ,
\tag{4.13}
\]
where \(\delta v(t) = \bar{v}(t) - v(t)\), \(g_{3}(u) = \frac{1}{6} u^{3} - \frac{1}{2} \pi u^{2} + \frac{\pi^{2}}{3} u\) and \(\epsilon_{a} = 1\) for \(a = 1, 2\) and \(\epsilon_{a} = -1\) for \(a = 3, 4\). We also assumed that \(\Delta_{a}\) are real and
\[
0 < -\epsilon_{a} \delta v(t) + \Delta_{a} < 2\pi \tag{4.14}
\]

\(^{43}\)We used the of polylogarithm identities given in footnote 37 in order to recombine the terms in (3.33), and discarded terms that do not contribute to the Bethe equations (3.37). We also introduced an extra minus sign in the definition of \(W\) in order to match the original conventions in [3, 89]. It is easy to check directly that the equations (4.10) give the position of the poles of the integrand after we sum the geometric series in \(m_{i}\) and \(\tilde{m}_{i}\) and that, with the given definition of \(W\), (4.10) are equivalent to (3.37).

\(^{44}\)This is similar to other matrix models solved using localization in three and five dimensions [90–93].
for all $a$. The first term in $\mathcal{W}$ comes from the Chern-Simons interaction and the second is the contribution of matter fields. The derivation of (4.13) is given in [3]. Here we just mention few facts.

- $\mathcal{W}$ is local because of the exponential terms $e^{i(\tilde{u}_j-u_i+\Delta_a)}$ in the arguments of polylogs in (4.9). Due to (4.11), for $j > i$ the polylogs are exponentially suppressed in the large $N$ limit. For $i > j$ the exponential is large but we can use the identity $\text{Li}_2(e^{iu}) + \text{Li}_2(e^{-iu}) = \frac{1}{2}u^2 - \pi u + \frac{\pi^2}{3}$, valid for $\Re u \in [0, 2\pi]$, (see footnote 37) to transform it into a polynomial plus exponentially suppressed terms. As a consequence, up to polynomial terms, the main contribution comes for values of the indices $i \sim j$ and makes the functional local.

- Terms with higher powers of $N$ cancel. For more general $\mathcal{N} = 2$ theories this is not automatic and imposes conditions on the matter content of the theories for which this method works [89].

- Polynomial terms coming from this manipulation or Chern-Simons terms that are not in (4.13) happily combine into a contribution $\sum_{i=1}^{4} 2\pi n_i u_i + 2\pi \tilde{n}_i \tilde{u}_i$ to $\mathcal{W}$, where $n_i$ and $\tilde{n}_i$ are integers. These angular ambiguities disappear in the Bethe equations (3.37).

In general, the two contributions in (4.13) have different powers of $N$. They compete and give a sensible functional with a minimum only for $\alpha = 1/2$. We then see that $\mathcal{W}$ scales as $N^{3/2}$ as predicted by holography for AdS$_4$ black holes. We will then set $\alpha = 1/2$ from now on.

In order to extremize (4.13) we add to $\mathcal{W}$ a Lagrange multiplier term

$$-iN^{3/2}\mu \left( \int \rho(t) dt - 1 \right)$$

that enforces the normalization condition (4.12). Differentiating $\mathcal{W}$ with respect to $\delta v(t)$ and $\rho(t)$, we obtain a pair of algebraic equations

$$t - \rho(t) \sum_{a=1}^{4} \epsilon_a g_3'(-\epsilon_a \delta v(t) + \Delta_a) = 0, \quad (4.16)$$

$$t\delta v(t) + 2\rho(t) \sum_{a=1}^{4} g_3(-\epsilon_a \delta v(t) + \Delta_a) = \mu, \quad (4.17)$$

which can be easily solved in terms of rational functions of $t$. The solution is depicted in figure 1 together with the numerical solution for large $N$. From the figure we see that $\rho(t)$ and $\delta v(t)$ are piece-wise continuous functions of $t$. The solution in (4.16) and (4.17) only covers the central part of these functions. Numerics suggest that there are two external intervals, which we dub tails, where $\delta v(t)$ is actually constant in the large $N$ limit. It turns out that such constant value corresponds to the saturation of the inequality (4.14) for some value of $a$, $\delta v(t) = \epsilon_a \Delta_a$. The inequalities (4.14)
are necessary to restrict to a particular determination of the multi-valued polylog functions and the saturation corresponds to the position of the cuts. The numerics suggest that, once \( v_i \) and \( \tilde{v}_i \) hit the cut, their value is frozen. The value for \( \rho(t) \) in the tails can be obtained by solving its equation of motion, (4.17), setting \( \delta v(t) \) to the constant value \( \epsilon_a \Delta_a \) and ignoring the equation of motion for \( \delta v(t) \), (4.16), which would be inconsistent. The end-points of the interval, \( t_- \) and \( t_+ \), are finally determined by \( \rho(t_{\pm}) = 0 \).

Obviously, the equation of motion for \( \delta v(t) \), (4.16), must be satisfied at finite \( N \). The main correction to \( \delta v(t) \) and to its equation comes from the terms with \( i = j \) in (4.9). Such terms contribute

\[
\delta W = N \int dt \rho(t) \left[ \sum_{a=3,4} \text{Li}_2 \left( e^{i(\delta v(t)+\Delta_a)} \right) - \sum_{a=1,2} \text{Li}_2 \left( e^{i(\delta v(t)-\Delta_a)} \right) \right].
\]

(4.18)

Notice that these terms are suppressed compared to \( W \).\(^{45}\) They contribute a term

\[- \frac{1}{\sqrt{N}} \left[ \sum_{a=1}^{4} \epsilon_a \log \left( 1 - e^{i(\delta v(t)-\epsilon_a \Delta_a)} \right) \right],
\]

(4.19)

to the right-hand side of the equation (4.16) for \( \delta v(t) \). Such a correction is generically of order \( 1/\sqrt{N} \). However, on the tails, since \( \delta v(t) = \epsilon_b \Delta_b \) for some \( b \), one of the logarithms blows up and the correction can be effectively of order one. Indeed, the equation of motion for \( \delta v(t) \) can be satisfied if

\[
\delta v(t) = \epsilon_b \left( \Delta_b - e^{-\sqrt{N}Y_b(t)} \right),
\]

(4.20)

where \( Y_b(t) \) is a quantity of order one. In this case, on the tail, the equation becomes

\[
t - \rho(t) \sum_{a=1}^{4} \epsilon_a g_a(-\epsilon_a \delta v(t) + \Delta_a) = \epsilon_b Y_b(t),
\]

(4.21)

\(^{45}\)This is a standard argument in the context of matrix models: \( \sum_{i\neq j} N = O(N^2) \) while \( \sum_{i=1}^{N} = O(N) \). Notice that the contribution to \( W \) comes from terms with \( i \) almost equal to \( j \) and contributions to \( \delta W \) from terms with \( i = j \).
not summed over \(b\), which determines the value of \(Y_b(t)\). Notice that, quite remarkably, the correction to \(\delta v(t)\) is not power-like but exponentially small. The equations for \(\rho(t)\) and the value of \(\mathcal{W}\) on the solution are not affected by these corrections in the large \(N\) limit since \(\text{Li}_2(z)\) is finite for \(z \to 1\). However, these corrections are important for evaluating the index.

The explicit solution is as follows [3]. Let us first take \(\Delta_a\) real. Using the periodicity of \(\Delta_a\), we can always restrict to the case where \(0 \leq \Delta_a \leq 2\pi\). We will also assume that \(\Delta_1 \leq \Delta_2, \Delta_3 \leq \Delta_4\). The constraint (4.6) can be satisfied only for \(\sum a \Delta_a = 0, 2\pi, 4\pi, 6\pi, 8\pi\) and we need to consider all possible cases. We find a solution for \(\sum a \Delta_a = 2\pi\). We have a central region where

\[
\rho = \frac{2\pi\mu + t(\Delta_3\Delta_4 - \Delta_1\Delta_2)}{(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_3)(\Delta_1 + \Delta_4)(\Delta_2 + \Delta_4)}, \quad -\frac{\mu}{\Delta_4} < t < \frac{\mu}{\Delta_2},
\]

(4.22)

When \(\delta v\) hits \(-\Delta_3\) on the left the solution becomes

\[
\rho = \frac{\mu + t\Delta_3}{(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_3)(\Delta_1 - \Delta_3)}, \quad \delta v = -\Delta_3, \quad -\frac{\mu}{\Delta_3} < t < -\frac{\mu}{\Delta_4},
\]

(4.23)

with the exponentially small correction \(Y_3 = (t\Delta_4 - \mu)/(\Delta_4 - \Delta_3)\), while when \(\delta v\) hits \(\Delta_1\) on the right the solution becomes

\[
\rho = \frac{\mu - t\Delta_1}{(\Delta_1 + \Delta_3)(\Delta_2 + \Delta_3)(\Delta_2 - \Delta_1)}, \quad \delta v = \Delta_1, \quad \frac{\mu}{\Delta_2} < t < \frac{\mu}{\Delta_1},
\]

(4.24)

with \(Y_1 = (t\Delta_2 - \mu)/(\Delta_2 - \Delta_1)\). It turns out that, for \(\sum a=1 \Delta_a = 0, 2\pi, 4\pi, 8\pi\), equations (4.16) and (4.17) have no regular solutions. There is also a solution for \(\sum a \Delta_a = 6\pi\) which, however, is obtained by the previous one by a discrete symmetry of the index: \(\Delta_a \to 2\pi - \Delta_a (y_a \to y_a^{-1})\).

We can also evaluate the twisted superpotential on the solution and find

\[
\widetilde{W}(\Delta) = \frac{2iN^{3/2}}{3} \sqrt{2\Delta_1\Delta_2\Delta_3\Delta_4}, \quad \Delta_a \in [0, 2\pi], \quad \sum_{a=1}^4 \Delta_a = 2\pi.
\]

(4.25)

The result for a different determination of the \(\Delta_a\) is obtained by periodicity: just replace \(\Delta_a\) with \([\Delta_a] = (\Delta_a \text{mod } 2\pi)\). We can also extend by holomorphicity the result to complex \(\Delta_a\).

It is interesting to observe that \(\widetilde{W}(\Delta)\) has the same functional dependence of two important physical quantities appearing in the study of ABJM and its dual \(\text{AdS}_4 \times S^7\). One is of purely field theory origin. It is known that, for any \(\mathcal{N} = 2\) theory, there exists a family of supersymmetric Lagrangians on \(S^3\) parameterized by arbitrary R-charges of the chiral fields [94, 95]. When the theory is superconformal,
the resulting partition function has the property that it is extremized precisely at the exact R-symmetry of the theory \cite{95}. For ABJM at \( k = 1 \), the \( S^3 \) free energy reads \cite{91}

\[
F_{S^3}(r) = \frac{4\pi N^{3/2}}{3} \sqrt{2r_1 r_2 r_3 r_4},
\]

where \( r_a \) are a general assignment of R-charges for the fields \( A_1, A_2, B_1, B_2 \) satisfying \( \sum_{a=1}^4 r_a = 2 \). We see that \cite{89}

\[
\tilde{W} (\Delta_a) = \frac{\pi i}{2} F_{S^3} \left( \frac{\Delta_a}{\pi} \right).
\]

The second quantity, of supergravity origin, is the prepotential of the \( N = 2 \) gauged supergravity describing the low energy theory of the holographic dual. Restricted to the \( U(1)^4 \) gauge sector, the prepotential is given by (2.36) and we see that

\[
\tilde{W} (\Delta_a) = -\sqrt{2} \frac{N^3}{2} \frac{F(\Delta_a)}{3}.
\]

This will be important for comparison with the attractor mechanism.

4.2 The large \( N \) limit of the ABJM index

Using (3.38), up to an irrelevant overall factor, the index is given by

\[
\left( \det_{AB} \frac{\partial^2 W}{\partial u_A \partial u_B} \right)^{g-1} \prod_{i \neq j}^N \left( 1 - \frac{x_i^*}{x_j^*} \right)^{1-g} \left( 1 - \frac{\tilde{x}_i^*}{\tilde{x}_j^*} \right)^{1-g} \times \prod_{i,j=1}^4 \prod_{a=1,2} \left( 1 - \frac{x_i^* y_a / \tilde{x}_j^*}{1 - \tilde{x}_i^* / x_j^*} \right)^{-p_a+1-g} \prod_{b=3,4} \left( \frac{\sqrt{\tilde{x}_j^* y_b / \tilde{x}_i^*}}{1 - \tilde{x}_j^* / x_i^*} \right)^{-p_b+1-g},
\]

where \( x_i^* \) and \( \tilde{x}_i^* \) is the large \( N \) Bethe vacuum found in the previous section. By taking the large \( N \) limit of (4.29), after some manipulations, one finds

\[
\log Z = -N \frac{3}{2} \int dt \rho(t)^2 \left[ (1 - g) \frac{x_1^2}{2} + \sum_{a=1}^4 (p_a - 1 + g) g_a ( - \epsilon_a \delta v(t) + \Delta_a ) \right] -N \frac{3}{2} \sum_{a=1}^4 p_a \int_{\delta v \approx \epsilon_a \Delta_a} dt \rho(t) Y_a(t),
\]

up to corrections of order \( N \log N \). The first contribution in the first line of (4.30) comes from the Vandermonde determinant and the second from the matter contribution. The second line in (4.30) comes from the tails. Since the logarithm of the one-loop determinant of the chiral fields is singular on such regions, we need to take into account the exponentially small corrections \( Y_a \) to the tails. The exponent \(-p_a + 1 - g\) of the one-loop determinant is corrected to \(-p_a\) by an analogous and subtle contribution from the determinant of the Hessian of \( W \).

\[46\] See \cite{3} for details.
By plugging in the explicit solution for $\rho(t), \delta v(t)$ and $Y_a(t)$ we find
\[
\log Z = -\frac{2\sqrt{2}}{3}N^2 \sum_{a=1}^{4} p_a \frac{\partial}{\partial \Delta_a} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}.
\] (4.31)

Notice that
\[
\log Z = i \sum_{a=1}^{4} p_a \frac{\partial}{\partial \Delta_a} \tilde{W}(\Delta).
\] (4.32)

One can show that this remarkable identity can be extended to other $\mathcal{N} = 2$ quiver theories. Indeed it can be proved using only the equations of motion for $\rho$ and $\delta v$, and taking into account the extra terms on the tails [89].

### 4.3 Matching index and entropy for ABJM

We can now extract the degeneracy of states from the index using (2.17). We introduce four electric charges $q_a$, adapted to the democratic basis of charges we are using, and we set $j_i = 0$ since we are interested in static black holes. The degeneracy of states with given electric charge is then obtain by extremizing the functional
\[
\mathcal{I}(\Delta) = \log Z(\Delta) - i \sum_{a=1}^{4} q_a \Delta_a,
\] (4.33)

which implicitly depends on the magnetic charges $p_a$ trough $Z$. This principle has been called $\mathcal{I}$-extremization in [3, 96]. For purely magnetic black holes we just extremize $\log Z$. For a generic dyonic static black hole, we obtain
\[
S_{\text{micro}}(p_a, q_a) = \log Z - i \sum_{a=1}^{4} q_a \Delta_a = i \sum_{a=1}^{4} \left( p_a \frac{\partial}{\partial \Delta_a} \tilde{W}(\Delta) - q_a \Delta_a \right),
\] (4.34)

evaluated at its critical point in $\Delta_a$ with the constraints $\sum_{a=1}^{4} \Delta_a = 2\pi$ and $\Re \Delta_a \in [0, 2\pi]$.

By an explicit computation one can see that this expression correctly reproduces the entropy of the black holes in [23–27]. This can be checked more easily by comparing with the attractor equations (2.35). Using (2.33) we find
\[
S_{\text{BH}}(p^\Lambda, q_\Lambda) = -\frac{i \text{Vol}(\Sigma_g)}{4G_N^{(3)}} \frac{q_\Lambda X^\Lambda - p^\Lambda F_\Lambda}{g_\Lambda X^\Lambda - g^\Lambda F_\Lambda} = i \sum_{\Lambda=0}^{3} \left( \hat{p}^\Lambda \frac{\partial}{\partial \hat{X}_\Lambda} \tilde{W}(\hat{X}) - \hat{q}_\Lambda \hat{X}^\Lambda \right). \] (4.35)

Here we used the data of the relevant gauged supergravity (see (2.36))
\[
g_\Lambda \equiv g, \quad g^\Lambda = 0, \quad \mathcal{F} = -2i \sqrt{X^0 X^1 X^2 X^3},
\] (4.36)

we defined adapted scalar fields
\[
\hat{X}^\Lambda = \frac{2\pi X^\Lambda}{\sum_{\Lambda=0}^{3} X^\Lambda},
\] (4.37)
satisfying $\sum_{\Lambda=0}^{3} \hat{X}^\Lambda = 2\pi$, and enforced the Dirac quantization condition of fluxes (2.29) by defining the integers $\hat{p}^\Lambda$ and $\hat{q}_\Lambda$,

$$\text{Vol}(\Sigma_\hat{g}) p^\Lambda g_\Lambda = -2\pi \hat{p}^\Lambda, \quad \frac{\text{Vol}(\Sigma_\hat{g}) q_\Lambda}{4G_N g_\Lambda} = 2\pi \hat{q}_\Lambda.$$  \hfill (4.38)

Finally, we used the known holographic dictionary (see for example [8])

$$\frac{1}{2g^2 G_N^{(4)}} = \frac{2\sqrt{2}}{3} N^{3/2}.$$ \hfill (4.39)

Using the identification\footnote{Notice that the identification is consistent with the twisting condition (2.31) and the constraint $\sum_{a=1}^{4} \Delta_a = 2\pi$.} $a = 1, 2, 3, 4 \to \Lambda = 0, 1, 2, 3$, $\Delta_a \to \hat{X}^\Lambda$, $\Delta_a (p_a, q_a) \to (\hat{p}^\Lambda, \hat{q}_\Lambda)$, we see that $S_{\text{micro}}(p_a, q_a) = S_{\text{BH}}(p^\Lambda, q_\Lambda)$. The extremization of (4.34) is thus completely equivalent to the attractor mechanism in gauged supergravity. The correspondence, up to constants, is

$$\tilde{W}(\Delta) \to F(X^\Lambda), \quad I(\Delta) \to I_{\text{sugra}}(X^\Lambda).$$ \hfill (4.41)

Notice that the functional $I(\Delta)$ is only defined up to integer multiples of $2\pi i$, due to the presence of $\log Z$. So the microscopical entropy should be properly defined as $S_{\text{micro}}(p_a, q_a) = I (\text{mod } 2\pi i)$. In field theory, $I$ only depends on three independent chemical potentials, which we can choose to be $\Delta_a$ with $a = 1, 2, 3$. The extremization of $I$ determines them in terms of three electric charges, $q_a - q_4$,

$$\frac{\partial \log Z}{\partial \Delta_a} = i(q_a - q_4), \quad a = 1, 2, 3.$$ \hfill (4.42)

The entropy can be then expressed in terms of $q_a - q_4$ as

$$\log Z - i \sum_{a=1}^{3} (q_a - q_4) \Delta_a \ (\text{mod } 2\pi i).$$ \hfill (4.43)

In these derivations we used that $q_a$ are integers and $\sum_{a=1}^{4} \Delta_a = 2\pi$.

Correspondingly, in gravity, the family of black holes depends only on three independent electric charges, as discussed in section 2.2. Indeed, the requirement that $S_{\text{BH}}(p^\Lambda, q_\Lambda)$ is real gives a constraints on the charges. For given magnetic charges
$p_a$ and flavor electric charges $q_a - q_4$, there is at most one value of $q_4$ that leads to black hole with regular horizon. The expression (4.43) unambiguously determines the entropy of this black hole.

We can see the left hand side of (4.42) as determining the average electric charge $q_a - q_4$ in our statistical ensemble at large $N$. The index only depends on the global symmetries of the theory and there are three of them. Correspondingly, with our method, we can not determine the average electric charge associated with the $R$-symmetry. However, this value is fixed by the BPS equations in gravity. It would be interesting to find a purely field theoretical method for testing this prediction for the fourth charge. It is interesting to notice that, with the permutationally invariant definition of $I$ given in (4.33), the value of $q_4$ predicted by gravity is precisely the one that makes the critical value of $I$ real.

Let us conclude this section with a comment. We extracted the entropy from the Legendre transform of the grand canonical partition function, according to a natural microscopic point of view. However, in holography, the partition function of the boundary field theory must be identified with the renormalized on-shell action of the bulk theory. Therefore, after performing holographic renormalization, we should be able to extract the entropy of supersymmetric black holes from the gravitational on-shell action. The agreement of these points of view has been discussed in [97–99].

### 4.4 Other examples and generalizations

The large $N$ limit of the topologically twisted index can be evaluated for other $\mathcal{N} = 2$ quiver Chern-Simons theories with an $\text{AdS}_4$ dual. There is a large class of Yang-Mills-Chern-Simons theories with fundamental and bi-fundamental chiral fields that have been proposed as duals of M-theory and massive type IIA compactifications. Most of these are obtained by dimensionally reducing a *parent* four-dimensional quiver gauge theory with an $\text{AdS}_5 \times SE_5$ dual, where $SE_5$ is a five-dimensional Sasaki-Einstein manifold, and then adding Chern-Simons terms and flavoring with fundamentals. Holography predicts that the twisted index scales as $N^{3/2}$ for theories dual to M-theory on seven-dimensional Sasaki-Einstein manifolds, and as $N^{5/3}$ for a class of theories dual to massive type IIA on warped six-manifolds.\(^\text{48}\) For a large class of these theories the large $N$ method discussed in section 4.1 applies and the twisted index has been evaluated in [89, 111].\(^\text{49}\) The twisted index depends on chemical potentials for the global symmetries. As for ABJM, we can use a redundant basis where we assign a chemical potential $\Delta_a$ to each field $\phi_a$. For each term $W_I$ in the

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\(^\text{48}\)See [90, 100–108] for examples of theories with M-theory dual and [109, 110] for theories with massive type IIA dual.

\(^\text{49}\)For theories with M-theory dual the method only works when the bi-fundamental fields transform in a real representation of the gauge group and the total number of fundamentals is equal to the total number of anti-fundamentals. The same restriction has been found in [91] for the large $N$ evaluation of the $S^3$ free energy.
superpotential we must require
\[
\sum_{a \in W_I} \Delta_a = 2\pi p, \quad p \in \mathbb{Z},
\]
(4.44)
for all the fields \(\phi_a\) appearing in \(W_I\). It turns out that, as for ABJM, a large \(N\) solution exists only for \(p = 1\) (up to equivalent solutions). Quite remarkably, for all these theories is true that the on-shell twisted superpotential coincides with the corresponding \(S^3\) free energy [89]
\[
\tilde{W}(\Delta_a) = \frac{i\pi}{d} F_{S^3} \left( \frac{\Delta_a}{\pi} \right),
\]
(4.45)
where \(d = 2\) for M-theory examples and \(d = 3\) in massive type IIA ones. Since the \(S^3\) free energy is a function of the R-charges of the fields, and these are constrained to satisfy
\[
\sum_{a \in W_I} r_a = 2,
\]
(4.46)
for each term in the superpotential, the identity (4.45) makes sense only because of (4.44) with \(p = 1\). Moreover, it is always possible to partially solve (4.44) in terms of a set of (still redundant) chemical potentials such that \(\tilde{W}\) becomes a homogeneous functions, as it is for ABJM. For this choice the twisted index is given again by
\[
\log Z = i \sum_a p_a \frac{\partial}{\partial \Delta_a} \tilde{W}(\Delta).
\]
(4.47)
For a generic choice of chemical potentials (redundant or not) satisfying (4.44) we can write the more complicated expression
\[
\log Z = (1 - g) \left( \frac{di}{\pi} \tilde{W}(\Delta_I) + i \sum_I \left[ \left( \frac{p_I}{1 - g} - \frac{\Delta_I}{\pi} \right) \frac{\partial \tilde{W}(\Delta_I)}{\partial \Delta_I} \right] \right).
\]
(4.48)
This result has been called index theorem in [89]. Since \(d\) is also the degree of homogeneity of \(\tilde{W}\), this equation obviously reduces to (4.47) when \(\tilde{W}\) is homogeneous.

The entropy of dyonic static black holes in such theories is obtained by taking the Legendre transform of (4.47). Comparison with the attractor mechanism in the form (2.34) then requires that
\[
\tilde{W}(\Delta) \sim \mathcal{F}(X^\Delta),
\]
(4.49)
under some map \(\Delta_a \to X^\Delta\). (4.49) is however oversimplified. The effective low energy theory for M or type IIA compactifications on general manifolds is an \(\mathcal{N} = 2\) gauged supergravity containing massless vector multiplets associated with the global

\[\text{See [112] for more details and applications.}\]
symmetries of the CFT. In general, such gauged supergravity contains also hyper-multiplets and a number of vector multiplet that is larger than the number of global symmetries. Therefore, in (4.49), the number of $\Delta_a$ does not match the number of sections $X^\Lambda$. In general, the hypermultiplets have a VEV at the horizon and give a mass to some of the vector fields. If $n_V$ is the number of vector multiplets and $n_H$ is the number of hypermultiplets, $n_V - n_H$ is the number of massless vectors, corresponding to the number of global symmetries and therefore to the number of independent $\Delta_a$. The BPS equations for the hyperinos are typically algebraic and give linear constraints among the sections $X^\Lambda$. By solving these constraints we can obtain an effective prepotential $F_{\text{eff}}(X^\Lambda)$ for the massless vector multiplets only. If we believe that this further truncated theory correctly describes the horizon of the black hole, (4.49) should hold with $F(X^\Lambda)$ replaced by $F_{\text{eff}}(X^\Lambda)$. This happens for example for the mass deformed ABJM model [57], where the value of one $\Delta_a$ is fixed by the mass deformation and, in gravity, one $X^\Lambda$ is fixed by the hyperino conditions.

Two notable applications where this works perfectly are the following. A well-known example of massive type IIA background is the warped AdS$_4 \times Y_6$ flux vacua of massive type IIA dual to the $\mathcal{N} = 2$ $U(N)$ gauge theory with three adjoint multiplets and a Chern-Simons coupling $k$ [109]. It corresponds to an internal manifold $Y_6$ with the topology of $S^6$. The theory has two global symmetries and there exists a family of black holes depending on two magnetic charges [113]. The entropy of such black holes has been matched with the prediction of the twisted index in [55, 56]. For this example

$$\tilde{W}(\Delta_a) \sim F_{S^3}(\Delta_a) \sim F_{\text{eff}}(\Delta_a) \sim (\Delta_1\Delta_2\Delta_3)^{2/3}, \quad \sum_{a=1}^{3} \Delta_a = 2\pi.$$ (4.50)

The second example involves the so-called universal black hole [97]. This is dual to the universal twist [114, 115] defined by a set of magnetic fluxes $p_a$ proportional to the exact R-symmetry of the CFT. This black hole is a solution of minimal gauged supergravity and, as such, can be embedded in all M-theory and massive type IIA compactifications, thus explaining the name universal. Hence it provides an infinite family of examples. Since $\tilde{W}$ is proportional to the $S^3$ free energy (see (4.45)) and the latter is extremized at the exact R-symmetry, it follows easily from (4.48) that

$$S_{\text{BH}}(p_a) = (g - 1)F_{S^3}.$$ (4.51)

This relation agrees with the gravitational prediction based on minimal gauged supergravity [97].

It would be interesting to derive also the entropy of the dyonic rotating black holes recently found in [29]. The chemical potential dual to rotation in the bulk is just an Omega-background for the boundary theory on $S^2 \times S^1$. The topologically twisted index in an Omega-background is known [60] but it gives rise to a complicated matrix
model. For particular values of the $\Omega$-deformation, it has been recently written as a sum over Bethe vacua [71]. This could be useful to solve the matrix model in the large $N$ limit.

For asymptotically flat black holes, there is a huge literature including higher derivative corrections and highly detailed precision tests. For asymptotically AdS black holes, the story is just begun. At the moment, $1/N$ corrections to the twisted index for ABJM and the above massive type IIA background have been computed numerically focusing in particular on the universal logarithmic correction [116–119]. The ABJM matrix model provides a quantum corrected entropy functional that would be interesting to study further. In particular, it would be interesting to find the analog of standard results and conjectures about the quantum entropy of asymptotically flat black holes, like the OSV conjecture [40], the Sen’s quantum entropy functional [41, 42] and localization in supergravity [120, 121], to cite only few of them. In particular, some interesting results about localization in $\text{AdS}_2 \times S^2$ have already been found in [122].

Finally, the topologically twisted index has been extended to five-dimensions [123, 124] and successfully compared with the entropy of $\text{AdS}_6$ black holes with horizon $\text{AdS}_2 \times \Sigma_{g_1} \times \Sigma_{g_2}$ [125–129]. The expression for the entropy is given by a generalization of the index theorem (4.47) and (4.48).

Other interesting related results and solutions can be found in [130–132].

5 Black strings in $\text{AdS}_5$

The methods introduced in section 4 can be generalized to other dimensions and can be used to provide general tests of holography. In particular, they can be applied to domain wall solutions interpolating between $\text{AdS}_{d+n}$ and $\text{AdS}_d \times \mathcal{M}_n$, with a topological twist along the $n$-dimensional compact manifold $\mathcal{M}_n$. In this section, making a brief digression from the main subject of these notes, we discuss the example of black strings in $\text{AdS}_5$.

More precisely, we consider a family of black strings in $\text{AdS}_5$ with near horizon geometry $\text{AdS}_3 \times \Sigma_g$. They correspond holographically to a twisted compactification of a four-dimensional theory that flows in the IR to a two-dimensional CFT. The properties of this CFT, and, in particular, its elliptic genus, can be computed using a topologically twisted index for four-dimensional theories on $\Sigma_g \times T^2$. We can connect these domain-wall solutions with the physics of black holes by compactifying the black string on a circle. In this way we also make contact with a more traditional way of deriving the entropy of stringy black holes, which involves using the Cardy formula, similarly to the original example [1]. Other examples of similar tests of holography related to twisted compactifications can be found in [123, 124, 131].
5.1 Black string solutions in $\text{AdS}_5 \times S^5$

We consider a five-dimensional $\mathcal{N} = 2$ effective gauged supergravity with abelian vector multiplets and solutions of the form

$$ds^2 = e^{2f(r)}(-dt^2 + dz^2) + e^{-2f(r)}dr^2 + e^{2g(r)}ds^2_{\Sigma_g},$$

$$A^\Lambda = p^\Lambda A_{\Sigma_g},$$

where $A_{\Sigma_g}$ is the gauge potential for a magnetic flux on $\Sigma_g$. In general, there are also scalar fields with radial dependence and supersymmetry is preserved via a twist along $\Sigma_g$. We are interested in solutions that are asymptotic to $\text{AdS}_5$ for large values of the radial coordinate,

$$e^{f(r)} \sim r, \quad e^{g(r)} \sim r, \quad r \gg 1$$

and approach a regular horizon $\text{AdS}_3 \times \Sigma_g$ at some fixed value $r = r_0$,

$$e^{f(r)} \sim r - r_0, \quad e^{g(r)} \sim \text{constant}, \quad r \sim r_0.$$ (5.2)

These solutions can be interpreted as black strings extended in the direction $z$ or, equivalently, as domain walls interpolating between $\text{AdS}_5$ and $\text{AdS}_3 \times \Sigma_g$. They are holographically dual to a twisted compactification of a SCFT$_4$ on $\Sigma_g$ that flows in the IR to a two-dimensional $(0,2)$ SCFT$_2$ associated with the horizon factor $\text{AdS}_3$.

Solutions of this kind that can be embedded in $\text{AdS}_5 \times S^5$ have been found in [133], using a five-dimensional gauged supergravity with three abelian vectors associated with the isometries $U(1)^3 \subset SO(6)$. Correspondingly, the solution depends on three fluxes $p_a$ constrained by the twisting condition

$$p_1 + p_2 + p_3 = 2 - 2g.$$ (5.3)

This solution corresponds to a twisted compactification of $\mathcal{N} = 4$ SYM on $\Sigma_g$ depending on the fluxes $p_a$, which parametrize a family of inequivalent twists. Holography suggests that this theory flows to an IR two-dimensional SCFT. The value of the central charge of the IR SCFT$_2$ in the large $N$ limit can be extracted from the solution using standard arguments [134, 135] and reads [133]

$$c = \frac{3R_{\text{AdS}_3}}{2G_N^{(3)}} = 12N^2 \frac{p_1p_2p_3}{p_1^2 + p_2^2 + p_3^2 - 2p_1p_2 - 2p_2p_3 - 2p_3p_1}. $$ (5.4)

This result has been successfully compared to field theory expectations using the following argument [133]. The central charge of the $(0,2)$ SCFT$_2$ can be obtained using a technique called $c$-extremization [136]. Let us briefly explain how it works. Consider $\mathcal{N} = 4$ SYM as an $\mathcal{N} = 1$ theory with three chiral multiplets $\phi_i$ and a superpotential

$$W = \text{Tr} \left( \phi_3 [\phi_1, \phi_2] \right)$$ (5.5)
We introduce a generic R-charge assignment $\Delta_a$ for the three chiral fields $\phi_i$, with $\Delta_1 + \Delta_2 + \Delta_3 = 2$. It is known that the exact R-symmetry of an $\mathcal{N} = 1$ superconformal theory can be obtained by extremizing a trial $a$-charge

$$a(\Delta_a) = \frac{9}{32} \text{Tr} R(\Delta_a)^3 - \frac{3}{32} \text{Tr} R(\Delta_a), \quad (5.6)$$

where $R(\Delta_a)$ is the matrix of R-charges of the fermionic fields. This construction is known as $a$-maximization [137]. For $\mathcal{N} = 4$ SYM at large $N$ we find

$$a(\Delta_a) = \frac{9}{32} N^2 \left( 1 + \sum_{a=1}^{3} (\Delta_a - 1)^3 \right) = \frac{27}{32} N^2 \Delta_1 \Delta_2 \Delta_3 , \quad (5.7)$$

where the first contribution comes from the gauginos and the second from the fermions in the chiral multiplets. This expression is trivially extremized for $\Delta_1 = \Delta_2 = \Delta_3 = 2/3$, the exact R-charges of the fields $\phi_i$. The critical value of $a(\Delta)$ is the central charge $a$ of the SCFT $4$. Similarly, the exact R-symmetry and the right-moving central charge of the twisted compactification of $\mathcal{N} = 4$ SYM is obtained by extremizing the trial quantity [136]

$$c_r(\Delta_a) = 3 \text{Tr} \gamma_3 R(\Delta_a)^2 , \quad (5.8)$$

where $\gamma_3$ is the two-dimensional chirality operator and $R(\Delta_a)$ the matrix of R-charges of the massless fermionic fields in the SCFT $2$. This quantity is a linear combination of t’Hooft anomalies and can be computed using topological arguments only. We just need to know the difference between the number of fermionic zero modes with positive and negative chirality. This is easily computed from the Riemann-Roch theorem as in section 3.3.1. We then find at large $N$ [133]

$$c_r(\Delta_a) = -3N^2 \left( 1 - g + \sum_{a=1}^{3} (p_a - 1 + g)(\Delta_a - 1)^2 \right)$$

$$= -3N^2 (\Delta_1 \Delta_2 p_3 + \Delta_2 \Delta_3 p_1 + \Delta_3 \Delta_1 p_2) , \quad (5.9)$$

where again the first contribution in the first line comes from gauginos and the second from matter fields. The extremization of (5.9) with respect to $\Delta_a$ indeed reproduces (5.4). The agreement is valid in the large $N$ limit where $c = c_l = c_r$.\footnote{For $\mathcal{N} = 4$ SYM Tr $R$ is identically zero. For theories with an AdS dual, Tr $R = 0$ in the large $N$ limit.}

Notice that, in the large $N$ limit, the $c_r$ trial central charge can be written as [5]

$$c_r(\Delta_a) = \frac{32}{9} \sum_{a=1}^{3} p_a \frac{\partial a(\Delta_a)}{\partial \Delta_a} . \quad (5.10)$$

\footnote{$c_r - c_l$ is equal to the gravitational anomaly $k = \text{Tr} \gamma_3$ which is of order one in the large $N$ limit [136].}
It is interesting to observe that formula (5.10), with a suitable parameterization for the fluxes and the R-charges, holds for all the twisted compactifications of $\mathcal{N} = 1$ SCFT$_d$ dual to $\text{AdS}_5 \times \text{SE}_5$, where $\text{SE}_5$ is a toric Sasaki-Einstein manifold [5].

In section 4 we extracted the information about the entropy of supersymmetric black holes from the Witten index of the quantum mechanics describing the horizon. In the case of black strings the information about states is encoded in the elliptic genus of the two-dimensional $(0,2)$ CFT

$$Z(y,q) = \text{Tr}(-1)^F y_I^{J_I} q^{L_0},$$

(5.11)

where $q = e^{2\pi i \tau}$, with $\tau$ the modular parameter of $T^2$, $y_I$ are fugacities for the global symmetries and $L_0$ the left-moving Virasoro generator. Since the index is independent of the scale, we can evaluate it in the UV, where it becomes the topologically twisted index, defined as the partition function on $\Sigma_g \times T^2$, with a topological twist along $\Sigma_g$.

### 5.2 The topologically twisted index on $T^2 \times \Sigma_g$

For an $\mathcal{N} = 1$ four-dimensional gauge theory with a non-anomalous $U(1)$ R-symmetry, the topologically twisted index is a function of $q = e^{2\pi i \tau}$, where $\tau$ is the modular parameter of $T^2$, fugacities $y_I$ for the global symmetries and flavor magnetic fluxes $m^F_I$ on $\Sigma_g$ parameterizing the twist. It can be computed using localization and it is given by an elliptic generalization of the formulae in section 3.2 [60, 140]. Explicitly, for a theory with gauge group $G$ and a set of chiral multiplets transforming in representations $\mathcal{R}_I$ of $G$ with R-charge $r_I$, the topologically twisted index is given by a contour integral of a meromorphic form

$$Z(p,y) = \frac{1}{|W|} \sum_{m \in \Gamma_h} \oint \prod_{C} \frac{dx}{2\pi i x} \eta(q)^{2(1-g)} \prod_{\alpha \in G} \left[ \frac{\theta_1(x^{\alpha}; q)}{i \eta(q)} \right]^{1-g} \times \prod_{I} \prod_{\rho_I \in \mathcal{R}_I} \left[ \frac{i \eta(q)}{\theta_1(x^{\rho_I}; q)} \right]^{\rho_I(m) + \rho^{(g-1)(r_I - 1) + m^F_I}} \left( \frac{\det \partial_i^2 \log Z_{\text{pert}}(u, m)}{\partial u_i \partial m_j} \right)^g,$$

(5.12)

where $\alpha$ are the roots of $G$, $\rho_I$ the weights of the representation $\mathcal{R}_I$ and $|W|$ denotes the order of the Weyl group. In this formula, $\theta_1(x; q)$ is a Jacobi theta function and $\eta(q)$ is the Dedekind eta function. The zero-mode gauge variables $x = e^{iu}$ parameterize the Wilson lines on the two directions of the torus

$$u = 2\pi \oint_{A\text{-cycle}} A - 2\pi \tau \oint_{B\text{-cycle}} A,$$

(5.13)
and are defined modulo
\[ u_i \sim u_i + 2\pi n + 2\pi m \tau, \quad n, m \in \mathbb{Z}. \]  
(5.14)

As in three dimensions the result is summed over a lattice of gauge magnetic fluxes \( \mathbf{m} \) living in the co-root lattice \( \Gamma_\mathbf{h} \) of the gauge group \( G \) and the contour of integration selects the Jeffrey-Kirwan prescription for taking the residues. In four dimensions there is a one-loop contribution from the Cartan components of the vector multiplets. One can show that the integrand in (5.12) is a well-defined meromorphic function of \( x \) on the torus provided that the gauge and the gauge-flavor anomalies vanish. The index has a trace interpretation as a sum over a Hilbert space of states on \( \Sigma_g \times S^1 \)

\[ Z(n, y, q) = \text{Tr}_{\Sigma_g \times S^1} (-1)^F q^{H_L} \prod_I y_I^{j_I}. \]  
(5.15)

This trace reduces to the elliptic genus of the two-dimensional theory obtained by the twisted compactification on \( \Sigma_g \).

We now consider the index for \( N = 4 \) SYM. The superpotential (5.5) imposes the following constraints on the chemical potentials \( \Delta_a \) and the flavor magnetic fluxes \(-p_a + 1 - g = (g - 1)(r_a - 1) + m_a^F \) associated with the fields \( \phi_a \),

\[ \sum_{a=1}^3 \Delta_a \in 2\pi \mathbb{Z}, \quad \sum_{a=1}^3 p_a = 2 - 2g. \]  
(5.16)

The topologically twisted index for the SYM theory with gauge group \( SU(N) \) is then given by

\[ Z = \frac{1}{N!} \sum_{\mathbf{m} \in \mathbb{Z}^N, \sum_{\mathbf{m}} = 0} \int \prod_{i=1}^{N-1} \frac{dx_i}{2\pi i x_i} \eta(q)^{2(N-1)(1-g)} \prod_{i,j=1}^{N} \left( \frac{\theta_1 \left( \frac{x_i}{x_j}; q \right)}{i\eta(q)} \right)^{1-g} \prod_{i,j=1}^{N} \left[ \frac{i\eta(q)}{\theta_1 \left( \frac{x_i}{x_j}; q \right)} \right]^{m_i - m_j - p_a + 1 - g} \left( \frac{\det_{ab} \partial^2 \log Z_{\text{pert}}(u, \mathbf{m})}{\partial u_a \partial m_b} \right)^g, \]  
(5.17)

with \( y_a = e^{i\Delta_a} \). The Bethe vacua are given by

\[ e^{i\frac{\partial Z}{\partial u_a}} = \prod_{j=1}^{N} \prod_{a=1}^{3} \theta_1 \left( \frac{x_j}{x_a}; y_a; q \right) \frac{\theta_1 \left( \frac{x_j}{x_a}; y_a; q \right)}{\theta_1 \left( \frac{x_j}{x_a}; y_a; q \right)} = 1. \]  
(5.18)

These Bethe vacua will play a role also in the physics of \( \text{AdS}_5 \) black holes [6, 86], as discussed in section 6.2.

This time the index is too hard to solve in the large \( N \) limit. We can study instead the high temperature limit corresponding to a shrinking of the torus given by
\[ \tau = i\beta/(2\pi) \text{ with } \beta \to 0^+ \text{ [5].} \]

This limit is also particularly interesting from the field theory point of view because it controls the asymptotic growing of the number of states with the dimension [141].

Using the asymptotic expansions of theta and Dedekind functions, one can easily show that, in the \( \beta \to 0 \) limit, the twisted superpotential can be written as

\[
W = \beta \sum_{i \neq j} \sum_{a=1}^{3} \frac{i}{2} \left( g_3(u_i - u_j + \Delta_a) + g_3(u_j - u_i + \Delta_a) \right) + i(N - 1) \beta \sum_{a=1}^{3} g_3(\Delta_a).
\]  
(5.19)

The expansion is valid for \( \Re \Delta_a \in [0,2\pi] \) so that \( \sum_a \Delta_a = 0, 2\pi, 4\pi, 6\pi \). As in the three-dimensional case, we set \( \sum_a \Delta = 2\pi \). The case \( 4\pi \) is related by a discrete symmetry and the cases \( 0, 6\pi \) are inconsistent. The solutions of the Bethe equations \( \partial_{\Delta} W + 2\pi n_i = 0 \) are then [5]

\[
u_k = \frac{i\beta}{N} \left( n_k - \frac{1}{N} \sum_{j=1}^{N} n_j \right) = -\frac{i\beta}{N} \left( k - \frac{N + 1}{2} \right).
\]
(5.20)

In writing the second equality, we used the fact that the integers \( n_i \) should be all different to avoid zeros of the Vandermonde determinant and that \( u_i \) live on the torus and are therefore periodic of period \( i\beta \). These conditions uniquely determine \( n_k = k \). As noticed in [142], the solution (5.20) is actually exact at all order in \( \beta \).\[56\] Using (5.20) it is easy to compute, at leading order in \( \beta \), [5]

\[
\tilde{W}(\Delta) = \frac{i(N^2 - 1)}{2\beta} \Delta_1 \Delta_2 \Delta_3,
\]
(5.21)

\[
\log Z(p, \Delta) = i \sum_{a=1}^{3} p_a \frac{\partial \tilde{W}}{\partial \Delta_a},
\]

where \( \tilde{W}(\Delta) \) is the on-shell value of the twisted superpotential. We see a striking similarity with the black hole case, in particular equation (4.47). Moreover, the twisted superpotential is proportional to the trial \( a \)-central charge (5.7) of the four-dimensional SCFT. This statement is the four-dimensional analog of (4.45).

Comparing with (5.10) we find the Cardy formula

\[
\log Z(p, \Delta) = \frac{\pi^2}{6\beta} c_r(\Delta) = \frac{\pi i}{12\tau} c_r(\Delta),
\]
(5.22)

\[54\] Notice that \( \beta \) is not really a temperature, but rather a parameterization of the modular parameter of the torus.

\[55\] One should use a Lagrange multiplier to enforce the \( SU(N) \) constraint \( \sum_{i=1}^{N} u_i = 0 \). The function \( g_3(u) \) is defined in footnote 37.

\[56\] Using \( SL(2, \mathbb{Z}) \), the authors of [142] also conjectured the form of all solutions of the Bethe equations for \( N = 4 \) SYM at finite \( \beta \). Notice that the exactness of the solution at all orders does not extend to more general quivers.
valid at leading order in $\beta$ and at large $N$.

Similar results are valid for all $\mathcal{N} = 1$ SCFT$_4$ dual to $\text{AdS}_5 \times SE_5$ vacua [5]. The example discussed in this section can be actually generalized to many other flows interpolating between $\text{AdS}_{3+n}$ and $\text{AdS}_3 \times \mathcal{M}_n$ where supersymmetry is preserved along the compact manifold $\mathcal{M}_n$ with a topological twist. Examples of compactifications of the $(2,0)$ theory in six dimensions on the product of two Riemann surfaces $\Sigma_{g_1} \times \Sigma_{g_2}$ are discussed for example in [123].

5.3 Back to Cardy

Similarly to what is done for generic CFT$_2$ [141], we can extract information on the growing of the number of supersymmetric states with the energy from the asymptotic behavior (5.22) of the elliptic genus.

From the definition of the index as a trace (5.15), we see that the number of supersymmetric states with momentum $n$ and electric charge $q_a$ under the Cartan subgroup of $SO(6)$ can be extracted as a Fourier coefficient

$$d(p, n, q) = -\frac{i}{(2\pi)^2} \int_{i\mathbb{R}} d\beta \int_0^{2\pi} d\Delta \ Z(p, \Delta) e^{\beta n - i \sum_{a=1}^3 \Delta a q_a},$$  \hspace{1cm} (5.23)

where $\beta = -2\pi i \tau$ and the corresponding integration is over the imaginary axis.

In the limit of large charges, we may use the saddle point approximation. Consider first $q = 0$. The number of supersymmetric states with charges $(p, n)$ can be obtained by extremizing

$$I(\beta, \Delta) \equiv \log Z(p, \Delta) + \beta n$$  \hspace{1cm} (5.24)

with respect to $\Delta$ and $\beta$. Given (5.22), we see that the extremization with respect to $\Delta$ is the $c$-extremization principle [133, 136] and sets the trial right-moving central charge $c_r(\Delta)$ to its exact value $c_{\text{CFT}}(p)$ given in (5.4). Extremizing $I(\beta, \Delta)$ with respect to $\beta$ yields

$$\bar{\beta}(p, n) = \pi \sqrt{\frac{c_{\text{CFT}}(p)}{6n}}.$$  \hspace{1cm} (5.25)

Plugging back (5.25) into $I(\beta, \Delta)$, we find for the entropy of states

$$S(p, n) \equiv \log d(p, n, 0) = I|_{\text{crit}}(\beta, \Delta) = 2\pi \sqrt{n \frac{c_{\text{CFT}}(p)}{6}}.$$  \hspace{1cm} (5.26)

This is obviously nothing else than Cardy formula [141].$^{57}$

One can generalize the previous computation to the case $q_a \neq 0$ by extremizing

$$I(\beta, \Delta) \equiv \log Z(p, \Delta) + \beta n - i \sum_a q_a \Delta_a.$$  \hspace{1cm} (5.27)

$^{57}$For a further discussion about the asymptotic behavior see [112].
After some manipulations, the result can be expressed in the form

\[ S(p, q, n) = 2\pi \sqrt{h(p) \left( n + \frac{1}{2} \sum_{I,K} q_I A_{IK}^{-1} q_K \right)} , \]  

(5.28)

where the index \( I \) runs over a set of independent global symmetries \( J_K \) and \( A_{IK} = \text{Tr} \gamma_3 J_I J_K \) is the 't Hooft anomaly matrix. The particular combination of \( n \) and electric charges appearing in (5.28) is related to the elliptical properties of the elliptic genus and is familiar from the Rademacher expansion for asymptotically flat black holes \([143, 144]\).58

### 5.4 Cardy formula and black holes

There is standard argument to obtain a black holes from a black string: add a momentum \( n \) along the circle inside \( \text{AdS}_3 \) and do a compactification along this circle. More precisely, one replaces the near-horizon geometry of the five-dimensional black string with \( \text{BTZ} \times \Sigma_{g_1} \), where the metric for the extremal BTZ reads \([146]\)

\[ ds^2_3 = \frac{1}{4} \left( -dt^2 + \frac{dr^2}{r^2} \right) + \rho \left[ dz + \left( -\frac{1}{4} + \frac{1}{2\rho r} \right) dt \right]^2 . \]  

(5.29)

Here, the parameter \( \rho \) is related to the electric charge \( n \). This solution is \textit{locally} equivalent to \( \text{AdS}_3 \), since there exists locally only one constant curvature metric in three dimensions, and solves the same BPS equations; however, BTZ and \( \text{AdS}_3 \) are inequivalent globally. Compactifying the full five-dimensional black string of \([133]\) on the circle with the extra momentum we obtain a static BPS black hole in four dimensions, with magnetic charges \( p_a \) and electric charge \( n \). This can be thought as a domain wall that interpolates between an \( \text{AdS}_2 \times \Sigma_{g_1} \) near-horizon region and a complicated asymptotic \textit{non-AdS}_4 vacuum \([147]\).

By another standard argument, the entropy of such a black hole is given by the number of states of the CFT with momentum \( n \), and is therefore given by the Cardy formula (5.26)

\[ S_{\text{BH}}(p, n) = \mathcal{I}_{\text{crit}}(\beta, \Delta) = 2\pi \sqrt{\frac{n_{\text{CFT}}(p)}{6}} . \]  

(5.30)

The consistency of this result with the gravity prediction has been checked in \([147]\).

It is interesting to observe that, after dimensional reduction, the field theory extremization of \( \mathcal{I} \) becomes again equivalent to the attractor mechanism in four-dimensional gauged supergravity. The dimensional reduction from the five-dimensional gauged...

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58 Our elliptic genus is actually a meromorphic function of \( \Delta_a \) and this leads to a more complicated structure of the corresponding modular forms, related to wall-crossing phenomena. For asymptotically flat black holes this leads to interesting behaviours (see for example \([145]\)). It would be interesting to see if the same happens for the black holes discussed in 5.4 obtained by compactifying the black string.
supergravity used in [133] to $\mathcal{N} = 2$ four-dimensional gauged supergravity has been performed in [147]. The five-dimensional theory has three vector fields and the compactification on a circle provides a new one. One obtains the STU model with prepotential

$$F = \frac{X^1 X^2 X^3}{X^0},$$

(5.31)

and purely electric FI parameters $g_0 = 0$ and $g_i = g$. The black hole has magnetic charges $p_a$ with respect to the vectors $1, 2, 3$ and electric charge $n$ with respect to the vector $A^0$. We see that, under the identifications

$$\hat{X}^0 \to \beta, \quad \hat{X}^i \to i\Delta_a,$$

$$\left( q_A, p^A \right) \to \left( -n, q_1, q_2, q_3, 0, p_1, p_2, p_3 \right),$$

(5.32)

the rescaled prepotential $F(\hat{X})$ is mapped to the twisted superpotential $\tilde{W}(\Delta)$, and the attractor functional (2.34)

$$p^A F_A(\hat{X}) - q_A \hat{X}^A \to i \sum_{a=1}^3 p_a \frac{\partial \tilde{W}}{\partial \Delta_a} + n\beta - iq_a \Delta_a,$$

(5.33)

to the $\mathcal{I}$-functional (5.27).

6 Electrically charged rotating black holes

In this section we come back to the problem of electrically charged rotating black holes with zero magnetic charge. As discussed in section 2, they are qualitatively different from the magnetically charged black holes that we have discussed so far. The main difference is the absence of twist in the dual CFT.

In general, we should be able to recover the entropy of the electrically charged rotating black holes in $\text{AdS}_d$ from the BPS partition function (2.14) that counts supersymmetric states of the dual CFT on $S^{d-2} \times \mathbb{R}$. Since the black holes preserve just two real supercharges, we need to count $1/16$ BPS states and this is a hard problem. Old attempts to evaluate the BPS partition function of $\mathcal{N} = 4$ SYM [148–150] reached the somehow disappointing conclusion that there is a subset of states whose number grows with $N$ but slower than the entropy of the black holes. Alternatively, one may try to replace the BPS partition function with the corresponding index (2.20), that can be expressed and computed as a supersymmetric Euclidean path integral on $S^{d-2} \times S^1$. This defines the so-called superconformal index of the CFT [43, 151]. It is known that the superconformal index, for generic real fugacities, is a quantity of order one in the large $N$ limit [43] and, as such, it does not reproduce the entropy which grows with powers of $N$. As a difference with the twisted

\footnote{We are cavalier about constants and normalizations.}
index, already in the large $N$ limit, there is a large cancellation between bosonic and fermionic supersymmetric states and $Z_{\text{index}}(\Delta_a, \omega_i) \neq Z(\Delta_a, \omega_i)$. All these (partial) results have stood as puzzles about supersymmetric electrically charged rotating black holes in the past.

However, the entropy of many electrically charged rotating black holes in different dimensions can be expressed in terms of quantities with a clear field theory interpretation [47–49], thus suggesting that the entropy can be reproduced by a field theory computation. Moreover, as stressed in [6, 39], the computation in [43] is valid only for real fugacities but the relevant saddle point at large $N$ could be complex, as also suggested by the entropy functionals derived in [47–49]. Recent results for AdS$_5$ confirm this point of view and lead to various derivations of the entropy using the index or the supersymmetric partition function [6, 38, 39, 50, 86, 152], as we discuss in this section.

6.1 The entropy functional for electrically charged rotating black holes

For many electrically charged rotating black holes, the entropy, as a function of electric charges $q_a$ and angular momenta $j_i$, can be written as a Legendre transform

$$S(q_a, j_i) = \log Z(\Delta_a, \omega_i) - i(\Delta_a q_a + \omega_i j_i)\bigg|_{\Delta_a, \omega_i}$$

of a quantity $\log Z(\Delta_a, \omega_i)$ related to anomalies or free energies of the dual CFT. This was derived in [47] for five-dimensional black holes and generalized to other dimensions in [48, 49]. We will refer to (6.1) as the entropy functional for electrically charged rotating black holes. The expression for such entropy functionals is schematically given in table 1.

The content of the table refers to electrically charged rotating black holes in each dimension that can be embedded in a maximally supersymmetric string theory or M-theory background. There are the familiar AdS$_5 \times S^5$ black holes in type IIB found in [10–14] and the analogous AdS$_7 \times S^4$ and AdS$_4 \times S^7$ black holes in M-theory [17–19]. The dual field theories are well known: the ABJM theory in three dimensions, $\mathcal{N} = 4$ SYM in four dimensions and the $(2, 0)$ theory in six dimensions. In addition, there are black holes in the warped AdS$_6 \times W$S$^4$ background of massive type IIA [20, 153]. The dual CFT is the $\mathcal{N} = 1$ five-dimensional fixed point associated with D4-D8-O8 branes in type IIA found in [154]. Notice that, in six dimensions, the maximal superconformal algebra has only sixteen supercharges instead of thirty-two and the superconformal theory with such an algebra is not unique. Among the theories with an AdS dual, the D4-D8-O8 system is somehow the simplest and most studied.

The chemical potentials in the table refer to the isometries of the internal manifold. Notice that the chemical potentials are always subject to a constraint. Indeed, as already discussed in section 2, in all dimensions, we expect the existence of a family

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60In the case of the D4-D8-O8 system, the field theory has $SU(2)_R \times SU(2) \times E_{n_f+1}$ symmetry,
Table 1. Entropy functionals for electrically charged rotating black holes. For simplicity of notations, we opted for a uniform notation for all dimensions, involving some sign redefinitions in $q_a$ and $j_i$ in comparison to [47–49], to which we refer for more precise statements.

In all other dimensions the relation (6.1) has been checked for the solutions available in the literature [48, 49].

In the second column of the table, there is a quantity, $F(\Delta_a)$, with a clear field theory interpretation. The reader has probably recognized the $S^3$ free energy of ABJM in the first row and the trial $a$-charge of $N = 4$ SYM in the second row, see (4.26) and (5.7), which are both functions of trial R-charges satisfying

$$\sum_a \Delta_a = 2.$$  \hfill (6.2)

In general, $F(\Delta_a)$ is the sphere free energy for odd-dimensional CFTs, and a particular combination of t’Hooft anomaly coefficients in even dimensions.\(^{62}\) In all cases, where $SU(2)_R \times SU(2)$ is realized by the isometry of the warped $S^4$ and $E_{n+r+1}$ by the theory on the $N_f$ physical D8 branes. We introduced a symmetric notation for the chemical potentials associated with $SU(2)_R \times SU(2)$ following the notations of [123]. The entropy functional discussed in [49] refers to the case $\Delta_1 = \Delta_2$.

\(^{61}\) Supersymmetric hairy AdS$_5$ black holes with diverging tidal forces depending on all the charges has been recently found in [15, 16].

\(^{62}\) It is also curious to observe that $F(\Delta)$ is the on-shell value of the twisted superpotential of the CFT in three and four-dimensions, as discussed in sections 4 and 5, and the on-shell prepotential

| $AdS \times S^7$ | $F(\Delta_a) = \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}$ | $\log Z(\Delta_a, \omega_i) = \frac{4 \sqrt{N_{3/2}} \sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}}{\omega_1}$ |
|-----------------|-----------------|-----------------|
| $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 = 2$ | $\Delta_1 + \Delta_2 + \Delta_3 + \Delta_4 + \omega_1 = 2\pi$ | |

| $AdS \times S^5$ | $F(\Delta_a) = \Delta_1 \Delta_2 \Delta_3$ | $\log Z(\Delta_a, \omega_i) = -i \frac{N^2}{2} \frac{\Delta_1 \Delta_2 \Delta_3}{\omega_1 \omega_2}$ |
|-----------------|-----------------|-----------------|
| $\Delta_1 + \Delta_2 + \Delta_3 = 2$ | $\Delta_1 + \Delta_2 + \Delta_3 + \omega_1 + \omega_2 = 2\pi$ | |

| $AdS_6 \times W S^4$ | $F(\Delta_a) = (\Delta_1 \Delta_2)^{3/2}$ | $\log Z(\Delta_a, \omega_i) \sim N^{5/2} \frac{(\Delta_1 \Delta_2)^{3/2}}{\omega_1 \omega_2}$ |
|-----------------|-----------------|-----------------|
| $\Delta_1 + \Delta_2 = 2$ | $\Delta_1 + \Delta_2 + \omega_1 + \omega_2 = 2\pi$ | |

| $AdS_7 \times S^4$ | $F(\Delta_a) = (\Delta_1 \Delta_2)^{3}$ | $\log Z(\Delta_a, \omega_i) = -i \frac{N^3 (\Delta_1 \Delta_2)^3}{24 \omega_1 \omega_2 \omega_3}$ |
|-----------------|-----------------|-----------------|
| $\Delta_1 + \Delta_2 = 2$ | $\Delta_1 + \Delta_2 + \omega_1 + \omega_2 + \omega_3 = 2\pi$ | |
the quantity \( \log Z(\Delta_a, \omega_i) \) can be obtained by taking the quotient of \( F(\Delta) \) by the product of all angular momentum chemical potentials and by replacing the R-charge constraint (6.2) with

\[
\sum_a \Delta_a + \sum_i \omega_i = 2\pi.
\] (6.3)

This constraint is strongly reminiscent of the analogous constraint (4.25) for magnetically charged black holes. Notice that \( \log Z(\Delta_a, \omega_i) \) is a sort of equivariant generalization of \( F(\Delta) \) with respect to rotation. Indeed, the expression for \( \log Z(\Delta_a, \omega_i) \) for AdS\(_5\) and AdS\(_7\) can be also directly obtained by an equivariant integration of the six-dimensional and eight-dimensional anomaly polynomial for \( \mathcal{N} = 4 \) SYM and the (2,0) theory in six dimensions, respectively [155].

Let us make some general comments. First, notice that the quantity in (6.1) is extremized for complex values of the chemical potentials \( \Delta_a \) and \( \omega_i \). However, quite remarkably, the on-shell value (6.1) is real, as an entropy must be. The fact that \( \Delta_a \) and \( \omega_i \) are complex at the extremum is important for the field theory interpretation of the result.

Secondly, the expression of \( \log Z(\Delta_a, \omega_i) \) for AdS\(_5\) black holes can be explicitly derived in gravity by taking the zero-temperature limit of the on-shell action of a family of supersymmetric, complexified Euclidean solutions [38]. In this approach, one can also derive the complex value for the chemical potentials at the saddle point and explain the constraint \( \sum_a \Delta_a + \sum_i \omega_i = 2\pi \) with a regularity condition for the Killing spinors [38].

Finally, we expect that (6.1) corresponds to the attractor mechanism in the relevant gauged supergravity. Unfortunately, the attractor mechanism in generic dimensions and, specifically, for electrically charged rotating black holes is not known. AdS\(_5\) black holes with equal angular momenta can be dimensionally reduced to static black holes in four dimensions and, in this case, one can show that (6.1) corresponds to the attractor mechanism in four-dimensional gauged supergravity [47]. This was actually the observation that led to write the entropy functional (6.1).

### 6.2 Recent progresses on the quantum field theory side

Various field theory interpretations of the extremization principle (6.1) have been recently proposed for AdS\(_5\) [6, 38, 39, 50, 86, 152] (and partially for AdS\(_7\) [39]).
The most natural place where to look for the entropy of the electrically charged rotating black holes in AdS$_5$ is the superconformal index (6.4). It is defined as

$$I(\Delta_a, \omega_i) = \text{Tr} \left|_{Q=0} (-1)^F e^{i(\Delta_a \mathcal{R}_a + \omega_i \mathcal{J}_i)} \right.,$$

for a choice of supercharge $Q$. As in section 2, $\mathcal{R}_a$ denotes the charge operator for R symmetries and $\mathcal{J}_i$ are the angular momenta. The index can be only refined with fugacities corresponding to the conserved charges that commute with $Q$. Assuming that the spectrum of $\mathcal{R}_a$ and $\mathcal{J}_i$ is half-integral, this imposes a constraint of the form $\sum_a \Delta_a + \sum_i \omega_i \in 4\pi \mathbb{Z}$ on the fugacities. Such a constraint is compatible with the constraint on charges of the black holes we are interested in, and suggests that the superconformal index is appropriate for studying their entropy. Due to cancellations between bosonic and fermionic supersymmetric states, the result obtained from the index can only be a lower bound on the number of BPS states. However, we may expect that, as for magnetically charged black holes, for large $N$, the result saturates the entropy. Most of the computations for the superconformal index in the literature has been performed for real fugacities and give results of order $O(1)$ for large $N$. However, the extremization principle discussed above strongly suggests that we should look at the behaviors of the index as a function of complex chemical potentials.

It has been shown in [6, 86] that, in the large $N$ limit, the index has a Stokes behavior as a functions of the chemical potentials, and it can give a contribution to the entropy of order $O(N^2)$ along the right direction in the complex plane. The author of [6, 86] were able to reproduce the entropy of the AdS$_5$ black holes and confirm the extremization formula (6.1) in the case of equal angular momenta. A crucial technical ingredient in their analysis involves writing the superconformal index as a sum over Bethe vacua. As shown in [69–71], the supersymmetric partition function of many three- and four-dimensional manifolds can be expressed as a sum over two-dimensional Bethe vacua using the formalism that we briefly discussed in section 3.3.2. A formula for the superconformal index was obtained in [70] and generalized to unequal fugacities for the angular momenta in [86]. Schematically, it allows to write the index as in (3.38)

$$I = \sum_{x^*} \text{det}_{ij} \left( -\partial^2_{u_i u_j} W(x^*) \right)^{Q(x^*)},$$

where $x^*$ are the Bethe vacua of the two-dimensional theory obtained by reduction on $T^2$, and $Q(x)$ is a suitable function whose expression can be found in [70, 86].

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65. For theories with less than thirty two supercharges we can also introduce charges for the global symmetries.

66. The two circles in $T^2$ corresponds to $S^1$ and a circle inside $S^3$ that allow to think about the four-manifold $S^1 \times S^3$ as a torus fibration over a two-dimensional manifold. For details see [70, 86].
Notice that the Bethe vacua for $\mathcal{N} = 4$ SYM on $T^2$ have been already discussed in section 5.2 for a different purpose and are explicitly given by solutions of (5.18). It is shown in [6, 86] that such sum over Bethe vacua exhibits a Stokes behavior and a particular contribution in the large $N$ limit exactly reproduces the entropy of the AdS$_5$ black holes. In this context, the constraint (6.3) emerges from the explicit computation. It is intriguing to see how the formalism of Bethe vacua enters in almost all computations for the entropy of AdS black holes and black objects in different dimensions.

A different calculation has been performed in [39, 50, 152, 165] using a certain scaling limit for the chemical potentials. It has been originally argued in [39, 152] that the imaginary parts of the fugacities at the saddle point, introduce phases that optimally obstruct the cancellation between bosonic and fermionic states. As derived in [39, 152], the number of states accounted by the index in an appropriate Cardy limit correctly reproduces the entropy of large AdS$_5$ black holes and the extremization formula (6.1). This approach has been further refined and generalized in [50, 165].

A somehow different explanation has been proposed in [38] using a localization computation for supersymmetric partition function on $S^{d-2} \times \mathbb{R}$. To understand this proposal, we must notice that the relation between the supersymmetric partition function on $S^{d-2} \times \mathbb{R}$, with $d$ odd, and the superconformal index $I$ is subtle. There is indeed a proportionality coefficient due to regularization,

$$Z_{S^{d-2} \times \mathbb{R}}(\Delta_{a}, \omega_{i}) = e^{E_{c.e}} I(\Delta_{a}, \omega_{i}),$$

(6.6)

where $E_{c.e}$, called supersymmetric Casimir energy [166–170], scales with the same power of $N$ as the entropy. As noticed in [47], the value for the supersymmetric Casimir energy $E_{c.e}$ quoted in the literature [155], but valid for $\sum_{a} \Delta_{a} + \sum_{i} \omega_{i} = 0$, matches the expression for $\log Z(\Delta_{a}, \omega_{i})$ given in the table with the precise coefficient for both $\mathcal{N} = 4$ SYM and the $(2, 0)$ theory. The authors of [38] argued that, in the case of $\mathcal{N} = 4$ SYM in four dimensions, one can consistently define supersymmetric partition functions with $\sum_{a} \Delta_{a} + \sum_{i} \omega_{i} = 2\pi n, \ n \in \mathbb{Z}$, and that, for both $n = 0$ and $n = 1$, one recovers the result given in the table. Comparing with the gravity solution, they also argued that the partition function with $n = 1$ is the correct one to use for an holographic description of the AdS$_5$ black holes. The supersymmetric Casimir energy can be interpreted as the energy of the vacuum of the CFT [167]. From this point of view, it is less clear why it should be related to the entropy of the black hole and the relation with the other approaches still to be clarified.

All these results are very recent and the connection between different approaches still to be understood, but all seems to indicate that the entropy is correctly accounted by the large $N$ limit of the index/partition function. At this point, we also expect that similar analyses in different dimensions would reproduce the full content of table 1. Moreover, as observed in [6, 39, 152], in all these computations, there seems to exist instabilities when decreasing the charges, which might suggest the contribution
of other types of black holes. Given also the recently found supersymmetric hairy black holes in AdS$_5$ [15, 16], we may expect a rich structure of the index/partition function still to be uncovered.

### 6.3 Some general comments

It is interesting to observe that, in all dimensions, a single function controls the entropy of most of the black holes and black objects asymptotic to a maximally supersymmetric AdS vacuum, with or without magnetic charges or rotation. For example, for black objects asymptotic to AdS$_5 \times S^5$, this function is the trial $a$-charge in the large $N$ limit,

$$a(\Delta) \sim \text{Tr} R(\Delta)^3 \sim 3N^2\Delta_1\Delta_2\Delta_3,$$

using the notations of section 5.1 and table 1. As discussed in section 5.3, it controls the asymptotic growing of states for the two-dimensional CFT associated with a black string in AdS$_5 \times S^5$ (see equations (5.22) and (5.10))

$$\log Z(p, \Delta) = \frac{\pi i}{12} c_r(\Delta) = -\frac{8\pi i}{27\tau} \sum_{\alpha=1}^3 p_\alpha \frac{\partial a(\Delta_\alpha)}{\partial \Delta_\alpha},$$

and the entropy of the corresponding dimensionally reduced four-dimensional static black holes (see (5.27))

$$I(\beta, \Delta) \equiv \log Z(p, \Delta) + \beta n - i \sum a q_a \Delta_a.$$  

Similarly, as discussed in section 6.1 and reported in table 1, the equivariant generalization of the central charge

$$\log Z(\Delta_a, \omega_i) = -i \frac{N^2}{2} \frac{\Delta_1\Delta_2\Delta_3}{\omega_1\omega_2},$$

controls the entropy of the electrically charged rotating black holes in AdS$_5 \times S^5$ through the Legendre transform (6.1).

We expect that the entropy of black objects in more general backgrounds AdS$_5 \times SE_3$ is similarly controlled by the trial $a$-charge

$$a(\Delta) \sim \text{Tr} R(\Delta)^3 \sim \sum f_{abc} N^2\Delta_a\Delta_b\Delta_c,$$

where $f_{abc}$ are the cubic t’Hooft anomaly coefficients. The coefficients $f_{abc}$ have a natural dual gravitational interpretation. They arise as intersection numbers of

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We assume that we can use a basis of (possible redundant) R-charges such that $a(\Delta)$ is a homogeneous function, as it is natural to do in compactifications on Sasaki-Einstein manifolds [171].
cycles in the internal manifold [171], and, from an effective field theory perspective, as Chern-Simons terms in the corresponding gauged supergravity in five dimensions. In particular, they determine completely the structure of $\mathcal{N} = 1$ gauged supergravity including the prepotential.

Similarly, the function $\sqrt{\Delta_1 \Delta_2 \Delta_3 \Delta_4}$ and its equivariant generalization given in table 1 controls the entropy of dyonic static black holes (see section 4.3 and (4.31)) and electrically charged rotating black holes in $\text{AdS}_4 \times S^7$ (see table 1). It is also related to the prepotential of the dual $\mathcal{N} = 2$ gauged supergravity. It would be interesting to see if also the entropy of dyonic rotating black holes found in [29] can be written in terms of this function.

An analogous situation holds in five and six dimensions. The functions $(\Delta_1 \Delta_2)^{3/2}$ and $(\Delta_1 \Delta_2)^2$ appear in the description of both twisted [123, 126] and untwisted [48, 49] black holes and black strings in $\text{AdS}_6 \times S^4$ and $\text{AdS}_7 \times S^4$.

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