CYCLICITY OF LUSZTIG’S STRATIFICATION OF GRASSMANNIANS AND POISSON GEOMETRY

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Abstract. We prove that the standard Poisson structure on the Grassmannian $\text{Gr}(k, n)$ is invariant under the action of the Coxeter element $c = (12\ldots n)$. In particular, its symplectic foliation is invariant under $c$. As a corollary, we obtain a second, Poisson geometric proof of the result of Knutson, Lam, and Speyer that the Coxeter element $c$ interchanges the Lusztig strata of $\text{Gr}(k, n)$. We also relate the main result to known anti-invariance properties of the standard Poisson structures on cominuscule flag varieties.

1. Introduction

For the purpose of the study of canonical bases, Lusztig defined [4] the totally nonnegative part $(G/P)_{\geq 0}$ of an arbitrary complex flag variety $G/P$. He also constructed an algebro-geometric stratification of $G/P$ and conjectured that intersecting this stratification with $(G/P)_{\geq 0}$ is producing a cell decomposition of $(G/P)_{\geq 0}$. This was latter proved by Rietsch in [5]. Both the non-negative part $(G/P)_{\geq 0}$ and the Lusztig stratification of a flag variety were studied in recent years from many different combinatorial and Lie theoretic points of view.

In a recent work Knutson, Lam, and Speyer proved that the Lusztig stratification of the Grassmannian $\text{Gr}(k, n)$ has a remarkable cyclicity property. If $c$ denotes the Coxeter element $(12\ldots n)$ of $S_n$ and the permutation matrix in $\text{GL}_n(\mathbb{C})$ which represents it, then $c$ permutes the strata of the Lusztig stratification of $\text{Gr}(k, n)$.

In this note we give a Poisson geometric proof of this fact. We also prove a stronger invariance property of a Poisson structure on $\text{Gr}(k, n)$. In [2], jointly with Goodearl, we found a Poisson geometric interpretation of the Lusztig stratification of any flag variety $G/P$. For a choice of opposite Borel subgroups $B$ and $B^-$ of $G$ such that $B \subset P$ one defines the standard Poisson structure $\pi_{G/P}$ on $G/P$ which is invariant under the action of the maximal torus $T = B \cap B^-$, see [2] for details. According to [2] Theorem 0.4] the $T$-orbits of symplectic leaves of $\pi_{G/P}$ are exactly the Lusztig strata.

In the case of the complex Grassmannian $\text{Gr}(k, n)$ the standard Poisson structure is given by

$$\pi_{k,n} = - \sum_{1 \leq i < j \leq n} \chi(E_{ij}) \wedge \chi(E_{ji})$$

where $\chi: \mathfrak{gl}_n(\mathbb{C}) \to \text{Vect}(\text{Gr}(k, n))$ denotes the induced infinitesimal action from the left action of $\text{GL}_n(\mathbb{C})$ on $\text{Gr}(k, n)$ and $E_{ij}$ denote the elementary matrices. This Poisson structure is invariant under the action of the maximal torus $T_n$ of diagonal matrices in $\text{GL}_n(\mathbb{C})$. For each $w \in S_n$ denote by the same letter the
Corollary 1.2. The actions of \( w_0, c, \) and \( T_n \) generate an action of \( I_2(n) \times T_n \) by Poisson and anti-Poisson automorphisms of \( (\text{Gr}(k,n), \pi_{k,n}) \) where \( I_2(n) \) denotes the dihedral group of order \( 2n \).

The Lusztig stratification of the Grassmannian \( \text{Gr}(k,n) \) is defined as follows, see [4] for details. Let \( B \) and \( B_- \) be the standard Borel subgroups of \( \text{GL}_n(\mathbb{C}) \) consisting of upper and lower triangular matrices. Denote the maximal parabolic subgroup

\[
P_{k,n} = \{ (a_{ij}) \in \text{GL}_n(\mathbb{C}) \mid a \in M_{k,k}, b \in M_{k,n-k}, c \in M_{n-k,n-k} \}
\]

of \( \text{GL}_n(\mathbb{C}) \) and the induced map

\[
q: \text{GL}_n(\mathbb{C})/B \to \text{GL}_n(\mathbb{C})/P_{k,n} \cong \text{Gr}(k,n).
\]

The strata in the Lusztig stratification of \( \text{Gr}(k,n) \) are given by

\[
R_{v,w} = q(B_\cdot vB \cap B \cdot wB), \quad v \in (S_n)^{S_k \times S_{n-k}}, w \in S_n, v \leq w.
\]

Here \( \leq \) refers to the Bruhat order. We denote by \( S_k \times S_{n-k} \) the subgroup of \( S_n \) consisting of those \( u \in S_n \) such that \( u(i) \leq k \) for \( i \leq k \) and \( u(i) \geq k + 1 \) for \( i \geq k + 1 \). Finally, \( (S_n)^{S_k \times S_{n-k}} \) denotes the set of maximal length representatives for the cosets \( S_n/(S_k \times S_{n-k}) \).

The symplectic foliation of a Poisson structure is uniquely determined by it. Thus the \( T_n \)-orbits of leaves of \( \pi_{k,n} \) (which are exactly the Lusztig strata) are an invariant of the pair \( (\pi_{k,n}, T_n \text{-action}) \). Therefore Theorem 1.1 gives a second proof of the result of Knutson, Lam, and Speyer that the action of the Coxeter element \( c \) on \( \text{Gr}(k,n) \) interchanges the Lusztig strata. In fact Theorem 1.1 is equivalent to the stronger statement:

"The action of \( c \) on \( \text{Gr}(k,n) \) restricts to Poisson isomorphisms between various Lusztig strata \( (R_{v,w}, \pi_{k,n}, R_{v,w}) \) considered as regular Poisson varieties."

Finally we trace the roots of this phenomenon from a Poisson geometric point of view. It is well known that on any flag variety \( G/P \) the standard Poisson structure \( \pi_{G/P} \) is anti-invariant under the action of any representative \( \hat{w}_o \) of the longest element of the Weyl group \( W \) of \( G \). If, in addition, \( P \) is cominuscule, then [2, Proposition 4.2] implies that the standard Poisson structure on \( G/P \) is anti-invariant under the action of any representative \( \hat{w}_o^P \) of the longest element of the corresponding parabolic subgroup of \( W \). In the special case of the Grassmannian the specific Coxeter element \( c \) happens to be a \( k \)-th root of \( \hat{w}_o^P \hat{w}_o \). Thus Theorem 1.1 claiming that the standard Poisson structure on \( \text{Gr}(k,n) \) is invariant under \( c \) is a strengthening of [2, Proposition 4.2]. See Section 2 for more details. We do not know of good Poisson properties of roots of \( \hat{w}_o^P \hat{w}_o \) for any other cominuscule
flag varieties.

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## 2. Proof of Theorem 1.1

**Proof of Theorem 1.1.** The statement is equivalent to showing that

\[
\sum_{1 \leq i < j \leq n} \chi(\text{Ad}_c(E_{ij})) \wedge \chi(\text{Ad}_c(E_{ji})) - \sum_{1 \leq i < j \leq n} \chi(E_{ij}) \wedge \chi(E_{ji}) = 0;
\]

that is

\[
V := \sum_{i=2}^{n} \chi(E_{1i}) \wedge \chi(E_{i1}) = 0.
\]

We will check this on the open Schubert cell \(B_- \cdot P_{k,n} \subset \text{Gr}(k,n)\). Since \(V\) is an algebraic bivector field, this will establish (2.1). Identify

\[
M_{n-k,k} \cong B_- \cdot P_{k,n} \subset \text{Gr}(k,n), \quad X \mapsto \begin{pmatrix} I_{k} & 0 \\ X & I_{n-k} \end{pmatrix} \cdot P_{k,n}
\]

where \(M_{n-k,k}\) denotes the space of \((n-k) \times k\) complex matrices. Applying [1, eq. (3.17)] we get that under (2.2)

\[
\chi(E_{1,i+k}) \mapsto -\sum_{p=1}^{n-k} \sum_{q=1}^{k} x_{pi}x_{iq} \frac{\partial}{\partial x_{pq}}, \quad \text{for } i = 1, \ldots, n-k.
\]

It is obvious that

\[
\chi(E_{i+k,1}) \mapsto \frac{\partial}{\partial x_{i1}}, \quad \text{for } i = 1, \ldots, n-k.
\]

Let \(1 \leq i, j \leq k\). Then

\[
\text{Ad}_{\exp(sE_{ij})} \left( \begin{pmatrix} I_{k} & 0 \\ X & I_{n-k} \end{pmatrix} \right) \left( \begin{pmatrix} I_{k} \\ X \sum_{p=1}^{n-k} x_{pi}E_{pq} I_{n-k} \end{pmatrix} \right),
\]

which implies that under (2.2)

\[
\chi(E_{i,j}) \mapsto -\sum_{p=1}^{n-k} x_{pi} \frac{\partial}{\partial x_{pj}}.
\]
The summation in (2.1) can be taken from $i = 1$ since $\chi(E_{1,1}) \wedge \chi(E_{1,1}) = 0$. Applying (2.3), (2.4), and (2.5) we obtain that under the identification (2.2)

$$V|_{B_-P_{k,n}} \mapsto \sum_{i=1}^{k} \sum_{p=1}^{n-k} \sum_{q=1}^{n-k} x_{pq} \frac{\partial}{\partial x_{pi}} \wedge \frac{\partial}{\partial x_{qj}}$$

$$- \sum_{i=1}^{n-k} \sum_{p=1}^{k} \sum_{q=1}^{n-k} x_{pq} \frac{\partial}{\partial x_{pi}} \wedge \frac{\partial}{\partial x_{qj}} = 0.$$  

This implies (2.1) and the statement of the Theorem. □

For an arbitrary complex simple group $G$ and a maximal parabolic subgroup $P$, one defines the standard Poisson structure

$$\pi_{G/P} = -\chi(r_G)$$

on the flag variety $G/P$ induced from a compatible triangular decomposition of $G$ (a pair of Borel subgroups $B$ and $B_-$, such that $B \cap B_-=T$ is a maximal torus of $G$ and $B \subset P$), see e.g. [2]. Here $\chi: \wedge^2 \text{Lie} (G) \to \Gamma( TG/P, G/P)$ denotes the induced action from the infinitesimal action of $\text{Lie}(G)$. The standard $r$-matrix $r_G \in \wedge^2 \text{Lie}(G)$ obtained from the triangular decomposition of $G$ is given by:

$$r_G = \sum_{\alpha \in \Delta_+} e_\alpha \wedge f_\alpha$$

where $e_\alpha$ and $f_\alpha$ are appropriately normalized root vectors of $\text{Lie}(G)$ and $\Delta_+$ is the set of positive roots of $G$, cf. [2]. It is obvious that the action of any representative $\hat{w}_0$ of the longest element of the Weyl group $W$ of $G$ on $(G/P, \pi_{G/P})$ is anti-Poisson, since $\text{Ad}_{\hat{w}_0}$ interchanges $e_\alpha$ and $f_\alpha$.

Denote the Levi factor of $P$ containing $T$ by $L$, and the longest element of the subgroup of $W$ corresponding to $L$ by $w^P_0$. Let $\hat{w}_0^P$ be any representative of $w^P_0$ in the normalizer of $T$.

Recall that among several equivalent definitions/characterizations of cominuscule parabolic subgroups: a parabolic subgroup $P$ of $G$ is cominuscule if and only if its unipotent radical is abelian. According to [2, Proposition 4.2], if $P$ cominuscule, then $\pi_{G/P}$ is also given by

$$\pi_{G/P} = -\chi(r_L).$$

where $r_L \in \wedge^2 \text{Lie}(L)$ is the standard $r$-matrix of $L$. Thus the action of $\hat{w}_0^P$ on $(G/P, \pi_{G/P})$ is anti-Poisson as well. So:

**Proposition 2.1.** For any cominuscule parabolic subgroup $P$ of a complex simple Lie group $G$, the action of $\hat{w}_0^P \hat{w}_0$ on $(G/P, \pi_{G/P})$ is Poisson.

In the special case of the Grassmannian $\text{Gr}(k,n)$

$$w^P_0 w_0 = c^k$$

for the particular Coxeter element $c$. Taking powers of this product, we see that the action of $w^{\text{gcd}(k,n)}_0$ on $\text{Gr}(k,n)$ is Poisson. In the case when $k$ and $n$ are relatively prime this gives yet another proof of Theorem [2]. One could argue that Theorem [1.1] holds because it is true for relatively prime $k$ and $n$, and its
statement (cf. also its proof) is independent of the numerical properties of \(k\) and \(n\).

Conceptually the invariance of \(\pi_{k,n}\) under \(c\) is the result of a two step process:

1. From Proposition 2.1 one has the invariance of \(\pi_{G/P}\) under the product \(\dot{w}_c P \dot{w}_c\) of the longest elements of the Weyl groups of \(G\) and the Levi subgroup \(L\), for arbitrary cominuscule flag variety \(G/P\).

2. In the case of the Grassmannian the Coxeter element \(c\) which is a \(k\)-th root of \(\dot{w}_c P \dot{w}_c\) acts by Poisson automorphisms of \((\text{Gr}(k,n), \pi_{k,n})\) as well.

The special property of the Coxeter element \(c = (12\ldots n)\) is that the other Coxeter elements of \(S_n\) are not roots of \(\dot{w}_c P \dot{w}_c\).

3. Corollaries

The symplectic foliation of a Poisson manifold \((M, \pi)\) is an invariant of it. Similarly if a group \(H\) act on \((M, \pi)\) by Poisson automorphisms the partition of \(M\) into \(H\)-orbits of symplectic leaves is an invariant of \((M, \pi)\) considered as a Poisson \(H\)-space. Since the partition of \(\text{Gr}(k,n)\) into \(T_n\)-orbits of symplectic leaves of \((\text{Gr}(k,n), \pi_{k,n})\) is exactly the Lusztig stratification of \(\text{Gr}(k,n)\) due to [2, Theorem 0.4] and \(c\) normalizes \(T_n\), Theorem 1.1 implies the following Theorem of Knutson, Lam, and Speyer:

**Theorem 3.1.** The action of the permutation matrix corresponding to the Coxeter element \(c = (12\ldots n)\) on \(\text{Gr}(k,n)\) permutes the strata \(R_{v,w} = q(B_\cdot vB \cap B \cdot wB)\) of the Lusztig stratification.

As we pointed out, in addition, when \(c\) maps one Lusztig stratum \(R_{v_1,w_1}\) to another \(R_{v_2,w_2}\) it matches the regular Poisson structures \(\pi_{k,n} |_{R_{v_1,w_1}}\) and \(\pi_{k,n} |_{R_{v_2,w_2}}\).

Finally let us point out that all constructions and invariance properties are valid over the reals since all constructions are derived from the real split form \(\text{GL}_n(\mathbb{R})\).

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