ON THE VILKOVISKY UNIQUE EFFECTIVE ACTION IN QUANTUM GRAVITY

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ABSTRACT

The divergent part of the one-loop unique effective action for quantum Einstein gravity is evaluated in the general parametrization of the quantum field, including the separated conformal factor. The output of the calculation explicitly verifies the independence on the field parametrization. The version of effective action introduced by Vilkovisky is unique if the metric in the space of quantum fields is chosen in a “natural” way. The uniqueness of the effective action enables constructing well-defined, individual renormalization group equations for both Newton and cosmological constants, which describe the running of these effective charges between the GUT scale in the UV and the extremely low energy scale in the IR.

Keywords: Unique effective action, parametrization independence, one-loop divergences, quantum gravity

1 Introduction

The off-shell effective action in gauge theories depends on the choice of the gauge-fixing and the parametrization of quantum fields. One of the important consequences of this ambiguity is that, even in the framework of effective low-energy quantum gravity, one cannot have well-defined individual renormalization group equations for the Newton constant $G$ and the cosmological constant $\Lambda$. There is only one unambiguous equation, for the dimensionless combination of these constants. On the other hand, in the modified versions of effective action proposed by Vilkovisky [1] and DeWitt [2] there is no gauge or parametrization ambiguity. The purpose of the present work is to evaluate the divergent part of the one-loop Vilkovisky effective action for the quantum version of Einstein gravity in a general parametrization of the quantum field, and explicitly verify the independence of this construction on the parametrization.

The classical action of the theory of our interest has the form

$$S(g_{\mu\nu}) = -\frac{1}{\kappa^2} \int d^Dx \sqrt{|g|} (R + 2\Lambda),$$

(1)
where \( G = \kappa^2/(16\pi) \) is the \((D\text{-dimensional})\) Newton constant and \( \Lambda \) is the cosmological constant. There is an extensive literature on the derivation and analysis of one-loop and two-loop divergences in the theory (1). The first calculations were performed in [3] for gravity coupled with the minimal scalar field and in [4] for gravity coupled to an electromagnetic field. The calculation in the nonminimal gauge was pioneered in [5]. The parametrization dependence was explored in [6–8] and, in a more general form, in the more recent Ref. [9]. In what follows we shall use some technical developments of the latter work, which can be also consulted for further references.

The unique effective action of Vilkovisky is independent of the parametrization of quantum fields by construction. On the other hand, this construction becomes complicated in gauge theories, where one has to combine corrections compensating gauge and parametrization ambiguities. In this regard, a special case is the two-dimensional quantum gravity. It was noted in [1] that, in this particular example, the gauge and parametrization ambiguities mix in such a way that the unique effective action may turn out to depend on the gauge fixing. Later on, this feature has been confirmed by a direct calculation in [10]. The origin of this contradictory result is that the unique effective action depends on the choice of the metric in the configuration space, or the space of the quantum fields, in the background field formalism. In gravity, the configuration-space metric has one arbitrary parameter \( a \). And it happens that in \( D = 2 \) this parameter depends on the gauge fixing, because of the reduced number of the physical degrees of freedom. The \( D = 4 \) quantum gravity in the conformal parametrization has a lot of technical similarity with the \( D = 2 \) case, so one can suspect that some gauge or parametrization dependence may persist in this case too. This possibility makes the explicit verification of the full parametrization independence in \( D = 4 \) quantum gravity a decent problem to solve.

Another aspect of the unique effective action, which was explored earlier in [11], is the possibility to construct the well-defined, unambiguous, separate renormalization group equations for both Newton and cosmological constants in the theory (1). In what follows we consider these equations in a slightly different manner, i.e. within the framework of effective quantum gravity.

The outline of the paper is as follows. Sec. 2 briefly reviews the formalism of Vilkovisky’s effective action. The main objective of this section is to make the paper self-consistent and to fix the notations. In Sec. 3 we formulate the one-loop quantum gravity using the background field method in a general non-conformal parametrization of quantum field and a special minimal gauge. The metric in the space of the fields, the Christoffel symbols and the improved bilinear form of the classical action are derived in Sec. 4. It is shown that the coefficients related to the parametrization nonlinearity are compensated by this correction. The corresponding one-loop divergences of the Vilkovisky effective action are computed, in the minimal DeWitt gauge, in Sec. 5. In Sec. 6 the result is generalized to the most general, conformal parametrization of the quantum metric. In Sec. 7 we construct, solve, and discuss the renormalization group equations for the Newton and cosmological constants. Using the framework of effective quantum gravity, it is shown that these equations are applicable in the extensive interval of energies, but do not provide the dramatically strong running. Finally, in Sec. 8 we draw our conclusions.

In this paper we adopt the condensed notations of Refs. [12] and [13].
2 Vilkovisky effective action: a short review

Vilkovisky’s proposal for defining a parametrization-independent effective action [1] is based on the following observation: even though the classical action $S(\phi)$ is a scalar in the space $\mathcal{M}$ of fields $\phi^i$, the generating functional of vertex functions (effective action) is not a scalar functional of the corresponding mean fields. In the simplest, one-loop approximation the effective action depends on the Hessian of the action, $S_{ij} = \frac{\delta^2 S}{\delta \phi^i \delta \phi^j}$, which does not transform as a tensor under field redefinitions $\phi^i = \phi^i(\phi'^i)$.

To provide the scalar nature of the effective action, in Ref. [1] it was introduced an affine structure compatible with the metric $G_{ij}$ in the space $\mathcal{M}$. For given two close points $\phi^i$ and $\phi'^i$, there exists a unique geodesic curve $x^i(\lambda) \subset \mathcal{M}$ with affine parameter $\lambda \in [0, 1]$ connecting them, $x^i(0) = \phi^i$ and $x^i(1) = \phi'^i$. Then, defining the two-point quantity $\sigma^i(\phi', \phi) = \frac{dx^i(\lambda)}{d\lambda} \bigg|_{\lambda=1}$ (the tangent vector to the geodesic at $\phi'^i$, see e.g. [12,14]), the modified definition of the effective action has the form

$$\exp i \Gamma(\phi) = \int D\phi' \mu(\phi') \exp \left\{ i \left[ S(\phi') + \sigma^i(\phi, \phi') \Gamma_{,i}(\phi) \right] \right\},$$

where $\mu(\phi')$ is an invariant functional measure and the comma denotes functional differentiation with respect to $\phi^i$. Because $\sigma^i(\phi, \phi')$ behaves as a vector with respect to $\phi^i$ and as a scalar with regard to $\phi'^i$, the effective action $\Gamma(\phi)$ constructed in this way is a scalar under field reparametrizations.

A qualitatively similar construction can be done for gauge theories, to restore the off-shell gauge independence, given that the effective actions calculated in different gauges are connected by changes of variables (in general, in the form of a canonical transformation [15–17]). However, in this case, the prescription (2) cannot be used directly since it is necessary to factor out the gauge group $G$ in the functional integral. Namely, one has to take into account the gauge orbits and define an affine connection in the configuration space $\mathcal{M}/G$ of physical fields. For the sake of simplicity, we assume that the generators $R^i_\alpha$ of gauge transformations are linearly independent and their algebra is closed, $R^i_\beta R^i_\nu - R^i_\alpha R^i_\nu = F^i_\gamma R^i_\gamma$, with the structure functions $F^i_\gamma$ being independent of the fields. Let the classical action be invariant under gauge transformations $\delta \phi^i = R^i_\alpha \xi^\alpha$,

$$\varepsilon_i R^i_\alpha = 0, \quad \varepsilon_i \equiv S_{,i}. \quad (3)$$

Given a metric $G_{ij}$ on $\mathcal{M}$ one can define the projection operator on $\mathcal{M}/G$ [11,18]

$$P^i_j = \delta^i_j - R^i_\alpha N^{\alpha\beta} R^k_\beta G_{kj},$$

where $N^{\alpha\beta}$ is the inverse of the metric on $G$,

$$N_{\alpha\beta} = R^i_\alpha G_{ij} R^j_\beta. \quad (5)$$

Then the projected metric is

$$G^\perp_{ij} \equiv P^k_i G_{kl} P^l_j = G_{ij} - G_{ik} R^k_\alpha N^{\alpha\beta} R^l_\beta G_{lj}. \quad (6)$$
The affine connection $\mathcal{T}_{ij}^k$ on the physical configuration space can then be obtained by requiring its compatibility with the metric $G_{ij}^{\perp\perp}$ i.e. $\nabla_k G_{ij}^{\perp\perp} = 0$ (see e.g. [19,20]). This yields
\begin{equation}
\mathcal{T}_{ij}^k = \Gamma_{ij}^k + T_{ij}^k,
\end{equation}
which consists of the Christoffel symbol $\Gamma_{ij}^k$ calculated with the metric $G_{ij}$,
\begin{equation}
\Gamma_{ij}^k = \frac{1}{2} G_{ik}^{\alpha\beta} (G_{kj}^{\beta\gamma} + G_{kj}^{\gamma\beta} - G_{ij}^{\beta\gamma}),
\end{equation}
and a non-local part $T_{ij}^k$ related to the gauge constraints on the connection,
\begin{equation}
T_{ij}^k = -2 G_{i[l} \Gamma_{j]k}^l N_{\alpha\beta} \mathcal{D}_{\alpha\beta} - 2 G_{ij}^{\alpha\beta} R_{\beta k} + G_{i[l} \Gamma_{j]k}^l N_{\alpha\beta} \mathcal{D}_{\alpha\beta} R_{k}^m N_{\gamma\delta} R_{\delta m} G_{n]j}).
\end{equation}
The parenthesis in the indices represent symmetrization in the pair $(i, j)$ and $\mathcal{D}$ denotes the covariant derivative calculated with the Christoffel connection $\Gamma_{ij}^k$. The non-locality of (9) is due to the fact that $N_{\alpha\beta}$ is a differential operator and thus its inverse $N_{\alpha\beta}^{-1}$ is formally a Green’s function. In addition to that, this procedure provides the measure $\mu(\chi)$ of the Faddeev-Popov quantization, see e.g. [21,22]. The effective action (2) constructed using the geodesic distance based on the connection $\mathcal{T}_{ij}^k$ is, therefore, reparametrization invariant, gauge invariant and gauge independent. For this reason this object is often called unique effective action.

Performing the loop expansion of the Vilkovisky effective action (2) one gets
\begin{equation}
\Gamma(\phi) = S(\phi) + \bar{\Gamma}^{(1)}(\phi) + \bar{\Gamma}^{(2)}(\phi) + \cdots, \quad \hbar = 1,
\end{equation}
where the one-loop quantum contribution is given by [1]
\begin{equation}
\bar{\Gamma}^{(1)} = \frac{i}{2} \text{Tr} \ln G_{i[l} \mathcal{D}_{j]} S - T_{i[l} \mathcal{D}_{j]} - \chi_{i[l}^{\alpha} Y_{\alpha\beta} \chi_{j]}^{\beta} - i \text{Tr} \ln M_{\alpha\beta}.
\end{equation}
As usual, in pure quantum gravity we can use $\kappa$ as a loop expansion parameter, instead of $\hbar$. Here $\chi^{\alpha}$ is a gauge condition introduced by the gauge-fixing action
\begin{equation}
S_{GF} = -\frac{1}{2} \chi^{\alpha} Y_{\alpha\beta} \chi^{\beta},
\end{equation}
$Y_{\alpha\beta}$ is a non-degenerate weight function (the $\chi^{\alpha}$-space metric) and $M_{\alpha\beta} = \chi^{\alpha} R_{\beta i}^{j}$ is the Faddeev-Popov ghost matrix. Comparing (11) to the loop expansion of the standard effective action, one notes that the second functional derivative of the classical action has been replaced by the second covariant variational derivative.

From the technical side, the computation of (11) is, in general, a very complicated task because of the non-localities of the term $T_{ij}^k$. For this reason, most of the evaluations found in the literature use some kind of DeWitt gauge [25], for which
\begin{equation}
\chi_{i[l}^{\alpha} = Y_{\alpha\beta} G_{\beta j} R_{i]}^{j}.
\end{equation}

1Another gauge- and parametrization-invariant effective action was proposed by DeWitt [2] and subsequently discussed in Refs. [23,24]. Since both definitions coincide at the one-loop level, we do not present this construction.
The purpose of the present work is to evaluate the divergent part of (11) for the quantum gravity based on the general relativity. In this calculation, we follow the reduction method introduced in Ref. [13], which mainly consists in making a power series expansion in the equations of motion $\varepsilon_i$ and applying the generalized Schwinger-DeWitt technique. By using the DeWitt gauge [13] and the Ward identities, it is possible to write (11) in the form

$$\bar{\Gamma}^{(1)} = \frac{i}{2} \text{Tr} \ln \hat{H} - i \text{Tr} \ln \hat{N} - \frac{i}{2} \left( \text{Tr} \hat{U}_1 - \text{Tr} \hat{U}_2 \right) - \frac{i}{4} \text{Tr} \hat{U}_1^2 + O(\varepsilon^3),$$

(14)

where $\hat{N} = Y^\alpha\gamma N_{\gamma\beta}$ and $N_{\alpha\beta}$ was defined in (5),

$$\hat{H} = G^{ik}(\mathcal{D}_i \mathcal{D}_j S - \chi^\alpha_k N_{\alpha\beta} \chi^\beta_j)$$

(15)

takes into account the nontrivial geometry of the space of fields $\mathcal{M}$, and

$$\hat{U}_1 = N^{\alpha\gamma} R_k^l(\mathcal{D}_i R_j^k) \varepsilon_j N^{\delta\gamma} Y_{\delta\beta},$$

(16)

$$\hat{U}_2 = N^{\alpha\gamma} (\mathcal{D}_i R_k^l) \varepsilon_k (H^{-1})^{ij} (\mathcal{D}_j R_k^l) \varepsilon_i N^{j\delta} Y_{\delta\beta}$$

(17)

are two nonlocal operators responsible for restoring the off-shell gauge independence of the one-loop effective action. In (17), $H^{-1}$ is defined by the relation $H \cdot H^{-1} = -1$. In the case of our interest, the terms of orders higher than $\varepsilon^2$ do not contribute to the divergent part of the one-loop effective action and, therefore, are not considered here.

It is worth noting that the latter feature is not true for other models of quantum gravity.

In fact, in the higher-derivative fourth-order gravity only linear terms in $\varepsilon_i$ contribute to the divergences [27, 28], while in quantum general relativity in higher dimensions other terms are necessary. For explicit expressions of the $O(\varepsilon^3)$-terms, see [29]. Calculations of the unique effective action in $D \neq 4$ gravity models can be found, e.g., in [20, 29–33]. Even though we are mainly interested in $D = 4$ results, for the sake of generality we let the space-time dimension $D$ arbitrary in our intermediate calculations.

### 3 Field parametrizations and bilinear form of the action

In the traditional background field method the original field $g_{\mu\nu}'$ is split into a sum of a classical background $g_{\mu\nu}$ and a quantum field $h_{\mu\nu}$, i.e. $g_{\mu\nu}' = g_{\mu\nu} + \kappa h_{\mu\nu}$. As in the present work we are interested in evaluating the one-loop divergences in a general parametrization of the quantum field, instead of performing the usual linear shift, we shall consider $g_{\mu\nu}' = f_{\mu\nu}(g_{\alpha\beta}, \phi_{\alpha\beta})$.

Here the indices are lowered and raised with the external metric $g_{\mu\nu}$ (and its inverse $g^{\mu\nu}$) and $f$ depends on the quantum field $\phi_{\mu\nu}$ possibly in a nonlinear way. Assuming that $f$ has a series expansion, we can define the most general (at one-loop order) parametrization of the quantum metric in the form [9]

$$g_{\mu\nu}' = g_{\mu\nu} + \kappa A_{(1)\mu\nu}^\alpha \phi_{\alpha\beta} + \kappa^2 A_{(2)\mu\nu}^{\lambda\rho\sigma} \phi_{\lambda\gamma} \phi_{\rho\sigma} + O(\kappa^3),$$

(18)

where $A_{(n)\mu\nu}$ are tensor structures depending only on the background metric, and $\kappa$ is the loop-expansion parameter. Through covariance and symmetry arguments, the coefficient functions
in (18) have the general tensor form

\[ A^{\alpha \beta}_{\mu \nu} = \gamma_1 \delta^{\alpha \beta}_{\mu \nu} + \gamma_2 g^{\alpha \beta} g_{\mu \nu}, \]

\[ A^{\lambda \tau, \rho \omega}_{\mu \nu} = \gamma_3 \frac{\gamma_4}{2} \frac{\delta^{\lambda \tau \rho \omega}_{\mu \nu}}{\delta^{\lambda \tau \rho \omega}_{\mu \nu}} + \gamma_5 \frac{\delta^{\lambda \tau \rho \omega}_{\mu \nu}}{\delta^{\lambda \tau \rho \omega}_{\mu \nu}} + \gamma_6 \frac{\delta^{\lambda \tau \rho \omega}_{\mu \nu}}{\delta^{\lambda \tau \rho \omega}_{\mu \nu}} + O(\kappa^3). \]

In these expressions

\[ \delta^{\mu \nu}_{\alpha \beta} = \frac{1}{2} (\delta^{\mu}_{\alpha} \delta^{\nu}_{\beta} + \delta^{\mu}_{\beta} \delta^{\nu}_{\alpha}) \]

and \( \gamma_i \) (\( i = 1, \ldots, 6 \)) are six arbitrary coefficients parameterizing the choice of the quantum variable. The restrictions \( \gamma_1 \neq 0 \) and \( D \gamma_2 \neq 0 \) have to be imposed, to provide that the change of coordinates from \( g'_{\mu \nu} \) to \( \phi_{\mu \nu} \) do not be degenerate. Terms of order \( O(\kappa^3) \) in (18) contribute only at the two- and higher-loop orders, hence are irrelevant and will be omitted in what follows. The one-loop contribution requires a functional integration of a quadratic form in \( \phi_{\mu \nu} \), hence it is evaluated taking \( \kappa \to 0 \) in Eq. (14). Inserting expressions (19) and (20) in Eq. (18) we get

\[ g'_{\mu \nu} = g_{\mu \nu} + \kappa (\gamma_1 \phi_{\mu \nu} + \gamma_2 \phi g_{\mu \nu}) + \kappa^2 \left( \gamma_3 \phi_{\mu \rho} \phi^\rho_{\nu} + \gamma_4 \phi_{\mu \nu} \phi_{\rho \sigma} \phi^\rho_{\tau} + \gamma_5 \phi_{\mu \nu} + \gamma_6 \phi^2 \right) + O(\kappa^3), \]

where \( g^{\mu \nu} \phi_{\mu \nu} \equiv \phi \) denotes the trace of the quantum metric. The Eq. (22) represents a general parametrization of the quantum metric for one-loop calculations. Other choices of quantum variables based on the expansions of \( |g'|_p^p g'_{\mu \nu} \) and \( |g'|^q g'^{\mu \nu} \) (see, e.g., Refs. [7,8,34]) can be reduced to particular cases of (22). The explicit values of \( \gamma_i \) for these parametrizations are displayed in the Table 1. Let us note that it is possible to construct a parametrization of the more general type \( g'_{\mu \nu} = e^{2\kappa \sigma} (g_{\mu \nu} + \cdots) \), in which the conformal factor \( \sigma(x) \) of the metric is explicitly separated. Calculations using the conformal parametrization can be found, e.g., in [6,8,9]. We postpone the discussion on this choice to Sec. 6.

| \( |g'|_p^p g'_{\mu \nu} \) | \( \gamma_1 \) | \( \gamma_2 \) | \( \gamma_3 \) | \( \gamma_4 \) | \( \gamma_5 \) | \( \gamma_6 \) |
|---|---|---|---|---|---|---|
| 1 | \( p \) | 0 | \(-p/2\) | 0 | \( p^2/2 \) |
| \( |g'|^q g'^{\mu \nu} \) | \(-1\) | \(-q\) | 1 | \( q/2 \) | \( q \) | \( q^2/2 \) |

Table 1: Values of the parameters in (22) for the covariant and contravariant densitized parametrizations.

The bilinear form of the action can be obtained by expanding (11) in powers of \( \phi_{\mu \nu} \) by means of (22). This yields

\[ S(g'_{\mu \nu}) = S(g_{\mu \nu}) + S^{(1)} + S^{(2)} + \cdots, \]
It is worth noticing that all the dependencies on the parameters are implicit (for more details, see [9]).

Following ones, we may present expressions in a compact form in which all algebraic symmetries of the field splitting (22) is encoded in the tensor $M_{\text{geomorphism group}}^g$ coset $M_{\text{field splitting}}$. The non-degeneracy of $G_{\text{field splitting}}$ is known as the space of spacetime geometries. In quantum gravity the invariant configuration space metric is defined, up to an arbitrary real parameter $\kappa$, by

$$S^{(2)} = \frac{1}{\kappa} \int d^4x \sqrt{|g|} \left\{ \frac{\gamma_1 R_{\mu\nu} \phi_{\mu\nu}}{2} - \frac{1}{2} \left[ \gamma_1 + (D - 2) \gamma_2 \right] R\phi - \left( \gamma_1 + D \right) \gamma_2 \Lambda \phi \right\},$$

and unnecessary superficial terms have been omitted. In the last formula

$$\beta = -\frac{1}{2} \left[ \gamma_1 + (D - 2) \gamma_2 \right]$$

and the tensor objects are defined as

$$K^{\mu\nu,\alpha\beta} = \frac{1}{2} \left\{ \gamma_1^2 \delta^{\mu\nu,\alpha\beta} - \frac{1}{2} \left[ \gamma_1^2 + 2(D - 2) \gamma_1 \gamma_2 + D(D - 2) \gamma_2^2 \right] g^{\mu\nu} g^{\alpha\beta} \right\},$$

$$M_{1}^{\mu\nu,\alpha\beta} = \gamma_1^2 P_{\mu\nu}^{\alpha\beta} + \gamma_2^2 \gamma_{\alpha\beta} R^{\alpha\beta} - \frac{x_1}{2} \left( g^{\mu\nu} R^{\alpha\beta} + g^{\alpha\beta} R^{\mu\nu} \right) - \frac{x_2}{4} g^{\mu\nu} g^{\alpha\beta} R + \frac{x_2}{4} g^{\mu\nu} g^{\alpha\beta} R,$$

$$M_{2}^{\mu\nu,\alpha\beta} = -2(3) g^{\mu\nu} R^{\alpha\beta} - \gamma_5 (g^{\mu\nu} R^{\alpha\beta} + g^{\alpha\beta} R^{\mu\nu}) + \left[ \gamma_3 + (D - 2) \gamma_4 \right] g^{\mu\nu} g^{\alpha\beta} R + 2(\gamma_3 + D\gamma_4) g^{\mu\nu} g^{\alpha\beta} \Lambda + 2(\gamma_5 + D\gamma_6) g^{\mu\nu} g^{\alpha\beta} \Lambda,$$

with

$$x_1 = \gamma_1^2 + (D - 4) \gamma_1 \gamma_2, \quad x_2 = \gamma_1^2 + 2(D - 4) \gamma_1 \gamma_2 + (D - 2)(D - 4) \gamma_2^2.$$ 

It is worth noticing that all the dependencies on the parameters $\gamma_3, \ldots, 6$ of the nonlinear part of the field splitting (22) is encoded in the tensor $M_{2}^{\mu\nu,\alpha\beta}$. In the above-given formulas, and in the following ones, we may present expressions in a compact form in which all algebraic symmetries are implicit (for more details, see [9]).

Finally, from Eq. [23] it follows that the equations of motion read

$$\varepsilon^{\mu\nu} = \frac{1}{\sqrt{|g|}} \delta S = \frac{1}{\kappa} \left\{ \gamma_1 R_{\mu\nu} - \frac{1}{2} \left[ \gamma_1 + (D - 2) \gamma_2 \right] R g_{\mu\nu} - \left( \gamma_1 + D \right) \gamma_2 \Lambda g_{\mu\nu} + O(\kappa) \right\}. \tag{31}$$

Now we have all basic elements to perform the desired calculation.

4 The improved bilinear form of the action

General relativity and other metric theories of gravity are gauge theories based on the diffeomorphism group $\mathcal{G}$. The configuration space $\mathcal{M}$ is the set of all spacetime metrics, and the coset $\mathcal{M}/\mathcal{G}$ is known as the space of spacetime geometries. In quantum gravity the invariant configuration space metric is defined, up to an arbitrary real parameter $a$, by

$$\delta s^2 = \int d^4x \sqrt{|g|} G^{\mu\nu,\alpha\beta} \delta g_{\mu\nu}(x) \delta g^{\alpha\beta}(x), \quad G^{\mu\nu,\alpha\beta} = \frac{1}{2} \left( \delta^{\mu\nu,\alpha\beta} + a g^{\mu\nu} g^{\alpha\beta} \right). \tag{32}$$

The non-degeneracy of $G^{\mu\nu,\alpha\beta}$ is ensured by the condition $a \neq -1/D$. Explicit calculations have shown that the Vilkovisky effective action depends on the choice of $a$ [20][26][27]. The ambiguity owed to the parameter $a$ can be fixed by an additional prescription.
A differential operator is said to be minimal if its highest derivative term is given by a power of the $\Box$ operator. In quantum gravity models, the minimal operator almost always has the form of $G^{\mu\nu,\alpha\beta}\Box^n$ with the parameter $a$ unambiguously fixed by the choice of classical Lagrangian and the parametrization of the quantum field. In Ref. [1], it was proposed that $a$ should be chosen correspondingly, namely, the field-space metric should be the expression in the highest-derivative term in the minimal version of the bilinear part of the classical action. For the quantum general relativity $n = 1$ and, in the standard simplest parametrization, this “natural” condition for choosing the configuration-space metric fixes the value $a = -1/2$. However, even in the minimal gauge, the coefficient $a$ may be changed by modifying the parametrization of the quantum metric, that is by changing the coefficients $\gamma_i$ in Eq. (22). The purpose of this work is to check whether this change does not produce a modification in the divergent part of the one-loop unique effective action. But, for the sake of generality, in most of the paper, we regard $a$ an arbitrary parameter.

The field-space metric in terms of the variable $\phi_{\mu\nu}$ can be obtained by performing a change of variables in Eq. (32), which gives

$$\delta s^2 = \int D^D x \sqrt{|g|} G^{\mu\nu,\alpha\beta} \delta \phi_{\mu\nu}(x) \delta \phi_{\alpha\beta}(x),$$

(33)

where

$$G^{\mu\nu,\alpha\beta} = G^{\mu\nu,\alpha\beta(0)} + \kappa G^{\mu\nu,\alpha\beta(1)} + O(\kappa^2),$$

(34)

$$G^{\mu\nu,\alpha\beta(0)} = \frac{1}{2} \left( g^{\mu\nu} \delta^{\alpha\beta} + \bar{a} g^{\mu\nu} g^{\alpha\beta} \right), \quad \bar{a} \equiv \gamma_2 (2\gamma_1 + D\gamma_2) + a(\gamma_1 + D\gamma_2)^2,$$

(35)

$$G^{\mu\nu,\alpha\beta(1)} = g_1 \delta^{\mu\nu} \delta^{\alpha\beta} + g_2 \delta^{\mu\nu,\alpha\beta} + g_3 \left( g^{\mu\nu} \delta^{\alpha\beta} + g^{\alpha\beta} \delta^{\mu\nu} \right) + g_4 g^{\mu\nu} g^{\alpha\beta},$$

(36)

with the coefficients

$$g_1 = -\gamma_1^3 + 2\gamma_1 \gamma_3, \quad g_2 = \frac{\gamma_1^2}{4} \left[ \gamma_1 + (D - 4)\gamma_2 \right] + \gamma_1 \gamma_5,$$

$$g_3 = \frac{-\gamma_1^2}{2} \left[ 2\gamma_2 + a(\gamma_1 + D\gamma_2) \right] + \gamma_2 \gamma_3 + (\gamma_1 + D\gamma_2)[\gamma_4 + a(\gamma_3 + D\gamma_4)] + \frac{\gamma_1 \gamma_5}{2},$$

$$g_4 = \frac{\bar{a}}{4} \left[ \gamma_1 + (D - 4)\gamma_2 \right] - \gamma_1 \gamma_2 \left[ \gamma_2 + a(\gamma_1 + D\gamma_2) \right] + 2(\gamma_1 \gamma_6 + \gamma_2 \gamma_5 + D\gamma_6) + a(\gamma_1 + D\gamma_2)(\gamma_5 + D\gamma_6).$$

(37)

Formula (35) can be rewritten using the definition of Eq. (27),

$$G^{\mu\nu,\alpha\beta(0)} = K^{\mu\nu,\alpha\beta} + \frac{1}{4} (1 + 2a)(\gamma_1 + D\gamma_2)^2 g^{\mu\nu} g^{\alpha\beta}.$$  

(38)

One can see that for $a = -1/2$ the background configuration space metric reduces to the factor of the d’Alembertian in Eq. (25). This agrees with the Vilkovisky’s prescription [1] for fixing the ambiguity in the one-parameter family of metrics, even for the general parametrization (22).

The Christoffel symbol $\Gamma^{\mu\nu,\alpha\beta}_{\rho\sigma}$ associated with the metric (34) has the form

$$\Gamma^{\mu\nu,\alpha\beta}_{\rho\sigma} = \frac{1}{2} G^{\mu\nu,\alpha\beta}_{\rho\sigma,\lambda\tau} \left( \frac{\partial G^{\lambda\tau,\alpha\beta}}{\partial \phi_{\mu\nu}} + \frac{\partial G^{\mu\nu,\lambda\tau}}{\partial \phi_{\alpha\beta}} - \frac{\partial G^{\mu\nu,\alpha\beta}}{\partial \phi_{\lambda\tau}} \right),$$

(39)

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where the inverse of the configuration-space metric (34) is

\[ G_{\mu\nu,\alpha\beta} = K_{\mu\nu,\alpha\beta}^{-1} + \frac{2(1 + 2a)}{(D - 2)(1 + aD)(\gamma_1 + D\gamma_2)^2} g_{\mu\nu}g_{\alpha\beta} + O(\kappa) \]  

(40)

and \(K_{\mu\nu,\alpha\beta}^{-1}\) is the inverse of (27).

\[ K_{\mu\nu,\alpha\beta}^{-1} = h_1 \delta_{\mu\nu,\alpha\beta} + h_2 g_{\mu\nu}g_{\alpha\beta}, \]  

(41)

with

\[ h_1 = \frac{2}{\gamma_1}, \quad h_2 = -\frac{2}{D\gamma_1^2} - \frac{4}{D(D - 2)(\gamma_1 + D\gamma_2)^2}. \]  

(42)

A straightforward calculation of (39) yields

\[ \Gamma^{\mu\nu,\alpha\beta} = \kappa c_1 \delta_{\mu\sigma} g^{\nu\beta} + c_2 (\delta_{\rho\sigma} g^{\alpha\beta} + \delta_{\rho\sigma} g^{\alpha\nu}) + c_3 \delta^{\mu\nu,\alpha\beta} g_{\rho\sigma} + c_4 g^{\mu\nu} g^{\alpha\beta} g_{\rho\sigma} \]  

+ O(\kappa^2), \]  

(43)

where the coefficients are

\[ c_1 = -\gamma_1 + \frac{2\gamma_3}{\gamma_1}, \quad c_2 = \frac{1}{4} \left[ \gamma_1 + (D - 4)\gamma_2 \right] + \frac{\gamma_5}{\gamma_1}, \]

\[ c_3 = \frac{1}{2(D - 2)(\gamma_1 + D\gamma_2)} \left[ \gamma_1^2 + 2(D - 2)\gamma_1\gamma_2 - \frac{(1 + 2a)D\gamma_2^2}{2(1 + aD)} \right] + \frac{2\gamma_3\gamma_4 - 2\gamma_2\gamma_3}{\gamma_1(\gamma_1 + D\gamma_2)}, \]

\[ c_4 = -\frac{1}{4(D - 2)(\gamma_1 + D\gamma_2)} \left[ \gamma_1^2 + 2(D - 4)\gamma_1\gamma_2 + (D - 2)(D - 4)\gamma_2^2 - \frac{(1 + 2a)\gamma_1^2}{(1 + aD)} \right] \]

+ \frac{2\gamma_6\gamma_2 - 2\gamma_5\gamma_3}{\gamma_1(\gamma_1 + D\gamma_2)}. \]

Using Eqs. (31) and (43), the Christoffel correction term in the second covariant derivative \(\partial_i \partial_j S = S_{,ij} - \Gamma_{ij}^k \varepsilon_k\) reads

\[ \Gamma^{\mu\nu,\alpha\beta} \varepsilon_{\rho\sigma} |_{\kappa \to 0} = \frac{x_1}{4} (g^{\mu\nu} R^{\alpha\beta} + g^{\alpha\beta} R^{\mu\nu}) - \frac{\gamma_1^2}{4} g^{\mu\nu} R^{\alpha\beta} + \frac{\gamma_1^2}{4} \delta^{\mu\nu,\alpha\beta} R - \frac{x_2}{8} g^{\mu\nu} g^{\alpha\beta} R \]

\[ - M^{\mu\nu,\alpha\beta}_2 + \frac{D - 4}{D - 2} K^{\mu\nu,\alpha\beta}_2 \Lambda + \frac{(1 + 2a)D\gamma_2^2}{8(1 + aD)} \left( R + \frac{2D}{D - 2} \Lambda \right) \left( \delta^{\mu\nu,\alpha\beta} - \frac{1}{D} g^{\alpha\beta} g^{\mu\nu} \right), \]  

(44)

where \(M^{\mu\nu,\alpha\beta}_2\) and \(x_{1,2}\) were defined in Eqs. (21) and (30), respectively. We remark that the parameters \(\gamma_3,\ldots,6\), which are related to the nonlinear terms in the parametrization (22), only occur in \(M^{\mu\nu,\alpha\beta}_2\), just like in (25). Because of this, the second functional covariant derivative of the action (23) only depends on the parameters \(\gamma_1\) and \(\gamma_2\),

\[ - \frac{\partial^2 S}{\partial \phi_{\mu\nu} \partial \phi_{\alpha\beta}} |_{\kappa \to 0} = \frac{\gamma_2}{4} \delta_{\mu\nu,\alpha\beta} \partial^2 - \frac{d_1}{4} g^{\mu\nu} g^{\alpha\beta} \partial + \frac{d_2}{4} (g^{\mu\nu} \nabla^{\alpha\beta} + g^{\alpha\beta} \nabla^{\mu\nu}) \]

\[ - \frac{\gamma_1^2}{2} g^{\mu\nu} \nabla^{\alpha\beta} + \frac{\gamma_1^2}{2} R^{\mu\nu,\alpha\beta} - \frac{x_1}{4} (g^{\mu\nu} R^{\alpha\beta} + g^{\alpha\beta} R^{\mu\nu}) - \frac{\gamma_2^2}{4} \delta_{\mu\nu,\alpha\beta} R + \frac{x_2}{8} g^{\mu\nu} g^{\alpha\beta} R \]

\[ - \frac{D}{D - 2} K^{\mu\nu,\alpha\beta}_0 \Lambda + \frac{(1 + 2a)D\gamma_2^2}{8(1 + aD)} \left( \delta^{\mu\nu,\alpha\beta} - \frac{1}{D} g^{\alpha\beta} g^{\mu\nu} \right) \left( R + \frac{2D}{D - 2} \Lambda \right), \]  

(45)

where

\[ d_1 = \gamma_1^2 + 2(D - 2)\gamma_1\gamma_2 + (D - 1)(D - 2)\gamma_2^2, \quad d_2 = \gamma_1^2 + (D - 2)\gamma_1\gamma_2. \]  

(46)
It is clear that the Christoffel symbol derived from the metric (34) should suffice to compensate the dependence of \( S_{ij} \) on the nonlinearity of the field parametrization. In fact, for \( \kappa \to 0 \) all the parameters \( \gamma_3, \ldots, \gamma_6 \) only contribute to the last term in the r.h.s. of

\[
\frac{\delta^2 S'}{\delta g'_{\mu\nu} \delta g'_{\alpha\beta}} = \frac{\delta \phi_{\lambda\tau}}{\delta g'_{\mu\nu}} \frac{\delta \phi_{\rho\sigma}}{\delta g'_{\alpha\beta}} \frac{\delta^2 S}{\delta \phi_{\lambda\tau} \delta \phi_{\rho\sigma}} + \frac{\delta^2 \phi_{\lambda\tau}}{\delta g'_{\mu\nu} \delta g'_{\alpha\beta}} \frac{\delta S}{\delta \phi_{\lambda\tau}},
\]

that represents the non-tensor nature of this transformation.

5 One-loop divergences of Vilkovisky effective action

Up to this point, we have considered the part of the Vilkovisky effective action based on the Christoffel symbols on the space \( \mathcal{M} \) of field parametrization. However, it is still necessary to introduce the gauge fixing for the diffeomorphism invariance and take into account the contribution of the Faddeev-Popov ghosts as well the terms (16) and (17) related to the gauge constraints on the affine connection.

The standard general form of the gauge-fixing action in quantum general relativity is

\[
S_{GF} = -\frac{1}{2} \int d^Dx \sqrt{|g|} \chi_\mu g^{\mu\nu} \chi_\nu,
\]

where \( \chi_\mu \) is the background gauge condition. The use of a linear gauge-fixing is not a necessary condition to ensure the invariance of the Vilkovisky effective action [18, 23]. Nonetheless, as explained in Sec. 2, the DeWitt gauge [13] is crucial for deriving the expanded formula (14). In our parametrization it assumes the form

\[
\chi_\alpha = G^{\mu\nu,\lambda\tau} R_{\mu\nu,\alpha} \phi_{\lambda\tau} = \gamma_1 \nabla_\rho \phi^{\mu}_\rho + [\gamma_2 + a (\gamma_1 + D \gamma_2)] \nabla_\mu \phi + O(\kappa),
\]

where we used the explicit expression for the generators of the gauge transformations \( R_{\mu\nu,\alpha} \) of the field \( \phi_{\mu\nu} \), presented in the Appendix.

Comparing Eqs. (49) and (25) it is easy to see that the choice \( a = -1/2 \) provides the minimal form of the operator (15),

\[
\hat{H} = G^{\mu\nu,\rho\sigma} \left( \frac{\partial^2 S}{\delta \phi_{\rho\sigma} \delta \phi_{\alpha\beta}} - \frac{\delta \chi_\lambda}{\delta \phi_{\rho\sigma}} g^{\lambda\tau} \frac{\delta \chi_\tau}{\delta \phi_{\alpha\beta}} \right) |_{\kappa \to 0}.
\]

Let us remark that another possible way of making the operator \( H^{\mu\nu,\alpha\beta} \) minimal is through the use of a specific parametrization, namely, \( \gamma_1 = -D \gamma_2 \). However, as explained in Sec. 3, this is not acceptable since it makes the metric in the space of the quantum fields singular, see Eq. (40), and the operator \( \hat{H} \) in (50) undefined. Thus, \( a = -1/2 \) is the sole reasonable choice. For this value of \( a \), the operator gets reduced to the standard form

\[
\hat{H} = - (\hat{1} \Box + \hat{\Pi}),
\]

\[2\text{See Ref. [38] for a recent discussion on nonlinear gauges within the framework of the background field method in the standard definition of the effective action.} \]
where \( \hat{1} = \delta_{\alpha\beta}^{\mu\nu} \) is the identity operator \([21]\) on the space of symmetric rank-2 tensors and

\[
\Pi = 2R_{\alpha,\beta}^{\mu,\nu} - \frac{p_1}{2} g^{\mu\nu} R_{\alpha\beta} - \frac{p_2}{D-2} g_{\alpha\beta} R^{\mu\nu} + \frac{p_3}{2(D-2)} g^{\mu\nu} g_{\alpha\beta} R + \delta_{\alpha\beta}^{\mu\nu} \left( \frac{DA}{D-2} - \frac{1}{2} R \right),
\]

with

\[
p_1 = 1 + \frac{\gamma_2 (D-4)}{\gamma_1}, \quad p_2 = \frac{\gamma_1 + 2(D-2) \gamma_2}{\gamma_1 + D \gamma_2}, \quad p_3 = p_2 + (D-2)(D-4) \gamma_2^2 \frac{1}{\gamma_1 (\gamma_1 + D \gamma_2)}.
\]

Furthermore, with the gauge condition \([19]\), the ghost matrix reads

\[
\hat{N} = g^{\alpha\lambda} R_{\mu\nu,\lambda} G^{\mu\nu,\rho\sigma} R_{\rho\sigma,\beta} = \delta_{\alpha\beta}^{\mu\nu} \square + (1 + 2a) \nabla^\alpha \nabla_\beta + R_{\alpha\beta}^\alpha + O(\kappa).
\]

Notice that in the DeWitt gauge all the dependence on the parametrization is cancelled in the ghost operator, and that \( a = -1/2 \) makes it also minimal. Hereafter, we choose this value for \( a \), such that both \( \hat{H} \) and \( \hat{N} \) assume minimal forms.

The correction which is responsible to restore the gauge invariance of the effective action is based on the nonlocal operators \( \hat{U}_1 \) and \( \hat{U}_2 \), defined in \([16]\) and \([17]\). These operators depend on the two new vertices

\[
(V_1)_{\alpha\beta} = (\mathcal{D}_1 R^\alpha_1) \varepsilon_j \quad \text{and} \quad (V_2)_{\alpha\beta} = R^\alpha_1 (\mathcal{D}_1 R^\beta_1) \varepsilon_j.
\]

Particularizing the formulas above for the gravity theory in the parametrization \([22]\) and using the gauge generators \([20]\) given in Appendix, after some algebra we get

\[
(V_1)^{\mu\nu}_\gamma = \frac{\gamma_1}{2} (R^\mu_\gamma \nabla^\nu + R^\nu_\gamma \nabla^\mu) - \frac{\gamma_1}{2} (\delta^\mu_\gamma R^{\nu\lambda} + \delta^\nu_\gamma R^{\mu\lambda}) \nabla_\lambda + \gamma_1 (\nabla_\gamma R^{\mu\nu})
\]

\[
+ \frac{\gamma_1}{2} R^{\mu\nu} \nabla_\gamma - \frac{1}{2} (\gamma_1 + D \gamma_2) R^{\mu\nu} \nabla_\lambda + \frac{\gamma_1}{4} R (\delta^{\mu\nu}_\gamma + \delta^{\nu\mu}_\gamma)
\]

\[
- \frac{1}{2} [\gamma_1 + (D-2) \gamma_2] g^{\mu\nu} (\nabla_\gamma R) - \frac{1}{4} [\gamma_1 + (D-4) \gamma_2] g^{\mu\nu} R \nabla_\gamma
\]

\[
+ \frac{D \gamma_1}{2(D-2)} \Lambda (\delta^{\mu\nu}_\gamma + \delta^{\nu\mu}_\gamma) - \frac{D [\gamma_1 + (D-4) \gamma_2]}{2(D-2)} g^{\mu\nu} \Lambda \nabla_\gamma + O(\kappa)
\]

and

\[
(V_2)_{\alpha\beta} = R_{\alpha\beta,\gamma} \square + \frac{1}{2} g_{\alpha\beta} R \square - g_{\alpha\beta} R^{\lambda\tau} \nabla_\lambda \nabla_\tau + (\nabla^\lambda R_{\alpha\beta}) \nabla_\lambda - (\nabla_\gamma R^{\lambda}_{\alpha\beta}) \nabla_\lambda + (\nabla_\gamma R^{\lambda}_{\alpha\beta}) \nabla_\lambda
\]

\[
- R_{\alpha\beta,\rho} \nabla_\rho + R_{\alpha\beta} R_\rho^\lambda + \frac{1}{2} RR_{\alpha\beta} + \frac{DA}{D-2} (g_{\alpha\beta} \square + R_{\alpha\beta}) + O(\kappa).
\]

We see that the dependence on the parameters \( \gamma_3, \ldots, 6 \) corresponding to the nonlinear part of the field splitting \([22]\) gets cancelled in \((V_1)_{\gamma}^{\mu\nu}\), while the vertex \((V_2)_{\alpha\beta}\) is parametrization-independent automatically.

The operators \( \hat{U}_1 \) and \( \hat{U}_2 \) can be obtained by substituting the two previous equations into the formulas \([16]\) and \([17]\), together with the propagators

\[
N^{\alpha\beta} = g^{\alpha\beta} \frac{1}{\square} - R^{\alpha\beta} \frac{1}{\square^2} + O(|m|^2), \quad H^{-1}_{\mu\nu,\alpha\beta} = K^{-1}_{\mu\nu,\alpha\beta} \frac{1}{\square} + O(|m|^2).
\]

11
Here $O(|m|^k)$ denotes a series of inessential terms of higher background dimension $k$. Remember that, according to [13], for a functional universal trace

$$\text{Tr} \hat{C}^{\mu_1 \ldots \mu_k} \nabla_{\mu_1} \cdots \nabla_{\mu_k} \frac{1}{\Box^n},$$  \hspace{1cm} (58)

the background dimension (in mass units) is defined as the dimension of the tensorial coefficient $\hat{C}^{\mu_1 \ldots \mu_k}$, and its superficial degree of divergence is expressed by the relation $\omega = D - 2n + k$. Thus, in four dimensions only the traces with background dimension 0, 1, 2, 3 and 4 contribute to the ultraviolet (UV) divergences.

With all these ingredients in hand, it is possible to evaluate the contribution of each term in (14), up to background dimension $O(|m|^4)$, to the effective action. In the case of the operators $\hat{H}$ and $\hat{N}$ (respectively given by Eqs. (21) and (63)), this can be obtained from the functional trace of the coefficient $\hat{a}_2$ of the Schwinger-DeWitt expansion [12]. On the other hand, the functional traces of the nonlocal operators $\hat{U}_1, \hat{U}_2^2$ and $\hat{U}_2$ can be evaluated using the table of universal functional traces within the generalized Schwinger-DeWitt technique [13]. For example, one can easily show that

$$\text{Tr} \hat{U}_2 = \int d^D x \text{tr} \left[ h_1 (V_1^2)_{\beta}^\alpha + h_2 (V_2^2)_{\beta}^\alpha \right] \frac{1}{\Box^n} |x' - x| + O(|m|^5),$$  \hspace{1cm} (59)

where $h_{1,2}$ were defined in Eq. (42) and we used the notations

$$(V_1^2)_{\beta}^\alpha = g^\gamma_\sigma \delta_{\mu_\nu, \rho\sigma} (V_1)_{\gamma}^{\mu\nu} (V_1)_{\beta}^{\rho\sigma}, \quad (\bar{V}_1)_{\gamma} = g_{\mu\nu} (V_1)^{\mu\nu}_{\gamma}, \quad (\bar{V}_2)_{\beta}^\alpha = g^\gamma_{\sigma} (\bar{V}_1)_{\gamma} (\bar{V}_1)_{\beta}.$$

Skipping the algebra, the contributions of the terms in (14) to the pole of the Vilkovisky unique effective action is presented in Table 2. It is important to recall that only in $D \to 4$ the displayed coefficients correspond to one-loop divergences; nonetheless, our calculation in arbitrary dimension shows that they do not depend on the field parametrization even for $D \neq 4$. Moreover, one can see that the parametrization dependence which remained after the Christoffel correction was taken into account is cancelled in the functional trace of each operator on its turn, as none of the coefficients depend on $\gamma_{1,2}$.

Since the object of our interest is the one-loop logarithmically divergent part of the Vilkovisky effective action, in the framework of dimensional regularization we can take the limit $D \to 4$ in the coefficient of the pole term, to obtain

$$\bar{\Gamma}_{\text{div}}^{(4)} = - \mu^{D-4} \int d^4 x \sqrt{|g|} \left\{ \frac{53}{45} R^2_{\mu\nu\alpha\beta} - \frac{61}{90} R^2_{\mu\nu} + \frac{25}{36} R^2 + 8 \Lambda R + 12 \Lambda^2 \right\}. \hspace{1cm} (60)$$

As usual, $\mu$ is the renormalization parameter. Formula (60) reproduces the results for the Vilkovisky effective action for general relativity calculated in the standard, particular, parametrization of the quantum variables in [13] (the coefficients of the terms related to the cosmological constant were calculated for the first time in [18]). Moreover, it is straightforward to verify that, on the classical mass shell, the divergences of Eq. (60) correctly reduce to the coefficients of the usual on-shell effective action [3,39].

$$\bar{\Gamma}_{\text{div\ on-shell}}^{(4)} = - \mu^{D-4} \int d^4 x \sqrt{|g|} \left\{ \frac{53}{45} R^2_{\mu\nu\alpha\beta} - \frac{58}{5} \Lambda^2 \right\}. \hspace{1cm} (61)$$
\begin{table}[h]
\centering
\begin{tabular}{|c|c|c|c|c|c|c|}
\hline
Invariant & \(-i\,\text{Tr} \ln \hat{H}^{i}\) & \(-i\,\text{Tr} \ln \hat{N}^{i}\) & \(-i\,\text{Tr} \hat{U}_{1}^{i}\) & \(-i\,\text{Tr} \hat{U}_{2}^{i}\) & \(-i\,\text{Tr} \hat{U}_{3}^{i}\) & \(\Gamma^{(1)}\) \\
\hline
\(R^{2}\) & \(-\frac{D^{2} - 2 D \pm 4 \sqrt{15}}{360 (D - 2)}\) & \(-\frac{D}{360}\) & \(-\frac{D}{360}\) & \(-\frac{3 D^{2} - 16}{360}\) & \(-\frac{3 D^{2} - 16}{360}\) & \(-\frac{D^{3} + 5 D^{2} + 29 D + 44}{360 (D - 2)}\) \\
\(R^{2\nu}\) & \(-\frac{D}{360 (D - 2)}\) & \(-\frac{D}{360}\) & \(-\frac{D}{24}\) & \(-\frac{3 D^{2} - 16}{360}\) & \(-\frac{3 D^{2} - 16}{360}\) & \(-\frac{D^{3} + 5 D^{2} + 29 D + 44}{360 (D - 2)}\) \\
\(R^{2}\) & \(-\frac{3}{480}\) & \(-\frac{1}{48}\) & \(-\frac{1}{48}\) & \(-\frac{1}{48}\) & \(-\frac{1}{48}\) & \(-\frac{1}{48}\) \\
\(\Lambda R\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) \\
\(\Lambda^{2}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) & \(-\frac{D}{480}\) \\
\hline
\end{tabular}
\caption{Contribution of each operator in (14) to the coefficients of each curvature invariant in the divergent (at \(D \to 4\)) part of the one-loop Vilkovisky effective action. Each invariant enters the effective action multiplied by the overall coefficient as in Eq. (60). The final coefficients, which are the sum of the coefficients of columns 2–6, are presented in the last column.}
\end{table}

This is an expected result since the Vilkovisky correction term is proportional to the equations of motion. On the other hand, this result is known to be gauge-fixing and parametrization independent [9].

It is interesting to compare the result for the unique effective action (60) and the one-loop divergences of the standard (usual) effective action in an arbitrary parametrization (22), derived in [9]. It turns out that the two expressions coincide if the parameters satisfy the conditions

\[
\gamma_4 = \frac{1}{48} \left[ (6 \pm \sqrt{15}) \gamma_1^2 - 12 \gamma_3 \right],
\]

\[
\gamma_5 = \frac{1}{12} \left[ -6 \gamma_3 \pm \left( 1 + \frac{4 \gamma_2}{\gamma_1} \right) \sqrt{6 \left( 12 \gamma_3^2 - 5 \gamma_1^2 \right)} \right],
\]

\[
\gamma_6 = -\frac{1}{64} \left[ 5 \left( \gamma_1 + 4 \gamma_2 \right)^2 + 4 \left( \gamma_3 + 4 \left( \gamma_4 + \gamma_5 \right) \right) \right].
\]

In this case, the one-loop divergences of the conventional effective action calculated in the minimal gauge coincide to those of the Vilkovisky effective action (60). Curiously, this result can be achieved only if the parametrization is nonlinear. The last can be readily seen from Eq. (63), which implies \(\gamma_3 \neq 0\). Let us note that the observation formulated above can be seen as a parametrization-dependence counterpart for the result of [40], where it was derived a gauge for which the one-loop divergences of the conventional effective action (in the particular simplest parametrization) reproduce those of the unique effective action. In this vein, it is also worth pointing out that the \(\Lambda\)-dependent terms in (60) can be obtained by means of the Landau-DeWitt gauge within the usual definition of the effective action [18]. Nevertheless, the simple use of this particular singular gauge in the standard effective action cannot give the other divergent terms of the unique effective action for Einstein gravity because the space of fields is not flat [18,23,25].
6 Conformal parametrization of the metric

Let us now consider a more general parametrization of the metric, which explicitly splits its conformal factor, namely,

\[ g'_{\mu\nu} = e^{2r}\kappa \left[ g_{\mu\nu} + \kappa (\gamma_1 \phi_{\mu\nu} + \gamma_2 \phi g_{\mu\nu}) \right. \]
\[ \left. + \kappa^2 (\gamma_3 \phi_{\mu\rho} \phi^\rho + \gamma_4 g_{\mu\nu} \phi^2 + \gamma_5 \phi \phi_{\mu\nu} + \gamma_6 \phi^2 g_{\mu\nu}) + O(\kappa^3) \right], \tag{65} \]

where \( g_{\mu\nu} \) is the background metric, \( \phi_{\mu\nu} \) and \( \phi \) are the quantum fields and \( \gamma_1, \ldots, 6 \) and \( r \) are arbitrary parameters. The one-loop divergences of the standard effective action for Einstein gravity were evaluated in this parametrization in Ref. [9].

It turns out, however, that it is not possible to construct the Vilkovisky effective action directly in this parametrization. The reason is that the insertion of the conformal factor \( \sigma \) as a new field increases the total number of scalar modes and, as a consequence, the quantum theory has an artificial conformal symmetry, which introduces an extra degeneracy making the transformation singular. For example, in this case we have the metric in the space of the field configurations

\[ G^{AB} = \begin{pmatrix} G^{\mu\nu,\alpha\beta(0)} & r(\gamma_1 + D\gamma_2)(1 + aD)g^{\mu\nu} \\ r(\gamma_1 + D\gamma_2)(1 + aD)g^{\mu\nu} & 2r^2D(1 + aD) \end{pmatrix} + O(\kappa), \tag{66} \]

where \( A, B, \ldots \) take the labels \( \phi_{\mu\nu}, \sigma, \) and \( G^{\mu\nu,\alpha\beta(0)} \) coincides with Eq. (38). The determinant of the \( O(\kappa^0) \)-term of this metric reads

\[ \left| G^{AB(0)} \right| = \left\{ 2r^2D(1 + aD) - r^2(\gamma_1 + D\gamma_2)^2(1 + aD)^2 g^{\mu\nu} g^{\alpha\beta} G^{(0)}_{\mu\nu,\alpha\beta} \right\} \times \left| G^{\mu\nu,\alpha\beta(0)} \right|. \tag{67} \]

It is straightforward to verify that the term inside curly brackets is equal to zero, proving that the field-space metric is degenerate. Therefore, it is not possible to evaluate the Christoffel symbols.

In view of this observation it is necessary to impose, from the beginning, the additional conformal gauge fixing

\[ \sigma = \lambda \phi \tag{68} \]

with \( \lambda \) being the gauge-fixing parameter. Expanding the exponential in Eq. (65) one can see that, up to order \( \kappa^2 \), this parametrization reduces to (22) via the substitutions

\[ \gamma_2 \longrightarrow \gamma_2 + 2r\lambda, \quad \gamma_5 \longrightarrow \gamma_5 + 2r\lambda \gamma_1, \quad \gamma_6 \longrightarrow \gamma_6 + 2r\lambda. \tag{69} \]

Then, all calculations that we carried out for (22) also apply for the conformal parametrization (65).

An alternative approach is to split the field \( \phi_{\mu\nu} \) in the trace and traceless part, that is,

\[ \phi_{\mu\nu} = \bar{\phi}_{\mu\nu} + \frac{1}{D} g_{\mu\nu} \phi. \tag{70} \]

It is clear that \( g^{\mu\nu} \bar{\phi}_{\mu\nu} = 0 \). We now have a parametrization in terms of two independent quantum fields: \( \bar{\phi}_{\mu\nu} \) and \( \phi \). Applying (68) and (70) in (65) we get

\[ g'_{\alpha\beta} = g_{\alpha\beta} + \kappa (\gamma_1 \bar{\phi}_{\alpha\beta} + \gamma_2 \phi g_{\alpha\beta}) + \kappa^2 (\gamma_3 \bar{\phi}_{\alpha\rho} \bar{\phi}^\rho + \gamma_4 \phi \phi_{\rho\sigma} g_{\alpha\beta} + \gamma_5 \bar{\phi}_{\alpha\beta} + \gamma_6 \phi^2 g_{\alpha\beta}) + O(\kappa^3), \tag{71} \]
where the new coefficients are
\[
\begin{align*}
\hat{\gamma}_2 &= \frac{\gamma_1}{D} + \gamma_2 + 2r\lambda, \\
\hat{\gamma}_5 &= \frac{2\gamma_3}{D} + \gamma_5 + 2\gamma_1 r\lambda, \\
\hat{\gamma}_6 &= \frac{1}{D^2} \left[ \gamma_3 + D(\gamma_4 + \gamma_5) + D^2\gamma_6 + 2D(\gamma_1 + D\gamma_2) r\lambda \right] + 2r^2\lambda^2.
\end{align*}
\]
(72)

Now it is possible to define a nonsingular metric in the space of the fields\(^\text{3}\),
\[
\begin{align*}
G^\phi_{\mu\nu,\phi,\alpha\beta} &= \gamma_1^2 \hat{\delta}_{\alpha\beta}^\mu + \kappa \left[ \zeta_1 (\gamma_2+\gamma_3) + \gamma_2 + \gamma_6 (\gamma_1+D\gamma_2) r\lambda \right] + O(\kappa^2), \\
G^\phi_{\alpha\beta,\phi} &= \kappa \zeta_3 \hat{\delta}_{\alpha\beta} + O(\kappa^2), \\
G^\phi_{\phi,\phi} &= \gamma_2 (1 + aD) + \kappa \zeta_4 \phi + O(\kappa^2),
\end{align*}
\]
(73)

where \( \hat{\delta}_{\alpha\beta}^\mu = \delta_{\alpha\beta} - \frac{1}{D} g_{\alpha\beta} \) is the identity operator in the space of traceless symmetric rank-2 tensors, and the coefficients read
\[
\begin{align*}
\zeta_1 &= -2\gamma_1 (\gamma_1^2 - 2\gamma_3), \\
\zeta_2 &= -\frac{D-4}{2} \gamma_1^2 \gamma_2 + 2\gamma_1 \gamma_5, \\
\zeta_3 &= 2\gamma_2 (1 + aD) (\gamma_3 + D\gamma_4) + \gamma_1 \gamma_5 - \gamma_1^2 \gamma_2 (2 + aD), \\
\zeta_4 &= \gamma_2 (1 + aD) \left( \frac{D-4}{2} \gamma_2^2 + 4\gamma_6 \right).
\end{align*}
\]
(74)

The inverse metric \((G^{-1})_{AB}(A, B, \cdots = \phi_{\mu\nu}, \phi)\) is given by
\[
(G^{-1})_{AB} = \begin{pmatrix}
\frac{1}{\gamma_1} & 0 \\
0 & \frac{1}{\gamma_2 (1 + aD)}
\end{pmatrix} + O(\kappa).
\]
(75)

With these ingredients, we can proceed the evaluation of the Christoffel symbols, whose non-zero components are
\[
\begin{align*}
\Gamma^\phi_{\mu\nu,\phi,\alpha\beta} &= \frac{\kappa \zeta_1}{\gamma_1^2} \delta_{\alpha\beta}^\mu + O(\kappa^2), \\
\Gamma^\phi_{\phi,\phi} &= \kappa \left[ \frac{2(\gamma_3 + D\gamma_4)}{D\gamma_2} - \frac{\gamma_1^2 (4 + D + 4aD)}{4D (1 + aD) \gamma_2} \right] \delta_{\alpha\beta}^\mu + O(\kappa^2), \\
\Gamma^\phi_{\phi,\phi} &= \kappa \left( \frac{D - 4}{4} \gamma_2 + \frac{\gamma_5}{\gamma_1} \right) \delta_{\alpha\beta} \phi_{\phi} \phi + O(\kappa^2), \\
\Gamma^\phi_{\phi} &= \kappa \left( \frac{D - 4}{4} \gamma_2 \phi_{\phi} + \frac{\gamma_6}{\gamma_2} \phi_{\phi} \phi + O(\kappa^2).
\end{align*}
\]
(76)

For the second covariant derivative of the action we have
\[
\begin{align*}
\frac{\partial^2 S}{\partial \phi_{\mu\nu} \partial \phi_{\alpha\beta}}_{\kappa \to 0} &= \gamma_1^2 \left[ g^\beta \nabla^\alpha \nabla_{\phi} - \frac{1}{2} \delta_{\mu\nu,\alpha\beta} \Box - R^\mu\alpha\nu\beta - \frac{1}{4(1 + aD)} \left( \frac{D - 2}{2} R + D\Lambda \right) \delta_{\mu\nu,\alpha\beta} \right], \\
\frac{\partial^2 S}{\partial \phi_{\mu\nu} \partial \phi} &_{\kappa \to 0} = \gamma_1 \gamma_2 \left( \frac{D - 2}{2} \nabla^\mu \nabla_{\nu} + \frac{D - 4}{4} R^\mu_{\nu} \right), \\
\frac{\partial^2 S}{\partial \phi \partial \phi} &_{\kappa \to 0} = \gamma_2 \left[ \frac{(D - 2)(D - 1)}{2} \Box - \frac{(D - 4)(D - 2)}{8} R - \frac{D^2 - 4}{4} \Lambda \right].
\end{align*}
\]
(77)

\(^3\)Here, to avoid any kind of ambiguity, we made use of a more explicit notation for the indices.
At this stage, it is clear that the dependence on the nonlinear quantum field parametrization was compensated by the Christoffel correction, just like in (45). In addition, the use of the parametrization in terms of the traceless and trace parts reveals that the improved bilinear operator can be written as constant matrix times a differential operator independent of γ1 and γ2, thus this dependence is trivial.

We point out that the conformal gauge fixing (68) does not require Faddeev-Popov ghosts because the conformal transformation has no derivatives [41]. Moreover, under the diffeomorphism (87) the field σ transforms as δσ = −∇µσ ξµ, and the terms in the ghost operator associated with the generators Rµ = −∇µσ can be safely ignored at one-loop level since they produce third-order contributions in quantum field; as a consequence we get (53). Therefore, even in the conformal parametrization, the final result matches the one presented in Eq. (60).

7 Renormalization group based on the unique effective action

One can use the result (60) and its generalization in Table 2 for analyzing the renormalization group equations in the low-energy (infrared, IR) sectors of the theory. Such a construction has a direct physical sense. In the high-energy domain (UV) the theory (1) cannot be applied without restrictions, as it is non-renormalizable. As we explained above, at high energies the contributions of massive degrees of freedom, related to higher derivative terms, are supposed to modify the beta-functions. However, since the quantum gravity based on general relativity is a massless theory, it makes sense to explore the renormalization group running in the IR. Differently from the fourth- and higher-derivative models, in the present case there is no chance to meet an IR decoupling of massive degrees of freedom [41] (see also the discussion of this issue in [42] and [43]).

Since the theory is massless, the quantum gravity based on general relativity can be regarded as an effective theory of quantum gravity at the energies between the Planck scale, where the massive degrees of freedom related to higher derivatives can become relevant, and far IR. Thus, the Vilkovisky-DeWitt unique effective action enables one to explore the scale dependence in this vast region in a gauge-fixing and parametrization independent manner.

From the classical action (1) and the expression for the divergences (60), it is easy to obtain the renormalization relations (we use dimensional regularization)

\[ \frac{1}{\kappa_0^2} = \mu^{D-4} \left( \frac{1}{\kappa^2} - \frac{k_R}{(4\pi)^2(D-4)} \Lambda \right), \quad \Lambda_0 = \Lambda \left( 1 + \frac{2k_R - k_\Lambda}{2(4\pi)^2(D-4)} \Lambda \kappa^2 \right), \]

(78)

where we introduced the notations for the coefficients depending on \( D = 4 + \epsilon \) and disregarded \( O(\epsilon^2) \)-terms,

\[ k_R = \frac{D(2D^2 + D + 12)}{12(D-2)} = 8 + \frac{5\epsilon}{6}, \quad k_\Lambda = \frac{D^3(D-1)}{4(D-2)^2} = 12 + \epsilon. \]

(79)

The bare quantities \( \kappa_0^2 \) and \( \Lambda_0 \) are \( \mu \)-independent, as it is the case for the renormalized effective action. Applying the operator \( \mu \frac{d}{d\mu} \) to both sides of each of the relations (78), after a small algebra
we arrive at the renormalization group equations
\[
\begin{align*}
\mu \frac{d}{d\mu} \frac{1}{\kappa^2} &= -\frac{\epsilon}{\kappa^2} + \frac{k_R \Lambda}{(4\pi)^2}, \\
\mu \frac{d\Lambda}{d\mu} &= \left(\frac{k_\Lambda}{2} - k_R\right) \frac{\Lambda^2 \kappa^2}{(4\pi)^2}.
\end{align*}
\]
(80)
(81)

In the \(D = 4\) limit these equations are equivalent to those obtained in [11,37].

One can explore the \(4 + \epsilon\) version of the renormalization group equations, similar to what was done in the two-dimensional case (see e.g. [44, 45]) and also in the four-dimensional fourth-derivative models [42]. However, in the present case the main results do not change and we restrict ourselves to the strict \(D = 4\) consideration.

To solve Eqs. (80) and (81), we define the dimensionless quantity \(\gamma = \kappa^2 \Lambda\). Due to the uniqueness of this dimensionless combination of \(\kappa^2\) and \(\Lambda\), the equation for \(\gamma\) gets factorized,
\[
\mu \frac{d\gamma}{d\mu} = \left(\frac{k_\Lambda}{2} - 2k_R\right) \frac{\gamma^2}{(4\pi)^2} = -\frac{10\gamma^2}{(4\pi)^2}.
\]
(82)

The solution of this equation has the standard form
\[
\gamma(\mu) = \frac{\gamma_0}{1 + \frac{10}{(4\pi)^2} \gamma_0 \ln \frac{\mu}{\mu_0}},
\]
(83)

where \(\gamma_0 = \gamma(\mu_0)\) and \(\mu_0\) marks a fiducial energy scale. We assume the initial values of the renormalization group trajectories of the cosmological constant \(\Lambda_0 = \Lambda(\mu_0)\) and the gravitational constant \(G_0 = G(\mu_0)\) as it is useful to come back from \(\kappa^2\) to \(G\) at this stage.

Now, using (83) in (80) and (81), it is an easy exercise to obtain the final solutions
\[
G(\mu) = \frac{G_0}{\left[1 + \frac{10}{(4\pi)^2} \gamma_0 \ln \frac{\mu}{\mu_0}\right]^{4/5}}
\]
(84)

and
\[
\Lambda(\mu) = \frac{\Lambda_0}{\left[1 + \frac{10}{(4\pi)^2} \gamma_0 \ln \frac{\mu}{\mu_0}\right]^{1/5}},
\]
(85)

which are certainly consistent with (83).

The solutions (84) and (85) are remarkable in several aspects. First of all, such independent solutions for the two effective charges are impossible in quantum gravity based on the usual effective action neither in quantum general relativity nor the fourth-derivative gravity, as the individual equations for \(G(\mu)\) and \(\Lambda(\mu)\) are completely ambiguous. In the latter model, only the solution for the dimensionless quantity in (83) is gauge-fixing and parametrization independent.

Here we have a well-defined running for the two parameters only because of the use of the Vilkovisky unique effective action.

Let us note that the unambiguous solutions for \(G(\mu)\) and \(\Lambda(\mu)\) exist in the superrenormalizable gravity model [43], but there are two relevant differences. The advantage of the equations and

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4In quantum Einstein gravity based on the usual effective action, on the other hand, only by using the on-shell version of renormalization group it is possible to define an unambiguous equation for \(\gamma\) [41].
solutions of \([43]\) is that those can be exact, in the sense of not depending on the order of the loop expansion. On the other hand, the higher-derivative models that lead to such an exact result imply the functional integration over massive degrees of freedom, which can be ghosts or healthy modes. This means that the corresponding equations are valid only in the UV for the quantum gravity energy scale, \(i.e.,\) only in the trans-Planckian region. Below the Planck scale the massive degrees of freedom decouple and we are left with the quantum effects of effective quantum gravity, such as the ones of quantum general relativity (see \(e.g., [46]\), the review \([47]\) and the recent discussion of the decoupling in gravity in \([48,49]\)).

On the contrary, the running described by (84) and (85) comes from the quantum effects of the purely massless degrees of freedom. Up to some extent, the running should be described by the same equations in both UV and IR. The equations (80) and (81) gain extra contributions at higher loops, but in the region of asymptotic freedom these contributions may be not very relevant.

It is clear that the physical interpretation of the solutions (84) and (85) depend on the sign of \(\gamma_0\). Since the positive sign of \(G\) is fixed by the positive definiteness of the theory, the sign of \(\gamma_0\) depends on the one of \(\Lambda_0\). Due to the cosmological observations, we know that the sign of the observed cosmological constant is positive in the present-day Universe \([50,51]\). For a positive \(\gamma_0\) the solutions (84) and (85) indicate the asymptotic freedom in the UV. In case of a moderate cosmological constant (remember \(\kappa \propto M_P^{-1}\)) the value of \(\gamma_0\) is very small. This implies a very weak running, that is irrelevant from the physical viewpoint. In particular, the running (84) and (85) is not essential for the cosmological constant problem between the electroweak scale and the present day, low-energy cosmic scale.

On the other hand, at the electroweak energy scale, the early Universe probably passed through the corresponding phase transition. At that epoch, the observable value of the cosmological constant could dramatically change because of the symmetry restoration. Does this change \(\Lambda\) in the action (1)? The answer to this question is negative. Let us remember that the observable cosmological constant is a sum of the two parts: one is the vacuum parameter in the gravitational action (1) and another is the induced counterpart, the main part of it coming from the symmetry breaking of the Higgs potential. The main relations are (see, \(e.g., [52]\) or [53])

\[
\rho^{\text{obs}}_\Lambda = \rho^{\text{ind}}_\Lambda + \rho^{\text{vac}}_\Lambda, \quad \rho^{\text{ind}}_\Lambda = \frac{\Lambda^{\text{ind}}}{8\pi G_{\text{ind}}} = -\lambda v_0^4, \tag{86}
\]

where \(\lambda\) is the self-coupling and \(v_0\) the vacuum expectation value of the Higgs field. As far as \(\rho^{\text{ind}}_\Lambda\) is negative and the magnitude of \(\rho^{\text{obs}}_\Lambda\) is negligible, the sign of \(\rho^{\text{vac}}_\Lambda = \frac{4\Lambda}{\kappa}\) is positive, independent of the electroweak phase transition.

Thus, we conclude that the sign of \(\gamma_0\) is always positive, at least between the present-day cosmic scale in the IR and the GUT scale in the UV, where the considerations based on the Minimal Standard Model formulas, such as (86), may become invalid. In all this interval, the value of \(\gamma_0\) is numerically small, such that the running in (84) and (85) is not physically relevant.

One can imagine a situation in which another phase transition occurs at the GUT scale (that means about \(10^{14–10^{16}}\) GeV), such that the new vacuum \(\Lambda\) between this scale and the Planck
scale \( M_P \approx 10^{19} \text{GeV} \) is negative. Then, the solutions (84) and (85) indicate the asymptotic freedom in the IR. Furthermore, if the cosmological constant in this energy scale interval has the order of magnitude of \( M_P \), these solutions describe the situation of a dramatically strong running of both constants \( G \) and \( \Lambda \), which are strongly decreasing in the IR. It might happen that in this case one needs to use higher loop approximation, that can change the form of the running. Further discussion of this possibility and the construction of the corresponding model of GUT is beyond the scope of this work, so we just want to note that our results indicate this possibility.

8 Conclusions

We performed the calculations of the one-loop divergences of the Vilkovisky unique effective action in quantum general relativity in an arbitrary, most general, parametrization of quantum metric, including the conformal parametrization and the corresponding gauge fixing. Due to the similarity between conformal parametrization and the two-dimensional quantum gravity, one could suspect that the unique effective action may lose its invariance and universality. We have shown that this does not happen and the one-loop divergences are universal.

The dependence of the unique effective action on the parameter \( a \) of the configuration space metric is fixed by an additional requirement that this metric is chosen as a bilinear form of the action in the minimal gauge, in consonance with [1]. We have shown that this parameter changes under modified parametrization of quantum metric, but the one-loop unique effective action does not change. This confirms the consistency of the mentioned additional requirement.

Using the unique effective action in quantum general relativity we considered the renormalization group equations for the Newton and cosmological constants separately, as it was done earlier in [11], but our analysis is done from a different perspective. The one-loop equations come from the quantum effects of the purely massless modes and, therefore, can be used in both UV and IR. In the UV the renormalization group trajectories can be used only until the scale where the massive degrees of freedom coming from higher derivatives become active. However, in the IR there are no restrictions. In this respect the renormalization group equations under discussion strongly differ from the ones in renormalizable and superrenormalizable models of quantum gravity which are valid only in the UV regime, usually with respect to the Planck scale. Finally, using these equations we have shown that the running of both Newton and cosmological constants, caused by the quantum gravity, does not produce an essential numerical change for these effective charges, at least between GUT scale in the UV and the present-day cosmic scale in the IR.

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Appendix. Generators of gauge transformations

The gauge generators for the field $\phi_{\mu\nu}$ have been evaluated in Ref. [9] up to the zeroth order in $\kappa$. Nonetheless, we need the expansion up to the next order. The reason is that the terms (16) and (17) depend on the covariant variational derivative of $R_{\mu\nu,\alpha}$ with respect to $\phi_{\mu\nu}$, requiring the $O(\kappa)$-approximation.

Consider the infinitesimal coordinate transformation

$$x^\mu \rightarrow x'^\mu = x^\mu + \xi^\mu.$$  

(87)

In the standard parametrization $g'_{\mu\nu}$ the generator reads

$$R'_{\mu\nu,\gamma}(g') = -(g'_{\mu\gamma} \nabla'_{\nu} + g'_{\nu\gamma} \nabla'_{\mu}).$$  

(88)

The generators of gauge transformation for the quantum field $\phi_{\mu\nu}$ can be obtained through a vector change of coordinates in the space of the field representations,

$$R_{\mu\nu,\gamma}(\phi) = \frac{\partial (\kappa \phi_{\mu\nu})}{\partial g'^{\mu\nu}} R'_{\rho\sigma,\gamma}(g').$$  

(89)

By using Eqs. (22), (88) and (89), it is possible to show that

$$R_{\mu\nu,\gamma}(\phi) = R_{\mu\nu,\gamma}^{(0)} + \kappa R_{\mu\nu,\gamma}^{(1)} + O(\kappa^2),$$  

(90)

where

$$R_{\mu\nu,\gamma}^{(0)} = \frac{1}{\gamma_1} (g_{\mu\gamma} \nabla_{\nu} + g_{\nu\gamma} \nabla_{\mu}) + \frac{2\gamma_2}{\gamma_1(\gamma_1 + D\gamma_2)} g_{\mu\nu} \nabla_{\gamma}$$  

(91)

and

$$R_{\mu\nu,\gamma}^{(1)} = (r_1 - 1) (\phi_{\mu\gamma} \nabla_{\nu} + \phi_{\nu\gamma} \nabla_{\mu}) + r_1 (g_{\mu\gamma} \phi_{\nu}^{\lambda} + g_{\nu\gamma} \phi_{\mu}^{\lambda}) \nabla_{\lambda} + r_2 g_{\mu\nu} \phi_{\gamma}^{\lambda} \nabla_{\lambda} + r_3 \phi_{\mu\nu} \nabla_{\gamma} - (\nabla_{\gamma} \phi_{\mu\nu}) + r_4 \phi(g_{\mu\gamma} \nabla_{\nu} + g_{\nu\gamma} \nabla_{\mu}) + r_5 g_{\mu\nu} \phi \nabla_{\gamma},$$  

(92)

with the coefficients

$$r_1 = \frac{\gamma_3}{\gamma_1^3}, \quad r_2 = \frac{2\gamma_1^2 \gamma_2 - 4(\gamma_2 \gamma_3 - \gamma_1 \gamma_4)}{\gamma_1^2(\gamma_1 + D\gamma_2)}, \quad r_3 = -\frac{2(2\gamma_2 \gamma_3 - \gamma_1 \gamma_5)}{\gamma_1^2(\gamma_1 + D\gamma_2)},$$

$$r_4 = \frac{\gamma_5 - \gamma_1 \gamma_1}{\gamma_1^2}, \quad r_5 = \frac{2\gamma_1^2 \gamma_2^2 + 4\gamma_2(\gamma_2 \gamma_3 - \gamma_1 \gamma_4) - 2\gamma_2 \gamma_5(3\gamma_1 + D\gamma_2) + 4\gamma_1^2 \gamma_6}{\gamma_1^2(\gamma_1 + D\gamma_2)}.$$  

The expressions (91) and (92) are sufficient for the one-loop calculations reported in the main part of the paper.

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