Renormalization - group analysis of dilute Bose system in $d$ -
dimension at finite temperature

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Abstract

We study the $d$ - dimensional Bose gas at finite temperature using the renormalization group method. The flow - equations and the free energy have been obtained for dimension $d$, and the cases $d < 2$ and $d = 2$ have been analysed in the limit of low and high temperatures. The critical temperature, the coherence length and the specific heat of a two dimensional Bose gas have been obtained using a solution for the coupling constant which does not present a singular behavior.
I. INTRODUCTION

Renewed interest for two dimensional dilute Bose gas has been recently determined by the discovery of high - temperature superconductors.

The Bose - Einstein condensation is known not to occur at finite temperature for the interacting boson system in one and two dimensions (the Mermin - Wagner theorem); the absence of a condensate does not necessary imply the absence of a phase transition (in the universality class as $XY$ model) and it is expected to be in the superfluid state.

In the last years the study of two dimensional (2D) Bose condensation was stimulated by new experimental results concerning scattering in dilute gases of alkaline atoms, spin polarized hydrogen recombining and adsorbed helium monoalloys [1].

The theory of weekly interacting Bose gas have been developed first at $T = 0$ by Bogolibov [2] and Lee, Huang and Yang [3] at finite temperature. Using the $T = 0$ results Hohenberg and Martin [4], Popov and Fadeev [5], Singh [6] and Cheung and Griffin [7] developed the field - theoretical methods for finite temperature. Using this method it was possible to calculate the observable physical quantities, but these calculations are quite difficult because of the influence of correlations induced by the condensed phase.

The many - body theory developed by Popov [8,9] is manly based on the $t$ - matrix approximation, also applied for the 2D Bose gas by Schick [11]. We mention that the $t$ - matrix for the scattering in the Bose system involves a sum over all ladder diagrams to infinite order in interaction, taking the repeated scattering of two particles.

The purpose of this paper is the study of the critical region of 2D interacting Bose gas, which was first done in the famous paper of Fisher and Hohenberg [12], using $t$ - matrix method and the Renormalization Group (RG) method. The problem was also studied by Kolomeisky and Straley [13] for a $D$ dimensional Bose system at $T = 0$. The influence of a gauge - field on a superfluid phase transition in a two dimensional model at finite temperature has been studied by Ubbens et al [14] and for the case of three dimensional dilute Bose gas by Biglsma and Stoof [15].
As we mentioned, the results of RG study from Ref. [12] have been used for the problem of the quantum phase transitions in the spin systems [16], or proposed [17] to study the large scale behavior in the crossover problem from high temperature superconductivity. All these papers used the approximations from Ref. [12], which will be reconsidered especially concerning the solutions of the RG scaling equations. We will also recalculate the number of condensate bosons for \( d = 2 \) and the coherence length using the new result obtained by solving more accurate the equations for the coupling constants.

The paper is organized as follows. In Sec. II we present the model and write down the RG scaling equations. In Sec. III we solve the equations in the low temperature region and calculate the number of condensate bosons and the coherence length \( \xi(T) \) for the general case of dimension \( d \). The interesting case \( d = 2 \) will be treated in Sec. IV where we also calculate the temperature dependence of the specific heat. The high temperature behavior will be analyzed in Sec. V. Sec. VI contain the relevance of our results compared with the results from literature.

**II. MODEL AND SCALING EQUATIONS**

We consider the dilute Bose system in \( d \) dimensions at finite temperature \( T \) described by the action:

\[
S_{eff} = S_{eff}^{(2)} + S_{eff}^{(4)}
\]

where

\[
S_{eff}^{(2)} = \frac{1}{2} \sum_{k} \left[ \frac{h^2 k^2}{2m} - \mu - \frac{\omega_n}{\Gamma} \right] |\phi(k)|^2
\]

and

\[
S_{eff}^{(4)} = \frac{u}{4} \sum_{k_1} \ldots \sum_{k_4} \phi(k_1) \cdots \phi(k_4) \delta \left( \sum_{i=1}^{4} k_i \right)
\]

and the following notation have been used:
\[ \sum_{k} \cdots \to k_B T \sum_{n} \int \frac{d^d k}{(2\pi)^d} \cdots \]  \hspace{1cm} (4)

\[ \omega_n = 2\pi n k_B T \] being the bosonic frequencies.

In Eq. (2) \( \mu \) is the chemical potential of the bosonic system, described by the scalar field \( \phi(k) \) and \( \Gamma \) is an energy parameter which controls the strength of the quantum fluctuations, the classical limit being \( \Gamma = 0 \). The Eq. (3) represent the interaction between fluctuations, \( u \ (u > 0) \) being the coupling constant.

The basic propagator of the bosonic system has the form:

\[ G(k, \omega_n) = \left[ \frac{k^2 k^2}{2m} - \mu - \frac{\omega_n}{\Gamma} \right]^{-1} \]  \hspace{1cm} (5)

and the renormalization transformations are carried out by integrating over a momentum shell and summing over all frequencies.

Regarding this procedure we mention that the summation over the bosonic frequencies (transformed in integral) has to be done for all frequencies and not on a shell as in the method applied in [18,19] for bosonic excitations. The physical argument is that in this case the presence of the chemical potential is important because we expect different contributions on the energy scale, which is in fact identical with a change of physical behavior in different temperatures domain. On the other hand, the limit cases \( \mu = \pm \infty \) correspond to the condensate, respectively normal states.

After integrating out the degrees of freedom in the momentum shell, we rescale the variables as:

\[ k = \frac{k'}{b} , \ \omega_n = \frac{\omega'_n}{b^z} , \ T = \frac{T'}{b^z} \]  \hspace{1cm} (6)

and the field operator as:

\[ \phi'(k', \omega'_n) = b^{-(d+z+2)/2} \phi \left( \frac{k'}{b} , \frac{\omega'_n}{b^z} \right) \]  \hspace{1cm} (7)

where \( z \) is the dynamic critical coefficient and we will introduce for \( b \) the parameterization \( b = e^l \ (b > 0 , \ l > 0) \). The scaling equations obtained in Ref. [12,14] for \( T \neq 0 \) and in [13] for \( T = 0 \) will be written as:
\[
\frac{d\Gamma(l)}{dl} = -(2 - z)\Gamma(l) \\
\frac{dT(l)}{dl} = zT(l)
\]  

(8)  

(9)  

\[
\frac{d\mu(l)}{dl} = 2\mu(l) - K_d F_\mu[\mu(l), T(l), u(l), \Gamma(l)] \\
= 2\mu(l) - K_d \frac{\Lambda^2 \Gamma(l)}{\exp \left[ \frac{\Gamma(l)}{k_B T(l)} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right) \right]}
\]

(10)  

(13)  

\[
\frac{du(l)}{dl} = [4 - (d + z)]u(l) - \frac{1}{4} K_d \left\{ 8 F_{p-p}[\mu(l), T(l), u(l), \Gamma(l)] \\
+ 2 F_{p-a}[\mu(l), T(l), u(l), \Gamma(l)] \right\} u^2(l)
\]  

(11)  

\[
\frac{dF(l)}{dl} = (d + z) F(l) + K_d F_f[\mu(l), T(l), u(l), \Gamma(l)]
\]  

(12)  

where \( F(l) \) is the free energy and \( K_d = \frac{\pi d^{d/2}}{2^{d-1} \Gamma(d/2)} \). The constants \( F_\mu, F_{p-p}, F_{p-a} \) and \( F_f \) from Eqs. (10 - 12) are given by the expressions:

\[
F_\mu = F_\mu[\mu(l), T(l), \Gamma(l)] \\
= 2\mu(l) - K_d \frac{\Lambda^2 \Gamma(l)}{\exp \left[ \frac{\Gamma(l)}{k_B T(l)} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right) \right]}
\]

(13)  

\[
F_{p-a} = F_{p-a}[\mu(l), T(l), \Gamma(l)] \\
= \frac{1}{2\Gamma(l)} \frac{1}{\left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right)} \coth \left[ \frac{\Gamma(l)}{2k_B T(l)} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right) \right]
\]

(14)  

\[
F_{p-p} = F_{p-p}[\mu(l), T(l)] \\
= \frac{1}{4k_B T(l)} \frac{1}{\sinh^2 \left[ \frac{1}{2k_B T(l)} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right) \right]}
\]

(15)  

\[
F_f = F_f[\mu(l), T(l), \Gamma(l)] \\
= k_B T(l) \ln \left\{ 1 - \exp \left[ -\frac{\Gamma(l)}{k_B T(l)} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right) \right] \right\} \\
- k_B T(l) \ln \left[ 1 - \exp \left( -\frac{\Gamma(l)}{k_B T(l)} \right) \right]
\]

(16)  

Following the Ref. [12,13] we will solve these equations for \( d < 2 \) (\( \epsilon = 2 - d \)) and for the case \( d = 2 \) which is of special interest.
III. BELOW TWO DIMENSIONS

In order to solve the scaling equations (8) - (12) given in the previous section we start with the equation (11) for the coupling constant at $T = 0$. Introducing the notation:

$$C = \frac{mK_d\Lambda^{d-2}\Gamma^2(l)}{2\hbar^2}$$

(17)

this equation becomes:

$$\frac{du(l)}{dl} = \epsilon u(l) - Cu^2(l)$$

(18)

with the initial condition $u(l = 0) = u_0$. The fixed point of Eq. (18) has the simple form:

$$u^* = \frac{\epsilon}{C}$$

(19)

and the general solution:

$$u(l) = \frac{u^*}{1 - \left(1 - \frac{\epsilon}{u_0}\right) e^{-\frac{\epsilon}{T}}}$$

(20)

This solution satisfies the conditions:

$$u(l = 0) = u_0$$

(21)

$$u(l \to \infty) = u^*$$

(22)

and in order to perform the calculation of physical quantities in the critical region we will use the linearized form of Eq. (20)

$$u(l) \simeq u^* + \frac{u^*}{u_0}(u_0 - u^*)e^{-\epsilon l}$$

(23)

In the limit of low temperatures the chemical potential $\mu(l)$ given by Eq. (10) becomes:

$$\mu(l) = -K_d\lambda^d e^{-2l} \int_0^l dl' \frac{u(l') e^{-2l'}}{\exp \left[ \frac{\hbar^2\Gamma(l')\lambda^2}{2mk_BT(l')} e^{-2l'} - \frac{\Gamma(l')\mu(l')}{2mk_BT(l')} \right] - 1}$$

(24)

From Eq. (8) and (11) we obtain:
\[ \Gamma(l) = \Gamma e^{-(2-d)l} \]  \hspace{1cm} (25)  

\[ T(l) = T e^{2l} \]  \hspace{1cm} (26)  

and using for \( \mu(l) \) the lowest approximation \( (\mu(l) = \mu e^{2l}) \) we calculate the second term from the exponential of Eq. (24) as:

\[ \frac{\Gamma\mu}{2mk_BT} \ll 1 \]  \hspace{1cm} (27)  

This inequality is satisfied even in the low temperatures domain, because \( \Gamma^{-1} \) is the energy scale of the quantum fluctuations and in this domain these are important. Using these considerations, Eq. (24) we get:

\[ \mu(l) \simeq -K_d\Lambda^d e^{2l} \int_0^l \frac{e^{-2l'}u(l')}{\exp \left( \frac{\hbar^2 \Lambda^2}{2mk_BT} \right) - 1} dl' \]  \hspace{1cm} (28)  

and in order to calculate the condensed density \( n \) and the coherence length \( \xi \) we will use Eq. (23) for \( u(l) \). The renormalization procedure will be stopped for \( l = l^* \), given by the equation:

\[ \mu(l^*) = -\alpha \frac{\hbar^2 \Lambda^2}{2m} \]  \hspace{1cm} (29)  

where \( \alpha \ll 1 \). If we introduce the notations:

\[ a = \frac{\hbar^2 \Lambda^2 \Gamma}{2mk_BT} \]  \hspace{1cm} (30)  

\[ b = 1 + \frac{\epsilon}{2} \]  \hspace{1cm} (31)  

we obtain from Eqs. (28) and (29)

\[ e^{-2l^*} \left( 1 + \frac{2\epsilon}{\alpha} \right) \simeq \frac{4\epsilon}{\alpha} I(l^*) \]  \hspace{1cm} (32)  

where

\[ I(l^*) = \int_0^{l^*} dl' \frac{e^{-2l'}}{\exp (ae^{-2l'}) - 1} \]  \hspace{1cm} (33)
This integral can be expressed as:

\[ I(l^*) \simeq \frac{1}{2a^b} \ln \frac{1}{A} \left[ 1 - \frac{\epsilon \cdot F(A)}{2 \ln \frac{1}{A}} \right] \]  

(34)

where \( A = 2\epsilon \) and

\[ F(A) = \int_A^\infty dx \ x^{\frac{\epsilon}{2} - 1} \ln \left( 1 - e^{-x} \right) \]  

(35)

In the limit of low temperatures, Eq. (32) gives:

\[ e^{-2l'} = 2\epsilon \left[ \frac{2mk_BT}{\hbar^2} \right]^{1+\epsilon} \ln \frac{1}{2\epsilon} \]  

(36)

Using these result we can calculate the density of the condensed bosons as:

\[ n = e^{-d_l^*} \int \frac{d^d k}{(2\pi)^d} \frac{1}{\exp \left[ \frac{1}{k_BT(l^*)} \left( \frac{\hbar^2 k^2}{2m} - \mu(l^*) \right) \right] - 1} \]  

(37)

The integral from Eq. (37),

\[ A = K_d \int_0^\infty dk \frac{k^{d-1}}{\exp \left[ \frac{1}{k_BT(l^*)} \left( \frac{\hbar^2 k^2}{2m} - \mu(l^*) \right) \right] - 1} \]  

(38)

will be transformed using the notation

\[ x = \frac{\hbar^2 k^2}{2mk_BT(l^*)} - \frac{\mu(l^*)}{k_BT(l^*)} \]  

(39)

as

\[ A = \frac{1}{2} K_d \left[ \frac{2mk_BT(l^*)}{\hbar^2} \right]^{(d/2)} \int_{|x_m|}^\infty dx \frac{x^{(d-2)/2}}{e^x - 1} \]  

(40)

where \( |x_m| = \mu(l^*)/k_BT(l^*) \). In order to perform the integral given by Eq. (40) we consider \( \epsilon \) small (\( d \) close to \( d = 2 \)) and in this case

\[ A \simeq -\frac{1}{2} K_d \left[ \frac{2mk_BT(l^*)}{\hbar^2} \right]^{d/2} \ln \left[ 1 - \exp \left( -\frac{\mu(l^*)}{k_BT(l^*)} \right) \right] \]  

(41)

Using \( T(l^*) = T e^{2l^*} \) and Eq. (29) we obtain for the condensed density

\[ n(T) \simeq \frac{1}{4\pi} \left[ \frac{2mk_BT}{\hbar^2} \right] \ln \frac{1}{2\epsilon \ln \frac{1}{2\epsilon}} \]  

(42)
The temperature dependence of the coherence length $\xi(T)$ will be calculated using

$$\xi^{-2}(T) = \frac{2m}{\hbar^2} |\mu(l^*)|$$  \hspace{1cm} (43)$$

where $\mu(l^*)$ is given by Eq. (29). This gives

$$\xi(T) = \frac{1}{\Lambda \alpha^{1/2}} e^{l^*}$$  \hspace{1cm} (44)$$

and using Eq. (30)

$$\xi(T) = \frac{1}{\left(2\epsilon \ln \frac{1}{2\epsilon}\right)^{1/2} (2mk_B)^{1/2} T^{-\frac{1}{2}(1+\frac{1}{z})}}$$  \hspace{1cm} (45)$$

if we take $\alpha \simeq 1$.

This behavior appear as universal and is not surprising because as it was mentioned in Ref. [13] the first contribution to the calculation of chemical potential in the fixed point gives a first term as universal.

IV. TWO DIMENSIONS

The dilute Bose gas at $T = 0$ was studied in Ref. [13] and is known as marginal. In this case we consider the system at finite temperature in the low temperatures approximation. The free energy and the specific heat will be also calculated for this system taking $z = 2$. The high temperature limit will be also analised.

A. Low temperature limit

1. Scaling equations

In this case the scaling equations have the form:

$$\frac{d\Gamma(l)}{dl} = 0$$  \hspace{1cm} (46)$$

$$\frac{dT(l)}{dl} = 2T(l)$$  \hspace{1cm} (47)$$
\[
\frac{du(l)}{dl} = -\frac{mK_2}{2\hbar^2} u^2(l)
\] (48)

\[
\frac{d\mu(l)}{dl} = 2\mu(l) - \frac{\Lambda^2 K_2 u^2(l)}{\exp\left(\frac{\hbar^2 \Lambda^2}{2m k_B T(l)}\right) - 1}
\] (49)

The Eqs. (46) - (48) have the solutions:

\[
\Gamma(l) = \Gamma
\] (50)

\[
T(l) = T e^{2l}
\] (51)

\[
u(l) = \frac{4\pi \hbar^2}{m} \frac{1}{l + l_0}
\] (52)

where $\Gamma$ will be take as $\Gamma = 1$ and $l_0$ has been calculated as:

\[
l_0 = \frac{4\pi \hbar^2}{m u_0}
\] (53)

The solution of Eq. (50) has the form:

\[
\mu(l) = -\frac{4\Lambda^2 \hbar^2}{2m} \int_0^l \frac{dl'}{l' + l_0} \frac{e^{-2l'}}{\exp\left(\frac{\hbar^2 \Lambda^2}{2m k_B T(l')}e^{-2l'}\right) - 1}
\] (54)

which will be written as:

\[
\mu(l) = -\frac{4\Lambda^2 \hbar^2}{2m} e^{2l} \frac{2m k_B T}{\hbar^2 \Lambda^2} \left\{ \frac{1}{2l_0} \ln \left[ 1 - \exp\left( -\frac{\hbar^2 \Lambda^2}{2m k_B T} \right) \right] \right. \\
- \frac{1}{2l_0} \left( \frac{l}{l_0} \right)^{-1} \ln \left[ 1 - \exp\left( -\frac{\hbar^2 \Lambda^2}{2m k_B T} e^{-2l} \right) \right]\left\} - \frac{2m k_B T}{\hbar^2 \Lambda^2} F(l)
\] (55)

where

\[
F(l) = \int_0^{2l} \frac{dx}{(x + 2l_0)^2} \ln \left[ 1 - \exp\left( -\frac{\hbar^2 \Lambda^2}{2m k_B T} e^{-x} \right) \right]
\] (56)

The renormalization procedure will be stoped at $l^*$ defined also by:

\[
\mu(l^*) = -\frac{\hbar^2 \Lambda^2}{2m}
\]

but in this case $\mu(l)$ is given by Eq. (55). Following the same procedure we calculate
\[ e^{-2l^*} = \frac{4}{\alpha} \left[ \frac{4}{\alpha} - \ln \frac{4}{\alpha} \right] \frac{2mk_B T}{\hbar^2 \Lambda^2} \frac{1}{\ln \frac{\alpha}{\frac{\hbar^2 \Lambda^2}{2mk_B T}}} \]  \hspace{1cm} (57)

and if we introduce the effective temperature \( T_0 = \hbar^2 \Lambda^2 \alpha / 8mk_B \) Eq. (57) will be written as:

\[ e^{-2l^*} = \frac{C(\alpha)}{4} \frac{T}{T_0} \frac{1}{\ln \frac{T_0}{T}} \]  \hspace{1cm} (58)

Using Eq. (44) we get for coherence length

\[ \xi(T) \sim \left| \ln \frac{T_0}{T} \right|^{1/2} \]  \hspace{1cm} (59)

The number of condensate bosons has been calculated using a similar relation with (57) and we get:

\[ n(T) = \frac{2mk_B T}{4\pi \hbar^2} \ln \frac{1}{1 - \exp \left( - \frac{\hbar^2 \Lambda^2}{2mk_B T} e^{-2l^*} \right)} \]  \hspace{1cm} (60)

which gives

\[ n(T) = \frac{2mk_B T}{4\pi \hbar^2} \ln \ln \left( \frac{T_0}{T} \right) \]  \hspace{1cm} (61)

2. The free energy and specific heat

The free energy given by general Eq. (1) has the form:

\[ \frac{dF(l)}{dl} = 4F(l) + C_f(\mu(l), T(l)) \]  \hspace{1cm} (62)

with

\[ C_f = K_2 \Lambda^2 k_B T \ln \left\{ 1 - \exp \left[ - \frac{\Gamma(l)}{k_B T} \left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right) \right] \right\} \]  \hspace{1cm} (63)

Using the substitution \( T = T e^{2x} \) we write the solution of Eq. (62) as:

\[ F(l) = \int_0^l dx \ e^{-4x} C_f \left( T e^{2x} \right) \]  \hspace{1cm} (64)

where we approximate \( C_f \) as:
\[
C_f = K_2 \Lambda^2 k_B T \ln \left[ 1 - \exp \left( -\frac{\Gamma(l)}{k_B T} - \mu(l) \right) \right]
\]  
(65)

The expression for \( F(l^*) \) becomes:
\[
F(l^*) = \frac{\Lambda^2}{2\pi} k_B T \int_{0}^{l^*} dx \ e^{-2x} \ln \left( 1 - e^{-\frac{A}{T^2} e^{-2x}} \right)
\]  
(66)

where
\[
A = \frac{\Gamma}{k_B} \left( \frac{k^2 \Lambda^2}{2m} - \mu \right)
\]

In the low temperature limit Eq. (4.21) will be approximated as:
\[
F = \frac{\Lambda^2}{4\pi} k_B T \ln \left[ \frac{A}{T} \ln \left( \frac{T}{T_0} \right) \right]
+ \frac{\Lambda^2}{4\pi} (k_B T)^2 \ln \left( \frac{T}{T_0} \right) \ln \left( \frac{T}{T_0} \right)
+ \frac{\Lambda^2}{4\pi} k_B T \left[ \frac{T}{T_0} - 1 \right]
\]  
(67)

From this equation we calculate the specific heat \( C_v(T) = -T \partial^2 T / \partial T^2 \) and the dominant contribution in temperature has the form:
\[
C_v(T) = C_0 \left| \frac{T}{T_0} \right| \ln^3 \left| \frac{T}{T_0} \right|
\]  
(68)

where \( C_0 = C_0(\Lambda) \) which shows that the result is \( \Lambda \) - dependent.

B. High temperature limit

The high temperature domain is called "classical" domain, because it is dominated by the classical fluctuations of the \( \phi(x, \tau) \) and is usually defined by \( z = 0 \).

1. Scaling equations

From the general equations \([\text{S}] - [\text{III}]\) we write these equations in the high temperature limit as:
\[
\frac{d[T(l)\Gamma^{-1}(l)]}{dl} = 2 \left[ T(l)\Gamma^{-1}(l) \right] 
\]

(69)

\[
\frac{d\mu(l)}{dl} = 2\mu(l) - \frac{1}{2\pi} \tilde{F}_\mu[\mu(l), T(l)]v(l) 
\]

(70)

\[
\frac{dv(l)}{dl} = 2v(l) - \frac{5}{4\pi} \tilde{F}_v[\mu(l), T(l)]v^2(l) 
\]

(71)

where

\[
v(l) = k_B T u(l) 
\]

(72)

and \( \tilde{F}_\mu, \tilde{F}_v \) have been obtained from Eqs. \((13) - (15)\) in the limit \( T \to \infty \) as:

\[
\tilde{F}_\mu[\mu(l), T(l)] \simeq \frac{\Lambda^2}{\frac{\hbar^2 \Lambda^2}{2m} - \mu(l)} 
\]

(73)

\[
\tilde{F}_v[\mu(l), T(l)] \simeq \frac{\Lambda^2}{\left( \frac{\hbar^2 \Lambda^2}{2m} - \mu(l) \right)^2} 
\]

(74)

In order to solve the Eqs. \((69) - (71)\) we have to define \( \bar{l} \), the value of \( l \) at which the flow enters in the classical regime defined by

\[
\frac{T(l)}{\Gamma(l)} \gg 1 
\]

(75)

and introduce

\[
\mu(\bar{l}) = \bar{\mu}_0, \ u(\bar{l}) = \bar{u}_0, \ v(\bar{l}) = \bar{v}_0 
\]

(76)

In order to simplify the calculation we perform a simple transformation \( l' = l - \bar{l} \) which make the flow to start at \( l' = 0 \).

The new scaling equations describing the classical regime will be:

\[
\frac{d[T(l')\Gamma^{-1}(l')]}{dl'} = 2[T(l')\Gamma^{-1}(l')] 
\]

(77)

\[
\frac{d\mu(l')}{dl'} = 2\mu(l') - \frac{1}{2\pi} \frac{\Lambda^2 v(l')}{\frac{\hbar^2 \Lambda^2}{2m} - \mu(l')} 
\]

(78)
\[
\frac{dv(l')}{dl'} \simeq 2v(l') - \frac{5m^2 \nu^2(l')}{\pi \hbar^4 \Lambda^2}
\] (79)

The equation (73) has been obtained neglecting \(\mu(l')\) in Eq. (74). In this form the equation can be solved and we obtain the exact solution

\[
v(l') = \frac{2\tilde{\nu}_0}{B\tilde{\nu}_0 + (2 - B\tilde{\nu}_0) \exp(-2l')}
\] (80)

where \(B = \frac{5m^2}{\pi \hbar^4 \Lambda^2}\).

We define \(l'_*\) a value of \(l'\) for which we stop the scaling by a similar condition as in the low temperature case,

\[
v(l'_*) = 1
\] (81)

which gives

\[
\exp(2l'_*) = \frac{2 - B\tilde{\nu}_0}{B\tilde{\nu}_0 - 2\tilde{\nu}_0}
\] (82)

and from this equation we calculate

\[
l'_* \simeq \frac{1}{2} \ln \frac{1}{\tilde{\nu}_0}
\] (83)

where we used \(uT(l'_*) = 1\).

The Eq. (78) for the chemical potential can also be solved using for \(v(l')\) the expression (80) and we get:

\[
\mu(l') = e^{2\nu} \left[ \tilde{\mu}_0 - \frac{2m}{\pi \hbar^2 B} l' - \frac{m}{\pi \hbar^2 B} \ln \left( e^{-2\nu} + \frac{B\tilde{\nu}_0}{2 - B\tilde{\nu}_0} \right) \right]
\] (84)

where \(\tilde{\mu}_0\) is given by:

\[
\tilde{\mu}_0 = \frac{m}{\pi \hbar^2 B} l'_* + \frac{m}{\pi \hbar^2 B} \ln \left( e^{-2\nu} + \frac{B\tilde{\nu}_0}{2 - B\tilde{\nu}_0} \right)
\] (85)

Using this equation \(\mu(l')\) is approximated as:

\[
\mu(l') \simeq e^{2\nu} \left[ \frac{2m\tilde{\nu}_0}{\pi \hbar^2} (l'_* - l') \right]
\] (86)
2. Density of bosons

In order to calculate the bosonic density, we will use the general equations:

\[ n = -\frac{\partial F}{\partial \mu} \quad (87) \]

where the free energy will be written as:

\[ F(l') = \frac{1}{2\pi} \int_0^{l'} dx \, e^{-2\pi T(x)} \ln \left\{ 1 - \exp \left[ -\frac{\Gamma(x)}{k_B T(x)} \left( \frac{\hbar^2 \lambda^2}{2m} - \mu(x) \right) \right] \right\} \quad (88) \]

Using the relations:

\[ n = -\int_0^{l'*} dl' \frac{\partial}{\partial \mu(l')} \left( \frac{\partial F}{\partial l'} \right) \frac{d\mu(l')}{d\mu} \quad (89) \]

and for \( \mu(l') \) the simple approximation

\[ \mu(l') = \mu(\bar{l}) e^{2(l'+\bar{l})} \quad (90) \]

where \( \mu(\bar{l}) \approx \mu \), the Eq. (89) becomes:

\[ n = -\int_0^{l'*} dl' \, e^{2(l'+\bar{l})} \frac{\partial}{\partial \mu(l')} \left( \frac{\partial F(l')}{\partial l'} \right) \quad (91) \]

From Eqs. (88) and (91) we calculate the general equation for bosonic density \( n \) as:

\[ \frac{n}{T} = \frac{1}{2\pi} \int_0^{l'*} dl_1 \frac{\Gamma(l_1)/k_B T(l_1)}{\exp \left\{ \Gamma(l_1)/k_B T(l_1) \left[ \frac{\hbar^2 \Lambda^2}{2m} - \mu(l_1) \right] \right\} - 1} \quad (92) \]

This integral will be approximated in the high temperature domain as:

\[ \frac{n}{T} \approx \frac{1}{2\pi} \int_0^{l'*} dl_1 \frac{2m}{\hbar^2 \Lambda^2} \left[ 1 + \frac{\hbar^2 \Lambda^2}{2m} \mu(l_1) + \cdots \right] \approx \frac{1}{2\pi} \frac{2m}{\hbar^2 \Lambda^2} l'* \quad (93) \]

and from Eq. (88) we calculate:

\[ \frac{n}{T} \approx \frac{m}{\pi \hbar^2} \ln \frac{1}{\bar{u}_0} \quad (94) \]

which is the density of bosons in the classical regime. This result gives us the possibility to perform a matching with the low temperatures regime. Indeed, using \( \bar{u}_0 \sim 1/\ln T_0 \) for \( T_0 \ll 1 \) we reobtain the expression for \( n_c \sim T \ln \ln 1/T \) the well-known result from Ref. [5].
V. DISCUSSIONS

We reconsidered the problem of dilute Bose gas at finite temperature using the RG method in $d = 2$. This problem has been considered first in the well-known paper by Fisher and Hohenberg [12], but their calculation may give divergencies in the integrals contained in the physical quantities calculated in the quantum regime, because of an approximation for the coupling constant which is taken of the form $v(l) \sim 1/l$.

We solved the problem by using a linearization of the exact solution near the fixed point for $d < 2$ and taking the exact solution in $d = 2$. We also calculate the coherence length $\xi(T)$ and the specific heat from the free energy. As a common point of view with the classical Fisher and Hohenberg paper we mention the non-universal behavior of the system. The new features which appear can be concluded as follows for $d < 2$:

- the bosonic density $n(T)$ present the specific $T^{d/2}$ dependence as well as $\ln(1/\epsilon) \ln 1/\epsilon$ dependence;
- the coherence length $\xi(T)$ calculated also has $1/[T^{1/2(1+\hat{z})}\epsilon \ln 1/\epsilon]^{1/2}$ dependence;

For the two dimensional case $d = 2$ we used the exact solution for the coupling constant $u(l) \sim 1/(l + l_0)$ and we showed that:

- the density $n(T)$ is non-universal and $n(T) \sim T \ln 1/T$, which gives for the critical temperature the Fisher and Hohenberg result, $\ln \ln 1/na^2$, where $\Lambda^2 = 1/2a^2$, $a$ being the potential range;
- the coherence length $\xi(T)$ is non-universal and depends of $T$ as $\xi(T) \sim |(\ln T)/T|^{1/2};$
- the specific heat $C(T)$ is also non-universal and $C_v(T) \sim T/|\ln T|^3$;

The RG scaling equation have been used also in the high temperature domain for $d = 2$. The main result of this treatment is the calculation of the bosonic density which can be matched with the result from the low temperature regime by choosing the coupling constant as the solution of the scaling equations from the low temperature regime.
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