On the closest stable/unstable nonnegative matrix and related stability radii *

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Abstract

We consider the problem of computing the closest stable/unstable non-negative matrix to a given real matrix. This problem is important in the study of linear dynamical systems, numerical methods, etc. The distance between matrices is measured in the Frobenius norm. The problem is addressed for two types of stability: the Schur stability (the matrix is stable if its spectral radius is smaller than one) and the Hurwitz stability (the matrix is stable if its spectral abscissa is negative). We show that the closest unstable matrix can always be explicitly found. For the closest stable matrix, we present an iterative algorithm which converges to a local minimum with a linear rate. It is shown that the total number of local minima can be exponential in the dimension. Numerical results and the complexity estimates are presented.

Keywords: positive linear system, stability, non-negative matrix, Frobenius norm, gradient relaxation

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1. Introduction

The problems of finding the closest stable matrix (stabilizing problem) or the closest unstable matrix (destabilizing problem) are very important in many applications such as the analysis of differential equations, linear dynamical systems, electrodynamics, etc. In this paper we focus on those problems in the set of non-negative matrices and call them positive stabilizing/destabilizing problems. They are needed in the study of positive linear systems which are widely applied in the multiagent problems, population dynamics (matrix

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population models), mathematical economics (Leontief model), etc. Non-negativity ensures certain advantages for this problem such as the special spectral properties of matrices guaranteed by the Perron-Frobenius theory. On the other hand, it brings extra constraints (the non-negativity of $d^2$ entries, where $d$ is the dimension) which complicates the problem a lot.

For the Schur stability, the problem of non-negative stabilizing of a given matrix $A$ consists in finding an entriwise non-negative matrix $X$ such such that $\rho(X) \leq 1$ and the distance to the matrix $\|X - A\|$ is minimal. As usual, $\rho(X)$ denotes the spectral radius of the matrix, which is the maximum modulus of its eigenvalues. As a rule, such stabilizing problems are notoriously hard due to properties of the spectral radius $\rho(X)$ as a function of the matrix $X$. The function $\rho(X)$ is neither convex nor concave, it may be non-differentiable and even non-Lipschitz at some points. This makes all methods of convex or smooth optimization hardly applicable. Basically, even finding a locally closest stable matrix is hard. In many situations we cannot hope for the global optimality due to a large number of local minima. Methods of matrix stabilization (without non-negativity assumption) were presented in [1, 2, 4, 8, 15, 16, 17].

In the stabilizing problem a lot depends on the norm we measure the distance $\|X - A\|$. Usually it is either Euclidean or Frobenius norm; the problem is hard in both those norms [2, 8, 17]. In the recent paper [17] it has been shown that in the $L_\infty$ matrix norm (equal to the largest $L_1$ norm of rows of the matrix) the problems of positive stabilizing/distabilizing both have surprisingly simple solutions and there exist efficient algorithms that find their global minima. In applications, however, the $L_\infty$ matrix norm has some disadvantages: it is non-smooth, badly correlated with the Euclidean norm, etc. That is why, many researches prefer the Frobenius norm, which is the sum of squares of the matrix components. Actually, the Frobenius norm is merely the vector Euclidean norm in the $d^2$ dimensional space of matrices. In this paper we deal with the positive stabilizing and destabilizing problems in the Frobenius norm. We show how to find explicitly the closest non-negative unstable matrix. In fact, it was observed in the literature that the destabilizing problem is usually simpler than the stabilizing one. In some favorable cases our method can be extended for finding the closest stable matrix as well, but in general we may hope on finding local minima only. To this end, we develop an iterative relaxation scheme that converges to a local minimum. It is computationally simple and uses only a standard quadratic programming routine. This makes the method applicable even in high dimensions. Another advantage is the stability of the method with respect to matrices: the algorithm works equally well even if the matrix in a current iteration has a multiple leading eigenvalue (which happens often) and its spectral radius is non-Lipschitz. In practice, the new method converges extremely fast, which is demonstrated in numerical examples. We prove that the rate of convergence is always linear and, moreover, if the limit matrix is strictly positive then it gives a global minimum. In general the limit matrix may have zero components, in which case it gives only a local minimum in general. In this case, a question arises about the possible number of local minima. We construct an example of a positive $d \times d$ matrix for which the positive stabilizing problem has at least $2^d$ local minima. This may be an argument for the high algorithmic complexity of the problem.

The paper is organized as follows. In Section 3 we solve the positive destabilizing prob-
lem and find the closest non-negative unstable matrix. Section 4 deals with the positive stabilizing problem. We present the iteration algorithm for computing the local minima and prove its convergence with a linear rate. We show that if it converges to a positive matrix, then that matrix gives the global minimum. In Section 5 we analyse possible number of local minima and for each $d$, give an example of $d \times d$ matrix for which this number is at least $2^d$. Then we apply our results for finding closest Hurwitz stable and unstable matrices (Section 6).

Finally, let us note that in the problem of finding the closest stable non-negative matrix to a matrix $A$, the matrix $A$ itself does not have to be non-negative. For any real-valued matrix $A$, this problem can be reduced to the case of non-negative $A$ by considering the matrix $A_+ = \max\{A, 0\}$ (the entrywise maximum). It is shown easily that closest stable matrix to the matrices $A$ and $A_+$ are the same. Therefore, in what follows we assume everywhere the initial matrix $A$ is non-negative.

2. Framework

Let us first introduce necessary notation. The Frobenius norm of any rectangular matrix $X$ is $\|X\| = \sqrt{\text{tr} X^T X} = \sqrt{\sum_{i,j} |x_{i,j}|^2}$. This is the standard Euclidean norm in the space of matrices regarded as vectors with the scalar product $(X, Y) = \text{tr} X^T Y$. We write $X \perp Y$ for the case $(X, Y) = 0$. We keep the notation $\|\|_F$ for the Frobenius norm and by $\|X\|_2 = \sqrt{\rho(X^T X)}$ denote the Euclidean operator norm of the matrix $X$.

For arbitrary rectangular matrices, $A$ and $B$ for which the product $AB$ is well defined we have $\|AB\| \leq \|A\| \|B\|$. In case when $A$ is a co-vector and $B$ is a vector, this becomes an equality. If $A$ is a matrix and $B = x$ is a vector, this inequality implies

$$\|Ax\|_2 = \|Ax\| \leq \|A\| \|x\| = \|A\| \|x\|_2.$$

Thus, $\|A\| \geq \|Ax\|_2/\|x\|_2$ for all $x \in \mathbb{R}^d \setminus \{0\}$, therefore $\|A\| \geq \|A\|_2$.

By the Perron-Frobenius theorem, a non-negative matrix possesses a non-negative eigenvector corresponding to a non-negative eigenvalue which is bigger than or equal to moduli of all other eigenvalues. This eigenvalue is called leading, and the corresponding eigenvector is the leading eigenvector.

By the support of a non-negative matrix $(\text{supp} X)$ we mean the set of positions of its positive entries.

3. The closest unstable matrix

For a given non-negative matrix $A$ with $\rho(A) < 1$, we consider the problem

$$\begin{cases} \|X - A\| \to \min \\ \rho(X) \geq 1. \end{cases} \tag{1}$$
As we show below in Theorem 1, the solution $X$ of this problem is always a non-negative matrix. Therefore, the search for a closest unstable non-negative matrix has the same result. Thus, for a non-negative matrix, the destabilizing problem is equivalent to the positive destabilising problem. We begin with some auxiliary notation and facts.

3.1. Auxiliary facts

For a given $d \times d$ matrix $A$ such that $\rho(A) \neq 1$ we denote

$$
M = (I - A^T)(I - A) ; \quad N = (I - A)(I - A^T)
$$

Since $M$ and $N$ are both symmetric and positive definite, all their eigenvalues are non-negative. Take an arbitrary eigenvector $v$ of $M$ associated to some eigenvalue $\mu \neq 0$ and normalize it as $\|v\| = 1$.

Choosing $r \in \{\sqrt{\mu}, -\sqrt{\mu}\}$, we denote

$$
u = \frac{1}{r} (I - A)v \quad \text{and} \quad X = A + r \, u \, v^T.
$$

Lemma 1 The vector $u$ defined in (3) satisfies $N u = \mu u$ and $\|u\| = 1$. Moreover, $v$ and $u$ are the right and the left eigenvectors respectively of $X$ associated to the eigenvalue one.

Proof. Combining the equality $(I - A^T)(I - A)v = r^2 v$ with the definition of $u$ we get

$$
\begin{cases}
(I - A) \, v = r \, u \\
(I - A)^T u = r \, v.
\end{cases}
$$

Since $\|(I - A)v\|^2 = ((I - A)(I - A^T)v, v) = r^2(v, v) = r^2$, taking into account the first equation in (4) we see that $\|r \, u\|^2 = r^2$ and hence $\|u\| = 1$. Furthermore,

$$
N u = (I - A)(I - A^T)u = (I - A)r v = r^2 v.
$$

Finally, $X v = Av + r u (v, v)$. Since $(v, v) = 1$ and $Av + r u = v$, which follows from the first equation of (4), we obtain $X v = v$. In the same way one shows that $u^T X = u^T$. \qed

3.2. A formula for the closest unstable matrix

The following theorem provides an explicit solution to the non-negative destabilization problem.
Theorem 1 Let $A$ be an arbitrary non-negative matrix such that $\rho(A) < 1$ and let $M = (I - A^T)(I - A)$. Let $\mu$ be the smallest eigenvalue of $M$. Then $M$ possesses a non-negative eigenvector $v$ associated to $\mu$; moreover, for $r = \sqrt{\mu}$, both the vector $u$ and the matrix $X$ defined by (3) are non-negative. Finally the matrix $X$ is the closest unstable matrix to $A$ and $\|X - A\| = r$.

Thus, to find the closest unstable matrix to a non-negative matrix $A$ one needs to take the smallest singular value $r$ of the matrix $I - A$ and take the corresponding normalized singular vector $v$. There may be a subspace of such vectors, in the case when $r$ is multiple, but it always contains a non-negative singular vector $v \neq 0$, as it is guaranteed by Theorem 1. Take this vector and normalize it as $\|v\| = 1$. Then the solution $X$ is readily available by formula (3). By Theorem 1 the matrix $X$ is non-negative and is the closest unstable matrix to $A$ among all matrices, not only non-negative ones.

Proof of Theorem 1 Observe that

$$
(I - A)^{-1} = \sum_{k=0}^{\infty} A^k \geq 0; \quad (I - A^T)^{-1} = \sum_{k=0}^{\infty} (A^T)^k \geq 0
$$

(both those series converge since $\rho(A^T) = \rho(A) < 1$). Therefore, the matrix $M^{-1}$ is non-negative as well. Consequently, its biggest by modulo eigenvalue is non-negative and is realized with a non-negative eigenvector $v$. The reciprocal to this eigenvalue is the smallest by modulo non-negative eigenvalue of $M$. Denote this eigenvalue by $\mu$ and take $r = \sqrt{\mu}$. The second equation of the system (4) yields $u = r(I - A^T)^{-1}v$, hence $u \geq 0$, because $(I - A)^{-1} \geq 0$. Therefore, $X = A + ruv^T \geq 0$. Moreover, Lemma 1 implies that $\|u\| = 1$. Hence

$$
\|X - A\| = r\|uv^T\| = r\|u\|\|v^T\| = r.
$$

On the other hand, for every matrix $Y$ with $\rho(Y) = 1$, we have $\|Y - A\| \geq r$, which proves the optimality of $X$. To show this we first assume that $Y \geq 0$. In this case $Y$ has a leading eigenvector $z$, for which $Yz = z$ and $\|z\| = 1$. Then

$$
\|Y - A\| = \|Y - A\|\|z\| \geq \|Y - A\|z\| = \|z - Az\| = \|(I - A)z\| = \|(I - A)z\|_2.
$$

However, since all singular values of the matrix $I - A$ are bigger than or equal to $r$, it follows that $\|(I - A)z\|_2 \geq r\|z\|_2 = r$. Thus, $\|Y - A\| \geq r$, which proves that the matrix $X$ is the closest stable matrix for $A$ among non-negative matrices.

Take now an arbitrary matrix $\Delta$ such that $\|\Delta\| = r$ and show that $\rho(A + \Delta) \leq 1$. This will prove the optimality of $X$ among all matrices. The matrix $|\Delta|$ composed by the moduli of the entries of $\Delta$ has the same norm $r$. On the other hand, $\|(A + |\Delta|)^k\| \geq \|(A + \Delta)^k\|$ for every $k$, which in view of Gelfand’s formula for the spectral radius implies that $\rho(A + |\Delta|) \geq \rho(A + \Delta)$. On the other hand, since $X$ is optimal among non-negative matrices, we see that $\rho(A + |\Delta|) \leq \rho(X) = 1$, and therefore $\rho(A + \Delta) \leq 1$, which completes the proof.

Remark 1 Note that since the difference $X - A$ has rank one, $X$ is also the closest stable matrix to $A$ in the spectral (Euclidean) norm.
Remark 2 The simplicity of solution of the non-negative destabilization problem is explained by the fact this problem is actually unconstrained. Indeed, By Theorem 1 if a matrix $A$ is non-negative, then its closest unstable matrix is also non-negative. Hence, the non-negativity constraints turn out to be redundant here. On the other hand, we can exploit all advantages of the non-negativity of the solution provided by the Perron-Frobenius theory. In contrast, in the stabilization problem the non-negativity constraints are significant, which makes that problem much more difficult.

3.3. Illustrative example

Consider the matrix

$$
A = \begin{pmatrix}
0.4 & 0.4 & 0.1 \\
0.5 & 0.3 & 0.3 \\
0.1 & 0.1 & 0.5 \\
\end{pmatrix}, \quad \text{with} \quad \rho(A) = 0.8960.
$$

The minimal eigenvalue of the matrix $M$ is 0.0102, which gives $r = 0.1009$; the computation of the vectors $v$ and $u$ gives (to a five digit precision):

$$
u = \begin{pmatrix}
0.6484 \\
0.5452 \\
0.5314 \\
\end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix}
0.6275 \\
0.6852 \\
0.3698 \\
\end{pmatrix}.
$$

This yields

$$
X = A + r \, u \, v^T = \begin{pmatrix}
0.4410 & 0.4448 & 0.1242 \\
0.5345 & 0.3377 & 0.3203 \\
0.1336 & 0.1367 & 0.5198 \\
\end{pmatrix}.
$$

Theorem 2 yields that $X$ is the closest unstable matrix, i.e., provides a global minimum to the destabilizing problem. Note that applying the general purpose algorithm in [4] to compute the stability radii, one gets, as expected, the same matrix $X$.

4. The closest stable matrix

For a given non-negative matrix $A$ with $\rho(A) > 1$, we consider the problem

$$
\begin{cases}
\|X - A\| \rightarrow \min \\
\rho(X) \leq 1, \quad X \geq 0.
\end{cases}
$$

We indicate by locmin the set of local minima for problem (5). Simple examples show that the constraint $X \geq 0$ is significant here. The reason is that even if a matrix $A$ is positive, then its closest stable matrix (in the space of all matrices) may have some negative elements as the following example demonstrates:
Example 1 For the matrix $A = \begin{pmatrix} 2 & 2 \\ 0 & 0 \end{pmatrix}$, the closest non-negative stable matrix is $X = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}$. Indeed, for any other non-negative matrix $Y$, we have $\rho(Y) \geq \begin{pmatrix} y_{11} & y_{12} \\ 0 & 0 \end{pmatrix} = y_{11}$. Hence, if $\rho(Y) \leq 1$, then $y_{11} \leq 1$, and consequently $\|Y - A\| \geq |y_{11} - 2| \geq 1$. Thus, $\|Y - A\| \|X - A\|$, and $X$ is the closest stable non-negative matrix to $A$. On the other hand, there exists a closer stable matrix, which is not non-negative: $Y' = \begin{pmatrix} 2 & 2 \\ -1/2 & 0 \end{pmatrix}$, for which $\|Y' - A\| = \frac{1}{2}$. Hence, for the matrix $A$, the closest stable matrix is not non-negative.

We see that the stabilization of the matrix can set some of its entries to zero, and this set of zeros can influence the spectral properties of $X$. That is why, in problem 5, the combinatorics of the matrix $X$ plays a role. It can be primitive, imprimitive, irreducible, reducible, etc. All this properties have to be considered. This explains the algorithmic complexity of the problem. As we will see in Section 5, problem 5 may have exponentially many local minima, all with different combinatorics. Explicit solutions can still be obtained, but under special assumptions (Subsection 4.1). In general, we may hope only for algorithmic solutions of finding local minima. This problem requires some preparation; we begin with some simple observations.

Lemma 2 Suppose $A$ is a reducible matrix, i.e., there exists a permutation matrix $\Pi$ which factorizes $A$ to block upper triangular form,

$$
\Pi A \Pi^T = \tilde{A} = \begin{pmatrix}
\tilde{A}_{1,1} & \tilde{A}_{1,2} & \ldots & \ldots & \tilde{A}_{1,m} \\
0 & \tilde{A}_{2,2} & \ldots & \ldots & \tilde{A}_{2,m} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \tilde{A}_{m-1,m-1} & \tilde{A}_{m-1,m} \\
0 & \ldots & 0 & 0 & \tilde{A}_{m,m}
\end{pmatrix}
$$

Then the closest stable non-negative matrix $\tilde{X}$ to $\tilde{A}$ is given as follows:

$$
X = \Pi^T \tilde{X} \Pi \text{ with } \tilde{X} = \begin{pmatrix}
\tilde{X}^{(1)} & \tilde{A}_{1,2} & \ldots & \ldots & \tilde{A}_{1,m} \\
0 & \tilde{X}^{(2)} & \ldots & \ldots & \tilde{A}_{2,m} \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
0 & \ldots & 0 & \tilde{X}^{(m-1)} & \tilde{A}_{m-1,m} \\
0 & \ldots & 0 & 0 & \tilde{X}^{(m)}
\end{pmatrix}
$$

(6)

where $\tilde{X}^{(i)}$ is the closest stable non-negative matrix to $\tilde{A}_{i,i}$ (for $i = 1, \ldots, m$). In particular if $\rho(\tilde{A}_{i,i}) \leq 1$, then $\tilde{X}^{(i)} = \tilde{A}_{i,i}$.

Proof. Since $\tilde{A}$ is similar to $A$ and $\Pi$ defines an isometry in the Frobenius norm, considering the problem of finding the closest stable matrix $X$ to $A$ is equivalent to that of finding the closest stable matrix $\tilde{X}$ to $\tilde{A}$. 


Let $\tilde{X}$ be the matrix constructed in (6). Since each block $\tilde{X}^{(i)}$ is stable, we have $\rho(\tilde{X}) = \max_i \rho(\tilde{X}^{(i)}) = 1$, hence $\tilde{X}$ is stable. Any change of some element off the diagonal blocks would increase the distance to $\tilde{A}$. On the other hand, this change would not reduce the spectral radius of $\tilde{X}$. Finally, any change of a diagonal block which keeps this block stable would not reduce the distance to $\tilde{A}$. Hence, $\tilde{X}$ gives the global minimum. □

Lemma 2 reduces problem (5) to several similar problems of smaller dimensions whenever $A$ is reducible. Hence, we do not consider this case any more.

In the sequel we assume that $A \geq 0$, $\rho(A) < 1$, and that $A$ is irreducible.

Lemma 3

If $X$ is a local minimum for (5), then $X \leq A$ and $\rho(X) = 1$.

Proof. If $X$ has a component $x_{ij}$ bigger than $a_{ij}$, then we slightly reduce $x_{ij}$. The distance $\|X - A\|$ decreases, while the spectral radius $\rho(X)$, as a monotone function on the set of non-negative matrices, does not increase. Hence the condition $\rho(X) \leq 1$ remains true, which contradicts to the local optimality of $X$. If $\rho(A) > 1$, then $X \neq A$, and hence $x_{ij} < a_{ij}$ at least for one component. If $\rho(X) < 1$, then we can slightly increase $x_{ij}$ so that the condition $\rho(X) \leq 1$ remains true.

Lemma 3 ensures that we can restrict our search to matrices that are entrywise smaller or equal to $A$ and have spectral radius one. So, in the sequel we assume that $X \leq A$. Thus, we look for solutions of the problem (5) on the set of matrices $X$ such that $0 \leq X \leq A$ and $\rho(X) = 1$.

Thus, the leading eigenvector of $X$ is equal to one. If this eigenvalue is simple, in particular, if the matrix $X$ is irreducible, then the spectral radius is differentiable at the point $X$, and problem (5) is smooth. Then we can apply the Lagrange theorem and derive the following condition for the local minimum:

Proposition 1

Suppose a matrix $X$ is a local minimum for (5) and its leading eigenvalue 1 is simple. Suppose $u$ and $v$ are respectively left and right leading eigenvectors of $X$, associated to the eigenvalue 1; then there exists a number $r > 0$ and a matrix $\Lambda \geq 0$ such that $\Lambda \perp X$ and

$$A = X + ruv^T - \Lambda. \quad (7)$$

Remark 3

Under the assumptions of the proposition, we have $(v, u) > 0$. Indeed, if $(v, u) = 0$, then the supports of the vectors $v$ and $u$ are disjoint. After a permutation of the basis the matrix $X$ obtains the block diagonal form $X = \begin{pmatrix} X^{(1,1)} & 0 \\ 0 & X^{(2,2)} \end{pmatrix}$ with the blocks corresponding to the supports of $v$ and $u$ respectively and with $\rho(X^{(1,1)}) = \rho(X^{(2,2)}) = 1$. Hence 1 is not a simple eigenvalue.

Proof. The derivative of the function $\|X - A\|^2$ at $X$ is equal to $2(X - A)$. Since the leading eigenvalue $\lambda_{\max}$ of $X$ is simple, the function $\lambda_{\max}(X)$ is differentiable at $X$ and the
gradient is equal to \( \mathbf{u} \mathbf{v}^T \) (see [10]). Now applying the Lagrange theorem to the problem (5) we see that there are non-negative multipliers \( \alpha_0, \alpha_1 \) and \( \alpha_{ij}, i, j = 1, \ldots, d \), such that

\[
2 \alpha_0 (X - A) + \alpha_1 \mathbf{u} \mathbf{v}^T - \Lambda = 0,
\]

where \( \Lambda = (\alpha_{ij})_{i,j} \) is the matrix of multipliers corresponding to the constraints \( x_{ij} \geq 0 \). The complementary slackness conditions give \( \alpha_{ij} x_{ij} = 0 \) for all \( i, j \), and hence \( \Lambda \perp X \).

If \( \alpha_0 = 0 \), then \( \mathbf{u} \mathbf{v}^T \perp X \). In this case

\[
0 = (\mathbf{u} \mathbf{v}^T, X) = \text{tr} \left( (\mathbf{u} \mathbf{v}^T)^TX \right) = \text{tr} \left( \mathbf{v} \mathbf{u}^TX \right).
\]

Since \( \mathbf{u}^TX = \mathbf{u}^T \), we see that \( \text{tr} \left( \mathbf{v} \mathbf{u}^TX \right) = \text{tr} \left( \mathbf{v} \mathbf{u}^T \right) = \langle \mathbf{v}, \mathbf{u} \rangle \). Thus, if \( \alpha_0 = 0 \), then \( \mathbf{v} \perp \mathbf{u} \). This means that the leading eigenvalue of \( X \) is multiple, which contradicts the assumption. Thus, \( \alpha_0 > 0 \), and we can set \( \alpha_0 = \frac{1}{2} \) and arrive at (7).

\[\Box\]

Equation (7) is not simple to solve, because it involves an unknown matrix \( X \) together with its left and right eigenvectors. Nevertheless, if it possesses a positive solution \( X \), then it can be found explicitly (Subsection 4.1). In general, \( X \) can have zero entries, and therefore additional unknowns occur: each zero entry of \( X \) generates the corresponding unknown element of the matrix \( \Lambda \). In Subsection 4.2 we present an algorithm for the numerical solution of problem (7). However, in some cases this solution is not able to identify a point of local minimum in a unique way. This happens when the matrix \( X \) is non-primitive. Moreover, if the eigenvalue 1 is multiple for \( X \), then Proposition 4 may not hold at all. In this case the matrix \( X \) must be reducible (see, for instance, [7, chapter 13, §2, theorem 2]), and this case is considered in the end of this section.

We see that the sparsity pattern of the matrix \( X \), i.e., the location of zero components, is crucial in the solution of (7), because it defines the set of extra variables in the matrix \( \Lambda \). That is why the solution involves the combinatorics of the matrix \( X \).

Let us recall that a matrix \( X \geq 0 \) is called primitive if some of its power is strictly positive. If \( X \) is non-primitive, but irreducible, it is called imprimitive. We have to analyse conditions for the local minimum in the three separate cases: 1) \( X \) is primitive; 2) \( X \) is imprimitive (a quite unusual case in our experiments); 3) \( X \) is reducible.

**Case 1.** \( X \) is primitive. We call a primitive matrix \( X \) satisfying conditions of Proposition 4 a stationary point of problem (5).

**Case 2.** \( X \) is imprimitive. In this case one more necessary condition to local minimality appears. By the Perron-Frobenius theorem, for an imprimitive matrix \( X \), there is a disjoint partition of the set \( \Omega = \{1, \ldots, d\} \) into \( r \geq 2 \) nonempty sets \( \Omega_1, \ldots, \Omega_r \), such that the matrix \( X_k \) defines a cyclic permutation of those sets: \( \Omega_1 \to \Omega_2 \to \cdots \to \Omega_r \to \Omega_1 \). This means that if \( \text{supp} \, \mathbf{a} \subset \Omega_i \), then \( \text{supp}(X \mathbf{a}) \subset \Omega_{i+1}, \, i = 1, \ldots, r \) (we set \( \Omega_{r+1} = \Omega_1 \)). After renumbering
the basis vectors, \( X \) gets the form of cyclic permutation of primitive blocks \( X^{[1]}, \ldots, X^{[r]} \).

\[
X = \begin{pmatrix}
0 & 0 & 0 & \ldots & 0 & X^{[r]} \\
X^{[1]} & 0 & 0 & \ldots & 0 & 0 \\
0 & X^{[2]} & 0 & \ldots & 0 & 0 \\
\vdots & \vdots & \ddots & \ddots & \ddots & \ddots \\
0 & 0 & 0 & \ldots & X^{[r-1]} & 0 \\
\end{pmatrix}.
\] (8)

For a positive vector of weights \((s_1, \ldots, s_r)\), denote by \(X^{[s_1, \ldots, s_r]}\) the same matrix with blocks \( s_1 X^{[1]}, \ldots, s_r X^{[r]} \). The spectral radius of this matrix is equal to \( s_1 \cdots s_r \rho(X) \). Then we optimize the weights \( s_1, \ldots, s_r \) by solving the problem

\[
\begin{cases}
\|X^{[s_1, \ldots, s_r]} - A\| \to \min \\
s_1 \cdots s_r = 1
\end{cases}
\] (9)

This problem always admits a unique point of minimum, the details of the solution are outlined in Sections 4.3. Now we can extend the notion of stationary point to all irreducible matrices.

**Definition 1** An irreducible matrix is stationary for problem (5) if it satisfies the equation (7) and, if it is imprimitive, has the optimal weights of the blocks \( s_1, \ldots, s_m \) obtained by solving problem (9).

The case \( r = 1 \) corresponds to a primitive matrix, the case \( r \geq 2 \) does to imprimitive one.

**Case 3. \( X \) is reducible.** In this case the matrix \( X \) admits a unique, up to a permutation of the basis vectors, Frobenius form. This means that there exists a reordering of the basis vectors, after which \( X \) gets the following block upper-triangular form:

\[
X = \begin{pmatrix}
X^{(1)} & * & \cdots & * \\
0 & X^{(2)} & * & \vdots \\
\vdots & \ddots & \ddots & * \\
0 & \cdots & 0 & X^{(m)} \\
\end{pmatrix}.
\] (10)

where all matrices \( X^{(i)} \) in the diagonal blocks are irreducible. For an irreducible matrix \( X \), we have \( m = 1 \), otherwise \( m \geq 2 \). Now we can define the notion of stationary point for a general non-negative matrix \( X \).

**Definition 2** A matrix \( X \) is said stationary for problem (5) if its Frobenius form (10) is such that above the diagonal blocks we have \( X = A \), and for the diagonal blocks \( X^{(i)} \) we have:

(1) if \( \rho(A^{(i)}) \leq 1 \), then \( X^{(i)} = A^{(i)} \);

(2) if \( \rho(A^{(i)}) > 1 \), then \( \rho(X^{(i)}) = 1 \) and \( X^{(i)} \) is a stationary point (by Definition 1) of the problem \( \|X^{(i)} - A^{(i)}\| \to \min \), \( \rho(X^{(i)}) \leq 1 \).
If $m = 1$, then this definition is reduced to the cases 1) and 2).

**Theorem 2**  If $X \in \text{locmin}$ for problem (5), then $X$ is stationary.

**Proof.** Consider the Frobenius form of the matrix $X$. If some element $x_{ij}$ over the diagonal block is not equal to $a_{ij}$, then $x_{ij} < a_{ij}$ (Lemma 3). If we slightly increase this element, the distance $\|X - A\|$ is reduced while the spectral radius of $X$ does not change, which contradicts the local optimality of $X$. Hence, $X = A$ beyond the diagonal blocks. Consider now any diagonal block $X^{(i)}$. If $\rho(A^{(i)}) \leq 1$, but $X^{(i)} \neq A^{(i)}$, then we consider a matrix $X^{(i)}(t) = (1 - t) X^{(i)} + t A^{(i)}$. Since $X^{(i)} \leq A^{(i)}$, all entries of the matrix $X^{(i)}(t)$ increase in $t$, and therefore so does the spectral radius $\rho(X^{(i)}(t))$. Hence, for every $t \in (0, 1)$, the spectral radius of the matrix $X^{(i)}(t)$ does not exceed one, while this matrix is closer to $A^{(i)}$ than $X^{(i)}$. This again contradicts the local optimality of $X$. Finally, if $\rho(A^{(i)}) > 1$, then $X^{(i)}$ must be the local minimum of the problem $\|X^{(i)} - A^{(i)}\| \to \min$, $\rho(X^{(i)}) \leq 1$. Since $X^{(i)}$ is irreducible, by Proposition 1 it satisfies equation (7) and, if it is imprimitive, has the optimal weights of the blocks $s_1, \ldots, s_m$ obtained by solving problem (9). Hence, $X^{(i)}$ is stationary. \qed

Thus, if a local minimum is attained at a matrix $X$, then $X$ is stationary: if $X$ is primitive, then it satisfies equation (7), if it is imprimitive, then it also satisfies an additional optimality condition (9), if it is reducible, then its Frobenius form consists of stationary blocks (primitive or imprimitive) and satisfies the requirements stated in Definition 2. In Subsection 4.2 we will see how to construct the stationary matrices and to find local minima of problem (5) algorithmically.

We begin with a special case when the stationary matrix is strictly positive. In this case it satisfies equation (7) with $\Lambda = 0$. It turns out that under some extra assumptions it provides the global minimum to problem (5).

### 4.1. A positive local minimum is a global minimum

We show here that if a strictly positive matrix gives a local minimum to problem (5), then it gives its global minimum. Moreover, this matrix can be explicitly found.

Consider the solution of the destabilization problem in Section 3. Can we apply the same reasoning to the stabilization problem, assuming $\rho(A) > 1$? We define the matrices $M$ and $N$ by the same formula (2), then define $u$ as an eigenvector of $M$ corresponding to its smallest eigenvalue, then define $v$ and $X$ by the same formula (3) with $r = -\sqrt{\mu}$ instead of $\sqrt{\mu}$ (this is the only difference!). Then we repeat the proof of Theorem 1 to establish that $X$ is the closest stable matrix. However, here we cannot show the positivity of the matrix $M^{-1}$ (this already may not be true) and hence, the positivity of $X$.

Thus, in general, Theorem 1 cannot be extended to the stabilization problem. Nevertheless, if the obtained vectors $v, u$ and the matrix $X$ are non-negative, then $X$ is the global minimum for problem (5), and the corresponding proof is literally the same as the proof of Theorem 1. We formulate it in the following
**Theorem 3** Assume the matrix $M$ (see (2)) possesses a non-negative eigenvector $v$, corresponding to its smallest eigenvalue $\mu$. Assume also that for $r = -\sqrt{\mu}$, the vector $u$ and the matrix $X$ defined by (3) are both non-negative. Then $X$ is the closest stable non-negative matrix for $A$ and $\|X - A\| = r$.

Now we formulate the main result of this subsection.

**Theorem 4** If a matrix $X > 0$ provides a local minimum for problem (5), then it provides a global minimum. Moreover, in this case all assumptions of Theorem 3 are fulfilled and $X$ coincides with the corresponding matrix from that theorem.

**Proof.** Each point of local minimum has the form (7), where in the case $X > 0$, we have $\Lambda = 0$. Thus, $X = A - ruv^T$. Multiplying this equality by $v$ from the right and taking into account that $v^Tv = (v, v) = 1$ and that $Xv = v$, we obtain $(I - A)v = -ru$ and hence $r^2 = \|(I - A)v\| = (v, Mv)$. On the other hand, $\|X - A\| = r$. Thus, $\|X - A\|^2 = (v, Mv)$. Substituting $u = -\frac{1}{r}(I - A)v$ in the formula for $X$ we get

$$X = A - (A - I)v v^T. \quad (11)$$

If $v$ is not an eigenvector of $M$ corresponding to its smallest eigenvalue, then there is a vector $\tilde{v}$ close to $v$ such that $\|\tilde{v}\| = 1$ and $(\tilde{v}, M\tilde{v}) < (v, Mv)$. Define the matrix

$$\tilde{X} = A - (A - I)\tilde{v} \tilde{v}^T.$$

If $\tilde{v}$ is close enough to $v$, then $\tilde{X} > 0$. Moreover, $\tilde{X}\tilde{v} = \tilde{v}$, hence the spectral radius of $\tilde{X}$ is one. Finally, $\|\tilde{X} - A\|^2 = \|(A - I)\tilde{v}\|^2 = (\tilde{v}, M\tilde{v})$, which is smaller than $(v, Mv) = \|X - A\|^2$. Thus, $\tilde{X}$ is closer to $A$ than $X$, hence $X \notin \text{locmin}$. The contradiction proves that the leading eigenvector $v$ of $X$ is a positive eigenvector of $M$ corresponding to the smallest eigenvalue of $M$. By the same argument we show that the right leading eigenvector $u$ of $X$ is a positive eigenvector of $N$ corresponding to its smallest eigenvalue. Thus, all assumptions of Theorem 3 are fulfilled and $X$ coincides with the corresponding matrix from that theorem. 

Thus, problem (5) does not have strictly positive local minima except for those constructed by Theorem 3. In particular, we have proved

**Corollary 1** If the assumptions of Theorem 3 are not fulfilled, then problem (5) does not have strictly positive local minima.

**Remark 4** Theorem 4 admits the case when there are infinitely many closest stable matrices for $A$. This happens when the smallest eigenvalue of $M$ is multiple.

Note that problem (5) may have positive stationary points different from local minima. They have the same form $X = A - rvu^T$, but with $\mu = r^2$ to be a non-minimal eigenvalue of $M$. On the other hand, problem (5) may possess positive stationary points corresponding to at most one eigenvalue of $M$. Indeed, if there are two stationary points $X$ and $X'$
corresponding to different eigenvalues \( \mu \) and \( \mu' \), then the corresponding eigenvectors \( v \) and \( v' \) are orthogonal to each other. On the other hand, they are both non-negative, hence \( v \) must have at least one zero component. This is impossible, because \( v \) is an eigenvector of a strictly positive matrix \( X \). We collect those observations in the following corollary.

**Corollary 2** If problem (5) possesses positive stationary points \( X \), they all have the form (11) with \( v \) being an eigenvector of the matrix \( M \).

**Remark 5** Note that since the difference \( X - A \) has rank one in this case \( X \) is also the closest stable matrix to \( A \) in the spectral norm.

**Example 2** Consider the matrix

\[
A = \begin{pmatrix}
0.6 & 0.4 & 0.1 \\
0.5 & 0.5 & 0.3 \\
0.1 & 0.1 & 0.7
\end{pmatrix}, \quad \text{with } \rho(A) = 1.0960.
\]

The minimal eigenvalue of the matrix \( M \) is 0.0082, which gives \( r = 0.0903 \); the computation of the vectors \( v \) and \( u \) gives (to a five digit precision):

\[
u = \begin{pmatrix}
0.6193 \\
0.4888 \\
0.6144
\end{pmatrix} \quad \text{and} \quad v = \begin{pmatrix}
0.6438 \\
0.7166 \\
0.2684
\end{pmatrix}.
\]

This yields

\[
X = A - r u v^T = \begin{pmatrix}
0.5640 & 0.3599 & 0.0850 \\
0.4716 & 0.4684 & 0.2881 \\
0.0643 & 0.0602 & 0.6851
\end{pmatrix},
\]

which is stable and, by Theorem 4, has minimal distance to \( A \).

**Example 3** Fix the dimension \( d \geq 2 \) and denote by \( E \) the matrix of all ones. We find the closest stable matrix to the matrix \( A = \alpha E \) depending on the parameter \( \alpha \).

For \( \alpha \leq \frac{1}{d} \), the matrix \( \alpha E \) is stable, hence \( X = \alpha E \) is the global minimum (and a unique local minimum). If \( \alpha > \frac{1}{d} \), the matrix \( M = (A - I)(A^T - I) = (\alpha E - I)^2 \) has eigenvalues \( (d\alpha - 1)^2, 1, \ldots, 1 \). Hence, if \( \alpha \in \left[ \frac{1}{d}, \frac{2}{d} \right] \), then \( (d\alpha - 1)^2 \leq 1 \), and therefore the eigenvector \( v = \frac{1}{\sqrt{d}} e \) is associated to the smallest eigenvalue of \( M \). Hence the matrix \( X = \frac{1}{d} E \) provides a global minimum. Indeed, \( X = A - r u v^T \) with \( u = v = \frac{1}{\sqrt{d}} e \) and \( r = (\alpha - \frac{1}{d}) \). It remains to refer to Theorem 3. However, for each \( \alpha > \frac{2}{d} \), the vector \( v \) does not correspond to the smallest eigenvalue of \( X \), hence, in view of Theorem 3, the same matrix \( X = \frac{1}{d} E \) does not provide even a local minimum. Although this matrix is still a stationary point since it is positive and has the form \( X = A - r u v^T \).

Surprisingly, the natural answer \( X = \frac{1}{d} E \) turns out to be wrong for all \( \alpha > \frac{2}{d} \): the matrix \( X \) is not even locally closest stable matrix for \( \alpha E \).
Example 4 Let $\alpha > 0$ and

$$A = \begin{pmatrix} \alpha & \alpha \\ \alpha & \alpha \end{pmatrix} ; \quad X_0 = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}$$

In view of Example 3 if $\alpha \in \left[0, \frac{1}{2}\right]$, then $X = A$ is the global minimum, if $\alpha \in \left[\frac{1}{2}, 1\right)$, then $X = X_0$ is the global minimum and a unique local minimum. For $\alpha = 1$, the matrix $M = 4I$ has equal eigenvalues, and hence $A$ has infinitely many closest stable matrices: every matrix

$$X_t = A - u_t v_t^T,$$

where $u_t = v_t = (\cos t \sin t)^T$, is a global minimum with $\|X_t - A\| = 1$.

Finally, if $\alpha > 1$, then, in view of Corollary 2, $A$ does not have positive local minima, although it has an obvious stationary point $X_0$, which is not locally closest any more. Hence, the closest stable matrix has a zero entry. Considering two possible cases, when this zero is either off the diagonal (in this case $X$ is reducible, and hence has ones on the diagonal and $\alpha$ and $0$ off the diagonal) or on the diagonal (this case does not provide minima), we conclude that there are two global minima:

$$\begin{pmatrix} 1 & \alpha \\ 0 & 1 \end{pmatrix} ; \quad \begin{pmatrix} 1 & 0 \\ \alpha & 1 \end{pmatrix}$$

4.2. The general relaxation scheme

Now we are going to tackle the general case: for a non-negative irreducible matrix $A$ such that $\rho(A) > 1$, solve the stabilization problem (5). The idea of the algorithm is the following. We take a matrix $X_0 \geq 0$ whose support is not smaller than the support of $A$, and normalize it so that $\rho(X_0) = 1$. Then we compute its leading eigenvector $v_0$, for which $X_0 v_0 = v_0$, and solve the problem

$$\begin{cases} \|X - A\| \to \min \\ X v_0 \leq v_0, \quad X \geq 0. \end{cases}$$

Its solution is denoted as $X_1$.

Then we compute the left leading eigenvector $u_1$ of $X_1$, for which $u_1^T X_1 = u_1^T$, and solve the problem

$$\begin{cases} \|X - A\| \to \min \\ u_1^T X \leq u_1^T, \quad X \geq 0. \end{cases}$$

Its solution is denoted as $X_2$. Then we loop by alternating between problems of the form (12) and (13), that is we compute the right leading eigenvector $v_2$ of $X_2$ and continue.

To summarize we make a consecutive relaxation of the objective function $\|X - A\|$ every time alternating the right and left leading eigenvectors of the matrix $X$. For even $k$, we
optimize $X_k$ with respect to the fixed leading eigenvector, and for odd, we do it with respect to the fixed leading right eigenvector.

We shall prove that if the value $\|X_k - A\|$ is the same for two consecutive iterations, then the algorithm halts at the matrix $X_k$.

Otherwise, the objective function (the distance to $A$) decreases in each iteration. The complexity and convergence analysis will be done in the following subsection.

Before introducing the algorithm, we recall some auxiliary facts. If a current matrix $X_{k-1}$ is reducible, then after a suitable renumbering of the basis vectors it gets the form

$$ X_k = \begin{pmatrix} X^{(1,1)} & X^{(1,2)} \\ 0 & X^{(2,2)} \end{pmatrix}. $$

(14)

We denote by $A^{(i,j)}$ the corresponding blocks of the matrix $A$ (after the same renumbering).

4.3. The Algorithm

The Algorithm is based on an inner iteration which implements an iterative method to minimize the function $\|X - A\|$ under the constraints $\rho(X) \leq 1$ and $X \geq 0$, and an outer iteration which takes into account about the reducibility/imprimitivity of the results provided by the inner iteration and is able to further refine the construction of a locally optimal solution to the problem. The outer iteration has a recursive structure, which makes use of the inner iteration possibly several times, until it halts on a stationary point.

The inner iterative optimization Algorithm

The optimization Algorithm is a descent method whose flowchart follows. It constructs a sequence of matrices $\{X_k\}$ which may converge to an irreducible/reducible matrix and in
the first case to a primitive/imprimitive matrix.

Algorithm 1: The iterative optimization Algorithm

Data: $A, X_0$
Result: $X_{k+}$, Reduce

begin

for $k = 0, \ldots, k_{\text{max}}$ do

if $k$ is odd then

1. Compute the right leading eigenvector $v_k$ of $X_k$
   if $v_k > 0$ then
       Solve the optimization problem
       \[
       \begin{cases}
       \|X - A\| \rightarrow \min \\
       Xv_k \leq v_k, \; X \geq 0.
       \end{cases}
       \]
   else
       Set Reduce=True

2. Compute the left leading eigenvector $u_k$ of $X_k$
   if $u_k > 0$ then
       Solve the optimization problem
       \[
       \begin{cases}
       \|X - A\| \rightarrow \min \\
       u_k^T X \leq u_k^T, \; X \geq 0.
       \end{cases}
       \]
   else
       Set Reduce=True

else

3. Set $k^+ = k + 1$ and $k^- = k - 1$
4. Let $X_{k+}$ be the solution of the optimization problem
5. if Reduce=True then
   return
   if $\|X_{k+} - X_k\| < \text{tol}$ and $\|X_{k-} - X_k\| < \text{tol}$ then
   return

end

Basically Algorithm 1 works as follows. If $v_k$ has zeros, then it stops and returns a reducible matrix. Otherwise, if $v_k > 0$, it computes the unique non-negative solution of the problem
\[
\begin{cases}
\|X - A\|^2 \rightarrow \min \\
Xv_k \leq v_k
\end{cases}
\] (15)
and proceeds to next iteration.

Similarly, if $u_k > 0$, then it computes the unique non-negative solution of the problem
\[
\begin{cases}
\|X - A\|^2 \rightarrow \min \\
u_k^T X \leq u_k^T
\end{cases}
\] (16)
and proceeds to next iteration.

If $v_k$ or $u_k$ has zeros, in fact, the matrix $X_k$ is reducible.
The outer recursive Algorithm

The outer Algorithm follows. We take an arbitrary initial matrix $X_0$ with $\rho(X_0) = 1$ and with the same support of $A$. Then we apply the algorithm recursively until a local minimum is found.

**Algorithm 2:** The main recursive Algorithm

| Data: $A, X_0$ | Result: $X$ |
|---------------|-------------|
| begin         |            |
| 1             | Apply Algorithm 1 with inputs $A$ and $X_0$ and outputs $X$ and Reduce |
| if Reduce=True then | Reorder components to get |
| $X = \begin{pmatrix} X^{(1,1)} & X^{(1,2)} \\ X^{(2,1)} & X^{(2,2)} \end{pmatrix}$, $A = \begin{pmatrix} A^{(1,1)} & A^{(1,2)} \\ A^{(2,1)} & A^{(2,2)} \end{pmatrix}$ |
| if $\rho(A^{(2,2)}) < 1$ then | Set $X^{(2,2)} = A^{(2,2)}$ |
| 2             | Apply Algorithm 2 with inputs $A^{(1,1)}$ and $X^{(1,1)}$ and output $X^{(1,1)}$ |
| else          | Apply Algorithm 2 with inputs $A^{(1,1)}$ and $X^{(1,1)}$ and output $X^{(1,1)}$ |
| if $X$ is imprimitive then | Apply Algorithm 2 with inputs $A^{(2,2)}$ and $X^{(2,2)}$ and output $X^{(2,2)}$ |
| 3             | find the sets $\Omega_1, \ldots, \Omega_r$ and compute the optimal weights $s_1, \ldots, s_r$ by solving the problem \( (\ref{eq17}) \) |
| 4             | Set $X = X[s_1, \ldots, s_r]$ |
| return        |            |

If the matrix $X$ is reducible, it gets the block upper triangular form \( (14) \), where $X^{(1,1)}$ is an $m \times m$-matrix and $X^{(2,2)}$ is a $(d - m) \times (d - m)$-matrix respectively.

Consider the case when the right leading eigenvector $v$ has zeros, the case of $u$ is similar. After renumbering it can be assumed that the first $m$ entries of $v$ are positive and the other $d - m$ are zeros.

Note that $\max \{ \rho(X^{(1,1)}), \rho(X^{(2,2)}) \} = \rho(X) = 1$. Moreover, $\rho(X^{(1,1)}) \geq \rho(X^{(2,2)})$, otherwise, the leading eigenvector $v$ cannot have zeros in the last $d - m$ positions (in that case the corresponding eigenvalue would be $\lambda_{\text{max}}(X^{(1,1)}) < 1$). Thus,

\[
\rho(X^{(1,1)}) = 1, \quad \rho(X^{(2,2)}) \leq 1. \tag{17}
\]

Then we first set $X^{(1,2)} = A^{(1,2)}$. This reduces the distance $\|X - A\|$ and does not change the spectral radius of $X$. Second the Algorithm proceeds following one of the following two cases.

Case Red-1. $\rho(A^{(2,2)}) < 1$. We set $X^{(2,2)} = A^{(2,2)}$. This reduces the distance $\|X - A\|$ and due to \( (17) \) does not change the spectral radius of $X$. Then we solve the $m$-dimensional
problem
\[ \begin{cases} \|X^{(1,1)} - A^{(1,1)}\| \to \min \\ \rho(X^{(1,1)}) \leq 1 \end{cases} \] (18)

In other words, we apply the Algorithm to the same problem of a smaller dimension. Then we denote its solution by \(X^{(1,1)}\) and set

\[ X = \begin{pmatrix} X^{(1,1)} & A^{(1,2)} \\ 0 & A^{(2,2)} \end{pmatrix}. \] (19)

**Case Red-2.** \(\rho(A^{(2,2)}) \geq 1\). In this case we apply the Algorithm independently to the blocks \((1,1)\) and \((2,2)\). Thus, we solve two independent problems of dimensions \(m\) and \(d - m\):

\[ \begin{cases} \|X^{(1,1)} - A^{(1,1)}\| \to \min \\ \rho(X^{(1,1)}) \leq 1 \end{cases} ; \begin{cases} \|X^{(2,2)} - A^{(2,2)}\| \to \min \\ \rho(X^{(2,2)}) \leq 1 \end{cases} \] (20)

Then we denote the solutions as \(X^{(1,1)}\) and \(X^{(2,2)}\) respectively and set the matrix \(X\):

\[ X = \begin{pmatrix} X^{(1,1)} & A^{(1,2)} \\ 0 & X^{(2,2)} \end{pmatrix}. \] (21)

This is a stationary point (see Proposition 4 in the next section), and the algorithm terminates.

### 4.4. Illustrative example

Consider the matrix

\[ A = \begin{pmatrix} 0.7 & 0.2 & 0.1 & 0.5 & 1.0 \\ 0.3 & 0.6 & 0.2 & 0.8 & 0.3 \\ 0.5 & 0.7 & 0.9 & 1.0 & 0.5 \\ 0.1 & 0.1 & 0.3 & 0.8 & 0.3 \\ 0.8 & 0.2 & 0.9 & 0.3 & 0.2 \end{pmatrix}, \quad \text{with} \quad \rho(A) = 2.4031. \]

After the first inner optimization step, the following matrix is found:

\[ \tilde{X}_1 = \begin{pmatrix} 0.4349 & 0.1406 & 0.0652 & 0.4912 & 0.9345 \\ 0 & 0.3751 & 0.0682 & 0.7668 & 0.0518 \\ 0 & 0.3383 & 0.6881 & 0.9466 & 0.1009 \\ 0 & 0 & 0 & 0.5917 & 0 \\ 0.2989 & 0.0878 & 0.8343 & 0.2834 & 0.0762 \end{pmatrix}, \]

which is reducible and has distance 1.1894 from \(A\).
A reordering allows to obtain

\[
X_1 = \begin{pmatrix}
0.0762 & 0.2989 & 0.0878 & 0.8343 & 0.2834 \\
0.9345 & 0.4349 & 0.1406 & 0.0652 & 0.4912 \\
0.0518 & 0 & 0.3751 & 0.0682 & 0.7668 \\
0.1009 & 0 & 0.3383 & 0.6881 & 0.9466 \\
0 & 0 & 0 & 0 & 0.5917
\end{pmatrix}, \quad A_1 = \begin{pmatrix}
0.2 & 0.8 & 0.2 & 0.9 & 0.3 \\
1.0 & 0.7 & 0.2 & 0.1 & 0.5 \\
0.3 & 0.3 & 0.6 & 0.2 & 0.8 \\
0.5 & 0.5 & 0.7 & 0.9 & 1.0 \\
0.3 & 0.1 & 0.1 & 0.3 & 0.8
\end{pmatrix}.
\]

We are in the first case, Case Red-1, that is \(\rho(A_1^{(2,2)}) = 0.8 < 1\). So we continue applying the optimization Algorithm to the matrix \(A_1^{(1,1)}\). This gives the matrix

\[
X_{1}^{(1,1)} = \begin{pmatrix}
0 & 0.4204 & 0.1759 & 0.6770 \\
0.7343 & 0.3796 & 0.1797 & 0 \\
0.0274 & 0 & 0.5791 & 0.0069 \\
0.1334 & 0.0580 & 0.6719 & 0.6403
\end{pmatrix},
\]

which is irreducible and primitive. Hence the final matrix is given

\[
X_2 = \begin{pmatrix}
0 & 0.4204 & 0.1759 & 0.6770 & 0.3 \\
0.7343 & 0.3796 & 0.1797 & 0 & 0.5 \\
0.0274 & 0 & 0.5791 & 0.0069 & 0.8 \\
0.1334 & 0.0580 & 0.6719 & 0.6403 & 1
\end{pmatrix},
\]

which optimally approximates the matrix \(A_1\), that is \(A\) expressed in the new coordinates. The distance is 1.1037, which is indeed smaller than 1.1894. In the old coordinates, the computed optimal solution is \(X^* = A - \Delta\) is

\[
X^* = \begin{pmatrix}
0.3796 & 0.1797 & 0 & 0.5 & 0.7343 \\
0 & 0.5791 & 0.0069 & 0.8 & 0.0274 \\
0.0580 & 0.6719 & 0.6403 & 1.0 & 0.1334 \\
0 & 0 & 0 & 0.8 & 0
\end{pmatrix}, \quad \Delta = \begin{pmatrix}
0.3204 & 0.0203 & 0.1000 & 0 & 0.2657 \\
0.3000 & 0.0209 & 0.1931 & 0 & 0.2726 \\
0.4420 & 0.0281 & 0.2597 & 0 & 0.3666 \\
0.1000 & 0.1000 & 0.3000 & 0 & 0.3000 \\
0.3796 & 0.0241 & 0.2230 & 0 & 0.2000
\end{pmatrix}.
\]

### 4.5. Realization and computational costs

An advantage of the algorithm is a relatively low computational cost of each iteration. Problem (15) is solved as \(d\) separate problems, one in each row. In the \(i\)-th row we find the minimum of \(\|x_i - a_i\|^2\) under the constraints \((x_i, v) \leq v_i, x_i \geq 0 (a_i, x_i)\) are the \(is\) rows of the matrices \(A\) and \(X\) respectively, and \(v = v_{k-l}\). This is a \(d\)-dimensional convex quadratic problem and is easily solved by quadratic programming. The same problem of minimizing positively definite quadratic form on the positive orthant under one linear constrained arises
in many applications. Since the objective function is strictly convex, the solution $x_i$ is unique and by the Karush-Kuhn-Tukker theorem (see e.g [6]), is characterized by the equation:

\[ x_i = \begin{cases} a_i, & \text{if } (a_i, v) \leq v_i \\ a_i - \lambda v + \Lambda_i, & \text{if } (a_i, v) > v_i, \end{cases} \]  

(22)

where $\lambda > 0$ is a multiplier and $\Lambda_i \geq 0$ is a vector orthogonal to $x_i$, i.e., it has zeros on the positions of positive components of $x_i$. If $x_i > 0$, then $\Lambda_i = 0$ and $\lambda = \frac{(a_i, v) - v_i}{\|v\|^2}$. So, in this case $x_i$ is explicitly computed:

\[ x_i = \begin{cases} a_i, & \text{if } (a_i, v) \leq v_i \\ a_i - \frac{(a_i, v) - v_i}{\|v\|^2} v, & \text{if } (a_i, v) > v_i. \end{cases} \]  

(23)

In general, if $x_i$ has zeros, it is characterized by equation (22) and is computed numerically. For even iterations $k$, with the problem (16), the formulas are the same, with replacing rows by columns and $v$ by $u$. The most expansive operation is the computing the leading eigenvector of $X_{k-1}$ (left or right one depending on $k$) in each step. Further conclusions from formulas (22) are the following.

**Corollary 3** For each $k$, there is a vector $\ell_k \in \mathbb{R}^d_+$ and a matrix $\Lambda_k \geq 0$, $\Lambda_k \perp X_k$ such that $X_k = A - \ell_k v_{k-1}^T + \Lambda_k$ if $k$ is odd, and $X_k = A - u_{k-1} \ell_k^T + \Lambda_k$ if $k$ is even.

**Corollary 4** For each $k$, we have $\text{rank } (X_k - A - \Lambda_k) = 1$.

If $X_k$ is primitive, then the $k$-th iteration of the algorithm is complete. Otherwise $X_k$ has a block cyclic form [8]. We multiply each block with a positive weight $s_i$ and minimize the distance to $A$ by optimizing those weight. Thus we get the matrix $X_{k+1}$ and go to the next iteration. The problem (9) of optimizing weights is also easily solvable. We omit the index $k$ and denote by $X^{(m)}$ the $m$th block of the matrix $X$ and by $A^{(m)}$ the corresponding pattern of the matrix $A$. Problem (9) becomes

\[ \sum_{m=1}^r \| s_m X^{(m)} - A^{(m)} \|^2 \rightarrow \min \]  

(24)

Its solution satisfies the system of Lagrangian equations

\[ s_m^2 \| X^{(m)} \|^2 - s_m \left( X^{(m)} , A^{(m)} \right) + \lambda, \quad m = 1, \ldots, r. \]

This is a union of univariate quadratic equations depending on one parameter $\lambda > 0$. Each equation has two positive roots. Taking every time the smallest one as $s_m$, we then find the numbers $s_1(\lambda), \ldots, s_r(\lambda)$. Then we find the smallest $\lambda$ for which $s_1 \cdots s_r = 1$, we find the optimal weights.

**Corollary 5** If the matrix $X$ has the optimal weights of the blocks, i.e., for that matrix $s_1 = \ldots = s_m = 1$, then all the scalar products $\left( X^{(m)} , X^{(m)} - A^{(m)} \right)$ are the same for $m = 1, \ldots, r$. 

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4.6. Optimal stabilization at a stationary point

Now we are going to show that if Algorithm 1 stabilizes at some matrix \( X \), i.e., \( X_k = X \) in several subsequent iterations, then \( X \) is a stationary point (Definition 2). In this case the algorithm terminates within finite time. The next step is to prove the convergence to a local minimum, this is done in the next subsection.

Clearly, the distance \( \|X_k - A\| \) does not increase in \( k \). Moreover, since each of the problems (15) and (16) possesses a unique solution, it follows that the distance \( \|X_k - A\| \) strictly decreases, unless \( X_k = X_{k-1} \). If this happens two times in a row, and \( X_k \) is primitive, then the algorithm stabilizes at \( X_k \). If \( X_k \) is imprimitive, then one more iteration is needed: finding optimal weights of the blocks, i.e., solving problem (24). If it does not change \( \|X_k - A\| \), then \( X_k \) possesses the optimal weights, that is satisfies conditions of Corollary 5. Hence, the algorithms stabilizes. Thus, we have proved the following

**Proposition 2** The value \( \|X_k - A\| \) does not increase in \( k \). If it does not change for two subsequent iterations at a primitive matrix \( X_k \), or three subsequent iterations at an imprimitive one, then the algorithm stabilizes.

We see that if three consecutive iterations with the same value of the objective function \( \|X - A\| \) mean that the algorithm stabilizes, provided the matrices are irreducible. Now we are going to show that \( X \) is a stationary point.

**Theorem 5** If Algorithm 1 stabilizes at a matrix \( X_k \), then \( X_k \) is a stationary matrix in the sense of Definition 2.

To prove the theorem we need some auxiliary results. We write \( a \sim b \) for two collinear (proportional) vectors.

**Lemma 4** Let \( X \) be a primitive matrix and \( a, \tilde{a}, b, \tilde{b} \) be non-negative vectors. If the rank-one matrices \( ab^T \) and \( \tilde{a}\tilde{b}^T \) are equal on \( \text{supp} \ X \), then \( \tilde{a} \sim a \) and \( \tilde{b} \sim b \).

The proof is in Appendix. Thus, a rank-one matrix \( C \) has a unique, up to multiplication by a constant, presentation \( C = ab^T \) on a support of any primitive matrix.

**Proposition 3** If in the Case Red-1 of the algorithm, the matrix \( X_{k+1} \) has the same upper triangular form as \( X_k \) (with the same sizes and positions of blocks, but with possibly new matrices in those blocks), then \( X_k \) is stationary.

**Proof.** In the \((k+1)\)-th iteration of the algorithm we compute the left eigenvector \( \mathbf{u}_k \) of the matrix \( X_k \) given by formula (19) and solve the problem \( \|X - A\| \rightarrow \min, \mathbf{u}_k^TX \leq \mathbf{u}_k^TA, X \geq 0 \). This problem is solved separately in each column: \( \|X_j - A_j\| \rightarrow \min, (\mathbf{u}_k, \mathbf{x}_j) \leq u_{k,j} \). For \( j = d - m + 1, \ldots, d \) we already have an optimal solution \( \mathbf{x}_j = \mathbf{a}_j \), which will not change, because it is unique. If the first \( m \) columns of \( X_{k+1} \) are concentrated in the block \((1,1)\), then \( X_{k+1}^{(1,1)} = X_k^{(1,1)} \), because \( X_{k}^{(1,1)} \) is the solution for this block obtained in the
previous iteration. Thus, $X_k = X_{k+1}$. On the other hand, in the next iteration we will have $v_{k+1} = v_{k-1}$, and hence again $X_{k+2} = X_{k+1}$. Thus, the matrix stays the same for two iterations in a row, hence it is stationary.

\[ \Box \]

**Proposition 4** If in the case Case Red-2 of the algorithm, both $X_k^{(1,1)}$ and $X_k^{(2,2)}$ are local minima for their problems, then $X_k$ is a local minimum for the original problem. If they both stationary for their problems, then $X_k$ is stationary for the original problem.

**Proof.** We prove the first part (for the local minima); the proof for stationary matrices is the same. It suffices to consider the case when the matrices $X_k^{(1,1)}$ and $X_k^{(2,2)}$ are both irreducible. If one of them is reducible, then we prove the same argument to it and the proposition follows by induction in the dimension. Adding an arbitrary nonzero matrix $\Delta$ multiplied with small $t > 0$ such that $X_k + t\Delta$ is an admissible matrix. This means $X_k + t\Delta \geq 0$ and $\rho(X_k + t\Delta) \leq 1$. Denote by $\Delta^{(i,j)}$ the corresponding blocks of the matrix $\Delta$. Since $\Delta$ is admissible, it follows that $\Delta^{(2,1)} \geq 0$. If $\Delta^{(2,1)} = 0$, then $\|X_k + t\Delta - A\| \leq \|X_k - A\|$, whenever $t$ is small enough, and hence for variations $\Delta$ with $\Delta^{(2,1)} = 0$ the matrix $X_k$ is a local minimum. Indeed, if $\Delta^{(2,1)} = 0$, then the spectral radii of both blocks $(1, 1)$ and $(2, 2)$ of the matrix $X_k + t\Delta$ do not exceed one (because its spectral radius is equal to the maximal spectral radius of those two blocks). Hence adding $t\Delta$ with a small $t$ do not reduce both $\|X_k^{(1,1)} - A^{(1,1)}\|$ and $\|X_k^{(2,2)} - A^{(2,2)}\|$, because $X_k^{(1,1)}$ and $X_k^{(2,2)}$ are both local minima. The value $\|X_k^{(1,2)} - A^{(1,2)}\|$ cannot be reduced either, because this is zero. Therefore, it remains to consider the case $\Delta^{(2,1)} \neq 0$. Denote by $\tilde{\Delta}^{(i,j)}$ the block $(i, j)$ extended by zeros to the whole matrix $\Delta$. Since $X^{(1,1)}$ and $X^{(2,2)}$ are both irreducible and $A^{(1,2)} \neq 0$ (otherwise $A$ is reducible), we have

\[
\rho(X_k + t \tilde{\Delta}^{(2,1)}) = 1 + C \sqrt{t} + O(t) \quad \text{as} \quad t \to 0,
\]

where $C > 0$. On the other hand,

\[
\rho(X_k + t \tilde{\Delta}^{(i,j)}) - 1 = O(t(\|\Delta^{(i,j)}\|)),
\]

for every $(i, j) \neq (2, 1)$, because the spectral radius is differentiable with respect to $\Delta^{(i,j)}$ for every $(i, j) \neq (2, 1)$. Therefore, $\|\Delta^{(i,j)}\| \geq C/\sqrt{t}$. On the other hand, $A^{(2,1)} \neq 0$ and hence

\[
\|X_k + t \tilde{\Delta}^{(2,1)} - A\|^2 = O(t),
\]

while in the other three blocks the square of the distance to $A$ increases at least as $C_0 \sqrt{t}$, where $C_0$ is a constant. Hence $\|X_k + t \Delta - A\| \geq \|X_k - A\|$.

\[ \Box \]

**Proof of Theorem** Assume $X$ is irreducible. Then the eigenvalue 1 is simple, the leading eigenvectors $u, v$ are well-defined up to multiplication by positive constants, and Corollary yields

\[
\begin{align*}
X & = A - \ell_1 v^T + \Lambda_1 \\
X & = A - u \ell_2^T + \Lambda_2
\end{align*}
\]
for some vectors \( \ell_i \geq 0 \) and matrices \( \Lambda_i \geq 0 \), \( \Lambda_i \perp X \) \( i = 1, 2 \).

Since in the support of the matrix \( X \) we have \( \Lambda_1 = \Lambda_2 = 0 \), it follows that \( \ell_1 v^T = u \ell_2^T \) on the support of \( X \). If \( X \) is primitive, we apply Lemma 4 and conclude that \( \ell_1 = ru \) for some \( r > 0 \). Hence \( X = A - ru v^T + \Lambda_1 \), which in view of Proposition 1 implies that \( X \in \text{locmin} \). This completes the proof for primitive \( X \).

If \( X \) is imprimitive, then we transfer the matrix \( X \) to the cyclic block form \( \mathbf{S} \). Respectively, the vector \( v \) is split into \( r \) blocks \( v = (v_1, \ldots, v_r) \), where \( v_i = v |_{\Omega_i} \), and the same for \( u = (u_1, \ldots, u_r) \), where \( u_i = u |_{\Omega_i} \). Similarly to the primitive case, we show that \( A(i) - X(i) = \mu_i u_{i+1} v_i^T \), on the support of \( X(i) \), where \( \mu_i \) are some multipliers, \( i = 1, \ldots, r \). From Corollary 3 it follows that \( (X(i), \mu_i u_{i+1} v_i^T) \) is the same for all \( i \). This scalar product is equal to the trace of the matrix \( \mu_i X(i)^T u_{i+1} v_i^T = \mu_i u_i v_i^T = \mu_i (u_i, v_i) \). We used the fact that \( X(i)v_i = v_{i+1} \) and \( u_{i+1}^T X(i) = u_i^T \). On the other hand, \( u_{i+1}^T v_{i+1} = u_{i+1}^T X(i)v_i = u_i^T v_i \). Thus, all the scalar products \( (u_i, v_i) \), \( i = 1, \ldots, r \) are equal. Therefore, all the numbers \( \mu_i \) are equal, hence \( X = A - ru v^T - \Lambda \), and so \( X \in \text{locmin} \).

\[ \square \]

**4.7. Convergence of the algorithm**

In the previous subsection we showed that if the Algorithm stabilizes, then the point of stabilization is a stationary point. Since the value \( \|X_k - A\| \) decreases in \( k \) and bounded below, it converges as \( k \to \infty \). In general, however, it does not imply that the algorithm converges. Theorem 6 below claims that the Algorithm indeed converges to a stationary point and, moreover, the rate of convergence is at least linear. However, it may not converge to a local minimum. For instance, if it starts at a stationary primitive matrix \( X_0 \), then it stays at \( X_0 \) forever and stabilizes after the first iteration. Say, consider Example 4 If \( A \) is 2 matrix with all entries equal to two and \( X_0 = \frac{1}{4} A \). Then \( X_0 \) is a stationary point and \( X_k = X_0 \) for all \( k \). Hence, the Algorithm converges to \( X_0 \), although \( X_0 \) is not a local minimum. Of course, this situation is not generic and a small variation of \( X_0 \) may lead to the convergence to a local minimum. In practice, because of roundings, tolerance parameters, etc. such small variations occur in each iterations. Hence, we can define the following notion of stable convergence.

**Definition 3** Assume the Algorithm converges to a matrix \( X \). This convergence is called stable (or \( X_k \) steadily converges to \( X \)) if there is a number \( \varepsilon > 0 \) and a number \( N \in \mathbb{N} \) such that for every \( k > N \), the Algorithm starting with a matrix \( X_k \) such that \( \|X_k - X_k\| < \varepsilon \) converges to the same matrix \( X \).

In fact, the stable convergence already implies that the limit point is a local minimum.

**Proposition 5** If the convergence is stable, then \( X \in \text{locmin} \).
Proof. Assume $X_k$ steadily converges to $X$ and $\|A - X\| = r$, but $X$ is not a local minimum. In this case, we can move $X$ to a distance at most $\varepsilon/2$ so that the distance $\|X - A\|$ decreases by some number $\delta > 0$. This means that for all sufficiently big $k$, we have $\|X_k - A\| < r$. Since the convergence is stable, the algorithm starting at $X_k$ has to converge to the same limit $X$. However, this is impossible, because the distance to $A$ does not increase each iteration, but finally must increase from $\|X_k - A\|$ to $r$.

We denote $f(X) = \|X - A\|^2$. The following lemma, whose proof is outlined to Appendix, plays a key role in the proof of convergence.

**Lemma 5** For every $k$, we have $\|X_k - X_{k-1}\|^2 \leq f(X_{k-1}) - f(X_k)$.

**Theorem 6** For arbitrary $A$, and for an arbitrary choice of the initial matrix $X_0$, the algorithm converges to a stationary point $X$ (which may depend on $X_0$) with the linear rate. This means that there are constants $q \in (0, 1)$ and $C > 0$ such that $\|X_k - X\| \leq C q^k$, $k \in \mathbb{N}$. If the convergence is stable, then $X \in \text{locmin}$.

Proof. First we show that each limit point of the sequence $\{X_k\}_{k \in \mathbb{Z}}$ is a point of local minimum. Then we prove that this sequence converges to that limit point with a linear rate. It suffices to consider the case when the limit point is a primitive matrix, the other cases are reduced to this one by the same argument as in the proof of Theorem 5.

Applying Lemma 5 and the fact that the sequence $f(X_k)$ has a limit as $k \to \infty$, we see that $\|X_k - X_{k-1}\| \to 0$ as $k \to \infty$. By compactness, the sequence $\{X_k\}$ has a limit point $X$. We assume $X$ is primitive. Let $v$ and $u$ be the right and the left leading eigenvectors of $X$. They are both strictly positive. For an arbitrary small $\varepsilon > 0$ and for an arbitrary large $M \in \mathbb{N}$, there is a number $m$ such that $\|X_{k-1} - X\| < \varepsilon$ for all $k = m, \ldots, m + 2M$. Taking $\varepsilon$ small enough, we obtain that, on the support of $X$, the values $\|X - A + \ell_k v_{k-1}^T\|$ and $\|X - A - \ell_k v_{k-1}^T\|$ are both small and $\|v_{k-1} - v\|$ and $\|u_{k-1} - u\|$ are both small for all $k = m, \ldots, m + 2M$. Lemma 4 and the primitivity of $X$ imply that $\|\ell_k - u\|$ and $\|\ell_k - v\|$ are both small as well. Hence, $\Lambda_k$ has a limit $\Lambda$ as $k \to \infty$ and $\Lambda \perp X$. Thus, $X = A - r u v^T$ on the support of $X$. Hence, $X = A - r u v^T + \Lambda$. Thus, $X$ is a stationary point.

Now we show that $\|X_k - X\| \leq C q^k$ for some $q \in (0, 1)$ and $C$. If $\varepsilon$ is smaller than the smallest positive entry of $X$, then $\text{supp } X \subset \text{supp } X_k$ for all $k = 1, \ldots, i + N$. Denote by $\bar{A}$ and $\bar{X}_k$ the restrictions of those matrices to $\text{supp } X$, i.e., we put all other entries of those matrices equal to zero. Similarly, for each $i$, we denote $\bar{a}_i$ and $\bar{v}_i$ is the restriction of $a_i$ to $\text{supp } x_i$. The next matrix $X_{k+1}$ is defined from the problem

$$\begin{align*}
\|\bar{x}_i - \bar{a}_i\|^2 &\to \min \\
(\bar{x}_i, \bar{v}_i) &= v_i \quad (26)
\end{align*}$$

For the solution, we have $\bar{x}_i - \bar{a}_i = -\ell_i \bar{v}_i^T$. The extra Lagrangian term $\Lambda_i$ vanishes, since $\bar{x}_i$ does not have zeros on the support. Multiplying by $\bar{v}_i$, we get $$(\bar{x}_i, \bar{v}_i) - (\bar{a}_i, \bar{v}_i) = -\ell_i \|\bar{v}_i\|^2.$$
where \( \| \bar{v}_i \|^2 = \sum_{(x_i)_n > 0} v_m^2 \). Since \((\bar{x}_i, \bar{v}_i) = v_i, (\bar{a}_i, \bar{v}_i) = (\bar{a}_i, v)\), we have

\[
(\bar{a}_i, \bar{v}_i) - v_i = -\ell_i\|\bar{v}_i\|^2,
\]

Therefore,

\[
\ell_i = \frac{(\bar{a}_i, \bar{v}_i) - v_i}{\|\bar{v}_i\|^2} = \frac{[ (\bar{A} - I) v ]_i}{\|\bar{v}_i\|^2}
\] (27)

Define the \( d \times d \) matrix \( B \) as follows: the \( i \)th row of \( B \) is equal to the \( i \)th row of the matrix \( \bar{A} - I \) divided by \( \|\bar{v}_i\|^2 \). Since, as we have shown above, \( \ell = u + o(1) \) as \( k \to \infty \), equality (26) yields \( u_{k+1} = B v_k + o(1) \). Similarly, defining the matrix \( C \): the \( j \)th row of \( C \) is equal to \( j \)th column of \( \bar{A} - I \) divided by \( \|\bar{u}_i\|^2 \), we obtain \( v_{k+2} = C^T u_{k+1} + o(1) \). Iterating we get \( v_{k+2} = C^T B v_k + o(1) \). Note that \( B \) and \( C \) are both independent of \( k \). Again assuming that \( \epsilon \) is small enough we obtain that the distance between \( v_{m+2M} \) and \( (C^T B)^M v_m \) is small. Taking \( m \) and \( M \) large enough we see that \( v \) is an eigenvector of the matrix \( C^T B \) corresponding to its eigenvalue 1 and that all other eigenvalues of this matrix restricted to its eigenspace containing all corresponding vectors \( v_k \) is smaller than one by modulo. If \( q \) is the biggest modulus of those eigenvalues, then \( q < 1 \) and \( \|v_k - v\| \leq C q^k, k \in \mathbb{N} \). Arguing similarly for \( u_k \) and taking into account that the matrix \( BC^T \) has the same eigenvalues, we conclude that \( \|u_k - u\| \leq C q^k, k \in \mathbb{N} \).

Thus, both \( v_k \) and \( u_k \) converge to \( v \) and \( u \) respectively with the linear rate as \( k \to \infty \). Invoking now Corollary 3 and Lemma 4 we see that \( \ell_k \to u_k \) with the same rate, and hence \( X_k \) converges linearly to \( X \).

\[ \square \]

**Remark 6** In the proof we see that the rate of linear convergence, i.e., the constant \( q \), is determined by the eigenvalues of the matrix \((C^T B)\). If the convergence is stable, then the rate is the ratio between the first and the second largest eigenvalues of this matrix. When the descent of the function \( \|X_k - A\| \) becomes very small, we can compute approximations for \( B \) and \( C \) and hence, can estimate the \( q \).

### 4.8. A favorable case: convergence to a positive matrix

Denote \( r_k = \|X_k - A\| \). If the \( k \)th matrix \( X_k \) in the Algorithm is strictly positive, then all formulas are simplified. Assume \( k \) is odd (for even \( k \) the situation is similar); then in Corollary 3 we have \( \Lambda_k = 0 \) and therefore,

\[
X_k = A - \ell_k v_{k-1}^T
\] (28)

where \( \|\ell_k\| = r_k \). Indeed, the eigenvector \( v_{k-1} \) is normalized to have the unit length, hence \( \|u_k \ell_{k+1}^T\| = \|u_k\| \|\ell_{k+1}\| = \|\ell_{k+1}\| \).
Proposition 6 If $X_k > 0$ for some $k$, then
\[ r_k = \begin{cases} \| (I - A) v_{k-1} \|, & k \text{ is odd} \\ \| (I - A^T) u_k \|, & k \text{ is even} \end{cases} \quad (29) \]

Proof. Multiplying both parts of equality (28) by $v_{k-1}$ from the right, we get
\[ X_k v_{k-1} = A v_{k-1} - \ell_k (v_{k-1}, v_{k-1}) = A v_{k-1} - \ell_k. \]
On the other hand, $X_k v_{k-1} = v_{k-1}$. Hence, $(I - A) v_{k-1} = -\ell_k$. Consequently, $r_k = \| \ell_k \| = \| (I - A) v_{k-1} \|$. For even $k$ the proof is the same. \hfill \Box

Proposition 7 If during $2n + 1$ iterations of the Algorithm, all matrices $X_0, \ldots, X_{2n}$, are strictly positive, then
\[ v_{2n} = a_{2n} M^{-n} v_0; \quad u_{2n+1} = a_{2n+1} N^{-n} u_1 \quad (30) \]
where $a_{2n}, a_{2n+1}$ are normalizing constants.

Proof. Writing (28) for $k$ and $k + 1$ we get the system
\[ \begin{cases} X_k = A - \ell_k v_{k-1}^T \\ X_{k+1} = A - u_k \ell_{k+1}^T \end{cases} \quad (31) \]
Multiplying the first equation by $u_k^T$ from the left and keeping in mind that $u_k$ is the left leading eigenvector for $X_k$, we obtain $u_k^T = u_k^T A - (u_k, \ell_k) v_{k-1}^T$. Therefore
\[ \mu_k v_{k-1} = (A^T - I) u_k, \quad (32) \]
where $\mu_k = (u_k, \ell_k)$. Similarly, multiplying the second equation of (31) from the right by $v_{k+1}$ we get $v_{k+1} = A v_{k+1} - u_k (\ell_{k+1}, v_{k+1})$, and hence
\[ \mu_{k+1} u_k = (A - I) v_{k+1}, \quad (33) \]
where $\mu_{k+1} = (\ell_{k+1}, v_{k+1})$. Substituting $u_k$ from (32) to (33) we obtain $\mu_k \mu_{k+1} v_{k-1} = (A^T - I)(A - I) v_{k+1}$. Therefore, $\mu_k \mu_{k+1} v_{k-1} = (A^T - I)(A - I) v_{k+1} = M v_{k+1}$. Thus, $v_{k+1} = c_k M^{-1} v_{k-1}$, where $c_k$ is a constant. Applying this equality successively for $k = 1, 3, \ldots, 2n - 1$ we prove the first assertion in (30). The second one is established in the same way. \hfill \Box

Theorem 7 If the Algorithm steadily converges to a strictly positive matrix $X$, then this matrix is a point of global minimum and is explicitly constructed by Theorem 3. Moreover, in this case
\[ \| X_n - X \| \leq C \left( \frac{\sigma_1}{\sigma_2} \right)^n, \]
where $\sigma_1, \sigma_2$ are the smallest and the second smallest singular value respectively of the matrix $I - A$. 26
Proof. If the limit matrix \( X \) is strictly positive then all the matrices \( X_k \) are positive for sufficiently large \( k \). Hence, we may assume that the Algorithm starts with a positive matrix \( X_0 \) and produces only positive matrices. Proposition 7 implies that \( v_{2n} = a_{2n} M^{-n} v_0 \).

Since the convergence is stable the vector \( v_{2n} \) tends to a vector \( v \), which is an eigenvector of \( M^{-1} \) corresponding to its largest eigenvalue, i.e., an eigenvector of \( M \) corresponding to its smallest eigenvalue. Similarly, \( u_{2n+1} \to u \), where \( u \) is an eigenvector of \( N \) corresponding to its smallest eigenvalue. However, \( X_k \to X \), hence \( u \) and \( v \) are left and right eigenvectors of \( X \) respectively. Since each stationary point has the form \(~\), the matrix \( X \) has the same form with that \( u \) and \( v \) and with \( \Lambda = 0 \), because \( X > 0 \). Thus, \( X = A - r u v^T \). We see that all assumptions of Theorem \( \star \) are satisfied, hence \( X \) is point of global minimum.

Remark 7 The condition of stability of the convergence can not be omitted. For instance, if \( d \geq 3 \) and \( A = E \) (the matrix of ones), then the matrix \( X_0 = \frac{1}{d} E \) is a stationary point, although not a local minimum (Example \( \star \)). The Algorithm starting at \( X_0 \) immediately stabilizes on this matrix, i.e., \( X_k = X_0 \) for all \( k \), hence it converges to \( X_0 \). We see that the Algorithm may converge to a positive matrix which is not a local minimum. Nevertheless, this convergence is unstable, and a small perturbation of the matrix \( X_k \) in some iteration leads to a different limit. That is why in practice the Algorithm converges to a local minimum. This is natural in view of Proposition 5.

5. How many local minima can occur?

Consider an arbitrary ordered partition of the set \( \Omega = \{1, \ldots, d\} \) to \( m \) nonempty subsets \( \{\Omega_1, \ldots, \Omega_m\} \), where \( 1 \leq m \leq d \). Denote \( d_j = |\Omega_j| \). For an arbitrary \( d \times d \) matrix \( X \), we denote by \( X^{(j)} \) the corresponding \( d_i \times d_j \) block in the intersections of rows from \( \Omega_i \) and of columns from \( \Omega_j \).

Let a non-negative matrix \( A \) be fixed. For an arbitrary non-negative matrix \( X \), we formulate the following properties:

1) \( X \) coincides with \( A \) above the diagonal blocks and is zero below them;
2) for each \( j = 1, \ldots, m \), the \( j \)th diagonal block \( X^{(j,j)} \) is a stable non-negative matrix locally closest to \( A^{(j,j)} \).
3) for each \( j = 1, \ldots, m \), the \( j \)th diagonal block \( X^{(j,j)} \) is the closest stable non-negative matrix to \( A^{(j,j)} \).

In items 2) and 3) closeness is in the set of \( d_j \times d_j \) matrices. Of course, property 3) is stronger than 2).
Property 1) requires that $X$ is a block upper triangular matrix with blocks corresponding to the partition of $\Omega$ and coincides with $A$ above the block diagonal. So, the matrix $X$ is uniquely defined out of the diagonal blocks by property 1). The diagonal blocks are not defined uniquely even if 3) is satisfied. Property 2) implies that $\rho(X) = \max_{j=1,\ldots,m} \rho(X^{(j,j)}) \leq 1$.

**Proposition 8** Let a matrix $A$ and an ordered partition $\{\Omega_i\}_{i=1}^m$ be given. If $A$ is strictly positive and $\rho(A^{(j,j)}) > 1$ for all $i = 1, \ldots, m$, then every $X$ satisfying 1) and 2) is a locally closest stable non-negative matrix to $A$.

**Proof.** This proposition follows from Proposition 4 directly by applying induction in the number of blocks $m$. □

We call a matrix $A$ lower dominant if $A_{ij} > A_{ji}$ whenever $i > j$. In other words, each component of $A$ below the main diagonal is bigger than its reflection above the diagonal.

**Proposition 9** Let a matrix $A$ be strictly positive, lower dominant, and have all its diagonal entries bigger than one. Then for an arbitrary ordered partition $\{\Omega_i\}_{i=1}^m$, every matrix $X$ satisfying 1) and 3) is a locally closest non-negative stable matrix for $A$. Moreover, those matrices $X$ are different for different partitions.

**Proof.** Since $a_{ii} > 1$ for all $i$, it follows that the diagonal blocks satisfy $\rho(A^{(i,i)}) > 1$ for any partition of $\Omega$. Hence, Proposition 8 implies that $X$ is a locally closest stable non-negative matrix to $A$. It remains to show that all those matrices are different for different partitions. Assume that the same matrix $X$ corresponds to a different partition $\{\Omega'_i\}_{i=1}^m$. Then either one of the sets $\Omega_j$ is split by the partition $\{\Omega'_i\}_{i=1}^m$ into several parts, or one of the sets $\Omega'_j$ is split by the partition $\{\Omega_i\}_{i=1}^m$. Assume the first case (the second one is considered in the same way). In this case, the matrix $X^{(j,j)}$ is block upper triangular, according to the corresponding part of the partition $\{\Omega'_i\}_{i=1}^m$ that splits the set $\Omega_j$. Thus, the matrix $X^{(j,j)}$ is zero below the diagonal blocks and coincides with $A^{(j,j)}$ above them. Denote by $\tilde{X}^{(j,j)}$ the matrix with the same diagonal blocks as $X^{(j,j)}$ but equal to zero above the diagonal blocks and equal to $A^{(j,j)}$ below them. Clearly, $\rho(\tilde{X}^{(j,j)}) = \rho(X^{(j,j)}) = 1$. Since $A$ is lower dominant, so is $A^{(j,j)}$, and hence $\|\tilde{X}^{(j,j)} - A^{(j,j)}\| < \|X^{(j,j)} - A^{(j,j)}\|$. Hence, $X^{(j,j)}$ is not the closest stable matrix to $A^{(j,j)}$, which contradicts to property 3) in the assumption. □

Thus, for every matrix $A$ satisfying the assumptions of Proposition 8 each ordered partition of the set $\{1, \ldots, d\}$ generates its own local minimum of the problem 5) and they are different for different partitions. The total number of ordered partitions for a $d$-element set is equal to $2^d$, hence the problem 5) has at least $2^d$ different points of local minima. Thus, we come to the following conclusion, which justifies the complexity of the problem:

**Corollary 6** A strictly positive lower dominant matrix that has all diagonal entries bigger than one possesses at least $2^d$ locally closest stable non-negative matrices.
Example 5 For a $d \times d$ matrix that has all twos on the diagonal and below it and all ones above the diagonal, problem (5) has at least $2^d$ local minima.

6. Positive Hurwitz stability

All our results can be modified to the Hurwitz stability of positive systems in a straightforward manner. We will describe the main constructions without penetrating the details.

We recall that a matrix is Hurwitz stable if its spectral abscissa (the maximal real part of eigenvalues) is negative. A matrix is called Metzler if all its off-diagonal elements are non-negative.

Since positive linear systems are defined by Metzler matrices, the corresponding problem are formulated as follows: find the closest Hurwitz stable/unstable Metzler matrix to a given matrix $A$.

First of all, the problem of finding the closest stable Metzler matrix to a matrix $A$ can be reduced to the case when $A$ is Metzler. Otherwise we make the same trick as in the last paragraph of the Introduction for non-negative matrices: we define the matrix $A_{\text{Metz}}$ entrywise: on the diagonal $A_{\text{Metz}} = A$, and off the diagonal $A_{\text{Metz}} = \max \{ A, 0 \}$. Thus, $A_{\text{Metz}}$ is a Metzler matrix. Then it is shown easily that the matrices $A$ and $A_{\text{Metz}}$ have the same closest Hurwitz stable Metzler matrix.

The following analogue of the Perron-Frobenius theorem takes place for Metzler matrices: the maximal spectral abscissa of a Metzler matrix is always attained at a real eigenvalue with a non-negative eigenvector (leading eigenvector). That is why, the Algorithm presented in Section 4 is naturally modified for computing the closest Hurwitz stable Metzler matrix. In equations (12) the inequality constraints $Xv_0 \leq v_0$ and $X \geq 0$ are replaced by $Xv_0 \leq 0$ and $x_{i,j} \geq 0$, $i \neq j$, respectively. The same for equations (13) and for all iterations of the algorithm. All convergence results from Section 4 stay the same and the example from Section 5 is also easily modified for the Hurwitz stability problem.

The closest Hurwitz unstable Metzler matrix is found by the explicit formula $X = A + ruv^T$, where $v$ and $u$ are the eigenvectors of the matrices $M = A^T A$ and $N = AA^T$ corresponding to their smallest eigenvalues.

Example 6 We apply our modified algorithm to the following matrix considered in [1]:

$$A = \begin{pmatrix} 0.6470 & 0.1720 & -0.7490 & 0.7280 & 0.7170 \\ -0.3540 & -0.0620 & -0.9360 & -0.7730 & -0.7780 \\ 0.0460 & 1.1990 & -1.2690 & 0.8370 & 0.3160 \\ -0.7930 & 0.8020 & 0.4980 & -1.1280 & 1.4070 \\ -1.5510 & 1.0530 & 2.7890 & -1.4250 & 0.4010 \end{pmatrix}$$

The matrix is not Metzler and is unstable since it has 3 eigenvalue in the right complex half-plane and spectral abscissa $\alpha(A) \approx 0.5317$. Aiming to compute the closest stable Metzler
matrix to $A$, Anderson found the matrix

$$X_{\text{Anderson}} = \begin{pmatrix}
-0.0590 & 0.1700 & 0.0030 & 0.6650 & 0.6552 \\
0 & -0.1730 & 0.0300 & 0 & 0 \\
0 & 1.1800 & -1.3160 & 0.0080 & 0 \\
0 & 0.8010 & 0.4950 & -1.1780 & 1.3570 \\
0 & 1.0400 & 2.7560 & 0 & -0.1830
\end{pmatrix}$$

whose eigenvalues are all contained in the left complex half-plane and whose spectral abscissa is $-0.0590$. The distance $\|A - X_{\text{Anderson}}\|_F \approx 9.485$. The algorithm proposed by Anderson makes use of the theory of dissipative Hamiltonian systems, which provide a helpful characterization of the feasible set of stable matrices.

Applying our algorithm yields instead the matrix

$$X^* = \begin{pmatrix}
0 & 0.1720 & 0 & 0.7280 & 0.7170 \\
0 & -0.0620 & 0 & 0 & 0 \\
0 & 1.1990 & -1.3444 & 0.1114 & 0 \\
0 & 0.8020 & 0.4910 & -1.1956 & 1.2779 \\
0 & 1.0530 & 2.7535 & 0 & -0.2525
\end{pmatrix}$$

which is quite different from $A_{\text{Anderson}}$. Its whose eigenvalues are still contained in the left complex half-plane and its spectral abscissa is $0$. The distance $\|A - X^*\|_F \approx 9.332$, which slightly improves the bound from \cite{1}.

**Example 7** We next apply our algorithm to the following randomly generated Metzler matrix:

$$A = \begin{pmatrix}
0.5700 & 0.4900 & 0.4700 & 0.7300 & 0.0500 & 0.0200 \\
0.1400 & -1.1300 & 0.9600 & 0.6700 & 0.3200 & 0.9100 \\
0.9100 & 0.4500 & -1.7000 & 0.9800 & 0.6000 & 0.1100 \\
0.8000 & 0.6000 & 0.0400 & 0 & 0.5200 & 0.1400 \\
0.4800 & 0.5400 & 0.7700 & 0.3600 & -1.0200 & 0.4600 \\
0.4300 & 0.3300 & 0.9200 & 1.0000 & 0.7600 & 0.0700
\end{pmatrix}$$

The matrix is very unstable: its spectral abscissa is $\alpha(A) \approx 2.1425$.

Applying the first step of our algorithm yields the matrix

$$X_1 = \begin{pmatrix}
-0.4074 & 0 & 0 & 0 & 0 \\
0 & -1.2885 & 0.7655 & 0.1879 & 0.0782 & 0.6328 \\
0.6033 & 0.2551 & -1.9391 & 0.3873 & 0.3028 & 0 \\
0.4651 & 0.3872 & 0 & -0.6471 & 0.1955 & 0 \\
0.2254 & 0.3782 & 0.5716 & 0 & -1.2666 & 0.1772 \\
0 & 0.0376 & 0.5613 & 0.1108 & 0.3141 & -0.4412
\end{pmatrix}$$
whose eigenvalues are all contained in the left complex half-plane and whose spectral abscissa is 0. However the matrix is reducible so that we can further optimize it and get

\[
X^* = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 \\
0.1400 & -1.2885 & 0.7655 & 0.1879 & 0.0782 & 0.6328 \\
0.9100 & 0.2551 & -1.9391 & 0.3873 & 0.3028 & 0 \\
0.8000 & 0.3872 & 0 & -0.6471 & 0.1955 & 0 \\
0.4800 & 0.3782 & 0.5716 & 0 & -1.2666 & 0.1772 \\
0.4300 & 0.0376 & 0.5613 & 0.1108 & 0.3141 & -0.4412 \\
\end{pmatrix}
\]

whose eigenvalues are still contained in the left complex half-plane and whose distance from \( A \) is improved to \( \| A - X^* \|_F^2 \approx 4.690 \).

### Appendix

**Proof of Lemma 4.** We call two indices \( i, j \in \Omega = \{1, \ldots, d\} \) equivalent if the ratio \( a_i : a_j \) is uniquely defined by the equation \( a b^T = C \) on the set \( \text{supp} \ X \). Thus, the whole set \( \Omega \) is split into several equivalence classes \( \Omega_1, \ldots, \Omega_r \). Denote \( X(\Omega_k) = \{ j \mid \exists i \in \Omega_k (i, j) \in \text{supp} \ X \} \). Since for each \( i \in \Omega_k \), we have \( a_j b_i = Y_{ij}, (i, j) \in \text{supp} \ X \), it follows that the ratios of all \( a_j \) for all \( j \in \text{supp} \ X \) are uniquely defined, hence, the ratios of all \( a_j, j \in \Omega_k \) are uniquely defined as well. Therefore, the sets \( X(\Omega_k), k = 1, \ldots, r \) are just some permutation of the sets \( \Omega_1, \ldots, \Omega_r \). Thus, the matrix \( X \) and all its powers define permutations of those sets, and hence \( X \) cannot be primitive, unless \( r = 1 \). Consequently, the ratios of all entries of the vector \( a \) is uniquely defined, and hence \( \sim a \). The same with \( b \sim b \).

\[ \square \]

**Proof of Lemma 5.** We prove this inequality in each row. For every \( i = 1, \ldots, d \), we denote by \( x_{i,k-1} \) and \( x_{i,k} \) the \( i \)th rows of \( X_{k-1} \) and \( X_k \) respectively. We are going to show that

\[
\| x_{i,k} - x_{i,k-1} \|_2^2 \leq \| x_{i,k-1} - a_i \|_2^2 - \| x_{i,k} - a_i \|_2^2
\]

and then take the sum of those inequalities over \( i = 1, \ldots, d \). Geometrically, \( (34) \) means that the angle \( \angle x_{i,k} x_{i,k-1} a_i \) is not acute. Invoking equation \( (22) \) we see that either \( x_{k,i} = a_i \), in which case \( (34) \) is obvious, or \( x_{k,i} = a_i - \lambda v_{k-1} + \Lambda_i \) where \( \lambda > 0 \) and \( \Lambda_i \perp x_{k,i} \). We have

\[
(a_i - x_{k,i}, x_{k-1,i} - x_{k,i}) = (\lambda v_{k-1} - \Lambda_i, x_{k-1,i} - x_{k,i}) = \lambda (v_{k-1}, x_{k-1,i} - x_{k,i}) - (\Lambda_i, x_{k-1,i})
\]

Note that \( (v_{k-1}, x_{k-1,i}) = (v_{k-1}, x_{k,i}) = (v_{k-1})_i \), therefore

\[
(a_i - x_{k,i}, x_{k-1,i} - x_{k,i}) = - (\Lambda_i, x_{k-1,i}) \leq 0,
\]

since the vectors \( x_{k-1,i} \) and \( \Lambda_i \) are both non-negative. This means that \( \angle x_{i,k} x_{i,k-1} a_i \geq 90^\circ \), which completes the proof.

\[ \square \]
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