On higher-power moments of $\Delta(x)$ (III)

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Abstract

Let $\Delta(x)$ be the error term of the Dirichlet divisor problem. An asymptotic formula with the error term $O(T^{\frac{53}{28}+\varepsilon})$ is established for the integral $\int_1^T \Delta^4(x)dx$. Similar results are also established for some other well-known error terms in the analytic number theory.

1 Introduction and main results

Let $d(n)$ denote the Dirichlet divisor function and $\Delta(x)$ denote the error term of the sum $\sum_{n \leq x} d(n)$ for a large real variable $x$. Dirichlet first proved that $\Delta(x) = O(x^{1/2})$. The exponent $1/2$ was improved by many authors. The latest result reads

$$\Delta(x) \ll x^{131/416} (\log x)^{26947/8320}, \quad (1.1)$$

proved by Huxley[3]. It is conjectured that

$$\Delta(x) = O(x^{1/4+\varepsilon}), \quad (1.2)$$

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which is supported by the classical mean-square result
\[
\int_1^T \Delta^2(x) dx = \frac{(\zeta(3/2))^4}{6\pi^2\zeta(3)} T^{3/2} + O(T \log^5 T) \tag{1.3}
\]
proved by Tong[10].

Tsang[11] studied the third- and fourth-power moments of \( \Delta(x) \). He proved that the asymptotic formulas
\[
\int_2^T \Delta^3(x) dx = \frac{3c_1}{28\pi^4} T^{7/4} + O(T^{7/4-\delta_1+\varepsilon}) \tag{1.4}
\]
and
\[
\int_2^T \Delta^4(x) dx = \frac{3c_2}{64\pi^4} T^2 + O(T^{2-\delta_2+\varepsilon}) \tag{1.5}
\]
hold, where \( \delta_1 = 1/14, \delta_2 = 1/23, \)
\[
c_1 := \sum_{\alpha,\beta,h \in \mathbb{N}} (\alpha\beta(\alpha + \beta))^{-3/2} h^{-9/4} |\mu(h)| d(\alpha^2 h) d(\beta^2 h) d((\alpha + \beta)^2 h),
\]
\[
c_2 := \sum_{n,m,k,l \in \mathbb{N}} \sqrt{n + \sqrt{m}} = \sqrt{k + \sqrt{l}} (nmkl)^{-3/4} d(n) d(m) d(k) d(l).
\]

Recently in [12] the author proved that (1.4) holds for \( \delta_1 = 1/4 \). Ivić and Sargos[7] proved that (1.4) holds for \( \delta_1 = 7/20 \). The author got this exponent independently. However, Professor Ivić kindly informed the author that the exponent \( \delta_1 = 7/20 \) had already been obtained by Professor Tsang several years ago but he had never published this result.

Following Tsang’s approach , in [12] the author proved that (1.5) holds for \( \delta_2 = 2/41 \). This approach used the method of exponential sums. Especially if the exponent pair conjecture is true, namely, if \( (\varepsilon, 1/2 + \varepsilon) \) is an exponent pair, then (1.5) holds for \( \delta_2 = 1/14 \). However, in [7] Ivić and Sargos ingeniously proved a substantially better result . They proved that (1.5) holds for \( \delta_2 = 1/12 \).

In this paper, combining the method of [7] and a recent deep result of Robert and Sargos[9], we shall prove the following

\textbf{Theorem 1.} We have
\[
\int_2^T \Delta^4(x) dx = \frac{3c_2}{64\pi^4} T^2 + O(T^{53/28+\varepsilon}). \tag{1.6}
\]

The theorem is also true for other error terms . Let \( P(x) \) denotes the error term of the Gauss circle problem , which is an error term similar to
Let \( a(n) \) be the Fourier coefficients of a holomorphic cusp form of weight \( \kappa = 2n \geq 12 \) for the full modular group and define

\[
A(x) := \sum_{n \leq x} a(n), \quad x \geq 2.
\]

We then have the following two corollaries, which improve the previous results([2], [11], [12]).

**Corollary 1.** We have

\[
\int_2^T P^4(x) dx = CT^2 + O(T^{53/28 + \varepsilon}). \tag{1.7}
\]

**Corollary 2.** We have

\[
\int_1^T A^4(x) dx = B_\kappa T^{2\kappa} + O(T^{2\kappa - 3/28 + \varepsilon}). \tag{1.8}
\]

Now let’s consider \( E(t) \), defined by

\[
E(t) := \int_0^t |\zeta(\frac{1}{2} + it)|^2 du - t \log(t/2\pi) - (2\gamma - 1)t, \quad t \geq 2. \tag{1.9}
\]

Tsang[11] also studied the fourth-power moment of \( E(t) \) by using Atkinson’s formula[1] and proved that

\[
\int_2^T E^4(t) dt = \frac{3}{8\pi} c_2 T^2 + O(T^{2 - \delta_3 + \varepsilon}) \tag{1.10}
\]

with some unspecified constant \( \delta_3 > 0 \).

Ivić[4] used a different way to study the higher power moments of \( E(t) \). The following is his approach. Let

\[
\Delta^*(x) := \frac{1}{2} \sum_{n \leq 4x} (-1)^n d(n) - x(\log x + 2\gamma - 1), \quad x \geq 1. \tag{1.11}
\]

Then for \( 1 \ll N \ll x \), we have[6]

\[
\Delta^*(x) = \frac{1}{\pi \sqrt{2}} \sum_{n \leq N} (-1)^n d(n)n^{-3/4}x^{1/4} \cos(4\pi \sqrt{nx} - \pi/4) + O(x^{1/2 + \varepsilon} N^{-1/2}) \tag{1.12}
\]
Jutila [8] proved that
\[
\int_0^T (E(t) - 2\pi \Delta^*(\frac{t}{2\pi}))^2 dt \ll T^{4/3} \log^3 T, \tag{1.13}
\]
which means that \(E(t)\) is well approximated by \(2\pi \Delta^*(\frac{t}{2\pi})\) at least in the mean square sense. From (1.13) Ivić [4] deduced that
\[
\int_0^T E^4(t) dt = (2\pi)^5 \int_0^T (\Delta^*(t))^4 dt + O(T^{23/12} \log^{3/2} T). \tag{1.14}
\]
Thus the fourth power moment of \(E(t)\) was transformed into the fourth power moment of \(\Delta^*(t)\), which can be dealt with in the same way as the fourth power moment of \(\Delta(x)\). By Tsang’s result [11], Ivić deduced from (1.14) that (1.10) holds for \(\delta_3 = 1/23\). In [7], Ivić and Sargos proved that one can take \(\delta_3 = 1/12\).

It is easy to see that 1/12 is the limit of this approach since it is the limit of Jutila’s result (1.13). In this paper, we shall use a different way to prove the following

**Theorem 2.** We have
\[
\int_2^T E^4(t) dt = \frac{3}{8\pi} c_2 T^2 + O(T^{53/28 + \varepsilon}). \tag{1.15}
\]

**Remark.** The proof of Theorem 2 doesn’t use (1.13) and it is actually a generalization of the approach used in the author [13]. In [14] the author used a similar method to study the third power moment of \(E(t)\).

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**Notations.** Throughout this paper, \([x]\) denotes the integer part of \(x\), \(\|x\|\) denotes the distance from \(x\) to the integer nearest to \(x\), \(n \sim N\) means \(N < n \leq 2N\), \(n \asymp N\) means \(C_1 N < n \leq C_2 N\) for positive constants \(C_1 < C_2\). \(\varepsilon\) always denotes a small positive constant which may be different at different places. We shall use the estimate \(d(n) \ll n^\varepsilon\) freely.

## 2 The spacing problem of the square roots

In the proofs of Theorem 1 and 2, the sums and differences of square roots will appear in the exponential. Thus we should study the spacing problem of the square roots.
We need the following Lemmas. Lemma 1 is a special case of a new result proved in Robert and Sargos [9], which also plays an important role in this paper. Lemma 2 is Lemma 3 of Tsang [11]. Lemma 3 provides an upper bound of the number of solutions of the inequality

\[ |n_1^{1/2} + n_2^{1/2} \pm n_3^{1/2} - n_4^{1/2}| < \Delta, \quad n_j \sim N_j (j = 1, 2, 3, 4), \quad (2.1) \]

where \( N_j \geq 2(j = 1, 2, 3, 4) \) are real numbers. Lemma 4 is essentially Lemma 3 of Ivić and Sargos [7], but we added the case \( \alpha \ll 1 \). Lemma 5 is essentially Lemma 5 of [7], but the term \( K \min(M, M', L) \) therein is superfluous since we add the condition \( |\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| > 0 \) in Lemma 5, and so we give a new proof here. Lemma 6 is Lemma 6 of [7].

**Lemma 1.** Suppose \( N \geq 2, \Delta > 0 \). Let \( A(N; \Delta) \) denote the number of solutions of the inequality

\[ |n_1^{1/2} + n_2^{1/2} - n_3^{1/2} - n_4^{1/2}| < \Delta, \quad n_j \sim N(j = 1, 2, 3, 4), \]

then

\[ A(N; \Delta) \ll (\Delta N^{7/2} + N^2) N^\varepsilon. \]

**Lemma 2.** If \( n, m, k, l \in \mathbb{N} \) such that \( \sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{l} \neq 0 \), then respectively,

\[ |\sqrt{n} + \sqrt{m} \pm \sqrt{k} - \sqrt{l}| \gg \max(n, m, k, l)^{-7/2}. \]

**Lemma 3.** Suppose \( N_j \geq 2(j = 1, 2, 3, 4), \Delta > 0 \). Let \( A_\pm(N_1, N_2, N_3, N_4; \Delta) \) denote the number of solutions of the inequality \( (2.1) \), then we have

\[ A_\pm(N_1, N_2, N_3, N_4; \Delta) \ll \prod_{j=1}^{4} (\Delta^{1/4} N_j^{7/8} + N_j^{1/2}) N_j^\varepsilon. \]

**Proof.** We shall use a combinatorial argument to prove this Lemma. Let \( \{a_i\} \) and \( \{b_i\} \) be two finite sequences of real numbers. Let \( \Delta > 0 \). Suppose \( u_0 \) and \( J \) (a positive integer) are chosen so that \( \{a_i\} \subset (u_0, u_0 + J\Delta], \{b_i\} \subset (u_0, u_0 + J\Delta] \). Divide this interval into the abutting subintervals \( I_j := (u_0 + j\Delta, u_0 + (j + 1)\Delta] \) for \( j = 0, 1, \ldots, J - 1 \) and then let

\[ N_j(A) := \#\{i : a_i \in I_j\}, N_j(B) := \#\{i : b_i \in I_j\}. \]
If $|a_r - b_s| \leq \Delta$, then either both $a_r$ and $b_s$ lie in the same subinterval $I_j$, or they lie in adjacent subintervals $I_j$ and $I_{j+1}$. Hence

\[
\sum_j N_j(A)N_j(B) + \sum_j N_j(A)N_{j+1}(B) + \sum_j N_{j+1}(A)N_{j}(B)
\]

by Cauchy-Schwarz's inequality. On the other hand, we have

\[
\sum_j N_j(A)^2 = \sum_j \#\{(r, r') : a_r, a_{r'} \in I_j\}
\]

and similarly for $\sum_j N_j(B)^2$. Thus we get the bound

\[
\#\{(r, s) : |a_r - b_s| \leq \Delta\} \leq \#\{(r, r') : |a_r - a_{r'}| \leq \Delta\}
\]

(2.2)

Suppose $n_j, n_j' \sim N_j(j = 1, 2, 3, 4)$. Applying (2.2) to the sequences $A = \{\sqrt{n_1 + n_2}\}$ and $B = \{\sqrt{n_3 + n_4}\}$, we get

\[
\mathcal{A}_-(N_1, N_2, N_3, N_4) = \#\{(n_1, n_2, n_3, n_4) : |n_1^{1/2} + n_2^{1/2} - n_3^{1/2} - n_4^{1/2}| \leq \Delta\}
\]

(2.3)

\[
\leq 3(\#\{(n_1, n_2, n_1', n_2') : |n_1^{1/2} + n_2^{1/2} - n_1'^{1/2} - n_2'^{1/2}| \leq \Delta\})^{1/2}
\]

\[
\times (\#\{(n_3, n_4, n_3', n_4') : |n_3^{1/2} + n_4^{1/2} - n_3'^{1/2} - n_4'^{1/2}| \leq \Delta\})^{1/2}
\]

Using the previous bound to the sequences $A_1 = \{n_1^{1/2} - n_1^{1/2}\}, B_1 = \{n_2^{1/2} - n_2^{1/2}\}, A_2 = \{n_3^{1/2} - n_3^{1/2}\}, B_2 = \{n_4^{1/2} - n_4^{1/2}\}$, respectively, we get

\[
\mathcal{A}_-(N_1, N_2, N_3, N_4) \leq 9 \prod_{j=1}^4 \mathcal{A}_-(N_j, N_j, N_j, N_j)^{1/4}
\]

(2.4)

which combining Lemma 1 gives Lemma 3 for the case "-". The proof for the case "+" is similar.
Lemma 4. Suppose $K \geq 10, \alpha, \beta \in \mathbb{R}, 2K^{-1/2} \leq |\alpha| \ll K^{1/2}$ and $0 < \delta < 1/2$. Then we have
$$\#\{k \sim K : \|\beta + \alpha \sqrt{k}\| < \delta\} \ll K\delta + K^{1/2+\varepsilon}.$$  

Proof. Without loss of generality, suppose $\alpha > 0$. Let $N = \#\{k \sim K : \|\beta + \alpha \sqrt{k}\| < \delta\}$. If $1 \ll \alpha \ll K^{1/2}$, from Lemma 3 of Ivić and Sargos[7] we get
$$N \ll K\delta + |\alpha|^{1/2}K^{1/4+\varepsilon} + K^{1/2+\varepsilon} \ll K\delta + K^{1/2+\varepsilon}.$$  

Now suppose $2K^{-1/2} \leq \alpha \ll 1$. Since $\|t\|$ is a periodic function with period 1, we suppose $0 < \beta < 1$. If $\|\beta + \alpha \sqrt{k}\| < \delta$, then there exists a unique $l \in [\alpha\sqrt{K}, 2\alpha\sqrt{K} + 2)$ such that
$$(l - \beta - \delta)^2/\alpha^2 < k \leq (l - \beta + \delta)^2/\alpha^2,$$
which implies
$$N \ll \sum_{l \sim \alpha\sqrt{K}} ((l - \beta + \delta)^2/\alpha^2 - (l - \beta - \delta)^2/\alpha^2) \ll \sum_{l \sim \alpha\sqrt{K}} ((l - \beta + \delta)^2/\alpha^2 - (l - \beta - \delta)^2/\alpha^2 + 1) \ll K\delta + K^{1/2}$$ if we notice $\alpha \ll 1$.  

Lemma 5. Suppose $1 \leq N \leq M, 1 \leq L \leq K, N \leq L, M \asymp K, 0 < \Delta \ll K^{1/2}$. Let $A_1(N, M, K, L; \Delta)$ denote the number of solutions of the inequality
$$0 < |\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| < \Delta$$ with $n \sim N, m \sim M, k \sim K, l \sim L$. Then we have
$$A_1(N, M, K, L; \Delta) \ll \Delta K^{1/2}NML + NLK^{1/2+\varepsilon}.$$ Especially if $\Delta K^{1/2} \gg 1$, then
$$A_1(N, M, K, L; \Delta) \ll \Delta K^{1/2}NML.$$  

Proof. If $(n, m, k, l)$ satisfies $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| < \Delta$, then we get
$$m = k + 2k^{1/2}(\sqrt{l} - \sqrt{n}) + (\sqrt{l} - \sqrt{n})^2 + u$$
with $|u| \leq C\Delta K^{1/2}$ for some absolute constant $C > 0$. Hence $A_1(N, M, K, L; \Delta)$ does not exceed the number of solutions of the inequality

$$|2k^{1/2}(\sqrt{l} - \sqrt{m}) + (\sqrt{l} - \sqrt{m})^2 + k - m| < C\Delta K^{1/2}$$

with $n \sim N, m \sim M, k \sim K, l \sim L$.

If $\Delta K^{1/2} \gg 1$, then for fixed $(n, k, l)$, the number of $m$ for which (2.5) holds is $\ll 1 + \Delta K^{1/2} \ll \Delta K^{1/2}$ if we notice $K \asymp M$. Hence

$$A_1(N, M, K, L; \Delta) \ll \Delta K^{1/2}NML.$$

Now suppose $\Delta K^{1/2} \leq 1/4C$. For fixed $(n, k, l)$, there is at most one $m$ such that (2.14) holds. If such $m$ exists, then we have

$$\|2k^{1/2}(\sqrt{l} - \sqrt{m}) + (\sqrt{l} - \sqrt{m})^2\| < C\Delta K^{1/2}. \quad (2.6)$$

We shall use Lemma 4 to bound the number of solutions of (2.6) with $\alpha = 2(\sqrt{l} - \sqrt{m}), \beta = (\sqrt{l} - \sqrt{m})^2$. Let $C_1$ denote the number of solutions of (2.6) with $|\alpha| \geq 2K^{-1/2}$, and $C_2$ the number of solutions with $|\alpha| < 2K^{-1/2}$, respectively. By Lemma 4 we get

$$C_1 \ll \Delta K^{1/2}NML + NLK^{1/2+\epsilon}$$

if we notice $M \asymp K$. Now we estimate $C_2$. From $|\alpha| < 2K^{-1/2}$, we get $N \asymp L$. If $l = n$, then from (2.5) we get $k = m$. This contradicts to $|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| > 0$. Thus $l \neq n$. From

$$2K^{-1/2} > |\sqrt{l} - \sqrt{m}| = \frac{|l - n|}{\sqrt{l} + \sqrt{m}} \geq \frac{1}{\sqrt{l} + \sqrt{m}} \geq 1/2\sqrt{2L}$$

we get $L \gg K$ and thus $N \asymp M \asymp K \asymp L$. So we have

$$C_2 \ll \#\{(l, n) : |\alpha| < 2K^{-1/2}\} \times \#\{k\} \ll K^2,$$

which can be absorbed into the estimate of $C_1$. This completes the proof of Lemma 5. \qed

**Lemma 6.** Suppose $1 \leq N \leq M \leq K \asymp L$, $0 < \Delta \ll L^{1/2}$. Let $A_2(N, M, K, L; \Delta)$ denote the number of solutions of the inequality

$$|\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l}| < \Delta$$

with $n \sim N, m \sim M, k \sim K, l \sim L$. Then we have

$$A_2(N, M, K, L; \Delta) \ll \Delta L^{1/2}NM + NMK^{1/2+\epsilon}.$$ 

Especially if $\Delta L^{1/2} \gg 1$, then

$$A_2(N, M, K, L; \Delta) \ll \Delta L^{1/2}NMK.$$
3 Proof of Theorem 1

Suppose $T \geq 10$. It suffices to evaluate the integral $\int_T^{2T} \Delta^4(x) dx$. Suppose $y = T^{3/4}$. For any $T \leq x \leq 2T$, by the truncated Voronoi’s formula we get

$$\Delta(x) = \frac{1}{\sqrt{2\pi}} R + (x^{1/2+\varepsilon} y^{-1/2}), \quad (3.1)$$

where

$$R := R(x) = x^{1/4} \sum_{n \leq y} \frac{d(n)}{n^{3/4}} \cos(4\pi \sqrt{xn} - \pi/4).$$

We have

$$\int_T^{2T} \Delta^4(x) dx = \frac{1}{4\pi^4} \int_T^{2T} R^4 dx + O(T^{9/4+\varepsilon} y^{-1/2} + T^{3+\varepsilon} y^{-2}) \quad (3.2)$$

$$= \frac{1}{4\pi^4} \int_T^{2T} R^4 dx + O(T^{15/8+\varepsilon}).$$

Let

$$g = g(n, m, k, l) := (nmkl)^{-1/4} d(n) d(m) d(k) d(l), \text{for } n, m, k, l \leq y,$$

and $g = 0$ otherwise.

The equation (3.4) of Tsang[11] reads

$$R_4^4 = S_1(x) + S_2(x) + S_3(x) + S_4(x), \quad (3.3)$$

where

$$S_1(x) := \frac{3}{8} \sum_{\sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}} gx,$$

$$S_2(x) := \frac{3}{8} \sum_{\sqrt{n} + \sqrt{m} \neq \sqrt{k} + \sqrt{l}} gx \cos(4\pi(\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}) \sqrt{x}),$$

$$S_3(x) := \frac{1}{2} \sum gx \sin(4\pi(\sqrt{n} + \sqrt{m} + \sqrt{k} - \sqrt{l}) \sqrt{x}),$$

$$S_4(x) := -\frac{1}{8} \sum gx \cos(4\pi(\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l}) \sqrt{x}).$$

From (3.7) of [11] we get

$$\int_T^{2T} S_1(x) dx = \frac{3c_2}{8} \int_T^{2T} x dx + O(T^{2-3/16+\varepsilon}). \quad (3.4)$$
From the first derivative test we get

\[ \int_T^{2T} S_4(x)dx \ll T^{3/2+\varepsilon} y^{1/2} \ll T^{15/8+\varepsilon}. \] (3.5)

Now let us consider the contribution of \( S_2(x) \). By the first derivative test we get

\[ \int_T^{2T} S_2(x)dx \ll \sum_{n,m,k,l \leq y} g \min \left( T^2, \frac{T^4}{|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}|} \right) \] (3.6)

\[ \ll T^\varepsilon G(N, M, K, L), \]

where

\[
G(N, M, K, L) = \sum_{n,m,k,l \leq y} g \min \left( T^2, \frac{T^4}{|\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}|} \right),
\]

\[
SC(\Sigma_1) : \sqrt{n} + \sqrt{m} \neq \sqrt{k} + \sqrt{l}, 1 \leq N \leq M \leq y, 1 \leq L \leq K \leq y,
\]

\[
N \leq L, n \sim N, m \sim M, k \sim K, l \sim L.
\]

If \( M \geq 100K \), then \( |\sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l}| \gg M^{1/2} \), so the trivial estimate yields

\[ G(N, M, K, L) \ll \frac{T^{4+\varepsilon} NMKL}{(NMLK)^{3/4}M^{1/2}} \ll T^{3+\varepsilon} y^{1/2} \ll T^{15/8+\varepsilon}. \]

If \( K > 100M \), we get the same estimate. So later we always suppose that \( M \sim K \).

Let \( \eta = \sqrt{n} + \sqrt{m} - \sqrt{k} - \sqrt{l} \). Write

\[ G(N, M, K, L, R) = G_1 + G_2 + G_3, \] (3.7)

where

\[
G_1 := T^2 \sum_{|\eta| \leq T^{-1/2}} g,
\]

\[
G_2 := T^{\frac{3}{2}} \sum_{T^{-1/2} < |\eta| \leq 1} g|\eta|^{-1},
\]

\[
G_3 := T^{\frac{3}{2}} \sum_{|\eta| \gg 1} g|\eta|^{-1}.
\]

10
We estimate $G_1$ first. From $|\eta| \leq T^{-1/2}$ we get $M \times K \gg T^{1/7}$ via Lemma 2. By Lemma 5 we get

$$G_1 \ll \frac{T^{2+\varepsilon}}{(N M K L)^{3/4}} A_1(N, M, K, L; T^{-1/2})$$  \hspace{1cm} (3.8)

$$\ll \frac{T^{2+\varepsilon}}{(N M K L)^{3/4}} \left( T^{-1/2} K^{1/2} N M L + N L K^{1/2} \right)$$

$$\ll \frac{T^{3/2+\varepsilon} (N L)^{1/4} + T^{2+\varepsilon} (N L)^{1/4} K^{-1}}{N M K L}$$

By Lemma 3 we get (notice $N \leq L \leq K$)

$$G_1 \ll \frac{T^{2+\varepsilon}}{(N M K L)^{3/4}} A_\pm(N, M, K, L; T^{-1/2})$$  \hspace{1cm} (3.9)

$$\ll \frac{T^{2+\varepsilon}}{(N M K L)^{3/4}} \left( T^{-1/8} N^{7/8} + N^{1/2} \right) \left( T^{-1/8} L^{7/8} + L^{1/2} \right) \left( T^{-1/4} K^{7/4} + K \right)$$

$$\ll T^{2+\varepsilon} \left( T^{-1/8} N^{1/8} + N^{-1/4} \right) \left( T^{-1/8} L^{1/8} + L^{-1/4} \right) \left( T^{-1/4} K^{1/4} + K^{-1/2} \right)$$

$$\ll T^{2+\varepsilon} \left( T^{-1/4} (N L)^{1/8} + T^{-1/8} L^{1/8} N^{-1/4} + (N L)^{-1/4} \right) \left( T^{-1/4} K^{1/4} + K^{-1/2} \right)$$

$$\ll T^{2+\varepsilon} T^{-1/4} (N L)^{1/8} \left( T^{-1/4} K^{1/4} + K^{-1/2} \right)$$

$$+ T^{2+\varepsilon} \left( T^{-1/8} L^{3/8} (N L)^{-1/4} + (N L)^{-1/4} \right) \left( T^{-1/4} K^{1/4} + K^{-1/2} \right)$$

$$\ll T^{3/2+\varepsilon} y^{1/2} + T^{7/4+\varepsilon} K^{-1/4}$$

$$+ T^{2+\varepsilon} \left( T^{-1/4} K^{1/4} + K^{-1/2} \right) \left( T^{-1/8} K^{3/8} + 1 \right) (N L)^{-1/4}$$

$$\ll T^{15/8+\varepsilon} + T^{2+\varepsilon} K^{-1/2} \left( T^{-1/4} K^{3/4} + 1 \right) \left( T^{-1/8} K^{3/8} + 1 \right) (N L)^{-1/4}$$

$$\ll T^{15/8+\varepsilon} + T^{2+\varepsilon} K^{-1/2} \left( T^{-3/8} K^{9/8} + 1 \right) (N L)^{-1/4}.$$  \hspace{1cm} (3.10)

From (3.8) and (3.9) we get

$$G_1 \ll T^{15/8+\varepsilon} + T^{2+\varepsilon} \min \left( (N L)^{1/4} K^{-1}, K^{-1/2} \left( T^{-3/8} K^{9/8} + 1 \right) (N L)^{-1/4} \right)$$

$$\ll T^{15/8+\varepsilon} + T^{2+\varepsilon} \left( (N L)^{1/4} K^{-1} \right)^{1/2} \left( K^{-1/2} \left( T^{-3/8} K^{9/8} + 1 \right) (N L)^{-1/4} \right)^{1/2}$$

$$\ll T^{15/8+\varepsilon} + T^{2+\varepsilon} K^{-3/4} \left( T^{-3/16} K^{9/16} + 1 \right)$$

$$\ll T^{15/8+\varepsilon} + T^{2+\varepsilon} K^{-3/4} \ll T^{53/28+\varepsilon}$$
if we notice $K \gg T^{1/7}$.

Now we estimate $G_2$. By a splitting argument we get that the estimate

$$G_2 \ll \frac{T^{3/2+\epsilon}}{(NMKL)^{3/4}} \sum_{\substack{\delta < |\eta| \leq 2\delta \\ \eta \neq 0}} 1$$  \hspace{1cm} (3.11)

holds for some $T^{-1/2} \leq \delta \leq 1$. By Lemma 5 we get that

$$G_2 \ll \frac{T^{3/2+\epsilon}}{(NMKL)^{3/4}} A_1(N, M, K, L; 2\delta)$$  \hspace{1cm} (3.12)

$$\ll \frac{T^{3/2+\epsilon}}{(NMKL)^{3/4}} (\delta^1 L^{1/4} + N^{1/4} + L^{-1/4} + N L^{-1/4} + (NL)^{-1/4} + 1) (K^{1/4} + K^{-1/2} \delta^{-1/2})$$

$$\ll T^{3/2+\epsilon} (NL)^{1/4} K^{1/4} + T^{3/2+\epsilon} (NL)^{1/4} K^{-1/2} \delta^{-1/2}$$

$$\ll T^{3/2+\epsilon} (NL)^{1/4} K^{1/4} + T^{3/2+\epsilon} (NL)^{1/4} K^{-1/2} \delta^{-1/2}$$

$$\ll T^{3/2+\epsilon} (NL)^{1/4} K^{1/4} + T^{3/2+\epsilon} (NL)^{1/4} K^{-1/2} \delta^{-1/2}$$

$$\ll T^{3/2+\epsilon} (NL)^{1/4} K^{1/4} + T^{3/2+\epsilon} (NL)^{1/4} K^{-1/2} \delta^{-1/2}$$

where the bound $\delta \gg T^{-1/2}$ was used to the term $T^{3/2+\epsilon} \delta^{-1/2}$.

From (3.12) and (3.13) we get

$$G_2 \ll T^{15/8+\epsilon} + \frac{T^{3/2+\epsilon}}{\delta} \min \left( \frac{(NL)^{1/4}}{K}, \frac{K^{9/8} \delta^{3/4} + 1}{K^{1/2} (NL)^{1/4}} \right)$$  \hspace{1cm} (3.14)

$$\ll T^{15/8+\epsilon} + \frac{T^{3/2+\epsilon}}{\delta} \left( \frac{(NL)^{1/4}}{K} \right)^{1/2} \left( \frac{K^{9/8} \delta^{3/4} + 1}{K^{1/2} (NL)^{1/4}} \right)^{1/2}$$

$$\ll T^{15/8+\epsilon} + T^{3/2+\epsilon} \delta^{-1} K^{-3/4} (K^{9/16} \delta^{3/8} + 1).$$
If \( \delta \gg K^{-3/2} \), then (3.14) implies (recall \( \delta \gg T^{-1/2} \))

\[
G_2 \ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon} K^{-3/16} \delta^{-5/8} \ll T^{15/8+\varepsilon}.
\] (3.15)

If \( \delta \ll K^{-3/2} \), then (3.14) becomes

\[
G_2 \ll T^{15/8+\varepsilon} + T^{3/2+\varepsilon} \delta^{-1} K^{-3/4}.
\] (3.16)

Since we have \( \delta \gg K^{-7/2} \) by Lemma 2 and \( \delta \gg T^{-1/2} \), we get \( \delta^{-1} \ll \min(K^{7/2}, T^{1/2}) \) and thus from (3.16) we get

\[
G_2 \ll T^{15/8+\varepsilon} + \min\left(T^{2+\varepsilon} K^{-3/4}, T^{3/2+\varepsilon} K^{11/4}\right) \ll T^{3/28+\varepsilon}.
\] (3.17)

For \( G_3 \), by a splitting argument and Lemma 5 again (notice \( |\eta| \gg 1 \)) we get

\[
G_3 \ll \frac{T^{3/2+\varepsilon}}{(N M K L)^{3/4} \delta} \sum_{\delta < |\eta| \leq 2\delta, \delta \gg 1} 1
\]

\[
\ll \frac{T^{3/2+\varepsilon}}{(N M K L)^{3/4}} K^{1/2} N M L \ll T^{3/8+\varepsilon} y^{1/4} \ll T^{15/8+\varepsilon}.
\] (3.18)

Combining (3.6), (3.7), (3.10) and (3.15)-(3.18) we get

\[
\int_T^{2T} S_2(x)dx \ll T^{53/28+\varepsilon}.
\] (3.19)

In the same way we can show that

\[
\int_T^{2T} S_3(x)dx \ll T^{53/28+\varepsilon}
\] (3.20)

by Lemma 3 and Lemma 6.

From (3.2)-(3.5) , (3.19) and (3.20) we get

\[
\int_T^{2T} \Delta^4(x)dx = \frac{3c_2}{32\pi^2} \int_T^{2T} xdx + O(T^{53/28+\varepsilon}),
\] (3.21)

which implies Theorem 1 immediately.
4 Preliminary Lemmas for Theorem 2

In order to prove Theorem 2, we need the following Lemmas.

Lemma 7. We have

\[ E(t) = \Sigma_1(t) + \Sigma_2(t) + O(\log^2 t) \]

with

\[ \Sigma_1(t) := \frac{1}{\sqrt{2}} \sum_{n \leq N} h(t, n) \cos(f(t, n)), \tag{4.1} \]

\[ \Sigma_2(t) := -2 \sum_{n \leq N'} d(n)n^{-1/2}(\log \frac{t}{2\pi n})^{-1} \cos(t\log \frac{t}{2\pi n} - t + \frac{\pi}{4}), \tag{4.2} \]

\[ h(t, n) := (-1)^n d(n)n^{-1/2}(\frac{t}{2\pi n} + \frac{1}{4})^{-1/4}(g(t, n))^{-1}, \tag{4.3} \]

\[ g(t, n) := \text{arsinh}(\frac{\pi n}{2t})^{1/2}, \tag{4.4} \]

\[ f(t, n) := 2tg(t, n) + (2\pi nt + \pi^2 n^2)^{1/2} - \pi/4, \tag{4.5} \]

\[ At \leq N \leq A't, N' := t/2\pi + N/2 - (N^2/4 + Nt/2\pi)^{1/2}, \tag{4.6} \]

where \( 0 < A < A' \) are any fixed constants.

Proof. This is the famous Atkinson’s formula, see Atkinson [1] or Ivić[5, Theorem 15.1]. \qed

Lemma 8. Suppose \( Y > 1 \). Define

\[ c_2^* := \sum_{\sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}, \]

\[ c_2^*(Y) := \sum_{\sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}, \]

\[ c_2(Y) := \sum_{n,m,k,l \leq Y} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}. \]

Then we have

\[ c_2 = c_2^*, \quad c_2(Y) = c_2^*(Y), \quad |c_2 - c_2(Y)| \ll Y^{-1/2+\varepsilon}. \]
Proof. The estimate \(|c_2 - c_2(Y)| \ll Y^{-1/2+\varepsilon}\) is a special case of Lemma 3.1 of the author[14]. The equalities \(c_2 = c_2^*\) and \(c_2(Y) = c_2^*(Y)\) follow from the fact that if \(\sqrt{n_1} + \sqrt{n_2} = \sqrt{n_3} + \sqrt{n_4}\), then \(n_1 + n_2 + n_3 + n_4\) must be an even number.

\[\tag{9}\]

Lemma 9. Suppose \(Y > 1\), then we have

\[H_1(Y) := \sum_{\sqrt{n} \in \mathbb{R}, \sqrt{m} \in \mathbb{R}, \sqrt{k}, \sqrt{l} \leq Y} \frac{d(n)d(m)d(k)d(l) \max(n, m, k, l)^3}{(nmlk)^{3/4}} \ll Y^{5/2+\varepsilon}.\]

Proof. Suppose \(\sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}\), then we have

1. \(n = k, m = l\) or \(n = l, m = k\);

or

2. \(n \neq k, l\).

If the case (2) holds, then by a classical result of Besicovitch, we know that

\[n = n_1^2h, m = m_1^2h, k = k_1^2h, l = l_1^2h, n_1 + m_1 = k_1 + l_1, \mu(h) \neq 0.\]

Thus we get

\[H_1(Y) \ll \Sigma_1 + \Sigma_2,\]

\[\Sigma_1 \ll \sum_{n, k \leq Y} \frac{d^2(n)d^2(m) \max(n, k)^3}{(nk)^{3/2}} \ll Y^{5/2} \log^3 Y,\]

\[\Sigma_2 \ll Y^\varepsilon \sum_{h < Y} \sum_{n_1 + m_1 = k_1 + l_1} \frac{\max(n_1, m_1, k_1, l_1)^6}{(n_1m_1k_1l_1)^{3/2}} \ll Y^\varepsilon \sum_{h < Y} \frac{k_1^{9/2}}{(n_1m_1l_1)^{3/2}} \ll Y^\varepsilon \sum_{h < Y} \frac{1}{l_1^{3/2}} \sum_{n_1 + m_1 = k_1 + l_1} \frac{k_1^{9/2}}{(n_1m_1)^{3/2}} \ll Y^\varepsilon \sum_{h < Y} \frac{1}{l_1^{3/2}} \sum_{n_1 + m_1 = k_1 + l_1} \frac{k_1^{9/2}}{(n_1m_1)^{3/2}} \ll Y^\varepsilon \sum_{h < Y} \frac{1}{l_1^{3/2}} \sum_{n_1 + m_1 = k_1 + l_1} \frac{k_1^{9/2}}{(n_1m_1)^{3/2}} \ll Y^\varepsilon \sum_{h < Y} (Y^{1/2}h^{-1/2})^5 \ll Y^{5/2+\varepsilon}.\]
Lemma 10. Suppose $Y > 1$, then we have
\[ H_2(Y) := \sum_{n,m,k,l \leq Y} \frac{d(n)d(m)d(k)d(l)^{3/4}}{(nmk)^{3/4}} \ll Y^{1/2+\varepsilon}. \]

Proof. If $\sqrt{n} + \sqrt{m} + \sqrt{k} = \sqrt{l}$, then we have
\[ n = n_1^2 h, m = m_1^2 h, k = k_1^2 h, l = l_1^2 h, n_1 + m_1 + k_1 = l_1, \mu(h) \neq 0. \]
Thus we get
\[ H_2(Y) \ll Y^{\varepsilon} \sum_{h(n_1 + m_1 + k_1)^2 \leq Y} \frac{(n_1 + m_1 + k_1)^{3/2}}{h^{3/2}(n_1m_1k_1)^{3/2}} \ll Y^{1/2+\varepsilon}. \]

Lemma 11. Suppose $f_j(t)(1 \leq j \leq k)$ and $g(t)$ are continuous, monotonic real-valued functions on $[a, b]$ and let $g(t)$ have a continuous, monotonic derivative on $[a, b]$. If $|f_j(t)| \leq A_j(1 \leq j \leq k), |g'(t)| \geq \Delta$ for any $t \in [a, b]$, then
\[ \int_a^b f_1(t) \cdots f_k(t) e(g(t)) dt \ll A_1 \cdots A_k \Delta^{-1}. \]

Proof. This is Lemma 15.3 of Ivić[5].

5 Proof of Theorem 2
Suppose $T \geq 10$. It suffices to evaluate $\int_T^{2T} E^4(t) dt$. Let $y := T^{1/3-\varepsilon}$ . For any $T \leq t \leq 2T$, define
\[ E_1(t) := \frac{1}{\sqrt{2}} \sum_{n \leq y} h(t, n) \cos(f(t, n)), \quad E_2(t) := E(t) - E_1(t). \]
From the inequality $(a + b)^4 - a^4 \ll |b|^3|a| + |b|^4$, we get
\[ \int_T^{2T} E^4(t) dt = \int_T^{2T} E_1^4(t) dt + O\int_T^{2T} |E_1(t)|^3 |E_2(t)| dt + O\int_T^{2T} |E_2(t)|^4 dt. \]
(5.1)
5.1 Evaluation of $\int_T^{2T} \mathcal{E}_1^4(t) dt$

In this subsection, we shall evaluate the integral $\int_T^{2T} \mathcal{E}_1^4(t) dt$. Similar to Tsang[11], we can write

$$\mathcal{E}_1^4(t) = \frac{3}{32} S_5(t) + \frac{3}{32} S_6(t) + \frac{1}{8} S_7(t) + \frac{1}{8} S_8(t) + \frac{1}{32} S_9(t), \quad (5.2)$$

where

$$S_5(t) := \sum_{n,m,k,l \leq y, \sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}} H(t; n, m, k, l) \cos(F_1(t; n, m, k, l)),$$

$$S_6(t) := \sum_{n,m,k,l \leq y, \sqrt{n} + \sqrt{m} \neq \sqrt{k} + \sqrt{l}} H(t; n, m, k, l) \cos(F_1(t; n, m, k, l)),$$

$$S_7(t) := \sum_{n,m,k,l \leq y, \sqrt{n} + \sqrt{m} + \sqrt{k} = \sqrt{l}} H(t; n, m, k, l) \cos(F_2(t; n, m, k, l)),$$

$$S_8(t) := \sum_{n,m,k,l \leq y, \sqrt{n} + \sqrt{m} + \sqrt{k} \neq \sqrt{l}} H(t; n, m, k, l) \cos(F_2(t; n, m, k, l)),$$

$$S_9(t) := \sum_{n,m,k,l \leq y} H(t; n, m, k, l) \cos(F_3(t; n, m, k, l)).$$

We first estimate the integral $\int_T^{2T} S_5(t) dt$. For $n \leq y$, it is easy to check that

$$h(t, n) = \frac{2^{3/4} \pi \pi d(n) n^{3/4} t^{1/4} (1 + O(n t))}{\pi^{1/4} n^{3/4} t^{1/4}} \quad (5.3)$$

$$f(t, n) = 2^{3/2} (\pi n t)^{1/2} - \pi/4 + O(n^{3/2} t^{-1/2}) \quad (5.4)$$

$$f'(t, n) = 2^{1/2} (\pi n)^{1/2} t^{-1/2} + O(n^{3/2} t^{-3/2}) \quad (5.5)$$

If $\sqrt{n} + \sqrt{m} = \sqrt{k} + \sqrt{l}$, then

$$\cos(F_1(n, m, k, l)) = \cos(O(D^{3/2} t^{1/2})) = 1 + O(D^3 t) \quad (5.6)$$
where $D := \max(n, m, k, l)$. So from (5.3), (5.6), Lemma 8 and Lemma 9 we get

$$
\int_T^{2T} S_5(t) \, dt = \sum_{n, m, k, l \leq y / \sqrt{n + m = \sqrt{k + \sqrt{l}}} \sqrt{m} + \sqrt{m} = \sqrt{k} + \sqrt{l}} H(t; n, m, k, l) \cos(F_1(t; n, m, k, l)) \, dt
$$

(5.7)

$$
= \frac{8}{\pi} \sum_{n, m, k, l \leq y / \sqrt{n + \sqrt{m} = \sqrt{k} + \sqrt{l}}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}
\times \int_T^{2T} t(1 + O(D^3 / t)) \, dt
\times \int_T^{2T} t(1 + (D^3 / t)) \, dt
$$

(5.8)

$$
= \frac{8}{\pi} \sum_{n, m, k, l \leq y / \sqrt{n + \sqrt{m} = \sqrt{k} + \sqrt{l}}} \frac{(-1)^{n+m+k+l}d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \int_T^{2T} t \, dt
$$

$$
+ O(TH_1(y))
$$

$$
= \frac{8c_2}{\pi} \int_T^{2T} t \, dt + O(T^{1+\varepsilon}y^{5/2} + T^{2+\varepsilon}y^{-1/2})
$$

$$
= \frac{8c_2}{\pi} \int_T^{2T} t \, dt + O(T^{11/6+\varepsilon}).
$$

Now we estimate $\int_T^{2T} S_6(t) \, dt$. From (5.5) we get

$$
F_1'(t; n, m, k, l) = (2\pi)^{1/2} \eta t^{-1/2} + O(D^3 t^{-3/2}),
$$

where $\eta = \sqrt{n + \sqrt{m} - \sqrt{k} - \sqrt{l}}$. Write

$$
\int_T^{2T} S_6(t) \, dt = \int_{|\eta| \leq T^{-1/2}} S_6(t) \, dt + \int_{|\eta| > T^{-1/2}} S_6(t) \, dt.
$$

(5.8)

If $|\eta| \leq T^{-1/2}$, the by (5.3) and the trivial estimate we get

$$
\int_{|\eta| \leq T^{-1/2}} S_6(t) \, dt \ll T^2 \sum_{n, m, k, l \leq y / |\eta| \leq T^{-1/2} / \sqrt{n + \sqrt{m} = \sqrt{k} + \sqrt{l}}} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}}.
$$

(5.9)
If $|\eta| > T^{-1/2}$, then $|F'(t; n, m, k, l)| > |\eta|T^{-1/2}$, thus from (5.3) and Lemma 11 we get

$$\int_{|\eta| > T^{-1/2}} S_6(t) dt \ll T^{3/2} \sum_{n, m, k, l \leq y, |\eta| > T^{-1/2}} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}|\eta|}. \quad (5.10)$$

From (5.9), (5.10) and the estimate in Section 3 we get

$$\int_{2T}^{T} S_6(t) dt \ll \sum_{n, m, k, l \leq y} \frac{d(n)d(m)d(k)d(l)}{(nmkl)^{3/4}} \min(T^2, T^{3/2}|\eta|^{-1}) \ll T^{53/28+\varepsilon}. \quad (5.11)$$

If $\sqrt{n} + \sqrt{m} + \sqrt{k} = \sqrt{l}$, then from (5.4) we have

$$F_2(t; n, m, k, l) = -\pi/2 + O(t^{3/2}T^{-1/2}), \quad \cos F_2(t; n, m, k, l) = t^{3/2}T^{-1/2}.$$  

Thus from (5.3), the trivial estimate and Lemma 10 we get

$$\int_{T}^{2T} S_7(t) dt \ll T^{3/2}H_2(y) \ll T^{3/2}y^{1/2+\varepsilon} \ll T^{5/3+\varepsilon}. \quad (5.12)$$

Similar to the integral $\int_{T}^{2T} S_6(t) dt$, we have

$$\int_{T}^{2T} S_8(t) dt \ll T^{53/28+\varepsilon}. \quad (5.13)$$

From (5.5) we get

$$F'_3(t; n, m, k, l) \gg (\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l})T^{-1/2},$$

which combining (5.3) and Lemma 11 implies

$$\int_{T}^{2T} S_9(t) dt \ll \sum_{n, m, k, l} \frac{d(n)d(m)d(k)d(l)T^{3/2}}{(nmkl)^{3/4}(\sqrt{n} + \sqrt{m} + \sqrt{k} + \sqrt{l})} \ll T^{3/2+\varepsilon}y^{1/2} \ll T^{5/3+\varepsilon}. \quad (5.14)$$

From (5.2), (5.7), (5.11)-(5.14) we get

$$\int_{T}^{2T} \mathcal{E}_1^4(t) dt = \frac{3c_2}{4\pi} \int_{T}^{2T} t dt + O(T^{53/28+\varepsilon}). \quad (5.15)$$
5.2 Completion of proof of Theorem 2

Let $A_0 = 35/8$. Ivić [5, Thm 15.7] proved the estimate

$$\int_1^T |E(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \quad (5.16)$$

By his method we can show

$$\int_T^{2T} |E_1(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \quad (5.17)$$

Thus

$$\int_T^{2T} |E_2(t)|^{A_0} dt \ll T^{1+A_0/4+\varepsilon}. \quad (5.18)$$

We also have

$$\int_T^{2T} |E_2(t)|^2 dt \ll T^{3/2+\varepsilon} y^{-1/2}, \quad (5.19)$$

which is the formula (4.15) of the author [14]. From (5.18), (5.19) and the Hölder’s inequality we get that the estimate

$$\int_T^{2T} |E_2(t)|^A dt \ll T^{1+A/4+\varepsilon} y^{-(A_0-A)/2(A_0-2)} \quad (5.20)$$

holds for any $2 < A < A_0$. The details of the above estimates can be found in the author [14].

From (5.17), (5.20) and the Hölder’s inequality we get

$$\int_T^{2T} |E_1(t)|E_2(t)| dt \ll \left( \int_T^{2T} |E_1(t)|^{A_0} dt \right)^{3/A_0} \left( \int_T^{2T} |E_2(t)|^{A_0/(A_0-3)} dt \right)^{(A_0-3)/A_0} \quad (5.21)$$

$$\ll T^{2+\varepsilon} y^{-(A_0-4)/2(A_0-2)} \ll T^{2-19/108+\varepsilon}.$$  

From (5.1), (5.15), (5.20) with $A = 4$ and (5.21) we get

$$\int_T^{2T} E^4(t) dt = \frac{3c_2}{4\pi} \int_T^{2T} t dt + O(T^{53/28+\varepsilon}) \quad (5.22)$$

and Theorem 2 follows.
References

[1] F. V. Atkinson, the mean value of the Riemann zeta-function, Acta Math. 1949, 81: 353-376.

[2] Cai Yingchun, On the third and fourth power moments of Fourier coefficients of cusp forms. Acta Math. Sinica (N.S.) 13 (1997), no. 4, 443–452.

[3] M. N. Huxley, Exponential sums and Lattice points III, Proc. London Math. Soc., Vol.87(3)(2003), 591-609.

[4] A. Ivić, On some problems involving the mean square of $\zeta(\frac{1}{2} + it)$. Bull. Cl. Sci. Math. Nat. Sci. Math. 1998, 23: 71–76.

[5] A. Ivić, The Riemann zeta-function. John. Wiley and Sons, 1985.

[6] A. Ivić, Lectures on mean values of the Riemann zeta-function, Lectures On Math. and Physics 82, Tata Inst. Fund. Res., Bombay, 1991.

[7] A. Ivić and P. Sargos, On the higher power moments of the error term in the divisor problem, Illinois Journal of Math. 51(2)(2007), 353-377.

[8] M. Jutila, On a formula of Atkinson. Topics in classical number theory, Vol. I, II (Budapest, 1981), 807–823, Colloq. Math. Soc. Jos Bolyai, 34, North-Holland, Amsterdam, 1984.

[9] O. Robert and P. Sargos, Three-dimensional exponential sums with monomials, J. Reine angew. Math. 591(2006), 1-20.

[10] K. C. Tong, On divisor problem III, Acta math. Sinica 6 (1956), 515-541.

[11] Kai-Man Tsang, Higher-power moments of $\Delta(x), E(t)$ and $P(x)$, Proc. London Math. Soc.(3)65(1992), 65-84.

[12] Wenguang Zhai, On higher-power moments of $\Delta(x)$, Acta Arith. Vol.112(2004), 1-24.

[13] Wenguang Zhai, On higher-power moments of $\Delta(x)(II)$, Acta Arith. Vol.114(2004), 35-54.

[14] Wenguang Zhai, On higher-power moments of $E(t)$, Acta Arith. Vol.115(2004), 329-348.