Boundary terms in Nambu–Goto string action

Leszek Hadasz†
Paweł Węgrzyn‡

Jagellonian University, Institute of Physics
Reymonta 4, 30–059 Kraków, Poland

Abstract

We investigate classical strings defined by the Nambu-Goto action with the boundary term added. We demonstrate that the latter term has a significant bearing on the string dynamics. It is confirmed that new action terms that depend on higher order derivatives of string coordinates cannot be considered as continuous perturbations from the starting string functional. In the case the boundary term reduces to the Gauss-Bonnet term, a stability analysis is performed on the rotating rigid string solution. We determine the most generic solution that the fluctuations grow to. Longitudinal string excitations are found. The Regge trajectories are nonlinear.

TPJU–15/94
August 1994

*Work supported in part by grant KBN 2 P302 049 05.
†E–mail: hadasz@ztc386a.if.uj.edu.pl
‡E–mail: wegrzyn@ztc386a.if.uj.edu.pl
A simple non–interacting string sweeps out a time–like surface of minimal area in four–dimensional Minkowski spacetime. The minimal surface can be parametrized in such a way that non–linear equations of motion turn into linear wave equations, and the string model becomes mathematically tractable. It was found [1] that the most general model of strings, which gives critical worldsheets of minimal area, is defined by the following action:

\[ S = -\gamma A - \frac{\alpha}{2} S_{GB} - \beta S_{Ch}, \]

where \( \gamma \) is string tension, \( A \) denotes worldsheet area, \( \alpha \) and \( \beta \) are dimensionless parameters (\( \gamma, \alpha > 0 \)). \( S_{GB} \) and \( S_{Ch} \) are pseudoeuclidean Gauss–Bonnet and Chern terms (related to Euler characteristics and surface self–intersections in Euclidean geometry).

String equations of motion derived from (1) are Nambu–Goto equations supplemented by some edge conditions, which depend on the action parameters.

The string action (1), that depends on two arbitrary dimensionless parameters, represents a generic form allowed by reparametrization and Poincare symmetries [1] and the requirement that the variational problem results in minimal surfaces. This statement is true as long as we do not consider additional objects that could couple to strings, like internal fields living on the worldsheet or constant external fields in the target spacetime. Obviously, there exist also “point–like” actions, being functionals of the trajectories of string endpoints. The simplest example is given by [2–6]:

\[ S_p = -mL_1 - mL_2, \]

where \( L_1 \) and \( L_2 \) are invariant lengths of the trajectories of string ends. Such “non–stringy” terms modify edge conditions in the variational problem for critical string worldsheets, but they cannot be represented as reparametrization invariant surface terms and do not modify local distributions of energies, momenta and angular momenta along strings.

The choice of the worldsheet parametrization can be defined by the following conditions [3]:

\[ (\dot{X} \pm X')^2 = 0, \]
\[ (\ddot{X} \pm \dot{X}')^2 = -\frac{1}{4} q^2, \]
where the dot and the prime stand for derivatives of worldsheet coordinates $X_\mu(\tau, \sigma)$ with respect to internal string parameters $\tau$ and $\sigma$. The parameter $q$ can be considered as a momentum scale unit. This parameter is freely adjustable. In the above parametrization, bulk equations of motion get linearized and their general solution reads:

$$X_\mu(\tau, \sigma) = X_{L\mu}(\tau + \sigma) + X_{R\mu}(\tau - \sigma). \quad (4)$$

To solve boundary problem at string ends $\sigma = \pm \frac{\pi}{2}$, we make use of the correspondence between minimal surfaces $X_\mu$ parametrized according to (3) and solutions $\Phi$ of the complex Liouville equation:

$$\ddot{\Phi} - \Phi'' = 2q^2 e^\Phi. \quad (5)$$

Any solution of either Nambu–Goto equations together with (3) or complex Liouville equation (5) can be presented in the following form:

$$e^\Phi = -\frac{4}{q^2} \frac{f_L'(\tau + \sigma)f_R'(\tau - \sigma)}{|f_L(\tau + \sigma) - f_R(\tau - \sigma)|^2},$$

$$\dot{X}_{L,R}^\mu = \frac{1}{4|f_{L,R}'|}(1 + |f_{L,R}|^2, 2Re f_{L,R}, 2Im f_{L,R}, 1 - |f_{L,R}|^2), \quad (6)$$

where $f_L$ and $f_R$ are arbitrary complex functions.

As the derivatives of left– and right–movers are light–like vectors, we can interprete $f_L$ and $f_R$ as their coordinates on the complex plane, on which the stereographical projection of the sphere of null directions in four–dimensional spacetime has been performed. Let us also note that modular transformations of $f_{L,R}$ induce Lorentz transformations of worldsheet coordinates while Liouville field $\Phi$ remains unchanged.

Now, we can present edge conditions following from the string action (4) as:

$$e^\Phi = -\frac{1}{q} \sqrt{\frac{\gamma}{\alpha}} e^{-i\theta},$$

$$Im \Phi' = 0 \quad \text{for} \quad \sigma = \pm \frac{\pi}{2}, \quad (7)$$

where the angle parameter $\theta \in [-\pi, \pi]$ is defined by:

$$\tan \frac{\theta}{2} = \frac{\beta}{\alpha}. \quad (8)$$
In this paper we consider only the case $\theta = 0$ ($\beta = 0$). This model has been investigated earlier in papers [4, 7, 8]. Then, there exists a well known solution corresponding to a rotating rigid rod,

$$X^\mu = \frac{q}{\lambda^2} (\lambda \tau, \cos \lambda \tau \sin \lambda \sigma, \sin \lambda \tau \sin \lambda \sigma, 0), \quad (9)$$

where the angular frequency $\lambda$ satisfies the relation:

$$\frac{\lambda^2}{\cos^2 \frac{\lambda \pi}{2}} = q \sqrt{\frac{\gamma}{\alpha}}. \quad (10)$$

Note that the velocity of string ends is smaller than the velocity of light and tends to it in the limit $\alpha \to 0$ ($\lambda \to 1$).

We can compute the energy and the angular momentum of the rotating string (9) (for relevant general formulae see ref. [1]),

$$E \equiv P^0 = \frac{\gamma q \pi}{\lambda} \left( 1 + \sin \frac{\pi \lambda}{\pi \lambda} \right),$$

$$J \equiv M^{12} = \frac{\gamma q^2 \pi}{2 \lambda^3} \left[ 1 + 2 \sin \frac{\pi \lambda}{\pi \lambda} + \sin \frac{2 \pi \lambda}{2 \pi \lambda} \right]. \quad (11)$$

The total momentum and other components of the total angular momentum vanish.

The pertinent Regge trajectory is plotted in Fig.1. Regge trajectories represent the angular momentum $J$ versus the squared mass $E^2$ relationships for given string configurations. We have compared trajectories for a rotating rigid rod obtained (a) in the standard Nambu–Goto open string model and (b) for the Nambu–Goto string with massive ends (due to the point–like terms (2) — see ref. [2]). Asymptotically, in the region of large masses, the trajectory can be approximated by the formula:

$$J = \frac{1}{2 \pi \gamma} E^2 + \frac{5}{4} \left( \frac{\alpha}{\pi^6 \gamma^3} \right)^{1/4} E^{3/2}. \quad (12)$$

We see that it is slightly raised in comparison with the Nambu–Goto open string trajectory. At low masses, unlike the case (b) where the appearance of point like masses at string ends curves the trajectory downwards and the intercept is lowered, we find here approximately a linear dependence:

$$J = \sqrt{\frac{\alpha}{\gamma}} E. \quad (13)$$
It is interesting to note that the energy distribution along the string has been also changed. For the Nambu–Goto rotating rigid string the energy density is constant. In the modified model, the energy density (plotted in Fig.2) is given by the formula:

\[ p^0 = \frac{\gamma q}{\lambda} \left[ 1 + \cos^4 \frac{\lambda \pi}{2} \left( \frac{3}{\cos^4 \lambda \sigma} - \frac{2}{\cos^2 \lambda \sigma} \right) \right]. \]  \hspace{1cm} (14)

We now turn to study small fluctuations around the solution (9). This solution is associated to a static solution of Liouville equation (5),

\[ e^{\Phi_0} = -\frac{1}{q} \frac{\lambda^2}{\cos^2 \lambda \sigma}. \]  \hspace{1cm} (15)

A small perturbation \( \Phi_1 \) from the static solution \( \Phi_0 \) satisfies the following linear equation:

\[ \ddot{\Phi}_1 - \Phi_1'' = -V(\sigma)\Phi_1, \] \hspace{1cm} (16)

where we have denoted

\[ V(\sigma) = \frac{2\lambda^2}{\cos^2 \lambda \sigma}. \] \hspace{1cm} (17)

The solution \( \Phi_1 \) is subject to the following boundary conditions at \( \sigma = \pm \frac{\pi}{2} \):

\[ \Phi_1 = 0, \] \hspace{1cm} (18)

\[ Im \Phi_1' = 0. \] \hspace{1cm} (19)

One can prove that the imaginary part of \( \Phi_1 \) must vanish at any worldsheet point. Thus, the Liouville field \( \Phi_1 \) is real. We can separate variables to find a solution satisfying the equation (16) together with the boundary conditions (18):

\[ \Phi_1(\tau, \sigma) = T(\tau)\Sigma(\sigma). \]

We obtain a system of ordinary differential equations:

\[ \ddot{T} + \varepsilon T = 0, \] \hspace{1cm} (20)

\[ \left(-\frac{d^2}{d\sigma^2} + V(\sigma)\right) \Sigma = \varepsilon \Sigma, \] \hspace{1cm} (21)

together with boundary conditions:

\[ \Sigma = 0 \text{ for } \sigma = \pm \frac{\pi}{2}. \] \hspace{1cm} (22)
The solutions of Schrödinger equation (21) that obey periodic boundary conditions (22) can exist only if
\[ E > 2\lambda^2, \]
where \( 2\lambda^2 \) is the minimal value of the potential \( V(\sigma) \). It implies that the separation constant \( E \) must be positive. For convenience we introduce new variable \( \omega \) defined as:
\[ E = \omega^2. \]
The Schrödinger equation (21) with the potential (17) can be exactly solved. The solutions exist only for discrete values of the separation constant \( \omega = \omega_n \) \((n = 1, 2, \ldots)\), being roots of the following equation (see Fig.3):
\[ \omega_n \tan \left( \frac{\pi(\omega_n + n)}{2} \right) = \lambda \tan \left( \frac{\pi \omega_n}{2} \right). \tag{23} \]
The general solution of the equation (16) satisfying boundary conditions (18, 19) reads:
\[ \Phi = 2\sum_{n=1}^{\infty} D_n \sin(\omega_n \tau + \varphi_n) \left[ \tan \lambda \cos \left( \omega_n \sigma + \frac{n\pi}{2} \right) - \frac{\omega_n}{\lambda} \sin \left( \omega_n \sigma + \frac{n\pi}{2} \right) \right], \tag{24} \]
where \( D_n \) and \( \varphi_n \) are arbitrary real constants.
To visualize string worldsheets that correspond to Liouville fields \( \Phi = \Phi_0 + \Phi_1 \) we must employ the relations (6). Taking into account that \( e^\Phi \) is real, we can make functions \( f_L \) and \( f_R \) unimodular (by some modular transformation — it is equivalent to specifying some reference frame in Minkowski spacetime). Then, it is convenient to introduce new real fields \( F_L \) and \( F_R \),
\[ f_{L,R} = e^{iF_{L,R}}, \]
and the relations (6) go over into:
\[ e^\Phi = -\frac{1}{q^2} \frac{F'_L F'_R}{\sin^2 \frac{F_{L,R}}{2}}, \tag{25} \]
\[ \dot{x}_{L,R}^\mu = \frac{1}{2F_{L,R}} (1, \cos F_{L,R}, \sin F_{L,R}, 0). \tag{26} \]
The static field \( \Phi_0 \) corresponds to:
\[ F_L^{(0)} = \lambda(\tau + \sigma), \quad F_R^{(0)} = \lambda(\tau - \sigma) + \pi, \tag{27} \]
while the first order fluctuations $\Phi_1$ are associated to the following corrections:

$$F_{L,R}^{(1)} = \pm \sum_{n/1}^{\infty} D_n \sin \left[ \omega_n (\tau \pm \sigma) + \varphi_n \pm \frac{n \pi}{2} \right],$$

(28)

where plus and minus signs correspond to left– and right–movers respectively.

In contrast to the Nambu–Goto case, there appear longitudinal excitations of the string. Moreover, only such kind of fluctuations come out at the first order. With the help of formulae above, the total string length $L$ can be evaluated at some fixed time $X^0$:

$$\lambda^2 L = 2q \left[ \omega_n^2 + \lambda^2 + 2\lambda^2 \tan^2 \left( \frac{\pi \lambda}{2} \right) \right] \cos \left( \frac{\pi \lambda}{2} \right).$$

(29)

Let us now calculate the contribution to the energy coming from fluctuations. The general formula for the total string energy reads:

$$P^0 = \frac{\gamma q \pi}{\lambda} \left( 1 + \sin \frac{\pi \lambda}{\pi \lambda} \right) + \frac{\gamma q \pi}{2\lambda^3} \sum_{n/1}^{\infty} D_n^2 \omega_n^2 \left[ 1 + \frac{\sin \pi \lambda}{\pi \lambda} + 2(-1)^{n+1} \cos \frac{\pi \lambda \sin \pi \omega_n}{\pi \omega_n} \right].$$

(31)

One can easy check that the energy of fluctuations is always positive. It means that the solution (10) is stable against small perturbations. In fact, to calculate the total string energy (31) up to the second order we need also to consider second order corrections to the zero order solution. It can be proved by straightforward calculations that they do not contribute to the energy at the second order.

Finally, we want to summarize our results. We examined a classical string model in four–dimensional Minkowski spacetime defined by the Nambu–Goto
action with some boundary term added. It warrants that critical worldsheets are minimal surfaces, but some non-linear, third order in time derivatives equations hold at string ends. It is evident from this paper that additional terms to the action functional depending on the second order derivatives of string coordinates cannot be regarded as higher order corrections to the starting Nambu–Goto action. In the limit of vanishing coupling constants \((\alpha, \beta \to 0)\) our model does not revert to the original Nambu–Goto string model. There are still higher order equations (7) to be satisfied. This is an unavoidable consequence of employing theoretical framework for string actions with second order derivatives. The number of boundary conditions for dynamical equations of motion is doubled. The same is true for the number of initial data necessary to formulate properly the Cauchy problem. Roughly speaking, passing to dynamical models, that are governed by the variational principle with actions depending on second order derivatives of dynamical variables, doubles the number of independent degrees of freedom.

A generic minimal wordsheet model (1) has been investigated for \(\beta = 0\). We have found a classical ground state solution, that corresponds to a rotating rigid rod. Unlike for the analogous Nambu–Goto configuration, string ends move with the velocity smaller than the velocity of light and non-relativistic limit can be defined. It has been shown that the mass distribution along the string has been changed. Regge trajectories in this model are non-linear. The ground state solution is stable against small perturbations. Eigenfrequencies for each fluctuation mode are found to be solutions of some simple transcendental equation. The excitations give a positive contribution to the total energy of the string. Another interesting property is that perturbations do not disturb the string from the planar motion, the shape of the string lies in a plane. But its total length measured in the laboratory frame oscillates, in contrast to other classical string models [4].

This work was supported in part by the KBN under grant 2 P302 049 05.

References

[1] P.Węgrzyn, Phys.Rev. D50, No.2 /in print/.
[2] A.Chodos, C.B.Thorn, Nucl.Phys. B72, 509 (1974).
[3] P.H.Frampton, Phys.Rev. D12, 538 (1975).
[4] B.M.Barbashov, A.L.Koshkarov, Lett. in Math. Phys. 3, 39 (1979).

[5] B.Barbashov, V.V.Nesterenko, Introduction to the relativistic string theory, Singapore, World Scientific 1990.

[6] B.M.Barbashov, A.M.Chervyakov, J.Phys. A24, 2443 (1991).

[7] A.A.Zheltukhin, Sov.J.Nucl.Phys. 34, 311 (1981).

[8] B.M.Barbashov, A.M.Chervyakov, Dubna preprint P2–86–572 /in Russian, unpublished/.

[9] V.V.Nesterenko, Z.Phys. C47, 111 (1990).

Figure Captions

Fig. 1. Regge trajectories for various string models:
   (a) Nambu–Goto string,
   (b) Rebbi–Thorn–Chodos string with massive ends,
   (c) string model with Gauss–Bonnet boundary term.

Fig. 2. The shape of energy density distribution along the string.

Fig. 3. Graphical solution of the equation \( \lambda \tan \left( \frac{\pi}{2} \right) = \omega \tan \left[ \frac{\pi(\omega+n)}{2} \right] \)
for eigenfrequencies \( \omega \) of fluctuation modes. The parameter \( \lambda \)
is a fixed value of angular frequency the rigid rod rotates with.
This figure "fig1-1.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9408072v1
This figure "fig1-2.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9408072v1
This figure "fig1-3.png" is available in "png" format from:

http://arxiv.org/ps/hep-th/9408072v1
Fig. 3