No Time Machines from Lightlike Sources in 2+1 Gravity

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Abstract

We extend the argument that spacetimes generated by two timelike particles in \( D=3 \) gravity (or equivalently by parallel-moving cosmic strings in \( D=4 \)) permit closed timelike curves (CTC) only at the price of Misner identifications that correspond to unphysical boundary conditions at spatial infinity and to a tachyonic center of mass. Here we analyze geometries one or both of whose sources are lightlike. We make manifest both the presence of CTC at spatial infinity if they are present at all, and the tachyonic character of the system: As the total energy surpasses its tachyonic bound, CTC first begin to form at spatial infinity, then spread to the interior as the energy increases further. We then show that, in contrast, CTC are entirely forbidden in topologically massive gravity for geometries generated by lightlike sources.

Among the many fundamental contributions by Charlie Misner to general relativity is his study of pathologies of Einstein geometries, particularly NUT spaces, which in his words are "counterexamples to almost everything"; in particular they can possess closed timelike curves (CTC). As with other farsighted results of his which were only appreciated later, this 25-year old one finds a resonance in very recent studies of conditions under which CTC can appear in apparently physical settings, but in fact require unphysical boundary conditions engendered by identifications very similar to those he discovered. In this paper, dedicated to him on his 60th birthday, we review and extend some of this current work. We hope it brings back pleasant memories.

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1. Introduction

Originally constructed by Gödel [1], but foreshadowed much earlier [2], spacetimes possessing CTC in general relativity came as a surprise to relativists. The shock was softened by the fact that these solutions required unphysical stress tensor sources, and in this sense should not have been so unexpected: it is after all a tautology that any spacetime is the solution of the Einstein equations with some stress tensor, as often emphasized by Synge. Indeed, Einstein himself, while elaborating general relativity, apparently worried about geometries with loss of causality and CTC, and hoped that they would be excluded by physically acceptable sources. Almost two decades after Gödel’s work, Misner in his pioneering studies [3] of NUT space showed how CTC could be generated as global effects, by taking local expressions for a metric and making appropriate identifications among the points.

Very recently, the subject of CTC was revived in two quite different contexts. The first, which we shall not discuss, involves tunneling through wormholes in $D=4$ gravity. The second, which will be our subject here, concerns solutions to $D=3$ Einstein theory with point sources or equivalently, $D=4$ gravity with infinite parallel cosmic strings (since the latter system is cylindrically symmetric). We will therefore operate entirely in the reduced dimensionality. The dramatic simplification at $D=3$ is that the Einstein and Riemann tensors are equivalent, so spacetime is locally flat wherever sources are absent; consequently there is neither gravitational radiation nor any Newtonian force between particles. For these reasons, the sign of the Einstein constant is not physically determined, unlike in $D=4$. We will adopt the usual sign here, but mention the opposite sign, “ghost” Einstein theory, at the end. Local flatness in $D=3$ means that all properties are encoded in the global structure, i.e., in the way the locally flat patches are sewn together. Indeed, we will use this geometric approach to analyze the CTC problem. However, for orientation we will begin with a brief discussion in terms of the analytic form of the metric.

Consider the general solution outside a localized physical source as given by the “Kerr” metric [4]

\[
ds^2 = -d(t + J\theta)^2 + dr^2 + \alpha^2 r^2 d\theta^2.
\]

Our units are $\kappa^2 = c = 1$, and $\alpha \equiv 1 - M/2\pi$. The constants of motion are the energy $M$ and angular momentum $J$ (space translations not being well defined [5]. This interval is manifestly locally flat in terms of the redefined coordinates $\Theta = \alpha \theta$, $T = t + J\theta$, but the global content lies in the different range, $0 \leq \Theta \leq 2\pi \alpha$, of $\Theta$ corresponding to the usual conical identification, and in the time-helical structure.

\[\text{We thank John Stachel for telling us this.}\]
resulting from the fact that the two times $T$ and $T + 2\pi J$ are to be identified whenever a closed spatial circuit is completed. The interval (1.1) can clearly support CTC; for example, the interval traced by a circle at constant $r$ and $t$,

$$\Delta s^2 = (2\pi \alpha)^2 (r^2 - J^2 / \alpha^2) < 0,$$

is timelike for $r < |J|/\alpha$. However, the relevant physical question is whether the constituent particles are ever confined within this radius; otherwise the CTC criterion (1.2) ceases to apply. To be sure, if we simply insert the metric (1.1) into the Einstein equations, it is valid down to $r = 0$; the “source” is a spinning particle with $T_0^0 \sim m \delta^2(r)$, $T_0^i \sim J \epsilon^{ij} \partial_j \delta^2(r)$. But we do not accept classical spinning particles as physical, precisely because of their singular stress tensors, any more than we do Gödel’s sources. Instead, one must check whether a system of moving spinless particles with orbital angular momentum can support CTC. This was the question that was initially raised in [4] and answered in the negative, on the simple physical grounds that the point particles, being essentially free, will — both initially and finally — be dispersed so that the constant $J$ would have been exceeded at $t = \pm \infty$ by the radius at which the exterior metric (1.1) is valid. Thus, CTC, if present at all, would have to appear and then disappear spontaneously in time in an otherwise normal Cauchy evolution, and it seemed unlikely that this violation of Cauchy causality would occur in a finite time region for an otherwise non-pathological system.

It was therefore quite surprising when, a year ago, Gott [6] gave an explicit construction of a geometry generated by an apparently acceptable source consisting of two massive particles passing by each other at subluminal velocities, in which CTC appear only during a limited time interval [7]. However, it was then shown both that, in these spaces CTC will also be present at spatial infinity, which constitutes an unphysical boundary condition, and that the spacetimes have an imaginary (tachyonic) total mass [8]. This is in contrast to the globally flat space of special relativity, where a collection of subluminal particles cannot of course be tachyonic. Indeed, the fact that in $D=3$ everything lies in the global properties raises a cautionary, and as we shall see, decisive, note. Let us illustrate this with one simple object lesson for the case of static sources. It is clear from (1.1) that a single particle cannot have a mass greater than $2\pi$ (in our units); indeed, $m = 2\pi$ corresponds to a cylindrical rather than conical 2-space. One might suppose that, since there are no interactions in $2+1$ dimensions, two stationary particles should give rise to a perfectly well-defined metric as long as each one separately satisfies the above inequality. This in fact is not so; the sum of the two masses must also not exceed $2\pi$. If it does, the total mass must then jump to the value $4\pi$ and at least one further particle is required to be present, the total system now having an $S_2$ — rather than an open — topology: $G^0_0$
is essentially the Euler density of the 2-space [4]. This example reflects the presence of effective global constraints in $2 + 1$ dimensions, even though the theory is locally trivial, so that a source distribution consisting of several individually acceptable particles is not thereby guaranteed to be itself physically acceptable. The moral applies to the Gott pair and, as we shall see, also to its lightlike extension. We emphasize that the pathology here is not merely that there is a total spacelike momentum, but more importantly, that the latter implies a “boost-identified” exterior geometry, namely one in which CTC will always be present at spatial infinity. But if one allows pathology in the boundary conditions of any system, then it is no surprise that it will be present in the interior as well! Indeed, this is just the sort of behavior that the Misner identifications [3] gave, and can be seen in the metric form of the interval as well. For, to say that the effective source is a tachyon, really means that the exterior geometry is that generated by an effective pointlike stress-tensor which replaces the $T^{00} \sim \delta(x)\delta(y)$ of a particle with a $T^{yy} \sim \delta(x)\delta(t)$, etc. Consequently, in (the cartesian coordinate form of) the Kerr line element (1.1), the $(x, y)$ space is replaced by $(x, t)$ with a jump in $t$ replaced by one in $y$ [8]. The resulting metric shows that CTC do not really appear and disappear spontaneously in some finite region where the particles pass each other, but rather that they are always present at spatial infinity; thence they close in (very rapidly!) on the finite interaction region.

Attempts to remedy these difficulties by adding more particles [9] to the system fail; it has been shown that the total momentum of any system containing the Gott time machine is necessarily tachyonic [10]; thus, two particles constituting the Gott system cannot arise from the decay of a pair of static particles (of allowed mass less than $2\pi$) since the latter’s momentum is timelike [11]. If the mass of the initial static particles exceeds $2\pi$, the universe closes; but as was shown in [12], a closed universe will end in a big crunch just before the CTC appear. Since there is no spatial infinity, the pathology there has been transmuted into a singularity!

In this paper, we extend the Gott construction to systems involving lightlike particles (“photons”). Here too, CTC will appear just as the the system becomes tachyonic. In Section 2, we review the geometries due to a two-photon source and to the “mixed” system consisting of one photon and one massive particle [13]. These systems will be our testing ground for the existence of CTC. In Section 3, we calculate the mass for these two-particle systems, thereby obtaining the condition for their total momentum to be non-tachyonic. In Section 4, the condition for CTC to arise is derived and is shown to coincide exactly with the condition that the system be tachyonic. Furthermore, it will be manifest that (since they first occur there) CTC exist at spatial infinity if they are present at all. In Section 5, the analysis is extended to a more general model, topologically massive gravity. Its two-photon solution is constructed
and is shown to exclude CTC for all positive values of the photons’ energies. This is true for the ghost Einstein theory as well.

2. Spacetimes Generated by Lightlike Sources

In this section, we review two systems, involving lightlike sources, from which we will attempt to build a time machine. The first consists of two non-colliding photons, the second of one photon and one massive particle. Each can be obtained by pasting together the appropriate one-particle solutions, which we first describe.

In \( D = 3 \), a vacuum spacetime is specified by the way in which locally flat patches are sewn together. Different patches are identified using Poincaré transformations, since these define the symmetries of flat space. This method of constructing solutions, in which the particle parameters (mass, velocity, and location) determine the transformation generators, was presented\(^2\) in [4]. This procedure is, of course, completely equivalent to the standard analytic approach of obtaining the metric from the field equations.

The simplest example is the conical spacetime describing a particle of mass \( m \) at rest at the origin of the \( x-y \) plane. This solution is obtained by excising a wedge of angle \( \Omega \) with vertex at the origin and identifying the two edges according to \( x' = \Omega m x \), where \( \Omega m \) is a rotation by \( \Omega m \)

\[
\Omega_m = \begin{pmatrix}
1 & 0 & 0 \\
0 & \cos m & \sin m \\
0 & -\sin m & \cos m
\end{pmatrix},
\] (2.1)

whose rows and columns are labelled by \( (t, x, y) \). This description is completely equivalent to the metric form (1.1) with \( J = 0 \).

The geometric description of the one-photon solution can be found from the analytic solution, or by an infinite boost of the conical static metric [13]. Consider a single photon moving along the \( x \)-axis with energy \( E \) and energy-momentum tensor \( T_{\mu\nu} = E\delta(u)\delta(y)l_\mu l_\nu \), \( l_\mu = \partial_\mu u \) where \( u = t - x \), \( v = t + x \) are the usual lightcone coordinates. The Einstein equations \( G_{\mu\nu} = T_{\mu\nu} \) can be solved with a plane-wave ansatz

\[
ds^2 = ds_0^2 + F(u, y)du^2,
\] (2.2)

where \( ds_0^2 = -dudv + dy^2 \) is the flat metric. This ansatz simplifies the Einstein tensor to \( G_{\mu\nu} = -\frac{1}{2} \partial^2 F l_\mu l_\nu \), and reduces the Einstein equations to the ordinary differential equations

\(^2\)Such procedures are described more formally in [4].
equation \( \frac{\partial^2 F}{\partial y^2} = -2E\delta(u)\delta(y) \). Solving for \( F \) yields the general one-photon solution:

\[
ds^2 = ds_0^2 - 2Ey\theta(y)\delta(u)du^2
\]

(2.3)

up to a homogeneous solution, of the form \( F = B(u)y + C(u) \), that can be absorbed by a coordinate transformation.

If we now apply the coordinate transformation \( v \rightarrow v - 2Ey\theta(y)\theta(u) \), the metric becomes

\[
ds^2 = \theta(u)\{-dud(v - 2Ey\theta(y)) + dy^2\} + \theta(-u)\{-dudv + dy^2\}.
\]

(2.4)

In this form, the geometric description of the one-photon solution becomes clear. It corresponds to making a cut along the \( u = 0, y > 0 \) halfplane extending from the photon’s worldline to infinity and then identifying \( v \) on the \( u = 0^- \) side with \( v - 2Ey \) on the \( u = 0^+ \) side. It is easily checked that the points being identified are in fact related by the Lorentz transformation

\[
N_E = \begin{pmatrix}
1 + \frac{1}{2}E^2 & -\frac{1}{2}E^2 & E \\
\frac{1}{2}E^2 & 1 - \frac{1}{2}E^2 & E \\
E & -E & 1
\end{pmatrix}.
\]

(2.5)

This matrix corresponds to the \( \beta \rightarrow 1, m \rightarrow 0 \) fixed energy \( E = \frac{m}{\sqrt{1-\beta^2}} \), limit of the boost-conjugated rotation matrix \( \Lambda_\beta \Omega_m \Lambda_\beta^{-1} \). Here \( \Lambda_\beta \) is a Lorentz boost in the \( x \)-direction,

\[
\Lambda_\beta = \frac{1}{\sqrt{1-\beta^2}} \begin{pmatrix}
1 & \beta & 0 \\
\beta & 1 & 0 \\
0 & 0 & 1
\end{pmatrix}.
\]

(2.6)

[This geometric construction of the one-photon solution is analogous to that of the Aichelburg–Sexl one-photon geometry [15] in \( D=4 \), our null boost being the analog of their null shift.] The above formulation is not unique however; an equivalent one, more analogous to the conical solution, is obtained by boosting the cone along its bisector rather than perpendicular to it [16]. The physics is of course independent of such choices.

The solution for two non-colliding photons can now be constructed by pasting together the individual one-photon solutions. [It is of course not possible to construct the two-photon solution in this way in \( D=4 \), since spacetime is not flat between sources.] We consider two non-parallel\(^3\) photons in their center-of-momentum frame,\(^6\)

\(^3\)The solution for parallel photons can also be constructed, but it does not admit CTC for the same reason as that given below for the one-photon solution.
where the photons are taken to be moving with energy $E$ respectively in the positive $x$-direction along $y = a$, $(a > 0)$ and in the negative $x$-direction along $y = -a$. The spacetime associated with the first photon is obtained by making a cut along the $u = t - x = 0$, $y > a$ halfplane and then identifying the point $(x, x, y)$ on the $u = 0^-$ side with the point $(x - E(y - a), x - E(y - a), y)$ on the $u = 0^+$ side. For the second photon, one makes a cut along $v = t + x = 0$, $y < -a$ and identifies $(-x, x, y)$ on $v = 0^-$ with $(-x + E(y + a), (x - E(y + a), y)$ on $v = 0^+$. The complete two-photon geometry then consists of these two one-photon solutions simply pasted together along the $y = 0$ plane. In contrast to the massive Gott pair, no relative boost between the two particles’ halfspaces is necessary, since the photons’ motion is already encoded in the identification made on their respective halfplanes.

The second system, consisting of a photon and a massive particle, can also be obtained by pasting together the respective one-particle solutions. We use a frame in which the massive particle is at rest at the origin in the $x - y$ plane, and the photon is moving in the positive $x$-direction along $y = a > 0$. For the static massive particle we excise from the $y < a/2$ halfspace a wedge of angle $m$ whose vertex is at the origin and which is oriented in the negative $y$-direction, then identify the edges as usual. For the photon, we simply translate the one-photon solution given above from $y = 0$ to $y = a$. If $m < \pi$, then the orientation of the wedge ensures that there is no intersection with the photon’s halfplane, so that the two-particle solution is obtained by pasting together these two halfspaces along $y = a/2$, again with no relative boost. If $m > \pi$, the solution is no longer obtainable by simple gluing, as the tails of the two sources would now overlap.

3. Total Energy and Tachyon Conditions

In this section, we calculate the total mass of the two previously described systems. In the following section, we will see that CTC arise precisely when the total mass becomes imaginary, i.e., the system becomes tachyonic and non-physical Misner identifications emerge. The mass can be found by composing the one-particle identifications and writing the result for the complete system in the form of the general spacetime identification

$$x' = a + \mathcal{L}(x - a) + b.$$  \hspace{1cm} (3.1)

The spatial vector $a = (0, a)$ describes the location of the center-of-mass; the direction and magnitude of the timelike vector $b$ define, respectively, the time axis and the time-shift along it. For a system of total mass $M$ and velocity $\beta$ in the $x$ direction, $\mathcal{L}$ will be a Lorentz transformation of the form $\mathcal{L} = \Lambda_\beta \Omega_M \Lambda_\beta^{-1}$, implying in particular that

$$\cos M = \frac{1}{2} (\text{Tr} \mathcal{L} - 1).$$  \hspace{1cm} (3.2)
The system is non-tachyonic provided $M$ is real, implying that the right-hand side lies in the range $[-1, 1]$. We now proceed to calculate $\mathcal{L}$, and from it $M$, in terms of the constituent parameters of the systems constructed in the previous section.

For the two-photon system, the one-particle identifications are given by

$$
x_1' = a + N_E(x_1 - a) \\
x_2' = -a + \Omega \pi N_E \Omega_{-\pi}(x_2 + a).
$$

(3.3)

The conjugation of $N_E$ by a $\pi$-rotation in the second equation reflects the fact that the second photon is moving in the negative $x$-direction. Composing the two identifications in (3.3), we find $\mathcal{L} = N_E\Omega_\pi N_E\Omega_{-\pi}$, and $\text{Tr} \mathcal{L} = 3 - 4E^2 + E^4$. Comparing with (3.2), we obtain the condition

$$
E > E_{\text{max}} = 2
$$

(3.4)

for the system to be tachyonic. This condition can also be formally obtained as the limit of the original Gott condition [3] for two masses $m$ moving subluminally: there, the tachyonic threshold is given by

$$
\frac{\sin \frac{1}{2}m}{\sqrt{1 - \beta^2}} > 1.
$$

(3.5)

Clearly, the limit $m \to 0, \beta \to 1$ in this equation (with the energy fixed) yields (3.4).

For the mixed system, the one-particle identifications are given by

$$
x_1' = a + N_E(x_1 - a), \\
x_2' = \Omega_m x_2.
$$

(3.6)

Composing these yields $\mathcal{L} = N_E\Omega_m$; comparison of its trace with (3.2) implies

$$
\cos M = \cos m - (\sin m)E + \frac{1}{4}(1 - \cos m)E^2.
$$

(3.7)

The criterion for tachyonic $M$ can be expressed as a condition on $E$ for fixed $m$,

$$
E > E_{\text{max}} = 2 \frac{\sin m + \sqrt{2(1 - \cos m)}}{1 - \cos m}.
$$

(3.8)
4. Closed Timelike Curves

We now find the conditions for CTC to be present in the two-photon and mixed systems. We first show that the one-photon spacetime (like the conical spacetime for a particle of non-zero mass) does not admit CTC. This may not be obvious, since the identification involves a timeshift, which as in the case of the Kerr solution (1.1), could potentially lead to CTC. Recall that the one-photon solution is characterized by the shift \((x, x, y) \to (x', x', y') = (x - Ey, x - Ey, y)\) upon crossing the \(u = t - x = 0, y > 0\) null halfplane from \(u = 0^-\) to \(u = 0^+\). A CTC, \(\gamma\), would have to cross this halfplane to take advantage of the timeshift (there being no CTC within flat spacetime patches). Irrespective of where \(\gamma\) enters the halfplane, the shifted point from which \(\gamma\) emerges is separated from the entry point by a lightlike interval. Therefore only particles travelling faster than the speed of light can complete the loop in time, showing that the one-photon solution does not admit CTC.

Now consider the two-photon solution described by the identifications (3.3). Since, as shown above, the one-photon solution does not permit CTC, we can restrict ourselves to curves \(\gamma\) that enclose both photons’ worldlines and therefore intersect both of their halfplanes. In order that it not become spacelike, \(\gamma\) must be directed opposite to the photon whose plane it is about to cross; this sense will automatically yield a gain in time upon crossing the halfplanes. Label the point at which \(\gamma\) intersects the \(u = 0, y > a\) halfplane by \(x^\mu_1 = (x_1, x_1, y_1)\) on the \(u = 0^-\) side and hence by \(x^\mu_1 = (x_1 - E(y_1 - a), x_1 - E(y_1 - a), y_1)\) on the \(u = 0^+\) side, and the point at which \(\gamma\) intersects the other halfplane by \(x^\mu_2 = (-x_2, x_2, y_2)\) on the \(v = 0^-\) side and hence by \(x^\mu_2 = (-x_2 + E(y_2 + a), x_2 - E(y_2 + a), y_2)\) on the \(v = 0^+\) side. For \(\gamma\) to be timelike, the total traversed distance,

\[
d = |x_2 - x'_1| + |x_1 - x'_2| \tag{4.1}
\]

must be less than the total elapsed time,

\[
T = (t_2 - t'_1) + (t_1 - t'_2) = E(y_1 - y_2 - 2a) \tag{4.2}
\]

For a given \(T\), we can find the minimum value of \(d\) as a function of its arguments. The extremization occurs at \(x_1 - x_2 = T/2\) and \(y_1 + y_2 = 0\), with the result that \(d_{\text{min}} = 4y_1\), with \(T = 2E(y_1 - a)\). Therefore CTC will be present if

\[
y_1/a > E/(E - 2) \tag{4.3}
\]
Recalling that $y_1 > a > 0$, we see that the lowest allowed value is $E = 2$, precisely the tachyon threshold $E_{\text{max}}$ of (3.4); there, $y_1$ (and therefore also $-y_2$) becomes infinite, i.e., CTC first arise at spatial infinity. As the energy increases, the CTC spread into the interior as well, but they are always present at spatial infinity, if present at all. [The requirement that $\gamma$ be everywhere future-directed imposes no relevant conditions: The individual time segments $(t_2 - t'_1)$ and $(t_1 - t'_2)$ must each be positive, implying the inequalities $E(y_1 - a) > (x_1 + x_2) > E(y_2 + a)$. At the minimum, they read $|x_1 + x_2| < E(y_1 - a)$, and are easily satisfied, since $(x_1 + x_2)$ is otherwise unconstrained.]

Let us now find the condition for which the “mixed” system admits CTC. Since neither individual one-particle solution alone admits CTC, a potential CTC, $\gamma$, must again enclose both particles, thereby intersecting both the photon’s halfplane and the static particle’s wedge. Let the point of intersection with the halfplane be labelled by $x_1^\mu = (x_1, x_1, y_1)$ on the $u = 0^-$ side and by $x_1'^\mu = (x_1 - E(y_1 - a), x_1 - E(y_1 - a), y_1)$ on the $u = 0^+$ side, and that with the wedge by $x_2'^\mu = (t_2, y_2 \tan \frac{m}{2}, y_2)$ on one edge and by the rotated values $x_2''^\mu = (t_2, -y_2 \tan \frac{m}{2}, y_2)$ on the other edge. The curve $\gamma$ is timelike provided the distance traversed,
\[
d = \sqrt{(x_1 - E(y_1 - a) - y_2 \tan \frac{m}{2})^2 + (y_1 - y_2)^2} + \sqrt{(x_1 + y_2 \tan \frac{m}{2})^2 + (y_1 - y_2)^2},
\]
is less than the elapsed time $T = E(y_1 - a)$. Here, we minimize $d$ with respect to $x_1$ and $y_2$ for fixed $T$ (or $y_1$); the extremum occurs at $x_1 = T/2$, $y_2 = y_1 \cos \frac{m}{2} - \frac{1}{2}E(y_1 - a)\sin \frac{m}{2}$ and is given by $d_{\text{min}} = 2a \sin \frac{m}{2} + (y_1 - a)(E \cos \frac{m}{2} + 2 \sin \frac{m}{2})$. Therefore, existence of CTC requires
\[
y_1/a > E/\left(E - 2 \frac{\sin \frac{m}{2}}{1 - \cos \frac{m}{2}}\right).
\]
Since $y_1 > 0$, $E$ must equal or exceed the threshold tachyon value $E_{\text{max}}$ of (3.8), as is easily seen using half-angle formulas. Again, $y_1$ is infinite at $E = E_{\text{max}}$, and as the energy increases, CTC begin to move into the finite region. [Here the requirements that the travel segments be future-directed reduce to $E(y_1 - a) > (x_1 - t_2) > 0$, which can always be fulfilled by adjusting $t_2$.]

5. Topologically Massive Gravity

Topologically massive gravity (TMG) is of interest because, in contrast to pure $D=3$ gravity, it is a dynamical theory. In exact solutions for lightlike sources, including (for certain orientations) two-photon solutions, were found. Here we show
that they do not admit CTC for any values of the photons’ energies. [We cannot construct the analog of the “mixed” system since the exact solution for a massive source is not known in TMG.] The field equations for TMG are the ghost \((i.e., \text{with the opposite sign of } \kappa^2)\) Einstein equations, to which is added the conformally invariant, conserved, symmetric Cotton tensor \(C_{\mu\nu} \equiv \varepsilon^{\alpha\beta\gamma\delta} D_\alpha (R_\beta \gamma - \frac{1}{4} \delta_\beta^\gamma R)\):

\[ E_{\mu\nu} \equiv G_{\mu\nu} + \frac{1}{\mu} C_{\mu\nu} = -\kappa^2 T_{\mu\nu}. \] (5.1)

Here \(\mu\) is a parameter (whose sign is arbitrary) with dimensions of mass or inverse length. These equations can be solved exactly for a photon source \([13]\), using the plane-wave ansatz \((2.2)\). For a photon with energy \(E\) moving along the positive \(x\)−axis, the resulting spacetime metric is given by

\[ ds^2 = -du dv + dy^2 + 2\kappa^2 Ef(y) \delta(u) du^2, \quad f \equiv (y + \frac{1}{\mu}(e^{-\mu y} - 1))\theta(y). \] (5.2)

Observe that as \(\mu \to \infty\), \(f(y) \to y\theta(y)\) and \((5.2)\) reduces to the ghost gravitational one-photon solution \((i.e., (2.3) \text{ with the opposite sign of } \kappa^2)\). Like its Einstein counterpart, the metric \((5.2)\) can also be obtained by a cut-and-paste procedure, albeit not by using Poincaré transformations, since the spacetime is not flat along the \(u = 0, y > 0\) null halfplane. After applying the coordinate transformation \(v \to v + 2\kappa^2 Ef(y)\theta(u)\) to remove the \(\delta(u)\) factor in \((5.2)\), it takes the form

\[ ds^2 = -du dv + dy^2 - 2\kappa^2 E\theta(u)f'(y)dudy \]

\[ = \theta(u)\{-du(dv + 2\kappa^2 Ef(y)) + dy^2\} + \theta(-u)\{-du dv + dy^2\}. \] (5.3)

Clearly, this corresponds to identifying \(v\) on \(u = 0^-\) with \(v + 2\kappa^2 Ef(y)\) on \(u = 0^+\).

We can construct the two-photon spacetime, in the convenient frame where one photon moves in the positive \(x\)-direction along \(y = a > 0\) and the other in the negative \(x\)-direction along \(y = -a\), by pasting together the one-photon solutions along the \(y = 0\) hyperplane. This pasting is possible since, as in Einstein gravity, each of the one-particle solutions is both flat and has zero extrinsic curvature on this hyperplane. The resulting spacetime consists in making cuts along the \(u = 0, y > a\) and \(v = 0, y < -a\) halfplanes and then identifying the point \((x_1, x_1, y_1)\) on the \(u = 0^-\) side with \((x_1 + \kappa^2 Ef(y_1 - a), x_1 + \kappa^2 Ef(y_1 - a), y_1)\) on the \(u = 0^+\) side, and the point \((-x_2, x_2, y_2)\) on the \(v = 0^-\) side with \((-x_2 - \kappa^2 Ef(-y_2 - a), x_2 + \kappa^2 Ef(-y_2 - a), y_2)\) on the \(v = 0^+\) side. [We note that if the directions of both photons were reversed (corresponding to the parity operation \(x \to -x\)), then this simple pasting prescription is not possible, since the individual one-photon solutions are no longer flat on the \(y = 0\) hyperplane. This reflects the parity violation implicit in the dependence.
of the field equations (5.1) on $\epsilon_{\mu\nu\rho}$. The absence of CTC for all positive values of $E$ can be seen as follows. Since $f(y) \geq 0$, the time shift upon crossing a halfplane has opposite sign relative to that of the pure gravity case, (2.3). Hence, to gain a timeshift one would have to cross the (null) halfplane from the $u > 0$ side, rather than from the $u < 0$ side, which is impossible for any timelike (or lightlike) curve. These conclusions obviously also apply to ghost Einstein gravity as well, since the latter is just the $\mu \to \infty$ limit of TMG.

6. Conclusion

We have examined, in the case where one or both of their sources are lightlike, the physical difficulties associated with geometries that permit CTC in $2+1$ Einstein gravity. We first obtained the total energy in terms of the constituent parameters, using the geometric approach in which flat patches are identified through null boosts, and found the conditions for the systems to be tachyonic. We then showed that, as the energy of a system first surpasses its tachyonic bound, CTC initially emerge at spatial infinity, then spread into the interior, but always remain present at infinity. This is, of course, the manifestation of the unphysical spatial boundary conditions that are the price paid for CTC, and are analogous to the Misner identifications that give rise to CTC in NUT space. We also demonstrated, using the known two-photon solution in TMG, that this dynamical model, and therefore also its limit, ghost Einstein theory, never admits CTC.

All sources of the $2+1$ Einstein equations considered to date thus share the property that if they are physical—nontachyonic—they do not engender acausal geometries. These results add evidence in favor of both Einstein's original hope and its recent avatar, that this is a universal property of general relativity.

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References

[1] K. Gödel, Rev. Mod. Phys. 21 (1949) 447.

As we mentioned at the outset, there is no a priori physical requirement that $\kappa^2$ be positive within the $D=3$ picture, except if one wishes to regard it as a reduction from $D=4$ in order to make contact with cosmic strings.
[2] K. Lanczos, Zeits. f. Phys. 21 (1924) 73; W.J. van Stockum, Proc. R. Soc. Edin. 57 (1937) 135; W. B. Bonnor, J. Phys. A 13 (1980) 2121.

[3] C.W. Misner, in Lectures in Applied Mathematics, vol. 8, J. Ehlers, ed., American Mathematical Society, Providence, R.I., 1967; C.W. Misner and A. Taub, Soviet Physics JETP 28 (1969) 122.

[4] S. Deser, R. Jackiw, G. ’t Hooft, Ann. Phys. 152 (1984) 220.

[5] M. Henneaux, Phys. Rev. D 29 (1984) 2766; S. Deser, Class. Quantum Grav. 2 (1985) 489.

[6] J. Gott, Phys. Rev. Lett. 66 (1991) 1126.

[7] A. Ori, Phys. Rev. D 44 (1991) R2214; C. Cutler, Phys. Rev. D 45 (1992) 487.

[8] S. Deser, R. Jackiw, and G. ’t Hooft, Phys. Rev. Lett. 68 (1992) 267.

[9] D. Kabat, Phys. Rev. D 46 (1992) 2720.

[10] S. Carroll, E. Farhi, and A. Guth, MIT preprint CTP-2117 (1992).

[11] S. Carroll, E. Farhi, and A. Guth, Phys. Rev. Lett. 68 (1992) 263, (E)3368.

[12] G. ’t Hooft, Class. Quantum Grav. 9 (1992) 1335.

[13] S. Deser and A. Steif, Gravity Theories with Lightlike Sources in $D=3$, Class. Quantum Grav., (in press).

[14] D.C. Duncan and E.C. Ihrig, Gen. Rel. Grav. 23 (1991) 381.

[15] P. C. Aichelburg and R. U. Sexl, J. Gen. Rel. Grav. 2 (1971) 303.

[16] G. ’t Hooft, private communication.

[17] S. Deser, R. Jackiw, S. Templeton, Ann. Phys. 140 (1982) 372.

[18] S.W. Hawking, Phys. Rev. D 46 (1992) 603.