(G,P)-OPERS AND GLOBAL SLODOWY SLICES

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ABSTRACT. In this paper we introduce a generalization of G-opers for arbitrary parabolic subgroups P < G. For parabolic subgroups associated to "even nilpotents" we parameterize (G,P)-opers by an object generalizing the base of the Hitchin fibration. In particular, we describe families of opers associated to higher Teichmüller spaces.

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A complex projective structure on a Riemann surface $X$ is a maximal atlas of charts with values in $\mathbb{CP}^1$ and transition functions which are restrictions of Möbius transformations. Historically, these structures arose from the study of monodromies of second order ordinary differential equations on $\mathbb{CP}^1$ minus a finite number of points. These structures were related to the uniformization problem, since a hyperbolic metric on $X$ induces a complex projective structure via the natural inclusion of the upper half plane $\mathbb{H} \subset \mathbb{CP}^1$. Interest later turned to higher order ordinary differential equations on $\mathbb{H} \subset \mathbb{CP}^1$ and their relationship to uniformization. Imposing certain invariance properties on these equations led to what are now known as developing maps of $\text{PSL}_n \mathbb{C}$-opers on $X$.

For $G$ a connected, complex semisimple Lie group, the notion of a $G$-oper structure on $X$ was codified by Beilinson-Drinfeld [2]. In their language, a $G$-oper is a triple $(E_\mathbb{G}, E_\mathbb{B}, \omega)$, where $E_\mathbb{G}$ is a holomorphic principal $G$-bundle on $X$, $E_\mathbb{B}$ is a holomorphic reduction to a Borel subgroup $B < G$, and $\omega$ is a holomorphic connection on $E_\mathbb{G}$ which satisfies certain transversality and nowhere vanishing properties with respect to the reduction $E_\mathbb{B}$. When $G = \text{PSL}_2 \mathbb{C}$, a $G$-oper on $X$ is equivalent to a $\mathbb{CP}^1$-structure on $X$.

Around the same time, Hitchin [24] introduced a special component of representations $\pi_1(X) \to G_\mathbb{R}$, where $G_\mathbb{R} < G$ is a split real form and $X$ is a compact Riemann surface of genus $g \geq 2$. For example, $\text{PSL}_n \mathbb{R}$ is the split real form of $\text{PSL}_n \mathbb{C}$. This component is now called the Hitchin component. When $G = \text{PSL}_2 \mathbb{C}$, the Hitchin component corresponds to holonomy representations of hyperbolic structures on the underlying topological surface, and so, is identified with Teichmüller space.

Although representations in Hitchin components and $G$-opers are very different geometrically, these spaces share many common features. Most notably, both spaces are parameterized by a vector space of holomorphic differentials on $X$ called the base of the Hitchin fibration [2, 24]. More recently, in [15], the authors showed that an operation called the conformal limit identifies these parameterizations, proving a conjecture of Gaiotto [17].

In [30], Labourie proved that the representations in Hitchin components generalize many geometric features of Teichmüller space; in particular, they are discrete and faithful. Moreover, he showed that Hitchin representations satisfy a very strong dynamical property called the Anosov property. Since Labourie’s work, there has been intense activity aimed at organizing and classifying Anosov representations. Notable among these spaces is the class of so called higher Teichmüller spaces (see [10, 22]). In the theory of Anosov representations, a parabolic subgroup $P < G$ plays a fundamental role; for Hitchin components the appropriate parabolic is the Borel subgroup.

Motivated by these spaces of representations and the conformal limit correspondence, in this paper, we introduce the notion of a $(G, P)$-oper for $P < G$ a parabolic subgroup. When $P = B$ is a Borel subgroup, we recover
Beilinson-Drinfeld’s notion of G-opers. Our main results (Theorems 5.8, 5.11 and 5.13) generalize the parameterization of Hitchin components and G-opers by the base of the Hitchin fibration.

Although many of the notions introduced involve a significant amount of Lie theory, in most cases these objects can be made relatively explicit with matrices and vector bundles. Throughout the paper, we have included many such examples to guide the reader. In a subsequent paper, we will develop the theory of (G, P)-opers using developing maps of geometric structures.

(G,P)-opers. Let $X$ be a compact Riemann surface of genus $g \geq 2$ and let $P < G$ be a parabolic subgroup of a connected complex semisimple Lie group $G$. Roughly, a $(G, P)$-oper on $X$ is a triple $(E_G, E_P, \omega)$, where $E_G$ is a holomorphic principal $G$-bundle on $X$, $E_P$ is a holomorphic reduction to the parabolic $P < G$, and $\omega$ is a holomorphic connection on $E_G$ which satisfies certain transversality and nowhere vanishing properties with respect to the reduction $E_P$. The subtle point is defining the correct notion of nowhere vanishing (see Definition 4.1).

A simple example is given by $G = \text{SL}_2$ and $P$ is the stabilizer of a half dimensional subspace $\mathbb{C}^n \subset \mathbb{C}^{2n}$. In this case, a $(G, P)$-oper can be defined as a triple $(V_2, V_n, \nabla)$, where $V_2$ is a rank 2 holomorphic vector bundle with trivial determinant, $V_n \subset V_2$ is a rank $n$ holomorphic subbundle and $\nabla$ is a holomorphic connection on $V_2$ such that the induced map $\nabla : V_n \to \mathcal{K} \otimes V_2/V_n$ is an isomorphism.

We briefly describe the picture we wish to generalize. In [23], Hitchin introduced the moduli space $\mathcal{M}_X^0(G)$ of polystable $G$-Higgs bundles on $X$.

- (1) There is a homeomorphism $\mathcal{T} : \mathcal{M}_X^0(G) \to \mathcal{M}_X^1(G)$ from Higgs bundles to the moduli space of holomorphic $G$-connections on $X$ [13, 14, 23, 34]. This is known as the nonabelian Hodge correspondence.
- (2) There is a fibration $\mathcal{M}_X^0(G) \to \bigoplus_{j=1}^{rkG} H^0(K^{m_j+1})$ [25], where $\mathcal{K}$ is the canonical bundle of $X$ and $\{m_j\}$ are the exponents of $g$.
- (3) The fibration has a section $s_h : \bigoplus_{j=1}^{rkG} H^0(K^{m_j+1}) \to \mathcal{M}_X^1(G)$ and the Hitchin components are the holonomy representations of $\mathcal{T} \circ s_h$ [24].
- (4) Each connected component of isomorphism classes of G-opers is parameterized by $\bigoplus_{j=1}^{rkG} H^0(K^{m_j+1})$ [2].

We summarize this in the following diagram:

$$
\begin{array}{ccc}
\mathcal{M}_X^0(G) & \xrightarrow{\mathcal{T}} & \mathcal{M}_X^1(G) \\
\bigoplus_{j=1}^{rkG} H^0(K^{m_j+1}) \downarrow \scriptstyle{s_h} & & \downarrow \scriptstyle{op} \\
\end{array}
$$

In fact, for $\lambda \in \mathbb{C}$, the above moduli spaces fit into a family of moduli spaces $\mathcal{M}_X^\lambda(G)$ of $\lambda$-connections. Moreover, there is a family of maps $op_0 = s_h$ and $op_1 = op$. Roughly, the
maps $\text{op}_{\lambda}$ are defined by choosing a base point (depending on a $\text{PSL}_2\mathbb{C}$-oper) and adding holomorphic differentials as coefficients in certain highest weight spaces of $\mathfrak{sl}_2\mathbb{C}$ representations.

**Remark 1.1.** It is important to note that $\text{op} \neq \mathcal{T} \circ s_h$. Indeed, for $G = \text{PSL}_2\mathbb{C}$, opers correspond to holonomies of $\mathbb{C}P^1$-structures on $X$ while $\mathcal{T} \circ s_h$ corresponds to holonomies of hyperbolic structures.

Our parameterization theorems do not apply to all parabolic subgroups but to a class of parabolics called **even Jacobson-Morozov parabolics** (see Definition 2.8). Such parabolics $P$ are associated to embeddings of $\text{PSL}_2\mathbb{C}$ in the adjoint Lie group, or equivalently, even $\mathfrak{sl}_2\mathbb{C}$ subalgebras of the Lie algebra $\mathfrak{g}$. There are many examples of such parabolics, for instance, the Borel subgroup $B = \mathcal{B}_0(G)$ and the stabilizer of a subspace $\mathbb{C}^n \subset \mathbb{C}^{2n}$. The object which parameterizes such $(G, P)$-opers (i.e., the generalization of the Hitchin base) is a groupoid which we call the **Slodowy category**. The root of this additional structure stems from the centralizer of the associated $\mathfrak{sl}_2\mathbb{C}$ subalgebra. Briefly, the centralizer of an $\mathfrak{sl}_2\mathbb{C}$ subalgebra is discrete if and only if it is a so called principal subalgebra, in which case, the associated parabolic is the Borel $B = \mathcal{B}_0(G)$.

Given an even $\mathfrak{sl}_2\mathbb{C}$ subalgebra $\mathfrak{s} \subset \mathfrak{g}$, let $S = G$ be the connected Lie group with Lie algebra $\mathfrak{s}$ and $C < G$ be the centralizer of $S$. The $\mathfrak{s}$-Slodowy category $\mathcal{B}_s(G)$ has objects $(E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})$, where $E_C$ is a holomorphic $C$-bundle, $\psi_0$ is a holomorphic connection on $E_C$ and, for $1 \leq j \leq N$, $\psi_{m_j}$ is a holomorphic section of an associated vector bundle $E_C[Z_{2m_j}] \otimes K^{m_j+1}$; the vector spaces $Z_{2m_j}$ are zero weight spaces of certain $\mathfrak{sl}_2\mathbb{C}$ representations (see Definition 5.1). When $B$ is the Borel, $C$ is the center of $G$ and $\mathcal{B}_s$ is the product of the Hitchin base with the finite set of $C$-bundles on $X$. When $P < \text{SL}_2\mathbb{C}$ is the stabilizer of a subspace $\mathbb{C}^n \subset \mathbb{C}^{2n}$, objects of the Slodowy category correspond to triples $(W, \psi_0, \psi_1)$, where $W$ is a rank $n$ holomorphic vector bundle on $X$ with $\det(W) \otimes^2$ trivial, $\psi_0$ is a holomorphic connection on $W$ and $\psi_1 \in H^0(K^2 \otimes \text{End}(W))$.

To avoid complications with moduli spaces we work with groupoids. Denote the category of $(G, P)$-opers on $X$ by $\mathcal{O}_pX(G, P)$. For an even $\mathfrak{sl}_2\mathbb{C}$ subalgebra $\mathfrak{s} \subset \mathfrak{g}$, let $S = G$ be the associated connected Lie group and $P = G$ be the associated even Jacobson-Morozov parabolic. In Theorem 5.5 we establish the following: For each $S$-oper $\Theta$ on $X$, there is a natural functor (the $(\Theta, \mathfrak{s})$-Slodowy functor)

$$F_{\Theta} : \mathcal{B}_s(G) \to \mathcal{O}_pX(G, P),$$

which is a bijection on isomorphism classes (see also Theorem 5.11). The Slodowy functor is defined in Definition 5.3. Roughly, the functor uses the fixed $S$-oper to interpret the sections $\psi_{m_j}$ as coefficients in certain highest weight spaces of $\mathfrak{sl}_2\mathbb{C}$ representations. When $P = B$ is a Borel subgroup, this is analogous to the Hitchin section.
To remove the choice of S-oper, we define a subcategory \( \hat{B}_s(G) \) of \( B_s(G) \) which is analogous to setting the quadratic differential in the Hitchin base to be zero. We now have a functor \( F : Op_X(S,B_s) \times \hat{B}_s(G) \to Op_X(G,P) \) defined by \( F(\Theta,\Xi) = F_{\Theta}(\Xi) \) which we call the \( s \)-Slodowy functor.

**Theorem 1.2.** (Theorem 5.8) Let \( X \) be a compact Riemann surface of genus \( g \geq 2 \) and \( G \) be a complex connected semisimple Lie group. Let \( s \subset g \) be an even \( \mathfrak{sl}_2 \mathbb{C} \) subalgebra, \( S < G \) be the associated connected subgroup and \( P < G \) be the associated even Jacobson-Morozov parabolic. Then the \( s \)-Slodowy functor

\[
F : Op_X(S,B_s) \times \hat{B}_s(G) \to Op_X(G,P) ; (\Theta,\Xi) \mapsto F_{\Theta}(\Xi)
\]

is an equivalence of categories when \( S \cong PSL_2\mathbb{C} \) and essentially surjective and full when \( S \cong SL_2\mathbb{C} \). In particular, the Slodowy functor induces a bijection on isomorphism classes.

**Remark 1.3.** For \( G = SL_{kr}\mathbb{C} \) and the parabolic \( P \) stabilizing a partial flag \( \mathbb{C}^r \subset \mathbb{C}^{2r} \subset \cdots \subset \mathbb{C}^{kr} \), the above theorem recovers results of Biswas in [4]. Partially motivated by the work presented in the current article, the Biswas-Schaposnik-Wang recently considered certain symplectic and orthogonal analogues of such partial flags, see [5].

The category of \((G,P)\)-opers and the Slodowy category have natural \( \lambda \)-connection generalizations, denoted by \( Op_{\lambda \chi}X(G,P) \) and \( B_{\lambda \chi}(G) \) respectively. The Slodowy functor also naturally generalizes to this context, and in Theorem 5.13 we establish the analogue of the above theorem in this setting.

For \( \lambda = 0 \), a \((\lambda,G,P)\)-oper is a special type of Higgs bundle. For \( G = SL_{2n}\mathbb{C} \) and \( P \) the stabilizer of a subspace \( \mathbb{C}^n \subset \mathbb{C}^{2n} \), the above objects can be made very explicit. The objects of \( B_0^s(G) \) consist of triples \((W,\psi_0,\psi_1)\), where \( W \) is a holomorphic vector bundle of rank \( n \) such that \( \det(W) \otimes^2 \) is trivial, \( \psi_0 \in H^0(K \otimes \text{End}(W)) \) is a \( K \)-twisted traceless endomorphism and \( \psi_1 \in H^0(K^2 \otimes \text{End}(W)) \) is a \( K^2 \)-twisted endomorphism. A triple \((W,\psi_0,\psi_1)\) is an object of \( B_0^s(G) \) if and only if \( \text{Tr}(\psi_1) = 0 \).

For the \((0,SL_2\mathbb{C},B)\)-oper \( \Theta = (K^{\frac{1}{2}} \oplus K^{-\frac{1}{2}}, (0_1 0)), \) the Slodowy functor is defined by

\[
(1.1) \quad F_{\Theta}(W,\psi_0,\psi_1) = (W \otimes K^{\frac{1}{2}} \oplus W \otimes K^{-\frac{1}{2}}, \left( \begin{array}{cc} \psi_0 & q_2 \otimes \text{Id}_W \psi_1 \\ \text{Id}_W \psi_0 & \psi_1 \end{array} \right)).
\]

When \( \psi_0 = 0 \), this is an \( SU_{n,n} \)-Higgs bundle, and the above description recovers the so called Cayley correspondence for maximal \( SU_{n,n} \)-Higgs bundles of [7]. Moreover, when the associated Higgs bundles are polystable, the non-abelian Hodge correspondence identifies them with the higher Teichmüller spaces known as maximal \( SU_{n,n} \)-representations.

**Remark 1.4.** The relation of our construction with other higher Teichmüller spaces is described in §6. While there are \((G,P)\)-opers associated to each expected higher Teichmüller space, there are many examples of \((G,P)\)-opers which are not related to higher Teichmüller spaces. For \( P \) an arbitrary even
Jacobson-Morozov parabolic, it would be interesting to understand whether the representations associated to applying nonabelian Hodge to the set of $(0,G,P)$-opers have any special geometric significance.

Finally, we discuss the relationship between $(G,P)$-opers, Simpson’s partial oper stratification and the conformal limit. In [15], the authors showed that the space of $(G,B)$-opers are related to the Hitchin section by an operation called the conformal limit, proving a conjecture of Gaiotto [17]. For $G = \text{SL}_n \mathbb{C}$, the space of $B$-opers and the Hitchin section are each a closed stratum of Bialynicki-Birula stratifications introduced by Simpson [33]. Moreover, under a smoothness assumption, the conformal limit correspondence for arbitrary strata was established in [11]. In the generality of this paper, it is natural to expect that the conformal limit also identifies $(0,G,P)$-opers with $(1,G,P)$-opers. Indeed, for the case $G = \text{SL}_2 \mathbb{C}$ and $P$ the stabilizer of a subspace $\mathbb{C}^n \subset \mathbb{C}^{2n}$, when the bundle $W$ in (1.1) is a stable bundle, this follows from the work in [11].

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## 2. Lie Theory Preliminaries

In this section, we recall most of the Lie theory which will be used throughout the paper. For this section, $G$ will be a connected, complex semi-simple Lie group with Lie algebra $\mathfrak{g}$ and Killing form $B_\mathfrak{g}$.

### 2.1. Parabolics

A Borel subgroup $B < G$ is a maximal solvable subgroup. There is a unique Borel subgroup up to conjugation. A subgroup $P < G$ is called a parabolic subgroup if it contains a Borel subgroup. This is equivalent to the homogeneous space $G/P$ being a projective variety.

For a parabolic $P < G$, let $U < P$ be the unipotent radical. There is a canonical filtration of $U = U^1$,

$$U^m < \cdots < U^2 < U^1 < P.$$ 

The quotient $L = P/U$ is a reductive group called the Levi factor of $P$. Choosing a splitting of the sequence $1 \to U \to P \to L \to 1$ defines a Levi subgroup of $P$.

Let $\mathfrak{u} \subset \mathfrak{p}$ be the Lie subalgebra associated to the subgroup $U < P$. There is a canonical $P$-invariant Lie algebra filtration of $\mathfrak{g}$ defined by

$$0 \subset \mathfrak{g}^m \subset \cdots \subset \mathfrak{g}^1 \subset \mathfrak{g}^0 \subset \mathfrak{g}^{-1} \subset \cdots \subset \mathfrak{g}^{-m} = \mathfrak{g},$$

where $\mathfrak{g}^j = \{ x \in \mathfrak{g} \mid \text{ad}_x(u) \subset \mathfrak{g}^{j+1} \}$. In particular, $\mathfrak{g}^1 = \mathfrak{u}$ and $\mathfrak{g}^0 = \mathfrak{p}$.
A choice of a Levi subgroup $L < P$ defines a splitting $p = l \oplus u$, where $l$ is the Lie algebra of $L$. Moreover, $\mathfrak{g}$ acquires an $L$-invariant grading

$$\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j,$$

which is $L$-equivariantly isomorphic to the associated graded of the canonical filtration defined above. Thus, $\mathfrak{g}_j \cong \mathfrak{g}^j/\mathfrak{g}^{j+1}$ as $L$-modules and $\mathfrak{g}^k \cong \bigoplus_{j \geq k} \mathfrak{g}_j$ as $P$-modules. In particular,

$$p \cong \bigoplus_{j \geq 0} \mathfrak{g}_j, \quad l \cong \mathfrak{g}_0 \quad \text{and} \quad u \cong \bigoplus_{j > 0} \mathfrak{g}_j.$$

By a theorem of Vinberg (see [27, Theorem 10.19]), each graded piece $\mathfrak{g}_j$ has a unique dense open $L$-orbit $O_j \subset \mathfrak{g}_j$. Since the action of $P$ and $L$ on $\mathfrak{g}^j/\mathfrak{g}^{j+1}$ are the same, there is a unique of open dense $P$-orbit in $\mathfrak{g}^j/\mathfrak{g}^{j+1}$.

**Example 2.1.** For the group $G = \text{SL}_4\mathbb{C}$, the group of unit determinant upper triangle matrices defines a Borel subgroup. The parabolic subgroups which contain this choice of $B$ consist of block upper triangle matrices. Geometrically, $B$ is the stabilizer of a complete flag in $\mathbb{C}^4$ and other parabolics are stabilizers of partial flags in $\mathbb{C}^4$. Here are some explicit examples:

1. The group $P = B$ which stabilizes a complete flag $C \subset C^2 \subset C^3 \subset C^4$ consists of unit determinant upper triangular matrices. The unipotent radical $U < B$ consists upper triangular matrices with 1’s on the diagonal. The diagonal matrices define a Levi subgroup $L < B$, and, with this choice, the graded pieces $\mathfrak{g}_j$ are given by the $j^{th}$-super diagonal. In particular, $\mathfrak{g}_{-1}$ is given by matrices of the form

$$\begin{pmatrix} \mathbf{0} & x_1 & 0 & 0 \\ x_2 & \mathbf{0} & 0 & 0 \\ x_3 & 0 & \mathbf{0} & 0 \\ x_4 & 0 & 0 & \mathbf{0} \end{pmatrix},$$

The unique open $L$-orbit $O_{-1} \subset \mathfrak{g}_{-1}$ is defined by $x_j \neq 0$ for all $j$.

2. The group $P < \text{SL}_4\mathbb{C}$ which stabilizes a partial flag $C^2 \subset C^3 \subset C^4$ consists of unit determinant $(2 \times 2)$-block matrices of the form

$$\begin{pmatrix} A & B \\ 0 & D \end{pmatrix}.$$ 

The unipotent radical $U < P$ is the subgroup where $A = \text{Id} = D$, and subgroup where $B = 0$ is defines a Levi. This choice gives

$$\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \begin{pmatrix} \mathbf{0} & \mathbf{0} \\ C & \mathbf{0} \end{pmatrix} \oplus \begin{pmatrix} A & \mathbf{0} \\ 0 & D \end{pmatrix} \oplus \begin{pmatrix} \mathbf{0} & B \\ 0 & \mathbf{0} \end{pmatrix}.$$

The unique open $L$-orbit $O_{-1} \subset \mathfrak{g}_{-1}$ is defined by $\det(C) \neq 0$.

3. The group $P < \text{SL}_4\mathbb{C}$ which stabilizes a partial flag $C^1 \subset C^3 \subset C^4$ is given by unit determinant matrices of the form

$$\begin{pmatrix} \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \\ \ast & \ast & \ast & \ast \end{pmatrix}.$$
Block diagonal matrices define a Levi subgroup, and, with this choice,\[
\mathfrak{p} = \mathfrak{g}_0 \oplus \mathfrak{g}_1 \oplus \mathfrak{g}_2 = \begin{pmatrix}
* & 0 & 0 & 0 \\
* & * & 0 & 0 \\
* & 0 & * & 0 \\
* & * & * & *
\end{pmatrix} \oplus \begin{pmatrix}
0 & * & 0 & 0 \\
0 & 0 & * & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
0 & 0 & * & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
The unique open L-orbit \(O_{-1} \subset \mathfrak{g}_{-1}\) is given by\[
O_{-1} = \left\{ \begin{pmatrix} 0 & 0 & 0 & c \\
0 & 0 & 0 & d \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \end{pmatrix} \bigg| (c, d) \neq 0 \right\}.
\]
(4) For the group \(P < SL_4\mathbb{C}\) which stabilizes a partial flag \(\mathbb{C} \subset \mathbb{C}^4\), block diagonal matrices define a Levi. With this choice, we have\[
\mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 = \begin{pmatrix}
0 & * & 0 & 0 \\
* & 0 & 0 & 0 \\
* & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix} \oplus \begin{pmatrix}
* & * & * & * \\
* & * & * & * \\
* & * & * & * \\
* & * & * & *
\end{pmatrix} \oplus \begin{pmatrix}
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0
\end{pmatrix}.
\]
The unique open L-orbit \(O_{-1} \subset \mathfrak{g}_{-1}\) consists of nonzero vectors.

The next example involves some basic root theory.

**Example 2.2.** For \(P = B\), a Levi subgroup \(L < B\) is a choice of Cartan subgroup. This data defines a set of positive simple roots \(\{\alpha_1, \ldots, \alpha_{rk(g)}\}\). The spaces \(\mathfrak{g}_j\) consist of direct sums of root spaces \(\mathfrak{g}_\alpha\) associated to roots \(\alpha = \sum_{i=1}^{rk(g)} n_i \alpha_i\) with \(\sum n_i = j\). In particular, the \(\mathfrak{g}_{-1}\) space consists of a direct sum of negative simple root spaces. In this case, an element \(x = (x_1, \ldots, x_{rk(g)}) \in \mathfrak{g}_{-\alpha_1} \oplus \cdots \oplus \mathfrak{g}_{-rk(g)}\) is in the open dense L-orbit \(O_{-1} \subset \mathfrak{g}_{-1}\) if and only if \(x_j \neq 0\) for all \(j\).

### 2.2. Nilpotents and the Jacobson-Morozov Theorem

An element \(e \in \mathfrak{g}\) is a nilpotent element if \(ad_e : \mathfrak{g} \to \mathfrak{g}\) is a nilpotent endomorphism of \(\mathfrak{g}\). Denote the centralizer of a nilpotent \(e \in \mathfrak{g}\) by \(V(e) = \ker(ad_e : \mathfrak{g} \to \mathfrak{g})\).

The Jacobson-Morozov theorem defines a bijective correspondence between conjugacy classes of \(\mathfrak{sl}_2\mathbb{C}\) subalgebras of \(\mathfrak{g}\) and conjugacy classes of nonzero nilpotents in \(\mathfrak{g}\) (see for example [12, §3]). Namely, every nonzero nilpotent \(e \in \mathfrak{g}\) can be completed to an \(\mathfrak{sl}_2\mathbb{C}\)-triple \(\langle f, h, e \rangle \subset \mathfrak{g}\), where\[
[h,e] = 2e, \quad [h,f] = -2f \quad \text{and} \quad [e,f] = h,
\]
and if \(\langle f, h, e \rangle\), \(\langle f', h, e \rangle\) are two such \(\mathfrak{sl}_2\)-triples then \(f = f'\). Throughout the paper, we will use the letter \(s\) to denote an \(\mathfrak{sl}_2\)-subalgebra.

Fix an \(\mathfrak{sl}_2\)-triple \(\langle f, h, e \rangle = s \subset \mathfrak{g}\). This data decomposes \(\mathfrak{g}\) into as \(\mathfrak{sl}_2\mathbb{C}\)-modules and \(ad_h\)-weight spaces. Namely,\[
\mathfrak{g} = \bigoplus_{j \geq 0} W_j,
\]
where \(W_j\) is isomorphic to a direct sum of \(n_j\)-copies (with \(n_j \geq 0\)) of the unique \((j+1)\)-dimensional \(\mathfrak{sl}_2\mathbb{C}\) representation, and\[
\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_j,
\]
where \( g_j = \{ x \in g \mid \text{ad}_h(x) = jx \} \). Note that \( e \in g_2 \) and \( f \in g_{-2} \). Note also that the trivial representation \( W_0 \) is a reductive subalgebra since it is the centralizer of subalgebra \( \mathfrak{s} \). We will denote this subalgebra by \( W_0 = \mathfrak{c}(\mathfrak{s}) = \mathfrak{c} \).

Let \( \mathfrak{b}_4 = \langle h, e \rangle \subset g \) to be the Borel subalgebra of \( \mathfrak{s} \). The centralizer \( V \) of \( e \) decomposes \( (\mathfrak{c} \oplus \mathfrak{b}_4) \)-invariantly as

\[
V = \bigoplus_{j \geq 0} V_j,
\]

where \( V_j = g_j \cap W_j \) is the highest weight space of \( W_j \). The affine space

\[
f + V \subset g.
\]

is called the Slodowy slice through \( e \). It is transverse to the \( G \)-orbit of \( e \) in \( g \) [37, §7.4].

**Definition 2.3.** Fix an \( \mathfrak{sl}_2 \)-triple \( \langle f, h, e \rangle \). We call the \( (\mathfrak{c} \oplus \mathfrak{b}_4) \)-invariant decomposition (2.5) of the centralizer \( V \) of \( e \) the Slodowy data of \( \langle f, h, e \rangle \).

The following proposition is immediate and will be useful later.

**Proposition 2.4.** Let \( \bigoplus_j V_j \) be the Slodowy data of an \( \mathfrak{sl}_2 \)-triple \( \langle f, h, e \rangle \). Then the subspace \( V_2 \) has an \( \mathfrak{c} \oplus \mathfrak{b}_4 \)-invariant decomposition

\[
V_2 = \langle e \rangle \oplus \hat{V}_2,
\]

where \( \hat{V}_2 \) is the kernel of the linear functional \( B_g(f, -)|_{V_2} \).

The decomposition (2.4) associates a parabolic subalgebra \( \mathfrak{p} \) to an \( \mathfrak{sl}_2 \)-triple \( \langle f, h, e \rangle \),

\[
\mathfrak{p} = \bigoplus_{j \geq 0} g_j.
\]

Moreover, this parabolic comes with a choice of Levi subalgebra \( \mathfrak{l} = \mathfrak{g}_0 \subset \mathfrak{p} \). Note that \( \mathfrak{p} \) also decomposes as

\[
\mathfrak{p} = V \oplus \text{ad}_f(u).
\]

In particular, for all \( j \geq 1 \) and \( u \in U \) we have

\[
\text{Ad}(u)(f) = f + a + b,
\]

where \( a \in \text{ad}_f(u^j) \) is nonzero and \( b \in u^j \).

The following theorem relates the open dense \( L \)-orbit \( O_2 \subset g_2 \) with the \( \mathfrak{sl}_2 \)-triple. It is a combination of results of Kostant and Malcev, see [27, §10] or [12, §3.4] for details.

**Theorem 2.5.** Let \( \langle f, h, e \rangle \subset g \) be an \( \mathfrak{sl}_2 \)-triple, and consider the \( \mathbb{Z} \)-grading \( g = \bigoplus_{j \in \mathbb{Z}} g_j \) from (2.4). Let \( \mathfrak{p} \) and \( \mathfrak{l} \) be the associated parabolic and Levi subalgebras with associated Lie subgroups \( L < P < G \). Then the nilpotent elements \( e \) and \( f \) are contained in the unique open dense \( L \)-orbits \( O_2 \subset g_2 \) and \( O_{-2} \subset g_{-2} \), respectively. Moreover, the \( L \)-stabilizer of \( e \) is the Lie group \( C < G \) which centralizes the \( \mathfrak{sl}_2 \)-subalgebra.
2.3. **Even Jacobson-Morozov parabolics.** We will call a parabolic which arises from an $\mathfrak{sl}_2$-subalgebra a *Jacobson-Morozov parabolic* (abbreviated JM-parabolic). Not all parabolics are JM-parabolics. For example, the stabilizer of a line in $\mathbb{C}^n$ is not a JM-parabolic when $n > 2$. Moreover, different conjugacy classes of $\mathfrak{sl}_2$-triples can define the same JM-parabolic.

**Definition 2.6.** An $\mathfrak{sl}_2$-triple $\langle f, h, e \rangle \subset g$ is called *even* if $g_1 = 0$ in the decomposition (2.4).

**Remark 2.7.** An $\mathfrak{sl}_2$-triple $\langle f, h, e \rangle \subset g$ is an even if and only if $g_{2j+1} = 0$ for all $j$ in decomposition (2.4). Equivalently, the $\mathfrak{sl}_2 \mathbb{C}$-module decomposition from (2.3) consists only of odd dimensional irreducible $\mathfrak{sl}_2$-representations.

**Definition 2.8.** A parabolic subgroup $P < G$ will be called an *even JM-parabolic* if it arises from an even $\mathfrak{sl}_2$-triple.

**Remark 2.9.** Since the JM-parabolic $p$ associated to an $\mathfrak{sl}_2$-triple $\langle f, h, e \rangle$ comes with a choice of Levi subalgebra, we have two $\mathbb{Z}$-gradings. Namely, the $\text{ad}_h$-weight space decomposition (2.4) and the associated graded of the canonical filtration (2.2). For even JM-parabolics, the $j^{th}$-graded piece of (2.2) is the $2j^{th}$-graded piece of (2.4).

The following proposition is an easy consequence of [12, §3.8].

**Proposition 2.10.** There is a one-to-one correspondence between conjugacy classes of even $\mathfrak{sl}_2$-triples and conjugacy classes of even JM-parabolics.

Fix an even $\mathfrak{sl}_2$-triple $\langle f, h, e \rangle \subset g$. Let $1 = m_1 < m_2 < \cdots < m_N$ be such that the $\mathfrak{sl}_2$-representations $W_{2m_j}$ from (2.3) are nonzero. We have

$$g = \mathfrak{c} \oplus \bigoplus_{j=1}^N W_{2m_j}. \quad (2.8)$$

The Slodowy data from (2.5) is

$$V = \mathfrak{c} \oplus \bigoplus_{j=1}^N V_{2m_j}. \quad (2.9)$$

Note that $\text{ad}^{m_j}_{f_j}(V_{2m_j}) \subset \mathfrak{g}_0$, the defines an $\mathfrak{c} \oplus \langle h \rangle$-invariant decomposition of the Levi subalgebra of the even JM-parabolic. This decomposition will be important later.

**Proposition 2.11.** The Slodowy data of an even JM-parabolic defines an $\mathfrak{c} \oplus \langle h \rangle$-invariant decomposition of the Levi subalgebra $\mathfrak{g}_0$

$$\mathfrak{g}_0 = \mathfrak{c} \oplus \bigoplus_{j=1}^N Z_{2m_j}. \quad (2.10)$$
where $Z_{2m_j} = \text{ad}_f^{m_j}(V_{2m_j}) = g_0 \cap W_{2m_j}$. Furthermore, $Z_2$ decomposes as
$$Z_2 = \text{ad}_f(e) \oplus \hat{V}_2 = \langle h \rangle \oplus \hat{Z}_2.$$ 

**Example 2.12.** The first three parabolic subgroups of $\text{SL}_4 C$ described in Example 2.1 are even JM-parabolics. In fact these are the only even JM-parabolic subgroups of $\text{SL}_4 C$. Below, we use the same numbering as the items in Example 2.1.

1. An associated $\mathfrak{sl}_2$-triple is
   $$\mathfrak{s} = \langle f, h, e \rangle = \left( \begin{array}{ccc} 0 & 3 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 1 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & -3 \end{array} \right), \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array} \right).$$
   The centralizer of $\mathfrak{s}$ is the center, $C = \langle \sqrt{-1} \text{Id} \rangle$. The associated $\mathfrak{s}$-module decomposition (2.3) is
   $$\mathfrak{sl}_4 C = W_0 \oplus W_2 \oplus W_4,$$
   with multiplicities $(n_0, n_2, n_4) = (1, 1, 1)$. The Slodowy data is
   $$V_0 \oplus V_2 \oplus V_4 = \left( \begin{array}{ccc} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right),$$
   and $\hat{V}_2 = 0$.

2. An associated $\mathfrak{sl}_2$-triple is given by $2 \times 2$-block matrices
   $$\mathfrak{s} = \langle f, h, e \rangle = \left( \begin{array}{ccc} 0 & 3 & 0 \\ 0 & 0 & 4 \\ 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$
   The centralizer of $\mathfrak{s}$ is given by
   $$C = \{ ( \alpha \alpha \alpha ) \mid \det(A)^2 = 1 \} \cong \text{SL}_2^\pm C.$$
   The associated $\mathfrak{s}$-module decomposition (2.3) is
   $$\mathfrak{sl}_4 C = W_0 \oplus W_2,$$
   with multiplicities $(n_0, n_2) = (3, 4)$. The Slodowy data is
   $$V_0 \oplus V_2 = \left( \begin{array}{ccc} X & 0 \\ 0 & Y \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & 0 \\ 0 & 0 \end{array} \right),$$
   where $\text{Tr}(X) = 0$, and the subspace $\hat{V}_2$ is given by $\text{Tr}(Y) = 0$.

3. An associated $\mathfrak{sl}_2$-triple is given by
   $$\mathfrak{s} = \langle f, h, e \rangle = \left( \begin{array}{ccc} 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 2 & 0 \\ 0 & 0 & 2 \\ 0 & 0 & 0 \end{array} \right), \left( \begin{array}{ccc} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right).$$
   The centralizer of $\mathfrak{s}$ is
   $$C = \{ ( \alpha \alpha \alpha ) \mid \alpha \in \mathbb{C}^* \} \cong \text{GL}_1 C.$$
   The associated $\mathfrak{s}$-module decomposition (2.3) is
   $$\mathfrak{sl}_4 C = W_0 \oplus W_2 \oplus W_4,$$
   with multiplicities $(n_0, n_2, n_4) = (1, 3, 1)$. The Slodowy data is
   $$V_0 \oplus V_2 \oplus V_4 = \left( \begin{array}{ccc} a & 0 \\ a & 0 \\ a & 0 \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & x & 0 \\ 0 & 0 & x \\ 0 & 0 & 0 \end{array} \right) \oplus \left( \begin{array}{ccc} 0 & 0 & w \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{array} \right),$$
and the subspace $\hat{V}_2$ is defined by setting $x = 0$.

The next example sketches the construction and states some properties of principal $\mathfrak{sl}_2$-triples, for more information see [28] or [12, §4].

**Example 2.13.** The Borel subgroup $B < G$ is always an even JM-parabolic. An associated even $\mathfrak{sl}_2$-triple is called a principal $\mathfrak{sl}_2$-subalgebra. Fix a Cartan subalgebra $\mathfrak{g}_0 \subset \mathfrak{g}$ and a set of simple roots $\{\alpha_1, \cdots, \alpha_{rk(\mathfrak{g})}\}$. For each $\alpha_j$ choose $e_j \in \mathfrak{g}_{\alpha_j}$ and $f_j \in \mathfrak{g}_{-\alpha_j}$ so that $\langle f_j, [e_j, f_j], e_j \rangle$ is an $\mathfrak{sl}_2$-triple. Choose $h \in \mathfrak{g}_0$ so that $\alpha_j(h) = 2$ for each $j$ and write

$$s = \langle f, h, e \rangle = \sum_j f_j, h, \sum_j a_j e_j$$

is a principal $\mathfrak{sl}_2$-triple. In particular, note that the projection of $f$ onto each $\mathfrak{g}_{-\alpha_j}$ is nonzero, so $f \in \mathcal{O}_{-1}$.

The centralizer $C$ of $s$ is the center of the group $G$, thus $\mathfrak{c} = W_0 = \{0\}$.

The associated $\mathfrak{s}$-module decomposition is $\mathfrak{g} = \bigoplus_j W_{m_j}$ where $\{m_j\}$ are the exponents of $\mathfrak{g}$. The multiplicities satisfy $n_{m_j} = 1$ for all $j$, with the exception that $n_{4n-2} = 2$ for $\mathfrak{g} = \mathfrak{so}_{2n}$.

Finally, we recall a generalization of a theorem of Kostant proven by Lynch in [31, Theorem 1.2]. It will play a crucial role later in the paper.

**Theorem 2.14.** Let $P < G$ be an even JM-parabolic subgroup determined by an $\mathfrak{sl}_2$-triple $(f, h, e)$. Let $U^m \subset U^{m-1} \subset \cdots \subset U^1 \subset P$ be the filtration of $P$ by unipotent subgroups and $\mathfrak{g}^m \subset \cdots \subset \mathfrak{g}^0 = \mathfrak{p}$ be the corresponding canonical filtration of $\mathfrak{p}$. Then, for $V = \ker(ad_e : \mathfrak{g} \to \mathfrak{g})$, the map

$$U^i \times \{f + V\} \xrightarrow{\text{Ad}(u)(f + v)} f + V + \mathfrak{g}^{i-1}$$

is an isomorphism of affine algebraic varieties for every $0 < i \leq m$.

**Remark 2.15.** In [18, Lemma 2.1], a more general version of the above theorem is proven for any (not necessarily even) $\mathfrak{sl}_2$-triples with a choice of isotropic subspace of $\mathfrak{g}_1$ from (2.4). It would be interesting to consider how such data could be used to generalize the results of this paper to arbitrary parabolics.

### 3. Geometric preliminaries

In this section we recall the necessary definitions concerning principal bundles, connections and flag varieties.

#### 3.1. Bundles and connections

Throughout this paper, all principal bundles are taken to be right principal bundles. If $G$ is a Lie group and $E_G$ is a principal $G$-bundle, then we denote the $G$-invariant vertical vector field on $E_G$ induced by $Y \in \mathfrak{g}$ by $Y^\dagger$. If $V$ is a complex vector space equipped with a holomorphic $G$-action, we will suppress the action and denote the
associated holomorphic vector bundle by $E_G(V)$. For example, $E_G(g)$ is the holomorphic vector bundle determined by the adjoint action of $G$ on $g$.

For the rest of this section, $E_G$ will be a holomorphic principal $G$-bundle over a compact connected Riemann surface $X$ and $K$ will denote the holomorphic cotangent bundle of $X$.

**Definition 3.1.** A holomorphic connection $\omega$ on $E_G$ is a $g$-valued holomorphic 1-form $\omega : TE_G \rightarrow g$ such that

1. $R^g_\omega \omega = \text{Ad}(g^{-1}) \circ \omega$ for all $g \in G$.
2. $\omega(Y^\sharp) = Y$ for all $Y \in g$.

On a Riemann surface, the curvature $F(\omega) = d\omega + \frac{1}{2}[\omega,\omega]$ of a holomorphic connection $\omega$ always vanishes. Thus, all holomorphic connections on $E_G$ are flat, and we refer to the pair $(E_G, \omega)$ as holomorphic flat $G$-bundle.

**Definition 3.2.** The category $\mathcal{F}(G)$ of holomorphic flat $G$-bundles on $X$ has objects $(E_G, \omega)$, where

- $E_G$ is a holomorphic $G$-bundle on $X$,
- $\omega$ is a holomorphic connection on $E_G$.

Morphisms $\Phi : (E_G, \omega) \rightarrow (F_G, \eta)$ are given by $G$-equivariant biholomorphism $\Phi : E_G \rightarrow F_G$ covering the identity map on $X$, such that $\Phi^* \eta = \omega$.

Let $\{U_\alpha\}$ be a trivializing open cover of for a holomorphic bundle $E_G$ defined by local sections $s_\alpha : U_\alpha \rightarrow E_G$ with transition functions $g_{\alpha \beta} : U_{\alpha \beta} \rightarrow G$. For a holomorphic connection $\omega$ on $E_G$, define the local connection forms $\omega_\alpha : = s_\alpha^* \omega \in \Omega^1(U_\alpha, g)$. These local connection forms satisfy the relation

$$\omega_\alpha = \text{Ad}(g_{\alpha \beta}) \circ \omega_\beta + g_{\alpha \beta}^* \theta_G,$$

where $\theta_G$ is the left-invariant Mauer-Cartan form on $G$.

Finally, let $E_G, F_G$ be holomorphic principal $G_1$-bundles and $\eta$ be a holomorphic connection on $F_G$. Let $s : U \rightarrow E_G$ and $t : U \rightarrow F_G$ be local sections over an open set $U \subset M$. Consider a morphism $\Phi : E_G \rightarrow F_G$ and the local connection forms $\eta_U := t^* \eta$ and $\eta_U^\Phi := t^* (\Phi^* \eta)$. Then, with respect to the trivializations induced by $s$ and $t$, $\Phi$ is determined by a holomorphic map $\phi : U \rightarrow G$ and

$$\eta_U^\phi = \text{Ad}(\phi^{-1}) \circ \eta_U + \phi^* \theta_G.$$

**3.2. Extensions and reductions of structure group.** Given a holomorphic principal $G_1$-bundle $E_{G_1}$ and a homomorphism $b : G_1 \rightarrow G_2$ between complex Lie groups, there is an associated holomorphic principal $G_2$-bundle

$$E_{G_1}(G_2) = (E_{G_1} \times G_2)/\sim,$$

where $(p, g_2) \sim (p \cdot g_1, b(g_1^{-1}) \cdot g_2)$ for $g_1 \in G_1$ and the action of $G_2$ is given by right multiplication on the second factor. Transition functions of $E_{G_1}(G_2)$

\footnote{This is of course false for higher dimensional manifolds}
are defined by post-composing transition functions of $E_{G_1}$ with the map $b : G_1 \to G_2$. This process is usually called extending the structure group.

Let $\omega$ be a holomorphic connection on $E_{G_1}$ and $\theta_{G_2}$ be the left invariant Maurer-Cartan form of $G_2$. Then the $G_2$-equivariant map

$$\omega_b : T(E_{G_1}) \times T G_2 \to g_2$$

$$(v, w) \mapsto b(\eta(v)) + \theta_{G_2}(w)$$

descends to define a connection on the extended bundle $E_{G_1}(G_2)$. We will denote the connection on $E_{G_1}(G_2)$ induced by $\omega$ by the same symbol $\omega$.

If $E_{G_1}$, $E_{G_2}$ are holomorphic $G_1$, $G_2$ bundles respectively, then their fiber product defines a holomorphic principal $(G_1 \times G_2)$-bundle $E_{G_1 \times G_2}$. Moreover, holomorphic connections $\omega, \eta$ on $E_{G_1}, E_{G_2}$ respectively, induce a holomorphic connection $\omega \oplus \eta$ on $E_{G_1 \times G_2}$. When $G_1, G_2 < G$ are two subgroups which commute with each other, then the multiplication map $m : G_1 \times G_2 \to G$ is a group homomorphism. In this case, two holomorphic bundles with holomorphic connection $(E_{G_1}, \omega), (E_{G_2}, \eta)$ induce a holomorphic $G$-bundle with connection which we denote by

$$(3.3) \quad (E_{G_1} \star E_{G_2}(G), \omega \star \eta) = (E_{G_1 \times G_2}(G), \omega \oplus \eta).$$

Note that the transition functions of $E_{G_1} \star E_{G_2}(G)$ are just the product of the transition functions of $E_{G_1}$ and $E_{G_2}$ in $G$ and that locally $\omega \star \eta$ is just a sum of the connection forms.

Let $V$ be a complex vector space with a $G$-action. A holomorphic $V$-valued 1-form $\beta$ is called horizontal if the contraction $\iota_Y \beta = 0$ for every $Y \in g$, and equivariant if $R_h^g \beta = g^{-1} \cdot \beta$. There is a canonical isomorphism between equivariant, horizontal $V$-valued holomorphic 1-forms and holomorphic sections the vector bundle $K \otimes E_G(V)$ over $X$.

If $H < G$ is a complex Lie subgroup, then a structure group reduction of a holomorphic principal $G$-bundle $E_G$ to $H$ is a holomorphic subbundle $E_H \subset E_G$. Given a holomorphic connection $\omega$ on $E_G$, the second fundamental form of a structure group reduction $E_H \subset E_G$ measures the failure of $\omega$ to be induced by a connection on $E_H$.

**Definition 3.3.** Suppose $E_H \subset E_G$ is a holomorphic reduction of structure to a closed, complex subgroup $H < G$. The second fundamental form $\Psi \in H^0(X, K \otimes E_H(g/h))$ is the holomorphic section determined by the $H$-equivariant, horizontal 1-form

$$TE_H \to TE_G \xrightarrow{\omega} g \to g/h.$$ 

Finally, we define a notion of relative position for a holomorphic principal with holomorphic connection with respect to a reduction of structure.

**Definition 3.4.** Let $(E_G, \omega)$ be a flat $G$-bundle and $E_H \subset E_G$ be a holomorphic reduction to a closed, complex subgroup $H < G$ with second fundamental form $\Psi$. Let $\mathcal{O} \subset g/h$ be a $\mathbb{C}^* \times H$-invariant open subset. Then
the position of $\omega$ relative to $E_{H}$ is equal to $O$, denoted $\text{pos}_{E_{H}}(\omega) = O$, if and only if $\Psi(v) \in E_{H}(O)$ for all non-zero vectors $v \in TM$.

3.3. Flag varieties. Let $G$ be a connected, complex semisimple Lie group and $P < G$ a parabolic subgroup. The associated flag variety $G/P$ is a smooth complex projective algebraic variety. In particular, $G/P$ a compact complex manifold, and $G \rightarrow G/P$ is a holomorphic principal $P$-bundle. There is a canonical $G$-equivariant isomorphism between the holomorphic tangent bundle $T(G/P)$ and the associated vector bundle $G(g/p)$. Namely,

$$T(G/P) \cong (G \times (g/h))/P := G(g/p),$$

where the $P$-action is defined by $(g, v) \cdot p = (g \cdot p, \text{Ad}(p^{-1})(v))$.

Every $P \times C^{*}$-invariant subset $O \subset g/p$ determines a holomorphic fiber bundle $\hat{O} := G \times_{P} O \rightarrow G/P$

which is equipped with a canonical $C^{*}$-equivariant bundle morphism

$$\hat{O} \rightarrow T(G/P).$$

**Definition 3.5.** Let $M$ be a complex manifold, $f : M \rightarrow G/P$ be a holomorphic map, and $O \subset g/p$ be a $P \times C^{*}$-invariant subset. We say that $f$ is in relative position $O$, denoted $\text{pos}(f) = O$, if the derivative

$$df : TM \backslash Z \rightarrow f^{*}T(G/P)$$

admits a lift to the fiber bundle $f^{*}\hat{O}$. Here, $Z \subset TM$ is the zero section of $TM$.

**Example 3.6.** Let $O = (g/p) \setminus \{0\}$ be the complement of the zero vector in the vector space $g/p$. A holomorphic map $f : M \rightarrow G/P$ satisfies $\text{pos}(f) = O$ if and only if $f$ is an immersion.

3.4. $\lambda$-connections and Higgs bundles. In this subsection we generalize the notion of a holomorphic connection to depend on a parameter $\lambda \in C$. These objects have been studied by Simpson [32], and many others. Let $G$ be a complex semisimple Lie group and $E_{G}$ be a holomorphic principal $G$-bundle on a closed Riemann surface $X$.

**Definition 3.7.** Let $\lambda \in C$, a holomorphic $\lambda$-connection on $E_{G}$ is a holomorphic 1-form $\omega : T E_{G} \rightarrow g$ satisfying

1. $R_{g}^{\ast} \omega = \text{Ad}(g^{-1}) \circ \omega$ for all $g \in G$.
2. $\omega(Y^{\sharp}) = \lambda Y$ for all $Y \in g$.

**Remark 3.8.** A holomorphic 1-connection is just an ordinary holomorphic connection, and if $\omega$ is a $\lambda$-connection for $\lambda \neq 0$, then $\frac{1}{\lambda} \omega$ is an ordinary connection. In contrast, a 0-connection $\omega$ vanishes on vertical vectors, thus $\omega$ is equivalent to a holomorphic section $\hat{\omega} \in H^{0}(X, K \otimes E_{G}(g))$, such a section is usually called a Higgs field.
Remark 3.9. Recall that local connection forms transform as in (3.1) and (3.2). Local \(\lambda\)-connections transform the same way with the Maurer-Cartan form \(\theta_G\) in (3.1) and (3.2) replaced by \(\lambda \theta_G\).

Definition 3.10. The category of holomorphic \((G, \lambda)\)-connections \(\mathcal{QF}(G)\) on a Riemann surface \(X\) has objects \((\lambda, E_G, \omega)\), where

- \(\lambda \in \mathbb{C}\),
- \(E_G \to X\) is a holomorphic principal \(G\)-bundle,
- \(\omega\) is a holomorphic \(\lambda\)-connection on \(E_G\).

A morphism between two objects \((\lambda_1, E_G, \omega)\) and \((\lambda_2, F_G, \eta)\) is a holomorphic bundle isomorphism \(\Phi : E_G \to F_G\) covering the identity on \(X\) such that \(\Phi^* \eta = \omega\). In particular, there are no such morphisms when \(\lambda_1 \neq \lambda_2\).

Note that there is a natural projection map \(\pi : \mathcal{QF}(G) \to \mathbb{C}\) defined by \(\pi(\lambda, E_G, \omega) = \lambda\). The fibers \(\pi^{-1}(\lambda) = \mathcal{F}^\lambda(G)\) are the categories of \(\lambda\)-connections for a fixed \(\lambda\). There is also a \(\mathbb{C}^*\)-action on \(\mathcal{QF}(G)\):

\[(3.4) \; \xi \cdot (\lambda, E_G, \omega) = (\xi \lambda, E_G, \xi \omega)\]

The map \(\pi\) is equivariant with respect to the standard action of \(\mathbb{C}^*\) on \(\mathbb{C}\).

When \(\lambda = 0\), the category of 0-connections \(\mathcal{F}^0(G)\) is also called the category of \(G\)-Higgs bundles on \(X\). In this case a holomorphic zero connection \((0, E_G, \omega)\) is equivalent to the data \((E_G, \hat{\omega})\) where \(\hat{\omega} \in H^0(K \otimes E_G(\mathfrak{g}))\).

Let \(\{p_1, \cdots, p_{\text{rk}(\mathfrak{g})}\}\) be a basis of the \(G\)-invariant polynomials on \(\mathfrak{g}\) with \(\text{deg}(p_j) = m_j + 1\). With this choice we can define the Hitchin fibration

\[(3.5) \; H : \mathcal{F}^0(G) \longrightarrow \bigoplus_{j=1}^{\text{rk}(\mathfrak{g})} H^0(K^{m_j} + 1)^\ast.
\]

\((E_G, \hat{\omega}) \longmapsto (p_1(\hat{\omega}), \cdots, p_{\text{rk}(\mathfrak{g})}(\hat{\omega}))\)

The vector space \(\bigoplus_{j=1}^{\text{rk}(\mathfrak{g})} H^0(K^{m_j} + 1)^\ast\) is usually called the Hitchin base. In [24], Hitchin used the special features of principal \(\mathfrak{sl}_2\)-triples to construct the section of this map.

Since the new objects introduced in this paper generalize the Hitchin section, we sketch the construction here. Let \((f, h, e) \subset \mathfrak{g}\) be a principal \(\mathfrak{sl}_2\)-triple from Example 2.13. Recall that the Lie algebra decomposes into \(\text{ad}_h\)-eigenspaces \(\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} \mathfrak{g}_{2j}\). Consider the holomorphic Lie algebra bundle

\[\mathcal{E}_\mathfrak{g} = \bigoplus_{j \in \mathbb{Z}} K^j \otimes \mathfrak{g}_{2j},\]

the holomorphic frame bundle \(E_{G_{\text{Ad}}}\) has structure group \(G_{\text{Ad}}\). The Lie algebra bundle \(E_{G_{\text{Ad}}}[\mathfrak{g}]\) is isomorphic to \(\mathcal{E}_\mathfrak{g}\).

Recall also that \(V = \ker(\text{ad}_e) = \bigoplus_{j=1}^{\text{rk}(\mathfrak{g})} V_{2m_j}\) where \(V_{2m_j} = V \cap \mathfrak{g}_{2m_j}\). Choose a vector \(e_{2m_j}\) in each \(V_{2m_j}\). Note that \(f\) defines a holomorphic section of \(K \otimes (K^{-1} \otimes \mathfrak{g}_{-2})\) and \(q_{m_j + 1} \otimes e_{2m_j}\) defines a holomorphic section.
of \( \mathcal{K} \otimes (K^{m_j} \otimes V_{2m_j}) \subset \mathcal{K} \otimes (K^{m_j} \otimes g_{2m_j}) \) for each \( q_{m_j+1} \in H^0(K^{m_j+1}) \).

Consider the map \( s_H : \bigoplus_{j=1}^{\text{rk}(g)} H^0(K^{m_j+1}) \to \mathcal{F}^0(G_{\text{Ad}}) \)

\[
s_H(q_{m_1+1}, \cdots, q_{m_{\text{rk}(g)}+1}) = (E_{G_{\text{Ad}}}, f + \sum_{j=1}^{\text{rk}(g)} q_{m_j+1} \otimes e_{2m_j}).
\]

In [29], Kostant proved that there is a basis \((p_1, \cdots, p_{\text{rk}(g)})\) of the \(G\)-invariant polynomials on \(g\) with \(\text{deg}(p_j) = m_j + 1\) such that

\[
p_j(f + \sum_{j=1}^{\text{rk}(g)} a_{2m_j} e_{2m_j}) = a_{2m_j}.
\]

Defining the Hitchin fibration with respect to this choice of basis for the invariant polynomials makes \(s_H\) a section.

4. The category of \((G,P)\)-opers

In this section we define the main new object of the paper, \((G,P)\)-opers. Let \(X\) be a compact, connected Riemann surface of genus at least two, \(G\) a connected complex semisimple Lie group, and \(P < G\) a parabolic subgroup. Let \(\mathcal{O} \subset g^{-1}/p\) be the unique open \(P\)-orbit on \(g^{-1}/p\) from §2.1. Note that \(\mathcal{O}\) is also \(\mathbb{C}^*\)-invariant.

**Definition 4.1.** A \((G,P)\)-oper on \(X\) is a triple \((E_G, E_P, \omega)\) such that

1. \((E_G, \omega)\) is a holomorphic flat \(G\)-bundle over \(X\),
2. \(E_P \subset E_G\) is a holomorphic reduction of structure to the parabolic subgroup \(P < G\),
3. the position of \(\omega\) relative to \(E_P\) satisfies \(\text{pos}_{E_P}(\omega) = \mathcal{O}\).

**Remark 4.2.** When \(P = B\), using Example 2.13, the third condition if Definition 4.1 is that the projection of \(\omega\) to every negative simple root space is nowhere vanishing. This recovers Beilinson-Drinfeld’s definition of an oper in [2].

A morphism between two \((G,P)\)-opers \((E_G, E_P, \omega), (F_G, F_P, \eta)\) is an isomorphism \(\Phi : E_G \to F_G\) of holomorphic principal bundles such that \(\Phi|_{E_P} : E_P \to F_P\) is an isomorphism and \(\Phi^* \eta = \omega\). The category of \((G,P)\)-opers over \(X\) is denoted \(\mathcal{O}(G,P)\).

**Remark 4.3.** The definition of a \((G,P)\)-oper extends immediately to the context of \(\lambda\)-connections. We denote the category of \((G,P,\lambda)\)-opers over \(X\) as \(\mathcal{Q}\mathcal{O}(G,P)\). Namely, the objects of \(\mathcal{Q}\mathcal{O}(G,P)\) consists of tuples \((\lambda, E_G, E_P, \omega)\), where \(\lambda \in \mathbb{C}\) and \((E_G, E_P, \omega)\) is a \(\lambda\)-oper.

Using Example 2.1, we can rephrase the above definition for \(G = \text{SL}_4\mathbb{C}\) in terms of rank 4-vector bundles.

**Example 4.4.** Given a holomorphic \(\text{SL}_4\mathbb{C}\)-bundle \(E\) with connection \(\omega\), let \((\mathcal{V}, \nabla)\) be the associated holomorphic vector bundle with connection.
(1) Since \( B \) is the stabilizer of full flag in \( \mathbb{C}^4 \), a holomorphic \( B \)-reduction \( E_B \subset E \) defines a holomorphic filtration

\[
0 = V_0 \subset V_1 \subset V_2 \subset V_3 \subset V_4 = V,
\]

where \( \text{rk}(V_i) = i \). The third condition of Definition 4.1 is that, for each \( i \), \( \nabla(V_i) \subset V_{i+1} \otimes \mathcal{K} \) and \( \nabla \) induces a bundle isomorphism

\[
V_i/V_{i-1} \cong V_{i+1}/V_i \otimes \mathcal{K}.
\]

(2) For \( P \) the stabilizer of a subspace \( \mathbb{C}^2 \subset \mathbb{C}^4 \), a holomorphic \( P \)-reduction \( E_P \subset E \) defines a holomorphic filtration

\[
0 = V_0 \subset V_2 \subset V_4 = V,
\]

where \( \text{rk}(V_i) = i \). The third condition of Definition 4.1 is that \( \nabla \) induces a bundle isomorphism

\[
V_1 \cong V/\mathcal{K}.
\]

(3) For \( P \) the stabilizer of a partial flag \( \mathbb{C}^1 \subset \mathbb{C}^3 \subset \mathbb{C}^4 \), a holomorphic \( P \)-reduction \( E_P \subset E \) defines a holomorphic filtration

\[
0 = V_0 \subset V_1 \subset V_3 \subset V_4 = V,
\]

where \( \text{rk}(V_i) = i \). The third condition of Definition 4.1 is that \( \nabla \) induces a bundle isomorphism

\[
V_1 \rightarrow V_1/\mathcal{K}^2
\]

(4) For \( P \) the stabilizer of a subspace \( \mathbb{C}^1 \subset \mathbb{C}^4 \), a holomorphic \( P \)-reduction \( E_P \subset E \) defines a holomorphic line subbundle \( V_1 \subset V \).

The third condition of Definition 4.1 is that the map \( V_1 \rightarrow V/V_1 \otimes \mathcal{K} \) induced by \( \nabla \) is an injective bundle map.

Remark 4.5. Each of the above examples generalizes naturally to higher rank holomorphic vector bundles. For the first example, we recover the classical vector bundle definition of an oper. Namely a \((\text{SL}_n \mathbb{C}, B)\)-oper consists of a rank \( n \) holomorphic vector bundle \( V \) equipped with a holomorphic filtration \( V_1 \subset \cdots \subset V_{n-1} \subset V \) with \( \text{rk}(V_i) = i \) and a holomorphic connection \( \nabla \) satisfying \( \nabla(V_i) \subset V_{i+1} \otimes \mathcal{K} \) and \( \nabla : V_i/V_{i-1} \rightarrow V_{i+1}/V_i \otimes \mathcal{K} \) is an isomorphism for all \( i \).

For \( G = S \) is isomorphic to \( \text{PSL}_2 \mathbb{C} \) or \( \text{SL}_2 \mathbb{C} \), the Borel \( BS < S \) is the only proper parabolic subgroup. In terms of vector bundles, an \( \text{SL}_2 \mathbb{C} \)-oper is a \((V, V_1, \nabla)\), where \( \text{rk}(V) = 2 \) with \( \Lambda^2 V \cong \mathcal{O} \), \( \text{rk}(V_1) = 1 \) and \( \nabla \) is a holomorphic connection such that the induced map \( \nabla : V_1 \rightarrow V/V_1 \otimes \mathcal{K} \) is an isomorphism. Thus, \( V_1 \cong K^{\frac{1}{2}} \) is one of the \( 2^g \) square roots of \( \mathcal{K} \) and \( V/V_1 \cong K^{-\frac{1}{2}} \). The associated holomorphic vector bundle \( V \) is isomorphic to the unique non-split extension

\[
K^{\frac{1}{2}} \xrightarrow{\mathcal{K}} V \xrightarrow{-\mathcal{K}^{\frac{1}{2}}}
\]
Choose a smooth identification of $\mathcal{V}$ with $K^\frac{1}{2} \oplus K^{-\frac{1}{2}}$. In this splitting, an explicit example of an oper is given by the following Dolbeault operator and holomorphic connection

$$\bar{\partial}_\mathcal{V} = \begin{pmatrix} \partial_{1/2} & h \\ 0 & \partial_{-1/2} \end{pmatrix} \quad \text{and} \quad \nabla = \begin{pmatrix} \nabla_{1/2}^h & q \\ 1 & \nabla_{-1/2}^h \end{pmatrix}.$$  

Here, $h$ is the uniformizing hyperbolic metric on $X$, $\partial_{1/2}$ and $\partial_{-1/2}$ are Dolbeault operators on $K^\frac{1}{2}$ and $K^{-\frac{1}{2}}$ respectively, $\partial_{1/2}^h$ and $\partial_{-1/2}^h$ are the Chern connections of the induced metric on $K^\frac{1}{2}$ and $K^{-\frac{1}{2}}$, respectively, and $q \in H^0(K^\frac{1}{2})$. In fact, every $\text{SL}_2 \mathbb{C}$-oper is isomorphic to (4.1). This will be generalized in Theorem 5.11 for $(G, P)$-opers when $P$ is an even JM-parabolic.

We can also use (4.1) to construct sections of $\mathcal{QOp}(\text{SL}_2 \mathbb{C}) \to \mathbb{C}$. Two such sections are given by

$$\tau_1(\lambda) = (\lambda, \bar{\partial}_\mathcal{F}, \nabla) = (\lambda, \begin{pmatrix} \partial_{1/2} & \lambda h \\ 0 & \partial_{-1/2} \end{pmatrix}, \begin{pmatrix} \lambda \nabla_{1/2}^h & \lambda^2 q \\ 1 & \lambda \nabla_{-1/2}^h \end{pmatrix}),$$  

$$\tau_2(\lambda) = (\lambda, \bar{\partial}_\mathcal{F}, \nabla) = (\lambda, \begin{pmatrix} \partial_{1/2} & \lambda h \\ 0 & \partial_{-1/2} \end{pmatrix}, \begin{pmatrix} \lambda \nabla_{1/2}^h & q \\ 1 & \lambda \nabla_{-1/2}^h \end{pmatrix}).$$  

Note that $\tau_1(\lambda)$ is isomorphic to $\lambda \cdot (1, \bar{\partial}_\mathcal{F}, \nabla)$ for the action (3.4).

**Remark 4.6.** In the above discussion, we have a choice of a square root of $K^\frac{1}{2}$. Each such choice defines the same $\text{PSL}_2 \mathbb{C}$-oper.

### 5. The global Slodowy slice for even JM-parabolics

In this section we define the $s$-Slodowy category $\mathcal{B}_s(G)$, associated to any even $\mathfrak{sl}_2$-triple $s = \langle f, h, e \rangle \subset \mathfrak{g}$. We first prove that the category $\mathcal{B}_s(G)$ is equivalent to the category $\mathcal{Op}(G, P)$ of $(G, P)$-opers, where $P$ is the even JM-parabolic associated to $s$. This equivalence can be interpreted as a "global slice theorem". When $s$ is a principal subalgebra and $G$ is the adjoint group, $\mathcal{B}_s(G)$ is the Hitchin base and we recover the Hitchin section and more generally Beilinson-Drinfeld’s parameterization of $(G, B)$-opers.

Let $S < G$ be the connected subgroup with Lie algebra $s$. The functor realizing the equivalence depends on the choice of an $S$-oper. We then show that there is a subcategory $\widehat{\mathcal{B}}_s(G)$ of $\mathcal{B}_s(G)$ and a natural functor $\mathcal{Op}(S, B_S) \times \widehat{\mathcal{B}}_s(G) \to \mathcal{Op}(G, P)$ which is an equivalence when $S \cong \text{PSL}_2 \mathbb{C}$ and essentially surjective and full when $S \cong \text{SL}_2 \mathbb{C}$. Finally we extend these results to the category of $(\lambda, G, P)$-opers.

#### 5.1. The $s$-Slodowy category

Let $G$ be a connected, complex semi-simple Lie group. Fix an even $\mathfrak{sl}_2$-triple $s = \langle f, h, e \rangle \subset \mathfrak{g}$, let $b_s = \langle h, e \rangle \subset \mathfrak{g}$ and let
Let $G$ be a complex semisimple Lie group and $\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{g}$ be an even $\mathfrak{sl}_2$-triple with centralizer $C < G$. The $\mathfrak{s}$-Slodowy category $\mathcal{B}_\mathfrak{s}(G)$ has objects $(\mathcal{E}_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})$, where

- $\mathcal{E}_C$ is a holomorphic $C$-bundle,
- $\psi_0$ is a holomorphic $C$-connection on $\mathcal{E}_C$,
- $\psi_{m_j} \in H^0(E_C(Z_{2m_j}) \otimes K^{m_j+1})$,

and morphisms given by isomorphisms of holomorphic flat $C$-bundles which identify holomorphic sections of $E_C(Z_{2m_j}) \otimes K^{m_j+1}$ for every $j$.

Remark 5.2. For a principal $\mathfrak{sl}_2$-triple $\mathfrak{s} \subset \mathfrak{g}$, $C < G$ is the center of $G$ and $\psi_{m_j} \in H^0(K^{m_j+1})$. In this case, $\mathcal{B}_\mathfrak{s}(G)$ is the product of the Hitchin base with finite set of holomorphic $C$-bundles on $X$. In particular, for an adjoint group $G$, $\mathcal{B}_\mathfrak{s}(G)$ is the Hitchin base.

Let $\Theta = (F_S, F_{B_\mathfrak{s}}, \eta)$ be an $(S, B_\mathfrak{s})$-oper. For every holomorphic $C$-bundle $E_C$, the second fundamental form of $\eta$ induces a holomorphic isomorphism

\begin{equation}
E_C \star F_{B_\mathfrak{s}}(V_{2m_j}) \otimes K \cong E_C(Z_{2m_j}) \otimes K^{m_j+1},
\end{equation}

where we recall that $\star$-notation from (3.3).

Definition 5.3. Let $G$ be a complex semisimple Lie group, $\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{g}$ be an even $\mathfrak{sl}_2$-triple and $S < G$ the associated connected subgroup. For each $(S, B_\mathfrak{s})$-oper $\Theta = (F_S, F_{B_\mathfrak{s}}, \eta)$, the $\Theta$-Slodowy functor

\begin{equation}
F_{\Theta} : \mathcal{B}_\mathfrak{s}(G) \rightarrow \mathfrak{D}p(G, P)
\end{equation}

is defined by

\begin{equation}
F_{\Theta}(\Xi) = (E_C \star F_S(G), E_C \star F_{B_\mathfrak{s}}(P), \eta \star \psi_0 + \psi_{m_1} + \cdots + \psi_{m_N}),
\end{equation}

where $\Xi = (E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})$ is an object in $\mathcal{B}_\mathfrak{s}(G)$ and we have identified each $\psi_{m_j}$ with a holomorphic section of $F_{B_\mathfrak{s}} \star E_C(V_{2m_j}) \otimes K$ using the isomorphism (5.1). If $f : E_C \rightarrow E'_C$ is a morphism in $\mathfrak{D}p(G, P)$ then $F_{\Theta}(f) = f \star \text{Id}_{F_S} : E_C \star F_S(G) \rightarrow E'_C \star F_S(G)$. 

B_\mathfrak{s} < S < G$ be the associated connected subgroups of $G$. Recall from (2.8) that we have $\mathfrak{s}$-module decomposition of $\mathfrak{g}$

$$\mathfrak{g} = c \oplus \bigoplus_{j=1}^{N} W_{2m_j}.$$ 

Recall also that the Slodowy data and Levi data of $\mathfrak{s}$ given by

$$V = c \oplus \bigoplus_{j=1}^{N} V_{2m_j} \quad \text{and} \quad \mathfrak{g}_0 = c \oplus \bigoplus_{j=1}^{N} Z_{2m_j}.$$ 

Moreover, from Proposition 2.4, we have $V_2 = \langle e \rangle \oplus \hat{V}_2$ and $Z_2 = \langle h \rangle \oplus \hat{Z}_2$.

The Slodowy data of $\mathfrak{s}$ is the Lie theoretic mechanism which defines the generalization of the Hitchin base.
**Remark 5.4.** The Slodowy functor can be extended to a functor

\[(5.3) \quad F : \mathfrak{Op}(S, B_S) \times B_\mathfrak{s}(G) \longrightarrow \mathfrak{Op}(G, P). \]

Given morphisms \(f_1 : \Theta \rightarrow \Theta'\) and \(f_2 : \Xi \rightarrow \Xi'\), we have \(F(f_1, f_2) = f_1 \star f_2 : F(\Theta, \Xi) \rightarrow F(\Theta', \Xi').\)

Recall that a functor defines an equivalence of categories if it is essentially surjective and fully faithful, i.e. every isomorphism class is in the image and the functor induces a bijection on morphisms.

**Theorem 5.5.** Let \(G\) be a complex connected semisimple Lie group, \(\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{g}\) be an even \(\mathfrak{sl}_2\)-triple and \(P < G\) be the associated even JM-parabolic. For every \((S, B_S)\)-oper \(\Theta\), the \(\Theta\)-Slodowy functor

\[F_\Theta : B_\mathfrak{s}(G) \rightarrow \mathfrak{Op}(G, P)\]

is an equivalence of categories.

**Remark 5.6.** When \(\mathfrak{s}\) is a principal nilpotent and \(P = B\) is the Borel subgroup, Theorem 5.5 is due to Beilinson-Drinfeld [2]. Note that, in this case, the Slodowy functor is defined similarly to the Hitchin section (3.6), see (5.9) for the explicit relation.

To remove the choice of \((S, B_S)\)-oper from Theorem 5.5, we introduce a subcategory \(\hat{B}_\mathfrak{s}(G)\) of \(B_\mathfrak{s}(G)\). Since, the space \(Z_2\) decomposes C-invariantly as \(\langle h \rangle \oplus \hat{Z}_2\), for any holomorphic C-bundle \(E_C\) we have

\[E_C(Z_2) \otimes K^2 \cong K^2 \oplus E_C(\hat{Z}_2) \otimes K^2.\]

Thus, for any object \((E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})\) in \(B_\mathfrak{s}(G)\), \(\psi_{m_1}\) decomposes as

\[(5.4) \quad \psi_{m_1} = q \oplus \hat{\psi}_{m_1},\]

where \(q \in H^0(K^2)\) and \(\hat{\psi}_{m_1} \in H^0(E_C(\hat{Z}_2) \otimes K^2)\). Thus, the \(\mathfrak{s}\)-Slodowy category \(B_\mathfrak{s}(G)\), factors as the product of the

\[H^0(K^2) \times \hat{B}_\mathfrak{s}(G).\]

**Definition 5.7.** Let \(\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{g}\) be an even \(\mathfrak{sl}_2\)-triple. Define the **traceless quadratic \(\mathfrak{s}\)-Slodowy category** to be the subcategory \(\hat{B}_\mathfrak{s}(G)\) of \(B_\mathfrak{s}(G)\) whose objects are \((E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})\) with \(\psi_{m_1} = \hat{\psi}_{m_1}\).

**Theorem 5.8.** Let \(G\) be a complex connected simple Lie group, \(\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{g}\) be an even \(\mathfrak{sl}_2\)-triple, \(S < G\) be the connected subgroup with Lie algebra \(\mathfrak{s}\) and \(P < G\) be the associated even JM-parabolic. Then the functor

\[F : \mathfrak{Op}(S, B_S) \times \hat{B}_\mathfrak{s}(G) \longrightarrow \mathfrak{Op}(G, P)\]

is an equivalence of categories when \(S \cong \text{PSL}_2\mathbb{C}\) and essentially surjective and full when \(S \cong \text{SL}_2\mathbb{C}\).
Remark 5.9. Recall that when $G$ is an adjoint group, the subgroup $S \triangleleft G$ is always isomorphic to $\text{PSL}_2 \mathbb{C}$. Thus, for adjoint groups, the above functor always defines an equivalence of categories.

5.2. Proofs of Theorems 5.5 and 5.8. In this section we will prove Theorems 5.5 and 5.8. The key technical result is Theorem 5.11. For this section we fix an even $\mathfrak{sl}_2$-triple $\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{g}$ and retain the notation from the previous sections. We begin with a local result.

Lemma 5.10. Let $\mathbb{D}$ be the unit disk, and consider holomorphic 1-forms $\omega : T\mathbb{D} \to O_{-1} + p$ and $\eta : T\mathbb{D} \to O_\mathfrak{s} + \mathfrak{b}_\mathfrak{s}$. Then there exists a holomorphic map $\Psi : \mathbb{D} \to P$ and a holomorphic 1-form $\psi : T\mathbb{D} \to V$ such that

\[
(5.5) \quad \text{Ad}(\Psi^{-1}) \circ \omega + \Psi^* \theta_P = \eta + \psi,
\]

where $\theta_P$ is the left invariant Mauer-Cartan form on $P$. Moreover, $\Psi$ is unique up to right multiplication by a holomorphic map $\mathbb{D} \to C$.

Proof. Let $L < P$ be the Levi subgroup with Lie algebra $\mathfrak{g}_0$. Since $L$ acts transitively on $O_{-1}$, $L$ preserves $p$, and $f \in O_\mathfrak{s}$, there is a holomorphic map $\Psi_0 : \mathbb{D} \to L$ such that

\[
\text{Ad}(\Psi_0^{-1}) \circ \omega - \eta : T\mathbb{D} \to \langle h \rangle + p.
\]

Since $C < L$ is the stabilizer of $f$, $\Psi_0$ is unique up to right multiplication by any holomorphic map $\mathbb{D} \to C$.

Define

\[
\omega_0 := \text{Ad}(\Psi_0^{-1}) \circ \omega + \Psi_0^* \theta_L : T\mathbb{D} \to O_\mathfrak{s} + p.
\]

By Theorem 2.14, there exists a unique holomorphic $\Psi_1 : \mathbb{D} \to U^1$ such that

\[
\text{Ad}(\Psi_1^{-1}) \circ \omega_0 : \mathbb{D} \to O_\mathfrak{s} + V
\]

Then, define

\[
\omega_1 := \text{Ad}(\Psi_1^{-1}) \circ \omega_0 + \Psi_1^* \theta_{U^1} : T\mathbb{D} \to O_\mathfrak{s} + V + u^1.
\]

Using Theorem 2.14 inductively for each $i \geq 2$, we obtain a unique $\Psi_i : \mathbb{D} \to U^i$ such that $\text{Ad}(\Psi_i^{-1}) \circ \omega_{i-1} : T\mathbb{D} \to O_\mathfrak{s} + V$, and the resulting

\[
\omega_i := \text{Ad}(\Psi_i^{-1}) \circ \omega_{i-1} + \Psi_i^* \theta_{U^i} : T\mathbb{D} \to O_\mathfrak{s} + V + u^i.
\]

When $i = m_N$, since $\mathfrak{g}_{m_N} \subset V$, we obtain $\omega_{m_N} : T\mathbb{D} \to O_\mathfrak{s} + V$.

Now, $\omega_{m_N} - \eta : T\mathbb{D} \to \langle h \rangle + V$. To complete the process, we claim there exists a unique $\Psi_{m_N+1} : \mathbb{D} \to U_S$ such that

\[
\text{Ad}(\Psi_{m_N+1}^{-1}) \circ \omega_{m_N} + \Psi_{m_N+1}^* \theta_{U_S} - \eta : T\mathbb{D} \to V.
\]

Then $\Psi = \Psi_0 \cdots \Psi_{m_N+1} : \mathbb{D} \to P$ is the desired map.
A direct calculation shows
\[ \text{Ad}(\Psi^{-1}) \circ \omega + \Psi^* \theta_P = \text{Ad}(\Psi_{m_N+1}^{-1}) \circ \omega_{m_N} - \text{Ad}(\Psi_{m_N+1}^{-1} \cdots \cdot \cdot \cdot \Psi_1^{-1}) \circ \Psi^*_1 \theta_L \]
\[ - \sum_{i=1}^{m_N} \text{Ad}(\Psi_{m_N+1}^{-1} \cdots \cdot \cdot \cdot \Psi_i^{-1}) \circ \Psi_{i+1}^* \theta_{U_{i+1}} + \Psi^* \theta_P \]
\[ = \text{Ad}(\Psi_{m_N+1}^{-1}) \circ \omega_{m_N} + \Psi^*_{m_N+1} \theta_{U_S}. \]

For the uniqueness of \( \Psi \), any other map satisfying (5.5) can be written as \( \Psi \cdot \Phi \) for \( \Phi : \mathbb{D} \rightarrow \mathbb{P} \). By (2.7), the projection of \( \Phi \) onto \( U \) is the identity. Moreover, \( \text{Ad}(\Phi^{-1}) \) acts trivially on \( O_\mathbb{D} \). Thus, the projection of \( \Phi \) to \( L \) is contained in \( C \). Hence, \( \Psi \) is unique up to right multiplication by a holomorphic map \( \mathbb{D} \rightarrow C \). \( \square \)

The proofs of Theorems 5.5 and 5.8 follow almost immediately from the following parameterization theorem.

**Theorem 5.11.** Let \( G \) be a connected complex semisimple Lie group, \( s = \langle f, h, e \rangle \subset g \) be an even \( s_{\mathbb{L}} \)-triple, \( S \subset G \) be the connected subgroup with Lie algebra \( s \), \( C < G \) be the centralizer of \( s \) and \( \mathbb{P} < G \) be the associated even \( J_{\mathbb{M}} \)-parabolic. Let \( \Theta = (F_S, F_{B_S}, \eta) \) be an \((S, B_S)\)-oper and \((E_G, E_P, \omega)\) a \((G, \mathbb{P})\)-oper. Then, there is a unique (up to isomorphism) flat \( C \)-bundle \( (E_C, \psi_0) \) and a holomorphic map \( \Psi : (E_C \ast F_{B_S})(P) \rightarrow E_P \) such that
\[ \Psi^* \omega - \eta \ast \psi_0 \in \bigoplus_{j=1}^N H^0(X, K \otimes (E_C \ast F_{B_S}(V_{m_j}))) . \]

**Proof.** Let \( \{U_\alpha\} \) be a trivializing open cover of both \( E_P \) and \( F_{B_S} \) and let
\[ \{p_{\alpha \beta} : U_{\alpha \beta} \rightarrow \mathbb{P}\} \quad \text{and} \quad \{b_{\alpha \beta} : U_{\alpha \beta} \rightarrow B_S\} \]
be the transition functions of \( E_P \) and \( F_{B_S} \) respectively. The restrictions of \( \omega \) and \( \eta \) to \( U_{\alpha \beta} \) satisfy
\[ \omega_\alpha = \text{Ad}(p_{\alpha \beta}^{-1}) \circ \omega_\beta + p_{\alpha \beta}^* \theta_P \quad \text{and} \quad \eta_\alpha = \text{Ad}(b_{\alpha \beta}^{-1}) \circ \eta_\beta + b_{\alpha \beta}^* \theta_{B_S} . \]

By Lemma 5.10, we have \( \{\Psi_\alpha : U_\alpha \rightarrow \mathbb{P}\} \) so that
\[ \text{Ad}(\Psi^{-1}_\alpha) \circ \omega_\alpha + \Psi^*_\alpha \theta_P - \eta_\alpha = \psi_\alpha : U_\alpha \rightarrow V. \]

Let \( E_Q \) be the principal \( P \)-bundle defined by the transition functions \( Q := \{q_{\alpha \beta} = \Psi^{-1}_\beta p_{\alpha \beta} \Psi_\alpha : U_{\alpha \beta} \rightarrow \mathbb{P}\} \). By construction, the locally defined \( \Psi_\alpha \) patch to a globally defined isomorphism \( \Psi : E_Q \rightarrow E_P \).

Next, let \( \omega^\Psi_\alpha = \text{Ad}(\Psi^{-1}_\alpha) \circ \omega_\alpha + \Psi^*_\alpha \theta_P \) denote the local expression of \( \Psi^* \omega \) is the coordinate chart \( U_\alpha \). Then
\[ \psi_\alpha + \eta_\alpha = \omega^\Psi_\alpha = \text{Ad}(q_{\alpha \beta}^{-1}) \circ \omega_\beta + q_{\alpha \beta}^* \theta_P = \text{Ad}(q_{\alpha \beta}^{-1}) \circ (\psi_\beta + \eta_\beta) + q_{\alpha \beta}^* \theta_P . \]

In particular,
\[ \psi_\alpha - \text{Ad}(q_{\alpha \beta}^{-1}) \circ \psi_\beta + \eta_\alpha - \text{Ad}(q_{\alpha \beta}^{-1}) \circ \eta_\beta = q_{\alpha \beta}^* \theta_P . \]
Equation (5.6) is a direct sum of graded pieces with respect to the canonical grading (2.2). The $-1$-graded piece is the first non-zero term. Since the $(-1)$-graded pieces of $\psi_\alpha$, $\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta$ and $q_{\alpha^{-1}} \theta_P$ vanish, we have

\[(\eta_0)_{-1} = (\text{Ad}(q_{\alpha^{-1}}) \circ \eta_\beta)_{-1} = 0.\]

This implies that the image of $q_{\alpha\beta}$ is contained in the subgroup of $P$ generated by $C$, $B_S$ and $U$. In particular, the Maurer-Cartan form $q_{\alpha\beta}^* \theta_P$ is valued in the subalgebra $c \oplus b_s \oplus u$.

For the zeroth graded piece we have

\[(\eta_0)_0 - (\text{Ad}(q_{\alpha^{-1}}) \circ \eta_\beta)_0 + (\psi_\alpha)_0 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_0 = (q_{\alpha\beta}^* \theta_P)_0.\]

The term $(\psi_\alpha)_0 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_0$ takes values in $c$. Since $(q_{\alpha\beta}^* \theta_P)_0$ is valued in $\langle h \rangle \oplus c$, the term $(\eta_0)_0 - (\text{Ad}(q_{\alpha^{-1}}) \circ \eta_\beta)_0$ is valued in $\langle h \rangle$. By (2.7), this implies that the image of $q_{\alpha\beta}$ is contained in the subgroup of $P$ generated by $C$, $B_S$ and $U^2$. The Maurer-Cartan form $q_{\alpha\beta}^* \theta_P$ is thus valued in $c \oplus b_s \oplus u^2$. Moreover, the $b_s$ part of $q_{\alpha\beta}^* \theta_P$ is $b_{\alpha\beta}^* \theta_{b_s}$.

For the first graded piece we have

\[(\eta_1)_1 - (\text{Ad}(q_{\alpha^{-1}}) \circ \eta_\beta)_1 + (\psi_\alpha)_1 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_1 = (q_{\alpha\beta}^* \theta_P)_1.\]

The term $(\psi_\alpha)_1 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_1$ is valued in $V_2$. So, using (2.7), it follows that $q_{\alpha\beta}$ takes values in the group generated by $B_S$, $C$ and $U^3$. Moreover, we have

\[(\eta_1)_1 - (\text{Ad}(q_{\alpha^{-1}}) \circ \eta_\beta)_1 = (\eta_0)_1 - (\text{Ad}(b_{\alpha\beta}^{-1}) \circ \eta_\beta)_1.\]

Since $(q_{\alpha\beta}^* \theta_P)_1 = (b_{\alpha\beta}^* \theta_{b_s})_1$, we have $(\psi_\alpha)_1 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_1 = 0$.

For the second graded piece we have

\[-(\text{Ad}(q_{\alpha^{-1}}) \circ \eta_\beta)_2 + (\psi_\alpha)_2 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_2 = 0\]

since $(q_{\alpha\beta}^* \theta_P)_2 = 0$. The term $(\psi_\alpha)_2 - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_2$ is valued in $V_4$. Thus, using (2.7), $(\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_2 = 0$ and the is image of $q_{\alpha\beta}$ is contained in the subgroup of $P$ generated by $C$, $B_S$ and $U^4$.

Now, for $j \geq 2$ assume $0 = (q_{\alpha\beta}^* \theta_P)_j$. The same argument as in the previous case implies that the image of $q_{\alpha\beta}$ is contained in the subgroup generated by $C$, $B_S$ and $U^{j+2}$ and

\[(\psi_\alpha)_j - (\text{Ad}(q_{\alpha^{-1}}) \circ \psi_\beta)_j = 0.\]

Since $U^{mN+1} = 0$, we conclude that $q_{\alpha\beta}$ is valued in the subgroup generated by $B_S$ and $C$.

Since $B_S$ and $C$ commute we have

\[q_{\alpha\beta} = b_{\alpha\beta} \cdot c_{\alpha\beta},\]

where $c_{\alpha\beta} : U_{\alpha\beta} \rightarrow C$ are the transition functions of a holomorphic $C$-bundle $E_C$. Thus, $E_Q = E_C * F_{B_S}(P)$ and, by (5.7), $\{(\psi_\alpha)_0\}$ defines a holomorphic
connection on $E_C$. Moreover, by (5.8) each $\{ (\psi_\alpha)_m \}$ defines a section $\psi_m \in H^0(\mathbb{C} \ast F_{\mathbb{B}_S}(V_{2m}))$. Thus,

$$\Psi^* \omega = \eta \ast \psi_0 + \psi_{m_1} + \cdots + \psi_{m_N}.$$ 

The uniqueness of $(E_C, \psi_0)$ up to isomorphism follows from Lemma 5.5. Namely, the maps $\Psi_\alpha$ are unique up to right multiplication by a map $C_\alpha : U_\alpha \rightarrow C$. The different choices define isomorphic $C$-bundles. \hfill $\square$

We now prove Theorem 5.5 and Theorem 5.8.

**Proof of Theorem 5.5.** By Theorem 5.11, for each $(S, B_S)$-oper $\Theta$, the Slodowy functor $F_\Theta$ is essentially surjective. Given a morphism $f : (E_C, \psi_0) \rightarrow (E'_C, \psi'_0)$ in $\mathcal{B}_S(G)$, $F_\Theta(f)$ is defined by $f \ast \text{Id}_{B_S} : E_C \ast F_{B_S}(G) \rightarrow E'_C \ast F_{B_S}(G)$. This is clearly faithful. To see that it is full, consider

$$g : E_C \ast F_{B_S}(P) \rightarrow E'_C \ast F_{B_S}(P)$$

such that $g^*(\eta \ast \psi'_0) = \eta \ast \psi_0$. Locally, $g = \{ g_\alpha : U_\alpha \rightarrow P \}$, where each $g_\alpha$ acts trivially on $\mathfrak{s}$. Thus $g_\alpha : U_\alpha \rightarrow C$, and $g : E_C \rightarrow E'_C$ with $g^* \psi'_0 = \psi_0$. \hfill $\square$

**Proof of Theorem 5.8.** Fix an $(S, B_S)$-oper $\Theta = (F_S, B_S, \eta)$ and let $(E_G, E_P, \omega)$ be a $(G, P)$-oper. By Theorem 5.11, there is $\Xi = (E_C, \psi_0, \psi_1, \cdots, \psi_{m_N})$ in $\mathcal{B}_S(G)$ such that

$$(E_G, E_P, \omega) \cong (F_S \ast E_C(G), F_{B_S} \ast E_C(P), \eta \ast \psi_0 + \psi_{m_1} + \cdots + \psi_{m_N}),$$

where, using the isomorphism (5.1), $\psi_{m_j} \in H^0(F_{B_S} \ast E_C(V_{2m_j}) \otimes K)$.

By (5.4), $\psi_{m_1}$ decomposes as $\psi_{m_1} = q + \hat{\psi}_{m_1}$, where

$$q \in H^0(E_C((h)) \otimes K^2) \cong H^0(F_{B_S}((e)) \otimes K),$$

$$\hat{\psi}_{m_1} \in H^0(E_C(Z_2) \otimes K^2) \cong H^0(E_C \ast F_{B_S}(V_2) \otimes K).$$

Let $\hat{\Xi} = (E_C, \psi_0, \hat{\psi}_{m_1}, \psi_{m_2}, \cdots, \psi_{m_N})$ be the associated object in $\mathcal{B}_S(G)$.

Note that $\Theta_q = (F_S, F_{B_S}, \eta + q)$ is another $(S, B_S)$-oper and

$$F_\Theta(\Xi) = F_\Theta(\hat{\Xi}).$$

Thus, the functor $F : \mathcal{O}p(S, B_S) \times \mathcal{B}_S(G) \rightarrow \mathcal{O}p(G, P)$ defined by $F(\Theta, \hat{\Xi}) = F_\Theta(\Xi)$ is essentially surjective.

For fullness, consider

$$g : E_C \ast F_{B_S}(P) \rightarrow E'_C \ast F_{B_S}(P)$$

such that $g^*(\eta \ast \psi'_0 + \hat{\psi}'_1 + \cdots + \psi'_{m_N}) = \eta \ast \psi_0 + \hat{\psi}_1 + \cdots + \psi_{m_N}$. Locally, $g = \{ g_\alpha : U_\alpha \rightarrow P \}$, where each $g_\alpha$ preserves $\mathfrak{s} + V$. By (2.7), we have $g_\alpha : U_\alpha \rightarrow C \ast B_S$, and $g : E_C \ast F_{B_S} \rightarrow E'_C \ast F_{B_S}$. Such a $g$ can be written as a product $g_C \cdot g_{B_S}$ where $g_C : E_C \rightarrow E'_C$ and $g_{B_S} : F_{B_S} \rightarrow F_{B_S}$.

For faithfulness, note that the multiplication map $S \times C \rightarrow S \ast C < G$ is injective if and only if $S \cong \text{PSL}_2 \mathbb{C}$. Thus, the functor $F$ is faithful if and only if $S \cong \text{PSL}_2 \mathbb{C}$. \hfill $\square$
5.3. **Explicit models for $\text{SL}_4 \mathbb{C}$-opers.** Recall examples 2.1, 2.12 and 4.4. For $\text{SL}_4 \mathbb{C}$, the $\mathfrak{s}$-Slodowy categories are defined as follows.

1. For $\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{sl}_4 \mathbb{C}$ a principal $\mathfrak{sl}_2$, we have $C = (\sqrt{-1}\text{Id})$. The objects of $B_{\mathfrak{s}}(\text{SL}_4 \mathbb{C})$ consists of tuples 

$$(\mathcal{L}, \psi_1, \psi_2, \psi_3),$$

where $\mathcal{L}$ is a holomorphic line bundle on $X$ such that $\mathcal{L}^4 = \mathcal{O}$ and $\psi_j \in H^0(K^{j+1})$ for $j = 1, 2, 3$.

2. For $\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{sl}_4 \mathbb{C}$ the even $\mathfrak{sl}_2$ whose associated JM-parabolic is the stabilizer of subspace $\mathbb{C}^2 \subset \mathbb{C}^4$, we have $C \cong \text{SL}_2^+ \mathbb{C}$. Using Example 2.12, the objects of $B_{\mathfrak{s}}(\text{SL}_4 \mathbb{C})$ are tuples

$$(\mathcal{W}, \nabla_{\mathcal{W}}, \psi_1),$$

where $\mathcal{W}$ is a rank two holomorphic vector bundle with $\det(\mathcal{W})^2 = \mathcal{O}$, $\nabla_{\mathcal{W}}$ is a holomorphic connection on $\mathcal{W}$ and $\psi_1 \in H^0(\text{End}(\mathcal{W}) \otimes K^2)$. The decomposition (5.4) is $\psi_1 = q \otimes \text{Id}_{\mathcal{W}} + \hat{\psi}_1$, where $q \in H^0(K^2)$ and $\hat{\psi}_1 \in H^0(\text{End}(\mathcal{W}) \otimes K^2)$ is a traceless.

3. For $\mathfrak{s} = \langle f, h, e \rangle \subset \mathfrak{sl}_4 \mathbb{C}$, the even $\mathfrak{sl}_2$ whose associated JM-parabolic is the stabilizes a partial flag $\mathbb{C} \subset \mathbb{C}^3 \subset \mathbb{C}^4$, we have $C \cong \text{GL}_1 \mathbb{C}$. Using Example 2.12, the objects of $B_{\mathfrak{s}}(\text{SL}_4 \mathbb{C})$ are tuples

$$(\mathcal{L}, \nabla_{\mathcal{L}}, \psi_1, \psi_2),$$

where $\mathcal{L}$ is a degree zero line bundle, $\nabla_{\mathcal{L}}$ is a holomorphic connection on $\mathcal{L}$, $\psi_1 \in H^0(K^2) \oplus H^0(L^4 K^2) \oplus H^0(L^{-4} K^2)$ and $\psi_2 \in H^0(K^3)$. The decomposition (5.4) of $\psi_1$ is $\psi_1 = q + \hat{\psi}_1$, where $q \in H^0(K^2)$ and $\hat{\psi}_1 \in H^0(L^4 K^2) \oplus H^0(L^{-4} K^2)$.

We now describe the Slodowy functor for each of these examples. To make the descriptions more concrete we will choose a smooth isomorphism between filtered objects and their associated graded. The holomorphic bundles with connections will then be expressed as Dolbeault operators and connections on the smooth bundle.

Recall Examples 4.4 and Example 2.12. Let $\Theta$ the $\text{SL}_2 \mathbb{C}$-oper from (4.1). The ordering of the below cases is the same as the previous examples.

1. The $\Theta$-Slodowy functor $F_{\Theta} : B_{\mathfrak{s}}(\text{SL}_4 \mathbb{C}) \rightarrow \mathcal{O}p(\text{SL}_4 \mathbb{C}, B)$ is

$$F_{\Theta}(\mathcal{L}, \psi_1, \psi_2, \psi_3) = (\bar{\partial}_\nu, \nabla)$$

where $(\bar{\partial}_\nu, \nabla)$ are Dolbeault operators and holomorphic connections on the smooth bundle $\mathcal{L}K^{\frac{1}{2}} \oplus LK^{\frac{1}{2}} \oplus LK^{-\frac{1}{2}} \oplus LK^{-\frac{1}{2}}$ given by

$$\bar{\partial}_\nu = \begin{pmatrix}
\bar{\partial}_\nu \otimes \bar{\partial}_3/2 & 3h & 0 & 0 \\
0 & \bar{\partial}_\nu \otimes \bar{\partial}_{1/2} & 4h & 0 \\
0 & 0 & \bar{\partial}_\nu \otimes \bar{\partial}_{-1/2} & 3h \\
0 & 0 & 0 & \bar{\partial}_\nu \otimes \bar{\partial}_{-3/2}
\end{pmatrix}$$
\[ \nabla = \begin{pmatrix} 0 & 3q + 3\psi_1 & \psi_2 & \psi_3 \\ 1 & \psi_3 & \psi_2 & 4q + 4\psi_1 \\ 0 & 1 & \psi_2 & 3q + 3\psi_1 \\ 0 & 0 & 1 & \psi_2 \end{pmatrix}, \]

where \( \nabla_l \) is the holomorphic connection which induces the trivial connection on \( L^4 = \mathcal{O} \).

(2) The \( \Theta \)-Slodowy functor \( F_\Theta : \mathcal{B}_s(\text{SL}_4 \mathbb{C}) \to \mathfrak{Op}(\text{SL}_4 \mathbb{C}, P) \) is

\[ F_\Theta(W, \nabla_W, \psi_1) = (\tilde{\partial}_\nu, \nabla), \]

where \((\tilde{\partial}_\nu, \nabla)\) are Dolbeault operators and holomorphic connections on the smooth bundle \( W \otimes \mathcal{K}_\lambda^+ \otimes \mathcal{W} \otimes \mathcal{K}_\lambda^- \) given by

\[ \tilde{\partial}_\nu = \begin{pmatrix} \partial_W \otimes \partial_{1/2} & h \otimes \text{Id}_W \\ 0 & \partial_W \otimes \partial_{1/2} \end{pmatrix} \quad \text{and} \quad \nabla = \begin{pmatrix} \partial_W \otimes \partial_{1/2} & q \otimes \text{Id}_W + \psi_1 \\ \text{Id}_W & \partial_W \otimes \partial_{1/2} \end{pmatrix}. \]

(3) The \( \Theta \)-Slodowy functor \( F_\Theta : \mathcal{B}_s(\text{SL}_4 \mathbb{C}) \to \mathfrak{Op}(\text{SL}_4 \mathbb{C}, P) \) is

\[ F_\Theta(L, \nabla_L, \psi_1, \psi_2) = (\tilde{\partial}_\nu, \nabla) \]

where \((\tilde{\partial}_\nu, \nabla)\) are Dolbeault operators and holomorphic connections on the smooth bundle \( \mathcal{L}_K \otimes L \otimes \mathcal{L}^{-3} \otimes \mathcal{L}^{-1} \) given by

\[ \tilde{\partial}_\nu = \begin{pmatrix} \tilde{\partial}_L \otimes \tilde{\partial}_1 & h & 0 & 0 \\ 0 & \tilde{\partial}_L & 0 & h \\ 0 & 0 & \tilde{\partial}_L^{-3} & 0 \\ 0 & 0 & 0 & \tilde{\partial}_L \otimes \tilde{\partial}_{-1} \end{pmatrix} \]

\[ \nabla = \begin{pmatrix} \nabla_l \otimes \nabla_{1/2} & 2q + 2\alpha & \beta & \psi_3 \\ 1 & \nabla_l & 0 & 2q + 2\alpha \\ 0 & 0 & \nabla_l^{-3} & \gamma \\ 0 & 1 & 0 & \nabla_l \otimes \nabla_{-1} \end{pmatrix}, \]

where \( \psi_1 = (\alpha, \beta, \gamma) \in H^0(\mathcal{K}^2) \oplus H^0(\mathcal{L}^4 \mathcal{K}^2) \oplus H^0(\mathcal{L}^{-4} \mathcal{K}^2) \).

5.4. The \( (s, \lambda) \)-Slodowy functor. We now describe appropriate generalizations of Theorems 5.5, 5.8 and 5.11 to the category \( \mathcal{Q}\mathfrak{Op}(G, P) \) of \((\lambda, G, P)\)-opers. Since the proofs and statements of the results are almost identical to the \( \lambda = 1 \) case, we mostly omit the details.

First note that the definition of the \( s \)-Slodowy category \( \mathcal{B}_s(G) \) generalizes immediately to a \((\lambda, s)\)-Slodowy category \( \mathcal{Q}\mathcal{B}_s(G) \) whose objects consist of tuples \((\lambda, E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})\), where \((\lambda, E_C, \psi_0)\) is an object in \( \mathcal{Q}\mathcal{F}(C) \) and \( \psi_{m_j} \in H^0(E_C(Z_{2m_j}) \otimes \mathcal{K}^{m_j+1}) \). Similarly, we define the traceless quadratic part \( \mathcal{Q}\mathcal{B}_s(G) \) to be the locus where \( \psi_{m_1} = \psi_{m_1} \).

There are natural projection maps,

\[ \mathcal{Q}\mathfrak{Op}(G, P) \to \mathbb{C} \quad \text{and} \quad \mathcal{Q}\mathcal{B}_s(G) \to \mathbb{C}. \]

Denote the fibers over \( \lambda \in \mathbb{C} \) by \( \mathfrak{Op}^\lambda(G, P) \) and \( \mathcal{B}_s^\lambda(G) \), respectively.
The Slodowy functor from Definition 5.3 also immediately generalizes to λ-connections. For each \((\lambda, S, B_S)\)-oper \(\Theta_\lambda = (\lambda, F_S, F_{B_S}, \eta)\), define the \(\Theta_\lambda\)-Slodowy functor \(F_{\Theta_\lambda} : \mathcal{B}_\lambda^S(G) \to \mathcal{O}p^\lambda(G, P)\) by

\[
F_{\Theta_\lambda}(\Xi_\lambda) = (\lambda, E_C \star F_S(G), E_C \star F_{B_S}(P), \eta \star \psi_0 + \psi_{m_1} + \cdots + \psi_{m_N}),
\]

where \(\Xi_\lambda = (\lambda, E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N})\).

Lemma 5.10 is valid when the Maurer-Cartan form \(\theta_P\) in (5.5) is replaced by \(\lambda \theta_P\). Moreover, the natural \(\lambda\)-connection generalization of Theorem 5.11 is proven by replacing the use of Lemma 5.10 by the \(\lambda\)-version. In particular, this implies that the \(\lambda\)-Slodowy functor \(F_{\Theta_\lambda} : \mathcal{B}_\lambda^S(G) \to \mathcal{O}p^\lambda(G, P)\) is an equivalence of categories. Moreover, each section \(\tau : C \to \mathcal{Q}\mathcal{O}p(S, B_S)\) defines an equivalence of categories

\[
F_\tau : \mathcal{Q}B_s(G) \longrightarrow \mathcal{Q}\mathcal{O}p(G, P) \\
\Xi_\lambda \longrightarrow F_\tau(\lambda)(\Xi_\lambda)
\]

Remark 5.12. Recall that one such sections \(\tau_1 : C \to \mathcal{Q}\mathcal{O}p(S, B_S)\) is defined in (4.2). When \(\langle f, h, e \rangle\) is a principal \(sl_2\)-triple, the functor

\[
(5.9) \quad F_{\tau_1(0)} : \mathcal{B}_0^S(G) \to \mathcal{O}p^0(G, B) \hookrightarrow \mathcal{F}(G)
\]

recovers the Hitchin section from (3.6).

For the group \(SL_4\mathbb{C}\), explicit models of the \((\lambda, s)\)-Slodowy category are given by replacing the \(C\)-connection in §5.3 with a \((\lambda, C)\)-connection. Explicit descriptions of the \(\lambda\)-Slodowy functor are defined by replacing the \(SL_2\mathbb{C}\)-oper from (4.1) with the \((\lambda, SL_2\mathbb{C})\)-oper \(\tau_1(\lambda)\) or \(\tau_2(\lambda)\) defined above. In particular, the associated Higgs bundles are obtained by setting \(\lambda = 0\).

Finally, analogous to Theorem 5.8, the above equivalence can be upgraded to remove choice of section \(\tau\). There are natural \(\mathbb{C}^*\)-actions on \(\mathcal{Q}B_s(G)\) and \(\mathcal{Q}\mathcal{O}p(G, P)\) defined by

\[
(5.10) \quad \xi \cdot (\lambda, E_G, E_P, \omega) = (\xi \lambda, E_G, E_P, \xi \omega) \\
\xi \cdot (\lambda, E_C, \psi_0, \psi_{m_1}, \cdots, \psi_{m_N}) = (\xi \lambda, E_C, \xi \psi_0, \xi \psi_{m_1}, \cdots, \xi \psi_{m_N})
\]

Denote the fiber product of the categories \(\mathcal{Q}\mathcal{O}p(S, B_S)\) and \(\mathcal{Q}\mathcal{B}_s(G)\) with respect to the natural projections to \(C\) by

\[
\mathcal{Q}\mathcal{O}p(S, B_S) \times_C \mathcal{Q}\mathcal{B}_s(G).
\]

The diagonal \(\mathbb{C}^*\)-action on \(\mathcal{Q}\mathcal{O}p(S, B_S) \times \mathcal{Q}\mathcal{B}_s(G)\) induces a natural \(\mathbb{C}^*\)-action on the fiber product.

The proof of the following theorem is almost identical to the proof of Theorem 5.8.

**Theorem 5.13.** Let \(G\) be a connected complex semisimple Lie group, \(s = \langle f, h, e \rangle \subset \mathfrak{g}\) be an even \(sl_2\)-triple, \(S < G\) the connected subgroup with Lie
algebra $s$ and $P < G$ be the associated even JM-parabolic. Let $F_{\Theta_\lambda}$ be the $\Theta_\lambda$-Slodowy functor associated to an $(\lambda, S, B_S)$-oper $\Theta_\lambda$. Then, the functor

$$QF : \mathcal{OD}(S, B_S) \times \mathbb{C} \mathcal{OB}(G) \to \mathcal{OD}(G, P)$$

defined by $F(\Theta_\lambda, \bar{\Xi}_\lambda) = F_{\Theta_\lambda}(\bar{\Xi}_\lambda)$ is an equivalence of categories when $S \cong \text{PSL}_2 \mathbb{C}$ and essentially surjective and full when $S \cong \text{SL}_2 \mathbb{C}$. Moreover, $QF$ is equivariant with respect to the natural $\mathbb{C}^*$-actions from (5.10).

6. More explicit examples and related objects

So far all of our explicit examples have been for $G = \text{SL}_4 \mathbb{C}$ or the parabolic being the Borel subgroup. In this section we discuss a few more explicit examples which generalize the $\text{SL}_4 \mathbb{C}$-examples discussed throughout the paper. We also discuss the relation between some of the examples below and so called higher Teichmüller spaces.

The parabolics of the groups $\text{SO}_n \mathbb{C}$ and $\text{Sp}_{2m} \mathbb{C}$ will be described in terms of the subspaces they stabilize in $\mathbb{C}^n$ ($n = 2m$ for $\text{Sp}_{2m} \mathbb{C}$) equipped with a nondegenerate bilinear form which is symmetric for $\text{SO}_n \mathbb{C}$ and skew-symmetric for $\text{Sp}_{2m} \mathbb{C}$. Also a holomorphic vector bundle $\mathcal{V}$ will sometimes be equipped with a holomorphic symplectic structure $\Omega_\mathcal{V}$ or a holomorphic orthogonal structure $Q_\mathcal{V}$. By this we mean,

- $\Omega_\mathcal{V} \in H^0(\Lambda^2 \mathcal{V})$ with $\det \Omega_\mathcal{V} \in H^0(\det \mathcal{V}) \setminus \{0\}$;
- $Q_\mathcal{V} \in H^0(\text{Sym}^2 \mathcal{V})$ with $\det Q_\mathcal{V} \in H^0((\det \mathcal{V})^2) \setminus \{0\}$.

6.1. B-opers for classical groups. An $(\text{SL}_n \mathbb{C}, B, \lambda)$-oper is a tuple

$$(\lambda, \mathcal{V}, \mathcal{V}_1 \subset \mathcal{V}_2 \subset \cdots \subset \mathcal{V}_n, \nabla_\mathcal{V}),$$

where $\lambda \in \mathbb{C}$, $\mathcal{V}$ is a rank $n$ holomorphic vector bundle with $\det \mathcal{V} \cong \mathcal{O}$, $\mathcal{V}_i$ is a rank $i$ holomorphic subbundle and $\nabla_\mathcal{V}$ is a $\lambda$-connection on $\mathcal{V}$ such that $\nabla(\mathcal{V}_i) \subset \mathcal{V}_{i+1} \otimes \mathcal{K}$ and $\nabla_\mathcal{V}$ induces an isomorphism $\mathcal{V}_i/\mathcal{V}_{i-1} \cong \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K}$.

For $\text{SO}_{2n+1} \mathbb{C}$ (resp. $\text{Sp}_{2n} \mathbb{C}$), a $(B, \lambda)$-oper is an $(\text{SL}_{2n+1} \mathbb{C}, B, \lambda)$-oper (resp. $(\text{SL}_{2n} \mathbb{C}, B, \lambda)$-oper), where $\mathcal{V}$ is equipped with a holomorphic orthogonal (resp. symplectic) structure which is preserved by $\nabla_\mathcal{V}$ and with respect to which $\mathcal{V}_i$ is isotropic and $\mathcal{V}_{n-i} = \mathcal{V}_i^\perp$ for $1 \leq i \leq n$. For $G = \text{SO}_{2n} \mathbb{C}$, $(B, \lambda)$-opers are described in §6.3 when $2n = 2k + 2$.

An object in the category $\mathcal{QB}(\text{SL}_n \mathbb{C})$ is a tuple $(\lambda, \mathcal{L}, \psi_1, \cdots, \psi_{n-1})$, where $\lambda \in \mathbb{C}$, $\mathcal{L}$ is a holomorphic line bundle with $L^n \cong \mathcal{O}$ and $\psi_j \in H^0(K^{n+1})$. For the groups $\text{SO}_{2n+1} \mathbb{C}$ and $\text{Sp}_{2n} \mathbb{C}$ one has $L^2 \cong \mathcal{O}$ and $\psi_{2j} = 0$.

For the section $\tau_2$ of $\mathcal{OD}_{\text{PSL}_2 \mathbb{C}}(X)$ from (4.2), the Slodowy functor $F_{\tau_2} : \mathcal{QB}(\text{SL}_n \mathbb{C}) \to \mathcal{OD}(\text{SL}_n \mathbb{C}, B)$ is given by

$$F_{\tau_2}(\lambda, \mathcal{L}, \psi_1, \cdots, \psi_{n-1}) = (\lambda, \partial_\mathcal{L} \otimes \bar{\partial}_\mathcal{V}, \nabla_\mathcal{L} \otimes \nabla_\mathcal{V}),$$

where, $\nabla_\mathcal{L}$ is the holomorphic $\lambda$-connection inducing the trivial connection on $L^n = \mathcal{O}$, and $(\partial_\mathcal{L} \otimes \bar{\partial}_\mathcal{V}, \nabla_\mathcal{L} \otimes \nabla_\mathcal{V})$ are Dolbeault operators and...
\( \lambda \)-connections on the smooth bundle \( \mathcal{K}_{-1}^{1} \oplus \mathcal{K}_{-2}^{1} \oplus \cdots \oplus \mathcal{K}_{-n}^{1} \) given by

\[
\bar{\nabla}_\mathcal{V} = \begin{pmatrix}
\partial_{\bar{\mu}_{1}^{1}} & \partial_{\bar{\mu}_{2}^{1}} & \ldots & \partial_{\bar{\mu}_{n}^{1}} \\
\mu_1(q+\psi_1) & \mu_2(q+\psi_1) & \ldots & \mu_n(q+\psi_1) \\
1 & \lambda \nabla_{\bar{h}_{2}} & \mu_2(q+\psi_1) & \ldots & \mu_n(q+\psi_1) \\
\ldots & \ldots & \ldots & \ldots & \ldots \\
1 & \lambda \nabla_{\bar{h}_{2}} & \mu_n(q+\psi_1) & \ldots & \mu_n(q+\psi_1) \\
1 & \lambda \nabla_{\bar{h}_{2}} & \mu_n(q+\psi_1) & \ldots & \mu_n(q+\psi_1)
\end{pmatrix}
\]

Here \( \mu_i = i(n-i) \) and the coefficient on each \( \psi_i \) is 1 for \( i > 1 \). The \( \text{SO}_{2n+1} \mathbb{C} \) (resp. \( \text{Sp}_{2n} \mathbb{C} \)) Slodowy functor is given by the restriction of the \( \text{SL}_{2n+1} \mathbb{C} \) (resp. \( \text{SL}_{2n} \mathbb{C} \)) Slodowy functor. Note that \( \mathcal{V} \) has natural orthogonal (resp. symplectic) structure when \( \text{rk}(\mathcal{V}) = 2n+1 \) (resp. \( \text{rk}(\mathcal{V}) = 2n \)) which is preserved by \( \nabla_\mathcal{V} \) when \( \psi_{2j} = 0 \) for all \( j \).

### 6.2. Compact dual of Tube-type Hermitian symmetric space

A family of even JM-parabolics \( P \subset G \) arise when \( G/P \) is the compact dual of a non-compact Hermitian symmetric space of tube type. There are five such pairs \( (G, P) \):

1. \( G = \text{SL}_{2n} \mathbb{C} \) and \( P \) the stabilizer of a \( \mathbb{C}^n \subset \mathbb{C}^{2n} \);
2. \( G = \text{Sp}_{2n} \mathbb{C} \) and \( P \) the stabilizer of an isotropic subspace \( \mathbb{C}^n \subset \mathbb{C}^{2n} \);
3. \( G = \text{SO}_{4n} \mathbb{C} \) and \( P \) the stabilizer of an isotropic subspace \( \mathbb{C}^{2n} \subset \mathbb{C}^{4n} \);
4. \( G = \text{SO}_n \mathbb{C} \) and \( P \) the stabilizer of an isotropic line \( \mathbb{C} \subset \mathbb{C}^n \);
5. \( G = \text{E}_7 \) and if \( \sum_{i=1}^{7} n_i \alpha_i \) is an expression of the longest root with respect to a choice of simple roots, \( P \) is the maximal parabolic subgroup associated to the unique simple root \( \alpha_k \) with \( n_k = 1 \).

Each such parabolic is a maximal parabolic which corresponds to a simple root \( \alpha_k \) with \( n_k = 1 \), where the longest root is given by \( \sum_{i=1}^{n} n_i \alpha_i \). Thus, the \( \mathbb{Z} \)-gradings (2.2) are \( \mathfrak{g} = \mathfrak{g}_{-1} \oplus \mathfrak{g}_0 \oplus \mathfrak{g}_1 \) and the \( \text{sl}_2 \)-module decompositions (2.3) are \( \mathfrak{g} = W_0 \oplus W_2 \). The centralizers \( C \) and multiplicities \( (n_0, n_2) \) are

| \( G \) | \( \text{SL}_{2n} \mathbb{C} \) | \( \text{Sp}_{2n} \mathbb{C} \) | \( \text{SO}_{4n} \mathbb{C} \) | \( \text{SO}_n \mathbb{C} \) | \( \text{E}_7 \) |
|---|---|---|---|---|---|
| \( n_0 \) | \( n^2 - 1 \) | \( \frac{n(n-1)}{2} \) | \( n(2n+1) \) | \( \frac{n(n-1)}{2} \) | 27 |
| \( n_2 \) | \( n^2 \) | \( \frac{n(n+1)}{2} \) | \( n(2n-1) \) | \( n-2 \) | 52 |

We now describe the vector bundle definition of an \( (G, P, \lambda) \)-oper in cases (1)-(3), case (4) will be described in the next subsection. An \( (\text{SL}_{2n} \mathbb{C}, P, \lambda) \)-oper consists of a tuple \( (\lambda, \mathcal{V}, \mathcal{V}_n, \nabla_\mathcal{V}) \), where \( \lambda \in \mathbb{C} \), \( \mathcal{V} \) is a rank \( 2n \) holomorphic vector bundle with \( \Lambda^{2n} \mathcal{V} \cong \mathcal{O} \), \( \nabla_\mathcal{V} \) is a \( \lambda \)-connection on \( \mathcal{V} \) and \( \mathcal{V}_n \subset \mathcal{V} \) is a holomorphic rank \( n \) subbundle such that the induced map
$\nabla_V : \mathcal{V}_n \to \mathcal{V}/\mathcal{V}_n \otimes \mathcal{K}$ is an isomorphism. For case (2), an (Sp$_{2n}$, C, P, $\lambda$)-oper is an (SL$_{2n}$, C, P, $\lambda$)-oper ($\lambda, \mathcal{V}, \mathcal{V}_n, \nabla_V$) such that $\mathcal{V}$ is equipped holomorphic symplectic structure $\Omega$ which is preserved by $\nabla_V$ and with respect to which $\mathcal{V}_n$ is isotropic. For case (3), an (SO$_{4n}$, C, P, $\lambda$)-oper is an (SL$_{4n}$, C, P, $\lambda$)-oper ($\lambda, \mathcal{V}, \mathcal{V}_{2n}, \nabla_V$) such that $\mathcal{V}$ is equipped holomorphic orthogonal structure $Q_\mathcal{V}$ which is preserved by $\nabla_V$ and with respect to which $\mathcal{V}_{2n}$ is isotropic.

For case (1), the objects $\mathcal{QB}_2(\text{SL}_{2n}, \mathbb{C})$ consist of tuples $(\lambda, \mathcal{V}, \nabla_V, \psi_1)$, where $\lambda \in \mathbb{C}$, $\mathcal{V}$ is a rank $n$ holomorphic vector bundle with $\text{det}(\mathcal{V})^2 \cong \mathcal{O}$, $\nabla_V$ is a holomorphic $\lambda$-connection on $\mathcal{V}$ and $\psi_1 \in H^0(\text{End}(\mathcal{V}) \otimes \mathcal{K}^2)$. For the section $\tau_2$ of $\mathcal{QD}_{\text{PSL}_2}(X)$ from (4.2), the Slodowy functor is $F_{\tau_2}(\lambda, \mathcal{V}, \nabla_V, \psi_1) = (\lambda, \tilde{\nabla}_V, \nabla_V)$, where $(\tilde{\nabla}_V, \nabla_V)$ are Dolbeault operators and $\lambda$-connections on the smooth bundle $\mathcal{V} \otimes \mathcal{K}^{1/2} \oplus \mathcal{V} \otimes \mathcal{K}^{-1/2}$ given by (6.2)

\[ \tilde{\nabla}_V = \begin{pmatrix} \tilde{\nabla}_V \otimes \tilde{\nabla}_{1/2} & \lambda h \otimes \text{Id}_W \\ 0 & \tilde{\nabla}_V \otimes \tilde{\nabla}_{-1/2} \end{pmatrix}, \quad \nabla_V = \begin{pmatrix} \nabla_V \otimes \lambda \nabla_{1/2}^h & q \otimes \text{Id}_W + \psi_1 \\ \text{Id}_W & \nabla_V \otimes \lambda \nabla_{-1/2}^h \end{pmatrix}. \]

For case (2), the objects of $\mathcal{QB}_2(\text{Sp}_{2n}, \mathbb{C})$ are objects $(\lambda, \mathcal{V}, \nabla_V, \psi_1)$ of $\mathcal{QB}_2(\text{SL}_{2n}, \mathbb{C})$ where $\mathcal{V}$ is equipped with a holomorphic orthogonal structure $Q_\mathcal{V}$ which is preserved by $\nabla_V$ and with respect to which $\psi_1$ is symmetric. The Slodowy functor $F_{\tau_2}$ is the restriction of functor on $\mathcal{QB}_2(\text{SL}_{2n}, \mathbb{C})$. The symplectic structure on $\mathcal{V} \otimes \mathcal{K}^{1/2} \oplus \mathcal{V} \otimes \mathcal{K}^{-1/2}$ is defined by

\[ \Omega_\mathcal{V} = \begin{pmatrix} 0 & Q_\mathcal{V} \\ -Q_\mathcal{V} & 0 \end{pmatrix}. \]

For case (3), the objects of $\mathcal{QB}_3(\text{SO}_{4n}, \mathbb{C})$ are objects $(\lambda, \mathcal{V}, \nabla_V, \psi_1)$ of $\mathcal{QB}_2(\text{SL}_{4n}, \mathbb{C})$ where $\mathcal{V}$ is equipped with a holomorphic symplectic structure $\Omega_\mathcal{V}$ which is preserved by $\nabla_V$ and with respect to which $\psi_1$ is symmetric. The Slodowy functor $F_{\tau_2}$ is the restriction of functor on $\mathcal{QB}_3(\text{SL}_{4n}, \mathbb{C})$. The orthogonal structure on $\mathcal{V} \otimes \mathcal{K}^{1/2} \oplus \mathcal{V} \otimes \mathcal{K}^{-1/2}$ is defined by

\[ Q_\mathcal{V} = \begin{pmatrix} 0 & \Omega_\mathcal{V} \\ -\Omega_\mathcal{V} & 0 \end{pmatrix}. \]

6.3. **Some more opers for SL$_n$ and SO$_n$.** Another family of even JM-parabolics of SL$_n$ for are given by the stabilizers of partial flags

\[ \mathbb{C}^1 \subset \mathbb{C}^2 \subset \cdots \subset \mathbb{C}^k \subset \mathbb{C}^{n-k} \subset \cdots \subset \mathbb{C}^{n-1} \subset \mathbb{C}^n \]

where $2k < n$. When $\mathbb{C}^n$ is equipped with an orthogonal structure and the subspaces $\mathbb{C}^i$ are isotropic for $1 \leq i \leq k$, the associated parabolic P < SO$_n$ is also an even JM-parabolic. In particular, for the orthogonal group, the parabolic P is the Borel subgroup when $n = 2k + 1$ and $n = 2k + 2$, and P is the parabolic from case (4) of the previous section when $k = 1$.

For the above parabolic P, an (SL$_n$, C, P, $\lambda$)-oper consists of a tuple

\[ (\lambda, \mathcal{V}, \mathcal{V}_1 \subset \cdots \mathcal{V}_k \subset \mathcal{V}_{n-k} \subset \cdots \subset \mathcal{V}, \nabla_V), \]
where $\lambda \in \mathbb{C}$, $\mathcal{V}$ is a rank $n$ holomorphic vector bundle with $\Lambda^n \mathcal{V} \cong \mathcal{O}$, $\nabla_\mathcal{V}$ is a $\lambda$-connection on $\mathcal{V}$ and $\mathcal{V}_i \subset \mathcal{V}$ is a holomorphic rank $i$ subbundle such that:

- for $i \neq k$, $\nabla_\mathcal{V}(\mathcal{V}_i) \subset \mathcal{V}_{i+1} \otimes \mathcal{K}$ and $\nabla_\mathcal{V}$ induces an isomorphism $\mathcal{V}_i/\mathcal{V}_{i-1} \cong \mathcal{V}_{i+1}/\mathcal{V}_i \otimes \mathcal{K}$;
- $\nabla_\mathcal{V}(\mathcal{V}_k) \subset \mathcal{V}_{n-k} \otimes \mathcal{K}$, and $\nabla_\mathcal{V}^2$ induces an isomorphism $\mathcal{V}_k/\mathcal{V}_{k-1} \cong \mathcal{V}_{n-k+1}/\mathcal{V}_{n-k} \otimes \mathcal{K}^2$.

When $G = \text{SO}_n \mathbb{C}$, an $(\text{SO}_n \mathbb{C}, P, \lambda)$-oper is an $(\text{SL}_n \mathbb{C}, P, \lambda)$-oper where the holomorphic bundle $\mathcal{V}$ is equipped with an orthogonal structure $Q_\mathcal{V}$ which is preserved by $\nabla_\mathcal{V}$ and with respect to which $\mathcal{V}_i$ is isotropic and $\mathcal{V}_{n-i} = \mathcal{V}^i_{\perp Q_\mathcal{V}}$ for $1 \leq i \leq k$.

We will describe the $s$-Slodowy category and functor for $G = \text{SO}_n \mathbb{C}$, the $G = \text{SL}_n \mathbb{C}$ case is left to the reader. The $\mathfrak{sl}_2$-module decomposition of the even $\text{JM}$-parabolic is

$$\text{so}_n \mathbb{C} = W_0 \oplus W_{2k} \oplus \bigoplus_{j=1}^{k} W_{4j-2}.$$  

The centralizer $C$ of the even $\mathfrak{sl}_2$ is isomorphic to $\text{O}_{n-2k+1} \mathbb{C}$, and the multiplicities are $n_0 = \frac{(n-2k-1)(n-2k-2)}{2}$, $n_{4j-2} = 1$ for $4j - 2 \neq 2k$, and

$$n_{2k} = \begin{cases} n - 2k & \text{if } k \text{ odd} \\ n - 2k - 1 & \text{if } k \text{ even}. \end{cases}$$

When $k$ is odd, the $C$-representation space $Z_{2k}$ decomposes as a direct sum of the one dimensional trivial representation and the standard $(n - 2k - 1)$-dimensional representation twisted by the determinant representation.

The objects of the category $\mathcal{QB}_s(\text{SO}_n \mathbb{C})$ consists of tuples

$$(\lambda, \mathcal{W}, \nabla_\mathcal{W}, \psi_1, \cdots, \psi_{2k-1}),$$

where $\lambda \in \mathbb{C}$, $\mathcal{W}$ is a rank $(n - 2k - 1)$ holomorphic vector bundle equipped with an orthogonal structure $Q_\mathcal{W}$, $\nabla_\mathcal{W}$ is a $\lambda$-connection on $\mathcal{W}$ which preserves $Q_\mathcal{W}$, $\psi_{2j-1} \in H^0(K^{2j})$ for $2j - 1 \neq k$ and

$$\psi_k = \begin{cases} \langle q_k, \hat{\psi}_k \rangle \in H^0(K^{k+1}) \oplus H^0(\mathcal{W} \otimes \text{det}(\mathcal{W}) \otimes K^{k+1}) & \text{if } k \text{ odd} \\ \hat{\psi}_k \in H^0(\mathcal{W} \otimes \text{det}(\mathcal{W}) \otimes K^{k+1}) & \text{if } k \text{ even}. \end{cases}$$

The Slodowy functor $F_{\tau_2} : \mathcal{QB}_s(\text{SO}_n \mathbb{C}) \to \mathcal{Op}(\text{SO}_n \mathbb{C}, P)$ is given by

$$F_{\tau_2}(\lambda, \mathcal{W}, \nabla_\mathcal{W}, \psi_1, \cdots, \psi_{2k-1}) = (\lambda, \hat{\partial}_\mathcal{V}, \nabla_\mathcal{V}),$$

where $(\hat{\partial}_\mathcal{V}, \nabla_\mathcal{V})$ are Dolbeault operators and $\lambda$-connections on the bundle $\mathcal{V} = \mathcal{W} \oplus \mathcal{W}_{2k+1} \otimes \text{det} \mathcal{W}$ given by

$$(6.3) \quad \hat{\partial}_\mathcal{V} = \begin{pmatrix} \hat{\partial}_\mathcal{W} & 0 \\ 0 & \hat{\partial}_{\mathcal{W}_{2k+1}} \otimes \hat{\partial}_{\text{det} \mathcal{W}} \end{pmatrix}, \quad \nabla_\lambda = \begin{pmatrix} \nabla_\mathcal{W} & \hat{\psi}_k^T \\ \hat{\psi}_k & \nabla_{\mathcal{W}_{2k+1}} \otimes \nabla_{\text{det} \mathcal{W}} \end{pmatrix}.$$
\((\lambda, \mathcal{W}_{2k+1}, \nabla_{\mathcal{W}_{2k+1}})\) denotes the \((\text{SO}_{2k+1}\mathbb{C}, B, \lambda)\)-oper \((6.1)\) associated to the Slodowy functor evaluated on
\[
\begin{cases}
(\psi_1, \psi_3, \psi_{k-2}, q_k, \psi_{k+2}, \cdots, \psi_{2k}) & \text{if } k \text{ is odd} \\
(\psi_1, \psi_3, \cdots, \psi_{2k}) & \text{if } k \text{ is even};
\end{cases}
\]
• \(\hat{\Psi}_k : \mathcal{W}_{2k+1} \otimes \det \mathcal{W} \to \mathcal{W} \otimes \mathcal{K}\) is a holomorphic bundle map which, in the smooth splitting of \(\mathcal{W}_{2k+1}\) from \((6.1)\), is given by
\[
\hat{\Psi}_k = \begin{pmatrix}
0 & \cdots & 0 & \hat{\psi}_k
\end{pmatrix} \colon (\mathcal{K}^k \oplus \cdots \oplus \mathcal{K}^{-k}) \otimes \det \mathcal{W} \to \mathcal{W} \otimes \mathcal{K}.
\]

The orthogonal structure \(Q_V\) on \(\mathcal{W} \oplus \mathcal{W}_{2k+1} \otimes \det \mathcal{W}\) is given by
\[
Q_V = \begin{pmatrix}
Q_W & 0 \\
0 & Q_{\mathcal{W}_{2k+1}} \otimes \det Q_W
\end{pmatrix}.
\]

6.4. **Remarks on Nonabelian Hodge and Higher Teichmüller spaces.**

So far, we have avoided the discussion of moduli spaces and stability. However, for every \(\lambda\), both the \(B^\lambda(G)\) and \(\text{Op}^\lambda(G, P) \subset F^\lambda(G)\) have natural stability conditions and corresponding coarse moduli spaces. One would expect the natural stability conditions on \(B^\lambda(G)\) and \(F^\lambda(G)\) to be related in a way such that the Slodowy functor \(F_{\lambda_0}\) induces a well defined map on moduli spaces. This is especially desirable when \(\lambda = 0\) because one can associate a holomorphic connection to a point in the moduli space of Higgs bundles via the Nonabelian Hodge correspondence.

Denote the moduli spaces of polystable \(G\)-Higgs bundles and holomorphic connections on a Riemann surface \(X\) by \(\mathcal{M}^0(G)\) and \(\mathcal{M}^1(G)\) respectively. We refer the reader to \([35, 36]\) for the construction of this moduli space. Furthermore, the moduli space \(\mathcal{M}^1(G)\) is analytically isomorphic to the character variety \(\mathcal{X}(G)\) of conjugacy classes of reductive representations of the fundamental group of \(X\) in \(G\):
\[
\mathcal{X}(G) = \text{Hom}(\pi_1(X), G) / G.
\]

Note that \(\mathcal{X}(G)\) only depends on the topological surface underlying the Riemann surface \(X\).

The nonabelian Hodge correspondence (see \([13, 14, 23, 34]\)) defines a real analytic isomorphism between these spaces:
\[
\mathcal{T} : \mathcal{M}^0(G) \to \mathcal{M}^1(G) \cong \mathcal{X}(G).
\]

For \((0, \text{PSL}_2\mathbb{C}, B)\)-opers, the Higgs bundles are always stable and the image under the map \(\mathcal{T}\) can be identified with the Teichmüller space of the underlying topological surface \([23]\). More precisely, the image of \((0, \text{PSL}_2\mathbb{C}, B)\)-opers under the nonabelian Hodge correspondence \(\mathcal{T}\) consists of all conjugacy classes of holonomies of hyperbolic structures on the underlying topological surface.

For \((0, G, B)\)-opers (i.e., the Hitchin section) it is not hard to show that the Higgs bundles are always stable and so define points in \(\mathcal{M}^0(G)\). Moreover, applying nonabelian Hodge to this locus defines a (union of) connected
component of the character variety of representations into the split real form of $G$ called the Hitchin component \cite{24}. Moreover, the surface group representations in this component generalize many features of Teichmüller space. In particular, they are all discrete and faithful \cite{16,30} and are holonomies of certain geometric structures on closed manifolds \cite{21,26}. The Hitchin component was the first example of what is now referred to as a higher Teichmüller component.

The family of $(0, G, P)$-opers described in §6.2 are related to a family of higher Teichmüller spaces known as maximal representations into a real Hermitian Lie group of tube type. In these cases, when the Higgs field $\psi_0$ is identically zero and $q = 0$, the Slodowy functor $F_{\tau_1(0)}$ recovers the Cayley correspondence of \cite[§5]{3} (see also \cite{7,8,19}). This correspondence relates Higgs bundles with so called maximal Toledo invariant for a real\textsuperscript{2} Hermitian Lie group of tube type $G^\mathbb{R}$ with $\mathcal{K}^2$-twisted Higgs bundles for another group. For the complex groups $\text{SL}_{2n}\mathbb{C}$, $\text{Sp}_{2n}\mathbb{C}$, $\text{SO}_{4n}\mathbb{C}$, $\text{SO}_n\mathbb{C}$ and $E_7$, the Hermitian Lie group $G^\mathbb{R}$ is $\text{SU}_{n,n}$, $\text{Sp}_{2n}\mathbb{R}$, $\text{SO}_{4n}^\mathbb{R}$, $\text{SO}_{2,n-2}$ and $E_7^{-25}$, respectively.

Under the nonabelian Hodge correspondence, the isomorphism classes such Higgs bundles which are polystable are in bijective correspondence with the set of so called maximal representations. Such representations have been studied by many authors, and correspond to unions of connected components of the character variety which consist entirely of discrete and faithful representations \cite{9} and carries a properly discontinuous action of the mapping class group \cite{38}.

For the family of $(0, \text{SL}_n\mathbb{C}, P)$-opers from §6.3 are not related to related to unions of connected components of the character variety of a real form of $\text{SL}_n\mathbb{C}$. However, the $(0, \text{SO}_n\mathbb{C}, P)$-opers from §6.3 are related to connected components of character varieties. In this case, when the Higgs field $\psi_0$ is identically zero and $q = 0$, the Slodowy functor $F_{\tau_1(0)}$ recovers the generalized Cayley correspondence of \cite{1}. Similar to the case of maximal representations, under the nonabelian Hodge correspondence, to the isomorphism classes such Higgs bundles which are polystable are identified with unions of connected components of the character variety for the real form $\text{SO}(k-1, n-k+1)$. From the results of \cite{1} and Guichard-Wienhard’s work on $\Theta$-positivity \cite{22} one expects such components to define higher Teichmüller spaces.

Remark 6.1. For a general even JM-parabolic, when $\lambda = 0$ and $\psi_0 = 0$ the Slodowy functor will not be related to Higgs bundles for a real group. However, using Guichard-Wienhard’s work on $\Theta$-positivity \cite{22}, one expects one more such family for certain real forms of $F_4$, $E_6$, $E_7$ and $E_8$. This is indeed the case and the relationship between the Slodowy functor and higher Teichmüller spaces will be described in \cite{6}.

\textsuperscript{2}We have not discussed Higgs bundles for real groups, for appropriate definitions see for example \cite{20}.
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