The Viability of Phantom Dark Energy as a Quantum Field in 1st-Order FLRW Space

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Abstract

In the standard cosmological framework of the 0th-order FLRW metric and the use of perfect fluids in the stress-energy tensor, dark energy with an equation-of-state parameter \( w < -1 \) (known as phantom dark energy) implies negative kinetic energy and vacuum instability when modeled as a scalar field. However, the accepted values for present-day \( w \) from Planck and WMAP9 include a significant range of values less than \(-1\). A flip of the sign in front of the kinetic energy term in the Lagrangian remedies the negative kinetic energy but introduces ghostlike instabilities, which perhaps may be rendered unobservable, but certainly not without great cost to the theory. Staying within the confines of observational constraints and general relativity, for which there is good experimental validation, we consider a reasonable departure from the standard 0th-order framework in an attempt to see if negative kinetic energy can be avoided despite an apparent \( w < -1 \), all without flipping the sign of the kinetic energy term. We consider a more accurate description of the universe through the perturbing of the isotropic and homogeneous FLRW metric and the components of the stress-energy tensor to 1st order. We treat dark energy as a quantum scalar field in the background of this 1st-order FLRW space-time, find an approximation for the Green’s function, and calculate the expectation value of the field’s kinetic energy for \( w < -1 \) using adiabatic expansion to renormalize and obtain a finite value. We find that the kinetic energy is positive for values of \( w \) less than \(-1\) in 0th- or 1st-order FLRW space, thus giving more theoretical credence to observational values of \( w < -1 \) and demonstrating that phantom dark energy does not categorically have negative kinetic energy. Also, our results are generally applicable for a massive free field or a field with a small potential in a 0th- or 1st-order FLRW background dominated by a fluid with a constant \( w \).
Introduction

A recent milestone in observational cosmology happened when the High-z Supernova Search Team in 1998 [1] and the Supernova Cosmology Project in 1999 [2] published observations of the emission spectra of Type Ia supernovae indicating that the universe’s rate of outward expansion is increasing. Galaxy surveys and the late-time integrated Sachs-Wolfe effect also give evidence for the universe’s acceleration. Thus, ”dark energy” was proposed as the pervasive energy in the universe necessary to produce the outward force that causes this acceleration, which has been observationally tested and vetted since its discovery. The 2011 Nobel Prize in Physics was awarded to Schmidt, Riess, and Perlmutter for their pioneering work leading to the discovery of dark energy. The present-day equation-of-state parameter $w$ from the equation of state most frequently tested by cosmological probes, $p = w \rho$ with constant $w$, assuming a flat universe and a perfect fluid representing dark energy, has been constrained by Planck in early 2015 to be $w = -1.006 \pm 0.045$ [3], and Planck’s 2013 value was $w = -1.13^{+0.13}_{-0.10}$ [4]. The value from the Nine-Year Wilkinson Microwave Anisotropy Probe (WMAP9), combining data from WMAP, the cosmic microwave background (CMB), baryonic acoustic oscillations (BAO), supernova measurements, and $H_0$ measurements, is $w = -1.084 \pm 0.063$ [5]. From these reported values, the prospect of $w < -1$ is clearly a distinct possibility, and under other assumptions (such as a spatially curved universe), the window reported for $w$ does not always include the value for the cosmological constant (CC) model, $w = -1$.

However, dark energy modeled as a perfect fluid with $w < -1$ leads to a field theory with negative kinetic energy (a ghost field theory), which implies vacuum instability. Either the phantom ghost has positive density and violates unitarity, rendering it unphysical, or unitarity is satisfied and the density is negative, which leads to vacuum instability [6]. This phantom dark energy with a wrong-sign kinetic term described as an effective field theory may be able to make this instability unobservable, but not without great difficulty and perhaps sacrifice of well-accepted physical principles [6, 7].

One deduces the ghost nature of phantom dark energy from $w < -1$ within the standard cosmological framework of the 0th-order Friedmann-Lemaître-Robertson-Walker (FLRW) metric with the use of perfect fluids in the stress-energy tensor, but the condition for negative kinetic energy is different for different frameworks. Given that our universe is not perfectly
isotropic and homogeneous, we examine the possibility for positive kinetic energy with \( w < -1 \) in light of first-order perturbations to the FLRW metric and the components of the stress-energy tensor. In earlier work \[8\], we found that for certain classical scalar field models of dark energy with non-constant \( w < -1 \), it is possible to have positive kinetic energy for certain length and time scales. We also found that it was not possible to have positive kinetic energy for constant \( w < -1 \) because that condition implied that one of the perturbations of the stress-energy tensor would have to be bigger than 1, violating the assumption of perturbation. In this work, we treat dark energy as a quantum scalar field with constant \( w < -1 \) and calculate the expectation value of the kinetic energy using the method of adiabatic subtraction for renormalization, and we find that the kinetic energy is positive for all relevant length and time scales.

**Dark Energy as a Scalar Field**

We consider the Einstein-Hilbert action for general relativity with a real scalar field for dark energy \((c = 1)\):

\[
S = \int d^4x \sqrt{-g} \left[ \frac{R}{16\pi G} - \frac{1}{2} g^{\mu\nu} \nabla_\mu \phi \nabla_\nu \phi - \frac{1}{2} m^2 \phi^2 - \frac{1}{2} \xi R \phi^2 - V(\phi) \right] + S_m, \tag{1}
\]

where the first term is the usual contribution to the Einstein tensor, the second term is the kinetic energy term, the third term is the mass term, the fourth term is the non-minimal coupling term (usually included for its utility in the renormalization of a scalar field in a curved background), the fifth term is the dark energy potential, and \( S_m \) is the action for the rest of the components of the stress-energy tensor \( T_{\mu\nu} \). We would like to calculate the expectation value of the kinetic energy, which is, for a free-field theory,

\[
-\frac{1}{2} g^{\sigma\rho} < \partial_\rho \phi \partial_\sigma \phi > = -\frac{1}{2} g^{\sigma\rho} \partial_\sigma \partial_\rho iG(x, x')|_{x' \to x}, \tag{2}
\]

and we will renormalize via adiabatic subtraction. To obtain the Green’s function \( G(x, x') \), we must solve the equation of motion \[9\]

\[
(-\Box_x + m^2 + \xi R^2)G(x, x') = g^{-1/4}(x)\delta(x - x')g^{-1/4}(x'), \tag{3}
\]

where the operator \( \Box_x = g(x)^{-1/2} \frac{\partial}{\partial x^\mu} \left[ g(x)^{1/2} g(x)^{\mu\nu} \frac{\partial}{\partial x^\nu} \right] \) when applied to a scalar. The data that support a value of \( w < -1 \) are for late-time redshift values for which dark energy
dominates, as it does currently, so we consider dark energy to be the only component of the
universe, which therefore specifies the background space. The self-interaction terms in $V(\phi)$
specify the equation of state of dark energy, and for constant $w$ satisfying $|w+1| \ll 1$, it can
be shown that the potential is slowly varying and small in 0th-order FLRW space, and it
should also be small in 1st-order FLRW space. So we will ignore the terms from $V(\phi)$ in our
calculation of the kinetic energy, so we will simply evaluate Eq. (2) to find the renormalized
kinetic energy.

**The 1st-Order FLRW Metric**

We take the 1st-order flat FLRW metric with scalar perturbations in the synchronous
gauge to be our fixed background (using the notation of [10]),

$$ds^2 = a^2(\tau) \left[ -d\tau^2 + (\delta_{ij} + h_{ij})dx^idx^j \right],$$

with the scalar mode of $h_{ij}$ written in $k$-space as

$$h_{ij}(\vec{x}, \tau) = \int d^3k e^{i\vec{k}\cdot\vec{x}} \left\{ \hat{k}_i\hat{k}_j h(k, \tau) + (\hat{k}_i\hat{k}_j - \delta_{ij}\frac{3}{3})6\eta(k, \tau) \right\}.$$  

The Friedmann equations resulting from solving Einstein’s equation for the flat FLRW
metric (0th order) are

$$\mathcal{H}^2 = \frac{8\pi G}{3} a^2 \rho,$$  

$$\dot{\mathcal{H}} = -\frac{4\pi G}{3} a^2 (\rho + 3P),$$

where $\cdot$ denotes differentiation with respect to $\tau$, $\mathcal{H} \equiv \frac{\dot{a}}{a}$, and $\rho$ and $P$ represent the total
density and pressure respectively. These equations lead to the evolution equation for each
density component:

$$\dot{\rho} = -3\mathcal{H}(\rho + P).$$

The equations resulting from solving the perturbed Einstein equation in $k$-space to first
order are

$$k^2\eta - \frac{1}{2} \mathcal{H}\dot{h} = 4\pi Ga^2 \delta T^0_0,$$  

$$k^2\dot{\eta} = 4\pi Ga^2 (\rho + P)\theta,$$  

$$\ddot{h} + 2\mathcal{H}\dot{h} - 2k^2\eta = -8\pi Ga^2 \delta T^i_i,$$  

$$\ddot{h} + 6\ddot{\eta} + 2\mathcal{H}(\dot{h} + 6\dot{\eta}) - 2k^2\dot{\eta} = -24\pi Ga^2 (\rho + P)\sigma,$$
where $\theta$ is the divergence of the fluid velocity $v_i$, $(\rho + P)\sigma \equiv -(\hat{k}_i \hat{k}_j - \frac{1}{3} \delta_{ij})\Sigma^i_j$ where $\Sigma^i_j$ is the anisotropic shear perturbation, and $h$ and $\eta$ are the scalar modes of the metric perturbation. The stress-energy tensor for a perturbed perfect fluid is given by

$$
T^0_0 = - (\rho + \delta\rho),
$$
$$
T^0_i = (\rho + P)v_i,
$$
$$
T^i_j = (\rho + \delta P)\delta^i_j + \Sigma^i_j, \quad \Sigma^i_i = 0.
$$

The conservation of energy-momentum, $T^\mu_\nu ; \mu = 0$, gives (using $\delta \equiv \delta\rho/\rho$)

$$
\dot{\delta} = -(1 + w) \left( \theta + \frac{\dot{h}}{2} \right) - 3\mathcal{H} \left( \frac{\delta P}{\delta\rho} - w \right) \delta,
$$

(11a)

$$
\dot{\theta} = -\mathcal{H}(1 - 3w)\theta - \frac{\dot{w}}{1 + w}\theta + \frac{\delta P/\delta\rho}{1 + w} k^2 \delta - k^2 \sigma.
$$

(11b)

Eq. (11) is valid when considering each fluid component or the total fluid, but Eq. (9) is valid only for the total fluid. The anisotropic shear stress is 0 ($\sigma = 0$) throughout, and in what follows, we use $c = G = 1$.

$\delta P/\rho$ for a given fluid component is in general given by

$$
\frac{\delta P}{\rho} = c_s^2 \delta + (c_a^2 - c_s^2) 3\mathcal{H}(1 + w) \frac{\theta}{k^2},
$$

(12)

where $c_s$ is the fluid’s sound speed and $c_a^2 \equiv \dot{P}/\dot{\rho} = w + \dot{w}\rho/\dot{\rho}$ is defined as the square of the fluid’s adiabatic sound speed [11]. For a barotropic fluid, $c_s^2 = c_a^2$, and $c_a^2 = w$ for constant $w$. Even though dark energy can have a barotropic equation of state, treating it like an adiabatic fluid (for which Eq. (12) reduces to $\delta P = c_s^2 \delta\rho$) would imply imaginary sound speed and instabilities in dark energy, so we use this general relation between $\delta P$ and $\delta\rho$.

Since the dynamics of the universe is currently dominated by dark energy, we will assume dark energy is the only fluid component with an equation of state $P = w\rho$ where $w < -1$ and is a constant. In this case, $\mathcal{H}$ can be written as

$$
\mathcal{H} = \frac{2}{(3w + 1)\tau}.
$$

(13)

For $w < -1/3$, $\tau \in (-\infty, 0)$ for $a \in (0, \infty)$. We find it more convenient to work in terms of $a$ rather than $\tau$. $\tau$ can be related to $a$ via Eq. [6]. Using Eqs. [9a], [9b], [11a], and [11b], and keeping the relevant growing modes of perturbations, one finds that [8]

$$
h(k, \tau) = 2^{\frac{3(1 + w)}{1 + 3w}} S \left( \frac{\rho_{DE0}\pi}{3} \right)^{\frac{1 - 3w}{2 + 6w}} 4\pi\rho_{DE0} k^{-\frac{2}{3w + 1} - 1} a^{-\frac{2(w + 1)}{3}}
$$

(14)
and $\eta(\mathbf{k}, \tau) = 0$, where $\rho_{DE0} = \frac{3H_0^2}{8\pi} \Omega_{DE0}$ is the present-day density of dark energy and $S$ is a dimensionless constant of integration. As a scaling factor for the perturbation magnitude, and from observational constraints, $|S| \ll 1$.

We can then use Eq. (5) to arrive at the metric perturbations in spatial coordinates of the metric. Using $r = \sqrt{x^2 + y^2 + z^2}$, the result is

$$h_{ij}(\mathbf{x}, \tau) = \partial_i \partial_j \left[ \frac{2(1+w)}{\pi^{1+3w}} S \left( \frac{\rho_{DE0}}{3} \right)^{\frac{1-3w}{2}} \frac{8\pi^2 \rho_{DE0} a}{y^2} \frac{3^{\frac{3w+1}{2}}}{\Gamma(1 - \frac{2}{1+3w})} \right]$$

where we have ignored the oscillatory upper bound of the integral with respect to $k$, and this is valid especially since we are only interested in large scales over which dark energy is relevant.

The Green’s Function in Riemann Normal Coordinates

One can see that the equation of motion, Eq. (3), cannot be solved exactly for our 1st-order FLRW metric, so we will express the equation of motion as a series expansion using Riemann normal coordinates \cite{12}. Assuming for any point $P$ in the neighborhood of $Q$ that there is a unique geodesic joining these points, we can use Riemann normal coordinates of that point $P$: $y^\mu = \lambda \xi^\mu$, where $\xi^\mu$ is the tangent to the geodesic at the point $Q$, and $\lambda$ is a parameter representing how far off along the geodesic we are from $Q$. We take the origin $Q$ at the space-time point $x'$, where $y^\mu = 0$, and we denote $x^\mu$ to be its normal coordinate $y^\mu$. We can therefore write $g_{\mu\nu}(y = 0) = \eta_{\mu\nu}$, and its determinant is $|g(y = 0)| = 1$. We expand $g_{\mu\nu}(y)$, $R(y)$, and $g(y)$ about $y^\mu = 0$. And the limit we will take in solving for the expectation value of the kinetic energy in Eq. (2) $x' \rightarrow x$ is equivalent to $y \rightarrow 0$. Eq. (3.180) of \cite{12} gives a series expansion of a tensor field about the origin in Riemann normal coordinates:

$$W_{\alpha_1 \cdots \alpha_p}(y) = W_{\alpha_1 \cdots \alpha_p}(0) + W_{\alpha_1 \cdots \alpha_p;\mu}(0)y^\mu$$

$$+ \frac{1}{2!} \left[ W_{\alpha_1 \cdots \alpha_p;\mu\nu} - \frac{1}{3} \sum_{k=1}^p R^\nu_{\mu\alpha_k\omega} W_{\alpha_1 \cdots \alpha_k-1\nu\alpha_{k+1} \cdots \alpha_p} \right]_0 y^\mu y^\nu$$

$$+ \frac{1}{3!} \left[ W_{\alpha_1 \cdots \alpha_p;\mu\nu\sigma} - \sum_{k=1}^p R^\nu_{\mu\alpha_k\omega} W_{\alpha_1 \cdots \alpha_k-1\nu\alpha_{k+1} \cdots \alpha_p} - \frac{1}{2} \sum_{k=1}^p R^\nu_{\mu\alpha_k\omega;\sigma} W_{\alpha_1 \cdots \alpha_k-1\nu\alpha_{k+1} \cdots \alpha_p} \right]_0 y^\mu y^\nu y^\sigma + \ldots ,$$

(16)
where each coefficient in front of the factors of \( y \) are evaluated at \( y = 0 \). From [13], the metric expansion about the origin in Riemann normal coordinates to 6th order is

\[
g_{\alpha_1 \alpha_2}(y) = g_{\alpha_1 \alpha_2}(0) + \frac{1}{2!} R_{\alpha_1 \beta_1 \beta_2 \alpha_2}(0) y^{\beta_1} y^{\beta_2} \\
+ \frac{1}{3!} \left[ \nabla_{\beta_1} R_{\alpha_1 \beta_2 \beta_3 \alpha_2} \right]_0 y^{\beta_1} ... y^{\beta_3} \\
+ \frac{1}{4!} \left[ \nabla_{\beta_1} \nabla_{\beta_2} R_{\alpha_1 \beta_3 \beta_4 \alpha_2} + \frac{8}{9} R_{\alpha_1 \beta_1 \beta_2 \rho} R_{\beta_3 \beta_4 \alpha_2} \right]_0 y^{\beta_1} ... y^{\beta_4} \\
+ \frac{1}{5!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_5} R_{\alpha_1 \beta_6 \alpha_2} + 2 \left( \nabla_{\beta_1} R_{\alpha_1 \beta_2 \beta_3 \rho} R_{\beta_4 \beta_5 \alpha_2} + \alpha_1 \leftrightarrow \alpha_2 \right) \right]_0 y^{\beta_1} ... y^{\beta_5} \\
+ \frac{1}{6!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_6} R \right]_0 y^{\beta_1} ... y^{\beta_6} + ... \tag{17}
\]

and the inverse metric is

\[
g^{\alpha_1 \alpha_2}(y) = g^{\alpha_1 \alpha_2}(0) - \frac{1}{2!} R^{\alpha_1 \beta_1 \beta_2 \alpha_2}(0) y^{\beta_1} y^{\beta_2} \\
- \frac{1}{3!} \left[ \nabla_{\beta_1} R^{\alpha_1 \beta_2 \beta_3 \alpha_2} \right]_0 y^{\beta_1} ... y^{\beta_3} \\
- \frac{1}{4!} \left[ \nabla_{\beta_1} \nabla_{\beta_2} R^{\alpha_1 \beta_3 \beta_4 \alpha_2} - \frac{4}{3} R^{\alpha_1 \beta_1 \beta_2 \rho} R^{\rho \beta_3 \beta_4 \alpha_2} \right]_0 y^{\beta_1} ... y^{\beta_4} \\
- \frac{1}{5!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_5} R^{\alpha_1 \beta_6 \alpha_2} - 3 \left( \nabla_{\beta_1} R^{\alpha_1 \beta_2 \beta_3 \rho} R^{\rho \beta_4 \beta_5 \alpha_2} + \alpha_1 \leftrightarrow \alpha_2 \right) \right]_0 y^{\beta_1} ... y^{\beta_5} \\
- \frac{1}{6!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_6} R^{\alpha_1 \beta_6 \alpha_2} - 5 \left( \nabla_{\beta_1} \nabla_{\beta_2} R^{\alpha_1 \beta_3 \beta_4 \rho} R^{\rho \beta_5 \beta_6 \alpha_2} + \alpha_1 \leftrightarrow \alpha_2 \right) \right]_0 y^{\beta_1} ... y^{\beta_6} + ... \tag{18}
\]

Using Eq. (16), it follows that

\[
R(y) = R(0) + \left[ \nabla_{\beta_1} R \right]_0 y^{\beta_1} + \frac{1}{2!} \left[ \nabla_{\beta_1} \nabla_{\beta_2} R \right]_0 y^{\beta_1} y^{\beta_2} + \frac{1}{3!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_3} R \right]_0 y^{\beta_1} ... y^{\beta_3} \\
+ \frac{1}{4!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_4} R \right]_0 y^{\beta_1} ... y^{\beta_4} + \frac{1}{5!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_5} R \right]_0 y^{\beta_1} ... y^{\beta_5} + \frac{1}{6!} \left[ \nabla_{\beta_1} \cdots \nabla_{\beta_6} R \right]_0 y^{\beta_1} ... y^{\beta_6} + ... \tag{19}
\]

To obtain a series expansion for the determinant \( g(y) \), we can use the relation \( \ln(\det X) = \text{tr}(\ln X) \) for a matrix \( X \) and the following relations for an invertible matrix \( A \) and matrix \( B \):

\[
\det(A + B) = e^{\ln \det(A + B)} = e^{\ln \det A \det(I + A^{-1}B)} = e^{\ln \det A + \ln \det(I + A^{-1}B)} = \det A e^{\text{tr}(\ln(I + A^{-1}B))}
\]

\[
\ln(I + A^{-1}B) = A^{-1}B - \frac{(A^{-1}B)^2}{2} + \frac{(A^{-1}B)^3}{3} - ... \tag{20}
\]

\[6\]
Via Eq. (17), \( g_{\mu\nu}(y) = A + B \), where \( A = g_{\mu\nu}(0) = \eta_{\mu\nu} \) and \( B \) is the matrix represented by all the other terms in Eq. (17). Then using the expansion \( e^x = 1 + \frac{x^2}{2!} + \frac{x^3}{3!} + \ldots \), we can obtain \( g(y) \). It turns out we only need our expansions to 5th order (as will be explained later). With the help of the Mathematica package xTensor [14, 15], we obtain

\[
g(y) = 1 - \frac{1}{3} R_{\beta_1\beta_2}(0) y^{\beta_1} y^{\beta_2} \\
- \frac{1}{6} \left[ \nabla_{\beta_3} R_{\beta_1\beta_2} \right]_0 y^{\beta_1} \ldots y^{\beta_3} \\
+ \frac{1}{360} \left[ 20 R_{\beta_1\beta_2} R_{\beta_3\beta_4} + 16 R_{\beta_1\beta_2\beta_3} R_{\beta_4} + 18 \nabla_{\beta_3} \nabla_{\beta_4} R_{\beta_1\beta_2} - 20 R_{\beta_1\beta_2\beta_3} R_{\beta_4} + 20 R_{\beta_1\beta_2} \nabla_{\beta_3} R_{\beta_1\beta_2} - 20 R_{\beta_1\beta_2\beta_3} \nabla_{\beta_3} R_{\beta_1\beta_2} \right]_0 y^{\beta_1} \ldots y^{\beta_4} \\
- \frac{1}{360} \left[ 16 R_{\beta_1\beta_4} \sigma \nabla_{\beta_1} R_{\beta_2\beta_3} R_{\beta_4} - 4 \nabla_{\beta_1} \nabla_{\beta_2} \nabla_{\beta_3} R_{\beta_1\beta_2} + 20 R_{\beta_1\beta_2} \nabla_{\beta_3} R_{\beta_1\beta_2} + 16 R_{\beta_1\beta_2} \nabla_{\beta_3} R_{\beta_1\beta_2} - 20 R_{\beta_1\beta_2\beta_3} \nabla_{\beta_3} R_{\beta_1\beta_2} \right]_0 y^{\beta_3} \ldots y^{\beta_5} + \ldots.
\]

(21)

For mathematical simplification, as is done in [12], we define

\[
G(x, x') = g(x)^{-1/4} \tilde{G}(x, x') g(x')^{-1/4}
\]

(22)

and make use of the generalized Fourier transformation

\[
\tilde{G}(x, x') = \int \frac{d^n k}{(2\pi)^n} e^{iky} G(k),
\]

(23)

where \( ky = \eta_{\mu\nu} k_\mu y_\nu \). Then we express \( \tilde{G}(k) \) as

\[
\tilde{G}(k) = \tilde{G}_0(k) + \tilde{G}_1(k) + \tilde{G}_2(k) + \ldots,
\]

(24)

where \( \tilde{G}_i(k) \) involves \( i \) derivatives of the metric.

For a given interval \( x \) to \( x' \), our adiabatic assumption is that the rate of change of \( a(t) \) is sufficiently slow, or adiabatic. So each higher-order term in metric derivative should be smaller than the previous.

Using these expansions about the origin, we may express Eq. (3) in momentum space and solve iteratively for \( \tilde{G}_i \) of \( i \)th adiabatic order. We obtain

\[
\tilde{G}_0(k) = (k^2 + m^2)^{-1}.
\]

(25)

At 0th order, the Green’s function is that of a free field in Minkowski space, as expected. We obtain \( \tilde{G}_1(k) = 0 \). In solving for higher-order \( \tilde{G}_i \), we use the fact that \( \partial / \partial y^\mu \rightarrow ik_\mu \) in momentum space, and \( y^\mu \rightarrow i\partial / \partial k_\mu \), which follows from integration by parts as the
We see that \( \bar{G}_i(k) \) is of order \( k^{-(2+i)} \). It turns out that we need solve iteratively at least up to adiabatic order \( i = 5 \) in 4 dimensions in order to subtract out all ultraviolet divergences and keep some finite part for the kinetic energy. Using Eqs. (22) and (23) in Eq. (2), the kinetic energy is

\[
\mathcal{G}_2(k) = \left( \frac{1}{6} - \xi \right) R(0)(k^2 + m^2)^{-2}.
\]

(26)

We obtain

\[
\mathcal{G}_i(k) = \left( \frac{1}{6} - \xi \right) R(0)(k^2 + m^2)^{-2} + \sum_{2 \leq i \leq 5} \left( \frac{1}{6} - \xi \right) R^i(0)(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\alpha}(k^2 + m^2)^{-1} R_{\alpha \beta}(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\beta}
\]

\[
+ \sum_{i > 5} \left( \frac{1}{6} - \xi \right) R^i(0)R_{\alpha \beta}(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\alpha} R_{\beta \gamma}(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\gamma}
\]

where \( \frac{\partial}{\partial x^\rho} \rightarrow ik_\rho \) and \( x' \) refers to the parts evaluated at \( y = 0 \), and the \( x' \)-dependent parts of \( \mathcal{G}(k) \) are the curvature factors that are evaluated at \( y = 0 \). The integral diverges for \( i \leq 3 \) and converges for \( i = 4 \) or bigger, and since the integral of the term involving \( \mathcal{G}_4 \) gives a contribution of 0, we must go to at least order \( i = 5 \). So using the method of adiabatic subtraction, we calculate a physically meaningful (renormalized) approximation of the kinetic energy \( KE_{phys} \) as

\[
KE_{phys} = \sum KE_{all \ orders} - \sum KE_{divergent} \approx KE_{i=4 \ and \ i=5},
\]

(28)

and \( i > 5 \) contributions would be smaller than the \( i = 5 \) contribution. After solving iteratively up to \( i = 5 \), we obtain (with the understanding that all the metric curvature factors in the following are evaluated at \( y = 0 \))

\[
\bar{G}(k) = (k^2 + m^2)^{-1} + \left( \frac{1}{6} - \xi \right) R(k^2 + m^2)^{-2} + iC_\alpha(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\alpha}(k^2 + m^2)^{-1} \\
+ \left( \frac{1}{6} - \xi \right)^2 R^2(k^2 + m^2)^{-3} - E_{\alpha \beta}(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\alpha} \frac{\partial}{\partial k_\beta}(k^2 + m^2)^{-1} \\
+ iC_\alpha(k^2 + m^2)^{-1} \left( \frac{1}{6} - \xi \right) R \frac{\partial}{\partial k_\alpha}(k^2 + m^2)^{-2} + i \left( \frac{1}{6} - \xi \right) RC_\alpha(k^2 + m^2)^{-2} \frac{\partial}{\partial k_\alpha}(k^2 + m^2)^{-1} \\
+ \frac{i}{3} R^{\nu}_{\alpha \beta}(k^2 + m^2)^{-1} k_\nu C_\beta \frac{\partial}{\partial k_\alpha} \left[ (k^2 + m^2)^{-1} \frac{\partial}{\partial k_\beta}(k^2 + m^2)^{-1} \right] \\
+ \frac{i}{3} R^{\mu \nu}_{\alpha \beta}(k^2 + m^2)^{-1} k_\mu k_\nu C_\gamma \frac{\partial}{\partial k_\alpha} \frac{\partial}{\partial k_\beta} \frac{\partial}{\partial k_\gamma} \left[ (k^2 + m^2)^{-1} \frac{\partial}{\partial k_\gamma}(k^2 + m^2)^{-1} \right] \\
- iI_{\alpha \beta \gamma}(k^2 + m^2)^{-1} \frac{\partial}{\partial k_\alpha} \frac{\partial}{\partial k_\beta} \frac{\partial}{\partial k_\gamma}(k^2 + m^2)^{-1},
\]

(29)
where, letting the semicolon denote the application of the covariant derivative, we use the following:

\[
C_\alpha \equiv \frac{1}{6} R^\beta_{\alpha \beta} + \frac{1}{12} R_{\alpha} - \xi R_{\alpha}
\]

\[
E_{\alpha\beta} \equiv -\frac{1}{12} R_{\alpha\gamma} R^\gamma_{\beta} + \frac{1}{15} R^\beta R_{\alpha\gamma\beta} - \frac{1}{90} R^\gamma_{\alpha\beta;\gamma} + \frac{1}{180} R_{\alpha} R^\gamma_{\alpha\beta} + \frac{1}{20} R_{\alpha} R^\gamma_{\beta;\gamma} + \frac{1}{90} R^\gamma_{\alpha\beta;\gamma} R_{\beta\gamma\epsilon}
\]

\[
I_{\alpha\beta\gamma} \equiv \frac{1}{30} R_{\alpha\delta\epsilon\beta;\gamma} R^\delta_{\gamma} - \frac{1}{12} R_{\alpha\delta;\gamma} R^\delta_{\gamma} + \frac{1}{90} R_{\alpha\delta\epsilon;\gamma} R^\delta_{\epsilon} + \frac{1}{90} R_{\alpha\delta\epsilon;\beta;\gamma} R^\delta_{\epsilon}
\]

\[
\int \frac{d^4k}{(2\pi)^4} (k^2 + m^2)^{-3} = \frac{i}{32\pi^2 m^2}.
\]

After applying the \(x'\)-derivative in Eq. (27) (keeping only the physically relevant \(i = 4\) and \(i = 5\) terms), we use integration by parts to express all the \(k\)-dependence in the integral as \((k^2 + m^2)^{-3}\). We then take the limit \(y \to 0\) (which makes \(e^{iky} \to 1\) and \(g^{\sigma\rho}(y) \to \eta^{\sigma\rho}\)), and all the \(k\)-dependence becomes

\[
\int \frac{d^4k}{(2\pi)^4} (k^2 + m^2)^{-3} = \frac{i}{32\pi^2 m^2}.
\]

We obtain the following expression for \(KE_{\text{phys}}\):

\[
KE_{\text{phys}}(x) = \frac{1}{2} \eta^{\rho\sigma} \frac{i}{32\pi^2 m^2} \left[ i \left( \frac{1}{6} - \xi \right) \partial_\sigma [RC_\alpha] \left( -\frac{1}{3} \delta^{\alpha}_\rho \right) + \frac{i}{3} \partial_\sigma [R_{\alpha\beta} C_\beta] \left( \frac{1}{3} \delta^{\alpha}_\rho \delta^{\beta}_\sigma + \frac{1}{3} \delta^{\alpha}_\rho \delta^{\beta}_\sigma - \frac{4}{3} \delta^{\beta}_\rho \delta^{\alpha}_\sigma - \frac{4}{3} \delta^{\beta}_\rho \delta^{\alpha}_\sigma - \frac{4}{3} \eta^{\beta\alpha} \eta_{\rho\sigma} \right) + \frac{i}{3} \partial_\sigma [R_{\alpha\beta} C_\beta] \left( -\frac{8}{5} \left[ \eta_{\rho\mu} \left( \eta^{\beta\gamma} \delta^{\alpha}_\mu + \eta^{\alpha\gamma} \delta^{\beta}_\mu + \eta^{\gamma\beta} \delta^{\alpha}_\mu \right) + \eta_{\mu\nu} \left( \eta^{\beta\gamma} \delta^{\alpha}_\mu + \eta^{\alpha\gamma} \delta^{\beta}_\mu + \eta^{\gamma\beta} \delta^{\alpha}_\mu \right) \right) \right]
\]

\[
+ 2 \left[ \eta^{\alpha\beta} \delta^{\beta}_\rho \eta_{\mu\nu} + \delta^{\alpha}_\rho \eta_{\mu\nu} + \delta^{\alpha}_\rho \eta_{\mu\nu} + \delta^{\beta}_\rho \eta_{\mu\nu} + \delta^{\beta}_\rho \eta_{\mu\nu} + \delta^{\beta}_\rho \eta_{\mu\nu} \right) + \eta^{\gamma\beta} \left( \delta^{\alpha}_\rho \eta_{\mu\nu} + \delta^{\alpha}_\rho \eta_{\mu\nu} + \delta^{\alpha}_\rho \eta_{\mu\nu} + \delta^{\beta}_\rho \eta_{\mu\nu} + \delta^{\beta}_\rho \eta_{\mu\nu} \right)
\]

\[
+ i \partial_\sigma [C_\alpha \left( \frac{1}{6} - \xi \right) R] \left( -\frac{2}{3} \delta^{\beta}_\rho \right) + i \partial_\sigma [I_{\alpha\beta\gamma}] \left[ \frac{20}{3} \left( \delta^{\beta}_\rho \eta^{\gamma\alpha} + \delta^{\beta}_\rho \eta^{\gamma\beta} + \delta^{\beta}_\rho \eta^{\beta\alpha} \right) \right].
\]

It is a function of \(x\) since \(y \to 0\) was equivalent to \(x' \to x\).
The expression is only valid for a non-zero $m$, and a very small mass is expected since dark energy acts on large scales. With the help of xTensor in Mathematica, we then evaluate this expression for the 1st-order FLRW metric. After a lengthy calculation, we arrive at a lengthy expression for $KE_{phys}(x)$ that depends on scale factor $a$, radial distance $r$, $w$, mass $m$, non-minimal coupling $\xi$, and the constant of integration $S$ mentioned earlier, and we have kept all terms to first order in $S$. The expression is far too long to show, but the relevant Mathematica files containing the expression are posted at https://drive.google.com/drive/folders/1fdLS7YxNiJqPxDAbQwSZ-9OX_1xiEHH6.

**Kinetic Energy Results**

We exhibit the behavior of $KE_{phys}$ in the figures in this section, varying different parameters involved in the expression for kinetic energy. For all of these plots, we use $\rho_{DE0} = 4.12 \times 10^{-9} \text{Mpc}^{-2}$, obtained from the best-fit values from Planck [3]. Most data sets of Type Ia supernovae data run from redshift $z \approx 0.3$ to $z \approx 2$ [8], corresponding to distances that light travels (along a null geodesic in FLRW space dominated by dark energy with constant $w < -1$) of about 1250 Mpc and 5410 Mpc respectively. All the plots indicate that the renormalized kinetic energy of the dark energy field with a constant $w < -1$ is positive during dark energy domination.

Figure (1) shows positive kinetic energy over the relevant time range from the beginning of dark energy domination until the present, and one can see that the kinetic energy increases in the future ($a > 1$). The other free parameters are chosen to be reasonable: a very small mass value to correspond to a very long range for the dark energy field $\phi$, $w < -1$ but close to $-1$, a small value for $\xi$, and a small $S$ value ($|S| \sim 10^{-5}$ from our previous work in [8]). This plot is for $r = 5410 \text{ Mpc}$ (corresponding to $z = 3$), the upper bound of length scale covered by Type Ia supernovae data, but the plot is virtually unchanged for all length scales covered by the data, from $z = 0.3$ to $z = 2$. Figures (2 - 4) have the same choices of reasonable parameter values, and all the conclusions from the plots are upheld for various values of $m$. Figure (2) is plotted for present day ($a = 1$) and shows how little the first-order terms of the metric (which are all linearly proportional to $S$) affect the kinetic energy. Clearly, the dominant contribution is from the terms that are 0th-order in $S$, and this is in fact true for any relevant value of $a$. So the kinetic energy can therefore be positive even
FIG. 1. $KE_{\text{phys}}$ vs. $a$ for $r = 5410$ Mpc (corresponding to $z = 2$). $KE_{\text{phys}}$ is plotted in units of Mpc$^{-1}$, and $m = 3.2 \times 10^{-34}$ eV, $\xi = 0$, $w = -1.1$, $S = 10^{-5}$. The range of $a$ is from roughly the start of dark energy domination to the future when $a = 10$. The plot looks virtually the same when evaluated at $r = 1250$ Mpc (corresponding to $z = 0.3$). $KE_{\text{phys}}$ scales up as mass is lowered in 0th-order FLRW space (i.e., even when $S = 0$). Figure (3) shows positive kinetic energy for a range of values for $w$ at present day, with the kinetic energy being negative around $w \leq -1.22$ (and this is true for $a$ at the beginning of dark energy domination as well). The range for $w$ that gives positive kinetic energy varies slightly depending on the choices for the other parameters. Figure (4) shows the kinetic energy over a range of values of $\xi$. It is generally expected that $\xi$ is small, and we see that kinetic energy is positive for small values of $\xi$. We do not show it here, but we have checked and verified that the kinetic energy is positive for $w > -1$, as expected.

Conclusion

For standard cosmology, it is well-known that dark energy as a scalar field is ill-defined for $w < -1$ in 0th-order FLRW space, and the condition for positive kinetic energy for the dark energy scalar field in 1st-order FLRW implies that the perturbation $|\delta| > 1$, which is inconsistent with the perturbative assumption of 1st-order FLRW space [8].

In this work, we have treated the scalar field as a quantum field with a small mass, obtained an approximate expression for the Green’s function to 5th order (Eq. (29)) using
FIG. 2. $K_{E_{\text{phys}}}$ vs. $S$ for $a = 1$ (present day), $r = 5410$ Mpc (corresponding to $z = 2$). $K_{E_{\text{phys}}}$ is plotted in units of Mpc$^{-1}$, and $m = 3.2 \times 10^{-34}$ eV, $\xi = 0$, $w = -1.1$. The plot line may be hard to see, but it is present, and it lies right on the horizontal axis. From matching with data, $S$ is expected be small, and $K_{E_{\text{phys}}}$ is virtually unaffected by its size, not only for $a = 1$ but virtually any $a$ value. So the contribution that is 0th order in $S$ in the kinetic energy dominates.

FIG. 3. $K_{E_{\text{phys}}}$ vs. $w$ for $a = 1$ (present day), $r = 5410$ Mpc (corresponding to $z = 2$). $K_{E_{\text{phys}}}$ is plotted in units of Mpc$^{-1}$, and $m = 3.2 \times 10^{-34}$ eV, $\xi = 0$, $S = 10^{-5}$. The kinetic energy is negative for $w$ a little less than $-1.2$, and the same holds for $a$ at the beginning of dark energy domination.
FIG. 4. $KE_{phys}$ vs. $\xi$ for $a = 1$ (present day), $r = 5410$ Mpc (corresponding to $z = 2$). $KE_{phys}$ is plotted in units of Mpc$^{-1}$, and $m = 3.2 \times 10^{-34}$ eV, $w = -1.1$, $S = 10^{-5}$. In general, the dimensionless $\xi$ is expected to be small.

adiabatic expansion in Riemann normal coordinates, and calculated an expression for the kinetic energy that has been renormalized via adiabatic subtraction (and the expression is available in Mathematica files at the link quoted earlier in this work). We find that the kinetic energy, somewhat surprisingly, is positive for constant $w < -1$, as long as $w$ is not too much less than $-1$. And we found this to be the case even at 0th-order FLRW (as illustrated by Figure (2)). We also confirmed that the dark energy field has positive kinetic energy for $w > -1$, as expected.

This result gives credence and a more natural framework for observational data that suggest $w < -1$. Without modifying gravity, flipping the sign in front of the kinetic term, or leaving the confines of general relativity, we have shown that a dark energy field with $w < -1$ is a viable option. In principle, one could go further and keep more terms in the adiabatic expansion of the Green’s function, and one could take into account effects due to the interaction potential $V(\phi)$ of the field in the calculation of the kinetic energy. We expect these differences to be small, though, as discussed in the other sections. Also, our results are generally applicable for a massive free field or a field with a small potential in a 0th- or 1st-order FLRW background dominated by a fluid with a constant $w$.

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