Existence criterion of such a cycle that the set of vertices beyond the cycle is independent

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1 Introduction

The problem of Hamiltonian cycle searching in graphs is well-known and well explored. Apart from classical Dirac’s and Ore’s criteria, which establish Hamiltonian cycle existence by simple restrictions on graph’s vertices degree, Chvátal’s theorems, which are presented in Bondy’s and Murty’s book [1], are worthy of mention. These theorems establish Hamiltonian cycle existence by its closure, degree sequence and vertex connectivity. In addition to these results, there is a problem whether graph power is Hamiltonian. There are classical results here: well-known Fleischner’s theorem, which is presented in Diestel’s book [2] and states that square of biconnected graph is Hamiltonian, and Chartrand’s and Kapoor’s theorem about Hamiltonian cycle existence in cube of connected graph.

As for long cycles search problem, first we mention classical Linial’s theorem [3] which contains estimation on the biggest simple cycle length in biconnected graphs and complements numerous results on Hamiltonian cycles. In particular, our work intellectually originates in the theorem.

Theorem (N. Linial, 1975) Let $G$ be a biconnected graph and

$$m = \min_{xy \notin E(G)} d_G(x) + d_G(y).$$

Then the length of the longest cycle of $G$ is at least $\min(m, v(G))$.

Thomassen’s article [4] contains, inter alia, existence criterion in planar graphs of a cycle with special condition: the set of vertices not included in it must be
independent. Our work contains existence criterion of such a cycle in terms of graph’s minimum degree.

**Theorem:** Let $G$ be a biconnected graph, $v(G) = n$ and $\delta(G) \geq \frac{n+2}{3}$. Then $G$ has such a cycle that the set of vertices beyond the cycle is independent.

2 Proof of the main result

We will invoke the following classic lemma in our proof, which are used in the proofs of Ore’s theorem and several Chvátal’s theorems.

**Lemma 1:** Let $m > 2$, $u_1...u_m$ is a path of maximal length in graph $G$ and $d_G(u_1) + d_G(u_m) \geq m$. Then graph $G$ has a cycle of length $m$. 
**Proof of the theorem:** The proof is long, so 5 claims are highlighted for the sake of convenience.

From the beginning, we highlight that we will use the estimation of $\delta(G) \geq \frac{n}{3}$. We will use estimation of $\delta(G) \geq \frac{n+2}{3}$ in a couple of places only.

Our proof is negative, so we assume that there exists such a graph, which meets all the conditions and does not contain such a cycle that the set of vertices not included in it is independent.

Firstly, let us consider a case when $n < 6$. We have $\delta(G) \geq 2$ due to biconnectivity. If $n = 3$ or $n = 4$, then from $\delta(G) \geq 2$ we can easily say that the graph contains a cycle. This cycle satisfies all the requirements because the set of vertices beyond the cycle contains 1 vertex at most. If $n = 5$, then the graph also contains a cycle. If the cycle length is 4 or 5, then, likewise, this cycle meets all the requirements. Hence, the cycle length is 3. Let us denote cycle’s vertices by $r_1, r_2$ and $r_3$ and the remaining vertices by $r_4, r_5$. If $r_4r_5 \notin E(G)$, then cycle $r_1r_2r_3$ obviously meets all the conditions. Otherwise, $r_4r_5 \in E(G)$. Whereas the graph is connected, there is an edge between sets of vertices $\{r_1, r_2, r_3\}$ and $\{r_4, r_5\}$. Without loss of generality, let $r_1r_4 \in E(G)$. Since the graph is biconnected, the graph remains connected after removing $r_1$, therefore, there exists an edge between sets of vertices $\{r_2, r_3\}$ and $\{r_4, r_5\}$. Let $r_4$ be an endpoint of the edge, so, without loss of generality, $r_2r_4 \in E(G)$. Then $r_1r_4r_2r_3$ is a suitable cycle of length 4 (see figure 1a). If $r_5$ is an endpoint of the edge, then, without loss of generality, $r_2r_5 \in E(G)$. Then $r_1r_4r_5r_2r_3$ is a suitable Hamiltonian cycle (see figure 1b).

![Diagram](image-url)

**Figure 1:** Illustration of case $n = 5$.

From now on we consider that

$$n \geq 6.$$  \hspace{1cm} (1)
If all aforementioned properties (biconnectivity, minimum degree, lack of the required cycle) remain, let us add an edge to the graph. Obviously, biconnectivity and minimum degree properties remain after adding any edge, so we add edge if absence of the demanded cycle remains. Let us perform the procedure as long as we can. Hence, we stop the procedure if we cannot add any edge. We do not want to make new designations, so we consider that graph $G$ is the graph we get after stopping the procedure for simplicity’s sake. Which means graph $G$ have such marvelous properties:

a) $G$ is biconnected.
b) $\delta(G) \geq \frac{n}{3}$.
c) For any $d_1, d_2 \in V(G)$ such that $d_1d_2 \notin E(G)$ there is such a path between $d_1$ and $d_2$ that the set of vertices not included in it is independent.

Property c) is formed right after finishing the procedure.

**Claim 1:** $G$ has a Hamiltonian path.

**Proof:**

![Figure 2:](image)

Figure 2: If $z_t$ is red, $z_{t-2}$ is green and $N_G(z_{t-1}) \subseteq L_0$, then cycle $z_1z_2...z_{t-2}z_kz_{k-1}...z_t$ (bold in the figure) is suitable.

Our proof is negative. Consider the longest path $L_0$ in the graph such that its endpoints are not connected by an edge and the set of vertices beyond the path is independent. Let it not be a Hamiltonian path, then there are $k < n$ vertices in $L_0$. Let us number them in the order of traversing the path: $z_1, z_2, ..., z_k$. We say that vertex $z_i$ is on the right of vertex $z_j$ if $i > j$ and vertex $z_i$ is on the left of vertex $z_j$ if $i < j$. Also, we say that vertex is red, if it is a neighbour of $z_1$ and we say
that vertex is green, if it is a neighbour of \( z_k \) (we do not have any problems with colouring’s overlapping, for instance, vertex can be green and red simultaneously).

It is clear that all the red vertices and green vertices are located in the path \( L_0 \) (otherwise, \( L_0 \) may be lengthened due to red or green vertex which is not located in \( L_0 \)). Consequently, there are at least \( \frac{n}{3} \) red or green vertices in the path \( L_0 \), inasmuch as \( d_G(z_1), d_G(z_k) \geq \frac{n}{3} \).

Note that \( \forall t \) the following property is satisfied: if \( z_t \) is red, then \( z_{t-2} \) cannot be green. Let us show it by contradiction. If \( N_G(z_{t-1}) \subseteq L_0 \), then the cycle of length \( k - 1 \) is formed (see figure 2). Set of vertices not included in the cycle consists of set of vertices beyond \( L_0 \) (which is independent) and vertex \( z_{t-1} \). Since \( N_G(z_{t-1}) \subseteq L_0 \), vertex \( z_{t-1} \) does not have neighbours among set of vertices not included in \( L_0 \). Therefore, suitable cycle is found. Otherwise, \( N_G(z_{t-1}) \not\subseteq L_0 \), then let \( h \) be the neighbour of \( z_{t-1} \), which is not located in \( L_0 \). Then there is path of length \( k + 1 \) such that set of vertices beyond the path is independent: \( z_1 z_2 ... z_{t-2} z_k z_{k-1} z_{k-2} ... z_t z_{t-1} h \) (see figure 3). This is a contradiction with the suggestion that \( L_0 \) is the longest path such that the set of remaining vertices is independent. The set of remaining vertices is independent insofar as this is the set of vertices not included in \( L_0 \) (which is independent) without vertex \( h \).

![Figure 3: If \( z_t \) is red, \( z_{t-2} \) is green and \( N_G(z_{t-1}) \not\subseteq L_0 \), then there is a path (bold in the illustration) such that it is longer than \( L_0 \) and set of vertices beyond the path is independent.](image)

Since the path \( L_0 \) is not Hamiltonian, \( k < n \) and there exists vertex \( y \not\in L_0 \). \( N_G(y) \subseteq V(L_0) \), inasmuch as set of vertices beyond \( L_0 \) is independent. Let \( \mu = |N_G(y)| \) and we say that vertex \( z_t \) is black if \( z_{t+1} y \in E(G) \). Whereas \( \delta(G) \geq \frac{n}{3} \), \( \mu \geq \frac{n}{3} \) and there are at least \( \mu \) black vertices (there are as many black vertices as there are neighbours of \( y \), insofar as \( z_1 \) cannot be a neighbour of \( y \), otherwise the
path $L_0$ can be obviously lengthened from $z_1 \ldots z_k$ to $yz_1 \ldots z_k$ with maintaining the condition about independence of the set of remaining vertices).

Also, it is clear that there is no such thing as a black and green vertex (see figure 4).

Figure 4: If $z_t$ is black and green, then $z_{t+1}y, z_t z_k \in E(G)$. Then the path (bold in the illustration) $z_1 z_2 \ldots z_t z_k z_{k-1} \ldots z_{t+1} y$ has length $k + 1$ and the set of vertices not included in this path is independent, a contradiction with choosing $L_0$.

Let us piece together all the facts about colours of vertices which we gained:

1) There are at least $n^3$ black, red and green vertices.
2) Red vertex can be green, but black vertex cannot be green.
3) If $z_t$ is red, then $z_{t-2}$ cannot be green.

We say that vertex $z_t$ is *blocked* if $z_{t+2}$ is red. From fact 3) it follows that blocked vertex cannot be green. Also, blocked vertex cannot be black. Indeed, suppose that $z_t$ is a blocked black vertex. Then $z_{t+1}y \in E(G)$ and $z_{t+2}$ is red, therefore $z_1 z_{t+2} \in E(G)$. Then path $yz_{t+1}z_t z_{t-1} \ldots z_1 z_{t+2} z_{t+3} \ldots z_k$ is longer than the path $L_0$ and the remaining vertices are independent (it is similar to what happens in the figure 4). Every red vertex (quantity of red vertices is $n^3$ at least) *generates* the blocked one by definition except $z_2$ ($z_1$ is not red, because it is not neighbour to itself). Therefore, there are at least $n^3 - 1$ blocked vertices.

Thus, three sets \{*blocked vertices*\}, \{*black vertices*\} \{*green vertices*\} are pairwise disjoint and their cardinalities are at least $n^3 - 1, \frac{n^3}{3}, \frac{n^3}{3}$, respectively. Since all these three sets are subsets of \{$z_1, \ldots, z_{k-1}$\} (obviously, $z_k$ is neither blocked, nor black, nor green), consequently, $k - 1 \geq \frac{n^3}{3} - 1 + \frac{n^3}{3} + \frac{n^3}{3} = n - 1$. But $k < n$, insofar as the path is not Hamiltonian, a contradiction. □

All right, $G$ has a Hamiltonian path. Let us number path’s vertices in the order of traversal of the path: $x_1, x_2, \ldots, x_n$. As before, we say that vertex $x_i$ is on the *right* of vertex $x_j$ if $i > j$ and vertex $x_i$ is on the *left* of vertex $x_j$ if $i < j$. Since
\[ d_G(x_1) \geq \frac{n}{3}, \] there exists such an index \( i \geq \frac{n}{3} + 1 \) that \( x_1x_i \in E(G) \). Vertices \( x_1x_2...x_i \) form a cycle (in this order of traversal) and all the remaining vertices \( x_{i+1}x_{i+2}...x_n \) form a path (in this order of traversal) (see figure 5). Hence, graph \( G \) contains such a cycle with length of at least \( \frac{n}{3} + 1 \) that all the remaining vertices form a path.

Figure 5: The graph \( G \) includes a cycle \( x_1x_2...x_i \), all the remaining vertices constitute a path \((x_{i+1}x_{i+2}...x_n)\).

\[ H \]

\[ T \]

Figure 6: Vertices of the graph \( G \) are divided into 2 disjoint sets: cycle \( T \) and path \( H \).

Let \( r \) be such the highest number that there exists such a cycle with length of \( r \) (let this cycle be \( T \)) that all the remaining vertices form a path (let this path be \( H \)). We have just proved that \( r \geq \frac{n}{3} + 1 \). The cycle \( T \) has \( r \) vertices, therefore the path \( H \) has \( n - r \) vertices and \( n - r - 1 \) edges (see figure 6).

Claim 2: \( r \geq \frac{n}{2} \).
Proof: Our proof is negative again, so we assume that $r < \frac{n}{2}$. Let $x_{\alpha_1}, \ldots, x_{\alpha_f}$ be neighbours of $x_1$ and let $x_{\beta_1}, \ldots, x_{\beta_g}$ be neighbours of $x_n$. $f \geq \frac{n}{3}$ and $g \geq \frac{n}{3}$, inasmuch as $\delta(G) \geq \frac{n}{3}$. Note that for any $1 \leq i \leq f$ and $1 \leq j \leq g$ property $\alpha_i - 1 \neq \beta_j + 1$ holds because otherwise a cycle of length $n - 1$ is formed (it contains all the vertices except $x_{\alpha_{i-1}} = x_{\beta_{j+1}}$, that is: $x_1 x_2 \ldots x_{\beta_j} x_n \ldots x_{\alpha_i}$). This cycle is suitable because set of vertices beyond the cycle (that is, one vertex) is independent. Also note that for any $1 \leq i \leq f$ and $1 \leq j \leq g$ inequalities $\alpha_i < \frac{n}{2}$ and $\beta_j > \frac{n}{2}$ hold (otherwise there is such a desired cycle with length of at least $\frac{n}{2}$ that all the remaining vertices form a path: it goes along the edges of path $L_0$ from $x_1$ to $x_{\alpha_i}$ and the edge $x_1 x_{\alpha_i}$ (or from $x_n$ to $x_{\beta_j}$ and the edge $x_{\beta_j} x_n$, respectively)). Consider vertex $x_{\lceil\frac{n}{2}\rceil}$. We have just proved that it is on the right of all the neighbours of $x_1$ and it is on the left of all the neighbours of $x_n$. Note that for any $1 \leq i \leq f$ and $1 \leq j \leq g$ properties $x_{\lceil\frac{n}{2}\rceil} x_{\alpha_{i-1}} \notin E(G)$ and $x_{\lceil\frac{n}{2}\rceil} x_{\beta_{j+1}} \notin E(G)$ hold (otherwise there is such a cycle of length of at least $\frac{n}{2}$ that all the remaining vertices form a path: see figures 7a and 7b, respectively). Whereas all the vertices $x_{\alpha_{i-1}}$ and $x_{\beta_{j+1}}$ are distinct, $x_{\lceil\frac{n}{2}\rceil}$ is not adjacent to at least $\frac{2n}{3}$ different vertices and it is not adjacent to itself. There are $n$ vertices, hence, $d_G(x_{\lceil\frac{n}{2}\rceil}) \leq \frac{n}{3} - 1$, a contradiction to $\delta(G) \geq \frac{n}{3}$. □

Let us come back directly to the proof of the theorem and consider 2 cases.

Case 1: There are such 2 distinct vertices of the cycle $T$ that the first one is adjacent to some endpoint (let this endpoint be $a$) of the path $H$ and the second one is adjacent to the other one (let this endpoint of the path $H$ be $b$).

Let $\rho_a$ be the quantity of vertices of the cycle $T$ which are adjacent to $a$ and are not adjacent to $b$ (let these vertices be $a$-vertices or vertices of type $a$), let $\rho_b$ be the quantity of vertices of the cycle $T$ which are adjacent to $b$ and are not adjacent to $a$ (let these vertices be $b$-vertices or vertices of type $b$). Also, let $\rho_{ab}$ be the quantity of vertices of the cycle $T$ which are adjacent to $a$ and $b$ (let these vertices be $ab$-vertices or vertices of type $ab$). From the definition above, a couple simple properties may immediately be seen. Firstly, every vertex of the cycle $T$ cannot pertain to 2 types.
simultaneously but it can pertain to none of types (if it is not adjacent to $a$ and $b$). Secondly, two vertices of the same type cannot be neighbours in the order of traversing the cycle (otherwise the cycle can be lengthened by vertex $a$ (or $b$) and the remaining vertices form a path; or a Hamiltonian cycle exists, see figure 8).

![Diagram](image)

Figure 8: If an endpoint of $H$ (vertex $b$ in the illustration) is adjacent to two neighbours in the order of traversal of the cycle, then the cycle $T$ may be lengthened due to this endpoint and the remaining vertices form a path, a contradiction to choosing cycle $T$.

Note that there is no such thing as 2 distinct vertices of cycle $T$ with three conditions:

1) first vertex is adjacent to $a$;
2) second vertex is adjacent to $b$;
3) distance between these vertices by edges of cycle $T$ is at most $n - r$.

If the opposite holds, then note that there is such a cycle (it goes along the edges of the major half of the cycle $T$ between these vertices, endpoints of this section are adjacent to $a$ and $b$, and it passes along the path $H$, see figure 9) that the set of vertices beyond the cycle form a path and its cardinality is at most $n - r - 1$. Hence, there are at least $r + 1$ vertices in the cycle, a contradiction to the definition of $r$: we have found a cycle which satisfies all the requirements of the cycle in the definition of $r$ but it is larger.

Therefore, if there are 2 distinct vertices of different types (or both of them have type $ab$), then the distance between them is at least $n - r + 1$.

Let us have 2 sections.

Section 1a: The cycle $T$ has no $a$-vertices or $b$-vertices.
Figure 9: If one vertex of the cycle $T$ is adjacent to $a$, the other one is adjacent to $b$ and the distance between them on the edges of the cycle $T$ (unbold edges in the figure) is at most $n - r$, then there exists another cycle (bold in the figure) whose existence is contrary to the definition of $r$.

From the condition of the section 1a we have $\rho_a = \rho_b = 0$. Then we get that $\rho_{ab} \geq 2$ because we are within the framework of the case 1.

The vertex $a$ is adjacent to $\rho_{ab}$ vertices of the cycle $T$ and may be adjacent to vertices of the path $H$, not including itself. Which means it is adjacent to at most $\rho_{ab} + n - r - 1$ vertices. Since $\delta(G) \geq \frac{n+2}{3}$, $\rho_{ab} + n - r - 1 \geq \frac{n+2}{3} \Rightarrow \frac{2n-5}{3} + \rho_{ab} \geq r$.

Whereas distance between every $ab$—vertices on the edges of the cycle $T$ is at least $n - r + 1$, the cycle $T$ contains at least $\rho_{ab}(n - r + 1)$ edges. The cycle $T$ has $r$ edges, hence, $r \geq \rho_{ab}(n - r + 1) \Rightarrow (\rho_{ab} + 1)r \geq \rho_{ab}(n + 1)$. By applying the estimation of $r$ from the previous paragraph, we get that

$$(\rho_{ab} + 1)(\frac{2n - 5}{3} + \rho_{ab}) \geq \rho_{ab}(n + 1) \Rightarrow \frac{2n - 5}{3}\rho_{ab} + \frac{2n - 5}{3} + \rho_{ab}^2 \geq n\rho_{ab} \Rightarrow$$

$$\Rightarrow 3\rho_{ab}^2 - 5\rho_{ab} - 5 \geq (\rho_{ab} - 2)n.$$

Note that if $\rho_{ab} = 2$, then this inequality does not hold. Therefore, $\rho_{ab} \geq 3 \Rightarrow \rho_{ab} - 2 > 0$, insofar as $\rho_{ab} \geq 2$. Hence,

$$3\rho_{ab}^2 - 5\rho_{ab} - 5 \geq (\rho_{ab} - 2)n \Rightarrow \frac{3\rho_{ab}^2 - 5\rho_{ab} - 5}{\rho_{ab} - 2} \geq n.$$

It is clear that

$$3\rho_{ab} + 1 > \frac{3\rho_{ab}^2 - 5\rho_{ab} - 5}{\rho_{ab} - 2}.$$
Then we get that

\[ 3\rho_{ab} + 1 > n \Rightarrow \rho_{ab} > \frac{n-1}{3}. \]

Inequality \( r \geq \rho_{ab}(n-r+1) \) is obtained within the frame of the section 1a. Using \( \rho_{ab} > \frac{n-1}{3} \) (it is possible to do that because inequality \( n-r+1 \geq 0 \) holds because otherwise \( r = n \), which means the cycle \( T \) is Hamiltonian and, hence, suitable), we get that

\[ r > \frac{n-1}{3}(n-r+1). \]

It is clear that \( r \leq n \). Then \( n-r+1 > 0 \). If \( n-r+1 = 1 \), then \( r = n \), which means the cycle \( T \) is Hamiltonian and, hence, suitable. If \( n-r+1 = 2 \), then \( r = n-1 \), which means the cycle \( T \) does not contain only one vertex, so it is suitable too. If \( n-r+1 \geq 3 \), then \( r > \frac{n-1}{3}(n-r+1) \Rightarrow r > n-1 \), this case is handled.

**Section 1b: The cycle \( T \) contains an a-vertex or b-vertex.**

*Section of type a or a-section* is such consecutive (in the order of traversing the cycle \( T \)) set of vertices which, firstly, is formed by all the vertices between two a-vertices (hence, extreme vertices of a-section are a-vertices), secondly, cannot be lengthened, thirdly, does not include vertices of other types (a-section can consist of one vertex). *Section of type b or b-section, section of type ab or ab-section* have analogous definitions. Edges of the cycle are naturally divided into 4 groups: edges in a-sections, edges in b-sections, edges in ab-sections and edges between 2 different sections. Let \( \gamma_a \) be the quantity of a—sections, let \( \gamma_b \) be the quantity of b—sections and let \( \gamma_{ab} \) be the quantity of ab—sections. From the formulation of case 1 and section 1b, there are at least 2 positive figures amongst three numbers \( \rho_a, \rho_b, \rho_{ab} \) (respectively, there are at least 2 positive figures among three numbers \( \gamma_a, \gamma_b, \gamma_{ab} \)). Hence, \( \gamma_a + \gamma_b + \gamma_{ab} \geq 2 \). Which means the cycle \( T \) cannot be composed of only one section.

From the observation, which is formulated before the section 1a, it follows that there are at least \( n-r+1 \) edges (in the order of traversal of the cycle \( T \)) between 2 distinct sections. Also, there are at least \( n-r+1 \) edges (in the order of traversing the cycle \( T \)) between 2 distinct ab-vertices in an ab-section. Let us consider all the edges of the cycle \( T \), which does not constitute edges from a-sections and b-sections. For this purpose, we fix some direction of the cycle \( T \). Walking around cycle \( T \) in this direction, there are at least \( n-r+1 \) edges between distinct sections after passing some a-section or b-section. Also, there are at least \( n-r+1 \) edges between distinct sections or in ab-section after ab-vertex. Thus, there are at least
\((n - r + 1)(\rho_{ab} + \gamma_a + \gamma_b)\) edges from cycle \(T\), which do not constitute edges inside a-section or b-section.

Consider any a-section, let it be \(s\) a-vertices in this section. It is clear that if \(s = 1\), then this section includes \(2s - 2\) edges at least. Since an a-vertex does not have a-vertices and b-vertices among its 2 neighbours in the order of traversal of the cycle, if \(s \geq 2\), then this section has at least \(2s - 1\) edges. Hence, for any \(s\) a-section has at least \(2s - 2\) edges. Therefore, there are at least \(2(\rho_a + \rho_b) - 2(\gamma_a + \gamma_b)\) edges inside a-section and \(b\)-sections.

Thus, there are at least \((n - r + 1)(\gamma_a + \gamma_b + \rho_{ab}) + 2(\rho_a + \rho_b) - 2(\gamma_a + \gamma_b)\) edges in the cycle \(T\). Whereas the cycle \(T\) includes \(r\) edges, the following inequality holds:

\[
 r \geq (n - r + 1)(\gamma_a + \gamma_b + \rho_{ab}) + 2(\rho_a + \rho_b) - 2(\gamma_a + \gamma_b) \geq
\]

\[
\geq (n - r - 2)(\gamma_a + \gamma_b + \rho_{ab}) + 2(\rho_a + \rho_b) + 3\rho_{ab}.\]

\(\gamma_a + \gamma_b + \rho_{ab} \geq 2\), inasmuch as \(\gamma_a + \gamma_b + \rho_{ab} \geq 2\) and, obviously, \(\rho_{ab} \geq \gamma_{ab}\). Also, \(n - r - 2 \geq 0\), because otherwise \(r = n\) or \(r = n - 1\), these cases are handled right before section 1b. It therefore follows that

\[
r \geq 2(n - r - 2) + 2(\rho_a + \rho_b) + 3\rho_{ab} \Rightarrow 3r \geq 2(n - 2) + 2(\rho_a + \rho_b) + 3\rho_{ab} \Rightarrow
\]

\[
\Rightarrow r \geq \frac{2}{3}(\rho_a + \rho_b) + \frac{2}{3}n - \frac{4}{3} + \rho_{ab}.
\]

The vertex \(a\) has \(\rho_a + \rho_{ab}\) neighbours amongst vertices of the cycle \(T\). Also, it can be adjacent to all the vertices of the path \(H\) except itself. Which means it is adjacent to at most \(n - r - 1 + \rho_a + \rho_{ab}\) vertices. On the other hand, from the formulation of our theorem, its degree is at least \(\frac{n+2}{3}\). Therefore, \(\frac{n+2}{3} \leq n - r - 1 + \rho_a + \rho_{ab}\). The same is true for the vertex \(b\), hence, \(\frac{n+2}{3} \leq n - r - 1 + \rho_b + \rho_{ab}\). Consequently, \(\frac{n+2}{3} \leq n - r - 1 + \min(\rho_a, \rho_b) + \rho_{ab}\). Thus,

\[
r \leq \frac{2n - 5}{3} + \min(\rho_a, \rho_b) + \rho_{ab}.
\]

From this and the previous inequalities it follows that

\[
\frac{2n - 5}{3} + \min(\rho_a, \rho_b) + \rho_{ab} \geq \frac{2}{3}(\rho_a + \rho_b) + \frac{2}{3}n - \frac{4}{3} + \rho_{ab} \Rightarrow
\]
\[ \Rightarrow \min(\rho_a, \rho_b) \geq \frac{2}{3}(\rho_a + \rho_b) + \frac{1}{3}. \]

Since \(\frac{2}{3}(\rho_a + \rho_b) \geq \frac{4}{3}\min(\rho_a, \rho_b)\), it follows that
\[ \min(\rho_a, \rho_b) \geq \frac{4}{3}\min(\rho_a, \rho_b) + \frac{1}{3} \Rightarrow 0 \geq \frac{1}{3}\min(\rho_a, \rho_b) + \frac{1}{3}. \]

This is impossible, a contradiction.

**Case 2:** There are no such 2 distinct vertices of the cycle \(T\) that one of them is adjacent to \(a\) and the second one is adjacent to \(b\).

Which means that \(a\) or \(b\) has at most 1 neighbour amongst vertices of the cycle \(T\). Without loss of generality, let \(a\) be this vertex, so \(a\) has at most 1 neighbour among vertices of the cycle \(T\). Since \(d_G(a) \geq \frac{n}{3}\), \(a\) has at least \(\frac{n}{3} - 1\) neighbours amongst vertices of the path \(H\). Hence, the path \(H\) contains (including \(a\)) at least \(\frac{n}{3} - 1 + 1 = \frac{n}{3}\) vertices. The path \(H\) has \(n - r\) vertices, thus,
\[ n - r \geq \frac{n}{3} \Rightarrow r \leq \frac{2n}{3}. \quad (2) \]

**Claim 3:** a) Any endpoint of \(H\) has at most \(\frac{r}{3}\) neighbours amongst vertices of the cycle \(T\).

b) If the graph \(G(H)\) is Hamiltonian, then any vertex of the graph \(G(H)\) has at most \(\frac{r}{3}\) neighbours amongst vertices of the cycle \(T\).

**Proof:** a) Firstly, let us prove that for the vertex \(a\). Using the method of the proof by contradiction, we assume that \(a\) has more than \(\frac{r}{3}\) neighbours among vertices of the cycle \(T\). Since \(a\) has at most one neighbour amongst vertices of the cycle \(T\), \(1 > \frac{r}{3} \Rightarrow r < 3\). By claim 2, \(r \geq \frac{n}{2}\). Hence, \(3 > \frac{n}{2} \Rightarrow n < 6\), a contradiction with \((1)\).

Now we prove that for the vertex \(b\). Let us consider \(k_b\) neighbours (of the vertex \(b\)) in the cycle \(T\) (keeping terminology, they are *vertices of the type b* or *b-vertices*). Note that b-vertices cannot be neighbours in the order of traversing the cycle, because otherwise the cycle \(T\) may be lengthened due to the vertex \(b\) and the remaining vertices form a path (this was illustrated in the figure 8). Let it be two b-vertices \(b_1\) and \(b_3\), let them have exactly one vertex (we denote it by \(b_2\)) between them in the order of traversal of the cycle \(T\). It has just been proved that \(bb_2 \notin E(G)\). Note that \(b_2\) is not adjacent to some vertex of the path \(H\), because otherwise a cycle is formed, which is larger than \(T\), and the remaining vertices form a path (see figure 10). Hence, all the neighbours of \(b_2\) are located in the cycle \(T\). Note that if two
Figure 10: $b_2$ cannot be adjacent to some vertex of the path $H$, because otherwise there is a cycle (bold in the illustration) whose existence runs counter to the choosing the cycle $T$.

neighbours of $b_2$ are adjacent by the edge of the cycle $T$, then $b_2$ may be ’inserted’ between these neighbours and then a cycle is formed which includes all the vertices of the cycle $T$ and the vertex $b$ (which means it is larger than $T$), and the set of vertices not included in the cycle forms a path (see figure 11). Hence, there are no two neighbours of $b_2$, which are neighbours in the order of traversing the cycle $T$. The cycle $T$ includes $r - 1$ vertices (except $b_2$), which means $b_2$ is adjacent to at most $\lceil \frac{r-1}{2} \rceil$ vertices. Thus, $\lceil \frac{r-1}{2} \rceil \geq \frac{n+2}{3} \Rightarrow \frac{r}{2} \geq \frac{n+2}{3} \Rightarrow r \geq \frac{2(n+2)}{3}$, inasmuch as $d_G(b_2) \geq \frac{n+2}{3}$. This is a contradiction with (2).

Therefore, there are no two neighbours (in the cycle $T$) of $b$, which are neighbours in the cycle $T$ or have exactly one vertex between them in the cycle $T$. Which means that $b$ has at most $\frac{r}{3}$ neighbours amongst vertices of the cycle $T$, insofar as $T$ includes $r$ vertices.

b) If the graph $G(H)$ is Hamiltonian, then any vertex of the path $H$ may serve as the endpoint of Hamiltonian path in the graph $G(V(H))$. Applying claim 3, item a) to this vertex and this Hamiltonian path, we acquire the desired property. □

**Claim 4:** Vertices of the path $H$ form a cycle.

**Proof:** Let us consider 2 cases.

**Case a:** The vertex $a$ has at least one neighbour in the cycle $T$ and the same is true for the vertex $b$. 

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Figure 11: If $b_2$ is adjacent to two vertices of the cycle $T$, which are adjacent by an edge of the cycle $T$, then there is a cycle (bold in the figure) whose existence runs counter to the choosing the cycle $T$.

Whereas we are within the framework of the case 2, $a$ and $b$ have exactly one neighbour in the cycle $T$ and this neighbour is the same for $a$ and $b$ (which means this neighbour is one vertex). We denote this neighbour by $c$. Consider subgraph $G(H)$. Since $a$ and $b$ have one neighbour in the cycle $T$, $d_{G(H)}(a) \geq \frac{n}{3} - 1$, $d_{G(H)}(b) \geq \frac{n}{3} - 1$. The graph $G(H)$ has a Hamiltonian path $H$. Applying lemma 1 to the path $H$, which has $n - r$ vertices: if $d_{G(H)}(a) + d_{G(H)}(b) \geq n - r$, then graph $G(H)$ has a Hamiltonian cycle (that is the desired property). Otherwise, $n - r > d_{G(H)}(a) + d_{G(H)}(b) \geq \frac{n}{3} - 1 + \frac{n}{3} - 1 = \frac{2n}{3} - 2 \Rightarrow r < \frac{n}{3} + 2$. Note that the path $H$ and the vertex $c$ form such a cycle of $n - r + 1$ vertices that the set of vertices beyond this cycle forms a path (see figure 12). Then, by the definition of $r$, the following holds: $r \geq n - r + 1 \Rightarrow r \geq \frac{n+1}{2}$. If $r > \frac{n+1}{2}$, then $r \geq \frac{n}{2} + 1$. Whereas $r < \frac{n}{3} + 2$, we get that $\frac{n}{3} + 2 > \frac{n}{2} + 1 \Rightarrow n < 6$, a contradiction with (1). If $r = \frac{n+1}{2}$, then $n$ is odd, and since $r < \frac{n}{3} + 2$, we have $\frac{n}{3} + 2 > \frac{n+1}{2} \Rightarrow n < 9$. Since $n$ is odd and (1) holds, $n = 7$, $r = \frac{n+1}{2} = 4$. From the previous observations it follows that the graph has two cycles (let them be $T_1$ and $T_2$) of 4 vertices which have common vertex $c$. And there is no such a cycle of 5 vertices that the remaining 2 vertices are adjacent (moreover, there is no such a cycle of 5 vertices that the remaining 2 vertices are not adjacent, because otherwise this cycle would meet all the requirements). So, graph does not have a cycle of 5 vertices. Note that if the vertex $c$ is removed, then graph remains connected, inasmuch as graph $G$ is biconnected. Hence, there is an edge $c_1c_2 \in E(G)$ for some $c_1$ and $c_2$ from cycles $T_1$ and $T_2$, respectively (see figure 13). Since $T_1$ and $T_2$ are the cycles of size 4, there exist paths from $c_1$ and $c_2$ to $c$, whose lengths are at least 2 and which move along the edges of the cycles $T_1$ and
Figure 12: The path $H$ and the vertex $c$ form such a cycle (bold in the illustration) that set of vertices beyond this cycle forms a path.

$T_2$, respectively. Concatenation of these paths and the edge $c_1c_2$ form a cycle whose size is at least 5. Whereas the graph does not have a cycle of size 5, this cycle’s size is 6 or 7, which means it is Hamiltonian or contains all the vertices except one and, hence, it is suitable.

**Case b:** There is such a vertex in a couple $(a, b)$ that does not have any neighbours in the cycle $T$.

Without loss of generality, let $a$ be this vertex, so $a$ does not have neighbours in the cycle $T$. Keeping previous denotations, let $b$ have $k_b$ neighbours in the cycle $T$. Consider subgraph $G(H)$. Since the vertex $a$ does not have neighbours amongst vertices of the cycle $T$ and the vertex $b$ have $k_b$ such neighbours, $d_{G(H)}(a) \geq \frac{n}{3}$, $d_{G(H)}(b) \geq \frac{n}{3} - k_b$. The graph $G(H)$ contains a Hamiltonian path $H$. Applying lemma 1 to the path $H$ (which has $n - r$ vertices): if $d_{G(H)}(a) + d_{G(H)}(b) \geq n - r$, then the graph $G(H)$ is Hamiltonian (that is the desired property). Otherwise, $n - r > d_{G(H)}(a) + d_{G(H)}(b) \geq \frac{n}{3} + \frac{n}{3} - k_b = \frac{2n}{3} - k_b \Rightarrow k_b > r - \frac{n}{3}$. Meanwhile, by the claim 3, item a) the following inequality holds $k_b \leq \frac{r}{3}$. Thus, the following inequalities are deduced $k_b \leq \frac{r}{3}$ and $k_b > r - \frac{n}{3} \Rightarrow \frac{r}{3} > r - \frac{n}{3} \Rightarrow n > 2r$. That is a contradiction with the claim 2. □

So, vertices of the path $H$ form a cycle. From now on we will name $H$ a cycle.

Let $p$ and $q$ be such two vertices of the cycle $H$ that they have the shortest path between them on the edges of the cycle $H$ among such couples of the vertices of the
cycle \( H \) that they have distinct neighbours in the cycle \( T \) (there exist such couples or a couple of vertices, because \( G \) is biconnected). Let \( p_0 \) and \( q_0 \) be distinct (so, \( p_0 \neq q_0 \)) neighbours of \( p \) and \( q \), respectively (among vertices of the cycle \( T \)). Let \( V_H \) be set of vertices of the cycle \( H \). Then \( V_H \) is the union \( V_H = V_{H_1} \cup V_{H_2} \cup \{p\} \cup \{q\} \), where \( H_1 \) and \( H_2 \) – two sections of the cycle \( H \), to which cycle is divided by the vertices \( p \) and \( q \) (see figure 14), and \( V_{H_1} \) and \( V_{H_2} \) are the sets of the vertices \( H_1 \) and \( H_2 \), respectively. Without loss of generality, inequality \(|V_{H_1}| \leq |V_{H_2}|\) is assumed. Note that for any vertex \( e \in V_{H_1} \) inequality \( d_{G(V_H)}(e) \geq \frac{n}{3} \) holds. Indeed, because otherwise, since \( d_G(e) \geq \frac{n}{3} \), \( e \) has a neighbour among the vertices of the cycle \( T \) (let it be \( e_1 \)). Then \( e_1 \neq p_0 \) or \( e_1 \neq q_0 \), without loss of generality, the first holds. Then \( e \) and \( p \) have two distinct vertices in \( T \), hence, the path between \( p \) and \( q \) is not the shortest path on the edges of the cycle \( H \) among all the couples of the vertices of \( H \) which have distinct neighbours in \( T \), because path between \( e \) and \( p \) on the edges of the cycle \( H \) is shorter.

**Claim 5:** The graph \( G(V_H) \) has such a Hamiltonian path that its endpoints are \( p \) and \( q \).

**Proof:** Consider the longest path with endpoints \( p \) and \( q \) (let it be \( L_2 \) and let it be \( \eta \) vertices in it) among all such paths that if some vertex \( d \) does not belong to this path, then \( d_{G(V_H)}(d) \geq \frac{n}{3} \), and all the vertices of the path \( H \), which do not belong to this path, form a path (let this path be \( L_1 \) with endpoints \( p_1 \) and \( q_1 \)). Such
a path $L_2$ exists, because from the proven before it follows that the path between $p$ and $q$, which moves along $H_2$, is suitable (it has at least $\frac{n-r}{2} + 1$ vertices, hence, $\eta \geq \frac{n-r}{2} + 1$). Assuming the opposite, let $L_2$ be not a Hamiltonian path. Then the path $L_1$ contains $n-r-\eta \geq 1$ vertices. Whereas from the condition on $L_2$ inequality $d_{G(V_H)}(p_1) \geq \frac{n}{3}$ holds, there are at least $\frac{n}{3} - (n-r-\eta - 1) = r + \eta + 1 - \frac{2n}{3}$ neighbours of the vertex $p_1$ among vertices of the path $L_2$. Note that neighbours of the vertex $p_1$ cannot be neighbours in the order of traversal of the path $L_2$ (otherwise, the vertex $p_1$ can be 'inserted' between these neighbours into the path $L_2$ (see figure 15) — then the path $L_2$ would be longer and all the remaining vertices form a path and their degrees in $G(V_H)$ are at least $\frac{n}{3}$, a contradiction with the choosing the $L_2$). Thus, $\eta \geq 2(r + \eta + 1 - \frac{2n}{3}) - 1 \Rightarrow \eta \leq \frac{4n}{3} - 2r - 1$.

We are going to prove that vertices of $L_1$ form a cycle. Assuming the opposite, they do not form a cycle. Firstly, let us consider the case where $L_1$ contains 3 vertices at least. Whereas it has $n-r-\eta$ vertices and $L_1$ is a path, by lemma 1: $d_{G(L_1)}(p_1) + d_{G(L_1)}(q_1) < n-r-\eta$, so, without loss of generality, we get that $d_{G(L_1)}(p_1) < \frac{n-r-\eta}{2}$. Then there are at least $\frac{n}{3} - \frac{n-r-\eta-1}{2} = \frac{r+\eta+1}{2} - \frac{n}{6}$ neighbours of the vertex $q_1$ amongst vertices of the path $L_2$. By similar consideration, as above, neighbours of $q_1$ cannot be neighbours in the order of traversal of the path $L_2$, so $\eta \geq 2(\frac{r+\eta+1}{2} - \frac{n}{6}) - 1 \Rightarrow \eta \geq r + \eta - \frac{n}{3} \Rightarrow \frac{n}{3} \geq r$, a contradiction with the claim 2.
Figure 15: $H$’s partition to two paths: $L_1$ and $L_2$. Demonstration that $p_1$ cannot have two consecutive (in the order of traversing the path $L_2$) neighbours.

Now, consider the case where $L_1$ comprises one vertex or two vertices. Then $\eta \geq n - r - 2$, and because $\eta \leq \frac{4n}{3} - 2r - 1$, we have $\frac{4n}{3} - 2r - 1 \geq n - r - 2 \Rightarrow r \leq \frac{n}{3} + 1$. But by the claim 2, $r \geq \frac{n}{2}$. Then $\frac{n}{3} + 1 \geq \frac{n}{2} \Rightarrow n \leq 6$. Then from (1) we get that $n = 6$, which means all the inequalities in this paragraph turn into equality, thus, $n = 6, r = \frac{n}{2} = 3, \eta = n - r - 2 = 1$. But $\eta \geq 2$, inasmuch as $L_2$ includes vertices $p$ and $q$, a contradiction.

So, vertices of the path $L_1$ form a cycle. From now on we will name $L_1$ a cycle.

Now, we are going to prove that there exist such 2 vertices of the path $L_2$ that one of them is adjacent to $p_1$, the other one is adjacent to $q_1$ and they are neighbours in the path $L_2$ or they have exactly one vertex between them in the order of traversing the path $L_2$. Suppose the opposite they do not exist. We fix some direction of the path $L_2$ (for instance, from $p$ to $q$). It should be recalled that there are at least $r + \eta + 1 - \frac{2n}{3}$ neighbours of $p_1$ among vertices of the path $L_2$. For each of them, we mark the following two vertices in accordance with the fixed order of traversal (see figure 16). Since it has been already proved (figure 15 was dedicated to that) that all the neighbours of the vertex $p_1$ are not neighbours in the order of traversing the path $L_2$, there is no such thing as a vertex which was marked twice. Hence, there are at least $2(r + \eta + 1 - \frac{2n}{3} - 1)$ marked vertices (for $q$ we do not mark anything, so we lose two vertices, and for the neighbour of $q$ in the path $L_2$ we mark only one vertex, so we lose one, hence, we lose at most two vertices, inasmuch as $q$ and the neighbour of $q$ cannot be neighbours of the vertex $p_1$ simultaneously). Whereas the opposite is supposed, all the neighbours of $q_1$ in the path $L_2$ (there are at least $r + \eta + 1 - \frac{2n}{3}$
Figure 16: For each neighbour of $p_1$ we mark the following two vertices in accordance with the fixed order of traversal.

such neighbours) cannot be marked. Thus,

$$\eta \geq r + \frac{2n}{3} + 2(r + \frac{2n}{3} - 1) \Rightarrow$$

$$\Rightarrow 0 \geq r + 1 - \frac{2n}{3} + 2r + 2\eta - \frac{4n}{3} \Rightarrow 2n \geq 3r + 2\eta + 1.$$  

Since $\eta \geq \frac{n-r}{2} + 1$, we have $2n \geq 3r + n - r + 2 + 1 \Rightarrow n \geq 2r + 3 \Rightarrow r < \frac{n}{2}$, a contradiction with the claim 2.

So, there exist such 2 vertices of the path $L_2$ that one of them is adjacent to $p_1$, the other one is adjacent to $q_1$ and they are neighbours in the path $L_2$ or they have exactly one vertex between them in the order of traversal of the path $L_2$. Let them be $y_1$ and $y_2$ (and let $y_1$ be closer to the vertex $p$ in the order of traversing the path $L_2$, and $y_2$ - closer to $q$, and, without loss of generality, let $y_1$ be adjacent to $p_1$, $y_2$ be adjacent to $q_1$). Note that if $y_1$ and $y_2$ are neighbours in the order of traversal of the path $L_2$, then there exists a Hamiltonian path in $V_H$ with endpoints $p$ and $q$ (see figure 17).

Therefore, there is a vertex (we denote it by $z$) between $y_1$ and $y_2$ in the path $L_2$. Since $z$ is a neighbour of both $y_1$ and $y_2$, as it was already mentioned (figure 15 was dedicated to that), $z$ is not adjacent to $p_1$ and $q_1$. Let us assume that $z$ is adjacent
Figure 17: There is a Hamiltonian path in $V_H$ with endpoints $p$ and $q$ if $y_1$ and $y_2$ are neighbours in the order of traversing the path $L_2$.

to some vertex of the cycle $L_1$. Then note that there is a path with endpoints $p$ and $q$ which includes $L_2$ and at least one another vertex (it goes along $L_2$ from $p$ to $z$, then it enters the cycle $L_1$ and moves along the edges of the cycle $L_1$ to the vertex $q_1$, goes to the vertex $y_2$ and then moves along the edges of the path $L_2$ to the vertex $q$, see figure 18), a contradiction with the choosing $L_2$. Hence, $z$ does not have neighbours amongst vertices of the cycle $L_1$. Note that if $z$ has two consecutive neighbours in the order of traversing the path $L_2$, then there is a Hamiltonian path in $V_H$ with endpoints $p$ and $q$, it looks like this: it goes from $p$ to $y_1$ on the edges of the path $L_2$, it goes along the edge $y_1p_1$, then it goes on the edges of the path $L_1$ to $q_1$, then enters the vertex $y_2$ and then it goes to $q$ on the edges of the path $L_2$ and, in so doing, we insert $z$ somewhere in this path between these neighbours (see figure 19). Thus, the vertex $z$ does not have two consecutive neighbours in the order of traversal of the path $L_2$. Let it be $\tau_1$ vertices in the path $L_2$ between $p$ and $y_1$ (first section of the path $L_2$) and let it be $\tau_2$ vertices in the path $L_2$ between $y_2$ and $q$ (second section of the path $L_2$). Then $\tau_1 + \tau_2 = \eta - 1$. Whereas the vertex $z$ does not have two consecutive neighbours amongst the vertices of the path $L_2$ in the
order of traversing the path $L_2$, it has at most $\frac{n+1}{2}$ neighbours in the first section and it has at most $\frac{r+1}{2}$ neighbours in the second section, which means it has at most $\frac{n+1}{2} + \frac{r+1}{2} = \frac{1}{2} + \frac{n+r-1}{2} = \frac{n+1}{2}$ neighbours among vertices of the path $L_2$.

![Diagram](image-url)

Figure 18: If $z$ has a neighbor from the cycle $L_1$, then there is a path (marked in bold) from $p$ to $q$ whose existence contradicts the choice of $L_2$.

By the claim 3, item b), the vertex $z$ has at most $\frac{r}{3}$ neighbours in the cycle $T$. Applying this observation, we get that $d_G(z) \leq \frac{n+1}{2} + \frac{r}{3}$. Whereas $d_G(z) \geq \frac{n+2}{3}$, it follows that $\frac{n+2}{3} \leq \frac{n+1}{2} + \frac{r}{3} \Rightarrow 2n + 4 \leq 2\eta + 3 + 2r \Rightarrow \eta \geq \frac{2n}{3} + \frac{1}{3} - \frac{2r}{3}$. Since $\eta \leq \frac{4n}{3} - 2r - 1$, we have $\frac{4n}{3} - 2r - 1 \geq \frac{2n}{3} + \frac{1}{3} - \frac{2r}{3} \Rightarrow \frac{2n}{3} - \frac{4}{3} \geq \frac{2r}{3} \Rightarrow \frac{n}{2} > r$, a contradiction with the claim 2.

Finally, we denote the Hamiltonian path in the graph $G(V_H)$ with endpoints $p$ and $q$ by $H_0$. Then there is such a cycle: the path $H_0$, the edges $pp_0$ and $qq_0$ and the major half of cycle $T$ between $p_0$ and $q_0$ (see figure 20). The length of such a cycle is at least $\frac{r}{2} + 2 + n - r - 1 = n - \frac{r}{2} + 1$ (2 edges $pp_0$, $qq_0$, the greater half of the cycle $T$ has at least $\frac{r}{2} + 2$ edges and a Hamiltonian path in the graph $G(V_H)$; since there are exactly $n - r$ vertices, there are $n - r - 1$ edges). Note that this is a cycle such that all other vertices not included in this cycle are one of the halves of the cycle $T$, that is, they form a path. Then, from the choice of the cycle $T$, the following must be fulfilled: $r \geq n - \frac{r}{2} + 1 \Rightarrow \frac{3r}{2} > n \Rightarrow r > \frac{2n}{3}$, a contradiction with (2). □
Figure 19: If $z$ has two consecutive neighbours in the order of traversing the path $L_2$, then there is a Hamiltonian path in $V_H$ (marked in bold).

Figure 20: From the part of the cycle $T$ and the path $H_0$, a cycle is obtained (marked in bold), the existence of which contradicts the choice of the cycle $T$. 

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3 Precision of the estimation

Claim: Our $\frac{n+2}{3}$ estimate is accurate. More specifically, for any $n \geq 8$ and for any $2 < \nu < \frac{n+2}{3}$ there exists a biconnected graph $G$ such that $v(G) = n$ and $\delta(G) = \nu$ and in which there is no such a cycle that the set of vertices not included in the cycle is independent.

Proof:

Let us take some 2 vertices, call them $f_1$ and $f_2$ and divide the remaining vertices into 3 groups: the first two groups contain exactly $\nu - 1$ vertices (let them be $A_1$ and $A_2$), and the third has $n - 2\nu$ vertices (let it be $A_3$). There are such edges in this graph: any two vertices in the same group are adjacent to each other, any two vertices in distinct groups are not adjacent, and the vertices $f_1$ and $f_2$ are adjacent to all vertices, but not to each other (see figure 21). Note that $\delta(G) = \nu$, because if vertex is located in $A_1$ or in $A_2$, then its degree is $\nu$, degrees of $f_1$ and $f_2$ are $n - 2 \geq \nu$ (whereas $n \geq 3$, $n - 2 \geq \frac{n}{3} > \nu$) and the degree of any vertex in $A_3$ is $n - 2\nu + 1 \geq \nu$. Now, let us prove that this graph is biconnected. Indeed, when removing any (except $f_1$) vertex, any remaining vertex is adjacent to $f_1$ ($f_2$ is adjacent to some vertex from $A_1$ or $A_2$, which in turn is adjacent to $f_1$), thus the graph without the removed vertex is connected. If the vertex $f_1$ is removed, then any remaining vertex is connected to $f_2$, that is, the remaining graph is connected. It remains to prove that there is no such a cycle in this graph that the set of vertices beyond the cycle is independent. Let us show it by contradiction, so we assume that there exists such a cycle. Then note that it must contain some vertices from all groups $A_1$, $A_2$, $A_3$ (otherwise, some entire group is not contained in the cycle, but any group has edges, since $A_1$ and $A_2$ have $\nu - 1 > 1$ vertices, and $A_3$ has $n - 2\nu > 1$ vertices). Let us start traversing the edges of the cycle, starting from some vertex

![Figure 21: Our example for $n = 8, \nu = 3$.](image-url)
from $A_1$. Without loss of generality, moving along the edges of the cycle, at some point we will find ourselves at the vertex from $A_2$, then at some point at the vertex from $A_3$, and then we will return to the original position. Passing from the group $A_i$ to the group $A_j$ for $i \neq j$, the cycle must visit $f_1$ or $f_2$. But there are at least 3 of these transitions between groups, that is, $f_1$ or $f_2$ was visited by the cycle twice, a contradiction. □
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