Non-Hermitian Disorder in Two-Dimensional Optical Lattices

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In this paper, we study the properties of two-dimensional lattices in the presence of non-Hermitian disorder. In the context of coupled mode theory, we consider random gain-loss distributions on every waveguide channel (on site disorder). Our work provides a systematic study of the interplay between disorder and non-Hermiticity. In particular, we study the eigenspectrum in the complex frequency plane and we examine the localization properties of the eigenstates, either by the participation ratio or the level spacing, defined in the complex plane. A modified level distribution function vs disorder seems to fit our computational results.

I. INTRODUCTION

The study of crystalline solids is based on Bloch’s theorem1,2 which assumes a perfect periodicity in the positions of the atoms and in the density of electrons. However, in actual crystalline solids there are always deviations from periodicity, such as point defects, linear faults (e.g. dislocations), 2D defects (e.g. interfaces of crystallites); if the concentration of these deviations becomes high enough Bloch’s theorem breaks down and a new paradigm emerges featuring novel properties such as the possibility of localized eigenstates3,4. The concept of this so-called Anderson localization, which claims that an electronic wave can be trapped in a finite region of a disordered lattice, has been at the center of the attention of the solid state physics community for more than sixty years5–10. The localization phenomenon appears due to the interference between multiple scattering processes of the electronic waves by random variations in the potential of the crystal lattice. As a result of this interference, the previously extended eigenmodes of the system, the Bloch waves, may now become localized decaying exponentially for large distances. The phenomenon of Anderson localization has also been studied experimentally, indirectly, by measurements of macroscopic quantities such as the conductance11–14 and the transmission15–18. In solid state systems though, the existence of many body interactions and temperature dependent effects, such as inelastic scattering, makes the interpretation of these experiments rather uncertain. In order to overcome this difficulty, the topic of localization was extended to the regime of optics, acoustics and elastics where its consequences were not clouded by other effects producing similar observations19–27. Such an extension is naturally valid, since the concept of localization is based on nothing else than wave scattering and interference28,29. The only difficulty with classical waves, such EM ones, is that they usually exhibit very weak scattering not enough to produce localization. Several ideas were proposed to circumvent this difficulty, some of which30,31 led through different paths to photonic crystals32–37 and phononic crystals38–44.

In recent years the problem of localization attracted renewed interest as research moved to two formerly unexplored areas: (a) Many body localization, meaning Anderson localization in the presence of many body interactions45–47 and (b) Non-Hermitian systems, such as those obeying parity and time-reversal (PT) symmetry48–52. This recently introduced concept of constructing parity-time (PT) symmetric systems is more appropriate for photonic rather than solid state systems since the former can easily incorporate and realize complex potentials that require physical gain and loss. Thus PT-symmetry in optics53–58 has been studied extensively over the past few years, leading to the development of the new field of non-Hermitian photonics59–74. The possibility for the potential (in our case index of refraction) to possess imaginary part makes the fundamental topic of localization even more complicated and at the same time gives rise to a whole lot of unanswered question about the properties of non-Hermitian disordered systems.

In particular, non-Hermitian random matrices is a topic of high research interest in the context of mathematical physics75, and disordered photonics76. More specifically, random lasers77,78 where the decay of the cavity modes and the gain material leads naturally to dissipation and amplification, respectively, is a prototypical system in the framework of disordered complex media, where non-Hermiticity plays a crucial role.

Apart from random lasers, most works regarding non-Hermitian disorder physics are devoted to Hatano-Nelson matrices79, PT-symmetric random lattices80 and dissipative arrays81; these works focus mostly on mostly one-dimensional systems with correlated disorder or special symmetries. In this work, we will try to answer some of the main questions that stem from the interplay of disorder and non-Hermiticity in two-spatial dimensions, while comparing our results with the corresponding, well known characteristics of the Hermitian case. In particular, we examine physically realistic Anderson type of non-Hermitian waveguide lattices with the most general uncorrelated disorder that includes gain and/or loss.

Our paper is organized as follows: In the next section, Sect. II, we introduce our model possessing diagonal disorder in the real or imaginary or both parts of the potential term $n$; we present also qualitative data concerning the
distribution of eigenfrequencies and the extent of their corresponding eigenfunctions, based on their participation ratio. In Sect. III, we give more quantitative numerical results regarding the level spacing/density of states and comment on the comparisons among the three types of disorder. Finally in Sect. IV, we present computational results and comments on the extent of the eigenfunctions in two different ways.

II. NON-HERMITIAN DISORDER IN COUPLED SYSTEMS

We begin our analysis by considering optical wave propagation in a disordered non-Hermitian model, in the context of coupled mode theory\textsuperscript{56}. We consider a 2D square lattice of $N \times N$ waveguides in the xy plane, with a field propagation constant per waveguide \{\(n_{p,q}\), \((p, q = 1, \ldots, N)\}, which here plays the role of the optical potential. The light propagation along the z axis is described by the following normalized paraxial equation of diffraction:

\[
\frac{i}{\partial_z} \psi_{p,q} = \nabla V \psi_{p,q} + \nabla \psi_{p,q} + \psi_{p,q} + \psi_{p,q} + \psi_{p,q} + \psi_{p,q} - \nabla n_{p,q} \psi_{p,q} = 0
\]  

(1)

where \(p, q = 1, 2, \ldots, N\), with \(N \times N\) being the total number of the waveguides, \(\psi_{p,q}\) the modal amplitudes, \(V\) the coupling coefficient between two neighboring channels, and \(n_{p,q}\) the complex potential strength (field propagation constant) at each waveguide channel. Here we have considered only nearest neighbors interactions and we assume (without loss of generality) that \(V=1\). For guided non-Hermitian structures, \(n_{p,q}\) is complex and this physically means that each waveguide is characterized by either gain (\(Im\{n_{p,q}\} < 0\)) or loss (\(Im\{n_{p,q}\} > 0\)) and by its real part \(Re\{n_{p,q}\}\).

In order to find the eigenmodes of the system, we substitute \(\psi_{p,q} = \phi_{p,q} e^{i(-\omega_j z)}\) in the evolution equation (Eq.1) and get the eigenvalue problem:

\[
-\omega_j \phi_{p,q} = (\phi_{p+1,q} + \phi_{p-1,q} + \phi_{p,q+1} + \phi_{p,q-1}) + n_{p,q} \phi_{p,q}
\]  

(2)

where \(\omega_j\) is the complex eigenvalue of the \(j\)th eigenmode, with \(j = 1, 2, \ldots, N \times N\).

So far the discussion was devoted to the general framework of the spatial coupled mode theory that describes the evolution of paraxial waves in two-dimensional optical waveguide lattices. In this paper though, we are interested to investigate the main features of disordered non-Hermitian lattices, and therefore we consider the potential strength \(n_{p,q}\) to be a random variable (on-site disorder). Let’s start by considering the modal problem. In particular, we are interested in finding how these eigenmodes and their corresponding eigenvalues change when the system becomes disordered. More specifically, we will examine phenomena related to Anderson localization in three different cases of disorder: (a) Real disorder, where the potential strength is real; \(n = n_R\), and possesses a random distribution \(n_R \in [-\frac{W}{2},\frac{W}{2}]\), (b) Imaginary disorder, where the potential strength is imaginary with a distribution of the form: \(n = in_I\), \(n_I \in [-\frac{W}{2},\frac{W}{2}]\) and finally, (c) Both real and imaginary disorder, where the potential strength is complex, \(n = n_R + in_I\) with \(n_R \in [-\frac{W}{2},\frac{W}{2}]\) and \(n_I \in [-\frac{W}{2},\frac{W}{2}]\). In the above expressions, \(W\) is defined as the disorder strength: a uniform distribution within the range \([-\frac{W}{2},\frac{W}{2}]\) in all the cases is assumed.

A value indicative to the localization of the eigenmodes is the participation ratio of each mode (\(PR_j\)), which is given by the relation:

\[
PR_j = \frac{\int_A |\phi_j(\vec{r})|^2 d^2(\vec{r})}{\int_A |\phi_j(\vec{r})|^4 d^2(\vec{r})}
\]  

(3)

where in the above \(A = N^2\) is the area of the system. Generally speaking, the PR measures the spread of a state \(|\phi\rangle\) over a basis \(|i\rangle\}_{i=1}^N\). For weak disorder strength, \(PR \sim A\), all lattice sites participate almost equally to the eigenfunction. For higher values of \(W\), the \(PR\) decreases, which means that the eigenmodes become more and more localized.

We also define the extent length of each mode, which is an easily experimentally measured and convenient in some cases quantity, by the relation:

\[
\lambda_j = \frac{\sqrt{PR_j}}{2}
\]  

(4)

To begin with, we will illustrate the eigenvalue spectrum of Eq.2 in the complex eigenfrequency plane and superimpose the values of \(PR\) (of the corresponding eigenmodes) by denoting them with different colors. We do not present any results for the case of real \(n\), since they are well known: For weak disorder the density of states (DOS) follows more or less the unperturbed DOS (see\textsuperscript{14}, p. 92) with the rounding of the discontinuities at the band edges and the logarithmic singularity at the band center; states remain essentially extended except at the extreme tails. For strong disorder, the DOS tends to follow the distribution of the potential strength and all states become strongly localized. In Fig.1, we show the calculated eigenvalue spectra in the complex frequency plane for various disorder strengths and relate the color of each eigenvalue with the logarithm of the corresponding participation ratio, as seen in the colorbar of the graphs. In this figure we restrict ourselves to only the cases of imaginary disorder (case (b), left column) and the case of disorder in both the real and the imaginary part of the potential (case (c), right column), and for three different disorder strengths.

We can observe that, on both cases, the eigenvalue spectrum forms an approximate ellipse on the complex plane, with a different ellipticity in each case. As expected, the ellipse is widened for increasing disorder. This elliptical pattern, instead of a circular one, is a consequence of the existence of the real nearest-neighbor couplings\textsuperscript{82,83}. We can also observe that, in general, the modes around the center of the elliptical pattern are the most extended.
Figure 1. Eigenvalue spectra of a 50 × 50 waveguide lattice \( A = 2.5 \times 10^3 \) in the complex frequency plane, for a particular realization of the random system and for various disorder strengths. In the left column, disorder is applied only to the imaginary part of the potential strength, while \( \text{Re}(n_{p,q})=0 \); in the right column, disorder of the same \( W \) is applied in both the real and the imaginary part of \( n_{p,q} \). Note that the color of each eigenvalue is related to the participation ratio of the corresponding eigenstate, as seen in the colorbar of the figures.

One of the interesting features in these plots is that the eigenvalues near the edges are the most localized, similarly to the case of real disorder.

### III. LEVEL SPACING AND DOS

Fig.1 provides a general semi-qualitative picture of the whole spectrum, as well as information about how extended (or localized) are the corresponding eigenstates. However, quantitative information about how dense is the spectrum in each sub-region of the complex frequency plane cannot be easily inferred from these plots. To remedy this missing information we consider first the density of states \( \text{DOS} \) in the complex frequency plane (averaged over realizations of the random system). To define the \( \text{DOS} \) we count the number of states \( \delta N \) with eigenfrequencies located within an elementary square of area \( \delta A = \delta \omega_R \times \delta \omega_I \) and centered at the point \( \omega = \omega_R + i \omega_I \) of the complex frequency plane; then we have by definition:

\[
\text{DOS}(\omega) \equiv \frac{\delta N}{\delta A} \tag{5}
\]

Usually, we implicitly assume that the \( \text{DOS} \) is averaged over many realizations of the random system. The \( \text{DOS} \), as expected, depends on the disorder strength. For small values of disorder, all the eigenvalues are concentrated near the real axis, while their density is higher in the center and decays as we move towards the edges of the spectrum. On the other hand, for the case of strong disorder, the eigenvalues become almost uniformly distributed in the whole spectrum and tend to follow the same distribution as the diagonal matrix elements, in the limit \( W \gg V \), where \( V \) is the coupling coefficient (the \( \text{DOS} \) at the edges drop to zero not discontinuously as the matrix elements).

Most interesting is the case of an intermediate value of disorder, which is shown in Fig.2, for \( W = 3 \). In this plot we can observe that the \( \text{DOS} \) shows four weak peaks located on the real and the imaginary axis and near the edges of the spectrum. The peaks are found symmetrically over the center of the complex plane. In addition, the \( \text{DOS} \) appears to decay as we further move away from these two axis.

Regarding the level spacing statistics, there are several ways to define the level spacing, \( S \), in the complex frequency plane. One way, termed “1” and for a particular realization of the disorder, is the following:

\[
S_1(\omega) = \sqrt{\frac{\delta A}{\delta N}} \tag{6}
\]
which is directly related with the $DOS$: $S = (DOS)^{-\frac{5}{2}}$. (This direct relation between $DOS$ and $S$ acquires an extra numerical coefficient if both quantities are averaged over many realizations of the random system).

Another way, termed "2", to define the level spacing, $S$, at each eigenfrequency $\omega$, is as the minimum distance in the complex frequency plane between two neighboring eigenfrequencies averaged over many realizations of the disorder:

$$S_2(\omega)|_{\omega=\omega_j} \equiv |\omega_j - \omega_{j-1}|$$  \hspace{1cm} (7)

In the above expression, $\omega_{j-1}$ is the eigenvalue which is nearest to the eigenvalue $\omega_j$, on the real axis (Hermitian case) or on the complex plane (non-Hermitian case).

Definition 1 gives the nearest level spacing in the complex frequency plane averaged essentially over all directions in this plane; definition 2 gives the nearest level spacing along only one direction, the direction which at each eigenfrequency gives the minimum nearest level spacing. It is obvious that the second definition will result systematically in a smaller level spacing, as shown in Fig.3. In Fig.3(a) we plot the level spacing $S$, according to both definitions 1 and 2, for reasons of comparison, and for $W = 3$ in both the real and the imaginary part of the diagonal matrix elements, vs the real frequency axis (i.e. vs $\omega_R$ and for $\omega_I=0$), while in Fig. 3(b) along the imaginary frequency axis, (i.e. vs $\omega_I$ and for $\omega_R=0$).

IV. SPATIAL EXTENT OF THE EIGENFUNCTIONS

In Fig.4 we present our results concerning the linear extent, $\lambda$, averaged over all the eigenmodes of the system, as a function of the disorder strength: $\lambda = \langle \lambda_j \rangle$. We consider the three different cases: (1) disorder only in the real part of the diagonal matrix elements; (2) same disorder only in the imaginary, and (3) same disorder in both the real and the imaginary parts. The comparison of the three cases reveals a very interesting feature: While case (1), real disorder, exhibits a monotonic drop of the extent of the eigenfunctions with increasing disorder, as expected, cases (2) and (3), complex disorder, exhibit a surprising increase of the extent of the eigenfunctions with increasing disorder (for small disorder) before they eventually drop even faster than case (1). We attribute this anomaly for weak disorder to the fact that an imaginary part in the Hamiltonian breaks time reversal symmetry (TRS); it is well known that in Hermitian systems the breaking of TRS (usually by the presence of a static magnetic field) favors delocalization (see pp. 510-513). In the present case imaginary disorder acts in a dual way: its implicit TRS breaking favors more extended states, while its disorder nature favors more localized states; the first aspect seems to dominate for only weak disorder, which is not actually surprising: for weak imaginary disorder its breaking of TRS is enough to randomize the phases of closed paths transversed in opposite directions; beyond this point the breaking of TRS has nothing to offer while the further increase of $W$ contributes only to localization.

The distribution of the level spacing averaged over the whole spectrum provides in the Hermitian case valid information about the localization or not of the eigenfunctions. This distribution of nearest-neighbor level spacing often takes one of several universal forms, depending only on the disorder strength and not on the details.
Figure 5. Probability density $P$ of normalized level spacings $S$, as defined by $7$, averaging over the whole spectrum. In this plot, disorder is applied in both the real and the imaginary part of the potential. The results for three values of disorder strength are shown: $W = 1$ (black dotted line), $W = 3$ (blue dashed line) and $W = 5$ (red dash-dot line). An ensemble of 50 realizations of disorder is used in each case.

For $W = 1$, the exponent is $b \approx 3$ and as we raise the disorder this exponent is gradually lowered (for $W = 3$, $b \approx 1.5$). For $W = 5$, $b = 1$ and the level spacing distribution obeys the Wigner-Dyson probability distribution $P_{WD}(S)$.

Fig. 5 shows an interesting and rather surprising result: In spite of the large disorder, $W=5$ (both in the real and imaginary parts), the Poisson distribution is far from being realized (although the low $S$ region is enhanced as the disorder is increased); this suggests that the overlap of the eigenmodes is more robust (with increasing disorder) in the random non-Hermitian case as compared with the Hermitian one. A possible explanation of this feature is a particular correlation induced by the non-Hermitian nature, in the sense that, at the sites of gain, $Im(n) < 0$, is more probable to have relatively larger values for most eigenfunctions; this is expected to keep the overlap of the eigenfunctions on the average rather appreciable and prevent the appearance of the Poisson distribution unless the disorder is reached unphysically large values.

V. CONCLUSIONS

In conclusion, we have examined the localization phenomenon of the eigenmodes of two-dimensional random optical lattices, in the presence of non-Hermitian diagonal disorder. We have found that the spectrum of such a system forms an approximate ellipse on the complex plane, with the eigenvalues located near the middle of the ellipse to correspond to the less localized eigenfunctions. In addition, the breaking of time reversal symmetry in the non-Hermitian case, leads to delocalization for weak disorder, while, as disorder is increased, the localization induced by the non-Hermitian disorder appears to be even stronger than in the Hermitian case. Finally, the level spacing distribution, averaging over the whole spectrum, seems to obey a sub-Wigner probability distribution, when non-Hermitian disorder is applied. For greater values of disorder, the corresponding distribution turns to obey the Wigner distribution, instead of the Poisson one which occurs in the Hermitian case, something that implies that the overlap of the eigenmodes of a non-Hermitian system is more robust.

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