A Yukawa coupling parameterization for type I + II seesaw formula and applications to lepton flavor violation and leptogenesis

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Abstract

In the type I + II seesaw formula the mass matrix of light neutrinos $m_\nu$ receives contributions from the exchanges of both heavy Majorana neutrinos and $SU(2)_L$-triplet Higgs bosons. We propose a new parameterization for the Dirac-type Yukawa coupling matrix of neutrinos in this case, which generalizes the well known Casas-Ibarra parameterization to type I + II seesaw and is useful when the triplet term in $m_\nu$ is known. Neutrino masses and mixing, lepton flavor violation in decays like $\mu \rightarrow e\gamma$ within mSUGRA models and leptogenesis can then be studied within this framework. We illustrate the usefulness of our new parameterization using a number of simple examples.

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1 Introduction

The seesaw mechanism [1, 2] provides a very natural and attractive explanation of the smallness of neutrino mass [3] as being due to exchanges of heavy particles. In the most commonly considered type I seesaw, these are heavy sterile (electroweak-singlet) Majorana neutrinos [1]; another well studied case is type II seesaw, where the small neutrino mass is generated by the induced vacuum expectation value (VEV) of an $SU(2)_L$-triplet Higgs boson [2]. In both cases, the light neutrinos are Majorana particles with the effective mass matrix $m_{\nu}$, which in the basis where the mass matrix of the charged leptons is diagonal and real, is diagonalized according to

$$m_{\nu} = U^* m_{\nu}^{\text{diag}} U^\dagger, \quad m_{\nu}^{\text{diag}} = \text{diag}(m_1, m_2, m_3). \quad (1)$$

Here $U$ is the leptonic mixing matrix, which depends on three mixing angles, one Dirac-type and two Majorana-type CP-violating phases. Although both sterile neutrinos and Higgs triplets can be freely added to the standard model, they are most natural in its partially unified or grand unified extensions, such as left-right symmetric models or $SO(10)$ grand unified theories (GUTs), where both type I and type II contributions to the neutrino mass are typically present. In that case the neutrino mass matrix is a sum of two terms

$$m_{\nu} = m_{\nu}^{II} + m_{\nu}^I = v_L f_L - \frac{v_u^2}{v_R} Y_D f_R^{-1} Y_D^T. \quad (2)$$

Here the first term is the $SU(2)_L$-triplet Higgs contribution with $v_L$ the VEV of the triplet and $f_L$ the triplet Yukawa coupling matrix. The triplet VEV $v_L \simeq \mu v_u^2/M_\Delta^2$, where $\mu$ is the trilinear Higgs coupling, $v_u$ is the VEV of the up-type Higgs doublet $H_u$, and $M_\Delta$ is the mass of the triplet. The second term in (2) is the conventional type I seesaw term, in which $v_u Y_D$ is the Dirac mass matrix $m_D$. Having in mind extensions of the standard model, we have written the Majorana mass matrix of heavy neutrinos $M_R$ as $v_R f_R$, with $f_R$ being the relevant coupling matrix. In particular, in left-right symmetric gauge theories $f_R$ is the Yukawa coupling matrix of an $SU(2)_R$-triplet Higgs and $v_R$ is its VEV, which is related to the VEV of the $SU(2)_L$-triplet via $v_L v_R \propto v_u^2$. Regardless of the variant of the seesaw mechanism and barring unnaturally small Yukawa couplings or strong cancellations, the typical mass scale of the neutrino mass generation ($M_\Delta$ or $v_R$ or both) exceeds $10^9$ GeV, which is way beyond the reach of direct experimental tests. Hence, the seesaw mechanism of neutrino mass generation can only be probed indirectly.

One way of indirectly probing the seesaw is provided by cosmology, where the observed baryon number of the universe can be generated through the baryogenesis via leptogenesis mechanism [4, 5]. Leptogenesis can work successfully within both type I [6, 7] and type II [8] seesaw scenarios [5], as well as in the combined type I + II seesaw [9] (for earlier works see

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1Note that sometimes in the literature the mechanism leading to the entire Eq. (2) (rather than only to the triplet contributions to $m_{\nu}$) is called type II seesaw.

2In the case of pure type II seesaw more than one Higgs triplet is necessary for leptogenesis to work.
Another possibility of testing the seesaw is through lepton flavor violation (LFV) within supersymmetric theories, which has also been discussed in both type I \cite{13} and type II \cite{14,15,16} seesaw frameworks. Here LFV is induced by off-diagonal entries in the slepton mass matrices, which can be generated radiatively. In many well motivated scenarios, the size of these entries depends on the seesaw parameters. Leptogenesis and LFV depend on combinations of the Yukawa coupling matrices that are different from those entering into the seesaw formula (2), and this can be used – at least in principle – to reconstruct the seesaw parameters.

An important issue in these approaches is that the number of high energy parameters, i.e. of those contained in $m^{II}_{\nu}$, $m_D$ and $M_R$, exceeds the number of low energy parameters contained in $m_{\nu}$, simply because the heavy degrees of freedom are integrated out at low energies. An exception is the case of type II dominance, when $m_{\nu}$ coincides (or approximately coincides) with $m^{II}_{\nu}$. In the general case, however, and without a specific model at hand, one can only parameterize the unknown high energy quantities. Within the pure type I seesaw, one such parameterization, which proved to be especially useful and convenient, was suggested by Casas and Ibarra \cite{17}. This is the parameterization of the Dirac-type Yukawa coupling matrix $Y_D$ in which it is written as

$$Y_D = i U^* \sqrt{m^\text{diag}_\nu} R \sqrt{M^\text{diag}_R} = i \sqrt{v_R} U^* \sqrt{m^\text{diag}_\nu} R \sqrt{f^\text{diag}_R}.$$  \hspace{1cm} (3)

Here $R$ is a complex orthogonal matrix that contains the parameters which are integrated out when $m_{\nu}$ is obtained, and which therefore cannot be determined from low energy neutrino data without additional input. Many analyses of neutrino mixing, LFV and/or leptogenesis in the type I seesaw framework have been performed using this parameterization \cite{18,19}. However, to the best of our knowledge, no parameterization of this kind has been suggested for the general case of type I+II seesaw. The purpose of the present paper is to generalize the Casas-Ibarra parameterization to the case of the combined type I+II seesaw, when the mass matrix of light neutrinos is given by Eq. (2), and to demonstrate the usefulness of the proposed parameterization.

The paper is organized as follows. We summarize the main aspects of lepton flavor violation, leptogenesis and neutrino mixing within type I+II seesaw in Section 2. Section 3 contains our central results. Here we introduce our parameterization of the Dirac-type Yukawa coupling matrix in the case of the combined type I+II seesaw. We also give simple examples on its usage, two of which are based on the approximate tri-bimaximality of neutrino mixing. We conclude in Section 4.

## 2 Formalism

We will work in the basis in which the mass matrix of charged leptons is real and diagonal. The mass matrix of light neutrinos is then diagonalized according to Eq. (1), with $U$ the
leptonic mixing matrix, for which we will use the parameterization

\[
U = \begin{pmatrix}
  c_{12} c_{13} & s_{12} c_{13} & s_{13} e^{-i\delta} \\
  -c_{23} s_{12} - s_{23} s_{13} c_{12} e^{i\delta} & c_{23} c_{12} - s_{23} s_{13} s_{12} e^{i\delta} & -s_{23} c_{12} - c_{23} s_{13} s_{12} e^{i\delta} \\
  s_{23} s_{12} - c_{23} s_{13} c_{12} e^{i\delta} & -s_{23} c_{12} - c_{23} s_{13} s_{12} e^{i\delta} & c_{23} c_{13}
\end{pmatrix} P. \quad (4)
\]

Here \(c_{ij} = \cos \theta_{ij}\), \(s_{ij} = \sin \theta_{ij}\), \(\delta\) is the Dirac-type CP-violating phase, and the Majorana phases \(\alpha\) and \(\beta\) are contained in the matrix

\[
P = \text{diag}(1, e^{-i\alpha}, e^{-i\beta}). \quad (5)
\]

The analyses of neutrino experiments revealed the following best-fit values and 3\(\sigma\) ranges of the oscillation parameters \([20]\):

\[
\Delta m^2_\odot \equiv m_2^2 - m_1^2 = (7.9^{+1.1}_{-0.8}) \cdot 10^{-5} \text{ eV}^2, \\
\sin^2 \theta_{12} = 0.31^{+0.07}_{-0.05}, \\
\Delta m^2_A \equiv |m_3^2 - m_1^2| = (2.6^{+0.6}_{-0.6}) \cdot 10^{-3} \text{ eV}^2, \\
\sin^2 \theta_{23} = 0.47^{+0.17}_{-0.15}, \\
|U_{e3}|^2 = 0^{+0.040}_{-0.000}. \quad (6)
\]

Depending on the sign of \(m^2_3 - m^2_1\), the neutrino masses are normally or inversely ordered:

- normal: \(m_3 > m_2 > m_1\) with \(m_2 = \sqrt{m_1^2 + \Delta m^2_\odot}\); \(m_3 = \sqrt{m_1^2 + \Delta m^2_A}\);
- inverted: \(m_2 > m_1 > m_3\) with \(m_2 = \sqrt{m_3^2 + \Delta m^2_\odot + \Delta m^2_A}\); \(m_1 = \sqrt{m_3^2 + \Delta m^2_A}\).

The overall scale of neutrino masses is not known, except for the upper limit of order 1 eV coming from direct mass search experiments and cosmology.

The type I seesaw mechanism \([1]\) corresponds to the situation when the light neutrino masses are induced by their coupling with heavy Majorana neutrinos. Introducing the Dirac mass matrix \(m_D = v_u Y_D\) with \(v_u = v \sin \beta\) being the VEV of \(H_u\), and the Majorana mass matrix \(M_R = v_R f_R\) for heavy neutrinos, one finds for \(v_R \gg v_u\) the light neutrino mass matrix

\[
m^I_\nu = \frac{-v_u^2}{v_R} Y_D f_R^{-1} Y_D^T. \quad (7)
\]

The masses of light neutrinos can also be generated through their coupling with an \(SU(2)_L\)-triplet Higgs, which gives the triplet, or type II seesaw \([2]\):

\[
m^{II}_\nu = v_L f_L. \quad (8)
\]

In this case the neutrino mass matrix is directly given by the triplet Yukawa coupling matrix \(f_L\), up to an overall scale which is just the triplet VEV \(v_L\). In left-right symmetric models and their GUT extensions both seesaw contributions to \(m_\nu\) are naturally present,
leading to the type I + II seesaw expression of Eq. (2). Moreover, in these models there is a relation between the VEVs of the neutral components of the two triplets $v_L$ and $v_R$:

$$v_L v_R = \gamma v_u^2,$$

(9)

where $\gamma$ depends on the parameters of the Higgs potential. Type I + II seesaw can, of course, also be realized without extending the gauge group of the standard model.

In general, the matrices $m_\nu$, $f_L$ and $f_R$ are complex symmetric, whereas $Y_D$ is a general complex matrix (of dimension $3 \times 3$ for three generations of light and heavy neutrinos). In left-right symmetric models and their extensions, in addition to the gauge symmetry, a discrete left-right symmetry is often assumed, which can be realized either as $C$-conjugation or as a parity symmetry. This leads to additional constraints on the entries of the seesaw relation (2). Namely, in the case of $C$-conjugation symmetry, one has $f_L = f_R$, $Y_D = Y_D^T$, while for parity symmetry $f_L = f_R^T$, $Y_D = Y_D^T$. In both cases the seesaw exhibits a curious duality property (21) (see also (22, 23)). In our study, however, we will not assume any additional constraints on the entries of Eq. (2). As the neutrino mass matrix given by this formula contains two terms, it leads to a number of interesting possibilities for explaining the features of neutrino mixing (24).

Let us now briefly summarize the LFV formulae relevant to our discussion. In supersymmetric scenarios LFV is triggered by off-diagonal entries in the slepton mass matrix $\tilde{m}_L^2$. The branching ratios for radiative decays of the charged leptons $\ell_i = e, \mu, \tau$ are

$$\text{BR}(\ell_i \to \ell_j \gamma) = \text{BR}(\ell_i \to \ell_j \nu \bar{\nu}) \frac{\alpha^3}{G_F^2 m_S^2} \left| \left(\tilde{m}_L^2\right)_{ij}\right|^2 \tan^2 \beta,$$

(10)

where $m_S$ is a typical mass scale of SUSY particles. The values of the branching ratios $\text{BR}(\ell_i \to \ell_j \nu \bar{\nu})$ are $\text{BR}(\mu \to e \nu \bar{\nu}) \approx 1$, $\text{BR}(\tau \to \mu \nu \bar{\nu}) \approx 0.174$ and $\text{BR}(\tau \to e \nu \bar{\nu}) \approx 0.178$ [25]. Current limits on the branching ratios for $\ell_i \to \ell_j \gamma$ are $\text{BR}(\mu \to e \gamma) \leq 1.2 \cdot 10^{-11}$ [26], $\text{BR}(\tau \to e \gamma) \leq 1.1 \cdot 10^{-7}$ [27] and $\text{BR}(\tau \to \mu \gamma) \leq 6.8 \cdot 10^{-8}$ [28]. One expects to improve these bounds by two to three orders of magnitude for $\text{BR}(\mu \to e \gamma)$ [29] and by one to two orders of magnitude for the other branching ratios [30].

To satisfy the requirement that the LFV branching ratios $\text{BR}(\ell_i \to \ell_j \gamma)$ be below their experimental upper bounds, one typically assumes that $\tilde{m}_L^2$ and all other slepton mass and trilinear coupling matrices are diagonal at the scale $M_X$. Such a situation occurs for instance in mSUGRA scenarios. Off-diagonal terms get induced at low energy scales radiatively, which explains their smallness. In this case a very good approximation for the typical SUSY mass appearing in Eq. (10) is $m_S^2 = 0.5 \frac{m_0^2}{\tan^2 \beta} \left(3m_0^2 + A_0^2\right) \left(Y_D Y_D^T\right)_{ij}$, where $m_0$ is the universal scalar mass and $m_{1/2}$ is the universal gaugino mass at $M_X$. In a supersymmetric seesaw framework the radiative entries giving rise to LFV depend on the same parameters as the neutrino masses. If there is only the type I seesaw term in $m_\nu$, the well-known result is [13]

$$\left(\tilde{m}_L^2\right)_{ij}^I = -\frac{3m_0^2 + A_0^2}{8 \pi^2} \left(Y_D Y_D^T\right)_{ij},$$

(11)

where $L_{ij} = \delta_{ij} \ln \frac{M_X}{M_i}$.
In case when only the triplet term \( m_{III}^{\nu} \) contributes to \( m_{\nu} \), one finds \[14\]

\[
(\tilde{m}_e^{2L})_{ij}^{II} = -3 \left( \frac{3m_0^2 + A_0^2}{8\pi^2} \right) \left( f_L f_L^\dagger \right)_{ij} \ln \frac{M_X}{M_{\Delta}}. \tag{12}
\]

Here and in Eq. \((11)\) \( A_0 \) is the universal trilinear coupling. When both terms in the mass matrix Eq. \((2)\) are present, their contributions to \((\tilde{m}_e^{2L})_{ij}\) sum up:

\[
(\tilde{m}_e^{2L})_{ij} = (\tilde{m}_e^{2L})_{ij}^I + (\tilde{m}_e^{2L})_{ij}^{II}
\]

\[
= -\left( \frac{3m_0^2 + A_0^2}{8\pi^2} \right) \left[ (Y_D)_{ik} \left( Y_D^\dagger \right)_{kj} \ln \frac{M_X}{M_k} + 3 \left( f_L f_L^\dagger \right)_{ij} \ln \frac{M_X}{M_{\Delta}} \right]. \tag{13}
\]

As the LFV branching ratios depend on the absolute value squared of this quantity, there will be an interference term between the contributions from the triplet term and from the type I seesaw term if both of them have off-diagonal entries. We will now compare the structures of two expressions:

\[
m_{\nu} = v_L f_L - \frac{v_{\nu}^2}{v_R} Y_D f_R^{-1} Y_D^T \quad \text{versus} \quad (\tilde{m}_e^{2L})_{ij} \propto \left( f_L f_L^\dagger \right)_{ij} + \left( Y_D Y_D^\dagger \right)_{ij}, \tag{14}
\]

where we have omitted logarithmic corrections to \((\tilde{m}_e^{2L})_{ij}\). There are several possibilities, depending on the relative magnitudes of the two contributions to \((\tilde{m}_e^{2L})_{ij}\) and \( m_{\nu} \):

(i) in the neutrino mass matrix the type I seesaw term \( m_{\nu}^{I}\nu \) dominates, and in the off-diagonal entries of the RG-induced slepton mass matrix \( (\tilde{m}_e^{2L})_{ij}^I \) dominates. This situation is the one best studied in the literature [17, 18, 19], for a recent review see [31]). We have nothing new to add in this case;

(ii) in the neutrino mass matrix the triplet term \( m_{\nu}^{I}\nu \) dominates, and in the off-diagonal entries of the RG-induced slepton mass matrix \( (\tilde{m}_e^{2L})_{ij}^{II} \) dominates. This situation has also been studied [15, 16], though less often than (i);

(iii) in the neutrino mass matrix the triplet term \( m_{\nu}^{I}\nu \) dominates, while in the off-diagonal entries of the RG-induced slepton mass matrix \( (\tilde{m}_e^{2L})_{ij}^{II} \) dominates. This situation, to our knowledge, has not been studied yet. However, there are hardly any useful statements to be made, as there is no link between neutrino masses and LFV, even if \( f_L \) and \( f_R \) are related by \( f_L = f_R \) or \( f_L = f_R^* \);

(iv) in the neutrino mass matrix the conventional seesaw term \( m_{\nu}^I \) dominates, whereas in the off-diagonal entries of the RG-induced slepton mass matrix \( (\tilde{m}_e^{2L})_{ij}^{II} \) dominates. Again, this situation remains to be investigated. However, as in case (iii), there is hardly any link between neutrino masses and LFV, even if \( f_L \) and \( f_R \) are related;

(v) both terms are of comparable magnitude both in \( m_{\nu} \) and in the off-diagonal entries of the slepton mass matrix. This case will be of prime interest to us.
In the next section we will propose a Yukawa coupling parameterization to deal with case (v), which in principle can also be applied to cases (iii) and (iv).

Before we turn to the Yukawa coupling parameterization, let us summarize the relevant leptogenesis formulae. We will assume, as it has been done in most studies, that the heavy Majorana neutrinos are lighter than the Higgs triplets. In that case it is sufficient to consider only the decay of heavy neutrinos into lepton and Higgs doublets (and similarly for the SUSY partners), while the decays of the triplets into two lepton doublets can be neglected. The CP-violating decay asymmetries of heavy neutrinos \( N_i \) contain two contributions. The first one is the same as in the case of pure type I seesaw [7, 5]:

\[
(\varepsilon_i^o)_N = \frac{1}{8\pi} \frac{1}{(Y^\dagger_D Y_D)_{ii}} \sum_{j\neq i} \text{Im} \left[ (Y^\dagger_D)_{ia} (Y_D)_{aj} (Y^\dagger_D Y_D)_{ij} \right] f(M^2_j/M^2_i) 
\]

\[
+ \frac{1}{8\pi} \frac{1}{(Y^\dagger_D Y_D)_{ii}} \sum_{j\neq i} \text{Im} \left[ (Y^\dagger_D)_{ia} (Y_D)_{aj} (Y^\dagger_D Y_D)_{ji} \right] \frac{1}{1 - M^2_j/M^2_i},
\]

where

\[
f(x) = \sqrt{x} \left[ \frac{2}{1-x} - \ln \left( \frac{1+x}{x} \right) \right].
\]

We have indicated here that flavor effects [32] might play a role, i.e., \( \varepsilon_i^o \) describes the decay of the heavy neutrino of mass \( M_i \) into leptons of flavor \( \alpha = e, \mu, \tau \). We will focus here on the case when the lowest-mass heavy neutrino is much lighter than the other two, i.e. \( M_1 \ll M_{2,3} \); the lepton asymmetry is then dominated by the decay of this lightest neutrino. In this case \( f(M^2_j/M^2_1) \approx -3 M_1/M_j \), and in addition the second term in Eq. (15) is strongly suppressed, therefore we will neglect it in what follows.

The second type of asymmetry is encountered when a Higgs triplet is exchanged in the loop diagrams [9]:

\[
(\varepsilon_i^o)_\Delta = \frac{3}{8\pi} \frac{M_i v_L}{v_u^2} \frac{1}{(Y^\dagger_D Y_D)_{ii}} \text{Im} \left[ (f^\dagger_L Y_D)_{ai} (Y^\dagger_D)_{ia} \right] g(M^2_\Delta/M^2_i),
\]

where

\[
g(x) = x \ln \left( \frac{1+x}{x} \right).
\]

In the limit \( M_\Delta \gg M_{1,2,3} \), which we will assume, one has \( g(x) \approx 1 - \frac{1}{2x} = 1 - \frac{1}{2} (M_i/M_\Delta)^2 \). The total asymmetries \( (\varepsilon_i)_N \) and \( (\varepsilon_i)_\Delta \) are obtained by summing \( (\varepsilon_i^o)_N \) and \( (\varepsilon_i^o)_\Delta \) over the flavor index \( \alpha \).

The baryon asymmetry of the universe \( (\eta_B = n_B/n_\gamma = 6.1 \cdot 10^{-10}) \) is finally found as

\[
\eta_B \simeq -0.96 \cdot 10^{-2} \sum_\alpha \varepsilon_i^o \kappa^\alpha,
\]

where the washout factors \( \kappa^\alpha \) are obtained by solving the relevant Boltzmann equations. The approximate expression we use is [33]

\[
\kappa^\alpha \simeq \frac{2}{K^\alpha z_B(K^\alpha)} \left\{ 1 - \exp \left[ -K^\alpha z_B(K^\alpha)/2 \right] \right\},
\]

\[
7
\]
where $K^\alpha = \sum K_i^\alpha$ with $K_i^\alpha = |(Y_D)_{ai}|^2 K_i/(Y_D^\dagger Y_D)_{ii}$ and

$$z_B(K^\alpha) = 2 + 4 (K^\alpha)^{0.13} \exp (-2.5/K^\alpha) . \quad (20)$$

The parameter $K_i$ in the expression for $K_i^\alpha$ is defined as $K_i = \Gamma_i/H(T)|_{T=M_i}$, with the tree-level decay width of the $i$th heavy neutrino $\Gamma_i = (Y_D^\dagger Y_D)_{ii} M_i/(8 \pi)$ and the Hubble parameter $H(T) = 1.66 \sqrt{g^* T^2}/M_{Pl}$. The out-of-equilibrium decay condition for $N_i$ is essentially $K_i < 1$.

## 3 Dirac-type Yukawa coupling parameterization for type I + II seesaw and its applications

When the triplet term $m_I^{II}$ is present in the seesaw relation, the procedure that led to the Casas-Ibarra parameterization $[3]$ of the matrix $Y_D$ cannot be directly applied. However, as we shall show, a simple transformation of Eq. (2) makes it possible to generalize the parameterization $[3]$ to the case of type I + II seesaw.

First, we move the type II contribution to the left hand side of Eq. (2), which gives

$$m_\nu - v_L f_L = -\frac{v_u^2}{v_R} Y_D f_R^{-1} Y_D^T \quad . \quad (21)$$

It is convenient to introduce the notation

$$X_\nu \equiv m_\nu - v_L f_L \, , \quad \text{diagonalized as} \quad X_\nu = V_\nu^* X_\nu^{\text{diag}} V_\nu^\dagger$$

with a unitary matrix $V_\nu$. Multiplying both sides of Eq. (21) by $X_\nu^{-1/2}$, we find

$$\mathbb{1} = -\frac{v_u^2}{v_R} \left( X_\nu^{-1/2} Y_D f_R^{-1/2} \right) \left( f_R^{-1/2} Y_D^T X_\nu^{-1/2} \right) . \quad (23)$$

Next, we note that although a square root of a symmetric matrix is not always automatically symmetric, it can always be chosen to be symmetric. We will make such a choice for the matrices $X_\nu^{-1/2}$ and $f_R^{-1/2}$, i.e. we assume them to be symmetric. One can then rewrite Eq. (23) as

$$\mathbb{1} = R R^T \quad \text{with} \quad R = \pm i \frac{v_u}{\sqrt{v_R}} \left( X_\nu^{-1/2} Y_D f_R^{-1/2} \right) . \quad (24)$$

Eq. (24) means that the type I + II seesaw relation requires $R$ to be an (in general complex) orthogonal matrix, but otherwise does not constrain it. Thus, for the Dirac-type Yukawa coupling $Y_D$ we have

$$v_u Y_D = \pm i \sqrt{v_R} X_\nu^{1/2} R f_R^{1/2} , \quad (25)$$

\textsuperscript{3}This can, e.g., be achieved by diagonalizing $X_\nu$ and $f_R$ by complex orthogonal transformations and then taking square roots.
where $R$ is an arbitrary complex orthogonal matrix. It can be parameterized as

$$R = R_{12} R_{13} R_{23},$$

(26)

where $R_{ij}$ is the matrix of rotation by a complex angle $\omega_{ij} = \rho_{ij} + i\sigma_{ij}$ in the $ij$-plane. The parameterization of the Yukawa coupling matrix $Y_D$ in Eq. (25) is the most general one satisfying the combined type I + II seesaw formula.

As was pointed out above, when the underlying theory possesses a discrete left-right symmetry, type I + II seesaw exhibits a duality property [21]. In that case the seesaw relation (2) is invariant with respect to the duality transformation $f_R \rightarrow \hat{f}_R = m_\nu/v_L - f_L$.

It is interesting to note that in terms of $f_R$ and its dual $\hat{f}_R$ Eq. (25) can be rewritten as

$$v_u Y_D = \pm i \sqrt{v_L v_R} \left( \hat{f}_R^{1/2} R f_R^{1/2} \right)^T.$$

(27)

For practical applications, it proves to be convenient to use a slightly modified version of Eq. (25). First, we note that for discussions of both LFV and leptogenesis one has to go to the basis where the mass matrix of heavy Majorana neutrinos $M_R$ is diagonal and real. As $M_R = f_R v_R$, this also diagonalizes the matrix $f_R$. The corresponding transformation is

$$V_R^T f_R V_R = f_R^{\text{diag}},$$

(28)

with a unitary matrix $V_R$. Note that $f_R^{\text{diag}} = \text{diag}(M_1, M_2, M_3)/v_R$. The transformation (28) amounts to replacing the Yukawa coupling matrix $Y_D$ in the seesaw relation according to $Y_D \rightarrow Y_D V_R$, i.e. it fixes its right-handed basis. In what follows we will be assuming that the matrix $M_R$ has been diagonalized and consider $Y_D$ in this basis, except in example 3 below, where this diagonalization will be carried out explicitly. Next, it is convenient to express $X_\nu$ through its eigenvalues. To this end, using Eq. (22) we rewrite $X_\nu$ on the left hand side of Eq. (21) as

$$X_\nu = V_\nu^* X_\nu^{\text{diag}} V_\nu^T = \left[ V_\nu^* (X_\nu^{\text{diag}})^{1/2} \right] \left[ V_\nu^* (X_\nu^{\text{diag}})^{1/2} \right]^T.$$

(29)

Multiplying then Eq. (21) by $[V_\nu^* (X_\nu^{\text{diag}})^{1/2}]^{-1}$ on the left and by $\{[V_\nu^* (X_\nu^{\text{diag}})^{1/2}]^T\}^{-1}$ on the right and following the same steps as above, one readily finds

$$v_u Y_D = \pm i \sqrt{v_R} V_\nu^* \sqrt{X_\nu^{\text{diag}}} R \sqrt{f_R^{\text{diag}}}.$$

(30)

This parameterization is the main point of the present paper and we will be using it in the subsequent discussion. Note that the matrix $R$ here is in general not the same as the matrix $R$ in Eq. (25). This is of no concern to us, as both are arbitrary complex symmetric matrices.

In the remainder of this section we will give simple examples demonstrating the usefulness of the parameterization (30). In the first two examples we consider tri-bimaximal neutrino mixing [34], which describes very well the current status of global fits to the low
energy neutrino data. The neutrino mass matrix giving rise to tri-bimaximal mixing can be written as

\[
m_\nu = \frac{m_1}{6} \begin{pmatrix} 4 & -2 & -2 \\
\cdot & 1 & 1 \\
\cdot & \cdot & 1 \end{pmatrix} + \frac{m_2 e^{2i\alpha}}{3} \begin{pmatrix} 1 & 1 & 1 \\
\cdot & 1 & 1 \\
\cdot & \cdot & 1 \end{pmatrix} + \frac{m_3 e^{2i\beta}}{2} \begin{pmatrix} 0 & 0 & 0 \\
\cdot & 1 & -1 \\
\cdot & \cdot & 1 \end{pmatrix}. \tag{31}
\]

If neutrinos enjoy the normal mass hierarchy, one can neglect \(m_1\), so that the first term in Eq. (31) vanishes, and in addition one has \(m_2 = \sqrt{\Delta m^2_\odot}\) and \(m_3 = \sqrt{\Delta m^2_A}\). An appealing possibility in this case is that the two remaining individual matrices in Eq. (31) correspond to \(m'_I\) and \(m''_I\), respectively [24]. The moderate ratio of the two terms in \(m_\nu\) is therefore \(\frac{3}{2} \sqrt{\Delta m^2_A/\Delta m^2_\odot} \simeq 8.4\). We will investigate this possibility and apply our parameterization of \(Y_D\) to this case in the following two examples. The third example will be based on a perturbation of bimaximal leptonic mixing [35] in the type I + II seesaw framework.

### 3.1 First example

Suppose first that the triplet term \(m''_I\) is the term proportional to \(m_3\) in Eq. (31), i.e.

\[
f_L = \begin{pmatrix} 0 & 0 & 0 \\
\cdot & 1 & -1 \\
\cdot & \cdot & 1 \end{pmatrix} e^{2i\beta} \quad \text{and} \quad v_L = \sqrt{\Delta m^2_A/2}. \tag{32}
\]

The second, flavor democratic term proportional to \(m_2\), is then provided by the conventional type I seesaw. Due to the seesaw relation (21) it determines \(X_\nu\):

\[
X_\nu = -\frac{v^2_u}{v_R} Y_D^{-1} Y_D^T = \frac{m_2 e^{2i\alpha}}{3} \begin{pmatrix} 1 & 1 & 1 \\
\cdot & 1 & 1 \\
\cdot & \cdot & 1 \end{pmatrix}.
\]

Consequently, one can write \(X_\nu^{\text{diag}} = \text{diag}(0, 0, m_2)\) and

\[
V_\nu = \begin{pmatrix} 1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\
-1/\sqrt{2} & 1/\sqrt{6} & 1/\sqrt{3} \\
0 & -2/\sqrt{6} & 1/\sqrt{3} \end{pmatrix} \text{diag}(1, 1, e^{-i\alpha}). \tag{33}
\]

The scales involved are \(v_L \simeq 0.025\) eV, \(v_R = 3 v^2_u/\sqrt{\Delta m^2_A} \simeq 1.0 \cdot 10^{16}\) GeV (assuming \(v_u = 174\) GeV, which is an excellent approximation as long as \(\tan \beta \gtrsim 5\)), and \(\gamma \simeq 8.4\). Note that we have rather arbitrarily decomposed the second and third terms in Eq. (31) into the VEVs and Yukawa couplings or their combinations.

We have now all ingredients to express \(Y_D\) through Eq. (30), and the result is

\[
Y_D = -ie^{i\alpha} \frac{\sqrt{m_2}}{\sqrt{3} v_u} \begin{pmatrix} \sqrt{M_1} \sin \omega_{13} & \sqrt{M_2} \cos \omega_{13} \sin \omega_{23} & -\sqrt{M_3} \cos \omega_{13} \cos \omega_{23} \\
\sqrt{M_1} \sin \omega_{13} & \sqrt{M_2} \cos \omega_{13} \sin \omega_{23} & -\sqrt{M_3} \cos \omega_{13} \cos \omega_{23} \\
\sqrt{M_1} \sin \omega_{13} & \sqrt{M_2} \cos \omega_{13} \sin \omega_{23} & -\sqrt{M_3} \cos \omega_{13} \cos \omega_{23} \end{pmatrix}. \tag{34}
\]
Interestingly, the complex angle $\omega_{12}$ drops out of this expression.

Let us now discuss LFV in the considered example. Eq. (32) yields

$$f_L f_L^\dagger = 2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & 1 & -1 \\ 0 & -1 & 1 \end{pmatrix},$$

(35)

from which it follows that only the decay $\tau \to \mu \gamma$ is influenced by the triplet term. The decays $\mu \to e \gamma$ and $\tau \to e \gamma$ depend only on $Y_D Y_D^\dagger$, which has a democratic structure with all terms equal to each other. Consequently, $\mu \to e \gamma$ and $\tau \to e \gamma$ decays are governed by the same quantity:

$$|(Y_D Y_D^\dagger)_{21} + 3(f_L f_L^\dagger)_{21}|^2 = |(Y_D Y_D^\dagger)_{31} + 3(f_L f_L^\dagger)_{31}|^2$$

$$= \frac{m_2^2}{9 v_u^2} (M_1 |\sin \omega_{13}|^2 + |\cos \omega_{13}|^2 (M_2 |\sin \omega_{23}|^2 + M_3 |\cos \omega_{23}|^2))^2.$$  

(36)

This equality implies that $\text{BR}(\tau \to e \gamma) = 0.178 \text{BR}(\mu \to e \gamma)$. With the current limit of $1.2 \cdot 10^{-11}$ on $\text{BR}(\mu \to e \gamma)$, and an expected improvement of two orders of magnitude on the limit of $\text{BR}(\tau \to e \gamma) \leq 1.1 \cdot 10^{-7}$, it follows that in this scenario $\tau \to e \gamma$ will not be observed in a foreseeable future. The branching ratio of the decay $\tau \to \mu \gamma$ depends on

$$\left(Y_D Y_D^\dagger\right)_{32} + 3 \left(f_L f_L^\dagger\right)_{32} = \left(Y_D Y_D^\dagger\right)_{21} - 6.$$  

(37)

We have omitted here the logarithmic dependence on the masses of the triplet and of the heavy Majorana neutrinos. In the plots to be shown in the following we use the full expressions, however. As follows from Eqs. (36) and (37), the matrix $Y_D Y_D^\dagger + 3 f_L f_L^\dagger$ depends in general on two complex angles, $\omega_{23}$ and $\omega_{13}$. If degenerate heavy Majorana masses are assumed, $M_1 = M_2 = M_3$, then the real part of $\omega_{23}$ drops out of this matrix.

Turning to leptogenesis, the first thing to note is that all $(\epsilon_i^i)_{\Delta}$ vanish, which is a consequence of the fact that the matrix $f_L Y_D$ vanishes identically. The decay asymmetry is therefore the same as for pure type I seesaw. The individual flavored asymmetries $(\epsilon_i^i)_N$ are all identical and equal to one third of the total asymmetry. For hierarchical heavy neutrinos we find

$$(\epsilon_1^1)_N = (\epsilon_1^2)_N = (\epsilon_1^3)_N = \frac{1}{3} \epsilon_1^N \approx \frac{1}{16\pi} \frac{m_2}{v_u^2} \frac{M_1}{|\sin \omega_{13}|^2} \sin 2\rho_{13} \sinh 2\sigma_{13} \frac{\sin 2\rho_{13} \sinh 2\sigma_{13}}{|\sin \omega_{13}|^2}.$$  

(38)

Hence, only the complex angle $\omega_{13}$ plays a role here. Terms containing $\omega_{23}$ appear in the decay asymmetry multiplied by $f(M_2^2/M_1^2) M_2 - f(M_3^2/M_1^2) M_3$, which vanishes in the limit of hierarchical heavy neutrinos. If $\omega_{13}$ is zero, then $N_1$ decouples (see Eq. (34)), and $N_2$
will be responsible for leptogenesis. The low energy (Majorana) phases \( \alpha \) and \( \beta \) do not contribute to either \( \varepsilon_1 \) or to \( \varepsilon_2 \), i.e. play no role in leptogenesis.

Choosing \( M_X = 2 \cdot 10^{16} \) GeV, \( M_\Delta = 5 \cdot 10^{15} \) GeV and the masses of heavy Majorana neutrinos \( M_1 = 10^{10} \) GeV, \( M_2 = 10^{12} \) GeV, and \( M_3 = 10^{15} \) GeV, we show in Fig. 1 the baryon asymmetry against the imaginary part of \( \omega_{13} \). All free parameters were varied, the baryon asymmetry was required to be positive and the branching ratios of \( \mu \to e\gamma \) (which in the considered example coincides with \( BR(\tau \to e\gamma)/0.178 \)) and of \( \tau \to \mu\gamma \) were required to lie below their current upper limits. The supersymmetric parameters we have used correspond to the SPS benchmark point 2 of Ref. [36] and are \( \tan \beta = 10, m_0 = 1450 \) GeV, \( m_{1/2} = 300 \) GeV and \( A_0 = 0 \). The apparent symmetry of Fig. 1 around the value \( \text{Im}(\omega_{13}) = \sigma_{13} = 0 \) can be explained by the dependence of the decay asymmetry \( 38 \) on \( \omega_{13} \). For all other parameters fixed, changing the sign of \( \sigma_{13} \) would also change the sign of the decay asymmetry. To regain the correct sign of the baryon asymmetry one would then also have to flip the sign of \( \sin 2\rho_{13} \) (recall that \( \rho_{13} \) is varied as a free parameter in this scatter plot), leading to the apparent symmetry of the figure.

The branching ratio \( BR(\tau \to \mu\gamma) \) is basically independent of the parameters of \( Y_D \), because the constant term in Eq. (37) turns out to be much larger than the \( Y_D \)-dependent one. The ratio \( BR(\mu \to e\gamma)/BR(\tau \to \mu\gamma) \) is of order \( 10^{-4} \), implying that \( \tau \to \mu\gamma \) is observable as long as \( BR(\mu \to e\gamma) \) is close to its current limit. Fixing in addition \( \rho_{23} = 1.7, \sigma_{23} = -0.3 \) and \( \sigma_{13} = -0.7 \), we show in Fig. 2 the branching ratio of \( \mu \to e\gamma \) against the remaining free parameter \( \rho_{13} = \text{Re}(\omega_{13}) \). For this particular point \( BR(\tau \to \mu\gamma) \approx 5.05 \cdot 10^{-8} \), which is very close to its current upper limit. Fig. 3 shows a scatter plot for the branching ratio of \( \mu \to e\gamma \) against the real part of \( \omega_{23} \) when the baryon asymmetry is within its allowed range. The symmetry around the value \( \text{Re}(\omega_{23}) = \rho_{23} = \pi \) of this plot can be understood by noting that the term proportional to \( M_3|\cos \omega_{23}|^2 \) is the leading one in Eq. (36). This term depends on \( \rho_{23} \) through \( \cos 2\rho_{23} \).

### 3.2 Second example

Let us now consider the situation in which the triplet term is flavor democratic, i.e.,

\[
f_L = \begin{pmatrix} 1 & 1 & 1 \\ \cdot & 1 & 1 \\ \cdot & \cdot & 1 \end{pmatrix} e^{2i\alpha} \quad \text{and} \quad v_L = \sqrt{\Delta m^2}/3.
\]

The remaining term in Eq. (31) is then

\[
X_\nu = -\frac{v_w^2}{v_R} Y_D f_R^{-1} Y_D^T = \frac{m_3 e^{2i\beta}}{2} \begin{pmatrix} 0 & 0 & 0 \\ \cdot & 1 & -1 \\ \cdot & \cdot & 1 \end{pmatrix}.
\]
The involved scales are \( v_L \simeq 0.003 \text{ eV} \), \( v_R = 2 v_u^2/\sqrt{\Delta m^2_L} \simeq 1.2 \cdot 10^{15} \text{ GeV} \), and \( \gamma \simeq 0.12 \). Here we have taken \( v_u^2/v_R = m_3/2 = \sqrt{\Delta m^2_L}/2 \). The matrix \( X_\nu \) is diagonalized by

\[
V_\nu = \begin{pmatrix}
1 & 0 & 0 \\
0 & 1/\sqrt{2} & -1/\sqrt{2} \\
0 & 1/\sqrt{2} & 1/\sqrt{2}
\end{pmatrix} \text{ diag}(1, 1, e^{-i\beta}). \tag{39}
\]

with \( X_\nu^{\text{diag}} = \text{diag}(0, 0, m_3) \). From Eq. (30) we then obtain

\[
Y_D = -i e^{i\beta} \frac{\sqrt{m_3}}{\sqrt{2} v_u} \begin{pmatrix}
\sqrt{M_1} \sin \omega_{13} & \sqrt{M_2} \cos \omega_{13} \sin \omega_{23} & -\sqrt{M_3} \cos \omega_{13} \cos \omega_{23} \\
-\sqrt{M_1} \sin \omega_{13} & -\sqrt{M_2} \cos \omega_{13} \sin \omega_{23} & \sqrt{M_3} \cos \omega_{13} \cos \omega_{23}
\end{pmatrix}. \tag{40}
\]

Note that, as in the previous example, \( Y_D \) does not depend on \( \omega_{12} \). The matrix \( Y_D Y_D^\dagger \) has zero first row and first column, therefore \( \mu \to e\gamma \) and \( \tau \to e\gamma \) decays are governed by the triplet contribution, and depend on the the same quantity, namely \((f_L f_L^\dagger)_{21} = (f_L f_L^\dagger)_{31} = 3 \) (note that \( f_L f_L^\dagger = 3 f_L e^{-2i\alpha} \), i.e. is flavor democratic and has no dependence on any of the free parameters). Consequently, the decay \( \tau \to e\gamma \) will not be observed in a near future. The fact that \( \mu \to e\gamma \) and \( \tau \to e\gamma \) decays depend on the same quantity in both our examples is a consequence of the \( \mu-\tau \) symmetry of the involved mass matrices. Finally,

\[
\left( Y_D Y_D^\dagger \right)_{32} + 3 \left( f_L f_L^\dagger \right)_{32} = \frac{1}{2} v_u^2 \left| 18 v_u^2 - m_3 \left( M_1 | \sin \omega_{13} |^2 + | \cos \omega_{13} |^2 (M_2 | \sin \omega_{23} |^2 + M_3 | \cos \omega_{23} |^2) \right) \right|. \tag{41}
\]

Because the 12- and 13-entries of \( f_L f_L^\dagger \) are independent of any free parameters, it is not possible to suppress them, and in general the branching ratios of LFV decays are too large unless the SUSY masses are around or above 10 TeV.

Let us now turn to leptogenesis. As in the previous example, all \((\varepsilon^0_i)_\Delta \) vanish because the matrix \( f_L f_L^\dagger \) vanishes identically. The decay asymmetry \((\varepsilon_1^0)_N \) is also zero, whereas \((\varepsilon_1^0)_N \) and \((\varepsilon_1^0)_N \) are identical, and equal to \( \frac{1}{2} (\varepsilon_1)_N \). In the limit of the hierarchical heavy neutrinos we find

\[
(\varepsilon_1^0)_N = (\varepsilon_1^0)_N = \frac{1}{2} (\varepsilon_1)_N = \frac{3}{32\pi} \frac{m_3 M_1 \sin 2\rho_{13} \sinh 2\sigma_{13}}{v_u^2 | \sin \omega_{13} |^2} \approx 5 \cdot 10^{-8} \left( \frac{M_1}{10^9 \text{ GeV}} \right) \frac{\sin 2\rho_{13} \sinh 2\sigma_{13}}{| \sin \omega_{13} |^2}. \tag{42}
\]

The dependence of these asymmetries on the complex angle \( \omega_{13} \) is identical to that in the first example considered above. Note that here the decay asymmetry is proportional to the mass of the heaviest of light neutrinos \( m_3 \), whereas it was proportional to \( m_2 \) in the first example.
3.3 Third example

Our final example is based on the following observation [37, 24]: if the triplet term corresponds to bimaximal mixing [35] (\(U_{e3} = 0\) and \(\theta_{12} = \theta_{23} = \pi/4\)), then a small contribution from the conventional type I seesaw term may shift \(\theta_{12}\) sufficiently away from the maximal mixing value to make it agree with data. Non-zero \(\theta_{13}\) and non-maximal mixing in the 2-3 sector are also generated. It was assumed in [37, 24] that \(m_D\) is hierarchical, symmetric and coincides with the up-type quark mass matrix. The triplet term \(v_L f_L\) alone would generate bimaximal neutrino mixing and a normal mass ordering with a non-vanishing smallest neutrino mass. A discrete left-right symmetry is also assumed, such that \(f_L = f_R\). It is easy to see that in this case the type I seesaw term contributes to \(m_\nu\) mainly a 33 entry \(v_L \eta\), which is suppressed with respect to the leading (order \(v_L\)) term of \(m_{\nu}^{III}\) [11]. The other elements of \(m_{\nu}^I\) are much smaller than \(v_L \eta\), and we will neglect them. It should be noted that many other Dirac mass matrices can also give the desired form \(m_{\nu}^I \propto \text{diag}(0, 0, 1)\), and our parameterization allows to study them all.

The triplet contribution is

\[
f_L = f_R = \begin{pmatrix}
\epsilon & B \epsilon & B \epsilon \\
\frac{1}{2} (\epsilon + e^{i\phi}) & \frac{1}{2} (\epsilon - e^{i\phi}) & \frac{1}{2} (\epsilon + e^{i\phi}) \\
\end{pmatrix}
\quad \text{and} \quad
v_L = \frac{\sqrt{\Delta m^2}}{2}.
\]

For simplicity we assume the order one parameter \(B\) and \(\epsilon \ll 1\) to be real. The product \(v_L \epsilon\) is of the order of \(\sqrt{\Delta m^2}\).

The type I contribution we require is

\[
X_\nu = -\frac{v_u^2}{v_R} Y_D^{-1} Y_D^T = v_L \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & \eta \\
0 & \eta & 0 \\
\end{pmatrix}.
\]

The involved scales are \(v_L = 0.025\) eV and \(v_R = 2 v_u^2/\sqrt{\Delta m^2} \simeq 1.2 \cdot 10^{15}\) GeV. As a consequence of non-zero \(\eta\), the zeroth-order values \(U_{e3} = 0\) and \(\theta_{12} = \theta_{23} = \pi/4\) are modified to \(|U_{e3}| \simeq B \epsilon \eta/\sqrt{2}, \, \tan^2 \theta_{23} \simeq 1 - 2 \eta\) and \(\tan 2\theta_{12} \simeq 4\sqrt{2} B \epsilon/\eta\), where for simplicity also \(\eta\) is assumed to be real. The value \(\sin^2 \theta_{12} = \frac{1}{3}\) is achieved for \(B \epsilon = \eta/2\). The ratio of the neutrino mass squared differences \(\Delta m^2_{\odot}/\Delta m^2_A\) is approximately \(\frac{3}{4} \eta (4 \epsilon + \eta)\).

A choice of parameters which leads to neutrino properties that agree with the data, and which we will use in what follows, is \(B = 1.1, \, \eta = 0.1194\) and \(\epsilon = 0.0542\). The low energy phase \(\phi\) is the Dirac-type CP violation phase which can influence neutrino oscillations.

Since \(X_\nu\) is diagonal, we have \(V_\nu = 1\), whereas the matrix diagonalizing \(f_R\) via \(V_R^T f_R V_R = f_R^{\text{diag}}\) is

\[
V_R = \begin{pmatrix}
\sqrt{\frac{1}{2}} & \sqrt{\frac{1}{2}} & 0 \\
-\frac{1}{2} & \frac{1}{2} & -\sqrt{\frac{1}{2}} \\
-\frac{1}{2} & \frac{1}{2} & \sqrt{\frac{1}{2}} \\
\end{pmatrix} P_R,
\]

where \(P_R\) is the phase matrix. The CP-odd phase in the mass matrix is determined by the Majorana phase. The phases in \(V_R\) which are left undetermined are the phases in \(P_R\). We will assume that \(P_R = 1\) for simplicity. The Dirac phase \(\phi\) is the Dirac-type CP violation phase which can influence neutrino oscillations.
with $P_R = \text{diag}(i, 1, e^{-i\phi/2})$. The eigenvalues of $f_R$ are $\epsilon (1 - \sqrt{2} B)$, $\epsilon (1 + \sqrt{2} B)$ and $e^{i\phi}$. Since we started in a basis in which $f_R = f_L$ is not diagonal, we have to use a modified parameterization for $Y_D$, which is obtained from Eq. (30) by multiplying it on the right by $V_R$. The Dirac-type Yukawa coupling matrix is then found to be

$$ Y_D = i(\sqrt{v_R/v_u})v_\nu^* \sqrt{X_\nu}^{\text{diag}} R \sqrt{f_R}^{\text{diag}} V_R = i(\sqrt{v_R v_L/v_u}) \text{diag}(0, 0, \sqrt{\eta}) R \sqrt{f_R}^{\text{diag}} V_R $$

where the non-zero entries are

$$
(Y_D)_{31} = \frac{\sqrt{v_L \eta}}{2 v_u} \left( \sqrt{2 M_1} \sin \omega_{13} + \cos \omega_{13} \left( \sqrt{M_3 \cos \omega_{23} - \sqrt{M_2 \sin \omega_{23}}} \right) \right),
$$

$$
(Y_D)_{32} = -i \frac{\sqrt{v_L \eta}}{2 v_u} \left( \sqrt{2 M_1} \sin \omega_{13} - \cos \omega_{13} \left( \sqrt{M_3 \cos \omega_{23} - \sqrt{M_2 \sin \omega_{23}}} \right) \right),
$$

$$
(Y_D)_{33} = i \frac{\sqrt{v_L \eta}}{\sqrt{2} v_u} e^{-i \phi/2} \cos \omega_{13} \left( \sqrt{M_3 \cos \omega_{23} + \sqrt{M_2 \sin \omega_{23}}} \right).
$$

Note that in all three examples we have considered so far, $Y_D$ does not depend on $\omega_{12}$. This is related to the fact that in all these examples the matrix $X_\nu^{\text{diag}}$ has only one (namely, third) non-vanishing diagonal entry.

The result for LFV in the present example is that the branching ratios of the decays $\ell_i \to \ell_j \gamma$ depend only on $f_L f_L^\dagger$, namely, $(f_L f_L^\dagger)_{12} = (f_L f_L^\dagger)_{13} = 2 \epsilon^2 B$ and $(f_L f_L^\dagger)_{23} = -\frac{1}{2} [1 - \epsilon^2 (1 + 2 B^2)]$. As in the previous two examples, $\tau \to e \gamma$ is too rare to be observable. The ratio $\text{BR}(\mu \to e \gamma)/\text{BR}(\tau \to \mu \gamma)$ is approximately $2 \epsilon^2 B/\beta^2/0.174 \simeq 10^{-3}$.

Turning to leptogenesis, only $(\varepsilon^* \tau)_N$ and $(\varepsilon^* \tau)_{\Delta}$ are non-zero. The corresponding expressions are rather lengthy and we do not give them here. Figs. 4 and 5 show scatter plots of the baryon asymmetry against the imaginary parts of $\omega_{13}$ and $\omega_{23}$ for fixed values of the LFV branching ratios. For definiteness, we have chosen again the SUSY parameters $\tan \beta = 10$, $m_0 = 1450$ GeV, $m_{1/2} = 300$ GeV and $A_0 = 0$, which gives $\text{BR}(\mu \to e \gamma) = 3.0 \cdot 10^{-12}$, $\text{BR}(\tau \to e \gamma) = 5.4 \cdot 10^{-13}$ and $\text{BR}(\tau \to \mu \gamma) = 3.1 \cdot 10^{-9}$.

### 4 Summary and conclusions

We have considered lepton flavor violation and leptogenesis in the case of type I + II seesaw, when the exchanges of both heavy Majorana neutrinos and $SU(2)_L$-triplet Higgs bosons contribute to the mass matrix of light neutrinos. We have proposed a parameterization of the Dirac-type neutrino Yukawa coupling matrix $Y_D$ in this framework, which generalizes the Casas-Ibarra parameterization suggested for type I seesaw. Our parameterization automatically takes into account the type I + II seesaw formula and, like the Casas-Ibarra
one, involves an arbitrary complex orthogonal matrix \( R \). This matrix depends in general on six real parameters and can be parameterized in terms of three complex angles. We have given simple examples illustrating the usefulness of the proposed parameterization. In particular, we have considered LFV decays \( \ell_i \to \ell_j \gamma \) and leptogenesis in the case when the type I and type II contributions to both the light neutrino mass matrix \( m_\nu \) and the slepton mass matrix \( \tilde{m}_L^2 \) governing the LFV decays are of the same order. We considered two examples leading to the tri-bimaximal leptonic mixing and an example based on a relatively small but phenomenologically viable deviation from bimaximal mixing. In all the examples we have studied we found that the matrix \( Y_D \) depends only on two out of the three complex angles parameterizing the matrix \( R \), which is related to the fact that the matrix \( X_\nu \equiv m_\nu - f_L v_L \) had only one non-zero eigenvalue.

In each of the three examples that we considered, we have found that the decays \( \mu \to e\gamma \) and \( \tau \to e\gamma \) are governed by the same quantity, and the corresponding branching ratios are related by \( \text{BR}(\tau \to e\gamma) \approx 0.178 \text{BR}(\mu \to e\gamma) \), which is a consequence of the approximate \( \mu-\tau \) symmetry of the involved mass matrices.

In the first two examples based on tri-bimaximal leptonic mixing we found that leptogenesis is essentially governed by one of the three complex angles parameterizing the matrix \( R \). This can be traced back to the facts that the masses of heavy Majorana neutrinos were assumed to be hierarchical and that the loops with the triplet exchange gave no contribution to lepton asymmetry in these examples.

To conclude, we proposed a new parameterization of the Dirac-type neutrino Yukawa coupling matrix \( Y_D \) which is the most general one satisfying the combined type I + II seesaw formula. It expresses the matrix \( Y_D \) through both low energy and high energy parameters and can be useful for studies of lepton flavor violation and leptogenesis in the type I + II seesaw framework.

Acknowledgments

We thank S. Antusch, S. Davidson, M. Frigerio, E. Nardi and Y. Nir for useful discussions. This work was supported in part by the Deutsche Forschungsgemeinschaft in the Transregio 27 “Neutrinos and beyond – weakly interacting particles in physics, astrophysics and cosmology” (W.R.).

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Figure 1: Scatter plot for the baryon asymmetry $\eta_B$ against the imaginary part of $\omega_{13}$ for the first example of Section 3.1. The observed value of $\eta_B$ corresponds to the region between the horizontal lines.

Figure 2: The branching ratio of $\mu \rightarrow e\gamma$ decay against the real part of $\omega_{13}$ for a particular point in the parameter space of the first example of Section 3.1 (see the text for details).
Figure 3: First example from Section 3.1 scatter plot for the branching ratio of $\mu \rightarrow e\gamma$ decay against the real part of $\omega_{23}$ when the baryon asymmetry $\eta_B$ is within its experimental range.

Figure 4: Scatter plot for the baryon asymmetry $\eta_B$ against the imaginary part of $\omega_{13}$ for the third example of Section 3.3. The observed value of $\eta_B$ corresponds to the region between the horizontal lines.
Figure 5: Scatter plot for the baryon asymmetry $\eta_B$ against the imaginary part of $\omega_{23}$ for the third example of Section 3.3. The observed value of $\eta_B$ corresponds to the region between the horizontal lines.